

Centre d'Etudes Doctorales : Sciences et Techniques de l'Ingénieur

Résumé :

La présente thèse porte sur l'étude des structures des anneaux non commutatifs via le comportement de certaines applications additives sur l'anneau en considération ou localement sur un sous-ensemble approprié de l'anneau. En plus d'une introduction bien détaillée qui donne d'une part des motivations sur le choix des problèmes abordés et d'autre part un inventaire sur nos contributions, ce document est composé d'un chapitre de préliminaires et de cinq chapitres qui peuvent être résumés comme suit :

Le premier et le deuxième chapitre sont dédiés à l'étude de la relation entre les identités différentielles et les propriétés algébriques des anneaux premiers à involution. L'objet fondamental de cette partie est de généraliser et/ou améliorer quelques résultats qui existent dans la littérature. Plus précisément, nous avons caractérisé la commutativité d'un anneau premier admettant une dérivation généralisée satisfaisant des identités algébriques spécifiques. En outre, dans certaines situations, nous avons prouvé que l'existence d'une telle dérivation généralisée est impossible et nous avons validé ses résultats par des contre-exemples qui montrent la nécessité des hypothèses imposées dans les différents théorèmes.

Vu son rôle primordial dans la catégorie des anneaux non commutatifs, le chapitre 3 est consacré à l'étude du centre d'un anneau. En effet, Nous avons introduit des sous-ensembles définis par des relations faisant intervenir des applications additives et qui coïncident avec le centre dans le cas d'un anneau premier. En outre, notre contribution est illustrée par des contre-exemples qui prouvent que les résultats obtenus ne peuvent pas s'étendre aux anneaux semi-premiers.

Dans le quatrième chapitre, nous avons pu déterminer la forme explicite des dérivations d'une extension amalgamée ce qui nous a permis de conclure pour des cas particuliers d'anneaux (La duplication, le produit direct de deux anneaux). De plus, nous avons montré que la dérivation d'une extension amalgamée $A \bowtie^f I$ est d'un type spécial lorsque l'anneau A est premier. Finalement, sous certaines conditions sur la dérivation, nous avons montré que l'extension amalgamée $A \bowtie^f I$ est isomorphe à $D[X]/(X^2)$ où D est un anneau à division.

Dans le cinquième et dernier chapitre, nous nous sommes intéressés à l'étude de la structure d'un anneau quotient R/P via le comportement des dérivations généralisées de R sur l'idéal premier P de R . Dans ce contexte, nous avons exhibé des conditions nécessaires et suffisantes pour que l'anneau R/P soit commutatif ainsi que des hypothèses sous lesquelles une dérivation laisse invariant des idéaux premiers minimaux d'un anneau semi-premier. De plus, cette approche nous a permis de généraliser quelques définitions et résultats classiques.

Mots clés : Anneaux (semi-)premiers, commutativité, dérivations (généralisées), involutions, homomorphismes d'anneaux, extension d'anneaux, idéaux premiers.

IDENTITÉS DIFFÉRENTIELLES SUR LES ANNEAUX PREMIERS À INVOLUTION

DÉDICACE

Je dédie ce travail:

A la mémoire de mon père, que DIEU garde son
âme dans son vaste paradis.

A ma mère qui m'a tout donné sans compter.
Aucun hommage ne saurait transmettre à ta
juste valeur, le dévouement et le respect que
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Contents

Introduction	vii
0 Preliminaries (In French)	1
0.1 Anneaux non commutatifs et involutions	1
0.2 Anneaux des quotients de Martindale	4
0.3 Dérivations et dérivations généralisées	8
0.4 Amalgamation	10
1 On generalized derivations and commutativity of prime rings with involution	12
1.1 Introduction	12
1.2 Identities with commutator	13
1.3 Identities with anti-commutator	16
1.4 Identities with commutator and anti-commutator	19
1.5 Examples proving necessity of our conditions	22
2 Some commutativity theorems for rings with involution involving generalized derivations	23
2.1 Introduction	23
2.2 Main results	24
2.2.1 Identities with commutator	25
2.2.2 Identities with anti-commutator	28
2.2.3 Identities with commutator and anti-commutator	30
2.2.4 Examples proving necessity of our conditions	32
3 Center-like subsets in rings with derivations or endomorphisms	34
3.1 Introduction	34
3.2 Center-like subsets involving derivations	35
3.3 Center-like subsets involving endomorphisms	39

CONTENTS

4	Derivations over amalgamated algebras along an ideal	42
4.1	Introduction	42
4.2	Derivations of $A \bowtie^f I$	43
4.3	Derivations of $A \bowtie^f I$ in case A is prime	48
4.4	Applications	49
5	Prime ideals and generalized derivations, invariance of minimal prime ideals under derivations	52
5.1	Introduction	52
5.2	P-Centralizing and P-Commuting maps	53
5.3	Algebraic identities involving prime ideal	56
5.4	Invariance of minimal prime ideals under derivations	62
	Bibliography	65

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Introduction

Motivée par les propriétés de base d'un opérateur différentiel, la notion de dérivation était introduite sur les anneaux. Une application additive $d : R \longrightarrow R$ est dite une dérivation sur l'anneau R si $d(xy) = d(x)y + xd(y)$ pour tout $x, y \in R$ (la formule de Leibnitz).

L'exemple le plus naturel d'une dérivation est la dérivée usuelle sur l'anneau des polynômes $K[x]$ sur le corps K .

La théorie des dérivations était introduite par Newton 1642-1727 et Leibnitz 1646-1717 et puis développée, dans les années trente du siècle dernier, par Hasse, Jacobson, Teichmüller, Kaplansky, Kolchin et Ritt. Cette théorie et celle des automorphismes d'anneaux associatifs sont des descendants directs du développement de la théorie classique de Galois. Elles jouent un rôle très important, non seulement en théorie des anneaux, mais aussi en analyse fonctionnelle et au traitement des équations différentielles linéaires. Ainsi, la théorie des dérivations figure dans l'inventaire des outils utilisées dans l'étude de la théorie des corps et des algèbres différentielles. Récemment, un grand intérêt a été accordé à l'étude des dérivations en algèbre. Cela est justifié par leurs applications dans plusieurs domaines scientifiques, à savoir la physique quantique et les nouvelles technologies de l'information (cryptographie, codage,...). Par ailleurs, l'introduction des dérivations en algèbre non commutative est récente.

Les dérivations ont donné naissance à une forte relation entre les identités différentielles et les propriétés algébriques des anneaux. Le premier résultat dans cette direction est dû à Posner qui a montré en 1957 qu'un anneau premier ayant une dérivation non nulle centralisante est nécessairement commutatif. Rappelons qu'une application $f : R \longrightarrow R$ est dite centralisante sur un sous-ensemble S de R si $[f(x), x] \in Z(R)$ pour tout $x \in S$. Dans le cas particulier où, $[f(x), x] = 0$ pour tout $x \in S$, on dit que f est commutante sur S . Plusieurs chercheurs ont caractérisé la commutativité de certains an-

neaux par l'existence des applications commutantes ou centralisantes sur des sous-ensembles particuliers de ces anneaux. L'un des plus célèbres résultats dans cette voie est le fameux théorème de Mayne prouvé en 1976. En effet, il a montré qu'un anneau premier ayant un automorphisme non trivial centralisant est commutatif. En 1984, le même auteur a montré que tout anneau premier ayant un automorphisme non trivial centralisant sur un idéal (ou un idéal quadratique de Jordan) est commutatif. Ces résultats ont été généralisés dans des contextes divers et variés par plusieurs auteurs.

L'étude des applications qui préservent la commutativité a toujours été une zone active de recherche dans la théorie des matrices, la théorie des opérateurs et celle des anneaux (Voir [19, 76]). Une application $f : R \longrightarrow R$ préserve la commutativité si $[f(x), f(y)] = 0$ à chaque fois que $[x, y] = 0$ où $x, y \in R$. Une application $f : R \longrightarrow R$ préserve fortement la commutativité (SCP) sur un sous ensemble S de R si $[f(x), f(y)] = [x, y]$ pour tout $(x, y) \in S^2$.

Dans [10], Bell et Daif ont étudié la commutativité d'un anneau admettant une dérivation qui préserve fortement la commutativité sur un idéal à droite non nul de R . Plus précisément, ils ont prouvé que si un anneau semi-premier R admet une dérivation d qui satisfait $[d(x), d(y)] = [x, y]$ pour tout x, y dans un idéal non nul I de R , alors $I \subset Z(R)$. En particulier, R est commutatif si $I = R$. Plus tard, Deng et Ashraf [31] ont prouvé que : si R est un anneau semi-premier ayant une dérivation d et une application $f : I \rightarrow R$ définie sur un idéal I de R telles que $[f(x), d(y)] = [x, y]$ pour tout $x, y \in I$, alors R contient un idéal central non nul. Plus loin, Ali et Huang ont montré que si R est un anneau semi-premier sans 2- torsion et d est une dérivation de R vérifiant $[d(x), d(y)] + [x, y] = 0$ pour tout x, y éléments d'un idéal non nul I de R , alors R contient un idéal non nul central. Plusieurs généralisations de ces résultats peuvent être trouvées dans la littérature (Voir par exemple [21]).

Plus récemment, un nouveau axe de recherche très attractif a été initié par plusieurs auteurs. Il s'agit de l'étude des structures de certains anneaux non commutatifs à involution via le comportement d'une application additive convenablement construite et agissante sur des sous-ensembles appropriés de l'anneau en considération. Signalons que tout anneau premier à involution $(R, *)$ est $*$ -premier, mais la réciproque n'est pas toujours vraie. En outre tout anneau premier R , n'ayant pas nécessairement une involution, s'injecte dans l'anneau $*$ -premier $R \times R^\circ$ où $*$ désigne l'involution d'échange définie par $(x, y)^* = (y, x)$. Cette remarque a conduit plusieurs chercheurs, M. Ashraf, L. Oukhtite, A. Mamouni, A. Shakir, ..., à étendre divers travaux cités ci-dessus aux anneaux $*$ -premiers.

Cette thèse présente notre contribution à l'étude des identités différentielles sur les anneaux premiers à involution. Ce domaine regroupe différents axes de recherche d'importance remarquable qui se reflète dans le grand nombre de chercheurs attirés par cet axe de recherche. En effet, nous nous sommes intéressés aux axes suivants :

1. Structure des anneaux vérifiant des identités différentielles concernant les dérivations et des applications additives particulières.
2. Description de la forme explicite de certains types de dérivations d'une extension amalgamée.
3. Initiation d'une nouvelle approche qui consiste à étudier la structure d'un anneau quotient R/P par des identités algébriques faisant intervenir l'idéal premier P .
4. Étude de l'invariance des idéaux premiers minimaux par les dérivations.
5. Étude de nouveaux sous-ensembles d'un anneau qui, sous certaines hypothèses, coïncident avec le centre de l'anneau.

La présente thèse regroupe six chapitres couvrant quatre articles publiés et un article soumis. Elle incarne notre contribution à la théorie des dérivations, des dérivations généralisées et des applications additives sur des anneaux. La thèse est organisée comme suit:

Le *premier chapitre* est consacré aux notions préliminaires et aux définitions de base. Des résultats clés et des théorèmes les plus frappants à la matière de la thèse ont été également présentés.

Le *deuxième chapitre* est destiné à l'étude des dérivations généralisées d'un anneau premier à involution. En effet, motivé par les travaux de Ashraf et Dhara ([6], [32]), qui ont considéré les identités suivantes : $[F(x), d(y)] = \pm[x, y]$, $[F(x), d(y)] = 0$, $F(x) \circ d(y) = \pm x \circ y$, $F(x) \circ d(y) = 0$, $[F(x), d(y)] = \pm x \circ y$ et $F(x) \circ d(y) = \pm[x, y]$ pour tout $x, y \in R$, nous avons examiné des identités algébriques plus générales, dans le cas d'un anneau à involution, faisant intervenir une seule variable au lieu de deux comme suit : $[F(x), d(x^*)] = \pm[x, x^*]$, $[F(x), d(x^*)] = 0$, $F(x) \circ d(x^*) = \pm x \circ x^*$, $F(x) \circ d(x^*) = 0$, $F(x) \circ d(x^*) = \pm[x, x^*]$ et $[F(x), d(x^*)] = \pm x \circ x^*$ pour tout $x \in R$. Nous avons ainsi montré que dans certains cas une telle dérivation généralisée n'existe pas et dans d'autres cas on a pu caractériser la commutativité de

l'anneau. Comme applications, nous avons retrouvé les résultats de [6] et [32] comme des corollaires pour la classe des anneaux à involution. Finalement, nous avons justifié la nécessité des hypothèses considérées dans les différents théorèmes par la construction des contre-exemples.

Le but fondamental du *chapitre 3* est d'établir des critères de commutativité pour un anneau premier à involution admettant une dérivation généralisée satisfaisant certaines identités algébriques. Herstein [41] a prouvé la commutativité d'un anneau premier sans 2-torsion ayant une dérivation d vérifiant $[d(x), d(y)] = 0$ pour tout $x, y \in R$. Motivé par le travail de Herstein, Bell et Daif [11] ont obtenu la même conclusion en considérant l'identité différentielle $d[x, y] = 0$ localement sur un idéal non nul de R . Ces travaux ont été étendus par Emine Koç et Öznur Gölbaşı [50] en considérant une dérivation généralisée F agissante sur un idéal de Lie stable par carré. Dans la première section de ce chapitre, nous avons traité le cas d'une dérivation généralisée F satisfaisant l'identité différentielle plus générale $F([x, x^*]) \in Z(R)$ pour tout $x \in R$. Ce qui nous a permis de généraliser les travaux de [11] et [50] pour les anneaux à involution. En considérant l'identité algébrique $F([x, y]) \pm [x, y] \in Z(R)$ pour une dérivation généralisée F , nous avons pu donner une version plus générale des travaux de N. Rehman [73] et Quadri [69].

Dans la deuxième section nous avons d'une part introduit de nouvelles identités différentielles faisant intervenir à la fois le commutateur et l'anti-commutateur. D'autre part, nous avons réussi à améliorer et/ou généraliser des travaux de [50] et ceux de d'autres auteurs. Dans la dernière section, nous avons montré via des contre-exemples que les restrictions imposées dans nos différents théorèmes ne sont pas superflues. En particulier, nos théorèmes ne peuvent pas s'étendre aux anneaux semi-premiers.

Le *quatrième chapitre* est dédié à l'introduction de nouveaux sous-ensembles d'un anneau qui, sous certaines hypothèses, coïncident avec son centre. Le premier résultat dans cet axe est l'hypercentre $T(R)$ de Herstein [38]. Dans un anneau semi-premier, Chacron [23] a introduit le co-hypercentre $G(R)$. Giambruno [35] a donné des généralisations du centre d'un anneau R comme suit : $\{a \in R \mid ax^n = x^m a \forall x \in R, \text{ où } n = n(a, x) \geq 1, m = m(a, x) \geq 1\}$. Dans le théorème principal de [35], Giambruno a prouvé que cet ensemble n'est que le centre de l'anneau dans le cas où le seul idéal nilpotent de R est $\{0\}$. En outre, Giambruno a montré que si l'anneau R n'admet pas d'idéal à droite

nilpotent non nul, alors $Z(R)$ coïncide avec le nouveau sous-ensemble

$$\{a \in R \mid (ax)^n = (xa)^m \forall x \in R, \text{ pour un certain } n = n(a, x) \geq 1, m = m(a, x) \geq 1\}.$$

Ce dernier résultat de Giambruno est équivalent à la conjecture de Köethe. Une nouvelle approche qui consiste à construire de nouveaux sous-ensembles dépendants fondamentalement de l'existence d'une application additive appropriée sur l'anneau est introduite par Herstein en considérant le sous-ensemble $\{a \in R \mid [a, d(x)] = 0 \ \forall x \in R\}$ où d est une dérivation non nulle. Herstein a montré que cet ensemble coïncide avec $Z(R)$ lorsque R est un anneau premier sans 2-torsion. Notre contribution dans ce domaine repose sur l'introduction d'un concept plus général consistant à considérer les sous-ensembles suivants:

$$Z^+(R; d) = \{y \in R \mid [d(x), d(y)] + [x, y] \in Z(R) \ \forall x \in R\};$$

$$Z^-(R; d) = \{y \in R \mid [d(x), d(y)] - [x, y] \in Z(R) \ \forall x \in R\};$$

$$Z^{*-}(R; d) = \{y \in R \mid [d(x), d(y)] - [y, d(x)] - [d(y), x] \in Z(R) \ \forall x \in R\};$$

où d désigne une dérivation ou un endomorphisme non nécessairement surjectif. Rappelons que Bell et Daif [12] ont examiné et étudié les sous-ensembles suivants où f est une dérivation ou un endomorphisme surjectif:

$$Z^*(R; f) = \{y \in R \mid [x, y] = [f(y), f(x)] \ \forall x \in R\};$$

$$Z^{**}(R; f) = \{y \in R \mid [x, y] = [f(x), f(y)] \ \forall x \in R\};$$

$$Z_1(R; f) = \{y \in R \mid [f(x), f(y)] = [f(y), x] + [y, f(x)] \ \forall x \in R\}.$$

Dans notre situation, nous avons montré que les sous-ensembles ci-dessus coïncident avec le centre quand l'anneau R est premier. Nous avons validé notre contribution par la construction des contre-exemples qui montrent que nos résultats ne peuvent pas s'étendre aux cas des anneaux semi-premiers. Comme applications, les résultats de Bell et Daif [12] découlent immédiatement de nos théorèmes.

Dans le *cinquième chapitre*, nous avons abordé une étude profonde sur les dérivations d'une extension amalgamée. Dans cette direction, nous avons déterminé la forme explicite des dérivations et des dérivations intérieures d'une extension amalgamée. Nous avons traité la forme générale d'une dérivation sur la duplication et sur le produit direct de deux anneaux. Nous avons abordé aussi la primalité et la semiprimalité d'une extension amalgamée et nous avons montré que la dérivation d'une extension amalgamée $A \bowtie^f I$ est d'une forme plus spéciale lorsque l'anneau A est premier. Comme applications du théorème

principale de ce travail, nous avons examiné la structure d'une extension amalgamée $A \bowtie^f I$ ayant une dérivation d vérifiant $d(x) = 0$ ou $d(x)$ est inversible pour tout $x \in A \bowtie^f I$. Nous avons réussi à exhiber non seulement la structure des anneaux A et $A \bowtie^f I$ mais on a donné la forme exacte de la dérivation d .

Théorème. (Theorem 4.4.3).

Soient $f : A \rightarrow B$ un homomorphisme d'anneaux, I un idéal non nul de B et d une dérivation non nulle de $A \bowtie^f I$. On suppose que A est premier unitaire d'unité 1 et que pour tout $x \in A \bowtie^f I$, $d(x) = 0$ ou $d(x)$ est inversible dans $A \bowtie^f I$. Alors A est un anneau à division et $A \bowtie^f I$ s'identifie à $D[x]/(x^2)$, où D est un anneau à division, $\text{char} D = 2$, $d(D) = \{0\}$ et $d(x) = 1 + ax$ pour un certain $a \in Z(D)$.

Le *sixième et dernier chapitre* consiste à introduire une nouvelle technique permettant d'établir une relation entre la structure d'un anneau quotient R/P et le comportement de quelques applications additives sur l'idéal premier P de R . L'originalité de notre travail réside dans le fait que nous avons considéré une dérivation généralisée sur R (et non sur R/P) qui vérifie une propriété algébrique sur R . Ce qui nous a permis de retrouver divers résultats fondamentaux prouvés dans la littérature pour les anneaux premiers ou semi-premiers. Nous avons initié deux nouveaux concepts concernant les applications additives d'un anneau à savoir les applications I -centralisante et les applications I -commutantes où I désigne un idéal de l'anneau en considération. Notre premier résultat fondamental est une généralisation du fameux théorème de Posner comme suit

Théorème. (Theorem 5.2.1).

Soient R un anneau, P un idéal premier de R et F une dérivation généralisée de R associée à une dérivation d . Si F est P -centralisante, alors $d(R) \subseteq P$ ou R/P est un anneau commutatif intègre.

Une application immédiate de ce théorème nous permet de retrouver le Théorème de Posner en prenant $P = (0)$ et $F = d$ quand l'anneau R est premier. Notre deuxième résultat est motivé par les travaux de Bell et Daif concernant les dérivations préservant fortement la commutativité [10]. En effet, ils ont prouvé que si un anneau semi-premier R admet une dérivation d vérifiant $[d(x), d(y)] - [x, y] = 0$ pour tout $x, y \in R$, alors R est commutatif. En s'inspirant des travaux de Bell et Daif, nous avons abordé le cas d'une dérivation généralisée F telle que $[F(x), d(y)] - [x, y] \in P$ pour tout $x, y \in R$ où P

est un idéal premier de R . Plus exactement, notre résultat est comme suit

Théorème. (Theorem 5.3.1).

Soient R un anneau et P un idéal premier de R . Si R admet une dérivation généralisée F associée à une dérivation d , alors les conditions suivantes sont équivalentes:

- 1) $[F(x), d(y)] + [x, y] \in P$ pour tout $x, y \in R$,
- 2) $[F(x), d(y)] - [x, y] \in P$ pour tout $x, y \in R$,
- 3) R/P est un anneau commutatif intègre.

Le théorème suivant constitue une autre contribution, qui nous donne non seulement la commutativité de R/P mais détermine également sa caractéristique.

Théorème. (Theorem 5.3.3).

Soient R un anneau, P un idéal premier de R et F une dérivation généralisée de R associée à une dérivation d . Si F satisfait l'une des conditions suivantes:

- 1) $F(x) \circ d(y) + x \circ y \in P$ pour tout $x, y \in R$,
- 2) $F(x) \circ d(y) - x \circ y \in P$ pour tout $x, y \in R$,
- 3) $[F(x), d(y)] + x \circ y \in P$ pour tout $x, y \in R$,
- 4) $[F(x), d(y)] - x \circ y \in P$ pour tout $x, y \in R$,

alors R/P est un anneau commutatif intègre et $\text{char}(R/P) = 2$.

Nous avons traité aussi les identités algébriques $\left(F(x) \circ d(y) + [x, y] \in P \right.$
 $\left. \text{pour tout } x, y \in R \right)$ et $\left(F(x) \circ d(y) - [x, y] \in P \text{ pour tout } x, y \in R \right)$. Notre résultat est, un peu, plus précis dans cette situation car il nous a permis de montrer que R/P est un anneau commutatif intègre de plus si R/P est sans 2-torsion alors d est à image dans P .

Dans la dernière partie de ce chapitre, nous avons traité un problème très ancien appelé problème d'invariance des idéaux minimaux par une dérivation. En effet, étant donné une dérivation d d'un anneau R et I un idéal de R , alors I est dit invariant par d ou simplement I est un d -idéal si $d(I) \subseteq I$. Si I est un d -idéal de R , alors d induit une dérivation sur l'anneau quotient R/I qui préserve plusieurs propriétés de d . En particulier, si on a une famille d'idéaux d -invariants J_i à intersection triviale $\bigcap_i J_i = (0)$, alors l'anneau R peut être injecté dans le produit direct des R/J_i et les propriétés de d peuvent provenir des propriétés des dérivations induites sur R/J_i . De cette manière,

nous pouvons atteindre l'objectif fondamental consistant à limiter les questions relatives aux dérivations sur les anneaux semi-premiers aux questions concernant les anneaux premiers. Ainsi, la véritable préoccupation concernant un anneau semi-premier donné est l'existence d'idéaux d -invariant avec une intersection triviale. Nous avons donc abordé une partie de ce problème comme étant un champ d'applications de nos résultats. Plus précisément, nous avons appliqué nos théorèmes pour donner des conditions nécessaires et suffisantes sous lesquelles une dérivation laisse invariant des idéaux premiers minimaux d'un anneau semi-premier. Nous avons montré que si R est un anneau semi-premier ayant une dérivation généralisée F centralisante sur R , alors R admet un idéal premier minimal invariant par d , en outre d est à valeurs dans $Z(R)$. En particulier, si un anneau semi-premier R admet une dérivation non nulle centralisante d , alors il existe un idéal premier minimal P invariant par d et d est à valeurs dans $Z(R)$.

Pour terminer ce chapitre, nous avons donné une réponse partielle au second problème suivant "étant donné un anneau semi-premier R , est-ce que P est d -invariant pour tout idéal premier minimal P de R et pour toute dérivation d de R ?".

Plus précisément, pour un anneau semi-premier ayant une dérivation généralisée (F, d) nous avons établi des conditions sous lesquelles tout idéal premier minimal P est invariant par la dérivation d .

PRELIMINARIES (IN FRENCH)

0.1 Anneaux non commutatifs et involutions

Dans tout ce qui suit $(R, +, \cdot)$ désigne un anneau associatif non nécessairement commutatif.

Définition 0.1.1 *Le centre d'un anneau R , noté $Z(R)$, est l'ensemble $Z(R) = \{x \in R \mid rx = xr \text{ pour tout } r \in R\}$.*

Exemple : Considérons l'anneau $R = M_2(\mathbb{Z})$. On a $Z(R) = \left\{ \begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix} \mid a \in \mathbb{Z} \right\}$.

Définition 0.1.2 *Un anneau R est dit simple si ses seuls idéaux bilatères sont $\{0\}$ et R .*

Exemples :

- 1) Si E est un espace vectoriel de dimension finie sur un corps K alors $\text{End}_K(E)$ est simple.
- 2) Si R est un anneau simple alors $M_n(R)$ est simple.

La notion d'intégrité a été généralisée de cette façon dans le cas d'un anneau non commutatif.

Définition 0.1.3 *Un idéal propre P de R est dit premier si pour tout $x, y \in R$, on a : $xRy \subseteq P \implies x \in P$ ou $y \in P$.
L'anneau R est dit premier si l'idéal $\{0\}$ est premier.*

Proposition 0.1.1 *Soit R un anneau. Les assertions suivantes sont équivalentes:*

- (1) R est premier.
- (2) Pour tout $a, b \in R$, $aRb = \{0\}$ implique $a = 0$ ou $b = 0$.
- (3) Pour tout I et J deux idéaux de R , $IJ = \{0\}$ implique $I = \{0\}$ ou $J = \{0\}$.

Exemple :

Si R est un anneau simple alors $M_n(R)$ est un anneau premier.

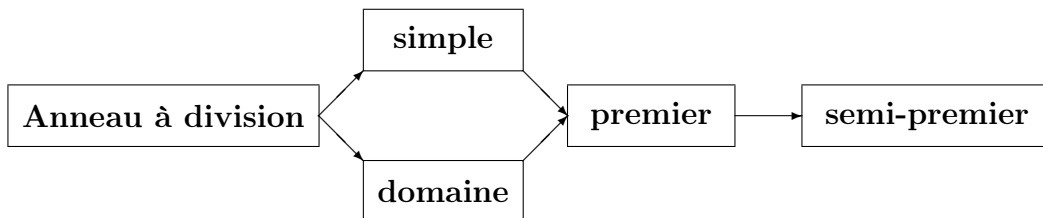
Définition 0.1.4 Un anneau est dit semi-premier si l'intersection de ses idéaux premiers est réduite à son idéal trivial $\{0\}$.

Proposition 0.1.2 Soit R un anneau. Les assertions suivantes sont équivalentes:

- (1) R est semi-premier.
- (2) Pour tout $a \in R$, $aRa = \{0\}$ implique $a = 0$.
- (3) R n'a pas d'idéal nilpotent non nul.

Remarque : Soit I un idéal non nul d'un anneau premier R donc I est un anneau premier. De même, un idéal d'un anneau semi-premier est un anneau semi-premier.

Les relations suivantes entre les classes d'anneaux introduites jusqu'à présent sont évidentes à partir des définitions :



Les implications réciproques ne sont pas en général vraies.

Définition 0.1.5 Soit $(R, +, *)$ un anneau non commutatif. L'anneau opposé de R noté $(R^o, +, \circ)$ est tel que $R^o = R$ et $x \circ y = y * x$ pour tout $x, y \in R$.

Tout anneau premier est semi-premier, mais la réciproque n'est pas en général vraie.

Exemple

Soit A un anneau premier et considérons l'anneau $R = A \times A^0$.

Du fait que $(1, 0)R(0, 1) = 0$, il résulte que l'anneau R n'est pas premier. En outre, si $(a, b) \in R$ est tel que $(a, b)R(a, b) = 0$, alors

$$aAa = 0 \text{ et } bAb = 0.$$

Or A est premier, ceci entraîne que $a = 0$ et $b = 0$. Par suite, $(a, b) = 0$ prouvant ainsi que R est semi-premier.

Définition 0.1.6 Une involution sur un anneau R est une application

$*$: $R \longrightarrow R$ vérifiant les conditions suivantes :

$$(a + b)^* = a^* + b^*, (ab)^* = b^*a^* \text{ et } (a^*)^* = a \text{ pour tout } a, b \in R.$$

Autrement dit, $*$ est un anti-automorphisme d'ordre 2.

Exemples :

1- Soit $\mathbb{C}$ le corps des nombres complexes. L'application $\sigma : \mathbb{C} \longrightarrow \mathbb{C}$ définie par $\sigma(z) = \bar{z}$ est une involution.

2- Si K est un corps, alors l'application $t : M_n(K) \longrightarrow M_n(K)$ définie par : $t(a_{ij}) = (a_{ji})$ est une involution de $M_n(K)$.

3- Si $*$ est une involution d'un anneau R , alors $*$ se prolonge à une involution de $M_n(R)$ par : $(a_{ij}) \longmapsto (a_{ji}^*)$.

4- L'application $*$: $M_2(\mathbb{R}) \longrightarrow M_2(\mathbb{R})$ définie par :

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}^* = \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$$

est une involution de $M_2(\mathbb{R})$.

4- Soit R un anneau non commutatif et R^o son anneau opposé. L'application

$$\begin{aligned} \tau_{ex} : R \times R^o &\longrightarrow R \times R^o \\ (x, y) &\longmapsto (y, x) \end{aligned}$$

est une involution appelée l'*involution d'échange*.

Définitions 0.1.7 Soit $(R, *)$ un anneau à involution.

1) Un élément $x \in R$ est dit *symétrique* (resp. *anti-symétrique*) si $x^* = x$ (resp. $x^* = -x$).

L'ensemble des éléments symétriques (resp. anti-symétriques) est noté $H(R)$ (resp. $S(R)$).

2) Un élément $x \in R$ est dit *normal* si $xx^* = x^*x$. Si tous les éléments de R sont normaux, on dit que R est un anneau normal.

Définition 0.1.8 Une involution est dite de première espèce si sa restriction au centre de l'anneau R est l'application identité. Dans le cas contraire, l'involution est dite de deuxième espèce.

Remarque. Une involution $*$ est de deuxième espèce si $S(R) \cap Z(R) \neq \{0\}$. Elle est de première espèce si $Z(R) \subset H(R)$. Effet, il existe $z_0 \in Z(R)$ tel que $z_0 - z_0^* \in Z(R) \cap S(R) \setminus \{0\}$.

Exemples :

- 1- La conjugaison est une involution de deuxième espèce sur $\mathbb{C}$.
- 2- Si K est un corps, alors l'application $*$: $M_2(K) \longrightarrow M_2(K)$ définie par :

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}^* = \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$$

est une involution de première espèce.

0.2 Anneaux des quotients de Martindale

En théorie des anneaux, un domaine commutatif peut être injecté dans un corps, son corps des fractions. Il est donc naturel de chercher une analogie pour les anneaux non commutatifs. Le manque de commutativité engendre des difficultés considérables. Néanmoins, diverses généralisations de la construction du corps des fractions existent en théorie des anneaux non commutatifs. Nous en discuterons quelques-unes : anneaux centraux des quotients, anneaux classiques des fractions et anneaux des quotients de Martindale.

Une vision sur la construction du corps des fractions $\widehat{Z}$ d'un domaine commutatif Z montre que cela dépend fortement de la commutativité. Il n'y a pas de moyen évident de l'éviter. Cependant, dans cette première discussion, nous allons utiliser essentiellement la même construction pour un anneau non commutatif R , mais ne permet que les éléments dans le centre $Z = Z(R)$ en tant que «dénominateur». Plus précisément, on suppose que les éléments non nuls de Z satisfont à la condition (évidemment nécessaire) de la prochaine définition, et élargir R de manière à ce que ces éléments deviennent inversibles dans le nouveau anneau plus grand.

Définition 0.2.1 *Un élément non nul dans un anneau R est dit **régulier** s'il n'est pas un diviseur de zéro à gauche ou à droite.*

✂ Construction de l'anneau central des quotients $Q_Z(R)$.

Nous supposons que R est un anneau quelconque tel que son centre Z soit non nul, et que tous les éléments de $Z \setminus \{0\}$ soient réguliers; en particulier, Z est un domaine commutatif. On définit une relation binaire sur $R \times Z \setminus \{0\}$ par :

$$(r, z) \sim (r', z') \iff rz' = r'z.$$

Il est facile de vérifier que $\sim$ est une relation d'équivalence. En outre l'ensemble quotient $Q_Z(R)$ muni de l'addition et la multiplication :

$$\begin{aligned} rz^{-1} + sw^{-1} &:= (rw + sz)(zw)^{-1}, \\ rz^{-1} \cdot sw^{-1} &:= rs(zw)^{-1}. \end{aligned}$$

est un anneau.

Définition 0.2.2 $Q_Z(R)$ est appelé anneau central des quotients de R .

Remarque : Peu importe si R est unitaire ou non, $Q_Z(R)$ l'est. Son unité est $z_0 z_0^{-1}$, où z_0 est un élément arbitraire dans $Z \setminus \{0\}$. Le centre de $Q_Z(R)$, que nous désignons par $\widehat{Z}$, est facilement vu comme composé d'éléments de la forme zw^{-1} ; $z, w \in Z$, $w \neq 0$. Si $z \neq 0$, alors wz^{-1} est l'inverse de zw^{-1} . Cependant, $\widehat{Z}$ est un corps qui n'est autre que le corps des fractions de Z .

La construcion précédente a résolu le problème dans un cas particulier, ce qui donne un autre volet pour la recherche. Étant donné un anneau R , est-il possible de construire un anneau plus grand Q tel que tous les éléments réguliers dans R , non seulement ceux du centre, sont inversibles dans Q ? Plus précisément, pouvons-nous construire Q de telle sorte que l'ensemble des éléments réguliers jouant le rôle de $Z \setminus \{0\}$ est valable pour Q ? La réponse n'est pas aussi simple que pour l'anneau central de quotients. C'est certainement positive pour certains anneaux (pour un domaine commutatif, par exemple). Mais cela peut être aussi impossible, comme le Théorème suivant l'indique.

Théorème (Théorème de Ore)

Un anneau R a un anneau classique des fractions à droite ssi, R satisfait la condition de Ore à droite i.e. $rS \cap sR \neq \emptyset$ pour tout $r \in R$ $s \in S$.

Définition 0.2.3 Soit R un anneau et S l'ensemble de tous les éléments réguliers de R . Un anneau Q est appelé anneau classique des fractions à droite de R s'il possède les propriétés suivantes :

- (a) Q est un anneau unitaire contenant R comme un sous-anneau.
- (b) Chaque élément dans S est inversible dans Q .
- (c) Chaque élément dans Q est de la forme rs^{-1} , où $r \in R$ et $s \in S$.

Il est clair que les éléments non réguliers de R ne peuvent pas être inversés dans un anneau plus grand quelconque. L'existence d'un anneau classique des fractions à droite de R est donc la meilleure possibilité que nous espérons si nous souhaitons rendre autant d'éléments dans R inversibles que possible.

Remarque : L'anneau classique des fractions à gauche de R est défini de manière analogue, on substitue rs^{-1} à $s^{-1}r$ dans (c) de la définition précédente.

Bien entendu, des résultats similaires sont valables pour l'anneau classique des fractions à gauche. On définit la condition de Ore à gauche comme $Sr \cap Rs \neq \emptyset$ pour tout $r \in R$ et $s \in S$, et pour la construction de l'anneau classique des fractions à gauche $Q_{lc}(R)$ de R . En d'autres termes, si les conditions de Ore à gauche et à droite sont réalisées, alors $Q_{rc}(R) \cong Q_{lc}(R)$.

Dans cette partie, nous allons remplacer le rôle des idéaux à droite contenant des éléments réguliers par les idéaux bilatéraux (plus faciles à manipuler) non nuls. Malgré les similitudes techniques, la philosophie derrière ces deux constructions est différente. Le but dans la construction suivante n'est pas de rendre les éléments réguliers inversibles, mais de fournir un anneau plus grand qui peut être utilisé comme un outil pour résoudre les problèmes concernant l'anneau original.

✂ Construction d'anneau des quotients de Martindale à droite $Q_r(R)$

Soit $R \neq \{0\}$ un anneau premier. L'ensemble $\mathcal{J}$ de tous les idéaux non nuls de R est stable par multiplication et donc également sous des intersections finies. Nous définissons sur l'ensemble de toutes les paires (f, I) , où $I \in \mathcal{J}$ et $f : I \rightarrow R$ un homomorphisme de R -module à droite, la relation binaire suivante: $(f, I) \sim (g, J)$ si et seulement si, f et g coïncident sur un certain $K \in \mathcal{J}$ tel que $K \subseteq I \cap J$. On vérifie aisément que $\sim$ est une relation d'équivalence. Désignons par $[f, I]$ la classe d'équivalence associée à (f, I) , et par $Q_r(R)$ l'ensemble des classes d'équivalences pour cette relation. On définit sur $Q_r(R)$ une addition et une multiplication de la manière suivante :

$$[f_1, I_1] + [f_2, I_2] := [f_1 + f_2, I_1 \cap I_2],$$

$$[f_1, I_1] \cdot [f_2, I_2] := [f_1 f_2, I_2 I_1]$$

Ces opérations confèrent à $Q_r(R)$ une structure d'anneau.

Définition 0.2.4 $Q_r(R)$ s'appelle l'anneau des quotients de Martindale à droite de R .

Notons que l'anneau des quotients de Martindale à gauche $Q_l(R)$ se construit de manière analogue par l'homomorphisme de R -module à gauche et en général, $Q_r(R) \not\cong Q_l(R)$.

Théorème 0.2.5 ([20], Theorem 7.23)

Soit $R \neq \{0\}$ un anneau premier, et soit $\mathcal{J}$ l'ensemble des idéaux non nuls de R . L'anneau $Q_r(R)$ possède les propriétés suivantes:

- (a) $Q_r(R)$ est un anneau unitaire contenant R comme un sous anneau.
- (b) Pour tout $q \in Q_r(R)$ il existe $I \in \mathcal{J}$ tel que $qI \subseteq R$.
- (c) Pour tout $q \in Q_r(R)$ et $I \in \mathcal{J}$, $qI = 0$ implique $q = 0$.
- (d) Si $I \in \mathcal{J}$ et $f : I \rightarrow R$ un homomorphisme de R -module à droite, alors il existe $q \in Q_r(R)$ tel que $f(x) = qx$ pour tout $x \in I$.

Les propriétés ci-dessous caractérisent $Q_r(R)$ à isomorphisme près.

Exemple : Si R est un anneau simple unitaire, alors $Q_r(R) = R$. En effet, R a évidemment les propriétés (a) – (c), et vérifions qu'il satisfait également (d) : Si $f : R \rightarrow R$ est un homomorphisme de R -module à droite, alors $f(x) = f(1x) = f(1)x$ pour tout $x \in R$.

Définition 0.2.6 Le centre de $Q_r(R)$ est appelé le **centroïde étendu** de R et noté C .

Théorème 0.2.7 ([20], Theorem 7.33)

Le centroïde étendu C d'un anneau premier non nul R est un corps.

Bresar (1993), en utilisant le centroïde étendu, a donné la description complète d'une application additive arbitraire centralisante sur un anneau premier comme suit:

Théorème 0.2.8 ([18], Theorem 3.2)

Soit R un anneau premier. Si $F : R \rightarrow R$ une application additive commutante sur R , alors il existe un élément λ du centroïde étendu C de R et une application additive de $\zeta : R \rightarrow C$ tels que $F(x) = \lambda x + \zeta(x)$, $x \in R$ pour tout $x \in R$.

Dans la même référence, Breasr a caractérisé les dérivations d, g et h d'un anneau premier vérifiant $f(x) = ag(x) + h(x)b$ pour tout $x \in R$ où a et b des éléments de R . Plus exactement il a montré le résultat suivant :

Théorème 0.2.9 ([18], Theorem 2.1)

Soit R un anneau premier et soient f, g, h des dérivations de R . On suppose qu'il existe $a, b \in R$ tels que $f(x) = ag(x) + h(x)b$ pour tout $x \in R$. Si $a \notin Z(R)$ et $b \notin Z(R)$, alors il existe $\lambda \in C$ (Centrôide étendu de R), tels que $d(x) = [\lambda ab, x]$, $g(x) = [\lambda b, x]$ et $h(x) = [\lambda a, x]$ pour tout $x \in R$.

0.3 Dérivations et dérivations généralisées

Dans les dernières années plusieurs auteurs ont étudié de manière intensive la relation entre la commutativité d'un anneau R ou la structure explicite de l'anneau et le comportement de certains types spéciaux d'applications sur R . Le premier résultat dans cette direction est dû à Divinsky [33], qui a prouvé qu'un anneau artinian simple est commutatif s'il admet un automorphisme non trivial commutant. Deux années après, Posner [68] a prouvé une version du théorème de Divinsky pour les dérivations comme suit:

Théorème 0.3.1 ([68], Theorem 2)

Soit R un anneau premier et d une dérivation non nulle de R telle que, pour tout $a \in R$, $ad(a) - d(a)a$ est dans le centre de R . Alors R est commutatif.

Les résultats de Posner et Divinsky ont été par la suite raffinés et étendus par plusieurs auteurs dans différentes directions (Voir [7],[10],[14]).

Définition 0.3.2 *Soit R un anneau. Une application additive $d : R \rightarrow R$ est appelée une dérivation si $d(xy) = d(x)y + xd(y)$ pour tout $x, y \in R$.*

Exemples :

- 1) Soit $a \in R$ un élément fixé. L'application $d_a : R \rightarrow R$ définie par $d_a(x) = [a, x] = ax - xa$ est une dérivation, appelée dérivation intérieure associée à a .
- 2) Soit $R = M_2(\mathbb{Z})$. L'application

$$d : R \longrightarrow R$$

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \longmapsto \begin{pmatrix} 0 & b \\ -c & 0 \end{pmatrix}$$

est une dérivation.

- 3) Toute dérivation d d'un anneau R se prolonge en une dérivation D de l'anneau $R \times R^\circ$, de la façon suivante $D(x, y) = (d(x), d(y))$ pour tout $x, y \in R$.

Plusieurs résultats dans la littérature montrent l'existence d'une relation forte entre la structure explicite d'un anneau et le comportement de ses dérivations. En effet, plusieurs auteurs ont pu déterminer la structure explicite d'un anneau ayant une dérivation vérifiant des identités algébriques spécifiques.

Théorème 0.3.3 ([17], Theorem)

Soient R un anneau unitaire et d une dérivation sur R vérifiant, pour tout $x \in R$, $d(x) = 0$ ou $d(x)$ est inversible dans R . Alors

- 1) R est un anneau à division D , ou

-
- 2) $R = M_2(D)$ où D est un anneau à division, ou
3) $R = D[x]/(x^2)$ où D est un anneau à division, $\text{char} D = 2$, $d(D) = \{0\}$, et $d(x) = 1 + ax$, pour un certain $a \in Z(D)$.

Théorème 0.3.4 ([36], Theorem, p. 47)

Soit R un anneau semi-premier sans 2-torsion à involution. Soit d une dérivation de R telle que $d(S) \neq \{0\}$ et les éléments non nuls de $d(S)$ sont inversibles dans R . Alors R est:

- 1) un anneau à division D , ou
- 2) $M_2(D)$, ou
- 3) $D \oplus D^\circ$ relativement à l'involution d'échange, ou
- 4) $M_2(D) \oplus (M_2(D))^\circ$ avec l'involution d'échange, ou
- 5) $M_4(F)$ où F est un corps à involution symplectique.

Définition 0.3.5 Une application additive $F : R \rightarrow R$ est une dérivation généralisée s'il existe une dérivation $d : R \rightarrow R$ telle que $F(xy) = F(x)y + xd(y)$, pour tout $x, y \in R$. Dans ce cas, on dit que F est une dérivation généralisée associée à la dérivation d .

Exemples :

- 1) L'application identité est une dérivation généralisée associée à la dérivation nulle.
- 2) Soit d une dérivation d'un anneau R . Alors $F = d + id_R$ est une dérivation généralisée associée à d .
- 3) Soit $R = \left\{ \begin{pmatrix} a & 0 \\ b & c \end{pmatrix} \mid a, b, c \in \mathbb{Z} \right\}$. L'application F définie par :

$$\begin{aligned} F : R &\longrightarrow R \\ \begin{pmatrix} a & 0 \\ b & c \end{pmatrix} &\longmapsto \begin{pmatrix} 0 & 0 \\ a+b & 0 \end{pmatrix}, \end{aligned}$$

est une dérivation généralisée associée à la dérivation d ainsi définie par :

$$\begin{aligned} d : R &\longrightarrow R \\ \begin{pmatrix} a & 0 \\ b & c \end{pmatrix} &\longmapsto \begin{pmatrix} 0 & 0 \\ b & 0 \end{pmatrix}. \end{aligned}$$

Remarque. Il est clair que toute dérivation d'un anneau R est une dérivation généralisée associée à elle même. La réciproque n'est pas toujours vraie. En effet, soient R un anneau et $a, b \in R$. L'application $F_{a,b}$ définie par $F_{a,b}(x) = ax + xb$ est une dérivation généralisée associée à d_b (d_b est la

dérivation intérieure associée à b), appelée la dérivation intérieure généralisée.

Théorème 0.3.6 ([20], Theorem 2.1)

Soient R un anneau et f une dérivation généralisée de R telle que, pour tout $x \in R$, $f(x) = 0$ ou $f(x)$ est inversible. Si $\text{Ker} f$ ne contient pas d'idéaux à droite non nuls, alors on a l'un des cas suivants :

- 1) *R est un anneau à division.*
- 2) *R est isomorphe à $M_2(D)$ pour un anneau à division D*
- 3) *R est de caractéristique 2, $D = \{a \in R \mid f(a) = f(1)a\}$ est un sous-anneau à division de R et il existe $0 \neq t \in Z(R)$, $\alpha \in Z(D)$ et $c \in D$ tels que $R = D \oplus tD$, $t^2 = 0$, et $f(a + tb) = (ca + b) + t(\alpha + c)(ca + b)$ pour tout $a, b \in D$.*

Théorème 0.3.7 ([51], Theorem 2.2)

Soient R un anneau et f une dérivation généralisée de R telle que, pour tout $x \in R$, $f(x) = 0$ ou $f(x)$ est inversible. Si $\text{Ker} f$ contient un idéal à droite non nul, alors R est à Radical de Jacobson J non nul avec $J^2 = 0$, $D = \{a \in R \mid f(a) = f(1)a\}$ est un sous-anneau à division de R tel que $R = D \oplus J$ et il existe $0 \neq \alpha \in Z(D)$ et $w \in J$ tels que $f(a + x) = (\alpha + w)a$ pour tout $a \in D$ et pour tout $x \in J$.

0.4 Amalgamation

En 1956, M. Nagata a introduit la notion d'extension triviale d'un anneau R par un module M (aussi appelé l'idéalisation de M) comme suit:

Définition 0.4.1 *Soient R un anneau et M un R -module. L'extension triviale de R par M est l'anneau $A := R \ltimes M$ où l'addition est définie de façon naturelle et la multiplication est donnée par:*

$$(r, m)(r_0, n) = (rr_0, rn + r_0m), \text{ pour tout } r, r_0 \in R \text{ et } m, n \in M.$$

Cette notion a fait l'objet de plusieurs travaux dans la littérature et constitue un champ de recherche fertile dans la théorie des anneaux. Les extensions triviales d'anneaux ont été largement étudiées; et un travail considérable, dont une partie résumée dans le livre de Glaz et le livre de Huckaba, a été concerné par ces extensions. Principalement, ces extensions ont été utiles pour la résolution de nombreux problèmes ouverts et pour les conjectures en théorie d'anneaux

commutatifs et non-commutatifs. En 2006, M. D'Anna et M. Fontana ont introduit une nouvelle construction dite la duplication amalgamée d'un anneau R le long d'un R -module M , qui est un idéal dans un certain sur-anneau de R , et noté $R \bowtie M$

$$R \bowtie M := \{(a, a + m) \mid a \in R, m \in M\}.$$

Quand $M^2 = \{0\}$ cette construction coïncide avec l'extension triviale de R par M . Récemment, dans plusieurs travaux, sont discutées les motivations ainsi que les applications de cette duplication amalgamée $R \bowtie M$ de R le long du R -module M , essentiellement dans le cas particulier où $M := I$ est un idéal de R . En 2009, M. D'Anna, M. Fontana and C. A. Finocchiaro ont étendu la notion de la duplication amalgamée $R \bowtie I$ de R le long d'un idéal I de R au contexte général des extensions d'homomorphisme d'anneau comme suit:

Définition 0.4.2 *Soient A et B deux anneaux, J un idéal de B et soit $f : A \rightarrow B$ un homomorphisme d'anneaux. On appelle l'amalgamation de A et B suivant J et respectant f , le sous-anneau de $A \times B$:*

$$A \bowtie^f J := \{(a, f(a) + j) \mid a \in A, j \in J\}.$$

Cette construction est une généralisation de la duplication amalgamée d'un anneau le long d'un idéal. Par ailleurs, d'autres constructions classiques (tels que $A + XB[X]$, $A + XB[[X]]$, et les constructions $D + M$) peuvent être étudiées comme des cas particuliers de l'amalgamation. D'autres constructions classiques, telle que l'idéalisation du Nagata, également appelé l'anneau extension trivial, et les extensions CPI (dans le sens de Boisen et Sheldon) sont strictement liées. En 2017, S. Kabbaj, K. Louartiti et M. Tamekkante, (voir [57]) ont étendu la notion de l'amalgamée algébrique $A \bowtie J$ de A le long de l'idéal J de B au contexte général des extensions d'homomorphisme d'anneau comme suit:

Définition 0.4.3 *Soient $f : A \rightarrow B$ et $g : A \rightarrow C$ deux homomorphismes d'anneaux et soit J (resp., J_0) un idéal de B (resp., C) tel que $f^{-1}(J) = g^{-1}(J_0)$. On considère le sous anneau de $B \times C$ suivant :*

$$A \bowtie^{f,g} (J, J_0) := \{(f(a) + j, g(a) + j_0) \mid a \in A, j \in J, j_0 \in J_0\}$$

appelé la bi-amalgamation de A avec (B, C) le long de (J, J_0) par rapport à (f, g) .

Cette construction a été introduite comme une nouvelle classe d'anneaux qui couvre celle des amalgamées algébriques comme cas particulier.

ON GENERALIZED DERIVATIONS AND COMMUTATIVITY OF PRIME RINGS WITH INVOLUTION

M. A. Idrissi , A. Mamouni and L. Oukhtite, *On generalized derivations and commutativity of prime rings with involution*, De Gruyter Proc. Math.(2018) 91-100, Berlin, 2018.

Abstract. In this chapter we investigate commutativity in rings with involution admitting a generalized derivation satisfying certain algebraic identities. Some well-known results characterizing commutativity of prime rings have been generalized. Moreover, we provide examples to show that the assumed restrictions cannot be relaxed.

1.1 Introduction

Many results in literature indicate how the global structure of a ring R is often tightly connected to the behavior of additive mappings defined on R . A well known result due to Posner [68] states that if d is a derivation of a prime ring R such that $[d(x), x] \in Z(R)$ for all $x \in R$, then either $d = 0$ or R is commutative. In [52] Lanski generalizes the result of Posner by considering a derivation d such that $[d(x), x] \in Z(R)$ for all x in a nonzero Lie ideal U of R .

More recently several authors consider similar situation in the case where the derivation d is replaced by a generalized derivation. Generalized derivations have been primarily studied on operator algebras thus any investigation from

the algebraic point of view might be interesting (see for example [30], [44] and [54]).

During the last two decades, many authors have studied commutativity of prime and semi-prime rings admitting suitably constrained additive mappings acting on appropriate subsets of the rings. Moreover, many of obtained results extend other ones previously proven just for the action of the considered mapping on the whole ring. In this direction, the recent literature contains numerous results on commutativity in prime and semi-prime rings admitting suitably constrained derivations and generalized derivations, and several authors have improved these results by considering rings with involution (for example, see [2], [3], [58], [65] and [66]).

The purpose of the present chapter is to continue this line of investigation and study the structure of a prime ring with involution of the second kind admitting a generalized derivation satisfying more specific algebraic identities.

1.2 Identities with commutator

We first fix the following lemma which shall be used frequently throughout the text.

Lemma 1.2.1 ([63], Fact 1.) *Let $(R, *)$ be a 2-torsion free prime ring with involution provided with a derivation d . Then $d(h) = 0$ for all $h \in H(R) \cap Z(R)$ implies that $d(z) = 0$ for all $z \in Z(R)$.*

Theorem 1.2.2 *Let $(R, *)$ be a 2-torsion free prime ring with involution of the second kind. If R admits a generalized derivation F associated with a nonzero derivation d , then the following assertions are equivalent:*

- (1) $[F(x), d(x^*)] = \pm[x, x^*]$ for all $x \in R$.
- (2) $[F(x), d(x^*)] = 0$ for all $x \in R$.
- (3) R is commutative.

Proof. Obviously, (3) $\implies$ (1) and (3) $\implies$ (2).

(1) $\implies$ (3) For all $x \in R$, we suppose that

$$[F(x), d(x^*)] = [x, x^*] \tag{1.1}$$

Linearizing (1.1), we get $[F(x), d(y^*)] + [F(y), d(x^*)] = [x, y^*] + [y, x^*]$, replace y by y^* we find that

$$[F(x), d(y)] + [F(y^*), d(x^*)] = [x, y] + [y^*, x^*] \quad \text{for all } x, y \in R. \tag{1.2}$$

Substituting yh for y in (1.2), where $h \in Z(R) \cap H(R) \setminus \{0\}$ and using (1.2) we obtain

$$([F(x), y] + [y^*, d(x^*)])d(h) = 0 \quad \text{for all } x, y \in R. \quad (1.3)$$

Using the primeness hypothesis, it follows that $[F(x), y] + [y^*, d(x^*)] = 0$ or $d(h) = 0$.

Suppose that

$$[F(x), y] + [y^*, d(x^*)] = 0 \quad \text{for all } x, y \in R. \quad (1.4)$$

Replacing y by ys in (1.4), where $s \in Z(R) \cap S(R) \setminus \{0\}$, it's easy to verify that

$$[F(x), y] - [y^*, d(x^*)] = 0 \quad \text{for all } x, y \in R. \quad (1.5)$$

Adding relations (1.4) and (1.5) we find that

$$[F(x), y] = 0 \quad \text{for all } x, y \in R$$

thus $[F(x), x] = 0$ for all $x \in R$, hence the commutativity of R follows from ([67], Theorem 3).

Now if $d(h) = 0$ for all $h \in Z(R) \cap H(R)$, then according to Lemma 1.2.1 one obtains $d(s) = 0$ for all $s \in Z(R) \cap S(R)$.

Putting ys for y in equation (1.2), we get

$$[F(x), d(y)] - [F(y^*), d(x^*)] = [x, y] - [y^*, x^*] \quad \text{for all } x, y \in R. \quad (1.6)$$

Comparing equations (1.2) and (1.6) we arrive at

$$[F(x), d(y)] = [x, y] \quad \text{for all } x, y \in R. \quad (1.7)$$

In view of ([6], Theorem 2.11), one can conclude that R is commutative.

Further, if $[F(x), d(x^*)] = -[x, x^*]$ for all $x \in R$, then using the same techniques as used above with necessary modifications we get the required result.

(2) $\implies$ (3) Suppose that

$$[F(x), d(x^*)] = 0 \quad \text{for all } x \in R. \quad (1.8)$$

Linearizing equation (1.8), we get $[F(x), d(y^*)] + [F(y), d(x^*)] = 0$ so that

$$[F(x), d(y)] + [F(y^*), d(x^*)] = 0 \quad \text{for all } x, y \in R. \quad (1.9)$$

Substituting yh for y in (1.9), where $h \in Z(R) \cap H(R) \setminus \{0\}$ and using (1.9) one obtains

$$([F(x), y] + [y^*, d(x^*)])d(h) = 0 \quad \text{for all } x, y \in R. \quad (1.10)$$

Since R is prime, then it follows that either $[F(x), y] + [y^*, d(x^*)] = 0$ or $d(h) = 0$.

Suppose that

$$[F(x), y] + [y^*, d(x^*)] = 0 \quad \text{for all } x, y \in R. \quad (1.11)$$

Replacing y by ys in (1.11), where $s \in Z(R) \cap S(R) \setminus \{0\}$, it's obvious to verify that

$$[F(x), y] - [y^*, d(x^*)] = 0 \quad \text{for all } x, y \in R. \quad (1.12)$$

Adding relations (1.11) and (1.12) we obtain $[F(x), y] = 0$ for all $x, y \in R$, in such a way that $[F(x), x] = 0$ for all $x \in R$. Therefore the commutativity of R follows from ([67], Theorem 3).

Now if $d(h) = 0$ for all $h \in Z(R) \cap H(R)$, then Lemma 1.2.1 yields that $d(s) = 0$ for all $s \in Z(R) \cap S(R)$.

Writing ys instead of y in equation (1.9), one obtains

$$[F(x), d(y)] - [F(y^*), d(x^*)] = 0 \quad \text{for all } x, y \in R. \quad (1.13)$$

From (1.9) and (1.13) one obtains

$$[F(x), d(y)] = 0 \quad \text{for all } x, y \in R. \quad (1.14)$$

In view of ([6], Theorem 2.6), one can conclude that R is commutative. ■

Remark. To prove (1) $\iff$ (3) the condition $d \neq 0$ is not necessary. Indeed, if $F = 0$ or $d = 0$ then R is normal and the required result follows from ([63], Lemma 2.1).

Corollary 1.2.3 *Let $(R, *)$ be a 2-torsion free prime ring with involution of the second kind. If R admits a generalized derivation F associated with a nonzero derivation d , then the following assertions are equivalent:*

- (1) $[F(x), d(y)] = \pm[x, y]$ for all $x, y \in R$.
- (2) $[F(x), d(y)] = 0$ for all $x, y \in R$.
- (3) R is commutative.

Corollary 1.2.4 ([3], [4] and [40]) *Let $(R, *)$ be a 2-torsion free prime ring with involution of the second kind. If R admits a nonzero derivation d , then the following assertions are equivalent:*

- (1) $[d(x), d(x^*)] = \pm[x, x^*]$ for all $x \in R$.
- (2) $[d(x), d(x^*)] = 0$ for all $x \in R$.
- (3) R is commutative.

1.3 Identities with anti-commutator

It is natural to ask what can we say about the commutativity of R if the commutator in the preceding theorem is replaced by anti-commutator. In the following theorem, we have investigated this problem for (1) $\iff$ (3) and our result is of a different kind. Indeed, we have proved that the commutativity cannot be characterized by the same condition on anti-commutator.

Theorem 1.3.1 *Let $(R, *)$ be a 2-torsion free prime ring with involution of the second kind. There is no generalized derivation F associated with a derivation d satisfying $F(x) \circ d(x^*) = \pm x \circ x^*$ for all $x \in R$.*

Proof. Assume that R admits a generalized derivation F associated with a derivation d such that

$$F(x) \circ d(x^*) = x \circ x^* \quad \text{for all } x \in R. \quad (1.15)$$

If $d = 0$, then (1.15) reduces to $x \circ x^* = 0$ for all $x \in R$ which gives

$$x \circ y^* + y \circ x^* = 0 \quad \text{for all } x, y \in R.$$

Replacing y by h where h is a nonzero element in $Z(R) \cap H(R)$, one obtains $x + x^* = 0$.

Similarly, putting s for y where s is a nonzero element in $Z(R) \cap S(R)$, we find that $x - x^* = 0$. Comparing the last two equations we conclude that $R = 0$, a contradiction.

Accordingly, we may suppose that $d \neq 0$. Now linearizing equation (1.15) one obtains $F(x) \circ d(y^*) + F(y) \circ d(x^*) = x \circ y^* + y \circ x^*$ so that

$$F(x) \circ d(y) + F(y^*) \circ d(x^*) = x \circ y + y^* \circ x^* \quad \text{for all } x, y \in R. \quad (1.16)$$

Replacing y by yh in (1.16), where $h \in Z(R) \cap H(R) \setminus \{0\}$, we arrive at

$$(F(x) \circ y + y^* \circ d(x^*))d(h) = 0 \quad \text{for all } x, y \in R. \quad (1.17)$$

In light of primeness, the above expression assures that either

$$d(h) = 0 \quad \text{or} \quad F(x) \circ y + y^* \circ d(x^*) = 0.$$

Suppose that

$$F(x) \circ y + y^* \circ d(x^*) = 0 \quad \text{for all } x, y \in R. \quad (1.18)$$

Putting $y = h$ where $h \in Z(R) \cap H(R) \setminus \{0\}$, we obtain $2(F(x) + d(x^*))h = 0$ which leads to $F(x) + d(x^*) = 0$ for all $x \in R$.

On the other hand, replacing y by s in (1.18) with $s \in Z(R) \cap S(R) \setminus \{0\}$, we arrive at $F(x) - d(x^*) = 0$ for all $x \in R$. Therefore $F(x) = 0$ for all $x \in R$ and a fortiori $d = 0$, a contradiction.

Now suppose that $d(h) = 0$ for all $h \in Z(R) \cap H(R)$, from Lemma 1.2.1 it follows that $d(s) = 0$ for all $s \in Z(R) \cap S(R)$.

Substituting ys for y in (1.16) for a nonzero element $s \in Z(R) \cap S(R)$, one can see that

$$F(x) \circ d(y) - F(y^*) \circ d(x^*) = x \circ y - y^* \circ x^* \quad \text{for all } x, y \in R. \quad (1.19)$$

Adding equations (1.19) and (1.16), we conclude that

$$F(x) \circ d(y) = x \circ y \quad \text{for all } x, y \in R. \quad (1.20)$$

So the commutativity of R follows from ([6], Theorem 2.7). Hence equation (1.20) reduces to

$$F(x)d(y) = xy \quad \text{for all } x, y \in R.$$

Replacing y by yt in the last equation, we get $F(x)yd(t) = 0$ and thus

$$F(x)Rd(t) = \{0\} \quad \text{for all } x, y \in R$$

so that $d = 0$. However, in this cases $xy = 0$ for all $x, y \in R$ so that $R = \{0\}$, a contradiction. ■

Using similar arguments as in the first case, it is obvious to show that there is no generalized derivation F associated with a derivation d satisfying $F(x) \circ d(x^*) = -(x \circ x^*)$ for all $x \in R$.

Corollary 1.3.2 *Let $(R, *)$ be a 2-torsion free prime ring with involution of the second kind. There is no generalized derivation F associated with a derivation d satisfying $F(x) \circ d(y) = \pm x \circ y$ for all $x, y \in R$.*

Corollary 1.3.3 *Let $(R, *)$ be a 2-torsion free prime ring with involution of the second kind. There is no derivation d satisfying $d(x) \circ d(x^*) = \pm x \circ x^*$ for all $x \in R$.*

Now if we replace the commutator by anti-commutator in (2) of Theorem 1.2.2 then it does not constitute a commutativity criterion. However, we can classify generalized derivations satisfying the hypothesis $F(x) \circ d(x^*) = 0$ for all $x \in R$.

Theorem 1.3.4 *Let $(R, *)$ be a 2-torsion free prime ring with involution of the second kind and F a generalized derivation associated with a derivation d . Then the following assertions are equivalent:*

- (1) $F(x) \circ d(x^*) = 0$ for all $x \in R$.
- (2) F is a left multiplier.

Proof. We need only prove that (1) $\implies$ (2). We are given that

$$F(x) \circ d(x^*) = 0 \quad \text{for all } x \in R. \quad (1.21)$$

Suppose that F is not a left multiplier that is $d \neq 0$.

Linearizing (1.21) we get

$$F(x) \circ d(y^*) + F(y) \circ d(x^*) = 0 \quad \text{for all } x, y \in R,$$

so that

$$F(x) \circ d(y) + F(y^*) \circ d(x^*) = 0 \quad \text{for all } x, y \in R. \quad (1.22)$$

Replacing y by yh in (1.22), where $h \in Z(R) \cap H(R) \setminus \{0\}$, we obtain

$$(F(x) \circ y + y^* \circ d(x^*))d(h) = 0 \quad \text{for all } x, y \in R. \quad (1.23)$$

Invoking primeness, it follows that either $F(x) \circ y + y^* \circ d(x^*) = 0$ or $d(h) = 0$.

Suppose that

$$F(x) \circ y + y^* \circ d(x^*) = 0 \quad \text{for all } x, y \in R. \quad (1.24)$$

Taking $y = h$ where $h \in Z(R) \cap H(R) \setminus \{0\}$, one can easily verify that $2(F(x) + d(x^*))h = 0$ and thus

$$F(x) + d(x^*) = 0 \quad \text{for all } x \in R.$$

Replacing y by s in (1.24) where s is a nonzero element in $Z(R) \cap S(R)$, we obviously get

$$F(x) - d(x^*) = 0 \quad \text{for all } x \in R.$$

Comparing the last two relations, we conclude that $F = d = 0$, a contradiction. Therefore $d(h) = 0$ for all $h \in Z(R) \cap H(R)$ in which case Lemma 1.2.1 forces $d(s) = 0$ for all $s \in Z(R) \cap S(R)$.

Putting ys for y in (1.22) where $s \in Z(R) \cap S(R) \setminus \{0\}$, one obtains

$$F(x) \circ d(y) - F(y^*) \circ d(x^*) = 0 \quad \text{for all } x, y \in R. \quad (1.25)$$

Adding equations (1.25) and (1.22) we conclude that

$$F(x) \circ d(y) = 0 \quad \text{for all } x, y \in R. \quad (1.26)$$

Hence by ([6], Theorem 2.5) we conclude that R is commutative. Then equation (1.26) becomes

$$F(x)d(y) = 0 \quad \text{for all } x, y \in R$$

in consequence of which $d = 0$, a contradiction. ■

Corollary 1.3.5 *Let $(R, *)$ be a 2-torsion free prime ring with involution of the second kind and F a generalized derivation associated with a derivation d . Then the following assertions are equivalent:*

- (1) $F(x) \circ d(y) = 0$ for all $x, y \in R$.
- (2) F is a left multiplier.

Corollary 1.3.6 *Let $(R, *)$ be a 2-torsion free prime ring with involution of the second kind and d is a derivation of R . Then the following assertions are equivalent:*

- (1) $d(x) \circ d(x^*) = 0$ for all $x \in R$.
- (2) $d = 0$.

1.4 Identities with commutator and anti-commutator

Theorem 1.4.1 *Let $(R, *)$ be a 2-torsion free prime ring with involution of the second kind. There is no generalized derivation F associated with a derivation d satisfying $[F(x), d(x^*)] = \pm x \circ x^*$ for all $x \in R$.*

Proof. If R is commutative or $d = 0$, then we get $x \circ x^* = 0$ for all $x \in R$. Using the same argument as before we have $R = \{0\}$. Hence we may suppose that neither R is commutative nor $d = 0$.

Assume that

$$[F(x), d(x^*)] = x \circ x^* \quad \text{for all } x \in R. \quad (1.27)$$

Linearizing (1.27), we get $[F(x), d(y^*)] + [F(y), d(x^*)] = x \circ y^* + y \circ x^*$ so that

$$[F(x), d(y)] + [F(y^*), d(x^*)] = x \circ y + y^* \circ x^* \quad \text{for all } x, y \in R. \quad (1.28)$$

Substituting yh for y in (1.28), where $h \in Z(R) \cap H(R) \setminus \{0\}$, we obtain

$$([F(x), y] + [y^*, d(x^*)])d(h) = 0 \quad \text{for all } x, y \in R. \quad (1.29)$$

Since R is prime, then either $d(h) = 0$ or $[F(x), y] + [y^*, d(x^*)] = 0$.

Suppose that

$$[F(x), y] + [y^*, d(x^*)] = 0 \quad \text{for all } x, y \in R. \quad (1.30)$$

this is just equation (1.4) in the proof of Theorem 1.2.2 it follows that R is commutative, a contradiction.

Now if $d(h) = 0$ for all $h \in Z(R) \cap H(R)$, then $d(s) = 0$ for all $s \in Z(R) \cap S(R)$, by Lemma 1.2.1.

On the other hand, replacing y by ys in (1.28), one obtains

$$[F(x), d(y)] - [F(y^*), d(x^*)] = x \circ y - y^* \circ x^* \quad \text{for all } x, y \in R. \quad (1.31)$$

Adding equations (1.28) and (1.31) we find that

$$[F(x), d(y)] = x \circ y \quad \text{for all } x, y \in R. \quad (1.32)$$

Hence by ([32], Theorem 2.2) we conclude that R is commutative, a contradiction. ■

The proof of the case $[F(x), d(x^*)] = -x \circ x^*$ for all $x \in R$ is similar and requires only slight modifications.

Corollary 1.4.2 *Let $(R, *)$ be a 2-torsion free prime ring with involution of the second kind. There is no generalized derivation F associated with a derivation d satisfying $[F(x), d(y)] = \pm x \circ y$ for all $x, y \in R$.*

Corollary 1.4.3 *Let $(R, *)$ be a 2-torsion free prime ring with involution of the second kind. There is no derivation d satisfying $[d(x), d(x^*)] = \pm x \circ x^*$ for all $x \in R$.*

Theorem 1.4.4 *Let $(R, *)$ be a 2-torsion free prime ring with involution of the second kind. If R admits a generalized derivation F associated with a derivation d such that $F(x) \circ d(x^*) = \pm[x, x^*]$ for all $x \in R$ then R is commutative. Furthermore, F is a left multiplier.*

Proof. Assume that

$$F(x) \circ d(x^*) = [x, x^*] \quad \text{for all } x \in R. \quad (1.33)$$

If $F = 0$ then the preceding relation reduces to $[x, x^*] = 0$ for all $x \in R$ hence R is commutative by ([63], Lemma 2.1). Therefore, we may assume that $F \neq 0$. Linearizing (1.33), one obtains

$$F(x) \circ d(y^*) + F(y) \circ d(x^*) = [x, y^*] + [y, x^*] \quad \text{for all } x, y \in R \quad (1.34)$$

thereby

$$F(x) \circ d(y) + F(y^*) \circ d(x^*) = [x, y] + [y^*, x^*] \quad \text{for all } x, y \in R. \quad (1.35)$$

Substituting yh for y in (1.35) where h is a nonzero element in $Z(R) \cap H(R)$ and using (1.35), we find that

$$(F(x) \circ y + y^* \circ d(x^*))d(h) = 0 \quad \text{for all } x, y \in R. \quad (1.36)$$

Using the primeness hypothesis, it follows that $F(x) \circ y + y^* \circ d(x^*) = 0$ or $d(h) = 0$.

Suppose that

$$F(x) \circ y + y^* \circ d(x^*) = 0 \quad \text{for all } x, y \in R. \quad (1.37)$$

Putting $y = h$ in (1.37) where $h \in Z(R) \cap H(R) \setminus \{0\}$, one obtains

$$F(x) + d(x^*) = 0 \quad \text{for all } x \in R.$$

Similarly, replacing y by s in (1.37) where $s \in Z(R) \cap S(R) \setminus \{0\}$, we arrive at

$$F(x) - d(x^*) = 0 \quad \text{for all } x \in R.$$

Comparing the two last relations we conclude that $F = 0$, a contradiction. Thus we need only consider $d(h) = 0$ for all $h \in Z(R) \cap H(R)$. Replacing y by ys in (1.35) where s is a nonzero element in $Z(R) \cap S(R)$, because of $d(s) = 0$, we have

$$(F(x) \circ d(y) - F(y^*) \circ d(x^*))s = ([x, y] + [y^*, x^*])s \quad \text{for all } x, y \in R \quad (1.38)$$

so the primeness yields

$$F(x) \circ d(y) - F(y^*) \circ d(x^*) = [x, y] - [y^*, x^*] \quad \text{for all } x, y \in R. \quad (1.39)$$

Combining equations (1.35) and (1.39), we get

$$F(x) \circ d(y) = [x, y] \quad \text{for all } x, y \in R. \quad (1.40)$$

Hence we conclude that R is commutative by ([32], Theorem 2.5). Accordingly, equation (1.40) becomes $F(x) \circ d(y) = 0$ which leads to $d = 0$, proving that F is a left multiplier. ■

Corollary 1.4.5 *Let $(R, *)$ be a 2-torsion free prime ring with involution of the second kind. If R admits a generalized derivation F associated with a derivation d such that $F(x) \circ d(y) = \pm[x, y]$ for all $x, y \in R$ then R is commutative. Furthermore, F is a left multiplier.*

Corollary 1.4.6 *Let $(R, *)$ be a 2-torsion free prime ring with involution of the second kind. If R admits a derivation d such that $d(x) \circ d(x^*) = \pm[x, x^*]$ for all $x \in R$ then R is commutative. Furthermore, $d = 0$.*

1.5 Examples proving necessity of our conditions

The following example proves that the primeness hypothesis in Theorems 1.2.2 and 1.4.4 is not superfluous.

Example 1.

Let us consider $R = M_2(\mathbb{Z})$ and define

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}^* = \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$$

and

$$F \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} a & b \\ 0 & 0 \end{pmatrix}.$$

Then F is a left multiplier and $(R, *)$ is a prime ring with involution of the first kind such that $[x, x^*] = 0 \forall x \in R$.

Set $\mathcal{R} = R \times \mathbb{C}$, then it is obvious to verify that $(\mathcal{R}, \sigma)$ is a semi-prime ring with involution of the second kind where $\sigma(r, z) = (r^*, \bar{z})$.

Moreover, if we put

$$\mathcal{F}(r, z) = (F(r), 0)$$

then $\mathcal{F}$ is a left multiplier satisfying the condition (1) of Theorem 1.2.2 and Theorem 1.4.4 but $\mathcal{R}$ is not commutative.

The following example proves that the condition ”* is of the second kind” is necessary in Theorems 1.2.2, and 1.4.4.

Example 2.

Let us consider $R = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \mid a, b, c, d \in \mathbb{Z} \right\}$ and $\begin{pmatrix} a & b \\ c & d \end{pmatrix}^* = \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$.

It is straightforward to check that $(R, *)$ is a prime ring with involution of the first kind such that

$$[x, x^*] = 0 \text{ for all } x \in R.$$

Furthermore, the mapping $F : R \longrightarrow R$ defined by

$$F \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} a & b \\ 0 & 0 \end{pmatrix}$$

is a left multiplier that satisfies conditions of Theorems 1.2.2 and 1.4.4 however R is not commutative.

SOME COMMUTATIVITY THEOREMS FOR RINGS WITH INVOLUTION INVOLVING GENERALIZED DERIVATIONS

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Abstract. Our purpose in this chapter is to investigate commutativity of a ring with involution $(R, *)$ which admits a generalized derivation satisfying certain algebraic identities. Some well-known results characterizing commutativity of prime rings have been generalized. Moreover, we provide examples to show that the assumed restrictions cannot be relaxed..

2.1 Introduction

Long ago Herstein [41] proved that if a prime ring R of characteristic different from two admits a derivation d such that $d(x)d(y) = d(y)d(x)$ holds for all $x, y \in R$, then R is commutative. Motivated by this result Bell and Daif [11] obtained the same result by considering the identity $d[x, y] = 0$ for all x, y in a nonzero ideal of R . Later, Emine Koç and Öznur Gölbaşı [50] have extended the mentioned result by considering a generalized derivation F acting on a Lie

ideal U . More precisely, they proved that if R is a 2-torsion free prime ring equipped with a nonzero generalized derivation F satisfying $F([x, y]) = 0$ for all x, y in a nonzero square-closed Lie ideal U , then $U \subseteq Z(R)$.

2.2 Main results

Throughout this section id will denote the identity map defined by $id(r) = r$ for all $r \in R$. The following Lemmas are essential for developing the proofs of our results.

Lemma 2.2.1 *Let R be a 2-torsion free prime ring. If R admits a nonzero generalized derivation F such that $F([x, y]) \in Z(R)$ for all $x, y \in R$, then R is commutative.*

Proof. Assume that R is not commutative and let F is a nonzero generalized derivation F associated with a derivation d such that

$$F([x, y]) \in Z(R) \quad \text{for all } x, y \in R. \quad (2.1)$$

If $d = 0$, then F is a left multiplier and (2.1) becomes

$$F(x)y - F(y)x \in Z(R) \quad \text{for all } x, y \in R. \quad (2.2)$$

Replacing y by yx in (2.2) and using (2.2) itself, it is obvious to see that

$$\left(F(x)y - F(y)x \right) R[x, s] = \{0\} \quad \text{for all } s, x, y \in R. \quad (2.3)$$

Hence we have either $F(x)y - F(y)x = 0$ or $[x, s] = 0$ for all $s, x, y \in R$. The sets of x for which theses conditions hold are additive subgroups of R with union equal to R ; thus either $F(x)y = F(y)x$ for all $x, y \in R$ or $[R, R] = \{0\}$. In the latter case R is commutative; therefore we assume henceforth that $F(x)y = F(y)x$ for all $x, y \in R$.

Under this assumption replacing y by yr we obtain $F(x)yr = F(y)rx = F(y)xr$ and thus

$$F(y)R[x, r] = \{0\} \quad \text{for all } r, x, y \in R. \quad (2.4)$$

Once again employing the primeness, equation (2.4) reduces to $[R, R] = \{0\}$. Consequently R is commutative, a contradiction.

Now assume that $d \neq 0$; if in ([30], Theorem 1) we put $L = R$, then $d(R)[R, R] = \{0\}$ in consequence of which $[R, R] = \{0\}$ proving that R is commutative, a contradiction.

In conclusion, the ring R is necessarily commutative. ■

Lemma 2.2.2 *Let $(R, *)$ be a 2-torsion free prime ring with involution of the second kind. If $x \circ y + y^* \circ x^* \in Z(R)$ for all $x, y \in R$, then R is commutative.*

Proof. Assume that

$$x \circ y + y^* \circ x^* \in Z(R) \quad \text{for all } x, y \in R. \quad (2.5)$$

If in (2.5) we put $y = s$, where $s \in Z(R) \cap S(R) \setminus \{0\}$, we get

$$2(x - x^*)s \in Z(R) \quad \text{for all } x \in R. \quad (2.6)$$

In light of primeness, because of 2-torsion freeness, equation (2.6) assures that

$$x - x^* \in Z(R) \quad \text{for all } x \in R. \quad (2.7)$$

Replacing y by h in (2.5), where $h \in Z(R) \cap H(R) \setminus \{0\}$, we obtain

$$2(x + x^*)h \in Z(R) \quad \text{for all } x \in R \quad (2.8)$$

in such a way that

$$x + x^* \in Z(R) \quad \text{for all } x \in R. \quad (2.9)$$

Using (2.7) together with (2.9) we conclude that R is commutative. ■

2.2.1 Identities with commutator

In the following theorem we will consider the situation of a prime ring $(R, *)$ with involution having a generalized derivation F satisfying the more general condition $F([x, x^*]) \in Z(R)$ for all $x \in R$.

Theorem 2.2.3 *Let $(R, *)$ be a 2-torsion free prime ring with involution of the second kind. If R admits a nonzero generalized derivation F associated with a derivation d , then R is commutative if and only if $F([x, x^*]) \in Z(R)$ for all $x \in R$.*

Proof. For the nontrivial implication, suppose that $d = 0$ and

$$F([x, x^*]) \in Z(R) \quad \text{for all } x \in R. \quad (2.10)$$

Linearizing equation (2.10), we get $F([x, y^*]) + F([y, x^*]) \in Z(R)$ so that

$$F([x, y]) + F([y^*, x^*]) \in Z(R) \quad \text{for all } x, y \in R. \quad (2.11)$$

Substituting ys for y in (2.11), where $s \in Z(R) \cap S(R) \setminus \{0\}$, we obtain

$$(F([x, y]) - F([y^*, x^*]))s \in Z(R) \quad \text{for all } x, y \in R \quad (2.12)$$

which leads to

$$F([x, y]) - F([y^*, x^*]) \in Z(R) \quad \text{for all } x, y \in R. \quad (2.13)$$

Combining equations (2.11) with (2.13) and using 2-torsion freeness hypothesis, we arrive at

$$F([x, y]) \in Z(R) \quad \text{for all } x, y \in R. \quad (2.14)$$

Using Lemma 2.2.1, from equation (2.14) it follows that R is commutative. To complete the proof, we need only consider the case $d \neq 0$. Replacing y by yh in (2.11), where $h \in Z(R) \cap H(R) \setminus \{0\}$, we get

$$([x, y] + [y^*, x^*])d(h) \in Z(R) \quad \text{for all } x, y \in R. \quad (2.15)$$

Using the primeness hypothesis, it follows that $d(h) = 0$ or $[x, y] + [y^*, x^*] \in Z(R)$.

Suppose that

$$[x, y] + [y^*, x^*] \in Z(R) \quad \text{for all } x, y \in R. \quad (2.16)$$

Replacing y by ys in (2.16), where $s \in Z(R) \cap S(R) \setminus \{0\}$, it's obvious to verify that

$$([x, y] - [y^*, x^*])s \in Z(R) \quad \text{for all } x, y \in R \quad (2.17)$$

in consequence of which

$$[x, y] - [y^*, x^*] \in Z(R) \quad \text{for all } x, y \in R. \quad (2.18)$$

Invoking (2.16), equation (2.18) yields

$$[x, y] \in Z(R) \quad \text{for all } x, y \in R$$

which assures the commutativity of R .

Now if $d(h) = 0$ for all $h \in Z(R) \cap H(R)$, then in particular $d(s^2) = 0$ for all $s \in Z(R) \cap S(R)$, in such a way that $d(s) = 0$.

Replacing y by ys in equation (2.11), we get $(F([x, y]) - F([y^*, x^*]))s \in Z(R)$, which reduces to

$$F([x, y]) - F([y^*, x^*]) \in Z(R) \quad \text{for all } x, y \in R. \quad (2.19)$$

Using equations (2.11) together with (2.19), we conclude that

$$F([x, y]) \in Z(R) \quad \text{for all } x, y \in R$$

and thus R is commutative by Lemma 2.2.1. ■

Corollary 2.2.4 *Let $(R, *)$ be a 2-torsion free prime ring with involution of the second kind. If R admits a nonzero generalized derivation F associated with a derivation d , then R is commutative if and only if $F([x, y]) \in Z(R)$ for all $x, y \in R$.*

Corollary 2.2.5 *Let $(R, *)$ be a 2-torsion free prime ring with involution of the second kind. If R admits a nonzero derivation d , then R is commutative if and only if $d([x, x^*]) \in Z(R)$ for all $x \in R$.*

Proposition 2.2.6 *Let $(R, *)$ be a 2-torsion free prime ring with involution of the second kind. If R admits a generalized derivation F associated with a derivation d such that $F \neq id$ and $F([x, x^*]) - [x, x^*] \in Z(R)$ (resp. $F \neq -id$ and $F([x, x^*]) + [x, x^*] \in Z(R)$) for all $x \in R$, then R is commutative.*

Proof. If $F \neq id$ (resp. $F \neq -id$) then $\psi = F - id$ (resp. $\chi = F + id$) is a nonzero generalized derivation satisfying $\psi([x, x^*]) \in Z(R)$ (resp. $\chi([x, x^*]) \in Z(R)$) for all $x \in R$. Applying Theorem 2.2.3, we conclude that R is commutative. ■

In [73] Rehman established that if a 2-torsion free prime ring admits a generalized derivation F associated with a nonzero derivation such that $F([x, y]) \pm [x, y] = 0$ for all x, y in a nonzero square closed Lie ideal U , then $U \subseteq Z(R)$. Quadri et al. [69], without 2-torsion freeness hypothesis, proved that a prime ring must be commutative if it admits a generalized derivation F , associated with a nonzero derivation, such that $F([x, y]) \pm [x, y] = 0$ for all x, y in a nonzero ideal I .

As an application of our theorem, the following corollary improves the result of [69] for the case when the underlying identity belongs to the center of a prime ring with involution of the second kind.

Corollary 2.2.7 *Let $(R, *)$ be a 2-torsion free prime ring with involution of the second kind. If R admits a generalized derivation F associated with a derivation d such that $F \neq 0$ and $F \neq \pm id$, then the following assertions are equivalent:*

- (1) $F([x, y]) \in Z(R)$ for all $x, y \in R$;
- (2) $F([x, y]) \pm [x, y] \in Z(R)$ for all $x, y \in R$;
- (3) R is commutative.

Corollary 2.2.8 *Let $(R, *)$ be a 2-torsion free prime ring with involution of the second kind. If R admits a derivation d such that $d([x, x^*]) - [x, x^*] \in Z(R)$ (resp. $d([x, x^*]) + [x, x^*] \in Z(R)$) for all $x \in R$, then R is commutative.*

2.2.2 Identities with anti-commutator

In ([50], Corollary 2.1) it is proved that, if R is a 2-torsion free prime ring provided with a multiplicative generalized F associated with a nonzero derivation d such that $F(x \circ y) = 0$ for all x, y in a square closed Lie ideal U of R , then $U \subseteq Z(R)$. This result is false, because in the particular case $U = R$, we get R is commutative. Hence the hypothesis reduces to $F(xy) = 0$ for all $x, y \in R$ thereby $d = 0$, which is a contradiction.

In the following theorem we consider the more general condition $F(x \circ x^*) \in Z(R)$ for all $x \in R$ for the category of prime rings with involution.

Theorem 2.2.9 *Let $(R, *)$ be a 2-torsion free prime ring with involution of the second kind. If R admits a nonzero generalized derivation F associated with a derivation d , then R is commutative if and only if $F(x \circ x^*) \in Z(R)$ for all $x \in R$.*

Proof. For the nontrivial implication suppose that $d = 0$ and

$$F(x \circ x^*) \in Z(R) \quad \text{for all } x \in R. \quad (2.20)$$

Linearizing this equation, we get

$$F(x \circ y^*) + F(y \circ x^*) \in Z(R) \quad \text{for all } x, y \in R \quad (2.21)$$

thereby obtaining

$$F(x \circ y) + F(y^* \circ x^*) \in Z(R) \quad \text{for all } x, y \in R. \quad (2.22)$$

In (2.22) we substitute ys for y , where $s \in Z(R) \cap S(R) \setminus \{0\}$, to get

$$(F(x \circ y) - F(y^* \circ x^*))s \in Z(R) \quad \text{for all } x, y \in R \quad (2.23)$$

in consequence of which

$$F(x \circ y) - F(y^* \circ x^*) \in Z(R) \quad \text{for all } x, y \in R. \quad (2.24)$$

Combining equations (2.22) and (2.24), we obviously get

$$F(x \circ y) \in Z(R) \quad \text{for all } x, y \in R. \quad (2.25)$$

Taking $y = h$ in (2.25), where $h \in Z(R) \cap H(R) \setminus \{0\}$, we obtain $2F(x)h \in Z(R)$ which leads to

$$F(x) \in Z(R) \quad \text{for all } x \in R \quad (2.26)$$

The fact that $F(xr) = F(x)r \in Z(R)$ implies that $F(x)[r, t] = 0$ for all $x, r, t \in R$ proving that R is commutative.

Now assume that $d \neq 0$; writing yh instead of y in (2.22), where $h \in Z(R) \cap H(R) \setminus \{0\}$, we obtain

$$(x \circ y + y^* \circ x^*)d(h) \in Z(R) \quad \text{for all } x, y \in R. \quad (2.27)$$

In view of the primeness, the last equation reduces to $d(h) = 0$ or $x \circ y + y^* \circ x^* \in Z(R)$. In the last case R is commutative by Lemma 2.2.2. Thus, we need only consider the case $d(h) = 0$ for all $h \in Z(R) \cap H(R)$. Under this assumption, one can easily verify that $d(s) = 0$ for all $s \in S(R) \cap Z(R)$.

Now replacing y by ys in (2.22), where $s \in Z(R) \cap S(R) \setminus \{0\}$, we find that

$$(F(x \circ y) - F(y^* \circ x^*))s \in Z(R) \quad \text{for all } x, y \in R \quad (2.28)$$

and therefore

$$F(x \circ y) - F(y^* \circ x^*) \in Z(R) \quad \text{for all } x, y \in R. \quad (2.29)$$

Using (2.22), equation (2.29) becomes

$$F(x \circ y) \in Z(R) \quad \text{for all } x, y \in R. \quad (2.30)$$

Substituting h for y in (2.30), where $h \in Z(R) \cap H(R) \setminus \{0\}$, we get

$$F(x) \in Z(R) \quad \text{for all } x \in R. \quad (2.31)$$

In particular, $[F(x), x] = 0$ for all $x \in R$ which means that F is commuting and thus R is commutative by ([67], Theorem 2). This completes the proof of our theorem. ■

Corollary 2.2.10 *Let $(R, *)$ be a 2-torsion free prime ring with involution of the second kind. If R admits a nonzero derivation d , then R is commutative if and only if $d(x \circ x^*) \in Z(R)$ for all $x \in R$.*

Proposition 2.2.11 *Let $(R, *)$ be a 2-torsion free prime ring with involution of the second kind. If R admits a generalized derivation F associated with a derivation d such that $F \neq id$ and $F(x \circ x^*) - (x \circ x^*) \in Z(R)$ (resp. $F \neq -id$ and $F(x \circ x^*) + (x \circ x^*) \in Z(R)$) for all $x \in R$, then R is commutative.*

Proof. Using the same arguments as in the proof of Proposition 2.2.6 and applying Theorem 2.2.9 we obtain the desired conclusion. ■

As an application of Theorem 2.2.9 we have the following result.

Corollary 2.2.12 *Let $(R, *)$ be a 2-torsion free prime ring with involution of the second kind. If R admits a generalized derivation F associated with a derivation d such that $F \neq 0$ and $F \neq \pm id$, then the following assertions are equivalent:*

- (1) $F(x \circ y) \in Z(R)$ for all $x, y \in R$;
- (2) $F(x \circ y) \pm (x \circ y) \in Z(R)$ for all $x, y \in R$;
- (3) R is commutative.

Corollary 2.2.13 *Let $(R, *)$ be a 2-torsion free prime ring with involution of the second kind. If R admits a derivation d such that $d(x \circ x^*) - (x \circ x^*) \in Z(R)$ (resp. $d(x \circ x^*) + (x \circ x^*) \in Z(R)$) for all $x \in R$, then R is commutative.*

2.2.3 Identities with commutator and anti-commutator

Theorem 2.2.14 *Let $(R, *)$ be a 2-torsion free prime ring with involution of the second kind. If R admits a generalized derivation F , associated with a derivation d , then the following assertions are equivalent:*

- (1) $F(x \circ x^*) \pm [x, x^*] \in Z(R)$ for all $x \in R$;
- (2) $F([x, x^*]) \pm x \circ x^* \in Z(R)$ for all $x \in R$;
- (3) R is commutative.

Proof. It is obvious that (3) implies both (1) and (2).

(1) $\implies$ (3) Assume that

$$F(x \circ x^*) + [x, x^*] \in Z(R) \quad \text{for all } x \in R. \quad (2.32)$$

If $F = 0$, then our condition reduces to $[x, x^*] \in Z(R)$ for all $x \in R$ and R is commutative by ([63], Lemma 2.1). Hence, we may assume that $F \neq 0$.

If $d = 0$, then a linearization of (2.32) yields

$$F(x \circ y^*) + F(y \circ x^*) + [x, y^*] + [y, x^*] \in Z(R)$$

so that

$$F(x \circ y) + F(y^* \circ x^*) + [x, y] + [y^*, x^*] \in Z(R) \quad \text{for all } x, y \in R. \quad (2.33)$$

Replacing y by h in (2.33), where $h \in Z(R) \cap H(R) \setminus \{0\}$, we obtain

$$2(F(x) + F(x^*))h \in Z(R) \quad \text{for all } x \in R. \quad (2.34)$$

which leads to

$$F(x) + F(x^*) \in Z(R) \quad \text{for all } x \in R. \quad (2.35)$$

If in (2.33) we put $y = s$, where $s \in Z(R) \cap S(R) \setminus \{0\}$, we find that

$$2(F(x) - F(x^*))s \in Z(R) \quad \text{for all } x \in R \quad (2.36)$$

thereby obtaining

$$F(x) - F(x^*) \in Z(R) \quad \text{for all } x \in R. \quad (2.37)$$

Invoking (2.35), the last equation yields

$$F(x) \in Z(R) \quad \text{for all } x \in R \quad (2.38)$$

As Eq. (2.38) is the same as Eq. (2.26), arguing as in the proof of Theorem 2.2.9, we conclude that R is commutative.

Now assume that $d \neq 0$; replacing y by yh in (2.33), where $h \in Z(R) \cap H(R) \setminus \{0\}$, we obtain

$$(x \circ y + y^* \circ x^*)d(h) \in Z(R) \quad \text{for all } x, y \in R. \quad (2.39)$$

In light of primeness, the last equation implies that $x \circ y + y^* \circ x^* \in Z(R)$, in which case R is commutative by Lemma 2.2.2, or $d(h) = 0$.

Assume that $d(h) = 0$ for all $h \in Z(R) \cap H(R)$, then as we have already seen it follows that $d(s) = 0$ for all $s \in Z(R) \cap S(R)$.

Replacing y by s in (2.33), where $s \in Z(R) \cap S(R) \setminus \{0\}$, one can easily verify that

$$F(x) - F(x^*) \in Z(R) \quad \text{for all } x \in R. \quad (2.40)$$

Taking $y = h$, where $h \in Z(R) \cap H(R) \setminus \{0\}$ in (2.33), we get

$$F(x) + F(x^*) \in Z(R) \quad \text{for all } x \in R \quad (2.41)$$

and therefore $F(x) \in Z(R)$ for all $x \in R$. Accordingly, F is centralizing so R is commutative by ([67], Theorem 2).

Using similar arguments as above with a slight modification, it is easy to prove that

$$F(x \circ x^*) - [x, x^*] \in Z(R) \quad \text{for all } x \in R$$

assures the commutativity of R .

(2) $\implies$ (3) Suppose that

$$F([x, x^*]) + x \circ x^* \in Z(R) \quad \text{for all } x \in R \quad (2.42)$$

If $F = 0$ then our hypothesis reduces to $x \circ x^* \in Z(R)$ for all $x \in R$ and R is commutative by ([63], Lemma 2.2).

Hence we may assume that F is a nonzero generalized derivation.

Linearizing (2.42), we get

$$F([x, y^*]) + F([y, x^*]) + x \circ y^* + y \circ x^* \in Z(R) \quad \text{for all } x, y \in R. \quad (2.43)$$

Replacing y by s in the last equation, where $s \in Z(R) \cap S(R) \setminus \{0\}$, we obtain

$$x \circ s - s \circ x^* \in Z(R) \quad \text{for all } x \in R \quad (2.44)$$

and therefore

$$x - x^* \in Z(R) \quad \text{for all } x \in R. \quad (2.45)$$

Similarly, taking $y = h$ in (2.43), where $h \in Z(R) \cap H(R) \setminus \{0\}$, we find that

$$x + x^* \in Z(R) \quad \text{for all } x \in R. \quad (2.46)$$

Using (2.45) together with (2.46), we conclude that R is commutative and this completes our proof. ■

Corollary 2.2.15 *Let $(R, *)$ be a 2-torsion free prime ring with involution of the second kind. If R admits a nonzero generalized derivation F , then the following assertions are equivalent:*

- (1) $F(x \circ y) \pm [x, y] \in Z(R)$ for all $x, y \in R$;
- (2) $F([x, y]) \pm x \circ y \in Z(R)$ for all $x, y \in R$;
- (3) R is commutative.

Corollary 2.2.16 *Let $(R, *)$ be a 2-torsion free prime ring with involution of the second kind. If R admits a derivation d , then the following assertions are equivalent:*

- (1) $d(x \circ x^*) \pm [x, x^*] \in Z(R)$ for all $x \in R$;
- (2) $d([x, x^*]) \pm x \circ x^* \in Z(R)$ for all $x \in R$;
- (3) R is commutative.

2.2.4 Examples proving necessity of our conditions

The following example proves that the condition "'' $*$ is of the second kind'' is necessary in Theorems 2.2.3, 2.2.9 and 2.2.14.

Example 1.

Let us consider $R = M_2(\mathbb{Z})$ and $\begin{pmatrix} a & b \\ c & d \end{pmatrix}^* = \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$. It is straightforward to check that $(R, *)$ is a prime ring with involution of the first kind. Moreover, for all $x \in R$ we have

$$[x, x^*] = 0 \quad \text{and} \quad x \circ x^* = 2(ad - bc)I_2 \in Z(R)$$

and thus any generalized derivation satisfies the conditions of Theorems 2.2.3 and 2.2.14. For Theorem 2.2.9 it suffices to consider the generalized derivation $F = id$ with its associated derivation $d = 0$. However, R is a non commutative ring.

The following example proves that the primeness hypothesis in Theorems 2.2.3, 2.2.9 and 2.2.14 is not superfluous. In particular, our theorems cannot be extended to semi-prime rings.

Example 2.

Let us consider $(R, *)$ and F as in the preceding example. Let (S, σ) be a commutative ring with involution of the second kind (for example the field of complex numbers with the conjugation involution). If we set $\mathcal{R} = R \times S$, then it is obvious to verify that $(\mathcal{R}, \tau)$ is a semi-prime ring with involution of the second kind where

$$\tau(r, s) = (r^*, \sigma(s)) \quad \text{for all } (r, s) \in R \times S.$$

Moreover, if the generalized derivation F is associated with a derivation d , then the map

$$\begin{aligned} \mathcal{F} : \quad \mathcal{R} &\longrightarrow \mathcal{R} \\ (r, s) &\longmapsto (F(r), 0) \end{aligned}$$

is a generalized derivation associated with the derivation

$$\begin{aligned} D : \quad \mathcal{R} &\longrightarrow \mathcal{R} \\ (r, s) &\longmapsto (d(r), 0) \end{aligned}$$

Furthermore, $\mathcal{F}$ satisfies the conditions of Theorems 2.2.3, 2.2.9 and 2.2.14 but $\mathcal{R}$ is a non commutative ring.

CENTER-LIKE SUBSETS IN RINGS WITH DERIVATIONS OR ENDOMORPHISMS

M . A. Idrissi, L. Oukhtite and N. Muthana, Center-like subsets in rings with derivations or endomorphisms, Communications in Algebra, Vol. 47 (2019), no. 1, 3794-3799.

Abstract. Our purpose in this chapter is to show that certain subsets, defined by commutativity conditions involving derivations or endomorphisms, coincide with the center in prime rings. Moreover, we provide examples to show that the assumed restrictions cannot be relaxed.

3.1 Introduction

Many results in the literature assert that certain subsets of a ring R , defined by some sort of commutativity condition, must coincide with its center $Z(R)$. Herstein [38] proved that for a ring R with no nil ideals the hypercenter

$$S(R) = \{a \in R \mid ax^n = x^n a \text{ for all } x \in R, \text{ for some } n = n(a, x) \geq 1\}$$

coincides with the center $Z(R)$ of R . Motivated by Herstein's hypercenter, Chacron [23] introduced a more general concept that he called the *cohypercenter* $T(R)$ of R defined by

$$T(R) = \{a \in R \mid [a, x - x^2 p(x)] = 0 \text{ for all } x \in R, \text{ and } p(x) \in \mathbb{Z}[x] \text{ depends on } (a, x)\}.$$

More precisely, he proved ([23], Theorem 1) that the cohypercenter of a semiprime ring R is exactly the center of R . Herstein [42] introduced the set

$$H(R; d) = \{a \in R \mid ad(x) = d(x)a \text{ for all } x \in R\}$$

and proved that if R is a 2-torsion free prime ring and d a nonzero derivation, then $H(R; d) = Z(R)$.

Recall that for a nonempty subset S of a ring R , a map $T : R \rightarrow R$ is called strong commutativity preserving on S if $[T(x), T(y)] = [x, y]$ for all $x, y \in S$, where the symbol $[x, y]$ denote $xy - yx$. The concept of strong commutativity preserving maps was first introduced by Bell and Mason [15].

Recently, H. E. Bell and M. N. Daif [12] introduced and studied the following new center-like subsets :

$$Z^*(R; f) = \{y \in R \mid [x, y] = [f(y), f(x)] \text{ for all } x \in R\};$$

$$Z^{**}(R; f) = \{y \in R \mid [x, y] = [f(x), f(y)] \text{ for all } x \in R\};$$

$$Z_1(R; f) = \{y \in R \mid [f(x), f(y)] = [f(y), x] + [y, f(x)] \text{ for all } x \in R\}$$

where f is either an epimorphism or a derivation. In fact, they succeeded to compare each one of the above subsets with the center of R for the class of prime (semi-prime) rings with some additional assumptions.

In this chapter we continue this line of investigation regarding the study of center-like subsets in prime rings involving more general identities.

3.2 Center-like subsets involving derivations

Motivated by the results of [12], we introduced a more general concept that consists to consider the following subsets

$$Z^+(R; d) = \{y \in R \mid [d(x), d(y)] + [x, y] \in Z(R) \text{ for all } x \in R\};$$

$$Z^-(R; d) = \{y \in R \mid [d(x), d(y)] - [x, y] \in Z(R) \text{ for all } x \in R\};$$

$$Z^{*-}(R; d) = \{y \in R \mid [d(x), d(y)] - [y, d(x)] - [d(y), x] \in Z(R) \text{ for all } x \in R\}$$

where d is a derivation of R .

We start by recalling the following lemma that will be often used throughout this chapter.

Lemma 3.2.1 ([53, Theorem 1]) *Let $d \neq 0$ be a derivation of the prime ring R with $\text{char}(R) \neq 2$ and let $a \in R$ be such that $[a, d(R)] \subseteq Z(R)$. Then $a \in Z(R)$.*

Theorem 3.2.2 *Let R be a 2-torsion free prime ring and d a derivation of R . Then $Z^+(R; d) = Z(R)$.*

Proof. It's obvious to see that $Z(R) \subseteq Z^+(R; d)$, thus we only need to show that $Z^+(R; d) \subseteq Z(R)$. Let us fix $y \in Z^+(R; d)$, i.e

$$[d(x), d(y)] + [x, y] \in Z(R) \quad \text{for all } x \in R. \quad (3.1)$$

Substituting $(d(y))^2$ for x , we obtain

$$([d^2(y), d(y)] + [d(y), y])d(y) \in Z(R) \quad (3.2)$$

which, in light of primeness, yields that either $[d^2(y), d(y)] + [d(y), y] = 0$ or $d(y) \in Z(R)$.

If $d(y) \in Z(R)$, then equation (3.1) reduces to $[x, y] \in Z(R)$ for all $x \in R$ which leads to $y \in Z(R)$.

Now assume that $d(y) \notin Z(R)$ and $[d^2(y), d(y)] + [d(y), y] = 0$; replacing x by $xd(y)$ in (3.1), we get

$$([d(x), d(y)] + [x, y])d(y) + [x, d(y)]d^2(y) \in Z(R) \quad \text{for all } x \in R. \quad (3.3)$$

Similarly, substituting $d(y)x$ for x in (3.1), one can easily verify that

$$([d(x), d(y)] + [x, y])d(y) + d^2(y)[x, d(y)] \in Z(R) \quad \text{for all } x \in R. \quad (3.4)$$

Comparing (3.3) with (3.4), we obtain

$$[x, d(y)], d^2(y)] \in Z(R) \quad \text{for all } x \in R.$$

Let us define $f(x) = [x, d(y)]$ for all $x \in R$. Then f is a nonzero derivation of R , moreover the last relation reduces to $[f(x), d^2(y)] \in Z(R)$ for all $x \in R$ so that $[d^2(y), f(R)] \subseteq Z(R)$. Applying Lemma 3.2.1, we get $d^2(y) \in Z(R)$ and therefore $[d(y), y] = 0$.

Replacing x by xy in (3.1), one can see that

$$([d(x), d(y)] + [x, y])y + [x, d(y)]d(y) \in Z(R) \quad \text{for all } x \in R. \quad (3.5)$$

On the other hand, writing yx instead of x in (3.1), we obviously get

$$([d(x), d(y)] + [x, y])y + d(y)[x, d(y)] \in Z(R) \quad \text{for all } x \in R. \quad (3.6)$$

Subtracting (3.5) from (3.6), we obtain $[x, d(y)], d(y)] \in Z(R)$ for all $x \in R$. Accordingly, $[d(y), f(R)] \subseteq Z(R)$ and once again using Lemma 3.2.1 we are forced to conclude that $d(y) \in Z(R)$, a contradiction. Consequently, $y \in Z(R)$ proving that $Z^+(R; d) = Z(R)$. ■

A slight modification in the proof of Theorem 3.2.2 yields the following result.

Theorem 3.2.3 *Let R be a 2-torsion free prime ring and d a derivation of R . Then $Z^-(R; d) = Z(R)$.*

In [10], Bell and Daif investigated commutativity in rings admitting a derivation d strong commutativity preserving on a nonzero right ideal. More precisely, they proved that if a semi-prime ring R admits a derivation d satisfying $[d(x), d(y)] = [x, y]$ for all x, y in a right ideal I of R , then $I \subseteq Z(R)$. In particular, R is commutative if $I = R$. Applying our Theorems and considering the more general situation $[d(x), d(y)] \pm [x, y] \in Z(R)$, we get the following result.

Corollary 3.2.4 *Let R be a 2-torsion free prime ring. If R admits a derivation d , then the following assertions are equivalent:*

- (1) $[d(x), d(y)] + [x, y] \in Z(R)$ for all $x, y \in R$;
- (2) $[d(x), d(y)] - [x, y] \in Z(R)$ for all $x, y \in R$;
- (3) R is a commutative integral domain.

Theorem 3.2.5 *Let R be a 2-torsion free prime ring and d a nonzero derivation of R . Then $Z^{*-}(R; d) = Z(R)$.*

Proof. We only need to prove that $Z^{*-}(R; d) \subseteq Z(R)$; let $y \in Z^{*-}(R; d)$ that is

$$[d(x), d(y)] - [d(y), x] - [y, d(x)] \in Z(R) \quad \text{for all } x \in R. \quad (3.7)$$

Substituting xy for x in the above expression, we obtain

$$H_y(x)y + [x, d(y)]d(y) + d(x)[y, d(y)] - [y, x]d(y) \in Z(R) \quad \text{for all } x \in R \quad (3.8)$$

where $H_y(x) = [d(x), d(y)] - [d(y), x] - [y, d(x)]$.

Replacing x by yx in (3.7), one can see that

$$H_y(x)y + d(y)[x, d(y)] + [y, d(y)]d(x) - d(y)[y, x] \in Z(R) \quad \text{for all } x \in R. \quad (3.9)$$

Using (3.8) and (3.9) we obtain that

$$[[x, d(y) + y], d(y)] + [d(x), [y, d(y)]] \in Z(R) \quad \text{for all } x \in R. \quad (3.10)$$

Now Substituting $xd(y)$ for x in equation (3.7), we easily get

$$H_y(x)d(y) + x[d^2(y), d(y) + y] - d(x)[y, d(y)] + [x, d(y) + y]d^2(y) \in Z(R). \quad (3.11)$$

Replacing x by $d(y)x$ in (3.7), it's obvious to verify that

$$H_y(x)d(y) + [d^2(y), d(y) + y]x - [y, d(y)]d(x) + d^2(y)[x, d(y) + y] \in Z(R) \quad (3.12)$$

Putting $d(y)$ for x in (3.7), we may write $[d^2(y), d(y) + y] \in Z(R)$.

Using (3.11) together with (3.12), we are forced to conclude that

$$[[x, d(y) + y], d^2(y)] - [d(x), [y, d(y)]] \in Z(R) \quad \text{for all } x \in R \quad (3.13)$$

Invoking equation (3.10), the preceding relation forces

$$[x, d(y) + y], d^2(y) + d(y)] \in Z(R) \quad \text{for all } x \in R \quad (3.14)$$

If $d(y) + y \in Z(R)$, then (3.7) becomes $[d(y), x] \in Z(R)$ so that $d(y) \in Z(R)$.

Once again using (3.7) we get $[y, d(x)] \in Z(R)$ which leads to $y \in Z(R)$.

Now assume that $d(y) + y \notin Z(R)$; then using Lemma 3.2.1 together with (3.14) we are forced to conclude that $d^2(y) + d(y) \in Z(R)$. Putting $x = d(y) + y$, then $d(x) \in Z(R)$ and (3.7) forces $[d(y), y] \in Z(R)$. Invoking Lemma 3.2.1, from (3.13) it follows that $d^2(y) \in Z(R)$ and then $d(y) \in Z(R)$ which as above implies that $y \in Z(R)$. This completes our proof. ■

Now consider the set

$$Z^{*+}(R; d) = \{y \in R \mid [d(x), d(y)] + [y, d(x)] + [d(y), x] \in Z(R) \text{ for all } x \in R\};$$

Applying Theorem 3.2.5 for the derivation " $-d$ " we get the following result.

Theorem 3.2.6 *Let R be a 2-torsion free prime ring and d a nonzero derivation of R . Then $Z^{*+}(R; d) = Z(R)$.*

As a consequence of Theorems 3.2.5 and 3.2.6 we get:

Corollary 3.2.7 *Let R be a 2-torsion free prime ring and d a nonzero derivation of R . The following assertions are equivalent:*

- (1) $[d(x), d(y)] - d[x, y] \in Z(R)$ for all $x, y \in R$;
- (2) $[d(x), d(y)] + d[x, y] \in Z(R)$ for all $x, y \in R$;
- (3) R is a commutative integral domain.

The following example shows that Theorems 3.2.5 and 3.2.6 cannot be extended to semi-prime rings.

Example 1.

Consider the semi-prime ring $\mathcal{R} := M_2(\mathbb{Z}) \times \mathbb{Z}[X]$ and define

$$d : \begin{array}{ccc} \mathcal{R} & \longrightarrow & \mathcal{R} \\ (M, P) & \longmapsto & (0, P'). \end{array}$$

Then, d is a derivation of $\mathcal{R}$ moreover if we set $M = \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix}$ then $(M, 0) \notin Z(\mathcal{R})$. However, $(M, 0) \in Z^{*+}(R; d)$ and $(M, 0) \in Z^{*-}(R; d)$.

3.3 Center-like subsets involving endomorphisms

Bell and Daif ([12], Theorem 3.4) showed that if T is an epimorphism of a 2-torsion free prime ring R , then $Z^{**}(R; T) = Z(R)$ where

$$Z^{**}(R; T) = \{y \in R \mid [x, y] = [T(x), T(y)] \text{ for all } x \in R\}.$$

Our aim in the present section is to generalize this result of Bell and Daif in two directions. First of all, we will omit the condition that T is an epimorphism by assuming that T is an endomorphism. Secondly we will treat the more general algebraic identity $[x, y] - [T(x), T(y)] \in Z(R)$ rather than the particular case $[x, y] = [T(x), T(y)]$.

Let T be an endomorphism of a ring R and let us start by introducing the sets

$$Z^-(R; T) = \{y \in R \mid [T(x), T(y)] - [x, y] \in Z(R) \text{ for all } x \in R\};$$

$$Z^+(R; T) = \{y \in R \mid [T(x), T(y)] + [x, y] \in Z(R) \text{ for all } x \in R\}.$$

Theorem 3.3.1 *Let R be a 2-torsion free prime ring and T a non-trivial endomorphism of R . Then $Z^-(R; T) = Z(R)$.*

Proof. Clearly $Z(R) \subseteq Z^-(R; T)$ since $[T(x), T(z)] = T[x, z] = 0$; therefore we need only to show that $Z^-(R; T) \subseteq Z(R)$. Let $y \in Z^-(R; T)$, then

$$[T(x), T(y)] - [x, y] \in Z(R) \text{ for all } x \in R. \quad (3.15)$$

Replacing x by $[x, y]$ in (3.15) and using the fact that $[T([x, y]), T(y)] = [[x, y], T(y)]$, one can easily see that

$$[[x, y], T(y) - y] \in Z(R) \text{ for all } x \in R. \quad (3.16)$$

Invoking Lemma 3.2.1, it follows that either $y \in Z(R)$ or $T(y) - y \in Z(R)$.

Assume that $T(y) - y \in Z(R)$; substituting xy for x in (3.15), one can verify that $[[x, y], y](T(y) - y) \in Z(R)$ for all $x \in R$. Consequently, either $[[x, y], y] \in Z(R)$ in which case $y \in Z(R)$ or $T(y) = y$.

If $T(y) = y$, then (3.15) becomes

$$[T(x), y] - [x, y] \in Z(R) \text{ for all } x \in R. \quad (3.17)$$

Replacing x by xy in (3.17), we find that

$$([T(x), y] - [x, y])y \in Z(R) \text{ for all } x \in R \quad (3.18)$$

which assures that $[T(x), y] - [x, y] = 0$ for all $x \in R$.

Now substituting rx for x in the last equation, we obtain

$$(T(r) - r)[x, y] + [r, y](T(x) - x) = 0 \quad \text{for all } x, r \in R. \quad (3.19)$$

Putting $[r, y]$ for r in (3.19), it follows that $[[r, y], y](T(x) - x) = 0$ for all $x, r \in R$ which leads to $[[r, y], y]R(T(x) - x) = \{0\}$ for all $x, r \in R$. Using the primeness of R together with the assumption that T is nontrivial we are forced to conclude that $[[r, y], y] = 0$ for all $r \in R$ proving that $y \in Z(R)$. ■

Remark. If T is strong commutativity preserving, then $Z^-(R; T) = R$. Applying Theorem 3.3.1, we conclude that R is a commutative integral domain.

As an application of Theorem 3.3.1, we get an improved version of ([12], Theorem 3.4) without the assumption that T is an epimorphism.

Corollary 3.3.2 *Let R be a prime ring with $\text{char}(R) \neq 2$ and let T be an endomorphism of R which is not the identity map. Then $Z^{**}(R, T) = Z$.*

The following example proves that Theorem 3.3.1 cannot be extended to semi-prime rings.

Example 2.

Consider $\mathcal{R} := M_2(\mathbb{Z}) \times \mathbb{Z}$ and define

$$\begin{aligned} T : \quad \mathcal{R} &\longrightarrow \mathcal{R} \\ (M, z) &\longmapsto (M, 0). \end{aligned}$$

Clearly $\mathcal{R}$ is semi-prime, T is a non-identity map, and for $M = \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix}$, we have $(M, 0) \notin Z(\mathcal{R})$ but $(M, 0) \in Z^-(\mathcal{R}; T)$.

A slight modification in the proof of Theorem 3.3.1 yields the following result.

Theorem 3.3.3 *Let R be a 2-torsion free prime ring and T an endomorphism of R such that T^2 is not the identity map. Then $Z^+(R; T) = Z(R)$.*

In ([12], Theorem 3.6) it is proved that, if T is an epimorphism of a 2-torsion free prime ring R such that T^2 is not the identity map, then $Z^*(R, T) = Z$. Applying Theorem 3.3.3, we get an improved result as follows:

Corollary 3.3.4 *Let R be a prime ring with $\text{char}(R) \neq 2$ and T an endomorphism of R such that T^2 is not the identity map. Then $Z^*(R, T) = Z$.*

DERIVATIONS OVER AMALGAMATED ALGEBRAS ALONG AN IDEAL

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Abstract. Let $f : A \rightarrow B$ be a ring homomorphism and I be an ideal of B . The amalgamated algebra of A and B along I with respect to f is the subring of $A \times B$ given by

$$A \bowtie^f I := \{(a, f(a) + i) \mid a \in A, i \in I\}.$$

In this chapter, we give a complete description of derivations over $A \bowtie^f I$. The obtained results cover some recent results over duplication of a ring along an ideal. Furthermore, it's shown that if A is a prime ring with 1 and d is a nonzero derivation of $A \bowtie^f I$ such that, for each $x \in A \bowtie^f I$, $d(x) = 0$ or $d(x)$ is invertible in $A \bowtie^f I$, then A is a division ring and $A \bowtie^f I$ is identified to $D[x]/(x^2)$, where D is a division ring, $\text{char}(D) = 2$, $d(D) = \{0\}$ and $d(x) = 1 + ax$ for some $a \in Z(D)$.

4.1 Introduction

Throughout, all rings considered are associative. By an ideal I of a ring A , we shall always mean a two-sided ideal. A mapping $f : A \rightarrow A$, where A is an arbitrary ring, is called centralizing on A if $[f(x), x] \in Z(A)$ holds for all

$x \in A$; in the special case where $[f(x), x] = 0$ for all $x \in A$, the mapping f is said to be commuting on A . Let R and T be rings and let θ and ϕ be homomorphisms of T into R . Let X be an R -bimodule. Following [22], an additive mapping $d : T \rightarrow X$ is called a (θ, ϕ) -derivation if

$$d(xy) = d(x)\phi(y) + \theta(x)d(y) \text{ for all } x, y \in T.$$

Let A and B be two rings, I be an ideal of B and $f : A \rightarrow B$ be a ring homomorphism. In this setting, we can consider the following subring of $A \times B$:

$$A \bowtie^f I := \{(a, f(a) + i) \mid a \in A, i \in I\}$$

called the amalgamation of A and B along I with respect to f (see [28] and [29] for more details). Recall that the particular case called amalgamated duplication of A along I (introduced and studied in [34]) is the subring of $A \times A$ defined by

$$A \bowtie I := \{(a, a + i) \mid a \in A, i \in I\}.$$

Given the importance of some additive mappings and their relations with the global structure of the ring, a considerable amount of work has been done on this line, for example, derivations, homomorphisms and related maps during the last decades (see for example [3], [42] and [53]). A classical result of Posner (Posner's second theorem, see [68]) asserts that the existence of a nonzero derivation $d : A \rightarrow A$, where A is a prime ring, which is centralizing on A , forces A to be commutative. Recently, in [17], the authors examine the following problem: Suppose that R is a ring with unit element, 1, and that d is a derivation of R such that for every $x \in R$, $d(x) = 0$ or $d(x)$ is invertible in R ; must R then have a very special structure?

The aim of this chapter is to give a complete description of derivations in the amalgamation of A and B along I with respect to f . Furthermore, we will show that if A is prime then the structure of $A \bowtie^f I$ is very tightly determined by the imposition of a special behavior on one of its derivations. More precisely, we will prove that if d is a nonzero derivation of $A \bowtie^f I$ such that, for each $x \in A \bowtie^f I$, either $d(x) = 0$ or $d(x)$ is invertible in $A \bowtie^f I$, then A is a division ring and $A \bowtie^f I$ must be the ring $D[x]/(x^2)$, where D is a division ring, $\text{char } D = 2$, $d(D) = \{0\}$ and $d(x) = 1 + ax$ for some a in $Z(D)$.

4.2 Derivations of $A \bowtie^f I$

For clarity, it is interesting to elucidate some of the terminology to be used throughout this section. Let $f : A \rightarrow B$ be a ring homomorphism and I be

an ideal of B . By π_1 and π_2 we denote, respectively, the naturel surjections of $A \bowtie^f I$ into A and B defined by

$$\pi_1(a, f(a) + i) = a \quad \text{and} \quad \pi_2(a, f(a) + i) = f(a) + i \quad \text{for all } a \in A, i \in I.$$

For an additive map $d : A \bowtie^f I \rightarrow A \bowtie^f I$, we consider the maps $d_1 : A \rightarrow A$, $d_2 : A \rightarrow B$, $s_1 : I \rightarrow A$, and $s_2 : I \rightarrow B$ defined by

$$d_1(a) = \pi_1 \circ d(a, f(a)) \quad \text{and} \quad s_1(i) = \pi_1 \circ d(0, i).$$

$$d_2(a) = \pi_2 \circ d(a, f(a)) \quad \text{and} \quad s_2(i) = \pi_2 \circ d(0, i).$$

for all $a \in A$ and all $i \in I$. It is clear that d_1 , d_2 , s_1 and s_2 are additive.

The following lemma will be frequently used in the sequel.

Lemma 4.2.1 *Let $f : A \rightarrow B$ be a ring homomorphism and I be an ideal of B . A map $d : A \bowtie^f I \rightarrow A \bowtie^f I$ is a derivation if and only if $\pi_1 \circ d$ is a (π_1, π_1) -derivation and $\pi_2 \circ d$ is a (π_2, π_2) -derivation.*

Proof. $(\Rightarrow)$ Clearly, $\pi_i \circ d$ is additive for each $i \in \{1, 2\}$, and for all $x, y \in A \bowtie^f I$ we have

$$\pi_i \circ d(xy) = \pi_i(d(x)y + xd(y)) = \pi_i \circ d(x)\pi_i(y) + \pi_i(x)\pi_i \circ d(y).$$

Hence, $\pi_i \circ d$ is a (π_i, π_i) -derivation.

$(\Leftarrow)$ For all $a \in A$ and $i \in I$, we have

$$d(a, f(a) + i) = (\pi_1 \circ d(a, f(a) + i), \pi_2 \circ d(a, f(a) + i)).$$

Since $\pi_1 \circ d$ and $\pi_2 \circ d$ are additive then so is d . Consider $x = (a, f(a) + i)$ and $y = (b, f(b) + j)$ in $A \bowtie^f I$. We have

$$\begin{aligned} d(xy) &= (\pi_1 \circ d(xy), \pi_2 \circ d(xy)) \\ &= (\pi_1 \circ d(x)\pi_1(y) + \pi_1(x)\pi_1 \circ d(y), \pi_2 \circ d(x)\pi_2(y) + \pi_2(x)\pi_2 \circ d(y)) \\ &= (\pi_1 \circ d(x)b + a\pi_1 \circ d(y), \pi_2 \circ d(x)(f(b) + j) + (f(a) + i)\pi_2 \circ d(y)) \\ &= (\pi_1 \circ d(x), \pi_2 \circ d(x))y + x(\pi_1 \circ d(y), \pi_2 \circ d(y)) \\ &= d(x)y + xd(y). \end{aligned}$$

Hence, d is a derivation. ■

Next, we characterize an additive map d of $A \bowtie^f I$ to be a derivation.

Theorem 4.2.2 *Let $f : A \rightarrow B$ be a ring homomorphism, I be an ideal of B , and $d : A \bowtie^f I \rightarrow A \bowtie^f I$ be an additive map. Then, d is a derivation if and only if*

- 1) d_1 and s_2 are derivations.
- 2) d_2 is a (f, f) -derivation.
- 3) For all $a \in A$ and all $i \in I$ we have:

$$(a) \quad s_1(f(a)i) = as_1(i), \quad s_1(if(a)) = s_1(i)a, \quad \text{and} \quad s_1(ij) = 0.$$

$$(b) \quad s_2(f(a)i) = d_2(a)i + f(a)s_2(i), \quad \text{and} \quad s_2(if(a)) = s_2(i)f(a) + id_2(a).$$

Moreover, $d(a, f(a) + i) = (d_1(a) + s_1(i), d_2(a) + s_2(i))$ for all $(a, f(a) + i) \in A \bowtie^f I$.

Proof. $(\Rightarrow)$ Following Lemma 4.2.1, $\pi_1 \circ d$ is a (π_1, π_1) -derivation and $\pi_2 \circ d$ is a (π_2, π_2) -derivation. Hence, for all $a, b \in A$, we have

$$\begin{aligned} d_1(ab) &= \pi_1 \circ d(ab, f(ab)) \\ &= \pi_1 \circ d((a, f(a))(b, f(b))) \\ &= \pi_1 \circ d(a, f(a))b + a\pi_1 \circ d(b, f(b)) \\ &= d_1(a)b + ad_1(b), \end{aligned}$$

and

$$\begin{aligned} d_2(ab) &= \pi_2 \circ d(ab, f(ab)) \\ &= \pi_2 \circ d((a, f(a))(b, f(b))) \\ &= \pi_2 \circ d(a, f(a))f(b) + f(a)\pi_2 \circ d(b, f(b)) \\ &= d_2(a)f(b) + f(a)d_2(b). \end{aligned}$$

Now, for all $i, j \in I$, we have

$$\begin{aligned} s_2(ij) &= \pi_2 \circ d(0, ij) \\ &= \pi_2 \circ d((0, i)(0, j)) \\ &= \pi_2 \circ d(0, i)\pi_2(0, j) + \pi_2(0, i)\pi_2 \circ d(0, j) \\ &= s_2(i)j + is_2(j). \end{aligned}$$

Hence, (1) and (2) hold.

Let $a \in A$ and $i \in I$. We have

$$\begin{aligned} s_1(f(a)i) &= \pi_1 \circ d(0, f(a)i) \\ &= \pi_1 \circ d((a, f(a))(0, i)) \\ &= \pi_1 \circ d(a, f(a))\pi_1(0, i) + \pi_1(a, f(a))\pi_1 \circ d(0, i) \\ &= as_1(i). \end{aligned}$$

Similarly, $s_1(if(a)) = s_1(i)a$.

Now, for all $i, j \in I$, we have

$$\begin{aligned}
 s_1(ij) &= \pi_1 \circ d(0, ij) \\
 &= \pi_1 \circ d((0, i)(0, j)) \\
 &= \pi_1 \circ d(0, i)\pi_1(0, j) + \pi_1(0, i)\pi_1 \circ d(0, j) \\
 &= 0.
 \end{aligned}$$

Let $a \in A$ and $i \in I$, we have

$$\begin{aligned}
 s_2(f(a)i) &= \pi_2 \circ d(0, f(a)i) \\
 &= \pi_2 \circ d((a, f(a))(0, i)) \\
 &= \pi_2 \circ d(a, f(a))\pi_2(0, i) + \pi_2(a, f(a))\pi_2 \circ d(0, i) \\
 &= d_2(a)i + f(a)s_2(i),
 \end{aligned}$$

and

$$\begin{aligned}
 s_2(if(a)) &= \pi_2 \circ d(0, if(a)) \\
 &= \pi_2 \circ d((0, i)(a, f(a))) \\
 &= \pi_2 \circ d(0, i)\pi_2(a, f(a)) + \pi_2(0, i)\pi_2 \circ d(a, f(a)) \\
 &= s_2(i)f(a) + id_2(a).
 \end{aligned}$$

Thus, (3) holds.

($\Leftarrow$) For all $a \in A$ and $i \in I$, we have

$$\pi_1 \circ d(a, f(a) + i) = \pi_1 \circ d(a, f(a)) + \pi_1 \circ d(0, i) = d_1(a) + s_1(i),$$

and

$$\pi_2 \circ d(a, f(a) + i) = \pi_2 \circ d(a, f(a)) + \pi_2 \circ d(0, i) = d_2(a) + s_2(i).$$

Using Lemma 4.2.1, we need only to prove that $\pi_1 \circ d$ is a (π_1, π_1) -derivation and $\pi_2 \circ d$ is a (π_2, π_2) -derivation. Let $x = (a, f(a) + i)$ and $y = (b, f(b) + j)$ in $A \bowtie^f I$. We have

$$\begin{aligned}
 \pi_1 \circ d(xy) &= \pi_1 \circ d(ab, f(a)f(b) + f(a)j + if(b) + ij) \\
 &= d_1(ab) + s_1(f(a)j + if(b) + ij) \\
 &= d_1(a)b + ad_1(b) + as_1(j) + s_1(i)b \\
 &= (d_1(a) + s_1(i))b + a(d_1(b) + s_1(j)) \\
 &= \pi_1 \circ d(x)\pi_1(y) + \pi_1(x)\pi_1 \circ d(y),
 \end{aligned}$$

and

$$\begin{aligned}
 \pi_2 \circ d(xy) &= \pi_2 \circ d(ab, f(a)f(b) + f(a)j + if(b) + ij) \\
 &= d_2(ab) + s_2(f(a)j + if(b) + ij) \\
 &= d_2(a)f(b) + f(a)d_2(b) + d_2(a)j + f(a)s_2(i) + s_2(i)f(b) + id_2(b) \\
 &\quad + s_2(i)j + is_2(j) \\
 &= (d_2(a) + s_2(i))(f(b) + j) + (f(a) + i)(d_2(b) + s_2(j)) \\
 &= \pi_2 \circ d(x)\pi_2(y) + \pi_2(x)\pi_2 \circ d(y).
 \end{aligned}$$

Hence, we have the desired result. ■

Using Theorem 4.2.2 and its proof, we can easily deduce the following corollary that characterizes inner derivations of $A \bowtie^f I$.

Corollary 4.2.3 (Inner derivations of $A \bowtie^f I$) *Let A be a ring, I be an ideal of B , and $d : A \bowtie^f I \rightarrow A \bowtie^f I$ be an additive map. Then, d is the inner derivation induced by $(b, f(b) + j) \in A \bowtie^f I$ if and only if*

- 1) d_1 is the inner derivation induced by b .
- 2) d_2 is the (f, f) -inner derivation induced by $f(b) + j$ and $s_2(i) = [i, f(b) + j]$ for all $i \in I$.

Note that, for $f = 0$ and $I = B$, we get $A \bowtie^f I = A \times B$. Hence, the next result use Theorem 4.2.2 to characterize derivation of the direct product of rings A and B provided A (resp. B) is either a semi-prime or a unitary ring.

Corollary 4.2.4 *Let A (resp. B) be either a semi-prime or a unitary ring and $d : A \times B \rightarrow A \times B$ be an additive map. Then, d is a derivation if and only if there exist derivations $\alpha : A \rightarrow A$ and $\beta : B \rightarrow B$ such that*

$$d(a, b) = (\alpha(a), \beta(b)) \text{ for all } (a, b) \in A \times B.$$

Proof. $\Rightarrow$) Using Theorem 4.2.2, we have, for all $(a, b) \in A \times B$, $as_1(b) = 0$ and $bd_2(a) = 0$ since $f = 0$. Hence, since A (resp. B) is either a semi-prime or a unitary ring, we get $s_1 = 0$ and $d_2 = 0$. Thus, for all $(a, b) \in A \times B$, we have

$$d(a, b) = (d_1(a), s_2(b)).$$

So, it suffices to take $\alpha = d_1$ and $\beta = s_2$.

$\Leftarrow$ Clear. ■

Corollary 4.2.5 (Proposition 2.2, [56]) *Let A be a ring, I be an ideal of A , and $d : A \bowtie I \rightarrow A \bowtie I$ be an additive map. Then, d is a derivation if and only if*

- 1) d_1 , d_2 , and s_2 are derivations.
- 2) For all $a \in A$ and all $i \in I$ we have:
 - (a) $s_1(ai) = as_1(i)$, $s_1(ia) = s_1(i)a$, and $s_1(ij) = 0$.
 - (b) $s_2(ai) = d_2(a)i + as_2(i)$ and $s_2(ia) = s_2(i)a + id_2(a)$.

4.3 Derivations of $A \bowtie^f I$ in case A is prime

We start this section by treating the primeness and the semi-primeness of $A \bowtie^f I$.

Lemma 4.3.1 *Let $f : A \rightarrow B$ be a ring homomorphism and I be a nonzero ideal of B . Then, the following conditions are equivalent:*

- 1) $A \bowtie^f I$ is a prime ring.
- 2) $f(A) + I$ is a prime ring and $f^{-1}(I) = \{0\}$.

Proof. 1) $\implies$ 2) Let $a \in f^{-1}(I)$. For each $(b, f(b) + j) \in A \bowtie^f I$ and i an nonzero element of I , we have $(a, 0)(b, f(b) + j)(0, i) = (0, 0)$. Hence, since $A \bowtie^f I$ is prime, $(a, 0) = (0, 0)$. Thus $f^{-1}(I) = \{0\}$. In this case, $A \bowtie^f I \cong f(A) + I$ (by (Proposition 5.1(3), [26])), and so $f(A) + I$ is prime.
 2) $\implies$ 1) It is obvious since $f^{-1}(I) = \{0\}$ implies that $A \bowtie^f I \cong f(A) + I$. ■

We leave the proof of the following easy lemma to the reader.

Lemma 4.3.2 *Let $f : A \rightarrow B$ be an homomorphism and I an ideal of B such that $f^{-1}(I) = \{0\}$. Then, the following conditions are equivalent:*

- 1) $A \bowtie^f I$ is a semi-prime ring.
- 2) $f(A) + I$ is semi-prime.

The following example proves that the condition $f^{-1}(I) = \{0\}$ in Lemma 4.3.2 is crucial.

Example 2.

Consider the naturel surjection $f : \mathbb{Z}_{18} \rightarrow \mathbb{Z}_6$ and set $I = 3\mathbb{Z}_6$. The ring

$$f(\mathbb{Z}_{18}) + I = \mathbb{Z}_6 + I = \mathbb{Z}_6$$

is semi-prime (but not prime). However, $\mathbb{Z}_{18} \bowtie^f I$ is not semi-prime. Indeed, for each $x \in \mathbb{Z}_{18} \bowtie^f I$ we have $(\bar{6}, \tilde{0})x(\bar{6}, \tilde{0}) = (\bar{0}, \tilde{0})$.

Remark. Note that $A \bowtie I$, the amalgamated duplication of the ring A along the ideal I , is never prime unless $I = 0$. However, a standard argument shows that $A \bowtie I$ is a semi-prime ring if and only if A is semi-prime.

The following proposition shows that derivations of $A \bowtie^f I$, in case A is a prime ring, are with a special kind.

Proposition 4.3.3 *Let A be a prime ring, $f : A \rightarrow B$ be a homomorphism of rings, and I be an ideal of B with $f^{-1}(I) \neq \{0\}$. An additive map $d : A \bowtie^f I \rightarrow A \bowtie^f I$ is a derivation if and only if there exist derivations $\alpha : A \rightarrow A$ and $s : I \rightarrow I$ and an (f, f) -derivation $\beta : A \rightarrow B$ such that $s(f(a)i) = \beta(a)i + f(a)s(i)$, $s(if(a)) = s(i)f(a) + i\beta(a)$, and*

$$d(a, f(a) + i) = (\alpha(a), \beta(a) + s(i)) \text{ for all } a \in A, i \in I.$$

Proof. Suppose that d is a derivation, and let b be a nonzero element of $f^{-1}(I)$. By Theorem 4.2.2, we have

$$0 = s_1(f(b)f(a)i) = bas_1(i) \text{ for all } a \in A, i \in I.$$

Consequently $s_1 = 0$. Hence, the desired form following by taken $\alpha = d_1$, $s = s_2$, and $\beta = d_2$.

The converse implication is clear. ■

4.4 Applications

In order to study the structure of $A \bowtie^f I$ we need to determine the center of an amalgamated algebra. We leave the proof of the following easy lemma to the reader.

Lemma 4.4.1 *Let $f : A \rightarrow B$ be a ring homomorphism and I be an ideal of B . Then,*

$$Z(A \bowtie^f I) = A \bowtie^f I \cap (Z(A) \times Z(f(A) + I)).$$

In particular,

$$Z(A \bowtie^f I) = Z(A) \bowtie^f I \text{ if and only if } I \subset Z(f(A) + I).$$

Proposition 4.4.2 *Let $f : A \rightarrow B$ be a ring homomorphism and I be an ideal of B such that $I \subseteq Z(f(A) + I)$. If A is a prime ring, J a nonzero ideal of A and d a derivation of $A \bowtie^f I$ satisfying $[d(x), x] \in Z(A \bowtie^f I)$ for all $x \in J \bowtie^f I$, then either $A \bowtie^f I$ is a commutative ring or $\pi_1 \circ d = 0$.*

Proof. It's obvious to verify that d is centralizing if and only if $\pi_1 \circ d$ and $\pi_2 \circ d$ are centralizing derivations.

Since A is prime, if $J \subseteq Z(A)$ then A is a commutative integral domain and Lemma 4.4.1 yields that $A \bowtie^f I$ is a commutative ring. Hence we may assume that $J \not\subseteq Z(A)$.

As $\pi_1 \circ d$ is centralizing, then $[d_1(a) + s_1(i), a] \in Z(A)$ for all $a \in J$ and $i \in I$. In particular, $[d_1(a), a] \in Z(A)$ so that d_1 is centralizing on J .

If $d_1 \neq 0$, then invoking [14, Theorem 3] it follows that $Z(A)$ contains the nonzero ideal $d_1(J)$ and therefore A is commutative. So is $A \bowtie^f I$.

Now, assume that $d_1 = 0$ and let us fix an element $a \in J \setminus Z(A)$. We are given that $[s_1(i), a] \in Z(A)$ for all $i \in I$. Replacing i by $if(a)$, we obtain $[s_1(i), a]a \in Z(A)$ for all $i \in I$. Consequently $[s_1(i), a] = 0$ for all $i \in I$. Then, for all $b \in A$ and $i \in I$

$$s_1(i)[b, a] = [s_1(i)b, a] = [s_1(if(b)), a] = 0$$

which leads to $s_1 = 0$. Since $d(a, f(a) + i) = (d_1(a) + s_1(i), d_2(a) + s_2(i))$ for all $(a, f(a) + i) \in A \bowtie^f I$ (by Theorem 4.2.2), then $\pi_1 \circ d = 0$. ■

It is proved in ([17], Theorem) that, if R is a ring with 1 and d a derivation of R such that, for each $x \in R$, $d(x) = 0$ or $d(x)$ is invertible in R , then R is either

1. a division ring D , or
2. $M_2(D)$ where D is a division ring, or
3. $D[x]/(x^2)$ where D is a division ring with $\text{char} D = 2$, $d(D) = \{0\}$, and $d(x) = 1 + ax$, for some a in $Z(D)$.

The following theorem treats the result of [17] over $R = A \bowtie^f I$. Our result is with a different kind because the structures of both $A \bowtie^f I$ and A are more precise. Indeed, we will prove $A \bowtie^f I$ is necessarily $D[x]/(x^2)$ and A is a division ring.

Theorem 4.4.3 *Let $f : A \rightarrow B$ be a ring homomorphism and I be a nonzero ideal of B . If A is a prime ring with 1 and d a nonzero derivation of $A \bowtie^f I$ such that, for each $x \in A \bowtie^f I$, $d(x) = 0$ or $d(x)$ is invertible in $A \bowtie^f I$. Then A is a division ring and $A \bowtie^f I$ is identified to $D[x]/(x^2)$, where D is a division ring, $\text{char} D = 2$, $d(D) = \{0\}$ and $d(x) = 1 + ax$ for some a in $Z(D)$.*

Proof. If $A \bowtie^f I$ is a division ring D or $M_2(D)$, then the only prime ideal of $A \bowtie^f I$ is (0) ([75, Theorem 1]). However, $A \cong \frac{A \bowtie^f I}{0 \times I}$ is prime, and so $0 \times I$ is a prime ideal of $A \bowtie^f I$. Thus, $I = (0)$, a contradiction. Hence, $A \bowtie^f I$ can be seen as $D[x]/(x^2)$, where D is a division ring, $\text{char} D = 2$, $d(D) = \{0\}$ and $d(x) = 1 + ax$ for some a in the center Z of D .

Since $D[x]/(x^2)$ admit only one ideal, this is also the case for $A \bowtie^f I$. Hence, if J an ideal of A , then $J \bowtie^f I = 0 \bowtie^f I = 0 \times I$. Thus, $J = (0)$. Consequently, A is a division ring. ■

PRIME IDEALS AND GENERALIZED DERIVATIONS, INVARIANCE OF MINIMAL PRIME IDEALS UNDER DERIVATIONS

Article soumis

Abstract. The purpose of this chapter is to continue the line of investigation regarding the relationship between the structure of a ring R and certain special types of maps on R . More precisely, we introduced a class of additive mappings whose concept covers the ones of centralizing and commuting mappings. In this direction, our approach consists to consider algebraic identities involving a prime ideal without the primeness (semiprimeness) assumption on the considered ring. As an application of our results, we will study the problem of invariance of minimal prime ideals of a semiprime ring under derivations.

5.1 Introduction

Throughout, R denotes an associative ring with center $Z(R)$. Recall that an ideal P of R is said to be prime if $P \neq R$ and for $x, y \in R$, $xRy \subseteq P$ implies that $x \in P$ or $y \in P$. A prime ideal P of R minimal if P does not properly include any prime ideals of R . Therefore, R is called a prime ring if and only if (0) is the only minimal prime ideal of R .

Over the last years, several authors have investigated the relationship between the commutativity, the structure of the ring R and certain special types of maps on R (viz.; [17], [39], [74]). Herstein [41] showed that a prime ring R with nonzero derivation d satisfying $d(x)d(y) = d(y)d(x)$ for all $x, y \in R$, must be a commutative integral domain if its characteristic is not two, and, if the characteristic equals two, must be commutative or an order in a simple algebra which is 4-dimensional over its center. Several authors have proved commutativity theorems for prime rings admitting derivations which are centralizing on R . We first recall that a mapping $f : R \rightarrow R$ is called centralizing on R if $[f(x), x] \in Z(R)$ for all $x \in R$; in the special case where $[f(x), x] = 0$ for all $x \in R$, the mapping f is said to be commuting on R . In [68], Posner proved that if a prime ring R admits a nonzero derivation d such that $[d(x), x] \in Z(R)$ for all $x \in R$, then R is commutative. Since then many authors have extended the Posner's result in several directions. A considerable number of researchers have investigated the ideals (Jordan ideal, Lie ideal...) in rings as well as the commutativity of rings that consider derivations and generalized derivations. For example, in [52] Lanski generalizes the result of Posner by considering a derivation d such that $[d(x), x] \in Z(R)$ for all x in a nonzero Lie ideal U of R .

One may observe that the concept of generalized derivations includes the concept of derivations, also of the left multipliers when $d = 0$. Hence it should be interesting to extend some results concerning these notions to generalized derivations. More specifically an additive map $F : R \rightarrow R$ is said to be a generalized derivation if there exists a derivation d of R such that $F(xy) = F(x)y + xd(y)$ for all $x, y \in R$. Generalized derivations have been primarily studied on operator algebras thus any investigation from the algebraic point of view might be interesting (see for example [30], [44], [54] and [67]).

In the present chapter we will adopt a new approach by considering algebraic identities involving a prime ideal without the primeness (semi-primeness) assumption on the considered ring. Furthermore, as an application of our results, we will study the problem of invariance of minimal prime ideals of a semi-prime ring under derivations.

5.2 P-Centralizing and P-Commuting maps

The motivation for this section lies in an attempt to extend in some way the well known first Posner's Theorem contained in [68]: there Posner proved that the existence of a nonzero centralizing derivation on a prime ring forces the ring to be commutative. Moreover, this result of Posner has been extended

by several authors in various directions.

Our purpose here is to continue this line of investigation by considering a class of additive mappings whose concept covers the ones of centralizing and commuting mappings. More precisely, motivated by the notion of centralizing and commuting mappings, we initiate the concepts of I -centralizing and I -commuting mappings where I is an ideal of the considered ring as follows.

Definition. Let I be an ideal of a ring R .

- 1) A map $f : R \rightarrow R$ is I -centralizing if $\overline{[f(x), x]} \in Z(R/I)$ for all $x \in R$ or equivalently $[[F(x), x], y] \in I$ for all $x, y \in R$.
- 2) A map $f : R \rightarrow R$ is I -commuting if $[f(x), x] \in I$ for all $x \in R$.

We start with the following observations:

Remark.

- 1) A map f is (0)-centralizing (resp. (0)-commuting) if and only if f is centralizing (resp. commuting).
- 2) if a map f is commuting then f is I -commuting.
- 3) if a map f is centralizing then f is I -centralizing.

The fundamental aim of the following example is to show that these new notions does not coincide with the classical concepts of centralizing and commuting mappings.

Example.

Let R be a non-commutative prime ring, I be a nonzero ideal of R and $d(x) = ax - xa$ for all $x \in R$ where a is an element of $I \setminus Z(R)$.

- 1) d is I -commuting but non-commuting.
- 2) d is I -centralizing but non-centralizing.

Our goal in the following theorem is to confirm that there is a relationship between the structure of the ring R/P and generalized derivations of R that are P -centralizing, where P is a prime ideal of R .

Theorem 5.2.1 *Let P be a prime ideal of a ring R and F be a generalized derivation of R associated with a derivation d . If F is P -centralizing then $d(R) \subseteq P$ or R/P is a commutative integral domain.*

Proof. We are given that

$$\overline{[F(x), x]} \in Z(R/P) \quad \text{for all } x \in R. \quad (5.1)$$

If $Z(R/P) \neq \{\bar{0}\}$, then there exists $\bar{z}_0 \in Z(R/P)$ and $z_0 \notin P$. A linearization of the above relation leads to

$$\overline{[F(x+z_0), x+z_0]} = \overline{[F(x), \bar{z}_0]} + \overline{[F(z_0), \bar{x}]} \in Z(R/P) \quad \text{for all } x \in R \quad (5.2)$$

thereby obtaining

$$\overline{[F(z_0), \bar{x}]} \in Z(R/P) \quad \text{for all } x \in R. \quad (5.3)$$

If we replace x by $xF(z_0)$, the primeness of R/P assures that

$$\overline{F(z_0)} \in Z(R/P).$$

Replacing x by z_0x in (5.1), we get

$$\overline{[F(z_0)x, \bar{z}_0\bar{x}]} + \overline{[z_0d(x), z_0x]} \in Z(R/P) \quad \text{for all } x \in R \quad (5.4)$$

and therefore

$$\overline{[d(x), x]} \bar{z}_0^2 \in Z(R/P) \quad \text{for all } x \in R. \quad (5.5)$$

Since $\bar{z}_0^2 \neq \bar{0}$, because of $z_0 \notin P$, it follows that $[[d(x), x], y] \in P$ for all $x \in R$. Invoking ([5], Theorem 2.2), then we conclude that $d(R) \subseteq P$ or R/P is commutative.

Now assume that $Z(R/P) = \{\bar{0}\}$, thus our hypothesis reduces to

$$[F(x), x] \in P \quad \text{for all } x \in R. \quad (5.6)$$

Linearizing equation (5.6), we obviously get

$$[F(x), y] + [F(y), x] \in P \quad \text{for all } x, y \in R. \quad (5.7)$$

Replacing y by yx in (5.7), one can easily verify that

$$y[d(x), x] + [y, x]d(x) \in P \quad \text{for all } x, y \in R. \quad (5.8)$$

Taking $y = ry$ in (5.8), we have

$$ry[d(x), x] + r[y, x]d(x) + [r, x]yd(x) \in P \quad \text{for all } x, y, r \in R. \quad (5.9)$$

Right multiplying (5.8) by r and comparing with (5.9) we may write

$$[r, x]yd(x) \in P \quad \text{for all } x, y, r \in R. \quad (5.10)$$

Since P is prime it follows that for all $x \in R$ either $[r, x] \in P$ or $d(x) \in P$. Consequently, R is a union of two additive subgroups G_1 and G_2 , where

$$G_1 = \{x \in R \mid d(x) \in P\} \quad \text{and} \quad G_2 = \{x \in R \mid [R, x] \subset P\}.$$

Since a group cannot be a union of two of its proper subgroups, we are forced to conclude that $R = G_1$ or $R = G_2$. Therefore, $d(R) \subset P$ or R/P is a commutative integral domain. ■

In ([1], Theorem 3.7), it is proved that if a prime ring R admits a nonzero generalized derivation F associated with a nonzero derivation d such that F is centralizing, then R is commutative. With the further assumption that the ring is 2-torsion free, Oukhtite and Mamouni in [67] obtained the commutativity of the ring by considering a local condition. More precisely, they proved that if the generalized derivation is assumed to be centralizing on a nonzero Jordan ideal J rather than on the whole ring, then R is a commutative integral domain.

As an application of Theorem 5.2.1, we get the following results with no further assumption on the characteristic of R .

Corollary 5.2.2 [1, Theorem 3.7] *Let R be a prime ring. If R admits a generalized derivation F associated with a nonzero derivation d such that $[F(x), x] \in Z(R)$ for all $x \in R$, then R is a commutative integral domain.*

Corollary 5.2.3 [68, Theorem 2.] *Let R be a prime ring. If R admits a nonzero derivation d such that $[d(x), x] \in Z(R)$ for all $x \in R$, then R is a commutative integral domain.*

5.3 Algebraic identities involving prime ideal

In [10], Bell and Daif investigated commutativity in rings admitting a derivation d strong commutativity preserving (SCP) on a nonzero right ideal. More precisely, they proved that if a semi-prime ring R admits a derivation d satisfying $[d(x), d(y)] - [x, y] = 0$ for all $x, y \in R$, then R is commutative.

Motivated by the above result we here investigate a more general context of differential identities involving a prime ideal by omitting the semi-primeness assumption imposed on the ring. This approach allows us to generalize the preceding result of Bell and Daif in two directions. First of all, we will assume that $[d(x), d(y)] - [x, y]$ belong to a prime ideal P rather than $[d(x), d(y)] - [x, y] = 0$. Secondly we will investigate the behaviour of the more general expression $[F(x), d(y)] - [x, y]$ involving a generalized derivation instead of the expression $[d(x), d(y)] - [x, y]$.

Theorem 5.3.1 *Let R be a ring and P a prime ideal of R . If R admits a generalized derivation F associated with a derivation d , then the following conditions are equivalent:*

- 1) $[F(x), d(y)] + [x, y] \in P$ for all $x, y \in R$,
- 2) $[F(x), d(y)] - [x, y] \in P$ for all $x, y \in R$,
- 3) R/P is a commutative integral domain.

Proof. It's obvious that (3) implies both of (1) and (2).

(1) $\Rightarrow$ (3) Assume that

$$[F(x), d(y)] + [x, y] \in P \quad \text{for all } x, y \in R. \quad (5.11)$$

If $d(R) \subseteq P$, then R/P is commutative so we may suppose that $d(R) \not\subseteq P$. Substituting ry for y in (5.11) and using (5.11), we obtain

$$d(r)[F(x), y] + [F(x), r]d(y) \in P \quad \text{for all } x, y, r \in R. \quad (5.12)$$

Replacing r by $d(x)r$ in (5.12), we get

$$d^2(x)r[F(x), y] + d(x)d(r)[F(x), y] + d(x)[F(x), r]d(y) \in P. \quad (5.13)$$

Right multiplying (5.12) by $d(x)$ and comparing with equation (5.13) we find that

$$d^2(x)r[F(x), y] \in P \quad \text{for all } x, y, r \in R.$$

In light of primeness of P , the last relation yields that

$$[F(x), y] \in P \quad \text{or} \quad d^2(x) \in P \quad \text{for all } x, y \in R. \quad (5.14)$$

Let us set $R_1 = \{x \in R \mid [F(x), R] \subseteq P\}$ and $R_2 = \{x \in R \mid d^2(x) \in P\}$. Clearly R_1 and R_2 are additive subgroups of R such that $R = R_1 \cup R_2$. According to Brauer's trick, it follows that either $R = R_1$ or $R = R_2$.

Assume that $R = R_1$, then $[F(x), y] \in P$ for all $x, y \in R$ and therefore

$$[F(x), x] \in P \quad \text{for all } x \in R. \quad (5.15)$$

Since equation (5.15) is the same as equation (5.6), reasoning in the same manner as in the proof of Theorem 5.2.1, we get the required result.

Assume now that $d^2(R) \subseteq P$; replacing y by $d(x)$ in (5.11) we obviously get

$$[d(x), x] \in P \quad \text{for all } x \in R.$$

Since d is a generalized derivation associated with d , hence the last equation is exactly equation (5.15) for the particular generalized derivation $\mathcal{F} = d$. Arguing as above, we conclude that R/P is a commutative integral domain.

(2) $\Rightarrow$ (3) Follows from the first implication with a slight modification. ■

Let R be a prime ring. Letting $P = (0)$ in the previous theorem, we deduce that, if $[F(x), d(y)] + [x, y] = 0$ (resp. $[F(x), d(y)] - [x, y] = 0$) for all $x, y \in R$, then R is commutative. The following corollary shows that the same conclusion remains satisfied for semi-prime rings.

Corollary 5.3.2 *Let R be a semi-prime ring and F a generalized derivation associated with a derivation d , then the following assertions are equivalent:*

- 1) $[F(x), d(y)] + [x, y] = 0$ for all $x, y \in R$,
- 2) $[F(x), d(y)] - [x, y] = 0$ for all $x, y \in R$,
- 3) R is a commutative integral domain.

Proof. For the non-trivial implication assume that $[F(x), d(y)] + [x, y] = 0$ for all $x, y \in R$. According to semi-primeness, there exists a family Γ of prime ideals such that $\bigcap_{P \in \Gamma} P = (0)$ thereby obtaining $[F(x), d(y)] + [x, y] \in P$ for all $P \in \Gamma$. Invoking Theorem 5.3.1, we conclude that R/P is a commutative integral domain which, because of $\bigcap_{P \in \Gamma} P = (0)$, assures that R is commutative. Similarly, if $[F(x), d(y)] - [x, y] = 0$ for all $x, y \in R$, then the same reasoning proves the required result. ■

The aim of the next theorem is to study the case when the generalized derivation satisfies some conditions involving commutator and anti-commutator. More specifically, we will treat the special identities

$$\left(F(x) \circ d(y) - x \circ y \in P \text{ for all } x, y \in R \right) \text{ and } \left(F(x) \circ d(y) + x \circ y \in P \text{ for all } x, y \in R \right).$$

Our results are of a different kind because we will characterize not only the structure of the ring R/P but we will also determine its characteristic. More precisely we will prove the following result.

Theorem 5.3.3 *Let R be a ring, P a prime ideal of R and F a generalized derivation of R associated with a derivation d . If F satisfies one of the following conditions:*

- 1) $F(x) \circ d(y) + x \circ y \in P$ for all $x, y \in R$,
- 2) $F(x) \circ d(y) - x \circ y \in P$ for all $x, y \in R$,
- 3) $[F(x), d(y)] + x \circ y \in P$ for all $x, y \in R$,
- 4) $[F(x), d(y)] - x \circ y \in P$ for all $x, y \in R$,

then R/P is a commutative integral domain and $\text{char}(R/P) = 2$.

Proof. (1) We are given that

$$F(x) \circ d(y) + x \circ y \in P \quad \text{for all } x, y \in R. \quad (5.16)$$

Replacing x by $xd(y)$ in (5.16), we obtain

$$xd^2(y)d(y) + d(y)xd^2(y) + x[d(y), y] \in P \quad \text{for all } x, y \in R. \quad (5.17)$$

As a special case of (5.17) when $x = rx$, we may write

$$rx d^2(y)d(y) + d(y)rx d^2(y) + rx[d(y), y] \in P \quad \text{for all } x, y, r \in R. \quad (5.18)$$

Right multiplying (5.17) by r and comparing with (5.18), we conclude that

$$[d(y), r]xd^2(y) \in P \quad \text{for all } x, y, r \in R$$

and by the primeness of P either $[d(y), r] \in P$ for all $y, r \in R$ or $d^2(R) \subseteq P$. In the first case, it's obvious that $[[d(x), x], y] \in P$ for all $x, y \in R$ proving that d is P -centralizing. Since d is a generalized derivation associated with d itself, by virtue of Theorem 5.2.1 either $d(R) \subset P$ or R/P is an integral domain.

Now assume that $d^2(R) \subseteq P$; substituting $d(y)$ for y in (5.16), we easily obtain

$$x \circ d(y) \in P \quad \text{for all } x, y \in R. \quad (5.19)$$

Writing zrx instead of x in the last relation, one can obviously verify that $zr[x, d(y)] + ((zr) \circ d(y))x \in P$ which, in light of (5.19), yields that $zR[x, d(y)] \subset P$. In particular, we may write

$$[x, d(y)]R[x, d(y)] \in P \quad \text{for all } x, y \in R.$$

Since P is prime, it follows that $[x, d(y)] \in P$ for all $x, y \in R$. Arguing as above, we have either $d(R) \subseteq P$ or R/P is a commutative integral domain.

Suppose $d(R) \subseteq P$; then Eq.(5.16) reduces to $x \circ y \in P$ for all $x, y \in R$. Replacing y by yr we find that $[x, y]r \in P$ and using the fact that a prime ideal is proper, we conclude that $[R, R] \subset P$ so that R/P is commutative. Consequently, in both cases R/P is a commutative integral domain.

Finally, we claim that R/P is with characteristic 2. Indeed, assume that R/P is 2-torsion free. As we have already seen that R/P is commutative then Eq.(5.16) assures that $2(F(x)d(y) + xy) \in P$ for all $x, y \in R$. If we replace y by ry , we get $F(x)rd(y) \in P$ for all $x, y, r \in R$. Accordingly, $d(R) \subseteq P$. Once again using 2-torsion freeness together with equation (5.16), we find that $xy \in P$ for all $x, y \in R$ and thus $P = R$, a contradiction. Consequently, R/P is an integral domain with characteristic 2.

(2) If $F(x) \circ d(y) - x \circ y \in P$ for all $x, y \in R$, then using similar argument as

in the first case with a suitable slight modification, we get the required result.

(3) Assume that

$$[F(x), d(y)] + x \circ y \in P \quad \text{for all } x, y \in R. \quad (5.20)$$

Substituting $xd(y)$ for x in (5.20), we obtain

$$xd^2(y)d(y) + x[d(y), y] - d(y)xd^2(y) \in P \quad \text{for all } x, y \in R. \quad (5.21)$$

In particular, setting $x = rx$ in (5.21) we have

$$rx d^2(y)d(y) + rx[d(y), y] - d(y)rx d^2(y) \in P \quad \text{for all } x, y, r \in R. \quad (5.22)$$

Right multiplying (5.21) by r and combining with (5.22), we obviously get

$$[d(y), r]xd^2(y) \in P \quad \text{for all } r, x, y \in R. \quad (5.23)$$

Invoking the primeness of P , the last relation assures that either $[d(y), r] \in P$ for all $y, r \in R$ or $d^2(R) \subseteq P$. By an argument similar to that in the proof of (1), we again conclude that either $d^2(R) \subseteq P$ or R/P is commutative.

Assume that $d^2(R) \subseteq P$; replacing y by $d(y)$ in (5.20) we obtain

$$x \circ d(y) \in P \quad \text{for all } x, y \in R$$

this relation is exactly (5.19), then as above R/P is commutative or $d(R) \subseteq P$. In the later case, our hypothesis reduces to $x \circ y \in P$ for all $x, y \in R$ therefore in both cases R/P is a commutative integral domain.

Suppose that R/P is a 2-torsion free prime ring; as we have already seen that R/P is commutative then our hypothesis becomes $x \circ y \in P$ for all $x, y \in R$ and therefore $P = R$, contradicting the fact that a prime ideal is proper. Consequently, R/P is necessarily with characteristic 2.

(4) Assume that $[F(x), d(y)] - x \circ y \in P$ for all $x, y \in R$, then using the same techniques as used above with necessary variations we get the required result. ■

In ([43], Proposition 1), M. Hongan proved that if R is a 2-torsion free ring with identity 1, then there is no derivation d of R such that $d(x \circ y) + x \circ y = 0$ for all $x, y \in R$ or $d(x \circ y) - x \circ y = 0$ for all $x, y \in R$. Applying Theorem 5.3.3, we get a similar result as follows:

Corollary 5.3.4 *Let R be a 2-torsion free ring which is either a semi-prime or a unitary ring. There is no generalized derivation F associated with a*

derivation d satisfying one of the following conditions:

- 1) $F(x) \circ d(y) + x \circ y = 0$ for all $x, y \in R$,
- 2) $F(x) \circ d(y) - x \circ y = 0$ for all $x, y \in R$,
- 3) $[F(x), d(y)] + x \circ y = 0$ for all $x, y \in R$,
- 4) $[F(x), d(y)] - x \circ y = 0$ for all $x, y \in R$.

Proof. Assume that R is a 2-torsion free semi-prime ring. Then there exists a family Γ of prime ideals such that $\bigcap_{P \in \Gamma} P = (0)$. Since the proof is similar and does not depend on the chosen identity we may assume existence of a generalized derivation (F, d) satisfying 1) that is $F(x) \circ d(y) + x \circ y = 0$ for all $x, y \in R$. Accordingly, $F(x) \circ d(y) + x \circ y \in P$ for all $P \in \Gamma$ and Theorem 5.3.3 yields that R/P is commutative. Therefore, for all $x, y \in R$ we have $[x, y] \in P$ (for all $P \in \Gamma$) so that $[x, y] = 0$ proving that R is commutative. Hence our hypothesis reduces to $F(x)d(y) + xy = 0$ for all $x, y \in R$. Replacing x by xy in the above relation, we find that $d(y)xd(y) = 0$ for all $x, y \in R$ and the semi-primeness forces $d = 0$. Consequently, $xy = 0$ for all $x, y \in R$ so that $R = \{0\}$, a contradiction. In the case R is an arbitrary 2-torsion free unitary ring, setting $y = 1$ our hypothesis reduces to $x = 0$ for all $x \in R$, a contradiction. ■

Theorem 5.3.5 *Let R be a ring, P a prime ideal and F a generalized derivation associated with a derivation d . If F satisfies one of the following conditions:*

- 1) $F(x) \circ d(y) + [x, y] \in P$ for all $x, y \in R$,
- 2) $F(x) \circ d(y) - [x, y] \in P$ for all $x, y \in R$,

then R/P is a commutative integral domain. Moreover, if R/P is 2-torsion free then $d(R) \subseteq P$.

Proof. (1) We are given that

$$F(x) \circ d(y) + [x, y] \in P \quad \text{for all } x, y \in R. \quad (5.24)$$

Substituting $xd(y)$ for x in (5.24), we obtain

$$xd^2(y)d(y) + d(y)xd^2(y) + x[d(y), y] \in P \quad \text{for all } x, y \in R \quad (5.25)$$

in particular, we may write

$$rxd^2(y)d(y) + d(y)rx d^2(y) + rx[d(y), y] \in P \quad \text{for all } x, y, r \in R. \quad (5.26)$$

Right multiplying (5.25) by r and comparing with (5.26), one can easily see that

$$[d(y), r]xd^2(y) \in P \quad \text{for all } r, x, y \in R. \quad (5.27)$$

Invoking arguments used in the proof of Theorem 5.3.3 it follows that either R/P is commutative or $d^2(R) \subseteq P$. In the latter case, writing $d(y)$ instead of y in (5.24), we get

$$[d(y), x] \in P \quad \text{for all } x, y \in R. \quad (5.28)$$

Therefore d is P -centralizing and Theorem 5.2.1 yields that either R/P is commutative or $d(R) \subset P$ in which case our hypothesis reduces to $[x, y] \in P$ for all $x, y \in R$ proving again that R/P is a commutative integral domain.

(2) Proceeding on the same lines with necessary variations, we can prove that $F(x) \circ d(y) - [x, y] \in P$ for all $x, y \in R$ assures the commutativity of R/P .

Now assume that R/P is 2-torsion free, then $F(x)d(y) \in P$ for all $x, y \in R$. Replacing x by $xd(r)$ we get $xd^2(r)d(y) \in P$ so that $d^2(R) \subset P$. Since $d^2(xy) = d^2(x)y + xd^2(y) + 2d(x)d(y)$, it follows that $d(x)d(y) \in P$ for all $x, y \in R$ proving that $d(R) \subset P$. ■

Following is an immediate corollary of the above theorem.

Corollary 5.3.6 *Let R be a semi-prime ring and F a generalized derivation of R associated with a derivation d . If F satisfies one of the following conditions:*

- 1) $F(x) \circ d(y) + [x, y] = 0$ for all $x, y \in R$,
- 2) $F(x) \circ d(y) - [x, y] = 0$ for all $x, y \in R$,

then R is a commutative integral domain. Moreover, if R is 2-torsion free then F is a left multiplier.

5.4 Invariance of minimal prime ideals under derivations

Given a derivation d of R , an ideal I of R is said to be invariant under d or d -invariant for short if $d(I) \subseteq I$. If I is such an ideal of R then d induces a derivation on R/I which preserves many properties of d . In particular, if we have a family of d -invariant ideals J_i with trivial intersection $\bigcap_i J_i = (0)$, then the ring R can be injected in the direct product of R/J_i and properties of d can arise from properties of induced derivations on R/J_i . In this way, we expect to realize the fundamental goal consisting to restrict questions regarding derivations in semi-prime rings to questions about prime rings. Thus the real concern about a given semi-prime ring is existence of prime d -ideals with trivial intersection. This leads us to the famous problem

Problem 1. Given a semi-prime ring R , does there exist a collection of prime ideals P_i , such that $\bigcap_i P_i = (0)$ and such that each P_i is invariant under all derivations of R ?

An affirmative answer to Problem 1 has been presented in ([25], Theorem 2.1) for a countable ring R . Furthermore, it is well known that every minimal prime ideal of a torsion-free semi-prime ring is invariant under all derivations [37]. In this direction Herstein raised the following problem:

Problem 2. Given a semi-prime ring R , does $d(P) \subseteq P$ hold for any minimal prime ideal P of R and for any derivation d of R ?

Therefore, the fundamental preoccupation regarding a semi-prime ring is whether the intersection of the invariant minimal prime ideals under a given derivation is 0 or not. This problem was also mentioned in ([16] and [55]). The best result of the conjecture is due to Beidar and Mikhal'ev ([9], Theorem 8.16): Let R be a ring of bounded index m (i.e. m is a positive integer such that $x^m = 0$ for all nilpotent elements $x \in R$) such that the additive order of every nonzero torsion element of R , if any, is strictly larger than m . Then all minimal prime ideals of R are invariant under all derivations of R . In particular, every minimal prime ideal of a reduced ring is invariant under derivations of the ring ([24], p. 614). However, Problem 2 turns out to be false in general. In fact, Chuang and Lee [6] constructed a semi-prime ring R which possesses a minimal prime ideal not invariant under a derivation of the ring.

Our fundamental aim in this section is to study the problem of invariance of minimal prime ideal of a semi-prime ring under derivations. More precisely, we will apply our previous theorems to give some sufficient conditions under which a derivation leaves invariant minimal prime ideals of a semi-prime ring.

Corollary 5.4.1 *Let R be a semi-prime ring and F a generalized derivation of R associated with a derivation d . If F is centralizing generalized derivation of R , then*

- 1) *there exists a minimal prime ideal P invariant under d .*
- 2) *d has its range in $Z(R)$.*

Proof. 1) Assume on the contrary that $d(P) \not\subseteq P$ for any minimal prime ideal P of R . According to Theorem 5.2.1, R/P is commutative and combining this fact with $\bigcap_{P \text{ minimal}} P = \{0\}$ it follows that R is commutative. Hence, R is a reduced ring and [16, p. 614] assures that every minimal prime ideal of R is

d -invariant, a contradiction.

2) Let x be an element in R and P be a minimal prime ideal of R . Using Theorem 5.2.1, we have $d(R) \subseteq P$ or R/P is commutative so that $[d(x), y] \in P$ for all $y \in R$ and therefore $d(x) \in Z(R)$. ■

Remark. In particular, if a semi-prime ring R admits a nonzero centralizing derivation d , then there exists a minimal prime ideal P invariant under d and d has its range in $Z(R)$.

Now we are in a position to prove the second result of this section which answers (at least partially) Problem 2. Indeed, for a semi-prime ring provided with a generalized derivation (F, d) we establish some conditions under which every minimal prime ideal of R is invariant under the derivation d .

Corollary 5.4.2 *Let R be a semi-prime and F a generalized derivation associated with a derivation d . If F satisfies one of the following conditions:*

- 1) $F(x) \circ d(y) + x \circ y = 0$ for all $x, y \in R$,
- 2) $F(x) \circ d(y) - x \circ y = 0$ for all $x, y \in R$,
- 3) $[F(x), d(y)] + x \circ y = 0$ for all $x, y \in R$,
- 4) $[F(x), d(y)] - x \circ y = 0$ for all $x, y \in R$,

then $d(P) \subseteq P$ for any minimal prime ideal of R . Moreover, there exists a minimal prime ideal P such that $\text{char}(R/P) = 2$.

Proof. According to Theorem 5.3.3, R/P is commutative for any minimal prime ideal and then R is commutative. Therefore R is a reduced ring and [[16], p. 614] assures that $d(P) \subseteq P$ for any minimal prime ideal of R .

Now assume that R/P is 2-torsion free for any minimal prime ideal P of R . Once again using Theorem 5.3.3, we have R/P is commutative and $d(R) \subseteq P$. Therefore, our hypothesis reduces to $xy \in P$ for all $x, y \in R$ and combining this fact together with $\bigcap_{P \text{ minimal}} P = \{0\}$ we arrive at $xy = 0$ for all $x, y \in R$ and thus $R = \{0\}$, a contradiction. ■

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