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Exploring Holomorphic and n-Analytic Automorphic Functions: Poincaré Theta Operator, Creation Operator & Spectral Analysis on Mixed Automorphic Functions.

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L.IMLAL

Abstract

In this thesis, we have dedicated our efforts to the study of the theory of holomorphic automorphic functions associated with a given discrete subgroup in $(\mathbb{C}, +)$, as well as to the detailed characterization of the space of mixed automorphic functions associated with a given discrete subgroup of the semidirect group $U(1) \ltimes \mathbb{C}$. Our focus revolves around the creation of automorphic functions through the utilization of Poincaré's theta series operator, delving into their characteristics. To be specific, we establish the rigorous definition of this operator within designated spaces and provide an intricate account of its kernel by employing special functions. We provide a precise characterization for the class of planar poly-analytic automorphic functions, which are manifested as images of holomorphic functions using the differential creation operator $\mathcal{U}_\nu^{(n)}$. We characterize the image of the operator $\mathcal{U}_\nu^{(n)}$ on the Hilbert space $H_\Gamma^{2,\nu}(\mathbb{C})$, deal with the surjectivity problem and show that it preserves orthogonality in $L_\Gamma^{2,\nu}(\mathbb{C})$. Furthermore, as a direct application of this operator, we reconstruct the construction of orthogonal bases for the generalized Bargmann-Fock spaces $\mathcal{F}_{\Gamma,n}^{2,\nu}(\mathbb{C})$ for discrete one- and two-rank subgroups.

In the context of mixed automorphic functions, we encounter the automorphic factor in relation to a given equivariant pair (ρ, τ). We have partially characterized the equivariant pairs, in our research and explored the potential for extending these findings to higher dimensions. To conduct this examination, we employ an elevation theorem to classical automorphic functions. In addition, we concduct a comprehensive spectral analysis of a magnetic invariant Schrödinger operator acting on mixed automorphic functions.

keywords: Automorphic functions; Mixed automorphic functions; Poincaré's theta series; Automorphy factor; Discrete subgroup; Poly-Bargmann spaces; Magnetic Laplacian; Poly-analytical functions.

Résumé

Dans cette thèse, nous nous sommes consacrés à l'étude de la théorie des fonctions automorphes holomorphes associés à un sous-groupe discret donné dans $(\mathbb{C},+)$, ainsi qu'à la caractérisation détaillée de l'espace des fonctions automorphes de seconde espèce, communément appelées fonctions automorphes "mixed" associe à un sous-groupe discret donné du groupe semi-direct $U(1) \ltimes \mathbb{C}$. En abordant la construction de fonctions automorphes à l'aide de la série (Opérateur) thêta de Poincaré, nous explorons leurs propriétés. Plus précisément, nous démontrons que cet opérateur est rigoureusement défini dans des espaces spécifiques et nous fournissons une description détaillée de son noyau en utilisant des fonctions spéciales. Nous proposons une caractérisation précise pour la catégorie des fonctions automorphes poly-analytiques planaires, qui se manifestent en tant qu'images de fonctions holomorphes par l'intermédiaire de l'opérateur différentiel de création $\mathcal{U}_{\nu}^{(n)}$. Nous donnons la caractérisation de l'image de l'opérateur $\mathcal{U}_{\nu}^{(n)}$ sur l'espace de Hilbert $H_{\Gamma}^{2,\nu}(\mathbb{C})$, nous traitons le problème de la surjectivité et montrons qu'il préserve l'orthogonalité dans $L_{\Gamma}^{2,\nu}(\mathbb{C})$. Enfin, nous utiliserons cet opérateur comme un outil de correspondance pour détailler la construction des bases orthogonales des espaces de Bargmann–Fock généralisés $\mathcal{F}_{\Gamma,n}^{2,\nu}(\mathbb{C})$ pour les sous-groupes discrets à un et deux rangs.

En ce qui concerne la seconde espèce, le facteur d'automorphie est en relation avec une paire équivariante donnée (ρ,τ). Nous avons partiellement caractérisé les paires équivariantes dans le cadre de notre étude et envisagé la possibilité d'une généralisation à des dimensions supérieures. Pour mener cette analyse, nous utilisons un théorème d'élévation vers des fonctions automorphes classiques. De plus, nous effectuons une analyse spectrale approfondie d'un opérateur de Schrödinger magnétique invariant agissant sur des fonctions automorphes "mixed".

Mots Clés: Fonctions automorphes; Fonctions automorphes "mixed"; La série thêta de Poincaré; Facteur d'automorphie; Sous groupe discret; Espaces poly-Bargmann; Laplacien magnétique; Fonctions poly-analytiques.

Résumé de la thèse

La théorie traitée dans cette thèse est associée à la donnée d'un sous-groupe discret Γ du groupe additif $(\mathbb{C},+)$ agissant sur le plan complexe muni de son produit scalaire hermitien standard $\langle z,w\rangle = z\overline{w}$, et la donnée d'un facteur d'automorphie, c'est-à-dire une fonction z-holomorphe définie sur $\Gamma\times\mathbb{C}$ vérifiant la condition du cocycle. Ainsi, une fonction f sur $\mathbb{C}$ est dite (Γ,χ,ν)-automorphe (ou simplement automorphe) si elle satisfait l'équation fonctionnelle

$$f(\gamma+z) = \chi(\gamma)e^{\nu\left\langle z+\frac{\gamma}{2},\gamma\right\rangle}f(z)$$

pour tout $z\in D$ et $\gamma\in\Gamma$, où ν est un paramètre réel strictement positive, et χ une application de Γ à valeurs dans le cercle unité $U(1)$. L'espace fonctionnel $Aut^{\nu}_{\Gamma,\chi}(\mathbb{C})$ de ces fonctions automorphes est non trivial si est seulement si le triplet (Γ,χ,ν) satisfait la condition suivante [24]

$$\chi\left(\gamma_1+\gamma_2\right) = \chi\left(\gamma_1\right)\chi\left(\gamma_2\right)e^{i\nu\omega(\gamma_1,\gamma_2)},\ \gamma_1,\gamma_2\in\Gamma.$$

De plus, il s'agit d'un espace vectoriel complexe de dimension infinie lorsque Γ est de rang 1, et de dimension finie donnée par $(\nu/\pi)\,\mathrm{vol}(\Lambda(\Gamma))$ quand Γ est cocompact, où $\Lambda(\Gamma)$ est un domaine fondamental de Γ représentant dans $\mathbb{C}$ le groupe orbital $\mathbb{C}/\Gamma$. Pour $f\in Aut^{\nu}_{\Gamma,\chi}(\mathbb{C})$, la quantité

$$\|f\|^2_{\Gamma,\nu} = \int_{\Lambda(\Gamma)}|f(z)|^2e^{-\nu|z|^2}d\lambda(z),$$

est indépendante du domaine fondamental $\Lambda(\Gamma)$ choisi et définit une norme sur l'espace des fonctions automorphes holomorphes $\mathcal{O}^{\nu}_{\Gamma,\chi}(\mathbb{C}) = \mathcal{H}ol(\mathbb{C})\cap Aut^{\nu}_{\Gamma,\chi}(\mathbb{C})$, où $\mathcal{H}ol(\mathbb{C})$ désigne l'espace de fonctions holomorphes sur $\mathbb{C}$. Il a été montré dans [24] que $\mathcal{O}^{\nu}_{\Gamma,\chi}(\mathbb{C})$ est un espace de Hilbert dont le noyau reproduisant est donné par

$$K^{\nu}_{\Gamma,\chi}(z,w) := \left(\frac{\nu}{\pi}\right)e^{\nu\langle z,w\rangle}\sum_{\gamma\in\Gamma}\chi(\gamma)e^{-\frac{\nu}{2}|\gamma|^2+\nu(\langle z,\gamma\rangle-\overline{\langle w,\gamma\rangle})}.$$

lorsque Γ est de rang deux.

Le présent travail a pour objectif d'étudier ces fonctions dans le cadre holomorphe via l'opérateur theta de Poincaré

$$\mathcal{P}_\Gamma(f)(z) := \sum_{\gamma\in\Gamma} \overline{\chi(\gamma)} e^{-2\nu\left\langle z+\frac{\gamma}{2},\gamma\right\rangle} f(z+\gamma), \tag{0.0.1}$$

et dans le cadre polya-nalytique moyennant un opérateur de création

$$\mathcal{U}_\nu^{(n)}(f)(z) := (-1)^n e^{\nu|z|^2} \frac{\partial^n}{\partial z^n}\left(e^{-\nu|z|^2} f\right)(z), \tag{0.0.2}$$

ainsi que la description de leurs propriétés spectrales de celles qui sont mixtes.

Dans le Chapitre 2, nous étudions l'opérateur thêta de Poincaré $\mathcal{P}_\Gamma$ défini dans l'équation (0.0.1) dans l'espace fonctionnel

$$\mathcal{F}^{1,\nu}(\mathbb{C}) := \left\{ f \in \mathcal{H}ol(\mathbb{C}),\ \int_{\mathbb{C}} |f(z)| e^{-\nu|z|^2} d\lambda(z) < +\infty \right\}.$$

Nous montrons que la série $\mathcal{P}_\Gamma(f)(z)$ converge absolument et uniformément sur les parties compacts de $\mathbb{C}$, et que $\mathcal{P}_\Gamma$ est un opérateur continu sur $\mathcal{F}^{1,\nu}(\mathbb{C})$ et que son image $\mathcal{P}_\Gamma\left(\mathcal{F}^{1,\nu}(\mathbb{C})\right)$ est dense dans

$$\mathcal{F}^{1,\nu}_{\Gamma,\chi}(\mathbb{C}) := \left\{ f \in \mathcal{O}^{2\nu}_{\Gamma,\chi}(\mathbb{C}),\ \|f\|_{1,\Gamma} := \int_{\Lambda(\Gamma)} |f(z)| e^{-\nu|z|^2} d\lambda(z) < +\infty \right\}.$$

Nous prouvons également que $\mathcal{P}_\Gamma : \mathcal{F}^{1,\nu}(\mathbb{C}) \longrightarrow \mathcal{F}^{1,\nu}_{\Gamma,\chi}(\mathbb{C})$ est surjective, et que l'espace dual de l'espace de Banach $\mathcal{F}^{1,\nu}_{\Gamma,\chi}(\mathbb{C})$ correspond exactement à l'espace

$$\mathcal{F}^{\infty,\nu}_{\Gamma,\chi}(\mathbb{C}) := \left\{ f \in \mathcal{O}^{2\nu}_{\Gamma,\chi}(\mathbb{C}),\ \|f\|_{\infty,\Gamma} := \sup\left\{ |f(z)| e^{-\nu|z|^2}, z \in \mathbb{C} \right\} < +\infty \right\},$$

par rapport au crochet de dualité

$$\langle f, g\rangle_\Gamma := \int_{\Lambda(\Gamma)} f(z)\overline{g(z)} e^{-2\nu|z|^2} d\lambda(z), \quad \text{pour } f \in \mathcal{F}^{1,\nu}_{\Gamma,\chi}(\mathbb{C}) \text{ et } g \in \mathcal{F}^{\infty,\nu}_{\Gamma,\chi}(\mathbb{C}).$$

Nous donnons la description du noyau de $\mathcal{P}_\Gamma$, qui est caractérisé en terme d'une famille spécifique de fonctions, et elle dépend du rang du sous-groupe discret Γ. Plus explicitement, le noyau de $\mathcal{P}_\Gamma$ est déterminé par

$$\ker(\mathcal{P}_\Gamma) = \left\{ e_n^{\alpha,\nu}(z) := e^{\nu z^2 + 2i\pi(\alpha+n)z}; n \in \mathbb{Z} \right\}^{\perp_{\langle\cdot,\cdot\rangle_\Gamma}}$$

quand $\Gamma = \mathbb{Z}$, où α est un nombre réel fixe tel que $\chi(1) = e^{2i\pi\alpha}$. Pour de cas de $\Gamma = \mathbb{Z} + \tau\mathbb{Z}$ avec $\Im m(\tau) > 0$, on a $\ker(\mathcal{P}_\Gamma) = \left\{\theta_r^{\nu,\alpha,\beta}; r = 1, \cdots, m\right\}^{\perp_{\langle\cdot,\cdot\rangle_\Gamma}}$ où

$$\theta_r^{\nu,\alpha,\beta}(z) := e^{\nu z^2} \sum_{\substack{k\in\mathbb{Z} \\ k=r \mod m}} e^{i\frac{\pi(k+\alpha)^2\tau}{m} + 2i\pi(\alpha+k)(z-\frac{\beta}{m})}; \qquad r = 1, 2, \cdots, m.$$

On donne ensuite une condition nécessaire et suffisante sur les coefficients de Fourier pour qu'une fonction donnée soit dans tel noyau.

Chapitre 3 étudie attentivement la structure automorphe poly-analytique, provenant en utilisant l'opérateur différentiel linéaire $\mathcal{U}_\nu^{(n)}$ défini par

$$\mathcal{U}_\nu^{(n)}(f)(z) := (-1)^n e^{\nu|z|^2} \frac{\partial^n}{\partial z^n} \left(e^{-\nu|z|^2} f \right)(z). \tag{0.0.3}$$

Partant du fait que l'image d'une fonction holomorphe par $\mathcal{U}_\nu^{(n)}$ est une fonction dans l'espace $A_n(\mathbb{C})$ de fonctions poly-analytique d'ordre n (voir Proposition 3.1), l'objectif est de caractériser les fonctions n-analytiques provenant de celles satisfont l'équation fonctionnelle d'automorphie (1.2.2). Nous donnons des conditions sur une fonction n-analytique pour qu'elle soit l'image par $\mathcal{U}_\nu^{(n)}$ d'une fonction dans $H_\Gamma^{2,\nu}(\mathbb{C}) = \mathcal{O}_{\Gamma,\chi}^\nu(\mathbb{C}) \cap L_\Gamma^{2,\nu}(\mathbb{C})$. Plus précisément, si $F = \sum\limits_{k=0}^{n} \overline{z}^k \psi_k$, avec ψ_k est une fonction analytique, alors il existe alors une fonction holomorphe f tel que $F = \mathcal{U}_\nu^{(n)}(f)$ si et seulement si les fonctions composantes ψ_k satisfaisant

$$(k+1)\frac{\partial \psi_{k+1}}{\partial z} = -\nu(n-k)\psi_k. \tag{0.0.4}$$

Il est également démontré que l'opérateur $\mathcal{U}_\nu^{(n)}$ préserve l'équation d'automorphie (Proposition 3.2) et l'orthogonalité des fonctions (Lemme 3.4). Enfin, il convient de noter que l'opérateur $(\sqrt{\nu^n !})^{-1}\mathcal{U}_\nu^{(n)}$ est unitaire de $H_\Gamma^{2,\nu}(\mathbb{C})$ à $L_\Gamma^{2,\nu}(\mathbb{C})$. Une caractérisation explicite de l'image de l'opérateur $\mathcal{U}_\nu^{(n)}$ est obtenue, plus exactement on montre que

$$\mathcal{U}_\nu^{(n)}\left(H_\Gamma^{2,\nu}(\mathbb{C})\right) = Aut_\Gamma^{2,\nu}(\mathbb{C}) \cap A_n(\mathbb{C}) \cap \ker(\mathcal{D}_\nu^{(n)} - \nu^n n!) = \mathcal{F}_{\Gamma,n}^{2,\nu}(\mathbb{C}).$$

avec $Aut_\Gamma^{2,\nu}(\mathbb{C}) = Aut_{\Gamma,\chi}^\nu(\mathbb{C}) \cap L_\Gamma^{2,\nu}(\mathbb{C})$ et

$$\mathcal{D}_\nu^{(n)} := \Delta_\nu(\Delta_\nu - \nu)(\Delta_\nu - 2\nu)\cdots(\Delta_\nu - (n-1)\nu).$$

Comme application directe des résultats obtenus, nous reproduisons des bases orthogonales pour les espaces généralisés de thêta Bargmann-Fock $\mathcal{F}_{\Gamma,n}^{2,\nu}(\mathbb{C})$ pour les sous-groupes discrets de rang un et de rang deux.

Dans un deuxième temps on s'intéresse à l'espace $\mathcal{M}_{\rho,\tau}^{\nu,\mu}(\mathbb{C})$ des fonctions automorphes "mixed" associées à un sous-groupe discret donné Γ et une paire équivariante (ρ, τ). Il s'agit des fonctions satisfaisant l'équation fonctionnelle

$$F(\gamma \cdot z) = \chi(\gamma) j^\nu(\gamma, z) j^\mu(\rho(\gamma), \tau(z)) F(z).$$

On commence cette étude en rappelant les résultats obtenus pour $\chi \equiv 1$ et Γ un sous-groupe discret de $(\mathbb{C}, +)$. Nous donnons une condition nécessaire et suffisant pour assurer la non-trivialité d'un tel espace et nous montrons que $\mathcal{M}_{\rho,\tau}^{\nu,\mu}(\mathbb{C})$ est isomorphe à l'espace des fonctions automorphes considérées dans la première partie $\mathcal{F}^{B^{\nu,\mu}_\tau}_{\Gamma,\chi_\tau}(\mathbb{C})$, avec χ_τ étant le pseudo caractère donné dans (1.3.16) et

$$B^{\nu,\mu}_\tau = \nu + \mu\left(\left|\frac{\partial\tau}{\partial z}\right|^2 - \left|\frac{\partial\tau}{\partial\bar{z}}\right|^2\right).$$

Notre contribution porte sur le cas d'un sous groupe discret du groupe semi-direct $G := U(1) \ltimes \mathbb{C}$. Nous discutons quelques propriétés des paires équivariantes (ρ, τ) dans ce cas, y compris leurs caractérisations comme celle pour laquelle ρ se décompose sous la forme $\rho = \tilde{\rho} \ltimes \tau$. Par suite on décrit concrètement les propriétés spectrales de l'opérateur de Schrödinger magnétique laissant invariant l'espace des fonctions automorphes "mixtes" appartenant à l'espace de Hilbert $L^{2,\nu}_{\mu,\tau}(\mathbb{C}/\Gamma)$. On montre ainsi que son problème aux valeurs propres de l'hamiltonian de Landau $\Delta^{\nu,\mu}_\tau$ sur $\mathcal{M}^{\nu,\mu}_{\rho,\tau}(\mathbb{C})$ est équivalent au problème aux valeurs propres sur l'espace des fonctions automorphes classiques $Aut^{\nu}_{\Gamma,\chi}(\mathbb{C})$. On montre par suite que

$$\mathcal{W}^{\nu,\mu}_\tau\left(\mathcal{E}^{\nu,\mu}_{\tau;\lambda}\right) = \left\{F \in \mathcal{F}^{B^{\mu,\nu}_\tau}_{\Gamma,\chi_\tau}(\mathbb{C}); \quad \Delta_{B^{\mu,\nu}_\tau} F = \lambda F\right\} =: \mathcal{E}^{B^{\mu,\nu}_\tau}_\lambda$$

où $\left[\mathcal{W}^{\mu,\nu}_\tau f\right](z) := e^{i\varphi^{\nu,\mu}_\tau(z)} f(z)$ et $\mathcal{E}^{\nu,\mu}_{\tau;\lambda}$ est le sous-espace propre de $\Delta^{\nu,\mu}_\tau$ associé à la valeur propre λ, ainsi nous montons que le spectre ponctuel du Laplacian considéré agissant sur $L^2(\mathbb{C}, d\lambda)$ est discret et se réduit au niveaux de Landau $\lambda_k = -2B^{\mu,\nu}_\tau(2k+1); k = 0, 1, \cdots$, et que les L^2-espaces propres associés

$$A^2_k\left(\Delta^{\nu,\mu}_\tau\right) = \left\{f \in L^2(\mathbb{C}, d\lambda); \quad \Delta^{\nu,\mu}_\tau f = \lambda_k f\right\}$$

conduisent à la décomposition orthogonale de $L^2(\mathbb{C}, d\lambda)$. On déduit également l'expression explicite de leur noyau reproduisant $K^{\nu,\mu}_{\tau;k}$ qui est donné par

$$K^{\nu,\mu}_{\tau;k}(z,w) = \frac{B^{\mu,\nu}_\tau}{\pi} e^{\psi^{\nu,\mu}_\tau(z,w)} e^{iB^{\mu,\nu}_\tau \Im m\langle z,w\rangle} e^{-\frac{B^{\mu,\nu}_\tau}{2}|z-w|^2} L_k\left(2B^{\mu,\nu}_\tau|z-w|^2\right),$$

où $L_k(x)$ désigne le polynôme de Laguerre usuel.

Contents

Introduction

The theory of holomorphic automorphic functions is widely treated in the literature and their properties have been a subject of extensive study since its first introduction by Poincaré in the late nineteenth century [50]. Generally speaking, a holomorphic function on an appropriate domain $D \subseteq \mathbb{C}^n$, on which a discrete group Γ acts, is called an automorphic function if it satisfies the functional equation

$$f(\gamma \cdot z) = J(\gamma, z)f(z); \quad z \in D, \gamma \in \Gamma,$$

where $J(\gamma, z)$ is a given automorphic factor, that is a z-holomorphic function satisfying the cocycle equation

$$J(\gamma_1\gamma_2, z) = J(\gamma_1, \gamma_2 \cdot z)J(\gamma_2, z).$$

This analytical definition has, as well, a geometrical relevance. It is closely connected to number theory and abelian varieties [57, 54, 14], representation theory [30, 60], spectral analysis [24, 25], cryptography and coding theory [56], chaoticity of a shift operator [39, 40] and quantum field theory [23]. The theory dealt in this context, related to the choice of the automorphic factor J, of the group Γ and the set D. For example if $J \equiv 1$ we fall on Γ-periodic functions [41].

Thought the construction of such functions remain hard to deal with. The well-known method of constructing automorphic functions is due to Poincaré and consists of considering

$$\mathcal{P}_\Gamma(f) := \sum_{\gamma \in \Gamma} f(\gamma \cdot z)J(\gamma, z)^{-1},$$

often called Poincaré theta series, provided that the series converges absolutely and uniformly on compact sets. Accordingly, important questions arose naturally. The first one concerns the classes of functions on which $\mathcal{P}_\Gamma$ is well-defined. The second aspect involves identifying the collection of functions f where $\mathcal{P}_\Gamma f$ equals zero entirely, along with those where $\mathcal{P}_\Gamma f$ does not equal zero (The vanishing Poincaré problem raised by Poincaré himself in 1882). As far as we know, related works treating these questions are done in the context of the unit complex hyperbolic disc $\mathbb{D}$ or the upper half-plane $\mathcal{H} = \{z \in \mathbb{C}/\Im m(z) > 0\}$ with Γ being a Fuchsian group acting on by linear fractional transformations and $J(\gamma, z) = \gamma'(z)^k$, for a fixed positive integer k. In this context, Bers has introduced in [11] a class of holomorphic integrable functions on $\mathbb{D}$, on which $\mathcal{P}_\Gamma$ is well-defined. Concerning the kernel of $\mathcal{P}_\Gamma$, viewed as an operator on the Bers spaces, Metzger has given in [46] a set of functions that closely spans $\ker(\mathcal{P}_\Gamma)$. Later, Kra obtained in [42] a characterization of the kernel using the Eichler cohomology for finitely generated Fuchsian group. The result is then improved by Masumoto who gave in [48] a concrete characterization of $\ker(\mathcal{P}_\Gamma)$ on Hardy spaces for finitely generated Fuchsian group of the first kind. Ghanmi and Intissar in [24] considering the case of $J(\gamma, z) = \chi(\gamma)e^{i\nu\Im m<z,\gamma>}$ for given real number $\nu > 0$, a lattice Γ of $\mathbb{C}^n$ and a given map $\chi : \Gamma \to U(1)$. They show that the space acting on the space $\mathcal{F}^\nu_{\Gamma,\chi}$ of (Γ, χ)-automotphic functions on $\mathbb{C}^n$, comprising $\mathcal{C}^\infty$ functions that satisfy the functional equation

$$f(z+\gamma) = \chi(\gamma)e^{i\nu\Im m\langle z,\gamma\rangle}f(z); \quad z \in \mathbb{C}^n, \gamma \in \Gamma$$

is a nonzero complex vector space if and only if the triplet (ν, Γ, χ) satisfies the following riemann dirac quantization (RDQ) condition

$$\chi\left(\gamma_1+\gamma_2\right) = \chi\left(\gamma_1\right)\chi\left(\gamma_2\right)e^{i\nu\omega(\gamma_1,\gamma_2)}$$

for every $\gamma_1, \gamma_2 \in \Gamma$ (see also the Proposition 1.2). So they show that the eigenspace $\mathcal{E}^\nu_{\Gamma,\chi}(0)$ associated with the lowest Landau level of $\mathbb{L}^\nu$ acting on the space $\mathcal{F}^\nu_{\Gamma,\chi}$ is isomorphic to the space $\mathcal{O}^\nu_{\Gamma,\chi}(\mathbb{C}^n)$, of holomorphic functions on $\mathbb{C}^n$ satisfying the functional equation

$$g(z+\gamma) = \chi(\gamma)e^{\frac{\nu}{2}|\gamma|^2+\nu\langle z,\gamma\rangle}g(z),$$

and that $\mathcal{O}^\nu_{\Gamma,\chi}(\mathbb{C}^n)$ is a finite dimensional Hilbert space, when Γ is rank two. Its concrete description, including the explicit construction of an orthonormal basis, and the explicit expression of its reproducing kernel, are investigated in [25] for $n = 1$ and generalized to high dimension in [58].

In have ben discussed some analytical and arithmetic properties of the reproducting kernel function $K^\nu_{\Gamma,\chi}(z, w)$ which is defined in 1.2.6, in a previous work [17]. Specifically, we have demonstrated that the set $\mathcal{Z}(K^\nu_{\Gamma,\chi})$ of zeros of $K^\nu_{\Gamma,\chi}$ exhibits several noteworthy characteristics. It is symmetric, non-isolated, and uniformly distributed. In other words, $\mathcal{Z}(K^\nu_{\Gamma,\chi})$ consists of the Γ-translation of the zeros of $K^\nu_{\Gamma,\chi}(z, w)$ contained in a cartesian product of fundamental cells. The systematic study of the zero set of $K^\nu_{\Gamma,\chi}$ inside a cartesian product of fundamental cells leads to a characterization of the common zeros (analytic set) contained in a fundamental cell of all holomorphic functions belonging to the theta Bargmann-Fock Hilbert space $\mathcal{O}^\nu_{\Gamma,\chi}(\mathbb{C})$. It has been proven that this analytical collection has a finite number of elements, and its cardinality is less than or equal to $(\nu/\pi)\,S(\Gamma)$. Finding the points where $K^\nu_{\Gamma,\chi}$ becomes zero leads to discovering interesting identities. Additionally, we establish more noteworthy lattice sums by computing the complex Hermite-Gauss coefficients, which are the coefficients of $K^\nu_{\Gamma,\chi}(z, w)$

when it is expanded as a power series, expanding on specific arithmetic relationships derived by Perelomov [53] within the context of coherent states, we generalize these for the particular scenario of von Neumann lattice [62]. These relationships were subsequently rediscovered by Boon and Zak [13].

The mixed automorphic functions we are dealing with in in the second part of this thesis can be seen as another way to generalize the classical automorphic functions. They include the classical ones as a special case. Nontrivial examples of them have been constructed in [15, 44]. Roughly speaking, they are a class of functions defined on a given (Hermitian symmetric) space X and satisfying a functional equation of type

$$F(\gamma \cdot z) = j^{\nu}(\gamma, z) j^{\mu}(\rho(\gamma), \tau(z)) F(z)$$

for every $z \in X$ and $g \in G$. Here, $j^{\alpha}(\gamma, x); \alpha \in \mathbb{R}$ is an automorphic factor associated to an appropriate action of a group G on X, and (ρ, τ) is an equivariant pair for the data (G, X). Such notion was introduced by Stiller [61] and extensively studied by Lee in the case of X being the Poincaré upper half-plane $\mathcal{H}$. They arise naturally as holomorphic forms on elliptic varieties [34]. We refer to [45] for an exhaustive list of references. In the case of the complex plane $X = \mathbb{C}$ and the group $G = U(1) \ltimes \mathbb{C}$, we are aware that Ghanmi in [27] attempted to characterize this space. In [21] it is introduced and realized some concrete spectral properties of a class of magnetic Schrödinger operators leaving invariant such space. Our contribution within this framework primarily focuses on spectral analysis.

Description of chapters: This thesis is organized into a comprehensive structure comprising five distinct chapters. Each chapter serves a unique purpose and contributes to the overall research endeavor.

- Chapter 1: We begin by recalling basic properties of certan particular family of discrete subgroups Γ of $\mathbb{C}$. Aside from introducing the basic terminology and notations concerning the classical automorphics fonctions, among the various functional spaces, we need to consider the vector space. $Aut^{\nu}_{\Gamma,\chi}(\mathbb{C})$ $(\mathcal{O}^{\nu}_{\Gamma,\chi}(\mathbb{C}))$ of functions (holomorphic functions) satisfying the functional equation

  $$f(\gamma + z) = \chi(\gamma) e^{\nu < z + \frac{\gamma}{2}, \gamma >} f(z). \tag{0.0.5}$$

  Then, we give a condition of non-triviality, denoted by (RDQ), so that the spaces $\mathcal{O}^{\nu}_{\Gamma,\chi}(\mathbb{C})$ is a Hilbirt space endowed with the scalar product $< \cdot, \cdot >_{\Gamma,\nu}$(see Eq: 1.2.5). In the case of Γ of rank-two, this space has finite dimension, which is equal to $\frac{\nu}{\pi}\operatorname{vol}(\Lambda(\Gamma))$, or $\Lambda(\Gamma)$ is a fundamontal domain of Γ. The formula for its reproducing kernel is provided by

  $$K^{\nu}_{\Gamma,\chi}(z, w) := \left(\frac{\nu}{\pi}\right) e^{\nu\langle z, w\rangle} \sum_{\gamma \in \Gamma} \chi(\gamma) e^{-\frac{\nu}{2}|\gamma|^2 + \nu(\langle z, \gamma\rangle - \overline{\langle w, \gamma\rangle})}.$$

We review from [27, 21] in the space of mixed automorphic forms $\mathscr{M}^{\nu,\mu}_{\Gamma,\tau}(\mathbb{C})$ defined on the complex plane $\mathbb{C}$ considering a specific lattice Γ in $\mathbb{C}$ and an equivariant pair (ρ, τ). It is isomorphic to the space of Landau automorphic forms,

$$F(z+\gamma) = \chi_\tau(\gamma) J^{B^{\nu,\mu}_\tau}(\gamma, z) F(z); \quad z \in \mathbb{C}, \gamma \in \Gamma,$$

of "weight" $B^{\nu,\mu}_\tau$ defined by

$$B^{\nu,\mu}_\tau(z) = \nu + \mu \left(\left|\frac{\partial \tau}{\partial z}\right|^2 - \left|\frac{\partial \tau}{\partial \bar{z}}\right|^2 \right) = B^{\nu,\mu}_\tau$$

with respect to a special pseudocharacter χ_τ defined on Γ. The crucial point in the proof was to that $B^{\nu,\mu}_\tau$ is a real constant independent of the complex variable z.

- Chapter 2: We study the Poincaré theta series operator $\mathcal{P}_\Gamma(f)(z) := \sum\limits_{\gamma\in\Gamma} \overline{\chi(\gamma)} e^{-2\nu\left\langle z+\frac{\gamma}{2},\gamma\right\rangle} f(z+\gamma)$, where Γ is a discrete subgroup of $(\mathbb{C}, +)$, $\nu > 0$ and $\chi : \Gamma \longrightarrow U(1)$ satisfying the condition (RDQ). As first result of this chapter, we show that also the series $\mathcal{P}_\Gamma(f)(z)$ converges absolutely and uniformly on compact sets. As $\mathcal{P}_\Gamma$ is a continuous operator from $\mathcal{F}^{1,\nu}(\mathbb{C})$ to $\mathcal{F}^{1,\nu}_{\Gamma,\chi}(\mathbb{C})$ (Theorem 2.1), with

  $$\mathcal{F}^{1,\nu}(\mathbb{C}) := \left\{ f \in \mathcal{H}ol(\mathbb{C}),\ \int_{\mathbb{C}} |f(z)| e^{-\nu|z|^2} d\lambda(z) < +\infty \right\}$$

  and

  $$\mathcal{F}^{1,\nu}_{\Gamma,\chi}(\mathbb{C}) := \left\{ f \in \mathcal{O}^{2\nu}_{\Gamma,\chi}(\mathbb{C}),\ \|f\|_{1,\Gamma} := \int_{\Lambda(\Gamma)} |f(z)| e^{-\nu|z|^2} d\lambda(z) < +\infty \right\}.$$

  Moreover, the $\mathcal{P}_\Gamma$ is shown to be onto (Theorem 2.2), and that the dual space of the Banach space $\mathcal{F}^{1,\nu}_{\Gamma,\chi}(\mathbb{C})$ is exactly the space $\mathcal{F}^{\infty,\nu}_{\Gamma,\chi}(\mathbb{C})$ of functions $f \in \mathcal{O}^{2\nu}_{\Gamma,\chi}(\mathbb{C})$, such that

  $$\|f\|_{\infty,\Gamma} := \sup_{z\in\mathbb{C}} \left\{ |f(z)| e^{-\nu|z|^2} \right\} < +\infty.$$

  Description of the kernel of $\mathcal{P}_\Gamma$ is then given in terms of the orthogonal of some concrete family of functions and depends of the rank of the discrete subgroup Γ, we distinguish two cases, the case of $\Gamma = \mathbb{Z}$ and the case of $\Gamma = \mathbb{Z} + \tau\mathbb{Z}$; $\Im m(\tau) > 0$. It comes down to giving an explicit condition to be satisfied by the Fourier coefficients of a given function to belong in such kernel (see Theorem 2.4).

- Chapter 3: In this chapter we examine those automorphic function that are poly-analytic. This follows using a linear differential operator.

  $$\mathcal{U}^{(n)}_\nu(f)(z) = \mathcal{U}^{(n,z)}_\nu(f)(z) := (-1)^n e^{\nu|z|^2} \frac{\partial^n}{\partial z^n} \left(e^{-\nu|z|^2} f \right)(z). \tag{0.0.6}$$

The key tools and auxiliary results required for a concrete description of the action of $\mathcal{U}_\nu^{(n)}$ on $H^{2,\nu}_\Gamma(\mathbb{C}) = \mathcal{O}^\nu_{\Gamma,\chi}(\mathbb{C}) \cap L^{2,\nu}_\Gamma(\mathbb{C})$ can be summarized as follows: Initially, it's important to highlight that when a holomorphic function undergoes the operation of $\mathcal{U}_\nu^{(n)}$, the outcome is a poly-analytic function of order n (See Proposition 3.1). Next, we give the necessary and sufficient condition on a Poly-analytic function F of order n to be be equal to $F = \mathcal{U}_\nu^{(n)}(f)$ for some $f \in H^{2,\nu}_\Gamma(\mathbb{C})$ is given in Theorem 3.1. It is also proved that the operator $\mathcal{U}_\nu^{(n)}$ preserves the automorphic equation (Proposition 3.2) and the orthogonality of the functions (Lemma 3.4). Finally, it should be noted that the operator $(\sqrt{\nu^n !})^{-1}\mathcal{U}^{(n)}\nu$ is unitary from $H^{2,\nu}_\Gamma(\mathbb{C})$ to $L^{2,\nu}_\Gamma(\mathbb{C})$. This property is crucial to guarantee the consistency and validity of calculations performed using this operator in order to provide a concrete characterization of the range of the operator $\mathcal{U}_\nu^{(n)}$ on the Hilbert space $H^{2,\nu}_\Gamma(\mathbb{C})$. Now, in Theorem 3.2, it is shown that

$$\mathcal{U}_\nu^{(n)}\left(H^{2,\nu}_\Gamma(\mathbb{C})\right) = Aut^{2,\nu}_\Gamma(\mathbb{C}) \cap A_n(\mathbb{C}) \cap \ker(\mathcal{D}_\nu^{(n)} - \nu^n n!) = \mathcal{F}^{2,\nu}_{\Gamma,n}(\mathbb{C}).$$

The construction of orthogonal bases of the theta generalized Bargmann–Fock spaces $\mathcal{F}^{2,\nu}_{\Gamma,n}(\mathbb{C})$ for one and two rank discrete subgroups is considered as a direct application of the obtained results.

- Chapter 4: In this chapter, we focus on the space of mixed automorphic forms $\mathscr{M}^{\nu,\mu}_{\Gamma,\tau}(\mathbb{C})$. We begin with a detailed analysis of the essential properties of equivariant pairs (ρ,τ), including their characterizations, such as cases in which ρ separates variables, as in $\rho = \tilde{\rho} \ltimes \tau$. This exploration aims to reveal their fundamental characteristics and illustrate their relevance to our field of study. We then provided a brief overview of the space of mixed automorphic functions and delved into the geometric representation of the magnetic Schrödinger operator Δ, given by

$$\Delta^{\nu,\mu}_\tau = 4\frac{\partial^2}{\partial z \partial \bar{z}} + 2\left(S^{\nu,\mu}_\tau \frac{\partial}{\partial z} - \overline{S^{\nu,\mu}_\tau}\frac{\partial}{\partial \bar{z}}\right) - |S^{\nu,\mu}_\tau|^2 + \mu(\tau\Delta\bar{\tau} - \tau\Delta\tau)$$

  and we investigate their variance property with respect to the automorphic action, as well as the connection to the Landau Hamiltonian. Furthermore, we can demonstrate that the eigenvalue problem of Δ on $\mathcal{M}^{\nu,\mu}_{\rho,\tau}(\mathbb{C})$ is equivalent to the eigenvalue problem of Δ_α on the space of classical automorphic functions $Aut^\nu_{\Gamma,\chi}(\mathbb{C})$ for a specific α. The lifting theorem is discussed and proven. It is then employed to describe and derive the concrete spectral analysis of this class, demonstrating that the same isomorphic transformation maps $\mathcal{M}^{\nu,\mu}_{\rho,\tau}(\mathbb{C})$ to the space of classical automorphic functions $\mathcal{F}^{B^{\mu,\nu}_\tau}_{\Gamma,\chi\tau}(\mathbb{C})$, thanks to the transformation $\mathcal{W}^{\mu,\nu}_\tau$ gevin by $\left[\mathcal{W}^{\mu,\nu}_\tau f\right](z) := e^{i\varphi^{\nu,\mu}_\tau(z)} f(z)$. Lastly, our effort to extend the main results to higher dimensions is presented.

Chapter 1

Classical automorphic function space associated with maximal rank discrete subgroups of $(\mathbb{C},+)$

In this chapter, we will give some study on group theory, especially on discrete subgroups on $\mathbb{C}$. And we introduce and discuss some basic properties of classical automorphic functions. To this end, we begin by fixing notations in this context. We begin by introducing the general context in which we shall consider automorphic forms and functions. Before moving on to discuss the objective of this part, let us give a short overview of the subject. Let $\mathbb{C}^n$ denote the n-dimensional complex space, equipped with its Hermitian form $\langle z, w\rangle = z_1\bar{w}_1 + \cdots + z_n\bar{w}_n$, and let $\omega(z,w) = \mathrm{Im}(\langle z, w\rangle)$ be the corresponding symplectic form. Given a fixed $\nu > 0$.

1.1 On the discrete subgroups

Given a topological space X and a group G, we say that $(G,*)$ acts continuously on X (on the left) if there exists a map $\cdot : G \to F(X,X)$ (functions from X to X), $g \mapsto g\cdot$ which satisfies:

- (i) $x \mapsto g\cdot x$ is a continuous function of x for all $g \in G$;
- (ii) $g\cdot(g'\cdot x) = (g*g')\cdot x, \quad$ for all $g, g' \in G, x \in X$;
- (iii) $e\cdot x = x$, for any element $x \in X$ and where e represents the identity element in the group G.

Example 1.1. *Let $G = \mathbb{Z}$ be the additive group of integers. Then it is easy to verify that the group $\mathbb{Z}$ acts continuously on complex numbers $\mathbb{C}$ with the group action $\cdot$ defined by $n\cdot z = n+z$, for all $n \in \mathbb{Z}$, $z \in \mathbb{C}$.*

For a given continuous action of a group on a topological space X, we define the equivalence relation $(\mathrm{mod} G)$, where for all elements x and y in X, there exists an element $g \in G$ such that $y = g \cdot x$. We set $Gx := \{g \cdot x / g \in G\}$ to be the equivalence class or orbit of x, and let $G/X = \{Gx; x \in X\}$ denote the set of orbits. We say that the action of G on X is transitive if there is only one G-orbit in X (which is necessarily X), which can also be expressed as

$$\forall (x, y) \in X \times X, \exists g \in G; \quad y = g \cdot x.$$

In this case, we say that X is a homogeneous G-set. In this opposite case, we say that G is intransitive on X, or that G is intransitive on X. We take the action given in the example 1.1. This action is also intransitive.

To define the notion of a discrete subgroup, we recall that each neighborhood of a given point in a topological group G is homeomorphic to a neighborhood of the identity (See [28]).

Definition 1.1. *Let G a group acting continuously on a topological space X. We say a subset $\Gamma \subset G$ is discrete if for any two compact subsets $A, B \subset X$, there are only finitely many $g \in \Gamma$ such that $(g \cdot A) \cap B \neq \phi$, where ϕ denotes the empty set.*

For example, if we consider the set Ω_f, which consists of the periods of a non-constant periodic function f defined on the complex plane $\mathbb{C}$. It can be observed that Ω_f forms a discrete additive subgroup within $\mathbb{C}$[41].

1.1.1 Discrete subgroups of $(\mathbb{C}, +)$

Theorem 1.1. *[41] If Ω is a discrete subgroup of $\mathbb{C}$ then it is one of the following cases:*

(i) $\Omega = \{0\}$*;*

(ii) $\Omega = \{n\omega_1 \mid n \in \mathbb{Z}\}$ *for some fixed* $\omega_1 \in \mathbb{C}\backslash\{0\}$*, and so* Ω *is isomorphic to* $\mathbb{Z}$*;*

(iii) $\Omega = \{m\omega_1 + n\omega_2 \mid m, n \in \mathbb{Z}\}$ *for some fixed* $\omega_1, \omega_2 \in \mathbb{C}$*, where* ω_1 *and* ω_2 *are* $\mathbb{R}$*-linearly independent, that is, in this case,* Ω *is isomorphic to* $\mathbb{Z} \times \mathbb{Z}$*.*

Remark 1.1. *A function f with a set Ω_f of type (ii) is simply periodic. However, if Ω_f is of type (iii), it is doubly periodic. Groups Ω of type (iii) are referred to as lattices, and any pair ω_1, ω_2 such that $\Omega = \{m\omega_1 + n\omega_2 \mid m, n \in \mathbb{Z}\}$ is called a basis for the lattice. The results above demonstrate that periodic functions are either simply or doubly periodic, with the exceptions of constant functions (where $\Omega = \mathbb{C}$) and non-meromorphic functions (where Ω can be any subgroup of $\mathbb{C}$, not necessarily discrete). In the case where Ω is a lattice, the action of Ω on $\mathbb{C}$ is transitive.*

Remark 1.2. *We shon notice that any $\Gamma = \mathbb{Z}\omega_1 + \mathbb{Z}\omega_2$ with ω_1 and ω_2 are $\mathbb{R}$-linearly independent, if and only if Γ is isomorphic to $\mathbb{Z} + \tau\mathbb{Z} = \mathbb{Z}[\tau]$ for some $\tau \in \mathbb{C}$ such that $\Im m(\tau) > 0$).*

1.1.2 Fundamental domains

Given a discrete subgroup Γ of $\mathbb{C}$, we say that $z_1, z_2 \in \mathbb{C}$ at congruent $\operatorname{mod}(\Gamma)$, and we write $z_1 \sim z_2$, if $z_1 - z_2 \in \Gamma$. Congruence $\operatorname{mod}(\Gamma)$ is easily seen to be an equivalence relation on $\mathbb{C}$, and the equivalence classes are the cosets $z + \Gamma$ Γ-orbit at z of Γ in the additive group $\mathbb{C}$. Then two points $z_1, z_2 \in \mathbb{C}$ are congruent $\operatorname{mod}(\Gamma)$ this happens if and only if they belong to the same group of points affected by the action of Γ. Alternatively, we may regard Γ as acting on $\mathbb{C}$ as a transformation group, each $\omega \in \Gamma$ inducing the translation map

$$t_\omega : z \mapsto z + \omega,$$

such that

$$t_{(\omega_1+\omega_2)} = t_{\omega_1} \circ t_{\omega_2}.$$

We have then a group isomorphism $\Gamma \cong \{t_\omega \mid \omega \in \Gamma\}$.

Definition 1.2. *A closed, connected subset* $\Lambda(\Gamma)$ *of* $\mathbb{C}$ *is defined to be a fundamental domain for* Γ *if*

1. *for each* $z \in \mathbb{C}, \Lambda(\Gamma)$ *contains at least one point in the same* Γ*-orbit at* z *(i.e. every point* $z \in \mathbb{C}$ *is congruent to some point in* $\Lambda(\Gamma)$ *),*
2. *no two points in the interior of* $\Lambda(\Gamma)$ *are in the same* Γ*-orbit (i.e. no pair of points in the interior of* $\Lambda(\Gamma)$ *are congruent).*

It is also true to say that we have $\mathbb{C} = \bigcup\limits_{\gamma\in\Gamma} \Lambda(\Gamma) + \gamma$.

For example for a lattice Γ, the domain of Dirichlet

$$D(\Gamma) = \{z \in \mathbb{C}/|z| \leqslant |z - w| \text{ for all } w \in \Gamma\}$$

is known to be fundamental for Γ. This is the set of points which are at least as close to 0 as they are to any other lattice-point. Clearly, $0 \in D(\Gamma)$, and as Γ is discrete $D(\Gamma)$ contains some neighbourhood of 0. For each $\omega \in \Gamma \backslash \{0\}$ the set $\{z \in \mathbb{C} | |z| \leqslant |z - \omega \ |\}$ is a closed half-plane bounded by the perpendicular bisector of the line segment joining 0 and ω. Thus $D(\Gamma)$ is an intersection of half-planes, each of which is convex, and so $D(\Gamma)$ is convex. Since $D(\Gamma)$ is an intersection of just finitely many half-planes, and since $D(\Gamma)$ has non-empty interior (containing 0 for example) it follows that $D(\Gamma)$ is a polygon. For this reason the domain of Dirichlet is often called the Dirichlet polygon see [9].

Figs 1.1 and 1.2 illustrate two examples of Dirichlet polygons. In the most general case, as in Fig 1.2, $D(\Gamma)$ is a hexagon with its opposite sides parallel and of equal lengths, but when Γ is "rectangular", that is, Γ has a pair of perpendicular generators, as in Fig 1.2, then $D(\Gamma)$ is a rectangle.

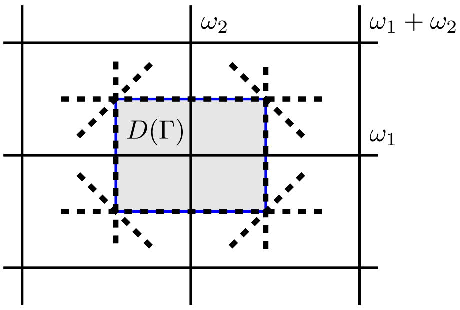

Figure 1.1

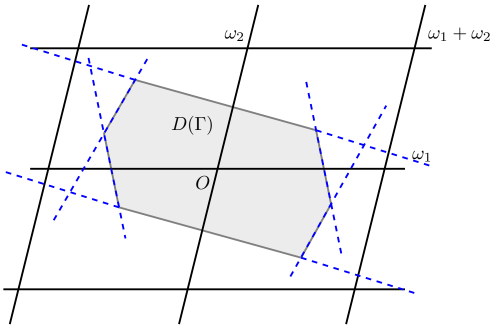

Figure 1.2

Remark 1.3. *We have seen that a fundamental region for a lattice is not unique.*

If, as is usually the case, $\Lambda(\Gamma)$ is also a Euclidean polygon, with a finite number of sides, then we call $\Lambda(\Gamma)$ a fundamental polygon for Γ, in particular, if $\Lambda(\Gamma)$ is a parallelogram, then it is called a fundamental parallelogram for Γ. For example, the parallelogram $\Lambda(\Gamma)$, shown in the figure 1.3, with vertices $0, \omega_1, \omega_2, \omega_1 + \omega_2$, is a fundamental parallelogram for the lattice $\Gamma = \mathbb{Z}\omega_1 + \mathbb{Z}\omega_2$. Thus, figure 1.4 represents the fundamental domain in the case of $\Gamma = \mathbb{Z}\omega$ with $\omega \in \mathbb{R}$.

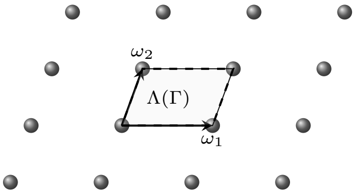

Figure 1.3

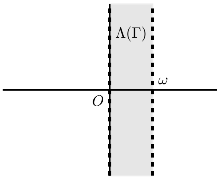

Figure 1.4

Remark 1.4. *Specifically, for the group* $\Gamma = \mathbb{Z}$, *the corresponding fundamental domain is the rectangular strip* $S = \mathbb{R} \times [0, 1[$.

The figures above were generated in PDF format using LaTeX, with some modifications to the codes taken from thesis [20]. These adjustments were necessary to adapt the graphics to our own research context. This approach enabled us to customize the figures while benefiting from the foundations provided by thesis [20], thus ensuring a certain visual coherence while meeting our specific needs.

1.1.3 Discrete subgroups of $U(1) \ltimes \mathbb{C}$

In this section, we recall a detailed description of the discrete subgroups Γ of the group $U(1) \ltimes \mathbb{C}$, a description that proves to be essential in Chapter 4. Let G denote the groupe $U(1) \ltimes \mathbb{C}$ defined by

$$G := U(1) \ltimes \mathbb{C} = \left\{ g := [a,b] \equiv \begin{pmatrix} a & b \\ 0 & 1 \end{pmatrix}; \quad a \in U(1), b \in \mathbb{C} \right\} \tag{1.1.1}$$

The law on $U(1) \ltimes \mathbb{C}$ is defined by $[a,b][a',b'] = [aa', ab'+b]$, and therefore in this context, the inverse of an element $[a,b]$ in $U(1) \ltimes \mathbb{C}$ is $[a,b]^{-1} = \left[\frac{a}{b}, -\frac{b}{a}\right]$. The semidirect group $U(1) \ltimes \mathbb{C}$ acts on the complex plane $\mathbb{C}$ by the holomorphic mappings

$$[a,b] \cdot z = az + b, \quad \text{for all } [a,b] \in G, z \in \mathbb{C}.$$

Subsequently, the subgroups Γ within the group $U(1) \ltimes \mathbb{C}$ take the form $\Gamma = U \ltimes L$, where U represent a finite-order subgroup of $U(1)$ and L denote a lattice in $\mathbb{C}$. Furthermore, it can be readily verified that the condition $UL \subset L$ is both necessary and sufficient. To illustrate this necessity, for any arbitrary $a \in U$ and $b \in L$, we have

$$[a,0][1,b] = [a,ab] \in \Gamma$$

so that $ab \in L$.Therefore, using Theorem 1.1 we can assert the following

Proposition 1.1. *[20] Let* $\Gamma = U \times L$ *be a discrete subgroup of* $U(1) \ltimes \mathbb{C}$*. Then, L is in one of the forms of Theorem 1.1. With in case* (iii)*, U is the group of q-th roots of the unity for some positive integer q that depends on L,* $U = U_q := \{z \in \mathbb{C} \mid z^q = 1\}$.

Subsequently, by fixing L in one of the three forms given in Proposition 1.1, we can determinate the corresponding q for which $\Gamma = \mathrm{U}_q \times L$ is a subgroup of G, i.e, the integers q for which $\mathrm{U}_q L \subset L$:

1. The case $L = \{0\}$: It's simple to observe that $U_q\{0\} \subset \{0\}$ for every $q = 0, 1, 2, \cdots$, and then, $U_q \times \{0\}$ is a subgroup of G for every $q = 0, 1, 2, \cdots$.
2. The case $L = \mathbb{Z}\omega_1$: For $q = 1$ or 2 , we have $\mathrm{U}_q\mathbb{Z}\omega_1 \subset \mathbb{Z}\omega_1$. This is trivially true because $\mathrm{U}_1 = \{1\}$ and $\mathrm{U}_2 = \{-1, 1\}$. If we suppose the existence of some $q \geq 3$ such $\mathrm{U}_q\{0\} \subset \mathbb{Z}\omega_1$, it follows that $e^{\frac{2\pi}{9}}\omega_1 = n\omega_1$ and therefore $e^{\frac{2\pi}{9}} = n$ for some $n \in \mathbb{Z}$, which is impossible. Thus, the only values that can be taken by q are 1 and 2 . Hence,

$$\Gamma = \{1\} \ltimes \mathbb{Z}\omega_1 \quad \text{or} \quad \Gamma = \{1,-1\} \ltimes \mathbb{Z}\omega_1$$

3. The case $L = \mathbb{Z}\omega_1 + \mathbb{Z}\omega_2$: We can show that the condition $\mathrm{U}_q(\mathbb{Z}\omega_1 + \mathbb{Z}\omega_1) \subset \mathbb{Z}\omega_1 + \mathbb{Z}\omega_1$ gives a necessary condition on q that restrict it to take no values other than $\{1, 2, 3, 4$ and $6\}$.

Remark 1.5. *Uniform lattice* Γ *of the additive group* $(\mathbb{C}, +)$ *can be seen as a discrete subgroup of* $U(1) \ltimes \mathbb{C}$ *by the identification*

$$\gamma \in \Gamma \mapsto [1, \gamma] = \begin{pmatrix} 1 & \gamma \\ 0 & 1 \end{pmatrix}$$

1.2 Spaces of classical automorphic functions

Let Γ is a discrete subgroup of the complex plane $\mathbb{C}$, and let $\Lambda(\Gamma)$ represent a fundamental region for Γ. Furthermore, consider a real number $\nu > 0$. We examine the mapping χ defined on Γ with its range restricted to the unit circle $U(1)$. A (Γ, χ, ν)-automorphic function is a complex valued function f on $\mathbb{C}$ whichin solution of the functional equation

$$f(\gamma + z) = \chi(\gamma) e^{\nu < z + \frac{\gamma}{2}, \gamma >} f(z) \tag{1.2.2}$$

for every $z \in \mathbb{C}$ and $\gamma \in \Gamma$. We also say f is an automorphic function on $\mathbb{C}$ associated to the triplet (Γ, χ, ν). We denote by $Aut^{\nu}_{\Gamma,\chi}(\mathbb{C})$ thir space.

1.2.1 Basic properties of the space $\mathcal{H}^{\nu}_{\Gamma,\chi}(\mathbb{C})$

We consider the functional space $\mathcal{H}^{\nu}_{\Gamma,\chi}(\mathbb{C})$ of holomorphic functions f defined on the complex plane $\mathbb{C}$ and subject to the following functional equation: (1.2.2).

$$\mathcal{H}^{\nu}_{\Gamma,\chi}(\mathbb{C}) := \left\{ f \in \mathcal{H}ol(\mathbb{C}) / f(z+\gamma) = \chi(\gamma) e^{\nu \left\langle z + \frac{\gamma}{2}, \gamma \right\rangle} f(z) \quad \forall z \in \mathbb{C} \quad \forall \gamma \in \Gamma \right\}$$

with $\mathcal{H}ol(\mathbb{C})$ denotes the space of holomorphic functions on $\mathbb{C}$. Wereview from [24], we get

Proposition 1.2. *[24] The complex vector space* $Aut^{\nu}_{\Gamma,\chi}(\mathbb{C})$ *is not empty if and only if the triplet* (ν, Γ, χ) *meets the following condition*

$$\chi(\gamma_1 + \gamma_2) = \chi(\gamma_1) \chi(\gamma_2) e^{i\nu\omega(\gamma_1, \gamma_2)} \tag{1.2.3}$$

for every $\gamma_1, \gamma_2 \in \Gamma$. *The condition satisfied by the triplet* (ν, Γ, χ) *in* (1.2.3) *is called "Riemann-Dirac Quantization", noted by* (***RDQ***).

Note that in case of $\Gamma = \mathbb{Z}\omega$, the condition (**RDQ**) holds again, and it is both a necessary and sufficient condition for the corresponding $Aut^{\nu}_{\Gamma,\chi}(\mathbb{C})$ and $\mathcal{H}^{\nu}_{\Gamma,\chi}(\mathbb{C})$ spaces to be non-trivial. Therefore, χ is a character of $\mathbb{Z}\omega$ and is thus entirely determined, as follows:

$$\chi(\gamma) = e^{2i\pi\alpha m}$$

for a specific constant real number α, where $\gamma = m\omega \in \mathbb{Z}\omega$.

The key tool for the sufficient condition of proof is the Poincaré operator, defined as follows:

$$\left[\mathcal{P}^{\nu}_{\Gamma,\chi}\psi\right](z) = \sum_{\gamma \in \Gamma} \overline{\chi(\gamma)} e^{-\nu \left\langle z + \frac{\gamma}{2}, \gamma \right\rangle} \psi(z + \gamma).$$

This operator is properly defined as a smooth function on the complex plane $\mathbb{C}$. Additionally, as Γ is a discrete set, and given that ψ has its support contained within $\Lambda(\Gamma)$, it is evident that for any z within the support of ψ, we obtain the following,

$$\mathcal{P}^{\nu}_{\Gamma,\chi}\psi(z) = \psi(z)$$

It follows $\mathcal{P}^{\nu}_{\Gamma,\chi}\psi/\operatorname{Supp}\psi = \psi \not\equiv 0$. This is equivalent to saying that the function $\mathcal{P}^{\nu}_{\Gamma,\chi}\psi$ is nonzero function on $\mathbb{C}$ and characterize automorphic functions.

According to the (RDQ) condition, the map χ satisfies the following characteristics:

$$\chi(0) = 1 \quad \text{and} \quad \chi(-\gamma) = \overline{\chi(\gamma)}$$

Moreover, if we switch the positions of γ_1 and γ_2 in the context of (RDQ) and consider the antisymmetric property of the symplectic form $\omega(\cdot,\cdot)$, we inevitably arrive at the following

$$\nu\omega\left(\gamma_1,\gamma_2\right) \in \pi\mathbb{Z}$$

for every $\gamma_1, \gamma_2 \in \Gamma$.

Moreover, the mapping $J_{\nu,\chi}(\gamma, z) := \chi(\gamma)e^{\nu z\overline{\gamma}+\frac{\nu}{2}|\gamma|^2}$ called automorphy factor, satisfies the cocycle identity,

$$J_{\nu,\chi}(\gamma_1+\gamma_2, z) = J_{\nu,\chi}(\gamma_1, \gamma_2+z)J_{\nu,\chi}(\gamma_2, z).$$

for everything $\gamma_1, \gamma_2 \in \Gamma$ and $z \in \mathbb{C}$. We have $J_{\nu,\chi} = \chi j^{\nu}$ where j^{ν} is the Appell-Humbert holomorphy factor

$$j^{\nu}(\gamma, z) := e^{\nu\left(z+\frac{\gamma}{2}\right)\overline{\gamma}}. \tag{1.2.4}$$

Lemma 1.1. *Let f and g be two (Γ,χ,ν)-automorphic functions, then the function $\phi : z \longmapsto f(z)\overline{g(z)}e^{-\nu|z|^2}$ is Γ-periodic on $\mathbb{C}$. i.e. $\phi(z+\gamma) = \phi(z)$ for all $z \in \mathbb{C}$ and $\gamma \in \Gamma$.*

Proof. Let f, g satisfies the condition 1.2.2. Let $\gamma \in \Gamma$, and $z \in \mathbb{C}$. We have that

$$\begin{aligned}
\phi(\gamma+z) &= f(\gamma+z)\overline{g(\gamma+z)}e^{-\nu|\gamma+z|^2}\\
&= \chi(\gamma)e^{\nu<z+\frac{\gamma}{2},\gamma>}\overline{\chi(\gamma)e^{\nu<z+\frac{\gamma}{2},\gamma>}}f(z)\overline{g(z)}e^{-\nu(|\gamma|^2+2\Re e<\gamma,z>+|z|^2)}\\
&= e^{\nu\left(2\Re e<z+\frac{\gamma}{2},\gamma>\right)}f(z)\overline{g(z)}e^{-\nu(|\gamma|^2+2\Re e<\gamma,z>+|z|^2)}\\
&= f(z)\overline{g(z)}e^{-\nu|z|^2} = \phi(z).
\end{aligned}$$

□

1.2.2 Square integrable (Γ,χ,ν)-automorphic functions

Denote by $d\lambda(z) = dxdy$; $z = x+iy \in \mathbb{C}$, the Lebesgue measure on $\mathbb{C}$. Let's also consider $\Lambda(\Gamma)$ as a fundamental region in $\mathbb{C}$, which represents the orbit group $\mathbb{C}/\Gamma$. We refer to the vector space $\mathcal{L}^{2,\nu}_{\Gamma,\chi}(\mathbb{C})$ as the space of measurable functions that are (Γ,χ,ν)-automorphic on $\mathbb{C}$ and satisfy the following condition

$$\|f\|^2_{\Gamma,\nu} = \int_{\Lambda(\Gamma)} |f(z)|^2 e^{-\nu|z|^2} d\lambda(z) < +\infty.$$

Note that for f satisfying the functional equation (1.2.2), the value of the expression $\int_{\Lambda(\Gamma)} |f(z)|^2 e^{-\nu|z|^2} d\lambda(z)$ is meaningful and does not depend on the fundamental domain $\Lambda(\Gamma)$ (see Lemma 1.1), and $\|f\|_{\Gamma,\nu}$ can define a norm whichs is associated to the hermitian inner scalar product

$$\langle f_1, f_2\rangle_{\Gamma,\nu} := \int_{\Lambda(\Gamma)} f_1(z)\overline{f_2(z)} e^{-\nu|z|^2} d\lambda(z). \tag{1.2.5}$$

The space $\mathcal{L}^{2,\nu}_{\Gamma,\chi}(\mathbb{C})$ is a Hilbert space equipped with a Hermitian inner scaler product $\langle\cdot,\cdot\rangle_{\Gamma,\nu}$ [59].

Assume that the triplet (Γ, χ, ν) is verified the (RDQ) condition, we also examine the function $K^{\nu}_{\Gamma,\chi}(z, w)$, which is defined on $\mathbb{C} \times \mathbb{C}$ as the convergent series:

$$K^{\nu}_{\Gamma,\chi}(z, w) := \left(\frac{\nu}{\pi}\right) e^{\nu\langle z,w\rangle} \sum_{\gamma\in\Gamma} \chi(\gamma) e^{-\frac{\nu}{2}|\gamma|^2+\nu(\langle z,\gamma\rangle-\overline{\langle w,\gamma\rangle})} \tag{1.2.6}$$

Theorem 1.2. *[24] The conditions below, we have:*

- $i)$ *For any pair of complex numbers z and w, we have*

$$K^{\nu}_{\Gamma,\chi}(z, w) = \overline{K^{\nu}_{\Gamma,\chi}(w, z)}$$

- $ii)$ *For every $\gamma_1, \gamma_2 \in \Gamma$ and $z, w \in \mathbb{C}$, we have*

$$K^{\nu}_{\Gamma,\chi}(z + \gamma_1, w + \gamma_2) = \chi(\gamma_1) e^{\nu\left\langle z+\frac{\gamma_1}{2},\gamma_1\right\rangle} \overline{\chi(\gamma_2)} e^{\nu\left\langle \gamma_2,w+\frac{\gamma_2}{2}\right\rangle} K^{\nu}_{\Gamma,\chi}(z, w).$$

  Specifically, the function $z \longmapsto K^{\nu}_{\Gamma,\chi}(z, w)$ belongs to $\mathcal{H}^{\nu}_{\Gamma,\chi}(\mathbb{C})$ for every $w \in \mathbb{C}$ fixed.

- $iii)$ *We use $\mathrm{vol}(\Gamma)$ to represent the Lebesgue volume of the fundamental domain of the lattice Γ. Thus, we have:*

$$\int_{\Lambda(\Gamma)} K^{\nu}_{\Gamma,\chi}(z, z) e^{-\nu\omega(z,z)} d\lambda(z) = \left(\frac{\nu}{\pi}\right) \mathrm{vol}(\Gamma)$$

The proof of this theorem is straightforward, except for the third point, which is based on the following equality

$$\int_{\Lambda(\Gamma)} e^{2i\nu\omega(z,\gamma)} d\lambda(z) = 0$$

for every $\gamma \in \Gamma\backslash\{0\}$.

Theorem 1.3. *1. Let $f \in \mathcal{H}^{\nu}_{\Gamma,\chi}(\mathbb{C})$, then a constant $C > 0$ exists, such that*

$$|f(z)| \leqslant C e^{\frac{\nu}{2}|z|^2}$$

for every $z \in \mathbb{C}$.

2. *$\mathcal{H}^{\nu}_{\Gamma,\chi}(\mathbb{C})$ is a reproducing Hilbert space with the reproducing kernel defined as $K^{\nu}_{\Gamma,\chi}(\cdot,\cdot)$, i.e.,*

$$f(z) = \int_{\Lambda(\Gamma)} K^{\nu}_{\Gamma,\chi}(z,w) f(w) e^{-\nu|w|^2} d\lambda(w).$$

For every $f \in \mathcal{H}^{\nu}_{\Gamma,\chi}(\mathbb{C})$.

Thanks to the first part of this theorem, it becomes evident that $\mathcal{H}^{\nu}_{\Gamma,\chi}(\mathbb{C}) \subset \mathcal{L}^{2,\nu}_{\Gamma,\chi}(\mathbb{C})$. The second part can be proven by employing the following lemma.

Lemma 1.2. *[63] For any holomorphic function g in the set of holomorphic functions (denoted as $\mathcal{H}ol(\mathbb{C})$) that meets the growth condition $|g(z)| \leqslant Ce^{\frac{\nu}{2}|z|^2}$, we get the following reproducing formula:*

$$g(z) = \frac{\nu}{\pi} \int_{\mathbb{C}} e^{\nu\langle z,w\rangle} g(w) e^{-\nu|w|^2} d\lambda(w).$$

We observe that we can determine the dimension of the Hilbert space $\mathcal{H}^{\nu}_{\Gamma,\chi}(\mathbb{C})$ by integrating its reproducing kernel $K^{\nu}_{\Gamma,\chi}(z,w)$ along the diagonal (as described in [63]). This is expressed as:

$$\dim \mathcal{H}^{\nu}_{\Gamma,\chi}(\mathbb{C}) = \int_{\Lambda(\Gamma)} K^{\nu}_{\Gamma,\chi}(z,z) e^{-\nu|z|^2} d\lambda(z)$$

So we will note these results in the following corollary.

Corollary 1.1. *$\mathcal{H}^{\nu}_{\Gamma,\chi}(\mathbb{C})$ is a space of finite dimension, with a dimension is*

$$\dim \mathcal{H}^{\nu}_{\Gamma,\chi}(\mathbb{C}) = \frac{\nu}{\pi} \operatorname{vol}(\Lambda(\Gamma))$$

Remark 1.6. *[25] Regarding the case of $\Gamma = \mathbb{Z}$, the space $\mathcal{H}^{\nu}_{\mathbb{Z},\chi_\alpha}(\mathbb{C})$ is of infinite dimension. Furthermore, for any function f in $\mathcal{H}^{\nu}_{\mathbb{Z},\chi_\alpha}(\mathbb{C})$, there exists a unique sequence of complex numbers $(a_n)_{n\in\mathbb{Z}}$ such that*

$$f(z) = \sum_{n\in\mathbb{Z}} a_n e^{\frac{\nu}{2}z^2 + 2i\pi(\alpha+n)z}$$

, where α is the real number such that $\chi_\alpha(m) = e^{2i\pi\alpha m}$ for all $m \in \mathbb{Z}$.
If we define $e^{\alpha,\nu}_n(z) = e^{\frac{\nu}{2}z^2 + 2i\pi(\alpha+n)z}$, we can further observe that the family of functions $\{e^{\alpha,\nu}_n; n \in \mathbb{Z}\}$ forms an orthogonal basis for the space $\mathcal{O}^{2,\nu}_{\Gamma,\alpha}(\mathbb{C}) = \mathcal{H}^{\nu}_{\mathbb{Z},\chi_\alpha}(\mathbb{C}) \cap \mathcal{L}^{2,\nu}_{\mathbb{Z},\chi_\alpha}(\mathbb{C})$. With

$$\|e^{\alpha,\nu}_n\|^2_{\mathbb{Z},\nu} = \left(\frac{\pi}{2\nu}\right) e^{\frac{2\pi^2}{\nu}(n+\alpha)^2}.$$

Moreover, the space $\mathcal{O}^{2,\nu}_{\Gamma,\alpha}(\mathbb{C})$ is a Hilbert space whose reproducing kernel function is $K_{\alpha,\nu}(\cdot,\cdot)$, which is explicitly defined as follows:

$$K_{\alpha,\nu}(z,w) = \left(\frac{2\nu}{\pi}\right)^{1/2} e^{\frac{\nu}{2}\left(z^2+\bar{w}^2\right)} \theta_{\alpha,0}\left(z-\bar{w} \mid \frac{2i\pi}{\nu}\right).$$

where $\theta_{\alpha,0}$ represents the Riemann theta function, defined as

$$\theta_{\alpha,\beta}(z \mid \tau) := \sum_{n\in\mathbb{Z}} e^{i\pi(n+\alpha)^2\tau + 2i\pi(n+\alpha)(z+\beta)}.$$

The space $\mathcal{H}^{\nu}_{\Gamma,\chi}(\mathbb{C})$ is of interest on its own because it is somehow connected to the classical Bargmann-Fock space

$$\mathcal{B}^{2,\nu}(\mathbb{C}) = \left\{ g \in \mathcal{H}ol(\mathbb{C}); \quad \int_{\mathbb{C}} |g(z)|^2 e^{-\nu|z|^2} d\lambda(z) < +\infty \right\}.$$

In fact, based on the explicit expression of the reproducing kernel $K^{\nu}_{\Gamma,\chi}$ of the space $\mathcal{H}^{\nu}_{\Gamma,\chi}(\mathbb{C})$, it appears as a (Γ,χ)-periodic function, influenced by the automorphic factor $\chi(\gamma)e^{\frac{\nu}{2}|\gamma|^2+\nu\langle w,\gamma\rangle}$, of the reproducing kernel $\frac{\nu}{\pi}e^{\nu\langle w,\gamma\rangle}$ of the Bargmann-Fock space $\mathcal{B}^{2,\nu}(\mathbb{C})$ ([38]).
Finally, let us notice that everything that has been done in this chapter in $\mathbb{C}$ is thus valid for $\mathbb{C}^n$, for $n \geqslant 2$ integer.

1.3 Mixed automorphic functions

Mixed automorphic functions (MAFs) generalize ellipticmodular forms and can be tracked back to Stiller [61]. They arise naturally in the description of holomorphic forms of highest degree on family of abelian varieties presented as Riemann surfaces [34], and are defined on a manifold M, on which a discrete group Γ acts properly and discontinuously, as functions satisfying the functional equation

$$f(\gamma\cdot z) = J(\gamma,z)J(\rho(\gamma),\tau(z))f(z)$$

for given equivariant pair (ρ,τ) (see Definition 1.3 below) and automorphic factor $J(\gamma,z)$. The case of the hyperbolic plane was extensively studied by Min Ho Lee [45] and his co-workers, whom analyzed different aspects of associated theory. The planar euclidean case $M=\mathbb{C}$ with respect to full rank lattice Γ of $(\mathbb{C},+)$ and given automorphic factor $j^{v}(\gamma,z) := \chi(\gamma)e^{iv\Im m\langle z,\gamma\rangle}$, was considered in [21, 27]. Here $v>0$, $\langle z,w\rangle = z\bar{w}$ is the usual hermitian inner product on $\mathbb{C}$, and χ is a specific mapping (called a pseudo-character) on Γ that takes values in the unitary group $U(1)$ and meets the following conditions

$$\chi(\gamma\gamma') = e^{iv\Im m\langle\gamma',\gamma\rangle}\chi(\gamma)\chi(\gamma') \text{ for all } \gamma,\gamma'\in\Gamma.$$

We denote by G the group $U(1)\ltimes\mathbb{C}$ (1.1.1), that we realize in matrix representation as

$$G := U(1)\ltimes\mathbb{C} = \left\{ g := [a,b] \equiv \begin{pmatrix} a & b \\ 0 & 1 \end{pmatrix}; \quad a\in U(1), b\in\mathbb{C} \right\} \tag{1.3.7}$$

G acts on the complex plane $\mathbb{C}$ by the holomorphic mappings $g\cdot z = az+b$, where $g=[a,b]\in G$, and $z\in\mathbb{C}$. We notice that this action is transitive.

Definition 1.3. *By G-equivariant pair (ρ,τ), we mean a G-endomorphism ρ and a compatible mapping $\tau:\mathbb{C}\longrightarrow\mathbb{C}$, in the sense that*

$$\tau(g\cdot z) = \rho(g)\cdot\tau(z) \tag{1.3.8}$$

holds true for every $g\in G$ and $z\in\mathbb{C}$.

Now, for given equivariant pair (ρ, τ), we define $J^{\nu,\mu}_{\rho,\tau}$ to be the complex-valued mapping

$$J^{\nu,\mu}_{\rho,\tau}(g,z) := j^{\nu}(g,z) j^{\mu}(\rho(g), \tau(z)); \quad (g,z) \in G \times \mathbb{C}$$

where, ν, μ are the real numbers given, and $j^{\alpha}; \alpha \in \mathbb{R}$ is the "automorphic factor" given by

$$j^{\alpha}(g,z) = e^{2i\alpha\omega(z, g^{-1}\cdot 0)}$$

In this sense we have, the automorphic factor $j^{\alpha}(\cdot,\cdot)$ is satisfied with the following equation

$$j^{\alpha}(gg', z) = e^{2i\alpha\omega\left(g^{-1}\cdot 0, g'\cdot 0\right)} j^{\alpha}(g, g'\cdot z)\, j^{\alpha}(g', z) \tag{1.3.9}$$

for all $g, g' \in G$, and $z \in \mathbb{C}$ [27].

Proposition 1.3. *[27] The mapping $J^{v,\mu}_{\rho,\tau}$ satisfies the chain rule*

$$J^{v,\mu}_{\rho,\tau}(gg', z) = e^{2i\phi^{\nu,\mu}_{\rho}(g,g')} J^{\nu,\mu}_{\rho,\tau}(g, g'\cdot z)\, J^{\nu,\mu}_{\rho,\tau}(g', z) \tag{1.3.10}$$

where $\phi^{\nu,\mu}_{\rho}(g,g')$ is the real-valued function defined on $G \times G$ by

$$\phi^{v,\mu}_{\rho}(g,g') := \Im m\left(v\left\langle g^{-1}\cdot 0, g'\cdot 0\right\rangle + \mu\left\langle \rho\left(g^{-1}\right)\cdot 0, \rho(g')\cdot 0\right\rangle\right) \tag{1.3.11}$$

Let Γ be a lattice of the additive group $(\mathbb{C}, +)$ that can be seen as a discrete subgroup of G (see remark 1.5). So that the action of Γ on $\mathbb{C}$ is the one induced from this of G, that is,

$$\gamma \cdot z = z + \gamma$$

given the strictly positive real numbers ν, μ, and an equivariant pair (ρ, τ), we consider $\mathcal{M}^{\nu,\mu}_{\rho,\tau}(\mathbb{C})$ the vector space of smooth complex-valued functions f on $\mathbb{C}$ satisfying the functional equation

$$f(\gamma \cdot z) = J^{\nu,\mu}_{\rho,\tau}(\gamma, z) F(z) = j^{\nu}(\gamma, z) j^{\mu}(\rho(\gamma), \tau(z)) F(z)$$

for every $\gamma \in \Gamma$ and $z \in \mathbb{C}$.

Definition 1.4. *The space $\mathcal{M}^{\nu,\mu}_{\rho,\tau}(\mathbb{C})$ is called the space of mixed automorphic functions of bi-weight (ν, μ) with respect to the equivariant pair (ρ, τ) and the lattice Γ.*

As what we have in the cas of classical automorphic functions (see 1.2). The proposition below provides a condition that is both sufficient and necessary for the map $\phi^{v,\mu}_{\rho}$ defined in (1.3.11), so that the space $\mathcal{M}^{\nu,\mu}_{\rho,\tau}(\mathbb{C})$ be a nonzero space. Namely, we have

Proposition 1.4. *[27] The functional space $\mathcal{M}^{\nu,\mu}_{\rho,\tau}(\mathbb{C})$ is nontrivial if and only if the real-valued function $\phi^{v,\mu}_{\rho}$ takes its values in $\pi\mathbb{Z}$ on $\Gamma \times \Gamma$.*

We introduce the functions $B^{\nu,\mu}_{\tau}$, and $S^{\nu,\mu}_{\tau}$ defineds by

$$B^{\nu,\mu}_{\tau}(z) = \nu + \mu \left(\left| \frac{\partial \tau}{\partial z} \right|^2 - \left| \frac{\partial \tau}{\partial \bar{z}} \right|^2 \right) \tag{1.3.12}$$

$$S^{\nu,\mu}_{\tau}(z) := \nu z + \mu \left(\tau \frac{\partial \bar{\tau}}{\partial \bar{z}} - \bar{\tau} \frac{\partial \tau}{\partial \bar{z}} \right) \tag{1.3.13}$$

The function $z \longmapsto B^{\nu,\mu}_{\tau}(z)$ is a real-valued constant function. Noted by $B^{\nu,\mu}_{\tau}$.

Proposition 1.5. *[27] The first-order differential equation*

$$\frac{\partial \widetilde{\varphi^{\nu,\mu}_{\tau}}}{\partial \bar{z}} = -i\mu \left(\left(\tau \frac{\partial \bar{\tau}}{\partial \bar{z}} - \bar{\tau} \frac{\partial \tau}{\partial \bar{z}} \right) - \left(\left| \frac{\partial \tau}{\partial z} \right|^2 - \left| \frac{\partial \tau}{\partial \bar{z}} \right|^2 \right) z \right) \tag{1.3.14}$$

admits a solution $\widetilde{\varphi^{\nu,\mu}_{\tau}} : \mathbb{C} \longrightarrow \mathbb{C}$ such that $\Im m \widetilde{\varphi^{\nu,\mu}_{\tau}}$ is constant.

Remark 1.7. *The partial differential equation (1.3.14) satisfied by $\widetilde{\varphi^{\nu,\mu}_{\tau}}$ can be reduced further to the following:*

$$\frac{\partial \psi^{\nu,\mu}_{\tau}}{\partial \bar{z}} = \bar{\tau} \frac{\partial \tau}{\partial \bar{z}}, \quad \frac{\partial \psi^{\nu,\mu}_{\tau}}{\partial z} = \frac{1}{\mu} \left(\nu - B^{\nu,\mu}_{\tau} \right) \bar{z} + \bar{\tau} \frac{\partial \tau}{\partial z}$$

with $\widetilde{\varphi^{\nu,\mu}_{\tau}}(z) = i \left(\left[B^{\nu,\mu}_{\tau} - \nu \right] |z|^2 - \mu |\tau(z)|^2 \right) + 2i\mu \psi^{\nu,\mu}_{\tau}(z)$.

Let $\varphi^{\nu,\mu}_{\tau}$ be the real part of $\widetilde{\varphi^{\nu,\mu}_{\tau}} - \widetilde{\varphi^{\nu,\mu}_{\tau}}(0)$. We define $\mathcal{W}^{\nu,\dot{\mu}}_{\tau}$ to be the special transformation given by

$$\left[\mathcal{W}^{\nu,\mu}_{\tau}(f) \right](z) := e^{i \varphi^{\nu,\mu}_{\tau}(z)} f(z) \tag{1.3.15}$$

Theorem 1.4. *[27] The image of $\mathcal{M}^{\nu,\mu}_{\Gamma,\tau}(\mathbb{C})$ by the transform $\mathcal{W}^{\upsilon,\mu}_{\tau}$ defined in 1.3.15 is the space of (Γ, X_{τ})-automorphic functions. More exactly, one has*

$$\mathcal{W}^{\nu,\mu}_{\tau} \left(\mathcal{M}^{\nu,\mu}_{\rho,\tau}(\mathbb{C}) \right) = \left\{ F \in \mathcal{C}^{\infty}(\mathbb{C}), F(z+\gamma) = \chi_{\tau}(\gamma) j^{B^{\nu,\mu}_{\tau}}(\gamma, z) F(z) \right\}$$

with χ_{τ} is the pseudocharacter defined on Γ by

$$\chi_{\tau}(\gamma) = e^{2i \left(\varphi^{\nu,\mu}_{\tau}(\gamma) - \mu \omega \left(\tau(0), \rho(\gamma)^{-1} \cdot 0 \right) \right)} \tag{1.3.16}$$

Corollary 1.2. *The function $\chi_{\tau}(\gamma) = e^{2i \left(\varphi^{\nu,\mu}_{\tau}(\gamma) - \mu \omega \left(\tau(0), \rho(\gamma)^{-1} \cdot 0 \right) \right)}$ satisfies the following pseudocharacter property:*

$$\chi_{\tau}\left(\gamma + \gamma'\right) = e^{2iB^{\nu,\mu}_{\tau} \omega(\gamma, \gamma')} \chi_{\tau}(\gamma) \chi_{\tau}\left(\gamma'\right)$$

for everything $(\gamma, \gamma') \in \Gamma^2$, if and only if $\phi^{\nu,\mu}_{\rho}$ takes its values in $\pi\mathbb{Z}$ on $\Gamma \times \Gamma$.

Chapter 2

On the Poincaré theta operator $\mathcal{P}_\Gamma$

In the present chapter, we emphasize to discuss similar results D being the whole complex plane $\mathbb{C}$ and Γ being a discrete subgroup of $(\mathbb{C},+)$ of rank 1 or 2. for this we consider Γ as a discrete subgroup within $(\mathbb{C},+)$, and ν be a nonnegative real parameter. We perform the space $\mathcal{H}^{2\nu}_{\Gamma,\chi}(\mathbb{C})$ of all holomorphic functions on $\mathbb{C}$ satisfying the functional equation

$$f(z+\gamma)=\chi(\gamma)e^{2\nu\left\langle z+\frac{\gamma}{2},\gamma\right\rangle}f(z), \tag{2.0.1}$$

where χ is a specific mapping applied to Γ, producing values within the unit circle, such that

$$\chi(\gamma_1+\gamma_2)=\chi(\gamma_2)\chi(\gamma_2)e^{2i\nu\omega(\gamma_1,\gamma_2)}. \tag{2.0.2}$$

this condition on χ ensures that the space $\mathcal{H}^{2\nu}_{\Gamma,\chi}(\mathbb{C})$ is nontrivial (as mentioned in Proposition 1.2). The use of 2ν instead of ν is present here to simplify later formulas and notations (used in the first chapter). Consequently, the function $z\longmapsto|f(z)|e^{-\nu|z|^2}$ is Γ-periodic on $\mathbb{C}$ (see the proposition 1.1), for every $f\in\mathcal{H}^{2\nu}_{\Gamma,\chi}(\mathbb{C})$, and therefore the quantity

$$\|f\|_{1,\Gamma}:=\int_{\Lambda(\Gamma)}|f(z)|e^{-\nu|z|^2}d\lambda(z)$$

this is meaningful and remains independent of the chosen fundamental domain $\Lambda(\Gamma)$. Furthermore, it establishes a norm on $\mathcal{H}^{2\nu}_{\Gamma,\chi}(\mathbb{C})$.

Associated to the automorphic factor involved in (2.0.1), the Poincaré series operator reads

$$\mathcal{P}_\Gamma(f)(z):=\sum_{\gamma\in\Gamma}\overline{\chi(\gamma)}e^{-2\nu\left\langle z+\frac{\gamma}{2},\gamma\right\rangle}f(z+\gamma) \tag{2.0.3}$$

for a given entire function f, it is necessary for $\mathcal{P}_\Gamma(f)$ to exhibit absolute and uniform convergence on every compact subset of $\mathbb{C}$ to qualify $\mathcal{P}_\Gamma(f)$ as an element of $\mathcal{H}^{2\nu}_{\Gamma,\chi}(\mathbb{C})$. To achieve this, we examine the impact of $\mathcal{P}_\Gamma$ on

$$\mathcal{F}^{1,\nu}(\mathbb{C}):=\left\{f\in\mathcal{H}ol(\mathbb{C}),\ \int_{\mathbb{C}}|f(z)|e^{-\nu|z|^2}d\lambda(z)<+\infty\right\}$$

that can be seen as a specific case of the Banach space

$$\mathcal{F}^{1,\nu}_{\Gamma,\chi}(\mathbb{C}) := \left\{ f \in \mathcal{H}^{2\nu}_{\Gamma,\chi}(\mathbb{C}),\ \|f\|_{1,\Gamma} := \int_{\Lambda(\Gamma)} |f(z)| e^{-\nu|z|^2} d\lambda(z) < +\infty \right\}$$

by taking to $\Gamma = \{0\}$ and $\chi = 1$.

The main results targeted by this chapter are presented in the following: They concern the concrete description of $\mathcal{P}_\Gamma$ when acting on $\mathcal{F}^{1,\nu}(\mathbb{C})$. Mainely, we prove a density theorem for the operator $\mathcal{P}_\Gamma$. Moreover, the $\mathcal{P}_\Gamma$ is shown to be onto (Theorem 2.2). The kernel is also characterized in terms of some special functions and depends of the rank of the discrete subgroup Γ (Theorem 2.4). Theorem 2.5 describes the explicit condition to be satisfied by the Fourier coefficients of a given function to belong in such kernel. To do we have determinate the dual of $\mathcal{F}^{1,\nu}_{\Gamma,\chi}(\mathbb{C})$ (Theorem 2.3).

2.1 The continuity and density of $\mathcal{P}_\Gamma$

Theorem 2.1. *The operator $\mathcal{P}_\Gamma$ is a continuous mapping of $\mathcal{F}^{1,\nu}(\mathbb{C})$ into $\mathcal{F}^{1,\nu}_{\Gamma,\chi}(\mathbb{C})$.*

Proof: The function $\mathcal{P}_\Gamma(f)$ satisfies the condition (2.0.1) for everything $f \in \mathcal{F}^{1,\nu}(\mathbb{C})$. Indeed, let $\gamma \in \Gamma$ and $z \in \mathbb{C}$, we have

$$\begin{aligned} \left[\mathcal{P}^\nu_{\Gamma,\chi} f\right](z+\gamma) &= \sum_{\delta\in\Gamma} \chi(\delta) e^{-\nu|\delta|^2 - 2\nu(z+\gamma)\overline{\delta}} f(z+\gamma-\delta) \\ &= \sum_{\sigma\in\Gamma} \chi(\gamma+\sigma) e^{-\nu|\gamma+\sigma|^2 - 2\nu(z+\gamma)\overline{\gamma+\sigma}} f(z-\sigma); \quad [\delta = \gamma+\sigma] \end{aligned} \tag{2.1.4}$$

However, according to the (1.2.3) condition, we have

$$\chi(\gamma+\sigma) = \chi(\gamma)\chi(\sigma)e^{i\nu\omega(\gamma,\sigma)}.$$

Then

$$\begin{aligned} \left[\mathcal{P}^\nu_{\Gamma,\chi} f\right](z+\gamma) &= \chi(\gamma) e^{\nu|\gamma|^2 + 2\nu z\gamma} \sum_{\sigma\in\Gamma} \chi(k) e^{-\nu|\sigma|^2 + 2\nu z\bar{\sigma}} f(z-\sigma) \\ &= \chi(\gamma) e^{2\nu\left\langle z+\frac{\gamma}{2},\gamma\right\rangle} \left[\mathcal{P}^\nu_{\Gamma,\chi} f\right](z) \end{aligned}$$

We conclude that

$$\mathcal{P}^\nu_{\Gamma,\chi}(f) \in Aut^{2\nu}_{\Gamma,\chi}(\mathbb{C}).$$

We will show the continue, let $f \in \mathcal{F}^{1,\nu}(\mathbb{C})$. Then

$$\begin{aligned} \|\mathcal{P}_\Gamma(f)\|_{1,\Gamma} &:= \int_{\Lambda(\Gamma)} \left| \sum_{\gamma\in\Gamma} \overline{\chi(\gamma)} e^{-2\nu\left\langle z+\frac{\gamma}{2},\gamma\right\rangle} f(z+\gamma) \right| e^{-\nu|z|^2} d\lambda(z) \\ &\leqslant \int_{\Lambda(\Gamma)} \sum_{\gamma\in\Gamma} \left| e^{-2\nu\left\langle z+\frac{\gamma}{2},\gamma\right\rangle} f(z+\gamma) \right| e^{-\nu|z|^2} d\lambda(z) \\ &\leqslant \sum_{\gamma\in\Gamma} \int_{\Lambda(\Gamma)} \left| e^{-2\nu\left\langle z+\frac{\gamma}{2},\gamma\right\rangle} f(z+\gamma) \right| e^{-\nu|z|^2} d\lambda(z) \end{aligned}$$

The last inequality follows by means of the monotone convergence theorem. Making the change of variable $w = z + \gamma$ taking into account that $\Lambda(\Gamma)$ is a fundamental domain, the family $(\Lambda(\Gamma) + \gamma)_{\gamma\in\Gamma}$ forms a partition of $\mathbb{C}$, we obtain

$$\begin{aligned}\|\mathcal{P}_\Gamma(f)\|_{1,\Gamma} &\leq \sum_{\gamma\in\Gamma} \int_{\Lambda(\Gamma)+\gamma} |f(w)|e^{-\nu|w|^2} d\lambda(w) \\ &\leq \int_{\mathbb{C}} |f(w)|e^{-\nu|w|^2} d\lambda(w) = \|f\|_1 \,. \end{aligned} \tag{2.1.5}$$

Now, since the L^1 convergence implies the normal convergence of holomorphic function, the inequality (2.1.5) implies that the series (2.0.3) converges uniformly on every compact subset of the fundamental domain $\Lambda(\Gamma)$. But, for every compact subset D of $\mathbb{C}$, we can find a compact subset D_0 of $\Lambda(\Gamma)$ and finite sequence $\gamma_0, \cdots, \gamma_n$ such that $D \subset \bigcup\limits_{j=0}^{n} (D_0 + \gamma_j)$. Hence $\mathcal{P}_\Gamma(f)$ is a holomorphic function on the whole $\mathbb{C}$. □

In the previous proof we have that $\mathcal{P}_\Gamma$ is well-defined and continuous from $\mathcal{F}^{1,\nu}(\mathbb{C})$ into $\mathcal{F}^{1,\nu}_{\Gamma,\chi}(\mathbb{C})$, of the norm less than or equal to 1.

2.2 The density of $\mathcal{P}_\Gamma(\mathcal{F}^{1,\nu}(\mathbb{C}))$ in $\mathcal{F}^{1,\nu}_{\Gamma,\chi}(\mathbb{C})$

Theorem 2.2. *The image $\mathcal{P}_\Gamma(\mathcal{F}^{1,\nu}(\mathbb{C}))$ is dense in $\mathcal{F}^{1,\nu}_{\Gamma,\chi}(\mathbb{C})$.*

We use the following lemma, which can be found in [32], to prove this theorem.

Lemma 2.1. *Let $\phi : \mathbb{C} \to \mathbb{R}$ be a C^2-smooth map with $\Delta\phi > 0$ everywhere and $f \in L^2(\mathbb{C}, \mathrm{e}^{2\phi})$ such that $\int_{\mathbb{C}} f(z)g(z)d\lambda(z) = 0$ for all $g \in \mathcal{F}^{2,\phi}(\mathbb{C})$. Then there exists a solution to the $\bar{\partial}$-equation $\frac{\partial u}{\partial \bar{z}} = f$ with*

$$\int_{\mathbb{C}} |u|^2 e^{2\phi} \Delta\phi d\lambda(z) \leqslant \frac{1}{2} \int_{\mathbb{C}} |f|^2 e^{2\phi} d\lambda(z).$$

Furthermore, if in addition ϕ is C^4-smooth and meets the curvature-type condition

$$\frac{1}{\Delta\phi} \Delta \log \Delta\phi \geqslant -2 \quad on \quad \mathbb{C},$$

then the solution u is unique.

Notice that for any continuous linear functional ℓ on $\mathcal{F}^{1,\nu}_{\Gamma,\chi}(\mathbb{C})$, there exists a bounded Borel measurable function g on $\Lambda(\Gamma)$ such that ℓ can be represented as

$$\ell(F) = \ell^g_{\Lambda(\Gamma)}(F) := \int_{\Lambda(\Gamma)} F(z)\overline{g(z)} e^{-\nu|z|^2} d\lambda(z). \tag{2.2.6}$$

This follows by Hahn-Banach and Riesz theorem. Next, since the fundamental domain $\Lambda(\Gamma)$ is arbitrary, we can extend g to the whole $\mathbb{C}$ by setting

$$g(z+\gamma) = \chi(\gamma) e^{2i\nu \Im m\langle z,\gamma\rangle} g(z). \tag{2.2.7}$$

This extension is well-defined since $\chi(\gamma)e^{2i\nu \Im m\langle z,\gamma\rangle}$ is an automorphic factor, and for $z \in \mathbb{C}$ there exists a unique $z_0 \in \Lambda(\Gamma)$ and a unique $\gamma \in \Gamma$ such that $z = z_0 + \gamma$. Associated to such g, we consider

$$\varphi_g(z) = \overline{g(z)} e^{-\nu|z|^2}$$

which clearly satisfies the functional equation

$$\varphi_g(z) = \chi(\gamma)\, \mathrm{e}^{2\nu\left\langle z+\frac{\gamma}{2},\gamma\right\rangle}\, \varphi_g(z+\gamma). \tag{2.2.8}$$

Moreover, if we define

$$\ell^{\varphi_g}_{\mathbf{C}} := \int_{\mathbf{C}} f(z)\varphi_g(z) d\lambda(z),$$

we can prove the following

Lemma 2.2. *For every* $f \in \mathcal{F}^{1,\nu}(\mathbb{C})$*, we have*

$$\ell^{g}_{\Lambda(\Gamma)}(\mathcal{P}_\Gamma(f)) = \ell^{\varphi_g}_{\mathbb{C}}(f).$$

Proof of Lemma 2.2. From $f \in \mathcal{F}^{1,\nu}(\mathbb{C})$, we get

$$\begin{aligned}
\ell^{g}_{\Lambda(\Gamma)}(\mathcal{P}_\Gamma(f)) &= \int_{\Lambda(\Gamma)} \mathcal{P}_\Gamma(f)(z)\overline{g(z)} e^{-\nu|z|^2} d\lambda(z)\\
&= \int_{\Lambda(\Gamma)} \left(\sum_{\gamma\in\Gamma} \overline{\chi(\gamma)} e^{-2\nu\left\langle z+\frac{\gamma}{2},\gamma\right\rangle} f(z+\gamma)\right) \overline{g(z)} e^{-\nu|z|^2} d\lambda(z)\\
&= \sum_{\gamma\in\Gamma} \overline{\chi(\gamma)} \int_{\Lambda(\Gamma)} e^{-2\nu\left\langle z+\frac{\gamma}{2},\gamma\right\rangle} f(z+\gamma)\overline{g(z)} e^{-\nu|z|^2} d\lambda(z)
\end{aligned}$$

and making the change of variable $w = z + \gamma$, we obtain

$$\ell^{g}_{\Lambda(\Gamma)}(\mathcal{P}_\Gamma(f)) = \sum_{\gamma\in\Gamma} \int_{\Lambda(\Gamma)+\gamma} f(w)\overline{g(w)} e^{-\nu|w|^2} d\lambda(w) = \int_{\mathbb{C}} f(w)\varphi_g(w) d\lambda(w).$$

□

Lemma 2.3. *Let g be a holomorphic function satisfying the automorphy condition* (2.0.1)*. Then*

$$\ell^{g}_{\Lambda(\Gamma)}(e^{-\nu|z|^2}\mathcal{P}_\Gamma(f)) = \ell^{\varphi_g}_{\mathbb{C}}(e^{-\nu|z|^2} f) \tag{2.2.9}$$

for all $f \in \mathcal{F}^{1,\nu}(\mathbb{C})$.

Proof. Let $f \in \mathcal{F}^{1,\nu}(\mathbb{C})$ and g be a function such that $g(z+\gamma) = \chi(\gamma)e^{2\nu\left\langle z+\frac{\gamma}{2},\gamma\right\rangle} g(z)$. Then, we have

$$\begin{aligned}
\ell^g_{\Lambda(\Gamma)}(e^{-\nu|z|^2}\mathcal{P}_\Gamma(f)) &= \int_{\Lambda(\Gamma)} \mathcal{P}_\Gamma(f)(z)\overline{g(z)}\, \mathrm{e}^{-2\nu|z|^2}\, d\lambda(z) \\
&= \int_{\Lambda(\Gamma)} \left(\sum_{\gamma\in\Gamma} \overline{\chi(\gamma)} e^{-2\nu\left\langle z+\frac{\gamma}{2},\gamma\right\rangle} f(z+\gamma) \right) \overline{g(z)}\, \mathrm{e}^{-2\nu|z|^2}\, d\lambda(z) \\
&= \sum_{\gamma\in\Gamma} \int_{\Lambda(\Gamma)} \overline{\chi(\gamma)} e^{-2\nu\left\langle z+\frac{\gamma}{2},\gamma\right\rangle} f(z+\gamma)\overline{g(z)} e^{-2\nu|z|^2} d\lambda(w).
\end{aligned}$$

Using the change of variable $w = z+\gamma$ as well as the facts $\chi(-\gamma) = \overline{\chi(\gamma)}$, it follows

$$\begin{aligned}
\ell^g_{\Lambda(\Gamma)}(e^{-\nu|z|^2}\mathcal{P}_\Gamma(f)) &= \sum_{\gamma\in\Gamma} \int_{\Lambda(\Gamma)+\gamma} f(z)\overline{g(z)}\, \mathrm{e}^{-2\nu|z|^2}\, d\lambda(z) \\
&= \int_{\mathbb{C}} f(z)\overline{g(z)}\, \mathrm{e}^{-2\nu|z|^2}\, d\lambda(z) \\
&= \ell^{\varphi_g}_{\mathbb{C}}(e^{-\nu|z|^2} f).
\end{aligned}$$

This completes the proof. $\square$

With this approach, we can establish the proof for Theorem 2.2.

Proof of Theorem 2.2.

Suppose that $\ell^g_{\Lambda(\Gamma)}(\mathcal{P}_\Gamma(f)) = 0$ for all $f \in \mathcal{F}^{1,\nu}(\mathbb{C})$ and let show that $\ell^g_{\Lambda(\Gamma)}(F) = 0$ for all $F \in \mathcal{F}^{1,\nu}_{\Gamma,\chi}(\mathbb{C})$. By Lemma 2.2, the assumption $\ell^g_{\Lambda(\Gamma)}(\mathcal{P}_\Gamma(f)) = 0$ reads

$$\int_{\mathbb{C}} f(z)\varphi_g(z)d\lambda(z) = 0.$$

for all $f \in \mathcal{F}^{2,\nu}$, since $\mathcal{F}^{2,\nu} := \mathcal{H}ol(\mathbb{C}) \cap L^2\left(\mathbb{C}; e^{-\nu|z|^2}d\lambda(z)\right) \subset \mathcal{F}^{1,\nu}(\mathbb{C})$. Now, making appeal to Lemma 2.1 with $\phi(z) := \frac{\nu}{2}|z|^2$, we conclude that there exists h such that $\frac{\partial h}{\partial \bar{z}} = \varphi_g$. Then, we claim that h satisfies the functional equation

$$h(z+\gamma) = \overline{\chi(\gamma)}\, \mathrm{e}^{-2\nu\left\langle z+\frac{\gamma}{2},\gamma\right\rangle}\, h(z).$$

This follows by uniqueness of h, since $z \to \chi(\gamma)\, \mathrm{e}^{2\nu\left\langle z+\frac{\gamma}{2},\gamma\right\rangle}\, h(z+\gamma)$ is also a solution of $\frac{\partial h}{\partial \bar{z}} = \varphi_g$, thanks to (2.2.8). Accordingly, the holomorphy of F we have

$$\ell^g_{\Lambda(\Gamma)}(F) = \int_{\Lambda(\Gamma)} F(z)\varphi_g(z)d\lambda(z) = \int_{\Lambda(\Gamma)} F(z)\frac{\partial h}{\partial \bar{z}}(z)d\lambda(z) = \int_{\Lambda(\Gamma)} \frac{\partial F h}{\partial \bar{z}}(z)d\lambda(z).$$

By Stokes formula we find

$$\ell^g_{\Lambda(\Gamma)}(F) = \int_{\partial(\Lambda(\Gamma))} F(z)h(z)d\lambda(z).$$

In addition, the function $F \cdot h$ is invariant by Γ. Indeed

$$F(z+\gamma)h(z+\gamma) = \chi(\gamma)\,\mathrm{e}^{2\nu\left\langle z+\frac{\gamma}{2},\gamma\right\rangle}\,F(z)\overline{\chi(\gamma)}\,\mathrm{e}^{-2\nu\left\langle z+\frac{\gamma}{2},\gamma\right\rangle}\,h(z) = F(z)h(z).$$

Moreover, using the symmetry of the boundary $\partial(\Lambda(\Gamma))$, we deduce that $\ell^{g}_{\Lambda(\Gamma)}(F) = 0$ for all $F \in \mathcal{F}^{1,\nu}_{\Gamma,\chi}(\mathbb{C})$, and therefore $\mathcal{P}_\Gamma(\mathcal{F}^{1,\nu}(\mathbb{C}))$ is dense in $\mathcal{F}^{1,\nu}_{\Gamma,\chi}(\mathbb{C})$.

$\square$

2.3 The duality of space $\mathcal{F}^{1,\nu}_{\Gamma,\chi}(\mathbb{C})$

In order to give explicit description of $\mathcal{P}_\Gamma$ when acting on $\mathcal{F}^{1,\nu}(\mathbb{C})$, we determine first the dual space of the Banach space $\mathcal{F}^{1,\nu}_{\Gamma,\chi}(\mathbb{C})$. Namely, we prove

Theorem 2.3. *The dual space of the Banach space $\mathcal{F}^{1,\nu}_{\Gamma,\chi}(\mathbb{C})$ is the space*

$$\mathcal{F}^{\infty,\nu}_{\Gamma,\chi}(\mathbb{C}) := \left\{ f \in, \mathcal{H}^{2\nu}_{\Gamma,\chi}(\mathbb{C});\, \|f\|_{\infty,\Gamma} := \sup_{z\in\mathbb{C}} \left\{ |f(z)|e^{-\nu|z|^2} \right\} < +\infty \right\}$$

under the scalar product

$$\langle f, g\rangle_\Gamma := \int_{\Lambda(\Gamma)} f(z)\overline{g(z)}e^{-2\nu|z|^2}d\lambda(z), \quad f \in \mathcal{F}^{1,\nu}_{\Gamma,\chi}(\mathbb{C}), g \in \mathcal{F}^{\infty,\nu}_{\Gamma,\chi}(\mathbb{C}). \tag{2.3.10}$$

Proof. Recall first that for any continuous linear functional ℓ on $\mathcal{F}^{1,\nu}_{\Gamma,\chi}(\mathbb{C})$, the bounded function g in (2.2.7) satisfies

$$\ell(f) = \int_{\Lambda(\Gamma)} f(z)\overline{g(z)}e^{-\nu|z|^2}d\lambda(z).$$

Accordingly, we define the ψ_g function to be

$$\widetilde{\psi_g}(z) := \left(\frac{2\nu}{\pi}\right)\int_{\mathbb{C}} K(z,w)g(w)e^{-\nu|w|^2}d\lambda(w),$$

where $K(z,w) = e^{2\nu z\overline{w}}$ is the Fock reproducing kernel function. Then, $\widetilde{\psi_g}$ is holomorphic on $\mathbb{C}$ and verifies

$$\begin{aligned}\left|\widetilde{\psi_g}\right| &\leqslant \left(\frac{2\nu}{\pi}\right)\int_{\mathbb{C}} |K(z,w)g(w)|e^{-\nu|w|^2}d\lambda(w)\\ &\leqslant \left(\frac{2\nu}{\pi}\right)||g||_\infty \int_{\mathbb{C}} |K(z,w)|e^{-\nu|w|^2}d\lambda(w)\\ &= 2||g||_\infty e^{\nu|z|^2}.\end{aligned}$$

Now, since the function g follows the condition in (2.2.7) and we know the specific form of $K(z,w)$, we can show that

$$\begin{aligned}
\widetilde{\psi_g}(z+\gamma) &= \left(\frac{2\nu}{\pi}\right)\int_{\mathbb{C}} K(z+\gamma,w)g(w)e^{-\nu|w|^2}d\lambda(w)\\
&= \left(\frac{2\nu}{\pi}\right)\int_{\mathbb{C}} e^{2\nu(z+\gamma)\overline{w}}g(w-\gamma)\chi(\gamma)e^{-i\nu\Im((w-\gamma)\overline{\gamma})}e^{-\nu|w|^2}d\lambda(w)\\
&= \left(\frac{2\nu}{\pi}\right)\chi(\gamma)\int_{\mathbb{C}} e^{2\nu(z+\gamma)\overline{w+\gamma}}g(w)e^{-i2\nu\Im((w)\overline{\gamma})-2\nu\Re e(w\overline{\gamma})}e^{-\nu|\gamma|^2}e^{-\nu|w|^2}d\lambda(w)\\
&= \left(\frac{2\nu}{\pi}\right)\chi(\gamma)e^{2\nu z\overline{\gamma}+\nu|\gamma|^2}\int_{\mathbb{C}} g(w)e^{2\nu z\overline{w}}e^{-\nu|w|^2}d\lambda(w)\\
&= \chi(\gamma)e^{2\nu\left\langle z+\frac{\gamma}{2},\gamma\right\rangle}\widetilde{\psi_g}(z).
\end{aligned}$$

Thus, $\widetilde{\psi_g}$ belongs to $\mathcal{F}^{\infty,\nu}_{\Gamma,\chi}(\mathbb{C})$. Lemma 2.2 implies that

$$\ell(\mathcal{P}_\Gamma(f)) = \int_{\mathbb{C}} f(z)\overline{g(z)}e^{-\nu|z|^2}d\lambda(z).$$

Using Fubini theorem, we obtain

$$\begin{aligned}
\left\langle f,\widetilde{\psi_g}\right\rangle_{\{0\}} &= \left(\frac{2\nu}{\pi}\right)\int_{\mathbb{C}} f(z)\overline{\int_{\mathbb{C}} K(z,w)g(w)e^{-\nu|w|^2}d\lambda(w)}e^{-2\nu|z|^2}d\lambda(z)\\
&= \left(\frac{2\nu}{\pi}\right)\int_{\mathbb{C}}\overline{g(w)}\int_{\mathbb{C}} f(z)K(w,z)e^{-2\nu|z|^2}d\lambda(z)e^{-\nu|w|^2}d\lambda(w).
\end{aligned}$$

Now, by the fact (see [63, Theorem 2.10, p. 39])

$$\mathcal{F}^{1,\nu}(\mathbb{C})\subset\mathcal{F}^{2,2\nu}(\mathbb{C}) := \left\{f\text{ holomorphic}, \int_{\mathbb{C}}|f(z)|^2e^{-2\nu|z|^2}d\lambda(z)<\infty\right\}$$

and by using the special property of the reproducing kernel $K(z,w)$ in the space $\mathcal{F}^{2,2\nu}(\mathbb{C})$, we get

$$\ell(\mathcal{P}_\Gamma(f)) = \int_{\mathbb{C}} f(w)\overline{g(w)}e^{-\nu|w|^2}d\lambda(w) = \left\langle f,\widetilde{\psi_g}\right\rangle_{\{0\}} \tag{2.3.11}$$

for all $f\in\mathcal{F}^{1,\nu}(\mathbb{C})$ and, by Lemma 2.3 we conclude that

$$\ell(\varphi) = \left\langle\varphi,\widetilde{\psi_g}\right\rangle_\Gamma$$

whenever $\varphi\in\mathcal{P}_\Gamma(\mathcal{F}^{1,\nu}(\mathbb{C}))$. By density (Theorem 2.2), we get

$$\ell(f) = \left\langle f,\widetilde{\psi_g}\right\rangle_\Gamma;\ f\in\mathcal{F}^{1,\nu}_{\Gamma,\chi}(\mathbb{C}).$$

For the uniqueness, let $g\in\mathcal{F}^{\infty,\nu}_{\Gamma,\chi}(\mathbb{C})$, such that $\langle f,g\rangle_\Gamma = 0$ for all $f\in\mathcal{F}^{1,\nu}_{\Gamma,\chi}(\mathbb{C})$. Then by Lemma 2.3 implies that

$$\langle\mathcal{P}_\Gamma(\phi),g\rangle_\Gamma = \langle\phi,g\rangle_{\{0\}} = 0$$

for all $\phi\in\mathcal{F}^{1,\nu}(\mathbb{C})$ and by ([63, Corollary 2.25]) we have that $g=0$. The proof of Theorem 2.3 is completed. □

Corollary 2.1. *The Poincaré theta operator* $\mathcal{P}_\Gamma : \mathcal{F}^{1,\nu}(\mathbb{C}) \longrightarrow \mathcal{F}^{1,\nu}_{\Gamma,\chi}(\mathbb{C})$ *is onto. Moreover, for* $F \in \mathcal{F}^{1,\nu}_{\Gamma,\chi}(\mathbb{C})$*, the function*

$$\psi_F(z) := \left(\frac{2\nu}{\pi}\right) \int_{\mathbb{C}} K(z,w) \mathbb{1}_{\Lambda(\Gamma)}(w) F(w) e^{-2\nu|w|^2} d\lambda(w).$$

where $K(z,w) = e^{2\nu z\overline{w}}$*, is to verify that* $\psi_F \in \mathcal{F}^{1,\nu}(\mathbb{C})$*, and* $\mathcal{P}_\Gamma(\psi_F) = F$*.*

Proof. To show that $\mathcal{P}_\Gamma : \mathcal{F}^{1,\nu}(\mathbb{C}) \longrightarrow \mathcal{F}^{1,\nu}_{\Gamma,\chi}(\mathbb{C})$ is onto. We denote by $\mathbb{1}_{\Lambda(\Gamma)}$ the characteristic function of a fundamental domain $\Lambda(\Gamma)$, and by where $K(z,w) = e^{2\nu z\overline{w}}$ the reproducing kernel of the holomorphic Fock–Bargmann space $\mathcal{F}^{2,\nu}(\mathbb{C})$. Then, for given let $F \in \mathcal{F}^{1,\nu}_{\Gamma,\chi}(\mathbb{C})$, the function

$$\psi_F(z) := \left(\frac{2\nu}{\pi}\right) \int_{\mathbb{C}} K(z,w) \mathbb{1}_{\Lambda(\Gamma)}(w) F(w) e^{-2\nu|w|^2} d\lambda(w).$$

is holomorphic on $\mathbb{C}$ and verifies

$$\int_{\mathbb{C}} |\psi_F(z)| e^{-\nu|z|^2} d\lambda(z) \leqslant \|F\|_{1,\Gamma}$$

indeed

$$\begin{aligned}
\int_{\mathbb{C}} |\psi_F(z)| e^{-\nu|z|^2} d\lambda(z) &\leqslant \left(\frac{2\nu}{\pi}\right) \int_{\mathbb{C}} \int_{\Lambda(\Gamma)} |K(z,w)||F(w)| e^{-2\nu|w|^2} d\lambda(w) e^{-\nu|z|^2} d\lambda(z) \\
&\leqslant \left(\frac{2\nu}{\pi}\right) \int_{\mathbb{C}} \int_{\Lambda(\Gamma)} e^{2\nu\Re(z\overline{w})} |F(w)| e^{-2\nu|w|^2} e^{-\nu|z|^2} d\lambda(w) d\lambda(z) \\
&= \left(\frac{2\nu}{\pi}\right) \int_{\mathbb{C}} \int_{\Lambda(\Gamma)} |F(w)| e^{-\nu|w|^2} e^{-\nu|z+w|^2} d\lambda(w) d\lambda(z) \\
&= \left(\frac{2\nu}{\pi}\right) \|F\|_{1,\Gamma} \int_{\mathbb{C}} e^{-\nu|z|^2} d\lambda(z) \\
&= \|F\|_{1,\Gamma}
\end{aligned}$$

This implies that $\psi_F \in \mathcal{F}^{1,\nu}(\mathbb{C})$. Then, $\mathcal{P}_\Gamma(\psi_F)$ is well-defined and lies in $\mathcal{F}^{1,\nu}_{\Gamma,\chi}(\mathbb{C})$. We shall show that $\mathcal{P}_\Gamma(\psi_F) = F$. Then by using the Lemma 2.3 we get

$$\begin{aligned}
\langle \mathcal{P}_\Gamma(\psi_F), g\rangle_\Gamma = \langle \psi_F, g\rangle_{\{0\}} &= \left(\frac{2\nu}{\pi}\right) \int_{\mathbb{C}} \overline{g(z)} \int_{\Lambda(\Gamma)} K(z,w) F(w) e^{-2\nu|w|^2} d\lambda(w) e^{-2\nu|z|^2} d\lambda(z) \\
&= \left(\frac{2\nu}{\pi}\right) \int_{\Lambda(\Gamma)} F(w) \int_{\mathbb{C}} \overline{g(z)} K(z,w) e^{-2\nu|z|^2} d\lambda(z) e^{-2\nu|w|^2} d\lambda(w) \\
&= \langle F, g\rangle_\Gamma .
\end{aligned}$$

for every $g \in \mathcal{F}^{\infty,\nu}_{\Gamma,\chi}(\mathbb{C})$. The last equality follows from the fact that $K(z,w)$ reproduces holomorphic functions satisfying the growth condition $|g(z)| \leqslant Ce^{-\nu|z|^2}$ for some positive constant C (see Lemma 1.2). Hence $\mathcal{P}_\Gamma(\psi_F) = F$ by the duality theorem 2.3. □

2.4 The characterization of the kernel of $\mathcal{P}_\Gamma$

Description of the kernel of $\mathcal{P}_\Gamma$ is given in terms of the orthogonal of some concrete family of functions. We distinguish two cases of the discrete subgroup Γ, which can be represented by $\Gamma = \mathbb{Z}$ for the non co-compact case and by $\Gamma = \mathbb{Z}[\tau] := \mathbb{Z} + \tau\mathbb{Z}$; where $\tau \in \mathbb{C}$ with $\Im m(\tau) > 0$, for the co-compact case.

Theorem 2.4. *The kernel of* $\mathcal{P}_\Gamma$ *is given by* $\ker(\mathcal{P}_\Gamma) = \{e_n^{\alpha,\nu}; n \in \mathbb{Z}\}^{\perp_{\langle\cdot,\cdot\rangle_\Gamma}}$ *for* $\Gamma = \mathbb{Z}$*, where* α *is a fixed real number such that* $\chi(1) = e^{2i\pi\alpha}$ *and*

$$e_n^{\alpha,\nu}(z) := e^{\nu z^2 + 2i\pi(\alpha+n)z}; \qquad n \in \mathbb{Z}, \tag{2.4.12}$$

while when $\Gamma = \mathbb{Z}[\tau] = \mathbb{Z} + \tau\mathbb{Z}$*;* $\Im m(\tau) > 0$*, we have* $\ker(\mathcal{P}_\Gamma) = \left\{\theta_r^{\nu,\alpha,\beta}; r = 1, \cdots, m\right\}^{\perp_{\langle\cdot,\cdot\rangle_\Gamma}}$*, where*

$$\theta_r^{\nu,\alpha,\beta}(z) := e^{\nu z^2} \sum_{\substack{k\in\mathbb{Z} \\ k=r \mod m}} e^{i\frac{\pi(k+\alpha)^2\tau}{m} + 2i\pi(\alpha+k)(z-\frac{\beta}{m})}; \qquad r = 1, 2, \cdots, m. \tag{2.4.13}$$

Proof. Notice that for $\Gamma = \mathbb{Z}$, the map χ reduces to be a character of $\mathbb{Z}$, and therefore, it is uniquely determined by its value at 1, $\chi(1) = e^{2i\pi\alpha}$ for fixed real α. Then, the following functions [25]

$$e_n^{\alpha,\nu}(z) := e^{\nu z^2 + 2i\pi(\alpha+n)z}; \qquad n \in \mathbb{Z}, \tag{2.4.14}$$

satisfy the functional equation (2.0.1). Moreover, the family $\{e_n^{\alpha,\nu}\}_{n\in\mathbb{Z}}$ belongs to $\mathcal{F}_{\Gamma,\chi}^{\infty,\nu}(\mathbb{C})$. Then, in view of Lemma 2.3 we see that $\mathcal{P}_\Gamma(f) \equiv 0$ if and only if f is orthogonal to $\{e_n^{\alpha,\nu}\}_{n\in\mathbb{Z}}$. Indeed, if $\mathcal{P}_\Gamma(f) = 0$ then f is orthogonal to the family $\{e_n^{\alpha,\nu}\}_n$. The converse follows since $\{e_n^{\alpha,\nu}\}_n$ is dense in $F_{\Gamma,\chi}^{\infty,\nu}(\mathbb{C})$.

The proof of the second assertion for $\Gamma = \mathbb{Z}[\tau]$ is similar. We need only to point out that the assumption (2.0.2) implies in particular that $2\nu\Im m(\tau) \in \pi\mathbb{Z}^+$, and the quantity

$$N_{\nu,\tau} := \frac{2\nu\Im m(\tau)}{\pi}$$

turns to be the dimension of $\mathcal{F}_{\Gamma,\chi}^{\infty,\nu}(\mathbb{C})$. Moreover, a basis for this space is given by the following functions [49, 20, 37]

$$\theta_r^{\nu,\alpha,\beta}(z) := e^{\nu z^2} \sum_{\substack{k\in\mathbb{Z} \\ k=r \mod N_{\nu,\tau}}} e^{i\frac{\pi(k+\alpha)^2\tau}{N_{\nu,\tau}} + 2i\pi(\alpha+k)\left(z-\frac{\beta}{N_{\nu,\tau}}\right)}; \qquad r = 1, 2, \cdots, N_{\nu,\tau}. \tag{2.4.15}$$

□

The last main result to stated is a variant of Theorem 2.4 and gives an explicit sufficient and necessary condition on the Fourier coefficients of a function f to belong to the kernel of the Poincaré theta operator.

Theorem 2.5. *The function f belongs to $\ker \mathcal{P}_\Gamma$ if and only if its Taylor coefficients satisfy the condition*

$$\sum_{k\geqslant 0} a_k \left(\frac{i}{2\sqrt{\nu}}\right)^k H_k\left(\frac{\pi}{\sqrt{\nu}}(\alpha+n)\right) = 0$$

for each $n \in \mathbb{Z}$ for rank one discrete subgroup $\Gamma = \mathbb{Z}$, and the condition

$$\sum_{k\geqslant 0} a_k \left(\frac{i}{2\sqrt{\nu}}\right)^k \sum_{\substack{n\in\mathbb{Z} \\ n=r \mod N_{\nu,\tau}}} e^{-i\frac{\pi(n+\alpha)^2\bar{\tau}}{N}_{\nu,\tau} - 2i\pi(\alpha+n)\frac{\beta}{N_{\nu,\tau}}} H_k\left(\frac{\pi}{\sqrt{\nu}}(\alpha+n)\right) = 0$$

for $r = 0, 1, \cdots N_{\nu,\tau} - 1$, if $\Gamma = \mathbb{Z}[\tau]$; with $\Im m(\tau) > 0$. Here H_k are the usual Hermite polynomials, are given by

$$H_k(x) = (-1)^k e^{x^2} \frac{d^k}{dx^k} e^{-x^2}.$$

Proof. Let f be a function in $\mathcal{F}^{1,\nu}(\mathbb{C})$ with the Taylor expansion

$$f(z) = \sum_{k=0}^{+\infty} a_k z^k.$$

Then, Theorem 2.5 follows by combining the result of Theorem 2.4 with the following integral formula

$$\int_{\mathbb{C}} z^k \overline{e^{\nu z^2 + 2i\pi(\alpha+n)z}} e^{-2\nu|z|^2} d\lambda(z) = \left(\frac{\pi}{2\nu 2}\right) \left(\frac{-i}{2\sqrt{\nu}}\right)^k H_k\left(\frac{\pi}{\sqrt{\nu}}(\alpha+n)\right).$$

The integral in the left hand-side is in fact the inverse Bargmann transform of the monomials (see [63, p. 222-223]). □

Chapter 3

Complex creation operator and planar automorphic functions

In the present chapter, we consider the construction of the planar poly-analytic automorphic by means of the linear differential operator

$$\mathcal{U}_\nu^{(n)}(f)(z) = \mathcal{U}_\nu^{(n,z)}(f)(z) := (-1)^n e^{\nu|z|^2} \frac{\partial^n}{\partial z^n} \left(e^{-\nu|z|^2} f \right)(z). \tag{3.0.1}$$

In fact, a large class of non-analytic automorphic functions are obtained from the fundamental analytic automorphic functions (the theta functions). More precisely, we are concerned with the space of single-valued poly-analytic automorphic functions arising as the images by $\mathcal{U}_\nu^{(n)}$ in (3.0.1) of those belonging to the Hilbert space $H^{2,\nu}_{\Gamma,\chi}(\mathbb{C})$.

Their construction makes appeal to the second order elliptic differential operator

$$\Delta_\nu = -\frac{\partial^2}{\partial z \partial \bar{z}} + \nu \bar{z} \frac{\partial}{\partial \bar{z}}, \quad \nu > 0, \tag{3.0.2}$$

where $\nu > 0$ and

$$\frac{\partial}{\partial z} = \frac{1}{2}\left(\frac{\partial}{\partial x} - i\frac{\partial}{\partial y}\right) =: \partial_z \text{ and } \frac{\partial}{\partial \bar{z}} = \frac{1}{2}\left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\right) =: \partial_{\bar{z}}$$

Indeed, appear as special subclasses of poly-analytic automorphic functions. Notice for instance that the intertwined operator $e^{-\frac{\nu}{2}|z|^2}\Delta_\nu e^{\frac{\nu}{2}|z|^2}$ is the classical magnetic Laplacian represents the traditional magnetic Laplacian, which interprets the Hamiltonian of a non-relativistic quantum particle moving in the (x, y)-plane affected by an external uniform magnetic field with strength ν, and is closely connected to the sub-Laplacian

$$\mathcal{L} = 4\frac{\partial^2}{\partial z \partial \bar{z}} + 2i\Big(z\frac{\partial}{\partial z} - \bar{z}\frac{\partial}{\partial \bar{z}}\Big)\frac{\partial}{\partial t} + |z|^2 \frac{\partial^2}{\partial t^2}$$

on the Heisenberg group $H^3_{eis} = \mathbb{C}_z \times \mathbb{R}_t$ by making use of the Fourier transform in t.

Accordingly, we perform the Hilbert space of holomorphic functions ($f \in Hol(\mathbb{C})$) belonging to $Aut^{2,\nu}_{\Gamma}(\mathbb{C}) := Aut^{\nu}_{\Gamma}(\mathbb{C}) \cap \mathcal{L}^{2,\nu}_{\Gamma,\chi}(\mathbb{C})$, denoted $H^{2,\nu}_{\Gamma}(\mathbb{C}) := Aut^{2,\nu}_{\Gamma}(\mathbb{C}) \cap Hol(\mathbb{C})$. Moreover, under the (RDQ) condition the space $H^{2,\nu}_{\Gamma,\chi}(\mathbb{C})$ coincides with the null space of Δ_ν, to wit $H^{2,\nu}_{\Gamma}(\mathbb{C}) = \ker(\Delta_\nu)$ (see [24, 25, 37]). The concrete description of the spectral analysis of Δ_ν in (3.0.2) acting densely on $Aut^{2,\nu}_{\Gamma}(\mathbb{C})$ gives rise to the so-called theta generalized Bargmann-Fock spaces defined as the L^2-eigenspaces

$$\mathcal{F}^{2,\nu}_{\Gamma,\ell}(\mathbb{C}) = \ker(\Delta_\nu - \nu\ell) \cap \mathcal{L}^{2,\nu}_{\Gamma,\chi}(\mathbb{C}), \quad \ell = 0, 1, 2, \cdots ,$$

with $\mathcal{F}^{2,\nu}_{\Gamma,0}(\mathbb{C}) = H^{2,\nu}_{\Gamma,\chi}(\mathbb{C})$, leading to the Hilbertian orthogonal decomposition

$$\mathcal{L}^{2,\nu}_{\Gamma,\chi}(\mathbb{C}) = \bigoplus_{\ell=0}^{+\infty} \mathcal{F}^{2,\nu}_{\Gamma,\ell}(\mathbb{C}). \tag{3.0.3}$$

The concrete description of $\mathcal{F}^{2,\nu}_{\Gamma,\ell}(\mathbb{C})$ was discussed in [24] for the case of the co-compact discrete subgroup.

3.1 Poly-analytic automorphic functions

Definition 3.1. *A function f defined on a subset of $\mathbb{C}$ satisfies the generalized Cauchy-Riemann equations, specifically $\frac{\partial^{n+1} f}{\partial \bar{z}^{n+1}} = 0$, then it is classified as a polyanalytic function of order n (n-analytic functions).*

To this purpose, we denote by $A_n(\mathbb{C})$ the class of the so-called n-analytic functions. This is equivalent to saying that $A_n(\mathbb{C})$ is the set of functions with real partial derivatives continuous up to order $n+1$ and such that $\frac{\partial^n f}{\partial \bar{z}^n}$ is holomorphic.

Remark 3.1. *The following statements are equivalent:*

1. *f is a polyanalytic function of order n.*
2. *f is written as a polynomial function in $\bar{z}$ of degree n whose coefficients are holomorphic functions. i.e. $f(z) = \sum\limits_{k=0}^{n} \psi_k(z)\bar{z}^k$, with $(\psi_k)_{0 \leqslant k \leqslant n}$ are analytical functions.*

Contrary to analytical functions, polyanalytic functions deviate from some fundamental properties. These properties, which are frequently connected to analytic functions, occasionally do not apply to polyanalytic functions. Take a polyanalytic function of order 1 as an illustration.

$$F(z, \bar{z}) = 1 - |z|^2 = 1 - z\bar{z}$$

Since

$$\frac{\partial F}{\partial \bar{z}}(z, \bar{z}) = \bar{z} \text{ and } \frac{\partial^2 F}{\partial \bar{z}^2}(z, \bar{z}) = 0$$

The function F is polyanalytic of order 1, not analytical in z. One of the reasons why polyanalytic functions may have distinct features from analytical functions is already illustrated by this straightforward example. Nevertheless, many analytic function features have been extended to polyanalytic functions, frequently in a nontrivial form. The function is referred to as a simple polyanalytic function in this instance. Their theory is a part of complex analysis and one refers to [1, 5] and the references therein for details. However. We prove here an explicit description of the n-analytic functions arising by means of $\mathcal{U}_\nu^{(n)}$ (interesting in its own).

We begin by proving the result describing the regularity and the automorphy of the functions arising as $\mathcal{U}_\nu^{(n)}(f)$ for certain holomorphic function $f \in Hol(\mathbb{C}) := A_0(\mathbb{C})$. Which we provide in the following propositions;

Proposition 3.1. *If f is a holomorphic function on $\mathbb{C}$, then $\mathcal{U}_\nu^{(n)}(f)$ is poly-analytic of order n.*

Proof. Let f a holomorphic function on $\mathbb{C}$. Using the formula of Leibniz, give by

$$\frac{\partial^n}{\partial \bar{z}^n}(fg) = \sum_{k=0}^{n} \binom{n}{k} \frac{\partial^k f}{\partial \bar{z}^k} \frac{\partial^{n-k} g}{\partial \bar{z}^{n-k}},$$

with $\binom{n}{k} = \frac{n!}{k!(n-k)!}$. Then, we have

$$\begin{aligned} \mathcal{U}_\nu^{(n)}(f)(z) &= (-1)^n e^{\nu|z|^2} \frac{\partial^n}{\partial z^n}\left(e^{-\nu|z|^2} f\right)(z) \\ &= (-1)^n e^{\nu|z|^2} \sum_{k=0}^{n} \binom{n}{k} \frac{\partial^k e^{-\nu|z|^2}}{\partial z^k} \frac{\partial^{n-k} f(z)}{\partial z^{n-k}} \\ &= (-1)^n \sum_{k=0}^{n} \binom{n}{k} (-\nu)^k \bar{z}^k \frac{\partial^{n-k} f}{\partial z^{n-k}}(z) \\ &= \sum_{k=0}^{n} \varphi_k(z) \bar{z}^k \end{aligned} \tag{3.1.4}$$

with, for all $0 \leqslant k \leqslant n$, $\varphi_k(z) = (-1)^n \binom{n}{k} (-\nu)^k \frac{\partial^{n-k} f}{\partial z^{n-k}}(z)$, which are holomorphic functions. it can be shown also as follows

$$\begin{aligned} \frac{\partial^{n+1}}{\partial \bar{z}^{n+1}}\left(\mathcal{U}_\nu^{(n)}(f)(z)\right) &= \sum_{k=0}^{n} \frac{\partial^{n+1}}{\partial \bar{z}^{n+1}}\left(\varphi_k(z)\bar{z}^k\right) \\ &= 0 \end{aligned}$$

$\frac{\partial^{n+1}}{\partial \bar{z}^{n+1}}(\varphi_k(z)) = 0$ and $\frac{\partial^{n+1}}{\partial \bar{z}^{n+1}}\left(\bar{z}^k\right) = 0$ for any integer k ranging from 0 to n. □

Proposition 3.2. *The operator $\mathcal{U}_\nu^{(n)}$ maps injectively the space $Hol(\mathbb{C}) \cap Aut_\Gamma^\nu(\mathbb{C})$ into $A_n(\mathbb{C}) \cap Aut_\Gamma^\nu(\mathbb{C})$.*

Proof. We start by pointing out that $\mathcal{U}_\nu^{(n)}(f)$ is a poly-analytic of order n whenever f is a holomorphic function on $\mathbb{C}$ (Proposition3.1). In fact, the functions

$$\psi_k(z) := \frac{(-1)^{n+k} n! \nu^k}{k!(n-k)!} \frac{\partial^{n-k}}{\partial z^{n-k}} f \tag{3.1.5}$$

are clearly holomorphic and we have $\mathcal{U}_\nu^{(n)}(f) = \overline{z}^n \psi_n + \overline{z}^{n-1}\psi_{n-1} + \cdots + \overline{z}\psi_1 + \psi_0$, which proves that the function $\mathcal{U}_\nu^{(n)}(f)$ is poly-analytic of order n. To obtain the invariance property of $Aut_\Gamma^\nu(\mathbb{C})$ under the action of the operator $\mathcal{U}_\nu^{(n)}$, we have

$$\begin{aligned}
\mathcal{U}_\nu^{(n)}(f)(z+\gamma) &= (-1)^n e^{\nu|z+\gamma|^2} \frac{\partial^n}{\partial w^n} \left(w \mapsto e^{-\nu|w+\gamma|^2} f(w+\gamma) \right) (w=z) \\
&= (-1)^n e^{\nu|z+\gamma|^2} \frac{\partial^n}{\partial z^n} \left(\chi(\gamma) \overline{j^{-\nu}(\gamma,z)} e^{-\nu|z|^2} f \right)(z) \\
&= \chi(\gamma) j^\nu(\gamma,z) \mathcal{U}_\nu^{(n)}(f)(z)
\end{aligned}$$

for every $f \in Aut_\Gamma^\nu(\mathbb{C})$. This follows since the automorphic factor in (1.2.4) is holomorphic and satisfies

$$j^\nu(\gamma,z) e^{-\nu|z+\gamma|^2} = \overline{j^{-\nu}(\gamma,z)} e^{-\nu|z|^2}.$$

To conclude, let prove the uniqueness of the holomorphic solution of $\mathcal{U}_\nu^{(n)}(f) = F$, when exists for given n-poly-analytic function F. To this end, assume that $n \geq 1$ and observe first that $\mathcal{U}_\nu^{(n)}(f) = \mathcal{U}_\nu^{(n)} g$ holds if and only if $f = g + e^{\nu|z|^2} h$ for certain h such that $\bar{h} \in A_{n-1}(\mathbb{C})$. However, when $f, g \in Hol(\mathbb{C})$, such function is in addition solution of

$$\frac{\partial h}{\partial \overline{z}} = -\nu z h. \tag{3.1.6}$$

By iterating (3.1.6), one easily shows that the unique anti-poly-analytic solution is $h = 0$. □

Remark 3.2. *The surjectivity problem of $\mathcal{U}_\nu^{(n)}$ (constituting the converse assertion of Proposition 3.2), i.e., the existence of certain holomorphic function f for every given n-analytic function F such that $F = \mathcal{U}_\nu^{(n)}(f)$, requires further conditions on their holmorphic component functions which are provided in Theorem 3.1.*

Theorem 3.1. *Let $F = \overline{z}^n \psi_n + \overline{z}^{n-1}\psi_{n-1} + \cdots + \overline{z}\psi_1 + \psi_0$ be a n-analytic function on $\mathbb{C}$. Then, the following are equivalent*

- *(i) We have $F = \mathcal{U}_\nu^{(n)}(f)$ for certain holomorphic function f.*
- *(ii) The component function ψ_k, $k = 0, 1, \cdots, n-1$, satisfies*

$$(k+1) \frac{\partial \psi_{k+1}}{\partial z} = -\nu (n-k) \psi_k. \tag{3.1.7}$$

(iii) For every $k = 0, 1, \cdots, n-1$, the function $\dfrac{\partial^k F}{\partial \overline{z}^k}$ is a eigenfunction of

$$\Delta^{(n)}_{\nu,k} := -\sum_{j=0}^{n-k-1} \frac{(-1)^j}{j!}\overline{z}^j \left[\frac{\partial}{\partial z} - \frac{\nu(n-k)}{j+1}\overline{z}\right] \frac{\partial^{j+1}}{\partial \overline{z}^{j+1}} \tag{3.1.8}$$

with $\nu(n-k)$ as corresponding eigenvalue.

Before we begin the proof of these results, we can state the following lemma obtained in [6, Theorem 2.3] for quaternionic S-poly-regular functions, which remains valid for classical poly-analytic functions. Which consists in giving a characterization of a poly-analytic function.

Lemma 3.1 ([6])**.** *A function F is poly-analytic on $\mathbb{C}$ of order n if and only if there exists an analytical function ψ_0 such that*

$$F(z) = \psi_0(z) + \sum_{j=1}^{n}\sum_{k=0}^{n-j} \frac{(-1)^k}{j!k!}\overline{z}^{j+k} \frac{\partial^{j+k} F}{\partial \overline{z}^{j+k}}(z) \tag{3.1.9}$$

Proof. We need only to prove "only if". Clearly, we have $\partial^k_{\overline{z}} f = 0$ whenever $k > n$ and f is a given n-analytic function. However, for $k \leq n$ we have

$$\partial^k_{\overline{z}} f = \sum_{j=k}^{n} \frac{j!}{(j-k)!}\overline{z}^{j-k}\varphi_j,$$

which follows using the classical characterization $f(z) = \overline{z}^n \psi_n(z) + \cdots + \overline{z}\psi_1(z) + \psi_0(z)$ for certain holomorphic functions ψ_k. Thus, by considering successively the particular cases $k = n$, $k = n-1$, $k = n-2$ and $k = n-3$, and reasoning by induction one gets

$$(n-k)!\varphi_{n-k} = \sum_{s=0}^{k} (-1)^s \frac{\overline{z}^s}{s!} \partial^{n-k+s}_{\overline{z}} f, \quad k < n.$$

This yields

$$f = \varphi_0 + \sum_{j=1}^{n}\sum_{k=0}^{n-j} (-1)^k \frac{\overline{z}^{j+k}}{j!k!} \partial^{j+k}_{\overline{z}}(f).$$

□

Proof of Theorem 3.1. The assertion "(i) implies (ii)" follows since the holomorphic components of the poly-analytic function $\mathcal{U}^{(n)}_{\nu}(f)$, with f being holomorphic, are given by (3.1.5) and clearly obey the system in (3.1.7). The proof of the converse implication follows starting from the assumption that (3.1.7) holds for the holomorphic components ψ_k of a given n-analytic function F. In fact, one gets

$$\psi_k = \frac{(-1)^{n+k} n! \nu^{k-n}}{k!(n-k)!} \frac{\partial^{n-k}\psi_n}{\partial z^{n-k}}$$

for every $k = 0, 1, \cdots, n$. Subsequently, the holomorphic function $g := \nu^{-n}\psi_n$ satisfies

$$F = \sum_{k=0}^{n} \overline{z}^k \psi_k = (-1)^n \sum_{k=0}^{n} \binom{n}{k} \frac{(-\nu\overline{z})^k}{\nu^n} \frac{\partial^{n-k}\psi_n}{\partial z^{n-k}} = (-1)^n \left(\frac{\partial}{\partial z} - \nu\overline{z}\right)^n (g) = \mathcal{U}_\nu^{(n)}(g).$$

The proof of "(ii) is equivalent to (iii)" relies essentially on an adapted characterization of finite order poly-analytic functions in terms of its successive derivatives, initially obtained in Lemma 3.1. Namely, a complex-valued function F is poly-analytic on $\mathbb{C}$ of order n if and only if

$$F - \left[\sum_{j=1}^{n}\sum_{k=0}^{n-j} \frac{(-1)^k}{j!k!} \overline{z}^{j+k} \frac{\partial^{j+k}F}{\partial \overline{z}^{j+k}}\right] \tag{3.1.10}$$

is a holomorphic function. In this case, the holomorphic component functions of F are given by

$$\psi_k = \frac{1}{k!} \sum_{s=0}^{n-k} \frac{(-1)^s}{s!} \overline{z}^s \frac{\partial^{k+s}F}{\partial \overline{z}^{k+s}} \tag{3.1.11}$$

for $1 \leq k \leq n$, while the constant holomorphic component ψ_0 is exactly the one given through (3.1.10). However, in view of (3.1.11), the condition on the logarithmic derivative of ψ_k in (ii), i.e.,

$$k! \frac{\partial \psi_k}{\partial z} = -\nu(n-k+1)(k-1)!\psi_{k-1},$$

holds if and only if we have

$$\sum_{s=0}^{n-k} \frac{(-1)^s}{s!} \overline{z}^s \left[\frac{\partial}{\partial z} - \frac{\nu(n-k+1)}{s+1}\overline{z}\right] \frac{\partial^{k+s}F}{\partial \overline{z}^{k+s}} = -\nu(n-k+1) \frac{\partial^{k-1}F}{\partial \overline{z}^{k-1}},$$

which is equivalent to assertion (iii). This completes the proof of Theorem 3.1. ☐

Corollary 3.1. *Let f be a holomorphic function. Then, the function $\dfrac{\partial^{n-1}}{\partial \overline{z}^{n-1}}[\mathcal{U}_\nu^{(n)}(f)]$ is a eigefunction of Δ_ν with ν as corresponding eigenvalue.*

Proof. This readily follows by specifying $k = n-1$ in (3.1.8). ☐

Remark 3.3. *The operator in* (3.1.8) *generalizes in some how to high-order the magnetic Laplacian Δ_ν in* (3.0.2)*. Indeed, for $k = n-1$ it reduces further to $-\Delta_\nu$. Notice also that if we denote by Ψ the column vector $\Psi := [\psi_{n-1}, \cdots, \psi_1, \psi_0]^t$ and by S the shifting operator defined by $S([\psi_{n-1}, \cdots, \psi_1, \psi_0]^t) = [\psi_n, \cdots, \psi_2, \psi_1]^t$ then the matrix representation of the system in* (3.1.7) *reads*

$$\frac{\partial}{\partial z}(S \circ \Psi) = -\nu M_n^\nu \Psi,$$

where $M_n^\nu := diag(d_k^\nu)$ is the diagonal matrix whose coefficients being the ratios $d_k^\nu := s_{k+1}/s_{n-k}$ for varying $0 \leq k \leq n-1$. Here $s_j = \nu j$, $j = 1, 2, \cdots$, is the arithmetic sequence of eigenvalues of the Laplacian Δ_ν.

3.2 A characterization of the rang of the creation operator

The main result of this section concerns a concrete characterization of the scope of the operator $\mathcal{U}_\nu^{(n)}$ on the Hilbert space $H^{2,\nu}_\Gamma(\mathbb{C})$ by means of the automorphic poly-analytic eigenfunctions of the high order differential operator

$$\mathcal{D}_\nu^{(n)} := \Delta_\nu(\Delta_\nu - s_1)(\Delta_\nu - s_2)\cdots(\Delta_\nu - s_{n-1}). \tag{3.2.12}$$

We begin by showing that the $\mathcal{D}_\nu^{(n)}$ is closely connected to the linear differential operators

$$Af = -\left(\frac{\partial}{\partial z} - \nu\bar{z}\right) f = -e^{\nu|z|^2}\frac{\partial}{\partial z}\left(e^{-\nu|z|^2} f\right) \quad \text{and} \quad Bf = \frac{\partial f}{\partial \bar{z}}.$$

Lemma 3.2. *Let* $\mathcal{D}_\nu^{(n)}$ *be as in* (3.2.12). *Then*

(i) We have $\mathcal{D}_\nu^{(n)} = A^n B^n$.

(ii) For every $F \in \mathcal{F}^{2,\nu}_{\Gamma,\ell}(\mathbb{C})$ *with* $0 \leq \ell \leq n$ *we have* $\mathcal{D}_\nu^{(n)} F = \nu^n n! \delta_{n,\ell} F$.

Proof. Notice first that AB is exactly the elliptic Laplacian Δ_ν in (3.0.2), while $A^2B^2 = \Delta_\nu^2 - \nu\Delta_\nu$. For arbitrary $n \geq 1$, we make use of the algebraic identity $A^nB = BA^n - s_nA^{n-1}$ with $s_n = n\nu$, which follows easily by successive application of the identity $AB = BA - \nu$. Indeed, we get

$$A^{n+1}B^{n+1} = A(A^nB)B^n = A(BA^n - s_nA^{n-1})B^n = (AB - s_n)A^nB^n.$$

Mathematical induction leads to

$$A^{n+1}B^{n+1} = (AB - s_n)(AB - s_{n-1})\cdots(AB - s_1)AB =: \mathcal{D}_\nu^{(n+1)}.$$

This proves (i). For (ii), we have $\Delta_\nu F = s_\ell F$ whenever $F \in \mathcal{F}^{2,\nu}_{\Gamma,\ell}(\mathbb{C})$ from which it follows

$$\mathcal{D}_\nu^{(n)} F = \Delta_\nu(\Delta_\nu - s_1)(\Delta_\nu - s_2)\cdots(\Delta_\nu - s_{n-1})F = \prod_{j=0}^{n-1}(s_\ell - s_j)F.$$

Accordingly, we have $\mathcal{F}^{2,\nu}_{\Gamma,\ell}(\mathbb{C}) \subset \ker(\mathcal{D}_\nu^{(n)})$ for every $0 \leq \ell \leq n-1$. But, since

$$\prod_{j=0}^{n-1}(s_\ell - s_j) = \prod_{j=0}^{n-1} s_{\ell-j} = \frac{\nu^n \ell!}{(\ell-n)!}$$

for every $\ell \geq n$, it becomes clear that $\mathcal{D}_\nu^{(n)} F = \nu^n n! F$ whenever $F \in \mathcal{F}^{2,\nu}_{\Gamma,n}(\mathbb{C})$. □

Remark 3.4. *The operators $AB - s_j$, $j = 0, 1, 2, \cdots, n-1$, involved in the expression of operator $\mathcal{D}_\nu^{(n)}$ commute. Therefore, the latter one can be seen as a polynomial in Δ_ν with simple zeros being exactly the first n eigenvalues of Δ_ν. The explicit polynomial expression of $\mathcal{D}_\nu^{(n)}$ involves the elementary symmetric functions in n variables given by*

$$S_{n-j} = (-1)^j \sum_{0 \leq k_1 < k_2 < \cdots < k_j \leq n-1} s_{k_1} s_{k_2} \cdots s_{k_j}$$

for $1 \leq j \leq n$ with the convention that $S_n = 1$. Namely, we have

$$\mathcal{D}_\nu^{(n)} = \sum_{j=0}^{n} S_j \Delta_\nu^j. \tag{3.2.13}$$

Lemma 3.3. *The Green-Riemann formula is written in the following form:*

$$\int_{\partial\Omega} A dz + B d\overline{z} = -2i \int_\Omega \left(\frac{\partial B}{\partial z} - \frac{\partial A}{\partial \overline{z}} \right) dxdy \tag{3.2.14}$$

With Ω is a elementary domain. particularly

$$\int_{\partial\Omega} f(z)dz = 2i \int_\Omega \frac{\partial f}{\partial \overline{z}}(z) dxdy \tag{3.2.15}$$

Lemma 3.4. *The operator $\mathcal{U}_\nu^{(n)}$ preserves orthogonality in $\mathcal{L}^{2,\nu}_{\Gamma,\chi}$. Moreover, the operator $(\sqrt{\nu^n n!})^{-1}\mathcal{U}_\nu^{(n)}$ is unitary from $H^{2,\nu}_\Gamma(\mathbb{C})$ into $L^{2,\nu}_\Gamma(\mathbb{C})$.*

Proof. We begin by claiming that

$$I_\nu^{(n)}(f,g) := \int_{\Lambda(\Gamma)} \frac{\partial}{\partial z} \left(e^{-\nu|z|^2} \mathcal{U}_\nu^{(n)}(f)(z) \overline{\mathcal{U}_\nu^{(n)}(g)(z)} \right) d\lambda(z) = 0$$

holds for every $n \geq 0$ and every sufficiently differentiable functions f and g belonging to $Aut^{2,\nu}_{\Gamma,\chi}$. For its proof we distinguish two cases. The case of the co-compact discrete subgroup $\Gamma = \mathbb{Z}\omega_1 + \mathbb{Z}\omega_2$ can be handled easily making use of the Green formula since in this case the function

$$(x,y) = z \longmapsto F(x,y) = F(z) := e^{-\nu|z|^2} \mathcal{U}_\nu^{(n)}(f)(z) \overline{\mathcal{U}_\nu^{(n)}(g)(z)}$$

is doubly periodic (for $\mathcal{U}_\nu^{(n)}(f)$ and $G = \mathcal{U}_\nu^{(n)}(g)$ being in $Aut^{\nu}_{\Gamma,\chi}(\mathbb{C})$), and the boundary $\partial\Lambda(\Gamma)$ of $\Lambda(\Gamma)$ is a piecewise smooth closed and symmetric path with summits are ω_1, $\omega_1 + \omega_2$, ω_2 and the origin O. Indeed, we have

$$\int_{\Lambda(\Gamma)} \frac{\partial F}{\partial z}(z) d\lambda(z) = \oint_{\partial\Lambda(\Gamma)} F(\xi) d\xi = \int_O^{\omega_1} + \int_{\omega_1}^{\omega_1+\omega_2} + \int_{\omega_1+\omega_2}^{\omega_2} + \int_{\omega_2}^{O} = 0,$$

which follows since

$$\int_O^{\omega_1} F(\xi)d\xi + \int_{\omega_1+\omega_2}^{\omega_2} F(\xi)d\xi = \omega_1 \int_0^1 \left(F(t\omega_1) - F(t\omega_2 + \omega_2) \right) dt = 0,$$

and

$$\int_{\omega_1}^{\omega_1+\omega_2} F(\xi)d\xi + \int_{\omega_2}^{O} F(\xi)d\xi = \omega_2 \int_0^1 \left(F(t\omega_2+\omega_1) - F(t\omega_2+\omega_2)\right) dt = 0.$$

For the rank one discrete subgroup $\Gamma = \mathbb{Z}$, the fundamental cell is a strip, that we can identify with $[0,1[\times\mathbb{R}$. Thus, the periodicity of the considered function in the x-direction with period 1 (so that $F(1,y) = F(0,y)$ for every fixed $y \in \mathbb{R}$) is employed to get

$$J^{(n)}_{\nu,1}(f,g) = \int_{\mathbb{R}} \left(\int_0^1 \frac{\partial}{\partial x}\left(F(x,y)\right) dx \right) dy = \int_{\mathbb{R}} \left(F(1,y) - F(0,y)\right) dy = 0$$

Now, since $\lim\limits_{|y|\to+\infty} F(x,y) = 0$, we get

$$J^{(n)}_{\nu,2}(f,g) = \int_0^1 \left(\int_{\mathbb{R}} \frac{\partial}{\partial y}\left(F(x,y)\right) dy \right) dx = \int_0^1 \Big[F(x,y) \Big]_{y=-\infty}^{y=+\infty} dx = 0.$$

This proves that $2I^{(n)}_{\nu}(f,g) := J^{(n)}_{\nu,1}(f,g) - iJ^{(n)}_{\nu,2}(f,g) = 0$.

Subsequently, starting from

$$\begin{aligned} \left\langle \mathcal{U}^{(n)}_{\nu} f, G \right\rangle_{\nu,\Gamma} &= \left\langle \mathcal{U}^{(n-1)}_{\nu} f, BG \right\rangle_{\nu,\Gamma} - \int_{\Lambda(\Gamma)} \frac{\partial}{\partial z} \left(e^{-\nu|z|^2} \overline{G(z)} \mathcal{U}^{(n-1)}_{\nu}(f)(z) \right) d\lambda(z) \\ &= \left\langle \mathcal{U}^{(n-1)}_{\nu} f, BG \right\rangle_{\nu,\Gamma}, \end{aligned}$$

for any $n \geq 1$, we obtain $\left\langle \mathcal{U}^{(n)}_{\nu} f, G \right\rangle_{\nu,\Gamma} = \langle f, (B)^n G\rangle_{\nu,\Gamma}$ by proceeding by induction. Therefore, we get

$$\left\langle \mathcal{U}^{(n)}_{\nu} f, \mathcal{U}^{(n)}_{\nu} g \right\rangle_{\nu,\Gamma} = \nu^n n! \left\langle f, g \right\rangle_{\nu,\Gamma}.$$

This completes the proof. □

The next Lemma concerns the action of $\mathcal{U}^{(n)}_{\nu}$ on the eigenfunctions of Δ_ν.

Lemma 3.5. *The operator $\mathcal{U}^{(n)}_{\nu}$ leaves invariant the set of eigenfunctions of Δ_ν belonging to Aut^{ν}_{Γ}.*

Proof. The operators A and B preserve the space $Aut^{\nu}_{\Gamma}(\mathbb{C})$, so is the operator $\Delta_\nu = AB$, i.e.,

$$(\Delta_\nu f)(z+\gamma) := \chi(\gamma) j^{\nu}(\gamma, z)(\Delta_\nu f)(z)$$

for every f satisfying (1.2.2), the corresponding eigenvalue problem $\Delta^{\nu} f = \nu\lambda f$ in $Aut^{2,\nu}_{\Gamma}(\mathbb{C})$ is well defined. Moreover, from the facts $\Delta_\nu = AB = BA - \nu$ and $\mathcal{U}^{(n+1)}_{\nu} = A\mathcal{U}^{(n)}_{\nu}$, one gets

$$\Delta_\nu \mathcal{U}^{(n+1)}_{\nu} = (AB)A\,\mathcal{U}^{(n)}_{\nu} = A(\nu + \Delta_\nu)\mathcal{U}^{(n)}_{\nu}. \tag{3.2.16}$$

Thus, the proof can be obtained by induction in n. Indeed, if $\mathcal{U}^{(n)}f$ is a eigenfunction of Δ_ν whose corresponding eigenvalue is λ, then $\Delta_\nu \mathcal{U}_\nu^{(n+1)} f = (\lambda + \nu)\mathcal{U}_\nu^{(n+1)} f$ by means of (3.2.16). Accordingly, if $\Delta_\nu f = \lambda f$, then $\Delta_\nu \mathcal{U}_\nu^{(n)} f = (\lambda + n\nu) f$. One concludes thanks to Proposition 3.2 proving that the operator $(1/\sqrt{\nu^n n!})\mathcal{U}_\nu^{(n)}$ leaves invariant the space $Aut_\Gamma^\nu(\mathbb{C})$. □

Theorem 3.2. *We have*

$$\mathcal{U}_\nu^{(n)}\left(H_\Gamma^{2,\nu}(\mathbb{C})\right) = Aut_\Gamma^{2,\nu}(\mathbb{C}) \cap A_n(\mathbb{C}) \cap \ker(\mathcal{D}_\nu^{(n)} - \nu^n n!) = \mathcal{F}_{\Gamma,\chi,n}^{2,\nu}(\mathbb{C}).$$

Proof of Theorem 3.2. According to Proposition 3.2, the function $F := \mathcal{U}_\nu^{(n)}(f)$ is poly-analytic of order n and belongs to $Aut_\Gamma^\nu(\mathbb{C})$, whenever $f \in Aut_\Gamma^\nu(\mathbb{C}) \cap Hol(\mathbb{C})$. Thus, from its expansion in $A_n(\mathbb{C})$

$$F = \mathcal{U}_\nu^{(n)}(f) = \sum_{k=0}^{n} \binom{n}{k} (-1)^{n+k} \nu^k \bar{z}^k \frac{\partial^{n-k} f}{\partial z^{n-k}},$$

it is clear that

$$\frac{\partial^n F}{\partial \bar{z}^n} = \nu^n n! f.$$

Subsequently, we get

$$\nu^n n! F = \mathcal{U}_\nu^{(n)}\left(\frac{\partial^n F}{\partial \bar{z}^n}\right) = (-1)^n \left(\frac{\partial}{\partial z} - \nu \bar{z}\right)^n \frac{\partial^n F}{\partial \bar{z}^n} = A^n B^n(F) = \mathcal{D}_\nu^{(n)}(F)$$

by means of (i) in Lemma 3.2. The second assertion in Lemma 3.4 proves that $F = \mathcal{U}_\nu^{(n)}(f) \in L_\Gamma^{2,\nu}(\mathbb{C})$ and completes the proof of

$$\mathcal{U}_\nu^{(n)}\left(H_\Gamma^{2,\nu}(\mathbb{C})\right) \subset Aut_\Gamma^{2,\nu}(\mathbb{C}) \cap A_n(\mathbb{C}) \cap \ker(\mathcal{D}_\nu^{(n)} - \nu^n n!).$$

The above discussion contains the elements of the proof of the inverse inclusion. In fact, for given $F \in Aut_\Gamma^{2,\nu}(\mathbb{C}) \cap A_n(\mathbb{C}) \cap \ker(\mathcal{D}_\nu^{(n)} - \nu^n n!)$, we have

$$\nu^n n! F = \mathcal{D}_\nu^{(n)}(F) = \mathcal{U}_\nu^{(n)}(B^n F) \text{by (i) of Lemma 3.2.}$$

Clearly, the function $f := B^n(F)$ is holomorphic (since $F \in A_n(\mathbb{C})$). Moreover, it satisfies the automorphy equation. Indeed, starting from $\mathcal{U}_\nu^{(n)}(f) = F$, keeping in mind that F satisfies (1.2.2), we get

$$f(z+\gamma) = \frac{\partial^n}{\partial \bar{\xi}^n}(\xi \mapsto F(\xi+\gamma))(z) = \frac{\partial^n}{\partial \bar{z}^n}(\chi(\gamma) j^\nu(\gamma, z) F)(z) = \chi(\gamma) j^\nu(\gamma, z) f(z).$$

To conclude, we only need to observe from Lemma 3.4 that $F = \mathcal{U}_\nu^{(n)}(\nu^n n! f)$ belongs to $L_\Gamma^{2,\nu}(\mathbb{C})$ when $f \in L_\Gamma^{2,\nu}(\mathbb{C})$.

For the proof of the second equality in Theorem 3.2, notice that $\mathcal{F}_{\Gamma,\chi,n}^{2,\nu}(\mathbb{C}) \subset Aut_\Gamma^{2,\nu}(\mathbb{C}) \cap A_n(\mathbb{C})$ by definition of $\mathcal{F}_{\Gamma,\chi,n}^{2,\nu}(\mathbb{C})$ and $\mathcal{F}_{\Gamma,\chi,n}^{2,\nu}(\mathbb{C}) \subset \ker(\mathcal{D}_\nu^{(n)} - \nu^n n!)$ by means of (ii) of Lemma

3.2. This proves $\mathcal{F}^{2,\nu}_{\Gamma,\chi,n}(\mathbb{C}) \subset \mathcal{U}^{(n)}_{\nu}\left(H^{2,\nu}_{\Gamma}(\mathbb{C})\right)$. The proof of $\mathcal{U}^{(n)}_{\nu}\left(H^{2,\nu}_{\Gamma}(\mathbb{C})\right) \subset \mathcal{F}^{2,\nu}_{\Gamma,\chi,n}(\mathbb{C})$ makes appeal to the fact that

$$Aut^{2,\nu}_{\Gamma}(\mathbb{C}) \cap A_n(\mathbb{C}) = \bigoplus_{\ell=0}^{n} \mathcal{F}^{2,\nu}_{\Gamma,\chi,\ell}(\mathbb{C}).$$

In fact, for given $F \in Aut^{2,\nu}_{\Gamma}(\mathbb{C}) \cap A_n(\mathbb{C}) \cap \ker(\mathcal{D}^{(n)}_{\nu} - \nu^n n!)$, we write $F = \varphi_0 + \varphi_1 + \cdots + \varphi_n$ for certain φ_j in $\mathcal{F}^{2,\nu}_{\Gamma,\chi,j}(\mathbb{C})$, $j = 0, 1, \cdots, n$. Therefore, from (ii) of Lemma 3.2, we get

$$F = \frac{1}{\nu^n n!}\mathcal{D}^{(n)}_{\nu} F = \frac{1}{\nu^n n!}\mathcal{D}^{(n)}_{\nu}\left(\sum_{j=0}^{n}\varphi_j\right) = \frac{1}{\nu^n n!}\mathcal{D}^{(n)}_{\nu}\varphi_n = \varphi_n \in \mathcal{F}^{2,\nu}_{\Gamma,\chi,n}(\mathbb{C}).$$

□

The following result appears as immediate consequence of the previous ones and provides sufficient conditions on a n-analytic automorphic function to belong to the range of $\mathcal{U}^{(n)}_{\nu}$.

Theorem 3.3. *Let F be a n-analytic function in $Aut^{2,\nu}_{\Gamma}(\mathbb{C})$ such that $\mathcal{D}^{(n)}_{\nu} F = \nu^n n! F$ and*

$$\Delta^{(n)}_{\nu,k}\left(\frac{\partial^k F}{\partial \overline{z}^k}\right) = \nu(n-k)\frac{\partial^k F}{\partial \overline{z}^k}$$

for every $k = 0, 1, \cdots, n-1$. Then $F \in \mathcal{F}^{2,\nu}_{\Gamma,n}(\mathbb{C})$ and there exists a unique $f \in H^{2,\nu}_{\Gamma}(\mathbb{C})$ such that $F = \mathcal{U}^{(n)}_{\nu}(f)$.

Proof. Let F be a complex-valued function satisfying the assumptions of the theorem. The existence of certain holomorphic function f such that $F = \mathcal{U}^{(n)}_{\nu}(f)$ follows by applying Theorem 3.1 since F is n-analytic and satisfies

$$\Delta^{(n)}_{\nu,k}\left(\frac{\partial^k F}{\partial \overline{z}^k}\right) = \nu(n-k)\frac{\partial^k F}{\partial \overline{z}^k}, \ k = 0, 1, \cdots, n-1$$

But, by Lemma 3.4 combined with the fact that $Aut^{2,\nu}_{\Gamma,\chi}$ we deduce that $f \in H^{2,\nu}_{\Gamma,\chi}(\mathbb{C})$. Hence $F \in \mathcal{F}^{2,\nu}_{\Gamma,\chi,n}(\mathbb{C})$ holds making use of Theorem 3.2. Uniqueness follows using Proposition 3.2. □

3.3 Construction of orthogonal basis

3.3.1 The rank one case:

Here, we are concerned with the rank one discrete subgroup $\Gamma = \mathbb{Z}$ and the action of $\mathcal{U}^{(n)}_{\nu}$ on the functions

$$E^{\alpha,\nu}_{k}(z) = \left(\frac{2\nu}{\pi}\right)^{1/4} e^{\frac{\nu}{2}z^2 + 2i\pi(\alpha+k)z - \frac{\pi^2}{\nu}(\alpha+k)^2}, \quad k \in \mathbb{Z}, \tag{3.3.17}$$

forming an orthonormal basis of $H^{2,\nu}_{\mathbb{Z}}(\mathbb{C})$ (see [25]).

Proposition 3.3. *Let $n \in \mathbb{N}$ and $k \in \mathbb{Z}$, we have*

$$\mathcal{U}_\nu^{(n)}\left(E_k^{\alpha,\nu}\right)(z)=\left(\frac{2\nu}{\pi}\right)^{1/4}(-i)^n e^{\frac{\nu}{2}z^2+2i\pi(\alpha+k)z-\frac{\pi^2}{\nu}(\alpha+k)^2}\left(\frac{\nu}{2}\right)^{\frac{n}{2}} H_n\left(\sqrt{2\nu}\Im(z)+\sqrt{\frac{2}{\nu}}\pi(\alpha+k)\right). \tag{3.3.18}$$

where H_m is a Hermite polynomial.

The proof of these results is based on the following lemma

Lemma 3.6. *[7] For every given a real number β and complex number ξ, we have*

$$\begin{aligned} I_n^{\nu,\beta}(z,\bar{z}\mid\xi) &= (-1)^n e^{\nu z\bar{z}-\beta z^2-\xi z}\frac{\partial^n}{\partial z^n}\left(e^{-\nu z\bar{z}+\beta z^2+\xi z}\right)\\ &= (-i)^n\beta^{\frac{n}{2}}H_n\left(\frac{2\beta z-\nu\bar{z}+\xi}{2i\beta^{1/2}}\right)\end{aligned} \tag{3.3.19}$$

Proof. of Proposition 3.3 When taking $\beta=\frac{\nu}{2}$ and $\xi=2i\pi(\alpha+k)$ in the above lemma, we have

$$\begin{aligned}\mathcal{U}_\nu^{(n)}(E_k^{\alpha,\nu})(z) &= \left(\frac{2\nu}{\pi}\right)^{1/4}(-1)^n e^{\nu|z|^2}\frac{\partial^n}{\partial z^n}\left(e^{-\nu|z|^2}e^{\frac{\nu}{2}z^2+2i\pi(\alpha+k)z-\frac{\pi^2}{\nu}(\alpha+k)^2}\right)\\ &= \left(\frac{2\nu}{\pi}\right)^{1/4}(-1)^n e^{\nu|z|^2}\frac{\partial^n}{\partial z^n}\left(e^{-\nu|z|^2+\frac{\nu}{2}z^2+2i\pi(\alpha+k)z-\frac{\pi^2}{\nu}(\alpha+k)^2}\right)\\ &= \left(\frac{2\nu}{\pi}\right)^{1/4} e^{\frac{\nu}{2}z^2+2i\pi(\alpha+k)z-\frac{\pi^2}{\nu}(\alpha+k)^2} I_n^{\nu,\frac{\nu}{2}}(z,\bar{z}\mid 2i\pi(\alpha+k))\\ &= \left(\frac{2\nu}{\pi}\right)^{1/4}(-i)^n e^{\frac{\nu}{2}z^2+2i\pi(\alpha+k)z-\frac{\pi^2}{\nu}(\alpha+k)^2}\left(\frac{\nu}{2}\right)^{\frac{n}{2}} H_n\left(i\sqrt{\frac{\nu}{2}}(\overline{z}-z)+\sqrt{\frac{2}{\nu}}\pi(\alpha+k)\right).\end{aligned}$$

□

The following corollary is a consequence of the above proof and Theorem 5.2 in [7], which is the integral representation of $\mathcal{U}_\nu^{(n)}(E_k^{\alpha,\nu})(z)$.

Corollary 3.2. *For all k in $\mathbb{Z}$, we have*

$$\mathcal{U}_\nu^{(n)}(E_k^{\alpha,\nu})(z)=2^{1/4}\left(\frac{\nu}{\pi}\right)^{5/4}e^{-\frac{\pi^2}{\nu}(\alpha+k)^2+\nu|z|^2}\int_{\mathbb{C}}(-\nu\bar{\zeta})^n e^{-\nu|\zeta|^2+\frac{\nu}{2}\zeta^2+2i\pi(\alpha+k)\zeta}e^{2i\nu\Im\langle z,\zeta\rangle}d\mu(\zeta) \tag{3.3.20}$$

Proposition 3.4. *The functions $(\nu^n n!)^{-1/2}\mathcal{U}_\nu^{(n)}(E_k^{\alpha,\nu})$, for varying $n\in\mathbb{Z}^+$ and $k\in\mathbb{Z}$, form an orthogonal system of eigenfunctions of Δ_ν on $L_{\mathbb{Z}}^{2,\nu}(\mathbb{C})$ with νn as associated eigenvalue.*

Proof. By using the proposition 3.3, thus, direct computation infers

$$\begin{aligned}\left\langle\mathcal{U}_\nu^{(m)}\left(E_j^{\alpha,\nu}\right),\mathcal{U}_\nu^{(n)}\left(E_k^{\alpha,\nu}\right)\right\rangle_{\nu,\Gamma} &= (-1)^n(i)^{n+m}\left(\frac{2\nu}{\pi}\right)^{\frac{1}{2}}\left(\frac{\nu}{2}\right)^{\frac{n+m}{2}}e^{-\frac{\pi^2}{\nu}\left[(\alpha+j)^2+(\alpha+k)^2\right]}\left(\int_0^1 e^{2i\pi(j-k)x}dx\right)\\ &\times\int_{\mathbb{R}}e^{-2\nu y^2-2\pi(2\alpha+j+k)y}H_m\left(\sqrt{2\nu}y+\sqrt{\frac{2}{\nu}}\pi(\alpha+j)\right)H_n\left(\sqrt{2\nu}y+\sqrt{\frac{2}{\nu}}\pi(\alpha+k)\right)dy\end{aligned}$$

Making the change of variable $u = \sqrt{2\nu}y + \sqrt{\frac{2}{\nu}}\pi(\alpha + j)$, we find that

$$\begin{aligned}\left\langle \mathcal{U}_\nu^{(m)}\left(E_j^{\alpha,\nu}\right), \mathcal{U}_\nu^{(n)}\left(E_k^{\alpha,\nu}\right)\right\rangle_{\nu,\Gamma} &= (-1)^n (i)^{n+m} \left(\frac{1}{\pi}\right)^{\frac{1}{2}} \left(\frac{\nu}{2}\right)^{\frac{n+m}{2}} e^{-2\frac{\pi^2}{\nu}\left[(\alpha+j)^2\right]}\delta_{j,k} \\ &\times \int_{\mathbb{R}} e^{2\frac{\pi^2}{\nu}\left[(\alpha+j)^2\right]} e^{-u^2} H_n(u) H_m(u) du \\ &= (-1)^n (i)^{n+m} \left(\frac{1}{\pi}\right)^{\frac{1}{2}} \left(\frac{\nu}{2}\right)^{\frac{n+m}{2}} \delta_{j,k} \times \int_{\mathbb{R}} e^{-u^2} H_n(u) H_m(u) du \\ &= (-1)^n (i)^{n+m} \left(\frac{1}{\pi}\right)^{\frac{1}{2}} \left(\frac{\nu}{2}\right)^{\frac{n+m}{2}} \delta_{j,k} \int_{\mathbb{R}} e^{-u^2} H_m(u) H_n(u) du \\ &= \nu^n n! \delta_{m,n} \delta_{j,k}\end{aligned}$$

This proves that $\mathcal{U}_\nu^{(n)}(E_k^{\alpha,\nu})$ form an orthogonal system in $L^{2,\nu}_{\mathbb{Z}}(\mathbb{C})$ with explicit square norm. The fact that $\mathcal{U}_\nu^{(n)}(f) \in L^{2,\nu}_{\mathbb{Z}}(\mathbb{C})$ whenever f is in $H^{2,\nu}_{\mathbb{Z}}(\mathbb{C})$ or moreover that the operator $(1/\sqrt{\nu^n n!})\mathcal{U}_\nu^{(n)}$ is unitary from $H^{2,\nu}_{\mathbb{Z}}(\mathbb{C})$ into $L^{2,\nu}_{\mathbb{Z}}(\mathbb{C})$ is immediate from Lemma 3.4. However, we provide below a direct proof based on the fact that the functions $E_k^{\alpha,\nu}$ in (3.3.17) form an orthonormal basis of $H^{2,\nu}_{\mathbb{Z}}(\mathbb{C})$. Indeed, for any $f \in H^{2,\nu}_{\mathbb{Z}}(\mathbb{C})$ we have

$$\mathcal{U}_\nu^{(n)}(f) = \sum_{k=0}^{+\infty} \langle f, E_k^{\alpha,\nu}\rangle_{\nu,\mathbb{Z}} \mathcal{U}_\nu^{(n)}(E_k^{\alpha,\nu}).$$

But, by Parsevel identity, one obtains

$$\left\|\mathcal{U}_\nu^{(n)}(f)\right\|^2_{\nu,\mathbb{Z}} = \sum_{k=0}^{+\infty} |\langle f, E_k^{\alpha,\nu}\rangle_{\nu,\mathbb{Z}}|^2 \left\|\mathcal{U}_\nu^{(n)}(E_k^{\alpha,\nu})\right\|^2_{\nu,\mathbb{Z}} = \nu^n n! \left\|f\right\|^2_{\nu,\mathbb{Z}}.$$

Since for $n \geq 1$, we have $BA^n = A^n B + n\nu A^{n-1}$, then

$$\Delta_\nu(\mathcal{U}_\nu^{(n)}(E_k^{\alpha,\nu})) = ABA^n(E_k^{\alpha,\nu}) = \left(A^{n+1}B + n\nu A^n\right)(E_k^{\alpha,\nu}) = n\nu A^n(E_k^{\alpha,\nu}) = n\nu \mathcal{U}_\nu^{(n)}(E_k^{\alpha,\nu}).$$

□

Remark 3.5. *More generally, we note that, if f is a holomorphic function on $\mathbb{C}$, then $\mathcal{U}_\nu^{(n)}(f)$ is an eigenfunction of Δ_ν with associated with the eigenvalue $n\nu$. Indeed, since $BA = AB + \nu$, four $n \geqslant 1$, we have $BA^n = A^n B + n\nu A^{n-1}$, then*

$$\Delta_\nu(\mathcal{U}_\nu^{(n)}(f)) = ABA^n(f) = \left(A^{n+1}B + n\nu A^n\right)(f) = n\nu A^n(f) = n\nu \mathcal{U}_\nu^{(n)}(f).$$

Remark 3.6. *Proposition 3.4 reproves that $\mathcal{U}_\nu^{(n)}(H^{2,\nu}_{\mathbb{Z}}(\mathbb{C})) = \mathcal{F}^{2,\nu}_{\mathbb{Z},n}(\mathbb{C})$ for the rank one discrete subgroup, since the functions $\mathcal{U}_\nu^{(n)}(E_k^{\alpha,\nu})$ coincide with those the functions $\psi^{\alpha,\nu}_{\ell,n}(z,\overline{z})$ in [25] (see also [37]) constituting a complete orthonormal basis of $\mathcal{F}^{2,\nu}_{\mathbb{Z},n}(\mathbb{C})$.*

As immediate consequence, we prove that the reproducing kernel of $\mathcal{F}^{2,\nu}_{\mathbb{Z},n}(\mathbb{C})$ is closely connected to the reproducing kernel of $H^{2,\nu}_{\Gamma}(\mathbb{C})$ This connection is expressed using the generalized theta function $\theta_{\alpha,0}$ with characteristic $[\alpha, 0]$, and explicitly given by [25]

$$K^{\alpha,\nu}(z,w) = \left(\frac{2\nu}{\pi}\right)^{1/2} e^{\frac{\nu}{2}(z^2+\overline{w}^2)}\theta_{\alpha,0}\left(z-\overline{w}\middle|\frac{2i\pi}{\nu}\right).$$

To this purpose, the z variable is added to $\mathcal{U}^{(n,z)}_{\nu}$ to mean that the derivation is down with respect to that variable.

Corollary 3.3. *For* $\Gamma = \mathbb{Z}$*, the function* $K^{\alpha,\nu}_{\mathbb{Z},n}(z,w) = (1/\nu^n n!)\mathcal{U}^{(n,z)}_{\nu} \circ \mathcal{U}^{(n,\overline{w})}_{\nu} K^{\nu}_{\mathbb{Z}}(z,w)$ *is the reproducing kernel of* $\mathcal{F}^{2,\nu}_{\mathbb{Z},n}(\mathbb{C})$

Proof. In view of the Remark 3.6, the functions $\mathcal{U}^{(n)}_{\nu}(E^{\alpha,\nu}_k)$, $k = 0, 1, 2, \cdots$, form an orthogonal basis of $\mathcal{F}^{2,\nu}_{\mathbb{Z},n}(\mathbb{C})$. Then its reproducing kernel is given by

$$\begin{aligned}\sum_{k=0}^{+\infty}\frac{\mathcal{U}^{(n)}_{\nu}(E^{\alpha,\nu}_k)(z)\overline{\mathcal{U}^{(n)}_{\nu}(E^{\alpha,\nu}_k)(w)}}{\nu^n n!} &= \frac{1}{\nu^n n!}\mathcal{U}^{(n,z)}_{\nu}\circ\mathcal{U}^{(n,\overline{w})}_{\nu}\left(\sum_{k=0}^{+\infty}E^{\alpha,\nu}_k(z)\overline{E^{\alpha,\nu}_k(w)}\right)\\ &= \frac{1}{\nu^n n!}\mathcal{U}^{(n,z)}_{\nu}\circ\mathcal{U}^{(n,\overline{w})}_{\nu}K^{\nu}_{\mathbb{Z}}(z,w),\end{aligned}$$

where $K^{\nu}_{\mathbb{Z}}$ is the reproducing kernel of $H^{2,\nu}_{\mathbb{Z}}(\mathbb{C})$. □

3.3.2 The full rank case :

For rank two discrete subgroup, the eigenspaces $\mathcal{F}^{2,\nu}_{\Gamma,\chi,n}(\mathbb{C})$ are of equal dimension. We have $\dim(\mathcal{F}^{2,\nu}_{\Gamma,\chi,n}(\mathbb{C})) = 1$ for $\nu = \pi/\mathrm{vol}(\mathbb{C}/\Gamma)$. In this case, the Γ-theta function

$$\theta^{\nu;n}_{\Gamma,\chi}(z) = \nu^n\sum_{\gamma\in\Gamma}\chi(\gamma)e^{-\nu|\gamma|^2}j^{\nu}(\gamma,z)(\overline{z-\gamma})^n \tag{3.3.21}$$

generates the eigenspace $\mathcal{F}^{2,\nu}_{\Gamma,\chi,n}(\mathbb{C})$. This follows by means of Theorem 3.2 since $\theta^{\nu}_{\Gamma,\chi}(z) := \theta^{\nu,0}_{\Gamma,\chi}(z) \in H^{2,\nu}_{\Gamma,\chi}(\mathbb{C})$ and

$$\mathcal{U}^{(n)}_{\nu}(\theta^{\nu}_{\Gamma,\chi}) = \sum_{\gamma\in\Gamma}\chi(\gamma)e^{-\nu|\gamma|^2}\mathcal{U}^{(n)}_{\nu}\left(j^{\nu}(\gamma,z)\right) = \theta^{\nu,n}_{\Gamma,\chi}(z),$$

thanks to $\mathcal{U}^{(n)}_{\nu}\left(j^{\nu}(\gamma,z)\right) = \nu^n\overline{(z-\gamma)}^n j^{\nu}(\gamma,z)$. This can also be handled directly using the explicit expression of $\theta^{\nu,n}_{\Gamma,\chi}$ in (3.3.21) keeping in mind that the triplet (Γ,χ,ν) satisfies the (RDQ) condition or also by observing that ∇'_z commutes with the transformation

$$[T^{\nu}_{\gamma}f](z) := j^{-\nu}(\gamma,z)f(z+\gamma).$$

To conclude, one only needs to assure that $\mathcal{U}_\nu^{(n)}(\theta^\nu_{\Gamma,\chi})$ is not identically zero, which readily follows from the discussion in Section 3.2. This also follows by direct computation starting from (3.3.21). Indeed, it is not hard to get $(\nabla''_z)^n \theta^{\nu,n}_{\Gamma,\chi} = \nu^n n! \theta^\nu_{\Gamma,\chi}$. But since $\Delta_\nu \theta^{\nu,n}_{\Gamma,\chi} = n\nu\theta^{\nu,n}_{\Gamma,\chi}$, it follows

$$(\nabla''_z)^n[\Delta_\nu \theta^{\nu,n}_{\Gamma,\chi}] = n \cdot \nu^{n+1} n! \theta^\nu_{\Gamma,\chi}.$$

This infers that $\theta^{\nu,n}_{\Gamma,\chi} \not\equiv 0$ for $\theta^\nu_{\Gamma,\chi}$ being a nonzero function on $\mathbb{C}$.

Chapter 4

Extension to the mixed automorphic functions

We assume that Γ is a discrete subgroup of $G = U(1) \ltimes \mathbb{C}$, and (ρ, τ) is an equivariant pair such that $\rho(\Gamma) \subset \Gamma$. Additionally, let μ and ν be two positive real numbers. Associated with the data $(\Gamma, \chi, \nu, \mu, \rho, \tau)$, we define $\mathcal{M}^{\nu,\mu}_{\rho,\tau}(\mathbb{C})$ as the vector space of smooth complex-valued functions F on $\mathbb{C}$ which satisfies the functional equation:

$$F(\gamma \cdot z) = J^{\nu,\mu}_{\rho,\tau}(\gamma, z) F(z) \tag{4.0.1}$$

for every $\gamma \in \Gamma$ and $z \in \mathbb{C}$, where

$$J^{\nu,\mu}_{\rho,\tau}(\gamma, z) := \chi(\gamma) j^{\nu}(\gamma, z) j^{\mu}(\rho(\gamma), \tau(z)) \tag{4.0.2}$$

and

$$j^{\alpha}(\gamma, z) := \mathrm{e}^{-\alpha i \Im \left\langle z, \gamma^{-1} \cdot 0 \right\rangle} \tag{4.0.3}$$

The equivariant condition (4.1.6) ensures that $j^{\alpha}(\rho(\cdot), \tau(\cdot))$ is also an automorphic factor when $j^{\alpha}(\cdot, \cdot)$ is one too. Indeed, for any $\gamma, \gamma' \in \Gamma$, we have

$$\begin{aligned} j^{\alpha}\left(\rho\left(\gamma\gamma'\right), \tau(z)\right) &= j^{\alpha}\left(\rho(\gamma)\rho\left(\gamma'\right), \tau(z)\right) \\ &= j^{\alpha}\left(\rho(\gamma), \rho\left(\gamma'\right) \cdot \tau(z)\right) j^{\alpha}\left(\rho\left(\gamma'\right), \tau(z)\right) \\ &= j^{\alpha}\left(\rho(\gamma), \tau\left(\gamma' \cdot z\right)\right) j^{\alpha}\left(\rho\left(\gamma'\right), \tau(z)\right) \end{aligned}$$

In the second line, we utilized the co-cycle equation satisfied by j^{α}, which is an automorphic facto, and the third line follows by means of (4.1.6). The space of $\mathcal{M}^{\nu,\mu}_{\rho,\tau}(\mathbb{C})$ is called the space of planar mixed automorphic functions, or also of bi weight (ν, μ), with respect to the equivalent pair. (ρ, τ) and the discrete subgroup Γ.

The space $\mathcal{M}^{\nu,\mu}_{\rho,\tau}(\mathbb{C})$ can be realized as holomorphic sections of the line bundle $\Gamma \backslash (\mathbb{C} \times \mathbb{C})$ over $\Gamma \backslash \mathbb{C}$ with fiber $\mathbb{C}$. This is achieved by taking into account the action

$$\gamma \cdot (z, w) = (\gamma \cdot z, J^{\nu,\mu}_{\rho,\tau}(\gamma, z) w).$$

As in the classical setting, the data $(\Gamma, \chi, \nu, \mu, \rho, \tau)$ cannot be chosen freely or else the space will be trivial. Namely, using similar arguments as in [[18], [21], [27]], and we use the results of the proposition 1.4, we can show the following.

From proposition 1.4, we can notice that the functional space $\mathcal{M}^{\nu,\mu}_{\rho,\tau}(\mathbb{C})$ is nontrivial if and only if the condition

$$\chi\left(\gamma\gamma'\right) = \chi(\gamma)\chi(\gamma)e^{-2i\phi^{\nu,u}_{\rho}(\gamma,\gamma')} \tag{4.0.4}$$

holds for every $\gamma, \gamma' \in \Gamma$. Where the phase function $\phi^{v,\mu}_{\rho}\left(\gamma,\gamma'\right)$ is the real-valued function defined on $G \times G$ by

$$\phi^{\nu,\mu}_{\rho}\left(\gamma,\gamma'\right) := \Im m\left(\nu\left\langle\gamma^{-1}\cdot 0, \gamma'\cdot 0\right\rangle + \mu\left\langle\rho\left(\gamma^{-1}\right)\cdot 0, \rho\left(\gamma'\right)\cdot 0\right\rangle\right)$$

The equivariant pair are important for our definition of the mixed automorphic functions of second species. Therefore, in the first section of this chapter, we will focus on their characterization in our specific one-dimensional complex context. After in the other sections, we emphasize to continue in this direction by investigating the general case of arbitrary discrete subgroup of Γ the semi-direct group $U(1) \ltimes \mathbb{C}$ of the unitary group $U(1)$ and $\mathbb{C}$. Our main purpose is the concrete spectral properties of the invariant magnetic Schrödinger operator

$$H_{\theta^{\nu,\mu}_{\tau'}} = \left(d + \operatorname{ext}\theta^{v,\mu}_{\tau}\right)^{*}\left(d + \operatorname{ext}\theta^{v,\mu}_{\tau}\right)$$

associated to the potential vector

$$\theta^{v,\mu}_{\tau}(z) := -\frac{v}{2}(\bar{z}dz - zd\bar{z}) - \frac{\mu}{2}(\bar{\tau}d\tau - \tau d\tau)$$

and acting on Γ-mixed automorphic functions with respect to given equivariant pair (ρ, τ) belonging to the Hilbert space $L^{2,\nu}_{\mu,\tau}(C/\Gamma)$ of square integrable functions with respect to the scalar product

$$\langle f, g\rangle_{\nu,\mu,\tau} := \int_{\mathbf{C}/\Gamma} f(z)\overline{g(z)}e^{-\nu|z|^2}d\lambda_{\tau}(z) \tag{4.0.5}$$

involving the special gaussian density

$$d\lambda_{\tau}(z) := e^{-\mu\left(|E\tau(z)|^2 - |\bar{E}\tau(z)|^2\right)}d\lambda(z),$$

where $\overline{E} = \bar{z}\frac{\partial}{\partial\bar{z}}$ is the complex conjugate of the complex Euler operator $E = z\frac{\partial}{\partial z}$, and $d\lambda(z) = dxdy, z = x + iy$, is the Lesgue measure on the complex plane.

4.1 On G-equivariant pairs

We denote by G the group $U(1) \ltimes \mathbb{C}$, that we realize in matrix representation (1.3.7) and let Γ be a discrete subgroup of it. The action of G on the complex plane $\mathbb{C}$ is transitive and given by the holomorphic mappings $g \cdot z = az + b$, where $g = [a, b] \in G$, and $z \in \mathbb{C}$. By G-equivariant pair (ρ, τ), we mean a G-endomorphism ρ and a compatible mapping $\tau : \mathbb{C} \longrightarrow \mathbb{C}$, in the sense of the definition 1.3. $i.e.$

$$\tau(g \cdot z) = \rho(g) \cdot \tau(z) \tag{4.1.6}$$

holds true for every $g \in G$ and $z \in \mathbb{C}$.

The first main objective is an explicit characterization of such equivariant pairs. Thus, it is natural to look for how to exploit (4.1.6) in order to exhibit suitable maps τ for fixed ρ, or vice versa to determinate possible ρ when τ is fixed. Such characterization seems to be far of being realizable without further assumptions. Thus, we digress into a partial characterization and provide such bridge between the choice of ρ and τ.

Theorem 4.1. *Let ρ be a G-endomorphism and define $\Xi_\rho := \{\beta \in \mathbb{C} \mid G_\beta \supset \rho(G_0)\}$, where $G_x := \{g \in G \mid g \cdot x = x\}$. Then, a given mapping $\tau : \mathbb{C} \longrightarrow \mathbb{C}$ satisfies the equivariant condition (4.1.6) if and only if*

$$\tau(z) = \tau_\beta(z) := \rho([1, z]) \cdot \beta \tag{4.1.7}$$

for certain fixed $\beta \in \Xi_\rho$ ($\tau_\beta(0) = \beta$).

Proof. For the proof of "only if", notice that Eq. 4.1.6 implies that $\rho(G_x) \subset G_{\tau(x)}$ for every fixed $x \in \mathbb{C}$. As a special case, we have

$$\rho(G_0) \subset G_{\tau(0)} \tag{4.1.8}$$

Next, using the fact that G acts transitively on $\mathbb{C}$, one can exhibit for every $z \in \mathbb{C}$ an element $g_z \in G$ such that $z = g_z \cdot 0$. Hence, one can check that $\tau(z) = \rho(g_z) \cdot \tau(0)$, and therefore we have

$$\tau(z) = \tau_\beta(z) := \rho([1, z]) \cdot \tau(0) \tag{4.1.9}$$

Conversely, it can be checked that (4.1.9) defines all mappings from $\mathbb{C}$ onto $\mathbb{C}$ that satisfy the equivariant condition (4.1.6) for a given G-endomorphism $\rho : G \longrightarrow G$. To this end, fix $\beta \in \Xi_\rho$ and note that τ_β as defined by (4.1.9), satisfies

$$\tau_\beta(z) = \rho(g_z) \cdot \beta \tag{4.1.10}$$

for every $g_z \in G$, such that $g_z \cdot 0 = z$. Indeed, since $g_z = [a, z] = [1, z][a, 0]$ for some arbitrary $a \in U(1)$, it follows

$$\rho(g_z) \cdot \beta = \rho([1, z][a, 0]) \cdot \beta = \rho([1, z]) \cdot (\rho([a, 0]) \cdot \beta)$$

Thus, using the fact that $\beta \in \Xi_\rho$, which reads here as $\rho([a, 0]) \cdot \beta = \beta$ for every $a \in U(1)$, one then gets

$$\rho(g_z) \cdot \beta = \rho([1, z]) \cdot \beta := \tau_\beta(z).$$

Now, since for every $g \in G$ we have $(gg_z) \cdot 0 = g \cdot z$, we obtain

$$\tau(g \cdot z) \overset{4.1.10}{=} \rho(gg_z) \cdot \beta = \rho(g) \cdot (\rho(g_z) \cdot \beta) = \rho(g) \cdot \tau_\beta(z)$$

which means that the equivariant condition (4.1.6) holds. This completes the proof. $\square$

Remark 4.1. *Note that when $\rho(G_0)$ is trivial; i.e $\rho(G_0) = \{1\}$, we have $\Xi_\rho = \mathbb{C}$, and therefore, for the particular case $\rho = Id_G$ (Id_G being the identity map of G), the mapping τ such that the pair (Id_G, τ) satisfies the equivariant condition (4.1.6) are the translations $\tau_\beta(z) = z + \beta$ with $\beta \in \mathbb{C}$. Note also that such translations satisfy (4.1.6) in the case when the G-endomorphism ρ verifies $\rho([1, z]) = (a_z, z)$.*

Example 4.1. *Let $h = [\alpha_h, \beta_h]$ be a fixed element in G. Consider the G-endomorphism ρ_h given by*

$$\rho_h(g) := hgh^{-1} = [a, (1-a)\beta_h + b\alpha_h]; \quad g = [a, b] \in G, \tag{4.1.11}$$

and the map $\tau_h : \mathbb{C} \longrightarrow \mathbb{C}$ defined by

$$\tau_h(z) := h \cdot z = \alpha_h z + \beta_h. \tag{4.1.12}$$

The pair (ρ_h, τ_h) defines an "alteration of $(Id_G, Id_\mathbb{C})$" in the sense of [31], and satisfies the equivariant condition (4.1.6). This is in correlation with what precede, since for such ρ_h, we have $\rho_h([a, 0]) \cdot \beta_h = \beta_h$, i.e. $\beta_h \in \Xi_\rho$. Therefore, one can use (4.1.7) to reproduce again τ_h given through (4.1.12).

Example 4.2. *Consider the conjugate endomorphism under G,*

$$\rho([a, b]) = \overline{[a, b]} := [\bar{a}, \bar{b}].$$

In this case, one can see easily that $\Xi_\rho = \{0\}$. According to Theorem (4.1), $([a, b] \longmapsto \overline{[a, b]}, \tau)$ is an equivariant pair if and only if :

$$\tau(z) := [1, \bar{z}] \cdot 0 = \bar{z} \tag{4.1.13}$$

We conclude this section by considering the case of the mapping $\rho : G \longrightarrow G$ separating variables, in the sense that it takes the form $\rho = \tilde{\rho} \ltimes \tilde{\tau}$, meaning $\rho([a, b]) := [\tilde{\rho}(a), \tilde{\tau}(b)]$, with $\tilde{\rho} : U(1) \longrightarrow U(1)$ and $\tilde{\tau} : \mathbb{C} \longrightarrow \mathbb{C}$.

Theorem 4.2. *Keep notations as above and assume that the pair $(\tilde{\rho} \ltimes \tilde{\tau}, \tau)$ satisfies the equivariant condition (4.1.6). Then, we have two possible cases :*

1. *$\tilde{\rho}$ is a $U(1)$-endomorphism, $\tilde{\tau}$ is additive such that $\tilde{\tau}(ab+c) = \tilde{\rho}(a)\tilde{\tau}(b) + \tilde{\tau}(c)$, $a \in U(1)$, $b, c \in \mathbb{C}$ and $\tau(z) = \tilde{\tau}(z)$.*

2. *$\tilde{\rho} \equiv 1$, $\tilde{\tau}$ is additive such that $\tilde{\tau}(ab+c) = \tilde{\tau}(b) + \tilde{\tau}(c)$, $a \in U(1)$, $b, c \in \mathbb{C}$ and $\tau(z) = \tilde{\tau}(z) + \beta$ for certain $\beta \in \mathbb{C}$.*

Proof. Notice first that it is clear that ρ is a G-endomorphism if and only if $\tilde{\rho}$ is a $U(1)$-endomorphism and $\tilde{\tau}$ satisfies

$$\tilde{\tau}(ab+c) = \tilde{\rho}\tilde{\tau}(b) + \tilde{\tau}(c) \tag{4.1.14}$$

for every $a \in U(1)$ and $b, c \in \mathbb{C}$. Therefore, $\tilde{\tau}$ becomes additive and satisfies

$$\tilde{\tau}(ab) = \tilde{\rho}(a)\tilde{\tau}(b) \tag{4.1.15}$$

since $\tilde{\tau}(0) = 0$. According to Theorem (4.1), the mappings $\tau : \mathbb{C} \longrightarrow \mathbb{C}$, satisfying (4.1.6) relatively to $\rho = \tilde{\rho} \ltimes \tilde{\tau}$, are those given by

$$\tau_\beta(z) = \tilde{\tau}(z) + \beta \tag{4.1.16}$$

for certain $\beta \in \Xi_\rho$. Moreover, we can show that for all $\rho = \tilde{\rho} \ltimes \tilde{\tau}$ we have $0 \in \Xi_\rho$. Then, from (4.1.16) with $\beta = 0$, we get that the compatible map is $\tau_0(z) = \tilde{\tau}(z)$. On the other side, it is easy to see that $\beta \neq 0$ belongs to Ξ_ρ if and only if $\tilde{\rho} \equiv 1$. Indeed,

$$\rho([a,0]) \cdot \beta := [\tilde{\rho}(a), \tilde{\tau}(0)] \cdot \beta = [\tilde{\rho}(a), 0] \cdot \beta = \tilde{\rho}(a)\beta$$

In this case, $\rho(G_0)$ is trivial and $\Xi_\rho = \mathbb{C}$. Furthermore, (4.1.14) reads simply as

$$\tilde{\tau}(ab+c) = \tilde{\tau}(b) + \tilde{\tau}(c), \tag{4.1.17}$$

and clearly yields

$$\tau_\beta(az+b) = \tilde{\tau}(z) + \tilde{\tau}(b) + \beta \tag{4.1.18}$$

for all $a \in U(1)$ and $z, b \in \mathbb{C}$. □

Remark 4.2. *In the context of Example 4.1, we have*

$$\tau_{\mathfrak{h}}(az+b) = \tilde{\rho}(a)\left(\tau_{\mathfrak{h}}(z) - \tau_{\mathfrak{h}}(0)\right) + \tau_{\mathfrak{h}}(b) \tag{4.1.19}$$

which is clearly similar to (4.1.18).

Remark 4.3. *The equivariant pair given in Example 4.2 by $\left([a,b] \longmapsto \overline{[a,b]}, z \longmapsto \overline{z}\right)$ fits the first point of Theorem 4.2. In fact we have, for all $a \in U(1)$ and $z \in \mathbb{C}$*

$$\tilde{\rho}(a) = \bar{a}, \text{ and } \tau(z) = \tilde{\tau}(z) = \bar{z}.$$

4.2 An invariant schrödinger operator

Starting from the next section, we will be interested in the spectral analysis on the space of mixed automorphic functions. To this end, we deal with a specific invariant magnetic Schrödinger operator. Recall, for instance, that a magnetic Schrödinger operator on a complete oriented Riemannian manifold (M, g) can defined by

$$H_\theta = (d + \operatorname{ext}\theta)^*(d + \operatorname{ext}\theta), \tag{4.2.20}$$

where θ is a given $\mathcal{C}^1$ real differential 1-form on M (potential vector). d stands for the usual exterior derivative acting on the space of differential p-forms $\Omega^p(M)$, $\operatorname{ext}\theta$ is the operator of exterior left multiplication by θ, $i.e.$ $(\operatorname{ext}\theta)w = \theta \wedge w$, and $(d + \operatorname{ext}\theta)^*$ is the formal adjoint of $d + \operatorname{ext}\theta$ with respect to the Hermitian product

$$< \alpha, \beta >_{\Omega^p} = \int_M \alpha \wedge \star\beta \tag{4.2.21}$$

induced by the metric g on $\Omega^p(M)$, where $\star$ denotes the Hodge star operator associated with the volume form. In our case, M is the complex plane $\mathbb{C}$ equipped with its Kähler metric

$$g = ds^2 = -(i/2)dz \otimes d\bar{z} = dx \otimes dy$$

and the corresponding volume form is $Vol = dxdy$. Thus, associated with the parameters $\mu, \nu, \alpha > 0$ and given equivariant pair (ρ, τ), we consider the potential vector in (4.2.22) which explicitly reads

$$\theta_\tau^{\nu,\mu}(z) = -\left(\frac{\overline{S_\tau^{\nu,\mu}(z)}}{2}dz - \frac{S_\tau^{\nu,\mu}(z)}{2}dz\right), \tag{4.2.22}$$

where defened in equation 1.3.13, $i.e.$ $S_\tau^{\nu,\mu}(z) := \nu z + \mu\left(\tau\frac{\partial\bar{\tau}}{\partial\bar{z}} - \bar{\tau}\frac{\partial\tau}{\partial\bar{z}}\right)$. Straightforward computation shows that the corresponding explicit expression of the Schrödinger operator $\Delta_\tau^{\nu,\mu} := -H_{\theta_\tau^{\nu,\mu}(z)}$ in (4.2.20), for given $\theta_\tau^{\nu,\mu}(z)$ in (4.2.22), is given by

$$\Delta_\tau^{\nu,\mu} = -\left\{\frac{\partial^2}{\partial z\partial\bar{z}} + \left(S_\tau^{\nu,\mu}\frac{\partial}{\partial z} - \overline{S_\tau^{\nu,\mu}}\frac{\partial}{\partial\bar{z}}\right) - |S_\tau^{\nu,\mu}|^2 + \mu(\tau\Delta\bar{\tau} - \bar{\tau}\Delta\tau)\right\}, \tag{4.2.23}$$

which generalizes, to some extent, the one $\Delta_\nu = H_{\theta_\nu}$ represented by:

$$\Delta_\nu = -\left\{\frac{\partial^2}{\partial z\partial\bar{z}} + \frac{\nu}{2}\left(z\frac{\partial}{\partial z} - \bar{z}\frac{\partial}{\partial\bar{z}}\right) - \frac{\nu^2}{4}|z|^2\right\}, \tag{4.2.24}$$

where the potential vector $i\theta_\nu(z) := (i\nu/2)(\bar{z}dz - zd\bar{z})$, representing the standard Landau gauge associated with a constant magnetic field of intensity 2ν. It should be mentioned here that, from physics standpoint, the considered Schrödinger operators are the Hamiltonians governing the behavior of a charge in the complex plane under the influence of a magnetic fields [4].

When they are acting on classical automorphic functions, they can be viewed geometrically as the Bochner Laplacian defined on the smooth cross sections of a line bundle over $\Gamma\backslash\mathbb{C}$.

The choice of $\theta_\tau^{\nu,\mu}$ is made to ensure that the underling $\Delta_\tau^{\nu,\mu}$ be invariant with respect to the projective representation

$$\left[\mathcal{T}_g^{\nu,\mu} f\right](z) := \overline{J_{\rho,\tau}^{\nu,\mu}(g,z)} f(g\cdot z) \tag{4.2.25}$$

associated with the mixed automorphic factor given by (4.0.2). This manifests in the following result.

Proposition 4.1. *The operator $\Delta_\tau^{\nu,\mu}$ commutes with $\mathcal{T}_g^{\nu,\mu}$, that is we have*

$$\mathcal{T}_g^{\nu,\mu}\Delta_\tau^{\nu,\mu} = \Delta_\tau^{\nu,\mu}\mathcal{T}_g^{\nu,\mu}; g\in G$$

Proof. The proof is similar to the one given in [21, Théorème 3.4]. Though, it involves some more computational difficulties. For instance, we find ourselves in need for the following handy formulas driven from the equivariant condition

$$\tau(g\cdot z) = \rho(g)\cdot\tau(z) = \phi(g)\tau(z) + \psi(g)$$

for given G-endomorphism $\rho : G \longrightarrow G = \mathrm{U}(1) \ltimes \mathbb{C}; \rho(g) = [\phi(g), \psi(g)]$. Indeed, by differentiating the equivariant condition, we get the identities

$$\frac{\partial\tau}{\partial z}(g\cdot z) = \bar{a}\phi(g)\frac{\partial\tau}{z}(\partial z) \tag{4.2.26a}$$

$$\frac{\partial\bar{\tau}}{\partial z}(\gamma\cdot z) = \bar{a}\overline{\phi(g)}\frac{\partial\bar{\tau}}{\partial z}(z) \tag{4.2.26b}$$

$$\frac{\partial\tau}{\partial\bar{z}}(g\cdot z) = a\phi(g)\frac{\partial\tau}{\partial\bar{z}}(z) \tag{4.2.26c}$$

$$\frac{\partial\bar{\tau}}{\partial\bar{z}}(g\cdot z) = a\overline{\phi(g)}\frac{\partial\bar{\tau}}{\partial z}(\partial\bar{z}) \tag{4.2.26d}$$

Therefore, the transform $\mathcal{T}_g^{\nu,\mu}$, can be naturally extended to the space of differential forms by considering

$$\mathcal{T}_g^{\nu,\mu} w = \overline{J_{\rho,\tau}^{\nu,\mu}(g,z)} g^* w, \tag{4.2.27}$$

where g^* denotes the pull-back mapping on the space of differential form of $\mathbb{C}$.

Now, by component-wise straightforward computations, making use of the equations (4.2.26a)-(4.2.26d), we show that the potential vector $\theta_\tau^{\nu,\mu}$ in (4.2.22) satisfies

$$g^*\theta_{v,\mu} = \theta_{v,\mu} + \overline{d\log\left(J^{v,\mu}(g,z)\right)} \tag{4.2.28}$$

Subsequently, we have

$$\begin{aligned}
\mathcal{T}_g^{v,\mu}\left((d+\operatorname{ext}\theta_{v,\mu})\, f\right) &= \overline{J^{v,\mu}(g,z)}\left[g^*\left((d+\operatorname{ext}\theta_{v,\mu})\, f\right)\right] \\
&= \overline{J^{v,\mu}(g,z)}\left(d\,[g^*f] + [g^*\theta_{v,\mu}]\wedge [g^*f]\right) \\
&\overset{(4.2.28)}{=} \frac{\overline{J^{v,\mu}}(g,z)}{d}\,[g^*f] + \overline{J^{\nu,\mu}(g,z)}\theta_{v,\mu}\,[g^*f] + d\left(\overline{J^{v,\mu}(g,z)}\right)[g^*f] \\
&= d\left(\overline{J^{v,\mu}(g,z)}\,[g^*f]\right) + \theta_{v,\mu}\overline{J^{v,\mu}(g,z)}\,[g^*f] \\
&= (d+\operatorname{ext}\theta_{v,\mu})\left(\mathcal{T}_g^{v,\mu} f\right)
\end{aligned}$$

For these algebraic computations we have made use of the facts $g^*d = dg^*$ as well as $g^*(\alpha\wedge\beta) = g^*\alpha\wedge g^*\beta$, together with the identity (4.2.28). This completes the proof, since $\mathcal{T}_g^{\nu,\mu}$ commutes also with $(d+\operatorname{ext}\theta_{\nu,\mu})^*$ for $\mathcal{T}_g^{\nu,\mu}$ being a unitary transformation. □

The invariance property shows in particular that the eigenvalue problem for $\Delta_\tau^{\nu,\mu}$ on the space of $\mathcal{M}_{\rho,\tau}^{\nu,\mu}(\mathbb{C})$ is well defined. Moreover, we may prove that the eigenvalue problemof $\Delta_\tau^{\nu,\mu}$ on $\mathcal{M}_{\rho,\tau}^{\nu,\mu}(\mathbb{C})$ is equivalent to the eigenvalue problem of Δ_α on the space of classical automorphic functions$Aut_{\Gamma,\chi}^{\nu}(\mathbb{C})$ for for some special α that depends on the parameters ν,μ and the equivariant pairs (ρ,τ). The proof of this assertion is contained Lemma (4.1) and Theorem (4.3) below.

Lemma 4.1. *Keep notations as above. Then, the following assertions hold trues*

(i) The quantity

$$B_\tau^{\mu,\nu}(z) := \nu + \mu\left(\left|\frac{\partial\tau}{\partial z}(z)\right|^2 - \left|\frac{\partial\tau}{\partial\bar z}(z)\right|^2\right) \tag{4.2.29}$$

is a real constant on $\mathbb{C}$, denoted as $B_\tau^{\mu,\nu}$.

(ii) Let $\alpha = \alpha_\tau^{\mu,\nu} = B_\tau^{\mu,\nu}$. Then, the operators Δ_α and $\Delta_\tau^{\nu,\mu}$ are intertwining, i.e.,

$$\Delta_\tau^{\nu,\mu} = e^{-i\varphi_\tau^{\nu,\mu}(z)}\Delta_{B_\tau^{\mu,\nu}}e^{i\varphi_\tau^{\nu,\mu}}(z) \tag{4.2.30}$$

for some smooth mapping $\varphi_\tau^{\nu,\mu}$.

(iii) There exists a smooth real mapping $\varphi_\tau^{\nu,\mu}$ such that (4.2.30) is fulfilled.

Proof. The fact that $z\longmapsto\mathbb{B}$ is constant on $\mathbb{C}$ is an immediate consequence of its invariance under the action of G. This follows by using (4.2.26a) and (4.2.26c), as well as the fact that

$$\phi(g)\frac{\partial(g\cdot z)}{\partial z}\in \mathrm{U}(1) \tag{4.2.31}$$

To prove (4.2.30), it is enough to show that the exterior derivatives of their potential vectors are equal. Indeed, it can be checked that $d\theta_\tau^{\nu,\mu}(z) = d\theta_{B_\tau^{\mu,\nu}}(z)$. Therefore, there exists a smooth mapping $\varphi_\tau^{\nu,\mu}$ such that

$$\theta_\tau^{\nu,\mu}(z) = \theta_{B_\tau^{\mu,\nu}}(z) + d\varphi_\tau^{\nu,\mu} \tag{4.2.32}$$

The map $\varphi_\tau^{\nu,\mu}$ can be freely chosen up to a translation by constants. By means of the last equation we can write

$$(d + \operatorname{ext}\theta_\tau^{\nu,\mu}) = e^{-i\varphi_\tau^{\nu,\mu}}\,(d + \operatorname{ext}\theta_{B_\tau^{\mu,\nu}})\,e^{i\varphi_\tau^{\nu,\mu}} \tag{4.2.33}$$

Accordingly, its adjoint $(d + \operatorname{ext}\theta_\tau^{\nu,\mu})^*$ holds similar equality. Subsequently,

$$\Delta_\tau^{\nu,\mu} = e^{-i\varphi_\tau^{\nu,\mu}(z)}\Delta_{B_\tau^{\mu,\nu}}e^{i\varphi_\tau^{\nu,\mu}(z)} \tag{4.2.34}$$

The last assertion follows easily by making the observation that

$$2id\Im m\,(\varphi_\tau^{\nu,\mu}) = d\left(\varphi_\tau^{\nu,\mu} + \overline{\varphi_\tau^{\nu,\mu}}\right) = 0 \tag{4.2.35}$$

since $\overline{\theta_\tau^{\nu,\mu}} = -\theta_\tau^{\nu,\mu}$ and $\overline{\theta_{B_\tau^{\mu,\nu}}} = -\theta_{B_\tau^{\mu,\nu}}$. Therefore, one can check that $\varphi_\tau^{\nu,\mu}$ has a constant purely imaginary part, and then can be translated such that it is real. $\square$

Remark 4.4. *The intertwining property for the Schrödinger operators $\Delta_{B_\tau^{\mu,\nu}}$ and $\Delta_\tau^{\nu,\mu}$ can be argued using physical interpretation (see for example [4]). Indeed, by (4.2.29), such operators are observables for the quantum behavior of a charged particle under the influence of such constant magnetic field, which means that they are related by a gouge transformation. Furthermore, the two operators would rather be unitary equivalent if we had considered a different convention for the Schrödinger operator. Namely, using the connection $(d \pm i\theta)$ instead of $(d \pm \theta)$, which is common in physic literature.*

4.3 Lifting theorem for $\mathcal{M}_{\rho,\tau}^{\nu,\mu}(\mathbb{C})$

The function $\varphi_\tau^{\nu,\mu}$ in $iii)$ of Lemma (4.1) turns out to be the key to establish, as well, a bijection between classical automorphic functions and mixed automorphic functions. Indeed, let us define $\mathcal{W}_\tau^{\mu,\nu}$ to be the transformation given by

$$[\mathcal{W}_\tau^{\mu,\nu}f]\,(z) := e^{i\varphi_\tau^{\nu,\mu}(z)}f(z). \tag{4.3.36}$$

This transformation serves as the bridge connecting classical automorphic functions to their mixed counterparts. The ensuing section delve into the details of this connection.

The following theorem illuminates the relationship between these spaces and the role played by $\mathcal{W}_\tau^{\mu,\nu}$:

Theorem 4.3. *The range of the space of mixed automorphic functions $\mathcal{M}_{\rho,\tau}^{\nu,\mu}(\mathbb{C})$ by the transform $\mathcal{W}_\tau^{\mu,\nu}$ is exactly the space of classical (Γ, χ_τ)-automorphic functions*

$$\mathcal{F}_{\Gamma,\chi_\tau}^{B_\tau^{\mu,\nu}}(\mathbb{C}) := \left\{F \in \mathcal{C}^\infty(\mathbb{C}), \quad F(\gamma\cdot z) = \chi_\tau(\gamma)j^{B_\tau^{\mu,\nu}}(\gamma, z)F(z)\right\}$$

associated to the specific pseudo-character defined on Γ by

$$\chi_\tau(\gamma) = \chi(\gamma)\exp\left(\varphi_\tau^{\nu,\mu}(\gamma\cdot 0) - 2i\mu\Im m\left\langle \tau(0), \rho(\gamma)^{-1}\cdot 0\right\rangle\right). \tag{4.3.37}$$

For the proof, we begin with the following assertion concerning the function $\widehat{\chi_\tau}$ defined on $\mathbb{C} \times \Gamma$ by

$$\widehat{\chi_\tau}(z;\gamma) := e^{\left(\varphi_\tau^{\nu,\mu}(\gamma\cdot z)-\varphi_\tau^{\nu,\mu}(z)\right)}\chi(\gamma) j^{\nu-B_\tau^{\mu,\nu}}(\gamma,z) j^{\mu}(\rho(\gamma),\tau(z)). \tag{4.3.38}$$

Here $j^\alpha = e^{-i\alpha\Im m\left\langle z,g^{-1}\cdot 0\right\rangle}$.

Lemma 4.2. *The function $\widehat{\chi_\tau}$ is independent of the variable z and we have*

$$\widehat{\chi_\tau}(z;\gamma) = \chi_\tau(\gamma). \tag{4.3.39}$$

Proof. Differentiation of $\widehat{\chi_\tau}(z;\gamma)$ with respect to the z-variable gives

$$\frac{\partial \log \widehat{\chi_\tau}}{\partial z} = S_1 + S_2 \tag{4.3.40}$$

where S_1 and S_2 stand for

$$S_1 := \frac{\partial \varphi_\tau^{\nu,\mu}}{\partial z}(\gamma\cdot z) - \frac{\partial \varphi_\tau^{\nu,\mu}}{\partial z}(z)$$

and

$$S_2 := [\nu - B_\tau^{\mu,\nu}]\,\frac{\partial i\Im m\left\langle z,\gamma^{-1}\cdot 0\right\rangle}{\partial z} + \mu\frac{\partial i\Im m\left\langle \tau(z),\rho(\gamma)^{-1}\cdot 0\right\rangle}{\partial z}.$$

Thus, it is easy to check that

$$S_2 = \frac{1}{2}\left[(B_\tau^{\mu,\nu} - v)\, a\bar{b} - \mu\left(\phi(\gamma)\overline{\psi(\gamma)}\frac{\partial \tau}{\partial z} - \overline{\phi(\gamma)}\psi(\gamma)\frac{\partial \bar{\tau}}{\partial z}\right)\right] \tag{4.3.41}$$

For the explicit computation, we have used (here and elsewhere) the notation $\gamma := [a,b]$ and $\rho(\gamma) = [\phi(\gamma),\varphi(\gamma)]$. For the evaluation of the component S_1, we make use of

$$\frac{\partial \varphi_\tau^{\nu,\mu}}{\partial z}(z) = \frac{1}{2}\left[B_\tau^{\mu,\nu}\bar{z} - S_\tau^{\nu,\mu}(z)\right] = \overline{\left(\frac{\partial \varphi_\tau^{\nu,\mu}}{\partial \bar{z}}(z)\right)} \tag{4.3.42}$$

which follows by direct computation, to get

$$\frac{\partial \varphi_\tau^{\nu,\mu}(\gamma\cdot z)}{\partial z} = \frac{a}{2}\left[(B_\tau^{\mu,\nu} - \nu)\,\overline{(\gamma\cdot z)} - \mu\left(\overline{\tau(\gamma\cdot z)}\frac{\partial \tau}{\partial z}(\gamma\cdot z) - \tau(\gamma\cdot z)\frac{\partial \bar{\tau}}{\partial z}(\gamma\cdot z)\right)\right]$$

Now, direct computations, mainly by making use of the equations (4.2.26a)–(4.2.26d), infers

$$\frac{\partial \varphi_\tau^{v,\mu}(\gamma\cdot z)}{\partial z} = \frac{B_\tau^{\mu,v} - v}{2}(\bar{z} + a\bar{b}) - \frac{\mu}{2}\left(\bar{\tau}\frac{\partial \tau}{\partial z} + \phi(\gamma)\overline{\psi(\gamma)}\frac{\partial \tau}{\partial z} - \left(\tau(z)\frac{\partial \bar{\tau}}{\partial z} + \overline{\phi(\gamma)}\psi(\gamma)\frac{\partial \bar{\tau}}{\partial z}\right)\right)$$

which reduces further to

$$\begin{aligned}\frac{\partial \varphi_\tau^{v,\mu}(\gamma\cdot z)}{\partial z} &= \frac{\partial \varphi_\tau^{v,\mu}}{\partial z}(z) + \frac{1}{2}\left[(B_\tau^{\mu,\nu} - v)\, a\bar{b} - \mu\left(\phi(\gamma)\overline{\psi(\gamma)}\frac{\partial \tau}{\partial z} - \overline{\phi(\gamma)}\psi(\gamma)\frac{\partial \bar{\tau}}{\partial z}(z)\right)\right]\\ &= \frac{\partial \varphi_\tau^{v,\mu}}{\partial z}(z) + S_2\end{aligned}$$

thanks to (4.3.41). This shows that $S_1 = S_2$ and therefore $\frac{\partial \widehat{\chi_\tau}}{\partial z} = 0$. Moreover, using the facts that $\overline{j^\alpha(\gamma, z)} = j^{-\alpha}(\gamma, z)$ and

$$\frac{\partial \varphi_\tau^{\nu,\mu}(\gamma \cdot z)}{\partial \bar{z}} = \frac{\overline{\partial \overline{\varphi_\tau^{\nu,\mu}(\gamma \cdot z)}}}{\partial z} = \frac{\overline{\partial \varphi_\tau^{\nu,\mu}(\gamma \cdot z)}}{\partial z},$$

it follows that $\frac{\partial \widehat{\chi_\tau}}{\partial \bar{z}} = 0$. This proves that $\widehat{\chi_\tau}$ is independent of the variable z and therefore we have

$$\widehat{\chi_\tau}(z;\gamma) = \widehat{\chi_\tau}(z;0) = \chi_\tau(\gamma). \tag{4.3.43}$$

This ends the proof. □

Proof of Theorem 4.3 . We shall prove that $\mathcal{W}_\tau^{\mu,\nu} F$ belongs to $\mathcal{F}_{\Gamma,\chi_\tau}^{B_\tau^{\mu,\nu}}(\mathbb{C})$ whenever $F \in \mathcal{M}_\tau^{\nu,\mu}(\mathbb{C})$, where $\chi_\tau(\gamma)$ is the one defined through (4.3.37). Indeed, for fixed $\gamma \in \Gamma$, we have

$$\begin{aligned}
\left[\mathcal{W}_\tau^{\nu,\mu} F\right](\gamma \cdot z) &:= e^{i\varphi_\tau^{\nu,\mu}(\gamma\cdot z)} F(\gamma \cdot z) \\
&= e^{i\varphi_\tau^{\nu,\mu}(\gamma\cdot z)} \chi(\gamma) j^\nu(\gamma, z) j^\mu(\rho(\gamma), \tau(z)) F(z) \\
&= e^{i\left(\varphi_\tau^{\nu,\mu}(\gamma\cdot z) - \phi_\tau^{\nu,\mu}(z)\right)} \chi(\gamma) j^\nu(\gamma, z) j^\mu(\rho(\gamma), \tau(z)) \left[\mathcal{W}_\tau^{\nu,\mu} F\right](z) \\
&= \widehat{\chi_\tau}(z;\gamma) j^{B_\tau^{\mu,\nu}}(\gamma, z) \left[\mathcal{W}_\tau^{\nu,\mu} F\right](z)
\end{aligned}$$

where $\widehat{\chi_\tau}(z;\gamma)$ is defined by (4.3.38). This completes the first part of the proof, thanks to the Lemma 4.2, asserting $\widehat{\chi_\tau}(z;\gamma) = \chi_\tau(\gamma)$.

Now, following the same computation from the bottom up, we can show that the transformation $\mathcal{W}^{\nu,\mu}\mathcal{T} : \mathcal{M}_{\rho,\tau}^{\nu,\mu}(\mathbb{C}) \to \mathcal{F}_{\Gamma,\chi_\tau}^{B_\tau^{\nu,\mu}}(\mathbb{C})$ is a bijection. The inverse transformation is given by

$$\left[\overline{\mathcal{W}_\tau^{\mu,\nu}} f\right](z) := e^{-i\varphi(z)} f(z). \tag{4.3.44}$$

This concludes the proof of Theorem 4.3. □

Remark 4.5. *For $\mathcal{M}_{\rho,\tau}^{\nu,\mu}(\mathbb{C})$ being nontrivial, which is equivalent to χ be a pseudo-character (Proposition 1.4), the data $(\Gamma; B_\tau^{\mu,\nu}, \chi_\tau)$ satisfies the Riemann-Dirac quantization type condition provided in [24] ensuring that the corresponding space of classical automorphic functions is nontrivial. Here $B_\tau^{\mu,\nu}$ and χ_τ are those defined by (4.3.37) and (4.2.29), respectively.*

4.4 Applications: Spectral analysis of L^2-mixed automorphic functions

In the sequel, we provide immediate application of the lifting Theorem 4.3. Namely, we are concerned with the concrete description of the spectral properties of $\Delta_\tau^{\nu,\mu}$ acting on mixed automorphic functions belonging to the Hilbert space $L^{2,\nu}_{\mu,\tau}(\mathbb{C}/\Gamma)$ of square integrable functions with respect to the scalar product $\langle\cdot,\cdot\rangle_{\nu,\mu,\tau}$ given in 4.0.5. To this end, we begin by determining the image by $\mathcal{W}_{\tau}^{\mu,\nu}$ of the eigenspace consisting of the eigenfunctions of $\Delta_{\tau}^{\nu,\mu}$ in $\mathcal{M}_{\tau}^{\nu,\mu}(\mathbb{C})$ corresponding to a given eigenvalue λ:

$$\mathcal{E}_{\tau;\lambda}^{\nu,\mu} := \{F \in \mathcal{M}_{\tau}^{\nu,\mu}(\mathbb{C}); \quad \Delta_{\tau}^{\nu,\mu} F = \lambda F\} \tag{4.4.45}$$

The following proposition establishes the connection between this eigenspace and a corresponding subspace in the classical automorphic functions.

Proposition 4.2. *We have*

$$\mathcal{W}_{\tau}^{\nu,\mu}\left(\mathcal{E}_{\tau;\lambda}^{\nu,\mu}\right) = \left\{F \in \mathcal{F}_{\Gamma,\chi_{\tau}}^{B_{\tau}^{\mu,\nu}}(\mathbb{C}); \quad \Delta_{B_{\tau}^{\mu,\nu}} F = \lambda F\right\} =: \mathcal{E}_{\lambda}^{B_{\tau}^{\mu,\nu}}.$$

Proof. The proof follows using $ii)$ and $iii)$ (intertwining) in Lemma 4.1 together with the lifting Theorem 4.3 (invariance). ☐

The next result involves the $\Delta_{\tau}^{\nu,\mu}$ eigenspaces of L^2, defined by

$$A_{\lambda}^{2,\nu,\mu,\tau}(\mathbb{C}) = \mathcal{E}_{\tau;\lambda}^{\nu,\mu} \cap L_{\mu,\tau}^{2,\nu}(\mathbb{C}/\Gamma)$$

The following proposition characterizes the range of the restriction of the transformation $\mathcal{W}_{\tau}^{\mu,\nu}$ to $A_{\lambda}^{2,\nu,\mu,\tau}(\mathbb{C})$ and establishes the spectral properties of $\Delta_{\tau}^{\nu,\mu}$.

Proposition 4.3. *The range of the restriction of the transform $\mathcal{W}_{\tau}^{\mu,\nu}$ in (4.3.36) to $A_{\lambda}^{2,\nu,\mu,\tau}(\mathbb{C})$ is exactly the space of $L^2-\Gamma$-automorphic functions $\mathcal{E}_{\lambda}^{B_{\tau}^{\mu,\nu}} \cap L_{0,Id_{\mathbb{C}}}^{2,B_{\tau}^{\mu,\nu}}(\mathbb{C}/\Gamma)$.*

Proof. The proof follows by means of Proposition 4.2 together with the fact that the transform $\mathcal{W}_{\tau}^{\mu,\nu}$ defines an isometric mapping from $L_{\mu,\tau}^{2,\nu}(\mathbb{C}/\Gamma)$ into $L^{2,\nu}(\mathbb{C}/\Gamma)$, up to multiplication constant, and $iii)$ in Lemma 4.1. ☐

Remark 4.6. *Subsequently, keeping in mind the spectral analysis of $\Delta_{B_{\tau}^{\mu,\nu}}$ on $L^{2,B_{\tau}^{\mu,\nu}}(\mathbb{C}/\Gamma)$, one concludes that the spectrum of $\Delta_{\tau}^{\nu,\mu}$ is purely discrete and given by the Landau levels*

$$\lambda_m = \frac{B_{\tau}^{\mu,\nu}}{2}(2m+1); \ m = 0,1,2,\cdots. \tag{4.4.46}$$

Subsequently, for rank one of the discrete subgroup that we identify to $\Gamma = \mathbb{Z}$, so that $\mathbb{C}/\mathbb{Z}$ is a strip, the space $L_{\mu,\tau}^{2,\nu}(\mathbb{C}/\Gamma)$ possesses a Hilbertian orthogonal decomposition in terms of $A_{\lambda_k}^{2,B_{\tau}^{\mu,\nu},\mu,\nu}(\mathbb{C})$; $k = 0,1,2,\cdots$. An orthogonal basis of each $A_{\lambda_m}^{2,\nu,\mu,\tau}(\mathbb{C})$ can be shown to be given in terms of the Intissar–Hermite polynomials $I_n^{\nu,\alpha}(z,\bar{z}|\xi)$ in 3.3.19 by the functions

$$\begin{aligned}\psi_{m,n}^{\alpha,B_{\tau}^{\mu,\nu}}(z,\bar{z}) &= e^{i\varphi(z)}\psi_n^{\alpha,B_{\tau}^{\mu,\nu}}(z)\exp\left(-\frac{\mathbb{B}}{2}|z|^2\right) I_m^{B_{\tau}^{\mu,\nu},B_{\tau}^{\mu,\nu}/2}(z,\bar{z}|2i\pi(\alpha+n)) \\ &= e^{i\varphi_{\tau}^{\nu,\mu}(z)}\psi_n^{\alpha,B_{\tau}^{\mu,\nu}}(z) H_m\left((2B_{\tau}^{\mu,\nu})^{1/2}\,\Im m(z) + (2/B_{\tau}^{\mu,\nu})^{1/2}\,\pi(n+\alpha)\right)\end{aligned}$$

for varying $m \in \mathbb{Z}^+$ and $n \in \mathbb{Z}$, where $\psi_n^{\alpha, B_\tau^{\mu,\nu}}$ stands for

$$\psi_n^{\alpha, B_\tau^{\mu,\nu}}(z) := \left(\frac{2\mathbb{B}}{\pi}\right)^{1/4} \exp\left(\frac{B_\tau^{\mu,\nu}}{2} z^2 + 2i\pi(\alpha+n)z - \frac{\pi^2}{B_\tau^{\mu,\nu}}(n+\alpha)^2\right); \quad n \in \mathbb{Z}. \quad (4.4.47)$$

We conclude this section by providing the concrete description of spectral properties of the Laplacian $\Delta_\tau^{\nu,\mu}$ on the free Hilbert space $L^2(\mathbb{C}; d\lambda)$.
We recall the usual Laguerre polynomial of degree n with parameter $\alpha \in \mathbb{R}$ is given by the Rodrigues formula

$$L_n^{(\alpha)}(x) = \frac{e^x x^{-\alpha}}{n!} \frac{d^n}{dx^n}\left(e^{-x} x^{n+\alpha}\right).$$

Theorem 4.4. *The point spectrum of $\Delta_\tau^{\nu,\mu}$ acting on $L^2(\mathbb{C}; d\lambda)$ is discrete and reduces to the Landau levels $\lambda_m = \frac{B_\tau^{\mu,\nu}}{2}(2m+1); m = 0, 1, \cdots$. Moreover, the corresponding L^2-eigenspaces*

$$A_m^2(\Delta_\tau^{\nu,\mu}) = \left\{f \in L^2(\mathbb{C}; d\lambda); \quad \Delta_\tau^{\nu,\mu} f = \lambda_m f\right\}$$

are reproducing kernel Hilbert spaces. The explicit closed expression of the eigenprojector is given by

$$K_{\tau;m}^{\nu,\mu}(z,w) = \frac{B_\tau^{\mu,\nu}}{\pi} \exp\left(\psi_\tau^{\nu,\mu}(z,w) - \frac{B_\tau^{\mu,\nu}}{2}\left(|z|^2 + |w|^2 - 2z\overline{w}\right)\right) L_m(\mathbb{B}|z-w|^2),$$

where $L_m(x) = L_m^{(0)}(x)$ denotes the usual Laguerre polynomial.

Proof. This is immediate thanks to $ii)$ and $iii)$ in the Lemma 4.1, combined with the observation that the transform $\mathcal{W}_\tau^{\mu,\nu}$, in (4.3.36) defines an isometric mapping of $L^2(\mathbb{C}, d\lambda)$, up to a multiplicative constant, so that $\Delta_\tau^{\nu,\mu} f = \lambda f$ for $f \in L^2(\mathbb{C}; d\lambda)$ becomes equivalent to

$$\Delta_{B_\tau^{\mu,\nu}} g = \lambda g \text{ for } g = e^{i\psi_\tau^{\nu,\mu}(z)} f \in L^2(\mathbb{C}, d\lambda).$$

Moreover, if $K_{B_\tau^{\mu,\nu}}$ denotes the reproducing kernel of the generalized Bargmann space [24], we get

$$e^{i\varphi_\tau^{v,\mu}(z)} f(z) = \int_C K_{B_\tau^{\mu,v}}(z,w) e^{i\varphi_\tau^{v,\mu}(w)} f(w) d\lambda(w)$$

which implies that $e^{-i\varphi_\tau^{\nu,\mu}(z)} K_{B_\tau^{\mu,\nu}}(z,w) e^{i\varphi_\tau^{\nu,\mu}(w)}$ is the reproducing of $A_{\lambda_k}^2(\Delta_\tau^{\nu,\mu})$. We conclude for Theorem 4.4 by making appeal to the results giving the spectrum and closed expression of $K_{B_\tau^{\mu,\nu}}(z,w)$ in [24]. □

Remark 4.7. *The eigenprojector kernel of $A_k^2(\Delta_\tau^{\nu,\mu})$ satisfies the invariance property*

$$K_{\tau;k}^{\nu,\mu}(z,w) = e^{\left(\psi_\tau^{\nu,\mu}(g\cdot z, g\cdot w)\psi_\tau^{\nu,\mu}(z,w)\right)} e^{iB_\tau^{\mu,\nu}\Im m\left(z-w, g^{-1}\cdot 0\right)} K_{\tau;k}^{\nu,\mu}(g\cdot z, g\cdot w); g \in G,$$

where $\psi_\tau^{\nu,\mu}(z,w) := \varphi_\tau^{\nu,\mu}(z) - \varphi_\tau^{\nu,\mu}(w)$.

Remark 4.8. *The L^2-eigenspaces $A_m^2(\Delta_\tau^{\nu,\mu})$ leads to an orthogonal decomposition similar to the one related to the magnetic Laplacian Δ_ν on the free Hilbert space $L^2(\mathbb{C}; d\lambda)$. Namely, we have*

$$L^2(\mathbb{C}; d\lambda) = \bigoplus_{m=0}^{\infty} A_m^2(\Delta_\tau^{\nu,\mu}).$$

4.5 Lifting theorem in high dimensions

In the previous section, a one-dimensional lifting theorem was proved connecting mixed automorphic functions to classical automorphic functions. It turns out that a generalization to high dimension fails in general. Below, we provide necessary and sufficient conditions on the equivariant map toward a lifting result in high dimensions. Thus, following the same scheme as in the 1-dimensional, the different constructions for mixed automorphic functions and the corresponding invariant magnetic Laplacians remain valid in high dimension. A discrete subgroup of the semi-direct group $G = U(1) \ltimes \mathbb{C}^n$, of the unitary group and $\mathbb{C}^n$, acts by $\gamma \cdot z = Az + b$ on $\mathbb{C}^n$, for $\gamma := [A, b] \in G$. Such action can be expanded componentwise as

$$(\gamma \cdot z)_i = \sum_{j=1}^{n} a_{i,j} z_j + b_i,$$

with $A := (a_{i,j})_{1 \leqslant i,j \leqslant n} \in U(n)$, $b := (b_i)_{1 \leqslant i \leqslant n} \in \mathbb{C}^n$, and $z := (z_1, \cdots, z_n) \in \mathbb{C}^n$. As expected, the mixed automorphic functions

$$F(Az + b) = J^{\nu,\mu}_{\rho,\tau}(\gamma, z) F(z) = \chi(\gamma) j^{\nu}(\gamma, z) j^{\mu}(\rho(\gamma), \tau(z)) F(z)$$

corresponds to the equivariant pair (ρ, τ) such that $\rho : G \longrightarrow G$ is a G-endomorphism verifyingt $\rho(\Gamma) \subset \Gamma$ for given discrete subgroup Γ of G, and $\tau : \mathbb{C}^n \longrightarrow \mathbb{C}^n$ a compatible $\mathcal{C}^1$ mapping such that

$$\tau(g \cdot z) = \rho(g) \cdot \tau(z); \quad g \in G, z \in \mathbb{C}^n, \tag{4.5.48}$$

with $\tau(z) = (\tau_1(z), \tau_2(z), \cdots, \tau_n(z))$. In terms of the linear and translational counterparts of $\rho : \gamma \longmapsto [\alpha(\gamma), \beta(\gamma)]$ with $\alpha := (\alpha_{i,j})_{i,j}$ is a matrix in $U(n)$ et $\beta := (\beta_i)_i$ is a column vector in $\mathbb{C}^n$, the condition (4.5.48) reads equivalently as

$$\tau_l = \sum_{j=1}^{n} \alpha_{l,j} \tau_j + \beta_l. \tag{4.5.49}$$

The components $\alpha_{i,j}$ and β_j are functions in γ.

The magnetic Schrödinger operator $H_{\theta^{\nu,\mu}_{\tau}} = (d + \operatorname{ext} \theta^{\nu,\mu}_{\tau}) * (d + \operatorname{ext} \theta^{\nu,\mu}_{\tau})$ corresponding to the potential vector form

$$\theta^{\nu,\mu}_{\tau} := -\frac{1}{2} \sum_{\ell=1}^{n} \left\{ v \left(\bar{z}_\ell dz_\ell - z_\ell d\bar{z}_\ell \right) + \mu \left(\overline{\tau_\ell} d\tau_\ell - \tau_\ell d\overline{\tau_\ell} \right) \right\} \tag{4.5.50}$$

is the one leaving invariant the space of mixed automorphic functions. The computations are quite straightforward and make use of the facts

$$\sum_{j=1}^{n} a_{j,k} \frac{\partial \tau_\ell}{\partial z_j}(\gamma \cdot z) = \sum_{j=1}^{n} \alpha_{\ell,j} \frac{\partial \tau_j}{\partial z_k} \quad \text{et} \quad \sum_{j=1}^{n} \overline{a_{j,k}} \frac{\partial \tau_\ell}{\partial \overline{z_j}}(\gamma \cdot z) = \sum_{j=1}^{n} \alpha_{\ell,j} \frac{\partial \tau_j}{\partial \overline{z_k}} \tag{4.5.51}$$

for all $k, \ell = 1, 2, \cdots, n$, which follow by identification after differentiating the right-hand sides of the identities obtained by differentiating left-hand side of (4.5.49) and with the usage of chain rule, for functions of several complex variables. However, the description of spectral analysis of the constructed Laplacian can not be handled by adopting similar approach as for one dimensional case. Here the corresponding magnetic field is not necessary constant, which was crucial in establishing the lifting Theorem 4.3.We claim that the intertwining property between $H_{B_\tau^{\mu,\nu}}$ and Δ_ν does not hold in general in high dimensions. More precisely, we prove

Theorem 4.5. *The intertwining property for $H_{B_\tau^{\mu,\nu}}$ and Δ_ν holds true if and only if and only if $|\partial_{\overline{z_k}}\tau_\ell|^2 - |\partial_{z_k}\tau_\ell|^2$ is constant and*

$$\det\begin{pmatrix} \partial_{z_i}\overline{\tau_\ell} & \partial_{z_j}\overline{\tau_\ell} \\ \partial_{z_i}\tau_\ell & \partial_{z_j}\tau_\ell \end{pmatrix} = \det\begin{pmatrix} \partial_{z_i}\overline{\tau_\ell} & \partial_{\overline{z_j}}\overline{\tau_\ell} \\ \partial_{z_i}\tau_\ell & \partial_{\overline{z_j}}\tau_\ell \end{pmatrix} = 0$$

for everyt $i, j, k, \ell = 1, 2, \cdots,$

Proof. For the proof, let us focus on the factor $w_\tau(z) := d(\overline{\tau_\ell}d\tau_\ell - \tau_\ell d\overline{\tau_\ell})$, where τ is present. It will be of importance in the squeal to express the information within the equivariant condition (4.5.48) on the partial derivatives of the components of τ. Therefore, we get

$$\begin{aligned}\omega_\tau(z) = &\sum_{\ell=1}^{n}\sum_{k=1}^{n} F_{\tau_\ell,k}(z)d\overline{z_k} \wedge d\overline{z_k} \\ &+ \sum_{\ell=1}^{n}\left(\sum_{j=1}^{n}\sum_{i=1}^{j-1} A_{\tau_\ell,ij}dz_i \wedge dz_j + B_{\tau_\ell ij}dz_i \wedge d\bar{z}_j + C_{\tau_\ell ij}d\overline{z_i} \wedge dz_j + D_{\tau_\ell,ij}d\overline{z_i} \wedge d\bar{z}_j\right)\end{aligned}$$

where the quantities $A_{\tau_\ell,ij}, B_{\tau_\ell,ij}, C_{\tau_\ell,ij}, D_{\tau_{\ell,ij}}$ and $F_{\tau_\ell,k}$ are given respectively by

$$\begin{aligned}
A_{\tau_\ell ij} &:= \partial_{z_i}\bar{\tau}_\ell\partial_{z_j}\tau_\ell - \partial_{z_j}\bar{\tau}_\ell\partial_{z_i}\tau_\ell = \det\begin{pmatrix} \partial_{z_i}\overline{\tau_\ell} & \partial_{z_j}\overline{\tau_\ell} \\ \partial_{z_i}\tau_\ell & \partial_{z_j}\tau_\ell \end{pmatrix} \\
B_{\tau_\ell,ij} &:= \partial_{z_i}\bar{\tau}_\ell\partial_{\bar{z}j}\tau_\ell - \partial_{\overline{z_j}}\overline{\tau_\ell}\partial_{z_i}\tau_\ell = \det\begin{pmatrix} \partial_{z_i}\overline{\tau_\ell} & \partial_{\bar{z}}\bar{\tau}_\ell \\ \partial_{z_i}\tau_\ell & \partial_{\bar{z}_j}\tau_\ell \end{pmatrix} \\
C_{\tau_\ell,ij} &:= \partial_{\bar{z}_i}\overline{\tau_\ell}\partial_{z_j}\tau_\ell - \partial_{z_j}\overline{\tau_\ell}\partial_{\bar{z}_i}\tau_\ell = \det\begin{pmatrix} \partial_{\bar{z}_i}\overline{\tau_\ell} & \partial_{z_j}\overline{\tau_\ell} \\ \partial_{\overline{z_i}}\tau_\ell & \partial_{z_j}\tau_\ell \end{pmatrix} \\
D_{\tau_\ell ij} &:= \partial_{\bar{z}_i}\bar{\tau}_\ell\partial_{\bar{z}_j}\tau_\ell - \partial_{\bar{z}_j}\bar{\tau}_\ell\partial_{\bar{z}_i}\tau_\ell = \det\begin{pmatrix} \partial_{\overline{z_i}}\overline{\tau_\ell} & \partial_{\bar{z}_j}\overline{\tau_\ell} \\ \partial_{\overline{z_i}}\tau_\ell & \partial_{\overline{z_j}}\tau_\ell \end{pmatrix} \\
F_{\tau_\ell,k} &:= |\partial_{\bar{z}_k}\tau_\ell|^2 - |\partial_{z_k}\tau_\ell|^2 = \det\begin{pmatrix} \partial_{z_k}\overline{\tau_\ell} & \partial_{\overline{z_k}}\bar{\tau}_\ell \\ \partial_{z_k}\tau_\ell & \partial_{\overline{z_k}}\tau_\ell \end{pmatrix}.
\end{aligned}$$

It is worth to observe that $D_{\tau_\ell,ij} = -\overline{A_{\tau_\ell,ij}}$ and $C_{\tau_\ell,ij} = -\overline{B_{\tau_\ell,ij}}$. Therefore, the operator $H_{B_\tau^{\mu,\nu}}$ is associated with a constant magnetic field if and only if $d\theta_{B_\tau^{\mu,\nu}}$ is a Kähler 2-form. This is equivalent to $A_{\tau_\ell,ij} = B_{\tau_\ell,ij} = 0$ and $F_{\tau_\ell,ij}$ is constant. □

Corollary 4.1. *The Lifting theorem holds true for holomorphic equivariant mapping* τ *such that* $\left|\partial_{z_j}\tau_\ell\right|^2$ *is constant and* $\partial_{\bar{z}_j}\overline{\tau_\ell}\partial_{z_i}\tau_\ell = 0$ *for every* $j, k, \ell = 1, 2, \cdots$.

Conclusion and Perspectives

The thesis delves deeply into the theory of automorphic functions in the context of discrete groups acting on the complex plane. By leveraging the functional equations of automorphy and associated differential operators, we have studied the structural and spectral properties of these functions in various settings. By examining the properties of automorphic functions in holomorphic and poly-analytic frameworks, we have demonstrated important theoretical results regarding their structure and behavior. The use of the Poincaré theta operator and the poly-analytic creation operator has been crucial in characterizing their spectral properties and behaviors under different transformations. In a second phase, the thesis addresses mixed automorphic functions, defined by more complex functional equations. The study focuses on the spectral properties of these functions and their behavior under specific differential operators. In particular, we examine the case of semi-direct groups and develop a detailed spectral theory for magnetic Schrödinger operators using the transformation theorem.

In summary, this thesis makes a significant contribution to the theory of automorphic functions by exploring their structural and spectral properties in various contexts. It paves the way for future research in the field and offers potential applications in various areas of mathematics, mathematical physics, and number theory.

Although the obtained results are interesting and encouraging, there are numerous points and ideas briefly discussed in this report that provide directions for future research:

- The characterization of the basis of the space $H_{\Gamma}^{2,\nu}(\mathbb{C})$ in the case of rank two. Additionally, exploring the transformations and properties of these bases under the action of the creation operator.
- Using fixed-point methods to further improve the properties of the Poincaré operator.
- In the last chapter, we leveraged the existence of the function $\varphi_{\tau}^{\nu,\mu}$. It is possible to explore the properties and concrete characterization of this function.

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Abstract :

In this thesis, we have dedicated our efforts to the study of the theory of holomorphic automorphic functions associated with a given discrete subgroup in $(\mathbb{C},+)$, as well as to the detailed characterization of the space of mixed automorphic functions associated with a given discrete subgroup of the semidirect group $U(1)\ltimes\mathbb{C}$. Our focus revolves around the creation of automorphic functions through the utilization of Poincaré's theta series operator, delving into their characteristics. To be specific, we establish the rigorous definition of this operator within designated spaces and provide an intricate account of its kernel by employing special functions. We provide a precise characterization for the class of planar poly-analytic automorphic functions, which are manifested as images of holomorphic functions using the differential creation operator $U_\nu^{(n)}$. We characterize the image of the operator $U_\nu^{(n)}$ on the Hilbert space $\mathrm{H}_\Gamma^{2,\nu}(\mathbb{C})$, deal with the surjectivity problem and show that it preserves orthogonality, Furthermore, as a direct application of this operator, we reconstruct the construction of orthogonal bases for the generalized Bargmann-Fock spaces for discrete one- and two-rank subgroups.

In the context of mixed automorphic functions, we encounter the automorphic factor in relation to a given equivariant pair $(\boldsymbol{\rho},\boldsymbol{\tau})$. We have partially characterized the equivariant pairs, in our research and explored the potential for extending these findings to higher dimensions. To conduct this examination, we employ an elevation theorem to classical automorphic functions. In addition, we concduct a comprehensive spectral analysis of a magnetic invariant Schrödinger operator acting on mixed automorphic functions.

Mots-clés : *Automorphic functions; Mixed automorphic functions; Poincaré's theta series; Automorphy factor; Discrete subgroup; Poly-Bargmann spaces; Magnetic Laplacian; Poly-analytical functions.*

Résumé :

Dans cette thèse, nous nous sommes consacrés à l'étude de la théorie des fonctions automorphes holomorphes associés à un sous-groupe discret donné dans $(\mathbb{C},+)$, ainsi qu'à la caractérisation détaillée de l'espace des fonctions automorphes de seconde espèce, communément appelées fonctions automorphes "mixed" associe à un sous-groupe discret donné du groupe semi-direct $U(1)\ltimes\mathbb{C}$. En abordant la construction de fonctions automorphes à l'aide de la série (Opérateur) thêta de Poincaré, nous explorons leurs propriétés. Plus précisément, nous démontrons que cet opérateur est rigoureusement défini dans des espaces spécifiques et nous fournissons une description détaillée de son noyau en utilisant des fonctions spéciales. Nous proposons une caractérisation précise pour la catégorie des fonctions automorphes poly-analytiques planaires, qui se manifestent en tant qu'images de fonctions holomorphes par l'intermédiaire de l'opérateur différentiel de création $U_\nu^{(n)}$**.** Nous donnons la caractérisation de l'image de l'opérateur $U_\nu^{(n)}$ sur l'espace de Hilbert $\mathrm{H}_\Gamma^{2,\nu}(\mathbb{C})$, nous traitons le problème de la surjectivité et montrons qu'il préserve l'orthogonalité. Enfin, nous utiliserons cet opérateur comme un outil de correspondance pour détailler la construction des bases orthogonales des espaces de Bargmann--Fock généralisés pour les sous-groupes discrets à un et deux rangs.

En ce qui concerne la seconde espèce, le facteur d’automorphie est en relation avec une paire équivariante donnée $(\boldsymbol{\rho},\boldsymbol{\tau})$. Nous avons partiellement caractérisé les paires équivariantes dans le cadre de notre étude et envisagé la possibilité d'une généralisation à des dimensions supérieures. Pour mener cette analyse, nous utilisons un théorème d'élévation vers des fonctions automorphes classiques. De plus, nous effectuons une analyse spectrale approfondie d'un opérateur de Schrödinger magnétique invariant agissant sur des fonctions automorphes "mixed".

Mots-clés : *Fonctions automorphes; Fonctions automorphes "mixed"; La série thêta de Poincaré; Facteur d’automorphie; Sous-groupe discret; Espaces poly-Bargmann; Laplacien magnétique; Fonctions poly-analytiques.*

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