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SUR CERTAINS SYSTÈMES QUASILINÉAIRES ELLIPTIQUES ET PARABOLIQUES

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Dédicace

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....leur soutien et leurs prières tout au long de mes études,

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Résumé

Ce travail traite des Systèmes non linéaires elliptiques et paraboliques d'équations aux dérivées partielles dont la partie principale qui s'écrit sous forme d'une divergence. Dans les deux parties, on s'intéresse à l'existence d'une solution faible de certains systèmes elliptiques et paraboliques de types suivants:

$$\text{(QES)}_{(f,g)} \begin{cases} -div\sigma\left(x,u\left(x\right),Du\left(x\right)\right) &= v(x)+f(x,u,Du)+divg(x,u) \text{ dans } \Omega \\ u &= 0 \qquad\qquad \text{sur } \partial\Omega, \end{cases}$$

$$\text{(QES)}_{(\mu)} \begin{cases} -div\sigma\left(x,u\left(x\right),Du\left(x\right)\right) &= \mu \text{ dans } \Omega \\ u &= 0 \qquad \text{sur}\partial\Omega, \end{cases}$$

$$(QPS)_{(\omega)} \begin{cases} \partial_t u - div\sigma(x,t,u,Du) &= v(x,t)+f(x,t,u,Du)+divg(x,t,u) \text{ dans } \Omega_T \\ u(x,t) &= 0 \qquad\qquad \text{sur } \partial\Omega\times(0,T) \\ u(x,0) &= u_0(x) \qquad\qquad \text{sur } \Omega \end{cases}$$

où f et g sont des fonctions vérifiant quelques conditions de la continuité et de la croissance, v est supposée dans le dual $W^{-1,p'}(\Omega,\omega^*)$ (réspectivement $L^{p'}(0,T,W^{-1,p'}(\Omega,\omega^*,I\!R^m))$) dans $(QES)_{(f,g)}$ (réspectivement $(QPS)_{(\omega)}$ où μ une mesure de Radon. Dans ce travail l'application σ vérifie des conditions, de la régularité, de croissance et de coercivité. Cependant les hypothèses de monotonie sur σ sont ici plus faibles. Ainsi, σ ne doit pas être obligatoirement monotone dans ses dernières variables ou encore satisfaire la condition de la stricte monotonie de Leray- Lions. L'approche principale qu'on utilise pour prouver l'existence d'une solution faible u de $(QES)_{(f,g)}$ et de $(QPS)_{(\omega)}$ est basée sur les conséquences de théorème de Ball concernant la mesure de Young: A partir d'un schéma de Galerkin, on construit une suite d'approximations u_k dont Du_k est controlée; ce qui nous permet le passage à la limite. Dans ce travail nous généralisons les travaux de Dolzmenn, Hungerbühler pour les problèmes $(QPS)_{(f,g)}$ et $(QES)_{(\omega)}$ ou v est un vecteur de distributions régulières à des problèmes dont le membre droit peut également dépendre de la fonction inconnue u qui appartient à un espace de Sobolev avec poids; cette extension peut constituer une avancée significative.

Summary

This work deals with elliptical and parabolic nonlinear systems of partial lying equations the main part, which is written in the form of a divergence. In both parts, there is an interest in the existence of a weak solution for some elliptical and parabolic systems of the following types:

$$\text{(QES)}_{(f,g)} \begin{cases} -div\sigma\left(x,u\left(x\right),Du\left(x\right)\right) &= v(x)+f(x,u,Du)+divg(x,u) \text{ in } \Omega \\ u &= 0 \qquad\qquad\qquad \text{on } \partial\Omega, \end{cases}$$

$$\text{(QES)}_{(\mu)} \begin{cases} -div\sigma\left(x,u\left(x\right),Du\left(x\right)\right) &= \mu \text{ in } \Omega \\ u &= 0 \text{ on } \partial\Omega, \end{cases}$$

$$(QPS)_{(\omega)} \begin{cases} \partial_t u - div\sigma(x,t,u,Du) &= v(x,t)+f(x,t,u,Du)+divg(x,t,u) \text{ in } \Omega_T \\ u(x,t) &= 0 \text{ on } \partial\Omega\times(0,T) \\ u(x,0) &= u_0(x) \text{ on } \Omega \end{cases}$$

where f and g functions check some conditions of continuity and growth, v is assumed in the dual $W^{-1,p'}(\Omega,\omega^*)$ (resp $L^{p'}(0,T,W^{-1,p'}(\Omega,\omega^*,I\!R^m))$ in $\text{(QES)}_{(f,g)}$ (resp $(QPS)_{(\omega)}$) and - μ is a measure of Radon. In this work the application of the application of the σ verifies conditions, regularity, growth and coercivity. However, the monotony assumptions on σ are weaker here. Thus, σmust not be monotonous in its last variables or satisfy the condition of the strict monotony of Leray-Lions. The main approach used to prove the existence of a low solution u of $(QES)_{(f,g)}$ and $(QPS)_{(\omega)}$ is based on the consequences of Ball's theorem regarding Young's measurement: From a Galerkin diagram, a Galerkin diagram is constructed following approximations u of which Du_k is controlled; which allows us to move to the limit. In this work we generalize the work of Dolzmenn, Hungerb for problems $(QPS)_{(f,g)}$ and $(QES)_{(\omega)}$ unknown u that belongs to a Space of Sobolev with weight; this extension can be a significant step forward.

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Introduction Générale

Introduction

En analyse mathématique, les espaces de Sobolev sont des espaces fonctionnels particulièrement adaptés à la résolution des problèmes des équations aux dérivées partielles. Ils doivent leur nom au mathématicien russe Serguel Luovitch sobolev. Plus précisément, un espace de Sobolev est un espace vectoriel de fonctions, muni de la norme obtenue par la combinaison de la norme L^p de la fonctionelle elle même et de ses dérivées jusqu'à un certain ordre. Les dérivées sont comprises dans un sens faible au sens des distributions afin de rendre l'espace complet. Les espaces de Sobolev sont donc des espaces de Banach; qui sont des espaces de fonctions pouvant être dérivées suffisamment de fois (pour donner sens par exemple à une équation aux dérivées partielles) et muni d'une norme qui mesure à la fois la taille et la régularité de la fonction.

Les espaces de Sobolev sont un outil essentiel pour l'étude des équations aux dérivées partielles. En effet, les solutions de ces équations appartiennent à un espace de Sobolev qu'à un espace de fonctions continues partiellement dérivables au sens classique. Les espaces de Sobolev sont un cadre fonctionnel omniprésent dans l'étude des EDPs elliptiques et paraboliques. Leur compréhension est donc une étape nécessaire avant d'aborder les équations en question.

L'étude des équations elliptiques avec second membre mesure est un thème très riche depuis une vingtaine d'années. Le premier résultat à ce sujet est celui de G. Stampacchia [57] qui concerne les équations linéaires de la forme:

$$\begin{cases} -div(A(\nabla u)) &= \mu \qquad dans \ \ \Omega \\ u &= 0, \quad \text{sur} \quad \partial\Omega. \end{cases} \tag{0.0.1}$$

Où Ω est un ouvert borné de $I\!R^n$ et $A : \Omega \to M_n(I\!R)$ est une application matricielle bornée qui vérifie le critère d'ellipticités usuel $A(x)\zeta.\zeta \geq \alpha \mid \zeta \mid^2$ pour prèsque $x \in \Omega$ et pour tout $\zeta \in I\!R^n$. Le cas linéaire est donc entièrement résolu depuis 1965, le résultat complet dans [57] concerne des équations semi-linéaires, de la forme:

$-\triangle u + g(u) = f$ avec $f \in L^1(\Omega)$

a été abordé peu après, dans [9] par exemple.

Si l'on considère des équations elliptiques non-linéaires plus générales, comme:

$$\left\{ \begin{array}{rcl} -div(a(x, \nabla u)) & = & \mu \qquad dans \ \ \Omega \\ u & = & 0, \quad \text{sur} \quad \partial\Omega. \end{array} \right. \tag{0.0.2}$$

dont un exemple modèle est le $p-$ Laplacien: $a(x, \nabla u) = |\nabla u|^{p-2}\nabla u$ avec $(1 < p < \infty)$, le premier résultat d'existence général d'une solution lorsque μ est une mesure est dû à, T. Gallouet et L. Boccardo [12], et est daté, en 1989, la formulation pour laquelle une solution classique de $(0.0.2)$ a été obtenue en multipliant l'equation par une fonction-test assez régulière et en intègrant par parties, la solution est alors dans les espaces de Sobolev $W^{1,q}(\Omega)$ pour tout $q < \frac{n(p-1)}{n-1}$; (ceci pour $p > 2 - \frac{1}{n}$, afin que les espaces en question soient bien définis).

Cependant, on sait que même dans le cas linéaire, cette formulation ne permet pas d'avoir l'unicité de solution voir [51], et ce problème d'unicité reste encore majoritairement ouvert jusqu' aujourd'hui.

Des nouvelles formulations ont alors vu le jour, avec l'espoir d'arriver à une définition de solution qui permettrait d'obtenir l'unicité. On peut par exemple citer la formulation entropique, introduite dans [13], qui donne l'unicité mais n'est adaptée qu'au second membre dans $L^1(\Omega) + W^{-1,p'}(\Omega)$; c-à-d, si on considère les seconds membres qui ne chargent pas les ensembles $W_0^{1,p}-$ capacité nulle), "la formulation" SOLA venue de [17], qui consiste à considérer uniquement des solutions de $(0.0.2)$ approchables par les solutions de problèmes, avec second membre plus réguliers, mais avec des fortes restrictions sur l'opérateur considéré (en particulier, $p = 2$) voir [14].

Il est fort probable qu'en 1999 ait vu un grand pas s'accomplir dans la recherche d'une formulation qui donne l'existence et l'unicité de solution $(0.0.2)$ avec l'introduction dans [18] de la notion de solution renormalisée qui accepte toute mesure au second membre et tout opérateur elliptique non-linéaire, et quelques résultats d'unicité partiels (lorsque deux solutions sont comparables) sont aussi donnés. Mais un résultat d'unicité vraiment satisfaisant est toujours en attente, même s'il parait clair à beaucoup de spécialistes du domaine que la formulation de [18] a éliminé toutes les causes de non-unicité.

L'étude des systèmes elliptiques non-linéaires s'est révélée très importante dans plusieurs problèmes

physiques, citons à titre d'exemples:

-Dans les modèles de Thomas-Fermis en physique atomique [9].

-Dans la modélisation d'écoulements poreux en réservoir [29].

-Dans les fluides dynamiques.

-Dans la mécanique des milieux continus.

-Dans le mouvement des planètes.

-Dans l'infographie.

-Dans la restauration de l'image.

Parmi les travaux qui ont été effectués relativement aux problèmes elliptiques de type Dirichlet, on trouve [1]; les auteurs ont traité l'équation dite fortement non-linéaires dégénérées de la forme

$$Au + g(x, u, \nabla u) = f$$

dans le cadre des espaces de Sobolev avec poids où A est un opérateur de second ordre à structure divergentièlle défini par,

$$Au = -diva(x, u, \nabla u),$$

où $a : \Omega \times I\!R \times I\!R^n \to I\!R^n$ est une fonction de Carathéodory vérifiant les conditions de Leray-Lions, avec Ω un ouvert borné de $I\!R^n \quad (n > 1)$. La fonction $g : \Omega \times I\!R \times I\!R^n \to I\!R$ est une non-linéarité forte vérifiant la condition de signe $g(x, s, \zeta)s \geq 0$ et une condition de croissance par rapport à ζ, mais aucune hypothèse de croissance en s n'est supposée. Un travail initiatif a été fait par Leray-Lions [48] dans les espaces de Sobolev classiques $W^{m,p}(\Omega)$ lorsque:

$$Au = \Sigma_{|\alpha|\leq m}(-1)^{|\alpha|}D^{\alpha}A_{\alpha}(x, u, ..., \nabla^m u)$$

est un opérateur d'ordre $2m$ avec les coefficients A_α vérifiant une condition de croissance polynomiale en u et toutes ses dérivées et une condition d'ellipticité de type:

$$\sum_{|\alpha|\leq m} A_\alpha(x, \zeta)\zeta_\alpha \geq c|\zeta|^p.$$

Ces conditions jouent un rôle très important dans le développement de la théorie générale des solutions faibles des équations elliptiques non-linéaires. L'impulsion décisive de l'étude du problème de second ordre de type $Au = f$, où le deuxième membre est une mesure ou dans L^1 a été donnée par Boccardo et Gallouet dans [12] and [11], et par J. M. Rakotosom dans [52]. Ils ont montré l'existence de solutions faibles dans $W_0^{1,q}(\Omega)$ pour tout réel q tel que $1 < q < q'$ où $q' = \frac{n(p-1)}{n-1}$.

Ces solutions faibles n'appartiennent pas en général à $W_0^{1,q}(\Omega)$, et pour atteindre cette régularité limite, les auteurs[12] ont supposé que $|f|log|f| \in L^1(\Omega)$ et $2 - \frac{1}{n} < p < n$.

Historiquement, considérons le problème elliptique suivant:

$$\begin{cases} Au = f \quad \text{dans } \Omega, \\ u = 0 \text{ sur } \partial\Omega, \end{cases} \tag{0.0.3}$$

avec $Au = -\text{div}\,(a(x, \nabla u))$ est de type Leray-Lions de $W_0^{1,p}(\Omega)$ vers son dual $W^{-1,p'}(\Omega)$,
et où la fonction de Carathédory a vérifie les hypothèses de croissance, de monotonie, et d'ellipticité sur Ω, ouvert borné de $I\!R^n$ et où f est une mesure. L'étude se limite au cas où $1 < p \leq n$ car si $p > n$ alors $W_0^{1,p}(\Omega) \hookrightarrow C^0(\Omega)$, et par suite $(C^0(\Omega))' \hookrightarrow W^{-1,p'}(\Omega)$; donc si f est une mesure de Radon, alors elle devient un élément de dual $W^{-1,p'}(\Omega)$.

Dans le cas $p > 2 - \frac{1}{n}$, avec f une mesure de Radon, Boccardo et Gallouet [12] ont montré que la solution faible du problème (0.0.3) appartient à $W_0^{1,q}$ où $1 < q < p^* = \frac{n(p-1)}{n-1}$.

Or si p est proche de 1, plus précisement si $1 < p \leq 2 - \frac{1}{n}$ alors la solution u n'appartient pas à $W_{loc}^{1,1}(\Omega)$.

Pour surmonter cette difficulté, Benilan, Boccardo, Gallouet, Gariepy, Pierre et Vazquez ont introduit dans [8] un nouvel espace $T_{loc}^{1,1}(\Omega)$ dans lequel on peut donner un sens au ∇u, qui n'est pas en général localement intégrable. L'idée consiste à introduire $T_k(u)$, troncature de u, et de considérer $\nabla T_k u$, et ceci est introduit dans le but de choisir les solutions qui s'approchent le plus à des solutions physiques.

En 1965, Stampacchia, voir [57], avait consideré le problème elliptique linéaire suivant:
$-div(A^t \nabla u) = f$, pour $f \in \mathcal{M}(\overline{\Omega})$, espace des mesures. Il a proposé une méthode donnant l'existence et l'unicité de la solution de la dite équation elliptique voir [51]; cette méthode utilise la dualité et conduit au problème variationnel suivant:

$$\begin{cases} u \in \bigcap\limits_{q < \frac{n}{n-1}} W_0^{1,q}(\Omega), \\ -\langle div(A^t \nabla \varphi), u \rangle = \int_\Omega \varphi df, \forall \varphi \in H_0^1(\Omega) \text{ tel que } -div(A^t \nabla \varphi) \in \bigcup\limits_{r > n} W_0^{1,r}(\Omega). \end{cases} \tag{0.0.4}$$

Cependant, pour les problèmes elliptiques et $f \in \mathcal{M}(\overline{\Omega})$, la méthode de Stampacchia ne fonctionne plus avec des opérateurs non linéaires; ainsi Boccardo et Gallouet, dans [12] ont construit, par approximation, une solution pour les équations elliptiques avec des conditions aux limites homogènes de Dirichlet.

La formulation obtenue pour le problème

$$\begin{cases} -div(a(x, u, \nabla u)) = \mu \quad \text{dans } \Omega, \\ u = 0 \qquad\qquad\qquad\quad \text{sur } \partial\Omega, \end{cases} \tag{0.0.5}$$

pour μ mesure est:

$$\begin{cases} u \in \bigcap\limits_{q<\frac{n(p-1)}{n-1}} W_0^{1,q}(\Omega), \\ -\langle div(A^t\nabla\varphi), u\rangle = \int_\Omega \varphi df, \forall \varphi \in H_0^1(\Omega) \text{ tel que } -div(A^t\nabla\varphi) \in \bigcup\limits_{r>n} W_0^{1,r}(\Omega). \end{cases} \tag{0.0.6}$$

Le résultat d'existence, lorsque a dépend de u et ∇u a été démontré en [29], [20], [51].
Dans [50] l'auteur a prouvé, dans le cas linéaire et en utilisant la méthode par approximation, l'existence de la même solution que celle de Stampacchia.

Dans cette direction citons aussi les travaux [13], [45] and [53]. Ainsi dans un cadre plus général des espaces d'Orlicz Sobolev, on trouve les travaux d'Azroul, Benkirane et Tienari [3] où les auteurs ont étudié la régularité de solution de l'équation du Poisson

$$\Delta u = f.$$

On trouve aussi un travail récent qui concerne l'existence de solution de problèmes elliptiques non linéaires non homogènes ou anisotropies avec second membre mesure, dans un espace de Sobolev à exposent variable $p(x)$, et μ est une mesure de Radon. Ce travail a été traité par Barbara. A, Azroul. E, M. B. Benboubker, H. Redwane, voir [6] et [7].

Dans notre travail on s'intéresse aux problèmes de Dirichlet sous formes des systèmes elliptiques et paraboliques d'équations aux dérivées partielles. Un travail qui a été fait sur les systèmes quasi-linéaires est celle de Hungerbüehler, N [36] dans l'espace de Sobolev classique $W^{1,p}(\Omega)$ sous les formes suivantes:

$$(\text{QES})_{(f,g)} \begin{cases} -div\sigma\left(x, u\left(x\right), Du\left(x\right)\right) &= v(x) + f(x,u,Du) + divg(x,u) \text{ in } \Omega \\ u &= 0 \qquad\qquad\qquad\qquad \text{on } \partial\Omega, \end{cases} \tag{0.0.7}$$

$$\begin{cases} -div\sigma(x,u,Du) &= \mu \text{ in } \Omega \\ u &= 0 \text{ on } \partial\Omega \end{cases} \tag{0.0.8}$$

où v est dans $W^{-1,p'}(\Omega, R^m)$, $\left(\frac{1}{p}+\frac{1}{p'}=1,\ p>1\right)$, f et g satisfont les conditions de continuité et de croissance. σ est une fonction de Carathéodory vérifie les conditions de croissance, de coercivité et non pas obligatoirement la monotonie large.

Dans cette thèse on traite des systèmes non linéaires d'équations aux dérivées partielles dont la partie principale s'écrit sous forme d'une divergence, pour un domaine borné Ω dans $I\!R^n$ et une fonction $u = (u_1, ..., u_m)$ à valeurs dans $I\!R^m$.
Ce travail est constitué de deux parties.

La première comporte quatre chapitres. qui traitent, avec un minimum des conditions, l'existence d'une solution faible de systemes; $(QES)(v)$, $(QES)_{(f,g)}$, $(QES)_{(f,g)'}$ et $(QES)_{(\mu)}$ (indiqués ci dessous) dans l'espace de Sobolev avec poids $W_0^{1,p}(\Omega,\omega,I\!R^m)$ (où ω est une famille de fonctions poids), et dans l'une des types de "monotonie" suivantes:

a) pour tout $x\in\Omega$, et tout $u\in I\!R^m$, l'application $F\longmapsto\sigma(x,u,F)$ est de classe C^1, et monotone(i-e, $(\sigma(x,u,F)-\sigma(x,u,G)):(F-G)\geq 0$, pour tout $x\in\Omega,$ tout $u\in I\!R^m$ et tout $F,G\in I\!M^{m\times n}$).

b) elle existe une fonction $W:\Omega\times I\!R^m\times I\!M^{m\times n}\longrightarrow I\!M^{m\times n}$ tel que $\sigma(x,u,F)=\dfrac{\partial W}{\partial F}(x,u,F)$ et $F\longmapsto W(x,u,F)$ convexe, et de classe C^1.

c) For all $x\in\Omega$, et pour tout $u\in I\!R^m$ l'application $F\longmapsto\sigma(x,u,F)$ est srictement monotone (i.e, $\sigma(x,u,.)$ est monotone et :$[(\sigma(x,u,F)-\sigma(x,u,G)):(F-G)=0]\Longrightarrow F=G$).

d) $\sigma(x,u,F)$ est strictement p-quasi-monotone en F, i.e,

$$\int_{I\!M^{m\times n}}(\sigma(x,u,\lambda)-\sigma(x,u,\overline{\lambda})):(\lambda-\overline{\lambda}))d\vartheta(\lambda)>0,$$

ϑ est une mesure de Radon sur Ω de masse finie

de chacun des systèmes quasi-lineaire elliptiques suivants:

$$\text{(QES)}(v)\begin{cases}-div\sigma\left(x,u\left(x\right),Du\left(x\right)\right) &=& v(x)\text{ in }\Omega\\ u &=& 0 \qquad\qquad \text{on }\partial\Omega,\end{cases}\tag{0.0.9}$$

$$\text{(QES)}_{(f)}\begin{cases}-div\sigma\left(x,u\left(x\right),Du\left(x\right)\right) &=& v(x)+f(x,u)\text{ in }\Omega\\ u &=& 0 \qquad\qquad \text{on }\partial\Omega,\end{cases}\tag{0.0.10}$$

$$\text{(QES)}_{(f,g)}\begin{cases}-div\sigma\left(x,u\left(x\right),Du\left(x\right)\right) &=& v(x)+f(x,u,Du)+divg(x,u)\text{ in }\Omega\\ u &=& 0 \qquad\qquad \text{on }\partial\Omega,\end{cases}\tag{0.0.11}$$

$$\text{(QES)}_{(\mu)}\begin{cases}-div\sigma\left(x,u\left(x\right),Du\left(x\right)\right) &=& \mu\text{ in }\Omega\\ u &=& 0 \qquad\qquad \text{on }\partial\Omega,\end{cases}\tag{0.0.12}$$

où l'application σ vérifie les conditions de régularité, de croissance et de coercivité. Ainsi σ ne doit pas être obligatoirement monotone dans ses dernières variables ou encore satisfaire la condition de stricte

monotonie d'un opérateur de typique de Leray-Lions comme c'est par exemple le cas dans la théorie classique des opérateurs monotones développés par Višik, Minty, Browder[16], Brézis[15], Lions[46], et bien d'autres...

La clé que nous utilisons pour prouver l'existence d'une solution faible u de chacun des systèmes au dessus est l'approche par des mesures de Young. En appliquant le schéma de Galerkin pour construire une suite d'approximations u_k; puis on applique les conséquences de théorème de Ball[5] qui donne alors un contrôle suffisant sur la suite Du_k pour passer à la limite.

La deuxième partie concerne à l'existence d'une solution faible des systèmes paraboliques dans l'espace de Sobolev avec poids $L^1(0,T,W^{1,1}(\Omega,\omega,I\!R^m))$ suivants:

$$(\mathcal{QPS})\begin{cases} \partial_t u - div\sigma(x,t,u,Du) &= v(x,t) \quad \text{in} \quad \Omega_T \\ u(x,t) &= 0 \quad \text{on} \quad \partial\Omega\times(0,T) \\ u(x,0) &= u_0(x) \quad \text{on} \quad \Omega \end{cases}$$

$$(QPS)_{(\omega)}\begin{cases} \partial_t u - div\sigma(x,t,u,Du) &= v(x,t)+f(x,t,u)+divg(x,t,u) \quad \text{in} \quad \Omega_T \\ u(x,t) &= 0 \quad \text{on} \quad \partial\Omega\times(0,T) \\ u(x,0) &= u_0(x) \quad \text{on} \quad \Omega \end{cases}$$

$$(QPS)_{(\omega)}\begin{cases} \partial_t u - div\sigma(x,t,u,Du) &= v(x,t)+f(x,t,u,Du)+divg(x,t,u) \quad \text{in} \quad \Omega_T \\ u(x,t) &= 0 \quad \text{on} \quad \partial\Omega\times(0,T) \\ u(x,0) &= u_0(x) \quad \text{on} \quad \Omega \end{cases}$$

Avec v dans $L^{p'}(0,T,W^{-1,p'}(\Omega,\omega^*,I\!R^m))$ et $u_0 \in L^2(\Omega,\omega_0,I\!R^m)$, f et g satisfont les conditions de continuité et de croissance. L'application σ est une fonction de Carathéodory vérifie les conditions de croissance, de coercivité, et l'une des types de "monotonie" a), b), c), ou d) indiqués ci- dessus. On utilise aussi dans cette partie l'approche par des mesures de Young; (après avoir construit une suite dite approximation de Galerkin) pour passer à la limite.

Ce travail est le premier qui concerne les systèmes quasi-linéaires elliptiques et paraboliques dans un espace de Sobolev avec poids, et avec les conditions de "monotonie" a), b), c), ou d).

Articles

Articles Publiés

1 - RAMI. E, AZROUL. E and ELLEKHLIFI. M: *Quasi-linear degenerated elliptic system in divergence form with mild monotonicity in weighted Sobolev spaces. Journal Afrika Matematika, Springer. 708 (2019) 1-16.*

2 *- RAMI. E, BARBARA. A and AZROUL. E: Existence of a weak solution of some quasi-linear elliptical system in a weighted Sobolev space. International Journal of Mathematics Trends and Technology (IJMTT) - Volume 66 Issue 2 - Feb 2020.*

3 *- AZROUL. E, BARBARA. A and RAMI. E: Existence of solution for some quasi-linear parabolic systems with weight and weak monotonicity. Proyeccioness Journal of Mathematics. Volume.39, n.3(2020)/pp.529-557.*

4 *- RAMI. E, AZROUL. E and BARBARA. A : Existence of solutions for some quasi-linear parabolic systems with weight and weak monotonicity general data. (SEMA Journal) Boletin de la Sociedad Española de Matemática Aplicada. https://doi.org/10.1007/s40324-020-00226-x june (2020).*

5 *-RAMI. E, AZROUL. E and BARBARA. A: Quasi-linear degenerated elliptic system in divergence form with mild monotonicity in weighted Sobolev spaces and general data, Accepted in Applied Mathematics 29/10/2020.*

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1 -RAMI. E, AZROUL. E and BARBARA. A: *A weak solution of a parabolic non linear system in weighted Sobolev space . Manuscripta Mathematica.*

2 *-RAMI. E, AZROUL. E and BARBARA. A: Existence of solution for some quasi-linear elliptic system with weight and measure valued right hand side. Afrika Mathematika Journal.*

3 *-RAMI. E, AZROUL. E and BARBARA. A: Existence of weak solution for some quasi-linear degenerated elliptic system in weighted Sobolev space. SN Partial Differential Equations and Applications.*

4 *-RAMI. E, AZROUL. E and BARBARA. A: Existence of T-nu -p(x) solution of non homogeneous elliptic problem with right hand side measure. International Journal of Nonlinear Analysis and applications.*

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6 *-RAMI. E, AZROUL. E and BARBARA. A: The existence of a weak solution to a parabolic non-linear system . Journal of Mathematical Analysis and applications.*

7 *-RAMI. E, AZROUL. E and BARBARA. A: Existence of a weak solutions for some degenerated quasi-linear parabolic systems with weak monotonicity. Discrete Contin. Dyn. Syst. Ser.*

8 *-RAMI. E, AZROUL. E and BARBARA. A: The existence of a weak solution of a quasilinear elliptical system in a Sobolev space with weight.*

9 *-RAMI. E, AZROUL. E and BARBARA. A: Study of a Parabolic quasilinear Systems. Mathematical Communication.*

Chapter 1

Preliminaries.

This part presents some basic elements on function spaces.

1.1 Notations and standard spaces

In this thesis, Ω we will denote an open domain in Euclidean space $\mathbb{R}^n$. Points in $\mathbb{R}^n$ are denoted by $x = (x_1, ..., x_n)$, where $x_i \in \mathbb{R}, \quad 1 \leq i \leq n$. If $x, y \in \mathbb{R}^n$, the inner product of x and y is

$$x.y = \sum_{i=1}^{n} x_i.y_i$$

and the Euclidean norm of x is $\sqrt{x.x}$.

Next, we denote by $M^{m\times n}$ the real vector space of $m \times n$ matrix equipped with the inner product $M : N = \sum_{i,j} M_{ij}N_{ij}$ and the tensor product $a \otimes b$ between two vectors $a, b \in \mathbb{R}^m$ is defined to be the matrix $(a_ib_j)_{i,j=1,....,m}$.

More generally, if X and Y, respectively, are a Banach and a Hilbert space, $\|.\|_X$ and $(,)_Y$ will denote the norm in X and the inner product in Y. Moreover, X' will denote the topological dual of X. In this case, we denote the dual pairing between $x' \in X'$ and $x \in X$ by $\langle x', x\rangle$. As usual, we use the notations$\rightharpoonup$ and $\rightharpoonup^*$ to indicate weak convergence and weak-$*$ convergence respectively.

If $u : \Omega \to \mathbb{R}^m, \ \ with \ \ m \in \mathbb{N}^*$ is a function defined on Ω, the support of u is defined by

$$spt(u) = \Omega \cap \overline{\{x \in \mathbb{R}^n | \quad u(x) \neq 0\}},$$

where the closure of a set $S \subset \mathbb{R}^n$ is denoted by $\overline{S}$. The boundary of a set $S \subset \mathbb{R}^n$ is defined by

$$\partial S = \overline{S} \cap \overline{\mathbb{R}^n \setminus S}$$

and the characteristic function of S is denoted by χ_S. Moreover, $B_r(x_0)$ will denote the open ball of radius r centered at x_0 in $\mathbb{R}^n$.

If $\alpha = (\alpha_1, ..., \alpha_n)$ is an $n - tuple$ of non-negative integers, α is called a multi-index and the length of α is

$$|\alpha| = \Sigma_{i=1}^n \alpha_i.$$

If $x = (x_1, ..., x_n) \in \mathbb{R}^n$, we will use the notation $x^\alpha = x_1^{\alpha_1}....x_n^{\alpha_n}$ and $\alpha! = \alpha_1!....\alpha_n!$. The partial derivatives operators are denoted by $D_i = \partial/\partial x_i$, for $1 \leq i \leq n$, and the higher order derivatives by

$$D^\alpha = D^{\alpha_1}...D^{\alpha_n} = \frac{\partial^{|\alpha|}}{\partial x_1^{\alpha_1}...\partial x_1^{\alpha_n}}.$$

The Jacobian matrix of a function $u : \Omega \to \mathbb{R}^m$ (which is the gradient as $m = 1$) is denoted by

$$Du(x) = (Du_1(x), ..., Du_n(x)) = \begin{pmatrix} D_1u_1(x).....D_nu_1(x) \\ \\ D_1u_m(x).....D_nu_m(x) \end{pmatrix}.$$

We denote by $C(\Omega) \equiv C^0$ the space of continuous real-valued functions on Ω.

More generally, if k is a non-negative integer, possibly ∞, let

$$C^k(\Omega) = \left\{ u : \Omega \to \mathbb{R} \mid \ D^\alpha u \in C(\Omega); \quad 0 \leq |\alpha| \leq k \right\}$$

$$C_c^k(\Omega) = \left\{ u \in C^k(\Omega) \ \mid \ sptu \ compact, \ \ sptu \subset \Omega \right\}$$

$$C^k(\overline{\Omega}) = \left\{ u \in C^k(\Omega) \ \mid \ D^\alpha u \ \ has \ a \ continuous \ extension \ to \ \overline{\Omega}, \quad 0 \leq |\alpha| \leq k \right\}$$

Moreover, $D(\Omega)$ denotes the space $C_c^\infty(\Omega)$ endowed with a topology so that sequence (u_k) converge to an element $u \in D(\Omega)$ if and only if there exists a compact $K \subset \Omega$ such that $sptu_k \subset K$ for all k and $D^\alpha u_k \to D^\alpha u$ uniformly on K for each multi-index α. Finally, $C_0(\mathbb{R}^n)$ denotes the Banach space of continuous functions $u : \mathbb{R}^n \to \mathbb{R}$ satisfying $lim_{|\xi| \to \infty} u = 0$ equipped with the $L^\infty-$ norm.

We will use $C_c^k(\Omega, \mathbb{R}^m)$ to denote the class of functions $u : \Omega \to \mathbb{R}^m$ defined on Ω whose coordinate functions belong to $C^k(\Omega)$. Similar notations are used for other function spaces whose elements are vector-valued.

as one of the aims of this work is to solve a parabolic system of PDEs, we will consider vector -valued functions of the form $u : \Omega \times (0, T) \to \mathbb{R}^m$. In this context, for $T > 0$ finite, Ω_T will designate the cylinder $\Omega \times (0, T)$ and the Jacobian matrix of $u : \Omega_T \to \mathbb{R}^m$ will be denoted by

$$\nabla u(x) = (Du(x), \partial_t u(x)) = \begin{pmatrix} D_1u_1(x).....D_nu_1(x) & \partial_t u_1(x) \\ \\ D_1u_m(x).....D_nu_m(x) & \partial_t u_m(x) \end{pmatrix}.$$

In the theory of parabolic PDEs, it is often useful to regard functions $u : \Omega_T \to I\!R^m$ as vector-valued functions of only one variable, namely the time variable.

Definition 1.1.1 *Let Xbe a Banach space and $0 < T < \infty$. The space $C^k([0,T],X)$ with $k \in I\!N$ consists of all continuous functions $u : [0,T] \to X$ that have continuous derivatives up to order k on $[0,T]$ with the norm*

$$\| u \| = \sum_{i=0}^{k} \max_{0 \le t \le T} \| \frac{d^i u}{dt}(t) \|_X$$

Here, only the right hand and the left hand derivatives need to exist at the boundary point $t = 0$ and $t = T$ respectively. We write $C^k([0,T],X)$ instead of $C^0([0,T],X)$. Equipped with the norm $\|.\|$, $C^k([0,T],X)$ is a Banach space (proposition 23.2 [65]).

1.2 Lebesgue spaces.

For $p \in]0,\infty]$ and $m \in I\!N^*$, we consider the usual Lebesgue space $L^p(\Omega, I\!R^m$ consisting of all measurable functions (identified modulo the relation "equal almost everywhere"$u : \Omega \to I\!R^m$ for which

$$\|u\|_p = \left\{ \begin{array}{l} (\int_\Omega |u(x)|^p dx)^{\frac{1}{p}} \quad if \;\; p > 0 \\ esssup_{x \in \Omega} |u(x)| \quad if \;\; p = \infty \end{array} \right\} < \infty$$

hold. Here dx denoted the usual Lebesgue measure. Note that we may also regard $L^p(\Omega, I\!R^m)$ as the m-cartesian product of the space $L^p(\Omega, I\!R)$. In this sense, one may equipped $L^p(\Omega, I\!R^m)$ with various equivalent norms and in particular with $|.|_p$ given by

$$|u|_p = \Big(\sum_{j=1}^{m} \|u_j\|_p^p \Big)^{\frac{1}{p}} \quad \forall \;\; u = (u_1, ..., u_m) \in \prod_{j=1}^{m} L^p(\Omega, I\!R).$$

Moreover, when Ω has finite measure we have that $L^q(\Omega, I\!R^m) \subset L^p(\Omega, I\!R^m)$ for $0 < p \le q \le \infty$ and

$$\|u\|_p \le c \|u\|_q \quad \forall u \in L^q(\Omega, I\!R^m),$$

for a constant $c = c(\mid \Omega \mid, p, q)$, where $\mid \Omega \mid = \int_\Omega \chi_\Omega dx$ denotes the volume of Ω. The embedding $L^q(\Omega, I\!R^m) \subset L^p(\Omega, I\!R^m)$ is then conditions. In this case, we will adopt the following notation:

$$L^q(\Omega, I\!R^m) \hookrightarrow L^p(\Omega, I\!R^m).$$

When $p \geq 1$, $\|.\|_p$ is a norm and $(L^p(\Omega, I\!R^m), \|.\|_p)$ a Banach space. In the special case $p = 2$, $L^2(\Omega, I\!R^m)$ equipped with the inner product

$$(u, v)_2 = \int_\Omega u(x).v(x)dx \quad u, v \in L^2(\Omega, I\!R^m)$$

is a Hilbert space. Of course,

$$L^p(\Omega, I\!R^m) \subset L^1_{loc}(\Omega, I\!R^m) = \{u : \Omega \to I\!R^m; \quad u \in L^1(A, I\!R^m) \quad \forall A \subset \Omega \quad \text{measurable and bounded}\}$$

holds. Again, note that in the particular case where Ω is bounded, we have an interesting property.

Theorem 1.2.1 *(Vitali)*
Let $\Omega \subset I\!R^n$ be a open bounded domain and let (u_n) be a sequence in $L^p(\Omega, I\!R^m)$ with $1 \leq p < \infty$. Then (u_n) is a Cauchy sequence in the $L^p - norm$ if and only if the two following conditions hold:

- (i) (u_n) *is a Cauchy in measure* $(i.e. \forall \varepsilon > 0 |\{x \in \Omega : \quad |u_n(x) - u_m(x)| \geq \varepsilon\}| \to 0 \quad$ *as* $m, n \to \infty)$
- (ii) $(|u_n|^p)$ *is equi-integrable (i.e.* $\sup_n \int_\Omega |u_n|^p dx < \infty$ *and* $\forall \varepsilon > 0; \exists \delta > 0$ *such that* $|\int_E |u_n|^p dx| < \epsilon$ *for all* n *whenever* $E \subset \Omega$ *and* $|E| < \delta$*).*

For the proof, see e.g [33]. Note that this theorem is also valid with a third condition when Ω is not bounded [30], page 180.
Next, we denote by p' the Hölder conjugate of $p \in [1, \infty]$ which is defined by:

$$p' = \left\{ \begin{matrix} {\scriptstyle (\infty \quad if \quad p=1), (1 \quad if \quad p=\infty)} \\ \\ {\scriptstyle \frac{p}{p-1} \quad if \quad 1<p<\infty} \end{matrix} \right., \quad \text{or equivalently } \frac{1}{p} + \frac{1}{p'} = 1 \Bigg\}$$

if we adopt the conventions $\frac{1}{0} = \infty$ and $\frac{1}{\infty} = 0$ if $u \in L^p(\Omega, I\!R^m)$ and $v \in L^{p'}(\Omega, I\!R^m)$, then $u.v \in L^1(\Omega, I\!R)$ and the Hölder inequality hold

$$\| u.v \|_1 \leq \| u \|_p \, . \, \| u \|_{p'} \, .$$

moreover it is will known that if $1 \leq p < \infty, \quad L^p(\Omega, I\!R^m)$ is separable and $C^\infty_c(\Omega, I\!R^m)$ is dense in $L^p(\Omega, I\!R^m)$. In the case $1 < p < \infty, \quad L^p(\Omega, I\!R^m)$ is uniformly convex and therefore reflexive. In fact,

$$(L^p(\Omega, I\!R^m))' \cong L^{p'}(\Omega, I\!R^m), \quad \forall p \in]1, \infty[.$$

Note that $(L^1(\Omega, I\!R^m))' \cong L^\infty(\Omega, I\!R^m)$ also hold but even though $L^1(\Omega, I\!R^m) \subset (L^\infty(\Omega, I\!R^m))', \quad (L^\infty(\Omega, I\!R^m))'$ is not isomorphic to $L^1(\Omega, I\!R^m)$. In the treatment of non-linear parabolic PDEs systems, we will consider the Lebesgue space $L^p(0,T,X)$ consisting of vector-valued functions of the time variable.

Definition 1.2.1 *Let X be a Banach space and $0 < T < \infty$. The space $L^p(0,T,X)$ with $1 \leq p \leq \infty$ consists of all measurable functions (identified modulo the relation "equal almost everywhere") $u :]0,T[\to X$ for which*

$$\|u\|_{L^p(0,T,X)} = \left\{ \begin{array}{l} (\int_0^T \|u(t)\|_X^p dt)^{\frac{1}{p}}, \quad if \;\; p \geq 1 \\ ess Sup_{t \in]0,T[} \|u(t)\|_X \quad if \;\; p = \infty \end{array} \right\} < \infty$$

hold.

Again, equipped with the norm $\|.\|_{L^p(0,T,X)}, \quad L^p(0,T,X)$ is a Banach space. In the special case where $p = 2$ and X is a Hilbert space, $L^p(0,T,X)$ equipped with the inner product

$$(u,v)_{L^2(0,T,X)} = \int_0^T (u(x), v(x))_X dt \qquad u, v \in L^2(0,T,X)$$

is a Hilbert space. Moreover, if $1 \leq p \leq q \leq \infty$ the following Hölder inequality holds for all $v \in L^{p'}(0,T,X')$ and all $u \in L^p(0,T,X)$

$$\int_0^T |(u(x), v(x))_X| dt \leq \|v\|_{L^{p'}(0,T,X')} \|u\|_{L^p(0,T,X)}$$

and $L^q(0,T,X) \hookrightarrow L^p(0,T,Y)$ for all Banach space $X,\ Y$ such that $X \hookrightarrow Y$.

Next, the set of all simple function $u : [0,T] \to X$ and also $C^k([0,T],X)$ are dense in $L^p(0,T,X)$ as $p < \infty$. Moreover, the embedding $C^k([0,T],X) \hookrightarrow L^p(0,T,X)$ is continuous. If $1 < p < \infty$, $L^p(0,T,X)$ is uniformly convex and reflexive and separable when X is a reflexive and separable Banach space. In fact

$$(L^p(0,T,X))' \cong L^{p'}(0,T,X') \qquad \forall p \in]1, \infty[.$$

As above $(L^\infty(0,T,X))'$ is not isomorphic to $L^1(0,T,X')$.

When Ω is bounded, the particular case $X = L^q(0,T,X)$ is interest in the theory of non-linear parabolic PDEs system. for all $1 \leq r \leq p \leq \infty$ and all $1 \leq s \leq q \leq \infty$, we have

$$L^p(0,T,L^q(\Omega, I\!R^m)) \hookrightarrow L^r(0,T,L^s(\Omega, I\!R^m))$$

Moreover, if $p = q$,

$$L^p(0,T,L^p(\Omega, I\!R^m)) \hookrightarrow L^p(\Omega \times]0,T[, I\!R^m).$$

And in the case $1 < p, \;\; q < \infty$, we have

$$(L^p(0,T,L^q(\Omega, I\!R^m)))' \cong L^{p'}(0,T,L^{q'}(\Omega, I\!R^m))$$

1.3 Sobolev spaces.

For $1 \leq p \leq \infty$ and $k \in I\!N$, we consider the standard Sobolev spaces

$$\bullet H^{k,p}(\Omega, I\!R^m) \equiv \ the\ the\ closure\ of\ \{u \in C^k(\Omega, I\!R^m) \mid \ \|u\|_{k,p} < \infty\}\ in\ L^p(\Omega, I\!R^m)$$

$$\bullet W^{k,p}(\Omega, I\!R^m) \equiv \{u \in L^p(\Omega, I\!R^m) \mid D^\alpha u \in L^p(\Omega, I\!R^m) \ \ for\ \ 0 \leq |\alpha| \leq k\} \ \ (D^\alpha u \ \ is\ the\ weak\ derivate\ of\ u)$$

$$\bullet W_0^{k,p}(\Omega, I\!R^m) \equiv \ the\ the\ closure\ of\ C_c^\infty(\Omega, I\!R^m)\ in\ W^{k,p}(\Omega, I\!R^m),$$

where the norm $\|.\|_{k,p}$ is given by

$$\|u\|_{k,p} = \begin{cases} \left(\sum_{0 \leq |\alpha| \leq k} \|D^\alpha u\|_p^p \right)^{\frac{1}{p}} & if \ \ p \geq 1 \\ \max_{0 \leq |\alpha| \leq k} \|D^\alpha u\|_\infty & if \ \ p = \infty \end{cases}.$$

We recall that $D^\alpha f$ is the weak (partial) derivative of $f \in L^1(\Omega, I\!R^m)$ if there exists a $g_\alpha \in L^1(\Omega, I\!R^m)$ such that $D^\alpha f = g_\alpha$ hold in the distributional sense, i.e. such that

$$\int f(x) D^\alpha \psi(x) dx = (-1)^{|\alpha|} \int g_\alpha \psi(x) dx \quad \forall \psi \in C_c^\infty(\Omega, I\!R^m).$$

$W^{k,p}(\Omega, I\!R^m)$ is a Banach space for all $p \in [1, \infty]$ and all $k \in I\!N$. In particular, equipped with the inner product

$$(u, v)_{k,2} = \sum_{0 \leq |\alpha| \leq k} (D^\alpha u, D^\alpha v)_2,$$

$W^{k,2}(\Omega, I\!R^m)$ is a Hilbert space.

Next, $W^{0,p}(\Omega, I\!R^m) = L^p(\Omega, I\!R^m)$ holds clearly for all $p \in [1, \infty]$. For any $k \in I\!N$, the embedding

$$W_0^{k,p}(\Omega, I\!R^m) \hookrightarrow W^{k,p}(\Omega, I\!R^m) \hookrightarrow L^p(\Omega, I\!R^m)$$

are also clear. Moreover, we have $H^{k,p}(\Omega, I\!R^m) \subset W^{k,p}(\Omega, I\!R^m)$. If $1 \leq p < \infty$,
$W_0^{0,p}(\Omega, I\!R^m) = L^p(\Omega, I\!R^m)$. We identified the dual of $W_0^{k,p}(\Omega, I\!R^m)$ for $p \in]1, \infty[$, with $W^{-k,p'}(\Omega, I\!R^m)$,

$$(W_0^{k,p}(\Omega, I\!R^m))' \cong W^{-k,p'}(\Omega, I\!R^m)$$

where $W^{-k,p'}(\Omega, I\!R^m)$ denotes the completion of $L^{p'}(\Omega, I\!R^m)$ with respect to the norm $\| \ . \ \|_{-k,p'}$ given by

$$\| \ v \ \|_{-k,p'} = \sup\{| \int_\Omega u.v dx|; \quad u \in W_0^{k,p}(\Omega, I\!R^m) \quad with \quad \| \ u \ \|_{k,p} \leq 1\}$$

Equipped with the norm $\| \ . \ \|_{-k,p'}$ which satisfies

$$\| \ v \ \|_{-k,p'} \leq \| \ v \ \|_{p'} \quad \forall v \in L^{p'}(\Omega, I\!R^m)$$

and (generalized Hölder inequality)

$$|\langle v,u\rangle| \leq \| u \|_{k,p} \| v \|_{-k,p'} \quad \forall u \in W_0^{k,p}(\Omega, I\!R^m), \quad v \in W^{-k,p'}(\Omega, I\!R^m).$$

$W^{-k,p'}(\Omega, I\!R^m)$ is a separable and reflexive Banach space.

Theorem 1.3.1 *(Rellich-Kondrachov) Let Ω be an open bounded domain in $I\!R^n$. Then, if $kp < n$ and $p \geq 1$, $W_0^{k,p}(\Omega, I\!R^m)$ is a compactly embedded in $L^q(\Omega, I\!R^m)$ where $q < p^* = \frac{np}{n-kp}$. If $kp > n + dp$, $W_0^{k,p}(\Omega, I\!R^m)$ is compactly embedded in $C^d(\overline{\Omega}, I\!R^m)$. see e.g.[63] theorem2.4.4.*

1.4 Weighted Sobolev spaces.

Let $1 < p < \infty$, Ω bounded open domain in $I\!R^n$ and $\omega = \{\omega_{ij} \mid \ 0 \leq i \leq n,\ 1 \leq j \leq m\}$, and $\overline{\omega_0} = (\omega_{0j})$ for all $1 \leq j \leq m$ the weight functions system defined in Ω. The weighted Lebesgue space $L^p(\Omega, \gamma, I\!R^m)$ is the set of functions

$$L^p(\Omega, \gamma, I\!R^m) = \left\{ u = u(x) \mid \ | u | \gamma^{\frac{1}{p}} \in L^p(\Omega, I\!R^m) \right\},$$

with $\gamma = \{\gamma_j \mid \ 1 \leq j \leq m\}$ is a weighted function on Ω i-e. $\gamma_j(x)$ is a non-negative measurable functions e.a; in $\gamma(x)$, for all $j = 1; ....., m$ The weighted Lebesgue space $L^p(\Omega, \gamma, I\!R^m)$ equipped by the norm:

$$\|u\|_{p,\gamma}^p = \sum_{j=1}^m \int_\Omega |u_j|^p \gamma_j.$$

The topological dual space of $L^p(\Omega, \gamma, I\!R^m)$ is equivalent to

$$\left(L^p(\Omega, \gamma, I\!R^m)\right)' \cong L^{p'}(\Omega, \gamma^*, I\!R^m),$$

with $\gamma^* = \left\{ (\gamma_j)^* = (\gamma_j)^{1-P'} \mid \ 1 \leq j \leq m \right\}$.

The space $W^{1,p}(\Omega, \omega, I\!R^m)$ is the set of functions

$$\left\{ u = u(x) \mid u \in L^p(\Omega, \overline{\omega_0}, I\!R^m),\ D_{ij}u = \frac{\partial u^i}{\partial x_j} \in L^p(\Omega, \omega_{ij}, I\!R^m),\ 1 \leq i \leq n,\ 1 \leq j \leq m \} \right\}$$

with

$$L^p(\Omega, \omega_{ij}, I\!R^m) = \left\{ u = u(x) \mid \ | u | \omega_{ij}{}^{\frac{1}{p}} \in L^p(\Omega, I\!R^m) \right\}.$$

The weighted space $W^{1,p}(\Omega, \omega, I\!R^m)$ can be equipped by the norm

$$\|u\|_{1,p,\omega} = \left(\sum_{j=1}^m \int_\Omega |u_j|^p \omega_{0j} dx + \sum_{1 \leq i \leq n, 1 \leq j \leq m} \int_\Omega |D_{ij}u|^p \omega_{ij} dx \right)^{\frac{1}{p}}.$$

The norm $\|.\|_{1,\omega,p}$ is equivalent to the norm $|||\ .\ |||$ on $W_0^{1,p}(\Omega,\omega,I\!R^m)$, defined by
$|||\ u\ |||= \left(\sum_{1\leq i\leq n_{1\leq j\leq m}} \int_\Omega |D_{ij}u|^p \omega_{ij} dx\right)^{\frac{1}{p}}$.
The topological dual of $(W^{1,p}(\Omega,\omega,I\!R^m))'$ is equivalent to

$$(W^{1,p}(\Omega,\omega,I\!R^m))' \cong W^{-1,p'}(\Omega,\omega^*,I\!R^m)$$

with $\omega^* = \{\omega_{ij}^* = \omega_{ij}^{1-p'},\ \ 0\leq i\leq n,\ 1\leq j\leq m\}$.
If $\omega_{ij}^{\frac{-1}{p-1}} \in L^1_{loc}(\Omega),$ for some $p\ \in]1,\infty[,$ for all $0\leq i\leq n, \quad 1\leq j\leq m$ then The weighted Sobolev space $W^{1,p}(\Omega,\omega,I\!R^m)$ is a Banach space, separable and reflexive, if more $\omega_{ij}\in L^1_{loc}(\Omega)$ for all $0\leq i\leq n, \quad 1\leq j\leq m$, the weighted Sobolev space $W_0^{1,p}(\Omega,\omega,I\!R^m)$ is the closure of $C_c^\infty(\Omega,I\!R^m)$ in $W^{1,p}(\Omega,\omega,I\!R^m)$ equipped by the norm $\|.\|_{1,p,\omega}$.
If the conditions satisfying $\omega_{ij}\in L^1_{loc}(\Omega)$ and $\exists s>0$ such that $\omega_{ij}^{-s}\in L^1(\Omega)$, then $W_0^{1,p}(\Omega,\omega,I\!R^m)$ is reflexive and $W_0^{1,p}(\Omega,\omega)\hookrightarrow L^p(\Omega)$ See [4], [24].

1.5 Lorentz spaces.

In this subsection, we consider measurable functions $f:\Omega\to I\!R^m$ on the measure space $(\Omega, Lebesgue\ \ measure)$ which are finite-valued almost everywhere and such that

$$|\{x\in\Omega:\quad |f(x)|>y\}|<\infty$$

for some $y>0$. As usual, we identify functions which are equal almost everywhere. For such functions f, we define
• $the\ distribution\ function\ \lambda_f:]0,\infty[\to I\!R$ of f by

$$\lambda_f(y)=|\{x\in\Omega:\quad |f(x)|>y\}|$$

and
• the non-increasing rearangement $f^*:]0,\infty[\to I\!R$ of f by

$$f^*(t)=\inf\{y>0:\quad \lambda_f(y)\leq t\}.$$

Clearly, the distribution function λ_f is non-negative and non-increasing and continuous from the right. Moreover, since $\lambda_f(y)<\infty$ for some $y>0$ and f is finite valued a.e., we have that $\lambda_f(y)\to 0$ as $y\rightharpoonup\infty$. It follows that the function f^* is well defined and it is easy to see that f^* is non-negative and

non-increasing.

Next, it follows immediately from the definition of f^* that

$$\lambda_f(f^*(t)) \leq y, \quad \forall y > 0.$$

Since λ_f is continuous from the right, we have also

$$\lambda_f(f^*(t)) \leq t, \quad \forall t > 0.$$

In fact, if λ_f is continuous and strictly decreasing, then f^* is the inverse function of λ_f and

$$\lambda_{f^*}(y) = \lambda_f(y), \quad \forall y > 0$$

Now for $p, q \in]0, \infty]$, we consider the Lorentz space $L^{p,q}(\Omega, I\!R^m)$ consisting of all measurable functions (identified modulo. the relation "equal almost everywhere ") $f : \Omega \to I\!R^m$ for which $\|f\|^*_{p,q} < \infty$, where

$$\|f\|^*_{p,q} = \begin{cases} \left(\frac{q}{p} \int_0^\infty (t^{\frac{1}{p}} f^*(t))^q \frac{dt}{t} \right)^{\frac{1}{q}} & if \quad 1 \leq p, q < \infty \\ \sup_{t>0} t^{\frac{1}{p}} f^*(t) \quad if \quad 1 \leq p < \infty, \quad q = \infty \end{cases}$$

Note that the case $p = \infty$ and $1 \leq q < \infty$ is note of interest since $\displaystyle\int_0^\infty (t^{\frac{1}{p}} f^*(t))^q \frac{dt}{t} < \infty$ implies $f = 0$ almost everywhere. The $L^{p,q}(\Omega, I\!R^m)$ is a linear space. In fact, equipped with the topology induced by $\|.\|^*_{p,q}$ via the relation $(n \to \infty)$,

$$f_n \to f \quad in \quad L^{p,q}(\Omega, I\!R^m) \Leftrightarrow \|f_n - f\|^*_{p,q} \to 0$$

$L^{p,q}(\Omega, I\!R^m)$ is a topological vector space. With Hardy's inequality, one can prove that this space is metrisable and even complete, see [40] and $L^{p,q}(\Omega, I\!R^m)$ with $1 < p \leq \infty$ and $1 \leq q \leq \infty$ may be equipped with a norm which defines an equivalent topology. Thus these space are Banach space.

In any case, if $1 < p < \infty$ and $1 \leq q \leq \infty$, the following Hölder inequality for Lorentz space holds

$$\| f.g \|_1 \leq c(p, q) \|f\|^*_{p,q} \|g\|^*_{p',q'}$$

for all $f \in L^{p,q}(\Omega, I\!R^m)$ and $g \in L^{p',q'}(\Omega, I\!R^m)$. Moreover $1 < p, q < \infty$, we have the following duality relation

$$\left(L^{p,q}(\Omega, I\!R^m)\right)' \cong L^{p',q'}(\Omega, I\!R^m)$$

and so $L^{p,q}(\Omega, I\!R^m)$ is reflexive. Again, even though $L^{p,1}(\Omega, I\!R^m)$ is not reflexive for $1 \leq p < \infty$, we have

$$\left(L^{p,1}(\Omega, I\!R^m)\right)' \cong L^{p',\infty}(\Omega, I\!R^m).$$

If $0 \leq p \leq \infty$, we have

$$L^{p,p}(\Omega, I\!R^m) = L^p(\Omega, I\!R^m)$$

and

$$L^{p,q_1}(\Omega, I\!R^m) \hookrightarrow L^{p,q_2}(\Omega, I\!R^m)$$

for $q_1 \leq q_2$. In particular

$$L^{p,q_1}(\Omega, I\!R^m) \hookrightarrow L^p(\Omega, I\!R^m) \hookrightarrow L^{p,q_2}(\Omega, I\!R^m) \hookrightarrow L^{p,\infty}(\Omega, I\!R^m)$$

for $0 < q_1 \leq p \leq q_2 \leq \infty$. Note that since f^* is essentially the inverse function of λ_f, we have:

$$\|f\|^*_{p,\infty} = \sup_{t>0} t^{\frac{1}{p}} f^*(t) = \sup_{y>0} y(\lambda_f(y))^{\frac{1}{p}}.$$

Theorem 1.5.1 *Let be Ω a bounded open domain in $I\!R^n$ and $p > 1$. Then*

$$L^{p,\infty}(\Omega, I\!R^m) \hookrightarrow L^q(\Omega, I\!R^m) \quad \forall q \in]0, p[.$$

See e.g.[50] for the proof.

1.6 Radon and Young measures.

This part presents the main tools we use to prove the existence of a solution of systems of quasi-linear partial differential equations(in particular with measure -valued right hand side), namely approximation of Radon measure and Young measure.

1.6.1 Bounded Radon measure

In this section, we recall first the basic elements of this theory, in particular the standard characterization of such measures. Then, we present a version of usual approximation of bounded Radon measure by smooth functions. Refer to the classical literature, e.g. [19] , [23], [30] and [55].
Let $M(\Omega)$ be the set of bounded real (signed) measure on an open domain Ω in $I\!R^n$, equipped with the norm $\|\|_M$ defined by

$$\|\mu\|_M = |\mu|(\Omega) \quad \forall\, \mu \in M(\Omega),$$

where $|\mu| = \mu^+ + \mu^-$ denotes the total variation of μ, $M(\Omega)$ is a Banach space. Moreover, it is well known that $M(\Omega)$ coincides with the victor space of bounded Radon measures.
Next, we denote by $M(\Omega, I\!R^m)$, with $m \in I\!N^*$, the Banach space of vectors μ of bounded Radon measures

$$\mu = (\mu_1, ...\mu_m) \quad with \quad \mu_i \in M(\Omega) \quad for \ \ i = 1, ..., m.$$

equipped with the norm $\sum_{j=1}^m \|\mu_j\|_M$. Moreover, if $f : \Omega \to I\!R^m$, we set

$$\int_\Omega f d\mu = \sum_{j=1}^m \int_\Omega f_j d\mu_j.$$

Proposition 1.6.1 *The application ϕ which to each element $\mu \in M(\Omega, I\!R^m)$ associates the continuous linear form $I_\mu : f \mapsto \int_\Omega f d\mu$ in $C_c(\Omega, I\!R^m)'$ is an isometric isomorphism and therefore*

$$C_c(\Omega, I\!R^m)' \cong M(\Omega, I\!R^m).$$

For the proof see [30](theorem 7.17). The isomorphism ϕ implies a natural definition of the $weak \rightharpoonup^*$ convergence in $M(\Omega, I\!R^m)$.

Definition 1.6.1 *Let Ω be a measurable subset of $I\!R^n$, $(\mu_k)_{k\in I\!N}$ and μ be a sequence and respectively an element in $M(\Omega, I\!R^m)$. We say that (μ_k) converges $(weakly - *)$ to μ, which is denoted by $(\mu_k) \rightharpoonup^* \mu$, if*

$$\int_\Omega f d\mu_k \to \int_\Omega f d\mu \quad \forall \ f \in C_c(\Omega, I\!R^m).$$

Proposition 1.6.2 *(Approximation of Bounded Radon measures)*
Let Ω be a bounded open measurable set in $I\!R^n$. For $m \geq 1$, let $\mu \in M(\Omega, I\!R^m)$ be a vector of Radon measures. Then there exists a sequence $(v_k)_{k\in I\!N}$ of functions belonging to
$C^\infty(\Omega, I\!R^m) \cap L^\infty(\Omega, I\!R^m)$for all $k \in I\!N$ such that

- $(i) \ \ v_k \rightharpoonup^* \mu \quad in \ \ M(\Omega, I\!R^m) \ \ as \ \ k \to \infty$
- $(ii) \ \ \| v_k \|_{L^1} \leq \| \mu \|_M \quad for \ \ all \ \ k \in I\!N$

For proof see e.g.[39], (section 4).

1.6.2 Young measures.

Weak convergence is a basic tool of modern nonlinear analysis because it enjoys the same compactness properties that convergence infinite dimensional space does. Nonetheless, this notion does not behave as one would desire with respect to nonlinear functionals and operations. Young measures are a device to understand and to control these difficulties.

In this section, we introduce Young measures as disintegration families resulting from the Ball's theorem. Note that this definition of Young measure does not entirely correspond to the original one. The link between the useful formulation due to Ball and the probabilistic point of view is outlined. Then, we present applications of Ball's Theorem to non-linear PDEs theory and to Fourier series.Finally, we restate some facts about gradient Young measure which will be used throughout this thesis. [60],[36],...

Theorem 1.6.1 *(Ball's theorem)[5]*

Let Ω be Lebesgue measurable of $\mathbb{R}^n$, let $K \subset \mathbb{R}^m$ be closed, and let

$u_j : \Omega \to \mathbb{R}^m, j \in \mathbb{N}$, be a sequence of Lebesgue measurable functions satisfying $u_j \to K$ in measure as $j \to \infty$, i.e: given any open neighborhood U of K in $\mathbb{R}^m$ such that:

$\lim_{j\to\infty} |\{x \in \Omega : u_j(x) \notin U\}| = 0.$

Then there exist a sequences u_k of u_j and a family ϑ_x, of positive measure on $\mathbb{R}^m$, depending measurably on x such that:

- *i)* $\| \vartheta_x \| = \int_\Omega d\vartheta_x \leq 1$, *for a.e.* $x \in \Omega$.
- *ii)* $Supp\vartheta_x \subset K$, *for a.e.* $x \in \Omega$.
- *iii)* $f(u_k) \rightharpoonup^* \varphi$ *with* $\varphi(x) = \int_{\mathbb{R}^m} f(\lambda) d\vartheta_x(\lambda)$, *in* $L^\infty(\Omega)$; *for each* f *continuous functions*

$f : \mathbb{R}^m \to \mathbb{R}$ *satisfying* $\lim_{|\lambda|\to\infty} f(\lambda) = 0$.

Theorem 1.6.2 *(Young, Tartar, Ball) [5]*

Let Ω be Lebesgue measurable of $\mathbb{R}^n$,(not necessarily bounded) and $z_j : \Omega \to \mathbb{R}^m, j \in \mathbb{N}$, be a sequence of Lebesgue measurable functions. Then there exists a subsequence z_k and a family $(\vartheta_x)_{x\in\Omega}$ of non negative Radon measure on $\mathbb{R}^n$, such that

- *i)* $\| \vartheta_x \| = \int_\Omega d\vartheta_x \leq 1$, *for a.e.* $x \in \Omega$.

- *ii)* $\varphi(z_k) \rightharpoonup^* \overline{\varphi}$ *weakly in* L^∞. *for all* $\varphi \in C_0^0(I\!R^n)$, *where* $\overline{\varphi}(x) = \langle \vartheta_x, \varphi \rangle$

- *iii) If for all* $R > 0$

$$\lim_{L\to\infty} \sup_{k\in I\!N} (|\{x \in \Omega \cap B(0,R) :| \ z_k(x) \ | \geq L\}|) = 0$$

then $\| \ \vartheta_x \ \| = 1$ *for almost every* $x \in \Omega$, *and for all measurable* $A \subset \Omega$ *there hold* $\varphi(z_k) \rightharpoonup \overline{\varphi} = \langle \vartheta_x, \varphi \rangle$ *weakly in* $L^1(A)$ *for continuous* φ *provided the sequence* $\varphi(z_k)$ *is weakly precompact in* $L^1(A)$.

Theorem 1.6.3 *(Hungerbühler)*

Let Ω, u_j *and* ϑ_x *be as in Ball's theorem. Then the following assertions are equivalents*

(i) - For all $R > 0$, $\lim_{L\to\infty} \sup_{k\in I\!N} | \ \{x \in \Omega \cap B_R : \ |u_k(x)| \geq L\} \ | = 0$, *where* $B_R = B_R(0)$.

(ii) - $\| \ \vartheta_x \ \|_M = 1$ *for a.e.* $x \in \Omega$.

(iii) - For any measurable $A \subset \Omega$ *and any continuous function* $f : I\!R^n \to I\!R$, *such that* $f(u_k)$ *is sequentially weakly relatively compact in* $L^1(A)$, *we have* $f(u_k) \rightharpoonup \langle \vartheta_x, f \rangle$ *in* $L^1(A)$.

For proof, see[38].

Remark 1.6.1 *Choosing* $K = I\!R^n$, *the assumptions of Ball's theorem are always fulfilled. Thus a family* $(\vartheta_x)_{x\in\Omega}$ *satisfying* $(i) - (iii)$ *always exists. Moreover, once the subsequence* (u_k) *of* (u_j) *is fixed,* ϑ_x *obtained by this way is unique and is a sub-probability family on* $I\!R^m$ *by* (i). *Such a family* ϑ_x *is called a Young measure on* $\Omega \times I\!R^m$. *Thus, in this sense, each sequence generates a Young measure.*

Proposition 1.6.3 *If* $|\Omega| < \infty$ *and* ϑ_x *is the Young measure generated by the (whole) sequence* (u_j), *then there holds*

$$u_j \to u \Leftrightarrow \vartheta_x = \delta_{u(x)} \quad for a.e. \ x \in \Omega$$

.

For the proof, see [38], (proposition 1.3).

Proposition 1.6.4 *Let* $|\Omega| < \infty$. *If the sequences* $u_j : \Omega \to I\!R^m$ *and* $v_j : \Omega \to I\!R^d$ *generate the Young measure* $\delta_{u(x)}$ *and* ϑ_x *respectively, then* (u_j, v_j) *generates the Young measure* $\delta_{u(x)} \otimes \vartheta_x$.

Lemma 1.6.1 *Let $u_j : \Omega \to I\!R^m$ be a sequence of measurable functions and $g : \Omega \to I\!R$ a measurable function. If a Young measure ϑ_x generate by u_j satisfies $\|\vartheta_x\| = 1$ a.e. in Ω, then $(g.u_j)$ generates a Young measure μ_x such that$\|\mu_x\| = 1$ and*

$$\langle \mu_x, id\rangle = g(x)\langle \vartheta_x, id\rangle \quad a.e.\ in\ \ \Omega.$$

In particular,

$$if\ \langle \vartheta_x, id\rangle = 0\ \ a.e.in\ \ \Omega,\ \ then\ \ \langle \mu_x, id\rangle = 0\ \ a.e.in\ \ \Omega.$$

For proof, applying the Ball's theorem and Hungerbühler theorem's.

Theorem 1.6.4 *(Kinderlehrer-Pedregal)*
Let $\vartheta_x, \quad x \in \Omega$, be a family of probability measures in $(C(M^{n\times m}))'$. Then, $(\vartheta_x)_{x\in\Omega}$ is a $W^{1,p}$ Young measure if and only if:

(i) - there is a $u \in W^{1,p}(\Omega, I\!R^n)$ such that $Du(x) = \int_{M^{n\times m}} A d\vartheta_x(A) \quad a.e.\ in\ \Omega.$

(ii) -Jensen's inequality $\phi(Du(x) \leq \int_{M^{n\times m}} \phi(A) d\vartheta_x(A) \quad holds \ \ for\ all\ \phi \in X^p quasiconvex,\ and$

(iii) -the function $\psi(x) = \int_{M^{n\times m}} |A|^p d\vartheta_x \ \ A \in L^1(\Omega)$. Here, X^p denotes the (not separable) space

$$X^p = \{\psi \in C(M^{n\times m}) : \ |\ \psi(A)\ | \leq c(1+\ |\ A\ |^p), for\ all\ A \in M^{n\times m}\}$$

See [42], (subsection 2.3).

Part I

Quasi-linear Elliptic Systems in weighted Sobolev spaces

Introduction of part I

In this part, the main point is that we not require monotonicity in the or strict monotonicity of a typical Leray-Lions operators as it is usually assumed in previous papers. The aims of this text is to prove analogous existence results under relaxed monotonicity, in particular under strict quasi-monotonicity. The main technical tool we advocate and use Throughout the proof are Young measures. By applying a Galerkin schema, we obtain easily an approximating sequence u_k. The Ball theorem [5] and especially the resulting tools mode available by Hungerbühler to partial differential equation theory give then a sufficient control on the gradient approximating sequence Du_k to pass to the limit.This method is used by Dolzmann [39] , Muller [32] and Mainly by Hungurbühler to get the existence of a weak solution for the quasi-linear elliptic system [36]. This part can be seen as generalization of Hungurbühler and as a continuation of Y- Akdim [2].

Chapter 2

Quasi-linear degenerated elliptic system in divergence form with mild monotonicity in weighted sobolev spaces

Abstract

We consider, for a bounded open domain Ω in $\mathbb{R}^n$ and a function u: $\Omega \to \mathbb{R}^m$, the quasi-linear elliptic system

$$\text{(QES)} \begin{cases} -\text{div}\sigma\left(x, u\left(x\right), Du\left(x\right)\right) &= f(x) \text{ in } \Omega \\ u &= 0 \quad \text{ on } \partial\Omega, \end{cases} \tag{2.0.1}$$

where f belongs to the dual space $W^{-1,p'}\left(\Omega, \omega^*, \mathbb{R}^m\right)$ of $W_0^{1,p}\left(\Omega, \omega, \mathbb{R}^m\right)$. We prove existence of a regularity, growth and coercivity conditions for σ, but with only very mild monotonicity.

2.1 Introduction.

Throughout all this work, Ω will denotes a bounded open domain in $\mathbb{R}^n$. This work is devoted to establish existence results for the Dirichlet problem in divergence form of type(2.0.1) below. We study this quasi-linear elliptic system in a weighted sobolev space and with mild monotonicity assumptions on σ and

various hypotheses of monotonicity. We consider the quasi-linear elliptic system with $W^{-1,p'}-$data:

$$(\text{QES}) \begin{cases} -\text{div}\sigma\left(x, u\left(x\right), Du\left(x\right)\right) &= f(x) \text{ in } \Omega \\ u &= 0 \quad \text{ on } \partial\Omega, \end{cases} \tag{2.1.1}$$

where f belongs to the dual space $W^{-1,p'}\left(\Omega, \omega^*, I\!R^m\right)$ of $W_0^{1,p}\left(\Omega, \omega, I\!R^m\right)$ $\left(\frac{1}{p}+\frac{1}{p'}=1,\ p>1\right)$ and $\omega = \{\omega_{ij} \mid 0 \leq i \leq n,\ \ 1 \leq j \leq m\}$ is a family of weight functions defined on Ω with $\omega_{ij}(x) > 0$ for almost every $x \in \Omega$ and $\omega^* = \{\omega_{ij}^* = \omega_{ij}^{1-p'},\ \ 0 \leq i \leq n,\ 1 \leq j \leq m\}$, $\sigma = (\sigma_{rs})_{1\leq s\leq n,\ 1\leq r\leq m}$ and which satisfies some hypotheses (see below).

The Jacobian matrix of a function $u : \Omega \longrightarrow I\!R^m$ is denoted by

$$Du(x) = (D_1u(x), D_2u(x), ..., D_nu(x)) \ \text{ with } D_i = \partial/\partial(x_i).$$

The first goal of this chapter is that we treat a class of problems for which the classical monotone operator methods developed by Visik [61], Brezis [15], Browder [16], Lions [46], Minty [49] and others do not apply. The reason for this is that σ does not need to satisfy the strict monotonicity condition of typical Leray-Lions operator. The tool we use in order to prove the needed compactness of approximating solutions is Young measures. The second goal of this chapter is to treat the degenerated or singular case. To do this, we replace the classical Sobolev spaces $W^{1,p}(\Omega, I\!R^m)$ by a general setting of weighted Sobolev spaces $W^{1,p}(\Omega, \omega, I\!R^m)$. So the existence results are proved in this general setting. This chapter can be seen as a generalization of Quasi-linear elliptic systems with $W^{-1,p'}$-data, by N. Hungerbühler [36], [34] and as a continuation of Y. Akdim [2].

Let $\omega = \{\omega_{ij} \mid \ \ 0 \leq i \leq n,\ 1 \leq j \leq m\}$, and $\overline{\omega_0} = (\omega_0 j)$ for all $1 \leq j \leq m$ the weight functions system defined in Ω satisfying the following integrability conditions:

$$\omega_{ij} \in L^1_{loc}(\Omega), \quad \omega_{ij}^{\frac{-1}{p-1}} \in L^1_{loc}(\Omega), \text{ for some } p \in]1, \infty[\text{ and } \exists s > Max(\frac{n}{p}, \frac{1}{p-1}) \text{ such that } \omega_{ij}^{-s} \in L^1(\Omega). \tag{2.1.2}$$

The space $W^{1,p}(\Omega, \omega, I\!R^m)$ is the set of functions

$$\left\{u = u(x) \mid u \in L^p(\Omega, \overline{\omega_0}, I\!R^m),\ D_{ij}u = \frac{\partial u^i}{\partial x_j} \in L^p(\Omega, \omega_{ij}, I\!R^m),\ 1 \leq i \leq n,\ 1 \leq j \leq m\}\right\}$$

which

$$L^p(\Omega, \omega_{ij}, I\!R^m) = \left\{u = u(x) \mid \ \mid u \mid {\omega_{ij}}^{\frac{1}{p}} \in L^p(\Omega, I\!R^m)\right\}.$$

The weighted space $W^{1,p}(\Omega, \omega, I\!R^m)$ can be equipped by the norm

$$\|u\|_{1,p,\omega} = \left(\sum_{j=1}^{m}\int_\Omega |u_j|^p\omega_{0j}dx + \sum_{1\leq i\leq n, 1\leq j\leq m}\int_\Omega |D_{ij}u|^p\omega_{ij}dx\right)^{\frac{1}{p}}.$$

The norm $\|.\|_{1,\omega,p}$ is equivalent to the norm $|||\ .\ |||$ on $W_0^{1,p}(\Omega,\omega,I\!R^m)$, defined by

$$|||\ u\ |||=(\sum_{1\leq i\leq n,1\leq j\leq m}\int_\Omega |D_{ij}u|^p\omega_{ij}dx)^{\frac{1}{p}}.$$

Proposition 2.1.1 *The weighted Sobolev space $W^{1,p}(\Omega,\omega,I\!R^m)$ is a Banach space, separable and reflexive. The weighted Sobolev space $W_0^{1,p}(\Omega,\omega,I\!R^m)$ is the closure of $C_0^\infty(\Omega,I\!R^m)$ in $W^{1,p}(\Omega,\omega,I\!R^m)$ equipped by the norm $\|.\|_{1,p,\omega}$.*

Proof: see [25] (with a slight modification).

Definition 2.1.1 *[31] A Young measure $(\vartheta_x)_{x\in\Omega}$ is called $W^{1,p}$-gradient young measures $(1\leq p<\infty)$ if it is associated to a sequence of gradients Du_k such that u_k is bounded in $W^{1,p}(\Omega)$.*

Definition 2.1.2 *The $W^{1,p}$-gradient young measures $(\vartheta_x)_{x\in\Omega}$ is called homogeneous, if it does not depend on x, i.e, if $\vartheta_x=\vartheta$ for a.e. $x\in\Omega$.*

Definition 2.1.3 *Let $u:\Omega\to I\!R^m$ be a measurable function. We say that u is approximately differentiable at $x\in\Omega$ if there exists a matrix $F_x\in I\!M^{m\times n}$ such that for all $\varepsilon>0$ we have*

$$\lim_{r\to 0}\frac{1}{r^n}|y\in B_r(x):|\ u(y)-u(x)-F_x(y-x)\ |>r\varepsilon|=0.$$

In this case, F_x is unique and write $apDu(x)=F_x$.

Hypothesis

(H_0) (Hardy-Type inequalities):

There exist some constant $c>0$, some weight function γ and some real $q\ \ (1<q<\infty)$ such that

$$\left(\sum_{j=1}^m\int_\Omega |u_j(x)|^q\gamma_j(x)dx\right)^{\frac{1}{q}}\leq c\left(\sum_{1\leq i\leq n;\ 1\leq j\leq m}\int_\Omega |D_{ij}u|^p\omega_{ij}\right)^{\frac{1}{p}}$$

for all $u\in W_0^{1,p}(\Omega,\omega,I\!R^m)$, with $\gamma=\{\gamma_j\mid 1\leq j\leq m\}$.

The injection $W_0^{1,p}(\Omega,\omega,I\!R^m)\hookrightarrow\hookrightarrow L^q(\Omega,\gamma,I\!R^m)$ is compact, and

$W_0^{1,p}(\Omega,\omega,I\!R^m)\hookrightarrow\hookrightarrow L^r(\Omega,I\!R^m)$ is compact, with

$$\begin{cases} 1\leq r<\frac{nps}{n(s+1)-ps} & \text{if}\quad ps<n(s+1)\\ r\geq 1 & \text{if}\quad n(s+1)\leq ps\end{cases}$$

(H_1) Continuity:

$\sigma : \Omega \times I\!R^m \times I\!M^{m\times n} \longrightarrow I\!M^{m\times n}$ is a Carathéodory function

(i.e: $x \longmapsto \sigma(x,u,F)$ is measurable for every $(u,F) \in I\!R^m \times I\!M^{m\times n}$ and $(u,F) \longmapsto \sigma(x,u,F)$ is continuous for almost every $x \in \Omega$).

(H_2) Growth and coercivity conditions:

There exist $c_1 \geq 0,\ \ c_2 > 0,\ \ \lambda_1 \in L^{p'}(\Omega),\ \ \lambda_2 \in L^1(\Omega),\ \lambda_3 \in L^{(p/\alpha)'}(\Omega),\ \ 0 < \alpha < p,\ \ 1 < q < \infty$ and $\beta > 0$ such that for all $1 \leq r \leq n,\ \ 1 \leq s \leq m$, we have:

$$|\sigma_{rs}(x,u,F)| \leq \beta w_{rs}^{1/p}[\lambda_1(x) + c_1 \sum_{j=1}^{m} |\gamma_j|^{1/p'} \cdot |u_j|^{q/p'} + c_1 \sum_{1\leq i\leq n;\ 1\leq j\leq m} \omega_{ij}^{1/p'} |D_{ij}u|^{p-1}] \qquad (2.1.3)$$

and

$$\sigma(x,u,F) : F \geq -\lambda_2(x) - \sum_{j=1}^{m} \omega_{0j}(x)^{\alpha/p} \lambda_3(x) |u_j|^{\alpha} + c_2 \sum_{1\leq i\leq n;\ 1\leq j\leq m} \omega_{ij}(x) \cdot |F_{ij}|^p \qquad (2.1.4)$$

(H_3) Monotonicity conditions: σ satisfies one of the following conditions:

a) For all $x \in \Omega$, and all $u \in I\!R^m$, the map $F \longmapsto \sigma(x,u,F)$ is a C^1-function and is monotone i.e, $(\sigma(x,u,F) - \sigma(x,u,G)) : (F-G) \geq 0$, for all $x \in \Omega,$ all $u \in I\!R^m$ and all $F, G \in I\!M^{m\times n}$).

b) There exists a function $W : \Omega \times I\!R^m \times I\!M^{m\times n} \longrightarrow I\!M^{m\times n}$ such that $\sigma(x,u,F) = \dfrac{\partial W}{\partial F}(x,u,F)$ and $F \longmapsto W(x,u,F)$ is convex and C^1 function.

c) For all $x \in \Omega$, and for all $u \in I\!R^m$ the map $F \longmapsto \sigma(x,u,F)$ is strictly monotone i.e, $\sigma(x,u,.)$ is monotone and, $(\sigma(x,u,F) - \sigma(x,u,G)) : (F-G) = 0$ implies that $F = G$).

d) $\sigma(x,u,F)$ is strictly p-quasi-monotone in $F,$ i.e:

$$\int_{I\!M^{m\times n}} (\sigma(x,u,\lambda) - \sigma(x,u,\overline{\lambda})) : (\lambda - \overline{\lambda})) d\vartheta(\lambda) > 0,$$

for all homogeneous $W^{1,p}$-gradient young measures ϑ with center of mass $\overline{\lambda} = \langle \vartheta, id \rangle$ which are not a single Dirac mass.

The main point is that we do not require strict monotonicity or monotonicity in the variables (u,F) in (H_3) as it is usually assumed in previous work see ([41] or [43]).

Our aim of this chapter is to prove the existence of the solution of the problem (QES) in the space $W_0^{1,p}(\Omega,\omega,I\!R^m)$, when the second member f is lying in $W_0^{-1,p'}(\Omega,\omega^*,I\!R^m)$.

Theorem 2.1.1 *If $p \in (1,\infty)$ and σ satisfies the conditions $(H_1)-(H_3)$, then the Dirichlet problem (QES) has a weak solution $u \in W_0^{1,p}(\Omega,\omega,I\!R^m)$, for every $f \in W^{-1,p'}(\Omega,\omega^*,I\!R^m)$.*

Lemma 2.1.1 *For arbitrary $u \in W_0^{1,p}(\Omega,\omega,I\!R^m)$ and $f \in W^{-1,p'}(\Omega,\omega^*,I\!R^m)$, the functional*

$$\begin{array}{rcl} F(u): W_0^{1,p}(\Omega,\omega,I\!R^m) & \longrightarrow & I\!R \\ v & \longmapsto & \displaystyle\int_\Omega \sigma(x,u(x),Du(x)) : Dv(x)dx - \langle f,v\rangle, \end{array}$$

is well defined, linear and bounded.

Proof For all $u,v \in W_0^{1,p}(\Omega,\omega,I\!R^m)$, we denote

$$F(u)(v) = I_1 - I_2,$$

with

$$I_1 = \int_\Omega \sigma(x,u(x),Du(x)) : Dv(x)dx,$$

and

$$I_2 = \langle f,v\rangle.$$

We define

$$I_{rs} = \int_\Omega \sigma_{rs}(x,u(x),Du(x)) : D_{rs}v(x)dx.$$

Firstly, by virtue of the growth conditions (H_2) and the Hölder inequality, one has

$$\begin{array}{rcl} |I_{rs}| & \le & \displaystyle\int_\Omega |\sigma_{rs}(x,u(x),Du(x))| : |D_{rs}v(x)|dx \\ & \le & \displaystyle\int_\Omega \beta\omega_{rs}^{1/p}(x)[\lambda_1(x) + c_1\sum_{j=1}^m |\gamma_j(x)|^{1/p'}|u(x)|^{q/p'} \\ & + & \displaystyle c_1 \sum_{1\le i\le n;\ 1\le j\le m} \omega_{ij}^{1/p'}|D_{ij}|^{p-1}]|D_{rs}v|dx \\ & \le & \displaystyle\beta\Big[(\int_\Omega |\lambda_1(x)|^{p'}dx)^{1/p'}(\int_\Omega |D_{rs}v(x)|^p\omega_{rs}dx)^{1/p} \\ & + & \displaystyle c_1(\int_\Omega |D_{rs}v(x)|^p\omega_{rs})^{1/p}(\sum_{j=1}^m \int_\Omega |u_j|^q\gamma_j dx)^{1/p'} \\ & + & \displaystyle c_1(\sum_{1\le i\le n;\ 1\le j\le m}\int_\Omega |D_{ij}u|^p\omega_{ij}dx)^{1/p'}(\int_\Omega |D_{rs}v|^p w_{rs}dx)^{1/p}\Big] \end{array}$$

with $p = p'(p-1)$, and thanks to the Hardy-Type inequalities (H_0) we have:

$$\begin{array}{rcl} |I_{rs}| & \le & \displaystyle c\beta\Big[\|\lambda_1\|_{p'}\|v\|_{1,p,\omega_{rs}} + c_1\|Dv\|_{p,w_{rs}}(\int_\Omega |u|^q\gamma dx)^{1/p'} \\ & & \displaystyle +c_1\sum_{ij}\|Dv\|_{p,\omega_{ij}}\|Du\|_{p,\omega_{rs}}\Big] \\ & \le & \displaystyle c'\beta[\|\lambda_1\|_{p'}\|v\|_{1,p,\omega_{rs}} + \|v\|_{1,p,\omega_{rs}}\|u\|_{q,\gamma} + \|u\|_{1,p,\omega}\|v\|_{1,p,\omega_{rs}} \end{array}$$

with $c' = \max(c, 1)$, we have:

$$|I_1| \leq c'\beta[\|\lambda_1\|_{p'} + \|u\|_{1,p,w}^{q/p'} + \|u\|_{1,p,w}]\|v\|_{1,p,\omega} < \infty$$

and

$$|I_2| \leq \int_\Omega |f||v|dx \leq \|f\|_{-1,p',\omega*} \|v\|_{1,p,\omega} < \infty.$$

Finally the functional $F(\cdot)$ is bounded. □

Lemma 2.1.2 *The restriction of F to a finite dimensional linear subspace V of $W_0^{1,p}(\Omega, \omega, I\!R^m)$ is continuous.*

Proof Let d be the dimension of V and $(e_1, e_2, ..., e_d)$ a basis of V. Let $u_j = \sum_{1\leq i\leq d} a_j^i e_i$ be a sequence in V which converges to $u = \sum_{1\leq i\leq d} a^i e_i$ in V. The sequence (a_j) converge to $a \in I\!R^d$, so $u_j \longrightarrow u$ and $Du_j \longrightarrow Du$ a.e. On the other hand $\|u_j\|_{1,p,\omega}$ and $\|Du_j\|_{p,\omega}$ are bounded by a constant c. Thus, it follows by the continuity conditions (H_1), that

$$\sigma(x, u_j, Du_j) : Dv \longrightarrow \sigma(x, u, Du) : Dv$$

for all $v \in W_0^{1,p}(\Omega, \omega, I\!R^m)$ and a.e. in Ω. Let Ω' be a measurable subset of Ω and let $v \in W_0^{1,p}(\Omega, \omega, I\!R^m)$. Thanks to the condition (H_2), we get

$$\int_{\Omega'} |\sigma(x, u_j, Du_j) : Dv|dx < \infty,$$

which implies that $\sigma(x, u_j, Du_j) : Dv$ is equi-integrable. And by applying the Vitali's theorem, it follows that

$$\int_\Omega \sigma(x, u_j, Du_j) : Dvdx \longrightarrow \int_\Omega \sigma(x, u, Du) : Dvdx$$

for all $v \in W_0^{1,p}(\Omega, \omega, I\!R^m)$.

Finally

$$\lim_{j\to\infty} \langle F(u_j), v\rangle = \langle F(u), v\rangle$$

which means that

$$F(u_j) \longrightarrow F(u) \text{ in } W^{-1,p'}(\Omega, \omega^*, I\!R^m).$$

□

2.2 Galerkin approximation

Let $V_1 \subset V_2 \subset ..... \subset W_0^{1,p}(\Omega, \omega, I\!R^m)$ be a sequence of finite dimensional subspaces with $\bigcup_{k\in I\!N} V_k$ dense in $W_0^{1,p}(\Omega, \omega, I\!R^m)$. Let us fix some k, We assume that V_k has dimension d and that $(e_1, e_2, ..., e_d)$ is a basis of V_k. Then, we define the map

$$\begin{array}{rccl} G: & I\!R^d & \longrightarrow & I\!R^d \\ & (a_1, ..., a_d)^t & \longmapsto & \left(\left\langle F(\sum_{i=1}^d a_i e_i), e_1 \right\rangle, ..., \left\langle F(\sum_{i=1}^d a_i e_i), e_d \right\rangle \right)^t. \end{array}$$

Proposition 2.2.1 *The map G is continuous and $G(a)\cdot a$ tends to infinity when $\|a\|_{I\!R^k}$ tends to infinity.*

Proof Since F restricted to V_k is continuous by Lemma 2.1.2, so G is continuous.
Let $a = (a_1, ..., a_d)^t$ and $u = \sum_{1\le i\le d} a^i e_i$, Then $G(a)\cdot a = \langle F(u), u\rangle$ and the fact that $\|a\|_{I\!R^d}$ tends to infinity is equivalent to the fact that $\|u\|_{1,p,\omega}$ tends to infinity. In fact, we have:

$$G(a)\cdot a = \sum_{1\le i\le d} \langle F(u), a^i e_i\rangle = \langle F(u), u\rangle$$

and

$$\begin{aligned} \|u\|_{1,p,\omega}^p = \left\| \sum_{1\le i\le d} a^i e_i \right\|_{1,p,\omega}^p &\le \left(\sum_{1\le i\le d} |a^i| \cdot \|e_i\|_{1,p,\omega} \right)^p \\ &\le \max_{1\le i\le d} \left(\|e_i\|_{1,p,\omega}^p \right) \cdot \left(\sum_{1\le i\le d} |a^i| \right)^p \\ &\le c\cdot \|a\|_{I\!R^d}, \end{aligned}$$

which implies that $\|a\|_{I\!R^d}$ tends to infinity if $\|u\|_{1,p,\omega}$ tends to infinity. Now, it suffices to prove that

$$\langle F(u), u\rangle \to \infty \quad \text{when} \quad \|u\|_{1,p,\omega} \to \infty.$$

Indeed, thanks to the first coercivity condition and the Hölder inequality, we obtain

$$I = \int_\Omega \sigma(x,u,Du) : Du dx \ge -\|\lambda_2\|_1 - \int_\Omega \lambda_3 \omega_{0j}^{\alpha/p} |u_j|^\alpha dx + c_2 \sum_{1\le i<n,\ 1<j\le m} \int_\Omega |D_{ij}u|^p \omega_{ij} dx.$$

By the Hölder inequality, we have

$$\begin{aligned} \int_\Omega \lambda_3 |u_j|^\alpha \omega_{0j}^{\alpha/p} dx &\le \|\lambda_3\|_{(p/\alpha)'} \left(\int_\Omega \omega_{0j} |u_j|^{(p/\alpha)\cdot\alpha} dx \right)^{\alpha/p} \\ &\le c' \|\lambda_3\|_{(p/\alpha)'} \|u_j\|_{1,p,\omega_{0j}}, \end{aligned}$$

where c' is a constant positive. For $\|u\|_{1,p,\omega}$ large enough, we can write

$$\begin{aligned} I &\geq -\|\lambda_2\|_1 - c' \|\lambda_3\|_{(p/\alpha)'} \cdot \|u_j\|^{\alpha}_{1,p,\omega_{0j}} + c_2 \cdot \textstyle\sum_{1\leq i\leq n, 1\leq j\leq m} \|Du_j\|^{p}_{1,p,\omega_{ij}} \\ &\geq -\|\lambda_2\|_1 - c' \|\lambda_3\|_{(p/\alpha)'} \cdot \|u\|^{\alpha}_{1,p,\omega} + c_2 c' \cdot \|u\|^{p}_{1,p,\omega} . \end{aligned}$$

And since

$$I' = |\langle f, u\rangle| \leq \|f\|_{-1,p',\omega^*} \cdot \|u\|_{1,p,\omega} ,$$

with $0 < \alpha < p$ and $p > 1$, we get

$$\langle F(u), u\rangle = I - I' \geq c_2 c' \cdot \|u\|^{p}_{1,p,\omega} + \|f\|_{-1,p',\omega^*} \|u\|_{1,p,\omega} - c' \|\lambda_3\|_{(p/\alpha)'} \|u\|^{\alpha}_{1,p,\omega} - \|\lambda_2\|_1 . \qquad (2.2.1)$$

Consequently, by using (2.2.1), we deduce

$$\langle F(u), u\rangle = I - I' \longrightarrow \infty \text{ as } \|u\|_{1,p,\omega} \longrightarrow \infty.$$

The properties of G allows us to construct our Galerkin approximations.

Corollary 2.2.1 *For all $k \in \mathbb{N}$, there exists $(u_k) \subset V_k$ such that $\langle F(u_k), v\rangle = 0$, for all $v \in V_k$.*

Proof By the proposition 2.2.1, there exists $R > 0$, such that for all
$a \in \partial B_R(0) \subset \mathbb{R}^d$, we have $G(a) \cdot a > 0$. And the usual topological argument see e.g [46] or [49], implies that $G(x) = 0$ has a solution $x \in B_R(0)$. Hence, for all $k \in \mathbb{N}$, there exists $(u_k) \subset V_k$, such that

$$\langle (Fu_k), v\rangle = 0, \text{ for all } v \in V_k.$$

Taking $(u_k = \Sigma_{i=1}^{d} x_k^i e_i)$ □

Proposition 2.2.2 *The Galerkin approximations sequence constructed in Corollary 2.2.1 is uniformly bounded in $W_0^{1,p}(\Omega, \omega, \mathbb{R}^m)$; i.e:*

there exists a constant $R > 0$, such that $\|u_k\|_{1,p,\omega} \leq R$, for all $k \in \mathbb{N}$.

Proof As in the proof of Proposition 2.2.1, we can see that $\langle F(u), u\rangle \longrightarrow \infty$ as $\|u\|_{1,p,\omega} \longrightarrow \infty$. Then, there exists R satisfying $\langle F(u), u\rangle > 1$ when $\|u\|_{1,p,\omega} > R$. Now, for the sequence of Galerkin approximations $(u_k) \subset V_k$ of Corollary 2.2.1, which satisfy $\langle F(u_k), u_k\rangle = 0$, we have the uniform bound $\|u_k\|_{1,p,\omega} \leq R$ for all $k \in \mathbb{N}$. And there exists a subsequence (u_k) of the sequence $(u_k) \subset V_k$, such that:

$$u_k \rightharpoonup u \text{ in } W_0^{1,p}(\Omega, \omega, \mathbb{R}^m) \text{ and } u_k \longrightarrow u \text{ in measure in } L^r(\Omega, \mathbb{R}^m).$$

The gradient sequence (Du_k) generates the Young measure ϑ_x. Since $u_k \longrightarrow u$ in measure,We have that (u_k, Du_k) generates the Young measure $(\delta_{u(x)} \otimes \vartheta_x)$, see e.g [5]. Moreover, for almost all x in Ω, we have,

(i) ϑ_x is a probability measure, i.e, $\|\vartheta_x\|_{mes} = 1$.

(ii) ϑ_x is a homogeneous $W^{1,p}$ - gradient Young measure.

(iii) $\langle \vartheta_x, id \rangle = Du(x)$, see e.g [16].

Proof. See [39] and [21].

2.3 Passage to the limit in (QES)

Now, we are in a position to prove our main result under convenient hypotheses.
Let

$$I_k = (\sigma(x, u_k, Du_k) - \sigma(x, u, Du)) : (Du_k - Du). \tag{2.3.1}$$

Lemma 2.3.1 *(Type-Fatou lemma)(See [39]) Let $F : \Omega \times \mathbb{R}^m \times \mathbb{M}^{m\times n} \longrightarrow \mathbb{R}$ be a Carathéodory function, and $u_k : \Omega \longrightarrow \mathbb{R}^m$ a measurable sequence, such that (Du_k) generates the Young measure ϑ_x, with $\|\vartheta_x\|_{mes} = 1$, for a.e. $x \in \Omega$. Then*

$$\liminf_{k\to\infty} \int_\Omega F(x, u_k, Du_k)dx \geq \int_\Omega \int_{\mathbb{M}^{m\times n}} F(x, u, \zeta)d\vartheta_x(\zeta)dx, \tag{2.3.2}$$

provided that the negative part of $F(x, u_k, Du_k)$ is equi-integrable.

Lemma 2.3.2 *The sequence $(I_k)_k$ is equi-integrable.*

Proof We have

$$\begin{aligned} I_k &= (\sigma(x, u_k, Du_k) - \sigma(x, u, Du)) : (Du_k - Du) \\ &= [\sigma(x, u_k, Du_k) : Du_k] - [\sigma(x, u_k, Du_k) : Du] - [\sigma(x, u, Du) : Du_k] + [\sigma(x, u, Du) : Du] \\ &= I_k^1 + I_k^2 + I_k^3 + I_k^4. \end{aligned} \tag{2.3.3}$$

We denote $(I_k^1)^- = -[\sigma(x, u_k, Du_k : Du_k]^-$. Thanks to the coercivity condition (H_2), we have

$$\begin{aligned} \int_{\Omega'} |(I_k^1)^-|dx &\leq \int_\Omega |\lambda_2| + c_2 \sum_{1\leq j\leq m} \omega_{0j}^{\frac{\alpha}{p}} |\lambda_3| \cdot |u_{kj}|^\alpha + c \sum_{1\leq i<n,\ 1<j\leq m} \omega_{ij} |D_{ij}u_k|^p dx \\ &\leq \|\lambda_2\|_1 + \int_{\Omega'} (\sum_{1\leq j\leq m} \omega_{0j}^{\alpha/p} |u_{kj}|^\alpha)^{p/\alpha} \|\lambda_3\|_{(p/\alpha)'} + c_2 \|u_k\|_{1,\omega,p}^p \end{aligned} \tag{2.3.4}$$

which $p/\alpha \geq 1$. Therefore,

$$\begin{aligned}\int_{\Omega'} |(I_k^1)^-|dx &\leq \|\lambda_2\|_1 + \Big(\sum_{1\leq j\leq m} \omega_{0j}|u_{kj}|^p\Big)^{\alpha/p} \|\lambda_3\|_{(p/\alpha)'} + c_2\,\|u_k\|^p_{1,\omega,p} \\ &\leq \|\lambda_2\|_1 + \|u_k\|^\alpha_{p,\overline{\omega}_{00}} \|\lambda_3\|_{(p/\alpha)'} + c_2\,\|u_k\|^p_{1,\omega,p} \\ &< \infty\end{aligned}$$

for all $\Omega' \subset \Omega$.

Similarly for $(|(I_k^4)^-|$.

Now, by using the growth condition (H_2) and the Hardy-Type inequalities (H_0), we have

$$\begin{aligned}\int_{\Omega'} |(I_k^2)^-|dx &= \int_{\Omega'} |\sigma(x,u_k,Du_k) : Du_k|dx \\ &\leq \beta \int_{\Omega'} \omega_{rs}^{1/p}\lambda_1(x) D_{rs}u_k dx \\ &+ c_1\beta \sum_{1\leq j\leq m} \int_{\Omega'} \gamma_j^{1/p'} |u_{kj}|^{q/p'} D_{rs}u_k dx \\ &+ c_1\beta \sum_{1\leq i<n,\ 1<j\leq m} \int_{\Omega'} \omega_{ij}^{1/p'} |D_{ij}u_k|^{p-1} D_{rs}u_k dx.\end{aligned} \tag{2.3.5}$$

Thus, by the Hölder inequality, we obtain

$$\begin{aligned}\int_{\Omega'} |(I_k^2)^-|dx &\leq \beta\Bigg[\|\lambda_1\|_{p'} \left(\int_{\Omega'} |D_{rs}u_k|^p \omega_{rs} dx\right)^{1/p} \\ &+ c_1 \left(\int_{\Omega'} |D_{rs}u_k|^p \omega_{rs} dx\right)^{1/p} \left(\int_{\Omega'} \left(\sum_{1\leq j\leq m} \gamma_j^{1/p'} |u_{kj}|^{q/p'}\right)^{p'} dx\right)^{1/p'} \\ &+ c_1 \left(\sum_{1\leq i<n\ 1<j\leq m} \int_{\Omega'} \Big(|D_{ij}u_k(x)|^{p'(p-1)} \omega_{ij} dx\Big)^{1/p'}\right) \\ &\times \left(\int_{\Omega'} |D_{rs}u_k|^p \omega_{rs} dx\right)^{1/p}\Bigg].\end{aligned} \tag{2.3.6}$$

So, by combining $(2.3.5)$ and $(2.3.9)$, we deduce that

$$\int_{\Omega'} |\sigma(x,u_k,Du_k) : Du_k|dx \leq c'\beta\,(\|\lambda_1\|_{p'}\|\|u_k\|_{1,p,\omega} + \|u_k\|_{1,p,\omega}) < \infty. \tag{2.3.7}$$

Similarly to $(|(I_k^2)^-|$, we obtain $(|(I_k^3)^-|$. Finally: I_k is equi-integrable.

Now, we consider $(I_k)' = (\sigma(x,u_k,Du_k) : (Du_k - Du)$. We have that I_k' is equi-integrable because I_k is.

So, we define

$$X = \liminf_k \int_\Omega I_k dx = \liminf_k \int_\Omega (I_k)' dx \geq \int_\Omega \int_{\mathbb{M}^{m\times n}} (\sigma(x,u,\lambda) : (\lambda - Du) d\vartheta_x(\lambda).$$

So to prove $(2.3.2)$, it suffices to prove that

$$X \leq 0. \tag{2.3.8}$$

Let $\varepsilon > 0$, so there exists $k_0 \in I\!N$ such that, for all $k > k_0$, we have

dist$(u, V_k) < \varepsilon$ since $\liminf_{v_k \in V_k} \|u - v_k\|_{1,p,\omega} < \varepsilon$, $(u_k \rightharpoonup u)$.

Or in an equivalent manner dist$(u_k - u, V_k) < \varepsilon, \forall k > k_0$ Then for all $v_k \in V_k$ we have

$$\begin{aligned} X &= \liminf_{k\to\infty} \int_\Omega \sigma(x, u_k, Du_k) : (Du_k - Du)dx \\ &= \liminf_{k\to\infty} \left[\int_\Omega \sigma(x, u_k, Du_k) : D(u_k - u - v_k)dx + \int_\Omega \sigma(x, u_k, Du_k) : D(v_k)dx \right] \end{aligned}$$

Combining (H_2) and $(2.1.1)$, we get

$$\begin{aligned} X &\leq \lim\inf_{k\to\infty} \Big(\int_\Omega \beta\omega_{rs}^{1/p} [\lambda_1 + c_1 \sum_{1\leq j\leq m} \gamma_j^{1/p'} |u_{k_j}|^{q/p'} + c_1 \sum_{i=1,j=1}^{n,m} \omega_{ij}^{1/p'} |D_{ij}u_k|^{p-1}] |D_{rs}(u_k - u - v_k)|dx \Big) \\ &+ \langle f, v_k \rangle. \end{aligned}$$

For all $\varepsilon > 0$, we choice $v_k \in V_k$ such that

$$\|u_k - u - v_k\|_{1,p,\omega} \leq 2\varepsilon,$$

for all $k \geq k_0$. Which implies that

$$|\langle f, v_k \rangle| \leq |\langle f, v_k + (u - u_k) \rangle| + |\langle f, u_k - u \rangle| \leq 2\varepsilon \|f\|_{-1,p',\omega^*} + o(k).$$

Hence $\lim_{k\to\infty} \langle f, u_k - v \rangle = 0$. According to the Hölder and Hardy-Type inequalities, and by lemma $(2.3.1)$ we deduce that

$$\begin{aligned} X &\leq \liminf_{k\to\infty} c\beta \Bigg(\Big\| \lambda_1 \|_{p'} \Big(\int_\Omega |D_{rs}(u_k - u - v_k)|^p \omega_{rs} dx \Big)^{1/p} \\ &\quad + c_1 \left(\int_\Omega |u_k|^q \gamma \right)^{1/p'} \left(\int_\Omega |D_{rs}(u_k - u - v_k)|^p \omega_{rs} dx \right)^{1/p} \\ &\quad + c_1 \left(\sum \int_\Omega \omega_{ij} |D_{ij}u|^{p'(p-1)} \right)^{1/p'} \left(\int_\Omega \omega_{rs} |D_{rs}(u_k - u - v_k)|^p \right)^{1/p} \Bigg) + |\langle f, v_k \rangle| \\ &\leq \liminf_{k\to\infty} c \left(\|\lambda_1\|_{p'} \|u_k - u - v_k\|_{1,p,\omega} \right) + \|u_k\|_{1,p,\omega}^q \|u_k - u - v_k\|_{1,p,\omega} + 2\varepsilon \|f\|_{-1,p',\omega^*} + o(k). \end{aligned}$$

Therefore,

$$X \leq 2\varepsilon c\beta \left(\|\lambda_1\|_{p'} + \|u\|_{1,p,\omega}^q + \|f\|_{-1,p',\omega*} \right).$$

This proves that $X \leq 0$, and finally

$$\int_\Omega \int_{I\!M^{mn}} \sigma(x, u, \lambda) : \lambda d\vartheta_x dx \leq \int_\Omega \int_{I\!M^{mn}} \sigma(x, u, \lambda) : Du d\vartheta_x(\lambda) dx. \tag{2.3.9}$$

Proof of Theorem 2.1.1 We will start with the easiest case

$$(d) \qquad F \longmapsto \sigma(x,u,F) \text{ is stictly } p\text{-quasi-monotone.} \tag{2.3.10}$$

Indeed, we assume that ϑ_x is not a Dirac mass on the set M with $x \in M$ of positive Lebesgue measure $|M| > 0$. Moreover, by the strict p-quasi-monotonicity of $\sigma(x,u,\cdot)$ and ϑ_x is a homogeneous $W^{1,p}$-gradient Young measure for a.e. $x \in M$. So, for a.e. $x \in M$, with $\bar{\lambda} = \langle \vartheta_x, Id\rangle = apDu(x))$, Where $apDu(x))$ is the differentiable approximation in x. We get

$$\begin{aligned}
\int_{\mathbb{M}^{m\times n}} \sigma(x,u,\lambda) : (\lambda - Du) d\vartheta_x(\lambda) &> \int_{\mathbb{M}^{m\times n}} \sigma(x,u,Du) : (\lambda - Du) d\vartheta_x(\lambda) \\
&> \sigma(x,u,Du) : \int_{\mathbb{M}^{m\times n}} \lambda d\vartheta_x(\lambda) \\
&- \sigma(x,u,Du) : Du \int_{\mathbb{M}^{m\times n}} d\vartheta_x(\lambda) \\
&> (\sigma(x,u,Du) : Du - \sigma(x,u,Du) : Du) = 0 \\
&> 0.
\end{aligned}$$

On the other hand by $(2.3.9)$, integrating over Ω, and using the div-cul inequality(See [36]) we have:

$$\begin{aligned}
\int_\Omega \int_{\mathbb{M}^{m\times n}} \sigma(x,u,\lambda) : \lambda d\vartheta_x(\lambda) dx &> \int_\Omega \int_{\mathbb{M}^{m\times n}} \sigma(x,u,\lambda) : Du d\vartheta_x(\lambda) dx \\
&\geq \int_\Omega \int_{\mathbb{M}^{m\times n}} \sigma(x,u,\lambda) : \lambda d\vartheta_x(\lambda) dx.
\end{aligned}$$

This contradicts $(2.3.9)$. Thus $\vartheta_x = \delta_{\bar{\lambda}} = \delta_{Du(x)}$ for a.e. $x \in \Omega$. Therefore, $Du_k \longrightarrow Du$ in measure when k tends to infinity. And thus, $\sigma(x,u_k,Du_k) \longrightarrow \sigma(x,u,Du)$ almost everywhere. Since, by the growth condition in (H_2), $\sigma(x,u_k,Du_k)$ is equi-integrable, it follows that $\sigma(x,u_k,Du_k) \longrightarrow \sigma(x,u,Du)$ in $L^1(\Omega)$ by the Vitali's theorem. The other hand, for all $v \in \bigcup_{k\in\mathbb{N}} V_k$ we have that $\sigma(x,u_k,Du_k) : Dv \longrightarrow \sigma(x,u,Du) : Dv$ a.e. $x \in \Omega$. Moveover, for all $\Omega' \subset \Omega$ measurable, it is easy to see that:

$$\int_{\Omega'} \sigma(x,u_k,Du_k) : Dv dx \leq c\beta \left(\|\lambda_1\|_{p'} + \|u_k\|_{1,p,\omega}^{q/p'} + \|u_k\|_{1,p,\omega}^{p/p'} \right) \|u\|_{1,p,\omega} < \infty,$$

because $\|u_k\|_{1,p,\omega} \leq R$. And thanks to Vitali's theorem, we obtain:

$$\langle F(u), v\rangle = 0, \text{ for all } v \in \bigcup_{k\in\mathbb{N}} V_k.$$

Which proves the theorem in this case. □

Remark 2.3.1 *Before treating the cases (a), (b) and (c) of (H_3), we note that*

$$\int_\Omega \int_{\mathbb{M}^{m\times n}} (\sigma(x,u,\lambda) - \sigma(x,u,Du)) : (\lambda - Du) d\vartheta_x(\lambda) dx \leq 0. \tag{2.3.11}$$

We conclude that

$$\int_\Omega \int_{\mathbb{M}^{m\times n}} \sigma(x,u,\lambda) : (\lambda - Du) d\vartheta_x(\lambda) dx = 0,$$

thanks to the $(2.3.9)$. On the other hand, the integrand in $(2.3.11)$ is non negative, by the monotonicity of σ. Consequently, the integrating should be null, a.e., with respect to the product measure $d\vartheta_x \otimes dx$, which means

$$(\sigma(x,u,\lambda) - \sigma(x,u,Du)) : (\lambda - Du) = 0 \text{ in spt}\vartheta_x. \tag{2.3.12}$$

Thus,

$$\text{spt}\vartheta_x \subset \left\{\lambda \in \mathbb{M}^{m\times n} \mid (\sigma(x,u,\lambda) - \sigma(x,u,Du)) : (\lambda - Du) = 0\right\}. \tag{2.3.13}$$

Case c: We prove that the map $F \longmapsto \sigma(x,u,F)$ is strictly monotone for all $x \in \Omega$ and for all $u \in \mathbb{R}^m$. Since σ is strictly monotone, and according to $(2.3.13)$,

$$\text{spt}\vartheta_x = \{Du\}, \text{ i.e, } \vartheta_x = \delta_{Du}, \quad \text{a.e. in } \Omega,$$

which implies that, $Du_k \longrightarrow Du$ in measure. The rest of our prove is similarly to case d.

Case b: We start by showing that for almost all $x \in \Omega$, the support of ϑ_x is contained in the set where W agrees with the supporting hyper-plane.

$$L = \left\{(\lambda, W(x,u,\overline{\lambda}) + \sigma(x,u,\overline{\lambda}) : (\lambda - \overline{\lambda}))\right\} \text{ with } \overline{\lambda} = Du(x).$$

So, it suffices to prove that

$$\text{spt}\vartheta_x \subset K_x = \left\{\lambda \in \mathbb{M}^{m\times n} \mid W(x,u,\lambda) = W(x,u,\overline{\lambda}) + \sigma(x,u,\overline{\lambda}) : (\lambda - \overline{\lambda})\right\}. \tag{2.3.14}$$

If $\lambda \in \text{spt}\vartheta_x$, thanks to $(2.3.13)$, we have

$$(1-t)\left(\sigma(x,u,Du) - \sigma(x,u,\lambda)\right) : (Du - \lambda) = 0 \text{ , for all } t \in [0,1]. \tag{2.3.15}$$

On the other hand, since σ is monotone, for all $t \in [0,1]$ we have

$$(1-t)\left(\sigma(x,u,Du + t.(\lambda - Du)) - \sigma(x,u,\lambda)\right) : (Du - \lambda) \geq 0. \tag{2.3.16}$$

By subtracting $(2.3.15)$ from $(2.3.16)$, we get

$$(1-t)\left[\sigma(x,u,\overline{\lambda} + t(\lambda - \overline{\lambda})) - \sigma(x,u,\overline{\lambda})\right] : (\overline{\lambda} - \lambda) \geq 0, \tag{2.3.17}$$

for all $t \in [0,1]$. Doing the same by the monotonicity in $(2.3.17)$, we obtain

$$(1-t)\left[\sigma(x,u,\overline{\lambda} + t(\lambda - \overline{\lambda})) - \sigma(x,u,\overline{\lambda})\right] : (\overline{\lambda} - \lambda) \leq 0. \tag{2.3.18}$$

Combining $(2.3.17)$ and $(2.3.18)$, we conclude that

$$(1-t)\left[\sigma(x,u,\overline{\lambda}+t(\lambda-\overline{\lambda}))-\sigma(x,u,\overline{\lambda})\right]:(\overline{\lambda}-\lambda)=0, \tag{2.3.19}$$

for all $t\in[0,1]$ and for all $\lambda\in \text{spt}\vartheta_x$.

Now, it follows from $(2.3.18)$ that

$$\begin{aligned} W(x,u,\lambda) &= W(x,u,\overline{\lambda})+(W(x,u,\lambda)-W(x,u,\overline{\lambda})) \\ &= W(x,u,\overline{\lambda})+\int_0^1[\sigma(x,u,\overline{\lambda}+t(\lambda-\overline{\lambda}))]:(\lambda-\overline{\lambda})dt \\ &= W(x,u,\overline{\lambda})+\sigma(x,u,\overline{\lambda}):(\lambda-\overline{\lambda}). \end{aligned}$$

This proves $(2.3.14)$.

Now, by the convexity of W, we get

$$W(x,u,\lambda)\geq W(x,u,\overline{\lambda})+\sigma(x,u,\overline{\lambda}):(\lambda-\overline{\lambda}),$$

for all $\lambda\in \mathbb{M}^{m\times n}$. Therefore,

$$L \text{ is a supporting hyper-plane, for all } \lambda\in K_x. \tag{2.3.20}$$

Moveover, the mapping $\lambda\longmapsto W(x,u,\lambda)$ is continuously differentiable, so we obtain

$$\sigma(x,u,\lambda)=\sigma(x,u,\overline{\lambda}), \text{ for all } \lambda\in K_x. \tag{2.3.21}$$

Thus,

$$\overline{\sigma}(x)=\int_{\mathbb{M}^{m\times n}}\sigma(x,u,\lambda)d\vartheta_x(\lambda)=\sigma(x,u,\overline{\lambda}). \tag{2.3.22}$$

Now, we consider the Carathéodory function

$$g(x,u,\rho)=|(\sigma(x,u,\rho)-\overline{\sigma}(x))|,$$

and observe that $g_k(x)=g(x,u_k,Du_k)$ is equi-integrable. Thus, thanks to Ball's theorem, see [32]and [35] $g_k\rightharpoonup\overline{g}$ weakly in $L^1(\Omega)$, and the weak limit of $\overline{g}$ is given by

$$\begin{aligned} \overline{g}(x) &= \int\int_{\mathbb{R}^m\times\mathbb{M}^{m\times n}}|\sigma(x,\eta,\lambda)-\overline{\sigma}(x)|d\delta_{u(x)}(\eta)\otimes d\vartheta_x(\lambda) \\ &= \int_{\text{spt}\vartheta_x}|\sigma(x,u(x),\lambda)-\overline{\sigma}(x)|d\vartheta_x(\lambda) \\ &= 0. \end{aligned}$$

According to $(2.3.21)$ and $(2.3.22)$, and since $g_k \geq 0$, it follow that $g_k \longrightarrow 0$ strongly in $L^1(\Omega)$ by Fatou's lemma, which gives

$$\lim_{k\to\infty} \int_\Omega \sigma(x, u_k, Du_k) : Dv.dx = \int_\Omega \sigma(x, u, Du) : Dvdx.$$

Thus

$$\langle F(u), v\rangle = 0, \ \ \forall v \in \bigcup_{k\in I\!N} V_k.$$

This completes the proof of the case (b). □

Case (a): In this case, on sptϑ_x, we affirm that,

$$\sigma(x, u, \lambda) : M = \sigma(x, u, Du) : M + (\nabla_F \sigma(x, u, Du) : M) : (Du - \lambda), \tag{2.3.23}$$

for all $M \in I\!M^{m\times n}$, where ∇_F is the derivative with respect to the third variable of σ and $\overline{\lambda} = Du(x)$. Thanks to the monotonicity of σ, we have

$$(\sigma(x, u, \lambda) - \sigma(x, u, Du + tM)) : (\lambda - Du - tM) \geq 0, \text{ for all } t \in I\!R.$$

and we obtain

$$-\sigma(x, u, \lambda) : (tM) \geq -\sigma(x, u, Du) : (\lambda - Du) + \sigma(x, u, Du + tM) : (\lambda - Du - tM).$$

On the other hand, $F \longmapsto \sigma(x, u, F)$ is a C^1 function, so

$$\sigma(x, u, Du + tM) = \sigma(x, u, Du) + \nabla_F \sigma(x, u, Du).(tM) + o(t).$$

Thus

$$-\sigma(x, u, \lambda) : (tM) \geq -\sigma(x, u, Du) : (tM) + \nabla_F \sigma(x, u, Du)(tM) : (\lambda - Du) + o(t),$$

which gives

$$-\sigma(x, u, \lambda) : (tM) \geq t[(\nabla_F \sigma(x, u, Du) : (M) : (\lambda - Du) - \sigma(x, u, Du) : (M)] + o(t).$$

The claim follows from this inequality since the sign of t is arbitrary in $(2.3.23)$. Finally for all $v \in \bigcup_{k\in I\!N} V_k$ the sequence $\sigma(x, u_k, Du_k) : Dv$ is equi-integrable. Then, by the Ball's theorem, see [32] and [35] the weak limit is $\int_{\text{spt}\vartheta_x} \sigma(x, u, \lambda) : Dvd\vartheta_x(\lambda)$ By choosing $M = Dv$ in $(2.3.23)$, we obtain

$$\int_{\text{spt}\vartheta_x} \sigma(x, u, \lambda) : Dvd\vartheta_x(\lambda)$$

$$= (\sigma(x,u,Du):Dv)\int_{\text{spt}\vartheta_x} d\vartheta_x(\lambda) = \sigma(x,u,Du):Dv.$$

Hence we have

$$\sigma(x,u_k,Du_k):Dv \longrightarrow \sigma(x,u,Du):Dv \text{ strongly }.$$

This proves that

$$\langle F(u),v\rangle = 0 \text{ for all } v \in \bigcup V_k.$$

And since $\bigcup V_k$ is dense in $W_0^{1,P}(\Omega,\omega,I\!R^m)$, u is a weak solution of (QES), as desired. □

Remark 2.3.2 *In case (b) we have $\sigma(x,u_k,Du_k) \longrightarrow \sigma(x,u,Du)$ in $L^1(\Omega,M^{m\times n})$, but in the case (c) and (d) $Du_k \longrightarrow Du$ in measure and thus $\sigma(x,u_k,Du_k) \longrightarrow \sigma(x,u,Du)$ almost everywhere. In case a) $\sigma(x,u_k,Du_k) \rightharpoonup \sigma(x,u,Du)$ in $L^1(\Omega,M^{m\times n})$.*

Example 2.3.1 *We shall suppose that the weight functions satisfy: $\omega_{i_0 j} = 0,\ j = 1,2,...,m$ for some $i_0 \in I^c$; and $\omega_{ij}(x) = \omega(x); x \in \Omega$, with $I^c \bigcup I = \{0;1;2;.....;n\}$, for all $i \in I \sqcup I^c,\ j = 1,2,...,m$, and $i \neq i_0$ with $\omega(x) > 0$ a.e in Ω. Let us consider the Carathéodory functions σ defined by*

$$\sigma_{ij}(x,\eta,\xi_I) = \omega(x)|\xi_{ij}|^{p-1}sign(\xi_{ij}),\ j = 1,2,....m\ , i \in I$$

$$\sigma_{ij}(x,\eta,\xi_{I^c}) = \omega(x)|\xi_{ij}|^{p-1}sign(\xi_{ij}),\ j = 1,2,....m\ , i \in I^c, i \neq i_0$$

$$\sigma_{i_0 j}(x,\eta,\xi_{I^c}) = 0\ , j = 1,2,...,m$$

The functions σ satisfies the growth conditions (H_2).
In the case of b it is considered the function W defined by

$$W_{rs}(x,u,F) = \omega(x)\frac{1}{p} \mid F_{rs} \mid^p sign(F_{rs})$$

In particular, let us the special weight function ω, γ expressed in term of the distance to the boundary $\partial\Omega$ denoted by $d(x) = dist(x;\partial\Omega)$ and $\omega(x) = d^\lambda(x)$, $\gamma_j(x) = d^\mu(x)$.The Hardy-Type inequalities reads:

$$\left(\sum_{j=1}^m \int_\Omega |u_j(x)|^q d^\mu(x)dx\right)^{\frac{1}{q}} \leq c\left(\sum_{1\leq i\leq n;\ 1\leq j\leq m} \int_\Omega |D_{ij}u|^p d^\lambda(x)\right)^{\frac{1}{p}},$$

for every $u \in W_0^{1,p}(\Omega,\omega,I\!R^m)$ with a constant $c > 0$ independent of u and for some $q > p'$ and the corresponding embedding $W_0^{1,p}(\Omega;\omega;I\!R^m) \hookrightarrow L^q(\Omega;\gamma;I\!R^m)$ is compact if:
i) For $1 < p \leq q < \infty$

$$\lambda < p-1,\ \ \frac{n}{q} - \frac{n}{p} + 1 \geq 0,\ \ \frac{\mu}{q} - \frac{\lambda}{p} + \frac{n}{q} - \frac{n}{p} + 1 > 0.$$

ii) For $1 \leq q < p < \infty$

$$\lambda < p-1, \;\; \frac{n}{q} - \frac{n}{p} + 1 \geq 0, \;\; \frac{\mu}{q} - \frac{\lambda}{p} + \frac{1}{q} - \frac{1}{p} + 1 > 0.$$

iii) For $q > 1$

$\mu(q'-1) < 1$*, by the simple modifications of the example in [2]. Moreover, the monotonicity condition are satisfied:*

$$\sum_{ij} (\sigma_{ij}(x,\eta,\xi_I) - \sigma_{ij}(x,\eta,\xi'_I)(\xi_{ij} - \xi'_{ij})$$

$$= \omega(x) \sum_{ij} (\mid \xi_{ij} \mid^{p-1} sign(\xi_{ij}) - \mid \xi'_{ij} \mid^{p-1} sign(\xi'_{ij}))(\xi_{ij} - \xi'_{ij}) \geq 0$$

for almost all $x \in \Omega$ *and for all,* $\xi, \xi' \in M^n$*. This last inequality can not be strict, since for* $\xi_{I^c} \neq \xi'_{I^c}$ *with* $\xi_{i_0 j} \neq \xi'_{i_0 j}$ *for all* $j = 1, 2, ..., m$*. But* $\xi_{ij} = \xi'_{ij}$ *for* $i \in I^c, \; i \neq i_0, \; j = 1, 2, ..., m$ *the corresponding expression is zero.*

Chapter 3

Existence of a weak solution of some quasi-linear elliptical system in a weighted Sobolev space

Abstract

We consider, for a bounded open domain Ω in $I\!R^n$ and a function $u : \Omega \to I\!R^m$; the quasi-linear elliptic system :

$$(\text{QESW})_{(f,g)} \begin{cases} -div\sigma\left(x, u\left(x\right), Du\left(x\right)\right) &= v(x) + f(x,u) \text{ in } \Omega \\ u &= 0 \qquad\qquad \text{on } \partial\Omega, \end{cases} \tag{3.0.1}$$

Which is a Dirichlet problem. Here, v belongs to the dual space $W^{-1,p'}(\Omega, \omega^*, R^m)$, $\left(\frac{1}{p} + \frac{1}{p'} = 1,\ p > 1\right)$, f satisfy some standard continuity and growth conditions. We prove existence of a regularity, growth and coercivity conditions for σ, but with only very mild monotonicity assumptions.

3.1 Preliminaries.

Let $\omega = \{\omega_{ij} :\ \ 0 \leq i \leq n;\ \ 1 \leq j \leq m\}$ and $\overline{\omega_0} = (\omega_{0j})_{1\leq j\leq m}$. The weight function systems defined in Ω and satisfying the following integrability conditions:

$$\omega_{ij} \in L^1_{loc}(\Omega), \quad \omega_{ij}^{\frac{-1}{p-1}} \in L^1_{loc}(\Omega), \text{ for some } p \in]1, \infty[\text{ and } \exists s > Max(\frac{n}{p}, \frac{1}{p-1}) \text{ such that } \omega_{ij}^{-s} \in L^1(\Omega). \tag{3.1.1}$$

with $\omega^* = \{\omega_{ij}^* = \omega_{ij}^{1-p'} : \quad 0 \leq i \leq n, \quad 1 \leq j \leq m\}$, $\sigma = (\sigma_{rs})$ with $1 \leq s \leq n, \quad 1 \leq r \leq m$ and which satisfies some hypotheses (see below).

The space $W^{1,p}(\Omega, \omega, I\!R^m)$ is the set of functions

$$\{u = u(x) |\ u \in L^p(\Omega, \overline{\omega_0}, I\!R^m)\}\ ,\ D_{ij}u = \frac{\partial u^i}{\partial x_j} \in L^p(\Omega, \omega_{ij}, I\!R^m),\ 1 \leq i \leq n,\ 1 \leq j \leq m\}.$$

With

$$L^p(\Omega, \omega_{ij}, I\!R^m) = \{u = u(x) |\quad | \ u \ | \ {\omega_{ij}}^{\frac{1}{p}} \in L^p(\Omega, I\!R^m)\}$$

The weighted space $W^{1,p}(\Omega, \omega, I\!R^m)$ can be equipped by the norm :

$$\|u\|_{1,p,\omega} = \left(\sum_{j=1}^{m} \int_\Omega |u_j|^p \omega_{0j} dx + \sum_{1 \leq i \leq n, 1 \leq j \leq m} \int_\Omega |D_{ij}u|^p \omega_{ij} dx \right)^{\frac{1}{p}}$$

The norm $\|.\|_{1,\omega,p}$ is equivalent to the norm $|||\ .\ |||$, on $W_0^{1,p}(\Omega, \omega, I\!R^m)$, such that,
$|||\ u\ ||| = (\sum_{1 \leq i \leq n, 1 \leq j \leq m} \int_\Omega |D_{ij}u|^p \omega_{ij} dx)^{\frac{1}{p}}$

Proposition 3.1.1 *The weighted Sobolev space $W^{1,p}(\Omega, \omega, I\!R^m)$ is a Banach space, separable and reflexive. The weighted Sobolev space $W_0^{1,p}(\Omega, \omega, I\!R^m)$ is the closure of $C_0^\infty(\Omega, \omega, I\!R^m)$ in $W^{1,p}(\Omega, \omega, I\!R^m)$ equipped by the norm $\|.\|_{1,p,\omega}$.*

Proof: The prove of proposition is a slight modification of the analogous one in [25] [Kufner-Drabek].

Definition 3.1.1 *A Young measure $(\vartheta_x)_{x \in \Omega}$ is called $W^{1,p}$-gradient young measures $(1 \leq p < \infty)$ if it is associated to a sequence of gradients Du_k such that u_k is bounded in $W^{1,p}(\Omega)$.*
The $W^{1,p}$-gradient young measures $(\vartheta_x)_{x \in \Omega}$ is called homogeneous, if it doesn't depend on x, i-e, if $\vartheta_x = \vartheta$ for a.e. $x \in \Omega$.

Hypothesis (H_0) (Hardy inequality): There exist some constant $c > 0$, some weighted function γ and some real q $(1 < q < \infty)$ such that,

$$\left(\sum_{j=1}^{m} \int_\Omega |u_j(x)|^q \gamma_j(x) dx \right)^{\frac{1}{q}} \leq c \left(\sum_{1 \leq i \leq n 1 \leq j \leq m} \int_\Omega |D_{ij}u|^p \omega_{ij} \right)^{\frac{1}{p}},$$

for all $u \in W_0^{1,p}(\Omega, \omega, I\!R^m)$, with $\gamma = \{\gamma_j / 1 \leq j \leq m\}$.
The injection $W_0^{1,p}(\Omega, \omega, I\!R^m) \hookrightarrow\hookrightarrow L^q(\Omega, \gamma, I\!R^m)$ is compact, and $W_0^{1,p}(\Omega, \omega, I\!R^m) \hookrightarrow\hookrightarrow L^r(\Omega, I\!R^m)$ is compact, (by [25]) with

$$\begin{cases} 1 \leq r \prec \frac{nps}{n(s+1)-ps} & \text{if} \quad ps \prec n(s+1) \\ r \geq 1 & \text{if} \quad n(s+1) \prec ps \end{cases}$$

(H_1) Continuity: $\sigma : \Omega \times I\!R^m \times I\!M^{m\times n} \longrightarrow I\!M^{m\times n}$ is a Carathéodory function (i-e $x \longmapsto \sigma(x,u,F)$ is measurable for every $(u,F) \in I\!R^m \times I\!M^{m\times n}$ and $(u,F) \longmapsto \sigma(x,u,F)$ is continuous for almost every $x \in \Omega$).

(H_2) Growths and coercivity conditions: there exist $c_1 \geq 0, c_2 > 0$ $\lambda_1 \in L^{p'}(\Omega),\ \lambda_2 \in L^1(\Omega),\ \lambda_3 \in L^{(p/\alpha)'}(\Omega),\ 0 < \alpha < p,\ 1 < q < \infty$ and $\beta > 0$ such that for all $1 \leq r \leq n,\ 1 \leq s \leq m$, we have:

$$|\sigma_{rs}(x,u,F)| \leq \beta\omega_{rs}^{1/p}[\lambda_1(x) + c_1 \sum_{j=1}^{m} |\gamma_j|^{1/p'} \cdot |u_j|^{q/p'} + c_1 \sum_{1\leq i\leq N1\leq j\leq m} \omega_{ij}^{1/p'} |F_{ij}|^{p-1}] \tag{3.1.2}$$

and

$$\sigma(x,u,F) : F \geq -\lambda_2(x) - \sum_{j=1}^{m} \omega_{0j}(x)^{\alpha/p} \lambda_3(x) |u_j|^{\alpha} + c_2 \sum_{1\leq i\leq n1\leq j\leq m} \omega_{ij}(x) \cdot |F_{ij}|^p \tag{3.1.3}$$

(H_3) Monotonicity conditions: σ satisfies one of the following conditions:

a) For all $x \in \Omega$, and all $u \in I\!R^m$, the map $F \longmapsto \sigma(x,u,F)$ is a C^1-function and is monotone(i-e, $(\sigma(x,u,F) - \sigma(x,u,G)) : (F-G) \geq 0$, for all $x \in \Omega,$ all $u \in I\!R^m$ and all $F, G \in I\!M^{m\times n}$).

b) There exists a function $W : \Omega \times I\!R^m \times I\!M^{m\times n} \longrightarrow I\!M^{m\times n}$ such that $\sigma(x,u,F) = \dfrac{\partial W}{\partial F}(x,u,F)$ and $F \longmapsto W(x,u,F)$ is convex and C^1 function.

c) For all $x \in \Omega$, and for all $u \in I\!R^m$ the map $F \longmapsto \sigma(x,u,F)$ is strictly monotone (i.e, $\sigma(x,u,.)$ is monotone and :$[(\sigma(x,u,F) - \sigma(x,u,G)) : (F-G) = 0] \Longrightarrow F = G$).

d) $\sigma(x,u,F)$ is strictly p-quasi-monotone in F, i.e,

$$\int_{I\!M^{m\times n}} (\sigma(x,u,\lambda) - \sigma(x,u,\overline{\lambda})) : (\lambda - \overline{\lambda}))d\vartheta(\lambda) > 0,$$

for all homogeneous $W^{1,p}$-gradient young measures ϑ with center of mass $\overline{\lambda} = \langle \vartheta, id \rangle$ which are not a single Dirac mass.

The main point is that we do not require strict monotonicity or monotonicity in the variables (u,F) in (H_3) as it is usually assumed in previous work see ([41] or [43]).

(F_0):(continuity) $f : \Omega \times I\!R^m \to I\!R^m$ is a Carathéodory function i-e: $x \mapsto f(x,u)$ is measurable for every $u \in I\!R^m$, and , $u \mapsto f(x,u)$ is continuous for almost every $x \in \Omega$.

(F_1): (growth condition): The exist : $b_1 \in L^{p'}(\Omega)$ such that :

$$|f_j(x,u)| \leq [b_1(x) + \gamma_j^{\frac{1}{p'}} |u_j|^{\frac{q}{p'}}]\omega_{0j}^{\frac{1}{P}};$$

$\forall 1 \leq j \leq m$

Our aim of this chapter is to prove the existence of the problem $(QES)_{f,g}$ in the space $W_0^{1,P}(\Omega, \omega, I\!R^m)$.

Remark 3.1.1 *-The condition (F_0) ensure the measurability of f for all measurable function u.*
-(F_1) ensure that growths conditions, in particularly: if $u \in W_0^{1,P}(\Omega, \omega, I\!R^m)$ then $f(., u).u$ is in $L^1(\Omega, \omega)$.

Theorem 3.1.1 *If $p \in (1, \infty)$ and σ satisfies the conditions $(H_1) - (H_3)$, then the Dirichlet problem $(QESW)_{f,g}$ has a weak solution $u \in W_0^{1,p}(\Omega, \omega, I\!R^m)$, for every $v \in W^{-1,p'}(\Omega, \omega^*, I\!R^m)$, f satisfies (F_0) and (F_1).*

Lemma 3.1.1 *For arbitrary $u \in W_0^{1,p}(\Omega, \omega, I\!R^m)$ and $v \in W^{-1,p'}(\Omega, \omega^*, I\!R^m)$, the functional*

$$\begin{aligned} F(u) : W_0^{1,p}(\Omega, \omega, I\!R^m) &\longrightarrow I\!R \\ \varphi &\longmapsto \int_\Omega \sigma(x, u(x), Du(x)) : D\varphi(x)dx \\ - \langle v, \varphi \rangle - \int_\Omega f(x, u) : \varphi dx. & \end{aligned}$$

is well defined, linear and bounded.

Proof For all $\varphi \in W_0^{1,p}(\Omega, \omega, I\!R^m)$, we denote

$$F(u)(\varphi) = I_1 + I_2 + I_3$$

with

$$I_1 = \int_\Omega \sigma(x, u(x), Du(x)) : D\varphi(x)dx,$$

and

$$I_2 = - \langle v, \varphi \rangle .$$

$$I_3 = - \int_\Omega f(x, u) : \varphi dx$$

We define

$$I_{rs} = \int_\Omega \sigma_{rs}(x, u(x), Du(x)) : D_{rs}\varphi(x)dx$$

Firstly, by virtue of the growth conditions (H_2) and the Hölder inequality, one has

$$\begin{aligned} |I_{rs}| &\leq \int_\Omega |\sigma_{rs}(x, u(x), Du(x))| : |D_{rs}\varphi(x)|dx \\ &\leq \int_\Omega \beta w_{rs}^{1/p}(x)[\lambda_1(x) + c_1 \sum_{j=1}^m |\gamma_j(x)|^{1/p'}|u(x)|^{q/p'} + c_1 \sum_{1\leq i\leq n 1\leq j\leq m} \omega_{ij}^{1/p'}|D_{ij}|^{p-1}]|D_{rs}\varphi|dx. \\ &\leq \beta[(\int_\Omega |\lambda_1(x)|^{p'}dx)^{1/p'}(\int_\Omega |D_{rs}\varphi(x)|^p w_{rs}dx)^{1/p} + (\int_\Omega |D_{rs}\varphi(x)|^p w_{rs})^{1/p}(\sum_{j=1}^m \int_\Omega |u_j|^q \gamma_j dx)^{1/p'} \\ &\quad + (\sum_{1\leq i\leq n 1\leq j\leq m} \int_\Omega |D_{ij}u|^p \omega_{ij}dx)^{1/p'}(\int_\Omega |D_{rs}\varphi|^p w_{rs}dx)^{1/p}] \end{aligned}$$

with $(p = p'(p-1))$, and thanks to Hardy-Type inequalities we have:

$$\begin{aligned} |I_{rs}| &\leq c\beta\Big[\|\lambda_1\|_{p'} \|\varphi\|_{1,p,\omega_{rs}} + c_1 \|D\varphi\|_{p,\omega_{rs}} (\int_\Omega |u|^q \gamma dx)^{1/p'} \\ &\quad + c_1 \sum_{ij} \|D\varphi\|_{p,\omega_{ij}} \|Du\|_{p,\omega_{rs}} \Big] \\ &\leq c'\beta[\|\lambda_1\|_{p'} \|\varphi\|_{1,p,\omega_{rs}} + \|\varphi\|_{1,p,\omega_{rs}} \|u\|_{q,\gamma} + \|u\|_{1,p,\omega} \|\varphi\|_{1,p,\omega_{rs}} \end{aligned}$$

with $c' = \max(c, 1)$. Which gives

$$|I_1| \leq c'\beta[\|\lambda_1\|_{p'} + \|u\|_{1,p,\omega}^{q/p'} + \|u\|_{1,p,\omega}]\|\varphi\|_{1,p,\omega} < \infty.$$

and

$$|I_2| \leq \int_\Omega |v||\varphi| dx \leq \|v\|_{-1,p',\omega*} \|\varphi\|_{1,p,\omega} < \infty.$$

$$I_3 = \sum_j \int_\Omega f_j(x,u)\varphi_j(x)dx$$

We denote $I_{3,j} = |\int_\Omega f_j(x,u)\varphi_j(x)dx|$.

$$\begin{aligned} I_{3,j} &\leq \int_\Omega |f_j(x,u)||\varphi_j(x)|dx| \\ &\leq \int_\Omega b_1(x)|\varphi_j(x)|\omega_{0j}^{\frac{1}{p}}dx + \int_\Omega \gamma_j^{\frac{1}{p'}}|u_j|^{\frac{q}{p'}}|\varphi_j(x)|\omega_{0j}^{\frac{1}{p}}dx \\ &\leq (\int_\Omega |b_1(x)|^{p'})^{\frac{1}{p'}}(\int_\Omega |\varphi_j(x)|^p\omega_{0j}dx)^{\frac{1}{p}} + (\int_\Omega \gamma_j(x)|u_j|^q dx)^{\frac{1}{p'}}(\int_\Omega |\varphi_j(x)|^p\omega_{0j}dx)^{\frac{1}{p}} \\ &\leq \| b_1 \|_{p'} \| \varphi \|_{1,p,\omega} + (\sum_j \int_\Omega \gamma_j(x)|u_j|^q dx)^{\frac{1}{p'}} \| \varphi \|_{1,p,\omega} \\ &\leq \| b_1 \|_{p'} \| \varphi \|_{1,p,\omega} + c \| Du \|_{1,p,\omega} \| \varphi \|_{1,p,\omega} \end{aligned}$$

$$\leq (\| b_1 + c \| Du \|_{1,p,\omega}) \| \varphi \|_{1,p,\omega} .$$

hence $I \leq c_4 \| \varphi \|_{1,p,\omega}$. With $c_4 < \infty$.

Finally the functional $F(.)$ is bounded.

Lemma 3.1.2 *The restriction of F to a finite dimensional linear subspace V of $W_0^{1,p}(\Omega, \omega, I\!R^m)$ is continuous.*

Proof Let d be the dimension of V and $(e_1, e_2, ..., e_d)$ a basis of V. Let $u_j = \sum_{1\leq i\leq d} a_j^i.e_i$ be a sequence in V which converges to $u = \sum_{1\leq i\leq d} a^i e_i$ in V. The sequence (a_j) converge to $a \in I\!R^d$, so $u_j \longrightarrow u$ and $Du_j \longrightarrow Du$ a.e., on the other hand $\|u_j\|_p$ and $\|Du_j\|_p$ are bounded by a constant c. Thus, it follows by the continuity conditions (H_1), that

$$\sigma(x, u_j, Du_j) : D\varphi \longrightarrow \sigma(x, u, Du) : D\varphi$$

for all $\varphi \in W_0^{1,p}(\Omega, \omega, I\!R^m)$ and a.e. in Ω. Let Ω' be a measurable subset of Ω and let $\varphi \in W_0^{1,p}(\Omega, \omega, I\!R^m)$. Thanks to the condition (H_2), we get

$$\int_{\Omega'} |\sigma(x, u_j, Du_j) : D\varphi| dx < \infty,$$

By the continuity conditions (F_0) we have :

$$f(x, u_j).\varphi \to f(x, u).\varphi$$

almost everywhere. Moreover we infer from the growth conditions (F_1) that the sequences:
$(\sigma(x, u_j, Du_j) : D\varphi)$ and $(f(x, u_j).\varphi)$
Are equi-integrable. Indeed , if $\Omega' \subset \Omega$ is a measurable subset and $\varphi \in W_0^{1,p}(\Omega, \omega, I\!R^m)$ then:
$\int_{\Omega'} |f(x, u_j).\varphi| dx < \infty$ (by (F_1) and Hölder inequality).
$\int_{\Omega'} |\sigma(x, u_j, Du_j) : D\varphi| dx < \infty$ (by Hölder inequality)
which implies that $\sigma(x, u_j, Du_j) : D\varphi$ is equi-integrable. And by applying the Vitali's theorem, it follows that

$$\int_\Omega \sigma(x, u_j, Du_j) : D\varphi dx \longrightarrow \int_\Omega \sigma(x, u, Du) : D\varphi dx,$$

for all $\varphi \in W_0^{1,p}(\Omega, \omega, I\!R^m$.
Finally

$$\lim_{j\to\infty} \langle F(u_j), \varphi\rangle = \langle F(u), \varphi\rangle ,$$

which means that

$$F(u_j) \longrightarrow F(u) \text{ in } W^{-1,p'}(\Omega, \omega^*, I\!R^m).$$

Remark 3.1.2 *Now, the problem $(QES)_{f,g}$ is equivalent to find a solution $u \in W_0^{1,p}(\Omega, \omega, I\!R^m)$ such that $\langle F(u), \varphi\rangle = 0$, for all $\varphi \in W_0^{1,p}(\Omega, \omega, I\!R^m)$.*
In order to find such a solution we apply a Galerkin schema .

3.2 Galerkin approximation

Remark 3.2.1 (*Galerkin Schema*)

Let $V_1 \subset V_2 \subset ..... \subset W_0^{1,p}(\Omega, \omega, I\!R^m)$ be a sequence of finite dimensional subspaces with $\bigcup_{k\in I\!N} V_k$ dense in $W_0^{1,p}(\Omega, \omega, I\!R^m)$. The sequence V_k exists since $W_0^{1,p}(\Omega, \omega, I\!R^m)$ is separable.

Let us fix some k, we assume that V_k has a dimension d and that $(e_1, e_2, ..., e_d)$ is a basis of V_k. Then, we define the map

$$\begin{array}{rccl} G: & I\!R^k & \longrightarrow & I\!R^k \\ & (a_1, ..., a_k) & \longmapsto & \left(\left\langle F(\sum_{i=1}^d a_i e_i), e_1 \right\rangle, ..., \left\langle F(\sum_{i=1}^d a_i e_i), e_d \right\rangle \right)^t. \end{array}$$

Proposition 3.2.1 *The map G is continuous and $G(a)\cdot a$ tends to infinity when $\|a\|_{I\!R^k}$ tends to infinity.*

Proof. Since F restricted to V_k is continuous by Lemma 3.1.2, so G is continuous.
Let $a \in I\!R^d$ and $u = \sum_{1\le i\le d} a^i \cdot e_i$ in V_k, then $G(a)\cdot a = \langle F(u), u\rangle$ and which implies that $\|a\|_{I\!R^d}$ tends to infinity if $\|u\|_{1,p,\omega}$ tends to infinity.

$$G(a)\cdot a = \sum_{1\le i\le d} \langle F(u), a^i \cdot e_i\rangle = \langle F(u), u\rangle$$

and

$$\begin{aligned} \|u\|_{1,p,\omega}^p = \left\| \sum_{1\le i\le d} a^i \cdot e_i \right\|_{1,p,\omega}^p &\le \left(\sum_{1\le i\le d} |a^i| \cdot \|e_i\|_{1,p,\omega} \right)^p \\ &\le \max_{1\le i\le d} \left(\|e_i\|_{1,p,\omega}^p \right) \cdot \left(\sum_{1\le i\le d} |a^i| \right)^p \\ &\le c \cdot \|a\|_{I\!R^p}, \end{aligned}$$

which implies that $\|a\|_{I\!R^p}$ tends to infinity if $\|u\|_{1,p,\omega}$ tends to infinity.

Now, it suffices to prove that

$$\langle F(u), u\rangle \to \infty \quad \text{when} \quad \|u\|_{1,p,\omega} \to \infty.$$

Indeed, thanks to the first coercivity condition and the Hölder inequality, we obtain

$$I = \int_\Omega \sigma(x, u, Du) : Du dx \ge - \|\lambda_2\|_1 - \int_\Omega \lambda_3 \omega_{0j}^{\alpha/p} |u_j|^\alpha dx + c_2 \sum_{1\le i,j\le n,m} \int_\Omega |D_{ij} u|^p \omega_{ij} dx$$

By the Hölder inequality, we have

$$\begin{aligned}\int_\Omega \lambda_3 |u_j|^\alpha \omega_{0j}^{\alpha/p} dx &\leq \|\lambda_3\|_{(p/\alpha)'} \left(\int_\Omega \omega_{0j} |u_j|^{(p/\alpha)\cdot\alpha} \, dx \right)^{\alpha/p} \\ &\leq c' \, \|\lambda_3\|_{(p/\alpha)'} \, \|u_j\|_{1,p,\omega_{0j}} .\end{aligned}$$

where c' is a constant positive. For $\|u\|_{1,p,\omega}$ large enough, we can write

$$\begin{aligned}| \, I \, | &\geq -\|\lambda_2\|_1 - c' \, \|\lambda_3\|_{(p/\alpha)'} \cdot \|u_j\|^\alpha_{1,p,\omega_{0j}} + c_2 \cdot \textstyle\sum_{1\leq i,j\leq n,m} \|Du_j\|^p_{1,p,\omega_{ij}} \\ &\geq -\|\lambda_2\|_1 - c' \, \|\lambda_3\|_{(p/\alpha)'} \cdot \|u\|^\alpha_{1,p,\omega} + c_2 c' \cdot \|u\|^p_{1,p,\omega} .\end{aligned}$$

And since

$$|I'| = | \, \langle v, u \rangle \, | \leq \|v\|_{-1,p',\omega^*} \cdot \|u\|_{1,p,\omega}$$

Finally, it follows from the growth condition (F_1) that:

$$|I''| = | \int_\Omega f(x,u).u dx| \leq (\|b_1\|_{p'} + c.\|Du\|_{1,p,\omega}).\|u\|_{1,p,\omega}$$

.

$$\leq c_3.\|u\|_{1,p,\omega}$$

With; $0 < \alpha < p$ and $p > 1$, we get :

$$I - I' - I'' \geq c_2.c'.\cdot\|u\|^p_{1,p,\omega} - \|v\|_{-1,p',\omega^*} \cdot \|u\|_{1,p,\omega} - c' \, \|\lambda_3\|_{(p/\alpha)'} \cdot \|u\|^\alpha_{1,p,\omega} - \|\lambda_2\|_1 - c_3. \, \|u\|_{1,p,\omega} . \quad (3.2.1)$$

Consequently, by using (3.2.1), we deduce

$$I - I' - I'' \longrightarrow \infty \text{ as } \|u\|_{1,p,\omega} \longrightarrow \infty$$

.

$$\langle F(u), u \rangle \longrightarrow \infty \text{ as } \|u\|_{1,p,\omega} \longrightarrow \infty$$

.

Remark 3.2.2 *The properties of G allows us to construct our Galerkin approximations.*

Corollary 3.2.1 *For all $k \in \mathbb{N}$, there exists $(u_k) \subset V_k$ such that $\langle F(u_k), \varphi \rangle = 0$, for all $\varphi \in V_k$.*

Proof By the proposition 3.2.1, there exists $R > 0$, such that for all $a \in \partial B_R(0) \subset \mathbb{R}^d$, we have $G(a) \cdot a > 0$. And the usual topological argument see e.g. [proposition 2.8] [64] implies that $G(x) = 0$ has a solution $x \in B_R(0)$. So, for all $k \in \mathbb{N}$, there exists $(u_k) \subset V_k$, such that

$$\left\langle F(x^j e_j), e_j \right\rangle = 0 \text{ for all } 1 \leq j \leq d, \text{ with } d = \dim V_k$$

Taking $u_k = (x_k^i e_i),\ e_i \in V_k$, so we obtain:

$$\langle (F(u_k)), \varphi \rangle = 0, \text{ for all } \varphi \in V_k.$$

Proposition 3.2.2 *The Galerkin approximations sequence constructed in corollary 3.2.1 is uniformly bounded in $W_0^{1,p}(\Omega, \omega, I\!R^m)$; i.e.,*

$$\textit{there exists a constant } R > 0, \textit{ such that } \|u_k\|_{1,p,\omega} \leq R, \textit{ for all } k \in I\!N.$$

Proof Like in the proof of proposition 3.2.1, we can see that

$$\langle F(u), u \rangle \longrightarrow \infty \text{ as } \|u\|_{1,p,\omega} \longrightarrow \infty$$

. Then, there exists R satisfying $\langle F(u), u \rangle > 1$ when $\|u\|_{1,p,\omega} > R$. Now, for the sequence of Galerkin approximations $(u_k) \subset V_k$ of corollary 3.2.1, which satisfying $\langle F(u_k), u_k \rangle = 0$, we have the uniform bound $\|u_k\|_{1,p,\omega} \leq R$, for all $k \in I\!N$.

Remark 3.2.3 *There exists a subsequence (u_k) of the sequence $(u_k) \subset V_k$, such that:*

$$u_k \rightharpoonup u \textit{ in } W_0^{1,p}(\Omega, \omega, I\!R^m)$$

and

$$u_k \longrightarrow u \textit{ in measure in } L^r(\Omega, I\!R^m)$$

; with

$$\begin{cases} 1 \leq r \prec \frac{nps}{n(s+1)-ps} & \textit{if} \quad ps \prec n(s+1) \\ r \geq 1 & \textit{if} \quad n(s+1) \prec ps \end{cases}$$

The gradient sequence (Du_k) generates the young measure ϑ_x. Since $u_k \longrightarrow u$ in measure, then (u_k, Du_k) generates the Young measure $(\delta_{u(x)} \otimes \vartheta_x)$, see e.g [4]*. Moreover, for almost x in Ω, we have,*

(i) *ϑ_x is the probability measure, i.e, $\|\vartheta_x\|_{mes} = 1$.*

(ii) *ϑ_x is the $W^{1,p,\omega}$ - gradient homogeneous young measure.*

(iii) *$\langle \vartheta_x, id \rangle = Du(x)$, see e.g* [3].

Proof. For proof see [39].(Dolzmann, N.Humgerbühler s Muller. Non linear elliptic system....)

3.3 Passage to the limit

Now, we are in a position to prove our main result under convenient hypotheses.

Let

$$I_k = (\sigma(x, u_k, Du_k) - \sigma(x, u, Du)) : (Du_k - Du). \tag{3.3.1}$$

Lemma 3.3.1 *Let $p > 1$ and u_k be a sequence which is uniformly bounded in $W_0^{1,p}(\Omega, \omega, I\!R^m)$. There exists a subsequence of u_k (for convenience not relabeled) and a function $u \in W_0^{1,p}(\Omega, \omega, I\!R^m)$ such that $u_k \rightharpoonup u$ in $W_0^{1,p}(\Omega, \omega, I\!R^m)$*

And such that $u_k \to u$ in measure on Ω and in $L^r(\Omega, I\!R^m)$, with :

$$\begin{cases} 1 \le r \prec \frac{nps}{n(s+1)-ps} & \text{if} \quad ps \prec n(s+1) \\ r \ge 1 & \text{if} \quad n(s+1) \prec ps \end{cases}$$

Proof. see [38].

Lemma 3.3.2 *The sequence (I_k) is equi-integrable.*

Proof

We have

$$\begin{aligned} I_k &= (\sigma(x, u_k, Du_k) - \sigma(x, u, Du)) : (Du_k - Du) \\ &= [\sigma(x, u_k, Du_k) : Du_k] - [\sigma(x, u_k, Du_k) : Du] - [\sigma(x, u, Du) : Du_k] + [\sigma(x, u, Du) : Du] \\ &= I_k^1 + I_k^2 + I_k^3 + I_k^4 \end{aligned} \tag{3.3.2}$$

We denote $(I_k^1)^- = -[\sigma(x, u_k, Du_k : Du_k]^-$. Thanks to the coercivity condition (H_2), we have

$$\begin{aligned} \int_{\Omega'} |(I_k^1)^-| dx &\le \int_{\Omega} |\lambda_2| + c_2 \sum_{1 \le j \le m} \omega_{0j}^{\frac{\alpha}{p}} |\lambda_3| \cdot |u_{kj}|^\alpha + c \sum_{1 \le i,j \le n,m} \omega_{ij} |D_{ij} u_k|^p dx \\ &\le \|\lambda_2\|_1 + \int_{\Omega'} (\sum_{1 \le j \le m} \omega_{0j}^{\alpha/p} |u_{kj}|^\alpha)^{p/\alpha} \|\lambda_3\|_{(p/\alpha)'} + c_2 \|u_k\|_{1,\omega,p}^p \end{aligned} \tag{3.3.3}$$

with $p/\alpha \ge 1$. Therefore,

$$\begin{aligned} \int_{\Omega'} |(I_k^1)^-| dx &\le \|\lambda_2\|_1 + \Big(\sum_{1 \le j \le m} \omega_{0j} |u_{kj}|^p \Big)^{\alpha/p} \|\lambda_3\|_{(p/\alpha)'} + c_2 \|u_k\|_{1,\omega,p}^p \\ &\le \|\lambda_2\|_1 + \|u_k\|_{p,\overline{\omega}_{00}}^\alpha \|\lambda_3\|_{(p/\alpha)'} + c_2 \|u_k\|_{1,\omega,p}^p \\ &< \infty, \end{aligned}$$

for all $\Omega' \subset \Omega$.

Similarly for $(|(I_k^4)^-|$.

Now, by using the growth condition (H_2) and the Hardy -Type inequalities (H_0), we have

$$\begin{aligned}\int_{\Omega'} |(I_k^2)^-|dx &= \int_{\Omega'} |\sigma(x,u_k,Du_k) : Du_k|dx \\ &\leq \beta \int_{\Omega'} \omega_{rs}^{1/p} \left(\lambda_1 + c_1 \sum_{1\leq j\leq m} \gamma_j^{1/p'}.|u_{kj}|^{q/p'} + c_1 \sum_{1\leq i,j\leq n,m} \omega_{ij}^{1/p'} |D_{ij}u_k|^{p-1} \right) D_{rs}u_k dx. \end{aligned} \tag{3.3.4}$$

Thus, by the Hölder inequality, we obtain

$$\begin{aligned}\int_{\Omega'} |(I_k^2)^-|dx \leq \beta \Bigg[\|\lambda_1\|_{p'} \left(\int_{\Omega'} |D_{rs}u_k|^p \omega_{rs} dx \right)^{1/p} \\ + c_1 \left(\int_{\Omega'} |D_{rs}u_k|^p \omega_{rs} dx \right)^{1/p} \left(\int_{\Omega'} \left(\sum_{1\leq j\leq m} \gamma_j^{1/p'} |u_{kj}|^{q/p'} \right)^{p'} dx \right)^{1/p'} \\ + c_1 \left(\sum_{1\leq j\leq m} \int_{\Omega'} \left(|D_{ij}u_k(x)|^{p'(p-1)} \omega_{ij} dx \right)^{1/p'} \right) \left(\int_{\Omega'} |D_{rs}u_k|^p \omega_{rs} dx \right)^{1/p} \Bigg]. \end{aligned} \tag{3.3.5}$$

So, by combining $(3.3.4)$ and $(3.3.5)$, we deduce that

$$\int_{\Omega'} |\sigma(x,u_k,Du_k) : Du_k|dx \leq c'\beta \left(\|\lambda_1\|_{p'} \| \|u_k\|_{1,p,\omega} + \|u_k\|_{1,p,\omega} \right) < \infty. \tag{3.3.6}$$

Similarly to $(|(I_k^2)^-|$, we obtain $(|(I_k^3)^-|$. Finally: I_k is equi-integrable.

We choose a sequence φ_k such that φ_k belongs to the same space V_k and $\varphi_k \to \varphi$ in $W_0^{1,p}(\Omega,\omega,I\!R^m)$ this allows us in particular, to use $u_k - \varphi_k$ as a test function in (3.2.1). We have :

$$\begin{aligned}\int_{\Omega} |\sigma(x,u_k,Du_k) : (Du_k - D\varphi_k)|dx = \langle v, u_k - \varphi_k \rangle \\ + \int_{\Omega} f(x,u_k)(u_k - \varphi_k)dx. \end{aligned} \tag{3.3.7}$$

The first term on the right in $(3.3.7)$ converge to zero since $(u_k - \varphi_k) \rightharpoonup 0$ in $W_0^{1,p}(\Omega,\omega,I\!R^m)$.

By the choice of φ_k, the sequence φ_k uniformly bounded in $W_0^{1,p}(\Omega,\omega,I\!R^m)$. And lemma $(3.3.1)$. Next, for the second term: $II_k = \displaystyle\int_{\Omega} f(x,u_k)(u_k - \varphi_k)dx$ in $(3.3.7)$ it follows from the growth condition (F_1) and the Hölder inequality that:

$$|II_k| \leq (\|b_1\|_{p'} + c.\|D(u_k - \varphi_k)\|_{1,p,\omega}) \|u_k - \varphi_k\|_{1,p,\omega}$$

$$\leq (\|b_1\|_{p'} + c.\|D(u_k - \varphi_k)\|_{1,p,\omega}).\|u_k - \varphi_k\|_{1,p,\omega}.$$

By the equivalence of the norm in $W_0^{1,p}(\Omega,\omega,I\!R^m)$ and the sequence (u_k is uniformly bounded in $W_0^{1,p}(\Omega,\omega,I\!R^m)$, $\|u_k\|_{1,p,\omega}$ is bounded. Moreover, by the construction of φ_k, and lemma 3.3.1 we have:

$$\|u_k-\varphi_k\|_{1,p,\omega}\leq\|u_k-u\|_{1,p,\omega}+\|u-\varphi_k\|_{1,p,\omega}$$

$$(\|u_k-u\|_{1,p,\omega}+\|u-\varphi_k\|_{1,p,\omega})\to 0$$

We infer that the second term in $(3.3.7)$ vanishes as $k\to\infty$. Now, we consider $(I_k)'=(\sigma(x,u_k,Du_k):(Du_k-Du)$. We have, I_k' is equi-integrable because I_k it is. So, we define

$$X=\liminf\int_\Omega I_k dx=\liminf\int_\Omega (I_k)'dx\geq\int_\Omega\int_{I\!M^{m\times n}}(\sigma(x,u,\lambda):(\lambda-Du)d\vartheta_x(\lambda)$$

So to prove (**??**), it suffices to prove that:

$$X\leq 0. \tag{3.3.8}$$

Let $\varepsilon>0$, so there exists $k_0\in I\!N$ such that, for all $k>k_0$, we have $dist(u,V_k)<\varepsilon$ since:
$(\liminf_{\varphi_k\in V_k}(\|u-\varphi_k\|_{1,p,\omega}<\varepsilon$, $(u_k\rightharpoonup u)$
Or in an equivalent manner $dist(u_k-u,V_k)<\varepsilon$; $\forall k>k_0$ then for all $v_k\in V_k$, we have

$$\begin{aligned}X&=\liminf_{k\to\infty}\int_\Omega(\sigma(x,u_k,Du_k):(Du_k-Du)dx\\&=\liminf_{k\to\infty}\left[\int_\Omega(\sigma(x,u_k,Du_k):D(u_k-u-\varphi_k)dx+\int_\Omega(\sigma(x,u_k,Du_k):D(\varphi_k)\right]\end{aligned}$$

Combining (H_2) and $(3.0.1)$, we get

$$X\leq\liminf_{k\to\infty}\int_\Omega\beta\omega_{rs}^{1/p}\left[\lambda_1+c_1\sum_{1\leq j\leq m}\gamma_j^{1/p'}|u_{k_j}|^{q/p'}+c_1\sum_{1\leq i,j\leq n,m}\omega_{ij}^{1/p'}|D_{ij}u_k|^{p-1}\right]$$

$$\times|D_{rs}(u_k-u-\varphi_k)|dx+\langle v,\varphi_k\rangle.$$

For all $\varepsilon>0$, we choice $\varphi_k\in V_k$ such that

$$()\ \|u_k-u-\varphi_k\|_{1,p,\omega}\leq 2\varepsilon, \tag{3.3.9}$$

for all $k\geq k_0$. Which implies that

$$|\langle v,\varphi_k\rangle|\leq|\langle v,\varphi_k+(u-u_k)\rangle|+|\langle v,u_k-u\rangle|\leq 2\varepsilon\|v\|_{-1,p',\omega^*}+o(k)$$

Hence $\lim_{k\to\infty}\langle v, u_k - u\rangle = 0$. According to Hölder and Hardy type inequalities, and by (3.3.9) we deduce that

$$\begin{aligned} X &\leq \liminf_{k\to\infty} c\beta\Bigg(\Big\|\lambda_1\|_{p'}\Big(\int_\Omega |D_{rs}(u_k - u - \varphi_k)|^p.\omega_{rs}dx\Big)^{1/p} \\ &\quad + c_1\left(\int_\Omega |u_k|^q.\gamma\right)^{1/p'}.\left(\int_\Omega |D_{rs}(u_k - u - \varphi_k)|^p\omega_{rs}dx\right)^{1/p} \\ &\quad + c_1\left(\sum\int_\Omega \omega_{ij}|D_{ij}u|^{p'(p-1)}\right)^{1/p'}.\left(\int_\Omega \omega_{rs}|D_{rs}(u_k - u - \varphi_k)|^p\right)^{1/p}\Bigg) + |\langle v, \varphi_k\rangle| \\ &\leq \liminf_{k\to\infty} c\left(\|\lambda_1\|_{p'}.\|u_k - u - \varphi_k\|_{1,p,\omega}\right) + \|u_k\|^q_{1,p,\omega}\|u_k - u - \varphi_k\|_{1,p,\omega} + 2\varepsilon\|v\|_{-1,p',\omega^*} + o(k) \end{aligned}$$

Therefore,

$$X \leq 2\varepsilon c\beta\left(\|\lambda_1\|_{p'} + \|u\|^q_{1,p,\omega} + \|v\|_{-1,p',\omega*}\right).$$

Which proves that $X \leq 0$, and finally

$$\int_\Omega\int_{\mathbb{M}^{mn}} \sigma(x,u,\lambda) : \lambda d\vartheta_x dx \leq \int_\Omega\int_{\mathbb{M}^{mn}} \sigma(x,u,\lambda) : Du d\vartheta_x(\lambda)dx.$$

Proof of theorem3.1.1

For arbitrary φ in $W^{1,p}_0(\Omega,\omega,\mathbb{R}^m)$. It follows from the continuity condition (F_0) and (G_0) that

$$f(x,u_k).\varphi(x) \to f(x,u).\varphi(x)$$

almost everywhere. Since, by the growth conditions (F_1) and the uniform bound of u_k, $f(x,u_k).\varphi(x)$ is equi-integrable, it follows that the Vitali's theorem. This implies that:

$$\lim_{k\to\infty}\int_\Omega f(x,u_k).\varphi(x)dx = \int_\Omega f(x,u).\varphi(x)dx$$

for all$\varphi \in \cup_{k=1}^\infty V_k$ We will start with the easiest case

$$(d): \qquad F \longmapsto \sigma(x,u,F) \text{ is stictly } p\text{-quasi-monotone.} \tag{3.3.10}$$

Indeed, we assume that ϑ_x is not a Dirac mass on the set M with $x \in M$ of positive Lebesgue measure $|M| > 0$. Moreover, by the strict p-quasi-monotonicity of $\sigma(x,u,\cdot)$ and ϑ_x is an homogeneous $W^{1,p}$ gradient young measure for a.e. $x \in M$. So, for a.e. $x \in M$, with $\bar{\lambda} = \langle\vartheta_x, Id\rangle = apDu(x))$, with $apDu(x))$ is the differentiable approximation in x. We get

$$\begin{aligned} \int_{\mathbb{M}^{m\times n}} \sigma(x,u,\lambda) : (\lambda - Du)d\vartheta_x(\lambda) &> \int_{\mathbb{M}^{m\times n}} \sigma(x,u,Du) : (\lambda - Du)d\vartheta_x(\lambda) \\ &> \sigma(x,u,Du) : \int_{\mathbb{M}^{m\times n}} \lambda d\vartheta_x(\lambda) - \sigma(x,u,Du) : Du\int_{\mathbb{M}^{m\times n}} d\vartheta_x(\lambda) \\ &> (\sigma(x,u,Du) : Du - \sigma(x,u,Du) : Du)) = 0 \\ &> 0 \end{aligned}$$

On the other hand $(4.3.8)$, integrating over Ω, and using the div-cul inequality we have:

$$\begin{aligned}\int_\Omega \int_{I\!M^{m\times n}} \sigma(x,u,\lambda) : \lambda d\vartheta_x(\lambda)dx &> \int_\Omega \int_{I\!M^{m\times n}} \sigma(x,u,\lambda) : Du d\vartheta_x(\lambda)dx \\ &\geq \int_\Omega \int_{I\!M^{m\times n}} \sigma(x,u,\lambda) : \lambda d\vartheta_x(\lambda)dx.\end{aligned}$$

Which is a contradiction with $(3.3.7)$. Thus $\vartheta_x = \delta_{\overline{\lambda}} = \delta_{Du(x)}$ for a.e. $x \in \Omega$. Therefore, $Du_k \longrightarrow Du$ in measure when k tends to infinity. Then, we get $\sigma(x,u_k,Du_k) \longrightarrow \sigma(x,u,Du)$ for all $x \in \Omega$. In the other hand, for all $\varphi \in \bigcup_{k\in I\!N} V_k$; $\sigma(x,u_k,Du_k) : D\varphi \longrightarrow \sigma(x,u,Du) : D\varphi$ a.e. $x \in \Omega$. Moveover, for all $\Omega' \subset \Omega$ measurable, it is easy to see that:

$$\int_{\Omega'} \sigma(x,u_k,Du_k) : D\varphi dx \leq c\beta \left(\|\lambda_1\|_{p'} + \|u_k\|_{1,p,\omega}^{q/p'} + \|u_k\|_{1,p,\omega}^{p/p'} \right) \|u\|_{1,p,\omega} < \infty,$$

because $\|u_k\|_{1,p,\omega} \leq R$. And thanks to Vitali's theorem, we obtain:

$$\langle F(u), \varphi \rangle = 0, \text{ for all } \varphi \in \bigcup_{k\in I\!N} V_k.$$

Which proves the theorem in this case.

Remark 3.3.1 *Before treating the cases (a), (b) and (c) of (H_3), we note that*

$$\int_\Omega \int_{I\!M^{m\times n}} (\sigma(x,u,\lambda) - \sigma(x,u,Du)) : (\lambda - Du) d\vartheta_x(\lambda)dx \leq 0 \tag{3.3.11}$$

Since

$$\int_\Omega \int_{I\!M^{m\times n}} \sigma(x,u,\lambda) : (\lambda - Du) d\vartheta_x(\lambda)dx = 0,$$

thanks to the div-Curl inequality in $(4.3.8)$. On the other hand, the integrand in $(3.3.11)$ is non negative, by the monotonicity of σ. Consequently, the integrating should be null, a.e., with respect to the product measure $d\vartheta_x \otimes dx$, which mean

$$(\sigma(x,u,\lambda) - \sigma(x,u,Du)) : (\lambda - Du) = 0 \text{ in } spt\vartheta_x. \tag{3.3.12}$$

Thus,

$$spt\vartheta_x \subset \left\{ \lambda \in I\!M^{m\times n} / \, (\sigma(x,u,\lambda) - \sigma(x,u,Du)) : (\lambda - Du) = 0 \right\}. \tag{3.3.13}$$

Case c: We prove that, the map $F \longmapsto \sigma(x,u,F)$ is strictly monotone, for all $x \in \Omega$ and for all $u \in I\!R^m$. Sine σ is strict monotone, and according to $(3.3.13)$,

$$spt\vartheta_x = \{Du\}, \text{ i.e, } \vartheta_x = \delta_{Du}, \quad \text{a.e. in } \Omega,$$

which implies that, $Du_k \longrightarrow Du$ in measure. For the rest of our prove is similarly to case d.

Cases a and b: For cases (a) resp (b) we use almost the same techniques that we used in (a) resp(b) chapter3, then we move to limit $\square$

Example 3.3.1 *Let Ω is a bounded open domain in $\mathbb{R}^n$. In this example we suppose that the weighted functions: $(\omega_{ij}),\ 0 \leq i \leq n;\ \ 1 \leq j \leq m$ and σ satisfies the conditions in example 2.3.1. Hence we have the Hardy-Type inequalities in (P_0) for all $u \in W_0^{1,p}(\Omega, \omega); q > p'$.*

Let us consider the functions f defined by:

$f_k(x,u) = \omega_{0k}^{\frac{1}{p}} e^{-|x|} + \mid u_j \mid^{p-1} sign(u_j)\omega_{0k}^{\frac{1}{p'}}$, *the function f satisfies $F_0 - F_1$.*

Chapter 4

Quasi-linear degenerated elliptic systems with weighted in divergence form with weak monotonicity with general data

Abstract

we consider, for a bounded open domain Ω in $I\!R^n$ and a function $u : \Omega \to I\!R^m$, the quasi-linear elliptic system :

$$\text{(QES)}_{(f,g)} \begin{cases} -div\sigma\left(x, u\left(x\right), Du\left(x\right)\right) &= v(x) + f(x, u, Du) + divg(x, u) \text{ in } \Omega \\ u &= 0 \qquad\qquad\qquad\qquad\qquad\qquad \text{on } \partial\Omega, \end{cases} \tag{4.0.1}$$

We generalize the system $(QES)_{f,g}$ in considering a right hand side depending, on the jacobian matrix Du. Here ,the star in $(QES)_{f,g}$ indicates that f may depend on Du. In the right hand side, v belongs to the dual space $W^{-1,p'}(\Omega, \omega^*, R^m)$, $\left(\frac{1}{p} + \frac{1}{p'} = 1,\ p > 1\right)$, $f$and g satisfy some standard continuity and growth conditions. We prove existence of a regularity, growth and coercivity conditions for σ, but with only very mild monotonicity assumptions.

4.1 Preliminaries

Let $\omega = \{\omega_{ij} \mid \ 0 \leq i \leq n; 1 \leq j \leq m\}$ the weight function systems defined in Ω and satisfying the following integrability conditions:

$$\omega_{ij} \in L^1_{loc}(\Omega) \ , \ \omega_{ij}^{\frac{-1}{p-1}} \in L^1_{loc}(\Omega), \text{ for some } p \in]1, \infty[\text{ and } \exists s > 0 \text{ such that } \omega_{ij}^{-s} \in L^1(\Omega). \qquad (4.1.1)$$

with $\omega^* = \{\omega^*_{ij} = \omega_{ij}^{1-p'}, \ \ 0 \leq i \leq n, \ 1 \leq j \leq m\}$, $\sigma = (\sigma_{rs})$ with $1 \leq s \leq n, \ 1 \leq r \leq m$ and which satisfies some hypotheses (see below).

The space $W^{1,p}(\Omega, \omega, I\!R^m)$ is the set of functions

$$\{u = u(x)/u \in L^p(\Omega, \overline{\omega_0}, I\!R^m)\}, \ D_{ij}u = \frac{\partial u^i}{\partial x_j} \in L^p(\Omega, \omega_{ij}, I\!R^m), \ 1 \leq i \leq n, \ 1 \leq j \leq m\}.$$

with

$$L^p(\Omega, \omega_{ij}, I\!R^m) = \{u = u(x)/ \mid u \mid \omega_{ij}{}^{\frac{1}{p}} \in L^p(\Omega, I\!R^m)\}$$

The weighted space $W^{1,p}(\Omega, \omega, I\!R^m)$ can be equipped by the norm:

$$\|u\|_{1,p,\omega} = \left(\sum_{j=1}^{m} \int_\Omega |u_j|^p \omega_{0j} dx + \sum_{1 \leq i \leq n, 1 \leq j \leq m} \int_\Omega |D_{ij}u|^p \omega_{ij} dx \right)^{\frac{1}{p}}$$

where $\overline{\omega_0} = (\omega_{0j})$ and $1 \leq j \leq m$. the norm $\|.\|_{1,\omega,p}$ is equivalent to the norm $|||\ .\ |||$, on $W^{1,p}_0(\Omega, \omega, I\!R^m)$, such that, $|||\ u\ ||| = (\sum_{1 \leq i \leq n, 1 \leq j \leq m} \int_\Omega |D_{ij}u|^p \omega_{ij} dx)^{\frac{1}{p}}$

Proposition 4.1.1 *The weighted Sobolev space $W^{1,p}(\Omega, \omega, I\!R^m)$ is a Banach space, separable and reflexive. The weighted Sobolev space $W^{1,p}_0(\Omega, \omega, I\!R^m)$ is the closure of $C^\infty_0(\Omega, \omega, I\!R^m)$ in $W^{1,p}(\Omega, \omega, I\!R^m)$ equipped by the norm $\|.\|_{1,p,\omega}$.*

Proof: The prove of proposition is a slight modification of the analogous one in [25] [Kufner-Drabek].

Definition 4.1.1 *A Young measure $(\vartheta_x)_{x \in \Omega}$ is called $W^{1,p}$-gradient young measures $(1 \leq p < \infty)$ if it is associated to a sequence of gradients Du_k such that u_k is bounded in $W^{1,p}(\Omega)$.*
The $W^{1,p}$-gradient young measures $(\vartheta_x)_{x \in \Omega}$ is called homogeneous, if it doesn't depend on x, i-e, if $\vartheta_x = \vartheta$ for a.e. $x \in \Omega$.

Hypothesis (H_0) (Hardy-Type inequalities): There exist some constant $c > 0$, some weighted function γ and some real q $(1 < q < \infty)$ such that,

$$\left(\sum_{j=1}^{m} \int_\Omega |u_j(x)|^q \gamma_j(x) dx \right)^{\frac{1}{q}} \leq c \left(\sum_{1 \leq i \leq n 1 \leq j \leq m} \int_\Omega |D_{ij}u|^p \omega_{ij} \right)^{\frac{1}{p}},$$

for all $u \in W_0^{1,p}(\Omega, \omega, I\!R^m)$, with $\gamma = \{\gamma_j / 1 \leq j \leq m\}$.

The injection $W_0^{1,p}(\Omega, \omega, I\!R^m) \hookrightarrow\hookrightarrow L^q(\Omega, \gamma, I\!R^m)$ is compact, and $W_0^{1,p}(\Omega, \omega, I\!R^m) \hookrightarrow\hookrightarrow L^r(\Omega, I\!R^m)$ is compact, (by [25]) with

$$\begin{cases} 1 \leq r \prec \frac{nps}{n(s+1)-ps} & \text{if} \quad ps \prec n(s+1) \\ r \geq 1 & \text{if} \quad n(s+1) \prec ps \end{cases}$$

(H_1) Continuity: $\sigma : \Omega \times I\!R^m \times I\!M^{m\times n} \longrightarrow I\!M^{m\times n}$ is a Carathéodory function (i-e $x \longmapsto \sigma(x, u, F)$ is measurable for every $(u, F) \in I\!R^m \times I\!M^{m\times n}$ and $(u, F) \longmapsto \sigma(x, u, F)$ is continuous for almost every $x \in \Omega$). (H_2) Growths and coercivity conditions: There exist $c_1 \geq 0, c_2 > 0$ $\lambda_1 \in L^{p'}(\Omega),\ \lambda_2 \in L^1(\Omega),\ \lambda_3 \in L^{(p/\alpha)'}(\Omega),\ 0 < \alpha < p,\ 1 < q < \infty$ and $\beta > 0$ such that for all $1 \leq r \leq n,\ 1 \leq s \leq m$, we have:

$$|\sigma_{rs}(x, u, F)| \leq \beta\omega_{rs}^{1/p}[\lambda_1(x) + c_1 \sum_{j=1}^{m} |\gamma_j|^{1/p'} \cdot |u_j|^{q/p'} + c_1 \sum_{1\leq i\leq N 1\leq j\leq m} \omega_{ij}^{1/p'} |F_{ij}|^{p-1}] \tag{4.1.2}$$

and

$$\sigma(x, u, F) : F \geq -\lambda_2(x) - \sum_{j=1}^{m} \omega_{0j}(x)^{\alpha/p} \lambda_3(x) |u_j|^{\alpha} + c_2 \sum_{1\leq i\leq n 1\leq j\leq m} \omega_{ij}(x) \cdot |F_{ij}|^p \tag{4.1.3}$$

(H_3) Monotonicity conditions: σ satisfies one of the following conditions:

a) For all $x \in \Omega$, and all $u \in I\!R^m$, the map $F \longmapsto \sigma(x, u, F)$ is a C^1-function and is monotone(i-e, $(\sigma(x, u, F) - \sigma(x, u, G)) : (F - G) \geq 0$, for all $x \in \Omega,$ all $u \in I\!R^m$ and all $F, G \in I\!M^{m\times n}$).

b) There exists a function $W : \Omega \times I\!R^m \times I\!M^{m\times n} \longrightarrow I\!M^{m\times n}$ such that $\sigma(x, u, F) = \dfrac{\partial W}{\partial F}(x, u, F)$ and $F \longmapsto W(x, u, F)$ is convex and C^1 function.

c) For all $x \in \Omega$, and for all $u \in I\!R^m$ the map $F \longmapsto \sigma(x, u, F)$ is strictly monotone (i.e, $\sigma(x, u, .)$ is monotone and :$[(\sigma(x, u, F) - \sigma(x, u, G)) : (F - G) = 0] \Longrightarrow F = G$).

d) $\sigma(x, u, F)$ is strictly p-quasi-monotone in F, i.e,

$$\int_{I\!M^{m\times n}} (\sigma(x, u, \lambda) - \sigma(x, u, \overline{\lambda})) : (\lambda - \overline{\lambda})) d\vartheta(\lambda) > 0,$$

for all homogeneous $W^{1,p,\omega}$-gradient young measures ϑ with center of mass $\overline{\lambda} = \langle \vartheta, id \rangle$ which are not a single Dirac mass.

The main point is that we do not require strict monotonicity or monotonicity in the variables (u, F) in (H_3) as it is usually assumed in previous work see ([41] or [43]) and [44].

$(F_0)^*$:(continuity) $f : \Omega \times I\!R^m \times I\!M^{m\times n} \to I\!R^m$ is a Carathéodory function i-e :$x \mapsto f(x, u, F)$ is measurable for every $u \in I\!R^m$, and $F \in I\!M^{m\times n}$, $(u, F) \mapsto f(x, u, F)$ is continuous for almost every $x \in \Omega$.

$(F_1)^*$: (growth condition): The exist : $b_1 \in L^{p'}(\Omega)$,$c_1' > 0$,$c_2' > 0$ such that :

$$|f_j(x, u, F)| \leq [b_1(x) + c_1'\gamma_j^{\frac{1}{p'}}|u_j|^{\frac{q}{p'}} + c_2' \sum_{r,s} \omega_{rs}^{\frac{1}{P'}}|F_{rs}|^{p-1}]\omega_{0j}^{\frac{1}{P}}$$

;$\forall 1 \leq j \leq m$ (G_0) : (continuity) the map $g : \Omega \times I\!R^m \to I\!M^{m\times n}$ is a Carathéodory function . (G_1): (growth condition) There exist : $b_2 \in L^{p'}(\Omega)$

$$|g_{rs}| \leq \omega_{rs}^{\frac{1}{p}}[b_2 + \sum_j \gamma_j^{\frac{1}{p'}}|u_j|^{\frac{q}{p'}}]$$

For all $1 \leq r \leq n$ and $1 \leq s \leq m$.

Our aim of this chapter is to prove the existence of the problem $(QES)_{f,g}$ in the space $W_0^{1,P}(\Omega, \omega, I\!R^m)$.

Remark 4.1.1 *The condition $(F_0)^*$ and (G_0) ensure the measurability of f and g for all measurable function u. (F_1) and $(G_1)^*$ ensure that growths conditions, in particularly : if $u \in W_0^{1,P}(\Omega, \omega, I\!R^m)$ then $f(., u(.), D(.)).u(.)$ and $g(., u) : Du$ is in $L^1(\Omega, \omega)$.*

Exploiting the convergence in measure of the gradients of the approximating solutions, we will prove the following theorem.

Theorem 4.1.1 *If $p \in (1, \infty)$ and σ satisfies the conditions $(H_0) - (H_3)$, then the Dirichlet problem $(QES)^*_{f,g}$ has a weak solution $u \in W_0^{1,p}(\Omega, \omega, I\!R^m)$, for every $v \in W^{-1,p'}(\Omega, \omega^*, I\!R^m)$, f satisfies $(F_0)^*$ and $(F_1)^*$ and g satisfies (G_0) and (G_1).*

In order to prove theorems, we will apply a Galerkin scheme, with this aim in view, we establish in the following subsections, the key ingredient to pass to the limit for this, we assume that the conditions:$(H_0)-(H_3)$,$(F_0)^*$,$(F_1)^*$, (G_0) and (G_1).

Lemma 4.1.1 *For arbitrary $u \in W_0^{1,p}(\Omega, \omega, I\!R^m)$ and $v \in W^{-1,p'}(\Omega, \omega^*, I\!R^m)$, the functional*

$$\begin{aligned} F(u) : W_0^{1,p}(\Omega, \omega, I\!R^m) &\longrightarrow I\!R \\ \varphi &\longmapsto \int_\Omega \sigma(x, u(x), Du(x)) : D\varphi(x)dx \end{aligned}$$

$$- \langle v, \varphi\rangle - \int_\Omega f(x, u, Du) : \varphi dx + \int_\Omega g(x, u) : D\varphi dx.$$

is well defined, linear and bounded.

Proof For all $\varphi \in W_0^{1,p}(\Omega, \omega, I\!R^m)$, we denote

$$F(u)(\varphi) = I_1 + I_2 + I_3 + I_4$$

with

$$I_1 = \int_\Omega \sigma(x, u(x), Du(x)) : D\varphi(x)dx,$$

and

$$I_2 = -\langle v, \varphi \rangle .$$

$$I_3 = -\int_\Omega f(x, u, Du) : \varphi dx$$

$$I_4 = \int_\Omega g(x, u) : D\varphi dx$$

We define

$$I_{rs} = \int_\Omega \sigma_{rs}(x, u(x), Du(x)) : D_{rs}\varphi(x)dx$$

Firstly, by virtue of the growth conditions (H_2) and the Hölder inequality, one has

$$\begin{aligned}
|I_{rs}| &\leq \int_\Omega |\sigma_{rs}(x, u(x), Du(x))| : |D_{rs}\varphi(x)|dx \\
&\leq \int_\Omega \beta\omega_{rs}^{1/p}(x)[\lambda_1(x) + c_1\sum_{j=1}^m |\gamma_j(x)|^{1/p'}.|u_j(x)|^{q/p'} + c_1 \sum_{1\leq i\leq n1\leq j\leq m} \omega_{ij}^{1/p'}|D_{ij}u|^{p-1}]|D_{rs}\varphi|dx. \\
&\leq \beta[(\int_\Omega |\lambda_1(x)|^{p'}dx)^{1/p'}(\int_\Omega |D_{rs}\varphi(x)|^p\omega_{rs}dx)^{1/p} + (\int_\Omega |D_{rs}\varphi(x)|^p\omega_{rs})^{1/p}.(\sum_{j=1}^m \int_\Omega |u_j|^q\gamma_j dx)^{1/p'} \\
&\quad +(\sum_{1\leq i\leq n1\leq j\leq m} \int_\Omega |D_{ij}u|^p\omega_{ij}dx)^{1/p'}(\int_\Omega |D_{rs}\varphi|^p\omega_{rs}dx)^{1/p}]
\end{aligned}$$

with $(p = p'(p-1))$, and thanks to Hardy inequality we have:

$$\begin{aligned}
|I_{rs}| &\leq c\beta\Bigg[\|\lambda_1\|_{p'} . \|\varphi\|_{1,p,\omega_{rs}} . + c_1 \|D\varphi\|_{p,\omega_{rs}} (\int_\Omega |u|^q\gamma dx)^{1/p'} \\
&\qquad +c_1\sum_{ij} \|D\varphi\|_{p,\omega_{ij}} \|Du\|_{p,\omega_{rs}}\Bigg] \\
&\leq c'\beta[\|\lambda_1\|_{p'} \|\varphi\|_{1,p,\omega_{rs}} + \|\varphi\|_{1,p,\omega_{rs}} \|u\|_{q,\gamma} + \|u\|_{1,p,\omega} \|\varphi\|_{1,p,\omega_{rs}}
\end{aligned}$$

with $c' = \max(c, 1)$. Which gives

$$|I_1| \leq c'\beta[\|\lambda_1\|_{p'} + \|u\|_{1,p,w}^{q/p'} + \|u\|_{1,p,w}]\|\varphi\|_{1,p,w} < \infty.$$

and

$$|I_2| \leq \int_\Omega |v||\varphi|dx \leq \|v\|_{-1,p',w*} \|\varphi\|_{1,p,w} < \infty.$$

$$I_3 = \sum_j \int_\Omega f_j(x,u,Du)\varphi_j(x)dx$$

We denote $I_{3,j} = |\int_\Omega f_j(x,u,Du)\varphi_j(x)dx|$.

$$I_{3,j} \leq \int_\Omega |f_j(x,u,Du)||\varphi_j(x)|dx|$$

$$\leq \int_\Omega b_1(x)|\varphi_j(x)|\omega_{0j}^{\frac{1}{P}}dx + c_1' \int_\Omega \gamma_j^{\frac{1}{p'}}|u_j|^{\frac{q}{p'}}|\varphi_j(x)|\omega_{0j}^{\frac{1}{P}} + c_2' \int_\Omega \sum_{rs}|\varphi_j(x)||D_{rs}u|^{p-1}\omega_{0j}^{\frac{1}{P}}dx$$

$$\leq (\int_\Omega |b_1(x)|^{p'})^{\frac{1}{p'}}.(\int_\Omega |\varphi_j(x)|^p\omega_{0j}dx)^{\frac{1}{p}} + (\int_\Omega \gamma_j(x)|u_j|^q dx)^{\frac{1}{p'}}.(\int_\Omega |\varphi_j(x)|^p\omega_{0j}dx)^{\frac{1}{p}}$$
$$+\sum_{rs}(\int_\Omega |\varphi_j|^p\omega_{0j})^{\frac{1}{p}}.(\int_\Omega |D_{rs}u|^{p'(p-1)}\omega_{rs})^{\frac{1}{p'}}$$

$$\leq \| b_1 \|_{p'} \| \varphi \|_{1,p,\omega} + c_1'(\sum_j \int_\Omega \gamma_j(x)|u_j|^q dx)^{\frac{1}{p'}}. \| \varphi \|_{1,p,\omega} + c_2'\|\varphi\|_{1,\omega,p}.\|Du\|_{1,p,\omega}^{\frac{p}{p'}}$$

$$\leq \| b_1 \|_{p'} \| \varphi \|_{1,p,\omega} + c_1'. \| Du \|_{1,p,\omega} . \| \varphi \|_{1,p,\omega} + c_2'\|\varphi\|_{1,\omega,p}.\|Du\|_{1,p,\omega}^{\frac{p}{p'}}$$

$$\leq (\| b_1 + c_1'. \| Du \|_{1,p,\omega} + c_2'.\|Du\|_{1,p,\omega}^{\frac{p}{p'}}). \| \varphi \|_{1,p,\omega}$$

.

$$I_4 = \sum_{rs}\int_\Omega g_{rs}(x,u)D_{rs}\varphi dx$$

$$\int_\Omega |g_{rs}| : |D_{rs}\varphi|dx \leq \int_\Omega b_2\omega_{rs}^{\frac{1}{p}}.D_{rs}\varphi dx + \sum_j \int_\Omega \gamma_j^{\frac{1}{p'}}(x)|u_j|^{\frac{q}{p'}}\omega_{rs}^{\frac{1}{p}}D_{rs}\varphi dx$$

$$\leq (\int_\Omega |b_2|^{p'}dx)^{\frac{1}{p'}}.(\int_\Omega |D_{rs}\varphi|^p\omega_{rs}dx)^{\frac{1}{p}} + \sum_j(\int_\Omega |u_j|^q\gamma_j(x)dx)^{\frac{1}{p'}}.(\int_\Omega |D_{rs}\varphi|^p\omega_{rs}(x)dx)^{\frac{1}{p}}$$

$$\leq \|b_2\|_{p'}\|D_{rs}\varphi\|_{1,p,\omega_{rs}} + \|u\|_{q,\gamma}^{\frac{q}{p'}}(\int_\Omega |D_{rs}\varphi|^p\omega_{rs}dx)^{\frac{1}{p}}$$

$$I_4 \leq \| b_2 \|_{p'} . \| D_{rs}\varphi \|_{1,p,\omega_{rs}} + \|u\|_{q,\gamma}^{\frac{q}{p'}}(\int_\Omega |D_{rs}\varphi|^p\omega_{rs}dx)^{\frac{1}{p}}$$

$$\leq \| b_2 \|_{p'} . \| D\varphi \|_{1,p,\omega} + \|u\|_{q,\gamma}^{\frac{q}{p'}}. \| D\varphi \|_{1,p,\omega}$$

$$\leq c'' \parallel \varphi \parallel_{1,p,\omega} .$$

hence $I \leq c_4 \parallel \varphi \parallel_{1,p,\omega}$.With $c_4 < \infty$.

Finally the functional $F(.)$ is bounded.

Lemma 4.1.2 *The restriction of F to a finite dimensional linear subspace V of $W_0^{1,p}(\Omega, \omega, I\!R^m)$ is continuous.*

Proof Let d be the dimension of V and $(e_1, e_2, ..., e_d)$ a basis of V. Let $u_j = \sum_{1 \leq i \leq d} a_j^i . e_i$ be a sequence in V which converges to $u = \sum_{1 \leq i \leq d} a^i e_i$ in V. The sequence (a_j) converge to $a \in I\!R^d$, so $u_j \longrightarrow u$ and $Du_j \longrightarrow Du$ a.e., on the other hand $\|u_j\|_p$ and $\|Du_j\|_p$ are bounded by a constant c. Thus, it follows by the continuity conditions (H_1), that

$$\sigma(x, u_j, Du_j) : D\varphi \longrightarrow \sigma(x, u, Du) : D\varphi$$

for all $\varphi \in W_0^{1,p}(\Omega, \omega, I\!R^m)$ and a.e. in Ω. Let Ω' be a measurable subset of Ω and let $\varphi \in W_0^{1,p}(\Omega, \omega, I\!R^m)$. Thanks to the condition (H_2), we get

$$\int_{\Omega'} |\sigma(x, u_j, Du_j) : D\varphi| dx < \infty,$$

By the continuity conditions $(F_0)^*$ and (G_0) we have :

$$f(x, u_j, Du_j).\varphi \to f(x, u, Du).\varphi$$

And

$$g(x, u_j).D\varphi \to g(x, u).D\varphi$$

almost everywhere. Moreover we infer from the growth conditions $(F_1)^*$ and (G_1) that the sequences:
$(\sigma(x, u_j, Du_j) : D\varphi)$,$(f(x, u_j, Du_j).\varphi)$ and $(g(x, u_j).D\varphi)$
Are equi-integrable. Indeed, if $\Omega' \subset \Omega$ is a measurable subset and
$\varphi \in W_0^{1,p}(\Omega, \omega, I\!R^m)$ then: $\int_{\Omega'} |f(x, u_j, Du_j).\varphi| dx < \infty$ (by $(F_1)^*$ and Hölder inequality).
$\int_{\Omega'} |g(x, u_j).D\varphi| dx < \infty$ (by (G_1) and Hölder inequality)
$\int_{\Omega'} |\sigma(x, u_j, Du_j) : D\varphi| dx < \infty$ (by Hölder inequality)
which implies that $\sigma(x, u_j, Du_j) : D\varphi$ is equi-integrable. And by applying the Vitali's theorem, it follows that

$$\int_{\Omega} \sigma(x, u_j, Du_j) : D\varphi dx \longrightarrow \int_{\Omega} \sigma(x, u, Du) : D\varphi dx,$$

for all $\varphi \in W_0^{1,p}(\Omega, \omega, I\!R^m$.

Finally

$$\lim_{j\to\infty} \langle F(u_j), \varphi\rangle = \langle F(u), \varphi\rangle ,$$

which means that

$$F(u_j) \longrightarrow F(u) \text{ in } W^{-1,p'}(\Omega, \omega^*, I\!R^m).$$

Remark 4.1.2 *Now, the problem $(QES)^*_{f,g}$ is equivalent to find a solution $u \in W_0^{1,p}(\Omega, \omega, I\!R^m)$ such that $\langle F(u), \varphi\rangle = 0$, for all $\varphi \in W_0^{1,p}(\Omega, \omega, I\!R^m)$.*
In order to find such a solution we apply a Galerkin scheme .

4.2 Galerkin approximation

Remark 4.2.1 $(Galerkin\ Schema)$

Let $V_1 \subset V_2 \subset ..... \subset W_0^{1,p}(\Omega, \omega, I\!R^m)$ be a sequence of finite dimensional subspaces with $\bigcup_{k\in I\!N} V_k$ dense in $W_0^{1,p}(\Omega, \omega, I\!R^m)$. The sequence V_k exists since $W_0^{1,p}(\Omega, \omega, I\!R^m)$ is separable.

Let us fix some k, we assume that V_k has a dimension d and that $(e_1, e_2, ..., e_d)$ is a basis of V_k. Then, we define the map,

$$\begin{array}{rccl} G: & I\!R^k & \longrightarrow & I\!R^k \\ & (a_1, ..., a_k) & \longmapsto & (\langle F(u), e_1\rangle , ..., \langle F(u), e_k\rangle); u = \sum_{i=1}^d a_i e_i. \end{array}$$

Proposition 4.2.1 *The map G is continuous and $G(a)\cdot a$ tends to infinity when $\|a\|_{I\!R^k}$ tends to infinity.*

Proof. Since F restricted to V_k is continuous by Lemma 4.1.2, so G is continuous.
Let $a \in I\!R^d$ and $u = \sum_{1\le i\le d} a^i \cdot e_i$ in V_k, then $G(a)\cdot a = \langle F(u), u\rangle$ and which implies that $\|a\|_{I\!R^d}$ tends to infinity if $\|u\|_{1,p,\omega}$ tends to infinity.

$$G(a)\cdot a = \sum_{1\le i\le d} \langle F(u), a^i \cdot e_i\rangle = \langle F(u), u\rangle$$

and

$$\begin{aligned} \|u\|^p_{1,p,\omega} = \left\| \sum_{1\le i\le d} a^i \cdot e_i \right\|^p_{1,p,\omega} &\le \left(\sum_{1\le i\le d} |a^i| \cdot \|e_i\|_{1,p,\omega} \right)^p \\ &\le \max_{1\le i\le d} \left(\|e_i\|^p_{1,p,\omega} \right) \cdot \left(\sum_{1\le i\le d} |a^i| \right)^p \\ &\le c \cdot \|a\|_{I\!R^p} , \end{aligned}$$

which implies that $\|a\|_{I\!R^p}$ tends to infinity if $\|u\|_{1,p,\omega}$ tends to infinity.

Now, it suffices to prove that

$$\langle F(u), u\rangle \to \infty \quad \text{when} \quad \|u\|_{1,p,\omega} \to \infty.$$

Indeed, thanks to the first coercivity condition and the Hölder inequality, we obtain

$$I = \int_\Omega \sigma(x,u,Du) : Dudx \geq -\|\lambda_2\|_1 - \int_\Omega \lambda_3 \omega_{0j}^{\alpha/p} |u_j|^\alpha dx + c_2 \sum_{1\leq i,j\leq n,m} \int_\Omega |D_{ij}u|^p \omega_{ij} dx$$

By the Hölder inequality, we have

$$\begin{aligned} \int_\Omega \lambda_3 |u_j|^\alpha \omega_{0j}^{\alpha/p} dx &\leq \|\lambda_3\|_{(p/\alpha)'} \left(\int_\Omega \omega_{0j} |u_j|^{(p/\alpha)\cdot\alpha}\, dx \right)^{\alpha/p} \\ &\leq c' \|\lambda_3\|_{(p/\alpha)'} \|u_j\|_{1,p,\omega_{0j}}. \end{aligned}$$

where c' is a constant positive. For $\|u\|_{1,p,\omega}$ large enough, we can write

$$\begin{aligned} |\ I\ | &\geq -\|\lambda_2\|_1 - c' \|\lambda_3\|_{(p/\alpha)'} \cdot \|u_j\|^\alpha_{1,p,\omega_{0j}} + c_2 \cdot \textstyle\sum_{1\leq i,j\leq n,m} \|Du_j\|^p_{1,p,\omega_{ij}} \\ &\geq -\|\lambda_2\|_1 - c' \|\lambda_3\|_{(p/\alpha)'} \cdot \|u\|^\alpha_{1,p,\omega} + c_2 c' \cdot \|u\|^p_{1,p,\omega}. \end{aligned}$$

And since

$$|I'| = |\ \langle v, u\rangle\ | \leq \|v\|_{-1,p',\omega^*} \cdot \|u\|_{1,p,\omega}$$

Finally, it follows from the growth condition $(F_1)^*$ and G_1 that:

$$\begin{aligned} |I''| = |\int \Omega f(x,u,Du).udx| &\leq (\|b_1\|_{p'} + c_1'.\|Du\|_{1,p,\omega} + c_2'\|Du\|_{1,p,\omega}).\|u\|_{1,p,\omega}. \\ &\leq c_3.\|u\|_{1,p,\omega} \\ |I'''| = |\int \Omega g(x,u).Dudx| &\leq (\|b_2\|_{p'} + \|u\|^{\frac{q}{p'}}_{q,\gamma}).\|Du\|_{1,p,\omega} \leq c_4.\|u\|_{1,p,\omega}; \end{aligned}$$

with c_4 is a constant. With; $0 < \alpha < p$ and $p > 1$, we get:

$$I - I' - I'' \geq c_2.c'.\cdot\|u\|^p_{1,p,\omega} - \|v\|_{-1,p',\omega^*} \cdot \|u\|_{1,p,\omega} - c' \|\lambda_3\|_{(p/\alpha)'} \cdot \|u\|^\alpha_{1,p,\omega} - \|\lambda_2\|_1 - c_3.\ \|u\|_{1,p,\omega} \quad (4.2.1)$$

Consequently, by using (4.2.1), we deduce

$$I - I' - I'' \longrightarrow \infty \text{ as } \|u\|_{1,p,\omega} \longrightarrow \infty.$$

and

$$I''' \longrightarrow \infty \text{ as } \|u\|_{1,p,\omega} \longrightarrow \infty.$$

$$\langle F(u), u\rangle \longrightarrow \infty \text{ as } \|u\|_{1,p,\omega} \longrightarrow \infty.$$

Remark 4.2.2 *The properties of G allows us to construct our Galerkin approximations.*

Corollary 4.2.1 *For all $k \in \mathbb{N}$, there exists $(u_k) \subset V_k$ such that $\langle F(u_k), \varphi \rangle = 0$, for all $\varphi \in V_k$.*

Proof By the proposition 4.2.1, there exists $R > 0$, such that for all $a \in \partial B_R(0) \subset \mathbb{R}^d$, we have $G(a) \cdot a > 0$. And the usual topological argument see.e.g [Zei 86 proposition 2.8] [64] implies that $G(x) = 0$ has a solution $x \in B_R(0)$. So, for all $k \in \mathbb{N}$, there exists $(u_k) \subset V_k$, such that

$$\left\langle F(x^j e_j), e_j \right\rangle = 0 \text{ for all } 1 \leq j \leq d, \text{ with } d = \dim V_k$$

Taking $u_k = (x_k^i e_i),\ e_i \in V_k$, so we obtain:

$$\langle (F(u_k)), \varphi \rangle = 0, \text{ for all } \varphi \in V_k.$$

Proposition 4.2.2 *The Galerkin approximations sequence constructed in corollary $(4.2.1)$ is uniformly bounded in $W_0^{1,p}(\Omega, \omega, \mathbb{R}^m)$; i.e.,*

$$\textit{there exists a constant } R > 0, \textit{ such that } \|u_k\|_{1,p,\omega} \leq R, \textit{ for all } k \in \mathbb{N}.$$

Proof Like in the proof of proposition $(4.2.1)$, we can see that

$$\langle F(u), u \rangle \longrightarrow \infty \text{ as } \|u\|_{1,p,\omega} \longrightarrow \infty.$$

Then, there exists R satisfying $\langle F(u), u \rangle > 1$ when $\|u\|_{1,p,\omega} > R$. Now, for the sequence of Galerkin approximations $(u_k) \subset V_k$ of corollary $(4.2.1)$, which satisfying $\langle F(u_k), u_k \rangle = 0$, we have the uniform bound $\|u_k\|_{1,p,\omega} \leq R$, for all $k \in \mathbb{N}$.

Remark 4.2.3 *There exists a subsequence (u_k) of the sequence $(u_k) \subset V_k$, such that:*

$$u_k \rightharpoonup u \textit{ in } W_0^{1,p}(\Omega, \omega, \mathbb{R}^m)$$

and

$$u_k \longrightarrow u \textit{ in measure in } L^r(\Omega, \mathbb{R}^m);$$

with

$$\begin{cases} 1 \leq r \prec \frac{nps}{n(s+1)-ps} & \textit{if} \quad ps \prec n(s+1) \\ r \geq 1 & \textit{if} \quad n(s+1) \prec ps \end{cases}$$

The gradient sequence (Du_k) generates the young measure ϑ_x. Since $u_k \longrightarrow u$ in measure, then (u_k, Du_k) generates the Young measure $(\delta_{u(x)} \otimes \vartheta_x)$, see e.g [39]. Moreover, for almost x in Ω, we have,

(i) *ϑ_x is the probability measure, i.e,* $\|\vartheta_x\|_{mes} = 1$.

(ii) *ϑ_x is the $W^{1,p,\omega}$ - gradient homogeneous young measure.*

(iii) $\langle \vartheta_x, id \rangle = Du(x)$*, see e.g* [3] .

Proof. See [39].(Dolzmann, N.Humgerbuhler S. Muller, Non linear elliptic system....)

4.3 Passage to the limit

Now, we are in a position to prove our main result under convenient hypotheses.
Let

$$I_k = (\sigma(x, u_k, Du_k) - \sigma(x, u, Du)) : (Du_k - Du). \tag{4.3.1}$$

Proof. see [39].

Lemma 4.3.1 *Let $p > 1$ and u_k be a sequence which is uniformly bounded in $W_0^{1,p}(\Omega, \omega, I\!R^m)$. There exists a subsequence of u_k (for convenience not relabeled) and a function $u \in W_0^{1,p}(\Omega, \omega, I\!R^m)$ such that $u_k \rightharpoonup u$ in $W_0^{1,p}(\Omega, \omega, I\!R^m)$*
And such that $u_k \to u$ in measure on Ω and in $L^r(\Omega, I\!R^m)$, with :

$$\begin{cases} 1 \leq r \prec \frac{nps}{n(s+1)-ps} & \text{if} \quad ps \prec n(s+1) \\ r \geq 1 & \text{if} \quad n(s+1) \prec ps \end{cases}$$

Proof. see [38]. with a slight modification.

Lemma 4.3.2 *The sequence (I_k) is equi-integrable.*

Proof

We have

$$\begin{aligned} I_k &= (\sigma(x, u_k, Du_k) - \sigma(x, u, Du)) : (Du_k - Du) \\ &= [\sigma(x, u_k, Du_k) : Du_k] - [\sigma(x, u_k, Du_k) : Du] - [\sigma(x, u, Du) : Du_k] + [\sigma(x, u, Du) : Du] \\ &= I_k^1 + I_k^2 + I_k^3 + I_k^4 \end{aligned} \tag{4.3.2}$$

We denote $(I_k^1)^- = -[\sigma(x,u_k,Du_k : Du_k]^-$. Thanks to the coercivity condition (H_2), we have

$$\begin{aligned}\int_{\Omega'} |(I_k^1)^-|dx &\leq \int_{\Omega} |\lambda_2| + c_2 \sum_{1\leq j\leq m} \omega_{0j}^{\frac{\alpha}{p}} |\lambda_3| \cdot |u_{kj}|^{\alpha} + c \sum_{1\leq i,j\leq n,m} \omega_{ij} |D_{ij}u_k|^p dx \\ &\leq \|\lambda_2\|_1 + \int_{\Omega'} (\sum_{1\leq j\leq m} \omega_{0j}^{\alpha/p} |u_{kj}|^{\alpha})^{p/\alpha} \|\lambda_3\|_{(p/\alpha)'} + c_2 \|u_k\|_{1,\omega,p}^p \end{aligned} \tag{4.3.3}$$

with $p/\alpha \geq 1$. Therefore,

$$\begin{aligned}\int_{\Omega'} |(I_k^1)^-|dx &\leq \|\lambda_2\|_1 + \Big(\sum_{1\leq j\leq m} \omega_{0j} |u_{kj}|^p \Big)^{\alpha/p} \|\lambda_3\|_{(p/\alpha)'} + c_2 \|u_k\|_{1,\omega,p}^p \\ &\leq \|\lambda_2\|_1 + \|u_k\|_{p,\overline{\omega}_{00}}^{\alpha} \|\lambda_3\|_{(p/\alpha)'} + c_2 \|u_k\|_{1,\omega,p}^p \\ &< \infty, \end{aligned}$$

for all $\Omega' \subset \Omega$.

Similarly for $(|(I_k^4)^-|$.

Now, by using the growth condition (H_2) and the Hardy inequality (H_0), we have

$$\begin{aligned}\int_{\Omega'} |(I_k^2)^-|dx &= \int_{\Omega'} |\sigma(x,u_k,Du_k) : Du_k|dx \\ &\leq \beta \int_{\Omega'} \omega_{rs}^{1/p} \left(\lambda_1 + c_1 \sum_{1\leq j\leq m} \gamma_j^{1/p'} . |u_{kj}|^{q/p'} + c_2 \sum_{1\leq i,j\leq n,m} \omega_{ij}^{1/p'} |D_{ij}u_k|^{p-1} \right) D_{rs}u_k dx. \end{aligned} \tag{4.3.4}$$

Thus, by the Hölder inequality, we obtain

$$\begin{aligned}&\int_{\Omega'} |(I_k^2)^-|dx \leq \beta [\|\lambda_1\|_{p'} \left(\int_{\Omega'} |D_{rs}u_k|^p \omega_{rs} dx \right)^{1/p} \\ &+ c_1 \left(\int_{\Omega'} |D_{rs}u_k|^p \omega_{rs} dx \right)^{1/p} \left(\int_{\Omega'} \left(\sum_{1\leq j\leq m} \gamma_j^{1/p'} |u_{kj}|^{q/p'} \right)^{p'} dx \right)^{1/p'} \\ &+ c_2 \left(\sum_{1\leq j\leq m} \int_{\Omega'} \left(|D_{ij}u_k(x)|^{p'(p-1)} \omega_{ij} dx \right)^{1/p'} \right) \left(\int_{\Omega'} |D_{rs}u_k|^p \omega_{rs} dx \right)^{1/p}]. \end{aligned} \tag{4.3.5}$$

So, by combining $(4.3.4)$ and $(4.3.5)$, we deduce that

$$\int_{\Omega'} |\sigma(x,u_k,Du_k) : Du_k|dx \leq c'\beta \left(\|\lambda_1\|_{p'} \| \|u_k\|_{1,p,\omega} + \|u_k\|_{1,p,\omega} \right) < \infty. \tag{4.3.6}$$

Similarly to $(|(I_k^2)^-|$, we obtain $(|(I_k^3)^-|$. Finally: I_k is equi-integrable.

We choose a sequence φ_k such that φ_k belongs to the same space V_k and $\varphi_k \to \varphi$ in $W_0^{1,p}(\Omega,\omega,I\!R^m)$ this allows us in particular, to use $u_k - \varphi_k$ as a test function in $(4.2.1)$. We have:

$$\begin{aligned}\int_{\Omega} |\sigma(x,u_k,Du_k) : (Du_k - D\varphi_k)|dx &= \langle v, u_k - \varphi_k \rangle \\ + \int_{\Omega} f(x,u_k,Du_k).(u_k - \varphi_k)dx - \int_{\Omega} g(x,u_k) : (Du_k - D\varphi_k)dx. \end{aligned} \tag{4.3.7}$$

The first term on the right in $(4.3.7)$ converge to zero since $(u_k - \varphi_k) \rightharpoonup 0$ in $W_0^{1,p}(\Omega, \omega, I\!R^m)$.

By the choice of φ_k, the sequence φ_k uniformly bounded in $W_0^{1,p}(\Omega, \omega, I\!R^m)$. And lemma $(4.3.1)$. Next ,for the second term :$II_k = \int_\Omega f(x, u_k, Du_k).(u_k - \varphi_k)dx$ in $(4.3.7)$ it follows from the growth condition F_1^* and the Hölder inequality that:

$$|II_k| \leq (\|b_1\|_{p'} + c_1'.\|D(u_k - \varphi_k)\|_{1,p,\omega} + c_2'.\|D(u_k - \varphi_k)\|^{\frac{p}{p'}}).\|u_k - \varphi_k\|_{1,p,\omega}$$

$$\leq (\|b_1\|_{p'} + c.\|D(u_k - \varphi_k)\|_{1,p,\omega}).\|u_k - \varphi_k\|_{1,p,\omega}$$

. by the equivalence of the norm in $W_0^{1,p}(\Omega, \omega, I\!R^m)$ and the sequence (u_k is uniformly bounded in $W_0^{1,p}(\Omega, \omega, I\!R^m)$, $\|u_k\|_{1,p,\omega}$ is bounded.

Moreover, by the construction of φ_k, and lemma $(4.3.1)$ we have :

$$\|u_k - \varphi_k\|_{1,p,\omega} \leq \|u_k - u\|_{1,p,\omega} + \|u - \varphi_k\|_{1,p,\omega}$$

$$(\|u_k - u\|_{1,p,\omega} + \|u - \varphi_k\|_{1,p,\omega}) \to 0$$

We infer that the second term in $(4.3.7)$ vanishes as $k \to \infty$. Finally, for the last term

$$III_k = \int_\Omega g(x, u_k) : D(u_k - \varphi_k)dx$$

in $(4.3.7)$, we note that

$$g(x, u_k) \to g(x, u)$$

Strongly in $L^{p'}(\Omega, M^{m\times n})$ by (G_0),(G_1) and lemma$(4.3.1)$.

Indeed we may assure that $u_k \to u$ almost everywhere.

$$III_k \leq (\|b_2\|_{p'} + \|u_k - \varphi_k\|_{q,\gamma}^{\frac{q}{p'}}).\|D(u_k - \varphi_k)\|_{1,p,\omega}$$

$$\leq c'.(\|b_2\|_{p'} + \|u_k - \varphi_k\|_{q,\gamma}^{\frac{q}{p'}}).\|(u_k - \varphi_k)\|_{1,p,\omega}$$

$$\leq c'.(\|b_2\|_{p'} + \|u_k - \varphi_k\|_{q,\gamma}^{\frac{q}{p'}}).(\|u_k - u\|_{1,p,\omega} + \|\varphi_k - u\|_{1,p,\omega})$$

$\|\varphi_k - u\|_{1,p,\omega} \to 0$, $\|u_k - u\|_{1,p,\omega} \to 0$ and $\|u_k - \varphi_k\|_{q,\gamma}^{\frac{q}{p'}} \to 0$

. Now, we consider $(I_k)' = (\sigma(x, u_k, Du_k) : (Du_k - Du)$. We have, I_k' is equi-integrable because I_k it is. So, we define

$$X = \liminf \int_\Omega I_k dx = \liminf \int_\Omega (I_k)' dx \geq \int_\Omega \int_{I\!M^{m\times n}} (\sigma(x, u, \lambda) : (\lambda - Du)d\vartheta_x(\lambda)$$

So to prove (**??**), it suffices to prove that:

$$X \leq 0. \tag{4.3.8}$$

Let $\varepsilon > 0$, so there exists $k_0 \in I\!N$ such that, for all $k > k_0$, we have $dist(u, V_k) < \varepsilon$ since :$(\liminf_{\varphi_k \in V_k}(\|u - \varphi_k\|_{1,p,\omega} < \varepsilon$, $(u_k \rightharpoonup u)$ Or in an equivalent manner $dist(u_k - u, V_k) < \varepsilon$, $\forall k > k_0$ then for all $v_k \in V_k$, we have

$$\begin{aligned} X &= \liminf_{k\to\infty} \int_\Omega (\sigma(x, u_k, Du_k) : (Du_k - Du)dx \\ &= \liminf_{k\to\infty} \left[\int_\Omega (\sigma(x, u_k, Du_k) : D(u_k - u - \varphi_k)dx + \int_\Omega (\sigma(x, u_k, Du_k) : D(\varphi_k) \right] \end{aligned}$$

Combining (H_2) and $(4.0.1)$, we get

$$\begin{aligned} X \leq \liminf_{k\to\infty} \int_\Omega \beta \omega_{rs}^{1/p} \left[\lambda_1 + c_1 \sum_{1\leq j\leq m} \gamma_j^{1/p'} |u_{k_j}|^{q/p'} + c_1 \sum_{1\leq i,j\leq n,m} \omega_{ij}^{1/p'} |D_{ij} u_k|^{p-1} \right] \\ \times |D_{rs}(u_k - u - \varphi_k)|dx + \langle v, \varphi_k \rangle. \end{aligned}$$

For all $\varepsilon > 0$, we choice $\varphi_k \in V_k$ such that

$$\|u_k - u - \varphi_k\|_{1,p,\omega} \leq 2\varepsilon, \tag{4.3.9}$$

for all $k \geq k_0$. Which implies that

$$|\langle v, \varphi_k \rangle| \leq |\langle v, \varphi_k + (u - u_k) \rangle| + |\langle v, u_k - u \rangle| \leq 2\varepsilon \|v\|_{-1,p',\omega^*} + o(k)$$

Hence $\lim_{k\to\infty} \langle v, u_k - u \rangle = 0$. According to Hölder and Hardy type inequalities, and by $(4.3.9)$ we deduce that

$$\begin{aligned} X &\leq \liminf_{k\to\infty} c\beta \Bigg(\Big\| \lambda_1 \|_{p'} \Big(\int_\Omega |D_{rs}(u_k - u - \varphi_k)|^p . \omega_{rs} dx \Big)^{1/p} \\ &\quad + c_1 \left(\int_\Omega |u_k|^q . \gamma \right)^{1/p'} . \left(\int_\Omega |D_{rs}(u_k - u - \varphi_k)|^p \omega_{rs} dx \right)^{1/p} \\ &\quad + c_1 \left(\sum \int_\Omega \omega_{ij} |D_{ij} u|^{p'(p-1)} \right)^{1/p'} . \left(\int_\Omega \omega_{rs} |D_{rs}(u_k - u - \varphi_k)|^p \right)^{1/p} \Bigg) + |\langle v, \varphi_k \rangle| \\ &\leq \liminf_{k\to\infty} c \left(\|\lambda_1\|_{p'} . \|u_k - u - \varphi_k\|_{1,p,\omega} \right) + \|u_k\|_{1,p,\omega}^q \|u_k - u - \varphi_k\|_{1,p,\omega} + 2\varepsilon \|v\|_{-1,p',\omega^*} + o(k) \end{aligned}$$

Therefore,

$$X \leq 2\varepsilon c\beta \left(\|\lambda_1\|_{p'} + \|u\|_{1,p,\omega}^q + \|v\|_{-1,p',\omega*} \right).$$

Which proves that $X \leq 0$, and finally

$$\int_\Omega \int_{I\!M^{m\times n}} \sigma(x, u, \lambda) : \lambda d\vartheta_x dx \leq \int_\Omega \int_{I\!M^{m\times n}} \sigma(x, u, \lambda) : Du d\vartheta_x(\lambda) dx.$$

Proof of theorem:

For arbitrary φ in $W_0^{1,p}(\Omega,\omega,I\!R^m)$. It follows from the continuity condition (F_0^*) and (G_0) that

$$f(x,u_k,Du_k).\varphi(x) \to f(x,u,Du).\varphi(x)$$

and

$$g(x,u_k) : D\varphi(x) \to g(x,u) : D\varphi(x)$$

almost everywhere. Since, by the growth conditions (F_1^*), (G_1) and the uniform bound of u_k, $f(x,u_k,Du_k).\varphi(x)$ and $g(x,u_k) : D\varphi(x)$ are equi-integrable, it follows that the Vitali's theorem . This implies that:

$$\lim_{k\to\infty}\int_\Omega f(x,u_k,Du_k).\varphi(x)dx = \int_\Omega f(x,u,Du).\varphi(x)dx$$

for all$\varphi \in \cup_{k=1}^{\infty} V_k$ and

$$\lim_{k\to\infty}\int_\Omega g(x,u_k) : D\varphi(x)dx = \int_\Omega g(x,u) : D\varphi(x)dx$$

for all$\varphi \in \cup_{k=1}^{\infty} V_k$ We will start with the easiest case

$$(d): \qquad F \longmapsto \sigma(x,u,F) \text{ is stictly p-quasi-monotone.} \tag{4.3.10}$$

Indeed, we assume that ϑ_x is not a Dirac mass on the set M with $x \in M$ of positive Lebesgue measure $|M| > 0$. Moreover, by the strict p-quasi-monotonicity of $\sigma(x,u,\cdot)$ and ϑ_x is an homogeneous $W^{1,p}$ gradient young measure for a.e. $x \in M$. So, for a.e. $x \in M$, with $\bar{\lambda} = \langle \vartheta_x, Id\rangle = apDu(x))$, with $apDu(x))$ is the differentiable approximation in x. We get

$$\begin{aligned}
\int_{I\!M^{m\times n}} \sigma(x,u,\lambda) : (\lambda - Du)d\vartheta_x(\lambda) &> \int_{I\!M^{m\times n}} \sigma(x,u,Du) : (\lambda - Du)d\vartheta_x(\lambda) \\
&> \sigma(x,u,Du) : \int_{I\!M^{m\times n}} \lambda d\vartheta_x(\lambda) - \sigma(x,u,Du) : Du \int_{I\!M^{m\times n}} d\vartheta_x(\lambda) \\
&> (\sigma(x,u,Du) : Du - \sigma(x,u,Du) : Du)) = 0 \\
&> 0
\end{aligned}$$

On the other hand $(4.3.8)$, integrating over Ω, and using the div-cul inequality we have:

$$\begin{aligned}
\int_\Omega \int_{I\!M^{m\times n}} \sigma(x,u,\lambda) : \lambda d\vartheta_x(\lambda)dx &> \int_\Omega \int_{I\!M^{m\times n}} \sigma(x,u,\lambda) : Dud\vartheta_x(\lambda)dx \\
&\geq \int_\Omega \int_{I\!M^{m\times n}} \sigma(x,u,\lambda) : \lambda d\vartheta_x(\lambda)dx.
\end{aligned}$$

Which is a contradiction with $(4.3.8)$. Thus $\vartheta_x = \delta_{\bar{\lambda}} = \delta_{Du(x)}$ for a.e. $x \in \Omega$. Therefore, $Du_k \longrightarrow Du$ in measure when k tends to infinity. Then, we get $\sigma(x,u_k,Du_k) \longrightarrow \sigma(x,u,Du)$ for all $x \in \Omega$. In the

other hand, for all $\varphi \in \bigcup_{k\in I\!N} \vartheta_k$; $\sigma(x,u_k,Du_k) : D\varphi \longrightarrow \sigma(x,u,Du) : D\varphi$ a.e. $x \in \Omega$. Moveover, for all $\Omega' \subset \Omega$ measurable, it is easy to see that:

$$\int_{\Omega'} \sigma(x,u_k,Du_k) : D\varphi dx \leq c\beta \left(\|\lambda_1\|_{p'} + \|u_k\|^{q/p'}_{1,p,\omega} + \|u_k\|^{p/p'}_{1,p,\omega} \right) \|u\|_{1,p,\omega} < \infty,$$

because $\|u_k\|_{1,p,\omega} \leq R$. And thanks to Vitali's theorem, we obtain:

$$\langle F(u), \varphi \rangle = 0, \text{ for all } \varphi \in \bigcup_{k\in I\!N} V_k.$$

Which proves the theorem in this case.

Remark 4.3.1 *Before treating the cases* (a), (b) *and* (c) *of* (H_3)*, we note that*

$$\int_\Omega \int_{I\!M^{m\times n}} (\sigma(x,u,\lambda) - \sigma(x,u,Du)) : (\lambda - Du) d\vartheta_x(\lambda) dx \leq 0 \tag{4.3.11}$$

Since

$$\int_\Omega \int_{I\!M^{m\times n}} \sigma(x,u,\lambda) : (\lambda - Du) d\vartheta_x(\lambda) dx = 0,$$

thanks to the div-Curl inequality in $(4.3.8)$. On the other hand, the integrand in $(4.3.11)$ is non negative, by the monotonicity of σ.Consequently, the integrating should be null, a.e., with respect to the product measure $d\vartheta_x \otimes dx$, which mean

$$(\sigma(x,u,\lambda) - \sigma(x,u,Du)) : (\lambda - Du) = 0 \text{ in } spt\vartheta_x. \tag{4.3.12}$$

Thus,

$$spt\vartheta_x \subset \left\{ \lambda \in I\!M^{m\times n} / \, (\sigma(x,u,\lambda) - \sigma(x,u,Du)) : (\lambda - Du) = 0 \right\}. \tag{4.3.13}$$

Case c: We prove that, the map $F \longmapsto \sigma(x,u,F)$ is strictly monotone, for all $x \in \Omega$ and for all $u \in I\!R^m$. Sine σ is strict monotone, and according to $(4.3.13)$,

$$spt\vartheta_x = \{Du\}, \text{ i.e, } \vartheta_x = \delta_{Du}, \quad \text{a.e. in } \Omega,$$

which implies that, $Du_k \longrightarrow Du$ in measure. For the rest of our prove is similarly to case d.

Cases (a) and (b): For the case (a) resp (b) we use near the same techniques that we used in a) chapter3, then we go to limit we find the proof of theorem $\square$

Example 4.3.1 *Let* Ω *is a bounded open domain in* $I\!R^n$*. In this example we suppose that the weighted functions:* (ω_{ij}), $0 \leq i \leq n$; $1 \leq j \leq m$ *and* σ *satisfies the condition in example 2.3.1. Hence we have*

the Hardy-Type inequalities in (P_0) *for all* $u \in W_0^{1,p}(\Omega, \omega)$; $q > p'$.

Let us consider the functions f *and* g *defined by:*

$$f_k(x, u, \zeta) = \omega_{0k}^{\frac{1}{p}} e^{-|x|} + \sum_j \mid u_j \mid^{p-1} sign(u_j) \omega_{0j}^{\frac{1}{p'}} \omega_{0k}^{\frac{1}{p}} + \sum_{i,j} \omega_{ij}^{\frac{1}{p'}} |\zeta_{ij}|^{p-1} sign(\zeta_{ij}).\omega_{0k}^{\frac{1}{p}}$$

$$g_{rs} = \omega_{rs}^{\frac{1}{p}} \cos \mid x \mid + \sum_j \mid u_j \mid^{p-1} sign(u_j) \omega_{0j}^{\frac{1}{p'}} \omega_{rs}^{\frac{1}{p}}$$

The functions satisfies the conditions $(F_0)^* - (F_1)^*$ *The functions satisfies the conditions* $(G_0) - (G_1)$.

Chapter 5

Existence of solutions for some quasi-linear elliptic system with weight and measure-valued right hand side

Abstract

We prove the existence of a solution u for the nonlinear elliptic system:

$$\begin{aligned} -div\sigma(x,u,Du) &= \mu \text{ in } \Omega \\ u &= 0 \text{ on } \partial\Omega \end{aligned}$$

where μ is Radon measure on Ω with finite mass. In particular, we show that if the coercivity rate of σ lies in the range $]\frac{s+1}{s}, (\frac{s+1}{s})(2-\frac{1}{n})]$ then u is approximately differentiable and the equation holds with Du replaced by $apDu$. The proof relies on an approximation of μ by smooth functions f_k and a compactness result for the corresponding solutions u_k. This follows from a detailed analysis of the Young measure $\{\delta_u(x) \otimes \vartheta(x)\}$ generated by the sequence (u_k, Du_k), and the div-curl type inequality $< \vartheta(x), \sigma(x,u,\cdot) > \leq \overline{\sigma}(x) < \vartheta(x), . >$ for the weak limit $\overline{\sigma}$ of the sequence.

5.1 Introduction

We consider the existence and compactness questions for elliptic systems of the form

$$-div\sigma(x,u(x),Du(x)) = \mu \text{ in } \Omega \tag{5.1.1}$$

$$u = 0 \text{ on } \partial\Omega \tag{5.1.2}$$

with measure-valued right hand side $\mu \in M(\Omega, \omega^*, I\!R^m)$ on an open, bounded domain Ω in $I\!R^n$. And the weight functions ω^* defined by $\omega = \{\omega_{ij}, 0 \leq i \leq n, 1 \leq j \leq m\}$ with $\omega^*_{ij} = \omega_{ij}^{1-p'}$ on this chapter we are interested in the solution u in the Sobolev space $W_0^{1,p}(\Omega, w, I\!R^m)$, and estimation of the weak Lebesgues spaces. We assume that σ satisfies the following hypotheses $(H_0) - (H_3)$. Let $\omega = \{\omega_{ij}|\ 0 \leq i \leq n;\ 1 \leq j \leq m\}$ the system for weight functions on Ω satisfying:

$$\omega_{ij} \in L^1_{loc}(\Omega), \quad \omega_{ij}^{\frac{-1}{p-1}} \in L^1_{loc}(\Omega) \tag{5.1.3}$$

$$\omega_{ij}^{-s} \in L^1(\Omega) \tag{5.1.4}$$

for some $s \in]Max(\frac{1}{p-1}; \frac{n}{p})\ ;\ \infty[$. The Sobolev space $W^{1,p}(\Omega, \omega, I\!R^m)$ defined by:

$$W^{1,p}(\Omega, \omega, I\!R^m) = \{u \in L^p(\Omega, \overline{\omega}_0, I\!R^m)|\ \overline{\omega}_0 = \omega_{0j};\ 1 \leq j \leq m;\ \frac{\partial u_j}{\partial x_i} \in L^p(\Omega, \omega_{ij});\ 1 \leq j \leq m;\ 1 \leq i \leq n\}$$

The Sobolev space $W^{1,p}(\Omega, \omega, I\!R^m)$ equipped with the norm

$$\| u \|^p_{1,p,\omega} = \sum_{j=1}^{m} \int_\Omega \omega_{0j} |u_j|^p dx + \sum_{i,j} \int_\Omega \omega_{ij} |D_{ij} u|^p dx.$$

The condition (5.1.3) we give $(W^{1,p}(\Omega, \omega, I\!R^m), \|..\|_{1,p,\omega})$ Banach space and $C_c^\infty(\Omega, I\!R^m)$ subspace of $(W^{1,p}(\Omega, \omega, I\!R^m)$. The space $(W_0^{1,p}(\Omega, \omega, I\!R^m)$ is the fermenter of $C_c^\infty(\Omega, I\!R^m)$ in $(W^{1,p}(\Omega, \omega, I\!R^m)$ for the norm $\|..\|^p_{1,p,\omega})$. The condition (5.1.4), we give

$$W^{1,p}(\Omega, \omega, I\!R^m) \hookrightarrow W^{1,p_s}(\Omega, I\!R^m)\ and\ (W^{1,p}(\Omega, \omega, I\!R^m) \hookrightarrow\hookrightarrow L^r(\Omega, I\!R^m)) \tag{5.1.5}$$

for all $1 \leq r \prec p^*_s$ with $p_s = \frac{p.s}{s+1}$
$p^*_s = \frac{n.p.s}{n(s+1)-p.s}$ if $p.s \prec n(s+1)$, and p^*_s is arbitrary otherwise.

5.2 Hypothesis

(H_0) (Continuity) $\sigma : \Omega \times I\!R^m \times I\!M^{m\times n} \longrightarrow I\!M^{m\times n}$ is a Carathéodory function, i.e:
$x \longmapsto \sigma(x, u, p)$ is measurable for every (u, p) and $(u, p) \longmapsto \sigma(x, u, p)$ is continuous for almost every $x \in \Omega$.

(H_1) (Coercivity and growth): There exist constants c_1, c_2, $\beta > 0$ and $\lambda_1 \in L^{p'}(\Omega)$, $\lambda_2 \in L^1(\Omega)$, $\lambda_3 \in L^{(\frac{p}{\nu})'}(\Omega)$, $0 < \nu < p$, $1 < q < \frac{p^2}{\nu}$, $0 < \theta < \frac{n(p_s-1)}{n-1}$, such that, for all $1 \leq r \leq n$, and

$1 \leq s \leq m$:

$$|\sigma_{rs}(x,u,F)| \leq \beta\omega_{rs}^{\frac{1}{p}}[\lambda_1 + c_1\sum_{j=1}^{m}\gamma_j^{\frac{1}{p'}}|u_j|^{\frac{q}{p'}} + c_1\sum_{i,j}\omega_{ij}^{\frac{1}{p'}}|F_{ij}|^{\theta}]$$

$$\sigma(x,u,F) : F \geq -\lambda_2 - \sum_{j=1}^{m}\lambda_3\gamma_j^{\frac{\nu}{p}}|u_j|^{\frac{q\nu}{p}} + c_2\sum_{i,j}\omega_{ij}|F_{ij}|^p.$$

(H_2) (Monotonicity) σ satisfies one following conditions:

a) For all $x \in \Omega$, $u \in I\!R^m$ the function $F \longmapsto \sigma(x,u,F)$ is a C^1 and monotone function, which means $(\sigma(x,u,F) - \sigma(x,u,G)) : (F-G) \geq 0$, for all $x \in \Omega$, $u \in I\!R^m$, and $F, G \in I\!M^{m\times n}$.

b) There exist a function $W : \Omega \times I\!R^m \times I\!M^{m\times n} \to R$ such that: $\sigma(x,u,F) = \frac{\partial W}{\partial F}(x,u,F)$, and the function $F \longmapsto W(x,u,F)$ is a C^1 and convex function.

c) σ is strictly monotone, i-e; σ is monotone and $(\sigma(x,u,F) - \sigma(x,u,G)) : (F-G) = 0.$ implies $F = G$.

d) A function $F \longmapsto \sigma(x,u,F)$ is strictly $p-$quasi-monotone, i-e:
$\int_{I\!M^{m\times n}}(\sigma(x,u,\lambda) - \sigma(x,u,\bar{\lambda})) : (\lambda - \bar{\lambda})d\nu(\lambda) > 0$, for all homogeneous $W^{1,p} - gradient$ Young measures ν with center of mass $\bar{\lambda} =< \nu; Id >= \int_{I\!M^{m\times n}} \lambda d\nu(\lambda)$ which are not a single Dirac mass.

(H_3) (structure conditions) i)- (Angle condition) for all $x \in \Omega$, $u \in I\!R^m$ and $F \in I\!M^{m\times n}$ there holds $\sigma(x,u,F) : MF \geq 0$, for all matrix $M \in I\!M^{m\times m}$ of the form $M = Id - a \otimes a$ with $|a| \leq 1$.
ii)- (The sign condition) for all $x \in \Omega$, $u \in I\!R^m$ and $F \in I\!M^{m\times n}$, $\sigma_j(x,u,F) : F_j \geq 0$, for all $1 \leq j \leq m$ with F_j and σ_j are the cologne j of matrix F and σ.

(H_4) (The Hardy-Type Inequality) There exist $c > 0$, one weighted function $\gamma = (\gamma_j)_{1\leq j\leq m}$, and the parameter $1 < q < \frac{p^2}{\alpha}$ (H_1), such that:

$$\left(\sum_{j=1}^{m}\int_\Omega \gamma_j \mid u_j \mid^q dx\right)^{\frac{1}{q}} \leq c\left(\sum_{i,j}\int_\Omega \omega_{ij} \mid D_{ij}u \mid^p dx\right)^{\frac{1}{p}}$$

and the expression $\left(\sum_{i,j}\int_\Omega \omega_{ij} \mid D_{ij}u \mid^p dx\right)^{\frac{1}{p}}$ is a norm equivalent to the norm $\|.\|_{1;p:\omega}$ in $W_0^{1,p}(\Omega,\omega,I\!R^m)$.

The injection $W_0^{1,p}(\Omega,\omega,I\!R^m) \hookrightarrow\hookrightarrow L^q(\Omega,\gamma,I\!R^m)$ is compact, and
$W_0^{1,p}(\Omega,\omega,I\!R^m) \hookrightarrow\hookrightarrow L^r(\Omega,I\!R^m)$ is compact, with

$$\begin{cases} 1 \leq r < \frac{nps}{n(s+1)-ps} & \text{if} \quad ps < n(s+1) \\ r \geq 1 & \text{if} \quad n(s+1) \leq ps \end{cases}$$

Remark 5.2.1 *1. Assumption (H_0) ensures that $\sigma(x, u(x), U(x))$ is measurable on Ω for measurable function $u : \Omega \longrightarrow I\!R^m$ and $U : \Omega \longrightarrow I\!M^{m\times n}$. A typical example for a function σ satisfying (H_0) is $\sigma(x, u, \rho) = \alpha(x, u, \rho)\rho$ with a real valued non-negative function α.*

2. A serious technical obstacle is that for $p_s \in (1, 2 - 1/n]$ solutions of the system (5.1.1) in general do not belongs to the Sobolev space $W^{1,1}(\Omega, \omega, I\!R^m)$.

This fact has led to the use of normalized solutions in [39] and generalized entropy solutions in [10] for elliptic equations of the above type. We will use a notion of solution where the weak derivative Du is replaced by the approximate derivative $apDu$. Recall that a measurable function u is said to be approximately differentiable at $x \in \Omega$ if there exists a matrix
$F_x \in I\!M^{m\times n}$ such that:
for all $\epsilon \succ 0$, $\lim_{r\to 0} \frac{1}{r^n} meas\{y \in B(x, r) : |u(y) - u(x) - F_x(y - x)| > \epsilon r\} = 0$. We write:
$apDu(x) = F_x$.

Definition 5.2.1 *A measurable function $u : \Omega \longrightarrow I\!R^m$ is called a solution of the system (5.1.1) if:*

(i) *u is almost everywhere approximately differentiable.*

(ii) *$\eta \circ u \in W^{1,1}(\Omega, \omega, I\!R^m)$, for all, $\eta \in C^1_c(I\!R^m, I\!R^m)$.*

(iii) *$\sigma(x, u, apDu) \in L^1(\Omega, I\!M^{m\times n})$*

(iv) *The equation: $-div\sigma(x, u(x), apDu(x)) = \mu$ holds in the sense of distributions. Moreover we say that u satisfies the boundary condition $(5.1.2)$ if $\eta\circ u \in W^{1,1}_0(\Omega, \omega, I\!R^m)$, for all, $\eta \in C^1(I\!R^m, I\!R^m)\cap L^\infty(I\!R^m, I\!R^m)$ with $\eta = id$ on $B(0, r)$, for some $r \succ 0$, and $|D\eta(y)| \leq c.(1 + |y|)^{-1}$, with $c < \infty$.*

Remark 5.2.2 *1. The conditions in Definition (5.2.1) (except (ii)) are the weakest possible in order to define the system (5.1.1) in the sense of distributions. Note that if u is approximately differentiable, then $apDu$ is measurable, so $\sigma(, u, apDu)$ is measurable.*

2. The assumption $\eta ou \in W^{1,1}(\Omega, I\!R^m)$ ensures minimal regularity of u. For example, if $\mu = 0$, and $\sigma(x, u, \rho) = \sigma(\rho)$ with $\sigma(0) = 0$, then piecewise constant functions u satisfy $apDu = 0$ a,e, but are not admissible solutions. The following theorem is the main result in this chapter.

Theorem 5.2.1 *Let Ω be a bounded, open set. We suppose that the hypotheses $(H_0)-(H_2)-(H_3)$ and the coercivity condition in (H_1) are satisfied. Let μ denote a $\mathbb{R}^m$- valued Radon measure on Ω with finite mass. Then the system (5.1.1)-(5.1.2) has a solution u in the sense of definition1, which satisfies the weak Lebesgue space estimate:*

$$\|u\|^*_{L^{t^*_{p_s},\infty}(\Omega,\mathbb{R}^m)} + \|apDu\|^*_{L^{t_{p_s},\infty}(\Omega,\mathbb{M}^{m\times n})} \leq C, \tag{5.2.1}$$

with the constant C depends of $|\Omega|, c, c_2,$ and $\| \lambda_3 \|_{L^{(\frac{p}{\alpha})\cdot,'}(\Omega)}$, with $t_{p_s} = \frac{n(p_s-1)}{n-1}$ and $t^*_{p_s} = \frac{n(p_s-1)}{n-p_s}$ is the Sobolev exposed of t_{p_s}. If $c_2 = 0$ the right hand side of (5.1.3) reduces to $C(c_1)\left\|\mu^{\frac{1}{p-1}}\right\|_M$.

Remark 5.2.3

1. *If $p_s > 2-\frac{1}{n}$, then $t_{p_s} > 1$ and $Du \in L^1(\Omega, M^{m\times n})$.*

2. *If $p \succ n$ one can replace the $L^{s,\infty}$-norm of u in (1.3) by the $C^0,\ \beta$-norm with $\beta = 1-\frac{n}{p}$. For $p = \theta + 1 = n$ it is an open question whether $Du \in L^{n,\infty}$.*

3. *The exponent in (5.1.3) are optimal as can be seen from the nonlinear Green's function $G_p(x) = c\,|x|^{\frac{-n}{s^*}}$ for the p-Laplace equation: $-div(|Du|^{p-2}\,Du) = \delta_0$ in $\mathbb{R}^m$, $n \geq 3$. In particular, $L^{s,\infty}$ cannot be replaced by L^s. with ($L^{s,\infty}$), is a Laurent space.*

4. *The pointiest monotonicity condition can be replaced by a weaker integrated version, called quasi-monotonicity,*

   The Key point in the proof of the theorem, is the div-curl inequality for the Young measure $\{\{\vartheta_x\}_{x\epsilon\Omega}\}$ generated by a sequence Du_k of gradients of approximate solutions. Together with the identity. (5.1.5):

   $apDu(x) =< \vartheta_x, Id >$. The div-curl inequality implies easily that $\sigma(., u_k, Du_k)$ converges weakly in L^1 to $\sigma(., u, apDu)$. (5.1.5) is a consequence of general properties of young measures if $p_s > 2-\frac{1}{n}$ since in this case Du_k is bounded in L^r for some $r > 1$. If $1 < p_s \leq 2-\frac{1}{n}$ one only has the weaker bounds.

5.3 Some preliminary lemmas

In this section, we will also use the Young measures, and Inequality div-curl for assume the convergence of subsequence $u_k \longrightarrow u$ in measure and for almost every subsequence, with u is approximately differentiable, and $apDu =< \nu_x, id >$, ν_x is the Young measures generated by a sequence Du_k. In the following

T_α denotes the truncation function

$$T_\alpha(y) = min(1; \frac{\alpha}{\mid y \mid})y, \quad \alpha > 0.$$

By definition $\mid T_\alpha \mid \leq \alpha$ and

$$DT_\alpha(y) = Id \quad for \ ; \mid y \mid < \alpha,$$

$$DT_\alpha(y) = \frac{\alpha}{\mid y \mid}(Id - \frac{y}{\mid y \mid} \otimes \frac{y}{\mid y \mid}) \quad for \ ; \mid y \mid > \alpha.$$

Lemma 5.3.1 *Let $u_{k:} : \Omega \longrightarrow I\!R^m$ a sequence of measurable functions such that:*

$$\sup_{k \in I\!N} \int_\Omega |u_k|^s dx < +\infty \ \textit{for some } s > 0. \tag{5.3.1}$$

We suppose that for each $\alpha > 0$ the sequence of truncated functions $\{T_\alpha(u_k)\}_{k \in I\!N}$ is precompact in $L^1(\Omega, I\!R^m)$. Then there exists a measurable function u on Ω such that for a subsequence $u_k \longrightarrow u$ in measure.

Proof

Choose a subsequence of $\{u_k\}$ (not relabeled) which generates a Young measure $\{\vartheta_x\}_{x \in \Omega}$. By 5.3.1 and Theorem (Young, Tartar, Ball) the measure ν_x are probability measure for almost every a $x \in \Omega$ and $T_\alpha(u_k) \longrightarrow v_\alpha =< \nu_x; T_\alpha >$, weakly in $L^1(\Omega, I\!R^m)$ and in fact strongly since $T_\alpha(u_k)$ is precompact in L^1. Consequently there exists a subsequence such that: $T_\alpha(u_{k_l}) \longrightarrow v_\alpha$ almost uniformly, i-e:

$$T_\alpha(u_{k_l}) \longrightarrow v_\alpha \ \text{ uniformly up to a set of arbitrary small measure.} \tag{5.3.2}$$

Let $M_\alpha = \{x \in \Omega : |v_\alpha(x)| < \alpha\}$. Then for each $\epsilon > 0$ and $\delta > 0$ there exists a set E_ϵ of measure meas$(E_\epsilon) < \epsilon$ and an index $l_0(\epsilon; \delta)$ such that: $|T_\alpha(u_{k_l})| < |v_\alpha(x)| + \delta$ for all $x \in M_\alpha \setminus E_\epsilon$ and all $l > l_0$. It follows that $u_{k_l}(x) \longrightarrow v_\alpha(x)$ for almost every$x \in M_\alpha \setminus E_\epsilon$ consider first $x \in M_\beta; \ \beta < \alpha$ and then the union over $\beta < \alpha$). Since $\varepsilon > 0$ was arbitrary it follows that $\nu_x = \delta_{v_\alpha}(x)$ for almost every $x \in M_\alpha$ In view of the Ball's theorem it suffices to show that $\cup M_\alpha$ has full measure. Now clearly $M_\alpha \subset M_\beta$ for $\alpha < \beta$ since $T_\beta(u_{k_l}) \longrightarrow T_\beta(v_\alpha) = v_\alpha$ almost everywhere in M_α and therefore $v_\alpha = v_\beta$ on M_α. By (5.3.2) there exists for each $\epsilon > 0$ a set E_ϵ, and an index $l_0(\epsilon, \alpha)$ such that meas $(E_\epsilon) < \epsilon$ and $\mid u_{k_l} \mid \geq \mid T_\alpha(u_{k_l}) \mid \geq \frac{\alpha}{2}$ on $(\Omega \setminus E_\epsilon) \setminus M_\alpha$ for all $l \geq l_0$. In view of (5.3.2) this implies meas $((\Omega \setminus E_\epsilon) \setminus M_\alpha) \leq \frac{c}{\alpha^s}\epsilon \longrightarrow 0$ we deduce meas$(\Omega \setminus \cup M_\alpha)$=$\lim_{\alpha \longrightarrow \infty} meas(\Omega \setminus M_\alpha) = 0$ $\square$

Lemma 5.3.2 *Let Ω be a domain in $\mathbb{R}^n$ with $|\Omega| < \infty$ and $u_k \in W^{1,1}(\Omega, \mathbb{R}^m)$. Suppose that there exist $\rho > 1$ and $s > 0$ such that:*

$$\sup_k \sum_{i,j} \int_{|u_k| \leq \alpha} \omega_{ij} |D_{ij} u_k|^\rho dx \leq c(\alpha) < \infty, \ \forall \alpha > 0, \tag{5.3.3}$$

and $\sup_{k \in \mathbb{N}} \int_\Omega |u_k|^s dx \leq c < \infty$. Then there exist a subsequence u_{k_j} and a measurable function $u : \Omega \longrightarrow \mathbb{R}^m$ such that $u_{k_j} \longrightarrow u$ in measure. Moreover u is for almost every $x \in \Omega$ approximately differentiable, for all $\eta \in C_c^1(\Omega, \mathbb{R}^m)$ there holds $\eta \circ u \in W^{1,p}(\Omega, \omega, \mathbb{R}^m)$. if $u_k \in W_0^{1,1}(\Omega, \mathbb{R}^m)$ then $\eta \circ u \in W_0^{1,1}(\Omega, \mathbb{R}^m) \cap W^{1,p}(\Omega, \omega, \mathbb{R}^m)$ provided that $\eta = id$ on $B(0, r)$ for some $r > 0$.

Proof

Choose

$$(u_k)_\alpha = \begin{cases} u_k & \text{if} \quad |u_k| \leq \alpha, \\ 0 & \text{if} \quad |u_k| > \alpha. \end{cases}$$

For the hypotheses:

$$\sum_{i,j} \int_\Omega \omega_{ij} |D_{ij} (u_k)_\alpha|^\rho dx = \sum_{i,j} \int_{|u_k| \leq \alpha} \omega_{ij} |D_{ij} u_k|^\rho dx \leq c(\alpha) < \infty.$$

Then, $(u_k)_\alpha \in W_0^{1,1}(\Omega, w, \mathbb{R}^m)$ and for (5.1.5), (H_4) and $|D|u|| \leq |Du|$ we have:

$$\int_\Omega |DT_\alpha(|u_k|)|^{\rho_s} dx = \int_{|u_k| \leq \alpha} |D|u_k|)|^{\rho_s} dx \leq$$

$$c \sum_{i,j} \int_\Omega \omega_{ij} |D_{ij} (u_k)_\alpha|^\rho dx \leq c(\alpha) < +\infty$$

Hence by the compact Sobolev embedding $W_0^{1,\rho_s}(\Omega) \hookrightarrow\hookrightarrow L^{\rho_s}(\Omega)$, we have $\{T_\alpha(|u_k|)\}$ is precompact in $L^1(\Omega)$. And, if $\eta \in C_c^\infty(B(0, 3\alpha), \mathbb{R}^m)$ a symmetric radial such that $\eta = id$ on $B(0, 2\alpha)$, then by (5.1.3) and (5.3.3): $\sum_{i,j} \int_\Omega \omega_{ij} |D_{ij}(\eta(u_k))|^\rho dx = \sum_{ij} \int_{|u_k| \leq 2\alpha} \omega_{ij} |D_{ij}(u_k)|^\rho dx + \sum_{i,j} \int_{2\alpha < |u_k| \leq 3\alpha} \omega_{ij} |D_{ij}(\eta(u_k))|^\rho dx \leq$ $c(\alpha) + c'(\alpha) < \infty$. Then, by (5.1.5), $\eta(u_k)$ is precompact in $L^{\rho_s}(\Omega, \mathbb{R}^m)$, and as in Lemma 8[39], there exist a measurable function $u : \Omega \longrightarrow \mathbb{R}^m$ such that $u_k \longrightarrow u$ in measure, with $apDu(x) =< \vartheta_x, id >$ for almost every $x \in \Omega$ and u is approximately differentiable because $\eta(u_k) \rightharpoonup \eta(u)$ in $W^{1,\rho}(\Omega, \omega, \mathbb{R}^m)$ and $apDu = apD(\eta \circ u)$. □

Lemma 5.3.3 *Let u_k be as in Lemma (5.3.2) with $p \succ 1$. Then the Young measure ϑ_x generated by (a subsequence of) Du_k has the following properties:*

(a) *ϑ_x is a probability measure for almost every $x \in \Omega$.*

(b) *ϑ_x has finite p_s-th moment for almost every $x \in \Omega$, i-e $\int_{M^{m\times n}} \mid \lambda \mid^{p_s} d\vartheta_x(\lambda)$ is finite for almost every $x \in \Omega$.*

(c) *ϑ_x satisfies $< \vartheta_x, id >= apDu(x)$ almost everywhere in Ω.*

(d) *ϑ_x is a homogeneous $W^{1,P_s}-$ gradient young measure for almost every $x \in \Omega$.*

Proof Let $\widetilde{\vartheta_x}$ denote the Young measure generated by (a subsequence of) the sequence $\{(u_k, Du_k)\}$. By Lemma 5.3.2 we have :

$$\widetilde{\vartheta_x} = \delta_{u(x)} \otimes \vartheta_x.$$

Let $\eta \in C_c^\infty(B(0,2\alpha),\ I\!R^m)$, $\eta = Id$ on $B(0,\alpha)$, and let ϑ^η be the Young measure generated by

$$D(\eta \circ u_k) = (D\eta)(u_k)Du_k(x),$$

then ϑ^η is a probability measure, has finite $p-$th moment and

$$< \vartheta^\eta, Id >= (D(\eta \circ u))(x) = D\eta(u(x))apDu(x).$$

It follows for $\varphi \in C_c^\infty(M^{m\times n})$, that:

$$\varphi(D(\eta \circ u_k)) \rightharpoonup < \vartheta^\eta, \varphi >= \int_{M^{m\times n}} \varphi(\lambda) d\vartheta_x^\eta(\lambda).$$

Based on the proof (5.3.2), we have $\sum_{i,j} \int_\Omega |\omega_{ij} D_{ij}(\eta \circ u_k)|^p dx \prec \infty$, and by (5.1.5) $\sup_{k\in I\!N} \int_\Omega |D(\eta \circ u_k)|^{p_s} dx \prec \infty$, and the (Ball's Theorem, proof lemma 9 [39]) we conclude: (a)-(b)-(c)- and (d) □

5.4 Approximate Solutions and a priori bounds.

We introduce the following approximating problems:

$$-div\sigma(x, u_k, Du_k) = f_k \ in\, \Omega. \tag{5.4.1}$$

$$u_k = 0 \ \ on\, \partial\Omega. \tag{5.4.2}$$

With $f_k \in W^{-1,p'}(\Omega, \omega^*, I\!R^m) \cap L^1(\Omega, I\!R^m)$ and $f_k \rightharpoonup^* \mu \ in\ M(\Omega, \omega^*, I\!R^m)$ such that: $\parallel f_k \parallel_{L^1(\Omega, I\!R^m)} \leq \parallel \mu \parallel_{M(\Omega,\omega^*, I\!R^m)}$. By [54], and using the assumptions: (H_0), (H_1),(H_2) and (H_4), the problem (5.4.1)-(5.4.2) has a solution $u_k \in W_0^{1,p}(\Omega, \omega, I\!R^m)$ with u_k is the subsequence approximates solutions of (5.1.1)-(5.1.2). The results of theorems (5.2.1) is the consequence of the following proposition:

Proposition 5.4.1 *Let, $f \in L^1(\Omega, I\!R^m)$ and σ satisfies (H_0), the coercivity of (H_1) and (H_3). If $u \in W_0^{1,p}(\Omega, \omega, I\!R^m)$ is a solution of:*

$$-div\sigma(x, u, Du) = f \quad in\,\Omega. \tag{5.4.3}$$

in the sense of distributions. Then

$$u \in L^{t_{p_s}^*,\infty}(\Omega, I\!R^m), \qquad Du \in L^{t_{p_s},\infty}(\Omega, I\!R^m)$$

and

$$\| u \|^*_{L^{t_{p_s}^*,\infty}(\Omega,I\!R^m)} + \| Du \|^*_{L^{t_{p_s},\infty}(\Omega,M^{m\times n})}$$

$$\leq C\bigg(|\Omega|, \| \lambda_1 \|_{L^1(\Omega)}, \| \lambda_3 \|_{L^{(\frac{p}{\alpha})'}(\Omega)}, \| f \|_{L^1(\Omega;I\!R^m)}\bigg) \sum_{i,j} \int_{|u|\leq\alpha} \omega_{ij}|D_{ij}u|^p dx \leq M\alpha + L, \quad \forall \alpha \succ 0,$$

M and L are the constants dependents on:

$$\| \lambda_1 \|_{L^1(\Omega)}, \| \lambda_3 \|_{L^{(\frac{p}{\alpha})'}(\Omega)}; \| f \|_{L^1(\Omega,I\!R^m)}, c_2$$

Proof i)- We suppose the condition of l'angle in (H_3). Let $\alpha \succ 0$. Testing $T_\alpha(u)$ in (5.4.3) and we use the coercivity condition in (H_1), and Hölder inequality, we have:

$$c_2 . \sum_{i,j} \int_{|u|\leq\alpha} \omega_{ij}|D_{ij}u|^p dx \leq \alpha \| f \|_{L^1(\Omega,I\!R^m)} + \| \lambda_2 \|_{L^1(\Omega)} + c \| \lambda_3 \|_{L^{(\frac{p}{\nu})'}} \left(\sum_{j=1}^m \int_{|u|\leq\alpha} \gamma_j \mid u_j \mid^q dx \right)^{\frac{\nu}{p}} \tag{5.4.4}$$

Choose :

$$(u)_\alpha = \begin{cases} u & \text{if} \quad |u| \leq \alpha, \\ 0 & \text{if} \quad |u| > \alpha. \end{cases}$$

Then $u \in W_0^{1,p}(\Omega, \omega, I\!R^m)$ because $u \in W_0^{1,p}(\Omega, \omega, I\!R^m)$ and by Hardy-Type inequality:

$$\sum_j^m \int_{|u|\leq\alpha} \gamma_j |u_j|^q dx = \sum_j^m \int_{|u|\leq\alpha} \gamma_j |(u_\alpha)_j|^q dx$$

$$\leq c. \left(\sum_{i,j} \int_\Omega \omega_{ij} |D_{ij}u_\alpha|^p dx \right)^{\frac{q}{p}}$$

$$\leq c.\left(\sum_{i;j}\int_{|u|\leq\alpha}\omega_{ij}|D_{ij}u_\alpha|^p dx\right)^{\frac{q}{p}}$$

By (5.4.4)

$$c_2.\left(\sum_{ij}\int_{|u|\leq\alpha}\omega_{ij}|D_{ij}u|^p dx\right)\leq \alpha\parallel f\parallel_{L^1(\Omega,I\!R^m)}+\parallel\lambda_2\parallel_{L^1(\Omega)}$$

$$+c\parallel\lambda_3\parallel_{L^{(\frac{p}{\alpha})'}}.\left(\sum_{i,j}\int_{|u|\leq\alpha}\omega_{ij}|D_{ij}u|^p dx\right)^{\frac{\alpha q}{p^2}}$$

and $\frac{\alpha q}{p^2}\prec 1$. Then

$$\left(\sum_{ij}\int_{|u|\leq\alpha}\omega_{ij}|D_{ij}u|^p dx\right)\leq c.\left(\alpha\parallel f\parallel_{L^1(\Omega,I\!R^m)}+\parallel\lambda_2\parallel_{L^1(\Omega)}\right)\leq M\alpha+L, \tag{5.4.5}$$

with $L=L(c,\parallel\lambda_2\parallel_{L^1(\Omega)},c\parallel\lambda_3\parallel_{L^{(\frac{p}{\alpha})\cdot}})$ and $M=M(c_1,c_2\parallel\lambda_3\parallel,\parallel f\parallel_{L^1(\Omega,I\!R^m)})$, we choose:
$u^\alpha=min(|u|,\alpha)$, then by $|D|u||\leq|Du|$
$\int_\Omega|Du^\alpha|^{p_s}dx=\int_{|u|\leq\alpha}|D|u||^{p_s}dx+0\leq\int_{|u|\leq\alpha}|Du|^{p_s}dx=\int_\Omega|Du_\alpha|^{p_s}dx$
$\leq\left(\sum_{i,j}\int_\Omega\omega_{ij}|D_{ij}u_\alpha|^p dx\right)^{\frac{p_s}{p}}=\left(\sum_{i,j}\int_{|u|\leq\alpha}\omega_{ij}|D_{ij}u|^p dx\right)^{\frac{p_s}{p}}$ And by (5.4.5), and $p_s\leq p$, we have:

$$\int_\Omega|Du^\alpha|^{p_s}dx\leq c.(\alpha\parallel f\parallel_{|L^1(\Omega,I\!R^m)}+\parallel\lambda_2\parallel_{L^1(\Omega)}). \tag{5.4.6}$$

By (5.1.5) and (5.4.6), we have:

$$\int_\Omega|u^\alpha|^{p_s^*}dx\leq c.\left(\int_\Omega|Du^\alpha|^{p_s}dx\right)^{\frac{p_s^*}{p}}\leq c.\left(\alpha\parallel f\parallel_{L^1(\Omega;I\!R^m)}+\parallel\lambda_1\parallel_{L^1(\Omega)}\right)^{\frac{p_s^*}{p}} \tag{5.4.7}$$

Then:

$$\lambda_{|u|}(\alpha)=\alpha^{-p_s^*}\int_{|u|\succ\alpha}\alpha^{p_s^*}dx\leq\alpha^{-p_s^*}\int_{|u|\succ\alpha}|u^\alpha|^{p_s^*}dx\leq c.\alpha^{-p_s^*}\left(\alpha\parallel f\parallel_{L^1(\Omega,I\!R^m)}+\parallel\lambda_2\parallel_{L^1(\Omega)}\right)^{\frac{p_s^*}{p}} \tag{5.4.8}$$

and we continue in the same way as in a case that is non-degenerated [39] by replacing p by p_s as well as

$$\begin{aligned}\|u\|^*_{L^{t^*_{ps},\infty}}(\Omega,I\!R^m) &= \sup_{\alpha\succ 0}\alpha|\lambda_{|u|}(\alpha)|^{\frac{1}{t^*_{ps}}}\\ &\leq |\Omega|+\sup_{\alpha\succ 1}\alpha|\lambda_{|u|}(\alpha)|^{\frac{1}{t^*_{ps}}}\\ &\leq |\Omega|+c.(\parallel f\parallel^{\frac{1}{p_s-1}}_{L^1(\Omega,I\!R^m)},\parallel\lambda_2\parallel^{\frac{1}{p_s-1}}_{L^1(\Omega)})\end{aligned}$$

i-e:

$$\|u\|^*_{L^{t^*_{ps},\infty}}(\Omega,I\!R^m)\leq c.\left(|\Omega|,\parallel\lambda_2\parallel_{L^1(\Omega)},\parallel\lambda_3\parallel_{L^{(\frac{p}{\alpha})'_{(\Omega)^\cdot}}}c_2,\parallel f\parallel_{L^1(\Omega,I\!R^m)}\right), \tag{5.4.9}$$

on the other hen, by using $(p_s \leq p)$ and thinks to (5.1.5), we obtain:

$$\begin{aligned}
\lambda_{|Du|}(s) &\leq s^{-p_s} \int_{|u|\leq\alpha} |Du|^{p_s} dx + \lambda_{|u|}(\alpha) \\
&= s^{-p_s} \int_{|u|\leq\alpha} |Du_\alpha|^{p_s} dx + \lambda_{|u|}(\alpha) \\
&\leq s^{-p_s} \left(\sum_{i,j} \int_{|u|\leq\alpha} \omega_{ij} |D_{ij} u_\alpha|^p dx \right) + \lambda_{|u|}(\alpha) \\
&\leq s^{-p_s} \left(\sum_{i,j} \int_{|u|\leq\alpha} \omega_{ij} |D_{ij} u|^p dx \right) + \lambda_{|u|}(\alpha)
\end{aligned}$$

By (5.4.5) and (5.4.8):

$$\lambda_{|Du|}(s) \leq c. \left(\max(\frac{\alpha}{s^{p_s}}, \frac{1}{s^{p_s}}) + \max(\alpha^{-p_s^*}, \alpha^{\frac{p_s^*}{p_s} - p_s^*}) \right)$$

or $-t_{p_s}^* = \frac{p_s^*}{p_s} - p_s^*$, so as in [28]

$$\| Du \|_{L^{t_{p_s},\infty}(\Omega, M^{m\times n})}^* \leq c. \left(|\Omega|, \| \lambda_2 \|_{L^1(\Omega)}, \| \lambda_3 \|_{L^{(\frac{p}{\nu})'_{(\Omega)}}}, c_2, \| f \|_{L^1(\Omega;\mathbb{R}^m)} \right). \tag{5.4.10}$$

From (5.4.5)-(5.4.9) and (5.4.10), we obtain the result of the proposition (5.4.1) in case i). ii)-Suppose the condition (of l'angle in H_3): Let $S_\alpha(y) = (T_\alpha(y_1); T_\alpha(y_2); .....T_\alpha(y_m))$, $y \in \mathbb{R}^m$,the cubic truncation, we have $Ds_\alpha(y) = Id$ if $|y|_{max} = \max_{1\leq i\leq m} |y_i| \leq \alpha$, in the same way as in i)- by testing $S_\alpha(u)$ in (5.4.3). Then $\int_\Omega \sigma(x,u,Du) : D(S_\alpha(u))dx = \int_\Omega f.S_\alpha(u)dx$

or

$$\begin{aligned}
\int_\Omega \sigma(x,u,Du) : D(S_\alpha(u))dx &= \sum_{i=1}^m \int_{|u_i|\leq\alpha} \sigma_i(x,u,Du) : Du_i dx \\
&\geq \int_{|u|=\max_{1\leq j\leq m}(|u_j|)} \sum_{i=1}^m \sigma_i(x,u,Du) : Du_i dx
\end{aligned}$$

and like $\sum_{i=1}^m \sigma_i(x,u,Du) : Du_i dx = \sigma(x,u,Du) : Du$. By the coercivity condition in (H_1) and the Hölder Inequality we obtain:

$$c_2. \sum_{i,j} \int_{|u|\leq\alpha} \omega_{ij} |D_{ij} u|^p dx \leq \sqrt{m}.\alpha \| f \|_{L^1(\Omega;\mathbb{R}^m)} + \| \lambda_2 \|_{L^1(\Omega)} + c. \| \lambda_3 \|_{L^{(\frac{p}{\alpha})'_{(\Omega)}}} \left(\sum_{j=1}^m \int_{|u|\leq\alpha} \gamma_j |u_j|^q dx \right)^{\frac{q}{p}}$$

and we continue in the same way as in i), this completes the proof of the proposition. □

5.5 A div-curl inequality

The result of this section is the key ingredient for the proof that one can pass to the limit in the equation (5.4.1) for the solution $\{u_k\}_{k\in\mathbb{N}}$ of approximating problems. Since it is independent of the differential

equation we state it a more general form using only the hypotheses (5.5.1)-(5.5.8) below:

$$\sigma; \tau : \Omega \times I\!R^m \times M^{m\times n} \longrightarrow M^{m\times n}, \tag{5.5.1}$$

is a Carathéodory function.

$$\sigma \text{ and } \tau \text{ satisfing one of the fellowing conditions:} \tag{5.5.2}$$

(i) $\sigma(x,u,F) : MF \geq 0, \tau(x,u,F) : MF \geq 0$; $M = Id - b \otimes b \in M^{m\times n}$, with $|b| \leq 1$.

(ii) $\sigma_j(x;u;F) : F_j \geq 0$, and $\tau_j(x,u,F) : F_j \geq 0$; $1 \leq j \leq m$, σ_j, τ_j and F_j is the j^{eme} colones of $\sigma,\ \tau,\ F$.

$$u_k \in W^{1;1}(\Omega, I\!R^m) \text{ and there exists an } s \geq 0 \text{ such that } \int_\Omega \mid Du_k \mid^s dx \leq c \text{ uniformly in } k \tag{5.5.3}$$

$$\text{The sequence } \sigma_k(x) = \sigma(x, u_k, Du_k) \text{ is equi-integrale.} \tag{5.5.4}$$

$$\begin{gathered}\text{The sequence } u_k \text{ converges in measure to some function } u,\\ \text{and } u \text{ is almost everywhere approximately differentiable.}\end{gathered} \tag{5.5.5}$$

$$\text{The sequence } f_k = -div(\sigma_k + \tau_k) - \mu \text{ is bounded in } L^1(\Omega, I\!R^m). \tag{5.5.6}$$

$$\begin{gathered}D_{ij}u_k \in L^r_{loc}(\Omega, \omega_{ij}, M^{m\times n}) \text{ and } (\sigma_k + \tau_k) \in L^{r,}_{loc}(\Omega, w^*, M^{m\times n}), \text{ for some}\\ 1 \leq r \prec \infty \text{ and } 1 \leq i \leq n,\ 1 \leq j \leq m.\end{gathered} \tag{5.5.7}$$

$$\text{The sequence } \tau_k(x) = \tau[k](x, u, Du_k) \text{ converges to weakly to } 0 \text{ in } L^1(\Omega, M^{m\times n}). \tag{5.5.8}$$

Lemma 5.5.1 *Suppose (5.5.1)-(5.5.8). Then (after passage to a subsequence) the sequence σ_k converges weakly in $L^1(\Omega, M^{m\times n})$ and the weak limit $\overline{\sigma}$ is given by $\overline{\sigma}(x) =< \nu_x; \sigma(x, u(x),\ .) >$. Moreover the following inequality holds:*

$$\int_{M^{m\times n}}, \sigma(x, u(x), \lambda) : \lambda d\nu_x(\lambda) \leq \overline{\sigma}(x) : apDu(x) \textit{ for a.e. } x \in \Omega. \tag{5.5.9}$$

Proof See [28]. □

5.6 Passage to the limit

Proposition 5.6.1 *Suppose that the sequence $(u_k)_{k\in I\!N}$ satisfies the hypotheses (5.5.1)-(5.5.7), (H_2) and that the Young measure ν generated by the sequence $(Du_k)_{k\in I\!N}$ satisfies: a)-c) and d)- in lemma (5.3.3).*

Then the sequence (σ_k) is weakly converge in $L^1(\Omega, M^{m\times n})$, with $\overline{\sigma}$ is the limit and $\overline{\sigma(x)} =< \nu_x, \sigma(x, u(x), apDu(x))$. If in H_2 b)- c)-or d)-holds, $\sigma(x, u_k, Du_k) \to \sigma(x, u, apDu)$ strongly in $L^1(\Omega, M^{m\times n})$.

In the cases (c) and (d) it follows addition that $Du_k \to apDu$ in measure.

Proof See[28].

Proof of the theoreme5.2.1

Case: $\theta = p - 1$

For using the results of proposition (5.6.1): we assume that (5.5.1)-(5.5.7) and the Young measure ν_x generated by the sequence Du_k satisfies :(i), (ii) and (iii) in Lemma(5.3.3), for the approximate systems (5.4.1)-(5.4.2). By the proposition (5.6.1), with $u_k \in W_0^{1,p}(\Omega, \omega, I\!R^m)$, we have: $\| u \|_{L^{t^*_{p_s},\infty}(\Omega, I\!R^m)} \leq c.\left(|\Omega|, \| \lambda_2 \|_{L^1(\Omega)}, \| \lambda_3 \|_{L^{(\frac{p}{\alpha})'_{(\Omega)}}}, c_2, \| \mu \|_{M(\Omega,\omega^*, I\!R^m)}\right)$, and

$$\sum_{i,j} \int_{|u_k|\leq\alpha} \omega_{ij} |D_{ij} u_k|^p dx \leq M\alpha + L < \infty. \tag{5.6.1}$$

By $L^{t^*_{p_s},\imath\infty}(\Omega, I\!R^m) \hookrightarrow L^p(\Omega, I\!R^m)$ for all $1 \prec p \prec t^*_{p_s}$, then

$$\| u_k \|_{L^p(\Omega, I\!R^m)} \leq c < \infty. \tag{5.6.2}$$

Now

- (5.5.1) is (H_0)
- (5.5.2) is (H_3)
- (5.5.3): $u_k \in W_0^{1,p}(\Omega, \omega, I\!R^m) \hookrightarrow W_0^{1,p_s}(\Omega, I\!R^m)$ with $p_s > 1$, then $u_k \in W^{1,1}(\Omega, I\!R^m)$. Moreover, by the proposition 5.6

  $$\| Du_k \|_{L^{t_{p_s},\infty}(\Omega, I\!R^m)} \leq c.\left(|\Omega|, \| \lambda_2 \|_{L^1(\Omega)}, \| \lambda_3 \|_{L^{(\frac{p}{\alpha})'_{(\Omega)}}}, c_2, \| \mu \|_{M(\Omega,\omega^*, I\!R^m)}\right)$$

  hence

  $$\| Du_k \|_{L^s(\Omega, M^{m\times n})} \leq c > \infty, \ \forall\, 1 < s < t_{p_s}$$

  with $\sup_{k\in I\!N} \int_\Omega |Du_k|^s dx < \infty$.
- (5.5.4): Let A a measurable in Ω, by (H_1) and Hölder we have

  $$\begin{aligned}\int_A |\sigma(x, u_k, Du_k)| dx \ &\leq \ c\left(\sum_{r,s} \int_\Omega \omega_{rs} dx\right)^{\frac{1}{p}} .\Big\{ \| \lambda_1 \|_{L^{p'}(\Omega)} + \\ &\left(\sum_{j=1}^m \int_\Omega \gamma_j |(u_k)_j|^q dx\right)^{\frac{1}{p'}} + \left(\sum_{i;j} \int_\Omega \omega_{ij} |D_{ij} u_k|^p dx\right)^{\frac{1}{p'}} \Big\},\end{aligned}$$

and with (5.1.5) and (5.1.3):

$$\begin{aligned}\int_A |\sigma(x,u_k,Du_k)|dx &\leq c\left(\sum_{r,s} \| \omega_{rs} \|^{\frac{1}{p}}_{L^1_{loc}(\Omega)}\right)\\ &\times\left\{ \| \lambda_1 \|_{L^{p'}(\Omega)} + \| u_k \|^{\frac{q}{pp'}}_{1,p,\omega} + \| u_k \|^{\frac{p}{p'}}_{1,p,w} \right\} < \infty.\end{aligned}$$

- (5.5.5): By (5.6.1) and (5.6.2) and Lemma (5.3.2).

- (5.5.6): $\| f_k \|_{L^1(\Omega,\mathbb{R}^m)} \leq \| \mu \|_{M(\Omega,\omega^*,\mathbb{R}^m)}$.

- (5.5.7): $\forall \varepsilon \succ 0$ and $x_0 \in \Omega$ $\int_{B(x_0,\varepsilon)} |D_{ij}u_k|^p \omega_{ij} dx \leq \| u_k \|^p_{1,p,\omega} < \infty$. And by (H_1)

$$\begin{aligned}\int_{B(x_0,\varepsilon)} |\sigma_{rs}(x,u_k,Du_k)|^{p'}\omega^*_{rs}dx &= \int_{B(x_0,\varepsilon)} |\sigma_{rs}(x,u_k,Du_k)|^{p'}\omega_{rs}^{1-p'}dx\\ &\leq c.\int_{B(x_0,\varepsilon)} \omega_{rs}^{1-p'+\frac{p'}{p}}\left[|\lambda_1|^{p'} + \sum_{j=1}^m \gamma_j |(u_k)_j|^q + \sum_{i,j}\omega_{ij}|D_{ij}u_k|^p\right]dx\\ &\leq c\left(\| \lambda_1 \|^{p'}_{L^{p'}(\Omega)} + \| u_k \|^{\frac{q}{p}}_{1,p,w} + \| u_k \|^p_{1,p,\omega}\right) < \infty.\end{aligned}$$

Then, by the proposition (5.6.1) $\sigma(x,u_k,Du_k) \longrightarrow \sigma(x,u,apDu)$ in $L^1(\Omega,\mathbb{M}^{m\times n})$ and $\forall \varphi \in C^\infty_c(\Omega,\mathbb{R}^m); D\varphi \in L^\infty(\Omega,\mathbb{M}^{m\times n})$ hence:

$$\int_\Omega \sigma(x,u_k,Du_k) : D\varphi dx \longrightarrow \int_\Omega \sigma(x,u,apDu) : D\varphi dx$$

i.e:

$$-div\sigma(x,u_k,apDu_k) \longrightarrow -div\sigma(x,u,apDu)$$

In the sense of distributions. On the other hand $f_k \overset{*}{\longrightarrow} \mu$ in $L^1(\Omega,\mathbb{R}^m)$.
Then $\int_\Omega f_k.\varphi dx \longrightarrow \int_\Omega \mu.\varphi dx \quad \forall \varphi \in C^\infty_c(\Omega,\mathbb{R}^m)$ so u is the solution in $W^{1,p}_0(\Omega,w,\mathbb{R}^m)$ of the system:

$$-div\sigma(x,u,apDu) = \mu \ \ in \ \ \Omega$$

$$u = 0 \ \ on \ \ \partial\Omega$$

to show the estimation (5.2.1), we take the function η in $C^1_c(B(0,2\alpha),\mathbb{R}^m); \eta = Id$ in $B(0,\alpha)$ and $|D\eta| \leq c$, then:

$$\begin{aligned}\sum_{i,j}\int_\Omega \omega_{ij}|D_{ij}\eta(u_k)|^p dx &= \sum_{i,j}\int_\Omega \omega_{ij}|(D_{ij}\eta)(u_k)|^p |Du_k\|^p dx\\ &\leq c.\sum_{ij}\int_{|u_k|\leq\alpha}\omega_{ij}|D_{ij}u_k)|dx + c^p.\sum_{i,j}\int_{|u_k|\leq 2\alpha}\omega_{ij}|D_{ij}u_k|^p dx\\ &\leq c.c(\alpha) + c.c(2\alpha) < \infty,\end{aligned}$$

thanks to (5.6.1).

Now, we have $\eta(u_k) \longrightarrow \eta(u)$, for every $x \in \Omega$ because η is C^∞. Then $\eta(u_k) \rightharpoonup \eta(u)$, in $W_0^{1,p}(\Omega, \omega, I\!R^m)$, and $apDu = apD(\eta \circ u)$ on $\{|u| \prec \alpha\}$. Hence,

$$\begin{aligned}
\sum_{i,j} \int_\Omega \omega_{ij} |D_{ij}(\eta \circ u)|^p dx &\leq \liminf_{k \longrightarrow \infty} \sum_{i,j} \int_\Omega \omega_{ij} |D(\eta \circ u_k)|^p dx \\
&\leq \liminf_{k \longrightarrow \infty} \sum_{ij} \int_{|u_k| \leq 2\alpha} |D_{ij}\eta(u_k)|^p |D_{ij}u_k| \omega_{ij} dx \\
&\leq \leq c. \liminf_{k \longrightarrow \infty} \int_{|u_k| \leq 2\alpha} \omega_{ij} |D_{ij}u_k)|^p dx \\
&\leq c.c(2\alpha) < \infty.
\end{aligned}$$

Then:

$$\sum_{i,j} \int_{|u| \leq \alpha} \omega_{ij} |apDu|^p dx = \sum_{i,j} \int_{|u_k| \leq 2\alpha} \omega_{ij} |D(\eta \circ u)|^p dx < \infty,$$

in the same as in the proof of the proposition (5.6) by replacing u_k by u and f_k by μ, we obtain the estimation (5.2.1) and this completes the proof of the theorem 5.2.1 $\square$

Case: $0 < \theta < \frac{n(p_s - 1)}{n-1}$ (the general case) The idea is to consider the regularized problems:

$$-div\phi_\varepsilon(x, u_\varepsilon, Du_\varepsilon) = \mu \text{ in } \Omega, \tag{5.6.3}$$

$$u_\varepsilon = 0 \text{ on } \partial\Omega \tag{5.6.4}$$

With

$$\phi_{\varepsilon,r,s}(x, u, F) = \sigma_{rs}(x, u, F) + \varepsilon\beta \left(\sum_{ij} \omega_{ij}^{\frac{1}{p'}} |F_{ij}|^{S-2} \right) \omega_{rs}^{\frac{1}{p}} F_{rs},$$

$\forall 1 \leq r \leq n,\ \forall 1 \leq s \leq m$, with $S \geq (\frac{s+1}{s})(n+1)$, and $\varepsilon < \frac{1}{2}$, we have $p < S$, then $S' < p'$, and $(\frac{S}{\nu})' < (\frac{p}{\nu})'$. Moreover $\exists c > 0$ which doesn't depend on p, S, such that $\omega_{rs}^{\frac{1}{p}} \leq c.\omega_{rs}^{\frac{1}{S}}$

$\forall\ 1 \leq r \leq n$ and $1 \leq s \leq m$.

By (H_1) for σ, we obtain

$$\begin{aligned}
|\phi_{\varepsilon,r,s}(x, u, F)| &\leq \beta'.|\omega_{rs}|^{\frac{1}{p}} \left[\lambda_1 + \sum_{j=1}^m \gamma_j^{\frac{1}{p'}} |u_j|^{\frac{q}{p'}} + \sum_{i,j} \omega_{ij}^{\frac{1}{p'}} |F_{ij}|^\theta \right] \\
&\quad + \varepsilon\beta\omega_{rs}^{\frac{1}{p}} \left(\sum_{i,j} \omega_{ij}^{\frac{1}{p'}} |F_{ij}|^{S-1} \right) \left(\theta < \tfrac{n(p_s-1)}{n-1} < n < (S-1) \right) \\
&\leq \leq \beta' \omega_{rs}^{\frac{1}{p}} \left[\lambda_1 + \sum_{j=1}^m \gamma_j^{\frac{1}{p'}} |u_j|^{\frac{q}{p'}} + \sum_{i,j} \omega_{ij}^{\frac{1}{p'}} |F_{ij}|^{S-1} \right].
\end{aligned}$$

And $p < S$, then $\frac{1}{p'} < \frac{1}{S'}$ and like $\omega_{rs}^{\frac{1}{p}} \leq c.\omega_{rs}^{\frac{1}{S}}$, then:

$|\phi_{\varepsilon,r,s}(x,u,F)| \leq \beta'.|\omega_{rs}|^{\frac{1}{S}} \left[\lambda_1 + \sum_{j=1}^{m} \gamma_j^{\frac{1}{S'}} |u_j|^{\frac{q}{S'}} + \sum_{ij} \omega_{ij}^{\frac{1}{S'}} |F_{ij}|^{S-1}\right]$, and by (H_3), we conclude that

$$\phi_\varepsilon(x,u,F) : F = \sigma(x,u,F) : F + \varepsilon \sum_{i,j,r,s} \omega_{ij}^{\frac{1}{p'}} \omega_{rs}^{\frac{1}{p}} |F_{ij}|^{S-2} F_{ij}.F_{rs}$$

$\geq -\lambda_2 - \sum_{j=1}^{m} \lambda_3 \gamma_j^{\frac{q}{S}}.|u_j|^{\frac{q\nu}{S}} + \varepsilon \sum_{ij} \omega_{ij}|F_{ij}|^S$.

On the other hand, $0 < \nu < p-1 < S-1$, $1 < q < \frac{p^2}{\nu} < \frac{S^2}{\nu'}$, $\lambda_1 \in L^{p'}(\Omega) \hookrightarrow L^{S'}(\Omega)$, and $\lambda_3 \in L^{(\frac{p}{\nu})'}(\Omega) \hookrightarrow L^{(\frac{S}{\nu})'}(\Omega)$ and as σ_ε verifies the conditions of the structures (of l'angle and sign), the strict monotony, the s-quasi monotonous with regard to F is a C^1 monotony in relation with F or accepting a convex potential because: $F \longrightarrow \varepsilon\beta \left(\sum_{ij} \omega_{ij}^{\frac{1}{p'}} |F_{ij}|^{S-2}\right) \omega_{rs}^{\frac{1}{p}} F_{rs}$ verify them as well, hence σ_ε verifies the hypotheses $(H_0) - (H_5)$, for the regularized problems (5.6.3)-(5.6.4), thus for the previous case, of theorem 5.2.1, there exists a solution, $u_\varepsilon \in W_0^{1,S}(\Omega, \omega, I\!R^m)$, of the system (5.6.3)-(5.6.4). Now showing that the conditions: i), ii) and iii), of lemma (5.3.3), and the hypotheses (5.5.1)-(5.5.8) of the div-curl inequality are verified for u_ε with order S in the place of p.

We suppose the condition of l'angle verifying that ϕ_ε by testing, $T_\alpha(u_\varepsilon)$, $\alpha \succ 0$ in (5.5.3)-(5.5.4), we get: $\int_\Omega \phi_\varepsilon(x,u_\varepsilon,Du_\varepsilon) : DT_\alpha(u_\varepsilon)dx = \langle \mu, T_\alpha(u_\varepsilon)\rangle$, so

$$\int_{|u_\varepsilon|\leq\alpha} \sigma(x,u_\varepsilon,Du_\varepsilon) : Du_\varepsilon dx + \int_{|u_\varepsilon|>\alpha} \frac{\alpha}{|u_\varepsilon|}\sigma_\varepsilon(x,u_\varepsilon,Du_\varepsilon) : \left(Id - \frac{u_\varepsilon}{|u_\varepsilon|} \otimes \frac{u_\varepsilon}{|u_\varepsilon|}\right) Du_\varepsilon dx$$

$$+\varepsilon\beta \int_{|u_\varepsilon|\leq\alpha} \sum_{i,j} \omega_{ij}^{\frac{1}{p'}} |D_{ij}u_\varepsilon|^{S-2} \sum_{r,s} \omega_{rs}^{\frac{1}{p}} |D_{rs}u_\varepsilon|^2 dx$$

$$+\varepsilon\beta \int_{|u_\varepsilon|>\alpha} \sum_{ij} \omega_{ij}^{\frac{1}{p'}} |D_{ij}u_\varepsilon|^{S-2} \sum_{r,s} \omega_{rs} D_{rs}u_\varepsilon \left(Id - \frac{u_\varepsilon}{|u_\varepsilon|} \otimes \frac{u_\varepsilon}{|u_\varepsilon|}\right)$$

$$\leq \alpha. \parallel \mu \parallel_{M(\Omega,\omega^*,I\!R^m)} .$$

since

$$\sum_{rs} |D_{rs}u_\varepsilon|^{S-2} D_{rs}u_\varepsilon \left(Id - \frac{\alpha}{|u_\varepsilon|}(\frac{u_\varepsilon}{|u_\varepsilon|} \otimes \frac{u_\varepsilon}{|u_\varepsilon|})\right) \geq 0$$

so

$$\int_{|u_\varepsilon|\leq\alpha} \sigma(x,u_\varepsilon,Du_\varepsilon) : Du_\varepsilon dx \leq \alpha \parallel \mu \parallel_{M(\Omega,\omega^* I\!R^m)} .$$

And by the coercivity condition of σ in (H_1) and Hölder inequality, we get as in the proof of the proposition

$$\sum_{ij} \int_{|u_\varepsilon|\leq\alpha} \omega_{ij}|D_{ij}u_\varepsilon|^p dx \leq M'\alpha + L', \tag{5.6.5}$$

And the following a priori estimation:

$$\| u_\varepsilon \|^*_{L^{t^*_{ps};\infty}(\Omega,I\!R^m)} + \| Du_\varepsilon \|^*_{L^{t_{ps},\infty}(\Omega,I\!R^{m\times n})} < c < \infty, \tag{5.6.6}$$

and by the injection $L^{\beta',\infty} \hookrightarrow L^{\alpha'}$, $\forall\ 0 < \alpha' < \beta'$, then $\forall$, $0 < r < t^*_{p_s}$, $0 < \rho < t_{p_s}$

$$\| u_\varepsilon \|_{L^r(\Omega,I\!R^m)} + \| Du_\varepsilon \|_{L^\rho(\Omega,I\!M^{m\times n})} + \| Du_\varepsilon \|^*_{L^{t_{ps},\infty}(\Omega,I\!M^{m\times n})} < \infty. \tag{5.6.7}$$

We suppose that the condition of the sign is verify.

As in the same way in the proof of the proposition (5.6), we test $S_\alpha(u_\varepsilon)$ in (5.6.3)-(5.6.4), we obtain (5.6.5) and (5.6.7).

Starting with verifying that i), ii) et iii) of lemma (5.3.3) and the hypotheses (5.5.1) and(5.5.7) for σ_ε. By (5.6.5)and(5.6.7), the points i), ii) et iii) are a direct consequence of lemma(5.3.2) and lemma (5.3.3). On the other hand:

-(5.5.1): for σ is (H_0) and $\tau_{rs}(x,u,F) = \varepsilon\beta\left(\sum_{i,j}\omega_{ij}^{\frac{1}{p'}}|F_{ij}|^{s-2}\right)\omega_{rs}^{\frac{1}{p}}F_{rs}$ is a Carathéodory function, because $x \longmapsto \omega_{ij}(x)$, is measurable, so σ_ε is a Cathéodory function.

-(5.5.2)

(i) $\phi_\varepsilon(x,u,F) : MF = \sigma(x,u,F) : MF + \left(\sum_{rs}(\varepsilon\sum_{i,j}\omega_{ij}^{\frac{1}{p'}}|F_{ij}|^{S-2})\omega_{rs}^{\frac{1}{p}}F_{rs}\right)(MF)_{rs} \geq 0,$

with $M = Id - a\otimes a$ and $|a| \leq 1$.

(ii)

$$\begin{aligned}\phi_{rs}(x,u,F).F_j &= \sigma_j(x,u,F) : F_j + \tau_j(x,u,F).F_j \\ &= \sigma_j(x,u,F) : F_j + \sum_{l=1}^m \varepsilon\beta\left(\sum_{i,j}\omega_{ij}^{\frac{1}{p'}}|F_{ij}|^{S-1}\right).w_{lj}^{\frac{1}{p}}|F_{lj}|^2 \geq 0,\end{aligned}$$

$\forall 1 \leq j \leq m$.

-(5.5.3): $u_\varepsilon \in W_0^{1,S}(\Omega,\omega,I\!R^m) \hookrightarrow W_0^{1,S_s}(\Omega,I\!R^m)$, $S_s > 1$, so $u_\varepsilon \in W^{1,1}(\Omega,I\!R^m)$, and by (5.6.7) $\sup_{\varepsilon>0}\int_\Omega |Du_\varepsilon|^\rho dx < \infty$, $\forall$, $0 < \rho < t_{p_s}$.

(5.5.4): $\sigma(x,u_\varepsilon,Du_\varepsilon)$ is equi-integrable as previously $\forall\ \Omega' \subset \Omega$, measurable, we have:

$$\begin{aligned}&\int_{\Omega'} |\sum_{i,j}(\omega_{ij}^{\frac{1}{p'}}|D_{ij}u_\varepsilon|^{S-2})\omega_{rs}^{\frac{1}{p}}D_{rs}u_\varepsilon|dx \\ &\leq \left(\sum_{i,j}\int_{\Omega'}\omega_{ij}|D_{ij}u_\varepsilon|^{S-1}dx\right) \\ &\leq c.\sum_{ij}\int_{\Omega'}\omega_{ij}|D_{ij}u_\varepsilon|^S dx \leq c.\ \| u_\varepsilon \|^S_{1,S,\omega}.\end{aligned}$$

-(5.5.5): by (5.6.7) and the lemma (5.3.2).

-(5.5.6): by (5.6.3), $-div(\sigma_l + \tau_k) - \mu = 0$, with $\mu \in M(\Omega, \omega^*, I\!R^m)$ is bounded in $L^1(\Omega, I\!R^m)$.

-(5.5.7): $\forall\ \varepsilon > 0$ and $x_0 \in \Omega$, by the growth condition of σ_ε and previously with s in the place of p,
$\int_{B(x,\varepsilon)} |\sigma_\varepsilon(x, u_\varepsilon, Du_\varepsilon)|^S \omega^*_{rs} dx < \infty$
and

-(5.5.8): $\int_{B(x,\varepsilon)} |D_{ij}u_\varepsilon|^S \omega_{rs} dx <\| u_\varepsilon \|^\varepsilon_{1,S,\omega} < \infty$.

Testing that u_ε in (5.6.3)-(5.6.4)

$$\begin{aligned} \varepsilon\beta \int_\Omega \left(\sum_{ij} \omega_{ij}^{\frac{1}{p'}} |D_{ij}u_\varepsilon|^{S-2} \right) \left(\sum_{rs} \omega_{rs}^{\frac{1}{p}} |D_{rs}u_\varepsilon|^2 \right) dx \\ \leq \| u_\varepsilon \|_{L^\infty(\Omega, I\!R^m)} \| \mu \|_{M(\Omega,\omega^*, I\!R^m)} \end{aligned} \tag{5.6.8}$$

We have $W_0^{1,s}(\Omega, \omega, I\!R^m) \hookrightarrow W_0^{1,S_s}(\Omega, I\!R^m) \hookrightarrow L^\infty(\Omega, I\!R^m)$. Then

$$\begin{aligned} \| u_\varepsilon \|_{L^\infty(\Omega, I\!R^m)} &\leq c. \left(\sum_{i,j} \int_\Omega \omega_{ij} |D_{ij}u_\varepsilon|^s dx \right)^{\frac{1}{S}} \\ &\leq c. \left(\sum_{i,j} \int_\Omega \omega_{ij}^{\frac{1}{p'}} |D_{ij}u_\varepsilon|^{S-2} \omega_{ij}^{\frac{1}{p}} |D_{ij}u_\varepsilon|^2 dx \right)^{\frac{1}{s}} \\ &\leq c(\int_\Omega (\sum_{i,j} \omega_{ij}^{\frac{1}{p'}} |D_{ij}u_\varepsilon|^{S-2}).(\sum_{r,s} \omega_{rs}^{\frac{1}{p}} |D_{rs}u_\varepsilon|^2 dx)^{\frac{1}{S}}. \end{aligned} \tag{5.6.9}$$

Thanks to (5.6.8) and (5.6.9), we have

$$\int_\Omega \sum_{ij} \omega_{ij}^{\frac{1}{p'}} |D_{ij}u_\varepsilon|^{S-2} \sum_{rs} \omega_{rs}^{\frac{1}{p}} |D_{rs}u_\varepsilon|^2 dx$$

$$\leq \frac{c.\|\mu\|_{M(\Omega,\omega^*, I\!R^m)}}{\varepsilon} \left(\int_\Omega (\sum_{i,j} \omega_{ij}^{\frac{1}{p'}} |D_{ij}u_\varepsilon|^{S-2}).(\sum_{rs} \omega_{rs}^{\frac{1}{p}} |D_{rs}u_\varepsilon|^2 dx \right) \text{ So:}$$

$$\left(\int_\Omega (\sum_{i,j} \omega_{ij}^{\frac{1}{p'}} |D_{ij}u_\varepsilon|^{S-2}).(\sum_{r,s} \omega_{rs}^{\frac{1}{p}} |D_{rs}u_\varepsilon|^2 dx \right)^{\frac{S-1}{S}} \leq \frac{c \| \mu \|_M}{\varepsilon},$$

which mean that

$$\left(\int_\Omega (\sum_{i,j} \omega_{ij}^{\frac{1}{p'}} |D_{ij}u_\varepsilon|^{S-2}).(\sum_{r,s} \omega_{rs}^{\frac{1}{p}} |D_{rs}u_\varepsilon|^2 dx \right)^{\frac{1}{S}} \leq \frac{c \| \mu \|_M}{\varepsilon}, \tag{5.6.10}$$

and

$$\| u_\varepsilon \|_{L^\infty(\Omega, I\!R^m)} \leq c. \left(c. \frac{\| \mu \|_M}{\varepsilon} \right)^{\frac{1}{S-1}}. \tag{5.6.11}$$

On the other hand and $\forall\ 1<\rho<\frac{S}{S-1}$, can write

$$\begin{aligned}
&\| \epsilon \sum_{i,j} \omega_{ij}^{\frac{1}{p'}} |D_{ij}u_\varepsilon|^{S-2}\omega_{rs}^{\frac{1}{p}}|F_{rs}| \|_{L^{\frac{S}{S-1}}(\Omega,\mathbb{M}^{m\times n})} \\
&\leq \varepsilon^{\frac{S}{S-1}} \left(\int_\Omega |\sum_{i,j} \omega_{ij}^{\frac{1}{p'}} |D_{ij}u_\varepsilon|^{S-1}\omega_{rs}^{\frac{1}{p}}|^{\frac{S}{S-1}} dx \right)^{\frac{S-1}{S}} \\
&\leq c\varepsilon^{\frac{S}{S-1}} \left(|\sum_{i,j} \int_\Omega \omega_{ij}^{\frac{1}{p'}} |D_{ij}u_\varepsilon|^{S-1}\omega_{rs}^{\frac{1}{p}}|^{\frac{S}{S-1}} dx \right)^{\frac{S-1}{S}} \\
&\leq c.\varepsilon^{\frac{S}{S-1}} \left(\sum_{i,j} \int_\Omega \omega_{ij}|D_{ij}u_\varepsilon|^{S-2} \sum_{r,s} \omega_{rs}^{\frac{S}{(S-1)p}} |D_{rs}u_\varepsilon|^2 dx \right) < \infty.
\end{aligned}$$

thanks to (5.6.10). Now, since $u_\varepsilon \in W_0^{1,S}(\Omega,\omega,I\!R^m) \hookrightarrow W_0^{1,S_s}(\Omega,I\!R^m) \hookrightarrow W_0^{1,p_s}(\Omega,I\!R^m)$, so by testing $T_\alpha(u_\varepsilon)$ in (5.6.3)-(5.6.4), we obtain as in the proof of the proposition (5.4.1)

$$\| Du_\varepsilon \|^*_{L^{\frac{n(p_s-1)}{n-1},\infty}(\Omega,\mathbb{M}^{m\times n})} \leq c. \tag{5.6.12}$$

By the Hölder inequality for the exponent a with a and ξ are the solutions of systems:

$$\begin{cases} a'\xi = \tau & > & \frac{n(p_s-1)}{n-1} \\ a\left((S-1)\rho-\xi\right) & = & s \end{cases}$$

a given system accepting the solution when $\rho<\frac{S}{S-1}$. So

$$\int_\Omega |\varepsilon \sum_{i,j} \omega_{ij}^{\frac{1}{p'}} |D_{ij}u_\varepsilon|^{S-1}\omega_{ij}^{\frac{1}{p}}|^\rho dx$$

$$\leq c.\int_\Omega \varepsilon^\rho \left(\sum_{i,j} \omega_{ij}^{\frac{\rho}{p'}} |D_{ij}u_\varepsilon|^{(S-1)\rho-\xi}\omega_{ij}^{\frac{\rho}{p}}|D_{ij}u_\varepsilon|^\xi \right)^\rho dx$$

$$\leq c.\varepsilon^\rho \left(\sum_{i,j} \int_\Omega \omega_{ij}^{a\rho}|D_{ij}u_\varepsilon|^{a((S-1)\rho-\xi)}dx \right)^{\frac{1}{a}} . \left(\int_\Omega |Du_\varepsilon|^{a'\xi}dx \right)^{\frac{1}{a'}}$$

$$\leq c.\varepsilon^\rho \left(\sum_{i,j} \int_\Omega \omega_{ij}^{\frac{1}{p'}} |D_{ij}u_\varepsilon|^{S-2} \sum_{r,s} \omega_{rs}^{\frac{1}{p}}|D_{rs}u_\varepsilon|^2 \right)^{\frac{1}{a}} . \| Du_\varepsilon \|^{\frac{\tau}{a}}_{L^\tau(\Omega,\mathbb{M}^{m\times n})}.$$

And by the injection: $L^{\frac{n(p_s-1)}{n-1}} \hookrightarrow L^\tau \qquad \forall\tau > \frac{n(p_s-1)}{n-1}$ and thanks to (5.6.10)-(5.6.12), we get:

$$\begin{aligned}
\int_\Omega |\varepsilon \sum_{i,j} \omega_{ij}^{\frac{1}{p'}} |D_{ij}u_\varepsilon|^{S-1}\omega_{ij}^{\frac{1}{p}}|^\rho dx &\leq c.\varepsilon^\rho \left(\frac{c\|\mu\|_M}{\varepsilon} \right)^{\frac{S}{(S-1)a}} .c^{\frac{\tau}{a}} \\
&\leq c.c^{\frac{\tau}{a}}\varepsilon^{\frac{a((S-1)\rho-S))}{a(S-1)}} \\
&\leq c.c^{\frac{\tau}{a}}\varepsilon^{\frac{a\xi}{a(S-1)}} \\
&\leq c.c^{\frac{\tau}{a}}.\varepsilon^{\frac{\xi}{S-1}}
\end{aligned}$$

with $\frac{\xi}{S-1} > 0$. Hence

$$\lim_{\varepsilon \longrightarrow} \| \varepsilon \sum_{i,j} \omega_{ij}^{\frac{1}{p'}} |D_{ij}u_\varepsilon|^{S-1} \omega_{rs}^{\frac{1}{p}} D_{rs}u_\varepsilon \|_{L^\rho(\Omega, I\!M^{m\times n})} = 0, \quad \forall \rho < \frac{S}{S-1}.$$

In particular for $\rho = 1$

$$\lim_{\varepsilon \longrightarrow} \int_\Omega |\varepsilon \sum_{i,j} \omega_{ij}^{\frac{1}{p'}} |D_{ij}u_\varepsilon|^{S-1} \omega_{rs}^{\frac{1}{p}} D_{rs}u_\varepsilon| dx = 0,$$

which mean that

$$\tau[\varepsilon](x, u_\varepsilon, Du_\varepsilon) = \varepsilon \sum_{i,j}^{\frac{1}{p'}} |D_{ij}u_\varepsilon|^{S-2} \omega_{rs}^{\frac{1}{p}} D_{rs}u_\varepsilon \rightharpoonup 0$$

in $L^1(\Omega, I\!M^{m\times n})$.

As well as by the proposition (5.6.1), $div\sigma(x, u_\varepsilon, Du_\varepsilon)$ converges to $div\sigma(x, u, apDu)$, in the sense of the distributions, and as

$$\tau[\varepsilon](x, u_\varepsilon, Du_\varepsilon) = \varepsilon \sum_{i,j} \omega_{ij}^{\frac{1}{p'}} |D_{ij}u_\varepsilon|^{S-2} \omega_{rs}^{\frac{1}{p}} D_{rs}u_\varepsilon \rightharpoonup 0,$$

in $L^1(\Omega, I\!M^{m\times n})$. Then $div\sigma_\varepsilon(x, u_\varepsilon, Du_\varepsilon)$ converge to $div\sigma(x, u, apDu)$ in the sense of distributions, i-e: u is the solution of the system

$$\begin{cases} -div\sigma(x, u, apDu) = \mu & \in \Omega \\ u = 0, \text{ on } \partial\Omega. \end{cases}$$

In the same way as in the case of $\theta = p_s - 1$, we have

$$\int_{|u|\leq\alpha} |apDu|^S dx < c(\alpha) < \infty \text{ and } p < S.$$

So we conclude as in the proof of the proposition (5.6.1), in order to get the estimation of theorem (5.2.1). This completes the proof of the theorem $\square$

Example 5.6.1 *We consider in this example the weights functions $\omega_{ij} = 1$ for all $i = 0; 1; ...n$ and $J = 1; 2; ...m$ The function $\sigma(x, u, F) = a(u) \mid F \mid^{p-2} F$ where a is a continuous functions, which is bounded from above and below by positive constants, satisfies the angle and the sign structure conditions and $(H_0) - (H_3)$.*

Example 5.6.2 *We consider in this example the weights functions $\omega_{ij} = 1$ for all $i = 0; 1; ...n$ and $j = 1; 2; ...m$. Let*

$$\sigma(x, u, F) = ((FA) : F)^{\frac{p-2}{2}} FA$$

where $A = A(x,u) : \Omega \times I\!R^m \to M^{n\times m}$ *is a symmetric matrix satisfying the ellipticity condition*

$$\lambda_1|\zeta|^2 \leq \sum_{i,j} A_{ij}(x,u)\zeta_i\zeta_j \leq \lambda_2|\zeta|^2$$

with constants $\lambda_1, \lambda_2 > 0$*. It is not difficult to verify that this example satisfy the sign condition. Next, we present a linear system which satisfies the conditions (Carathéodory function, Growth, coercivity and sign condition), but does not satisfy structure condition. Since the sign condition requires that the components of the matrix* σ *can be only weakly - complied, this shows the limitation of (structure condition). Therefore, it would be interesting to prove the same results under weaker conditions than (structure conditions).*

Example 5.6.3 *Let* $F = (F_{ij})i, j = 1; 2 \in M^{2\times 2}$ *and* $\sigma = (\sigma_{ij})i, j = 1, 2$ *with:* $\sigma_{11} = \varepsilon F_{11}$, $\sigma_{12} = \varepsilon^\alpha F_{12}$, $\sigma_{21} = F_{21}$ *and* $\sigma_{22} = F_{22}$, *with* $(\alpha \geq 1), (0 < \varepsilon \leq \frac{1}{5})$. *It is obvious that* $\sigma(F)$ *satisfies the conditions* $(H_0) - (H_3)$. *and the sign condition. Choosing* $a = (\varepsilon^{\frac{1}{2}}, (1-\varepsilon)^{\frac{1}{2}}) \in I\!R^2$, *we have* $\mid a \mid = 1$ *and*
$M = Id - a \bigotimes a = (M_{ij})_{i,j=1,2}$ *with* $M_{11} = 1-\varepsilon$, $M_{22} = \varepsilon$, $M_{21} = -\varepsilon^{\frac{1}{2}}(1-\varepsilon)^{\frac{1}{2}}$ *and* $M_{12} = -\varepsilon^{\frac{1}{2}}(1-\varepsilon)^{\frac{1}{2}}$. *Now choosing* $F = (F_{ij})i, j = 1, 2$ *with* $F_{ij} = 1$ *for all* $i, j = 1, 2$, *we have* $\sigma(F)_{11} = \varepsilon$, $\sigma(F)_{21} = 1$, $\sigma(F)_{12} = \varepsilon^\alpha$, $\sigma(F)_{22} = 1$ *and*

$$(MF)_{11} = 1 - \varepsilon - \varepsilon^{\frac{1}{2}}(1-\varepsilon)^{\frac{1}{2}}$$

$$(MF)_{12} = 1 - \varepsilon - \varepsilon^{\frac{1}{2}}(1-\varepsilon)^{\frac{1}{2}}$$

$$(MF)_{21} = \varepsilon - \varepsilon^{\frac{1}{2}}(1-\varepsilon)^{\frac{1}{2}}$$

$$(MF)_{22} = \varepsilon - \varepsilon^{\frac{1}{2}}(1-\varepsilon)^{\frac{1}{2}}$$

therefore,

$$\sigma(F) : MF = (\varepsilon + \varepsilon^\alpha)(1 - \varepsilon - \varepsilon^{\frac{1}{2}}(1-\varepsilon)^{\frac{1}{2}} + 2(\varepsilon - \varepsilon^{\frac{1}{2}}(1-\varepsilon)^{\frac{1}{2}}$$

$$\leq \varepsilon + \varepsilon^\alpha + 2\varepsilon - 2\varepsilon^{\frac{1}{2}}(1-\varepsilon)^{\frac{1}{2}}$$

$$< 4\varepsilon - 2\varepsilon^{\frac{1}{2}}(1-\varepsilon)^{\frac{1}{2}}$$

$$< 2\varepsilon^{\frac{1}{2}}(2\varepsilon^{\frac{1}{2}} - (1-\varepsilon)^{\frac{1}{2}} \leq 0$$

which implies that the structure conditions not hold for this example.

Part II

Quasi-linear Parabolic Systems in weighted Sobolev spaces

Introduction of part II

Let Ω be a bounded open domain in $\mathbb{R}^n$, $n \geq 2$ and $0 < T < \infty$ is given; we recall that Ω_T is defined as the cylinder $\Omega \times (0;T)$ this part is devoted to establish Leray-lions existence results for parabolic problems in divergence form of type $(QPS)_{(\omega)}$. In this part, the aims of this text is to prove existence results under relaxed monotonicity, in particular under strict quasi-monotonicity. The main technical tool we advocate and use throughout the proof are Young measures. By applying a Galerkin schema, we obtain easily an approximating sequence u_k. The Ball's theorem and especially the resulting tools mode available by Hungerbühler [38] to partial differential equation theory give then a sufficient control on the gradient approximating sequence Du_k to pass to the limit. This method is used by Dolzmann, Muller and Mainly by Hungurbühler to get the existence of a weak solution for the quasi-linear elliptic system and quasi-linear parabolic. This paper can be seen as generalization of Hungurbühler. This problem was made by, Hungerbühler N, see [38] in a space of sobolev without weight. In this part we will demonstrate the result of existence in a Sobolev space with weight.

Chapter 6

Existence of solutions for some degenerated or singular quasi-linear parabolic systems and weak monotonicity in weighted sobolev spaces

Abstract

We prove the existence of weak solution u for the nonlinear parabolic systems:

$$(\mathcal{QPS})\begin{cases} \partial_t u - div\sigma(x,t,u,Du) &= v(x,t) \quad \text{in} \quad \Omega_T \\ u(x,t) &= 0 \quad \text{on} \quad \partial\Omega\times(0,T) \\ u(x,0) &= u_0(x) \quad \text{on} \quad \Omega \end{cases}$$

which is a Dirichlet Problem. In this system, v belongs to $L^{p'}(0,T,W^{-1,p'}(\Omega,\omega^*,I\!R^m))$ and $u_0 \in L^2(\Omega,\omega_0,I\!R^m)$, we prove existence of a weak solution of different variants of this system under classical regularity for some $p \in](\frac{s+1}{s})\frac{2n}{n+2}\ ;\ \infty[$, growth and coercivity for σ but with only very mild monotonicity assumptions.

6.1 functional framework

Definition 6.1.1 *: Let X be a Banach space and $0 < T < \infty$, the space $L^p(0,T,X)$ with $1 \le p \le \infty$ consists of all measurable functions (identified modulo the relation "equal almost everywhere") $u : (0;T) \to X$ for which*

$\|u\|_{L^p(0,T,X)} = \left(\int_0^T \|u(t)\|_X^p dt \right)^{\frac{1}{p}}$, *if* $p \geq 1$;
$\|u\|_{L^p(0,T,X)} = ess \sup_{t\in(0;T)} \|u(t)\|_X$, *if* $p = \infty$ *holds.*

Again, equipped with the norm $\|.\|_{L^p(0,T,X)}$; $L^p(0,T,X)$ is a Banach space.

In the special case where $p = 2$ and X is a Hilbert space $L^2(0,T,X)$ equipped with the inner product $\langle u, v\rangle = \int_0^T \langle u(t), v(t)\rangle_X dt$; for all $u; v \in L^2(0,T,X)$ is a Hilbert space, Moreover; if $1 \leq q \leq \infty$ the following Hölder inequality holds for all $v \in L^{p'}(0,T,X')$ and all

$u \in L^p(0,T,X)$.
$\int_0^T \mid \langle u(t), v(t)\rangle \mid dt \leq \| v \|_{L^{p'}(0,T,X')} . \| u \|_{L^p(0,T,X)}$
and $L^q(0,T,X) \hookrightarrow L^p(0,T,Y)$, for all X, Y such that $X \hookrightarrow Y$.

$C^k([0;T],X)$ are dense in $L^p(0,T,X)$ as $p \leq \infty$.

$C^k([0;T],X) \hookrightarrow L^p(0,T,X)$ is continuous.

If $1 \leq p < \infty$ and X is separable; so is $L^p(0,T,X)$.

If $1 < p < \infty$, $L^p(0,T,X)$ is uniformly convex in the case where X is uniformly convex, and, reflexive, separable when X is reflexive and separable Banach space.

Definition 6.1.2 *(Lebesgue and Sobolev spaces with weight):*

For $1 \leq p \leq \infty$ *and* $k \in I\!N$; *we consider the weight spaces:*

$$L^p(\Omega, \omega_0, I\!R^m) \equiv \left\{ u = u(x);\ u_j \omega_{0j}^{\frac{1}{p}} \in L^p(\Omega, I\!R^m) \right\}$$

with, $\omega_0 = (\omega_{0j})_{1\leq j\leq m}$. *The weighted functions* ω_α *defined by:*

$$\omega_\alpha = \begin{cases} \omega_0 = (\omega_{0j})_{1\leq j\leq m} \ \ if \quad \mid \alpha \mid = 0, \\ \omega = (\omega_{ij})_{1\leq i\leq n;\ 1\leq j\leq m} \ \ if \quad \mid \alpha \mid = 1, \\ arbitrary\ weight\ functions \qquad if\ \ 2 \leq \mid \alpha \mid \leq k. \end{cases}$$

$$W^{k,p}(\Omega, \omega_\alpha, I\!R^m) \equiv \left\{ u \in L^p(\Omega, \omega_\alpha, I\!R^m);\ \ D^\alpha u \in L^p(\Omega, \omega_\alpha, I\!R^m);\ \ for\ \ 0 \leq \mid \alpha \mid \leq k \right\}.$$

$D^\alpha u$ *is the weak derivative of* u. *And* $\omega_{ij}^* = \omega_{ij}^{1-p'}$ *is the conjugate of* ω_{ij}.
We defined the inner product in $L^2(\Omega, \omega_0, I\!R^m)$ *by:* $\langle u, v\rangle_{2,\omega_0} = \sum_{j=1}^m \int_\Omega ((\omega_{0j})^{\frac{1}{2}} u_j)((\omega_{0j})^{\frac{1}{2}} v_j) dx$, *for all* $u; v$ *in* $L^2(\Omega, \omega_0, I\!R^m)$. *And for all* $u; v$ *in* $W_0^{r,2}(\Omega, \omega_\alpha, I\!R^m)$ *we denotes the inner product by:*
$\langle u, v\rangle_{W^{r,2}(\omega_\alpha)} = \langle u, v\rangle_{2,\omega_0} + \sum_{i=j=1}^{n;m} \int_\Omega (\omega_{ij}^{\frac{1}{2}} D_{ij} u)(\omega_{ij}^{\frac{1}{2}} D_{ij} v) dx + \sum_{2\leq|\alpha|\leq r} \int_\Omega (\omega_\alpha^{\frac{1}{2}} D^\alpha u)(\omega_\alpha^{\frac{1}{2}} D^\alpha v) dx.$

We denote $W_0^{k,p}(\Omega,\omega_\alpha,I\!R^m)$ the closure of $C_c^\infty(\Omega,I\!R^m)$ in $W^{k,p}(\Omega,\omega_\alpha,I\!R^m)$, where the norm is $\|.\|_{k,p,\omega_\alpha}$ is given by:

$\|u\|_{k,p,\omega_\alpha}=\Big(\sum\limits_{0\leq|\alpha|\leq k}\|\ D^\alpha u\ \|_p\ \omega_\alpha\Big)^{\frac{1}{p}}$.

We have:

$W_0^{k,p}(\Omega,\omega_\alpha,I\!R^m)\hookrightarrow W^{k,p}(\Omega,\omega_\alpha,I\!R^m)\hookrightarrow L^p(\Omega,\omega_0,I\!R^m)$.

$(W_0^{k,p}(\Omega,\omega_\alpha,I\!R^m))'\equiv W^{-k,p'}(\Omega,\omega_\alpha^*,I\!R^m)$.

$I\!M^{m\times n}$ denotes the space of real $m\times n$ matrices equipped with inner product $M:N=M_{ij}N_{ij}$ (we use the usual summation convention) and the tensor product $a\otimes b$ of two vectors $a,b\in I\!R^n$ is defined to be the matrix $(a_ib_j)_{i,j=1,...,n}$.

Definition 6.1.3 *let Ω be a measurable subset of $I\!R^n$, $(\mu_k)_{k\in I\!N}$ and μ be a sequence and respectively an element in $M(\Omega,I\!R^m)$. we say that (μ_k) converges weakly $*$ to μ, which is denoted by $\mu_k\rightharpoonup^*\mu$, if:* $\int_\Omega f d\mu_k\to\int_\Omega f d\mu$, *for all* $f\in C_c(\Omega,I\!R^m)$.

Lemma 6.1.1 *(Fatou) Let $F:\Omega\times I\!R^m\times I\!M^{m\times n}\to I\!R$ be a Carathéodory functions and $u_k:\Omega\to I\!R^m$ a sequence of measurable functions such that Du_k generates the Young measure ϑ_x with $\|\ \vartheta_x\ \|=1$ for almost every $x\in\Omega$. Then:*

$\liminf\limits_{k\to\infty}\int_\Omega F(x,u_k,Du_k)dx\geq\int_\Omega\int_{I\!M^{m\times n}}F(x,u,\xi)d\vartheta_x(\xi)dx$.

Provided that the negation $F^-(x,u_k,Du_k)$ is equi-integrable. More general versions of the this lemma may be found in [59], [60] and [42], proof see [37] (lemma 1-5).

6.2 Hypothesis

We assume that

$$\omega_{ij}\in L^1(\Omega),\quad \omega_{ij}^{\frac{-1}{p-1}}\in L^1_{loc}(\Omega)\ \ and\ \ \omega_{0j}is\ \ bounded\ a.e.\ i\text{`}n\ \Omega \tag{6.2.1}$$

$$\omega_{ij}^{-s}\in L^1(\Omega) \tag{6.2.2}$$

for some $s\in]max(\frac{1}{p-1};\frac{n}{p})\ ;\ \infty[$.

The Sobolev space $W^{1,p}(\Omega,\omega,I\!R^m)$ equipped with the norm

$$\|\ u\ \|^p_{1,p,\omega}=\sum_{j=1}^m\int_\Omega\omega_{0j}|u_j|^pdx+\sum_{i,j}\int_\Omega\omega_{ij}|D_{ij}u|^pdx.$$

is a Banach space, reflexive, and $C_0^\infty(\Omega, I\!R^m)$ is a subspace of $W^{1,p}(\Omega,\omega, I\!R^m)$. The space $W_0^{1,p}(\Omega,\omega, I\!R^m)$ is the fermenter of $C_0^\infty(\Omega, I\!R^m)$ in $W^{1,p}(\Omega,\omega, I\!R^m)$ for the norm $\|.\|_{1,p,\omega}^p$,

(by the condition 6.2.1).

The norm $\| \ .. \ \|_{1,p,w}^p$ equivalent to the norm $|||u|||_{W_0^{1,p}(\Omega,\omega,I\!R^m)} = (\sum_{i,j}\int_\Omega \omega_{ij}|D_{ij}u|^p dx)^{\frac{1}{p}}$, with $D_{ij}u = \frac{\partial u^j}{\partial x_i}$

(by: The condition 6.2.2).

By the condition 6.2.2, we give:

$$(W^{1,p}(\Omega,\omega, I\!R^m) \hookrightarrow (W^{1,p_s}(\Omega, I\!R^m) resp(W^{1,p}(\Omega,\omega, I\!R^m) \hookrightarrow\hookrightarrow L^r(\Omega, I\!R^m)), \tag{6.2.3}$$

for all $1 \leq r < p_s^*$ with $p_s = \frac{p.s}{s+1}$

$p_s^* = \frac{n.p.s}{n(s+1)-p.s}$ if $p.s < n(s+1)$, and p_s^* is arbitrary otherwise.

with The Hardy-Type inequalities in condition (P_0) is verifies for $\gamma_j = 1; j = 1; 2; .....; m$

(P_1) $\sigma : \Omega\times]0,T[\times I\!R^m \times I\!M^{m\times n} \longrightarrow I\!M^{m\times n}$ is a Carathéodory functions, i.e:
$(x,t) \longmapsto \sigma(x,t,u,F)$ is measurable for every $(u,F) \in I\!R^m \times I\!M^{m\times n}$ and
$(u,F) \in I\!R^m \times I\!M^{m\times n} \longmapsto \sigma(x,t,u,F)$ is continuous for almost every $(x,t) \in \Omega_T$.

(P_2) (Coercivity and growth): There exist constants c_1, c_2, $\beta > 0$ and $\lambda_1 \in L^{p'}(\Omega_T)$, $\lambda_2 \in L^1(\Omega_T)$, $\lambda_3 \in L^{(\frac{p}{\alpha})'}(\Omega_T)$, $0 < \alpha < p$, $1 < q < \infty$, such that, for all $1 \leq r \leq n$, and $1 \leq s \leq m$:

$$|\sigma_{rs}(x,t,u,F)| \leq \beta\omega_{rs}^{\frac{1}{p}}[\lambda_1 + c_1\sum_{j=1}^m \gamma_j^{\frac{1}{p'}}|u_j|^{\frac{q}{p'}} + c_2\sum_{i,j}\omega_{ij}^{\frac{1}{p'}}|F_{ij}|^{p-1}]$$

$$\sigma(x,t,u,F) : F \geq -\lambda_2(x,t) - \sum_{j=1}^m \lambda_3(x,t)\omega_{1j}^{\frac{\alpha}{p}}|u_j|^\alpha + c_2\sum_{i,j}\omega_{ij}|F_{ij}|^p.$$

(P_3) (Monotonicity): σ satisfies one of the following conditions:

- (a) For all $(x,t) \in \Omega_T$ and all $u \in I\!R^m$, the map:$F \mapsto \sigma(x,t,u,F)$ is a C^1- functions and is monotone.
- (b) There exist a function: $W : \Omega_T \times I\!R^m \times I\!M^{m\times n} \to I\!M^{m\times n}$, such that $\sigma(x,t,u,F) = \frac{\partial W}{\partial F}(x,t,u,F)$, and $F \mapsto W(x,t,u,F)$ is convex and C^1- function.
- (c) For all $(x,t) \in \Omega_T$ and all $u \in I\!R^m$, the map: $F \mapsto \sigma(x,t,u,F)$ is strictly monotone .
- (d) σ is strictly -p quasi monotone in F i.e:
$\int_{I\!M^{m\times n}} (\sigma(x,t,u,\lambda) - \sigma(x,t,u,\overline{\lambda}) : (\lambda - \overline{\lambda})d\vartheta(\lambda) > 0$

(P_0) (The Hardy-Type inequalities); There exist $c > 0$, one weighted function $\gamma = (\gamma_j)_{1\leq j\leq m}$, and the parameter $1 < q$, such that:

$$\left(\sum_{j=1}^{m}\int_{\Omega}\gamma_j \mid u_j \mid^q dx\right)^{\frac{1}{q}} \leq c\left(\sum_{i,j}\int_{\Omega}\omega_{ij} \mid D_{ij}u \mid^p dx\right)^{\frac{1}{p}}$$

and the expression $\left(\sum_{i,j}\int_{\Omega}\omega_{ij} \mid D_{ij}u \mid^p dx\right)^{\frac{1}{p}}$ is a norm equivalent to the norm $\|.\|_{1;p;\omega}$ in $W_0^{1,p}(\Omega,\omega,I\!R^m)$. The injection $W_0^{1,p}(\Omega,\omega,I\!R^m) \hookrightarrow\hookrightarrow L^q(\Omega,\gamma,I\!R^m)$ is compact, and $W_0^{1,p}(\Omega,\omega,I\!R^m) \hookrightarrow\hookrightarrow L^r(\Omega,I\!R^m)$ is compact, with

$$\begin{cases} 1 \leq r < \frac{nps}{n(s+1)-ps} & \text{if} \quad ps < n(s+1) \\ r \geq 1 & \text{if} \quad n(s+1) \leq ps \end{cases}$$

Remark 6.2.1

(P_1) *(Carathéodory condition) ensures that $\sigma(x,t,u(x,t),U(x,t))$ is measurable on $\partial\Omega_T$ for measurable functions:*

$u : \Omega_T \to I\!R^m$

$U : \Omega_T \to I\!M^{m\times n}$.

The growth condition and coercivity

(P_2) *is standard it is used in the construction of approximate solutions by a Galerkin method and when we pass to the limit.*

Although the strict monotonicity condition (c) in (P_3) ensure existence of a weak solution by standard method see e-g [46], the main point is that we do not require strict monotonicity or monotonicity in the variables (u,F) in $(a),(b)$ and (d) as it is usually assumed in previous works see-e-g [16], [46],[1].

Under these mild assumptions and using again Young measures, Hungerbühler shows in [38] that initial and boundary value problem (QPS) has a weak solution $u \in L^p(0,T,W_0^{1,p}(\Omega,\omega,I\!R^m)$ in following sense.

Definition 6.2.1 *We say that: $u : \Omega_T \to I\!R^m$ is weak solutions of:*

$$(\mathcal{QPS})\begin{cases} \partial_t u - div\sigma(x,t,u,Du) &= v(x,t) \quad \textit{in} \quad \Omega_T \\ u(x,t) &= 0 \quad \textit{on} \quad \partial\Omega\times(0,T) \\ u(x,0) &= u_0(x) \quad \textit{on} \quad \Omega \end{cases}$$

where $v \in L^{p'}(0,T,W^{-1,p'}(\Omega,\omega^*,I\!R^m))$ if:

- *(i) u belongs to $L^1(0,T,W_0^{1,1}(\Omega,I\!R^m))$.*
- *(ii) $\sigma(.,u(.),Du(.))$ belongs to $L^1(\Omega_T,I\!M^{m\times n})$.*
- *(iii) The initial condition $u(x,0)=u_0(x)$ on Ω is fulfilled.*
- *(iv) The equation:*
  $\sum_j \int_{\Omega_T} \partial_t u^j.\varphi^j(x,t)\omega_{0j}dxdt + \int_{\Omega_T} \sigma(x,t,u,Du):D\varphi(x,t)dxdt = \langle v,\varphi\rangle$,
  holds for every function $\varphi \in C^\infty(\overline{\Omega_T},I\!R^m)$ which is zero in a neighborhood of $(\partial\Omega\times]0,T[)\cup(\Omega\times\{0;T\})$.

Remark 6.2.2
- *(a) In definition the boundary condition $u=0$ on $(\partial\Omega\times]0,T[)$ is interpreted in the sense of (i) .*
- *(b) The condition (iv) is understood in the sense that the Galerkin approximations converge to u_0 in $L^2(\Omega,\omega_0)$.*

Theorem 6.2.1 *If $p\in]\frac{s+1}{s}\frac{2n}{n+2}\ ,\infty[$ and if σ satisfies the conditions $(P_1)-(P_3)$ then the parabolic system (QPS) has a weak solution $u\in L^p(0,T,W_0^{1,p}(\Omega,\omega,I\!R^m))$ for every $v\in L^{p'}(0,T,W_0^{-1,p'}(\Omega,\omega^*,I\!R^m))$ and $(n\geq 2)$.*

6.3 A first parabolic div-curl inequality

We state here the first set of hypothesis:

- (A_1): The sequence (u_k) is uniformly bounded in $L^p(0,T,W_0^{1,p}(\Omega,\omega,I\!R^m))$ for some $p_s\in]\frac{2n}{n+2}\ ,\ \infty[$ and hence (Bay the Elberlem Suljan theorem: see [15].a subsequence converges weakly in $L^p(0,T,W_0^{1,p}(\Omega,\omega,I\!R^m))$ to an element denoted by u.
- (A_2): The sequence $\partial_t u_k$ is uniformly bounded in $L^{p'}(0,T,(W_0^{r,2}(\Omega,\omega,I\!R^m))')$ for some $2<r<p_s^*$ if $p_s<n$ and $2<r<\infty$ if $p_s\geq n$.
- (A_3): $\sigma:\Omega\times]0,T[\times I\!R^m\times I\!M^{m\times n}\longrightarrow I\!M^{m\times n}$ is a Carathéodory function.

- (A_4): The sequence $\sigma_k(x,t) \equiv \sigma(x,t,u_k(x,t), Du_k(x,t))$ is uniformly bounded in $L^{p'}(\Omega_T, I\!M^{m\times n})$ and therefore equi-integrable (the equi-integrability follows from, the Hölder inequality).

- (A_5): The sequence $(\sigma_k(x,t) : Du_k)^-$ is equi-integrable.

- (A_6): There exists a sequence (v_k) such that $v_k \to u$ in $L^p(0,T,(W_0^{1,p}(\Omega,\omega,I\!R^m))$ and $\int_{\Omega_T} \sigma_k(x,t) : (Du_k - Dv_k)dxdt \to 0; k \to \infty$.

Remark 6.3.1 *Note that the assumption (A_1) and(A_2) ensure a strangle convergence in some Lebesgue spaces and in measure. .*

Lemma 6.3.1 *Let B, B_0 and B_1 be Banach spaces, B_0 and B_1 reflexive. Let $\imath : B_0 \to B$ be a compact linear map and:*
$\jmath : B \to B_1$ an injective bounder linear operator. For T finite and: $1 < p_i < \infty$ with $i = 0; 1$
$W = \{v \in L^{p_0}(0;T;B_0) : \frac{d}{dt}(\jmath \, o \, \imath \, o \, v) \in L^{p_1}(0;T;B_1)\}$ is a Banach space under the norm:
$\| \, v \, \|_{L^{p_0}(0;T;B_0)} + \| \, \jmath \, o \, \imath \, o \, v \, \|_{L^{p_1}(0;T;B_1)}$.
Then, if $V \subset W$is bounded, the set $\{\imath \, o \, v; v \in V\}$ is precompact in $L^{p_0}(0;T;B_0)$.

Proof of lemma6.3.1

For the proof, see [37] (Appendix I). For generalization of lemma, see [56]. Now, we apply lemma to obtain the desired bound. □

Proposition 6.3.1 *Let $p_s \in]\frac{2n}{n+2} \ , \ \infty[$ and (u_k) be a sequence satisfying (A_1) and (A_2). Then there exists a subsequence of (u_k) (for convention not relabeled) and a functions:*
$u \in L^p(0,T,W_0^{1,p}(\Omega,\omega,I\!R^m))$ such that: $u_k \to u$ in measure on Ω_T and in

$$L^p(0,T,L^r(\Omega,\omega,I\!R^m)), \forall r < p_s^*. \tag{6.3.1}$$

Proof of prop 6.3.1

Let $B_0 \equiv W_0^{1,p}(\Omega,\omega,I\!R^m)); B \equiv L^r(\Omega,\omega_0,I\!R^m))$, for some r with:
$2 < r < p^*$ if $p_s < n$ and $2 < r < \infty$ if $p_s \geq n$.
and $B_1 \equiv (W_0^{r,2}(\Omega,\omega_\alpha,I\!R^m))'$. Since we assumed that $p_s \in]\frac{2n}{n+2} \ , \ \infty[$. we have the following chain of continuous injections:

$$B_0 \hookrightarrow^i B \hookrightarrow^{i_0} L^2(\Omega,\omega_0,I\!R^m)) \cong (L^2(\Omega,\omega_0,I\!R^m))' \hookrightarrow^{i_1} B_1; \tag{6.3.2}$$

here $(L^2(\Omega,\omega_0,I\!R^m))' \equiv (L^2(\Omega,\omega_0,I\!R^m))$ is the canonical isomorphism θ of the Hilbert space $L^2(\Omega,\omega_0,I\!R^m)$ and its, for

$\imath : B_0 \to B$ we take simply the injection mapping, and for $\jmath : B \to B_1$ we take the concatenation of injections and the canonical isomorphism given bay (6.3.2) i.e: $\jmath = \imath_1 \; o \; \theta \; o \; \imath_0$. Then (u_k) is a bounded sequence in $L^p(0,T,B_0)$ by (A_1). Next observe that $(\partial_t u_k)$ or rather $(\partial_t \; j \; o \; i \; o \; u_k)$ is a bounded sequence in $(L^{p'}(0,T,(W_0^{r,2}(\Omega,\omega_\alpha,I\!R^m))')$ by (A_2). Hence, from 6.3.1, we may conclude that there exists a subsequence, which we still denote by u_k, having the property that the second part of (6.3.1) hold. Notice that in order to have the strong convergence simultaneously for all $r < p_s^*$, the usual diagonal sequence procedure applies finally, as $p > 1$, the sequence converges in measure see, e-g.[30] (proposition 2-29). Which end the proof. □

Lemma 6.3.2 *(div-curl inequality) Suppose $(A_1) - (A_6)$ and assume (after passing to a suitable subsequence if necessary) that (Du_k) generates the young measure $\vartheta_{(x,t)}$. Then the following inequality hold:*

$$\int_{\Omega_T}\int_{I\!M^{m\times n}} \sigma(x,t,u,\lambda) : \lambda d\vartheta_{(x,t)}(\lambda)dxdt \leq \int_{\Omega_T}\int_{I\!M^{m\times n}} \sigma(x,t,u,\lambda) : Du d\vartheta_{(x,t)}(\lambda)dxdt \qquad (6.3.3)$$

Proof of lemma 6.3.2

Let us consider the sequence:

$I_k = \Big(\sigma(x,t,u_k,Du_k) : (Du_k - Du)\Big) = \Big(\sigma_k : Du_k\Big) - \Big(\sigma_k : Du\Big)$.

By conditions (A_4) and (A_5), the negative part I_k^- of I_k is equi-integrable. Hence, in view of (A_3), we may use the (Fatou - Lemma) which gives that

$X = \liminf_{k\to\infty}(\int_{\Omega_T} I_k dxdt \geq \int_{\Omega_T}\int_{I\!M^{m\times n}} \sigma(x,t,u,\lambda) : (\lambda - Du)d\vartheta_{(x,t)}(\lambda)dxdt$

$X = \liminf_{k\to\infty}(\int_{\Omega_T} \sigma_k : (Du_k - Dv_k)dxdt + \int_{\Omega_T} \sigma_k : (Dv_k - Du)dxdt)$

$= \liminf_{k\to\infty}(\int_{\Omega_T} \sigma_k : (Dv_k - Du)dxdt \leq c.\liminf_{k\to\infty} \parallel \sigma_k \parallel_{L^{p'}(\Omega_T,\omega^*)} \parallel v_k - u \parallel_{L^p(0,T,W_0^{1,p}(\Omega,\omega,I\!R^m))} = 0$ by the Hölder inequality and (A_4). Thus the conclusion follows.

6.4 A second parabolic div-curl inequality

We state here the second set of conditions:

- $(c_1) : u_k \in L^1(0,T,W_0^{1,p}(\Omega,\omega,I\!R^m))$ for all; $k \in I\!N$, and there exists some $c > 0$ and $\rho > 0$ such

that:
$\sup_k \sum_{ij} \int_{\Omega_T} |D_{ij}u_k|^\rho \omega_{ij} dxdt \leq c.$

- (c_2): The sequence u_k converges in measure to some function u and $u(.,t)$ is almost everywhere approximately differentiable a.e. $t \in]0,T[$.

- $(c_3) : \sigma : \Omega_T \times I\!R^m \times I\!M^{m\times n} \longrightarrow I\!M^{m\times n}$ and
$\tau[e] : \Omega_T \times I\!R^m \times I\!M^{m\times n} \longrightarrow I\!M^{m\times n}; e \in I\!N$ are Carathéodory conditions.

- (c_4) : σ and $\tau[e]$, satisfy all the angle structure conditions or satisfies all the sign structure conditions:
(Angle condition) For all $(x,t) \in \Omega_T, u \in I\!R^m$ and $M; F \in I\!M^{m\times n}$, with M of the form $Id - a \otimes a$ where $a \in I\!R^m$ satisfies $\mid a \mid \leq 1$; $\sigma(x,t,u,F) : MF \geq 0$, for all matrix.
$M \in I\!M^{m\times m}$ of the form $M = Id - a \otimes a$ with $|a| \leq 1$.
(The sign condition) For all $(x,t) \in \Omega_T, u \in I\!R^m$ and $F \in I\!M^{m\times n}$, we have $\sigma_j(x,t,u,F) : F_j \geq 0$, for all $1 \leq j \leq m$ with
F_j and σ_j are the cologne j of matrix F and σ.(this conditions is verifies for $\tau[e]$)

- (c_5) : The sequence $\sigma_k(x,t) = \sigma(x,t,u_k,Du_k)$ is equi-integrable.

- (c_6) : The sequence $\tau_k(x,t) \equiv \tau[k](x,t,u_k,Du_k)$; and converges weakly to 0 in the space $L^1(\Omega_T, I\!M^{m\times n})$.

- (c_7): The sequence $g_k = -div\sigma_k$ is bounded in $L^1(\Omega_T, I\!R^m)$.

- (c_8): $Du_k \in L^i_{loc}(\Omega_T, \omega, I\!M^{m\times n})$ and $(\sigma_k + \tau_k) \in L^{i'}_{loc}(\Omega_T, I\!M^{m\times n})$.
for some $i; 1 < i < \infty$.

Lemma 6.4.1 *(Second div-curl inequality) Suppose $(C_1) - (C_2)$, and assure (after passing to a suitable subsequence if necessary) that Du_k generates a Young measure $\vartheta_{(x,t)}$. Then (after passing to a subsequence), the σ_k converges weakly in $L^1(\Omega_T, I\!M^{m\times n})$ and the weak limit $\overline{\sigma}$ is given by:*
$\overline{\sigma}(x,t) = \langle \vartheta_{(x,t)}; \sigma(x,t,u(x,t),.)\rangle$, *moreover the following inequality hold*
$\int_{I\!M^{m\times n}} \sigma(x,t,u(x,t),\lambda) : \lambda d\vartheta_{(x,t)}(\lambda) \leq \overline{\sigma}(x,t) : apDu(x,t)$. *for a.e.* $(x,t) \in \Omega_T$.

Proof of lemma6.4.1

Proof of lemma is a slightly modification of the corresponding in lemma $(6-2-1)$ in [28] pages (54-61)
□

Remark 6.4.1 *If the sequence Du_k is equi-integrable then $Du(x,t) = apDu(x,t) = \langle \vartheta_{(x,t)}; id \rangle$ almost everywhere. (By the Ball's theorem).*

The parabolic div-curl inequality will be the key ingredient to pass the limit in the different cases, but we need some additional information on the Young measure $\vartheta_{(x,t)}$ generated by the sequence of the gradients Du_K to exploit lemma (div-curl inequality) these properties are the following:

- (N_1): $\vartheta_{x,t}$ is a probability measure for almost every $(x,t) \in \Omega_T$.
- (N_2): $\vartheta_{x,t}$ is a homogeneous $W^{1,P}-$ gradient young measure for almost every $(x,t) \in \Omega_T$.
- (N_3): $\vartheta_{x,t}$ satisfies $< \vartheta_{(x,t)}, id >= apDu(x,t)$ almost everywhere in Ω_T.

Note that these properties hold if the condition (A_1) is fulfilled.

Lemma 6.4.2 *Assume that the sequence u_k satisfies (A_1);*
then the Young measure $\vartheta_{(x,t)}$ generated by the sequence of the gradient Du_k satisfies $(N_1) - (N_3)$.

Proof of lemma6.4.2

Noting that (A_1) implies that $apDu(x,t) = Du(x,t)$, follows exactly as in [37] (prop 4-3).
We mention that the properties, also hold if the conditions $(C_1) - (C_2)$ and a particular estimate are fulfilled as shown in [39] (lemma 9), with a slight modifications. □

Lemma 6.4.3 *Assume that the sequence (u_k) satisfies $(C_1) - (C_2)$ and*
$\sup_{k \in I\!N} \sum_{i,j} \int_{|u_k|<R} \mid D_{ij} u_k \mid^{\rho} \omega_{ij} dxdt \leq c$, *for all $R > 0$; for some $\rho > 1$.*
then the Young measure $\vartheta_{(x,t)}$ generated by the sequence of the gradients Du_k satisfies $(N_1) - (N_3)$.

Proof of lemma6.4.3

The proof of lemma is a slightly modification of the corresponding in lemma (11-3-2) in [28].

6.5 Passage to the limit for σ

Proposition 6.5.1 *Suppose that $(A_1) - (A_6)$ or $(C_1) - (C_8)$ hold further assume that the Young measure $\vartheta_{(x,t)}$ generated by the gradients Du_k satisfies $(N_1) - (N_3)$ and that one of the following conditions*

holds $(a)-(b)-(c)-(d)$ *then(after passage to a subsequence) the sequence* σ_k *converges weakly in* $L^1(\Omega_T;\mathbb{M}^{m\times n})$ *and the weak limit* $\overline{\sigma}$ *is given by:*

$\overline{\sigma}(x,t)=\sigma(x,t,u(x,t),Du(x,t))$. *If* $(b)-(c)$ *or* (d) *holds, then:*

$\sigma(x,t,u_k,Du_k)\to\sigma(x,t,u,Du)$ *in* $L^1(\Omega_T;\mathbb{M}^{m\times n})$

In case (c) *and* (d)*; it follows in addition that* $Du_k\to Du$ *in measure and thus also almost everywhere in* Ω_T.

Before to prove proposition, we note that by proceeding exactly as in the elliptic case, we get the analogous result to lemma $(6-4-2)$ in [28] and [1].

Lemma 6.5.1 *Suppose that* $(A_1)-(A_6)$ *or* $(C_1)-(C_8)$ *hold. Further assume that* $\vartheta_{(x,t)}$ *is the Young measure generated by the gradients* Du_k *and satisfies* $(N_1)-(N_3)$. *If the map* $F\mapsto\sigma(x,t,u,F)$ *is monotone for all* $(x,t,u)\in\Omega\times]0;T[\times\mathbb{R}^m$ *then:*

$$Spt\vartheta_{(x,t)}\subset\{\lambda/[\sigma(x,t,u,\lambda)-\sigma(x,t,u,Du)]:(\lambda-Du)=0\}. \tag{6.5.1}$$

Proof of lemma6.5.1

The proof of lemma is a slightly modification of the corresponding in lemma (11-4-2) in [28].

Proof of proposition 6.5.1

We start with the easiest case:

Case (c)*: since* σ *is monotone by assumption, (6.5.1) hold by lemma, by strict monotonicity, it follows from (6.5.1) that*

$\vartheta_{(x,t)}=\delta_{Du(x,t)}$ *for almost all* $(x,t)\in\Omega_T$; *and hence* $Du_k\to Du$ *in measure for* $k\to\infty$ *by proposition* $(4-2-1)$ *in [28], since we have already that* $u_k\to u$ *in measure for* $k\to\infty$ *by* (A_1); (A_2) *and proposition 6.3.1 or by* (C_2) *we may infer that (after extraction of a suitable subsequence, if necessary see: [30] (theorem 2-30)* $u_k\to u$ *and* $Du_k\to Du$ *almost everywhere in* Ω *for* $k\to\infty$.

thus, from the continuity condition (A_3) *or* (C_3) *it follows that:* $\sigma(x,t,u_k,Du_k)\to\sigma(x,t,u,Du)$ *almost everywhere in* Ω *for* $k\to\infty$. *Since, by assumption* (A_4) *or* $(A_5),\sigma_k(x,t)$ *is equi-integrable, it follows from the Vitali's converge theorem* $(1-2-1)$ *in [28]*

that; $\sigma(x,t,u_k,Du_k)\to\sigma(x,t,u,Du)$ *in* $L^1(\Omega_T;\mathbb{M}^{m\times n})$ *which proves the proposition in this case.* □

Case (d):Assume that $\vartheta_{(x,t)}$ *is not a Dirac mass on a set* $(x,t)\in M\subset\Omega_T$ *of positive Lebesgue measure* $\mid M\mid>0$. *Then, by the strict p-quasi-monotonicity of* $\sigma(x,t,u,.)$ *and* (N_2), *we have for a.e.* $(x,t)\in M$

with: $\overline{\lambda} = \langle \vartheta_{(x,t)}; id \rangle = apDu(x,t)$, *by* (N_3) :

$$\begin{aligned} \int_{\mathbb{M}^{m\times n}} \sigma(x,t,u,\lambda) : \lambda d\vartheta_{(x,t)}(\lambda) \quad &> \int_{\mathbb{M}^{m\times n}} \sigma(x,t,u,\overline{\lambda}) : \lambda d\vartheta_{(x,t)}(\lambda) \\ &- \int_{\mathbb{M}^{m\times n}} \sigma(x,t,u,\overline{\lambda}) : \overline{\lambda} d\vartheta_{(x,t)}(\lambda) \\ &+ \int_{\mathbb{M}^{m\times n}} \sigma(x,t,u,\lambda) : \overline{\lambda} d\vartheta_{(x,t)}(\lambda) \\ = \int_{\mathbb{M}^{m\times n}} \sigma(x,t,u,\lambda) : \overline{\lambda} d\vartheta_{(x,t)}(\lambda) & \end{aligned} \tag{6.5.2}$$

Where we used (N_1), *we claim now that we obtain a contradiction. Indeed, if* $(C_1) - (C_8)$ *hold, it follows by second div-curl inequality that:*

$$\begin{aligned} \int_{\mathbb{M}^{m\times n}} \sigma(x,t,u,\lambda) : \lambda d\vartheta_{(x,t)}(\lambda) \quad &> \int_{\mathbb{M}^{m\times n}} \sigma(x,t,u,\lambda) : \overline{\lambda} d\vartheta_{(x,t)}(\lambda) \\ &\geq \int_{\mathbb{M}^{m\times n}} \sigma(x,t,u,\lambda) : \lambda d\vartheta_{(x,t)}(\lambda). \end{aligned}$$

On the other hand if $(A_1) - (A_6)$ *hold, by integrating (6.5.2) over* Ω *and using the div-curl inequality in lemma (6.3.2), we get:*

$$\begin{aligned} \int_{\Omega_T}\int_{\mathbb{M}^{m\times n}} \sigma(x,t,u,\lambda) : \lambda d\vartheta_{(x,t)}(\lambda)dxdt \quad &> \int_{\Omega_T}\int_{\mathbb{M}^{m\times n}} \sigma(x,t,u,\lambda) : \overline{\lambda} d\vartheta_{(x,t)}(\lambda)dxdt \\ &\geq \int_{\Omega_T}\int_{\mathbb{M}^{m\times n}} \sigma(x,t,u,\lambda) : \lambda d\vartheta_{(x,t)}(\lambda)dxdt. \end{aligned}$$

As desired, hence, we have $\vartheta_{(x,t)} = \delta_\lambda = \delta_{Du(x,t)}$ *for almost every* $(x,t) \in \Omega_T$; *thus, it follows again by proposition* $(4-2-1)$ *in [28] that* $Du_k \to Du$ *in measures, The reminder of the proof in this case is exactly as in case (c)* $\square$

Case (b): We start by noting that for almost all $(x,t) \in \Omega_T$, *the support of* $\vartheta_{(x,t)}$ *is contained in the set where* W *agrees with the supporting hyper-plane.*

$L = \{(\lambda, W(x,t,u,\overline{\lambda}) + \sigma(x,t,u,\lambda)) : (\lambda - \overline{\lambda})\}$

in $\overline{\lambda} = apDu(x,t)$ *i.e, it may be shown see [38]. as in the proof of proposition (6.5.1) that* $spt\vartheta_{(x,t)} \subset K_{(x,t)}$ *where.*

$K_{(x,t)} = \{(\lambda \in \mathbb{M}^{m\times n} / W(x,t,u,\lambda) = W(x,t,u,\overline{\lambda}) + \sigma(x,t,u,\overline{\lambda}) : (\lambda - \overline{\lambda})\}.$

By the convexity of W *we have:*

$W(x,t,u,\lambda) \geq W(x,t,u,\overline{\lambda}) + \sigma(x,t,u,\overline{\lambda}) : (\lambda - \overline{\lambda})\}.$

For all $\lambda \in \mathbb{M}^{m\times n}$*and thus* L *is a supporting hyper-plane for all* $\lambda \in K_{(x,t)}$. *Since the mapping* $\lambda \to W(x,t,u,\lambda)$ *is by assumptions continuously differentiable we obtain:*

$$\sigma(x,t,u,\lambda) = \sigma(x,t,u,\overline{\lambda}), forall \lambda \in K_{(x,t)} \supset spt\vartheta_{(x,t)} \tag{6.5.3}$$

and thus:

$$\overline{\sigma}(x,t) \equiv \int_{\mathbb{M}^{m\times n}} \sigma(x,t,u,\lambda) d\vartheta_{(x,t)}(\lambda) = \sigma(x,t,u,\overline{\lambda}) \tag{6.5.4}$$

Now consider the Carathéodory function:

$\psi(x,t,u,\rho) =\mid \sigma(x,t,u,\rho) - \overline{\sigma}(x,t) \mid$, *the sequence:*

$\psi_k(x,t) = \psi(x,t,u_k(x,t),\overline{\lambda}_k(x,t))$ *is equi-integrable and thus* $\psi_k \to \overline{\psi}$ *weakly in* $L^1(\Omega_T)$ *and the weak limit* $\overline{\psi}$ *is given by:*

$\overline{\psi}(x,t) = \int_{I\!R^{m\times n}\times I\!M^{m\times n}} \mid \sigma(x,t,\eta,\lambda) - \overline{\sigma}(x,t) \mid d\delta_{u(x,t)}(\eta) \otimes d\vartheta_{(x,t)}(\lambda)$

$= \int_{spt} \vartheta_{(x,t)} \mid \sigma(x,t,u(x,t),\lambda) - \overline{\sigma}(x,t) \mid d\vartheta_{(x,t)}(\lambda) = 0$ *by (6.5.3) and (6.5.4). Since* $\psi_k \geq 0$ *it follows that* $\psi_k \to 0$ *strongly in* $L^1(\Omega_T)$.

This again to pass to the limit and the proof of the case (b)is finished $\square$

Case (a): We claim that in this case for almost all $(x,t) \in \Omega\times]0;T[$ *the following identity hold for all* $M \in I\!M^{m\times n}$*on the support of* $\vartheta_{(x,t)}$:

$$\sigma(x,t,u,\lambda) : M = \sigma(x,t,u,\overline{\lambda}) : M + (\nabla_F \sigma(x,t,u,\overline{\lambda})M) : (\lambda - \overline{\lambda}), \tag{6.5.5}$$

where ∇_F *is the derivative with respect to the third variable of* σ *and* $\overline{\lambda} = apDu(x,t)$.

Indeed, by the monotonicity of σ *we have for all* $\tau \in I\!R$

$(\sigma(x,t,u,\lambda) - \sigma(x,t,u,\overline{\lambda} + \tau M)) : (\lambda - \overline{\lambda} - \tau M) \geq 0,$

whence, by (6.5.1),

$-\sigma(x,t,u,\lambda) : (\tau M) \geq \tau((\nabla_F \sigma(x,t,u,\overline{\lambda})M)(\lambda - \overline{\lambda}) - \sigma(x,t,u,\overline{\lambda}) : M) + \circ(\tau).$

The claim follows from this inequality since the sign of τ *is arbitrary. Since the sequence* $\sigma_k(x,t)$ *is equi-integrable by* (A_4)*, its weak* L^1*-limit* $\overline{\sigma}$ *is given by*

$\overline{\sigma}(x,t) = \int_{spt\vartheta_{(x,t)}} \sigma(x,t,u,\lambda) d\vartheta_{(x,t)}(\lambda)$

$= \int_{spt\vartheta_{(x,t)}} \sigma(x,t,u,\overline{\lambda}) d\vartheta_{(x,t)}(\lambda) + (\nabla_F \sigma(x,t,u,\overline{\lambda}))^t \int_{spt\vartheta_{(x,t)}} (\overline{\lambda} - \lambda) d\vartheta_{(x,t)}(\lambda)$

$= \sigma(x,t,u,\overline{\lambda}),$

where we used (6.5.5) in this calculation. This finishes the proof of the case (c) and hence of the proposition. $\square$

Remark 6.5.1 *In case, (b), we remark, that the relation (6.5.4) already states that* $\sigma(x,t,u,\overline{\lambda})$ *is the weak* L^1*- limit of* $\sigma_k(x,t)$*, which is enough to pass to the limit in an equation which must hold in the distributional sense. However, we wanted to point out that in this case, the convergence is even strong in* $L^1(\Omega_T, I\!M^{m\times n})$.

6.6 Galerkin Scheme

Lemma 6.6.1 *Let Ω bounded domain in $\mathbb{R}^n$, $p \in]\frac{2n}{(n+2)}\ ,\ 2]$ and $\omega_{ij} \in L^1(\Omega)$, for all $j = 1; 2; ....; m$, and $i = 0; 1; .....; n$.Then $(W_0^{r,2}(\Omega, \omega_\alpha, \mathbb{R}^m) \hookrightarrow W_0^{1,p}(\Omega, \omega, \mathbb{R}^m)$ For $r \geq 1 + n(\frac{1}{2} - \frac{1}{P})$.*

Proof of lemma6.6.1

The proof of this lemma is a slightly modification of the corresponding in a space of Sobolev without weight $(W_0^{r,2}(\Omega, \mathbb{R}^m) \hookrightarrow W_0^{1,p}(\Omega, \mathbb{R}^m)$.

By the theorem $(1 - 3 - 1)$ in [28] and proceeding as [38] (section 4-2) we may choose a set $(\delta_i)_{i\in\mathbb{N}}$ of functions which is an orthonormal basis in $L^2(\Omega, \omega_0, \mathbb{R}^m)$ and at the some time an orthogonal basis in $W_0^{r,2}(\Omega, \omega_\alpha, \mathbb{R}^m)$. In particular, $(\delta_i)_{i\in\mathbb{N}}$ is uniformly bounded in $W_0^{1,p}(\Omega, \omega, \mathbb{R}^m)$ and then: $\parallel \delta_i \parallel_{1,\omega,p}$ $+ \parallel \delta_i \parallel_{p^*} \leq c$ for all $i \in \mathbb{N}$.

we will need bellow the $L^2(\omega_0)$ - orthogonal projector

$p_k : L^2(\Omega, \omega_0, \mathbb{R}^m) \to L^2(\Omega, \omega_0, \mathbb{R}^m)$ on to spam $(\delta_1, \delta_2, ..., \delta_k)_{k\in\mathbb{N}}$:

of course, the operator norm $\parallel p_k \parallel_{\Im(W_0^{r,2}(\omega_\alpha), W_0^{r,2}(\omega_\alpha))} = 1$.

(The Galerkin scheme): New we build our Galerkin approximations, we make the following ansate for approximating solution of (QPS), $u_k(x,t) = \sum_{i=1}^k a_{k,i}(t)\delta_i(x)$. Where

$a_{k,i} : [0; T) \to \mathbb{R}$ are supposed to be continuous bounded functions. such that u_k satisfies the boundary condition in (QPS) by constructions in the sense that $u_k \in L^p(0, T, W_0^{1,p}(\Omega, \omega, \mathbb{R}^m))$, we take core of the initial conditions in (QPS) by choosing the initial coefficients:

$a_{k,i}(0) = \langle u_0, \delta_i\rangle_{2,\omega_0}$ such that:

$u_k(., 0) = \sum_{i=1}^k a_{k,i}(0)\delta_i(.) \to u_0$, with $u_0 \in L^2(\Omega, \omega_0, \mathbb{R}^m)$.

we try to determine the coefficients $a_{k,i}(t)$ in such that a way, that for every $k \in \mathbb{N}$ the system of ordinary differential equations

$$\langle \partial_t u_k, \delta_j\rangle_{2,\omega_0} + \int_\Omega \sigma(x, t, u_k, Du_k) : D\delta_j dx = \langle v(t), \delta_j\rangle$$

for $j \in \{1; 2; ....; k\}$.

Now we fix to $k \in \mathbb{N}$for the moment. Let $0 < \varepsilon < T$ and $J = [0; \varepsilon]$. Moreover we chose $r > 0$ large enough; such that the set $B_r(0) \subset \mathbb{R}^k$ contains the vector $(a_{1k}(0), ..., a_{kk}(0))$, and we set $E = \overline{B_r(0)}$. Then we have the following result.

Lemma 6.6.2 *The application: $F : J \times E \mapsto \mathbb{R}^k$ given by:*

$$(t, a_1, ...a_k) \mapsto (\langle v(t), \delta_j\rangle - \int_\Omega (\sigma(x, t, \Sigma_{i=1}^k a_i\delta_i, \Sigma_{i=1}^k a_i D\delta_i) : D\delta_j)dx)_{j=1;2;....;k}$$

is a Carathéodory function. Moreover, each components F_j may be estimated on $J \times E$ by

$$\mid F_j(t, a_1, ...a_k) \mid \leq C(r,k).M(t) \tag{6.6.1}$$

uniformly on $J \times E$, where $C(r,k)$ is a constant which depends on r and k. and here M is a functions dependent of $j; k$ and r belonging to $L^1(J)$.

Proof of lemma 6.6.2

The first claim follows from the condition (P_1).

For the second claim; we note that the generalized Hölder inequality implies that:

$\mid I_j \mid = \mid \langle v(t), \delta_j \rangle \mid \leq \parallel v(t) \parallel_{-1,p';\omega^*} \parallel \delta_j \parallel_{1,p;\omega}$

Moreover, on the one hand, if; $II_j = \int_\Omega (\sigma(x,t,\Sigma_{i=1}^k a_i\delta_i, \Sigma_{i=1}^k a_i D\delta_i) : D\delta_j)dx$ by the growth condition in (P_2) and the Hölder inequality

$\mid II_j \mid \leq \beta[\parallel \lambda_1(t) \parallel_{p'} + c_1 \parallel \sum_{i=1}^k a_i\delta_j \parallel_{1,p,\gamma}^{\frac{q}{p'}} + c_2 \parallel \sum_{i=1}^k a_i D\delta_j u \parallel_{1,p,\omega}^{p-1}] \parallel D\delta_j \parallel_{1,p,\omega}$

$\leq \beta \max(1, c_1, c_2).[\parallel \lambda_1 \parallel_{p'} + \parallel \sum_{i=1}^k a_i\delta_j \parallel_{1,p,\gamma}^{\frac{q}{p'}} + \parallel \sum_{i=1}^k a_i D\delta_j u \parallel_{1,p,\omega}^{p-1}] \parallel D\delta_j \parallel_{1,p,\omega}$

Then, in view of $\parallel \delta_i \parallel_{1,\omega,p} + \parallel \delta_i \parallel_{p^*} \leq c$ for all $i \in \mathbb{N}$ and since each variable a_i satisfies: $\mid a_i \mid \leq r$; for all $i = 1; .....; k$ fora constant r, each component F_j may be estimated in such a way that

$$\mid F_j(t, a_1, ...a_k) \mid \leq \mid II_j \mid + \mid I_j \mid \leq C(r,k).M(t) \tag{6.6.2}$$

uniformly on $J \times E$, where $C(r,k)$ is a constant which depends on r and k, and where $M(t) \in L^1(J)$ (independent of j, k) and r Then, in view of lemma 6.6.2 the Carathéodory existence result on ordinary differential equation applied to the system:

$$a_j^{'}(t) = F_j(t, a_1(t), ..., a_k(t)) \tag{6.6.3}$$

$$a_j(0) = a_{k,j}(0); j \in 1; ...; k \tag{6.6.4}$$

ensures existence of a distributional, continuous solutions a_j (depending on k) of (6.6.3) - (6.6.4) on a time interval $[0; \varepsilon'[$ where $\varepsilon' > 0$, a priori, may depend on k. Moreover, the corresponding integral equation

$$a_j(t) = a_j(0) + \int_0^t F_j(\tau, a_1(\tau), ...a_k(\tau))d\tau \tag{6.6.5}$$

holds on $[0; \varepsilon'[$. Then, $u_k = \sum_{j=1}^k a_j(t)\delta_j$ is the desired (short time) solution of

$\langle \partial_t u_k, \delta_k \rangle_{2,\omega_0} + \int_\Omega \sigma(x,t,u_k,Du_k) : D\delta_k dx = \int_\Omega v(x,t)\delta_k dx$, *with initial condition*

$u_k(0) = \sum_{i=1}^k a_{k,i}(0)\delta_i(.) \to u_0$, *with $u_0 \in L^2(\Omega, \omega_0, \mathbb{R}^m)$. Now, we want to show that the local solution*

constructed above can be extended to the whole interval $[0;T[$ *independent of* k, *As a word of warning we should mention, that the solution need not be unique.*

Lemma 6.6.3 *We have:*

$A = \left\{ t \in [0;T[:\ there\ exist\ a\ weak\ solutions\ of\ (6.6.3)-(6.6.4)\ on\ [0;T[\right\} = [0;T[.$

Proof of lemma 6.6.3

The first thing we want to establish is a uniform bound on the coefficients $(a_{ki}(t))$. *Since*

$\langle \partial_t u_k, \delta_k \rangle_{2,\omega_0} + \int_\Omega \sigma(x,t,u_k,Du_k) : D\delta_k dx = \int_\Omega v(x,t)\delta_k dx$

is linear in δ_k, *it is allowed to use* u_k *as a test function in precedent equation in place of* δ_k. *This gives for a an arbitrary time* τ *in the existence interval*

$\int_0^\tau \langle \partial_t u_k, u_k \rangle_{2,\omega_0} dt + \int_0^\tau \int_\Omega \sigma(x,t,u_k,Du_k) : Du_k dx dt = \int_0^\tau \langle v(t), u_k \rangle dt.$

For the first term we have:

$\int_0^\tau \langle \partial_t u_k, u_k \rangle_{2,\omega_0} dt = \sum_j \frac{1}{2} \int_0^\tau \partial_t (u_k^j)^2 \omega_{0j} dt$

$= \sum_j \frac{1}{2} \parallel u_k^j(\tau) \parallel^2_{2,\omega_0} - \sum_j \frac{1}{2} \parallel u_k^j(0) \parallel^2_{2,\omega_0} \geq \frac{1}{2} \parallel u_k(\tau) \parallel^2_{2,\omega_0} - c$

Since $u_k(.,0)$ *converges in* $L^2(\Omega,\omega_0,I\!R^m)$. *Using the coercivity in* (P_2) *for the second term and the Hölder inequality, we obtain:*

$\int_0^\tau \int_\Omega \sigma(x,t,u_k,Du_k) : Du_k dx dt \geq - \parallel \lambda_2 \parallel_{L^1(\Omega_T)} - \parallel \lambda_3 \parallel_{L^{(\frac{p}{\alpha})'}(\Omega_T)} \parallel u_k \parallel^\alpha_{L^p(0,T,W_0^{1,p}(\Omega,\omega,I\!R^m))} + c_2 \parallel u_k \parallel^p_{L^p(0,T,W_0^{1,p}(\Omega,\omega,I\!R^m))}$

By the the Hölder inequality for the third term we get

$\int_0^\tau \langle v(t), u_k \rangle dt \leq \parallel v \parallel_{L^{p'}(0,T,W^{-1,p'}(\Omega,\omega))} \parallel u_k \parallel_{L^p(0,T,W_0^{1,p}(\Omega,\omega))}$. *By the Hölder inequality we obtain:*

$\frac{1}{2} \parallel u_k(\tau) \parallel^2_2 \leq c + \parallel \lambda_2 \parallel_{L^1(\Omega_T)} + \parallel \lambda_3 \parallel_{L^{(\frac{p}{\alpha})'}(\Omega_T)} \parallel u_k \parallel^\alpha_{L^p(0,T,W_0^{1,p}(\Omega,\omega,I\!R^m))} - c_2 \parallel u_k \parallel^p_{L^p(0,T,W_0^{1,p}(\Omega,\omega,I\!R^m))}$

$\equiv \psi(\parallel u_k \parallel_{L^p(0,T,W_0^{1,p}(\Omega,\omega,I\!R^m))})$

Since $p > 1$, *we have:*

$$\psi(y) \to -\infty;\ y \to +\infty \tag{6.6.6}$$

and so $\psi(\parallel u_k \parallel_{L^p(0,T,W_0^{1,p}(\Omega,\omega,I\!R^m))})$ *is bounded, since* ψ *is continuous and* $\parallel u_k \parallel^2_2 \geq 0$.

Thus, in view of the definition of $a_{k,i}$ *we may conclude that:*

$$\mid a_{k,i}(\tau) \mid^2_{I\!R^k} = \parallel u_k(\tau) \parallel^2_{2,\omega_0} \leq \overline{c} \tag{6.6.7}$$

for a constant $\bar{c}$ which is independent of τ and k. Now, we prove that A is non empty, open and closed in $[0;T]$. First, A is clearly non- empty since we proved local existence above. Moreover, let $t \in A$ and $0 < \tau_1 < \tau_2 < T$. Then by (6.6.5), we have:

$$| a_{k,j}(\tau_1) - a_{k,j}(\tau_2) | \leq \int_{\tau_1}^{\tau_2} | F_j(\tau, a_{k,i}(\tau), ..., a_{k,k}(\tau)) | d\tau$$

$$\leq C(\bar{c}, k) \int_{\tau_1}^{\tau_2} | M(\tau) | d\tau;$$

since $M \in L^1(0,T)$, this implies that $\tau \mapsto a_{k,j}(\tau)$ is uniformly continuous. Then, we can restart to solve

$$\langle \partial_t u_k, \delta_k \rangle_{2,\omega_0} + \int_\Omega \sigma(x,t,u_k,Du_k) : D\delta_k dx = \int_\Omega v(x,t)\delta_k dx$$

for $k \in \{1;2;....;m\}$.

At time t with initial data $\lim_{\tau \nearrow \longrightarrow t} u_k(\tau)$ and hence get a solution of (6.6.3) - (6.6.4) on $[0, \tau + \varepsilon]$. This proves that A is open. Finally, we consider a sequence $\tau_i \nearrow t; \tau_i \in A$. Let $a_{k,j,i}$ denote the solution of (6.6.3) - (6.6.4) we constructed on $[0, \tau_i]$ and define:

$$\tilde{a_{k,j,i}} \equiv \begin{cases} a_{k,j,i}(\tau) & \text{if} \quad \tau \in [0;\tau_i] \\ a_{k,j,i}(\tau_i) & \text{if } \tau \in]\tau_i;t]. \end{cases}$$

The sequence $(\tilde{a_{k,j,i}})_i$ is bounded and equicontinuous on $[0;t[$, as seen above. Hence, by the Arsela-Ascoli Theorem, a subsequence (again denoted by $\tilde{a_{k,j,i}}(\tau)$ converges uniformly in τ on $[0;t[$ to a continuous functions $a_{k,j}(\tau)$). Using Lebesgue convergence theorem in (6.6.5) it is now easy to see that $a_{k,j}(\tau)$ solves (6.6.3) on $[0;t[$. Hence $t \in A$ and thus A is indeed closed, and as claimed, it follows $A = [0;T[$. □

Proposition 6.6.1 *The sequence of the Galerkin approximations (u_k) is bounded in $L^\infty(0,T,L^2(\Omega,\omega_0,I\!R^m)) \cap L^p(0,T,W_0^{1,p}(\Omega,\omega,I\!R^m)))$ and satisfies (A_1) and (A_2).*

Proof of proposition6.6.1

By testing equation:

$$\langle \partial_t u_k, \delta_j \rangle_{2,\omega_0} + \int_\Omega \sigma(x,t,u_k,Du_k) : D\delta_j dx = \langle v(t), \delta_j \rangle$$

for:$j \in \{1,2,...,k\}$.

With u_k; we obtained in the proof of lemma (6.6.3) that (6.6.7) holds for arbitrary time $\tau \in [0;T[$. So (u_k) is bounded in the space $L^\infty(0,T,L^2(\Omega,\omega_0,I\!R^m))$. Moreover we have proved that $\psi(\| u_k \|_{L^p(0,T,W_0^{1,p}(\Omega,\omega,I\!R^m))})$ is positive, bounded and satisfies (6.6.6).Then there exists a constant $c > 0$ such that

$$\| u_k \|_{L^p(0,T,W_0^{1,p}(\Omega,\omega,I\!R^m))} \leq c; \quad \forall k \in I\!N. \tag{6.6.8}$$

Therefore, by extracting a suitable subsequence which is again denoted by u_k, we may assume by the Elberlein-Smuljan see e-g, [62] (Page 141), that

$u_k \rightharpoonup^* u$ *in* $L^\infty(0,T,L^2(\Omega,\omega_0,I\!R^m))$

$u_k \rightharpoonup u$ *in* $L^p(0,T,W_0^{1,p}(\Omega,\omega,I\!R^m))$

which proves in particular (A_1). *To verify* (A_2), *observe that the time derivative*

$\frac{d}{dt}(j\circ i\circ u_k) : [0;T[\to B_1 = (W_0^{r,2}(\Omega,\omega_\alpha,I\!R^m)))'$ *is according to*

$\langle \partial_t u_k, \delta_j\rangle_{2,\omega_0} + \int_\Omega \sigma(x,t,u_k,Du_k) : D\delta_j dx = \langle v(t),\delta_j\rangle$

for: $j \in \{1;2;....;k\}$. *Given by: we recall that the projection operators* P_k *are self adjoint with respect to the* L^2 *inner product.*

$t \mapsto (\phi \mapsto \langle v(t), P_k\phi\rangle - \int_\Omega \sigma(x,t,u_k,Du_k) : D(P_k\phi)dx)$. *Now we have:*

$\langle \frac{d}{dt}(\jmath\circ\imath\circ u_k)(t),\phi\rangle = I + II$ *where*

$I = \int_0^T \langle v(t), P_k\phi\rangle dt \leq \| v \|_{L^{p'}(0,T,(W^{-1,p'}(\Omega,\omega^*))} \| P_k\phi \|_{L^p(0,T,(W^{1,p}(\Omega,\omega))}$ *by the generalized Hölder inequality,*

$II = -\int_0^T \int_\Omega \sigma(x,t,u_k,Du_k) : D(P_k\phi)dxdt$

$\leq (\| \lambda_1 \|_{L^{p'}(\Omega_T)} + C(c_1) \| u_k \|^{p-1}_{L^p(0,T,W_0^{1,p}(\Omega,\omega)}) \| P_k\phi \|_{L^p(0,T,W_0^{1,p}(\Omega,\omega))}$ *by the growth conditions in* (P_2) *and the Hölder inequality. Since*

$\| P_k\phi \|_{L^p(0,T,W_0^{1,p}(\Omega,\omega))} \leq \| P_k\phi \|_{L^p(0,T,W_0^{r,2}(\Omega,\omega_\alpha))} \leq \| \phi \|_{L^p(0,T,W_0^{r,2}(\Omega,\omega))}$, *the claim follows by the uniform bounded (6.6.8).* □

Remark 6.6.1 *For further use, we note that from the bounded obtained for* $\langle \frac{d}{dt}(\jmath\circ\imath\circ u_k)(t),\phi\rangle$ *in the proof of proposition we can conclude that* $\partial_t u$ *(or rather* $\partial_t(\jmath\circ\imath\circ u)$*) is an element of the space* $L^{p'}(0,T,(W_0^{-1,p'}(\Omega,\omega^*))$. *This follows easily from the fact, that the set*

$\{\phi \in L^p(0,T,W_0^{1,p}(\Omega,\omega)) : \exists k \in I\!N / P_k\phi = \phi\}$ *is dense in* $L^p(0,T,W_0^{1,p}(\Omega,\omega))$, *as proved in [37] (Appendix II) with a slight modification.* □

Remark 6.6.2 *Recall that the space*

$\{u \in L^p(0,T,W_0^{1,p}(\Omega,\omega)) : \partial_t(\jmath\circ\imath\circ u) \in L^{p'}(0,T,(W^{-1,p'}(\Omega,\omega^*)$ *is continuously embedded in* $C([0;T];L^2(\Omega,\omega_0))$ *see e-g [65] [proposition23-23], with slight assimilation.*

Hence, we have that $u \in C^0([0;T];L^2(\Omega,\omega_0))$ *after possible modification of* u *on a Lebesgue Zero-set of* $[0;T]$. *This gives* $u(t,.) \in L^2(\Omega,\omega_0)$ *a pointiest interpretation for all* $t \in [0;T]$ *and allows in particular to state that* $u(t,.)$ *attains its initial value*

$$u(.,0) = u_0 \tag{6.6.9}$$

continuously in $L^2(\Omega,\omega_0)$ see [37] (Appendix III). In the two following subsections, we prove that u is really a solution of the system (QPS). □

6.7 Passage to the limit in (QPS)

In this subsection, we prove theorem (6.2.1). First, in order to apply proposition (6.6.1), we verify that, under our assumptions, the conditions $(A_1)-(A_6)$ and $(N_1)-(N_3)$ holds for the Galerkin approximations solution u_k. (A_1) and (A_2) holds by proposition (6.6.1). Moreover, it follows then by lemma (6.6.1) that $(N_1)-(N_3)$ holds. The condition (A_3) is equivalent to (P_1). Next,to prove (A_4), we note that by the growth condition in (P_2) that

$$\int_{\Omega_T} \mid \sigma_{rs}(x,t,u_k,Du_k) \mid^{p'} \omega^*_{rs} dxdt$$
$$\leq C.(\int_{\Omega_T} \mid \lambda_1 \mid^{p'} dxdt + \sum_{j=1}^{m} \mid u_{kj} \mid^{q} \gamma_j dxdt + \sum_{i,j} \mid D_{ij}u_k \mid^{p} \omega_{ij} dxdt);$$

with (c is a constants) which is uniformly bounded in k by (6.6.8). To verify the equi-integrability of $(\sigma_k : Du_k)$, we fix an arbitrary measurable subset $\Omega' \subset \Omega_T$. Then, on the one hand, the growth condition in (P_2) implies that

$$\int_{\Omega'} \mid \min(\sigma_k : Du_k; 0) \mid dxdt \leq \int_{\Omega'} \mid \lambda_2 \mid dxdt + \int_{\Omega'} \mid \lambda_3 \mid \sum_{j=1}^{m} \omega_{1j}^{\frac{\alpha}{p}} \mid u_{kj} \mid^{\alpha}.$$

by the Hölder inequality and (6.6.8). Thus (A_5) follows. Finally, we prove that (A_6) hold. According to Mazur's Theorem see e-g [62] (Theorem2, page 120), there exists a sequence v_k in $L^p(0,T,W_0^{1,p}(\Omega,\omega))$ where each v_k is a convex linear combination of $u_1,...,u_k$ such that $v_k \to u$ in $L^p(0,T,W_0^{1,p}(\Omega,\omega))$. In particular, $v_k(t,.) \in span(\delta_1,\delta_2,...,\delta_k)$ for all $t \in [0,T]$. Now, we have

$$\int_{\Omega_T} \sigma_k : (Du_k - Dv_k) dxdt = \langle v, u_k - v_k \rangle - \sum_j \int_{\Omega_T} (u_k^j - v_k^j) \partial_t(u_k^j) \omega_{0j} dxdt. \tag{6.7.1}$$

Observe that $(u_k - v_k) \in span(\delta_1,\delta_2,...,\delta_k)$ which allowed to use
$\langle \partial_t u_k, \delta_k \rangle_2 + \int_\Omega \sigma(x,t,u_k,Du_k) : D\delta_k dx = \int_\Omega v(x,t)\delta_k dx$ *in the equality above. The term $\langle v, u_k - v_k \rangle$ in (6.7.1) converge to zero, since $u_k - v_k \rightharpoonup 0$ in $L^p(0,T,W_0^{1,p}(\Omega,\omega))$ by the choice of v_k and (A_1).*

For the last therm in (6.7.1), we have

$$\begin{aligned} -\sum_j \int_{\Omega_T} (u_k^j - v_k^j)\partial_t(u_k^j)\omega_{0j} dxdt &= -\sum_j \int_{\Omega_T} \frac{1}{2}\partial_t(u_k^j)^2 \omega_{0j} dxdt + \sum_j \int_{\Omega_T} v_k^j \partial_t(u_k)\omega_{0j} dxdt \\ &= -\tfrac{1}{2} \parallel u_k(T) \parallel^2_{2,\omega_0} + \tfrac{1}{2} \parallel u_k(0) \parallel^2_{2,\omega_0} + \sum_j \int_{\Omega_T} v_k^j \partial_t(u_k^j)\omega_{0j} dxdt. \end{aligned} \tag{6.7.2}$$

Concerning the last term in (6.7.1) we claim that for $k \to \infty$ we have:

$$\sum_j \int_{\Omega_T} v_k^j \partial_t(u_k^j)\omega_{0j} dxdt \to \sum_j \int_{\Omega_T} u^j \partial(u^j)\omega_{0j} dxdt = \frac{1}{2} \parallel u(T) \parallel^2_{2,\omega_0} - \frac{1}{2} \parallel u_0 \parallel^2_{2,\omega_0} \tag{6.7.3}$$

To see this, let $\varepsilon > 0$ be given. Then, there exists M such that for all $l \geq m \geq M$ we have:

- *i)- $\sum_j \mid \int_{\Omega_T} (u^j - v_m^j)\partial_t(u^j)\omega_{0j} dxdt \mid \leq \varepsilon$. This is possible, since $\partial(j \circ i \circ u) \in L^{p'}(0,T,W_0^{-1,p'}(\Omega,\omega)$ and $v_m \to u \ \ in \ \ L^p(0,T,W_0^{1,p}(\Omega,\omega))$*
- *ii)- $\sum_j \mid \int_{\Omega_T} (v_l^j - v_m^j)\partial_t(u^j)\omega_{0j} dxdt \mid \leq \varepsilon$. This is possible by the bound obtained for $\langle \frac{d}{dt}(j \circ i \circ u_k)(t), \phi \rangle$ in the proof of proposition 6.6.1 since $(v_l - v_k) \in span(\delta_1, \delta_2 ....., \delta_k)$ for all fixed $t \in]0,T[$.*

Now, we fix $m \geq M$ and choose $m_0 \geq m$ such that for all $l \geq m_0$;
$\sum_j \mid \int_{\Omega_T} v_m^j(\partial_t(u^j - u_l^j)\omega_{0j} dxdt \mid \leq \varepsilon$. This is possible, since $\partial_t u_l \rightharpoonup^* \partial_t u$ in $L^{p'}(0,T,(W_0^{r,2}(\Omega,\omega_\alpha))')$. combination yields for all $l = l(\varepsilon) \geq m_0(\varepsilon)$

$$\mid \sum_j \int_{\Omega_T} v_l^j \partial_t(u_l^j)\omega_{0j} dxdt - \sum_j \int_{\Omega_T} u^j \partial_t(u^j)\omega_{0j} dxdt \mid$$
$$\leq \sum_j \mid \int_{\Omega_T} (v_l^j - v_m^j)\partial_t(u_l^j)\omega_{0j} \mid dxdt + \sum_j \mid \int_{\Omega_T} v_m^j(\partial_t u_l^j - \partial_t u^j)\omega_{0j} dxdt \mid$$
$$+ \sum_j \mid \int_{\Omega_T} (v_m^j - u^j)\partial_t u^j \omega_{0j} dxdt$$
$$\leq 3\varepsilon.$$ *This establishes (6.7.3).on the other hand, since $(u_k)_k$ is bounded in $L^\infty(0,T,L^2(\Omega,\omega_0))$ by proposition 6.6.1 we have that $(u_k(T) \rightharpoonup u(T))$ in $L^2(\Omega,\omega_0))$. Hence,*

$$\liminf_{k\to\infty} \parallel u_k(T) \parallel_{2,\omega_0} \geq \parallel u(T) \parallel_{2,\omega_0} . \tag{6.7.4}$$

By construction of u_k we also have

$$\lim_{k\to\infty} \parallel u_k(0) \parallel_{2,\omega_0)} = \parallel u_0 \parallel_{2,\omega_0} . \tag{6.7.5}$$

Using (6.7.5), (6.7.4) and (6.7.3), in (6.7.2), we conclude that
$\liminf_{k\to\infty}(-\sum_j \int_{\Omega_T} (u_k^j - v_k^j)\partial_t(u_k^j)\omega_0) dxdt) \leq 0$, *thus (A_6) is fulfilled.*
So,$(A_1) - (A_6)$ and $(N_1) - (N_3)$ hold and we may infer from proposition (6.6.1) that $\sigma_k(x,t,u_k,Du_k)$ converge to $\sigma_k(x,t,u,Du)$ at least weakly in $L^1(\Omega_T, I\!M^{m\times n})$, for $k \to \infty$. Since we have that $u_k \to u$ in measure on Ω_T by proposition (6.3.1), we may infer that $u_k \to u$ almost everywhere on Ω_T, for $k \to \infty$. Now, we take a test function $\varphi \in \cup_{i\in I\!N} span(\delta_1, \delta_2 ....., \delta_i)$ and $\phi \in C_0^\infty([0,T])$ in

$\langle \partial_t u_k, \delta_j \rangle_{2,\omega_0} + \int_\Omega \sigma(x,t,u_k,Du_k) : D\delta_j dx = \langle v(t), \delta_j \rangle$
for: $j \in \{1,2,...,k\}$.

And integrate over the interval $]0,T[$ *and pass to the limit* $k \to \infty$*. The resulting equation is*
$\sum_j \int_{\Omega_T} \partial_t u^j(x,t)\phi^j(t)\varphi(x)\omega_{0j})dxdt + \int_{\Omega_T} \sigma_k(x,t,u,Du) : D\varphi(x)\phi(t)dxdt = \langle v, \phi\varphi \rangle$
For arbitrary $\varphi \in \cup_{i\in I\!N} span(\delta_1,\delta_2,...,\delta_i)$ *and* $\phi \in C_0^\infty([0,T])$*. By density of the linear span of these functions in* $L^p(0,T,W_0^{1,p}(\Omega,\omega))$*, this proves that* u *is in fact a weak solution of* (QPS)*. Hence the theorem is proved.* □

Chapter 7

Existence of solution for some quasi-linear parabolic systems with weight and weak monotonicity

Abstract

We prove the existence of weak solution u for the nonlinear parabolic systems:

$$(QPS)_{(\omega)} \begin{cases} \partial_t u - div\sigma(x,t,u,Du) &= v(x,t) + f(x,t,u) + divg(x,t,u) \ \textit{in} \quad \Omega_T \\ u(x,t) &= 0 \quad \textit{on} \quad \partial\Omega \times (0,T) \\ u(x,0) &= u_0(x) \quad \textit{on} \quad \Omega \end{cases}$$

which is a Dirichlet Problem. In this system, v belongs to $L^{p'}(0,T,W^{-1,p'}(\Omega,\omega^*,I\!R^m))$; and $u_0 \in L^2(\Omega,\omega_0,I\!R^m)$, f and g satisfy some standards continuity and growth conditions. We prove existence of a weak solution of different variants of this system under classical regularity for some $p \in]\frac{2n(s+1)}{(n+2)s}\ ;\ \infty[$, growth and coercivity for σ but with only very mild monotonicity assumptions.

7.1 functional framework

Definition 7.1.1 *(Lebesgues and Sobolev spaces with weight):*

For $1 \leq p \leq \infty$ and $k \in I\!N$; we consider the weight spaces:

$$L^p(\Omega,\omega_0,I\!R^m)\equiv\left\{u=u(x);\ u_j\omega_{0j}^{\frac{1}{p}}\in L^p(\Omega,I\!R^m)\right\}$$

with, $\omega_0=(\omega_{0j})_{1\leq j\leq m}$.

$$W^{k,p}(\Omega,\omega_\alpha,I\!R^m)\equiv\left\{u\in L^p(\Omega,\omega_\alpha,I\!R^m);\ \ D^\alpha u\in L^p(\Omega,\omega_\alpha,I\!R^m);\ \ for\ \ 0\leq\mid\alpha\mid\leq k\right\}.$$

$D^\alpha u$ is the weak derivative of u. The weighted functions ω_α defined by:

$$\omega_\alpha=\begin{cases}\omega_0=(\omega_{0j})_{1\leq j\leq m}\ \ if\quad\mid\alpha\mid=0,\\ \omega=(\omega_{ij})_{1\leq i\leq n;\ 1\leq j\leq m}\ \ if\quad\mid\alpha\mid=1,\\ arbitrary\ weight\ functions\qquad if\ \ 2\leq\mid\alpha\mid\leq k.\end{cases}$$

And $\omega_{ij}^*=\omega_{ij}^{1-p'}$ is the conjugate of ω_{ij}.

We defined the inner product in $L^2(\Omega,\omega_0,I\!R^m)$ by: $\langle u,v\rangle_{2,\omega_0}=\sum_{j=1}^m\int_\Omega((\omega_{0j})^{\frac{1}{2}}u_j)((\omega_{0j})^{\frac{1}{2}}v_j)dx$, for all $u;v$ in $L^2(\Omega,\omega_0,I\!R^m)$. And for all $u;v$ in $W_0^{r,2}(\Omega,\omega_\alpha,I\!R^m)$ we denotes the inner product by:

$$\langle u,v\rangle_{W^{r,2}(\omega_\alpha)}=\langle u,v\rangle_{2,\omega_0}+\sum_{i;j=1}^{n;m}\int_\Omega(\omega_{ij}^{\frac{1}{2}}D_{ij}u)(\omega_{ij}^{\frac{1}{2}}D_{ij}v)dx+\sum_{2\leq|\alpha|\leq r}\int_\Omega(\omega_\alpha^{\frac{1}{2}}D^\alpha u)(\omega_\alpha^{\frac{1}{2}}D^\alpha v)dx.$$

We denote $W_0^{k,p}(\Omega,\omega_\alpha,I\!R^m)$ the closure of $C_0^\infty(\Omega,I\!R^m)$ in $W^{k,p}(\Omega,\omega_\alpha,I\!R^m)$, where the norm is $\|.\|_{k,p,\omega_\alpha}$ is given by:

$$\|u\|_{k,p,\omega_\alpha}=\left(\sum_{0\leq|\alpha|\leq k}\|\ D^\alpha u\ \|_p\ \omega_\alpha\right)^{\frac{1}{p}}.$$

We have:

$W_0^{k,p}(\Omega,\omega_\alpha,I\!R^m)\hookrightarrow W^{k,p}(\Omega,\omega_\alpha,I\!R^m)\hookrightarrow L^p(\Omega,\omega_0,I\!R^m)$.

$(W_0^{k,p}(\Omega,\omega_\alpha,I\!R^m))'\equiv W^{-k,p'}(\Omega,\omega_\alpha^*,I\!R^m)$.

Definition 7.1.2 *let Ω be a measurable subset of $I\!R^n$, $(\mu_k)_{k\in I\!N}$ and μ be a sequence and respectively an element in $M(\Omega,I\!R^m)$. we say that (μ_k) converges weakly $*$ to μ, which is denoted by $\mu_k\rightharpoonup^*\mu$, if:* $\int_\Omega f d\mu_k\to\int_\Omega f d\mu$, *for all $f\in C_c(\Omega,I\!R^m)$.*

Lemma 7.1.1 *(Fatou) Let $F:\Omega\times I\!R^m\times I\!M^{m\times n}\to I\!R$ be a carathéodory functions and $u_k:\Omega\to I\!R^m$ a sequence of measurable functions such that Du_k generates the Young measure ϑ_x with $\|\ \vartheta_x\ \|=1$ for almost every $x\in\Omega$. Then:*

$$\liminf_{k\to\infty}\int_\Omega F(x,u_k,Du_k)dx\geq\int_\Omega\int_{I\!M^{m\times n}}F(x,u,\xi)d\vartheta_x(\xi)dx.$$

Provided that the negation $F^-(x,u_k,Du_k)$ is equi-integrable. More general versions of the this lemma may be found in [59], [60] and [42], proof see [37] (lemma 1-5).

7.2 Hypothesis

We assume that:

$$\omega_{ij} \in L^1(\Omega), \quad \omega_{ij}^{\frac{-1}{p-1}} \in L^1_{loc}(\Omega) \text{ and } \omega_{0j} \text{ is bounded a.e. in } \Omega \tag{7.2.1}$$

$$\omega_{ij}^{-s} \in L^1(\Omega) \tag{7.2.2}$$

for some $s \in]Max(\frac{1}{p-1}; \frac{n}{p}) \ ; \ \infty[$.

The Sobolev space $W^{1,p}(\Omega, \omega, I\!R^m)$ *equipped with the norm*

$$\| \ u \ \|^p_{1,p,\omega} = \sum_{j=1}^{m} \int_\Omega \omega_{0j} |u_j|^p dx + \sum_{i,j} \int_\Omega \omega_{ij} |D_{ij} u|^p dx.$$

is a Banach space, reflexive, and $C_0^\infty(\Omega, I\!R^m)$ *is a subspace of* $(W^{1,p}(\Omega, \omega, I\!R^m)$. *The space* $(W_0^{1,p}(\Omega, \omega, I\!R^m)$ *is the fermenter of* $C_0^\infty(\Omega, I\!R^m)$ *in* $(W^{1,p}(\Omega, \omega, I\!R^m)$ *for the norm* $\|.\|^p_{1,p,\omega})$,

(by the condition 7.2.1).

The norm $\| \ .. \ \|^p_{1,p,\omega}$ *equivalent to the norm* $|||u|||_{W_0^{1,p}(\Omega,\omega,I\!R^m)} = (\sum_{i,j} \int_\Omega \omega_{ij} |D_{ij} u|^p dx)^{\frac{1}{p}}$, *with* $D_{ij} u = \frac{\partial u^j}{\partial x_i}$

(by: The condition 7.2.2).

By the condition (7.2.2), we give:

$$(W^{1,p}(\Omega, \omega, I\!R^m) \hookrightarrow (W^{1,p_s}(\Omega, I\!R^m) \text{ and } (W^{1,p}(\Omega, \omega, I\!R^m \hookrightarrow\hookrightarrow L^r(\Omega, I\!R^m), \tag{7.2.3}$$

for all $1 \leq r < p_s^*$ *with* $p_s = \frac{p.s}{s+1}$

$p_s^* = \frac{n.p.s}{n(s+1)-p.s}$ *if* $p.s < n(s+1)$, *and* p_s^* *is arbitrary otherwise.*

The Hardy-Type inequalities in condition (P_0) *is verifies for* $\gamma_j = 1$; $j = 1; 2; .....; m$

(P_1) $\sigma : \Omega \times]0, T[\times I\!R^m \times I\!M^{m \times n} \longrightarrow I\!M^{m \times n}$ *is a Carathéodory functions, i.e:*
$(x, t) \longmapsto \sigma(x, t, u, F)$ *is measurable for every* $(u, F) \in I\!R^m \times I\!M^{m \times n}$ *and*
$(u, F) \in I\!R^m \times I\!M^{m \times n} \longmapsto \sigma(x, t, u, F)$ *is continuous for almost every* $(x, t) \in \Omega_T$.

(P_2) *(Coercivity and growth): There exist constants* c_1, c_2, $\beta > 0$ *and* $\lambda_1 \in L^{p'}(\Omega_T, I\!R^m)$, $\lambda_2 \in L^1(\Omega_T, I\!R^m)$, $\lambda_3 \in L^{(\frac{p}{\alpha})'}(\Omega_T)$, $0 < \alpha < p$, $1 < q < \infty$, *such that, for all* $1 \leq r \leq n$, *and* $1 \leq s \leq m$:

$$|\sigma_{rs}(x, t, u, F)| \leq \beta \omega_{rs}^{\frac{1}{p}} [\lambda_1 + c_1 \sum_{j=1}^{m} \gamma_j^{\frac{1}{p'}} |u_j|^{\frac{q}{p'}} + c_1 \sum_{i,j} \omega_{ij}^{\frac{1}{p'}} |F_{ij}|^{p-1}]$$

$$\sigma(x,t,u,F) : F \geq -\lambda_2(x,t) - \sum_{j=1}^{m} \lambda_3(x,t)\omega_{1j}^{\frac{\alpha}{p}}|u_j|^{\alpha} + c_2 \sum_{i,j} \omega_{ij}|F_{ij}|^p.$$

(P_3) *(Monotonicity): σ satisfies one of the following conditions:*

- *(a) For all $(x,t) \in \Omega_T$ and all $u \in I\!R^m$, the map: $F \mapsto \sigma(x,t,u,F)$ is a C^1- functions and is monotone.*
- *(b) There exist a function: $W : \Omega_T \times I\!R^m \times I\!M^{m\times n} \to I\!M^{m\times n}$, such that $\sigma(x,t,u,F) = \frac{\partial W}{\partial F}(x,t,u,F)$, and $F \mapsto W(x,t,u,F)$ is convex and C^1- function.*
- *(c) For all $(x,t) \in \Omega_T$ and all $u \in I\!R^m$, the map: $F \mapsto \sigma(x,t,u,F)$ is strictly monotone .*
- *(d) σ is strictly -p quasi monotone in F i.e:* $\int_{I\!M^{m\times n}} (\sigma(x,t,u,\lambda) - \sigma(x,t,u,\overline{\lambda}) : (\lambda - \overline{\lambda})d\vartheta(\lambda) > 0$

(P_0) *(The Hardy-Type inequalities); There exist $c > 0$, one weighted function $\gamma = (\gamma_j)_{1\leq j\leq m}$, and the parameter $1 < q$, such that:*

$$\left(\sum_{j=1}^{m} \int_{\Omega} \gamma_j \mid u_j \mid^q dx\right)^{\frac{1}{q}} \leq c \left(\sum_{i,j} \int_{\Omega} \omega_{ij} \mid D_{ij}u \mid^p dx\right)^{\frac{1}{p}}$$

and the expression $\left(\sum_{i,j} \int_{\Omega} \omega_{ij} \mid D_{ij}u \mid^p dx\right)^{\frac{1}{p}}$ is a norm equivalent to the norm $\|.\|_{1;p:\omega}$ in $W_0^{1,p}(\Omega,\omega,I\!R^m)$. The injection $W_0^{1,p}(\Omega,\omega,I\!R^m) \hookrightarrow\hookrightarrow L^q(\Omega,\gamma,I\!R^m)$ is compact, and $W_0^{1,p}(\Omega,\omega,I\!R^m) \hookrightarrow\hookrightarrow L^r(\Omega,I\!R^m)$ is compact, with

$$\begin{cases} 1 \leq r < \frac{nps}{n(s+1)-ps} & \text{if } \ ps < n(s+1) \\ r \geq 1 & \text{if } \ n(s+1) \leq ps \end{cases}$$

(F_0) *(continuity) $f : \Omega_T \times I\!R^n \to I\!R^m$ is a Carathéodory function, i.e.$(x,t) \mapsto f(x,t,u)$ is measurable for every $u \in I\!R^n$ and $u \mapsto f(x,t,u)$ is continuous for almost every $(x,t) \in \Omega_T$.*

(F_1) *(Growth) There exist $0 < \gamma < p-1$, $b_1 \in L^{p'}(\Omega_T)$ and $b_2 \in L^{\gamma^{\mp}}(0,T,L^{\frac{n}{p}}(\Omega))$, such that:* $\mid f_k(x,t,u) \mid \leq b_1\omega_{0k}^{\frac{1}{p}} + \sum_{j=1}^{m} b_2|u_j|^{\gamma}\omega_{0j}^{\frac{p-1}{p^*}}\omega_{0k}^{\frac{1}{p^*}} : \quad k = 1;2;....;m. \quad \gamma^{\mp} = (\frac{p}{p-1-\gamma})'$ *is the Hölder conjugate of $\frac{p}{\gamma+1}$.*

(G_0) *(continuity)$g : \Omega_T \times I\!R^n \to I\!M^{m\times n}$ is a Carathéodory function.*

(G_1) *(Growth) There exist $0 < \eta < p-1$, $b_4 \in L^{p'}(\Omega_T)$ and $b_5 \in L^{\infty}(0,T,L^{\frac{n}{p-1}}(\Omega))$, such that:* $\mid g_{rs}(x,t,u) \mid \leq b_4\omega_{rs}^{\frac{1}{p}} + \sum_{j=1}^{m} b_5|u_j|^{\eta}\omega_{0j}^{\frac{n-p+1}{np'}}\omega_{rs}^{\frac{n-p+1}{np}}$

Remark 7.2.1

(P_1) *(Carathéodory condition) ensures that $\sigma(x,t,u(x,t),U(x,t))$ is measurable on $\partial\Omega_T$ for measurable functions:*

$u : \Omega_T \to I\!R^m$

$U : \Omega_T \to I\!M^{m\times n}$.

The growth condition and coercivity

(P_2) *is standard it is used in the construction of approximate solutions by a Galerkin method and when we pass to the limit.*

Although the strict monotonicity condition (c) in (P_3) ensure existence of a weak solution by standard method see e-g [46], the main point is that we do not require strict monotonicity or monotonicity in the variables (u,F) in $(a),(b)$ and (d) as it is usually assumed in previous works see-e-g [16], [46],[1].

Under these mild assumptions and using again Young measures, Hungerbühler shows in [38] that initial and boundary value problem $(\mathcal{QPS})_\omega$ has a weak solution $u \in L^p(0,T,W_0^{1,p}(\Omega,\omega,I\!R^m)$ in following sense.

Definition 7.2.1 *We say that: $u : \Omega_T \to I\!R^m$ is weak solutions of:*

$$(\mathcal{QPS})_\omega \begin{cases} \partial_t u - div\sigma(x,t,u,Du) &= v(x,t) + f(x,t,u) + divg(x,t,u) \quad \textit{in} \quad \Omega_T \\ u(x,t) &= 0 \quad \textit{on} \quad \partial\Omega\times(0,T) \\ u(x,0) &= u_0(x) \quad \textit{on} \quad \Omega \end{cases}$$

where $v \in L^{p'}(0,T,W^{-1,p'}(\Omega,\omega^*,I\!R^m))$ if:

- *(i) u belongs to $L^1(0,T,W_0^{1,1}(\Omega,I\!R^m))$.*
- *(ii) $\sigma(.,u(.),Du(.))$ belongs to $L^1(\Omega_T,I\!M^{m\times n})$.*
- *(iii) The initial condition $u(x,0)=u_0(x)$ on Ω is fulfilled.*
- *(iv) The equation:*
  $\sum_j \int_{\Omega_T} \partial_t u^j.\varphi^j(x,t)\omega_{0j}dxdt + \int_{\Omega_T} \sigma(x,t,u,Du) : D\varphi(x,t)dxdt = \langle v,\varphi\rangle + \int_{\Omega_T} f(x,t,u).\varphi(x,t)dxdt -$
  $\int_{\Omega_T} g(x,t,u) : D\varphi(x,t)dxdt,$
  holds for every function $\varphi \in C^\infty(\overline{\Omega_T},I\!R^m)$ which is zero in a neighborhood of
  $(\partial\Omega\times]0,T[) \cup (\Omega\times\{0;T\})$.

Remark 7.2.2

- *(a) In definition the boundary condition $u = 0$ on $(\partial\Omega\times]0,T[)$ is interpreted in the sense of (i) .*
- *(b) The condition (iv) is understood in the sense that the Galerkin approximations converge to u_0 in $L^2(\Omega,\omega_0)$.*

Theorem 7.2.1 *If $p \in]\frac{2n}{n+2}(\frac{s+1}{s})\ ;\infty[$ and if σ satisfies the conditions $(P_1)-(P_3)$ then the parabolic system $(QPS)_\omega$ has a weak solution $u \in L^p(0,T,W_0^{1,p}(\Omega,\omega,I\!R^m))$ for every $v \in L^{p'}(0,T,W_0^{1,p}(\Omega,\omega^*,I\!R^m))$ every f satisfying $(F_0)-(F_1)$, g satisfying $(G_0)-(G_1)$ and $(n \geq 2)$.*

Remark 7.2.3

- *(a) For $p_s = n$, Theorem 7.2.1 hold if the condition $b_2 \in L^\infty(\Omega_T)$ in (F_1) and $b_5 \in L^\infty(\Omega_T)$ in (G_1) are assumed.*
- *(b) For $p_s > n$, Theorem 7.2.1 remains valid if $b_2 \in L^{\gamma^\mp}(0,T,L^1(\Omega))$ in (F_1) and $b_5 \in L^\infty(0,T,L^1(\Omega))$ in (G_1) hold.*
- *(c) The strict bound $p-1$ for γ and η in the growth conditions (F_1) and (G_1) ensures the coercivity of the function ψ However, the limit value $p-1$ is admissible in some particular cases.*

7.3 A first parabolic div-curl inequality

We state here the first set of hypothesis:

- (A_1)*: The sequence (u_k) is uniformly bounded in $L^p(0,T,W_0^{1,p}(\Omega,\omega,I\!R^m))$ for some $p \in]\frac{2n(s+1)}{(n+2)s}\ ;\ \infty[$ and hence (Bay the Elberlem Suljan theorem: see [15].a subsequence converges weakly in $L^p(0,T,W_0^{1,p}(\Omega,\omega,I\!R^m))$ to an element denoted by u.*
- (A_2)*: The sequence $\partial_t u_k$ is uniformly bounded in $L^{p'}(0,T,(W_0^{r,2}(\Omega,\omega,I\!R^m))')$ for some ($2 < r < p_s^*$ if $p_s < n$) and ($2 < r < \infty$ if $p_s \geq n$.)*
- (A_3)*: $\sigma : \Omega\times]0,T[\times I\!R^m \times I\!M^{m\times n} \longrightarrow I\!M^{m\times n}$ is a Carathéodory function.*
- (A_4)*: The sequence $\sigma_k(x,t) \equiv \sigma(x,t,u_k(x,t),Du_k(x,t))$ is uniformly bounded in $L^{p'}(\Omega_T,I\!M^{m\times n})$ and therefore equi-integrable (the equi-integrability follows from, the Hölder inequality).*
- (A_5)*: The sequence $(\sigma_k(x,t) : Du_k)^-$ is equi-integrable.*

- (A_6): *There exists a sequence* (v_k) *such that* $v_k \to u$ *in* $L^p(0,T,(W_0^{1,p}(\Omega,\omega,I\!R^m))')$ *and* $\int_{\Omega_T} \sigma_k(x,t) : (Du_k - Dv_k)dxdt \to 0; \quad k \to \infty$.

Remark 7.3.1 *Note that the assumption* (A_1) *and*(A_2) *ensure a strangle convergence in some Lebesgue spaces and in measure.*

Lemma 7.3.1 *Let* B, B_0 *and* B_1 *be Banach spaces,* B_0 *and* B_1 *reflexive. Let* $\imath : B_0 \to B$ *be a compact linear map and:*
$\jmath : B \to B_1$ *an injective bounder linear operator. For* T *finite and:* $1 < p_i < \infty$ *with* $i = 0; 1$
$W = \{v \in L^{p_0}(0;T;B_0) : \frac{d}{dt}(\jmath \, o \, \imath \, o \, v) \in L^{p_1}(0;T;B_1)\}$ *is a Banach space under the norm:*
$\| \, v \, \|_{L^{p_0}(0;T;B_0)} + \| \, \jmath \, o \, \imath \, o \, v \, \|_{L^{p_1}(0;T;B_1)}$.
Then, if $V \subset W$*is bounded, the set* $\{\imath \, o \, v; v \in V\}$ *is precompact in* $L^{p_0}(0;T;B_0)$.

Proof of lemma7.3.1
For the proof, see [37] (Appendix I). For generalization of lemma, see [56]. Now, we apply lemma to obtain the desired bound.

Proposition 7.3.1 *Let* $p \in]\frac{2n(s+1)}{(n+2)s} \; ; \; \infty[$ *and* (u_k) *be a sequence satisfying* (A_1) *and* (A_2). *Then there exists a subsequence of* (u_k) *(for convention not relabeled) and a functions:*
$u \in L^p(0,T,W_0^{1,p}(\Omega,\omega,I\!R^m))$ *such that:* $u_k \to u$ *in measure on* Ω_T *and in*

$$L^p(0,T,L^r(\Omega,I\!R^m)), \forall r < p_s^*. \tag{7.3.1}$$

Proof of prop 7.3.1
Let $B_0 \equiv W_0^{1,p}(\Omega,\omega,I\!R^m)); B \equiv L^r(\Omega,\omega_0,I\!R^m))$, *for some* r *with:*
$2 < r < p_s^*$,
and $B_1 \equiv (W_0^{r,2}(\Omega,\omega_\alpha,I\!R^m))'$. *Since we assumed that* $p_s \in]\frac{2n}{n+2} \; ; \; \infty[$, *we have the following chain of continuous injections:*

$$B_0 \hookrightarrow^i B \hookrightarrow^{i_0} L^2(\Omega,\omega_0,I\!R^m)) \cong (L^2(\Omega,\omega_0,I\!R^m))' \hookrightarrow^{i_1} B_1; \tag{7.3.2}$$

here $(L^2(\Omega,\omega_0,I\!R^m))' \equiv (L^2(\Omega,\omega_0,I\!R^m))$ *is the canonical isomorphism* θ *of the Hilbert space* $L^2(\Omega,\omega_0,I\!R^m)$ *and its, for*
$\imath : B_0 \to B$ *we take simply the injection mapping, and for* $\jmath : B \to B_1$ *we take the concatenation of*

injections and the canonical isomorphism given bay (7.3.2) i.e: $\jmath = \imath_1 \ o \ \theta \ o \ \imath_0$. *Then* (u_k) *is a bounded sequence in* $L^p(0,T,B_0)$ *by* (A_1). *Next observe that* $(\partial_t u_k)$ *or rather* $(\partial_t \ j \ o \ i \ o \ u_k)$ *is a bounded sequence in* $(L^{p'}(0,T,(W_0^{r,2}(\Omega,\omega_\alpha,I\!R^m))'))$ *by* (A_2). *Hence, from (7.3.1), we may conclude that there exists a subsequence, which we still denote by* u_k, *having the property that the second part of (7.3.1) hold. Notice that in order to have the strong convergence simultaneously for all* $r < p_s^*$, *the usual diagonal sequence procedure applies finally, as* $p > 1$, *the sequence converges in measure see, e-g.[30] (proposition 2-29). Which end the proof.*

Lemma 7.3.2 *(div-curl inequality) Suppose* $(A_1 - A_6)$ *and assume (after passing to a suitable subsequence if necessary) that* (Du_k) *generates the young measure* $\vartheta_{(x,t)}$. *Then the following inequality hold:*

$$\int_{\Omega_T}\int_{I\!M^{m\times n}} \sigma(x,t,u,\lambda) : \lambda d\vartheta_{(x,t)}(\lambda)dxdt \leq \int_{\Omega_T}\int_{I\!M^{m\times n}} \sigma(x,t,u,\lambda) : Du d\vartheta_{(x,t)}(\lambda)dxdt \qquad (7.3.3)$$

Proof of lemma 7.3.2

Let us consider the sequence:

$I_k = \Big(\sigma(x,t,u_k,Du_k) : (Du_k - Du)\Big) = \Big(\sigma_k : Du_k\Big) - \Big(\sigma_k : Du\Big)$.

By conditions (A_4) *and* (A_5), *the negative part* I_k^- *of* I_k *is equi-integrable. Hence, in view of* (A_3), *we may use the (Fatou - Lemma) which gives that*

$X = \liminf_{k\to\infty}(\int_{\Omega_T} I_k dxdt \geq \int_{\Omega_T}\int_{I\!M^{m\times n}} \sigma(x,t,u,\lambda) : (\lambda - Du)d\vartheta_{(x,t)}(\lambda)dxdt$

$X = \liminf_{k\to\infty}(\int_{\Omega_T} \sigma_k : (Du_k - Dv_k)dxdt + \int_{\Omega_T} \sigma_k : (Dv_k - Du)dxdt)$

$= \liminf_{k\to\infty}(\int_{\Omega_T} \sigma_k : (Dv_k - Du)dxdt \leq c.\liminf_{k\to\infty} \parallel \sigma_k \parallel_{L^{p'}(\Omega_T,\omega^*)} \parallel v_k - u \parallel_{L^p(0,T,W_0^{1,p}(\Omega,\omega,I\!R^m))}$

$= 0$ *by the Hölder inequality and* (A_4). *Thus the conclusion follows.* □

7.4 A second parabolic div-curl inequality

We state here the second set of conditions:

- $(c_1) : u_k \in L^1(0,T,W_0^{1,p}(\Omega,\omega,I\!R^m))$ *for all;* $k \in I\!N$, *and there exists some* $c > 0$ *and* $\rho > 0$ *such that:*
  $\sup_k \sum_{ij} \int_{\Omega_T} |D_{ij}u_k|^\rho \omega_{ij} dxdt \leq c.$
- (c_2)*: The sequence* u_k *converges in measure to some function* u *and* $u(.,t)$ *is almost everywhere approximately differentiable a.e.* $t \in]0,T[$.

- (c_3) : $\sigma : \Omega_T \times I\!R^m \times I\!M^{m\times n} \longrightarrow I\!M^{m\times n}$ *and*
  $\tau[e] : \Omega_T \times I\!R^m \times I\!M^{m\times n} \longrightarrow I\!M^{m\times n};\ e \in I\!N$ *are Carathéodory conditions.*

- (c_4) : σ *and* $\tau[e]$*, satisfy all the angle structure conditions or satisfies all the sign structure conditions:*
  (Angle condition) For all $(x,t) \in \Omega_T, u \in I\!R^m$ *and* $M; F \in I\!M^{m\times n}$*, with* M *of the form* $Id - a \otimes a$
  where $a \in I\!R^m$ *satisfies* $\mid a \mid \leq 1$; $\sigma(x,t,u,F) : MF \geq 0$*, for all matrix.*
  $M \in I\!M^{m\times m}$ *of the form* $M = Id - a \otimes a$ *with* $|a| \leq 1$.
  (The sign condition) For all $(x,t) \in \Omega_T, u \in I\!R^m$ *and* $F \in I\!M^{m\times n}$*, we have*
  $\sigma_j(x,t,u,F) : F_j \geq 0$*, for all* $1 \leq j \leq m$ *with*
  F_j *and* σ_j *are the cologne j of matrix* F *and* σ*.(this conditions is verifies for* $\tau[e]$ *)*

- (c_5) : *The sequence* $\sigma_k(x,t) = \sigma(x,t,u_k,Du_k)$ *is equi-integrable.*

- (c_6)*: The sequence* $\tau_k(x,t) \equiv \tau[k](x,t,u_k,Du_k)$*; and converges weakly to* 0 *in the space* $L^1(\Omega_T, I\!M^{m\times n})$.

- (c_7)*: The sequence* $g_k = -div(\sigma_k + g(x,t,u_k) + \tau_k) - v - f$ *is bounded in* $L^1(\Omega_T, I\!R^m)$.

- (c_8)*:* $Du_k \in L^i_{loc}(\Omega_T, \omega, I\!M^{m\times n})$ *and* $(\sigma_k + g(x,t,u_k) + \tau_k) \in L^{i'}_{loc}(\Omega_T, I\!M^{m\times n})$.
  for some $i; 1 < i < \infty$.

Lemma 7.4.1 *(Second div-curl inequality) Suppose* $(C_1) - (C_2)$*, and assure (after passing to a suitable subsequence if necessary) that* Du_k *generates a Young measure* $\vartheta_{(x,t)}$*. Then (after passing to a subsequence), the* σ_k *converges weakly in* $L^1(\Omega_T, I\!M^{m\times n})$ *and the weak limit* $\overline{\sigma}$ *is given by:*
$\overline{\sigma}(x,t) = \langle \vartheta_{(x,t)}; \sigma(x,t,u(x,t),.)\rangle$*, moreover the following inequality hold*
$\int_{I\!M^{m\times n}} \sigma(x,t,u(x,t),\lambda) : \lambda d\vartheta_{(x,t)}(\lambda) \leq \overline{\sigma}(x,t) : apDu(x,t)$*. for a.e.* $(x,t) \in \Omega_T$.

Proof of lemma7.4.1

Proof of lemma is a slightly modification of the corresponding in lemma $(6-2-1)$ *in [28] pages (54-61).*

Remark 7.4.1 *If the sequence* Du_k *is equi-integrable then* $Du(x,t) = apDu(x,t) = \langle \vartheta_{(x,t)}; id\rangle$ *almost everywhere. by the Ball's theorem.*

The parabolic div-curl inequality will be the key ingredient to pass the limit in the different cases, but we need some additional information on the Young measure $\vartheta_{(x,t)}$ *generated by the sequence of the gradients* Du_K *to exploit lemma (div-curl inequality) these properties are the following:*

- (N_1): *$\vartheta_{x,t}$ is a probability measure for almost every $(x,t) \in \Omega_T$.*
- (N_2): *$\vartheta_{x,t}$ is a homogeneous $W^{1,P}-$ gradient young measure for almost every $(x,t) \in \Omega_T$.*
- (N_3): *$\vartheta_{x,t}$ satisfies $< \vartheta_{(x,t)}, id >= apDu(x,t)$ almost everywhere in Ω_T.*

Note that these properties hold if the condition (A_1) is fulfilled.

Lemma 7.4.2 *Assume that the sequence u_k satisfies (A_1);*
then the Young measure $\vartheta_{(x,t)}$ generated by the sequence of the gradient Du_k satisfies $(N_1) - (N_3)$.

Proof of lemma7.4.2

Noting that (A_1) implies that $apDu(x,t) = Du(x,t)$, follows exactly as in [37] (prop 4-3).
We mention that the properties, also hold if the conditions $(C_1) - (C_2)$ and a particular estimate are fulfilled as shown in [39] (lemma 9), with a slight modifications □

Lemma 7.4.3 *Assume that the sequence (u_k) satisfies $(C_1) - (C_2)$ and*
$\sup_{k \in I\!N} \sum_{i,j} \int_{|u_k|<R} \mid D_{ij}u_k \mid^{\rho} \omega_{ij} dxdt \leq c$, *for all $R > 0$; for some $\rho > 1$.*
then the Young measure $\vartheta_{(x,t)}$ generated by the sequence of the gradients Du_k satisfies $(N_1) - (N_3)$.

Proof of lemma7.4.3

The proof of lemma is a slightly modification of the corresponding in lemma (11-3-2) in [28]

7.5 Passage to the limit for σ

Proposition 7.5.1 *Suppose that $(A_1) - (A_6)$ or $(C_1) - (C_8)$ hold further assume that the Young measure $\vartheta_{(x,t)}$ generated by the gradients Du_k satisfies $(N_1) - (N_3)$ and that one of the following conditions holds $(a) - (b) - (c) - (d)$ then(after passage to a subsequence) the sequence σ_k converges weakly in $L^1(\Omega_T; I\!M^{m \times n})$ and the weak limit $\overline{\sigma}$ is given by:*
$\overline{\sigma}(x,t) = \sigma(x,t,u(x,t),Du(x,t))$. *If $(b) - (c)$ or (d) holds, then:*

$\sigma(x,t,u_k,Du_k) \to \sigma(x,t,u,Du)$ *in* $L^1(\Omega_T;\mathbb{M}^{m\times n})$

In case (c) *and* (d)*; it follows in addition that* $Du_k \to Du$ *in measure and thus also almost everywhere in* Ω_T.

Before to prove proposition, we note that by proceeding exactly as in the elliptic case, we get the analogous result to lemma $(6-4-2)$ *in [28] and [1].*

Lemma 7.5.1 *Suppose that* $(A_1)-(A_6)$ *or* $(C_1)-(C_8)$ *hold. Further assume that* $\vartheta_{(x,t)}$ *is the Young measure generated by the gradients* Du_k *and satisfies* $(N_1)-(N_3)$*. If the map* $F \mapsto \sigma(x,t,u,F)$ *is monotone for all* $(x,t,u) \in \Omega\times]0;T[\times\mathbb{R}^m$ *then:*

$$Spt\vartheta_{(x,t)} \subset \{\lambda/[\sigma(x,t,u,\lambda)-\sigma(x,t,u,Du)] : (\lambda - Du) = 0\}. \tag{7.5.1}$$

Proof of lemma7.5.1

The proof of lemma is a slightly modification of the corresponding in lemma (11-4-2) see:[28].

Proof of proposition 7.5.1

For the proof see the proof of analogous proposition in chapter7. □

7.6 Galerkin Scheme

Lemma 7.6.1 *Let* Ω *bounded domain in* $\mathbb{R}^n$, $p \in]\frac{2n}{n+2}\ ;\ \infty[$ *and* $\omega_{ij} \in L^1(\Omega)$*, for all* $j = 1;2;....;m$*, and* $i = 0;1;.....;n$*.Then* $(W_0^{r,2}(\Omega,\omega_\alpha,\mathbb{R}^m) \hookrightarrow W_0^{1,p}(\Omega,\omega,\mathbb{R}^m)$ *For* $r \geq 1+n(\frac{1}{2}-\frac{1}{P})$.

Proof of lemma7.6.1 *For the case* $p \leq 2$ *it's easy (using the Hölder inequalities), for the case* $p \geq 2$ *The proof of this lemma is a slightly modification of the corresponding in a space of Sobolev without weight* $(W_0^{r,2}(\Omega,\mathbb{R}^m) \hookrightarrow W_0^{1,p}(\Omega,\mathbb{R}^m)$.

By the theorem $(1-3-1)$ *in [28] and proceeding as [37] (section 4-2) we may choose a set* $(\delta_i)_{i\in\mathbb{N}}$ *of functions which is an orthonormal basis in* $L^2(\Omega,\omega_0,\mathbb{R}^m)$ *and at the some time an orthogonal basis in* $W_0^{r,2}(\Omega,\omega_\alpha,\mathbb{R}^m)$*. In particular,* $(\delta_{(i)})_{i\in\mathbb{N}}$ *is uniformly bounded in* $W_0^{1,p}(\Omega,\omega,\mathbb{R}^m)$ *and then:* $\parallel \delta_i \parallel_{1,\omega,p}$ $+ \parallel \delta_i \parallel_{p^*} \leq c$ *for all* $i \in \mathbb{N}$.

we will need bellow the $L^2(\omega_0)$ *- orthogonal projector*

$p_k : L^2(\Omega, \omega_0, I\!R^m) \to L^2(\Omega, \omega_0, I\!R^m)$ *on to spam* $(\delta_1, \delta_2, ..., \delta_k)_{k\in I\!N}$*:*

of course, the operator norm $\| p_k \|_{\Im(W_0^{r,2}(\omega_\alpha), W_0^{r,2}(\omega_\alpha))} = 1$.

(The Galerkin scheme): New we build our Galerkin approximations, we make the following ansate for approximating solution of $(QPS)_\omega$, $u_k(x,t) = \sum_{i=1}^k a_{k,i}(t)\delta_i(x)$. *Where*

$a_{k,i} : [0;T) \to I\!R$ *are supposed to be continuous bounded functions. such that* u_k *satisfies the boundary condition in* $(QPS)_\omega$ *by constructions in the sense that* $u_k \in L^p(0,T,W_0^{1,p}(\Omega,\omega,I\!R^m))$, *we take core of the initial conditions in* $(QPS)_\omega$ *by choosing the initial coefficients:*

$a_{k,i}(0) = \langle u_0, \delta_i\rangle_{2,\omega_0}$ *such that:*

$u_k(.,0) = \sum_{i=1}^k a_{k,i}(0)\delta_i(.) \to u_0$. *with* $u_0 \in L^2(\Omega,\omega_0,I\!R^m)$.

we try to determine the coefficients $a_{k,i}(t)$ *in such that a way, that for every* $k \in I\!N$ *the system of ordinary differential equations*

$$\langle \partial_t u_k, \delta_j\rangle_{2,\omega_0} + \int_\Omega \sigma(x,t,u_k,Du_k) : D\delta_j dx$$
$$= \langle v(t), \delta_j\rangle + \int_\Omega f(x,t,u_k)\delta_j dx - \int_\Omega g(x,t,u_k) : D\delta_j dx \text{ for } j \in \{1;2;....;k\}.$$

Now we fix to $k \in I\!N$ *for the moment. Let* $0 < \varepsilon < T$ *and* $J = [0;\varepsilon]$. *Moreover we chose* $r > 0$ *large enough; such that the set* $B_r(0) \subset I\!R^k$ *contains the vector* $(a_{1k}(0), ..., a_{kk}(0))$, *and we set* $k = \overline{B_r(0)}$. *Then we have the following result.*

Lemma 7.6.2 *The application:* $F : J \times k \mapsto I\!R^k$ *given by:*

$$(t, a_1, ...a_k) \mapsto (\langle v(t), \delta_j\rangle + \int_\Omega f(x,t,\Sigma_{i=1}^k a_i\delta_i)\delta_j dx - \int_\Omega g(x,t,\Sigma_{i=1}^k a_i\delta_i) : D\delta_j dx$$
$$- \int_\Omega (\sigma(x,t,\Sigma_{i=1}^k a_i\delta_i, \Sigma_{i=1}^k a_i D\delta_i) : D\delta_j)dx)_{j=1;2;....k}$$

is a Carathéodory function.Moreover, each components F_j *may be estimated on* $J \times k$ *by*

$$| F_j(t, a_1, ...a_k) | \leq C(r,k).M(t) \tag{7.6.1}$$

uniformly on $J \times k$, *where* $C(r,k)$ *is a constant which depends on* r *and* k. *And here* M *is a functions dependent of* $j; k$ *and* r *belonging to* $L^1(J)$.

Proof of lemma 7.6.2

The first claim follows from the condition (P_1).

For the second claim; we note that the generalized Hölder inequality implies that:

$$| I_j | = | \langle v(t), \delta_j\rangle | \leq \| v(t) \|_{-1,p';\omega^*} \| \delta_j \|_{1,p;\omega}$$

Moreover, on the one hand, if; $II_j = \int_\Omega (\sigma(x,t,\Sigma_{i=1}^k a_i\delta_i, \Sigma_{i=1}^k a_i D\delta_i) : D\delta_j)dx$, *by the growth condition in* (P_2) *and the Hölder inequality*

$\mid II_j \mid \leq \beta[\parallel \lambda_1(t) \parallel_{p'} + c_1 \parallel \sum_{i=1}^k a_i\delta_j \parallel_{1,p,\gamma}^{\frac{q}{p'}} + c_2 \parallel \sum_{i=1}^k a_i D\delta_j u \parallel_{1,p,\omega}^{p-1}] \parallel D\delta_j \parallel_{1,p,\omega}$

$\leq \beta \max(1, c_1, c_2).[\parallel \lambda_1 \parallel_{p'} + \parallel \sum_{i=1}^k a_i\delta_j \parallel_{1,p,\gamma}^{\frac{q}{p'}} + \parallel \sum_{i=1}^k a_i D\delta_j u \parallel_{1,p,\omega}^{p-1}] \parallel D\delta_j \parallel_{1,p,\omega}$.

And by Hölder inequality we have:

$\mid III_j \mid = \mid \int_\Omega g(x,t,\Sigma_{i=1}^k a_i\delta_i) : D\delta_j dx \mid$

$\leq (\parallel b_4(t) \parallel_{p'} + \parallel b_5(t) \parallel_{\frac{n}{p-1}} \parallel \Sigma_{i=1}^k a_i\delta_i \parallel_{p^*,\omega_0}^{\eta}) \parallel D\delta_j \parallel_{p,\omega}$

$\mid IV_j \mid = \mid \int_\Omega f(x,t,\Sigma_{i=1}^k a_i\delta_i) : \delta_j dx \mid$

$\leq \parallel b_1(t) \parallel_{p'} \parallel \delta_j \parallel_{p,\omega_0} + \parallel b_2(t) \parallel_{\frac{n}{p}} \parallel \Sigma_{i=1}^k a_i\delta_i \parallel_{p^*,\omega_0}^{\gamma} \parallel \delta_j \parallel_{p,\omega_0}$ *Then, in view of*

$\parallel \delta_i \parallel_{1,\omega,p} + \parallel \delta_i \parallel_{1,p^*,\omega} \leq c$ *for all* $i \in \mathbb{N}$ *and since each variable* a_i *satisfies:* $\mid a_i \mid \leq r$; *for all* $i = 1; .....; k$ *fora constant* r, *each component* F_j *may be estimated in such a way that*

$$\mid F_j(t, a_1, ...a_k) \mid \leq \mid II_j \mid + \mid I_j \mid + \mid III_j \mid + \mid IV_j \mid \leq C(r,k).M(t) \tag{7.6.2}$$

uniformly on $J \times K$, *where* $C(r,k)$ *is a constant which depends on* r *and* k, *and where* $M(t) \in L^1(J)$ *(independent of* j, k*)and* r *Then, in view of lemma 7.6.2 the Carathéodory existence result on ordinary differential equation applied to the system:*

$$a_j^{'}(t) = F_j(t, a_1(t), ..., a_k(t)) \tag{7.6.3}$$

$$a_j(0) = a_{k,j}(0); j \in 1; ...; k \tag{7.6.4}$$

ensures existence of a distributional, continuous solutions a_j *(depending on* k*) of (7.6.3) - (7.6.4) on a time interval* $[0; \varepsilon'[$ *where* $\varepsilon' > 0$, *a priori, may depend on* k. *Moreover, the corresponding integral equation*

$$a_j(t) = a_j(0) + \int_0^t F_j(\tau, a_1(\tau), ...a_k(\tau))d\tau \tag{7.6.5}$$

holds on $[0; \varepsilon'[$. *Then,* $u_k = \sum_{j=1}^k a_j(t)\delta_j$ *is the desired (short time) solution of*

$\langle \partial_t u_k, \delta_k \rangle_{2,\omega_0} + \int_\Omega \sigma(x,t,u_k,Du_k) : D\delta_k dx$

$= \int_\Omega v(x,t)\delta_k dx + \int_\Omega f(x,t,u_k).u_k dx - \int_\Omega g(x,t,u_k) : Du_k dx$ *with initial condition*

$u_k(0) = \sum_{i=1}^k a_{k,i}(0)\delta_i(.) \to u_0$, *with* $u_0 \in L^2(\Omega, \omega_0, \mathbb{R}^m)$. *Now, we want to show that the local solution constructed above can be extended to the whole interval* $[0; T[$ *independent of* k, *as a - word of warning we should mention, that the solution need not be unique.*

Lemma 7.6.3 *We have:*

$A = \left\{ t \in [0; T[: \ there\ exist\ a\ weak\ solutions\ of\ 7.6.3 - 7.6.4\ on\ [0; T[\right\} = [0; T[.$

Proof of lemma 7.6.3

The first thing we want to establish is a uniform bound on the coefficients $(a_{ki}(t))$. *Since*

$$\langle \partial_t u_k, \delta_k \rangle_{2,\omega_0} + \int_\Omega \sigma(x,t,u_k,Du_k) : D\delta_k dx$$
$$= \int_\Omega v(x,t)\delta_k dx + \int_\Omega f(x,t,u_k).u_k dx - \int_\Omega g(x,t,u_k) : Du_k dx.$$

is linear in δ_k, *it is allowed to use* u_k *as a test function in precedent equation in place of* δ_k. *This gives for a an arbitrary time* τ *in the existence interval*

$$\int_0^\tau \langle \partial_t u_k, u_k \rangle_{2,\omega_0} dt + \int_0^\tau \int_\Omega \sigma(x,t,u_k,Du_k) : Du_k dx dt$$
$$= \int_0^\tau \langle v(t), u_k \rangle dt + \int_0^\tau \int_\Omega f(x,t,u_k).u_k dx dt - \int_0^\tau \int_\Omega g(x,t,u_k) : Du_k dx dt.$$

For the first term we have:

$$\int_0^\tau \langle \partial_t u_k, u_k \rangle_{2,\omega_0} dt = \sum_j \frac{1}{2} \int_0^\tau \partial_t (u_k^j)^2 \omega_{0j} dt$$
$$= \sum_j \frac{1}{2} \parallel u_k^j(\tau) \parallel^2_{2,\omega_0} - \sum_j \frac{1}{2} \parallel u_k^j(0) \parallel^2_{2,\omega_0} \geq \frac{1}{2} \parallel u_k(\tau) \parallel^2_{2,\omega_0} - c$$

Since $u_k(.,0)$ *converges in* $L^2(\Omega,\omega_0,I\!R^m)$ *. Using the coercivity and the Hölder inequality in* (P_2) *for the second term, we obtain:*

$$\int_0^\tau \int_\Omega \sigma(x,t,u_k,Du_k) : Du_k dx dt \geq - \parallel \lambda_2 \parallel_{L^1(\Omega_T)} - A'^\alpha \parallel \lambda_3 \parallel_{L^{(\frac{p}{\alpha})'}(\Omega_T)} \parallel u_k \parallel^\alpha_{L^p(0,T,W_0^{1,p}(\Omega,\omega,I\!R^m))}$$
$$+ c_2 \parallel u_k \parallel^p_{L^p(0,T,W_0^{1,p}(\Omega,\omega,I\!R^m))}.$$

By the the Hölder inequality for the third term we get

$$\int_0^\tau \langle v(t), u_k \rangle dt \leq \parallel v \parallel_{L^{p'}(0,T,W^{-1,p'}(\Omega,\omega))} \parallel u_k \parallel_{L^p(0,T,W_0^{1,p}(\Omega,\omega))}$$ *. Using the growth condition*(F_1) *and the Hölder inequality for the fourth term, we obtain:*

$$\int_0^\tau \int_\Omega f(x,t,u_k).u_k dx dt$$
$$\leq \parallel b_1(t) \parallel_{L^{p'}(\Omega_T)} . \parallel u_k \parallel_{L^p(0,\tau,L^p(\Omega,\omega_0))} + \int_0^\tau \parallel b_2(t) \parallel_{\frac{n}{p}} . \parallel u_k \parallel^{\gamma+1}_{p^*,\omega_0} dt.$$
$$\leq A' \parallel b_1 \parallel_{L^{p'}(\Omega_T)} . \parallel u_k \parallel_{L^p(0,\tau,W_0^{1,p}(\Omega,\omega))} + A'^{\gamma+1} \parallel b_2 \parallel_{L^1(0,T,L^{\frac{n}{p}(\Omega))}} . \parallel u_k \parallel^{\gamma+1}_{L^p(0,\tau,W_0^{1,p}(\Omega,\omega))} .$$

Finally, using the growth condition (G_1) *for the last term, we obtain:*

$$\int_0^\tau \int_\Omega g(x,t,u_k).Du_k dx dt$$
$$\leq \parallel b_4(t) \parallel_{L^{p'}(\Omega_T)} . \parallel Du_k \parallel_{L^p(0,\tau,L^p(\Omega,\omega))} + \int_0^\tau \parallel b_5(t) \parallel_{\frac{n}{p-1}} . \parallel u_k \parallel^\eta_{p^*,\omega_0} . \parallel Du_k \parallel_{p,\omega} dt$$
$$\leq A'. \parallel b_4 \parallel_{L^{p'}(\Omega_T)} . \parallel u_k \parallel_{L^p(0,\tau,W_0^{1,p}(\Omega,\omega))} + A'^\eta \parallel b_5 \parallel_{L^\infty(0,\tau,L^{\frac{n}{p-1}}(\Omega))} \parallel u_k \parallel^{\eta+1}_{L^p(0,\tau,W_0^{1,p}(\Omega,\omega))} .$$

The combination of these five estimates gives:

$$\tfrac{1}{2} \parallel u_k(\tau) \parallel_2^2 \leq A'(\parallel b_4 \parallel_{L^{p'}(\Omega_T)} + \parallel b_1 \parallel_{L^{p'}(\Omega_T)} + \parallel v \parallel_{L^{p'}(0,T,W^{-1,p'}(\Omega,\omega))}) \parallel u_k \parallel_{L^p(0,\tau,W_0^{1,p}(\Omega,\omega))} + (\parallel$$
$$\lambda_3 \parallel_{L^{(\frac{p}{\alpha})'}(\Omega_T)} + A'^{\gamma+1} \parallel b_2 \parallel_{L^{\gamma\mp}(0,T,L^{\frac{n}{p}(\Omega))}} + A'^\eta \parallel b_5 \parallel_{L^\infty(0,\tau,L^{\frac{n}{p-1}}(\Omega))}) \parallel u_k \parallel^{\kappa'}_{L^p(0,T,W_0^{1,p}(\Omega,\omega,I\!R^m))} + c + \parallel$$
$$\lambda_2 \parallel_{L^1(\Omega_T)} - c_2 \parallel u_k \parallel^p_{L^p(0,T,W_0^{1,p}(\Omega,\omega,I\!R^m))}$$

$\equiv \psi(\| u_k \|_{L^p(0,\tau,W_0^{1,p}(\Omega,\omega,I\!R^m))})$

where $\kappa' = \max(\alpha, \gamma+1, \eta+1)$ *Since* $p > 1$, *and* $c_2 > 0$ *we have:*

$$\psi(y) \to -\infty; y \to +\infty \tag{7.6.6}$$

and so $\psi(\| u_k \|_{L^p(0,T,W_0^{1,p}(\Omega,\omega,I\!R^m))})$ *is bounded, since* ψ *is continuous and* $\| u_k \|_2^2 \geq 0$.
Thus, in view of the definition of $a_{k,i}$ *we may conclude that:*

$$| a_{k,i}(\tau) |_{I\!R^k}^2 = \| u_k(\tau) \|_2^2 \leq \bar{c} \tag{7.6.7}$$

for a constant $\bar{c}$ *which is independent of* τ *and* k. *Now, we prove that* A *is non empty, open and closed in* $[0;T]$. *First,* A *is clearly non- empty since we proved local existence above. Moreover, let* $t \in A$ *and* $0 < \tau_1 < \tau_2 < T$. *Then by (7.6.5), we have:*
$| a_{k,j}(\tau_1) - a_{k,j}(\tau_2) | \leq \int_{\tau_1}^{\tau_2} | F_j(\tau, a_{k,i}(\tau), ..., a_{k,k}(\tau)) | d\tau$
$\leq C(\bar{c}, k) \int_{\tau_1}^{\tau_2} | M(\tau) | d\tau;$
since $M \in L^1(0,T)$, *this implies that* $\tau \mapsto a_{k,j}(\tau)$ *is uniformly continuous.Then, we can restart to solve*
$\langle \partial_t u_k, \delta_k \rangle_{2,\omega_0} + \int_\Omega \sigma(x,t,u_k,Du_k) : D\delta_k dx$
$= \int_\Omega v(x,t)\delta_k dx + \int_\Omega f(x,t,u_k).\delta_k dx - \int_\Omega g(x,t,u_k) : D\delta_k dx$
for $k \in \{1;2;....;m\}$.
At time t *with initial data* $\lim_{\tau \nearrow \longrightarrow T} u_k(\tau)$ *and hence get a solution of (7.6.3) - (7.6.4) on* $[0, \tau+\varepsilon]$.*This proves that* A *is open. Finally, we consider a sequence* $\tau_i \nearrow t; \tau_i \in A$. *Let* $a_{k,j,i}$ *denote the solution of (7.6.3) - (7.6.4) we constructed on* $[0,\tau_i]$ *and define:*

$$\tilde{a_{k,j,i}} \equiv \begin{cases} a_{k,j,i}(\tau) & \text{if} \quad \tau \in [0;\tau_i] \\ a_{k,j,i}(\tau_i) & \text{if } \tau \in]\tau_i; t]. \end{cases}$$

The sequence $(\tilde{a_{k,j,i}})_i$ *is bounded and equi-continuous on* $[0;t[$, *as seen above. Hence, by the Arsela-Ascoli Theorem, a subsequence (again denoted by* $\tilde{a_{k,j,i}}(\tau)$ *converges uniformly in* τ *on* $[0;t[$ *to a continuous functions* $a_{k,j}(\tau)$*). Using Lebesgues convergence theorem in (7.6.5) it is now easy to see that* $a_{k,j}(\tau)$ *solves (7.6.3) on* $[0;t[$. *Hence* $t \in A$ *and thus* A *is indeed closed, and as claimed, it follows* $A = [0;T[$. $\square$

Proposition 7.6.1 *The sequence of the Galerkin approximations* (u_k) *is bounded in* $L^\infty(0,T,L^2(\Omega,\omega_0,I\!R^m)) \cap L^p(0,T,W_0^{1,p}(\Omega,\omega,I\!R^m)))$ *and satisfies* (A_1) *and* (A_2).

Proof of proposition7.6.1 *By testing equation:*

$\langle \partial_t u_k, \delta_j \rangle_{2,\omega_0} + \int_\Omega \sigma(x,t,u_k,Du_k) : D\delta_j dx$

$= \langle v(t), \delta_j \rangle + \int_\Omega f(x,t,u_k).\delta_j dx - \int_\Omega g(x,t,u_k) : D\delta_j dx$

for:$j \in \{1,2,...,k\}$.

With u_k*; we obtained in the proof of lemma (7.6.3) that (7.6.7) holds for arbitrary time*$\tau \in [0;T[$. *So* (u_k) *is bounded in the space* $L^\infty(0,T,L^2(\Omega,I\!R^m))$. *Moreover we have proved that*

$\psi(\parallel u_k \parallel_{L^p(0,T,W_0^{1,p}(\Omega,\omega,I\!R^m))})$ *is positive, bounded and satisfies 7.6.6. Then there exists a constant* $c > 0$ *such that*

$$\parallel u_k \parallel_{L^p(0,T,W_0^{1,p}(\Omega,\omega,I\!R^m))} \leq c; \quad \forall k \in I\!N. \tag{7.6.8}$$

Therefore, by extracting a suitable subsequence which is again denoted by u_k*, we may assume by the Elberlein-Smuljan see e.g, [62] (Page 141), that*

$u_k \rightharpoonup^* u$ *in* $L^\infty(0,T,L^2(\Omega,\omega_0,I\!R^m))$

$u_k \rightharpoonup u$ *in* $L^p(0,T,W_0^{1,p}(\Omega,\omega,I\!R^m))$

which proves in particular (A_1). *To verify* (A_2), *observe that the time derivative*

$\frac{d}{dt}(j \circ i \circ u_k) : [0;T[\to B_1 = (W_0^{s,2}(\Omega,\omega_\alpha,I\!R^m)))'$ *is according to*

$\langle \partial_t u_k, \delta_j \rangle_{2,\omega_0} + \int_\Omega \sigma(x,t,u_k,Du_k) : D\delta_j dx$

$= \langle v(t), \delta_j \rangle + \int_\Omega f(x,t,u_k).\delta_j dx - \int_\Omega g(x,t,u_k) : D\delta_j dx$

for: $j \in \{1;2;....;k\}$. *Given by: we recall that the projection operators* P_k *are self adjoint with respect to the* L^2 *inner product.*

$t \mapsto (\phi \mapsto \langle v(t), P_k\phi \rangle - \int_\Omega \sigma(x,t,u_k,Du_k) : D(P_k\phi)dx + \int_\Omega f(x,t,u_k).(P_k\phi)dx - \int_\Omega g(x,t,u_k) : D(P_k\phi)dx)$. *Now we have:*

$\langle \frac{d}{dt}(\jmath \circ \imath \circ u_k)(t), \phi \rangle = I + II + III + IV$ *where*

$I = \int_0^T \langle v(t), P_k\phi \rangle dt \leq \parallel v \parallel_{L^{p'}(0,T,(W^{-1,p'}(\Omega,\omega^*)))} \parallel P_k\phi \parallel_{L^p(0,T,(W^{1,p}(\Omega,\omega)))}$ *by the generalized Hölder inequality,*

$II = -\int_0^T \int_\Omega \sigma(x,t,u_k,Du_k) : D(P_k\phi)dxdt$

$\leq (\parallel \lambda_1 \parallel_{L^{p'}(\Omega_T)} + C(c_1) \parallel u_k \parallel^{p-1}_{L^p(0,T,W_0^{1,p}(\Omega,\omega))}) \parallel P_k\phi \parallel_{L^p(0,T,W_0^{1,p}(\Omega,\omega))}$ *by the growth conditions in* (P_2) *and the Hölder inequality.* $III = \int_0^T \int_\Omega f(x,t,u_k).(P_k\phi)dxdt$

$\leq (\parallel b_1 \parallel_{L^{p'}(\Omega_T)} + A'^{\gamma+1} \parallel b_2 \parallel_{L^{\gamma^\mp}(0,T,L^{\frac{n}{p}(\Omega))}} . \parallel u_k \parallel^\gamma_{L^p(0,\tau,W_0^{1,p}(\Omega,\omega))}) . \parallel P_k\phi \parallel_{L^p(0,\tau,W_0^{1,p}(\Omega,\omega))}$.

By the growth condition(F_1) $IV = -\int_0^\tau \int_\Omega g(x,t,u_k).D(P_k\phi)dxdt$

$\leq (\| b_4 \|_{L^{p'}(\Omega_T)} + A'^{\eta} \| b_5 \|_{L^{\infty}(0,\tau,L^{\frac{n}{p-1}}(\Omega))} . \| u_k \|^{\eta}_{L^p(0,\tau,W_0^{1,p}(\Omega,\omega))}). \| P_k\phi \|_{L^p(0,\tau,W_0^{1,p}(\Omega,\omega))}$.

By the growth condition (G_1)

Since

$\| P_k\phi \|_{L^p(0,T,W_0^{1,p}(\Omega,\omega))} \leq \| P_k\phi \|_{L^p(0,T,W_0^{r,2}(\Omega,\omega_\alpha))} \leq \| \phi \|_{L^p(0,T,W_0^{r,2}(\Omega,\omega))}$,

the claim follows by the uniform bounded (7.6.8). □

Remark 7.6.1 *For further use, we note that from the bounded obtained for* $\langle \frac{d}{dt}(\jmath \circ \imath \circ u_k)(t), \phi \rangle$ *in the proof of proposition we can conclude that* $\partial_t u$ *(or rather* $\partial_t(\jmath \circ \imath \circ u)$*) is an element of the space* $L^{p'}(0,T,(W_0^{-1,p'}(\Omega,\omega^*))$. *This follows easily from the fact, that the set*

$\{\phi \in L^p(0,T,W_0^{1,p}(\Omega,\omega)) : \exists k \in I\!N / P_k\phi = \phi\}$ *is dense in* $L^p(0,T,W_0^{1,p}(\Omega,\omega))$, *as proved in [37] (Appendix II, with slight assimilation)* □

Remark 7.6.2 *Recall that the space*

$\{u \in L^p(0,T,W_0^{1,p}(\Omega,\omega)) : \partial_t(\jmath \circ \imath \circ u) \in L^{p'}(0,T,(W^{-1,p'}(\Omega,\omega^*)$ *is continuously embedded in* $C([0;T];L^2(\Omega,\omega_0))$ *see e-g [65] [proposition23-23], with slight assimilation.*

Hence, we have that $u \in C^0([0;T];L^2(\Omega,\omega_0))$ *after possible modification of* u *on a Lebesgue Zero-set of* $[0;T]$. *This gives* $u(t,.) \in L^2(\Omega,\omega_0)$ *a pointiest interpretation for all* $t \in [0;T]$ *and allows in particular to state that* $u(t,.)$ *attains its initial value*

$$u(.,0) = u_0 \tag{7.6.9}$$

continuously in $L^2(\Omega,\omega_0)$ *see [37] (Appendix III). In the two following subsections, we prove that* u *is really a solution of the system* $(QPS)_\omega$. □

7.7 Passage to the limit in $(\mathcal{QPS})_\omega$

In this subsection, we prove theorem (7.2.1). First, in order to apply proposition (7.6.1), we verify that, under our assumptions, the conditions $(A_1)-(A_6)$ *and* $(N_1)-(N_3)$ *holds for the Galerkin approximations solution* u_k. (A_1) *and* (A_2) *holds by proposition (7.6.1). Moreover, it follows then by lemma (7.6.1) that* $(N_1)-(N_3)$ *holds. The condition* (A_3) *is equivalent to* (P_1). *Next, to prove* (A_4), *we note that by the growth condition in* (P_2) *that*

$\int_{\Omega_T} | \sigma_{rs}(x,t,u_k,Du_k) |^{p'} \omega^*_{rs} dxdt$

$\leq C.(\int_{\Omega_T} \mid \lambda_1 \mid^{p'} dxdt + \sum_{j=1}^{m} \mid u_{kj} \mid^q \gamma_j dxdt + \sum_{i,j} \mid D_{ij}u_k \mid^p \omega_{ij} dxdt);$

with (c is a constants) which is uniformly bounded in k by (7.6.8). To verify the equi-integrability of $(\sigma_k : Du_k)^-$, we fix an arbitrary measurable subset $\Omega' \subset \Omega_T$. Then, on the one hand, the growth condition in P_2 implies that

$\int_{\Omega'} \mid \min(\sigma_k : Du_k; 0) \mid dxdt \leq \int_{\Omega'} \mid \lambda_2 \mid dxdt + \int_{\Omega'} \mid \lambda_3 \mid \sum_{j=1}^{m} \omega_{1j}^{\frac{\alpha}{p}} \mid u_{kj} \mid^{\alpha}.$

by the Hölder inequality and (7.6.8). Thus (A_5) follows. Finally, we prove that (A_6) hold. According to Mazur's Theorem see e-g [62] (Theorem2, page 120),there exists a sequence v_k in $L^p(0,T,W_0^{1,p}(\Omega,\omega))$ where each v_k is a convex linear combination of $u_1,...,u_k$ such that $v_k \to u$ in $L^p(0,T,W_0^{1,p}(\Omega,\omega))$. In particular, $v_k(t,.) \in span(\delta_1,\delta_2,...,\delta_k)$ for all $t \in [0,T]$. Now, we have

$$\begin{aligned}\int_{\Omega_T} \sigma_k : (Du_k - Dv_k)dxdt &= \langle v, u_k - v_k\rangle - \sum_j \int_{\Omega_T} (u_k^j - v_k^j)\partial_t(u_k^j)\omega_{0j}dxdt \\ &+ \int_{\Omega_T} f(x,t,u_k).(u_k - v_k)dxdt - \int_{\Omega_T} g(x,t,u_k) : (Du_k - Dv_k)dxdt.\end{aligned} \tag{7.7.1}$$

Observe that $(u_k - v_k) \in span(\delta_1,\delta_2,...,\delta_k)$ which allowed to use

$\langle \partial_t u_k, \delta_k\rangle_2 + \int_\Omega \sigma(x,t,u_k,Du_k) : D\delta_k dx$

$= \int_\Omega v(x,t).\delta_k dx + \int_\Omega f(x,t,u_k).\delta_k dxdt - \int_\Omega g(x,t,u_k) : D\delta_k dxdt.$ *in the equality above. The term $\langle v, u_k - v_k\rangle$ in (7.7.1) converge to zero, since $u_k - v_k \rightharpoonup 0$ in $L^p(0,T,W_0^{1,p}(\Omega,\omega))$ by the choice of v_k and (A_1).*

Next, defining $II_k = \int_{\Omega_T} f(x,t,u_k).(u_k - v_k)dxdt$, we infer by the Hölder inequality and the growth condition (F_1) that

$|II_k| \leq \int_0^T (\| b_1(t) \|_{p'} \| u_k(t) - v_k(t) \|_{p,\omega_0} + \| b_2(t) \|_{\frac{n}{p}} \| u_k(t) \|_{p^*,\omega_0} \| u_k(t) - v_k(t) \|_{\frac{p^*}{p-\gamma}})dt$

$\leq \| b_1 \|_{L^{p'}(\Omega_T)} \| u_k - v_k \|_{L^p(0,T,L^p(\Omega,\omega_0))}$

$+ \| b_2 \|_{L^{\gamma^\mp}(0,T,L^{\frac{n}{p}})} \| u_k \|^{\gamma}_{L^p(0,T,L^{p^*}(\Omega,\omega))} \| u_k - v_k \|_{L^p(0,T,L^{\frac{p^*}{p-\gamma}}(\Omega,\omega))}.$

Since the expression $\| u_k(t) \|_{p^*,\omega_0}$ is bounded and we have $u_k - v_k \to 0$ in $L^p(0,T,L^r(\Omega,\omega_0))$ for all $r < p^*$ by the proposition (7.3.1), the second term in (7.7.1) vanishes in the limit. For the third term in (7.7.1), we note that $g_k(x,t) = g(x,t,u_k) \to g(x,t,u)$ strongly in $L^{p'}(\Omega_T)$ by $(G_0),(G_1)$ and proposition 7.3.1. To see this, we may assume by proposition (7.3.1) that $u_k \to u$ almost everywhere. Since by (G_1), $\mid g_k \mid^{p'}$ is bounded by an integrals function, the claim follows from (G_0) and the Dominated Convergence Theorem. We infer then that

$| \int_{\Omega_T} g_k : (Dv_k - Du_k)dxdt | \leq \| g_k - g(.,.,u) \|_{L^{p'}(\Omega_T)} \| u_k - v_k \|_{L^p(0,T,W_0^{1,p}(\Omega,\omega))}$

$+ \mid \int_{\Omega_T} g : D(v_k - u_k)dxdt \mid \to 0,$

since $v_k - u_k \rightharpoonup 0$ *in* $L^p(0,T,W_0^{1,p}(\Omega,\omega))$. *Thus the third term in (7.7.1) disappears also. Finally,For the last therm in (7.7.1), we have:*

$$-\sum_j \int_{\Omega_T} (u_k^j - v_k^j)\partial_t(u_k^j)\omega_{0j}dxdt = -\sum_j \int_{\Omega_T} \frac{1}{2}\partial_t(u_k^j)^2\omega_{0j}dxdt + \sum_j \int_{\Omega_T} v_k^j \partial_t(u_k)\omega_{0j}dxdt$$
$$= -\tfrac{1}{2} \parallel u_k(T) \parallel^2_{2,\omega_0} + \tfrac{1}{2} \parallel u_k(0) \parallel^2_{2,\omega_0} + \textstyle\sum_j \int_{\Omega_T} v_k^j \partial_t(u_k^j)\omega_{0j}dxdt. \tag{7.7.2}$$

Concerning the last term in (7.7.1) we claim that for $k \to \infty$ *we have:*

$$\sum_j \int_{\Omega_T} v_k^j \partial_t(u_k^j)\omega_{0j}dxdt \to \sum_j \int_{\Omega_T} u^j \partial(u^j)\omega_{0j}dxdt = \frac{1}{2} \parallel u(T) \parallel^2_{2,\omega_0} - \frac{1}{2} \parallel u_0 \parallel^2_{2,\omega_0} \tag{7.7.3}$$

To see this, let $\varepsilon > 0$ *be given. Then, there exists* M *such that for all* $l \geq m \geq M$ *we have:*

- $(i) \sum_j \mid \int_{\Omega_T} (u^j - v_m^j)\partial_t(u^j)\omega_{0j}dxdt \mid \leq \varepsilon$. *This is possible, since* $\partial(j\circ i\circ u) \in L^{p'}(0,T,W_0^{-1,p'}(\Omega,\omega)$ *and* $v_m \to u \;\; in \;\; L^p(0,T,W_0^{1,p}(\Omega,\omega))$
- $(ii) \sum_j \mid \int_{\Omega_T} (v_l^j - v_m^j)\partial_t(u^j)\omega_{0j}dxdt \mid \leq \varepsilon$. *This is possible by the bound obtained for* $\langle \frac{d}{dt}(j \circ i \circ u_k)(t), \phi\rangle$ *in the proof of proposition (7.6.1) since* $(v_l - v_k) \in span(\delta_1, \delta_2.....,\delta_k)$ *for all fixed* $t \in]0,T[$.

Now, we fix $m \geq M$ *and choose* $m_0 \geq m$ *such that for all* $l \geq m_0$;
$\sum_j \mid \int_{\Omega_T} v_m^j(\partial_t(u^j - u_l^j)\omega_{0j}dxdt \mid \leq \varepsilon$. *This is possible, since* $\partial_t u_l \rightharpoonup^* \partial_t u$ *in* $L^{p'}(0,T,(W_0^{r,2}(\Omega,\omega_\alpha))')$.
combination yields for all $l = l(\varepsilon) \geq m_0(\varepsilon)$
$\mid \sum_j \int_{\Omega_T} v_l^j \partial_t(u_l^j)\omega_{0j}dxdt - \sum_j \int_{\Omega_T} u^j \partial_t(u^j)\omega_{0j}dxdt \mid$
$\leq \sum_j \mid \int_{\Omega_T} (v_l^j - v_m^j)\partial_t(u_l^j)\omega_{0j} \mid dxdt + \sum_j \mid \int_{\Omega_T} v_m^j(\partial_t u_l^j - \partial_t u^j)\omega_{0j}dxdt \mid$
$+ \sum_j \mid \int_{\Omega_T} (v_m^j - u^j)\partial_t u^j \omega_{0j}dxdt$
$\leq 3\varepsilon$. *This establishes (7.7.3). on the other hand, since* $(u_k)_k$ *is bounded in* $L^\infty(0,T,L^2(\Omega,\omega_0))$ *by proposition (7.6.1) we have that* $(u_k(T) \rightharpoonup u(T))$ *in* $L^2(\Omega,\omega_0))$. *Hence,*

$$\liminf_{k\to\infty} \parallel u_k(T) \parallel_{2,\omega_0} \geq \parallel u(T) \parallel_{2,\omega_0} . \tag{7.7.4}$$

By construction of u_k *we also have*

$$\lim_{k\to\infty} \parallel u_k(0) \parallel_{2,\omega_0)} = \parallel u_0 \parallel_{2,\omega_0} . \tag{7.7.5}$$

Using (7.7.5), (7.7.4) and (7.7.3), in (7.7.2), we conclude that
$\liminf_{k\to\infty}(-\sum_j \int_{\Omega_T}(u_k^j - v_k^j)\partial_t(u_k^j)\omega_0)dxdt) \leq 0$*, thus* (A_6) *is fulfilled.*
So, $(A_1)-(A_6)$ *and* $(N_1)-(N_3)$ *hold and we may infer from proposition 7.6.1 that* $\sigma_k(x,t,u_k,Du_k)$ *converge to* $\sigma_k(x,t,u,Du)$ *at least weakly in* $L^1(\Omega_T,\mathbb{M}^{m\times n})$*, for* $k\to\infty$*. Since we have that* $u_k\to u$ *in measure on* Ω_T *by proposition 7.3.1, we may infer that* $u_k\to u$ *almost everywhere on* Ω_T*, for* $k\to\infty$*.*
Thus, for arbitrary $\varphi\in W_0^{1,p}(\Omega,\omega,\mathbb{R}^m)$*, it follows from the continuity conditions* (F_0) *and* (G_0) *that* $f(x,t,u_k).\varphi(x)\to f(x,t,u).\varphi(x)$ *and* $g(x,t,u_k).D\varphi(x)\to g(x,t,u):D\varphi(x)$ *almost everywhere on* Ω_T*. Since, by the growth conditions in* (F_1) *and* (G_1) *and the uniform bound in proposition 7.6.1,* $f(x,t,u_k).\varphi(x)$ *and* $g(x,t,u_k).D\varphi(x)$ *are equi-integrable, it follows that*
$f(x,t,u_k).\varphi(x)\to f(x,t,u).\varphi(x)$ *and* $g(x,t,u_k).D\varphi(x)\to g(x,t,u):D\varphi(x)$ *in* $L^1(\Omega_T)$ *by Vitali converge theorem.*
Now, we take a test function $\varphi\in\cup_{i\in\mathbb{N}}span(\delta_1,\delta_2.....,\delta_i)$ *and* $\phi\in C_0^\infty([0,T])$ *in*
$\langle\partial_t u_k,\varphi(x)\rangle_{2,\omega_0}+\int_\Omega\sigma(x,t,u_k,Du_k):D\delta_j dx=\langle v(t),\delta_j\rangle+\int_\Omega f(x,t,u_k).\delta_j dx+\int_\Omega g(x,t,u_k):D\delta_j dx$
for:$j\in\{1,2,...,k\}$.
And integrate over the interval $]0,T[$ *and pass to the limit* $k\to\infty$*. The resulting equation is*
$\sum_j\int_{\Omega_T}\partial_t u^j(x,t)\phi^j(t)\varphi(x)\omega_{0j})dxdt+\int_{\Omega_T}\sigma_k(x,t,u,Du):D\varphi(x)\phi(t)dxdt$
$=\langle v,\phi\varphi\rangle+\int_{\Omega_T}f(x,t,u).\varphi(x)\phi(t)dxdt+\int_{\Omega_T}g(x,t,u):D\varphi(x)\phi(t)dxdt$
For arbitrary $\varphi\in\cup_{i\in\mathbb{N}}span(\delta_1,\delta_2,...,\delta_i)$ *and* $\phi\in C_0^\infty([0,T])$*. By density of the linear span of these functions in* $L^p(0,T,W_0^{1,p}(\Omega,\omega))$*, this proves that* u *is in fact a weak solution of* $(QPS)_\omega$*. Hence the theorem is proved.* $\square$

Chapter 8

Existence of solutions for some quasi-linear parabolic systems with weight and weak monotonicity general data

Abstract

We prove the existence of weak solution u for the nonlinear parabolic systems:

$$(\mathcal{QPS})_{(\omega)}(f,g)\begin{cases} \partial_t u - div\sigma(x,t,u,Du) &= v(x,t) + f(x,t,u,Du) + divg(x,t,u) \ \textit{in} \quad \Omega_T \\ u(x,t) &= 0 \quad \textit{on} \quad \partial\Omega\times(0,T) \\ u(x,0) &= u_0(x) \quad \textit{on} \quad \Omega \end{cases}$$

which is a Dirichlet Problem. In this system, v belongs to $L^{p'}(0,T,W^{-1,p'}(\Omega,\omega^*,I\!R^m))$; and $u_0\in L^2(\Omega,\omega_0,I\!R^m)$, f and g satisfy some standards continuity and growth conditions. We prove existence of a weak solution of different variants of this system under classical regularity for some $p\in]\frac{2n(s+1)}{(n+2)s}\ ;\ \infty[$, growth and coercivity for σ but with only very mild monotonicity assumptions.

8.1 functional framework

Definition 8.1.1 *: Let X be a Banach space and $0<T<\infty$, the space $L^p(0,T,X)$ with $1\le p\le\infty$ consists of all measurable functions (identified modulo the relation "equal almost everywhere") $u:(0;T)\to X$ for which*

$\|u\|_{L^p(0,T,X)} = \left(\int_0^T \|u(t)\|_X^p dt \right)^{\frac{1}{p}}$, *if* $p \geq 1$;
$\|u\|_{L^p(0,T,X)} = ess \sup_{t\in(0;T)} \|u(t)\|_X$, *if* $p = \infty$ *holds.*

Again, equipped with the norm $\|.\|_{L^p(0,T,X)}$; $L^p(0,T,X)$ *is a Banach space.*

In the special case where $p = 2$ *and* X *is a Hilbert space* $L^2(0,T,X)$ *equipped with the inner product* $\langle u, v\rangle = \int_0^T \langle u(t), v(t)\rangle_X dt$; *for all* $u; v \in L^2(0,T,X)$ *is a Hilbert space, Moreover; if* $1 \leq q \leq \infty$ *the following Hölder inequality holds for all* $v \in L^{p'}(0,T,X')$ *and all* $u \in L^p(0,T,X)$.
$\int_0^T | \langle u(t), v(t)\rangle | dt \leq \| v \|_{L^{p'}(0,T,X')} . \| u \|_{L^p(0,T,X)}$
and $L^q(0,T,X) \hookrightarrow L^p(0,T,Y)$, *for all* X, Y *such that* $X \hookrightarrow Y$.

$C^k([0;T],X)$ *are dense in* $L^p(0,T,X)$ *as* $p \leq \infty$.

$C^k([0;T],X) \hookrightarrow L^p(0,T,X)$ *is continuous.*

If $1 \leq p < \infty$ *and* X *is separable; so is* $L^p(0,T,X)$.

If $1 < p < \infty$, $L^p(0,T,X)$ *is uniformly convex in the case where* X *is uniformly convex, and, reflexive, separable when* X *is reflexive and separable Banach space.*

Definition 8.1.2 *(Lebesgues and Sobolev spaces with weight):*

For $1 \leq p \leq \infty$ *and* $k \in \mathbb{N}$; *we consider the weight spaces:*

$$L^p(\Omega, \omega_0, \mathbb{R}^m) \equiv \left\{ u = u(x);\ u_j \omega_{0j}^{\frac{1}{p}} \in L^p(\Omega, \mathbb{R}^m) \right\}$$

with, $\omega_0 = (\omega_{0j})_{1\leq j\leq m}$.

$$W^{k,p}(\Omega, \omega_\alpha, \mathbb{R}^m) \equiv \left\{ u \in L^p(\Omega, \omega_\alpha, \mathbb{R}^m);\ \ D^\alpha u \in L^p(\Omega, \omega_\alpha, \mathbb{R}^m);\ \ for\ \ 0 \leq | \alpha | \leq k \right\}.$$

$D^\alpha u$ *is the weak derivative of* u. *The weighted functions* ω_α *defined by:*

$$\omega_\alpha = \begin{cases} \omega_0 = (\omega_{0j})_{1\leq j\leq m} \ \ if \quad | \alpha | = 0, \\ \omega = (\omega_{ij})_{1\leq i\leq n;\ 1\leq j\leq m} \ \ if \quad | \alpha | = 1, \\ arbitrary\ weight\ functions \qquad if\ \ 2 \leq | \alpha | \leq k. \end{cases}$$

And $\omega_{ij}^* = \omega_{ij}^{1-p'}$ *is the conjugate of* ω_{ij}.

We defined the inner product in $L^2(\Omega, \omega_0, \mathbb{R}^m)$ *by:* $\langle u, v\rangle_{2,\omega_0} = \sum_{j=1}^m \int_\Omega ((\omega_{0j})^{\frac{1}{2}} u_j)((\omega_{0j})^{\frac{1}{2}} v_j) dx$, *for all* $u; v$ *in* $L^2(\Omega, \omega_0, \mathbb{R}^m)$. *And for all* $u; v$ *in* $W_0^{r,2}(\Omega, \omega_\alpha, \mathbb{R}^m)$ *we denotes the inner product by:* $\langle u, v\rangle_{W^{r,2}(\omega_\alpha)} = \langle u, v\rangle_{2,\omega_0} + \sum_{i;j=1}^{n;m} \int_\Omega (\omega_{ij}^{\frac{1}{2}} D_{ij} u)(\omega_{ij}^{\frac{1}{2}} D_{ij} v) dx + \sum_{2\leq|\alpha|\leq r} \int_\Omega (w_\alpha^{\frac{1}{2}} D^\alpha u)(w_\alpha^{\frac{1}{2}} D^\alpha v) dx.$

We denote $W_0^{k,p}(\Omega,\omega_\alpha,I\!R^m)$ the closure of $C_0^\infty(\Omega,I\!R^m)$ in $W^{k,p}(\Omega,\omega_\alpha,I\!R^m)$, where the norm is $\|.\|_{k,p,\omega_\alpha}$ is given by:

$$\|u\|_{k,p,\omega_\alpha}=\Big(\sum_{0\le|\alpha|\le k}\|\ D^\alpha u\ \|_p\ \omega_\alpha\Big)^{\frac{1}{p}}.$$

We have:

$W_0^{k,p}(\Omega,\omega_\alpha,I\!R^m)\hookrightarrow W^{k,p}(\Omega,\omega_\alpha,I\!R^m)\hookrightarrow L^p(\Omega,\omega_0,I\!R^m)$.

$(W_0^{k,p}(\Omega,\omega_\alpha,I\!R^m))'\equiv W^{-k,p'}(\Omega,\omega_\alpha^*,I\!R^m)$.

$I\!M^{m\times n}$ denotes the space of real $m\times n$ matrices equipped with inner product $M:N=M_{ij}N_{ij}$ (we use the usual summation convention) and the tensor product $a\otimes b$ of two vectors $a,b\in I\!R^n$ is defined to be the matrix $(a_ib_j)_{i,j=1,\dots,m}$.

Definition 8.1.3 *let Ω be a measurable subset of $I\!R^n$, $(\mu_k)_{k\in I\!N}$ and μ be a sequence and respectively an element in $M(\Omega,I\!R^m)$. we say that (μ_k) converges weakly $*$ to μ, which is denoted by $\mu_k\rightharpoonup^*\mu$, if:*
$\int_\Omega f d\mu_k\to\int_\Omega f d\mu$, *for all* $f\in C_c(\Omega,I\!R^m)$.

8.2 Hypothesis

We assume that:

$$\omega_{ij}\in L^1(\Omega),\quad \omega_{ij}^{\frac{-1}{p-1}}\in L^1_{loc}(\Omega)\ and\ \omega_{ij}\ is bounded\ a.e.\ in\ \Omega \tag{8.2.1}$$

$$\omega_{ij}^{-s}\in L^1(\Omega) \tag{8.2.2}$$

for some $s\in]max(\frac{1}{p-1};\frac{n}{p})\ ;\ \infty[$.

The Sobolev space $W^{1,p}(\Omega,\omega,I\!R^m)$ equipped with the norm

$$\|\ u\ \|^p_{1,p,\omega}=\sum_{j=1}^m\int_\Omega\omega_{0j}|u_j|^p dx+\sum_{i,j}\int_\Omega\omega_{ij}|D_{ij}u|^p dx.$$

is a Banach space, reflexive, and $C_0^\infty(\Omega,I\!R^m)$ is a subspace of $(W^{1,p}(\Omega,\omega,I\!R^m)$. The space $(W_0^{1,p}(\Omega,\omega,I\!R^m)$ is the fermenter of $C_0^\infty(\Omega,I\!R^m)$ in $(W^{1,p}(\Omega,\omega,I\!R^m)$ for the norm $\|.\|^p_{1,p,\omega}$,

(by the condition 8.2.1).

The norm $\|\ ..\ \|^p_{1,p,\omega}$ equivalent to the norm $|||u|||_{W_0^{1,p}(\Omega,\omega,I\!R^m)}=(\sum_{i,j}\int_\Omega\omega_{ij}|D_{ij}u|^p dx)^{\frac{1}{p}}$, with $D_{ij}u=\frac{\partial u^j}{\partial x_i}$

(by: The condition 8.2.2).

By the condition 8.2.2, we give:

$$(W^{1,p}(\Omega,\omega,I\!R^m)\hookrightarrow(W^{1,p_s}(\Omega,I\!R^m)\ and\ (W^{1,p}(\Omega,\omega,I\!R^m\hookrightarrow\hookrightarrow L^r(\Omega,I\!R^m), \tag{8.2.3}$$

for all $1 \leq r < p_s^*$ *with* $p_s = \frac{p.s}{s+1}$

$p_s^* = \frac{n.p.s}{n(s+1)-p.s}$, *if* $p.s < n(s+1)$ *and* p_s^* *is arbitrary otherwise. The Hardy-Type inequalities in condition* (P_0) *is verifies for* $\gamma_j = 1$*;* $j = 1; 2; .....; m$

(P_1) $\sigma : \Omega \times]0, T[\times I\!R^m \times I\!M^{m\times n} \longrightarrow I\!M^{m\times n}$ *is a Carathéodory functions, i.e:*
$(x,t) \longmapsto \sigma(x,t,u,F)$ *is measurable for every* $(u,F) \in I\!R^m \times I\!M^{m\times n}$ *and*
$(u,F) \in I\!R^m \times I\!M^{m\times n} \longmapsto \sigma(x,t,u,F)$ *is continuous for almost every* $(x,t) \in \Omega_T$.

(P_2) *(Coercivity and growth): There exist constants* c_1, c_2, $\beta > 0$ *and* $\lambda_1 \in L^{p'}(\Omega_T)$, $\lambda_2 \in L^1(\Omega_T)$, $\lambda_3 \in L^{(\frac{p}{\alpha})'}(\Omega_T)$, $0 < \alpha < p$, $1 < q < \infty$, *such that, for all* $1 \leq r \leq n$, *and* $1 \leq s \leq m$*:*

$$|\sigma_{rs}(x,t,u,F)| \leq \beta \omega_{rs}^{\frac{1}{p}} [\lambda_1 + c_1 \sum_{j=1}^{m} \gamma_j^{\frac{1}{p'}} |u_j|^{\frac{q}{p'}} + c_1 \sum_{i,j} \omega_{ij}^{\frac{1}{p'}} |F_{ij}|^{p-1}]$$

$$\sigma(x,t,u,F) : F \geq -\lambda_2(x,t) - \sum_{j=1}^{m} \lambda_3(x,t) \omega_{1j}^{\frac{\alpha}{p}} |u_j|^{\alpha} + c_2 \sum_{i,j} \omega_{ij} |F_{ij}|^p.$$

(P_3) *(Monotonicity):* σ *satisfies one of the following conditions:*

- *(a) For all* $(x,t) \in \Omega_T$ *and all* $u \in I\!R^m$*, the map:* $F \mapsto \sigma(x,t,u,F)$ *is a* C^1*- functions and is monotone.*
- *(b) There exist a function:* $W : \Omega_T \times I\!R^m \times I\!M^{m\times n} \to I\!M^{m\times n}$*, such that* $\sigma(x,t,u,F) = \frac{\partial W}{\partial F}(x,t,u,F)$*, and* $F \mapsto W(x,t,u,F)$ *is convex and* C^1*- function.*
- *(c) For all* $(x,t) \in \Omega_T$ *and all* $u \in I\!R^m$*, the map:* $F \mapsto \sigma(x,t,u,F)$ *is strictly monotone .*
- *(d)* σ *is strictly -p quasi monotone in* F *i.e:*
$\int_{I\!M^{m\times n}} (\sigma(x,t,u,\lambda) - \sigma(x,t,u,\overline{\lambda}) : (\lambda - \overline{\lambda}) d\vartheta(\lambda) > 0$

(P_0) *(The Hardy-Type inequalities); There exist* $c > 0$*, one weighted function* $\gamma = (\gamma_j)_{1\leq j\leq m}$*, and the parameter* $1 < q$*, such that:*

$$\left(\sum_{j=1}^{m} \int_{\Omega} \gamma_j \mid u_j \mid^q dx \right)^{\frac{1}{q}} \leq c \left(\sum_{i,j} \int_{\Omega} \omega_{ij} \mid D_{ij} u \mid^p dx \right)^{\frac{1}{p}}$$

and the expression $\left(\sum_{i,j} \int_{\Omega} \omega_{ij} \mid D_{ij} u \mid^p dx \right)^{\frac{1}{p}}$ *is a norm equivalent to the norm* $\|.\|_{1;p:\omega}$ *in* $W_0^{1,p}(\Omega, \omega, I\!R^m)$.

(F_0) *(continuity)* $f : \Omega_T \times I\!R^m \times I\!M^{m\times n} \to I\!R^m$ *is a Carathéodory function.*

(F_1) *(Growth) There exist* $0 < \gamma;\ \nu < p-1, \quad b_1 \in L^{p'}(\Omega_T), \quad b_2 \in L^{\gamma^{\mp}}(0,T,L^{\frac{n}{p}}(\Omega))$ *and* $b_3 \in L^{\nu^{\mp}}(0,T,L^{n}(\Omega))$ *such that:*

$$| f_k(x,t,u,F) | \leq b_1 \omega_{0k}^{\frac{1}{p}} + \sum_{j=1}^{m} b_2 |u_j|^{\gamma} \omega_{0j}^{\frac{p-1}{p^*}} \omega_{0k}^{\frac{1}{p^*}} + b_3 \sum_{i,j=1}^{n,m} |F_{ij}|^{\nu} . \omega_{ij}^{\frac{n-1}{np}} \omega_{0k}^{\frac{n-1}{np'}};$$

for all $k = 1;2;....;m.$ *, with* $\frac{1}{p*} = \frac{1}{p} - \frac{1}{n}$ *and* $\gamma^{\mp} = (\frac{p}{p-1-\gamma})'$ *is the Hölder conjugate of* $\frac{p}{\gamma+1}$.

(G_0) *(continuity)* $g : \Omega_T \times I\!R^n \to I\!M^{m\times n}$ *is a Carathéodory function.*

(G_1) *(Growth) There exist* $0 < \eta < p-1,\ \ b_4 \in L^{p'}(\Omega_T)$ *and* $b_5 \in L^{\infty}(0,T,L^{\frac{n}{p-1}}(\Omega))$*, such that:*

$$| g_{rs}(x,t,u) | \leq b_4 \omega_{rs}^{\frac{1}{p}} + \sum_{j=1}^{m} b_5 |u_j|^{\eta} \omega_{0j}^{\frac{n-p+1}{np'}} \omega_{rs}^{\frac{n-p+1}{np}}$$

Remark 8.2.1

(P_1) *(Carathéodory condition) ensures that* $\sigma(x,t,u(x,t),U(x,t))$ *is measurable on* $\partial\Omega_T$ *for measurable functions:*

$u : \Omega_T \to I\!R^m$

$U : \Omega_T \to I\!M^{m\times n}$.

The growth condition and coercivity

(P_2) *is standard it is used in the construction of approximate solutions by a Galerkin method and when we pass to the limit.*

Although the strict monotonicity condition (c) *in* (P_3) *ensure existence of a weak solution by standard method see e-g [46], the main point is that we do not require strict monotonicity or monotonicity in the variables* (u,F) *in* $(a),(b)$ *and* (d) *as it is usually assumed in previous works see-e-g [16], [46], [1].*

Under these mild assumptions and using again Young measures, Hungerbühler shows in [38] that initial and boundary value problem $(\mathcal{QPS})_{\omega}(f,g)$ *has a weak solution* $u \in L^p(0,T,W_0^{1,p}(\Omega,\omega,I\!R^m)$ *in following sense.*

Definition 8.2.1 *We say that:* $u : \Omega_T \to I\!R^m$ *is weak solutions of:*

$$(\mathcal{QPS})_{\omega}(f,g) \begin{cases} \partial_t u - div\sigma(x,t,u,Du) &= v(x,t) + f(x,t,u,Du) + divg(x,t,u) \quad \textit{in} \quad \Omega_T \\ u(x,t) &= 0 \quad \textit{on} \quad \partial\Omega \times (0,T) \\ u(x,0) &= u_0(x) \quad \textit{on} \quad \Omega \end{cases}$$

where $v \in L^{p'}(0,T,W_0^{-1,p'}(\Omega,\omega^*,I\!R^m))$ *if:*

- *(i)* u *belongs to* $L^1(0,T,W_0^{1,1}(\Omega,\omega,I\!R^m))$.

- *(ii)* $\sigma(.,u(.),Du(.))$ *belongs to* $L^1(\Omega_T,\mathbb{M}^{m\times n})$.

- *(iii) The initial condition* $u(x,0)=u_0(x)$ *on* Ω *is fulfilled.*

- *(iv) The equation:*

$$\sum_j \int_{\Omega_T} \partial_t u^j.\varphi^j(x,t)\omega_{0j}dxdt + \int_{\Omega_T} \sigma(x,t,u,Du):D\varphi(x,t)dxdt = \langle v,\varphi\rangle$$

$$+\int_{\Omega_T} f(x,t,u,Du).\varphi(x,t)dxdt - \int_{\Omega_T} g(x,t,u):D\varphi(x,t)dxdt,$$

holds for every function $\varphi\in C^\infty(\overline{\Omega_T},\mathbb{R}^m)$ *which is zero in a neighborhood of* $(\partial\Omega\times]0,T[)\cup(\Omega\times\{0;T\})$.

Remark 8.2.2

- *(a) In definition the boundary condition* $u=0$ *on* $(\partial\Omega\times]0,T[)$ *is interpreted in the sense of (i) .*

- *(b) The condition (iv) is understood in the sense that the Galerkin approximations converge to* u_0 *in* $L^2(\Omega,\omega_0)$.

Theorem 8.2.1 *If* $p\in]\frac{2n(s+1)}{(n+2)s}\ ;\infty[$ *and if* σ *satisfies the conditions* $(P_1)-(P_2)$ *and* (c) *or* (d) *in* (P_3) *then the parabolic system* $(QPS)_\omega(f,g)$ *has a weak solution* $u\in L^p(0,T,W_0^{1,p}(\Omega,\omega,\mathbb{R}^m))$ *for every* $v\in L^{p'}(0,T,W_0^{-1,p'}(\Omega,\omega^*,\mathbb{R}^m))$, *every* f *satisfying* $(F_0)-(F_1)$ *and* g *satisfying* $(G_0)-(G_1)$ *with* $(n\geq 2)$.

Remark 8.2.3

- *(a) For* $p_s=n$, *Theorem 8.2.1 hold if the condition* $b_2\in L^\infty(\Omega_T)$ *in* (F_1) *and* $b_5\in L^\infty(\Omega_T)$ *in* (G_1) *are assumed.*

- *(b) For* $p>n$, *Theorem 8.2.1 remains valid if* $b_2\in L^{\gamma^\mp}(0,T,L^1(\Omega))$ *in* (F_1) *and* $b_5\in L^\infty(0,T,L^1(\Omega))$ *in* (G_1) *hold.*

- *(c) The strict bound* $p-1$ *for* γ *and* η *in the growth conditions* (F_1) *and* (G_1) *ensures the coercivity of the function* ψ. *However, the limit value* $p-1$ *is admissible in some particular cases.*

8.3 A first parabolic div-curl inequality

We state here the first set of hypothesis:

- (A_1)*: The sequence* (u_k) *is uniformly bounded in* $L^p(0,T,W_0^{1,p}(\Omega,\omega,I\!R^m))$ *for some* $p \in]\frac{2n(s+1)}{(n+2)s}\ ;\ \infty[$ *and hence (Bay the Elberlem Suljan theorem: see [15].a subsequence converges weakly in* $L^p(0,T,W_0^{1,p}(\Omega,\omega,I\!R^m))$ *to an element denoted by* u.
- (A_2)*: The sequence* $\partial_t u_k$ *is uniformly bounded in* $L^{p'}(0,T,(W_0^{r,2}(\Omega,\omega,I\!R^m))')$ *for some (* $2 < r < p_s^*$ *if* $p_s < n$*) and (*$2 < r < \infty$ *if* $p_s \geq n$.*)*
- (A_3)*:* $\sigma : \Omega\times]0,T[\times I\!R^m \times I\!M^{m\times n} \longrightarrow I\!M^{m\times n}$ *is a Carathéodory function.*
- (A_4)*: The sequence* $\sigma_k(x,t) \equiv \sigma(x,t,u_k(x,t),Du_k(x,t))$ *is uniformly bounded in* $L^{p'}(\Omega_T, I\!M^{m\times n})$ *and therefore equi-integrable (the equi-integrability follows from, the Hölder inequality).*
- (A_5)*: The sequence* $(\sigma_k(x,t) : Du_k)^-$ *is equi-integrable.*
- (A_6)*: There exists a sequence* (v_k) *such that* $v_k \to u$ *in* $L^p(0,T,(W_0^{1,p}(\Omega,\omega,I\!R^m))')$ *and* $\int_{\Omega_T} \sigma_k(x,t) : (Du_k - Dv_k)dxdt \to 0;\ \ k \to \infty$.

Lemma 8.3.1 *Let* B, B_0 *and* B_1 *be Banach spaces,* B_0 *and* B_1 *reflexive. Let* $\imath : B_0 \to B$ *be a compact linear map and:*
$\jmath : B \to B_1$ *an injective bounder linear operator. For* T *finite and:* $1 < p_i < \infty$ *with* $i = 0;1$
$W = \{v \in L^{p_0}(0;T;B_0) : \frac{d}{dt}(\jmath\ o\ \imath\ o\ v) \in L^{p_1}(0;T;B_1)\}$ *is a Banach space under the norm:*
$\parallel v \parallel_{L^{p_0}(0;T;B_0)} + \parallel \jmath\ o\ \imath\ o\ v \parallel_{L^{p_1}(0;T;B_1)}$.
Then, if $V \subset W$ *is bounded, the set* $\{\imath\ o\ v; v \in V\}$ *is precompact in* $L^{p_0}(0;T;B_0)$.

Proof of lemma8.3.1
For the proof, see [37] (Appendix I). For generalization of lemma, see [56]. Now, we apply lemma to obtain the desired bound. □

Proposition 8.3.1 *Let* $p \in]\frac{2n}{n+2}\ ;\ \infty[$ *and* (u_k) *be a sequence satisfying* (A_1) *and* (A_2)*. Then there exists a subsequence of* (u_k) *(for convention not relabeled) and a functions:*
$u \in L^p(0,T,W_0^{1,p}(\Omega,\omega,I\!R^m))$ *such that:* $u_k \to u$ *in measure on* Ω_T *and in*

$$L^p(0,T,L^r(\Omega,\omega,I\!R^m)),\ \ \forall r < p_s^*. \tag{8.3.1}$$

Proof of prop 8.3.1

See the proof of the analogous proposition in chapter7.

Lemma 8.3.2 *(div-curl inequality) Suppose* $(A_1 - A_6)$ *and assume (after passing to a suitable subsequence if necessary) that* (Du_k) *generates the young measure* $\vartheta_{(x,t)}$*. Then the following inequality hold:*

$$\int_{\Omega_T}\int_{\mathbb{M}^{m\times n}} \sigma(x,t,u,\lambda) : \lambda d\vartheta_{(x,t)}(\lambda)dxdt \leq \int_{\Omega_T}\int_{\mathbb{M}^{m\times n}} \sigma(x,t,u,\lambda) : Dud\vartheta_{(x,t)}(\lambda)dxdt \qquad (8.3.2)$$

Proof of lemma 8.3.2

See the proof of analogous lemma in chapter7. □

8.4 A second parabolic div-curl inequality

We state here the second set of conditions:

- $(c_1) : u_k \in L^1(0,T,W_0^{1,p}(\Omega,\omega,\mathbb{R}^m))$ *for all;* $k \in \mathbb{N}$*, and there exists some* $c > 0$ *and* $\rho > 0$ *such that:*
  $\sup_k \sum_{ij} \int_{\Omega_T} |D_{ij}u_k|^\rho \omega_{ij} dxdt \leq c.$

- (c_2)*: The sequence* u_k *converges in measure to some function* u *and* $u(.,t)$ *is almost everywhere approximately differentiable a.e.* $t \in]0,T[$.

- $(c_3) : \sigma : \Omega_T \times \mathbb{R}^m \times \mathbb{M}^{m\times n} \longrightarrow \mathbb{M}^{m\times n}$ *and*
  $\tau[e] : \Omega_T \times \mathbb{R}^m \times \mathbb{M}^{m\times n} \longrightarrow \mathbb{M}^{m\times n};\ e \in \mathbb{N}$ *are Carathéodory conditions.*

- $(c_4) : \sigma$ *and* $\tau[e]$*, satisfy all the angle structure conditions or satisfies all the sign structure conditions:*
  (Angle condition) For all $(x,t) \in \Omega_T, u \in \mathbb{R}^m$ *and* $M; F \in \mathbb{M}^{m\times n}$*, with* M *of the form* $Id - a\otimes a$ *where* $a \in \mathbb{R}^m$ *satisfies* $\mid a \mid \leq 1$*;* $\sigma(x,t,u,F) : MF \geq 0$*, for all matrix.*
  $M \in \mathbb{M}^{m\times m}$ *of the form* $M = Id - a\otimes a$ *with* $|a| \leq 1$.
  (The sign condition) For all $(x,t) \in \Omega_T, u \in \mathbb{R}^m$ *and* $F \in \mathbb{M}^{m\times n}$*, we have*
  $\sigma_j(x,t,u,F) : F_j \geq 0$*, for all* $1 \leq j \leq m$ *with*
  F_j *and* σ_j *are the cologne j of matrix* F *and* σ*.(this conditions is verifies for* $\tau[e]$ *)*

- (c_5)*: The sequence* $\sigma_k(x,t) = \sigma(x,t,u_k,Du_k)$ *is equi-integrable.*

- (c_6): *The sequence* $\tau_k(x,t) \equiv \tau[k](x,t,u_k,Du_k)$*; and converges weakly to* 0 *in the space* $L^1(\Omega_T, I\!M^{m\times n})$.
- (c_7): *The sequence* $g_k = -div(\sigma_k + g(x,t,u_k) + \tau_k) - v - f$ *is bounded in* $L^1(\Omega_T, I\!R^m)$.
- (c_8): $Du_k \in L^i_{loc}(\Omega_T, \omega, I\!M^{m\times n})$ *and* $(\sigma_k + g(x,t,u_k) + \tau_k) \in L^{i'}_{loc}(\Omega_T, I\!M^{m\times n})$.
  for some $i; 1 < i < \infty$.

Lemma 8.4.1 *(Second div-curl inequality) Suppose* $(C_1) - (C_2)$*, and assure (after passing to a suitable subsequence if necessary) that* Du_k *generates a Young measure* $\vartheta_{(x,t)}$*. Then (after passing to a subsequence), the* σ_k *converges weakly in* $L^1(\Omega_T, I\!M^{m\times n})$ *and the weak limit* $\overline{\sigma}$ *is given by:*
$\overline{\sigma}(x,t) = \langle \vartheta_{(x,t)}; \sigma(x,t,u(x,t),.)\rangle$*, moreover the following inequality hold*
$\int_{I\!M^{m\times n}} \sigma(x,t,u(x,t),\lambda) : \lambda d\vartheta_{(x,t)}(\lambda) \leq \overline{\sigma}(x,t) : apDu(x,t)$*. for a.e.* $(x,t) \in \Omega_T$.

Proof of lemma8.4.1

Proof of lemma is a slightly modification of the corresponding in lemma $(6-2-1)$ *in [28] pages (54-61).*
□

Remark 8.4.1 *If the sequence* Du_k *is equi-integrable then* $Du(x,t) = apDu(x,t) = \langle \vartheta_{(x,t)}; id\rangle$ *almost everywhere. by the Ball's theorem.*

The parabolic div-curl inequality will be the key ingredient to pass the limit in the different cases, but we need some additional information on the Young measure $\vartheta_{(x,t)}$ *generated by the sequence of the gradients* Du_K *to exploit lemma (div-curl inequality) these properties are the following:*

- (N_1): $\vartheta_{x,t}$ *is a probability measure for almost every* $(x,t) \in \Omega_T$.
- (N_2): $\vartheta_{x,t}$ *is a homogeneous* $W^{1,P}-$ *gradient young measure for almost every* $(x,t) \in \Omega_T$.
- (N_3): $\vartheta_{x,t}$ *satisfies* $< \vartheta_{(x,t)}, id >= apDu(x,t)$ *almost everywhere in* Ω_T.

Note that these properties hold if the condition (A_1) *is fulfilled.* □

Lemma 8.4.2 *Assume that the sequence* u_k *satisfies* (A_1);
then the Young measure $\vartheta_{(x,t)}$ *generated by the sequence of the gradient* Du_k *satisfies* $(N_1) - (N_3)$.

Proof of lemma8.4.2

Noting that (A_1) *implies that* $apDu(x,t) = Du(x,t)$*, follows exactly as in [37] (prop 4-3).*
We mention that the properties, also hold if the conditions $(C_1) - (C_2)$ *and a particular estimate are fulfilled as shown in [39] (lemma 9), with a slight modifications.* □

Lemma 8.4.3 *Assume that the sequence* (u_k) *satisfies* $(C_1) - (C_2)$ *and*
$\sup_{k\in I\!N} \sum_{i,j} \int_{|u_k|<R} \mid D_{ij}u_k \mid^{\rho} \omega_{ij} dxdt \leq c$*, for all* $R > 0$*; for some* $\rho > 1$.
then the Young measure $\vartheta_{(x,t)}$ *generated by the sequence of the gradients* Du_k *satisfies*
$(N_1) - (N_3)$.

Proof of lemma8.4.3

The proof of lemma is a slightly modification of the corresponding in lemma (11-3-2) in [28]. □

8.5 Passage to the limit for σ

Proposition 8.5.1 *Suppose that* $(A_1) - (A_6)$ *or* $(C_1) - (C_8)$ *hold further assume that the Young measure* $\vartheta_{(x,t)}$ *generated by the gradients* Du_k *satisfies* $(N_1) - (N_3)$ *and that one of the following conditions holds* $(a) - (b) - (c) - (d)$ *then(after passage to a subsequence) the sequence* σ_k *converges weakly in* $L^1(\Omega_T; I\!M^{m\times n})$ *and the weak limit* $\overline{\sigma}$ *is given by:*
$\overline{\sigma}(x,t) = \sigma(x,t,u(x,t), Du(x,t))$*. If* $(b) - (c)$ *or* (d) *holds, then:*
$\sigma(x,t,u_k,Du_k) \to \sigma(x,t,u,Du)$ *in* $L^1(\Omega_T; I\!M^{m\times n})$
In case (c) *and* (d)*; it follows in addition that* $Du_k \to Du$ *in measure and thus also almost everywhere in* Ω_T.

Before to prove proposition, we note that by proceeding exactly as in the elliptic case, we get the analogous result to lemma $(6-4-2)$ *in [28] and [1].*

Lemma 8.5.1 *Suppose that* $(A_1) - (A_6)$ *or* $(C_1) - (C_8)$ *hold. Further assume that* $\vartheta_{(x,t)}$ *is the Young measure generated by the gradients* Du_k *and satisfies* $(N_1) - (N_3)$*. If the map* $F \mapsto \sigma(x,t,u,F)$ *is monotone for all* $(x,t,u) \in \Omega\times]0;T[\times I\!R^m$ *then:*

$$Spt\vartheta_{(x,t)} \subset \{\lambda/[\sigma(x,t,u,\lambda) - \sigma(x,t,u,Du)] : (\lambda - Du) = 0\}. \tag{8.5.1}$$

Proof of lemma 8.5.1

The proof of lemma is a slightly modification of the corresponding in lemma (11-4-2) see:[28].

Proof of proposition 8.5.1

See proof of the analogous proposition in chapter7. □

8.6 Galerkin Scheme

Lemma 8.6.1 *Let Ω bounded domain in $\mathbb{R}^n$, $p \in]\frac{2n(s+1)}{(n+2)s}\ ;\ \infty[$ and $\omega_{ij} \in L^1(\Omega)$, for all $j = 1; 2; ....; m$, and $i = 0; 1; .....; n$.Then $(W_0^{r,2}(\Omega, \omega_\alpha, \mathbb{R}^m) \hookrightarrow W_0^{1,p}(\Omega, \omega, \mathbb{R}^m)$ For $r \geq 1 + n(\frac{1}{2} - \frac{1}{P})$.*

Proof of lemma 8.6.1 *For the case $p \leq 2$ it's easy using (Hölder inequalities), for the case $p > 2$ the proof of this lemma is a slightly modification of the corresponding in a space of Sobolev without weight $(W_0^{r,2}(\Omega, \mathbb{R}^m) \hookrightarrow W_0^{1,p}(\Omega, \mathbb{R}^m)$.*

By the theorem $(1 - 3 - 1)$ in [28] and proceeding as [37] (section 4-2) we may choose a set $(\delta_i)_{i\in\mathbb{N}}$ of functions which is an orthogonal basis in $L^2(\Omega, \omega_0, \mathbb{R}^m)$ and at the some time an orthogonal basis in $W_0^{r,2}(\Omega, \omega_\alpha, \mathbb{R}^m)$. In particular, $\delta_{(i)_{i\in\mathbb{N}}}$ is uniformly bounded in $W_0^{1,p}(\Omega, \omega, \mathbb{R}^m)$ and then: $\parallel \delta_i \parallel_{1,\omega,p}$ $+ \parallel \delta_i \parallel_{p^*} \leq c$ for all $i \in \mathbb{N}$.

we will need bellow the $L^2(\omega_0)$ - orthogonal projector

$p_k : L^2(\Omega, \omega_0, \mathbb{R}^m) \to L^2(\Omega, \omega_0, \mathbb{R}^m)$ on to spam $(\delta_1, \delta_2, ..., \delta_k)_{k\in\mathbb{N}}$:

of course, the operator norm $\parallel p_k \parallel_{\Im(W_0^{r,2}(\omega_\alpha), W_0^{r,2}(\omega_\alpha))} = 1$.

(The Galerkin scheme): New we build our Galerkin approximations, we make the following ansate for approximating solution of $(QPS)_\omega(f, g)$, $u_k(x, t) = \sum_{i=1}^k a_{k,i}(t)\delta_i(x)$. Where

$a_{k,i} : [0; T) \to \mathbb{R}$ are supposed to be continuous bounded functions. such that u_k satisfies the boundary condition in $(QPS)_\omega$ by constructions in the sense that $u_k \in L^p(0, T, W_0^{1,p}(\Omega, \omega, \mathbb{R}^m))$, we take core of the initial conditions in $(QPS)_\omega$ by choosing the initial coefficients:

$a_{k,i}(0) = \langle u_0, \delta_i\rangle_{2,\omega_0}$ such that:

$u_k(., 0) = \sum_{i=1}^k a_{k,i}(0)\delta_i(.) \to u_0$, with $u_0 \in L^2(\Omega, \omega_0, \mathbb{R}^m)$.

we try to determine the coefficients $a_{k,i}(t)$ in such that a way, that for every $k \in \mathbb{N}$ the system of ordinary differential equations

$$\langle \partial_t u_k, \delta_j\rangle_{2,\omega_0} + \int_\Omega \sigma(x, t, u_k, Du_k) : D\delta_j dx$$

$= \langle v(t), \delta_j \rangle + \int_\Omega f(x,t,u_k)\delta_j dx - \int_\Omega g(x,t,u_k) : D\delta_j dx$ *for* $j \in \{1;2;....;k\}$.

Now we fix to $k \in I\!N$ *for the moment. Let* $0 < \varepsilon < T$ *and* $J = [0;\varepsilon]$. *Moreover we chose* $r > 0$ *large enough; such that the set* $B_r(0) \subset I\!R^k$ *contains the vector* $(a_{1k}(0), ..., a_{kk}(0))$, *and we set* $k = \overline{B_r(0)}$. *Then we have the following result.*

Lemma 8.6.2 *The application:* $F : J \times k \mapsto I\!R^k$ *given by:*

$(t, a_1, ...a_k) \mapsto (\langle v(t), \delta_j \rangle + \int_\Omega f(x,t,\Sigma_{i=1}^k a_i\delta_i)\delta_j dx - \int_\Omega g(x,t,\Sigma_{i=1}^k a_i\delta_i) : D\delta_j dx$

$- \int_\Omega (\sigma(x,t,\Sigma_{i=1}^k a_i\delta_i, \Sigma_{i=1}^k a_i D\delta_i) : D\delta_j)dx)_{j=1;2;....k}$

is a Carathéodory function.Moreover, each components F_j *may be estimated on* $J \times k$ *by*

$$\mid F_j(t, a_1, ...a_k) \mid \leq C(r,k).M(t) \tag{8.6.1}$$

uniformly on $J \times k$, *where* $C(r,k)$ *is a constant which depends on* r *and* k *and here* M *is a functions dependent of* $j; k$ *and* r *belonging to* $L^1(J)$.

Proof of lemma 8.6.2 The first claim follows from the condition (P_1).

For the second claim; we note that the generalized Hölder inequality implies that:

$\mid I_j \mid = \mid \langle v(t), \delta_j \rangle \mid \leq \parallel v(t) \parallel_{-1,p';\omega^*} \parallel \delta_j \parallel_{1,p;\omega}$

Moreover, on the one hand, if; $II_j = \int_\Omega (\sigma(x,t,\Sigma_{i=1}^k a_i\delta_i, \Sigma_{i=1}^k a_i D\delta_i) : D\delta_j)dx$, *by the growth condition in* (P_2) *and the Hölder inequality*

$\mid II_j \mid \leq \beta[\parallel \lambda_1(t) \parallel_{p'} + c_1 \parallel \sum_{i=1}^k a_i\delta_j \parallel_{1,p,\gamma}^{\frac{q}{p'}} + c_2 \parallel \sum_{i=1}^k a_i D\delta_j u \parallel_{1,p,\omega}^{p-1}] \parallel D\delta_j \parallel_{1,p,\omega}$

$\leq \beta \max(1, c_1, c_2).[\parallel \lambda_1 \parallel_{p'} + \parallel \sum_{i=1}^k a_i\delta_j \parallel_{1,p,\gamma}^{\frac{q}{p'}} + \parallel \sum_{i=1}^k a_i D\delta_j u \parallel_{1,p,\omega}^{p-1}] \parallel D\delta_j \parallel_{1,p,\omega}$.

And by Hölder inequalities we have:

$\mid III_j \mid = \mid \int_\Omega g(x,t,\Sigma_{i=1}^k a_i\delta_i) : D\delta_j dx \mid$

$\leq (\parallel b_4(t) \parallel_{p'} + \parallel b_5(t) \parallel_{\frac{n}{p-1}} \parallel \Sigma_{i=1}^k a_i\delta_i \parallel_{p^*,\omega_0}^{\eta}) \parallel D\delta_j \parallel_{p,\omega}$

$\mid IV_j \mid = \mid \int_\Omega f(x,t,\Sigma_{i=1}^k a_i\delta_i) : \delta_j dx \mid$

$\leq \parallel b_1(t) \parallel_{p'} \parallel \delta_j \parallel_{p,\omega_0} + \parallel b_2(t) \parallel_{\frac{n}{p}} \parallel \Sigma_{i=1}^k a_i\delta_i \parallel_{p^*,\omega_0}^{\gamma} \parallel \delta_j \parallel_{p,\omega_0}$ *Then, in view of*

$\parallel \delta_i \parallel_{1,\omega,p} + \parallel \delta_i \parallel_{1,p^*,\omega} \leq c$ *for all* $i \in I\!N$ *and since each variable* a_i *satisfies:* $\mid a_i \mid \leq r$; *for all* $i = 1; .....; k$ *fora constant* r, *each component* F_j *may be estimated in such a way that*

$$\mid F_j(t, a_1, ...a_k) \mid \leq \mid II_j \mid + \mid I_j \mid + \mid III_j \mid + \mid IV_j \mid \leq C(r,k).M(t) \tag{8.6.2}$$

uniformly on $J \times K$, *where* $C(r,k)$ *is a constant which depends on* r *and* k, *and where* $M(t) \in L^1(J)$ *(independent of* j, k*)and* r *Then, in view of lemma 8.6.2 the Carathéodory existence result on ordinary*

differential equation applied to the system:

$$a_j^{'}(t) = F_j(t, a_1(t), ..., a_k(t)) \tag{8.6.3}$$

$$a_j(0) = a_{k,j}(0); j \in 1; ...; k \tag{8.6.4}$$

ensures existence of a distributional, continuous solutions a_j (depending on k) of 8.6.3 - (8.6.4) on a time interval $[0; \varepsilon'[$ where $\varepsilon' > 0$, a priori, may depend on k. Moreover, the corresponding integral equation

$$a_j(t) = a_j(0) + \int_0^t F_j(t, a_1(t), ...a_k(t))dt \tag{8.6.5}$$

holds on $[0; \varepsilon'[$. Then, $u_k = \sum_{j=1}^k a_j(t)\delta_j$ is the desired (short time) solution of
$\langle \partial_t u_k, \delta_k \rangle_{2,\omega_0} + \int_\Omega \sigma(x, t, u_k, Du_k) : D\delta_k dx$
$= \int_\Omega v(x,t)\delta_k dx + \int_\Omega f(x,t,u_k).u_k dx - \int_\Omega g(x,t,u_k) : Du_k dx$ *with initial condition*
$u_k(0) = \sum_{i=1}^k a_{k,i}(0)\delta_i(.) \to u_0$, *with* $u_0 \in L^2(\Omega, \omega_0, I\!R^m)$. *Now, we want to show that the local solution constructed above can be extended to the whole interval $[0; T[$ independent of k, as a - word of warning we should mention, that the solution need not be unique.*

Lemma 8.6.3 *We have:*
$$A = \left\{ t \in [0; T[: \ there\ exist\ a\ weak\ solutions\ of\ (8.6.3) - (8.6.4)\ on\ [0; T[\right\} = [0; T[.$$

Proof of lemma 8.6.3

The first thing we want to establish is a uniform bound on the coefficients $|a_{ki}(t)|$. Since
$\langle \partial_t u_k, \delta_k \rangle_{2,\omega_0} + \int_\Omega \sigma(x, t, u_k, Du_k) : D\delta_k dx$
$= \int_\Omega v(x,t)\delta_k dx + \int_\Omega f(x,t,u_k).\delta_k dx - \int_\Omega g(x,t,u_k) : D\delta_k dx.$
is linear in δ_k, it is allowed to use u_k as a test function in precedent equation in place of δ_k. This gives for a an arbitrary time τ in the existence interval
$\int_0^\tau \langle \partial_t u_k, u_k \rangle_{2,\omega_0} dt + \int_0^\tau \int_\Omega \sigma(x, t, u_k, Du_k) : Du_k dx dt$
$= \int_0^\tau \langle v(t), u_k \rangle dt + \int_0^\tau \int_\Omega f(x,t,u_k).u_k dx dt - \int_0^\tau \int_\Omega g(x,t,u_k) : Du_k dx dt.$
For the first term we have:
$\int_0^\tau \langle \partial_t u_k, u_k \rangle_{2,\omega_0} dt = \sum_j \frac{1}{2} \int_0^\tau \partial_t (u_k^j)^2 \omega_{0j} dt$
$= \sum_j \frac{1}{2} \parallel u_k^j(\tau) \parallel_{2,\omega_0}^2 - \sum_j \frac{1}{2} \parallel u_k^j(0) \parallel_{2,\omega_0}^2 \geq \frac{1}{2} \parallel u_k(\tau) \parallel_{2,\omega_0}^2 - c$

Since $u_k(.,0)$ converges in $L^2(\Omega,\omega_0,I\!R^m)$. Using the coercivity and the Hölder inequality in (P_2) for the second term, we obtain:

$$\int_0^\tau\int_\Omega \sigma(x,t,u_k,Du_k)\,:\,Du_k dxdt \geq -\parallel \lambda_2 \parallel_{L^1(\Omega_T)} -A'^{\alpha} \parallel \lambda_3 \parallel_{L^{(\frac{p}{\alpha})'}(\Omega_T)} \parallel u_k \parallel^{\alpha}_{L^p(0,T,W_0^{1,p}(\Omega,\omega,I\!R^m))}$$
$$+c_2 \parallel u_k \parallel^p_{L^p(0,T,W_0^{1,p}(\Omega,\omega,I\!R^m))}.$$

By the the Hölder inequality for the third term we get

$$\int_0^\tau \langle v(t),u_k\rangle dt \leq \parallel v \parallel_{L^{p'}(0,T,W^{-1,p'}(\Omega,\omega))} \parallel u_k \parallel_{L^p(0,T,W_0^{1,p}(\Omega,\omega))}$$ *. Using the growth condition(F_1) and the Hölder inequality for the fourth term, we obtain:*

$$\int_0^\tau\int_\Omega f(x,t,u_k).u_k dxdt$$
$$\leq \parallel b_1(t) \parallel_{L^{p'}(\Omega_T)} . \parallel u_k \parallel_{L^p(0,\tau,L^p(\Omega,\omega_0))} + \int_0^\tau \parallel b_2(t) \parallel_{\frac{n}{p}} . \parallel u_k \parallel^{\gamma+1}_{p^*,\omega_0} dt.$$
$$\leq A' \parallel b_1 \parallel_{L^{p'}(\Omega_T)} \parallel u_k \parallel_{L^p(0,\tau,W_0^{1,p}(\Omega,\omega))} + A'^{\gamma+1} \parallel b_2 \parallel_{L^1(0,T,L^{\frac{n}{p}(\Omega))}} \parallel u_k \parallel^{\gamma+1}_{L^p(0,\tau,W_0^{1,p}(\Omega,\omega))} .$$

Finally, using the growth condition (G_1) for the last term, we obtain:

$$\int_0^\tau\int_\Omega g(x,t,u_k).Du_k dxdt$$
$$\leq \parallel b_4(t) \parallel_{L^{p'}(\Omega_T)} . \parallel Du_k \parallel_{L^p(0,\tau,L^p(\Omega,\omega))} + \int_0^\tau \parallel b_5(t) \parallel_{\frac{n}{p-1}} . \parallel u_k \parallel^{\eta}_{p^*,\omega_0} . \parallel Du_k \parallel_{p,\omega} dt$$
$$\leq A'. \parallel b_4 \parallel_{L^{p'}(\Omega_T)} . \parallel u_k \parallel_{L^p(0,\tau,W_0^{1,p}(\Omega,\omega))} + A'^{\eta} \parallel b_5 \parallel_{L^\infty(0,\tau,L^{\frac{n}{p-1}}(\Omega))} . \parallel u_k \parallel^{\eta+1}_{L^p(0,\tau,W_0^{1,p}(\Omega,\omega))}.$$

The combination of these five estimates gives:

$$\tfrac{1}{2} \parallel u_k(\tau) \parallel_2^2 \leq A'(\parallel b_4 \parallel_{L^{p'}(\Omega_T)} + \parallel b_1 \parallel_{L^{p'}(\Omega_T)} + \parallel v \parallel_{L^{p'}(0,T,W^{-1,p'}(\Omega,\omega))}) \parallel u_k \parallel_{L^p(0,\tau,W_0^{1,p}(\Omega,\omega))} + (\parallel$$
$$\lambda_3 \parallel_{L^{(\frac{p}{\alpha})'}(\Omega_T)} + A'^{\gamma+1} \parallel b_2 \parallel_{L^{\gamma\mp}(0,T,L^{\frac{n}{p}(\Omega))}} + A'^{\eta} \parallel b_5 \parallel_{L^\infty(0,\tau,L^{\frac{n}{p-1}}(\Omega))}) \parallel u_k \parallel^{\kappa'}_{L^p(0,T,W_0^{1,p}(\Omega,\omega,I\!R^m))} + c + \parallel$$
$$\lambda_2 \parallel_{L^1(\Omega_T)} - c_2 \parallel u_k \parallel^p_{L^p(0,T,W_0^{1,p}(\Omega,\omega,I\!R^m))}$$
$$\equiv \psi(\parallel u_k \parallel_{L^p(0,\tau,W_0^{1,p}(\Omega,\omega,I\!R^m))})$$

where $\kappa' = \max(\alpha,\gamma+1,\eta+1)$ Since $p>1$, and $c_2>0$ we have:

$$\psi(y) \to -\infty; y \to +\infty \tag{8.6.6}$$

and so $\psi(\parallel u_k \parallel_{L^p(0,T,W_0^{1,p}(\Omega,\omega,I\!R^m))})$ is bounded, since ψ is continuous and $\parallel u_k \parallel_2^2 \geq 0$.

Thus, in view of the definition of $a_{k,i}$ we may conclude that:

$$\mid a_{k,i}(\tau) \mid^2_{I\!R^k} = \parallel u_k(\tau) \parallel_2^2 \leq \overline{c} \tag{8.6.7}$$

for a constant $\overline{c}$ which is independent of τ and k. Now, we prove that A is non empty, open and closed in $[0;T]$. First, A is clearly non- empty since we proved local existence above. Moreover, let $t \in A$ and $0<\tau_1<\tau_2<T$. Then by (8.6.5), we have:

$$\mid a_{k,j}(\tau_1) - a_{k,j}(\tau_2) \mid \leq \int_{\tau_1}^{\tau_2} \mid F_j(\tau, a_{k,i}(\tau), ..., a_{k,k}(\tau)) \mid d\tau$$
$$\leq C(\overline{c},k) \int_{\tau_1}^{\tau_2} \mid M(\tau) \mid d\tau;$$

since $M \in L^1(0,T)$, this implies that $\tau \mapsto a_{k,j}(\tau)$ is uniformly continuous.Then, we can restart to solve
$\langle \partial_t u_k, \delta_k \rangle_{2,\omega_0} + \int_\Omega \sigma(x,t,u_k,Du_k) : D\delta_k dx$
$= \int_\Omega v(x,t)\delta_k dx + \int_\Omega f(x,t,u_k).\delta_k dx - \int_\Omega g(x,t,u_k) : D\delta_k dx$
for $k \in \{1;2;....;m\}$.

At time t with initial data $\lim_{\tau \nearrow \longrightarrow T} u_k(\tau)$ and hence get a solution of (8.6.3) - (8.6.4) on $[0, \tau+\varepsilon]$.This proves that A is open. Finally, we consider a sequence $\tau_i \nearrow t; \tau_i \in A$. Let $a_{k,j,i}$ denote the solution of (8.6.3) - (8.6.4) we constructed on $[0,\tau_i]$ and define:

$$\tilde{a_{k,j,i}} \equiv \begin{cases} a_{k,j,i}(\tau) & \text{if} \quad \tau \in [0;\tau_i] \\ a_{k,j,i}(\tau_i) & \text{if } \tau \in]\tau_i;t]. \end{cases}$$

The sequence $(\tilde{a_{k,j,i}})_i$ is bounded and equicontinuous on $[0;t[$, as seen above. Hence, by the Arsela-Ascoli Theorem, a subsequence (again denoted by $\tilde{a_{k,j,i}}(\tau)$ converges uniformly in τ on $[0;t[$ to a continuous functions $a_{k,j}(\tau)$). Using Lebesgues convergence theorem in 8.6.5 it is now easy to see that $a_{k,j}(\tau)$ solves (8.6.3) on $[0;t[$. Hence $t \in A$ and thus A is indeed closed, and as claimed, it follows $A = [0;T[$.
□

Proposition 8.6.1 *The sequence of the Galerkin approximations (u_k) is bounded in $L^\infty(0,T,L^2(\Omega,\omega_0,I\!R^m)) \cap L^p(0,T,W_0^{1,p}(\Omega,\omega,I\!R^m)))$ and satisfies (A_1) and (A_2).*

Proof of proposition8.6.1

By testing equation:
$\langle \partial_t u_k, \delta_j \rangle_{2,\omega_0} + \int_\Omega \sigma(x,t,u_k,Du_k) : D\delta_j dx$
$= \langle v(t), \delta_j \rangle + \int_\Omega f(x,t,u_k).\delta_j dx - \int_\Omega g(x,t,u_k) : D\delta_j dx$
for: $j \in \{1,2,...,k\}$.

With u_k; we obtained in the proof of lemma 8.6.3 that (8.6.7) holds for arbitrary time $\tau \in [0;T[$. So (u_k) is bounded in the space $L^\infty(0,T,L^2(\Omega,I\!R^m))$. Moreover we have proved that $\psi(\| u_k \|_{L^p(0,T,W_0^{1,p}(\Omega,\omega,I\!R^m))})$ is positive, bounded and satisfies (8.6.6). Then there exists a constant $c>0$ such that

$$\| u_k \|_{L^p(0,T,W_0^{1,p}(\Omega,\omega,I\!R^m))} \leq c; \quad \forall k \in I\!N. \tag{8.6.8}$$

Therefore, by extracting a suitable subsequence which is again denoted by u_k, we may assume by the Elberlein-Smuljan see e.g, [62] (Page 141), that

$u_k \rightharpoonup^* u$ *in* $L^\infty(0,T,L^2(\Omega,\omega_0,I\!R^m))$

$u_k \rightharpoonup u$ *in* $L^p(0,T,W_0^{1,p}(\Omega,\omega,I\!R^m))$

which proves in particular (A_1). *To verify* (A_2), *observe that the time derivative*

$\frac{d}{dt}(j\circ i\circ u_k) : [0;T[\to B_1 = (W_0^{r,2}(\Omega,\omega_\alpha,I\!R^m)))'$ *is according to*

$\langle \partial_t u_k, \delta_j\rangle_{2,\omega_0} + \int_\Omega \sigma(x,t,u_k,Du_k) : D\delta_j dx$

$= \langle v(t),\delta_j\rangle + \int_\Omega f(x,t,u_k).\delta_j dx - \int_\Omega g(x,t,u_k) : D\delta_j dx$

for: $j \in \{1;2;....;k\}$. *Given by: we recall that the projection operators* P_k *are self adjoint with respect to the* L^2 *inner product.*

$t \mapsto (\phi \mapsto \langle v(t), P_k\phi\rangle - \int_\Omega \sigma(x,t,u_k,Du_k) : D(P_k\phi)dx + \int_\Omega f(x,t,u_k).(P_k\phi)dx - \int_\Omega g(x,t,u_k) : D(P_k\phi)dx)$. *Now we have:*

$\langle \frac{d}{dt}(\jmath\circ\imath\circ u_k)(t),\phi\rangle = I + II + III + IV$ *where*

$I = \int_0^T \langle v(t), P_k\phi\rangle dt \leq \| v \|_{L^{p'}(0,T,(W^{-1,p'}(\Omega,\omega^*))} \| P_k\phi \|_{L^p(0,T,(W^{1,p}(\Omega,\omega))}$ *by the generalized Hölder inequality,*

$II = -\int_0^T \int_\Omega \sigma(x,t,u_k,Du_k) : D(P_k\phi)dxdt$

$\leq (\| \lambda_1 \|_{L^{p'}(\Omega_T)} + C(c_1) \| u_k \|^{p-1}_{L^p(0,T,W_0^{1,p}(\Omega,\omega)}) \| P_k\phi \|_{L^p(0,T,W_0^{1,p}(\Omega,\omega))}$ *by the growth conditions in* (P_2) *and the Hölder inequality.* $III = \int_0^T \int_\Omega f(x,t,u_k).(P_k\phi)dxdt$

$\leq (\| b_1 \|_{L^{p'}(\Omega_T)} + A'^{\gamma+1} \| b_2 \|_{L^{\gamma\mp}(0,T,L^{\frac{n}{p}(\Omega))}} . \| u_k \|^\gamma_{L^p(0,\tau,W_0^{1,p}(\Omega,\omega))}) . \| P_k\phi \|_{L^p(0,\tau,W_0^{1,p}(\Omega,\omega))}$.

By the growth condition (F_1) $IV = -\int_0^\tau \int_\Omega g(x,t,u_k).D(P_k\phi)dxdt$

$\leq (\| b_4 \|_{L^{p'}(\Omega_T)} + A'^\eta \| b_5 \|_{L^\infty(0,\tau,L^{\frac{n}{p-1}}(\Omega))} . \| u_k \|^\eta_{L^p(0,\tau,W_0^{1,p}(\Omega,\omega))}) . \| P_k\phi \|_{L^p(0,\tau,W_0^{1,p}(\Omega,\omega))}$.

By the growth condition (G_1)

Since

$\| P_k\phi \|_{L^p(0,T,W_0^{1,p}(\Omega,\omega))} \leq \| P_k\phi \|_{L^p(0,T,W_0^{r,2}(\Omega,\omega_\alpha))} \leq \| \phi \|_{L^p(0,T,W_0^{r,2}(\Omega,\omega))}$,

the claim follows by the uniform bounded (8.6.8). □

Remark 8.6.1 *For further use, we note that from the bounded obtained for* $\langle \frac{d}{dt}(\jmath\circ\imath\circ u_k)(t),\phi\rangle$ *in the proof of proposition we can conclude that* $\partial_t u$ *(or rather* $\partial_t(\jmath\circ\imath\circ u)$*) is an element of the space* $L^{p'}(0,T,(W_0^{-1,p'}(\Omega,\omega^*))$. *This follows easily from the fact, that the set*

$\{\phi \in L^p(0,T,W_0^{1,p}(\Omega,\omega)) : \exists k \in I\!N / P_k\phi = \phi\}$ *is dense in* $L^p(0,T,W_0^{1,p}(\Omega,\omega))$, *as proved in [37] (*

Appendix II, with slight assimilation).

Remark 8.6.2 *Recall that the space*

$\{u \in L^p(0,T,W_0^{1,p}(\Omega,\omega)) : \partial_t(\jmath \circ \imath \circ u) \in L^{p'}(0,T,(W^{-1,p'}(\Omega,\omega^*)$ *is continuously embedded in* $C([0;T];L^2(\Omega,\omega_0))$ *see e-g [65] [proposition23-23], with slight assimilation.*

Hence, we have that $u \in C^0([0;T];L^2(\Omega,\omega_0))$ *after possible modification of* u *on a Lebesgue Zero-set of* $[0;T]$. *This gives* $u(t,.) \in L^2(\Omega,\omega_0)$ *a pointiest interpretation for all* $t \in [0;T]$ *and allows in particular to state that* $u(t,.)$ *attains its initial value*

$$u(.,0) = u_0 \tag{8.6.9}$$

continuously in $L^2(\Omega,\omega_0)$ *see [37] (Appendix III). In the two following subsections, we prove that* u *is really a solution of the system* $(QPS)_\omega(f,g)$.

8.7 Passage to the limit in $(\mathcal{QPS})_\omega$

In this subsection, we prove theorem 8.2.1. First, in order to apply proposition 8.6.1, we verify that, under our assumptions, the conditions $(A_1)-(A_6)$ *and* $(N_1)-(N_3)$ *holds for the Galerkin approximations solution* u_k. (A_1) *and* (A_2) *holds by proposition 8.6.1. Moreover, it follows then by lemma 8.6.1 that* $(N_1)-(N_3)$ *holds. The condition* (A_3) *is equivalent to* (P_1). *Next, to prove* (A_4), *we note that by the growth condition in* (P_2) *that*

$$\int_{\Omega_T} \mid \sigma_{rs}(x,t,u_k,Du_k) \mid^{p'} \omega_{rs}^* dxdt$$

$$\leq C.(\int_{\Omega_T} \mid \lambda_1 \mid^{p'} dxdt + \sum_{j=1}^{m} \mid u_{kj} \mid^q \gamma_j dxdt + \sum_{i,j} \mid D_{ij}u_k \mid^p \omega_{ij} dxdt);$$

with (c *is a constants) which is uniformly bounded in* k *by (8.6.8). To verify the equi-integrability of* $(\sigma_k : Du_k)^-$, *we fix an arbitrary measurable subset* $\Omega' \subset \Omega_T$. *Then, on the one hand, the growth condition in* P_2 *implies that*

$$\int_{\Omega'} \mid \min(\sigma_k : Du_k;0) \mid dxdt \leq \int_{\Omega'} \mid \lambda_2 \mid dxdt + \int_{\Omega'} \mid \lambda_3 \mid \sum_{j=1}^{m} \omega_{1j}^{\frac{\alpha}{p}} \mid u_{kj} \mid^\alpha.$$

by the Hölder inequality and (8.6.8). Thus (A_5) *follows. Finally, we prove that* (A_6) *hold. According to Mazur's Theorem see e-g [62] (Theorem2, page 120),there exists a sequence* v_k *in* $L^p(0,T,W_0^{1,p}(\Omega,\omega))$ *where each* v_k *is a convex linear combination of* $u_1,...,u_k$ *such that* $v_k \to u$ *in* $L^p(0,T,W_0^{1,p}(\Omega,\omega))$. *In particular,* $v_k(t,.) \in span(\delta_1,\delta_2,...,\delta_k)$ *for all* $t \in [0,T]$. *Now, we have*

$$\int_{\Omega_T} \sigma_k : (Du_k - Dv_k)dxdt = \langle v, u_k - v_k\rangle - \sum_j \int_{\Omega_T} (u_k^j - v_k^j)\partial_t(u_k^j)\omega_{0j}dxdt$$
$$+ \int_{\Omega_T} f(x,t,u_k).(u_k - v_k)dxdt - \int_{\Omega_T} g(x,t,u_k) : (Du_k - Dv_k)dxdt. \tag{8.7.1}$$

Observe that $(u_k - v_k) \in span(\delta_1, \delta_2, ..., \delta_k)$ *which allowed to use*

$\langle \partial_t u_k, \delta_k\rangle_2 + \int_\Omega \sigma(x,t,u_k,Du_k) : D\delta_k dx$

$= \int_\Omega v(x,t).\delta_k dx + \int_\Omega f(x,t,u_k).\delta_k dxdt - \int_\Omega g(x,t,u_k) : D\delta_k dxdt.$ *in the equality above. The term* $\langle v, u_k - v_k\rangle$ *in (8.7.1) converge to zero, since* $u_k - v_k \rightharpoonup 0$ *in* $L^p(0,T,W_0^{1,p}(\Omega,\omega))$ *by the choice of* v_k *and* (A_1).

Next, defining $II_k = \int_{\Omega_T} f(x,t,u_k).(u_k - v_k)dxdt$, *we infer by the Hölder inequality and the growth condition* (F_1) *that*

$|II_k| \leq \int_0^T (\| b_1(t) \|_{p'} \| u_k(t) - v_k(t) \|_{p,\omega_0} + \| b_2(t) \|_{\frac{n}{p}} \| u_k(t) \|_{p^*,\omega_0} \| u_k(t) - v_k(t) \|_{\frac{p^*}{p-\gamma}})dt$

$\leq \| b_1 \|_{L^{p'}(\Omega_T)} \| u_k - v_k \|_{L^p(0,T,L^p(\Omega,\omega_0))}$

$+ \| b_2 \|_{L^{\gamma^\mp}(0,T,L^{\frac{n}{p}})} \| u_k \|^\gamma_{L^p(0,T,L^{p^*}(\Omega,\omega))} \| u_k - v_k \|_{L^p(0,T,L^{\frac{p^*}{p-\gamma}}(\Omega,\omega))}$.

Since the expression $\| u_k(t) \|_{p^*,\omega_0}$ *is bounded and we have* $u_k - v_k \to 0$ *in* $L^p(0,T,L^r(\Omega,\omega_0))$ *for all* $r < p^*$ *by the proposition 8.3.1, the second term in (8.7.1) vanishes in the limit. For the third term in (8.7.1), we note that* $g_k(x,t) = g(x,t,u_k) \to g(x,t,u)$ *strongly in* $L^{p'}(\Omega_T)$ *by* $(G_0), (G_1)$ *and proposition 8.3.1. To see this, we may assume by proposition 8.3.1 that* $u_k \to u$ *almost everywhere. Since by* (G_1), $| g_k |^{p'}$ *is bounded by an integrals function, the claim follows from* (G_0) *and the Dominated Convergence Theorem. We infer then that*

$| \int_{\Omega_T} g_k : (Dv_k - Du_k)dxdt | \leq \| g_k - g(.,.,u) \|_{L^{p'}(\Omega_T)} \| u_k - v_k \|_{L^p(0,T,W_0^{1,p}(\Omega,\omega))}$

$+ | \int_{\Omega_T} g : D(v_k - u_k)dxdt | \to 0,$

since $v_k - u_k \rightharpoonup 0$ *in* $L^p(0,T,W_0^{1,p}(\Omega,\omega))$. *Thus the third term in (8.7.1) disappears also. Finally,For the last therm in (8.7.1), we have:*

$$\begin{aligned} -\sum_j \int_{\Omega_T}(u_k^j - v_k^j)\partial_t(u_k^j)\omega_{0j}dxdt &= -\sum_j \int_{\Omega_T} \tfrac{1}{2}\partial_t(u_k^j)^2\omega_{0j}dxdt + \sum_j \int_{\Omega_T} v_k^j\partial_t(u_k)\omega_{0j}dxdt \\ &= -\tfrac{1}{2}\| u_k(T) \|^2_{2,\omega_0} + \tfrac{1}{2}\| u_k(0) \|^2_{2,\omega_0} + \sum_j \int_{\Omega_T} v_k^j\partial_t(u_k^j)\omega_{0j}dxdt. \end{aligned} \tag{8.7.2}$$

Concerning the last term in (8.7.1) we claim that for $k \to \infty$ *we have:*

$$\sum_j \int_{\Omega_T} v_k^j\partial_t(u_k^j)\omega_{0j}dxdt \to \sum_j \int_{\Omega_T} u^j\partial(u^j)\omega_{0j}dxdt = \frac{1}{2}\| u(T) \|^2_{2,\omega_0} - \frac{1}{2}\| u_0 \|^2_{2,\omega_0} \tag{8.7.3}$$

To see this, let $\varepsilon > 0$ *be given. Then, there exists* M *such that for all* $l \geq m \geq M$ *we have:*

- $(i)\sum_j \mid \int_{\Omega_T}(u^j - v_m^j)\partial_t(u^j)\omega_{0j}dxdt \mid \leq \varepsilon$. *This is possible, since* $\partial(j\circ i\circ u) \in L^{p'}(0,T,W_0^{-1,p'}(\Omega,\omega)$ *and* $v_m \to u \;\; in \;\; L^p(0,T,W_0^{1,p}(\Omega,\omega))$
- $(ii)\sum_j \mid \int_{\Omega_T}(v_l^j - v_m^j)\partial_t(u^j)\omega_{0j}dxdt \mid \leq \varepsilon$. *This is possible by the bound obtained for* $\langle \frac{d}{dt}(j\circ i\circ u_k)(t), \phi\rangle$ *in the proof of proposition 8.6.1 since* $(v_l - v_k) \in span(\delta_1, \delta_2.....,\delta_k)$ *for all fixed* $t \in]0,T[$.

Now, we fix $m \geq M$ *and choose* $m_0 \geq m$ *such that for all* $l \geq m_0$;
$\sum_j \mid \int_{\Omega_T} v_m^j(\partial_t(u^j - u_l^j)\omega_{0j}dxdt \mid \leq \varepsilon$. *This is possible, since* $\partial_t u_l \rightharpoonup^* \partial_t u$ *in* $L^{p'}(0,T,(W_0^{r,2}(\Omega,\omega_\alpha))')$.
combination yields for all $l = l(\varepsilon) \geq m_0(\varepsilon)$
$\mid \sum_j \int_{\Omega_T} v_l^j \partial_t(u_l^j)\omega_{0j}dxdt - \sum_j \int_{\Omega_T} u^j\partial_t(u^j)\omega_{0j}dxdt \mid$
$\leq \sum_j \mid \int_{\Omega_T}(v_l^j - v_m^j)\partial_t(u_l^j)\omega_{0j} \mid dxdt + \sum_j \mid \int_{\Omega_T} v_m^j(\partial_t u_l^j - \partial_t u^j)\omega_{0j}dxdt \mid$
$+ \sum_j \mid \int_{\Omega_T}(v_m^j - u^j)\partial_t u^j\omega_{0j}dxdt$
$\leq 3\varepsilon$. *This establishes (8.7.3). on the other hand, since* $(u_k)_k$ *is bounded in* $L^\infty(0,T,L^2(\Omega,\omega_0))$ *by proposition 8.6.1 we have that* $(u_k(T) \rightharpoonup u(T))$ *in* $L^2(\Omega,\omega_0))$. *Hence,*

$$\liminf_{k\to\infty} \parallel u_k(T) \parallel_{2,\omega_0} \geq \parallel u(T) \parallel_{2,\omega_0} . \tag{8.7.4}$$

By construction of u_k *we also have*

$$\lim_{k\to\infty} \parallel u_k(0) \parallel_{2,\omega_0)} = \parallel u_0 \parallel_{2,\omega_0} . \tag{8.7.5}$$

Using (8.7.5), (8.7.4) and (8.7.3), in (8.7.2), we conclude that
$\liminf_{k\to\infty}(-\sum_j \int_{\Omega_T}(u_k^j - v_k^j)\partial_t(u_k^j)\omega_0)dxdt) \leq 0$, *thus* (A_6) *is fulfilled.*
So, $(A_1) - (A_6)$ *and* $(N_1) - (N_3)$ *hold and we may infer from proposition 8.6.1 that* $\sigma_k(x,t,u_k,Du_k)$ *converge to* $\sigma_k(x,t,u,Du)$ *at least weakly in* $L^1(\Omega_T, I\!M^{m\times n})$, *for* $k \to \infty$. *Since we have that* $u_k \to u$ *in measure on* Ω_T *by proposition 8.3.1, we may infer that* $u_k \to u$ *almost everywhere on* Ω_T, *for* $k \to \infty$.
Thus, for arbitrary $\varphi \in W_0^{1,p}(\Omega,\omega,I\!R^m)$, *it follows from the continuity conditions* (F_0) *and* (G_0) *that* $f(x,t,u_k).\varphi(x) \to f(x,t,u).\varphi(x)$ *and* $g(x,t,u_k).D\varphi(x) \to g(x,t,u) : D\varphi(x)$ *almost everywhere on* Ω_T. *Since, by the growth conditions in* (F_1) *and* (G_1) *and the uniform bound in proposition 8.6.1,* $f(x,t,u_k).\varphi(x)$ *and* $g(x,t,u_k).D\varphi(x)$ *are equi-integrable, it follows that*
$f(x,t,u_k).\varphi(x) \to f(x,t,u).\varphi(x)$ *and* $g(x,t,u_k).D\varphi(x) \to g(x,t,u) : D\varphi(x)$ *in* $L^1(\Omega_T)$ *by Vitali's Converge Theorem.*

Now, we take a test function $\varphi \in \cup_{i\in I\!N} span(\delta_1, \delta_2 ....., \delta_i)$ *and* $\phi \in C_0^\infty([0,T])$ *in*

$\langle \partial_t u_k, \varphi(x)\rangle_{2,\omega_0} + \int_\Omega \sigma(x,t,u_k,Du_k) : D\delta_j dx = \langle v(t), \delta_j\rangle + \int_\Omega f(x,t,u_k).\delta_j dx + \int_\Omega g(x,t,u_k) : D\delta_j dx$

for:$j \in \{1,2,...,k\}$.

And integrate over the interval $]0,T[$ *and pass to the limit* $k \to \infty$. *The resulting equation is*

$\sum_j \int_{\Omega_T} \partial_t u^j(x,t)\phi^j(t)\varphi(x)\omega_{0j})dxdt + \int_{\Omega_T} \sigma_k(x,t,u,Du) : D\varphi(x)\phi(t)dxdt$

$= \langle v, \phi\varphi\rangle + \int_{\Omega_T} f(x,t,u).\varphi(x)\phi(t)dxdt + \int_{\Omega_T} g(x,t,u) : D\varphi(x)\phi(t)dxdt$

For arbitrary $\varphi \in \cup_{i\in I\!N} span(\delta_1, \delta_2, ..., \delta_i)$ *and* $\phi \in C_0^\infty([0,T])$. *By density of the linear span of these functions in* $L^p(0,T,W_0^{1,p}(\Omega,\omega))$, *this proves that* u *is in fact a weak solution of* $(QPS)_\omega(f,g)$. *Hence the theorem is proved.* □

Example 8.7.1 *Let* Ω *is a bounded open domain in* $I\!R^n$. *In this example we suppose that the weighted functions:* $(\omega_{ij}),\ 0 \le i \le n;\ \ 1 \le j \le m$ *and* σ *satisfies the condition in example 2.3.1 with* $\partial_t u = 0$. *Hence we have the Hardy-Type inequalities in* (P_0) *for all* $u \in W_0^{1,p}(\Omega,\omega)$; $q > p'$.

σ *is a Carathéodory functions in* (P_1); σ_{rs} *verify* $(P_2) - (P_3)$.

$f_k(x,t,u,F) = \omega_{0k}^{\frac{1}{p}} + \frac{1}{T^{p-1}} \sum_j \mid tu_j \mid^{p-1} \omega_{0j}^{\frac{1}{p'}} \omega_{0k}^{\frac{1}{p}} + \frac{1}{T^{p-1}} \sum_{i,j} \omega_{ij}^{\frac{1}{p'}} |tF_{ij}|^{p-1}.\omega_{0k}^{\frac{1}{p}}$

$g_{rs} = \omega_{rs}^{\frac{1}{p}} + \frac{1}{T^{p-1}} \sum_j \mid tu_j \mid^{p-1} \omega_{0j}^{\frac{1}{p'}} \omega_{rs}^{\frac{1}{p}}$

f *resp(* g *) verify the conditions* (F_0) *and* (F_1) *resp(*(G_0) *and* (G_1).*)*

Example 8.7.2 *We suppose that* $p = n = m = 2$. $\Omega = \{x \in I\!R^2; |x| < 1\}$

$\omega = \omega_{11}(x) = \omega_{22}(x) = d^\lambda(x) = |x - \frac{x}{|x|}|^\lambda$. $\omega_{12} \equiv \omega_{21} \equiv 0$ *and* $\gamma(x) = |x - \frac{x}{|x|}|^\mu$.

Let $\sigma_{rs}(x,t,\eta,\xi) = \omega \mid \xi_{rs} \mid^{p-1} sign(\xi_{rs})$ *for all* $1 \le r \le n$ *and* $1 \le s \le m$; $\sigma = (\sigma_{rs})_{r,s}$, *with* $\partial_t \eta = 0$

$f = g = 0$. *With* $q > 1$; $\lambda \ne p-1$; $\mu > \frac{q\lambda}{2} - 2$, *by this conditions the hypothesis* $(P_1) - (P_3)$ *are verify.*

Perspectives

- *Introduce a new functional framework that can be overlooked exhibitor ps.*
- *Study of Dirichlet-type systems, in Sobolev spaces with weight and variable exposure.*
- *Study of Neumann-type problems with data Radon measure in the spaces of Sobolev with weight.*
- *Study of parabolic systems with conditions at the edges of Neumann in the spaces of Sobolev with weight.*
- *Study of second-member parabolic systems measured, in Sobolev spaces with variable exposure weight.*
- *Study of elliptical and parabolic systems, in Sobolev spaces with variable exposure weight using Minty's lemma.*
- *Introduce a new functional framework that can be replace the differential approximation with lower differentiation.*
- *Study of second-member measure elliptical and Parabolic systems, that does not break down in weight Sobolev spaces or in Sobolev spaces with variable exposure.*

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