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Some nonlinear elliptic inclusion problems within the non-variational framework

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DEDICATION

To my beloved parents: *Yamna* and *Mohamed*.

To my lovely wife *Wassima* and my little daughter *Julia*.

To my dear brothers *Fouad, Hicham* and *Hamza*.

This work dedicated to you today is only the fruit of your appreciation and continued efforts.

Your words of encouragement for patience and success still and will always ring in my ears.

I thank you for your support, your love and your affection which gives me the strength and the courage for facing difficulties and overcoming all challenges.

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MORAD OUBOUFETTAL
University of Sidi Mohamed Ben Abdellah
Saturday, March 02, 2024

ABSTRACT

Our goal in this thesis is to study some nonlinear elliptic inclusion problems with L^1-data in a bounded domain with boundary conditions of the Dirichlet type in the setting of Sobolev Spaces, Sobolev spaces with variable exponent or Orlicz-Sobolev spaces.

This work, composed of six chapters, present results of existence (uniqueness and regularity) of weak, entropy or renormalized solutions for the nonlinear problems of the type mentioned below.

After a brief presentation of definitions and results needed for the work, we prove in chapter 2 the existence of weak solutions for a strongly nonlinear elliptic problem of the form

$$\beta(u) - \operatorname{div}\left(a(x,u,\nabla u)\right) + g(x,u,\nabla u) \ni f.$$

In chapter 3, we study the existence and regularity of solution for nonlinear elliptic inclusion problems with degenerate coercivity of the type

$$\beta(u) - \operatorname{div}\left(\frac{a(x,\nabla u)}{(1+|u|)^{\theta(p-1)}}\right) \ni f.$$

Next, in chapter 4, we establish some existence results for the nonlinear multivalued elliptic problems having unbounded coefficients whose prototype is

$$\beta(u) - \operatorname{div}\left(\frac{|\nabla u|^{p-2}\nabla u}{(1-|u|)^{\rho}}\right) \ni f.$$

In chapter 5, we study the problem of chapter 3 in the functional framework involving Sobolev spaces with variable exponent.

The last results presented in chapter 6, are the existence and uniqueness of renormalized solutions in the Orlicz-Sobolev space for nonlinear multivalued elliptic problems of type

$$\beta(u) - \operatorname{div}(a(x,\nabla u) + F(u)) \ni f.$$

Keywords: Elliptic equation, Leray-Lions operator, Weak solution, Entropy solution, Renormalized solution, Inclusion problems, Maximal monotone graph, Sobolev spaces with variable exponent, Orlicz-Sobolev spaces, Degenerate coercivity, Unbounded coefficients, Regularity.

RÉSUMÉ

Notre objectif dans cette thèse est d'étudier certains problèmes d'inclusion elliptique non linéaire avec données L^1 dans un domaine borné avec des conditions aux limites de type Dirichlet dans le cadre des espaces de Sobolev, des espaces de Sobolev à exposant variable ou des espaces d'Orlicz-Sobolev.

Ce travail, composé de six chapitres, présente des résultats d'existence (unicité et régularité) de solutions faibles, entropiques ou renormalisées pour les problèmes non linéaires du type mentionné ci-dessous.

Après une brève présentation des définitions et des résultats nécessaires au travail, nous prouvons au chapitre 2 l'existence de solutions faibles pour un problème elliptique fortement non linéaire de la forme

$$\beta(u) - \operatorname{div}\left(a(x,u,\nabla u)\right) + g(x,u,\nabla u) \ni f.$$

Dans le chapitre 3, nous étudions l'existence et la régularité de solution pour des problèmes d'inclusion elliptique non linéaire avec coercitivité dégénérée du type

$$\beta(u) - \operatorname{div}\left(\frac{a(x,\nabla u)}{(1+|u|)^{\theta(p-1)}}\right) \ni f.$$

Ensuite, dans le chapitre 4, nous établissons quelques résultats d'existence pour des problèmes elliptiques multivoques non linéaires ayant des coefficients non bornés dont le prototype est

$$\beta(u) - \operatorname{div}\left(\frac{|\nabla u|^{p-2}\nabla u}{(1-|u|)^{\rho}}\right) \ni f.$$

Dans le chapitre 5, nous étudions les problèmes du chapitre 3 dans le cadre fonctionnel impliquant des espaces de Sobolev à exposant variable.

Les derniers résultats présentés dans le chapitre 6, sont l'existence et l'unicité de solutions renormalisées dans l'espace d'Orlicz-Sobolev pour des problèmes elliptiques multivoques non linéaires de type

$$\beta(u) - \operatorname{div}(a(x,\nabla u) + F(u)) \ni f.$$

Mots clés: Équation elliptique, Opérateur de Leray-Lions, Solution faible, Solution entropique, Solution renormalisée, Problèmes d'inclusion, Graphe maximal monotone, Espaces de Sobolev à exposant variable, Espaces d'Orlicz-Sobolev, Coercitivité dégénérée, Coefficients non bornés, Régularité.

LIST OF PUBLICATIONS

The publications that constitute the basis of the PhD thesis can be found in

Published works and works to appear

1. Y. Akdim, M. Belayachi and **M. Ouboufettal**, Existence of solution for nonlinear elliptic inclusion problems with degenerate coercivity and L^1-data, **J. Elliptic Parabol. Equ.**, 8(1), 2022, 127–150.

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3. Y. Akdim and **M. Ouboufettal**, Solvability of nonlinear elliptic inclusion problems in Orlicz spaces and L^1-data, **Rend. Circ. Mat. Palermo.**, 73(1), 2024, 489–511.

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4. Y. Akdim, M. Belayachi and **M. Ouboufettal**, Solvability of nonlinear multivalued elliptic problems involving degenerate coercivity with variable exponents and L^1-data, **Filomat**, 2024.

5. Y. Akdim and **M. Ouboufettal**, Existence of Solution for a General Class of Strongly Nonlinear Elliptic Problems, **Nonlinear Dyn. Syst. Theory.**, 2024.

Submitted works

1. Y. Akdim, M. Belayachi and **M. Ouboufettal**, Existence results for nonlinear multivalued elliptic problems involving unbounded coefficients and L^1-data.

LIST OF SYMBOLS

Ω	Bounded domain of $\mathbb{R}^N$ for $N \geq 1$
$\partial\Omega$	Boundary of Ω
$x = (x_1, \ldots, x_N)$	Point in $\mathbb{R}^N$ for $N \geq 1$
$dx = dx_1 \ldots dx_N$	Lebesgue measure in Ω
∇u	Gradient of a function u
div$u$	$\sum_{i=1}^{N} \frac{\partial u}{\partial x_i}$
$D(\beta)$	Domain of a multivalued mapping β
T_k	Truncation function $k > 0$
$supp(u)$	Support of a function u
u^+, u^-	$\max(u, 0), \max(-u, 0)$
$\rightarrow$	Strong convergence
$\rightharpoonup$	Weak convergence
$\overset{*}{\rightharpoonup}$	Weak-$*$ convergence
$\mathcal{D}(\Omega)$	Test functions in Ω
$L^p(\Omega)$	$\{u : \Omega \to \mathbb{R} \mid u \text{ measurable}, \int_\Omega \lvert u(x)\rvert^p dx < \infty\}$ for $1 \leq p < \infty$
$L^\infty(\Omega)$	$\{u : \Omega \to \mathbb{R} \mid u \text{ measurable, ess sup}_{x\in\Omega} \lvert u(x)\rvert < \infty\}$
$W^{1,p}(\Omega)$	$\left\{u : \Omega \to \mathbb{R} \mid u \in L^p(\Omega), \nabla u \in (L^p(\Omega))^N\right\}$ for $1 \leq p \leq +\infty$
$W_0^{1,p}(\Omega)$	Closure of $\mathcal{D}(\Omega)$ in $W^{1,p}(\Omega)$ for $1 \leq p < +\infty$
$W^{-1,p'}(\Omega)$	Dual space of $W_0^{1,p}(\Omega)$
$L^{p(.)}(\Omega)$	Variable exponent Lebesgue space
$W^{1,p(.)}(\Omega)$	Variable exponent Sobolev space
$W_0^{1,p(.)}(\Omega)$	Closure of $\mathcal{D}(\Omega)$ in $W^{1,p(.)}(\Omega)$
$L_M(\Omega)$	Orlicz space
$W^1L_M(\Omega)$	Orlicz-Sobolev space
$W_0^1L_M(\Omega)$	Closure of $\mathcal{D}(\Omega)$ in $W^1L_M(\Omega)$

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Introduction

The phenomena of melting and solidification occurs in a multitude of natural and industrial situations, from the melting of the polar ice caps or the solidification of lava from a volcano to the manufacture of ice cream or the production of steel. For a material to undergo a solid-liquid phase change, thermal energy has to be delivered to the solid to break the bonds that maintain its molecules or atoms in an organized lattice structure. For the opposite process, energy must be taken from the liquid phase to slow down the motion of its molecules and organize them back into a stable lattice structure. The mathematical formulation describing this intuitively simple physical process is known as ***the Stefan problem***, named after the Slovene physicist Josef Stefan. It is one of the most important ***free boundary problems*** for the heat-diffusion equation. A complete bibliography may be found in [103].

Partial differential equations (**PDEs**) are in fact involved in many models of physics, engineering or biology, such as the propagation of heat or sound, the flow of fluids, electrodynamics, the propagation of epidemics. They are also used for weather forecasting models and for climate models [51, 54, 78, 83, 84, 99, 101, 104]. PDEs came into being after the work of Leibniz and Newton on infinitesimals in the early 18th century; it was Euler who in 1757 formulated the first PDE system (known as Euler's equations) to describe the motion of an incompressible fluid. PDEs can be linear, as is the case with the heat equation for example, or nonlinear, such as the Euler or Navier-Stokes equations which describe the motion of a fluid.

The mathematician's primary concern when confronted with a partial differential equation is to make sense of the partial differential equation in appropriate functional spaces and to demonstrate the existence and uniqueness of the solution. During this process, various difficulties linked to the type of equation of the equation (elliptic,

parabolic, hyperbolic, degenerate, ...), to the regularity of the data and also to the definition of appropriate spaces.

This work is devoted to study, develop and deeply understand a set of problems, tools, or simply questions framed in the huge field of PDEs. In particular, we focus on the study of a class of nonlinear elliptic inclusion problem in a bounded domain with boundary conditions of the Dirichlet type, of the form

$$(S,f)\quad \begin{cases} \beta(u)+A(u)+g(x,u,\nabla u)\ni f & \text{in } \Omega,\\ \\ u=0 & \text{on } \partial\Omega, \end{cases}$$

where Ω is a bounded domain in $\mathbb{R}^N$, $\beta:\mathbb{R}\to 2^{\mathbb{R}}$ is a set valued, maximal monotone mapping such that $0\in\beta(0)$, A is a monotone operator of divergence form, while $g(x,s,\xi)$ is a nonlinear term which has a growth condition with respect to ξ and no growth with respect to s but it satisfies a sign-condition on s. The right hand side f is assumed to belong to $L^1(\Omega)$.

Note that, since β may be multivalued, problems of type (S,f) appear in various phenomena with changes of state like multiphase Stefan problem (cf. [43, 55, 69, 88, 89, 90]) and in the weak formulation of the mathematical model of the so called Hele–Shaw problem (cf. [60] and [66]).

Many works have been dedicated to the questions of existence, uniqueness and regularity of solutions for this type of problem (S,f). Let us describe their results in some detail:

The equation (S,f) can be viewed as generalization of the $p-$Laplacian equation

$$(L,f)\quad \begin{cases} -\mathrm{div}(|\nabla u|^{p-2}\nabla u)=f & \text{in } \Omega,\\ \\ u=0 & \text{on } \partial\Omega, \end{cases}$$

In case of $1<p<\infty$ and $f\in W^{-1,p'}(\Omega)$ it follows from Minty-Browder Theorem that there exists a unique solution $u\in W_0^{1,p}(\Omega)$ to (L,f) in the sense of distributions (cf. [82] and [86]).

For $1<p<N$ and right-hand side $f\in L^1(\Omega)$ we can not expect solutions $u\in W_0^{1,p}(\Omega)$. Indeed, supposing that for each $f\in L^1(\Omega)$ there exists a solution $u\in W_0^{1,p}(\Omega)$ of (L,f), as $-\mathrm{div}(|\nabla u|^{p-2}\nabla u)\in W^{-1,p'}(\Omega)$, it follows that $L^1(\Omega)\subset$

$W^{-1,p'}(\Omega)$. By duality, this implies $W_0^{1,p}(\Omega) \subset L^\infty(\Omega)$. From Sobolev embedding theorems we know that this is not true in general if $1 < p < N$.

Supposing $1 < p < 2$ and that there exists a solution $u \in W_0^{1,1}(\Omega)$ of (L, f), as $|\nabla u|^{p-2}\nabla u \in \left(L^{1/(p-1)}(\Omega)\right)^N$ we get $-\mathrm{div}(|\nabla u|^{p-2}\nabla u) \in W^{-1,1/(p-1)}(\Omega)$ and therefore $L^1(\Omega) \subset W^{-1,1/(p-1)}(\Omega)$. By duality, this implies $W_0^{1,1/(2-p)}(\Omega) \subset L^\infty(\Omega)$. By Sobolev embedding theorems, this is true if $p > 2 - \frac{1}{N}$. Therefore we can not even expect solutions $u \in W_0^{1,1}(\Omega)$ of (L, f) for $p \leq 2 - \frac{1}{N}$ and $f \in L^1(\Omega)$. In the case of $p > 2 - \frac{1}{N}$ the existence of a distributional solution u of (L, f) in the space

$$\bigcap_{q < \frac{N(p-1)}{N-1}} W_0^{1,q}(\Omega)$$

has been shown in [37].

As it has been shown in [100] and [96], the distributional solution u is in general not unique. In order to get well-posedness for $L^1(\Omega)-$data, the notion of an entropy solution for problem (L, f) was introduced by Bénilan et al. in [25]. Moreover, existence and uniqueness of an entropy solution of (L, f) has been established for $1 < p < N$.

An equivalent notion of solution for problem (L, f) is called renormalized solution. The concept of renormalized solutions was introduced by DiPerna and Lions in [64]. This notion was then extended to the study of various problems of partial differential equations of parabolic, elliptic-parabolic and hyperbolic type, we refer to [14, 15, 19, 26, 36, 52, 97] and the references therein for more details.

In the absence of the nonlinearity term $\beta \equiv 0$ and in the case where $Au := -\mathrm{div}\,(a(x, u, \nabla u))$ such that $a(x, s, \xi) : \Omega \times \mathbb{R}^N \times \mathbb{R}^N \to \mathbb{R}^N$ is a Carathéodory function satisfying the classical Leray-Lions-type conditions, our equation becomes:

$$(G, f) \quad \begin{cases} -\mathrm{div}\,(a(x, u, \nabla u)) + g(x, u, \nabla u) = f & \text{in } \Omega, \\ \\ u = 0 & \text{on } \partial\Omega. \end{cases}$$

We found in the variational case (i.e. $f \in W^{-1,p'}(\Omega)$), the work of Bensoussan, Boccardo and Murat [29], where the existence of a weak solution of the strongly nonlinear problem associated to (G, f) was given by using the strong convergence of the positive (resp. negative) part of the sequence u_ε of the approached problem.

The same problem was studied by Boccardo, Gallouët and Murat [39] by using the strong convergence of truncations.

The extension of these results when $f \in L^1(\Omega)$ was treated in particular in [38], but under a coercivity condition on the nonlinearity of type:

$$|g(x,s,\xi)| \geq \gamma|\xi|^p \text{ for } |s| \geq \sigma \text{ where } \sigma > 0 \text{ and } \gamma > 0. \tag{1}$$

The use of (1) appears in particularly at level of $u \in W_0^{1,p}(\Omega)$.

Next, Porretta [94] deleted the condition (1) and proved the existence of solution of the strongly nonlinear problem associated to (G,f) with $u \in W_0^{1,q}(\Omega)$ for all $q < \frac{N(p-1)}{N-1}$.

More recently, the same author [95] studied the problem (G,f) without assuming the sign condition, but by changing the classical growth condition of the nonlinearity g by

$$|g(x,s,\xi)| \leq h(x) + \rho(s)|\xi|^p, \text{ where } \rho \in L^1(\Omega) \text{ and } \rho > 0.$$

On the other side, for $g \equiv 0$ and $Au := -\text{div}\,(a(x,u,\nabla u))$, our problem (S,f) is written:

$$(D,f) \quad \begin{cases} \beta(u) - \text{div}\,(a(x,u,\nabla u)) \ni f & \text{in } \Omega, \\ \\ u = 0 & \text{on } \partial\Omega. \end{cases}$$

Although doubly nonlinear equations of the type (D,f) are of mathematical interest in their own right, they arise as a model for a variety of diffusion problems. In particular, they comprise in a unifying scheme, free-boundary problems of different natures. An example situation is nonstationary saturated-unsaturated flow of an incompressible fluid through a porous medium in the case of time-dependent water levels. A weak formulation of this problem leads to equations of the type (D,f); see [12].

However, a large held of application for our results is the diffusion problem involving a solid-liquid phase change of the Stefan type. An architypical example is heat transfer during solidification in a nonhomogeneous medium with heat capacity and thermal conductivities dependent on temperature and convection in the liquid phase [30].

There is a wide literature in which one can find existence results for problem (D,f). Let us mention some parallel developments. First, The work of Bénilan, Crandall and Sacks [27] was pioneer in this kind of problems. They study problem (D,f) for the Laplacian operator, i.e., for $a(x,s,\xi) = \xi$.

In the case of nonlinear operator and $f \in W^{-1,p'}(\Omega)$, it is known (cf. [106]) that the problem (D, f) has a solution in the standard sense, the so-called weak solution, that is a couple $(u, b) \in W_0^{1,p}(\Omega) \times W^{-1,p'}(\Omega) \cap L^1(\Omega)$ such that $b \in \beta(u)$ a.e. in Ω and

$$\int_\Omega b\varphi dx + \int_\Omega a(x, u, \nabla u) \cdot \nabla \varphi dx = \int_\Omega f\varphi dx, \quad \forall \varphi \in W_0^{1,p}(\Omega).$$

However, in the L^1-theory for $p-$Laplacian type equations the important work is [25]. The authors considered $Au := -\text{div}\,(a(x, \nabla u))$ and proved that, for any $f \in L^1(\Omega)$, there exists a unique, named entropy solution, $(u, b) \in \mathcal{T}_0^{1,p}(\Omega) \times L^1(\Omega)$, $b(x) \in \beta(u(x))$ a.e. in Ω, such that

$$\int_\Omega bT_k(u - \varphi)dx + \int_\Omega a(x, \nabla u) \cdot \nabla T_k(u - \varphi)dx \leq \int_\Omega fT_k(u - \varphi)dx, \quad \forall k > 0,$$

for all $\varphi \in W_0^{1,p}(\Omega) \cap L^\infty(\Omega)$.

Recently, in another important work [105], Wittbold and Zimmermann adapted the notion of renormalized solution to a new and interesting elliptic problem type diffusion-convection in the framework of variable exponent Sobolev spaces

$$(E, f) \quad \begin{cases} \beta(u) - \text{div}\,(a(x, \nabla u) + F(u)) \ni f & \text{in } \Omega, \\ \\ u = 0 & \text{on } \partial\Omega. \end{cases}$$

They proved for F locally Lipschitz continuous, β maximal monotone mapping with $0 \in \beta(0)$, $f \in L^1(\Omega)$ and a continuous exponent $p(\cdot)$, the existence and uniqueness of a renormalized solution.

The equations of the form (E, f) have been studied extensively by many authors, for example, see [2] for the weighted variable exponent case and [8] for the anisotropic case.

In dealing with nonlinear elliptic inclusion problems, there are several works in this direction, we refer to [7, 16, 17, 18, 20, 56, 74, 76, 77, 87].

On a parallel direction, nonlinear problems with principal part having degenerate coercivity have been extensively studied in the recent years, see for instance [13, 32, 34, 35] for the elliptic case, and [42, 44, 58, 65, 85, 107, 91] for parabolic case.

In this context, for $\beta \equiv 0$ and $Au := -\text{div}\left(\frac{a(x,\nabla u)}{(1+|u|)^{\theta(p-1)}}\right)$ the problem (S, f) can be seen as follow

$$(E_\theta^p, f) \quad \begin{cases} -\text{div}\left(\frac{a(x,\nabla u)}{(1+|u|)^{\theta(p-1)}}\right) = f & \text{in } \Omega, \\ \\ u = 0 & \text{on } \partial\Omega. \end{cases}$$

For different summability of the data f, Alvino et al. in the pioneering paper [13] have done the existence and regularity of solutions (or entropy solutions) to problem (E_θ^p, f). More precisely, they proved some existence and regularity results for all $0 \leq \theta \leq 1$, and a non-existence result for $\theta > 1$, even if the datum $f \in L^\infty(\Omega)$. Moreover, if f is less regular, then problem (E_θ^p, f) admits only distributional or entropy solutions for $0 \leq \theta < 1$ (for more details, see also [31, 32, 34, 35]). The obstacle problems governed by degenerate elliptic equations are studied in [3, 33, 110, 111], but the obtained results concerning the regularity still incomplete.

Recently, Zhang and Fu [108] established a partial generalization of results from the pioneering paper [13] of Alvino et al. to the variable exponents case. More precisely, they considered the problems of the form

$$(E_\theta^{p(x)}, f) \quad \begin{cases} -\text{div}\left(\frac{a(x,\nabla u)}{(1+|u|)^{\theta(p(x)-1)}}\right) = f & \text{in } \Omega, \\ \\ u = 0 & \text{on } \partial\Omega, \end{cases}$$

where $a(x, \xi) : \Omega \times \mathbb{R}^N \to \mathbb{R}^N$ is a Carathéodory function which verifies the natural extensions of Leray-Lions assumptions to the variable exponents case such that $p(\cdot) \in C(\overline{\Omega})$ such that $1 < p^- \leq p^+ < N$ and $f \in L^{m(x)}(\Omega)$. Based on the theory of variable exponent Sobolev space, if $m^- > \frac{N}{p^-}$, they proved the existence of a bounded weak energy solution to $(E_\theta^{p(x)}, f)$ for $0 \leq \theta \leq \frac{p^- - 1}{p^+ - 1}$. While if $m^- \leq \frac{N}{p^-}$, the existence of weak or entropic solutions is achieved only for $0 \leq \theta < \frac{p^- - 1}{p^+ - 1}$.

On another hand, we want to mention the important work in [57], Della Pietra and di Blasio have investigated the existence and regularity of weak or entropy solutions to

the following problem

$$\begin{cases} -\operatorname{div}\left(\dfrac{|\nabla u|^{p-2}\nabla u}{(1-|u|)^{\rho}}\right) = f & \text{in } \Omega, \\ \\ u = 0 & \text{on } \partial\Omega, \end{cases}$$

for the data $f \in L^m(\Omega)$ with $m \geq 1$.

Each of the above mentioned subjects is a difficult one to handle and it can be quite challenging to deal with nonlinear multivalued elliptic problems which can enlarge the scale of the applications.

Therefore, the present thesis generalizes some results as described before. More precisely, it deals with the existence (uniqueness and regularity results) for weak, entropy or renormalized solutions of a class of nonlinear elliptic inclusion problem type (S, f) in the setting of Sobolev Spaces, Sobolev spaces with variable exponent or Orlicz-Sobolev spaces.

Finally, we briefly summarize the organization of this thesis and the main results contained in every chapter. The thesis is conformed by six chapters:

In chapter 1, we give some basic notations and necessary results that we will use in the accomplishment of the work. We start by presenting the standard Lebesgue and Sobolev spaces, we also recall the variable exponent Lebesgue spaces and the Orlicz-Sobolev spaces. Moreover, we will introduce some notation and functions which will be used frequently, we will give the general analytic framework, which is needed to study multivalued differential equations with maximal monotone drift and at last we recall an abstract result of surjectivity.

Chapter 2 is devoted to investigate the existence of weak solutions for the following strongly nonlinear elliptic problem

$$(S, f) \quad \begin{cases} \beta(u) - \operatorname{div}\left(a(x, u, \nabla u)\right) + g(x, u, \nabla u) = f & \text{in } \Omega, \\ \\ u = 0 & \text{on } \partial\Omega, \end{cases}$$

where $\beta : \mathbb{R} \to 2^{\mathbb{R}}$ is a set valued, maximal monotone mapping such that $0 \in \beta(0)$, $a(x, s, \xi) : \Omega \times \mathbb{R}^N \times \mathbb{R}^N \to \mathbb{R}^N$ is a Carathéodory function satisfying the classical Leray-Lions-type conditions, $g(x, s, \xi)$ is a nonlinear term which has a growth condition with respect to ξ and no growth with respect to s but it satisfies a sign-condition on s, while the right-hand side f is assumed to belong either to $L^{\infty}(\Omega)$ or to $L^1(\Omega)$. The

work presented can be seen as a generalization of [39] and [106]. Furthermore, the results are obtained using a classic theorem of J. L. Lions [86] through utilizing the Yosida approximation of the graph and via the strong convergence of the truncations. The results of this chapter are based on the papers [9] and [10].

Chapter 3 is based on the paper [6], in which we have motivated by the two works [13] and [25], we study the existence of entropy solution to nonlinear elliptic inclusion problems with degenerate coercivity for any maximal monotone graph β with L^1-data. More precisely we deal with problem of the form

$$(P,f)\quad \begin{cases} \beta(u) - \operatorname{div}\left(\dfrac{a(x,\nabla u)}{(1+|u|)^{\theta(p-1)}}\right) \ni f & \text{in } \Omega, \\ \\ u = 0 & \text{on } \partial\Omega. \end{cases}$$

Here $Au := -\operatorname{div}\left(\dfrac{a(x,\nabla u)}{(1+|u|)^{\theta(p-1)}}\right)$ is a monotone operator acting on $W_0^{1,p}(\Omega)$ such that $1 < p < N$, where $a(x,\xi) : \Omega \times \mathbb{R}^N \to \mathbb{R}^N$ is a Carathéodory function satisfying the classical Leray-Lions-type conditions.

For $0 < \theta < 1$, the item $A(u)$ degenerates, hence, a slow diffusion effect may appear as soon as the solution u becomes large. The lack of coercivity implies that the standard Leray-Lions surjectivity theorem can not be used even in the case $f \in W^{-1,p'}(\Omega)$. However, by cutting the nonlinearity and using the technique of approximation, a pseudo-monotone and coercive differential operator on $W_0^{1,p}(\Omega)$ can be applied to establish a priori estimates on approximating solutions. As a result, the existence of weak or entropy solutions to problems (P,f), can be obtained by taking limitation (see, e.g., [13]).

If $f \in L^\infty(\Omega)$, we proved the existence and uniqueness of a bounded weak energy solution of the problem (P,f). While if $f \in L^1(\Omega)$, we established the existence and some regularity results for the so-called entropy solution of the problem (P,f) only for $0 \le \theta(p-1) < \frac{N(p-1)}{N-1} - 1$. Moreover, we showed that $u \in W_0^{1,q}(\Omega)$, with $q \in \left(1, \frac{N(p-1)(1-\theta)}{N-1-\theta(p-1)}\right)$.

In chapter 4, we analyze the existence results to nonlinear multivalued elliptic problems involving unbounded coefficients and L^1- data. More precisely, we will deal with the following nonlinear multivalued elliptic problem with Dirichlet boundary

conditions:

$$(U,f)\quad \begin{cases} \gamma(u) - \operatorname{div}\left(\dfrac{|\nabla u|^{p-2}\nabla u}{(1-|u|)^{\rho}}\right) \ni f & \text{in } \Omega, \\ \\ u = 0 & \text{on } \partial\Omega, \end{cases}$$

where $\gamma : \mathbb{R} \to 2^{\mathbb{R}}$ is a set valued, maximal monotone mapping such that $0 \in \gamma(0)$ and $Au := \operatorname{div}\left(\dfrac{|\nabla u|^{p-2}\nabla u}{(1-|u|)^{\rho}}\right)$ is a monotone operator from $W_0^{1,p}(\Omega)$ into its dual $W^{-1,p'}(\Omega)$.

For the data $f \in L^m(\Omega)$ with $m \geq 1$, and in the case where $\gamma \equiv 0$, Della Pietra and di Blasio have investigated the existence and regularity of weak or entropy solutions to problems (U,f) in [57].

For $\rho > 0$, it is obvious that the differential operator $A(u)$ is defined on $W_0^{1,p}(\Omega)$, but that it may not be coercive on the same space as u near to ± 1. On account of this lack of coercivity, the standard Leray-Lions surjectivity theorem can not be applied even in the case $f \in W^{-1,p'}(\Omega)$. Nevertheless, by cutting the nonlinearity near the critical value 1 and applying the technique of approximation, a coercive and pseudo-monotone differential operator on $W_0^{1,p}(\Omega)$ give a meaning to the equation when the solutions near to ± 1 and can be applied to establish a priori estimates on approximating solutions also to manage the set $\{x \in \Omega : |u(x)| = 1\}$. As a result, the existence of weak or entropy solutions to problems (U,f), can be obtained. The results presented in this chapter are contained in [4].

In chapter 5, we established a partial generalization of results from [6] to the variable exponents case. More precisely, we will deal with the following nonlinear multivalued elliptic problem with Dirichlet boundary conditions:

$$(P^*,f)\quad \begin{cases} \gamma(u) - \operatorname{div}\left(\dfrac{a(x,\nabla u)}{(1+|u|)^{\theta(p(x)-1)}}\right) \ni f & \text{in } \Omega, \\ \\ u = 0 & \text{on } \partial\Omega, \end{cases}$$

where γ is a maximal monotone graph such that $0 \in \gamma(0)$ and $a(x,\xi) : \Omega \times \mathbb{R}^N \to \mathbb{R}^N$ is a Carathéodory function which verifies the natural extensions of Leray-Lions assumptions to the variable exponents case such that $p(\cdot)$ satisfies the log-Hölder continuity condition (see (1.1) below) such that $1 < p^- \leq p^+ < N$. As well as the right-hand side $f \in L^1(\Omega)$.

We will study the existence and some $q(x)$-regularity results of entropy solution to the nonlinear multivalued elliptic problem (P^*, f) with L^1-data and for any maximal monotone graph γ where the functional setting involves Sobolev spaces with variable exponents.

We want to mention that there are some difficulties associated with these kinds of problems:

Firstly, $Au := -\text{div}\left(\frac{a(x,\nabla u)}{(1+|u|)^{\theta(p(x)-1)}}\right)$ is a monotone operator from $W_0^{1,p(x)}(\Omega)$ into its dual. However, when $0 < \theta < 1$, the term $A(u)$ becomes degenerate, which can result in a slow diffusion effect when the solution u becomes large. As a result of this degeneracy, classical methods for elliptic equations are not applicable, even when the datum f is regular.

The second difficulty arises when a variable exponential growth condition is given for $a(\cdot,\cdot)$ in the equation. In this case, the operator A exhibits a more complex nonlinearity, making some techniques used in the constant exponent case inapplicable.

We utilize the approximation procedure to obtain the results. Specifically, we focus on a sequence of nondegenerate Dirichlet problems, which ensures the existence of solutions. By obtaining a priori estimates on the approximate solutions and subsequently taking the limit, we are able to find a solution for (P^*, f). The results of this chapter are based on the paper [5].

The last results presented in chapter 6 are based on the paper [11], we establish the existence and uniqueness of renormalized solutions in the Orlicz-Sobolev space to the nonlinear multivalued elliptic problem (E, f).

As mentioned above, the function a is supposed to satisfy a polynomial growth and coerciveness condition. When trying to generalize this restriction on a i.e. in the case where a satisfies a more general growth condition with respect to ∇u (such growth to relax the coefficients of the operator A), the adequate space in which (E, f) can be studied is the Orlicz-Sobolev spaces $W_0^1 L_M(\Omega)$ where the $N-$function M is related to the actual growth of a.

Nevertheless, there is no literature has considered the problem (E, f) in the setting of Orlicz-Sobolev spaces $W_0^1 L_M(\Omega)$. It is our purpose to prove an existence theorem for (E, f) under the assumption that the $N-$function $\overline{M}$ conjugate to M satisfies the Δ_2 condition and it is not clear whether the present proof can be further adapted to obtain the same result for general $N-$functions. In this case, existence of approximate

solutions follows from the theory of monotone operators in Orlicz-Sobolev spaces as developed by Gossez and Mustonen in [73]. The arguments used to prove the convergence of such approximate solutions to a renormalized solution of (E, f) are based on an approximation property in Orlicz-Sobolev spaces proved by Gossez in [71, Theorem 4].

We want to mention that in this chapter, we extend the recent works [2, 8, 105] by using some ideas developed in [53, 74, 105]. The main difficulties we encounter are how to establish the a priori estimates and the convergence results.

This thesis ends with a conclusion and further perspectives of our works.

Chapter 1

Some preliminary tools and basic results

In this chapter, we present the functional setting and some auxiliary results that will play an important role throughout this thesis.

Unless otherwise required, in this chapter, $\Omega \in \mathbb{R}^N$ is a regular bounded open set equipped with N-dimensional Lebesgue measure. We denote by $\mathcal{D}(\Omega)$ the space of functions of class $\mathcal{C}^\infty$ with compact support in Ω.

1.1 Standard Lebesgue and Sobolev spaces

In this section, we recall some basic facts on classical Lebesgue and Sobolev spaces with constant exponent that we will use in the remainder of this thesis. For further details on this topic, we refer to [1, 47, 79, 80].

Let $1 \leq p < \infty$, the Lebesgue space $L^p(\Omega)$ is the space of all measurable functions $u : \Omega \to \mathbb{R}$, for which

$$\|u\|_{L^p(\Omega)} = \left(\int_\Omega |u(x)|^p \mathrm{d}x \right)^{1/p}$$

is finite.

The space $L^\infty(\Omega)$ consists all Lebesgue measurable functions that are essentially bounded on Ω. The norm of u in $L^\infty(\Omega))$ is its essential supremum, i.e.

$$\|u\|_{L^\infty(\Omega)} = \inf\{c > 0 : |u(x)| \leq c \text{ a.e. } x \in \Omega\}.$$

Endowed with the norm $\|.\|_{L^p(\Omega)}$, the space $L^p(\Omega)$, $1 \leq p \leq \infty$, is a Banach space, separable for $1 \leq p < \infty$ and reflexive for $1 < p < \infty$.

For all $1 \leq p < \infty$, the topological dual space of $L^p(\Omega)$ is isometrically identified with $L^{p'}(\Omega)$ where p' is the Hölder conjugate exponent, i.e., $\frac{1}{p} + \frac{1}{p'} = 1$ (being $(L^p(\Omega))' = L^{p'}(\Omega)$). On the other hand, $(L^\infty(\Omega))'$ is strictly bigger than $L^1(\Omega)$.

Let us just recall Hölder's inequality stating that, if $u \in L^p(\Omega)$ and $v \in L^{p'}(\Omega)$, then

$$\left| \int_\Omega u(x)v(x)\mathrm{d}x \right| \leq \|u\|_{L^p(\Omega)} \|v\|_{L^{p'}(\Omega)}.$$

For $1 \leq p \leq \infty$, we define the Sobolev space

$$W^{1,p}(\Omega) = \{u \in L^p(\Omega) : \nabla u \in (L^p(\Omega))^N\},$$

which is a Banach space for the norm

$$\|u\|_{W^{1,p}(\Omega)} = \left(\|u\|^p_{L^p(\Omega)} + \|\nabla u\|^p_{L^p(\Omega)} \right)^{\frac{1}{p}} \text{ if } 1 \leq p < \infty,$$

or

$$\|u\|_{W^{1,\infty}(\Omega)} = \max\left(\|u\|_{L^\infty(\Omega)}, \|\nabla u\|_{L^\infty(\Omega)} \right) \text{ if } p = \infty,$$

while $W^{1,p}_0(\Omega)$ will denote the closure of $\mathcal{D}(\Omega)$ with respect to this norm.

The space $W^{1,p}(\Omega)$ is separable for $1 \leq p < \infty$ and reflexive for $1 < p < \infty$. Note that if Ω is sufficiently smooth, then $W^{1,\infty}(\Omega)$ identifies with the space of locally Lipschitz functions.

For $1 \leq p < \infty$, we designate by $W^{-1,p'}(\Omega)$ the dual space of $W^{1,p}_0(\Omega)$, by a well-known result, each element T in $W^{-1,p'}(\Omega)$ can be written in the form $T = \mathrm{div}(F)$ where F belongs to $\left(L^{p'}(\Omega)\right)^N$.

For functions in Sobolev spaces we will often use Sobolev's theorem stating that, if $1 \leq p < N$, $W^{1,p}_0(\Omega)$ continuously injects into $L^{p^*}(\Omega)$ with $p^* = \frac{Np}{N-p}$; if $p = N$, $W^{1,p}_0(\Omega)$ continuously injects into $L^q(\Omega)$ for every $p \leq q < \infty$, while if $p > N$, $W^{1,p}_0(\Omega)$ continuously injects into $L^\infty(\Omega)$. In particular, for every $u \in W^{1,p}_0(\Omega)$, there exists a positive constant S depending only on N and p such that

$$\|u\|_{L^{p^*}(\Omega)} \leq S\|\nabla u\|_{L^p(\Omega)},$$

this inequality is called Sobolev's inequality.

Let us also recall Rellich's theorem stating that, if $1 \leq p < N$, the injection of $W^{1,p}_0(\Omega)$ into $L^q(\Omega)$ is compact for every $1 \leq q < p^*$, and Poincaré's inequality

declaring that, for every $u \in W_0^{1,p}(\Omega)$, there exists a positive constant C depending only on Ω and p such that

$$\|u\|_{L^p(\Omega)} \leq C\|\nabla u\|_{L^p(\Omega)}.$$

So that $\|\nabla u\|_{L^p(\Omega)}$ can be used as an equivalent norm in $W_0^{1,p}(\Omega)$.

Moreover, we will use several times the following result due to G. Stampacchia (see [80]): Let $G : \mathbb{R} \to \mathbb{R}$ be a Lipschitz function such that $G(0) = 0$. Then for every $u \in W_0^{1,p}(\Omega)$ we have $G(u) \in W_0^{1,p}(\Omega)$ and $\nabla G(u) = G'(u)\nabla u$ almost everywhere in Ω.

We recall here some results on the convergence of sequences of measurable functions.

Theorem 1.1.1. *(Vitali's convergence theorem [48]) Let (u_n) be a sequence of functions in $L^p(\Omega)$ with $1 \leq p < \infty$ such that*

- *$u_n \to u$ a.e. in Ω.*
- *(u_n) is equi-integrable, that is, for every $\varepsilon > 0$, there exists $\delta > 0$ such that*

$$\int_E |u_n|^p dx \leq \varepsilon,$$

for all n and for every measurable set $E \subset \Omega$ with $meas(E) \leq \delta$.

Then $u_n \to u$ in $L^p(\Omega)$.

Proposition 1.1.2. *(Weak compactness [48]) Assume that $1 < p < \infty$ and the sequence (u_n) is bounded in $L^p(\Omega)$. Then there exists a subsequence (u_{n_k}) and a function $u \in L^p(\Omega)$ such that $u_{n_k} \rightharpoonup u$ weakly in $L^p(\Omega)$, i.e.,*

$$\int_\Omega u_{n_k} v dx \to \int_\Omega uv dx \text{ as } k \to \infty, \text{ for all } v \in L^{p'}(\Omega).$$

Similarly, if (u_n) is bounded in $L^\infty(\Omega)$. Then there exists a subsequence (u_{n_k}) and a function $u \in L^\infty(\Omega)$ such that $u_{n_k} \rightharpoonup u$ weakly- in $L^\infty(\Omega)$, i.e.,*

$$\int_\Omega u_{n_k} v dx \to \int_\Omega uv dx \text{ as } k \to \infty, \text{ for all } v \in L^1(\Omega).$$

It is important to note that the above theorem is false when $p = 1$, since a bounded sequence in $L^1(\Omega)$ has in general no weak convergence property.

Theorem 1.1.3. *(Dunford–Pettis [48]) Let (u_n) be a bounded sequence of $L^1(\Omega)$ functions. Assume that for every measurable subset $E \subset \Omega$, one has $\int_E |u_n| \to 0$, as $meas(E) \to 0$, uniformly with respect to n. Then (u_n) has a subsequence weakly converging to some $u \in L^1(\Omega)$.*

1.2 Variable exponent Lebesgue spaces

In the following, we will briefly review some definitions and fundamental properties of Lebesgue and Sobolev spaces with variable exponents for the reader's convenience. We suggest the reader to [61, 62, 63, 68, 67, 75, 109] and the references therein for more background about these spaces.

Let Ω be a bounded open subset of $\mathbb{R}^N$ ($N \geq 2$), we say that a real-valued continuous function $p(\cdot)$ is log-Hölder continuous on $\overline{\Omega}$ if

$$|p(x) - p(y)| \leq \frac{C}{-log|x-y|} \text{ for all } x, y \in \overline{\Omega} \text{ such that } 0 < |x-y| \leq \frac{1}{2}, \tag{1.1}$$

where C is a constant.

We denote

$$C_+(\overline{\Omega}) = \left\{\text{log-Hölder continuous function } p : \overline{\Omega} \to \mathbb{R} \text{ with } 1 < p^- \leq p^+ < N\right\},$$

where $p^- = \min_{x\in\overline{\Omega}} p(x)$ and $p^+ = \max_{x\in\overline{\Omega}} p(x)$.

We define the variable exponent Lebesgue space for $p(\cdot) \in C_+(\overline{\Omega})$ by

$$L^{p(x)}(\Omega) = \left\{u : \Omega \to \mathbb{R} \text{ measurable} : \int_\Omega |u(x)|^{p(x)} dx < \infty\right\}.$$

The space $L^{p(x)}(\Omega)$ under the norm

$$\|u\|_{p(x)} = \inf\left\{\lambda > 0 : \int_\Omega \left|\frac{u(x)}{\lambda}\right|^{p(x)} dx \leq 1\right\}$$

is a uniformly convex Banach space and therefore reflexive. The conjugate space of $L^{p(x)}(\Omega)$ will be denoted by $L^{p'(x)}(\Omega)$ where $\frac{1}{p(x)} + \frac{1}{p'(x)} = 1$.

Proposition 1.2.1 (See [67], [109]).

i) For any $u \in L^{p(x)}(\Omega)$ and $v \in L^{p'(x)}(\Omega)$, we have the Hölder type inequality

$$\left|\int_\Omega uv dx\right| \leq \left(\frac{1}{p^-} + \frac{1}{p'^-}\right) \|u\|_{p(x)} \|v\|_{p'(x)}.$$

ii) For all $p_1(\cdot), p_2(\cdot) \in C_+(\overline{\Omega})$ such that $p_1(x) \leq p_2(x)$, we have

$$L^{p_2(x)}(\Omega) \hookrightarrow L^{p_1(x)}(\Omega)$$

and the embedding is continuous.

Proposition 1.2.2 (See [67], [109]). *If we denote*

$$\rho(u) = \int_\Omega |u(x)|^{p(x)} dx \quad \text{for all } u \in L^{p(x)}(\Omega),$$

then, the following assertions hold

i) $\|u\|_{p(x)} < 1$ *(resp.* $= 1, > 1$*)* $\Leftrightarrow \rho(u) < 1$ *(resp.* $= 1, > 1$*).*

ii) $\|u\|_{p(x)} > 1 \Rightarrow \|u\|^{p^-}_{p(x)} \leq \rho(u) \leq \|u\|^{p^+}_{p(x)}$.

iii) $\|u\|_{p(x)} < 1 \Rightarrow \|u\|^{p^+}_{p(x)} \leq \rho(u) \leq \|u\|^{p^-}_{p(x)}$.

iv) $\|u\|_{p(x)} \to 0 \Leftrightarrow \rho(u) \to 0$ *and* $\|u\|_{p(x)} \to \infty \Leftrightarrow \rho(u) \to \infty$.

Now, we define the Sobolev space with variable exponent by

$$W^{1,p(x)}(\Omega) = \left\{ u \in L^{p(x)}(\Omega) : |\nabla u| \in L^{p(x)}(\Omega) \right\}$$

and it can be equipped with the norm

$$\|u\|_{1,p(x)} = \|u\|_{p(x)} + \|\nabla u\|_{p(x)}. \tag{1.2}$$

By $W^{1,p(x)}_0(\Omega)$, we denote the subspace of $W^{1,p(x)}(\Omega)$, which is the closure of $C^\infty_0(\Omega)$ with respect to the norm (1.2).

The critical Sobolev exponent is defined as $p^*(x) = \frac{Np(x)}{N-p(x)}$ for $p(x) < N$, and the dual space of $W^{1,p(x)}_0(\Omega)$ will be denoted by $W^{-1,p'(x)}(\Omega)$.

Proposition 1.2.3 (See [67], [109]).

i) The spaces $W^{1,p(x)}(\Omega)$ *and* $W^{1,p(x)}_0(\Omega)$ *are separable and reflexive Banach spaces.*

ii) If $q(\cdot) \in C_+(\overline{\Omega})$ *and* $q(x) < p^*(x)$ *for evey* $x \in \Omega$*, then the embedding*

$$W^{1,p(x)}_0(\Omega) \hookrightarrow\hookrightarrow L^{q(x)}(\Omega)$$

is continuous and compact.

iii) Poincaré's inequality: there exists a constant $C > 0$*, such that*

$$\|u\|_{p(x)} \leq C\|\nabla u\|_{p(x)} \quad \text{for every } u \in W^{1,p(x)}_0(\Omega). \tag{1.3}$$

iv) Sobolev's inequality: there exists another constant $C > 0$, such that

$$\|u\|_{p^*(x)} \leq C\|\nabla u\|_{p(x)} \text{ for every } u \in W_0^{1,p(x)}(\Omega).$$

Remark 1.2.4. The log-Hölder continuous condition (1.1) is used to obtain several regularity results for Sobolev spaces with variable exponents; specially, $C^\infty(\overline{\Omega})$ is dense in $W^{1,p(x)}(\Omega)$ and $W_0^{1,p(x)}(\Omega) = W^{1,p(x)}(\Omega) \cap W_0^{1,1}(\Omega)$. Furthermore, the Sobolev embedding also holds (see [61]) for $q(x) = p^*(x)$, i.e., $W^{1,p(x)}(\Omega) \subset L^{p^*(x)}(\Omega)$.

Remark 1.2.5. By (1.3), we deduce that $\|u\|_{1,p(x)}$ and $\|\nabla u\|_{p(x)}$ are equivalent norms on $W_0^{1,p(x)}(\Omega)$. Note that the following inequality

$$\int_\Omega |u(x)|^{p(x)} dx \leq C \int_\Omega |\nabla u(x)|^{p(x)} dx,$$

in general does not hold (see [68]). But by Proposition 1.2.2 and (1.3), we have

$$\int_\Omega |u(x)|^{p(x)} dx \leq C \max\left\{ \|\nabla u\|_{p(x)}^{p^+}, \|\nabla u\|_{p(x)}^{p^-} \right\}.$$

1.3 Orlicz-Sobolev spaces

In this section, we briefly review some basic facts about Sobolev-Orlicz space which we will need in our analysis in chapter 6. A comprehensive presentation of Sobolev-Orlicz space can be found in [1, 81].

Let $M : \mathbb{R}^+ \to \mathbb{R}^+$ be an $N-$function, i.e. M is continuous, convex, with $\dfrac{M(t)}{t} \to 0$ as $t \to 0$ and $\dfrac{M(t)}{t} \to +\infty$ as $t \to +\infty$.

Equivalently, M admits the representation $M(t) = \displaystyle\int_0^t a(s)ds$, where $a : \mathbb{R}^+ \to \mathbb{R}^+$ is nondecreasing, right continuous, with $a(0) = 0$, $a(t) > 0$ for $t > 0$ and $a(t) \to +\infty$ as $t \to +\infty$.

The $N-$function $\overline{M}$ conjugate to M is defined by $\overline{M}(t) = \displaystyle\int_0^t \overline{a}(s)ds$, where $\overline{a} : \mathbb{R}^+ \to \mathbb{R}^+$ is given by $\overline{a}(t) = \sup_{s \geq 0}\{s : a(s) \leq t\}$ (see [81]).

The $N-$function M is said to satisfy the Δ_2 condition if for some $k > 0$

$$M(2t) \leq kM(t) \text{ for all } t \geq 0, \tag{1.4}$$

when (1.4) holds only for $t \geq t_0$ for some $t_0 > 0$ then M is said to satisfy the Δ_2 condition near infinity.

We will extend these $N-$functions into even functions on all $\mathbb{R}$.

Let Ω be an open subset of $\mathbb{R}^N$. The Orlicz class $\mathcal{K}_M(\Omega)$ (resp. the Orlicz space $L_M(\Omega)$) is defined as the set of (equivalence classes of) real valued measurable functions u on Ω such that

$$\int_\Omega M(|u(x)|)dx < +\infty \left(\text{resp. } \int_\Omega M\left(\frac{|u(x)|}{\lambda}\right)dx < +\infty \text{ for some } \lambda > 0\right).$$

The set $L_M(\Omega)$ is Banach space under the norm

$$\|u\|_M = \inf\left\{\lambda > 0 : \int_\Omega M\left(\frac{|u(x)|}{\lambda}\right)dx \leq 1\right\}$$

and $\mathcal{K}_M(\Omega)$ is a convex subset of $L_M(\Omega)$. The closure in $L_M(\Omega)$ of the set of bounded measurable functions with compact support in $\overline{\Omega}$ is denoted by $E_M(\Omega)$.

The equality $E_M(\Omega) = L_M(\Omega)$ holds if and only if M satisfies the Δ_2 condition, for all t or for t large according to whether Ω has an infinite measure or not.

The dual space of $E_M(\Omega)$ can be identified with $L_{\overline{M}}(\Omega)$ by means of the pairing $\int_\Omega uvdx$ and the dual norm of $L_{\overline{M}}(\Omega)$ is equivalent to $\|.\|_{\overline{M}}$.

The space $L_M(\Omega)$ is reflexive if and only if M and $\overline{M}$ satisfy the Δ_2 condition, for all t or for t large according to whether Ω has an infinite measure or not (see [1]).

We now turn to the Orlicz-Sobolev space, $W^1L_M(\Omega)$ (resp. $W^1E_M(\Omega)$) is the space of all functions u such that u and its distributional derivatives up to order 1 lie in $L_M(\Omega)$ (resp. $E_M(\Omega)$). It is a Banach space under the norm

$$\|u\|_{1,M} = \sum_{|\alpha|\leq 1} \|D^\alpha u\|_M.$$

Thus, $W^1L_M(\Omega)$ and $W^1E_M(\Omega)$ can be identified with subspaces of product of $N+1$ copies of $L_M(\Omega)$. Denoting this product by ΠL_M we will use the weak topologies $\sigma(\Pi L_M, \Pi E_{\overline{M}})$ and $\sigma(\Pi L_M, \Pi L_{\overline{M}})$. The space $W_0^1E_M(\Omega)$ is defined as the (norm) closure of the Schwartz space $\mathcal{D}(\Omega)$ in $W^1E_M(\Omega)$ and the space $W_0^1L_M(\Omega)$ as the $\sigma(\Pi L_M, \Pi E_{\overline{M}})$ closure of $\mathcal{D}(\Omega)$ in $W^1L_M(\Omega)$.

We say that u_n converges to u for the modular convergence in $W^1L_M(\Omega)$ if for some $\lambda > 0$ it holds

$$\int_\Omega M\left(\frac{D^\alpha u_n - D^\alpha u}{\lambda}\right)dx \to 0 \text{ for all } |\alpha| \leq 1.$$

This implies convergence for $\sigma(\Pi L_M, \Pi L_{\overline{M}})$.

If M satisfies the Δ_2 condition on $\mathbb{R}^+$ (near infinity only if Ω has finite measure), then modular convergence coincides with norm convergence.

Let $W^{-1}L_{\overline{M}}(\Omega)$ (resp. $W^{-1}E_{\overline{M}}(\Omega)$) denote the space of distributions on Ω which can be written as sums of derivatives of order ≤ 1 of functions in $L_{\overline{M}}(\Omega)$ (resp. $E_{\overline{M}}(\Omega)$). It is a Banach space under the norm

$$\|f\|_{-1,\overline{M}} = \sup_{\substack{u \in W_0^1 L_M(\Omega) \\ \|u\|_{1,M} \leq 1}} |\langle f, u \rangle|.$$

If the open Ω has the segment property then the space $\mathcal{D}(\Omega)$ is dense in $W^1L_M(\Omega)$ for the modular convergence and thus for the topology $\sigma(\Pi L_M, \Pi L_{\overline{M}})$. Consequently, the action of a distribution S in $W^{-1}L_{\overline{M}}(\Omega)$ on an element u of $W_0^1 L_M(\Omega)$ is well defined. It will be denoted by $\langle S, u \rangle$ (for more details see [1]).

Lemma 1.3.1. *Let $G : \mathbb{R} \longrightarrow \mathbb{R}$ be uniformly Lipschitzian, with $G(0) = 0$. Let M be an $N-$function and let $u \in W^1L_M(\Omega)$ (resp. $W^1E_M(\Omega)$). Then $G(u) \in W^1L_M(\Omega)$ (resp. $W^1E_M(\Omega)$). Moreover, if the set D of discontinuity points of G' is finite, we have*

$$\frac{\partial}{\partial x_i} G(u) = \begin{cases} G'(u) \frac{\partial}{\partial x_i} u & \text{a.e. in } \{x \in \Omega : u(x) \notin D\}, \\ 0 & \text{a.e. in } \{x \in \Omega : u(x) \in D\}. \end{cases}$$

Proof. It is easily adapted from that given in [72] in the case $W^1L_M(\Omega)$, by Theorem 1 of [71] instead of Theorem 4 of [71] (see also Remark 5 of [72]). □

Lemma 1.3.2. *Let $G : \mathbb{R} \longrightarrow \mathbb{R}$ be uniformly Lipschitzian, with $G(0) = 0$. Suppose that the set D of discontinuity points of G' is finite. Let M be an $N-$function. Then the mapping $G : W^1L_M(\Omega) \longrightarrow W^1L_M(\Omega)$ is sequentially continuous with respect to the weak* topology $\sigma(\Pi L_M, \Pi E_{\overline{M}})$.*

Proof. We have by the last Lemma that $G(u) \in W^1L_M(\Omega)$ for all $u \in W^1L_M(\Omega)$ and that

$$||G(u)||_{1,M} \leq ||u||_{1,M},$$

which easily gives the result. □

Lemma 1.3.3. ([71, Theorem 4]) *Let Ω have the segment property. Then for each $v \in W_0^1 L_M(\Omega)$, there exists a sequence $(v_j)_j \subset \mathcal{D}(\Omega)$ which converges to v for the modular convergence in $W_0^1 L_M(\Omega)$.*
Furthermore, if $v \in W_0^1 L_M(\Omega) \cap L^\infty(\Omega)$, then $||v_j||_\infty \leq (N+1)||v||_\infty$.

Proposition 1.3.4. ([98, Corollary 5]) *Let M be an N-function such that $\overline{M}$ satisfies the Δ_2 condition. Then there exist $c > 0$ and $p > 1$ such that*

$$M(\xi) \geq c|\xi|^p \tag{1.5}$$

for all $\xi \in \mathbb{R}^N$ such that $|\xi| \geq |\xi_0|$.

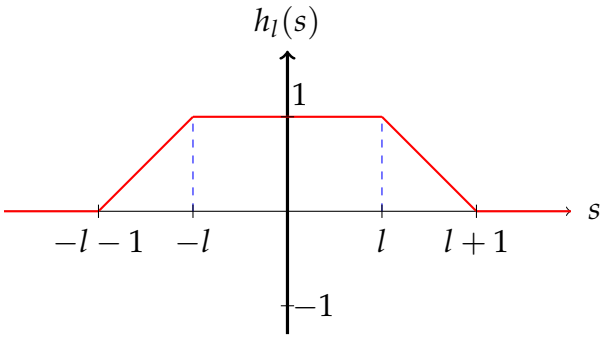
1.4 Notations and functions

Throughout the thesis, we will use the following notations. Giving $E \subset \Omega$ a Lebesgue measurable set, we will denote its characteristic function by χ_E and its Lebesgue measure by $|E|$.

For any $j \geq 0$ and $v : \Omega \to \mathbb{R}$, the set $\{x \in \Omega : |v(x)| \geq (>, \leq, <, =) j\}$ will be written by $\{|v| \geq (>, \leq, <, =) j\}$.

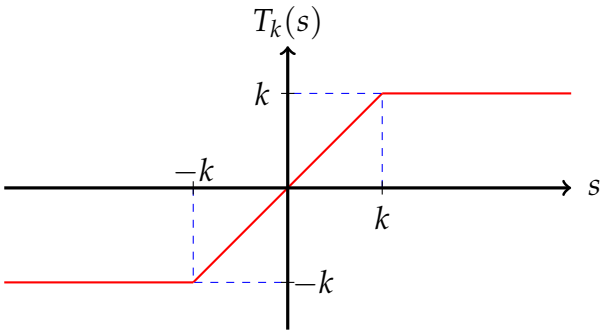
For any given $l > 0$, let $h_l : \mathbb{R} \to \mathbb{R}$ be the function defined by

$$h_l(s) = \begin{cases} 0 & \text{if } |s| \geq l+1, \\ l+1-|s| & \text{if } l < |s| < l+1, \\ 1 & \text{if } |s| \leq l. \end{cases}$$



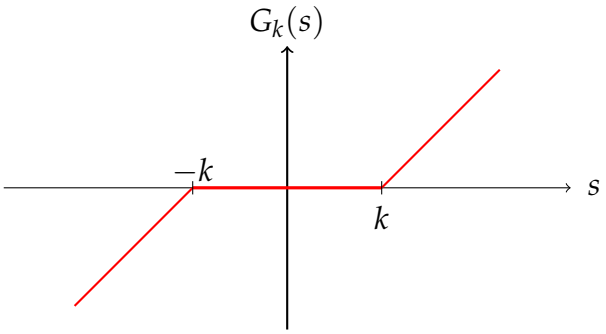
For $k > 0$, the standard truncation function $T_k : \mathbb{R} \to \mathbb{R}$ is defined as

$$T_k(s) = \begin{cases} k & \text{if } s \geq k, \\ s & \text{if } |s| < k, \\ -k & \text{if } s \leq -k. \end{cases}$$



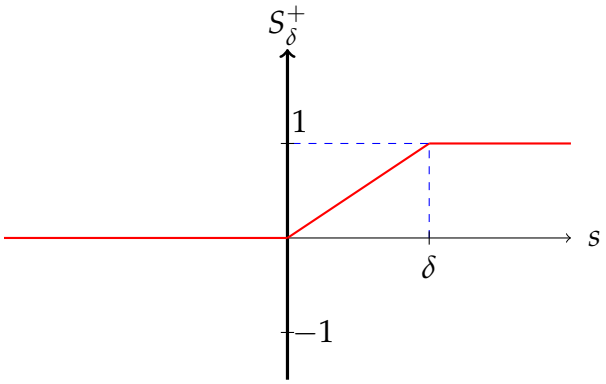
and $G_k : \mathbb{R} \to \mathbb{R}$ is defined by

$$G_k(r) = \begin{cases} r-k & \text{if } r \geq k, \\ 0 & \text{if } |r| < k, \\ r+k & \text{if } r \leq -k. \end{cases}$$



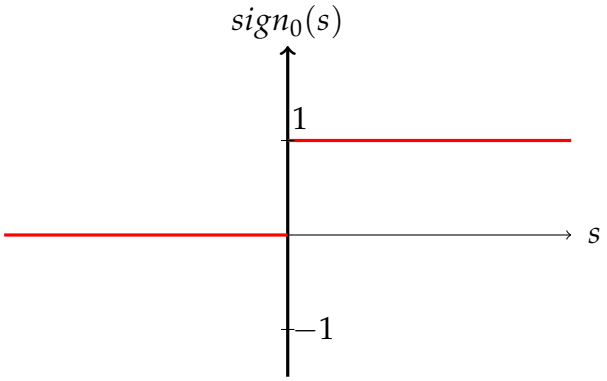
Let $\delta > 0$ and $S_\delta^+ : \mathbb{R} \longrightarrow \mathbb{R}$ be defined by

$$S_\delta^+(s) = \begin{cases} 1 & \text{if } s \geq \delta, \\ \dfrac{s}{\delta} & \text{if } 0 < s < \delta, \\ 0 & \text{if } s \leq 0. \end{cases}$$



Obviously, S_δ^+ is an approximation of the function $sign_0^+ : \mathbb{R} \to \mathbb{R}$ defined as

$$sign_0^+(s) = \begin{cases} 1 & \text{if } s > 0, \\ 0 & \text{if } s \leq 0. \end{cases}$$



1.5 Maximal monotone graphs

In this section, we introduce the general analytic framework, which is needed to study multivalued differential equations with maximal monotone drift. For a detailed overview on the theory of maximal monotone operators, we refer to [21, 22, 46, 50, 59, 92, 93].

1.5.1 Definitions and fundamental results

Let H and H' be two vector spaces and A a mapping from H into the power set $2^{H'}$ of H'. The mapping A is called a *multivalued mapping* or *multivalued operator* from H to H'. In this case, from $A(x_1) = x_1^{'}$ and $A(x_1) = y_1^{'}$ it does not follow that $x_1^{'} = y_1^{'}$, as happens with single-valued functions.

Considering A as a subset of $H \times H'$, we can write

$$A(x) = \{y \in H'; (x, y) \in A\}.$$

The subset

$$D(A) = \{x \in H; A(x) \neq \emptyset\}$$

is called the *domain* of A. Unless otherwise noted the domain of A is assumed to be non-empty. We write $A : D(A) \subset H \to 2^{H'}$ and say A is a multivalued mapping on H if $H' = H$.

As usual, the *range* $R(A) \subset H'$ is defined by

$$R(A) = \bigcup_{x \in H} A(x).$$

By convention, we identify the multivalued mapping $A : H \to 2^{H'}$ with its *graph* in $H \times H'$, denoted

$$G(A) = \{(x, y) \in H \times H'; x \in D(A), y \in A(x)\},$$

by writing $(x, y) \in A$ for $y \in A(x)$. Then $G(A_1) \subset G(A_2)$ if and only if $A_1(x) \subset A_2(x)$ for all $x \in H$.

Suppose further that H and H' are placed in separating duality by means of the duality pairing $\langle y, x \rangle$ for $x \in H$ and $y \in H'$.

The multivalued mapping $A : H \to 2^{H'}$ is said to be *monotone* if

$$\langle y - y', x - x' \rangle \geq 0,$$

$$\forall (x, x') \in D(A), \forall y \in A(x), \forall y' \in A(x').$$

Definition 1.5.1. Let $f : H \to \overline{\mathbb{R}}$. The vector $y \in H'$, for which

$$f(x') - f(x) \geq \langle y, x' - x \rangle, \ \forall x' \in H \tag{1.6}$$

holds, where $f(x)$ is finite at $x \in H$, is called the *subgradient* of f at x. The set of all $y \in H'$ satisfying (1.6) is called the *subdifferential* of f at x and is denoted by of $\partial f(x)$. We then write $y \in \partial f(x)$.

Proposition 1.5.2. ([92, Proposition 2.6.1.])

Let f be a convex proper functional on H. Then ∂f is a monotone multivalued mapping from H into $2^{H'}$.

A monotone operator $A : H \to 2^{H'}$ is called *maximal monotone* if and only if $G(A) \subset G(B)$ implies that $A = B$, where $B : H \to 2^{H'}$ is an arbitrary monotone operator, i.e., if and only if $G(A)$ is not properly contained in any other monotone subset of $H \times H'$.

The operator A^{-1} is defined by its graph: $(x,y) \in A^{-1}$ if, and only if, $(y,x) \in A$. Observe that if A is monotone (resp. maximal monotone) then so are A^{-1} and λA if $\lambda > 0$.

Suppose that $A : H \to H$, where H is a Hilbert space, and let $I : H \to H$ be the identity operator.

Proposition 1.5.3. ([92, Proposition 2.6.2.])
If $A + I$ is surjective, i.e., $R(A+I) = H$, and A is monotone, then A is maximal monotone.

The converse is also true [46]. For the generalization of this result, see [23].

Proposition 1.5.4. ([92, Proposition 2.6.3.])
The subdifferential ∂f of a convex, l.s.c., proper functional f on X, where X is a Banach space, is a maximal monotone operator.

1.5.2 Maximal monotone graphs in $\mathbb{R}^2$.

In this thesis, we consider the class of the monotone operators $\beta : \mathbb{R} \to 2^{\mathbb{R}}$. It is well known (cf. [59, p. 253]), that any m.m.g. β of $\mathbb{R}^2$ can be characterized through a nondecreasing real function b to which some vertical segments are added at every jump of b, i.e.

$$\beta(r) = \begin{cases} [b(r^-), b(r^+)] & \text{if } -\infty < b(r^-) \le b(r^+) < +\infty, \\ (-\infty, b(r^+)] & \text{if } -\infty = b(r^-) < b(r^+) < +\infty, \\ [b(r^-), +\infty) & \text{if } -\infty < b(r^-) < b(r^+) = +\infty. \end{cases}$$

The following property of a maximal monotone graph in $\mathbb{R}^2$ can be found, e.g. in [46, 59, 92].

Proposition 1.5.5. *Let $\beta : \mathbb{R} \to 2^{\mathbb{R}}$ be any maximal monotone graph. Then, there exists a convex, lower semi-continuous and proper function $j : \mathbb{R} \to (-\infty, +\infty]$ such that β is the subdifferential of j, i.e.*

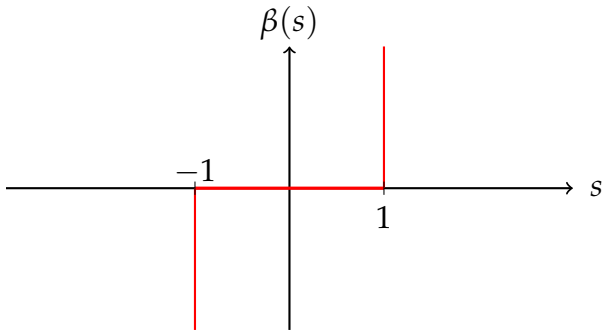
$$\beta = \partial j.$$

The function j is uniquely determined up to an additive constant and it is superpositionally measurable.

Note that the proof of Proposition 1.5.5 provides a method for the determination of j, when β is given.

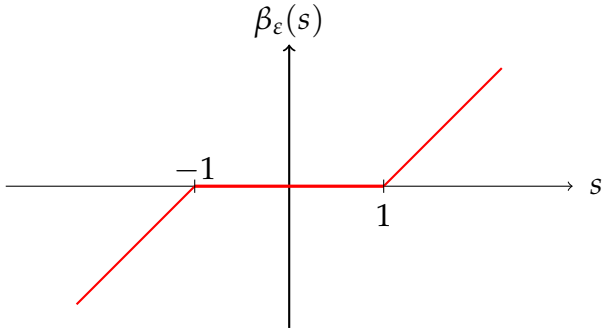
In the following simple example, yet another method is illustrated:
Let β be given for $x \in \mathbb{R}$ by

$$\beta(x) = \begin{cases} \emptyset & \text{if } x < -1, \\ (-\infty, 0] & \text{if } x = -1, \\ 0 & \text{if } -1 < x < 1, \\ [1, +\infty) & \text{if } x = 1, \\ \emptyset & \text{if } x > 1. \end{cases}$$



It is obvious that β is maximal monotone. For the determination of f, we define a continuous and single-valued function β_ε. (called a regularized operator), which depends on $\varepsilon > 0$. One possible form of β_ε is

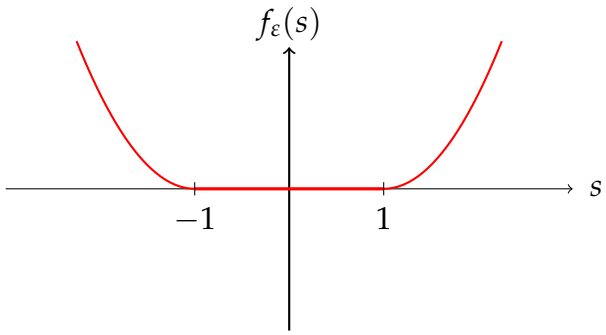
$$\beta_\varepsilon(x) = \begin{cases} \frac{x+1}{\varepsilon} & \text{if } x \leq -1, \\ 0 & \text{if } -1 < x < 1, \\ \frac{x-1}{\varepsilon} & \text{if } x \geq 1. \end{cases}$$



i.e., $\beta_\varepsilon \in C^0(\mathbb{R})$. We can also apply another regularization procedure such that $\beta_\varepsilon \in C^1(\mathbb{R})$, etc. For $\varepsilon \to 0^+$ the graph $G(\beta_\varepsilon)$ "coincides" with the graph $G(\beta)$ and f_ε results

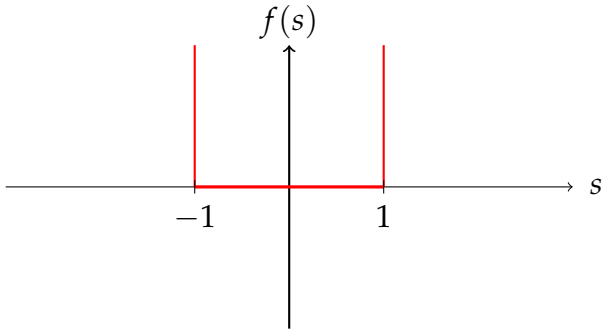
from β_ε through integration:

$$f_\varepsilon(x) = \begin{cases} \dfrac{(x+1)^2}{2\varepsilon} & \text{if } x \le -1, \\ 0 & \text{if } -1 < x < 1, \\ \dfrac{(x-1)^2}{2\varepsilon} & \text{if } x \ge 1. \end{cases}$$



From f_ε we obtain f for $\varepsilon \to 0^+$. Thus

$$f(x) = \begin{cases} 0 & \text{if } |x| \le 1, \\ +\infty & \text{if } |x| > 1. \end{cases}$$



The method of regularization is closely related to *Yosida's approximation* of maximal monotone operators (see [46]).

1.5.3 The Yosida approximation

Let I be the identity mapping on H. For $\lambda > 0$, the *resolvent* of A is the operator

$$J_\lambda^A = (I + \lambda A)^{-1}.$$

The *Yosida approximation* of A is the single-valued maximal monotone operator A_λ, $\lambda > 0$, defined by

$$A_\lambda = \frac{1}{\lambda}(I - J_\lambda^A).$$

We recall the definition of the *main section* A^0 of A

$$A^0(s) = \begin{cases} \text{the element of minimal absolute value of } A(s) & \text{if } A(s) \neq \emptyset, \\ +\infty & \text{if } [s, +\infty) \cap D(A) = \emptyset, \\ -\infty & \text{if } (-\infty, s] \cap D(A) = \emptyset. \end{cases}$$

If $s \in D(A)$, $|A_\lambda(s)| \leq |A^0(s)|$ and $A_\lambda(s) \to A^0(s)$ as $\lambda \to 0^+$, and if $s \notin D(A)$, $|A_\lambda(s)| \to +\infty$ as $\lambda \to 0^+$.

To regularize A, we consider

$$j_\varepsilon(s) = \min_{r \in \mathbb{R}} \left\{ \frac{1}{2\varepsilon}|s - r|^2 + j(r) \right\}, \text{ for all } s \in \mathbb{R} \text{ and any } \varepsilon > 0.$$

Thanks to [46, Proposition 2.11], we have the next results:

Proposition 1.5.6.

i) $D(A) \subset D(j) \subset \overline{D(j)} \subset \overline{D(A)}$.

ii) $j_\varepsilon(s) = \frac{\varepsilon}{2}|A_\varepsilon(s)|^2 + j(J_\varepsilon(s))$ where $J_\varepsilon := (I + \varepsilon A)^{-1}$.

iii) j_ε is a convex, Frechet-differentiable function and $A_\varepsilon = \partial j_\varepsilon$.

iv) $j_\varepsilon(s) \uparrow j(s)$ as $\varepsilon \downarrow 0$ for all $s \in \mathbb{R}$.

Moreover, for any $\varepsilon > 0$, A_ε is a nondecreasing and Lipschitz-continuous function.

1.6 Surjectivity theorems

In this section, we recall an abstract result of surjectivity which allows to prove an existence result for nonlinear problems involving pseudo-monotone operators.

Throughout this section, let $A : V \to V'$ be a nonlinear, possibly multivalued operator, where V will is a separable reflexive Banach space and V' its dual space.

Definition 1.6.1.

i) A is called *bounded* if it maps bounded sets in V into bounded sets in V'.

ii) A is *hemi-continuous* if for any $u, v, w \in V$, the mapping

$$t \mapsto \langle A(u + tv), w \rangle$$

is continuous.

iii) A is *coercive* if

$$\frac{\langle A(u), u \rangle}{\|u\|_V} \to +\infty \text{ when } \|u\|_V \to +\infty.$$

Theorem 1.6.2. *[86]*

Any A bounded, monotone, hemi-continuous and coercive is surjective; this means, for any $f \in V'$, there is at least one solution $u \in V$ to the equation $A(u) \ni f$.

Moreover, if A is strictly monotone, then u is unique.

Definition 1.6.3. [34, 45, 86]

The mapping A is said to be pseudo-monotone if A is bounded and if whenever

$$\begin{cases} u_k \rightharpoonup u \text{ in } V \text{ as } k \to \infty, \\ \limsup_{k\to\infty} \langle A(u_k), u_k - u \rangle \leq 0, \end{cases}$$

it follows that, for all $v \in V$,

$$\liminf_{k\to\infty} \langle A(u_k), u_k - v \rangle \geq \langle A(u), u - v \rangle.$$

Proposition 1.6.4. *[86]*

If A is bounded, monotone and hemi-continuous, then A is pseudo-monotone.

The following theorem stemming from [45], provides a surjectivity result for pseudo-monotone operators.

Theorem 1.6.5. *[86]*

Any A pseudo-monotone and coercive is surjective.

Chapter 2

On a general class of strongly nonlinear elliptic problems having natural growth terms

In this chapter, we mainly study a strongly nonlinear elliptic inclusion problem with L^∞ or L^1-data. We obtain the existence of weak solutions in appropriate Sobolev spaces by using the approximation method and via the strong convergence of the truncations. The results of this chapter are based on the papers [9] and [10].

2.1 Setting of the problem

Let Ω be a bounded domain in $\mathbb{R}^N (N \geq 1)$. Our aim is to show existence of solution for the following strongly nonlinear elliptic inclusion problem with Dirichlet boundary conditions

$$(S,f) \quad \begin{cases} \beta(u) + A(u) + g(x,u,\nabla u) \ni f \ \text{ in } \mathcal{D}'(\Omega), \\ \\ u \in W_0^{1,p}(\Omega),\ g(x,u,\nabla u) \in L^1(\Omega), \end{cases}$$

with right-hand side f is assumed to belong either to $L^\infty(\Omega)$ or to $L^1(\Omega)$, A is a nonlinear operator from $W_0^{1,p}(\Omega)$ into its dual, defined by $A(u) = -\text{div}(a(x,u,\nabla u))$, where $a : \Omega \times \mathbb{R} \times \mathbb{R}^N \to \mathbb{R}^N$ is a Carathéodory function satisfies the following assumptions:

$$a(x,s,\xi) \cdot \xi \geq \lambda |\xi|^p \ \text{ where } \lambda > 0, p > 1, \tag{2.1}$$

$$|a(x,s,\xi)| \leq C_0(a_0(x) + |s|^{p-1} + |\xi|^{p-1}) \ \text{ where } a_0 \in L^{p'}(\Omega), a_0 \geq 0, C_0 > 0, \tag{2.2}$$

$$(a(x,s,\xi) - a(x,s,\eta)) \cdot (\xi - \eta) > 0 \ \text{ for } \xi \neq \eta \in \mathbb{R}^N. \tag{2.3}$$

Moreover, $g : \Omega \times \mathbb{R} \times \mathbb{R}^N \to \mathbb{R}$ is Carathéodory function such that

$$g(x,s,\xi)s \geq 0, \tag{2.4}$$

$$|g(x,s,\xi)| \leq h(|s|)(c(x)+|\xi|^p), \tag{2.5}$$

$$\text{there exist } \sigma>0 \text{ and } \gamma>0 \text{ such that } |g(x,s,\xi)| \geq \gamma|\xi|^p \text{ when } |s|\geq\sigma, \tag{2.6}$$

where $h:\mathbb{R}^+\to\mathbb{R}^+$ is a continuous increasing function and $c(x)$ a positive function wich is in $L^1(\Omega)$.

The nonlinearity $\beta:\mathbb{R}\to 2^{\mathbb{R}}$ is a set valued, maximal monotone mapping such that $0\in\beta(0)$.

Definition 2.1.1. A weak solution of the problem (S,f) is a couple of functions $(u,b)\in W_0^{1,p}(\Omega)\times L^1(\Omega)$ such that $b(x)\in\beta(u(x))$ a.e in Ω, $g(x,u,\nabla u)\in L^1(\Omega)$ and

$$b-\operatorname{div}(a(x,u,\nabla u))+g(x,u,\nabla u)=f \text{ in } \mathcal{D}'(\Omega).$$

2.2 Existence for L^∞-data

Our first aim in this chapter is to prove the following existence Theorem:

Theorem 2.2.1. *Under the assumptions* (2.1)-(2.5) *and for $f\in L^\infty(\Omega)$. There exists at least one solution of (S,f) in the sens of Definition 2.1.1. Moreover, $g(x,u,\nabla u)u\in L^1(\Omega)$.*

The proof of Theorem 2.2.1 will be divided into several steps.

2.2.1 Approximation problem for L^∞-data

First, we introduce the approximate problem

$$(S_\varepsilon,f)\quad\begin{cases}\beta_\varepsilon(T_{\frac{1}{\varepsilon}}(u_\varepsilon))-\operatorname{div}(a(x,u_\varepsilon,\nabla u_\varepsilon))+g_\varepsilon(x,u_\varepsilon,\nabla u_\varepsilon)=f,\\ \\ u_\varepsilon\in W_0^{1,p}(\Omega),\end{cases}$$

where for each $\varepsilon\in]0;1]$, $\beta_\varepsilon:\mathbb{R}\to\mathbb{R}$ is the Yosida approximation of β and

$$g_\varepsilon(x,s,\xi)=\frac{g(x,s,\xi)}{1+\varepsilon|g(x,s,\xi|}$$

satisfies

$$g_\varepsilon(x,s,\xi)s\geq 0\ ,|g_\varepsilon(x,s,\xi)|\leq|g(x,s,\xi)| \text{ and } |g_\varepsilon(x,s,\xi)|\leq\frac{1}{\varepsilon}.$$

See that, for any $u \in W_0^{1,p}(\Omega)$ the function $\beta_\varepsilon(\cdot)$ satisfies the following conditions:

$$\langle \beta_\varepsilon(u), u\rangle \geq 0, \ \ |\beta_\varepsilon(u)| \leq \frac{1}{\varepsilon}|u| \ \text{ and } \ \lim_{\varepsilon\to 0}\beta_\varepsilon(u) = \beta(u).$$

We recommend that the reader consult the section 1.5, for additional information on the maximal monotone mapping.

For any fixed $\varepsilon > 0$, since $\beta_\varepsilon o T_{\frac{1}{\varepsilon}}$ and g_ε are bounded, there exists at least one solution u_ε of (S_ε, f) (cf.[82], [86]), i.e: for each $0 < \varepsilon \leq 1$ and $f \in W^{-1,p'}(\Omega)$ there exists at least one solution $u_\varepsilon \in W_0^{1,p}(\Omega)$ such that

$$\int_\Omega \beta_\varepsilon(T_{\frac{1}{\varepsilon}}(u_\varepsilon))\varphi dx + \int_\Omega a(x, u_\varepsilon, \nabla u_\varepsilon)\cdot\nabla\varphi dx + \int_\Omega g_\varepsilon(x, u_\varepsilon, \nabla u_\varepsilon)\varphi dx = \langle f, \varphi\rangle \quad (2.7)$$

holds for all $\varphi \in W_0^{1,p}(\Omega)$, where $\langle .,.\rangle$ denotes the duality pairing between $W_0^{1,p}(\Omega)$ and $W^{-1,p'}(\Omega)$.

2.2.2 A priori estimates

Taking u_ε as a test function in (2.7), we obtain

$$\int_\Omega \beta_\varepsilon(T_{\frac{1}{\varepsilon}}(u_\varepsilon))u_\varepsilon dx + \int_\Omega a(x, u_\varepsilon, \nabla u_\varepsilon)\cdot\nabla u_\varepsilon dx + \int_\Omega g_\varepsilon(x, u_\varepsilon, \nabla u_\varepsilon)u_\varepsilon dx = \int_\Omega f u_\varepsilon dx. \quad (2.8)$$

As the first term on the left hand side is nonnegative and since g_ε verifies the sign condition, by (2.1) we have

$$\lambda||u_\varepsilon||^p_{W_0^{1,p}(\Omega)} \leq C||f||_{L^\infty(\Omega)}||u_\varepsilon||_{W_0^{1,p}(\Omega)},$$

where C is a positve constant coming from the Hölder's and Poincaré's inequalities.

Then

$$||u_\varepsilon||_{W_0^{1,p}(\Omega)} \leq C_1. \quad (2.9)$$

Moreover, from (2.8) and (2.9), we infer that

$$0 \leq \int_\Omega g_\varepsilon(x, u_\varepsilon, Du_\varepsilon)u_\varepsilon dx \leq C_2. \quad (2.10)$$

On the other hand, using $S_\delta^+(\beta_\varepsilon(T_{\frac{1}{\varepsilon}}(u_\varepsilon)) - k)$ as a test function in (2.7). Since β_ε is monotone increasing with $\beta_\varepsilon(0) = 0$ and by (2.1), we get

$$\int_\Omega a(x, u_\varepsilon, \nabla u_\varepsilon)(S_\delta^+)'(\beta_\varepsilon(T_{\frac{1}{\varepsilon}}(u_\varepsilon)) - k)\beta_\varepsilon'(T_{\frac{1}{\varepsilon}}(u_\varepsilon))\cdot\nabla u_\varepsilon dx \geq 0.$$

Since g_ε verifies the sign condition, then

$$\int_\Omega g_\varepsilon(x, u_\varepsilon, \nabla u_\varepsilon) S_\delta^+(\beta_\varepsilon(T_{\frac{1}{\varepsilon}}(u_\varepsilon)) - k)dx \geq 0.$$

Consequently, we obatin

$$\int_\Omega (\beta_\varepsilon(T_{\frac{1}{\varepsilon}}(u_\varepsilon)) - k) S_\delta^+(\beta_\varepsilon(T_{\frac{1}{\varepsilon}}(u_\varepsilon)) - k)dx \leq \int_\Omega (f - k) S_\delta^+(\beta_\varepsilon(T_{\frac{1}{\varepsilon}}(u_\varepsilon)) - k)dx.$$

Taking $\delta \to 0$ yields

$$\int_\Omega (\beta_\varepsilon(T_{\frac{1}{\varepsilon}}(u_\varepsilon)) - k)^+ dx \leq \int_\Omega (f - k)^+ dx. \tag{2.11}$$

Similarly, one can show

$$\int_\Omega (\beta_\varepsilon(T_{\frac{1}{\varepsilon}}(u_\varepsilon)) + k)^- dx \leq \int_\Omega (f + k)^- dx. \tag{2.12}$$

Combinining (2.11) and (2.12) gives

$$\int_\Omega (|\beta_\varepsilon(T_{\frac{1}{\varepsilon}}(u_\varepsilon))| - k)^+ dx \leq \int_\Omega (|f| - k)^+ dx. \tag{2.13}$$

Choosing $k > ||f||_\infty$, we obtain

$$||\beta_\varepsilon(T_{\frac{1}{\varepsilon}}(u_\varepsilon))||_\infty \leq ||f||_\infty. \tag{2.14}$$

2.2.3 Basic convergence results

By (2.14), there exists $b \in L^\infty(\Omega)$ such that

$$\beta_\varepsilon(T_{\frac{1}{\varepsilon}}(u_\varepsilon)) \overset{*}{\rightharpoonup} b \text{ in } L^\infty(\Omega). \tag{2.15}$$

Since u_ε remains bounded in $W_0^{1,p}(\Omega)$, we can extract a subsequence, still denoted by u_ε, such that

$$u_\varepsilon \rightharpoonup u \text{ weakly in } W_0^{1,p}(\Omega) \text{ and } u_\varepsilon \to u \text{ a.e. in } \Omega.$$

We already know that for any fixed $k \in \mathbb{R}^{*+}$,

$$T_k(u_\varepsilon) \rightharpoonup T_k(u) \text{ weakly in } W_0^{1,p}(\Omega).$$

Our objective is to prove that

$$T_k(u_\varepsilon) \to T_k(u) \text{ strongly in } W_0^{1,p}(\Omega).$$

We shall use in (2.7) the test function

$$v_\varepsilon = \varphi(z_\varepsilon)$$

where

$$z_\varepsilon = T_k(u_\varepsilon) - T_k(u) \text{ and } \varphi(s) = se^{\lambda s^2}.$$

We get

$$\int_\Omega \beta_\varepsilon(T_{\frac{1}{\varepsilon}}(u_\varepsilon))v_\varepsilon dx + \int_\Omega a(x,u_\varepsilon,\nabla u_\varepsilon)\cdot\nabla v_\varepsilon dx + \int_\Omega g_\varepsilon(x,u_\varepsilon,\nabla u_\varepsilon)v_\varepsilon dx = \int_\Omega f v_\varepsilon dx.$$

From now on, we denote by $\eta^1(\varepsilon), \eta^2(\varepsilon), ...$ various sequences of real numbers which converge to zero when ε tends to zero.

Since v_ε converges to zero weakly$*$ in $L^\infty(\Omega)$, we have

$$\int_\Omega f v_\varepsilon dx \to 0$$

this implies that

$$\eta^1(\varepsilon) = \int_\Omega \beta_\varepsilon(T_{\frac{1}{\varepsilon}}(u_\varepsilon))v_\varepsilon dx + \int_\Omega a(x,u_\varepsilon,\nabla u_\varepsilon)\cdot\nabla v_\varepsilon dx + \int_\Omega g_\varepsilon(x,u_\varepsilon,\nabla u_\varepsilon)v_\varepsilon dx.$$

Note that

$$\int_\Omega \beta_\varepsilon(T_{\frac{1}{\varepsilon}}(u_\varepsilon))v_\varepsilon dx = \int_{\{|u_\varepsilon|\le k\}} \beta_\varepsilon(T_{\frac{1}{\varepsilon}}(u_\varepsilon))v_\varepsilon dx + \int_{\{|u_\varepsilon|>k\}} \beta_\varepsilon(T_{\frac{1}{\varepsilon}}(u_\varepsilon))v_\varepsilon dx.$$

The second term on the right hand is nonnegative. Also $\chi_{\{|u_\varepsilon|\le k\}}\beta_\varepsilon(T_{\frac{1}{\varepsilon}}(u_\varepsilon))$ is uniformly bounded, together with the Lebesgue Dominated Convergence Theorem provides that

$$\int_{\{|u_\varepsilon|\le k\}} \beta_\varepsilon(T_{\frac{1}{\varepsilon}}(u_\varepsilon))v_\varepsilon dx \to 0.$$

This implies that

$$\int_\Omega a(x,u_\varepsilon,\nabla u_\varepsilon)\cdot\nabla v_\varepsilon dx + \int_\Omega g_\varepsilon(x,u_\varepsilon,\nabla u_\varepsilon)v_\varepsilon dx \le \eta^2(\varepsilon).$$

Using the same arguments as in [39], we can find that

$$0 \le \int_\Omega [a(x,T_k(u_\varepsilon),\nabla T_k(u_\varepsilon)) - a(x,T_k(u_\varepsilon),\nabla T_k(u))]\cdot\nabla(T_k(u_\varepsilon) - T_k(u))dx \le \eta^3(\varepsilon).$$

Finally, a result in [41] (see also [49]) implies

$$T_k(u_\varepsilon) \to T_k(u) \text{ strongly in } W_0^{1,p}(\Omega). \tag{2.16}$$

2.2.4 Passing to the limit

In vertue of (2.16), we have for a subsequence

$$\nabla u_\varepsilon \to \nabla u \text{ a.e. in } \Omega$$

which with

$$u_\varepsilon \to u \text{ a.e. in } \Omega$$

yields, since $a(x, u_\varepsilon, \nabla u_\varepsilon)$ is bounded in $(L^{p'}(\Omega))^N$,

$$a(x, u_\varepsilon, \nabla u_\varepsilon) \rightharpoonup a(x, u, \nabla u) \text{ weakly in } (L^{p'}(\Omega))^N \tag{2.17}$$

as well as

$$g_\varepsilon(x, u_\varepsilon, \nabla u_\varepsilon) \to g(x, u, \nabla u) \text{ a.e. in } \Omega. \tag{2.18}$$

We now use the classical trick in order to prove that $g_\varepsilon(x, u_\varepsilon, \nabla u_\varepsilon)$ is uniformly equi-integrable.

For any measurable subset E of Ω and for any $m \in \mathbb{R}^+$, we have

$$\begin{aligned}\int_E |g_\varepsilon(x, u_\varepsilon, \nabla u_\varepsilon)|dx &= \int_{E\cap\{|u_\varepsilon|\le m\}} |g_\varepsilon(x, u_\varepsilon, \nabla u_\varepsilon)|dx + \int_{E\cap\{|u_\varepsilon|> m\}} |g_\varepsilon(x, u_\varepsilon, \nabla u_\varepsilon)|dx \\ &\le \int_E |g_\varepsilon(x, T_m(u_\varepsilon), \nabla T_m(u_\varepsilon))|dx + \frac{1}{m}\int_E g_\varepsilon(x, u_\varepsilon, \nabla u_\varepsilon)u_\varepsilon dx.\end{aligned}$$

Using (2.5) and (2.10), we obtain

$$\int_E |g_\varepsilon(x, u_\varepsilon, \nabla u_\varepsilon)|dx \le h(m)\int_E (c(x) + |\nabla T_m(u_\varepsilon)|^p)dx + \frac{C_2}{m}.$$

Since the sequence $(\nabla T_m(u_\varepsilon))$ converge strongly in $(L^p(\Omega))^N$ the above inequality implies the equi-integrability of $g_\varepsilon(x, u_\varepsilon, \nabla u_\varepsilon)$.

In view of (2.18), we thus have

$$g_\varepsilon(x, u_\varepsilon, \nabla u_\varepsilon) \to g(x, u, \nabla u) \text{ strongly in } L^1(\Omega). \tag{2.19}$$

From (2.15), (2.17) and (2.19), we can pass to the limit in (2.7), we obtain

$$\int_\Omega b\varphi dx + \int_\Omega a(x, u, \nabla u)\cdot\nabla\varphi dx + \int_\Omega g(x, u, \nabla u)\varphi dx = \int_\Omega f\varphi dx$$

holds for any $\varphi \in W_0^{1,p}(\Omega)\cap L^\infty(\Omega)$.

Moreover, since $g_\varepsilon(x, u_\varepsilon, \nabla u_\varepsilon)u_\varepsilon \ge 0$ a.e. in Ω, by (2.10), (2.18) and Fatou's lemma, we have

$$g(x, u, \nabla u)u \in L^1(\Omega).$$

2.2.5 Subdifferential argument

It remains to prove that $u(x) \in D(\beta(x))$ and $b(x) \in \beta(u(x))$ for almost all $x \in \Omega$.

In view of Proposition 1.5.5, since $\beta : \mathbb{R} \to 2^{\mathbb{R}}$ is a maximal monotone graph, then, there exists a convex, l.s.c. and proper function $j : \mathbb{R} \to [0, \infty]$ such that

$$\beta(r) = \partial j(r) \text{ for all } r \in \mathbb{R}.$$

According to the Proposition 1.5.6, for $0 < \varepsilon \leq 1$, $j_\varepsilon : \mathbb{R} \to \mathbb{R}$ defined by $j_\varepsilon(r) = \int_0^r \beta_\varepsilon(s)ds$ has the following properties:

i) j_ε is convex and differentiable for all $r \in \mathbb{R}$, such that

$$j'_\varepsilon(r) = \beta_\varepsilon(r) \text{ for all } r \in \mathbb{R} \text{ and any } 0 < \varepsilon \leq 1.$$

ii) $j_\varepsilon(r) \to j(r)$ for all $r \in \mathbb{R}$ as $\varepsilon \to 0$.

From $i)$ it follows that for any $0 < \varepsilon \leq 1$

$$j_\varepsilon(r) \geq j_\varepsilon(T_{\frac{1}{\varepsilon}}(u_\varepsilon)) + (r - T_{\frac{1}{\varepsilon}}(u_\varepsilon))\beta_\varepsilon(T_{\frac{1}{\varepsilon}}(u_\varepsilon)) \tag{2.20}$$

holds for all $r \in \mathbb{R}$ and almost everywhere in Ω.

Let $E \subset \Omega$ be an arbitrary measurable set. We fix $\varepsilon_0 > 0$, multiplying (2.20) by $h_l(u_\varepsilon)\chi_E$, integrating over Ω and using $ii)$, we obtain

$$j(r) \int_E h_l(u_\varepsilon)dx \geq \int_E \left(j_{\varepsilon_0}(T_{l+1}(u_\varepsilon))h_l(u_\varepsilon) + (r - T_{l+1}(u_\varepsilon))h_l(u_\varepsilon)\beta_\varepsilon(T_{\frac{1}{\varepsilon}}(u_\varepsilon)) \right) dx \tag{2.21}$$

for all $r \in \mathbb{R}$ and all $0 < \varepsilon < \min\left(\varepsilon_0, \frac{1}{l}\right)$. As $\varepsilon \to 0$, taking into account that E arbitrary, we obtain from (2.21),

$$j(r)h_l(u) \geq j_{\varepsilon_0}(T_{l+1}(u))h_l(u) + bh_l(u)(r - T_{l+1}(u)) \tag{2.22}$$

for all $r \in \mathbb{R}$ and almost everywhere in Ω.

Passing to the limit with $l \to \infty$ and then with $\varepsilon_0 \to 0$ in (2.22), finally yields

$$j(r) \geq j(u(x)) + b(x)(r - u(x))$$

for all $r \in \mathbb{R}$ and almost everywhere in Ω, hence $u \in D(\beta)$ and $b \in \beta(u)$ for almost everywhere in Ω.

With this last step the proof of Theorem 2.2.1 is concluded.

2.3 Existence for L^1-data

Our second aim in this chapter is to prove the following existence Theorem:

Theorem 2.3.1. *Under the assumptions* (2.1)-(2.6) *and for* $f \in L^1(\Omega)$. *There exists at least one solution of* (S, f) *in the sens of Definition 2.1.1.*

Remark 2.3.2. We shall prove the existence of a solution in $W_0^{1,p}(\Omega)$, but it should be emphasized that for $\beta \equiv 0$ and $\gamma = 0$ in condition (2.6), the existence of u in such a space cannot expected, if $p \leq N$. In [38] the existence of a solution has been proved only in $W_0^{1,q}(\Omega)$ for all $q < (N(p-1))/(N-1)$.

The proof of Theorem 2.3.1 will be divided into several steps.

2.3.1 Approximation problem for L^1-data

Consider the sequence of approximate equations

$$(S, f_n) \quad \begin{cases} \beta(u_n) - \operatorname{div}(a(x, u_n, \nabla u_n)) + g(x, u_n, \nabla u_n) \ni f_n, \\ \\ u_n \in W_0^{1,p}(\Omega), \end{cases}$$

where f_n is a sequence of $L^\infty-$functions which converges strongly to f in $L^1(\Omega)$ and $|f_n| \leq |f|$.

From Theorem 2.2.1, there exists a solution $(u_n, b_n) \in W_0^{1,p}(\Omega) \times L^\infty(\Omega)$ of (E, f_n) such that

$$\int_\Omega b_n \varphi dx + \int_\Omega a(x, u_n, \nabla u_n) \cdot \nabla \varphi dx + \int_\Omega g(x, u_n, \nabla u_n) \varphi dx = \int_\Omega f_n \varphi dx \tag{2.23}$$

holds for all $n \in \mathbb{N}$ and $\varphi \in W_0^{1,p}(\Omega) \cap L^\infty(\Omega)$.

2.3.2 A priori estimates

Lemma 2.3.3. *For* $n \in \mathbb{N}$, *let* $(u_n, b_n) \in W_0^{1,p}(\Omega) \times L^\infty(\Omega)$ *be a solution of* (S, f_n). *Then, there exists a constant* C, *not depending on* n, *such that*

$$||u_n||_{W_0^{1,p}(\Omega)} \leq C \tag{2.24}$$

and

$$||b_n||_{L^1(\Omega)} \leq ||f||_{L^1(\Omega)}. \tag{2.25}$$

Indeed, taking $T_k(u_n)$ as a test funtion in (2.23), we obtain

$$\int_\Omega b_n T_k(u_n)dx + \int_\Omega a(x,u_n,\nabla T_k(u_n))\cdot\nabla T_k(u_n)dx + \int_\Omega g(x,u_n,\nabla u_n)T_k(u_n)dx = \int_\Omega f_n T_k(u_n)dx, \tag{2.26}$$

as the first term on the left hand side is nonnegative, g verifies the sign condition and by (2.1), we have

$$\lambda\int_\Omega |\nabla T_k(u_n)|^p dx \le \int_\Omega f_n T_k(u_n)dx \le k||f||_{L^1(\Omega)} \tag{2.27}$$

and

$$k\int_{\{|u_n|>k\}} |g(x,u_n,\nabla u_n)|dx \le \int_\Omega |f_n||T_k(u_n)|dx \le k||f||_{L^1(\Omega)}. \tag{2.28}$$

Hence from (2.6), (2.27), (2.28) and for $k>\sigma$, we obtain

$$\begin{aligned}\int_\Omega |\nabla u_n|^p dx &= \int_{\{|u_n|>k\}} |\nabla u_n|^p dx + \int_\Omega |\nabla T_k(u_n)|^p dx\\ &\le \frac{1}{\gamma}\int_{\{|u_n|>k\}} |g(x,u_n,\nabla u_n)|dx + \frac{k}{\lambda}||f||_{L^1(\Omega)}\\ &\le (\frac{1}{\gamma}+\frac{k}{\lambda})||f||_{L^1(\Omega)}.\end{aligned}$$

Then

$$||u_n||_{W_0^{1,p}(\Omega)} \le C.$$

We neglect in (2.26) the positive terms

$$a(x,u_n,\nabla T_k(u_n))\cdot\nabla T_k(u_n) \text{ and } g(x,u_n,\nabla u_n)T_k(u_n),$$

and keep

$$\int_\Omega b_n T_k(u_n)dx \leqslant \int_\Omega f_n u_n dx.$$

Since $b_n \in \beta(u_n)$ a.e. in Ω

$$\int_{\{|u_n|>k\}} |b_n|dx \leqslant \int_\Omega |f|dx.$$

Passing the limit as $k\downarrow 0$ and using the Fatou's Lemma, we find

$$\int_\Omega |b_n|dx \leqslant ||f||_{L^1(\Omega)}.$$

2.3.3 Basic convergence results

Lemma 2.3.4. *For $n \in \mathbb{N}$, let $(u_n, b_n) \in W_0^{1,p}(\Omega) \times L^\infty(\Omega)$ be a solution of (S, f_n). We have*

$$b_n \rightharpoonup b \text{ weakly in } L^1(\Omega). \tag{2.29}$$

$$u_n \rightharpoonup u \text{ weakly in } W_0^{1,p}(\Omega) \text{ and } u_n \to u \text{ a.e. in } \Omega. \tag{2.30}$$

$$T_k(u_n) \to T_k(u) \text{ strongly in } W_0^{1,p}(\Omega). \tag{2.31}$$

Proof.

Let $(u_n^\varepsilon, b_n^\varepsilon)$ be a solution for the problem

$$\begin{cases} \beta_\varepsilon(T_{\frac{1}{\varepsilon}}(u_n^\varepsilon)) - div(a(x, u_n^\varepsilon, \nabla u_n^\varepsilon)) + g_\varepsilon(x, u_n^\varepsilon, \nabla u_n^\varepsilon) = f_n, \\ \\ u_n^\varepsilon \in W_0^{1,p}(\Omega). \end{cases}$$

By (2.13), we have

$$\int_\Omega (|\beta_\varepsilon(T_{\frac{1}{\varepsilon}}(u_n^\varepsilon))| - k)^+ dx \leq \int_\Omega (|f_n| - k)^+ dx.$$

Using the fact that $\beta_\varepsilon(T_{\frac{1}{\varepsilon}}(u_n^\varepsilon)) \overset{*}{\rightharpoonup} b_n$ in $L^\infty(\Omega)$, we get

$$\int_\Omega (|b_n| - k)^+ dx \leq \int_\Omega (|f_n| - k)^+ dx. \tag{2.32}$$

The sequence b_n is weakly sequentialy compact in $L^1(\Omega)$. This follows from the following creterion for weak sequential compactness of subset F of $L^1(\mu)$ where μ is a finite measure

$$\lim_{k\to\infty} \sup_F \int_\Omega (|f| - k)^+ d\mu = 0.$$

Indeed, this condition is easily seen to be equivalent to the uniform integrability of the family F (that is, for every $\varepsilon > 0$ there is a $\delta > 0$ such that $\int_A |f| d\mu < \varepsilon$ if $\mu(A) < \delta$) and this implies the weak sequential compactness.

Since we have (2.32) and f_n convergent in $L^1(\Omega)$ by assumption, implies that b_n is weakly precompact in $L^1(\Omega)$.

Then

$$b_n \rightharpoonup b \text{ weakly in } L^1(\Omega).$$

From (2.24) we deduce that for a subsequence still indexed by n, (2.30) hold as $n \to \infty$, where u is a mesurable function defined on Ω.

We already know that for any fixed $k \in \mathbb{R}^{*+}$

$$T_k(u_n) \rightharpoonup T_k(u) \text{ weakly in } W_0^{1,p}(\Omega).$$

Our objective is to prove that

$$T_k(u_n) \to T_k(u) \text{ strongly in } W_0^{1,p}(\Omega).$$

We shall use in (2.23) the test function

$$v_n = \varphi(z_n)$$

where

$$z_n = T_k(u_n) - T_k(u) \text{ and } \varphi(s) = se^{\lambda s^2}.$$

We get

$$\int_\Omega b_n v_n dx + \int_\Omega a(x, u_n, \nabla u_n) \cdot \nabla v_n dx + \int_\Omega g(x, u_n, \nabla u_n) v_n dx = \int_\Omega f_n v_n dx.$$

From now on, we denote by $\varepsilon_1(n), \varepsilon_2(n), \ldots$ various sequences of real numbers which converge to zero when $n \to \infty$.

Since v_n converges to zero weakly$*$ in $L^\infty(\Omega)$, f_n converges strongly to f in $L^1(\Omega)$, we get

$$\int_\Omega f_n v_n dx \to 0.$$

In the other hand, We have

$$\int_\Omega b_n v_n dx = \int_{\{|u_n| \le k\}} b_n v_n dx + \int_{\{|u_n| > k\}} b_n v_n dx.$$

Since $b_n \in \beta(u_n)$, the second term on the right hand is nonnegative. Also $\chi_{\{|u_n| \le k\}} b_n$ is uniformly bounded, together with the Lebesgue Dominated Convergence Theorem provides that

$$\int_{\{|u_n| \le k\}} b_n v_n dx \to 0.$$

This implies that

$$\int_\Omega a(x, u_n, \nabla u_n) \cdot \nabla v_n dx + \int_\Omega g(x, u_n, \nabla u_n) v_n dx \le \varepsilon_1(n).$$

Using same arguments as in [39], we obtain

$$0 \le \int_\Omega [a(x, T_k(u_n), \nabla T_k(u_n)) - a(x, T_k(u_n), \nabla T_k(u))] \cdot \nabla (T_k(u_n) - T_k(u)) dx \le \varepsilon_2(n).$$

Then

$$\int_\Omega [a(x, T_k(u_n), \nabla T_k(u_n)) - a(x, T_k(u_n), \nabla T_k(u))] \cdot \nabla (T_k(u_n) - T_k(u)) dx \to 0 \text{ as } n \to \infty.$$

Finally, a result in [41] (see also [49]) implies

$$T_k(u_n) \to T_k(u) \text{ strongly in } W_0^{1,p}(\Omega).$$

2.3.4 Conclusion of the proof of Theorem 2.3.1

In vertue of (2.31), we have for a subsequence

$$\nabla u_n \to \nabla u \text{ a.e. in } \Omega$$

which with

$$u_n \to u \text{ a.e. in } \Omega$$

yields, since $a(x, u_n, \nabla u_n)$ is bounded in $(L^{p'}(\Omega))^N$,

$$a(x, u_n, \nabla u_n) \rightharpoonup a(x, u, \nabla u) \text{ weakly in } (L^{p'}(\Omega))^N \tag{2.33}$$

as well as

$$g(x, u_n, \nabla u_n) \to g(x, u, \nabla u) \text{ a.e. in } \Omega. \tag{2.34}$$

We now use the classical trick in order to prove that $g(x, u_n, \nabla u_n)$ is uniformly equi-integrable.

For any measurable subset E of Ω and for any $m \in \mathbb{R}^+$, we have

$$\begin{aligned}\int_E |g(x, u_n, \nabla u_n)|dx &= \int_{E\cap\{|u_n|\le m\}} |g(x, u_n, \nabla u_n)|dx + \int_{E\cap\{|u_n|>m\}} |g(x, u_n, \nabla u_n)|dx \\ &\le \int_E |g(x, T_m(u_n), \nabla T_m(u_n))|dx + \int_{\{|u_n|>m\}} |g(x, u_n, \nabla u_n)|dx.\end{aligned}$$

Using (2.5), we obtain

$$\int_E |g(x, u_n, \nabla u_n)|dx \le h(m) \int_E (c(x) + |\nabla T_m(u_n)|^p)dx + \int_{\{|u_n|>m\}} |g(x, u_n, \nabla u_n)|dx. \tag{2.35}$$

For fixed m, the first integral of the right hand side of (2.35) is small uniformly in n when the measure of E is small (due to $\nabla T_m(u_n)$ converges strongly in $L^p(\Omega)$).

We now discuss the behaviour of the second integral of the right hand side of (2.35). We use in (2.23) the test function $\psi_m(u_n)$, where for $m > 1$

$$\begin{cases} \psi_m(s) = 0 & \text{if } |s| \le m-1, \\ \psi_m(s) = \dfrac{|s|}{s} & \text{if } |s| \ge m, \\ \psi_m'(s) = 1 & \text{if } m-1 \le |s| \le m. \end{cases}$$

This yields

$$\int_\Omega b_n \psi_m(u_n)dx + \int_\Omega a(x,u_n,\nabla u_n)\cdot\nabla u_n \psi_m'(u_n)dx \\ + \int_\Omega g(x,u_n,\nabla u_n)\psi_m(u_n)dx = \int_\Omega f_n\psi_m(u_n)dx.$$

Wich implies

$$\int_{\{|u_n|>m\}} |g(x,u_n,\nabla u_n)|dx \leq \int_{\{|u_n|>m-1\}} |f_n|dx,$$

and thus

$$\underset{n\to\infty}{limsup} \int_{\{|u_n|>m\}} |g(x,u_n,\nabla u_n)|dx \leq \int_{\{|u|>m-1\}} |f|dx.$$

We have proved that the seconde terme of the right hand side of (2.35) is small, uniformly in n and in E, when p is sufficiently large. This completes the proof of the uniforme equi-integrability of $g(x,u_n,\nabla u_n)$.

In view of (2.34), we thus have

$$g(x,u_n,\nabla u_n) \to g(x,u,\nabla u) \text{ strongly in } L^1(\Omega). \tag{2.36}$$

From (2.29), (2.33) and (2.36), we can pass to the limit in (2.23), we obtain

$$\int_\Omega b\varphi dx + \int_\Omega a(x,u,\nabla u)\cdot\nabla\varphi dx + \int_\Omega g(x,u,\nabla u)\varphi dx = \int_\Omega f\varphi dx \tag{2.37}$$

holds for any $\varphi \in W_0^{1,p}(\Omega)\cap L^\infty(\Omega)$.

Next, we plug $T_k(u)$ as a test function in (2.37) neglecting nonnegative terms, we obtain

$$\int_\Omega bT_k(u)dx \leqslant \int_\Omega fudx.$$

Since $b \in \beta(u)$ a.e. in Ω

$$\int_{\{|u|>k\}} |b|dx \leqslant \int_\Omega |f|dx. \tag{2.38}$$

Passing the limit as $k \downarrow 0$ in (2.38) and using the Fatou's Lemma, we find

$$\int_\Omega |b|dx \leqslant ||f||_{L^1(\Omega)}$$

and $b \in L^1(\Omega)$.

With this last step the proof of Theorem 2.3.1 is concluded.

2.4 Example

Let Ω be a bounded domain of $\mathbb{R}^{\mathbb{N}}(N \geq 1)$. Let us consider the Carathéodory functions

$$a(x,s,\xi) = |\xi|^{p-2}\xi,$$

$$g(x,s,\xi) = \rho s|s|^r|\xi|^p,\ \rho > 0,\ r > 0,$$

and β the maximal monotone graph defined by

$$\beta(r) = \begin{cases} \emptyset & \text{if } r < 0, \\ (-\infty, 0] & \text{if } r = 0, \\ 0 & \text{if } r > 0. \end{cases}$$

It is easy to show that the Carathéodory function $a(x,s,\xi)$ satisfies the growth condition (2.2), the coercivity (2.1) and the strict monotonie condition (2.3). Also the Carathéodory function $g(x,s,\xi)$ satisfies the conditions (2.4), (2.5) and (2.6) with $|s| > \sigma = 1$ and $\gamma = \rho$.

Finally, the hypotheses of Theorem 2.3.1 are satisfied, therefore for all $f \in L^1(\Omega)$ the following problem

$$(E,f) \quad \begin{cases} \beta(u) - \Delta_p(u) + g(x,u,\nabla u) \ni f \text{ in } \mathcal{D}'(\Omega), \\ \\ u \in W_0^{1,p}(\Omega),\ g(x,u,\nabla u) \in L^1(\Omega), \end{cases}$$

has at least one solution.

Chapter 3

Existence of solution for nonlinear elliptic inclusion problems with degenerate coercivity

This chapter is devoted to the study of a nonlinear elliptic inclusion problems with degenerate coercivity and L^1- data. First, we introduce and solve approximating problems for any L^∞-data by using the Yosida approximation of the graph and cutting the nonlinearity, a pseudo-monotone and coercive differential operator on $W_0^{1,p}(\Omega)$ can be applied to establish the existence of bounded weak solutions. Next we prove the existence of entropy solutions for data in L^1 and finally, we derive some properties of these solutions. The results of this chapter are based on the paper [6].

3.1 Problem formulation and main result

Let Ω be a bounded domain in $\mathbb{R}^N (N \geq 2)$ with sufficiently smooth boundary $\partial\Omega$ and let us consider the nonlinear elliptic inclusion problem with Dirichlet boundary conditions:

$$(P,f) \quad \begin{cases} \beta(u) - \operatorname{div}\left(\dfrac{a(x,\nabla u)}{(1+|u|)^{\theta(p-1)}}\right) \ni f & \text{in } \Omega, \\ \\ u = 0 & \text{on } \partial\Omega, \end{cases}$$

where $a(x,\xi) : \Omega \times \mathbb{R}^N \to \mathbb{R}^N$ is a Carathéodory function (i.e. measurable with respect to x in Ω for every ξ in $\mathbb{R}^N$ and continuous with respect to ξ in $\mathbb{R}^N$ for almost every x in Ω) such that

Assumption (H_1)

$$a(x,\xi)\cdot\xi \geq \alpha|\xi|^p, \tag{3.1}$$

$$|a(x,\xi)| \leq C_0(a_0(x) + |\xi|^{p-1}), \tag{3.2}$$

$$(a(x,\xi) - a(x,\eta)) \cdot (\xi - \eta) > 0, \tag{3.3}$$

for almost every $x \in \Omega$ and every $\xi, \eta \in \mathbb{R}^N$ with $\xi \neq \eta$, where $\alpha > 0$, $C_0 > 0$ and a_0 is a nonnegative function in $L^{p'}(\Omega)$ with $p' = \frac{p}{p-1}$.

Assumption (H_2)

The nonlinearity $\beta : \mathbb{R} \to 2^{\mathbb{R}}$ is a set valued, maximal monotone mapping such that $0 \in \beta(0)$.

Throughout this paper, without special statements, we always assume that

Assumption (H_3)

$$2 - \frac{1}{N} < p < N \text{ and } 0 \leq \theta(p-1) < \frac{N(p-1)}{N-1} - 1. \tag{3.4}$$

Definition 3.1.1. Let $f \in L^\infty(\Omega)$. A pair of functions (u, b) is a weak solution of the problem (P, f) if $u \in W_0^{1,p}(\Omega) \cap L^\infty(\Omega)$, $b \in \beta(u)$ a.e. in Ω with $b \in L^\infty(\Omega)$ and

$$\int_\Omega b\varphi dx + \int_\Omega \frac{a(x, \nabla u)}{(1+|u|)^{\theta(p-1)}} \cdot \nabla\varphi dx = \int_\Omega f\varphi dx \text{ holds for all } \varphi \in W_0^{1,p}(\Omega).$$

In this chapter, since the problem (P, f) involves data $f \in L^1(\Omega)$, it is natural to consider the notion of entropy solution, which will be introduce in the following definition.

Definition 3.1.2. Let $f \in L^1(\Omega)$. An entropy solution to (P, f) is a pair of functions (u, b) satisfying the following conditions

(C1) $u : \Omega \to \mathbb{R}$ is measurable and $b \in L^1(\Omega)$ such that $u(x) \in D(\beta(x))$ and $b(x) \in \beta(u(x))$ for a.e. x in Ω.

(C2) For each $k > 0$, $T_k(u) \in W_0^{1,p}(\Omega)$ and

$$\int_\Omega bT_k(u-\varphi)dx + \int_\Omega \frac{a(x, \nabla u)}{(1+|u|)^{\theta(p-1)}} \cdot \nabla T_k(u-\varphi)dx \leq \int_\Omega fT_k(u-\varphi)dx \tag{3.5}$$

holds for all $\varphi \in W_0^{1,p}(\Omega) \cap L^\infty(\Omega)$.

Remark 3.1.3. We recall that a function u such that $T_k(u) \in W_0^{1,p}(\Omega)$, for all $k > 0$, does not necessarily belong to $W_0^{1,1}(\Omega)$. However, it is possible to define its generalized gradient (still denoted by ∇u) as the unique measurable function $v : \Omega \to \mathbb{R}^N$ such that

$$\nabla T_k(u) = v\chi_{\{|u|<k\}}$$

for a.e. $x \in \Omega$ and for all $k > 0$, Moreover, if $u \in W_0^{1,1}(\Omega)$, then v coincides with the standard distributional gradient of u (See [25] for more details).

Remark 3.1.4. We can easily check that all the terms in (C2) make sense. Indeed, the second integral in the left-hand side is well defined since the integral is only on the set $|u - \varphi| \leq k$, and on this set $|u| \leq k + ||\varphi||_{L^\infty(\Omega)} = M$, so that we have

$$\int_\Omega \frac{a(x, \nabla u)}{(1 + |u|)^{\theta(p-1)}} \cdot \nabla T_k(u - \varphi)dx = \int_\Omega \frac{a(x, \nabla T_M(u))}{(1 + |T_M(u)|)^{\theta(p-1)}} \cdot \nabla T_k(u - \varphi)dx,$$

which is finite by the growth assumptions (3.2) on a.

Our aim in this chapter is to prove the following existence Theorem:

Theorem 3.1.5. *Under the assumptions (H_1)-(H_3) and for $f \in L^1(\Omega)$. There exists at least one entropy solution (u, b) of the problem (P, f) in the sens of Definition 3.1.2. Moreover, $u \in W_0^{1,q}(\Omega)$, with $q \in \left(1, \frac{N(p-1)(1-\theta)}{N-1-\theta(p-1)}\right)$.*

3.2 Existence of bounded weak solutions for L^∞-data

To prove Theorem 3.1.5, we will introduce and solve approximating problems defined in the subsection 3.2.1. For $f \in L^1(\Omega)$ and $m, n \in \mathbb{N}$ we define $f_{m,n} : \Omega \to \mathbb{R}$ by $f_{m,n}(x) = max(-n, min(f(x), m))$ for almost every $x \in \Omega$. Clearly, $f_{m,n} \in L^\infty(\Omega)$ for each $m, n \in \mathbb{N}$, $|f_{m,n}(x)| \leq |f(x)|$ a.e. in Ω, hence $\lim_{m\to\infty} \lim_{n\to\infty} f_{m,n} = f$ in $L^1(\Omega)$ and a.e. in Ω. The next Theorem will give us existence of weak solutions $(u_{m,n}, b_{m,n})$ of $(P, f_{m,n})$ for each $m, n \in \mathbb{N}$:

Theorem 3.2.1. *Under the assumptions (H_1)-(H_3) and for $f \in L^\infty(\Omega)$. There exists at least one weak solution (u, b) of the problem (P, f).*

Remark 3.2.2. Note that the assumption on f given in the preceding theorem yields $L^\infty(\Omega)$ solutions for nonlinear coercive elliptic inclusion problem. The result (which

is independent of θ) is not surprising, since if one looks for bounded solutions then the lack of coercivity of the operator $A(u)$ (which is created by unbounded functions) "disappears".

The proof of Theorem 3.2.1 will be divided into several steps.

3.2.1 Approximate problems for L^∞-data

Let $0 < \varepsilon \leq 1$, we consider the sequence of approximate equations

$$(P_\varepsilon, f) \quad \begin{cases} \beta_\varepsilon(T_{\frac{1}{\varepsilon}}(u_\varepsilon)) + \varepsilon arctan(u_\varepsilon) + A_\varepsilon(u_\varepsilon) = f & \text{in } \Omega, \\ \\ u_\varepsilon = 0 & \text{on } \partial\Omega, \end{cases}$$

where $A_\varepsilon(u) = -\text{div}\left(\dfrac{a(x,\nabla u)}{(1+|T_{\frac{1}{\varepsilon}}(u)|)^{\theta(p-1)}}\right)$ and $\beta_\varepsilon(\cdot) : \mathbb{R} \to \mathbb{R}$ is the Yosida approximation of $\beta(\cdot)$. Note that, for any $u \in W_0^{1,p}(\Omega)$ we have

$$\langle \beta_\varepsilon(u), u\rangle \geq 0, \ \ |\beta_\varepsilon(u)| \leq \frac{1}{\varepsilon}|u| \ \text{ and } \ \lim_{\varepsilon\to 0}\beta_\varepsilon(u) = \beta(u).$$

We refer the reader to [46] for more details about the maximal monotone mapping.

We introduce the operator $B_\varepsilon : W_0^{1,p}(\Omega) \to W^{-1,p'}(\Omega)$, defined by

$$\langle B_\varepsilon u, v\rangle = \int_\Omega \left(\beta_\varepsilon(T_{\frac{1}{\varepsilon}}(u)) + \varepsilon arctan(u) + A_\varepsilon(u)\right) v dx, \text{ for any } u, v \in W_0^{1,p}(\Omega).$$

Lemma 3.2.3. *The operator $B_\varepsilon : W_0^{1,p}(\Omega) \to W^{-1,p'}(\Omega)$ is bounded, coercive and pseudo-monotone.*

Proof. Because of (3.2) and (3.3), B_ε is well-defined and monotone. Since $\beta_\varepsilon \circ T_{\frac{1}{\varepsilon}}$ and *arctan* are bounded and continuous, also by the growth condition (3.2), it follows that B_ε is pseudo-monotone (see [86], p.157).

For any $u \in W_0^{1,p}(\Omega)$ we have

$$\langle B_\varepsilon u, u\rangle = \int_\Omega \frac{a(x,\nabla u)\cdot\nabla u}{(1+|T_{\frac{1}{\varepsilon}}(u_\varepsilon)|)^{\theta(p-1)}} dx + \int_\Omega \left(\beta_\varepsilon(T_{\frac{1}{\varepsilon}}(u)) + \varepsilon arctan(u)\right) u dx,$$

as the second term on the right hand side is nonnegative, by (3.1) we obtain

$$\langle B_\varepsilon u, u\rangle \geq \frac{\alpha}{(1+\frac{1}{\varepsilon})^{\theta(p-1)}} \int_\Omega |\nabla u|^p dx.$$

Hence, B_ε is coercive on $W_0^{1,p}(\Omega)$. □

In view of Lemma 3.2.3, for any fixed $\varepsilon > 0$, there exists at least one solution u_ε of (E_ε, f) (cf.[82], [86]), i.e: for each $0 < \varepsilon \leq 1$ and $f \in W^{-1,p'}(\Omega)$ there exists at least one solution $u_\varepsilon \in W_0^{1,p}(\Omega)$ such that

$$\int_\Omega \left(\beta_\varepsilon(T_{\frac{1}{\varepsilon}}(u_\varepsilon)) + \varepsilon arctan(u_\varepsilon)\right)\varphi dx + \int_\Omega \frac{a(x,\nabla u_\varepsilon)\cdot\nabla\varphi}{(1+|T_{\frac{1}{\varepsilon}}(u_\varepsilon)|)^{\theta(p-1)}}dx = \langle f,\varphi\rangle \tag{3.6}$$

holds for all $\varphi \in W_0^{1,p}(\Omega)$, where $\langle .,.\rangle$ denotes the duality pairing between $W_0^{1,p}(\Omega)$ and $W^{-1,p'}(\Omega)$.

In the next proposition, we discuss uniqueness of solutions u_ε of (P_ε, f) with right-hand side $f \in L^\infty(\Omega)$ through a comparison principle that will play an important role in the approximation of solutions to (P, f) with $f \in L^1(\Omega)$.

Proposition 3.2.4. *For $0 < \varepsilon \leq 1$ fixed and $f, \tilde{f} \in L^\infty(\Omega)$ let $u_\varepsilon, \tilde{u}_\varepsilon \in W_0^{1,p}(\Omega)$ be solutions of (P_ε, f) and $(P_\varepsilon, \tilde{f})$, respectively. Then, the following comparison principle holds*

$$\varepsilon\int_\Omega (arctan(u_\varepsilon) - arctan(\tilde{u}_\varepsilon))^+ dx \leq \int_\Omega (f-\tilde{f})sign_0^+(u_\varepsilon - \tilde{u}_\varepsilon)dx. \tag{3.7}$$

Proof. We use the test function $S_\delta^+(u_\varepsilon - \tilde{u}_\varepsilon)$ in the weak formulation (3.6) for u_ε and $\tilde{u}_\varepsilon$. Subtracting the resulting inequalities, we obtain

$$I_{\varepsilon,\delta}^1 + I_{\varepsilon,\delta}^2 + I_{\varepsilon,\delta}^3 = I_{\varepsilon,\delta}^4$$

where

$$I_{\varepsilon,\delta}^1 = \int_\Omega \left(\beta_\varepsilon(T_{\frac{1}{\varepsilon}}(u_\varepsilon)) - \beta_\varepsilon(T_{\frac{1}{\varepsilon}}(\tilde{u}_\varepsilon))\right) S_\delta^+(u_\varepsilon - \tilde{u}_\varepsilon)dx,$$

$$I_{\varepsilon,\delta}^2 = \varepsilon\int_\Omega (arctan(u_\varepsilon) - arctan(\tilde{u}_\varepsilon))\, S_\delta^+(u_\varepsilon - \tilde{u}_\varepsilon)dx,$$

$$I_{\varepsilon,\delta}^3 = \int_\Omega \left(\frac{a(x,\nabla u_\varepsilon)}{(1+|T_{\frac{1}{\varepsilon}}(u_\varepsilon)|)^{\theta(p-1)}} - \frac{a(x,\nabla \tilde{u}_\varepsilon)}{(1+|T_{\frac{1}{\varepsilon}}(\tilde{u}_\varepsilon)|)^{\theta(p-1)}}\right)\cdot\nabla S_\delta^+(u_\varepsilon - \tilde{u}_\varepsilon)dx,$$

$$I_{\varepsilon,\delta}^4 = \int_\Omega (f-\tilde{f})S_\delta^+(u_\varepsilon - \tilde{u}_\varepsilon)dx.$$

Note that, for $b \geq a \geq 0$ and $\lambda \geq 0$, we have the inequality

$$\left|\frac{1}{(1+b)^\lambda} - \frac{1}{(1+a)^\lambda}\right| = \left|\frac{\lambda(b-a)}{(1+c)^{\lambda+1}}\right| \leq \lambda|b-a| \text{ for some } c \in [a,b]. \tag{3.8}$$

By using (3.2) and (3.8), we have

$$
\begin{aligned}
I^3_{\varepsilon,\delta} &\geq \frac{1}{\delta}\int_{\{(u_\varepsilon-\tilde{u}_\varepsilon)^+<\delta\}}\left(\frac{1}{(1+|T_{\frac{1}{\varepsilon}}(u_\varepsilon)|)^{\theta(p-1)}}-\frac{1}{(1+|T_{\frac{1}{\varepsilon}}(\tilde{u}_\varepsilon)|)^{\theta(p-1)}}\right)a(x,\nabla\tilde{u}_\varepsilon)\cdot\nabla(u_\varepsilon-\tilde{u}_\varepsilon)^+dx\\
&\geq \frac{-C\delta}{\delta}\int_{\{(u_\varepsilon-\tilde{u}_\varepsilon)^+<\delta\}}(a_0(x)+|\nabla\tilde{u}_\varepsilon|^{p-1})\cdot|\nabla(u_\varepsilon-\tilde{u}_\varepsilon)^+|dx\\
&\geq -C\int_\Omega\left(a_0(x)+|\nabla\tilde{u}_\varepsilon|^{p-1}\right)\cdot|\nabla(u_\varepsilon-\tilde{u}_\varepsilon)^+|\chi_{\{(u_\varepsilon-\tilde{u}_\varepsilon)^+<\delta\}}dx
\end{aligned}
$$

then $\liminf_{\delta\to 0} I^3_{\varepsilon,\delta}\geq 0$.

Passing to the limit with $\delta\downarrow 0$, (3.7) follows.

□

Remark 3.2.5. Let $f,\tilde{f}\in L^\infty(\Omega)$ be such that $f\leq\tilde{f}$ a.e. in Ω, $\varepsilon>0$ and $u_\varepsilon,\tilde{u}_\varepsilon\in W^{1,p}_0(\Omega)$ be solutions of (P_ε,f) and $(P_\varepsilon,\tilde{f})$, respectively. Then it is an immediate consequence of Proposition 3.2.4 that $u_\varepsilon\leq\tilde{u}_\varepsilon$ a.e. in Ω. Furthermore, from the monotonicity of $\beta_\varepsilon\circ T_{\frac{1}{\varepsilon}}$ it follows that also $\beta_\varepsilon(T_{\frac{1}{\varepsilon}}(u_\varepsilon)\leq\beta_\varepsilon(T_{\frac{1}{\varepsilon}}(\tilde{u}_\varepsilon)$ a.e. in Ω.

3.2.2 A priori estimates

In this step, we will give some estimates on weak solutions of approximate problems. By taking $G_k(H(u_\varepsilon))$ as a test function in (3.6), where

$$H(s)=\int_0^s\frac{dt}{(1+t)^\theta},$$

we obtain

$$
\begin{aligned}
\int_{X_k}\left(\beta_\varepsilon(T_{\frac{1}{\varepsilon}}(u_\varepsilon))+\varepsilon arctan(u_\varepsilon)\right)G_k(H(u_\varepsilon))dx+\int_{X_k}\frac{a(x,\nabla u_\varepsilon)\cdot\nabla u_\varepsilon}{(1+|T_{\frac{1}{\varepsilon}}(u_\varepsilon)|)^{\theta p}}dx\\
=\int_{X_k}fG_k(H(u_\varepsilon))dx,
\end{aligned}
$$

where $X_k=\{|H(u_\varepsilon)|>k\}$.

As the first term on the left hand side is nonnegative, by (3.1) we have

$$\alpha\int_{X_k}|\nabla H(u_\varepsilon)|^p dx\leq\int_{X_k}|f||G_k(H(u_\varepsilon))|dx. \tag{3.9}$$

Inequality (3.9) is exactly the starting point of Stampacchia's L^∞-regularity proof (see [102, Lemma 4.1]), so that there exists a constant C such that

$$||H(u_\varepsilon)||_{L^\infty(\Omega)}\leq C.$$

Indeed, using the Sobolev embedding $W_0^{1,p}(\Omega) \subset L^{p^*}(\Omega)$ and Hölder's inequality, we have

$$\begin{aligned}\left(\int_{X_k} |G_k(H(u_\varepsilon))|^{p^*} dx\right)^{\frac{p}{p^*}} &\leq S^p \int_{X_k} |\nabla H(u_\varepsilon)|^p dx \\ &\leq \frac{S^p}{\alpha} \int_{X_k} |f||G_k(H(u_\varepsilon))| dx \\ &\leq C \int_{X_k} |G_k(H(u_\varepsilon))| dx \\ &\leq C|X_k|^{1-\frac{1}{p^*}} \left(\int_{X_k} |G_k(H(u_\varepsilon))|^{p^*} dx\right)^{\frac{1}{p^*}}.\end{aligned}$$

Then

$$\left(\int_{X_k} |G_k(H(u_\varepsilon))|^{p^*} dx\right)^{\frac{p-1}{p^*}} \leq C|X_k|^{1-\frac{1}{p^*}}.$$

Let $k' > k$, using the fact that $X_{k'} \subset X_k$ and

$$|G_k(H(u_\varepsilon))| \geq k' - k$$

on the set $X_{k'}$, we obtain

$$(k'-k)^{p-1}|X_{k'}|^{\frac{p-1}{p^*}} \leq C|X_k|^{1-\frac{1}{p^*}}.$$

Hence

$$|X_{k'}| \leq \frac{C}{(k'-k)^{p^*}}|X_k|^{\frac{p^*-1}{p-1}}.$$

We have $\frac{p^*-1}{p-1} > 1$, so that by Satmpacchia's Lemma $H(u_\varepsilon)$ is uniformly bounded. The properties of the function H (in particular the fact that $\lim_{s\to\infty} H(s) = \infty$) yield a bound for u_ε in $L^\infty(\Omega)$, that is

$$||u_\varepsilon||_{L^\infty(\Omega)} \leq C_\infty, \tag{3.10}$$

with C_∞ is a constant that don't depend on ε.

Now, we choose u_ε as a test function in (3.6), we obtain

$$\int_\Omega \left(\beta_\varepsilon(T_{\frac{1}{\varepsilon}}(u_\varepsilon)) + \varepsilon arctan(u_\varepsilon)\right) u_\varepsilon dx + \int_\Omega \frac{a(x,\nabla u_\varepsilon)\cdot\nabla u_\varepsilon}{(1+|T_{\frac{1}{\varepsilon}}(u_\varepsilon)|)^{\theta(p-1)}} dx = \int_\Omega f u_\varepsilon dx,$$

as the first term on the left hand side is nonnegative, by (3.1) we have

$$\alpha \int_\Omega \frac{|\nabla u_\varepsilon|^p}{(1+|u_\varepsilon|)^{\theta(p-1)}} dx \leq \int_\Omega |f||u_\varepsilon| dx.$$

Since the norm of u_ε in $L^\infty(\Omega)$ is bounded, which implies

$$\int_\Omega |\nabla u_\varepsilon|^p dx \leq C. \tag{3.11}$$

On the other hand, for $\delta > 0$, we use $\varphi_{\delta,\varepsilon} = \frac{1}{\delta}\left(T_{k+\delta}(\beta_\varepsilon(T_{\frac{1}{\varepsilon}}(u_\varepsilon))) - T_k(\beta_\varepsilon(T_{\frac{1}{\varepsilon}}(u_\varepsilon)))\right)$ as a test function in (3.6), we obtain

$$\int_\Omega \left(\beta_\varepsilon(T_{\frac{1}{\varepsilon}}(u_\varepsilon)) + \varepsilon arctan(u_\varepsilon)\right)\varphi_{\delta,\varepsilon} dx + \int_\Omega \frac{a(x,\nabla u_\varepsilon)\cdot\nabla\varphi_{\delta,\varepsilon}}{(1+|T_{\frac{1}{\varepsilon}}(u_\varepsilon)|)^{\theta(p-1)}} dx = \int_\Omega f\varphi_{\delta,\varepsilon} dx.$$

Since $|\varphi_{\delta,\varepsilon}| \leq 1$, β_ε monotone increasing with $\beta_\varepsilon(0) = 0$ and by (3.1), we obtain

$$\frac{1}{\delta}\int_{\{k\leq|\beta_\varepsilon(T_{\frac{1}{\varepsilon}}(u_\varepsilon))|\leq k+\delta\}} \frac{a(x,\nabla u_\varepsilon)\cdot\nabla T_{\frac{1}{\varepsilon}}(u_\varepsilon)}{(1+|T_{\frac{1}{\varepsilon}}(u_\varepsilon)|)^{\theta(p-1)}}\beta_\varepsilon'(T_{\frac{1}{\varepsilon}}(u_\varepsilon))dx \geq 0.$$

Then

$$\frac{1}{\delta}\int_{\{|\beta_\varepsilon(T_{\frac{1}{\varepsilon}}(u_\varepsilon))|\geq k\}} \beta_\varepsilon(T_{\frac{1}{\varepsilon}}(u_\varepsilon))\left(T_{k+\delta}(\beta_\varepsilon(T_{\frac{1}{\varepsilon}}(u_\varepsilon))) - T_k(\beta_\varepsilon(T_{\frac{1}{\varepsilon}}(u_\varepsilon)))\right)dx \leq \int_{\{|\beta_\varepsilon(T_{\frac{1}{\varepsilon}}(u_\varepsilon)|\geq k\}} |f|dx.$$

Having in mind that $\varphi_{\delta,\varepsilon}$ has the same sign as u_ε and using the monotonicity of the opertaor β_ε, we conclude that

$$\begin{aligned}
&k|\{|\beta_\varepsilon(T_{\frac{1}{\varepsilon}}(u_\varepsilon))| \geq k+\delta\}| \\
&\leq \int_{\{|\beta_\varepsilon(T_{\frac{1}{\varepsilon}}(u_\varepsilon))|\geq k+\delta\}} |\beta_\varepsilon(T_{\frac{1}{\varepsilon}}(u_\varepsilon))|dx \\
&\leq \frac{1}{\delta}\int_{\{|\beta_\varepsilon(T_{\frac{1}{\varepsilon}}(u_\varepsilon))|\geq k\}} \beta_\varepsilon(T_{\frac{1}{\varepsilon}}(u_\varepsilon))\left(T_{k+\delta}(\beta_\varepsilon(T_{\frac{1}{\varepsilon}}(u_\varepsilon))) - T_k(\beta_\varepsilon(T_{\frac{1}{\varepsilon}}(u_\varepsilon)))\right)dx \\
&\leq \int_{\{|\beta_\varepsilon(T_{\frac{1}{\varepsilon}}(u_\varepsilon))|\geq k\}} |f|dx \\
&\leq ||f||_\infty|\{|\beta_\varepsilon(T_{\frac{1}{\varepsilon}}(u_\varepsilon))| \geq k\}|.
\end{aligned}$$

Choosing $k > ||f||_\infty$ and passing to the limit with $\delta \to 0$, we obtain

$$|\{|\beta_\varepsilon(T_{\frac{1}{\varepsilon}}(u_\varepsilon))| \geq k\}| = 0 \text{ for any } k > ||f||_\infty.$$

Therefore

$$||\beta_\varepsilon(T_{\frac{1}{\varepsilon}}(u_\varepsilon))||_\infty \leq ||f||_\infty. \tag{3.12}$$

3.2.3 Basic convergence results

By (3.12), there exist $b \in L^\infty(\Omega)$ such that

$$\beta_\varepsilon(T_{\frac{1}{\varepsilon}}(u_\varepsilon)) \overset{*}{\rightharpoonup} b \text{ in } L^\infty(\Omega). \tag{3.13}$$

Thanks to (3.11), u_ε remains bounded in $W_0^{1,p}(\Omega)$, we can extract a subsequence, still denoted by u_ε, such that

$$u_\varepsilon \rightharpoonup u \text{ weakly in } W_0^{1,p}(\Omega) \text{ and a.e. in } \Omega. \tag{3.14}$$

We shall prove that

$$u_\varepsilon \to u \text{ strongly in } W_0^{1,p}(\Omega). \tag{3.15}$$

Using $u_\varepsilon - u$ as test function in (3.6), we obtain for $\dfrac{1}{\varepsilon} > C_\infty$

$$\int_\Omega \left(\beta_\varepsilon(T_{\frac{1}{\varepsilon}}(u_\varepsilon)) + \varepsilon arctan(u_\varepsilon)\right)(u_\varepsilon - u)dx + \int_\Omega \frac{a(x,\nabla u_\varepsilon)}{(1+|u_\varepsilon|)^{\theta(p-1)}} \cdot (\nabla u_\varepsilon - \nabla u)dx = \int_\Omega f(u_\varepsilon - u)dx. \tag{3.16}$$

For the first term on the left-hand side of (3.16). In view of (3.14) we have $u_\varepsilon \to u$ in $L^1(\Omega)$ and thanks to (3.13), then

$$\lim_{\varepsilon\to 0} \int_\Omega \left(\beta_\varepsilon(T_{\frac{1}{\varepsilon}}(u_\varepsilon)) + \varepsilon arctan(u_\varepsilon)\right)(u_\varepsilon - u)dx = 0. \tag{3.17}$$

Similarly, we have f belongs to $L^\infty(\Omega)$, then

$$\lim_{\varepsilon\to 0} \int_\Omega f(u_\varepsilon - u)dx = 0. \tag{3.18}$$

On the other hand, by (3.17) and (3.18) one has

$$\lim_{\varepsilon\to 0} \int_\Omega \frac{a(x,\nabla u_\varepsilon)}{(1+|u_\varepsilon|)^{\theta(p-1)}} \cdot (\nabla u_\varepsilon - \nabla u)dx = 0.$$

Using (3.2) and Vitali's Theorem, we obtain that

$$\frac{a(x,\nabla u)}{(1+|u_\varepsilon|)^{\theta(p-1)}} \to \frac{a(x,\nabla u)}{(1+|u|)^{\theta(p-1)}} \text{ strongly in } (L^{p'}(\Omega))^N.$$

It follows by (3.14) that

$$\lim_{\varepsilon\to 0}\int_\Omega \frac{a(x,\nabla u)}{(1+|u_\varepsilon|)^{\theta(p-1)}}\cdot(\nabla u_\varepsilon-\nabla u)dx=0.$$

Therefore, we obtain

$$\lim_{\varepsilon\to 0}\int_\Omega \left(\frac{a(x,\nabla u_\varepsilon)}{(1+|u_\varepsilon|)^{\theta(p-1)}}-\frac{a(x,\nabla u)}{(1+|u_\varepsilon|)^{\theta(p-1)}}\right)\cdot(\nabla u_\varepsilon-\nabla u)dx=0. \tag{3.19}$$

Thanks to (3.3), the integrand function in the left-hand side in (3.19) is nonnegative, therefore

$$\left(\frac{a(x,\nabla u_\varepsilon)}{(1+|u_\varepsilon|)^{\theta(p-1)}}-\frac{a(x,\nabla u)}{(1+|u_\varepsilon|)^{\theta(p-1)}}\right)\cdot(\nabla u_\varepsilon-\nabla u)\to 0 \text{ strongly in } L^1(\Omega).$$

Thus, up to a subsequence still indexed by ε, one has

$$\left(\frac{a(x,\nabla u_\varepsilon(x))}{(1+|u_\varepsilon(x)|)^{\theta(p-1)}}-\frac{a(x,\nabla u(x))}{(1+|u_\varepsilon(x)|)^{\theta(p-1)}}\right)\cdot(\nabla u_\varepsilon(x)-\nabla u(x))\to 0 \text{ for a.e. } x \text{ in } \Omega.$$

There exists a subset Z of Ω of zero measure, such that for all x in $\Omega\backslash Z$ we have

$$D_\varepsilon(x)=\left(\frac{a(x,\nabla u_\varepsilon(x))}{(1+|u_\varepsilon(x)|)^{\theta(p-1)}}-\frac{a(x,\nabla u(x))}{(1+|u_\varepsilon(x)|)^{\theta(p-1)}}\right)\cdot(\nabla u_\varepsilon(x)-\nabla u(x))\to 0,$$

$|u(x)|<\infty$, $|\nabla u(x)|<\infty$, $|k(x)|<\infty$ and $u_\varepsilon(x)\to u(x)$.

Then, by (3.2), (3.1) and (3.10) we can write

$$D_\varepsilon(x)\geq \frac{\alpha}{(1+|C_\infty|)^{\theta(p-1)}}|\nabla u_\varepsilon(x)|^p-c(x)(1+|\nabla u_\varepsilon(x)|+|\nabla u_\varepsilon(x)|^{p-1}), \tag{3.20}$$

where $c(x)$ is a constant which depends on x but does not depend on ε, which shows, thanks to (3.20), that the sequence $|\nabla u_\varepsilon(x)|$ is uniformly bounded in $\mathbb{R}^N$ with respect to ε. We argue similarly as in Lemma 5 in [41], to obtain

$$u_\varepsilon\to u \text{ strongly in } W_0^{1,p}(\Omega).$$

3.2.4 Proof of Theorem 3.2.1

We can now pass to the limit going back to the equation (3.6). To do this, let $\varphi\in W_0^{1,p}(\Omega)$. For every $\frac{1}{\varepsilon}>C_\infty$ one has

$$\int_\Omega\left(\beta_\varepsilon(T_{\frac{1}{\varepsilon}}(u_\varepsilon))+\varepsilon arctan(u_\varepsilon)\right)\varphi dx+\int_\Omega\frac{a(x,\nabla u_\varepsilon)\cdot\nabla\varphi}{(1+|u_\varepsilon|)^{\theta(p-1)}}dx=\int_\Omega f\varphi dx. \tag{3.21}$$

In vertue of (3.15), we have for a subsequence

$$\nabla u_\varepsilon \to \nabla u \text{ a.e. in } \Omega,$$

which with

$$u_\varepsilon \to u \text{ a.e. in } \Omega$$

yields, since $\dfrac{a(x,\nabla u_\varepsilon)}{(1+|u_\varepsilon|)^{\theta(p-1)}}$ is bounded in $(L^{p'}(\Omega))^N$,

$$\frac{a(x,\nabla u_\varepsilon)}{(1+|u_\varepsilon|)^{\theta(p-1)}} \rightharpoonup \frac{a(x,\nabla u)}{(1+|u|)^{\theta(p-1)}} \text{ weakly in } (L^{p'}(\Omega))^N.$$

Then, passing to the limit in (3.21), we obtain

$$\int_\Omega b\varphi dx + \int_\Omega \frac{a(x,\nabla u)}{(1+|u|)^{\theta(p-1)}} \cdot \nabla\varphi dx = \int_\Omega f\varphi dx \text{ for any } \varphi \in W_0^{1,p}(\Omega).$$

3.2.5 Subdifferential argument

Using the same arguments as in the subsection 2.2.5, we can find that $u \in D(\beta)$ and $b \in \beta(u)$ for almost everywhere in Ω.

With this last step the proof of Theorem 3.2.1 is concluded.

3.3 Existence and regularity of entropy solutions for L^1-data

3.3.1 Approximation problem for L^1-data and a priori estimates

For $f \in L^1(\Omega)$ and $m,n \in \mathbb{N}$ let $f_{m,n} \in L^\infty(\Omega)$ be defined as in the beginning of Section 3.2.

From Theorem 3.2.1, there exists a solution $(u_{m,n}, b_{m,n}) \in W_0^{1,p}(\Omega) \cap L^\infty(\Omega) \times L^\infty(\Omega)$ of $(P, f_{m,n})$ such that

$$\int_\Omega b_{m,n}\varphi dx + \int_\Omega \frac{a(x,\nabla u_{m,n})}{(1+|u_{m,n}|)^{\theta(p-1)}} \cdot \nabla\varphi dx = \int_\Omega f_{m,n}\varphi dx \tag{3.22}$$

holds for all $m,n \in \mathbb{N}$ and $\varphi \in W_0^{1,p}(\Omega)$.

Lemma 3.3.1. *Suppose $(u_{m,n}, b_{m,n})$ is a solution of $(P, f_{m,n})$. Then, for $k > 0$, there exists a constant $C(k) > 0$, not depending on $m, n \in \mathbb{N}$, such that*

$$||T_k(u_{m,n})||_{W_0^{1,p}(\Omega)} \leq C(k), \tag{3.23}$$

$$||b_{m,n}||_{L^1(\Omega)} \leq ||f||_{L^1(\Omega)}. \tag{3.24}$$

Proof. By taking $T_k(u_{m,n})$ as a test function in (3.22), we have

$$\int_\Omega b_{m,n} T_k(u_{m,n}) dx + \int_\Omega \frac{a(x, \nabla u_{m,n})}{(1 + |u_{m,n}|)^{\theta(p-1)}} \cdot \nabla T_k(u_{m,n}) dx = \int_\Omega f_{m,n} T_k(u_{m,n}) dx. \tag{3.25}$$

Since $b_{m,n}$ has the same sign as $u_{m,n}$ and thanks to (3.1), we obtain

$$\alpha \int_\Omega |\nabla T_k(u_{m,n})|^p dx \leq (1 + k)^{\theta(p-1)} k ||f||_{L^1(\Omega)}.$$

To prove (3.24), we neglect in (3.25) the positive term and keep

$$\int_\Omega b_{m,n} T_k(u_{m,n}) dx \leq \int_\Omega f_{m,n} T_k(u_{m,n}) dx \leq k ||f||_{L^1(\Omega)}.$$

Since $b_{m,n} \in \beta(u_{m,n})$ a.e. in Ω, we get

$$\int_{\{|u_{m,n}| > k\}} |b_{m,n}| dx \leq ||f||_{L^1(\Omega)}.$$

Passing the limit as $k \downarrow 0$ and using the Fatou's Lemma, we find

$$\int_\Omega |b_{m,n}| dx \leq ||f||_{L^1(\Omega)}.$$

□

3.3.2 Weak convergence of $T_k(u_n)$ in $W_0^{1,p}(\Omega)$

By definition, $f_{m,n}$ is a bi-monotone approximation of f in $L^1(\Omega)$,

$$f_{m,n} \leq f_{m+1,n} \text{ and } f_{m,n+1} \leq f_{m,n}.$$

From Proposition 3.2.4 it follows that

$$u^\varepsilon_{m,n} \leq u^\varepsilon_{m+1,n} \text{ and } u^\varepsilon_{m,n+1} \leq u^\varepsilon_{m,n} \tag{3.26}$$

almost everywhere in Ω for any $m, n \in \mathbb{N}$ and all $\varepsilon > 0$.

Hence passing to the limit with $\varepsilon \downarrow 0$ in (3.26) yields

$$u_{m,n} \leq u_{m+1,n} \text{ and } u_{m,n+1} \leq u_{m,n} \tag{3.27}$$

almost everywhere in Ω for any $m, n \in \mathbb{N}$.

Setting $b^\varepsilon := \beta_\varepsilon(T_{\frac{1}{\varepsilon}}(u_\varepsilon))$, using (3.26), Remark 3.2.5, the fact that $b^\varepsilon_{m,n} \overset{*}{\rightharpoonup} b_{m,n}$ in $L^\infty(\Omega)$ and this convergence preserves order, we get

$$b_{m,n} \leq b_{m+1,n} \text{ and } b_{m,n+1} \leq b_{m,n} \tag{3.28}$$

almost everywhere in Ω for any $m, n \in \mathbb{N}$.

By (3.24) and (3.28), we obtain

$$b_{m,n} \uparrow_m b^n \downarrow_n b \text{ in } L^1(\Omega),$$

where $b^n, b : \Omega \to \mathbb{R}$ are measurable functions, finite a.e. in Ω. Here, we use the notation $\uparrow_m$ respectively $\downarrow_m$, to denote convergence of a sequence which is monotone increasing, respectively decreasing, in m.

By (3.27), we get

$$u_{m,n} \uparrow_m u^n \downarrow_n u \text{ almost everywhere in } \Omega,$$

where $u^n, u : \Omega \to \overline{\mathbb{R}}$ are measurable functions.

In order to show that u is finite almost everywhere, we will give an estimate on the level sets of $u_{m,n}$ in the next Lemma 3.3.2.

Lemma 3.3.2. *For $m, n \in \mathbb{N}$, let $(u_{m,n}, b_{m,n})$ be a solution of $(P, f_{m,n})$. Then, there exist a constant $C > 0$, not depending on $m, n \in \mathbb{N}$ such that*

$$|\{|u_{m,n}| \geq h\}| \leq Ch^{-(p-1)(1-\theta)} \quad \textit{for all } h \geq 1. \tag{3.29}$$

Proof. Note that, as $(u_{m,n})_m$ is pointwise increasing with respect to m, we have

$$\lim_{m\to\infty} |\{u_{m,n} > h\}| = |\{u^n > h\}| \tag{3.30}$$

and

$$\lim_{m\to\infty} |\{u_{m,n} \leq -h\}| = |\{u^n \leq -h\}|. \tag{3.31}$$

Combining (3.29) with (3.30) and (3.31), we get

$$|\{u^n \leq -h\}| + |\{u^n > h\}| \leq Ch^{-(p-1)(1-\theta)} \tag{3.32}$$

for all $h \geq 1$, hence u^n is finite almost everywhere for any $n \in \mathbb{N}$.

By the same arguments, we get

$$|\{u \leq -h\}| + |\{u > h\}| \leq Ch^{-(p-1)(1-\theta)} \tag{3.33}$$

from (3.32), hence u is finite almost everywhere.

Now, since $b_{m,n} \in \beta(u_{m,n})$ almost everywhere in Ω it follows by a subdifferential argument that $b^n \in \beta(u^n)$ and $b \in \beta(u)$ almost everywhere in Ω.

□

In the rest of this section, for $m, n \in \mathbb{N}$, let $(u_{m,n}, b_{m,n})$ be a solution of $(P, f_{m,n})$. Applying the diagonal principle in $L^1(\Omega)$, we may assume that, for some sequence $(m(n))_n$, we have (with $f_n := f_{m(n),n}$, $b_n := b_{m(n),n}$ and $u_n := u_{m(n),n}$)

$$f_n \to f \text{ in } L^1(\Omega),$$

$$b_n \to b \text{ in } L^1(\Omega)$$

and

$$u_n \to u \text{ a.e. in } \Omega. \tag{3.34}$$

Combining (3.23) with (3.34) we get for any $k > 0$

$$T_k(u_n) \rightharpoonup T_k(u) \text{ in } W_0^{1,p}(\Omega). \tag{3.35}$$

3.3.3 Some regularity results

Lemma 3.3.3. *There exists a subsequence of u_n such that, for each $q \in \left(1, \frac{N(p-1)(1-\theta)}{N-1-\theta(p-1)}\right)$, we have*

$$||u_n||_{W_0^{1,q}(\Omega)} \leq C. \tag{3.36}$$

Proof. Let $k > 0$. Define $D_k = \{|u_n| \leq k\}$ and $B_k = \{k < |u_n| \leq k+1\}$. By taking $T_k(u_n)$ as a test function in (3.22), we have

$$\int_\Omega b_n T_k(u_n)dx + \int_\Omega \frac{a(x,\nabla u_n)}{(1+|u_n|)^{\theta(p-1)}} \cdot \nabla T_k(u_n)dx = \int_\Omega f_n T_k(u_n)dx,$$

since b_n has the same sign as u_n and thanks to (3.1), we obtain

$$\alpha \int_{D_k} \frac{|\nabla u_n)|^p}{(1+|u_n|)^{\theta(p-1)}}dx \leq k||f||_{L^1(\Omega)}. \tag{3.37}$$

Taking the function $T_1(u_n - T_k(u_n))$ as a test function in (3.22), we obtain

$$\int_\Omega b_n T_1(u_n - T_k(u_n))dx + \int_\Omega \frac{a(x,\nabla u_n)\cdot \nabla T_1(u_n - T_k(u_n))}{(1+|u_n|)^{\theta(p-1)}}dx = \int_\Omega f_n T_1(u_n - T_k(u_n))dx,$$

as the first term on the left hand side is nonnegative, by (3.1) we have

$$\alpha \int_{B_k} \frac{|\nabla u_n|^p}{(1+|u_n|)^{\theta(p-1)}}dx \leq ||f||_{L^1(\Omega)}. \tag{3.38}$$

Let $q \in \left(1, \frac{N(p-1)(1-\theta)}{N-1-\theta(p-1)}\right)$ and $r = \frac{q\theta(p-1)}{p}$. Noting that $q < p$ and $\frac{pr}{p-q} < q^*$.

For all $k > 0$, we estimate $\int_{B_k} |\nabla u_n|^q dx$. By using Hölder's inequality and (3.38), we have

$$\begin{aligned}
\int_{B_k} |\nabla u_n|^q dx &= \int_{B_k} \frac{|\nabla u_n|^q}{(1+|u_n|)^r}(1+|u_n|)^r dx \\
&\leq \left(\int_{B_k} \frac{|\nabla u_n|^p}{(1+|u_n|)^{\theta(p-1)}}dx\right)^{\frac{q}{p}} \left(\int_{B_k} (1+|u_n|)^{\frac{pr}{p-q}}dx\right)^{\frac{p-q}{p}} \\
&\leq C|B_k|^{\frac{p-q}{p}} + C\left(\int_{B_k} |u_n|^{\frac{pr}{p-q}}dx\right)^{\frac{p-q}{p}} \\
&\leq C|B_k|^{\frac{p-q}{p}} + C\left(\int_{B_k} |u_n|^{q^*}dx\right)^{\frac{r}{q^*}} |B_k|^{\frac{p-q}{p}-\frac{r}{q^*}}.
\end{aligned}$$

Since $|B_k| \leq \frac{1}{k^{q^*}} \int_{B_k} |u_n|^{q^*}dx$ for $k \geq k_0 \geq 1$, we have

$$\begin{aligned}
\int_{B_k} |\nabla u_n|^q dx &\leq C\left(\frac{1}{k^{q^*}}\int_{B_k} |u_n|^{q^*}dx\right)^{\frac{p-q}{p}} + C\frac{1}{k^{q^*(\frac{p-q}{p}-\frac{r}{q^*})}}\left(\int_{B_k} |u_n|^{q^*}dx\right)^{\frac{p-q}{p}} \\
&\leq \frac{2C}{k^{q^*(\frac{p-q}{p}-\frac{r}{q^*})}}\left(\int_{B_k} |u_n|^{q^*}dx\right)^{\frac{p-q}{p}}.
\end{aligned}$$

Summing up from $k = k_0$ to $k = K$ and using Hölder's inequality, we have

$$\sum_{k=k_0}^{K} \int_{B_k} |\nabla u_n|^q dx \leq C \left(\sum_{k=k_0}^{K} \frac{1}{k^{q^*(\frac{p-q}{p} - \frac{r}{q^*})\frac{p}{q}}} \right)^{\frac{q}{p}} \left(\sum_{k=k_0}^{K} \int_{B_k} |u_n|^{q^*} dx \right)^{\frac{p-q}{p}}. \tag{3.39}$$

Note that

$$\int_{\{|u_n| \leq K\}} |\nabla u_n|^q dx = \int_{D_{k_0}} |\nabla u_n|^q dx + \sum_{k=k_0}^{K} \int_{B_k} |\nabla u_n|^q dx.$$

By using Hölder's inequality and (3.37), we obtain

$$\begin{aligned} \int_{D_{k_0}} |\nabla u_n|^q dx &\leq \left(\int_{D_{k_0}} \frac{|\nabla u_n|^p}{(1 + |u_n|)^{\theta(p-1)}} dx \right)^{\frac{q}{p}} \left(\int_{D_{k_0}} (1 + |u_n|)^{\frac{pr}{p-q}} dx \right)^{\frac{p-q}{p}} \\ &\leq C. \end{aligned} \tag{3.40}$$

Note that $\sum\limits_{k=k_0}^{K} \frac{1}{k^{q^*(\frac{p-q}{p} - \frac{r}{q^*})\frac{p}{q}}}$ converges since $q^*(\frac{p-q}{p} - \frac{r}{q^*})\frac{p}{q} > 1$. Combining (3.39) and (3.40), for k_0 large enough, we get

$$\int_{\{|u_n| \leq K\}} |\nabla u_n|^q dx \leq C + C \left(\int_{\{|u_n| \leq K\}} |u_n|^{q^*} dx \right)^{\frac{p-q}{p}}. \tag{3.41}$$

Using the Sobolev embedding $W_0^{1,q}(\Omega) \subset L^{q^*}(\Omega)$, we have

$$\int_{\{|u_n| \leq K\}} |u_n|^{q^*} dx \leq C \left(\int_{\{|u_n| \leq K\}} |\nabla u_n|^q dx \right)^{\frac{q^*}{q}}. \tag{3.42}$$

From (3.41) and (3.42) we obtain

$$\int_{\{|u_n| \leq K\}} |\nabla u_n|^q dx \leq C + C \left(\int_{\{|u_n| \leq K\}} |\nabla u_n|^q dx \right)^{\frac{q^*}{q}\frac{p-q}{p}}.$$

Since $p < N$ one has $\frac{q^*}{q}\frac{p-q}{p} < 1$. Consequently, we conclude that, for k_0 large enough, $\int\limits_{\{|u_n| \leq K\}} |\nabla u_n|^q dx$ is bounded independently of n and K.

Letting $K \to \infty$, we deduce that $||\nabla u_n||_q$ is uniformly bounded independently of n.

Therefore, there exists a subsequence of u_n and a function $v \in W_0^{1,q}(\Omega)$ such that $u_n \rightharpoonup v$ weakly in $W_0^{1,q}(\Omega)$.

By (3.34), $u_n \to u$ a.e in Ω, we conclude that $u = v$ and $u \in W_0^{1,q}(\Omega)$. □

3.3.4 Almost everywhere convergence of gradients

Lemma 3.3.4. *There exist a subsequence of u_n and a measurable function u such that*

$$\nabla u_n \to \nabla u \text{ a.e. in } \Omega. \tag{3.43}$$

Proof. The proof of this Lemma is essentially the same as the proof of [13, Theorem 4.1]. Here we sketch only the main steps due to slight modifications.

For $s > 1$, let $\lambda = \frac{q}{ps} < 1$, where q is the same as in Lemma 3.3.3. We Define $\tilde{a}(x,u,\xi) = \dfrac{a(x,\xi)}{(1+|u|)^{\theta(p-1)}}$ (for simplicity, we omit the dependence of $\tilde{a}(x,u,\xi)$ on x) and

$$I_n = \int_\Omega ((\tilde{a}(u_n, \nabla u_n) - \tilde{a}(u_n, \nabla u)) \cdot \nabla(u_n - u))^\lambda dx.$$

We split I_n on the sets $\{|u| > k\}$ and $\{|u| \le k\}$, obtaining

$$I_n = I_1(n,k) + I_2(n,k),$$

where

$$I_1(n,k) = \int_{\{|u|>k\}} ((\tilde{a}(u_n, \nabla u_n) - \tilde{a}(u_n, \nabla u)) \cdot \nabla(u_n - u))^\lambda dx$$

and

$$I_2(n,k) = \int_{\{|u|\le k\}} ((\tilde{a}(u_n, \nabla u_n) - \tilde{a}(u_n, \nabla u)) \cdot \nabla(u_n - u))^\lambda dx.$$

For $I_2(n,k)$, we have

$$I_2(n,k) \le I_3(n,k) = \int_\Omega ((\tilde{a}(u_n, \nabla u_n) - \tilde{a}(u_n, \nabla T_k(u))) \cdot \nabla(u_n - T_k(u)))^\lambda dx.$$

Fix $h > 0$ and split $I_3(n,k)$ on the sets $\{|u_n - T_k(u)| > h\}$ and $\{|u_n - T_k(u)| \le h\}$, obtaining

$$I_4(n,k,h) = \int_{\{|u_n - T_k(u)|>h\}} ((\tilde{a}(u_n, \nabla u_n) - \tilde{a}(u_n, \nabla T_k(u))) \cdot \nabla(u_n - T_k(u)))^\lambda dx$$

and

$$\begin{aligned} I_5(n,k,h) &= \int_{\{|u_n - T_k(u)|\le h\}} ((\tilde{a}(u_n, \nabla u_n) - \tilde{a}(u_n, \nabla T_k(u))) \cdot \nabla(u_n - T_k(u)))^\lambda dx \\ &= \int_\Omega ((\tilde{a}(u_n, \nabla u_n) - \tilde{a}(u_n, \nabla T_k(u))) \cdot \nabla T_h(u_n - T_k(u)))^\lambda dx \\ &\le |\Omega|^{1-\lambda} \left(\int_\Omega (\tilde{a}(u_n, \nabla u_n) - \tilde{a}(u_n, \nabla T_k(u))) \cdot \nabla T_h(u_n - T_k(u)) dx \right)^\lambda \\ &= |\Omega|^{1-\lambda} \left(I_6(n,k,h) \right)^\lambda. \end{aligned}$$

For $I_6(n,k,h)$, we can split it as the difference $I_7(n,k,h) - I_8(n,k,h)$, where

$$I_7(n,k,h) = \int_\Omega \tilde{a}(u_n, \nabla u_n) \cdot \nabla T_h(u_n - T_k(u))dx$$

and

$$I_8(n,k,h) = \int_\Omega \tilde{a}(u_n, T_k(u)) \cdot \nabla T_h(u_n - T_k(u))dx.$$

Note that $|\nabla u_n|$ is bounded in $L^q(\Omega)$ and $\lambda ps = q$. Due to (3.34), (3.35), (3.36) and in the same way as Theorem 4.1 in [13], we can get that

$$\lim_{k\to\infty} \limsup_{n\to\infty} I_1(n,k) = 0,$$

$$\lim_{h\to\infty} \limsup_{k\to\infty} \limsup_{n\to\infty} I_4(n,k,h) = 0,$$

$$\lim_{n\to\infty} I_8(n,k,h) = 0.$$

For $I_7(n,k,h)$, taking the function $T_h(u_n - T_k(u))$ as a test function in (3.22), we obtain

$$\int_\Omega b_n T_h(u_n - T_k(u))dx + \int_\Omega \frac{a(x,\nabla u_n)\cdot\nabla T_h(u_n - T_k(u))}{(1+|u_n|)^{\theta(p-1)}}dx = \int_\Omega f_n T_h(u_n - T_k(u))dx.$$

Using the strong convergence of f_n and b_n in $L^1(\Omega)$, we have

$$\lim_{n\to\infty} I_7(n,k,h) = \int_\Omega f T_h(u - T_k(u))dx - \int_\Omega b T_h(u - T_k(u))dx.$$

It follows that

$$\lim_{k\to\infty} \lim_{n\to\infty} I_7(n,k,h) = 0.$$

Putting together all the limitations and noting that $I_n \geq 0$, we have

$$\lim_{n\to\infty} I_n = 0.$$

The same arguments as in [13] give that $\nabla u_n(x) \to \nabla u(x)$ a.e. in Ω. □

3.3.5 Conclusion of the proof of Theorem 3.1.5

Let $k > 0$ be fixed, $\varphi \in W_0^{1,p}(\Omega) \cap L^\infty(\Omega)$ and choose $T_k(u_n - \varphi)$ as a test function in (3.22), we have

$$\int_\Omega b_n T_k(u_n - \varphi)dx + \int_\Omega \frac{a(x,\nabla u_n)\cdot\nabla T_k(u_n - \varphi)}{(1+|u_n|)^{\theta(p-1)}}dx = \int_\Omega f_n T_k(u_n - \varphi)dx. \quad (3.44)$$

Since b_n and f_n are strongly convergent in $L^1(\Omega)$, while $T_k(u_n - \varphi)$ converges both weakly$*$ in $L^\infty(\Omega)$ and almost everywhere to $T_k(u - \varphi)$, we have

$$\lim_{n\to\infty} \int_\Omega b_n T_k(u_n - \varphi)dx = \int_\Omega bT_k(u - \varphi)dx$$

and

$$\lim_{n\to\infty} \int_\Omega f_n T_k(u_n - \varphi)dx = \int_\Omega fT_k(u - \varphi)dx.$$

Finally, we focus our attention on the second term of (3.44), we split it as the sum

$$\int_{\{|u_n-\varphi|\le k\}} \frac{a(x,\nabla u_n)\cdot\nabla u_n}{(1+|u_n|)^{\theta(p-1)}}dx - \int_{\{|u_n-\varphi|\le k\}} \frac{a(x,\nabla u_n)\cdot\nabla \varphi}{(1+|u_n|)^{\theta(p-1)}}dx.$$

Taking into account the fact that the second integral is on a subset of the set where $|u_n| \le k + ||\varphi||_{L^\infty(\Omega)} = M$, we can rewrite it as

$$\int_{\{|u_n-\varphi|\le k\}} \frac{a(x,\nabla T_M(u_n))\cdot\nabla \varphi}{(1+|T_M(u_n)|)^{\theta(p-1)}}dx.$$

Since $\dfrac{a(x,\nabla T_M(u_n))}{(1+|T_M(u_n)|)^{\theta(p-1)}}$ is bounded in $(L^{p'}(\Omega))^N$. Thus, by (3.34) and (3.43), we obtain

$$\frac{a(x,\nabla T_M(u_n))}{(1+|T_M(u_n)|)^{\theta(p-1)}} \rightharpoonup \frac{a(x,\nabla T_M(u))}{(1+|T_M(u)|)^{\theta(p-1)}} \quad \text{weakly in} (L^{p'}(\Omega))^N.$$

Hence, the second integral converges, as n tends to infinity, to

$$\int_{\{|u-\varphi|\le k\}} \frac{a(x,\nabla T_M(u))\cdot\nabla \varphi}{(1+|T_M(u)|)^{\theta(p-1)}}dx = \int_{\{|u-\varphi|\le k\}} \frac{a(x,\nabla u)\cdot\nabla \varphi}{(1+|u|)^{\theta(p-1)}}dx.$$

Thanks to (3.1), the integrand function of the first is nonnegative, by Fatou's Lemma (3.34) and (3.43), we get

$$\int_{\{|u-\varphi|\le k\}} \frac{a(x,\nabla u)\cdot\nabla u}{(1+|u|)^{\theta(p-1)}}dx \le \liminf_{n\to\infty} \int_{\{|u_n-\varphi|\le k\}} \frac{a(x,\nabla u_n)\cdot\nabla u_n}{(1+|u_n|)^{\theta(p-1)}}dx.$$

Putting all the terms together, we obtain

$$\int_\Omega bT_k(u - \varphi)dx + \int_{\{|u-\varphi|\le k\}} \frac{a(x,\nabla u)\cdot\nabla (u-\varphi)}{(1+|u|)^{\theta(p-1)}}dx \le \int_\Omega fT_k(u - \varphi)dx.$$

which is (3.5).

Hence, (u, b) is an entropy solution of the problem. The regularity of the entropy solution u is guaranteed by Lemma 3.3.3.

3.4 Example

Let Ω be a bounded domain of $\mathbb{R}^N (N \geq 2)$, $2 - \frac{1}{N} < p < N$ and

$$0 \leq \theta(p-1) < \frac{N(p-1)}{N-1} - 1.$$

Let us consider the Carathéodory function

$$a(x, \xi) = |\xi|^{p-2}\xi,$$

it is easy to show that a satisfies the coercivity (3.1), the growth condition (3.2) and the strict monotonie condition (3.3).

Let β the maximal monotone graph defined by

$$\beta(s) = (s-1)^+ - (s+1)^-,$$

or

$$\beta(r) = \begin{cases} \emptyset & \text{if } r < 0, \\ (-\infty, 0] & \text{if } r = 0, \\ 0 & \text{if } r > 0. \end{cases}$$

Finally, the hypotheses of Theorem 3.1.5 are satisfied, therefore for all $f \in L^1(\Omega)$ the following problem

$$(P, f) \quad \begin{cases} \beta(u) - \operatorname{div}\left(\dfrac{|\nabla u|^{p-2}\nabla u}{(1+|u|)^{\theta(p-1)}}\right) \ni f & \text{in } \Omega, \\ \\ u = 0 & \text{on } \partial\Omega, \end{cases}$$

has at least one entropy solution (u, b).

Chapter 4

Some results to nonlinear multivalued elliptic problems involving unbounded coefficients

This chapter is devoted to the study of a nonlinear elliptic inclusion problems involving unbounded coefficients and L^1-data. First, we study the existence of solutions of approximated problems, next we prove the existence of weak solutions for data in L^∞ and finally the existence of entropy solutions for data in L^1. The results of this chapter are based on the paper [4].

4.1 Assumptions on the data and statement of result

In the present chapter, we consider the following nonlinear multivalued elliptic problem with Dirichlet boundary conditions:

$$(U,f)\quad \begin{cases} \gamma(u) - \operatorname{div}(a(x,u,\nabla u)) \ni f & \text{in } \Omega, \\ \\ u = 0 & \text{on } \partial\Omega, \end{cases}$$

where Ω is a bounded domain in $\mathbb{R}^N$, $N \geq 2$, with sufficiently smooth boundary $\partial\Omega$, and γ is a maximal monotone graph such that $0 \in \gamma(0)$ and $a(x,s,\xi) : \Omega \times (-m,m) \times \mathbb{R}^N \to \mathbb{R}^N$, with $m > 0$, is a Carathéodory function satisfying the following assumptions:

There exists a positive function $l \in C^0([0,m))$ such that

$$\lim_{s\to m^-} l(s) = +\infty \text{ and } \lim_{s\to m^-} \int_0^s l(r)^{\frac{1}{p-1}} dr = +\infty, \tag{4.1}$$

$$a(x,s,\xi)\cdot\xi \geq l(|s|)|\xi|^p, \tag{4.2}$$

$$|a(x,s,\xi)| \leq C_0(a_0(x) + l(|s|)|\xi|^{p-1}), \tag{4.3}$$

$$(a(x,s,\xi) - a(x,s,\eta)) \cdot (\xi - \eta) > 0, \tag{4.4}$$

for almost every $x \in \Omega$, for all $s \in (-m,m)$ and all $\xi, \eta \in \mathbb{R}^N$ with $\xi \neq \eta$, where $1 < p < N$, $C_0 > 0$ and $a_0 \in L^{p'}(\Omega)$ with $p' = \frac{p}{p-1}$.

Let us now turn to our main result. We start with some classes of solutions.

Definition 4.1.1. Let $f \in L^\infty(\Omega)$. We say that a couple of functions $(u,w) \in W_0^{1,p}(\Omega) \times L^\infty(\Omega)$ is a weak solution of the problem (U,f) if $w \in \gamma(u)$ a.e. in Ω and

$$\int_\Omega wv dx + \int_\Omega a(x,u,\nabla u) \cdot \nabla v dx = \int_\Omega fv dx \text{ holds for all } v \in W_0^{1,p}(\Omega).$$

In this chapter, our assumptions on the problem (U,f) do not guarantee that a weak solution exists. So, it is reasonable to work with the concept of an entropy solution, which will be presented in the next Definition 4.1.2. In this way, we will obtain existence results for solutions that can achieve the critical values $\pm m$.

Definition 4.1.2. Let $f \in L^1(\Omega)$. We say that a couple of measurables functions (u,w) is an entropy solution of (U,f) if

(C1) $u \in W_0^{1,p}(\Omega)$ and $w \in L^1(\Omega)$ such that $|u| \leq m$, $u(x) \in D(\gamma(x))$ and $w(x) \in \gamma(u(x))$ for a.e. x in Ω, and

(C2) for all $0 < k < m$,

$$\int_\Omega wT_k(u-v)dx + \int_\Omega a(x,u,\nabla u) \cdot \nabla T_k(u-v)dx \leq \int_\Omega fT_k(u-v)dx \tag{4.5}$$

holds for any $v \in W_0^{1,p}(\Omega) \cap L^\infty(\Omega)$ such that $\|v\|_\infty < m - k$.

Remark 4.1.3. Observe that the use of $T_k(u-v)$ as a test function gives a meaning to all the terms in (C2). Indeed, the second integral in the left-hand side is only on the set $|u-v| \leq k$, then, the integral is well defined, and on this set $|u| \leq k + \|v\|_\infty = M < m$, as a consequence, we get

$$\int_\Omega a(x,u,\nabla u) \cdot \nabla T_k(u-v)dx = \int_\Omega a(x,T_M(u),\nabla T_M(u)) \cdot \nabla T_k(u-v)dx,$$

which is finite by the growth assumptions (4.3) on a.

We investigate in this work to demonstrate the following existence theorem:

Theorem 4.1.4. *Let us assume that the assumptions* (4.1)-(4.4) *hold, and for* $f \in L^1(\Omega)$. *Then there exists at least one entropy solution* (u, w) *of the problem* (U, f) *in the sense of Definition 4.1.2 and* $|\{|u| = m\}| = 0$.

To demonstrate Theorem 4.1.4, we will present and solve approximating problems. In order to achieve this, let f_n be a sequence of $L^\infty-$functions which converges strongly to f in $L^1(\Omega)$ and $|f_n| \leq |f|$. The following Theorem 4.2.1 shows the existence of weak solutions (u_n, w_n) of (U, f_n) for each $n \in \mathbb{N}$.

4.2 Existence of bounded weak solutions for L^∞-data

Theorem 4.2.1. *Let* $f \in L^\infty(\Omega)$ *and let us assume the assumptions* (4.1)-(4.4) *hold. Then there exists at least one weak solution* (u, w) *of the problem* (U, f) *such that* $\|u\|_\infty < m$.

Proof. We will divide our proof of the above Theorem 4.2.1 in five steps and we will use standard techniques.

4.2.1 Approximate solution for L^∞-data

Let $f \in L^\infty(\Omega)$, we consider the approximate problem of (U, f), for $0 < \varepsilon \leq 1$, by

$$(U_\varepsilon, f) \quad \begin{cases} \gamma_\varepsilon(T_{\frac{1}{\varepsilon}}(u_\varepsilon)) - \operatorname{div}(a_\varepsilon(x, u_\varepsilon, \nabla u_\varepsilon)) = f & \text{in } \Omega, \\ \\ u_\varepsilon = 0 & \text{on } \partial\Omega, \end{cases}$$

where $a_\varepsilon(x, u_\varepsilon, \nabla u_\varepsilon) = a(x, T_{m-\varepsilon}(u_\varepsilon), \nabla u_\varepsilon)$ and $\gamma_\varepsilon(\cdot) : \mathbb{R} \to \mathbb{R}$ is the Yosida approximation of $\gamma(\cdot)$.

See that, for any $u \in W_0^{1,p}(\Omega)$ the function $\gamma_\varepsilon(\cdot)$ satisfies the following conditions:

$$\langle \gamma_\varepsilon(u), u \rangle \geq 0, \ \ |\gamma_\varepsilon(u)| \leq \frac{1}{\varepsilon}|u| \ \text{ and } \ \lim_{\varepsilon \to 0} \gamma_\varepsilon(u) = \gamma(u).$$

We recommend that the reader consult [46], for additional information on the maximal monotone mapping.

Since $|\gamma_\varepsilon(T_{\frac{1}{\varepsilon}}(u_\varepsilon))| \leq \frac{1}{\varepsilon^2}$, according to the standard Leray-Lions surjectivity theorem (cf.[82], [86]), there exists at least one solution u_ε of (U_ε, f), i.e: for each $0 < \varepsilon \leq 1$ and $f \in W^{-1,p'}(\Omega)$ there exists at least one solution $u_\varepsilon \in W_0^{1,p}(\Omega)$ such that

$$\int_\Omega \gamma_\varepsilon(T_{\frac{1}{\varepsilon}}(u_\varepsilon))v dx + \int_\Omega a_\varepsilon(x, u_\varepsilon, \nabla u_\varepsilon) \cdot \nabla v dx = \langle f, v \rangle \tag{4.6}$$

holds for all $v \in W_0^{1,p}(\Omega)$, where $\langle ., . \rangle$ denotes the duality pairing between $W_0^{1,p}(\Omega)$ and $W^{-1,p'}(\Omega)$.

4.2.2 A priori estimates

Next, we will provide some estimates on weak solutions of approximate problems. We define

$$L_\varepsilon(s) = \int_0^s l_\varepsilon(r)^{\frac{1}{p-1}} dr$$

where

$$l_\varepsilon(s) = l(T_{m-\varepsilon}(s)) \text{ for all } s \in [0, +\infty).$$

In (4.6), if $G_k(L_\varepsilon(|u_\varepsilon|))sign(u_\varepsilon)$ is used as a test function, we get

$$\begin{aligned}\int_{X_k} \gamma_\varepsilon(T_{\frac{1}{\varepsilon}}(|u_\varepsilon|))G_k(L_\varepsilon(|u_\varepsilon|))dx + \int_{X_k} a_\varepsilon(x, |u_\varepsilon|, \nabla|u_\varepsilon|) \cdot \nabla L_\varepsilon(|u_\varepsilon|)dx \\ = \int_{X_k} f G_k(L_\varepsilon(|u_\varepsilon|))sign(u_\varepsilon)dx,\end{aligned}$$

where $X_k = \{L_\varepsilon(|u_\varepsilon|) > k\}$.

Since $\gamma_\varepsilon(\cdot)$ verifies the sign condition, by (4.2) we get

$$\int_{X_k} |\nabla L_\varepsilon(|u_\varepsilon|)|^p dx \leq \int_{X_k} |f||G_k(L_\varepsilon(|u_\varepsilon|))|dx. \tag{4.7}$$

Inequality (4.7) is exactly the starting point of Stampacchia's L^∞-regularity proof (see [102, Lemma 4.1]), consequently, there exists a constant C_f such that

$$\|L_\varepsilon(|u_\varepsilon|)\|_\infty \leq C_f. \tag{4.8}$$

Indeed, from the Sobolev embedding $W_0^{1,p}(\Omega) \subset L^{p^*}(\Omega)$ and Hölder's inequality, we get

$$\begin{aligned}\left(\int_{X_k} |G_k(L_\varepsilon(|u_\varepsilon|))|^{p^*} dx\right)^{\frac{p}{p^*}} &\leq S^p \int_{X_k} |\nabla L_\varepsilon(|u_\varepsilon|)|^p dx \\ &\leq S^p \int_{X_k} |f||G_k(L_\varepsilon(|u_\varepsilon|))| dx \\ &\leq C \int_{X_k} |G_k(L_\varepsilon(|u_\varepsilon|))| dx \\ &\leq C|X_k|^{1-\frac{1}{p^*}} \left(\int_{X_k} |G_k(L_\varepsilon(|u_\varepsilon|))|^{p^*} dx\right)^{\frac{1}{p^*}}.\end{aligned}$$

Then

$$\left(\int_{X_k} |G_k(L_\varepsilon(|u_\varepsilon|))|^{p^*} dx\right)^{\frac{p-1}{p^*}} \leq C|X_k|^{1-\frac{1}{p^*}}. \tag{4.9}$$

Let $k < k'$, regarding that $X_{k'} \subset X_k$ and

$$|G_k(L_\varepsilon(|u_\varepsilon|))| \geq k' - k$$

on the set $X_{k'}$, we get

$$(k'-k)^{p-1}|X_{k'}|^{\frac{p-1}{p^*}} \leq C|X_k|^{1-\frac{1}{p^*}}. \tag{4.10}$$

Therefore

$$|X_{k'}| \leq \frac{C}{(k'-k)^{p^*}}|X_k|^{\frac{p^*-1}{p-1}}. \tag{4.11}$$

We have $\frac{p^*-1}{p-1} > 1$, consequently that by Satmpacchia's Lemma $L_\varepsilon(|u_\varepsilon|)$ is uniformly bounded.

We can take $C_\infty = L^{-1}(C_f)$ because $L(s) := \int_0^s l(r)^{\frac{1}{p-1}} dr$ is unbounded in $[0, m)$, and then we choose $\varepsilon_0 > 0$ such that $l(s) \leq l(m - \varepsilon_0)$ for any $s \in [0, C_\infty]$.

According to the definitions l_ε and L_ε we have, for any $\varepsilon < \varepsilon_0$,

$$L_\varepsilon(s) = L(s), \quad s \in [0, C_\infty].$$

Moreover, being L_ε increasing, we have, for any $\varepsilon < \varepsilon_0$,

$$L_\varepsilon(s) \leq C_f \Leftrightarrow s \in [0, C_\infty],$$

so by (4.8), we obtain

$$\|u_\varepsilon\|_\infty \leq C_\infty < m. \tag{4.12}$$

Now, by choosing u_ε as a test function in (4.6), we get

$$\int_\Omega \gamma_\varepsilon(T_{\frac{1}{\varepsilon}}(u_\varepsilon))u_\varepsilon dx + \int_\Omega a_\varepsilon(x,u_\varepsilon,\nabla u_\varepsilon)\cdot\nabla u_\varepsilon dx = \int_\Omega f u_\varepsilon dx,$$

since the first term on the left hand side is nonnegative, by (4.2) and since the norm of u_ε in $L^\infty(\Omega)$ is bounded, we get

$$\lambda\int_\Omega |\nabla u_\varepsilon|^p dx \le \int_\Omega |f||u_\varepsilon| dx,$$

where $\lambda = \min\limits_{s\in[0,C_\infty]} l(s)$. In order to obtain that

$$\int_\Omega |\nabla u_\varepsilon|^p dx \le C. \tag{4.13}$$

In the other hand, taking $\frac{1}{\delta}(T_{k+\delta}(\gamma_\varepsilon(T_{\frac{1}{\varepsilon}}(u_\varepsilon))) - T_k(\gamma_\varepsilon(T_{\frac{1}{\varepsilon}}(u_\varepsilon))))$ as a test function in (4.6), passing to the limit with $\delta\to 0$ and choosing $k > \|f\|_\infty$, we get

$$\|\gamma_\varepsilon(T_{\frac{1}{\varepsilon}}(u_\varepsilon))\|_\infty \le \|f\|_\infty. \tag{4.14}$$

Indeed, for $\delta > 0$, we use $\varphi_{\delta,\varepsilon} = \frac{1}{\delta}(T_{k+\delta}(\gamma_\varepsilon(T_{\frac{1}{\varepsilon}}(u_\varepsilon))) - T_k(\gamma_\varepsilon(T_{\frac{1}{\varepsilon}}(u_\varepsilon))))$ as a test function in (4.6), we get

$$\int_\Omega \gamma_\varepsilon(T_{\frac{1}{\varepsilon}}(u_\varepsilon))\varphi_{\delta,\varepsilon} dx + \int_\Omega a_\varepsilon(x,u_\varepsilon,\nabla u_\varepsilon)\cdot\nabla\varphi_{\delta,\varepsilon} dx = \int_\Omega f\varphi_{\delta,\varepsilon} dx.$$

As $|\varphi_{\delta,\varepsilon}| \le 1$, γ_ε is monotone increasing with $\gamma_\varepsilon(0) = 0$ and by (4.2), we get

$$\frac{1}{\delta}\int_{\{k\le|\gamma_\varepsilon(T_{\frac{1}{\varepsilon}}(u_\varepsilon))|\le k+\delta\}} a_\varepsilon(x,u_\varepsilon,\nabla u_\varepsilon)\cdot\nabla T_{\frac{1}{\varepsilon}}(u_\varepsilon)\gamma_\varepsilon'(T_{\frac{1}{\varepsilon}}(u_\varepsilon))dx \ge 0.$$

Then

$$\frac{1}{\delta}\int_{\{|\gamma_\varepsilon(T_{\frac{1}{\varepsilon}}(u_\varepsilon))|\ge k\}} \gamma_\varepsilon(T_{\frac{1}{\varepsilon}}(u_\varepsilon))\left(T_{k+\delta}(\gamma_\varepsilon(T_{\frac{1}{\varepsilon}}(u_\varepsilon))) - T_k(\gamma_\varepsilon(T_{\frac{1}{\varepsilon}}(u_\varepsilon)))\right)dx \le \int_{\{|\gamma_\varepsilon(T_{\frac{1}{\varepsilon}}(u_\varepsilon)|\ge k\}} |f|dx.$$

Taking into account that $\varphi_{\delta,\varepsilon}$ has the same sign as u_ε and the fact that γ_ε is nondecreasing with $\gamma_\varepsilon(0) = 0$, we deduce that

$$\begin{aligned}
&k|\{|\gamma_\varepsilon(T_{\frac{1}{\varepsilon}}(u_\varepsilon))| \ge k+\delta\}| \\
&\le \int_{\{|\gamma_\varepsilon(T_{\frac{1}{\varepsilon}}(u_\varepsilon))|\ge k+\delta\}} |\gamma_\varepsilon(T_{\frac{1}{\varepsilon}}(u_\varepsilon))|dx \\
&\le \frac{1}{\delta}\int_{\{|\gamma_\varepsilon(T_{\frac{1}{\varepsilon}}(u_\varepsilon))|\ge k\}} \gamma_\varepsilon(T_{\frac{1}{\varepsilon}}(u_\varepsilon))\left(T_{k+\delta}(\gamma_\varepsilon(T_{\frac{1}{\varepsilon}}(u_\varepsilon))) - T_k(\gamma_\varepsilon(T_{\frac{1}{\varepsilon}}(u_\varepsilon)))\right)dx \\
&\le \int_{\{|\gamma_\varepsilon(T_{\frac{1}{\varepsilon}}(u_\varepsilon))|\ge k\}} |f|dx \\
&\le ||f||_\infty|\{|\gamma_\varepsilon(T_{\frac{1}{\varepsilon}}(u_\varepsilon))| \ge k\}|.
\end{aligned}$$

which gives, if we pass to the limit with $\delta \to 0$ and choose $k > \|f\|_\infty$,

$$|\{|\gamma_\varepsilon(T_{\frac{1}{\varepsilon}}(u_\varepsilon))| \geq k\}| = 0 \text{ for all } k > \|f\|_\infty.$$

Consequently, we obtain the result (4.14).

4.2.3 Basic convergence results

This step's goal is to present the convergence results that are required for the proof that solutions exist.

Taking into account (4.14), there exists $w \in L^\infty(\Omega)$ such that

$$\gamma_\varepsilon(T_{\frac{1}{\varepsilon}}(u_\varepsilon)) \overset{*}{\rightharpoonup} w \text{ in } L^\infty(\Omega), \tag{4.15}$$

According to (4.13), u_ε remains bounded in $W_0^{1,p}(\Omega)$, we deduce that, there exists a function u and a subsequence, still denoted by u_ε, such that

$$u_\varepsilon \rightharpoonup u \text{ weakly in } W_0^{1,p}(\Omega) \text{ and a.e. in } \Omega. \tag{4.16}$$

We shall demonstrate that

$$\nabla u_\varepsilon \to \nabla u \text{ a.e. in } \{|u| < m\}. \tag{4.17}$$

In order to prove (4.17), we will use the following result (see [37]):

Lemma 4.2.2. *Let (X, T, μ) a measurable space, such that $\mu(X) < \infty$. Let Φ be a measurable function $\Phi : X \to [0, +\infty)$ such that $\mu(\{x \in X : \Phi(x) = 0\}) = 0$. Then for any $\sigma > 0$ there exists $\delta > 0$ such that:*

$$\int_A \Phi d\mu \leq \delta \Rightarrow \mu(A) \leq \sigma.$$

First, by (4.16), we obtain

$$u_\varepsilon \to u \text{ in measure.} \tag{4.18}$$

We will demonstrate that

$$\nabla u_\varepsilon \to \nabla u \text{ in measure on } \{|u| < m\}. \tag{4.19}$$

Let $\lambda > 0$ and $j > 0$. For $0 < k < m$ and $M > 0$, we set

$$
\begin{aligned}
A_1 =& \{|u| < m\} \cap (\{|\nabla u_\varepsilon| > M\} \cup \{|\nabla u| > M\} \cup \{|u_\varepsilon| > k\} \cup \{|u| > k\}),\\
A_2 =& \{|u| < m\} \cap \{|u_\varepsilon - u| > j\},\\
A_3 =& \{|u| < m\} \cap \{|u_\varepsilon - u| \le j, |\nabla u_\varepsilon| \le M, |\nabla u| \le M, |u_\varepsilon| \le k, |u| \le k, |\nabla(u_\varepsilon - u)| \ge \lambda\}.
\end{aligned}
$$

Observe that

$$
\{|u| < m\} \cap \{|\nabla(u_\varepsilon - u)| \ge \lambda\} \subset A_1 \cup A_2 \cup A_3.
$$

Since u_ε and ∇u_ε are bounded in $L^1(\Omega)$, for any $\sigma > 0$ we can fix M and $k < m$ such that $|A_1| < \dfrac{\sigma}{3}$ independently of ε.

By the monotonicity assumption (4.4), there exists a real-valued function Φ such that

$$
|\{\Phi(x) = 0\}| = 0
$$

and

$$
(a(x, s, \xi) - a(x, s, \xi')) \cdot (\xi - \xi') \ge \Phi(x),
$$

for any $s \in (-m, m), \xi, \xi' \in \mathbb{R}^N, |s| \le k$, $|\xi|, |\xi'| \le M$, and $|\xi - \xi'| \ge \lambda$.

Denoting by χ_j the characteristic function of $[0, j]$, we deduce that

$$
\begin{aligned}
&\int_{A_3} \Phi(x)\, dx \le \int_{A_3} [a_\varepsilon(x, u_\varepsilon, \nabla u_\varepsilon) - a_\varepsilon(x, u_\varepsilon, \nabla u)] \cdot \nabla(u_\varepsilon - u)\, dx\\
&\le \int_{\{|u_\varepsilon| \le k, |u| \le k\}} (a_\varepsilon(x, u_\varepsilon, \nabla u_\varepsilon) - a_\varepsilon(x, u_\varepsilon, \nabla T_k(u))) \cdot \nabla(u_\varepsilon - T_k(u)) \chi_j(|u_\varepsilon - T_k(u)|) dx\\
&\le \int_\Omega (a_\varepsilon(x, u_\varepsilon, \nabla u_\varepsilon) - a_\varepsilon(x, u_\varepsilon, \nabla T_k(u))) \cdot \nabla(u_\varepsilon - T_k(u)) \chi_j(|u_\varepsilon - T_k(u)|) dx\\
&\le \int_\Omega a_\varepsilon(x, u_\varepsilon, \nabla u_\varepsilon) \cdot \nabla T_j(u_\varepsilon - T_k(u)) dx - \int_\Omega a_\varepsilon(x, u_\varepsilon, \nabla T_k(u)) \cdot \nabla T_j(u_\varepsilon - T_k(u)) dx\\
&:= I_1 - I_2.
\end{aligned}
$$

For the term I_1, taking $T_j(u_\varepsilon - T_k(u))$ as a test function in (4.6), we have

$$
\begin{aligned}
\int_\Omega \gamma_\varepsilon(T_{\frac{1}{\varepsilon}}(u_\varepsilon)) T_j(u_\varepsilon - T_k(u)) dx + \int_\Omega a_\varepsilon(x, u_\varepsilon, \nabla u_\varepsilon) \cdot \nabla T_j(u_\varepsilon - T_k(u)) dx\\
= \int_\Omega f T_j(u_\varepsilon - T_k(u)) dx.
\end{aligned}
$$

For the first term on the left-hand side, from (4.16), we have $u_\varepsilon \to u$ in $L^1(\Omega)$ and thanks to (4.15), we get

$$\lim_{j\to 0}\lim_{\varepsilon\to 0}\int_\Omega \gamma_\varepsilon(T_{\frac{1}{\varepsilon}}(u_\varepsilon))T_j(u_\varepsilon - T_k(u))dx = 0. \tag{4.20}$$

Similarly, since f belongs to $L^\infty(\Omega)$, we obtain

$$\lim_{j\to 0}\lim_{\varepsilon\to 0}\int_\Omega fT_j(u_\varepsilon - T_k(u))dx = 0. \tag{4.21}$$

Then,

$$\lim_{j\to 0}\lim_{\varepsilon\to 0} I_1 = 0. \tag{4.22}$$

In view to evaluate the other term I_2, we see that

$$\{|u_\varepsilon - T_k(u)| \le j\} \subset \{|u_\varepsilon| \le k + j\},$$

hence, we choose $j > 0$ such that $k + j < m$, there exists $\varepsilon_0 > 0$ such that for all $\varepsilon < \varepsilon_0$,

$$a_\varepsilon(x, u_\varepsilon, \nabla T_k(u)) = a(x, u_\varepsilon, \nabla T_k(u)) \quad \text{in } \{|u_\varepsilon - T_k(u)| \le j\}.$$

Therefore

$$\begin{aligned} I_2 &= \int_\Omega a(x, u_\varepsilon, \nabla T_k(u)) \cdot \nabla T_j(u_\varepsilon - T_k(u))\, dx \\ &= \int_\Omega a(x, T_{k+j}(u_\varepsilon), \nabla T_k(u)) \cdot \nabla T_{k+j}(u_\varepsilon - T_k(u))\chi_j(|u_\varepsilon - T_k(u)|)\, dx. \end{aligned}$$

By (4.16), we get that

$$T_{k+j}(u_\varepsilon) \rightharpoonup T_{k+j}(u) \quad \text{weakly in } W_0^{1,p}(\Omega),$$

on the other hand

$$|a(x, T_{k+j}(u_\varepsilon), \nabla T_k(u))| \le C_0(a_0(x) + b(|T_{k+j}(u_\varepsilon)|)|\nabla T_{k+j}(u)|^{p-1}),$$

using Vitali's Theorem, we get

$$a(x, T_{k+j}(u_\varepsilon), \nabla T_k(u)) \to a(x, T_{k+j}(u), \nabla T_k(u)) \quad \text{strongly in } (L^{p'}(\Omega))^N.$$

Then, pass to the limit with $\varepsilon \to 0$ and next $j \to 0$ in I_2 in order to obtain

$$\lim_{j\to 0}\lim_{\varepsilon\to 0} I_2 = 0. \tag{4.23}$$

From (4.22) and (4.23), it follow that

$$\int_{A_3} \Phi(x)dx \leq \delta.$$

Then, by Lemma 4.2.2, we can conclude that $|E_3| < \frac{\sigma}{3}$ independently of ε.

Lastly, fixing $j < \frac{\sigma}{3}$ and thanks to the fact that $u_\varepsilon \to u$ in measure, we can choose ε' such that $|A_2| < j$ for any $\varepsilon \leq \varepsilon'$.

So, we deduce that (4.19), consequently

$$\nabla u_\varepsilon \to \nabla u \quad \text{a.e. in } \{|u| < m\}.$$

4.2.4 Proof of Theorem 4.2.1

Now, we can show that (u, w) satisfies (4.5). To do this, coming back to (4.6) and let $v \in W_0^{1,p}(\Omega)$. There exists $\varepsilon_1 < \varepsilon_0$ for every $\varepsilon < \varepsilon_1$ such that $m - \varepsilon > C_\infty$ one has

$$\int_\Omega \gamma_\varepsilon(T_{\frac{1}{\varepsilon}}(u_\varepsilon))vdx + \int_\Omega a(x, u_\varepsilon, \nabla u_\varepsilon) \cdot \nabla v dx = \int_\Omega fvdx. \tag{4.24}$$

Thanks to (4.15), we have

$$\int_\Omega \gamma_\varepsilon(T_{\frac{1}{\varepsilon}}(u_\varepsilon))vdx \to \int_\Omega wvdx. \tag{4.25}$$

From (4.3) and (4.12), we obtain that

$$a(x, u_\varepsilon, \nabla u_\varepsilon) \text{ is bounded in } (L^{p'}(\Omega))^N,$$

in virtue of (4.17), which with (4.16) yields

$$a(x, u_\varepsilon, \nabla u_\varepsilon) \rightharpoonup a(x, u, \nabla u) \text{ weakly in } (L^{p'}(\Omega))^N.$$

Thus, by letting ε goes to zero in (4.24), we get that

$$\int_\Omega wvdx + \int_\Omega a(x, u, \nabla u) \cdot \nabla v dx = \int_\Omega fvdx \text{ for any } v \in W_0^{1,p}(\Omega).$$

Moreover,

$$\|u\|_\infty < m.$$

4.2.5 Subdifferential argument

To prove that $u(x) \in D(\gamma(x))$ and $w(x) \in \gamma(u(x))$ for almost all $x \in \Omega$, we use the same arguments as in the subsection 2.2.5.

We therefore complete the proof of Theorem 4.2.1 with this final step. □

4.3 Existence of entropy solutions for L^1-data

4.3.1 Approximation problem for L^1-data and a priori estimates

Let f_n be a sequence of $L^\infty-$functions which converges strongly to f in $L^1(\Omega)$ and $|f_n| \leq |f|$. Through Theorem 4.2.1, there exists a solution $(u_n, w_n) \in W_0^{1,p}(\Omega) \cap L^\infty(\Omega) \times L^\infty(\Omega)$ of (U, f_n) such that

$$\|u_n\|_\infty < m \tag{4.26}$$

and

$$\int_\Omega w_n v dx + \int_\Omega a(x, u_n, \nabla u_n) \cdot \nabla v dx = \int_\Omega f_n v dx \tag{4.27}$$

holds for any $n \in \mathbb{N}$ and $v \in W_0^{1,p}(\Omega)$.

First we show that w_n is bounded in $L^1(\Omega)$. In view of $w_n \in \gamma(u_n)$, we deduce $\chi_{\{|u_n|<m\}} w_n$ is uniformly bounded, then we obtain

$$\|w_n\|_{L^1(\Omega)} \leq C_1. \tag{4.28}$$

Next, taking u_n as a test function in (4.27), as w_n has the same sign as u_n, from (4.2) and (4.26), we get

$$\lambda \int_\Omega |\nabla u_n|^p dx \leq m \|f\|_{L^1(\Omega)},$$

where $\lambda = \min_{s \in [0,m)} l(s)$. We conclude

$$\|u_n\|_{W_0^{1,p}(\Omega)} \leq C_2. \tag{4.29}$$

4.3.2 Basic convergence results

We start to prove that w_n is weakly compact in $L^1(\Omega)$. Since $\chi_{\{|u_n|<m\}}w_n$ is uniformly bounded, hence it is uniformly integrable. There exists $\delta > 0$ such that for every $E \subset \Omega$ with $|E| < \delta$ we have

$$\int_E \chi_{\{|u_n|<m\}}|w_n|dx < \varepsilon.$$

We estimate

$$\begin{aligned}\int_E |w_n|dx &= \int_{E\cap\{|u_n|<m\}} |w_n|dx + \int_{E\cap\{|u_n|\geq m\}} |w_n|dx \\ &\leq \int_E \chi_{\{|u_n|<m\}}|w_n|dx \\ &< \varepsilon.\end{aligned}$$

According to Dunford-Pettis Theorem, we get

$$w_n \rightharpoonup w \text{ weakly in } L^1(\Omega). \tag{4.30}$$

Now, thanks to (4.29), u_n remains bounded in $W_0^{1,p}(\Omega)$, we can extract a subsequence, still denoted by u_n, such that

$$u_n \rightharpoonup u \text{ weakly in } W_0^{1,p}(\Omega) \text{ and a.e. in } \Omega. \tag{4.31}$$

By using the same arguments as in the proof of (4.17), also we obtain

$$\nabla u_n \to \nabla u \text{ a.e. in } \{|u| < m\}. \tag{4.32}$$

Next, since $w_n \in \gamma(u_n)$ almost everywhere in Ω, we conclude by a subdifferential argument that $w \in \gamma(u)$ almost everywhere in Ω.

4.3.3 Conclusion of the proof of Theorem 4.1.4

From (4.31), $u_n \to u$ a.e. in Ω, we have that

$$L(|u_n|) \to L(|u|) \text{ a.e. in } \Omega.$$

By using $L(|u_n|))sign(u_n)$ as a test function in (4.27), as b_n has the same sign as u_n and from (4.2), we get

$$\int_\Omega |\nabla L(|u_n|)|^p dx \leq \int_\Omega |f||L(|u_n|)|dx. \tag{4.33}$$

Then, $L(|u_n|)$ is bounded in $W_0^{1,p}(\Omega)$, so that

$$L(|u_n|) \to L(|u|) \text{ weakly in } W_0^{1,p}(\Omega).$$

As a result $L(|u|)$ is bounded in $L^1(\Omega)$ and

$$|\{|u| = m\}| = 0.$$

Now, let $0 < k < m$ be fixed, $v \in W_0^{1,p}(\Omega) \cap L^\infty(\Omega)$ such that $\|v\|_\infty < m - k$ and choose $T_k(u_n - v)$ as a test function in (4.27), we have

$$\int_\Omega w_n T_k(u_n - v)dx + \int_\Omega a(x, u_n, \nabla u_n) \cdot \nabla T_k(u_n - v)dx = \int_\Omega f_n T_k(u_n - v)dx. \quad (4.34)$$

Since f_n is strongly convergent in $L^1(\Omega)$ to f and $T_k(u_n - v)$ converges weakly$*$ in $L^\infty(\Omega)$ to $T_k(u - v)$, we obtain

$$\lim_{n\to\infty} \int_\Omega f_n T_k(u_n - v)dx = \int_\Omega f T_k(u - v)dx.$$

On the other hand, since w_n is weakly covergent in $L^1(\Omega)$ to b, $T_k(u_n - v)$ converges a.e. to $T_k(u - v)$ and $\|T_k(u_n - v)\|_\infty \leq k$, then

$$\lim_{n\to\infty} \int_\Omega w_n T_k(u_n - v)dx = \int_\Omega w T_k(u - v)dx.$$

At last, we concentrate on the second term of (4.34), we divide it as the sum

$$\int_{\{|u_n - v| \leq k\}} a(x, u_n, \nabla u_n) \cdot \nabla u_n dx - \int_{\{|u_n - v| \leq k\}} a(x, u_n, \nabla u_n) \cdot \nabla v dx.$$

By regarding that the second integral is on a subset of the set where $|u_n| \leq k + \|v\|_{L^\infty(\Omega)} = M < m$, then, it can be written as

$$\int_{\{|u - v| \leq k\}} a(x, T_M(u_n), \nabla T_M(u_n)) \cdot \nabla v dx.$$

Since $a(x, T_M(u_n), \nabla T_M(u_n))$ is bounded in $(L^{p'}(\Omega))^N$. Thus, by (4.31) and (4.32), we obtain

$$a(x, T_M(u_n), \nabla T_M(u_n)) \rightharpoonup a(x, T_M(u), \nabla T_M(u)) \text{ weakly in } (L^{p'}(\Omega))^N.$$

Therefore, the second integral converges, as n tends to infinity, to

$$\int_{\{|u - v| \leq k\}} a(x, T_M(u), \nabla T_M(u)) \cdot \nabla v dx = \int_{\{|u - v| \leq k\}} a(x, u, \nabla u) \cdot \nabla v dx.$$

From (4.2), the integrand function of the first integral is nonnegative, using Fatou's Lemma, (4.31) and (4.32), we obtain that

$$\int_{\{|u-v|\leq k\}} a(x,u,\nabla u)\cdot\nabla u dx \leq \liminf_{n\to\infty}\int_{\{|u_n-v|\leq k\}} a(x,u_n,\nabla u_n)\cdot\nabla u_n dx.$$

Putting all the terms together, we get that

$$\int_\Omega wT_k(u-v)dx + \int_{\{|u-v|\leq k\}} a(x,u,\nabla u)\cdot\nabla(u-v)dx \leq \int_\Omega fT_k(u-v)dx,$$

which leads to (4.5). As a consequence, the proof of the existence of entropy solutions (u,w) to the problem (U,f) is then complete.

4.4 Example

Let Ω be a bounded domain of $\mathbb{R}^N (N\geq 2)$ with sufficiently smooth boundary $\partial\Omega$ and let $\mathcal{A}(u) = -\text{div}(a(x,u,\nabla u))$ where $a(x,s,\xi) : \Omega\times(-1,1)\times\mathbb{R}^N \to \mathbb{R}^N$ is the Carathéodory function defined by

$$a(x,s,\xi) = l(s)|\xi|^{p-2}\xi,$$

such that $1<p<N$ and $l(s) = \dfrac{1}{(1-|s|)^{p-1}}$. So it is easy to show that (4.1)-(4.4) hold true.

Let γ be the maximal monotone graph given by

$$\gamma(r) = \begin{cases} \emptyset & \text{if } r<0, \\ (-\infty,0] & \text{if } r=0, \\ 0 & \text{if } r>0. \end{cases}$$

Finally, by Theorem 4.1.4, for all $f\in L^1(\Omega)$ there exists an entropy solution based on the Definition 4.1.2 to the following problem

$$\begin{cases} \mathcal{A}(u) = f & \text{in } \{u>0\}, \\ \mathcal{A}(u) \geq f & \text{in } \{u=0\}, \\ u=0 & \text{on } \partial\Omega. \end{cases}$$

Chapter 5

Solvability of nonlinear multivalued elliptic problems involving degenerate coercivity with variable exponents

The results of this chapter, which have been published in [5], generalize the corresponding ones given in chapter 3 to the functional framework involves Sobolev spaces with variable exponent.

5.1 Essential assumptions and statements of result

In this chapter, we deal with the following nonlinear multivalued elliptic problem with Dirichlet boundary conditions:

$$(P^*, f) \quad \begin{cases} \gamma(u) - \operatorname{div}\left(\dfrac{a(x, \nabla u)}{(1+|u|)^{\theta(p(x)-1)}}\right) \ni f & \text{in } \Omega, \\ \\ u = 0 & \text{on } \partial\Omega, \end{cases}$$

where Ω is a bounded domain of $\mathbb{R}^N$, $N \geq 2$, with sufficiently smooth boundary denoted by $\partial\Omega$, $p \in C_+(\overline{\Omega})$ and $a(x, \xi) : \Omega \times \mathbb{R}^N \to \mathbb{R}^N$ is a Carathéodory function (i.e. $a(x, \xi)$ is continuous in ξ for a.e. $x \in \Omega$ and measurable in x for every $\xi \in \mathbb{R}^N$) such that

Assumption (H_1)

$$a(x, \xi) \cdot \xi \geq \alpha |\xi|^{p(x)}, \tag{5.1}$$

$$|a(x, \xi)| \leq C_0 \left(a_0(x) + |\xi|^{p(x)-1}\right), \tag{5.2}$$

$$(a(x, \xi) - a(x, \xi')) \cdot (\xi - \xi') > 0, \tag{5.3}$$

for almost every $x \in \Omega$ and all $\xi, \xi' \in \mathbb{R}^N$ with $\xi \neq \xi'$, where $\alpha > 0$, $C_0 > 0$ and a_0 is a nonnegative function in $L^{p'(x)}(\Omega)$.

Assumption (H_2)

The nonlinearity $\gamma : \mathbb{R} \to 2^{\mathbb{R}}$ is a set valued, maximal monotone graph such that $0 \in \gamma(0)$.

Assumption (H_3)

Throughout this chapter, if not otherwise specified, we always assume that

$$2 - \frac{1}{N} < p^- \leq p(x) \leq p^+ < N \text{ and } 0 \leq \theta(p^+ - 1) < \frac{N(p^- - 1)}{N - 1} - 1. \tag{5.4}$$

Let us now turn to our main result. In the same spirit of [6], we introduce the following classes of solutions for the problem (P^*, f).

Definition 5.1.1. Let $f \in L^\infty(\Omega)$. A weak solution of the problem (P^*, f) is a couple of functions $(u, w) \in W_0^{1,p}(\Omega) \times L^\infty(\Omega)$ such that $w \in \gamma(u)$ a.e. in Ω and

$$\int_\Omega wv dx + \int_\Omega \frac{a(x, \nabla u)}{(1 + |u|)^{\theta(p(x)-1)}} \cdot \nabla v dx = \int_\Omega fv dx \text{ holds for all } v \in W_0^{1,p(x)}(\Omega). \tag{5.5}$$

In this chapter, our assumption on the data $f \in L^1(\Omega)$ does not assure the existence of a weak solution. This forces us to use the notion of an entropy solution, which will be presented in the definition that follows:

Definition 5.1.2. Let $f \in L^1(\Omega)$. A couple of measurable functions (u, w) is called entropy solution of (P^*, f) if this couple satisfies the two conditions:

(C1) $u : \Omega \to \mathbb{R}$ is measurable and $w \in L^1(\Omega)$ such that $u(x) \in D(\gamma(x))$ and $w(x) \in \gamma(u(x))$ for a.e. x in Ω.

(C2) For each $k > 0$, $T_k(u) \in W_0^{1,p(x)}(\Omega)$ and

$$\int_\Omega wT_k(u - v) dx + \int_\Omega \frac{a(x, \nabla u)}{(1 + |u|)^{\theta(p(x)-1)}} \cdot \nabla T_k(u - v) dx \leq \int_\Omega fT_k(u - v) dx \tag{5.6}$$

holds for all $v \in W_0^{1,p(x)}(\Omega) \cap L^\infty(\Omega)$.

Remark 5.1.3. We recall that the gradient of u which appears in (5.6) is defined as in Lemma 2.1 in [25] to be the unique measurable function $\phi : \Omega \to \mathbb{R}^N$ such that

$$\nabla T_k(u) = \phi \chi_{\{|u| < k\}}$$

for a.e. $x \in \Omega$ and for any $k > 0$.

Furthermore, if $u \in W_0^{1,1}(\Omega)$, then $\phi \equiv \nabla u$ in the usual weak sense.

Remark 5.1.4. Observe that the use of $T_k(u-v)$ as a test function gives meaning to all the terms in **(C2)**. Indeed, the second integral in the left-hand side is only on the set $\{|u-v| \leq k\}$, then, the integral is well defined, and on this set, we have $|u| \leq k + \|v\|_\infty = M$, as a consequence, we get

$$\int_\Omega \frac{a(x,\nabla u)}{(1+|u|)^{\theta(p(x)-1)}} \cdot \nabla T_k(u-v)dx = \int_\Omega \frac{a(x,\nabla T_M(u))}{(1+|T_M(u)|)^{\theta(p(x)-1)}} \cdot \nabla T_k(u-v)dx,$$

which is finite by the growth condition (5.2) on a.

The aim of this paper is to prove the following result stated bellow:

Theorem 5.1.5. *Assume that (H_1)-(H_3) hold true and $f \in L^1(\Omega)$, there exists at least one entropy solution (u,w) of the problem (P^*,f) in the sense of Definition 5.1.2. Furthermore, $u \in W_0^{1,q(x)}(\Omega)$, with $q(x) \in \left(1, \frac{(p(x)-1-\theta(p^+-1))N}{N-1-\theta(p^+-1)}\right)$.*

5.2 Existence of bounded weak solutions for L^∞-data

We shall introduce and resolve the approximate problems listed in the subsection 5.2.1 before establishing the aforementioned Theorem 5.1.5.

For any $f \in L^1(\Omega)$ and $m,n \in \mathbb{N}$, let $f_{m,n} : \Omega \to \mathbb{R}$ be the function defined by $f_{m,n}(x) = max(-n, min(f(x),m))$ for almost every $x \in \Omega$. Evidently, one has $f_{m,n} \in L^\infty(\Omega)$ for every $m,n \in \mathbb{N}$ and $|f_{m,n}(x)| \leq |f(x)|$ a.e. in Ω, as a result $\lim\limits_{m\to\infty}\lim\limits_{n\to\infty} f_{m,n} = f$ in $L^1(\Omega)$ and a.e. in Ω.

The following theorem will give us an existence of weak solutions $(u_{m,n}, w_{m,n})$ of $(P^*, f_{m,n})$ for every fixed $m,n \in \mathbb{N}$:

Theorem 5.2.1. *Assume that (H_1)-(H_3) hold true and $f \in L^\infty(\Omega)$, there exists at least one weak solution (u,w) of the problem (P^*,f).*

Remark 5.2.2. Observe that the assumption on f given in the above theorem yields $L^\infty(\Omega)$ solutions for nonlinear multivalued elliptic problems involving degenerate coercivity with variable exponents. The result (which is not depending on θ) is not surprising, since if one looks for bounded solutions then the lack of coercivity of the operator $A(u) := -\text{div}\left(\frac{a(x,\nabla u)}{(1+|u|)^{\theta(p(x)-1)}}\right)$ (which is created by unbounded functions) "disappears".

We will divide our proof of the above Theorem 5.2.1 in several steps and we will use standard techniques.

5.2.1 Approximate problems for L^∞-data

Let $f \in L^\infty(\Omega)$, we consider the approximate problem of (P^*, f), for $0 < \varepsilon \leq 1$, by

$$(P_\varepsilon^*, f) \quad \begin{cases} \gamma_\varepsilon(T_{\frac{1}{\varepsilon}}(u_\varepsilon)) + \varepsilon arctan(u_\varepsilon) + A_\varepsilon(u_\varepsilon) = f & \text{in } \Omega, \\ \\ u_\varepsilon = 0 & \text{on } \partial\Omega, \end{cases}$$

where $A_\varepsilon(u) = -\text{div}\left(\dfrac{a(x,\nabla u)}{(1+|T_{\frac{1}{\varepsilon}}(u)|)^{\theta(p(x)-1)}}\right)$ and $\gamma_\varepsilon(\cdot) : \mathbb{R} \to \mathbb{R}$ is the Yosida regularization of $\gamma(\cdot)$. Observe that, for every $u \in W_0^{1,p(x)}(\Omega)$, we obviously have

$$|\gamma_\varepsilon(u)| \leq \frac{1}{\varepsilon}|u|, \ \langle \gamma_\varepsilon(u), u\rangle \geq 0 \text{ and } \lim_{\varepsilon\to 0}\gamma_\varepsilon(u) = \gamma(u).$$

For a deeper treatment about the maximal monotone graph, we recommend that reader consult [46].

We introduce the operator $B_\varepsilon : W_0^{1,p(x)}(\Omega) \to W^{-1,p'(x)}(\Omega)$, defined by

$$\langle B_\varepsilon u, v\rangle = \int_\Omega \left(\gamma_\varepsilon(T_{\frac{1}{\varepsilon}}(u)) + \varepsilon arctan(u) + A_\varepsilon(u)\right) v dx, \text{ for any } u, v \in W_0^{1,p(x)}(\Omega).$$

Lemma 5.2.3. *The operator $B_\varepsilon : W_0^{1,p(x)}(\Omega) \to W^{-1,p'(x)}(\Omega)$ is bounded, coercive and pseudo-monotone.*

Proof. Due to (5.2) and (5.3), B_ε is monotone and well-defined. As *arctan* and $\gamma_\varepsilon \circ T_{\frac{1}{\varepsilon}}$ are continuous and bounded, again by the growth assumption (5.2), we conclude that B_ε is pseudo-monotone (see [86], p.157).

Next, for all $u \in W_0^{1,p(x)}(\Omega)$, one has

$$\langle B_\varepsilon u, u\rangle = \int_\Omega \frac{a(x,\nabla u)\cdot\nabla u}{(1+|T_{\frac{1}{\varepsilon}}(u_\varepsilon)|)^{\theta(p(x)-1)}} dx + \int_\Omega \left(\gamma_\varepsilon(T_{\frac{1}{\varepsilon}}(u)) + \varepsilon arctan(u)\right) u dx,$$

since the second term on the right hand side is nonnegative, from (5.1), we get

$$\begin{aligned} \langle B_\varepsilon u, u\rangle &\geq \frac{\alpha}{(1+\frac{1}{\varepsilon})^{\theta(p^+-1)}} \int_\Omega |\nabla u|^{p(x)} dx \\ &\geq \alpha' \|\nabla u\|_{p(x)}^\rho, \end{aligned}$$

with

$$\rho = \begin{cases} p^- & \text{if } \|\nabla u\|_{p(x)} \geq 1, \\ p^+ & \text{if } \|\nabla u\|_{p(x)} < 1. \end{cases}$$

Hence, B_ε is coercive on $W_0^{1,p(x)}(\Omega)$. □

In light of Lemma 5.2.3, for evey fixed $\varepsilon > 0$, the problem (S_ε, f) has at least one solution u_ε (see [82], [86]). In other word, for any $0 < \varepsilon \leq 1$ and $f \in W^{-1,p'(x)}(\Omega)$ there exists at least one solution $u_\varepsilon \in W_0^{1,p(x)}(\Omega)$ such that

$$\int_\Omega \left(\gamma_\varepsilon(T_{\frac{1}{\varepsilon}}(u_\varepsilon)) + \varepsilon arctan(u_\varepsilon)\right) v dx + \int_\Omega \frac{a(x, \nabla u_\varepsilon) \cdot \nabla v}{(1 + |T_{\frac{1}{\varepsilon}}(u_\varepsilon)|)^{\theta(p(x)-1)}} dx = \langle f, v \rangle \tag{5.7}$$

holds for all $v \in W_0^{1,p(x)}(\Omega)$, where $\langle ., . \rangle$ denotes the duality pairing between $W_0^{1,p(x)}(\Omega)$ and $W^{-1,p'(x)}(\Omega)$.

The following proposition gives the uniqueness of solutions u_ε of (P_ε^*, f) with the right-hand side $f \in L^\infty(\Omega)$ through a comparison principle that will play an important role in the approximation of solutions to (P^*, f) with $f \in L^1(\Omega)$.

Proposition 5.2.4. *For $0 < \varepsilon \leq 1$ fixed and $f, \tilde{f} \in L^\infty(\Omega)$, let $u_\varepsilon, \tilde{u}_\varepsilon \in W_0^{1,p(x)}(\Omega)$ be solutions of (P_ε^*, f) and $(P_\varepsilon^*, \tilde{f})$, respectively. Then, the following comparison principle holds*

$$\varepsilon \int_\Omega (arctan(u_\varepsilon) - arctan(\tilde{u}_\varepsilon))^+ dx \leq \int_\Omega (f - \tilde{f}) sign_0^+(u_\varepsilon - \tilde{u}_\varepsilon) dx. \tag{5.8}$$

Proof. By taking $S_\delta^+(u_\varepsilon - \tilde{u}_\varepsilon)$ as a test function in (5.7) for u_ε and $\tilde{u}_\varepsilon$. Subtracting the resulting inequalities, we get

$$I_{\varepsilon,\delta}^1 + I_{\varepsilon,\delta}^2 + I_{\varepsilon,\delta}^3 = I_{\varepsilon,\delta}^4$$

where

$$I_{\varepsilon,\delta}^1 = \int_\Omega \left(\gamma_\varepsilon(T_{\frac{1}{\varepsilon}}(u_\varepsilon)) - \gamma_\varepsilon(T_{\frac{1}{\varepsilon}}(\tilde{u}_\varepsilon))\right) S_\delta^+(u_\varepsilon - \tilde{u}_\varepsilon) dx,$$

$$I_{\varepsilon,\delta}^2 = \varepsilon \int_\Omega (arctan(u_\varepsilon) - arctan(\tilde{u}_\varepsilon)) S_\delta^+(u_\varepsilon - \tilde{u}_\varepsilon) dx,$$

$$I_{\varepsilon,\delta}^3 = \int_\Omega \left(\frac{a(x, \nabla u_\varepsilon)}{(1 + |T_{\frac{1}{\varepsilon}}(u_\varepsilon)|)^{\theta(p(x)-1)}} - \frac{a(x, \nabla \tilde{u}_\varepsilon)}{(1 + |T_{\frac{1}{\varepsilon}}(\tilde{u}_\varepsilon)|)^{\theta(p(x)-1)}}\right) \cdot \nabla S_\delta^+(u_\varepsilon - \tilde{u}_\varepsilon) dx,$$

$$I_{\varepsilon,\delta}^4 = \int_\Omega (f - \tilde{f}) S_\delta^+(u_\varepsilon - \tilde{u}_\varepsilon) dx.$$

It is easy to see that, for $\tau \geq 0$ and $b \geq a \geq 0$, the following inequality holds

$$\left|\frac{1}{(1+a)^\tau} - \frac{1}{(1+b)^\tau}\right| = \left|\frac{\tau(a-b)}{(1+c)^{\tau+1}}\right| \leq \tau|a-b| \text{ for some } c \in [a, b]. \tag{5.9}$$

By using (5.2) and (5.9), we have

$$
\begin{aligned}
I^3_{\varepsilon,\delta} &\geq \frac{1}{\delta}\int_{\{(u_\varepsilon-\tilde{u}_\varepsilon)^+<\delta\}}\left(\frac{a(x,\nabla\tilde{u}_\varepsilon)}{(1+|T_{\frac{1}{\varepsilon}}(u_\varepsilon)|)^{\theta(p(x)-1)}}-\frac{a(x,\nabla\tilde{u}_\varepsilon)}{(1+|T_{\frac{1}{\varepsilon}}(\tilde{u}_\varepsilon)|)^{\theta(p(x)-1)}}\right)\cdot\nabla(u_\varepsilon-\tilde{u}_\varepsilon)^+dx\\
&\geq \frac{-C\delta}{\delta}\int_{\{(u_\varepsilon-\tilde{u}_\varepsilon)^+<\delta\}}(a_0(x)+|\nabla\tilde{u}_\varepsilon|^{p^+-1})\cdot|\nabla(u_\varepsilon-\tilde{u}_\varepsilon)^+|dx\\
&\geq -C\int_\Omega\left(a_0(x)+|\nabla\tilde{u}_\varepsilon|^{p^+-1}\right)\cdot|\nabla(u_\varepsilon-\tilde{u}_\varepsilon)^+|\chi_{\{(u_\varepsilon-\tilde{u}_\varepsilon)^+<\delta\}}dx
\end{aligned}
$$

then $\liminf_{\delta\to 0} I^3_{\varepsilon,\delta}\geq 0$.

By letting $\delta\downarrow 0$, we deduce (5.8). □

Remark 5.2.5. Let $u_\varepsilon,\tilde{u}_\varepsilon\in W^{1,p(x)}_0(\Omega)$ be solutions of (P^*_ε,f) and $(P^*_\varepsilon,\tilde{f})$, respectively, where $f,\tilde{f}\in L^\infty(\Omega)$ be such that $f\leq\tilde{f}$ a.e. in Ω and $\varepsilon>0$. So, it is a direct result of Proposition 5.2.4 that $u_\varepsilon\leq\tilde{u}_\varepsilon$ a.e. in Ω. Moreover, by the monotonicity of $\gamma_\varepsilon\circ T_{\frac{1}{\varepsilon}}$ it follows that also $\gamma_\varepsilon(T_{\frac{1}{\varepsilon}}(u_\varepsilon))\leq\gamma_\varepsilon(T_{\frac{1}{\varepsilon}}(\tilde{u}_\varepsilon))$ a.e. in Ω.

Next, we need a priori estimates on the weak solution u_ε of approximate problems (P^*_ε,f) and show that the solution has the regularity needed to give sense to the weak formulation and that it is possible to pass to the limit in the part with the differential operator.

5.2.2 A priori estimates

If we choose the test function $G_k(H(u_\varepsilon))$ in the weak formulation (5.7), where

$$H(s)=\int_0^s\frac{dt}{(1+t)^\theta},$$

we obtain

$$
\begin{aligned}
\int_{X_k}\left(\gamma_\varepsilon(T_{\frac{1}{\varepsilon}}(u_\varepsilon))+\varepsilon arctan(u_\varepsilon)\right)G_k(H(u_\varepsilon))dx+\int_{X_k}\frac{a(x,\nabla u_\varepsilon)\cdot\nabla u_\varepsilon}{(1+|T_{\frac{1}{\varepsilon}}(u_\varepsilon)|)^{\theta p(x)}}dx\\
=\int_{X_k}fG_k(H(u_\varepsilon))dx,
\end{aligned}
$$

such that $X_k=\{|H(u_\varepsilon)|>k\}$.

By using (5.1) and the fact that γ_ε is nondecreasing with $\gamma_\varepsilon(0)=0$, it follows that

$$\alpha \int_\Omega |\nabla G_k(H(u_\varepsilon))|^{p(x)} dx \leq \int_\Omega |f||G_k(H(u_\varepsilon))| dx. \tag{5.10}$$

In view of the Sobolev inequality, we have

$$\int_\Omega |\nabla G_k(H(u_\varepsilon))|^{p(x)} dx \geq C_1 \left(\int_\Omega |G_k(H(u_\varepsilon))|^{p^*(x)} dx \right)^{\frac{\rho_2}{\rho_1}}, \tag{5.11}$$

where

$$\rho_1 = \begin{cases} p^{*+} & \text{if } \|G_k(H(u_\varepsilon))\|_{p^*(x)} \geq 1, \\ p^{*-} & \text{if } \|G_k(H(u_\varepsilon))\|_{p^*(x)} < 1. \end{cases} ; \rho_2 = \begin{cases} p^+ & \text{if } \|\nabla G_k(H(u_\varepsilon))\|_{p(x)} < 1, \\ p^- & \text{if } \|\nabla G_k(H(u_\varepsilon))\|_{p(x)} \geq 1. \end{cases}$$

By (5.10) and (5.11), we obtain

$$\begin{aligned} \left(\int_\Omega |G_k(H(u_\varepsilon))|^{p^*(x)} dx \right)^{\frac{\rho_2}{\rho_1}} &\leq C_2 \|\chi_{X_k}\|_{p^{*\prime}(x)} \|G_k(H(u_\varepsilon))\|_{p^*((x)} \\ &\leq C \|\chi_{X_k}\|_{p^{*\prime}(x)} \left(\int_\Omega |G_k(H(u_\varepsilon))|^{p^*(x)} dx \right)^{\frac{1}{\rho_1}}. \end{aligned}$$

Thus

$$\left(\int_\Omega |G_k(H(u_\varepsilon))|^{p^*(x)} dx \right)^{\frac{\rho_2 - 1}{\rho_1}} \leq C_3 |X_k|^{\frac{1}{\rho_3}}$$

where

$$\rho_3 = \begin{cases} p^{*\prime +} & \text{if } \|\chi_{X_k}\|_{p^{*\prime}(x)} < 1, \\ p^{*\prime -} & \text{if } \|\chi_{X_k}\|_{p^{*\prime}(x)} \geq 1. \end{cases}$$

Let $k' > k$, it is clear that $X_{k'} \subset X_k$ and

$$|G_k(H(u_\varepsilon))| \geq k' - k$$

on the set $X_{k'}$. Hence

$$|X_{k'}| \leq \frac{C}{(k'-k)^{p^{*-}}} |X_k|^{\frac{\rho_1}{(\rho_2 - 1)\rho_3}}.$$

We have $\frac{\rho_1}{(\rho_2 - 1)\rho_3} > 1$, as a result, by Stampacchia's Lemma, $H(u_\varepsilon)$ is uniformly bounded (see [102, Lemma 4.1]).

The properties of the function H (especially the fact that $\lim_{s \to \infty} H(s) = \infty$) give a bound for u_ε in $L^\infty(\Omega)$, that is

$$||u_\varepsilon||_{L^\infty(\Omega)} \leq C_\infty, \tag{5.12}$$

with C_∞ is a constant that don't depend on ε.

Next, by choosing u_ε as a test function in (5.7), we get

$$\int_\Omega \left(\gamma_\varepsilon(T_{\frac{1}{\varepsilon}}(u_\varepsilon)) + \varepsilon arctan(u_\varepsilon)\right) u_\varepsilon dx + \int_\Omega \frac{a(x,\nabla u_\varepsilon)\cdot\nabla u_\varepsilon}{(1+|T_{\frac{1}{\varepsilon}}(u_\varepsilon)|)^{\theta(p(x)-1)}} dx = \int_\Omega f u_\varepsilon dx,$$

since the first term on the left hand side is non-negative, from (5.1), we obtain

$$\alpha \int_\Omega \frac{|\nabla u_\varepsilon|^{p(x)}}{(1+|u_\varepsilon|)^{\theta(p^+-1)}} dx \leq \int_\Omega |f||u_\varepsilon| dx.$$

As the norm of u_ε in $L^\infty(\Omega)$ is bounded, we conclude that

$$\|\nabla u\|^{\rho_4}_{p(x)} \leq \int_\Omega |\nabla u_\varepsilon|^{p(x)} dx \leq C \tag{5.13}$$

where

$$\rho_4 = \begin{cases} p^- & \text{if } \|\nabla u\|_{p(x)} \geq 1, \\ p^+ & \text{if } \|\nabla u\|_{p(x)} < 1. \end{cases}$$

Now, arguing as in the same way as in [6], i.e. taking $\frac{1}{\delta}(T_{k+\delta}(\gamma_\varepsilon(T_{\frac{1}{\varepsilon}}(u_\varepsilon))) - T_k(\gamma_\varepsilon(T_{\frac{1}{\varepsilon}}(u_\varepsilon))))$ as a test function in (5.7), passing to the limit with $\delta \to 0$ and choosing $k > \|f\|_\infty$, we obtain

$$\|\gamma_\varepsilon(T_{\frac{1}{\varepsilon}}(u_\varepsilon))\|_\infty \leq \|f\|_\infty. \tag{5.14}$$

5.2.3 Basic convergence results

This step's goal is to present the convergence results that are required for the proof that solutions exist.

Taking into account (5.14), there exists $w \in L^\infty(\Omega)$ such that

$$\gamma_\varepsilon(T_{\frac{1}{\varepsilon}}(u_\varepsilon)) \overset{*}{\rightharpoonup} w \text{ in } L^\infty(\Omega). \tag{5.15}$$

According to (5.13), u_ε remains bounded in $W_0^{1,p(x)}(\Omega)$, we deduce that, there exist a subsequence (not relabeled) and a function u such that

$$u_\varepsilon \rightharpoonup u \text{ weakly in } W_0^{1,p(x)}(\Omega) \text{ and a.e. in } \Omega. \tag{5.16}$$

We will demonstrate that

$$u_\varepsilon \to u \text{ strongly in } W_0^{1,p(x)}(\Omega). \tag{5.17}$$

By choosing $u_\varepsilon - u$ as test function in (5.7), we get that, for $\frac{1}{\varepsilon} > C_\infty$,

$$\int_\Omega \left(\gamma_\varepsilon(T_{\frac{1}{\varepsilon}}(u_\varepsilon)) + \varepsilon arctan(u_\varepsilon)\right)(u_\varepsilon - u)dx + \int_\Omega \frac{a(x,\nabla u_\varepsilon)}{(1+|u_\varepsilon|)^{\theta(p(x)-1)}} \cdot (\nabla u_\varepsilon - \nabla u)dx = \int_\Omega f(u_\varepsilon - u)dx. \tag{5.18}$$

Concerning the first term on the left-hand side of (5.18). In virtue of (5.16), we get $u_\varepsilon \to u$ in $L^1(\Omega)$ and by (5.15), we obtain

$$\lim_{\varepsilon\to 0}\int_\Omega \left(\gamma_\varepsilon(T_{\frac{1}{\varepsilon}}(u_\varepsilon)) + \varepsilon arctan(u_\varepsilon)\right)(u_\varepsilon - u)dx = 0. \tag{5.19}$$

Likewise, since f belongs to $L^\infty(\Omega)$, we have

$$\lim_{\varepsilon\to 0}\int_\Omega f(u_\varepsilon - u)dx = 0. \tag{5.20}$$

On the other hand, by (5.19) and (5.20) one has

$$\lim_{\varepsilon\to 0}\int_\Omega \frac{a(x,\nabla u_\varepsilon)}{(1+|u_\varepsilon|)^{\theta(p(x)-1)}} \cdot (\nabla u_\varepsilon - \nabla u)dx = 0.$$

By using (5.2) and Vitali's Theorem, we get that

$$\frac{a(x,\nabla u)}{(1+|u_\varepsilon|)^{\theta(p(x)-1)}} \to \frac{a(x,\nabla u)}{(1+|u|)^{\theta(p(x)-1)}} \text{ strongly in } (L^{p'(x)}(\Omega))^N.$$

We deduce, by (5.16), that

$$\lim_{\varepsilon\to 0}\int_\Omega \frac{a(x,\nabla u)}{(1+|u_\varepsilon|)^{\theta(p(x)-1)}} \cdot (\nabla u_\varepsilon - \nabla u)dx = 0.$$

Then, we obtain

$$\lim_{\varepsilon\to 0}\int_\Omega \left(\frac{a(x,\nabla u_\varepsilon)}{(1+|u_\varepsilon|)^{\theta(p(x)-1)}} - \frac{a(x,\nabla u)}{(1+|u_\varepsilon|)^{\theta(p(x)-1)}}\right) \cdot (\nabla u_\varepsilon - \nabla u)dx = 0. \tag{5.21}$$

According to (5.3), the integrand function in the left-hand side in (5.21) is nonnegative, so,

$$\left(\frac{a(x,\nabla u_\varepsilon)}{(1+|u_\varepsilon|)^{\theta(p(x)-1)}} - \frac{a(x,\nabla u)}{(1+|u_\varepsilon|)^{\theta(p(x)-1)}}\right) \cdot (\nabla u_\varepsilon - \nabla u) \to 0 \text{ in } L^1(\Omega).$$

So, up to a subsequence still indexed by ε, we have

$$\left(\frac{a(x,\nabla u_\varepsilon(x))}{(1+|u_\varepsilon(x)|)^{\theta(p(x)-1)}} - \frac{a(x,\nabla u(x))}{(1+|u_\varepsilon(x)|)^{\theta(p(x)-1)}}\right) \cdot (\nabla u_\varepsilon(x) - \nabla u(x)) \to 0$$

for almost every x in Ω.

Consequently, there exists a subset Γ of Ω of zero measure, such that for every x in $\Omega\backslash\Gamma$, one has

$$D_\varepsilon(x) = \left(\frac{a(x, \nabla u_\varepsilon(x))}{(1+|u_\varepsilon(x)|)^{\theta(p(x)-1)}} - \frac{a(x, \nabla u(x))}{(1+|u_\varepsilon(x)|)^{\theta(p(x)-1)}} \right) \cdot (\nabla u_\varepsilon(x) - \nabla u(x)) \to 0,$$

$|u(x)| < \infty$, $|\nabla u(x)| < \infty$, $|a_0(x)| < \infty$ and $u_\varepsilon(x) \to u(x)$.

As a result, from (5.2), (5.1) and (5.12) we can write

$$D_\varepsilon(x) \geq \frac{\alpha}{(1+|C_\infty|)^{\theta(p^+-1)}} |\nabla u_\varepsilon(x)|^{p(x)} - c(x)(1 + |\nabla u_\varepsilon(x)| + |\nabla u_\varepsilon(x)|^{p(x)-1}), \quad (5.22)$$

where $c(x)$ is a constant which depends on x but does not depend on ε, which shows, in view of (5.22), that the sequence $|\nabla u_\varepsilon(x)|$ is uniformly bounded in $\mathbb{R}^N$ with respect to ε. Now, arguing as in the same way as in Lemma 3.4. in [24], we get

$$u_\varepsilon \to u \text{ strongly in } W_0^{1,p(x)}(\Omega).$$

5.2.4 Proof of Theorem 5.2.1

Now, we can show that (u, w) satisfies (5.5). To do this, coming back to (5.7) and let $v \in W_0^{1,p}(\Omega)$. For every $\dfrac{1}{\varepsilon} > C_\infty$ one has

$$\int_\Omega \left(\gamma_\varepsilon(T_{\frac{1}{\varepsilon}}(u_\varepsilon)) + \varepsilon arctan(u_\varepsilon) \right) v dx + \int_\Omega \frac{a(x, \nabla u_\varepsilon) \cdot \nabla v}{(1+|u_\varepsilon|)^{\theta(p(x)-1)}} dx = \int_\Omega f v dx. \quad (5.23)$$

In view of (5.17), we have for a subsequence

$$\nabla u_\varepsilon \to \nabla u \text{ a.e. in } \Omega,$$

which with

$$u_\varepsilon \to u \text{ a.e. in } \Omega$$

yields, since $\dfrac{a(x, \nabla u_\varepsilon)}{(1+|u_\varepsilon|)^{\theta(p(x)-1)}}$ is bounded in $(L^{p'(x)}(\Omega))^N$,

$$\frac{a(x, \nabla u_\varepsilon)}{(1+|u_\varepsilon|)^{\theta(p(x)-1)}} \rightharpoonup \frac{a(x, \nabla u)}{(1+|u|)^{\theta(p(x)-1)}} \text{ weakly in } (L^{p'(x)}(\Omega))^N.$$

Then, passing to the limit in (5.23), we get

$$\int_\Omega w v dx + \int_\Omega \frac{a(x, \nabla u)}{(1+|u|)^{\theta(p(x)-1)}} \cdot \nabla v dx = \int_\Omega f v dx \text{ for any } v \in W_0^{1,p(x)}(\Omega).$$

5.2.5 Subdifferential argument

To demonstrate that $u(x) \in D(\gamma(x))$ and $w(x) \in \gamma(u(x))$ for almost all $x \in \Omega$, we use the same arguments as in the subsection 2.2.5.

We therefore complete the proof of Theorem 5.2.1 with this final step.

5.3 Existence and regularity of entropy solutions for L^1-data

5.3.1 Approximation problems for L^1-data and a priori estimates

For $m, n \in \mathbb{N}$ and $f \in L^1(\Omega)$, let $f_{m,n} \in L^\infty(\Omega)$ be defined as in the beginning of section 5.2. In view of Theorem 5.2.1, there exists a solution $(u_{m,n}, w_{m,n})$ of $(P^*, f_{m,n})$ such that $u_{m,n} \in W_0^{1,p(x)}(\Omega) \cap L^\infty(\Omega)$, $w_{m,n} \in L^\infty(\Omega)$ and

$$\int_\Omega w_{m,n} v dx + \int_\Omega \frac{a(x, \nabla u_{m,n})}{(1 + |u_{m,n}|)^{\theta(p(x)-1)}} \cdot \nabla v dx = \int_\Omega f_{m,n} v dx \tag{5.24}$$

holds for all $v \in W_0^{1,p(x)}(\Omega)$ and $m, n \in \mathbb{N}$.

Claim 5.3.1. Suppose $(u_{m,n}, w_{m,n})$ is a solution of $(P^*, f_{m,n})$. Then, for $k > 0$, there exists a constant $C(k) > 0$, not depending on $m, n \in \mathbb{N}$, such that

$$||T_k(u_{m,n})||_{W_0^{1,p(x)}(\Omega)} \leq C(k), \tag{5.25}$$

and

$$||w_{m,n}||_{L^1(\Omega)} \leq ||f||_{L^1(\Omega)}. \tag{5.26}$$

Proof. By taking $T_k(u_{m,n})$ as a test function in (5.24), we obtain

$$\int_\Omega w_{m,n} T_k(u_{m,n}) dx + \int_\Omega \frac{a(x, \nabla u_{m,n})}{(1 + |u_{m,n}|)^{\theta(p(x)-1)}} \cdot \nabla T_k(u_{m,n}) dx = \int_\Omega f_{m,n} T_k(u_{m,n}) dx. \tag{5.27}$$

Since $w_{m,n}$ has the same sign as $u_{m,n}$ and from (5.1), we get

$$\alpha \int_\Omega |\nabla T_k(u_{m,n})|^{p(x)} dx \leq (1 + k)^{\theta(p^+ - 1)} k ||f||_{L^1(\Omega)}. \tag{5.28}$$

Now, to obtain (5.26), we drop in (5.27) the positive term and keep

$$\int_\Omega w_{m,n} T_k(u_{m,n}) dx \leq \int_\Omega f_{m,n} T_k(u_{m,n}) dx \leq k||f||_{L^1(\Omega)},$$

since $w_{m,n} \in \gamma(u_{m,n})$ a.e. in Ω, it follows that

$$\int_{\{|u_{m,n}|>k\}} |w_{m,n}| dx \leq ||f||_{L^1(\Omega)},$$

by passing to the limit with $k \downarrow 0$ and applying the Fatou's Lemma, one has

$$\int_\Omega |w_{m,n}| dx \leq ||f||_{L^1(\Omega)}.$$

□

5.3.2 Weak convergence of $T_k(u_n)$ in $W_0^{1,p(x)}(\Omega)$

By definition, $f_{m,n}$ is a bi-monotone approximation of f in $L^1(\Omega)$,

$$f_{m,n} \leq f_{m+1,n} \text{ and } f_{m,n+1} \leq f_{m,n}.$$

According to Proposition 5.2.4 it follows that

$$u^\varepsilon_{m,n} \leq u^\varepsilon_{m+1,n} \text{ and } u^\varepsilon_{m,n+1} \leq u^\varepsilon_{m,n} \tag{5.29}$$

a.e. in Ω for all $m, n \in \mathbb{N}$ and any $\varepsilon > 0$.

Then, passing the limit as $\varepsilon \downarrow 0$ in (5.29) yields

$$u_{m,n} \leq u_{m+1,n} \text{ and } u_{m,n+1} \leq u_{m,n} \tag{5.30}$$

a.e. in Ω for all $m, n \in \mathbb{N}$.

Setting $w^\varepsilon := \gamma_\varepsilon(T_{\frac{1}{\varepsilon}}(u_\varepsilon))$, using (5.29), Remark 5.2.5, the fact that $w^\varepsilon_{m,n} \overset{*}{\rightharpoonup} w_{m,n}$ in $L^\infty(\Omega)$ and this convergence preserves order, we obtain

$$w_{m,n} \leq w_{m+1,n} \text{ and } w_{m,n+1} \leq w_{m,n} \tag{5.31}$$

a.e. in Ω for all $m, n \in \mathbb{N}$.

From (5.26) and (5.31), we get

$$w_{m,n} \uparrow_m w^n \downarrow_n w \text{ in } L^1(\Omega),$$

where $w^n, w : \Omega \to \mathbb{R}$ are measurable functions, finite a.e. in Ω. Here, we use the notation $\downarrow_n$ respectively $\uparrow_n$, to denote convergence of a sequence which is monotone decreasing, respectively increasing, in n.

In view of (5.30), we conclude that

$$u_{m,n} \uparrow_m u^n \downarrow_n u \text{ a.e. in } \Omega,$$

where $u^n, u : \Omega \to \overline{\mathbb{R}}$ are measurable functions.

So as to prove that u is finite almost everywhere, we will give an estimate on the level sets of $u_{m,n}$ in the following Claim 5.3.2.

Claim 5.3.2. For $m, n \in \mathbb{N}$, let $(u_{m,n}, w_{m,n})$ be a solution of $(P^*, f_{m,n})$. Then, there exists a constant $C > 0$, not depending on $m, n \in \mathbb{N}$ such that

$$|\{|u_{m,n}| \geq l\}| \leq Cl^{-\Theta} \quad \text{for all } l \geq 1, \tag{5.32}$$

where $\Theta = (p^+ - 1)\left(\dfrac{p^- - 1}{p^+ - 1} - \theta\right)$.

Proof. By Chebyshev's inequality and Poincaré's inequality in $W_0^{1,p^-}(\Omega)$, also by (5.28), we have

$$\begin{aligned} l^{p^-}|\{|u_{m,n}| \geq l\}| &\leq \int_{\{|u_{m,n}| \geq l\}} |T_l(u_{m,n})|^{p^-} dx \\ &\leq C\left(\int_{\{|\nabla T_l(u_{m,n})| < 1\}} |\nabla T_l(u_{m,n})|^{p^-} dx + \int_{\{|\nabla T_l(u_{m,n})| \geq 1\}} |\nabla T_l(u_{m,n})|^{p(x)} dx\right) \\ &\leq C\left(|\Omega| + \int_\Omega |\nabla T_l(u_{m,n})|^{p(x)} dx\right) \\ &\leq Cl^{\theta(p^+ - 1)+1}. \end{aligned}$$

Then, we obtain (5.32). □

Clearly (5.4) in particular implies $\Theta > 0$. Note that, as $(u_{m,n})_m$ is pointwise increasing with respect to m, we get

$$\lim_{m\to\infty} |\{u_{m,n} > l\}| = |\{u^n > l\}| \tag{5.33}$$

and

$$\lim_{m\to\infty} |\{u_{m,n} \leq -l\}| = |\{u^n \leq -l\}|. \tag{5.34}$$

Combining (5.32) with (5.33) and (5.34), we obtain

$$|\{u^n \leq -l\}| + |\{u^n > l\}| \leq Cl^{-\Theta} \tag{5.35}$$

for all $l \geq 1$, thus u^n is finite a.e. for any $n \in \mathbb{N}$.

Using similar justifications, we conclude that

$$|\{u \leq -l\}| + |\{u > l\}| \leq Cl^{-\Theta} \tag{5.36}$$

from (5.35), it follows that u is finite a.e. in Ω. Next, since $w_{m,n} \in \gamma(u_{m,n})$ almost everywhere in Ω we deduce, by a subdifferential argument, that $w^n \in \gamma(u^n)$ and $w \in \gamma(u)$ a.e. in Ω.

In the sequel of this section, for $m, n \in \mathbb{N}$, let $(u_{m,n}, w_{m,n})$ be a solution of $(P^*, f_{m,n})$. We construct a subsequence $(m(n))_n$, by applying the diagonal principle in $L^1(\Omega)$, such that

$$w_n := w_{m(n),n} \to w \text{ in } L^1\Omega) \text{ and a.e. in } \Omega,$$

$$f_n := f_{m(n),n} \to f \text{ in } L^1\Omega) \text{ and a.e. in } \Omega,$$

it follows that

$$u_n := u_{m(n),n} \to u \text{ converges a.e. in } \Omega. \tag{5.37}$$

Using (5.25) and (5.37) together, for any $k > 0$, we obtain

$$T_k(u_n) \rightharpoonup T_k(u) \text{ in } W_0^{1,p(x)}(\Omega). \tag{5.38}$$

5.3.3 Some regularity results

Claim 5.3.3. There exists a subsequence of u_n such that, for each

$$q(x) \in \left(1, \frac{(p(x)-1-\theta(p^+-1))N}{N-1-\theta(p^+-1)}\right), \tag{5.39}$$

we have

$$||u_n||_{W_0^{1,q(x)}(\Omega)} \leq C. \tag{5.40}$$

Proof. Take $k > 0$ and denote $B_k = \{k < |u_n| \leq k+1\}$. We choose the test function $T_1(u_n - T_k(u_n))$ in the weak formulation (5.24), to obtain

$$\int_\Omega w_n T_1(u_n - T_k(u_n))dx + \int_\Omega \frac{a(x,\nabla u_n)\cdot\nabla T_1(u_n - T_k(u_n))}{(1+|u_n|)^{\theta(p-1)}}dx = \int_\Omega f_n T_1(u_n - T_k(u_n))dx,$$

using (5.1) and the fact that γ_ε is nondecreasing with $\gamma_\varepsilon(0) = 0$, we get

$$\alpha \int_{B_k} \frac{|\nabla u_n|^{p(x)}}{(1+|u_n|)^{\theta(p(x)-1)}}dx \leq ||f||_{L^1(\Omega)},$$

particularly, there exists $C_1 > 0$ such that

$$\int_{B_k} \frac{|\nabla u_n|^{p^-}}{(1+|u_n|)^{\theta(p^+-1)}}dx \leq C_1. \tag{5.41}$$

In a first step, let q^+ be a constant such that

$$1 \leq q^+ < \frac{(p^- - 1 - \theta(p^+-1))N}{N-1-\theta(p^+-1)}. \tag{5.42}$$

Note that (5.42) in particular implies $q^+ < p^-$ and $\dfrac{p^- r}{p^- - q^+} < q^{+*}$ where $r = \dfrac{q^+\theta(p^+-1)}{p^-}$.

Next, we will estimate $\displaystyle\int_{B_k} |\nabla u_n|^{q^+} dx$ for all $k > 0$. According to (5.41) and Hölder's

inequality, we conclude that

$$\begin{aligned}\int_{B_k}|\nabla u_n|^{q^+}dx &= \int_{B_k}\frac{|\nabla u_n|^{q^+}}{(1+|u_n|)^r}(1+|u_n|)^r dx\\ &\leq \left(\int_{B_k}\frac{|\nabla u_n|^{p^-}}{(1+|u_n|)^{\theta(p^+-1)}}dx\right)^{\frac{q^+}{p^-}}\left(\int_{B_k}(1+|u_n|)^{\frac{p^-r}{p^--q^+}}dx\right)^{\frac{p^--q^+}{p^-}}\\ &\leq C_2|B_k|^{\frac{p^--q^+}{p^-}}+C_2\left(\int_{B_k}|u_n|^{\frac{p^-r}{p^--q^+}}dx\right)^{\frac{p^--q^+}{p^-}}\\ &\leq C_2|B_k|^{\frac{p^--q^+}{p^-}}+C_3\left(\int_{B_k}|u_n|^{q^{+*}}dx\right)^{\frac{r}{q^{+*}}}|B_k|^{\frac{p^--q^+}{p^-}-\frac{r}{q^{+*}}}.\end{aligned}$$

Clearly $|B_k|\leq\frac{1}{k^{q^{+*}}}\int_{B_k}|u_n|^{q^{+*}}dx$ for all $k\geq k_0\geq 1$, one has

$$\begin{aligned}\int_{B_k}|\nabla u_n|^{q^+}dx &\leq C_2\left(\frac{1}{k^{q^{+*}}}\int_{B_k}|u_n|^{q^{+*}}dx\right)^{\frac{p^--q^+}{p^-}}+C_3\frac{1}{k^{q^{+*}\left(\frac{p^--q^+}{p^-}-\frac{r}{q^{+*}}\right)}}\left(\int_{B_k}|u_n|^{q^{+*}}dx\right)^{\frac{p^--q^+}{p^-}}\\ &\leq \frac{C_4}{k^{q^{+*}\left(\frac{p^--q^+}{p^-}-\frac{r}{q^{+*}}\right)}}\left(\int_{B_k}|u_n|^{q^{+*}}dx\right)^{\frac{p^--q^+}{p^-}}. \qquad (5.43)\end{aligned}$$

Now note that

$$\int_\Omega|\nabla u_n|^{q^+}dx=\sum_{k=0}^{k_0-1}\int_{B_k}|\nabla u_n|^{q^+}dx+\sum_{k=k_0}^{\infty}\int_{B_k}|\nabla u_n|^{q^+}dx. \qquad (5.44)$$

From (5.41) and Hölder's inequality, we deduce

$$\begin{aligned}\int_\Omega|\nabla u_n|^{q^+}dx &\leq C_{k_0}+\sum_{k=k_0}^{\infty}\int_{B_k}|\nabla u_n|^{q^+}dx\\ &\leq C_{k_0}+C_4\left(\sum_{k=k_0}^{\infty}\frac{1}{k^{q^{+*}\left(\frac{p^--q^+}{p^-}-\frac{r}{q^{+*}}\right)\frac{p^-}{q^+}}}\right)^{\frac{q^+}{p^-}}\left(\sum_{k=k_0}^{\infty}\int_{B_k}|u_n|^{q^{+*}}dx\right)^{\frac{p^--q^+}{p^-}}\\ &\leq C_{k_0}+C_4\left(\sum_{k=k_0}^{\infty}\frac{1}{k^{q^{+*}\left(\frac{p^--q^+}{p^-}-\frac{r}{q^{+*}}\right)\frac{p^-}{q^+}}}\right)^{\frac{q^+}{p^-}}\|u_n\|_{L^{q^{+*}}(\Omega)}^{\frac{q^{+*}(p^--q^+)}{p^-}}.\end{aligned}$$

Since $q^{+*}\left(\frac{p^- - q^+}{p^-} - \frac{r}{q^{+*}}\right)\frac{p^-}{q^+} > 1$ yields $\sum_{k=k_0}^{\infty} \frac{1}{k^{q^{+*}\left(\frac{p^- - q^+}{p^-} - \frac{r}{q^{+*}}\right)\frac{p^-}{q^+}}}$ converges.

Then

$$\int_\Omega |\nabla u_n|^{q^+} dx \leq C_{k_0} + C_5 \|u_n\|^{\frac{q^{+*}(p^- - q^+)}{p^-}}_{L^{q^{+*}}(\Omega)}. \tag{5.45}$$

Applying the Sobolev embedding $W_0^{1,q^+}(\Omega) \subset L^{q^{+*}}(\Omega)$, one gets

$$\begin{aligned}\left(\int_\Omega |u_n|^{q^{+*}} dx\right)^{\frac{q^+}{q^{+*}}} &\leq C_6 \int_\Omega |\nabla u_n|^{q^+} dx \\ &\leq C_{k_0} + C_7 \|u_n\|^{\frac{q^{+*}(p^- - q^+)}{p^-}}_{L^{q^{+*}}(\Omega)}.\end{aligned}$$

In other words,

$$\|u_n\|^{q^+}_{L^{q^{+*}}(\Omega)} \leq C_{k_0} + C_7 \|u_n\|^{\varrho}_{L^{q^{+*}}(\Omega)},$$

where $\varrho := \frac{q^{+*}(p^- - q^+)}{p^-}$. Since $p^- < N$ one has $\varrho < q^+$ and as a result, we get that

$$\|u_n\|_{W_0^{1,q^+}(\Omega)} \leq C,$$

for some constant $C > 0$. As a consequence, there exists a constant $C' > 0$ such that

$$\|u_n\|_{L^1(\Omega)} \leq C'. \tag{5.46}$$

Now let us consider a continuous variable exponent q on $\overline{\Omega}$ verifying the pointwise estimate (5.39). According to the continuity of $q(\cdot)$ and $p(\cdot)$ on $\overline{\Omega}$, there exists a constant $\delta > 0$ such that

$$\max_{y \in \overline{B(x,\delta)} \cap \Omega} q(y) < \min_{y \in \overline{B(x,\delta)} \cap \Omega} \frac{(p(y) - 1 - \theta(p^+ - 1))N}{N - 1 - \theta(p^+ - 1)} \text{ for all } x \in \Omega. \tag{5.47}$$

Note that $\overline{\Omega}$ is compact and therefore we can cover it with a finite number of balls $(B_i)_{i=1,...,k}$. Furthermore, there exists a constant $\rho > 0$ such that

$$|B_i \cap \Omega| > \rho \text{ for all } i = 1, ..., k. \tag{5.48}$$

We denote by q_i^- and q_i^+ (respectively p_i^- and p_i^+) the local minimum and the local maximum of q on $\overline{B_i \cap \Omega}$ (respectively the local minimum and the local maximum of p on $\overline{B_i \cap \Omega}$).

Using now the same reasoning as before locally, we get that inequalities (5.43) and (5.44) hold on $B_i \cap B_k$ and $B_i \cap \Omega$, respectively. In particular, it is easy to check that, instead of the global estimate (5.45), we find

$$\begin{aligned}\int_{B_i\cap\Omega} |\nabla u_n|^{q_i^+} dx &\leq C_{k_0} + \sum_{k=k_0}^{\infty} \int_{B_i\cap B_k} |\nabla u_n|^{q_i^+} dx \\ &\leq C_{k_0} + C_4 \left(\sum_{k=k_0}^{\infty} \frac{1}{k^{q_i^{+*}\left(\frac{p_i^- - q_i^+}{p_i^-} - \frac{r_i}{q_i^{+*}}\right)\frac{p_i^-}{q_i^+}}} \right)^{\frac{q_i^+}{p_i^-}} \left(\sum_{k=k_0}^{\infty} \int_{B_i\cap B_k} |u_n|^{q_i^{+*}} dx \right)^{\frac{p_i^- - q_i^+}{p_i^-}} \\ &\leq C_{k_0} + C_4 \left(\sum_{k=k_0}^{\infty} \frac{1}{k^{q_i^{+*}\left(\frac{p_i^- - q_i^+}{p_i^-} - \frac{r_i}{q_i^{+*}}\right)\frac{p_i^-}{q_i^+}}} \right)^{\frac{q_i^+}{p_i^-}} \|u_n\|_{L^{q_i^{+*}}(B_i\cap\Omega)}^{\frac{q_i^{+*}(p_i^- - q_i^+)}{p_i^-}}, \end{aligned} \tag{5.49}$$

where $r_i = \dfrac{q_i^+\theta(p_i^+ - 1)}{p_i^-}$.

Denote by $\overline{u}_{n_i}$ the average of u_{n_i} over $B_i \cap \Omega$:

$$\overline{u}_{n_i} = \frac{1}{|B_i \cap \Omega|} \int_{B_i\cap\Omega} u_n dx.$$

In view of (5.46) and (5.48), we deduce that

$$|\overline{u}_{n_i}| \leq \frac{C'}{\rho}. \tag{5.50}$$

By Poincaré-Wirtinger's inequality, we obtain

$$\|u_n - \overline{u}_{n_i}\|_{L^{q_i^{+*}}(B_i\cap\Omega)} \leq C_8 \|\nabla u_n\|_{L^{q_i^+}(B_i\cap\Omega)}, \tag{5.51}$$

for some constant $C_8 > 0$.

Keeping in mind (5.47), from (5.50), (5.51) and (5.49), we conclude that

$$\|u_n\|_{L^{q_i^{+*}}(B_i\cap\Omega)}^{q_i^+} \leq C_9 + C_{10} \|u_n\|_{L^{q_i^{+*}}(B_i\cap\Omega)}^{\varrho_i},$$

for some constants $C_9, C_{10} > 0$, where $\varrho_i := \frac{q_i^{+*}(p_i^- - q_i^+)}{p_i^-} < q_i^+$.

Clearly, this gives that, for some constant C_{11}, depending on $p(\cdot)$, $q(\cdot)$ and C_1,

$$\|u_n\|^{q_i^+}_{L^{q_i^{+*}}(B_i\cap\Omega)} \leq C_{11} \tag{5.52}$$

for all $i = 1, ..., k$.

Finally, since $q_i^{+*} \geq q^*(x) \geq q(x)$ and $q_i^+ \geq q(x)$ for all $x \in B_i \cap \Omega$ and for all $i = 1, ..., k$, we deduce from (5.49) and (5.52) that

$$\|u_n\|_{L^{q^*(\cdot)}(\Omega)} + \|u_n\|_{W_0^{1,q(\cdot)}(\Omega)} \leq C_{12},$$

for some constant C_{12} depending on $p(\cdot)$, $q(\cdot)$ and Ω. So, the proof of the Claim 5.3.3 is concluded. □

In order to pass to the limit in the weak formulation (5.24), the almost everywhere of the ∇u_n to ∇u is required.

5.3.4 Almost everywhere convergence of gradients

Claim 5.3.4. There exist a measurable function u and a subsequence of u_n, still denoted by u_n, such that

$$\nabla u_n \to \nabla u \text{ a.e. in } \Omega. \tag{5.53}$$

Proof. The proof for this claim is practically similar to that establishing in [108, Lemma 4.1]. Due to small modifications, we simply outline the main processes in this proof.

Let λ be a real number between 0 and 1 which will be determined later, define

$$\tilde{a}(x, u, \xi) = \frac{a(x, \xi)}{(1 + |u|)^{\theta(p(x)-1)}}$$

and

$$E_n = \int_\Omega ((\tilde{a}(x, u_n, \nabla u_n) - \tilde{a}(x, u_n, \nabla u)) \cdot \nabla(u_n - u))^\lambda dx.$$

It is clear that E_n is well defined and $E_n \geq 0$ thanks to (5.3). We fix $k > 0$ and split E_n on the sets $\{|u| \leq k\}$ and $\{|u| > k\}$, we obtain

$$E_n = E_1(n, k) + E_2(n, k),$$

where

$$E_1(n, k) = \int_{\{|u|>k\}} ((\tilde{a}(x, u_n, \nabla u_n) - \tilde{a}(x, u_n, \nabla u)) \cdot \nabla(u_n - u))^\lambda dx$$

and

$$E_2(n,k) = \int_{\{|u|\leq k\}} ((\tilde{a}(x,u_n,\nabla u_n) - \tilde{a}(x,u_n,\nabla u)) \cdot \nabla(u_n - u))^{\lambda} dx.$$

For $E_2(n,k)$, we have

$$E_2(n,k) \leq E_3(n,k) = \int_{\Omega} ((\tilde{a}(x,u_n,\nabla u_n) - \tilde{a}(x,u_n,\nabla T_k(u))) \cdot \nabla(u_n - T_k(u)))^{\lambda} dx.$$

For $h > k+1$, we split $E_3(n,k)$ on the sets $\{|u_n - T_k(u)| \leq h\}$ and $\{|u_n - T_k(u)| > h\}$, we get

$$E_4(n,k,h) = \int_{\{|u_n - T_k(u)|>h\}} ((\tilde{a}(x,u_n,\nabla u_n) - \tilde{a}(x,u_n,\nabla T_k(u))) \cdot \nabla(u_n - T_k(u)))^{\lambda} dx$$

and

$$\begin{aligned} E_5(n,k,h) &= \int_{\{|u_n - T_k(u)|\leq h\}} ((\tilde{a}(x,u_n,\nabla u_n) - \tilde{a}(x,u_n,\nabla T_k(u))) \cdot \nabla(u_n - T_k(u)))^{\lambda} dx \\ &= \int_{\Omega} ((\tilde{a}(x,u_n,\nabla u_n) - \tilde{a}(x,u_n,\nabla T_k(u))) \cdot \nabla T_h(u_n - T_k(u)))^{\lambda} dx \\ &\leq |\Omega|^{1-\lambda} \left(\int_{\Omega} (\tilde{a}(x,u_n,\nabla u_n) - \tilde{a}(x,u_n,\nabla T_k(u))) \cdot \nabla T_h(u_n - T_k(u)) dx \right)^{\lambda} \\ &= |\Omega|^{1-\lambda} \left(E_6(n,k,h) \right)^{\lambda}. \end{aligned}$$

The term $E_6(n,k,h)$ can be written as the difference $E_7(n,k,h) - E_8(n,k,h)$ such that

$$E_7(n,k,h) = \int_{\Omega} \tilde{a}(x,u_n,\nabla u_n) \cdot \nabla T_h(u_n - T_k(u)) dx$$

and

$$E_8(n,k,h) = \int_{\Omega} \tilde{a}(x,u_n,\nabla T_k(u)) \cdot \nabla T_h(u_n - T_k(u)) dx.$$

Now we choose $\lambda < 1$ such that $\lambda p^+ < q^-$ where q is the same as in Claim 5.3.3. Due to (5.37), (5.38), (5.40) and in similar fashion as Lemma 4.1. in [108], it yields that

$$\lim_{k\to\infty} \limsup_{n\to\infty} E_1(n,k) = 0,$$

$$\lim_{k\to\infty} \lim_{h\to\infty} \limsup_{n\to\infty} E_4(n,k,h) = 0,$$

$$\lim_{n\to\infty} E_8(n,k,h) = 0.$$

For $E_7(n,k,h)$, let us take, for $h > 0$, $T_h(u_n - T_k(u))$ as a test function in (5.24), to get

$$\int_{\Omega} w_n T_h(u_n - T_k(u)) dx + \int_{\Omega} \frac{a(x,\nabla u_n) \cdot \nabla T_h(u_n - T_k(u))}{(1+|u_n|)^{\theta(p(x)-1)}} dx = \int_{\Omega} f_n T_h(u_n - T_k(u)) dx.$$

By the strong convergence of f_n and w_n in $L^1(\Omega)$, we conclude that

$$\lim_{n\to\infty} E_7(n,k,h) = \int_\Omega f T_h(u - T_k(u))dx - \int_\Omega w T_h(u - T_k(u))dx.$$

As a result,

$$\lim_{k\to\infty}\lim_{n\to\infty} E_7(n,k,h) = 0.$$

Putting together all the limitations, one thus has

$$\lim_{n\to\infty} E_n = 0.$$

So, in the similar way as in [108], we obtain that

$$\nabla u_n(x) \to \nabla u(x) \text{ a.e. in } \Omega.$$

□

5.3.5 Conclusion of the proof of Theorem 5.1.5

We are now able to prove the result of existence of entropy solution of the problem (P^*, f) announced in Theorem 5.1.5.

Let $v \in W_0^{1,p(x)}(\Omega) \cap L^\infty(\Omega)$ and $k > 0$ be fixed, by taking $T_k(u_n - v)$ as a test function in (5.24), one has

$$\int_\Omega w_n T_k(u_n - v)dx + \int_\Omega \frac{a(x,\nabla u_n)\cdot\nabla T_k(u_n - v)}{(1+|u_n|)^{\theta(p(x)-1)}}dx = \int_\Omega f_n T_k(u_n - v)dx. \quad (5.54)$$

As w_n and f_n are strongly convergent in $L^1(\Omega)$, while $T_k(u_n - v)$ converges both weakly$*$ in $L^\infty(\Omega)$ and a.e. to $T_k(u - v)$, we obtain

$$\lim_{n\to\infty}\int_\Omega w_n T_k(u_n - v)dx = \int_\Omega w T_k(u - v)dx$$

and

$$\lim_{n\to\infty}\int_\Omega f_n T_k(u_n - v)dx = \int_\Omega f T_k(u - v)dx.$$

Lastly, let us examine the second term of (5.54), we split it as the sum

$$\int_{\{|u_n - v|\le k\}} \frac{a(x,\nabla u_n)\cdot\nabla u_n}{(1+|u_n|)^{\theta(p(x)-1)}}dx - \int_{\{|u_n - v|\le k\}} \frac{a(x,\nabla u_n)\cdot\nabla v}{(1+|u_n|)^{\theta(p(x)-1)}}dx.$$

Observe that the second integral is on a subset of the set where $|u_n| \leq k + ||v||_{L^\infty(\Omega)} = M$, thus, it can be written as

$$\int_{\{|u_n - v| \leq k\}} \frac{a(x, \nabla T_M(u_n)) \cdot \nabla v}{(1 + |T_M(u_n)|)^{\theta(p(x)-1)}} dx.$$

Since $\dfrac{a(x, \nabla T_M(u_n))}{(1 + |T_M(u_n)|)^{\theta(p(x)-1)}}$ is bounded in $(L^{p'(x)}(\Omega))^N$. Thus, by (5.37) and (5.53), we obtain

$$\frac{a(x, \nabla T_M(u_n))}{(1 + |T_M(u_n)|)^{\theta(p(x)-1)}} \rightharpoonup \frac{a(x, \nabla T_M(u))}{(1 + |T_M(u)|)^{\theta(p(x)-1)}} \text{ weakly in } (L^{p'(x)}(\Omega))^N.$$

Then, the second integral converges, as n goes to infinity, to

$$\int_{\{|u-v| \leq k\}} \frac{a(x, \nabla T_M(u)) \cdot \nabla v}{(1 + |T_M(u)|)^{\theta(p(x)-1)}} dx = \int_{\{|u-v| \leq k\}} \frac{a(x, \nabla u) \cdot \nabla v}{(1 + |u|)^{\theta(p(x)-1)}} dx.$$

Due to (5.1), the integrand function of the first integral is nonnegative, so, using Fatou's Lemma, (5.37) and (5.53), we obtain that

$$\int_{\{|u-v| \leq k\}} \frac{a(x, \nabla u) \cdot \nabla u}{(1 + |u|)^{\theta(p(x)-1)}} dx \leq \liminf_{n \to \infty} \int_{\{|u_n - v| \leq k\}} \frac{a(x, \nabla u_n) \cdot \nabla u_n}{(1 + |u_n|)^{\theta(p(x)-1)}} dx.$$

Finally, putting all the terms together, we get

$$\int_\Omega w T_k(u - v) dx + \int_{\{|u-v| \leq k\}} \frac{a(x, \nabla u) \cdot \nabla (u - v)}{(1 + |u|)^{\theta(p(x)-1)}} dx \leq \int_\Omega f T_k(u - v) dx.$$

which is (5.6). As a result, (u, w) is an entropy solution of the problem (P^*, f). Claim 5.3.3 ensures the regularity of the entropy solution u.

So that the proof of the Theorem 5.1.5 is now completed.

Chapter 6

Existence and uniqueness of renormalized solution of a nonlinear elliptic inclusion problems in Orlicz-Sobolev space

In this chapter, we study a nonlinear elliptic inclusion problems with L^1-data in the setting of Orlicz Sobolev space. We obtain the existence of the renormalized solutions by using truncation techniques and a generalized Minty method in the functional spaces. Under an additional strict monotonicity assumption uniqueness of the renormalized solution is established. The results of this chapter, which have been published in [11], generalize the corresponding ones given in [105].

6.1 Essential assumptions and statements of results

Let Ω be a bounded open subset of $\mathbb{R}^N$, $N \geq 1$, with the segment property. Let M be an N-function such that $\overline{M}$ satisfies the Δ_2 condition.

Our aim is to show the existence and uniqueness of renormalized solutions to the following nonlinear multivalued elliptic problem with Dirichlet boundary conditions

$$(E, f) \quad \begin{cases} \beta(u) - \operatorname{div}(a(x, \nabla u) + F(u)) \ni f & \text{in } \Omega, \\ \\ u = 0 & \text{on } \partial\Omega, \end{cases}$$

with right-hand side $f \in L^1(\Omega)$, $F : \mathbb{R} \to \mathbb{R}^N$ is locally Lipschitz continuous and the nonlinearity $\beta : \mathbb{R} \to 2^{\mathbb{R}}$ a set valued, maximal monotone graph such that $0 \in \beta(0)$. Furthermore, $a : \Omega \times \mathbb{R}^N \to \mathbb{R}^N$ is Carathéodory function satisfying for a.e. $x \in \Omega$ and all $\xi, \eta \in \mathbb{R}^N$ the following assumptions:

$$|a(x,\xi)| \leq c(x) + k_1 \overline{M}^{-1} M(k_2|\xi|), \tag{6.1}$$

$$(a(x,\xi) - a(x,\eta)) \cdot (\xi - \eta) \geq 0, \tag{6.2}$$

$$a(x,\xi) \cdot \xi \geq \alpha M\left(\frac{|\xi|}{\lambda}\right), \tag{6.3}$$

where $c(x) \in E_{\overline{M}}(\Omega)$, $c \geq 0$, $k_1, k_2 \geq 0$ and $\alpha, \lambda > 0$.

Remark 6.1.1. The following argument will be frequently used to treat the "*convection*" term $F(u) \cdot \nabla u$ in (E, f). Observe that for $u \in W_0^1 L_M(\Omega)$ and $F = (F_1, ..., F_N) \in L^\infty(\mathbb{R}, \mathbb{R}^N)$ such that $F(0) = 0$, we have

$$\int_\Omega F(u) \cdot \nabla u dx = 0. \tag{6.4}$$

Proof. Let us define $\int_0^s F(\tau)d\tau := \left(\int_0^s F_1(\tau)d\tau, ..., \int_0^s F_N(\tau)d\tau\right)$ and $\Psi : \mathbb{R} \to \mathbb{R}^N$, $\Psi(s) := \int_0^s F(\tau)d\tau$ for $s \in \mathbb{R}$.

Note that Ψ is a Lipschitz continuous function such that $\Psi(0) = 0$ and therefore $\Psi \circ u \in (W_0^1 L_M(\Omega))^N$ by Lemma 1.3.1. Furthermore

$$\frac{\partial}{\partial x_i}(\Psi \circ u) = F(u)\frac{\partial}{\partial x_i}u \text{ in } D'(\Omega) \text{ for any } i = 1, ..., N.$$

As a consequence,

$$-\text{div}\left(\int_0^u F(\tau)d\tau\right) = F(u) \cdot \nabla u \text{ in } \mathcal{D}'(\Omega). \tag{6.5}$$

Then, (6.4) follows by using (6.5) and the Gauss-Green Theorem for Sobolev functions from $u = 0$ almost everywhere on $\partial\Omega$. □

Definition 6.1.2. Let $f \in L^\infty(\Omega)$. A weak solution to (E, f) is a couple of functions $(u, b) \in W_0^1 L_M(\Omega) \times L^1_{loc}(\Omega)$ satisfying $F(u) \in (L^1_{loc}(\Omega))^N$, $b \in \beta(u)$ almost everywhere in Ω and

$$b - \text{div}(a(x, \nabla u) + F(u)) = f \ \ in \ \ \mathcal{D}'(\Omega).$$

Remark 6.1.3. Note that no growth hypothesis is assumed on the function F, which implies that for a solution $u \in W_0^1 L_M(\Omega)$ the term div $F(u)$ may be meaningless, even as a distribution. As in [40] we shall use the following definition of renormalized solutions for problem (E, f) in the following sense:

Definition 6.1.4. Let $f \in L^1(\Omega)$. A couple of measurable functions (u, b) is called a renormalized solution of (E, f) if

(R_1) $u : \Omega \to \mathbb{R}$ is measurable, $b \in L^1(\Omega)$ and $b(x) \in \beta(u(x))$ for a.e. $x \in \Omega$.

(R_2) For each $k > 0$, $T_k(u) \in W_0^1 L_M(\Omega)$, $a(x, \nabla T_k(u)) \in \left(L_{\overline{M}}(\Omega)\right)^N$ and

$$\int_\Omega bh(u)\varphi dx + \int_\Omega (a(x, \nabla u) + F(u)) \cdot \nabla(h(u)\varphi) dx = \int_\Omega fh(u)\varphi dx, \tag{6.6}$$

holds for all $h \in C_c^1(\mathbb{R})$ and all $\varphi \in W_0^1 L_M(\Omega) \cap L^\infty(\Omega)$.

(R_3)

$$\int_{\{l<|u|<l+1\}} a(x, \nabla u) \cdot \nabla u \, dx \to 0 \text{ as } l \to \infty.$$

Our main results in this chapter are the following Theorems:

Theorem 6.1.5. *For $f \in L^1(\Omega)$, there exists at least one renormalized solution (u, b) to (E, f).*

Theorem 6.1.6. *Let β be a strictly increasing, continuous function. For $f \in L^1(\Omega)$ let (u, b) and $(\tilde{u}, \tilde{b})$ be renormalized solutions to (E, f). Then $u = \tilde{u}$ and $b = \tilde{b}$.*

6.2 Existence and uniqueness of solutions in the case of $f \in L^\infty(\Omega)$

To prove Theorem 6.1.5, we will introduce and solve approximating problems. For $f \in L^1(\Omega)$ and $m, n \in \mathbb{N}$ we define $f_{m,n} : \Omega \to \mathbb{R}$ by $f_{m,n}(x) = max(-n, min(f(x), m))$ for almost every $x \in \Omega$. Clearly, $f_{m,n} \in L^\infty(\Omega)$ for each $m, n \in \mathbb{N}$, $|f_{m,n}(x)| \leq |f(x)|$ a.e. in Ω, hence $\lim_{m\to\infty} \lim_{n\to\infty} f_{m,n} = f$ in $L^1(\Omega)$ and a.e. in Ω. The next Theorem will give us existence of renormalized solutions $(u_{m,n}, b_{m,n})$ of $(E, f_{m,n})$ for each $m, n \in \mathbb{N}$:

Theorem 6.2.1. *For $f \in L^\infty(\Omega)$, there exists at least one renormalized solution (u, b) to (E, f).*

We will divide our proof of the above Theorem 6.2.1 in several steps.

6.2.1 Approximate problems for L^∞-data

For $0 < \varepsilon \leq 1$, let us consider the following approximate problem

$$(E_\varepsilon, f) \quad \begin{cases} \beta_\varepsilon(T_{\frac{1}{\varepsilon}}(u_\varepsilon)) + \varepsilon arctan(u_\varepsilon) - \operatorname{div}\Big(a(x, \nabla u_\varepsilon) + F(T_{\frac{1}{\varepsilon}}(u_\varepsilon))\Big) = f & \text{in } \mathcal{D}'(\Omega), \\ \\ u_\varepsilon \in W_0^1 L_M(\Omega), \end{cases}$$

where $\beta_\varepsilon(\cdot) : \mathbb{R} \to \mathbb{R}$ is the Yosida approximation of $\beta(\cdot)$. Note that, for any $u \in W_0^1 L_M(\Omega)$ we have

$$\langle \beta_\varepsilon(u), u\rangle \geq 0, \ \ |\beta_\varepsilon(u)| \leq \frac{1}{\varepsilon}|u| \ \text{ and } \ \lim_{\varepsilon\to 0}\beta_\varepsilon(u) = \beta(u),$$

we refer the reader to [46] for more details.

As $|T_{\frac{1}{\varepsilon}}(u)| \leq \frac{1}{\varepsilon}$, we have $|\beta_\varepsilon(T_{\frac{1}{\varepsilon}}(u_\varepsilon))| \leq \frac{1}{\varepsilon^2}$ and $|F(T_{\frac{1}{\varepsilon}}(u))| \leq C_\varepsilon$ since F is continuous. From Proposition 1 and Remark 2 due to [73], the problem (E_ε, f) has at least one solution u_ε. i.e., for each $0 < \varepsilon \leq 1$ and $f \in W^{-1}L_{\overline{E}}(\Omega)$ there exists a solution $u_\varepsilon \in W_0^1 L_M(\Omega)$ to the problem (E_ε, f) such that

$$\int_\Omega \Big(\beta_\varepsilon(T_{\frac{1}{\varepsilon}}(u_\varepsilon)) + \varepsilon arctan(u_\varepsilon)\Big)\varphi dx + \int_\Omega \Big(a(x, \nabla u_\varepsilon) + F(T_{\frac{1}{\varepsilon}}(u_\varepsilon))\Big)\cdot\nabla\varphi dx = \langle f, \varphi\rangle \tag{6.7}$$

holds for all $\varphi \in W_0^1 L_M(\Omega)$, where $\langle ., .\rangle$ denotes the duality pairing between $W_0^1 L_M(\Omega)$ and $W^{-1}L_{\overline{M}}(\Omega)$.

In the following proposition, we discuss uniqueness of solutions u_ε of (E_ε, f) with right-hand side $f \in L^\infty(\Omega)$ through a comparison principle that will play an important role in the approximation of solutions to (E, f) with $f \in L^1(\Omega)$.

Proposition 6.2.2. *For $0 < \varepsilon \leq 1$ fixed and $f, \tilde{f} \in L^\infty(\Omega)$ let $u_\varepsilon, \tilde{u}_\varepsilon \in W_0^{1,p}(\Omega)$ be solutions of (E_ε, f) and $(E_\varepsilon, \tilde{f})$, respectively. Then, the following comparison principle holds*

$$\varepsilon \int_\Omega (arctan(u_\varepsilon) - arctan(\tilde{u}_\varepsilon))^+ dx \leq \int_\Omega (f - \tilde{f}) sign_0^+(u_\varepsilon - \tilde{u}_\varepsilon) dx. \tag{6.8}$$

Proof. We use the test function $p_\delta^+(u_\varepsilon - \tilde{u}_\varepsilon)$ in the weak formulation (6.7) for u_ε and $\tilde{u}_\varepsilon$. Subtracting the resulting inequalities and passing to the limit with $\delta \downarrow 0$, (6.8) follows. □

Remark 6.2.3. Let $f, \tilde{f} \in L^\infty(\Omega)$ be such that $f \leq \tilde{f}$ a.e. in Ω, $\varepsilon > 0$ and $u_\varepsilon, \tilde{u}_\varepsilon \in W_0^{1,p}(\Omega)$ be solutions of (E_ε, f) and $(E_\varepsilon, \tilde{f})$, respectively. Then it is an immediate consequence of Proposition 6.2.2 that $u_\varepsilon \leq \tilde{u}_\varepsilon$ a.e. in Ω. Moreover, from the monotonicity of $\beta_\varepsilon \circ T_{\frac{1}{\varepsilon}}$ it follows that also $\beta_\varepsilon(T_{\frac{1}{\varepsilon}}(u_\varepsilon)) \leq \beta_\varepsilon(T_{\frac{1}{\varepsilon}}(\tilde{u}_\varepsilon))$ a.e. in Ω.

6.2.2 A priori estimates

Lemma 6.2.4. *For $0 < \varepsilon \leq 1$ and $f \in L^\infty(\Omega)$ let $u_\varepsilon \in W_0^1 L_M(\Omega)$ be a solution of (E_ε, f). Then there exist two constants C_1 and C_2, not depending on ε, such that*

$$\int_\Omega M\left(\frac{|\nabla u_\varepsilon|}{\lambda}\right)dx \leq C_1 \text{ for some } \lambda > 1, \tag{6.9}$$

$$(a(x, \nabla u_\varepsilon))_{\varepsilon>0} \text{ is bounded in } \left(L_{\overline{M}}(\Omega)\right)^N, \tag{6.10}$$

$$||\nabla u_\varepsilon||_p \leq C_2 \text{ for some } p > 1, \tag{6.11}$$

$$||\beta_\varepsilon(T_{\frac{1}{\varepsilon}}(u_\varepsilon))||_\infty \leq ||f||_\infty, \tag{6.12}$$

$$\int_{\{l<|u_\varepsilon|<l+1\}} a(x, \nabla u_\varepsilon) \cdot \nabla u_\varepsilon dx \leq \int_{\{|u_\varepsilon|>l\}} |f| dx \tag{6.13}$$

holds for all $l > 0$.

Proof. Taking u_ε as a test function in (6.7), we have

$$\int_\Omega \left(\beta_\varepsilon(T_{\frac{1}{\varepsilon}}(u_\varepsilon)) + \varepsilon arctan(u_\varepsilon)\right) u_\varepsilon dx + \int_\Omega a(x, \nabla u_\varepsilon) \cdot \nabla u_\varepsilon dx + \int_\Omega F(T_{\frac{1}{\varepsilon}}(u_\varepsilon)) \cdot \nabla u_\varepsilon dx = \int_\Omega f u_\varepsilon dx.$$

As the first term on the left-hand side is nonnegative and the integral over the convection term vanishes, we get

$$\int_\Omega a(x, \nabla u_\varepsilon) \cdot \nabla u_\varepsilon dx \leq \int_\Omega f u_\varepsilon dx.$$

By (6.3) and Young's inequality we get (6.9) and then $\int_\Omega a(x, \nabla u_\varepsilon) \cdot \nabla u_\varepsilon dx \leq C$, so one has by Proposition 5 of [73] that (6.10). By Proposition 1.3.4, we obtain (6.11).

Using the same arguments as in [6], i.e. taking $\varphi_{\delta,\varepsilon} = \frac{1}{\delta}(T_{k+\delta}(\beta_\varepsilon(T_{\frac{1}{\varepsilon}}(u_\varepsilon))) - T_k(\beta_\varepsilon(T_{\frac{1}{\varepsilon}}(u_\varepsilon))))$ as a test function in (6.7), passing to the limit with $\delta \to 0$ and choosing $k > ||f||_\infty$, we get (6.12).

Finally, for $l > 0$ fixed, by taking $T_1(u_\varepsilon - T_l(u_\varepsilon))$ as a test function in (6.7) we conclude (6.13). □

Remark 6.2.5. By the Chebyshev, the Poincarée inequality, (1.5) and (6.9), we get

$$\begin{aligned} |\{|u_\varepsilon| > l\}| &\leq \int_\Omega \frac{|T_l(u_\varepsilon)|^p}{l^p} dx \\ &\leq C(p,N) l^{-p} \int_\Omega |\nabla T_l(u_\varepsilon)|^p dx \\ &\leq C(p,N,f) l^{-p}. \end{aligned}$$

Then from Lemma 6.2.4, we deduce

$$\int_{\{l<|u_\varepsilon|<l+1\}} a(x,\nabla u_\varepsilon)\cdot\nabla u_\varepsilon dx \leq ||f||_\infty |\{|u_\varepsilon| > l\}| \leq C(p,N,f) l^{-p}. \tag{6.14}$$

6.2.3 Basic convergence results

Lemma 6.2.6. *For $0 < \varepsilon \leq 1$ and $f \in L^\infty(\Omega)$ let $u_\varepsilon \in W_0^1 L_M(\Omega)$ be a solution of (E_ε, f). There exist $u \in W_0^1 L_M(\Omega)$, $b \in L^\infty(\Omega)$ such that for a not relabeled subsequence of $(u_\varepsilon)_{\varepsilon>0}$ as $\varepsilon \to 0$*

$$u_\varepsilon \rightharpoonup u \text{ in } W_0^1 L_M(\Omega) \text{ for } \sigma(\Pi L_M, \Pi E_{\overline{M}}), \tag{6.15}$$

$$u_\varepsilon \to u \text{ in } E_M(\Omega) \text{ and a.e. in } \Omega, \tag{6.16}$$

$$u_\varepsilon \rightharpoonup u \text{ in } W_0^{1,p}(\Omega) \text{ for some } p > 1, \tag{6.17}$$

$$u_\varepsilon \to u \text{ in } L^p(\Omega), \tag{6.18}$$

$$\beta_\varepsilon(T_{\frac{1}{\varepsilon}}(u_\varepsilon)) \overset{*}{\rightharpoonup} b \text{ in } L^\infty(\Omega). \tag{6.19}$$

Moreover, for any $k > 0$,

$$a(x,\nabla T_k(u_\varepsilon)) \rightharpoonup a(x,\nabla T_k(u)) \text{ in } (L_{\overline{M}}(\Omega))^N \text{ for } \sigma(\Pi L_{\overline{M}}, \Pi E_M). \tag{6.20}$$

Proof. (6.15)-(6.19) follows directly from Lemma 6.2.4. It is left to prove (6.20). First, since $(a(x,\nabla u_\varepsilon))_{\varepsilon>0}$ is bounded in $\left(L_{\overline{M}}(\Omega)\right)^N$, there exists $\Phi_k \in (L_{\overline{M}}(\Omega))^N$ such that

$$a(x,\nabla T_k(u_\varepsilon)) \rightharpoonup \Phi_k \text{ in } (L_{\overline{M}}(\Omega))^N \text{ for } \sigma(\Pi L_{\overline{M}}, \Pi E_M).$$

We intend to show now that

$$\Phi_k = a(x,\nabla T_k(u)) \text{ a.e. in } \Omega.$$

By Lemma 1.3.3 there exists a sequence $(v_j)_j \subset \mathcal{D}(\Omega)$ which converges to u for the modular convergence in $W_0^1 L_M(\Omega)$.

For $l > k$, taking $\varphi = h_l(u_\varepsilon)(T_k(u_\varepsilon) - T_k(v_j))$ as test function in (6.7), we get

$$I^1_{\varepsilon,j,l} + I^2_{\varepsilon,j,l} + I^3_{\varepsilon,j,l} + I^4_{\varepsilon,j,l} + I^5_{\varepsilon,j,l} = I^6_{\varepsilon,j,l} \tag{6.21}$$

where

$$\begin{aligned}
I^1_{\varepsilon,j,l} &= \int_\Omega \left(\beta_\varepsilon(T_{\frac{1}{\varepsilon}}(u_\varepsilon)) + \varepsilon arctan(u_\varepsilon)\right) h_l(u_\varepsilon)(T_k(u_\varepsilon) - T_k(v_j))dx,\\
I^2_{\varepsilon,j,l} &= \int_\Omega a(x,\nabla u_\varepsilon)\cdot(\nabla T_k(u_\varepsilon) - \nabla T_k(v_j))h_l(u_\varepsilon)dx,\\
I^3_{\varepsilon,j,l} &= \int_\Omega a(x,\nabla u_\varepsilon)\cdot\nabla u_\varepsilon h_l'(u_\varepsilon)(T_k(u_\varepsilon) - T_k(v_j))dx,\\
I^4_{\varepsilon,j,l} &= \int_\Omega F(T_{\frac{1}{\varepsilon}}(u_\varepsilon))\cdot(\nabla T_k(u_\varepsilon) - \nabla T_k(v_j))h_l(u_\varepsilon)dx,\\
I^5_{\varepsilon,j,l} &= \int_\Omega F(T_{\frac{1}{\varepsilon}}(u_\varepsilon))\cdot\nabla u_\varepsilon h_l'(u_\varepsilon)(T_k(u_\varepsilon) - T_k(v_j))dx,\\
I^6_{\varepsilon,j,l} &= \int_\Omega f h_l(u_\varepsilon)(T_k(u_\varepsilon) - T_k(v_j))dx.
\end{aligned}$$

We will deal with each term. In view of (6.16) and (6.19) from Lemma 6.2.6, we have $h_l(u_\varepsilon)(T_k(u_\varepsilon) - T_k(v_j)) \to h_l(u)(T_k(u) - T_k(v_j))$ in $L^1(\Omega)$ and $\beta_\varepsilon(T_{\frac{1}{\varepsilon}}(u_\varepsilon)) \overset{*}{\rightharpoonup} b$ in $L^\infty(\Omega)$ as $\varepsilon \to 0$. Then

$$\lim_{\varepsilon\to 0} I^1_{\varepsilon,j,l} = \int_\Omega b h_l(u)(T_k(u) - T_k(v_j))dx.$$

Since $h_l(u)(T_k(u) - T_k(v_j)) \overset{*}{\rightharpoonup} 0$ in $L^\infty(\Omega)$ as $j \to \infty$, we get

$$\lim_{j\to\infty}\lim_{\varepsilon\to 0} I^1_{\varepsilon,j,l} = 0.$$

Similarly, we have

$$\lim_{j\to\infty}\lim_{\varepsilon\to 0} I^6_{\varepsilon,j,l} = 0.$$

For $\varepsilon > 0$ small enough, we can write

$$I^4_{\varepsilon,j,l} = \int_\Omega F(T_{l+1}(u_\varepsilon)) \cdot (\nabla T_k(u_\varepsilon) - \nabla T_k(v_j)) h_l(u_\varepsilon) dx,$$

therefore by (6.17), since $\nabla T_k(u_\varepsilon) \rightharpoonup \nabla T_k(u)$ weakly in $(L^p(\Omega))^N$, it follows that

$$\lim_{\varepsilon \to 0} I^4_{\varepsilon,j,l} = \int_\Omega F(T_{l+1}(u)) \cdot (\nabla T_k(u) - \nabla T_k(v_j)) h_l(u) dx.$$

Then

$$\lim_{j \to \infty} \lim_{\varepsilon \to 0} I^4_{\varepsilon,j,l} = 0.$$

Now, for $\varepsilon > 0$ sufficiently small, we can write

$$I^5_{\varepsilon,j,l} = \int_\Omega \operatorname{div} \left(\int_0^{T_{l+1}(u_\varepsilon)} F(r) h_l'(r) dr \right) (T_k(u_\varepsilon) - T_k(v_j)) dx,$$

hence from the Gauss–Green Theorem for Sobolev functions, it follows that

$$I^5_{\varepsilon,j,l} = -\int_\Omega \int_0^{T_{l+1}(u_\varepsilon)} F(r) h_l'(r) dr \cdot (\nabla T_k(u_\varepsilon) - \nabla T_k(v_j)) dx,$$

therefore we also get

$$\lim_{j \to \infty} \lim_{\varepsilon \to 0} I^5_{\varepsilon,j,l} = 0.$$

Moreover, in view of Remark 6.2.5, from (6.14), i.e.

$$\int_{\{l \le |u_\varepsilon| \le l+1\}} a(x, \nabla u_\varepsilon) . \nabla u_\varepsilon dx \le C l^{-p},$$

we infer

$$\begin{aligned}
\sup_j \sup_{\varepsilon \in (0,1]} |I^3_{\varepsilon,j,l}| &= \sup_j \sup_{\varepsilon \in (0,1]} \int_{\{l < |u_\varepsilon| < l+1\}} a(x, \nabla T_{l+1}(u_\varepsilon)) \cdot \nabla T_{l+1}(u_\varepsilon) |(T_k(u_\varepsilon) - T_k(v_j))| dx \\
&\le \sup_j \sup_{\varepsilon \in (0,1]} 2k \int_{\{l < |u_\varepsilon| < l+1\}} a(x, \nabla T_{l+1}(u_\varepsilon)) \cdot \nabla T_{l+1}(u_\varepsilon) dx \\
&\le 2kCl^{-p},
\end{aligned}$$

therefore

$$\lim_{l \to \infty} \sup_j \sup_{\varepsilon \in (0,1]} |I^3_{\varepsilon,j,l}| = 0.$$

Thus the above considerations of (6.21) provide

$$\limsup_{l \to \infty} \limsup_{j \to \infty} \limsup_{\varepsilon \to 0} \int_\Omega a(x, \nabla T_k(u_\varepsilon)) \cdot (\nabla T_k(u_\varepsilon) - \nabla T_k(v_j)) h_l(u_\varepsilon) dx \le 0, \quad (6.22)$$

since $l > k$, on the set $\{|u_\varepsilon| \leq k\}$, we have $h_l(u_\varepsilon) = 1$, as a consequence, we can write

$$\begin{aligned}&\int_\Omega a(x, \nabla T_k(u_\varepsilon)) \cdot (\nabla T_k(u_\varepsilon) - \nabla T_k(v_j)) h_l(u_\varepsilon) dx \\ &= \int_\Omega a(x, \nabla T_k(u_\varepsilon)) \cdot (\nabla T_k(u_\varepsilon) - \nabla T_k(v_j)) dx. \end{aligned} \tag{6.23}$$

Using the modular convergence of the sequence $(v_j)_j$, we can combine (6.23) with (6.22) and pass to the limit with $\varepsilon \to 0$ and next with $j \to \infty$ in order to obtain

$$\limsup_{\varepsilon \to 0} \int_\Omega a(x, \nabla T_k(u_\varepsilon)) \cdot \nabla T_k(u_\varepsilon) dx \leq \int_\Omega \Phi_k \cdot \nabla T_k(u) dx.$$

Now, Since $a(x, \cdot)$ is monotone, we have

$$\int_\Omega a(x, \nabla T_k(u_\varepsilon)) \cdot \nabla T_k(u_\varepsilon) dx \geq \int_\Omega a(x, \nabla T_k(u_\varepsilon)) \cdot \zeta dx + \int_\Omega a(x, \zeta) \cdot (\nabla T_k(u_\varepsilon) - \zeta) dx \tag{6.24}$$

for all $\zeta \in (L^\infty(\Omega))^N$.

Note that $a(x, \zeta) \in (E_{\overline{M}}(\Omega))^N$ and letting $\varepsilon \to 0$ in (6.24), we achieve

$$\int_\Omega (a(x, \zeta) - \Phi_k) \cdot (\zeta - \nabla T_k(u)) dx \geq 0. \tag{6.25}$$

Inequality (6.25) is exactly the starting point of monotonicity trick proof (see [70, Lemma 4.5]). Then, we obtain

$$\Phi_k = a(x, \nabla T_k(u)) \text{ a.e. in } \Omega.$$

□

Remark 6.2.7. As immediate consequence of (6.22), (6.23) and (6.2), we obtain

$$\lim_{\varepsilon \to 0} \int_\Omega (a(x, \nabla T_k(u_\varepsilon)) - a(x, \nabla T_k(u))) \cdot (\nabla T_k(u_\varepsilon) - \nabla T_k(u)) dx = 0. \tag{6.26}$$

Lemma 6.2.8. *The limit u of the approximate solution u_ε of (E_ε, f) satisfies*

$$\int_{\{l < |u| < l+1\}} a(x, \nabla u) \cdot \nabla u dx \to 0 \text{ as } l \to \infty.$$

Proof. Note that for any fixed $l \geq 0$, we have

$$\begin{aligned}&\int_{\{l < |u_\varepsilon| < l+1\}} a(x, \nabla u_\varepsilon) \cdot \nabla u_\varepsilon dx \\ &= \int_{\{l < |u_\varepsilon| < l+1\}} a(x, \nabla u_\varepsilon) \cdot (\nabla T_{l+1}(u_\varepsilon) - \nabla T_l(u_\varepsilon)) dx \\ &= \int_\Omega a(x, \nabla T_{l+1}(u_\varepsilon)) \cdot \nabla T_{l+1}(u_\varepsilon) dx - \int_\Omega a(x, \nabla T_l(u_\varepsilon)) \cdot \nabla T_l(u_\varepsilon) dx.\end{aligned}$$

According to (6.26) is at liberty passing the limit as $\varepsilon \to 0$ for fixed $l \geq 0$ and to obtain

$$\begin{aligned}
&\lim_{\varepsilon\to 0} \int_{\{l<|u_\varepsilon|<l+1\}} a(x, \nabla u_\varepsilon) \cdot \nabla u_\varepsilon dx \\
&= \int_\Omega a(x, \nabla T_{l+1}(u)) \cdot \nabla T_{l+1}(u) dx - \int_\Omega a(x, \nabla T_l(u)) \cdot \nabla T_l(u) dx \\
&= \int_{\{l<|u|<l+1\}} a(x, \nabla u) \cdot \nabla u dx. \qquad (6.27)
\end{aligned}$$

Taking the limit as $l \to \infty$ in (6.27) and using the estimate (6.14) one can show that u satisfies (R_3) and the proof of the Lemma is completed. □

6.2.4 Proof of existence

Now, we are able to conclude the proof of Theorem 6.2.1.

By virtue of Lemma 1.3.3 there exists a sequence $(v_j)_j \subset \mathcal{D}(\Omega)$ such that

$$v_j \to u \text{ in } W_0^1 L_M(\Omega) \text{ for the modular convergence and a.e. in } \Omega.$$

Let $h \in C_c^1(\mathbb{R})$ and $\varphi \in W_0^1 L_M(\Omega) \cap L^\infty(\Omega)$. Taking $h_l(u_\epsilon)h(v_j)\varphi$ as a test function in (6.7), we get

$$I_{\varepsilon,j,l}^1 + I_{\varepsilon,j,l}^2 + I_{\varepsilon,j,l}^3 = I_{\varepsilon,j,l}^4$$

where

$$\begin{aligned}
I_{\varepsilon,j,l}^1 &= \int_\Omega \left(\beta_\varepsilon(T_{\frac{1}{\varepsilon}}(u_\varepsilon)) + \varepsilon arctan(u_\varepsilon)\right) h_l(u_\varepsilon)h(v_j)\varphi dx, \\
I_{\varepsilon,j,l}^2 &= \int_\Omega a(x, \nabla u_\varepsilon) \cdot \nabla (h_l(u_\varepsilon)h(v_j)\varphi) dx, \\
I_{\varepsilon,j,l}^3 &= \int_\Omega F(T_{\frac{1}{\varepsilon}}(u_\varepsilon)) \cdot \nabla (h_l(u_\varepsilon)h(v_j)\varphi) dx, \\
I_{\varepsilon,j,l}^4 &= \int_\Omega f h_l(u_\varepsilon)h(v_j)\varphi dx.
\end{aligned}$$

Using the convergence results (6.18) and (6.19), we can immediately calculate the following limit

$$\lim_{\varepsilon\to 0} I_{\varepsilon,j,l}^1 = \int_\Omega b h_l(u)h(v_j)\varphi dx := I_{j,l}^1.$$

Applying the Lebesgue Dominated Convergence Theorem

$$\lim_{j\to\infty} I_{j,l}^1 = \int_\Omega b h_l(u)h(u)\varphi dx := I_l^1.$$

Since h has compact support on $\mathbb{R}$, there exists a real number $m > 0$, such that $supph \subset [-m, m]$. For all $l > m$, we obtain

$$I_l^1 = \int_\Omega bh(u)\varphi dx.$$

We write $I_{\varepsilon,j,l}^2 = I_{\varepsilon,j,l}^{2,1} + I_{\varepsilon,j,l}^{2,2}$ where

$$I_{\varepsilon,j,l}^{2,1} = \int_\Omega h_l'(u_\varepsilon)a(x, \nabla u_\varepsilon) \cdot \nabla u_\varepsilon h(v_j)\varphi dx$$

and

$$I_{\varepsilon,j,l}^{2,2} = \int_\Omega h_l(u_\varepsilon)a(x, \nabla u_\epsilon) \cdot \nabla(h(v_j)\varphi)dx.$$

Using (6.14), we get the estimate

$$|I_{\varepsilon,j,l}^{2,1}| \leq ||h||_\infty ||\varphi||_\infty C l^{-p}.$$

It follows that

$$\lim_{l\to\infty} \limsup_{j\to\infty} \limsup_{\varepsilon\to 0} |I_{\varepsilon,j,l}^{2,1}| = 0.$$

By (6.10) and (6.20), it follows that

$$a(x, \nabla T_{l+1}(u_\varepsilon)) \rightharpoonup a(x, \nabla T_{l+1}(u)) \text{ in } (L^1(\Omega))^N.$$

Moreover, $h_l(u_\varepsilon) \to h_l(u)$ a.e. in Ω, $|h_l(u_\varepsilon)| \leq 1$ and $\nabla(h(v_j)\varphi) \in (L^\infty(\Omega))^N$.

Using the Egorov Theorem to $(h_l(u_\varepsilon))_{\varepsilon>0}$ and applying the uniform integrability of the $\big(a(x, \nabla T_{l+1}(u_\varepsilon))h_l(u_\varepsilon) \cdot \nabla(h(v_j)\varphi)\big)_{\varepsilon>0}$, it follows that

$$\lim_{\varepsilon\to 0} I_{\varepsilon,j,l}^{2,2} = \int_\Omega a(x, \nabla T_{l+1}(u))h_l(u) \cdot \nabla(h(v_j)\varphi)dx := I_{j,l}^{2,2}.$$

Since $a(x, \nabla T_{l+1}(u))h_l(u) \in (E_{\overline{M}}(\Omega))^N$, we obtain

$$\lim_{j\to\infty} I_{j,l}^{2,2} = \int_\Omega a(x, \nabla T_{l+1}(u))h_l(u) \cdot \nabla(h(u)\varphi)dx := I_l^{2,2}.$$

For $l > m$, we get

$$I_l^{2,2} = \int_\Omega a(x, \nabla u) \cdot \nabla(h(u)\varphi)dx.$$

For ε such that $\frac{1}{\varepsilon} \geq l + 1$, one has

$$\begin{aligned} I_{\varepsilon,j,l}^3 = &\int_\Omega F(T_{l+1}(u_\varepsilon))h_l'(u_\varepsilon) \cdot \nabla T_{l+1}(u_\varepsilon)h(v_j)\varphi dx \\ &+ \int_\Omega F(T_{l+1}(u_\varepsilon))h_l(u_\varepsilon) \cdot \nabla(h(v_j)\varphi)dx := I_{\varepsilon,j,l}^{3,1} + I_{\varepsilon,j,l}^{3,2}. \end{aligned}$$

Since $\nabla T_{l+1}(u_\varepsilon) \rightharpoonup \nabla T_{l+1}(u)$ weakly in $(L^p(\Omega))^N$, we deduce

$$\lim_{\varepsilon\to 0} I^{3,1}_{\varepsilon,j,l} = \int_\Omega F(T_{l+1}(u))h_l'(u)\cdot\nabla T_{l+1}(u)h(v_j)\varphi dx.$$

By Lebesgue Dominated Convergence Theorem

$$\lim_{j\to\infty}\lim_{\varepsilon\to 0} I^{3,1}_{\varepsilon,j,l} = \int_\Omega F(T_{l+1}(u))h_l'(u)\cdot\nabla T_{l+1}(u)h(u)\varphi dx.$$

For $l > m$, we obtain

$$\lim_{j\to\infty}\lim_{\varepsilon\to 0} I^{3,1}_{\varepsilon,j,l} = 0.$$

Since $F(T_{l+1}(\cdot))h_l(\cdot)$ is uniformly bounded, the a.e. convergence of u_ε and the Vitali Theorem provides that $F(T_{l+1}(u_\varepsilon))h_l(u_\varepsilon) \to F(T_{l+1}(u))h_l(u)$ in $(L^r(\Omega))^N$ for any $r \in [1,\infty)$, thus

$$\lim_{\varepsilon\to 0} I^{3,2}_{\varepsilon,j,l} = \int_\Omega F(T_{l+1}(u))h_l(u)\cdot\nabla(h(v_j)\varphi)dx.$$

As $\nabla(h(v_j)\varphi) \rightharpoonup \nabla(h(u)\varphi)$ in $(L_M(\Omega))^N$ and F is locally Lipschitz continuous, we conclude that

$$\lim_{j\to\infty}\lim_{\varepsilon\to 0} I^{3,2}_{\varepsilon,j,l} = \int_\Omega F(T_{l+1}(u))h_l(u)\cdot\nabla(h(u)\varphi)dx.$$

Again, for $l > m$, we get

$$\lim_{j\to\infty}\lim_{\varepsilon\to 0} I^{3,2}_{\varepsilon,j,l} = \int_\Omega F(u)\cdot\nabla(h(u)\varphi)dx.$$

Applying the Lebesgue Dominated Convergence Theorem, we obtain

$$\lim_{l\to\infty}\lim_{j\to\infty}\lim_{\varepsilon\to 0} I^{4}_{\varepsilon,j,l} = \int_\Omega f h_l(u)h(u)\varphi dx.$$

Therefore, we get

$$\int_\Omega bh(u)\varphi dx + \int_\Omega (a(x,\nabla u)+F(u))\cdot\nabla(h(u)\varphi)dx = \int_\Omega fh(u)\varphi dx$$

for all $h \in C^1_c(\mathbb{R})$ and all $\varphi \in W^1_0 L_M(\Omega)\cap L^\infty(\Omega)$.

6.2.5 Subdifferential argument

Using the same arguments as in [105], we can find that $u \in D(\beta)$ and $b \in \beta(u)$ for almost everywhere in Ω.

With this last step the proof of Theorem 6.2.1 is concluded.

6.2.6 Weak solutions for L^∞-data

Remark 6.2.9. If (u, b) is a renormalized solution of (E, f) such that $u \in L^\infty(\Omega)$, it is a direct consequence of Lemma 6.2.6 that u is in $W_0^1 L_M(\Omega)$ and since (6.6) holds with the formal choice $h \equiv 1$, (u, b) is a weak solution.

Indeed, let us choose $\varphi \in \mathcal{D}(\Omega)$ and plug $h_l(u)\varphi$ as a test function in (6.6). Since $u \in L^\infty(\Omega)$, we can pass to the limit with $l \to \infty$ and find that u solves (E, f) in the sense of distributions.

In the next proposition we will show that renormalized solutions to (E, f) for right-hand side $f \in L^\infty(\Omega)$ are weak solutions.

Proposition 6.2.10. *Let (u, b) be a renormalized solution to (E, f) for $f \in L^\infty(\Omega)$. Then $u \in W_0^1 L_M(\Omega) \cap L^\infty(\Omega)$ and thus, in particular, (u, b) is a weak solution to (E, f).*

Proof. By Lemma 6.2.6 it follows that $u \in W_0^1 L_M(\Omega)$. According to Remark 6.2.9, it suffices to prove that $u \in L^\infty(\Omega)$. We choose $\frac{1}{\varepsilon} T_\varepsilon(u - T_k(u))$ as a test function in the renormalized formulation with $h = h_l$. Neglecting positive terms, using Proposition 1.3.4 and passing to the limit with $l \to \infty$, we get

$$\frac{c}{\varepsilon} \int_{\{k<|u|<k+\varepsilon\}} |\nabla u|^p dx \leq \int_{\{|u|>k\}} |f| dx. \tag{6.28}$$

Inequality (6.28) is exactly the starting point of Lemma L^∞-regularity proof (see [28, Lemma 2.5]), which implies that $u \in L^\infty(\Omega)$. □

6.3 Existence and uniqueness of solutions in the case of $f \in L^1(\Omega)$

6.3.1 Proof of the existence result

6.3.1.1 Approximation problem for L^1-data and a priori estimates

For $f \in L^1(\Omega)$ and $m, n \in \mathbb{N}$ let $f_{m,n} \in L^\infty(\Omega)$ be defined as in the beginning of Section 6.2.

From Theorem 6.2.1, there exist $u_{m,n} \in W_0^1 L_M(\Omega)$, $b_{m,n} \in L^\infty(\Omega)$ such that $(u_{m,n}, b_{m,n})$ is a renormalized solution of $(E, f_{m,n})$. On that account

$$\int_\Omega b_{m,n} h(u_{m,n})\varphi dx + \int_\Omega (a(x, \nabla u_{m,n}) + F(u_{m,n})) \cdot \nabla(h(u_{m,n})\varphi) dx = \int_\Omega f_{m,n} h(u_{m,n})\varphi dx \tag{6.29}$$

holds for all $m, n \in \mathbb{N}$, $h \in C_c^1(\mathbb{R})$ and $\varphi \in W_0^1 L_M(\Omega) \cap L^\infty(\Omega)$.

Lemma 6.3.1. *Suppose $(u_{m,n}, b_{m,n})$ is a renormalized solution of $(E, f_{m,n})$. Then, for $k > 0$, there exists a constant $C(k) > 0$, not depending on $m, n \in \mathbb{N}$, such that*

$$\int_\Omega M\left(\frac{|\nabla T_k(u_{m,n})|}{\lambda}\right)dx \leq C(k), \tag{6.30}$$

$$||\nabla T_k(u_{m,n})||_p \leq C \text{ for some } p > 1, \tag{6.31}$$

$$||b_{m,n}||_{L^1(\Omega)} \leq ||f||_{L^1(\Omega)}. \tag{6.32}$$

Proof. For $l, k > 0$, we plug $h_l(u_{m,n})T_k(u_{m,n})$ as a test function in (6.29). Then (6.30) follow with similar arguments as used in the proof of Lemma 6.2.4. By Proposition 1.3.4, we obtain (6.31). To prove (6.32), we neglect the positive term

$$\int_\Omega a(x, \nabla T_k(u_{m,n})) \cdot \nabla T_k(u_{m,n})dx$$

and keep

$$\int_\Omega b_{m,n}T_k(u_{m,n})dx \leq \int_\Omega f_{m,n}T_k(u_{m,n})dx \leq k||f||_{L^1(\Omega)}$$

since $b_{m,n} \in \beta(u_{m,n})$ a.e. in Ω

$$\int_{\{|u_{m,n}|>k\}} |b_{m,n}|dx \leq ||f||_{L^1(\Omega)},$$

we find (6.32) by passing to the limit with $k \downarrow 0$. □

6.3.1.2 Convergence results

By definition, $f_{m,n}$ is a bi-monotone approximation of f in $L^1(\Omega)$

$$f_{m,n} \leq f_{m+1,n} \text{ and } f_{m,n+1} \leq f_{m,n}.$$

From Remark 6.2.3 it follows that

$$u^\varepsilon_{m,n} \leq u^\varepsilon_{m+1,n} \text{ and } u^\varepsilon_{m,n+1} \leq u^\varepsilon_{m,n} \tag{6.33}$$

a.e. in Ω for any $m, n \in \mathbb{N}$ and all $\varepsilon > 0$, hence passing to the limit with $\varepsilon \downarrow 0$ in (6.33) yields

$$u_{m,n} \leq u_{m+1,n} \text{ and } u_{m,n+1} \leq u_{m,n} \tag{6.34}$$

a.e. in Ω for any $m, n \in \mathbb{N}$. Setting $b^\varepsilon := \beta_\varepsilon(T_{\frac{1}{\varepsilon}}(u_\varepsilon))$, using (6.33), Remark 6.2.3 and the fact that $b^\varepsilon_{m,n} \overset{*}{\rightharpoonup} b_{m,n}$ in $L^\infty(\Omega)$ and this convergence preserves order we obtain

$$b_{m,n} \leq b_{m+1,n} \text{ and } b_{m,n+1} \leq b_{m,n} \tag{6.35}$$

a.e. in Ω for any $m, n \in \mathbb{N}$. By (6.32) and (6.35)

$$b_{m,n} \uparrow_m b^n \downarrow_n b \text{ in } L^1(\Omega),$$

where $b^n, b : \Omega \to \mathbb{R}$ are measurable functions, finite a.e. in Ω. Here, we use the notation $\uparrow_m$ respectively $\downarrow_m$, to denote convergence of a sequence which is monotone increasing, respectively decreasing, in m.

By (6.34), we obtain

$$u_{m,n} \uparrow_m u^n \downarrow_n u \text{ a.e. in } \Omega.$$

where $u^n, u : \Omega \to \overline{\mathbb{R}}$ are measurable functions.

In order to show that u is finite a.e., we will give an estimate on the level sets of $u_{m,n}$ in the next Lemma.

Lemma 6.3.2. *For $m, n \in \mathbb{N}$, let $(u_{m,n}, b_{m,n})$ be a renormalized solution of $(E, f_{m,n})$. Then, there exists a constant $C > 0$, not depending on $m, n \in \mathbb{N}$ such that*

$$|\{|u_{m,n}| \geq h\}| \leq Ch^{-p} \quad \textit{for all } h \geq 1. \tag{6.36}$$

Proof. Using (6.31), we perform the same calculation as in Remark 6.2.5. □

Observe that, as $(u_{m,n})_m$ is pointwise increasing with respect to m,

$$\lim_{m\to\infty} |\{u_{m,n} > h\}| = |\{u^n > h\}| \tag{6.37}$$

and

$$\lim_{m\to\infty} |\{u_{m,n} \leq -h\}| = |\{u^n \leq -h\}|. \tag{6.38}$$

Combining (6.36) with (6.37) and (6.38), we obtain

$$|\{u^n \leq -h\}| + |\{u^n > h\}| \leq Ch^{-p}$$

for all $h \geq 1$, hence u^n is finite a.e. for any $n \in \mathbb{N}$.

By the same arguments, we conclude that

$$|\{u \leq -h\}| + |\{u > h\}| \leq Ch^{-p}.$$

Hence u is finite a.e.

Now, since $b_{m,n} \in \beta(u_{m,n})$ a.e. in Ω, it follows by a subdifferential argument that $b^n \in \beta(u^n)$ and $b \in \beta(u)$ a.e. in Ω.

Remark 6.3.3. If $(u_{m,n}, b_{m,n})$ is a renormalized solution of $(E, f_{m,n})$, using $h_\nu(u_{m,n})T_k(u_{m,n} - T_l(u_{m,n}))$ as a test function in (6.29), neglecting positive terms and passing to the limit with $\nu \to \infty$, we get

$$\int_{\{l<|u_{m,n}|<l+k\}} a(x, \nabla u_{m,n}) \cdot \nabla u_{m,n} dx \leq k \left(\int_{\{|u_{m,n}|>l\}\cap\{|f|\leq\sigma\}} |f| dx + \int_{\{|f|>\sigma\}} |f| dx \right)$$

for any $k, l, \sigma > 0$. Now, applying (6.36), we find that

$$\int_{\{l<|u_{m,n}|<l+k\}} a(x, \nabla u_{m,n}) \cdot \nabla u_{m,n} dx \leq \sigma k C l^{-p} + k \int_{\{|f|>\sigma\}} |f| dx \tag{6.39}$$

holds for any $k, \sigma > 0$, $l \geq 1$ uniformly in $m, n \in \mathbb{N}$.

Lemma 6.3.4. *For $m, n \in \mathbb{N}$, let $(u_{m,n}, b_{m,n})$ be a renormalized solution of $(E, f_{m,n})$. There exists a subsequence $(m(n))_n$ such that setting $f_n := f_{m(n),n}$, $b_n := b_{m(n),n}$ and $u_n := u_{m(n),n}$ we have*

$$u_n \to u \text{ a.e. in } \Omega. \tag{6.40}$$

Moreover, for any $k > 0$,

$$T_k(u_n) \to T_k(u) \text{ in } L^p(\Omega) \text{ for } p \in [1, +\infty) \text{ and a.e. in } \Omega, \tag{6.41}$$

$$\nabla T_k(u_n) \rightharpoonup \nabla T_k(u) \text{ in } (L^p(\Omega))^N, \tag{6.42}$$

$$\nabla T_k(u_n) \rightharpoonup \nabla T_k(u) \text{ in } (L_M(\Omega))^N \text{ for } \sigma(\Pi L_M, \Pi E_{\overline{M}}) \tag{6.43}$$

and

$$a(x, \nabla T_k(u_n)) \rightharpoonup a(x, \nabla T_k(u)) \text{ in } (L_{\overline{M}}(\Omega))^N \text{ for } \sigma(\Pi L_{\overline{M}}, \Pi E_M). \tag{6.44}$$

Proof. Applying the diagonal principle in $L^1(\Omega)$, we construct a subsequence $(m(n))_n$, such that

$$f_n := f_{m(n),n} \to f \text{ in } L^1(\Omega),$$
$$b_n := b_{m(n),n} \to b \text{ in } L^1(\Omega)$$

as $n \to \infty$ in $L^1(\Omega)$ and almost everywhere in Ω. It follows that $u_n := u_{m(n),n} \to u$ a.e. in Ω and (6.41) holds. Then this together with (6.31) provides (6.42).

Combining (6.41) with (6.30) we get $T_k(u) \in L_M(\Omega)$, $T_k(u_n) \to T_k(u)$ in $E_M(\Omega)$ and (6.43) holds for any $k > 0$.

From (6.30) and (6.1) it follows, that, for fixed $k > 0$, given any subsequence of $(a(x, \nabla T_k(u_n)))_n$, there exists a subsequence, still denoted by $(a(x, \nabla T_k(u_n)))_n$, such that

$$a(x, \nabla T_k(u_n)) \rightharpoonup \Phi_k \text{ in } (L_{\overline{M}}(\Omega))^N \text{ for } \sigma(\Pi L_{\overline{M}}, \Pi E_M) \text{ as } n \to \infty.$$

By virtue of Lemma 1.3.3 there exists a sequence $(v_j)_j \subset \mathcal{D}(\Omega)$ such that

$$v_j \to u \text{ in } W_0^1 L_M(\Omega) \text{ for the modular convergence and a.e. in } \Omega.$$

Since $h_l(u_n)(T_k(u_n) - T_k(v_j))$ is an admissible test function in (6.29), (6.44) follows with the same arguments as in the proof of Lemma 6.2.6. □

Remark 6.3.5. With the same arguments as in Remark 6.2.7, we have

$$\lim_{n\to\infty} \int_\Omega (a(x, \nabla T_k(u_n)) - a(x, \nabla T_k(u))) \cdot (\nabla T_k(u_n) - \nabla T_k(u)) dx = 0$$

and

$$\int_{\{l<|u|<l+1\}} a(x, \nabla u) \cdot \nabla u dx \to 0 \text{ as } l \to \infty.$$

6.3.1.3 Conclusion of the proof of Theorem 6.1.5

It is left to prove that (u, b) satisfies

$$\int_\Omega bh(u)\varphi dx + \int_\Omega (a(x, \nabla u) + F(u)) \cdot \nabla(h(u)\varphi) dx = \int_\Omega fh(u)\varphi dx \tag{6.45}$$

for all $h \in C_c^1(\mathbb{R})$ and all $\varphi \in W_0^1 L_M(\Omega) \cap L^\infty(\Omega)$.

By virtue of Lemma 1.3.3 there exists a sequence $(v_j)_j \subset \mathcal{D}(\Omega)$ such that

$$v_j \to u \text{ in } W_0^1 L_M(\Omega) \text{ for the modular convergence and a.e. in } \Omega.$$

Take $h \in C_c^1(\mathbb{R})$ and all $\varphi \in W_0^1 L_M(\Omega) \cap L^\infty(\Omega)$ arbitrary and plug $h_l(u_n)h(v_j)\varphi$ as a test function in (6.29) to get

$$I_{n,j,l}^1 + I_{n,j,l}^2 + I_{n,j,l}^3 = I_{n,j,l}^4 \tag{6.46}$$

where

$$I^1_{n,j,l} = \int_\Omega b_n h_l(u_n) h(v_j)\varphi dx,$$
$$I^2_{n,j,l} = \int_\Omega a(x, \nabla u_n) \cdot \nabla (h_l(u_n) h(v_j)\varphi) dx,$$
$$I^3_{n,j,l} = \int_\Omega F(u_n) \cdot \nabla (h_l(u_n) h(v_j)\varphi) dx,$$
$$I^4_{n,j,l} = \int_\Omega f_n h_l(u_n) h(v_j)\varphi dx.$$

Applying the Lebesgue Dominated Convergence Theorem, we obtain

$$\lim_{l\to\infty}\lim_{j\to\infty}\lim_{n\to\infty} I^1_{n,j,l} = \int_\Omega bh(u)\varphi dx \tag{6.47}$$

and

$$\lim_{l\to\infty}\lim_{j\to\infty}\lim_{n\to\infty} I^4_{n,j,l} = \int_\Omega fh(u)\varphi dx.$$

We write $I^2_{n,j,l} = I^{2,1}_{n,j,l} + I^{2,2}_{n,j,l}$ where

$$I^{2,1}_{n,j,l} = \int_\Omega h_l(u_n) a(x, \nabla u_n) \cdot \nabla (h(v_j)\varphi) dx$$

and

$$I^{2,2}_{n,j,l} = \int_\Omega h_l'(u_n) a(x, \nabla u_n) \cdot \nabla u_n h(v_j)\varphi dx.$$

With similar arguments as in the subsection 6.2.4 it follows that

$$\lim_{l\to\infty}\lim_{j\to\infty}\lim_{n\to\infty} I^{2,1}_{n,j,l} = \int_\Omega a(x, \nabla u) \cdot \nabla (h(u)\varphi) dx. \tag{6.48}$$

By (6.39), we infer

$$|I^{2,2}_{n,j,l}| \leq ||h||_\infty ||\varphi||_\infty \left(C\sigma l^{-p} + \int_{\{|f|>\sigma\}} |f| dx \right).$$

for all $\sigma > 0$ and $l \geq 1$.

Then,

$$\lim_{l\to\infty}\lim_{j\to\infty}\lim_{n\to\infty} |I^{2,2}_{n,j,l}| \leq ||h||_\infty ||\varphi||_\infty \int_{\{|f|>\sigma\}} |f| dx. \tag{6.49}$$

for all $\sigma > 0$.

Next, we write

$$I^3_{n,j,l} = \int_\Omega F(T_{l+1}(u_n))h_l'(u_n)\cdot\nabla T_{l+1}(u_n)h(v_j)\varphi dx \\ + \int_\Omega F(T_{l+1}(u_n))h_l(u_n)\cdot\nabla(h(v_j)\varphi)dx := I^{3,1}_{n,j,l} + I^{3,2}_{n,j,l}.$$

With the same arguments as in the subsection 6.2.4, it follows that

$$\lim_{l\to\infty}\lim_{j\to\infty}\lim_{n\to\infty} I^{3,1}_{n,j,l} = 0 \tag{6.50}$$

and

$$\lim_{l\to\infty}\lim_{j\to\infty}\lim_{n\to\infty} I^{3,2}_{n,j,l} = \int_\Omega F(u)\cdot\nabla(h(u)\varphi)dx. \tag{6.51}$$

Combining (6.46) with (6.47)-(6.51) and passing to the limit with $\sigma\to\infty$, we finally conclude that

$$\int_\Omega bh(u)\varphi dx + \int_\Omega (a(x,\nabla u)+F(u))\cdot\nabla(h(u)\varphi)dx = \int_\Omega fh(u)\varphi dx$$

holds for all $h\in C^1_c(\mathbb{R})$ and all $\varphi\in W^1_0L_M(\Omega)\cap L^\infty(\Omega)$.

Hence (u,b) satisfies (R_1), (R_2) and (R_3). Therefore, the proof of Theorem is completed.

6.3.2 Proof of the uniqueness result

In this subsection, we prove a uniqueness result for renormalized solutions to the problem (E,f) with $f\in L^1(\Omega)$.

Lemma 6.3.6. *For $f,\tilde f\in L^1(\Omega)$ let (u,b) and $(\tilde u,\tilde b)$ be renormalized solutions to (E,f) and $(E,\tilde f)$, respectively. Then*

$$\int_\Omega (b-\tilde b)sign_0^+(u-\tilde u)dx \le \int_\Omega (f-\tilde f)sign_0^+(u-\tilde u)dx. \tag{6.52}$$

Proof. we choose $p^+_\delta(T_{l+1}(u)-T_{l+1}(\tilde u))$ as a test function in the renormalized formulation with $h=h_l$ for (u,b) and for $(\tilde u,\tilde b)$ respectively. Subtracting the resulting equalities, we get

$$I^1_{l,\delta}+I^2_{l,\delta}+I^3_{l,\delta}+I^4_{l,\delta}+I^5_{l,\delta} = I^6_{l,\delta}$$

where

$$I^1_{l,\delta} = \int_\Omega (bh_l(u) - \tilde{b}h_l(\tilde{u}))p^+_\delta(T_{l+1}(u) - T_{l+1}(\tilde{u}))dx,$$
$$I^2_{l,\delta} = \int_\Omega \left(h_l'(u)a(x,\nabla u)\cdot\nabla u - h_l'(\tilde{u})a(x,\nabla\tilde{u})\cdot\nabla\tilde{u}\right) p^+_\delta(T_{l+1}(u) - T_{l+1}(\tilde{u}))dx,$$
$$I^3_{l,\delta} = \frac{1}{\delta}\int_K (h_l(u)a(x,\nabla u) - h_l(\tilde{u})a(x,\nabla\tilde{u}))\cdot\nabla(T_{l+1}(u) - T_{l+1}(\tilde{u}))dx,$$
$$I^4_{l,\delta} = \int_\Omega \left(h_l'(u)F(u)\cdot\nabla u - h_l'(\tilde{u})F(\tilde{u})\cdot\nabla\tilde{u}\right) p^+_\delta(T_{l+1}(u) - T_{l+1}(\tilde{u}))dx,$$
$$I^5_{l,\delta} = \frac{1}{\delta}\int_K (h_l(u)F(u) - h_l(\tilde{u})F(\tilde{u}))\cdot\nabla(T_{l+1}(u) - T_{l+1}(\tilde{u}))dx,$$
$$I^6_{l,\delta} = \int_\Omega (f - \tilde{f})p^+_\delta(T_{l+1}(u) - T_{l+1}(\tilde{u}))dx,$$

with $K = \{0 < T_{l+1}(u) - T_{l+1}(\tilde{u}) < \delta\}$.

Step 1: Passing to the limit with $\delta \downarrow 0$.

By applying the Dominated Convergence Theorem, it is obvious that

$$\lim_{\delta\downarrow 0} I^1_{l,\delta} = \int_\Omega (bh_l(u) - \tilde{b}h_l(\tilde{u}))sign^+_0(T_{l+1}(u) - T_{l+1}(\tilde{u}))dx := I^1_l \tag{6.53}$$

and

$$\lim_{\delta\downarrow 0} I^6_{l,\delta} = \int_\Omega (f - \tilde{f})sign^+_0(T_{l+1}(u) - T_{l+1}(\tilde{u}))dx := I^6_l. \tag{6.54}$$

From the energy dissipation condition (R_3), we have

$$\lim_{l\to\infty}\int_{\{l<|u|<l+1\}} a(x,\nabla u)\cdot\nabla u dx = 0$$

and obviously the counterpart of this holds for $\tilde{u}$. Hence the integrand of $I^2_{l,\delta}$ is bounded in $L^1(\Omega)$. Therefore,

$$\lim_{\delta\downarrow 0} I^2_{l,\delta} = I^2_l, \tag{6.55}$$

where

$$I^2_l = \int_\Omega \left(h_l'(u)a(x,\nabla u)\cdot\nabla u - h_l'(\tilde{u})a(x,\nabla\tilde{u})\cdot\nabla\tilde{u}\right) sign^+_0(T_{l+1}(u) - T_{l+1}(\tilde{u}))dx.$$

Next, we can write $I^3_{l,\delta} = I^{3,1}_{l,\delta} + I^{3,2}_{l,\delta}$ where

$$I^{3,1}_{l,\delta} = \frac{1}{\delta}\int_K (h_l(u) - h_l(\tilde{u}))\, a(x,\nabla T_{l+1}(u))\cdot\nabla(T_{l+1}(u) - T_{l+1}(\tilde{u}))dx$$

and

$$I_{l,\delta}^{3,2} = \frac{1}{\delta}\int_K h_l(\tilde{u})\left(a(x,\nabla T_{l+1}(u)) - a(x,\nabla T_{l+1}(\tilde{u}))\right)\cdot\nabla(T_{l+1}(u) - T_{l+1}(\tilde{u}))dx.$$

By the monotonicity assumption (6.2), $I_{l,\delta}^{3,2}$ is nonnegative. As $\|h_l\|_\infty = 1$, using Young's inequality, we get

$$\lim_{\delta\downarrow 0} I_{l,\delta}^{3,1} = 0.$$

Therefore,

$$\limsup_{\delta\downarrow 0} I_{l,\delta}^{3} \geq 0. \tag{6.56}$$

Now, let us write

$$\begin{aligned}
I_{l,\delta}^{4} &= \int_\Omega div\left(\int_{T_{l+1}(u)}^{T_{l+1}(\tilde{u})} h_l'(\tau)F(\tau)d\tau\right) p_\delta^+(T_{l+1}(u) - T_{l+1}(\tilde{u}))dx \\
&= -\int_\Omega \left(\int_{T_{l+1}(u)}^{T_{l+1}(\tilde{u})} h_l'(\tau)F(\tau)d\tau\right)\cdot\nabla p_\delta^+(T_{l+1}(u) - T_{l+1}(\tilde{u}))dx \\
&= -\frac{1}{\delta}\int_K \left(\int_{T_{l+1}(u)}^{T_{l+1}(\tilde{u})} h_l'(\tau)F(\tau)d\tau\right)\cdot\nabla(T_{l+1}(u) - T_{l+1}(\tilde{u}))dx.
\end{aligned}$$

Hence,

$$|I_{l,\delta}^{4}| \leq \max_{s\in[-l-1,l+1]}|F(s)|\int_K |\nabla(T_{l+1}(u) - T_{l+1}(\tilde{u}))|dx.$$

It follows that

$$\lim_{\delta\downarrow 0} I_{l,\delta}^{4} = 0. \tag{6.57}$$

Concerning $I_{l,\delta}^{5}$, we split it as the sum $I_{l,\delta}^{5} = I_{l,\delta}^{5,1} + I_{l,\delta}^{5,2}$ where

$$I_{l,\delta}^{5,1} = \frac{1}{\delta}\int_K h_l(u)\left(F(T_{l+1}(u)) - F(T_{l+1}(\tilde{u}))\right)\cdot\nabla(T_{l+1}(u) - T_{l+1}(\tilde{u}))dx$$

and

$$I_{l,\delta}^{5,2} = \frac{1}{\delta}\int_K \left(h_l(u) - h_l(\tilde{u})\right)F(\tilde{u})\cdot\nabla(T_{l+1}(u) - T_{l+1}(\tilde{u}))dx.$$

Let $L_F > 0$ be the Lipschitz constant of F. Then,

$$|I_{l,\delta}^{5,1}| \leq L_F\int_K |\nabla(T_{l+1}(u) - T_{l+1}(\tilde{u}))|dx$$

and

$$|I_{l,\delta}^{5,2}| \leq \max_{s\in[-l-1,l+1]}|F(s)|\int_K |\nabla(T_{l+1}(u) - T_{l+1}(\tilde{u}))|dx.$$

As the domain of integration shrink, it follows that

$$\lim_{\delta \downarrow 0} I^5_{l,\delta} = 0. \tag{6.58}$$

Step 2: Passing to the limit with $l \to \infty$.

Combining (6.53) with (6.54)-(6.58), we get

$$I^1_l + I^2_l \leq I^6_l, \text{ for all } l \geq 1.$$

We immediately notice that (R_3) implies that

$$\lim_{l \to \infty} I^2_l = 0.$$

Finally, since $sign^+_0(T_{l+1}(u) - T_{l+1}(\tilde{u})) \to sign^+_0(u - \tilde{u})$ as $l \to \infty$ a.e. in Ω and weakly-$*$ in $L^\infty(\Omega)$, we can pass to the limit with $l \to \infty$ and find (6.52).

□

Remark 6.3.7. Assuming $f = \tilde{f}$, from Lemma 6.3.6, we obtain

$$\int_\Omega (b - \tilde{b}) sign^+_0(u - \tilde{u}) dx \leq 0.$$

If β is a strictly increasing, continuous function, we arrive at $u = \tilde{u}$ a.e. in Ω. Now, we write the renormalized formulation for (u, b) and $(\tilde{u}, \tilde{b})$ respectively. Subtracting the resulting equalities, we get

$$\int_\Omega (b - \tilde{b}) h(u) \varphi dx = 0$$

for all $h \in C^1_c(\mathbb{R})$ and all $\varphi \in W^1_0 L_M(\Omega) \cap L^\infty(\Omega)$. Choosing $h(u) = h_l(u)$ and passing to the limit with $l \to \infty$, we find $b = \tilde{b}$ a.e. in Ω.

Conclusion and further perspectives

In this thesis, we focused our attention with the results of existence (uniqueness and regularity) to a class of nonlinear elliptic inclusion problems in divergence form within the non-variational framework in different settings, we have been able to prove the following results:

- We have established the existence of weak solutions for the strongly nonlinear elliptic problem (S, f) where $f \in L^1(\Omega)$ using a classic theorem of J. L. Lions [86] on operators of the pseudo-monotone type and via strong convergence of the truncations.

- We have established the existence and regularity results for the nonlinear elliptic inclusion problem (P, f) with degenerate coercivity for any maximal monotone graph β. Notably, if $f \in L^\infty(\Omega)$, we proved the existence and uniqueness of a bounded weak energy solution for the problem (P, f). While if $f \in L^1(\Omega)$, we established the existence of the so-called entropy solution for the problem (P, f) only for $0 \leq \theta(p-1) < \frac{N(p-1)}{N-1} - 1$. Moreover, we showed that $u \in W_0^{1,q}(\Omega)$, with $q \in \left(1, \frac{N(p-1)(1-\theta)}{N-1-\theta(p-1)}\right)$.

- We have investigated the existence and regularity of weak or entropy solutions to problem (U, f) involving unbounded coefficients by cutting the nonlinearity near the critical value and applying the technique of approximation.

- We have established a partial generalization of results from [6] to the variable exponents case. More precisely, if $f \in L^\infty(\Omega)$, we proved the existence and uniqueness of a bounded weak energy solution for the problem (P^*, f). While if $f \in L^1(\Omega)$, we established the existence of an entropy solution for the problem (P^*, f) only for $2 - \frac{1}{N} < p^- \leq p^+ < N$ and $0 \leq \theta(p^+ - 1) < \frac{N(p^- - 1)}{N-1} - 1$. Furthermore, we showed that $u \in W_0^{1,q(x)}(\Omega)$, with $q(x) \in \left(1, \frac{(p(x)-1-\theta(p^+-1))N}{N-1-\theta(p^+-1)}\right)$.

- The last results established are the existence and uniqueness of renormalized solutions to the nonlinear multivalued elliptic problem (E, f) in the setting of Orlicz-Sobolev spaces.

This work raises a number of questions that researchers can explore in future studies:

1. The main point in the proof of the existence results shows that the hypotheses: Ω is a bounded domain in $\mathbb{R}^N$, Dirichlet boundary conditions and f is belonging at least to $L^1(\Omega)$ allowed us to prove the almost everywhere convergence of the gradients of the solutions of the approximated problems. What may happens when Ω is a unbounded domain? is it possible to deal with general nonlinear boundary conditions of the form $a(x, \nabla u) \cdot \eta + \gamma(u) \ni g$? and what could occur when f is a measure?

2. The problems (S, f) considered in chapter 2, are studied under sign condition (2.4) on the nonlinear lower order term g. We are convinced that it possible to achieve some existence results for problems (S, f) without imposing the sign condition.

3. The problems (E, f) considered in chapter 6, are studied under $\overline{M}$ satisfies the Δ_2 condition. What may happens without imposing this condition?

4. At present, there are just few studies on nonlinear parabolic inclusion problems with principal part having degenerate coercivity. It seems that there were difficulties related essentially to the multivalued mapping β and to the dealing with this kind of degeneracy. Then, future work can be devoted to the generalization of the result of problems type (P, f) to the parabolic equation

$$(\mathcal{P}_t, f, b_0) \quad \begin{cases} \beta(u)_t - \operatorname{div}\left(\dfrac{a(x, \nabla u)}{(1+|u|)^{\theta(p-1)}}\right) \ni f & \text{in } (0,T)\times\Omega, \\ u = 0 & \text{on } (0,T)\times\partial\Omega, \\ \beta(u(0,\cdot)) \ni b_0 & \text{in } \Omega. \end{cases}$$

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