جامعة سيدي محمد بن عبد الله بفاس
ⵜⴰⵙⴷⴰⵡⵉⵜ ⵙⵉⴷⵉ ⵎⵓⵃⴰⵎⵎⴷ ⵏ ⵄⴰⴱⴷⵍⵍⴰⵀ ⴳ ⴼⴰⵙ
UNIVERSITÉ SIDI MOHAMED BEN ABDELLAH DE FES

كلية العلوم ظهر المهراز فاس
ⵜⴰⵖⵡⵍⵎⵜ ⵏ ⵜⵣⵔⴼⵜ ⵎⴰⵀⵔⴰⵣ ⴼⴰⵙ
FACULTE DES SCIENCES DHAR EL MAHRAZ FES

N° d'ordre : 31/2024

Centre des Etudes : ***Sciences et Techniques et Sciences Médicales***

Formation Doctorale : ***Mathématiques et Applications (MA)***

Discipline : ***Mathématiques Appliquées***

Spécialité : ***Probabilités et Statistique***

Laboratoire : ***LPAIS***

THÈSE DE DOCTORAT

Mathematical Analysis and Threshold Refinement of some Stochastic Epidemic Models with White Noises and Lévy Jumps

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Année Universitaire : 2023-2024

Summary

Mathematical epidemiology is a field based on the use of mathematical models and tools to understand and analyze the dynamics of infectious illnesses' dissemination in human populations. In clearer terms, it allows the formulation of equations that can describe the transmission and spread of epidemics and then gives a vision of suitable strategies to adopt in order to mitigate them. Generally, the use of mathematical models in epidemiology helps to enhance our ability to interpret empirical data, make informed predictions, and guide public health policies. Nevertheless, a growing number of experimental observations confirms the significant impact of external fluctuations on infectious disease dynamics, which makes their inclusion in the modeling and analysis phases a very good added value. For this reason, the adoption of probabilistic models is a highly favorable choice, if not one of the best, for studying communicable diseases. Indeed, the incorporation of stochasticity allows for a more realistic representation of the infectious disease spread, acknowledging the impact of unexpected events and demographic changes. However, analyzing this kind of model demands advanced analytical techniques owing to their complexity. In addressing this challenge, the current thesis is dedicated to exploring the propagation dynamics of some epidemic models influenced by continuous natural environmental noises and sudden unforeseen phenomena. Within an appropriate theoretical framework, we introduce and cultivate many novel and non-standard approaches to derive some asymptotic properties of perturbed models. These properties encompass stability, persistence in the mean, disease extinction, and long-term behavior in general. Each chapter concludes with numerical simulations validating the theoretical findings and vividly illustrating the precision of the obtained conditions and thresholds.

Broadly speaking, this thesis enhances various existing studies within the realm of stochastic mathematical epidemiology, offering alternative methodologies to address and analyze more intricate epidemiological and biological models.

Keywords: Asymptotic properties; Brownian Motion; Disease extinction; Environmental disturbances; Epidemics; Infectious diseases; Lévy jumps; Persistence in the mean; Stochastic systems.

Résumé

L'épidémiologie mathématique est un domaine basé sur l'utilisation de modèles et d'outils mathématiques pour comprendre et analyser la dynamique de propagation des maladies infectieuses au sein des populations humaines. En termes plus clairs, il permet de formuler des équations pouvant décrire la transmission et la propagation des épidémies et donne ensuite une vision des stratégies appropriées à adopter pour les atténuer. D'une manière générale, l'utilisation de modèles mathématiques en épidémiologie augmente notre capacité à interpréter les données empiriques, à faire des prédictions éclairées et à orienter les politiques de santé publique. Néanmoins, un nombre croissant d'observations expérimentales confirment l'impact significatif des fluctuations externes sur la dynamique des maladies infectieuses, ce qui fait de leur inclusion dans les phases de modélisation et d'analyse une très bonne valeur ajoutée. Pour cette raison, l'adoption de modèles probabilistes constitue un très bon choix, voire l'un des meilleurs pour étudier les maladies transmissibles. En effet, l'intégration de la stochasticité permet une représentation plus réaliste de la propagation des maladies infectieuses, en reconnaissant l'impact des événements inattendus et des changements démographiques. Cependant, l'analyse de ce type de modèle nécessite des techniques analytiques avancées en raison de leur complexité. Pour relever ce défi, cette thèse se consacre à l'exploration de la dynamique de propagation de quelques modèles épidémiques influencés par des bruits environnementaux naturels continus et des phénomènes soudains et imprévus. Dans un cadre théorique approprié, nous introduisons de nombreuses approches nouvelles et non standard pour dériver certaines propriétés asymptotiques des modèles perturbés. Ces propriétés englobent la stabilité, la persistance en moyenne, l'extinction de la maladie et le comportement à long terme en général. Chaque chapitre se termine par des simulations numériques validant les résultats théoriques et illustrant d'une manière claire la précision des conditions et des seuils obtenus.

D'une manière générale, cette thèse améliore diverses études existantes dans le domaine de l'épidémiologie mathématique stochastique, en proposant des méthodologies alternatives pour aborder et analyser des modèles épidémiologiques et biologiques plus compliqués.

Mots-clés: Epidémies; Extinction de la maladie; Maladies infectieuses; Mouvement Brownian; Persistance en moyenne; Perturbations environnementales; Propriétés asymptotiques; Sauts de Lévy; Systèmes stochastiques.

ملخص

علم الأوبئة الرياضي هو مجال يعتمد على استخدام النماذج والأدوات الرياضية لفهم وتحليل ديناميكيات انتشار الأمراض المعدية بين افراد ساكنة بشرية معينة. وبعبارات أكثر وضوحا، فهو يسمح بصياغة معادلات يمكنها وصف انتقال الأوبئة وانتشارها ومن ثم تقديم نظرة ثاقبة للاستراتيجيات المناسبة التي يجب اعتمادها للتخفيف من حدتها. بشكل عام، يساعد استخدام النماذج الرياضية في علم الأوبئة على تحسين قدرتنا على تفسير البيانات التجريبية، وإجراء تنبؤات مستنيرة، وتوجيه سياسات الصحة العامة. ومع ذلك، يؤكد عدد متزايد من الملاحظات التجريبية التأثير الكبير للتقلبات الخارجية على ديناميات الأمراض المعدية، مما يجعل إدراجها في مرحلتي النمذجة والتحليل قيمة مضافة جيدة للغاية. ولهذا السبب، فإن اعتماد النماذج الاحتمالية يشكل خيارا جيدا جدا، إن لم يكن واحدا من أفضل الخيارات لدراسة الأمراض المعدية. والواقع أن اعتبار العشوائية يسمح بتمثيل أكثر واقعية لانتشار الأمراض المعدية، كونها تعترف بتأثير الأحداث غير المتوقعة والتغيرات الديموغرافية. ومع ذلك، فإن تحليل هذا النوع من النماذج يتطلب تقنيات تحليلية متقدمة بسبب تعقيدها. ولمواجهة هذا التحدي، تم تخصيص هذه الأطروحة لاستكشاف ديناميكيات انتشار بعض النماذج الوبائية المتأثرة بالضوضاء البيئية المستمرة والظواهر المفاجئة والغير المتوقعة. بالبقاء ضمن إطار نظري مناسب، نقدم العديد من الأساليب الجديدة وغير الاعتيادية لاستخلاص خصائص مقاربة معينة للنماذج المضطربة. وتشمل هذه الخصائص الاستقرار، والثبات في المتوسط، وانقراض المرض، والسلوك طويل المدى بشكل عام. يختتم كل فصل من الاطروحة بمحاكاة عددية تدعم صحة النتائج النظرية وتظهر بوضوح دقة الشروط والعتبات التي تم التوصل إليها. بشكل عام، تعمل هذه الأطروحة على تجويدالدراسات المختلفة الموجودة في مجال علم الأوبئة الرياضي العشوائي من خلال اقتراح منهجيات بديلة تسمح بتحليل نماذج وبائية و بيولوجية أكثر تعقيدا و تركيبا.

الكلمات المفتاحية: الأمراض المعدية ؛ الأوبئة ؛ الاضطرابات البيئية ؛ الأنظمة العشوائية ؛ حركة براونية ؛ قفزات ليفي ؛ خصائص مقاربة ؛ الثبات في المتوسط ؛ انقراض المرض.

Acknowledgements

After five years of dedicated learning and research, I pen these acknowledgments today, marking the end of my thesis journey. This period has been marked by intense intellectual growth, not only in the scientific realm but also on a personal level. The process of writing this thesis has profoundly influenced me, and it is crucial to express gratitude to those who supported me throughout this transformative time.

First and foremost, my gratitude extends to God, who bestowed upon me the strength to believe in myself and pursue my aspirations. I am indebted to my parents for their unwavering love, wise counsel, and continuous support throughout my life. A heartfelt appreciation goes to my advisor, Professor Driss Kiouach, for his consistent guidance and valuable advice. Working under his direction has been an incredibly fulfilling experience.

I extend sincere thanks to all committee members for their meticulous review of my work and for providing professional and impartial feedback. Finally, I wish to express my gratitude to my brother and friend, Mr.Ibrahim El-fedany, for his outstanding collaboration and unwavering motivation.

Thank you all so much!

Table of contents

Summary . . . i
Summary (French version) . . . ii
Summary (Arabic version) . . . ii
Acknowledgements . . . iv
Table of contents . . . iv
List of abbreviations . . . vi
List of figures . . . vii
List of tables . . . viii

General Introduction **1**

I Mathematical Preliminaries **3**
1 Introduction . . . 3
2 Random processes: Some features and properties . . . 3
3 Filtrations: An understanding of sequential information . . . 5
4 Insight into the concept of stopping times . . . 7
5 Essentials of Continuous-time Martingales . . . 8
6 The Essence of Randomness: Brownian Motion in Focus . . . 10
7 Stochastic integration: Itô integral . . . 11
8 Itô's formula . . . 14
9 Key Concepts in Stochastic Differential Equations (SDE) . . . 15
10 Overview of Lévy Processes Fundamentals . . . 18
10.1 Poisson random measure and Lévy processes . . . 18
10.2 The Itô-Lévy formula . . . 20
10.3 Itô-Lévy stochastic differential equations . . . 21
11 Conclusion . . . 22

II Advanced and comprehensive research on the dynamics of COVID-19 under mass communication outlets intervention and quarantine strategy: a deterministic and probabilistic approach **23**
1 Introduction and model formulation . . . 23
2 Analysis of the deterministic Coronavirus model . . . 27
2.1 Well-posedness . . . 28
2.2 Invariant, absorbing and attracting regions . . . 29
2.3 The Basic reproduction number $\mathcal{R}_0$ . . . 31
2.4 Stability analysis of the disease-free steady state $\mathcal{E}^o$ . . . 33
2.5 Existence of endemic equilibrium and uniform persistence . . . 36
3 Analysis of the stochastic Coronavirus model . . . 37
3.1 Existence and uniqueness of the global positive solution . . . 37
3.2 Stochastically ultimate boundedness and permanence . . . 40

3.3 Extinction of Stochastic COVID-19 model . . . 42
3.4 Persistence in the mean of COVID-19 . . . 46
4 Numerical simulation examples . . . 48
5 Conclusion and discussion . . . 55

III A novel mathematical analysis and threshold reinforcement of a stochastic dengue epidemic model with Lévy jumps 56
1 Introduction and model formulation . . . 56
2 Existence, uniqueness and positivity of the global-in-time solution . . . 60
3 Extinction of the dengue disease . . . 63
4 Persistence in the mean of the dengue disease . . . 69
5 Numerical simulation examples . . . 72
5.1 The case of dengue fever extinction . . . 73
5.2 The case of dengue fever persistence in the mean . . . 74
5.3 The effect of Lévy jumps on dengue fever dynamics . . . 76
6 Conclusion and discussion . . . 77

IV Asymptotic behavior analysis and threshold sharpening of a staged progression AIDS/HIV epidemic model with Lévy jumps 79
1 Introduction and model formulation . . . 79
2 Existence, uniqueness and positivity of the global-in-time solution . . . 85
3 Extinction of AIDS. . . . 87
4 Persistence in the mean of the AIDS disease . . . 95
5 Numerical simulation . . . 100
5.1 AIDS extinction case . . . 100
5.2 Persistence in the mean case . . . 101
5.3 The effect of Lévy jumps on AIDS dynamics . . . 101
6 Conclusion and discussion . . . 108

V An improvement of the extinction sufficient conditions for a higher-order stochastically disturbed AIDS/HIV model 109
1 Introduction . . . 109
2 A new extinction theorem for the AIDS model (V.2) . . . 111
3 Comparison between our new results and the ones existing in the recent literature . . . 117
4 Numerical simulations . . . 120
5 Conclusion . . . 121

VI The impact of Lévy noise on the threshold dynamics of a stochastic SVIR epidemic model with general incidence functions 122
1 Introduction and model formulation . . . 122
2 Existence and unicity of the positive global-in-time solution . . . 125
3 Extinction, persistence in the mean and weak persistence . . . 126
4 Some examples and numerical simulations . . . 135
4.1 The case of bilinear incidence rates . . . 136
4.2 The case of saturated incidence rates . . . 138
4.3 The case of The Beddington-DeAngelis incidence rates . . . 140
5 Conclusion and discussion . . . 142

Conclusions and outlook 143

Index x

Bibliography xxiv

List of abbreviations

- SIR : Susceptible-Infected-Removed.
- SVIR : Susceptible-Vaccinated-Infected-Removed.
- AIDS :Acquired Immune Deficiency Syndrome.
- HIV : Human Immunodeficiency Virus.
- COVID-19 : Corona Virus Disease 2019.
- SARS : Severe Acute Respiratory Syndrome.
- WHO : World Health Organization.
- ODE : Ordinary Differential Equation.
- SDE : Stochastic Differential Equation.
- SDE-LJ : Stochastic Differential Equations with Lévy Jumps.
- cadlag : Continu à Droite, Limite à Gauche[a].
- HAART : Highly Active Antiretroviral Therapy.
- a.s. : almost surely.
- a.e. : almost everywhere.
- SBM : Standard Brownian Motion.

[a] French abbreviation signifying right-continuous with left limits.

List of Figures

II.1 Schematic diagram of the proposed COVID-19 epidemic model. 26
II.2 Solutions of COVID-19 deterministic model (II.1) taking $\beta_1 = 3.97 \times 10^{-6}$, $\beta_2 = 0.6\times\beta_1$ and $p = 0.6201$ ($\mathcal{R}_0 = 0.9180 < 1$). 50
II.3 Trajectories of COVID-19 deterministic model (II.1) taking $\beta_1 = 5 \times 10^{-6}$, $\beta_2 = 0.6\times\beta_1$ and $p = 0.6201$ ($\mathcal{R}_0 = 1.1562 > 1$). 51
II.4 Trajectories of COVID-19 stochastic model (II.2) taking $\beta_1 = 2.08 \times 10^{-9}$, $\beta_2 = 0.6\times\beta_1$ and $p = 0.6201$. 52
II.5 Trajectories of COVID-19 stochastic model (II.2) taking $\beta_1 = 4.1 \times 10^{-3}$, $\beta_2 = 0.1\times\beta_1$ and $p = 0.6201$ ($\rho_1(\widehat{\alpha}) = 1.0266 > 0.9694 = \rho_2$) 53
II.6 The impact of the quarantine parameters λ and q on the deterministic and stochastic trajectories of the total infected individuals $I_{\text{total}}(t) := E(t) + A(t) + I(t)$. The rest of the parameters is taken respectively as in Figure II.3 and Figure II.5. 54
II.7 The effect of the awareness rate β_2 on the deterministic trajectories of the total infected individuals $I_{\text{total}}(t) := E(t)+A(t)+I(t)$ when parameters are taken respectively as in Figure II.2 and Figure II.3. 54
II.8 Stochastic paths of the total infected individuals $I_{\text{total}}(t) := E(t) + A(t) + I(t)$ under different response intensities β_2. The other parameters are taken respectively as in Figure II.4 and Figure II.5. 54

III.1 Schematic diagram of the proposed dengue compartmental model. 60
III.2 Solutions' paths of the dengue fever model (III.5) when the numerical values are taken as shown in the third column of Table III.1 ($\mathcal{K} = -0.2044 < 0$). 75
III.3 Solutions' paths of the dengue fever model (III.5) when the numerical values are chosen as shown in the fourth column of Table III.1 ($\widetilde{\mathcal{R}}_0 = 1.0862 > 1$). 76
III.4 The paths of $I_{\text{total}}(t) = I(t)+\widehat{I}(t)$ for the deterministic model (III.29), the model (III.28) with Brownian motions, and the stochastic system with Lévy jumps (III.5). 77
III.5 The paths of $I_{\text{total}}(t) = I(t) + \widehat{I}(t)$ associated respectively to the models (III.29),(III.28) and (III.5). 77

IV.1 Flow diagram of the AIDS compartmental model (IV.1). 82
IV.2 Solutions' paths of the AIDS model (IV.5) when the numerical values are taken as shown in the third column of Table IV.2 ($\mathcal{T} = -0.1953 < 0$). In this case, the curves show the AIDS extinction phenomenon. 105
IV.3 Solutions' paths of the AIDS model (IV.5) when the numerical values are taken as shown in the fourth column of Table IV.2 ($\mathcal{R}_0^{\text{i}} = 1.0769 > 1$). In this case, the curves show the AIDS persistence phenomenon. 106
IV.4 The stochastic trajectory of $I_1(t) + T_1(t) + I_2(t) + T_2(t) + I_3(t)$ when $\mathcal{R}_0^{\text{i}} = 1.0769 > 1$. . . . 107
IV.5 The paths of $\left(\sum_{i=1}^{3} I_i(s) + \sum_{i=1}^{2} T_i(s)\right)$ for the deterministic model (IV.1), the model (IV.5) with Brownian motions, and the SDE-LJ (IV.4). 107

IV.6 The paths of $\left(\sum_{i=1}^{3} I_i(s) + \sum_{i=1}^{2} T_i(s)\right)$ associated respectively to the models (IV.1),(IV.5) and (IV.4). 107

V.1 Flow diagram of the AIDS compartmental model (V.2). 110
V.2 Solutions' paths of the AIDS model (V.2) when the numerical values are taken as shown in Table V.1. 121

VI.1 Flowchart of the proposed SVIR compartmental model. 124
VI.2 Solutions' paths of the SVIR model (VI.3) when the numerical values are taken as shown in the third column of Table VI.1. 137
VI.3 Solutions' paths of the SVIR model (VI.3) when the numerical values are taken as shown in the last column of Table VI.1. 137
VI.4 Solutions' paths of the SVIR model (VI.3) when the numerical values are taken as shown in the third column of Table VI.1. 139
VI.5 Solutions' paths of the SVIR model (VI.3) when the numerical values are taken as shown in the forth column of Table VI.1. 139
VI.6 Solutions' paths of the SVIR model (VI.3) when the numerical values are taken as shown in the third column of Table VI.1. 141
VI.7 Solutions' paths of the SVIR model (VI.3) when the numerical values are taken as shown in the forth column of Table VI.1. 141

List of Tables

II.1 Definitions and values (per day) of COVID-19 model parameters used in the simulation. 48

III.1 Definitions and nominal values, per day, of the system parameters and perturbations intensities used in the simulations. 72
III.2 Some quantities and their corresponding numerical values in the case of $p = 2.5 > 2$. 73
III.3 The corresponding values of some quantities in the case of $p = 2.5 > 2$. 74

IV.1 Description of the different classes constituting the total host population in the case of the AIDS deterministic model presented in [159]. 80
IV.2 Definitions and nominal values, per day, of the system parameters and perturbations intensities used in the simulations. 102
IV.3 Some quantities and their corresponding numerical values in the case of $q = 2.1 > 2$. . 103
IV.4 The corresponding values of some quantities in the case of $q = 2.1 > 2$. 104

V.1 Descriptions and numerical values of the system parameters used in the simulations. 120

VI.1 Descriptions and nominal values of the various system parameters and disturbance intensities taken in the numerical simulations. 135
VI.2 Some quantities and their associated numerical values in both cases, extinction and persistence ($k = 2.01 > 2$). 135

General Introduction

Throughout history, humanity has grappled with a multitude of highly perilous epidemics, distinguished by swift propagation and elevated mortality rates. Often instigated by unfamiliar bacteria or viruses of their time, these epidemics claimed millions of lives, leaving an indelible mark on the histories of numerous countries, societies, and Humanity as a whole. Delving into the past reveals numerous examples, including smallpox, tuberculosis, cholera, plague, the 1918 Spanish flu, and more recently Ebola, Zika virus, HIV, and the well-famed COVID-19. The substantial mortality rates during epidemics have far-reaching consequences for life expectancy and the economic systems of affected countries [1]. A notable illustration is the Coronavirus pandemic, which has triggered a profound international crisis, causing more than two million deaths and leading to global health and economic challenges in addition to shifts in lifestyles [2]. In such contexts and similar epidemics, the challenge lies essentially in containing the spread by understanding its dynamics. To this end, mathematical modeling emerges as a fitting tool, serving as an efficient method to address and navigate such crises and scenarios [3]. This adoption of mathematics to comprehend the dissemination dynamics of transmissible diseases gave rise to a new branch of biomathematics named mathematical epidemiology. However, the idea of having recourse to mathematics in the frame of epidemic propagation description and analysis is not recent, dating back to Daniel Bernoulli in 1760, who analyzed the impact of Smallpox on life expectancy using mathematical lifetime charts [4]. His study advocated for the benefits of inoculation against the epidemic. Despite many attempts to describe epidemic impacts, the nonlinear dynamics of transmission were not well understood until the previous century. In 1906, W. H. Hamer observed that a decreasing density of susceptible individuals could lead to the extinction of measles, proposing the first epidemiological system for disease modeling. Similarly, in 1911, S. R. Ross used mathematical modeling to assess intervention approaches against malaria. In the late twenties of the last century a public health doctor, W.O. Kermack, and a biochemist, A.G. McKendrick, introduced a simple model for the transmission of epidemics through direct contact. They validated their model by comparing it to real data from the Bombay plague outbreak of 1905-1906 [5]. The significance of their theory came to the forefront during the successful Smallpox eradication campaign in the 1970s. As the twentieth century drew to a close, mathematical epidemiology gained prominence as a vital tool in shaping public health policies. Since its emergence, the model of Kermack and McKendrick has served as a foundational framework, guiding researchers in understanding and formulating mathematical representations of the transmission dynamics of epidemics. Indeed, several contributions have been devoted to modeling various epidemics, with a predominant reliance on their model and a consideration of diverse additional assumptions. These latter are often tailored to the unique characteristics of each specific epidemic with the aim of having a more realistic view [6, 7].

The first wave of epidemiological models was based on deterministic formulations and addressed questions such as the potential exacerbation of the epidemic's impact across a broad population spectrum, the extent of individuals affected, and the effects of intervention strategies on the transmission of the disease. Over time, these standard models have been refined and diversified to better align with real-world dynamics [8]. Some improvements included considerations of irregular interpersonal contact and sudden environmental changes. This led to the development of stochastic

epidemiological models integrating these hypotheses [8]. Deterministic and stochastic epidemiological systems play an important role, but this thesis focuses principally on perturbed epidemiological models. In reality, epidemiological models consistently face perturbations from environmental noises inducing irregular and random fluctuations in parameters like birth rate, transmission rate, and carrying capacity which affects the model to varying degrees. Consequently, numerous studies have integrated white noise into deterministic formulations to illustrate the impacts of environmental fluctuations on population dynamics [9, 10, 11]. These systems are stochastic epidemic models with only white noise, resulting in continuous solutions. However, abrupt environmental shocks (hurricanes, earthquakes,...etc.) can disrupt this continuity. Hence, employing stochastic differential equations with jumps becomes essential to study the dynamic behavior of the epidemic model. In essence, jump times are inherently random, and the expected time of jumps aligns with Lévy jumps. Consequently, these stochastic perturbations can be effectively approximated and simulated using Lévy jumps.

This thesis strives to introduce innovative methods for enhancing the analysis of probabilistic epidemiological models. It puts forth several generalizations, techniques, and approaches to anticipate and manage the dynamics of infectious diseases within a well-defined assumptional framework. Consequently, this thesis serves as a good foundation for prospective studies in the field of stochastic mathematical epidemiology. In essence, this work is structured into six main chapters, each contributing to the comprehensive understanding of epidemiological dynamics. The first chapter serves as a concise introduction, providing a brief overview of crucial stochastic calculus concepts essential for the subsequent discussions. Emphasis is placed on presenting only the key definitions and results needed to comprehend the main findings, as a comprehensive theory is readily available in various references. In Chapter II, our focus shifts to the SQEAIHR (Susceptible-Quarantined-Exposed-Asymptomatically infective- Infected-Hospitalized-Recovered) epidemic model, a robust framework for describing the dynamics of COVID-19 under the influence of quarantine and media coverage strategies. Both deterministic and stochastic forms of the model are presented, followed by a meticulous mathematical analysis aimed at unraveling the dissemination behavior of COVID-19. Chapter III introduces an SIR-SI mathematical compartmental model, shedding light on the dissemination of dengue disease under continuous environmental disturbances and unexpected external perturbations. Employing a system of stochastic differential equations with proportional white noises and Lévy jumps, a thorough mathematical analysis is conducted to gain valuable insights into the propagation behavior of dengue fever. Moving to Chapter IV, our exploration extends to a generalized stochastic AIDS/HIV epidemic model that incorporates both Brownian motions and Lévy jumps. This model, structured as a staged progression compartmental system represented by an Itô-Lévy stochastic differential equations system, undergoes a sophisticated mathematical analysis to reveal insights into the long-run behavior of AIDS propagation, namely: extinction and persistence in the mean. Chapter V revisits the extinction conditions of the higher-order perturbed multi-stage AIDS/HIV model proposed by Liu and Jiang in [12]. Utilizing new analytical techniques, an extinction theorem stronger than the one presented in [12] is introduced. Notably, our theorem requires a weaker hypothetical framework, enhancing its robustness. The final chapter (Chapter VI) explores a stochastic SVIR epidemic model with general incidence rates, perturbed by both Brownian motions and Lévy jumps. The chapter establishes the well-posedness of the model, ensuring a unique positive and global in-time solution. Building upon some assumptions and nonstandard analytic techniques, we derive sufficient and almost necessary conditions for extinction, persistence in the mean, and weak persistence.

It is important to highlight that the studies discussed in the aforementioned chapters not only enhance numerous existing works in the field of mathematical epidemiology but also provide innovative techniques for the treatment and analysis of many other classes of epidemic models. In addition, each chapter from Chapters II to VI includes findings that have been previously published in reputable specialized journals with proper indexing (see [13, 14, 15, 16, 17]).

Chapter I

Mathematical Preliminaries

Abstract

In Biomathematics, the formulation and study of biological models incorporating stochasticity are essentially based on probability theory and stochastic analysis, the thing that made them key tools in the framework of stochastic epidemiology. For this reason, we will present in this chapter a brief overview of the stochastic calculus concepts that will be evoked in this thesis. We have attempted to compile only the definitions and results necessary to make the main findings of the thesis understandable, rather than providing a comprehensive theory, which can be found easily in many references (see for example [18, 19, 20, 21, 22, 23, 24]).

1 Introduction

In recent decades, the field of biomathematics, and more precisely mathematical epidemiology, has undergone profound changes through the introduction and inclusion of the concept of randomness, which remains an inherent character of any natural phenomenon such as the spread of infectious illnesses. This desire, or we can say even curiosity, to understand the effect of randomness has given rise to a new type of epidemic models called stochastic models. The latter are more difficult and complicated to study, requiring more sophisticated analytical tools often found in a field called stochastic calculus. In this chapter, we set off on a trip in this field, which has become crucial to understand the complexities of stochastic biological models. Roughly speaking, stochastic calculus provides a theoretical vision and a mathematical foundation for conceptualizing the dynamics of random occurrences and their effect on biological processes. However, this chapter is not meant to be an exhaustive treatise on the subject; rather, it just seeks to give a concise and accessible introduction, focusing on the key concepts and aspects that will be discussed and used later in our thesis. In other words, this chapter is guided by a philosophy of precision and accuracy; we present only the essential definitions and results, sparing the reader from the intricacies of the detailed theory that can easily be found in other references like [18, 23, 19, 20, 21, 22]. Our aim is to provide the necessary foundations, allowing a comprehensive understanding of the main conclusions and ideas that will be presented later.

2 Random processes: Some features and properties

In practice and in a few words, the notion of the random or stochastic process aims to model in a mathematical way the random and irregular events that can occur during a determined temporal interval. As usual, we introduce mathematically the aspect of randomness and unpredictability by the consideration of a measurable space $(\Omega, \mathcal{F})$, where Ω is a sample description set and $\mathcal{F}$ is a σ-field on Ω. In essence, a stochastic process is a collection, denoted $\{X_t\}_{t\in I}$, of random variables

taking values in a determined measurable space $(S, \mathcal{S})$. The latter is usually called state space, and in the remainder of this thesis we take it always as $(\mathbb{R}^n, \mathcal{B}(\mathbb{R}^n))$, where $n \in \mathbb{N} = \{1, 2 \cdots\}$ and $\mathcal{B}(\mathbb{R}^n)$ is the family of all Borel subsets of $\mathbb{R}^n$. The index set is often denoted I, and for the context of this thesis, we consider continuous-time stochastic processes, with I being the non-negative real numbers, $I = [0, \infty)$ [a].

For any fixed sample point $\omega \in \Omega$, the function $t \mapsto \mathcal{X}_t(\omega) \in \mathbb{R}^n$ (for $t \geqslant 0$) represents the trajectory of the process $\mathcal{X} = \{\mathcal{X}_t(\omega)\}_{t \geqslant 0}$ associated with that specific ω. To simplify notation, we often represent a stochastic process $\{\mathcal{X}_t(\omega)\}$ as $\mathcal{X}_t$. With this foundation in mind, we introduce the following definitions:

Definition 2.1. *Let $\{\mathcal{X}_t(\omega)\}_{t \geqslant 0}$ be an $\mathbb{R}^n$-valued stochastic process defined on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$. This process is said to be*

- ***Increasing*** *if $\mathcal{X}_s \leqslant \mathcal{X}_t$ for any $s \leqslant t$. In other terms, a stochastic process is increasing if almost all its paths are increasing.*
- ***Integrable*** *if, for all $t \geqslant 0$, $\mathcal{X}_t$ is integrable. Moreover, it is square-integrable if $\mathbb{E}|\mathcal{X}_t|^2 < \infty$ for every $t \geqslant 0$.*
- ***Measurable*** *if the stochastic process regarded as a function of two variables (t, ω) from $\mathbb{R}_+ \times \Omega$ to $\mathbb{R}^n$ is measurable with respect to the σ-algebra $\mathcal{B}(\mathbb{R}_+) \otimes \mathcal{F}$ [b], where $\mathcal{B}(\mathbb{R}_+)$ is the family of all Borel subsets of $\mathbb{R}_+$.*
- ***Gaussian*** *if all its finite-dimensional distributions[c] are normal.*
- ***Continuous*** *if for almost all $\omega \in \Omega$, the function $t \mapsto \mathcal{X}_t(\omega)$ is continuous on $t \geqslant 0$. Hence, the process $\mathcal{X}$ is continuous if almost all its paths are continuous. Furthermore, it is right-continuous (resp. left continuous) if $t \mapsto \mathcal{X}_t(\omega)$ is right-continuous (resp. left continuous) for almost $\omega \in \Omega$ and every $t \geqslant 0$.*
- ***Cadlag***[d] *if it is right-continuous and for almost all $\omega \in \Omega$, the left limit $\lim_{\substack{s \to t \\ s<t}} \mathcal{X}_s(\omega)$ exists finitely for any $t > 0$.*

When dealing with two stochastic processes,$\{\mathcal{X}_t(\omega)\}_{t \geqslant 0}$ and $\{\mathcal{Y}_t(\omega)\}_{t \geqslant 0}$ defined on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$, both of which have values in $\mathbb{R}^n$, we encounter two distinct concepts of equality:

Definition 2.2 ((see [25])). *Let $\{\mathcal{X}_t(\omega)\}_{t \geqslant 0}$ and $\{\mathcal{Y}_t(\omega)\}_{t \geqslant 0}$ be two $\mathbb{R}^n$-valued stochastic processes defined on a specified probability space $(\Omega, \mathcal{F}, \mathbb{P})$.*

- *$\{\mathcal{X}_t(\omega)\}_{t \geqslant 0}$ is a **version** or **modification** of $\{\mathcal{Y}_t(\omega)\}_{t \geqslant 0}$ if, for every $t \geqslant 0$, we have $\mathcal{X}_t = \mathcal{Y}_t$ almost surely (a.s. for short). In other terms, for each time $t \geqslant 0$ we have*

$$\mathbb{P}\big(\{\omega \in \Omega |\ \mathcal{X}_t(\omega) = \mathcal{Y}_t(\omega)\}\big) = 1.$$

- *$\{\mathcal{X}_t(\omega)\}_{t \geqslant 0}$ and $\{\mathcal{Y}_t(\omega)\}_{t \geqslant 0}$ are said to be stochastically **indistinguishable** or **equivalent** if they have the same sample paths almost surely. In other words, the set $\{\omega \in \Omega |\ \mathcal{X}_t(\omega) \neq \mathcal{Y}_t(\omega) \forall t \geqslant 0\}$ is $\mathbb{P}-$null .*

[a] In general, the indexing set I can be discrete or continuous, but in this thesis we will focus exclusively on continuous-time stochastic processes indexed by $[0, \infty)$.

[b] Given two measurable spaces $(X, \mathcal{A})$ and $(Y, \mathcal{B})$, where $\mathcal{A}$ and $\mathcal{B}$ are the respective σ-algebras, the product σ-field on $X \times Y$ is defined as $\mathcal{A} \otimes \mathcal{B} := \sigma(\{S_1 \times S_2 :\ S_1 \in \mathcal{A}_1,\ S_2 \in \mathcal{B}\})$.

[c] In the context of a stochastic process $\{\mathcal{X}_t(\omega)\}_{t \geqslant 0}$, which is defined on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$, we refer to the collective or joint distribution of a random vector $(\mathcal{X}(t_1), \cdots, \mathcal{X}(t_p))$ as a finite-dimensional distribution of the process $\mathcal{X}$, where $p \geqslant 1$ and $0 \leq t_1 < \cdots < t_p$.

[d] The term "cadlag," which might appear unclear, is a French abbreviation for "continu à droite, limite à gauche," essentially signifying right-continuous with left limits.

The fact that two indistinguishable stochastic processes are versions of each other is always true, but the converse is generally false. However, by adding certain conditions, this converse can indeed occur, as stated by the following theorem

Theorem 2.1. *Let $X = \{\mathcal{X}_t(\omega)\}_{t\geqslant 0}$ and $Y = \{\mathcal{Y}_t(\omega)\}_{t\geqslant 0}$ be two right-continuous (or left-continuous) stochastic processes such that X is a version of Y. Then, they are indistinguishable.*

3 Filtrations: An understanding of sequential information

In the theory of stochastic calculus, one often encounters nested collections of measured subsets, called filtrations. The latter are used to represent the information accessible at precise and particular moments in time and they therefore play an important role in the formalization of random processes.

Definition 3.1. *Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space, and $(I, \leqslant)$ a totally ordered set [a]. A filtration is a family $\{\mathcal{F}_t\}_{t\in I}$ of sub-σ-fields that verifies $\mathcal{F}_{t_1} \subset \mathcal{F}_{t_2}$ [b] for any $t_1 \leqslant t_2$, where $(t_1, t_2) \in I^2$. A probability space $(\Omega, \mathcal{F}, \mathbb{P})$ endowed with a filtration $\{\mathcal{F}_t\}_{t\in I}$ is called a* ***filtered probability space*** *, and it is often denoted by $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\in I}, \mathbb{P})$.*

Example 3.1. *Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space, and $\mathcal{X} = (\mathcal{X}_t)_{t\geqslant 0}$ a stochastic process on it. Then the family $\{\mathcal{F}_t^{\mathcal{X}}\}_{t\geqslant 0}$ defined by*

$$\mathcal{F}_t^{\mathcal{X}} \triangleq \sigma(\mathcal{X}_s \mid 0 \leqslant s \leqslant t) \triangleq \sigma\left\{\mathcal{X}_s^{-1}(D) \mid 0 \leqslant s \leqslant t,\ D \in \mathcal{B}(\mathbb{R}^n)\right\}$$

is a filtration (of $\mathcal{F}$) known as the canonical, natural or generated filtration associated with the process $(\mathcal{X}_t)_{t\geqslant 0}$.

Remark 3.1. *The σ-field $\mathcal{F}_t^{\mathcal{X}}$ can be viewed as the smallest one in which the measurability of $\mathcal{X}_s$ holds for every $s \in [0, t]$. In other terms, $\mathcal{F}_t^{\mathcal{X}}$ contains all the information that can be garnered from observing the trajectories of $\mathcal{X}$ over the period $[0, t]$. So, one can say that if an event E belongs to $\mathcal{F}_t^{\mathcal{X}}$, a tracker of $\mathcal{X}$ has knowledge concerning the occurrence or not of E. In a similar vein of thought, one can say that the available information over all time $t \in [0, +\infty)$, is encapsulated in the σ-algebra $\mathcal{F}_\infty^{\mathcal{X}}$ defined as $\mathcal{F}_\infty^{\mathcal{X}} \triangleq \sigma\big(\bigcup_{t\geq 0} \mathcal{F}_t^{\mathcal{X}}\big)$.*

Definition 3.2. *Let $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geqslant 0}, \mathbb{P})$ be a filtered probability space, the filtration $\{\mathcal{F}_t\}_{t\geqslant 0}$ is said to be right-continuous if $\mathcal{F}_t = \mathcal{F}_{t^+}$ for any $t \in [0, +\infty)$, where $\mathcal{F}_{t^+} \triangleq \bigcap_{s>t} \mathcal{F}_s = \bigcap_{\varepsilon\in\mathbb{R}_+} \mathcal{F}_{t+\varepsilon}$. If in addition, this filtration is $\mathbb{P}$-complete (or $\mathbb{P}$-augmented) in the sense that the first σ–field $\mathcal{F}_0$ contains all the $\mathbb{P}$-null sets of $\mathcal{F}$ [c], then we say that the filtration verifies the usual conditions.*

Remark 3.2. *The σ–field $\mathcal{F}_{t^+}$ represents somehow the information available when one examines a very small increment into the future. Accordingly, the right-continuity of a filtration is interpreted by saying that the available information in each instant t permits one to get an infinitesimal insight into the future. Regarding the completeness of a filtration, it implies that if an event is impossible, then we will be aware of its impossibility from the start, i.e. at time $t = 0$.*

Definition 3.3. *Let $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geqslant 0}, \mathbb{P})$ be a filtered probability space. The family $\{\mathcal{F}_{t^+}\}_{t\geqslant 0}$ is a filtration called the right-continuous version of $\{\mathcal{F}_t\}_{t\geqslant 0}$, and by construction, it is always right-continuous (see [23]).*

Definition 3.4. *Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a complete probability space [d], and $\{\mathcal{F}_t\}_{t\geqslant 0}$ a given filtration on it. The $\mathbb{P}$-augmentation (or the $\mathbb{P}$-completion) of $\{\mathcal{F}_t\}_{t\geqslant 0}$ is a new filtration denoted $\{\widehat{\mathcal{F}}_t\}_{t\geqslant 0}$, and defined by*

$$\widehat{\mathcal{F}}_t := \sigma\left(\mathcal{F}_t \cup N_{\mathbb{P}}\right), \quad \forall t \geqslant 0.$$

Here, $N_{\mathbb{P}}$ is referring to the collection of all $\mathbb{P}$–negligible sets.

[a] In practice, the set I is often taken as $\mathbb{N}$ or $[0, +\infty)$. In the rest of this thesis, we take $I = [0, +\infty)$.

[b] The inclusion $\mathcal{F}_{t_1} \subset \mathcal{F}_{t_2}$ means that $\mathcal{F}_{t_2}$ holds more information about the process than does $\mathcal{F}_{t_1}$, which means that the implication $t_1 \leqslant t_2 \Rightarrow \mathcal{F}_{t_1} \subset \mathcal{F}_{t_2},\ \forall (t_1, t_2) \in I^2$ reflects an accumulation of information over time.

[c] Let $B \subset \Omega$, we say that B is a $\mathbb{P}$-null set if there is some measurable set $\hat{B} \in \mathcal{F}$ such that $B \subset \hat{B}$ and $\mathbb{P}(\hat{B}) = 0$.

[d] A complete probability space is one in which any $\mathbb{P}$–null set is measurable.

Remark 3.3. *Needless to say, the filtration $\{\widehat{\mathcal{F}}_t\}_{t\geqslant 0}$ is $\mathbb{P}$-complete by construction.*

Remark 3.4. *Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a complete probability space, endowed with a filtration $\{\mathcal{F}_t\}_{t\geqslant 0}$. In practice, it is common to consider that the last filtration verifies the usual conditions, but when this is not the case, the solution is to replace it with the right-continuous version of its augmentation, i.e., $\{\widehat{\mathcal{F}}_{t^+}\}_{t\geqslant 0}$. For a stochastic process $\mathcal{X} = (\mathcal{X}_t)_{t\geqslant 0}$, the natural filtration's right-continuity does not hold in general, even if this process is right-continuous*[a]*. Therefore, it is rational to supplant the natural filtration $\{\mathcal{F}_t^{\mathcal{X}}\}_{t\geqslant 0}$ by $\{\widehat{\mathcal{F}}_{t^+}^{\mathcal{X}}\}_{t\geqslant 0}$, we mention that any process $\{\mathcal{X}_t\}_{t\geqslant 0}$ is always $\{\widehat{\mathcal{F}}_{t^+}^{\mathcal{X}}\}_{t\geqslant 0}-$adapted.*

In order to grasp the stochastic processes dynamic character and its relation with a given filtration $\{\mathcal{F}_t\}_{t\geqslant 0}$, it is necessary to address certain fundamental concepts like non-anticipation, progressive measurability, and predictability. For this reason, we introduce the following definitions:

Definition 3.5. *Let $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geqslant 0}, \mathbb{P})$ be a given filtered probability space. A stochastic process $\{\mathcal{X}_t\}_{t\geqslant 0}$ is said to be $\{\mathcal{F}_t\}_{t\geqslant 0}-$**adapted** or unanticipated if for each $t \geqslant 0$, $\mathcal{F}_t^{\mathcal{X}} \triangleq \sigma(\mathcal{X}_s \mid 0 \leqslant s \leqslant t) \subset \mathcal{F}_t$. In other words, $\{\mathcal{X}_t\}_{t\geqslant 0}$ is $\{\mathcal{F}_t\}_{t\geqslant 0}-$adapted if and only if $\mathcal{X}_t$ is $\mathcal{F}_t-$measurable for all $t \geqslant 0$.*

Remark 3.5. *In plainer language, an $\{\mathcal{F}_t\}_{t\geqslant 0}-$adapted process is one whose value at any date $t \geqslant 0$ is disclosed by the information carried by $\mathcal{F}_t$, i.e., for any t, $\{\mathcal{X}_t\}_{t\geqslant 0}$ is known at time t when the information loaded by $\{\mathcal{F}_t\}_{t\geqslant 0}$ is available and known. Needless to say, any stochastic process is adapted with respect to its canonical filtration.*

Definition 3.6. *Let $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geqslant 0}, \mathbb{P})$ be a given filtered probability space. A stochastic process $\{\mathcal{X}_t\}_{t\geqslant 0}$ is said to be $\{\mathcal{F}_t\}_{t\geqslant 0}$-**progressively measurable** or just $\{\mathcal{F}_t\}_{t\geqslant 0}$-progressive if the following function:*

$$\begin{array}{rccc} \mathcal{X}: & \left([0,T]\times\Omega\ ;\ \mathcal{B}([0,T])^{[b]}\otimes\mathcal{F}_T\right) & \longrightarrow & \left(\mathbb{R}^n\ ;\ \mathcal{B}(\mathbb{R}^n)\right) \\ & (t,\omega) & \longmapsto & \mathcal{X}(t,\omega). \end{array}$$

is measurable for every $T \geqslant 0$.

Proposition 3.1. *Let $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geqslant 0}, \mathbb{P})$ be a given filtered probability space. If a stochastic process $\{\mathcal{X}_t\}_{t\geqslant 0}$ is $\{\mathcal{F}_t\}_{t\geqslant 0}$-progressively measurable then it is adapted and measurable at the same time. The converse is not generally valid*[c]*.*

To make the transition from the property of adaptiveness to that of progressive measurability, we must add some conditions relating to trajectorial regularity, and this is exactly what is the subject of the next property's statement.

Proposition 3.2. *Let $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geqslant 0}, \mathbb{P})$ be a given filtered probability space. If a stochastic process $\{\mathcal{X}_t\}_{t\geqslant 0}$ is $\{\mathcal{F}_t\}_{t\geqslant 0}-$adapted and right-continuous (Or left-continuous) then it is $\{\mathcal{F}_t\}_{t\geqslant 0}-$progressively measurable.*

Remark 3.6. *Any left- or right-continuous process (even if it is not adapted) is measurable.*

In the case of discrete time, the notion of predictability is somehow concrete and easy to conceive, because quite simply, we say that a process $\mathcal{X} = \{\mathcal{X}_k\}_{k\in\mathbb{N}}$ is $\{\mathcal{F}_k\}_{k\in\mathbb{N}}$-predictable if $\mathcal{X}_k$ is $\mathcal{F}_{k-1}$-measurable for all $k \in \mathbb{N}_{\geqslant 2}$[d]. In other terms, If $\mathcal{F}_k$ represents the accumulation of our information over time, then we can glimpse $\mathcal{X}$ ahead by one step. But in the case of a continuous-time process, the definition is a little less trivial, and it can be given as follows:

Definition 3.7. *Let $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geqslant 0}, \mathbb{P})$ be a given filtered probability space, and let $\mathfrak{P}$ denotes the smallest σ-algebra on $\mathbb{R}_+ \times \Omega$ with respect to which every left continuous $\{\mathcal{F}_t\}_{t\geqslant 0}-$adapted process is measurable. A continuous-time stochastic process $\{\mathcal{X}_t\}_{t\geqslant 0}$ is said to be $\{\mathcal{F}_t\}_{t\geqslant 0}-$**predictable**[e] if the process regarded as a function of (t,ω) is $\mathfrak{P}$-measurable.*

[a] However, it is well known that the completion of the natural filtration of a Lévy process, and in particular Brownian motion, is always right-continuous.

[b] Here, $\mathcal{B}([0,T])$ refers to the $\sigma-$algebra containing all the Borel subsets of $[0,T]$.

[c] However, any adapted and measurable process admits a progressive modification.

[d] We mention that any predictable process is adapted, but not vice versa.

[e] The following implications hold: [Left-continuous + $\{\mathcal{F}_t\}_{t\geqslant 0}-$adapted] $\Rightarrow$ $\{\mathcal{F}_t\}_{t\geqslant 0}-$predictable $\Rightarrow$ $\{\mathcal{F}_t\}_{t\geqslant 0}-$progressive.

4 Insight into the concept of stopping times

When studying and tracking a given stochastic process, we are always interested in the time at which it presents a particular and distinguished comportment. For this reason, the notion of stopping times is important and must thus be discussed. For the rest of this section, unless otherwise stated, $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geq 0}, \mathbb{P})$ refers to a filtered probability space (without any additional property).

Definition 4.1. *An $\{\mathcal{F}_t\}_{t\geqslant 0}$-**stopping time** is a random variable $\mathcal{T} : \Omega \to [0,\infty]$[a] that verifies $\{\mathcal{T} \leqslant t\} \triangleq \{\omega \in \Omega | \mathcal{T}(\omega) \leqslant t\} \in \mathcal{F}_t$ for every $t \geqslant 0$.*

Remark 4.1. *From now on in this thesis, the standard convention $\inf \emptyset = \infty$ is adopted.*

Theorem 4.1. *Let $\mathcal{X} = \{\mathcal{X}_t\}_{t\geqslant 0}$ be an $\mathbb{R}^n$-valued cadlag and $\{\mathcal{F}_t\}_{t\geqslant 0}$-adapted process. Let also $\mathcal{U} \subset \mathbb{R}^n$ be an open set. Then, $\mathcal{T}_e^{\mathcal{U}} = \inf\{t \geqslant 0 | \mathcal{X}_t \notin \mathcal{U}\}$ is an $\{\mathcal{F}_t\}_{t\geqslant 0}$-stopping time called the first **exiting time** from $\mathcal{U}$. If in addition, the filtration $\{\mathcal{F}_t\}_{t\geqslant 0}$ is right-continuous, then $\mathcal{T}_h^{\mathcal{U}} = \inf\{t \geqslant 0 | \mathcal{X}_t \in \mathcal{U}\}$ is also an $\{\mathcal{F}_t\}_{t\geqslant 0}$-stopping time named the first **hitting time** of $\mathcal{U}$* [b].

Remark 4.2. *If we keep the same notations and hypotheses as in the last theorem, then for any $\{\mathcal{F}_t\}_{t\geqslant 0}$−stopping time τ, the infimum $\mathcal{T}_{e,\tau}^{\mathcal{U}} = \inf\{t \geqslant \tau | \mathcal{X}_t \notin \mathcal{U}\}$ is equally an $\{\mathcal{F}_t\}_{t\geqslant 0}$-stopping time*[c]*. Needless to say, τ can be chosen as a non-negative constant without any problem, since any random time equal to a non-negative constant is a stopping time.*

Now, we will turn to provide a brief overview of some classic properties associated with stopping times.

Lemma 4.1. *Let η be an $\{\mathcal{F}_t\}_{t\geqslant 0}$−stopping time, the set $\mathcal{F}_\eta$ defined by*

$$\mathcal{F}_\eta \triangleq \big\{S \in \mathcal{F} \mid S \cap \{\omega \in \Omega | \eta(\omega) \leqslant t\} \in \mathcal{F}_t,\ \forall t \geqslant 0\big\},$$

is a sub-σ-field of $\mathcal{F}$ and η is always $\mathcal{F}_\eta$-measurable. Moreover, for any $\{\mathcal{F}_t\}$-stopping times η_1, η_2, if $\eta_1 \leqslant \eta_2$ a.s, then $\mathcal{F}_{\eta_1} \subseteq \mathcal{F}_{\eta_2}$.

Proposition 4.1. *Let $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geq 0}, \mathbb{P})$ be a given filtered probability space*

- *If η_1 and η_2 are two $\{\mathcal{F}_t\}_{t\geq 0}$−stopping times, then so are $\eta_1 \vee \eta_2 \triangleq \max(\eta_1, \eta_2)$, $\eta_1 \wedge \eta_2 \triangleq \min(\eta_1, \eta_2)$ and $\eta_1 + \eta_2$.*
- *Let $(\eta_n)_{n\in\mathbb{N}}$ be a sequence of $\{\mathcal{F}_t\}_{t\geqslant 0}$− stopping times, then $\sup_{n\in\mathbb{N}} \eta_n$ is also an $\{\mathcal{F}_t\}_{t\geqslant 0}$− stopping time.*
- *Let $(\theta_n)_{n\in\mathbb{N}}$ be a sequence of stopping times. If the filtration $\{\mathcal{F}_t\}_{t\geqslant 0}$ is right-continuous, then $\liminf_{n\to\infty} \eta_n$ and $\limsup_{n\to\infty} \eta_n$ are in turn stopping times.*

We bring this section to a close by presenting the following useful theorem:

Theorem 4.2. *If η is a finite $\{\mathcal{F}_t\}_{t\geqslant 0}$-stopping time*[d]*, and $(X_t)_{t\geqslant 0}$ is an $\{\mathcal{F}_t\}_{t\geqslant 0}$-progressively measurable process, then X_η is $\mathcal{F}_\eta$-measurable.*

[a] For a stopping time, infinity is a possible value.

[b] See [18, page 7] and [19, page 11].

[c] $\mathcal{T}_{e,\tau}^{\mathcal{U}}$ is referring to the first exiting time from $\mathcal{U}$ after τ.

[d] For a stopping time τ (or a random variable generally), we say that it is bounded (almost surely) if there is some $K \in \mathbb{R}_+$ such that $\mathbb{P}[|\tau| \leqslant K] = 1$, while τ is finite (almost surely) if $\mathbb{P}[|\tau| < \infty] = 1$.

5 Essentials of Continuous-time Martingales

In this section, we will briefly recall the concept of continuous-time martingale which is a fundamental notion in probability theory and stochastic calculus. Generally speaking, it is a specific type of random process that has certain properties and is widely used in many fields, including mathematical epidemiology. In clearer and more precise terms, the continuous-time martingale is defined as follows:

Definition 5.1. *Let $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geqslant 0}, \mathbb{P})$ be a specified filtered probability space, an $\{\mathcal{F}_t\}$-adapted **integrable** process $\mathrm{M} = \{\mathrm{M}_t\}_{t\geqslant 0}$ is called a **continuous-time martingale** with respect to $\{\mathcal{F}_t\}_{t\geqslant 0}$ if*

$$\mathbb{E}(\mathrm{M}_t|\mathcal{F}_s) = \mathrm{M}_s \ a.s. \ \ \forall s \in [0,t) \subsetneq [0,\infty). \tag{I.1}$$

Remark 5.1. *Needless to say, the martingale M has constant expectation: $\forall t \geqslant 0$, $\mathbb{E}(\mathrm{M}_t) = \mathbb{E}(\mathrm{M}_0)$.*

Remark 5.2. *If "=" is substituted by "$\geqslant$" (resp. "$\leqslant$") in (I.1), then M will be called a submartingale (resp. Supermartingale).*

In general, it is preferable, if not crucial, to work with good versions[a] of stochastic processes as they showcase notable characteristics and tend to be more useful[b]. For this reason, we present in the case of martingales the following theorem:

Theorem 5.1. *Let $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geqslant 0}, \mathbb{P})$ be a filtered probability space. If the filtration $\{\mathcal{F}_t\}_{t\geqslant 0}$ is right-continuous, then every $\{\mathcal{F}_t\}_{t\geqslant 0}$-martingale possesses a cadlag version.*

Remark 5.3. *By virtue of this last theorem, we can always consider that a martingale M is cadlag, while bearing in mind that if it is not the case, then we will simply use its cadlag version.*

Remark 5.4. ***For the rest of this thesis, unless otherwise stated,** $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geq 0}, \mathbb{P})$ **refers to a filtered probability space with a filtration that verifies the usual conditions (see Definition 3.2).***

Before proceeding to the remaining content in this section, it is important to mention the famous Doob martingale stopping theorem (see [26]), which is stated as follows:

Theorem 5.2. *Let $\mathrm{M} = \{\mathrm{M}_t\}_{t\geqslant 0}$ be an $\{\mathcal{F}_t\}_{t\geqslant 0}$-martingale, and let τ_1, τ_2 be two bounded $\{\mathcal{F}_t\}_{t\geqslant 0}$-stopping times. Then*

$$\mathbb{E}(\mathrm{M}_{\tau_1}|\mathcal{F}_{\tau_2}) = \mathbb{E}(\mathrm{M}_{\tau_2}|\mathcal{F}_{\tau_1}) = \mathrm{M}_{\min(\tau_1,\tau_2)} \quad a.s.$$

Remark 5.5. *According to the above theorem, If η is a bounded $\{\mathcal{F}_t\}_{t\geqslant 0}$-stopping time, the equality $\mathbb{E}\left(\mathrm{M}_{\eta\wedge t} \mid \mathcal{F}_s\right) = \mathrm{M}_{\eta\wedge s}$ holds for all $0 \leqslant s < t < \infty$. This implies that the stopped process $\mathrm{M}^\tau = (\mathrm{M}_{\tau\wedge t})_{t\geqslant 0}$ is also an $\{\mathcal{F}_t\}_{t\geqslant 0}$-martingale (The $\{\mathcal{F}_t\}$-adaptability of M^τ is guaranteed by Theorem 4.2).*

To develop a good understanding of the martingales' significance, it is important to explore some of their prominent classes. One of these latter is that of real-valued, continuous square-integrable martingales. Among the several properties exhibited by the elements of this important class, we find the following one:

Proposition 5.1. *Let $\mathrm{M} = \{\mathrm{M}_t\}_{t\geqslant 0}$ be a real-valued $\{\mathcal{F}_t\}_{t\geqslant 0}$-martingale. if M is square-integrable and continuous, then, there is a unique continuous $\{\mathcal{F}_t\}_{t\geqslant 0}$-adapted, integrable and increasing process denoted by $\langle \mathrm{M}, \mathrm{M}\rangle = \{\langle \mathrm{M}, \mathrm{M}\rangle_t\}_{t\geqslant 0}$ such that $\{\mathrm{M}_t^2 - \langle \mathrm{M}, \mathrm{M}\rangle_t\}_{t\geqslant 0}$ is a continuous $\{\mathcal{F}_t\}_{t\geqslant 0}$-martingale vanishing at $t = 0$.*

Remark 5.6. *The process $\{\langle \mathrm{M}, \mathrm{M}\rangle_t\}_{t\geqslant 0}$ is called the quadratic variation of M, and for any finite stopping time η, we have $\mathbb{E}(\mathrm{M}_\eta^2) = \mathbb{E}\left(\langle \mathrm{M}, \mathrm{M}\rangle_\eta\right)$.*

[a] Like Left-or Right- continuous ones.

[b] For example, one of the benefits of choosing left or right-continuous versions is that it guarantees the measurability and the progressive measurability (see Remark 3.6).

Next, let us consider another square-integrable continuous martingale $\hat{M} = \{\hat{M}_t\}_{t\geqslant 0}$. The joint quadratic variation of M and $\hat{M}$ is the stochastic process given by

$$\langle M, \hat{M}\rangle_t = \frac{1}{2}\left(\langle M+\hat{M}, M+\hat{M}\rangle_t - \langle M, M\rangle_t - \langle \hat{M}, \hat{M}\rangle_t\right) = \frac{1}{4}\left(\langle M+\hat{M}, M+\hat{M}\rangle_t - \langle M-\hat{M}, M-\hat{M}\rangle_t\right).$$

This latter is the unique continuous, integrable, adapted process of finite variation[a], ensuring that $\{M_t\hat{M}_t - \langle M, \hat{M}\rangle_t\}_{t\geqslant 0}$ is a continuous martingale vanishing at $t = 0$. In addition, for any finite stopping time τ, $\mathbb{E}[M_\tau \hat{M}_\tau] = \mathbb{E}[\langle M, \hat{M}\rangle_\tau]$. Moving beyond martingales, we now focus on the concept of local martingale, a more generalized notion that enhances our understanding of the nuanced behavior of stochastic processes in continuous time.

Definition 5.2. *Let $M = \{M_t\}_{t\geqslant 0}$ be an $\{\mathcal{F}_t\}_{t\geqslant 0}$-adapted right-continuous process. We say that M is a **local martingale** if the following two conditions hold:*

- *There is a non-decreasing sequence of stopping times $(\eta_n)_{n\in\mathbb{N}}$ such that $\eta_n \xrightarrow[n\to\infty]{} \infty$ a.s.*
- *For each $n \in \mathbb{N}$, the stopped process $M^{\eta_n} := \{M_{\eta_n\wedge t}\}_{t\geqslant 0}$ is a martingale.*

Remark 5.7. *Although every martingale is a local martingale (as per Theorem 5.2), the reverse is not always true[b]. This is the case with what we call strict local martingales (i.e., local martingales that are not martingales). However, if we have a local martingale, we can add some conditions to ensure that it's a martingale. For example, a bounded continuous local martingale is a martingale (see [27] for more details).*

Proposition 5.2. *If M is a continuous $\{\mathcal{F}_t\}_{t\geqslant 0}$-local martingale of finite variation then $M \equiv M_0$ a.s.*

Not surprisingly, and akin to martingales, when we have two $\{\mathcal{F}_t\}_{t\geqslant 0}$-continuous square-integrable local martingales, denoted as L and N, their joint variation is defined as the unique continuous $\{\mathcal{F}_t\}_{t\geqslant 0}$-adapted process of finite variation such that $\{L_tN_t - \langle L_t, N_t\rangle_t\}_{t\geqslant 0}$ is a continuous local $\{\mathcal{F}_t\}_{t\geqslant 0}$-martingale vanishing at $t = 0$ [28, 19]. Now, one can introduce the following useful strong law of large numbers Theorem (see [19]).

Theorem 5.3. *Let L be a continuous local martingale such that $L_0 = 0$ a.s. Then*

$$\lim_{t\to\infty}\langle L, L\rangle_t = \infty \ \ a.s. \quad\Rightarrow\quad \lim_{t\to\infty}\frac{L_t}{\langle L, L\rangle_t} = 0 \ \ a.s.$$

On top of that,

$$\limsup_{t\to\infty}\frac{\langle L, L\rangle_t}{t} < \infty \ \ a.s. \quad\Rightarrow\quad \lim_{t\to\infty}\frac{L_t}{t} = 0 \ \ a.s.$$

We conclude this section by recalling the well-known Doob's theorem (see [29, 19]).

Theorem 5.4. *Let $\{M_t\}_{t\geqslant 0}$ be a right continuous $\mathbb{R}^n$-valued martingale. Let $[a, b] \subset \mathbb{R}_+$ be a bounded interval and $k \in \mathbb{R}_+$.*

- *If $k \geqslant 1$ and $M_t \in L^k(\Omega; \mathbb{R}^n)$, then*

$$\mathbb{P}\left\{\omega \middle| \sup_{t\in[a,b]} |M_t(\omega)| \geqslant c\right\} \leqslant \frac{\mathbb{E}|M_b|^k}{c^k}.$$

  holds for each $c > 0$.

- *If $k > 1$ and $M_t \in L^k(\Omega; \mathbb{R}^n)$, then[c]*

$$\mathbb{E}\left(\sup_{t\in[a,b]} |M_t|^k\right) \leqslant \left(\frac{k}{k-1}\right)^k \mathbb{E}|M_b|^k.$$

[a] A real-valued stochastic process is said to be of finite variation if it can be written as the difference of two non-decreasing stochastic processes (see [19]).

[b] Indeed, for a local martingale, the expectation can be distorted by large values of small probability.

[c] This inequality is sometimes called Doob's maximal inequality

6 The Essence of Randomness: Brownian Motion in Focus

In this section, we take a look at the concept of Brownian motion , an influential model historically used to represent disordered motion, from the erratic movements of microscopic particles in physics to the volatility of stock prices in finance. For a general introduction, see [19], and for an in-depth exploration of its main properties, go to [20]. Situated at the heart of stochastic processes, Brownian motion provides a versatile and widely applicable framework for mathematical characterization, making it an indispensable tool in theoretical and practical contexts.

Definition 6.1. *Let $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geq 0}, \mathbb{P})$ be a filtered probability space (without any additional property). A unidimensional $\{\mathcal{F}_t\}$-**standard Brownian motion** ($\{\mathcal{F}_t\}$-SBM, for short) is an $\{\mathcal{F}_t\}$-adapted and continuous process $\mathcal{B} \equiv \{\mathcal{B}_t\}_{t\geqslant 0}$ that verifies the following three properties:*

1. $\mathbb{P}(\mathcal{B}_0 = 0) = 1$.
2. *For $0 \leqslant s < t < \infty$, $\mathcal{B}_t - \mathcal{B}_s$ is normally distributed with mean zero and variance $t - s$. In other words, $\mathcal{B}_t - \mathcal{B}_s \sim \sqrt{t-s}\, N(0,1)$.*
3. *For $0 \leqslant s < t < \infty$, the increment $\mathcal{B}_t - \mathcal{B}_s$ is independent of $\mathcal{F}_s$. In clearer terms, for any $S_1 \in \mathcal{B}(\mathbb{R})$, $S_2 \in \mathcal{F}_s$,*

$$\mathbb{P}\Big(\big\{\omega \in \Omega : \ \mathcal{B}_t(\omega) - \mathcal{B}_s(\omega) \in S_1 \ \text{ and } \ \omega \in S_2\big\}\Big) = \mathbb{P}\big(S_2\big) \times \mathbb{P}\big(\mathcal{B}_t(\omega) - \mathcal{B}_s(\omega) \in S_1\big).$$

An $\{\mathcal{F}_t\}$-standard Brownian motion will be referred to as an $\{\mathcal{F}_t\}$-SBM for brevity.

Let $\mathcal{B}$ be a unidimensional $\{\mathcal{F}_t\}$-standard Brownian motion, and consider a sequence of increasing times $0 \leqslant t_0 < t_1 < \cdots < \infty$. The increments $\mathcal{B}_{t_i} - \mathcal{B}_{t_{i-1}}$, $1 \leqslant i \leqslant k$, are mutually independent. Consequently, we characterize $\mathcal{B}$ as having ***independent increments***. Moreover, the distribution of $\mathcal{B}_{t_i} - \mathcal{B}_{t_{i-1}}$ depends solely on the time difference $t_i - t_{i-1}$. In this context, we describe $\mathcal{B}$ as having ***stationary increments***[a].

Remark 6.1. *Note that the family of σ-algebras $\{\mathcal{F}_t\}_{t\geqslant 0}$ is integral to the definition of $\{\mathcal{F}_t\}_{t\geqslant 0}$-SBM. However, it is possible to define a standard Brownian motion $\mathcal{B}$ on the probability space $(\Omega, \mathcal{F}, \mathbb{P})$ without specifying a filtration. In such instances, $\mathcal{B}$ is a a real-valued continuous process that satisfy the first two properties (1. and 2.), although the third one (property 3.) is replaced by the characteristic of having independent increments* [b].

Remark 6.2. *In Definition 6.1, we do not require the probability space $(\Omega, \mathcal{F}, \mathbb{P})$ to be complete, and the filtration $\{\mathcal{F}_t\}_{t\geqslant 0}$ need not satisfy the usual conditions. However, for the purposes of this thesis, we will work with a complete probability space and a filtration that meets these conditions* [c].

Brownian motion possesses numerous notable traits and important properties. We highlight some of these in the list below. Let $\mathcal{B}$ be an $\{\mathcal{F}\}_{t\geqslant 0}$-standard Brownian motion (SBM):

- For any $\lambda > 0$, $\{\frac{\mathcal{B}_{\lambda t}}{\sqrt{\lambda}}\}_{t\geqslant 0}$ is an $\{\mathcal{F}_{\lambda t}\}_{t\geqslant 0}$-standard Brownian motion (Scaling property).
- The process $-\mathcal{B}$ is also an $\{\mathcal{F}_t\}_{t\geqslant 0}$-standard Brownian motion (Symmetry property).
- $\mathcal{B}$ is a continuous square-integrable martingale, and its quadratic variation is given by $\langle \mathcal{B}, \mathcal{B} \rangle_t = t$ for all $t \geqslant 0$.
- For any $t \geqslant 0$, we have $\mathbb{E}(\mathcal{B}_t) = 0$ and $\mathrm{Var}(\mathcal{B}_t) = \mathbb{E}(\mathcal{B}_t^2) = t$.
- $\lim\limits_{t\to\infty} (\frac{\mathcal{B}_t}{t}) = 0$ almost surely (The strong law of large numbers property).

[a] While $\mathcal{B}_s$ and $\mathcal{B}_t$ denote the position of the process at times s and t, respectively, the increment $\mathcal{B}_t - \mathcal{B}_s$ represents the cumulative noise over the interval $]s, t]$.

[b] In such context, $\mathcal{B}$ will be an $\{\mathcal{F}_t^{\mathcal{B}}\}_{t\geqslant 0}$-SBM in the sense of definition 6.1. In addition, for any filtration $\{\mathcal{F}_t\}_{t\geqslant 0}$ such that $\mathcal{F}_t^{\mathcal{B}} \subset \mathcal{F}_t$, and $\mathcal{B}_t - \mathcal{B}_s$ is independent of $\mathcal{F}_s$ ($\forall\ 0 \leqslant s < t$), $\mathcal{B}$ will be an $\{\mathcal{F}_t\}_{t\geqslant 0}$-Brownian motion.

[c] While bearing in mind that even if it is not the case, then we will simply use its augmentation (see Definition 3.4).

- For almost every $\omega \in \Omega$, the sample path $t \mapsto \mathcal{B}_t(\omega)$ is locally Hölder continuous with exponent $\kappa \in (0, \frac{1}{2})$[a], but it is nowhere differentiable.

For an n-dimensional standard Brownian motion, the following definition applies.

Definition 6.2. *An n-dimensional process $\{\mathcal{B}_t = (\mathcal{B}_t^1, \ldots, \mathcal{B}_t^n)\}_{t \geqslant 0}$ is termed an n-dimensional standard Brownian motion if each $\{\mathcal{B}_t^k\}$ is a scalar standard Brownian motion in the sense of Definition 6.1, and the processes $\{\mathcal{B}_t^1\}, \cdots, \{\mathcal{B}_t^n\}$ are mutually independent.*

An n-dimensional standard Brownian motion is characterized as an n-dimensional continuous martingale with the associated quadratic variations given by[b]

$$\langle \mathcal{B}^j, \mathcal{B}^k \rangle_t = \delta_{jk} \times t \quad \text{for } j, k \in \{1, \cdots, n\}.$$

Here, $\delta_{jk} = \begin{cases} 1 & \text{if } j = k \\ 0 & \text{if } j \neq k \end{cases}$ is the well-famed Kronecker delta.

7 Stochastic integration: Itô integral

Let us consider a scenario where the dynamics of a given population $\mathcal{N}$ are described by the following ordinary differential equation (ODE):

$$\begin{cases} \mathrm{d}\mathcal{N}(t) = \mathcal{R}(t)\mathcal{N}(t)\mathrm{d}t, \\ \mathcal{N}(0) = \mathcal{N}_0. \end{cases} \tag{I.2}$$

where $\mathcal{R}(t)$ represents the corresponding growth rate. Assuming that $\mathcal{R}(t)$ is prone to random fluctuations, we can represent it as $\mathcal{R}(t) + \sigma \cdot \text{noise}$, where σ denotes the intensity of the noise. Consequently, Equation (I.2) transforms into the perturbed ODE below:

$$\mathrm{d}\mathcal{N}(t) = \mathcal{R}(t)\mathcal{N}(t)\mathrm{d}t + \sigma\mathcal{N}(t) \cdot \text{noise} \cdot \mathrm{d}t.$$

Through integration, the last equality turns into:

$$\mathcal{N}(t) - \mathcal{N}(0) = \int_0^t \mathcal{R}(s)\mathcal{N}(s)\mathrm{d}s + \int_0^t \sigma\mathcal{N}(s) \cdot \text{noise} \cdot \mathrm{d}s.$$

The question arises: How can we define the expression $\int_0^t \mathcal{N}(s) \cdot \text{noise} \cdot \mathrm{d}s$? To tackle this, we opt for white noise[c] $\zeta(t) = \dot{\mathcal{B}}(t)$ as a rational choice to represent the noise. Therefore,

$$\int_0^t \sigma\mathcal{N}(s) \cdot \zeta(s) \cdot \mathrm{d}s = \int_0^t \sigma\mathcal{N}(s)\mathrm{d}\mathcal{B}(s). \tag{I.3}$$

Given that the Brownian motion $\mathcal{B}$ is almost surely nowhere differentiable, and its paths have infinite variation for almost every $\omega \in \Omega$, the integral (I.3) cannot be defined in the usual manner[d]. Consequently, there is a need to establish the integral for a broad class of stochastic processes, utilizing the stochastic characterization of $\mathcal{B}$. Particularly, our focus turns to the Itô stochastic integral with respect to $\mathcal{B}$, a concept initially introduced by Kiyosi Itô[e] in 1946. To achieve this,

[a] A real valued function Ψ is locally Hölder continuous with exponent κ, if for every interval (a, b) there is a constant $C_{(a,b)} > 0$ such that $|\Psi(x) - \Psi(y)| \leqslant C_{(a,b)}|x - y|^\kappa$ for every $x, y \in (a, b)$.

[b] This property is elucidated by the well-known Lévy theorem (refer to, for example, [19]).

[c] In engineering and applied sciences, it is common practice to denote white noise as a "derivative" of $\mathcal{B}(t)$, specifically as $\xi(t) = \dot{\mathcal{B}}(t)$. Consequently, the expression for the term "noise" multiplied by $\mathrm{d}t$ becomes $\zeta(t)\mathrm{d}t = \mathrm{d}\mathcal{B}(t)$, with the awareness that $\dot{\mathcal{B}}(t)$ is just a notation and not a well-defined concept since it lacks an ordinary interpretation.

[d] Brownian motion lacks bounded variation, making the standard definitions of Riemann-Stieltjes inapplicable for evaluating integrals along specific realized paths.

[e] Kiyosi Itô was a Japanese mathematician renowned for his fundamental contributions to the theory of stochastic processes. He is credited with inventing the stochastic integral and is recognized as the founder of Itô calculus.

consider $\emptyset \subsetneq [a,b] \subset [0,\infty)$, and define $\mathcal{M}^2([a,b];\mathbb{R})$ as the space of all real-valued, measurable $\{\mathcal{F}_t\}$-adapted processes $\varphi = \{\varphi(t)\}_{a\leqslant t\leqslant b}$ such that

$$||\varphi||_{[a,b]} = \sqrt{\mathbb{E}\left(\int_a^b |\varphi(t)|^2 \mathrm{d}t\right)} < \infty. \tag{I.4}$$

Our objective in the following is to establish the definition of the Itô integral $\mathcal{I}(\varphi) = \int_a^b \varphi(t)\mathrm{d}\mathcal{B}(t)$ for a stochastic process $\varphi \in \mathcal{M}^2([a,b];\mathbb{R})$. This involves initiating the definition for an elementary process class $\mathcal{M}_0^2([a,b];\mathbb{R}) \subset \mathcal{M}^2([a,b];\mathbb{R})$ and subsequently proving that each $\varphi \in \mathcal{M}^2([a,b];\mathbb{R})$ can be effectively approximated by processes f from $\mathcal{M}_0^2([a,b];\mathbb{R})$. Following this, we define $\mathcal{I}(\varphi)$ as the limit of $\mathcal{I}(f)$. It is worth noting that, under the metric (I.4), the space $\mathcal{M}^2([a,b];\mathbb{R})$ is identified as complete. A concise summary of this procedure unfolds in these three stages:

Stage 1. Consider a real-valued stochastic process $f = \{f(t)\}_{t\in[a,b]}$. We say that f is an elementary process if there is a partition $a = t_0 < t_1 < t_2 \cdots < t_k = b$ of $[a,b]$, and bounded random variables θ_i, $0 \leqslant i \leqslant k-1$, such that θ_i is $\mathcal{F}_{t_i}$-measurable, and

$$f(t) = \theta_0 \mathbb{1}_{[t_0,t_1]}(t) + \sum_{i=1}^{k-1} \theta_i \mathbb{1}_{(t_i,t_{i+1}]}(t). \tag{I.5}$$

Let us denote by $\mathcal{M}_0^2([a,b];\mathbb{R})$ the subspace of elementary processes in $\mathcal{M}^2([a,b];\mathbb{R})$, and define the stochastic integral of $f \in \mathcal{M}_0^2([a,b];\mathbb{R})$ with respect to the Brownian motion $\mathcal{B}$ as

$$\int_a^b f(t)\mathrm{d}\mathcal{B}(t) = \sum_{i=0}^{k-1} \theta_i\big(\mathcal{B}(t_{i+1}) - \mathcal{B}(t_i)\big). \tag{I.6}$$

Stage 2. Obviously, the stochastic integral (I.6), is $\mathcal{F}_b$-measurable. Additionally, θ_i exhibits $\mathcal{F}_{t_i}$-measurability, and the increments $\mathcal{B}(t_{i+1}) - \mathcal{B}(t_i)$ are independent of $\mathcal{F}_{t_i}$ for all $i \in \{0,1,2\ldots,k-1\}$. Furthermore, the integral $\int_a^b f(t)\mathrm{d}\mathcal{B}(t)$ belongs to $\mathrm{L}^2(\Omega;\mathbb{R})$ given that, for any f in $\mathcal{M}_0^2([a,b];\mathbb{R})$, we have:

$$\mathbb{E}\left(\int_a^b f(t)\mathrm{d}\mathcal{B}(t)\right) = 0 \quad \text{and} \quad \mathbb{E}\left(\left|\int_a^b f(t)\mathrm{d}\mathcal{B}(t)\right|^2\right) = \mathbb{E}\left(\int_a^b |f(t)|^2 dt\right)^{\text{a}}. \tag{I.7}$$

Moreover, when considering two processes $f_1, f_2 \in \mathcal{M}_0^2([a,b];\mathbb{R})$ and constants $\lambda_1, \lambda_2 \in \mathbb{R}$, the combination $\lambda_1 f_1 + \lambda_2 f_2 \in \mathcal{M}_0^2([a,b];\mathbb{R})$, and we have the linearity property

$$\int_a^b \big(\lambda_1 f_1(t) + \lambda_2 f_2(t)\big)\mathrm{d}\mathcal{B}(t) = \lambda_1 \int_a^b f_1(t)\mathrm{d}\mathcal{B}(t) + \lambda_2 \int_a^b f_2(t)\mathrm{d}\mathcal{B}(t). \tag{I.8}$$

Based on the properties in (I.7) and (I.8), we extend the definition of the integral from simple processes to processes in $\mathcal{M}^2([a,b];\mathbb{R})$. The rationale behind this extension is elaborated in the subsequent step.

Stage 3. Consider a stochastic process $\varphi \in \mathcal{M}^2([a,b];\mathbb{R})$, one can find a sequence $\{f_n\}$ of elementary processes such that (see Lemma 5.6 [19]):

$$\lim_{n\to\infty} ||\varphi(t) - f_n(t)||_{[a,b]}^2 = \lim_{n\to\infty} \mathbb{E}\left(\int_a^b |\varphi(t) - f_n(t)|^2 \mathrm{d}t\right) = 0. \tag{I.9}$$

Applying (I.7), (I.8), and (I.9), we obtain

$$\begin{aligned}
\lim_{n_1,n_2\to\infty} \left\|\int_a^b f_{n_1}(t)\mathrm{d}\mathcal{B}(t) - \int_a^b f_{n_2}(t)\mathrm{d}\mathcal{B}(t)\right\|_{[a,b]}^2 &= \lim_{n_1,n_2\to\infty} \mathbb{E}\left(\left|\int_a^b f_{n_1}(t)\mathrm{d}\mathcal{B}(t) - \int_a^b f_{n_2}(t)\mathrm{d}\mathcal{B}(t)\right|^2\right) \\
&= \lim_{n_1,n_2\to\infty} \mathbb{E}\left(\left|\int_a^b \big(f_{n_1}(t) - f_{n_2}(t)\big)\mathrm{d}\mathcal{B}(t)\right|^2\right) \\
&= \lim_{n_1,n_2\to\infty} \mathbb{E}\left(\int_a^b |f_{n_1}(t) - f_{n_2}(t)|^2 \mathrm{d}t\right) = 0.
\end{aligned}$$

[a] This property is commonly referred to as the Itô's isometry.

Hence, the sequence $\left\{\int_a^b f_n(t)\mathrm{d}\mathcal{B}(t)\right\}_{n\in\mathbb{N}}$ is a Cauchy sequence in $\mathrm{L}^2(\Omega;\mathbb{R})$. Consequently, and by the completeness of $\mathrm{L}^2(\Omega;\mathbb{R})$, its limit exists in $\mathrm{L}^2(\Omega;\mathbb{R})$, and we define it as the Itô stochastic integral of φ. In clearer terms

$$\int_a^b \varphi(t)\mathrm{d}\mathcal{B}(t) \triangleq \lim_{n\to\infty}\int_a^b f_n(t)\mathrm{d}\mathcal{B}(t) \quad \text{in } \mathrm{L}^2(\Omega;\mathbb{R}),$$

where $\{f_n\}$ is a sequence of elementary processes satisfying (I.9). We mention that this limit does not depend on the choice of $\{f_n\}$.

The stochastic Itô's integral exhibits a range of specific and important properties. In the list below, we highlight some of them as essential to our thesis. Let $\varphi_1, \varphi_2 \in \mathcal{M}^2([a,b];\mathbb{R})$ and $\lambda_1, \lambda_2 \in \mathbb{R}$:

- $\mathcal{F}_b$**-measurability**: The stochastic integral $\displaystyle\int_a^b \varphi_1(t)\mathrm{d}\mathcal{B}(t)$ is $\mathcal{F}_b$-measurable.
- **Linearity property**: $\displaystyle\int_a^b \big(\lambda_1\varphi_1(t)+\lambda_2\varphi_2(t)\big)\mathrm{d}\mathcal{B}(t) = \lambda_1\int_a^b \varphi_1(t)\mathrm{d}\mathcal{B}(t) + \lambda_2\int_a^b \varphi_2(t)\mathrm{d}\mathcal{B}(t).$
- **Expectation properties**: $\displaystyle\mathbb{E}\left(\int_a^b \varphi_1(t)\mathrm{d}\mathcal{B}(t)\right) = 0$ and $\displaystyle\mathbb{E}\left(\int_a^b \varphi_1(t)\mathrm{d}\mathcal{B}(t)|\mathcal{F}_a\right) = 0.$
- **Itô's isometry**: $\displaystyle\mathbb{E}\left(\left|\int_a^b \varphi_1(t)\mathrm{d}\mathcal{B}(t)\right|^2\right) = \mathbb{E}\left(\int_a^b |\varphi_1(t)|^2\mathrm{d}t\right).$
- **Itô's isometry for conditional expectation**:

$$\mathbb{E}\left(\left|\int_a^b \varphi_1(t)\mathrm{d}\mathcal{B}(t)\right|^2 \Big|\mathcal{F}_a\right) = \mathbb{E}\left(\int_a^b |\varphi_1(t)|^2\mathrm{d}t\Big|\mathcal{F}_a\right) = \int_a^b \mathbb{E}\left(|\varphi_1(t)|^2\big|\mathcal{F}_a\right)\mathrm{d}t.$$

- **Extended homogeneity property**: For a real-valued, bounded $\mathcal{F}_b$-measurable random variable θ, $\theta\times\varphi_1 \in \mathcal{M}^2([a,b];\mathbb{R})$, and

$$\int_a^b \theta\times\varphi_1(t)\mathrm{d}\mathcal{B}(t) = \theta\times\int_a^b \varphi_1(t)\mathrm{d}\mathcal{B}(t).$$

At this point, let $T\in(0,+\infty)$ be a given positive time. We introduce the indefinite Itô stochastic integral for a stochastic process $\varphi\in\mathcal{M}^2([0,T];\mathbb{R})$ as:

$$\mathcal{A}(t) = \int_0^t \varphi(s)\mathrm{d}\mathcal{B}(s), \quad t\in[0,T].$$

By definition, we set $\mathcal{A}(0) = \int_0^0 \varphi(s)\mathrm{d}\mathcal{B}(s) = 0$ to be zero. The indexed family $\{\mathcal{A}(t)\}_{t\in[0,T]}$ is an $\{\mathcal{F}_t\}_{t\geqslant 0}$-square-integrable martingale, and the following properties hold:

$$\mathbb{E}\big(\mathcal{A}(t)|\mathcal{F}_s\big) = \mathbb{E}\big(\mathcal{A}(s)|\mathcal{F}_s\big) + \mathbb{E}\left(\int_s^t \varphi(r)\mathrm{d}\mathcal{B}(r)\Big|\mathcal{F}_s\right) = \mathcal{A}(s), \quad 0\leqslant s<t\leqslant T.$$

Applying the second Doob's martingale inequality (see Theorem 5.4), we derive the following result:

$$\mathbb{E}\left(\sup_{t\in[0,T]}|\mathcal{A}(t)|^2\right) \leqslant 4\times\mathbb{E}\left(\int_0^T |\varphi(s)|^2\mathrm{d}s\right).$$

Moreover, $\mathcal{A} = \{\mathcal{A}(t)\}_{t\in[0,T]}$ possesses a continuous version, and its quadratic variation is expressed as

$$\langle\mathcal{A},\mathcal{A}\rangle_t = \int_0^t |\varphi(s)|^2\mathrm{d}s, \quad 0\leqslant t\leqslant T.$$

Now, let us extend the Itô stochastic integral to the multidimensional framework. Suppose we have an p-dimensional standard Brownian motion denoted by $\mathcal{B}(t) = \left\{(\mathcal{B}_1(t), \ldots, \mathcal{B}_p(t))^T\right\}_{t\geqslant 0}$. Consider $\varphi \in \mathcal{M}^2([0,T];\mathbb{R}^{n\times p})$ as the collection of all $n\times p$ matrix-valued measurable $\{\mathcal{F}_t\}$-adapted processes, given by $\varphi = \{(\varphi_{ij}(t))_{n\times p}\}_{0\leqslant t\leqslant T}$, such that

$$\mathbb{E}\left(\int_0^T |\varphi(s)|^2 \mathrm{d}t\right) < \infty^{\text{a}}.$$

By adopting a matrix representation, we define the multidimensional stochastic integral as

$$\int_0^t \varphi(s)\mathrm{d}\mathcal{B}(s) = \int_0^t \begin{pmatrix} \varphi_{11}(s) & \ldots & \varphi_{1p}(s) \\ \vdots & \ddots & \vdots \\ \varphi_{n1}(s) & \ldots & \varphi_{np}(s) \end{pmatrix} \begin{pmatrix} \mathrm{d}\mathcal{B}_1(s) \\ \vdots \\ \mathrm{d}\mathcal{B}_p(s) \end{pmatrix} = \begin{pmatrix} \sum_{k=1}^p \int_0^t \varphi_{1k}(s)\mathrm{d}\mathcal{B}_1(s) \\ \vdots \\ \sum_{k=1}^p \int_0^t \varphi_{nk}(s)\mathrm{d}\mathcal{B}_n(s) \end{pmatrix}.$$

Let $\mathcal{M}^2(\mathbb{R}_+;\mathbb{R}^{n\times p})$ represent the set of all processes $\varphi \in \mathfrak{L}^2(\mathbb{R}_+;\mathbb{R}^{n\times p})$[b] with the property that

$$\mathbb{E}\left(\int_0^T |\varphi(t)|^2 \,\mathrm{d}t\right) < \infty \quad \text{for any } T \in \mathbb{R}_+.$$

Notably, if $\varphi \in \mathcal{M}^2(\mathbb{R}_+;\mathbb{R}^{n\times p})$, then $\{\varphi\}_{0\leqslant t\leqslant T} \in \mathcal{M}^2([0,T];\mathbb{R}^{n\times p})$ for every $T > 0$, and thus, the integral $\int_0^t \varphi(s)\,\mathrm{d}\mathcal{B}(s)$ is well-defined for all $t \geqslant 0$. Our goal is now to extend this integral definition to all processes in $\mathfrak{L}^2(\mathbb{R}_+;\mathbb{R}^{n\times m})$. For a given $\varphi \in \mathfrak{L}^2(\mathbb{R}_+;\mathbb{R}^{n\times p})$, we define, for each integer $k \geqslant 1$, the stopping time below:

$$\eta_k = \inf\left\{t \geqslant 0 \middle| \int_0^t |\varphi(s)|^2 \,\mathrm{d}s \geqslant k\right\} \wedge k.$$

Obviously, $\eta_k \uparrow \infty$ almost surely. Furthermore, $\{\varphi(t)\mathbb{1}_{[0,\eta_k]}(t)\}_{t\geqslant 0} \in \mathcal{M}^2(\mathbb{R}_+;\mathbb{R}^{n\times p})$, and the integral

$$\mathcal{I}_k(t) = \int_0^t \varphi(s)\mathbb{1}_{[0,\eta_k]}(s)\,\mathrm{d}\mathcal{B}(s), \quad t \geqslant 0$$

is well-defined. For all $1 \leqslant k \leqslant m$ and $t \geqslant 0$, $\mathcal{I}_m(t\wedge\tau_k) = \mathcal{I}_k(t)$. Consequently, $\mathcal{I}_k(t) = \mathcal{I}_m(t)$ for all $0 \leqslant t \leqslant \tau_k$. In general, we define the indefinite stochastic integral $\mathcal{I}(t)$ as follows: $\mathcal{I}(t) = \mathcal{I}_k(t)$ for $0 \leqslant t \leqslant \tau_k$. For $\varphi \in \mathfrak{L}^2(\mathbb{R}_+;\mathbb{R}^{n\times p})$, the indefinite Itô integral $\int_0^t \varphi(s)\,\mathrm{d}\mathcal{B}(s)$ for $t \geqslant 0$ is an $\mathbb{R}^n$-valued continuous local martingale.

Remark 7.1. *It's crucial to note that in this thesis, all of the notations below are equivalent and will be used interchangeably:*

$$\int_0^t \varphi\,\mathrm{d}\mathcal{B} \triangleq \int_0^t \varphi(s)\,\mathrm{d}\mathcal{B}(s) \triangleq \left(\int_0^t \varphi(s,\omega)\,\mathrm{d}\mathcal{B}_s\right)(\omega).$$

8 Itô's formula

To effectively compute stochastic Itô integrals, we will make recourse to the well famed **Itô formula**, a pivotal concept in stochastic analysis. Let $\{\mathcal{B}(t)\}_{t\geqslant 0}$ represent an $\{\mathcal{F}_t\}_{t\geqslant 0}$-scalar standard Brownian motion defined on the complete probability space $(\Omega,\mathcal{F},\mathbb{P})$ endowed with a filtration $\{\mathcal{F}_t\}_{t\geqslant 0}$ verifying the usual conditions (it is right-continuous while $\mathcal{F}_0$ contains all $\mathbb{P}-$null sets).

[a] Here, $|A|$ is referring to the trace norm of A, i.e., $|A| = \sqrt{\mathrm{trace}(A^TA)}$.

[b] $\mathfrak{L}^2(\mathbb{R}_+;\mathbb{R}^{n\times p})$ denotes the collection of all $n\times p$-matrix-valued measurable $\{\mathcal{F}_t\}$-adapted processes $\varphi = \{\varphi(t)\}_{t\geqslant 0}$ such that $\int_0^T |\varphi(t)|^2\,\mathrm{d}t < \infty$ almost surely for every $T > 0$.

Additionally, consider $\psi \in \mathfrak{L}^1(\mathbb{R}_+;\mathbb{R})$[a] and $\varphi \in \mathfrak{L}^2(\mathbb{R}_+;\mathbb{R})$. A unidimensional continuous adapted process $\{\mathcal{X}(t)\}_{t\geqslant 0}$ that takes the form below

$$\mathcal{X}(t) = \mathcal{X}(0) + \int_0^t \varphi(s)\mathrm{d}s + \int_0^t \psi(s)\mathrm{d}\mathcal{B}(s), \tag{I.10}$$

is called an Itô process, and it can be also written in the form of the following differential representation

$$\mathrm{d}\mathcal{X}(t) = \varphi(t)\mathrm{d}t + \psi(t)\mathrm{d}\mathcal{B}(t).$$

The Itô formula asserts that if a function $\mathcal{U}$ belongs to the class $C^{1,2}(\mathbb{R}_+ \times \mathbb{R};\mathbb{R})$[b], then, $\mathcal{U}(t,\mathcal{X}(t))$ also qualifies as an Itô process, and it is characterized by the stochastic differential representation given by:

$$\begin{aligned}\mathrm{d}\mathcal{U}(t,\mathcal{X}(t)) = &\left(\mathcal{U}_t(t,\mathcal{X}(t)) + \mathcal{U}_x(t,\mathcal{X}(t))\varphi(t) + \frac{1}{2}\times\mathcal{U}_{xx}(t,\mathcal{X}(t))\psi^2(t)\right)\mathrm{d}t\\ &+\mathcal{U}_x(t,\mathcal{X}(t))\psi(t)\mathrm{d}\mathcal{B}(t) \quad \text{a.s.}\end{aligned} \tag{I.11}$$

Here, $\mathcal{U}_t \triangleq \frac{\partial \mathcal{U}}{\partial t}$, $\mathcal{U}_x \triangleq \frac{\partial \mathcal{U}}{\partial x}$, and $\mathcal{U}_{xx} \triangleq \frac{\partial^2 \mathcal{U}}{\partial x^2}$. In the context of multidimensional processes, consider an $\mathbb{R}^m$-valued Brownian motion $\mathcal{B} = \mathcal{B}(t) = \{(\mathcal{B}_1(t),\ldots,\mathcal{B}_m(t))^T\}_{t\geqslant 0}$ $(t \geqslant 0)$, and let $\mathcal{X}(t) = (\mathcal{X}_1(t),\ldots,\mathcal{X}_n(t))^T$ be an n-dimensional continuous adapted Itô process given by

$$\mathcal{X}(t) = \mathcal{X}(0) + \int_0^t \varphi(s)\mathrm{d}s + \int_0^t \psi(s)\mathrm{d}\mathcal{B}(s),$$

where $\varphi = (\varphi_1,\ldots,\varphi_n)^T \in \mathfrak{L}^1(\mathbb{R}_+;\mathbb{R}^n)$ and $\psi = (\psi_{jk})_{n\times m} \in \mathfrak{L}^2(\mathbb{R}_+;\mathbb{R}^{n\times m})$. Additionally, let $\mathcal{U} \in C^{1,2}(\mathbb{R}_+\times\mathbb{R}^n;\mathbb{R})$. Then, $\mathcal{U}(t,\mathcal{X}(t))$ is once again an Itô process, described by the stochastic equation:

$$\begin{aligned}\mathrm{d}\mathcal{U}(t,\mathcal{X}(t)) = &\left\{\mathcal{U}_t(t,\mathcal{X}(t)) + \mathcal{U}_x(t,\mathcal{X}(t))\varphi(t) + \frac{1}{2}\times\mathrm{trace}\left(\psi^T(t)\mathcal{U}_{xx}(t,\mathcal{X}(t))\psi(t)\right)\right\}\mathrm{d}t\\ &+\mathcal{U}_x(t,\mathcal{X}(t))\psi(t)\mathrm{d}\mathcal{B}(t) \quad \text{a.s.}\end{aligned} \tag{I.12}$$

Here,

$$\mathcal{U}_x = \left(\frac{\partial \mathcal{U}}{\partial x_1},\ldots,\frac{\partial \mathcal{U}}{\partial x_n}\right) \quad \text{and} \quad \mathcal{U}_{xx} = \left(\frac{\partial^2 \mathcal{U}}{\partial x_j \partial x_k}\right)_{n\times n} = \begin{pmatrix} \frac{\partial^2 \mathcal{U}}{\partial x_1 \partial x_1} & \cdots & \frac{\partial^2 \mathcal{U}}{\partial x_1 \partial x_n} \\ \vdots & \ddots & \vdots \\ \frac{\partial^2 \mathcal{U}}{\partial x_n \partial x_1} & \cdots & \frac{\partial^2 \mathcal{U}}{\partial x_n \partial x_n} \end{pmatrix}.$$

9 Key Concepts in Stochastic Differential Equations (SDE)

In simple terms, a stochastic differential equation (SDE) is a special kind of differential equation in which one or more components involve randomness or uncertainty. This randomness leads to a solution that is also random, in other terms a stochastic process. SDEs find applications in modeling various phenomena, ranging from unpredictable stock prices to biological systems affected by changes in the external environment. These equations often include a variable representing random white noise, which is like the unpredictable variations seen in Brownian motion. However, other types of randomness, like Lévy jumps, are also possible [21]. In the next example, we will look at a straightforward SDE:

[a] Here, $\mathfrak{L}^p(\mathbb{R}_+;\mathbb{R}^n)$ denotes the collection of all $\mathbb{R}^n$-valued measurable $\{\mathcal{F}_t\}$-adapted processes $\psi = \{\psi(t)\}_{t\geqslant 0}$ such that $\int_0^T |\psi(t)|^p \mathrm{d}t < \infty$ a.s. for every $T > 0$.

[b] In this context, $C^{1,2}(\mathbb{R}_+\times\mathbb{R}^n;\mathbb{R})$ represents the collection of all real-valued functions $\mathcal{U}(t,x)$ defined on $\mathbb{R}_+\times\mathbb{R}^n$ such that they exhibit continuous twice differentiability in x and once in t.

Example 9.1. *A scalar Brownian motion with constant drift*[a]*, is defined as the solution to the following Stochastic Differential Equation (SDE):*

$$\begin{cases} \mathrm{d}\mathcal{X}(t) = m\mathrm{d}t + \xi\mathrm{d}\mathcal{B}(t), \\ \mathcal{X}(0) = \mathcal{X}_0 \in \mathbb{R}. \end{cases} \tag{I.13}$$

Here, $\mathcal{X}(0) = \mathcal{X}_0$ is the initial condition. Through direct integration, we obtain the explicit expression of $\mathcal{X}$ given by:

$$\mathcal{X}(t) = \mathcal{X}_0 + mt + \xi\mathcal{B}(t),$$

indicating that $\mathcal{X}(t)$ follows a normal distribution with mean $\mu_{\mathcal{X}(t)} = \mathcal{X}_0 + m\times t$ and variance $\sigma_{\mathcal{X}(t)} = \xi^2\times t$.

To well dot the i's and cross the t's: Let $\{\mathcal{B}\}_{t\geqslant 0} = \{(\mathcal{B}_1(t), \cdots, \mathcal{B}_m(t))\}_{t\geqslant 0}$ be an $\{\mathcal{F}_t\}_{t\geqslant 0}$-multidimensional standard Brownian motion defined on the filtered probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geqslant 0}, \mathbb{P})$. We assume that this space is equipped with a filtration satisfying the standard conditions. Additionally, let $\varphi : [t_0, T] \times \mathbb{R}^n \to \mathbb{R}^n$ and $\psi : [t_0, T] \times \mathbb{R}^n \to \mathbb{R}^{n\times m}$ be two Borel-measurable functions.

This section aims to derive and analyze the solution of a stochastic differential equation expressed as follows:

$$\begin{cases} \mathrm{d}\mathcal{X}(t) = \varphi(t, \mathcal{X}(t))\mathrm{d}t + \psi(t, \mathcal{X}(t))\mathrm{d}\mathcal{B}(t), \quad t \in [t_0, T],\ t_0 \geqslant 0 \text{ and } T > t_0, \\ \mathcal{X}(t_0) = \mathcal{X}_0. \end{cases} \tag{I.14}$$

where $\mathcal{X}_0$ is an $\mathcal{F}_{t_0}$-measurable $\mathbb{R}^n$-valued random variable ensuring $\mathbb{E}|\mathcal{X}_0|^2 < \infty$. In this context, φ and ψ are referred to as the drift and diffusion coefficient functions, respectively. Now, our goal is to address the following three questions:

- What is exactly a solution of the equation (I.14)?
- Under what conditions does a solution of (I.14) exist?
- If a such solution exists, is it unique?

To this end, we start by introducing the following definition

Definition 9.1. *An $\mathbb{R}^n$-dimensional stochastic process $\mathcal{X}(t) = \{\mathcal{X}(t)\}_{t\in[t_0,T]}$ is deemed a solution*[b] *of (I.14) if the following criteria are met:*

- *$\mathcal{X}(t)$ is continuous and $\{\mathcal{F}_t\}_{t\geqslant 0}$-adapted.*
- *$\{\varphi(t, \mathcal{X}(t))\} \in \mathfrak{L}^1([t_0, T]; \mathbb{R}^n)$ and $\{\psi(t, \mathcal{X}(t))\} \in \mathfrak{L}^2(\mathbb{R}_+; \mathbb{R}^{n\times m})$.*
- *For every $t \in [t_0, T]$, :*

$$\mathcal{X}(t) = \mathcal{X}_0 + \int_{t_0}^{t} \varphi(s, \mathcal{X}(s))\mathrm{d}s + \int_{t_0}^{t} \psi(s, \mathcal{X}(s))\mathrm{d}\mathcal{B}(s) \text{ a.s.} \tag{I.15}$$

A solution $\mathcal{X}(t)$ is unique if any alternative solution $\hat{\mathcal{X}}(t)$ is indistinguishable from $\mathcal{X}(t)$, meaning that

$$\mathbb{P}\Big(\mathcal{X}(t) = \hat{\mathcal{X}}(t) \quad \textit{for all } t \in [t_0, T]\Big) = 1.$$

Now, we present some conditions ensuring the existence and uniqueness of the solution to equation (I.14).

Theorem 9.1 (see [18]). *If there is a positive constant $\mathfrak{A} \in \mathbb{R}_+$ such that for any $(x, y, t) \in (\mathbb{R}^n)^2\times[t_0, T]$, the coefficient functions φ, ψ appearing in the* SDE *(I.14) verify the following two conditions*

[a] More precisely, Brownian motion with constant drift m and intensity ξ.

[b] In this definition, we are specifically referring to the solution in the strong sense (see page 285 in [18]).

- ***Uniform Lipschitz condition**:*

$$|\varphi(t,x)-\varphi(t,y)|+|\psi(t,x)-\psi(t,y)|\leqslant \mathfrak{A}|x-y|. \tag{I.16}$$

- ***Linear growth condition**:*

$$|\varphi(t,x)|^2+|\psi(t,x)|^2\leqslant \mathfrak{A}(1+|x|^2). \tag{I.17}$$

Then, there exists a unique solution $\mathcal{X}(t)$ in $\mathcal{M}^2([t_0,T];\mathbb{R}^n)$ of (I.14). Furthermore, this solution verifies $\sup_{t\in[t_0,T]}\mathbb{E}|\mathcal{X}(t)|^2<\infty$.

Remark 9.1. *The Lipschitz condition (I.16) guarantees that for all $t\in[t_0,T]$, the functions $x\mapsto\varphi(t,x)$ and $x\mapsto\psi(t,x)$ do not vary faster than a linear function[a]. However, the linear growth condition (I.17) serves to uniformly bound φ and ψ with respect to $t\in[t_0,T]$. This restriction on the growth of φ and ψ guarantees that, almost surely, the solution $\mathcal{X}(t)$ does not "explode" (i.e., $||\mathcal{X}||\not\to+\infty$) in the interval $[t_0,T]$, notwithstanding $\mathcal{X}_0$ value.*

The conditions posed by Theorem 9.1 appear somehow restrictive, for this reason, an extension where the uniform Lipschitz condition is substituted with the more lenient local local Lipschitz condition is proposed.

Theorem 9.2. *Assuming the fulfillment of the linear growth condition, we introduce a modified Lipschitz condition in place of (I.16).*

- ***Localized Lipschitz condition**: If for each integer $\ell\geqslant 1$, there exists a constant $\mathcal{K}_\ell>0$ such that, for all $t\in[t_0,T]$ and $x,y\in\mathbb{R}^n$, with $\max\{|x|,|y|\}\leqslant\ell$, the following inequality holds:*

$$|\varphi(t,x)-\varphi(t,y)|+|\psi(t,x)-\psi(t,y)|\leqslant\mathcal{K}_\ell|x-y|. \tag{I.18}$$

Then, the SDE (I.14) admits a solution in $\mathcal{M}^2([t_0,T];\mathbb{R}^n)$ and in addition this solution is unique.

The local Lipschitz condition (I.18) allows us to consider a broad range of functions for φ and ψ, including those that are defined on $[t_0,T]\times\mathbb{R}$ and continuously differentiable in x . Similarly, the requirement of linear growth can be substituted with a monotonic condition, as stated in the following theorem:

Theorem 9.3. *Suppose that the linear growth condition (I.17) is replaced by a monotone condition (I.19), and the local Lipschitz condition (I.18) holds.*

- *The **monotone condition** is defined as follows: there exists a constant $\mathcal{K}$ such that for all $(t,x)\in[t_0,T]\times\mathbb{R}^n$,*

$$x^T\varphi(t,x)+\frac{1}{2}\times|\psi(t,x)|^2\leqslant\mathcal{K}(1+|x|^2). \tag{I.19}$$

Under these conditions, a unique solution $\mathcal{X}(t)$ to the SDE (I.14) exists in $\mathcal{M}^2([t_0,T];\mathbb{R}^n)$.

In general, the monotone condition (I.19) ensures the existence of the solution $\mathcal{X}(t)$ over the entire interval $[t_0,T]$. Additionally, it is worth noting that while the linear growth condition (I.17) implies the monotone condition (I.19), the reverse implication does not always hold.

Remark 9.2. *In the context of time-autonomy[b], the stochastic differential equation (I.14) is formulated as follows:*

$$\mathrm{d}\mathcal{X}(t)=\varphi(\mathcal{X}(t))\mathrm{d}t+\psi(\mathcal{X}(t))\mathrm{d}\mathcal{B}(t),\quad t\in[t_0,T],\ T>t_0\geqslant 0, \tag{I.20}$$

with an initial value $\mathcal{X}(t_0)=\mathcal{X}_0\in\mathbb{R}$. In this case, and under the conditions (I.16) and (I.17), the SDE (I.20) possesses a unique, continuous global[c] solution $\mathcal{X}(t)$ defined on $[t_0,\infty)$.

[a] This implies the Lipschitz continuity of $x\mapsto\varphi(t,x)$ and $x\mapsto\psi(t,x)$ for all $t\in[t_0,T]$.

[b] Time-autonomous means that both φ and ψ are independent of time, and are expressed as $\varphi(t,x)=\varphi(x)$, $\psi(t,x)=\psi(x)$.

[c] Under the assumptions of Theorem 9.1 on every finite sub-interval $[t_0,T]\subset[t_0,\infty)$, SDE $\mathrm{d}\mathcal{X}(t)=\varphi(t,\mathcal{X}(t))\mathrm{d}t+\psi(t,\mathcal{X}(t))\mathrm{d}\mathcal{B}(t)$, $t\in[t_0,\infty)$ has a unique global solution $\mathcal{X}(t)$ defined on the entirety of $[t_0,\infty)$.

10 Overview of Lévy Processes Fundamentals

Recently, many studies have highlighted the importance of considering the potential occurrence of rapid and substantial movements (jumps) in the number of infected individuals or deaths when analyzing certain epidemics. To mathematically model these phenomena, jump processes provide a suitable framework for describing and predicting the future comportment of epidemics dissemination under the existence of these brutal perturbations. In this section, we introduce the Lévy process, named after the French mathematician Paul Lévy (1886-1971), who made significant contributions to probability theory. We provide an overview of the category of processes that Lévy processes encompass and outline some fundamental concepts essential for the applied calculus of jump-diffusion, a key component in this thesis. While detailed theoretical explanations are available in various comprehensive works [30, 21] we present a summary here for context.

10.1 Poisson random measure and Lévy processes

Definition 10.1. *Let $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geqslant 0}, \mathbb{P})$ be a filtered probability space. A scalar $\{\mathcal{F}_t\}$-adapted process $\mathcal{L}(t) = \{\mathcal{L}_t\}_{t\geqslant 0}$ is called a Lévy process (or $\{\mathcal{F}_t\}_{t\geqslant 0}$−Lévy process more precisely) if it is vanishing at $t = 0$[a], stochastically continuous[b] and has stationary[c] and independent[d] increments.*

Remark 10.1. *Every Lévy process possesses a cadlag version that is also a Lévy process. Therefore, we can consistently regard a Lévy process as cadlag, with the understanding that if it is not originally cadlag, we will simply employ its cadlag version instead.*

In light of the previous definition, we present the following notable examples of Lévy processes:

Example 10.1.

- *A Brownian motion $\{\mathcal{B}(t)\}_{t\geqslant 0}$ exhibits stationary and independent increments. Consequently, it qualifies as a Lévy process.*
- *The Poisson process $\{\Theta(t)\}_{t\geqslant 0}$ with intensity $\theta \in \mathbb{R}_+$ is equally a Lévy process that takes values in the set of non-negative integers, $\mathbb{N}_{\geqslant 0} = \{0, 1, 2, \cdots\}$ such that*

$$\mathbb{P}\big(\Theta(t) = n\big) = \frac{(\theta t)^n}{n!} \times e^{-\theta t}, \quad \forall n \in \mathbb{N}_{\geqslant 0}$$

Let us denote by $\mathcal{B}_0$ the collection of Borel sets $S \subset \mathbb{R}$ such that the closure $\bar{S}$ does not include 0 (i.e., $0 \notin \bar{S}$). Then, one can associate to any Lévy process $\mathcal{L}(t) = \{\mathcal{L}_t\}_{t\geqslant 0}$ the counting measure $\mathcal{N}$ defined as follows:

$$\mathcal{N}(t, S) = \mathcal{N}(t, S, \omega) = \sum_{0<s\leqslant t} \mathbb{1}_S\big(\overbrace{\Delta\mathcal{L}(s)}^{\Delta\mathcal{L}(t)\triangleq\mathcal{L}(t)-\mathcal{L}(t^-)}\big), \quad \forall S \in \mathcal{B}_0 \text{ and } t > 0. \tag{I.21}$$

From an interpretational perspective, (I.21) corresponds to the *Poisson random measure*, signifying the count of jumps in S up to time t. In simpler terms, it represents the number of jumps with size $\Delta\mathcal{L}(t) \in S$ occurring before or at time t. Given the cadlag nature of the path, for any $S \in \mathcal{B}_0$, it is observed that $\mathcal{N}(t, S) < \infty$ almost surely. The mapping $S \mapsto \mathcal{N}(t, S, \omega)$ represents a σ-finite measure on $\mathcal{B}_0$ for any fixed values of $t \in \mathbb{R}_+$ and $\omega \in \Omega$, where the corresponding differential form is denoted by $\mathcal{N}(t, \mathrm{d}u)$. Additionally, when ω is held constant in Ω, the mapping $(a, b] \times S \mapsto \mathcal{N}(b, S, \omega) - \mathcal{N}(a, S, \omega)$, where $(a, b] \subset [0, \infty)$ and $S \in \mathcal{B}_0$, defines a σ-finite measure. The associated differential form is expressed as $\mathcal{N}(\mathrm{d}t, \mathrm{d}u)$. Now, we focus on the mean measure $\nu(S) = \mathbb{E}\big(\mathcal{N}(1, S)\big)$. This function also establishes a σ-finite measure on $\mathcal{B}_0$ and is referred to as the *Lévy measure* associated with $\mathcal{L}(t)$. By fixing S_0 in $\mathcal{B}_0$, the process

$$\vartheta_{S_0} := \{\vartheta_{S_0}(t)\}_{t\geqslant 0} := \{\mathcal{N}(t, S_0, \omega)\}_{t\geqslant 0},$$

acts as a Poisson process with an intensity $\lambda = \nu(S_0)$.

[a] In simpler terms, $\mathcal{L}(0) = 0$ a.s.

[b] In other words, for any $t \geqslant 0$ and all $\kappa > 0$, $\mathbb{P}\big(|\mathcal{L}(t+\varepsilon) - \mathcal{L}(t)| > \kappa\big) \underset{\varepsilon\to 0}{\to} 0$.

[c] As mentioned before, this signifies that he distribution of $\mathcal{L}_{t+\epsilon} - \mathcal{L}_t$ depends on ϵ, but not on t.

[d] As we have previously said, that means that $\mathcal{L}(t+s) - \mathcal{L}(t)$ is independent of $\mathcal{F}_t$

Example 10.2 (Compound Poisson Process). *Consider a sequence* $\{C(n)\}_{n\in\mathbb{N}}$*, comprising independent and identically distributed (i.i.d.) random variables in* $\mathbb{R}$ *with a common finite distribution denoted as* μ_C*. Let* $\vartheta(t)$ *be a Poisson process with intensity* $\theta > 0$*, independent of all the* $C(n)$*'s. The compound Poisson process* $C_\Sigma(t)$ *is defined as*

$$C_\Sigma(t) = C(1) + C(2) + \cdots + C\big(\vartheta(t)\big), \quad t > 0.$$

An increment of this process is given by

$$C_\Sigma(s) - C_\Sigma(t) = \sum_{k=\vartheta(t+1)}^{\vartheta(s)} C(k), \quad s > t. \tag{I.22}$$

It's noteworthy that (I.22) *is independent of* $C(1), \ldots, C\big(\vartheta(t)\big)$*, and its distribution depends solely on the difference* $(s - t)$*. Consequently,* $C_\Sigma(t)$ *is a Lévy process. To determine the Lévy measure* ν *of* $C_\Sigma(t)$*, consider* $S \in \mathrm{B}_0$*. Then,*

$$\nu(S) = \mathbb{E}\big(\mathcal{N}(1, S)\big) = \mathbb{E}\left(\sum_{0<s\leqslant 1} \mathbb{1}_S\big(\Delta C_\Sigma(s)\big)\right) = \mathbb{E}\big(\vartheta(1)\mathbb{1}_S(C)\big) = \theta \times \mu_C(S).$$

In broad terms, the preceding example highlights that a Lévy process can be represented as a compound Poisson process only when its Lévy measure is finite. However, there exist numerous Lévy processes featuring an infinite Lévy measure ν while satisfying $\int_{|u|\leqslant\rho} |u|\nu(\mathrm{d}u) = \infty$ for $\rho > 0$ (see [21]). Furthermore, it's worth noting that, for any fixed ρ, the processes

$$\mathcal{I}^{(k)}(t) \triangleq \int_{\frac{1}{k}\leqslant u\leqslant\rho} u\big[\mathcal{N}(t, \mathrm{d}u) - t\nu(\mathrm{d}u)\big], \quad k \in \{1, 2, \cdots\}$$

are L^2-martingales, and they converge in L^2 to a martingale $\mathcal{I}(t)$ denoted as:

$$\mathcal{I}(t) = \int_{|u|\leqslant\rho} u\big[\mathcal{N}(t, \mathrm{d}u) - t\nu(\mathrm{d}u)\big].$$

Indeed, we have

Proposition 10.1 (**Itô-Lévy Decomposition**). *Let* $\mathcal{L}(t)$ *be a Lévy process. Then,* $\mathcal{L}(t)$ *possesses the unique representation:*

$$\mathcal{L}(t) = \eta t + \sigma\mathcal{B}(t) + \int_{|u|<\rho} u\widetilde{\mathcal{N}}(t, \mathrm{d}u) + \int_{|u|\geqslant\rho} u\mathcal{N}(t, \mathrm{d}u). \tag{I.23}$$

Here, $\eta \in \mathbb{R}$*,* $\sigma \in \mathbb{R}$*,* $\rho \in [0, \infty]$*, and*

$$\widetilde{\mathcal{N}}(\mathrm{d}t, \mathrm{d}u) = \mathcal{N}(\mathrm{d}t, \mathrm{d}u) - \nu(\mathrm{d}u)\mathrm{d}t,$$

is the compensated Poisson random measure of $\mathcal{L}(\bullet)$*. We assume that the standard Brownian motion* $\mathcal{B}(t)$ *is independent of* $\widetilde{\mathcal{N}}$*. Note that for each* $D \in \mathcal{B}_0$*, the process* $\mathcal{M}(t) \triangleq \widetilde{\mathcal{N}}(t, D)$ *is a martingale. Specifically, when* $\eta = 0$ *and* $\rho = \infty$*, we refer to* $\mathcal{L}(t)$ *as a Lévy martingale. Furthermore, if* $\mathbb{E}|\mathcal{L}(t)| < \infty$ *for all* $t > 0$[a]*, then we can choose* $\rho = \infty$*. In this case,* $\mathcal{L}(t)$ *can be expressed in the reduced form below:*

$$\mathcal{L}(t) = \eta t + \sigma\mathcal{B}(t) + \int_{\mathbb{R}} u\widetilde{\mathcal{N}}(t, \mathrm{d}u).$$

Theorem 10.1 (**Lévy-Khintchine formula**). *Let* $\mathcal{L}(t)$ *be a Lévy process with associated Lévy measure* ν*. Then,*

$$\int_{\mathbb{R}} \min(1, |u|^2)\nu(\mathrm{d}u) < \infty \quad \textit{and} \quad \mathbb{E}\left(e^{i\zeta\mathcal{L}(t)}\right) = e^{t\Psi(\zeta)}, \quad \zeta \in \mathbb{R},$$

where,

$$\Psi(\zeta) = -\frac{1}{2}\sigma^2\zeta^2 + i\alpha\zeta + \int_{|u|<\rho} \left\{e^{i\zeta u} - 1 - i\zeta u\right\} \nu(\mathrm{d}u) + \int_{|u|\geqslant\rho} \left\{e^{i\zeta u} - 1\right\} \nu(\mathrm{d}u).$$

[a] If $\mathbb{E}|\mathcal{L}(t)| < \infty$ for all $\rho > 0$, then $\int_{|u|\geqslant\rho} |u|\nu(\mathrm{d}u)$ is finite.

Now, consider $\mathcal{D}^{sk}$ as the space of cadlag adapted processes equipped with the Skorohod topology[a]. Additionally, let $\mathcal{D}^{ucp}$ represent the space of cadlag adapted processes endowed with the topology based on uniform convergence on compacts in probability[b] (ucp, for brevity). By introducing an elementary process $f(t)$ in the form of (I.5) and considering a cadlag process $\mathcal{X}$, we define

$$\mathcal{J}_{\mathcal{X}}(f) \triangleq \int_0^t f(s)\mathrm{d}\mathcal{X}(s) := \theta_0\mathcal{X}_0 + \sum_{i=1}^{n} \theta_i \left(\mathcal{X}_{t_{i+1}\wedge t} - \mathcal{X}_{t_i\wedge t}\right), \quad t > 0,$$

where θ_i are $\mathcal{F}_{t_i}$–measurable random variables and $t_0 \leqslant t_1 \leqslant \cdots \leqslant t_{n+1} < \infty$. Suppose $\mathcal{X}$ is a semimartingale[c]. Then, the function $\mathcal{J}_{\mathcal{X}}$ is able to be prolonged to a continuous linear mapping $\mathcal{J}_{\mathcal{X}} : \mathcal{D}^{ucp} \to \mathcal{D}^{sk}$ [21]. Utilizing this construction, stochastic integrals of the form $\int_0^t h(s)\mathrm{d}\mathcal{X}(s)$ for all $h \in \mathcal{D}^{ucp}$ can be defined. From the decomposition (I.23), we derive the generalized stochastic integrals as follows:

$$\mathcal{X}(t,\omega) = \mathcal{X}_0 + \int_0^t \varphi(s,\omega)\mathrm{d}s + \int_0^t \psi(s,\omega)\mathrm{d}\mathcal{B}(s) + \int_0^t \int_{\mathbb{R}\setminus\{0\}} \Upsilon(s,u,\omega)\widetilde{\mathcal{N}}(\mathrm{d}s,\mathrm{d}u),$$

where the integrands, being $\{\mathcal{F}_t\}_{t\geqslant 0}$-predictable, verify the following assumption:

$$\int_0^t \left\{|\varphi(s,\omega)| + \psi^2(s,\omega) + \int_{\mathbb{R}\setminus\{0\}} \Upsilon^2(s,u,\omega)\nu(\mathrm{d}u)\right\}\mathrm{d}s < \infty \quad \text{a.s. for all; } t > 0.$$

Subsequently, and by adopting a differential notation, the *Itô-Lévy* process $\mathcal{X}$ can be written as follows:

$$\mathrm{d}\mathcal{X}(t,\omega) = \varphi(t,\omega)\mathrm{d}t + \psi(t,\omega)\mathrm{d}\mathcal{B}(t) + \int_{\mathbb{R}\setminus\{0\}} \Upsilon(t,u,\omega)\widetilde{\mathcal{N}}(\mathrm{d}t,\mathrm{d}u). \tag{I.24}$$

Clearly, when $\mathbb{E}\left(\int_0^T \int_{\mathbb{R}\setminus\{0\}} \Upsilon^2(t,u,\omega)\nu(\mathrm{d}u)\mathrm{d}t\right) < \infty$, the process $\mathcal{M}(t) = \int_0^t \int_{\mathbb{R}\setminus\{0\}} \Upsilon(t,u,\omega)\widetilde{\mathcal{N}}(\mathrm{d}s,\mathrm{d}u)$, $0 \leqslant t \leqslant T$, is a martingale. Furthermore, if $\int_0^T \int_{\mathbb{R}\setminus\{0\}} \Upsilon^2(t,u,\omega)\nu(\mathrm{d}u)\mathrm{d}t < \infty$ a.s., then the process $\mathcal{M}(t)$ is a local martingale for all $t \in [0,T]$.

10.2 The Itô-Lévy formula

Let us consider the Itô-Lévy process $\mathcal{X}$ defined through the equation (I.24), and assume that $W : \mathbb{R}_+ \times \mathbb{R}^n \to \mathbb{R}$ is a $C^{1,2}(\mathbb{R}_+ \times \mathbb{R}; \mathbb{R})$ function. Then, $\mathfrak{A}(t) := \mathcal{W}(t,\mathcal{X}(t))$ is, in turn, an Itô-Lévy process, and:

$$\begin{aligned}
\mathrm{d}\mathfrak{A}(t) = {} & \frac{\partial \mathcal{W}}{\partial t}(t,\mathcal{X}(t))\mathrm{d}t + \frac{\partial \mathcal{W}}{\partial x}(t,\mathcal{X}(t))\left\{\varphi(t,\omega)\mathrm{d}t + \psi(t,\omega)\mathrm{d}\mathcal{B}(t)\right\} + \frac{1}{2}\times\psi^2(t,\omega)\frac{\partial^2 \mathcal{W}}{\partial x^2}(t,\mathcal{X}(t))\mathrm{d}t \\
& + \int_{\mathbb{R}\setminus\{0\}} \left\{\mathcal{W}\big(t,\mathcal{X}(t) + \Upsilon(t,u,\omega)\big) - \mathcal{W}(t,\mathcal{X}(t)) - \frac{\partial \mathcal{W}}{\partial x}(t,\mathcal{X}(t))\Upsilon(t,u,\omega)\right\}\nu(\mathrm{d}u)\mathrm{d}t \\
& + \int_{\mathbb{R}\setminus\{0\}} \left\{\mathcal{W}\big(t,\mathcal{X}(t^-) + \Upsilon(t,u,\omega)\big) - \mathcal{W}(t,\mathcal{X}(t^-))\right\}\widetilde{\mathcal{N}}(\mathrm{d}t,\mathrm{d}u).
\end{aligned}$$

[a] The Skorohod topology on $\mathcal{D}^{sk}$ is established through the Skorohod metric d^{sk} defined by

$$\mathrm{d}^{sk}(h_1,h_2) = \inf_{\tau}\sup_{t\geqslant 0}\{|h_1(t) - h_2(\tau(t))| + |\tau(t) - t|\},$$

where τ ranges over all strictly increasing, continuous mappings of $\mathbb{R}_+$ to $\mathbb{R}_+$ with $\tau(0) = 0$.

[b] In this context, $h_n \overset{ucp}{\to} h_\infty$ if, for all $t > 0$, $\sup_{0\leqslant s\leqslant t} |h_n(s) - h_\infty(s)| \to 0$ in probability.

[c] A stochastic process $\mathcal{X}(t)$ is termed a semimartingale if it can be decomposed as $\mathcal{X} = \mathcal{Y}(t) + \mathcal{A}(t)$, where $\mathcal{Y}$ is a local martingale, and $\mathcal{A}$ is a cadlag adapted process of locally bounded variation.

We now redirect our attention to present the preceding result in the multi-dimensional context. Consider an $\mathbb{R}^n-$valued Itô-Lévy process $\mathcal{X}(t)$ represented by:

$$\mathrm{d}\mathcal{X}(t) = \varphi(t,\omega)\mathrm{d}t + \psi(t,\omega)\mathrm{d}\mathcal{B}(t) + \int_{\mathbb{R}^n\setminus\{0\}} \Upsilon(t,u,\omega)\widetilde{\mathcal{N}}(\mathrm{d}t,\mathrm{d}u), \tag{I.25}$$

where $\varphi : [0,T]\times\Omega \to \mathbb{R}^n$, $\psi : [0,T]\times\Omega \to \mathbb{R}^{n\times m}$, and $\Upsilon : [0,T]\times\mathbb{R}^n\times\Omega \to \mathbb{R}^{n\times\ell}$ are adapted integrable processes. Here, $\mathcal{B}(t)$ is an m-dimensional Brownian motion, and

$$\widetilde{\mathcal{N}}(\mathrm{d}t,\mathrm{d}u)^\top = \Big(\overbrace{\mathcal{N}_1(\mathrm{d}t,\mathrm{d}u_1) - \nu_1(\mathrm{d}u_1)\mathrm{d}t}^{\widetilde{\mathcal{N}}_1(\mathrm{d}t,\mathrm{d}u_1)}, \cdots, \overbrace{\mathcal{N}_\ell(\mathrm{d}t,\mathrm{d}u_\ell) - \nu_\ell(\mathrm{d}u_\ell)\mathrm{d}t}^{\widetilde{\mathcal{N}}_\ell(\mathrm{d}t,\mathrm{d}u_\ell)}\Big),$$

where $\{\mathcal{N}_j\}$ are independent Poisson random measures with Lévy measures ν_j. In detail, the i-th component $\mathcal{X}_i(t)$ of $\mathcal{X}(t)$ in (I.25), has the form:

$$\mathrm{d}\mathcal{X}_i(t) = \varphi_i(t,\omega)\mathrm{d}t + \sum_{j=1}^{m}\psi_{ij}(t,\omega)\mathrm{d}\mathcal{B}_j(t) + \sum_{j=1}^{\ell}\int_{\mathbb{R}}\Upsilon_{ij}(t,u_j,\omega)\widetilde{\mathcal{N}}_j(\mathrm{d}t,\mathrm{d}u_j), \quad 1\leqslant i\leqslant n.$$

Let $\mathcal{W}\in\mathcal{C}^{1,2}([0,T]\times\mathbb{R}^n;\mathbb{R})$, and define $\mathcal{A}(t) = \mathcal{W}(t,\mathcal{X}(t))$. Then,

$$\begin{aligned}
\mathrm{d}\mathcal{A}(t) = {}& \frac{\partial\mathcal{W}}{\partial t}\mathrm{d}t + \sum_{i=1}^{n}\frac{\partial\mathcal{W}}{\partial x_i}\Big(\varphi_i\mathrm{d}t + \psi_i\mathrm{d}\mathcal{B}(t)\Big) + \frac{1}{2}\sum_{i,j=1}^{n}\big(\psi\psi^\top\big)_{ij}\frac{\partial^2\mathcal{W}}{\partial x_i\partial x_j}\mathrm{d}t \\
&+ \sum_{k=1}^{\ell}\int_{\mathbb{R}\setminus\{0\}}\left\{\mathcal{W}\Big(t,\mathcal{X}(t^-)+\Upsilon^{(k)}(t,u_k)\Big) - \mathcal{W}(t,\mathcal{X}(t^-)) - \sum_{i=1}^{n}\frac{\partial\mathcal{W}}{\partial x_i}(t,\mathcal{X}(t^-))\Upsilon_i^{(k)}(t,u_k)\right\}\nu_k(\mathrm{d}u_k)\mathrm{d}t \\
&+ \sum_{k=1}^{\ell}\int_{\mathbb{R}\setminus\{0\}}\Big\{\mathcal{W}\big(t,\mathcal{X}(t^-)+\Upsilon^{(k)}(t,u_k)\big) - \mathcal{W}(t,\mathcal{X}(t^-))\Big\}\widetilde{\mathcal{N}}_k(\mathrm{d}t,\mathrm{d}u_k),
\end{aligned}$$

where $\Upsilon^{(k)}\in\mathbb{R}^n$ is the k-th column of the $n\times\ell$ matrix $\Upsilon = (\Upsilon)_{\substack{1\leqslant k\leqslant\ell\\ 1\leqslant i\leqslant n}}$, and $\Upsilon_i^{(k)} = \Upsilon_{ik}$ is the coordinate number i of $\Upsilon^{(k)}$. Generally, the Itô-Lévy isometries assert the following outcomes:

- $\mathbb{E}\left[\left(\int_0^T\psi(s,\omega)\mathrm{d}\mathcal{B}(s)\right)^2\right] = \mathbb{E}\left[\int_0^T\psi^2(s,\omega)ds\right]$[a].
- $\mathbb{E}\left[\left(\int_0^T\int_{\mathbb{R}\setminus\{0\}}\Upsilon(s,u,\omega)\widetilde{\mathcal{N}}(\mathrm{d}s,\mathrm{d}u)\right)^2\right] = \mathbb{E}\left[\int_0^T\int_{\mathbb{R}\setminus\{0\}}\Upsilon^2(s,u,\omega)\nu(\mathrm{d}u)\mathrm{d}s\right]$.[b]

If the last quantity is finite, then $\mathcal{M}(t) = \int_0^t\int_{\mathbb{R}\setminus\{0\}}\eta(s,u,\omega)\widetilde{\mathcal{N}}(\mathrm{d}s,\mathrm{d}u)$[c] becomes a martingale for $t\leqslant T$ with the following quadratic variation

$$\langle\mathcal{M},\mathcal{M}\rangle_t = \int_0^t\int_{\mathbb{R}\setminus\{0\}}\Upsilon^2(s,u,\omega)\nu(\mathrm{d}u)\mathrm{d}s.$$

10.3 Itô-Lévy stochastic differential equations

Theorem 10.2. *Consider the Lévy stochastic differential equation (SDE) in $\mathbb{R}^n$ given by:*

$$\mathrm{d}\mathcal{X}(t) = \varphi(t,\mathcal{X}(t))\mathrm{d}t + \psi(t,\mathcal{X}(t))\mathrm{d}\mathcal{B}(t) + \int_{\mathbb{R}^n\setminus\{0\}}\Upsilon(t,\mathcal{X}(t^-),u)\widetilde{\mathcal{N}}(\mathrm{d}t,\mathrm{d}u), \quad \mathcal{X}(0) = \mathcal{X}_0\in\mathbb{R}^n, \tag{I.26}$$

where $\varphi : [0,T]\times\mathbb{R}^n \to \mathbb{R}^n$, $\psi : [0,T]\times\mathbb{R}^n \to \mathbb{R}^{n\times m}$, and $\Upsilon : [0,T]\times\mathbb{R}^n\times\mathbb{R}^n \to \mathbb{R}^{n\times\ell}$ verify the following assumptions for some positive constants $\hat{\mathfrak{C}}_1,\hat{\mathfrak{C}}_2\in\mathbb{R}_+$ and for each $(t,x,y)\in\mathbb{R}_+\times\mathbb{R}^n\times\mathbb{R}^n$:

[a] The Itô isometry.
[b] The Lévy isometry.
[c] The martingale property.

► ***Lipschitz continuity condition**:*

$$|\varphi(t,x)-\varphi(t,y)|^2+||\psi(t,x)-\psi(t,y)||^2+\sum_{k=1}^{\ell}\int_{\mathbb{R}\setminus\{0\}}|\Upsilon^{(k)}(t,x,u_k)-\Upsilon^{(k)}(t,y,u_k)|^2\nu_k(\mathrm{d}u_k)\leqslant\hat{\mathfrak{C}}_1|x-y|^2. \tag{I.27}$$

► ***Linear growth condition**:*

$$|\varphi(t,x)|^2+||\psi(t,x)||^2+\int_{\mathbb{R}\setminus\{0\}}\sum_{k=1}^{\ell}|\Upsilon_k(t,x,u)|^2\nu_k(\mathrm{d}u_k)\leqslant\hat{\mathfrak{C}}_2\Big(1+|x|^2\Big). \tag{I.28}$$

Then, there is a unique cadlag adapted solution $\mathcal{X}(t)$ to the equation I.26, and $\mathbb{E}\left(|\mathcal{X}(t)|^2\right)<\infty$ for all $t\geqslant 0$.

Remark 10.2. *In the time-homogeneous case, where $\varphi(t,x)=\varphi(x)$, $\psi(t,x)=\psi(x)$, and $\Upsilon(t,x,u)=\Upsilon(x,u)$, the solutions of Lévy SDEs are referred to as Lévy jump diffusions.*

Now, let's examine this pair of one-dimensional stochastic differential equations driven by a Brownian motion $(\mathcal{B}_t)_{t\geqslant 0}$ and a Poisson process $(\mathcal{N}_t)_{t\geqslant 0}$:

$$\begin{aligned}\mathcal{X}_1(t)&=\mathcal{X}_1(0)+\int_0^t\varphi_1(s,\mathcal{X}_1(s))\mathrm{d}s+\int_0^t\psi_1(s,\mathcal{X}_1(s))\mathrm{d}\mathcal{B}(s)+\int_0^t\int_{\mathbb{R}^n\setminus\{0\}}\Upsilon_1(s,\mathcal{X}_1(s^-),u)\widetilde{\mathcal{N}}(\mathrm{d}s,\mathrm{d}u),\\ \mathcal{X}_2(t)&=\mathcal{X}_2(0)+\int_0^t\varphi_2(s,\mathcal{X}_2(s))\mathrm{d}s+\int_0^t\psi_2(s,\mathcal{X}_2(s))\mathrm{d}\mathcal{B}(s)+\int_0^t\int_{\mathbb{R}^n\setminus\{0\}}\Upsilon_2(s,\mathcal{X}_2(s^-),u)\widetilde{\mathcal{N}}(\mathrm{d}s,\mathrm{d}u),\end{aligned} \tag{I.29}$$

Here, $\mathcal{X}_1(0)$ and $\mathcal{X}_2(0)$ denote respectively the initial states of these two stochastic differential equations (SDEs) at the initial time $t=0$. The primary goal of the ensuing theorem is to identify a necessary and sufficient condition for the coefficients above that guarantees

$$\mathcal{X}_1(0)\leqslant\mathcal{X}_2(0)\implies\mathcal{X}_1(t)\leqslant\mathcal{X}_2(t),\quad\forall t\in[0,T],\quad\mathbb{P}-a.s. \tag{I.30}$$

Theorem 10.3 (Stochastic Comparison Theorem, [31]). *Assuming that (I.27) and (I.28) are satisfied, the following conditions are equivalent:*

1. *(I.30) holds for the stochastic differential equations (I.29);*
2. *For every $(t,x)\in[0,T]\times\mathbb{R}$, the coefficients φ_i, ψ_i, and Υ_i, $i=1,2$, satisfy:*

$$\begin{cases}\varphi_1(t,x)\leqslant\varphi_2(t,x),\\ \psi_1(t,x)=\psi_2(t,x),\\ \Upsilon_1(t,x,\bullet)=\Upsilon_2(t,x,\bullet),\quad\nu\text{-}a.s.\end{cases}$$

Remark 10.3. *For equations that do not incorporate the concept of jumps, the hypothetical framework of the last theorem simplifies to only the first two conditions, as the third one is automatically satisfied in this case (since $\Upsilon_1=\Upsilon_2=0$).*

11 Conclusion

In this chapter, we got an overview of the fundamentals of stochastic analysis and jump-diffusion theory, as well as some classic tools that will be used in subsequent chapters to examine various perturbed epidemic models. It is important to note that this chapter serves as a compilation of definitions, notations and theorems essential to understanding the main outcomes of this thesis. For more detailed information and deeper insights, readers are encouraged to consult the works referenced in this chapter (see [18, 23, 19, 20, 21, 22, 31]).

Chapter II

Advanced and comprehensive research on the dynamics of COVID-19 under mass communication outlets intervention and quarantine strategy: a deterministic and probabilistic approach

Abstract

The Coronavirus disease 2019 (COVID-19) is a major crisis that has significantly affected the healthcare sector and global economies, which made it the main subject of various fields in scientific and technical research. To properly understand and control this new epidemic, mathematical modeling is presented as a very effective tool that can illustrate the mechanisms of its propagation. In this regard, the use of compartmental models is the most prominent approach adopted in the literature to describe the dynamics of COVID-19. Along the same line, we aim in this chapter to present some generalizations and amelioration of many existing works that are consecrated to analyze the behavior of this epidemic. Precisely, we propose an SQEAIHR (Susceptible-Quarantined-Exposed-Asymptomatically infective- Infected-Hospitalized-Recovered) epidemic system for Coronavirus. Our constructed model is enriched by taking into account the media intervention and vital dynamics. By the use of the next-generation matrix method, the theoretical basic reproductive number $\mathcal{R}_0$ is obtained for COVID-19. Based on some nonstandard and generalized analytical techniques, the local and global stability of the disease-free equilibrium are proven when $\mathcal{R}_0 < 1$. Moreover, in the case of $\mathcal{R}_0 > 1$, the uniform persistence of COVID-19 model is also shown. In order to better adapt our epidemic model to reality, the randomness factor is taken into account by considering proportional white noises, which leads to a well-posed stochastic model. Under appropriate conditions, interesting asymptotic properties are proved, namely: extinction and persistence in the mean. The theoretical results show that the dynamics of the perturbed COVID-19 model are determined by parameters that are closely related to the magnitude of the stochastic noise. Finally, we present some numerical illustrations to confirm our theoretical results and to show the impact of media intervention and quarantine strategies.

1 Introduction and model formulation

The control of human and animal epidemics is principally based on modeling and simulation as the main decision-making tools [32, 3]. Anyhow, each epidemic is distinguished by its own biological

characteristics, which impose the adaptation of the dynamical models describing their propagation mechanisms to any specific case, and this in order to deal with real situations [33, 34, 35]. Presently, the whole world is under a tremendous threat due to the Coronavirus disease, which is a highly contagious virus that first appeared in China at the end of 2019, and spread rapidly to cover almost the entire globe [36]. Several researchers have discovered that this infectious disease, commonly called COVID-19 or 2019-nCoV, is caused by a new generation of beta-coronavirus [2], which affects the lungs and leads to the severe acute respiratory syndrome, the reason why the World Health Organization (WHO) renamed it SARS-CoV-2 [37]. Because of the exponential increase in cases and victims that it caused, the Coronavirus was declared to be an international public health emergency [38]. However, the danger did not stop at all, and the disease area carried on expanding to cover more than a hundred countries in March 2020, until the WHO had considered it as a pandemic in April of the same year when the statistics revealed that the number of infected populations and deaths surpassed respectively $800,000$ and $40,000$ [39]. Despite the existence of many suggested COVID-19 vaccines for which some national authorities have already permitted the emergency use [40], none is proven to be completely safe yet. According to the WHO, these vaccines are not rigorously tested and still in the phase of large clinical trial (see [41, 42]). The absence of an WHO-officially authorized or recommended treatment [42] remains a genuine challenge for all the governments, especially with the significant and noticeable repercussions that this epidemic presents in the economic and health realms. In the case of this new virus, the majority of transmissions is occurred by respiratory droplets that may be inhaled from close contact with an infected person when he exhales, sneezes or coughs [43]. Moreover, these droplets fall quickly on the floors and surfaces which makes them also a possible source of infection. Currently, the adoption of suitable strategies to tackle Coronavirus transportation presents big defiance for all the decision-makers around the world, and to overcome it, a good understanding of the pandemic evolution dynamics is really required.

Developing an appropriate mathematical model is a prominent method to purvey essential instructions and guidelines measures for disease mitigation. In this context, compartmental systems, like the simple SIR or the more advanced ones such as SIRS, SEIRS, SEIRQ and others, (see [8, 3]) can be an inspiring choice to deal with the critical situation that we are going through now. Since its discovery, a lot of models have been suggested for the study of COVID-19 dynamics [44, 45, 46]. Some works like [47], suggested a classical SIR system for predicting and analyzing the novel Coronavirus, other ones such as [48, 49] used a modified version of this system for the purpose of being more adapted to the dynamics transmission of SARS-CoV-2. With regards to this new virus characteristics, the last mentioned model was expanded to an SEIR one by taking into consideration a latency period during which the infected individuals are not infectious yet. Yang and Wang [50] used exactly this extended model to describe COVID-19 dissemination in Wuhan, China. They considered various transmission ways and incorporated the importance of the environmental reservoir. In the same context, Kucharski et al. [44] adopted an SEIR model to study the transmission variation of COVID-19 at the beginning of 2020. For a more rigorous vision of deaths number evolution, the authors in [51] developed a new form of the SEIR model by adding a deceased individuals class denoted (D). They utilized a fractional-order formulation SEIRD and arrived to the fact that this latter is more suitable than the classical one introduced in [52], and has less root mean square deviation. Inspired by the studies presented in [53, 54] and [55], Ivorra et al. [39] treated a fractional SEIHRD model to give successful forecasts about future variation in cases and deaths statistics, bearing in mind the existence of undetected infectious individuals. They proposed a new strategy that considers a proportion of the detected cases over total infected ones and exhibited the impact of this percentage, usually denoted θ, on the spread level of COVID-19. We mention that there is many interesting works related to the estimations of θ for the Coronavirus (see, e.g., [56, 57]). By considering the isolation strategy which played a significant role in controlling many diseases, Pal et al [58] proposed an SEQIR system to describe the outbreak of the Coronavirus. They determined the basic reproduction number $\mathcal{R}_0$ in terms of system parameters, and on the basis of its expression, they analyzed the stability dynamics of their model. In the same regard, Hu et al. [59] showed the effect of the strategy stated above on the

COVID-19 evolution by using an SEIRQ compartmental system that takes into account the input population. They discussed different scenarios of the epidemic spread on Guangdong province by using the officially published data. In order to be well adapted to the prevalence mechanisms of the current epidemic, Jia et al [60] suggested a new mathematical model that counts on the isolation and treatment besides home quarantine as prominent strategies to reduce the contagion intensity, without neglecting the existence of an asymptomatic transmission. In the case of COVID-19, it is important to point out that there seems to be a myriad of asymptomatic infected individuals [61] with a significant case fatality ratio, but it stills noticeably lower than MERS-CoV or SARS-CoV [62]. They also effectuated the parameters estimation building on officially published data and the well-known Least-Squares approach before passing to calculate the control reproduction number $\mathcal{R}_c$ for most Chinese provinces. It should be noted that the aforementioned study supposed the existence of meteorological factors effects on the virus activity, justifying this by the high level of genetic similarity between the current SARS-COV-2 and SARS-COV, and the role of temperature elevation in the disappear of this latter in 2003 [63]. In spite of its formulation complexity, the previously mentioned model has a realistic hypothetical framework, which made it a perfect basis for many interesting contributions [64, 65].

Along the same line, in this chapter, we will propose and analyze a generalized form of the model presented in [60]. Our new version will comprise two main extra hypotheses: the demographic variations and media intervention. The first addition's objective is to enhance this model and make it able to describe the behavior of the current pandemic over a long period of time by including natural birth and general mortality rates. For the second one, we consider the role of the various awareness campaigns and daily reports announced by different mass communication outlets like radio, television, newspaper, internet etc. In order to formulate the model mathematically under the previous assumptions, we consider a host population denoted at time t by $N(t)$ which is partitioned into seven classes of susceptible, quarantined, exposed, infectious with symptoms, asymptomatically infected, hospitalized and recovered individuals, with densities respectively denoted by $S(t), Q(t), E(t), A(t), I(t), H(t)$ and $R(t)$. The overall interactions between these classes are described through the system of deterministic ordinary differential equations below:

$$\begin{cases} \dfrac{\mathrm{d}S}{\mathrm{d}t} = \Lambda - \left(\beta_1 - \beta_2 \dfrac{I}{b+I}\right) S\,(I + \theta A) + \lambda Q - (\mu + q)S, \\ \dfrac{\mathrm{d}Q}{\mathrm{d}t} = qS - (\mu + \lambda)\,Q, \\ \dfrac{\mathrm{d}E}{\mathrm{d}t} = \left(\beta_1 - \beta_2 \dfrac{I}{b+I}\right) S\,(I + \theta A) - (\mu + \sigma)E, \\ \dfrac{\mathrm{d}A}{\mathrm{d}t} = (1-p)\,\sigma E - (\mu + \varepsilon_A + \gamma_A + d_A)\,A, \\ \dfrac{\mathrm{d}I}{\mathrm{d}t} = \sigma p E - (\mu + \varepsilon_I + \gamma_I + d_I)\,I, \\ \dfrac{\mathrm{d}H}{\mathrm{d}t} = \varepsilon_I I + \varepsilon_A A - (\mu + d_H + \gamma_H)\,H, \\ \dfrac{\mathrm{d}R}{\mathrm{d}t} = \gamma_H H + \gamma_I I + \gamma_A A - \mu R. \end{cases} \tag{II.1}$$

In this system, it was assumed that exposed individuals $E(t)$ are not infectious, and being in reality low-level virus carriers [39] explains the adoption of this assumption. Due to the exerted efforts by local authorities, like the closure of educational institutions, traffic control, travel restrictions, extension of vacations and postponing the return to work, we accept in (II.1) that there is no contact chance between home quarantined individuals $Q(t)$ and infectious population. For the hospitalized people $H(t)$, we suppose that they are isolated and in treatment process. Based on the studies presented in [66, 67, 68], it is admitted in (II.1) that there is no vertical transmission from mother to fetus, also, we presume that the total lockdown policy is applied. So, the recruitment rate Λ of uninfected population corresponds only to natural births. In addition, and similarly to

[65, 69], a media-induced incidence function is used to depict the disease infection mechanism in our system. We take first β_1 as the usual contact rate before media intervention and we suppose that it undergoes a reduction expressed by $\frac{\beta_2 I}{b+I}$, when infective people are reported in the media. Note that β_2 is none other than the maximum reduced contact[a] rate due to the presence of infective individuals $I(t)$ $\left(\text{i.e. } \max_{I\geqslant 0} \frac{\beta_2 I}{b+I}\right)$ and b is the half-saturation[b] constant that presents the effect of media alert on the transmission rate (see [70] for more details). It should be clarified that media coverage is able to reduce the intensity of the disease propagation, but cannot completely prevent it, therefore, we assume that $\beta_1 \geqslant \beta_2 > 0$. Logically speaking, symptomatic infected individuals should have a greater contagion rate than asymptomatic ones, and this is the clue why we have denoted the ration between these two rates as $\theta \in (0,1)$. Regarding the home confined population, we have used parameters p and λ in reference respectively to the quarantined and release rates. The constant σ in (II.1) is actually the passage rate from exposed individuals class to infected compartments A and I, but with a probability $p \in (0,1)$ of becoming symptomatic and $1-p$ for being asymptomatic. The parameters ε_A and ε_I stand respectively for the hospitalization rates of asymptomatic and symptomatic infective, whereas γ_A,γ_I and γ_H are the recovery rates of classes A, I, H. Moreover, the constants μ, d_A, d_I and d_H are representing, in this order, the natural death rate of the whole population and the disease-induced death rates affecting only from classes H, A and I. Based on the abovementioned modeling assumptions, the proposed system is illustrated by the following schematic flow diagram:

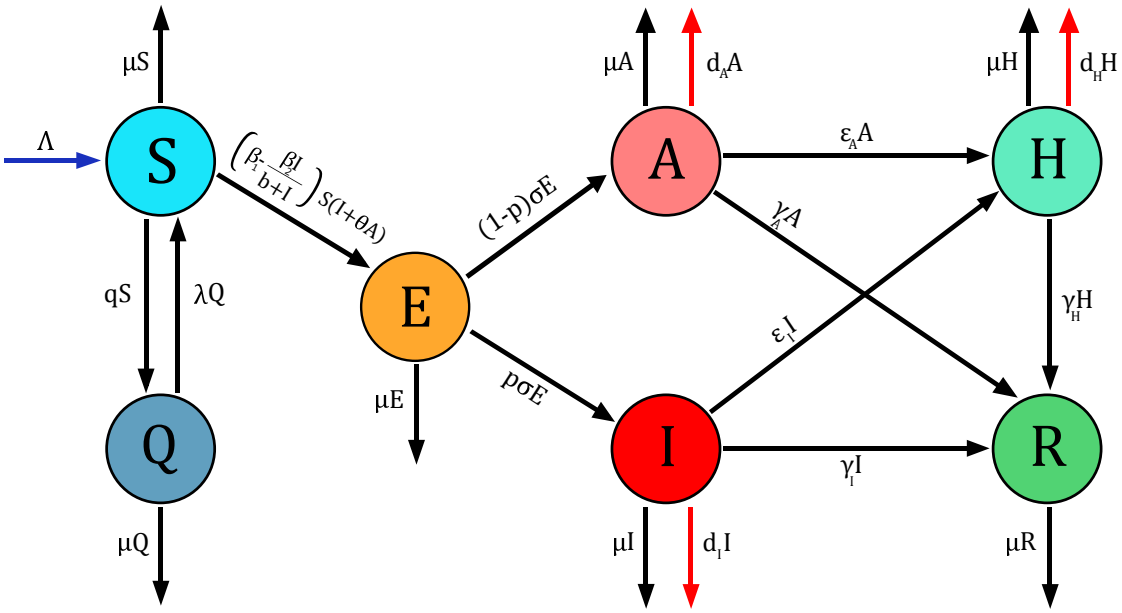


Figure II.1: Schematic diagram of the proposed COVID-19 epidemic model.

Generally speaking, the deterministic formulations analysis is very necessary and commonly used in the mathematical epidemiology, and it can be seen as a first tool for modeling new diseases spread and getting an overview of their asymptotic behavior. However, the real phenomena are not always deterministic and may be subject to some uncertainties and randomness due to fluctuations in the natural environment [71, 72, 73]. Therefore, an adapted version that considers this stochasticity is requisite in the case of COVID-19. For this purpose, and like several works [74, 75, 69], we extend the system (II.1) to the following probabilistic version that incorporates

[a] Here, "contact" designates the interaction between susceptible individuals (S) and infectious ones (A and I).

[b] The half-saturation constant b in the saturated function $f(I) = \frac{\beta_2 I}{b+I}$ represents the value of I at which the function reaches half of its maximum value β_2.

proportional Gaussian white noises:

$$\begin{cases} \mathrm{d}S = \left[\Lambda - \left(\beta_1 - \beta_2 \dfrac{I}{b+I}\right) S\,(I+\theta A) + \lambda Q - (\mu+q)S\right] \mathrm{d}t + \sigma_1 S\ \mathrm{d}B_1(t), \\ \mathrm{d}Q = [qS - (\mu+\lambda)\,Q]\,\mathrm{d}t + \sigma_2 Q\ \mathrm{d}B_2(t), \\ \mathrm{d}E = \left[\left(\beta_1 - \beta_2 \dfrac{I}{b+I}\right) S\,(I+\theta A) - (\mu+\sigma)E\right] \mathrm{d}t + \sigma_3 E\ \mathrm{d}B_3(t), \\ \mathrm{d}A = [(1-p)\,\sigma E - (\mu+\varepsilon_A+\gamma_A+d_A)\,A]\,\mathrm{d}t + \sigma_4 A\ \mathrm{d}B_4(t), \\ \mathrm{d}I = [\sigma p E - (\mu+\varepsilon_I+\gamma_I+d_I)\,I]\,\mathrm{d}t + \sigma_5 I\ \mathrm{d}B_5(t), \\ \mathrm{d}H = [\varepsilon_I I + \varepsilon_A A - (\mu+d_H+\gamma_H)\,H]\,\mathrm{d}t + \sigma_6 H\ \mathrm{d}B_6(t), \\ \mathrm{d}R = [\gamma_H H + \gamma_I I + \gamma_A A - \mu R]\,\mathrm{d}t + \sigma_7 R\ \mathrm{d}B_7(t). \end{cases} \tag{II.2}$$

Here and subsequently, σ_i $(i = 1, \cdots, 7)$ are denoting the positive intensities of the mutually independent Brownian motions B_i $(i = 1, \cdots, 7)$. These latter, and all the random variables that we will meet in this chapter, are presumed to be defined on a complete probability space $(\Omega, \mathfrak{F}, \mathbb{P})$ with a filtration $(\mathfrak{F}_t)_{t\geqslant 0}$, that satisfies the usual conditions, i.e., it is right-continuous while $\mathfrak{F}_0$ contains all $\mathbb{P}-$null sets.

The primary purpose of this chapter is to provide an overview of COVID-19 dynamics and examine its various properties over a relatively long period. Since the asymptotic analysis of epidemic models provides an excellent insight into the future pandemic situation, we decide in this study to treat it for our suggested system of COVID-19 in both deterministic and probabilistic levels. In the deterministic framework, we highlight several methods which make it possible to obtain, and under certain conditions, of course, the local and global stability of the disease-free equilibrium as well as the uniform persistence of the epidemic, without resorting to Lyapunov theory [76]. For the perturbed version of our model, we check in detail its well-posedness. Furthermore, we study the extinction case and present by using some new techniques a sufficient condition for the persistence of Coronavirus.

The rest of this chapter is arranged as follows: in Section 2, we demonstrate that the deterministic system (II.1) has a unique global positive solution. Next, we show the existence and uniqueness of disease-free and endemic equilibria of this system. Moreover, we investigate besides the local and global stability of disease-free equilibrium (DFE) the uniform persistence of the epidemic. In Section 3, we will first prove the existence and uniqueness of the positive solution of the stochastic system (II.2). Then, we show the extinction and persistence in the mean of COVID-19 under some conditions. In Section 4, we support our findings with some numerical simulations to illustrate our theoretical results and the effect of control strategies (coverage media and quarantine). Finally, the results are discussed in Section 5.

2 Analysis of the deterministic Coronavirus model

This section is devoted to the study of the deterministic COVID-19 model presented by the aforementioned differential system (II.1), but before starting our analysis, we will first associate to (II.1) the following initial-value problem:

$$\begin{cases} \overbrace{\dfrac{\mathrm{d}x(t)}{\mathrm{d}t}}^{7\times 1} = \overbrace{F\,(x(t))}^{7\times 1}, \\ x(0) = x_0 \in \mathbb{R}_+^7, \end{cases} \tag{II.3}$$

where

$$x(t) = \big(x_i(t)\big)_{1\leqslant i\leqslant 7} = \big(S(t), Q(t), E(t), A(t), I(t), H(t), R(t)\big),$$

and

$$F\left(x(t)\right)=\left(F_i\left(x(t)\right)\right)_{1\leqslant i\leqslant 7}=\begin{pmatrix}\Lambda-\left(\beta_1-\beta_2\dfrac{I}{b+I}\right)S\left(I+\theta A\right)+\lambda Q-(\mu+q)S\\ qS-(\mu+\lambda)\,Q\\ \left(\beta_1-\beta_2\dfrac{I}{b+I}\right)S\left(I+\theta A\right)-(\mu+\sigma)E\\ (1-p)\,\sigma E-(\mu+\varepsilon_A+\gamma_A+d_A)\,A\\ \sigma pE-(\mu+\varepsilon_I+\gamma_I+d_I)\,I\\ \varepsilon_I I+\varepsilon_A A-(\mu+d_H+\gamma_H)\,H\\ \gamma_H H+\gamma_I I+\gamma_A A-\mu R\end{pmatrix}. \tag{II.4}$$

2.1 Well-posedness

In this subsection, we will show that the Coronavirus model presented by system (II.1) is well posed, in the sense that if $S_0, Q_0, E_0, A_0, I_0, H_0$ and R_0 are positive, then the initial-value problem, or the Cauchy problem [77], (II.3) admits one and only one solution which is global in time, positive, and bounded.

Theorem 2.1. *If the initial value x_0 is in the positive orthant $\mathbb{R}^7_+$, then there exists a unique solution $x(t)$ to the system (II.3). Moreover, this solution remains bounded and positive for all $t\geqslant 0$.*

Proof. First of all, it is obvious that F is a locally Lipschitz continuous function on the open connected domain $\mathbb{R}^7_+$. Therefore, the famous Picard-Lindlöf theorem [78], called also the Cauchy-Lipschitz theorem [79], guarantees the existence of a unique maximal solution to the initial-value problem (II.3). This solution is defined on an interval $[0,\tau_e)$, where τ_e is the explosion time [80]. Moreover, it can be easily seen from (II.1) that for all $(S,Q,E,A,I,H,R)\in\mathbb{R}^7_+$,

$$\begin{cases}F_1\left(0,Q,E,A,I,H,R\right)=\Lambda+\lambda Q\geqslant 0,\\ F_2\left(S,0,E,A,I,H,R\right)=qS\geqslant 0,\\ F_3\left(S,Q,0,A,I,H,R\right)=\left(\beta_1-\beta_2\dfrac{I}{b+I}\right)S\left(I+\theta A\right)\geqslant 0,\\ F_4\left(S,Q,E,0,I,H,R\right)=\sigma(1-p)E\geqslant 0,\\ F_5\left(S,Q,E,A,0,H,R\right)=\sigma pE\geqslant 0,\\ F_6\left(S,Q,E,A,I,0,R\right)=\varepsilon_I I+\varepsilon_A A\geqslant 0,\\ F_7\left(S,Q,E,A,I,H,0\right)=\gamma_H H+\gamma_I I+\gamma_A A\geqslant 0.\end{cases}$$

So, by using Proposition 4.1 of [81], it follows immediately that $\mathbb{R}^7_+$ is positively invariant for the dynamical system (II.1), which means that any solution trajectory starting from the cone $\mathbb{R}^7_+$ will stay inside it for all the future time $t\in[0,\tau_e)$. On the other hand, for any initial value $(S_0,Q_0,E_0,A_0,I_0,H_0,R_0)\in\mathbb{R}^7_+$, the total population $N(t)$ satisfies the following equality for all $t\in[0,\tau_e)$,

$$\frac{\mathrm{d}N(t)}{\mathrm{d}t}=\Lambda-\mu N(t)-d_A A(t)-d_I I(t)-d_H H(t). \tag{II.5}$$

From the positivity of the solution, we obtain

$$\frac{\mathrm{d}N(t)}{\mathrm{d}t}\leqslant\Lambda-\mu N(t). \tag{II.6}$$

Applying the well-known Gronwall's inequality in its differential form (see [82, page 20]) to (II.6) yields

$$N(t)\leqslant\frac{\Lambda}{\mu}+\left(N_0-\frac{\Lambda}{\mu}\right)e^{-\mu t}\leqslant\frac{\Lambda}{\mu}+N_0. \tag{II.7}$$

Therefore, the solution of system (II.3) is bounded on $[0,\tau_e)$. In addition, this last fact together with Proposition A.1 of [83] lead to the globality in time of our solution. Thus, the initial-value problem (II.3) is well posed as required. ∎

Remark 2.1. *In addition to its interesting mathematical interpretation, the importance of the preceding theorem lies essentially in the fact that it makes our model logical, realistic and biologically meaningful.*

2.2 Invariant, absorbing and attracting regions

In the theory of ordinary differential equations, the first step after proving the well-posedness of a dynamical system is to get a first glimpse of its asymptotic behavior's nature. For example, to know if its trajectories are bounded on a neighborhood of infinity or not. For that, we will present in this subsection some special regions of $\mathbb{R}^7_+$ to which the solutions of (II.1) are belongs always, converge, or at least remain close as $t \to \infty$. This situation is often described in terms of invariant, absorbing and attracting subsets which will be defined in the following.

Definition 2.1 (Invariant region [84, 85]). *A set $\Delta \subset \mathbb{R}^7$ is said to be an invariant region for (II.1), or more precisely a positively invariant region with respect to (II.1), if $x(0) \in \Delta$ implies $x(t) \in \Delta$ for all $t \geqslant 0$. That is, whenever x_0 is in Δ, the solution $x(t)$ of (II.1) which starts from x_0 remains in Δ for all future time.*

Remark 2.2. *It is fairly easy to see from Theorem 2.1 that $\mathbb{R}^7_+$ is a positively invariant set for (II.1) in the sense of the previous definition.*

Definition 2.2 (Absorbing region [86]). *We say that a region $\Delta \subset \mathbb{R}^7_+$ is absorbing for (II.1) if there is an open set $\mathcal{U} \subset \mathbb{R}^7_+$ containing it such that $x(0) \in \mathcal{U}$ implies the existence of some $T > 0$ for which $x(t) \in \Delta$ whenever $t \geqslant T$. In other words, any solution of (II.1) that starts from $\mathcal{U}$, will necessarily enter and stay in Δ after a certain time. In the particular case of $\mathcal{U} = \mathbb{R}^7_+$, the set Δ is called globally absorbent for (II.1).*

Remark 2.3. *The existence of a bounded absorbing set is considered as a specific and interesting property of the system, and generally this property is called dissipativity [86].*

Definition 2.3 (Attracting region [86, 76]). *A set $\Delta \subset \mathbb{R}^7_+$ is called an attracting region for the system (II.1) if there exists an open set $\mathcal{U} \subset \mathbb{R}^7_+$ containing it such that*

$$\lim_{t\to\infty} \operatorname{dist}(x(t, x_0), \Delta) = 0 \quad \text{for all } x_0 \in \mathcal{U}.$$

Here, $x(t, x_0)$ denotes the solution of (II.1) that satisfies the initial condition $x(0) = x_0$ and $\operatorname{dist}(\mathrm{p}, \mathbb{B})$ is the distance from a point $\mathrm{p} \in \mathbb{R}^7$ to a subset $\mathbb{B} \subset \mathbb{R}^7$, that is, the smallest distance from p to any point in $\mathbb{B}$. More explicitly,

$$\operatorname{dist}(\mathrm{p}, \mathbb{B}) := \inf_{x \in \mathbb{B}} \|x - \mathrm{p}\|.$$

If $\mathcal{U} = \mathbb{R}^7_+$, the region Δ will be said to be globally attracting for (II.1).

Remark 2.4. *Clearly, an absorbent set is always attracting, but the converse is generally false (see [86]).*

After having defined the notions of invariant, absorbing and attracting sets, we will turn now to the main results of this subsection. But before doing so, let us first introduce these notations which will be used from now on to simplify the writing:

$$S^o = \frac{\Lambda}{\mu} \times \frac{\mu + \lambda}{\mu + q + \lambda} \quad \text{and} \quad Q^o = \frac{\Lambda}{\mu} \times \frac{q}{\mu + q + \lambda}. \tag{II.8}$$

Theorem 2.2. *Let $\eta, \eta' \geqslant 0$ be two nonnegative numbers, and let S^o, Q^o be defined by (II.8). The following regions:*

1. $\mathcal{D}^\eta = \left\{(S, Q, E, A, I, H, R) \in \mathbb{R}^7_+ \mid S + Q + E + A + I + H + R \leqslant \frac{\Lambda}{\mu} + \eta\right\}$,
2. $\mathcal{D}_{\eta'} = \left\{(S, Q, E, A, I, H, R) \in \mathbb{R}^7_+ \mid S + Q + E + A + I + H + R \geqslant \frac{\Lambda}{\mu + d_I + d_A + d_H} - \eta'\right\}$,

3. $\mathcal{D}_{\eta'}^{\eta} = \left\{(S,Q,E,A,I,H,R) \in \mathbb{R}_+^7 \mid \frac{\Lambda}{\mu+d_I+d_A+d_H} - \eta' \leqslant S+Q+E+A+I+H+R \leqslant \frac{\Lambda}{\mu} + \eta\right\}$,

4. $\mathcal{S} = \left\{(S,Q,E,A,I,H,R) \in \mathbb{R}_+^7 \mid S \leqslant S^o \ \text{ and } \ Q \leqslant Q^o\right\}$,

are positively invariant with respect to the dynamical system (II.1).

Proof. For the reader's convenience, we will divide the proof into four parts, each of which will be devoted to showing the invariance of one of the sets appearing in the last theorem's statement. Throughout this demonstration, $x_0 = (S_0, Q_0, E_0, A_0, I_0, H_0, R_0) \in \mathbb{R}_+^7$ is a given value and $x(t) = (S(t), Q(t), E(t), A(t), I(t), H(t), R(t))$ is the solution of (II.1) that starts from it.

1. Let $\eta \geqslant 0$ and suppose that $x_0 \in \mathcal{D}^{\eta}$. From (II.5), we have for all $t \geqslant 0$

$$N(t) \leqslant \frac{\Lambda}{\mu} + \left(N_0 - \frac{\Lambda}{\mu}\right) e^{-\mu t}, \tag{II.9}$$

and since $x_0 \in \mathcal{D}^{\eta}$, it follows that $N_0 \in \left(0, \frac{\Lambda}{\mu} + \eta\right)$, then

$$N(t) \leqslant \frac{\Lambda}{\mu} + \left(N_0 - \frac{\Lambda}{\mu}\right) e^{-\mu t} \leqslant \frac{\Lambda}{\mu} + \eta \times e^{-\mu t} \leqslant \frac{\Lambda}{\mu} + \eta.$$

Therefore, $x(t) \in \mathcal{D}^{\eta}$ for all $t \geqslant 0$, which means that $\mathcal{D}^{\eta}$ is positively invariant for (II.1).

2. According to (II.7), we have

$$\frac{\mathrm{d}N(t)}{\mathrm{d}t} = \Lambda - \mu N(t) - d_A A(t) - d_I I(t) - d_H H(t), \quad \forall t \geqslant 0,$$

and by using the positivity of solutions we get for all $t \geqslant 0$

$$\frac{\mathrm{d}N(t)}{\mathrm{d}t} \geqslant \Lambda - (\mu + d_A + d_H + d_I)\, N(t).$$

Thus,

$$N(t) \geqslant \frac{\Lambda}{\mu + d_A + d_H + d_I} + \left(N_0 - \frac{\Lambda}{\mu + d_A + d_H + d_I}\right) e^{-\mu t}. \tag{II.10}$$

Then, if $x_0 \in \mathcal{D}_{\eta'}$, we obtain

$$N(t) \geqslant \frac{\Lambda}{\mu + d_A + d_H + d_I} - \eta' e^{-\mu t} \geqslant \frac{\Lambda}{\mu + d_A + d_H + d_I} - \eta',$$

which is equivalent to saying that $\mathcal{D}_{\eta'}$ is positively invariant with respect to (II.1).

3. The invariance of region $\mathcal{D}_{\eta'}^{\eta} = \mathcal{D}_{\eta'} \cap \mathcal{D}^{\eta}$ with respect to system (II.1) follows immediately from the fact that the intersection of two positively invariant sets is also a positively invariant set (see Proposition 2.31, [81]).

4. Let $x_0 \in \mathcal{S}$. From the system (II.1), we remark that

$$\left.\frac{\mathrm{d}S}{\mathrm{d}t}\right|_{S=0} = \Lambda + \lambda Q \geqslant 0, \qquad \left.\frac{\mathrm{d}Q}{\mathrm{d}t}\right|_{Q=0} = qS \geqslant 0, \qquad \left.\frac{\mathrm{d}E}{\mathrm{d}t}\right|_{E=0} = \left(\beta_1 - \frac{\beta_2 I}{b+I}\right) S\,(I+\theta A) \geqslant 0,$$

$$\left.\frac{\mathrm{d}A}{\mathrm{d}t}\right|_{A=0} = \sigma(1-p)E \geqslant 0, \qquad \left.\frac{\mathrm{d}I}{\mathrm{d}t}\right|_{I=0} = \sigma p E \geqslant 0, \qquad \left.\frac{\mathrm{d}H}{\mathrm{d}t}\right|_{H=0} = \varepsilon_I I + \varepsilon_A A \geqslant 0,$$

$$\left.\frac{\mathrm{d}R}{\mathrm{d}t}\right|_{R=0} = \gamma_H H + \gamma_I I + \gamma_A A \geqslant 0, \qquad \left.\frac{\mathrm{d}Q}{\mathrm{d}t}\right|_{Q=Q^o} = q(S - S^o) \leqslant 0, \qquad \left.\frac{\mathrm{d}S}{\mathrm{d}t}\right|_{S=S^o} = \lambda\,(Q - Q^o) - \beta S^o\,(I+\theta A) \leqslant 0.$$

Thus, the vector field on each boundary point of the region $\mathcal{S}$ is pointing toward its interior, and this prevents any solution that starts in $\mathcal{S}$ from leaving it (see [85]). Therefore, $\mathcal{S}$ is an invariant region for (II.1).

Hence, the proof is completed. ■

Remark 2.5. *The method used to prove the last point of Theorem 2.2 is known as the Bony-Brezis theorem, and for more details on this, we refer the reader to [87].*

From now on, we will keep the notations of Theorem 2.2.

Theorem 2.3. *For any positive numbers η and η', the set $\mathcal{D}_{\eta'}^{\eta}$ is globally absorbent with respect to the system (II.1).*

Proof. Let $\eta, \eta' > 0$ be two positive numbers, and let $x_0 \in \mathbb{R}_+^7$. Based on (II.5) and (II.10), we observe that for all $t \geqslant 0$

$$\frac{\Lambda}{\mu + d_A + d_H + d_I} + \left(N_0 - \frac{\Lambda}{\mu + d_A + d_H + d_I}\right) e^{-\mu t} \leqslant N(t) \leqslant \frac{\Lambda}{\mu} + \left(N_0 - \frac{\Lambda}{\mu}\right) e^{-\mu t}. \tag{II.11}$$

By letting t tend to infinity in (II.11), we obtain

$$\frac{\Lambda}{\mu + d_A + d_H + d_I} \leqslant \lim_{t \to \infty} N(t) \leqslant \frac{\Lambda}{\mu}.$$

So, necessarily there is a certain time, depending in η and η', from which we have

$$\frac{\Lambda}{\mu + d_A + d_H + d_I} - \eta' \leqslant N(t) \leqslant \frac{\Lambda}{\mu} + \eta.$$

Therefore, $\mathcal{D}_{\eta'}^{\eta}$ is a globally absorbent region for (II.1). ■

Remark 2.6. *Needless to say, the last theorem implies that the system (II.1) is dissipative. Moreover $\mathcal{D}_{\eta'}$, $\mathcal{D}^{\eta}$, or generally any region containing $\mathcal{D}_{\eta'}^{\eta}$, is also going to be globally absorbent for (II.1).*

Now, let us mention an important consequence of the previous theorem.

Corollary 2.1. *With respect to the dynamical system (II.1), the region*

$$\mathcal{D}_0^0 = \left\{(S, Q, E, A, I, H, R) \in \mathbb{R}_+^7 \mid \frac{\Lambda}{\mu + d_I + d_A + d_H} \leqslant S + Q + E + A + I + H + R \leqslant \frac{\Lambda}{\mu}\right\}$$

is globally attracting.

Proof. The attraction property of the set $\mathcal{D}_0^0$ with respect to (II.1) comes easily from the following observation:

$$\mathcal{D}_0^0 = \bigcap_{\eta > 0} \mathcal{D}_{\eta}^{\eta},$$

and the fact that the region $\mathcal{D}_{\eta}^{\eta}$ is globally absorbent with respect to (II.1) for any positive number η (see Theorem 2.3 in the particular case $\eta = \eta'$). The last passage can be explained by saying that reducing the η value, and bringing it close to zero leaves no choice for the solutions trajectories except converging to $\mathcal{D}_0^0$, which is the desired conclusion. ■

2.3 The Basic reproduction number $\mathcal{R}_0$

Epidemiologically, the basic reproduction number $\mathcal{R}_0$ is the number of secondary cases produced by one infected individual in an entirely susceptible population during its period as an infective. In the literature, several techniques have been proposed for the calculation of $\mathcal{R}_0$ [88, chapter 5], but the most known is that of the next generation approach introduced by van den Driessche and Watmough in [89]. According to this method, which will be used in our case, $\mathcal{R}_0$ is the spectral radius ρ of the next generation matrix defined by $M = FV^{-1}$, where V and F are respectively the matrices expressing the infections transition and the emergence of new infected cases in the different contaminated compartments of the model. By observing our system (II.1), we can easily

notice that it has two essential and particular properties. The first is the fact that it admits one, and only one, disease-free equilibrium (DFE) $\mathcal{E}^o = (S^o, Q^o, 0, 0, 0, 0, 0)$, where S^o and Q^o are defined by (II.8). The second is the possibility of rearranging its equations and rewriting it in the modified form

$$\frac{\mathrm{d}\widetilde{x}(t)}{\mathrm{d}t} = \widetilde{F}\left(\widetilde{x}(t)\right), \tag{II.12}$$

infected components — equations of E,A and I

where $\widetilde{x} = (\overbrace{E, A, I}, S, Q, H, R)$ and $\widetilde{F} = (\overbrace{F_3, F_4, F_5}, F_1, F_2, F_6, F_7)$. Needless to say, the last arrangement makes $\widetilde{\mathcal{E}}^o = (0, 0, 0, S^o, Q^o, 0, 0)$ the unique disease-free equilibrium of (II.12). As stated in [89], we can split the right-hand side of (II.12) in the following way:

$$\frac{\mathrm{d}\widetilde{x}(t)}{\mathrm{d}t} = \widetilde{F}\left(\widetilde{x}(t)\right) = \mathcal{F}\left(\widetilde{x}(t)\right) - \mathcal{V}\left(\widetilde{x}(t)\right), \tag{II.13}$$

where $\mathcal{F}$ is the appearance rate of new infections vector and $\mathcal{V}$ is the remaining transitional terms vector represented respectively in this case by

$$\mathcal{F} = \begin{pmatrix} \left(\beta_1 - \beta_2 \dfrac{I}{b+I}\right) S\left(I + \theta A\right) \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} \quad \text{and} \quad \mathcal{V} = \begin{pmatrix} (\mu+\sigma)E \\ (\mu+\varepsilon_A+\gamma_A+d_A)A - (1-p)\sigma E \\ (\mu+\varepsilon_I+\gamma_I+d_I)I - \sigma p E \\ \left(\beta_1 - \beta_2 \dfrac{I}{b+I}\right) S\left(I+\theta A\right) + (\mu+q)S - \lambda Q - \Lambda \\ (\mu+\lambda)Q - qS \\ (\mu+d_H+\gamma_H)H - \varepsilon_A A - \varepsilon_I I \\ \mu R - \gamma_H H - \gamma_I I - \gamma_A A \end{pmatrix}.$$

The corresponding Jacobian matrices evaluated around the DFE are respectively block decomposed as follows:

$$D\mathcal{F}(\widetilde{\mathcal{E}}^o) = \left(\begin{array}{c|c} F & \mathbf{0}_{3,4} \\ \hline \mathbf{0}_{4,3} & \mathbf{0}_{4,4} \end{array}\right) \quad \text{and} \quad D\mathcal{V}(\widetilde{\mathcal{E}}^o) = \left(\begin{array}{c|c} V & \mathbf{0}_{3,4} \\ \hline W_3 & W_4 \end{array}\right), \tag{II.14}$$

where $\mathbf{0}_{n,p}$ denotes the null matrix (i.e, a matrix filled with zeros) of dimension $n \times p$ and

$$F = \begin{pmatrix} 0 & \theta\beta_1 S^o & \beta_1 S^o \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \qquad V = \begin{pmatrix} \mu+\sigma & 0 & 0 \\ -(1-p)\sigma & \mu+d_A+\varepsilon_A+\gamma_A & 0 \\ -\sigma p & 0 & \mu+d_I+\varepsilon_I+\gamma_I \end{pmatrix},$$

$$W_3 = \begin{pmatrix} 0 & \theta\beta_1 S^o & 0 \\ 0 & 0 & 0 \\ 0 & -\varepsilon_A & -\varepsilon_I \\ 0 & -\gamma_A & -\gamma_I \end{pmatrix}, \qquad \text{and} \qquad W_4 = \left(\begin{array}{c|c} \mathbf{w}_1 & \mathbf{0}_{2,2} \\ \hline \mathbf{0}_{2,2} & \mathbf{w}_4 \end{array}\right) = \begin{pmatrix} \mu+q & -\lambda & 0 & 0 \\ -q & \mu+\lambda & 0 & 0 \\ 0 & 0 & \mu+d_H+\gamma_H & 0 \\ 0 & 0 & -\gamma_H & \mu \end{pmatrix}.$$

So, the next generation matrix FV^{-1} is

$$FV^{-1} = \begin{pmatrix} \left(\dfrac{\theta(1-p)}{\mu+d_A+\varepsilon_A+\gamma_A} + \dfrac{p}{\mu+d_I+\varepsilon_I+\gamma_I}\right) \times \dfrac{\sigma\beta S^o}{\mu+\sigma} & \dfrac{\theta\beta_1 S^o}{\mu+d_A+\varepsilon_A+\gamma_A} & \dfrac{\beta_1 S^o}{\mu+d_I+\varepsilon_I+\gamma_I} \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

Consequently, the reproduction number associated to the Coronavirus model (II.1) is given by

$$\mathcal{R}_0 \triangleq \rho\left(FV^{-1}\right) = \left(\frac{\theta(1-p)}{\mu+d_A+\varepsilon_A+\gamma_A} + \frac{p}{\mu+d_I+\varepsilon_I+\gamma_I}\right)\frac{\sigma\beta_1 S^o}{\mu+\sigma}. \tag{II.15}$$

2.4 Stability analysis of the disease-free steady state $\mathcal{E}^o$

In this subsection, we will deal with the issue of disease-free equilibrium stability in its local and global levels, and we will show the effect of $\mathcal{R}_0$ value on each of them, but before doing so, let us first present these needed notations and terminologies to follow this part without ambiguity.

- I_n is the identity matrix of order n.
- The real part of a complex number $z \in \mathbb{C}$ is written as $\mathfrak{Re}(z)$.
- The set of eigenvalues of a square matrix A, is denoted by the symbol $\Sigma(A)$.
- For a square matrix with complex entries M, the greatest real part of eigenvalues is called the spectral bound, or the stability modulus, of M (see [90]) and we write it as $\alpha(M) \triangleq \max_{\omega \in \Sigma(M)} \mathfrak{Re}(\omega)$.
- Finally, we remind that $\rho(M)$ refers to the maximum modulus of a square matrix M eigenvalues, that is, $\rho(M) \triangleq \max_{\omega \in \Sigma(M)} |\omega|$.

2.4.1 Local stability of the disease-free steady state $\mathcal{E}^o$

Theorem 2.4. *Under the condition $\mathcal{R}_0 < 1$, the disease-free equilibrium of the system (II.1) is locally asymptotically stable, but it becomes unstable if $\mathcal{R}_0 > 1$.*

Proof. To show the local stability of $\mathcal{E}^o = (S^o, Q^o, 0, 0, 0, 0, 0)$ for system (II.1), we will go through that of $\widetilde{\mathcal{E}}^o = (0, 0, 0, S^o, Q^o, 0, 0)$ for system (II.12) because, as we have already mentioned in the previous subsection, (II.12) is just a slightly modified version of (II.1) obtained by an arrangement of its lines. Obviously, the Jacobian matrix of system (II.12) at $\widetilde{\mathcal{E}}^o$ is given by the following block representation:

$$D\widetilde{F}(\widetilde{\mathcal{E}}^o) = D\left[\mathcal{F} - \mathcal{V}\right](\widetilde{\mathcal{E}}^o) = D\mathcal{F}(\widetilde{\mathcal{E}}^o) - D\mathcal{V}(\widetilde{\mathcal{E}}^o) = \left(\begin{array}{c|c} F - V & \mathbf{0}_{3,4} \\ \hline -W_3 & -W_4 \end{array}\right). \tag{II.16}$$

So, and with the help of Theorem 5.2.10 of [91], one can easily conclude that the eigenvalues of $D\widetilde{F}(\widetilde{\mathcal{E}}^o)$ are exactly the combined characteristic roots of $F - V$ and $-W_4$. In other terms,

$$\Sigma\left(D\widetilde{F}(\widetilde{\mathcal{E}}^o)\right) = \Sigma(F - V) \cup \Sigma(-W_4), \tag{II.17}$$

with Σ is referring to the spectrum (i.e., the set of eigenvalues). By the same argument, we can draw the following equality for the matrix $-W_4$:

$$\Sigma(-W_4) = \Sigma(-\mathbf{w}_1) \cup \Sigma(-\mathbf{w}_4),$$

which together with (II.17) implies that

$$\Sigma\left(D\widetilde{F}(\widetilde{\mathcal{E}}^o)\right) = \Sigma(F - V) \cup \Sigma(-\mathrm{w}_1) \cup \Sigma(-\mathrm{w}_4).$$

Then

$$\alpha\left(D\widetilde{F}(\widetilde{\mathcal{E}}^o)\right) = \alpha(F - V) \vee \alpha(-\mathbf{w}_1) \vee \alpha(-\mathbf{w}_4), \tag{II.18}$$

where

$$-\mathbf{w}_1 = \begin{pmatrix} -\mu - q & \lambda \\ q & -\mu - \lambda \end{pmatrix} \quad \text{and} \quad -\mathbf{w}_4 = \begin{pmatrix} -\mu - \gamma_H - d_H & 0 \\ \gamma_H & -\mu \end{pmatrix}.$$

The matrix $-\mathbf{w}_4$ is upper triangular, then its eigenvalues are exactly the diagonal entries, and since they are all negative in this case we get

$$\alpha(-\mathbf{w}_4) < 0. \tag{II.19}$$

On the other hand, it is easy to check that the characteristic polynomial of $-\mathbf{w}_1$ is

$$P(w) = \det(-\mathrm{w}_1 - w\mathrm{I}_2) = w^2 + \underbrace{(q + 2\mu + \lambda)}_{a_1} w + \underbrace{\mu(\mu + \lambda + q)}_{a_2}.$$

So, and by observing the positivity of a_1 and a_2, we can immediately conclude that

$$\alpha(-\mathbf{w}_1) < 0. \tag{II.20}$$

According to Lemma 1 of [89], V is a non-singular M-matrix (a real matrix with nonpositive off-diagonal entries and positive spectral bound [92]), and since F is nonnegative (i.e., all of whose entries are nonnegative), we can deduce by using Varga's theorem [93, 90, 94] that

$$\alpha(F - V) < 0 \iff \rho(FV^{-1}) < 1. \tag{II.21}$$

Combining (II.18), (II.19) and (II.20) with (II.21) gives the following equivalence:

$$\mathcal{R}_0 < 1 \iff \alpha\left(D\widetilde{F}(\widetilde{\mathcal{E}}^o)\right) < 0, \tag{II.22}$$

which leads us thanks to the famous Lyapunov's linearisation theorem [76, page 139], to the fact that $\mathcal{R}_0 < 1$ implies the local asymptotic stability of the disease-free equilibrium.

Now, the only remaining point is to check the instability of the DFE when $\mathcal{R}_0 > 1$. This time, the task is somewhat easy, especially when we use the second version of Varga's theorem [88, 32] which asserts that

$$\alpha(F - V) > 0 \iff \rho(FV^{-1}) > 1. \tag{II.23}$$

The last equivalence together with (II.18) and the Lyapunov's linearisation theorem allows us to say that the DFE is unstable if $\mathcal{R}_0 > 1$. Hence, the theorem is proved. ■

Remark 2.7. *The adopted method in the previous proof is a little different from what we usually see in the literature, since it shows the local stability of the disease-free state without using the Routh-Hurwitz criterion (see[88, page 101]) to the characteristic polynomial $\chi(w) = \det(DF(\mathcal{E}^o) - w\mathrm{I}_7)$, and this enabled us to sidestep lengthy and laborious calculations.*

Remark 2.8. *The fact that $\mathcal{R}_0 < 1$ implies the local asymptotic stability of the disease-free equilibrium, is wrong in general, and to obtain it we should ensure in addition the stability condition of the matrix $-W_4$ (i.e., $\alpha(-W_4) < 0$). This condition was formulated implicitly in [89] by the assumption* (A5)*, and it plays an important role in the completion of the proof concerning the DFE local asymptotic stability. But despite this, we note that there is not any reason to use it in the demonstration of instability in the case of $\mathcal{R}_0 > 1$.*

2.4.2 Global stability of the disease-free steady state $\mathcal{E}^o$

In the following, we seek to proof the global asymptotic stability of $\mathcal{E}^0$ on the biologically feasible region $\mathcal{S}$ (see Theorem 2.2) by applying an approach adapted from [95]. But before this, let us first follow the notation of [32] and write system (II.12) in the form:

$$\begin{cases} \dfrac{\mathrm{d}\widetilde{x}_1(t)}{\mathrm{d}t} = \widetilde{F}_1\left(\widetilde{x}_1(t), \widetilde{x}_2(t)\right) = \mathbf{M}\widetilde{x}_1(t) - \widehat{f}\left(\widetilde{x}_1(t), \widetilde{x}_2(t)\right), \\ \dfrac{\mathrm{d}\widetilde{x}_2(t)}{\mathrm{d}t} = \widetilde{F}_2\left(\widetilde{x}_1(t), \widetilde{x}_2(t)\right), \end{cases} \tag{II.24}$$

where $\mathbf{M} = F - V$, $\widetilde{x} = (\widetilde{x}_1, \widetilde{x}_2) = (\overbrace{E, A, I}^{\text{infected components } \widetilde{x}_1}, \underbrace{S, Q, H, R}_{\text{uninfected components } \widetilde{x}_2})$, $\widetilde{F} = (\widetilde{F}_1, \widetilde{F}_2) = (\overbrace{F_3, F_4, F_5}^{\text{equations of } E,A \text{ and } I}, \underbrace{F_1, F_2, F_6, F_7}_{\text{equations of } S,Q,H \text{ and } R})$ and

$$\widehat{f}\left(\widetilde{x}_1, \widetilde{x}_2\right) = \begin{pmatrix} \left(\beta_1 S^o - \left(\beta_1 - \beta_2 \dfrac{I}{b + I}\right) S\right)(I + \theta A) \\ 0 \\ 0 \end{pmatrix}.$$

Lemma 2.1. *Let $\mathcal{M} \subset \mathbb{R}^7_+$, if the following conditions are satisfied:*

(a) *The set $\mathcal{M}$ is a positively invariant region with respect to system (II.1).*

(b) *For the disease-free system $\dfrac{\mathrm{d}\widetilde{x}_2(t)}{\mathrm{d}t} = \widetilde{F}_2\left(0, \widetilde{x}_2(t)\right)$, the equilibrium $\widetilde{\mathcal{E}}^o = \left(0,0,0,S^o,Q^o,0,0\right)$ is globally asymptotically stable.*

(c) $\mathbf{M}$ *is a stable Metzler matrix (i.e.,$-\mathbf{M}$ is a non-singular M-matrix [90]).*

(d) *For all $x \in \mathcal{M}$, $\widehat{f}\left(\widetilde{x}_1, \widetilde{x}_2\right) \geqslant 0$.*

Then, the disease-free equilibrium $\mathcal{E}^o := (S^o, Q^o, 0, 0, 0, 0, 0)$ is globally asymptotically stable on $\mathcal{M}$.

Proof. Suppose that the conditions of this lemma hold. Let $x(0) \in \mathcal{M}$ and $\left(\widetilde{x}_1(t), \widetilde{x}_2(t)\right)$ be respectively a given initial value and the solution of (II.24) that starts from it. By using the famous variation-of-constant formula [79] for the first equation of system (II.24), we get

$$\widetilde{x}_1(t) = e^{t\mathbf{M}}\widetilde{x}_1(0) - \int_0^t e^{(t-s)\mathbf{M}}\widehat{f}\left(\widetilde{x}_1(s), \widetilde{x}_2(s)\right) \mathrm{d}s.$$

From the third condition of this lemma, $\mathbf{M}$ is a stable Metzler matrix, then it can be expressed in the form $\mathbf{M} = \widehat{\mathrm{B}} - \kappa \mathrm{I}_7$ where $\widehat{\mathrm{B}} \geqslant 0$ is a nonnegative matrix (i.e., all of whose entries are nonnegative) and $\kappa > \rho(\mathbf{M})$ (see [92]). Since $\widehat{\mathrm{B}} \geqslant 0$, we have $e^{\xi\widehat{\mathrm{B}}} \geqslant 0$ for all $\xi \geqslant 0$, then $e^{\xi\mathbf{M}} = e^{\xi\widehat{\mathrm{B}}}e^{-\xi\kappa\mathrm{I}_7} = e^{-\xi\kappa}e^{\xi\widehat{\mathrm{B}}} \geqslant 0$. This together with the invariance of the region $\mathcal{M}$, and the fact that $\widehat{f}$ is a nonnegative function on this region (assumptions (a),(d)) implies that

$$\widetilde{x}_1(t) = e^{t\mathbf{M}}\widetilde{x}_1(0) - \int_0^t e^{(t-s)\mathbf{M}}\widehat{f}\left(\widetilde{x}_1(s), \widetilde{x}_2(s)\right) \mathrm{d}s \leqslant e^{t\mathbf{M}}\widetilde{x}_1(0).$$

The matrix $\mathbf{M}$ is stable (i.e, $\alpha(\mathbf{M}) < 0$), so $\lim\limits_{t\to\infty} e^{t\mathbf{M}}\widetilde{x}_1(0) = 0$. Combining this fact with the positivity of the solution $\widetilde{x}(t)$ (Theorem 2.1) gives $\lim\limits_{t\to\infty} \widetilde{x}_1(t) = 0$. By using the condition (b) and an argument similar to the proof of Theorem 1 in [95] (see also [96, page 94] for more details) we can conclude immediately that $\mathcal{E}^o := (S^o, Q^o, 0, 0, 0, 0, 0)$ is globally asymptotically stable on $\mathcal{M}$, which proves the lemma. ∎

Remark 2.9. *Clearly, if we take $\mathcal{M} = \mathbb{R}^7_+$, the condition (a) in the last lemma becomes unnecessary because $\mathbb{R}^7_+$ is already positively invariant, and in this case, we recover exactly the statement of Theorem 9.2 of [32], which allows us to say that Lemma 2.1 is a generalization of this theorem.*

Remark 2.10. *According to the previously mentioned Varga's theorem (see (II.21)), the stability of the matrix $\mathbf{M} := F - V$ (i.e., $\alpha(F-V) < 0$) is equivalent to $\mathcal{R}_0 := \rho(FV^{-1}) < 1$, so the last assumption of Theorem 9.2 in [32] ($\mathcal{R}_0 < 1$) is redundant and can therefore be dropped.*

Theorem 2.5. *If $\mathcal{R}_0 < 1$, then the disease-free equilibrium $\mathcal{E}^o$ is globally asymptotically stable on $\mathcal{S}$.*

Proof. To demonstrate this theorem, we will use Lemma 2.1, in other words, we are going to show the global stability of $\mathcal{E}^o$ by proving that all the conditions (a),(b),(c) and (d) are satisfied. As supposed in the statement of the theorem, let $\mathcal{R}_0 < 1$. From the equations of (II.1), we observe that the disease-free system is expressed by

$$\frac{\mathrm{d}\widetilde{x}_2(t)}{\mathrm{d}t} = \widetilde{F}_2\left(0, \widetilde{x}_2(t)\right) = -W_4\,\widetilde{x}_2(t) + \begin{pmatrix}\Lambda\\ \mathbf{0}_{3,1}\end{pmatrix}. \tag{II.25}$$

Obviously, (II.25) is a linear differential system with $\alpha(-W_4) < 0$ (see the proof of Theorem 2.4), so the fixed point $\widetilde{\mathcal{E}}^o$ is a globally asymptotic stable equilibrium of (II.25) [76, Theorem 4.5]. Thus, the condition (b) is satisfied. On the other hand, for any $x \in \mathcal{S} = \left\{(S, \cdots, R) \in \mathbb{R}^7_+ \mid S \leqslant S^o \text{ and } Q \leqslant Q^o\right\}$, we have

$$\left(\beta_1 S^o - \left(\beta_1 - \beta_2\frac{I}{b+I}\right)S\right)(I + \theta A) = \left(\beta_1 \overbrace{(S^o - S)}^{\geqslant 0} + \beta_2\frac{I}{b+I}S\right)(I + \theta A) \geqslant 0.$$

Hence, $\widehat{f}\left(\widetilde{x}_1, \widetilde{x}_2\right) \geqslant 0$, for all $x \in \mathcal{S}$, which means that the condition (d) is also satisfied. Apparently, $\mathbf{M} = F - V$ is a Metzler matrix, and since $\mathcal{R}_0 < 1$, one can say that it is also stable (see (II.21)), so the assumption (c) holds. According to Theorem 2.2, the set $\mathcal{S}$ is an invariant region for (II.1). Therefore, (a) is satisfied and the conclusion of the theorem follows immediately from Lemma 2.1. ■

2.5 Existence of endemic equilibrium and uniform persistence

The endemic equilibria of the proposed COVID-19 model are obtained by solving the system $F(x) = 0$ on the positive orthant $\mathbb{R}^7_+$, where F is the vector-valued function given by (II.4). In other words, a point $\mathcal{E}^\star = \left(S^\star, Q^\star, E^\star, A^\star, I^\star, H^\star, R^\star\right)$ is an endemic equilibrium of (II.1) if and only if all its components are strictly greater than zero and satisfy the following equalities:

$$S^\star = \frac{\mu+\lambda}{q}\left(c - mc'I^\star\right), \qquad Q^\star = c - mc'I^\star, \qquad E^\star = c'I^\star, \qquad A^\star = c'm'I^\star, \qquad H^\star = \frac{\left(\varepsilon_I + \varepsilon_A c'm'\right)I^\star}{\mu + d_H + \gamma_H},$$
$$R^\star = \frac{1}{\mu}\left(\gamma_A c'm' + \gamma_I + \frac{\gamma_H\left(\varepsilon_I + \varepsilon_A c'm'\right)}{\mu + d_H + \gamma_H}\right)I^\star,$$

where

$$c = \Lambda\left((\mu+q)\times\frac{\mu+\lambda}{q} - \lambda\right)^{-1}, \qquad m = (\mu+\sigma)\left((\mu+q)\times\frac{\mu+\lambda}{q} - \lambda\right)^{-1},$$
$$c' = \frac{\mu + \varepsilon_I + \gamma_I + d_I}{\sigma p}, \qquad m' = \frac{\sigma(1-p)}{\mu + \varepsilon_A + \gamma_A + d_A},$$

and $I^\star$ is defined as the positive solution of this equation derived from the fact that $F_3(\mathcal{E}^\star) = 0$:

$$\left(\beta_1 - \beta_2\frac{I^\star}{b + I^\star}\right)S^\star\left(I^\star + \theta A^\star\right) = (\mu+\sigma)E^\star. \tag{II.26}$$

Substituting the aforementioned expressions of $S^\star$, $A^\star$ and $E^\star$ into (II.26) leads us, and after some simplifications, to the following equation:

$$\mathfrak{A}I^{\star 2} + \mathfrak{B}I^\star + \mathfrak{C} = 0,$$

where

$$\mathfrak{A} = mc'(\beta_1 - \beta_2) > 0, \quad \mathfrak{B} = \beta_1 bmc' + \beta_1 c\left(\frac{1}{\mathcal{R}_0} - 1\right) + \beta_2 c, \text{ and } \mathfrak{C} = \frac{q(\sigma + \mu c'b)}{(\mu+\lambda)(1+\theta m'c')}\left(1 - \mathcal{R}_0\right).$$

If $\mathcal{R}_0 > 1$, then $\mathfrak{C} < 0$, and since $\mathfrak{A} > 0$, the equation (II.26) admits a unique strictly positive solution. Therefore (II.1) admits in turn a unique endemic equilibrium. On the other hand, when $\mathcal{R}_0 \leqslant 1$, we get $\mathfrak{A} > 0$,$\mathfrak{B} > 0$ and $\mathfrak{C} \geqslant 0$, so the equation (II.26) does not have any endemic equilibrium in this case, which implies the non-existence of an endemic equilibrium for (II.26).

Hence, we can summarize the above discussions in the following theorem.

Theorem 2.6. *The system (II.1) has a unique endemic equilibrium in the case of $\mathcal{R}_0 > 1$, but when $\mathcal{R}_0 \leqslant 1$ such an equilibrium can never exist.*

After having studied the existence of the endemic equilibrium, we will now explore the uniform persistence of system (II.1).

Definition 2.4 (Uniform persistence [97]). *Let $\Delta \subset \mathbb{R}^7_+$. We say that the system (II.1) is uniformly persistent in Δ if there is a positive constant $\varrho > 0$ such that for any initial value $\left(S_0, Q_0, E_0, A_0, I_0, H_0, R_0\right) \in \Delta$, the solution of (II.1) starting from this value satisfies*

$$\liminf_{t\to\infty} S(t) > \varrho, \quad \liminf_{t\to\infty} Q(t) > \varrho, \quad \liminf_{t\to\infty} E(t) > \varrho,$$
$$\liminf_{t\to\infty} A(t) > \varrho, \quad \liminf_{t\to\infty} I(t) > \varrho, \quad \liminf_{t\to\infty} H(t) > \varrho,$$
$$\liminf_{t\to\infty} R(t) > \varrho.$$

Theorem 2.7. *If $\mathcal{R}_0 > 1$, then the system (II.1) is uniformly persistent in $\mathcal{S}$.*

Proof. As assumed in the statement of the theorem, let $\mathcal{R}_0 > 1$. Plainly, the disease-free equilibrium $\mathcal{E}^o$ is on the boundary of $\mathbb{R}^7_+$, and as already mentioned in Theorem 2.4, this equilibrium is unstable when $\mathcal{R}_0 > 1$. Combining this fact with the dissipativity of system (II.1) (Theorem 2.3) leads directly by virtue of Theorem 4.3 in [98] to the uniform persistence of this system, which is the desired conclusion. ■

This result is interpreted by saying that when the basic reproductive number $\mathcal{R}_0$ is strictly greater than one, then all the individuals appearing in (II.1) and especially the infected ones, will stay above a certain positive threshold, which means that the Coronavirus disease will persist in the population in this case.

3 Analysis of the stochastic Coronavirus model

The intent of this section is to deal with the perturbed version of the COVID-19 model expressed by the stochastic differential system (II.2). For simplicity of notation, we write from now on the initial-value problem associated with (II.2) in the following form:

$$\begin{cases} \overbrace{\mathrm{d}X(t)}^{7\times1} = \overbrace{F\left(X(t)\right)\mathrm{d}t}^{7\times1} + \overbrace{G\left(X(t)\right)}^{7\times7}\overbrace{\mathrm{d}B(t)}^{7\times1}, \\ X(0) = X_0 \in \mathbb{R}^7_+ \quad \text{almost surely.} \end{cases} \tag{II.27}$$

Here, the function F is the same as in (II.4), with

$$X(t) = \left(X_i(t)\right)_{1\leqslant i\leqslant 7} = \left(S(t), Q(t), E(t), A(t), I(t), H(t), R(t)\right),$$

$$G\left(X(t)\right) = \operatorname{diag}\left(\left(\sigma_i X_i(t)\right)_{1\leqslant i\leqslant 7}\right) = \begin{pmatrix} \sigma_1 X_1 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & \sigma_7 X_7 \end{pmatrix},$$

and

$$B(t) = \left(B_1(t), B_2(t), \cdots, B_7(t)\right).$$

For the sake of simplicity, we will denote the temporary mean $\dfrac{1}{t}\displaystyle\int_0^t \varphi(s)\,\mathrm{d}s$ of a continuous function φ by $\langle\varphi\rangle_t$. Also, and for all $(x, y) \in \mathbb{R}^2$, we adopt the following notations $x \wedge y := \min(x, y)$ and $x \vee y := \max(x, y)$.

3.1 Existence and uniqueness of the global positive solution

To explore the dynamical properties of a population system, the first concern is to know if it admits a solution, and if this solution is unique, positive and global (in time). In what follows, we will give some conditions under which these four points above (existence, uniqueness, positivity and globality), are verified for the system (II.2).

Theorem 3.1. *For any initial value $X_0 \in \mathbb{R}^7_+$, there is a unique solution $X(t)$ to the system (II.27) on $t \geqslant 0$, and it will remain in $\mathbb{R}^7_+$ with probability one, which means that, if $\left(S(0), Q(0), \cdots, R(0)\right)$ is in $\mathbb{R}^7_+$, then $\left(S(t), Q(t), E(t), A(t), I(t), H(t), R(t)\right) \in \mathbb{R}^7_+$ for all $t \geqslant 0$ almost surely (a.s. for short).*

Proof. In the system (II.27), the coefficients F and G are continuously differentiables on their domains of definition, so they satisfy the local Lipschitz condition, and for this reason, there exists for any given initial value $X_0 \in \mathbb{R}^7_+$, a unique maximal local solution $X(t)$ on $t \in [0, \tau_e)$, where τ_e is the explosion time [19]. At this point, our goal will be to demonstrate that this solution is global, that is $\tau_e = \infty$ a.s.

To this purpose, let $k_0 \in \mathbb{N}$ be very large such that $X(0) \in [k_0^{-1}, k_0]$, and define for each integer $k \geqslant k_0$ the stopping time τ_k as follows:

$$\tau_k = \inf\left\{t \in [0,\tau_e) \mid (\exists i \in \{1,\cdots,7\}) : X_i(t) \notin \left(\frac{1}{k}, k\right)\right\} = \inf\left\{t \in [0,\tau_e) \mid X(t) \notin \left(\frac{1}{k}, k\right)^7\right\}$$
$$= \inf\left\{t \in [0,\tau_e) \mid \min_{1\leqslant i\leqslant 7} X_i(t) \leqslant \frac{1}{k} \text{ or } \max_{1\leqslant i\leqslant 7} X_i(t) \geqslant k\right\}. \tag{II.28}$$

Set $\tau_\infty = \lim\limits_{k\to\infty} \tau_k$, clearly, $(\tau_k)_{k\geqslant k_0}$ is increasing; hence, $\lim\limits_{k\to\infty} \tau_k = \sup\limits_{k\geqslant k_0} \tau_k$, and according to Lemma 2.11 of [18] $\sup\limits_{k\geqslant k_0} \tau_k$ is a stopping time, then so is τ_∞.
By adopting the convention $\inf \emptyset = \infty$ for the rest of this chapter, we can easily affirm that $\tau_\infty \leqslant \tau_e$ a.s., and this because

- if for all $k \geqslant k_0$
$$\left\{t \in [0,\tau_e) \mid X(t) \notin \left(\frac{1}{k}, k\right)^7\right\} \neq \emptyset,$$
then obviously $\tau_k \leqslant \tau_e$ for each $k \geqslant k_0$, which implies that $\lim\limits_{k\to\infty} \tau_k = \tau_\infty \leqslant \tau_e$.

- On the other hand, if there is an integer $k_1 \geqslant k_0$ such that
$$\left\{t \in [0,\tau_e) \mid X(t) \notin \left(\frac{1}{k_1}, k_1\right)^7\right\} = \emptyset, \tag{II.29}$$
then $\tau_{k_1} = \infty$, and as $(\tau_k)_{k\geqslant k_0}$ is increasing, we get $\tau_k = \infty$ for all $k \geqslant k_1$, which yields $\tau_\infty = \infty$. At the same time, it follows from (II.29) that the solution $X(t)$ is bounded and belongs to $\left(\frac{1}{k_1}, k_1\right)^7$ for every $t \in [0,\tau_e)$, namely
$$X(t) \in \left(\frac{1}{k_1}, k_1\right)^7 \text{ a.s.,} \quad \text{for all } t \in [0,\tau_e). \tag{II.30}$$
Therefore, $\tau_e = \infty$ and so $\tau_\infty \leqslant \tau_e$.

Hence, $\tau_e = \infty$ a.s. will follow directly if we show that $\tau_\infty = \infty$ a.s., and that is exactly what we are going to do to finish the proof.

Assume that $\tau_\infty = \infty$ a.s. is untrue, then there exists a positive constant T such that $\mathbb{P}(\tau_\infty \leqslant T) > 0$. Therefore, there exists an $\epsilon > 0$ for which
$$\mathbb{P}(\tau_k \leqslant T) > \epsilon \quad \text{for all } k \geqslant k_0. \tag{II.31}$$

Consider the C^2-function V defined for $x = (x_1,\cdots,x_7) \in \mathbb{R}_+^7$ by
$$V(x) = \left[x_1 - a - a\ln\left(\frac{x_1}{a}\right)\right] + \sum_{i=2}^{7}(x_i - 1 - \ln(x_i)),$$
where a is a positive constant to be chosen suitably later. The nonnegativity of this function can be deduced from the following inequality: $x - 1 - \ln(x) \geqslant 0,\ \forall x > 0$.
Applying the multi-dimensional Itô's formula (see [19, page 36]) to $V(X(t))$, we obtain for all $k \geqslant k_0$ and $t \in [0,\tau_k)$
$$\begin{aligned}
&\mathrm{d}V\big(S(t),Q(t),E(t),A(t),I(t),H(t),R(t)\big)\\
&= \mathcal{L}V\big(S(t),Q(t),E(t),A(t),I(t),H(t),R(t)\big)\ \mathrm{d}t + (S(t)-a)\,\sigma_1\ \mathrm{d}B_1(t)\\
&\quad + (Q(t)-1)\,\sigma_2\ \mathrm{d}B_2(t) + (E(t)-1)\,\sigma_3\ \mathrm{d}B_3(t) + (A(t)-1)\,\sigma_4\ \mathrm{d}B_4(t)\\
&\quad + (I(t)-1)\,\sigma_5\ \mathrm{d}B_5(t) + (H(t)-1)\,\sigma_6\ \mathrm{d}B_6(t) + (R(t)-1)\,\sigma_7\ \mathrm{d}B_7(t),
\end{aligned}$$

where $\mathcal{L}V : \mathbb{R}_+^7 \to \mathbb{R}$ is defined by

$$
\begin{aligned}
&\mathcal{L}V(S,Q,E,A,I,H,R)\\
&=\left(1-\frac{a}{S}\right)\times\left[\Lambda-\left(\beta_1-\beta_2\frac{I}{b+I}\right)S\,(I+\theta A)+\lambda Q-(\mu+q)S\right]\\
&\quad+\left(1-\frac{1}{Q}\right)\times\left[qS-(\mu+\lambda)\,Q\right]+\left(1-\frac{1}{E}\right)\times\left[\left(\beta_1-\beta_2\frac{I}{b+I}\right)S\,(I+\theta A)-(\mu+\sigma)E\right]\\
&\quad+\left(1-\frac{1}{A}\right)\times\left[(1-p)\,\sigma E-(\mu+\varepsilon_A+\gamma_A+d_A)\,A\right]+\left(1-\frac{1}{I}\right)\times\left[\sigma pE-(\mu+\varepsilon_I+\gamma_I+d_I)\,I\right]\\
&\quad+\left(1-\frac{1}{H}\right)\times\left[\varepsilon_I I+\varepsilon_A A-(\mu+d_H+\gamma_H)\,H\right]+\left(1-\frac{1}{R}\right)\times\left[\gamma_H H+\gamma_I I+\gamma_A A-\mu R\right]\\
&\quad+\frac{1}{2}\left[a\sigma_1^2+\sigma_2^2+\sigma_3^2+\sigma_4^2+\sigma_5^2+\sigma_6^2+\sigma_7^2\right]\\
&=\Lambda-\mu\,(S+Q+E+A+I+H+R)-d_A A-d_I I-d_H H\\
&\quad+\left[-\frac{a\Lambda}{S}+a\beta_1\,(I+\theta A)-a\beta_2\frac{I\,(I+\theta A)}{b+I}-a\frac{\lambda Q}{S}+a(\mu+q)\right]+\left[-q\frac{S}{Q}+(\mu+\lambda)\right]\\
&\quad+\left[-\left(\beta_1-\beta_2\frac{I}{b+I}\right)\frac{S}{E}\,(I+\theta A)+(\mu+\sigma)\right]+\left[-\,(1-p)\,\sigma\frac{E}{A}+(\mu+\varepsilon_A+\gamma_A+d_A)\right]\\
&\quad+\left[-\sigma p\frac{E}{I}+(\mu+\varepsilon_I+\gamma_I+d_I)\right]+\left[-\varepsilon_I\frac{I}{H}-\varepsilon_A\frac{A}{H}+(\mu+\gamma_H+d_H)\right]\\
&\quad+\left[-\gamma_H\frac{H}{R}-\gamma_I\frac{I}{R}-\gamma_A\frac{A}{R}+\mu\right]+\frac{1}{2}\left[a\sigma_1^2+\sigma_2^2+\sigma_3^2+\sigma_4^2+\sigma_5^2+\sigma_6^2+\sigma_7^2\right]\\
&\leqslant\left[\Lambda+6\mu+\lambda+\sigma+\varepsilon_A+\gamma_A+d_A+\varepsilon_I+\gamma_I+d_I+d_H+\gamma_H+a(\mu+q)\right.\\
&\quad\left.+\frac{1}{2}\left(a\sigma_1^2+\sigma_2^2+\sigma_3^2+\sigma_4^2+\sigma_5^2+\sigma_6^2+\sigma_7^2\right)\right]-\mu(S+Q+E+R)-\theta\beta_1\left(\frac{\mu+d_A}{\theta\beta_1}-a\right)A\\
&\quad-\beta_1\left(\frac{\mu+d_I}{\beta_1}-a\right)I.
\end{aligned}
$$

By choosing $a=\frac{1}{2}\min\left\{\frac{\mu+d_A}{\theta\beta_1},\frac{\mu+d_I}{\beta_1}\right\}$, the coefficients of A and I will be negatives, therefore

$$
\mathcal{L}V\,(S,Q,E,A,I,H,R)\leqslant\Lambda+6\mu+\lambda+\sigma+\varepsilon_A+\gamma_A+d_A+\varepsilon_I+\gamma_I+d_I+d_H+\gamma_H+a(q+\mu)+\frac{1}{2}\left(a\sigma_1^2+\sum_{i=2}^{7}\sigma_i^2\right)\triangleq\mathcal{K}.
$$

Hence, we get for all $k\geqslant k_0$ and $t\in[0,\tau_k)$

$$
\begin{aligned}
&\mathrm{d}V\big(S(t),Q(t),E(t),A(t),I(t),H(t),R(t)\big)\\
&\leqslant\mathcal{K}\ \mathrm{d}t+(S(t)-a)\,\sigma_1\ \mathrm{d}B_1(t)+(Q(t)-1)\,\sigma_2\ \mathrm{d}B_2(t)+(E(t)-1)\,\sigma_3\ \mathrm{d}B_3(t)\\
&\quad+(A(t)-1)\,\sigma_4\ \mathrm{d}B_4(t)+(I(t)-1)\,\sigma_5\ \mathrm{d}B_5(t)+(H(t)-1)\,\sigma_6\ \mathrm{d}B_6(t)\\
&\quad+(R(t)-1)\,\sigma_7\ \mathrm{d}B_7(t).
\end{aligned}
$$

Integrating from 0 to $\tau_k\wedge T$ and then taking the expectation on both sides of the above inequality leads to

$$
\mathbb{E}\left[V(X(T\wedge\tau_k)\right]\leqslant V(X(0))+\mathcal{K}\mathbb{E}\left[\tau_k\wedge T\right]\leqslant V(X(0))+\mathcal{K}T. \tag{II.32}
$$

We have $V(x)\geq 0$ for all $x>0$, then

$$
\mathbb{E}\left[V(X(T\wedge\tau_k)\right]=\mathbb{E}\left[V(X(T\wedge\tau_k)\times\mathbb{1}_{\{\tau_k\leqslant T\}}\right]+\mathbb{E}\left[V(X(t\wedge\tau_k)\times\mathbb{1}_{\{\tau_k>T\}}\right]\geqslant\mathbb{E}\left[V(X(\tau_k)\times\mathbb{1}_{\{\tau_k\leqslant T\}}\right], \tag{II.33}
$$

where $\mathbb{1}_A$ denotes the indicator function of a measurable set $A\in\mathfrak{F}$. Note that for every $\omega\in\{\omega\in\Omega\mid\tau_k(\omega)\leqslant T\}$, there is some component of $V\,(X(\tau_k))$ equals to k or $\frac{1}{k}$ so

$$
V\,(X(\tau_k))\geqslant\left(k-a-a\ln\left(\frac{k}{a}\right)\right)\wedge\left(\frac{1}{k}-a-a\ln\left(\frac{1}{ka}\right)\right)\wedge(k-1-\ln(k))\wedge\left(\frac{1}{k}-1-\ln\left(\frac{1}{k}\right)\right).
$$

Therefore

$$\mathbb{E}\left[V(X(\tau_k)\times \mathbb{1}_{\{\tau_k\leqslant T\}}\right] \geqslant \mathbb{P}\left(\tau_k \leqslant T\right)\left(k-a-a\ln\left(\frac{k}{a}\right)\right)\wedge\left(\frac{1}{k}-a-a\ln\left(\frac{1}{ka}\right)\right)\wedge\left(k-1-\ln\left(k\right)\right)$$
$$\wedge\left(\frac{1}{k}-1-\ln\left(\frac{1}{k}\right)\right). \tag{II.34}$$

Combining (II.32), (II.33) and (II.34) with (II.31), we conclude that

$$V\left(X(0)\right)+\mathcal{K}T \geqslant \varepsilon\left(k-a-a\ln\left(\frac{k}{a}\right)\right)\wedge\left(\frac{1}{k}-a-a\ln\left(\frac{1}{ka}\right)\right)\wedge\left(k-1-\ln\left(k\right)\right)\wedge\left(\frac{1}{k}-1-\ln\left(\frac{1}{k}\right)\right).$$

Letting $k\to\infty$ leads to the contradiction $V\left(X(0)\right)+\mathcal{K}T=\infty$, which completes the proof. ■

3.2 Stochastically ultimate boundedness and permanence

After having demonstrated the positivity and the globality of our system's solution, it is now time to discuss in more detail how it behaves in the positive cone $\mathbb{R}^7_+$. In the following, we are going to define the notions of stochastically ultimate boundedness and permanence, and we will subsequently show that the solution of system (II.2) verifies these properties.

Definition 3.1 (Stochastically ultimate boundedness [99]). *We say that the system (II.2) is stochastically ultimately bounded, or ultimately bounded in probability (see [19, page 395]), if to every $\varepsilon>0$, there corresponds a $\delta_\varepsilon>0$ such that*

$$\limsup_{t\to\infty}\mathbb{P}\left(\|X\left(t,X_0\right)\|>\delta_\varepsilon\right)\leqslant\varepsilon \quad \textit{for every } X_0\in\mathbb{R}^7_+,$$

where $X\left(t,X_0\right)$ here denotes the solution of (II.27) which satisfies the initial condition $X(0)=X_0$.

Remark 3.1. *Other authors, as in [100, 101], and [102] adopt a different definition for this last notion (Stochastically ultimate boundedness), according to which the system (II.27) is stochastically bounded in probability, if for any $\varepsilon\in(0,1)$ there is some $\delta_\varepsilon>0$ for which the following inequality:*

$$\limsup_{t\to\infty}\mathbb{P}\left(\|X\left(t,X_0\right)\|\leqslant\delta_\varepsilon\right)\geqslant 1-\varepsilon \tag{II.35}$$

is satisfied for any initial value $X_0\in\mathbb{R}^7_+$.We mention that the inequality (II.35) can equivalently be rewritten as:

$$\liminf_{t\to\infty}\mathbb{P}\left(\|X\left(t,X_0\right)\|>\delta_\varepsilon\right)\leqslant\varepsilon,$$

and this allows us to say that if our system is stochastically ultimate bounded in the sense of Definition 3.1, then it will also be so in the sense of (II.35).

Definition 3.2 (Stochastic persistence [103]). *Let $X\left(t,X_0\right)$ be the solution of (II.2) that verifies $X(0)=X_0$, we say that the system (II.2) is stochastically persistent, if for each $\varepsilon>0$, there exists a positive constant $\eta_\varepsilon>0$ such that the following property is satisfied:*

$$\liminf_{t\to\infty}\mathbb{P}\left(\|X\left(t,X_0\right)\|\geqslant\eta_\varepsilon\right)\geqslant 1-\varepsilon, \quad \textit{for all } X_0\in\mathbb{R}^7_+.$$

Definition 3.3 (Stochastic permanence [75]). *The system (II.27) is called stochastically permanent, if it is both stochastically ultimate bounded and persistent.*

Theorem 3.2. *The system (II.2) is stochastically ultimate bounded, and even stochastically permanent.*

Proof. Let $X_0\in\mathbb{R}^7_+$, by summing up the seven equations in (II.27) and denoting $N=S+Q+E+A+I+H+R$, we obtain for all $t\geqslant 0$

$$\mathrm{d}N(t)=\left(\Lambda-\mu N(t)-d_AA(t)-d_II(t)-d_HH(t)\right)\,\mathrm{d}t+\sigma_1S(t)\,\mathrm{d}B_1(t)+\sigma_2Q(t)\,\mathrm{d}B_2(t)+\sigma_3E(t)\,\mathrm{d}B_3(t)$$
$$+\sigma_4A(t)\,\mathrm{d}B_4(t)+\sigma_5I(t)\,\mathrm{d}B_5(t)+\sigma_6H(t)\,\mathrm{d}B_6(t)+\sigma_7R(t)\,\mathrm{d}B_7(t).$$

Define a C^2-function $\widetilde{V}:\mathbb{R}_+\to[2,\infty)$ by $\widetilde{V}(x)=x+\dfrac{1}{x}$, the Itô's formula shows that

$$\begin{aligned}\mathrm{d}\widetilde{V}(N(t)) &= \mathcal{L}\widetilde{V}(N(t))\,\mathrm{d}t+\left(1-\frac{1}{N(t)^2}\right)\big[\sigma_1 S(t)\ \mathrm{d}B_1(t)+\sigma_2 Q(t)\ \mathrm{d}B_2(t)+\sigma_3 E(t)\ \mathrm{d}B_3(t)+\sigma_4 A(t)\ \mathrm{d}B_4(t)\\ &\quad+\sigma_5 I(t)\ \mathrm{d}B_5(t)+\sigma_6 H(t)\ \mathrm{d}B_6(t)+\sigma_7 R(t)\ \mathrm{d}B_7(t)\big],\end{aligned}$$

where $\mathcal{L}\widetilde{V}(N)$ is given by

$$\begin{aligned}\mathcal{L}\widetilde{V}(N) &= \big(\Lambda-\mu N-d_A A-d_I I-d_H H\big)-\frac{\Lambda-\mu N-d_A A-d_I I-d_H H}{N^2}+\frac{1}{N^3}\Big(\sigma_1^2S^2+\sigma_2^2Q^2+\sigma_3^2E^2\\ &\quad+\sigma_4^2A^2+\sigma_5^2I^2+\sigma_6^2H^2+\sigma_7^2R^2\Big)\\ &\leqslant \big(\Lambda-\mu N\big)-\frac{\Lambda}{N^2}+\frac{\mu+d_A+d_I+d_H}{N}+\frac{1}{N}\sum_{i=1}^{7}\sigma_i^2\\ &\leqslant -\mu\left(N+\frac{1}{N}\right)+\Lambda+\frac{1}{N}\left(2\mu+d_A+d_I+d_H+\sum_{i=1}^{7}\sigma_i^2\right)-\frac{\Lambda}{N^2}\\ &\leqslant -\mu\widetilde{V}(N)-\left(\frac{\sqrt{\Lambda}}{N}-\frac{1}{2\sqrt{\Lambda}}\left(2\mu+d_A+d_I+d_H+\sum_{i=1}^{7}\sigma_i^2\right)\right)^2+\underbrace{\frac{1}{4\Lambda}\left(2\mu+d_A+d_I+d_H+\sum_{i=1}^{7}\sigma_i^2\right)^2+\Lambda}_{:=C}\\ &\leqslant -\mu\widetilde{V}\,(N)+C.\end{aligned}$$

Applying the integration by parts formula (see [19, page 37]) to $e^{\mu t}\widetilde{V}\,(N(t))$ gives

$$\begin{aligned}\mathrm{d}e^{\mu t}\widetilde{V}\,(N(t)) &= \mu e^{\mu t}\widetilde{V}\,(N(t))\ \mathrm{d}t+e^{\mu t}\,\mathrm{d}\widetilde{V}\,(N(t))\\ &\leqslant \mu e^{\mu t}\widetilde{V}\,(N(t))\,\mathrm{d}t+e^{\mu t}\Big[\left(-\mu\widetilde{V}\,(N(t))+C\right)\mathrm{d}t+\left(1-\frac{1}{N(t)^2}\right)[\sigma_1 S(t)\ \mathrm{d}B_1(t)+\sigma_2 Q(t)\ \mathrm{d}B_2(t)\\ &\quad+\sigma_3 E(t)\ \mathrm{d}B_3(t)+\sigma_4 A(t)\ \mathrm{d}B_4(t)+\sigma_5 I(t)\ \mathrm{d}B_5(t)+\sigma_6 H(t)\ \mathrm{d}B_6(t)+\sigma_7 R(t)\ \mathrm{d}B_7(t)]\Big]\\ &\leqslant Ce^{\mu t}\,\mathrm{d}t+e^{\mu t}\left(1-\frac{1}{N(t)^2}\right)[\sigma_1 S(t)\ \mathrm{d}B_1(t)+\sigma_2 Q(t)\ \mathrm{d}B_2(t)+\sigma_3 E(t)\ \mathrm{d}B_3(t)+\sigma_4 A(t)\ \mathrm{d}B_4(t)\\ &\quad+\sigma_5 I(t)\ \mathrm{d}B_5(t)+\sigma_6 H(t)\ \mathrm{d}B_6(t)+\sigma_7 R(t)\ \mathrm{d}B_7(t)]\,.\end{aligned}$$

By integrating from 0 to $t\wedge\tau_k$ (τ_k is already given in (II.28)), and then taking the expectation on both sides of this inequality, we get for all $t\geqslant 0$ and $k\geqslant k_0$

$$\mathbb{E}\left[e^{\mu(t\wedge\tau_k)}\widetilde{V}\,(N(t\wedge\tau_k))\right]\leqslant\widetilde{V}\,(N(0))+\mathbb{E}\left[\int_0^{t\wedge\tau_k}C\times e^{\mu s}\,\mathrm{d}s\right]\leqslant\widetilde{V}\,(N(0))+\frac{C}{\mu}\left(e^{\mu t}-1\right). \tag{II.36}$$

According to Theorem 3.1, $\tau_k\to\infty$ almost surely as $k\to\infty$, so extending k to ∞ in (II.36) leads to

$$\mathbb{E}\left[\widetilde{V}\,(N(t))\right]\leqslant\widetilde{V}\,(N(0))\times e^{-\mu t}+\frac{C}{\mu}\left(1-e^{-\mu t}\right)\leqslant\widetilde{V}\,(N(0))\times e^{-\mu t}+\frac{C}{\mu},$$

Let $\varepsilon>0$, and take $\delta_\varepsilon=\frac{C}{\varepsilon\mu}$, by making use of the well-known Markov's inequality (see for example [104] and the related bibliography), we get

$$\mathbb{P}\left(\widetilde{V}\,(N(t))>\delta_\varepsilon\right)\leqslant\frac{1}{\delta_\varepsilon}\times\mathbb{E}\big[\widetilde{V}\,(N(t))\,\big]\leqslant\frac{1}{\delta_\varepsilon}\left(\widetilde{V}\,(N(0))\times e^{-\mu t}\right)+\varepsilon.$$

Then

$$\mathbb{P}\left(N(t)+\frac{1}{N(t)}\leqslant\delta_\varepsilon\right)\geqslant 1-\varepsilon-\frac{1}{\delta_\varepsilon}\left(\widetilde{V}\,(N(0))\times e^{-\mu t}\right).$$

Therefore

$$\mathbb{P}\left(\frac{1}{\delta_\varepsilon}\leqslant N(t)\leqslant\delta_\varepsilon\right)\geqslant\mathbb{P}\left(N(t)+\frac{1}{N(t)}\leqslant\delta_\varepsilon\right)\geqslant 1-\varepsilon-\frac{1}{\delta_\varepsilon}\left(\widetilde{V}\,(N(0))\times e^{-\mu t}\right).$$

By noting that

$$\frac{1}{\sqrt{7}}\underbrace{(S+Q+E+A+I+H+R)}_{N}\leqslant\underbrace{\sqrt{S^2+Q^2+E^2+A^2+I^2+H^2+R^2}}_{\|X\|}\leqslant\underbrace{S+Q+E+A+I+H+R}_{N},$$

we obtain

$$1-\varepsilon-\frac{1}{\delta_\varepsilon}\left(\widetilde{V}\left(N(0)\right)\times e^{-\mu t}\right)\leqslant\mathbb{P}\left(\frac{1}{\delta_\varepsilon}\leqslant N(t)\leqslant\delta_\varepsilon\right)\leqslant\mathbb{P}\left(\frac{1}{\sqrt{7}\delta_\varepsilon}\leqslant\|X\left(t,X_0\right)\|\leqslant\delta_\varepsilon\right),$$

so

$$\liminf_{t\to\infty}\mathbb{P}\Big(\underbrace{\frac{1}{\sqrt{7}\delta_\varepsilon}}_{:=\eta_\varepsilon}\leqslant\|X\left(t,X_0\right)\|\leqslant\delta_\varepsilon\Big)\geqslant 1-\varepsilon.$$

Hence, the theorem has been proved. ■

3.3 Extinction of Stochastic COVID-19 model

In epidemiology, we are usually concerned about two things, the first, is to know when the disease will die out, and the second, is when it well persist. In this subsection, we will try our best to find a condition for the extinction of the disease expressed in terms of system parameters and intensities of noises, and for the persistence, it will be dealt with in the next subsection.

Definition 3.4 (Extinction [75]). *For system (II.2), the infected individuals $E(t), A(t)$ and $I(t)$ are said to be extinct if $\lim_{t\to\infty} E(t)+I(t)+A(t)=0$ almost surely.*

Before stating the result to be proved, we must firstly give the following useful lemma that was stated and proved as Lemma 3.1 in [73].

Lemma 3.1. *For any initial value $X_0\in\mathbb{R}^7_+$, the solution $X(t)=(S(t),Q(t),E(t),A(t),I(t),H(t),R(t))$ of system (II.27) verifies the following properties:*

(a) $\lim_{t\to\infty}\dfrac{X_k(t)}{t}=0$ *a.s.* $\quad\forall k\in\{1,2,\cdots,7\}$.

(b) *Moreover, if* $\mu>\frac{1}{2}\left(\sigma_1^2\vee\sigma_2^2\vee\sigma_3^2\vee\sigma_4^2\vee\sigma_5^2\vee\sigma_6^2\vee\sigma_7^2\right)$, *then*

$$\lim_{t\to\infty}\frac{\int_0^t X_k(s)\,\mathrm{d}B_k(s)}{t}=0\quad a.s.\quad\forall k\in\{1,2,\cdots,7\}.$$

Proof. The proof of this lemma is similar in spirit to that of lemmas 2.1 and 2.2 of [105] and therefore it is omitted here. ■

Theorem 3.3. *Let us denote by $X(t)=\big(S(t),Q(t),E(t),A(t),I(t),H(t),R(t)\big)$ the solution of system (II.2) that starts from a given value $X_0=\big(S(0),Q(0),E(0),A(0),I(0),H(0),R(0)\big)\in\mathbb{R}^7_+$.*
If $\mu>\frac{1}{2}\left(\sigma_1^2\vee\sigma_2^2\vee\sigma_3^2\vee\sigma_4^2\vee\sigma_5^2\vee\sigma_6^2\vee\sigma_7^2\right)$ and $\sigma_3^2\wedge\sigma_4^2\wedge\sigma_5^2>6\times(\beta_1 S^o-\mu)$, with $S^o=\frac{\Lambda}{\mu}\cdot\frac{\lambda+\mu}{\lambda+q+\mu}$, then

$$\limsup_{t\to\infty}\frac{\ln(E(t)+A(t)+I(t))}{t}\leqslant\beta_1 S^o-\mu-\frac{\sigma_3^2\wedge\sigma_4^2\wedge\sigma_5^2}{6}<0\quad a.s.,$$

which implies that the disease will die out exponentially with probability one.

Proof. From Itô's formula and system (II.2), we have

$$\begin{aligned}\mathrm{d}\ln(E+A+I)=&\left[\frac{1}{E+A+I}\left(\left(\beta_1-\beta_2\frac{I}{b+I}\right)S\left(I+\theta A\right)-\left(\varepsilon_A+\gamma_A+d_A\right)A-\left(\varepsilon_I+\gamma_I+d_I\right)I\right)-\mu\right.\\&\left.-\frac{\sigma_3^2\,E^2+\sigma_4^2\,A^2+\sigma_5^2\,I^2}{2\left(E+A+I\right)^2}\right]\mathrm{d}t+\frac{1}{E+A+I}\left(\sigma_3 E\,\mathrm{d}B_3(t)+\sigma_4 A\,\mathrm{d}B_4(t)+\sigma_5 I\,\mathrm{d}B_5(t)\right).\end{aligned}$$

Thus

$$\mathrm{d}\ln(E+A+I) \leqslant \left[\left(\beta_1 - \beta_2\frac{I}{b+I}\right)S - \mu - \frac{\sigma_3^2\wedge\sigma_4^2\wedge\sigma_5^2}{2}\times\frac{E^2+A^2+I^2}{(E+A+I)^2}\right]\mathrm{d}t + \sigma_3\frac{E}{E+A+I}\,\mathrm{d}B_3(t) + \sigma_4\frac{A}{E+A+I}\,\mathrm{d}B_4(t) + \sigma_5\frac{I}{E+A+I}\,\mathrm{d}B_5(t).$$

By using the famous Cauchy-Schwartz inequality (see for instance [106] and the references given there), we can assert that

$$\frac{E^2+A^2+I^2}{(E+A+I)^2} \geqslant \frac{1}{3}.$$

Hence

$$\mathrm{d}\ln(E+A+I) \leqslant \left[\beta_1 S - \mu - \frac{\sigma_3^2\wedge\sigma_4^2\wedge\sigma_5^2}{6}\right]\mathrm{d}t + \sigma_3\frac{E}{E+A+I}\,\mathrm{d}B_3(t) + \sigma_4\frac{A}{E+A+I}\,\mathrm{d}B_4(t) + \sigma_5\frac{I}{E+A+I}\,\mathrm{d}B_5(t), \tag{II.37}$$

Integrating (II.37) from 0 to t, and then dividing by t on both sides, we get

$$\frac{\ln(E(t)+A(t)+I(t))}{t} \leqslant \frac{\ln(E(0)+A(0)+I(0))}{t} + \beta_1\langle S\rangle_t - \mu - \frac{\sigma_3^2\wedge\sigma_4^2\wedge\sigma_5^2}{6} + \frac{\sigma_3}{t}\int_0^t \frac{E(s)}{E(s)+A(s)+I(s)}\,\mathrm{d}B_3(s) + \frac{\sigma_4}{t}\int_0^t \frac{A(s)}{E(s)+A(s)+I(s)}\,\mathrm{d}B_4(s) + \frac{\sigma_5}{t}\int_0^t \frac{I(s)}{E(s)+A(s)+I(s)}\,\mathrm{d}B_5(s). \tag{II.38}$$

On the other hand, the first equation of (II.2) gives

$$\begin{aligned} S(t) - S(0) &= \Lambda t - \int_0^t \left(\beta_1 - \beta_2\frac{I(s)}{b+I(s)}\right)S(s)\,(I(s)+\theta A(s))\,\mathrm{d}s + \lambda\int_0^t Q(s)\,\mathrm{d}s - (q+\mu)\int_0^t S(s)\,\mathrm{d}s \\ &\quad + \sigma_1\int_0^t S(s)\,\mathrm{d}B_1(s) \\ &\leqslant \Lambda t + \lambda\int_0^t Q(s)\,\mathrm{d}s - (q+\mu)\int_0^t S(s)\,\mathrm{d}s + \sigma_1\int_0^t S(s)\,\mathrm{d}B_1(s). \end{aligned}$$

Therefore

$$\begin{aligned} \langle S\rangle_t = \frac{1}{t}\int_0^t S(s)\,\mathrm{d}s &\leqslant \frac{1}{q+\mu}\left(\Lambda + \frac{\lambda}{t}\int_0^t Q(s)\,\mathrm{d}s + \frac{S(0)}{t} + \frac{\sigma_1}{t}\int_0^t S(s)\,\mathrm{d}B_1(s) - \frac{S(t)}{t}\right) \\ &\leqslant \frac{1}{q+\mu}\left(\Lambda + \lambda\langle Q\rangle_t + \frac{S(0)}{t} + \frac{\sigma_1}{t}\int_0^t S(s)\,\mathrm{d}B_1(s)\right). \end{aligned} \tag{II.39}$$

Also, the second one gives

$$Q(t) - Q(0) = q\int_0^t S(s)\,\mathrm{d}s - (\mu+\lambda)\int_0^t Q\,\mathrm{d}s + \sigma_2\int_0^t Q(s)\,\mathrm{d}B_2(s),$$

which shows that

$$\langle Q\rangle_t = \frac{1}{t}\int_0^t Q(s)\,\mathrm{d}s = \frac{1}{\lambda+\mu}\left(\frac{Q(0)-Q(t)}{t} + \frac{q}{t}\int_0^t S(s)\,\mathrm{d}s + \frac{\sigma_2}{t}\int_0^t Q(s)\,\mathrm{d}B_2(s)\right) \tag{II.40}$$

$$\leqslant \frac{1}{\lambda+\mu}\times\frac{Q(0)}{t} + \frac{q}{\lambda+\mu}\langle S\rangle_t + \frac{\sigma_2}{\lambda+\mu}\times\frac{1}{t}\int_0^t Q(s)\,\mathrm{d}B_2(s). \tag{II.41}$$

Combining (II.39) with (II.41) yields

$$\langle S(t)\rangle \leqslant \frac{1}{q+\mu}\left(\Lambda+\lambda\left(\frac{Q(0)}{(\lambda+\mu)\,t}+\frac{q}{\lambda+\mu}\langle S\rangle_t+\frac{\sigma_2}{\lambda+\mu}\times\frac{1}{t}\int_0^t Q(s)\,\mathrm{d}B_2(s)\right)+\frac{S(0)}{t}+\frac{\sigma_1}{t}\int_0^t S(s)\,\mathrm{d}B_1(s)\right).$$

Hence,

$$\begin{aligned}\langle S\rangle_t \leqslant{}& \frac{\Lambda(\lambda+\mu)}{\mu(q+\mu+\lambda)}+\frac{\lambda}{\mu(q+\mu+\lambda)}\frac{Q(0)}{t}+\frac{\lambda+\mu}{\mu(q+\mu+\lambda)}\frac{S(0)}{t}+\frac{\lambda\sigma_2}{\mu(q+\mu+\lambda)}\times\frac{1}{t}\int_0^t Q(s)\,\mathrm{d}B_2(s)\\&+\sigma_1\frac{(\lambda+\mu)}{\mu(q+\mu+\lambda)}\times\frac{1}{t}\int_0^t S(s)\,\mathrm{d}B_1(s).\end{aligned}\tag{II.42}$$

Since $\mu > \frac{1}{2}\left(\sigma_1^2\vee\sigma_2^2\vee\sigma_3^2\vee\sigma_4^2\vee\sigma_5^2\vee\sigma_6^2\vee\sigma_7^2\right)$, we can conclude by virtue of Lemma 3.1(b) and inequality (II.42) that

$$\lim_{t\to\infty}\langle S\rangle_t \leqslant \frac{\Lambda}{\mu}\times\frac{\lambda+\mu}{q+\mu+\lambda}=S^o.\tag{II.43}$$

According to the strong law of large numbers for local martingales (see [19, page 12]), we have

$$\begin{cases}\lim\limits_{t\to\infty}\dfrac{\sigma_3}{t}\displaystyle\int_0^t\frac{E(s)}{E(s)+A(s)+I(s)}\,\mathrm{d}B_3(s)=0 \quad \text{a.s.},\\[2ex] \lim\limits_{t\to\infty}\dfrac{\sigma_4}{t}\displaystyle\int_0^t\frac{A(s)}{E(s)+A(s)+I(s)}\,\mathrm{d}B_4(s)=0 \quad \text{a.s.},\\[2ex] \lim\limits_{t\to\infty}\dfrac{\sigma_5}{t}\displaystyle\int_0^t\frac{I(s)}{E(s)+A(s)+I(s)}\,\mathrm{d}B_5(s)=0 \quad \text{a.s.}\end{cases}\tag{II.44}$$

From (II.38), (II.43) and (II.44) we get

$$\limsup_{t\to\infty}\frac{\ln(E(t)+A(t)+I(t))}{t}\leqslant\beta_1 S^o-\mu-\frac{\sigma_3^2\wedge\sigma_4^2\wedge\sigma_5^2}{6}<0 \quad \text{a.s.},$$

which is exactly the desired conclusion. ■

Remark 3.2. *Unlike Theorem 4.1 of [74], the above proof uses just the second assertion of Lemma 3.1, and makes no appeal to the first one.*

Remark 3.3. *Needless to say, the preceding theorem implies the extinction of infected individuals, which implies in turn (by the positivity of the solution) that* $\lim\limits_{t\to\infty}E(t)=0, \lim\limits_{t\to\infty}A(t)=0$ *and* $\lim\limits_{t\to\infty}I(t)=0$ *a.s. (see for example [107, 108] and [109, page 5071]).*

Corollary 3.1. *Under the same notations and hypotheses as in Theorem 3.3, we have*

$$\lim_{t\to\infty}\langle S\rangle_t=S^o \text{ a.s.}, \quad \lim_{t\to\infty}\langle Q\rangle_t=Q^o \text{ a.s.}, \quad \lim_{t\to\infty}\langle R\rangle_t=0 \text{ a.s.}, \quad \text{and} \quad \lim_{t\to\infty}\langle H\rangle_t=0 \text{ a.s.}$$

Proof. Our proof starts with the observation that for all $t\geqslant 0$, we have

$$\mathrm{d}\big(S(t)+E(t)\big)=\left[\Lambda+\lambda Q(t)-(q+\mu)\,S(t)-(\mu+\sigma)\,E(t)\right]\mathrm{d}t+\sigma_1 S(t)\;\mathrm{d}B_1(t)+\sigma_3 E(t)\;\mathrm{d}B_3(t).\tag{II.45}$$

Integrating (II.45) from 0 to t, and then dividing by t on both sides gives

$$\begin{aligned}\frac{S(t)+E(t)}{t}-\frac{S(0)+E(0)}{t}={}&\Lambda+\lambda\langle Q\rangle_t-(q+\mu)\,\langle S\rangle_t-(\mu+\sigma)\,\langle E\rangle_t+\frac{\sigma_1}{t}\int_0^t S(s)\;\mathrm{d}B_1(s)\\&+\frac{\sigma_3}{t}\int_0^t E(s)\;\mathrm{d}B_3(s).\end{aligned}$$

Replacing in the last equality $\langle Q\rangle_t$ by its expression from (II.40) yields

$$\begin{aligned}\frac{S(t)+E(t)}{t}={}&\frac{S(0)+E(0)}{t}+\Lambda+\frac{\lambda}{\lambda+\mu}\left(q\langle S\rangle_t-\frac{Q(t)-Q(0)}{t}+\frac{\sigma_2}{t}\int_0^t Q(s)\,\mathrm{d}B_2(s)\right)-(q+\mu)\,\langle S\rangle_t\\&-(\mu+\sigma)\,\langle E\rangle_t+\frac{\sigma_1}{t}\int_0^t S(s)\;\mathrm{d}B_1(s)+\frac{\sigma_3}{t}\int_0^t E(s)\;\mathrm{d}B_3(s).\end{aligned}$$

Hence

$$\left(-\frac{\lambda\times q}{\lambda+\mu}+q+\mu\right)\langle S\rangle_t=\Lambda+\frac{S(0)+E(0)}{t}+\frac{\lambda}{\lambda+\mu}\times\frac{Q(0)-Q(t)}{t}-\frac{S(t)+E(t)}{t}-(\mu+\sigma)\langle E\rangle_t$$
$$+\frac{\lambda}{\lambda+\mu}\frac{\sigma_2}{t}\int_0^t Q(s)\,dB_2(s)+\frac{\sigma_1}{t}\int_0^t S(s)\,dB_1(s)+\frac{\sigma_3}{t}\int_0^t E(s)\,dB_3(s). \tag{II.46}$$

Letting t go to infinity on both sides of (II.46), then using (a) and (b) of Lemma 3.1, we obtain

$$\lim_{t\to\infty}\left(-\frac{\lambda\times q}{\lambda+\mu}+q+\mu\right)\langle S\rangle_t=\Lambda-(\mu+\sigma)\lim_{t\to\infty}\langle E\rangle_t\quad\text{a.s.,} \tag{II.47}$$

On the account of Remark 3.3, we have

$$\lim_{t\to\infty}E(t)=0\quad\text{a.s.,}$$

which implies by the continuous version of Cesàro's theorem [110, page 3] that

$$\lim_{t\to\infty}\langle E\rangle_t=0\quad\text{a.s.}$$

So

$$\lim_{t\to\infty}\left(-\frac{\lambda\times q}{\lambda+\mu}+q+\mu\right)\langle S\rangle_t=\Lambda.$$

Therefore

$$\lim_{t\to\infty}\langle S\rangle_t=\Lambda\left(-\frac{\lambda\times q}{\lambda+\mu}+q+\mu\right)^{-1}=\Lambda\times\frac{\lambda+\mu}{\mu\,(\lambda+q+\mu)}=S^o. \tag{II.48}$$

At the same time, we have

$$dQ(t)=[qS(t)-(\lambda+\mu)\,Q(t)]\,dt+\sigma_2Q(t)\;dB_2(t).$$

Then

$$\frac{Q(t)-Q(0)}{t}=q\langle S\rangle_t-(\lambda+\mu)\,\langle Q\rangle_t+\frac{\sigma_2}{t}\int_0^t Q(s)\,dB_2(s).$$

Hence

$$\langle Q\rangle_t=\frac{q}{\lambda+\mu}\langle S\rangle_t-\frac{Q(t)}{(\lambda+\mu)t}+\frac{Q(0)}{(\lambda+\mu)t}+\frac{\sigma_2}{\lambda+\mu}\times\frac{1}{t}\int_0^t Q(s)\,dB_2(s).$$

Consequently, and by a passage to the limit similar to the above, we get

$$\lim_{t\to\infty}\langle Q\rangle_t=\frac{q}{q+\mu}\times S^o=Q^o.$$

The same reasoning remains valid for the two last assertions of our theorem $\left(\langle R\rangle_t,\langle H\rangle_t\underset{n\to\infty}{\longrightarrow}0\;\text{ a.s.}\right)$, and this finishes the proof, the detailed verification being left to the reader. ■

Remark 3.4. *Obviously, if we keep the same notations and assumptions as in Theorem 3.3, the last result can be rewritten as follows:*

$$\lim_{t\to\infty}\langle X\rangle_t=\mathcal{E}^o\quad a.s.,$$

which implies by Cesàro's theorem [110, page 3], that if the solution $X(t)$ has a limit (finite or infinite) as t approaches infinity, almost everywhere (a.e.for brevity), then necessarily this limit will be equal to $\mathcal{E}^o$.

3.4 Persistence in the mean of COVID-19

In the following, we give a condition for the persistence in the mean of the disease, but before stating the main result, we shall first recall the concept of persistence in the mean.

Definition 3.5 (Persistence in the mean [108, 111]). *For system (II.2), the total mass of infectious individuals $I_{total} := (A+I)$ is said to be strongly persistent in the mean, or just persistent in the mean, if there is a positive constant c such that*

$$\liminf_{t\to\infty} \langle A+I\rangle_t \geqslant c > 0 \quad \textit{almost surely.}$$

Remark 3.5. *In the last definition, some authors use the word "Permanent" instead of "Persistent" [112], but others prefer to avoid this terminology so as not to confuse the notion of Persistence in the mean, with that of stochastic permanence which is completely different (see Definition 3.3).*

For brevity and simplicity in writing the next results, it will be convenient to adopt the following notations:

- $\rho_1(\alpha) = 3\times\sqrt[3]{\Lambda\,(\beta_1-\beta_2)\,\sigma}\times\left(\sqrt[3]{\theta(1-p)\times\alpha}+\sqrt[3]{p\times(1-\alpha)}\right), \quad \forall\alpha\in(0,1),$
- $\rho_2 = 7\mu+\sigma+(\varepsilon_A+\gamma_A+d_A)+(\varepsilon_I+\gamma_I+d_I)+(d_H+\gamma_H)+|\lambda-q|+\dfrac{1}{2}\sum_{i=1}^{7}\sigma_i^2,$
- $\widehat{\alpha} = \dfrac{\sqrt{\theta(1-p)}}{\sqrt{\theta(1-p)}+\sqrt{p}} \in (0,1)\,.$

Lemma 3.2. *For any $\alpha\in(0,1)$, the following inequality is satisfied*

$$\rho_1(\alpha) \leqslant \rho_1(\widehat{\alpha}). \tag{II.49}$$

In other terms, $\rho_1(\widehat{\alpha})$ is the maximum value of $\rho_1(\alpha)$ on the open interval $(0,1)$.

Proof. We start our proof by observing that the function $\rho_1(\alpha)$ is differentiable on $(0,1)$, with a first derivative given by

$$\begin{aligned}
\rho_1'(\alpha) := \frac{\mathrm{d}\rho_1(\alpha)}{\mathrm{d}\alpha} &= \sqrt[3]{\Lambda\,(\beta_1-\beta_2)\,\sigma}\left(\frac{\sqrt[3]{\theta(1-p)}}{\sqrt[3]{\alpha^2}}-\frac{\sqrt[3]{p}}{\sqrt[3]{(1-\alpha)^2}}\right)\\
&= \frac{\sqrt[3]{\Lambda\,(\beta_1-\beta_2)\,\sigma}}{\sqrt[3]{(\alpha\times(1-\alpha))^2}}\times\frac{\theta(1-p)(1-\alpha)^2-p\alpha^2}{\left(\sqrt[3]{\theta(1-p)\,(1-\alpha)^2}\right)^2+\sqrt[3]{\theta(1-p)p(1-\alpha)^2\alpha^2}+\left(\sqrt[3]{p\alpha^2}\right)^2}\\
&= \frac{\sqrt[3]{\Lambda\,(\beta_1-\beta_2)\,\sigma}\left(\sqrt{\theta(1-p)}(1-\alpha)+\sqrt{p}\alpha\right)\left(\sqrt{\theta(1-p)}+\sqrt{p}\right)}{\left((1-\alpha)\sqrt[3]{\theta(1-p)\alpha}\right)^2+\sqrt[3]{\theta p(1-p)\alpha^4(1-\alpha)^4}+\left(\alpha\sqrt[3]{p(1-\alpha)}\right)^2}\times(\widehat{\alpha}-\alpha)\,.
\end{aligned}$$

As it can be seen, the derivative $\rho_1'(\alpha)$ and the linear function $L(\alpha)=\widehat{\alpha}-\alpha$ have the same sign, so the function $\rho_1(\alpha)$ decreases for $\alpha\in(0,\widehat{\alpha})$ and increases for $\alpha\in(\widehat{\alpha},1)$. Therefore, the highest value of ρ_1 in the interval $(0,1)$ is $\rho_1(\widehat{\alpha})$, and this is precisely the assertion of the lemma. ■

Theorem 3.4. *If $\rho_1(\widehat{\alpha})>\rho_2$, then for any $X_0\in\mathbb{R}_+^7$, the solution $X(t)=\big(S(t),Q(t),E(t),\cdots,R(t)\big)$ of the initial-value problem (II.27) verifies the following property:*

$$\liminf_{t\to\infty}\langle I+A\rangle_t \geqslant \frac{1}{\beta_1}\left(\rho_1(\widehat{\alpha})-\rho_2\right) > 0 \quad a.s.,$$

which is to say that the infectious individuals $A(t)$ and $I(t)$ are persistent in the mean.

Proof. Consider the function

$$\begin{array}{rccc} \widehat{V}: & \mathbb{R}_+^7 & \longrightarrow & \mathbb{R} \\ & x & \longmapsto & \sum_{i=1}^{7} \ln(x_i). \end{array}$$

From Itô's formula and system (II.2), we have

$$\begin{aligned} \mathrm{d}\widehat{V}(X(t)) = & \left(\left[\frac{\Lambda}{S} - \left(\beta_1 - \beta_2 \frac{I}{b+I} \right) (I + \theta A) + \lambda \frac{Q}{S} - (q+\mu) \right] + \left[q \frac{S}{Q} - (\lambda + \mu) \right] \right. \\ & + \left[\left(\beta_1 - \beta_2 \frac{I}{b+I} \right) \frac{S}{E} (I + \theta A) - (\mu + \sigma) \right] + \left[(1-p)\, \sigma \frac{E}{A} - (\mu + \varepsilon_A + \gamma_A + d_A) \right] \\ & + \left[\sigma p \frac{E}{I} - (\mu + \varepsilon_I + \gamma_I + d_I) \right] + \left[\varepsilon_I \frac{I}{H} + \varepsilon_A \frac{A}{H} - (d_H + \gamma_H + \mu) \right] \\ & \left. + \left[\gamma_H \frac{H}{R} + \gamma_I \frac{I}{R} + \gamma_A \frac{A}{R} - \mu \right] - \frac{1}{2} \sum_{i=1}^{7} \sigma_i^2 \right) \mathrm{d}t + \sum_{i=1}^{7} \sigma_i \, \mathrm{d}B_i(t) \\ \geqslant & \left(\frac{\Lambda}{S} - \beta_1 (I + \theta A) + (\lambda \wedge q) \left(\frac{S}{Q} + \frac{Q}{S} \right) + (\beta_1 - \beta_2) \frac{S}{E} (I + \theta A) + (1-p)\, \sigma \frac{E}{A} + \sigma p \frac{E}{I} \right. \\ & \left. - \left[7\mu + \lambda + q + \sigma + (\varepsilon_A + \gamma_A + d_A) + (\varepsilon_I + \gamma_I + d_I) + (d_H + \gamma_H) + \frac{1}{2} \sum_{i=1}^{7} \sigma_i^2 \right] \right) \mathrm{d}t \\ & + \sum_{i=1}^{7} \sigma_i \, \mathrm{d}B_i(t). \end{aligned}$$

Noticing that $\lambda \wedge q = \dfrac{\lambda + q - |\lambda - q|}{2}$ and $\left(\dfrac{S}{Q} + \dfrac{Q}{S} \right) \geqslant 2$, we get for all $t \geqslant 0$

$$\begin{aligned} \mathrm{d}\widehat{V}(X(t)) \geqslant & \left(\left[\frac{(1-\widehat{\alpha})\Lambda}{S} + (\beta_1 - \beta_2) \frac{SI}{E} + \sigma p \frac{E}{I} \right] + \left[\frac{\widehat{\alpha}\Lambda}{S} + \theta (\beta_1 - \beta_2) \frac{SA}{E} + (1-p)\, \sigma \frac{E}{A} \right] \right. \\ & \left. - \beta_1 (I + \theta A) - \rho_2 \right) \mathrm{d}t + \sum_{i=1}^{7} \sigma_i \, \mathrm{d}B_i(t), \end{aligned}$$

and from the relation between arithmetic and geometric means (the first is greater than or equal to the second, see [113]), it results that

$$\begin{aligned} \mathrm{d}\widehat{V}(X(t)) & \geqslant \left(3 \times \sqrt[3]{(1-\widehat{\alpha})\Lambda (\beta_1 - \beta_2)\, \sigma p} + 3 \times \sqrt[3]{\widehat{\alpha}\Lambda (\beta_1 - \beta_2)\, \theta \sigma (1-p)} - \beta_1 (I + \theta A) - \rho_2 \right) \mathrm{d}t + \sum_{i=1}^{7} \sigma_i \, \mathrm{d}B_i(t) \\ & \geqslant \left((\rho_1(\widehat{\alpha}) - \rho_2) - \beta_1 (I + \theta A) \right) \mathrm{d}t + \sum_{i=1}^{7} \sigma_i \, \mathrm{d}B_i(t). \end{aligned} \tag{II.50}$$

Integrating from 0 to t and dividing by t on both sides of (II.50) gives

$$\frac{\widehat{V}(X(t)) - \widehat{V}(X(0))}{t} \geqslant (\rho_1(\widehat{\alpha}) - \rho_2) - \beta_1 \langle I(t) + \theta A(t) \rangle + \sum_{i=1}^{7} \sigma_i \frac{B_i(t)}{t}.$$

Hence

$$\langle I(t) + A(t) \rangle \geqslant \langle I(t) + \theta A(t) \rangle \geqslant \frac{1}{\beta_1} \left(\frac{\widehat{V}(X(0)) - \widehat{V}(X(t))}{t} + (\rho_1(\widehat{\alpha}) - \rho_2) \right) + \sum_{i=1}^{7} \frac{\sigma_i}{\beta_1} \frac{B_i(t)}{t}. \tag{II.51}$$

Since $\ln(y) \leqslant y-1 \leqslant y$ for all $y>0$, one can assert that $\widehat{V}(x) \leqslant \sum_{i=1}^{7} x_i$ for any $x \in \mathbb{R}_+^7$. Combining the last inequality with (II.51) yields

$$\langle I(t)+A(t)\rangle \geqslant \frac{1}{\beta_1}\left(\frac{\widehat{V}(X(0))}{t} - \frac{1}{t}\sum_{i=1}^{7} X_i(t) + (\rho_1(\widehat{\alpha})-\rho_2)\right) + \sum_{i=1}^{7}\frac{\sigma_i}{\beta_1}\,\frac{B_i(t)}{t}.$$

By using the strong law of large numbers for local martingales and the first assertion of Lemma 3.1, we obtain

$$\liminf_{t\to\infty}\langle I+A\rangle_t \geqslant \frac{1}{\beta_1}\,(\rho_1(\widehat{\alpha})-\rho_2) > 0 \quad \text{a.s.},$$

which is the required assertion. ■

Remark 3.6. *In the last proof, we can notice that any constant $\alpha \in (0,1)$ can play the role of $\widehat{\alpha}$, but the peculiarity of the latter lies essentially in its capacity to weaken the hypothesis of Theorem 3.4. Indeed, according to Lemma 3.2, if $\rho_1(\alpha) > \rho_2$ for some $\alpha \in (0,1)$ then necessarily $\rho_1(\widehat{\alpha}) > \rho_2$.*

4 Numerical simulation examples

In this section, and using the parameter values as shown in Table II.1, we present some numerical simulations to validate the various results proved in this chapter. Most of the parametric values appearing in this table (Table II.1) are selected from real data available in existing literature ([60, 114, 64, 115] more precisely) and the rest of them are just assumed for numerical calculations. The solution of our COVID-19 model, in its both stochastic and deterministic forms, is simulated in our case with the initial state given by $S(0) = 1.8\times10^6$, $Q(0) = 0$, $E(0) = 10$, $A(0) = 15$, $I(0) = 8$, $H(0) = 5$ and $R(0) = 0$ (see [114]). In what follows, the unity of time is one day and the number of individuals is expressed in one million population.

Parameter	Description	Nominal value
Λ	Recruitment rate	108.63
β_1	Contact rate in absence of media coverage	$(1.7\times10^{-9}, 5.2\times10^{-3})$
β_2	Awareness rate (or also response intensity)	$[0,\beta_1]$
b	Constant of media's half saturation	70
θ	Modification ratio of asymptomatic infectiousness	0.0494
q	Quarantine rate	0.071
λ	Rate of release from quarantine	0.1003
μ	Natural death rate	0.00029
σ	The transition rate of exposed individuals to the infective classes	0.2
p	Probability of having symptoms among infected individuals	(0,1)
ε_A	The hospitalization rate of asymptomatic infected individuals	0.1
γ_A	Recovery rate of asymptomatic infected individuals	0.15
d_A	Disease-induced death rate for asymptomatic infected individuals	0.005
ε_I	The hospitalization rate of symptomatic infected individuals	0.33
γ_I	Recovery rate of symptomatic infected individuals	0.1001
d_I	Disease-induced death rate for symptomatic infected individuals	0.008
γ_H	Recovery rate of hospitalized individuals	0.14
d_H	Disease-induced death rate for hospitalized individuals	0.004

Table II.1: Definitions and values (per day) of COVID-19 model parameters used in the simulation.

Example 4.1 (**Deterministic case**). *In this case, and adopting the parameter values listed in Table II.1, we will illustrate the theoretical results of the first section. Figures II.2 and II.3 present the dynamical behavior of the COVID-19 deterministic model when β_1, β_2 and p are conveniently fixed in their admissible ranges. In Figure II.2, we take $\beta_1 = 3.97 \times 10^{-6}$, $\beta_2 = 0.6\times\beta_1$, $p = 0.6201$, and we get $\mathcal{R}_0 = 0.9180 < 1$. From the curves appearing in this figure, it is clear that the disease is dying out, and in addition to that, the solution $\big(S(t), Q(t), E(t), A(t), I(t), H(t), R(t)\big)$ converges to the free-disease state $\mathcal{E}^o = \big(1.5563 \times 10^5, 2.0896 \times 10^5, 0, 0, 0, 0, 0\big)$ which supports the Theorem 2.5. On the other hand, and changing β_1 to 5×10^{-6}, we obtain a basic reproductive number $\mathcal{R}_0$ greater than one ($\mathcal{R}_0 = 1.1562 > 1$). From Figure II.3, we observe the COVID-19 persistence in this case, which agree well with Theorem 2.7.*

Example 4.2 (**Stochastic case**). *In order to exhibit the random fluctuations effect on COVID-19 dynamics, we present in Figures II.4 and II.5 a collection of numerical simulations. In the first instance, we take $\beta_1 = 2.08 \times 10^{-9}$, $\beta_2 = 0.6\times\beta_1$, $p = 0.6201$, and we choose the stochastic intensities as follows: $\sigma_1 = 0.024$, $\sigma_2 = 0.0235$, $\sigma_3 = 0.015$, $\sigma_4 = 0.0174$, $\sigma_5 = 0.019$, $\sigma_6 = 0.0213$, and $\sigma_7 = 0.0238$. Then,*

$$\frac{1}{2}\left(\sigma_1^2 \vee \sigma_2^2 \vee \sigma_3^2 \vee \sigma_4^2 \vee \sigma_5^2 \vee \sigma_6^2 \vee \sigma_7^2\right) = 0.000288 < 0.000290 = \mu,$$

and

$$\sigma_3^2 \wedge \sigma_4^2 \wedge \sigma_5^2 = 0.000225 > 0.000202 = 6\,(\beta_1 S^o - \mu)\,.$$

Hence, the assumptions of Theorem 3.3 are verified, and consequently

$$\limsup_{t\to\infty} \frac{\ln(E(t) + A(t) + I(t))}{t} \leqslant \beta_1 S^o - \mu - \frac{\sigma_3^2 \wedge \sigma_4^2 \wedge \sigma_5^2}{6} = -3.83 \times 10^{-6} < 0 \quad a.s.$$

That is to say that the COVID-19 dies out exponentially almost surely. Moreover, by Corollary 3.1 and Remark 3.4, the mean time of the solution converges to the deterministic free-disease equilibrium $\mathcal{E}^o$. These two last results are confirmed by the curves depicted in Figure II.4. To make the condition $\rho_1(\widehat{\alpha}) > \rho_2$ true, we take $\beta_1 = 4.1 \times 10^{-3}$, $\beta_2 = 0.1\times\beta_1$ and we select new values of stochastic intensities as follows: $\sigma_1 = 0.019$, $\sigma_2 = 0.0185$, $\sigma_3 = 0.014$, $\sigma_4 = 0.017$, $\sigma_5 = 0.0158$, $\sigma_6 = 0.0136$, and $\sigma_7 = 0.0182$. Thus, the main result of Theorem 3.4 is satisfied and this time, the COVID-19 persists in the mean as shown in Figure II.5.

Example 4.3 (**The effectiveness of media intervention and quarantine strategies**). *We aim during this example to examine numerically the impact of media intrusion and quarantine strategies on the COVID-19 spread. To this end, we simulate the progression of the total infected population number with various values of β_2, λ and q. Through Figure II.6, we can perceive that the increase of the quarantine rate and duration can delay the arrival of infection peak, reduce remarkably the impact of the disease, and even lead it to the extinction sometimes (see for example the last two curves presented in Figure II.6). On the other hand, and as it can be seen from figures II.7 and II.8, the media alert strategy is able also to diminish the severity of the COVID-19 spread, but it can not make it disappear, and we explain this theoretically by the absence of the parameters β_2 and b in the persistence and extinction conditions (for example $\mathcal{R}_0$ does not involve these parameters). Roughly speaking, the role of the quarantine and the information intervention about COVID-19 is critically important, particularly in its beginnings. The growth of the positive response in susceptible individuals leads to reduce the gravity of the infection and creates a conscious public able to overcome this new pandemic by respecting social distancing and self-isolation procedures.*

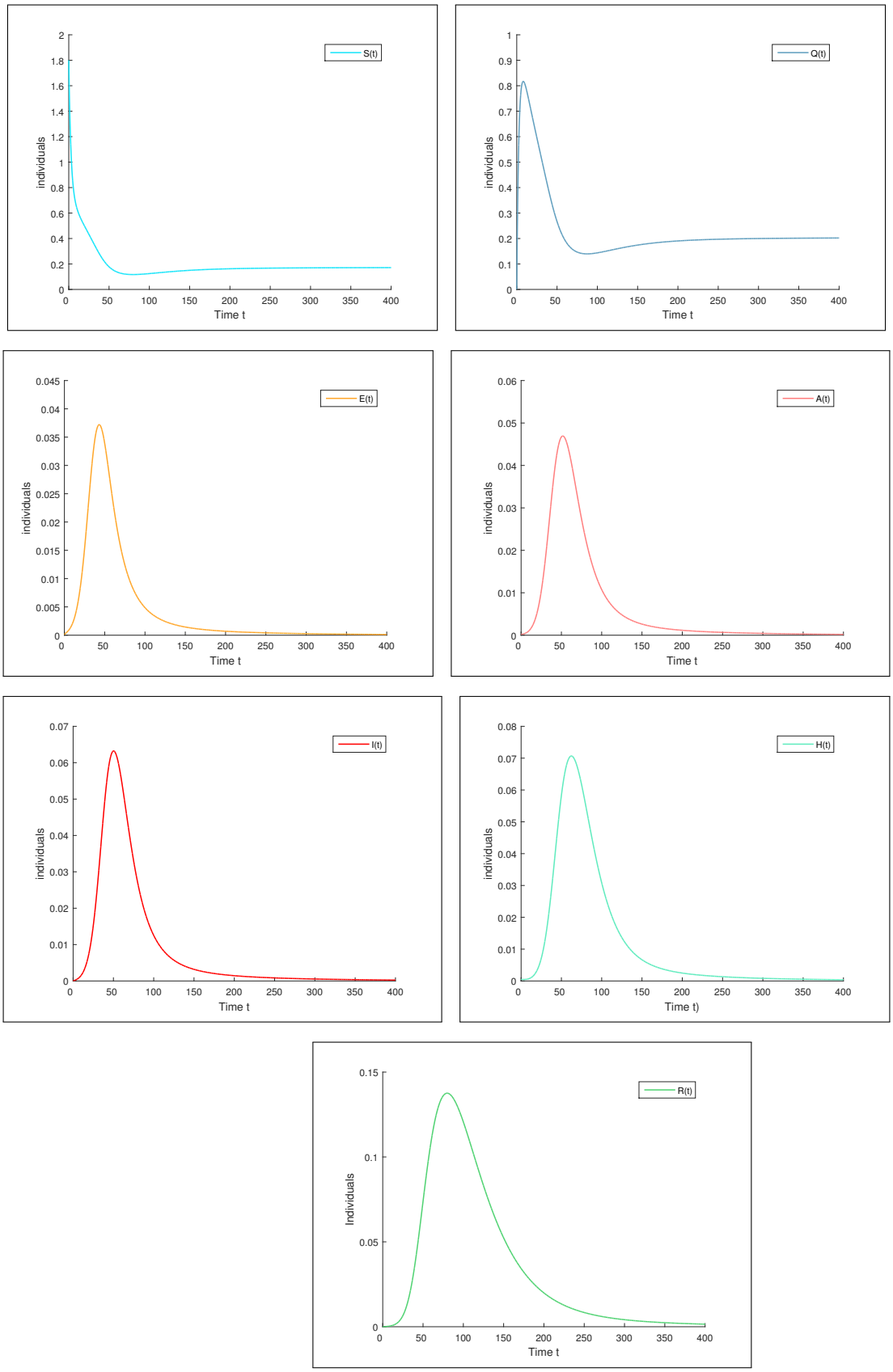


Figure II.2: Solutions of COVID-19 deterministic model (II.1) taking $\beta_1 = 3.97 \times 10^{-6}$, $\beta_2 = 0.6 \times \beta_1$ and $p = 0.6201$ ($\mathcal{R}_0 = 0.9180 < 1$).

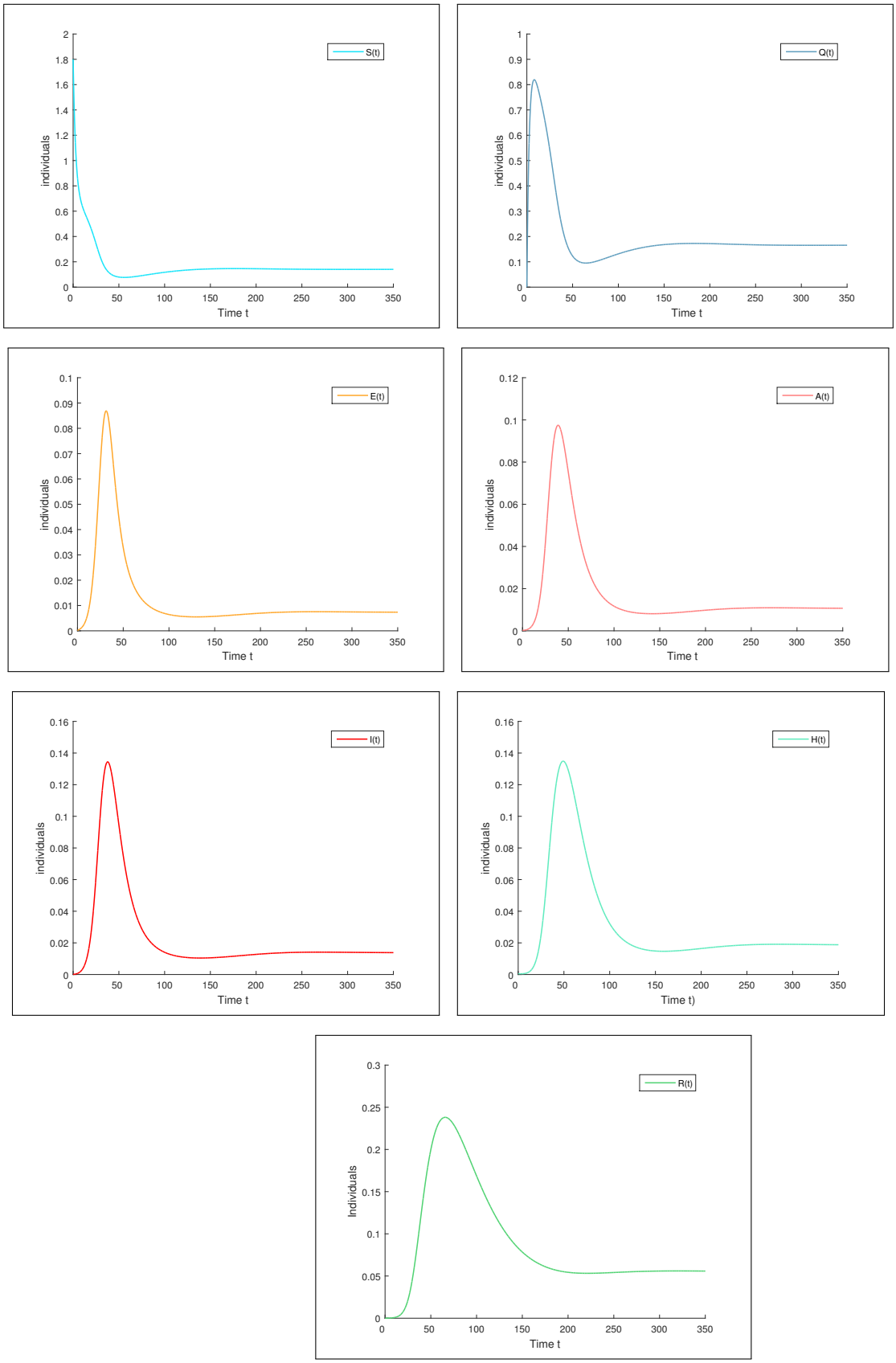


Figure II.3: Trajectories of COVID-19 deterministic model (II.1) taking $\beta_1 = 5 \times 10^{-6}$, $\beta_2 = 0.6 \times \beta_1$ and $p = 0.6201$ ($\mathcal{R}_0 = 1.1562 > 1$).

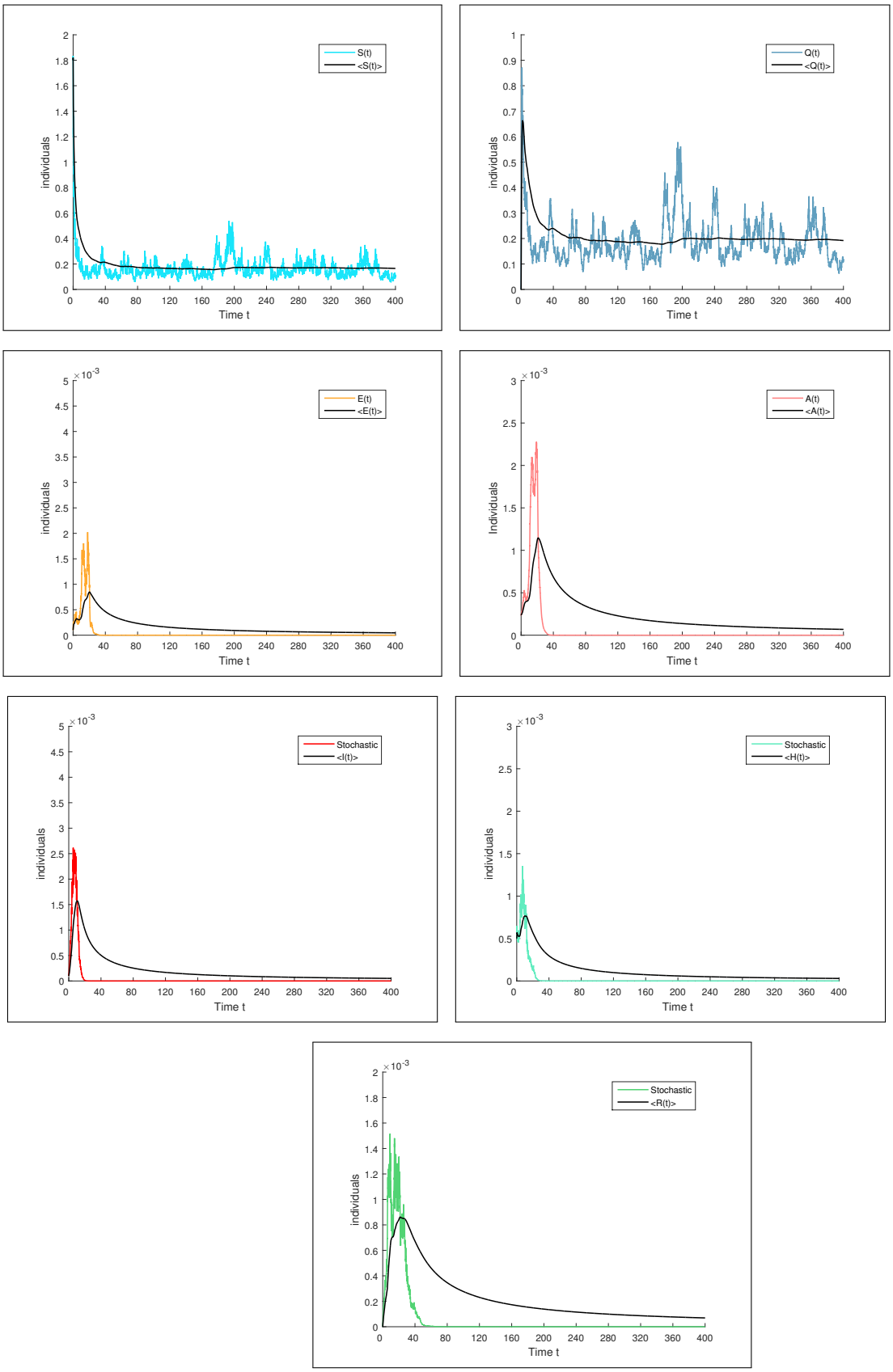


Figure II.4: Trajectories of COVID-19 stochastic model (II.2) taking $\beta_1 = 2.08 \times 10^{-9}$, $\beta_2 = 0.6 \times \beta_1$ and $p = 0.6201$.

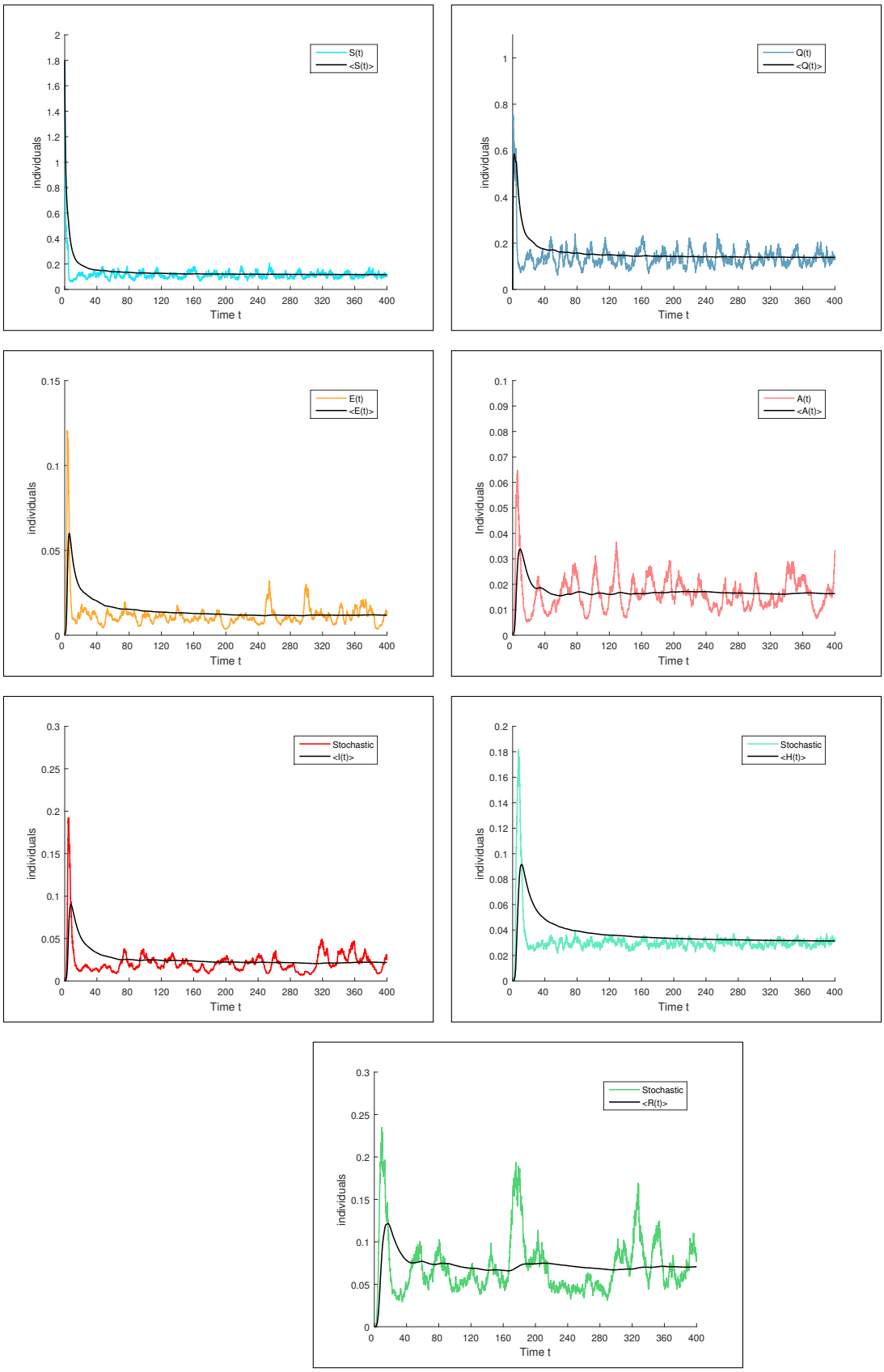


Figure II.5: Trajectories of COVID-19 stochastic model (II.2) taking $\beta_1 = 4.1 \times 10^{-3}$, $\beta_2 = 0.1 \times \beta_1$ and $p = 0.6201$ $(\rho_1(\widehat{\alpha}) = 1.0266 > 0.9694 = \rho_2)$.

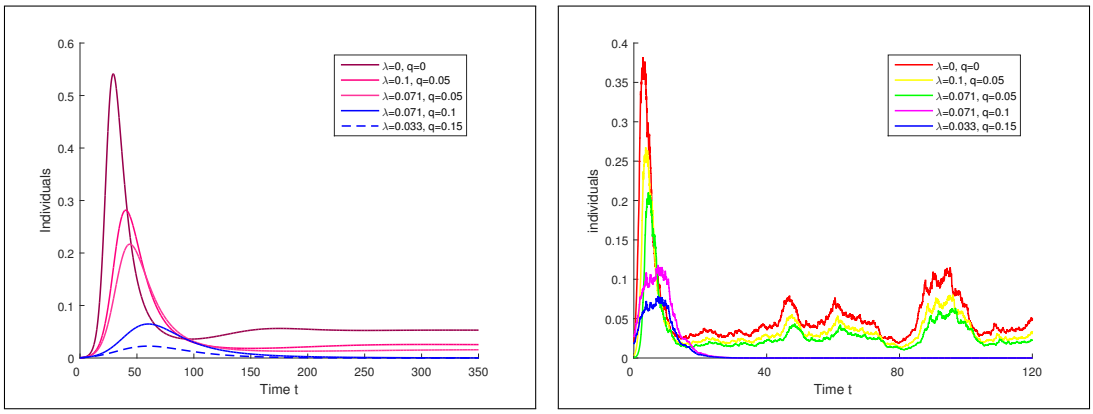


Figure II.6: The impact of the quarantine parameters λ and q on the deterministic and stochastic trajectories of the total infected individuals $I_{\text{total}}(t) := E(t) + A(t) + I(t)$. The rest of the parameters is taken respectively as in Figure II.3 and Figure II.5.

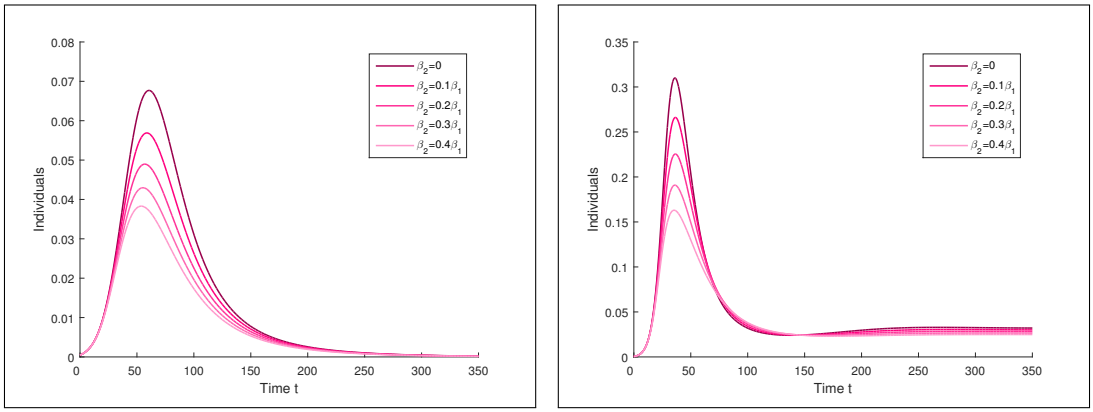


Figure II.7: The effect of the awareness rate β_2 on the deterministic trajectories of the total infected individuals $I_{\text{total}}(t) := E(t) + A(t) + I(t)$ when parameters are taken respectively as in Figure II.2 and Figure II.3.

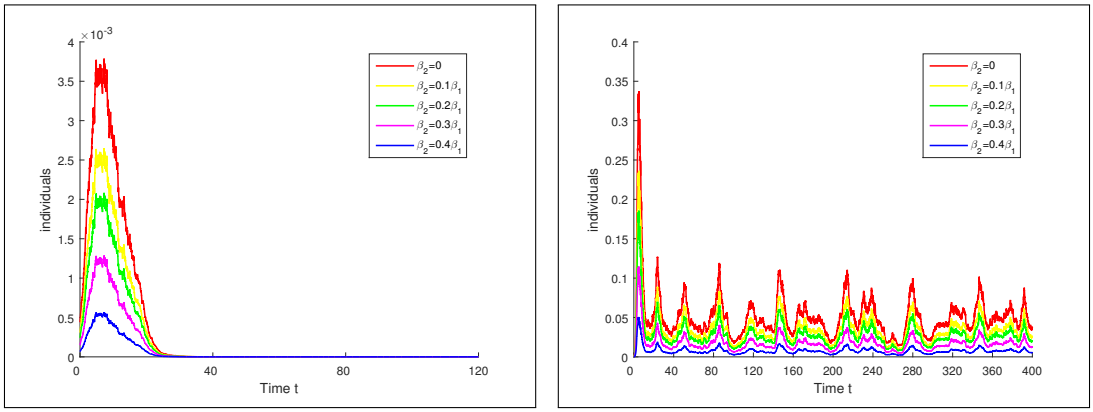


Figure II.8: Stochastic paths of the total infected individuals $I_{\text{total}}(t) := E(t) + A(t) + I(t)$ under different response intensities β_2. The other parameters are taken respectively as in Figure II.4 and Figure II.5.

5 Conclusion and discussion

The current Coronavirus disease is a major danger that threatens the whole world, and in this context, mathematical modeling is a very powerful tool for knowing more about how such a virus is transmitted within a host population of humans. In this regard, an SQEAIHR epidemic model that describes the COVID-19 dynamics under the application of quarantine and coverage media strategies is proposed in both deterministic and stochastic forms in this chapter. Moreover, a rigorous mathematical analysis of this model is performed to get an overview of COVID-19 dissemination behavior. The principal epidemiological and mathematical findings of our study are presented as follows:

- For the deterministic version of COVID-19 model, the basic reproduction number $\mathcal{R}_0$ is calculated by using the next-generation matrix approach and based on its expression, we have determined many dynamical properties of this version. More precisely, when $\mathcal{R}_0 < 1$, the COVID-19-free steady point $\mathcal{E}^o$ is the unique equilibrium of system (II.1), and it is globally asymptotically stable in this case. On the other side, when $\mathcal{R}_0 > 1$, the disease-free equilibrium $\mathcal{E}^o$ is still present but it becomes unstable, and another endemic one $\mathcal{E}^\star$ appears this time, which makes our system (II.1) uniformly persistent according to Theorem 2.7.

- For the stochastic version of COVID-19 model, we have demonstrated the existence and uniqueness of a global positive solution, and besides this, we have established that this latter is stochastically ultimately bounded, in other words, the probability of this solution exploding in the infinite time is very low (see Theorem 3.2). During our exploration of the perturbed system (II.2), we have derived the conditions for COVID-19 extinction and persistence, and we remarked that they are mainly depending on the magnitude of the noises intensities as well as the system parameters.

Compared to the existing literature, the novelty of our findings lies in new analysis techniques and improvements which are summarized in the following items:

- Our results shed some new light on the next-generation matrix method and applies the Varga's theorem [93, 90, 94] to show the local stability of the disease-free state without making recourse to the Routh-Hurwitz criterion, which enabled us to avoid many long calculations.

- By eliminating a hypothetical redundancy and bringing into play the notion of the positively invariant set, our results provide an improved and generalized version of Theorem 9.2 in [32], and use it to establish the global stability of the disease-free equilibrium $\mathcal{E}^o$.

- For the case of the stochastic COVID-19 model (II.2), in Lemma 3.2 and Theorem 3.4, we showed that the following inequality:

$$\rho_1(\widehat{\alpha}) > \rho_2,$$

  is a sufficient condition for the non-disappearance of COVID-19 infective individuals.

In order to support the theoretical results and clarify the role of quarantine and awareness strategies towards the COVID-19 spreading behavior, we have presented some numerical simulation examples. From the curves appearing in these simulations, and more precisely those where we have gradually varied the value of β_2, λ and q, we noticed that the quarantine and awareness strategies can effectively lower the infection and reduce the sizes of infected species. Despite its remarkable efficiency, the coverage media alone is unfortunately unable to prevent the COVID-19 from persisting. This last fact can be clearly remarked and confirmed for the deterministic case by observing that $\mathcal{R}_0$ does not contain any media coverage parameter. So, in short, we conclude that the first thing that must be done in the future during confronting a new and rapidly spreading disease like COVID-19 is to adopt quarantine and media intervention strategies, pending the emergence of an appropriate and safe treatment.

We believe that our findings can be a rich basis for future studies especially after the recent discovery of new and a stronger variants of COVID-19.

Chapter III

A novel mathematical analysis and threshold reinforcement of a stochastic dengue epidemic model with Lévy jumps

Abstract

The aggressive re-emergence of dengue fever during the last decade made it a principal topic of several research fields, especially biomathematics. In this vein, mathematical modeling is commonly viewed as a great option to get an idea of this disease's prevalence behavior. This chapter presents and analyzes a generalized stochastic dengue model that incorporates slight and huge environmental perturbations. In rigorous terms, our proposed model takes the form of a stochastic differential equations system that includes white noises and Lévy jumps. We first demonstrate its well-posedness then, and based on many novel analytical techniques, we provide some ameliorated necessary conditions for the extinction and the persistence in the mean. The theoretical findings show that the dynamics of our disturbed dengue model are mainly determined by the parameters related to the white noises and Lévy jumps intensities. In the end, certain numerical illustrative examples are depicted to confirm our findings and to highlight the effect of the adopted mathematical techniques on the results.

1 Introduction and model formulation

Since ancient times, mankind has had to deal with various very dangerous epidemics which are characterized by rapid spread and high death rate [116, 117, 118, 119, 120]. Often caused by some kind of bacteria or viruses unknown in their time, these epidemics killed millions of people, and thus marked the history of several countries, societies and even Humanity in general [121, 122, 123]. By looking a little into the past, we can find in this context many examples such as smallpox, tuberculosis, plague, cholera, typhus, the "Spanish flu" of 1918, and closer to us SARS, Ebola, Zika virus, HIV and most recently COVID-19 [124, 125, 126, 127, 128]. Seemingly, the list of all these diseases is very long, and we cannot write it all down here, but what we can guarantee is that the name of Dengue fever will undoubtedly appear in it [129]. The dengue fever is a mosquito-borne viral infection caused mainly by one of the four dengue virus stereotypes (DENV-1 to DENV-4). According to the World Health Organization (WHO) [130], a significant number of dengue infections produce just mild illness, but many others can lead to an acute flu-like illness which later turns into a potentially fatal complication named severe dengue. With thousands of mortalities and nearly four hundred million infections annually around the globe, this disease is considered to be the deadliest vector-borne epidemic after Malaria [131]. Despite the existence of some suggested developments regarding a remedy for the dengue virus [132], until now, no effective

vaccine or treatment against it are available in the market [133]. So, early-stage detection and access to appropriate medical care still the only possible solutions at hand to face this murderous disease. At present, more than one hundred countries are under the threat of dengue fever [134], and what makes matters worse is the ability of this epidemic to affect almost all age groups, that is why a good comprehension of its evolution dynamics is firmly required. In this vein, mathematical modeling can be presented as the most useful, efficient and applicable tool for appropriately describing the dengue fever prevalence and perceiving its effects on a host population, especially in the long term.

In order to comprehend and supervise the running of dengue infection, a considerable number of mathematical models, notably compartmental ones, have been suggested and treated in details by several works [135, 136, 137]. The first attempt to describe the dengue spread was introduced by Newton and Reiter [135] in the form of an SEIR model that does not take into consideration the mosquitoes populations. Later, and in the same context, Focks et al. [136, 137] closed this loophole by using dynamic table models to illustrate the evolution of these populations. Because of its continued re-emergence [138], the study of the dengue's spread has not stopped at this stage, and even it remains until now an active research subject that inspires several recent papers, see [139, 140, 141, 142] and the references given there. For example, in [139] the authors constructed a deterministic dengue's propagation model and parametrized it by employing real data from the 2017 dengue outbreak in Pakistan. In [140], Wang and Zhao studied the vaccination effect on the prevalence of dengue fever under the framework of coinfection with Zika virus. A brief discussion on the dengue modeling in both deterministic and stochastic levels is presented in [141]. The dengue dissemination dynamics with the mosquitoes control, temperature elevation and human mobility restriction are detailedly treated in [142]. In [143], Cai et al. drew up a dengue epidemic model with bilinear saturated incidence before going to investigate the global stability of the disease-free and the endemic equilibria. The formulation of their model can be presented by the following ordinary differential equations system:

$$\begin{cases} \dfrac{\mathrm{d}S}{\mathrm{d}t} = \Lambda - \dfrac{b\beta S\widehat{I}}{1+a\widehat{I}} - \mu S, \\ \dfrac{\mathrm{d}I}{\mathrm{d}t} = \dfrac{b\beta S\widehat{I}}{1+a\widehat{I}} - (\mu+\rho_0+r_1+r_2)\, I, \\ \dfrac{\mathrm{d}R}{\mathrm{d}t} = (r_1+r_2)\, I - \mu R, \\ \dfrac{\mathrm{d}\widehat{S}}{\mathrm{d}t} = \hat{\Lambda} - b\hat{\beta}\widehat{S}I - \hat{\mu}\widehat{S}, \\ \dfrac{\mathrm{d}\widehat{I}}{\mathrm{d}t} = b\hat{\beta}\widehat{S}I - \hat{\mu}\widehat{I}, \end{cases} \tag{III.1}$$

with initial conditions $S(0) > 0, I(0) > 0, R(0) > 0, \widehat{S}(0) > 0$ and $\widehat{I}(0) > 0$. Clearly, the previous system describes the simultaneous evolution of two populations, the first is the host population denoted by $N(t)$ at time t and subdivided into three compartments of susceptible, infected, and recovered individuals, with densities indicated respectively by $S(t)$, $I(t)$ and $R(t)$. The second population is the mosquitoes one which is denoted at time t by $\widehat{N}(t)$ and partitioned in turn into two classes of susceptible and infected individuals with densities $\widehat{S}(t)$ and $\widehat{I}(t)$ respectively. The recovered mosquitoes class is not considered in this model because of the short lifespan of these insects and the relatively long time required to recover from the dengue disease [143]. The eleven parameters appearing in system (III.1) are summarized in the following list:

- Λ and $\hat{\Lambda}$ are the susceptible humans and mosquitoes recruitment rates respectively.
- β is the transmission rate from mosquitoes to humans.
- a is a parameter measuring the inhibitory effect due to psychological and behavioral changes in the susceptible human population.
- b is the biting rate of mosquitoes, in other words, the average number of bites per mosquito in one day.

- ρ_0 is the disease-induced death rate for human individuals.
- r_1 and r_2 are respectively the human individuals recovery and treatment rates. For convenience in writing we denote from now on $r_1 + r_2$ by ρ_1.
- $\hat{\beta}$ is the transmission rate from humans to mosquitoes.
- μ and $\hat{\mu}$ are, in this order, the humans and mosquitoes natural death rates.

All parameters listed above are assumed to be in the positive real axis $\mathbb{R}_+ := \{x \in \mathbb{R} \mid x > 0\}$ except a that is supposed to be just nonnegative. Since $R(t)$ does not appear in the other equations of system (III.1), the dynamical behavior of dengue infection can be deduced just from the following simplified model:

$$\begin{cases} \dfrac{\mathrm{d}S}{\mathrm{d}t} = \Lambda - b\beta\dfrac{S\widehat{I}}{1+a\widehat{I}} - \mu S, \\ \dfrac{\mathrm{d}I}{\mathrm{d}t} = b\beta\dfrac{S\widehat{I}}{1+a\widehat{I}} - (\mu+\rho_0+\rho_1)\, I, \\ \dfrac{\mathrm{d}\widehat{S}}{\mathrm{d}t} = \hat{\Lambda} - b\hat{\beta}\widehat{S}I - \hat{\mu}\widehat{S}, \\ \dfrac{\mathrm{d}\widehat{I}}{\mathrm{d}t} = b\hat{\beta}\widehat{S}I - \hat{\mu}\widehat{I}. \end{cases} \tag{III.2}$$

According to the mathematical analyses effected in [143], the spread dynamics of the aforementioned dengue model is completely determined by the basic reproduction number which is expressed in this case by $\mathcal{R}_0 = \frac{b^2\beta\Lambda\hat{\beta}\hat{\Lambda}}{\mu(\mu+\rho_0+\rho_1)\hat{\mu}^2}$. More precisely, if $\mathcal{R}_0 \leqslant 1$, the system (III.2) admits a unique disease-free equilibrium $\mathcal{E}_o = \left(\frac{\Lambda}{\mu}, 0, 0, 0\right)$, and it is globally asymptotically stable in the invariant region $\mathcal{D} = \left\{\left(S, I, \widehat{S}, \widehat{I}\right) \in \mathbb{R}_+ \mid S + I \leqslant \frac{\Lambda}{\mu} \text{ and } \widehat{S} + \widehat{I} \leqslant \frac{\hat{\Lambda}}{\hat{\mu}}\right\}$. On the other hand if $\mathcal{R}_0 > 1$, the disease-free equilibrium is still present, but it becomes unstable. In this case, the system (III.2) admits another steady state $\mathcal{E}_\star = \left(S_\star, I_\star, \widehat{S}_\star, \widehat{I}_\star\right)$ called the endemic equilibrium, and it is globally asymptotically stable in the interior of $\mathcal{D}$. The endemic equilibrium components that we have just mentioned are given as follows: $S_\star = \frac{\hat{\mu}^2(\mu+\rho_0+\rho_1)(1+a\widehat{I}_\star)}{b^2\beta\hat{\beta}(\Lambda-\hat{\mu}\widehat{I}_\star)}$, $I_\star = \frac{\hat{\mu}^2\widehat{I}_\star}{b\hat{\beta}(\Lambda-\hat{\mu}\widehat{I}_\star)}$, $\widehat{S}_\star = \frac{\Lambda}{\mu} - \widehat{I}_\star$ and $\widehat{I}_\star = \frac{\mu\hat{\mu}(\mu+\rho_0+\rho_1)(\mathcal{R}_0-1)}{\hat{\mu}(\mu a+b\beta)(\mu+\rho_0+\rho_1)+b^2\beta\hat{\beta}\hat{\Lambda}}$.

It is well known that environmental uncertainties influence the spread of any disease and make the divination of its behavior more difficult [107]. In such a situation deterministic models, and despite their ability in providing highly informative results and predictions, are not suitable enough. Therefore, a new or modified mathematical formulation that takes into account this effect of randomness is really required, especially in the framework of dengue fever dissemination analysis. In other terms, and to be more explicit, it is necessary to resort to the stochastic models which are distinguished and differentiated from their deterministic counterparts by the ability to formulate and take into consideration the impact of randomness, the thing that makes them undoubtedly more reasonable and adapted to reality. In this regard, several authors have proposed and developed many stochastic models that are presenting the dengue infection dynamics from different viewpoints and perspectives. For example, in [144] Otero and Solari investigated a probabilistic model depicting the evolution of dengue fever within a spatially fixed humans population, that is to say that mosquitoes dispersal is the only one responsible for the disease's spreading. In order to highlight the role of human mobility on dengue prevalence, Barmak et al. [145] treated a perturbed model incorporating the infected mosquitoes and humans dynamics with different human mobility patterns. In [146], Liu at al. took into account the randomness's effect on the dengue fever pervasion by assuming that the solution of the model (III.2) fluctuates normally around its value. Following this point of view, they explored a disturbed version of the dengue epidemic model (III.2), in which the environmental perturbations are in the form of proportional whites noises to the variables. Hence, the stochastic dengue model that they studied is presented

as follows:

$$\begin{cases} \mathrm{d}S(t) = \left[\Lambda - \dfrac{b\beta S(t)\widehat{I}(t)}{1+a\widehat{I}(t)} - \mu S(t)\right]\mathrm{d}t + \sigma_1 S(t)\mathrm{d}\mathcal{B}_1(t), \\ \mathrm{d}I(t) = \left[\dfrac{b\beta S(t)\widehat{I}(t)}{1+a\widehat{I}(t)} - (\mu+\rho_0+\rho_1)\,I(t)\right]\mathrm{d}t + \sigma_2 I(t)\mathrm{d}\mathcal{B}_2(t), \\ \mathrm{d}\widehat{S}(t) = \left[\hat{\Lambda} - b\hat{\beta}\widehat{S}(t)I(t) - \hat{\mu}\widehat{S}(t)\right]\mathrm{d}t + \sigma_3\widehat{S}(t)\mathrm{d}\mathcal{B}_3(t), \\ \mathrm{d}\widehat{I}(t) = \left[b\hat{\beta}\widehat{S}(t)I(t) - \hat{\mu}\widehat{I}(t)\right]\mathrm{d}t + \sigma_4\widehat{I}(t)\mathrm{d}\mathcal{B}_4(t), \end{cases} \tag{III.3}$$

where, the nonnegative constants σ_i $(i = 1,2,3,4)$ denote the intensities of the mutually independent Brownian motions $\mathcal{B}_i$ $(i = 1,2,3,4)$. These latter, and even all the stochastic processes or random variables that will be met in this chapter, are supposed to be defined on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ equipped with a filtration $(\mathcal{F}_t)_{t\geqslant 0}$ that satisfies the usual conditions (it is right-continuous while $\mathcal{F}_0$ contains all $\mathbb{P}$-null sets).

The insertion of white noises is a very reasonable and prominent approach to model phenomena in which a given quantity is constantly subjected to slight variable fluctuations, for example, those of the environment during a certain disease's spread [147, 9, 148]. But unfortunately, this method is neither appropriate nor sufficient to represent the effect of strong and sudden external disturbances like climate changes, floods, earthquakes, tornadoes, etc [149, 150]. For this reason, we will make recourse to the well known Lévy processes which are able to formulate more adequately this kind of cases. By taking into account this type of random perturbations, we can extend the model (III.3) to the following system of stochastic differential equations with Lévy jumps (SDELJ for brevity):

$$\begin{cases} \mathrm{d}S(t) = \left[\Lambda - \dfrac{b\beta S(t)\widehat{I}(t)}{1+a\widehat{I}(t)} - \mu S(t)\right]\mathrm{d}t + \sigma_1 S(t)\mathrm{d}\mathcal{B}_1(t) + \displaystyle\int_{\mathbb{U}} \xi_1(u)S(t^-)\widetilde{\mathcal{N}}(\mathrm{d}t,\mathrm{d}u), \\ \mathrm{d}I(t) = \left[\dfrac{b\beta S(t)\widehat{I}(t)}{1+a\widehat{I}(t)} - (\mu+\rho_0+\rho_1)\,I(t)\right]\mathrm{d}t + \sigma_2 I(t)\mathrm{d}\mathcal{B}_2(t) + \displaystyle\int_{\mathbb{U}} \xi_2(u)I(t^-)\widetilde{\mathcal{N}}(\mathrm{d}t,\mathrm{d}u), \\ \mathrm{d}\widehat{S}(t) = \left[\hat{\Lambda} - b\hat{\beta}\widehat{S}(t)I(t) - \hat{\mu}\widehat{S}(t)\right]\mathrm{d}t + \sigma_3\widehat{S}(t)\mathrm{d}\mathcal{B}_3(t) + \displaystyle\int_{\mathbb{U}} \xi_3(u)\widehat{S}(t^-)\widetilde{\mathcal{N}}(\mathrm{d}t,\mathrm{d}u), \\ \mathrm{d}\widehat{I}(t) = \left[b\hat{\beta}\widehat{S}(t)I(t) - \hat{\mu}\widehat{I}(t)\right]\mathrm{d}t + \sigma_4\widehat{I}(t)\mathrm{d}\mathcal{B}_4(t) + \displaystyle\int_{\mathbb{U}} \xi_4(u)\widehat{I}(t^-)\widetilde{\mathcal{N}}(\mathrm{d}t,\mathrm{d}u). \end{cases} \tag{III.4}$$

Here and thereafter, $S(t^-), I(t^-), \widehat{S}(t^-)$ and $\widehat{I}(t^-)$ are respectively standing for the left limits of $S(t), I(t), \widehat{S}(t)$ and $\widehat{I}(t)$. $\mathcal{N}$ is a Poisson counting measure independent of $\mathcal{B}_i$ $(i = 1,2,3,4)$ with compensating martingale $\widetilde{\mathcal{N}}$ and finite characteristic measure ν that is defined on a measurable set $\mathbb{U} \subset \mathbb{R}_+$. Also, it is assumed that ν is a Lévy measure such that $\widetilde{\mathcal{N}}(\mathrm{d}t,\mathrm{d}u) = \mathcal{N}(\mathrm{d}t,\mathrm{d}u) - \nu(\mathrm{d}u)\mathrm{d}t$ and we suppose that the jumps intensities $\xi_i : \mathbb{U} \to \mathbb{R}$ are continuous functions on $\mathbb{U}$. In order to provide an additional degree of realism to our study, we will suppose that the contact between humans and mosquitoes is homogeneously mixed. In other words, and in analogy with a chemical reaction, we will adopt the mass action rates $\beta S\widehat{I}$ and $\hat{\beta}\widehat{S}I$ as the incidence functions of our epidemic model. The last fact will lead us to the following system which is none other than system (III.4) but in the case of the inhibitory effect absence:

$$\begin{cases} \mathrm{d}S(t) = \left[\Lambda - b\beta S(t)\widehat{I}(t) - \mu S(t)\right]\mathrm{d}t + \sigma_1 S(t)\mathrm{d}\mathcal{B}_1(t) + \displaystyle\int_{\mathbb{U}} \xi_1(u)S(t^-)\widetilde{\mathcal{N}}(\mathrm{d}t,\mathrm{d}u), \\ \mathrm{d}I(t) = \left[b\beta S(t)\widehat{I}(t) - (\mu+\rho_0+\rho_1)\,I(t)\right]\mathrm{d}t + \sigma_2 I(t)\mathrm{d}\mathcal{B}_2(t) + \displaystyle\int_{\mathbb{U}} \xi_2(u)I(t^-)\widetilde{\mathcal{N}}(\mathrm{d}t,\mathrm{d}u), \\ \mathrm{d}\widehat{S}(t) = \left[\hat{\Lambda} - b\hat{\beta}\widehat{S}(t)I(t) - \hat{\mu}\widehat{S}(t)\right]\mathrm{d}t + \sigma_3\widehat{S}(t)\mathrm{d}\mathcal{B}_3(t) + \displaystyle\int_{\mathbb{U}} \xi_3(u)\widehat{S}(t^-)\widetilde{\mathcal{N}}(\mathrm{d}t,\mathrm{d}u), \\ \mathrm{d}\widehat{I}(t) = \left[b\hat{\beta}\widehat{S}(t)I(t) - \hat{\mu}\widehat{I}(t)\right]\mathrm{d}t + \sigma_4\widehat{I}(t)\mathrm{d}\mathcal{B}_4(t) + \displaystyle\int_{\mathbb{U}} \xi_4(u)\widehat{I}(t^-)\widetilde{\mathcal{N}}(\mathrm{d}t,\mathrm{d}u). \end{cases} \tag{III.5}$$

For the reader's convenience, we summarize and illustrate the transmission mechanisms of the aforementioned dengue model with the help of the flowchart depicted in figure III.1.

Our principal objective in this chapter is to explore sufficient conditions for extinction and persistence in the mean of the dengue model (III.5). These two asymptotic properties are considered to be sufficient for having an excellent idea of the future dengue pandemic situation. The originality of our results lies essentially in the method that we adopt to estimate the limit values of the temporal averages $\frac{\int_0^t \Psi(s)\,\mathrm{d}s}{t}$, $\frac{\int_0^t \Psi^2(s)\,\mathrm{d}s}{t}$, $\frac{\int_0^t \widehat{\Psi}(s)\,\mathrm{d}s}{t}$ and $\frac{\int_0^t \widehat{\Psi}^2(s)\,\mathrm{d}s}{t}$, where Ψ and $\widehat{\Psi}$ are respectively the positive solutions of the following systems:

$$\begin{cases} \mathrm{d}\Psi(t) = [\Lambda - \mu\Psi(t)]\,\mathrm{d}t + \sigma_1\Psi(t)\mathrm{d}\mathcal{B}_1(t) + \displaystyle\int_{\mathbb{U}} \xi_1(u)\Psi(t^-)\widetilde{\mathcal{N}}(\mathrm{d}t,\mathrm{d}u), \quad \forall t > 0, \\ \Psi(0) = S(0) > 0. \end{cases} \tag{III.6}$$

and

$$\begin{cases} \mathrm{d}\widehat{\Psi}(t) = \left[\hat{\Lambda} - \hat{\mu}\widehat{\Psi}(t)\right]\mathrm{d}t + \sigma_3\widehat{\Psi}(t)\mathrm{d}\mathcal{B}_3(t) + \displaystyle\int_{\mathbb{U}} \xi_3(u)\widehat{\Psi}(t^-)\widetilde{\mathcal{N}}(\mathrm{d}t,\mathrm{d}u), \quad \forall t > 0, \\ \widehat{\Psi}(0) = \widehat{S}(0) > 0. \end{cases} \tag{III.7}$$

Our method permits us to bridge the gap left by the use of classical approaches presented for example in [151, 152]. Furthermore, we have developed a new and nonstandard analytical technique that can be used to determine a more precise threshold in the case of extinction. The performed analysis in this chapter seems to be very encouraging to study other related epidemic models and especially those which are perturbed with Lévy noises.

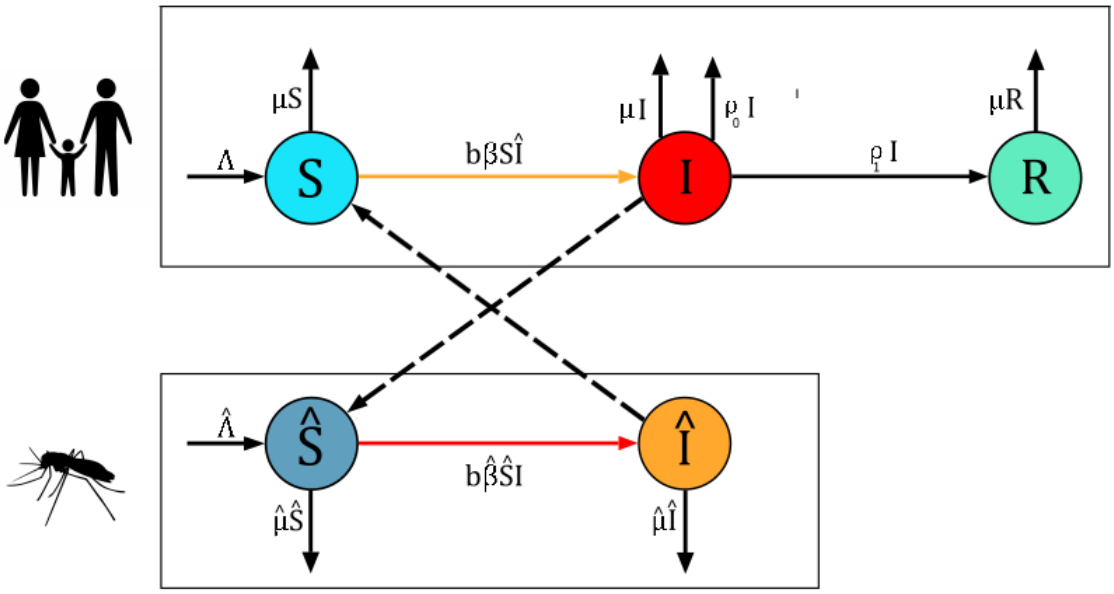


Figure III.1: Schematic diagram of the proposed dengue compartmental model.

The remainder of this chapter is organized as follows: in Section 2, we demonstrate the existence and uniqueness of a positive global-in-time solution to the stochastic dengue model (III.5). In Section 3, we provide some sufficient conditions for the dengue disease extinction, whereas those of its persistence in the mean are presented in Section 4. In Section 5, we support our theoretical results with the help of some numerical simulations before drawing the main conclusions in Section 6.

2 Existence, uniqueness and positivity of the global-in-time solution

The first step in exploring the dynamical characteristics of a mathematical population system is to know if it is well-posed or not, where the well-posedness here designates that the system admits a

unique and positive global-in-time solution. In what follows, we will provide some conditions and assumptions under which the well-posedness of the dengue disease model (III.5) is ensured. But before doing so, let us first introduce the following hypotheses:

- (A_1) : The jumps coefficients $(\xi_i)_{1\leqslant i\leqslant 4}$ verify $\mathfrak{M}_1 := \max\limits_{1\leqslant i\leqslant 4}\left(\int_{\mathbb{U}}\xi_i^2(u)\nu(\mathrm{d}u)\right) < \infty$.
- (A_2) : For any $i\in\{1,2,3,4\}$, $\xi_i(u) > -1$ and $\mathfrak{M}_2 := \max\limits_{1\leqslant i\leqslant 4}\left(\int_{\mathbb{U}}\Big(\xi_i(u) - \ln\left(1+\xi_i(u)\right)\Big)\nu(\mathrm{d}u)\right) < \infty$.

Theorem 2.1. *Let assumptions* (A_1) *and* (A_2) *hold. Then, for any initial data* $\big(S(0),I(0),\widehat{S}(0),\widehat{I}(0)\big)\in\mathbb{R}_+^4$, *there corresponds one and only one solution of the stochastic differential system* (III.5) *on* $t\geqslant 0$. *Moreover, for all* $t\geqslant 0$, *this solution will stay in* $\mathbb{R}_+^4$ *almost surely (a.s. for short).*

Proof. As supposed in the statement of the theorem, let the hypotheses (A_1) and (A_2) hold. From (A_1), we can easily observe that the coefficients of the system (III.5) are locally Lipschitz continuous. Hence, by using Theorem 1.19 of [21], one can immediately conclude that for any given initial value $\big(S(0),I(0),\widehat{S}(0),\widehat{I}(0)\big)\in\mathbb{R}_+^4$ there corresponds a unique maximal local solution $\big(S(t),I(t),\widehat{S}(t),\widehat{I}(t)\big)$ of (III.5) on an interval $[0,\tau_e)$, where τ_e is the explosion time [19]. At this point, our objective will be to show the globality in time of this solution, in other words, $\tau_e=\infty$ almost surely. To this end, let k_0 be a sufficiently large positive integer such that $\big(S(0),I(0),\widehat{S}(0),\widehat{I}(0)\big)\in\left(\frac{1}{k_0},k_0\right)$, and consider for any integer $k\geqslant k_0$ the following quantity, which is well defined by the adoption of the convention $\inf\emptyset=\infty$:

$$\tau_k = \inf\left\{t\in[0,\tau_e)\mid \big(S(t),I(t),\widehat{S}(t),\widehat{I}(t)\big)\notin\left(\frac{1}{k},k\right)^4\right\}\in\mathbb{R}_+\cup\{\infty\}\,. \tag{III.8}$$

Clearly, $\left(\frac{1}{k},k\right)^4$ is an open subset of $\mathbb{R}^4$, so it follows from Theorem 3.1 of [19] that τ_k is a stopping time for all $k\geqslant k_0$. Set $\tau_\infty=\lim\limits_{k\to\infty}\tau_k$, it is obvious that $(\tau_k)_{k\geqslant k_0}$ is increasing; hence, $\lim\limits_{k\to\infty}\tau_k=\sup\limits_{k\geqslant k_0}\tau_k$, and by virtue of Lemma 2.11 in [18] $\sup\limits_{k\geqslant k_0}\tau_k$ is a stopping time, then so is τ_∞. Evidently, $\tau_\infty\leqslant\tau_e$ (see [153] for more details), so $\tau_e=\infty$ will follow immediately once we show that $\tau_\infty=\infty$ a.s., and this is exactly what we are going to do for accomplishing our proof. Suppose that $\tau_\infty=\infty$ a.s. is false, then there is necessarily two positive constants ε and η such that

$$\mathbb{P}\left(\tau_k\leqslant\eta\right)>\varepsilon\quad\text{for all } k\geqslant k_0. \tag{III.9}$$

Consider the C^2-function $V:\mathbb{R}_+^4\to[0,+\infty)$ defined by

$$V(x_1,x_2,x_3,x_4) = \left(x_1-\alpha-\alpha\ln\left(\frac{x_1}{\alpha}\right)\right)+\big(x_2-1-\ln(x_2)\big)+\left(x_3-\alpha-\alpha\ln\left(\frac{x_3}{\alpha}\right)\right)+\big(x_4-1-\ln(x_4)\big),$$

where α is a to be determined suitably later. The nonnegativity of this function can be observed from the inequality $a-1-\ln(a)\geqslant 0,\ \forall a>0$. According to the general multi-dimensional Itô's formula (see [21, page 8]), we have for all $k\geqslant k_0$ and $t\in[0,\tau_k)$

$$
\begin{aligned}
&\mathrm{d}V\big(S(t),I(t),\widehat{S}(t),\widehat{I}(t)\big)\\
&=\mathcal{L}V\big(S(t),I(t),\widehat{S}(t),\widehat{I}(t)\big)\ \mathrm{d}t+(S(t)-\alpha)\,\sigma_1\ \mathrm{d}\mathcal{B}_1(t)+(I(t)-1)\,\sigma_2\ \mathrm{d}\mathcal{B}_2(t)\\
&\quad+\Big(\widehat{S}(t)-\alpha\Big)\,\sigma_3\mathrm{d}\mathcal{B}_3(t)+\Big(\widehat{I}(t)-1\Big)\,\sigma_4\mathrm{d}\mathcal{B}_4(t)+\int_{\mathbb{U}}\Big(\xi_1(u)S(t^-)-\alpha\ln\left(1+\xi_1(u)\right)\Big)\widetilde{\mathcal{N}}(\mathrm{d}t,\mathrm{d}u)\\
&\quad+\int_{\mathbb{U}}\Big(\xi_2(u)I(t^-)-\ln\left(1+\xi_2(u)\right)\Big)\widetilde{\mathcal{N}}(\mathrm{d}t,\mathrm{d}u)+\int_{\mathbb{U}}\Big(\xi_3(u)\widehat{S}(t^-)-\alpha\ln\left(1+\xi_3(u)\right)\Big)\widetilde{\mathcal{N}}(\mathrm{d}t,\mathrm{d}u)\\
&\quad+\int_{\mathbb{U}}\Big(\xi_4(u)\widehat{I}(t^-)-\ln\left(1+\xi_4(u)\right)\Big)\widetilde{\mathcal{N}}(\mathrm{d}t,\mathrm{d}u),
\end{aligned}
$$

where $\mathcal{L}V\big(S(t),I(t),\widehat{S}(t),\widehat{I}(t)\big)$ is given by

$$
\begin{aligned}
&\mathcal{L}V\big(S(t),I(t),\widehat{S}(t),\widehat{I}(t)\big)\\
&=\left(1-\frac{\alpha}{S(t)}\right)\times\Big(\Lambda-b\beta S(t)\widehat{I}(t)-\mu S(t)\Big)+\left(1-\frac{1}{I(t)}\right)\times\Big(b\beta S(t)\widehat{I}(t)-(\mu+\rho_0+\rho_1)\,I(t)\Big)\\
&\quad+\left(1-\frac{\alpha}{\widehat{S}(t)}\right)\Big(\hat{\Lambda}-b\hat{\beta}\widehat{S}(t)I(t)-\hat{\mu}\widehat{S}(t)\Big)+\left(1-\frac{1}{\widehat{I}(t)}\right)\Big(b\hat{\beta}\widehat{S}(t)I(t)-\hat{\mu}\widehat{I}(t)\Big)\\
&\quad+\frac{1}{2}\left(\alpha\sigma_1^2+\sigma_2^2+\alpha\sigma_3^2+\sigma_4^2\right)+\alpha\int_{\mathbb{U}}\Big(\xi_1(u)-\ln\left(1+\xi_1(u)\right)\Big)\nu(\mathrm{d}u)\\
&\quad+\int_{\mathbb{U}}\Big(\xi_2(u)-\ln\Big(1+\xi_2(u)\Big)\Big)\,\nu(\mathrm{d}u)+\alpha\int_{\mathbb{U}}\Big(\xi_3(u)-\ln\left(1+\xi_3(u)\right)\Big)\nu(\mathrm{d}u)\\
&\quad+\int_{\mathbb{U}}\Big(\xi_4(u)-\ln\left(1+\xi_4(u)\right)\Big)\nu(\mathrm{d}u).
\end{aligned}
$$

Therefore

$$
\begin{aligned}
&\mathcal{L}V\big(S(t),I(t),\widehat{S}(t),\widehat{I}(t)\big)\\
&=\Big(\Lambda-\mu S(t)-(\mu+\rho_0+\rho_1)\,I(t)+\hat{\Lambda}-\hat{\mu}\widehat{I}(t)-\hat{\mu}\widehat{S}(t)\Big)-\frac{\alpha\Lambda}{S(t)}+\alpha b\beta\widehat{I}(t)+\alpha\mu\\
&\quad-b\beta S(t)\frac{\widehat{I}(t)}{I(t)}+(\mu+\rho_0+\rho_1)-\frac{\alpha\hat{\Lambda}}{\widehat{S}(t)}+\alpha b\hat{\beta}I(t)+\alpha\hat{\mu}-b\frac{\hat{\beta}\widehat{S}(t)I(t)}{\widehat{I}(t)}+\hat{\mu}+\frac{1}{2}\big(\alpha\sigma_1^2\\
&\quad+\sigma_2^2+\alpha\sigma_3^2+\sigma_4^2\big)+\alpha\int_{\mathbb{U}}\Big(\xi_1(u)-\ln\left(1+\xi_1(u)\right)\Big)\nu(\mathrm{d}u)+\int_{\mathbb{U}}\Big(\xi_2(u)-\ln\left(1+\xi_2(u)\right)\Big)\nu(\mathrm{d}u)\\
&\quad+\alpha\int_{\mathbb{U}}\Big(\xi_3(u)-\ln\left(1+\xi_3(u)\right)\Big)\nu(\mathrm{d}u)+\int_{\mathbb{U}}\Big(\xi_4(u)-\ln\left(1+\xi_4(u)\right)\Big)\nu(\mathrm{d}u)\\
&\leqslant\Big(\alpha b\hat{\beta}-(\mu+\rho_0+\rho_1)\Big)\,I(t)+(\alpha b\beta-\hat{\mu})\,\widehat{I}(t)+\big(\Lambda+\hat{\Lambda}+\alpha\mu+(\mu+\rho_0+\rho_1)+\alpha\hat{\mu}+\hat{\mu}\big)\\
&\quad+\frac{1}{2}\left(\alpha\sigma_1^2+\sigma_2^2+\alpha\sigma_3^2+\sigma_4^2\right)+\alpha\int_{\mathbb{U}}\Big(\xi_1(u)-\ln\left(1+\xi_1(u)\right)\Big)\nu(\mathrm{d}u)\\
&\quad+\int_{\mathbb{U}}\Big(\xi_2(u)-\ln\left(1+\xi_2(u)\right)\Big)\nu(\mathrm{d}u)+\alpha\int_{\mathbb{U}}\Big(\xi_3(u)-\ln\left(1+\xi_3(u)\right)\Big)\nu(\mathrm{d}u)\\
&\quad+\int_{\mathbb{U}}\Big(\xi_4(u)-\ln\left(1+\xi_4(u)\right)\Big)\nu(\mathrm{d}u).
\end{aligned}
$$

By choosing $\alpha=\dfrac{\hat{\mu}}{b\beta}\wedge\dfrac{\mu+\rho_0+\rho_1}{b\hat{\beta}}$, we obtain

$$
\mathcal{L}V\big(S(t),I(t),\widehat{S}(t),\widehat{I}(t)\big)\leqslant\mathfrak{A},
$$

and $\mathfrak{A}$ here is to the positive constant given by

$$\mathfrak{A} = \left(\Lambda + \hat{\Lambda} + \alpha\mu + (\mu + \rho_0 + \rho_1) + \alpha\hat{\mu} + \hat{\mu}\right) + \frac{1}{2}\left(\alpha\sigma_1^2 + \sigma_2^2 + \alpha\sigma_3^2 + \sigma_4^2\right) + \alpha\int_{\mathbb{U}}\Big(\xi_1(u) - \ln\left(1 + \xi_1(u)\right)\Big)\nu(\mathrm{d}u)$$
$$+\int_{\mathbb{U}}\Big(\xi_2(u) - \ln\left(1 + \xi_2(u)\right)\Big)\nu(\mathrm{d}u) + \alpha\int_{\mathbb{U}}\Big(\xi_3(u) - \ln\left(1 + \xi_3(u)\right)\Big)\nu(\mathrm{d}u) + \int_{\mathbb{U}}\Big(\xi_4(u) - \ln\left(1 + \xi_4(u)\right)\Big)\nu(\mathrm{d}u).$$

The remainder of the proof runs on the same lines as the demonstration of Theorem 2.1 in [154], so we omit it here for the sake of space. ■

3 Extinction of the dengue disease

In mathematical epidemiology, our prime concern after proving the well-posedness is to know if the disease will disappear or it will continue to exist. In this section, we will try to find some conditions for the dengue disease extinction expressed in terms of noises intensities, jumps coefficients and system parameters. For the reader's convenience, the persistence in the mean will be covered and analyzed separately in the next section. For the sake of brevity, we will adopt from now on the following notations:

- $\Sigma := \max\left\{\sigma_i^2 \mid i \in \{1,2,3,4\}\right\}$.
- $\widetilde{\xi}(u) := \max\left\{\xi_i(u) \mid i \in \{1,2,3,4\}\right\}$.
- $\widetilde{\theta}_p(u) := \left(1 + \widetilde{\xi}(u)\right)^p - p \times \widetilde{\xi}(u) - 1$.
- $\theta_p(u) := \max\left\{\widetilde{\theta}_p(u), \underset{\sim}{\theta}_p(u)\right\}$.
- $\rho := \rho_0 + \rho_1$.
- $\underset{\sim}{\xi}(u) := \min\left\{\xi_i(u) \mid i \in \{1,2,3,4\}\right\}$.
- $\underset{\sim}{\theta}_p(u) := \left(1 + \underset{\sim}{\xi}(u)\right)^p - p \times \underset{\sim}{\xi}(u) - 1$.
- $\varrho_p = \displaystyle\int_{\mathbb{U}}\theta_p(u)\nu(\mathrm{d}u)$.

Before stating the main result of this section, we must firstly give the following useful lemma:

Lemma 3.1. *Let* $\left(S(t), I(t), \widehat{S}(t), \widehat{I}(t)\right)$ *be the solution of* (III.5) *that starts from a given initial value* $\left(S_0, I_0, \widehat{S}_0, \widehat{I}_0\right) \in \mathbb{R}_+^4$. *Let also* $\Psi(t)$ *and* $\widehat{\Psi}(t)$ *be respectively the solutions of* (III.6) *and* (III.7) *that satisfy* $\Psi(0) = S(0)$ *and* $\widehat{\Psi}(0) = \widehat{S}(0)$. *If the following conditions are satisfied:*

- (A_3) : *There exists a real constant* $p > 2$ *such that* $\Delta_p := \mu \wedge \hat{\mu} - \dfrac{p-1}{2}\Sigma - \dfrac{\varrho_p}{p} > 0$.
- (A_4) : $\mathfrak{M}_3 := \displaystyle\max_{1\leqslant i\leqslant 4}\left(\int_{\mathbb{U}}\left(\left(1 + \xi_i(u)\right)^2 - 1\right)^2\nu(du)\right) < \infty$.

Then

(a) $\displaystyle\lim_{t\to\infty}\frac{\Psi(t)}{t} = 0,\ \lim_{t\to\infty}\frac{\Psi^2(t)}{t} = 0,\ \lim_{t\to\infty}\frac{S(t)}{t} = 0,$ *and* $\displaystyle\lim_{t\to\infty}\frac{I(t)}{t} = 0$ *a.s.*

(b) $\displaystyle\lim_{t\to\infty}\frac{\widehat{\Psi}(t)}{t} = 0,\ \lim_{t\to\infty}\frac{\widehat{\Psi}^2(t)}{t} = 0,\ \lim_{t\to\infty}\frac{\widehat{S}(t)}{t} = 0,$ *and* $\displaystyle\lim_{t\to\infty}\frac{\widehat{I}(t)}{t} = 0$ *a.s.*

(c) $\displaystyle\lim_{t\to\infty}\frac{\int_0^t\Psi(s)\,\mathrm{d}\mathcal{B}_1(s)}{t} = 0,\ \lim_{t\to\infty}\frac{\int_0^t\Psi^2(s)\,\mathrm{d}\mathcal{B}_1(s)}{t} = 0,\ \lim_{t\to\infty}\frac{\int_0^t S(s)\,\mathrm{d}\mathcal{B}_1(s)}{t} = 0,$ *and* $\displaystyle\lim_{t\to\infty}\frac{\int_0^t I(s)\,\mathrm{d}\mathcal{B}_2(s)}{t} = 0$ *a.s.*

(d) $\displaystyle\lim_{t\to\infty}\frac{\int_0^t\widehat{\Psi}(s)\,\mathrm{d}\mathcal{B}_3(s)}{t} = 0,\ \lim_{t\to\infty}\frac{\int_0^t\widehat{\Psi}^2(s)\,\mathrm{d}\mathcal{B}_3(s)}{t} = 0,\ \lim_{t\to\infty}\frac{\int_0^t\widehat{S}(s)\,\mathrm{d}\mathcal{B}_3(s)}{t} = 0,$ *and* $\displaystyle\lim_{t\to\infty}\frac{\int_0^t\widehat{I}(s)\,\mathrm{d}\mathcal{B}_4(s)}{t} = 0$ *a.s.*

(e) $\displaystyle\lim_{t\to\infty}\frac{\int_0^t\int_{\mathbb{U}}\xi_1(u)\Psi(s^-)\widetilde{\mathcal{N}}(\mathrm{d}s,\mathrm{d}u)}{t} = 0$ *and* $\displaystyle\lim_{t\to\infty}\frac{\int_0^t\int_{\mathbb{U}}\left((1+\xi_1(u))^2-1\right)\Psi^2(s^-)\widetilde{\mathcal{N}}(\mathrm{d}s,\mathrm{d}u)}{t} = 0$ *a.s.*

(f) $\displaystyle\lim_{t\to\infty}\frac{\int_0^t\int_{\mathbb{U}}\xi_3(u)\widehat{\Psi}(s^-)\widetilde{\mathcal{N}}(\mathrm{d}s,\mathrm{d}u)}{t} = 0$ *and* $\displaystyle\lim_{t\to\infty}\frac{\int_0^t\int_{\mathbb{U}}\left((1+\xi_3(u))^2-1\right)\widehat{\Psi}^2(s^-)\widetilde{\mathcal{N}}(\mathrm{d}s,\mathrm{d}u)}{t} = 0$ *a.s.*

(g) $\displaystyle\lim_{t\to\infty}\frac{\int_0^t\int_{\mathbb{U}}\xi_1(u)S(s^-)\widetilde{\mathcal{N}}(\mathrm{d}s,\mathrm{d}u)}{t} = 0$ *and* $\displaystyle\lim_{t\to\infty}\frac{\int_0^t\int_{\mathbb{U}}\xi_2(u)I(s^-)\widetilde{\mathcal{N}}(\mathrm{d}s,\mathrm{d}u)}{t} = 0$ *a.s.*

(h) $\displaystyle\lim_{t\to\infty}\frac{\int_0^t\int_{\mathbb{U}}\xi_3(u)\widehat{S}(s^-)\widetilde{\mathcal{N}}(\mathrm{d}s,\mathrm{d}u)}{t} = 0$ *and* $\displaystyle\lim_{t\to\infty}\frac{\int_0^t\int_{\mathbb{U}}\xi_4(u)\widehat{I}(s^-)\widetilde{\mathcal{N}}(\mathrm{d}s,\mathrm{d}u)}{t} = 0$ *a.s.*

Proof. The proof of this lemma is similar in spirit to that of Lemma 2.5 in [150]. Hereby, it is omitted here. ■

Lemma 3.2. *Let $\Psi(t)$ and $\widehat{\Psi}(t)$ be, in this order, the solutions of systems (III.6) and (III.7) that start respectively from the given initial values $\Psi(0) > 0$ and $\widehat{\Psi}(0) > 0$. Assume that the conditions (A_3) and (A_4) hold. Then,*

(a) $\Upsilon := 2\mu - \sigma_1^2 - \int_{\mathbb{U}} \xi_1^2(u)\nu(\mathrm{d}u) > 0 \quad a.s.$

(b) $\lim\limits_{t\to\infty} \dfrac{\int_0^t \Psi(s)\,\mathrm{d}s}{t} = \dfrac{\Lambda}{\mu} \quad a.s.$

(c) $\lim\limits_{t\to\infty} \dfrac{\int_0^t \Psi^2(s)\,\mathrm{d}s}{t} = \dfrac{2\Lambda^2}{\mu\Upsilon} \quad a.s.$

(d) $\widehat{\Upsilon} := 2\hat{\mu} - \sigma_3^2 - \int_{\mathbb{U}} \xi_3^2(u)\nu(\mathrm{d}u) > 0 \quad a.s.$

(e) $\lim\limits_{t\to\infty} \dfrac{\int_0^t \widehat{\Psi}(s)\,\mathrm{d}s}{t} = \dfrac{\hat{\Lambda}}{\hat{\mu}} \quad a.s.$

(f) $\lim\limits_{t\to\infty} \dfrac{\int_0^t \widehat{\Psi}^2(s)\,\mathrm{d}s}{t} = \dfrac{2\hat{\Lambda}^2}{\hat{\mu}\widehat{\Upsilon}} \quad a.s.$

Proof. Let $\Psi(0)$ be a positive initial datum, by integrating the both sides of equation (III.6) from 0 to t, and then dividing by t, we get

$$\frac{\Psi(t) - \Psi(0)}{t} = \Lambda - \mu \times \frac{\int_0^t \Psi(s)\,\mathrm{d}s}{t} + \sigma_1 \times \frac{\int_0^t \Psi(s)\,\mathrm{d}\mathcal{B}_1(s)}{t} + \frac{\int_0^t \int_{\mathbb{U}} \xi_1(u)\Psi(s^-)\widetilde{\mathcal{N}}(\mathrm{d}s,\mathrm{d}u)}{t}.$$

Therefore

$$\frac{\int_0^t \Psi(s)\,\mathrm{d}s}{t} = \frac{\Lambda}{\mu} - \frac{\Psi(t) - \Psi(0)}{\mu t} + \sigma_1 \times \frac{\int_0^t \Psi(s)\,\mathrm{d}\mathcal{B}_1(s)}{\mu t} + \frac{\int_0^t \int_{\mathbb{U}} \xi_1(u)\Psi(s^-)\,\widetilde{\mathcal{N}}(\mathrm{d}s,\mathrm{d}u)}{\mu t}.$$

Letting t go to infinity in the last equality and then using Lemma 3.1 yields

$$\lim_{t\to\infty} \frac{\int_0^t \Psi(s)\,\mathrm{d}s}{t} = \frac{\Lambda}{\mu} \quad a.s. \tag{III.10}$$

Hence, the item (b) of the lemma is proved. We are now in a position to show the statements (a) and (c). By applying the generalized Itô's formula (see [21]) to Ψ^2 we obtain

$$\begin{aligned}\mathrm{d}\Psi^2(t) = &\left(2\Psi(t)\Big(\Lambda - \mu\Psi(t)\Big) + \sigma_1^2\Psi^2(t) + \int_{\mathbb{U}} \Psi^2(t)\Big((1+\xi_1(u))^2 - 1 - 2\xi_1(u)\Big)\nu(\mathrm{d}u)\right)\mathrm{d}t + 2\sigma_1\Psi^2(t)\,\mathrm{d}\mathcal{B}_1(t)\\ &+ \int_{\mathbb{U}} \Psi^2(t^-)\Big((1+\xi_1(u))^2 - 1\Big)\,\widetilde{\mathcal{N}}(\mathrm{d}t,\mathrm{d}u).\end{aligned} \tag{III.11}$$

Integrating both sides of (III.11) from 0 to t and then dividing by t gives

$$\begin{aligned}\frac{\Psi^2(t) - \Psi^2(0)}{t} = 2\Lambda \times \frac{\int_0^t \Psi(s)\mathrm{d}s}{t} - \left(2\mu - \sigma_1^2 - \int_{\mathbb{U}} \xi_1^2(u)\nu(\mathrm{d}u)\right) \times \frac{\int_0^t \Psi^2(s)\mathrm{d}s}{t} + 2\sigma_1 \times \frac{\int_0^t \Psi^2(s)\mathrm{d}\mathcal{B}_1(s)}{t}\\ + \frac{\int_0^t \int_{\mathbb{U}} \Psi^2(s^-)\Big((1+\xi_1(u))^2 - 1\Big)\widetilde{\mathcal{N}}(\mathrm{d}s,\mathrm{d}u)}{t}.\end{aligned}$$

So

$$\begin{aligned}\left(2\mu - \sigma_1^2 - \int_{\mathbb{U}} \xi_1^2(u)\nu(\mathrm{d}u)\right) \times \frac{\int_0^t \Psi^2(s)\mathrm{d}s}{t} = \frac{\Psi^2(0) - \Psi^2(t)}{t} + 2\Lambda \times \frac{\int_0^t \Psi(s)\mathrm{d}s}{t} + 2\sigma_1 \times \frac{\int_0^t \Psi^2(s)\mathrm{d}\mathcal{B}_1(s)}{t}\\ + \frac{\int_0^t \int_{\mathbb{U}} \Psi^2(s^-)\Big((1+\xi_1(u))^2 - 1\Big)\widetilde{\mathcal{N}}(\mathrm{d}s,\mathrm{d}u)}{t}.\end{aligned} \tag{III.12}$$

Clearly, the constant $\Upsilon := 2\mu - \sigma_1^2 - \int_{\mathbb{U}} \xi_1^2(u)\nu(\mathrm{d}u)$ is not zero, since if it is not the case, we will tend t to infinity and come across the contradictory equality $\frac{2\Lambda^2}{\mu} = 0$. For this reason, one can easily divide the both sides of (III.12) by Υ and deduce that

$$\frac{\int_0^t \Psi^2(s)\mathrm{d}s}{t} = \frac{\Psi^2(0) - \Psi^2(t)}{\Upsilon t} + 2\Lambda \times \frac{\int_0^t \Psi(s)\mathrm{d}s}{\Upsilon t} + 2\sigma_1 \times \frac{\int_0^t \Psi^2(s)\mathrm{d}\mathcal{B}_1(s)}{\Upsilon t} + \frac{\int_0^t \int_{\mathbb{U}} \Psi^2(s^-)\Big((1+\xi_1(u))^2 - 1\Big)\widetilde{\mathcal{N}}(\mathrm{d}s, \mathrm{d}u)}{\Upsilon t}.$$

Using Lemma 3.1 together with (III.10) implies that

$$\lim_{t\to\infty} \frac{\int_0^t \Psi^2(s)\,\mathrm{d}s}{t} = \frac{2\Lambda^2}{\mu\Upsilon} \quad a.s.$$

and by observing the positivity of $\frac{\int_0^t \Psi^2(s)\,\mathrm{d}s}{t}$, we can conclude at the same time that $\Upsilon > 0$. Hence, the claims (a) and (c) of the lemma are proved. By an analogous argument, the assertions (c), (d) and (e) can be drawn, and this finishes the proof. ■

Remark 3.1. *By making use of the famous stochastic comparison theorem (see [155]), we can easily assert, and for almost all $w \in \Omega$, that*

$$\begin{cases} S(t,w) \leqslant \Psi(t,w) \\ \widehat{S}(t,w) \leqslant \widehat{\Psi}(t,w) \end{cases} \quad \text{for all } t > 0. \tag{III.13}$$

Remark 3.2. *In the lévy jumps case, the previous lemma can be seen as an alternative method to overcome the inexistence of an explicit expression for the stationary distribution of (III.6) and (III.7). This problem remains an open question until now, and we can find in the literature several works (see for example [151] and [152]) that present the threshold analysis of their epidemiological model with a formulation incorporating an unknown stationary distribution.*

Definition 3.1 (Ordinary extinction [153]). *For system (III.5) the infected individuals I and $\widehat{I}$ are said to be extinct if $\lim_{t\to\infty}\Big(\widehat{I}(t) + I(t)\Big) = 0$ a.s.*

Definition 3.2 (Exponential extinction [148]). *The total mass of infected individuals $I_{\text{total}} = (I + \widehat{I})$ appearing in the system (III.5) is said to be exponentially extinct if there is a negative constant c such that*

$$\limsup_{t\to\infty} \frac{1}{t} \ln\Big(I(t) + \widehat{I}(t)\Big) \leqslant c < 0 \quad a.s.$$

Remark 3.3. *It is easily seen that the exponential extinction implies the ordinary one (see [109]), but the converse is not true in general.*

In order to simplify the writing of the next theorem, we introduce these conventions:

- For all $x \in \mathbb{R}$, $x^+ := \max\{0, x\} = \frac{x + |x|}{2}$ (this function is commonly known as the ramp function, see [156]).
- $\underline{\mathfrak{b}}(u) := \big(\ln(1 + \xi_2(u) \wedge \xi_4(u)) - \xi_2(u) \wedge \xi_4(u)\big) \times \mathbb{1}_{\{\xi_2(u)\wedge\xi_4(u) > 0\}}$.
- $\overline{\mathfrak{b}}(u) := \big(\ln(1 + \xi_2(u) \vee \xi_4(u)) - \xi_2(u) \vee \xi_4(u)\big) \times \mathbb{1}_{\{\xi_2(u)\vee\xi_4(u) \leqslant 0\}}$.
- $\mathfrak{B} := \int_{\mathbb{U}} \Big(\overline{\mathfrak{b}}(u) + \underline{\mathfrak{b}}(u)\Big)\nu(\mathrm{d}u)$.

- $\mathfrak{C} := \dfrac{(\sigma_2\sigma_4)^2}{2(\sigma_2^2+\sigma_4^2)}$.
- $\mathfrak{D} := [(\mu+\rho)\vee\hat{\mu}]\times\left(\sqrt{\mathcal{R}_0}-1\right)^+ - [(\mu+\rho)\wedge\hat{\mu}]\times\left(1-\sqrt{\mathcal{R}_0}\right)^+$.

Theorem 3.1. *Let* $\left(S_0, I_0, \widehat{S}_0, \widehat{I}_0\right) \in \mathbb{R}^4_+$, *and let* $\left(S(t), I(t), \widehat{S}(t), \widehat{I}(t)\right)$ *denote the solution of* (III.5) *that satisfies the*
initial condition $\left(S(0), I(0), \widehat{S}(0), \widehat{I}(0)\right) = \left(S_0, I_0, \widehat{S}_0, \widehat{I}_0\right)$. *If hypotheses* $(\mathrm{A}_1)-(\mathrm{A}_4)$ *hold, and if moreover we have*

- $(\mathrm{A}_5) : \mathfrak{M}_4 := \max\limits_{1\leqslant i\leqslant 4}\left(\int_{\mathbb{U}}\left(\ln\left(1+\xi_i(u)\right)\right)^2\nu(\mathrm{d}u)\right) < \infty.$

Then

$$\limsup_{t\to\infty}\frac{1}{t}\ln\left(\frac{b\hat{\beta}\hat{\Lambda}}{\hat{\mu}^2(\mu+\rho)}I(t)+\frac{\sqrt{\mathcal{R}_0}}{\hat{\mu}}\widehat{I}(t)\right)\leqslant \boldsymbol{\mathcal{K}} \quad a.s,$$

where

$$\boldsymbol{\mathcal{K}} := \mathfrak{D}-\mathfrak{C}+\mathfrak{B}+\frac{\hat{\mu}\sqrt{\mathcal{R}_0}}{2}\left(\frac{2\mu}{\Upsilon}-1\right)^{\frac{1}{2}}+\frac{(\mu+\rho)\sqrt{\mathcal{R}_0}}{2}\left(\frac{2\hat{\mu}}{\widehat{\Upsilon}}-1\right)^{\frac{1}{2}}.$$

In particular, if the condition $\boldsymbol{\mathcal{K}} < 0$ *is verified, then the disease will die out exponentially almost surely.*

Proof. First of all, let us define a C^2-function $W : \mathbb{R}^2_+ \to \mathbb{R}$ by

$$W(x_1,x_2) = \ln\left(\overbrace{\frac{b\hat{\beta}\hat{\Lambda}}{\hat{\mu}^2(\mu+\rho)}}^{:=\lambda_1}x_1+\overbrace{\frac{\sqrt{\mathcal{R}_0}}{\hat{\mu}}}^{:=\lambda_2}x_2\right).$$

Applying the Itô's formula to $W\left(I(t),\widehat{I}(t)\right)$ shows that for all $t\geqslant 0$ we have

$$\begin{aligned}\mathrm{d}W\left(I(t),\widehat{I}(t)\right) &= \mathcal{L}W\left(I(t),\widehat{I}(t)\right)\mathrm{d}t+\frac{\left(\lambda_1\sigma_2 I(t)\mathrm{d}\mathcal{B}_2(t)+\lambda_2\sigma_4\widehat{I}(t)\mathrm{d}\mathcal{B}_4(t)\right)}{\lambda_1 I(t)+\lambda_2\widehat{I}(t)}\\ &\quad+\int_{\mathbb{U}}\ln\left(1+\frac{\lambda_1\xi_2(u)I(t)+\lambda_2\xi_4(u)\widehat{I}(t)}{\lambda_1 I(t)+\lambda_2\widehat{I}(t)}\right)\widetilde{\mathcal{N}}(\mathrm{d}t,\mathrm{d}u),\end{aligned}$$

where

$$\begin{aligned}\mathcal{L}W\left(I(t),\widehat{I}(t)\right) &= \frac{\lambda_1\left(b\beta S(t)\widehat{I}(t)-(\mu+\rho_0+\rho_1)I(t)\right)+\lambda_2\left(b\hat{\beta}\widehat{S}(t)I(t)-\hat{\mu}\widehat{I}(t)\right)}{\lambda_1 I(t)+\lambda_2\widehat{I}(t)}-\frac{\left(\lambda_1^2\sigma_2^2(I(t))^2+\lambda_2^2\sigma_4^2(\widehat{I}(t))^2\right)}{2\left(\lambda_1 I(t)+\lambda_2\widehat{I}(t)\right)^2}\\ &\quad+\int_{\mathbb{U}}\left(\ln\left(1+\frac{\lambda_1\xi_2(u)I(t)+\lambda_2\xi_4(u)\widehat{I}(t)}{\lambda_1 I(t)+\lambda_2\widehat{I}(t)}\right)-\frac{\lambda_1\xi_2(u)I(t)+\lambda_2\xi_4(u)\widehat{I}(t)}{\lambda_1 I(t)+\lambda_2\widehat{I}(t)}\right)\nu(\mathrm{d}u).\end{aligned}\tag{III.14}$$

By virtue of the well-known Cauchy-Schwartz inequality (see for example [106]), we have

$$\left(\lambda_1 I(t)+\lambda_2\widehat{I}(t)\right)^2=\left(\frac{1}{\sigma_2}\lambda_1\sigma_2 I(t)+\frac{1}{\sigma_4}\lambda_2\sigma_4\widehat{I}(t)\right)^2\leqslant\left(\frac{1}{\sigma_2^2}+\frac{1}{\sigma_4^2}\right)\left(\lambda_1^2\sigma_2^2(I(t))^2+\lambda_2^2\sigma_4^2(\widehat{I}(t))^2\right).$$

So

$$-\frac{1}{\left(\lambda_1 I(t)+\lambda_2\widehat{I}(t)\right)^2}\left(\lambda_1^2\sigma_2^2(I(t))^2+\lambda_2^2\sigma_4^2(\widehat{I}(t))^2\right)\leqslant-\frac{(\sigma_2\sigma_4)^2}{\sigma_2^2+\sigma_4^2}.\tag{III.15}$$

At the same time, it follows from the monotonicity of the function $a \mapsto \ln(1+a) - a$ on $(-1, +\infty)$ (it is increasing over $(-1, 0)$ and decreasing over $[0, +\infty)$) that

$$\int_{\mathbb{U}} \left(\ln\left(1 + \frac{\lambda_1 \xi_2(u) I(t) + \lambda_2 \xi_4(u) \widehat{I}(t)}{\lambda_1 I(t) + \lambda_2 \widehat{I}(t)}\right) - \frac{\lambda_1 \xi_2(u) I(t) + \lambda_2 \xi_4(u) \widehat{I}(t)}{\lambda_1 I(t) + \lambda_2 \widehat{I}(t)} \right) \nu(\mathrm{d}u) \leqslant \mathfrak{B}. \tag{III.16}$$

Combining (III.15), (III.16) and (III.13) with (III.14) yields

$$\begin{aligned}
\mathcal{L}W\big(I(t), \widehat{I}(t)\big) &\leqslant \frac{\lambda_1 \left(b\beta \Psi(t) \widehat{I}(t) - (\mu + \rho_0 + \rho_1) I(t)\right) + \lambda_2 \left(b\hat{\beta} \widehat{\Psi}(t) I(t) - \hat{\mu} \widehat{I}(t)\right)}{\lambda_1 I(t) + \lambda_2 \widehat{I}(t)} - \mathfrak{C} + \mathfrak{B} \\
&\leqslant \frac{\lambda_1 \left(b\beta \frac{\Lambda}{\mu} \widehat{I}(t) - (\mu + \rho_0 + \rho_1) I(t)\right) + \lambda_2 \left(b\hat{\beta} \frac{\hat{\Lambda}}{\hat{\mu}} I(t) - \hat{\mu} \widehat{I}(t)\right)}{\lambda_1 I(t) + \lambda_2 \widehat{I}(t)} - \mathfrak{C} + \mathfrak{B} \\
&\quad + \frac{\lambda_1 b\beta \widehat{I}(t)}{\lambda_1 I(t) + \lambda_2 \widehat{I}(t)} \left(\Psi(t) - \frac{\Lambda}{\mu}\right) + \frac{\lambda_2 b\hat{\beta} I(t)}{\lambda_1 I(t) + \lambda_2 \widehat{I}(t)} \left(\widehat{\Psi}(t) - \frac{\hat{\Lambda}}{\hat{\mu}}\right) \\
&\leqslant \frac{\left(\lambda_1 b\beta \frac{\Lambda}{\mu} - \lambda_2 \hat{\mu}\right) \widehat{I}(t) + \left(\lambda_2 b\hat{\beta} \frac{\hat{\Lambda}}{\hat{\mu}} - \lambda_1 (\mu + \rho_0 + \rho_1)\right) I(t)}{\lambda_1 I(t) + \lambda_2 \widehat{I}(t)} - \mathfrak{C} + \mathfrak{B} \\
&\quad + \frac{\lambda_1 b\beta \widehat{I}(t)}{\lambda_1 I(t) + \lambda_2 \widehat{I}(t)} \left(\Psi(t) - \frac{\Lambda}{\mu}\right)^+ + \frac{\lambda_2 b\hat{\beta} I(t)}{\lambda_1 I(t) + \lambda_2 \widehat{I}(t)} \left(\widehat{\Psi}(t) - \frac{\hat{\Lambda}}{\hat{\mu}}\right)^+.
\end{aligned} \tag{III.17}$$

Since $\mathcal{R}_0 = \lambda_1 b\beta \frac{\Lambda}{\mu}$, $\sqrt{\mathcal{R}_0} = \lambda_2 \hat{\mu}$ and $\lambda_1(\mu + \rho) = b\hat{\beta} \frac{\hat{\Lambda}}{\hat{\mu}^2}$, we get

$$\begin{aligned}
\mathcal{L}W\big(I(t), \widehat{I}(t)\big) &\leqslant \frac{\left(\mathcal{R}_0 - \sqrt{\mathcal{R}_0}\right) \widehat{I}(t) + \left(\sqrt{\mathcal{R}_0} \lambda_1 (\mu + \rho) - \lambda_1 (\mu + \rho_0 + \rho_1)\right) I(t)}{\lambda_1 I(t) + \lambda_2 \widehat{I}(t)} - \mathfrak{C} + \mathfrak{B} \\
&\quad + \frac{\lambda_1 b\beta}{\lambda_2} \left(\Psi(t) - \frac{\Lambda}{\mu}\right)^+ + \frac{\lambda_2 b\hat{\beta}}{\lambda_1} \left(\widehat{\Psi}(t) - \frac{\hat{\Lambda}}{\hat{\mu}}\right)^+ \\
&\leqslant \frac{\left(\sqrt{\mathcal{R}_0} - 1\right) \left(\lambda_1 (\mu + \rho_0 + \rho_1) I(t) + \lambda_2 \hat{\mu} \widehat{I}(t)\right)}{\lambda_1 I(t) + \lambda_2 \widehat{I}(t)} - \mathfrak{C} + \mathfrak{B} + \frac{\lambda_1 b\beta}{\lambda_2} \left(\Psi(t) - \frac{\Lambda}{\mu}\right)^+ \\
&\quad + \frac{\lambda_2 b\hat{\beta}}{\lambda_1} \left(\widehat{\Psi}(t) - \frac{\hat{\Lambda}}{\hat{\mu}}\right)^+ \\
&\leqslant \mathfrak{D} - \mathfrak{C} + \mathfrak{B} + \frac{\lambda_1 b\beta}{\lambda_2} \left(\Psi(t) - \frac{\Lambda}{\mu}\right)^+ + \frac{\lambda_2 b\hat{\beta}}{\lambda_1} \left(\widehat{\Psi}(t) - \frac{\hat{\Lambda}}{\hat{\mu}}\right)^+.
\end{aligned}$$

Hence, we obtain

$$\begin{aligned}
&\mathrm{d}W\big(I(t), \widehat{I}(t)\big) \\
&\leqslant \left(\mathfrak{D} - \mathfrak{C} + \mathfrak{B} + \frac{\lambda_1 b\beta}{\lambda_2} \left(\Psi(t) - \frac{\Lambda}{\mu}\right)^+ + \frac{\lambda_2 b\hat{\beta}}{\lambda_1} \left(\widehat{\Psi}(t) - \frac{\hat{\Lambda}}{\hat{\mu}}\right)^+\right) \mathrm{d}t + \frac{\left(\lambda_1 \sigma_2 I(t) \mathrm{d}\mathcal{B}_2(t) + \lambda_2 \sigma_4 \widehat{I}(t) \mathrm{d}\mathcal{B}_4(t)\right)}{\lambda_1 I(t) + \lambda_2 \widehat{I}(t)} \\
&\quad + \int_{\mathbb{U}} \ln\big(1 + \xi_2(u) \vee \xi_4(u)\big) \widetilde{\mathcal{N}}(\mathrm{d}t, \mathrm{d}u)
\end{aligned}$$

Integrating the last inequality from 0 to t, and then dividing by t on both sides gives

$$\begin{aligned}\frac{W\big(I(t),\widehat{I}(t)\big)}{t} &\leqslant \frac{W\big(I(0),\widehat{I}(0)\big)}{t} + \mathfrak{D} - \mathfrak{C} + \mathfrak{B} + \frac{\lambda_1 b\beta}{\lambda_2 t}\int_0^t \left(\Psi(s) - \frac{\Lambda}{\mu}\right)^+ \mathrm{d}s + \frac{\lambda_2 b\hat{\beta}}{\lambda_1 t}\int_0^t \left(\widehat{\Psi}(s) - \frac{\hat{\Lambda}}{\hat{\mu}}\right)^+ \mathrm{d}s \\ &\quad + \frac{1}{t}\times \underbrace{\left(\int_0^t \frac{\lambda_1\sigma_2 I(s)}{\lambda_1 I(s) + \lambda_2\widehat{I}(s)}\mathrm{d}\mathcal{B}_2(s) + \int_0^t \frac{\lambda_2\sigma_4\widehat{I}(s)}{\lambda_1 I(s) + \lambda_2\widehat{I}(s)}\mathrm{d}\mathcal{B}_4(s)\right)}_{:=\mathrm{M}_1(t)} \\ &\quad + \frac{1}{t}\times \underbrace{\int_0^t\int_{\mathbb{U}} \ln\big(1+\xi_2(u)\vee\xi_4(u)\big)\widetilde{\mathcal{N}}(\mathrm{d}s,\mathrm{d}u)}_{:=\mathrm{M}_2(t)}. \end{aligned} \tag{III.18}$$

On the other hand, we can see by employing the classical Hölder's inequality that

$$\begin{aligned}\frac{1}{t}\int_0^t \left(\Psi(s) - \frac{\Lambda}{\mu}\right)^+ \mathrm{d}s &= \frac{1}{2t}\int_0^t \left(\Psi(s) - \frac{\Lambda}{\mu}\right)\ \mathrm{d}s + \frac{1}{2t}\int_0^t \left|\Psi(s) - \frac{\Lambda}{\mu}\right|\ \mathrm{d}s \\ &\leqslant \frac{1}{2t}\int_0^t \left(\Psi(s) - \frac{\Lambda}{\mu}\right)\ \mathrm{d}s + \frac{1}{2\sqrt{t}}\left(\int_0^t \left(\Psi(s) - \frac{\Lambda}{\mu}\right)^2 \mathrm{d}s\right)^{\frac{1}{2}} \\ &\leqslant \frac{1}{2}\left(\frac{1}{t}\int_0^t \Psi(s)\ \mathrm{d}s - \frac{\Lambda}{\mu}\right) + \frac{1}{2}\left(\frac{1}{t}\int_0^t \left(\Psi^2(s) - \frac{2\Lambda}{\mu}\Psi(s) + \frac{\Lambda^2}{\mu^2}\right)\mathrm{d}s\right)^{\frac{1}{2}}.\end{aligned}$$

This last fact together with Lemma 3.2 implies that

$$\lim_{t\to\infty}\frac{1}{t}\int_0^t \left(\Psi(s) - \frac{\Lambda}{\mu}\right)^+ \mathrm{d}s \leqslant \frac{1}{2}\left(\frac{2\Lambda^2}{\mu\Upsilon} - 2\frac{\Lambda^2}{\mu^2} + \frac{\Lambda^2}{\mu^2}\right)^{\frac{1}{2}} = \frac{\Lambda}{2\mu}\left(\frac{2\mu}{\Upsilon} - 1\right)^{\frac{1}{2}} \quad a.s. \tag{III.19}$$

By a similar argument, we can also assert that

$$\lim_{t\to\infty}\frac{1}{t}\int_0^t \left(\widehat{\Psi}(s) - \frac{\hat{\Lambda}}{\hat{\mu}}\right)^+ \mathrm{d}s \leqslant \frac{1}{2}\left(\frac{2\hat{\Lambda}^2}{\hat{\mu}\widehat{\Upsilon}} - 2\frac{\hat{\Lambda}^2}{\hat{\mu}^2} + \frac{\hat{\Lambda}^2}{\hat{\mu}^2}\right)^{\frac{1}{2}} = \frac{\hat{\Lambda}}{2\hat{\mu}}\left(\frac{2\hat{\mu}}{\widehat{\Upsilon}} - 1\right)^{\frac{1}{2}} \quad a.s. \tag{III.20}$$

It is fairly easy to see that $\mathrm{M}_1(t)$ is a local martingale with finite quadratic variation, and from the hypothesis (A_5) we can affirm that $\mathrm{M}_2(t)$ will be also so. Therefore, we conclude by the strong law of large numbers for local martingales that

$$\lim_{t\to\infty}\frac{\mathrm{M}_1(t)}{t} \quad \text{and} \quad \lim_{t\to\infty}\frac{\mathrm{M}_2(t)}{t} \qquad \text{a.s.} \tag{III.21}$$

Taking the superior limit on both sides of (III.18) and combining the resulting inequality with (III.19), (III.20) and (III.21) lead us to

$$\begin{aligned}\limsup_{t\to\infty}\frac{W\big(I(t),\widehat{I}(t)\big)}{t} &\leqslant \mathfrak{D} - \mathfrak{C} + \mathfrak{B} + \frac{\lambda_1 b\beta}{\lambda_2}\frac{\Lambda}{2\mu}\left(\frac{2\mu}{\Upsilon} - 1\right)^{\frac{1}{2}} + \frac{\lambda_2 b\hat{\beta}}{\lambda_1}\frac{\hat{\Lambda}}{2\hat{\mu}}\left(\frac{2\hat{\mu}}{\widehat{\Upsilon}} - 1\right)^{\frac{1}{2}} \\ &= \mathfrak{D} - \mathfrak{C} + \mathfrak{B} + \frac{\hat{\mu}\sqrt{\mathcal{R}_0}}{2}\left(\frac{2\mu}{\Upsilon} - 1\right)^{\frac{1}{2}} + \frac{(\mu+\rho)\sqrt{\mathcal{R}_0}}{2}\left(\frac{2\hat{\mu}}{\widehat{\Upsilon}} - 1\right)^{\frac{1}{2}} = \mathcal{K},\end{aligned}$$

which is exactly the desired conclusion. In addition, it goes without saying that if $\mathcal{K} < 0$ then the disease will die out exponentially almost surely. Thus, the theorem is proved. ■

Remark 3.4. *Compared to several existing works (see for instance [146, 74, 157, 158, 12, 159]), the statement of the last theorem is indeed stronger because it offers a sharper threshold that weakens the disease extinction condition. The precision of our threshold κ comes back essentially to inequality (III.17) in the previous proof where we used the ramp function $(x \mapsto x^+)$ instead of the absolute value always adopted in the literature to our best knowledge.*

4 Persistence in the mean of the dengue disease

After having studied the extinction of the dengue fever, we turn now to explore its persistence in the mean, but before doing so, let us first recall the definition of this notion.

Definition 4.1 (Persistence in the mean [153]). *For system (III.5), the total mass of infectious individuals $I_{total} := (I + \widehat{I})$ is said to be persistent in the mean if there is a positive constant c such that $\liminf_{t\to\infty} \frac{1}{t}\int_0^t \Big(I(s) + \widehat{I}(s)\Big)\,\mathrm{d}s \geqslant c > 0$ almost surely.*

For the sake of greater clarity and readability, we use from now on these notations:

- $M_1 := \mu + \dfrac{\sigma_1^2}{2} + \displaystyle\int_{\mathbb{U}} \Big(\xi_1(u) - \ln\big(1+\xi_1(u)\big)\Big)\nu(\mathrm{d}u).$
- $M_2 := (\mu+\rho) + \dfrac{\sigma_2^2}{2} + \displaystyle\int_{\mathbb{U}} \Big(\xi_2(u) - \ln\big(1+\xi_2(u)\big)\Big)\nu(\mathrm{d}u).$
- $M_3 := \hat{\mu} + \dfrac{\sigma_3^2}{2} + \displaystyle\int_{\mathbb{U}} \Big(\xi_3(u) - \ln\big(1+\xi_3(u)\big)\Big)\nu(\mathrm{d}u).$
- $M_4 := \hat{\mu} + \dfrac{\sigma_4^2}{2} + \displaystyle\int_{\mathbb{U}} \Big(\xi_4(u) - \ln\big(1+\xi_4(u)\big)\Big)\nu(\mathrm{d}u).$
- $\widetilde{\mathcal{R}}_0 := \dfrac{b^2\beta\hat{\beta}\Lambda\hat{\Lambda}}{M_1M_2M_3M_4}.$

Theorem 4.1. *Suppose that the assumptions $(\mathrm{A}_1)-(\mathrm{A}_5)$ are verified, and let $\big(S(t),I(t),\widehat{S}(t),\widehat{I}(t)\big)$ be the solution of system (III.5) that starts from an initial value $\big(S_0,I_0,\widehat{S}_0,\widehat{I}_0\big)$ belonging to the positive orthant $\mathbb{R}^4_+$. If the inequality $\widetilde{\mathcal{R}}_0 > 1$ is true, then the dengue disease presented by equation (III.5) will persist in the mean.*

Proof. Let us consider the function $\overline{W}$ defined by

$$\begin{array}{rccc} \overline{W}: & \mathbb{R}^4_+ & \longrightarrow & \mathbb{R} \\ & x & \longmapsto & -\displaystyle\sum_{i=1}^{4}\theta_i \ln(x_i), \end{array}$$

where $\theta_2 = 1$ and $\theta_1, \theta_3, \theta_4$ are three positive constants to be chosen suitably later. From Itô's formula, we have and for all $t \geqslant 0$

$$\begin{aligned} &\mathrm{d}\overline{W}\big(S(t),I(t),\widehat{S}(t),\widehat{I}(t)\big) \\ &= \mathcal{L}\overline{W}\big(S(t),I(t),\widehat{S}(t),\widehat{I}(t)\big)\ \mathrm{d}t - \theta_1\sigma_1\mathrm{d}\mathcal{B}_1(t) - \sigma_2\mathrm{d}\mathcal{B}_2(t) - \theta_3\sigma_3\mathrm{d}\mathcal{B}_3(t) - \theta_4\sigma_4\mathrm{d}\mathcal{B}_4(t) \\ &\quad - \theta_1\int_{\mathbb{U}}\ln\big(1+\xi_1(u)\big)\widetilde{\mathcal{N}}(\mathrm{d}t,\mathrm{d}u) - \int_{\mathbb{U}}\ln\big(1+\xi_2(u)\big)\widetilde{\mathcal{N}}(\mathrm{d}t,\mathrm{d}u) - \theta_3\int_{\mathbb{U}}\ln\big(1+\xi_3(u)\big)\widetilde{\mathcal{N}}(\mathrm{d}t,\mathrm{d}u) \\ &\quad - \theta_4\int_{\mathbb{U}}\ln\big(1+\xi_4(u)\big)\widetilde{\mathcal{N}}(\mathrm{d}t,\mathrm{d}u), \end{aligned}$$

where $\mathcal{L}\overline{W}\big(S(t), I(t), \widehat{S}(t), \widehat{I}(t)\big)$ is given by

$$\begin{aligned}\mathcal{L}\overline{W}\big(S(t), I(t), \widehat{S}(t), \widehat{I}(t)\big) = &-\theta_1\frac{\Lambda}{S(t)} + \theta_1 b\beta\widehat{I}(t) + \theta_1\mu + \theta_1\frac{\sigma_1^2}{2} + \theta_1\int_{\mathbb{U}}\Big(\xi_1(u) - \ln\big(1+\xi_1(u)\big)\Big)\nu(\mathrm{d}u)\\ &- \frac{b\beta S(t)\widehat{I}(t)}{I(t)} + (\mu+\rho) + \frac{\sigma_2^2}{2} + \int_{\mathbb{U}}\Big(\xi_2(u) - \ln\big(1+\xi_2(u)\big)\Big)\nu(\mathrm{d}u)\\ &- \theta_3\frac{\hat{\Lambda}}{\widehat{S}(t)} + \theta_3 b\hat{\beta}I(t) + \theta_3\hat{\mu} + \theta_3\frac{\sigma_3^2}{2} + \theta_3\int_{\mathbb{U}}\Big(\xi_3(u) - \ln\big(1+\xi_3(u)\big)\Big)\nu(\mathrm{d}u)\\ &- \frac{\theta_4 b\hat{\beta}\widehat{S}(t)I(t)}{\widehat{I}(t)} + \theta_4\hat{\mu} + \theta_4\frac{\sigma_4^2}{2} + \theta_4\int_{\mathbb{U}}\Big(\xi_4(u) - \ln\big(1+\xi_4(u)\big)\Big)\nu(\mathrm{d}u).\end{aligned}$$

After some rearrangements, we get

$$\begin{aligned}\mathcal{L}\overline{W}\big(S(t), I(t), \widehat{S}(t), \widehat{I}(t)\big) = &-\theta_1\frac{\Lambda}{S(t)} - \frac{b\beta S(t)\widehat{I}(t)}{I(t)} - \theta_3\frac{\hat{\Lambda}}{\widehat{S}(t)} - \frac{\theta_4 b\hat{\beta}\widehat{S}(t)I(t)}{\widehat{I}(t)} + \theta_1 b\beta\widehat{I}(t) + \theta_3 b\hat{\beta}I(t)\\ &+ \theta_1\left(\mu + \frac{\sigma_1^2}{2} + \int_{\mathbb{U}}\Big(\xi_1(u) - \ln\big(1+\xi_1(u)\big)\Big)\nu(\mathrm{d}u)\right)\\ &+ \left((\mu+\rho) + \frac{\sigma_2^2}{2} + \int_{\mathbb{U}}\Big(\xi_2(u) - \ln\big(1+\xi_2(u)\big)\Big)\nu(\mathrm{d}u)\right)\\ &+ \theta_3\left(\hat{\mu} + \frac{\sigma_3^2}{2} + \int_{\mathbb{U}}\Big(\xi_3(u) - \ln\big(1+\xi_3(u)\big)\Big)\nu(\mathrm{d}u)\right)\\ &+ \theta_4\left(\hat{\mu} + \frac{\sigma_4^2}{2} + \int_{\mathbb{U}}\Big(\xi_4(u) - \ln\big(1+\xi_4(u)\big)\Big)\nu(\mathrm{d}u)\right).\end{aligned}$$

From the relation between geometric and arithmetic means (the first is less than or equal to the second), it follows that

$$\begin{aligned}\mathcal{L}\overline{W}\big(S(t), I(t), \widehat{S}(t), \widehat{I}(t)\big) \leqslant &-4\sqrt[4]{\theta_1\theta_3\theta_4 b^2\beta\hat{\beta}\Lambda\hat{\Lambda}} + \theta_1 b\beta\widehat{I}(t) + \theta_3 b\hat{\beta}I(t)\\ &+ \theta_1\underbrace{\left(\mu + \frac{\sigma_1^2}{2} + \int_{\mathbb{U}}\Big(\xi_1(u) - \ln\big(1+\xi_1(u)\big)\Big)\nu(\mathrm{d}u)\right)}_{:=M_1}\\ &+ \underbrace{\left((\mu+\rho) + \frac{\sigma_2^2}{2} + \int_{\mathbb{U}}\Big(\xi_2(u) - \ln\big(1+\xi_2(u)\big)\Big)\nu(\mathrm{d}u)\right)}_{:=M_2}\\ &+ \theta_3\underbrace{\left(\hat{\mu} + \frac{\sigma_3^2}{2} + \int_{\mathbb{U}}\Big(\xi_3(u) - \ln\big(1+\xi_3(u)\big)\Big)\nu(\mathrm{d}u)\right)}_{:=M_3}\\ &+ \theta_4\underbrace{\left(\hat{\mu} + \frac{\sigma_4^2}{2} + \int_{\mathbb{U}}\Big(\xi_4(u) - \ln\big(1+\xi_4(u)\big)\Big)\nu(\mathrm{d}u)\right)}_{:=M_4}.\end{aligned}$$

By taking $\theta_1 = \dfrac{b^2\beta\hat{\beta}\Lambda\hat{\Lambda}}{M_1^2M_3M_4}$, $\theta_3 = \dfrac{b^2\beta\hat{\beta}\Lambda\hat{\Lambda}}{M_1M_3^2M_4}$ and $\theta_4 = \dfrac{b^2\beta\hat{\beta}\Lambda\hat{\Lambda}}{M_1M_3M_4^2}$, we obtain:

$$\mathcal{L}\overline{W}\big(S(t),I(t),\widehat{S}(t),\widehat{I}(t)\big) \leqslant \theta_1 b\beta\widehat{I}(t)+\theta_3 b\hat{\beta}I(t)+M_2-\frac{b^2\beta\hat{\beta}\Lambda\hat{\Lambda}}{M_1M_3M_4} = b\left(\theta_1\beta\widehat{I}(t)+\theta_3\hat{\beta}I(t)\right)-M_2\Big(\overbrace{\frac{b^2\beta\hat{\beta}\Lambda\hat{\Lambda}}{M_1M_2M_3M_4}}^{\widetilde{\mathcal{R}}_0}-1\Big).$$

Therefore

$$\begin{aligned}
&\mathrm{d}\overline{W}\big(S(t),I(t),\widehat{S}(t),\widehat{I}(t)\big)\\
&\leqslant\Big[b\left(\theta_1\beta\widehat{I}(t)+\theta_3\hat{\beta}I(t)\right)-M_2\Big(\widetilde{\mathcal{R}}_0-1\Big)\Big]\mathrm{d}t-\theta_1\sigma_1\mathrm{d}\mathcal{B}_1(t)-\sigma_2\mathrm{d}\mathcal{B}_2(t)-\theta_3\sigma_3\mathrm{d}\mathcal{B}_3(t)-\theta_4\sigma_4\mathrm{d}\mathcal{B}_4(t)\\
&\quad-\theta_1\int_{\mathbb{U}}\ln\big(1+\xi_1(u)\big)\widetilde{\mathcal{N}}(\mathrm{d}t,\mathrm{d}u)-\int_{\mathbb{U}}\ln\big(1+\xi_2(u)\big)\widetilde{\mathcal{N}}(\mathrm{d}t,\mathrm{d}u)-\theta_3\int_{\mathbb{U}}\ln\big(1+\xi_3(u)\big)\widetilde{\mathcal{N}}(\mathrm{d}t,\mathrm{d}u)\\
&\quad-\theta_4\int_{\mathbb{U}}\ln\big(1+\xi_4(u)\big)\widetilde{\mathcal{N}}(\mathrm{d}t,\mathrm{d}u).
\end{aligned}\tag{III.22}$$

Integrating (III.22) from 0 to t, and then dividing by t on both sides, we get

$$\begin{aligned}
\frac{\overline{W}\big(S(t),I(t),\widehat{S}(t),\widehat{I}(t)\big)}{t}-\frac{\overline{W}\big(S(0),I(0),\widehat{S}(0),\widehat{I}(0)\big)}{t} &\leqslant \frac{b\theta_1\beta}{t}\int_0^t\widehat{I}(s)\,\mathrm{d}s+\frac{b\theta_3\hat{\beta}}{t}\int_0^t I(s)\,\mathrm{d}s-M_2\Big(\widetilde{\mathcal{R}}_0-1\Big)\\
&\quad-\sum_{i=1}^4\theta_i\sigma_i\frac{\mathcal{B}_i(t)}{t}-\sum_{i=1}^4\frac{\theta_i}{t}\int_0^t\int_{\mathbb{U}}\ln\big(1+\xi_i(u)\big)\widetilde{\mathcal{N}}(\mathrm{d}s,\mathrm{d}u)
\end{aligned}$$

So

$$\begin{aligned}
\frac{\overline{W}\big(S(t),I(t),\widehat{S}(t),\widehat{I}(t)\big)}{t}-\frac{\overline{W}\big(S(0),I(0),\widehat{S}(0),\widehat{I}(0)\big)}{t} &\leqslant \frac{b\left(\theta_1\beta\vee\theta_3\hat{\beta}\right)}{t}\int_0^t\left(\widehat{I}(s)+I(s)\right)\mathrm{d}s-M_2\Big(\widetilde{\mathcal{R}}_0-1\Big)\\
&\quad-\sum_{i=1}^4\theta_i\sigma_i\frac{\mathcal{B}_i(t)}{t}-\sum_{i=1}^4\frac{\theta_i}{t}\int_0^t\int_{\mathbb{U}}\ln\big(1+\xi_i(u)\big)\widetilde{\mathcal{N}}(\mathrm{d}s,\mathrm{d}u).
\end{aligned}$$

Hence

$$\begin{aligned}
\frac{1}{t}\int_0^t\left(\widehat{I}(s)+I(s)\right)\mathrm{d}s &\geqslant \frac{M_2\Big(\widetilde{\mathcal{R}}_0-1\Big)}{b\left(\theta_1\beta\vee\theta_3\hat{\beta}\right)}+\frac{\overline{W}\big(S(t),I(t),\widehat{S}(t),\widehat{I}(t)\big)}{b\left(\theta_1\beta\vee\theta_3\hat{\beta}\right)t}-\frac{\overline{W}\big(S(0),I(0),\widehat{S}(0),\widehat{I}(0)\big)}{b\left(\theta_1\beta\vee\theta_3\hat{\beta}\right)t}\\
&\quad+\sum_{i=1}^4\theta_i\sigma_i\frac{\mathcal{B}_i(t)}{b\left(\theta_1\beta\vee\theta_3\hat{\beta}\right)t}+\sum_{i=1}^4\frac{\theta_i}{b\left(\theta_1\beta\vee\theta_3\hat{\beta}\right)t}\int_0^t\int_{\mathbb{U}}\ln\big(1+\xi_i(u)\big)\widetilde{\mathcal{N}}(\mathrm{d}s,\mathrm{d}u).
\end{aligned}\tag{III.23}$$

Since $\ln(a)\leqslant a-1\leqslant a$ for any $a>0$, one can conclude that $\overline{W}(x)\geqslant-\sum_{i=1}^4\theta_i x_i$ for all $x\in\mathbb{R}_+^4$. Combining this fact with (III.23) leads to

$$\begin{aligned}
\frac{1}{t}\int_0^t\left(\widehat{I}(s)+I(s)\right)\mathrm{d}s &\geqslant \frac{M_2\Big(\widetilde{\mathcal{R}}_0-1\Big)}{b\left(\theta_1\beta\vee\theta_3\hat{\beta}\right)}-\frac{1}{t}\times\frac{\overbrace{\theta_1S(t)+I(t)+\theta_3\widehat{S}(t)+\theta_1\widehat{I}(t)}^{:=\mathcal{M}_0(t)}}{b\left(\theta_1\beta\vee\theta_3\hat{\beta}\right)}-\frac{\overline{W}\big(S(0),I(0),\widehat{S}(0),\widehat{I}(0)\big)}{b\left(\theta_1\beta\vee\theta_3\hat{\beta}\right)t}\\
&\quad+\underbrace{\frac{1}{t}\times\sum_{i=1}^4\theta_i\sigma_i\frac{\mathcal{B}_i(t)}{b\left(\theta_1\beta\vee\theta_3\hat{\beta}\right)}}_{:=\mathcal{M}_1(t)}+\underbrace{\frac{1}{t}\times\sum_{i=1}^4\frac{\theta_i}{b\left(\theta_1\beta\vee\theta_3\hat{\beta}\right)}\int_0^t\int_{\mathbb{U}}\ln\big(1+\xi_i(u)\big)\widetilde{\mathcal{N}}(\mathrm{d}s,\mathrm{d}u)}_{:=\mathcal{M}_2(t)}.
\end{aligned}\tag{III.24}$$

Needless to say, $\mathcal{M}_1(t)$ is a local martingale with finite quadratic variation, and from the assumption (A_5) we can assert that $\mathcal{M}_2(t)$ is also so. Therefore, we deduce by the strong law of large numbers for local martingales that

$$\lim_{t\to\infty} \frac{\mathcal{M}_1(t)}{t} \quad \text{and} \quad \lim_{t\to\infty} \frac{\mathcal{M}_2(t)}{t} \qquad \text{a.s.} \tag{III.25}$$

On the other hand it is clear by virtue of Lemma 3.1 that

$$\lim_{t\to\infty} \frac{\mathcal{M}_0(t)}{t} \qquad \text{a.s.} \tag{III.26}$$

Taking the inferior limit on both sides of (III.24) and combining the resulting inequality with (III.25) and (III.26) yields

$$\liminf_{t\to\infty} \frac{1}{t} \int_0^t \left(\widehat{I}(s) + I(s)\right) \mathrm{d}s \geqslant \frac{M_2\left(\widetilde{\mathcal{R}}_0 - 1\right)}{b\left(\theta_1\beta \vee \theta_3\hat{\beta}\right)} > 0 \quad a.s. \tag{III.27}$$

So if $\widetilde{\mathcal{R}}_0 > 1$, then the disease will persist in the mean as claimed, which completes the proof. ■

5 Numerical simulation examples

In this section, and by taking the parameter values from the theoretical data presented in Table III.1, we set forth some numerical simulations to belay the various results proved in this chapter. The solution of our Dengue model, is simulated in our case with the initial condition given by $S(0) = 0.2, I(0) = 0.1, \widehat{S}(0) = 0.3$ and $\widehat{I}(0) = 0.4$. In what follows, we consider that the unity of time is one day and the number of individuals is expressed in one million population.

Parameters	Description	Numerical values	
Λ	Humans recruitment rate	0.5	0.85
b	Mosquitoes' biting rate	3	7
β	Transmission rate from mosquitoes to humans	0.15	0.65
μ	Humans' natural death rate	0.8	0.8
ρ_0	Dengue induced death rate	0.8	0.8
ρ_1	Heal rate, whether by treatment or naturally	0.02	0.25
$\hat{\Lambda}$	Mosquitoes recruitment rate	0.6	0.6
$\hat{\beta}$	Transmission rate from humans to mosquitoes	0.55	0.55
$\hat{\mu}$	Mosquitoes' natural death rate	0.9	0.88
σ_1	Intensity of the Brownian motion $\mathcal{B}_1$	0.269	0.269
σ_2	Intensity of the Brownian motion $\mathcal{B}_2$	0.25	0.25
σ_3	Intensity of the Brownian motion $\mathcal{B}_3$	0.25	0.245
σ_4	Intensity of the Brownian motion $\mathcal{B}_4$	0.13	0.14
ξ_1	Intensity of the Lévy jumps associated to S	-0.75	-0.75
ξ_2	Intensity of the Lévy jumps associated to I	0.8	0.78
ξ_3	Intensity of the Lévy jumps associated to $\widehat{S}$	-0.9	-0.9
ξ_4	Intensity of the Lévy jumps associated to $\widehat{I}$	0.85	0.85
		Figure III.2	Figure III.3

Table III.1: Definitions and nominal values, per day, of the system parameters and perturbations intensities used in the simulations.

5.1 The case of dengue fever extinction

In this case, and by selecting the parameter values appearing in the third column of Table III.1, and also taking $\mathbb{U} = \mathbb{R}_+$ with $\nu(\mathbb{U}) = 1$, we will shed some light on the theoretical results of Section 3. By a few simple calculations, it can be verified that we have the numerical values presented in Table III.2. From the latter, we observe easily that the condition $\mathcal{K} < 0$, and hypothesis (A_1), (A_2), (A_3) and (A_4) hold. So, and by the virtue of Theorem 3.1 the dengue epidemic dies out exponentially almost surely, which is exactly depicted in Figure III.2.

	Quantity	Expression	Value
Assumption (A_1)	$\mathfrak{M}_1$	$\max_{1\leqslant i\leqslant 4}\left(\int_{\mathbb{U}}\xi_i^2(u)\nu(\mathrm{d}u)\right)$	$0.81 < \infty$
Assumption (A_2)	$\mathfrak{M}_2$	$\max_{1\leqslant i\leqslant 4}\left(\int_{\mathbb{U}}\left(\xi_i(u)-\ln\left(1+\xi_i(u)\right)\right)\nu(\mathrm{d}u)\right)$	$1.4025 < \infty$
	Σ	$\max\left\{\sigma_i^2 \mid i\in\{1,2,3,4\}\right\}$	0.0724
	$\widetilde{\xi}$	$\max\left\{\xi_i(u) \mid i\in\{1,2,3,4\}\right\}$	0.85
	$\underset{\sim}{\xi}$	$\min\left\{\xi_i(u) \mid i\in\{1,2,3,4\}\right\}$	-0.9
	$\widetilde{\theta}_p$	$\left(1+\widetilde{\xi}\right)^p - p\times\widetilde{\xi} - 1$	1.5301
	$\underset{\sim}{\theta}_p$	$\left(1+\underset{\sim}{\xi}\right)^p - p\times\underset{\sim}{\xi} - 1$	1.2531
	θ_p	$\underset{\sim}{\theta}_p \vee \widetilde{\theta}_p$	1.5301
	ϱ_p	$\int_{\mathbb{U}}\theta_p(u)\nu(\mathrm{d}u)$	1.5301
Assumption (A_3)	Δ_p	$\mu\wedge\hat{\mu} - \frac{p-1}{2}\Sigma - \frac{\varrho_p}{p}$	$0.13366 > 0$
Assumption (A_4)	$\mathfrak{M}_3$	$\max_{1\leqslant i\leqslant 4}\left(\int_{\mathbb{U}}\left(\left(1+\xi_i(u)\right)^2-1\right)^2\nu(\mathrm{d}u)\right)$	$5.868 < \infty$
	Υ	$2\mu - \sigma_1^2 - \int_{\mathbb{U}}\xi_1^2(u)\nu(\mathrm{d}u)$	0.9651
	$\widehat{\Upsilon}$	$2\hat{\mu} - \sigma_3^2 - \int_{\mathbb{U}}\xi_3^2(u)\nu(\mathrm{d}u)$	0.9275
	$\underline{\mathfrak{b}}$	$\left(\ln\left(1+\xi_2\wedge\xi_4\right) - \xi_2\wedge\xi_4\right)\times\mathbb{1}_{\{\xi_2(u)\wedge\xi_4(u)>0\}}$	-0.2122
	$\overline{\mathfrak{b}}$	$\left(\ln\left(1+\xi_2\vee\xi_4\right) - \xi_2\vee\xi_4\right)\times\mathbb{1}_{\{\xi_2(u)\vee\xi_4(u)\leqslant 0\}}$	0
	$\mathfrak{B}$	$\int_{\mathbb{U}}\left(\overline{\mathfrak{b}}(u)+\underline{\mathfrak{b}}(u)\right)\nu(\mathrm{d}u)$	-0.2122
	$\mathfrak{C}$	$\frac{(\sigma_2\sigma_4)^2}{2(\sigma_2^2+\sigma_4^2)}$	0.026
	$\mathfrak{D}$	$[(\mu+\rho)\vee\hat{\mu}]\times\left(\sqrt{\mathcal{R}_0}-1\right)^+ - [(\mu+\rho)\wedge\hat{\mu}]\times\left(1-\sqrt{\mathcal{R}_0}\right)^+$	-0.4854
	$\mathcal{R}_0$	$\frac{b^2\beta\Lambda\hat{\beta}\hat{\Lambda}}{\mu\left(\mu+\rho_0+\rho_1\right)\hat{\mu}^2}$	0.2122
	$\mathcal{K}$	$\mathfrak{D}-\mathfrak{C}+\mathfrak{B}+\frac{\hat{\mu}\sqrt{\mathcal{R}_0}}{2}\left(\frac{2\mu}{\Upsilon}-1\right)^{\frac{1}{2}}+\frac{(\mu+\rho)\sqrt{\mathcal{R}_0}}{2}\left(\frac{2\hat{\mu}}{\widehat{\Upsilon}}-1\right)^{\frac{1}{2}}$	$-0.2044 < 0$

Table III.2: Some quantities and their corresponding numerical values in the case of $p = 2.5 > 2$.

5.2 The case of dengue fever persistence in the mean

In this subsection, we will turn to the dengue's persistence in the mean case. By keeping in mind the numerical values appearing in the last column of Table III.1, we can easily draw up the following list:

	Quantity	Expression	Value
Assumption (A_1)	$\mathfrak{M}_1$	$\max\limits_{1\leqslant i\leqslant 4}\left(\int_{\mathbb{U}}\xi_i^2(u)\nu(\mathrm{d}u)\right)$	$0.81<\infty$
Assumption (A_2)	$\mathfrak{M}_2$	$\max\limits_{1\leqslant i\leqslant 4}\left(\int_{\mathbb{U}}\xi_i(u)-\ln\left(1+\xi_i(u)\right)\nu(\mathrm{d}u)\right)$	$1.4025<\infty$
	Σ	$\max\left\{\sigma_i^2 \mid i\in\{1,2,3,4\}\right\}$	0.0724
	$\widetilde{\xi}$	$\max\left\{\xi_i(u) \mid i\in\{1,2,3,4\}\right\}$	0.85
	$\underset{\sim}{\xi}$	$\min\left\{\xi_i(u) \mid i\in\{1,2,3,4\}\right\}$	-0.9
	$\widetilde{\theta}_p$	$\left(1+\widetilde{\xi}\right)^p-p\times\widetilde{\xi}-1$	1.5301
	$\underset{\sim}{\theta}_p$	$\left(1+\underset{\sim}{\xi}\right)^p-p\times\underset{\sim}{\xi}-1$	1.2531
	θ_p	$\underset{\sim}{\theta}_p\vee\widetilde{\theta}_p$	1.5301
	ϱ_p	$\int_{\mathbb{U}}\theta_p(u)\nu(\mathrm{d}u)$	1.5301
Assumption (A_3)	Δ_p	$\mu\wedge\hat{\mu}-\frac{p-1}{2}\Sigma-\frac{\varrho_p}{p}$	$1.13366>0$
Assumption (A_4)	$\mathfrak{M}_3$	$\max\limits_{1\leqslant i\leqslant 4}\left(\int_{\mathbb{U}}\left(\left(1+\xi_i(u)\right)^2-1\right)^2\nu(\mathrm{d}u)\right)$	$5.868<\infty$
Assumption (A_5)	$\mathfrak{M}_4$	$\max\limits_{1\leqslant i\leqslant 4}\left(\int_{\mathbb{U}}\left(\ln\left(1+\xi_i(u)\right)\right)^2\nu(\mathrm{d}u)\right)$	0.378
	M_1	$\mu+\frac{\sigma_1^2}{2}+\int_{\mathbb{U}}\left(\xi_1(u)-\ln\left(1+\xi_1(u)\right)\right)\nu(\mathrm{d}u)$	1.4725
	M_2	$(\mu+\rho)+\frac{\sigma_2^2}{2}+\int_{\mathbb{U}}\left(\xi_2(u)-\ln\left(1+\xi_2(u)\right)\right)\nu(\mathrm{d}u)$	2.0935
	M_3	$\hat{\mu}+\frac{\sigma_3^2}{2}+\int_{\mathbb{U}}\left(\xi_3(u)-\ln\left(1+\xi_3(u)\right)\right)\nu(\mathrm{d}u)$	2.3338
	M_4	$\hat{\mu}+\frac{\sigma_4^2}{2}+\int_{\mathbb{U}}\left(\xi_4(u)-\ln\left(1+\xi_4(u)\right)\right)\nu(\mathrm{d}u)$	1.1433
	$\widetilde{\mathcal{R}}_0$	$\frac{b^2\beta\hat{\beta}\Lambda\hat{\Lambda}}{M_1M_2M_3M_4}$	$1.0862>1$

Table III.3: The corresponding values of some quantities in the case of $p=2.5>2$.

On the basis of this last table's numerical values, the assumptions $(A_1)-(A_5)$ hold, and the quantity $\widetilde{\mathcal{R}}_0$ outnumbers one. So, and by Theorem 4.1, the dengue fever is persistent in the mean, which agrees well with the pictorial curves presented in Figure III.3.

Remark 5.1. *Plainly, and in all the simulation examples mentioned above, assumptions* (A_1) *and* (A_2) *are satisfied. So, and since* $\big(S(0), I(0), \widehat{S}(0), \widehat{I}(0)\big) \in \mathbb{R}^4_+$*, Theorem 2.1 guarantees the positivity of the solution for all future time* $t > 0$*. This last fact is illustrated and clearly corroborated by the different curves depicted in figures III.2 and III.3.*

Remark 5.2. *If we kept the same approach as [146, 74, 157, 158, 12, 159] in our treatment, the threshold that we will obtain in Theorem 3.1 will be of the following form* $K := \mathfrak{D} - \mathfrak{C} + \mathfrak{B} + \hat{\mu} \times \sqrt{\mathcal{R}_0\left(\frac{2\mu}{\Upsilon} - 1\right)} + (\mu + \rho) \times \sqrt{\mathcal{R}_0\left(\frac{2\hat{\mu}}{\widehat{\Upsilon}} - 1\right)}$*. The latter, and in the case of the numerical values taken in Figure III.2, is positive, therefore, it will be unable to guarantee the disease's extinction. On the other hand, the new threshold* $\boldsymbol{\kappa}$ *that we have proposed is well capable of doing it, and this reflects clearly its sharpness.*

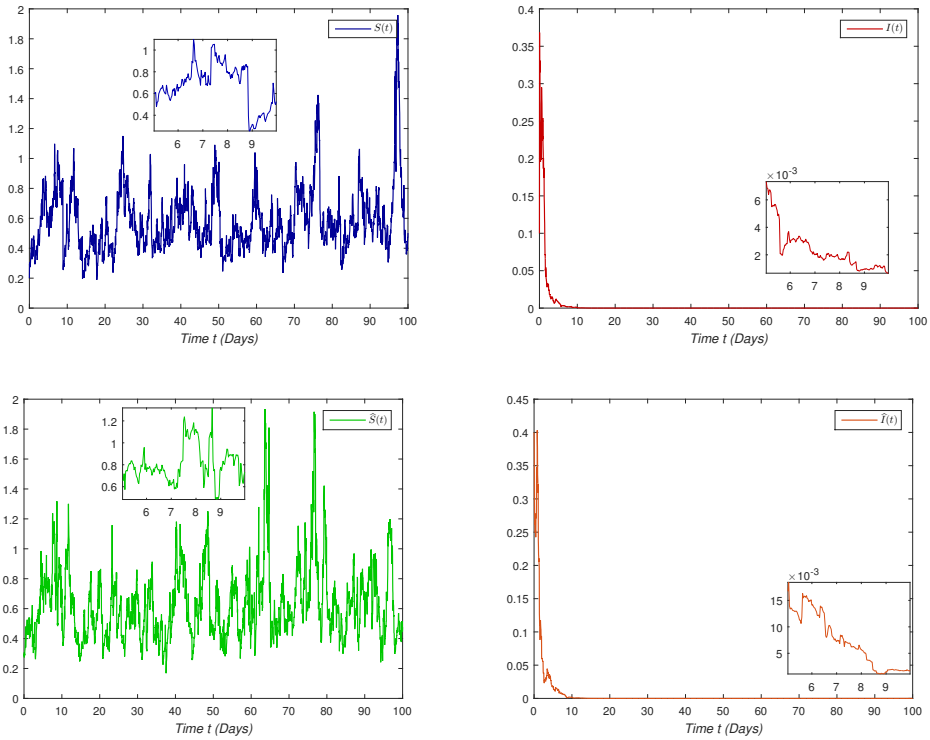


Figure III.2: Solutions' paths of the dengue fever model (III.5) when the numerical values are taken as shown in the third column of Table III.1 ($\boldsymbol{\kappa} = -0.2044 < 0$).

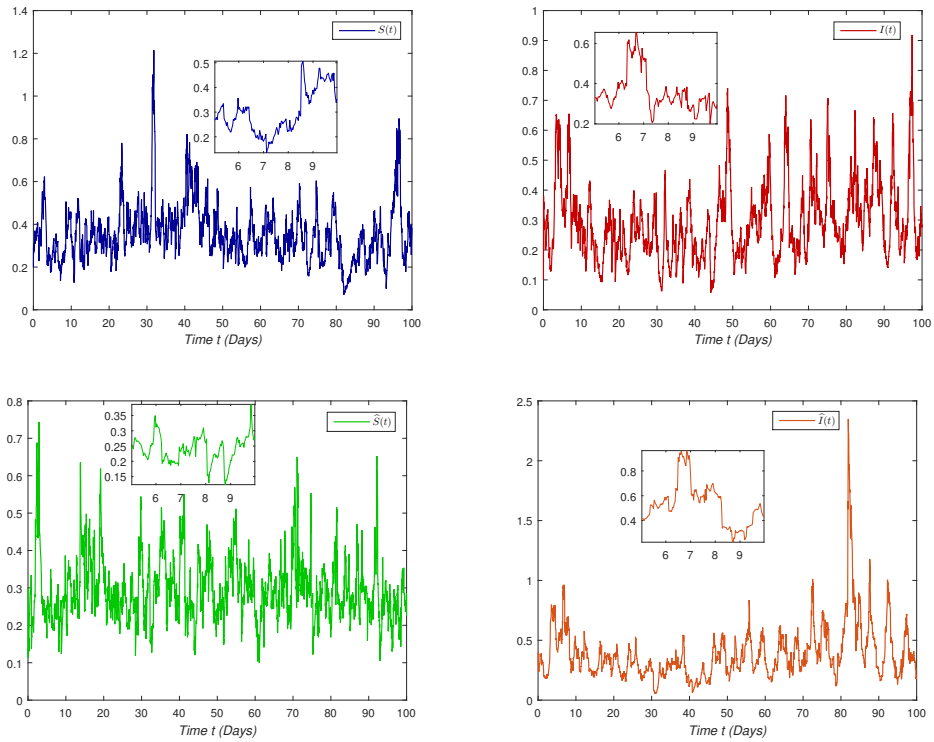


Figure III.3: Solutions' paths of the dengue fever model (III.5) when the numerical values are chosen as shown in the fourth column of Table III.1 ($\widetilde{\mathcal{R}}_0 = 1.0862 > 1$).

5.3 The effect of Lévy jumps on dengue fever dynamics

To explore the effect of white noises and Lévy jumps on the dengue fever dynamics, we will compare in this part, the trajectories of the following three systems:

- The stochastic system with Lévy jumps (III.5).
- The stochastic system with Brownian motions, which is none other than the system (III.5) but with $\sigma_i \neq 0$ and $\xi_i = 0$ $(i = 1, 2, 3, 4)$:

$$\begin{cases} \mathrm{d}S(t) = \left[\Lambda - b\beta S(t)\widehat{I}(t) - \mu S(t)\right]\mathrm{d}t + \sigma_1 S(t)\mathrm{d}\mathcal{B}_1(t), \\ \mathrm{d}I(t) = \left[b\beta S(t)\widehat{I}(t) - (\mu + \rho_0 + \rho_1)\, I(t)\right]\mathrm{d}t + \sigma_2 I(t)\mathrm{d}\mathcal{B}_2(t), \\ \mathrm{d}\widehat{S}(t) = \left[\hat{\Lambda} - b\hat{\beta}\widehat{S}(t)I(t) - \hat{\mu}\widehat{S}(t)\right]\mathrm{d}t + \sigma_3 \widehat{S}(t)\mathrm{d}\mathcal{B}_3(t), \\ \mathrm{d}\widehat{I}(t) = \left[b\hat{\beta}\widehat{S}(t)I(t) - \hat{\mu}\widehat{I}(t)\right]\mathrm{d}t + \sigma_4 \widehat{I}(t)\mathrm{d}\mathcal{B}_4(t). \end{cases} \tag{III.28}$$

- The deterministic model, that is none other than the system (III.5) but with $\sigma_i = 0$ and $\xi_i = 0$ $(i = 1, 2, 3, 4)$:

$$\begin{cases} \mathrm{d}S(t) = \left[\Lambda - b\beta S(t)\widehat{I}(t) - \mu S(t)\right]\mathrm{d}t, \\ \mathrm{d}I(t) = \left[b\beta S(t)\widehat{I}(t) - (\mu + \rho_0 + \rho_1)\, I(t)\right]\mathrm{d}t, \\ \mathrm{d}\widehat{S}(t) = \left[\hat{\Lambda} - b\hat{\beta}\widehat{S}(t)I(t) - \hat{\mu}\widehat{S}(t)\right]\mathrm{d}t, \\ \mathrm{d}\widehat{I}(t) = \left[b\hat{\beta}\widehat{S}(t)I(t) - \hat{\mu}\widehat{I}(t)\right]\mathrm{d}t. \end{cases} \tag{III.29}$$

For the sake of comparison, we keep the same initial data and parameter values as those used in the previous two numerical examples, but in the case of the model (III.5) extinction (that of Figure III.4) we apply the following modifications: $\beta = 0.64$, $b = 3.5$. From Figure III.4 we see

that the impact of Lévy jumps leads to the extinction of the dengue fever while the deterministic model (III.1) and the perturbed model with white noises both go to persistence. So, we can say that the jumps have a negative effect on the dengue spread in this case. To examine the effect of jumps on system (III.1) in the persistence case, we present the Figure III.5, in which we observe the persistence of the epidemic in all cases with a greater variation in the Lévy jumps case. Therefore, we can say in this case That the jumps have positive effects on the dengue prevalence.

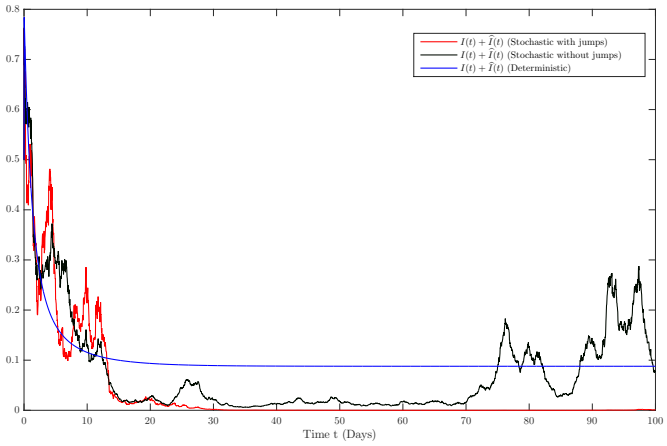


Figure III.4: The paths of $I_{\text{total}}(t) = I(t) + \widehat{I}(t)$ for the deterministic model (III.29), the model (III.28) with Brownian motions, and the stochastic system with Lévy jumps (III.5).

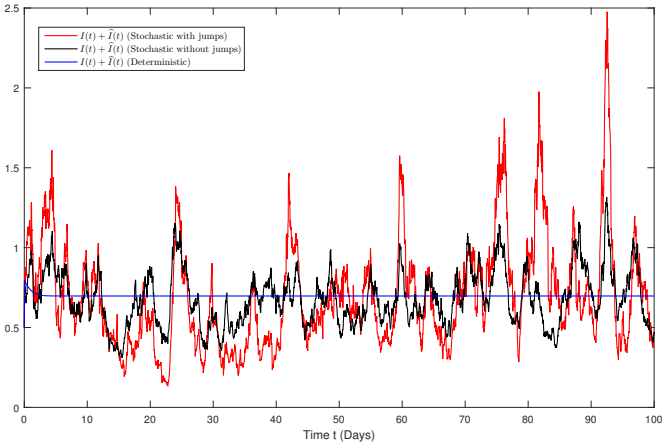


Figure III.5: The paths of $I_{\text{total}}(t) = I(t) + \widehat{I}(t)$ associated respectively to the models (III.29),(III.28) and (III.5).

6 Conclusion and discussion

Dengue fever is an arthropod-borne viral epidemic conveyed to humans through the bite of an infected mosquito with one of the dengue virus's serotypes which all belong to the Flaviviridae viruses family [133]. The dangerousness of this disease lies mainly in its ability to affect almost all age groups, ranging from infants to adults, and in its capacity to re-emerge rapidly in almost every human host population. The absence of an approved treatment or effective vaccine against dengue makes a good comprehension of its prevalence dynamics the only remaining solution to mitigate its severity. In this context, we have presented a mathematical compartmental model that describes

the dengue disease dissemination under environmental small disturbances and unexpected massive external perturbations. More explicitly, we have formulated the dengue spread mechanisms by using an SIR-SI stochastic differential equations system that includes both proportional white noises and Lévy jumps. After having drawn up our model, a rigorous mathematical analysis is performed to get an insight into the dengue fever propagation behavior, especially in the long-term. The principal mathematical and epidemiological findings of this chapter are listed as follows:

- We have demonstrated the existence and uniqueness of a positive global-in-time solution to our proposed dengue model.
- We have provided some sufficient conditions for the dengue fever extinction.
- An appropriate hypothetical framework for the persistence in the mean of the dengue disease is also established.

Compared to the existing works, the originality of our results resides in the following mathematical techniques and amelioration that we have used to accomplish our analysis:

- Our study adopted an alternative approach to estimate the time averages $\frac{\int_0^t \Psi(s)\,\mathrm{d}s}{t}$, $\frac{\int_0^t \Psi^2(s)\,\mathrm{d}s}{t}$, $\frac{\int_0^t \widehat{\Psi}(s)\,\mathrm{d}s}{t}$ and $\frac{\int_0^t \widehat{\Psi}^2(s)\,\mathrm{d}s}{t}$, without resorting to the explicit stationary distribution's formula which still unknown until now (see [151, 152]).
- In our analysis, we have used the inequality (III.16) instead of $\ln(x+1) \leqslant x$ for all $x > -1$, adopted usually in the literature (see for example [160]), which allowed us to obtain a sharper threshold for the dengue extinction case.
- Our results are distinguished from those of several previous works [146, 74, 157, 158, 12, 159] by the use of the ramp function in place of the absolute value in inequality (III.17), which remarkably improved the extinction threshold and hence strengthened the statement of Theorem 3.1.

Roughly speaking, our theoretical results show that the extinction and persistence conditions depend mainly on the white noise intensities, Lévy jumps magnitudes, and the system parameters of course. In order to elucidate the theoretical results and exhibit the effect of replacing the absolute value by the ramp function in the inequality (III.17), we have presented some numerical simulation examples. In the end, we point out that the obtained results generalize several previous works (for instance, [143] and [146]), and improve our understanding of the dengue's spreading comportment, which makes our findings a good basis for future studies, especially with the continuous reappearance of the dengue fever disease in many regions around the globe.

Chapter IV

Asymptotic behavior analysis and threshold sharpening of a staged progression AIDS/HIV epidemic model with Lévy jumps

Abstract

In this chapter, we present and investigate a generalized stochastic AIDS/HIV epidemic model that includes both Brownian motions and Lévy jumps. Our proposed model is a staged progression compartmental one that takes the form of an Itô-Lévy stochastic differential equations system. First, we demonstrate its well-posedness in the sense that it admits one and only one solution which is positive and global in time. Then, and based on some assumptions and nonstandard analytical techniques, we prove two principal asymptotic properties: extinction and persistence in the mean. The theoretical results show that the dynamical behavior of our AIDS/HIV model is mainly determined by the parameters that are closely related to the perturbations intensities. In the end, we provide some numerical simulation examples to corroborate our theoretical results and exhibit the effect of the new adopted mathematical techniques on the findings.

1 Introduction and model formulation

The Acquired Immunodeficiency Syndrome (AIDS) is a very dangerous infectious disease that has attracted much interest from a considerable number of scientists throughout the world. In accordance with several medical research, this disease is caused by the Human Immunodeficiency Virus (HIV) which harms the human body's immune system by attacking the $CD4^+$ T-lymphocytes [161, 162]. The said virus is principally transmitted through penetrative sex, sharing contaminated drug injection equipment (syringes, needles, or other), blood transfusion, and from a mother to her infant during pregnancy, childbirth, and breastfeeding [163, 164]. Based on the World Health Organization (WHO) reports, nearly 38 million people are living with HIV in 2019, and by the end of the same year, more than half a million people have died from AIDS-related illnesses [165]. Worse still, there is no cure for AIDS yet, and the prevention procedures like condom use, pre- and post-sexual exposure prophylaxis [166], and preventing mother-to-child transmission are still the only remaining solutions to lessen and reduce its severity [167]. Therefore, the understanding of the AIDS spreading mechanisms is tightly required since it will provide an overview of its future behavior, and hence help the different governments, authorities, and decision-makers around the globe to propose adequate guidelines and measures for this mortal disease mitigation. In this

respect, mathematical modeling is viewed as one of the most powerful tools for duly describe AIDS dissemination and see its impacts on a human population, especially in the long term [168].

With the view of saving human civilization from AIDS propagation, it is imperative to know more about its transmission process. For this purpose, many scholars have formulated a great number of mathematical models over the last two decades [169, 170]. For example, in [170], Lv and Huang established a deterministic AIDS model with Beddington-DeAngelis infection rate and showed the global stability of its two equilibria, the disease-free, and the endemic one. In [171] Hyman et al. investigated the efficiency of random screening and contact tracing measures on human immunodeficiency virus disease prevalence. They studied the efficacy of these control strategies by formulating and analyzing two deterministic epidemiological models based respectively on the staged progression (SP) and the differential infectivity (DI). In [172], Otuniga derived and discussed a $2n+1$-dimensional differential system that describes the transmission and the treatment of AIDS. First, the model was presented and analyzed in its deterministic version, then it was extended to a Stratonovich stochastic one by introducing white noises in the transmission rate. Through some personal investigations into the pathology of patients with AIDS, several authors have developed some models of this malady in the form of a system constituted by differential equations with delay (see for instance [173, 174], and the bibliography given there).

In early tries to describe it, the AIDS spread was typically represented by compartmental models of increasing infection. In these latter, the HIV taint goes through a number of levels before evolving into full-blown AIDS [175, 176]. Generally, these models suppose a monotonic progression via the infection stages, so that there is no return once an individual has surpassed one of them [177]. With the coming of highly active antiretroviral therapy (HAART for short) in the late 1990s [178], many researchers have noted that this assumption needs really to be reconsidered and modified. In this regard, McCluskey [177] treated an AIDS propagation model that incorporates multiple infection levels and amelioration, that is to say, infected individuals can move from advanced stages of infection back to less ones. He found a threshold parameter determining the local stability of the disease-free equilibrium and under some conditions on the system parameters, global stability is demonstrated using techniques involving compound matrices. In the same context, Cai and Fang [179] have considered also a staged-progression AIDS model, but this time by taking into consideration an imperfect vaccination. They derived the disease threshold and ensured the global asymptotic stability conditions of the equilibria corresponding to their model. Inspired by the studies depicted in [175], [176], [177] and [180], recently, Han et al. [159] suggested a very interesting and reasonable AIDS model that includes both differential infectivity and staged amelioration. They split the host population into seven classes, which are described as follows:

Class	Description
S	Susceptible individuals
I_1	HIV acute infection individuals
T_1	HIV acute infection individuals that are under treatment
I_2	Asymptomatic case patients
T_2	Asymptomatic case patients that are under treatment
I_3	AIDS sufferers without therapy
T_3	AIDS sufferers without therapy that are under treatment

Table IV.1: Description of the different classes constituting the total host population in the case of the AIDS deterministic model presented in [159].

Then, they formulated the interactions between them by the following ordinary differential equa-

tions (ODEs) system:

$$\begin{cases} \dfrac{\mathrm{d}S(t)}{\mathrm{d}t} = \Lambda - \Big(\sum\limits_{i=1}^{3} \beta_i I_i(t) + \sum\limits_{j=1}^{2} \alpha_j T_j(t)\Big) S(t) - \mu_1 S(t), \\ \dfrac{\mathrm{d}I_1(t)}{\mathrm{d}t} = \Big(\sum\limits_{i=1}^{3} \beta_i I_i(t) + \sum\limits_{j=1}^{2} \alpha_j T_j(t)\Big) S(t) - (\mu_2 + \delta_1 + \delta_2 + \delta_3) I_1(t), \\ \dfrac{\mathrm{d}T_1(t)}{\mathrm{d}t} = \delta_1 I_1(t) + \omega T_2(t) - (\mu_3 + \gamma_1)\, T_1(t), \\ \dfrac{\mathrm{d}I_2(t)}{\mathrm{d}t} = \delta_2 I_1(t) + \gamma_1 T_1(t) - (\mu_4 + \rho_1 + \rho_2)\, I_2(t), \\ \dfrac{\mathrm{d}T_2(t)}{\mathrm{d}t} = \rho_1 I_2(t) - (\mu_5 + \gamma_2 + \omega)\, T_2(t), \\ \dfrac{\mathrm{d}I_3(t)}{\mathrm{d}t} = \delta_3 I_1(t) + \rho_2 I_2(t) + \gamma_2 T_2(t) - (\mu_6 + \eta) I_3(t), \\ \dfrac{\mathrm{d}T_3(t)}{\mathrm{d}t} = \eta I_3(t) - \mu_7 T_3(t), \end{cases} \tag{IV.1}$$

with initial conditions $S(0) > 0, I_1(0) > 0, T_1(0) > 0, I_2(0) > 0, T_2(0) > 0, I_3(0) > 0$ and $T_3(0) > 0$. The seven variables of the system above are none other than the densities of the corresponding compartments at time t, whereas the parameters appearing therein are outlined in the following list:

- Λ is the susceptible recruitment rate.
- β_1, β_2, β_3, α_1 and α_2 denote, in this order, the contact rates between $S(t)$ and $I_1(t), I_2(t), I_3(t)$ $T_1(t), T_2(t)$.
- $\mu_1, \mu_2, \mu_3, \mu_4, \mu_5, \mu_6$ and μ_7 are separately, the death rates of the classes $S(t), I_1(t), I_2(t), I_3(t), T_1(t)$ $T_2(t)$ and $T_3(t)$.
- δ_1, δ_2 and δ_3 are respectively the rates at which the individuals of the class $I_1(t)$ develop into $T_1(t)$, $I_2(t)$ and $T_3(t)$. For more convenience in notation, let $\delta := \delta_1 + \delta_2 + \delta_3$.
- ρ_1 is the progression rate from $I_2(t)$ to $T_2(t)$ and ρ_2 is that from $I_2(t)$ to $I_3(t)$.
- γ_1 and γ_2 reflect the transmission rates from $T_1(t)$ to $I_2(t)$ and from $T_2(t)$ to $I_3(t)$ respectively.
- η represents the treatment rate of the AIDS patients, and ω is the effective treatment rate of the group $T_2(t)$.

All the above parameters are assumed to be in the positive real axis $\mathbb{R}_+ := \{x \in \mathbb{R} \mid x > 0\}$. On account of the mathematical results presented in [159], the spread behavior of the aforesaid AIDS/HIV model is entirely determined by the basic reproduction number $\mathfrak{R}_0$ which is expressed in this case by

$$\mathfrak{R}_0 = \frac{\Lambda\,(\beta_1 + \vartheta_1\beta_2 + \vartheta_2\beta_3 + \vartheta_3\alpha_1 + \vartheta_4\alpha_2)}{\mu_1(\mu_2 + \delta_1 + \delta_2 + \delta_3)}, \tag{IV.2}$$

where

$$\vartheta_4 = \frac{\rho_1\big((\mu_3+\gamma_1)\delta_2+\gamma_1\delta_1\big)}{(\mu_3+\gamma_1)(\mu_4+\rho_1+\rho_2)(\mu_5+\gamma_2+\omega)-\gamma_1\omega\rho_1}, \qquad \vartheta_3 = \frac{(\mu_4+\rho_1+\rho_2)(\mu_5+\gamma_2+\omega)\delta_1+\delta_2\omega\rho_1}{(\mu_3+\gamma_1)(\mu_4+\rho_1+\rho_2)(\mu_5+\gamma_2+\omega)-\gamma_1\omega\rho_1},$$
$$\vartheta_2 = \frac{\delta_3+\rho_2\vartheta_1+\gamma_2\vartheta_4}{(\mu_6+\eta)}, \qquad \vartheta_1 = \frac{(\mu_5+\gamma_2+\omega)\vartheta_4}{\rho_1}.$$

More explicitly, when $\mathfrak{R}_0 \leqslant 1$, the system (IV.1) has a unique disease-free equilibrium $E_o = \left(\frac{\Lambda}{\mu_1}, 0, 0, 0, 0, 0\right)$, and it is globally asymptotically stable in the invariant and attracting domain Π given by

$$\Pi = \left\{(S, I_1, T_1, I_2, T_2, I_3, T_3) \in \mathbb{R}_+^7 \mid S + \sum_{i=1}^{3}\Big(I_i + T_i\Big) \leqslant \frac{\Lambda}{\min\{\mu_k \mid k \in \{1, \cdots, 7\}\}}\right\}.$$

On the contrary, if $\mathfrak{R}_0 > 1$, the disease-free equilibrium stills present, but becomes unstable this time. In such case, the system (IV.1) admits another equilibrium $E^\star = \left(S^\star, I_1^\star, T_1^\star, I_2^\star, T_2^\star, I_3^\star, T_3^\star\right)$ called the endemic equilibrium, and it is globally asymptotically stable in Π (see [159]). For ease of reading the rest of this chapter, we recap the transmission mechanisms of the aforementioned AIDS/HIV model by the flow diagram shown in Figure IV.1.

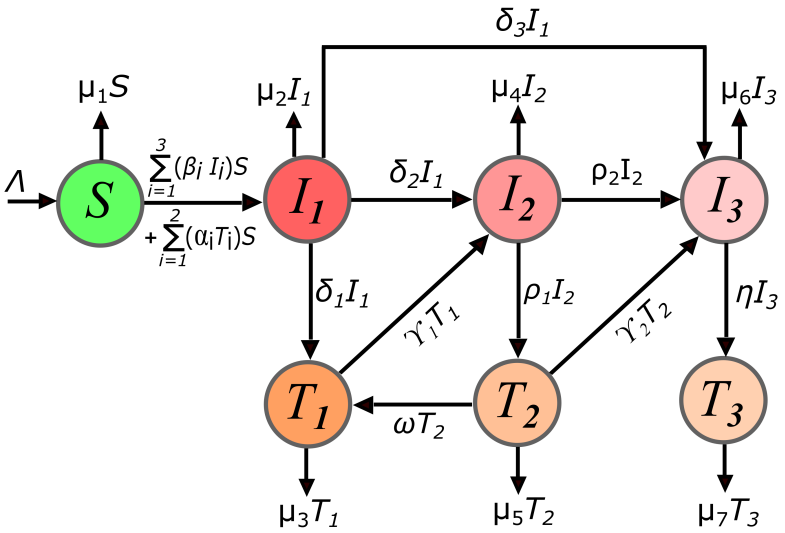


Figure IV.1: Flow diagram of the AIDS compartmental model (IV.1).

As far as we know, taking environmental disturbances into account brings an additional degree of realism to the spread analysis of an epidemic, and makes it interesting but more complicated, to predict its behavior [181]. In such cases, deterministic systems, while able to make very instructive predictions and forecasts, are not adequate enough [39]. Hence, there is a pressing need for a developed and sophisticated mathematical model that takes into consideration the randomness effect, especially in the context of studying the prevalence of a highly harmful infectious disease like AIDS. In this vein, a large number of authors have suggested and evolved many stochastic models that describe the AIDS infection dynamics from various angles and prospects [182, 183, 184, 185, 186, 187, 126, 188, 189]. For instance, in [182] the authors demonstrated the existence of a unique ergodic stationary distribution for an HIV system with a nonlinear infection function. Liu et al. [183] have proposed and studied a stochastic delayed HIV infection model with general incidence rate. They first proved the well-posedness of their model, then and by constructing some appropriate Lyapunov functions, they exhibited the asymptotic comportment of the solutions around the infection-free and endemic equilibria. More precisely, they showed that if the basic reproduction number associated to their model is less than or equal to one, then its unique solution fluctuates around the infection-free equilibrium, while it oscillates around the endemic equilibrium if it is not the case. In [184], Liu and Jiang explored the transmission of HIV through a stochastic

multigroup S-DI-A epidemic model that incorporates a generic incidence rate. They have offered sufficient conditions for the existence and uniqueness of an ergodic stationary distribution to their HIV system by constructing a suitable Lyapunov function. Then, they supported their results with many simulations which include some usual types of incidence rates like Saturated, Holling type IV and Beddington-DeAngelis ones. In all these works mentioned previously, the passage from the deterministic AIDS formulation to the stochastic one is done by supposing that the solution of this first wiggles normally around its value, which is often expressed by perturbing some system parameters with white noises. The addition of these latter is considered as one of the most logical and eminent ways to model any phenomenon in which a given quantity is subjected to slight and continuous fluctuations. For example, in the framework of infectious diseases pervasion, the environmental uncertainties are oftentimes represented in such a manner [148, 153, 190, 191, 192]. Unluckily, this approach is not suitable enough for modeling the impact of massive and sudden disturbances like floods, hurricanes, climate changes, earthquakes, etc [193, 150, 35, 194]. For this reason, we resort to the Lévy processes which are renowned for their ability to correctly formulate this type of case. So, we can expand the model (IV.1) to the system of stochastic differential equations with Lévy jumps (SDE-LJ for short) below:

$$
\begin{cases}
\mathrm{d}S(t) = \Big(\Lambda - \Big(\sum\limits_{i=1}^{3}\beta_i I_i(t) + \sum\limits_{j=1}^{2}\alpha_j T_j(t)\Big)S(t) - \mu_1 S(t)\Big)\mathrm{d}t + \sigma_1 S(t)\mathrm{d}\mathcal{B}_1(t) + \int_{\mathbb{U}}\zeta_1(u)S(t^-)\widetilde{\mathcal{N}}(\mathrm{d}t,\mathrm{d}u),\\
\mathrm{d}I_1(t) = \Big(\Big(\sum\limits_{i=1}^{3}\beta_i I_i(t) + \sum\limits_{j=1}^{2}\alpha_j T_j(t)\Big)S(t) - (\mu_2+\delta)I_1(t)\Big)\mathrm{d}t + \sigma_2 I_1(t)\mathrm{d}\mathcal{B}_2(t) + \int_{\mathbb{U}}\zeta_2(u)I_1(t^-)\widetilde{\mathcal{N}}(\mathrm{d}t,\mathrm{d}u),\\
\mathrm{d}T_1(t) = \Big(\delta_1 I_1(t) + \omega T_2(t) - (\mu_3+\gamma_1)\,T_1(t)\Big)\mathrm{d}t + \sigma_3 T_1(t)\mathrm{d}\mathcal{B}_3(t) + \int_{\mathbb{U}}\zeta_3(u)T_1(t^-)\widetilde{\mathcal{N}}(\mathrm{d}t,\mathrm{d}u),\\
\mathrm{d}I_2(t) = \Big(\delta_2 I_1(t) + \gamma_1 T_1(t) - (\mu_4+\rho_1+\rho_2)\,I_2(t)\Big)\mathrm{d}t + \sigma_4 I_2(t)\mathrm{d}\mathcal{B}_4(t) + \int_{\mathbb{U}}\zeta_4(u)I_2(t^-)\widetilde{\mathcal{N}}(\mathrm{d}t,\mathrm{d}u),\\
\mathrm{d}T_2(t) = \Big(\rho_1 I_2(t) - (\mu_5+\gamma_2+\omega)\,T_2(t)\Big)\mathrm{d}t + \sigma_5 T_2(t)\mathrm{d}\mathcal{B}_5(t) + \int_{\mathbb{U}}\zeta_5(u)T_2(t^-)\widetilde{\mathcal{N}}(\mathrm{d}t,\mathrm{d}u),\\
\mathrm{d}I_3(t) = \Big(\delta_3 I_1(t) + \rho_2 I_2(t) + \gamma_2 T_2(t) - (\mu_6+\eta)I_3(t)\Big)\mathrm{d}t + \sigma_6 I_3(t)\mathrm{d}\mathcal{B}_6(t) + \int_{\mathbb{U}}\zeta_6(u)I_3(t^-)\widetilde{\mathcal{N}}(\mathrm{d}t,\mathrm{d}u),\\
\mathrm{d}T_3(t) = \Big(\eta I_3(t) - \mu_7 T_3(t)\Big)\mathrm{d}t + \sigma_7 T_3(t)\mathrm{d}\mathcal{B}_7(t) + \int_{\mathbb{U}}\zeta_7(u)T_3(t^-)\widetilde{\mathcal{N}}(\mathrm{d}t,\mathrm{d}u).
\end{cases}
\tag{IV.3}
$$

Here, $\mathcal{B}_i$ $(i = 1, 2, \cdots, 7)$, are mutually independent Brownian motions defined on a filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geqslant 0}, \mathbb{P})$ endowed with a filtration that satisfies the usual conditions (it is right-continuous and $\mathcal{F}_0$ contains all $\mathbb{P}$-null sets). Unless otherwise stated, all random variables or stochastic processes that we will meet in this chapter, are defined on $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geqslant 0}, \mathbb{P})$. The positive constants $\sigma_i > 0$ $(i = 1, 2, \cdots, 7)$ are the intensities of the environmental random disturbance. $S(t^-), I_1(t^-), I_2(t^-), I_3(t^-), T_1(t^-), T_2(t^-)$ and $T_3(t^-)$ are denoting the left limits of $S(t), I_1(t), I_2(t), I_3(t), T_1(t)$, $T_2(t)$ and $T_3(t)$ respectively. $\mathcal{N}$ is a Poisson counting measure independent of $\mathcal{B}_i$ $(i = 1, 2, \cdots, 7)$ with a finite characteristic measure ν defined on a measurable set $\mathbb{U} \subset \mathbb{R}_+$. $\widetilde{\mathcal{N}}$ is a compensating martingale, and it is assumed that ν is a Lévy measure such that $\widetilde{\mathcal{N}}(\mathrm{d}t, \mathrm{d}u) = \mathcal{N}(\mathrm{d}t, \mathrm{d}u) - \nu(\mathrm{d}u)\mathrm{d}t$. Lastly, $\zeta_i : \mathbb{U} \to \mathbb{R}$ are the jumps magnitude functions which are supposed to be bounded and continuous on $\mathbb{U}$. Obviously, in the last system, the AIDS treated population $T_3(t)$ has no effect on the dynamics of the other compartments. So, the dynamics of the AIDS propagation progression

can be obtained just from the following simplified model:

$$\begin{cases} \mathrm{d}S(t) = \Big(\Lambda - \Big(\sum\limits_{i=1}^{3}\beta_i I_i(t) + \sum\limits_{j=1}^{2}\alpha_j T_j(t)\Big)S(t) - \mu_1 S(t)\Big)\mathrm{d}t + \sigma_1 S(t)\mathrm{d}\mathcal{B}_1(t) + \int_{\mathbb{U}}\zeta_1(u)S(t^-)\widetilde{\mathcal{N}}(\mathrm{d}t,\mathrm{d}u), \\ \mathrm{d}I_1(t) = \Big(\Big(\sum\limits_{i=1}^{3}\beta_i I_i(t) + \sum\limits_{j=1}^{2}\alpha_j T_j(t)\Big)S(t) - (\mu_2+\delta)I_1(t)\Big)\mathrm{d}t + \sigma_2 I_1(t)\mathrm{d}\mathcal{B}_2(t) + \int_{\mathbb{U}}\zeta_2(u)I_1(t^-)\widetilde{\mathcal{N}}(\mathrm{d}t,\mathrm{d}u), \\ \mathrm{d}T_1(t) = \Big(\delta_1 I_1(t) + \omega T_2(t) - (\mu_3+\gamma_1)\,T_1(t)\Big)\mathrm{d}t + \sigma_3 T_1(t)\mathrm{d}\mathcal{B}_3(t) + \int_{\mathbb{U}}\zeta_3(u)T_1(t^-)\widetilde{\mathcal{N}}(\mathrm{d}t,\mathrm{d}u), \\ \mathrm{d}I_2(t) = \Big(\delta_2 I_1(t) + \gamma_1 T_1(t) - (\mu_4+\rho_1+\rho_2)\,I_2(t)\Big)\mathrm{d}t + \sigma_4 I_2(t)\mathrm{d}\mathcal{B}_4(t) + \int_{\mathbb{U}}\zeta_4(u)I_2(t^-)\widetilde{\mathcal{N}}(\mathrm{d}t,\mathrm{d}u), \\ \mathrm{d}T_2(t) = \Big(\rho_1 I_2(t) - (\mu_5+\gamma_2+\omega)\,T_2(t)\Big)\mathrm{d}t + \sigma_5 T_2(t)\mathrm{d}\mathcal{B}_5(t) + \int_{\mathbb{U}}\zeta_5(u)T_2(t^-)\widetilde{\mathcal{N}}(\mathrm{d}t,\mathrm{d}u), \\ \mathrm{d}I_3(t) = \Big(\delta_3 I_1(t) + \rho_2 I_2(t) + \gamma_2 T_2(t) - (\mu_6+\eta)I_3(t)\Big)\mathrm{d}t + \sigma_6 I_3(t)\mathrm{d}\mathcal{B}_6(t) + \int_{\mathbb{U}}\zeta_6(u)I_3(t^-)\widetilde{\mathcal{N}}(\mathrm{d}t,\mathrm{d}u). \end{cases} \tag{IV.4}$$

Our method of modeling the complexity of fluctuations is based mainly on the addition of Lévy jumps, which is completely different from the vision adopted by Han et al. [159]. Depending on their viewpoint, the intricacy of the random disturbances is expressed by considering the white noises in a second-order form. More explicitly, they presumed that the disturbance results of the natural mortality rates μ_k $(k = 1,2,3,4,5,6)$ are given by

$$\begin{aligned} &\mu_1 \to \mu_1 - (\sigma_{11}S(t) + \sigma_{12})\dot{\mathcal{B}}_1(t), && \mu_2 \to \mu_2 - (\sigma_{21}I_1(t) + \sigma_{22})\dot{\mathcal{B}}_2(t), \\ &\mu_3 \to \mu_3 - (\sigma_{31}T_1(t) + \sigma_{32})\dot{\mathcal{B}}_3(t), && \mu_4 \to \mu_4 - (\sigma_{41}I_2(t) + \sigma_{42})\dot{\mathcal{B}}_4(t), \\ &\mu_5 \to \mu_5 - (\sigma_{51}T_2(t) + \sigma_{52})\dot{\mathcal{B}}_5(t), && \mu_6 \to \mu_6 - (\sigma_{61}I_3(t) + \sigma_{62})\dot{\mathcal{B}}_6(t). \end{aligned}$$

Despite this difference in the way of disturbing the system (IV.1), our proposed model coincides with that treated by Han et al. in the special case of the absence of pure jumps $\zeta_k = 0$ $(k = 1,2,3,4,5,6)$ and of higher-order perturbation intensities $\sigma_{k2} = 0)$ $(k = 1,2,3,4,5,6)$, which leads us to the following particular and common model:

$$\begin{cases} \mathrm{d}S(t) = \Big(\Lambda - \Big(\sum\limits_{i=1}^{3}\beta_i I_i(t) + \sum\limits_{j=1}^{2}\alpha_j T_j(t)\Big)S(t) - \mu_1 S(t)\Big)\mathrm{d}t + \sigma_1 S(t)\mathrm{d}\mathcal{B}_1(t), \\ \mathrm{d}I_1(t) = \Big(\Big(\sum\limits_{i=1}^{3}\beta_i I_i(t) + \sum\limits_{j=1}^{2}\alpha_j T_j(t)\Big)S(t) - (\mu_2+\delta_1+\delta_2+\delta_3)I_1(t)\Big)\mathrm{d}t + \sigma_2 I_1(t)\mathrm{d}\mathcal{B}_2(t), \\ \mathrm{d}T_1(t) = \Big(\delta_1 I_1(t) + \omega T_2(t) - (\mu_3+\gamma_1)\,T_1(t)\Big)\mathrm{d}t + \sigma_3 T_1(t)\mathrm{d}\mathcal{B}_3(t), \\ \mathrm{d}I_2(t) = \Big(\delta_2 I_1(t) + \gamma_1 T_1(t) - (\mu_4+\rho_1+\rho_2)\,I_2(t)\Big)\mathrm{d}t + \sigma_4 I_2(t)\mathrm{d}\mathcal{B}_4(t), \\ \mathrm{d}T_2(t) = \Big(\rho_1 I_2(t) - (\mu_5+\gamma_2+\omega)\,T_2(t)\Big)\mathrm{d}t + \sigma_5 T_2(t)\mathrm{d}\mathcal{B}_5(t), \\ \mathrm{d}I_3(t) = \Big(\delta_3 I_1(t) + \rho_2 I_2(t) + \gamma_2 T_2(t) - (\mu_6+\eta)I_3(t)\Big)\mathrm{d}t + \sigma_6 I_3(t)\mathrm{d}\mathcal{B}_6(t), \end{cases} \tag{IV.5}$$

Although we intersect in this case, our analysis provides sharp conditions especially in the case of extinction, and hence more ameliorated results are obtained. In other words, we develop the extinction criteria even in the said particular case. Regarding the present chapter, our principal intention is to find sufficient conditions for extinction and persistence in the mean of the AIDS model (IV.5). These two asymptotic properties are regarded to be sufficient for having an excellent view of the AIDS future pandemic situation. The novelty of our results resides principally in

the approach that we use to estimate the limit value of the temporal averages $\frac{\int_0^t \Phi(s)\,\mathrm{d}s}{t}$ and $\frac{\int_0^t \Phi^2(s)\,\mathrm{d}s}{t}$, where Φ is the positive solution of the following initial-value problem:

$$\begin{cases} \mathrm{d}\Phi(t) = [\Lambda - \mu_1\Phi(t)]\,\mathrm{d}t + \sigma_1\Phi(t)\mathrm{d}\mathcal{B}_1(t) + \displaystyle\int_{\mathbb{U}} \zeta_1(u)\Phi(t^-)\widetilde{\mathcal{N}}(\mathrm{d}t,\mathrm{d}u), \qquad \forall t > 0, \\ \Phi(0) = S(0) > 0. \end{cases} \tag{IV.6}$$

Our new approach allows us to span the gap[a] left by the use of standard methods presented for example in [152, 151, 146]. Also, we adopt some new analytic techniques to get a more precise threshold for the AIDS extinction. The analysis presented in this chapter sounds to be very promising to study other epidemic models, specially those which are perturbed with Lévy jumps.

The rest of this chapter is arranged as follows: in Section 2, we prove the existence and uniqueness of a global positive solution to the stochastic AIDS model (IV.5). In the sections 3 and 4 we respectively afford some sufficient criteria for the AIDS extinction and persistence in the mean. In Section 5, we reinforce our theoretical findings by some numerical simulations before deriving the main conclusions in Section 6.

2 Existence, uniqueness and positivity of the global-in-time solution

The first concern in analyzing the dynamical properties of a mathematical population system is to know whether it is well-posed or not, and we mean by well-posedness here that the system admits a unique, positive, and global-in-time solution. In this section, we will provide a suitable hypothetical framework under which the well-posedness of the AIDS model (IV.4) is guaranteed.

Theorem 2.1. *Let the following conditions be fulfilled:*

- $(\mathfrak{C}_1)$: *The jumps magnitudes* $(\zeta_i)_{1\leqslant i\leqslant 6}$ *verify* $\mathfrak{m}_1 := \max\limits_{1\leqslant i\leqslant 6}\left(\displaystyle\int_{\mathbb{U}} \zeta_i^2(u)\nu(\mathrm{d}u)\right) < \infty$.
- $(\mathfrak{C})_2$: *For any* $i \in \{1,\cdots,6\}$, $\zeta_i(u) > -1$ *and* $\mathfrak{m}_2 := \max\limits_{1\leqslant i\leqslant 6}\left(\displaystyle\int_{\mathbb{U}} \big(\zeta_i(u) - \ln(1+\zeta_i(u))\big)\nu(\mathrm{d}u)\right) < \infty$.

Then, for any initial value $\big(S(0), I_1(0), T_1(0), I_2(0), T_2(0), I_3(0)\big) \in \mathbb{R}_+^6$, *there is one and only one solution to the stochastic differential system* (IV.4) *on* $t \geqslant 0$. *Moreover, for all* $t \geqslant 0$, *this solution will stay in the positive cone* $\mathbb{R}_+^6$ *almost surely (a.s. for brevity).*

Proof. Let us suppose that the conditions $(\mathfrak{C}_6)$ and $(\mathfrak{C}_7)$ are satisfied. It is clear from $(\mathfrak{C}_6)$, that the system (IV.4) coefficients are locally Lipschitz continuous. So, and by virtue of [21, Theorem 1.19], we can assert that for any given initial value $\big(S(0), I_1(0), T_1(0), I_2(0), T_2(0), I_3(0)\big) \in \mathbb{R}_+^6$ there is a unique maximal local solution $\big(S(t), I_1(t), T_1(t), I_2(t), T_2(t), I_3(t)\big)$ of (IV.4). This latter is supposed to be defined on an interval $[0,\tau_e)$, where τ_e is its explosion time (see [19]). At this juncture, our aim will be to demonstrate that this solution is global in time, that is $\tau_e = \infty$ almost surely. To that end, let k_0 be a sufficiently large positive integer such that $S(0), I_1(0), T_1(0), I_2(0), T_2(0)$ and $I_3(0)$ lie within the interval $\left(\frac{1}{k_0}, k_0\right)$. For each integer $k \geqslant k_0$, let us consider the following quantity, which is well defined by the adoption of the convention $\inf\emptyset = \infty$:

$$\tau_k = \inf\left\{t \in [0,\tau_e) \mid \big(S(t), I_1(t), T_1(t), I_2(t), T_2(t), I_3(t)\big) \notin \left(\frac{1}{k}, k\right)^6\right\} \in \mathbb{R}_+ \cup \{\infty\}. \tag{IV.7}$$

[a] We will return to explain this gap in detail in Remark 3.2.

Obviously, $\left(\frac{1}{k},k\right)^6$ is an open domain of $\mathbb{R}^6$. Therefore, and by Theorem 3.1 of [19], τ_k is a stopping time for all $k \geqslant k_0$. Let us denote by τ_∞ the limit $\lim\limits_{k\to\infty}\tau_k$. Needless to say, the sequence $(\tau_k)_{k\geqslant k_0}$ is increasing; hence, $\lim\limits_{k\to\infty}\tau_k = \sup\limits_{k\geqslant k_0}\tau_k$, and by using Lemma 2.11 of [18] we deduce that $\tau_\infty = \sup\limits_{k\geqslant k_0}\tau_k$ is a stopping time. Plainly, $\tau_\infty \leqslant \tau_e$ (see [153] for more details), so we only need to show that $\tau_\infty = \infty$ in order to conclude that $\tau_e = \infty$. Suppose that $\tau_\infty = \infty$ a.s. is untrue, then there are definitely two positive constants ε and ϱ such that

$$\mathbb{P}\left(\tau_k \leqslant \varrho\right) > \varepsilon \quad \text{for all } k \geqslant k_0. \tag{IV.8}$$

Consider the C^2-function $\mathcal{V} : \mathbb{R}^6_+ \to [0,+\infty)$ defined by

$$\mathcal{V}\left(x_1,x_2,x_3,x_4,x_5,x_6\right) = \left(x_1 - \theta - \theta\ln\left(\frac{x_1}{\theta}\right)\right) + \sum_{i=2}^{6}\left(x_i - 1 - \ln(x_i)\right),$$

where θ is a positive real constant to be determined appropriately later. It is a simple matter to observe that this function is nonnegative because $x - 1 - \ln(x) \geqslant 0,\ \forall x > 0$. On account of the multi-dimensional Itô's formula [21], we have for all $k \geqslant k_0$ and $t \in [0,\tau_k)$

$$\begin{aligned}
&\mathrm{d}\mathcal{V}\big(S(t),I_1(t),T_1(t),I_2(t),T_2(t),I_3(t)\big)\\
&= \mathcal{L}\mathcal{V}\big(S(t),I_1(t),T_1(t),I_2(t),T_2(t),I_3(t)\big)\ \mathrm{d}t + \sigma_1\left(S(t)-\theta\right)\mathrm{d}\mathcal{B}_1(t) + \sigma_2\left(I_1(t)-1\right)\mathrm{d}\mathcal{B}_2(t)\\
&\quad + \sigma_3\left(T_1(t)-1\right)\mathrm{d}\mathcal{B}_3(t) + \sigma_4\left(I_2(t)-1\right)\mathrm{d}\mathcal{B}_4(t) + \sigma_5\left(T_2(t)-1\right)\mathrm{d}\mathcal{B}_5(t) + \sigma_6\left(I_3(t)-1\right)\mathrm{d}\mathcal{B}_6(t)\\
&\quad + \int_{\mathbb{U}}\Big(\zeta_1(u)S(t^-) - \theta\ln\left(1+\zeta_1(u)\right)\Big)\widetilde{\mathcal{N}}(\mathrm{d}t,\mathrm{d}u) + \int_{\mathbb{U}}\Big(\zeta_2(u)I_1(t^-) - \ln\left(1+\zeta_2(u)\right)\Big)\widetilde{\mathcal{N}}(\mathrm{d}t,\mathrm{d}u)\\
&\quad + \int_{\mathbb{U}}\Big(\zeta_3(u)T_1(t^-) - \ln\left(1+\zeta_3(u)\right)\Big)\widetilde{\mathcal{N}}(\mathrm{d}t,\mathrm{d}u) + \int_{\mathbb{U}}\Big(\zeta_4(u)I_2(t^-) - \ln\left(1+\zeta_4(u)\right)\Big)\widetilde{\mathcal{N}}(\mathrm{d}t,\mathrm{d}u)\\
&\quad + \int_{\mathbb{U}}\Big(\zeta_5(u)T_2(t^-) - \ln\left(1+\zeta_5(u)\right)\Big)\widetilde{\mathcal{N}}(\mathrm{d}t,\mathrm{d}u) + \int_{\mathbb{U}}\Big(\zeta_6(u)I_3(t^-) - \ln\left(1+\zeta_6(u)\right)\Big)\widetilde{\mathcal{N}}(\mathrm{d}t,\mathrm{d}u).
\end{aligned}$$

where $\mathcal{L}\mathcal{V}\big(S(t),I_1(t),T_1(t),I_2(t),T_2(t),I_3(t)\big)$ is given by

$$\begin{aligned}
&\mathcal{L}\mathcal{V}\big(S(t),I_1(t),T_1(t),I_2(t),T_2(t),I_3(t)\big)\\
&= \left(1-\frac{\theta}{S(t)}\right)\times\left[\Lambda - \Big(\sum_{i=1}^{3}\beta_i I_i(t) + \sum_{j=1}^{2}\alpha_j T_j(t)\Big)S(t) - \mu_1 S(t)\right]\\
&\quad + \left(1-\frac{1}{I_1(t)}\right)\times\left[\Big(\sum_{i=1}^{3}\beta_i I_i(t) + \sum_{j=1}^{2}\alpha_j T_j(t)\Big)S(t) - (\mu_2+\delta)I_1(t)\right]\\
&\quad + \left(1-\frac{1}{T_1(t)}\right)\times\left[\delta_1 I_1(t) + \omega T_2(t) - (\mu_3+\gamma_1)\,T_1(t)\right] + \left(1-\frac{1}{I_2(t)}\right)\times\left[\delta_2 I_1(t) + \gamma_1 T_1(t) - (\mu_4+\rho_1+\rho_2)\,I_2(t)\right]\\
&\quad + \left(1-\frac{1}{T_2(t)}\right)\times\left[\rho_1 I_2(t) - (\mu_5+\gamma_2+\omega)\,T_2(t)\right] + \left(1-\frac{1}{I_3(t)}\right)\times\left[\delta_3 I_1(t) + \rho_2 I_2(t) + \gamma_2 T_2(t) - (\mu_6+\eta)I_3(t)\right]\\
&\quad + \frac{1}{2}\left(\theta\sigma_1^2 + \sum_{i=2}^{6}\sigma_i^2\right) + \theta\int_{\mathbb{U}}\Big(\zeta_1(u) - \ln\left(1+\zeta_1(u)\right)\Big)\ \nu(\mathrm{d}u) + \sum_{i=2}^{6}\int_{\mathbb{U}}\Big(\zeta_i(u) - \ln\left(1+\zeta_i(u)\right)\Big)\ \nu(\mathrm{d}u).
\end{aligned}$$

Hence

$$\begin{aligned}
&\mathcal{LV}\big(S(t),I_1(t),T_1(t),I_2(t),T_2(t),I_3(t)\big)\\
&\leqslant \Big(\Lambda-\mu_1S(t)-\mu_2I_1(t)-\mu_3T_1(t)-\mu_4I_2(t)-\mu_5T_2(t)-\mu_6I_3(t)\Big)+\theta\Big(\sum_{i=1}^{3}\beta_iI_i(t)+\sum_{j=1}^{2}\alpha_jT_j(t)\Big)\\
&\quad+\theta\mu_1+(\mu_2+\delta)+(\mu_4+\rho_1+\rho_2)+(\mu_3+\gamma_1)+(\mu_5+\gamma_2+\omega)+(\mu_6+\eta)\\
&\quad+\frac{1}{2}\left(\theta\sigma_1^2+\sum_{i=2}^{6}\sigma_i^2\right)+\theta\int_{\mathbb{U}}\Big(\zeta_1(u)-\ln\left(1+\zeta_1(u)\right)\Big)\,\nu(\mathrm{d}u)+\sum_{i=2}^{6}\int_{\mathbb{U}}\Big(\zeta_i(u)-\ln\left(1+\zeta_i(u)\right)\Big)\,\nu(\mathrm{d}u)\\
&\leqslant K-(\mu_2-\theta\beta_1)\,I_1(t)-(\mu_3-\theta\alpha_1)\,T_1(t)-(\mu_4-\theta\beta_2)\,I_2(t)-(\mu_5-\theta\alpha_2)\,T_2(t)-(\mu_6-\theta\beta_3)\,I_3(t).
\end{aligned}$$

Here K is standing for the following positive constant:

$$\begin{aligned}
K=\Lambda+\theta\mu_1+(\mu_2+\delta)+(\mu_4+\rho_1+\rho_2)+(\mu_3+\gamma_1)+(\mu_5+\gamma_2+\omega)+(\mu_6+\eta)+\frac{1}{2}\left(\theta\sigma_1^2+\sum_{i=2}^{6}\sigma_i^2\right)\\
+\theta\int_{\mathbb{U}}\Big(\zeta_1(u)-\ln\left(1+\zeta_1(u)\right)\Big)\,\nu(\mathrm{d}u)+\sum_{i=2}^{6}\int_{\mathbb{U}}\Big(\zeta_i(u)-\ln\left(1+\zeta_i(u)\right)\Big)\,\nu(\mathrm{d}u).
\end{aligned}$$

By taking $\theta=\min\left\{\frac{\mu_2}{\beta_1},\frac{\mu_3}{\alpha_1},\frac{\mu_4}{\beta_2},\frac{\mu_5}{\alpha_2},\frac{\mu_6}{\beta_3}\right\}$, we get

$$\mathcal{LV}\big(S(t),I_1(t),T_1(t),I_2(t),T_2(t),I_3(t)\big)\leqslant K.$$

The remaining part of the proof is analogous to that of Theorem 1 in [193], and therefore it is skipped here. ■

Remark 2.1. *Besides its interesting mathematical interpretation, the importance of the last theorem lies basically in its ability to make the AIDS model (IV.4) rational, and biologically meaningful.*

3 Extinction of AIDS.

When analyzing a mathematical model describing the prevalence of a particular infectious disease, our primary interest after establishing its well-posedness is to know whether the disease will die out or continue to persist. In this section, we will try to find some conditions for the AIDS model (IV.4) extinction. These latter are expressed in terms of white noises intensities, Lévy jumps magnitudes, and system parameters of course. For the sake of brevity, we adopt the following conventions from here on:

- $\overline{\sigma}:=\max\left\{\sigma_i^2\mid i\in\{1,2,3,4,5,6\}\right\}$.
- $\overline{\zeta}(u):=\max\left\{\zeta_i(u)\mid i\in\{1,2,3,4,5,6\}\right\}$.
- $\overline{\lambda}_q(u):=\left(1+\overline{\zeta}(u)\right)^q-q\times\overline{\zeta}(u)-1$.
- $\mathcal{I}_q=\displaystyle\int_{\mathbb{U}}\max\left\{\overline{\lambda}_q(u),\underline{\lambda}_q(u)\right\}\nu(\mathrm{d}u)$.
- $\underline{\mu}=\min\left\{\mu_i\mid i\in\{1,2,3,4,5,6\}\right\}$.
- $\underline{\zeta}(u):=\min\left\{\zeta_i(u)\mid i\in\{1,2,3,4,5,6\}\right\}$.
- $\underline{\lambda}_q(u):=\left(1+\underline{\zeta}(u)\right)^q-q\times\underline{\zeta}(u)-1$.
- $\varphi_q:=\underline{\mu}-\dfrac{(q-1)\overline{\sigma}}{2}-\dfrac{\mathcal{I}_q}{q}$.

Before going the main results of this section, we first give the following useful lemmas which will be involved in the sequel:

Lemma 3.1. *Let $\big(S(0),I_1(0),T_1(0),I_2(0),T_2(0),I_3(0)\big)\in\mathbb{R}_+^6$ be a given initial value, and denote by $\big(S(t),I_1(t),T_1(t),I_2(t),T_2(t),I_3(t)\big)$ the solution of (IV.4) that starts from it. If the following assumptions hold:*

- $(\mathfrak{C}_3)$: *There is some real constant* $q > 2$ *such that* $\varphi_q := \underline{\mu} - \dfrac{(q-1)\overline{\sigma}}{2} - \dfrac{\mathcal{I}_q}{q} > 0.$
- $(\mathfrak{C}_4)$: $\mathfrak{m}_3 := \max\left\{\left(\displaystyle\int_{\mathbb{U}}\left(\left(1+\zeta_i(u)\right)^2-1\right)^2 \nu(du)\right) \mid i \in \{1,2,3,4,5,6\}\right\} < \infty.$

Then

(i) $\displaystyle\lim_{t\to\infty}\frac{\Phi(t)}{t}=0,\ \lim_{t\to\infty}\frac{\Phi^2(t)}{t}=0,\ \lim_{t\to\infty}\frac{\int_0^t \Phi(s)\,\mathrm{d}\mathcal{B}_1(s)}{t}=0,\ \lim_{t\to\infty}\frac{\int_0^t \Phi^2(s)\,\mathrm{d}\mathcal{B}_1(s)}{t}=0,$
$\displaystyle\lim_{t\to\infty}\frac{\int_0^t\int_{\mathbb{U}}\zeta_1(u)\Phi(s^-)\widetilde{\mathcal{N}}(\mathrm{d}s,\mathrm{d}u)}{t}=0,$ *and* $\displaystyle\lim_{t\to\infty}\frac{\int_0^t\int_{\mathbb{U}}\left((1+\zeta_1(u))^2-1\right)\Phi^2(s^-)\widetilde{\mathcal{N}}(\mathrm{d}s,\mathrm{d}u)}{t}=0$ *a.s.*

(ii) $\displaystyle\lim_{t\to\infty}\frac{S(t)}{t}=0,\ \lim_{t\to\infty}\frac{\int_0^t S(s)\,\mathrm{d}\mathcal{B}_1(s)}{t}=0,$ *and* $\displaystyle\lim_{t\to\infty}\frac{\int_0^t\int_{\mathbb{U}}\zeta_1(u)S(s^-)\widetilde{\mathcal{N}}(\mathrm{d}s,\mathrm{d}u)}{t}=0,$ *a.s.*

(iii) $\displaystyle\lim_{t\to\infty}\frac{I_i(t)}{t}=0,\ \lim_{t\to\infty}\frac{\int_0^t I_i(s)\,\mathrm{d}\mathcal{B}_{2i}(s)}{t}=0,$ *and* $\displaystyle\lim_{t\to\infty}\frac{\int_0^t\int_{\mathbb{U}}\zeta_{2i}(u)I_i(s^-)\widetilde{\mathcal{N}}(\mathrm{d}s,\mathrm{d}u)}{t}=0,$ *a.s.* $\forall i\in\{1,2\}.$

(iv) $\displaystyle\lim_{t\to\infty}\frac{T_i(t)}{t}=0,\ \lim_{t\to\infty}\frac{\int_0^t T_i(s)\,\mathrm{d}\mathcal{B}_{2i+1}(s)}{t}=0,\ \lim_{t\to\infty}\frac{\int_0^t\int_{\mathbb{U}}\zeta_{2i+1}(u)T_i(s^-)\widetilde{\mathcal{N}}(\mathrm{d}s,\mathrm{d}u)}{t}=0,$ *a.s.* $\forall i\in\{1,2,3\}.$

Proof. The proof is very akin to that of Lemma 2.5 in [150]. Hereby, we omit it here. ■

Remark 3.1. *By using the well-known comparison theorem in its stochastic version (see [155]), we can guarantee, and for almost all* $w\in\Omega$, *that*

$$S(t,w)\leqslant\Phi(t,w),\quad\forall t>0. \tag{IV.9}$$

Lemma 3.2. *Let* $\Phi(t)$ *be the solution of* (IV.6) *that starts from a positive initial value* $\Phi(0)$. *Assume that the conditions* $(\mathfrak{C}_3)$ *and* $(\mathfrak{C}_4)$ *are satisfied. Then,*

(i) $\phi := 2\mu_1-\sigma_1^2-\displaystyle\int_{\mathbb{U}}\zeta_1^2(u)\nu(\mathrm{d}u)>0$ *a.s.*

(ii) $\displaystyle\lim_{t\to\infty}\frac{\int_0^t\Phi(s)\,\mathrm{d}s}{t}=\frac{\Lambda}{\mu_1}$ *a.s.*

(iii) $\displaystyle\lim_{t\to\infty}\frac{\int_0^t\Phi^2(s)\,\mathrm{d}s}{t}=\frac{2\Lambda^2}{\mu_1\phi}$ *a.s.*

Proof. Let $\Phi(0)>0$ be a given initial value, by integrating (IV.6) from 0 to t on both sides, and then dividing by t, we obtain

$$\frac{\Phi(t)-\Phi(0)}{t}=\Lambda-\frac{\mu_1\int_0^t\Phi(s)\,\mathrm{d}s}{t}+\frac{\sigma_1\int_0^t\Phi(s)\,\mathrm{d}\mathcal{B}_1(s)}{t}+\frac{\int_0^t\int_{\mathbb{U}}\zeta_1(u)\Phi(s^-)\widetilde{\mathcal{N}}(\mathrm{d}s,\mathrm{d}u)}{t}.$$

Hence

$$\frac{\int_0^t\Phi(s)\,\mathrm{d}s}{t}=\frac{\Lambda}{\mu_1}-\frac{\Phi(t)-\Phi(0)}{\mu_1 t}+\frac{\sigma_1\int_0^t\Phi(s)\,\mathrm{d}\mathcal{B}_1(s)}{\mu_1 t}+\frac{\int_0^t\int_{\mathbb{U}}\zeta_1(u)\Phi(s^-)\,\widetilde{\mathcal{N}}(\mathrm{d}s,\mathrm{d}u)}{\mu_1 t}.$$

Letting t tend to infinity in the last equality, and then using Lemma 3.1 implies

$$\lim_{t\to\infty}\frac{\int_0^t\Phi(s)\,\mathrm{d}s}{t}=\frac{\Lambda}{\mu_1}\quad a.s. \tag{IV.10}$$

Hence, the second claim of the lemma is shown, we turn now to the proof of the first and the third ones. By a direct application of the generalized Itô's formula [21] to Φ^2 we get

$$\begin{aligned} \mathrm{d}\Phi^2(t) = &\Big(2\Phi(t)\times\big(\Lambda-\mu_1\Phi(t)\big)+\sigma_1^2\Phi^2(t)+\int_{\mathbb{U}}\Phi^2(t)\Big((1+\zeta_1(u))^2-1-2\times\zeta_1(u)\Big)\nu(\mathrm{d}u)\Big)\mathrm{d}t \\ &+2\sigma_1\Phi^2(t)\,\mathrm{d}\mathcal{B}_1(t)+\int_{\mathbb{U}}\Phi^2(t^-)\Big((1+\zeta_1(u))^2-1\Big)\widetilde{\mathcal{N}}(\mathrm{d}t,\mathrm{d}u). \end{aligned} \tag{IV.11}$$

Integrating both sides of (IV.11) from 0 to t and then dividing by t yields

$$\begin{aligned} \frac{\Phi^2(t)-\Phi^2(0)}{t} = &\frac{2\Lambda\int_0^t\Phi(s)\mathrm{d}s}{t}-\Big(2\mu_1-\sigma_1^2-\int_{\mathbb{U}}\zeta_1^2(u)\nu(\mathrm{d}u)\Big)\times\frac{\int_0^t\Phi^2(s)\mathrm{d}s}{t} \\ &+\frac{2\sigma_1\int_0^t\Phi^2(s)\mathrm{d}\mathcal{B}_1(s)}{t}+\frac{\int_0^t\int_{\mathbb{U}}\Phi^2(s^-)\Big((1+\zeta_1(u))^2-1\Big)\widetilde{\mathcal{N}}(\mathrm{d}s,\mathrm{d}u)}{t}. \end{aligned}$$

Consequently,

$$\frac{\phi\int_0^t\Phi^2(s)\mathrm{d}s}{t}=\frac{\Phi^2(0)-\Phi^2(t)}{t}+\frac{2\Lambda\int_0^t\Phi(s)\mathrm{d}s}{t}+\frac{2\sigma_1\int_0^t\Phi^2(s)\mathrm{d}\mathcal{B}_1(s)}{t}+\frac{\int_0^t\int_{\mathbb{U}}\Phi^2(s^-)\Big((1+\zeta_1(u))^2-1\Big)\widetilde{\mathcal{N}}(\mathrm{d}s,\mathrm{d}u)}{t}. \tag{IV.12}$$

It is easily seen that the constant $\phi := 2\mu_1-\sigma_1^2-\int_{\mathbb{U}}\zeta_1^2(u)\nu(\mathrm{d}u)$ is not zero, because if it is not the case, we will find the contradictory equality $\frac{2\Lambda^2}{\mu_1}=0$ after letting t go to infinity. Therefore, one can divide (IV.12) by ϕ and conclude that

$$\begin{aligned} \frac{\int_0^t\Phi^2(s)\mathrm{d}s}{t} = &\frac{\Phi^2(0)-\Phi^2(t)}{\phi t}+\frac{2\Lambda\int_0^t\Phi(s)\mathrm{d}s}{\phi t}+\frac{2\sigma_1\int_0^t\Phi^2(s)\mathrm{d}\mathcal{B}_1(s)}{\phi t} \\ &+\frac{\int_0^t\int_{\mathbb{U}}\Phi^2(s^-)\Big((1+\zeta_1(u))^2-1\Big)\widetilde{\mathcal{N}}(\mathrm{d}s,\mathrm{d}u)}{\phi t}. \end{aligned}$$

Using Lemma 3.1 and (IV.10) leads to

$$\lim_{t\to\infty}\frac{\int_0^t\Phi^2(s)\,\mathrm{d}s}{t}=\frac{2\Lambda^2}{\mu_1\phi}\quad a.s.$$

and since $\dfrac{\int_0^t\Phi^2(s)\,\mathrm{d}s}{t}$ is positive, we can deduce that $\phi>0$, and the lemma follows. ■

Remark 3.2. *In the case of Lévy jumps, the precedent lemma can be considered as an alternative approach to overcome the inexistence of a specific formula for the stationary distribution of* (IV.6)*. This matter stills an open question until now, and we may see in the literature many works (for instance [152] and [151]) that effectuate the threshold analysis of their epidemiological systems with expressions involving an undetermined stationary distribution.*

Definition 3.1 (Simple extinction [153])**.** *The infectious individuals I_1, T_1, I_2, T_2 and I_3 that appear in system* (IV.4) *are called extinct if*

$$\lim_{t\to\infty}\left(\sum_{i=1}^{3}I_i(t)+\sum_{i=1}^{2}T_i(t)\right)=0\quad a.s.$$

Definition 3.2 (Exponential extinction [148])**.** *For the system* (IV.4)*, The total mass of infected individuals $I_{\text{total}}=(I_1+T_1+I_2+T_2+I_3)$ is said to be exponentially extinct if there is some negative constant c such that*

$$\limsup_{t\to\infty}\frac{1}{t}\ln\left(\sum_{i=1}^{3}I_i(t)+\sum_{i=1}^{2}T_i(t)\right)\leqslant c<0\quad a.s.$$

For simplicity and clarity in writing the next result, it will be practical to introduce these notations:

- For all $a \in \mathbb{R}$, $a^+ := \max\{0, a\} = \dfrac{|a| + a}{2}$.
- $\underset{\sim}{\zeta}(u) := \min\left\{\zeta_i(u) \mid i \in \{2,3,4,5,6\}\right\}$.
- $\underline{\mathfrak{h}}(u) := \left(\ln\left(1 + \underset{\sim}{\zeta}(u)\right) - \underset{\sim}{\zeta}(u)\right) \times \mathbb{1}_{\{\underset{\sim}{\zeta}(u) > 0\}}$.
- $\mathfrak{S} := \dfrac{1}{2}\left(\sum\limits_{i=2}^{6} \dfrac{1}{\sigma_i^2}\right)^{-1}$.
- $\widetilde{\zeta}(u) := \max\left\{\zeta_i(u) \mid i \in \{2,3,4,5,6\}\right\}$.
- $\overline{\mathfrak{h}}(u) := \left(\ln\left(1 + \widetilde{\zeta}(u)\right) - \widetilde{\zeta}(u)\right) \times \mathbb{1}_{\{\widetilde{\zeta}(u) \leqslant 0\}}$.
- $\mathcal{H} := \displaystyle\int_{\mathbb{U}} \overline{\mathfrak{h}}(u) + \underline{\mathfrak{h}}(u)\ \nu(\mathrm{d}u)$.
- $\mathfrak{c}_4 := \dfrac{\Lambda\beta_3}{\mu_1(\mu_6 + \eta)}$.
- $\mathfrak{c}_3 := \dfrac{\Lambda\big(\alpha_1(\mu_4 + \rho_1 + \rho_2)\omega + (\mu_3 + \gamma_1)(\mu_4 + \rho_1 + \rho_2)(\alpha_2 + \mathfrak{c}_4\rho_2\mu_1) + \gamma_1\omega(\beta_2 + \mathfrak{c}_4\rho_2\mu_1)\big)}{\mu_1\big((\mu_3 + \gamma_1)(\mu_4 + \rho_1 + \rho_2)(\mu_5 + \gamma_2 + \omega) - \gamma_1\omega\rho_1\big)}$.
- $\mathfrak{c}_2 := \dfrac{\Lambda\big(\alpha_1\rho_1\omega + (\mu_3 + \gamma_1)\rho_1(\alpha_2 + \mathfrak{c}_4\rho_2\mu_1) + (\mu_3 + \gamma_1)(\mu_5 + \gamma_2 + \omega)(\beta_2 + \mathfrak{c}_4\rho_2\mu_1)\big)}{\mu_1\big((\mu_3 + \gamma_1)(\mu_4 + \rho_1 + \rho_2)(\mu_5 + \gamma_2 + \omega) - \gamma_1\omega\rho_1\big)}$.
- $\mathfrak{c}_1 := \dfrac{\Lambda\big(\alpha_1(\mu_4 + \rho_1 + \rho_2)(\mu_5 + \gamma_2 + \omega) + \gamma_1\rho_1(\alpha_2 + \mathfrak{c}_4\rho_2\mu_1) + \gamma_1(\mu_5 + \gamma_2 + \omega)(\beta_2 + \mathfrak{c}_4\rho_2\mu_1)\big)}{\mu_1\big((\mu_3 + \gamma_1)(\mu_4 + \rho_1 + \rho_2)(\mu_5 + \gamma_2 + \omega) - \gamma_1\omega\rho_1\big)}$.
- $\overline{\chi} := \max\left\{\dfrac{\beta_1}{\mathfrak{R}_0}, \dfrac{\beta_2}{\mathfrak{c}_2}, \dfrac{\beta_3}{\mathfrak{c}_4}, \dfrac{\alpha_1}{\mathfrak{c}_1}, \dfrac{\alpha_2}{\mathfrak{c}_3}\right\}$.
- $\underline{\chi} := \min\left\{\dfrac{\beta_1}{\mathfrak{R}_0}, \dfrac{\beta_2}{\mathfrak{c}_2}, \dfrac{\beta_3}{\mathfrak{c}_4}, \dfrac{\alpha_1}{\mathfrak{c}_1}, \dfrac{\alpha_2}{\mathfrak{c}_3}\right\}$.
- $\mathcal{U} = \dfrac{\Lambda}{\mu_1}\left(\overline{\chi}\,(\mathfrak{R}_0 - 1)^+ - \underline{\chi}\,(1 - \mathfrak{R}_0)^+\right)$.
- $\mathcal{T} = \mathfrak{R}_0 \times \overline{\chi} \times \dfrac{\Lambda}{2\mu_1}\sqrt{\dfrac{2\mu_1}{\phi} - 1} + \mathcal{U} + \mathcal{H} - \mathfrak{S}$.

Theorem 3.1. *Let $\big(S(t), I_1(t), T_1(t), I_2(t), T_2(t), I_3(t)\big)$ indicate the solution of (IV.4) that begins from an initial state $\big(S(0), I_1(0), T_1(0), I_2(0), T_2(0), I_3(0)\big) \in \mathbb{R}_+^6$. If the conditions $(\mathfrak{C}_1) - (\mathfrak{C}_4)$ hold, and if, on top of that, we have*

- $(\mathfrak{C}_5)$: $\mathfrak{m}_4 := \max\left\{\displaystyle\int_{\mathbb{U}} \Big(\ln\big(1 + \zeta_i(u)\big)\Big)^2 \nu(\mathrm{d}u) \mid i \in \{1,2,3,4,5,6\}\right\} < \infty$.

Then

$$\limsup_{t\to\infty} \frac{1}{t} \ln\left(\mathfrak{R}_0 I_1(t) + \mathfrak{c}_1 T_1(t) + \mathfrak{c}_2 I_2(t) + \mathfrak{c}_3 T_2(t) + \mathfrak{c}_4 I_3(t)\right) \leqslant \mathcal{T} \quad a.s.$$

In particular, when $\mathcal{T} < 0$, the disease will disappear exponentially almost surely.

Proof. Suppose that the conditions of this theorem hold, and consider the C^2-function $\mathcal{W}$ defined for all $x = (x_1, x_2, x_3, x_4, x_5) \in \mathbb{R}_+^5$ by

$$\mathcal{W}(x_1, x_2, x_3, x_4, x_5) = \ln\big(\psi_0 x_1 + \psi_1 x_2 + \psi_2 x_3 + \psi_3 x_4 + \psi_4 x_5\big),$$

where $\psi_0, \psi_1, \psi_2, \psi_3$ and ψ_4 are five positive parameters to be chosen properly later. By applying the Itô's formula to $\mathcal{W}\big(I_1(t), T_1(t), I_2(t), T_2(t), I_3(t)\big)$ we get, and for all $t \geqslant 0$, that

$$\begin{aligned}
&\mathrm{d}\mathcal{W}\big(S(t), I_1(t), T_1(t), I_2(t), T_2(t), I_3(t)\big) \\
&= \mathcal{L}\mathcal{W}\big(I_1(t), T_1(t), I_2(t), T_2(t), I_3(t)\big)\mathrm{d}t + \frac{\sigma_2\psi_0 I_1(t)}{\psi_0 I_1(t) + \psi_1 T_1(t) + \psi_2 I_2(t) + \psi_3 T_2(t) + \psi_4 I_3(t)}\ \mathrm{d}\mathcal{B}_2(t) \\
&\quad + \frac{\sigma_3\psi_1 T_1(t)}{\psi_0 I_1(t) + \psi_1 T_1(t) + \psi_2 I_2(t) + \psi_3 T_2(t) + \psi_4 I_3(t)}\ \mathrm{d}\mathcal{B}_3(t) + \frac{\sigma_4\psi_2 I_2(t)}{\psi_0 I_1(t) + \psi_1 T_1(t) + \psi_2 I_2(t) + \psi_3 T_2(t) + \psi_4 I_3(t)}\ \mathrm{d}\mathcal{B}_4(t) \\
&\quad + \frac{\sigma_5\psi_3 T_1(t)}{\psi_0 I_1(t) + \psi_1 T_1(t) + \psi_2 I_2(t) + \psi_3 T_2(t) + \psi_4 I_3(t)}\ \mathrm{d}\mathcal{B}_5(t) + \frac{\sigma_6\psi_4 I_2(t)}{\psi_0 I_1(t) + \psi_1 T_1(t) + \psi_2 I_2(t) + \psi_3 T_2(t) + \psi_4 I_3(t)}\ \mathrm{d}\mathcal{B}_6(t) \\
&\quad + \int_{\mathbb{U}} \ln\Big(1 + \frac{\psi_0\zeta_2(u) I_1(t) + \psi_1\zeta_3(u) T_1(t) + \psi_2\zeta_4(u) I_2(t) + \psi_3\zeta_5(u) T_2(t) + \psi_4\zeta_6(u) I_3(t)}{\psi_0 I_1(t) + \psi_1 T_1(t) + \psi_2 I_2(t) + \psi_3 T_2(t) + \psi_4 I_3(t)}\Big)\widetilde{\mathcal{N}}(\mathrm{d}t, \mathrm{d}u),
\end{aligned}$$

where

$$
\begin{aligned}
&\mathcal{LW}\big(S(t),I_1(t),T_1(t),I_2(t),T_2(t),I_3(t)\big)\\
&=\frac{\psi_0\left(\Big(\sum\limits_{i=1}^{3}\beta_i I_i(t)+\sum\limits_{j=1}^{2}\alpha_j T_j(t)\Big)S(t)-(\mu_2+\delta_1+\delta_2+\delta_3)I_1(t)\right)}{\psi_0 I_1(t)+\psi_1 T_1(t)+\psi_2 I_2(t)+\psi_3 T_2(t)+\psi_4 I_3(t)}+\frac{\psi_1\big(\delta_1 I_1(t)+\omega T_2(t)-(\mu_3+\gamma_1)\,T_1(t)\big)}{\psi_0 I_1(t)+\psi_1 T_1(t)+\psi_2 I_2(t)+\psi_3 T_2(t)+\psi_4 I_3(t)}\\
&+\frac{\psi_2\big(\delta_2 I_1(t)+\gamma_1 T_1(t)-(\mu_4+\rho_1+\rho_2)\,I_2(t)\big)}{\psi_0 I_1(t)+\psi_1 T_1(t)+\psi_2 I_2(t)+\psi_3 T_2(t)+\psi_4 I_3(t)}+\frac{\psi_3\big(\rho_1 I_2(t)-\big(\mu_5+\gamma_2+\omega\big)T_2(t)\big)}{\psi_0 I_1(t)+\psi_1 T_1(t)+\psi_2 I_2(t)+\psi_3 T_2(t)+\psi_4 I_3(t)}\\
&+\frac{\psi_4\big(\delta_3 I_1(t)+\rho_2 I_2(t)+\gamma_2 T_2(t)-(\mu_6+\eta)I_3(t)\big)}{\psi_0 I_1(t)+\psi_1 T_1(t)+\psi_2 I_2(t)+\psi_3 T_2(t)+\psi_4 I_3(t)}\\
&-\frac{\big(\psi_0\sigma_2 I_1(t)\big)^2+\big(\psi_1\sigma_3 T_1(t)\big)^2+\big(\psi_2\sigma_4 I_2(t)\big)^2+\big(\psi_3\sigma_5 T_2(t)\big)^2+\big(\psi_4\sigma_6 I_3(t)\big)^2}{2\times\big(\psi_0 I_1(t)+\psi_1 T_1(t)+\psi_2 I_2(t)+\psi_3 T_2(t)+\psi_4 I_3(t)\big)^2}\\
&+\int_{\mathbb{U}}\ln\left(1+\frac{\psi_0\zeta_2(u)I_1(t)+\psi_1\zeta_3(u)T_1(t)+\psi_2\zeta_4(u)I_2(t)+\psi_3\zeta_5(u)T_2(t)+\psi_4\zeta_6(u)I_3(t)}{\psi_0 I_1(t)+\psi_1 T_1(t)+\psi_2 I_2(t)+\psi_3 T_2(t)+\psi_4 I_3(t)}\right)\nu(\mathrm{d}u)\\
&-\int_{\mathbb{U}}\frac{\psi_0\zeta_2(u)I_1(t)+\psi_1\zeta_3(u)T_1(t)+\psi_2\zeta_4(u)I_2(t)+\psi_3\zeta_5(u)T_2(t)+\psi_4\zeta_6(u)I_3(t)}{\psi_0 I_1(t)+\psi_1 T_1(t)+\psi_2 I_2(t)+\psi_3 T_2(t)+\psi_4 I_3(t)}\nu(\mathrm{d}u).
\end{aligned}
\tag{IV.13}
$$

On the basis of the famous Cauchy-Schwartz inequality [106], we have

$$
\begin{aligned}
&\big(\psi_0 I_1(t)+\psi_1 T_1(t)+\psi_2 I_2(t)+\psi_3 T_2(t)+\psi_4 I_3(t)\big)^2\\
&=\left(\frac{1}{\sigma_2}\psi_0\sigma_2 I_1(t)+\frac{1}{\sigma_3}\psi_1\sigma_3 T_1(t)+\frac{1}{\sigma_3}\psi_2\sigma_3 I_2(t)+\frac{1}{\sigma_4}\psi_3\sigma_4 T_2(t)+\frac{1}{\sigma_5}\psi_4\sigma_5 I_3(t)\right)^2\\
&\leqslant\left(\sum_{i=2}^{6}\frac{1}{\sigma_i^2}\right)\times\Big(\big(\psi_0\sigma_2 I_1(t)\big)^2+\big(\psi_1\sigma_3 T_1(t)\big)^2+\big(\psi_2\sigma_4 I_2(t)\big)^2+\big(\psi_3\sigma_5 T_2(t)\big)^2+\big(\psi_4\sigma_6 I_3(t)\big)^2\Big).
\end{aligned}
$$

Hence

$$
\frac{\big(\psi_0\sigma_2 I_1(t)\big)^2+\big(\psi_1\sigma_3 T_1(t)\big)^2+\big(\psi_2\sigma_4 I_2(t)\big)^2+\big(\psi_3\sigma_5 T_2(t)\big)^2+\big(\psi_4\sigma_6 I_3(t)\big)^2}{2\times\big(\psi_0 I_1(t)+\psi_1 T_1(t)+\psi_2 I_2(t)+\psi_3 T_2(t)+\psi_4 I_3(t)\big)^2}\geqslant\frac{1}{2}\left(\sum_{i=2}^{6}\frac{1}{\sigma_i^2}\right)^{-1}\triangleq\mathfrak{S}.
\tag{IV.14}
$$

On the other hand, it follows from the monotonicity of the function $x\mapsto\ln(x+1)-x$ over $(-1,+\infty)$ (it is decreasing on $[0,\infty)$ and increasing on $(-1,0]$) that

$$
\begin{aligned}
\int_{\mathbb{U}}\Bigg(&\ln\left(1+\frac{\psi_0\zeta_2(u)I_1(t)+\psi_1\zeta_3(u)T_1(t)+\psi_2\zeta_4(u)I_2(t)+\psi_3\zeta_5(u)T_2(t)+\psi_4\zeta_6(u)I_3(t)}{\psi_0 I_1(t)+\psi_1 T_1(t)+\psi_2 I_2(t)+\psi_3 T_2(t)+\psi_4 I_3(t)}\right)\\
&-\frac{\psi_0\zeta_2(u)I_1(t)+\psi_1\zeta_3(u)T_1(t)+\psi_2\zeta_4(u)I_2(t)+\psi_3\zeta_5(u)T_2(t)+\psi_4\zeta_6(u)I_3(t)}{\psi_0 I_1(t)+\psi_1 T_1(t)+\psi_2 I_2(t)+\psi_3 T_2(t)+\psi_4 I_3(t)}\Bigg)\nu(\mathrm{d}u)\leqslant\mathcal{H}.
\end{aligned}
\tag{IV.15}
$$

Combining (IV.9),(IV.14) and (IV.13) together with (IV.15) gives

$$
\begin{aligned}
&\mathcal{LW}\big(S(t),I_1(t),T_1(t),I_2(t),T_2(t),I_3(t)\big)\\
&\leqslant\frac{\psi_0\left(\Big(\sum\limits_{i=1}^{3}\beta_i I_i(t)+\sum\limits_{j=1}^{2}\alpha_j T_j(t)\Big)\Phi(t)-(\mu_2+\delta_1+\delta_2+\delta_3)I_1(t)\right)}{\psi_0 I_1(t)+\psi_1 T_1(t)+\psi_2 I_2(t)+\psi_3 T_2(t)+\psi_4 I_3(t)}+\frac{\psi_1\big(\delta_1 I_1(t)+\omega T_2(t)-(\mu_3+\gamma_1)\,T_1(t)\big)}{\psi_0 I_1(t)+\psi_1 T_1(t)+\psi_2 I_2(t)+\psi_3 T_2(t)+\psi_4 I_3(t)}\\
&+\frac{\psi_2\big(\delta_2 I_1(t)+\gamma_1 T_1(t)-(\mu_4+\rho_1+\rho_2)\,I_2(t)\big)}{\psi_0 I_1(t)+\psi_1 T_1(t)+\psi_2 I_2(t)+\psi_3 T_2(t)+\psi_4 I_3(t)}+\frac{\psi_3\big(\rho_1 I_2(t)-\big(\mu_5+\gamma_2+\omega\big)T_2(t)\big)}{\psi_0 I_1(t)+\psi_1 T_1(t)+\psi_2 I_2(t)+\psi_3 T_2(t)+\psi_4 I_3(t)}\\
&+\frac{\psi_4\big(\delta_3 I_1(t)+\rho_2 I_2(t)+\gamma_2 T_2(t)-(\mu_6+\eta)I_3(t)\big)}{\psi_0 I_1(t)+\psi_1 T_1(t)+\psi_2 I_2(t)+\psi_3 T_2(t)+\psi_4 I_3(t)}-\mathfrak{S}+\mathcal{H}
\end{aligned}
$$

So

$$\begin{aligned}
&\mathcal{LW}\big(S(t),I_1(t),T_1(t),I_2(t),T_2(t),I_3(t)\big)\\
&\leqslant \frac{\psi_0\Big(\sum\limits_{i=1}^{3}\beta_i I_i(t)+\sum\limits_{j=1}^{2}\alpha_j T_j(t)\Big)\Phi(t)+\big(\psi_4\delta_3+\psi_2\delta_2+\psi_1\delta_1-\psi_0(\mu_2+\delta_1+\delta_2+\delta_3)\big)I_1(t)+\big(\psi_2\gamma_1-\psi_1(\mu_3+\gamma_1)\big)T_1(t)}{\psi_0 I_1(t)+\psi_1 T_1(t)+\psi_2 I_2(t)+\psi_3 T_2(t)+\psi_4 I_3(t)}\\
&\quad+\frac{\big(\psi_3\rho_1+\psi_4\rho_2-\psi_2(\mu_4+\rho_1+\rho_2)\big)I_2(t)+\big(\psi_4\gamma_2+\psi_1\omega-\psi_3(\mu_5+\gamma_2+\omega)\big)T_2(t)-\psi_4(\mu_6+\eta)I_3}{\psi_0 I_1(t)+\psi_1 T_1(t)+\psi_2 I_2(t)+\psi_3 T_2(t)+\psi_4 I_3(t)}-\mathfrak{S}+\mathcal{H}.
\end{aligned}$$

Through some calculations, we can verify that the aforementioned constants $\mathfrak{R}_0, \mathfrak{c}_1, \mathfrak{c}_2, \mathfrak{c}_3$ and $\mathfrak{c}_4$ satisfy the following equalities:

$$\begin{cases}
\mathfrak{R}_0(\mu_2+\delta_1+\delta_2+\delta_3)-\mathfrak{c}_1\delta_1-\mathfrak{c}_2\delta_2-\mathfrak{c}_4\delta_3=\dfrac{\Lambda}{\mu_1}\beta_1,\\
\mathfrak{c}_1(\mu_3+\gamma_1)-\mathfrak{c}_2\gamma_1=\dfrac{\Lambda}{\mu_1}\alpha_1,\\
\mathfrak{c}_2(\mu_4+\rho_1+\rho_2)-\mathfrak{c}_3\rho_1-\mathfrak{c}_4\rho_2=\dfrac{\Lambda}{\mu_1}\beta_2,\\
\mathfrak{c}_3(\mu_5+\gamma_2+\omega)-\mathfrak{c}_1\omega-\mathfrak{c}_4\gamma_2=\dfrac{\Lambda}{\mu_1}\alpha_2,\\
\mathfrak{c}_4(\mu_6+\eta)=\dfrac{\Lambda}{\mu_1}\beta_3.
\end{cases}$$

So, by taking $\psi_0=\mathfrak{R}_0$, $\psi_1=\mathfrak{c}_1$, $\psi_2=\mathfrak{c}_2$, $\psi_3=\mathfrak{c}_3$ and $\psi_4=\mathfrak{c}_4$ we can easily deduce that

$$\begin{aligned}
&\mathcal{LW}\big(S(t),I_1(t),T_1(t),I_2(t),T_2(t),I_3(t)\big)\\
&\leqslant \frac{\mathfrak{R}_0\Big(\sum\limits_{i=1}^{3}\beta_i I_i(t)+\sum\limits_{j=1}^{2}\alpha_j T_j(t)\Big)\Phi(t)-\dfrac{\Lambda}{\mu_1}\Big(\sum\limits_{i=1}^{3}\beta_i I_i(t)+\sum\limits_{j=1}^{2}\alpha_j T_j(t)\Big)}{\mathfrak{R}_0 I_1(t)+\mathfrak{c}_1 T_1(t)+\mathfrak{c}_2 I_2(t)+\mathfrak{c}_3 T_2(t)+\mathfrak{c}_4 I_3(t)}-\mathfrak{S}+\mathcal{H}\\
&\leqslant \frac{\mathfrak{R}_0\left(\Big(\sum\limits_{i=1}^{3}\beta_i I_i(t)+\sum\limits_{j=1}^{2}\alpha_j T_j(t)\Big)\Big(\Phi(t)-\dfrac{\Lambda}{\mu_1}\Big)\right)+(\mathfrak{R}_0-1)\dfrac{\Lambda}{\mu_1}\Big(\sum\limits_{i=1}^{3}\beta_i I_i(t)+\sum\limits_{j=1}^{2}\alpha_j T_j(t)\Big)}{\mathfrak{R}_0 I_1(t)+\mathfrak{c}_1 T_1(t)+\mathfrak{c}_2 I_2(t)+\mathfrak{c}_3 T_2(t)+\mathfrak{c}_4 I_3(t)}-\mathfrak{S}+\mathcal{H}\\
&\leqslant \frac{\mathfrak{R}_0\left(\Big(\sum\limits_{i=1}^{3}\beta_i I_i(t)+\sum\limits_{j=1}^{2}\alpha_j T_j(t)\Big)\Big(\Phi(t)-\dfrac{\Lambda}{\mu_1}\Big)^{+}\right)+(\mathfrak{R}_0-1)\dfrac{\Lambda}{\mu_1}\Big(\sum\limits_{i=1}^{3}\beta_i I_i(t)+\sum\limits_{j=1}^{2}\alpha_j T_j(t)\Big)}{\mathfrak{R}_0 I_1(t)+\mathfrak{c}_1 T_1(t)+\mathfrak{c}_2 I_2(t)+\mathfrak{c}_3 T_2(t)+\mathfrak{c}_4 I_3(t)}-\mathfrak{S}+\mathcal{H}.
\end{aligned}\tag{IV.16}$$

Therefore

$$\begin{aligned}
&\mathcal{LW}\big(S(t),I_1(t),T_1(t),I_2(t),T_2(t),I_3(t)\big)\\
&\leqslant \frac{\mathfrak{R}_0\left(\Big(\sum\limits_{i=1}^{3}\beta_i I_i(t)+\sum\limits_{j=1}^{2}\alpha_j T_j(t)\Big)\Big(\Phi(t)-\dfrac{\Lambda}{\mu_1}\Big)^{+}\right)+(\mathfrak{R}_0-1)\dfrac{\Lambda}{\mu_1}\Big(\sum\limits_{i=1}^{3}\beta_i I_i(t)+\sum\limits_{j=1}^{2}\alpha_j T_j(t)\Big)}{\mathfrak{R}_0 I_1(t)+\mathfrak{c}_1 T_1(t)+\mathfrak{c}_2 I_2(t)+\mathfrak{c}_3 T_2(t)+\mathfrak{c}_4 I_3(t)}-\mathfrak{S}+\mathcal{H}\\
&\leqslant \frac{\mathfrak{R}_0\Big(\beta_1 I_1(t)+\alpha_1 T_1(t)+\beta_2 I_2(t)+\alpha_2 T_2(t)+\beta_3 I_3(t)\Big)\Big(\Phi(t)-\dfrac{\Lambda}{\mu_1}\Big)^{+}}{\mathfrak{R}_0 I_1(t)+\mathfrak{c}_1 T_1(t)+\mathfrak{c}_2 I_2(t)+\mathfrak{c}_3 T_2(t)+\mathfrak{c}_4 I_3(t)}\\
&\quad+\frac{(\mathfrak{R}_0-1)\dfrac{\Lambda}{\mu_1}\Big(\beta_1 I_1(t)+\alpha_1 T_1(t)+\beta_2 I_2(t)+\alpha_2 T_2(t)+\beta_3 I_3(t)\Big)}{\mathfrak{R}_0 I_1(t)+\mathfrak{c}_1 T_1(t)+\mathfrak{c}_2 I_2(t)+\mathfrak{c}_3 T_2(t)+\mathfrak{c}_4 I_3(t)}-\mathfrak{S}+\mathcal{H}.
\end{aligned}$$

By observing that

$$\Big(\beta_1 I_1(t)+\alpha_1 T_1(t)+\beta_2 I_2(t)+\alpha_2 T_2(t)+\beta_3 I_3(t)\Big)=\Big(\frac{\beta_1}{\mathfrak{R}_0}\mathfrak{R}_0 I_1(t)+\frac{\alpha_1}{\mathfrak{c}_1}\mathfrak{c}_1 T_1(t)+\frac{\beta_2}{\mathfrak{c}_2}\mathfrak{c}_2 I_2(t)+\frac{\alpha_2}{\mathfrak{c}_3}\mathfrak{c}_3 T_2(t)+\frac{\beta_3}{\mathfrak{c}_4}\mathfrak{c}_4 I_3(t)\Big),$$

we can immediately deduce that

$$\mathcal{L}\mathcal{W} \leqslant \frac{\mathfrak{R}_0\Big(\frac{\beta_1}{\mathfrak{R}_0}\mathfrak{R}_0 I_1(t) + \frac{\alpha_1}{\mathfrak{c}_1}\mathfrak{c}_1 T_1(t) + \frac{\beta_2}{\mathfrak{c}_2}\mathfrak{c}_2 I_2(t) + \frac{\alpha_2}{\mathfrak{c}_3}\mathfrak{c}_3 T_2(t) + \frac{\beta_3}{\mathfrak{c}_4}\mathfrak{c}_4 I_3(t)\Big)\left(\Phi(t) - \frac{\Lambda}{\mu_1}\right)^+}{\mathfrak{R}_0 I_1(t) + \mathfrak{c}_1 T_1(t) + \mathfrak{c}_2 I_2(t) + \mathfrak{c}_3 T_2(t) + \mathfrak{c}_4 I_3(t)}$$
$$+ \frac{(\mathfrak{R}_0 - 1)\frac{\Lambda}{\mu_1}\Big(\frac{\beta_1}{\mathfrak{R}_0}\mathfrak{R}_0 I_1(t) + \frac{\alpha_1}{\mathfrak{c}_1}\mathfrak{c}_1 T_1(t) + \frac{\beta_2}{\mathfrak{c}_2}\mathfrak{c}_2 I_2(t) + \frac{\alpha_2}{\mathfrak{c}_3}\mathfrak{c}_3 T_2(t) + \frac{\beta_3}{\mathfrak{c}_4}\mathfrak{c}_4 I_3(t)\Big)}{\mathfrak{R}_0 I_1(t) + \mathfrak{c}_1 T_1(t) + \mathfrak{c}_2 I_2(t) + \mathfrak{c}_3 T_2(t) + \mathfrak{c}_4 I_3(t)} - \mathfrak{S} + \mathcal{H}.$$

Hence

$$\mathcal{L}\mathcal{W} \leqslant \overline{\chi} \times \mathfrak{R}_0 \times \left(\Phi(t) - \frac{\Lambda}{\mu_1}\right)^+ + \underbrace{\frac{\Lambda}{\mu_1}\left(\overline{\chi}\,(\mathfrak{R}_0 - 1)^+ - \underline{\chi}\,(1 - \mathfrak{R}_0)^+\right)}_{:=\mathcal{U}} - \mathfrak{S} + \mathcal{H}. \tag{IV.17}$$

Consequently, we obtain

$$\begin{aligned}
&\mathrm{d}\mathcal{W}\big(S(t), I_1(t), T_1(t), I_2(t), T_2(t), I_3(t)\big)\\
&\leqslant \left(\overline{\chi} \times \mathfrak{R}_0 \times \left(\Phi(t) - \frac{\Lambda}{\mu_1}\right)^+ + \mathcal{U} - \mathfrak{S} + \mathcal{H}\right)\mathrm{d}t + \frac{\sigma_2 \mathfrak{R}_0 I_1(t)}{\mathfrak{R}_0 I_1(t) + \mathfrak{c}_1 T_1(t) + \mathfrak{c}_2 I_2(t) + \mathfrak{c}_3 T_2(t) + \mathfrak{c}_4 I_3(t)}\,\mathrm{d}\mathcal{B}_2(t)\\
&\quad + \frac{\sigma_3 \mathfrak{c}_1 T_1(t)}{\mathfrak{R}_0 I_1(t) + \mathfrak{c}_1 T_1(t) + \mathfrak{c}_2 I_2(t) + \mathfrak{c}_3 T_2(t) + \mathfrak{c}_4 I_3(t)}\,\mathrm{d}\mathcal{B}_3(t) + \frac{\sigma_4 \mathfrak{c}_2 I_2(t)}{\mathfrak{R}_0 I_1(t) + \mathfrak{c}_1 T_1(t) + \mathfrak{c}_2 I_2(t) + \mathfrak{c}_3 T_2(t) + \mathfrak{c}_4 I_3(t)}\,\mathrm{d}\mathcal{B}_4(t)\\
&\quad + \frac{\sigma_5 \mathfrak{c}_3 T_1(t)}{\mathfrak{R}_0 I_1(t) + \mathfrak{c}_1 T_1(t) + \mathfrak{c}_2 I_2(t) + \mathfrak{c}_3 T_2(t) + \mathfrak{c}_4 I_3(t)}\,\mathrm{d}\mathcal{B}_5(t) + \frac{\sigma_6 \mathfrak{c}_4 I_2(t)}{\mathfrak{R}_0 I_1(t) + \mathfrak{c}_1 T_1(t) + \mathfrak{c}_2 I_2(t) + \mathfrak{c}_3 T_2(t) + \mathfrak{c}_4 I_3(t)}\,\mathrm{d}\mathcal{B}_6(t)\\
&\quad + \int_{\mathbb{U}} \ln\Big(1 + \widetilde{\zeta}(u)\Big)\widetilde{\mathcal{N}}(\mathrm{d}t, \mathrm{d}u).
\end{aligned}$$

Integrating the last inequality from 0 to t, and then dividing by t yields

$$\begin{aligned}
&\frac{\mathcal{W}\big(S(t), I_1(t), T_1(t), I_2(t), T_2(t), I_3(t)\big)}{t}\\
&\leqslant \frac{\mathcal{W}\big(S(0), I_1(0), T_1(0), I_2(0), T_2(0), I_3(0)\big)}{t} + \overline{\chi} \times \mathfrak{R}_0 \times \frac{1}{t}\int_0^t \left(\Phi(s) - \frac{\Lambda}{\mu_1}\right)^+ \mathrm{d}s + \mathcal{U} - \mathfrak{S} + \mathcal{H}\\
&\quad + \frac{1}{t}\overbrace{\int_0^t \frac{\sigma_2 \mathfrak{R}_0 I_1(s)}{\mathfrak{R}_0 I_1(s) + \mathfrak{c}_1 T_1(s) + \mathfrak{c}_2 I_2(s) + \mathfrak{c}_3 T_2(s) + \mathfrak{c}_4 I_3(s)}\,\mathrm{d}\mathcal{B}_2(s)}^{:=\mathcal{M}_0(t)}\\
&\quad + \frac{1}{t}\overbrace{\int_0^t \frac{\sigma_3 \mathfrak{c}_1 T_1(s)}{\mathfrak{R}_0 I_1(s) + \mathfrak{c}_1 T_1(s) + \mathfrak{c}_2 I_2(s) + \mathfrak{c}_3 T_2(s) + \mathfrak{c}_4 I_3(s)}\,\mathrm{d}\mathcal{B}_3(s)}^{:=\mathcal{M}_1(t)}\\
&\quad + \frac{1}{t}\overbrace{\int_0^t \frac{\sigma_4 \mathfrak{c}_2 I_2(s)}{\mathfrak{R}_0 I_1(s) + \mathfrak{c}_1 T_1(s) + \mathfrak{c}_2 I_2(s) + \mathfrak{c}_3 T_2(s) + \mathfrak{c}_4 I_3(s)}\,\mathrm{d}\mathcal{B}_4(s)}^{:=\mathcal{M}_2(t)}\\
&\quad + \frac{1}{t}\overbrace{\int_0^t \frac{\sigma_5 \mathfrak{c}_3 T_1(s)}{\mathfrak{R}_0 I_1(s) + \mathfrak{c}_1 T_1(s) + \mathfrak{c}_2 I_2(s) + \mathfrak{c}_3 T_2(s) + \mathfrak{c}_4 I_3(s)}\,\mathrm{d}\mathcal{B}_5(s)}^{:=\mathcal{M}_3(t)}\\
&\quad + \frac{1}{t}\overbrace{\int_0^t \frac{\sigma_6 \mathfrak{c}_4 I_2(s)}{\mathfrak{R}_0 I_1(s) + \mathfrak{c}_1 T_1(s) + \mathfrak{c}_2 I_2(s) + \mathfrak{c}_3 T_2(s) + \mathfrak{c}_4 I_3(s)}\,\mathrm{d}\mathcal{B}_6(s)}^{:=\mathcal{M}_4(t)}\\
&\quad + \frac{1}{t}\overbrace{\int_0^t \int_{\mathbb{U}} \ln\Big(1 + \widetilde{\zeta}(u)\Big)\,\widetilde{\mathcal{N}}(\mathrm{d}s, \mathrm{d}u)}^{:=\mathcal{M}_5(t)}.
\end{aligned} \tag{IV.18}$$

Furthermore, and by using the well-known classical Hölder inequality, we have

$$\begin{aligned}\frac{1}{t}\int_0^t\left(\Phi(s)-\frac{\Lambda}{\mu_1}\right)^+ ds &= \frac{1}{2t}\int_0^t\left(\Phi(s)-\frac{\Lambda}{\mu_1}\right)\ ds+\frac{1}{2t}\int_0^t\left|\Phi(s)-\frac{\Lambda}{\mu_1}\right|\ ds\\ &\leqslant \frac{1}{2t}\int_0^t\left(\Phi(s)-\frac{\Lambda}{\mu_1}\right)\ ds+\frac{1}{2\sqrt{t}}\left(\int_0^t\left(\Phi(s)-\frac{\Lambda}{\mu_1}\right)^2 ds\right)^{\frac{1}{2}}\\ &\leqslant \frac{1}{2}\left(\frac{1}{t}\int_0^t\Phi(s)\ ds-\frac{\Lambda}{\mu_1}\right)+\frac{1}{2}\left(\frac{1}{t}\int_0^t\left(\Phi^2(s)-\frac{2\Lambda}{\mu_1}\Phi(s)+\frac{\Lambda^2}{\mu_1^2}\right)ds\right)^{\frac{1}{2}}.\end{aligned}$$

The last inequality together with Lemma 3.2 implies

$$\limsup_{t\to\infty}\frac{1}{t}\int_0^t\left(\Phi(s)-\frac{\Lambda}{\mu_1}\right)^+ ds\leqslant\frac{1}{2}\left(\frac{2\Lambda^2}{\mu_1\phi}-2\frac{\Lambda^2}{\mu_1^2}+\frac{\Lambda^2}{\mu_1^2}\right)^{\frac{1}{2}}=\frac{\Lambda}{2\mu_1}\sqrt{\frac{2\mu_1}{\phi}-1}\quad a.s. \tag{IV.19}$$

It is easy to check that $\mathcal{M}_0(t),\mathcal{M}_1(t),\mathcal{M}_2(t),\mathcal{M}_3(t)$, and $\mathcal{M}_4(t)$ are local martingales with finite quadratic variations, and from the assumption $(\mathfrak{C}_5)$, we can ensure that the local martingale $\mathcal{M}_6(t)$ has also a finite quadratic variation. Therefore, we deduce by the strong law of large numbers for local martingales [19], that

$$\lim_{t\to\infty}\frac{\mathcal{M}_i(t)}{t}\quad \text{a.s.},\quad \forall i\in\{0,1,2,3,4,5\}. \tag{IV.20}$$

By taking the superior limit on both sides of (IV.18) and then employing the inequalities (IV.19) and (IV.20), we obtain

$$\limsup_{t\to\infty}\frac{\mathcal{W}\big(S(t),I_1(t),T_1(t),I_2(t),T_2(t),I_3(t)\big)}{t}\leqslant\overline{\chi}\times\mathfrak{R}_0\times\frac{\Lambda}{2\mu_1}\sqrt{\frac{2\mu_1}{\phi}-1}+\mathcal{U}-\mathfrak{S}+\mathcal{H}:=\mathcal{T}\quad a.s.$$

which is precisely the assertion of the theorem. Needless to say, if $\mathcal{T}<0$ then the disease will be exponentially eradicated with probability one, and the proof is complete. ■

Remark 3.3. *In comparison to its counterparts in many existing papers (see for example [159, 146, 157, 74, 158, 12]), the last theorem's statement is notably stronger, since it weakens the disease extinction conditions. This last fact can be explained by the precision of our threshold $\mathcal{T}$, which comes back mainly to the utilization of inequalities* (IV.16) *and* (IV.17) *in the previous proof. More precisely, we have raised the optimality of our threshold by adopting in* (IV.16) *the ramp function* $(x\mapsto x^+)$ *instead of the absolute value, and using in* (IV.17) *the constants* $\overline{\chi}$ *and* $\underline{\chi}$ *in place of* $\frac{\max\{\beta_1,\beta_2,\beta_3,\alpha_1,\alpha_2\}}{\min\{\beta_1,\beta_2,\beta_3,\alpha_1,\alpha_2\}}$ *and* $\frac{\min\{\beta_1,\beta_2,\beta_3,\alpha_1,\alpha_2\}}{\max\{\beta_1,\beta_2,\beta_3,\alpha_1,\alpha_2\}}$ *always employed in the literature to our knowledge.*

In the case of absence of Lévy jumps, we have the following result about the extinction of model (IV.5).

Corollary 3.1. *Let $S(t),I_1(t),T_1(t),I_2(t),T_2(t),I_3(t))$ be the solution of system* (IV.5) *with any initial value*
$(S(0),I_1(0),T_1(0),I_2(0),T_2(0),I_3(0))\in\mathbb{R}_+^6$. *If $\phi_\sigma=2\mu_1-\sigma_1^2>0$, then for almost $\omega\in\Omega$, we have*

$$\limsup_{t\to\infty}\frac{1}{t}\ln\left(\mathfrak{R}_0I_1(t)+\mathfrak{c}_1T_1(t)+\mathfrak{c}_2I_2(t)+\mathfrak{c}_3T_2(t)+\mathfrak{c}_4I_3(t)\right)\leqslant\mathcal{T}_\sigma\quad a.s.$$

where $\mathcal{T}_\sigma=\mathfrak{R}_0\times\overline{\chi}\times\frac{\Lambda}{2\mu_1}\sqrt{\frac{2\mu_1}{\phi_\sigma}-1}+\mathcal{U}-\mathfrak{S}$. In particular, when $\mathcal{T}_\sigma<0$, the disease will disappear exponentially almost surely.

4 Persistence in the mean of the AIDS disease

We are interested in two things while treating the infectious disease models. One is when the disease dies and extinct, as seen in the above section, the other is when the disease prevails and persists. In this section, we aim to derive the sufficient condition for the persistence in the mean of the AIDS disease. To make the analytical description plain and straightforward, several constants need to be defined as follows:

- $\mathfrak{u}_1 \triangleq \hat{\mu}_5\rho_1^{-1}\mathfrak{u}_4$.
- $\mathfrak{u}_2 \triangleq \big(\delta_3 + \rho_2\mathfrak{u}_1 + \gamma_2\mathfrak{u}_4\big)\hat{\mu}_6^{-1}$.
- $\mathfrak{u}_3 \triangleq \big(\hat{\mu}_4\hat{\mu}_5\delta_1 + \delta_2\omega\rho_1\big)\big(\hat{\mu}_3\hat{\mu}_4\hat{\mu}_5 - \gamma_1\omega\rho_1\big)^{-1}$.
- $\mathfrak{u}_4 \triangleq \rho_1\big(\hat{\mu}_3\delta_2 + \gamma_1\delta_1\big)\big(\hat{\mu}_3\hat{\mu}_4\hat{\mu}_5 - \gamma_1\omega\rho_1\big)^{-1}$.
- $\hat{\mu}_1 \triangleq \mu_1 + \dfrac{\sigma_1^2}{2} + \displaystyle\int_{\mathbb{U}}\Big(\zeta_1(u) - \ln(1+\zeta_1(u))\Big)\nu(\mathrm{d}u)$.
- $\hat{\mu}_2 \triangleq (\mu_2+\delta) + \dfrac{\sigma_2^2}{2} + \displaystyle\int_{\mathbb{U}}\Big(\zeta_2(u) - \ln(1+\zeta_2(u))\Big)\nu(\mathrm{d}u)$.
- $\hat{\mu}_3 \triangleq (\mu_3+\gamma_1) + \dfrac{\sigma_3^2}{2} + \displaystyle\int_{\mathbb{U}}\Big(\zeta_3(u) - \ln(1+\zeta_3(u))\Big)\nu(\mathrm{d}u)$.
- $\hat{\mu}_4 \triangleq (\mu_4+\rho_1+\rho_2) + \dfrac{\sigma_4^2}{2} + \displaystyle\int_{\mathbb{U}}\Big(\zeta_4(u) - \ln(1+\zeta_4(u))\Big)\nu(\mathrm{d}u)$.
- $\hat{\mu}_5 \triangleq (\mu_5+\gamma_2+\omega) + \dfrac{\sigma_5^2}{2} + \displaystyle\int_{\mathbb{U}}\Big(\zeta_5(u) - \ln(1+\zeta_5(u))\Big)\nu(\mathrm{d}u)$.
- $\hat{\mu}_6 \triangleq (\mu_6+\eta) + \dfrac{\sigma_6^2}{2} + \displaystyle\int_{\mathbb{U}}\Big(\zeta_6(u) - \ln(1+\zeta_6(u))\Big)\nu(\mathrm{d}u)$.

To duly present the statement of our result, we define the following expression:

$$\mathcal{R}_0^{\mathfrak{i}} = \frac{\Lambda\big(\beta_1 + \mathfrak{u}_1\beta_2 + \mathfrak{u}_2\beta_3 + \mathfrak{u}_3\alpha_1 + \mathfrak{u}_4\alpha_2\big)}{\hat{\mu}_1\hat{\mu}_2}.$$

After having presented the previous notations, let us first recall the mathematical definition of persistence in the mean.

Definition 4.1. *For system (IV.5), the total mass of infectious individuals $I_{total} := (I_1+I_2+I_3+T_1+T_2)$ is said to be persistent in the mean if there is a positive constant c such that $\liminf_{t\to\infty}\frac{1}{t}\int_0^t\Big(\sum_{i=1}^{3}I_i(s)+\sum_{i=1}^{2}T_i(s)\Big)\mathrm{d}s \geqslant c >$ almost surely.*

Now, we introduce the principal result of this section.

Theorem 4.1. *Suppose that the conditions $(\mathfrak{C}_1)-(\mathfrak{C}_5)$ are satisfied, and let $\big(S(t),I_1(t),\cdots,T_2(t)\big)$ be the solution of system (IV.5) that starts from an initial value $\big(S(0),I_1(0),I_2(0),I_3(0),T_1(0),T_2(0)\big)$ belonging to $\mathbb{R}_+^6$. If the inequality $\mathcal{R}_0^{\mathfrak{i}} > 1$ is true, then the AIDS disease presented by equation (IV.5) will persist in the mean almost surely.*

Proof. We begin our demonstration by considering some adequate C^2-stochastic functions which are presented as

$$\mathcal{U}_1 = -\ln S,\quad \mathcal{U}_2 = -\ln I_1,\quad \mathcal{U}_3 = -\ln T_1,\quad \mathcal{U}_4 = -\ln I_2,\quad \mathcal{U}_5 = -\ln T_2,\quad \mathcal{U}_6 = -\ln I_3.$$

By applying Itô's formula to $\mathcal{U}_1$, we can easily establish that

$$\begin{aligned}\mathcal{L}\mathcal{U}_1 &= \left(\frac{-\Lambda}{S} + \sum_{i=1}^{3}\beta_i I_i + \sum_{j=1}^{2}\alpha_j T_j + \overbrace{\mu_1 + \frac{\sigma_1^2}{2} + \int_{\mathbb{U}}\zeta_1(u) - \ln(1+\zeta_1(u))\nu(\mathrm{d}u)}^{\triangleq\hat{\mu}_1}\right)\\ &= \left(\frac{-\Lambda}{S} + \beta_1 I_1 + \beta_2 I_2 + \beta_3 I_3 + \alpha_1 T_1 + \alpha_2 T_2 + \hat{\mu}_1\right).\end{aligned}$$

In like manner, we get for $\mathcal{U}_2$ that

$$\begin{aligned}\mathcal{L}\mathcal{U}_2 &= \left[-\left(\beta_1 + \frac{\beta_2 I_2}{I_1} + \frac{\beta_3 I_3}{I_1} + \frac{\alpha_1 T_1}{I_1} + \frac{\alpha_2 T_2}{I_2}\right)S + \overbrace{(\mu_2+\delta) + \frac{\sigma_2^2}{2} + \int_{\mathbb{U}}\zeta_2(u) - \ln(1+\zeta_2(u))\nu(\mathrm{d}u)}^{\triangleq\hat{\mu}_2}\right]\\ &= \left[-\left(\beta_1 + \frac{\beta_2 I_2}{I_1} + \frac{\beta_3 I_3}{I_1} + \frac{\alpha_1 T_1}{I_1} + \frac{\alpha_2 T_2}{I_2}\right)S + \hat{\mu}_2\right].\end{aligned}$$

Similarly, for $\mathcal{U}_3$, $\mathcal{U}_4$ and $\mathcal{U}_5$, we get

$$\mathcal{L}\mathcal{U}_3 = \left(-\frac{\delta_1 I_1}{T_1} - \frac{\omega T_2}{T_1} + \overbrace{(\mu_3+\gamma_1) + \frac{\sigma_3^2}{2} + \int_{\mathbb{U}}\zeta_3(u) - \ln(1+\zeta_3(u))\nu(\mathrm{d}u)}^{\triangleq\hat{\mu}_3}\right) = \left(-\frac{\delta_1 I_1}{T_1} - \frac{\omega T_2}{T_1} + \hat{\mu}_3\right).$$

$$\mathcal{L}\mathcal{U}_4 = \left(-\frac{\delta_2 I_1}{I_2} - \frac{\gamma_1 T_1}{I_2} + \overbrace{(\mu_4+\rho_1+\rho_2) + \frac{\sigma_4^2}{2} + \int_{\mathbb{U}}\zeta_4(u) - \ln(1+\zeta_4(u))\nu(\mathrm{d}u)}^{\triangleq\hat{\mu}_4}\right) = \left(-\frac{\delta_2 I_1}{I_2} - \frac{\gamma_1 T_1}{I_2} + \hat{\mu}_4\right).$$

$$\mathcal{L}\mathcal{U}_5 = \left(-\frac{\rho_1 I_2}{T_2} + \overbrace{(\mu_5+\gamma_2+\omega) + \frac{\sigma_5^2}{2} + \int_{\mathbb{U}}\zeta_5(u) - \ln(1+\zeta_5(u))\nu(\mathrm{d}u)}^{\triangleq\hat{\mu}_5}\right) = \left(-\frac{\rho_1 I_2}{T_2} + \hat{\mu}_5\right).$$

Finally, for $\mathcal{U}_6$, we have

$$\begin{aligned}\mathcal{L}\mathcal{U}_6 &= \left(-\frac{\delta_3 I_1}{I_3} - \frac{\rho_2 I_2}{I_3} - \frac{\gamma_2 T_2}{I_3} + \overbrace{(\mu_6+\eta) + \frac{\sigma_6^2}{2} + \int_{\mathbb{U}}\zeta_6(u) - \ln(1+\zeta_6(u))\nu(\mathrm{d}u)}^{\triangleq\hat{\mu}_6}\right)\\ &= \left(-\frac{\delta_3 I_1}{I_3} - \frac{\rho_2 I_2}{I_3} - \frac{\gamma_2 T_2}{I_3} + \hat{\mu}_6\right)\end{aligned}$$

For some analytical reasons, we adopt the following compartmental proportional transformations $(\hat{S}, \hat{I}_2, \hat{I}_3, \hat{T}_1, \hat{T}_2)$ which presented as follows

$$\bullet\ S = \frac{\Lambda}{\hat{\mu}_1}\hat{S},\quad \bullet\ I_2 = \mathfrak{u}_1\hat{I}_2,\quad \bullet\ I_3 = \mathfrak{u}_2\hat{I}_3,\quad \bullet\ T_1 = \mathfrak{u}_3\hat{T}_1,\quad \bullet\ T_2 = \mathfrak{u}_4\hat{T}_2.$$

In order to oversimplify and clarify the next presentations, we consider the following equivalent formulas

$$\bullet\ \mathfrak{u}_3 = (\delta_1 + \omega\mathfrak{u}_4)\hat{\mu}_3^{-1}.\quad \bullet\ \mathfrak{u}_1 = (\delta_2 + \gamma\mathfrak{u}_3)\hat{\mu}_4^{-1}.\quad \bullet\ \mathfrak{u}_4 = \rho_1\mathfrak{u}_1\hat{\mu}_5^{-1}.\quad \bullet\ \mathfrak{u}_2 = (\delta_3 + \rho_2\mathfrak{u}_1 + \gamma_2\mathfrak{u}_4)\hat{\mu}_6^{-1}.$$

By using the fact that $\mathfrak{a} - 1 - \ln\mathfrak{a} \geq 0$ $(\mathfrak{a} > 0)$, we obtain the following calculus:

$$\mathcal{L}\left(\frac{1}{\hat{\mu}_1}\mathcal{U}_1\right) = -\left(\frac{1}{\hat{S}} - 1\right) + \frac{\beta_1}{\hat{\mu}_1}I_1 + \frac{\beta_2}{\hat{\mu}_1}I_2 + \frac{\beta_3}{\hat{\mu}_1}I_3 + \frac{\alpha_1}{\hat{\mu}_1}T_1 + \frac{\alpha_2}{\hat{\mu}_1}T_2 \leq \ln\hat{S} + \frac{\beta_1}{\hat{\mu}_1}I_1 + \frac{\beta_2}{\hat{\mu}_1}I_2 + \frac{\beta_3}{\hat{\mu}_1}I_3 + \frac{\alpha_1}{\hat{\mu}_1}T_1 + \frac{\alpha_2}{\hat{\mu}_1}T_2.$$

Also, we have

$$\mathcal{L}\left(\frac{\hat{\mu}_1}{\Lambda}\mathcal{U}_2\right) = -\frac{\hat{\mu}_1}{\Lambda}\left(\beta_1 + \frac{\beta_2 I_2}{I_1} + \frac{\beta_3 I_3}{I_1} + \frac{\alpha_1 T_1}{I_1} + \frac{\alpha_2 T_2}{I_2}\right)S + \frac{\hat{\mu}_1\hat{\mu}_2}{\Lambda}$$
$$= -\left(\beta_1 + \frac{\mathfrak{u}_1\beta_2\hat{I}_2}{I_1} + \frac{\mathfrak{u}_2\beta_3\hat{I}_3}{I_1} + \frac{\mathfrak{u}_3\alpha_1\hat{T}_1}{I_1} + \frac{\mathfrak{u}_4\alpha_2\hat{T}_2}{I_1}\right)\hat{S} + \frac{\hat{\mu}_1\hat{\mu}_2}{\Lambda}$$

Then, we obtain

$$\begin{aligned}\mathcal{L}\left(\frac{\hat{\mu}_1}{\Lambda}\mathcal{U}_2\right) &= -(\beta_1 + \mathfrak{u}_1\beta_2 + \mathfrak{u}_2\beta_3 + \mathfrak{u}_3\alpha_1 + \mathfrak{u}_4\alpha_2) + \frac{\hat{\mu}_1\hat{\mu}_2}{\Lambda} - \left[\beta_1(\hat{S}-1) + \mathfrak{u}_1\beta_2\left(\frac{\hat{S}\hat{I}_2}{I_1} - 1\right)\right.\\ &\quad \left. + \mathfrak{u}_2\beta_3\left(\frac{\hat{S}\hat{I}_3}{I_1} - 1\right) + \mathfrak{u}_3\alpha_1\left(\frac{\hat{S}\hat{T}_1}{I_1} - 1\right) + \mathfrak{u}_4\alpha_2\left(\frac{\hat{S}\hat{T}_2}{I_1} - 1\right)\right]\\ &\le -\frac{\hat{\mu}_1\hat{\mu}_2}{\Lambda}\left(\mathcal{R}_0^{\mathfrak{i}} - 1\right) - \left(\beta_1 + \mathfrak{u}_1\beta_2 + \mathfrak{u}_2\beta_3 + \mathfrak{u}_3\alpha_1 + \mathfrak{u}_4\alpha_2\right)\ln\hat{S} - \mathfrak{u}_1\beta_2\ln\hat{I}_2 - \mathfrak{u}_2\beta_3\ln\hat{I}_3\\ &\quad - \mathfrak{u}_3\alpha_1\ln\hat{T}_1 - \mathfrak{u}_4\alpha_2\ln\hat{T}_2 + \left(\beta_1 + \mathfrak{u}_1\beta_2 + \mathfrak{u}_2\beta_3 + \mathfrak{u}_3\alpha_1 + \mathfrak{u}_4\alpha_2\right)\ln I_1.\end{aligned}$$

In similar fashion, we derive

$$\mathcal{L}(\mathfrak{u}_3\mathcal{U}_3) = -\frac{\delta_1 I_1}{\hat{T}_1} - \frac{\mathfrak{u}_4\omega}{\hat{T}_1} + \mathfrak{u}_3\hat{\mu}_3 = -\delta_1\left(\frac{I_1}{\hat{T}_1} - 1\right) - \mathfrak{u}_4\omega\left(\frac{\hat{T}_2}{\hat{T}_1} - 1\right) \le -\delta_1\ln I_1 - \mathfrak{u}_4\omega\ln\hat{T}_2 + (\delta_1 + \mathfrak{u}_4\omega)\ln\hat{T}_1.$$
$$\mathcal{L}(\mathfrak{u}_1\mathcal{U}_4) = -\frac{\delta_2 I_1}{\hat{I}_2} - \frac{\gamma_1\hat{T}_1}{\hat{I}_2} + \mathfrak{u}_1\hat{\mu}_4 = -\delta_2\left(\frac{I_1}{\hat{I}_2} - 1\right) - \mathfrak{u}_3\gamma_1\left(\frac{\hat{T}_1}{\hat{I}_2} - 1\right) \le -\delta_2\ln I_1 - \mathfrak{u}_3\gamma_1\ln\hat{T}_1 + (\delta_2 + \mathfrak{u}_3\gamma_1)\ln\hat{I}_2.$$
$$\mathcal{L}(\mathfrak{u}_4\mathcal{U}_5) = -\frac{\mathfrak{u}_1\rho_1\hat{I}_2}{\hat{T}_2} + \mathfrak{u}_4\hat{\mu}_5 \le -\mathfrak{u}_1\rho_1\ln\hat{I}_2 + \mathfrak{u}_1\rho_1\ln\hat{T}_2.$$

Ultimately, we obtain for $\mathfrak{u}_2\mathcal{U}_6$ that

$$\begin{aligned}\mathcal{L}(\mathfrak{u}_2\mathcal{U}_6) &= -\frac{\delta_3 I_1}{\hat{I}_3} - \frac{\mathfrak{u}_1\rho_2\hat{I}_2}{\hat{I}_3} - \frac{\mathfrak{u}_4\gamma_2\hat{T}_2}{\hat{I}_3} + \mathfrak{u}_2\hat{\mu}_6 = -\delta_3\left(\frac{I_1}{\hat{I}_3} - 1\right) - \mathfrak{u}_1\rho_2\left(\frac{\hat{I}_2}{\hat{I}_3} - 1\right) - \mathfrak{u}_4\gamma_2\left(\frac{\hat{T}_2}{\hat{I}_3} - 1\right)\\ &\le -\delta_3\ln I_1 - \mathfrak{u}_1\rho_2\ln\hat{I}_2 - \mathfrak{u}_4\gamma_2\ln\hat{T}_2 + (\delta_3 + \mathfrak{u}_1\rho_2 + \mathfrak{u}_4\gamma_2)\ln\hat{I}_3.\end{aligned}$$

Now, we construct the following general C^2-function:

$$\mathcal{U} = \frac{\mathfrak{q}_6}{\hat{\mu}_1}\mathcal{U}_1 + \frac{\hat{\mu}_1}{\Lambda}\mathcal{U}_2 + \mathfrak{q}_2\mathfrak{u}_3\mathcal{U}_3 + \mathfrak{q}_3\mathfrak{u}_1\mathcal{U}_4 + \mathfrak{q}_4\mathfrak{u}_4\mathcal{U}_5 + \mathfrak{q}_5\mathfrak{u}_2\mathcal{U}_6,$$

where the constants $\mathfrak{q}_{\mathfrak{i}}$ $(i = 1, 2, 3, 4, 5)$ are determined later. According to the above calculus, we can conclude that

$$\begin{aligned}\mathcal{L}\mathcal{U} &\le -\frac{\hat{\mu}_1\hat{\mu}_2}{\Lambda}(\mathcal{R}_0^{\mathfrak{i}} - 1) + \left(\mathfrak{q}_1 - \left(\beta_1 + \mathfrak{u}_1\beta_2 + \mathfrak{u}_2\beta_3 + \mathfrak{u}_3\alpha_1 + \mathfrak{u}_4\alpha_2\right)\right)\ln\hat{S}\\ &\quad + \left(\left(\mathfrak{u}_1\beta_2 + \mathfrak{u}_2\beta_3 + \mathfrak{u}_3\alpha_1 + \mathfrak{u}_4\alpha_2\right) - \left(\mathfrak{q}_2\delta_1 + \mathfrak{q}_3\delta_2 + \mathfrak{q}_5\delta_3\right)\right)\ln I_1 + \left(\mathfrak{q}_2\left(\delta_1 + \mathfrak{u}_4\omega\right) - \mathfrak{u}_3\left(\alpha_1 + \mathfrak{q}_3\gamma_1\right)\right)\ln\hat{T}_1\\ &\quad + \left(\mathfrak{q}_3\left(\delta_2 + \mathfrak{u}_3\gamma_1\right) - \mathfrak{u}_1\left(\beta_2 + \mathfrak{q}_4\rho_1 + \mathfrak{q}_5\rho_2\right)\right)\ln\hat{I}_2 + \left(\mathfrak{u}_1\mathfrak{q}_4\rho_1 - \mathfrak{u}_4\left(\alpha_1 + \mathfrak{q}_2\omega + \mathfrak{q}_5\gamma_2\right)\right)\ln\hat{T}_2\\ &\quad + \left(\mathfrak{q}_5\left(\delta_3 + \mathfrak{u}_1\rho_2 + \mathfrak{u}_2\gamma_2\right) - \mathfrak{u}_2\beta_3\right)\ln\hat{I}_3 + \frac{\mathfrak{q}_1\beta_1}{\hat{\mu}_1}I_1 + \frac{\mathfrak{q}_1\beta_2}{\hat{\mu}_1}I_2 + \frac{\mathfrak{q}_1\beta_3}{\hat{\mu}_1}I_3 + \frac{\mathfrak{q}_1\alpha_1}{\hat{\mu}_1}T_1 + \frac{\mathfrak{q}_1\alpha_2}{\hat{\mu}_1}T_2.\end{aligned}$$

Let the constants $\mathfrak{q}_i$ $(i = 1, 2, 3, 4, 5)$ be the unique solution of the following system

$$\begin{cases}\mathfrak{q}_1 - \left(\beta_1 + \mathfrak{u}_1\beta_2 + \mathfrak{u}_2\beta_3 + \mathfrak{u}_3\alpha_1 + \mathfrak{u}_4\alpha_2\right) = 0,\\ \left(\mathfrak{u}_1\beta_2 + \mathfrak{u}_2\beta_3 + \mathfrak{u}_3\alpha_1 + \mathfrak{u}_4\alpha_2\right) - \left(\mathfrak{q}_2\delta_1 + \mathfrak{q}_3\delta_2 + \mathfrak{q}_5\delta_3\right) = 0,\\ \mathfrak{q}_2\left(\delta_1 + \mathfrak{u}_4\omega\right) - \mathfrak{u}_3\left(\alpha_1 + \mathfrak{q}_3\gamma_1\right) = 0,\\ \mathfrak{q}_3\left(\delta_2 + \mathfrak{u}_3\gamma_1\right) - \mathfrak{u}_1\left(\beta_2 + \mathfrak{q}_4\rho_1 + \mathfrak{q}_5\rho_2\right) = 0,\\ \mathfrak{u}_1\mathfrak{q}_4\rho_1 - \mathfrak{u}_4\left(\alpha_1 + \mathfrak{q}_2\omega + \mathfrak{q}_5\gamma_2\right) = 0,\\ \mathfrak{q}_5\left(\delta_3 + \mathfrak{u}_1\rho_2 + \mathfrak{u}_4\gamma_2\right) - \mathfrak{u}_2\beta_3 = 0.\end{cases} \tag{IV.21}$$

After an unpretentious calculation, we achieve the following explicit solution

$$\begin{cases} \mathfrak{q}_1 &= \beta_1 + \mathfrak{u}_1\beta_2 + \mathfrak{u}_2\beta_3 + \mathfrak{u}_3\alpha_1 + \mathfrak{u}_4\alpha_2 > 0, \\ \mathfrak{q}_2 &= \dfrac{\alpha_1\hat{\mu}_4\hat{\mu}_5 + \gamma_1\rho_1(\alpha_2 + \mathfrak{q}_5\gamma_2) + \gamma_1\hat{\mu}_5(\beta_2 + \mathfrak{q}_5\rho_2)}{\hat{\mu}_3\hat{\mu}_4\hat{\mu}_5 - \gamma_1\omega\rho_1} > 0, \\ \mathfrak{q}_3 &= \dfrac{\alpha_1\rho_1\omega + \rho_1\hat{\mu}_3(\alpha_2 + \mathfrak{q}_5) + \hat{\mu}_3\hat{\mu}_5(\beta_2 + \mathfrak{q}_5\rho_2)}{\hat{\mu}_3\hat{\mu}_4\hat{\mu}_5 - \gamma_1\omega\rho_1} > 0, \\ \mathfrak{q}_4 &= \dfrac{\alpha_1\omega\hat{\mu}_4 + \hat{\mu}_3\hat{\mu}_4(\alpha_2 + \mathfrak{q}_5\gamma_2) + \gamma_1\omega(\beta_2 + \mathfrak{q}_5\rho_2)}{\hat{\mu}_3\hat{\mu}_4\hat{\mu}_5 - \gamma_1\omega\rho_1} > 0, \\ \mathfrak{q}_5 &= \beta_3\hat{\mu}_6^{-1} > 0. \end{cases}$$

Consequently, one can get that

$$\mathcal{L}\mathcal{U} \leq -\frac{\hat{\mu}_1\hat{\mu}_2}{\Lambda}\big(\mathcal{R}_0^{\mathsf{i}} - 1\big) + \frac{\mathfrak{q}_1\beta_1}{\hat{\mu}_1}I_1 + \frac{\mathfrak{q}_1\beta_2}{\hat{\mu}_1}I_2 + \frac{\mathfrak{q}_1\beta_3}{\hat{\mu}_1}I_3 + \frac{\mathfrak{q}_1\alpha_1}{\hat{\mu}_1}T_1 + \frac{\mathfrak{q}_1\alpha_2}{\hat{\mu}_1}T_2.$$

In general and based on the Itô's formula, we show that

$$\begin{aligned} \mathrm{d}\mathcal{U}\big(S, I_1, I_2, I_3, T_1, T_2\big) \leq{}& \left(-\frac{\hat{\mu}_1\hat{\mu}_2}{\Lambda}\big(\mathcal{R}_0^{\mathsf{i}} - 1\big) + \frac{\mathfrak{q}_1\beta_1}{\hat{\mu}_1}I_1 + \frac{\mathfrak{q}_1\beta_2}{\hat{\mu}_1}I_2 + \frac{\mathfrak{q}_1\beta_3}{\hat{\mu}_1}I_3 + \frac{\mathfrak{q}_1\alpha_1}{\hat{\mu}_1}T_1 + \frac{\mathfrak{q}_1\alpha_2}{\hat{\mu}_1}T_2\right)\mathrm{d}t \\ &- \frac{\sigma_1}{\hat{\mu}_1}\mathrm{d}\mathcal{B}_1(t) - \frac{\hat{\mu}_1\sigma_2}{\Lambda}\mathrm{d}\mathcal{B}_2(t) - \mathfrak{u}_3\sigma_3\mathrm{d}\mathcal{B}_3(t) - \mathfrak{u}_1\sigma_4\mathrm{d}\mathcal{B}_4(t) - \mathfrak{u}_4\sigma_5\mathrm{d}\mathcal{B}_5(t) - \mathfrak{u}_2\sigma_6\mathrm{d}\mathcal{B}_6(t) \\ &- \frac{1}{\hat{\mu}_1}\int_{\mathbb{U}}\ln(1+\zeta_1(u))\widetilde{\mathcal{N}}(\mathrm{d}t,\mathrm{d}u) - \frac{\hat{\mu}_1}{\Lambda}\int_{\mathbb{U}}\ln(1+\zeta_2(u))\widetilde{\mathcal{N}}(\mathrm{d}t,\mathrm{d}u) \\ &- \int_{\mathbb{U}}\mathfrak{u}_3\ln(1+\zeta_3(u))\widetilde{\mathcal{N}}(\mathrm{d}t,\mathrm{d}u) - \int_{\mathbb{U}}\mathfrak{u}_1\ln(1+\zeta_4(u))\widetilde{\mathcal{N}}(\mathrm{d}t,\mathrm{d}u) \\ &- \int_{\mathbb{U}}\mathfrak{u}_4\ln(1+\zeta_5(u))\widetilde{\mathcal{N}}(\mathrm{d}t,\mathrm{d}u) - \int_{\mathbb{U}}\mathfrak{u}_2\ln(1+\zeta_6(u))\widetilde{\mathcal{N}}(\mathrm{d}t,\mathrm{d}u). \end{aligned}$$

Integrating the previous inequality from 0 to t, and then dividing by t on both sides, we get

$$\begin{aligned} &\frac{\mathcal{U}\big(S(t), I_1(t), I_2(t), I_3(t), T_1(t), T_2(t)\big)}{t} - \frac{\mathcal{U}\big(S(0), I_1(0), I_2(0), I_3(0), T_1(0), T_2(0)\big)}{t} \\ &\leq -\frac{\hat{\mu}_1\hat{\mu}_2}{\Lambda}\big(\mathcal{R}_0^{\mathsf{i}} - 1\big) + \frac{\mathfrak{q}_1\beta_1}{\hat{\mu}_1 t}\int_0^t I_1(s)\mathrm{d}s + \frac{\mathfrak{q}_1\beta_2}{\hat{\mu}_1 t}\int_0^t I_2(s)\mathrm{d}s \\ &\quad + \frac{\mathfrak{q}_1\beta_3}{\hat{\mu}_1 t}\int_0^t I_3(s)\mathrm{d}s + \frac{\mathfrak{q}_1\alpha_1}{\hat{\mu}_1 t}\int_0^t T_1(s)\mathrm{d}s + \frac{\mathfrak{q}_1\alpha_2}{\hat{\mu}_1 t}\int_0^t T_2(s)\mathrm{d}s \\ &\quad - \frac{\sigma_1}{\hat{\mu}_1 t}\mathcal{B}_1(t) - \frac{\hat{\mu}_1\sigma_2}{\Lambda t}\mathcal{B}_2(t) - \frac{\mathfrak{u}_3\sigma_3}{t}\mathcal{B}_3(t) - \frac{\mathfrak{u}_1\sigma_4}{t}\mathcal{B}_4(t) - \frac{\mathfrak{u}_4\sigma_5}{t}\mathcal{B}_5(t) - \frac{\mathfrak{u}_2\sigma_6}{t}\mathcal{B}_6(t) \\ &\quad - \frac{1}{\hat{\mu}_1 t}\int_0^t\int_{\mathbb{U}}\ln(1+\zeta_1(u))\widetilde{\mathcal{N}}(\mathrm{d}s,\mathrm{d}u) - \frac{\hat{\mu}_1}{\Lambda t}\int_0^t\int_{\mathbb{U}}\ln(1+\zeta_2(u))\widetilde{\mathcal{N}}(\mathrm{d}s,\mathrm{d}u) \\ &\quad - \frac{1}{t}\int_0^t\int_{\mathbb{U}}\mathfrak{u}_3\ln(1+\zeta_3(u))\widetilde{\mathcal{N}}(\mathrm{d}s,\mathrm{d}u) - \frac{1}{t}\int_0^t\int_{\mathbb{U}}\mathfrak{u}_1\ln(1+\zeta_4(u))\widetilde{\mathcal{N}}(\mathrm{d}s,\mathrm{d}u) \\ &\quad - \frac{1}{t}\int_0^t\int_{\mathbb{U}}\mathfrak{u}_4\ln(1+\zeta_5(u))\widetilde{\mathcal{N}}(\mathrm{d}s,\mathrm{d}u) - \frac{1}{t}\int_0^t\int_{\mathbb{U}}\mathfrak{u}_2\ln(1+\zeta_6(u))\widetilde{\mathcal{N}}(\mathrm{d}s,\mathrm{d}u). \end{aligned}$$

Therefore

$$\begin{aligned}
&\frac{\mathcal{U}\big(S(t),I_1(t),I_2(t),I_3(t),T_1(t),T_2(t)\big)}{t}-\frac{\mathcal{U}\big(S(0),I_1(0),I_2(0),I_3(0),T_1(0),T_2(0)\big)}{t}\\
&\leq -\frac{\hat{\mu}_1\hat{\mu}_2}{\Lambda}\big(\mathcal{R}_0^{\mathsf{i}}-1\big)+\overbrace{\max\left\{\frac{\mathfrak{q}_1\beta_1}{\hat{\mu}_1},\frac{\mathfrak{q}_1\beta_2}{\hat{\mu}_1},\frac{\mathfrak{q}_1\beta_3}{\hat{\mu}_1},\frac{\mathfrak{q}_1\alpha_1}{\hat{\mu}_1},\frac{\mathfrak{q}_1\alpha_1}{\hat{\mu}_1}\right\}}^{\triangleq\mathfrak{M}_0}\frac{1}{t}\int_0^t\left(\sum_{i=1}^3 I_i(s)+\sum_{i=1}^2 T_i(s)\right)\mathrm{d}s\\
&\quad-\frac{\sigma_1}{\hat{\mu}_1 t}\mathcal{B}_1(t)-\frac{\hat{\mu}_1\sigma_2}{\Lambda t}\mathcal{B}_2(t)-\frac{\mathfrak{u}_3\sigma_3}{t}\mathcal{B}_3(t)-\frac{\mathfrak{u}_1\sigma_4}{t}\mathcal{B}_4(t)-\frac{\mathfrak{u}_4\sigma_5}{t}\mathcal{B}_5(t)-\frac{\mathfrak{u}_2\sigma_6}{t}\mathcal{B}_6(t)\\
&\quad-\frac{1}{\hat{\mu}_1 t}\int_0^t\int_{\mathbb{U}}\ln(1+\zeta_1(u))\widetilde{\mathcal{N}}(\mathrm{d}s,\mathrm{d}u)-\frac{\hat{\mu}_1}{\Lambda t}\int_0^t\int_{\mathbb{U}}\ln(1+\zeta_2(u))\widetilde{\mathcal{N}}(\mathrm{d}s,\mathrm{d}u)\\
&\quad-\frac{1}{t}\int_0^t\int_{\mathbb{U}}\mathfrak{u}_3\ln(1+\zeta_3(u))\widetilde{\mathcal{N}}(\mathrm{d}s,\mathrm{d}u)-\frac{1}{t}\int_0^t\int_{\mathbb{U}}\mathfrak{u}_1\ln(1+\zeta_4(u))\widetilde{\mathcal{N}}(\mathrm{d}s,\mathrm{d}u)\\
&\quad-\frac{1}{t}\int_0^t\int_{\mathbb{U}}\mathfrak{u}_4\ln(1+\zeta_5(u))\widetilde{\mathcal{N}}(\mathrm{d}s,\mathrm{d}u)-\frac{1}{t}\int_0^t\int_{\mathbb{U}}\mathfrak{u}_2\ln(1+\zeta_6(u))\widetilde{\mathcal{N}}(\mathrm{d}s,\mathrm{d}u).
\end{aligned}$$

Accordingly

$$\begin{aligned}
&\frac{1}{t}\int_0^t\left(\sum_{i=1}^3 I_i(s)+\sum_{i=1}^2 T_i(s)\right)\mathrm{d}s\\
&\geq\frac{\mathcal{U}\big(S(t),I_1(t),I_2(t),I_3(t),T_1(t),T_2(t)\big)}{\mathfrak{M}_0 t}-\frac{\mathcal{U}\big(S(0),I_1(0),I_2(0),I_3(0),T_1(0),T_2(0)\big)}{\mathfrak{M}_0 t}\\
&\quad+\frac{\hat{\mu}_1\hat{\mu}_2}{\Lambda\mathfrak{M}_0}\big(\mathcal{R}_0^{\mathsf{i}}-1\big)+\frac{\sigma_1}{\hat{\mu}_1\mathfrak{M}_0 t}\mathcal{B}_1(t)+\frac{\hat{\mu}_1\sigma_2}{\Lambda\mathfrak{M}_0 t}\mathcal{B}_2(t)+\frac{\mathfrak{u}_3\sigma_3}{\mathfrak{M}_0 t}\mathcal{B}_3(t)+\frac{\mathfrak{u}_1\sigma_4}{\mathfrak{M}_0 t}\mathcal{B}_4(t)+\frac{\mathfrak{u}_4\sigma_5}{\mathfrak{M}_0 t}\mathcal{B}_5(t)+\frac{\mathfrak{u}_2\sigma_6}{\mathfrak{M}_0 t}\mathcal{B}_6(t)\\
&\quad+\frac{1}{\hat{\mu}_1\mathfrak{M}_0 t}\int_0^t\int_{\mathbb{U}}\ln(1+\zeta_1(u))\widetilde{\mathcal{N}}(\mathrm{d}s,\mathrm{d}u)+\frac{\hat{\mu}_1}{\Lambda\mathfrak{M}_0 t}\int_0^t\int_{\mathbb{U}}\ln(1+\zeta_2(u))\widetilde{\mathcal{N}}(\mathrm{d}s,\mathrm{d}u)\\
&\quad+\frac{1}{\mathfrak{M}_0 t}\int_0^t\int_{\mathbb{U}}\mathfrak{u}_3\ln(1+\zeta_3(u))\widetilde{\mathcal{N}}(\mathrm{d}s,\mathrm{d}u)+\frac{1}{\mathfrak{M}_0 t}\int_0^t\int_{\mathbb{U}}\mathfrak{u}_1\ln(1+\zeta_4(u))\widetilde{\mathcal{N}}(\mathrm{d}s,\mathrm{d}u)\\
&\quad+\frac{1}{\mathfrak{M}_0 t}\int_0^t\int_{\mathbb{U}}\mathfrak{u}_4\ln(1+\zeta_5(u))\widetilde{\mathcal{N}}(\mathrm{d}s,\mathrm{d}u)+\frac{1}{\mathfrak{M}_0 t}\int_0^t\int_{\mathbb{U}}\mathfrak{u}_2\ln(1+\zeta_6(u))\widetilde{\mathcal{N}}(\mathrm{d}s,\mathrm{d}u).
\end{aligned}$$

Since $\ln(\mathfrak{a})\leq\mathfrak{a}-1\leq\mathfrak{a}$ for any $\mathfrak{a}>0$, one can conclude that

$$\frac{1}{t}\int_0^t\left(\sum_{i=1}^3 I_i(s)+\sum_{i=1}^2 T_i(s)\right)\mathrm{d}s \tag{IV.22}$$

$$\begin{aligned}
&\geq-\frac{\overbrace{\frac{\mathfrak{q}_6}{\hat{\mu}_1}S(t)+\frac{\hat{\mu}_1}{\Lambda}I_1(t)+\mathfrak{q}_2\mathfrak{u}_3T_1(t)+\mathfrak{q}_3\mathfrak{u}_1I_2(t)+\mathfrak{q}_4\mathfrak{u}_4T_2(t)+\mathfrak{q}_5\mathfrak{u}_2I_3(t)}^{\triangleq\mathfrak{M}_1}}{\mathfrak{M}_0 t}\\
&\quad-\frac{\mathcal{U}\big(S(0),I_1(0),I_2(0),I_3(0),T_1(0),T_2(0)\big)}{\mathfrak{M}_0 t}+\frac{\hat{\mu}_1\hat{\mu}_2}{\Lambda\mathfrak{M}_0}\big(\mathcal{R}_0^{\mathsf{i}}-1\big)+\frac{\sigma_1}{\hat{\mu}_1\mathfrak{M}_0 t}\mathcal{B}_1(t)\\
&\quad+\frac{\hat{\mu}_1\sigma_2}{\Lambda\mathfrak{M}_0 t}\mathcal{B}_2(t)+\frac{\mathfrak{u}_3\sigma_3}{\mathfrak{M}_0 t}\mathcal{B}_3(t)+\frac{\mathfrak{u}_1\sigma_4}{\mathfrak{M}_0 t}\mathcal{B}_4(t)+\frac{\mathfrak{u}_4\sigma_5}{\mathfrak{M}_0 t}\mathcal{B}_5(t)+\frac{\mathfrak{u}_2\sigma_6}{\mathfrak{M}_0 t}\mathcal{B}_6(t)\\
&\quad+\frac{1}{\hat{\mu}_1\mathfrak{M}_0 t}\int_0^t\int_{\mathbb{U}}\ln(1+\zeta_1(u))\widetilde{\mathcal{N}}(\mathrm{d}s,\mathrm{d}u)+\frac{\hat{\mu}_1}{\Lambda\mathfrak{M}_0 t}\int_0^t\int_{\mathbb{U}}\ln(1+\zeta_2(u))\widetilde{\mathcal{N}}(\mathrm{d}s,\mathrm{d}u)\\
&\quad+\frac{1}{\mathfrak{M}_0 t}\int_0^t\int_{\mathbb{U}}\mathfrak{u}_3\ln(1+\zeta_3(u))\widetilde{\mathcal{N}}(\mathrm{d}s,\mathrm{d}u)+\frac{1}{\mathfrak{M}_0 t}\int_0^t\int_{\mathbb{U}}\mathfrak{u}_1\ln(1+\zeta_4(u))\widetilde{\mathcal{N}}(\mathrm{d}s,\mathrm{d}u)\\
&\quad+\frac{1}{\mathfrak{M}_0 t}\int_0^t\int_{\mathbb{U}}\mathfrak{u}_4\ln(1+\zeta_5(u))\widetilde{\mathcal{N}}(\mathrm{d}s,\mathrm{d}u)+\frac{1}{\mathfrak{M}_0 t}\int_0^t\int_{\mathbb{U}}\mathfrak{u}_2\ln(1+\zeta_6(u))\widetilde{\mathcal{N}}(\mathrm{d}s,\mathrm{d}u).
\end{aligned} \tag{IV.23}$$

For simplicity, we adopt the following notations

$$\mathfrak{M}_2\triangleq\frac{\sigma_1}{\hat{\mu}_1\mathfrak{M}_0}\mathcal{B}_1(t)+\frac{\hat{\mu}_1\sigma_2}{\Lambda\mathfrak{M}_0}\mathcal{B}_2(t)+\frac{\mathfrak{u}_3\sigma_3}{\mathfrak{M}_0}\mathcal{B}_3(t)+\frac{\mathfrak{u}_1\sigma_4}{\mathfrak{M}_0}\mathcal{B}_4(t)+\frac{\mathfrak{u}_4\sigma_5}{\mathfrak{M}_0}\mathcal{B}_5(t)+\frac{\mathfrak{u}_2\sigma_6}{\mathfrak{M}_0}\mathcal{B}_6(t),$$

and

$$\begin{aligned}\mathfrak{M}_3 \triangleq\ & \frac{1}{\hat{\mu}_1\mathfrak{M}_0}\int_0^t\int_{\mathbb{U}}\ln(1+\zeta_1(u))\widetilde{\mathcal{N}}(\mathrm{d}s,\mathrm{d}u)+\frac{\hat{\mu}_1}{\Lambda\mathfrak{M}_0}\int_0^t\int_{\mathbb{U}}\ln(1+\zeta_2(u))\widetilde{\mathcal{N}}(\mathrm{d}s,\mathrm{d}u)\\ &+\frac{1}{\mathfrak{M}_0}\int_0^t\int_{\mathbb{U}}\mathrm{u}_3\ln(1+\zeta_3(u))\widetilde{\mathcal{N}}(\mathrm{d}s,\mathrm{d}u)+\frac{1}{\mathfrak{M}_0}\int_0^t\int_{\mathbb{U}}\mathrm{u}_1\ln(1+\zeta_4(u))\widetilde{\mathcal{N}}(\mathrm{d}s,\mathrm{d}u)\\ &+\frac{1}{\mathfrak{M}_0}\int_0^t\int_{\mathbb{U}}\mathrm{u}_4\ln(1+\zeta_5(u))\widetilde{\mathcal{N}}(\mathrm{d}s,\mathrm{d}u)+\frac{1}{\mathfrak{M}_0}\int_0^t\int_{\mathbb{U}}\mathrm{u}_2\ln(1+\zeta_6(u))\widetilde{\mathcal{N}}(\mathrm{d}s,\mathrm{d}u).\end{aligned}$$

Patently, $\mathfrak{M}_2(t)$ is a local martingale with finite quadratic variation, and from the condition $(\mathfrak{C}_5)$ we can affirm that $\mathfrak{M}_3(t)$ is also so. Then, we deduce by the strong law of large numbers for local martingales that

$$\lim_{t\to\infty}\frac{\mathfrak{M}_2(t)}{t}=0\quad\text{and}\quad\lim_{t\to\infty}\frac{\mathfrak{M}_3(t)}{t}=0\qquad\text{a.s.}\tag{IV.24}$$

Furthermore, from Lemma 3.1, we establish that

$$\lim_{t\to\infty}\frac{\mathfrak{M}_1(t)}{t}=0\qquad\text{a.s.}\tag{IV.25}$$

Taking the inferior limit on both sides of (IV.23) and combining the resulting inequality with (IV.24) and (IV.25) yields

$$\liminf_{t\to\infty}\frac{1}{t}\int_0^t\left(\sum_{i=1}^{3}I_i(s)+\sum_{i=1}^{2}T_i(s)\right)\mathrm{d}s\geq\frac{\hat{\mu}_1\hat{\mu}_2}{\Lambda\mathfrak{M}_0}\left(\mathcal{R}_0^{\mathrm{i}}-1\right)>0\quad\text{a.s.}$$

■

5 Numerical simulation

In order to verify the various analytical results proved in this chapter, we perform some numerical simulations by taking the parameter values from the theoretical data presented in Table IV.2. The trajectories of the analyzed AIDS model are simulated in our case with the initial data given by $S(0)=0.2, I_1(0)=0.1, T_1(0)=0.1, I_2(0)=0.3, T_2(0)=0.3, I_3(0)=0.3$. In the following, the unity of time is expressed by one day and the number of individuals is one million population.

5.1 AIDS extinction case

In this example, and by choosing the parameter values presented in the third column of Table IV.2, and also taking $\mathbb{U}=\mathbb{R}_+$ with $\nu(\mathbb{U})=1$, we will examine the theoretical results of Section 3. By performing straightforward calculations, we obtain the numerical values presented in Table IV.3. Consequently, we conclude easily that the condition $\mathcal{T}<0$, and hypothesis $(\mathfrak{C}_6)$, $(\mathfrak{C}_7)$, $(\mathfrak{C}_8)$ and $(\mathfrak{C}_9)$ hold. So, and by the virtue of Theorem 3.1, the AIDS epidemic dies out exponentially almost surely, which is exactly shown in Figure IV.2.

5.1.1 Effect of the new proposed threshold on the AIDS extinction in model (IV.5)

If we used the same calculus as [159] in our treatment, the extinction threshold that we will obtain in the case of the stochastic model (IV.5) takes the following form

$$\mathcal{R}_0^{E,\sigma}=\int_0^\infty\left|x-\frac{\Lambda}{\mu_1}\right|\pi_\sigma(x)\mathrm{d}x+\frac{\Lambda(\mathcal{R}_0-1)}{\mu_1}\left(\bar{\chi}\mathbb{1}_{\{\mathcal{R}_0>1\}}+-\underline{\chi}\mathbb{1}_{\{\mathcal{R}_0\leq1\}}\right)-\frac{\min\{\sigma_2^2,\sigma_3^2,\sigma_4^2,\sigma_5^2,\sigma_6^2\}}{10}\approx0.3166>0.$$

Consequently, the last expression is positive. Therefore, it will be unable to guarantee the disease's extinction. On the other hand, by using our new analysis, we get

$$\mathcal{T}_\sigma=\mathfrak{R}_0\times\overline{\chi}\times\frac{\Lambda}{2\mu_1}\sqrt{\frac{2\mu_1}{\phi_\sigma}-1}+\mathcal{U}-\mathfrak{S}=-0.0507<0.$$

The new threshold $\mathcal{T}_\sigma$ that we have proposed is well capable of doing it, and this reflects clearly its sharpness.

5.2 Persistence in the mean case

In this part, we will illustrate the persistence in the mean case. By employing the numerical values appearing in the last column of Table IV.2, we can directly draw up the values listed in Table IV.4. Based on the last table's numerical values, the hypotheses $(\mathfrak{C}_6)-(\mathfrak{C}_5)$ hold, and the threshold $\mathcal{R}_0^{\mathrm{i}} > 1$ higher than one. Consequently, and by Theorem 4.1, the AIDS epidemic is persistent in the mean, which agrees well with the results shown in Figure IV.3. Furthermore, in Figure IV.4, we present the path of the sum $I_1(t)+T_1(t)+I_2(t)+T_2(t)+I_3(t)$ to illustrate its persistence in the mean.

5.3 The effect of Lévy jumps on AIDS dynamics

To examine the impact of white noises and Lévy jumps intensities on AIDS dynamics, in this numerical example, we will compare the trajectories of the following systems:

- The SDE-LJ (IV.4) ($\sigma_i \neq 0$ and $\zeta_i \neq 0$, $i = 1, 2 \cdots, 7$).
- The stochastic system with Brownian motions (IV.5) ($\sigma_i \neq 0$ and $\zeta_i = 0$, $i = 1, 2 \cdots, 7$).
- The deterministic model (IV.1) ($\sigma_i = 0$ and $\zeta_i = 0$, $i = 1, 2 \cdots, 7$).

For the sake of comparison, we take the same initial data and we adopt the same parameter values as used in the two preceding numerical examples, but each time we reduce to zero some intensities of randomness (σ_i and ζ_i) as mentioned in the above list. We see from Figure (IV.5) that the impact of Levy jumps lead to the extinction of the disease while the deterministic model (IV.1) and the perturbed model driven by Brownian motions both go to persistence. Hence, we say that the jumps have negative effects on the AIDS prevalence. This means that jumps can modify the asymptotic behavior of the epidemic model significantly. To explore the effect of jumps on system (IV.1) in the persistence case, we present the Figure (IV.6), in which we observe the persistence of the epidemic in all cases with a greater variation in the Levy jumps case. Therefore, we can say that the jumps have positive effects on the AIDS prevalence in this case.

Parameters	Description	Numerical values	
Λ	Susceptible recruitment rate	0.9	0.7
β_1	Contact rates between S and I_1	0.2	1.15
β_2	Contact rates between S and I_2	0.1	0.7
β_3	Contact rates between S and I_3	0.4	0.9
α_1	Contact rates between S and T_2	0.061	0.97
α_2	Contact rates between S and T_2	0.08	1.3
μ_1	Death rate of the classe S	0.6	0.6
μ_2	Death rate of the classe I_1	0.65	0.65
μ_3	Death rate of the classe I_2	0.61	0.61
μ_4	Death rate of the classe I_3	0.56	0.56
μ_5	Death rate of the classe T_1	0.67	0.67
μ_6	Death rate of the classe T_2	0.5	0.5
δ_1	Rate at which the individuals of the class I_1 develop into T_1	0.1	0.1
δ_2	Rate at which the individuals of the class I_1 develop into I_2	0.1	0.1
δ_3	Rate at which the individuals of the class I_1 develop into T_1	0.1	0.1
ρ_1	Progression rate from $I_2(t)$ to T_2	0.8	0.8
ρ_2	Progression rate from $I_2(t)$ to T_3	0.8	0.8
γ_1	Transmission rates from $I_1(t)$ to I_2	0.269	0.4
γ_2	Transmission rates from $T_2(t)$ to I_3	0.9	0.9
η	Treatment rate of the AIDS patients	0.7	0.7
ω	Effective treatment rate of the group T_2	0.1	0.1
σ_1	Intensity of the Brownian motion $\mathcal{B}_1$	0.269	0.269
σ_2	Intensity of the Brownian motion $\mathcal{B}_2$	0.25	0.25
σ_3	Intensity of the Brownian motion $\mathcal{B}_3$	0.24	0.24
σ_4	Intensity of the Brownian motion $\mathcal{B}_4$	0.23	0.23
σ_5	Intensity of the Brownian motion $\mathcal{B}_5$	0.275	0.275
σ_6	Intensity of the Brownian motion $\mathcal{B}_6$	0.213	0.213
ζ_1	Intensity of the Lévy jumps associated to S	0.75	0.75
ζ_2	Intensity of the Lévy jumps associated to I_1	-0.8	-0.8
ζ_3	Intensity of the Lévy jumps associated to T_1	-0.9	-0.9
ζ_4	Intensity of the Lévy jumps associated to I_2	-0.85	-0.85
ζ_5	Intensity of the Lévy jumps associated to T_2	-0.3	-0.3
ζ_6	Intensity of the Lévy jumps associated to T_3	-0.5	-0.5
		Figure IV.2	Figure IV.3

Table IV.2: Definitions and nominal values, per day, of the system parameters and perturbations intensities used in the simulations.

	Quantity	Expression	Value
Condition $(\mathfrak{C}_6)$	$\mathfrak{m}_1$	$\max\limits_{1\leqslant i\leqslant 6}\left(\int_{\mathbb{U}}\zeta_i^2(u)\nu(\mathrm{d}u)\right)$	0.81< ∞
Condition $(\mathfrak{C}_7)$	$\mathfrak{m}_2$	$\max\limits_{1\leqslant i\leqslant 6}\left(\int_{\mathbb{U}}\left(\zeta_i(u)-\ln\left(1+\zeta_i(u)\right)\right)\nu(\mathrm{d}u)\right)$	1.4026< ∞
	$\bar{\sigma}$	$\max\left\{\sigma_i^2 \mid i\in\{1,2,3,4,5,6\}\right\}$	0.0724
	$\underline{\mu}$	$\min\left\{\mu_i \mid i\in\{1,2,3,4,5,6\}\right\}$	0.5
	$\widetilde{\zeta}$	$\max\left\{\zeta_i(u) \mid i\in\{1,2,3,4,5,6\}\right\}$	0.75
	$\underset{\sim}{\zeta}$	$\min\left\{\zeta_i(u) \mid i\in\{1,2,3,4,5,6\}\right\}$	-0.9
	$\bar{\lambda}_q$	$\left(1+\widetilde{\zeta}\right)^q - q\times\widetilde{\zeta}-1$	0.6638
	$\underline{\lambda}_q$	$\left(1+\underset{\sim}{\zeta}\right)^q - q\times\underset{\sim}{\zeta}-1$	0.8979
	λ_q	$\underline{\lambda}_q\vee\bar{\lambda}_q$	0.8979
	$\mathcal{I}_q$	$\int_{\mathbb{U}}\lambda_q(u)\nu(\mathrm{d}u)$	0.8979
Condition $(\mathfrak{C}_8)$	φ_q	$\underline{\mu}-\frac{(q-1)\overline{\sigma}}{2}-\frac{\mathcal{I}_q}{q}$	0.0326> 0
Condition $(\mathfrak{C}_9)$	$\mathfrak{m}_3$	$\max\limits_{1\leqslant i\leqslant 6}\left(\int_{\mathbb{U}}\left(\left(1+\zeta_i(u)\right)^2-1\right)^2\nu(\mathrm{d}u)\right)$	4.2539< ∞
Condition $(\mathfrak{C}_5)$	$\mathfrak{m}_4$	$\max\limits_{1\leqslant i\leqslant 6}\left(\int_{\mathbb{U}}\left(\ln\left(1+\xi_i(u)\right)\right)^2\nu(\mathrm{d}u)\right)$	5.3019
	ϕ	$2\mu_1-\sigma_1^2-\int_{\mathbb{U}}\zeta_1^2(u)\nu(\mathrm{d}u)$	0.5651
	$\widetilde{\zeta}$	$\max\left\{\zeta_i(u) \mid i\in\{2,3,4,5,6\}\right\}$	-0.19
	$\underset{\sim}{\zeta}$	$\min\left\{\zeta_i(u) \mid i\in\{2,3,4,5,6\}\right\}$	-0.85
	$\overline{\mathfrak{h}}$	$\left(\ln(1+\widetilde{\zeta})-\widetilde{\zeta}\right)\times\mathbb{1}_{\left\{\underset{\sim}{\zeta}\leq 0\right\}}$	-0.0207
	$\mathcal{H}$	$\int_{\mathbb{U}}\left(\overline{\mathfrak{h}}(u)+\underline{\mathfrak{h}}(u)\right)\nu(\mathrm{d}u)$	-0.0207
	$\mathfrak{S}$	$\frac{1}{2}\left(\sum\limits_{i=2}^{6}\frac{1}{\sigma_i^2}\right)^{-1}$	0.0399
	$\bar{\chi}$	$\max\left\{\frac{\beta_1}{\mathfrak{R}_0},\frac{\beta_2}{\mathfrak{c}_2},\frac{\beta_3}{\mathfrak{c}_4},\frac{\alpha_1}{\mathfrak{c}_1},\frac{\alpha_2}{\mathfrak{c}_3}\right\}$	0.8
	$\underline{\chi}$	$\min\left\{\frac{\beta_1}{\mathfrak{R}_0},\frac{\beta_2}{\mathfrak{c}_2},\frac{\beta_3}{\mathfrak{c}_4},\frac{\alpha_1}{\mathfrak{c}_1},\frac{\alpha_2}{\mathfrak{c}_3}\right\}$	0.2657
	$\mathfrak{R}_0$	$\frac{\Lambda\left(\beta_1+\vartheta_1\beta_2+\vartheta_2\beta_3+\vartheta_3\alpha_1+\vartheta_4\alpha_2\right)}{\mu_1(\mu_2+\delta_1+\delta_2+\delta_3)}$	0.4347
	$\mathcal{U}$	$\frac{A}{\mu_1}\left(\bar{\chi}\left(\mathfrak{R}_0-1\right)^+-\underline{\chi}\left(1-\mathfrak{R}_0\right)^+\right)$	-0.2253
	$\mathcal{T}$	$\mathfrak{R}_0\times\overline{\chi}\times\frac{\Lambda}{2\mu_1}\sqrt{\frac{2\mu_1}{\phi}-1}+\mathcal{U}+\mathcal{H}-\mathfrak{S}$	-0.1952< 0

Table IV.3: Some quantities and their corresponding numerical values in the case of $q = 2.1 > 2$.

	Quantity	Expression	Value
Condition ($\mathfrak{C}_6$)	$\mathfrak{m}_1$	$\max\limits_{1\leqslant i\leqslant 6}\left(\int_{\mathbb{U}}\xi_i^2(u)\nu(\mathrm{d}u)\right)$	$0.81<\infty$
Condition ($\mathfrak{C}_7$)	$\mathfrak{m}_2$	$\max\limits_{1\leqslant i\leqslant 6}\left(\int_{\mathbb{U}}\xi_i(u)-\ln\left(1+\xi_i(u)\right)\nu(\mathrm{d}u)\right)$	$1.4026<\infty$
	$\bar{\sigma}$	$\max\left\{\sigma_i^2 \mid i\in\{1,2,3,4,5,6\}\right\}$	0.0724
	$\underline{\mu}$	$\min\left\{\mu_i \mid i\in\{1,2,3,4,5,6\}\right\}$	0.5
	$\widetilde{\zeta}$	$\max\left\{\zeta_i(u) \mid i\in\{1,2,3,4,5,6\}\right\}$	0.75
	$\underset{\sim}{\zeta}$	$\min\left\{\zeta_i(u) \mid i\in\{1,2,3,4,5,6\}\right\}$	-0.9
	$\bar{\lambda}_q$	$\left(1+\widetilde{\zeta}\right)^q - q\times\widetilde{\zeta}-1$	0.6638
	$\underline{\lambda}_q$	$\left(1+\underset{\sim}{\zeta}\right)^q - q\times\underset{\sim}{\zeta}-1$	0.8979
	λ_q	$\underline{\lambda}_q\vee\bar{\lambda}_q$	0.8979
	$\mathcal{I}_q$	$\int_{\mathbb{U}}\lambda_q(u)\nu(\mathrm{d}u)$	0.8979
Condition ($\mathfrak{C}_8$)	φ_q	$\underline{\mu}-\frac{(q-1)\overline{\sigma}}{2}-\frac{\mathcal{I}_q}{q}$	$0.0326>0$
Condition ($\mathfrak{C}_9$)	$\mathfrak{m}_3$	$\max\limits_{1\leqslant i\leqslant 6}\left(\int_{\mathbb{U}}\left(\left(1+\xi_i(u)\right)^2-1\right)^2\nu(\mathrm{d}u)\right)$	$4.2539<\infty$
Condition ($\mathfrak{C}_5$)	$\mathfrak{m}_4$	$\max\limits_{1\leqslant i\leqslant 6}\left(\int_{\mathbb{U}}\left(\ln\left(1+\xi_i(u)\right)\right)^2\nu(\mathrm{d}u)\right)$	5.3019
	$\mathcal{R}_0^{\mathrm{i}}$	$\frac{\Lambda\left(\beta_1+\mathfrak{u}_1\beta_2+\mathfrak{u}_2\beta_3+\mathfrak{u}_3\alpha_1+\mathfrak{u}_4\alpha_2\right)}{\hat{\mu}_1\hat{\mu}_2}$	$1.0769>1$

Table IV.4: The corresponding values of some quantities in the case of $q = 2.1 > 2$.

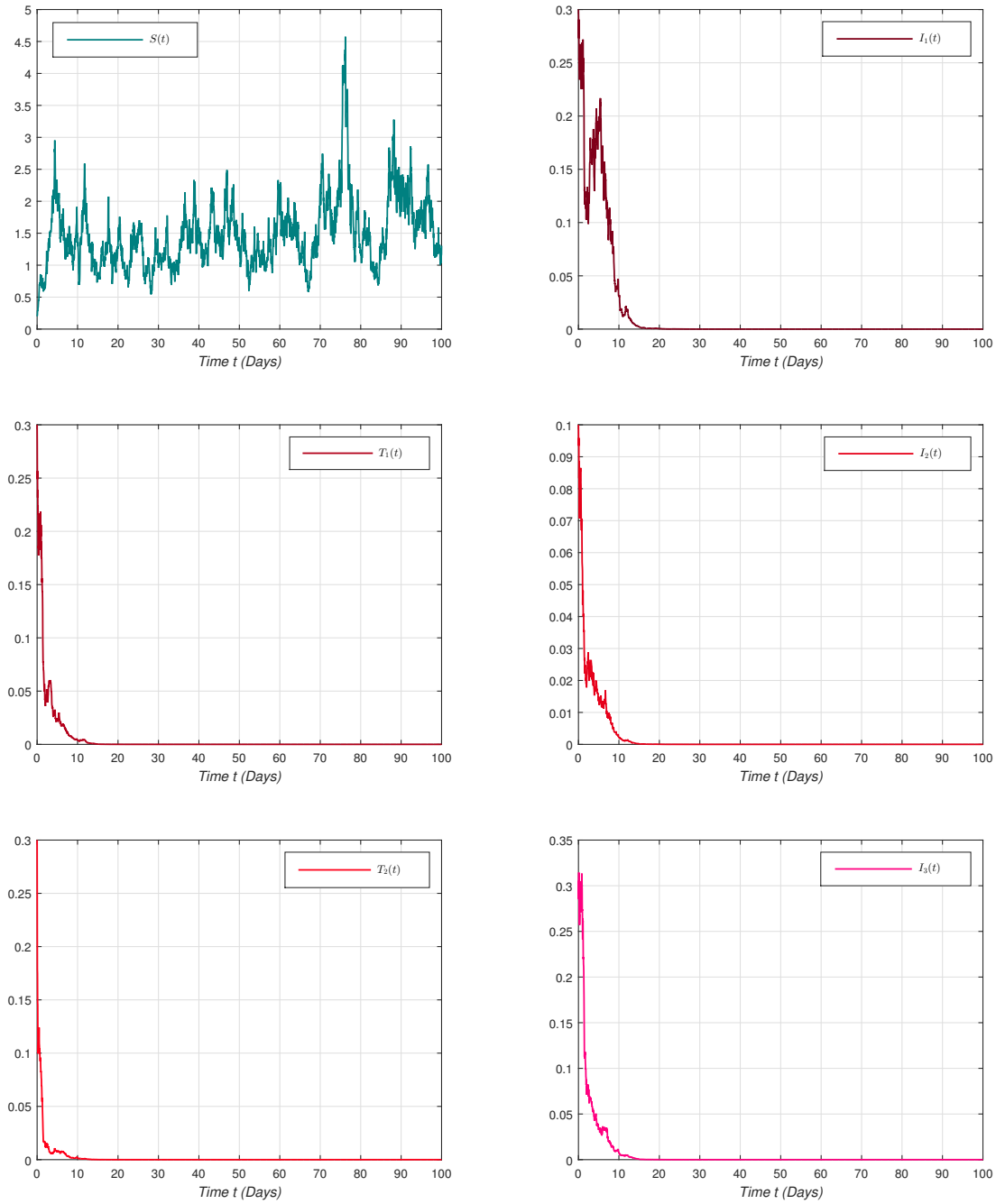


Figure IV.2: Solutions' paths of the AIDS model (IV.5) when the numerical values are taken as shown in the third column of Table IV.2 ($\mathcal{T} = -0.1953 < 0$). In this case, the curves show the AIDS extinction phenomenon.

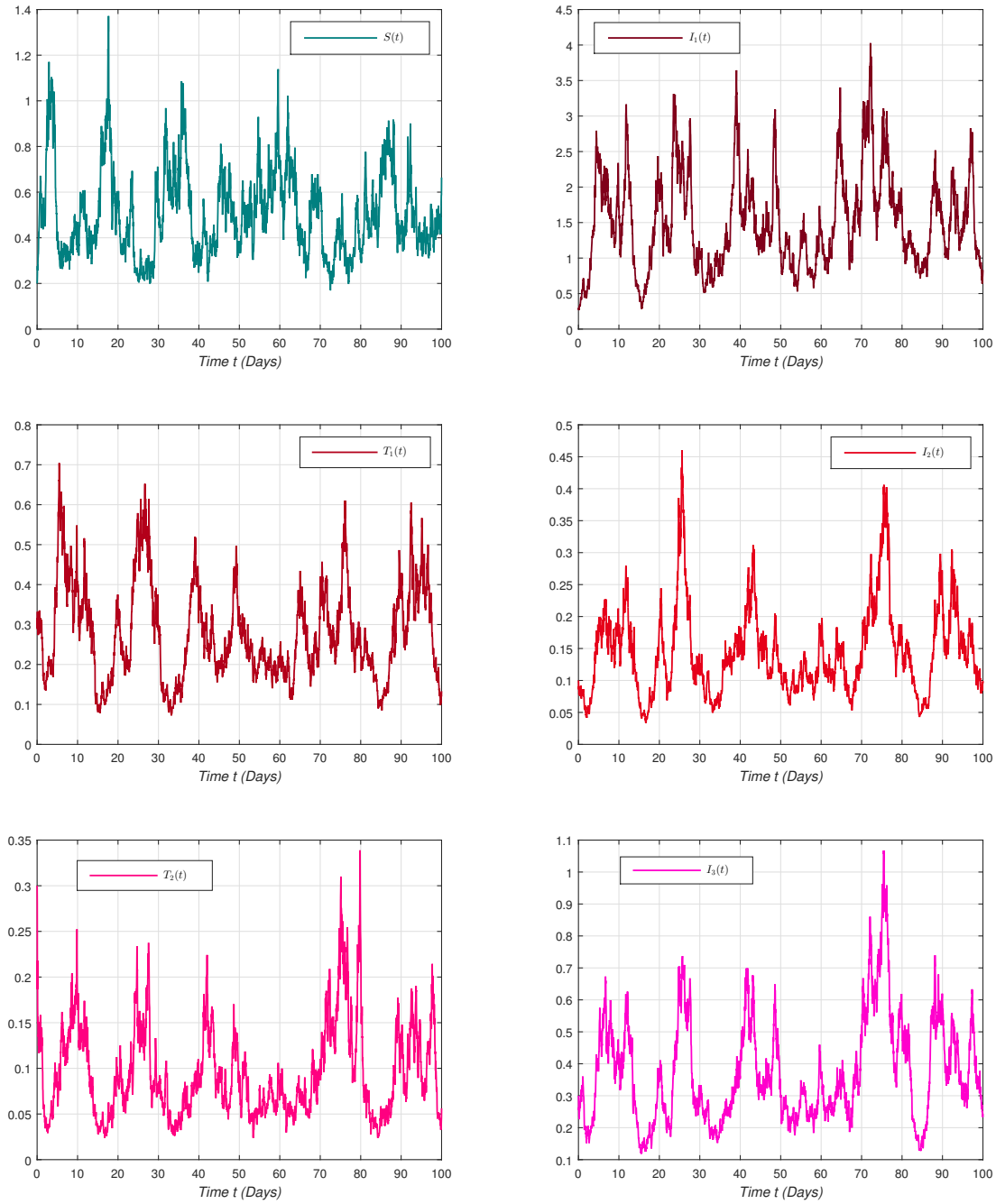


Figure IV.3: Solutions' paths of the AIDS model (IV.5) when the numerical values are taken as shown in the fourth column of Table IV.2 ($\mathcal{R}_0^{\mathrm{i}} = 1.0769 > 1$). In this case, the curves show the AIDS persistence phenomenon.

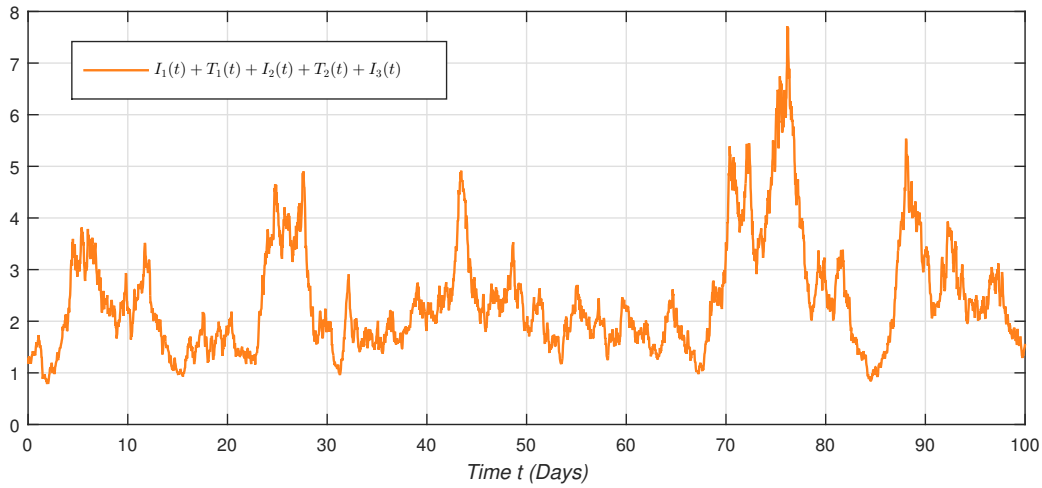


Figure IV.4: The stochastic trajectory of $I_1(t) + T_1(t) + I_2(t) + T_2(t) + I_3(t)$ when $\mathcal{R}_0^{\mathrm{i}} = 1.0769 > 1$.

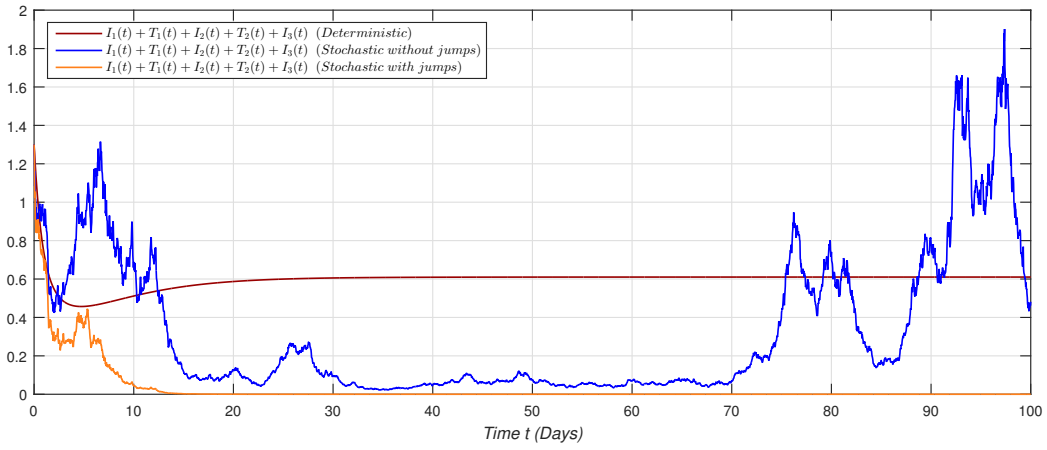


Figure IV.5: The paths of $\left(\sum_{i=1}^{3} I_i(s) + \sum_{i=1}^{2} T_i(s)\right)$ for the deterministic model (IV.1), the model (IV.5) with Brownian motions, and the SDE-LJ (IV.4).

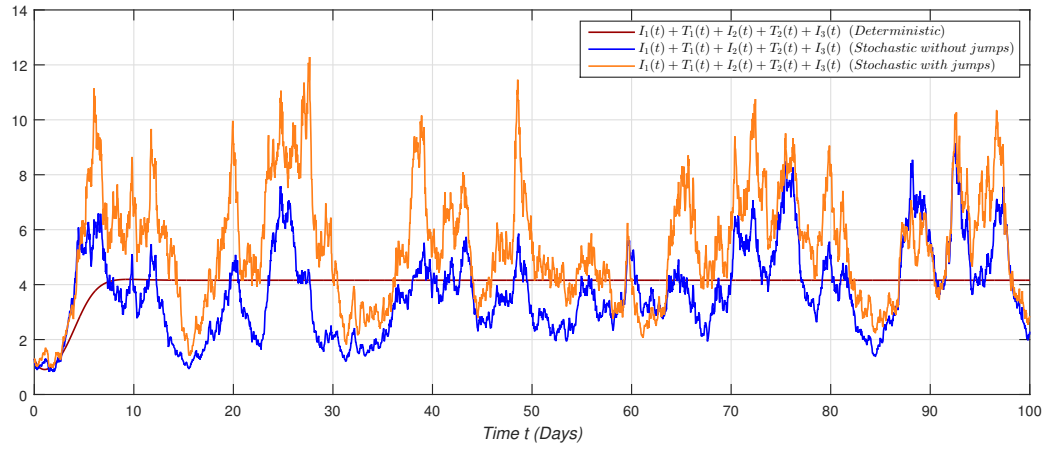


Figure IV.6: The paths of $\left(\sum_{i=1}^{3} I_i(s) + \sum_{i=1}^{2} T_i(s)\right)$ associated respectively to the models (IV.1),(IV.5) and (IV.4).

6 Conclusion and discussion

The greatly felled transmissible infection of Acquired Immunodeficiency Syndrome (AIDS) has caught much attention from worldwide scholars. It is caused by the Human Immunodeficiency Virus (HIV), which takes the CD 4-positive T lymphocytes in the bodily immune system as the prime base and destroys its mechanism. This epidemic has a long incubation period with a high fatality rate for patients and his treatment increases the predicted time for its transmission. This treatment strategy guarantees a more reasonable probability of affecting others. Consequently, we should edify the infected individuals who proactively seek treatment to assist the government to control the dissemination of HIV/AIDS.

The present chapter analyzed a generalized stochastic AIDS/HIV epidemic model that includes both Brownian motions and Lévy jumps. More explicitly, the treated system represents a staged progression compartmental system that takes the form of an Itô-Lévy stochastic differential equations system. After having presented the model, a sophisticated mathematical analysis is carried out to offer an insight into AIDS propagation, notably, its long-run behavior. The main mathematical and biological results of this chapter are arranged as follows:

- We have proved the well-posedness of the proposed AIDS model. Namely, we demonstrated the existence and uniqueness of a positive global-in-time solution.
- We have given some sufficient and sharper conditions for the AIDS extinction.
- We have established sufficient conditions for the persistence in the mean of the AIDS epidemic.

Compared to the previous works, the newness of our results resides in the following analytical techniques and improvements that we have employed to achieve our results:

- Since the explicit expression of the stationary distribution of equation (IV.6) is until now unknown, we have adopted an alternative approach to estimating its time averages.
- In order to boost the optimality of the conditions in the extinction case, we have adopted the inequality (IV.15) instead of $\ln(x+1) \leq x$ for all $x > -1$, used mostly in the literature (see for example [160]).
- We have raised the optimality of our threshold by adopting in (IV.16) the ramp function $(x \mapsto x^+)$ instead of the absolute value, and using in (IV.17) the constants $\overline{\chi}$ and $\underline{\chi}$ in place of $\dfrac{\max\{\beta_1,\beta_2,\beta_3,\alpha_1,\alpha_2\}}{\min\{\beta_1,\beta_2,\beta_3,\alpha_1,\alpha_2\}}$ and $\dfrac{\min\{\beta_1,\beta_2,\beta_3,\alpha_1,\alpha_2\}}{\max\{\beta_1,\beta_2,\beta_3,\alpha_1,\alpha_2\}}$ always employed in the literature (see for example [159]).

Generally speaking, our obtained results show that the extinction and permanence conditions rely fundamentally on white noises and Lévy jumps intensities. In order to clarify the analytical results and explain the effect of substituting the absolute value by the ramp function in the inequality (IV.16) and using others techniques, we have presented some numerical simulation examples. Furthermore, we indicate that the obtained results improve and extend many previous papers (for instance, [159] and [160]), and enhance our comprehension of the AIDS spreading behavior, which makes our findings a rich ground for further studies, essentially with the continuous reappearance of the AIDS disease in many contries around the globe.

In brief, and through some mathematical analysis and numerical examples, we have got an insight into epidemiological outcomes about the impacts of Lévy jumps on the AIDS spread. It is noted that the Lévy jumps are very suitable for modeling the impact of massive and sudden disturbances like floods, climate changes, earthquakes, etc. Furthermore, the results prophesied by deterministic and stochastic models can differ widely, especially in terms of persistence and extinction.

Chapter V

An improvement of the extinction sufficient conditions for a higher-order stochastically disturbed AIDS/HIV model

Abstract

The principal purpose of this chapter is to improve the extinction sufficient conditions of the higher-order perturbed multi-stage AIDS/HIV model proposed by Liu and Jiang in [12]. First, and by utilizing some novel and non-standard analytical techniques, we give a new extinction theorem of the said AIDS model. Then, we demonstrate that this new theorem is stronger and much more general than its homologue existing in [12]. Finally, we provide some simulations in order to bear out our theoretical results and clarify the impact of the adopted mathematical techniques on the findings.

1 Introduction

To maintain consistency and facilitate a smooth transition into this chapter's subject matter, we briefly recall some points about the AIDS disease mentioned earlier in the previous chapter. As has already been said, Acquired Immunodeficiency Syndrome (AIDS) is a very dangerous infection that has caught much attention of many scientists around the globe. Caused by the Human Immunodeficiency Virus (HIV), which attacks and destroys the bodily immune system, this disease has killed more than forty million people since its start [37]. Worse still, there is no effective treatment or vaccine for AIDS yet, and the prevention procedures like pre-sexual exposure prophylaxis, post-sexual exposure prophylaxis, condom use [166], etc., are the only remaining ways that can minimize its propagation severity [167]. With the view to protect Humanity from AIDS danger, it is vitally important to know more about its dissemination process. To this end, a great number of mathematical models have been formulated and investigated over the last three decades [169, 170]. As far as we know, environmental fluctuations influence the spread of infectious diseases and make it more complicated to foresee their behavior [181]. In such cases, deterministic systems, while able to make very informative forecasts and previsions, are not appropriate enough [39]. So, there is a pressing need for a developed mathematical model that can take into account the randomness effect, especially in the context of a harmful infectious disease like AIDS. In this context, many scholars have suggested and evolved a large number of stochastic models that describe the AIDS/HIV propagation dynamics from different angles and viewpoints [183, 184]. One of the most interesting works in this vein is the recent paper of Liu and Jiang [12]. In this latter, the authors have proposed the following higher order perturbed AIDS model:

$$\begin{cases} \mathrm{d}S(t) = \Big(r - c \times \sum_{k=1}^{p} b_k S(t) I_k(t) - mS(t)\Big)\mathrm{d}t + \big(\sigma_{01} + \sigma_{02} S(t)\big) S(t)\mathrm{d}\mathfrak{B}_0(t), \\ \mathrm{d}I_1(t) = \Big(c \times \sum_{k=1}^{p} b_k S(t) I_k(t) - (n_1 + m) I_1(t) + g_2 I_2(t)\Big)\mathrm{d}t + \big(\sigma_{11} + \sigma_{12} I_1(t)\big) I_1(t)\mathrm{d}\mathfrak{B}_1(t), \\ \mathrm{d}I_k(t) = \Big(n_{k-1} I_{k-1}(t) - (n_k + g_k + m) I_k(t) + g_{k+1} I_{k+1}(t)\Big)\mathrm{d}t + \big(\sigma_{k1} + \sigma_{k2} I_k(t)\big) I_k(t)\mathrm{d}\mathfrak{B}_k(t), \text{ for } k = 2, \cdots, p-1 \\ \mathrm{d}I_p(t) = \Big(n_{p-1} I_{p-1}(t) - (n_p + g_p + m) I_p(t)\Big)\mathrm{d}t + \big(\sigma_{p1} + \sigma_{p2} I_p(t)\big) I_p(t)\mathrm{d}\mathfrak{B}_p(t) \\ \mathrm{d}A(t) = \Big(n_p I_p(t) - mA(t)\Big)\mathrm{d}t + \big(\sigma_{(p+1)1} + \sigma_{(p+1)2} I_p(t)\big) I_p(t)\mathrm{d}\mathfrak{B}_p(t), \end{cases} \tag{V.1}$$

with a given initial data $\big(S(0), I_1(0), \cdots, I_p(0), A(0)\big) \in \mathbb{R}_+^{p+2}$. Here, the total population $N := S + A + \sum_{k=1}^{p} I_k$ was partitioned into $p+2$ classes. One of these classes, denoted S, is for the susceptible individuals, the other is for AIDS sufferers without therapy that are under treatment, and it is denoted A. The p remaining classes $I_1, \cdots, I_p$ are all of infectious individuals but with a decreasing level of infection. Since the individuals of A are not actively spreading the disease, this class can be removed and the dynamics of the AIDS propagation progression will be obtained just from the following simplified model:

$$\begin{cases} \mathrm{d}S(t) = \Big(r - c \times \sum_{k=1}^{p} b_k S(t) I_k(t) - mS(t)\Big)\mathrm{d}t + \big(\sigma_{01} + \sigma_{02} S(t)\big) S(t)\mathrm{d}\mathfrak{B}_0(t), \\ \mathrm{d}I_1(t) = \Big(c \times \sum_{k=1}^{p} b_k S(t) I_k(t) - (n_1 + m) I_1(t) + g_2 I_2(t)\Big)\mathrm{d}t + \big(\sigma_{11} + \sigma_{12} I_1(t)\big) I_1(t)\mathrm{d}\mathfrak{B}_1(t), \\ \mathrm{d}I_k(t) = \Big(n_{k-1} I_{k-1}(t) - (n_k + g_k + m) I_k(t) + g_{k+1} I_{k+1}(t)\Big)\mathrm{d}t + \big(\sigma_{k1} + \sigma_{k2} I_k(t)\big) I_k(t)\mathrm{d}\mathfrak{B}_k(t), \text{ for } k = 2, \cdots, p-1 \\ \mathrm{d}I_p(t) = \Big(n_{p-1} I_{p-1}(t) - (n_p + g_p + m) I_p(t)\Big)\mathrm{d}t + \big(\sigma_{p1} + \sigma_{p2} I_p(t)\big) I_p(t)\mathrm{d}\mathfrak{B}_p(t). \end{cases} \tag{V.2}$$

Regarding the parameters appearing in this last system, r is the recruitment rate, c is the average number of contacts made by a susceptible individual per unit of time, m is the mortality rate, n_k and g_{k+1} ($k \in \{1, \cdots, p-1\}$) are respectively the movement rates from I_k to I_{k+1} and from I_{k+1} to I_k and finally n_p is the movement rate from I_p to the removed class A. Needless to say, All the aforementioned parameters are presumed to be in the positive real axis $\mathbb{R}_+ := \{x \in \mathbb{R} \mid 0 < x\}$. For all $k \in \{0, \cdots, p+1\}$, σ_{k1} and σ_{k2} are the positive linear and high-order intensities associated respectively to the mutually independent Brownian motions $\mathfrak{B}_k$. These latter, and all the random variables that will be met in this chapter, are defined on a filtered probability space $(\Omega, F, (F_k)_{k\geqslant 0}, \mathrm{P})$ endowed with a filtration $(F_k)_{k\geqslant 0}$ that verifies the usual hypotheses $\big((F_k)_{k\geqslant 0}$ is right-continuous, and F_0 contains all the P−negligible subsets$\big)$. Concerning the details of their work in [12], Liu and Jiang have first proved the well-posedness of the system (V.2), then they found some criteria for the existence and uniqueness of an ergodic stationary distribution, before going to an extinction theorem (Theorem 4.1) in which they give some sufficient conditions to the complete disease disappearance. Our principal aim in this chapter is to generalize and improve this last-mentioned theorem by weakening considerably the conditions permitting to deduce the AIDS extinction.

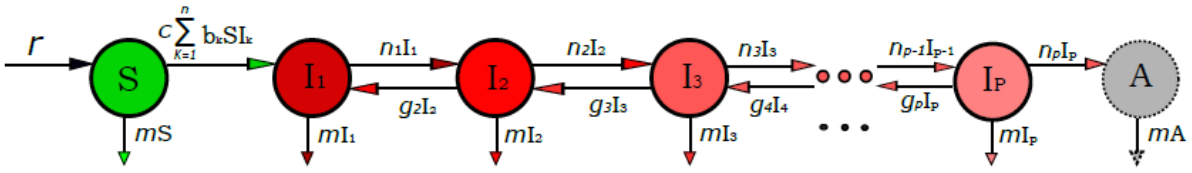


Figure V.1: Flow diagram of the AIDS compartmental model (V.2).

The remaining of this chapter is structured as follows: in Section 2, we give some preliminary results and we prove a new extinction theorem for the stochastic AIDS model (V.2). In section 3 we justify the fact that our theorem is much stronger than its counterpart proposed by Liu and Jiang in [12]. In Section 4, we support our findings with some computer simulations and we derive the main conclusions of this chapter in Section 5.

2 A new extinction theorem for the AIDS model (V.2)

Before stating the main result of this chapter, we must first introduce some useful lemmas and some conventions to lighten its statement. For convenience in writing, we denote from now on by M_1 and M_2 the two following tridiagonal square matrices of order $p-1$, where $p \in \mathbb{N}_{\geqslant 3} = \{3, 4, \cdots\}$:

$$M_1 = \begin{pmatrix} -(n_2+g_2+m) & n_2 & 0 & \cdots & 0 \\ g_3 & -(n_3+g_3+m) & n_3 & \ddots & \vdots \\ 0 & \ddots & \ddots & \ddots & 0 \\ \vdots & \ddots & g_{p-1} & -(n_{p-1}+g_{p-1}+m) & n_p \\ 0 & \cdots & 0 & g_p & -(n_p+g_p+m) \end{pmatrix}$$

and

$$M_2 = \begin{pmatrix} -\left(n_2+g_2+m+\frac{\sigma_{21}^2}{2}\right) & n_2 & 0 & \cdots & 0 \\ g_3 & -\left(n_3+g_3+m+\frac{\sigma_{31}^2}{2}\right) & n_3 & \ddots & \vdots \\ 0 & \ddots & \ddots & \ddots & 0 \\ \vdots & \ddots & g_{p-1} & -\left(n_{p-1}+g_{p-1}+m+\frac{\sigma_{p-11}^2}{2}\right) & n_p \\ 0 & \cdots & 0 & g_p & -\left(n_p+g_p+m+\frac{\sigma_{p1}^2}{2}\right) \end{pmatrix}$$

Lemma 2.1. *The matrices $-M_1$ and $-M_2$ are non-singular M-matrices.*

Proof. It is easily seen that $-M_1$ and $-M_2$ are real Z-matrices, i.e., their entries are real and the off-diagonal ones are non-positive. Also, for each non-negative diagonal matrix D, $-M_1+D$ and $-M_2+D$ are non-singular because they are strictly diagonally dominant matrices (see [94, page 24]). So, and by the item A_3 of Theorem 1 in [92] $(-M_1)$ and $(-M_2)$ are non-singular M-matrices, which completes the proof. ∎

Corollary 2.1. *Let $\widetilde{I}_1 = \widehat{I}_1 = 1$. Then each of the two following algebraic linear systems*

$$\begin{cases} n_{k-1}\widetilde{I}_{k-1} - (n_k+g_k+m)\,\widetilde{I}_k + g_{k+1}\widetilde{I}_{k+1} = 0, \ k \in \{2, \cdots, p-1\} \\ n_{r-1}\widetilde{I}_{r-1} - (n_r+g_r+m)\,\widetilde{I}_r = 0. \end{cases} \tag{V.3}$$

and,

$$\begin{cases} n_{k-1}\widehat{I}_{k-1} - \left(n_k+g_k+m+\dfrac{\sigma_{k1}^2}{2}\right)\widehat{I}_k + g_{k+1}\widehat{I}_{k+1} = 0, \ k \in \{2, \cdots, p-1\} \\ n_{r-1}\widehat{I}_{r-1} - \left(n_r+g_r+m+\dfrac{\sigma_{r1}^2}{2}\right)\widehat{I}_r = 0. \end{cases} \tag{V.4}$$

admits a unique solution denoted respectively $\left(\widetilde{I}_2, \cdots, \widetilde{I}_p\right)$ and $\left(\widehat{I}_2, \cdots, \widehat{I}_p\right)$. Moreover, $\left(\widetilde{I}_2, \cdots, \widetilde{I}_p\right) \in \mathbb{R}_+^{p-1}$ and $\left(\widehat{I}_2, \cdots, \widehat{I}_p\right) \in \mathbb{R}_+^{p-1}$.

Proof. Clearly, the system (V.3) can be expressed as follows

$$\left(\widetilde{I}_2, \cdots, \widetilde{I}_p\right) \times M_1 = (-n_1, 0, \cdots, 0).$$

So, and since $-M_1$ is a non-singular M-matrix (see Lemma 2.1), we get $(-M_1)^{-1} > 0$. The last fact implies that

$$\left(\widetilde{I_2}, \cdots, \widetilde{I_p}\right) = (-n_1, 0, \cdots, 0) \times M_1^{-1} = (n_1, 0, \cdots, 0) \times (-M_1)^{-1} > 0. \tag{V.5}$$

Hence, there is one solution $\left(\widetilde{I_2}, \cdots, \widetilde{I_p}\right)$ to the system (V.3), and $\left(\widetilde{I_2}, \cdots, \widetilde{I_p}\right) \in \mathbb{R}_+^{p-1}$. By a similar argument, we can also conclude the same for the system (V.4), which achieves the proof. ■

Before going to the principal theorem of this section, let us first start by a useful lemma concerning the following auxiliary subsystem obtained from the first equation of (V.2) when the infectious individuals are absent:

$$\begin{cases} \mathrm{d}X(t) = \Big(r - mX(t)\Big)\mathrm{d}t + \big(\sigma_{01} + \sigma_{02}X(t)\big)X(t)\mathrm{d}\mathfrak{B}_0(t), \\ X(0) = S(0) \in \mathbb{R}_+. \end{cases} \tag{V.6}$$

Lemma 2.2. *The ergodic property is verified for the system (V.6), and the invariant density in this case is given by*

$$\mu(x) = K \times x^{-2-\frac{2(2r\sigma_{02}+m\sigma_{01})}{\sigma_{01}^3}} \times (\sigma_{01} + \sigma_{02}x)^{-2+\frac{2(2r\sigma_{02}+m\sigma_{01})}{\sigma_{01}^3}} \times \exp\left(-\frac{2}{\sigma_{01}(\sigma_{01}+\sigma_{02}x)}\left(\frac{r}{x} + \frac{2r\sigma_{02}+m\sigma_{01}}{\sigma_{01}}\right)\right).$$

K is a positive constant such that $\displaystyle\int_0^\infty \mu(x)\mathrm{d}x = 1.$

Proof. The proof is identical to the demonstration of Theorem 3.1 in [195]. Hereby, we omit it here. ■

Remark 2.1. *By virtue of the stochastic comparison theorem (see [155]), we can assert that $S(t) \leqslant X(t)$ for all $t \geqslant 0$ a.s.*

Lemma 2.3. *Let $X(t)$ be the solution of the auxiliary system (V.6) that starts from a given initial datum $S(0) = X(0)$. Then,*

$$\lim_{t\to\infty} \frac{1}{t}\int_0^t X(u)\mathrm{d}u = \frac{r}{m} \quad a.s.$$

Proof. According to the ergodic property of system (V.6), we have

$$\lim_{t\to\infty} \frac{1}{t}\int_0^t X(u)\mathrm{d}u = \int_0^\infty x\mu(x)\mathrm{d}x \quad a.s.$$

By taking the expectation on both sides of (V.6) we get

$$\begin{aligned} 0 &= \lim_{t\to\infty} \frac{\mathbb{E}(X(t))}{t} = r - m\lim_{t\to\infty}\frac{1}{t}\int_0^t \mathbb{E}(X(u))\mathrm{d}u \\ &= r - m \times \mathbb{E}\left(\lim_{t\to\infty}\frac{1}{t}\int_0^t X(u)\mathrm{d}u\right) = r - m \times \mathbb{E}\left(\int_0^\infty x\mu(x)\mathrm{d}x\right) \\ &= r - m \times \left(\int_0^\infty x\mu(x)\mathrm{d}x\right). \end{aligned}$$

So

$$\lim_{t\to\infty} \frac{1}{t}\int_0^t X(u)\mathrm{d}u = \int_0^\infty x\mu(x)\mathrm{d}x = \frac{r}{m},$$

which finishes the proof. ■

Theorem 2.1. *Let $p \in \mathbb{N}_{\geqslant 3} = \{3, 4, \cdots\}$, and let $\big(S(t), I_1(t), \cdots, I_p(t)\big)$ be the solution of the system (V.2) that starts from a given initial value $\big(S(0), I_1(0), \cdots, I_p(0)\big) \in \mathbb{R}_+^{p+1}$. If $\mathfrak{R}_0 \triangleq \dfrac{\frac{cr}{m}\sum\limits_{k=1}^{p} b_k\widetilde{I_k} + g_2\widetilde{I_2}}{n_1+m} < 1$, then*

$$\limsup_{t\to\infty} \frac{1}{t} \ln\left(\sum_{k=1}^{p} \alpha_k I_k(t) + (1-\mathfrak{R}_0)\sum_{k=2}^{p} I_k(t)\right) \leqslant v^* \quad a.s,$$

where $\alpha_1 := 1$,

$$\big(\alpha_2, \cdots, \alpha_p\big) \triangleq -\frac{cr}{m}\left(b_2 + g_2\frac{m}{cr}, b_3, \cdots, b_p\right) \times M_1^{-1} \in \mathbb{R}_+^{p-1}, \tag{V.7}$$

and

$$v^* \triangleq c \times \max_{1\leqslant k\leqslant p}\left\{\frac{b_k}{\alpha_k}\right\} \times \frac{1}{2}\int_0^{\infty}\left|x - \frac{r}{m}\right|\mu(x)\mathrm{d}x - \frac{(1-\mathfrak{R}_0)m}{\max\limits_{1\leqslant k\leqslant p}\{\alpha_k\} + (1-\mathfrak{R}_0)} - \frac{1}{4\times\sum\limits_{i=1}^{p}\left(\frac{1}{\sigma_{i1}}\right)^2}. \tag{V.8}$$

Clearly, if the condition $v^* < 0$ holds, then the disease will vanish exponentially almost surely.

Proof. Let us consider the C^2-function

$$W\big(S, I_1, \cdots, I_p\big) = I_1 + \sum_{k=2}^{p}\left(\alpha_k + (1-\mathfrak{R}_0)\right) I_k.$$

By an application of the famous Itô's formula [19] to $W\big(S, I_1, \cdots, I_p\big)$, we get

$$\begin{aligned}\mathcal{L}W = &\left(c\times\sum_{k=1}^{p} b_k S I_k - (n_1+m)I_1 + g_2 I_2\right) + \sum_{k=2}^{p-1}\alpha_k\left(n_{k-1}I_{k-1} - (n_k+g_k+m)I_k + g_{k+1}I_{k+1}\right)\\ &+ \alpha_p\left(n_{p-1}I_{p-1} - (n_p+g_p+m)I_p\right) + (1-\mathfrak{R}_0)\left(n_1 I_1 - g_2 I_2 - m\sum_{k=2}^{p} I_k - n_p I_p\right).\end{aligned}$$

Hence

$$\begin{aligned}\mathcal{L}W = &\left(c\times\sum_{k=1}^{p} b_k\left(S - \frac{r}{m}\right)I_k + \left(c\times\sum_{k=1}^{p} b_k\frac{r}{m}I_k - (n_1+m)I_1 + g_2 I_2\right)\right)\\ &+ \sum_{k=2}^{p-1}\alpha_k\left(n_{k-1}I_{k-1} - (n_k+g_k+m)I_k + g_{k+1}I_{k+1}\right)\\ &+\alpha_p\left(n_{p-1}I_{p-1} - (n_p+g_p+m)I_p\right) + (1-\mathfrak{R}_0)\left(n_1 I_1 - m\sum_{k=2}^{p} I_k \overbrace{-(g_2 I_2 + n_p I_p)}^{\leqslant 0}\right)\\ \leqslant\ & c\times\sum_{k=1}^{p} b_k\left(S - \frac{r}{m}\right)I_k + \left(cb_1\frac{r}{m}I_1 + c\times\sum_{k=2}^{p} b_k\frac{r}{m}I_k - (n_1+m)I_1 + g_2 I_2\right)\\ &+ \alpha_2\left(n_1 I_1 - (n_2+g_2+m)I_2 + g_3 I_3\right) + \sum_{k=3}^{p-1}\alpha_k\left(n_{k-1}I_{k-1} - (n_k+g_k+m)I_k + g_{k+1}I_{k+1}\right)\\ &+ \alpha_p\left(n_{p-1}I_{p-1} - (n_p+g_p+m)I_p\right) + (1-\mathfrak{R}_0)\left(n_1 I_1 - m\sum_{k=2}^{p} I_k\right).\end{aligned}$$

Thus

$$\mathcal{L}W \leqslant \left(cb_1\frac{r}{m} - (n_1+m) + \alpha_2 n_1\right) I_1 + c\times\sum_{k=1}^{p} b_k\left(S-\frac{r}{m}\right) I_k + \left[c\times\sum_{k=2}^{p} b_k\frac{r}{m} I_k + g_2 I_2\right]$$
$$+ \left[\alpha_2\left(-(n_2+g_2+m)I_2 + g_3 I_3\right) + \sum_{k=3}^{p-1}\alpha_k\left(n_{k-1}I_{k-1} - (n_k+g_k+m)I_k + g_{k+1}I_{k+1}\right)\right.$$
$$\left. + \alpha_p\left(n_{p-1}I_{p-1} - (n_p+g_p+m)I_p\right)\right] + (1-\mathfrak{R}_0)\left(n_1 I_1 - m\sum_{k=2}^{p} I_k\right).$$

Clearly, the last inequality can be rewritten as follows

$$\mathcal{L}W \leqslant \left(cb_1\frac{r}{m} - (n_1+m) + \alpha_2 n_1\right) I_1 + c\times\sum_{k=1}^{p} b_k\left(S-\frac{r}{m}\right) I_k + \left[\frac{cr}{m}\left(b_2 + g_2\frac{m}{cr}, b_3, \cdots, b_p\right)\left(I_2,\cdots,I_p\right)^T\right]$$
$$+ \left[\left(\alpha_2,\cdots,\alpha_p\right) M_1\left(I_2,\cdots,I_p\right)^T\right] + (1-\mathfrak{R}_0)\left(n_1 I_1 - m\sum_{k=2}^{p} I_k\right).$$
$$\leqslant \left(cb_1\frac{r}{m} - (n_1+m) + \alpha_2 n_1\right) I_1 + c\times\sum_{k=1}^{p} b_k\left(S-\frac{r}{m}\right) I_k$$
$$+ \left[\frac{cr}{m}\left(b_2 + g_2\frac{m}{cr}, b_3, \cdots, b_p\right) + \left(\alpha_2,\cdots,\alpha_p\right) M_1\right]\left(I_2,\cdots,I_p\right)^T + (1-\mathfrak{R}_0)\left(n_1 I_1 - m\sum_{k=2}^{p} I_k\right). \quad \text{(V.9)}$$

According to the relation (V.7), we have

$$\left(\alpha_2,\cdots,\alpha_p\right) = \underbrace{-\frac{cr}{m}\left(b_2 + g_2\frac{m}{cr}, b_3, \cdots, b_p\right)\times M_1^{-1}}_{\text{since } -M_1 \text{ is a non-singular M-matrix}} \in \mathbb{R}_+^{p-1}.$$

So

$$\frac{cr}{m}\left(b_2 + g_2\frac{m}{cr}, b_3, \cdots, b_p\right) + \left(\alpha_2,\cdots,\alpha_p\right) M_1 = 0, \quad \text{(V.10)}$$

and

$$\begin{aligned}\alpha_2 &= \left(\alpha_2,\cdots,\alpha_p\right)(1,0,\cdots,0)^T\\ &= -\frac{cr}{m}\left(b_2 + g_2\frac{m}{cr}, b_3, \cdots, b_p\right)\times M_1^{-1}\times(1,0,\cdots,0)^T\\ &= -\frac{cr}{n_1 m}\left(b_2 + g_2\frac{m}{cr}, b_3, \cdots, b_p\right)\times\underbrace{(-M_1)^{-1}\times(n_1,0,\cdots,0)^T}_{=\left(\widetilde{I_2},\cdots,\widetilde{I_p}\right)^T \text{ (see (V.5))}}\\ &= \frac{1}{n_1}\left(\frac{cr}{m}\sum_{k=2}^{p} b_k\widetilde{I_k} + g_2\widetilde{I_2}\right). \end{aligned} \quad \text{(V.11)}$$

Combining (V.10) and (V.11) with (V.9) yields

$$\mathcal{L}W \leqslant \left(cb_1\frac{r}{m} - (n_1+m) + \frac{cr}{m}\sum_{k=2}^{p} b_k\widetilde{I_k} + g_2\widetilde{I_2}\right) I_1 + c\times\sum_{k=1}^{p} b_k\left(S-\frac{r}{m}\right) I_k + (1-\mathfrak{R}_0)\left(n_1 I_1 - m\sum_{k=2}^{p} I_k\right)$$
$$\leqslant \left(\frac{cr}{m}\sum_{k=1}^{p} b_k\widetilde{I_k} + g_2\widetilde{I_2} - (n_1+m)\right) I_1 + c\times\sum_{k=1}^{p} b_k\left(S-\frac{r}{m}\right) I_k + (1-\mathfrak{R}_0)\left(n_1 I_1 - m\sum_{k=2}^{p} I_k\right)$$
$$\leqslant (n_1+m)(\mathfrak{R}_0-1) I_1 + c\times\sum_{k=1}^{p} b_k\left(X-\frac{r}{m}\right) I_k + (1-\mathfrak{R}_0)\left(n_1 I_1 - m\sum_{k=2}^{p} I_k\right)$$
$$\leqslant -m(1-\mathfrak{R}_0)\sum_{k=1}^{p} I_k + c\times\sum_{k=1}^{p} b_k\left(X-\frac{r}{m}\right)^+ I_k. \quad \text{(V.12)}$$

Here and subsequently, the notation x^+ is referring to the ramp function defined by $x^+ = \max\{0, x\}$. Applying again the well-known Itô's formula to $\ln(W)$ gives

$$\begin{aligned}
\mathcal{L}\ln(W) &\leqslant \frac{-m(1-\mathfrak{R}_0)\sum_{k=1}^{p} I_k + c\times\sum_{k=1}^{p} b_k\left(X-\frac{r}{m}\right)^+ I_k}{\sum_{k=1}^{p}\alpha_k I_k + (1-\mathfrak{R}_0)\sum_{k=2}^{p} I_k} - \frac{\sum_{k=1}^{p}\alpha_k^2\left(\sigma_{k1}+\sigma_{k2}I_k\right)^2 I_k^2}{2W^2} \\
&\quad - (1-\mathfrak{R}_0)^2\times\frac{\sum_{k=2}^{p}\left(\sigma_{k1}+\sigma_{k2}I_k\right)^2 I_k^2}{2W^2} \\
&\leqslant \frac{-m(1-\mathfrak{R}_0)\sum_{k=1}^{p} I_k}{\sum_{k=1}^{p}\alpha_k I_k + (1-\mathfrak{R}_0)\sum_{k=1}^{p} I_k} + \frac{c\times\sum_{k=1}^{p}\frac{b_k}{\alpha_k}\alpha_k I_k}{\sum_{k=1}^{p}\alpha_k I_k}\left(X-\frac{r}{m}\right)^+ - \frac{\sum_{k=1}^{p}\alpha_k^2\sigma_{k1}^2 I_k^2}{2W^2} \\
&\quad - \frac{\sum_{k=1}^{p}\alpha_k^2\sigma_{k2}^2 I_k^4}{2W^2} - (1-\mathfrak{R}_0)^2\times\frac{\sum_{k=2}^{p}\sigma_{k1}^2 I_k^2}{2W^2} - (1-\mathfrak{R}_0)^2\times\frac{\sum_{k=2}^{p}\sigma_{k2}^2 I_k^4}{2W^2}
\end{aligned}$$

So,

$$\begin{aligned}
\mathcal{L}\ln(W) &\leqslant -\frac{(1-\mathfrak{R}_0)m}{\max_{1\leqslant k\leqslant p}\{\alpha_k\} + (1-\mathfrak{R}_0)} + c\times\max_{1\leqslant k\leqslant p}\left\{\frac{b_k}{\alpha_k}\right\}\left(X-\frac{r}{m}\right)^+ - \frac{\sum_{k=1}^{p}\alpha_k^2\sigma_{k1}^2 I_k^2}{2W^2} \\
&\quad - \frac{\sum_{k=1}^{p}\alpha_k^2\sigma_{k2}^2 I_k^4}{2W^2} - (1-\mathfrak{R}_0)^2\times\frac{\sum_{k=2}^{p}\sigma_{k1}^2 I_k^2}{2W^2} - (1-\mathfrak{R}_0)^2\times\frac{\sum_{k=2}^{p}\sigma_{k2}^2 I_k^4}{2W^2}. \qquad \text{(V.13)}
\end{aligned}$$

Furthermore, it is clear by virtue of the well-known Cauchy-Schwartz inequality (see [106] for example) that

$$\begin{aligned}
W^2 = \left(\sum_{k=1}^{p}\alpha_k I_k + (1-\mathfrak{R}_0)\sum_{k=2}^{p} I_k\right)^2 &\leqslant 2\times\left[\left(\sum_{k=1}^{p}\frac{1}{\sigma_{k1}}\sigma_{k1}\alpha_k I_k\right)^2 + \left((1-\mathfrak{R}_0)\sum_{k=2}^{p}\frac{1}{\sigma_{k1}}\sigma_{k1} I_k\right)^2\right] \\
&\leqslant 2\times\left[\sum_{k=1}^{p}\left(\sigma_{k1}\alpha_k I_k\right)^2\times\sum_{k=1}^{p}\frac{1}{\sigma_{k1}^2} + (1-\mathfrak{R}_0)^2\sum_{k=2}^{p}\left(\sigma_{k1} I_k\right)^2\times\sum_{k=2}^{p}\frac{1}{\sigma_{k1}^2}\right] \\
&\leqslant 2\times\sum_{k=1}^{p}\frac{1}{\sigma_{k1}^2}\times\left[\sum_{k=1}^{p}\left(\sigma_{k1}\alpha_k I_k\right)^2 + (1-\mathfrak{R}_0)^2\sum_{k=2}^{p}\left(\sigma_{k1} I_k\right)^2\right]. \qquad \text{(V.14)}
\end{aligned}$$

By using the inequality (V.14) in (V.13) we obtain

$$\begin{aligned}
\mathcal{L}\ln(W) &\leqslant -\frac{(1-\mathfrak{R}_0)m}{\max_{1\leqslant k\leqslant p}\{\alpha_k\} + (1-\mathfrak{R}_0)} + c\max_{1\leqslant k\leqslant p}\left\{\frac{b_k}{\alpha_k}\right\}\left(X-\frac{r}{m}\right)^+ - (1-\mathfrak{R}_0)^2\times\frac{\sum_{k=2}^{p}\sigma_{k2}^2 I_k^4}{2W^2} \\
&\quad - \frac{\sum_{k=1}^{p}\alpha_k^2\sigma_{k2}^2 I_k^4}{2W^2} - \frac{(1-\mathfrak{R}_0)^2\sum_{k=2}^{p}\left(\sigma_{k1} I_k\right)^2 + \sum_{k=1}^{p}\left(\alpha_k\sigma_{k1} I_k\right)^2}{2W^2} \\
&\leqslant -\frac{(1-\mathfrak{R}_0)m}{\max_{1\leqslant k\leqslant p}\{\alpha_k\} + (1-\mathfrak{R}_0)} + c\max_{1\leqslant k\leqslant p}\left\{\frac{b_k}{\alpha_k}\right\}\left(X-\frac{r}{m}\right)^+ - (1-\mathfrak{R}_0)^2\times\frac{\sum_{k=2}^{p}\sigma_{k2}^2 I_k^4}{2W^2} \\
&\quad - \frac{\sum_{k=1}^{p}\alpha_k^2\sigma_{k2}^2 I_k^4}{2W^2} - \frac{1}{4\times\sum_{k=1}^{p}\frac{1}{\sigma_{k1}^2}}.
\end{aligned}$$

So

$$\mathrm{d}\ln(W) \leqslant \Big(- \frac{(1-\mathfrak{R}_0)m}{\max\limits_{1\leqslant k\leqslant p}\{\alpha_k\} + (1-\mathfrak{R}_0)} + c \max_{1\leqslant k\leqslant p}\left\{\frac{b_k}{\alpha_k}\right\}\Big(X - \frac{r}{m}\Big)^+ - \frac{1}{4\sum\limits_{k=1}^{p}\sigma_{k1}^{-2}}$$
$$- \frac{\sum\limits_{k=1}^{p}\alpha_k^2\sigma_{k2}^2 I_k^4}{2W^2} - (1-\mathfrak{R}_0)^2 \times \frac{\sum\limits_{k=2}^{p}\sigma_{k2}^2 I_k^4}{2W^2}\Big)\mathrm{d}t + \frac{1}{W}\sum_{k=1}^{p}\alpha_k\big(\sigma_{k1}+\sigma_{k2}I_k\big)I_k\mathrm{d}\mathfrak{B}_k$$
$$+ \frac{(1-\mathfrak{R}_0)}{W}\sum_{k=2}^{p}\big(\sigma_{k1}+\sigma_{k2}I_k\big)I_k\mathrm{d}\mathfrak{B}_k. \tag{V.15}$$

Integrating (V.15) from zero to t, and then dividing by t, we get

$$\frac{\ln W(t)}{t} \leqslant \frac{\ln W(0)}{t} - \frac{(1-\mathfrak{R}_0)m}{\max\limits_{1\leqslant k\leqslant p}\{\alpha_k\} + (1-\mathfrak{R}_0)} + c\max_{1\leqslant k\leqslant p}\left\{\frac{b_k}{\alpha_k}\right\}\frac{1}{t}\int_0^t\Big(X(u)-\frac{r}{m}\Big)^+\mathrm{d}u - \frac{1}{4\sum\limits_{k=1}^{p}\sigma_{k1}^{-2}}$$
$$- \frac{1}{2t}\int_0^t\frac{\sum\limits_{k=1}^{p}\alpha_k^2\sigma_{k2}^2 I_k^4(u)}{W^2(u)}\mathrm{d}u - \frac{(1-\mathfrak{R}_0)^2}{2t}\times\int_0^t\frac{\sum\limits_{k=2}^{p}\sigma_{k2}^2 I_k^4(u)}{W^2(u)}\mathrm{d}u + \sum_{k=1}^{p}\frac{A_{1k}(t)}{t} + \sum_{k=2}^{p}\frac{A_{2k}(t)}{t}$$
$$+ \sum_{k=1}^{p}\frac{A_{3k}(t)}{t} + \sum_{k=2}^{p}\frac{A_{4k}(t)}{t}, \tag{V.16}$$

where

$$\begin{cases} A_{1k} = \displaystyle\int_0^t \frac{\alpha_k\sigma_{k1}I_k(u)}{W(u)}\mathrm{d}\mathfrak{B}_k(u), \quad 1\leqslant k\leqslant p \\ A_{2k} = \displaystyle\int_0^t \frac{(1-\mathfrak{R}_0)\sigma_{k1}I_k(u)}{W(u)}\mathrm{d}\mathfrak{B}_k(u), \quad 2\leqslant k\leqslant p \\ A_{3k} = \displaystyle\int_0^t \frac{\alpha_k\sigma_{k2}I_k^2(u)}{W(u)}\mathrm{d}\mathfrak{B}_k(u), \quad 1\leqslant k\leqslant p \\ A_{4k} = \displaystyle\int_0^t \frac{(1-\mathfrak{R}_0)\sigma_{k2}I_k^2(u)}{W(u)}\mathrm{d}\mathfrak{B}_k(u), \quad 2\leqslant k\leqslant p. \end{cases}$$

The quadratic variations of the above local martingales are respectively given by

$$\begin{cases} \langle A_{1k}, A_{1k}\rangle(t) = \displaystyle\int_0^t\left(\frac{\alpha_k\sigma_{k1}I_k(u)}{W(u)}\right)^2\mathrm{d}u, \quad 1\leqslant k\leqslant p \\ \langle A_{2k}, A_{2k}\rangle(t) = \displaystyle\int_0^t\left(\frac{(1-\mathfrak{R}_0)\sigma_{k1}I_k(u)}{W(u)}\mathrm{d}u\right)^2, \quad 2\leqslant k\leqslant p \\ \langle A_{3k}, A_{3k}\rangle(t) = \displaystyle\int_0^t\left(\frac{\alpha_k\sigma_{k2}I_k^2(u)}{W(u)}\mathrm{d}u\right)^2\mathrm{d}u, \quad 1\leqslant k\leqslant p \\ \langle A_{4k}, A_{4k}\rangle(t) = \displaystyle\int_0^t\left(\frac{(1-\mathfrak{R}_0)\sigma_{k2}I_k^2(u)}{W(u)}\right)^2\mathrm{d}u, \quad 2\leqslant k\leqslant p. \end{cases}$$

Clearly, $\frac{\langle A_{1k},A_{1k}\rangle(t)}{t} \leqslant \sigma_{k1}^2$ and $\frac{\langle A_{2k},A_{2k}\rangle(t)}{t} \leqslant \sigma_{k2}^2$. So, by the strong law of large numbers for local martingales we get

$$\begin{cases} \lim\limits_{t\to\infty}\dfrac{A_{1k}(t)}{t} = 0, \quad 1\leqslant k\leqslant p \\ \lim\limits_{t\to\infty}\dfrac{A_{2k}(t)}{t} = 0, \quad 2\leqslant k\leqslant p. \end{cases} \tag{V.17}$$

On the other hand, the exponential martingales inequality [19] guarantees that

$$\mathrm{P}\left\{\sup_{0\leqslant t\leqslant\tau}\left\{A_{jk} - \frac{\lambda_1}{2}\langle A_{jk}, A_{jk}\rangle(t)\right\} > \lambda_2\right\} \leqslant \exp(-\lambda_1\lambda_2), \quad j\in\{3,4\} \text{ and } 2\leqslant k\leqslant p,$$

for any $\tau, \lambda_1, \lambda_2 > 0$. By taking in particular $\tau \in \{2, 4, \cdots\}$, $\lambda_1 = 1$ and $\lambda_2 = \ln(\tau^2)$, we get

$$\mathrm{P}\left\{\sup_{0\leqslant t\leqslant\tau}\left\{A_{jk} - \frac{1}{2}\langle A_{jk}, A_{jk}\rangle(t)\right\} > \ln(\tau^2)\right\} \leqslant \tau^{-2}, \quad j \in \{3,4\} \quad \text{and} \quad 2 \leqslant k \leqslant p.$$

On the basis of the Borel-Cantelli lemma [19], we deduce that for almost all $\omega \in \Omega$, there is a random integer τ_0 such that for $\tau \geqslant \tau_0$, we have

$$\sup_{0\leqslant t\leqslant\tau}\left\{A_{jk} - \frac{1}{2}\langle A_{jk}, A_{jk}\rangle(t)\right\} \leqslant \ln(\tau^2), \quad j \in \{3,4\} \quad \text{and} \quad 2 \leqslant k \leqslant p. \tag{V.18}$$

Substituting (V.18) into (V.16) yields

$$\frac{\ln W(t)}{t} \leqslant \frac{\ln W(0)}{t} - \frac{(1-\mathfrak{R}_0)m}{\max\limits_{1\leqslant k\leqslant p}\{\alpha_k\} + (1-\mathfrak{R}_0)} + c\max_{1\leqslant k\leqslant p}\left\{\frac{b_k}{\alpha_k}\right\}\frac{1}{t}\int_0^t\left(X(u) - \frac{r}{m}\right)^+ \mathrm{d}u - \frac{1}{4\sum\limits_{k=1}^{p}\sigma_{k1}^{-2}}$$
$$+ \sum_{k=1}^{p}\frac{A_{1k}(t)}{t} + \sum_{k=2}^{p}\frac{A_{2k}(t)}{t} + \frac{4p\ln(\tau)}{t},$$

for all $0 \leqslant t \leqslant \tau$, $\tau \geqslant \tau_0$ a.s. By choosing $0 \leqslant \tau - 1 \leqslant t \leqslant \tau$ and using the fact that $x^+ = \dfrac{x + |x|}{2}$, we obtain

$$\frac{\ln W(t)}{t} \leqslant \frac{\ln W(0)}{t} - \frac{(1-\mathfrak{R}_0)m}{\max\limits_{1\leqslant k\leqslant p}\{\alpha_k\} + (1-\mathfrak{R}_0)} + c\max_{1\leqslant k\leqslant p}\left\{\frac{b_k}{\alpha_k}\right\}\frac{1}{2t}\int_0^t\left|X(u) - \frac{r}{m}\right|\mathrm{d}u$$
$$+ \frac{1}{2}\times c\max_{1\leqslant k\leqslant p}\left\{\frac{b_k}{\alpha_k}\right\}\underbrace{\frac{1}{t}\int_0^t\left(X(u) - \frac{r}{m}\right)\mathrm{d}u}_{=\frac{1}{t}\int_0^t X(u)\mathrm{d}u - \frac{r}{m}} - \frac{1}{4\sum\limits_{k=1}^{p}\sigma_{k1}^{-2}} + \sum_{k=1}^{p}\frac{A_{1k}(t)}{t} + \sum_{k=2}^{p}\frac{A_{2k}(t)}{t} + \frac{4p\ln(\tau)}{\tau - 1}. \tag{V.19}$$

Taking the superior limit in both sides of the last inequality and using (V.17) with Lemma 2.3 and the fact that $X(t)$ is ergodic implies

$$\limsup_{t\to\infty}\frac{\ln(W(t))}{t} \leqslant c\times\max_{1\leqslant k\leqslant p}\left\{\frac{b_k}{\alpha_k}\right\}\times\frac{1}{2}\int_0^\infty\left|x - \frac{r}{m}\right|\mu(x)\mathrm{d}x - \frac{(1-\mathfrak{R}_0)m}{\max\limits_{1\leqslant k\leqslant p}\{\alpha_k\} + (1-\mathfrak{R}_0)} - \frac{1}{4\times\sum\limits_{i=1}^{p}\left(\frac{1}{\sigma_{i1}}\right)^2} := \nu^*,$$

which is exactly the required conclusion. Moreover, it goes without saying that if $\nu^* < 0$ then the disease will vanish exponentially almost surely. Thus, the theorem is proved. ■

3 Comparison between our new results and the ones existing in the recent literature

The objective of this section is to clearly exhibit the fact that Theorem 2.1 is much stronger than its counterpart proposed by Liu and Jiang in [12]. To this end, we present first the following interesting lemma:

Lemma 3.1. *Let λ_1, λ_2, λ_3 and λ_4 be four positive constants. Then, the function $\varphi : x \mapsto \dfrac{\min\{\lambda_1 - \lambda_2 x, \lambda_3 x\}}{\lambda_4 + x}$ takes its maximum value at the point $x^* = \frac{\lambda_1}{\lambda_2+\lambda_3}$. In other terms, for any $x \in \mathbb{R}_{\geqslant 0} = [0, \infty)$*

$$\frac{\min\{\lambda_1 - \lambda_2 x, \lambda_3 x\}}{\lambda_4 + x} = \varphi(x) \leqslant \varphi\left(\frac{\lambda_1}{\lambda_2 + \lambda_3}\right) = \frac{\lambda_1\lambda_3}{\lambda_1 + \lambda_4(\lambda_2 + \lambda_3)}. \tag{V.20}$$

Proof. Let $x \in \mathbb{R}_{\geqslant 0}$, since $\min\{w,v\} = \dfrac{w+v-|w-v|}{2}$ for all $w,v \in \mathbb{R}$, we have

$$\begin{aligned}\min\{\lambda_1-\lambda_2 x, \lambda_3 x\} &= \frac{(\lambda_1-\lambda_2 x+\lambda_3 x)-|\lambda_1-\lambda_2 x-\lambda_3 x|}{2}\\ &= \frac{(\lambda_1+(\lambda_3-\lambda_2)\,x)-|\lambda_1-(\lambda_2+\lambda_3)\,x|}{2}\\ &= \frac{(\lambda_1+(\lambda_3-\lambda_2)\,x)-(\lambda_2+\lambda_3)\left|x-\frac{\lambda_1}{\lambda_2+\lambda_3}\right|}{2}.\end{aligned}$$

Therefore

$$\varphi(x) = \frac{\min\{\lambda_1-\lambda_2 x, \lambda_3 x\}}{\lambda_4+x} = \begin{cases}\varphi_1(x) \triangleq \dfrac{\lambda_3 x}{x+\lambda_4} = \lambda_3 - \dfrac{\lambda_3\lambda_4}{x+\lambda_4}, & \text{if } 0 \leqslant x \leqslant \dfrac{\lambda_1}{\lambda_2+\lambda_3}\\ \varphi_2(x) \triangleq \dfrac{\lambda_1-\lambda_2 x}{x+\lambda_4} = -\lambda_2 + \dfrac{\lambda_1+\lambda_2\lambda_4}{x+\lambda_4}, & \text{if } x \geqslant \dfrac{\lambda_1}{\lambda_2+\lambda_3}.\end{cases} \tag{V.21}$$

Plainly, the functions $\varphi_1 : x \mapsto \lambda_3 - \dfrac{\lambda_3\lambda_4}{x+\lambda_4}$ and $\varphi_2 : x \mapsto -\lambda_2 + \dfrac{\lambda_1+\lambda_2\lambda_4}{x+\lambda_4}$ are respectively increasing and decreasing on $[0,\infty)$. So, we get

$$\begin{cases}\varphi(x) = \varphi_1(x) \leqslant \varphi_1\left(\dfrac{\lambda_1}{\lambda_2+\lambda_3}\right) = \varphi\left(\dfrac{\lambda_1}{\lambda_2+\lambda_3}\right) = \dfrac{\lambda_1\lambda_3}{\lambda_1+\lambda_4(\lambda_2+\lambda_3)}, & \text{for all } 0 \leqslant x \leqslant \dfrac{\lambda_1}{\lambda_2+\lambda_3},\\ \text{and} & \\ \varphi(x) = \varphi_2(x) \leqslant \varphi_2\left(\dfrac{\lambda_1}{\lambda_2+\lambda_3}\right) = \varphi\left(\dfrac{\lambda_1}{\lambda_2+\lambda_3}\right) = \dfrac{\lambda_1\lambda_3}{\lambda_1+\lambda_4(\lambda_2+\lambda_3)}, & \text{for all } x \geqslant \dfrac{\lambda_1}{\lambda_2+\lambda_3}.\end{cases} \tag{V.22}$$

Consequently, $\varphi(x) \leqslant \varphi\left(\frac{\lambda_1}{\lambda_2+\lambda_3}\right)$ for any $x \geqslant 0$, and the proof is complete. ∎

Theorem 3.1. *If $\mathfrak{R}_0 < 1$, then, the function $\Psi : l \mapsto -\dfrac{\min\{(n_1+m)(1-\mathfrak{R}_0)-n_1 l, ml\}}{\max\limits_{1\leqslant k\leqslant p}\{\alpha_k\}+l}$ assumes its minimum value at the point $l^* = (1-\mathfrak{R}_0)$. In other terms, for any $l \geqslant 0$*

$$-\frac{(1-\mathfrak{R}_0)m}{\max\limits_{1\leqslant k\leqslant p}\{\alpha_k\}+(1-\mathfrak{R}_0)} = \Psi(1-\mathfrak{R}_0) \leqslant \Psi(l) = -\frac{\min\{(n_1+m)(1-\mathfrak{R}_0)-n_1 l, ml\}}{\max\limits_{1\leqslant k\leqslant p}\{\alpha_k\}+l}. \tag{V.23}$$

Proof. Let us assume that $\mathfrak{R}_0 < 1$. By taking $\lambda_1 = (n_1+m)(1-\mathfrak{R}_0) > 0$, $\lambda_2 = n_1$, $\lambda_3 = m$ and $\lambda_4 = \max\limits_{1\leqslant k\leqslant p}\{\alpha_k\}$, we can deduce by applying Lemma 3.1 that

$$\frac{\min\{(n_1+m)(1-\mathfrak{R}_0)-n_1 l, ml\}}{\max\limits_{1\leqslant k\leqslant p}\{\alpha_k\}+l} \leqslant \frac{(n_1+m)(1-\mathfrak{R}_0)m}{(n_1+m)(1-\mathfrak{R}_0)+\max\limits_{1\leqslant k\leqslant p}\{\alpha_k\}(n_1+m)}, \quad \forall l \geqslant 0.$$

Thus, for any $l \in \mathbb{R}_{\geqslant 0}$

$$\underbrace{-\frac{(1-\mathfrak{R}_0)m}{\max\limits_{1\leqslant k\leqslant p}\{\alpha_k\}+(1-\mathfrak{R}_0)}}_{\Psi(1-\mathfrak{R}_0)} \leqslant \underbrace{-\frac{\min\{(n_1+m)(1-\mathfrak{R}_0)-n_1 l, ml\}}{\max\limits_{1\leqslant k\leqslant p}\{\alpha_k\}+l}}_{\Psi(l)},$$

which is justly the desired inequality. ∎

In their recent work presented in [12], Liu and Jiang have demonstrated a very important result concerning the extinction of the AIDS disease. More precisely they provided in Theorem 4.1 the following sufficient conditions to have an exponential extinction of the system (V.2):

- (C_1) : $\mathfrak{R}_0 \triangleq \dfrac{\frac{cr}{m}\sum\limits_{k=1}^{p} b_k\widetilde{I_k} + g_2\widetilde{I_2}}{n_1+m} < 1.$
- (C_2) : There exists a positive constant l such that $l < (n_1+m)(1-\mathfrak{R}_0)/n_1$.
- (C_3) : $v \triangleq c\dfrac{\max\limits_{1\leqslant k\leqslant p}\{b_k\}}{\min\limits_{1\leqslant k\leqslant p}\{\alpha_k\}}\displaystyle\int_0^{\infty}\left|x-\frac{r}{m}\right|\mu(x)\mathrm{d}x - \frac{\min\{(n_1+m)(1-\mathfrak{R}_0)-n_1l, ml\}}{\max\limits_{1\leqslant k\leqslant p}\{\alpha_k\}+l} - \frac{1}{4\times\sum\limits_{i=1}^{p}\left(\frac{1}{\sigma_{i1}}\right)^2} < 0.$

Comparing those latter with the hypothetical framework of Theorem 2.1 which consists of the following two assumptions

- (C_1) : $\mathfrak{R}_0 \triangleq \dfrac{\frac{cr}{m}\sum\limits_{k=1}^{p} b_k\widetilde{I_k} + g_2\widetilde{I_2}}{n_1+m} < 1.$
- (C_3') : $v^* \triangleq c\times\max\limits_{1\leqslant k\leqslant p}\left\{\dfrac{b_k}{\alpha_k}\right\}\times\dfrac{1}{2}\displaystyle\int_0^{\infty}\left|x-\frac{r}{m}\right|\mu(x)\mathrm{d}x - \frac{(1-\mathfrak{R}_0)m}{\max\limits_{1\leqslant k\leqslant p}\{\alpha_k\}+(1-\mathfrak{R}_0)} - \frac{1}{4\times\sum\limits_{i=1}^{p}\left(\frac{1}{\sigma_{i1}}\right)^2} < 0.$

one can immediately affirm that our result improves and ameliorates that of Liu and Jiang [12]. Indeed, the assumption (C_2) is no longer imposed in our case. Also, and what is more important, the condition (C_3') is weaker than (C_3). This last fact comes from the inequality $v^* < v$, and which in turn results the from the following improvements adopted in our analysis:

1. We have used the quantity $\max\limits_{1\leqslant k\leqslant p}\left\{\dfrac{b_k}{\alpha_k}\right\}$ instead of $\dfrac{\max\limits_{1\leqslant k\leqslant p}\{b_k\}}{\min\limits_{1\leqslant k\leqslant p}\{\alpha_k\}}$, and as it is easy to see:

$$\max_{1\leqslant k\leqslant p}\left\{\frac{b_k}{\alpha_k}\right\} \leqslant \frac{\max\limits_{1\leqslant k\leqslant p}\{b_k\}}{\min\limits_{1\leqslant k\leqslant p}\{\alpha_k\}}.$$

2. We have reduced the integral $\displaystyle\int_0^{\infty}\left|x-\frac{r}{m}\right|\mu(x)\mathrm{d}x$ to the half, and needless to say:

$$\frac{1}{2}\int_0^{\infty}\left|x-\frac{r}{m}\right|\mu(x)\mathrm{d}x < \int_0^{\infty}\left|x-\frac{r}{m}\right|\mu(x)\mathrm{d}x.$$

   This amelioration comes back essentially to inequality (V.12) in the proof of Theorem 2.1 where we have used the ramp function $(x\mapsto x^+)$ instead of the absolute value adopted by Liu and Jiang in [12].

3. We have minimized the function $\Psi : l \mapsto -\dfrac{\min\{(n_1+m)(1-\mathfrak{R}_0)-n_1l, ml\}}{\max\limits_{1\leqslant k\leqslant p}\{\alpha_k\}+l}$ on $[0,\infty)$, and found that for any $l \geqslant 0$:

$$-\frac{(1-\mathfrak{R}_0)m}{\max\limits_{1\leqslant k\leqslant p}\{\alpha_k\}+(1-\mathfrak{R}_0)} = \Psi(1-\mathfrak{R}_0) \leqslant \Psi(l) = -\frac{\min\{(n_1+m)(1-\mathfrak{R}_0)-n_1l, ml\}}{\max\limits_{1\leqslant k\leqslant p}\{\alpha_k\}+l}.$$

   This last inequality, allowed us to choose the optimal $l \in \left(0, \frac{(n_1+m)(1-\mathfrak{R}_0)}{n_1}\right)$ (which is $l^* = (1-\mathfrak{R}_0)$ in this case) and that makes the quantity $-\frac{\min\{(n_1+m)(1-\mathfrak{R}_0)-n_1l, ml\}}{\max\limits_{1\leqslant k\leqslant p}\{\alpha_k\}+l}$ appearing in the sill v minimal, the thing that has permitted us to reduce and ameliorate considerably the extinction threshold.

More explicitly, the inequality $\nu^* < \nu$, makes the condition $\nu^* < 0$ weaker than $\nu < 0$. So, our new extinction theorem, which requires the condition $\nu^* < 0$ to guarantee the disease's disappearance, will be stronger than its counterpart proposed in [12], since the latter demands a stronger condition (i.e., $\nu < 0$).

In sum, the statement of Theorem 2.1 that we have proposed is considerably stronger than that of Liu and Jiang [12] because it weakens the conditions and hence permits us to deduce the AIDS extinction in some cases where the Theorem 4.1 of [12] is not capable of doing it.

4 Numerical simulations

Parameters	Description	Numerical values
p	Number of infected classes	3
r	Susceptible recruitment rate	0.11
b_1	Contact rate between S and I_1	0.12
b_2	Contact rate between S and I_2	0.17
b_3	Contact rate between S and I_3	0.09
c	Average contact number	0.1
m	Mortality rate (not directly related to the AIDS)	0.4
n_1	Movement rate from I_1 to I_2	0.28
n_2	Movement rate from I_2 to I_3	0.15
n_3	Movement rate from I_3 to the removed class A	0.17
g_2	Movement rate from I_2 to I_1	0.2
g_3	Movement rate from I_3 to I_2	0.1

Table V.1: Descriptions and numerical values of the system parameters used in the simulations.

In this section, and by picking the parameter values from the theoretical data depicted in Table V.1, we set forth some simulations to shed a little light on the strength of our new extinction theorem associated with the AIDS model (V.2). The solution of this latter, is simulated in our case with the initial condition $(S(0), I_1(0), I_2(0), I_3(0)) = (0.2, 0.31, 0.3, 0.25)$ and the stochastic intensities $\sigma_{01} = 0.85,\ \sigma_{02} = 0.51,\ \sigma_{11} = 0.79,\ \sigma_{12} = 0.31,\ \sigma_{21} = 0.73,\ \sigma_{22} = 0.25,\ \sigma_{31} = 0.83,\ \sigma_{32} = 0.27$. Henceforth, the units adopted for time and number of individuals are respectively one day and one million population.

If we try to use Theorem 4.1 of [12], then we will notice that the condition (C_1) is verified ($\mathfrak{R}_0 = 0.1278 < 1$), but the extinction threshold ν will be positive for all $0 < l < \dfrac{(n_1+m)(1-\mathfrak{R}_0)}{n_1}$:

$$\begin{aligned}
\nu &\triangleq c\frac{\max\limits_{1\leqslant k\leqslant p}\{b_k\}}{\min\limits_{1\leqslant k\leqslant p}\{\alpha_k\}}\int_0^\infty \left|x-\frac{r}{m}\right|\mu(x)\mathrm{d}x - \frac{\min\{(n_1+m)(1-\mathfrak{R}_0)-n_1 l, ml\}}{\max\limits_{1\leqslant k\leqslant p}\{\alpha_k\}+l} - \frac{1}{4\times\sum\limits_{i=1}^{p}\left(\frac{1}{\sigma_{i1}}\right)^2}\\
&\geqslant c\frac{\max\limits_{1\leqslant k\leqslant p}\{b_k\}}{\min\limits_{1\leqslant k\leqslant p}\{\alpha_k\}}\int_0^\infty \left|x-\frac{r}{m}\right|\mu(x)\mathrm{d}x - \frac{(1-\mathfrak{R}_0)m}{\max\limits_{1\leqslant k\leqslant p}\{\alpha_k\}+(1-\mathfrak{R}_0)} - \frac{1}{4\times\sum\limits_{i=1}^{p}\left(\frac{1}{\sigma_{i1}}\right)^2} \text{ (See the inequality (V.23))}\\
&\geqslant 0.5796 > 0.
\end{aligned}$$

Consequently, and however l is chosen, (C_3) does not hold. Hence we are unable to deduce the disease's extinction from Theorem 4.1 of [12] in this case. On the other hand, we have

$$\nu^* \triangleq c\times\max_{1\leqslant k\leqslant p}\left\{\frac{b_k}{\alpha_k}\right\}\times\frac{1}{2}\int_0^\infty \left|x-\frac{r}{m}\right|\mu(x)\mathrm{d}x - \frac{(1-\mathfrak{R}_0)m}{\max\limits_{1\leqslant k\leqslant p}\{\alpha_k\}+(1-\mathfrak{R}_0)} - \frac{1}{4\times\sum\limits_{i=1}^{p}\left(\frac{1}{\sigma_{i1}}\right)^2} = -0.02086 < 0.$$

So, the condition (C_3') is verified. Hence, and by using our new theorem (Theorem 2.1), we can easily conclude this time that the AIDS disease will be eradicated and this is what we see exactly in Figure V.2.

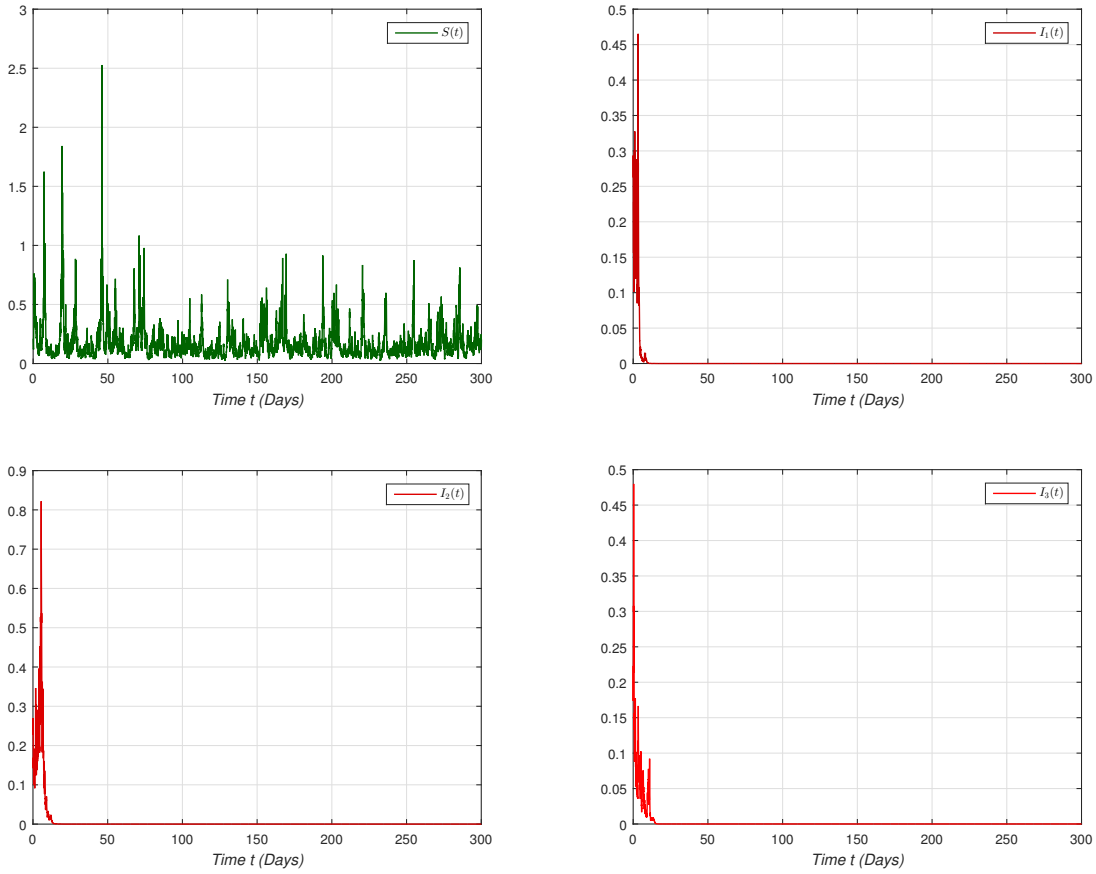


Figure V.2: Solutions' paths of the AIDS model (V.2) when the numerical values are taken as shown in Table V.1.

5 Conclusion

Acquired Immunodeficiency Syndrome (AIDS for short) is a highly dangerous infectious disease that has aroused the interest of a considerable number of scholars in the last few decades, the thing that made it the subject of many research fields like mathematical epidemiology. In this chapter, we have reconsidered the extinction sufficient conditions of the higher-order perturbed multi-stage AIDS/HIV model proposed by Liu and Jiang in [12]. First, and by adopting some new analytical techniques, we gave an extinction theorem other than the one presented in [12]. Then, we have demonstrated that our theorem is much stronger than this latter since it is requiring a weaker hypothetical framework. In order to show clearly and in a practical way the strength of our new result, we have introduced some numerical examples for which the Theorem 4.1 of [12] is unable to warranty the AIDS extinction, while ours is well capable to do so. Finally, we mention that the obtained results improve our comprehension of the AIDS spreading behavior, which makes it a rich basis for further studies concerning the multi-dimensional epidemic models.

Chapter VI

The impact of Lévy noise on the threshold dynamics of a stochastic SVIR epidemic model with general incidence functions

Abstract

In this chapter, we survey a stochastic SVIR epidemic model with general incidence rates that is disturbed by both Brownian motions and Lévy jumps. Initially, we establish its well-posedness in the sense that it has a unique positive and global in time solution. Then, we rely on some assumptions and nonstandard analytic techniques, to derive sufficient and almost necessary conditions for extinction, persistence in the mean and also weak persistence. More explicitly, we identify firstly under some hypotheses a threshold λ between extinction and persistence in the mean. In other phrases, if $\lambda < 0$ the infected population dies out while it persists in the mean when $\lambda > 0$. Then, and by modifying the hypothetical framework in order to cover more incidence rates, we prove that λ can act also as a threshold between extinction and weak persistence. At last, we provide some numerical simulations to corroborate our findings and cover some particular cases of response functions.

1 Introduction and model formulation

The supervision and controlling of infectious diseases are premised on modeling and simulation as the principal decision-making implements [3, 196, 197]. Nevertheless, each infectious disease is characterized by its own spreading behavior, so and in each particular case, the choice of an appropriate manner to describe suitably this latter is strongly required [32, 198, 199, 200]. On this matter, mathematical epidemiology, and particularly through compartmental models, is considered the most prominent and eminent tool to have a clear vision of an epidemic's prevalence dynamics [71, 148]. The majority of mathematical models currently adopted in the analysis of infectious diseases propagation are derived from the classical SIR model proposed by Kermack and McKendrick in the late twenties of the last century [5]. One of the mostly important of these latter is the well-known SVIR model which is none other than the usual SIR but with a consideration of the vaccination strategy, the reason that makes it very appropriate to analyze many diseases such as tuberculosis, avian influenza, diphtheria, polio, pertussis, etc. [201, 202]. However, the spread of any infectious disease is naturally influenced by environmental disturbances, and this makes it surely more difficult to predict its behavior [14, 157]. In such cases, the introduction of the random effect into the dynamical model, and the use of mathematical formulations based on stochastic differential equations is very convenient, the thing that has motivated several researchers in recent years to adopt this vision for many epidemic models including the SVIR one. For example, we find in the recent work of Zhang and Yang [10], a very intriguing stochastic version of the SVIR

model that is distinguished by the adoption of two general incidence rates $IF_1(S,I)$ and $IF_2(V,I)$, as well as the incorporation of proportional Brownian motions modeling the slight and continuous environmental fluctuations. This version covers a lot of special cases like the one studied in [203], and it is presented by the following stochastic differential equations system:

$$\begin{cases} \mathrm{d}S(t) = \big[(1-p)\Lambda - F_1\left(S(t),I(t)\right)\times I(t) - \mu S(t)\big]\mathrm{d}t + \sigma_1 S(t)\mathrm{d}\mathcal{B}_1(t), \\ \mathrm{d}V(t) = \big[p\Lambda - F_2\left(V(t),I(t)\right)\times I(t) - \mu V(t)\big]\mathrm{d}t + \sigma_2 V(t)\mathrm{d}\mathcal{B}_2(t), \\ \mathrm{d}I(t) = \big[\big(F_1\left(S(t),I(t)\right) + F_2\left(V(t),I(t)\right)\big)\times I(t) - (\mu+\delta_1+\delta_2)I(t)\big]\mathrm{d}t + \sigma_3 I(t)\mathrm{d}\mathcal{B}_3(t), \\ \mathrm{d}R(t) = \big[\delta_2 I(t) - \mu R(t)\big]\mathrm{d}t + \sigma_4 R(t)\mathrm{d}\mathcal{B}_4(t). \end{cases} \tag{VI.1}$$

Here, $S(t), V(t), I(t)$ and $R(t)$ are referring respectively to the intensities of susceptible, vaccinated infected and recovered individuals at the instant t. $\mathcal{B}_i$ $(i = 1,2,3,4)$ are independent Brownian motions of intensities σ_i $(i = 1,2,3,4)$ defined on a filtered probability space $(\Omega, \mathcal{E}, \{\mathcal{E}_t\}_{t\geqslant 0}, \mathbb{P})$. The latter is presumed to be equipped with an increasing filtration $\{\mathcal{E}_t\}_{t\geqslant 0}$ that is right-continuous while $\mathcal{E}_0$ contains all the $\mathbb{P}$-null sets. Regarding the system parameters, Λ designates the recruitment rate of the susceptible individuals, $p \in (0,1)$ is the prevalence rate of the vaccination program, μ and δ_1 are respectively the natural and the disease-induced death rates, lastly, δ_2 is the recovery rate. For convenience in writing, we denote from now on the constant $\delta_1 + \delta_2$ by δ. All the aforementioned parameters are supposed to be in the positive real semi-line $\mathbb{R}_+ := \{x \in \mathbb{R} \mid x > 0\}$. The functions $IF_2\left(V,I\right)$ and $IF_1\left(S,I\right)$ appearing in system (VI.1) denote the incidence rates representing the number of cases infected in unit time. However, the addition of white noises is a very logical manner to model any phenomenon in which some quantities are subject to slight and continuous aleatory fluctuations like those produced by the environment in the course of a certain infectious disease's propagation [9, 10]. But unluckily, this approach is not suitable enough to represent the impact of strong and brusque external perturbations like earth tremors, weather changes, tornadoes, floods, etc [150, 149]. For this reason, we will resort to Lévy processes which are known for their huge ability in formulating properly this type of case. By taking into consideration this kind of random perturbations, one can extend the model (VI.1) of Zhang and Yang to the following system of stochastic differential equations with Lévy jumps (SDE-LJ for short):

$$\begin{cases} \mathrm{d}S(t) = \big[(1-p)\Lambda - F_1\left(S(t),I(t)\right)\times I(t) - \mu S(t)\big]\mathrm{d}t + \sigma_1 S(t)\mathrm{d}\mathcal{B}_1(t) + \displaystyle\int_{\mathbb{U}} \eta_1(u)S(t^-)\widetilde{\mathcal{N}}(\mathrm{d}t,\mathrm{d}u), \\ \mathrm{d}V(t) = \big[p\Lambda - F_2\left(V(t),I(t)\right)\times I(t) - \mu V(t)\big]\mathrm{d}t + \sigma_2 V(t)\mathrm{d}\mathcal{B}_2(t) + \displaystyle\int_{\mathbb{U}} \eta_2(u)V(t^-)\widetilde{\mathcal{N}}(\mathrm{d}t,\mathrm{d}u), \\ \mathrm{d}I(t) = \big[\big(F_1\left(S(t),I(t)\right) + F_2\left(V(t),I(t)\right)\big)\times I(t) - (\mu+\delta)I(t)\big]\mathrm{d}t + \sigma_3 I(t)\mathrm{d}\mathcal{B}_3(t) + \displaystyle\int_{\mathbb{U}} \eta_3(u)I(t^-)\widetilde{\mathcal{N}}(\mathrm{d}t,\mathrm{d}u), \\ \mathrm{d}R(t) = \big[\delta_2 I(t) - \mu R(t)\big]\mathrm{d}t + \sigma_4 R(t)\mathrm{d}\mathcal{B}_4(t) + \displaystyle\int_{\mathbb{U}} \eta_4(u)R(t^-)\widetilde{\mathcal{N}}(\mathrm{d}t,\mathrm{d}u), \end{cases} \tag{VI.2}$$

Here and elsewhere, $S(t^-), V(t^-), I(t^-)$ and $R(t^-)$ are respectively indicating the left limits of $S(t), V(t), I(t)$ and $R(t)$. $\mathcal{N}$ is a Poisson counting measure independent of $\mathcal{B}_i$ $(i = 1,2,3,4)$ with a finite characteristic measure ν defined on a measurable set $\mathbb{U} \subset \mathbb{R}_+$. $\widetilde{\mathcal{N}}$ is a compensating martingale, and it is supposed that ν is a Lévy measure such that $\widetilde{\mathcal{N}}(\mathrm{d}t,\mathrm{d}u) = \mathcal{N}(\mathrm{d}t,\mathrm{d}u) - \nu(\mathrm{d}u)\mathrm{d}t$. Finally, η_i $(i = 1,2,3,4)$ are the jumps magnitude functions which are presumed to be continuous and bounded on $\mathbb{U}$. Clearly, the recovered individuals do not affect the compartments S, V and I, hence the dynamics of system (VI.1) can be concluded just from the following simplified model:

$$\begin{cases} \mathrm{d}S(t) = \big[(1-p)\Lambda - F_1\left(S(t),I(t)\right)\times I(t) - \mu S(t)\big]\mathrm{d}t + \sigma_1 S(t)\mathrm{d}\mathcal{B}_1(t) + \displaystyle\int_{\mathbb{U}} \eta_1(u)S(t^-)\widetilde{\mathcal{N}}(\mathrm{d}t,\mathrm{d}u), \\ \mathrm{d}V(t) = \big[p\Lambda - F_2\left(V(t),I(t)\right)\times I(t) - \mu V(t)\big]\mathrm{d}t + \sigma_2 V(t)\mathrm{d}\mathcal{B}_2(t) + \displaystyle\int_{\mathbb{U}} \eta_2(u)V(t^-)\widetilde{\mathcal{N}}(\mathrm{d}t,\mathrm{d}u), \\ \mathrm{d}I(t) = \big[\big(F_1\left(S(t),I(t)\right) + F_2\left(V(t),I(t)\right)\big)\times I(t) - (\mu+\delta)I(t)\big]\mathrm{d}t + \sigma_3 I(t)\mathrm{d}\mathcal{B}_3(t) + \displaystyle\int_{\mathbb{U}} \eta_3(u)I(t^-)\widetilde{\mathcal{N}}(\mathrm{d}t,\mathrm{d}u). \end{cases} \tag{VI.3}$$

For ease of reading the remainder of this chapter, we resume the transmission mechanisms of the abovementioned model by the flow diagram shown in Figure VI.1.

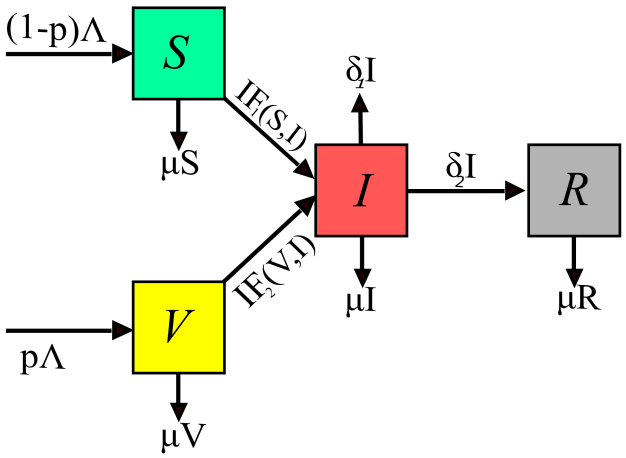


Figure VI.1: Flowchart of the proposed SVIR compartmental model.

Our primary objective in this chapter is to figure out sufficient conditions of extinction, persistence in the mean and also weak persistence of the SVIR model (VI.3). These asymptotic properties are deemed to be adequate for obtaining an excellent perception of the future behavior of any disease formulated by the said model. The newness of our results resides mainly in the adopted way to estimate the limits $\lim_{t\to\infty}\frac{1}{t}\int_0^t \overline{S}(s)\,\mathrm{d}s$ and $\lim_{t\to\infty}\frac{1}{t}\int_0^t \overline{V}(s)\,\mathrm{d}s$, where $\overline{S}$ and $\overline{V}$ are respectively denoting the solutions of the following auxiliary subsystems:

$$\begin{cases} \mathrm{d}\overline{S}(t) = \Big[(1-p)\Lambda - \mu\overline{S}(t)\Big]\,\mathrm{d}t + \sigma_1\overline{S}(t)\mathrm{d}\mathcal{B}_1(t) + \displaystyle\int_{\mathbb{U}} \eta_1(u)\overline{S}(t^-)\widetilde{\mathcal{N}}(\mathrm{d}t,\mathrm{d}u), \qquad \forall t>0, \\ \overline{S}(0) = S(0) > 0. \end{cases} \tag{VI.4}$$

and

$$\begin{cases} \mathrm{d}\overline{V}(t) = \Big[p\Lambda - \mu\overline{V}(t)\Big]\,\mathrm{d}t + \sigma_1\overline{V}(t)\mathrm{d}\mathcal{B}_1(t) + \displaystyle\int_{\mathbb{U}} \eta_2(u)\overline{V}(t^-)\widetilde{\mathcal{N}}(\mathrm{d}t,\mathrm{d}u), \qquad \forall t>0, \\ \overline{V}(0) = V(0) > 0. \end{cases} \tag{VI.5}$$

Our method permits us to fill the gap left by the use of classical approaches presented for instance in [152, 146, 151]. Also, and under some appropriate assumptions, we find a threshold λ between extinction and persistence in the mean. Then, we notice that these latter do not include some incidence rates often used in the literature. So, we modify the hypothetical framework adopted previously in order to cover more functions, and we prove by some non-standard analytical techniques that λ stills play the role of a threshold, but this time between the extinction and weak persistence. The performed analysis in this chapter sounds to be very hopeful to examine other epidemic models and especially those which are perturbed with Lévy noises.

The remaining of this chapter are structured as follows: in the first section we establish the existence and uniqueness of a global positive solution to the stochastic SVIR system (VI.3). In the second one, we provide under some assumptions a threshold λ between extinction and persistence in the mean. Then, we change the hypothetical framework adopted previously to cover more response functions and we prove that λ can act also as a threshold between extinction and weak persistence. Lastly we support our theoretical results by some numerical examples in Section 4. before drawing the main conclusions of this chapter in Section 5.

2 Existence and unicity of the positive global-in-time solution

The first concerning in exploring an epidemic model is to know whether it admits a unique and positive global-in-time solution or not. In this section, we will provide some conditions under which this property is ensured. But before doing so, let us first introduce the following assumptions:

- (H_0) : There exists a positive constant c such that for all $S, V, I \geqslant 0$

$$\underbrace{\frac{\partial F_1}{\partial I}(S,I)}_{(1)} \underbrace{\leqslant}_{} 0 \overbrace{\leqslant}^{(2)} \underbrace{\frac{\partial F_1}{\partial S}(S,I)}_{(3)} \underbrace{\leqslant}_{} c \quad \text{and} \quad \underbrace{\frac{\partial F_2}{\partial I}(V,I)}_{(4)} \underbrace{\leqslant}_{} 0 \overbrace{\leqslant}^{(5)} \underbrace{\frac{\partial F_2}{\partial V}(V,I)}_{(6)} \underbrace{\leqslant}_{} c. \tag{VI.6}$$

- (H_1) : The jumps coefficients $(\eta_i)_{1\leqslant i\leqslant 3}$ satisfy $M_1 := \max\limits_{1\leqslant i\leqslant 3}\left(\int_{\mathbb{U}} \eta_i^2(u)\nu(\mathrm{d}u)\right) < \infty$.

- (H_2) : For all $i \in \{1,2,3\}$, $\eta_i(u) > -1$ and $M_2 := \max\limits_{1\leqslant i\leqslant 3}\left(\int_{\mathbb{U}} \Big(\eta_i(u) - \ln\left(1+\eta_i(u)\right)\Big)\nu(\mathrm{d}u)\right) < \infty$.

Theorem 2.1. *Let hypotheses* (H_1) *and* (H_2) *hold. Then, for any initial value* $\big(S(0),V(0),I(0)\big) \in \mathbb{R}^3_+$, *there corresponds one and only one solution* $\big(S(t),V(t),I(t)\big)$ *to the stochastic differential system* (VI.3) *on* $t \geqslant 0$. *Furthermore, for all* $t \geqslant 0$, *this solution will stay in* $\mathbb{R}^3_+$ *almost surely (a.s. for brevity).*

Proof. As assumed in the statement of the theorem, let the assumptions (H_1) and (H_2) hold. From (H_1), the coefficients of the system (VI.3) are locally Lipschitz continuous. So, for any initial data $\big(S(0),V(0),I(0)\big) \in \mathbb{R}^3_+$ there corresponds a unique maximal solution $\big(S(t),V(t),I(t)\big)$ of (VI.3) that is defined on an interval $[0,\tau_e)$ [21], where τ_e is the explosion time [19]. At this stage, our goal will be to show that $\tau_e = \infty$ almost surely. Let k_1 be a sufficiently large positive integer such that $\big(S(0),V(0),I(0)\big) \in \left(\frac{1}{k_1},k_1\right)$, and consider for any integer $k \geqslant k_1$ the following quantity, which is well defined by the adoption of the convention $\inf\emptyset = \infty$:

$$\tau_k = \inf\left\{t \in [0,\tau_e) \mid \big(S(t),V(t),I(t)\big) \notin \left(\frac{1}{k},k\right)^4\right\} \in [0,\infty]\,. \tag{VI.7}$$

Clearly, τ_k is a stopping time for all $k \geqslant k_1$ ([19], Theorem 3.1). Hence, $\tau_\infty := \lim\limits_{k\to\infty}\tau_k = \sup\limits_{k\geqslant k_1}\tau_k$ is also a stopping time ([18], Lemma 2.11). Since $\tau_\infty \leqslant \tau_e$ (see [13]), $\tau_e = \infty$ will follow if we show that $\tau_\infty = \infty$ a.s., and this is what we are going to do to finish our proof. If $\tau_\infty = \infty$ a.s. is untrue, then there exist perforce two positive constants ξ and ε such that

$$\mathbb{P}\left(\tau_k \leqslant \xi\right) > \varepsilon, \quad \forall k \geqslant k_1. \tag{VI.8}$$

Consider the C^2-function $V : \mathbb{R}^3_+ \to [0,+\infty)$ defined by

$$V\left(S,V,I\right) = \left(S - \kappa - \kappa\ln\left(\frac{S}{\kappa}\right)\right) + \left(V - \kappa - \kappa\ln\left(\frac{V}{\kappa}\right)\right) + \big(I - 1 - \ln(I)\big),$$

where $\kappa > 0$ is a constant to be chosen rightly at a later time. On account of the general Itô's formula (see [21]), we have for all $k \geqslant k_0$ and $0 \leqslant t \leqslant \tau_k$

$$\begin{aligned}\mathrm{d}V\left(S,V,I\right) = {} & \mathcal{L}V\left(S,V,I\right)\mathrm{d}t + \sigma_1\left(S(t)-\kappa\right)\mathrm{d}\mathcal{B}_1(t) + \sigma_2\left(V(t)-\kappa\right)\mathrm{d}\mathcal{B}_2(t) + \sigma_3\left(I(t)-1\right)\mathrm{d}\mathcal{B}_3(t)\\ & + \int_{\mathbb{U}}\Big(\eta_1(u)S(t^-) - \kappa\ln\left(1+\eta_1(u)\right)\Big)\widetilde{\mathcal{N}}(\mathrm{d}t,\mathrm{d}u) + \int_{\mathbb{U}}\Big(\eta_2(u)V(t^-) - \kappa\ln\left(1+\eta_2(u)\right)\Big)\widetilde{\mathcal{N}}(\mathrm{d}t,\mathrm{d}u)\\ & + \int_{\mathbb{U}}\Big(\eta_3(u)I(t^-) - \ln\left(1+\eta_3(u)\right)\Big)\widetilde{\mathcal{N}}(\mathrm{d}t,\mathrm{d}u),\end{aligned}$$

where $\mathcal{L}V\,(S,V,I)$ is given by

$$\begin{aligned}
&\mathcal{L}V\,(S(t),V(t),I(t))\\
&=\left(1-\frac{\kappa}{S(t)}\right)\times\Big((1-p)\Lambda-F_1\,(S(t),I(t))\times I(t)-\mu S(t)\Big)\\
&\quad+\left(1-\frac{\kappa}{V(t)}\right)\times\Big(p\Lambda-F_2\,(V(t),I(t))\times I(t)-\mu V(t)\Big)\\
&\quad+\left(1-\frac{1}{I(t)}\right)\times\Big(\big(F_1\,(S(t),I(t))+F_2\,(V(t),I(t))\big)\times I(t)-(\mu+\delta)I(t)\Big)+\frac{\left(\kappa\sigma_1^2+\kappa\sigma_2^2+\sigma_3^2\right)}{2}\\
&\quad+\kappa\sum_{k=1}^{2}\int_{\mathbb{U}}\Big(\eta_k(u)-\ln\left(1+\eta_k(u)\right)\Big)\nu(\mathrm{d}u)+\int_{\mathbb{U}}\Big(\eta_3(u)-\ln\left(1+\eta_3(u)\right)\Big)\nu(\mathrm{d}u)\\
&=\Big(\Lambda-\mu\left(I(t)+V(t)+S(t)\right)-\delta I(t)\Big)-\left(\frac{\alpha(1-p)\Lambda}{S(t)}-\kappa\mu-\frac{\kappa F_1\,(S(t),I(t))}{S(t)}\times I(t)\right)\\
&\quad-\kappa\left(\frac{p\Lambda}{V(t)}-\mu-\frac{F_2\,(V(t),I(t)))}{V(t)}\times I(t)\right)-\left(F_1\,(S(t),I(t))+F_2\,(V(t),I(t))-(\mu+\sigma)\right)+K.
\end{aligned}$$

Here, K is the following constant:

$$K:=\frac{1}{2}\left(\kappa\sigma_1^2+\kappa\sigma_2^2+\sigma_3^2\right)+\kappa\int_{\mathbb{U}}\Big(\eta_1(u)-\ln\left(1+\eta_1(u)\right)\Big)\nu(\mathrm{d}u)+\kappa\int_{\mathbb{U}}\Big(\eta_2(u)-\ln\left(1+\eta_2(u)\right)\Big)\nu(\mathrm{d}u).$$

So

$$\begin{aligned}
\mathcal{L}V\,(S(t),V(t),I(t)) &\leqslant(\Lambda+(2\kappa+1)\mu+\delta+K)+\left(\kappa\left(\frac{F_1(S(t),I(t))}{S(t)}+\frac{F_2(V(t),I(t))}{V(t)}\right)-\mu-\delta\right)\times I(t)\\
&\leqslant(\Lambda+(2\kappa+1)\mu+\delta+K)+(2c\kappa-\mu-\delta)\times I(t).
\end{aligned}$$

The last inequality is obtained from the inequalities (3) and (6) of (VI.6). The remainder of the proof follows the same lines as the proof of Theorem 2.1 in [154], therefore we skip it here. ■

Remark 2.1. *Besides its interesting mathematical interpretation, the importance of the last theorem lies basically in its ability to make the model* (VI.3) *rational, and biologically meaningful [15].*

3 Extinction, persistence in the mean and weak persistence

When analyzing a mathematical model that describes the spread of a particular epidemic, our main preoccupation is to know if it will end or will last. In this section, we will try to find sufficient and almost necessary conditions for these two asymptotic properties in terms of noises intensities and model parameters.

Definition 3.1 (Simple and exponential extinction [13]). *The infectious individuals $I(t)$ of the system* (VI.3) *are said to be extinct if $\lim_{t\to\infty} I(t)=0$ a.s., and they are called exponentially extinct if there is a negative constant c such that*

$$\limsup_{t\to\infty}\frac{\ln I(t)}{t}\leqslant c<0\quad a.s.$$

Definition 3.2 (Weak persistence and persistence in the mean [153]). *The infected individuals $I(t)$ appearing in the system* (VI.3)*, are said to be persistent (or more precisely weakly persistent) if there is a positive constant c such that $\limsup_{t\to\infty} I(t)\geqslant c>0$ a.s., and they are called persistent in the mean if $\liminf_{t\to\infty}\frac{1}{t}\int_0^t I(s)\,\mathrm{d}s\geqslant c>0$ a.s.*

For simple writing of the next results, it will be practical to use the following notations throughout the rest of this chapter:

- $\overline{\eta}(u) := \max\left\{\eta_i(u) \mid i \in \{1,2,3\}\right\}$.
- $\overline{\chi}_k(u) := (1+\overline{\eta}(u))^k - k \times \overline{\eta}(u) - 1$.
- $\chi_k(u) := \max\left\{\overline{\chi}_k(u), \underline{\chi}_k(u)\right\}$.
- $\underline{\eta}(u) := \min\left\{\eta_i(u) \mid i \in \{1,2,3\}\right\}$.
- $\underline{\chi}_k(u) := \left(1+\underline{\eta}(u)\right)^k - k \times \underline{\eta}(u) - 1$.
- $\ell_k = \int_{\mathbb{U}} \chi_k(u)\nu(\mathrm{d}u)$.

Before passing to the main theorems of this section, let us first start by some useful lemmas concerning the following auxiliary subsystems which are obtained from the two first equations of (VI.3) when the infectious individuals are absent:

$$\begin{cases} \mathrm{d}\overline{S}(t) = \left[(1-p)\Lambda - \mu\overline{S}(t)\right]\mathrm{d}t + \sigma_1\overline{S}(t)\mathrm{d}\mathcal{B}_1(t) + \displaystyle\int_{\mathbb{U}} \eta_1(u)\overline{S}(t^-)\widetilde{\mathcal{N}}(\mathrm{d}t,\mathrm{d}u), \quad \forall t > 0, \\ \overline{S}(0) = S(0) > 0. \end{cases} \tag{VI.9}$$

and

$$\begin{cases} \mathrm{d}\overline{V}(t) = \left[p\Lambda - \mu\overline{V}(t)\right]\mathrm{d}t + \sigma_1\overline{V}(t)\mathrm{d}\mathcal{B}_1(t) + \displaystyle\int_{\mathbb{U}} \eta_2(u)\overline{V}(t^-)\widetilde{\mathcal{N}}(\mathrm{d}t,\mathrm{d}u), \quad \forall t > 0, \\ \overline{V}(0) = V(0) > 0. \end{cases} \tag{VI.10}$$

According to some analytic and mathematical reasons, we assume from now that (H_1), (H_2) and the following additionally hypotheses hold:

- (H_3) : There is a real constant $k > 2$ such that $\Phi_k := \mu - \dfrac{k-1}{2} \times (\sigma_1^2 \vee \sigma_2^2 \vee \sigma_3^2) - \dfrac{\ell_k}{k} > 0$.
- (H_4) : $M_3 := \max\limits_{1\leqslant i\leqslant 3}\left(\displaystyle\int_{\mathbb{U}}\left(\left(1+\eta_i(u)\right)^2 - 1\right)^2 \nu(\mathrm{d}u)\right) < \infty$.
- (H_5) : $M_4 := \left(\displaystyle\int_{\mathbb{U}}\left(\ln\left(1+\eta_3(u)\right)\right)^2 \nu(\mathrm{d}u)\right) < \infty$.

Lemma 3.1. *Let $\left(S(t),V(t),I(t)\right)$ denotes the solution of (VI.3) that starts from a given initial point $\left(S_0,I_0,V_0\right) \in \mathbb{R}_+^3$. Let also $\overline{S}(t)$ and $\overline{V}(t)$ be respectively the solutions of (VI.4) and (VI.5) that verify $\overline{S}(0) = S_0$ and $\overline{V}(0) = V_0$. Then*

(i) $\displaystyle\lim_{t\to\infty}\frac{\overline{S}(t)}{t} = 0, \ \lim_{t\to\infty}\frac{\overline{S}^2(t)}{t} = 0, \ \lim_{t\to\infty}\frac{\int_0^t \overline{S}(s)\,\mathrm{d}\mathcal{B}_1(s)}{t} = 0, \ \lim_{t\to\infty}\frac{\int_0^t \overline{S}^2(s)\,\mathrm{d}\mathcal{B}_1(s)}{t} = 0,$
and $\displaystyle\lim_{t\to\infty}\frac{\int_0^t\int_{\mathbb{U}} \eta_1(u)\overline{S}(s^-)\widetilde{\mathcal{N}}(\mathrm{d}s,\mathrm{d}u)}{t} = 0 \ \ a.s.$

(ii) $\displaystyle\lim_{t\to\infty}\frac{\overline{V}(t)}{t} = 0, \ \lim_{t\to\infty}\frac{\overline{V}^2(t)}{t} = 0, \ \lim_{t\to\infty}\frac{\int_0^t \overline{V}(s)\,\mathrm{d}\mathcal{B}_2(s)}{t} = 0, \ \lim_{t\to\infty}\frac{\int_0^t \overline{V}^2(s)\,\mathrm{d}\mathcal{B}_2(s)}{t} = 0,$
and $\displaystyle\lim_{t\to\infty}\frac{\int_0^t\int_{\mathbb{U}} \eta_2(u)\overline{V}(s^-)\widetilde{\mathcal{N}}(\mathrm{d}s,\mathrm{d}u)}{t} = 0 \ \ a.s.$

(iii) $\displaystyle\lim_{t\to\infty}\frac{S(t)}{t} = 0, \ \lim_{t\to\infty}\frac{\int_0^t S(s)\,\mathrm{d}\mathcal{B}_1(s)}{t} = 0, \ \textit{and} \ \lim_{t\to\infty}\frac{\int_0^t\int_{\mathbb{U}} \eta_1(u)S(s^-)\widetilde{\mathcal{N}}(\mathrm{d}s,\mathrm{d}u)}{t} = 0, \quad a.s.$

(iv) $\displaystyle\lim_{t\to\infty}\frac{V(t)}{t} = 0, \ \lim_{t\to\infty}\frac{\int_0^t V(s)\,\mathrm{d}\mathcal{B}_2(s)}{t} = 0, \ \textit{and} \ \lim_{t\to\infty}\frac{\int_0^t\int_{\mathbb{U}} \eta_2(u)V(s^-)\widetilde{\mathcal{N}}(\mathrm{d}s,\mathrm{d}u)}{t} = 0, \quad a.s.$

(v) $\displaystyle\lim_{t\to\infty}\frac{I(t)}{t} = 0, \ \lim_{t\to\infty}\frac{\int_0^t I(s)\,\mathrm{d}\mathcal{B}_3(s)}{t} = 0, \ \textit{and} \ \lim_{t\to\infty}\frac{\int_0^t\int_{\mathbb{U}} \eta_3(u)I(s^-)\widetilde{\mathcal{N}}(\mathrm{d}s,\mathrm{d}u)}{t} = 0, \quad a.s.$

Proof. The proof's principle of this lemma is identical to that of Lemma 2.5 in [150]. Hence, it is omitted here. ■

Lemma 3.2. *Let $\overline{S}(t)$ and $\overline{V}(t)$ denote the solutions of (VI.4) and (VI.5) that start respectively from $\overline{S}(0) > 0$ and $\overline{V}(0) > 0$. Then, each of the two auxiliary systems (VI.4) and (VI.5) admits a unique and ergodic stationary distribution concentrated on $\mathbb{R}_+$. We denote these two distributions respectively by π_1 and π_2.*

Proof. The proof is standard and the same as that of Theorem 2.2 in [152]. Hereby, we omit it here. ■

Remark 3.1. *From Theorem 3.3.1 of [204], any π_1-square-integrable function φ verifies*

$$\lim_{t\to\infty} \frac{1}{t}\int_0^t \varphi(\overline{S}(s))\mathrm{d}s = \int_{\mathbb{R}_+} \varphi(s)\pi_1(\mathrm{d}s) \quad a.s. \tag{VI.11}$$

Also, for all π_2-square-integrable function ψ we have:

$$\lim_{t\to\infty} \frac{1}{t}\int_0^t \psi(\overline{V}(s))\mathrm{d}s = \int_{\mathbb{R}_+} \psi(s)\pi_2(\mathrm{d}s) \quad a.s. \tag{VI.12}$$

Lemma 3.3. *Let $\overline{S}(t)$ and $\overline{V}(t)$ denote the solutions of (VI.4) and (VI.5) that are starting from $\overline{S}(0) \in \mathbb{R}_+$ and $\overline{V}(0) \in \mathbb{R}_+$. Then*

$$\lim_{t\to\infty} \frac{\int_0^t \overline{S}(s)\,\mathrm{d}s}{t} = \frac{(1-p)\Lambda}{\mu} \text{ and } \lim_{t\to\infty} \frac{\int_0^t \overline{V}(s)\,\mathrm{d}s}{t} = \frac{p\Lambda}{\mu} \quad a.s.$$

Proof. Let $\overline{S}(0) \in \mathbb{R}_+$. By integrating from 0 to t the both sides of the equality (VI.4), and then dividing by t, we obtain

$$\frac{\overline{S}(t)-\overline{S}(0)}{t} = (1-p)\Lambda - \mu\times\frac{\int_0^t \overline{S}(s)\,\mathrm{d}s}{t} + \sigma_1\frac{\int_0^t \overline{S}(s)\,\mathrm{d}\mathcal{B}_1(s)}{t} + \frac{\int_0^t\int_{\mathbb{U}} \eta_1(u)\overline{S}(s^-)\widetilde{\mathcal{N}}(\mathrm{d}s,\mathrm{d}u)}{t}.$$

Thus

$$\frac{\int_0^t \overline{S}(s)\,\mathrm{d}s}{t} = \frac{(1-p)\Lambda}{\mu} - \frac{\overline{S}(t)-\overline{S}(0)}{\mu t} + \sigma_1\frac{\int_0^t \overline{S}(s)\,\mathrm{d}\mathcal{B}_1(s)}{\mu t} + \frac{\int_0^t\int_{\mathbb{U}} \eta_1(u)\overline{S}(s^-)\,\widetilde{\mathcal{N}}(\mathrm{d}s,\mathrm{d}u)}{\mu t}. \tag{VI.13}$$

Letting t go to infinity in (VI.13) and then using Lemma 3.1 leads to

$$\lim_{t\to\infty} \frac{\int_0^t \overline{S}(s)\,\mathrm{d}s}{t} = \frac{(1-p)\Lambda}{\mu} \quad a.s.$$

By a similar argument, we can also conclude that

$$\lim_{t\to\infty} \frac{\int_0^t \overline{V}(s)\,\mathrm{d}s}{t} = \frac{p\Lambda}{\mu} \quad a.s.$$

Hence, the lemma is proved. ■

Remark 3.2. *The precedent lemma can be viewed as another way to calculate the limits $\lim_{t\to\infty} \frac{\int_0^t \overline{S}(s)\,\mathrm{d}s}{t}$ and $\lim_{t\to\infty} \frac{\int_0^t \overline{V}(s)\,\mathrm{d}s}{t}$ without resorting to the explicit stationary distributions' expressions of π_1 and π_2. Note that these expressions are still unknown in the Lévy jumps framework [151, 152].*

Remark 3.3. *According to the stochastic comparison theorem [155] we have and for all* $t \geqslant 0$

$$S(t) \leqslant \overline{S}(t) \text{ and } V(t) \leqslant \overline{V}(t) \quad a.s.$$

So, and by making use of Remark 3.1 and the inequalities (3) and (6) of (VI.6) we deduce that:

$$\int_{\mathbb{R}_+} F_1(s,0)\ \pi_1(\mathrm{d}s) \leqslant c\int_{\mathbb{R}_+} s\ \pi_1(\mathrm{d}s) = \frac{(1-p)\Lambda}{\mu} < \infty \text{ and } \int_{\mathbb{R}_+} F_2(v,0)\ \pi_2(\mathrm{d}v) \leqslant c\int_{\mathbb{R}_+} v\ \pi_2(\mathrm{d}v) = \frac{p\Lambda}{\mu} < \infty \ a.s.$$

Before moving on to the main results of this section, let us denote by λ the well-defined quantity given by

$$\lambda \triangleq \int_{\mathbb{R}_+} F_1(s,0)\ \pi_1(\mathrm{d}s) + \int_{\mathbb{R}_+} F_2(v,0)\ \pi_2(\mathrm{d}v) - \left(\mu+\delta+\frac{\sigma_3^2}{2} + \int_{\mathbb{U}} \Big(\eta_3(u) - \ln\left(1+\eta_3(u)\right)\Big)\nu(\mathrm{d}u)\right)$$

Theorem 3.1. *Let* $\big(S(t),V(t),I(t)\big)$ *be the solution of the system (VI.3) with the initial value* $\big(S(0),V(0),I(0)\big) \in \mathbb{R}^3_+$.

- *If* $\lambda < 0$, *then* $\limsup_{t\to\infty} \frac{\ln(I(t))}{t} \leqslant \lambda < 0$ *a.s., which signifies that any disease formulated by the equation (VI.3) will disappear exponentially almost surely.*
- *If* $\lambda > 0$ *and the following assumption holds*
  - (H_0') : *There is some* $c' > 0$ *such that for all* $S, V, I \geqslant 0$

$$-c' \leqslant \frac{\partial F_1}{\partial I}(S,I) \text{ and } -c' \leqslant \frac{\partial F_2}{\partial I}(V,I). \tag{VI.14}$$

  Then, $\liminf_{t\to\infty} \frac{1}{t}\int_0^t I(s)\,\mathrm{d}s \geqslant \frac{\mu\lambda}{(\mu+\delta)c+2c'\mu} > 0$ *a.s., which means that any disease presented by the equation (VI.3) will persist in the mean.*

Proof. From Itô's formula and the third equation of (VI.3), we have

$$\begin{aligned}
\mathrm{d}\ln\big(I(t)\big) &= \left(F_1\big(S(t),I(t)\big) + F_2\big(V(t),I(t)\big) - (\mu+\delta) - \frac{\sigma_3^2}{2} + \int_{\mathbb{U}} \big(\ln(1+\eta_3(u)) - \eta_3(u)\big)\nu(\mathrm{d}u)\right)\mathrm{d}t \\
&\quad + \sigma_3 \mathrm{d}\mathcal{B}_3(t) + \int_{\mathbb{U}} \Big(\ln\left(1+\eta_3(u)\right)\Big)\widetilde{\mathcal{N}}(\mathrm{d}t,\mathrm{d}u) \\
&\leqslant \left(F_1\big(S(t),0\big) + F_2\big(V(t),0\big) - (\mu+\delta+\frac{\sigma_3^2}{2}) + \int_{\mathbb{U}} \big(\ln(1+\eta_3(u)) - \eta_3(u)\big)\nu(\mathrm{d}u)\right)\mathrm{d}t \\
&\quad + \sigma_3 \mathrm{d}\mathcal{B}_3(t) + \int_{\mathbb{U}} \Big(\ln\left(1+\eta_3(u)\right)\Big)\widetilde{\mathcal{N}}(\mathrm{d}t,\mathrm{d}u) \\
&\leqslant \left(F_1\big(\overline{S}(t),0\big) + F_2\big(\overline{V}(t),0\big) - (\mu+\delta+\frac{\sigma_3^2}{2}) + \int_{\mathbb{U}} \big(\ln(1+\eta_3(u)) - \eta_3(u)\big)\nu(\mathrm{d}u)\right)\mathrm{d}t \\
&\quad + \sigma_3 \mathrm{d}\mathcal{B}_3(t) + \int_{\mathbb{U}} \Big(\ln\left(1+\eta_3(u)\right)\Big)\widetilde{\mathcal{N}}(\mathrm{d}t,\mathrm{d}u).
\end{aligned} \tag{VI.15}$$

Here, the two last inequalities are deduced from the inequalities (1), (2), (4) and (5) of (VI.6) and

also from the fact that $S(t) \leqslant \overline{S}(t)$ and $V(t) \leqslant \overline{V}(t)$ *a.s.* Integrating (VI.15) from 0 to t, and then dividing by t on both sides gives

$$\frac{\ln\big(I(t)\big)}{t} - \frac{\ln\big(I(0)\big)}{t} \leqslant \frac{1}{t}\int_0^t F_1\big(\overline{S}(s),0\big)\ \mathrm{d}s + \frac{1}{t}\int_0^t F_2\big(\overline{V}(s),0\big)\ \mathrm{d}s - (\mu+\delta+\frac{\sigma_3^2}{2}) + \int_{\mathbb{U}}\big(\ln(1+\eta_3(u)) - \eta_3(u)\big)\nu(\mathrm{d}u) + \frac{\sigma_3\mathcal{B}_3(t)}{t} + \frac{1}{t}\int_0^t\int_{\mathbb{U}}\Big(\ln\left(1+\eta_3(u)\right)\Big)\widetilde{\mathcal{N}}(\mathrm{d}s,\mathrm{d}u). \tag{VI.16}$$

On the basis of the assumption (H_5), $t \mapsto \displaystyle\int_0^t\int_{\mathbb{U}}\Big(\ln\left(1+\eta_3(u)\right)\Big)\widetilde{\mathcal{N}}(\mathrm{d}s,\mathrm{d}u)$ is a local martingale with finite quadratic variation. So, and by the strong law of large numbers for local martingales (see [19, page 11]), we have

$$\lim_{t\to\infty}\frac{1}{t}\int_0^t\int_{\mathbb{U}}\Big(\ln\left(1+\eta_3(u)\right)\Big)\widetilde{\mathcal{N}}(\mathrm{d}s,\mathrm{d}u) = 0 \quad \text{a.s.} \tag{VI.17}$$

Taking the superior limit on both sides of (VI.16) and combining the resulting inequality with (VI.11), (VI.12) and (VI.17) yields

$$\begin{aligned}\limsup_{t\to\infty}\frac{\ln\big(I(t)\big)}{t} &\leqslant \lim_{t\to\infty}\frac{1}{t}\int_0^t F_1\big(\overline{S}(s),0\big)\ \mathrm{d}s + \lim_{t\to\infty}\frac{1}{t}\int_0^t F_2\big(\overline{V}(s),0\big)\ \mathrm{d}s - \left(\mu+\delta+\frac{\sigma_3^2}{2}\right)\\ &\quad + \int_{\mathbb{U}}\big(\ln(1+\eta_3(u)) - \eta_3(u)\big)\nu(\mathrm{d}u)\\ &= \int_{\mathbb{R}_+}F_1\big(s,0\big)\pi_1(\mathrm{d}s) + \int_{\mathbb{R}_+}F_2\big(v,0\big)\pi_2(\mathrm{d}v) - \left(\mu+\delta+\frac{\sigma_3^2}{2}\right) + \int_{\mathbb{U}}\big(\ln(1+\eta_3(u)) - \eta_3(u)\big)\nu(\mathrm{d}u) \triangleq \lambda.\end{aligned}$$

Hence, the first statement of the theorem is proved, and the infected individuals $I(t)$ will extinguish once $\lambda < 0$.

Let us now turn to the proof of the second part concerning the persistence in the mean of the infection. Consider the C^2-function $\mathcal{V} : \mathbb{R}_+^3 \to \mathbb{R}$ defined by

$$\mathcal{V}\left(S,V,I,\overline{S},\overline{V}\right) = -\ln(I) + \frac{c}{\mu}\left(\overline{S}-S\right) + \frac{c}{\mu}\left(\overline{V}-V\right) - \frac{c}{\mu}I.$$

By an application of the Itô's formula, we get

$$\begin{aligned}&\mathrm{d}\mathcal{V}\left(S(t),V(t),I(t),\overline{S}(t),\overline{V}(t)\right)\\ &= \mathcal{L}\mathcal{V}\left(S(t),V(t),I(t),\overline{S}(t),\overline{V}(t)\right)\mathrm{d}t\\ &\quad + \left(\frac{c}{\mu}\sigma_1\left(\overline{S}(t)-S(t)\right)\mathrm{d}\mathcal{B}_1(t) + \frac{c}{\mu}\sigma_2\left(\overline{V}(t)-V(t)\right)\mathrm{d}\mathcal{B}_2(t) - \left(\sigma_3+\sigma_3\frac{c}{\mu}I(t)\right)\mathrm{d}\mathcal{B}_3(t)\right)\\ &\quad - \int_{\mathbb{U}}\Big(\ln\left(1+\eta_3(u)\right)\Big)\widetilde{\mathcal{N}}(\mathrm{d}t,\mathrm{d}u) + \frac{c}{\mu}\int_{\mathbb{U}}\eta_1(u)\left(\overline{S}(t^-)-S(t^-)\right)\widetilde{\mathcal{N}}(\mathrm{d}t,\mathrm{d}u)\\ &\quad + \frac{c}{\mu}\int_{\mathbb{U}}\eta_2(u)\left(\overline{V}(t^-)-V(t^-)\right)\widetilde{\mathcal{N}}(\mathrm{d}t,\mathrm{d}u) - \frac{c}{\mu}\int_{\mathbb{U}}\eta_3(u)I(t^-)\widetilde{\mathcal{N}}(\mathrm{d}t,\mathrm{d}u),\end{aligned} \tag{VI.18}$$

where

$$
\begin{aligned}
&\mathcal{L}\mathcal{V}\left(S(t),V(t),I(t),\overline{S}(t),\overline{V}(t)\right)\\
&=\left(\left(\mu+\delta+\frac{\sigma_3^2}{2}\right)-\int_{\mathbb{U}}\big(\ln(1+\eta_3(u))-\eta_3(u)\big)\nu(\mathrm{d}u)-F_1\big(S(t),I(t)\big)-F_2\big(V(t),I(t)\big)\right)\\
&\quad-c\left(\overline{S}(t)-S(t)\right)-c\left(\overline{V}(t)-V(t)\right)+\frac{c(\mu+\delta)}{\mu}I(t)\\
&\leqslant\left[\left(\mu+\delta+\frac{\sigma_3^2}{2}\right)-\int_{\mathbb{U}}\big(\ln(1+\eta_3(u))-\eta_3(u)\big)\nu(\mathrm{d}u)-F_1\big(\overline{S}(t),0\big)-F_2\big(\overline{V}(t),0\big)\right.\\
&\quad+\left(F_1\big(S(t),0\big)-F_1\big(S(t),I(t)\big)\right)+\left(F_2\big(V(t),0\big)-F_2\big(V(t),I(t)\big)\right)\\
&\quad\left.+\left(F_1\big(\overline{S}(t),0\big)-F_1\big(S(t),0\big)\right)+\left(F_2\big(\overline{V}(t),0\big)-F_2\big(V(t),0\big)\right)\right]\\
&\quad-c\left(\overline{S}(t)-S(t)\right)-c\left(\overline{V}(t)-V(t)\right)+\frac{c(\mu+\delta)}{\mu}I(t)\\
&\leqslant\left[-\lambda+\left(\int_{\mathbb{R}_+}F_1(s,0)\ \pi_1(\mathrm{d}s)-F_1\big(\overline{S}(t),0\big)\right)+\left(\int_{\mathbb{R}_+}F_2(v,0)\ \pi_2(\mathrm{d}v)-F_2\big(\overline{V}(t),0\big)\right)\right.\\
&\quad\left.+\left(F_1\big(S(t),0\big)-F_1\big(S(t),I(t)\big)\right)+\left(F_2\big(V(t),0\big)-F_2\big(V(t),I(t)\big)\right)\right]+\frac{c(\mu+\delta)}{\mu}I(t).
\end{aligned}
\tag{VI.19}
$$

Here, the last inequality is concluded from those carrying the numbers (3), (6) in (VI.6) and also from the fact that $S(t)\leqslant\overline{S}(t)$ and $V(t)\leqslant\overline{V}(t)$ almost surely. Combining (VI.18) with (VI.19) gives

$$
\begin{aligned}
&\mathrm{d}\mathcal{V}\left(S(t),V(t),I(t),\overline{S}(t),\overline{V}(t)\right)\\
&\leqslant\left[-\lambda+\left(\int_{\mathbb{R}_+}F_1(s,0)\ \pi_1(\mathrm{d}s)-F_1\big(\overline{S}(t),0\big)\right)+\left(\int_{\mathbb{R}_+}F_2(v,0)\ \pi_2(\mathrm{d}v)-F_2\big(\overline{V}(t),0\big)\right)\right.\\
&\quad\left.+\left(F_1\big(S(t),0\big)-F_1\big(S(t),I(t)\big)\right)+\left(F_2\big(V(t),0\big)-F_2\big(V(t),I(t)\big)\right)+\frac{c(\mu+\delta)}{\mu}I(t)\right]\mathrm{d}t\\
&\quad+\left(\frac{c}{\mu}\sigma_1\left(\overline{S}(t)-S(t)\right)\mathrm{d}\mathcal{B}_1(t)+\frac{c}{\mu}\sigma_2\left(\overline{V}(t)-V(t)\right)\mathrm{d}\mathcal{B}_2(t)-\left(\sigma_3+\sigma_3\frac{c}{\mu}I(t)\right)\mathrm{d}\mathcal{B}_3(t)\right)\\
&\quad-\int_{\mathbb{U}}\Big(\ln\left(1+\eta_3(u)\right)\Big)\widetilde{\mathcal{N}}(\mathrm{d}t,\mathrm{d}u)+\frac{c}{\mu}\int_{\mathbb{U}}\eta_1(u)\left(\overline{S}(t^-)-S(t^-)\right)\widetilde{\mathcal{N}}(\mathrm{d}t,\mathrm{d}u)\\
&\quad+\frac{c}{\mu}\int_{\mathbb{U}}\eta_2(u)\left(\overline{V}(t^-)-V(t^-)\right)\widetilde{\mathcal{N}}(\mathrm{d}t,\mathrm{d}u)-\frac{c}{\mu}\int_{\mathbb{U}}\eta_3(u)I(t^-)\widetilde{\mathcal{N}}(\mathrm{d}t,\mathrm{d}u).
\end{aligned}
\tag{VI.20}
$$

By integrating (VI.20) from 0 to t, and then dividing by t on both sides we get

$$\begin{aligned}
&\frac{\mathcal{V}\left(S(t),V(t),I(t),\overline{S}(t),\overline{V}(t)\right)}{t}\\
&\leqslant \frac{\mathcal{V}\left(S(0),V(0),I(0),\overline{S}(0),\overline{V}(0)\right)}{t} - \lambda + \left(\int_{\mathbb{R}_+} F_1(s,0)\pi_1(\mathrm{d}s) - \frac{1}{t}\int_0^t F_1\big(\overline{S}(s),0\big)\mathrm{d}s\right)\\
&\quad + \left(\int_{\mathbb{R}_+} F_2(v,0)\pi_2(\mathrm{d}v) - \frac{1}{t}\int_0^t F_2\big(\overline{V}(s),0\big)\mathrm{d}s\right) + \frac{c(\mu+\delta)}{\mu t}\int_0^t I(s)\,\mathrm{d}s\\
&\quad + \frac{1}{t}\int_0^t \Big(F_1\big(S(s),0\big) - F_1\big(S(s),I(s)\big)\Big)\mathrm{d}s + \frac{1}{t}\int_0^t \Big(F_2\big(V(s),0\big) - F_2\big(V(s),I(s)\big)\Big)\mathrm{d}s\\
&\quad + \frac{c\sigma_1}{\mu t}\int_0^t \Big(\overline{S}(s) - S(s)\Big)\,\mathrm{d}\mathcal{B}_1(s) + \frac{c\sigma_2}{\mu t}\int_0^t \Big(\overline{V}(s) - V(s)\Big)\,\mathrm{d}\mathcal{B}_2(s) - \sigma_3\frac{\mathcal{B}_3(t)}{t}\\
&\quad - \sigma_3\frac{c}{\mu t}\int_0^t I(s)\mathrm{d}\mathcal{B}_3(s) - \frac{1}{t}\int_0^t\int_{\mathbb{U}} \Big(\ln\left(1+\eta_3(u)\right)\Big)\widetilde{\mathcal{N}}(\mathrm{d}s,\mathrm{d}u)\\
&\quad + \frac{c}{\mu t}\int_0^t\int_{\mathbb{U}} \eta_1(u)\Big(\overline{S}(s^-) - S(s^-)\Big)\widetilde{\mathcal{N}}(\mathrm{d}s,\mathrm{d}u)\\
&\quad + \frac{c}{\mu t}\left(\int_0^t\int_{\mathbb{U}} \eta_2(u)\Big(\overline{V}(s^-) - V(s^-)\Big)\widetilde{\mathcal{N}}(\mathrm{d}s,\mathrm{d}u) - \int_0^t\int_{\mathbb{U}} \eta_3(u)I(s^-)\widetilde{\mathcal{N}}(\mathrm{d}s,\mathrm{d}u)\right). \qquad \text{(VI.21)}
\end{aligned}$$

Since $\ln(x) \leqslant x$ for all $x > 0$, one can assert that

$$\mathcal{V}\left(S,V,I,\overline{S},\overline{V}\right) = -\ln(I) + \frac{c}{\mu}\left(\overline{S} - S\right) + \frac{c}{\mu}\left(\overline{V} - V\right) - \frac{c}{\mu}I \geqslant -\left(1+\frac{c}{\mu}\right)I + \frac{c}{\mu}\left(\overline{S} - S\right) + \frac{c}{\mu}\left(\overline{V} - V\right),$$

which together with (VI.21) implies that

$$\begin{aligned}
&\frac{c(\mu+\delta)}{\mu t}\int_0^t I(s)\,\mathrm{d}s\\
&\geqslant \lambda + \frac{-\left(1+\frac{c}{\mu}\right)I(t) + \frac{c}{\mu}\left(\overline{S}(t) - S(t)\right) + \frac{c}{\mu}\left(\overline{V}(t) - V(t)\right)}{t} - \frac{\mathcal{V}\left(S(0),V(0),I(0),\overline{S}(0),\overline{V}(0)\right)}{t}\\
&\quad - \left(\int_{\mathbb{R}_+} F_1(s,0)\pi_1(\mathrm{d}s) - \frac{1}{t}\int_0^t F_1\big(\overline{S}(s),0\big)\mathrm{d}s\right) - \left(\int_{\mathbb{R}_+} F_2(v,0)\pi_2(\mathrm{d}v) - \frac{1}{t}\int_0^t F_2\big(\overline{V}(s),0\big)\mathrm{d}s\right)\\
&\quad - \frac{1}{t}\int_0^t \Big(F_1\big(S(s),0\big) - F_1\big(S(s),I(s)\big)\Big)\mathrm{d}s - \frac{1}{t}\int_0^t \Big(F_2\big(V(s),0\big) - F_2\big(V(s),I(s)\big)\Big)\mathrm{d}s\\
&\quad - \frac{c\sigma_1}{\mu t}\int_0^t \Big(\overline{S}(s) - S(s)\Big)\,\mathrm{d}\mathcal{B}_1(s) - \frac{c\sigma_2}{\mu t}\int_0^t \Big(\overline{V}(s) - V(s)\Big)\,\mathrm{d}\mathcal{B}_2(s) + \sigma_3\frac{\mathcal{B}_3(t)}{t}\\
&\quad + \sigma_3\frac{c}{\mu t}\int_0^t I(s)\mathrm{d}\mathcal{B}_3(s) - \frac{c}{\mu t}\int_0^t\int_{\mathbb{U}} \eta_1(u)\Big(\overline{S}(s^-) - S(s^-)\Big)\widetilde{\mathcal{N}}(\mathrm{d}s,\mathrm{d}u)\\
&\quad - \frac{c}{\mu t}\left(\int_0^t\int_{\mathbb{U}} \eta_2(u)\Big(\overline{V}(s^-) - V(s^-)\Big)\widetilde{\mathcal{N}}(\mathrm{d}s,\mathrm{d}u) - \int_0^t\int_{\mathbb{U}} \eta_3(u)I(s^-)\widetilde{\mathcal{N}}(\mathrm{d}s,\mathrm{d}u)\right)\\
&\quad + \frac{1}{t}\int_0^t\int_{\mathbb{U}} \Big(\ln\left(1+\eta_3(u)\right)\Big)\widetilde{\mathcal{N}}(\mathrm{d}s,\mathrm{d}u). \qquad \text{(VI.22)}
\end{aligned}$$

Combining (VI.14) with (VI.22) yields

$$\frac{c(\mu+\delta)}{\mu t}\int_0^t I(s)\,\mathrm{d}s \geqslant \lambda + \frac{-\left(1+\frac{c}{\mu}\right)I(t)+\frac{c}{\mu}\left(\overline{S}(t)-S(t)\right)+\frac{c}{\mu}\left(\overline{V}(t)-V(t)\right)}{t} - \frac{\mathcal{V}\left(S(0),V(0),I(0),\overline{S}(0),\overline{V}(0)\right)}{t}$$
$$-\left(\int_{\mathbb{R}_+}F_1(s,0)\pi_1(\mathrm{d}s)-\frac{1}{t}\int_0^t F_1\big(\overline{S}(s),0\big)\mathrm{d}s\right)-\left(\int_{\mathbb{R}_+}F_2(v,0)\pi_2(\mathrm{d}v)-\frac{1}{t}\int_0^t F_2\big(\overline{V}(s),0\big)\mathrm{d}s\right)$$
$$-\frac{2}{t}\int_0^t c'I(s)\mathrm{d}s-\frac{c\sigma_1}{\mu t}\int_0^t\left(\overline{S}(s)-S(s)\right)\mathrm{d}\mathcal{B}_1(s)-\frac{c\sigma_2}{\mu t}\int_0^t\left(\overline{V}(s)-V(s)\right)\mathrm{d}\mathcal{B}_2(s)+\sigma_3\frac{\mathcal{B}_3(t)}{t}$$
$$+\sigma_3\frac{c}{\mu t}\int_0^t I(s)\mathrm{d}\mathcal{B}_3(s)-\frac{c}{\mu t}\int_0^t\int_{\mathbb{U}}\eta_1(u)\left(\overline{S}(s^-)-S(s^-)\right)\widetilde{\mathcal{N}}(\mathrm{d}s,\mathrm{d}u)$$
$$-\frac{c}{\mu t}\left(\int_0^t\int_{\mathbb{U}}\eta_2(u)\left(\overline{V}(s^-)-V(s^-)\right)\widetilde{\mathcal{N}}(\mathrm{d}s,\mathrm{d}u)-\int_0^t\int_{\mathbb{U}}\eta_3(u)I(s^-)\widetilde{\mathcal{N}}(\mathrm{d}s,\mathrm{d}u)\right)$$
$$+\frac{1}{t}\int_0^t\int_{\mathbb{U}}\Big(\ln\left(1+\eta_3(u)\right)\Big)\widetilde{\mathcal{N}}(\mathrm{d}s,\mathrm{d}u). \tag{VI.23}$$

So

$$\frac{c(\mu+\delta)+2c'\mu}{\mu t}\int_0^t I(s)\,\mathrm{d}s$$
$$\geqslant \lambda + \frac{-\left(1+\frac{c}{\mu}\right)I(t)+\frac{c}{\mu}\left(\overline{S}(t)-S(t)\right)+\frac{c}{\mu}\left(\overline{V}(t)-V(t)\right)}{t} - \frac{\mathcal{V}\left(S(0),V(0),I(0),\overline{S}(0),\overline{V}(0)\right)}{t}$$
$$-\left(\int_{\mathbb{R}_+}F_1(s,0)\pi_1(\mathrm{d}s)-\frac{1}{t}\int_0^t F_1\big(\overline{S}(s),0\big)\mathrm{d}s\right)-\left(\int_{\mathbb{R}_+}F_2(v,0)\pi_2(\mathrm{d}v)-\frac{1}{t}\int_0^t F_2\big(\overline{V}(s),0\big)\mathrm{d}s\right)$$
$$-\frac{c\sigma_1}{\mu t}\int_0^t\left(\overline{S}(s)-S(s)\right)\mathrm{d}\mathcal{B}_1(s)-\frac{c\sigma_2}{\mu t}\int_0^t\left(\overline{V}(s)-V(s)\right)\mathrm{d}\mathcal{B}_2(s)+\sigma_3\frac{\mathcal{B}_3(t)}{t}$$
$$+\sigma_3\frac{c}{\mu t}\int_0^t I(s)\mathrm{d}\mathcal{B}_3(s)-\frac{c}{\mu t}\int_0^t\int_{\mathbb{U}}\eta_1(u)\left(\overline{S}(s^-)-S(s^-)\right)\widetilde{\mathcal{N}}(\mathrm{d}s,\mathrm{d}u)$$
$$-\frac{c}{\mu t}\left(\int_0^t\int_{\mathbb{U}}\eta_2(u)\left(\overline{V}(s^-)-V(s^-)\right)\widetilde{\mathcal{N}}(\mathrm{d}s,\mathrm{d}u)-\int_0^t\int_{\mathbb{U}}\eta_3(u)I(s^-)\widetilde{\mathcal{N}}(\mathrm{d}s,\mathrm{d}u)\right)$$
$$+\frac{1}{t}\int_0^t\int_{\mathbb{U}}\Big(\ln\left(1+\eta_3(u)\right)\Big)\widetilde{\mathcal{N}}(\mathrm{d}s,\mathrm{d}u). \tag{VI.24}$$

By taking the inferior limit on both sides of the last inequality and using Lemma 3.1 together with (VI.17) we obtain

$$\liminf_{t\to\infty}\frac{1}{t}\int_0^t I(s)\,\mathrm{d}s \geqslant \frac{\lambda\mu}{(\mu+\delta)c+2c'\mu}\ \ a.s.$$

Hence, the last statement of the theorem is proven and the infected individuals $I(t)$ will persist in the mean once $\lambda > 0$. ■

Remark 3.4. *Theorem 3.1 gives a threshold between persistence in the mean and extinction of the model (VI.3). In other words, it offers sufficient and almost necessary conditions for the disease's extinction and persistence in the mean.*

Remark 3.5. *The condition* (H_0') *that is imposed in Theorem 3.1 to obtain the persistence in the mean of the model (VI.3) includes many incidence functional responses adopted usually in the literature. For example*

- *The bilinear incidence rate [14]:* $IF_1(S,I)=\beta_1 SI$ *and* $IF_2(V,I)=\beta_2 VI$.
- *The Holling type II incidence rate [205]:* $IF_1(S,I)=\dfrac{\beta_1 SI}{m_1+S}$ *and* $IF_2(V,I)=\dfrac{\beta_2 VI}{m_2+V}$ $(m_1,m_2>0)$.

- *The Holling type III incidence rate [206]:* $IF_1(S,I) = \dfrac{\beta_1 S^2 I}{(m_1+S)\left(m_1'+S\right)} \quad (m_1 > 0, m_1' > 0)$ *and* $IF_2(V,I) = \dfrac{\beta_2 V^2 I}{(m_2+V)\left(m_2'+V\right)} \quad (m_2 > 0, m_2' > 0)$.

- *The Beddington-DeAngelis incidence rate [207]:* $IF_1(S,I) = \dfrac{\beta_1 SI}{1+m_1 S+m_1' I} \quad (m_1 > 0, m_1' > 0)$ *and* $IF_2(V,I) = \dfrac{\beta_2 VI}{1+m_2 V+m_2' I} \quad (m_2 > 0, m_2' > 0)$.

But, in contrast, it does not cover all of them, as an example the saturated response functions $IF_1(S,I) = \frac{\beta_1 SI}{1+m_1 I}$ *and* $IF_2(V,I) = \frac{\beta_2 VI}{1+m_2 I}$ *do not verify the said condition. For this reason, we will present in the next theorem another version of the persistence which requires a condition that covers more incidence functions like the saturated one just mentioned (see [10, 207]).*

Theorem 3.2. *Let* $\left(S(0),V(0),I(0)\right) \in \mathbb{R}^3_+$ *and* $\left(S(t),V(t),I(t)\right)$ *be respectively a given initial value and the solution of (VI.3) that starts from it. If the assumption* (H_0') *is replaced by the following hypothesis:*

- (H_0'') : $F_1\left(S(t),I(t)\right)$ *and* $F_2\left(V(t),I(t)\right)$ *are uniformly continuous at* $I = 0$*, which means that*

$$\limsup_{I\to 0}\left\{|F_1(S,I) - F_1(S,0)|\right\} = 0 \quad \text{and} \quad \limsup_{I\to 0}\left\{|F_2(V,I) - F_2(V,0)|\right\} = 0. \tag{VI.25}$$

Then, the disease will persist weakly once $\lambda > 0$*. In other terms, if* $\lambda > 0$*, then*

$$\limsup_{t\to\infty} I(t) > 0 \ a.s.$$

Proof. Let us suppose that $\lambda > 0$. We remind that (VI.22) hold for all $t \geqslant 0$ and for almost all $\omega \in \Omega$. Assume that $\limsup_{t\to\infty} I(t) > 0$ *a.s.* is untrue, then there is definitely a measurable subset $\Omega' \subset \Omega$ with $\mathbb{P}(\Omega') > 0$ such that for all $\omega \in \Omega'$, $\limsup_{t\to\infty} I(t,\omega) = 0$. This last fact together with the positivity of $I(t)$ and the uniform continuity of $F_1(S,I)$ and $F_2(V,I)$ at $I = 0$ (see (VI.25)), lead to

$$\lim_{t\to\infty} I(t,\omega) = 0, \lim_{t\to\infty}\Big(F_1\big(S(t,\omega),0\big) - F_1\big(S(t,\omega),I(t,\omega)\big)\Big) = 0, \lim_{t\to\infty}\Big(F_2\big(V(t,\omega),0\big) - F_2\big(V(t,\omega),I(t,\omega)\big)\Big) = 0,$$

which implies in turn by the continuous version of Cesàro's theorem [110] that

$$\begin{cases} \lim\limits_{t\to\infty} \dfrac{1}{t}\displaystyle\int_0^t I(s)\,\mathrm{d}s = 0, \\ \lim\limits_{t\to\infty} \dfrac{1}{t}\displaystyle\int_0^t \Big(F_1\big(S(s,\omega),0\big) - F_1\big(S(s,\omega),I(s,\omega)\big)\Big)\,\mathrm{d}s = 0, \\ \lim\limits_{t\to\infty} \dfrac{1}{t}\displaystyle\int_0^t \Big(F_2\big(V(s,\omega),0\big) - F_2\big(V(s,\omega),I(s,\omega)\big)\Big)\,\mathrm{d}s = 0. \end{cases} \tag{VI.26}$$

By taking the superior limit on both sides of (VI.22) and then using Lemma 3.1 together with (VI.11), (VI.12), (VI.17) and (VI.26) we obtain the contradiction $\lambda \leqslant 0$. Thus the proof is accomplished, and the disease will persist in the mean once $\lambda > 0$. ■

Remark 3.6. *The last theorem gives the threshold between extinction and weak persistence of the model (VI.3). In other words, it provides the sufficient and almost necessary conditions for the disease's extinction and weak persistence.*

Remark 3.7. *When the jumps coefficients* $\eta_i(u)$ $(i = 1,2,3,4)$ *become zero, we recover exactly the first result of Remark 5 in [10], which allows us to say that Theorem 3.1 is a generalized version of this latter.*

4 Some examples and numerical simulations

In this section, and by utilizing the parameter values of the theoretic data listed in Table VI.1, we set out some simulations to corroborate the different results established previously in this chapter. During the forthcoming numerical examples, the solutions of our SVIR compartmental system (VI.3) are supposed to be starting from the initial conditions $S(0) = 0.5, V(0) = 0.4$ and $I(0) = 0.5$. Also, we deem from now on that the unity of time is one day and the number of individuals is given in one million populations.

Parameters	Description	Numerical values	
Λ	Recruitment rate	0.6	0.65
p	Prevalence rate of the vaccination program	0.3	0.3
μ	Natural death rate	0.4	0.4
δ_1	Disease-induced death rate	0.04	0.04
δ_2	Healing or recovery rate	0.07	0.07
σ_1	Intensity of the Brownian motion $\mathcal{B}_1$	0.3	0.3
σ_2	Intensity of the Brownian motion $\mathcal{B}_2$	0.2	0.2
σ_3	Intensity of the Brownian motion $\mathcal{B}_3$	0.25	0.25
η_1	Intensity of the Lévy jumps related to S	-0.7	-0.7
η_2	Intensity of the Lévy jumps related to I	0.8	0.8
η_3	Intensity of the Lévy jumps related to V	0.75	0.75
		Extinction case	Persistence case

Table VI.1: Descriptions and nominal values of the various system parameters and disturbance intensities taken in the numerical simulations.

	Quantity	Expression	Value
(H_1)	M_1	$\max\limits_{1\leqslant i\leqslant 3}\left(\int_{\mathbb{U}}\eta_i^2(u)\nu(\mathrm{d}u)\right)$	$0.64<\infty$
(H_2)	M_2	$\max\limits_{1\leqslant i\leqslant 3}\left(\int_{\mathbb{U}}\Big(\eta_i(u)-\ln\left(1+\eta_i(u)\right)\Big)\nu(\mathrm{d}u)\right)$	$0.5040<\infty$
	$\overline{\eta}$	$\max\left\{\eta_i(u)\mid i\in\{1,2,3\}\right\}$	0.8
	$\underline{\eta}$	$\min\left\{\eta_i(u)\mid i\in\{1,2,3\}\right\}$	-0.7
	$\overline{\chi_k}$	$(1+\overline{\eta})^k-k\times\overline{\eta}-1$	0.6511
	$\underline{\chi_k}$	$\left(1+\underline{\eta}\right)^k-k\times\underline{\eta}-1$	0.4959
	χ_k	$\max\left\{\overline{\chi}_k,\underline{\chi}_k\right\}$	0.6511
	ℓ_k	$\int_{\mathbb{U}}\chi_k(u)\nu(\mathrm{d}u)$	0.6511
(H_3)	Φ_k	$\mu-\frac{k-1}{2}\times(\sigma_1^2\vee\sigma_2^2\vee\sigma_3^2)-\frac{\ell_k}{k}$	0.0306
(H_4)	M_3	$\max\limits_{1\leqslant i\leqslant 3}\left(\int_{\mathbb{U}}\left(\left(1+\eta_i(u)\right)^2-1\right)^2\nu(\mathrm{d}u)\right)$	$5.0176<\infty$
(H_5)	M_4	$\int_{\mathbb{U}}\left(\ln\left(1+\eta_3(u)\right)\right)^2\nu(\mathrm{d}u)$	$0.3132<\infty$

Table VI.2: Some quantities and their associated numerical values in both cases, extinction and persistence ($k = 2.01 > 2$).

4.1 The case of bilinear incidence rates

In this case, and as a first step we assign to our system parameters the numerical values appearing in the third column of Table VI.1. Also, we suppose that the incidence rates $IF_1(S,I)$ and $IF_2(V,I)$ are respectively the bilinear response functions $\beta_1 SI$ and $\beta_2 VI$ (β_1 and β_2 being in this order the contact rates between S, V and I). Needless to say, the conditions (H_0) and (H_0') are verified; in fact,

$$-\beta_1 \vee \beta_2 \leqslant \underbrace{\frac{\partial F_1}{\partial I}(S,I)}_{=0} \leqslant 0 \leqslant \underbrace{\frac{\partial F_1}{\partial S}(S,I)}_{=\beta_1} \leqslant \beta_1 \vee \beta_2 \quad \text{and} \quad -\beta_1 \vee \beta_2 \leqslant \underbrace{\frac{\partial F_2}{\partial I}(V,I)}_{=0} \leqslant 0 \leqslant \underbrace{\frac{\partial F_2}{\partial V}(V,I)}_{=\beta_2} \leqslant \beta_1 \vee \beta_2. \tag{VI.27}$$

By taking $\mathbb{U} = \mathbb{R}_+$, $\nu(\mathbb{U}) = 1$, $\beta_1 = 0.5$ and $\beta_2 = 0.4$ we obtain after some computations that

$$\begin{aligned}
\lambda &= \int_{\mathbb{R}_+} F_1(s,0)\ \pi_1(\mathrm{d}s) + \int_{\mathbb{R}_+} F_2(v,0)\ \pi_2(\mathrm{d}v) - \left(\mu + \delta + \frac{\sigma_3^2}{2} + \int_{\mathbb{U}} \Big(\eta_3(u) - \ln\left(1+\eta_3(u)\right)\Big)\nu(\mathrm{d}u)\right) \\
&= \frac{\beta_1(1-p)\Lambda}{\mu} + \frac{\beta_2 p\Lambda}{\mu} - \left(\mu + \delta + \frac{\sigma_3^2}{2} + \int_{\mathbb{U}} \Big(\eta_3(u) - \ln\left(1+\eta_3(u)\right)\Big)\nu(\mathrm{d}u)\right) \\
&= -0.0266\ .
\end{aligned} \tag{VI.28}$$

From (VI.28) and Table VI.2 we notice easily that the condition $\lambda < 0$, and assumptions $(H_1) - (H_5)$ hold. So, Theorem 3.1 maintains that any disease formulated by the system (VI.3) will disappear exponentially almost surely, which is exactly presented in Figure VI.2.

Now, we will wick our way to the SVIR persistence case. Keeping in mind the numerical values disclosed in the last column of Table VI.1, and taking again $\beta_1 = 0.5$, $\beta_2 = 0.4$, one can easily get

$$\begin{aligned}
\lambda &= \int_{\mathbb{R}_+} F_1(s,0)\ \pi_1(\mathrm{d}s) + \int_{\mathbb{R}_+} F_2(v,0)\ \pi_2(\mathrm{d}v) - \left(\mu + \delta + \frac{\sigma_3^2}{2} + \int_{\mathbb{U}} \Big(\eta_3(u) - \ln\left(1+\eta_3(u)\right)\Big)\nu(\mathrm{d}u)\right) \\
&= \frac{\beta_1(1-p)\Lambda}{\mu} + \frac{\beta_2 p\Lambda}{\mu} - \left(\mu + \delta + \frac{\sigma_3^2}{2} + \int_{\mathbb{U}} \Big(\eta_3(u) - \ln\left(1+\eta_3(u)\right)\Big)\nu(\mathrm{d}u)\right) \\
&= 0.0321\ .
\end{aligned} \tag{VI.29}$$

According to (VI.29) and the numerical values of Table VI.2, assumptions $(H_1) - (H_5)$ hold, and λ surpasses zero. Therefore, Theorem 3.1 guarantees the persistence in the mean of the SVIR model,

which concurs well with the curve presented in Figure VI.3.

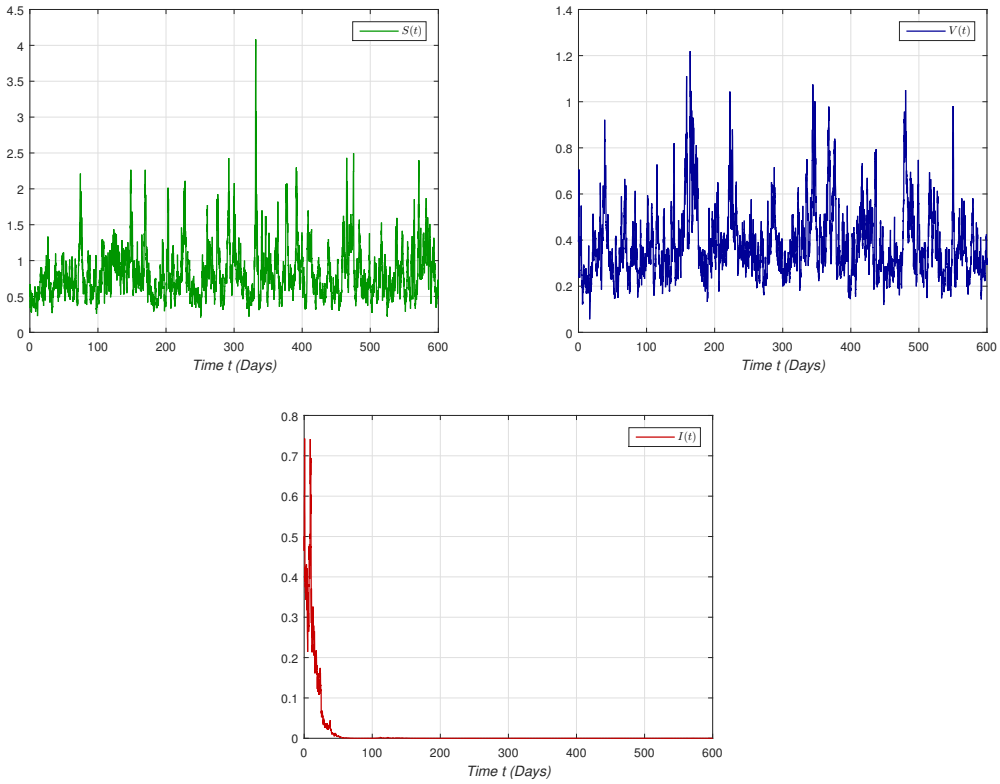


Figure VI.2: Solutions' paths of the SVIR model (VI.3) when the numerical values are taken as shown in the third column of Table VI.1.

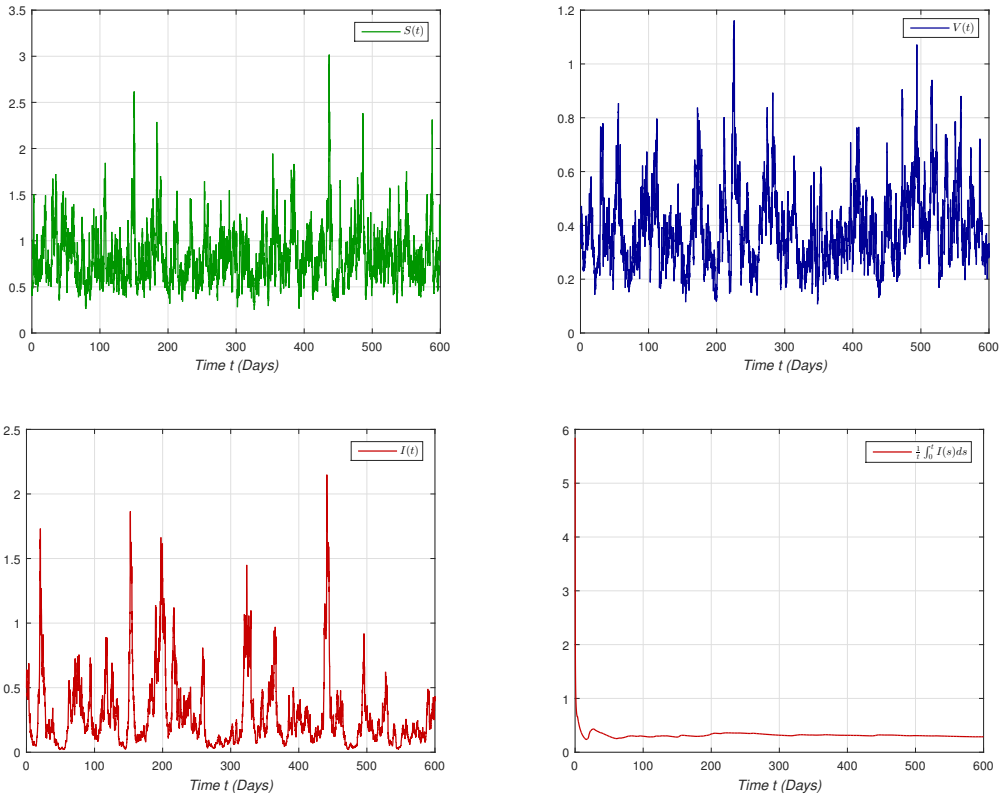


Figure VI.3: Solutions' paths of the SVIR model (VI.3) when the numerical values are taken as shown in the last column of Table VI.1.

Remark 4.1. *When calculating the value of λ, we did not use a numerical approximation approach to evaluate the quantities $\int_{\mathbb{R}_+} F_1(s,0)\ \pi_1(\mathrm{d}s)$ and $\int_{\mathbb{R}_+} F_2(v,0)\ \pi_2(\mathrm{d}v)$, because the latter can be obtained directly in this case by Lemma 3.3*

$$\begin{cases} \displaystyle\int_{\mathbb{R}_+} F_1(s,0)\ \pi_1(\mathrm{d}s) = \beta_1 \int_{\mathbb{R}_+} s\ \pi_1(\mathrm{d}s) = \beta_1 \lim_{t\to\infty} \frac{\int_0^t \overline{S}(s)\,\mathrm{d}s}{t} = \frac{\beta_1(1-p)\Lambda}{\mu}. \\ \displaystyle\int_{\mathbb{R}_+} F_2(v,0)\ \pi_2(\mathrm{d}v) = \beta_2 \int_{\mathbb{R}_+} v\ \pi_2(\mathrm{d}v) = \beta_2 \lim_{t\to\infty} \frac{\int_0^t \overline{V}(s)\,\mathrm{d}s}{t} = \frac{\beta_2 p\Lambda}{\mu}. \end{cases} \tag{VI.30}$$

4.2 The case of saturated incidence rates

This time, we will treat the case of the saturated response functionals $IF_1(S,I) = \frac{\beta_1 SI}{1+m_1 I}$ and $IF_2(V,I) = \frac{\beta_2 VI}{1+m_2 I}$. β_1 and β_2 designate always the contact rates between S, V and I, and m_1, m_2 are respectively the saturation factors reflecting the inhibitory effect. After reassigning to our system parameters the numerical values appearing in the third column of Table VI.1 and taking $\beta_1 = 0.5$, $\beta_2 = 0.4$ with $m_1 = 0.61$, $m_2 = 0.76$, we will reobtain by a similar procedure to that of (VI.28) the value $\lambda = -0.0266 < 0$. Needless to say, assumptions $(\mathrm{H}_1) - (\mathrm{H}_5)$ are satisfied (see Table VI.2), so one can assures by the first item of Theorem 3.1 that any disease formulated by the system (VI.3) will exponentially extinguish almost surely, what we find accuratetly depicted in Figure VI.4.

Now, we will cope with the SVIR persistence case. Plainly, the (H_0') condition is no longer satisfied this time; whereas (H_0'') holds (see [10, 207]). This last fact forced us to use Theorem 3.2 instead of Theorem 3.1. Hence, if we readopt the numerical values appearing in the forth column of Table VI.1, we can easily reobtain by an analogous argument to that of (VI.29) the value $\lambda = 0.0321 > 0$. By virtue of Table VI.2, hypotheses $(\mathrm{H}_1) - (\mathrm{H}_5)$ hold, and since λ is positive in this case, Theorem 3.2 assures the weak persistence of the SVIR model (VI.3), which completely agrees with the curves of Figure VI.5.

Remark 4.2. *As in the case of the bilinear incidence rate, this time also we did not use the numerical approximation approach when calculating the value of λ, and this is because the integrals $\int_{\mathbb{R}_+} F_1(s,0)\ \pi_1(\mathrm{d}s)$ and $\int_{\mathbb{R}_+} F_2(v,0)\ \pi_2(\mathrm{d}v)$ can be deduced directly by the lemma 3.3 (see (VI.30)).*

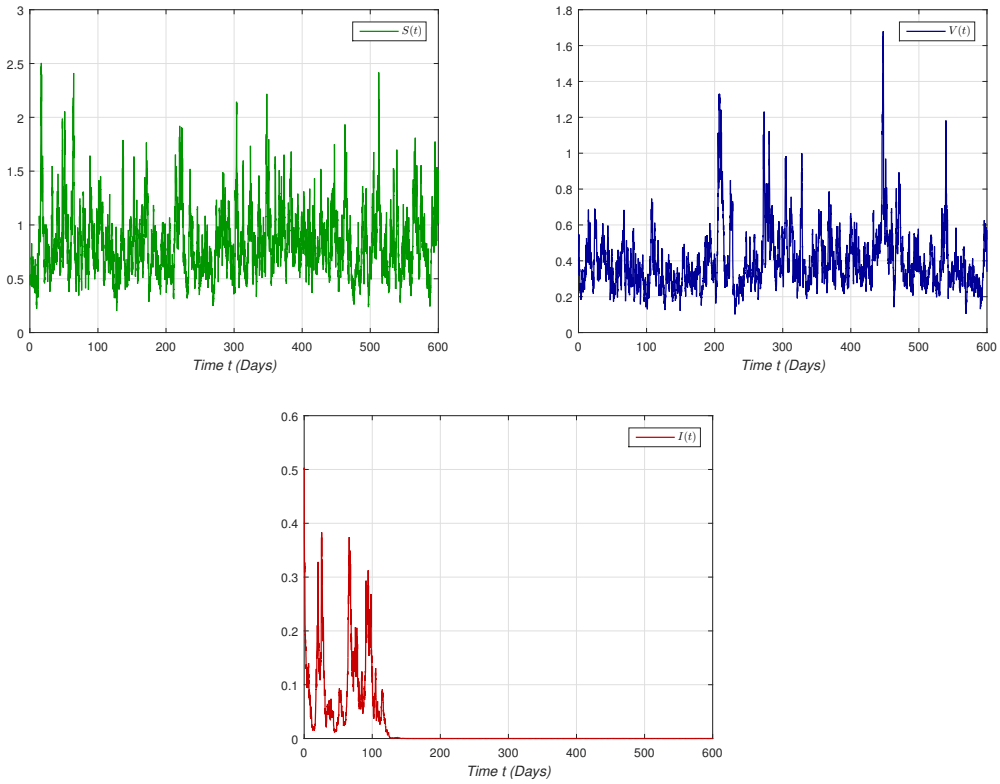


Figure VI.4: Solutions' paths of the SVIR model (VI.3) when the numerical values are taken as shown in the third column of Table VI.1.

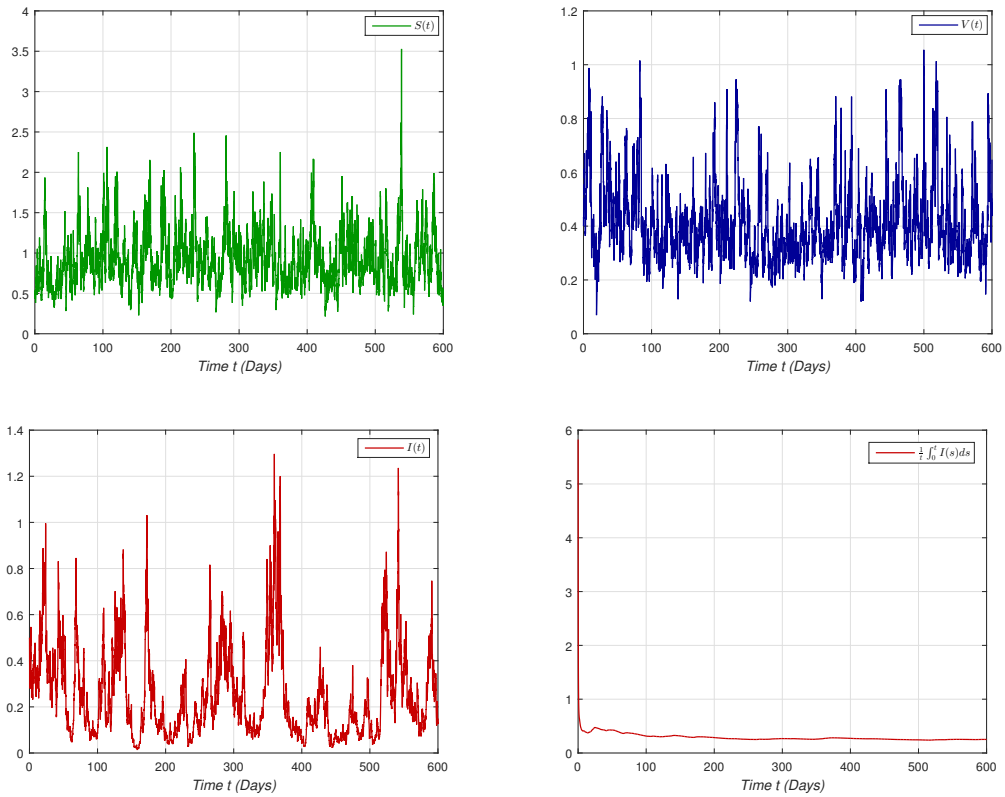


Figure VI.5: Solutions' paths of the SVIR model (VI.3) when the numerical values are taken as shown in the forth column of Table VI.1.

4.3 The case of The Beddington-DeAngelis incidence rates

At this time we suppose that the incidence rates $IF_1(S,I)$ and $IF_2(V,I)$ are of the Beddington-DeAngelis type, in other words, $IF_1(S,I) = \dfrac{\beta_1 SI}{1+m_1S+m_1'I}$ and $IF_2(V,I) = \dfrac{\beta_2 VI}{1+m_2V+m_2'I}$ $\Big($$\beta_1$, β_2 being the transmission rates between S, V and I, and the positive constants m_1, m_1', m_2, m_2' are the parameters of the inhibition effect $\Big)$. Obviously, the conditions (H_0) and (H_0') are verified; in fact,

$$-\frac{m_1'\beta_1}{m_1}\vee\frac{m_2'\beta_2}{m_2} \leqslant \underbrace{\frac{\partial F_1}{\partial I}(S,I)}_{=\frac{-m_1'\beta_1 S}{(1+m_1S+m_1'I)^2}} \leqslant 0 \leqslant \underbrace{\frac{\partial F_1}{\partial S}(S,I)}_{=\frac{\beta_1(1+m_1'I)}{(1+m_1S+m_1'I)^2}} \leqslant \beta_1\vee\beta_2 \quad \text{and} \quad -\frac{m_1'\beta_1}{m_1}\vee\frac{m_2'\beta_2}{m_2} \leqslant \underbrace{\frac{\partial F_2}{\partial I}(V,I)}_{=\frac{-m_2'\beta_2 V}{(1+m_2V+m_2'I)^2}} \leqslant 0 \leqslant \underbrace{\frac{\partial F_2}{\partial V}(V,I)}_{=\frac{\beta_2(1+m_2'I)}{(1+m_2V+m_2'I)^2}} \leqslant \beta_1\vee\beta_2.$$

By reassigning to our system parameters the numerical values appearing in the third column of Table VI.1 and by considering $\beta_1 = 1.78$, $\beta_2 = 1.49$ with $m_1 = 0.45$, $m_2 = 0.59$ and $m_1' = 1.1$, $m_2' = 1.3$, the condition $\lambda < 0$ becomes true

$$\begin{aligned}\lambda &= \int_{\mathbb{R}_+} F_1(s,0)\ \pi_1(\mathrm{d}s) + \int_{\mathbb{R}_+} F_2(v,0)\ \pi_2(\mathrm{d}v) - \left(\mu+\delta+\frac{\sigma_3^2}{2}+\int_{\mathbb{U}}\Big(\eta_3(u)-\ln\left(1+\eta_3(u)\right)\Big)\nu(\mathrm{d}u)\right)\\ &= \lim_{t\to\infty}\frac{1}{t}\int_0^t F_1\big(\overline{S}(s),0\big)\ \mathrm{d}s + \lim_{t\to\infty}\frac{1}{t}\int_0^t F_2\big(\overline{V}(s),0\big)\ \mathrm{d}s - \left(\mu+\delta+\frac{\sigma_3^2}{2}\right)+\int_{\mathbb{U}}\big(\ln(1+\eta_3(u))-\eta_3(u)\big)\nu(\mathrm{d}u)\\ &\approx -0.0419 < 0\,.\end{aligned}$$

So, Theorem 3.1 assures that any disease formulated by the system (VI.3) will exponentially extinguish almost surely, which is exactly depicted in Figure VI.6.

Now, we will deal with the SVIR persistence case. Keeping in mind the numerical values appearing in the last column of Table VI.1, we can easily notice that hypotheses $(H_1)-(H_5)$ hold (see Table VI.2), and λ outnumbers zero

$$\begin{aligned}\lambda &= \int_{\mathbb{R}_+} F_1(s,0)\ \pi_1(\mathrm{d}s) + \int_{\mathbb{R}_+} F_2(v,0)\ \pi_2(\mathrm{d}v) - \left(\mu+\delta+\frac{\sigma_3^2}{2}+\int_{\mathbb{U}}\Big(\eta_3(u)-\ln\left(1+\eta_3(u)\right)\Big)\nu(\mathrm{d}u)\right)\\ &= \lim_{t\to\infty}\frac{1}{t}\int_0^t F_1\big(\overline{S}(s),0\big)\ \mathrm{d}s + \lim_{t\to\infty}\frac{1}{t}\int_0^t F_2\big(\overline{V}(s),0\big)\ \mathrm{d}s - \left(\mu+\delta+\frac{\sigma_3^2}{2}\right)+\int_{\mathbb{U}}\big(\ln(1+\eta_3(u))-\eta_3(u)\big)\nu(\mathrm{d}u)\\ &\approx 0.0165 > 0\,.\end{aligned}$$

So, Theorem 3.1 guarantees the persistence in the mean of the SVIR model, which fully agrees with the curves presented in Figure VI.7.

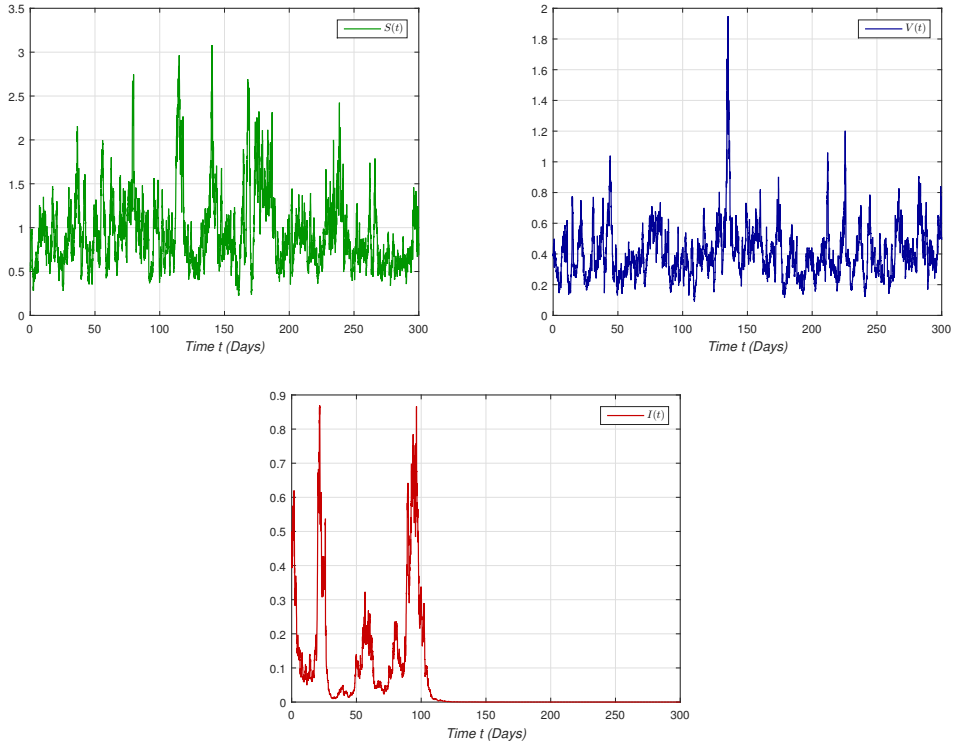


Figure VI.6: Solutions' paths of the SVIR model (VI.3) when the numerical values are taken as shown in the third column of Table VI.1.

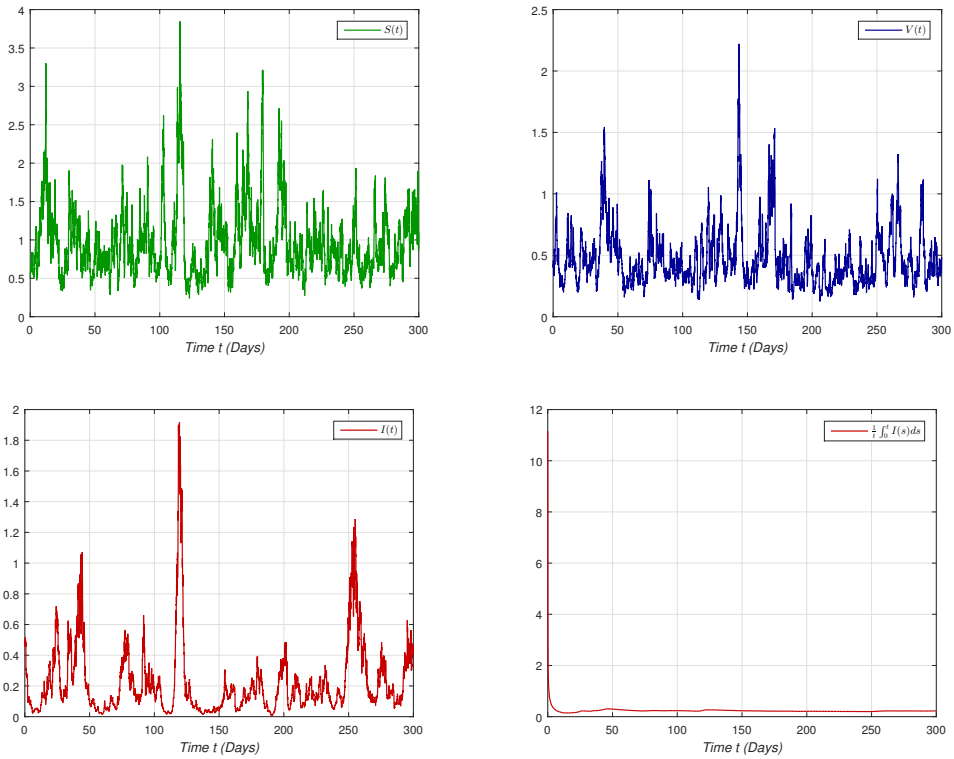


Figure VI.7: Solutions' paths of the SVIR model (VI.3) when the numerical values are taken as shown in the forth column of Table VI.1.

5 Conclusion and discussion

The vaccination strategy is one of the most effective means that can reduce the spread of diseases and even lead to their complete disappearance in some cases, which has prompted researchers in recent years to evoke the SVIR epidemic model [201]. The present chapter inspected a generalized stochastic SVIR epidemic model that comprehends both Brownian motions and Lévy jumps. More explicitly, the studied model takes the form of an Itô-Lévy stochastic differential equations system that includes two general response functions $IF_1(S,I)$ and $IF_2(V,I)$. After having presented our system, a detailed mathematical analysis has been performed to gain an insight into its long-run behavior. The principal mathematical and biological findings of this chapter are listed as follows:

- We have shown the well-posedness of the proposed SVIR model in the sense that it has a unique positive and global-in-time solution.
- We have identified under some hypotheses a common threshold λ between extinction and persistence in the mean.
- After noting that the assumptional framework used in the case of the persistence in the mean does not include some incidence rates often used in the literature, we modified it to cover more functions, and we found that λ stills acting as a threshold, but this time between the extinction and weak persistence.

Compared to existing works, the novelty of our findings lies in the following mathematical techniques that we have utilized:

- The explicit formulas of the stationary distributions of subsystems (VI.4) and (VI.5) are still unknown. So, we have used another manner to estimate the limits of their time averages.
- The assumptional frame of the persistence in the mean does not fit with all the classical response functions adopted in the literature. Thus, we have altered it to cover more functions, and we have used a non-standard analytical approach to find the common threshold between extinction and weak persistence.

Widely speaking, our findings show that the extinction and persistence of the studied SVIR model depend principally on white noises intensities, Lévy jumps magnitudes, and system parameters for sure. In order to support the theoretical results and cover some particular cases of incidence rates, we have introduced some numerical simulation examples. Lastly, we signalize that the obtained results ameliorate and extend many previous works (for instance, [10, 203]), and enhance our understanding of the SVIR model behavior, the thing that makes the results presented in this chapter a rich basis for further studies.

Conclusions and outlook

The objective of this thesis was to comprehensively investigate several stochastic epidemic models and derive some new results concerning their behavior. For this purpose, a novel analytical framework was formulated to address various questions and challenges, incorporating innovative techniques not commonly employed in existing literature. This final chapter provides a summary of the accomplishments in this thesis, emphasizes the significant discoveries, and explores their connections to the initial goals and the scientific research context.

- **Problematic context of the various studies conducted in this thesis and our proposed solutions**
  Exploring stochastic epidemic models demands the application of numerous inventive analytical approaches, given their complexity. The analytical endeavors in this thesis sought to offer new tools and techniques to fill the identified gaps in various recent publications.

  - **Broadening Epidemic Models to Include Lévy jumps**
    Throughout this thesis, numerous epidemic models have been expanded to encompass a more inclusive context. More explicitly, the effects of Lévy noises are incorporated into the treatment of AIDS, Dengue fever, and SVIR models proposed in Chapters III,IV and VI. Moreover, many intriguing asymptotic properties like extinction and persistence in the mean are established.
  - **Addressing of some open issues**
    In the case of Lévy jumps the determination of the stationary distribution expression is crucial for estimating the time averages of some auxiliary systems. Unfortunately, this expression remains elusive in the presence of Lévy noise, a question implicitly posed in various papers (see [152] and [151]). This thesis introduces an innovative approach to precisely define the threshold parameter without relying on the expression of this distribution or the ergodic theorem. The detailed exploration of this creative idea can be found in Chapters III,IV and VI.
  - **Sharpening of some thresholds and sills**
    We have raised the optimality of the thresholds associated with the treated epidemiological systems by adopting reinforced inequalities and new analytical techniques. This idea has been clearly mentioned and explained in different parts of the chapters III,IV, V and VI.

- **Comprehensive Overview and Prospects for Future Research**
  Upon reviewing prior mathematical analyses, we pinpointed some analytic gaps that underscore the significance of the work presented in this thesis. By addressing these gaps, our research contributes to an enhanced understanding of many probabilistic transmissible disease models. Consequently, considering the valuable insights obtained in this thesis, we contemplate several suggestions for future research endeavors:

  - **Exploration of More Intricate and Broad Biological Models.** Indeed, it is imperative to verify the applicability of the methods employed in this thesis to analyze additional

stochastic and realistic models, such as chemostat and prey-predator models. We defer these inquiries to our upcoming research works.

- **Delve into more sophisticated randomization methods.** In clearer words, further investigation should be conducted into more complex randomization approaches, such as incorporating high-order or even polynomial perturbations.
- **Utilization of Real Data and Validation against Real-world Observations.** The parameters of the model can be estimated through the incorporation of real data, allowing for a meaningful comparison between the obtained theoretical results and empirical findings. The exploration of this approach is a definite consideration for our future research endeavors.

- **Compilation Thesis Declaration: Originality and Third-Party Rights Assurance:**
  This thesis is a compilation thesis (or article-based thesis), incorporating findings previously published in reputable specialized journals with proper indexing [208]. We affirm that the works presented herein are original and do not violate any third-party rights.
- **Declaration of Ethical Enhancement of Language:** In adherence to contemporary research policies and ethical standards [209, 210, 211], we acknowledge the application of some AI-based technologies in the writing process of certain paragraphs within this thesis. This use is strictly limited to enhancing language proficiency and readability. It is essential to emphasize that this utilization respects both ethical and legal considerations [212, 213], and it was applied with rigorous human oversight and control.

Index

A
Absorbing region, 29
Absorbing set, 29
Adapted process, 6
AIDS/HIV, 56, 79, 80, 82, 109
Asymptomatic, 25, 26
Asymptotic analysis, 27
Asymptotic behavior, 26, 101
Asymptotic comportment, 82
Asymptotic properties, 23, 60, 79, 124
Asymptotic stability, 34
Asymptotically stable, 33, 82
Attracting region, 29
Attracting set, 29
Augmentation, 5

B
Basic reproduction number, 24, 32, 55, 58, 81, 82
Basic reproduction ratio, 31
Brownian motion, 10, 12
Brownian motions, 27, 59, 77, 101, 122

C
Canonical filtration, 5
Cauchy problem, 28
Cauchy-Lipschitz theorem, 28
Cauchy-Schwartz inequality, 43, 66, 91, 115
Compartment, 26, 31, 57, 81
Compartmental model, 23, 77, 80
Compartmental system, 24, 25, 108, 122
Complete, 10
Compound Poisson Process, 19
Confined population, 26
Continuous-time martingales, 8
Coronavirus, 23, 24, 37, 55
COVID-19, 1, 23, 24, 26, 27, 36, 37, 48, 49, 55, 56

D
Diffusion, 16
Disease-free equilibrium, 23, 27, 32–35, 55, 58, 80, 82
Dissemination, 24, 58, 78, 109
Dissemination behavior, 55
Dissemination dynamics, 57
Doob martingale stopping theorem, 8
Drift, 16

E
Ebola, 56
Endemic equilibrium, 27, 36, 58, 82
Epidemic, 23, 56
Epidemic model, 23, 57, 59, 60, 83, 122, 125
Exiting time, 7

F
Filtered probability space, 5
Filtration, 5, 10
Finite variation, 9
Fluctuations, 26, 49, 59, 123

H
Hölder continuous, 11
Hitting time, 7
Host population, 25

I
Indistinguishable, 4, 5
Infectious disease, 24, 79, 82, 83, 87, 109, 121, 122
Itô formula, 14
Itô process, 15

J
joint quadratic variation, 9
Jump processes, 18

L
Lévy jump diffusion, 22
Lipschitz condition, 17
Local Lipschitz condition, 17
Local martingale, 9

M
Monotone condition, 17

N
Negligible set, 5, 110
Null set, 5

P
P-null set, 4, 5
Population, 25, 26, 57, 100
Predictable, 6
Prevalence, 56, 57
Prevalence dynamics, 77, 122
Probability space, 5
Progressive, 6
Progressively measurable, 6
Propagation, 26, 57
Propagation behavior, 78
Propagation mechanisms, 24

Q
Quadratic variation, 8

R
right-continuous filtration, 5

S
SARS, 56
SIR model, 24
Spanish flu, 56
Stochastic differential equation, 15, 16
Stochastic process, 3, 16
Stopping time, 7
Submartingale, 8
Supermartingale, 8

U
Usual conditions, 5, 10

V
Vaccination, 57, 80
Vaccination program, 123
Vaccination strategy, 122, 142
Vaccine, 57, 77, 109
Version, 4

W
White noise, 15, 142
White noises, 123

Z
Zika virus, 56

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Résumé de la thèse

L'épidémiologie mathématique est un domaine basé sur l'utilisation de modèles et d'outils mathématiques pour comprendre et analyser la dynamique de propagation des maladies infectieuses au sein des populations humaines. En termes plus clairs, il permet de formuler des équations pouvant décrire la transmission et la propagation des épidémies et donne ensuite une vision des stratégies appropriées à adopter pour les atténuer. D'une manière générale, l'utilisation de modèles mathématiques en épidémiologie augmente notre capacité à interpréter les données empiriques, à faire des prédictions éclairées et à orienter les politiques de santé publique. Néanmoins, un nombre croissant d'observations expérimentales confirment l'impact significatif des fluctuations externes sur la dynamique des maladies infectieuses, ce qui fait de leur inclusion dans les phases de modélisation et d'analyse une très bonne valeur ajoutée. Pour cette raison, l'adoption de modèles probabilistes constitue un très bon choix, voire l'un des meilleurs pour étudier les maladies transmissibles. En effet, l'intégration de la stochasticité permet une représentation plus réaliste de la propagation des maladies infectieuses, en reconnaissant l'impact des événements inattendus et des changements démographiques. Cependant, l'analyse de ce type de modèle nécessite des techniques analytiques avancées en raison de leur complexité. Pour relever ce défi, cette thèse se consacre à l'exploration de la dynamique de propagation de quelques modèles épidémiques influencés par des bruits environnementaux naturels continus et des phénomènes soudains et imprévus. Dans un cadre théorique approprié, nous introduisons de nombreuses approches nouvelles et non standard pour dériver certaines propriétés asymptotiques des modèles perturbés. Ces propriétés englobent la stabilité, la persistance en moyenne, l'extinction de la maladie et le comportement à long terme en général. Chaque chapitre se termine par des simulations numériques validant les résultats théoriques et illustrant d'une manière claire la précision des conditions et des seuils obtenus.

D'une manière générale, cette thèse améliore diverses études existantes dans le domaine de l'épidémiologie mathématique stochastique, en proposant des méthodologies alternatives pour aborder et analyser des modèles épidémiologiques et biologiques plus compliqués.

Mots clés : Maladies infectieuses; Epidémies; Perturbations environnementales ; Systèmes stochastiques; Mouvement Brownian; Sauts de Lévy; Propriétés asymptotiques; Persistance en moyenne; Extinction de la maladie.