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ASPECTS HOMOLOGIQUES ET ALGÉBRIQUES DES EXTENSIONS D'ANNEAUX COMMUTATIFS

HOMOLOGICAL AND ALGEBRAIC ASPECTS OF COMMUTATIVE RING EXTENSIONS

Dédicace

Avec tous mes voeux de bonheur,
de santé et de réussite
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A mes parents

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(1) R. El Khalfaoui and N. Mahdou, *Commutative rings in which every pure ideal is projective*, Annali dell' Università di Ferrara, Vol. 65 (2019), 301-310.

(2) R. El Khalfaoui and N. Mahdou, *Rings in which every regular locally principal ideal is projective*, Studia Scientiarum Mathematicarum Hungarica, Vol. 56 (2019), No. 2, 241-251.

(3) R. El Khalfaoui, N. Mahdou and A. Mimouni, *On LPI rings*, Hacettepe Journal of Mathematics and Statistics, Accepted for publication.

(4) R.El Khalfaoui and N. Mahdou, *On rings with adequate range one*, Indian journal of mathematics, Vol. 61 (2019), No. 2, , 281-293.

(5) R. El Khalfaoui and N. Mahdou, *The ϕ-Krull dimension of some commutative extensions*, submitted for publication.

Other papers/Autres articles

(1) R. El Khalfaoui and N. Mahdou, *On stable range one property and strongly π-regular rings*, submitted for publication.

(2) R. El Khalfaoui, N. Mahdou, P. Sahandi and N. Shirmohammadi, *Amalgamated modules along an ideal*, submitted for publication.

(3) R. El Khalfaoui and N. Mahdou, *Rings in which every singly projective ideal is finitely projective*, in preparation.

Résumé

L'algèbre commutative est la discipline des mathématiques qui explore les anneaux commutatifs, leurs idéaux, les modules et les algèbres. Les travaux de cette thèse se situent dans un domaine établissant des liens communs entre l'algèbre commutative et l'algèbre homologique. Plus particulièrement, ils sont consacrés à l'étude de certaines propriétés algébriques dans différentes extensions d'anneaux commutatifs. Cette thèse comporte cinq chapitres rendant compte respectivement des travaux des cinq articles [49, 50, 51, 52, 53] qu'on a élaboré durant les années de recherche.

Dans un premier axe, on s'intéresse au transfert des propriétés homologiques telles que la *PIP*-propriété, la *LPP*-propriété et la *LPI*-propriété aux extensions d'anneaux. Cette étude a pour but d'étendre ces propriétés aux anneaux commutatifs contenant des diviseurs de zéro et de mieux distinguer les différentes classes d'anneaux en construisant des exemples montrant les inclusions strictes qui existent entre eux.

Dans un second axe, notre étude se porte sur la classe d'anneaux à rang adéquat un que nous caractérisons dans différents modèles de constructions à savoir les extensions triviales, l'amalgamation des anneaux ainsi que les produits fibrés (pullbacks). Cette étude enrichit la littérature courante avec des exemples originaux d'anneaux vérifiant cette propriété. Pour un anneau quelconque R, nous avons l'implication suivante:

$$R \text{ est adéquat} \Rightarrow R \text{ est à rang adéquat un.}$$

Mais cette implication est, en général, irréversible et rares sont les exemples qui prouvent cette irréversibilité. L'étude des extensions d'anneaux nous a permis de construire des nouveaux exemples, mettant au clair la distinction de ces deux propriétés.

Le troisième axe est consacré à la généralisation de la notion d'idéal premier.
Du fait que la dimension de Krull est définie à partir des idéaux premiers, la génération traitée dans cet axe nous a servi à définir une nouvelle dimension dite la ϕ-dimension de Krull.

Au terme de ce travail, quelques perspectives sont présentées pour nos travaux futurs.

Mots clés: *PIP*-anneau, *LPP*-anneau, *LPI*-anneau, ϕ-dimension de Krull, extension triviale, produit fibré, duplication amalgamée, amalgamation d'anneaux.

Summary

Commutative algebra is a branch of mathematics which studies commutative rings, their ideals, modules and algebras. This thesis consists of five chapters covering five papers [49, 50, 51, 52, 53]. It can be considered as a conjunction between commutative algebra and homological algebra.

In the first axis, we are interested with the study of the transfere of some homological properties such as the *PIP*-property, the *LPP*-property and the *LPI*-property. This study aims to extend the classes of commutative rings that verify these properties to rings with zero divisors and to better structure the different classes of rings by constructing examples showing the strict inclusions that exist between these classes.

In the second axis, our study focuses on the algebraic property "rings with adequate range one". we characterize these property in different models of constructions namely the trivial extensions, the amalgamation of rings and pullbacks. This study enriches the current literature with original examples of rings verifying this property. For a ring R, we have the following implication :

$$R \text{ is an adequate ring} \Rightarrow R \text{ has adequate range one.}$$

But these implication is usually irreversible and few are the examples that prove this irreversibility. The study of ring extensions has allowed us to construct new examples of rings clarifying the distinction of these properties.

The third axis is concerned with the generalization of the notion of prime ideal. The fact that the Krull dimension is defined from the prime ideals, the generalization treated in this axis allows to define a new dimension called the ϕ-Krull dimension.

Finally, we end this thesis by some open questions that we wish study in future.

Key Words: *PIP*-ring, *LPP*-ring, *LPI*-ring, ϕ-Krull dimension, trivial extension, pullback, amalgamated duplication, amalgamation of rings.

Introduction

Notre travail se base sur le transfert de certaines propriétés d'anneaux commutatifs dans différents contextes d'extension d'anneaux. Avant la description de la majorité des contributions de chaque chapitre, nous présentons d'abord quelques définitions, terminologies et notations. Tous les anneaux considérés sont supposés commutatifs, unitaires et les modules sont unitaires.

Définition 0.0.1 Soient R un anneau commutatif et M un R-module.

1. Pour un entier positif n, M est dit n-présenté s'il existe une suite exacte de R-modules:
$$F_n \to F_{n-1} \to \dots F_1 \to F_0 \to E \to 0$$
où chaque F_i est un R-module libre de type fini.
En particulier, un module 0-présenté est équivalent à un module de type fini et un module 1-présenté est dit de présentation finie.

2. M est dit un R-module cohérent s'il est de type fini et chaque sous-module de type fini de M est de présentation finie.

Définition 0.0.2 Soit R un anneau commutatif. R est dit un anneau cohérent si tout idéal de type fini de R est de présentation finie.

Définition 0.0.3 *Un idéal I de R est dit régulier s'il contient un élément régulier; i.e, un élément non diviseur de zéro.*

Définition 0.0.4 *Soit M un R-module.*
On dit que la dimension projective de M est inférieure ou égale à n s'il existe une résolution projective de la forme:

$$0 \to P_n \to \cdots \to P_0 \to M \to 0$$

où tous les P_i sont projectifs. On note $pd(M) \leq n$.
Si une telle résolution n'existe pas, on dit que M est de dimension projective infinie et on note $pd(M) = \infty$.
Si une telle résolution existe et n est le plus petit entier qui vérifie une telle résolution, on dit que la dimension projective de M est n. On note $pd_R(M) = n$.

Définition 0.0.5 *Soit R un anneau commutatif.*
La dimension de R notée $dim(R)$ (dite aussi dimension de Krull) est la longueur maximale n d'une chaîne d'idéaux premiers

$$P_0 \subset P_1 \cdots \subset P_n,$$

avec $P_i \neq P_{i+1}$ pour tout $i = 0, \cdots, n-1$.

En 1956 Nagata a introduit la notion d'extension triviale d'un anneau par un module de la manière suivante:

Définition 0.0.6 Soient A un anneau et E un $A-$module. L'extension triviale $R = A \propto E$ de A par E est l'ensemble $A \times E$ muni de l'addition définie de façon naturelle et de la multiplication donnée par :

$$(a,e)(b,f) = (ab, af + be).$$

Cette notion a fait l'objet de plusieurs travaux dans la littérature et constitue un champ de recherche fertile dans la théorie des anneaux (voir [58, 68]).

Définition 0.0.7 *Soit T un anneau et M un idéal de T et soit π la surjection naturelle $\pi : T \longrightarrow T/M$ et D un sous-anneau de T/M. Alors, $R := \pi^{-1}(D)$ est un sous-anneau de T et M est un idéal commun de R et T tel que $D = R/M$. R est dit l'anneau produit fibré ou pullback associé au carré commutatif suivant :*

$$\begin{array}{ccc} R := \pi^{-1}(D) & \xrightarrow{\pi|R} & D = R/M \\ \downarrow i & & \downarrow j \\ T & \xrightarrow{\pi} & T/M \end{array}$$

où i et j sont les injections naturelles.

comme cas particulier de produits fibrés, on a la $D+M$-construction, lorsque l'anneau T est de la forme $K+M$, où K est un corps et M est un idéal maximal de T, et R est de la forme $D+M$. Voir ([22], [29], [58]).

En 2006, D'Anna et Fontana [44] ont introduit une nouvelle construction, appelée la duplication amalgamée d'un anneau A le long d'un sous $A-$module E de $Q(A)$ (l'anneau total des fractions de A) vérifiant $E^2 \subseteq E$. Quand $E^2 = \{0\}$ cette construction coïncide avec l'extension triviale de A par E.

Dans [40, 41, 42, 43], sont discutées les motivations ainsi que les applications de cette duplication amalgamée $A \bowtie E$ de A le long du sous $A-$module E de $Q(A)$ essentiellement dans le cas particulier où E est un idéal de A. En 2010, dans [41] une extension de la construction de la duplication amalgamée $A \bowtie I$ de A le long de l'idéal I de A était donnée et définie comme suit:

Définition 0.0.8 Soient A et B deux anneaux, J un idéal de B et soit $f : A \to B$ un homomorphisme d'anneaux. On appelle l'amalgamation de A et B suivant J et respectant f, le sous-anneau de $A \times B$:

$$A \bowtie^f J := \{(a, f(a)+j) \mid a \in A, j \in J\}.$$

Soient R est un anneau et M un R-module. Nous utiliserons les notations suivantes:
- Si R est intègre, on note par $qf(R)$, le corps des fractions de R.
- $Nilp(R)$, l'ensemble des éléments nilpotents de R.
- $Rad(R)$, le radial de Jacobson de R.
- $Spec(R)$, l'ensemble des idéaux premiers de R.
- $Max(R)$, l'ensemble des idéaux maximaux de R.
- $Z(R)$, l'ensemble des diviseurs de zéro de R.
- $Reg(R)$, l'ensemble des élément réguliers de R.
- $gldim(R)$, la dimension globale de R.
- $pd_R(M)$, la dimension projective de M.

Dans ce qui suit, nous allons passer en revue les contributions majeures de chaque chapitre tout en assurant de les avoir illustrées par des exemples justifiants la nécessité des conditions imposées dans certains résultats ou montrant les inclusions strictes qui existent entre des classes d'anneaux.

Chapitre 1
Anneaux commutatifs où tout idéal pur est projectif

Dans ce chapitre [*], on étudie la classe d'anneaux où tout ideal pur est projectif. On cherche les conditions de stabilité de cette propriété sous l'image homomorphique, et son transfert à certaines constructions d'anneaux comme l'extension triviale, produit fibré et l'amalgamation d'anneaux.

1.1 Préliminaires

Soit R un anneau et I un idéal de R. On dit que I est un idéal pur de R si la suite $0 \longrightarrow I \otimes_R L \longrightarrow R \otimes_R L \longrightarrow \frac{R}{I} \otimes_R L \longrightarrow 0$ est exacte pour tout R-module L; d'une manière équivalente I est un idéal pur si $I_M = 0$ ou $I_M = R_M$ pour tout idéal maximal M de R. Dans [58, Theorem 1.2.15], il est démontré qu'un idéal I de R est pur si et seulement si R/I est plat, ce qui est aussi équivalent à $I \cap J = IJ$ pour tout idéal J de R. Par conséquent, tout idéal pur I de R est plat, voir [58, Theorem 1.2.15].

Ici on s'intéresse aux anneaux où tout idéal pur est projectif, appelés des PIP-anneaux. En 2016, les auteurs de [66] étudient la PIP-propriété dans un anneau arbitraire et montrent que les domaines et les anneaux locaux sont des PIP-anneaux. Ils montrent aussi que les F-anneaux (i.e., les anneaux où tout idéal pur est engendré par un idempotent), les anneaux parfaits et les anneaux Noethériens sont des PIP-anneaux [66, Corollary 2.6]. Les FF-anneaux (i.e., les anneaux où tout idéal plat est de type fini) et les FP-anneaux (i.e., les anneaux où tout idéal plat est projectif) sont aussi des PIP-anneaux.

[*] Ann. Un. Ferrara, Vol. 65 (2019), 301-310 (en collaboration avec N. Mahdou).

1.2 Principaux résultats

Nous commençons par le transfert de la *PIP*-propriété aux images homomorphiques, mais avant on présente l'exemple suivant concernant un cas particulier.

Exemple 0.0.9 *[66, Example 2.12.(1)] Soit R un domaine de Dedekind et I un idéal non-nul de R. Alors* $\frac{R}{I}$ *est un PIP-anneau.*

Théorème 0.0.10 *Soit R un anneau.*

1. *Si R est un PIP-anneau alors pour tout idéal pur I de R,* $\frac{R}{I}$ *est un PIP-anneau.*
2. *Si I est un idéal pur de type fini de R inclus dans tous les idéaux purs de R, alors R est un PIP-anneau si et seulement si* $\frac{R}{I}$ *est un PIP-anneau.*
3. *Soit I un idéal tel que* $I \subseteq Rad(R)$*. Si* $\frac{R}{I}$ *est Noethérien alors R est un PIP-anneau.*

La preuve de ce théorème nécessite le lemme suivant.

Lemme 0.0.11 *Un anneau R est un PIP-anneau si et seulement si tout R-module cyclique et plat a une dimension projective au plus égale à* 1.

Remarque 0.0.12 *En utilisant le Théorème 0.0.10(3), on peut montrer d'une autre façon que les anneaux Noethérien sont des PIP-anneaux.*

Maintenant on étudie le transfert de la *PIP*-propriété aux extensions triviales d'anneaux.

Soit A un anneau et E un R-module. Dans [39], un sous-module E' de E est dit sous-module pur si la suite $0 \longrightarrow E' \otimes M \longrightarrow E \otimes M$ est exacte pour tout A-module M. On sait que si $\frac{E}{E'}$ est plat alors E' est pur dans E, [55]. Dans [55, Corollaries 11.21 et 11.23], il est démontré que si E est un module plat alors pour tout sous-module E' de E les assertions suivantes sont équivalentes :

1. $\frac{E}{E'}$ est plat.
2. $IE' = IE \cap E'$ pour tout idéal I de A.
3. $rE' = rE \cap E'$ pour tout $r \in A$.

Chacune de ces conditions (équivalentes) peut être considérée comme une définition de sous-module pur dans le cas où E est plat. Mais en général ces conditions ne sont pas équivalentes, voir [71, p.158] pour quelques exemples.

Théorème 0.0.13 *Soit A un anneau et E un A-module, et soit* $R = A \propto E$ *l'extension triviale de A par E.*

1. *Si R est un PIP-anneau, alors A est un PIP-anneau.*
2. *Si tous les idéaux purs de R sont homogènes et E est un A-module plat, alors R est un PIP-anneau si et seulement si A est un PIP-anneau.*

Pour démontrer ce résultat on aura besoin des lemmes suivants.

Lemme 0.0.14 *Soit A un anneau et E un A-module et soit $R = A \propto E$ l'extension triviale de A par E. Alors,*

1. *Si $I \propto E'$ est un idéal pur de R, alors I est un idéal pur de A et E' est un sous-module pur de E et $E' = IE$.*
2. *I est un idéal pur de A si et seulement si $I \propto IE$ est un idéal pur de R.*
3. *Supposons que I est un idéal pur de A, alors $I \propto E'$ est un idéal pur de R si et seulement si $E' = IE$.*

Lemme 0.0.15 *Soit A un anneau, E un A-module et soit $R = A \propto E$ l'extension triviale de A par E. Si $I \propto IE$ (où I est un idéal non-nul de A) est un idéal projectif de R, alors I est un idéal projectif de A. L'inverse est vrai si E est plat.*

Dans le théorème suivant on donne un résultat sur le transfert de la *PIP*-propriété aux produits fibrés.

Théorème 0.0.16 *Soit $\varphi : R \longrightarrow T$ un homomorphisme d'anneaux injectif et Q un idéal de R tel que $QT = Q$.*

1. *Si R est un PIP-anneau, alors pour tout idéal pur J de R, le $\frac{R}{Q}$-module $\frac{J}{JQ}$ est projectif.*
2. *Si T est un PIP-anneau R-plat, alors R est un PIP-anneau si et seulement si pour tout idéal pur J de R, le $\frac{R}{Q}$-module $\frac{J}{JQ}$ est projectif.*

Pour démontrer ce théorème on aura besoin du lemme suivant.

Lemme 0.0.17 *[58, Theorem 5.1.1] Soit $\varphi : R \longrightarrow T$ un homomorphisme d'anneaux injectif tel qu'il existe un idéal Q de R qui satisfait $QT = Q$. Alors, un R-module E est projectif si et seulement si $E \otimes_R T$ est un T-module projectif et $\frac{E}{QE}$ est un $\frac{R}{Q}$-module projectif.*

Application. Soit T un anneau de la forme $T = K + M$, où K est un corps et M est un idéal maximal non-nul de T, D est un sous-anneau de K tel que $qf(D) = K$, et $R = D + M$. Alors, $T = S^{-1}R$ avec $S = D \setminus 0$.
Puisque T est un anneau local alors T est un *PIP*-anneau et il est R-plat (puisque $T = S^{-1}R$) et on obtient le corollaire suivant.

Corollaire 0.0.18 *En préservant les notations ci-dessus, R est un PIP-anneau si et seulement si pour tout idéal pur J de R, le $\frac{R}{Q}$-module $\frac{J}{JQ}$ est projectif.*

Après, on étudie le transfert de la *PIP*-propriété à l'amalgamation d'anneaux.

Rappelons qu'un idéal I d'un anneau R est dit idéal de multiplication si pour tout idéal $J \subseteq I$ il existe un idéal K tel que $J = IK$, on dit que J est un multiple de I. Un anneau R est dit de multiplication si tous ses idéaux sont de multiplication [10, 72].

On dit qu'un idéal I d'un anneau R est idempotent si $I^2 = I$. Aussi, un idéal I de $A \bowtie^f J$ est dit homogène si $I = K \bowtie^f J$ pour un certain idéal K de A. Si H est un idéal de $A \bowtie^f J$ tel que $0 \times J \subseteq H$ alors H est homogène [72].

Théorème 0.0.19 *Soit A et B deux anneaux, J un idéal de B et soit $f : A \to B$ un homomorphisme d'anneaux.*

1. *Supposons que J est un idéal plat de B. Si $A \bowtie^f J$ est un PIP-anneau alors A l'est aussi.*
2. *Supposons que J est un idéal idempotent de B et $A \bowtie^f J$ est un anneau de multiplication. Si $A \bowtie^f J$ est un PIP-anneau alors A l'est aussi.*
3. *Supposons que A est un anneau Noethérien et $0 \times J \subseteq Rad(A \bowtie^f J)$. Alors, $A \bowtie^f J$ est un PIP-anneau.*
4. *Supposons que tous les idéaux purs de $A \bowtie^f J$ sont homogènes. Si J est un idéal pur de type fini de B alors $A \bowtie^f J$ est un PIP-anneau si et seulement si A est un PIP-anneau.*

La démonstration de ce théorème nécessite les lemmes suivants.

Lemme 0.0.20 *Soit A et B deux anneaux et $f : A \longrightarrow B$ un homomorphisme d'anneaux rendant B un A-module fidèlement plat. Alors, I est un idéal pur de A si et seulement si IB est un idéal pur de B.*

Lemme 0.0.21 *[72, Lemma 2.10] Soit A et B deux anneaux, J un idéal idempotent de B et soit $f : A \to B$ un homomorphisme d'anneaux. Si K et L sont des idéaux de A, alors $(KL) \bowtie^f J = (K \bowtie^f J)(L \bowtie^f J)$ (est le produit de deux idéaux $K \bowtie^f J$ et $L \bowtie^f J$).*

Lemme 0.0.22 *[72, Proposition 2.8] Soit A et B deux anneaux, J un idéal idempotent de B, $f : A \to B$ un homomorphisme d'anneaux et soit I un idéal de multiplication de A. Alors, tout idéal homogène contenu dans $I \bowtie^f J$ est un multiple de $I \bowtie^f J$.*

Lemme 0.0.23 *Soit A et B deux anneaux, J un idéal de B, $f : A \to B$ un homomorphisme d'anneaux et soit I un idéal de A.*

1. *Supposons que J est plat. Alors, I est un idéal pur de A si et seulement si $I \bowtie^f (f(I)B)J$ est un idéal pur de $A \bowtie^f J$.*
2. *Si $I \bowtie^f J$ est un idéal pur de $A \bowtie^f J$, alors I est un idéal pur de A.*
3. *Si J est un idéal idempotent de B et $I \bowtie^f J$ est idéal de $A \bowtie^f J$ tel que tous ses sous-idéaux sont homogènes alors, $I \bowtie^f J$ est un ideal pur de $A \bowtie^f J$ si et seulement si I est un idéal pur de A.*
4. *Si J est un idéal idempotent de B et $A \bowtie^f J$ est un anneau de multiplication alors $I \bowtie^f J$ est un idéal pur de $A \bowtie^f J$ si et seulement si I est un idéal pur de A.*

Lemme 0.0.24 *Soit A et B deux anneaux, J un idéal de B, $f : A \to B$ un homomorphisme d'anneaux et soit I un idéal de A. Si $I \bowtie^f H$ est un idéal projectif de $A \bowtie^f J$, où I est un idéal de A tel que $f(I)J \subseteq H \subseteq J$, alors I est idéal projectif de A. En particulier, si $I \bowtie^f J$ (ou $I \bowtie^f (f(I)B)J$) est un idéal projectif de $A \bowtie^f J$ alors I est un idéal projectif de A.*

Exemple 0.0.25 *Soit $A = \mathbb{Z}$, $B = \frac{\mathbb{Z}}{12\mathbb{Z}}$ et $J = \frac{4\mathbb{Z}}{12\mathbb{Z}}$ et soit $f : A \to B$ la projection canonique. Alors, $A \bowtie^f J$ est un PIP-anneau.*

On termine ce chapitre avec les remarques et exemples suivants.

Remarques 0.0.26
1. *Si R est un anneau régulier au sens de Von Neumann non-héréditaire alors R n'est pas un PIP-anneau. Dans ce cas, même si R_P est un PIP-anneau pour tout $P \in Spec(R)$ (puisque c'est un corps), R n'est pas un PIP-anneau.*
2. *Si R est un anneau où tout idéal pur est de présentation finie, alors R est un PIP-anneau (puisque tout idéal pur est plat).*
3. *De la deuxième remarque, on peut dire qu'un anneau R où tout idéal pur est de type fini est un PIP-anneau.*

Exemple 0.0.27 *[66, Example 2.12 (3)] Soit $R = \Pi_{p \in S}(\mathbb{Z}/p\mathbb{Z})$, où S est l'ensemble de tous les entiers premiers. Alors R est un anneau régulier au sens de Von Neumann qui n'est pas héréditaire d'après [79]. Donc R n'est pas un PIP-anneau et R_P est un PIP-anneau pour tout $P \in Spec(R)$.*

On donne un exemple d'un *PIP*-anneau qui n'est pas cohérent.

Exemple 0.0.28 *Soit $R := \mathbb{Z} \propto \mathbb{Q}[X]$ l'extension triviale de l'anneau $\mathbb{Z}$ par $\mathbb{Q}[X]$. Alors*

1. *R est un PIP-anneau, puisque c'est un FF-anneau par [48, Example 3.13].*
2. *R n'est pas cohérent du [69, Theorem 2.8 (1)]. En particulier, R n'est pas Noethérien.*

Chapitre 2
Anneaux où tout idéal régulier localement principal est projectif

Dans ce chapitre [*], on étudie la classe d'anneaux où tout idéal régulier localement principal est projectif dits *LPP*-anneaux.

2.1 Préliminaires

On rappelle que dans le cas des domaines la notion d'idéal projectif est équivalente à celle d'idéal inversible. En général, un idéal inversible est projectif mais l'inverse n'est pas vrai. Dans un domaine D, les idéaux projectifs sont exactement les idéaux non-nuls de type fini qui sont localement principaux (voir [27]). Pour les domaines Noethériens les idéaux projectifs sont les idéaux non-nuls localement principaux. En 2011, D. D. Anderson and Muhammad Zafrullah [13], étudient les domaines où tout idéal localement principal est inversible (ce qui est équivalent à projectif), appelés les *LPI* domaines.

Puisque hors du contexte des domaines les idéaux projectifs ne sont pas nécessairement inversible, on donne la définition suivante :
Soit R un anneau. On dit que R est un *LPP*-anneau si tout idéal régulier localement principal est projectif. Les anneaux Noethériens, Artiniens et héréditaires sont des exemples de *LPP*-anneaux.

[*] Stud. Sci. Math. Hung, Vol. 56 (2019), No. 2, 241-251 (en collaboration avec N. Mahdou).

2.2 Principaux résultats

Notons qu'une caracterisation d'idéaux localement principaux était donnée dans [13]. La proposition suivante prolonge cette caractérisation au cas d'idéaux réguliers localement principaux dans les anneaux avec diviseurs de zéro. On rappelle qu'un idéal I dans un anneau R est dit d'annulation (en anglais "cancellation ideal") si $IJ \subseteq IK$ pour des idéaux J et K de R implique que $J \subseteq K$, et on dit que I est un idéal régulier si I contient un non-diviseur de zéro de R.

Proposition 0.0.29 *Soit R un anneau et soit I un idéal régulier de R. Alors, les assertions suivantes sont équivalentes :*

1. *I est localement principal.*
2. *I est fidèlement plat.*
3. *I est un idéal d'annulation.*

Remarques 0.0.30

1. *Les anneaux Noethériens, héréditaires et semi-simples (produit fini de corps) sont des exemples de LPP-anneaux.*
2. *Soit R un anneau Artinien. Tout élément de R est soit inversible soit un diviseur de zéro, un tel anneau admet un seul idéal régulier qui est R lui même. D'où, les anneaux Artiniens sont des LPP-anneaux.*
3. *Soit R un anneau. Alors, R_M est toujours un LPP-anneau indépendamment de R et cela pour tout $M \in Max(R)$ (puisque un idéal régulier principal est libre) mais l'inverse n'est pas vrai. Donc, les anneaux locaux sont des LPP-anneaux.*

Maintenant on va étudier le transfert de la *LPP*-propriété aux produits directs.

Théorème 0.0.31 *Soit $(R_i)_{i=1,\cdots,n}$ une famille d'anneaux commutatifs. Alors, $R = \prod_{i=1}^{n} R_i$ est un LPP-anneau si et seulement si R_i l'est aussi pour tout $i = 1, \cdots, n$.*

Pour démontrer ce résultat on aura besoin des lemmes suivants.

Lemme 0.0.32 *[73, Lemma 2.5] Soit $(R_i)_{i=1,2}$ deux anneaux et E_i un R_i-module pour $i = 1, 2$. Alors,*

$$pd_{R_1 \times R_2}(E_1 \times E_2) = sup\{pd_{R_1}(E_1), pd_{R_2}(E_2)\}.$$

Lemme 0.0.33 *Soit $(R_i)_{i=1,2}$ deux anneaux et I_i un idéal de R_i pour $i = 1, 2$. Alors, $I_1 \times I_2$ est un idéal régulier localement principal de $R_1 \times R_2$ si et seulement si I_1 et I_2 sont des idéaux réguliers localement principaux de R_1 et R_2 respectivement.*

Le résultat suivant montre que la *LPP*-propriété descend aux domaines fidèlement plats par un homomorphisme. On rappelle que dans un domaine, un idéal plat est de type fini si et seulement s'il est projectif.

Proposition 0.0.34 *Soit R et S deux domaines et $f : R \to S$ un homomorphisme d'anneaux rendant S un R-module fidèlement plat. Si S est un LPP-anneau, alors R l'est aussi.*

Le théorème suivant est sur le transfert de la *LPP*-properiété aux extensions triviales. On rappelle que si A est un anneau, E un A-module et $R = A \propto E$ l'extension triviale de A par E, un idéal J de R est dit homogène si $J = I \propto E'$ où I est un idéal de R, E' un sous-module de E et $IE \subseteq E'$ (voir par exemple [12]).

Théorème 0.0.35 *Soit A un anneau, E un A-module et soit $R = A \propto E$ l'extension triviale de A par E et soit $S = A \setminus (Z(A) \cup Z(E))$.*

1. *Supposons que pour tout idéal régulier localement principal I de A on a $I \cap S \neq \emptyset$. Alors, si R est un LPP-anneau alors A l'est aussi. En particulier, si $Z(E) \subseteq Z(A)$ (par exemple si on considère E sans torsion) alors R est un LPP-anneau implique que A l'est aussi.*

2. *Supposons que E est un A-module plat et divisible ($E = S^{-1}E$).*

   (a) *Si A est un LPP-anneau alors R l'est aussi.*

   (b) *Supposons que pour tout idéal régulier localement principal I de A, $I \cap S \neq \emptyset$ (en particulier, si $Z(E) \subseteq Z(A)$), alors R est un LPP-anneau si et seulement si A l'est aussi.*

   (c) *Si A est un domaine, alors R est un LPP-anneau si et seulement si A l'est aussi.*

La démonstration de ce théorème nécessite les lemmes suivants.

Lemme 0.0.36

1. *Soit A un anneau, E un A-module et I un idéal de A.*

   (a) *Si $I \propto IE$ est idéal régulier localement principal de $A \propto E$, alors I est un idéal régulier localement principal de A.*

   (b) *Si $I \cap S \neq \emptyset$, alors I est un idéal régulier localement principal de A si et seulement si $I \propto IE$ est un idéal régulier localement principal de $A \propto E$.*

   (c) *Si E est sans torsion, alors I est un idéal régulier localement principal de A si et seulement si $I \propto IE$ est un idéal régulier localement principal de $A \propto E$.*

2. *Soit $I \propto E'$ un idéal de $A \propto E$ où I est un idéal de A, E' un sous-module de E, et $IE \subseteq E'$. Si $I \propto E'$ est un idéal régulier localement principal de $A \propto E$ alors I l'est aussi.*

Lemme 0.0.37 *[4, Theorem 8] Soit A un anneau, E un A-module et I un idéal de A. Si $I \propto IE$ est un idéal projectif de $A \propto E$, alors I est projectif. L'inverse est vrai si E est plat.*

Lemme 0.0.38 *[12, Theorem 3.9] Soit A un anneau et E un A-module.*
Soit $S = A \setminus (Z(A) \cup Z(E))$. Les assertions suivantes sont équivalentes :

1. *Tout idéal régulier de $A \propto E$ est de la forme $I \propto E$ où I est idéal de A avec $I \cap S \neq \emptyset$.*
2. *Tout idéal régulier de $A \propto E$ est homogène.*
3. *$E = S^{-1}E$.*

Le Théorème 0.0.35 enrichit la littérature avec de nouveaux exemples de *LPP*-anneaux qui ne sont ni Noethériens ni de dimension zéro.

Exemple 0.0.39 *Soit K un corps, $A = K[x_1, x_2, \cdots, x_n]$ l'anneau des polynômes à n variables sur K et $E = qf(K[x_1, x_2, \cdots, x_n])$ le corps des fractions de A. Alors :*

1. *$A \propto E$ est un LPP-anneau puisque A est Noethérien (et donc A est un LPP-anneau) et en utilisant le Théorème 0.0.35(2,c).*
2. *$A \propto E$ n'est pas Noethérien puisque E n'est pas de type fini comme A-module et du [12, Theorem 4.8].*
3. *$A \propto E$ n'est pas de dimension zéro puisque A n'est pas de dimension zéro.*

Maintenant, on étudie le transfert de la *LPP*-propriété aux produits fibrés.

Théorème 0.0.40 *Soit $f : R \to S$ un homomorphisme d'anneaux injectif tel qu'il existe un idéal Q de R tel que $QS = Q$ et S est un R-module plat. Supposons que S est un LPP-anneau et $Reg(R) \subseteq Reg(S)$. R est un LPP-anneau si et seulement si pour tout idéal localement principal J de R, le R/Q-module J/JQ est projectif.*

La démonstration de ce théorème nécessite le lemme suivant.

Lemme 0.0.41 *[58, Theorem 5.1.1] Soit $f : R \to S$ un homomorphisme d'anneaux injectif tel qu'il existe un idéal Q de R tel que $QS = Q$. Alors, un R-module E est projectif si et seulement si $E \otimes_R S$ est un S-module projectif et E/QE est un R/Q-module projectif.*

Proposition 0.0.42 *Soit A et B deux anneaux, $f : A \to B$ un homomorphisme d'anneaux et J un idéal de B. Supposons que Jest un idéal plat de A et $f(Reg(A)) \subseteq Reg(B)$. Alors, si $A \bowtie^f J$ est un LPP-anneau alors A l'est aussi.*

La démonstration de cette proposition nécessite le lemme suivant.

Lemme 0.0.43 *Soit A et B deux anneaux, J un idéal de B, $f : A \to B$ un homomorphisme d'anneaux et I un idéal de A. Si $I \bowtie^f H$ est un idéal projectif de $A \bowtie^f J$, où I est un idéal de A tel que $f(I)J \subseteq H \subseteq J$, alors I est un idéal projectif de A. En particulier, si $I \bowtie^f J$ (ou $I \bowtie^f (f(I)B)J$) est un idéal projectif de $A \bowtie^f J$ alors I est idéal projectif de A.*

L'exemple suivant montre qu'il existe un homomorphisme d'anneaux qui satisfait la condition ($f(Reg(A)) \subseteq Reg(B)$) dans la proposition 0.0.42.

Exemple 0.0.44 *Soit I un idéal premier qui n'est pas régulier d'un anneau A et soit $B = A/I$. Alors, $p : A \to B$ vérifie $p(Reg(A)) \subseteq Reg(B)$.*

Maintenant, on va voir le transfert de la *LPP*-propriété à la duplication amalgamée. Rappelons qu'un idéal J de $A \bowtie I$ est dit homogène si $I = K \bowtie I$ pour un certain idéal K de A. Si K est un idéal de $A \bowtie I$ tel que $0 \times I \subseteq K$ alors K est homogène (voir [72, Lemma 2.9]).

Théorème 0.0.45 *Soit A un anneau et I un idéal de A.*

1. *Supposons que I est un idéal plat de A alors si $A \bowtie I$ est un LPP-anneau alors A l'est aussi.*

2. *Soit I un idéal plat non-régulier de type fini de A et $S = A \setminus Z(A)$ tels que $S^{-1}I = I$. Supposons que pour tout idéal régulier localement principal J de A on a $JI = I$. Alors, A est un LPP-anneau si et seulement si $A \bowtie I$ l'est aussi.*

Pour démontrer ce résultat on aura besoin des lemmes suivants.

Lemme 0.0.46 *Soit A un anneau et I un idéal de A. Si $(a, a+i)$ est un élément régulier de $A \bowtie I$ alors a est un élément régulier de A.*

Lemme 0.0.47 *Supposons que I est un idéal propre non-régulier d'un anneau A et $S = A \setminus (Z(A))$. Les assertions suivantes sont équivalentes :*

1. *Tous les idéaux réguliers de $A \bowtie I$ sont homogènes.*

2. *Pour tout $a \in S$ on a, $I = aI$ ce qui est équivalent à $I = S^{-1}I$.*

Chapitre 3
Sur les *LPI* anneaux

Dans ce chapitre [*] , on étend la notion de *LPI* domaine (les domaines où tout idéal localement principal est inversible) au anneaux avec diviseurs de zéro et on étudie la classe d'anneaux commutatifs dans lesquels tout idéal régulier localement principal est inversible dits *LPI* anneaux. On s'intéresse à la stabilité de cette propriété sous image homomorphique et son transfert à certaines constructions d'anneaux.

3.1 Préliminaires

En 2011, D. D. Anderson and Muhammad Zafrullah [13] introduisent et étudient la notion de *LPI* domaines, il s'agit des domaines où tout idéal non-nul localement principal est inversible.

Dans ce travail, on étend la notion de *LPI* domaine aux anneaux avec diviseurs de zéro. Un anneau R est appelé un *LPI* anneau si tout idéal régulier localement principal de R est inversible. Les anneaux Noethériens sont des *LPI* anneaux par [68, Lemma 18.1].

L'objectif de ce travail est de donner des exemples de *LPI* anneaux hors du cadre des domaines ou des anneaux Noethériens.

[*] Hacettepe J. Math. Stat., accepté pour publication (en collaboration avec N. Mahdou et A. Mimouni).

3.2 Principaux résultats

Notons qu'une caractérisation des idéaux localement principaux dans les domaines est donnée dans [13, Theorem 1]. La proposition suivante étend cette caractérisation aux idéaux réguliers localement principaux dans les anneaux avec diviseurs de zéros. la preuve est similaire à celle de [13, Theorem 1]. On rappelle qu'un idéal I dans un anneau R est dit d'annulation si $IJ \subseteq IK$ pour des idéaux J et K de R implique que $J \subseteq K$.

Proposition 0.0.48 *([13, Theorem 1]) Soit R un anneau et soit I un idéal régulier de R. Les assertions suivantes sont équivalentes :*

1. *I est localement principal.*
2. *I est fidèlement plat.*
3. *I est un idéal d'annulation.*

Maintenant, on va voir la propriété *LPI* dans les produits directs.

Théorème 0.0.49 *Soit $(R_i)_{i=1,\cdots,n}$ une famille d'anneaux commutatifs. Alors $R = \prod_{i=1}^{n} R_i$ est un LPI anneau si et seulement si R_i l'est pour chaque $i = 1, \cdots, n$.*

La démonstration de ce théorème nécessite les lemmes suivants.

Lemme 0.0.50 *Soit $(R_i)_{i=1,2}$ deux anneaux et I_i un idéal de R_i pour $i = 1, 2$. Alors, $I_1 \times I_2$ est un idéal régulier localement principal de $R_1 \times R_2$ si et seulement si I_1 et I_2 sont des idéaux réguliers localement principaux de R_1 et R_2 respectivement.*

Lemme 0.0.51 *([68, Lemma 18.1]) Soit I un idéal régulier localement principal d'un anneau A. Alors, I est inversible si est seulement s'il est de type fini.*

Proposition 0.0.52 *Soit R et S deux anneaux et $f : R \to S$ un homomorphisme d'anneaux rendant S un R-module fidèlement plat. Supposons que $f(Reg(R)) \subseteq Reg(S)$. Si S est un LPI anneau, alors R l'est aussi. En particulier, si R et S sont deux domaines et $f : R \to S$ est un homomorphisme d'anneaux rendant S un R-module fidèlement plat alors si S est un LPI domaine, alors R l'est aussi.*

Comme conséquence immédiate, on retrouve le résultat [13, Theorem 5(1)].

Corollaire 0.0.53 *([13, Theorem 5 (1)]) Soit R un domaine et soit X une indéterminée sur R. Si $R[X]$ (respectivement $R[[X]]$) est un LPI anneau, alors R l'est aussi.*

Maintenant, on va voir le transfert de la propriété *LPI* aux images homomorphiques. Rappelons qu'un idéal I d'un anneau R est un idéal pur si pour tout idéal maximal $M \in Max(R)$, $I_M = 0_M$ ou $I_M = R_M$.

Proposition 0.0.54 *Soit R un anneau.*

1. *Soit P un idéal premier de type fini non-régulier de R tel que P est contenu dans tous les idéaux réguliers localement principaux de R. Si $\frac{R}{P}$ est un LPI anneau, alors R l'est aussi.*

2. *Soit I un idéal pur régulier de R. Si R est un LPI anneau, alors $\frac{R}{I}$ est un LPI anneau.*

Pour démontrer ce résultat on aura besoin du lemme suivant.

Lemme 0.0.55 *Soit P un idéal premier non-régulier d'un anneau R.*
Si J est un idéal régulier localement principal de R tel que $P \subseteq J$, alors $\frac{J}{P}$ est un idéal régulier localement principal de $\frac{R}{I}$.

Exemple 0.0.56 *Soit K un corps et $R = K \propto K$ et soit $P = 0 \propto K$. Alors P est un idéal premier de type fini non-régulier de R qui est contenu dans tous les idéaux réguliers de R par le Lemme 0.0.59. Puisque $\frac{R}{P} = \frac{K \propto K}{0 \propto K} = K$, $K \propto K$ est un LPI anneau de la Proposition 0.0.54.*

Le résultat suivant concerne le transfert de la propriété *LPI* à l'extension triviale. Rappelons que si E est un A-module, alors $Z(E) = \{a \in A$ tel que $ae = 0$ pour un certain $0 \neq e \in E\}$.

Théorème 0.0.57 *Soit A un anneau, E un A-module, $R = A \propto E$ l'extension triviale de A par E et soit $S = A \setminus (Z(A) \cup Z(E))$.*

1. *Supposons que pour tout idéal régulier localement principal I de A, $I \cap S \neq \emptyset$. Si R est un LPI anneau, alors A l'est aussi.*

2. *Supposons que E est sans torsion et divisible. Alors A est un LPI anneau si et seulement si R l'est aussi. En particulier, si A est un domaine et E est un R-module plat et divisible, alors A est un LPI anneau si et seulement si R l'est aussi.*

3. *Supposons que E est de type fini et $E = S^{-1}E$. Si A est un LPI anneau, alors R l'est aussi. Si de plus, pour tout idéal régulier localement principal I de A, $I \cap S \neq \emptyset$, alors A est un LPI anneau si et seulement si R l'est aussi.*

La preuve de ce théorème nécessite les lemmes suivants.

Lemme 0.0.58 *1. Soit A un anneau, E un A-module et I un idéal de A et soit $S = A \setminus (Z(A) \cup Z(E))$.*

   (a) Si $I \propto IE$ est un idéal régulier localement principal de $A \propto E$, alors I est un idéal régulier localement principal de A.

(b) *Si $I \cap S \neq \emptyset$, alors I est un idéal régulier localement principal de A si est seulement si $I \propto IE$ est un idéal régulier localement principal de $A \propto E$.*

(c) *Si E est sans torsion, alors I est un idéal régulier localement principal de A si est seulement si $I \propto IE$ est un idéal régulier localement principal de $A \propto E$.*

2. *Soit E' un sous-module de E. Si $I \propto E'$ est un idéal régulier localement principal de $A \propto E$, alors I est un idéal régulier localement principal de A.*

Lemme 0.0.59 *([12, Theorem 3.9]) Soit A un anneau, E un A-module et $S = A \setminus (Z(A) \cup Z(E))$. Les assertions suivantes sont équivalentes :*

1. *Tout idéal régulier de $A \propto E$ est de la forme $I \propto E$ où I est un idéal de A avec $I \cap S \neq \emptyset$.*

2. *Tout idéal régulier de $A \propto E$ est homogène.*

3. *$E = S^{-1}E$.*

Lemme 0.0.60 *([3, Theorem 7]) Soit A un anneau, I un idéal non-nul de A et E un A-module. Si $I \propto IE$ est un idéal inversible de $A \propto E$, alors I est inversible.*

Lemme 0.0.61 *([4, Theorem 11]) Soit A un anneau, E un A-module sans torsion et divisible et $I \propto N$ un idéal homogène de $A \propto E$. Si I est un idéal inversible de A, alors $I \propto N$ est inversible.*

Lemme 0.0.62 *([3, Theorem 9]) Soit A un anneau, E un A-module, et $I \propto N$ un idéal homogène de $A \propto E$. Si I est un idéal de type fini de A et N un sous-module de type fini de E, alors $I \propto N$ est de type fini.*

Théorème 0.0.57 donne le résultat suivant.

Corollaire 0.0.63 *Soit A un domaine et E un K-espace vectoriel où $K = qf(A)$. Alors $A \propto E$ est un LPI anneau si est seulement si A l'est aussi.*

Exemple 0.0.64 *Soit $A = \mathbb{Z}_{(2)} = \{\frac{a}{b} \mid a, b \in \mathbb{Z}$ et b n'est pas divisible par $2\}$ et $E = \frac{\mathbb{Q}}{\mathbb{Z}_{(2)}}$. Alors $A \propto E$ est un LPI anneau.*

En utilisant le Théorème 0.0.57, on est capable de donner de nouveaux exemples de *LPI* anneaux qui ne sont pas Noethériens.

Exemple 0.0.65 *Soit K un corps et E un K-espace vectoriel de dimension infinie. Alors :*

1. *$K \propto E$ est un LPI anneau (du Corollaire 0.0.63).*

2. *$K \propto E$ n'est pas Noethérien (puisque E n'est pas de type fini et par [12, Theorem 4.8]).*

Maintenant, on va voir le transfert de la propriété *LPI* à la duplication amalgamée. Rappelons qu'un idéal J de $A \bowtie I$ est dit homogène si $J = K \bowtie I$ pour un certain idéal K de A. Si K est un idéal de $A \bowtie I$ tel que $0 \times I \subseteq K$, alors K est homogène (voir [72, Lemma 2.9]).

Théorème 0.0.66 *Soit A un anneau et I un idéal de A.*

1. *Supposons que I est plat. Si $A \bowtie I$ est un LPI anneau, alors A l'est aussi.*

2. *Supposons que I est un idéal de type fini non-régulier de A et $S^{-1}I = I$ où $S = A \setminus Z(A)$. Si A est un LPI anneau, alors $A \bowtie I$ l'est aussi. Si de plus I est plat, alors A est un LPI anneau si est seulement si $A \bowtie I$ l'est aussi.*

La démonstration de ce théorème nécessite les lemmes suivants.

Lemme 0.0.67 *Soit A un anneau et I un idéal de A. Si $(a, a+i)$ est un élément régulier de $A \bowtie I$, alors a est un élément régulier de A.*

Lemme 0.0.68 *Soit I un idéal propre d'un anneau A non-régulier et $S = A \setminus (Z(A))$. Les assertions suivantes sont équivalentes :*

1. *Tous les idéaux réguliers de $A \bowtie I$ sont homogènes.*

2. *Pour tout $a \in S$ on a $I = aI$ ce qui équivaut à $I = S^{-1}I$.*

Lemme 0.0.69 *Soit I et J deux idéaux d'un anneau A. Si I et J sont de type fini, alors $J \bowtie I$ est un idéal de type fini de $A \bowtie I$.*

Et voici un exemple d'un *LPI* anneau qui n'est pas Noethérien.

Exemple 0.0.70 *Soit $A = K \propto E$ où K est un corps et E un K-espace vectoriel de dimension infinie et soit $I = 0 \propto F$ où F un sous-espace vectoriel de dimension fini de E. Alors:*

1. *$A \bowtie I$ est un LPI anneau (du Théorème 0.0.66(2)).*

2. *$A \bowtie I$ n'est pas Noethérien (puisque A n'est pas Noethérien de l'exemple 0.0.65 et par [44, Corollary 2.9]).*

Chapitre 4
Sur les anneaux à rang adéquat un (rings with adequate range one)

Dans ce chapitre *, on s'intéresse aux classes d'anneaux à rang adéquat un. On étudie le transfert de cette propriété à certaines constructions d'anneaux pour construire des exemples qui nous permettrons de mieux classifier ces anneaux.

4.1 Préliminaires

Un élément c d'un anneau R est dit adéquat si pour tout $a \in R$ ils existent $r,s \in R$ tels que:

$$\begin{cases} c = rs; \\ rR + aR = R; \\ s'R + aR \neq R \text{ pour tout diviseur } s' \text{ de } s \text{ non-inversible} \end{cases}$$

Un anneau ou tout élément non-nul est adéquat est dit un anneau adéquat.

Un anneau R est à rang stable un si $aR + bR = R$ avec $a,b \in R$ implique qu'il existe $y \in R$ tel que $a + by \in U(R)$.

Un anneau R est à rang adéquat un si $aR + bR = R$ avec $a,b \in R$ implique qu'il existe $y \in R$ tel que $a + by$ est adéquat [47].

*Indian J. Math., Vol. 61 (2019), No. 2, , 281-293 (en collaboration avec N. Mahdou).

4.2 Principaux résultats

On commence par une caractérisation des anneaux à rang adéquat un.

Proposition 0.0.71 *Soit R un anneau. Les assertions suivantes sont équivalentes :*

1. *R est à rang adéquat un;*
2. $\forall a,x,b \in R$ *tels que* $ax+b=1$ *il existe* $y \in R$ *tel que* $a+by$ *est adéquat.*

Pour l'image homomorphique, on a le résultat suivant:

Théorème 0.0.72 *Soit R un anneau et I un idéal de R. Supposons que* $\forall x \in R \setminus I$, $I = xI$. *Alors,*

1. *Si R est à rang adéquat un alors* $\frac{R}{I}$ *l'est aussi.*
2. *Supposons que tous les éléments de I sont adéquats. Alors, R est à rang adéquat un si est seulement si* $\frac{R}{I}$ *l'est aussi.*

Pour démontrer ce théorème on aura besoin du lemme suivant.

Lemme 0.0.73 *Soit R un anneau et I un idéal de R. Supposons que* $\forall x \in R \setminus I$, $I = xI$. *Alors, a est un élément adéquat de* $R \setminus I$ *si et seulement si* $\overline{a}$ *est un élément adéquat de* $\frac{R}{I} \setminus 0$.

Du Théorème 0.0.72, on a le corollaire suivant.

Corollaire 0.0.74 *Soit R un anneau et P un idéal premier de R et* $S = R \setminus P$. *Supposons que* $S^{-1}P = P$. *Alors,*

1. *Si R est à rang adéquat un alors* $\frac{R}{P}$ *l'est aussi.*
2. *Supposons que tous les éléments de P sont adéquats. Alors, R est à rang adéquat un si et seulement si* $\frac{R}{P}$ *l'est aussi.*

Maintenant, on étudie le transfert de cette propriété à l'extension triviale.

Théorème 0.0.75 *Soit A un anneau, E un A-module et* $R = A \propto E$ *l'extension triviale de A par E. Supposons que pour tout* $k \in A \setminus \{0\}$, $kE = E$. *Alors, R est à rang adéquat un si et seulement si A l'est aussi.*

La démonstration de ce théorème nécessite le lemme suivant.

Lemme 0.0.76
Soit A un anneau, E un A-module et $R = A \propto E$ *l'extension triviale de A par E. Alors,*

1. *Soit $a,b,c \in A$ tels que $bE = E$ et soit $x,y \in E$. Alors, les assertions suivantes sont équivalentes:*
   (a) *$\exists z \in E$ tel que $(a,x) = (b,y)(c,z)$;*
   (b) *$a = bc$.*
2. *Soit $a,b \in A$ tel que $bE = E$ et soit $x,y \in E$. Alors, $(b,y)|(a,x)$ si et seulement si $b|a$.*
3. *Soit $a,b \in A$ tels que $bE = E$ et soit $x,y \in E$. Alors, $(a,x)R + (b,y)R = R$ si et seulement si $aA + bA = A$.*

Le corollaire suivant est une conséquence du Théorème 0.0.75.

Corollaire 0.0.77 *Soit A un domaine, E un A-module divisible et $R = A \propto E$ l'extension triviale de A par E. Alors R est à rang adéquat un si et seulement si A l'est aussi. En particulier, si $K := qf(A)$ est le corps des fractions de A et E est un K-espace vectoriel. Alors R est à rang adéquat un si et seulement si A l'est aussi.*

Remarque 0.0.78 *Un anneau local est un anneau adéquat d'après [75, Theorem 2.1] et donc les anneaux locaux sont à rang adéquat un.*

Exemple 0.0.79 *Soit $A = \mathbb{Z}_{(2)} = \{\frac{a}{b} \mid a,b \in \mathbb{Z}$ où b n'est pas divisible par $2\}$ et $E = \frac{\mathbb{Q}}{\mathbb{Z}_{(2)}}$. Alors, $A \propto E$ est à rang adéquat un.*

Pour le transfert de la propriété "à rang adéquat un" aux produits fibrés, on a pu obtenir le résultat suivant du Théorème 0.0.72.

Corollaire 0.0.80 *Soit R un anneau, $\varphi : R \longrightarrow T$ un homomorphisme d'anneaux injectif et Q un idéal de R tel que $QT = Q$. Supposons que $\forall x \in T \setminus Q$, $Q = xQ$. Alors,*

1. *Si R (resp. T) est à rang adéquat un alors $\frac{R}{Q}$ (resp $\frac{T}{Q}$) l'est aussi.*
2. *Supposons que tous les éléments de Q sont adéquats. Alors, R (resp T) est à rang adéquat un si et seulement si $\frac{R}{Q}$ (resp $\frac{T}{Q}$) l'est aussi.*

Pour l'amalgamation d'anneaux, on a le résultat suivant.

Théorème 0.0.81 *Soit A et B deux anneaux, J un idéal propre de B et $f : A \to B$ un homomorphisme d'anneaux.*

1. *Si $A \bowtie^f J$ est à rang adéquat un alors A l'est aussi.*
2. *Supposons que tous les éléments de J sont adéquats et $\forall x \in (f(A) + J) \setminus J$, $xJ = J$. Alors, $A \bowtie^f J$ est à rang adéquat un si et seulement si A l'est aussi.*

On rappelle qu'un anneau adéquat est à rang adéquat un, l'exemple suivant montre que la réciproque n'est pas vraie.

Exemple 0.0.82 *Soit A le localisé de l'anneau des entiers par les idéaux maximaux* $2\mathbb{Z}$ *et* $3\mathbb{Z}$, $A = \{\frac{a}{b} \in \mathbb{Q} |\ b$ *n'est divisible ni par* 2 *ni par* $3\}$ *et* $B = \dfrac{\mathbb{R}[[X]]}{(X^2+1)^4\mathbb{R}[[X]]}$ *et soit* $J = \dfrac{(X^2+1)^2\mathbb{R}[[X]]}{(X^2+1)^4\mathbb{R}[[X]]}$ *un idéal de B, et*

$$\begin{array}{rccl} f: & A & \to & B \\ & a & \to & f(a) = \overline{a} \end{array}$$

1. $A \bowtie^f J$ *est à rang adéquat un.*
2. $A \bowtie^f J$ *n'est pas un anneau adéquat.*

Chapitre 5
La ϕ-dimension de Krull de certaines extensions commutatives

Dans ce chapitre*, on étudie la généralisation de la notion de premier dans les idéaux pour quelques constructions d'anneaux commutatifs. Après, on définit une nouvelle notion qui est la ϕ-dimension de Krull inspirée de cette généralisation.

5.1 Préliminaires

On note par $Z(R)$ l'ensemble des diviseurs de zéro d'un anneau R et par $Reg(R)$ l'ensemble des éléments régulier de R (c'est à dire les éléments qui ne sont pas des diviseurs de zéro). On note l'ensemble de tous les idéaux (resp. les idéaux propres) d'un anneau R par $\mathscr{I}(R)$ (resp. $\mathscr{I}^*(R)$).

Un idéal propre P de R est dit faiblement premier si pour $a,b \in R$, $0 \neq ab \in P$ implique $a \in P$ ou $b \in P$. Un idéal propre P de R est dit presque premier si pour $a,b \in R$, $ab \in P \setminus P^2$, soit $a \in P$ soit $b \in P$.

Soit $\phi : \mathscr{I}(R) \to \mathscr{I}(R) \cup \{\emptyset\}$ une fonction. Un idéal propre P de R est dit ϕ-premier si pour $x,y \in R$, $xy \in P \setminus \phi(P)$ implique que $x \in P$ ou $y \in P$. Puisque $P \setminus \phi(P) = P \setminus (P \cap \phi(P))$, sans perte de généralité, on peut supposer que $\phi(P) \subseteq P$.

*Ce travail est soumis pour publication (en collaboration avec N. Mahdou).

5.2 La généralisation d'idéaux premiers dans l'extension triviale et l'amalgamation d'anneaux

La proposition suivante est un résultat donné dans [19, Proposition 2.7] qui concerne des cas particuliers où les idéaux de l'extension triviale sont ψ-premiers.

Rappelons que si A est un anneau, E un A-module et $R = A \propto E$ est l'extension triviale de A par E, un idéal J de R est dit homogène si $J = I \propto E'$ où I est un idéal de R, E' un sous-module de E et $IE \subseteq E'$. $A \propto E$ est dit un anneau homogène si tous ses idéaux sont homogènes (voir [12]).

Proposition 0.0.83 *[19, Proposition 2.7] Soit A un anneau, E un A-module et $A \propto E$ homogène. Soit $\phi : \mathscr{I}(A) \to \mathscr{I}(A) \cup \{\emptyset\}$ et $\psi : \mathscr{I}(A \propto E) \to \mathscr{I}(A \propto E) \cup \{\emptyset\}$ deux fonctions telles que $\psi(I \propto N) = \phi(I) \propto N$. Alors*

1. *Si $Q = P \propto N$ est un idéal ψ-premier de $A \propto E$, alors P est un idéal ϕ-premier de A.*
2. *P est un idéal ϕ-premier de A si et seulement si $P \propto E$ est un idéal ψ-premier de $A \propto E$.*

La proposition suivante montre dans quel cas les idéaux de la forme $0 \propto F$ sont ψ-premiers.

Proposition 0.0.84 *Soit A un anneau, E un A-module et $\phi : \mathscr{I}(A) \to \mathscr{I}(A) \cup \{\emptyset\}$ une fonction. Soit N un sous-module de E et $\psi : \mathscr{I}(A \propto E) \to \mathscr{I}(A \propto E) \cup \{\emptyset\}$ une fonction définie par :*

$$\psi(H) = \begin{cases} (\phi(I_H) \propto N) \cap H & si\ \phi(I_H) \neq \emptyset \\ \emptyset & si\ \phi(I_H) = \emptyset \end{cases}$$

où $I_H = \{a \in A \mid (a,e) \in H\}$. Soit F un sous-module de E et $J = 0 \propto F$. Supposons que $\psi(J) \neq \emptyset$ alors,

1. *Si $K = N \cap F \neq F$ et ils existent $a, b \in A \setminus \{0\}$ tels que $ab = 0$ et $f \in F \setminus K$ tel que $af \in F \setminus K$ alors J n'est pas ψ-premier.*
2. *Si $K = N \cap F = F$ alors J est ψ-premier.*
3. *Si A est un domaine, alors les assertions suivantes sont équivalentes :*
   - *(a) $0 \propto F$ est ψ-premier.*
   - *(b) $\forall a \in A \setminus \{0\}$, $\forall e \in E$, $ae \in F \setminus K$ implique que $e \in F \setminus K$.*

Si $\psi(J) = \emptyset$, alors $0 \propto F$ est ψ-premier si et seulement si $\{0\}$ est φ-premier et $F = E$.

Remarque 0.0.85 *Si $\psi : \mathscr{I}(A \propto E) \to \mathscr{I}(A \propto E) \cup \{\emptyset\}$ est une fonction définie par :*

$$\psi(H) = \begin{cases} (\phi(I_H) \propto E) \cap H & si\ \phi(I_H) \neq \emptyset \\ \emptyset & si\ \phi(I_H) = \emptyset \end{cases}$$

où $I_H = \{a \in A \mid (a,e) \in H$ pour certain $e \in E\}$. Alors,
(1) Si $\phi(\{0\}) = \emptyset$. Alors, $0 \propto E$ est ψ-premier si et seulement si $\{0\}$ est ϕ-premier [12, Theorem 3.2].
(2) Si $\phi(\{0\}) \neq \emptyset$. Alors, $0 \propto E$ est toujours ψ-premier et $\{0\}$ est toujours ϕ-premier.

Le théorème suivant donne une forme générale des idéaux ψ-premiers de $A \propto E$ dans des cas particuliers. Rappelons qu'un idéal propre I de R est dit semi-premier si $a \in R$ et $a^2 \in I$ implique que $a \in I$ (voir [18]).

Théorème 0.0.86 *Soit A un anneau, E un A-module et $\phi : \mathscr{I}(A) \to \mathscr{I}(A) \cup \{\emptyset\}$ une fonction. Soit N un sous-module de E et $\psi : \mathscr{I}(A \propto E) \to \mathscr{I}(A \propto E) \cup \{\emptyset\}$ une fonction définie par :*

$$\psi(H) = \begin{cases} (\phi(I_H) \propto N) \cap H & si\ \phi(I_H) \neq \emptyset \\ \emptyset & si\ \phi(I_H) = \emptyset \end{cases}$$

où $I_H = \{a \in A \mid (a,e) \in H\}$. Soit J un idéal ψ-premier de $A \propto E$ tel que $\psi(J) \neq \emptyset$. Si J est un idéal ψ-premier de $A \propto E$ tel que $\phi(I_J)$ est un idéal semi-premier de A, alors

1. *Soit $J \subseteq \phi(I_J) \propto E$ soit $J = I_J \propto E$.*
2. *Si on suppose de plus que $\psi(J)$ est semi-premier alors soit $J = \phi(I_J) \propto E$ soit $J = I_J \propto E$.*

Si $\psi(J) = \emptyset$ alors $J = P \propto E$ où P est un idéal ϕ-premier de A.

La démonstration de ce théorème nécessite le lemme suivant.
On rappelle qu'un élément a d'un anneau R est dit un zéro-incassable (unbreakable-zero) d'un idéal I si $a^2 = 0$ et $a \notin I$ [18, Definition 2.2].

Lemme 0.0.87 *Soit A un anneau et $\phi : \mathscr{I}(A) \to \mathscr{I}(A) \cup \{\emptyset\}$ une fonction et soit I un idéal ϕ-premier de A tel que $\phi(I) \neq \emptyset$. Supposons que a est un élément zéro-incassable de I. Alors, $(a+i)^2 \in \phi(I)$ et $(a-i)^2 \in \phi(I)$ pour tout $i \in I$.*

Dans le Théorème 0.0.86, si on prend $\phi(I_H) = 0$ et $N = 0$ on aura un résultat sur le cas particulier des idéaux faiblement premiers.

Corollaire 0.0.88 *Soit A un anneau, E un A-module et J un idéal faiblement premier de $A \propto E$ et $I_J = \{a \in A \mid (a,e) \in J$ pour un certain $e \in E\}$. Si $\{0\}$ est un idéal semi-premier de A, Alors,*

1. *Soit $J \subseteq 0 \propto E$ soit $J = I_J \propto E$.*

2. *Si on suppose de plus que $\{(0,0)\}$ est un idéal semi-premier de $A \propto E$ alors soit $J = 0 \propto E$ soit $J = I_J \propto E$.*

Maintenant, on étudie la généralisation d'idéal premier dans l'amalgamation d'anneaux.

Proposition 0.0.89 *Soit A et B deux anneaux, $f : A \to B$ un homomorphisme d'anneaux et J un idéal de B. Soit $\phi : \mathscr{I}(A) \to \mathscr{I}(A)$ et $\psi : \mathscr{I}(A \bowtie^f J) \to \mathscr{I}(A \bowtie^f J) \cup \{\emptyset\}$ deux fonctions telles que :*

$$\psi(P \bowtie^f J) = \begin{cases} \phi(P) \bowtie^f K & si\ \phi(P) \neq \emptyset \\ \emptyset & si\ \phi(P) = \emptyset \end{cases}$$

où P est un idéal de A et K un sous-idéal de J. Alors, $P \bowtie^f J$ est un idéal ψ-premier si et seulement si P est un idéal ϕ-premier et pour tous $a,b \notin P$ tels que $ab \in \phi(P)$ on a $(f(a)j + f(b)i + ij) \in K$ pour tous $i, j \in J$.

Remarque 0.0.90 *Soit $\psi : \mathscr{I}(A \bowtie^f J) \to \mathscr{I}(A \bowtie^f J) \cup \{\emptyset\}$ une fonction définie par :*

$$\psi(P \bowtie^f J) = \begin{cases} \phi(P) \bowtie^f J & si\ \phi(P) \neq \emptyset \\ \emptyset & si\ \phi(P) = \emptyset \end{cases}$$

(1) Si $\phi(\{0\}) = \emptyset$. Alors $0 \times J$ est ψ-premier si et seulement si $\{0\}$ est ϕ-premier [42, Corollary 2.5].
(2) Si $\phi(\{0\}) \neq \emptyset$. Alors $0 \times J$ est toujours ψ-premier et $\{0\}$ est toujours ϕ-premier.

Dans la proposition suivante, on verra quand les idéaux de la forme $\overline{H}^f = \{(a, f(a) + j) \in A \bowtie^f J \mid (f(a) + j) \in H\}$, où H est un idéal de B, sont ψ-premiers.

Proposition 0.0.91 *Soit A et B deux anneaux, $f : A \to B$ un homomorphisme d'anneaux et J un idéal de B. Soit $\phi : \mathscr{I}(A) \to \mathscr{I}(A) \cup \{\emptyset\}$, $\varphi : \mathscr{I}(f(A) + J) \to \mathscr{I}(f(A) + J) \cup \{\emptyset\}$ et $\psi : \mathscr{I}(A \bowtie^f J) \to \mathscr{I}(A \bowtie^f J) \cup \{\emptyset\}$ trois fonctions telles que :*

$$\varphi(H) = \emptyset \Leftrightarrow \phi(I_H) = \emptyset$$

$$\psi(\overline{H}^f) = \begin{cases} \{(a, f(a) + j) \mid a \in \phi(I_H)\ et\ f(a) + j \in \varphi(H)\} & si\ \varphi(H) \neq \emptyset \\ \emptyset & si\ \varphi(H) = \emptyset \end{cases}$$

pour tout idéal H de $f(A) + J$, où $I_H = \{a \in A \mid (a, f(a) + j) \in \overline{H}^f\ pour\ certain\ j \in J\}$. Alors, $\overline{H}^f$ est un idéal ψ-premier si et seulement si H est un idéal φ-premier de $f(A) + J$ et pour tous $(f(a) + i), (f(b) + j) \notin H$ tels que $(f(a) + i)(f(b) + j) \in \varphi(H)$ on a $ab \in \phi(I_H)$.

Le théorème suivant donne une forme générale aux idéaux ψ-premiers de l'amalgamation d'anneaux dans des cas particuliers.
Rappelons que I est un idéal radical de R si pour tout $x \in R$ et pour tout entier positif n, si $x^n \in I$ alors $x \in I$.

Théorème 0.0.92 *Soit A et B deux anneaux, $f : A \to B$ un homomorphisme d'anneaux et J un idéal de B. Soit $\phi : \mathscr{I}(A) \to \mathscr{I}(A) \cup \{\emptyset\}$ et $\psi : \mathscr{I}(A \bowtie^f J) \to \mathscr{I}(A \bowtie^f J) \cup \{\emptyset\}$ deux fonctions telles que :*

$$\psi(H) = \begin{cases} (\phi(I_H) \bowtie^f K) \cap H & si\ \phi(I_H) \neq \emptyset \\ \emptyset & si\ \phi(I_H) = \emptyset \end{cases}$$

où $I_H = \{a \in A | (a, f(a)+i) \in H$ pour certain $i \in J\}$ et K est un sous-idéal de J. Soit P un idéal ψ-premier de $A \bowtie^f J$ et $J \subseteq Nilp(B)$.
(I) Si $\phi(I_P) \neq \emptyset$ est un idéal radical, alors

1. *Soit $P \subseteq \phi(I_P) \times J$ soit $P = I_P \bowtie^f J$.*
2. *Si on suppose de plus que $\psi(P)$ est un idéal radical de $A \bowtie^f J$, alors soit $P = \phi(I_P) \times J$ (nécessairement dans ce cas $f(\phi(I_P)) \subseteq J$) soit $P = I_P \bowtie^f J$.*

(II) Si $\phi(P) = \emptyset$ alors $P = H \bowtie^f J$ où H est un idéal ϕ-premier de A.

Pour démontrer ce résultat on aura besoin du lemme suivant.

Lemme 0.0.93 *Soit A un anneau et $\phi : \mathscr{I}(A) \to \mathscr{I}(A) \cup \{\emptyset\}$ une fonction et soit I un idéal ϕ-premier de A tel que $\phi(I) \neq \emptyset$. Soit a un élément de A tel que $a \notin I$ et $a^n = 0$ pour certain entier $n \geq 2$. Alors, $(a+i)^n \in \phi(I)$ et $(a-i)^n \in \phi(I)$ pour tout $i \in I$.*

Lorsque $\phi(I_H) = \{0\}$ et $K = \{0\}$ dans le Théorème 0.0.92, on obtient le résultat pour les idéaux faiblement premiers.

Corollaire 0.0.94 *Soit A et B deux anneaux, $f : A \to B$ un homomorphisme d'anneaux et J un idéal de B. Soit P un idéal faiblement premier de $A \bowtie^f J$ et $I_P = \{a \in A | (a, f(a)+i) \in P$ pour certain $i \in J\}$. Si $\{0\}$ est un idéal radical de A et $J \subseteq Nilp(B)$, alors*

1. *Soit $P \subseteq 0 \times J$ soit $P = I_P \bowtie^f J$.*
2. *Si on suppose de plus que $\{(0,0)\}$ est un idéal radical de $A \bowtie^f J$, alors soit $P = 0 \times J$ soit $P = I_P \bowtie^f J$.*

Pour l'image homomorphique, on donne le résultat suivant.

Proposition 0.0.95 *Soit R un anneau, I un idéal de R et $\phi : \mathscr{I}(R) \to \mathscr{I}(R) \cup \{\emptyset\}$, $\psi : \mathscr{I}(R/I) \to \mathscr{I}(R/I) \cup \{\emptyset\}$ deux fonctions telles que ;*

$$\psi(J/I) = \begin{cases} \phi(J)/I & si \quad I \subseteq \phi(J) \\ \{\overline{0}\} & si \quad \phi(J) \neq \emptyset \text{ et } I \nsubseteq \phi(J) \\ \emptyset & si \quad \phi(J) = \emptyset \end{cases}$$

Soit P/I un idéal de R/I. Alors,

1. *Supposons que I est un idéal faiblement premier de R alors si P/I est un idéal ψ-premier de R/I alors P est un idéal ϕ-premier de R.*
2. *Si $I \subseteq \phi(P)$ alors P/I est un idéal ψ-premier de R/I si et seulement si P est un idéal ϕ-premier de R.*

5.3 La généralisation de la dimension de Krull

Dans cette section, On définit et étudie une nouvelle notion considérée comme une généralisation de la notion de dimension de Krull dans les anneaux commutatifs.

Définition 0.0.96 *Soit A un anneau et $\phi : \mathscr{I}(A) \to \mathscr{I}(A) \cup \{\emptyset\}$ une fonction. La ϕ-dimension de Krull de A (notée $\phi - dim(A)$) est le supremum des longueurs de toutes les chaines d'idéaux ϕ-premiers distincts de A.*

On commence avec le produit direct d'anneaux.

Théorème 0.0.97 *Soit $R = \prod_{i=1}^{n} R_i$ un produit direct d'anneaux et soit $\{\phi_i : \mathscr{I}(R_i) \to \mathscr{I}(R_i) \cup \{\emptyset\}\}_{i=1,\cdots,n}$ une famille de fonctions telles que pour tout idéal I_i de R_i, $\phi_i(I_i) \neq I_i$ $(i = 1, \cdots, n)$. On considère $\phi : \mathscr{I}(R) \to \mathscr{I}(R) \cup \{\emptyset\}$ une fonction telle que $\phi(\prod_{i=1}^{n} I_i) = \prod_{i=1}^{n} \phi_i(I_i)$ où I_i est un idéal de R_i pour tout $i = 1, \cdots, n$. Alors,*

$$\phi - dim(R) = \sup_{i=1,\cdots,n} \{\phi_i - dim(R_i)\}.$$

Comme corollaire nous avons le résultat connu sur la dimension de Krull du produit direct d'anneaux.

Corollaire 0.0.98 *Soit $R = \prod_{i=1}^{n} R_i$ un produit direct d'anneaux. Alors,*

$$dim(R) = \sup_{i=1,\cdots,n} \dim(R_i).$$

Pour l'image homomorphique, on a le résultat suivant.

Proposition 0.0.99 *Soit R un anneau, I un idéal de R et* $\phi : \mathscr{I}(A) \to \mathscr{I}(A) \cup \{\emptyset\}$, $\psi : \mathscr{I}(R/I) \to \mathscr{I}(R/I) \cup \{\emptyset\}$ *deux fonctions telles que :*

$$\psi(J/I) = \begin{cases} \phi(J)/I & si \quad \phi(J) \neq \emptyset \text{ et } I \subseteq \phi(J) \\ \{\overline{0}\} & si \quad \phi(J) \neq \emptyset \text{ et } I \nsubseteq \phi(J) \\ \emptyset & si \quad \phi(J) = \emptyset \end{cases}$$

Si une des deux conditions suivantes est vérifiées :

1. *I est un idéal faiblement premier de R.*
2. *Pour tout idéal* ψ*-premier* P/I *de* R/I *on a* $I \subseteq \phi(P)$.

Alors,

$$\psi - dim(R/I) \leq \phi - dim(R).$$

Corollaire 0.0.100 *Soit R un anneau et I un idéal de R.*

$$dim(R/I) \leq dim(R).$$

Maintenant, on traite le cas de l'extension triviale.

Théorème 0.0.101 *Soit A un anneau, E un A-module et* $\phi : \mathscr{I}(A) \to \mathscr{I}(A) \cup \{\emptyset\}$ *une fonction. Soit* $\psi : \mathscr{I}(A \propto E) \to \mathscr{I}(A \propto E) \cup \{\emptyset\}$ *une fonction définie par*

$$\psi(H) = \begin{cases} (\phi(I_H) \propto E) \cap H & si\ \phi(I_H) \neq \emptyset \\ \emptyset & si\ \phi(I_H) = \emptyset \end{cases}$$

où $I_H = \{a \in A \mid (a,e) \in H \text{ pour certain } e \in E\}$. *Alors,*

1. $\psi - dim(A \propto E) \geq \phi - dim(A)$.
2. *On suppose de plus que pour tout idéal* ψ*-premier P de* $A \propto E$, $\phi(I_P)$ *et* $\psi(P)$ *sont des idéaux semi-premiers alors,*

$$\psi - dim(A \propto E) = \phi - dim(A).$$

Si pour tout idéal ψ-premier P de $A \propto E$, $\phi(I_P) = \emptyset$ alors on aura le résultat connu suivant.

Corollaire 0.0.102 *[12, Theorem 3.2] Soit A un anneau, E un A-module. Alors,*

$$dim(A \propto E) = dim(A).$$

Pour l'amalgamation, on a le résultat suivant.

Théorème 0.0.103 *Soit A et B deux anneaux, $f : A \to B$ un homomorphisme d'anneaux et J un idéal de B. Soit $\phi : \mathscr{I}(A) \to \mathscr{I}(A) \cup \{\emptyset\}$ et $\psi : \mathscr{I}(A \bowtie^f J) \to \mathscr{I}(A \bowtie^f J) \cup \{\emptyset\}$ deux fonctions telles que :*

$$\psi(H) = \begin{cases} (\phi(I_H) \bowtie^f J) \cap H & si\ \phi(I_H) \neq \emptyset \\ \emptyset & si\ \phi(I_H) = \emptyset \end{cases}$$

où $I_H = \{a \in A | (a, f(a) + i) \in H$ pour certain $i \in J\}$. Alors,

1. $\psi - dim(A \bowtie^f J) \geq \phi - dim(A)$.
2. *Supposons que pour tout idéal ψ-premier P de $A \bowtie^f J$, $\phi(I_p)$ et $\psi(P)$ sont des idéaux radicaux et $J \subseteq Nilp(B)$ alors,*

$$\psi - dim(A \bowtie^f J) = \phi - dim(A).$$

Si pour tout idéal ψ-premier P de $A \bowtie^f J$, $\phi(I_P) = \emptyset$ et $J \subseteq Nilp(B)$, alors on a le résultat suivant.

Corollary 0.0.1 *Soit A et B deux anneaux, $f : A \to B$ un homomorphisme d'anneaux et J un idéal de B tel que $J \subseteq Nilp(B)$. Alors,*

$$dim(A \bowtie^f J) = dim(A).$$

Introduction

In this thesis, we study the transfer of some algebraic properties to various context of commutative rings such as amalgamation of rings, trivial ring extensions. Before describing our major contributions in each chapter, we first give some definitions, terminologies and notations used in the following chapters. Throughout this thesis, all rings considered are commutative with identity elements and all modules are unital.

Definition 0.0.2 Let R be a commutative ring and M be an R-module.

1. For a nonnegative integer n, M is called n-presented if there is an exact sequence of R-modules:

   $$F_n \to F_{n-1} \to \dots F_1 \to F_0 \to E \to 0$$

   where each F_i is a finitely generated free R-module.
   In particular, "0-presented" is equivalent to finitely generated and "1-presented" is equivalent to finitely presented.

2. M is called a coherent R-module if it is finitely generated and every finitely generated submodule of M is finitely presented.

Definition 0.0.3 Let R be a commutative ring. R is called coherent if every finitely generated ideal of R is finitely presented.

Definition 0.0.4 *An ideal I of R is called regular if it contains a regular element; i.e, a non zero-divisor element.*

Definition 0.0.5 *Let M be an R-module.*
The projective dimension of M is less than or equal to n if there is a projective resolution of the form:

$$0 \to P_n \to \cdots \to P_0 \to M \to 0$$

where all P_i are projective. We note $pd(M) \leq n$.
If there is no resolution, the projective dimension of M is infinite. We note $pd(M) = \infty$.
If there is a resolution and n is the smallest integer that satisfies such resolution, the projective dimension of M is n. We note $pd_R(M) = n$.

Definition 0.0.6 *Let R be a commutative ring. The dimension of a ring R, denoted by dimR, will be taken as its Krull dimension, is the maximum length n of a chain $P_0 \subset P_1 \cdots \subset P_n$ of prime ideals of R, with $p_i \neq p_{i+1}$ for all i.*

In 1956, Nagata introduced the notion of trivial ring extension of a ring A by a module E as follows :

Definition 0.0.7 *Let A be a ring and E an $A-$module. The trivial ring extension $R = A \propto E$ of A by E is the set of pairs (a,e) with $a \in A$ and $e \in E$ under coordinate-wise addition and adjusted multiplication defined by:*

$$(a,e)(b,f) = (ab, af + be).$$

This notion has been the subject of several works in the area, and it is a fertile field of research in the theory of rings (see [58, 68] and the reference therein). Trivial ring extensions have been studied extensively; and considerable work, part of is summarized in Glaz's book [58] and Huckaba's book [68], has been concerned with these extensions. Mainly, Trivial ring extensions have been useful for solving many open problems and conjectures in both commutative and non-commutative ring theory. See for instance [68].

Definition 0.0.8 *Let T be a ring and let M be an ideal of T and let π be the natural surjection $\pi : T \longrightarrow T/M$ and D be a subring of T/M. Then, $R := \pi^{-1}(D)$ is a subring of T and M is a common ideal of R and T, such that $D = R/M$. R is known by a pullback ring associated to the following pullback diagram:*

$$\begin{array}{ccc} R := \pi^{-1}(D) & \xrightarrow{\pi|R} & D = R/M \\ \downarrow i & & \downarrow j \\ T & \xrightarrow{\pi} & T/M \end{array}$$

where i and j are the natural injections.

As a particular case of this pullback is the $D+M$-construction, when the ring T is of the form $K+M$, where K is a field and M is a maximal ideal of T, and R has the form $D+M$. See for instance [22, 29, 58].

In 2006, M. D'Anna and M. Fontana [44] introduced a new construction, called amalgamated duplication of a ring A along an $A-$submodule E of $Q(A)$ (the total ring of fractions of A) such that $E^2 \subseteq E$. When $E^2 = \{0\}$, this construction coincides with the trivial ring extension of A by E. Motivations and more applications of the amalgamated duplication $A \bowtie E$ of A along an $A-$submodule E of $Q(A)$ are discussed in more details, especially in the particular case where E is an ideal of A, in recent papers, for instance, see [40, 41, 42, 43].
In 2010, D'Anna, Finocchiaro and Fontana [41] extended the notion of amalgamated duplication construction $A \bowtie I$ of a ring A along an ideal I of A to the general context of ring homomorphism extensions as follows:

Definition 0.0.9 *Let A and B be two rings with identity elements, J be an ideal of B and let $f : A \to B$ be a ring homomorphism. In this setting, we consider the following subring of $A \times B$; $A \bowtie^f J := \{(a, f(a)+j) \mid a \in A, j \in J\}$ called the amalgamation of A and B along J with respect to f.*

Other classical constructions (such as the $A+XB[X]$, $A+XB[[X]]$, and the $D+M$ constructions) can be studied as particular cases of the amalgamation (see [41, Examples 2.5 and 2.6]). A particular case of this construction is the amalgamated duplication of a ring along an ideal I (introduced and studied by D'Anna and Fontana in [41, 42, 43]). Let A be a ring, and let I be an ideal of A. $A \bowtie I := \{(a, a+i) : a \in A,\ i \in I\}$ is called the amalgamated duplication of A along the ideal I. See for instance [41, 42, 43, 44].

For a ring R. We use the following notations:
• $Z(R) := \{a \in R / ax := 0 \text{ for some } 0 \neq x \in R\}$, denotes the set of zero divisors of R.
• $Ann(M) := \{a \in R / ax := 0 \text{ for all } 0 \neq x \in M\}$ M. denotes the annihilator of
• $Q(R)$ denotes the total ring of quotients of R, that is, the localization of R by the set of all its non zero divisors.
• $qf(R)$ denotes the quotient field of R.
• $Nilp(R)$, the set of nilpotent elements of R.
• $Rad(R)$, the jacobson radical of R.
• $Spec(R)$, the set of prime ideals of R.
• $Max(R)$, the set of maximal ideals of R.
• $Reg(R)$, the set regular elements of R.

Summaries of Chapters

The thesis is divided into five chapters covering the following five papers [49, 50, 51, 52, 53]. In what follows we will review the major contributions of each chapter. Throughout this thesis, all rings considered are commutative with identity elements and all modules are unital.

Chapter 1 *

In this chapter, we study the class of rings in which every pure ideal is projective. We investigate the stability of this property under homomorphic image, and its transfer to various contexts of constructions such as pullbacks trivial ring extensions and amalgamation of rings. Our results generate examples which enrich the current literature with new and original families of rings that satisfy this property.

We start with the transfer of the *PIP*-proprety under homomorphic images, but first we give the following example concerning a particular case.

Example 0.0.10 *[66, Example 2.12.(1)] Let R be a Dedekind domain and I a nonzero ideal of R. Then $\frac{R}{I}$ is a PIP-ring.*

Theorem 0.0.11 *Let R be a ring.*

1. *If R is a PIP-ring then for every pure ideal I of R, $\frac{R}{I}$ is a PIP-ring.*
2. *If I is a finitely generated pure ideal of R included in all pure ideals of R, then R is a PIP-ring if and only if so is $\frac{R}{I}$.*
3. *Let I be an ideal such that $I \subseteq Rad(R)$. If $\frac{R}{I}$ is Noetherian then R is a PIP-ring.*

The proof of this theorem involves the following lemma.

Lemma 0.0.12 *A ring R is a PIP-ring if and only if every cyclic flat R-module has projective dimension at most* 1.

Remark 0.0.13 *By Theorem 0.0.11(3), we can prove by another way that Noetherian rings are PIP-rings.*

Theorem 0.0.14 *Let A be a ring and E an A-module, and let $R = A \propto E$ be the trivial ring extension of A by E.*

*Recall that in this chapter all rings are assumed to be commutative.

1. *If R is a PIP-ring, then so is A.*
2. *If all pure ideals of R are homogeneous and E is a flat A-module, then R is a PIP-ring if and only if so is A.*

The proof of this theorem involves the following lemmas.

Lemma 0.0.15 *Let A be a ring and E an A-module, and let $R = A \propto E$ be the trivial ring extension of A by E. Then*

1. *If $I \propto E'$ is a pure ideal of R, then I is a pure ideal of A and E' is a pure submodule of E and $E' = IE$.*
2. *I is a pure ideal of A if and only if $I \propto IE$ is a pure ideal of R.*
3. *Assume that I is a pure ideal of A, then $I \propto E'$ is a pure ideal of R if and only if $E' = IE$.*

Lemma 0.0.16 *Let A be a ring, E be an A-module and let $R = A \propto E$ be the trivial ring extension of A by E. If $I \propto IE$ (where I is a non-zero ideal of A) is a projective ideal of R, then I is a projective ideal of A. The converse is true if E is flat.*

Theorem 0.0.17 *Let $\varphi : R \longrightarrow T$ be an injective ring homomorphism and Q be an ideal of R such that $QT = Q$.*

1. *If R is a PIP-ring, then for every pure ideal J of R, the $\frac{R}{Q}$-module $\frac{J}{JQ}$ is projective.*
2. *If T is a PIP-ring R-flat, then R is a PIP-ring if and only if for every pure ideal J of R, the $\frac{R}{Q}$-module $\frac{J}{JQ}$ is projective.*

The proof of this theorem involves the following lemma.

Lemma 0.0.18 *[58, Theorem 5.1.1] Let $\varphi : R \longrightarrow T$ be an injective ring homomorphism satisfying the fact that there is an ideal Q of R such that $QT = Q$. Then, an R-module E is projectif if and only if $E \otimes_R T$ is a projective T-module and $\frac{E}{QE}$ is a projective $\frac{R}{Q}$-module.*

Application. Assume that T is a ring of the form $T = K + M$, where K is a field and M is a nonzero maximal ideal of T, D is a subring of K such that $qf(D) = K$, and $R = D + M$. Then, $T = S^{-1}R$ with $S = D \setminus 0$.
Since T is a local ring then T is a *PIP*-ring and it is R-flat (since $T = S^{-1}R$) and we obtain the following corollary.

Corollary 0.0.19 *Preserving the notation introduced above, R is a PIP-ring if and only if for every pure ideal J of R, the $\frac{R}{Q}$-module $\frac{J}{JQ}$ is projective.*

Theorem 0.0.20 *Let A and B be rings, J an ideal of B and let $f : A \to B$ be a ring homomorphism.*

1. *Assume that J is a flat ideal of B. If $A \bowtie^f J$ is a PIP-ring then so is A.*
2. *Assume that J is an idempotent ideal of B and $A \bowtie^f J$ is a multiplication ring. If $A \bowtie^f J$ is a PIP-ring then so is A.*
3. *Assume that A is a Noetherian ring and $0 \times J \subseteq Rad(A \bowtie^f J)$. Then, $A \bowtie^f J$ is a PIP-ring.*
4. *Assume that all pure ideals of $A \bowtie^f J$ are homogeneous. If J is a pure finitely generated ideal of B then $A \bowtie^f J$ is a PIP-ring if and only if so is A.*

The proof of this theorem involves the following series of lemmas.

Lemma 0.0.21 *Let A and B be two rings and $f : A \longrightarrow B$ be a ring homomorphism making B a faithfully flat A-module. Then,*

$$I \text{ is a pure ideal of } A \Leftrightarrow IB \text{ is a pure ideal of } B.$$

Lemma 0.0.22 *[72, Lemma 2.10] Let A and B be a rings, J an idempotent ideal of B and let $f : A \to B$ be a ring homomorphism. If K and L are ideals of A. Then, $(KL) \bowtie^f J = (K \bowtie^f J)(L \bowtie^f J)$ (is the product of the two ideals $K \bowtie^f J$ and $L \bowtie^f J$).*

Lemma 0.0.23 *[72, Proposition 2.8] Let A and B be a rings, J an idempotent ideal of B, $f : A \to B$ be a ring homomorphism and let I be a multiplication ideal of A. Then, every homogeneous ideal contained in $I \bowtie^f J$ is a multiple of $I \bowtie^f J$.*

Lemma 0.0.24 *Let A and B be rings, J an ideal of B, $f : A \to B$ be a ring homomorphism and let I be an ideal of A.*

1. *Assume that J is flat. Then, I is a pure ideal of A if and only if $I \bowtie^f (f(I)B)J$ is a pure ideal of $A \bowtie^f J$.*
2. *If $I \bowtie^f J$ is a pure ideal of $A \bowtie^f J$, then I is a pure ideal of A.*
3. *If J is an idempotent ideal of B and $I \bowtie^f J$ is an ideal of $A \bowtie^f J$ such that and all its subideals are homogeneous then, $I \bowtie^f J$ is a pure ideal of $A \bowtie^f J$ if and only if I is a pure ideal of A.*
4. *If J is an idempotent ideal of B and $A \bowtie^f J$ is a multiplication ring then $I \bowtie^f J$ is a pure ideal of $A \bowtie^f J$ if and only if I is a pure ideal of A.*

Lemma 0.0.25 *Let A and B be rings, J an ideal of B, $f : A \to B$ be a ring homomorphism and let I be an ideal of A. If $I \bowtie^f H$ is a projective ideal of $A \bowtie^f J$, where I is an ideal of A such that $f(I)J \subseteq H \subseteq J$, then I is a projective ideal of A. In particular, if $I \bowtie^f J$ (or $I \bowtie^f (f(I)B)J$) is a projective ideal of $A \bowtie^f J$ then I is a projective ideal of A.*

Example 0.0.26 *Let $A = \mathbb{Z}$, $B = \frac{\mathbb{Z}}{12\mathbb{Z}}$ and $J = \frac{4\mathbb{Z}}{12\mathbb{Z}}$ and let $f : A \to B$ be the canonical projection. Then, $A \bowtie^f J$ is a PIP-ring.*

Remark 0.0.27

1. *If R is a Von Neumann regular ring which is non-hereditary then R is a non-PIP-ring. In that case, even if R_P is a PIP-ring for every $P \in Spec(R)$ (since it is a field), R is not a PIP-ring.*
2. *If R is a ring where every pure ideal is finitely presented, then R is a PIP-ring (since every pure ideal is flat).*
3. *By the second remark, we can say that if R is a coherent ring such that every pure ideal is finitely generated, then R is PIP-ring.*

Example 0.0.28 *[66, Example 2.12 (3)] Let $R = \Pi_{p \in S}(\mathbb{Z}/p\mathbb{Z})$, where S is the set of all prime integers. Then R is a Von Neumann regular ring which is not hereditary by [79]. So R is not a PIP-ring and R_P is a PIP-ring for all $P \in Spec(R)$.*

We give an example of a *PIP*-ring which is not coherent.

Example 0.0.29 *Let $R := \mathbb{Z} \propto \mathbb{Q}[X]$ be the trivial extension ring of $\mathbb{Z}$ by the polynomial ring $\mathbb{Q}[X]$. Then*

1. *R is a PIP-ring, since it is an FF-ring by [48, Example 3.13].*
2. *R is not coherent by [69, Theorem 2.8 (1)]. In particular R is not Noetherian.*

Chapter 2 *

In this chapter, we study the class of rings in which every regular locally principal ideal is projective called *LPP*-rings. We investigate the transfer of this property to various constructions such as direct products, amalgamation of rings, and trivial ring extensions. Our aim is to provide examples of new classes of commutative rings satisfying the above-mentioned property.

*Recall that in this chapter all rings are assumed to be commutative.

Proposition 0.0.30 *Let R be a ring and let I be a regular ideal of R. Then, the following are equivalent:*

1. *I is locally principal.*
2. *I is faithfully flat.*
3. *I is a cancellation ideal.*

Remark 0.0.31

1. *Noetherian rings, hereditary rings and semi-simple rings (finite direct product of fields) are obviously examples of LPP-rings.*
2. *Let R be an Artinian ring. Each element of R is either invertible or a zero divisor. Such a ring only has one regular ideal, which is R. Therefore, Artinian rings are LPP-rings.*
3. *Let R be a ring. Then, R_M is always an LPP-ring independently of R for all $M \in Max(R)$ (since a principal regular ideal is free). But the converse is not true. Hence, local rings are LPP-rings.*

Theorem 0.0.32 *Let $(R_i)_{i=1,\cdots,n}$ be a family of commutative rings. Then $R = \prod_{i=1}^{n} R_i$ is an LPP-ring if and only if R_i is for each $i = 1, \cdots, n$.*

The proof of this theorem needs the following lemmas.

Lemma 0.0.33 *[73, Lemma 2.5] Let $(R_i)_{i=1,2}$ be two rings and E_i be an R_i-module for $i = 1, 2$. Then,*

$$pd_{R_1 \times R_2}(E_1 \times E_2) = sup\{pd_{R_1}(E_1), pd_{R_2}(E_2)\}.$$

Lemma 0.0.34 *Let $(R_i)_{i=1,2}$ be two rings and I_i be an ideal of R_i for $i = 1, 2$. Then, $I_1 \times I_2$ is a regular locally principal ideal of $R_1 \times R_2$ if and only if I_1 and I_2 are regular locally principal ideals of R_1 respectively R_2.*

Proposition 0.0.35 *Let R and S be two domains and $f : R \to S$ be a ring homomorphism making S a faithfully flat R-module. If S is an LPP-ring, then so is R.*

Theorem 0.0.36 *Let A be a ring, E be an A-module and let $R = A \propto E$ be the trivial ring extension of A by E and let $S = A \setminus (Z(A) \cup Z(E))$.*

1. *Assume that for every regular locally principal ideal I of A, $I \cap S \neq \emptyset$. Then, if R is an LPP-ring then so is A. In particular, if $Z(E) \subseteq Z(A)$ (for example we can consider E torsion free) then if R is an LPP-ring then so is A.*

2. *Assume that E is a divisible flat A-module ($E = S^{-1}E$ since E is divisible). Then,*

   (a) *If A is an LPP-ring then so is R.*

   (b) *Assume that for every regular locally principal ideal I of A, $I \cap S \neq \emptyset$ (in particular, if $Z(E) \subseteq Z(A)$), then R is an LPP-ring if and only if so is A.*

   (c) *If A is a domain, then R is an LPP-ring if and only if so is A.*

The proof of this theorem needs the following lemmas.

Lemma 0.0.37

1. *Let A be a ring, E an A-module and I an ideal of A.*

   (a) *If $I \propto IE$ is a regular locally principal ideal of $A \propto E$, then I is a regular locally principal ideal of A.*

   (b) *If $I \cap S \neq \emptyset$, then I is a regular locally principal ideal of A if and only if $I \propto IE$ is a regular locally principal ideal of $A \propto E$.*

   (c) *If E is torsion free, then I is a regular locally principal ideal of A if and only if $I \propto IE$ is a regular locally principal ideal of $A \propto E$.*

2. *Let $I \propto E'$ be an ideal of $A \propto E$ where I is an ideal of A, E' is a submodule of E, and $IE \subseteq E'$. If $I \propto E'$ is a regular locally principal ideal of $A \propto E$ then so is I.*

Lemma 0.0.38 *[4, Theorem 8] Let A be a ring, E be an A-module and I be an ideal of A. If $I \propto IE$ is a projective ideal of $A \propto E$, then I is projective. The converse is true if E is flat.*

Lemma 0.0.39 *[12, Theorem 3.9] Let A be a ring and E an A-module. Let $S = A \setminus (Z(A) \cup Z(E))$. Then the following conditions are equivalent.*

1. *Every regular ideal of $A \propto E$ has the form $I \propto E$ where I is an ideal of A with $I \cap S \neq \emptyset$.*

2. *Every regular ideal of $A \propto E$ is homogenous.*

3. *$E = S^{-1}E$.*

Example 0.0.40 *Let K be a field, $A = K[x_1, x_2, \cdots, x_n]$ be the polynomial ring in n variables over K and $E = qf(K[x_1, x_2, \cdots, x_n])$ the quotient field of A. Then:*

1. *$A \propto E$ is an LPP-ring since A is Noetherian (and so an LPP-ring) and by Theorem 0.0.36(2,c).*

2. *$A \propto E$ is not Noetherian since E is not finitely generated as A-module and by [12, Theorem 4.8].*

3. *$A \propto E$ is not zero-dimensional since A is not zero-dimensional.*

Theorem 0.0.41 *Let $f : R \to S$ be an injective ring homomorphism satisfying the property that there is an ideal Q of R such that $QS = Q$ and S is a flat R-module. Assume that S is an LPP-ring and $Reg(R) \subseteq Reg(S)$. R is an LPP-ring if and only if for every regular locally principal ideal J of R the R/Q-module J/JQ is projective.*

The proof of this theorem involves the following lemma.

Lemma 0.0.42 *[58, Theorem 5.1.1] Let $f : R \to S$ be an injective ring homomorphism satisfying the property that there is an ideal Q of R such that $QS = Q$. Then, an R-module E is projective if and only if $E \otimes_R S$ is a projective S-module and E/QE is a projective R/Q-module.*

Proposition 0.0.43 *Let A and B be two rings, $f : A \to B$ be a ring homomorphism and J be an ideal of B. Assume that J is a flat ideal of A and $f(Reg(A)) \subseteq Reg(B)$. Then, if $A \bowtie^f J$ is an LPP-ring then so is A.*

The proof of this theorem needs the following lemma.

Lemma 0.0.44 *Let A and B be rings, J an ideal of B, $f : A \to B$ be a ring homomorphism and let I be an ideal of A. If $I \bowtie^f H$ is a projective ideal of $A \bowtie^f J$, where I is an ideal of A such that $f(I)J \subseteq H \subseteq J$, then I is a projective ideal of A. In particular, if $I \bowtie^f J$ (or $I \bowtie^f (f(I)B)J$) is a projective ideal of $A \bowtie^f J$ then I is a projective ideal of A.*

Example 0.0.45 *Let I be a non-regular prime ideal of a ring A and let $B = A/I$. Then $p : A \to B$ verifies $p(Reg(A)) \subseteq Reg(B)$.*

Theorem 0.0.46 *Let A be a ring and I be an ideal of A.*

1. *Assume that I is a flat ideal of A then if $A \bowtie I$ is an LPP-ring then so is A.*

2. *Let I be a non-regular finitely generated flat ideal of A and $S = A \setminus Z(A)$ such that $S^{-1}I = I$. Assume that for every regular locally principal ideal J of A we have $JI = I$. Then, A is an LPP-ring if and only if so is $A \bowtie I$.*

The proof of this theorem needs the following lemmas.

Lemma 0.0.47 *Let A be a ring and I be an ideal of A. If $(a, a+i)$ is a regular element of $A \bowtie I$ then a is a regular element of A.*

Lemma 0.0.48 *Assume that I is a proper non-regular ideal of a ring A and* $S = A \setminus (Z(A))$. *The following are equivalent :*

1. *All regular ideals of* $A \bowtie I$ *are homogeneous .*
2. *For all* $a \in S$ *we have* $I = aI$ *which is equivalent to* $I = S^{-1}I$.

Chapter 3 *

In this chapter, we extend the *LPI* property (that is, every locally principal ideal in an integral domain is invertible) to rings with zero-divisors and we study the class of commutative rings in which every regular locally principal ideal is invertible called *LPI* rings. We investigate the stability of this property under homomorphic image, and its transfer to various contexts of constructions such as direct products, amalgamation of rings and trivial ring extensions. Our results generate examples which enrich the current literature with new and original families of rings that satisfy this property.

Proposition 0.0.1 *([13, Theorem 1]) Let R be a ring and let I be a regular ideal of R. The following conditions are equivalent:*

1. *I is locally principal.*
2. *I is faithfully flat.*
3. *I is a cancellation ideal.*

Theorem 0.0.2 *Let* $(R_i)_{i=1,\cdots,n}$ *be a family of commutative rings. Then* $R = \prod_{i=1}^{n} R_i$ *is an LPI ring if and only if so is* R_i *for each* $i = 1, \cdots, n$.

The proof of this theorem needs the following lemmas.

Lemma 0.0.3 *Let* $(R_i)_{i=1,2}$ *be two rings and* I_i *be an ideal of* R_i *for* $i = 1, 2$. *Then,* $I_1 \times I_2$ *is a regular locally principal ideal of* $R_1 \times R_2$ *if and only if* I_1 *and* I_2 *are regular locally principal ideals of* R_1 *respectively* R_2.

Lemma 0.0.4 *([68, Lemma 18.1]) Let I be a regular locally principal ideal of a ring A. Then, I is invertible if and only if it is finitely generated.*

*Recall that in this chapter all rings are assumed to be commutative.

Proposition 0.0.5 *Let R and S be two rings and $f : R \to S$ be a ring homomorphism making S a faithfully flat R-module. Assume that $f(Reg(R)) \subseteq Reg(S)$. If S is an LPI ring, then so is R. In particular, if R and S are two domains and $f : R \to S$ is a ring homomorphism making S a faithfully flat R-module then if S is an LPI domain, then so is R.*

Corollary 0.0.6 *([13, Theorem 5 (1)]) Let R be a domain and let X be an indeterminate over R. If $R[X]$ (resp.$R[[X]]$) is an LPI ring, then so is R.*

Proposition 0.0.7 *Let R be a ring.*

1. *Let P be a finitely generated non-regular prime ideal of R such that P is contained in all regular locally principal ideals of R. If $\frac{R}{P}$ is an LPI ring, then so is R.*

2. *Let I be a regular pure ideal of R. If R is an LPI ring, then $\frac{R}{I}$ is an LPI ring.*

The proof of this proposition needs the following lemma.

Lemma 0.0.8 *Let P be a non-regular prime ideal of a ring R.*
If J is a regular locally principal ideal of R such that $P \subseteq J$, then $\frac{J}{P}$ is a regular locally principal ideal of $\frac{R}{I}$.

Example 0.0.9 *Let K be a field and $R = K \propto K$ and let $P = 0 \propto K$. Then P is a finitely generated non-regular prime ideal of R that is contained in all regular ideals of R by Lemma 0.0.12. Since $\frac{R}{P} = \frac{K \propto K}{0 \propto K} = K$, $K \propto K$ is an LPI ring by Proposition 0.0.7.*

Theorem 0.0.10 *Let A be a ring, E an A-module, $R = A \propto E$ the trivial ring extension of A by E and let $S = A \setminus (Z(A) \cup Z(E))$.*

1. *Assume that for every regular locally principal ideal I of A, $I \cap S \neq \emptyset$. If R is an LPI ring, then so is A.*

2. *Assume that E is torsion-free and divisible. Then A is an LPI ring if and only if so is R. In particular, if A is a domain and E is a flat divisible R-module, then A is an LPI ring if and only if so is R.*

3. *Assume E is finitely generated and $E = S^{-1}E$. If A is an LPI ring, then so is R. If furthermore, for every regular locally principal ideal I of A, $I \cap S \neq \emptyset$, then A is an LPI ring if and only if so is R.*

The proof of this theorem needs the following lemmas.

Lemma 0.0.11 *1. Let A be a ring, E an A-module and I an ideal of A and let $S = A \setminus (Z(A) \cup Z(E))$.*

(a) *If $I \propto IE$ is a regular locally principal ideal of $A \propto E$, then I is a regular locally principal ideal of A.*

(b) *If $I \cap S \neq \emptyset$, then I is a regular locally principal ideal of A if and only if $I \propto IE$ is a regular locally principal ideal of $A \ltimes E$.*

(c) *If E is torsion free, then I is a regular locally principal ideal of A if and only if $I \propto IE$ is a regular locally principal ideal of $A \propto E$.*

2. *Let E' is a submodule of E. If $I \propto E'$ is a regular locally principal ideal of $A \propto E$, then so is I.*

Lemma 0.0.12 *([12, Theorem 3.9]) Let A be a ring, E an A-module and $S = A \setminus (Z(A) \cup Z(E))$. Then the following conditions are equivalent.*

1. *Every regular ideal of $A \propto E$ has the form $I \propto E$ where I is an ideal of A with $I \cap S \neq \emptyset$.*
2. *Every regular ideal of $A \propto E$ is homogenous.*
3. *$E = S^{-1}E$.*

Lemma 0.0.13 *([3, Theorem 7]) Let A be a ring, I a nonzero ideal of A and E an A-module. If $I \propto IE$ is an invertible ideal of $A \propto E$, then I is invertible.*

Lemma 0.0.14 *([4, Theorem 11]) Let A be a ring, E be a torsion-free and divisible A-module and $I \propto N$ a homogenous ideal of $A \propto E$. If I is an invertible ideal of A, then $I \propto N$ is invertible.*

Lemma 0.0.15 *([3, Theorem 9]) Let A be a ring, E an A-module, $I \propto N$ an homogeneous ideal of $A \propto E$. If I is a finitely generated ideal of A and N is a finitely generated submodule of E, then $I \propto N$ is finitely generated.*

Corollary 0.0.16 *Let A is a domain and E a K-vector space where $K = qf(A)$. Then $A \propto E$ is an LPI ring if and only if so is A.*

Example 0.0.17 *Let $A = \mathbb{Z}_{(2)} = \{\frac{a}{b} \mid a,b \in \mathbb{Z}$ and b is not divisible by $2\}$ and $E = \frac{\mathbb{Q}}{\mathbb{Z}_{(2)}}$. Then $A \propto E$ is an LPI ring.*

Example 0.0.18 *Let K be a field and E be an infinite dimensional vector space over K. Then :*

1. *$K \propto E$ is an LPI ring (by Corollary 0.0.16).*
2. *$K \propto E$ is not Noetherian (since E is not finitely generated and by [12, Theorem 4.8]).*

Theorem 0.0.19 *Let A be a ring and I be an ideal of A.*

1. *Assume that I is flat. If $A \bowtie I$ is an LPI ring, then so is A.*
2. *Assume that I is a non-regular finitely generated ideal of A and $S^{-1}I = I$ where $S = A \setminus Z(A)$. If A is an LPI ring, then so is $A \bowtie I$. If furthermore I is flat, then A is an LPI ring if and only if so is $A \bowtie I$.*

The proof of this theorem needs the following lemmas. .

Lemma 0.0.20 *Let A be a ring and I be an ideal of A. If $(a, a+i)$ is a regular element of $A \bowtie I$, then a is a regular element of A.*

Lemma 0.0.21 *Assume that I is a non-regular proper ideal of a ring A and $S = A \setminus (Z(A))$. The following are equivalent:*

1. *All regular ideals of $A \bowtie I$ are homogeneous.*
2. *For all $a \in S$ we have $I = aI$ which is equivalent to $I = S^{-1}I$.*

Lemma 0.0.22 *Let I and J be two ideals of a ring A. If I and J are finitely generated, then $J \bowtie I$ is a finitely generated ideal of $A \bowtie I$.*

Example 0.0.23 *Let $A = K \ltimes E$ where K is a field and E is an infinite dimensional vector space over K and let $I = 0 \ltimes F$ where F is a finite dimensional subspace of E . Then :*

1. *$A \bowtie I$ is an LPI ring (by Theorem 0.0.19(2)).*
2. *$A \bowtie I$ is not Noetherian (since A is not Noetherian by Example 0.0.18 and by [44, Corollary 2.9]).*

Chapter 4 *

In this chapter, we study the class of rings with adequate range one and investigate the transfer of this property to various contexts of constructions such as pullbacks, trivial ring extensions and amalgamation of rings. Our results provide new classes of commutative rings satisfying this property.

Proposition 0.0.49 *Let R be a ring. The following conditions are equivalent:*

*Recall that in this chapter all rings are assumed to be commutative.

1. *R satisfies adequate range one;*

2. *$\forall a,x,b \in R$ such that $ax+b=1$ there exists $y \in R$ such that $a+by$ is adequate.*

Theorem 0.0.50 *Let R be a ring and I be an ideal of R. Assume that $\forall x \in R \setminus I$, $I = xI$. Then,*

1. *If R has adequate range one then so does $\frac{R}{I}$.*

2. *Assume that all elements of I are adequate. Then, R has adequate range one if and only if so does $\frac{R}{I}$.*

The proof of this theorem involves the following lemma.

Lemma 0.0.51 *Let R be a ring and I be an ideal of R. Assume that $\forall x \in R \setminus I$, $I = xI$. Then, a is an adequate element of $R \setminus I$ if and only if $\overline{a}$ is an adequate element of $\frac{R}{I} \setminus 0$.*

Corollary 0.0.52 *Let R be a ring and P be a prime ideal of R and $S = R \setminus P$. Assume that $S^{-1}P = P$. Then,*

1. *If R has adequate range one then so does $\frac{R}{P}$.*

2. *Assume that all elements of P are adequate. Then, R has adequate range one if and only if so does $\frac{R}{P}$.*

Theorem 0.0.53 *Let A be a ring and, E be an A-module, and let $R = A \propto E$ be the trivial ring extension of A by E. Assume that for every $k \in A \setminus \{0\}$, $kE = E$. Then, R has adequate range one if and only if so does A.*

The proof of this theorem involves the following lemma.

Lemma 0.0.54
Let A be a ring and, E be an A-module, and let $R = A \propto E$ be the trivial ring extension of A by E. Then,

1. *Let $a,b,c \in A$ such that $bE = E$ and let $x,y \in E$. Then, the following conditions are equivalent:*

   (a) $\exists z \in E$ such that $(a,x) = (b,y)(c,z)$;

   (b) $a = bc$.

2. *Let $a,b \in A$ such that $bE = E$ and let $x,y \in E$. Then, $(b,y)|(a,x)$ if and only if $b|a$.*

3. *Let $a,b \in A$ such that $bE = E$ and let $x,y \in E$. Then, $(a,x)R + (b,y)R = R$ if and only if $aA + bA = A$.*

Corollary 0.0.55 *Let A be an integral domain, E a divisible A-module and $R = A \propto E$ be the trivial ring extension of A by E. Then R has adequate range one if and only if so does A.*
In particular, if $K := qf(A)$ is the quotient field of A and E is a K-vector space. Then R has adequate range one if and only if so does A.

Remark 0.0.56 *A local ring is an adequate ring by [75, Theorem 2.1] and so local rings have adequate range one.*

Example 0.0.57 *Let $A = \mathbb{Z}_{(2)} = \{\frac{a}{b} \mid a,b \in \mathbb{Z}$ and b is not divisible by $2\}$ and $E = \frac{\mathbb{Q}}{\mathbb{Z}_{(2)}}$. Then, $A \propto E$ has adequate range one.*

Corollary 0.0.58 *Let R be a ring, $\varphi : R \longrightarrow T$ be an injective ring homomorphism, and Q be an ideal of R such that $QT = Q$. Assume that $\forall x \in T \setminus Q$, $Q = xQ$. Then,*

1. *If R (resp. T) has adequate range one then so does $\frac{R}{Q}$ (resp $\frac{T}{Q}$).*
2. *Assume that all elements of Q are adequate. Then, R (resp T) has adequate range one if and only if so does $\frac{R}{Q}$ (resp $\frac{T}{Q}$).*

Theorem 0.0.59 *Let A and B be rings, J be a propre ideal of B, and let $f : A \to B$ be a ring homomorphism.*

1. *If $A \bowtie^f J$ has adequate range one, then so does A.*
2. *Assume that all elements of J are adequate and $\forall x \in (f(A)+J) \setminus J$, $xJ = J$. Then, $A \bowtie^f J$ has adequate range one if and only if so does A.*

Example 0.0.60 *Let A be the localization of the ring of integers by the maximal ideals $2\mathbb{Z}$ and $3\mathbb{Z}$, $A = \{\frac{a}{b} \in \mathbb{Q} |$ b is neither divisible by 2 nor by $3\}$ and $B = \dfrac{\mathbb{R}[[X]]}{(X^2+1)^4\mathbb{R}[[X]]}$ and let $J = \dfrac{(X^2+1)^2\mathbb{R}[[X]]}{(X^2+1)^4\mathbb{R}[[X]]}$ be an ideal of B, and*

$$\begin{array}{rccc} f: & A & \to & B \\ & a & \to & f(a) = \overline{a} \end{array}$$

1. *$A \bowtie^f J$ has adequate range one.*
2. *$A \bowtie^f J$ is not an adequate ring.*

Chapter 5 *

In this chapter, we define the ϕ-Krull dimension as a generalization of Krull dimension. This property is treated here as a part of a study on some constractions of rings such as direct products, amalgamation of rings, and trivial ring extensions.

The generalization of prime ideals in Trivial ring extension and amalgamation of rings

Proposition 0.0.61 *[19, Proposition 2.7] Let A be a ring, E an A-module, and $A \propto E$ a homogeneous ring. Let $\phi : \mathscr{I}(A) \to \mathscr{I}(A) \cup \{\emptyset\}$ and $\psi : \mathscr{I}(A \propto E) \to \mathscr{I}(A \propto E) \cup \{\emptyset\}$ be two functions such that $\psi(I \propto N) = \phi(I) \propto N$. Then*

1. *If $Q = P \propto N$ is a ψ-prime ideal of $A \propto E$, then P is a ϕ-prime ideal of A.*
2. *P is a ϕ-prime ideal of A if and only if $P \propto E$ is a ψ-prime ideal of $A \propto E$.*

Proposition 0.0.62 *Let A be a ring, E an A-module and $\phi : \mathscr{I}(A) \to \mathscr{I}(A) \cup \{\emptyset\}$ a function. Let N be a submodule of E and $\psi : \mathscr{I}(A \propto E) \to \mathscr{I}(A \propto E) \cup \{\emptyset\}$ be a function defined by :*

$$\psi(H) = \begin{cases} (\phi(I_H) \propto N) \cap H & \text{if } \phi(I_H) \neq \emptyset \\ \emptyset & \text{if } \phi(I_H) = \emptyset \end{cases}$$

where $I_H = \{a \in A \mid (a,e) \in H\}$. Let F be a submodule of E and $J = 0 \propto F$. Assume that $\psi(J) \neq \emptyset$ then,

1. *If $K = N \cap F \neq F$ and there exist $a, b \in A \setminus \{0\}$ such that $ab = 0$ and $f \in F \setminus K$ such that $af \in F \setminus K$ then J is not ψ-prime.*
2. *If $K = N \cap F = F$ then J is ψ-prime.*
3. *If A is a domain, then the following statements are equivalent :*
   (a) *$0 \propto F$ is ψ-prime.*
   (b) *$\forall a \in A \setminus \{0\}$, $\forall e \in E$, $ae \in F \setminus K$ implies that $e \in F \setminus K$.*

If $\psi(J) = \emptyset$, then $0 \propto F$ is ψ-prime if and only if $\{0\}$ is φ-prime and $F = E$.

*Recall that in this chapter all rings are assumed to be commutative.

Theorem 0.0.63 *Let A be a ring, E an A-module and $\phi : \mathscr{I}(A) \to \mathscr{I}(A) \cup \{\emptyset\}$ a function. Let N be a submodule of E and $\psi : \mathscr{I}(A \propto E) \to \mathscr{I}(A \propto E) \cup \{\emptyset\}$ be a function defined by :*

$$\psi(H) = \begin{cases} (\phi(I_H) \propto N) \cap H & \text{if } \phi(I_H) \neq \emptyset \\ \emptyset & \text{if } \phi(I_H) = \emptyset \end{cases}$$

where $I_H = \{a \in A \mid (a,e) \in H\}$. Let J be an ideal of $A \propto E$ such that $\psi(J) \neq \emptyset$. If J is a ψ-prime ideal of $A \propto E$ such that $\phi(I_J)$ is a semiprime ideal of A, then

1. *either $J \subseteq \phi(I_J) \propto E$ or $J = I_J \propto E$.*
2. *If we suppose further more that $\psi(J)$ is semiprime then either $J = \phi(I_J) \propto E$ or $J = I_J \propto E$.*

If $\psi(J) = \emptyset$ then $J = P \propto E$ where P is a ϕ-prime ideal of A.

The proof of this theorem needs the following lemma.

Lemma 0.0.64 *Let A be a ring and $\phi : \mathscr{I}(A) \to \mathscr{I}(A) \cup \{\emptyset\}$ a function and let I be a ϕ-prime ideal of A such that $\phi(I) \neq \emptyset$. Suppose that a is an unbreakble zero element of I. Then, $(a+i)^2 \in \phi(I)$ and $(a-i)^2 \in \phi(I)$ for all $i \in I$.*

Corollary 0.0.65 *Let A be a ring, E an A-module and J a weakly prime ideal of $A \propto E$ and $I_J = \{a \in A \mid (a,e) \in J \text{ for some } e \in E\}$. If $\{0\}$ is a semiprime ideal of A, Then*

1. *either $J \subseteq 0 \propto E$ or $J = I_J \propto E$.*
2. *If we suppose further more that $\{(0,0)\}$ is a semiprime ideal of $A \propto E$ then either $J = 0 \propto E$ or $J = I_J \propto E$.*

Proposition 0.0.66 *Let A and B be two rings, $f : A \to B$ be a ring homomorphism and J an ideal of B. Let $\phi : \mathscr{I}(A) \to \mathscr{I}(A) \cup \{\emptyset\}$ and $\psi : \mathscr{I}(A \bowtie^f J) \to \mathscr{I}(A \bowtie^f J) \cup \{\emptyset\}$ be two functions such that*

$$\psi(P \bowtie^f J) = \begin{cases} \phi(P) \bowtie^f K & \text{if } \phi(P) \neq \emptyset \\ \emptyset & \text{if } \phi(P) = \emptyset \end{cases}$$

where P is an ideal of A and K a subideal of J. Then, $P \bowtie^f J$ is ψ-prime if and only if P is ϕ-prime and for all $a, b \notin P$ such that $ab \in \phi(P)$ we have $(f(a)j + f(b)i + ij) \in K$ for all $i, j \in J$.

Proposition 0.0.67 *Let A and B be two rings, $f : A \to B$ be a ring homomorphism and J an ideal of B. Let $\phi : \mathscr{I}(A) \to \mathscr{I}(A) \cup \{\emptyset\}$, $\varphi : \mathscr{I}(f(A)+J) \to \mathscr{I}(f(A)+J) \cup \{\emptyset\}$ and $\psi : \mathscr{I}(A \bowtie^f J) \to \mathscr{I}(A \bowtie^f J) \cup \{\emptyset\}$ be three functions such that*

$$\varphi(H) = \emptyset \Leftrightarrow \phi(I_H) = \emptyset$$

$$\psi(\overline{H}^f) = \begin{cases} \{(a, f(a)+j) \mid a \in \phi(I_H) \text{ and } f(a)+j \in \varphi(H)\} & \text{if } \varphi(H) \neq \emptyset \\ \emptyset & \text{if } \varphi(H) = \emptyset \end{cases}$$

for every ideal H of $f(A)+J$, where $I_H = \{a \in A \mid (a, f(a)+j) \in \overline{H}^f \text{ for some } j \in J\}$. Then, $\overline{H}^f$ is a ψ-prime ideal if and only if H is a φ-prime ideal of $f(A)+J$ and for all $(f(a)+i), (f(b)+j) \notin H$ such that $(f(a)+i)(f(b)+j) \in \varphi(H)$ we have $ab \in \phi(I_H)$.

Theorem 0.0.68 *Let A and B be two rings, $f : A \to B$ be a ring homomorphism and J an ideal of B. Let $\phi : \mathscr{I}(A) \to \mathscr{I}(A) \cup \{\emptyset\}$ and $\psi : \mathscr{I}(A \bowtie^f J) \to \mathscr{I}(A \bowtie^f J) \cup \{\emptyset\}$ be two functions such that*

$$\psi(H) = \begin{cases} (\phi(I_H) \bowtie^f K) \cap H & \text{if } \phi(I_H) \neq \emptyset \\ \emptyset & \text{if } \phi(I_H) = \emptyset \end{cases}$$

where $I_H = \{a \in A | (a, f(a)+i) \in H \text{ for some } i \in J\}$ and K is a subideal of J. Let P be a ψ-prime ideal of $A \bowtie^f J$ and $J \subseteq Nilp(B)$.
(I) If $\phi(I_P) \neq \emptyset$ is a radical ideal then,

1. *either $P \subseteq \phi(I_P) \times J$ or $P = I_P \bowtie^f J$.*
2. *If we suppose further more that $\psi(P)$ is a radical ideal of $A \bowtie^f J$, then either $P = \phi(I_P) \times J$ (necessarily in that case $f(\phi(I_P)) \subseteq J$) or $P = I_P \bowtie^f J$.*

(II) If $\phi(P) = \emptyset$. Then $P = H \bowtie^f J$ where H is a ϕ-prime ideal of A.

The proof of this theorem needs the following lemma.

Lemma 0.0.69 *Let A be a ring and $\phi : \mathscr{I}(A) \to \mathscr{I}(A) \cup \{\emptyset\}$ a function and let I be a ϕ-prime ideal of A such that $\phi(I) \neq \emptyset$. Let a be an element of A such that $a \notin I$ and $a^n = 0$ for some integer $n \geq 2$. Then, $(a+i)^n \in \phi(I)$ and $(a-i)^n \in \phi(I)$ for all $i \in I$.*

Corollary 0.0.70 *Let A and B be two rings, $f : A \to B$ be a ring homomorphism and J an ideal of B. Let P be a weakly prime ideal of $A \bowtie^f J$ and $I_P = \{a \in A | (a, f(a)+i) \in P \text{ for some } i \in J\}$. If $\{0\}$ is a radical ideal of A and $J \subseteq Nilp(B)$, then*

1. *either $P \subseteq 0 \times J$ or $P = I_P \bowtie^f J$.*
2. *If we suppose further more that $\{(0,0)\}$ is a radical ideal of $A \bowtie^f J$, then either $P = 0 \times J$ or $P = I_P \bowtie^f J$.*

Proposition 0.0.71 *Let R be a ring, I an ideal of R and $\phi : \mathscr{I}(R) \to \mathscr{I}(R) \cup \{\emptyset\}$, $\psi : \mathscr{I}(R/I) \to \mathscr{I}(R/I) \cup \{\emptyset\}$ two functions such that*

$$\psi(J/I) = \begin{cases} \phi(J)/I & \text{if} \quad I \subseteq \phi(J) \\ \{\overline{0}\} & \text{if} \quad \phi(J) \neq \emptyset \text{ and } I \nsubseteq \phi(J) \\ \emptyset & \text{if} \quad \phi(J) = \emptyset \end{cases}$$

Let P/I be an ideal of R/I. Then,

1. *Assume that I is a weakly prime ideal of R then if P/I is a ψ-prime ideal of R/I then P is a ϕ-prime ideal of R.*

2. *If $I \subseteq \phi(P)$ then P/I is a ψ-prime ideal of R/I if and only if P is a ϕ-prime ideal of R.*

The generalization of Krull dimension

Definition 0.0.72 *Let A be a ring and $\phi : \mathscr{I}(A) \to \mathscr{I}(A)$ a function. The ϕ-Krull dimension of A (denoted by $\phi - dim(A)$) is the supremum of the lengths of all chains of distincts ϕ-prime ideals of A.*

Theorem 0.0.73 *Let $R = \prod_{i=1}^{n} R_i$ be a direct product of rings and let $\{\phi_i : \mathscr{I}(R_i) \to \mathscr{I}(R_i) \cup \{\emptyset\}\}_{i=1,\cdots,n}$ be a family of functions such that for all ideals I_i of R_i, $\phi_i(I_i) \neq I_i$ $(i = 1, \cdots, n)$. We consider $\phi : \mathscr{I}(R) \to \mathscr{I}(R) \cup \{\emptyset\}$ a function such that $\phi(\prod_{i=1}^{n} I_i) = \prod_{i=1}^{n} \phi_i(I_i)$ where I_i is an ideal of R_i for each $i = 1, \cdots, n$. Then,*

$$\phi - dim(R) = \sup_{i=1,\cdots,n} \{\phi_i - dim(R_i)\}.$$

Corollary 0.0.74 *Let $R = \prod_{i=1}^{n} R_i$ be a direct product of rings. Then,*

$$dim(R) = \sup_{i=1,\cdots,n} \dim(R_i).$$

Proposition 0.0.75 *Let R be a ring, I an ideal of R and $\phi : \mathscr{I}(R) \to \mathscr{I}(R) \cup \{\emptyset\}$, $\psi : \mathscr{I}(R/I) \to \mathscr{I}(R/I) \cup \{\emptyset\}$ two functions such that*

$$\psi(J/I) = \begin{cases} \phi(J)/I & \text{if} \quad \phi(J) \neq \emptyset \text{ and } I \subseteq \phi(J) \\ \{\overline{0}\} & \text{if} \quad \phi(J) \neq \emptyset \text{ and } I \nsubseteq \phi(J) \\ \emptyset & \text{if} \quad \phi(J) = \emptyset \end{cases}$$

If one of the two following conditions is verified :

1. *I is a weakly prime ideal of R.*

2. *for all ψ-prime ideals P/I of R/I we have $I \subseteq \phi(P)$.*

Then,

$$\psi - dim(R/I) \leq \phi - dim(R).$$

Corollary 0.0.76 *Let R be a ring and I an ideal of R.*

$$dim(R/I) \leq dim(R).$$

Theorem 0.0.77 *Let A be a ring, E an A-module and $\phi : \mathscr{I}(A) \to \mathscr{I}(A) \cup \{\emptyset\}$ a function. Let $\psi : \mathscr{I}(A \propto E) \to \mathscr{I}(A \propto E) \cup \{\emptyset\}$ be a function defined by*

$$\psi(H) = \begin{cases} (\phi(I_H) \propto E) \cap H & \text{if } \phi(I_H) \neq \emptyset \\ \emptyset & \text{if } \phi(I_H) = \emptyset \end{cases}$$

where $I_H = \{a \in A \mid (a,e) \in H \text{ for some } e \in E\}$. Then,

1. *$\psi - dim(A \propto E) \geq \phi - dim(A)$.*
2. *We suppose further more that for any ψ-prime ideal P of $A \propto E$, $\phi(I_P) \neq \emptyset$ and $\psi(P) \neq \emptyset$ are semiprime ideals, then*

$$\psi - dim(A \propto E) = \phi - dim(A).$$

Corollary 0.0.78 *[12, Theorem 3.2] Let A be a ring, E an A-module. Then,*

$$dim(A \propto E) = dim(A).$$

Theorem 0.0.79 *Let A and B be two rings, $f : A \to B$ be a ring homomorphism and J an ideal of B. Let $\phi : \mathscr{I}(A) \to \mathscr{I}(A) \cup \{\emptyset\}$ and $\psi : \mathscr{I}(A \bowtie^f J) \to \mathscr{I}(A \bowtie^f J) \cup \{\emptyset\}$ be two functions such that*

$$\psi(H) = \begin{cases} (\phi(I_H) \bowtie^f J) \cap H & \text{if } \phi(I_H) \neq \emptyset \\ \emptyset & \text{if } \phi(I_H) = \emptyset \end{cases}$$

where $I_H = \{a \in A | (a, f(a) + i) \in H \text{ for some } i \in J\}$. Then,

1. *$\psi - dim(A \bowtie^f J) \geq \phi - dim(A)$.*
2. *Suppose that for any ψ-prime ideal P of $A \bowtie^f J$, $\phi(I_p) \neq \emptyset$ and $\psi(P) \neq \emptyset$ are radical ideals and $J \subseteq Nilp(B)$ then,*

$$\psi - dim(A \bowtie^f J) = \phi - dim(A).$$

Corollary 0.0.80 *Let A and B be two rings, $f : A \to B$ be a ring homomorphism and J an ideal of B such that $J \subseteq Nilp(B)$. Then,*

$$dim(A \bowtie^f J) = dim(A).$$

CHAPTER 1

COMMUTATIVE RINGS IN WHICH EVERY PURE IDEAL IS PROJECTIVE

Abstract

In this chapter*, we study the class of rings in which every pure ideal is projective. We investigate the stability of this property under homomorphic image, and its transfer to various contexts of constructions such as pullbacks trivial ring extensions and amalgamation of rings. Our results generate examples which enrich the current literature with new and original families of rings that satisfy this property.

Key Words: *PIP*-ring, trivial extension, pullback, amalgamation of rings, pure ideal, projective ideal.

1.1 Introduction

Let R be a ring and I be an ideal of R. We say that I is a pure ideal of R if the sequence $0 \longrightarrow I \otimes_R L \longrightarrow R \otimes_R L \longrightarrow \frac{R}{I} \otimes_R L \longrightarrow 0$ is exact for any R-module L; equivalently $I_M = 0$ or $I_M = R_M$ for any maximal ideal M of R. In [58, Theorem 1.2.15], an ideal I of R is pure if and only if R/I is flat, which is also equivalent to $I \cap J = IJ$ for any ideal J of R. Consequently, any pure ideal I of R is flat, see for instance [58, Theorem 1.2.15].

In this chapter, we are interested in those rings in which every pure ideal is projective and which will be called *PIP*-rings.
In 2016, the authors of [66] study the *PIP*-property on an arbitrary ring and showed that domains and local rings are *PIP*-rings. Also they showed that F-rings (i.e., rings in which every pure ideal is generated by one idempotent), perfect rings and Noetherian rings are

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also PIP-rings [66, Corollary 2.6]. FF-rings (i.e., rings in which every flat ideal is finitely generated), and FP-rings (i.e., rings in which every flat ideal is projective) are also PIP-rings (since any FF-ring is an FP-ring by [38, Proposition 2.2.]). And in [66, Corollary 2.7] they prove that:

$$R \text{ is a hereditary ring} \Rightarrow R \text{ is an } FP\text{-ring} \Rightarrow R \text{ is a } PIP\text{-ring},$$

and if R is a von Neumann regular ring, then all the above conditions are equivalent.

The purpose of this study is to give some simple methods in order to construct PIP-rings. For this, we investigate the stability of the PIP-property under homomorphic image, and its transfer to various contexts of constructions such as pullbacks, trivial ring extensions and amalgamation of rings.

1.2 Main Results

We start with the transfer of the PIP-proprety under homomorphic images, but first we give the following example concerning a particular case.

Example 1.2.1 *[66, Example 2.12.(1)] Let R be a Dedekind domain and I a nonzero ideal of R. Then $\frac{R}{I}$ is a PIP-ring.*

Theorem 1.2.2 *Let R be a ring.*

1. *If R is a PIP-ring then for every pure ideal I of R, $\frac{R}{I}$ is a PIP-ring.*
2. *If I is a finitely generated pure ideal of R included in all pure ideals of R, then R is a PIP-ring if and only if so is $\frac{R}{I}$.*
3. *Let I be an ideal such that $I \subseteq Rad(R)$. If $\frac{R}{I}$ is Noetherian then R is a PIP-ring.*

The proof of this theorem involves the following lemma.

Lemma 1.2.3 *A ring R is a PIP-ring if and only if every cyclic flat R-module has projective dimension at most* 1.

Proof. By [66, Theorem 2.5].

Proof of Theorem 1.2.2. (1) Let $\frac{J}{I}$ be a pure ideal of $\frac{R}{I}$. $\dfrac{R/I}{J/I}$ is a cyclic flat $\frac{R}{I}$-module and since $\frac{R}{I}$ is a flat R-module (I pure), then $\dfrac{R/I}{J/I} \cong \dfrac{R}{J}$ is a cyclic flat R-module. As R is a *PIP*-ring then by Lemma 1.2.3, $pd_R(\frac{R}{J}) \leq 1$. Hence, J is a projective ideal of R. We have $\frac{J}{I} \cong J \otimes_R \frac{R}{I}$ (because I is pure, then $I = I \cap J = IJ$). Since J is a projective ideal of R, then $J \otimes_R \frac{R}{I}$ is a projective $\frac{R}{I}$-module. Therefore, $\frac{J}{I}$ is a projective ideal of $\frac{R}{I}$. So, $\frac{R}{I}$ is a *PIP*-ring.

(2) The first implication comes from (1).
Assume that $\frac{R}{I}$ is a *PIP*-ring. Let J be a pure ideal of R. Then, $\frac{J}{I}$ is a pure ideal of $\frac{R}{I}$. Indeed, let M be a maximal ideal of R such that $I \subseteq M$ and $\overline{M} = \frac{M}{I}$. So,

$$\left(\frac{J}{I}\right)_{\overline{M}} \cong \frac{J_M}{I_M} = \begin{cases} \frac{R_M}{I_M} \cong (\frac{R}{I})_{\overline{M}} \\ \quad or \\ \frac{0_M}{I_M} \cong 0_{\overline{M}} \end{cases}$$

Hence, $\frac{J}{I}$ is a pure ideal of $\frac{R}{I}$ and so it is projective (since $\frac{R}{I}$ is a *PIP*-ring). Then, there exists K an $\frac{R}{I}$-module such that

$$\tfrac{J}{I} \oplus K = \left(\tfrac{R}{I}\right)^{(L)}$$

where L is an index set.
In the other hand, I is a finitely generated pure ideal of R if and only if $\frac{R}{I}$ is a projective R-module (because $\frac{R}{I}$ is a projective R-module if and only if I is pure and $\frac{R}{I}$ is finitely presented and we have $\frac{R}{I}$ is finitely presented is equivalent to I is finitely generated). Then, there exists F an R-module such that

$$\tfrac{R}{I} \oplus F = R^{(H)}$$

where H is an index set. So,

$$(R^{(H)})^{(L)} = (\tfrac{R}{I} \oplus F)^{(L)} = \left(\tfrac{R}{I}\right)^{(L)} \oplus (F)^{(L)} = \tfrac{J}{I} \oplus K \oplus (F)^{(L)}.$$

Therefore, $\frac{J}{I}$ is a projective R-module.
Since $\frac{R}{I}$ is R-projective then I is also R-projective (by the exact sequence $0 \to I \to R \to \frac{R}{I} \to 0$). So, $J = I \oplus \frac{J}{I}$ is a projective R-module (by the exact sequence $0 \to I \to J \to \frac{J}{I} \to 0$ and since I and $\frac{J}{I}$ are R-projective). Hence, R is a *PIP*-ring.

(3) Assume that $\frac{R}{I}$ is a Noetherian ring and $I \subseteq Rad(R)$. Let J be a pure ideal of R. We have $J \otimes \frac{R}{I} = \frac{J}{IJ} = \frac{J}{I \cap J} = \frac{J+I}{I}$. Since $\frac{J}{IJ} = \frac{J+I}{I}$ is an ideal of $\frac{R}{I}$ then it is finitely generated and by the Nakayama's Lemma J is finitely generated. Therefore, $\frac{R}{J}$ is projective (since J is pure and finitely generated) and so J is projective. Hence, R is a *PIP*-ring.

Remark 1.2.4 *By Theorem 1.2.2(3), we can prove by another way that Noetherian rings are PIP-rings.*

Next we study the transfer of the *PIP*-property to the trivial ring extension of A by E.

Let A be a ring and E an R-module. In [39], a submodule E' of E is called a pure submodule if the sequence $0 \longrightarrow E' \otimes M \longrightarrow E \otimes M$ is exact for every A-module M. It is well known that if $\frac{E}{E'}$ is flat then E' is pure in E, [55]. It is proved in [55, Corollaries 11.21 and 11.23] that if E is a flat module then for any submodule E' of E the following are equivalent:

1. $\frac{E}{E'}$ is flat.
2. $IE' = IE \cap E'$ for each ideal I of A.
3. $rE' = rE \cap E'$ for each $r \in A$.

Each of these (equivalent) conditions can be taken as definition of purity when E is flat. But in general these definitions are not equivalent, see [71, p.158] for an example.

Theorem 1.2.5 *Let A be a ring and E an A-module, and let $R = A \propto E$ be the trivial ring extension of A by E.*

1. *If R is a PIP-ring, then so is A.*
2. *If all pure ideals of R are homogeneous and E is a flat A-module, then R is a PIP-ring if and only if so is A.*

The proof of this theorem involves the following lemmas.

Lemma 1.2.6 *Let A be a ring and E an A-module, and let $R = A \propto E$ be the trivial ring extension of A by E. Then*

1. *If $I \propto E'$ is a pure ideal of R, then I is a pure ideal of A and E' is a pure submodule of E and $E' = IE$.*
2. *I is a pure ideal of A if and only if $I \propto IE$ is a pure ideal of R.*
3. *Assume that I is a pure ideal of A, then $I \propto E'$ is a pure ideal of R if and only if $E' = IE$.*

Proof. (1) From [4, Theorem 9] and [3, Theorem 12].
(2) See [3, Theorem 7].
(3) The first implication comes from (1) and the second from (2).

Lemma 1.2.7 *Let A be a ring, E be an A-module and let $R = A \propto E$ be the trivial ring extension of A by E. If $I \propto IE$ (where I is a non-zero ideal of A) is a projective ideal of R, then I is a projective ideal of A. The converse is true if E is flat.*

Proof. See [4, Theorem 8].

Proof of Theorem 1.2.5. (1) Let I be a pure ideal of A.
$I \propto IE$ is a pure ideal of R by Lemma 1.2.6. Since R is a *PIP*-ring then $I \propto IE$ is a projective ideal of R. Therefore, I is a projective ideal of A by Lemma 1.2.7 and so A is a *PIP*-ring.

(2) Let A be a *PIP*-ring.
Assume that all pure ideals of R are homogeneous and E is a flat A-module. Let J be a nonzero pure ideal of R. By [12, Theorem 3.3], J has the form $J = I \propto E'$ where I is an ideal of A and E' is a submodule of E, but since J is pure, then $J = I \propto IE$ for some pure ideal I of A (by Lemma 1.2.6(1)). As A is a *PIP*-ring, then I is a projective ideal of A. Therefore J is a projective ideal of R by Lemma 1.2.7.

Now, we study the transfer of the *PIP*-property to pullbacks.

Theorem 1.2.8 *Let $\varphi : R \longrightarrow T$ be an injective ring homomorphism and Q be an ideal of R such that $QT = Q$.*

1. *If R is a PIP-ring, then for every pure ideal J of R, the $\frac{R}{Q}$-module $\frac{J}{JQ}$ is projective.*
2. *If T is a PIP-ring R-flat, then R is a PIP-ring if and only if for every pure ideal J of R, the $\frac{R}{Q}$-module $\frac{J}{JQ}$ is projective.*

The proof of this theorem involves the following lemma.

Lemma 1.2.9 *[58, Theorem 5.1.1] Let $\varphi : R \longrightarrow T$ be an injective ring homomorphism satisfying the fact that there is an ideal Q of R such that $QT = Q$. Then, an R-module E is projective if and only if $E \otimes_R T$ is a projective T-module and $\frac{E}{QE}$ is a projective $\frac{R}{Q}$-module.*

Proof of Theorem 1.2.8. (1) Assume that R is a *PIP*-ring.
Let J be a pure ideal of R. Then J is a projective ideal of R. Hence, $\frac{J}{JQ} = J \otimes_R \frac{R}{Q}$ is a projective $\frac{R}{Q}$-module.

(2) Assume that for every pure ideal J of R, $\frac{J}{JQ}$ is a projective $\frac{R}{Q}$-module. Let J be a pure ideal of R. Hence, $\frac{R}{J}$ is a flat R-module and so $\frac{R}{J} \otimes_R T = \frac{T}{JT}$ is a flat T-module. Therefore, JT is a pure ideal of T and so it is projective (since T is a *PIP*-ring).
As T is R-flat, then $JT = J \otimes_R T$. Thus, $J \otimes_R T$ is a projective T-module. Moreover, $\frac{J}{JQ}$ is a projective $\frac{R}{Q}$-module. By Lemma 1.2.9, J is a projective ideal of R, as desired.

Application. Assume that T is a ring of the form $T = K + M$, where K is a field and M is a nonzero maximal ideal of T, D is a subring of K such that $qf(D) = K$, and $R = D + M$. Then, $T = S^{-1}R$ with $S = D \setminus 0$.
Since T is a local ring then T is a *PIP*-ring and it is R-flat (since $T = S^{-1}R$) and we obtain the following corollary.

Corollary 1.2.10 *Preserving the notation introduced above, R is a PIP-ring if and only if for every pure ideal J of R, the $\frac{R}{Q}$-module $\frac{J}{JQ}$ is projective.*

Now, we study the transfer of the *PIP*-property to amalgamation of rings.

Recall that an ideal I of a ring R is called a multiplication ideal if for every ideal $J \subseteq I$ there exists an ideal K such that $J = IK$, we say that J is a multiple of I. A ring R is called a multiplication ring if all its ideals are multiplication ideals [6, 72].

We say that an ideal I of a ring R is idempotent if $I^2 = I$. Also, an ideal I of $A \bowtie^f J$ is called homogeneous if $I = K \bowtie^f J$ for some ideal K of A. If H is an ideal of $A \bowtie^f J$ such that $0 \times J \subseteq H$ then H is homogeneous [72].

Theorem 1.2.11 *Let A and B be rings, J an ideal of B and let $f : A \to B$ be a ring homomorphism.*

1. *Assume that J is a flat ideal of B. If $A \bowtie^f J$ is a PIP-ring then so is A.*
2. *Assume that J is an idempotent ideal of B and $A \bowtie^f J$ is a multiplication ring. If $A \bowtie^f J$ is a PIP-ring then so is A.*
3. *Assume that A is a Noetherian ring and $0 \times J \subseteq Rad(A \bowtie^f J)$, then $A \bowtie^f J$ is a PIP-ring.*
4. *Assume that all pure ideals of $A \bowtie^f J$ are homogeneous. If J is a pure finitely generated ideal of B then $A \bowtie^f J$ is a PIP-ring if and only if so is A.*

The proof of this theorem involves the following series of lemmas.

Lemma 1.2.12 *Let A and B be two rings and $f : A \longrightarrow B$ be a ring homomorphism making B a faithfully flat A-module. Then,*

$$I \text{ is a pure ideal of } A \Leftrightarrow IB \text{ is a pure ideal of } B.$$

Proof. We know that

$$\tfrac{A}{I} \otimes_A B \cong \tfrac{B}{IB}.$$

Since B is a faithfully flat A-module, then $\frac{A}{I}$ is a flat A-module if and only if $\frac{A}{I} \otimes_A B \cong \frac{B}{IB}$ is a flat B-module. So, I is a pure ideal of A if and only if IB is a pure ideal of B.

Lemma 1.2.13 *[72, Lemma 2.10] Let A and B be rings, J an idempotent ideal of B and let $f : A \to B$ be a ring homomorphism. If K and L are ideals of A, then $(KL) \bowtie^f J = (K \bowtie^f J)(L \bowtie^f J)$ (is the product of the two ideals $K \bowtie^f J$ and $L \bowtie^f J$).*

Lemma 1.2.14 *[72, Proposition 2.8] Let A and B be rings, J an idempotent ideal of B, $f : A \to B$ be a ring homomorphism and let I be a multiplication ideal of A. Then, every homogeneous ideal contained in $I \bowtie^f J$ is a multiple of $I \bowtie^f J$.*

Lemma 1.2.15 *Let A and B be rings, J an ideal of B, $f : A \to B$ be a ring homomorphism and let I be an ideal of A.*

1. *Assume that J is flat. Then, I is a pure ideal of A if and only if $I \bowtie^f (f(I)B)J$ is a pure ideal of $A \bowtie^f J$.*
2. *If $I \bowtie^f J$ is a pure ideal of $A \bowtie^f J$, then I is a pure ideal of A.*
3. *If J is an idempotent ideal of B and $I \bowtie^f J$ is an ideal of $A \bowtie^f J$ such that and all its subideals are homogeneous then, $I \bowtie^f J$ is a pure ideal of $A \bowtie^f J$ if and only if I is a pure ideal of A.*
4. *If J is an idempotent ideal of B and $A \bowtie^f J$ is a multiplication ring then $I \bowtie^f J$ is a pure ideal of $A \bowtie^f J$ if and only if I is a pure ideal of A.*

Proof. (1) Since J is a flat ideal of A, R is a faithfully flat A-module. Then, I is a pure ideal of A if and only if $IR = I(A \bowtie^f J)$ is a pure ideal of R (by Lemma 1.2.12) and by [42, Proposition 3.2 (2)] we have $IR = I(A \bowtie^f J) = (I \bowtie^f (f(I)B)J)$ and so $I \bowtie^f (f(I)B)J$ is a pure ideal of R.

(2) Let $I \bowtie^f J$ be a pure ideal of $A \bowtie^f J$ and let M be a maximal ideal of A. By [43, Proposition 2.6 (5)], $M \bowtie^f J$ is a maximal ideal of $A \bowtie^f J$. Then, by [42, Proposition 2.9 (3)] $(I \bowtie^f J)_{M \bowtie^f J} = I_M \bowtie^{f_M} J_S = A_M \bowtie^{f_M} J_S$, or $(I \bowtie^f J)_{M \bowtie^f J} = I_M \bowtie^{f_M} J_S = 0_{M \bowtie^f J}$ where $S = f(A \setminus M) + J$ and so $I_M = A_M$ or $I_M = 0_M$. Hence, I is a pure ideal of A.

(3) By (2), if $I \bowtie^f J$ is a pure ideal of $A \bowtie^f J$, then I is a pure ideal of A.
Conversely, assume that I is a pure ideal of A. By [5, Theorem 1.1], an ideal is pure if and only if it is multiplication and idempotent. So, we will show that $I \bowtie^f J$ is multiplication and idempotent. $I \bowtie^f J$ is multiplication by Lemma 1.2.14. By Lemma 1.2.13, we have

$$(I \bowtie^f J)^2 = (I \bowtie^f J)(I \bowtie^f J) = I^2 \bowtie^f J = I \bowtie^f J,$$

(since I is pure and so it is multiplication and idempotent) and so $I \bowtie^f J$ is idempotent. Therefore, $I \bowtie^f J$ is a pure ideal of $A \bowtie^f J$.

(4) If J is an idempotent ideal of B and $A \bowtie^f J$ is a multiplication ring then $I \bowtie^f J$ is a multiplication ideal and as we have seen in the proof of (3), $I \bowtie^f J$ is idempotent. Therefore, $I \bowtie^f J$ is a pure ideal of $A \bowtie^f J$.

Lemma 1.2.16 *Let A and B be rings, J an ideal of B, $f : A \to B$ be a ring homomorphism and let I be an ideal of A. If $I \bowtie^f H$ is a projective ideal of $A \bowtie^f J$, where I is an ideal of A such that $f(I)J \subseteq H \subseteq J$, then I is a projective ideal of A. In particular, if $I \bowtie^f J$ (or $I \bowtie^f (f(I)B)J$) is a projective ideal of $A \bowtie^f J$ then I is a projective ideal of A.*

Proof. By [42, Proposition 3.1(1)], $I \bowtie^f H$ is an ideal of $A \bowtie^f J$. Suppose that $I \bowtie^f H$ is a projective ideal of $A \bowtie^f J$. By the Dual Basis Lemma [55, Lemma 3.23], there exist a family of elements $\{a_i, b_i\} \subseteq I \bowtie^f H$ and $(A \bowtie^f J)$-homomorphisms $f_i : I \bowtie^f H \to A \bowtie^f J$ such that $(x,y) = \sum_i f_i(x,y)(a_i,b_i)$ for each $(x,y) \in I \bowtie^f H$. For each i we define $g_i : I \to A$ such that for all $c \in I$, $f_i(c, f(c)) = (g_i(c), v_i^{(c)})$ where $v_i^{(c)} \in (f(g_i(c)) + J)$. It is easy to show that g_i are A-homomorphisms since f_i and f are homomorphisms. Let $a \in I$. Then, $(a, f(a)) = \sum_i f_i(a, f(a))(a_i, b_i) = \sum_i (g_i(a), v_i^{(a)})(a_i, b_i) = (\sum_i g_i(a)a_i, \sum_i v_i^{(a)} b_i)$. Therefore, $a = \sum_i g_i(a)a_i$ and hence I is projective.

Proof of Theorem 1.2.11. (1) Assume that $A \bowtie^f J$ is a *PIP*-ring and J is a flat ideal of B. Let I be a pure ideal of A. $I \bowtie^f (f(I)B)J$ is a pure ideal of $A \bowtie^f J$ by Lemma 1.2.15(1). Since $A \bowtie^f J$ is a *PIP*-ring then $I \bowtie^f (f(I)B)J$ is a projective ideal of $A \bowtie^f J$. Therefore, I is a projective ideal of A by Lemma 1.2.13 and so A is a *PIP*-ring.

(2) Assume that $A \bowtie^f J$ is a *PIP*-ring and J. Let I be a pure ideal of A. $I \bowtie^f J$ is a pure ideal of $A \bowtie^f J$ by Lemma 1.2.15(4). Since $A \bowtie^f J$ is a *PIP*-ring then $I \bowtie^f J$ is a projective ideal of $A \bowtie^f J$. Therefore, I is a projective ideal of A by Lemma 1.2.16 and so A is a *PIP*-ring.

(3) Assume that A is Noetherian and $0 \times J \subseteq J(A \bowtie^f J)$. By [41, Proposition 5.1], $\frac{A \bowtie^f J}{0 \times J} = A$ and so $\frac{A \bowtie^f J}{0 \times J}$ is a Noetherian ring. Hence, $A \bowtie^f J$ is a *PIP*-ring by Theorem 1.2.2(3).

(4) Assume that J is a pure finitely generated ideal of B then $0 \times J$ is a pure finitely generated ideal of $A \bowtie^f J$ (since it is multiplication and idempotent). Then, $A = \frac{A \bowtie^f J}{0 \times J}$ is a *PIP*-ring if and only if $A \bowtie^f J$ is a *PIP*-ring by Theorem 1.2.2.

Example 1.2.17 *Let $A = \mathbb{Z}$, $B = \frac{\mathbb{Z}}{12\mathbb{Z}}$ and $J = \frac{4\mathbb{Z}}{12\mathbb{Z}}$ and let $f : A \to B$ be the canonical projection. Then, $A \bowtie^f J$ is a PIP-ring.*

Proof. Since J is generated by an idempotent so it is a pure finitely generated ideal of B. As A is a *PIP*-ring (since it is a domain) then $A \bowtie^f J$ is a *PIP*-ring by Theorem 1.2.11(4).

We end this work by the following remark and examples.

Remark 1.2.18 *1. If R is a Von Neumann regular ring which is non-hereditary then R is a non-PIP-ring. In that case, even if R_P is a PIP-ring for every $P \in Spec(R)$ (since it is a field), R is not a PIP-ring.*

2. *If R is a ring where every pure ideal is finitely presented, then R is a PIP-ring (since every pure ideal is flat).*

3. *By the second remark, we can say that if R is a coherent ring such that every pure ideal is finitely generated, then R is PIP-ring.*

Example 1.2.19 *[66, Example 2.12 (3)] Let $R = \Pi_{p \in S}(\mathbb{Z}/p\mathbb{Z})$, where S is the set of all prime integers. Then R is a Von Neumann regular ring which is not hereditary by [79]. So R is not a PIP-ring and R_P is a PIP-ring for all $P \in Spec(R)$.*

We give an example of a *PIP*-ring which is not coherent.

Example 1.2.20 *Let $R := \mathbb{Z} \propto \mathbb{Q}[X]$ be the trivial extension ring of $\mathbb{Z}$ by the polynomial ring $\mathbb{Q}[X]$. Then*

1. *R is a PIP-ring, since it is an FF-ring by [48, Example 3.13].*

2. *R is not coherent by [69, Theorem 2.8 (1)]. In particular R is not Noetherian.*

CHAPTER 2

RINGS IN WHICH EVERY REGULAR LOCALLY PRINCIPAL IDEAL IS PROJECTIVE

Abstract.

In this chapter *, we study the class of rings in which every regular locally principal ideal is projective called *LPP*-rings. We investigate the transfer of this property to various constructions such as direct products, amalgamation of rings, and trivial ring extensions. Our aim is to provide examples of new classes of commutative rings satisfying the above-mentioned property.

Key Words: Locally principal ideals, regular ideals, trivial extension, pullback, amalgamation of rings, *LPP*-rings.

2.1 Introduction

In this chapter, we denote by $Rad(R)$ the Jacobson radical of a ring R, $Z(R)$ the set of zero divisors of R and by $Reg(R)$ the set of regular elements in R.

We recall that in integral domains the notion of projective ideal is equivalent to the one of invertible ideal. In general an invertible ideal is projective but the converse is not true. It is well-known that in an integral domain D the projective ideals are exactly the nonzero finitely generated ideals that are also locally principal (see [27]). For Noetherian domains the projective ideals are exactly the nonzero locally principal ideals. S. Bazzoni

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in [23] conjectured that Prüfer domains for which, "an ideal is invertible (equivalently projective) if and only if it is a locally principal", are exactly the ones with finite character, i.e. each nonzero element of the domain belongs to only finitely many maximal ideals. This conjecture was first proved in [64] and then, in 2011, D. D. Anderson and Muhammad Zafrullah [13], studied locally principal ideals in ordinary integral domains, called *LPI* domains, in which every nonzero locally principal ideal is invertible (equivalently projective).

Since out of the context of domains projective ideals are not necessarily invertible, we give the following definition :
Let R be a ring. We said that R is an *LPP*-ring if every regular locally principal ideal is projective. Noetherian rings, Artinian rings and hereditary rings are examples of *LPP*-rings.

2.2 Main results

Notice that a characterization of locally principal ideals in integral domains is given in [13]. The following proposition extends this characterization to regular locally principal ideals in rings with zero-divisors. Recall that an ideal I in a ring R is called a cancellation ideal if $IJ \subseteq IK$ for ideals J and K of R implies that $J \subseteq K$, and we say that I is a regular ideal if I contains a non-zero divisor of R.

Proposition 2.2.1 *Let R be a ring and let I be a regular ideal of R. Then, the following are equivalent:*

1. *I is locally principal.*
2. *I is faithfully flat.*
3. *I is a cancellation ideal.*

Proof. $(1) \Rightarrow (2)$ Let I be a regular locally principal ideal of R. Then, I_M is free for all maximal ideals M of R. Indeed, I_M is a regular ideal of R_M for all $M \in Max(R)$ since I is regular. Therefore, I_M is a regular principal ideal of R_M for all $M \in Max(R)$ and so I_M is generated by a regular element and so it is free. Hence, I_M is faithfully flat for all $M \in Max(R)$. Then, I is faithfully flat.
$(2) \Rightarrow (3)$ Let I be a regular faithfully flat ideal of R. Let $IJ \subseteq IK$ for some ideals J and K of R. We concider the exact sequence,

$$0 \to K \to K+J \to (K+J)/K \to 0.$$

Since I is faithfully flat, then we have

$$0 \to I \otimes K \to I \otimes (K+J) \to I \otimes (K+J)/K \to 0.$$

Therefore, $I \otimes (K+J)/K = (I \otimes (K+J))/(I \otimes K) = I(K+J)/IK = IK/IK = 0$. Since I is faithfully flat, then $(K+J)/K = 0$ and so $J \subseteq K$.
$(3) \Leftrightarrow (1)$ By [10, Theorem].

Remark 2.2.2

1. *Noetherian rings, hereditary rings and semi-simple rings (finite direct product of fields) are obviously examples of LPP-rings.*

2. *Let R be an Artinian ring. Each element of R is either invertible or a zero divisor. Such a ring only has one regular ideal, which is R. Therefore, Artinian rings are LPP-rings.*

3. *Let R be a ring. Then, R_M is always an LPP-ring independently of R for all $M \in Max(R)$ (since a principal regular ideal is free). But the converse is not true. Hence, local rings are LPP-rings.*

Next, we study the transfer of the *LPP*-property to direct products.

Theorem 2.2.3 *Let $(R_i)_{i=1,\cdots,n}$ be a family of commutative rings. Then $R = \prod_{i=1}^{n} R_i$ is an LPP-ring if and only if R_i is for each $i = 1, \cdots, n$.*

The proof of this theorem needs the following lemmas.

Lemma 2.2.4 *[73, Lemma 2.5] Let $(R_i)_{i=1,2}$ be two rings and E_i be an R_i-module for $i = 1,2$. Then,*

$$pd_{R_1 \times R_2}(E_1 \times E_2) = sup\{pd_{R_1}(E_1), pd_{R_2}(E_2)\}.$$

Lemma 2.2.5 *Let $(R_i)_{i=1,2}$ be two rings and I_i be an ideal of R_i for $i = 1,2$. Then, $I_1 \times I_2$ is a regular locally principal ideal of $R_1 \times R_2$ if and only if I_1 and I_2 are regular locally principal ideals of R_1 respectively R_2.*

Proof. Assume that $I_1 \times I_2$ is a regular locally principal ideal of $R_1 \times R_2$. Then, I_1 and I_2 are regular ideals of R_1 and R_2 respectively. Indeed, let (a,b) be a regular element in $I_1 \times I_2$. Suppose that a is not regular. Then, there exists $x \in R_1 \setminus \{0\}$ such that $ax = 0$. So, $(a,b)(x,0) = (0,0)$ which is absurd since (a,b) is regular. Hence, a is a regular element in I_1. By the same way we show that b is a regular element of I_2. Therefore, I_1 and I_2 are regular ideals of R_1 and R_2 respectively.
Let M be a maximal ideal of R_1. Then, $M \times R_2$ is a maximal ideal of $R_1 \times R_2$. So, $(I_1 \times I_2)_{M \times R_2} = (I_1)_M$. Hence, I_1 is a locally principal ideal of R_1. By the same argument we show that I_2 is a locally principal ideal of R_2.

Conversely, assume that I_1 and I_2 are regular locally principal ideals of R_1 and R_2 respectively. Then, $I_1 \times I_2$ is a regular ideal of $R_1 \times R_2$.
Let $M \times R_2$ (resp. $R_1 \times M'$) be a maximal ideal of $R_1 \times R_2$ where M is a maximal ideal of R_1 (resp. M' is a maximal ideal of R_2). So, $(I_1 \times I_2)_{M\times R_2} = (I_1)_M$ (resp. $(I_1 \times I_2)_{R_1\times M'} = (I_2)_{M'}$). Since I_1 and I_2 are locally principal then $I_1 \times I_2$ is a locally principal ideal of $R_1 \times R_2$.

Proof of Theorem 2.2.3. The proof is done by induction on n and it suffices to check it for $n = 2$. Assume that $R_1 \times R_2$ is an *LPP*-ring. Let I_1 be a regular locally principal ideal of R_1. Then, $I_1 \times R_2$ is a regular locally principal ideal of $R_1 \times R_2$ by Lemma 2.2.5. Thus, $I_1 \times R_2$ is projective since $R_1 \times R_2$ is an *LPP*-ring. We have $pd_{R_1\times R_2}(I_1 \times R_2) = sup\{pd_{R_1}(I_1), pd_{R_2}(R_2)\} = 0$ by Lemma 2.2.4. Hence, $pd_{R_1}(I_1) = 0$ and so I_1 is a projective ideal of R_1. Therefore, R_1 is an *LPP*-ring. We have also that R_2 is an *LPP*-ring by the same argument.

Conversely, let R_1 and R_2 be two *LPP*-rings and let $I_1 \times I_2$ be a regular locally principal ideal of $R_1 \times R_2$. Then, I_1 and I_2 are regular locally principal ideals of R_1 and R_2 respectively by Lemma 2.2.5. Hence, I_1 and I_2 are projective since R_1 and R_2 are *LPP*-rings and so $pd_{R_1}(I_1) = 0$ and $pd_{R_2}(I_2) = 0$. We have $pd_{R_1\times R_2}(I_1 \times I_2) = sup\{pd_{R_1}(I_1), pd_{R_2}(I_2)\} = 0$ by Lemma 2.2.4 and this completes the proof of the theorem.

The next result shows that the *LPP*-property descends into a faithfully flat domain homomorphism. We recall that in integral domains, a flat ideal is finitely generated if and only if it is projective.

Proposition 2.2.6 *Let R and S be two domains and $f : R \to S$ be a ring homomorphism making S a faithfully flat R-module. If S is an LPP-ring, then so is R.*

Proof. Let I be a locally principal ideal of R. Then, I is a faithfully flat R-module by Proposition 2.1. So, $I \otimes S = IS$ is a faithfully flat S-module. Hence, IS is a locally principal ideal of S and so it is projective since S is an *LPP*-ring. Therefore, $I \otimes S = IS$ is finitely generated (since S is a domain and IS a flat ideal of S). Then, I is a finitely generated ideal of R. Hence, I is a projective ideal of R and so R is an *LPP*-ring.

Our next theorem develops a result on the transfer of the *LPP*-property to trivial ring extensions. Recall that if A is a ring, E is an A-module and $R = A \propto E$ is the trivial ring extension of A by E, an ideal J of R is said to be homogeneous if $J = I \propto E'$ where I is an ideal of R, E' is a submodule of E, and $IE \subseteq E'$ (see for instance [12]) .

Theorem 2.2.7 *Let A be a ring, E be an A-module and let $R = A \propto E$ be the trivial ring extension of A by E and let $S = A \setminus (Z(A) \cup Z(E))$.*

1. *Assume that for every regular locally principal ideal I of A, $I \cap S \neq \emptyset$. Then, if R is an LPP-ring then so is A. In particular, if $Z(E) \subseteq Z(A)$ (for example we can consider E torsion free) then if R is an LPP-ring then so is A.*

2. *Assume that E is a divisible flat A-module ($E = S^{-1}E$ since E is divisible). Then,*

   (a) *If A is an LPP-ring then so is R.*

   (b) *Assume that for every regular locally principal ideal I of A, $I \cap S \neq \emptyset$ (in particular, if $Z(E) \subseteq Z(A)$), then R is an LPP-ring if and only if so is A.*

   (c) *If A is a domain, then R is an LPP-ring if and only if so is A.*

The proof of this theorem needs the following lemmas.

Lemma 2.2.8

1. *Let A be a ring, E an A-module and I an ideal of A.*

   (a) *If $I \propto IE$ is a regular locally principal ideal of $A \propto E$, then I is a regular locally principal ideal of A.*

   (b) *If $I \cap S \neq \emptyset$, then I is a regular locally principal ideal of A if and only if $I \propto IE$ is a regular locally principal ideal of $A \propto E$.*

   (c) *If E is torsion free, then I is a regular locally principal ideal of A if and only if $I \propto IE$ is a regular locally principal ideal of $A \propto E$.*

2. *Let $I \propto E'$ be an ideal of $A \propto E$ where I is an ideal of A, E' is a submodule of E, and $IE \subseteq E'$. If $I \propto E'$ is a regular locally principal ideal of $A \propto E$ then so is I.*

Proof. (1) (a) I is locally principal if and only if $I \propto IE$ is a locally principal ideal of $A \propto E$ by [3, Theorem 7].
Let (a,e) be a regular element of $I \propto IE$. Then a is a regular element of I by [3, Lemma 6]. So, I is a regular ideal of A.

(b) Let $I \cap S \neq \emptyset$. Then there exists a a regular element in I such that $a \notin Z(E)$. Then $(a,0)$ is a regular element of $I \propto IE$. Indeed, assume that there exists $(0,0) \neq (b,e) \in I \propto IE$ such that $(a,0)(b,e) = (0,0)$. Then, $ab = 0$ and $ae = 0$. Since a is regular then $b = 0$ and since $a \notin Z(E)$ then $e = 0$. Hence, $(b,e) = (0,0)$ which is absurd. So, $I \propto IE$ is a regular ideal of $A \propto E$.
Let (a,e) be a regular element of $I \propto IE$. Then a is a regular element of I by [3, Lemma 6]. So, I is a regular ideal of A.

(c) By [4, Lemma 10].

(2) Let $I \propto E'$ be an ideal of $A \propto E$ where I is an ideal of A, E' is a submodule of E, and $IE \subseteq E'$. Assume that $I \propto E'$ is a regular locally principal ideal of $A \propto E$. Then, there exists a regular element $(a,e) \in I \propto E'$. By [3, Lemma 6], a is a regular element of A and

so I is a regular ideal of A. Since $I \propto E'$ is locally principal, then $(I \propto E')_M = \langle (x,e) \rangle$ where $M \in Max(A)$. So,

$$\begin{aligned}(I \propto E')_{M \propto E} &= I_M \propto E_M \\ &= (A_M \propto E_M)(x,e) \\ &= A_M x \propto (A_M e + E_M e)\end{aligned}$$

Hence, $I_M = A_M x$. Therefore, I is a locally principal ideal of A.

Lemma 2.2.9 *[4, Theorem 8] Let A be a ring, E be an A-module and I be an ideal of A. If $I \propto IE$ is a projective ideal of $A \propto E$, then I is projective. The converse is true if E is flat.*

Lemma 2.2.10 *[12, Theorem 3.9] Let A be a ring and E an A-module. Let $S = A \setminus (Z(A) \cup Z(E))$. Then the following conditions are equivalent.*

1. *Every regular ideal of $A \propto E$ has the form $I \propto E$ where I is an ideal of A with $I \cap S \neq \emptyset$.*
2. *Every regular ideal of $A \propto E$ is homogenous.*
3. *$E = S^{-1}E$.*

Proof of Theorem 2.2.7. (1) Let I be a regular locally principal ideal of A. By Lemma 2.2.8(1,b), $I \propto IE$ is a regular locally principal ideal of R. Since R is an *LPP*-ring, then $I \propto IE$ is a projective ideal of R. By Lemma 2.2.9, we conclude that I is projective. Therefore, A is an *LPP*-ring.

(2) (a) Let J be a regular locally principal ideal of R. By Lemma 2.2.10, J has the form $I \propto E$ where I is an ideal of A with $I \cap S \neq \emptyset$. Then, I is a regular locally principal ideal of A by Lemma 2.2.8 (2). Hence, I is projective since A is an *LPP*-ring. Therefore, $I \propto E$ is projective by Lemma 2.2.9 (since E is divisible then $IE = E$ and also E is flat). So, $A \propto E$ is an *LPP*-ring.

(b) Comes from (1) and $(2)(a)$.

(c) Assume that A is a domain. Then, E is torsion free since it is flat. Hence, the result is given by (b) (since $Z(E) \subseteq Z(A)$).

Theorem 2.2.7 enriches the literature with new examples of *LPP*-rings that are neither Noetherian nor zero-dimensional.

Example 2.2.11 *Let K be a field, $A = K[x_1, x_2, \cdots, x_n]$ be the polynomial ring in n variables over K and $E = qf(K[x_1, x_2, \cdots, x_n])$ the quotient field of A. Then:*

1. *$A \propto E$ is an LPP-ring since A is Noetherian (and so an LPP-ring) and by Theorem 2.2.7(2,c).*

2. *$A \propto E$ is not Noetherian since E is not finitely generated as A-module and by [12, Theorem 4.8].*

3. *$A \propto E$ is not zero-dimensional since A is not zero-dimensional.*

Now, we study the transfer of the *LPP*-property to pullbacks.

Theorem 2.2.12 *Let $f : R \to S$ be an injective ring homomorphism satisfying the property that there is an ideal Q of R such that $QS = Q$ and S is a flat R-module. Assume that S is an LPP-ring and $Reg(R) \subseteq Reg(S)$. R is an LPP-ring if and only if for every regular locally principal ideal J of R the R/Q-module J/JQ is projective.*

The proof of this theorem involves the following lemma.

Lemma 2.2.13 *[58, Theorem 5.1.1] Let $f : R \to S$ be an injective ring homomorphism satisfying the property that there is an ideal Q of R such that $QS = Q$. Then, an R-module E is projective if and only if $E \otimes_R S$ is a projective S-module and E/QE is a projective R/Q-module.*

Proof. Let J be a regular locally principal ideal of R. By Proposition 2.2.1, J is a faithfully flat ideal of R. Therefore, $J \otimes_R S = JS$ is a faithfully flat ideal of S. Since $Reg(R) \subseteq Reg(S)$, then $J \otimes_R S = JS$ is a regular ideal of S. By Proposition 2.2.1, JS is a regular faithfully flat ideal of S is equivalent to JS is a regular locally principal ideal of S. Hence, $J \otimes_R S = JS$ is projective since S is an *LPP*-ring. By Lemma 2.2.13, we conclude that J is a projective ideal of R. Therefore, R is an *LPP*-ring.

Conversely, let J be a regular locally principal ideal of R. Since R is an *LPP*-ring then J is projective. Hence, $J \otimes_R R/Q = J/JQ$ is a projective R/Q-module.

Now, we will see the transfer of the *LPP*-property to the amalgamation of rings.

Proposition 2.2.14 *Let A and B be two rings, $f : A \to B$ be a ring homomorphism and J be an ideal of B. Assume that J is a flat ideal of A and $f(Reg(A)) \subseteq Reg(B)$. Then, if $A \bowtie^f J$ is an LPP-ring then so is A.*

The proof of this theorem needs the following lemma.

Lemma 2.2.15 *Let A and B be rings, J an ideal of B, $f : A \to B$ be a ring homomorphism and let I be an ideal of A. If $I \bowtie^f H$ is a projective ideal of $A \bowtie^f J$, where I is an ideal of A such that $f(I)J \subseteq H \subseteq J$, then I is a projective ideal of A. In particular, if $I \bowtie^f J$ (or $I \bowtie^f (f(I)B)J$) is a projective ideal of $A \bowtie^f J$ then I is a projective ideal of A.*

Proof. By [42, Proposition 3.1(1)], $I \bowtie^f H$ is an ideal of $A \bowtie^f J$. Suppose that $I \bowtie^f H$ is a projective ideal of $A \bowtie^f J$. By the Dual Basis Lemma [55, Lemma 3.23], there exist a family of elements $\{a_i, b_i\} \subseteq I \bowtie^f H$ and $(A \bowtie^f J)$-homomorphisms $f_i : I \bowtie^f H \to$

$A \bowtie^f J$ such that $(x,y) = \sum_i f_i(x,y)(a_i,b_i)$ for each $(x,y) \in I \bowtie^f H$. For each i we define $g_i : I \to A$ such that for all $c \in I$, $f_i(c,f(c)) = (g_i(c), v_i^{(c)})$ where $v_i^{(c)} \in (f(g_i(c)) + J)$. It is easy to show that g_i are A-homomorphisms since f_i and f are homomorphisms. Let $a \in I$. Then, $(a,f(a)) = \sum_i f_i(a,f(a))(a_i,b_i) = \sum_i (g_i(a), v_i^{(a)})(a_i,b_i) = (\sum_i g_i(a)a_i, \sum_i v_i^{(a)} b_i)$. Therefore, $a = \sum_i g_i(a)a_i$ and hence I is projective.

Proof of proposition 2.2.14. Assume tha J is a flat ideal of A. Then, $A \bowtie^f J$ is a faithfully flat A-module (by the exact sequence $0 \to J \to A \bowtie^f J \to A \to 0$). Let I be a regular locally principal ideal of A. We have $I \otimes (A \bowtie^f J) = I(A \bowtie^f J)$ since $A \bowtie^f J$ is a faithfully flat A-module and so $I \otimes (A \bowtie^f J) = I(A \bowtie^f J) = I \bowtie (f(I)B)J$ by [42, Proposition 3.1(2)]. Since J is regular and $f(Reg(A)) \subseteq Reg(B)$ then $I \bowtie (f(I)B)J$ is a regular ideal of $A \bowtie^f J$. By Proposition 2.2.1, I is a faithfully flat ideal of A and so $I \otimes (A \bowtie^f J) = I \bowtie (f(I)B)J$ is a faithfully flat ideal of $(A \bowtie^f J)$ and so it is locally principal by Proposition 2.2.1. Since $(A \bowtie^f J)$ is an *LPP*-ring then $I \bowtie (f(I)B)I$ is projective. Hence, I is a projective ideal of A by Lemma 2.2.15 and so A is an *LPP*-ring.

The following example shows that there exist ring homomorphisms that satisfy the condition $(f(Reg(A)) \subseteq Reg(B))$ in Proposition 2.2.14.

Example 2.2.16 *Let I be a non-regular prime ideal of a ring A and let $B = A/I$. Then $p : A \to B$ verifies $p(Reg(A)) \subseteq Reg(B)$.*

Proof. Let $a \in A$ be a regular element of A. We have, $\forall x \in A$, $\overline{ax} = \overline{0}$ implies that $ax \in I$. Since I is non-regular then $a \notin I$. Hence $x \in I$ since I is prime. Then, $\overline{x} = 0$ and so $\overline{a}$ is a regular element in A/I. Therefore, $p(Reg(A)) \subseteq Reg(B)$.

Now, we will see the transfere of the *LPP*-property to the amalgamated duplication. Recall that an ideal J of $A \bowtie I$ is called homogeneous if $I = K \bowtie I$ for some ideal K of A. If K is an ideal of $A \bowtie I$ such that $0 \times I \subseteq K$ then K is homogeneous (see [72, Lemma 2.9]).

Theorem 2.2.17 *Let A be a ring and I be an ideal of A.*

1. *Assume that I is a flat ideal of A then if $A \bowtie I$ is an LPP-ring then so is A.*
2. *Let I be a non-regular finitely generated flat ideal of A and $S = A \setminus Z(A)$ such that $S^{-1}I = I$. Assume that for every regular locally principal ideal J of A we have $JI = I$. Then, A is an LPP-ring if and only if so is $A \bowtie I$.*

The proof of this theorem needs the following lemmas.

Lemma 2.2.18 *Let A be a ring and I be an ideal of A. If $(a, a+i)$ is a regular element of $A \bowtie I$ then a is a regular element of A.*

Proof. Let $(a,a+i)$ be a regular element of $A \bowtie I$. Suppose that a is not regular. Then, there exists $b \in A \setminus \{0\}$ such that $ab = 0$. Two cases are possible:

Case 1: There exists $j \in I$ such that $jb \neq 0$.
Then, $(a,a+i)(bj,0) = (0,0)$ which is absurd since $(a,a+i)$ is regular.

Case 2: For all $j \in I$ we have $jb = 0$.
Then, $(a,a+i)(b,b) = (0,0)$ which is also absurd since $(a,a+i)$ is regular.

Hence, in both cases we can not have a non-regular and so a is a regular element of A.

Lemma 2.2.19 *Assume that I is a proper non-regular ideal of a ring A and $S = A \setminus (Z(A))$. The following are equivalent :*

1. *All regular ideals of $A \bowtie I$ are homogeneous .*
2. *For all $a \in S$ we have $I = aI$ which is equivalent to $I = S^{-1}I$.*

Proof. $(1) \Rightarrow (2)$ Assume all regular ideals of $A \bowtie I$ are homogeneous. Let $a \in S$. Then, (a,a) is a regular element of $A \bowtie I$. Then, $\langle (a,a) \rangle$ is an homogeneous ideal of $A \bowtie I$. So, $0 \times I \subseteq \langle (a,a) \rangle = J \bowtie I$ for some ideal J of A. Let $i \in I$. Then, $(0,i) = (\alpha, \alpha + k)(a,a)$ where $\alpha \in A$ and $k \in I$. Since a is a regular element of A then $\alpha a = 0$ implies that $\alpha = 0$. So, $(0,i) = (0,k)(a,a)$. Therefore, $i = ka$ and so $I = aI$.

$(2) \Rightarrow (1)$ Let K be a regular ideal of $A \bowtie I$. Then, there exists $(a,a+i)$ a regular element of $A \bowtie I$ in K. By Lemma 2.2.18, a is a regular element of A. So, $\langle (a,a+i) \rangle = (a,a+i)A \bowtie I = aA \bowtie (iA + aI + iI))$. Since $I = aI$ then $\langle (a,a+i) \rangle = aA \bowtie I$. Hence, $0 \times I \subseteq \langle (a,a+i) \rangle \subseteq K$. Therefore, K is an homogeneous ideal of $A \bowtie I$.

Remark 2.2.20 *Let A be a ring and I be a proper regular ideal of A. Then there exists a regular ideal of $A \bowtie I$ which is not homogeneous. Indeed, if we suppose that all regular ideals of $A \bowtie I$ are homogeneous then in particular the ideal generated by (c,c), where c is a regular element in I, is homogeneous. By the same argument in the proof of Lemma 2.2.19 we show that $I = cI$ and so $c = c\lambda$ for some $\lambda \in I$ (since $c \in I$). So, $\lambda = 1$ since c is regular. Therefore, $I = A$ which is absurd since we suppose that I is a proper ideal of A.*

Proof of Theorem 2.2.17. (1) is a consequence of Proposition 2.2.14.

(2) Let K be a regular locally principal ideal of $A \bowtie I$. Then K is an homogeneous ideal by Lemma 2.2.19. So, $K = J \bowtie I$ for some ideal J of A. Let M be a maximal ideal of A. Two cases are possible:

Case 1: $I \subseteq M$.
$M \bowtie I$ is a maximal ideal of $A \bowtie I$ and so $(J \bowtie I)_{M \bowtie I} = J_M \bowtie I_M$ [44, Theorem 3.8] is a principal ideal of $(A \bowtie I)_{M \bowtie I}$. Hence, J_M is a principal ideal of A_M.

Case 2: $I \nsubseteq M$.
Then, $(J \bowtie I)_{M\bowtie I} = J_M$ [44, Theorem 3.5] is a principal ideal of $(A \bowtie I)_{M\bowtie I}$. Hence, J_M is a principal ideal of A_M.

In both cases J is locally principal. So, J is regular locally principal. Hence, J is projective. We have, $J \otimes A \bowtie I = J \bowtie JI = J \bowtie I = K$ (since $IJ = I$ and $A \bowtie I$ is a flat A-module). As J is A-projective then $J \otimes A \bowtie I$ is $A \bowtie I$-projective. Hence, $K = J \bowtie I$ is a projective ideal of $A \bowtie I$ and so $A \bowtie I$ is an *LPP*-ring.

The converse holds by (1).

CHAPTER 3

ON LPI RINGS

Abstract

In this chapter * , we extend the *LPI* property (that is, every locally principal ideal in an integral domain is invertible) to rings with zero-divisors and we study the class of commutative rings in which every regular locally principal ideal is invertible called *LPI* rings. We investigate the stability of this property under homomorphic image, and its transfer to various contexts of constructions such as direct products, amalgamation of rings and trivial ring extensions. Our results generate examples which enrich the current literature with new and original families of rings that satisfy this property.

Key Words: Locally principal ideals, regular ideals, trivial extension, pullback, amalgamation of rings, LPI-rings.

3.1 Introduction

It is well-known that a finitely generated flat module over a domain is projective, and over an integral domain, the notion of projective ideal is equivalent to the one of invertible ideal. In general, an invertible ideal is projective but the converse is not true. The notion of domains with flat ideals invertible was first studied by Sally and Vasconcelos in 1975 (see [81]) as domains with property $\mathscr{P}$. They showed that if a domain D has the property $\mathscr{P}$, then so does the polynomial ring $D[X]$. In 1977 Glaz and Vasconcelos studied the invertibility of faithfully flat ideals over an H-domain and conjectured that over an H-domain, a faithfully flat ideal is finitely generated (and hence invertible) see [59].
In [23], S. Bazzoni conjectured that Prüfer domains for which "an ideal is invertible if and only if it is a locally principal" are exactly the ones with the finite character property, i.e. each nonzero element of the domain belongs to finitely many maximal ideals.

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This conjecture was first resolved in the affirmative by Holland, Martinez, McGovern and Tesemma ([64]). Later, Halter-Koch stated and proved an analog of Bazzoni's conjecture using the language of ideal systems, that is, r-Prüfer monoids, which in the domain case are $PVMD$'s and include Prüfer domains ([62]). In 2010, Picozza and Tartarone introduced the notion of quasi-stable ideals as ideals I that are flat in their ring homomorphisms $(I : I)$. They also studied domains in which every ideal is quasi-stable, and proved that Glaz-Vascocncelos conjecture is false ([80]).
In 2011, D. D. Anderson and Muhammad Zafrullah [13] introduced and studied the notion of *LPI* domains as integral domains in which every nonzero locally principal ideal is invertible. They proved that a finite character intersection of *LPI* overrings is an *LPI*-domain and so if a domain D is a finite character intersection $D = \cap D_P$ for some set of prime ideals of D, then D is an *LPI* domain. In 2013, Kui, Wang and Chen answered positively a question raised by Anderson and Zafrullah of whetter a polynomial ring over an *LPI* domain is an *LPI* domain ([67, Theorem 1.8]). In 2014, D. D. Anderson and A. Mimouni studied *LPI* domains in pullbacks ([9]). Very recently, Xing and Wang F. answered negatively a question by Anderson-Zafurllah by showing that if R is an *LPI* domain and S is a multiplicatively closed set, then R_S need not be an *PLI* domain (see [86]).

The purpose of the present work is to extend the notion of *LPI* domain to an arbitrary ring with zero-divisors. A ring R is said to be an *LPI* ring if every regular locally principal ideal of R is invertible. Noetherian rings are obviously *LPI* rings by [68, Lemma 18.1]. Our aim is to give some simple methods in order to construct *LPI* rings outside the context of integral domains that are not Noetherian. For this, we investigate the stability of the *LPI* property under homomorphic image, and its transfer to various contexts of constructions such as direct products, amalgamation of rings and trivial ring extensions. Our results generate original examples which enrich the current literature with new families of rings satisfying the *LPI* property.

3.2 Main results

Notice that a characterization of locally principal ideals in integral domains is given in [13, Theorem 1]. The following proposition extends this characterization to regular locally principal ideals in rings with zero-divisors. The proof is similar to that one in [13, Theorem 1], and for the convenience of the reader we include it here. Recall that an ideal I in a ring R is called a cancellation ideal if $IJ \subseteq IK$ for ideals J and K of R implies that $J \subseteq K$.

Proposition 3.2.1 *([13, Theorem 1]) Let R be a ring and let I be a regular ideal of R. The following conditions are equivalent:*

1. *I is locally principal.*

2. *I is faithfully flat.*

3. *I is a cancellation ideal.*

Proof. $(1) \Rightarrow (2)$ Let I be a regular locally principal ideal of R. Then, I_M is free for all maximal ideals M of R. Indeed, I_M is a regular ideal of R_M for all $M \in Max(R)$ since I is regular. Therefore, I_M is a regular principal ideal of R_M for all $M \in Max(R)$ and so I_M is genereted by a regular element and so it is free. Hence, I_M is faithfully flat for all $M \in Max(R)$. Then, I is faithfully flat.
$(2) \Rightarrow (3)$ Let I be a regular faithfully flat ideal of R. Let $IJ \subseteq IK$ for some ideals J and K of R. We concider the exact sequence,

$$0 \to K \to K+J \to \tfrac{K+J}{K} \to 0.$$

Since I is faithfully flat, then we have

$$0 \to I \otimes K \to I \otimes (K+J) \to I \otimes \tfrac{K+J}{K} \to 0.$$

Therefore, $I \otimes \frac{K+J}{K} = \frac{I \otimes (K+J)}{I \otimes K} = \frac{I(K+J)}{IK} = \frac{IK}{IK} = 0$. Since I is faithfully flat, then $\frac{K+J}{K} = 0$ and so $J \subseteq K$.
$(3) \Leftrightarrow (1)$ By [10, Theorem].

Next, we study the transfer of the *LPI* property to direct products.

Theorem 3.2.2 *Let $(R_i)_{i=1,\cdots,n}$ be a family of commutative rings. Then $R = \prod_{i=1}^{n} R_i$ is an LPI ring if and only if so is R_i for each $i = 1, \cdots, n$.*

The proof of this theorem needs the following lemmas. The proof of the first lemma is straightforward and it is omitted, and the second lemma is a well-known result.

Lemma 3.2.3 *Let $(R_i)_{i=1,2}$ be two rings and I_i be an ideal of R_i for $i = 1,2$. Then, $I_1 \times I_2$ is a regular locally principal ideal of $R_1 \times R_2$ if and only if I_1 and I_2 are regular locally principal ideals of R_1 respectively R_2.*

Proof. Assume that $I_1 \times I_2$ is a regular locally principal ideal of $R_1 \times R_2$. Then, I_1 and I_2 are regular ideals of R_1 respectively R_2. Endeed, let (a,b) be a regular element in $I_1 \times I_2$. Suppose that a is not regular. Then, there exists $x \in R_1 \setminus \{0\}$ such that $ax = 0$. So, $(a,b)(x,0) = (0,0)$ which is absurd since (a,b) is regular. Hence, a is a regular element in I_1. By the same argument we show that b is a regular element of I_2. Therefore, I_1 and I_2 are regular ideals of R_1 respectively R_2.
Let M be a maximal ideal of R_1. Then, $M \times R_2$ is a maximal ideal of $R_1 \times R_2$. So, $(I_1 \times I_2)_{M \times R_2} = (I_1)_M$. Hence, I_1 is a locally principal ideal of R_1. By the same argument

we show that I_2 is a locally principal ideal of R_2.Conversely, assume that I_1 and I_2 are regular locally principal ideals of R_1 respectively R_2. Then, $I_1 \times I_2$ is a regular ideal of $R_1 \times R_2$.
Let $M \times R_2$ (resp. $R_1 \times M'$) be a maximal ideal of $R_1 \times R_2$ where M is a maximal ideal of R_1 (resp. M' is a maximal ideal of R_2). So, $(I_1 \times I_2)_{M\times R_2} = (I_1)_M$ (resp. $(I_1 \times I_2)_{R_1\times M'} = (I_2)_{M'}$). Since I_1 and I_2 are locally principal then $I_1 \times I_2$ is a locally principal ideal of $R_1 \times R_2$.

Lemma 3.2.4 *([68, Lemma 18.1]) Let I be a regular locally principal ideal of a ring A. Then, I is invertible if and only if it is finitely generated.*

Proof of Theorem 3.2.2. The proof is done by induction on n and it suffices to check it for $n = 2$. By Lemma 3.2.3, $I_1 \times I_2$ is a regular locally principal ideal of $R_1 \times R_2$ if and only if I_1 and I_2 are locally principal ideals of R_1 respectively R_2 and it is easy to see that $I_1 \times I_2$ is a finitely generated ideal of $R_1 \times R_2$ if and only if I_1 and I_2 are finitely generated ideals of R_1 respectively R_2 and we conclude by Lemma 3.2.4.

The next result shows that the *LPI*-property descends into a faithfully flat ring homomorphism.

Proposition 3.2.5 *Let R and S be two rings and $f : R \to S$ be a ring homomorphism making S a faithfully flat R-module. Assume that $f(Reg(R)) \subseteq Reg(S)$. If S is an LPI ring, then so is R. In particular, if R and S are two domains and $f : R \to S$ is a ring homomorphism making S a faithfully flat R-module then if S is an LPI domain, then so is R.*

Proof. Let I be a regular locally principal ideal of R. Then I is a faithfully flat R-module by Proposition 3.2.1. So $I \otimes S = IS$ is a faithfully flat S-module. Since $f(Reg(R)) \subseteq Reg(S)$, IS is a regular locally principal ideal of S and so it is invertible since S is an *LPI*-ring which, in turn, is equivalent to IS is finitely generated. Therefore $I \otimes S = IS$ is finitely generated. Hence I is a finitely generated ideal of R (as S is a faithfully flat R-module), and therefore I is an invertible ideal of R. Thus R is an *LPI* ring as desired.

As an immediate consequence, we recover [13, Teorem 5(1)].

Corollary 3.2.6 *([13, Theorem 5 (1)]) Let R be a domain and let X be an indeterminate over R. If $R[X]$ (resp.$R[[X]]$) is an LPI ring, then so is R.*

Now, we will see the transfer of the *LPI* property to homomorphic image. Recall that an ideal I of a ring R is a pure ideal if for every maximal ideal $M \in Max(R)$, $I_M = 0_M$ or $I_M = R_M$.

Proposition 3.2.7 *Let R be a ring.*

1. *Let P be a finitely generated non-regular prime ideal of R such that P is contained in all regular locally principal ideals of R. If $\frac{R}{P}$ is an LPI ring, then so is R.*

2. *Let I be a regular pure ideal of R. If R is an LPI ring, then $\frac{R}{I}$ is an LPI ring.*

The proof of this proposition needs the following lemma.

Lemma 3.2.8 *Let P be a non-regular prime ideal of a ring R.*
If J is a regular locally principal ideal of R such that $P \subseteq J$, then $\frac{J}{P}$ is a regular locally principal ideal of $\frac{R}{I}$.

Proof. Let J be a regular locally principal ideal of R such that $P \subseteq J$ and let a be a regular element of J. Necessarily $a \notin P$. Now, for every $x \in R$, $\overline{ax} = \overline{0}$ in $\frac{J}{P}$ implies that $ax \in P$. Hence $x \in P$ since P is prime. Then, $\overline{x} = 0$ and so $\overline{a}$ is a regular element in $\frac{J}{P}$. Therefore, $\frac{J}{P}$ is a regular ideal of $\frac{R}{I}$. Thus if J is locally principal, then the localization of $\frac{J}{P}$ at any maximal ideal $\frac{M}{P}$ of $\frac{R}{P}$ is isomorphic to $\frac{J_M}{P_M}$. Since J_M is principal, $\frac{J_M}{P_M}$ is principal and therefore $\frac{J}{P}$ is locally principal.

Proof of Proposition 3.2.7. (1) Let P be a finitely generated non-regular prime ideal of R contained in all regular locally principal ideals. Assume that $\frac{R}{P}$ is an *LPI* ring. Let J be a regular locally principal ideal of R. Then $\frac{J}{P}$ is a regular locally principal ideal of $\frac{R}{P}$ (Lemma 3.2.8) and so it is invertible. By Lemma 3.2.4, $\frac{J}{P}$ is finitely generated. Now, by the exact sequence,

$$0 \longrightarrow P \longrightarrow J \longrightarrow \frac{J}{P} \longrightarrow 0$$

J is finitely generated and by Lemma 3.2.4 we conclude that J is invertible. Hence R is an *LPI* ring.

(2) Let $\frac{J}{I}$ be a regular locally principal ideal of $\frac{R}{I}$. Then $\frac{J}{I}$ is a faithfully flat $\frac{R}{I}$-module. Since I is a regular ideal of R and $I \subseteq J$, then J is a regular ideal of R. Now, consider the following exact sequence:

$$0 \to I \to J \to \frac{J}{I} \to 0$$

Since I is a regular pure ideal of R, I is a regular locally principal ideal of R. So I is a faithfully flat ideal of R by Proposition 3.2.1. Thus $\frac{J}{I}$ is a flat R-module (since $\frac{R}{I}$ is a flat R-module). By the exact sequence, we conclude that J is a faithfully flat ideal of R. Hence J is a regular locally principal ideal of R. So J is invertible and so finitely generated. Therefore $J \otimes \frac{R}{I} = \frac{J}{I}$ is a finitely generated ideal of $\frac{R}{I}$ and so it is invertible by Lemma 3.2.4. Therefore $\frac{R}{I}$ is an *LPI* ring.

Example 3.2.9 *Let K be a field and $R = K \propto K$ and let $P = 0 \propto K$. Then P is a finitely generated non-regular prime ideal of R that is contained in all regular ideals of R by Lemma 3.2.12. Since $\frac{R}{P} = \frac{K \propto K}{0 \propto K} = K$, $K \propto K$ is an LPI ring by Proposition 3.2.7.*

Our next theorem develops a result on the transfer of the *LPI* property to trivial ring extension. Recall that if E is an A-module, then $Z(E) = \{a \in A$ such that $ae = 0$ for some $0 \neq e \in E\}$.

Theorem 3.2.10 *Let A be a ring, E an A-module, $R = A \propto E$ the trivial ring extension of A by E and let $S = A \setminus (Z(A) \cup Z(E))$.*

1. *Assume that for every regular locally principal ideal I of A, $I \cap S \neq \emptyset$. If R is an LPI ring, then so is A.*

2. *Assume that E is torsion-free and divisible. Then A is an LPI ring if and only if so is R. In particular, if A is a domain and E is a flat divisible R-module, then A is an LPI ring if and only if so is R.*

3. *Assume E is finitely generated and $E = S^{-1}E$. If A is an LPI ring, then so is R. If furthermore, for every regular locally principal ideal I of A, $I \cap S \neq \emptyset$, then A is an LPI ring if and only if so is R.*

The proof of this theorem needs the following lemmas.

Lemma 3.2.11 *1. Let A be a ring, E an A-module and I an ideal of A and let $S = A \setminus (Z(A) \cup Z(E))$.*

   (a) If $I \propto IE$ is a regular locally principal ideal of $A \propto E$, then I is a regular locally principal ideal of A.

   (b) If $I \cap S \neq \emptyset$, then I is a regular locally principal ideal of A if and only if $I \propto IE$ is a regular locally principal ideal of $A \propto E$.

   (c) If E is torsion free, then I is a regular locally principal ideal of A if and only if $I \propto IE$ is a regular locally principal ideal of $A \propto E$.

2. *Let E' is a submodule of E. If $I \propto E'$ is a regular locally principal ideal of $A \propto E$, then so is I.*

Proof. (1) (a) Clearly if (a,e) is a regular element of $I \propto IE$, then a is a regular element of I by [3, Lemma 6]; and I is a locally principal ideal of A if and only if $I \propto IE$ is a locally principal ideal of $A \propto E$ by [3, Theorem 7].

(b) Assume that $I \cap S \neq \emptyset$. Then there exists a regular element a in I such that $a \notin Z(E)$. Clearly $(a,0)$ is a regular element of $I \propto IE$. Indeed, let $(0,0) \neq (b,e) \in I \propto IE$ such that

$(a,0)(b,e)=(0,0)$. Then $ab=0$ and $ae=0$. Since a is regular, $b=0$ and since $a\notin Z(E)$, $e=0$. Thus $(b,e)=(0,0)$ and so $(a,0)$ is regular. Hence $I\propto IE$ is a regular ideal of $A\propto E$.

(c) Follows from [4, Lemma 10].

(2) Let E' is a submodule of E such that $I\propto E'$ is a regular locally principal ideal of $A\propto E$. Then I is a regular ideal of A and for every maximal ideal M of R,

$$\begin{aligned}(I\propto E')_{M\propto E} &= I_M\propto E_M\\ &= (A_M\propto E_M)(x,e)\\ &= A_Mx\propto(A_Me+E_Me)\end{aligned}$$

Hence $I_M=A_Mx$ and therefore I is a locally principal ideal of A.

Lemma 3.2.12 *([12, Theorem 3.9]) Let A be a ring, E an A-module and $S=A\setminus(Z(A)\cup Z(E))$. Then the following conditions are equivalent.*

1. *Every regular ideal of $A\propto E$ has the form $I\propto E$ where I is an ideal of A with $I\cap S\neq\emptyset$.*
2. *Every regular ideal of $A\propto E$ is homogenous.*
3. *$E=S^{-1}E$.*

Lemma 3.2.13 *([3, Theorem 7]) Let A be a ring, I a nonzero ideal of A and E an A-module. If $I\propto IE$ is an invertible ideal of $A\propto E$, then I is invertible.*

Lemma 3.2.14 *([4, Theorem 11]) Let A be a ring, E be a torsion-free and divisible A-module and $I\propto N$ a homogenous ideal of $A\propto E$. If I is an invertible ideal of A, then $I\propto N$ is invertible.*

Lemma 3.2.15 *([3, Theorem 9]) Let A be a ring, E an A-module, $I\propto N$ an homogeneous ideal of $A\propto E$. If I is a finitely generated ideal of A and N is a finitely generated submodule of E, then $I\propto N$ is finitely generated.*

Proof of Theorem 3.2.10. (1) Assume that for every regular locally principal ideal I of A, $I\cap S\neq\emptyset$ and let I be a regular locally principal ideal of A. By Lemma 3.2.11 (1)(b), $I\propto IE$ is a regular locally principal ideal of $A\propto E$. Since $A\propto E$ is an *LPI* ring, then $I\propto IE$ is invertible. By Lemma 3.2.13, I is invertible and so A is an *LPI* ring.

(2) Assume that E is torsion-free and divisible. Then $E=S^{-1}E$ where $S=A\setminus(Z(A)\cup Z(E))$ (since E is divisible).
Let J be a regular locally principal ideal of R. By Lemma 3.2.12, J has the form $I\propto E$ where I is an ideal of A with $I\cap S\neq\emptyset$. Thus I is a regular locally principal ideal of A by

Lemma 3.2.11 (2). Since A is an *LPI* ring, I is invertible. By Lemma 3.2.14, $I \propto E$ is an invertible ideal of $A \propto E$ and therefore $A \propto E$ is an *LPI* ring.

Conversely, assume that $A \propto E$ is an *LPI* ring. Since E is torsion free, $Z(E) \subseteq Z(A)$ and so for every regular locally principal ideal I of A, $I \cap S \neq \emptyset$. Hence A is an *LPI* ring by (1).

(3) Assume that E is finitely generated, $E = S^{-1}E$ and A is an *LPI* ring. Let $I \propto E$ be a regular locally principal ideal of R where I is an ideal of A with $I \cap S \neq \emptyset$. Then I is a regular locally principal ideal of A (Lemma 3.2.11 (2)) and thus I is invertible (and so is finitely generated). Since E is finitely generated, $I \propto E$ is a finitely generated ideal of $A \propto E$ (Lemma 3.2.15) and therefore it is invertible by Lemma 3.2.4. Thus $A \ltimes E$ is an *LPI* ring. If furthermore, for every regular locally principal ideal I of A, $I \cap S \neq \emptyset$, then the equivalence by (1).

Theorem 3.2.10 leads to the following result.

Corollary 3.2.16 *Let A is a domain and E a K-vector space where $K = qf(A)$. Then $A \propto E$ is an LPI ring if and only if so is A.*

Example 3.2.17 *Let $A = \mathbb{Z}_{(2)} = \{\frac{a}{b} \mid a,b \in \mathbb{Z}$ and b is not divisible by $2\}$ and $E = \frac{\mathbb{Q}}{\mathbb{Z}_{(2)}}$. Then $A \propto E$ is an LPI ring.*

Proof. $\mathbb{Z}_{(2)}$, as a *DVR*, is an *LPI* ring.
$\frac{\mathbb{Q}}{\mathbb{Z}_{(2)}}$ is a divisible $(\mathbb{Z}_{(2)})$-module since Q is a divisible $(\mathbb{Z}_{(2)})$-module. Also, $\frac{\mathbb{Q}}{\mathbb{Z}_{(2)}} = \{\frac{a}{b} \mid a,b \in \mathbb{Z}$ and b is divisible by $2\}$ is a flat $(\mathbb{Z}_{(2)})$-module since it is a free module generated by $\frac{1}{2}$. So $A3.2.11E$ is an *LP*-ring by Theorem 3.2.10 (2).

By Theorem 3.2.10, we are able to give new examples of *LPI* rings that are not Noetherian.

Example 3.2.18 *Let K be a field and E be an infinite dimensional vector space over K. Then :*

1. *$K \propto E$ is an LPI ring (by Corollary 3.2.16).*
2. *$K \propto E$ is not Noetherian (since E is not finitely generated and by [12, Theorem 4.8]).*

Next, we will see the transfer of the *LPI* property to the amalgamated duplication. Recall that an ideal J of $A \bowtie I$ is called homogeneous if $J = K \bowtie I$ for some ideal K of A. If K is an ideal of $A \bowtie I$ such that $0 \times I \subseteq K$, then K is homogeneous (see [72, Lemma 2.9]).

Theorem 3.2.19 *Let A be a ring and I be an ideal of A.*

1. *Assume that I is flat. If $A \bowtie I$ is an LPI ring, then so is A.*

2. *Assume that I is a non-regular finitely generated ideal of A and $S^{-1}I = I$ where $S = A \setminus Z(A)$. If A is an LPI ring, then so is $A \bowtie I$. If furthermore I is flat, then A is an LPI ring if and only if so is $A \bowtie I$.*

The proof of this theorem needs the following lemmas. The proofs of the first and third lemma are elementary proofs and for convenience we include them here.

Lemma 3.2.20 *Let A be a ring and I be an ideal of A. If $(a, a+i)$ is a regular element of $A \bowtie I$, then a is a regular element of A.*

Proof. Let $(a, a+i)$ be a regular element of $A \bowtie I$ and $b \in A$ such $ab = 0$. If there exists $j \in I$ such that $jb \neq 0$, then $(a, a+i)(bj, 0) = (0,0)$, which is absurd since $(a, a+i)$ is regular. Hence, for all $j \in I$, $jb = 0$. Thus $(a, a+i)(b, b) = (0,0)$ and so $(b, b) = (0,0)$. Hence a is a regular element of A.

Lemma 3.2.21 *Assume that I is a non-regular proper ideal of a ring A and $S = A \setminus (Z(A))$. The following are equivalent:*

1. *All regular ideals of $A \bowtie I$ are homogeneous.*

2. *For all $a \in S$ we have $I = aI$ which is equivalent to $I = S^{-1}I$.*

Proof. $(1) \Rightarrow (2)$ Assume that all regular ideals of $A \bowtie I$ are homogeneous and let $a \in S$. Then (a, a) is a regular element of $A \bowtie I$ and so $\langle (a, a) \rangle$ is a homogeneous ideal of $A \bowtie I$. Thus $0 \times I \subseteq \langle (a, a) \rangle = J \bowtie I$ for some ideal J of A. Let $i \in I$. Then, $(0, i) = (\alpha, \alpha + k)(a, a)$ where $\alpha \in A$ and $k \in I$. Since a is a regular element of A, $\alpha a = 0$ implies that $\alpha = 0$. So $(0, i) = (0, k)(a, a)$. Thus $i = ka$ and so $I = aI$.

$(2) \Rightarrow (1)$ Let K be a regular ideal of $A \bowtie I$ and let $(a, a+i) \in K$ a regular element of $A \bowtie I$. By Lemma 3.2.20, a is a regular element of A and so $\langle (a, a+i) \rangle = (a, a+i)A \bowtie I = aA \bowtie (iA + aI + iI))$. Since $I = aI$, $\langle (a, a+i) \rangle = aA \bowtie I$. Hence $0 \times I \subseteq \langle (a, a+i) \rangle \subseteq K$ and therefore K is a homogeneous ideal of $A \bowtie I$.

Remark 3.2.22 *Let A be a ring and I be a proper regular ideal of A. Then there exists a regular ideal of $A \bowtie I$ which is not homogeneous. Indeed, if we suppose that all regular ideals of $A \bowtie I$ are homogeneous, then the ideal generated by (c, c), where c is a regular element in I, is homogeneous. By the same argument in the proof of Lemma 3.2.21 we show that $I = cI$ and so $c = c\lambda$ for some $\lambda \in I$ (since $c \in I$). So, $\lambda = 1$ since c is regular. Therefore, $I = A$ which is absurd since I is a proper ideal of A.*

Lemma 3.2.23 *Let I and J be two ideals of a ring A. If I and J are finitely generated, then $J \bowtie I$ is a finitely generated ideal of $A \bowtie I$.*

Proof. Assume J is generated by a family of elements $\{a_1, a_2 \cdots, a_n\}$ and I is generated by a family of elements $\{k_1, k_2 \cdots, k_n\}$. Let $(a, a+h) \in J \bowtie I$. Then, $a = \sum_{i=1}^{n} \alpha_i a_i$ where $\alpha_i \in A$ for $i = 1, \cdots, n$, and since $h \in I$, then $h = \sum_{i=1}^{n} \beta_i k_i$ where $\beta_i \in A$ for $i = 1, \cdots, n$. Hence,

$$(a, a+h) = (a, a) + (0, h) = \sum_{i=1}^{n} (\alpha_i, \alpha_i)(a_i, a_i) + \sum_{i=1}^{n} (\beta_i, \beta_i)(0, k_i).$$

Hence $J \bowtie I$ is a finitely generated ideal of $A \bowtie I$ generated by

$$\{(a_i, a_i)\}_{i=1,\cdots,n} \cup \{(0, k_i)\}_{i=1,\cdots,n}.$$

Proof of Theorem 3.2.19. (1) Assume that I is a flat ideal of A. Then $A \bowtie I$ is a faithfully flat A-module. Let J be a regular locally principal ideal of A. Then $J \otimes (A \bowtie I) = J(A \bowtie I)$ since $A \bowtie I$ is a faithfully flat A-module. So $J \otimes (A \bowtie I) = J(A \bowtie I) = J \bowtie JI$ by [42, Proposition 3.1(2)]. Since J is regular, then $J \bowtie JI$ is a regular ideal of $A \bowtie I$. By Proposition 3.2.1, J is a faithfully flat ideal of A and so $J \otimes (A \bowtie I) = J \bowtie JI$ is a faithfully flat ideal of $(A \bowtie I)$. Thus it is locally principal by Proposition 3.2.1. Since $(A \bowtie I)$ is an *LPI* ring, then $J \otimes (A \bowtie I) = J \bowtie JI$ is invertible which is equivalent to $J \otimes (A \bowtie I)$ is finitely generated. Since $A \bowtie I$ is a faithfully flat A-module, J is a finitely generated ideal of A and so invertible. Hence A is an *LPI* ring.

(2) Let K be a regular locally principal ideal of $A \bowtie I$. Then K is a homogeneous ideal by Lemma 3.2.21. So $K = J \bowtie I$ for some ideal J of A. Let M be a maximal ideal of A. Two cases are possible:
Case 1: $I \subseteq M$. Then $M \bowtie I$ is a maximal ideal of $A \bowtie I$ and so $J_M \bowtie I_M = (J \bowtie I)_{M \bowtie I} = K_{M \bowtie I}$ ([44, Theorem 3.8]) is a principal ideal of $(A \bowtie I)_{M \bowtie I}$. Hence J_M is a principal ideal of A_M.
Case 2: $I \nsubseteq M$. Then $J_M = (J \bowtie I)_{M \bowtie I}$ ([44, Theorem 3.5]) is a principal ideal of $(A \bowtie I)_{M \bowtie I}$. Hence J_M is a principal ideal of A_M.
In the both cases J is locally principal. So J is regular locally principal. Hence J is invertible and so it is finitely generated. Thus $J \bowtie I$ is a finitely generated ideal of $A \bowtie I$ by Lemma 3.2.23, and therefore, $J \bowtie I$ is invertible. Hence $A \bowtie I$ is an *LPI* ring. If furthermore I is flat, then the converse follows from (1).

Example 3.2.24 *Let $A = K \propto E$ where K is a field and E is an infinite dimensional vector space over K and let $I = 0 \propto F$ where F is a finite dimensional subspace of E . Then :*

1. *$A \bowtie I$ is an LPI ring (by Theorem 3.2.19(2)).*
2. *$A \bowtie I$ is not Noetherian (since A is not Noetherian by Example 3.2.18 and by [44, Corollary 2.9]).*

CHAPTER 4

ON RINGS WITH ADEQUATE RANGE ONE

Abstract.

In this chapter*, we study the class of rings with adequate range one and investigate the transfer of this property to various contexts of constructions such as pullbacks, trivial ring extensions and amalgamation of rings. Our results provide new classes of commutative rings satisfying this property.

Key Words: Amalgamated algebra, Amalgamated duplication, Krull dimension, (n,d)-Krull ring, n-presentation, von Neumann regular ring.

4.1 Introduction

We denote by $rad(R)$ the Jacobson radical of a ring R and by $U(R)$ the set of all units in R. Let $a,b \in R$, $a|b$ means a divides b, that is $b = ac$ for some $c \in R$.

An element c of a ring R is adequate provided that for any $a \in R$ there exist some $r,s \in R$ such that :

$$\begin{cases} c = rs; \\ rR + aR = R; \\ s'R + aR \neq R \text{ for each noninvertible divisor } s' \text{ of } s. \end{cases}$$

A ring in which every nonzero element is adequate is called an adequate ring. This is the definition given in [47, 63, 75, 74, 76] and that we will use in this paper. In [88], they state that an adequate ring is a Bézout ring in which any nonzero element is adequate. A ring R has stable range one provided that $aR + bR = R$ with $a,b \in R$ implies that there

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exists some $y \in R$ such that $a+by \in U(R)$. Equivalently, R has stable range one if and only if for any $a,x,b \in R$ satisfying $ax+b=1$, there exists $y \in R$ such that $a+by \in U(R)$. Such a condition plays an important role in algebraic K-theory (see [33]). Rings having stable range one include semilocal rings, unit-regular rings, strongly π-regular rings, exchange rings with artinian primitive factors, fields or division rings and Artinian rings (in particular, any finite-dimensional algebra over a field has stable range one), (see [22]). The author of [82] proved that if I is an ideal of a ring R contained in its Jacobson radical $rad(R)$, then R has stable range one if and only if so does $\frac{R}{I}$. A ring R has adequate range one provided that $aR+bR=R$ with $a,b \in R$ implies that there exists some $y \in R$ such that $a+by$ is adequate [47].

The notion of an adequate domain was originally defined by Helmer [63]. By definition, every adequate domain is a Prüfer domain. Also, every principal ideal domain is adequate. An example of an adequate ring which is not a principal ideal domain is furnished by the set of integral functions with coeficients in a field. Also, it is clear to see that a local ring is adequate. For instance, see [63, 76].

We know that an elementary divisor ring is a Hermite ring. One of the most attractive problems is to investigate various conditions under which a Hermite ring is an elementary divisor ring. Many authors have studied this problem over some classes of rings, like Euclidean rings, 2-Euclidean rings, regular rings, adequate rings, and rings having almost stable range one (cf. [47, 70, 89]). Kaplansky showed that for the class of adequate domains being a Hermite ring was equivalent to being an elementary divisor ring. Gillman and Henriksen showed that this was also true for rings with zero-divisors. See for instance [63, 70, 76]. Domsha and Vasiunyk stadied the same problem and combined the stable range one condition with adequate condition together and define a new class of rings called rings having adequate range one. They showed in [47, Theorem 14] that every Bézout domain having adequate range one is an elementary divisor ring.

4.2 Main results

We start by giving a characterization of the adequate range one property.

Proposition 4.2.1 *Let R be a ring. The following conditions are equivalent:*

1. *R satisfies adequate range one;*
2. $\forall a,x,b \in R$ *such that* $ax+b=1$ *there exists* $y \in R$ *such that* $a+by$ *is adequate.*

Proof. $(1) \Rightarrow (2)$. Let $a,x,b \in R$ such that $ax+b=1$. Then, $aR+bR=R$ since $1 \in (aR+bR)$. Hence, there exists $y \in R$ such that $a+by$ is adequate.

$(2) \Rightarrow (1)$. Let $a,b \in R$ such that $aR+bR=R$. Then, there exist $r,s \in R$ such that $ar+bs=1$. By hypothesis, there exists $z \in R$ such that $a+bsz$ is adequate. So, there exists $y=sz$ such that $a+by$ is adequate.

For homomorphic image of adequate range one property, we have:

Theorem 4.2.2 *Let R be a ring and I be an ideal of R. Assume that $\forall x \in R \setminus I$, $I=xI$. Then,*

1. *If R has adequate range one then so does $\frac{R}{I}$.*
2. *Assume that all elements of I are adequate. Then, R has adequate range one if and only if so does $\frac{R}{I}$.*

The proof of this theorem involves the following lemma.

Lemma 4.2.3 *Let R be a ring and I be an ideal of R. Assume that $\forall x \in R \setminus I$, $I=xI$. Then, a is an adequate element of $R \setminus I$ if and only if $\overline{a}$ is an adequate element of $\frac{R}{I} \setminus 0$.*

Proof. Let a be an edequate element of $R \setminus I$ and let $\overline{k} \in \frac{R}{I}$. Then, $k \in R$ and since a is adequate in R, there exist some $r,s \in R$ such that : $\begin{cases} 1) & a=rs; \\ 2) & rR+kR=R; \\ 3) & s'R+kR \neq R \text{ for each noninvertible divisor } s' \text{ of } s \end{cases}$

1) $a=rs$ implies that $\overline{a}=\overline{rs} \neq \overline{0}$.
2) $rR+kR=R$ implies that there exsit $x,y \in R$ such that $rx+ky=1$. So, $\overline{rx+ky}=\overline{rx}+\overline{k}\overline{y}=\overline{1}$. Hence, $\overline{r}\frac{R}{I}+\overline{k}\frac{R}{I}=\frac{R}{I}$.
3) Let $\overline{s'} \notin U(\frac{R}{I})$ such that $\overline{s'}|\overline{s}$. Then, $s' \notin U(R)$. Indeed, suppose that $s' \in U(R)$. Then there exists $m \in R$ such that $s'm=1$. So, $\overline{s'm}=\overline{1}$ which is absurd since $\overline{s'} \notin U(\frac{R}{I})$. Since $\overline{s'}|\overline{s}$, then there exists $c \in R$ such that

$$\begin{aligned} \overline{s} &= \overline{c}\overline{s'}, \text{ and so} \\ s &= cs'+i, \text{ for some } i \in I \text{ and since } s' \notin I \text{ (because } \overline{s}=\overline{c}\overline{s'} \neq \overline{0}) \text{ then} \\ s &= cs'+ms', \text{ for some } m \in I \text{ (since } I=xI, \forall x \in R \setminus I) \\ &= (c+m)s'. \end{aligned}$$

Hence, $s'|s$. Therefore, $s'R+kR \neq R$ and so $\forall x,y \in R$, $s'x+ky \neq 1$. Suppose that there exist $x,y \in R$ such that $\overline{s'}\overline{x}+\overline{k}\overline{y}=\overline{1}$. Then,

$$\begin{aligned} s'x+ky+i &= 1, \text{ for some } i \in I \text{ and so} \\ s'x+ky+kg &= 1, \text{ for some } g \in I \text{ (since } I=xI, \forall x \in R \setminus I \text{ and } s \notin I) \\ s'x+k(y+g) &= 1 \end{aligned}$$

and so $s'R+kR=R$ which is absurd. Hence, $\overline{s'}\frac{R}{I}+\overline{k}\frac{R}{I} \neq \frac{R}{I}$.

By 1), 2) and 3), $\overline{a}$ is an adequate element of $\frac{R}{I}$.

Conversely, assume that $\overline{a}$ is an adequate element of $\frac{R}{I} \setminus 0$. Let $k \in R$. Since $\overline{a}$ is adequate in $\frac{R}{I}$, there exist some $\overline{r}, \overline{s} \in \frac{R}{I}$ such that :

$$\begin{cases} 1) & \overline{a} = \overline{rs}; \\ 2) & \overline{r}\frac{R}{I} + \overline{k}\frac{R}{I} = \frac{R}{I}; \\ 3) & \overline{s'}\frac{R}{I} + \overline{k}\frac{R}{I} \neq \frac{R}{I} \text{ for each noninvertible divisor } \overline{s'} \text{ of } \overline{s}. \end{cases}$$

1) $\overline{a} = \overline{rs}$ implies that $a = rs + i$ for some $i \in I$ and so $a = rs + rm$ for some $m \in I$ (since $I = xI$, $\forall x \in R \setminus I$ and $r \notin I$). Hence, $a = r(s+m)$.
2) $\overline{r}\frac{R}{I} + \overline{k}\frac{R}{I} = \frac{R}{I}$ implies that there exist $\overline{x}, \overline{y} \in \frac{R}{I}$ such that

$$\begin{aligned} \overline{rx} + \overline{ky} &= 1 \\ rx + ky + i &= 1, \text{ for some } i \in I \text{ and so} \\ rx + ky + rg &= 1, \text{ for some } g \in I \text{ (since } I = xI, \forall x \in R \setminus I \text{ and } r \notin I) \\ r(x+g) + ky &= 1 \end{aligned}$$

and so $rR + kR = R$.
3) Let $s' | (s+m)$ such that $s' \notin U(R)$. Then, $\overline{s'} | \overline{s}$. Indeed, $(s+m) = cs'$ for some $c \in R$ and since $m \in I$ then $\overline{s+m} = \overline{s} = \overline{c}\overline{s'}$. Also, $\overline{s'} \notin U(\frac{R}{I})$ since if $\overline{s'} \in U(\frac{R}{I})$ then there exists some $\overline{v}$ such that $\overline{s'}\overline{v} = \overline{1}$. Hence, $s'v + i = 1$ for some $i \in I$ and so $s'v + s'g = s'(v+g) = 1$ for some $g \in I$ (since $I = xI$, $\forall x \in R \setminus I$ and $s' \in U(\frac{R}{I})$ so $s' \notin I$) which is absurd. Therefore, $\overline{s'} \notin U(\frac{R}{I})$. So, $\overline{s'}\frac{R}{I} + \overline{k}\frac{R}{I} \neq \frac{R}{I}$. Suppose that there exist $x, y \in R$ such that $s'x + ky = 1$. Then, $\overline{s'}\overline{x} + \overline{k}\overline{y} = \overline{1}$ and so $\overline{s'}\frac{R}{I} + \overline{k}\frac{R}{I} = \frac{R}{I}$ which is absurd. Therefore, $s'R + kR \neq R$.

By 1), 2) and 3), a is an adequate element of $R \setminus I$.

Proof of Theorem 4.2.2. (1) Assume that R is a ring with adequate range one. Let $a, x, b \in R$ such that $\overline{ax+b} = \overline{1}$. Then, there exists $i \in I$ such that $ax + b + i = 1$. Since R has adequate range one then there exists $y \in R$ such that $a + (b+i)y$ is an adequate element of R (by Proposition 4.2.1). Two cases are possible :

Case 1: $a + (b+i)y \in I$, then $\overline{a+(b+i)y} = \overline{a+by+iy} = \overline{a+by} = \overline{0}$ and $\overline{0}$ is an adequete element of $\frac{R}{I}$.

Case 2: $a + (b+i)y \notin I$. $\overline{a+(b+i)y}$ is an adequate element of $\frac{R}{I} \setminus \{0\}$ by Lemma 4.2.3. So, $\overline{a+(b+i)y} = \overline{a+by+iy} = \overline{a+by}$ is adequate in $\frac{R}{I}$.
Hence, $\frac{R}{I}$ has adequate range one.

(2) Assume that $\frac{R}{I}$ has adequate range one and all elements of I are adequate. Let $a, x, b \in R$ such that $ax + b = 1$. Then, $\overline{ax+b} = \overline{1}$. Since $\frac{R}{I}$ has adequate range one then there exists $\overline{y} \in \frac{R}{I}$ such that $\overline{a+by}$ is adequate. Two cases are possible :

Case 1: $a + by \in I$. Then, $a + by$ is an adequete element of R by hypothesis.

Case 2: $a + by \notin I$. By Lemma 4.2.3, $a + by$ is an adequate element of $R \setminus I$.
Therefore, R has adequate range one.

By Theorem 4.2.2, we have the following corollary.

Corollary 4.2.4 *Let R be a ring and P be a prime ideal of R and $S = R \setminus P$. Assume that $S^{-1}P = P$. Then,*

1. *If R has adequate range one then so does $\frac{R}{P}$.*
2. *Assume that all elements of P are adequate. Then, R has adequate range one if and only if so does $\frac{R}{P}$.*

Now, we study the transfer of the adequate range one property in trivial ring extension.

Theorem 4.2.5 *Let A be a ring and, E be an A-module, and let $R = A \propto E$ be the trivial ring extension of A by E. Assume that for every $k \in A \setminus \{0\}$, $kE = E$. Then, R has adequate range one if and only if so does A.*

The proof of this theorem involves the following lemma.

Lemma 4.2.6
Let A be a ring and, E be an A-module, and let $R = A \propto E$ be the trivial ring extension of A by E. Then,

1. *Let $a,b,c \in A$ such that $bE = E$ and let $x,y \in E$. Then, the following conditions are equivalent:*
   - *(a) $\exists z \in E$ such that $(a,x) = (b,y)(c,z)$;*
   - *(b) $a = bc$.*
2. *Let $a,b \in A$ such that $bE = E$ and let $x,y \in E$. Then, $(b,y)|(a,x)$ if and only if $b|a$.*
3. *Let $a,b \in A$ such that $bE = E$ and let $x,y \in E$. Then, $(a,x)R + (b,y)R = R$ if and only if $aA + bA = A$.*

Proof. (1) and (2) comes from [75, Lemma 2.3].
(3) Let $a,b \in A$ such that $bE = E$ and let $x,y \in E$. Then, $(a,x)R + (b,y)R = R$ is equivalent to $(aA \propto (aE + xA)) + (bA \propto E) = A \propto E$ (since $bE = E$ by hypothesis) which is equivalent to $((aA + bA) \propto E) = A \propto E$ and so $aA + bA = A$.

Proof of Theorem 4.2.5. Suppose that R has adequate range one. Let $a,b \in A$ such that $aA + bA = A$. Since $\forall k \in A \setminus \{0\}$, $kE = E$ then $(a,0)R + (b,0)R = R$. So, there exists $(x,y) \in R$ such that $(a,0) + (b,0)(x,y) = (a + bx, by)$ is adequate.
Let $d \in A$. Then $\forall e \in E$, $(d,e) \in R$. Since $(a + bx, by)$ is adequate then there exist $(r,l),(s,m) \in R$ such that,

$$\begin{cases} 1) & (a+bx,by)=(r,l)(s,m); \\ 2) & (r,l)R+(d,e)R=R; \\ 3) & (s',z)R+(d,e)R\neq R \text{ for each noninvertible divisor } (s',z) \text{ of } (s,m). \end{cases}$$

1) $(a+bx,by)=(r,l)(s,m)$ implies that $a+bx=rs$.
2) $(r,l)R+(d,e)R=R$ implies that $rA+dA=A$ by Lemma 4.2.6(3).
3) Let $s'\in A\setminus U(A)$ such that $s'|s$. Then, $(s',z)|(s,m)$ for all $z,m\in E$ by Lemma 4.2.6(2) and $(s',z)\notin U(R)$ since $s'\notin U(A)$. So, $(s',z)R+(d,e)R\neq R$ and by Lemma 4.2.6(3) $s'A+dA\neq A$.

By 1), 2) and 3), $a+bx$ is an adequate element of A and so A has adequate range one.

Conversely, Assume that A has adequate range one. Let $(a,e)R+(b,e')R=R$. Then, $aA+bA=A$ by Lemma 4.2.6(3). so, there exists $y\in A$ such that $a+by$ is adequate. Let $x\in E$ and $(d,f)\in R$. Since $a+by$ is adequate then there exist some $r,s\in A$ such that :

$$\begin{cases} 1) & a+by=rs; \\ 2) & rA+dA=A; \\ 3) & s'A+dA\neq A \text{ for each noninvertible divisor } s' \text{ of } s. \end{cases}$$

1) By Lemma 4.2.6(1), $a+by=rs$ implies that $(a,e)+(b,e')(y,x)=(a+by,bx+e'y+e)=(rs,bx+e'y+e)$ and since $bx+e'y+e\in E$ there exists $m\in E$ such that $bx+e'y+e=rm$. Hence, $(a,e)+(b,e')(y,x)=(a+by,bx+e'y+e)=(rs,rm)=(r,0)(s,m)$.
2) $rA+dA=A$ implies that $(r,0)R+(d,f)R=R$ by Lemma 4.2.6(3).
3) Let $(s',n)\in R\setminus U(R)$ such that $(s',n)|(s,m)$. Then, $s'\notin U(A)$ and $s'|s$. So, $s'A+dA\neq A$. Hence, $(s',n)R+(d,f)R\neq R$.

By 1), 2) and 3), there exists $Y=(y,x)$ such that $(a,e)+(b,e')(y,x)$ is an adequate element of R. Therefore, R has adequate range one.

The following corollary is a consequence of Theorem 4.2.5.

Corollary 4.2.7 *Let A be an integral domain, E a divisible A-module and $R=A\propto E$ be the trivial ring extension of A by E. Then R has adequate range one if and only if so does A.*
In particular, if $K:=qf(A)$ is the quotient field of A and E is a K-vector space. Then R has adequate range one if and only if so does A.

Remark 4.2.8 *A local ring is an adequate ring by [75, Theorem 2.1] and so local rings have adequate range one.*

Example 4.2.9 *Let $A=\mathbb{Z}_{(2)}=\{\frac{a}{b}\mid a,b\in\mathbb{Z}$ and b is not divisible by $2\}$ and $E=\frac{\mathbb{Q}}{\mathbb{Z}_{(2)}}$. Then, $A\propto E$ has adequate range one.*

Proof. $A=\mathbb{Z}_{(2)}$ is a local domain and so has adequate range one.
$E=\frac{\mathbb{Q}}{\mathbb{Z}_{(2)}}$ is a divisible $(\mathbb{Z}_{(2)})$-module (since $\mathbb{Q}$ is divisible). Hence, for all $k\in\mathbb{Z}_{(2)}\setminus\{0\}$,

$kE = E$. So, $A \propto E$ has adequate range one by Theorem 4.2.5.

For the transfer of the adequate range one property to pullbacks, by Theorem 4.2.2 we can obtain the following result.

Corollary 4.2.10 *Let R be a ring, $\varphi : R \longrightarrow T$ be an injective ring homomorphism, and Q be an ideal of R such that $QT = Q$. Assume that $\forall x \in T \setminus Q$, $Q = xQ$. Then,*

1. *If R (resp. T) has adequate range one then so does $\frac{R}{Q}$ (resp $\frac{T}{Q}$).*
2. *Assume that all elements of Q are adequate. Then, R (resp T) has adequate range one if and only if so does $\frac{R}{Q}$ (resp $\frac{T}{Q}$).*

Now, we study the transfer of the adequate range one property to amalgamation of rings.

Theorem 4.2.11 *Let A and B be rings, J be a propre ideal of B, and let $f : A \to B$ be a ring homomorphism.*

1. *If $A \bowtie^f J$ has adequate range one, then so does A.*
2. *Assume that all elements of J are adequate and $\forall x \in (f(A)+J) \setminus J$, $xJ = J$. Then, $A \bowtie^f J$ has adequate range one if and only if so does A.*

Proof. (1) Assume that $A \bowtie^f J$ has adequate range one. Let $a, x, b \in A$ such that $ax+b = 1$. Then, $(ax+b, f(ax+b)) = (1,1)$. Therefore, $(ax+b, f(ax+b)) = (ax+b, f(a)f(x) + f(b)) = (a, f(a))(x, f(x)) + (b, f(b)) = (1,1)$. Since $A \bowtie^f J$ has adequate range one then there exists $u = (y, f(y)+j) \in A \bowtie^f J$ such that $(a, f(a)) + (b, f(b))(y, f(y)+j)$ is adequate. Thus, $(a+by, f(a)+f(by)+jf(b))$ is adequate and clearly $a+by$ is an adequate element of A. Therefore, A has adequate range one.

(2) Assume that $\forall x \in f(A)+J$, $xJ = J$. Since $\frac{A \bowtie^f J}{0 \times J} = A$ then by Theorem 4.2.2, $A \bowtie^f J$ has adequate range one if and only if so is A.

We recall that an adequate ring has adequate range one. By the following example we show that the convers is not true.

Recall that if a ring R has stable range one then R has adequate range one (see for instance [37]) and if I is an ideal of a ring R contained in its Jacobson radical $rad(R)$, then R has stable range one if and only if so does $\frac{R}{I}$ (see [82]).

Example 4.2.12 *Let A be the localization of the ring of integers by the maximal ideals $2\mathbb{Z}$ and $3\mathbb{Z}$, $A = \{\frac{a}{b} \in \mathbb{Q} |\ b$ is neither divisible by 2 nor by $3\}$ and $B = \frac{\mathbb{R}[[X]]}{(X^2+1)^4\mathbb{R}[[X]]}$ and let $J = \frac{(X^2+1)^2\mathbb{R}[[X]]}{(X^2+1)^4\mathbb{R}[[X]]}$ be an ideal of B, and*

$$\begin{array}{rccl} f: & A & \to & B \\ & a & \to & f(a)=\overline{a} \end{array}$$

1. *$A \bowtie^f J$ has adequate range one.*

2. *$A \bowtie^f J$ is not an adequate ring.*

Proof. (1) We have $J \subseteq Rad(B)$ (since B is a local ring) and so $0 \times J \subseteq Rad(A \bowtie^f J)$. So, $\frac{A \bowtie^f J}{0 \times J} = A$ has stable range one if and only if so does $A \bowtie^f J$ (by [82]) and since A is a semilocal ring then A has stable range one. Hence, $A \bowtie^f J$ has stable range one and so has adequate range one.

(2) In [74, Example 2.13], we replace the ring $A = \mathbb{Z}$ by the ring $A = \{\frac{a}{b} \in \mathbb{Q} | \ b$ is neither divisible by 2 nor by $3\}$. Then, by applying [74, Corollary 2.12] $A \bowtie^f J$ is an adequate ring if and only if A is a local ring. and since it is not the case then $A \bowtie^f J$ is not an adequate ring.

CHAPTER 5

THE ϕ-KRULL DIMENSION OF SOME COMMUTATIVE EXTENSIONS

Abstract.

In this chapter*, we define the ϕ-Krull dimension as a generalization of Krull dimension. This property is treated here as a part of a study on some constractions of rings such as direct products, amalgamation of rings, and trivial ring extensions.

Key Words: Semiprime ideal, ϕ-prime ideal, ϕ-Krull dimension, trivial extension, amalgamation of rings.

5.1 Introduction

We denote by $Nilp(R)$ the nilradical of a ring R, $Z(R)$ the set of zero divisors of R and by $Reg(R)$ the set of regular elements in R. We denote the set of all ideals (resp. proper ideals) of a ring R by $\mathscr{I}(R)$ (resp. $\mathscr{I}^*(R)$).

Anderson and Smith [10], defined a weakly prime ideal, i.e., a proper ideal P of R with the property that for $a,b \in R$, $0 \neq ab \in P$ implies $a \in P$ or $b \in P$. Then the authors of [24] defined the notion of almost prime ideal, i.e., an ideal $P \in \mathscr{I}^*(R)$ with the property that if $a,b \in R$, $ab \in P \setminus P^2$, then either $a \in P$ or $b \in P$. Thus a weakly prime ideal is almost prime and any proper idempotent ideal is also almost prime. Moreover, an ideal P of R is almost prime if and only if P/P^2 is a weakly prime ideal of R/P^2. Anderson and Bataineh in [13], extended these concepts to ϕ-prime ideals. Let $\phi : \mathscr{I}(R) \to \mathscr{I}(R) \cup \{\emptyset\}$ be a function. A proper ideal P of R is called ϕ-prime if for $x,y \in R$, $xy \in P \setminus \phi(P)$ implies $x \in P$ or $y \in P$. In fact, P is a ϕ-prime ideal of R if and only if $P/\phi(P)$ is a weakly prime ideal of $R/\phi(P)$.

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In 2017, J. Bagheri Harehdashti and H. Fazaeli Moghimi defined the ϕ-radical of an ideal I as the intersection of all ϕ-prime ideals of R containing I and investigate when the set of all ϕ-prime ideals of R has a Zariski topology analogous to that of the prime spectrum. Since $P \setminus \phi(P) = P \setminus (P \cap \phi(P))$, there is no loss of generality in assuming that $\phi(P) \subseteq P$.

The purpose of this stady is to open the door for new studies that concern what we define and call the ϕ-Krull dimension as a generalization of Krull dimension. This property is treated here as a part of a study on some constractions of rings such as direct products, amalgamation of rings, and trivial ring extensions.

5.2 The generalization of prime ideals in Trivial ring extension and amalgamation of rings

The following proposition is a result given in [19, Proposition 2.7] that concerns some particular cases when an ideal of the trivial ring extension is ψ-prime, and for the convenience of the reader we include it here. Recall that if A is a ring, E is an A-module and $R = A \propto E$ is the trivial ring extension of A by E, an ideal J of R is said to be homogeneous if $J = I \propto E'$ where I is an ideal of R, E' is a submodule of E, and $IE \subseteq E'$ and $A \propto E$ is said an homogeneous ring if all its ideals are homogeneous (see for instance [12]).

Proposition 5.2.1 *[19, Proposition 2.7] Let A be a ring, E an A-module, and $A \propto E$ a homogeneous ring. Let $\phi : \mathscr{I}(A) \to \mathscr{I}(A) \cup \{\emptyset\}$ and $\psi : \mathscr{I}(A \propto E) \to \mathscr{I}(A \propto E) \cup \{\emptyset\}$ be two functions such that $\psi(I \propto N) = \phi(I) \propto N$. Then*

1. *If $Q = P \propto N$ is a ψ-prime ideal of $A \propto E$, then P is a ϕ-prime ideal of A.*
2. *P is a ϕ-prime ideal of A if and only if $P \propto E$ is a ψ-prime ideal of $A \propto E$.*

The followin proposition shows in which case the ideals with the form $0 \propto F$ are ψ-prime.

Proposition 5.2.2 *Let A be a ring, E an A-module and $\phi : \mathscr{I}(A) \to \mathscr{I}(A) \cup \{\emptyset\}$ a function. Let N be a submodule of E and $\psi : \mathscr{I}(A \propto E) \to \mathscr{I}(A \propto E) \cup \{\emptyset\}$ be a function defined by :*

$$\psi(H) = \begin{cases} (\phi(I_H) \propto N) \cap H & \text{if } \phi(I_H) \neq \emptyset \\ \emptyset & \text{if } \phi(I_H) = \emptyset \end{cases}$$

where $I_H = \{a \in A \mid (a,e) \in H\}$. Let F be a submodule of E and $J = 0 \propto F$. Assume that $\psi(J) \neq \emptyset$ then,

1. *If $K = N \cap F \neq F$ and there exist $a,b \in A \setminus \{0\}$ such that $ab = 0$ and $f \in F \setminus K$ such that $af \in F \setminus K$ then J is not ψ-prime.*
2. *If $K = N \cap F = F$ then J is ψ-prime.*
3. *If A is a domain, then the following statements are equivalent :*
   (a) *$0 \propto F$ is ψ-prime.*
   (b) *$\forall a \in A \setminus \{0\}$, $\forall e \in E$, $ae \in F \setminus K$ implies that $e \in F \setminus K$.*

If $\psi(J) = \emptyset$, then $0 \propto F$ is ψ-prime if and only if $\{0\}$ is φ-prime and $F = E$.

Proof. (1) If $K = N \cap F \neq F$ then $\psi(J) = 0 \propto K$ since $\phi(I_J) \subseteq I_J = \{0\}$. We have $(a,0)(b,f) = (a,af) \in (0 \propto F) \setminus (0 \propto K)$ since $af \in F \setminus K$. But $(a,0) \notin (0 \propto F)$ so $0 \propto F$ is not ψ-prime.
(2) If $N \cap F = F$, then $\psi(J) = 0 \propto F = J$ and so $J \setminus \psi(J) = \emptyset$. Hence, J is ψ-prime.
(3) $(a) \Rightarrow (b)$ By contraposition, suppose that there exist $a \in A \setminus \{0\}$ and $e \in E$, such that $ae \in F \setminus K$ and $e \notin F \setminus K$. We have $\psi(J) = 0 \propto K$. If $e \in K$ then $ae \in K$ which is a contradiction with the fact that $ae \in F \setminus K$. So, e is surely in $E \setminus F$. We have $(a,f)(0,e) = (0,ae) \in J \setminus \psi(J)$ but $(0,e) \notin (0 \propto F) = J$. Therefore, J is not ψ-prime.
$(b) \Rightarrow (a)$ Let $(a,e)(b,f) \in J \setminus \psi(J)$. Then, $(ab, af+be) \in J \setminus \psi(J)$. So, $ab = 0$ and $(af+be) \in F \setminus K$. Hence, $a = 0$ or $b = 0$. If $a = 0$ then $be \in F \setminus K$ and so $e \in F \setminus K$ by hypothesis and so $(a,e) = (0,e) \in 0 \propto F$. Similarly, we show that if $b = 0$ then $(b,f) = (0,f) \in 0 \propto F$. Hence, J is ψ-prime.

If $\psi(J) = \emptyset$ then $\phi(I_J) = \emptyset$, then we are in the case of prime ideals. So $0 \propto F$ is prime if and only if $\{0\}$ is prime and $F = E$ by [12, Theorem 3.2].

Remark 5.2.3 *If $\psi : \mathscr{I}(A \propto E) \to \mathscr{I}(A \propto E) \cup \{\emptyset\}$ is a function defined by*

$$\psi(H) = \begin{cases} (\phi(I_H) \propto E) \cap H & \text{if } \phi(I_H) \neq \emptyset \\ \emptyset & \text{if } \phi(I_H) = \emptyset \end{cases}$$

where $I_H = \{a \in A \mid (a,e) \in H \text{ for some } e \in E\}$. Then,
(1) If $\phi(\{0\}) = \emptyset$. Then, $0 \propto E$ is ψ-prime if and only if $\{0\}$ is ϕ-prime [12, Theorem 3.2].
(2) If $\phi(\{0\}) \neq \emptyset$. Then, $0 \propto E$ is always ψ-prime and $\{0\}$ is always ϕ-prime.

The next theorem gives a general form of ψ-prime ideals of $A \propto E$ in some particular cases. Recall that a proper ideal I of R is called semiprime if $a \in R$ and $a^2 \in I$ implies that $a \in I$ (see for instance [18]).

Theorem 5.2.4 *Let A be a ring, E an A-module and $\phi : \mathscr{I}(A) \to \mathscr{I}(A) \cup \{\emptyset\}$ a function. Let N be a submodule of E and $\psi : \mathscr{I}(A \propto E) \to \mathscr{I}(A \propto E) \cup \{\emptyset\}$ be a function defined by :*

$$\psi(H) = \begin{cases} (\phi(I_H) \propto N) \cap H & \text{if } \phi(I_H) \neq \emptyset \\ \emptyset & \text{if } \phi(I_H) = \emptyset \end{cases}$$

where $I_H = \{a \in A \mid (a,e) \in H\}$. Let J be an ideal of $A \propto E$ such that $\psi(J) \neq \emptyset$. If J is a ψ-prime ideal of $A \propto E$ such that $\phi(I_J)$ is a semiprime ideal of A, then

1. *either $J \subseteq \phi(I_J) \propto E$ or $J = I_J \propto E$.*
2. *If we suppose further more that $\psi(J)$ is semiprime then either $J = \phi(I_J) \propto E$ or $J = I_J \propto E$.*

If $\psi(J) = \emptyset$ then $J = P \propto E$ where P is a ϕ-prime ideal of A.

The proof of this theorem needs the following lemma. We recall that an element a of a ring R is said an unbreakable-zero element of an ideal I if $a^2 = 0$ and $a \notin I$ [18, Definition 2.2].

Lemma 5.2.5 *Let A be a ring and $\phi : \mathscr{I}(A) \to \mathscr{I}(A) \cup \{\emptyset\}$ a function and let I be a ϕ-prime ideal of A such that $\phi(I) \neq \emptyset$. Suppose that a is an unbreakble zero element of I. Then, $(a+i)^2 \in \phi(I)$ and $(a-i)^2 \in \phi(I)$ for all $i \in I$.*

Proof. Let $i \in I$. We have $(a+i)^2 = a^2 + 2ai + i^2 = 0 + 2ai + i^2 \in I$ and $(a+i) \notin I$. Since I is ϕ-prime then $(a+i)^2 \in \phi(I)$. Similarly, we prove that $(a-i)^2 \in \phi(I)$.

Proof of Theorem 5.2.4. (1) Suppose that $J \nsubseteq \phi(I_J) \propto E$. Then, there exists $(a,f) \in J$ such that $a \notin \phi(I_J)$. Clearly, $J \subseteq I_J \propto E$. Let's prove that $I_J \propto E \subseteq J$. Let $(b,e) \in I_J \propto E$. Two cases are possible.

Case 1 : $b \in \phi(I_J)$. Then, there exists $e' \in E$ such that $(b,e') \in J$. We suppose by absurd that $(b,e) \notin J$. So, $(b,e) = (b,e') + (0,e-e') \notin J$ and since $(b,e') \in J$ then $(0,e-e') \notin J$. Hence, $(0,e-e')$ is an unbreakble zero element of J. Then, $[(a,f) + (0,e-e')]^2 \in \psi(J)$ by Lemma 5.2.5. Thereby, $a^2 \in \phi(I_J)$ and so $a \in \phi(I_J)$ since $\phi(I_J)$ is semiprime, which is a contradiction.

Case 2 : $b \notin \phi(I_J)$. As case 1, there exists $e' \in E$ such that $(b,e') \in J$. Our aim is to show that $(0,e-e') \in J$. Suppose that $(0,e-e') \notin J$. Then, $(0,e-e')$ is an unbreakble zero element of J. Hence, $[(b,e') + (0,e-e')]^2 \in \psi(J)$ by Lemma 5.2.5. So, $b^2 \in \phi(I_J)$ and so $b \in \phi(I_J)$ since $\phi(I_J)$ is semiprime, which is a contradiction. Hence, $(0,e-e') \in J$ and so $(b,e') + (0,e-e') = (b,e) \in J$.

In both cases $(b,e) \in J$. Therefore, $J = I_J \propto E$.

(2) We suppose further more that $\psi(J)$ is semiprime. Let $J \subseteq \phi(I_J) \propto E$. Let $(a,e) \in \phi(I_J) \propto E$. Since $\phi(I_J) \subseteq I_J$ then there exists $f \in E$ such that $(a,f) \in J$. Suppose that $(a,e) \notin J$. Then, $(a,e) = (a,f) + (0,e-f) \notin J$. Hence, $(0,e-f) \notin J$ and so $(0,e-f)$

is an unbreakble zero element of J. Then, $[(a,f)+(0,e-e')]^2 \in \psi(J)$ by Lemma 5.2.5. Thereby, $(a,e)^2 \in \psi(J)$ and so $(a,e) \in \psi(J)$ which is a contradiction since $\psi(J) \subseteq J$. Therefore, $(a,e) \in J$ and $J = \phi(I_J) \propto E$. If $J \nsubseteq \phi(I_J) \propto E$ then by the proof of (1), $J = I_J \propto E$.

If $\psi(J) = \emptyset$ we are in the case of prime ideals and $J = P \propto E$ where P is a prime ideal of A by [12, Theorem 3.2].

In Theorem 5.2.4, if we take $\phi(I_H) = 0$ and $N = 0$ we will find ourselves in the case of weakly prime ideals.

Corollary 5.2.6 *Let A be a ring, E an A-module and J a weakly prime ideal of $A \propto E$ and $I_J = \{a \in A \mid (a,e) \in J \text{ for some } e \in E\}$. If $\{0\}$ is a semiprime ideal of A, Then*

1. *either $J \subseteq 0 \propto E$ or $J = I_J \propto E$.*
2. *If we suppose further more that $\{(0,0)\}$ is a semiprime ideal of $A \propto E$ then either $J = 0 \propto E$ or $J = I_J \propto E$.*

Now, we study the generalization of prime ideals in the amalgamation of rings.

Proposition 5.2.7 *Let A and B be two rings, $f : A \to B$ be a ring homomorphism and J an ideal of B. Let $\phi : \mathscr{I}(A) \to \mathscr{I}(A) \cup \{\emptyset\}$ and $\psi : \mathscr{I}(A \bowtie^f J) \to \mathscr{I}(A \bowtie^f J) \cup \{\emptyset\}$ be two functions such that*

$$\psi(P \bowtie^f J) = \begin{cases} \phi(P) \bowtie^f K & \text{if } \phi(P) \neq \emptyset \\ \emptyset & \text{if } \phi(P) = \emptyset \end{cases}$$

where P is an ideal of A and K a subideal of J. Then, $P \bowtie^f J$ is ψ-prime if and only if P is ϕ-prime and for all $a,b \notin P$ such that $ab \in \phi(P)$ we have $(f(a)j + f(b)i + ij) \in K$ for all $i, j \in J$.

Proof. Suppose that $P \bowtie^f J$ is a ψ-prime ideal of $A \bowtie^f J$. Let $a,b \in A$ such that $ab \in P \setminus \phi(P)$. Then, $(ab, f(ab)) = (a,f(a))(b,f(b)) \in P \bowtie^f J \setminus \phi(P) \bowtie^f K$. Hence, $(a,f(a)) \in P \bowtie^f J$ or $(b,f(b)) \in P \bowtie^f J$. Therefore, $a \in P$ or $b \in P$ and so P is a ϕ-prime ideal of A. Let $a,b \notin P$ such that $ab \in \phi(P)$ and $(f(a)j + f(b)i + ij) \notin K$ for some i and j of J. Then, $(a,f(a)+i)(b,f(b)+j) = (ab, f(ab)+f(a)j+f(b)i+ij) \in P \bowtie^f J \setminus \phi(P) \bowtie^f K$ since $(f(a)j + f(b)i + ij) \notin K$. Thus, $(a,f(a)+i) \in P \bowtie^f J$ or $(b,f(b)+j) \in P \bowtie^f J$ which is a contradiction since $a,b \notin P$. So, for all $a,b \notin P$ such that $ab \in \phi(P)$ we have $(f(a)j + f(b)i + ij) \in K$ for all $i, j \in J$.
Conversely, let $(a,f(a)+i)(b,f(b)+j) = (ab, f(ab)+f(a)j+f(b)i+ij) \in P \bowtie^f J \setminus \phi(P) \bowtie^f K$. So, $ab \in P$. Two cases are possible :

Case 1 : $ab \in \phi(P)$. Then, $a \in P$ or $b \in P$. Indeed, suppose that $a,b \notin P$ then $(ab, f(ab)+f(a)j+f(b)i+ij) \in \phi(P) \bowtie^f K$ since $(f(a)j+f(b)i+ij) \in K$ for all $i, j \in J$

and so $(ab, f(ab)+f(a)j+f(b)i+ij) \notin P\bowtie^f J \setminus \phi(P)\bowtie^f K$ which is a contradiction. Hence, $a \in P$ or $b \in P$. Therefore, $(a, f(a)+i) \in P\bowtie^f J$ or $(b, f(b)+j) \in P\bowtie^f J$.

Case 2 : $ab \notin \phi(P)$. Then, $a \in P$ or $b \in P$ since P is ϕ-prime. Hence, $(a, f(a)+i) \in P\bowtie^f J$ or $(b, f(b)+j) \in P\bowtie^f J$.

In both cases we have $(a, f(a)+i) \in P\bowtie^f J$ or $(b, f(b)+j) \in P\bowtie^f J$ and so $P\bowtie^f J$ is a ψ-prime ideal of $A\bowtie^f J$.

Remark 5.2.8 *Let $\psi : \mathscr{I}(A\bowtie^f J) \to \mathscr{I}(A\bowtie^f J)\cup\{\emptyset\}$ be a function defined by*

$$\psi(P\bowtie^f J) = \begin{cases} \phi(P)\bowtie^f J & \text{if } \phi(P) \neq \emptyset \\ \emptyset & \text{if } \phi(P) = \emptyset \end{cases}$$

(1) If $\phi(\{0\}) = \emptyset$. Then $0\times J$ is ψ-prime if and only if $\{0\}$ is ϕ-prime [42, Corollary 2.5].
(2) If $\phi(\{0\}) \neq \emptyset$. Then $0\times J$ is always ψ-prime and $\{0\}$ is always ϕ-prime.

In the next proposition we will see when ideals with the form $\overline{H}^f = \{(a, f(a)+j) \in A\bowtie^f J \mid (f(a)+j) \in H\}$, where H is an ideal of B, are ψ-prime.

Proposition 5.2.9 *Let A and B be two rings, $f : A \to B$ be a ring homomorphism and J an ideal of B. Let $\phi : \mathscr{I}(A) \to \mathscr{I}(A)\cup\{\emptyset\}$, $\varphi : \mathscr{I}(f(A)+J) \to \mathscr{I}(f(A)+J)\cup\{\emptyset\}$ and $\psi : \mathscr{I}(A\bowtie^f J) \to \mathscr{I}(A\bowtie^f J)\cup\{\emptyset\}$ be three functions such that*

$$\varphi(H) = \emptyset \Leftrightarrow \phi(I_H) = \emptyset$$

$$\psi(\overline{H}^f) = \begin{cases} \{(a, f(a)+j) \mid a \in \phi(I_H) \text{ and } f(a)+j \in \varphi(H)\} & \text{if } \varphi(H) \neq \emptyset \\ \emptyset & \text{if } \varphi(H) = \emptyset \end{cases}$$

for every ideal H of $f(A)+J$, where $I_H = \{a \in A \mid (a, f(a)+j) \in \overline{H}^f \text{ for some } j \in J\}$. Then, $\overline{H}^f$ is a ψ-prime ideal if and only if H is a φ-prime ideal of $f(A)+J$ and for all $(f(a)+i), (f(b)+j) \notin H$ such that $(f(a)+i)(f(b)+j) \in \varphi(H)$ we have $ab \in \phi(I_H)$.

Proof. Suppose that $\overline{H}^f$ is a ψ-prime ideal of $A\bowtie^f J$. Let $x, y \in f(A)+J$ such that $xy \in H \setminus \varphi(H)$. Then, there exist $a, b \in A$ and $i, j \in J$ such that $x = f(a)+i$ and $y = f(b)+j$. Hence, $(a, f(a)+i)(b, f(b)+j) = (ab, f(ab)+f(a)j+f(b)i+ij) = (ab, xy) \in \overline{H}^f \setminus \psi(\overline{H}^f)$. So, $(a, f(a)+i) \in \overline{H}^f$ or $(b, f(b)+j) \in \overline{H}^f$. Therefore, $(f(a)+i) = x \in H$ or $(f(b)+j) = y \in H$ which means that H is a φ-prime ideal of $f(A)+J$. Let $f(a)+i, f(b)+j \notin H$ such that $(f(a)+i)(f(b)+j) \in \varphi(H)$ and $ab \notin \phi(I_H)$. Then, $(a, f(a)+i)(b, f(b)+j) = (ab, f(ab)+f(a)j+f(b)i+ij) \in \overline{H}^f \setminus \psi(\overline{H}^f)$ since $ab \notin \phi(I_H)$. Hence, $(a, f(a)+i) \in \overline{H}^f$ or $(b, f(b)+j) \in \overline{H}^f$ and so $(f(a)+i) \in H$ or $(f(b)+j) \in H$ which is a contradiction. Thus, for all $(f(a)+i), (f(b)+j) \notin H$ such that $(f(a)+i)(f(b)+j) \in \varphi(H)$ we have $ab \in \phi(I_H)$.

Conversely, let $(a, f(a)+i)(b, f(b)+j) \in \overline{H}^f \setminus \psi(\overline{H}^f)$. Then, $(ab, (f(a)+i)(f(b)+j)) \in \overline{H}^f \setminus \psi(\overline{H}^f)$. Hence, $(f(a)+i)(f(b)+j) \in H$. Two cases are possible :

Case 1 : $(f(a)+i)(f(b)+j) \in \varphi(H)$. Then, $(f(a)+i) \in H$ or $(f(b)+j) \in H$. Indeed, suppose that $(f(a)+i), (f(b)+j) \notin H$ then by hypothesis $ab \in \phi(I_H)$ and so $(ab, (f(a)+i)(f(b)+j)) \in \psi(\overline{H}^f)$ which is a contradiction since $(ab, (f(a)+i)(f(b)+j)) \in \overline{H}^f \setminus \psi(\overline{H}^f)$. Therefore, $(f(a)+i) \in H$ or $(f(b)+j) \in H$ and so $(a, f(a)+i) \in \overline{H}^f$ or $(b, f(b)+j) \in \overline{H}^f$.

Case 2 : $(f(a)+i)(f(b)+j) \notin \varphi(H)$. Then, $(f(a)+i)(f(b)+j) \in H \setminus \varphi(H)$. So, $(f(a)+i) \in H$ or $(f(b)+j) \in H$ since H is φ-prime. Hence, $(a, f(a)+i) \in \overline{H}^f$ or $(b, f(b)+j) \in \overline{H}^f$.

In both cases, $(a, f(a)+i) \in \overline{H}^f$ or $(b, f(b)+j) \in \overline{H}^f$ and so $\overline{H}^f$ is a ψ-prime ideal of $A \bowtie^f J$.

The next theorem gives a general form of ψ-prime ideals of the amalgamation of rings in some particular cases. Recall that I is a radical ideal of a ring R if for any $x \in R$ and for any positive integer n, if $x^n \in I$ then $x \in I$.

Theorem 5.2.10 *Let A and B be two rings, $f : A \to B$ be a ring homomorphism and J an ideal of B. Let $\phi : \mathscr{I}(A) \to \mathscr{I}(A) \cup \{\emptyset\}$ and $\psi : \mathscr{I}(A \bowtie^f J) \to \mathscr{I}(A \bowtie^f J) \cup \{\emptyset\}$ be two functions such that*

$$\psi(H) = \begin{cases} (\phi(I_H) \bowtie^f K) \cap H & \text{if } \phi(I_H) \neq \emptyset \\ \emptyset & \text{if } \phi(I_H) = \emptyset \end{cases}$$

where $I_H = \{a \in A | (a, f(a)+i) \in H \text{ for some } i \in J\}$ and K is a subideal of J. Let P be a ψ-prime ideal of $A \bowtie^f J$ and $J \subseteq Nilp(B)$.
(I) If $\phi(I_P) \neq \emptyset$ is a radical ideal then,

1. *either $P \subseteq \phi(I_P) \times J$ or $P = I_P \bowtie^f J$.*
2. *If we suppose further more that $\psi(P)$ is a radical ideal of $A \bowtie^f J$, then either $P = \phi(I_P) \times J$ (necessarily in that case $f(\phi(I_P)) \subseteq J$) or $P = I_P \bowtie^f J$.*

(II) If $\phi(P) = \emptyset$. Then $P = H \bowtie^f J$ where H is a ϕ-prime ideal of A.

The proof of this theorem needs the following lemma.

Lemma 5.2.11 *Let A be a ring and $\phi : \mathscr{I}(A) \to \mathscr{I}(A) \cup \{\emptyset\}$ a function and let I be a ϕ-prime ideal of A such that $\phi(I) \neq \emptyset$. Let a be an element of A such that $a \notin I$ and $a^n = 0$ for some integer $n \geq 2$. Then, $(a+i)^n \in \phi(I)$ and $(a-i)^n \in \phi(I)$ for all $i \in I$.*

Proof. Let $i \in I$. We have $(a+i)^n = a^n + \sum_{k=0}^{n-1} C_n^k a^k i^{n-k} = 0 + \sum_{k=0}^{n-1} C_n^k a^k i^{n-k} \in I$ and $(a+i) \notin I$. Since I is ϕ-prime then $(a+i)^n \in \phi(I)$. Similarly, we prove that $(a-i)^n \in \phi(I)$.

Proof of Theorem 5.2.10. (I) (1) Suppose that $P \nsubseteq \phi(I_P) \times J$. Then, there exists $(a, f(a)+i) \in P$ such that $a \notin \phi(I_P)$. Clearly, $P \subseteq I_P \bowtie^f J$. Let's prove that $I_P \bowtie^f J \subseteq P$. Let $(x, f(x)+j) \in I_P \bowtie^f J$. Two cases are possible.

Case 1 : $x \in \phi(I_P)$. Then, there exists $k \in J$ such that $(x, f(x)+k) \in P$ (since $\phi(I_P) \subseteq I_P$). We suppose by absurd that $(x, f(x)+j) \notin P$. So, $(x, f(x)+j) = (x, f(x)+k)+(0, j-k) \notin P$ and since $(x, f(x)+k) \in P$ then $(0, j-k) \notin P$ and we have $(0, j-k)^n = 0$ for some $n \geq 2$ as $J \subseteq Nilp(B)$. Hence, $[(a, f(a)+i)+(0, j-k)]^n \in \psi(P)$ by Lemma 5.2.11. Thereby, $a^n \in \phi(I_P)$ and so $a \in \phi(I_P)$ by hypothesis, which is a contradiction.

Case 2 : $x \notin \phi(I_P)$. As case 1, there exists $k \in J$ such that $(x, f(x)+k) \in P$. Our aim is to show that $(0, j-k) \in P$. Suppose that $(0, j-k) \notin P$. Then, $[(x, f(x)+k)+(0, j-k)]^n \in \psi(P)$ by Lemma 5.2.11. So, $x^n \in \phi(I_P)$ and so $x \in \phi(I_P)$ by hypothesis, which is a contradiction. Hence, $(0, j-k) \in P$ and so $(x, f(x)+k)+(0, j-k) = (x, f(x)+j) \in P$.

In both cases $(x, f(x)+j) \in P$. Therefore, $P = I_P \bowtie^f J$.

(2) We suppose further more that $b^n \in \psi(P)$ implies that $b \in \psi(P)$ for all $b \in A \bowtie^f J$ and $n \geq 2$. Let $P \subseteq \phi(I_P) \times J$. Then, $\forall x \in A$ such that $(x, f(x)+j) \in P$ we have $f(x) \in J$. Let $(a, i) \in \phi(I_P) \times J$. Since $\phi(I_P) \subseteq I_P$ then there exists $k \in J$ such that $(a, f(a)+k) \in P$ and $f(a) \in J$. Suppose that $(a, i) \notin P$. Then, $(a, i) = (a, f(a)+k)+(0, i-f(a)-k) \notin P$. Hence, $(0, i-f(a)-k) \notin P$ and there exists $n \geq 2$ such that $(0, i-f(a)-k)^n = 0$ (since $J \subseteq Nilp(B)$). Then, $[(a, f(a)+k)+(0, i-f(a)-k)]^n \in \psi(P)$ by Lemma 5.2.11. Thereby, $(a, i)^n \in \psi(P)$ and so $(a, i) \in \psi(P)$ which is a contradiction since $\psi(P) \subseteq P$. Therefore, $(a, i) \in P$ and $P = \phi(I_P) \times J$. If $P \nsubseteq \phi(I_P) \times J$ then by the proof of (1), $P = I_P \bowtie^f J$.

(II) If $\phi(P) = \emptyset$ then we are in the case of prime ideals. Since $J \subseteq Nilp(B)$ then $P = H \bowtie^f J$ where H is a prime ideal of A by [42, Corollary 2.5].

When $\phi(I_H) = \{0\}$ and $K = \{0\}$ in Theoreme 5.2.10, we obtain the result for weakly prime ideals.

Corollary 5.2.12 *Let A and B be two rings, $f : A \to B$ be a ring homomorphism and J an ideal of B. Let P be a weakly prime ideal of $A \bowtie^f J$ and $I_P = \{a \in A | (a, f(a)+i) \in P$ for some $i \in J\}$. If $\{0\}$ is a radical ideal of A and $J \subseteq Nilp(B)$, then*

1. *either $P \subseteq 0 \times J$ or $P = I_P \bowtie^f J$.*
2. *If we suppose further more that $\{(0,0)\}$ is a radical ideal of $A \bowtie^f J$, then either $P = 0 \times J$ or $P = I_P \bowtie^f J$.*

For homomorphic images, we give the following result.

Proposition 5.2.13 *Let R be a ring, I an ideal of R and $\phi : \mathscr{I}(R) \to \mathscr{I}(R) \cup \{\emptyset\}$, $\psi : \mathscr{I}(R/I) \to \mathscr{I}(R/I) \cup \{\emptyset\}$ two functions such that*

$$\psi(J/I) = \begin{cases} \phi(J)/I & \text{if} \quad I \subseteq \phi(J) \\ \{\overline{0}\} & \text{if} \quad \phi(J) \neq \emptyset \text{ and } I \nsubseteq \phi(J) \\ \emptyset & \text{if} \quad \phi(J) = \emptyset \end{cases}$$

Let P/I be an ideal of R/I. Then,

1. *Assume that I is a weakly prime ideal of R then if P/I is a ψ-prime ideal of R/I then P is a ϕ-prime ideal of R.*
2. *If $I \subseteq \phi(P)$ then P/I is a ψ-prime ideal of R/I if and only if P is a ϕ-prime ideal of R.*

Proof. (1) Let P/I be a ψ-prime ideal of R/I and let $a,b \in R$ such that $ab \in P \setminus \phi(P)$. Three cases are possible :

Case 1 : $I \subseteq \phi(P)$. Then $\overline{ab} \in (P/I) \setminus (\phi(P)/I)$ that is $\overline{ab} \in (P/I) \setminus \psi(P/I)$. Since P/I is ψ-prime then $\overline{a} \in P/I$ or $\overline{b} \in P/I$. Hence, $a \in P$ or $b \in P$.

Case 2 : $\phi(P) \neq \emptyset$ and $I \nsubseteq \phi(P)$. Two cases are possible :

Case 1 : $\overline{ab} = \overline{0}$. Then, $0 \neq ab \in I$ and so $a \in I \subseteq P$ or $b \in I \subseteq P$.

Case 2 : $\overline{ab} \neq \overline{0}$. Then, $\overline{ab} \in (P/I) \setminus \{\overline{0}\}$ that is $\overline{ab} \in (P/I) \setminus \psi(P/I)$. Since P/I is ψ-prime then $\overline{a} \in P/I$ or $\overline{b} \in P/I$. Hence, $a \in P$ or $b \in P$.

Case 3 : $\phi(P) = \emptyset$. It is the case of prime ideals. Hence, if P/I is a prime ideal of R/I then P is a prime ideal of R and so $a \in P$ or $b \in P$.

In all cases $a \in P$ or $b \in P$. Therefore, P is a ϕ-prime ideal of R.

(2) ($\Rightarrow$) It is the same proof in (1) **Case 1**, when $I \subseteq \phi(P)$.

($\Leftarrow$) Suppose that P is a ϕ-prime ideal of R. Let $\overline{a},\overline{b} \in R/I$ such that $\overline{ab} \in (P/I) \setminus \psi(P/I)$. Then, $\overline{ab} \in (P/I) \setminus (\phi(P)/I)$. Hence, $ab \in P \setminus \phi(P)$. So, $a \in P$ or $b \in P$. Therefore, $\overline{a} \in P/I$ or $\overline{b} \in P/I$.

5.3 The generalization of Krull dimension

In this section, we define and study a new notion considered as a generalization of Krull dimension in commutative rings.

Definition 5.3.1 *Let A be a ring and $\phi : \mathscr{I}(A) \to \mathscr{I}(A) \cup \{\emptyset\}$ a function. The ϕ-Krull dimension of A (denoted by $\phi - dim(A)$) is the supremum of the lengths of all chains of*

distincts ϕ-prime ideals of A.
Clearly,

$$\phi - dim(A) \geq dim(A).$$

We star with the direct product of rings.

Theorem 5.3.2 *Let $R = \prod_{i=1}^{n} R_i$ be a direct product of rings and let $\{\phi_i : \mathscr{I}(R_i) \to \mathscr{I}(R_i) \cup \{\emptyset\}\}_{i=1,\cdots,n}$ be a family of functions such that for all ideals I_i of R_i, $\phi_i(I_i) \neq I_i$ $(i = 1,\cdots,n)$. We consider $\phi : \mathscr{I}(R) \to \mathscr{I}(R) \cup \{\emptyset\}$ a function such that $\phi(\prod_{i=1}^{n} I_i) = \prod_{i=1}^{n} \phi_i(I_i)$ where I_i is an ideal of R_i for each $i = 1,\cdots,n$. Then,*

$$\phi - dim(R) = \sup_{i=1,\cdots,n} \{\phi_i - dim(R_i)\}.$$

Proof. The proof is done by induction on n and it suffices to check it for $n = 2$. Let $P_0 \subseteq P_1 \subseteq \cdots \subseteq P_k$ be a maximal chain of ϕ_1-prime ideals of R_1 and $Q_0 \subseteq Q_1 \subseteq \cdots \subseteq Q_h$ a maximal chain of ϕ_2-prime ideals of R_2. Let $m = sup\{k,h\}$. Then, $H_0 \subseteq H_1 \subseteq \cdots \subseteq H_m$ is a chain of ϕ-prime ideals of $\mathbb{R}_1 \times \mathbb{R}_2$ where $H_i = P_i \times R_2$ for all $i = 0,\cdots,k$ or $H_i = R_1 \times Q_i$ for all $i = 0,\cdots,h$, by [13, Theorem 16]. Hence, $\phi - dim(R_1 \times R_2) \geq \sup_{i=1,2} \{\phi_i - dim(R_i)\}$.

Now, let $H_0 \subseteq H_1 \subseteq \cdots \subseteq H_m$ be a chain of ϕ-prime ideals of $R_1 \times R_2$. By [13, Theorem 16], $H_i = P_i \times R_2$ or $H_i = R_1 \times Q_i$ where P_i is a ϕ_1-prime ideal of R_1 and Q_i is a ϕ_2-prime ideal of R_2 for $i = 0,\cdots,m$. It is clear that all H_i have one of the two forms, which means that $H_i = P_i \times R_2$ for all $i = 0,\cdots,m$ or $H_i = R_1 \times Q_i$ for all $i = 0,\cdots,m$. Hence, $\phi - dim(R_1 \times R_2) \leq \sup_{i=1,2} \{\phi_i - dim(R_i)\}$. Therefore, $\phi - dim(R_1 \times R_2) = \sup_{i=1,2} \{\phi_i - dim(R_i)\}$.

As a corollary we have the known result about the Krull dimension of a direct product of rings.

Corollary 5.3.3 *Let $R = \prod_{i=1}^{n} R_i$ be a direct product of rings. Then,*

$$dim(R) = \sup_{i=1,\cdots,n} \dim(R_i).$$

For homomorphic images, we give the following result.

Proposition 5.3.4 *Let R be a ring, I an ideal of R and $\phi : \mathscr{I}(R) \to \mathscr{I}(R) \cup \{\emptyset\}$, $\psi : \mathscr{I}(R/I) \to \mathscr{I}(R/I) \cup \{\emptyset\}$ two functions such that*

$$\psi(J/I) = \begin{cases} \phi(J)/I & \text{if} \quad \phi(J) \neq \emptyset \text{ and } I \subseteq \phi(J) \\ \{\overline{0}\} & \text{if} \quad \phi(J) \neq \emptyset \text{ and } I \nsubseteq \phi(J) \\ \emptyset & \text{if} \quad \phi(J) = \emptyset \end{cases}$$

If one of the two following conditions is verified :

1. *I is a weakly prime ideal of R.*
2. *for all ψ-prime ideals P/I of R/I we have $I \subseteq \phi(P)$.*

Then,

$$\psi - dim(R/I) \leq \phi - dim(R).$$

Proof. Let $H_0 \subseteq H_1 \subseteq \cdots \subseteq H_n$ be a chain of ψ-prime ideals of R/I. If one of the two conditions of the proposition is verified then H_i has the form P_i/I where P_i is a ϕ-prime ideal of R for $i = 0, \cdots, n$, by Proposition 5.2.13. Hence, $P_0 \subseteq P_1 \subseteq \cdots \subseteq P_n$ is a chain of ϕ-prime ideals of R. Therefore, $\psi - dim(R/I) \leq \phi - dim(R)$.

Corollary 5.3.5 *Let R be a ring and I an ideal of R.*

$$dim(R/I) \leq dim(R).$$

Now, we treat the case of the trivial ring extension.

Theorem 5.3.6 *Let A be a ring, E an A-module and $\phi : \mathscr{I}(A) \to \mathscr{I}(A) \cup \{\emptyset\}$ a function. Let $\psi : \mathscr{I}(A \propto E) \to \mathscr{I}(A \propto E) \cup \{\emptyset\}$ be a function defined by*

$$\psi(H) = \begin{cases} (\phi(I_H) \propto E) \cap H & \text{if } \phi(I_H) \neq \emptyset \\ \emptyset & \text{if } \phi(I_H) = \emptyset \end{cases}$$

where $I_H = \{a \in A \mid (a,e) \in H \text{ for some } e \in E\}$. Then,

1. *$\psi - dim(A \propto E) \geq \phi - dim(A)$.*
2. *We suppose further more that for any ψ-prime ideal P of $A \propto E$, $\phi(I_P) \neq \emptyset$ and $\psi(P) \neq \emptyset$ are semiprime ideals, then*

$$\psi - dim(A \propto E) = \phi - dim(A).$$

Proof. (1) Let $P_0 \subseteq P_1 \subseteq \cdots \subseteq P_n$ be a maximal chain of ϕ-prime ideals of A. We have $\psi(P_i \propto E) = (\phi(P_i) \propto E) \cap (P_i \propto E) = \phi(P_i) \propto E$ for all $i = 0, 1, \cdots, n$. Then, $P_i \propto E$ is a ψ-prime ideal of $A \propto E$ for each $i = 0, 1, \cdots, n$ by [19, Proposition 2.7] that is the proposition 5.2.1 cited in the beginning of the second section. So, $P_0 \propto E \subseteq P_1 \propto E \subseteq \cdots \subseteq P_n \propto E$ is a chain of ψ-prime ideals that has the same length of the chain $P_0 \subseteq P_1 \subseteq \cdots \subseteq P_n$. Hence, $\psi - dim(A \propto E) \geq \phi - dim(A)$.

(2) We suppose further more that for any ψ-prime ideal P of $A \propto E$, $\phi(I_P)$ and $\psi(P)$ are semiprime ideals. Let $H_0 \subseteq H_1 \subseteq \cdots \subseteq H_m$ be a maximal chain of ψ-prime ideals of $A \propto E$. By Theorem 5.2.4, $H_i = \phi(I_{H_i}) \propto E$ or $H_i = I_{H_i} \propto E$ where $\phi(I_{H_i})$ and

I_{H_i} are ϕ-prime ideals by proposition 5.2.1 for $i = 0, \cdots, m$. Then there exists a chain of ϕ-prime ideals $P_0 \subseteq P_1 \subseteq \cdots \subseteq P_m$ where $P_i = \phi(I_{H_i})$ or $P_i = I_{H_i}$ for $i = 0, \cdots, m$. Hence, $\phi - dim(A) \geq \psi - dim(A \propto E)$ and by (1) we conclude that $\psi - dim(A \propto E) = \phi - dim(A)$.

If for each ψ-prime ideal P of $A \propto E$, $\phi(I_P) = \emptyset$ then we have the following known result.

Corollary 5.3.7 *[12, Theorem 3.2] Let A be a ring, E an A-module. Then,*

$$dim(A \propto E) = dim(A).$$

For the amalgamation we have the following result.

Theorem 5.3.8 *Let A and B be two rings, $f : A \to B$ be a ring homomorphism and J an ideal of B. Let $\phi : \mathscr{I}(A) \to \mathscr{I}(A) \cup \{\emptyset\}$ and $\psi : \mathscr{I}(A \bowtie^f J) \to \mathscr{I}(A \bowtie^f J) \cup \{\emptyset\}$ be two functions such that*

$$\psi(H) = \begin{cases} (\phi(I_H) \bowtie^f J) \cap H & \text{if } \phi(I_H) \neq \emptyset \\ \emptyset & \text{if } \phi(I_H) = \emptyset \end{cases}$$

where $I_H = \{a \in A | (a, f(a) + i) \in H \text{ for some } i \in J\}$. Then,

1. $\psi - dim(A \bowtie^f J) \geq \phi - dim(A)$.
2. *Suppose that for any ψ-prime ideal P of $A \bowtie^f J$, $\phi(I_p) \neq \emptyset$ and $\psi(P) \neq \emptyset$ are radical ideals and $J \subseteq Nilp(B)$ then,*

$$\psi - dim(A \bowtie^f J) = \phi - dim(A).$$

Proof. (1) Let $P_0 \subseteq P_1 \subseteq \cdots \subseteq P_n$ be a maximal chain of ϕ-prime ideals of A. We have $\psi(P_i \bowtie^f J) = (\phi(P_i) \bowtie^f J) \cap (P_i \bowtie^f J) = \phi(P_i) \bowtie^f J$ for all $i = 0, 1, \cdots, n$. Then, $P_i \bowtie^f J$ is a ψ-prime ideal of $A \bowtie^f J$ for each $i = 0, 1, \cdots, n$ by proposition 5.2.7. So, $P_0 \bowtie^f J \subseteq P_1 \bowtie^f J \subseteq \cdots \subseteq P_n \bowtie^f J$ is a chain of ψ-prime ideals that has the same length of the chain $P_0 \subseteq P_1 \subseteq \cdots \subseteq P_n$. Hence, $\psi - dim(A \bowtie^f J) \geq \phi - dim(A)$.

(2) We suppose that for any ψ-prime ideal P of $A \bowtie^f J$, $\phi(I_p)$ and $\psi(P)$ are radical ideals and $J \subseteq Nilp(B)$. Let $H_0 \subseteq H_1 \subseteq \cdots \subseteq H_m$ be a maximal chain of ψ-prime ideals of $A \bowtie^f J$. By Theorem 5.2.10, $H_i = \phi(I_{H_i}) \times J$ or $H_i = I_{H_i} \bowtie^f J$ where $\phi(I_{H_i})$ and I_{H_i} are ϕ-prime ideals by proposition 5.2.7 for $i = 0, \cdots, m$. Then there exists a chain of ϕ-prime ideals $P_0 \subseteq P_1 \subseteq \cdots \subseteq P_m$ where $P_i = \phi(I_{H_i})$ or $P_i = I_{H_i}$ for $i = 0, \cdots, m$. Hence, $\phi - dim(A) \geq \psi - dim(A \bowtie^f J)$ and by (1) we conclude that $\psi - dim(A \bowtie^f J) = \phi - dim(A)$.

If for each ψ-prime ideal P of $A \bowtie^f J$, $\phi(I_P) = \emptyset$ and $J \subseteq Nilp(B)$, then we have the following result.

Corollary 5.3.9 *Let A and B be two rings, $f : A \to B$ be a ring homomorphism and J an ideal of B such that $J \subseteq Nilp(B)$. Then,*

$$dim(A \bowtie^f J) = dim(A).$$

Perspectives

To end this thesis, we introduce the following open questions :

1. Is there an exemple of rings that have π-regular one stable range but do not have regular one stable range ?

2. Can we find necessary and sufficient conditions for which the amalgamation of rings is an LPP-ring or an LPI ring ?

3. Is it possible to generalize all the algebraic and homological properties already studied in the amalagamation of rings to the amalgamation of modules ?

Bibliography

[1] A. G. Agargün, D. D. Anderson, and S. Valdes-Leon, *Unique factorization rings with zero divisors*, Comm. Algebra, **27**(4) (1999), 1967-1974

[2] K. Alaoui Ismaili and N. Mahdou, *Coherence in amalgameted algebra along an ideal*, Bull. Iranian Math. Soc., **41**(3) (2015), 1-9.

[3] M. M. Ali, *Idealization and Theorems of D.D. Anderson*, Comm. Algebra **34** (2006), 4479-4501.

[4] M. M. Ali, *Idealization and Theorems of D.D. Anderson II*, Comm. Algebra **35** (2007), 2767-2792.

[5] M.M. Ali and D. J. Smith, *Pure Submodules of Multiplication Modules*, Beiträge Algebra Geom. **45** (2004), 61-74.

[6] D.D. Anderson, *Multiplication Ideals, Multiplication rings, And the ring $R(X)$*, Can. J. Math, **45**(4) (1976), 760-768.

[7] D. D. Anderson, Some remarks on multiplication ideals II, Comm. in Algebra **28** (2000), 2577-2583.

[8] D. D. Anderson and M. Bataineh, *Generalization of prime ideals*, Comm. Algebra **36**(2) (2008), 686-696.

[9] D.D. Anderson and A. Mimouni, *LPI domains and Pullbacks*, Comm. Algebra **42** (2014), 2759-2768.

[10] D.D. Anderson and M. Roitman, A characterization of cancellation ideals, Proc. Amer. Math. Soc. **125** (1997), 2853-2854.

[11] D. D. Anderson and E. Smith, *Weakly prime ideals*, Houston J. Math. **29**(4) (2003), 831-840.

[12] D.D. Anderson and M. Winders, *Idealization of a module*, J. Commut. Algebra **1**(1) (2009), 3-56.

[13] D.D. Anderson and M. Zafrullah, *Integral domains in which nonzero locally principal ideals are invertible*, Comm. Algebra **39** (2011), 933-941.

[14] P. Ara, *Strongly-regular rings have stable range one*, Proc. Amer. Math. Soc. **124**(11) (1996), 3293-3298.

[15] N. Ashrafi and E. Nasibi, *r-clean rings*, Math. Rep. (Bucur.) **15** (65) (2013), 125-132.

[16] N. Ashrafi and E. Nasibi, *Rings with many regular elements*, Commun. Korean Math. Soc. **32**(2) (2017), 267-276.

[17] G. Azumaya, *Strongly π-regular rings*, J. Fac. Sci. Hokkaido Univ. Ser. I **13** (1954), 34-39.

[18] A. Badawi, *On weakly semiprime ideals of commutative rings*, Beitrge zur Algebra und Geometri / Contributions to Algebra and Geometry **57** (2016), 589-597.

[19] J. Bagheri Harehdashti and H. Fazaeli Moghimi, *A generalization of the prime radical of ideals in commutative rings*, Commun. Korean Math. Soc. **32**(3) (2017), 543-552.

[20] C. Bakkari S. Kabbaj and N. Mahdou, *Trivial extension definided by Prûfer conditions*, J. Pure App. Algebra **214** (2010), 53-60.

[21] H. Bass, *K-theory and stable algebra*, Publ. Math. IHES **22** (1964), 5-60.

[22] E. Bastida and R. Gilmer, Overrings and divisorial ideals of rings of the form $D + M$, Mich. Math. J. **20** (1973), 79-95.

[23] S. Bazzoni, *Class semigroups of Prüfer domains*, J. Algebra **184** (1996), 613-631.

[24] S. M. Bhatwadekar and P. K. Sharma, *Unique factorization and birth of almost primes*, Comm. Algebra **33**(1) (2005), 43-49.

[25] M. B. Boisen and P. B. Sheldon, *CPI-extension: Over rings of integral domains with special prime spectrum*, Canad. J. Math. **29** (1977), 722-737.

[26] E. M. Bouba, N. Mahdou and M. Tamekkante, *Duplication of a module along an ideal*, Acta Math. Hungar., **154**, (2018), 29-42.

[27] N. Bourbaki, *Commutative Algebra Chapters 1-7*, Springer, (1989).

[28] J. W. Brewer and E. A. Rutter, *A note on finitely generated ideals which are locally principal*, Proc. Amer. Math. Soc. **31**(2) (1972).

[29] J. W. Brewer and E. A. Rutter, $D+M$ constructions with general overrings, Mich. Math. J. **23** (1976), 33-42.

[30] V. P. Camillo and H. P. Yu, *Stable range one for rings with many idempotents*, Trans. Amer. Math. Soc. **347**(8) (1995), 3141-3147.

[31] M. C. Campany, M. G. Lozano, and M. S. Molina, *Exchange Morita rings*, Comm. Algebra **29**(2) (2001), 907-925.

[32] F. Cheniour and N. Mahdou, *About some flatness properties over commutative rings*, Acta Math Hungar **146**(1) (2015), 142-152.

[33] H. Chen, *Rings Related to Stable Range Conditions*, Series in Algebra **11** (2011).

[34] H. Chen, *Rings with many idempotents*, Int. J. Math. Math. Sci. **22**(3) (1999), 547-558.

[35] H. Chen and M. Chen, *On unit 1-stable range*, J. Appl. Algebra Discrete Struct. **1**(3) (2003), 189-196.

[36] H. Chen and F. Li, *Rings with many unit-regular elements*, Chinese J. Contemp. Math. **21**(1) (2000), 33-38.

[37] H. Chen and M. Sheibani, *Combining Euclidean and adequate rings*, Turk. J. Math. **40**(3) (2016), 506-5016.

[38] F. Cheniour and N. Mahdou, *When every flat ideal is projective*, Comment. Math. Univ. Carolina **55**(1) (2014), 1-7.

[39] P. M. Cohn, *On the free product of associative rings*, Math. Z. **7**(1) (1959), 380-398.

[40] M. D'Anna, *A construction of Gorenstein rings*, J. Algebra **306**(2) (2006), 507-519.

[41] M. D'Anna, C. Finocchiaro and M. Fontana, *Amalgamated algebras along an ideal*, in: M. Fontana, S. Kabbaj, B. Olberding, I. Swanson (eds.), Commutative Algebra and its Applications, Walter de Gruyter, Berlin, (2009), 155-172.

[42] M. D'Anna, C. A. Finocchiaro and M. Fontana, *New algebraic properties of an amalgamated algebra along an ideal*, Communications in Algebra **44** (2016), 1836-1851.

[43] M. D'Anna, C. A. Finocchiaro and M. Fontana, *Properties of chains of prime ideals in amalgamated algebras along an ideal*, J. Pure Appl. Algebra **214** (2010), 1633-1641.

[44] M. D'Anna and M. Fontana, *An amalgamated duplication of a ring along an ideal: the basic properties*, J. Algebra Appl. **6** (2007), 443-459.

[45] M. D'Anna and M. Fontana, *The amalgamated duplication of a ring along a multiplicative-canonical ideal*, Ark. Mat. **45**(2) (2007), 241-252.

[46] G. De Marco, *Projectivity of pure ideals*, Rend. Sem. Mat. Univ. Padova **68** (1983), 289-304.

[47] OV. Domsha and IS. Vasiunyk, *Combining local and adequate rings*, Book of abstracts of the International Algebraic Conference. Ukraine, Kyiv: Taras Shevchenko National University of Kyiv, (2014).

[48] S. El Baghdadi, A. Jhilal and N. Mahdou, *On FF-rings*, J. Pure and Appl. Algebra **216** (2012), 71-76.

[49] R. El Khalfaoui and N. Mahdou, *Commutative rings in which every pure ideal is projective*, Ann. Un. Ferrara **65** (2019), 301-310.

[50] R. El Khalfaoui and N. Mahdou, *On rings with adequate range one*, Indian J. Math. **61**(2) (2019), 281-293.

[51] R. El Khalfaoui and N. Mahdou, *Rings in which every regular locally principal ideal is projective*, Stud. Sci. Math. Hung. **56**(2) (2019), 241-251.

[52] R. El Khalfaoui and N. Mahdou, *The ϕ-Krull dimension of some commutative extensions*, submitted for publication.

[53] R. El Khalfaoui, N. Mahdou and A. Mimouni, *On LPI rings*, Hacettepe J. Math. Stat., accepted for publication.

[54] E. G. Evans, *Krull-Schmidt and cancellation over local rings*, Pacifica J. Math. **46** (1973), 115-121.

[55] C. Faith, Algebra: *Rings, Modules and Categories*, Springer-Verlag (1981).

[56] D. J. Fieldhouse, *Pure theories*, Math. Ann. **184** (1969), 1-18.

[57] S. Galovich, *Unique factorization rings with zero divisors*, Math. Mag. **51**(5) (1978), 276-283.

[58] S. Glaz, *Commutative coherent rings*, Springer-Verlag, Lecture Notes in Mathematics (1989), 13-71.

[59] S. Glaz and W. Vasconcelos, *Flat ideals II*, manuscripta math. **22**(4) (1977), 325-341.

[60] K. R. Goodearl and P. Mental, *Stable range one for rings with many units*, J. Pure Appl. Algebra **54**(2-3) (1988), 261-287.

[61] A. Haghany, *Hopfcity and co-Hopfcity for Morita contexts*, Comm. Algebra **27**(1) (1999), 477-492.

[62] F. Halter-Koch, *Clifford semigroups of ideals in monoids and domains*, Forum Math. **21** (2009), 1001-1020.

[63] O. Helmer, *The elementary divisor theorem for certain rings without chain condition*, Bull. Am. Math. Soc. **49** (1943), 225-236

[64] W.C. Holland, J. Martinez, W.Wm. McGovern and M.Tesemma, *Bazzoni's Conjecture*, J. Algebra **320** (2008), 1764-1768.

[65] J. S. Hu and N. Q. Ding, *Some results on torsionfree modules*, J. Algebra Appl. **12**(1) (2013), 1-20.

[66] J.S. Hu, H. Liu and Y. Geng, *When every pure ideal is projective*, J. Algebra Appl. **15**(2) (2016), 1-19.

[67] K. Hu, K. Wang, F.G. and Chen, H. *A note on LPI domains*, Bull. Korean Math. Soc. **50**(3) (2013), 719-725.

[68] J.A. Huckaba, *Commutative rings with zero divisors*, Marcel Dekker, New York-Basel, (1988).

[69] S. Kabbaj and N. Mahdou, *Trivial extensions defined by coherent-like conditions*, Comm. Algebra **32**(10) (2004), 3937-3953.

[70] I. Kaplansky, *Elementary divisors and modules*, Trans Amer Math Soc **66** (1949), 464-491.

[71] T. Y. Lam, *Lectures on Modules and Rings* (Springer, New York, 1999).

[72] K. Lourtiti and N. Mahdou, *Transfer of multiplication-like conditions in amalgamated algebra along an ideal*, African Diaspora J. Math. **14**(1) (2012), 119-125.

[73] N. Mahdou, *On Costa's conjecture*, Comm. Algebra **29** (2001), 2775-2785.

[74] N. Mahdou and M. Zennayi, *Adequate property in amalgamated algebra along an ideal*, Palestine T. Math. **4**(2) (2015), 242-250.

[75] N. Mahdou and M. Zennayi, *On adequate rings*, J. Taibah Univ. Sci. **9**(2-3) (2015), 320-325.

[76] WW. McGovern, *Bézout rings with almost stable range 1*, J. of Pure Appl. Algebra **212** (2008), 340-348.

[77] M. Nagata, *Local Rings*, Interscience, New York, (1962).

[78] W. K. Nicholson, *Lifting idempotent and exchange rings*, Trans. Amer. Math. Soc. **229** (1977), 269-278.

[79] B. L. Osofsky, *Noninjective cyclic modules*, Proc.Amer.Math.Soc. **19** (1968), 1383-1384.

[80] G. Picozza and F. Tartarone, *Flat ideals and stability in integral domains*, J. Algebra **324** (2010), 1790-1802.

[81] J. D. Sally and W. V. Vasconcelos, *Flat ideal I*, Comm. Algebra **3** (1975), 531-543.

[82] L. N. Vaserstein, *Bass's first stable range condition*, J. Pure Appl. Algebra **34**(2-3) (1984), no. 2-3, 319-330.

[83] L. N. Vaserstein, *Stable rank of rings and dimensionality of topological spaces*, Functional anal. Appl. **5** (1971), 102-110.

[84] R. B. Warfield, *Cancellation of modules and groups and stable range of endomorphism rings*, Pacific J. Math. **91**(2) (1980), 457-485.

[85] G. Xiao and W. Tong, *n-clean rings and weakly unit stable range rings*, Comm. Algebra **33**(5) (2005), 1501-1517.

[86] S. Xing and F. G. Wang, *Two questions on domains in which locally principal ideals are invertible*. J. Algebra Appl. **16**(6) (2017), 8.

[87] H. P. Yu, *Stable range one for exchange rings*, J. Pure Appl. Algebra **98**(1) (1995), 105-109.

[88] BV. Zabavsky and SI. Bilavska, *Every zero adequate ring is an exchange ring*, J Math Sci **187** (2012),153-156.

[89] BV. Zabavsky and SI. Bilavska, *Stable rank of adequate rings*, Math Stud **33** (2010), 212-214.