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Etude de quelques problèmes non linéaires dans les espaces d'Orlicz et de Musielak-Orlicz

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Dédicace

Je dédie cette thèse
à la mémoire de ma mère Aicha et de ma tante Aicha.
Qui ont été toujours dans mon esprit et dans mon coeur.
Je vous dédie aujourd'hui ma réussite.
Que Dieu ait vos âmes dans sa sainte miséricorde.

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Notations

Ω	Ouvert de $\mathbb{R}^N$, $N \in \mathbb{N}^*$
Q_T	$\Omega \times [0,T]$, $T \in \mathbb{R}^{+*}$
$\partial\Omega$	Frontière topologique de Ω
$x = (x_1, x_2, ..., x_N)$	Point générique de $\mathbb{R}^N$
$dx = dx_1 dx_2 ... dx_N$	Mesure de Lebesgue sur Ω
$\nabla u = (\frac{\partial u}{\partial x_1}, \frac{\partial u}{\partial x_2}, ..., \frac{\partial u}{\partial x_N})$	Gradient de u
$f^+ =$	$\max(f, 0)$
$f^- =$	$\max(-f, 0)$
$\mathscr{C}^\infty(\Omega)$:	Espace des fonctions indéfiniments différentiables dans Ω
$\mathscr{D}(\Omega)$:	Espace des fonctions différentiables et à support compact dans Ω
$\mathscr{D}(\overline{\Omega})$:	La restriction de $\mathscr{D}(\mathbb{R}^N)$ sur Ω
$L^p(\Omega)$:	$\{u : \Omega \to \mathbb{R} \mid u$ mesurable, $\int_\Omega \lvert u(x)\rvert^p dx < \infty\}$, $1 \leq p < \infty$
$L^\infty(\Omega)$:	$\{u : \Omega \to \mathbb{R} \mid u$ mesurable et $\exists C > 0$ tel que $\lvert u(x)\rvert \leq C$ dans p.p. $x \in \Omega\}$
$\lVert u\rVert_p =$	$\left(\int_\Omega \lvert u(x)\rvert^p dx\right)^{\frac{1}{p}}$
$W^{1,p}(\Omega)$:	$\{u \in L^p(\Omega);\ \nabla u \in (L^p(\Omega))^N\}$
$\lVert u\rVert_{1,p} =$	$\left(\lVert u\rVert_p^p + \lVert \nabla u\rVert_p^p\right)^{\frac{1}{p}}$
$W_0^{1,p}(\Omega)$:	Adhérence de $\mathscr{D}(\Omega)$ dans $W^{1,p}(\Omega)$
$W^{-1,p'}(\Omega)$:	Espace dual de $W_0^{1,p}(\Omega)$
$\mathrm{div} u = \sum_{i=1}^N \frac{\partial u}{\partial x_i}$	Divergence de u
$\mathscr{M}(\Omega)$:	Espace des mesures de Radon
T_k :	Fonction troncature de niveau $k > 0$

Résumé

Ce travail porte sur des résultats d'existence et parfois d'unicité pour des problèmes non linéaires de type Dirichlet dans des espaces modulaires non réflexifs à savoir les espaces d'Orlicz-Sobolev et les espaces de Musielak-Sobolev. La thèse se décompose en trois parties:

Dans la première partie, nous donnons brièvement quelques rappels sur les espaces d'Orlicz-Sobolev, puis on étudie l'existence de solutions entropiques pour des problèmes unilatéraux associés à l'équation elliptique non linéaire de type

$$-\mathrm{div}(a(x,u,\nabla u)) + K(x,u,\nabla u) = \mu + \mathrm{div}\Phi(x,u) \quad \text{dans} \quad \Omega,$$

où div$a(x,u,\nabla u)$ est un opérateur du type Leray-Lions, $K(x,u,\nabla u)$ est un terme non linéaire d'ordre inférieur vérifiant seulement la condition de croissance et Φ est une fonction de Carathéodory qui ne satisfait pas la condition de coercivité. Le problème est traité dans le cas non variationnel où le terme source μ est une mesure de la forme $\mu = f - div(F)$ avec $f \in L^1(\Omega)$, $F \in (E_{\overline{M}}(\Omega))^N$ et $\overline{M}$ est la fonction complémentaire de la N-fonction M.

Nous nous intéressons aussi aux équations paraboliques non linéaires de type

$$\frac{\partial b(u)}{\partial t} - \mathrm{div}(a(x,t,u,\nabla u)) - \mathrm{div}(\Phi(x,t,u)) = f \text{ dans } Q_T,$$

où b est une fonction strictement croissante tel que $b \in \mathscr{C}^1(\mathbb{R},\mathbb{R})$ et le terme source f appartient à $L^1(Q)$. On démontre d'abord l'existence de la solution renormalisée de l'équation, puis en imposant des conditions appropriées sur les termes b, a, et Φ on démontre l'unicité de cette solution.

Dans la seconde partie, on donne un préliminaire concernant les espaces de Musielak-Orlicz-Sobolev ainsi que des Lemmes et Théorèmes utiles pour la suite de la partie. Puis on cherche à établir des résultats d'existence de solutions des problèmes elliptiques non linéaires de type

$$-\mathrm{div}(a(x,u,\nabla u)) + K(x,u,\nabla u) = \mu,$$

où μ est une mesure de Radon bornée. On introduit deux Lemmes principales, afin de prouver que pour toute fonction P de $\mathscr{A}_M$ (voir Définition 3.1.1), la solution u de l'équation appartient à l'espace de Musielak $W_0^1 L_P(\Omega)$.

On finit cette partie par un chapitre qui serait dédié à l'étude des problèmes paraboliques non linéaires avec obstacle de type

$$\begin{cases} u \geq \zeta & \text{dans} \quad Q_T, \\ \frac{\partial u}{\partial t} + \mathrm{div}(a(x,t,u,\nabla u)) + K(x,t,u,\nabla u) = f + div(\Phi(x,t,u)) & \text{dans} \quad Q_T, \\ u(x,t) = 0 & \text{sur} \quad \partial\Omega \times (0,T), \\ u(x,t=0) = u_0(x) & \text{dans} \quad \Omega. \end{cases}$$

La donnée f est un élément de $L^1(Q)$, le terme Φ vérifie une condition de croissance non polynômiale. Nous prouvons l'existence de solutions en utilisant la technique de troncature, des fonctions tests adéquats et la méthode de pénalisation.

Enfin, dans la dernière partie, nous retournons au cadre des espaces d'Orlicz-Sobolev pour étudier l'existence et l'unicité de la solution numérique des problèmes paraboliques non linéaires associés au schéma numérique suivant:

$$\int_\Omega \Big[(\frac{b(u^n)-b(u^{n-1})}{\tau})v + a(\nabla u^n)\nabla v + \Phi(u^n)\nabla v \Big] dx = \int_\Omega f(.,t_n)v dx \quad \forall v \in V. \tag{1}$$

où $\{u^n\}_{n=1}^N \subset V = \{v \in L^2(\Omega) : \nabla v \in (E_M(\Omega))^d, \gamma_0 v = 0\}$.

On prouve d'abord l'existence d'une solution unique de l'équation sous certaines conditions sur ces termes, puis on démontre la convergence de la solution numérique vers la solution exacte de l'équation:

$$\frac{\partial b(u)}{\partial t} - \mathrm{div}(a(\nabla u) + \Phi(u)) = f. \tag{2}$$

Ensuite on détermine l'estimation de l'erreur de la semi-discrétisation temporelle et on finit par une simulation numérique d'un exemple.

Rappelons que tous ces résultats ne font pas recours ni au condition de croissance pour le terme d'ordre inférieur, ni au condition de $\triangle_2$ pour la N-fonction M et son complémentaire $\overline{M}$.

Mots clés : Espace d'Orlicz-Sobolev, espace de Musielak-Sobolev, non réflexive, non coercive, donnée mesure, problème elliptique et parabolique, solution entropique et renormalisée, pénalisation, troncature, solution numérique, existence, unicité, illustration numérique.

Abstract

This work focuses on the results of existence and sometimes uniqueness for non-linear Dirichlet-type problems in non-reflective modular spaces, namely the Orlicz-Sobolev spaces and the Musielak-Sobolev. The thesis is divided into three parts:

In the first part, we briefly give some reminders about the spaces of Orlicz-Sobolev, and then the existence of entropic solutions for associated unilateral problems is studied to the non-linear elliptical equation of the type

$$-\mathrm{div}(a(x,u,\nabla u)) + K(x,u,\nabla u) = \mu + \mathrm{div}\Phi(x,u) \quad \text{in} \quad \Omega,$$

where div$a(x,u,\nabla u)$ is an operator of the Leray-Lions type, $K(x,u,\nabla u)$ is a non-linear term of order checking only the growth condition and Φ is a Carathéodory function that does not meet the condition of coercivity. The problem is addressed in the non-changing case where the source term μ is a measure of the form $\mu = f - div(F)$ with $f \in L^1(\Omega)$, $F \in (E_{\overline{M}}(\Omega))^N$ and $\overline{M}$ is the complementary function of the N-function M.

We are also interested in non-linear parabolic equations of the type

$$\frac{\partial b(u)}{\partial t} - \mathrm{div}(a(x,t,u,\nabla u)) - \mathrm{div}(\Phi(x,t,u)) = f \text{ in } Q_T,$$

where b is a strictly increasing function such that $b \in \mathscr{C}^1(\mathbb{R},\mathbb{R})$ and the source term f belong to $L^1(Q)$. First, the existence of the renormalized solution of the equation is proved, then by imposing appropriate conditions on terms b, a, and Φ, we attest the uniqueness of this solution .

In the second part, we give a preliminary overview of the Musielak-Orlicz-Sobolev spaces as well as useful Lemmas and Theorems for the rest of this part. Then seeks to establish the results of the existence of solutions to nonlinear elliptic problems of the type

$$-\mathrm{div}(a(x,u,\nabla u)) + K(x,u,\nabla u) = \mu,$$

whereμ is a bounded Radon measure. We introduce two main Lemmas and their demonstrations, which will make it easier to prove that for any function P of $\mathscr{A}_M$ function (see definition 3.1.1), the solution u of equation belongs to the Musielak space $W_0^1 L_P(\Omega)$.

We end this part with a chapter that would be dedicated to the study of non-linear parabolic problems with obstacles of type

$$\begin{cases} u \geq \zeta & \text{in} \quad Q_T, \\ \frac{\partial u}{\partial t} + \mathrm{div}(a(x,t,u,\nabla u)) + K(x,t,u,\nabla u) = f + div(\Phi(x,t,u)) & \text{in} \quad Q_T, \\ u(x,t) = 0 & \text{on} \quad \partial\Omega \times (0,T), \\ u(x,t=0) = u_0(x) & \text{in} \quad \Omega. \end{cases}$$

The data f is an element of $L^1(Q)$, the term Φ verifies a non-polynomial growth condition. We prove the existence of solutions using the truncation technique, appropriate test functions and the penalty method.

Finally, in the last part, we return to the framework of the Orlicz-Sobolev spaces to study the existence and uniqueness of the numerical solution of nonlinear parabolic problems associated with the following numerical scheme:

$$\int_\Omega \Big[(\frac{b(u^n) - b(u^{n-1})}{\tau}) v + a(\nabla u^n)\nabla v + \Phi(u^n)\nabla v \Big] dx = \int_\Omega f(.,t_n) v dx \quad \forall v \in V,$$

where $\{u^n\}_{n=1}^N \subset V = \{v \in L^2(\Omega) : \nabla v \in (E_M(\Omega))^d, \gamma_0 v = 0\}$.

First, we prove the existence of an unique solution of equation under certain conditions on its terms, then we demonstrate the convergence of the numerical solution towards the exact solution of the equation:

$$\frac{\partial b(u)}{\partial t} - \operatorname{div}(a(\nabla u) + \Phi(u)) = f.$$

Then we determine the estimate of the error of the temporal semi-discretization and we end up with a numerical simulation of an example. It should be recalled that all these results do not use either the growth condition for the term of lower order nor to the condition of Δ_2 for the N-function M and its complementary $\overline{M}$.

Key words: Orlicz-Sobolev space, Musielak-Sobolev space, non-reflective, non-coercive, measure data, elliptical and parabolic problem, entropy and renormalized solution, penalization, truncation, numerical solution, existence, uniqueness, numerical illustration.

Introduction générale

La première véritable théorie des équations aux dérivées partielles (EDP) a été introduite par D'Alembert dans ses Réflexions sur la Cause des Vents (1746) et dans ses recherches sur les cordes vibrantes de (1747) qui l'implique dans une querelle (des cordes vibrantes) avec Euler et Daniel Bernoulli, sachant qu'à cet époque la notion de fonction n'est pas encore définie avec rigueur. Bien qu'Euler presentait, pour la mécanique des fluides, la notion de solution faible, l'étude systématique des EDP n'a commencé qu'au $20^{\text{ème}}$ siècle avec l'intégrale de Lebesgue et les distributions de Schwartz. Et depuis, l'étude des EDP n'a cessé d'évoluer et d'intervenir dans plusieurs domaines tels que les sciences, l'économie, l'aéronotique, la synthèse d'image, la météorologie....

On peut plus ou moins les classer en catégories elliptique, parabolique et hyperbolique mais la difficulté confrontée à ces équations est de leur donner un sens dans des espaces fonctionnels appropriés et d'y démontrer l'existence et parfois l'unicité de la solution en introduisant les définitions des solutions entropiques et renormalisées. En effet un problème non linéaire avec des conditions aux limites de Dirichlet assez réguliers admet des solutions classiques, mais dans le cas contraire, on ne peut chercher que les solutions dites faibles au sens des distributions, et naturellement, les chercheurs ont été obligé de réfléchir à ces solutions faibles qui sont trés proche aux solutions physiques d'une part et d'assurer l'existence et l'unicité d'une autre part. Cela les a amenés à adopter trois types de solutions:

1. Solutions Entropiques ont été developpées pour les équations elliptiques par P. Bénilan et al. [37] et également pour les equations paraboliques [94] par A. Prignet.

2. Solutions Renormalisées qui ont été introduites par R. Di Perna et P. L. Lions [58] pour l'étude des équations de Boltzmann et ensuite la notion a été adaptée pour des équations elliptiques et paraboliques.

3. Solutions appellés SOLA (Solution Obtenue comme Limite d'Approximations) adoptés par Dall'Aglio [53, 54].

Dans ce travail, les espaces modulaires non réflexifs, choisis pour étudier les équations aux dérivées partielles (EDP) sont respectivement les espaces d'Orlicz $L_M(\Omega)$ où M est une N-fonction et les espaces de Musielak-Orlicz, dit aussi les espaces d'Orlicz généralisés. Ces espaces sont semblables aux espaces d'Orlicz, mais ils sont définis par une fonction plus générale $M(x,t)$ qui peut varier avec la localisation dans l'espace, la norme est définie au moyen de l'intégrale $\int_\Omega M(x,|f(x)|)dx$. Ces espaces incluent plusieurs d'autres espaces comme les espaces de Lebesgue, les espaces de Lebesgue avec poids, les espaces d'Orlicz et les espaces de Lebesgue-Sobolev à exposant variables.

Historiquement, les espaces de Musielak-Orlicz ont été étudié dans les années quarantes du siécle dernier et depuis ils ont connu un développement assez considérable, d'abord l'analyse fonctionnelle dans ces espaces est donnée dans la monographie de celui qui porte leur nom J. Musielak et aussi par W. Orlicz, H. Hudzik dans [83, 92] puis entre autres par des travaux récents de notre laboratoire LAMA, qui ont traité avec soin plusieurs résultats d'approximations [17, 35, 36] et également par P. Gwiazda et al dans [78].

De nombreux phénomènes physiques sont des motivations des problèmes non linéaires dans ces espaces, à savoir les problèmes liés aux fluides non newtoniens de comportement fortement

inhomogène avec une grande capacité à augmenter leur viscosité dans des conditions différentes. Alors que les problèmes modèles considérés sont de type elliptique-parabolique non linéaires suivant:

$$(EP)\begin{cases} \frac{\partial b(u)}{\partial t} + A(u) + K(x,t,u,\nabla u) = f + div(\Phi(x,t,u)) & \text{dans} \quad Q_T, \\ u(x,t) = 0 & \text{sur} \quad \partial\Omega \times (0,T), \\ b(u)(x,0) = u_0(x) & \text{dans} \quad \Omega. \end{cases} \tag{3}$$

faisant intervenir l'opérateur divergentiel $A(u) = \text{div}a(x,t,u,\nabla u)$ où $a : Q_T \times \mathbb{R} \times \mathbb{R}^N \to \mathbb{R}^N$ est un opérateur de Leray-Lions qui vérifie les conditions de croissance, de monotonie et de coercivité. $K : Q_T \times \mathbb{R} \times \mathbb{R}^N \to \mathbb{R}$ est une fonction de Carathéodory qui ne vérifie pas la condition de signe mais elle satisfait la condition de croissance, $\Phi : Q_T \times \mathbb{R} \to \mathbb{R}^N$ est aussi une fonction de Carathéodory qui vérifie juste la condition de croissance non plynômiale mais pas la condition de coercivité. La donnée f est une mesure de Radon ou elle est plus restrictive si on prend $f \in L^1(\Omega)$.
Dans la suite l'équation (EP) sera notée par (E) dans le cas elliptique.
On rappelle que ce type de problèmes a été longuement étudié dans le contexte des espaces classiques de Lebesgue-Sobolev. Dans le cas elliptique (E) et variationnel, on remonte aux premières études de Leray-Lions [87] avec $K = \Phi = 0$ ensuite F. E. Browder [52] et P. Hess [82] qui ont abordé le cas $K = K(x,u)$ dans le cas non variationnel. De même dans [45], L. Boccardo et T. Gallouet ont généralisé le cas linéaire de (E) avec ($f \in \mathscr{M}(\Omega)$ ou $f \in L^1(\Omega)$). Dans le cas parabolique, on renvoit à R. Landes et V. Mustonen [86] ou $b = u, \Phi = 0, K = K(x,t,u)$ et $f \in L^{p'}(0,T,W^{-1,p'}(\Omega))$ et aussi [45] où f est une mesure.

Peu à peu, les travaux scientifiques ont commencé à apparaître et à traiter tous les aspects de l'équation (EP), soit en écartant ou en intervenant l'un ou plus des termes b, K et Φ ou en y associant certaines conditions. Mais toutes ces solutions aux sens de distribution ne suffisent pas pour répondre aux exigences physiques puisqu'elles ne sont pas uniques et donc ne sont pas réalistes. On peut citer les deux fameux contre-exemples, l'un de J. Serrin [99] dans le cas elliptique et l'autre une équation parabolique issue de la mécanique de fluide dite Equation de Burgers, qui montrent clairement la non unicité de la solution. Retenons aussi que la résolution numérique de ce type d'équation nécessite d'avoir une stabilité de la solution.

C'est au début des années 90 du dernier siècle que R. Di Perna et P. L. Lions ont introduit la notion de la solution renormalisée qui a été ensuite appliqué au type (E) dans [91] où $K = \Phi = 0$ et dans le cas parabolique on peut consulter [38, 39]. Après, un grand nombre d'articles ont été consacrés à l'étude de l'existence de cette solution (on peut citer [4, 7]). On se réfère aux articles traitant l'unicité de la solution renormalisée [41, 65]. Parallèlement d'autres chercheurs étaient intéréssés par la solution entropique du problème (EP) (on renvoie à [43, 94]).

Dans le cas des espaces d'Orlicz, la condition de croissance polynômiale de l'opérateur de Leray-Lions et du terme non linéaire inférieur, sera remplacée par une condition associée à une N-fonction. L'étude des problèmes elliptiques de type (E) avec $\Phi = 0$ dans ce cadre fonctionnel, a vu le jour avec les premiers travaux de J. P. Gossez [77] et A. Benkirane et J. P. Gossez [33] où le second membre f est un élèment de l'espace dual $W^{-m}E_{\overline{M}}(\Omega)$ et ensuite L. Boccardo et T. Gallouët [44] dans le cas où $f \in L^1(\Omega)$ (ou f est une mesure dans $\mathscr{M}(\Omega)$). Des théorèmes d'approximation impliquant la convergence modulaire sont prouvés par J. P. Gossez [74] et dans [33] par A. Benkirane et J. P. Gossez, qui améliorent les résultats de densité souvent utilisés dans

la théorie d'existence de la solution faible du problème (E) (on consulte aussi [31, 32]).

Dans le cas des problèmes paraboliques, A. Elmahi et D. Meskine dans [62] ont traité les résultats d'approximation et de compacité dans le cadre des espaces de Orlicz-Sobolev inhomogène $W^{-1,x}L_M(Q_T)$ qui généralisent ceux de [100] et [86]. Pour l'étude du problème (EP), on renvoie entre autres aux travaux de R. Aboulaich et al dans [8] avec $b = u$, $K = \Phi = 0$ et $f \in W^{-1,x}E_{\overline{M}}(Q_T)$. D. Meskine dans [89] avec la donnée mesure, $b = u$, et $K = \Phi = 0$ et H. Redwane dans [96] avec $K = 0$, $\Phi \in \mathscr{C}(\mathbb{R};\mathbb{R}^N)$. Et dans le cas non variationnel dans [29].

Concernant les espaces de Musielak, les membres de notre Laboratoire LAMA ont contribué par plusieurs travaux. En effet, ils ont enrichi ce cadre fonctionnel par plusieurs théorèmes d'approximation et aussi par des résultats d'existence de solution des problèmes fortement non linéaires (voir [15, 35, 36, 17]) et on renvoie aux travaux de P. Gwiazda et al. dans le cas où l'espace de Musielak est anisotropique ([78, 79, 101]).

A notre connaissance, les seuls travaux qui ont étudiés l'existence de la solution faible dans le cas variationnel de (E) et (EP) ont traité seulement le cas où $\Phi = 0$, $K = 0$ et $b(u) = u$, on se réfère à [101, 15], à [19] si $K \neq 0$ et il vérifie la condition de signe et à [13] dans le cas où la donnée est une mesure. Dans les mêmes conditions les résultats d'existence de la solution entropique ou renormalisée ainsi que l'unicité n'ont été abordés que récemment (voir [24]).

Les travaux présentés dans ce travail sont basés sur certains des articles cités. Leur analyse et interprétation nous ont inspiré des idées et nous ont motivé à ajouter des résultats d'existence ou d'unicité de solutions entropiques ou renormalisées pour des problèmes non linéaires du type mentionnés ci-dessus, en surmontant les difficultés rencontrées dans les travaux précédents et qui ne sont pas encore traitées. Ainsi, cette thèse comporte trois parties et elle est présentée comme suit.

On débute la 1ère partie, par quelques préliminaires concernant les espaces d'Orlicz et les espaces d'Orlicz-Sobolev inhomogène qui seront nécessaires pour aborder les chapitres 1, 2 et 5.
Le chapitre 1 est consacré à l'étude des problèmes elliptiques unilatérales non linéaires suivant:

$$(E)_\mu \begin{cases} u \geq \zeta \quad \text{p.p. dans} \quad \Omega, \\ -div(a(x,u,\nabla u)) - div(\Phi(x,u)) + K(x,u,\nabla u) = \mu \quad \text{dans} \quad \Omega, \\ u = 0 \quad \text{sur} \quad \partial\Omega, \end{cases} \tag{4}$$

où $\mu \in L^1(\Omega) + W^{-1}E_{\overline{M}}(\Omega)$.

Le premier terme non linéaire Φ ne vérifie pas la condition de coercivité, le deuxième terme d'ordre inférieur K est contrôlé par une croissance non polyômiale qui dépend de $|\nabla u|$ et aucune condition de signe n'y est supposée et l'espace d'Orlicz-Sobolev $W^1L_M(\Omega)$ n'est pas réflexif puisque La N-fonction M ne satisfait pas la condition Δ_2. Ce sont ces obstacles et autres auxquels nous avons été confrontés pendant l'étude de $(E)_\mu$. Pour contourner ces diffucltés, on a construit une suite de solutions d'un problème approché et avec des fonctions tests bien choisies, on a prouvé l'existence d'une solution entropique unilatérale de $(E)_\mu$.

Le contenu de ce chapitre est l'objet d'un article soumis [68] qui est l'étendue de [11, 31, 42].

Le chapitre 2, qui est l'objet du papier accepté [65], est consacré à l'étude des problèmes de type (EP) où $K = 0$ et $a : Q_T \times \mathbb{R} \times \mathbb{R}^N \to \mathbb{R}^N$ est un champ vérifiant la condition de la monotonie stricte de Leray-Lions ainsi que :

- La condition de croissance

il existe $\nu > 0$ tel que pour $x \in \Omega$ p.p. et pour toutes $s \in \mathbb{R}$ et $\xi \in \mathbb{R}^N$, on a

$$|a(x,t,s,\xi)| \leq \nu(a_0(x,t) + \overline{M}^{-1}P(|s|) + \overline{M}^{-1}M(|\xi|)) \text{ avec } a_0(.,.) \in E_{\overline{M}}(Q_T) \text{ et } P \ll M,$$

- La condition de coercivité

$$a(x,t,s,\xi).\xi \geq \alpha M(|\xi|).$$

$\Phi : Q_T \times \mathbb{R} \to \mathbb{R}^N$ est une fonction de Caratheodory telle que

$$|\Phi(x,t,s)| \leq c(x,t)\overline{M}^{-1}M(\frac{\alpha_0}{\lambda}|b(s)|),$$

Enfin la fonction $b : \mathbb{R} \to \mathbb{R}$ est strictement croissante et $\mathscr{C}^1(\mathbb{R})$-function, $b(0) = 0$,

$$b_0 < b'(s) < b_1, \quad \forall s \in \mathbb{R} \quad \text{such that} \quad b_1 < \frac{1}{\alpha_0}.$$

Premièrement, on aborde l'existence d'au moins une solution renormalisée de l'équation puis deuxièmement, on étudie l'unicité de la solution de ce problème en régularisant les données. L'importance de nos résultats réside dans la généralisation de quelques travaux qui ont été consacrés ces dernières décénnies à l'étude de l'unicité surtout si la donnée $f \in L^1(\Omega)$ ou $f \in \mathscr{M}(\Omega)$ dans le cas des espaces d'Orlicz (voir [3, 9, 39, 90]).

La deuxième partie est consacrée à l'étude des problèmes de type elliptique-parabolique dans le cadre des espaces de Musielak définit par une fonction de Musielak-Orlicz. On entame cette partie par des préliminaires à propos des espaces de Musielak-Orlicz et des espaces Musielak-Sobolev inhomogenes. Ensuite on introduit quelques Lemmes utiles pour la suite.

Le chapitre 3 est consacré aux résultats de l'article de A. Aberqi, J. Bennouna et M. Elmassoudi dans [67] qui est la généralisation de [57] dans le cas des espaces d'Orlicz. En effet dans ce travail, on a prouvé l'existence des solutions au sens des distributions dans les espaces de Musielak-Sobolev du problème elliptique fortement non linéaire, avec second membre qui est une mesure de Radon bornée, de type:

$$A(u) + K(x,u,\nabla u) = \mu, \tag{5}$$

où $A(u) = -\text{div}(a(x,u,\nabla u))$ est un opérateur de Leray-Lions, la fonction K est supposée avoir la condition de croissance suivante

$$|K(x,u,\nabla u)| \leq b(x) + \rho(u)\varphi(x,|\nabla u|),$$

mais ne satisfait ni la condition de coercivité ni la condition de signe. Le second membre $\mu \in \mathscr{M}(\Omega)$. D'autre part, on montre que chaque solution u du problème (5), appartient à $W_0^1 L_P(\Omega)$ pour tout fonction de Musielak P de $\mathscr{A}_M$ où

$$\mathscr{A}_M = \left\{ \begin{array}{l} P : \Omega \times \mathbb{R}^+ \to \mathbb{R}^+ \quad \text{est une fonction de Musielak-Orlicz tel que} \\ P \ll M \text{ et } \displaystyle\int_0^1 P(x, \beta H^{-1}(x, \frac{1}{r^{1-\frac{1}{N}}}))dr < \infty \text{ p.p. dans } \Omega, \ \forall \beta \geq 1. \end{array} \right\}$$

et $H(x,r) = \frac{M(x,r)}{r}$, $r \in \mathbb{R}^+$.

Pour parvenir à surmonter cette diffuclté, on a introduit et démontré deux Lemmes techniques.

L'objectif du chapitre 4 est d'étudier quelques résultats d'existence des solutions entropiques unilatérales pour des problèmes paraboliques de type

$$(EP)_\zeta \begin{cases} u \geq \zeta & \text{dans} \quad Q_T, \\ \frac{\partial u}{\partial t} + A(u) + K(x,t,u,\nabla u) = f + div(\Phi(x,t,u)) & \text{dans} \quad Q_T, \\ u(x,t) = 0 & \text{sur} \quad \partial\Omega \times (0,T), \\ u(x,t=0) = u_0(x) & \text{dans} \quad \Omega. \end{cases}$$

On va essayer de démontrer quelques théorèmes d'approximation dans les espaces de Musielak-Sobolev Inhomogènes, ensuite on va donner une définition de la solution entropique du problème $(EP)_\zeta$ pour aborder les résultats d'existence.

La condition de coercivité de l'opérateur $A(u) = a(x,t,u,\nabla u)$ est remplacée par

$$|a(x,t,u,\nabla u)| \geq \alpha \overline{M}(x,|\nabla u|) + \overline{M}(x,|s|),$$

afin de surmonter la difficulté causée par l'élimination de la condition de signe de K. Ajoutant aussi qu'aucune hypothèse de régularité n'etait supposée être vérifié par la fonction obstacle ζ.

L'originalité de cette étude réside dans le cadre fonctionnel qui est compliqué vu la dépendance de la fonction de Musielak de deux variables et aussi le manque de travaux qui ont traité ce type de problèmes. Enfin, on propose une méthode de pénalisation pour répondre à la difficuté de l'obstacle qui doit respecter la solution trouvée.

Signalons que ce chapitre est l'objet de l'article accepté [64].

Enfin, dans la dernière partie de ce travail qui est constituée du chapitre 5, on propose une autre approche de solutions à savoir la solution numérique. En effet partons du problème suivant:

$$\begin{cases} \frac{\partial b(u)}{\partial t} - \operatorname{div}(a(\nabla u) + \Phi(u)) = f, \\ u = 0 \quad \text{on} \quad \partial\Omega \times (0,T), \\ b(u(.,0)) = b(u_0(x)) \quad \text{in} \quad \partial\Omega. \end{cases} \tag{6}$$

où l'opérateur de gouvernance $a(\nabla u)$ est décrit par une non-linéarité dont la croissance isotrope et non polynomiale en termes de N-fonction M qui ne satisfait pas la condition $\triangle_2$, le terme non linéaire Φ est une fonction de carathéodory non coercive qui satisfait la condition de croissance et $f \in L^1([0,T];L^2(\Omega))$.

On étudie l'existence et l'unicité de solution du problème (6) dans le sens des distributions via une méthode numérique qui combine le schéma de Backward-Euler pour la discrétisation temporelle avec un schéma d'approximation interne généralisé pour la discrétisation spatiale. A partir de la condition initiale u_0, on construit une sous-suite de solutions numériques qui converge vers la solution théorique, puis on fournie une estimation a priori de l'erreur pour la semi-discrétisation temporelle et on termine par une illustration numérique pour un exemple simple en utilisant le logiciel MATLAB. Notons que, ces approximations numériques utilisent les méthodes des éléments finis dans l'espace et la méthode d'Euler dans le temps.

Le contenu de cette partie fait l'objet de l'article soumis [69].

Tout au long de ce tarvail, les conditions aux limites sont de type de Dirichlet et les techniques utilisées pendant la résolution des problèmes traités sont en premier lieu, une recherche d'estimation à priori puis utilisation des fonctions tests et des molifications adéquates pour les transformer (ces estimations) en des résultats d'existence de solutions.

Outils de base

Les résultats qu'on va présenter içi restent vrais dans un cadre général et on va s'en servir dans les trois parties. Les démonstrations des théorèmes et lemmes se trouvent dans plusieurs ouvrages. Dans la suite Ω est un ouvert borné de $\mathbb{R}^N$ muni de la mesure de Lebesgue usuelle et N est un entier naturel non nul.

Notons que $|E|$ ou $meas(E)$ désignera la mesure (de Lebesgue) du sous-ensemble $E \subset \mathbb{R}$.

Theorem 0.0.1. ***(Théorème de convergence dominée)***
Soit (f_n) une suite de fonctions dans $L^1(\Omega)$ telle que

i) *$f_n \to f$ p.p. dans Ω,*

ii) *il existe $g \in L^1(\Omega)$ tel que pour tout n, $|f_n| \leq g$ p.p. dans Ω.*

Alors, on a $f \in L^1(\Omega)$ et $\lim\limits_{n\to\infty} \int_\Omega f_n dx = \int_\Omega f dx$.

Lemma 0.0.1. ***(Lemme de Fatou)***
Soit (f_n) une suite de fonctions mesurables positives. Alors

$$\int_\Omega (\liminf_{n\to\infty} f_n) dx \leq \liminf_{n\to\infty} \int_\Omega f_n dx.$$

Lemma 0.0.2. *Soit $f_n, f \in L^1(\Omega)$ tels que*

i) *$f_n \geq 0$ p.p. dans Ω,*

ii) *$f_n \to f$ p.p. dans Ω,*

iii) $\lim\limits_{n\to\infty} \int_\Omega f_n dx = \int_\Omega f dx$.

Alors $f_n \to f$ fortement dans $L^1(\Omega)$.

Theorem 0.0.2. ***(Théorème de Vitali)***
Soit (f_n) une suite de fonctions dans $L^1(\Omega)$ telle que

(i) *$f_n \to f$ pp dans Ω,*

(ii) $\forall \varepsilon > 0,\ \exists \delta > 0 \,/\, |E| < \delta \Rightarrow \int_E |f_n| dx < \varepsilon,\ \forall n,$

(iii) $\forall \varepsilon > 0,\ \exists F \subset \Omega$ *avec* $|F| < \infty$ *et* $\int_{\Omega\setminus F} |f_n| dx < \varepsilon,\ \forall n.$

Alors $f \in L^1(\Omega)$ et $f_n \to f$ fortement dans $L^1(\Omega)$.

Theorem 0.0.3. *(Banach-Alaoglu-Bourbaki)([51])*
Soit E un espace de Banach et E' son espace dual.
Alors l'ensemble $B_{E'} = \{f \in E'; \|f\| \leq 1\}$ est compact pour la topologie faible- $\sigma(E', E)$.*

Definition 0.0.1. ***(Suite équi-integrable)***
On dit qu'une suite $f_n \in L^1(\Omega)$ est équi-integrable si pour tout $\varepsilon > 0$ il existe $\eta > 0$ et un ensemble mesurable $A \subset \Omega$ de mesure finie $(|A| < \infty)$ tels que

i) $\forall n \geq 1, \int_{\Omega \setminus A} |f_n| dx < \varepsilon,$

ii) $\forall E \subset \Omega, |E| < \eta \Rightarrow \int_{\Omega \setminus E} |f_n| dx < \varepsilon.$

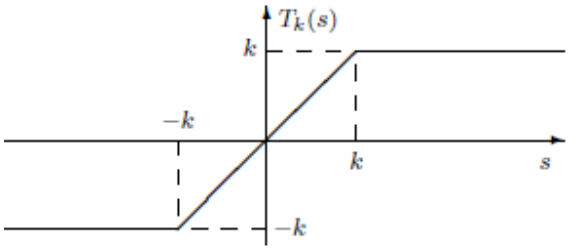
Definition 0.0.2. ***(Fonction de Carathéodory)***
Une fonction réelle $f = f(x,s)$, définie sur $\Omega \times \mathbb{R}^m$, est dite de Carathéodory si on a

i) *la fonction $f(x,.)$ est continue pour presque tout $x \in \Omega$,*

ii) *la fonction $f(.,s)$ est mesurable sur Ω pour tout $s \in \mathbb{R}$.*

Definition 0.0.3. ***(Fonction troncature)***
Soit $k > 0$, la fonction tronquante $T_k : \mathbb{R} \to \mathbb{R}$ est définie par $T_k(s) = \max(-k, \min(k,s))$.

$$T_k(s) = \max(-k, \min(k, s)) .$$



Notation On utilise la notation suivante.
Désignant par $\varepsilon(n, \eta, j, m)$ toute quantité telle que

$$\lim_{n \to +\infty} \lim_{\eta \to +\infty} \lim_{j \to +\infty} \lim_{m \to +\infty} \varepsilon(n, \eta, j, m) = 0.$$

Si la quantité que nous considérons ne dépend pas d'un paramètre parmi n, η, j ou m, nous omettons la dépendance du paramètre correspondant.

Part I

Some nonlinear problems in Orlicz-Sobolev spaces

0.1 Preliminaries of Orlicz-Sobolev spaces

We recall here some standard notations, properties and results which will be used through this thesis. Let Ω be a bounded open set of $\mathbb{R}^N$ and $Q_T = \Omega \times (0,T)$, T is a positive real number.
Let $M : \mathbb{R}^+ \to \mathbb{R}^+$ be an N-function, that is, M is continuous, convex, with $M(t) > 0$ for $t > 0$, $M(t)/t \to 0$ as $t \to 0$, and $M(t)/t \to +\infty$ as $t \to +\infty$. Equivalently, M admits the representation $M(t) = \int_0^t a(s)ds$, where $a : \mathbb{R}^+ \to \mathbb{R}^+$ is non-decreasing, right continuous, with $a(0) = 0$, $a(t) > 0$ for $t > 0$, and $a(t) \to +\infty$ as $t \to +\infty$. The N-function $\overline{M}$ conjugate to M is defined by $\overline{M}(t) = \int_0^t \overline{a}(s)ds$, where $\overline{a} : \mathbb{R}^+ \to \mathbb{R}^+$ is given by $\overline{a}(t) = sup\{s : a(s) \leq t\}$.

We will extend these N-functions into even functions on all $\mathbb{R}$.

Let P and Q be two N-functions. $P \ll Q$ means that P grows essentially less rapidly than Q, that is, for each $\varepsilon > 0$, $\lim_{t \to +\infty} \frac{P(t)}{Q(\varepsilon t)} = 0$. This is the case if and only if $\lim_{t \to +\infty} \frac{Q^{-1}(t)}{P^{-1}(t)} = 0$.

Proposition 0.1.1.
$P \ll M$ if and only if, for all $\varepsilon > 0$ there exists a constant c_ε such that,

$$P(t) \leq M(\varepsilon t) + c_\varepsilon, \forall t \geq 0. \tag{7}$$

Examples 0.1.1. *We give some example for a N-functions*
$M(t) = \frac{1}{p}|t|^p$ and $\overline{M}(\eta) = \frac{1}{q}|t|^q$ where $\frac{1}{p} + \frac{1}{q} = 1$;
$M(t) = (|t| - 1)\exp(|t|) + 1$ and $\overline{M}(\eta) = (|t(\eta)|^2 - |t(\eta)| + 1)\exp(|t(\eta)|) - 1$ where $t(\eta)$ is the solution of equation $|t(\eta)|\exp(|t(\eta)|) = \eta$.

The Orlicz space $L_M(\Omega)$ is defined as

$$L_M(\Omega) = \{u : \Omega \to \mathbb{R} : \int_\Omega M(\frac{u(x)}{\lambda})dx < +\infty \quad \text{for some} \quad \lambda > 0\}.$$

The set $L_M(\Omega)$ is Banach space under the norm $\|u\|_{M,\Omega} = \inf\{\lambda > 0 : \int_\Omega M(\frac{u(x)}{\lambda})dx \leq 1\}$.

The closure in $L_M(\Omega)$ of the set of bounded measurable functions with compact support in $\overline{\Omega}$ is denoted by $E_M(\Omega)$. It is a separable space and $(E_M(\Omega))^* = L_{\overline{M}}(\Omega)$. The dual $E_M(\Omega)$ can be identified with $L_{\overline{M}}(\Omega)$ by means of the pairing $\int_\Omega uvdx$, and the dual norm of $L_{\overline{M}}(\Omega)$ is equivalent to $\|u\|_{\overline{M},\Omega}$.

Now we turn to the Orlicz-Sobolev space, $W^1L_M(\Omega)$ (resp. $W^1E_M(\Omega)$) is the space of all functions u such that u and its distributional derivatives up to order 1 lie in $L_M(\Omega)$ (resp. $E_M(\Omega)$). It is a Banach space under the norm

$$\|u\|_{1,M} = \sum_{|\alpha| \leq 1} \|D^\alpha u\|_{M,\Omega}.$$

We recall the following lemmas:

Lemma 0.1.1. *(Dominated convergence) Let f_k, f in $L_M(\Omega)$.*
If $f_k \to f$ a.e. and $|f_k| \leq |g|$ a.e. and $\int_\Omega M(\lambda|g|)dx < \infty$ for every $\lambda > 0$, then $f_k \to f$ in $L_M(\Omega)$.

Lemma 0.1.2. *(see [77], [84, p.132]) If a sequence $g_n \in L_M(\Omega)$ converges a.e. to g and g_n remains bounded in $L_M(\Omega)$, then $g \in L_M(\Omega)$ and $g_n \to g$ in $\sigma(L_M, E_{\overline{M}})$.*

Lemma 0.1.3. *Let $F : \mathbb{R} \to \mathbb{R}$ be uniformly Lipschitzian, with $F(0) = 0$. Let $u \in W^1L_M(\Omega)$. Then $F(u) \in W^1L_M(\Omega)$.*
Moreover if the set D of discontinuity points of F' is finite, then

$$\frac{\partial}{\partial x_i}F(u) = \begin{cases} F'(u)\frac{\partial u}{\partial x_i} & a.e.\ in \quad \{x \in \Omega; u(x) \notin D\} \\ 0 & a.e.\ in \quad \{x \in \Omega; u(x) \in D\}. \end{cases}$$

Proof. It is easily adapted from that given in [76] in the case $W^1L_M(\Omega)$, by Theorem 1 of [74] instead of Theorem 4 of [74] (see also Remark 5 of [76]). □

Lemma 0.1.4. *(see [61]) Let $F : \mathbb{R} \to \mathbb{R}$ be uniformly Lipschitzian, with $F(0) = 0$. We suppose that the set of discontinuity points of F' is finite. Let M be an N-function. Then the mapping $F : W^1L_M(\Omega) \to W^1L_M(\Omega)$ is sequentially continuous with respect to the weak-* topology $\sigma(\Pi L_M, \Pi E_{\overline{M}})$.*

Lemma 0.1.5. *[74]*
Let Ω has the segment property. For each $v \in W_0^1L_M(\Omega)$, there exists a sequence $v_n \in \mathcal{D}(\Omega)$ such that v_n converges to v for the modular convergence in $W_0^1L_M(\Omega)$.
Furthermore, if $v \in W_0^1L_M(\Omega) \cap L^\infty(\Omega)$ then $\|v_n\|_{L^\infty(\Omega)} \leq (N+1)\|v\|_{L^\infty(\Omega)}$.

Lemma 0.1.6. *(See [74])*
For all $u \in W_0^1L_M(\Omega)$ with $meas(\Omega) < +\infty$ one has

$$\int_\Omega M(\frac{|u|}{\delta})dx \leq \int_\Omega M(|\nabla u|)dx, \tag{8}$$

where $\delta = diam(\Omega)$ is the diameter of Ω.

Inhomogeneous Orlicz-Sobolev spaces

For each $\alpha \in \mathbb{N}^N$, denote by ∇_x^α the distributional derivative on Q_T of order α with respect to the variable $x \in \mathbb{N}^N$.

The inhomogeneous Orlicz-Sobolev spaces are defined as follows,

$$W^{1,x}L_M(Q_T) = \{u \in L_M(Q_T) : \nabla_x^\alpha u \in L_M(Q_T), \forall \alpha \in \mathbb{N}^N, |\alpha| \leq 1\},$$

$$W^{1,x}E_M(Q_T) = \{u \in E_M(Q_T) : \nabla_x^\alpha u \in E_M(Q_T), \forall \alpha \in \mathbb{N}^N, |\alpha| \leq 1\}.$$

We can easily show that they form a complementary system when Ω satisfies the segment property. These spaces are considered as subspaces of the product space $\Pi L_M(Q_T)$ which have as many copies as there is α-order derivatives, $|\alpha| \leq 1$. We shall also consider the weak topologies $\sigma(\Pi L_M, \Pi E_{\overline{M}})$ and $\sigma(\Pi L_M, \Pi L_{\overline{M}})$. If $u \in W^{1,x}L_M(Q_T)$ then the function $: t \mapsto u(t) = u(t,.)$ is defined on $(0,T)$ with values $W^1L_M(\Omega)$. If, further, $u \in W^{1,x}E_M(Q_T)$, then the concerned function is a $W^1E_M(Q_T)$-valued and is strongly measurable. Furthermore the following imbedding holds:

$$W^1E_M(Q_T) \subset L^1(0,T,W^1E_M(\Omega)).$$

The space $W^{1,x}L_M(Q_T)$ is not in general separable, if $W^{1,x}L_M(Q_T)$, we can not conclude that the function $u(t)$ is measurable on $(0,T)$. However, the scalar function $t \mapsto \|u(t)\|_{M,\Omega}$, is in $L^1(0,T)$. The space $W_0^{1,x}E_M(Q_T)$ is defined as the (norm) closure $W^{1,x}E_M(Q_T)$ of $\mathscr{D}(Q_T)$. We can easily show as in [29] that when Ω has the segment property, then each element u of the closure of $\mathscr{D}(Q_T)$ with respect of the weak* topology $\sigma(\Pi L_M, \Pi E_{\overline{M}})$ is a limit, in $W_0^{1,x}E_M(Q_T)$, of some subsequence $(u_i) \subset \mathscr{D}(Q_T)$ for the modular convergence; i.e. there exists $\lambda > 0$ such that for all $|\alpha| \leq 1$

$$\int_{Q_T} M(\frac{\nabla_x^\alpha u_i - \nabla_x^\alpha u}{\lambda})dx\,dt \to 0 \quad \text{as} \quad i \to \infty.$$

This implies that (u_i) converge to u in $W^{1,x}L_M(Q_T)$ for the weak topology $\sigma(\Pi L_M, \Pi L_{\overline{M}})$.
Furthermore, $W_0^{1,x}E_M(Q_T) = W_0^{1,x}L_M(Q_T) \bigcap \Pi E_M$.
We have then the following complementary system

$$\begin{pmatrix} W_0^{1,x}L_M(Q_T) & F \\ W_0^{1,x}E_M(Q_T) & F_0 \end{pmatrix}$$

F being the dual space of $W_0^{1,x}E_M(Q_T)$. It is also, except for an isomorphism, the quotient of $\Pi L_{\overline{M}}$ by the polar set $W_0^{1,x}E_M(Q_T)^\perp$, and will be denoted by $F = W^{-1,x}L_{\overline{M}}(Q_T)$ and it is show that,

$$W^{-1,x}L_{\overline{M}}(Q_T) = \left\{ f = \sum_{|\alpha| \leq 1} \nabla_x^\alpha f_\alpha : \quad f_\alpha \in L_{\overline{M}}(Q_T) \right\}.$$

This space will be equipped with the usual quotient norm $\|f\| = \inf \sum_{|\alpha|\leq 1} \|f_\alpha\|_{\overline{M},Q_T}$, where the infimum is taken on all possible decompositions $f = \sum_{|\alpha|\leq 1} \nabla_x^\alpha f_\alpha, \quad f_\alpha \in L_{\overline{M}}(Q_T)$. The space F_0 is then given by, $F_0 = \left\{ f = \sum_{|\alpha|\leq 1} \nabla_x^\alpha f_\alpha : \quad f_\alpha \in E_{\overline{M}}(Q_T) \right\}$, and is denoted by $F_0 = W^{-1,x}E_{\overline{M}}(Q_T)$.

Remark 0.1.1. *We can easily check, using Lemma 0.1.3, that each uniformly Lipschitzian mapping F, with $F(0) = 0$, acts in inhomogeneous Orlicz-Sobolev spaces of order 1: $W^{1,x}L_M(Q_T)$ and $W_0^{1,x}L_M(Q_T)$.*

Lemma 0.1.7. *Let $\{u_n\}$ be a bounded sequence in $W^{1,x}L_M(Q_T)$ such that $\frac{\partial u_n}{\partial t} = \alpha_n + \beta_n$ in $\mathscr{D}'(Q_T)$, $u_n \rightharpoonup u$ weakly in $W^{1,x}L_M(Q_T)$ for $\sigma(\Pi L_M, \Pi E_{\overline{M}})$, with $\{\alpha_n\}$ and $\{\beta_n\}$ two bounded sequences respectively in $W^{-1,x}L_{\overline{M}}(Q_T)$ and in $\mathscr{M}(Q_T)$. Then $u_n \to u$ in $L^1_{loc}(Q_T)$.*
Furthermore, if $u_n \in W_0^{1,x}L_M(Q_T)$, then $u_n \to u$ strongly in $L^1(Q_T)$.

Lemma 0.1.8. *(see [57]) For all $u \in W_0^1 L_M(Q_T)$ with $meas(Q_T) < +\infty$ one has*

$$\int_{Q_T} M(\frac{|u|}{\lambda})dxdt \leq \int_{Q_T} M(|\nabla u|)dxdt \tag{9}$$

where $\lambda = diam(Q_T)$, is the diameter of Q_T.

1

Nonlinear elliptic equations with measure data

In this chapter we consider the following equation

$$Au - div(\Phi(x,u)) + K(x,u,\nabla u) = \mu,$$

where $Au = -\mathrm{div}(a(x,u,\nabla u))$ is a Leray−Lions operator defined on $D(A) \subset W_0^1 L_M(\Omega)$, $\mu \in L^1(\Omega) + W^{-1}E_{\overline{M}}(\Omega)$, and M and $\overline{M}$ are two complementary N-functions. we will be concerned with the existence result of the unilateral problem in Orlicz spaces, such that the first and the second lower terms Φ and K satisfies only the growth condition and any sign condition is assumed and $u \geq \zeta$ where ζ is a measurable function. The modular functions M and $\overline{M}$ do not require $M \in \triangle_2$.

1.1 Introduction

Let Ω be a bounded open domain in $\mathbb{R}^N$, $(N \geq 2)$ and consider the following strongly nonlinear Dirichlet problem with obstacle

$$\begin{cases} u \geq \zeta \quad \text{a.e. in} \quad \Omega, \\ -div(a(x,u,\nabla u)) - div(\Phi(x,u)) + K(x,u,\nabla u) = \mu \quad \text{on} \quad \Omega, \\ u = 0 \quad \text{on} \quad \partial\Omega, \end{cases} \tag{1.1}$$

where $\mu \in L^1(\Omega) + W^{-1}E_{\overline{M}}(\Omega)$.

Under our assumptions, the problem (1.1) does not admit, in general, a weak solution since the term Φ may not belong to $(L^1(\Omega))^N$ and a lack of coercivity for the two terms Φ and K. Thus, to overcome this difficulty we use in this chapter the framework of an entropy solution, which needs less regularity than the usual weak solutions. Knowing that the notion of entropy solutions have been developed by P. Bénilan et al. [37] for the study of nonlinear elliptic problems.

In [42], L. Boccardo et al. has study the existence and uniqueness of entropy solutions for the problem

$$\begin{cases} -div(a(x,u,\nabla u)) + K(x,u,\nabla u) = \mu \quad \text{on} \quad \Omega, \\ u = 0 \quad \text{on} \quad \partial\Omega, \end{cases}$$

in the Classical Sobolev space $W_0^{1,p}(\Omega)$, where K is a polynomial growth and μ is measure in $\mathscr{M}_b(\Omega)$, L. Boccardo et al. proved the existence results and in [47] have demonstrated that $\mu \in$

$\mathscr{M}(\Omega)$ if only if $\mu \in L^1(\Omega) + W^{-1,p'}(\Omega)$.

To our knowledge, it is not yet possible to decompose the Radon measure in setting Orlicz space, therefore the authors studied the problem (1.1) with the second member as the sum of an element of $W^{-1}E_{\overline{M}}(\Omega)$ and of a function of $L^1(\Omega)$. In the first J. P. Gossez and Mustonen in [75] proved Existence of entropy solutions for nonlinear problem (1.1) where $\mu \in W^{-1}E_{\overline{M}}(\Omega)$ and K satisfies the sign condition. A. Benkirane and J. Bennouna in [32] proved the existence and uniqueness of solutions of unilateral problem where $\Phi = K = 0$ and $\mu \in L^1(\Omega)$, L. Aharouch and M. Rhoudaf in [10] with $\Phi = 0$ and K satisfied the sign condition and L. Aharouch et al. in [11] have proved the existence results where $K = 0$, $\Phi \in \mathscr{C}^0(\mathbb{R}, \mathbb{R}^N)$ and $\mu \in L^1(\Omega) + W^{-1}E_{\overline{M}}(\Omega)$.

In this chapter, we are interested in proving the existence of entropy solution for unilateral problem associated to (1.1) where Φ depends on x, u and satisfies only the growth condition and K is a nonlinear lower-order term having natural growth with respect to $|\nabla u|$. the second member of (1.1) as $\mu = f - div(F)$ with $f \in L^1(\Omega)$ and $F \in W^{-1}E_{\overline{M}}(\Omega)$.

The main difficulties of this problem are in the first the lack of coercivity lower order term Φ that makes the operator that governs the equation, non coercive. The second lower order term K is controlled by a non-polynomial growth (see (1.6)) and no sign condition is assumed.

Note that Our result generalizes that of [11, 31, 32, 42] to the case of Orlicz Sobolev spaces.

1.2 Main result

Let M and P be two N-functions such that $P \ll M$.
$a : \Omega \times \mathbb{R} \times \mathbb{R}^N \to \mathbb{R}^N$ is Caratheodory function such that for a.e. $x \in \Omega$ and for all $s \in \mathbb{R}$, $\xi, \xi^* \in \mathbb{R}^N$, $\xi \neq \xi^*$

$$|a(x,s,\xi)| \leq \beta(a_0(x) + \overline{M}^{-1}P(k_1|s|)) + \overline{M}^{-1}M(k_2|\xi|)), \tag{1.2}$$

where $\beta > 0, a_0(.) \in E_{\overline{M}}(\Omega)$.

$$(a(x,s,\xi) - a(x,s,\xi^*)(\xi - \xi^*) > 0, \tag{1.3}$$

$$a(x,s,\xi).\xi \geq \alpha M(|\xi|) + M(|s|). \tag{1.4}$$

$\Phi : \Omega \times \mathbb{R} \to \mathbb{R}^N$ is a Carathéodory function such that

$$|\Phi(x,s)| \leq c(x)\overline{M}^{-1}M(\alpha_0|s|), \tag{1.5}$$

where $c(.) \in L^\infty(\Omega)$, $\|c(.)\|_{L^\infty(\Omega)} < \frac{\alpha}{2}$ and $0 < \alpha_0 < \min(1, \frac{1}{\alpha})$.
$K : \Omega \times \mathbb{R} \times \mathbb{R}^N \to \mathbb{R}$ is a Carathéodory function such that

$$|K(x,s,\xi)| \leq h(x) + \rho(s)M(|\xi|), \tag{1.6}$$

$\rho : \mathbb{R} \to \mathbb{R}^+$ is a continuous positive function which belongs to $L^1(\mathbb{R}) \cap L^\infty(\mathbb{R})$ and $h(.) \in L^1(\Omega)$.
Let $\mu \in L^1(\Omega) + W^{-1}E_{\overline{M}}(\Omega)$, such that

$$\mu = f - \operatorname{div}(F) \quad \text{with} \quad f \in L^1(\Omega) \quad \text{and} \quad F \in (E_{\overline{M}}(\Omega))^N. \tag{1.7}$$

Given a measurable obstacle function $\zeta : \Omega \to \mathbb{R}$,

$$K_\zeta = \{u \in W_0^1 L_M(\Omega) : u \geq \zeta \quad \text{a.e. in} \quad \Omega\}, \tag{1.8}$$

and we suppose that $K_\zeta \cap L^\infty(\Omega) \neq \emptyset$.

Definition 1.2.1. *A measurable function u defined on Ω is a entropy solution of problem (1.1), if it satisfies the following conditions:*

$$\begin{cases} u \in D(A) \cap W_0^1 L_M(\Omega), u \geq \zeta, \\ \int_\Omega a(x,u,\nabla u)\nabla T_k(u-v)dx + \int_\Omega \Phi(x,u)\nabla T_k(u-v)dx \\ + \int_\Omega K(x,u,\nabla u)\nabla T_k(u-v)dx \leq \int_\Omega f T_k(u-v)dx + \int_\Omega F\nabla T_k(u-v)dx \\ \forall v \in K_\zeta \cap L^\infty(\Omega), \quad \forall k > 0. \end{cases} \tag{1.9}$$

Theorem 1.2.1. *Assume that* $(1.2)-(1.8)$ *hold true.Then there exists at least one solution of the unilateral problem (1.1) in the sense of the Definition 1.2.1.*

Remark 1.2.1.

- *The condition (1.4) can be replaced by the weaker one*

$$a(x,s,\xi)\xi \geq \alpha M(|\xi|) + M(|s|) - b(x),$$

where $b(x)$ *is in* L^1*-function.*

- *The result obtained in Theorem 1.2.1, remains true if we replace (1.5) by the following growth condition*

$$|\Phi(x,s)| \leq c(x)\overline{P}^{-1}P(|s|), \tag{1.10}$$

where $c(.) \in E_P(\Omega)$ *and* $P \ll M$.

- *For any* $s \in \mathbb{R}$ *and* $\alpha' > 0$*, we have*

$$\exp(-\frac{\|\rho\|_{L^1(\Omega)}}{\alpha'}) \leq \exp(\pm G(s)) \leq \exp(\frac{\|\rho\|_{L^1(\Omega)}}{\alpha'}), \tag{1.11}$$

where $G(s) = \int_0^s \frac{\rho(r)}{\alpha'}dr$

Remark 1.2.2.

- *The explicit dependence in x of the functions a,* Φ *and K will be omitted so that* $a(x,u,\nabla u) = a(u,\nabla u)$*,* $\Phi(x,u) = \Phi(u)$ *and* $K(x,u,\nabla u) = K(u,\nabla u)$.
- *We will denote by* C_i *with* $i = 1,2,...$ *any constant which depends on the various quantities of the problem but not on n.*

Proof of Theorem 1.2.1:

Step 1: Existence for the problem Galerkin approximation.

For each $n > 0$, we define the following approximations

$$a_n(s,\xi) = a(T_n(s),\xi) \quad \text{a.e. } x \in \Omega, \ \forall\, s \in \mathbb{R}, \ \forall\, \xi \in \mathbb{R}^N,$$

$$\Phi_n(s) = \Phi(T_n(s)) \quad \text{a.e. } x \in \Omega, \ \forall\, s \in \mathbb{R},$$

$$K_n(s,\xi) = \frac{K(s,\xi)}{1+\frac{1}{n}|K(s,\xi)|}, \ \forall\, s \in \mathbb{R}, \ \forall\, \xi \in \mathbb{R}^N.$$

Let $(f_n)_n$ be a sequence of smooth functions such that $f_n \to f$ strongly in $L^1(\Omega)$.
Let us now consider the approximate problem :

$$\begin{cases} u_n \in K_\zeta \cap D(A) \\ \int_\Omega a_n(u_n, \nabla u_n)\nabla(u_n - v)dx + \int_\Omega \Phi_n(u_n)\nabla(u_n - v)dx \\ + \int_\Omega K_n(u_n, \nabla u_n)(u_n - v)dx \leq \int_\Omega f_n(u_n - v)dx + \int_\Omega F\nabla(u_n - v)dx \quad \forall v \in K_\zeta. \end{cases} \tag{1.12}$$

Since K_n is bounded for any fixed $n > 0$, there exists at least one solution $u_n \in W_0^1 L_M(\Omega)$ of (1.12) (see [75]).

Step 2: A priori estimates.

Lemma 1.2.1.
Let $\{u_n\}_n$ be a solution of the approximate problem (1.12). Then for all $k > 0$, there exists a two constants C_1 and C_2 such that

$$\int_\Omega M(|\nabla T_k(u_n)|)dx \leq kC_1 + C_2. \tag{1.13}$$

Proof. Let $v_0 \in K_\zeta \cap L^\infty(\Omega) \cap W_0^1 E_M(\Omega)$ and fix $k > 0$.

Let $u_n - \exp(G(u_n))T_k(u_n - v_0)^+$ as a test function in problem (1.12), where $G(s) = \int_0^s \frac{\rho(r)}{\alpha'}dr$ and $\alpha' > 0$ is a parameter to be specified later. We get

$$\begin{aligned} &\int_\Omega a_n(u_n, \nabla u_n)\nabla\big(\exp(G(u_n))T_k(u_n - v_0)^+\big)dx \\ &\quad + \int_\Omega \Phi_n(u_n)\nabla\big(\exp(G(u_n))T_k(u_n - v_0)^+\big)dx + \int_\Omega K_n(u_n, \nabla u_n)\exp(G(u_n))T_k(u_n - v_0)^+dx \\ &\qquad \leq k\exp\Big(\frac{\|\rho\|_{L^1}}{\alpha'}\Big)\|f_n\|_{L^1(\Omega)} + \int_\Omega F\nabla\big(\exp(G(u_n))T_k(u_n - v_0)^+\big)dx. \end{aligned} \tag{1.14}$$

* In the second term of the left-hand side of (1.14), we use (1.5), (2.5.8) and Young inequality to get

$$\begin{aligned} \int_\Omega \Phi_n(u_n)\nabla(\exp(G(u_n))T_k(u_n - v_0)^+)dx \leq \frac{\|c(.)\|_{L^\infty(\Omega)}}{\alpha'}\Big[\alpha_0 \int_\Omega M(|u_n|)\rho(u_n)\exp(G(u_n))T_k(u_n - v_0)^+dx \\ + \int_\Omega M(|\nabla u_n|)\rho(u_n)\exp(G(u_n))T_k(u_n - v_0)^+dx\Big] \\ + 2\|c(.)\|_{L^\infty(\Omega)}\alpha_0 \int_{|u_n - v_0| \leq k} M(|u_n|)\exp(G(u_n))dx \\ + \|c(.)\|_{L^\infty(\Omega)} \int_{\{0 \leq u_n - v_0 \leq k\}} M(|\nabla u_n|)\exp(G(u_n))dx + c_1. \end{aligned}$$

* For the third term of the left-hand side of (1.14), we have

$$\begin{aligned} \int_\Omega K_n(u_n, \nabla u_n)\exp(G(u_n))T_k(u_n - v_0)^+dx \leq k\exp\Big(\frac{\|\rho\|_{L^1}}{\alpha'}\Big)\|h\|_{L^1(\Omega)} \\ + \int_\Omega \rho(u_n)\exp(G(u_n))M(|\nabla u_n|)T_k(u_n - v_0)^+dx. \end{aligned}$$

* For the second term of the right-hand side of (1.14), we have

$$\int_\Omega F\nabla\big(\exp(G(u_n))T_k(u_n-v_0)^+\big)dx \le \frac{k}{\alpha'}\exp(\frac{\|\rho\|_{L^1}}{\alpha'})\|\rho\|_{L^\infty}\int_\Omega \overline{M}(\frac{|F|}{\varepsilon_1})dx$$
$$+\frac{\varepsilon_1}{\alpha'}\int_\Omega \rho(u_n)\exp(G(u_n))M(|\nabla u_n|)T_k(u_n-v_0)^+dx$$
$$+2\exp(\frac{\|\rho\|_{L^1}}{\alpha'})\int_\Omega \overline{M}(\frac{|F|}{\varepsilon_1})dx+\varepsilon_1\int_{\{0\le u_n-v_0)^+\le k\}}\exp(G(u_n))M(|\nabla u_n|)dx+c_2.$$

Finally, using the above and (1.4) in (1.14) we get

$$(\frac{1-\alpha_0\|c(.)\|_{L^\infty(\Omega)}}{\alpha'})\int_\Omega M(|u_n|)\rho(u_n)\exp(G(u_n)T_k(u_n-v_0)^+dx$$
$$+\big[\frac{\alpha}{\alpha'}-\frac{\|c(.)\|_{L^\infty(\Omega)}}{\alpha'}-1-\frac{\varepsilon_1}{\alpha'}\big]\int_\Omega \rho(u_n)\exp(G(u_n)M(|\nabla u_n|)T_k(u_n-v_0)^+dx$$
$$+\int_\Omega a(u_n,\nabla u_n)\exp(G(u_n))\nabla(T_k(u_n-v_0)^+)dx$$
$$\le 2\|c(.)\|_{L^\infty(\Omega)}\alpha_0\int_{\{0\le u_n-v_0\le k\}}M(|u_n|)\exp(G(u_n))dx$$
$$+(\|c(.)\|_{L^\infty(\Omega)}+\varepsilon_1)\int_{\{0\le u_n-v_0\le k\}}M(|\nabla u_n|)\exp(G(u_n))dx+c_3k+c_4.$$

where $c_4=c_1+c_2$.
If we choose α' and ε_1 such that $\alpha'=\dfrac{\alpha}{2}$, $\varepsilon_1<\alpha-\alpha'-\|c(.)\|_{L^\infty(\Omega)}$, we obtain

$$\int_{\{0\le u_n-v_0\le k\}}a(u_n,\nabla u_n)\exp(G(u_n))\nabla u_n dx$$
$$\le(\frac{\varepsilon_1+\|c(.)\|_{L^\infty(\Omega)}}{\alpha})\Big[\alpha_0\alpha\int_{\{0\le u_n-v_0\le k\}}M(|u_n|)\exp(G(u_n))dx+\alpha\int_{\{0\le u_n-v_0\le k\}}M(|\nabla u_n|)\exp(G(u_n))dx\Big]$$
$$+\int_{\{0\le u_n-v_0\le k\}}a(u_n,\nabla u_n)\exp(G(u_n))\nabla v_0dx+c_3k+c_4.$$

Since $\alpha_0\alpha<1$ and using (1.4) we get

$$[1-(\frac{\varepsilon_1+\|c(.)\|_{L^\infty(\Omega)}}{\alpha})]\int_{\{0\le u_n-v_0\le k\}}a(x,u_n,\nabla u_n)\exp(G(u_n))\nabla u_n dx$$
$$\le\int_{\{0\le u_n-v_0\le k\}}a(u_n,\nabla u_n)\exp(G(u_n))\nabla v_0dx+c_3k+c_4,$$

By using (1.3), we get for any $\alpha_1>0$ (α_1 is a parameter to be specified later).

$$\alpha_1\int_{\{0\le u_n-v_0\le k\}}a(u_n,\nabla u_n)\exp(G(u_n))\frac{\nabla v_0}{\alpha_1}dx$$
$$\le\alpha_1\Big[\int_{\{0\le u_n-v_0\le k\}}a(u_n,\nabla u_n)\exp(G(u_n))\nabla u_n dx-\int_{\{0\le u_n-v_0\le k\}}a(u_n,\frac{\nabla v_0}{\alpha_1})\exp(G(u_n))(\nabla u_n-\frac{\nabla v_0}{\alpha_1})dx\Big],$$

then

$$c' \int_{\{0\le u_n - v_0 \le k\}} a(x,u_n,\nabla u_n)\exp(G(u_n))\nabla u_n dx$$
$$\le \alpha_1 \Big[\int_{\{0\le u_n - v_0 \le k\}} |a(u_n, \frac{\nabla v_0}{\alpha_1})| \exp(G(u_n))|\nabla u_n| dx + \int_{\{0\le u_n - v_0 \le k\}} |a(u_n, \frac{\nabla v_0}{\alpha_1})| \exp(G(u_n))| \frac{\nabla v_0}{\alpha_1} | dx \Big]. \tag{1.15}$$

Taking $c' = [1 - (\frac{\varepsilon_1 + \|c(.)\|_{L^\infty(\Omega)}}{\alpha}) - \alpha_1]$ such that $\alpha_1 < 1 - (\frac{\varepsilon_1 + \|c(.)\|_{L^\infty(\Omega)}}{\alpha})$.
Using (1.2), (7) and Young inequality, we have

$$|a(u_n, \frac{\nabla v_0}{\alpha_1})| \exp(G(u_n))|\nabla u_n|$$
$$\le \exp(\frac{\|\rho\|_{L^1(\Omega)}}{\alpha'})\beta \Big[\overline{M}(a(x)) + M(\frac{|u_n|}{\delta}) + C' + M(k_2 \frac{|\nabla v_0|}{\alpha_1}) + 3M(|\nabla u_n|) \Big], \tag{1.16}$$

and

$$|a(u_n, \frac{\nabla v_0}{\alpha_1})| \exp(G(u_n))| \frac{\nabla v_0}{\alpha_1} |$$
$$\le \exp(\frac{\|\rho\|_{L^1(\Omega)}}{\alpha'})\beta \Big[\overline{M}(a(x)) + M(\frac{|u_n|}{\delta}) + C' + M(k_2 \frac{|\nabla v_0|}{\alpha_1}) + 3M(\frac{|\nabla v_0|}{\alpha_1}) \Big]. \tag{1.17}$$

Since $\frac{\nabla v_0}{\alpha_1} \in (E_M(\Omega))^N$, one pass to the integral in (1.16), (1.17) and using (2.5.8), (1.11) and by (1.15), we get

$$c' \int_{\{0\le u_n - v_0 \le k\}} a(x,u_n,\nabla u_n)\exp(G(u_n))\nabla u_n dx$$
$$\le 4\alpha_1 \beta \exp(\frac{\|\rho\|_{L^1(\Omega)}}{\alpha'}) \int_{\{0\le u_n - v_0 \le k\}} M(|\nabla u_n|) dx + c_5 k + c_6.$$

Taking also α_1 such that $4\alpha_1\beta \exp(\frac{\|\rho\|_{L^1(\Omega)}}{\alpha'}) < \frac{c'}{2}\exp(-\frac{\|\rho\|_{L^1(\Omega)}}{\alpha'})$ and using (1.4), we obtain

$$\int_{\{0\le u_n - v_0 \le k\}} M(|\nabla u_n|) dx \le c_7 k + c_8. \tag{1.18}$$

Similarly, taking $u_n + \exp(-G(u_n))T_k(u_n - v_0)^-$ as a test function in problem (1.12), we get

$$c' \int_{\{-k\le u_n - v_0 \le 0\}} a(x,u_n,\nabla u_n)\exp(-G(u_n))\nabla u_n dx$$
$$\le \frac{c'}{2} \int_{\{-k\le u_n - v_0 \le 0\}} \exp(-G(u_n)) M(|\nabla u_n|) dx + c_9 k + c_{10}. \tag{1.19}$$

and

$$\int_{\{-k\le u_n - v_0 \le 0\}} M(|\nabla u_n|) dx \le c_{11} k + c_{12}. \tag{1.20}$$

Combining now (1.18) and (1.20), we deduce (1.13).
Since $\{x \in \Omega; |u_n| \leq k\} \subset \{x \in \Omega; |u_n - v_0| \leq k + \|v_0\|_\infty\}$, we obtain

$$\begin{aligned}\int_\Omega M(|\nabla T_k(u_n)|)dx &= \int_{|u_n|\leq k} M(|\nabla T_k(u_n)|)dx \\ &\leq \int_{|u_n - v_0 \leq |\leq k+\|v_0\|_\infty} M(|\nabla T_k(u_n)|)dx \\ &\leq kC_1 + C_2.\end{aligned}$$

We conclude that $\{T_k(u_n)\}_n$ is bounded in $W_0^1 L_M(\Omega)$ independently of n and for any $k > 0$, so there exists a subsequence still denoted by u_n such that

$$T_k(u_n) \rightharpoonup \xi_k \quad \text{weakly in} \quad W_0^1 L_M(\Omega). \tag{1.21}$$

On the other hand, using (2.5.8), we have

$$\begin{aligned}M(\frac{k}{\delta})\text{meas}\{|u_n| > k\} &\leq \int_{\{|u_n|>k\}} M(\frac{|T_k(u_n)|}{\delta})dx \\ &\leq \int_\Omega M(|\nabla T_k(u_n)|)dx \leq kC_1 + C_2.\end{aligned}$$

Then $\text{meas}\{|u_n| > k\} \leq \dfrac{kC_1 + C_2}{M(\frac{k}{\delta})}$ for all n and for all k.
Thus, we get

$$\lim_{k\to\infty} \text{meas}\{|u_n| > k\} = 0.$$

□

Step 3:
Now we turn to prove the almost every convergence of u_n and convergence of $a_n(T_k(u_n), \nabla T_k(u_n))$.

Proposition 1.2.1. *Let u_n be a solution of the approximate problem, then there exists a measurable function u such that*

$$u_n \to u \quad a.e \text{ in} \quad \Omega, \tag{1.22}$$

and

$$a_n(T_k(u_n), \nabla T_k(u_n)) \rightharpoonup \varpi_k \text{ in } (L_{\overline{M}}(\Omega))^N, \text{ for } \sigma(\Pi L_{\overline{M}}, \Pi E_M), \text{ and for some } \varpi_k \in (L_{\overline{M}}(\Omega))^N. \tag{1.23}$$

Proof.
Proof of (1.22):
Let $\eta > 0$ and $\varepsilon > 0$ then

$$\text{meas}\{|u_n - u_m| > \eta\} \leq \text{meas}\{|u_n| > k\} + \text{meas}\{|u_m| > k\} + \text{meas}\{|T_k(u_n) - T_k(u_m)| > \eta\}.$$

By (4.43), we can assume that $\{T_k(u_n)\}_n$ is a Cauchy sequence in measure in Ω.
Thus, there exists $k(\varepsilon) > 0$ such that $\text{meas}\{|T_k(u_n) - T_k(u_m)| > \eta\} < \varepsilon$ for all $n, m > n_0$. Then $\{u_n\}_n$

is a Cauchy sequence in measure in Ω and then converges almost everywhere to some measurable function u.

Proof of (1.23):

We shall prove that $\{a(T_k(u_n),\nabla T_k(u_n))\}_n$ is bounded in $\big(L_{\overline{M}}(\Omega)\big)^N$.
Let $w\in(E_M(\Omega))^N$ be arbitrary. By condition (1.3) we have

$$(a(u_n,\nabla u_n)-a(u_n,w))(\nabla u_n-w)>0.$$

Then

$$\int_{\{|u_n|\le k\}} a(u_n,\nabla u_n)w dx \le \int_{\{|u_n|\le k\}} a(u_n,\nabla u_n)\nabla u_n dx+\int_{\{|u_n|\le k\}} a(u_n,w)(w-\nabla u_n)dx.$$

By (1.2) we have for $\nu>\beta$,

$$\begin{aligned}\int_{\{|u_n|\le k\}} \overline{M}\Big(\frac{a(u_n,\frac{w}{k_2})}{3\beta}\Big)dx &\le \frac{\beta}{3\nu}\Big[\int_\Omega \overline{M}(a_0(x))dx+\int_\Omega P(k_1|T_k(u_n)|)dx+M(|w|)dx\Big]\\ &\le \frac{\beta}{3\nu}\Big[\int_\Omega \overline{M}(a_0(x))dx+\int_\Omega P(k_1 k)dx+M(|w|)dx\Big].\end{aligned}$$

Thus, $\{a(T_k(u_n),\frac{w}{k_2})\}$ is bounded in $(L_{\overline{M}}(\Omega))^N$. By (1.13), (1.19) and by the Theorem of Banach-Steinhaus, the sequence $\{a(T_k(u_n),\nabla T_k(u_n))\}$ remains bounded in $(L_{\overline{M}}(\Omega))^N$ and we conclude (1.23). □

Step 4: Almost everywhere convergence of the gradients.

To have that the gradient converges almost everywhere, we need to prove this proposition

Proposition 1.2.2. *Let $\{u_n\}_n$ be a solution of the approximate problem (1.12), then*

$$\lim_{m\to\infty}\limsup_{n\to\infty}\int_{\{m\le|u_n|\le m+1\}} a(u_n,\nabla u_n)\nabla u_n dx=0. \tag{1.24}$$

For a subsequence as $n\to\infty$

$$\nabla u_n\to\nabla u \quad a.e.\ in \quad \Omega. \tag{1.25}$$

Proof.

Taking the function $Z_m(u_n)=u_n+\exp(-G(u_n))\psi_m(u_n)^-$, where $\psi_m(u_n)=T_1(u_n-T_m(u_n))$.
Multiplying the approximating equation (1.12) by the test function $Z_m(u_n)$, we get

$$\begin{cases}\displaystyle\int_\Omega a_n(u_n,\nabla u_n)\nabla(\exp(-G(u_n))\psi_m(u_n)^-)dx\\ \displaystyle+\int_\Omega \Phi_n(u_n)\nabla(\exp(-G(u_n))\psi_m(u_n)^-)dx+\int_\Omega K_n(u_n,\nabla u_n)\exp(-G(u_n))\psi_m(u_n)^- dx\\ \displaystyle\ge\int_\Omega f_n\exp(-G(u_n))\psi_m(u_n)^- dx+\int_\Omega F\nabla(\exp(-G(u_n))\psi_m(u_n)^-)dx.\end{cases} \tag{1.26}$$

* In the second term of the left-hand side of (1.26), we use (1.5), (2.5.8) and Young inequality to get

$$\int_\Omega \Phi_n(u_n)\nabla(\exp(-G(u_n))\psi_m(u_n)^-)dx \le \frac{\|c(.)\|_{L^\infty(\Omega)}}{\alpha'}\Big[\alpha_0\int_\Omega M(|u_n|)\rho(u_n)\exp(-G(u_n))\psi_m(u_n)^- dx + \int_\Omega M(\nabla u_n)\rho(u_n)\exp(-G(u_n))\psi_m(u_n)^- dx\Big] + \|c(.)\|_{L^\infty(\Omega)}\alpha_0\int_{\{-(m+1)\le u_n\le -m\}} M(|u_n|)\exp(-G(u_n))dx + \|c(.)\|_{L^\infty(\Omega)}\int_\Omega M(|\nabla\psi_m(u_n)^-|)\exp(-G(u_n))dx.$$

* For the second term of the right-hand side of (1.26), we have

$$\int_\Omega F\nabla\big(\exp(-G(u_n))\psi_m(u_n)^-\big)dx \le \frac{\|\rho\|_{L^\infty}}{\alpha'}\exp(\frac{\|\rho\|_{L^1}}{\alpha'})\int_\Omega \overline{M}(\frac{|F|}{\varepsilon_1})\psi_m(u_n)^- dx + \frac{\varepsilon_1}{\alpha'}\int_\Omega \rho(u_n)\exp(-G(u_n))M(|\nabla u_n|)\psi_m(u_n)^- dx + \exp(\frac{\|\rho\|_{L^1}}{\alpha'})\int_{\{-(m+1)\le u_n\le -m\}}\overline{M}(\frac{|F|}{\varepsilon_1})dx + \varepsilon_1\int_\Omega \exp(-G(u_n))M(|\nabla\psi_m(u_n)^-|)dx.$$

Using the same argument in step 2, we obtain

$$\int_\Omega a_n(u_n,\nabla u_n)\nabla(\psi_m(u_n)^-)dx \le \exp(\frac{\|\rho\|_{L^1}}{\alpha'})C\Big[\int_\Omega f_n Z_m(u_n)\,dx + \int_\Omega h(x)Z_m(u_n)dx + \frac{\|\rho\|_{L^\infty}}{\alpha'}\int_\Omega \overline{M}(\frac{|F|}{\varepsilon_1})Z_m(u_n)dx + \int_{\{-(m+1)\le u_n\le -m\}}\overline{M}(\frac{|F|}{\varepsilon_1})dx\Big]. \quad (1.27)$$

where $\frac{1}{C} = \Big[1-(\frac{\|c(.)\|_{L^\infty(\Omega)}+\varepsilon_1}{\alpha})\Big]\exp(-\frac{\|\rho\|_{L^1}}{\alpha'})$.
Passing to limit as $n\to+\infty$, since the pointwise convergence of u_n and strongly convergence in $L^1(\Omega)$ of f_n we get

$$\lim_{n\to+\infty}\int_\Omega a_n(u_n,\nabla u_n)\nabla(\psi_m(u_n)^-)dx \le \exp(\frac{\|\rho\|_{L^1}}{\alpha'})C\Big[\int_\Omega f_n Z_m(u)\,dx + \int_\Omega h(x)Z_m(u)dx + \frac{\|\rho\|_{L^\infty}}{\alpha'}\int_\Omega \overline{M}(\frac{|F|}{\varepsilon_1})Z_m(u)dx + \int_{\{-(m+1)\le u\le -m\}}\overline{M}(\frac{|F|}{\varepsilon_1})dx\Big].$$

Using Lebesgue's theorem and passing to limit as $m\to+\infty$, in the all term of the right-hand side, we get

$$\lim_{m\to+\infty}\lim_{n\to+\infty}\int_{\{-(m+1)\le u_n\le -m\}} a_n(u_n,\nabla u_n)\nabla u_n dx = 0.$$

In the same way we take $Z_m(u_n) = u_n - \exp(G(u_n))\psi_m(u_n)^+$ and multiplying the approximating equation (1.12) by the test function $Z_m(u_n)$ and we also obtain

$$\lim_{m\to+\infty}\lim_{n\to+\infty}\int_{\{m\le u_n\le m+1\}} a_n(u_n,\nabla u_n)\nabla u_n dx = 0.$$

On the above we get (1.24).

To show that $\nabla u_n \to \nabla u$ a.e. in Ω where $\{u_n\}_n$ is the solution of the approximate problem (1.12), we first call the following Lemma,

Lemma 1.2.2. *([29])*
Under assumptions (1.2)-(1.7) and let (z_n) be a sequence in $W_0^1 L_M(\Omega)$ such that:

$$z_n \rightharpoonup z \quad for \quad \sigma(\Pi L_M, \Pi E_{\overline{M}}), \tag{1.28}$$

$$(a(z_n, \nabla z_n)) \quad is\ bounded\ in\ (L_{\overline{M}}(\Omega))^N, \tag{1.29}$$

$$\int_\Omega [a(z_n, \nabla z_n) - a(z_n, \nabla z \chi_s)][\nabla z_n - \nabla z \chi_s] dx \to 0, \tag{1.30}$$

as n and s tend to $+\infty$ and where χ_s is the characteristic function of $\Omega^s = \{x \in \Omega; |\nabla z| \le s\}$ then,

$$\nabla z_n \to \nabla z \quad a.e.\ in\ \Omega, \tag{1.31}$$

$$\lim_{n\to+\infty} \int_\Omega a(z_n, \nabla z_n) \nabla z_n dx = \int_\Omega a(z, \nabla z) \nabla z dx, \tag{1.32}$$

$$M(|\nabla z_n|) \to M(|\nabla z|) \quad in\ L^1(\Omega). \tag{1.33}$$

Let $\upsilon_j \in \mathscr{D}(\Omega)$ be a sequence such that $\upsilon_j \to u$ in $W_0^1 L_M(\Omega)$ for the modular convergence.
We introduce a sequence of increasing $\mathbf{C}^1(\mathbb{R})$-functions S_m such that for any $m \ge 1$

$$\begin{cases} S_m(r) = 1 & \text{for } |r| \le m, \\ S_m(r) = m+1-|r| & \text{for } m \le |r| \le m+1, \\ S_m(r) = 0 & |r| \ge m+1. \end{cases}$$

And we denote by $\varepsilon(n, \eta, j, m)$ all quantities (possibly different) such that

$$\lim_{m\to+\infty} \lim_{j\to+\infty} \lim_{\eta\to+\infty} \lim_{\mu\to+\infty} \lim_{n\to+\infty} \varepsilon(n, \eta, j, m) = 0.$$

For fixed $k \ge 0$, let $W_\eta^{n,j} = T_\eta(T_k(u_n) - T_k(\upsilon_j))^+$ and $W_\eta^j = T_\eta(T_k(u) - T_k(\upsilon_j))^+$.
Multiplying the approximating equation by $u_n - \exp(G(u_n)) W_\eta^{n,j} S_m(u_n)$ and using (1.4) and (1.6) we obtain

$$\begin{cases} \displaystyle\int_\Omega a_n(u_n, \nabla u_n) \exp(G(u_n)) \nabla(W_\eta^{n,j}) S_m(u_n) dx + \int_\Omega a_n(u_n, \nabla u_n) \nabla u_n \exp(G(u_n)) W_\eta^{n,j} S_m'(u_n)\, dx \\ \displaystyle - \int_\Omega \Phi_n(u_n) \exp(G(u_n)) \nabla(W_\eta^{n,j}) S_m(u_n)\, dx - \int_\Omega \Phi_n(u_n) \nabla u_n \exp(G(u_n)) W_\eta^{n,j} S_m'(u_n) dx \\ \displaystyle \le \int_\Omega f_n \exp(G(u_n)) W_\eta^{n,j} S_m(u_n)\, dx + \int_\Omega h(x) \exp(G(u_n)) W_\eta^{n,j} S_m(u_n) dx \\ \displaystyle + \int_\Omega F \exp(G(u_n)) \nabla(W_\eta^{n,j}) S_m(u_n)\, dx + \int_\Omega F \nabla u_n \exp(G(u_n)) W_\eta^{n,j} S_m'(u_n) dx. \end{cases} \tag{1.34}$$

Now we pass to the limit in (1.34) for k real number fixed.
In order to perform this task we prove below the following results for any fixed $k \ge 0$:

$$\int_\Omega \Phi_n(u_n) S_m(u_n) \exp(G(u_n)) \nabla(W_\eta^{n,j})\, dx = \varepsilon(n, j) \quad \text{for any } m \ge 1, \tag{1.35}$$

$$\int_\Omega \Phi_n(u_n)\nabla u_n S_m'(u_n)\exp(G(u_n))W_\eta^{n,j}\,dx=\varepsilon(n,j)\qquad\text{for any } m\geq 1, \tag{1.36}$$

$$\int_\Omega a_n(u_n,\nabla u_n)\nabla u_n S_m'(u_n)\exp(G(u_n))W_\eta^{n,j}\,dx\leq\varepsilon(n,m), \tag{1.37}$$

$$\int_\Omega a_n(u_n,\nabla u_n)S_m(u_n)\exp(G(u_n))\nabla(W_\eta^{n,j})\,dx\leq C\eta+\varepsilon(n,j,m), \tag{1.38}$$

$$\int_\Omega f_n S_m(u_n)\exp(G(u_n))W_\eta^{n,j}\,dx+\int_\Omega h(x)\exp(G(u_n))W_\eta^{n,j}S_m(u_n)\,dx\leq C\eta+\varepsilon(n,\eta), \tag{1.39}$$

$$\int_\Omega F\exp(G(u_n))\nabla(W_\eta^{n,j})S_m(u_n)\,dx+\int_\Omega F\nabla u_n\exp(G(u_n))W_\eta^{n,j}S_m'(u_n)dx\leq\varepsilon(n,m,j,\eta), \tag{1.40}$$

$$\int_\Omega [a(T_k(u_n),\nabla T_k(u_n))-a(T_k(u_n),\nabla T_k(u))][\nabla T_k(u_n)-\nabla T_k(u)]dx\to 0. \tag{1.41}$$

Proof of (1.35):
If we take $n>m+1$, we get

$$\Phi_n(u_n)\exp(G(u_n))S_m(u_n)=\Phi(T_{m+1}(u_n))\exp(G(T_{m+1}(u_n)))S_m(T_{m+1}(u_n)),$$

then $\Phi_n(u_n)\exp(G(u_n))S_m(u_n)$ is bounded in $L_{\overline{M}}(\Omega)$, thus, by using the pointwise convergence of u_n and Lebesgue's theorem we obtain $\Phi_n(u_n)\exp(G(u_n))S_m(u_n)\to\Phi(u)\exp(G(u))S_m(u)$, with the modular convergence as $n\to+\infty$, then
$\Phi_n(u_n)\exp(G(u_n))S_m(u_n)\to\Phi(u)\exp(G(u))S_m(u)$ for $\sigma(\prod L_{\overline{M}},\prod L_M)$.
In the other hand $\nabla W_{\mu,\eta}^{n,j}=\nabla T_k(u_n)-\nabla(T_k(\upsilon_j))_\mu$ for $|T_k(u_n)-(T_k(\upsilon_j))_\mu|\leq\eta$ converge to $\nabla T_k(u)-\nabla(T_k(\upsilon_j))_\mu$ weakly in $(L_M(\Omega))^N$, then

$$\int_\Omega \Phi_n(u_n)\exp(G(u_n))S_m(u_n)\nabla W_{\mu,\eta}^{n,j}\,dx\to\int_\Omega\Phi(u)S_m(u)\exp(G(u))\nabla W_{\mu,\eta}^{j}\,dx\text{ as }n\to+\infty.$$

Using the modular convergence of $W_{\mu,\eta}^j$ as $j\to+\infty$ and letting μ tends to infinity, we get (1.35).
Proof of (1.36):
For $n>m+1>k$, we have $\nabla u_n S_m'(u_n)=\nabla T_{m+1}(u_n)$ a.e. in Ω. By the almost every where convergence of u_n we have $\exp(G(u_n))W_{\mu,\eta}^{n,j}\to\exp(G(u))W_{\mu,\eta}^j$ in $L^\infty(\Omega)$ weak-* and since the sequence $(\Phi_n(T_{m+1}(u_n)))_n$ converge strongly in $E_{\overline{M}}(\Omega)$, then

$$\Phi_n(T_{m+1}(u_n))\exp(G(u_n))\ W_{\mu,\eta}^{n,j}\to\Phi(T_{m+1}(u))\ \exp(G(u))W_{\mu,\eta}^j,\text{ converge strongly in }E_{\overline{M}}(\Omega),$$

as $n\to+\infty$. By virtue of $\nabla T_{m+1}(u_n)\to\nabla T_{m+1}(u)$ weakly in $(L_M(\Omega))^N$ as $n\to+\infty$ we have

$$\int_{m\leq|u_n|\leq m+1}\Phi_n(T_{m+1}(u_n))\nabla u_n S_m'(u_n)\exp(G(u_n))W_{\mu,\eta}^{n,j}\,dx\to\int_{m\leq|u|\leq m+1}\Phi(u)\nabla u\exp(G(u))W_{\mu,\eta}^j\,dx$$

as $n\to+\infty$ with the modular convergence of $W_{\mu,\eta}^j$ as $j\to+\infty$ and letting $\mu\to+\infty$ we get (1.36).
Proof of (1.37):
For (1.37), we have
$$\int_\Omega a_n(u_n,\nabla u_n)S_m'(u_n)\nabla u_n\exp(G(u_n))\exp(G(u_n))W_{\mu,\eta}^{n,j}\,dx$$

$$=\int_{m\leq|u_n|\leq m+1}a_n(u_n,\nabla u_n)S_m'(u_n)\nabla u_n\exp(G(u_n))W_{\mu,\eta}^{n,j}\,dx$$

$$\leq \eta C \int_{m\leq|u_n|\leq m+1} a_n(u_n,\nabla u_n)\nabla u_n\,dx.$$

Using (1.24), we get

$$\int_\Omega a_n(u_n,\nabla u_n)S_m'(u_n)\nabla u_n \exp(G(u_n))W_{\mu,\eta}^{n,j}\,dx \leq \varepsilon(n,\mu,m).$$

Proof of (1.39):
Since $S_m(r)\leq 1$ and $W_{\mu,\eta}^{n,j}\leq \eta$ we get

$$\int_\Omega f_n S_m(u_n)\exp(G(u_n))W_{\mu,\eta}^{n,j}\,dx \leq \varepsilon(n,\eta) \text{ and } \int_\Omega h(x)\exp(G(u_n))W_{\mu,\eta}^{n,j}S_m(u_n)\,dx \leq C\eta.$$

Proof of (1.40):

$$\int_\Omega F\exp(G(u_n))\nabla(W_\eta^{n,j})S_m(u_n)\,dx+\int_\Omega F\nabla u_n\exp(G(u_n))W_\eta^{n,j}S_m'(u_n)dx = K_{F,1}+K_{F,2}.$$

For the first integral we have

$$K_{F,1}\leq \exp(\frac{\|\rho\|_{L^1(\mathbb{R})}}{\alpha'})\int_\Omega F\,\nabla W_\eta^{n,j}dx \leq \varepsilon(\eta).$$

Since $T_k(u_n)$ and $T_k(v_j)$ converge weakly in $W_0^1L_M(\Omega)$, we deduce

$$K_{F,1}\leq \varepsilon(n,j,\eta).$$

For the first integral we know that $\nabla u_n S_m'(u_n)=T_{m+1}(u_n)$ and using (1.4) we get

$$K_{F,2}\leq \exp(\frac{\|\rho\|_{L^1(\mathbb{R})}}{\alpha'})\Big[\varepsilon_1\int_\Omega \overline{M}(\frac{F}{\varepsilon_1})W_\eta^{n,j}dx+\varepsilon_1\eta\int_{m\leq|u_n|\leq m+1}a_n(u_n,\nabla u_n)\nabla u_n dx\Big]$$

$$\leq \varepsilon(n,m,j,\eta).$$

Proof of (1.38):

$$\int_\Omega a_n(u_n,\nabla u_n)S_m(u_n)\exp(G(u_n))\nabla W_{\mu,\eta}^{n,j}\,dx$$

$$=\int_{\{|u_n|\leq k\}\cap\{0\leq T_k(u_n)-T_k(\upsilon_j)_\mu\leq\eta\}} a_n(T_k(u_n),\nabla T_k(u_n))S_m(u_n)\exp(G(u_n))(\nabla T_k(u_n)-\nabla T_k(\upsilon_j)_\mu)\,dx$$

$$-\int_{\{|u_n|>k\}\cap\{0\leq T_k(u_n)-T_k(\upsilon_j)_\mu\leq\eta\}} a_n(u_n,\nabla u_n)S_m(u_n)\exp(G(u_n))\nabla T_k(\upsilon_j)_\mu\,dx. \tag{1.42}$$

Since $a_n(T_{k+\eta}(u_n),\nabla T_{k+\eta}(u_n))$ is bounded in $(L_{\overline{M}}(\Omega))^N$, there exist some $\varpi_{k+\eta}\in(L_{\overline{M}}(\Omega))^N$ such that $a_n(T_{k+\eta}(u_n),\nabla T_{k+\eta}(u_n))\to\varpi_{k+\eta}$ weakly in $(L_{\overline{M}}(\Omega))^N$. Consequently,

$$\int_{\{|u_n|>k\}\cap\{0\leq T_k(u_n)-T_k(\upsilon_j)_\mu\leq\eta\}} a_n(u_n,\nabla u_n)S_m(u_n)\exp(G(u_n))\nabla T_k(\upsilon_j)_\mu\,dx$$

$$=\int_{\{|u|>k\}\cap\{0\leq T_k(u)-T_k(\upsilon_j)_\mu\leq\eta\}} S_m(u)\exp(G(u))\nabla T_k(\upsilon_j)_\mu\varpi_{k+\eta}\,dx+\varepsilon(n), \tag{1.43}$$

where we have used the fact that

$$S_m(u_n)\exp(G(u_n))\nabla T_k(\upsilon_j)_\mu)\chi_{\{|u_n|>k\}\cap\{0\le T_k(u_n)-T_k(\upsilon_j)_\mu)\le\eta\}}$$

$$\to S_m(u)\exp(G(u))\nabla T_k(\upsilon_j)_\mu)\chi_{\{|u|>k\}\cap\{0\le T_k(u)-T_k(\upsilon_j)_\mu)\le\eta\}},$$

strongly in $(E_M(\Omega))^N$.
Letting $j\to+\infty$, we obtain

$$\int_{\{|u|>k\}\cap\{0\le T_k(u)-T_k(\upsilon_j)_\mu)\le\eta\}} S_m(u)\exp(G(u))\nabla T_k(\upsilon_j)_\mu\varpi_{k+\eta}\,dx$$

$$=\int_{\{|u|>k\}\cap\{0\le T_k(u)-T_k(u)_\mu)\le\eta\}} S_m(u)\exp(G(u))\nabla T_k(u)_\mu\varpi_{k+\eta}\,dx+\varepsilon(n,j).$$

One easily has

$$\int_{\{|u|>k\}\cap\{0\le T_k(u)-T_k(u)_\mu)\le\eta\}} S_m(u)\exp(G(u))\nabla T_k(u)_\mu\varpi_{k+\eta}\,dx=\varepsilon(n,j,\mu).$$

By (1.34)-(1.39), (1.42) and (1.43) we obtain

$$\int_{\{|u_n|\le k\}\cap\{0\le T_k(u_n)-T_k(\upsilon_j)_\mu)|\le\eta\}} a_n(T_k(u_n),\nabla T_k(u_n))S_m(u_n)\exp(G(u_n))(\nabla T_k(u_n)-\nabla T_k(\upsilon_j)_\mu)\,dx$$

$$\le C\eta+\varepsilon(n,j,\mu,m),$$

we know that $\exp(G(u_n))\ge 1$ and $S_m(u_n)=1$ for $|u_n|\le k$, then

$$\int_{\{|u_n|\le k\}\cap\{0\le T_k(u_n)-T_k(\upsilon_j)_\mu)|\le\eta\}} a_n(T_k(u_n),\nabla T_k(u_n))(\nabla T_k(u_n)-\nabla T_k(\upsilon_j)_\mu)\,dx\le C\eta+\varepsilon(n,j,\mu,m). \tag{1.44}$$

Proof of (1.41):
Setting for $s>0$, $\Omega^s=\{x\in\Omega:|\nabla T_k(u)|\le s\}$ and $\Omega_j^s=\{x\in\Omega:|\nabla T_k(\upsilon_j)|\le s\}$ and denoting by χ^s and χ_j^s the characteristic functions of Ω^s and Ω_j^s respectively, we deduce that letting $0<\delta<1$, define

$$\Theta_{n,k}=(a(T_k(u_n),\nabla T_k(u_n))-a(T_k(u_n),\nabla T_k(u)))(\nabla T_k(u_n)-\nabla T_k(u)).$$

For $s>0$, we have

$$0\le\int_{\Omega^s}\Theta_{n,k}^\delta\,dx=\int_{\Omega^s}\Theta_{n,k}^\delta\chi_{\{|T_k(u_n)-T_k(\upsilon_j)_\mu|\le\eta\}}\,dx+\int_{\Omega^s}\Theta_{n,k}^\delta\chi_{\{|T_k(u_n)-T_k(\upsilon_j)_\mu|>\eta\}}\,dx.$$

The first term of the right-side hand, with the Hölder inequality,

$$\int_{\Omega^s}\Theta_{n,k}^\delta\chi_{\{|T_k(u_n)-T_k(\upsilon_j)_\mu|\le\eta\}}\,dx\le(\int_{\Omega^s}\Theta_{n,k}\chi_{\{|T_k(u_n)-T_k(\upsilon_j)_\mu|\le\eta\}}\,dx)^\delta(\int_{\Omega^s}dx)^{1-\delta}$$

$$\le C_1(\int_{\Omega^s}\Theta_{n,k}\chi_{\{|T_k(u_n)-T_k(\upsilon_j)_\mu|\le\eta\}}\,dx)^\delta.$$

Also using the Hölder inequality, the second term of the right-side hand is

$$\int_{\Omega^s}\Theta_{n,k}^\delta\chi_{\{|T_k(u_n)-T_k(\upsilon_j)_\mu|>\eta\}}\,dx\le(\int_{\Omega^s}\Theta_{n,k}\,dx)^\delta(\int_{\{|T_k(u_n)-T_k(\upsilon_j)_\mu|>\eta\}}dx)^{1-\delta},$$

since $a(T_k(u_n),\nabla T_k(u_n))$ is bounded in $(L_{\overline{M}}(\Omega))^N$, while $\nabla T_k(u_n)$ is bounded in $(L_M(\Omega))^N$ then

$$\int_{\Omega^s}\Theta_{n,k}^{\delta}\chi_{\{|T_k(u_n)-T_k(\upsilon_j)_\mu|>\eta\}}\,dx\le C_2 meas\{x\in\Omega:|T_k(u_n)-T_k(\upsilon_j)_\mu|>\eta\}^{1-\delta}.$$

We obtain

$$\int_{\Omega^s}\Theta_{n,k}^{\delta}\,dx\le C_1(\int_{Q^s}\Theta_{n,k}\chi_{\{|T_k(u_n)-T_k(\upsilon_j)_\mu|\le\eta\}}\,dx)^{\delta}+C_2 meas\{x\in\Omega:|T_k(u_n)-T_k(\upsilon_j)_\mu|>\eta\}^{1-\delta}.$$

On the other hand

$$\int_{\Omega^s}\Theta_{n,k}\chi_{\{|T_k(u_n)-T_k(\upsilon_j)_\mu|\le\eta\}}\,dx$$

$$\le\int_{\{|T_k(u_n)-T_k(\upsilon_j)_\mu|\le\eta\}}(a(T_k(u_n),\nabla T_k(u_n))-a(T_k(u_n),\nabla T_k(u)\chi_s))(\nabla T_k(u_n)-\nabla T_k(u)\chi_s)\,dx.$$

For each $s>r, r>0$, one has

$$0\le\int_{\Omega^r\cap\{|T_k(u_n)-T_k(\upsilon_j)_\mu|\le\eta\}}(a(T_k(u_n),\nabla T_k(u_n))-a(T_k(u_n),\nabla T_k(u)))(\nabla T_k(u_n)-\nabla T_k(u))\,dx$$

$$\le\int_{\Omega^s\cap\{|T_k(u_n)-T_k(\upsilon_j)_\mu|\le\eta\}}(a(T_k(u_n),\nabla T_k(u_n))-a(T_k(u_n),\nabla T_k(u)))(\nabla T_k(u_n)-\nabla T_k(u))\,dx$$

$$=\int_{\Omega^s\cap\{|T_k(u_n)-T_k(\upsilon_j)_\mu|\le\eta\}}(a(T_k(u_n),\nabla T_k(u_n))-a(T_k(u_n),\nabla T_k(u)\chi_s))(\nabla T_k(u_n)-\nabla T_k(u)\chi_s)\,dx$$

$$\le\int_{\Omega\cap\{|T_k(u_n)-T_k(\upsilon_j)_\mu|\le\eta\}}(a(T_k(u_n),\nabla T_k(u_n))-a(T_k(u_n),\nabla T_k(u)\chi^s))(\nabla T_k(u_n)-\nabla T_k(u)\chi^s)\,dx$$

$$=\int_{\{|T_k(u_n)-T_k(\upsilon_j)_\mu|\le\eta\}}(a(T_k(u_n),\nabla T_k(u_n))-a(T_k(u_n),\nabla T_k(\upsilon_j)\chi_j^s))(\nabla T_k(u_n)-\nabla T_k(\upsilon_j)\chi_j^s)\,dx$$

$$+\int_{\{|T_k(u_n)-T_k(\upsilon_j)_\mu|\le\eta\}}a(T_k(u_n),\nabla T_k(u_n))(\nabla T_k(\upsilon_j)\chi_j^s-\nabla T_k(u)\chi^s)\,dx$$

$$+\int_{\{|T_k(u_n)-T_k(\upsilon_j)_\mu|\le\eta\}}(a(T_k(u_n),\nabla T_k(\upsilon_j)\chi_j^s)-a(T_k(u_n),\nabla T_k(u)\chi^s))\nabla T_k(u_n)\,dx$$

$$-\int_{\{|T_k(u_n)-T_k(\upsilon_j)_\mu|\le\eta\}}a(T_k(u_n),\nabla T_k(\upsilon_j)\chi_j^s)\nabla T_k(\upsilon_j)\chi_j^s)\,dx$$

$$+\int_{\{|T_k(u_n)-T_k(\upsilon_j)_\mu|\le\eta\}}a(T_k(u_n),\nabla T_k(u)\chi^s)\nabla T_k(u)\chi^s)\,dx$$

$$=I_1(n,j,s)+I_2(n,j)+I_3(n,j)+I_4(n,j,\mu)+I_5(n,\mu).$$

We go to the limit as n, j, μ and $s\to+\infty$

$$I_1=\int_{\{|T_k(u_n)-T_k(\upsilon_j)_\mu|\le\eta\}}a(T_k(u_n),\nabla T_k(u_n))(\nabla T_k(u_n)-\nabla T_k(\upsilon_j)_\mu)\,dx$$

$$-\int_{\{|T_k(u_n)-T_k(\upsilon_j)_\mu|\le\eta\}}a(T_k(u_n),\nabla T_k(u_n))(\nabla T_k(\upsilon_j)\chi_j^s-\nabla T_k(\upsilon_j)_\mu)\,dx$$

$$-\int_{\{T_k(u_n)-T_k(\upsilon_j)_\mu|\le\eta\}} a(T_k(u_n),\nabla T_k(\upsilon_j)\chi_j^s))(\nabla T_k(u_n)-\nabla T_k(\upsilon_j)\chi_j^s))\,dx.$$

Using (1.44), the first term of the right-hand side, we get

$$\int_{\{|T_k(u_n)-T_k(\upsilon_j)_\mu|\le\eta\}} a(T_k(u_n),\nabla T_k(u_n))(\nabla T_k(u_n)-\nabla T_k(\upsilon_j)_\mu)\,dx$$

$$\le C\eta+\varepsilon(n,m,j,s)-\int_{\{|u|>k\}\cap\{|T_k(u)-T_k(\upsilon_j)_\mu|\le\eta\}} a(T_k(u),0)\nabla T_k(\upsilon_j)_\mu\,dx$$

$$\le C\eta+\varepsilon(n,m,j,\mu).$$

The second term of the right-hand side tends to

$$\int_{\{|T_k(u)-T_k(\upsilon_j)_\mu|\le\eta\}} \varpi_k(\nabla T_k(\upsilon_j)\chi_j^s-\nabla T_k(\upsilon_j)_\mu)\,dx,$$

since $a(T_k(u_n),\nabla T_k(u_n))$ is bounded in $(L_{\overline{M}}(\Omega))^N$, there exist some $\varpi_k\in(L_{\overline{M}}(\Omega))^N$ such that (for a subsequence still denoted by u_n

$$a(T_k(u_n),\nabla T_k(u_n))\to\varpi_k \quad\text{in}\quad (L_M(\Omega))^N \quad\text{for}\quad \sigma(\Pi L_{\overline{M}},\Pi E_M).$$

In view of the fact that

$$(\nabla T_k(\upsilon_j)\chi_j^s-\nabla T_k(\upsilon_j)_\mu)\chi_{\{|T_k(u_n)-T_k(\upsilon_j)_\mu|\le\eta\}}\to(\nabla T_k(\upsilon_j)\chi_j^s-\nabla T_k(\upsilon_j)_\mu)\chi_{\{|T_k(u)-T_k(\upsilon_j)_\mu|\le\eta\}},$$

strongly in $(E_M(\Omega))^N$ as $n\to+\infty$.
The third term of the right-hand side tends to

$$\int_{\{|T_k(u)-T_k(\upsilon_j)_\mu|\le\eta\}} a(T_k(u),\nabla T_k(\upsilon_j)\chi_j^s))(\nabla T_k(u)-\nabla T_k(\upsilon_j)\chi_j^s))\,dx.$$

Since

$$a(T_k(u_n),\nabla T_k(\upsilon_j)\chi_j^s))\chi_{\{|T_k(u_n)-T_k(\upsilon_j)_\mu|\le\eta\}}\to a(T_k(u),\nabla T_k(\upsilon_j)\chi_j^s))\chi_{\{|T_k(u)-T_k(\upsilon_j)_\mu|\le\eta\}},$$

in $(E_{\overline{M}}(\Omega))^N$ while $(\nabla T_k(u_n)-\nabla T_k(\upsilon_j)\chi_j^s))\to(\nabla T_k(u)-\nabla T_k(\upsilon_j)\chi_j^s))$, in $(L_M(\Omega))^N$ for $\sigma(\Pi L_{\overline{M}},\Pi E_M)$.
Passing to limit as $j\to+\infty$ and $\mu\to+\infty$ and using Lebesgue's theorem, we have

$$I_1\le C\eta+\varepsilon(n,j,s,\mu).$$

For what concerns I_2, by letting $n\to+\infty$, we have

$$I_2\to\int_{\{|T_k(u)-T_k(\upsilon_j)_\mu|\le\eta\}} \varpi_k(\nabla T_k(\upsilon_j)\chi_j^s-\nabla T_k(u)\chi^s)\,dx.$$

Since $a(T_k(u_n),\nabla T_k(u_n))\rightharpoonup\varpi_k$ in $(L_{\overline{M}}(\Omega))^N$, for $\sigma(\Pi L_{\overline{M}},\Pi E_M)$,
while

$$(\nabla T_k(\upsilon_j)\chi_j^s-\nabla T_k(u)\chi^s)\chi_{\{|T_k(u_n)-T_k(\upsilon_j)_\mu|\le\eta\}}\to(\nabla T_k(\upsilon_j)\chi_j^s-\nabla T_k(u)\chi^s)\chi_{\{|T_k(u)-T_k(\upsilon_j)_\mu|\le\eta\}},$$

strongly in $(E_M(\Omega))^N$.
Passing to limit $j \to +\infty$ and using Lebesgue's theorem, we have

$$I_2 = \varepsilon(n,j).$$

Similar ways as above give

$$I_3 = \varepsilon(n,j).$$

$$I_4 = \int_{\{|T_k(u)-T_k(u)_\mu|\leq\eta\}} a(T_k(u),\nabla T_k(u))\nabla T_k(u)\,dx + \varepsilon(n,j,\mu,s,m).$$

$$I_5 = \int_{\{|T_k(u)-T_k(u)_\mu|\leq\eta\}} a(T_k(u),\nabla T_k(u))\nabla T_k(u)\,dx + \varepsilon(n,j,\mu,s,m).$$

Finally, we obtain

$$\int_{\Omega^s} \Theta_{n,k}\,dx \leq C_1(C\eta + \varepsilon(n,\mu,\eta,m))^\delta + C_2(\varepsilon(n,\mu,))^{1-\delta}.$$

Which yields, by passing to the limit sup over n, j, μ , s and η

$$\int_{\{T_\eta(T_k(u_n)-T_k(\upsilon_j))\geq 0\}\cap\Omega^r} (a(x,T_k(u_n),\nabla T_k(u_n)) - a(x,T_k(u_n),\nabla T_k(u)))(\nabla T_k(u_n)-\nabla T_k(u))dx = \varepsilon(n). \tag{1.45}$$

On the other hand, taking the functions

$$W_\eta^{n,j} = T_\eta(T_k(u_n)-T_k(\upsilon_j))^- \text{ and } W_\eta^j = T_\eta(T_k(u)-T_k(\upsilon_j))^-.$$

Multiplying the approximating equation by $u_n + \exp(-G(u_n))W_\eta^{n,j}S_m(u_n)$, we obtain

$$\int_{\{T_\eta(T_k(u_n)-T_k(\upsilon_j))\leq 0\}\cap\Omega^r} (a(T_k(u_n),\nabla T_k(u_n)) - a(T_k(u_n),\nabla T_k(u)))(\nabla T_k(u_n)-\nabla T_k(u))dx = \varepsilon(n), \tag{1.46}$$

by (1.45) and (1.46) we get

$$\int_{\Omega^r} (a(T_k(u_n),\nabla T_k(u_n)) - a(T_k(u_n),\nabla T_k(u)))(\nabla T_k(u_n)-\nabla T_k(u))dx = \varepsilon(n)$$

Thus, passing to a subsequence if necessary, $\nabla u_n \to \nabla u$ a.e. in Ω^r and since r is arbitrary,

$$\nabla u_n \to \nabla u \quad \text{a.e. in} \quad \Omega.$$

□

Step 5: Compactness of the nonlinearities
We shall prove that $K_n(u_n,\nabla u_n) \to K(u,\nabla u)$ strongly in $L^1(\Omega)$.
Consider $g_0(u_n) = \int_0^{u_n} \rho(s)\chi_{\{s>h\}}ds$ and multiply (1.12) by $u_n - \exp(G(u_n))g_0(u_n)$, we get after

using the same technique in step 2,

$$\int_{\{u_n>h\}} a_n(u_n,\nabla u_n)\rho(u_n)\chi_{\{u_n>h\}}\nabla u_n dx$$
$$\leq C\Big(\int_h^{+\infty}\rho(s)ds\Big)\exp\Big(\frac{\|\rho\|_{L^1(\mathbb{R})}}{\alpha'}\Big)\Big[\|f\|_{L^1(\Omega)}+\|h(x)\|_{L^1(\Omega)}+\frac{\|\rho\|_{L^\infty(\mathbb{R})}}{\alpha'}\int_\Omega \overline{M}(\frac{F}{\varepsilon_1})dx\Big]$$
$$+\exp\Big(\frac{\|\rho\|_{L^1(\mathbb{R})}}{\alpha'}\|\rho\|_{L^\infty(\mathbb{R})}\int_{\{u_n>h\}}\overline{M}(\frac{F}{\varepsilon_1})dx.$$

where $\frac{1}{C}=\Big[1-(\frac{\varepsilon_1+\|c(.)\|_{L^\infty(\Omega)}}{\alpha})\Big]\exp(-\frac{\|\rho\|_{L^1}}{\alpha'})$ and by (1.4), we have

$$\int_{\{u_n>h\}}\rho(u_n)M(\nabla u_n)dx$$
$$\leq \frac{C}{\alpha}\Big(\int_h^{+\infty}\rho(s)ds\Big)\exp\Big(\frac{\|\rho\|_{L^1(\mathbb{R})}}{\alpha'}\Big)\Big[\|f\|_{L^1(\Omega)}+\|h(x)\|_{L^1(\Omega)}+\frac{\|\rho\|_{L^\infty(\mathbb{R})}}{\alpha'}\int_\Omega \overline{M}(\frac{F}{\varepsilon_1})dx\Big]$$
$$+\exp\Big(\frac{\|\rho\|_{L^1(\mathbb{R})}}{\alpha'}\|\rho\|_{L^\infty(\mathbb{R})}\int_{\{u_n>h\}}\overline{M}(\frac{F}{\varepsilon_1})dx.$$

Since $\rho\in L^1(\mathbb{R})$, we get

$$\limsup_{h\to\infty}\int_{\{u_n>h\}}\rho(u_n)M(\nabla u_n)dx=0.$$

Similarly, let $g_0(u_n)=\int_{u_n}^0\rho(s)\chi_{\{s<-h\}}ds$ and multiply (1.12) by $u_n+\exp(-G(u_n))g_0(u_n)$, we have also

$$\limsup_{h\to\infty}\int_{\{u_n<-h\}}\rho(u_n)M(\nabla u_n)dx=0,$$

and we conclude that

$$\limsup_{h\to\infty}\int_{\{|u_n|>h\}}\rho(u_n)M(\nabla u_n)dx=0. \tag{1.47}$$

Let $D\subset\Omega$ then

$$\int_D\rho(u_n)M(\nabla u_n)dx\leq\max_{\{|u_n|\leq h\}}(\rho(x))\int_{D\cap\{|u_n|\leq h\}}M(\nabla u_n)dx+\int_{D\cap\{|u_n|>h\}}\rho(u_n)M(\nabla u_n)dx.$$

Consequently, $\rho(u_n)M(\nabla u_n)$ is equi-integrable. Then $\rho(u_n)M(\nabla u_n)$ converge to $\rho(u)M(\nabla u)$ strongly in $L^1(\mathbb{R})$. By (1.6), we get our result.

Step 6: We show that u satisfies (1.9)

Let $v\in K_\zeta\cap L^\infty(\Omega)$, then by Lemma 0.1.5 there exists $v_j\in\mathscr{D}(\Omega)$ such that $v_j\to v$ in $W_0^1L_M(\Omega)$ for the modular convergence in $W_0^1L_M(\Omega)$, with $\|v_j\|_{L^\infty(\Omega)}\leq(N+1)\|v\|_{L^\infty(\Omega)}$ and we can take $v_j\in K_\zeta$.

Pointwise multiplication of the approximate equation (1.12) by $u_n-T_k(u_n-v_j)$, we get

$$\begin{cases}\int_\Omega a_n(u_n,\nabla u_n)\nabla T_k(u_n-v_j)dx+\int_\Omega\Phi_n(u_n)\nabla T_k(u_n-v_j)dx\\ +\int_\Omega K_n(u_n)\nabla T_k(u_n-v_j)dx=\int_\Omega f_nT_k(u_n-v_j)dx-\int_\Omega F\nabla T_k(u_n-v_j)dx.\end{cases}\tag{1.48}$$

We pass to the limit as in (1.48), n tend to $+\infty$ and j tend to $+\infty$:

- We follow the same way in [32] to prove that

$$\liminf_{j\to\infty}\liminf_{n\to\infty}\int_\Omega a(u_n,\nabla u_n)\nabla T_k(u_n-v_j)dxds \geq \int_\Omega a(u,\nabla u)\nabla T_k(u-v)dxds.$$

- For $n \geq k+(N+1)\|v\|_{L^\infty(\Omega)}$

$$\Phi_n(u_n)\nabla T_k(u_n-v_j) = \Phi(T_{k+(N+1)\|v\|_{L^\infty(\Omega)}}(u_n))\nabla T_k(u_n-v_j).$$

  The pointwise convergence of u_n to u as n tends to $+\infty$ and (1.10), then

$$\Phi(T_{k+(N+1)\|v\|_{L^\infty(\Omega)}}(u_n))\nabla T_k(u_n-v_j) \rightharpoonup \Phi(T_{k+(N+1)\|v\|_{L^\infty(\Omega)}}(u))\nabla T_k(u-v_j)$$

  weakly for $\sigma(\Pi L_M, \Pi L_{\overline{M}})$.

- In a similar way, we obtain

$$\begin{aligned}\lim_{j\to\infty}\int_\Omega \Phi(T_{k+(N+1)\|v\|_{L^\infty(\Omega)}}(u))\nabla T_k(u-v_j)dx &= \int_\Omega \Phi(T_{k+(N+1)\|v\|_{L^\infty(\Omega)}}(u))\nabla T_k(u-v)dx \\ &= \int_\Omega \Phi(u)\nabla T_k(u-v)dx.\end{aligned}$$

- Limit of $K_n(u_n,\nabla u_n)T_k(u_n-v_j)$: Since $K_n(u_n,\nabla u_n)$ converge strongly to $K(u,\nabla u)$ in $L^1(\Omega)$ and and the pointwise convergence of u_n to u as $n\to+\infty$, it is possible to prove that $K_n(u_n,\nabla u_n)T_k(u_n-v_j)$ converge to $K(u,\nabla u)T_k(u-v_j)$ in $L^1(\Omega)$ and

$$\lim_{j\to\infty}\int_\Omega K(u,\nabla u)T_k(u-v_j)dx = \int_\Omega K(u,\nabla u)T_k(u-v)dx.$$

- Since f_n converge strongly to f in $L^1(\Omega)$ and $T_k(u_n-v_j)\to T_k(u-v_j)$ weakly* in $L^\infty(\Omega)$, we have $\int_\Omega f_n T_k(u_n-v_j)dx \to \int_\Omega f T_k(u-v_j)dx$ as $n\to\infty$ and also we have

$$\int_\Omega f T_k(u-v_j)dx \to \int_\Omega f T_k(u-v)dx \quad \text{as} \quad j\to\infty.$$

- It easy to get

$$\lim_{j\to\infty}\lim_{n\to\infty}\int_\Omega F\nabla T_k(u_n-v_j)dx = \int_\Omega F\nabla T_k(u-v)dx.$$

As a conclusion, the proof of Theorem 1.2.1 is complete.

Example 1.2.1. *As an example of equations to which the present result renormalized solutions can be applied, we give For* $M(t)=tlog(1+t)$ *and* $a(x,u,\nabla u)=(1+|u|)^2\nabla u\dfrac{log(1+|\nabla u|)}{|\nabla u|}$, $F=0$ *and* $c(.)\in(L^\infty)^N$.

$$\begin{cases} -div(a(x,u,\nabla u)) - div(c(x)\exp\Big(\dfrac{\eta}{\|x\|+1}\Big)\overline{M}^{-1}M(\alpha_0|u|)) = f \quad in \quad \Omega, \\ u=0 \qquad\qquad\qquad\qquad on \qquad \partial\Omega. \end{cases}$$

2

Existence and Uniqueness of a renormalized Solution of parabolic problems in Orlicz Spaces

We will first focus on the existence of the result of the non-linear parabolic prototype equations

$$\begin{cases} \dfrac{\partial b(u)}{\partial t} - \operatorname{div}\big((1+|u|)^2 Du \dfrac{\log(e+Du)}{|Du|}\big) - \operatorname{div}(\overline{c}(x,t)\overline{M}^{-1} M(\dfrac{\alpha_0}{\lambda}|b(u)|)) = f & \text{in} \quad Q_T, \\ u(x,t) = 0 & \text{on} \quad \partial\Omega \times (0,T), \\ b(u)(t=0) = b(u_0) & \text{in} \quad \Omega, \end{cases}$$

where $\overline{c} \in (L^\infty(Q_T))^N$, $M(t) = t\log(e+t)$ is an N-function, and the data $f \in L^1(Q_T)$, $b(u_0) \in L^1(\Omega)$. While there are many mathematically correct solutions to this problem, we have difficulty determining who meets the physical conditions and who should be eliminated. In the second part of this chapter we will impose additional conditions on the terms of the problem that will allow us to show the uniqueness.

2.1 Introduction

Let Ω be a bounded open set of $\mathbb{R}^N$ $(N \geq 2)$, T is a positive real number, and $Q_T = \Omega \times (0,T)$. Consider the following nonlinear Dirichlet equation:

$$\begin{cases} \dfrac{\partial b(u)}{\partial t} - \operatorname{div}(a(x,t,u,\nabla u)) - \operatorname{div}(\Phi(x,t,u)) = f & \text{in} \quad Q_T, \\ u(x,t) = 0 & \text{on} \quad \partial\Omega \times (0,T), \\ b(u)(t=0) = b(u_0) & \text{in} \quad \Omega, \end{cases} \tag{2.1}$$

where $A(u) = -\operatorname{div}(a(x,t,u,\nabla u))$ is a Leary-Lions operator defined on the inhomogeneous Orlicz-Sobolev space $W_0^{1,x}L_M(Q_T)$, M is an N-function related to the growth of $A(u)$ (see assumptions (2.5)-(2.7)), and to the growth of the lower order Carathéodory function $\Phi(x,t,u)$ (see assumption (2.8)). $b : \mathbb{R} \longrightarrow \mathbb{R}$ is a strictly increasing C^1-function, the second term f in $L^1(Q_T)$.

In the classical Lebesgue spaces $L^p(0,T,W_0^{1,p}(\Omega))$, D. Blanchard, F. Murat and H. Redwane were adapted the renormalized solution to parabolic equations with L^1-data in [38], [39] where they treated the existence and uniqueness with $b(u)$ a linear function $(b(u) = u)$ and $a(x,t,u,\nabla u) + \Phi(u)$

with $\Phi \in \mathscr{C}^{\infty}(\mathbb{R})$, $u \in L^{\infty}(0,T,L^1(\Omega))$ and the source data is a measure $\mu = f - \operatorname{div}(G)$. Recently, A. Aberqi, J. Bennouna and M. Hammoumi in [3] where b is a general strictly increasing $\mathscr{C}^1(\mathbb{R})$-function. L. Aharouch and J. Bennouna in [9] have proved the existence and uniqueness of entropy solutions in the framework of Orlicz-Sobolev spaces $W_0^1 L_M(\Omega)$ assuming the $\triangle_2$-condition on the N-function M. Recently, F. KH. Mukminov in [90] proved the uniqueness of renormalized solutions to the Cauchy problem for parabolic equation using Kruzhkovis method of doubling the variable.

In the generalized-Orlicz spaces, the work [24] is a continuation of [9] where M. M. Al-Hawmi A. Benkirane H. Hjiaj and A. Touzani proved the existence and uniqueness of entropy solution for

$$\begin{cases} -\operatorname{div}(a(x,u,\nabla u)) = f & \text{in} \quad \Omega, \\ u(x) = 0 & \text{on} \quad \partial\Omega, \end{cases} \tag{2.2}$$

where $\Phi = 0$ and $\overline{M}$ satisfy the $\triangle_2$-condition. In [27] S. Antontsev and S. Shmarev proved theorems of existence and uniqueness of weak solutions of Dirichlet problem for a class of nonlinear parabolic equations with nonstandard anisotropic growth conditions in the variable exponent Lebesgue spaces. Equations of this class generalize the evolution $p(x,t)$-Laplacian of the type

$$\begin{cases} \dfrac{\partial u}{\partial t} - \displaystyle\sum_i \frac{\partial}{\partial x_i}[a_i(x,t,u)|D_i u|^{p_i(x,t)-2} D_i u + b_i(x,t,u)] = 0 \quad \text{in } Q_T, \\ u(x,t) = 0 \qquad\qquad \text{on } \partial\Omega \times (0,T), \\ u(x,0) = u_0(x) \qquad \text{in } \Omega. \end{cases} \tag{2.3}$$

2.2 Essential assumptions

Let M and P be two N-functions such that $P \ll M$.
$b : \mathbb{R} \to \mathbb{R}$ is a strictly increasing $\mathscr{C}^1(\mathbb{R})$-function, $b(0) = 0$,

$$b_0 < b'(s) < b_1, \quad \forall s \in \mathbb{R} \quad \text{such that} \quad b_1 < \frac{1}{\alpha_0}. \tag{2.4}$$

where α_0 is the constant appearing in (2.8).
$a : Q_T \times \mathbb{R} \times \mathbb{R}^N \to \mathbb{R}^N$ is Carathéodory function and there exists a constant $\nu > 0$ such that for a.e. $x \in \Omega$ and for all $s \in \mathbb{R}$, $\xi, \xi^* \in \mathbb{R}^N$, $\xi \neq \xi^*$:

$$|a(x,t,s,\xi)| \leq \nu(a_0(x,t) + \overline{M}^{-1}P(|s|) + \overline{M}^{-1}M(|\xi|)), a_0(.,.) \in E_{\overline{M}}(Q_T), \tag{2.5}$$

$$(a(x,t,s,\xi) - a(x,t,s,\xi^*))(\xi - \xi^*) > 0, \tag{2.6}$$

$$a(x,t,s,\xi).\xi \geq \alpha M(|\xi|). \tag{2.7}$$

$\Phi : Q_T \times \mathbb{R} \to \mathbb{R}^N$ is a Carathéodory function such that

$$|\Phi(x,t,s)| \leq c(x,t)\overline{M}^{-1}M(\frac{\alpha_0}{\lambda}|b(s)|), \tag{2.8}$$

where $\lambda = \operatorname{diam}(Q_T)$, $\|c(.,.)\|_{L^{\infty}(Q_T)} \leq \min(\frac{\alpha}{\alpha_0+1}; \frac{\alpha}{2(\alpha_0 b_1+1)})$ and $0 < \alpha_0 < 1$.

$$f \in L^1(Q_T), \tag{2.9}$$

$$u_0 \in L^1(\Omega) \text{ such that } b(u_0) \in L^1(\Omega). \tag{2.10}$$

Note that $\langle , \rangle$ means for either the pairing between $W_0^{1,x}L_M(Q_T) \cap L^\infty(Q_T)$ and $W^{-1,x}L_{\overline{M}}(Q_T) + L^1(Q_T)$ or between $W_0^{1,x}L_M(Q_T)$ and $W^{-1,x}L_{\overline{M}}(Q_T)$.

The definition of a renormalized solution of problem (2.1) can be stated as follows.

Definition 2.2.1. *A measurable function u defined on Q_T is a renormalized solution of problem* (2.1)*, if it satisfies the following conditions:*

$$b(u) \in L^\infty(0,T;L^1(\Omega)), \tag{2.11}$$

$$T_k(b(u)) \in W_0^{1,x}L_M(Q_T), \quad \forall k > 0, \tag{2.12}$$

$$\lim_{m\to+\infty} \int_{\{(x,t)\in Q_T : m \le |u| \le m+1\}} a(x,t,u,\nabla u)\nabla u dxdt = 0, \tag{2.13}$$

and if, for every function $S \in W^{2,\infty}(\mathbb{R})$ which is piecewise C^1 and such that S' has a compact support, we have in the sense of distributions

$$\frac{\partial S(b(u))}{\partial t} - div\Big(S'(b(u))(a(x,t,u,\nabla u) + \Phi(x,t,u))\Big) + S''(b(u))\Big(a(x,t,u,\nabla u) + \Phi(x,t,u)\Big) = fS'(b(u)) \quad in \quad \mathscr{D}(Q_T), \tag{2.14}$$

$$S(b(u))(t=0) = S(b(u_0)) \qquad in \quad \Omega. \tag{2.15}$$

2.3 The existence result

In this chapter we purpose to study the problem in the framework of renormalized solutions is motivated by the lack of regularity of the distributional formulation (it's not enough to assure the uniqueness, for more detail see [94] and the counterexample in [99]).

Our novelty is to give an existence and uniqueness result of renormalized solution of (2.1) in the general framework of inohomogeneous Orlicz spaces with a lower order term Φ which depends on x,t and u, that is with $a(x,t,u,\nabla u)$ replaced by $a(x,t,u,\nabla u) + \Phi(x,t,u)$. The difficulty encountered during the proof of the existence of the solution is that the term Φ does not satisfy the coercivity condition. Nonlinearities are characterized by N-functions M, for which $\triangle_2$-conditions are not imposed, losing the reflexivity of the spaces $L_M(Q_T)$ and $W_0^1L_M(Q_T)$.

Theorem 2.3.1. *Assume that* (2.4)–(2.10) *hold true. Then there exists at least one renormalized solution u of the problem* (2.1) *in the sense of the Definition 2.2.1.*

2.4 The stages of the proof of theorem 2.3.1

Truncated problem.

For each $n > 0$, we define the following approximations:

$$b_n(s) = b(T_n(s)), \ \forall\, s \in \mathbb{R}, \tag{2.16}$$

$$a_n(x,t,s,\xi) = a(x,t,T_n(s),\xi) \quad \text{a.e. } (x,t) \in Q_T, \ \forall\, s \in \mathbb{R}, \ \forall\, \xi \in \mathbb{R}^N, \tag{2.17}$$

$$\Phi_n(x,t,s) = \Phi(x,t,T_n(s)) \quad \text{a.e. } (x,t) \in Q_T,\ \forall\, s \in I\!R, \tag{2.18}$$

$$\text{Let } f_n \text{ be a smooth function such that } f_n \to f \text{ strongly in } L^1(Q_T) \tag{2.19}$$

and

$$u_{0n} \in \mathscr{C}_0^\infty(\Omega) \text{ such that } b_n(u_{0n}) \to b(u_0) \text{ strongly in } L^1(\Omega). \tag{2.20}$$

Consider the approximate problem:

$$\begin{cases} \dfrac{\partial b_n(u_n)}{\partial t} - div(a_n(x,t,u_n,\nabla u_n) + \Phi_n(x,t,u_n)) = f_n & \text{in} \quad Q_T, \\ u_n(x,t) = 0 & \text{on} \quad \partial\Omega \times (0,T), \\ b_n(u_n)(t=0) = b_n(x,u_{0n}) & \text{in} \quad \Omega. \end{cases} \tag{2.21}$$

Let $u_n \in W_0^{1,x} L_M(Q_T)$, then for any fixed $n > 0$, there exists at least one solution $u_n \in W_0^{1,x} L_M(Q_T)$ of (2.21), (see [85]).

Note that by Lemma 0.1.3 and Remark 0.1.1, we have $T_k(u_n) \in W_0^{1,x} L_M(Q_T)$, and by (2.4), (2.7), (2.8) and Young inequality, the quantity $\displaystyle\int_{Q_T} M(|\nabla T_k(u_n)|)dxdt$ is finite for all $k > 0$.

Remark 2.4.1. *The explicit dependence in x and t of the functions a and Φ will be omitted so that $a(x,t,u,\nabla u) = a(u,\nabla u)$ and $\Phi(x,t,u) = \Phi(u)$.*

Step 1: A priori estimates.

Lemma 2.4.1.
Let u_n be a solution of the approximate problem (2.21)*, then for all $k > 0$, there exists a constant C and for a subsequence, still indexed by n we have*

$$\int_{Q_T} M(|\nabla T_k(u_n)|)dxdt \le kC, \tag{2.22}$$

$$u_n \to u \text{ a.e in } Q_T, \text{ where } u \text{ is a measurable function on } Q_T, \tag{2.23}$$

$$b_n(u_n) \to b(u) \quad a.e \text{ in} \quad Q_T, \quad b(u) \in L^\infty(0,T,L^1(\Omega)), \tag{2.24}$$

$$a_n(T_k(u_n), \nabla T_k(u_n)) \quad \text{is bounded in} \quad (L_{\overline{M}}(Q_T))^N, \tag{2.25}$$

Proof. Fix $k > 0$ and $\tau \in (0,T)$. Let $T_k(u_n)\chi_{(0,\tau)}$ a test function in problem (2.21). Using the Young inequality we get

$$\int_\Omega B_n(u_n(\tau))dx + \int_{Q_\tau} a_n(u_n,\nabla u_n)\nabla T_k(u_n)dxdt + \int_{Q_\tau} \Phi_n(u_n)\nabla T_k(u_n)dxdt$$
$$\le k[\|f_n\|_{L^1(Q_T)} + \|b(u_0)\|_{L^1(Q_T)}], \tag{2.26}$$

where $B_n(r) = \displaystyle\int_0^r \frac{\partial b_n(s)}{\partial s} T_k(s)ds$.

By definition, we have $\displaystyle\int_\Omega B_n(u_n(\tau))dx \ge 0$ and $\displaystyle\int_\Omega B_n(u_n(0))dx \le kb_1\|b(u_0)\|_{L^1(Q_T)}$.

By (2.8), (2.4) and Young inequality we have

$$\int_{Q_\tau} \Phi_n(u_n)\nabla T_k(u_n)dxdt \le \|c(.,.)\|_{L^\infty(Q_T)}[\alpha_0 b_1 \int_{Q_\tau} M(\frac{|T_k(u_n)|}{\lambda})dxdt + \int_{Q_\tau} M(|\nabla T_k(u_n)|)dxdt],$$

thanks to Lemma 0.1.8, we obtain

$$\int_{Q_\tau} \Phi_n(u_n)\nabla T_k(u_n)dxdt \leq \|c(.,.)\|_{L^\infty(Q_T)}(\alpha_0 b_1+1)\int_{Q_\tau} M(|\nabla T_k(u_n)|)dxdt,$$

return to (2.26) and using (2.7) we get

$$\int_{Q_\tau} a_n(u_n,\nabla u_n)\nabla T_k(u_n)dxdt \leq \frac{(\alpha_0 b_1+1)}{\alpha}\int_{Q_\tau} a_n(u_n,\nabla u_n)\nabla T_k(u_n)dxdt + k[\|f_n\|_{L^1(Q_T)}+\|b(u_0)\|_{L^1(Q_T)}],$$

thus

$$[1-\frac{(\alpha_0 b_1+1)}{\alpha}\|c(.,.)\|_{L^\infty(Q_T)}]\int_{Q_T} a_n(u_n,\nabla u_n)\nabla T_k(u_n)dxdt \leq kc_1.$$

We take $\frac{1}{c_2} = [1-\frac{(\alpha_0 b_1+1)}{\alpha}\|c(.,.)\|_{L^\infty(Q_T)}]$.
By (2.8) we have $c_2 > 0$ and we obtain

$$\int_{Q_\tau} a(u_n,\nabla u_n)\nabla T_k(u_n)dxdt \leq kC,$$

where $C = c_1c_2$. And by (2.7) we have (2.22).

We conclude that $(T_k(u_n))$ is bounded in $W_0^{1,x}L_M(Q_T)$ independently of n. Since $(E_{\overline{M}}(Q_T))' = L_M(Q_T)$ then by Theorem 0.0.3, the set $\{(T_k(u_n))\}$ is compact for the weak topology $\sigma(\Pi L_M, \Pi E_{\overline{M}})$ so there exists a subsequence still denoted by u_n and there exists a measurable function ξ_k such that $T_k(u_n) \rightharpoonup \xi_k$ for the weak topology $\sigma(\Pi L_M, \Pi E_{\overline{M}})$.

On the other hand, using Lemma 0.1.8, we have

$$\begin{aligned} M(\frac{k}{\lambda})\text{meas}\{|u_n|>k\} &= \int_{\{|u_n|>k\}} M(\frac{|T_k(u_n)|}{\lambda})dxdt \\ &\leq \int_{Q_T} M(|\nabla T_k(u_n)|)dxdt \leq kC, \end{aligned}$$

then $\text{meas}\{|u_n|>k\} \leq \frac{kC}{M(\frac{k}{\lambda})}$ for all n and for all k.
Thus, we get $\lim_{k\to\infty}\text{meas}\{|u_n|>k\} = 0$. □

Proof of (2.23) **and** (2.24)**:**

For $k<n$, let $g_k \in W^{2,\infty}(\mathbb{R})$, such that g_k', has a compact support $\text{supp}(g_k') \subset [-k,k]$. We multiply the equation (2.21) by $g_k'(u_n)$, to obtain in $\mathscr{D}'(Q_T)$,

$$\frac{\partial B_{g_k}^n(u_n)}{\partial t} = div(g_k'(u_n)(a_n(u_n,\nabla u_n)+\Phi_n(u_n)) - g_k''(u_n)(a_n(u_n,\nabla u_n)+\Phi_n(u_n))\nabla u_n + f_n g_k'(u_n) \tag{2.27}$$

where $B_{g_k}^n(r) = \int_0^r g_k'(s)\frac{\partial b_n(s)}{\partial s}ds$.

Then, we show that

$$\left(B_{g_k}^n(u_n)\right) \quad \text{is bounded in} \quad W_0^{1,x}L_M(Q_T), \tag{2.28}$$

and

$$\Big(\frac{\partial B_{g_k}^n(u_n)}{\partial t}\Big) \quad \text{is bounded in} \quad L^1(Q_T)+W^{-1,x}L_{\overline{M}}(Q_T) \tag{2.29}$$

independently of n.

Indeed, first we have $|\nabla B_{g_k}^n(u_n)| \le b_1|\nabla T_k(u_n)|\|g_k'\|_{L^\infty(\mathbb{R})}$ a.e. in Q_T, and using (2.22) we obtain (2.28). To show that (2.29) holds true, since $\mathrm{supp}(g_k')$ and $\mathrm{supp}(g_k'')$ are both included in $[-k,k]$, u_n may be replaced by $T_k(u_n)$ in each of these terms. As a consequence, each term in the right hand side of (2.27) is bounded either in $W^{-1,x}L_{\overline{M}}(Q_T)$ or in $L^1(Q_T)$ which shows that (2.29) holds true. Arguing again as in [40] estimates (2.28), (2.29) and the following remark, imply that, for a subsequence, still indexed by n, we have $u_n \to u$ a.e in Q_T and $b(u) \in L^\infty(0,T,L^1(\Omega))$, where u is a measurable function defined on Q_T.

Remark 2.4.2.

For every $g \in W^{2,\infty}(\mathbb{R})$, nondecreasing function such that $supp(g') \subset [-k,k]$ and (2.4)*, we have*

$$b_0|g(r)-g(r')| \le |B_g(r)-B_g(r')| \le b_1|g(r)-g(r')| \quad \textit{for every} \quad \textit{in} \quad \mathbb{R}. \tag{2.30}$$

Proof of (2.25) **:**

As in [21], we may deduce that $a_n(T_k(u_n),\nabla T_k(u_n))$ is a bounded sequence in $(L_{\overline{M}}(Q_T))^N$, and we obtain (2.25).

Step 2: Almost everywhere convergence of the gradients.

In order to show that the gradient converges almost everywhere, we need to prove the next proposition.

Proposition 2.4.1. *Let u_n be a solution of the approximate problem* (2.21)*, then*

$$\lim_{m\to\infty}\limsup_{n\to\infty}\int_{\{m\le|u_n|\le m+1\}} a(u_n,\nabla u_n)\nabla u_n dxdt = 0, \tag{2.31}$$

$$\lim_{m\to\infty}\limsup_{n\to\infty}\int_{\{m\le|u_n|\le m+1\}} \Phi(u_n)\nabla u_n dxdt = 0. \tag{2.32}$$

For any $r>0$ and $0<\delta<1$, we have

$$\lim_{n\to+\infty}\int_{Q_T}[(a(T_k(u_n),\nabla T_k(u_n))-a(T_k(u_n),\nabla T_k(u)\chi_{\{|\nabla T_k(u)|\le k\}})$$

$$\times(\nabla T_k(u_n)-\nabla T_k(u)\chi_{\{|\nabla T_k(u)|\le k\}})]^\delta\, dxdt = 0, \tag{2.33}$$

$$\nabla u_n \to \nabla u \quad \textit{a.e. in} \quad Q_T. \tag{2.34}$$

Proof.

Taking $Z_m(u_n) = T_1(u_n - T_m(u_n))$ as a test function in the approximate equation (2.21) we get

$$\int_{\{m\le|u_n|\le m+1\}} a_n(u_n,\nabla u_n)\nabla u_n dxdt \le C[\int_{Q_T} f_n Z_m(u_n)\,dxdt + \int_{\{|u_{0n}|>m\}}|b_n(u_{0n})|dxdt],$$

where $\frac{1}{C} = [1 - \frac{(\alpha_0 b_1 + 1)}{\alpha}\|c(.,.)\|_{L^\infty(Q_T)}] > 0$.

Passing to the limit as $n \to +\infty$, using the pointwise convergence of u_n and strongly convergence in $L^1(Q_T)$ of f_n we get

$$\lim_{n\to+\infty}\int_{\{m\le|u_n|\le m+1\}} a_n(u_n,\nabla u_n)\nabla u_n dxdt \le C[\int_{Q_T} fZ_m(u)\,dxdt + \int_{\{|u_0|>m\}}|b(u_0)|dxdt].$$

Owning to Lebesgue's theorem and passing to the limit as $m \to +\infty$, in the all terms of the right-hand side, we get (2.31).
From (2.7), we deduce also

$$\lim_{m\to+\infty}\lim_{n\to+\infty}\int_{\{m\le|u_n|\le m+1\}} M(|\nabla Z_m(u_n)|)dxdt = 0. \tag{2.35}$$

On the other hand, we have

$$\lim_{m\to+\infty}\lim_{n\to+\infty}\int_{Q_T}\Phi_n(u_n)\nabla Z_m(u_n)dxdt \le \lim_{m\to+\infty}\lim_{n\to+\infty}\int_{Q_T} M(|\nabla Z_m(u_n)|)dxdt,$$

$$+\lim_{m\to+\infty}\lim_{n\to+\infty}\int_{\{m\le|u_n|\le m+1\}}\overline{M}(|\Phi_n(u_n)|)dxdt.$$

Using the pointwise convergence of u_n and by Lebegue's theorem, in the second term of the right side of this last expression, we get

$$\lim_{n\to+\infty}\int_{\{m\le|u_n|\le m+1\}}\overline{M}(|\Phi_n(u_n)|)dxdt = \int_{\{m\le|u|\le m+1\}}\overline{M}(|\Phi(u)|)dxdt,$$

and also, by Lebesgue's theorem

$$\lim_{m\to+\infty}\int_{\{m\le|u|\le m+1\}}\overline{M}(|\Phi(u)|)dxdt = 0. \tag{2.36}$$

Thus with (2.35) and (2.36), we get the (2.32).

The proof of (2.33) is the same as the corresponding result in ([63], [96]).

Finally, for the almost everywhere convergence of the gradients we use the following lemma and same techniques as in [21] and [63].

Lemma 2.4.2. *(see [29])*
Under the assumptions (2.4)–(2.10)*, let* (z_n) *be a sequence in* $W_0^{1,x}L_M(Q_T)$ *such that:*

$$z_n \rightharpoonup z \quad for \quad \sigma(\Pi L_M, \Pi E_{\overline{M}}), \tag{2.37}$$

$$(a(x,t,z_n,\nabla z_n)) \quad is\ bounded\ in\ (L_{\overline{M}}(Q_T))^N, \tag{2.38}$$

$$\int_{Q_T}[a(x,t,z_n,\nabla z_n) - a(x,t,z_n,\nabla z\chi_s)][\nabla z_n - \nabla z\chi_s]dxdt \to 0 \tag{2.39}$$

as n and s tend to $+\infty$*, and where* χ_s *is the characteristic function of* $Q^s = \{x \in Q_T; |\nabla z| \le s\}$*. Then,*

$$\nabla z_n \to \nabla z \quad a.e.\ in\ Q_T, \tag{2.40}$$

$$\lim_{n\to+\infty}\int_{Q_T} a(x,t,z_n,\nabla z_n)\nabla z_n dxdt = \int_{Q_T} a(x,t,z,\nabla z)\nabla z dxdt, \tag{2.41}$$

$$M(|\nabla z_n|) \to M(|\nabla z|) \quad in\ L^1(Q_T). \tag{2.42}$$

□

Step 3: We show that u satisfies the conditions of Definition 2.2.1

For this, let show that (2.13) holds.

We have for any $m > 0$,

$$\begin{aligned}\int_{\{m\le|u_n|\le m+1\}} a(u_n,\nabla u_n)\nabla u_n dxdt &= \int_{Q_T} a(u_n,\nabla u_n)[\nabla T_{m+1}(u_n)-\nabla T_m(u_n)]dxdt\\ &= \int_{Q_T} a(T_{m+1}(u_n),\nabla T_{m+1}(u_n))\nabla T_{m+1}(u_n)dxdt\\ &\quad -\int_{Q_T} a(T_m(u_n),\nabla T_m(u_n))\nabla T_m(u_n)dxdt.\end{aligned}$$

According to (2.41), we pass to the limit as n tends to $+\infty$ for fixed $m > 0$ and we obtain

$$\begin{aligned}\lim_{n\to+\infty}\int_{\{m\le|u_n|\le m+1\}} a(u_n,\nabla u_n)\nabla u_n dxdt &= \int_{Q_T} a(T_{m+1}(u),\nabla T_{m+1}(u))\nabla T_{m+1}(u)dxdt\\ &\quad -\int_{Q_T} a(T_m(u),\nabla T_m(u))\nabla T_m(u)dxdt\\ &= \int_{\{m\le|u|\le m+1\}} a(u,\nabla u)\nabla u dxdt,\end{aligned}$$

with (2.31), we easily obtain (2.13).

Similarly we deduce

$$\lim_{m\to+\infty}\int_{\{m\le|u|\le m+1\}}\Phi(u)\nabla u dxdt = 0. \tag{2.43}$$

Let $S\in W^{2,\infty}(\mathbb{R})$ which is a piecewise C^1-function and such that S' has a compact support, Let $K > 0$ such that $\mathrm{supp}(S')\subset[-K,K]$. Pointwise multiplication of the approximate problem (2.21) by $S'(u_n)$, we get

$$\begin{cases}\dfrac{\partial S(b(u_n))}{\partial t}+\mathrm{div}\Big(S'(b(u_n))(a(x,t,u_n,\nabla u_n)+\Phi(x,t,u_n))\Big)\\ +S''(b(u_n))\Big(a(x,t,u_n,\nabla u_n)+\Phi(x,t,u_n)\Big)\nabla b(u_n) = fS'(b(u_n)).\end{cases} \tag{2.44}$$

Now we will pass to the limit as $n\to+\infty$ of each term of (2.44),

- ***Limit of*** $\frac{\partial S(b(u_n))}{\partial t}$ since S is bounded, and $S(b(u_n))$ converges to $S(b(u))$ a.e. in Q_T and weakly in $L^\infty(Q_T)$, then $\frac{\partial S(b(u_n))}{\partial t}$ converges to $\frac{\partial S(b(u))}{\partial t}$ in $\mathscr{D}'(Q_T)$.
- ***Limit of*** $S'(b(u_n))a(u_n,\nabla u_n)$ since $\mathrm{supp}(S')\subset[-K,K]$ and (2.4) we have $S'(b(u_n))a(u_n,\nabla u_n) = S'(b(u_n))a(T_{\frac{k}{b_0}}(u_n),\nabla T_{\frac{k}{b_0}}(u_n))$ a.e. in Q_T.
  Owing to the pointwise convergence of u_n to u, the bounded character of S', and by Lemma 2.4.2 and Proposition 2.4.1, we conclude $a(T_{\frac{k}{b_0}}(u_n),\nabla T_{\frac{k}{b_0}}(u_n))$ converges to $a(T_{\frac{k}{b_0}}(u_n),\nabla T_{\frac{k}{b_0}}(u_n))$ weakly in $(L_{\overline{M}}(Q_T))^N$. This allows us to obtain $S'(b(u_n))a(T_{\frac{k}{b_0}}(u_n),\nabla T_{\frac{k}{b_0}}(u_n))$ converges to $S'(b(u))a(T_{\frac{k}{b_0}}(u),\nabla T_{\frac{k}{b_0}}(u))$ weakly for $\sigma(\Pi L_{\overline{M}},\Pi E_M)$, and $S'(b(u))a(T_{\frac{k}{b_0}}(u),\nabla T_{\frac{k}{b_0}}(u) = S'(b(u))a(u,\nabla u)$ a.e. in Q_T.

- ***Limit of*** $S''(b(u_n))a(u_n,\nabla u_n)\nabla b(u_n)$ since $\mathrm{supp}(S')\subset[-K,K]$ and (2.4), we get $S''(b(u_n))a(u_n,\nabla u_n)\nabla b(u_n)=S''(b(u_n))a(T_{\frac{k}{b_0}}(u_n),\nabla T_{\frac{k}{b_0}}(u_n))\nabla b(u_n)$ a.e. in Q_T. Owing to the pointwise convergence of $S''(b(u_n))$ to $S''(b(u_n))$ as n tends to $+\infty$, the bounded character of S'' and by Lemma 2.4.2 and Proposition 2.4.1, we conclude $S''(b(u_n))a(T_{\frac{k}{b_0}}(u_n),\nabla T_{\frac{k}{b_0}}(u_n))\nabla b(u_n)\rightharpoonup S''(b(u))a(T_{\frac{k}{b_0}}(u),\nabla T_{\frac{k}{b_0}}(u))\nabla b(u)$ weakly in $L^1(Q_T)$ as $n\to+\infty$, and $S''(b(u))a(T_{\frac{k}{b_0}}(u),\nabla T_{\frac{k}{b_0}}(u))\nabla b(u)=S''(b(u))a(u,\nabla u)\nabla b(u)$ a.e. in Q_T.

- ***Limit of*** $S'(b(u_n))\Phi(u_n)$ since $\mathrm{supp}(S')\subset[-K,K]$ and (2.4) we have $S'(b(u_n))\Phi(u_n)=S'(b(u_n))\Phi(T_{\frac{k}{b_0}}(u_n))$ a.e. in Q_T. In a similar way, we obtain $S'(b(u_n))\Phi(u_n)\to S'(b(u))\Phi(u)$ weakly for $\sigma(\Pi L_{\overline{M}},\Pi E_M)$.

- ***Limit of*** $S''(b(u_n))\Phi(x,t,u_n)\nabla b(u_n)$
  Also we have $S''(b(u_n))\Phi(u_n)\nabla b(u_n)=S''(b(u_n))\Phi(T_{\frac{k}{b_0}}(u_n))\nabla T_{\frac{k}{b_0}}(u_n)b'(u_n)$,
  using the weakly convergence of truncation, it is possible to prove that,
  $S''(b(u_n))\Phi(u_n)\nabla b(u_n)\to S''(b(u))\Phi(u)\nabla b(u)$ strongly in $L^1(Q_T)$.

- ***Limit of*** $f_nS'(b(u_n))$ we have $u_n\to u$ a.e. in Q_T, S' and b are piecewise C^1. It is enough to use (2.19) to get that $f_nS'(b(u_n))\to fS'(b(u))$ strongly in $L^1(Q_T)$.

Finally, to show (2.15), remark that S being bounded, $S(b(u_n))$ is bounded in $L^\infty(Q_T)$. the equation (2.44) allows to show that $\frac{\partial S(b(u_n))}{\partial t}$ is bounded in $W^{-1,x}L_{\overline{M}}(Q_T)+L^1(Q_T)$. By Lemma 5 in [62] this implies that $S(b(u_n))$ lies in a compact set of $C^0([0,T];L^\infty(\Omega))$. It follows that, on one hand, $S(b(u_n)(t=0))$ converges to $S(b(u)(t=0))$ strongly in $L^1(Q_T)$. On the other hand, the smoothness of S imply that $S(b(u)(t=0))=S(b(u_0))$ in Ω. This complete the proof of the existence result.

Example 2.4.1.

As an example of equations to which the present result on the existence of renormalized solutions can be applied, we give:

1. *For* $M(t)=\frac{1}{p}|u|^p$, $b(u)=|u|^{p-2}u$, $a(x,t,u,\nabla u)=|\nabla u|^{p-2}\nabla u$
   and $\Phi(x,t,u)=\exp\Big(\frac{\eta}{\|x\|+t+2}\Big)\beta(\frac{\alpha_0}{\lambda})^{\frac{p}{q}}|u|^{\frac{p}{q}}$, $\beta>0$.
$$\begin{cases}\dfrac{\partial b(u)}{\partial t}--div(|\nabla u|^{p-2}\nabla u)-div(\exp\Big(\dfrac{\eta}{\|x\|+t+2}\Big)\beta(\dfrac{\alpha_0}{\lambda})^{\frac{p}{q}}|u|^{\frac{p}{q}})=f & in \quad Q_T,\\ u(x,t)=0 & on \quad \partial\Omega\times(0,T),\\ b(u(x,0))=b(u_0)(x) & in \quad \Omega.\end{cases}$$

2. *For* $-\triangle_M=-div(\frac{m(|\nabla u|)}{|\nabla u|}.\nabla u)$ *where m is the derivative of M,* $b(u)=u$ *and* $\overline{c}\in(L^\infty(Q_T))^N$.
$$\begin{cases}\dfrac{\partial u}{\partial t}-\triangle_M-div(\overline{c}(x,t)\overline{M}^{-1}M(\dfrac{\alpha_0}{\lambda}|b(u)|))=f & in \quad Q_T,\\ u(x,t)=0 & on \quad \partial\Omega\times(0,T),\\ u(x,0)=u_0(x) & in \quad \Omega.\end{cases}$$

3. *For* $M(t)=t\log(1+t)$, $a(x,t,u,\nabla u)=(1+|u|)^2\nabla u\dfrac{\log(1+|\nabla u|)}{|\nabla u|}$ *and* $\bar{c}\in(L^\infty(Q_T))^N$.

$$\begin{cases} \dfrac{\partial b(u)}{\partial t}-div(1+|u|)^2\nabla u\dfrac{log(1+|\nabla u|)}{|\nabla u|}-div(\bar{c}(x,t)\overline{M}^{-1}M(\dfrac{\alpha_0}{\lambda}b(u)))=f \quad in \quad Q_T,\\ u(x,t)=0 \qquad\qquad on \quad \partial\Omega\times(0,T),\\ b(u(x,0))=b(u_0)(x) \qquad in \quad \Omega. \end{cases}$$

2.5 Uniqueness result

Before showing the uniqueness of the solution of the problem (2.1), we will give the following technical lemma.

Let u and v be two renormalized solutions of the problem (2.1) and let us define for any $0<k<s$,

$$\Gamma(u,v,s,k)=\int_{\substack{\{b(s)-k<b(u)<b(s)+k\}\\ \cup\{b(-s)-k<b(u)<b(-s)+k\}}} b'(u)\Big(a(u,\nabla u)\nabla u+|\Phi(u)||\nabla u|\Big)dxdt$$

$$+\int_{\substack{\{b(s)-k<b(v)<b(s)+k\}\\ \cup\{b(-s)-k<b(v)<b(-s)+k\}}} b'(v)\Big(a(v,\nabla v)\nabla v+|\Phi(v)||\nabla v|\Big)dxdt. \tag{2.45}$$

Lemma 2.5.1. *Assume that* (2.4)–(2.10) *hold. Then for any* $r>0$ *we have*

$$\liminf_{s\to+\infty}\limsup_{k\to 0}\frac{1}{k}\Gamma(u,v,s,k)=0. \tag{2.46}$$

Proof of Lemma 2.5.1

Define the functions

$$L_1(s)=\int_{\{0<b(u)<s\}}\Big(a(u,\nabla u)\nabla u+|\Phi(u)||\nabla u|\Big)dxdt+\int_{\{0<b(v)<s\}}\Big(a(v,\nabla v)\nabla v+|\Phi(v)||\nabla v|\Big)dxdt, \tag{2.47}$$

and

$$L_2(s)=\int_{\{-s<b(u)<0\}}\Big(a(u,\nabla u)\nabla u+|\Phi(u)||\nabla u|\Big)dxdt+\int_{\{-s<b(v)<0\}}\Big(a(v,\nabla v)\nabla v+|\Phi(v)||\nabla v|\Big)dxdt. \tag{2.48}$$

Due to (2.6) the functions L_1 and L_2 are monotone increasing. L_1 and L_2 are differentiable almost everywhere (see [98]), with L_1' and L_2' measurable and so we have for any $s>\eta>0$

$$L_1(s)-L_1(\eta)\geq\int_\eta^s L_1'(\xi)d\xi \quad \text{and} \quad L_2(s)-L_2(\eta)\geq\int_\eta^s L_2'(\xi)d\xi, \tag{2.49}$$

and for almost any $s>0$

$$L_1'(s)=\frac{1}{2}\limsup_{k\to 0}\frac{1}{k}\Big[\int_{\{s-k<b(u)<s+k\}}\Big(a(u,\nabla u)\nabla u+|\Phi(u)||\nabla u|\Big)dxdt$$

$$+\int_{\{s-k<b(v)<s+k\}}\Big(a(v,\nabla v)\nabla v+|\Phi(v)||\nabla v|\Big)dxdt\Big], \tag{2.50}$$

and

$$L_2'(s)=\frac{1}{2}\limsup_{k\to 0}\frac{1}{k}\int_{\{-s-k<b(u)<-s+k\}}\Big(a(u,\nabla u)\nabla u+|\Phi(u)||\nabla u|\Big)dxdt$$
$$+\int_{\{-s-k<b(v)<-s+k\}}\Big(a(v,\nabla v)\nabla v+|\Phi(v)||\nabla v|\Big)dxdt]. \tag{2.51}$$

If the thesis of the lemma is not true, let $\varepsilon_0>0$ and let $n_0>0$ be a real number such that for every real number $s\geq n_0$ we have

$$\limsup_{k\to 0}\frac{1}{k}\Gamma(u,v,s,k)\geq\varepsilon_0. \tag{2.52}$$

Since b' is a continuous and positive function, we have for almost $\xi\geq n_0$,

$$\limsup_{k\to 0}\frac{1}{k}\Gamma(u,v,s,k)\leq b'(s)\limsup_{k\to 0}\frac{1}{k}[\int_{\{b(s)-k<b(u)<b(s)+k\}}\Big(a(u,\nabla u)\nabla u+|\Phi(u)||\nabla u|\Big)dxdt$$
$$+\int_{\{b(s)-k<b(v)<b(s)+k\}}\Big(a(v,\nabla v)\nabla v+|\Phi(v)||\nabla v|\Big)dxdt]$$
$$+b'(-s)\limsup_{k\to 0}\frac{1}{k}[\int_{\{b(-s)-k<b(u)<b(-s)+k\}}\Big(a(u,\nabla u)\nabla u+|\Phi(u)||\nabla u|\Big)dxdt$$
$$+\int_{\{b(-s)-k<b(v)<b(-s)+k\}}\Big(a(v,\nabla v)\nabla v+|\Phi(v)||\nabla v|\Big)dxdt]. \tag{2.53}$$

From (2.50), (2.52) and (2.51) it follows that

$$b'(\xi)L_1'(b(\xi))+b'(-\xi)L_2'(-b(-\xi))\geq\frac{\varepsilon_0}{2}.$$

In view of (2.49), we deduce that for any $s>\eta>n_0$ we have

$$L_1(b(s))-L_1(b(\eta))+L_2(-b(-s))-L_2(-b(-\eta))\geq\frac{\varepsilon_0}{2}(s-\eta). \tag{2.54}$$

Taking $s=n+1$ and $\eta=n$ with $n>n_0$ we have

$$\int_{\{n\leq|u|\leq n+1\}}\Big(a(u,\nabla u)\nabla u+|\Phi(u)||\nabla u|\Big)dxdt+\int_{\{n\leq|v|\leq n+1\}}\Big(a(v,\nabla v)\nabla v+|\Phi(v)||\nabla v|\Big)dxdt\geq\frac{\varepsilon_0}{2}.$$

The last inequality contradicts (2.13) and (2.43).

Theorem 2.5.1. *Assume that assumptions* (2.4)–(2.10) *hold true and moreover that for any compact set $D\subset\mathbb{R}$, there exists $L_D\in E_{\overline{M}}(Q_T)$ and $\rho_D>0$ such that $\forall s,\overline{s}\in D$,*

$$|a(x,t,s,\xi)-a(x,t,\overline{s},\xi)|\leq\Big(L_D(x,t)+\rho_D\overline{P}^{-1}P(|\xi|)\Big)|s-\overline{s}|, \tag{2.55}$$

$$|\Phi(x,t,s)-\Phi(x,t,\overline{s})|\leq L_D(x,t)|s-\overline{s}|, \tag{2.56}$$

$$|b'(s)-b'(\overline{s})|\leq\beta_D|s-\overline{s}|, \tag{2.57}$$

for almost every $(x,t)\in Q_T$ and for every $\xi\in\mathbb{R}^N$. Then the problem (2.1) *has a unique renormalized solution.*

Proof. We define a smooth approximation of $\widetilde{T}_n(r) = \min(b(n), \max(r, b(-n)))$ by $\widetilde{T}_n^\sigma$ where $\widetilde{T}_n^\sigma(0) = 0$ and

$$(\widetilde{T}_n^\sigma)'(r) = \begin{cases} 0 & \text{for} \quad r \geq b(n) + \sigma, \\ \frac{b(n)+\sigma-r}{\sigma} & \text{for} \quad b(n) \leq r \leq b(n) + \sigma, \\ 1 & \text{for} \quad b(-n) \leq r \leq b(n), \\ \frac{r+\sigma-b(-n)}{\sigma} & \text{for} \quad b(-n) - \sigma \leq r \leq b(-n), \\ 0 & \text{for} \quad r \leq b(-n) - \sigma. \end{cases} \tag{2.58}$$

For a fixed $n > 0$, we have for any $z \in L^1(Q_T)$,

$$\lim_{\sigma \to 0} (\widetilde{T}_n^\sigma)'(b(z)) = \chi_{\{|z| \leq n\}} \quad \text{a.e. in} \quad Q_T, \tag{2.59}$$

and

$$\lim_{\sigma \to 0} \widetilde{T}_n^\sigma(b(z)) = \widetilde{T}_n(b(z)) \quad \text{a.e. in} \quad Q_T. \tag{2.60}$$

Consider now two renormalized solutions u and v of (2.11)- (2.14) for the data f and $b(u_0)$. Since $\widetilde{T}_n^\sigma \in W^{2,\infty}(\mathbb{R})$ and $supp((\widetilde{T}_n^\sigma)') \subset [b(-n) - \sigma, b(n) + \sigma]$, then we take $S = \widetilde{T}_n^\sigma$ and we use $\frac{1}{k} T_k(\widetilde{T}_n^\sigma(b(u)) - \widetilde{T}_n^\sigma(b(v)))$ as a test function in the difference of equations (2.14) for u and v, we get

$$\begin{aligned} \frac{1}{k} \int_0^T \int_0^t \langle \frac{\partial \left(\widetilde{T}_n^\sigma(b(u)) - \widetilde{T}_n^\sigma(b(v)) \right)}{\partial t}; T_k(\widetilde{T}_n^\sigma(b(u)) - \widetilde{T}_n^\sigma(b(v))) \rangle ds dt \\ + I_{1,n}^\sigma + I_{2,n}^\sigma + I_{3,n}^\sigma + I_{4,n}^\sigma = I_{5,n}^\sigma, \end{aligned} \tag{2.61}$$

where

$$I_{1,n}^\sigma = \frac{1}{k} \int_0^T \int_0^t \int_\Omega [(\widetilde{T}_n^\sigma)'(b(u)) a(u, \nabla u) - (\widetilde{T}_n^\sigma)'(b(v)) a(v, \nabla v)] \nabla T_k(\widetilde{T}_n^\sigma(b(u)) - \widetilde{T}_n^\sigma(b(v))) dx ds dt,$$

$$I_{2,n}^\sigma = \frac{1}{k} \int_0^T \int_0^t \int_\Omega [(\widetilde{T}_n^\sigma)''(b(u)) a(u, \nabla u) \nabla b(u) - (\widetilde{T}_n^\sigma)''(b(v)) a(v, \nabla v) \nabla b(v)] T_k(\widetilde{T}_n^\sigma(b(u)) - \widetilde{T}_n^\sigma(b(v))) dx ds dt,$$

$$I_{3,n}^\sigma = \frac{1}{k} \int_0^T \int_0^t \int_\Omega [(\widetilde{T}_n^\sigma)'(b(u)) \Phi(u) - (\widetilde{T}_n^\sigma)'(b(v)) \Phi(v)] \nabla T_k(\widetilde{T}_n^\sigma(b(u)) - \widetilde{T}_n^\sigma(b(v))) dx ds dt,$$

$$I_{4,n}^\sigma = \frac{1}{k} \int_0^T \int_0^t \int_\Omega [(\widetilde{T}_n^\sigma)''(b(u)) \Phi(u) \nabla b(u) - (\widetilde{T}_n^\sigma)''(b(v)) \Phi(v) \nabla b(v)] T_k(\widetilde{T}_n^\sigma(b(u)) - \widetilde{T}_n^\sigma(b(v))) dx ds dt,$$

$$I_{5,n}^\sigma = \frac{1}{k} \int_0^T \int_0^t \int_\Omega f[(\widetilde{T}_n^\sigma)'(b(u)) - (\widetilde{T}_n^\sigma)'(b(v))] T_k(\widetilde{T}_n^\sigma(b(u)) - \widetilde{T}_n^\sigma(b(v))) dx ds dt,$$

for any $k > 0$, $n > 0$, $\sigma > 0$.

The following lemma will be useful in the sequel,

Lemma 2.5.2.

$$\lim_{n \to +\infty} \lim_{k \to 0} \lim_{\sigma \to 0} \frac{1}{k} \int_0^T \int_0^t \langle \frac{\partial \left(\widetilde{T}_n^\sigma(b(u)) - \widetilde{T}_n^\sigma(b(v)) \right)}{\partial t}; T_k(\widetilde{T}_n^\sigma(b(u)) - \widetilde{T}_n^\sigma(b(v))) \rangle ds dt$$

$$= \int_{Q_T} |b(u) - b(v)| dxdt. \tag{2.62}$$

Proof.

Notice that

$$\widetilde{T}_n^\sigma(b(u))(t=0) = \widetilde{T}_n^\sigma(b(v))(t=0) = \widetilde{T}_n^\sigma(b(u_0)) \quad \text{a.e. in} \quad \Omega,$$

and

$$\frac{1}{k}\int_0^T \int_0^t \langle \frac{\partial\Big(\widetilde{T}_n^\sigma(b(u)) - \widetilde{T}_n^\sigma(b(v))\Big)}{\partial t}; T_k(\widetilde{T}_n^\sigma(b(u)) - \widetilde{T}_n^\sigma(b(v))\Big)\rangle dsdt$$
$$= \frac{1}{k}\int_{Q_T} \overline{T}_k(\widetilde{T}_n^\sigma(b(u)) - \widetilde{T}_n^\sigma(b(v)))dxdt,$$

where $\overline{T}_k(r) = \displaystyle\int_0^r T_k(z)dz$.

Passing to the limit we obtain

$$\lim_{k\to 0}\lim_{\sigma\to 0}\frac{1}{k}\int_{Q_T} \overline{T}_k(\widetilde{T}_n^\sigma(b(u)) - \widetilde{T}_n^\sigma(b(v)))dxdt = \int_{Q_T} |\widetilde{T}_n(b(u)) - \widetilde{T}_n(b(v))|dxdt,$$

and letting $n \to +\infty$ in this last equality, we deduce (2.62). □

Now, we analyze the limit of $I_{1,n}^\sigma, I_{2,n}^\sigma, I_{3,n}^\sigma, I_{4,n}^\sigma$ and $I_{5,n}^\sigma$ one by one.

The limit of $I_{1,n}^\sigma$: Notice that

$$I_{1,n}^\sigma = \frac{1}{k}\int_0^T \int_0^t \int_\Omega Q_n^\sigma dxdsdt = \frac{1}{k}\int_{Q_T} (T-t)Q_n^\sigma dxdt,$$

where $Q_n^\sigma = (\widetilde{T}_n^\sigma)'(b(u))a(u,\nabla u) - (\widetilde{T}_n^\sigma)'(b(v))a(v,\nabla v)]\nabla T_k(\widetilde{T}_n^\sigma(b(u)) - \widetilde{T}_n^\sigma(b(v)))$.
Since $\text{supp}((\widetilde{T}_n^\sigma)') \subset [b(-n)-\sigma, b(n)+\sigma]$,
then

$$(\widetilde{T}_n^\sigma)'(b(u))a(u,\nabla u) = (\widetilde{T}_n^\sigma)'(b(u))a(T_{n+1}(u),\nabla T_{n+1}(u))$$

and

$$(\widetilde{T}_n^\sigma)'(b(v))a(v,\nabla v) = (\widetilde{T}_n^\sigma)'(b(u))a(T_{n+1}(v),\nabla T_{n+1}(v)).$$

Then by (2.59), (2.60) and (2.58) one has

$$\begin{cases} Q_n^\sigma \quad \text{converges to} \quad [\chi_{\{|u|\le n\}}a(u,\nabla u) - \chi_{\{|v|\le n\}}a(v,\nabla v)]\nabla T_k(\widetilde{T}_n(b(u)) - \widetilde{T}_n(b(v))), \\ \\ |Q_n^\sigma| \le C_n^2 b_1[|a(T_{n+1}(u),\nabla T_{n+1}(u))| + |a(T_{n+1}(v),\nabla T_{n+1}(v))|] \\ \qquad\qquad \times(|\nabla T_{n+1}(u)| + |\nabla T_{n+1}(v)|)\chi_{\{|\widetilde{T}_n(b(u)) - \widetilde{T}_n(b(v))|\le k\}} = R_n. \end{cases}$$

where $C_n = \max(|b(-n)-\sigma|, b(n)+\sigma)$.

Since $R_n \in L^1(Q_T)$ we use the Lebesgue dominated convergence theorem to have

$$\lim_{\sigma\to 0} I_{1,n}^\sigma = \lim_{\sigma\to 0}\frac{1}{k}\int_{Q_T}(T-t)Q_n^\sigma dxdt$$

$$= \frac{1}{k}\int_{Q_T}(T-t)[\chi_{\{|u|\le n\}}a(u,\nabla u) - \chi_{\{|v|\le n\}}a(v,\nabla v)]\nabla T_k(T_n(b(u)) - T_n(b(v)))dxdt$$

$$= J_1 + J_2 + J_3 + J_4 + J_5, \tag{2.63}$$

where

$$J_1 = \frac{1}{k}\int_{\{|b(u)-b(v)|\le k,|u|\le n,|v|\le n\}} (T-t)\Big(a(u,\nabla u)-a(u,\nabla v)\Big)(\nabla u-\nabla v)b'(u)dxdt,$$
$$J_2 = \frac{1}{k}\int_{\{|b(u)-b(v)|\le k,|u|\le n,|v|\le n\}} (T-t)\Big(a(u,\nabla v)-a(v,\nabla v)\Big)(\nabla u-\nabla v)b'(u)dxdt,$$
$$J_3 = \frac{1}{k}\int_{\{|b(u)-b(v)|\le k,|u|\le n,|v|\le n\}} (T-t)\Big(a(u,\nabla u)-a(v,\nabla v)\Big)\nabla v(b'(u)-b'(v))dxdt,$$
$$J_4 = \frac{1}{k}\int_{\{|\widetilde{T}_n(b(u))-\widetilde{T}_n(b(v))|\le k,|u|>n,|v|\le n\}} (T-t)a(v,\nabla v)\nabla b(v)dxdt,$$
$$J_5 = \frac{1}{k}\int_{\{|\widetilde{T}_n(b(u))-\widetilde{T}_n(b(v))|\le k,|u|\le n,|v|>n\}} (T-t)a(u,\nabla u)\nabla b(u)dxdt.$$

Since $a(u,\nabla u)$ satisfies the condition (2.7), one has

$$J_1 \ge 0. \tag{2.64}$$

On the other hand by (2.55) we have

$$|J_2| \le \frac{b_1}{k}T\int_{Q_T} \chi_{\{|b(u)-b(v)|\le k\},|u|\le n,|v|\le n\}}|u-v|\Big(L_D(x,t)+\rho_D\overline{P}^{-1}P(|v|)\Big)\times(|\nabla u|+|\nabla v|)dxdt.$$

Using (2.4), one has $|u-v| \le \frac{1}{b_0}|b(u)-b(v)| \le \frac{1}{b_0}$, then

$$|J_2| \le \frac{b_1}{b_0}T\int_{\{|b(u)-b(v)|\le k\}} \Big(L_D(x,t)+\rho_D\overline{P}^{-1}P(|v|)\Big)(|\nabla u|+|\nabla v|)dxdt.$$

Since $L_D(x,t) \in E_{\overline{M}}(Q_T)$, u and v in $W^{1,x}L_M(Q_T)$ and using (**??**), one has

$$(L_D(x,t)+\rho_D\overline{P}^{-1}P(|v|))(|\nabla u|+|\nabla v|) \in L^1(Q_T)$$

The Lebesgue dominated convergence theorem allows us to conclude that for all $n \ge 1$

$$\limsup_{k\to 0} J_2 = 0. \tag{2.65}$$

We denote by C_n the compact subset $[-n-1, n+1]$. Due to (2.57), there exists a positive number β_n such that $|b'(r_1)-b'(r_2)| \le \beta_n|r_1-r_2|$ for all r_1 and r_2 lying in C_n. Using (2.4) and Rolle's theorem, we get

$$|b'(r_1)-b'(r_2)| \le \frac{\beta_n}{b_0}|b(r_1)-b(r_2)|.$$

Then $|b'(r_1)-b'(r_2)| \le k\frac{\beta_n}{b_0}$ on $\{|b(u)-b(v)| \le k, |u| \le n, |v| \le n\}$ and we get

$$|J_3| \le \frac{\beta_n}{b_0}T\int_{Q_T} \Big(a(T_n(u),\nabla T_n(u))-a(T_n(v),\nabla T_n(v))\Big)|\nabla v|\chi_{\{|b(u)-b(v)|\le k,|u|\le n,|v|\le n,u\ne v\}}dxdt.$$

Since $\begin{cases} \lim\limits_{k\to 0}\chi_{\{|b(u)-b(v)|\le k,|u|\le n,|v|\le n,u\neq v\}}=0 \quad \text{a.e. in} \quad Q_T, \\ |a(T_n(u),\nabla T_n(u))-a(T_n(v),\nabla T_n(v))||\nabla T_n(v)|\in L^1(Q_T), \end{cases}$
we use the Lebesgue dominated convergence theorem to conclude that, for all $n\ge 1$,

$$\lim_{k\to 0} J_3 = 0. \tag{2.66}$$

In view of the definition of $\widetilde{T}_n$, we have

$$J_4 = \frac{1}{k}\int_{\substack{\{b(n)-k\le b(v)\le b(n)\}\\ \cup\{b(-n)\le b(v)\le b(-n)+k\}}} (T-t)a(v,\nabla v)\nabla b(v)dxdt,$$

and using (2.7) we deduce

$$\liminf_{n\to+\infty}\limsup_{k\to 0} J_4 \ge 0. \tag{2.67}$$

Similarly we have

$$J_5 = \frac{1}{k}\int_{\substack{\{b(n)-k\le b(u)\le b(n)\}\\ \cup\{b(-n)\le b(u)\le b(-n)+k\}}} (T-t)a(u,\nabla u)\nabla b(u)dxdt,$$

and

$$\liminf_{n\to+\infty}\limsup_{k\to 0} J_5 \ge 0. \tag{2.68}$$

Now from (2.63)-(2.68) we obtain

$$\liminf_{n\to+\infty}\limsup_{k\to 0}\lim_{\sigma\to 0} I^{\sigma}_{1,n} \ge 0. \tag{2.69}$$

The limit of $I^{\sigma}_{2,n}$ and $I^{\sigma}_{4,n}$: Now we claim that

$$|I^{\sigma}_{2,n}| + |I^{\sigma}_{4,n}| \le \frac{T}{\sigma}\Gamma(u,v,n,\sigma). \tag{2.70}$$

From a simple derivation of the function $(\widetilde{T}^{\sigma}_n)'$ it yields that for any $\sigma>0$ and $k>0$

$$\begin{aligned} |I^{\sigma}_{2,n}| \le \frac{T}{\sigma}\int_{\substack{\{b(n)-\sigma\le b(u)\le b(n)\}\\ \cup\{b(-n)\le b(u)\le b(-n)+\sigma\}}} & a(u,\nabla u))\nabla b(u)dxdt \\ +\frac{T}{\sigma}\int_{\substack{\{b(n)-\sigma\le b(v)\le b(n)\}\\ \cup\{b(-n)\le b(v)\le b(-n)+\sigma\}}} & a(v,\nabla v))\nabla b(v)dxdt. \end{aligned} \tag{2.71}$$

Similarly we have

$$|I^{\sigma}_{4,n}| \le \frac{T}{\sigma}\int_{\substack{\{b(n)-\sigma\le b(u)\le b(n)\}\\ \cup\{b(-n)\le b(u)\le b(-n)+\sigma\}}} \Phi(u)\nabla b(u)dxdt$$

$$+\frac{T}{\sigma}\int_{\substack{\{b(n)-\sigma\le b(v)\le b(n)\}\\ \cup\{b(-n)\le b(v)\le b(-n)+\sigma\}}}\Phi(v)\nabla b(v)dxdt. \tag{2.72}$$

By combining (2.71) and (2.72) we readily deduce (2.70).

The limit of $I^{\sigma}_{3,n}$: There we prove that

$$\limsup_{\sigma\to 0}|I^{\sigma}_{3,n}|\le\frac{T}{k}\Gamma(u,v,n,k)+\varepsilon(k), \tag{2.73}$$

where $\varepsilon(k)$ is a positive function such that $\lim_{k\to 0}\varepsilon(k)=0$.

For $n\ge 0$ we have

$$\limsup_{\sigma\to 0}|I^{\sigma}_{3,n}|=|\frac{1}{k}\int_{Q_T}(T-t)(\chi_{\{|u|\le n\}}\Phi(u)-\chi_{\{|v|\le n\}}\Phi(v))\nabla T_k(\widetilde{T}_n(b(u))-\widetilde{T}_n(b(v)))dxdt|.$$

$$\le K_1+K_2+K_3,$$

where

$$K_1=\frac{T}{k}\int_{Q_T}\chi_{\{|u|\le n,|v|>n\}}|\Phi(u)||\nabla T_k(\widetilde{T}_n(b(u))-b(n.\ \mathrm{sgn}(v))|dxdt,$$

$$K_2=\frac{T}{k}\int_{Q_T}\chi_{\{|u|>n,|v|\le n\}}|\Phi(v)||\nabla T_k(\widetilde{T}_n(b(v))-b(n.\ \mathrm{sgn}(u))|dxdt,$$

$$K_3=\frac{T}{k}\int_{Q_T}\chi_{\{|u|\le n,|v|\le n\}}|\Phi(u)-\Phi(v)||\nabla T_k(\widetilde{T}_n(b(u))-\widetilde{T}_n(b(v)))|dxdt.$$

We estimate K_1 and K_2 by (2.8) we have

$$K_1\le\frac{T}{k}\int_{Q_T}\chi_{\{|u|\le n,|v|>n\}}\chi_{\{|b(u)-b(n.\ \mathrm{sgn}(v))|\le k\}}|\Phi(u)||\nabla b(u)|dxdt$$

$$\le\frac{T}{k}\int_{\substack{\{b(n)-k\le b(u)\le b(n)\}\\ \cup\{b(-n)\le b(u)\le b(-n)+k\}}}|\Phi(u)||\nabla b(u)|dxdt, \tag{2.74}$$

and similarly

$$K_2\le\frac{T}{k}\int_{\substack{\{b(n)-k\le b(v)\le b(n)\}\\ \cup\{b(-n)\le b(v)\le b(-n)+k\}}}|\Phi(v)||\nabla b(v)|dxdt. \tag{2.75}$$

On the other hand, by (2.57) one have since $L_D\in L_{\overline{M}}(Q_T)$,

$$K_3\le\frac{T}{k}\int_{\{|\widetilde{T}_n(b(u))-\widetilde{T}_n(b(v))|\le k\}\cap\{|u|\le n,|v|\le n\}}L_D(x,t)|u-v||\nabla T_k(\widetilde{T}_n(b(u))-\widetilde{T}_n(b(v)))|dxdt$$

by (2.4) we obtain

$$K_3\le\frac{T}{k}\int_{\{|\widetilde{T}_n(b(u))-\widetilde{T}_n(b(v))|\le k\}\cap\{|u|\le n,|v|\le n\}}\frac{1}{b_0}L_D(x,t)|b(u)-b(v)||\nabla T_k(\widetilde{T}_n(b(u))-\widetilde{T}_n(b(v)))|dxdt$$

$$= \frac{T}{k}\int_{\{|\widetilde{T}_n(b(u))-\widetilde{T}_n(b(v))|\le k\}\cap\{|u|\le n,|v|\le n\}} \frac{T}{b_0} L_D(x,t)|\widetilde{T}_n(b(u))-\widetilde{T}_n(b(v))||\nabla T_k(\widetilde{T}_n(b(u))-\widetilde{T}_n(b(v)))|dxdt$$

$$\le \frac{T}{b_0}\int_{\{|\widetilde{T}_n(b(u))-\widetilde{T}_n(b(v))|\le k\}\cap\{|u|\le n,|v|\le n\}} L_D(x,t)(|\nabla\widetilde{T}_n(b(u))|+|\nabla\widetilde{T}_n(b(v))|)dxdt = \varepsilon(k).$$

Since L_D in $L_{\overline{M}}(Q_T)$ and due to (2.12), the function $L_D(x,t)(|\nabla T_n(b(u))|+|\nabla T_n(b(v))|) \in L^1(Q_T)$. Using the Lebesgue dominated convergence theorem we obtain $\lim_{k\to 0}\varepsilon(k)=0$ and

$$\lim_{k\to 0}|K_3| = 0. \tag{2.76}$$

Estimates (2.74)-(2.76) imply (2.73).

The limit of $I^{\sigma}_{5,n}$: Using the Lebesgue theorem and (2.59) and (2.60), we obtain

$$\lim_{\sigma\to 0}|I^{\sigma}_{5,n}| \le \frac{T}{k}\int_{Q_T}|T_k(\widetilde{T}_n(b(u)-\widetilde{T}_n(b(v))|\times|f||\chi_{\{|u|\le n\}}-\chi_{\{|v|\le n\}}|dxdt.$$

Since $\lim_{k\to 0}\frac{T_k(z)}{k} = \operatorname{sgn}(z)$ in $\mathbb{R}$ and weakly-* in L^{∞} then

$$\lim_{k\to 0}\lim_{n\to+\infty}\lim_{\sigma\to 0}|I^{\sigma}_{5,n}| \le T\lim_{n\to+\infty}\Big(\int_{\{|u|\ge n\}}|f|dxdt+\int_{\{|v|\ge n\}}|f|dxdt\Big) = 0.$$

Then

$$\lim_{k\to 0}\liminf_{n\to+\infty}\lim_{\sigma\to 0} I^{\sigma}_{5,n} = 0. \tag{2.77}$$

Finally, going back to (2.61) and using Lemma 2.5.1, one may deduce that $u=v$ a.e. in Q_T. □

Part II

Existence results for some nonlinear problems in the Musielak-Orlicz spaces

Preliminaries of Musielak-Sobolev spaces

Let M be a real-valued function defined in $\Omega \times \mathbb{R}_+$ and satisfying conditions:

- $M(x,.)$ is a N-function for all $x \in \Omega$, (i.e. convex, non-decreasing, continuous, $M(x,0) = 0$, $M(x,0) > 0$ for $t > 0$, $\lim_{t\to 0} \sup_{x\in\Omega} \frac{M(x,t)}{t} = 0$ and $\lim_{t\to\infty} \inf_{x\in\Omega} \frac{M(x,t)}{t} = \infty$).
- $M(.,t)$ is a measurable function for all $t \geq 0$.

A function M which satisfies the above conditions is called a Musielak-Orlicz function.

Let $M_x(t) = M(x,t)$, we associate its non-negative reciprocal function M_x^{-1}, with respect to t, that is $M_x^{-1}(M(x,t)) = M(x,M_x^{-1}(t)) = t$.

Let M and P be two Musielak-Orlicz functions, we say that M dominates P, and we write $P \prec M$, near infinity (resp.globally) if there exist two positive constants c and t_0 such that
for a.e. $x \in \Omega$,
$P(x,t) \leq M(x,ct)$ for all $t \geq t_0$ (resp. for all $t \geq 0$). We say that M and P are equivalents, and we write $M \sim P$, if M dominates P and P dominates M. Finally, we say that P grows essentially less rapidly than M at 0 (resp. near infinity), and we write $P \ll M$, for every positive constant c, we have $\lim_{t\to 0} \big(\sup_{x\in\Omega} \frac{P(x,ct)}{M(x,t)}\big) = 0$ (respectively, $\lim_{t\to\infty} \big(\sup_{x\in\Omega} \frac{P(x,ct)}{M(x,t)}\big) = 0\big)$.

Remark 2.5.1. *[36] If $P \ll M$ near infinity, then $\forall \varepsilon > 0$ there exists $k(\varepsilon) > 0$ such that for almost all $x \in \Omega$ we have*

$$P(x,t) \leq k(\varepsilon) M(x,\varepsilon t) \quad \forall t \geq 0.$$

Proposition 2.5.1. *([64]) Let $P \ll M$ near infinity and $\forall t > 0$, $\sup_{x\in\Omega} P(x,t) < \infty$, then $\forall \varepsilon > 0$, $\exists C_\varepsilon > 0$ such that*

$$P(x,t) \leq M(x,\varepsilon t) + C_\varepsilon, \forall t > 0. \tag{2.78}$$

The Musielak-Orlicz space $L_M(\Omega)$ is define as

$$L_M(\Omega) = \{u : \Omega \to \mathbb{R} \quad \text{mesurable} : \quad \rho_{M,\Omega}(\frac{u}{\lambda}) < \infty, \quad \text{for some} \quad \lambda > 0\}.$$

where $\rho_{M,\Omega}(u) = \int_\Omega M(x,|u(x)|)dx$, equipped with the Luxemburg norm

$$\|u\|_M = \inf\big\{\lambda > 0 : \rho_{M,\Omega}(\frac{u}{\lambda}) \leq 1\big\}.$$

Denote $\overline{M}(x,s) = \sup_{t\geq 0}(st - M(x,s))$ the conjugate Musielak-Orlicz function of M.

We define $E_M(\Omega)$ as the subset of $L_M(\Omega)$ of all measurable functions $u : \Omega \mapsto \mathbb{R}$ such that $\rho_{M,\Omega}(\frac{u}{\lambda}) < \infty$ for all $\lambda > 0$. It is a separable space and $(E_M(\Omega))^* = L_{\overline{M}}(\Omega)$.
For two complementary Musielak-Orlicz functions M and $\overline{M}$, we have [35].

- The Young inequality:

$$st \leq M(x,s) + \overline{M}(x,t) \text{ for all } s,t \geq 0\ , x \in \Omega.$$

- The Hölder inequality:

$$\Big|\int_\Omega u(x)v(x)dx\Big| \le \|u\|_{M,\Omega}\||v|\|_{\overline{M},\Omega} \text{ for all } u \in L_M(\Omega), v \in L_{\overline{M}}(\Omega).$$

We define the Musielak-Orlicz-Sobolev space as

$$W^1L_M(\Omega) = \{u \in L_M(\Omega) : D^\alpha u \in L_M(\Omega), \quad \forall |\alpha| \le 1\},$$

endowed with the norm

$$\|u\|^1_{M,\Omega} = \inf\{\lambda > 0 : \sum_{|\alpha|\le 1} \rho_{M,\Omega}(\frac{D^\alpha u}{\lambda}) \le 1\}.$$

Lemma 2.5.3. *([18])(Approximation theorem) Let Ω be a bounded Lipschitz domain in $\mathbb{R}^N$ and let M and $\overline{M}$ be two complementary Musielak-Orlicz functions which satisfy the following conditions:*

1. *There exists a constant $c > 0$ such that $\inf_{x\in\Omega} M(x,1) > c$,*
2. *There exists a constant $A > 0$ such that for all $x, y \in \Omega$ with $|x-y| \le \frac{1}{2}$, we have*

$$\frac{M(x,t)}{M(y,t)} \le |t|^{\left(\frac{A}{\log(\frac{1}{|x-y|})}\right)} \quad \text{for all} \quad t \ge 1,$$

3. $\int_K M(y,\lambda)dx < \infty, \quad \forall \lambda > 0$ *and for every compact* $K \subset \Omega$,
4. *There exists a constant* $C > 0$ *such that* $\overline{M}(y,t) \le C$ *a.e. in* Ω.

Under this assumptions $\mathscr{D}(\Omega)$ is dense in $L_M(\Omega)$ with respect to the modular topology, $\mathscr{D}(\Omega)$ is dense in $W^1_0L_M(\Omega)$ for the modular convergence and $\mathscr{D}(\overline{\Omega})$ is dense in $W^1_0L_M(\Omega)$ for the modular convergence. Consequently, the action of a distribution S in in $W^{-1}L_{\overline{M}}(\Omega)$ on an element u of $W^1_0L_M(\Omega)$ is well defined. It will be denoted by $< S, u >$.

Example 2.5.1. *We give some example for a Musielak-Orlicz functions of approximation theorem*

- $M_1(x,t) = |t|^{p(x)}$ *with* $p : \Omega \to [1,\infty)$ *a measurable function with Log-Hölder continuite*

$$\frac{M_1(x,t)}{M_1(y,t)} = |t|^{p(x)-p(y)} \le t^{\left(\frac{A}{\log(\frac{1}{|x-y|})}\right)} \quad \text{for all} \quad t \ge 1.$$

- $M_2(x,t) = \alpha(x)(\exp(|t|) - 1 + |t|)$, $0 < \alpha(x) \in L^\infty(\Omega)$.

Remark that $M_1 \in \triangle_2$ if $p^+ := \operatorname{ess\,sup}_{x\in\Omega} p(x) < \infty$ while $M_2 \notin \triangle_2$.

We say that a sequence of functions u_n converges to u for the modular convergence in $W^1L_M(\Omega)$ (resp. in $W^1_0L_M(\Omega)$) if, for some $\lambda > 0$, $\lim_{n\to\infty} \overline{\rho}_{M,\Omega}\big(\frac{u_n-u}{\lambda}\big) = 0$.
The following spaces of distributions will also be used

$$W^{-1}L_{\overline{M}}(\Omega) = \big\{f \in \mathscr{D}'(\Omega) : f = \sum_{\alpha\le 1}(-1)^\alpha D^\alpha f_\alpha \quad \text{where} \quad f_\alpha \in L_{\overline{M}}(\Omega)\big\},$$

and

$$W^{-1}E_{\overline{M}}(\Omega) = \big\{f \in \mathscr{D}'(\Omega) : f = \sum_{\alpha\le 1}(-1)^\alpha D^\alpha f_\alpha \quad \text{where} \quad f_\alpha \in E_{\overline{M}}(\Omega)\big\}.$$

The following abstract lemmas will be applied to the truncation operators.

Lemma 2.5.4. *[36] Let $F : \mathbb{R} \to \mathbb{R}$ be uniformly lipschitzian,with $F(0) = 0$. Let M be an Musielak-Orlicz function and let $u \in W_0^1 L_M(\Omega)$(resp.$u \in W^1 E_M(\Omega)$). Then $F(u) \in W^1 L_M(\Omega)$(resp.$u \in W_0^1 E_M(\Omega)$). Moreover, if the set of discontinuity points D of F' is finite, then*

$$\frac{\partial}{\partial x_i} F(u) = \begin{cases} F'(x)\frac{\partial u}{\partial x_i} & a.e.\ in\ \{x \in \Omega;\ u(x) \notin D\}, \\ 0 & a.e.\ in\ \{x \in \Omega;\ u(x) \in D\}. \end{cases}$$

Lemma 2.5.5. *[35] Suppose that Ω satisfies the segment property and let $u \in W_0^1 L_M(\Omega)$. Then, there exists a sequence $u_n \in \mathscr{D}(\Omega)$ such that*

$$u_n \to u \text{ for modular convergence in } W_0^1 L_M(\Omega).$$

Furthermore, if $u \in W_0^1 L_M(\Omega) \cap L^\infty(\Omega)$ then $\|u_n\|_\infty \leq (N+1)\|u\|_\infty$.

Corollary 2.5.1. *[17] Let Ω be a bounded Lipschitz domain and let M be a Musielak-Orlicz function satisfying the conditions of lemma* (2.5.3)*, then $W_0^1 L_M(\Omega) \hookrightarrow\hookrightarrow L_M(\Omega)$.*

Lemma 2.5.6. *[36] If a sequence un $u_n \in L_M(\Omega)$ converges a.e. to u and if u_n remains bounded in $L_M(\Omega)$, then $u \in L_M(\Omega)$ and $u_n \rightharpoonup u$ for $\sigma(L_M(\Omega), E_{\overline{M}}(\Omega))$.*

Lemma 2.5.7. *Let $u_n, u \in L_M(\Omega)$. If $u_n \to u$ with respect to the modular convergence, then $u_n \rightharpoonup u$ for $\sigma(L_M(\Omega), L_{\overline{M}}(\Omega))$.*

Proof. Let $\lambda > 0$ such that $\int_\Omega M(x, \frac{u_n - u}{\lambda})dx \to 0$. Thus, for a subsequence, $u_n \to u$ a.e. in Ω. Take $v \in L_{\overline{M}}(\Omega)$ and multiplying v by a suitable constant, we can assume $\lambda v \in \mathscr{L}_{\overline{M}}(\Omega)$.
By Young's inequality, we have $|(u_n - u)v| \leq M(x, \frac{u_n - u}{\lambda}) + \overline{M}(x, \lambda v)$ which implies, by Vitali's theorem, that $\int_\Omega |(u_n - u)v|dx \to 0$. □

Lemma 2.5.8. *([16])(Modular Poincaré inequality) Under the assumptions of lemma 2.5.3, and by assuming that $M(x,.)$ decreases with respect to one of coordinate of x, there exists a constant $\delta > 0$ which depends only on Ω such that*

$$\int_\Omega M(x, |u|)dx \leq \int_\Omega M(x, \delta|\nabla u|)dx \quad for\ all \quad u \in W_0^1 L_M(\Omega). \tag{2.79}$$

Theorem 2.5.2.
Let M be a Musielak function and define the function ρ_M as

$$\rho_M(f) = \int_\Omega M(x, |f(x)|)dx.$$

1. *If $f_n \to f$ almost everywhere in Ω, then $\rho_M(f) \leq \liminf_{n\to\infty} \rho_M(f_n)$. (Fatou's lemma).*
2. *If $f_n \rightharpoonup f$ weakly in $L_M(\Omega)$, then $\rho_M(f) \leq \liminf_{n\to\infty} \rho_M(f_n)$. (Lower semicontinuity).*
3. *If $|f_n| \nearrow |f|$ almost everywhere in Ω, then $\rho_M(f) = \liminf_{n\to\infty} \rho_M(f_n)$. (Monotone convergence).*
4. *If $f_n \to f$ almost everywhere in Ω, $|f_n| \leq |g|$ almost everywhere, and $\rho_M(\lambda g) < \infty$ for every $\lambda > 0$, then $f_n \rightharpoonup f$ in $L_M(\Omega)$. (Dominated convergence).*

Inhomogeneous Musielak-Orlicz-Sobolev spaces

Let Ω an bounded open subset $\mathbb{R}^N$ and let $Q_T = \Omega\times]0,T[$ with some given $T>0$. Let M be an Musielak-Orlicz function, for each $\alpha \in \mathbb{N}^N$, denote by ∇_x^α the distributional derivative on Q_T of order α with respect to the variable $x \in \mathbb{N}^N$. The inhomogeneous Musielak-Orlicz-Sobolev spaces are defined as follows,

$$W^{1,x}L_M(Q_T) = \{u \in L_M(Q_T) : \nabla_x^\alpha u \in L_M(Q_T), \forall \alpha \in \mathbb{N}^N, |\alpha| \leq 1\},$$

$$W^{1,x}E_M(Q_T) = \{u \in E_M(Q_T) : \nabla_x^\alpha u \in E_M(Q_T), \forall \alpha \in \mathbb{N}^N, |\alpha| \leq 1\}.$$

The last space is a subspace of the first one, and both are Banach spaces under the norm

$$\|u\| = \sum_{|\alpha|\leq m} \|\nabla_x^\alpha u\|_{M,Q_T}.$$

We can easily show that they form a complementary system when Ω satisfies the Lipschitz domain [35]. These spaces are considered as subspaces of the product space $\Pi L_M(Q_T)$ which have as many copies as there is α-order derivatives,$|\alpha| \leq 1$. We shall also consider the weak topologies $\sigma(\Pi L_M, \Pi E_{\overline{M}})$ and $\sigma(\Pi L_M, \Pi L_{\overline{M}})$. If $u \in W^{1,x}L_M(Q_T)$ then the function : $t \mapsto u(t) = u(t,.)$ is defined on $(0,T)$ with values $W^1L_M(\Omega)$. If, further, $u \in W^{1,x}E_M(Q_T)$ then the concerned function is a $W^{1,x}E_M(\Omega)$-valued and is strongly measurable. Furthermore the following imbedding holds $W^{1,x}E_M(\Omega) \subset L^1(0,T,W^{1,x}E_M(\Omega))$.

The space $W^{1,x}L_M(Q_T)$ is not in general separable, if $W^{1,x}L_M(Q_T)$, we can not conclude that the function $u(t)$ is measurable on $(0,T)$. However, the scalar function $t \mapsto \|u(t)\|_{M,\Omega}$, is in $L^1(0,T)$. The space $W_0^{1,x}E_M(Q_T)$ is defined as the (norm) closure $W^{1,x}E_M(Q_T)$ of $\mathscr{D}(Q_T)$. We can easily show as in [35], that when Ω has the segment property, then each element u of the closure of $\mathscr{D}(Q_T)$ with respect of the weak* topology $\sigma(\Pi L_M, \Pi E_{\overline{M}})$ is a limit, in $W_0^{1,x}E_M(Q_T)$, of some subsequence $(u_i) \subset \mathscr{D}(Q_T)$ for the modular convergence, i.e. there exists $\lambda > 0$ such that for all $|\alpha| \leq 1$

$$\int_{Q_T} M(x, \frac{\nabla_x^\alpha u_i - \nabla_x^\alpha u}{\lambda})dx\,dt \to 0 \quad \text{as} \quad i \to \infty.$$

This implies that (u_i) converge to u in $W^{1,x}L_M(Q_T)$ for the topology $\sigma(\Pi L_M, \Pi L_{\overline{M}})$.

Consequently,

$$\overline{\mathscr{D}(Q_T)}^{\sigma(\Pi L_M, \Pi E_{\overline{M}})} = \overline{\mathscr{D}(Q_T)}^{\sigma(\Pi L_M, \Pi L_{\overline{M}})}.$$

This space will be denoted by $W_0^{1,x}L_M(Q_T)$. Furthermore, $W_0^{1,x}E_M(Q_T) = W_0^{1,x}L_M(Q_T) \cap \Pi E_M$.

We have the following complementary system $\begin{pmatrix} W_0^{1,x}L_M(Q_T) & F \\ W_0^{1,x}E_M(Q_T) & F_0 \end{pmatrix}$, F being the dual space of $W_0^{1,x}E_M(Q_T)$. It is also, except for an isomorphism, the quotient of $\Pi L_{\overline{M}}$ by the polar set $W_0^{1,x}E_M(Q_T)^\perp$, and will be denoted by $F = W^{-1,x}L_{\overline{M}}(Q_T)$ and it is show that,

$$W^{-1,x}L_{\overline{M}}(Q_T) = \left\{ f = \sum_{|\alpha|\leq 1} \nabla_x^\alpha f_\alpha : \quad f_\alpha \in L_{\overline{M}}(Q_T) \right\}.$$

This space will be equipped with the usual quotient norm $\|f\| = \inf \sum_{|\alpha|\leq 1} \|f_\alpha\|_{\overline{M},Q_T}$,

where the infimum is taken on all possible decompositions $f = \sum_{|\alpha|\leq 1} \nabla_x^\alpha f_\alpha, \quad f_\alpha \in L_{\overline{M}}(Q_T)$.

The space F_0 is then given by, $F_0 = \left\{ f = \sum_{|\alpha|\leq 1} \nabla_x^\alpha f_\alpha : \quad f_\alpha \in E_{\overline{M}}(Q_T) \right\}$, and is denoted by $F_0 = W^{-1,x} E_{\overline{M}}(Q_T)$.

Definition 2.5.1. *We say that $u_n \to u$ in $W^{-1,x} L_{\overline{M}}(Q_T) + L^1(Q_T)$ for the modular convergence if we can write $u_n = \sum_{|\alpha|\leq 1} D_x^\alpha u_n^\alpha + u_n^0$ and $u = \sum_{|\alpha|\leq 1} D_x^\alpha u^\alpha + u^0$ with $u_n^\alpha \to u^\alpha$ in $L_{\overline{M}}(Q_T)$ for modular convergence for all $|\alpha| \leq 1$, and $u_n^0 \to u^0$ strongly in $L^1(Q_T)$.*

Lemma 2.5.9. *Let $\{u_n\}$ be a bounded sequence in $W^{1,x} L_M(Q_T)$ such that*

$$\frac{\partial u_n}{\partial t} = \alpha_n + \beta_n \text{ in } \mathscr{D}'(Q_T),$$

$$u_n \rightharpoonup u, \quad \text{weakly in} \quad W^{1,x} L_M(Q_T), \quad \text{for} \quad \sigma(\Pi L_M, \Pi E_{\overline{M}}),$$

with $\{\alpha_n\}$ and $\{\beta_n\}$ two bounded sequences respectively in $W^{-1,x} L_{\overline{M}}(Q_T)$ and in $\mathscr{M}(Q_T)$. Then $u_n \to u$ in $L^1_{loc}(Q_T)$. Furthermore, if $u_n \in W_0^{1,x} L_M(Q_T)$, then $u_n \to u$ strongly in $L^1(Q_T)$.

Proof. It is easily adapted from that given in [46] by using Theorem 4.4 and Remark 4.3 instead of Lemma 8 of [100]. □

Theorem 2.5.3. *if $u \in W^{1,x} L_M(Q_T) \cap L^1(Q_T)$ (resp. $W_0^{1,x} L_M(Q_T) \cap L^1(Q_T)$) and $\frac{\partial u}{\partial t} \in W^{-1,x} L_{\overline{M}}(Q_T) + L^1(Q_T)$, then there exists a sequence (v_j) in $\mathscr{D}(\overline{Q}_T)$ (resp. $\mathscr{D}(\overline{I}, \mathscr{D}(Q_T))$) such that $v_j \to u$ in $W^{1,x} L_M(Q_T)$ and $\frac{\partial v_j}{\partial t} \to \frac{\partial u}{\partial t}$ in $W^{-1,x} L_{\overline{M}}(Q_T) + L^1(Q_T)$ for the modular convergence.*

Proof. Let $u \in W^{1,x} L_M(Q_T) \cap L^1(Q_T)$ and $\frac{\partial u}{\partial t} \in W^{-1,x} L_{\overline{M}}(Q_T) + L^1(Q_T)$, then for any $\varepsilon > 0$. Writing $\frac{\partial u}{\partial t} = \sum_{|\alpha|\leq 1} D_x^\alpha u^\alpha + u^0$, where $u^\alpha \in L_{\overline{M}}(Q_T)$ for all $|\alpha| \leq 1$ and $u^0 \in L^1(Q_T)$, we will show that there exits $\lambda > 0$ (depending Only on u and N) and there exists $v \in \mathscr{D}(\overline{Q}_T)$ for which we can write $\frac{\partial u}{\partial t} = \sum_{|\alpha|\leq 1} D_x^\alpha v^\alpha + v^0$ with $v^\alpha, v^0 \in \mathscr{D}(\overline{Q}_T)$ such that

$$\int_{Q_T} M(x, \frac{D_x^\alpha v - D_x^\alpha u}{\lambda}) dxdt \leq \varepsilon, \forall |\alpha| \leq 1, \tag{2.80}$$

$$\|v - u\|_{L^1(Q_T)} \leq \varepsilon, \tag{2.81}$$

$$\|v^0 - u^0\|_{L^1(Q_T)} \leq \varepsilon, \tag{2.82}$$

$$\int_{Q_T} \overline{M}(x, \frac{v^\alpha - u^\alpha}{\lambda}) dxdt \leq \varepsilon, \quad \forall |\alpha| \leq 1. \tag{2.83}$$

The equation (2.80) flows from a slight adaptation of the arguments [35], the equations (2.81)-(2.82) flows also from classical approximation results.
For The equation (2.83) we know that $\mathscr{D}(\overline{Q}_T)$ is dense in $L_{\overline{M}}(Q_T)$ for the modular convergence.

The case where $u \in W_0^{1,x} L_M(Q_T) \cap L^1(Q_T))$ can be handled similarly without essential difficulty as it mentioned [35]. □

Remark 2.5.2. *The assumption $u \in L^1(Q_T)$ in theorem* (2.5.3) *is needed only when Q_T has infinite measure, since else, we have $L_M(Q_T) \subset L^1(Q_T)$ and so $W^{1,x}L_M(Q_T) \cap L^1(Q_T) = W^{1,x}L_M(Q_T)$.*

Remark 2.5.3. *If in the statement of theorem* (2.5.3) *above, one takes $I = \mathbb{R}$, we have that $\mathscr{D}(\Omega \times \mathbb{R})$ is dense in $\{u \in W_0^{1,x}L_M(\Omega \times \mathbb{R}) \cap L^1(\Omega \times \mathbb{R}) : \frac{\partial u}{\partial t} \in W^{-1,x}L_{\overline{M}}(\Omega \times \mathbb{R}) + L^1(\Omega \times \mathbb{R})\}$ for the modular convergence. This trivially follows from the fact that $\mathscr{D}(\mathbb{R}, \mathscr{D}(\Omega)) = \mathscr{D}(\Omega \times \mathbb{R})$.*

Lemma 2.5.10. *Let $a < b \in \mathbb{R}$ and Ω be a bounded open subset of $\mathbb{R}^N$ with the segment property, then $\{u \in W_0^{1,x}L_M(\Omega \times (a,b)) \cap L^1(\Omega \times (a,b)) : \dfrac{\partial u}{\partial t} \in W^{-1,x}L_{\overline{M}}(\Omega \times (a,b)) + L^1(\Omega \times (a,b))\} \subset \mathscr{C}([a,b], L^1(\Omega))$.*

Proof. Let $u \in W_0^{1,x}L_M(\Omega \times (a,b))$ and $\frac{\partial u}{\partial t} \in W^{-1,x}L_{\overline{M}}(\Omega \times (a,b)) + L^1(\Omega \times (a,b))$.
After two consecutive reflections first with respect to $t = b$ and then with respect to $t = a$,

$$\hat{u}(x,t) = u(x,t)\chi_{(a,b)} + u(x,2b-t)\chi_{(b,2b-a)} \qquad \text{in} \qquad \Omega \times (b, 2b-a),$$

and

$$\tilde{u}(x,t) = \hat{u}(x,t)\chi_{(a,2b-a)} + \hat{u}(x,2a-t)\chi_{(3a-2b,a)} \qquad \text{in} \qquad \Omega \times (3a-2b, 2b-a).$$

We get function $\tilde{u} \in W_0^{1,x}L_M(\Omega \times (3a-2b, 2b-a))$ with $\frac{\partial \tilde{u}}{\partial t} \in W^{-1,x}L_{\overline{M}}(\Omega \times (3a-2b, 2b-a)) + L^1(\Omega \times (3a-2b, 2b-a))$. Now by letting a function $\eta \in \mathscr{D}(\mathbb{R})$ with $\eta = 1$ on $[a,b]$ and $\operatorname{supp}(\eta) \subset (3a-2b, 2b-a)$, we set $\overline{u} = \eta\tilde{u}$, therefore, by standard arguments (see [51]), we have $\overline{u} = u$ on $(\Omega \times (a,b))$, $\overline{u} \in W_0^{1,x}L_M(\Omega \times \mathbb{R}) \cap L^1(\Omega \times \mathbb{R})$ and $\frac{\partial \overline{u}}{\partial t} \in W_0^{-1,x}L_{\overline{M}}(\Omega \times \mathbb{R}) + L^1(\Omega \times \mathbb{R})$. Let now v_j the sequence given by theorem (2.5.3) corresponding to $\overline{u}$, that is,

$$v_j \to \overline{u} \qquad \text{in} \qquad W_0^{1,x}L_M(\Omega \times \mathbb{R}),$$

and

$$\frac{\partial v_j}{\partial t} \to \frac{\partial \overline{u}}{\partial t} \qquad \text{in} \qquad W_0^{-1,x}L_{\overline{M}}(\Omega \times \mathbb{R}) + L^1(\Omega \times \mathbb{R}),$$

for the modular convergence.

If we denote $S_k(s) = \displaystyle\int_0^s T_k(t)dt$ the primitive of T_k. We have,

$$\int_\Omega S_1(v_i - v_j)(\tau)dx = \int_\Omega \int_{-\infty}^r T_1(v_i - v_j)(\frac{\partial v_i}{\partial t} - \frac{\partial v_j}{\partial t})dxdt \to 0 \qquad \text{as} \qquad i, j \to 0,$$

from which, one deduces that v_j is a Cauchy sequence in $C(\mathbb{R}; L^1(\Omega))$ and hence $\overline{u} \in C(\mathbb{R}, L^1(\Omega))$. Consequently, $u \in C([a;b]; L^1(\Omega))$. □

3

Nonlinear elliptic equations with some measure data in Musielak-Orlicz spaces

Consider the following nonlinear Dirichlet problem:

$$A(u)+K(x,u,\nabla u)=f, \tag{3.1}$$

on a Lipschitz bounded domain Ω in $\mathbb{R}^N$. $A(u)=-\mathrm{div}(a(x,u,\nabla u))$ is a Leary-Lions operator defined on $D(A)\subset W_0^1L_M(\Omega)\to W^{-1}L_{\overline{M}}(\Omega)$ where M and $\overline{M}$ are two complementary Musielak-Orlicz functions and K is a nonlinear lower term which satisfies the growth condition and without the sign condition. In the framework of Sobolev spaces with variable exponents (the Musielak-Orlicz-function is $M(x,t)=|t|^{p(x)}$), a series of papers on nonlinear elliptic and parabolic equations without sign condition in the nonlinearity studied (see [22], [28]).

On the Orlicz spaces, many papers were devoted to the existence of solutions of (3.1). In fact, Gossez J.P. [76] solved the problem in the variational case, Elmahi A. et al. [60] proved the existence results for the unilateral problem of (3.1), where K satisfies the growth condition and the right-hand side belongs to $L^1(\Omega)$. Recently, Dong G. et al. in [57] have taken the source term as a bounded nonnegative Radon measure on Ω.

The study of nonlinear partial differential equations is motivated by numerous phenomena of physics, namely, the electrorheological fluids, the flow thought the porous media (see the monograph of A. Antsenov [26]).

As an example of operator for which the present result can be applied, we give

$$-div\Big(\frac{m(x,|\nabla u|).\nabla u}{|\nabla u|}\Big)+uM(x,|\nabla u|)=f,$$

where $m(x,s)$ is the derivative of $M(x,s)$ with respect to s.

The aim of this chapter is the study of the problem (3.1) in the setting of Musielak-Orlicz spaces overcoming two difficulties. Firstly, we do not assume the sign condition on the nonlinearity K, after we prove that there exist at least one solution for approximate equations. Secondly we show that solutions belongs to the Musielak-Sobolev spaces $W_0^1L_\phi(\Omega)$ where ϕ is in a special class of Musielak-Orlicz functions of $\mathscr{A}_M$ (see Definition 3.1.1).

3.1 Technical Lemmas

Throughout this chapter, we assume also that every Musielak-Orlicz function $M(.,.)$ is decreasing in x in the following sense.
For any $x \in \Omega$, let $\Omega_x = \{s \in \Omega / \|x\| \geq \|s\|\}$ and

$$\begin{cases} M(x,t) \leq M(s,t) & \text{if} \quad s \in \Omega_x, \\ M(x,t) = M(s,t) & \text{if} \quad s \in \Omega \backslash \Omega_x, \end{cases} \tag{3.2}$$

for any $t \in \mathbb{R}$.

Definition 3.1.1. *Let M be a Musielak-Orlicz function. We define the following set:*

$$\mathscr{A}_M = \left\{ \begin{array}{l} \phi : \Omega \times \mathbb{R}_+ \to \mathbb{R}_+ \quad \text{is a Musielak-Orlicz function such that} \\ \phi \ll M \quad \text{and} \quad \displaystyle\int_0^1 \phi(x, \beta H^{-1}(x, \frac{1}{r^{1-\frac{1}{N}}}))dr < \infty \quad \text{a.e. in } \Omega \end{array} \right\}$$

for all constant $\beta \geq 1$, where $H(x,r) = \dfrac{M(x,r)}{r}$.

The following lemma generalizes Lemma 2 in [102].

Lemma 3.1.1. *Let Ω be a open subset of $\mathbb{R}^N$ with finite measure.*
Let M be a Musielak-Orlicz function under assumption (3.2) *and the assumptions of Lemma 2.5.3.*
For any $u \in W_0^1 L_M(\Omega)$ such that $\displaystyle\int_\Omega M(x,|\nabla u|)dx < \infty$, we have for all $x \in \Omega$,

$$-\mu'(t) \geq N C_N^{\frac{1}{N}} \mu(t)^{1-\frac{1}{N}} C\Big(x, \frac{-1}{C_N^{\frac{1}{N}} \mu(t)^{1-\frac{1}{N}}} \frac{d}{dt} \int_{\{|u|>t\}} M(s,|\nabla u|)ds\Big) \tag{3.3}$$

for a.e. $t > 0$. Here μ is the distribution function of u, and the function $C(.,.)$ is defined by $C(x,t) = \frac{1}{H_x^{-1}(x,t)}$ with $H(x,t) = \frac{M(x,t)}{t}$, C_N is the measure of the unit ball of $\mathbb{R}^N$, and $\mu(t) = meas\{|u| > t\}$.

Proof. By definition of the Musielak-Orlicz function, M is an increasing convex function in t, then H is an increasing convex function in t, and $C(.,.)$ is a decreasing convex function in t.
Fix $x \in \Omega$. Jensen's inequality for a convex function gives

$$C\left(x, \frac{\displaystyle\int_{\{t<|u|\leq t+h\}} M(s,|\nabla u|)\,ds}{\displaystyle\int_{\{t<|u|\leq t+h\}} |\nabla u|\,ds}\right) = C\left(x, \frac{\displaystyle\int_{\{t<|u|\leq t+h\}} H(s,|\nabla u|)|\nabla u|\,ds}{\displaystyle\int_{\{t<|u|\leq t+h\}} |\nabla u|\,ds}\right)$$

$$\leq \frac{\displaystyle\int_{\{t<|u|\leq t+h\}} C\Big(x, H(s,|\nabla u|)\Big)|\nabla u|ds}{\displaystyle\int_{\{t<|u|\leq t+h\}} |\nabla u|\,ds}$$

$$\leq \frac{\displaystyle\int_{\{t<|u|\leq t+h\}\cap\Omega_x} C\Big(x, H(s,|\nabla u|)\Big)|\nabla u|ds}{\displaystyle\int_{\{t<|u|\leq t+h\}} |\nabla u|\,ds}$$

$$+\frac{\displaystyle\int_{\{t<|u|\le t+h\}\cap(\Omega\setminus\Omega_x)} C\Big(x,H(s,|\nabla u|)\Big)|\nabla u|\,ds}{\displaystyle\int_{\{t<|u|\le t+h\}}|\nabla u|ds}.$$

By (3.2) and for all $t>0$ we have
for $s\in\Omega_x$, $H(x,|\nabla u|)\le H(s,|\nabla u|)$ and $|\nabla u|\le H_x^{-1}(H(s,|\nabla u|))$ then
$C(x,H(s,|\nabla u|))=\frac{1}{H_x^{-1}(H(s,|\nabla u|))}\le\frac{1}{|\nabla u|}$,
for $s\in\Omega\setminus\Omega_x$, $H(s,|\nabla u|)=H(x,|\nabla u|)$ and $C(x,H(s,|\nabla u|))=\frac{1}{|\nabla u|}$.
Hence

$$C\left(x,\frac{\displaystyle\int_{\{t<|u|\le t+h\}}M(s,|\nabla u|)\,ds}{\displaystyle\int_{\{t<|u|\le t+h\}}|\nabla u|\,ds}\right)\le\frac{-\mu(t+h)+\mu(t)}{\displaystyle\int_{\{t<|u|\le t+h\}}|\nabla u|\,ds},$$

letting $h\to 0$, we have

$$C\left(x,\frac{(\frac{d}{dt})\displaystyle\int_{\{|u|>t\}}M(s,|\nabla u|)\,ds}{(\frac{d}{dt})\displaystyle\int_{\{|u|>t\}}|\nabla u|\,ds}\right)\le\frac{\mu^{'}(t)}{(\frac{d}{dt})\displaystyle\int_{\{|u|>t\}}|\nabla u|\,ds}\tag{3.4}$$

for all $t>0$.
On the other hand we can follow [71] to prove that

$$-\frac{d}{dt}\int_{\{|u|>t\}}|\nabla u|\,dx\ge NC_N^{\frac{1}{N}}\mu(t)^{1-\frac{1}{N}}\tag{3.5}$$

for a.e. $t>0$. Finally, combining (3.4), (3.5) and monotony of $C(.,.)$ we get (3.3). □

Lemma 3.1.2. *Let M be a Musielak-Orlicz function under assumption* (3.2) *and the assumptions of Lemma 2.5.3 and $\phi\in\mathscr{A}_M$ with $\phi\sim M$, there exists a constant $\beta\ge 1$ such that*

$$\frac{d}{dt}\int_{\{|u|>t\}}\phi(s,|\nabla u|)\,ds\le-\mu^{'}(t)\phi\left(x,\beta H_x^{-1}\Big(\frac{1}{NC_N^{\frac{1}{N}}\mu(t)^{1-\frac{1}{N}}}\frac{d}{dt}\int_{\{|u|>t\}}M(s,|\nabla u|)\,ds\Big)\right)$$

for each $x\in\Omega$, and for any $u\in W_0^1L_M(\Omega)$ such that $\displaystyle\int_\Omega M(x,|\nabla u|)dx<\infty$.

Proof. For $x\in\Omega$, let $C(x,t)=\dfrac{1}{H_x^{-1}(x,t)}$, then $C(x,t)=\dfrac{t}{M\circ H_x^{-1}(x,t)}$.
By (3.3), we have

$$-\mu^{'}(t)\ge NC_N^{\frac{1}{N}}\mu(t)^{1-\frac{1}{N}}C\left(x,\frac{-1}{NC_N^{\frac{1}{N}}\mu(t)^{1-\frac{1}{N}}}\frac{d}{dt}\int_{\{|u|>t\}}M(s,|\nabla u|)\,ds\right),$$

then

$$-\mu^{'}(t)M\circ H_x^{-1}\left(\frac{-1}{NC_N^{\frac{1}{N}}\mu(t)^{1-\frac{1}{N}}}\frac{d}{dt}\int_{\{|u|>t\}}M(s,|\nabla u|)\,ds\right)$$
$$\ge NC_N^{\frac{1}{N}}\mu(t)^{1-\frac{1}{N}}\Big(-\frac{1}{NC_N^{\frac{1}{N}}\mu(t)^{1-\frac{1}{N}}}\frac{d}{dt}\int_{\{|u|>t\}}M(s,|\nabla u|)\,ds\Big)$$

$$-\mu^{'}(t)M\circ H_x^{-1}\Big(\frac{-1}{NC_N^{\frac{1}{N}}\mu(t)^{1-\frac{1}{N}}}\frac{d}{dt}\int_{\{|u|>t\}}M(s,|\nabla u|)ds\Big)\geq-\frac{d}{dt}\int_{\{|u|>t\}}M(s,|\nabla u|)\,ds,$$

and also

$$\frac{1}{\mu^{'}(t)}\frac{d}{dt}\int_{\{|u|>t\}}M(s,|\nabla u|)\,ds\leq M\circ H_x^{-1}\left(\frac{-1}{NC_N^{\frac{1}{N}}\mu(t)^{1-\frac{1}{N}}}\frac{d}{dt}\int_{\{|u|>t\}}M(s,|\nabla u|)\,ds\right),$$

using the monotony of the function M_x^{-1}, we obtain

$$M_x^{-1}\Big(\tfrac{1}{\mu^{'}(t)}\tfrac{d}{dt}\int_{\{|u|>t\}}M(s,|\nabla u|)\,ds\Big)\leq H_x^{-1}\Big(\tfrac{-1}{NC_N^{\frac{1}{N}}\mu(t)^{1-\frac{1}{N}}}\tfrac{d}{dt}\int_{\{|u|>t\}}M(s,|\nabla u|)\,ds\Big).$$

Let $\phi\in\mathscr{A}_M$ and let $D(x,t)=M(x,\phi_x^{-1}(t))$, then D is convex and by Jensen's inequality we have

$$D\Big(x,\frac{\int_{\{t<|u|<t+h\}}\phi(s,|\nabla u|)\,ds}{-\mu(t+h)+\mu(t)}\Big)\leq\frac{\int_{\{t<|u|<t+h\}}D\Big(x,\phi(s,|\nabla u|)\Big)\,ds}{-\mu(t+h)+\mu(t)}.$$

Since $\phi\sim M$, there exists a constant $\lambda>0$ such that $M(x,t)\leq\lambda\phi(x,t)$.
Then for every $\phi\in\mathscr{A}_M$ with $\lambda\leq 1$ and by the monotony of the functions M_x and ϕ_x^{-1} for any x and s in Ω, we have

$$D\left(x,\phi(s,|\nabla u|)\right)=M\Big(x,\phi_x^{-1}(\phi(s,|\nabla u|))\Big)\leq\phi(s,|\nabla u|),$$

and by Remark 2.5.1, there exists $\beta>0$ such that $D\Big(x,\phi(s,|\nabla u|)\Big)\leq\beta M(s,|\nabla u|)$,
then

$$D\left(x,\frac{\int_{\{t<|u|<t+h\}}\phi(s,|\nabla u|)\,ds}{-\mu(t+h)+\mu(t)}\right)\leq\frac{\beta\int_{\{t<|u|<t+h\}}M\Big(s,|\nabla u|)\Big)\,ds}{-\mu(t+h)+\mu(t)},$$

using the definition of D(.,.) and the monotony of M_x^{-1} we have

$$\begin{aligned}\phi_x^{-1}\left(x,\frac{1}{\mu^{'}(t)}\frac{d}{dt}\int_{\{|u|>t\}}\phi(s,|\nabla u|)\,ds\right)&\leq\beta M_x^{-1}\left(x,\frac{1}{\mu^{'}(t)}\frac{d}{dt}\int_{\{|u|>t\}}M(s,|\nabla u|)\,ds\right),\\&\leq\beta H_x^{-1}\left(\frac{-1}{C_N^{\frac{1}{N}}\mu(t)^{1-\frac{1}{N}}}\frac{d}{dt}\int_{\{|u|>t\}}M(s,|\nabla u|)\,ds\right),\end{aligned}$$

which gives our result. □

3.2 Essential Assumptions

Let M and γ be two Musielak-Orlicz functions such that M and its complementary $\overline{M}$ satisfy the previous conditions and $\gamma\ll M$.

$A:D(A)\subset W_0^1L_M(\Omega)\to W^{-1}L_{\overline{M}}(\Omega)$ is defined by $A(u)=-\mathrm{div}(a(x,u,\nabla u))$, where $a:\Omega\times\mathbb{R}\times\mathbb{R}^N\to\mathbb{R}^N$ is a Carathéodory function such that for a.e. $x\in\Omega$ and for all $s\in\mathbb{R}$, $\xi,\xi^*\in\mathbb{R}^N$, $\xi\neq\xi^*$.

$$|a(x,s,\xi)|\leq\beta(c(x)+\overline{M}_x^{-1}(\gamma(x,\nu_1|s|))+\overline{M}_x^{-1}(M(x,\nu_2|\xi|))),\beta>0,c(x)\in E_{\overline{M}}(\Omega),\qquad(3.6)$$

$$(a(x,s,\xi)-a(x,s,\xi^*))(\xi-\xi^*)>0, \tag{3.7}$$

$$a(x,s,\xi).\xi \geq \alpha M(x,|\xi|), \tag{3.8}$$

$K(x,s,\xi):\Omega\times\mathbb{R}\times\mathbb{R}^N\to\mathbb{R}^N$ is a Carathéodory function such that

$$|K(x,s,\xi)|\leq b(x)+\rho(s)M(x,|\xi|), \tag{3.9}$$

$\rho:\mathbb{R}\to\mathbb{R}^+$ is a continuous positive function which belongs to $L^1(\mathbb{R})$ and b(x) belongs to $L^1(\Omega)$.

$$\mu\in\mathscr{M}_b(\Omega), \tag{3.10}$$

assume that there exists $\phi\in\mathscr{A}_M$ such that

$$\phi\circ H^{-1} \text{ is a Musielak-Orlicz function.} \tag{3.11}$$

3.3 Main Results

Let Ω be an open bounded subset of $\mathbb{R}^N$ $(N\geq 2)$, and let M and $\overline{M}$ be two complementary Musielak Orlicz functions. Define the set

$$\mathscr{T}_0^{1,M}(\Omega)=\left\{u:\Omega\mapsto\mathbb{R} \quad \text{is measurable and} \, T_k(u)\in D(A)\right\}.$$

Theorem 3.3.1. *Assume that* $(3.6)-(3.10)$ *hold true with* $\mathscr{A}_M\neq\emptyset$. *Then there exists at least one solution of the following problem:*

$$\begin{cases} u\in\mathscr{T}_0^{1,M}(\Omega)\cap W_0^1L_\phi(\Omega), \quad \forall\phi\in\mathscr{A}_M, \\ <A(u),v>+\displaystyle\int_\Omega K(x,u,\nabla u)v\,dx=<\mu,v>, \quad \forall v\in\mathscr{D}(\Omega). \end{cases} \tag{3.12}$$

Example 3.3.1.
We give an example of equations to which the present result can be applied

1. *We give an example of the Musielak-Orlicz-functions M for which the set* $\mathscr{A}_M$ *is not empty. Let* $a(.),b(.)$ *be two functions in* $L^\infty(\Omega)$ *such that* $a(.),b(.)$ *are decreasing strict positive and there exist two constants* $\lambda_1>0,\lambda_2>0$ *such that* $\lambda_1\leq\frac{a(x)}{b(x)}\leq\lambda_2$. *Take now* $M(x,t)=a(x)|t|^p$ *and* $\phi(x,t)=b(x)|t|^p$ *such that* $p>N$, *then* $\phi\in\mathscr{A}_M$.

2. *Let us take the functions mentioned above and consider the following problem:*

$$\begin{cases} div(a(x)|\nabla u|^{p-2}\nabla u)+b(x)+\rho(u)M(x,|\nabla u|)=\mu, & \text{in } \Omega, \\ u=0 & \text{on } \partial\Omega. \end{cases}$$

Here $a(x,u,\nabla u)=a(x)|\nabla u|^{p-2}\nabla u$ *satifies the hypotheses (3.6)-(3.8),* $K(x,u,\nabla u)=b(x)+\rho(u)M(x,|\nabla u|)$, *where* $\rho:\mathbb{R}\to\mathbb{R}^+$ *is a continuous positive function which belongs to* $L^1(\mathbb{R})$ *and* $\mu\in\mathscr{M}_b(\Omega)$.

Proof of theorem.

Step 1: Existence of weak solutions for approximate problems.

We consider the following approximate equation for any $n \in \mathbb{N}$,

$$\int_\Omega [a(x,u,\nabla u)\nabla v + K_n(x,u,\nabla u)v]dx = \int_\Omega \mu_n v dx, \quad \forall v \in W_0^1 L_M(\Omega), \tag{3.13}$$

where $K_n(x,u,\nabla u) = \dfrac{K(x,u,\nabla u)}{1+\frac{1}{n}|K(x,u,\nabla u)|}$, and $(\mu_n)_n \in \mathscr{D}(\Omega)$ is a sequence such that

$$\mu_n \to \mu \quad \text{in the sense of the distributions.} \tag{3.14}$$

We will prove that, for every n, there exists at least one bounded solution u_n of (3.13) with $u_n \in W_0^1 E_M(\Omega)$.

Proposition 3.3.1. *Let M and $\overline{M}$ be two complementary Musielak Orlicz functions satisfying the conditions of Lemma 2.5.3, assume that (3.6)-(3.10) hold, then, for any $n \in \mathbb{N}^*$, there exists at least one solution $u_n \in W_0^1 E_M(\Omega)$ of (3.13).*

Proof. See [57]. □

Step 2: Consider the following approximate problems:

$$\begin{cases} u_n \in \mathscr{T}_0^{1,M}(\Omega) \cap W_0^1 E_M(\Omega) \\ < A(u_n), v > + \displaystyle\int_\Omega K_n(x,u_n,\nabla u_n)v dx = < \mu_n, v >, \quad \forall v \in W_0^1 L_M(\Omega). \end{cases} \tag{3.15}$$

By proposition 3.3.1, there exists at least one solution u_n of (3.15).

Lemma 3.3.1. *Let u_n be a solution of the approximate problem (3.13), then*

1. *for all $k > 0$, there exists a constant C_i, $(i = 1,2)$ (which does not depend on n and k) such that*

$$\int_\Omega a(x,T_k(u_n),\nabla T_k(u_n))\nabla T_k(u_n)dx \le kC_1, \tag{3.16}$$

and

$$\int_\Omega M(x,|\nabla T_k(u_n)|)dx \le kC_2. \tag{3.17}$$

2. *There exists a measurable function u such that*

$$u_n \to u \quad a.e.\ in \quad \Omega. \tag{3.18}$$

3.

$$a(x,T_k(u_n),\nabla T_k(u_n)) \rightharpoonup \varpi_k \quad weakly\ in \quad \left(L_{\overline{M}}(\Omega)\right)^N \quad for \quad \sigma(\Pi L_{\overline{M}}, \Pi E_M). \tag{3.19}$$

Proof of lemma 3.3.1

(1) Let $v_0 \in W_0^1 L_M(\Omega) \cap L^\infty(\Omega)$ with $v_0 \geq 0$.

On the one hand, taking $\exp(G(u_n))v_0$ as a test function in 3.13, where $G(s) = \int_0^s \frac{1}{\alpha}\rho(r)dr$, we obtain

$$\int_\Omega a(x,u_n,\nabla u_n)\exp(G(u_n))\frac{\rho(u_n)}{\alpha}\nabla u_n v_0 dx + \int_\Omega a(x,u_n,\nabla u_n)\exp(G(u_n))\nabla v_0 dx$$

$$+ \int_\Omega K_n(x,u_n,\nabla u_n)\exp(G(u_n))v_0 dx = \int_\Omega \mu_n \exp(G(u_n))v_0 dx,$$

by (3.8) and (3.9) we simplify by the term $\int_\Omega \rho(u_n)M(x,|\nabla u_n|)v_0 dx$ and we have

$$\int_\Omega a(x,u_n,\nabla u_n)\exp(G(u_n))\nabla v_0 dx \leq \int_\Omega \mu_n \exp(G(u_n))v_0 dx + \int_\Omega b(x)\exp(G(u_n))v_0 dx. \quad (3.20)$$

On the other hand, taking $\exp(-G(u_n))v_0$ as a test function in (3.13), we deduce also

$$\int_\Omega a(x,u_n,\nabla u_n)\exp(-G(u_n))\nabla v_0 dx + \int_\Omega b(x)\exp(-G(u_n))v_0 dx \geq \int_\Omega \mu_n \exp(-G(u_n))v_0 dx. \quad (3.21)$$

By choosing $v_0 = T_k(u_n)^+$ in (3.20), we otain

$$\int_\Omega a(x,u_n,\nabla u_n)\exp(G(u_n)))\nabla T_k(u_n)^+ dx \leq \int_\Omega \mu_n \exp(G(u_n)))T_k(u_n)^+ dx + \int_\Omega b(x)\exp(G(u_n)))T_k(u_n)^+ dx.$$

Since $\rho \in L^1(\mathbb{R})$, we see that $G(-\infty) \leq G(s) \leq G(+\infty)$ and $|G(\pm\infty)| \leq \frac{1}{\alpha}\|\rho\|_{L^1(\mathbb{R})}$, then we have

$$\int_\Omega a(x,T_k(u_n),\nabla T_k(u_n))\nabla T_k(u_n)^+ dx \leq \exp\big(\frac{\|\rho\|_{L^1(\mathbb{R})}}{\alpha}\big)k[\|\mu\|_{\mathscr{M}_b(\Omega)} + \|b\|_{L^1(\Omega)}] = kC_3,$$

and using (3.8) we get

$$\int_\Omega M(x,|\nabla T_k(u_n)^+|)dx \leq kC_4.$$

Choosing again $v_0 = T_k(u_n)^-$ in (3.21) we get

$$-\int_{\{-k\leq u_n\leq 0\}} a(x,u_n,\nabla u_n)\exp(-G(u_n)))\nabla u_n dx + \int_\Omega b(x)\exp(-G(u_n)))T_k(u_n)^- dx$$

$$\geq \int_\Omega \mu_n \exp(-G(u_n)))T_k(u_n)^- dx.$$

Similarly we obtain

$$\int_{\{-k\leq u_n\leq 0\}} a(x,T_k(u_n),\nabla T_k(u_n))\nabla u_n dx \leq \exp\big(\frac{\|\rho\|_{L^1(\mathbb{R})}}{\alpha}\big)k[\|\mu\|_{\mathscr{M}_b(\Omega)} + \|b\|_{L^1(\Omega)}]$$

$$= kC_3.$$

and by (3.8) we have

$$\int_\Omega M(x,|\nabla T_k(u_n)^-|)dx \leq kC_5.$$

We deduce respectively the results (3.17) and (3.16).

(2) Using (2.79) we have

$$\inf_{x\in\Omega} M(x,\frac{k}{\delta})meas\{|u_n|>k\} \leq \int_{|u_n|>k} M(x,\frac{|T_k(u_n)|}{\delta})dx$$

$$\leq \int_\Omega M(x,|\nabla T_k(u_n)|)dx \leq kC_6.$$

Then

$$meas\{|u_n|>k\} \leq \frac{C_6 k}{\inf_{x\in\Omega} M(x,\frac{k}{\delta})}$$

for all n and for all k.

Assume that there exists a positive function Ψ such that $\lim_{t\to\infty}\frac{\Psi(t)}{t}=+\infty$ and $\Psi(t)\leq ess\inf_{x\in\Omega} M(x,t)$, $\forall t\geq 0$. Thus, we get

$$\lim_{k\to\infty} meas\{|u_n|>k\}=0.$$

By the property (1) of Lemma 3.3.1, we deduce that $T_k(u_n)$ is bounded in $W_0^1 L_M(\Omega)$ and then there exists some $\tau_k\in W_0^1 L_M(\Omega)$ such that $T_k(u_n)\rightharpoonup \tau_k$ weakly in $W_0^1 L_M(\Omega)$ for $\sigma(\Pi L_M,\Pi E_{\overline{M}})$, strongly in $E_M(\Omega)$ and a.e. in Ω And by (2) of Lemma 3.3.1, the sequence $(u_n)_n$ converges almost everywhere to some measurable function u.

Then we have

$$T_k(u_n)\rightharpoonup T_k(u) \quad \text{weakly in} \quad W_0^1 L_M(\Omega) \quad \text{for} \quad \sigma(\Pi L_M,\Pi E_{\overline{M}}),$$

$$\text{strongly in} \quad E_M(\Omega) \quad \text{and a.e. in} \quad \Omega.$$

(3) We shall prove that $\{a(x,T_k(u_n),\nabla T_k(u_n))\}_n$ is bounded in $\left(L_{\overline{M}}(\Omega)\right)^N$ for all $k>0$.
Let $w\in (E_M(\Omega))^N$ be arbitrary. By (3.7) we have

$$(a(x,u_n,\nabla u_n)-a(x,u_n,w))(\nabla u_n - w)\geq 0.$$

Then

$$\int_{\{|u_n|\leq k\}} a(x,u_n,\nabla u_n)w dx \leq \int_{\{|u_n|\leq k\}} a(x,u_n,\nabla u_n)\nabla u_n dx + \int_{\{|u_n|\leq k\}} a(x,u_n,w)(w-\nabla u_n)dx.$$

By (3.6) and according to Remark 2.5.1 there exists $k^{'}>0$ such that $\gamma(x,\nu_1 k)\leq k^{'}M(x,1)$ and for $\lambda>0$ being large enough

$$\int_{\{|u_n|\leq k\}} \overline{M}(\frac{a(x,u_n,\frac{w}{\nu_2})}{3\beta})dx \leq \frac{1}{3}[\int_\Omega \overline{M}(c(x))dx + \int_\Omega k^{'}M(x,1)dx + \int_\Omega M(x,w)dx] \leq C_6. \quad (3.22)$$

Thus $\{a(x,T_k(u_n),\frac{w}{\nu_2})\}$ is bounded in $(L_{\overline{M}}(\Omega))^N$, by (3.22), (3.16) and in view of the Banach-Steinhaus theorem, the sequence $\{a(x,T_k(u_n),\nabla T_k(u_n))\}$ remains bounded in $(L_{\overline{M}}(\Omega))^N$ and for a subsequence

$$a(x,T_k(u_n),\nabla T_k(u_n))\rightharpoonup \varpi_k \quad \text{weakly in} \quad \left(L_{\overline{M}}(\Omega)\right)^N \quad \text{for} \quad \sigma(\Pi L_{\overline{M}},\Pi E_M).$$

Step 3:Almost everywhere convergence of the gradients.

To have that the gradient converges almost everywhere, we need to prove the following proposition.

Proposition 3.3.2. *Let $\{u_n\}_n$ be a solution of the approximate problem 3.13, then*

1.

$$\lim_{m\to\infty}\limsup_{n\to\infty}\int_{\{-(m+1)\le u_n\le -m\}} a(x,u_n,\nabla u_n)\nabla u_n dx = 0; \tag{3.23}$$

2. for a subsequence as $n\to\infty$

$$\nabla u_n \to \nabla u \quad a.e.\ in \quad \Omega. \tag{3.24}$$

Proof.

(1) Take the function $v_0 = T_1(u_n - T_m(u_n))^-$ in (3.21), this function is admissible since $v_0 \in W_0^1 L_M(\Omega)\cap L^\infty(\Omega)$, and $v_0 \ge 0$, then we have

$$-\int_\Omega a(x,u_n,\nabla u_n)\exp(-G(u_n))\nabla T_1(u_n - T_m(u_n))^- dx \le \int_\Omega b(x)\exp(-G(u_n))T_1(u_n - T_m(u_n))^- dx,$$

since μ is nonnegative. We get

$$\int_{\{-(m+1)\le u_n\le -m\}} a(x,u_n,\nabla u_n)\exp(-G(u_n)))\nabla u_n dx \le \int_\Omega b(x)\exp(-G(u_n)))T_1(u_n - T_m(u_n))^- dx$$

$$\le \exp(\frac{\|\rho\|_{L^1_{I\!R}}}{\alpha})\int_\Omega |b(x)|T_1(u_n - T_m(u_n))^- dx.$$

By Lebesgue's theorem, we conclude the result (3.23).

(2) To show that $\nabla u_n \to \nabla u$ a.e. in Ω is true, simply adapt the proof from the chapter 1 and follow the same steps by taking $\Phi = 0$ and $\mu = f \in L^1(\Omega)$.

Step 4: Equi-integrability of the nonlinearity sequence.

We shall prove that

$$K_n(x,u_n,\nabla u_n) \to K(x,u,\nabla u) \quad \text{strongly in} \quad L^1(\Omega). \tag{3.25}$$

Consider $v_0 = \int_0^{u_n}\rho(s)\chi_{\{s>h\}}dx$ in (3.20), we get

$$\int_\Omega a(x,u_n,\nabla u_n)\exp(G(u_n))\nabla v_0 dx \le \int_\Omega \mu_n \exp(G(u_n))v_0 dx + \int_\Omega b(x)\exp(G(u_n))v_0 dx.$$

Then using (3.8) and (3.9) we have

$$\alpha\int_{\{u_n>h\}}\rho(u_n)M(x,\nabla u_n)dx \le (\int_h^{+\infty}\rho(s)dx)\exp\big(\frac{\|\rho\|_{L^1(I\!R)}}{\alpha}\big)[\|\mu\|_{\mathscr{M}_b(\Omega)} + \|b\|_{L^1(\Omega)}],$$

then

$$\int_{\{u_n>h\}}\rho(u_n)M(x,\nabla u_n)dx \le \frac{C_3}{\alpha}(\int_h^{+\infty}\rho(s)dx).$$

Since $\rho \in L^1(I\!R)$, we get

$$\lim_{h\to+\infty}\sup_{n\in I\!N}\int_{\{u_n>h\}}\rho(u_n)M(x,\nabla u_n)dx = 0.$$

Similarly, let $v_0 = \int_{u_n}^{0} \rho(s)\chi_{\{s<-h\}}dx$ in (3.21), we have also

$$\lim_{h\to+\infty}\sup_{n\in\mathbb{N}}\int_{\{u_n<-h\}}\rho(u_n)M(x,\nabla u_n)dx = 0.$$

We conclude that

$$\lim_{h\to+\infty}\sup_{n\in\mathbb{N}}\int_{\{|u_n|>h\}}\rho(u_n)M(x,\nabla u_n)dx = 0. \tag{3.26}$$

Let $D \subset \Omega$, then

$$\int_D \rho(u_n)M(x,\nabla u_n)dx \leq \max_{\{|u_n|\leq h\}}(\rho(x))\int_{D\cap\{|u_n|\leq h\}} M(x,\nabla u_n)dx + \int_{D\cap\{|u_n|>h\}}\rho(u_n)M(x,\nabla u_n)dx.$$

Consequently, $\rho(u_n)M(x,\nabla u_n)$ is equi-integrable, and since $\rho(u_n)M(x,\nabla u_n)$ converges to $\rho(u)M(x,\nabla u)$ strongly in $L^1(\mathbb{R})$, we get our result.

Step 5: We show that u satisfies (0.1).

- $\{u_n\}$ is bounded $W_0^1L_\phi(\Omega)$ and converges to u strongly in $L_\phi(\Omega)$, where $\phi \in \mathscr{A}_M$.

Firstly, we can take $T_\varepsilon(u_n - T_t(u_n)), \varepsilon > 0, t > 0$ as a test function in (3.15), from (3.9) and (4.72) we have

$$\int_{\{t\leq|u_n|\leq t+\varepsilon\}} a(x,u_n,\nabla u_n)\nabla u_n dx \leq \varepsilon C_7.$$

The constant C_7 is independent of n, ε and t, then

$$\frac{1}{\varepsilon}\int_{\{t\leq|u_n|\leq t+\varepsilon\}} M(x,\nabla u_n)dx \leq \frac{C_7}{\alpha}.$$

Let now $\varepsilon \to 0$, we have

$$-\frac{d}{dt}\int_{\{t\leq|u_n|\}} M(x,\nabla u_n)dx \leq \frac{C_7}{\alpha}. \tag{3.27}$$

Secondly, let $\phi \in \mathscr{A}_M$ and $\phi \sim M$. Using Lemma 3.1.1, Lemma 3.1.2, the equation (3.27) and the same techniques as in [31], we deduce that ∇u_n is bounded in $L_\phi(\Omega)$ for each $\phi \in \mathscr{A}_M$, then u_n is bounded in $W_0^1L_\phi(\Omega)$ for each $\phi \in \mathscr{A}_M$.

- $a(x,u_n,\nabla u_n) \rightharpoonup a(x,u,\nabla u)$ weakly for $\sigma(\Pi L_{\phi\circ H^{-1}}, \Pi E_{\varpi})$,
  where ϖ and $\phi\circ H^{-1}$ are two complementary Musielak-Orlicz functions.
  The first time, using (3.6) and Remark 2.5.1, we have

$$\int_\Omega \phi\circ H_x^{-1}\Big(\frac{|a(x,u_n,\nabla u_n)|}{6\beta}\Big)dx \leq \int_\Omega \phi\circ H_x^{-1}\Big(\frac{1}{6}[c(x)+k(\nu_1)\overline{M}_x^{-1}(M(x,|u_n|)) + \overline{M}_x^{-1}(M(x,\nu_2|\nabla u_n|))]\Big)dx.$$

  Since $\phi\circ H_x^{-1}$ is a Musielak-Orlicz function, we get

$$\int_\Omega \phi\circ H_x^{-1}\Big(\frac{|a(x,u_n,\nabla u_n)|}{6\beta}\Big)dx \leq \frac{1}{3}\int_\Omega [\phi\circ H_x^{-1}\Big(\frac{1}{2}(c(x)) + \phi\circ H_x^{-1}\Big(\frac{1}{2}(k(\nu_1)\overline{M}_x^{-1}(M(x,|u_n|)))\Big)$$

$$+\phi\circ H_x^{-1}\big(\frac{1}{2}\overline{M}_x^{-1}(M(x,\nu_2|\nabla u_n|))\big)]dx. \tag{3.28}$$

On the other hand, due to the definition of Musielak-Orlicz function, we can easily deduce

$$\frac{1}{2}\overline{M}_x^{-1}(M(x,t)) \leq \frac{M(x,t)}{t},$$

by definition of H we have

$$M(x,H_x^{-1}(\frac{t}{2})) \leq \overline{M}(x,t),$$

and hence, by Remark 2.5.1,

$$\phi\circ H_x^{-1}\big(\frac{1}{2}c(x)\big) \leq k_1 M(x,H_x^{-1}(\frac{c(x)}{2})) \leq k_1\overline{M}(x,c(x)), \tag{3.29}$$

also we have

$$\phi\circ H^{-1}\big(\frac{1}{2}\overline{M}_x^{-1}(k(\nu_1)M(x,|u_n|))\big) \leq \phi\circ H_x^{-1}\big(\frac{1}{2}\overline{M}_x^{-1}M(x,k_2|u_n|)\big) \leq \phi\circ H_x^{-1}\big(\frac{M(x,k_2|\nabla u_n|)}{k_2|\nabla u_n|}\big),$$

where $k_2=\max(1,k(\nu_1))$, then

$$\phi\circ H^{-1}\big(\frac{1}{2}\overline{M}_x^{-1}(k(\nu_1)M(x,|u_n|))\big) \leq k_3 M(x,|u_n|), \tag{3.30}$$

and

$$\begin{aligned}\phi\circ H_x^{-1}\big(\overline{M}_x^{-1}(\frac{1}{2}M(x,\nu_2|\nabla u_n|))\big) &\leq \phi\circ H_x^{-1}\big(\frac{M(x,\nu_2|\nabla u_n|)}{\nu_2|\nabla u_n|)}\big)\\ &= \phi(x,\nu_2|\nabla u_n|).\end{aligned}$$

Using the remark 2.5.1 we get

$$\phi\circ H_x^{-1}\big(\overline{M}_x^{-1}(\frac{1}{2}M(x,\nu_2|\nabla u_n|))\big) \leq k_4 M(x,|\nabla u_n|), \tag{3.31}$$

applying (3.29), (3.30) and (3.31) in (3.28) we obtain

$$\int_\Omega \frac{1}{3}\phi\circ H_x^{-1}\big(\frac{|a(x,u_n,\nabla u_n)|}{6\beta}\big)dx \leq \int_\Omega [k_1\overline{M}(x,c(x))dx + k_3 M(x,|u_n|) + k_4 M(x,|\nabla u_n|)]dx < C_8.$$

Consequently, $a(x,u_n,\nabla u_n) \rightharpoonup a(x,u,\nabla u)$ weakly for $\sigma(\Pi L_{\phi\circ H^{-1}},\Pi E_{\varpi})$.

- Take now $v\in\mathscr{D}(\Omega)$ as a test function in approximate equation (3.13), one has

$$\int_\Omega a(x,u_n,\nabla u_n)vdx + \int_\Omega K(x,u_n,\nabla u_n)vdx = \int_\Omega \mu_n vdx,$$

since we have $u_n\to u$ strongly in $(E_\varphi(\Omega))^N$, for every $\varphi\ll\phi$, $\forall\phi\in\mathscr{A}_M$.
Using (3.25), (3.14) and passing to the limit as $n\to+\infty$ to end the proof of Theorem 3.3.1.

□

4

Existence of Entropy Solutions in Musielak-Orlicz Spaces Via a Sequence of Penalized Equations

This chapter is devoted to an existence result of entropy unilateral solutions for the nonlinear parabolic problems with obstacle in Musielak- Orlicz–spaces:

$$\partial_t u + A(u) + K(x,t,u,\nabla u) = f + div(\Phi(x,t,u)),$$

and

$$u \geq \zeta \text{ a.e. in } Q_T.$$

Where A is a pseudomonotone operator of Leray-Lions type defined in the inhomogeneous Musielak-Orlicz space $W_0^{1,x}L_M(Q_T)$, K and Φ are only assumed to be Crathéodory's functions satisfying only the growth conditions prescribed by Musielak-Orlicz functions M and $\overline{M}$ which inhomogeneous and does not satisfies Δ_2-condition. The data f and u_0 are still taken in $L^1(Q_T)$ and $L^1(\Omega)$.

Statement of the problem: Consider the following nonlinear Dirichlet equation:

$$\begin{cases} u \geq \zeta & \text{in} \quad Q_T, \\ \frac{\partial u}{\partial t} + A(u) + K(x,t,u,\nabla u) = f + div(\Phi(x,t,u)) & \text{in} \quad Q_T, \\ u(x,t) = 0 & \text{on} \quad \partial\Omega \times (0,T), \\ u(x,t=0) = u_0(x) & \text{in} \quad \Omega, \end{cases} \tag{4.1}$$

where $A(u) = -div(a(x,t,u,\nabla u))$ is a Leary-Lions operator defined on the inhomogeneous Musielak-Orlicz-Sobolev space $W_0^{1,x}L_M(Q_T)$, M is a Musielak-Orlicz-function related to the growths of the Carathéodory functions $a(x,t,u,\nabla u)$, Φ and K (see assumptions (4.2), (4.5) and (4.6). The data f and u_0 in $L^1(Q_T)$ and $L^1(\Omega)$ respectively, and $u_0 \geq \zeta$.

The first prototype is taken from the Classical Sobolev spaces, having the following form:

$$\frac{\partial u}{\partial t} - \triangle_p(u) + div(c(.,t)|u|^{\gamma-1}u) + b|\nabla u|^{\delta} = f, \text{in} \quad Q_T.$$

Porzio et al. in [93] have proved the existence of weak solutions, with $c(.,.) \equiv 0$. For $c(.,.) \in L^2(Q_T)$ and $p = 2$, Boccardo et al. in [48] have proved the existence of entropy solutions, recently R. Di-Nardo et al. in [55] have proved an existence results of renormalized solutions in the case

where $p \geq 2$ and $c(.,.) \in L^r(Q_T)$ with $r = \frac{N+p}{p-1}$, and by Aberqi et al. in [6] for more general parabolic term. For the elliptic version of the problem (4.1), more results are obtained see e.g. [49, 50].

In the degenerate Sobolev-spaces an existence results is shown in [22] without sign condition in K.

In the Orlicz-Sobolev spaces, Rhoudaf et al. in [88] proved the existence of entropy solutions of the problem (4.1) where $K \equiv 0$ and the growth of the first lower order Φ prescribed by an isotropic N-function M defining space does not satisfy the $\triangle_2$-condition.

To our knowledge, differential equations in general Musielak–Sobolev spaces have been studied rarely see [16, 36, 17, 92], then our aim in this chapter is to overcome some difficulties encountered in these spaces and to generalize the result of [6, 21, 66, 88], and we prove an existence result of entropy solutions for the obstacle parabolic problem (4.1), with less restrictive growth, and no coercivity condition in the first lower order term Φ, and without sign condition in the second lower order K, in the general framework of inhomogeneous Musielak-Orlicz-Sobolev spaces $W_0^{1,x}L_M(Q_T)$, and the anisotropic N-function M, defining space does not satisfy the $\triangle_2$-condition.

Let us give an example of equations to which the present result can be applied:

$$\begin{cases} u \geq \zeta & \text{in} \quad Q_T, \\ \dfrac{\partial u}{\partial t} - \Delta_M(u) + uM(x,\nabla u) = f + c(x,t)\overline{M}_x^{-1}M(x,\alpha_0|u|) & \text{in} \quad Q_T, \\ u(x,t) = 0 & \text{on} \quad \partial\Omega \times (0,T), \end{cases}$$

where $-\Delta_M(u) = -div(\frac{m(x,|\nabla u|)}{|\nabla u|}.\nabla u)$, $m(x,s)$ is the derivative of $M(x,s)$ with respect to s, ζ is an admissible obstacle function.

4.1 Essential assumptions

Let Ω be an open subset of $\mathbb{R}^N$ $(N \geq 2)$ satisfying the segment property, and let M and γ be two Musielak-Orlicz functions such that M and its complementary $\overline{M}$ satisfies conditions of Lemma 2.5.3 and Lemma 2.5.8 and $\gamma \ll M$.
$A : D(A) \subset W_0^1 L_M(Q_T) \to W^{-1}L_{\overline{M}}(Q_T)$ defined by $A(u) = -diva(x,u,\nabla u)$, where $a : Q_T \times \mathbb{R} \times \mathbb{R}^N \to \mathbb{R}^N$ is Carathéodory function such that for a.e. $x \in \Omega$ and for all $s \in \mathbb{R}, \xi, \xi^* \in \mathbb{R}^N, \xi \neq \xi^*$,

$$|a(x,t,s,\xi)| \leq \beta(a_0(x,t) + \overline{M}_x^{-1}\gamma(x,k_1|s|) + \overline{M}_x^{-1}M(x,k_1|\xi|)), \tag{4.2}$$

with $a_0(.) \in E_{\overline{M}}(Q_T)$, and $\beta > 0$,

$$(a(x,t,s,\xi) - a(x,t,s,\xi^*))(\xi - \xi^*) > 0, \tag{4.3}$$

$$a(x,t,s,\xi).\xi \geq \alpha M(x,|\xi|) + M(x,|s|). \tag{4.4}$$

$\Phi : Q_T \times \mathbb{R} \to \mathbb{R}^N$ is a Carathéodory function such that

$$|\Phi(x,t,s)| \leq c(x,t)\overline{M}_x^{-1}M(x,\alpha_0|s|), \tag{4.5}$$

$$\text{where} \quad \|c(.,.)\|_{L^\infty(Q_T)} \leq \alpha \quad \text{and} \quad 0 < \alpha_0 < \min(1; \frac{1}{\alpha}).$$

$K : Q_T \times \mathbb{R} \times \mathbb{R}^N \to \mathbb{R}$ is a Carathéodory function such that

$$|K(x,t,s,\xi)| \leq h(x,t) + \rho(s)M(x,|\xi|), \tag{4.6}$$

$\rho : \mathbb{R} \to \mathbb{R}^+$ is continuous positive function which belongs to $L^1(IR)$ and $h(.,.)$ belongs to $L^1(Q_T)$.

$$f \in L^1(Q_T), \tag{4.7}$$

$$u_0 \in L^1(\Omega). \tag{4.8}$$

Let ζ a measurable function with values in $\mathbb{R}$ such that

$$\zeta \in W_0^1 E_M(Q_T) \cap L^\infty(Q_T), \quad \frac{\partial \zeta}{\partial t} \in L^1(Q_T) \quad \text{such that} \quad u_0 \geq \zeta,$$

and let $K_\zeta = \left\{ u \in W_0^{1,x} L_M(Q_T) : u \geq \zeta \text{ a.e. in } Q_T \right\}$.

Note that $<,>$ means for either the pairing between $W_0^{1,x} L_M(Q_T) \cap L^\infty(Q_T)$ and $W^{-1,x} L_{\overline{M}}(Q_T) + L^1(Q_T)$ or between $W_0^{1,x} L_M(Q_T)$ and $W^{-1,x} L_{\overline{M}}(Q_T)$.

The definition of a entropy solution of Problem (4.1) can be stated as follows.

Definition 4.1.1. *A measurable function u defined on Q_T is a entropy solution of Problem* (4.1)*, if it satisfies the following conditions:*

$$\begin{cases} \int_\Omega S_k(u(T) - v(T))dx + \int_{Q_T} \frac{\partial v}{\partial t} T_k(s - v(x,0))dxdt \\ + \int_{Q_T} a(x,t,u,\nabla u)\nabla T_k(u-v)dxdt \\ + \int_{Q_T} \Phi(x,t,u)\nabla T_k(u-v)dxdt + \int_{Q_T} K(x,t,u,\nabla u)T_k(u-v)dxdt \\ \leq \int_{Q_T} f T_k(u-v)dxdt + \int_\Omega S_k(u_0 - v(x,0))dx, \\ \forall k > 0, \quad \textit{and} \quad \forall v \in K_\zeta \cap L^\infty(Q_T) \quad \textit{such that} \quad \frac{\partial v}{\partial t} \in L_{\overline{M}}(0,T; W^{-1} L_{\overline{M}}(\Omega)). \end{cases} \tag{4.9}$$

4.2 Main result

Theorem 4.2.1. *Assume that* (4.2) – (4.8) *hold true . Then there exists at least one entropy solution u of the problem* (4.1) *in the sense of definition 4.1.1.*

Remark 4.2.1. *The results obtained in Theorem 4.2.1, remains true if we replace* (4.5) *by the growth condition $|\Phi(x,t,s)| \leq c(x)\overline{\gamma}_x^{-1}\gamma(x,|s|)$, where $c(.) \in E_\gamma(\Omega)$ and $\gamma \ll M$.*

Remark 4.2.2. *Condition* (4.4) *can be replaced by the weaker one*

$$a(x,s,\xi)\xi \geq \alpha M(x,|\xi|) - b(x),$$

where $b(x)$ is in L^1-function.

Proof.

Step 1: Approximate problem.

For each $n > 0$, we define the following approximations

$$a_n(x,t,s,\xi) = a(x,t,T_n(s),\xi) \quad \text{a.e. } (x,t) \in Q_T,\ \forall\, s \in I\!R,\ \forall\, \xi \in I\!R^N, \tag{4.10}$$

$$\Phi_n(x,t,s) = \Phi(x,t,T_n(s)) \quad \text{a.e. } (x,t) \in Q_T,\ \forall\, s \in I\!R, \tag{4.11}$$

$$K_n(x,t,s,\xi) = \frac{K(x,t,s,\xi)}{1+\frac{1}{n}|K(x,t,s,\xi)|}, \tag{4.12}$$

$$\text{Let } (f_n)_n \text{ be a sequence of smooth functions such that } f_n \to f \text{ strongly in } L^1(\Omega), \tag{4.13}$$

and

$$u_{0n} \in \mathscr{C}_0^\infty(\Omega) \text{such that } u_{0n} \to u_0 \text{ strongly in } L^1(\Omega). \tag{4.14}$$

We define $sg_n(s) = \frac{T_n(s)}{n}$.

Let us now consider the approximate problem :

$$\begin{cases} \dfrac{\partial u_n}{\partial t} - div(a_n(x,t,u_n,\nabla u_n)) + K_n(x,t,u_n,\nabla u_n) & \\ +nT_n(u_n-\zeta)^- sg_{\frac{1}{n}}(u_n) = f_n + div(\Phi_n(x,t,u_n)) & \text{in} \quad Q_T, \\ u_n(x,t) = 0 & \text{on} \quad \partial\Omega \times (0,T), \\ u_n(x,0) = u_{0n} & \text{in} \quad \Omega. \end{cases} \tag{4.15}$$

Since K_n is bounded for any fixed $n > 0$, there exists at last one solution $u_n \in W_0^{1,x} L_M(Q_T)$ of (4.15)(see [85]).

Step 2: A priori estimates.

Lemma 4.2.1. *Let $\{u_n\}_n$ be a solution of the approximate problem* (4.15)*, then for all $k > 0$, there exists a constants C_1 and C_2 such that*

$$\int_{Q_T} a(x,t,T_k(u_n),\nabla T_k(u_n))\nabla T_k(u_n)dx \le kC_1, \tag{4.16}$$

and

$$\int_{Q_T} M(x,|\nabla T_k(u_n)|)dx \le kC_2, \tag{4.17}$$

where C_1 and C_2 does not depend on the n and k.

Proof. Fixed $k > 0$,

Let $\tau \in (0,T)$ and using $\exp(G(u_n))T_k(u_n)^+\chi_{(0,\tau)}$ as a test function in problem (4.15), where $G(s) = \int_0^s \frac{\rho(r)}{\alpha'}dr$, and $\alpha' > 0$ is a parameter to be specified later. We get

$$\int_{Q_\tau} \frac{\partial u_n}{\partial t} \exp(G(u_n))T_k(u_n)^+ dxdt \tag{4.18}$$

$$+\int_{Q_\tau} a_n(x,t,u_n,\nabla u_n)\nabla\big(\exp(G(u_n))T_k(u_n)^+\big)dxdt \tag{4.19}$$

$$+\int_{Q_\tau}\Phi_n(x,t,u_n)\nabla\big(\exp(G(u_n))T_k(u_n)^+\big)dxdt \tag{4.20}$$

$$+\int_{Q_\tau}K(x,t,u_n,\nabla u_n)\exp(G(u_n))T_k(u_n)^+dxdt \tag{4.21}$$

$$+\int_{Q_\tau}nT_n(u_n-\zeta)^-sg_{\frac{1}{n}}(u_n)\exp(G(u_n))T_k(u_n)^+dxdt \tag{4.22}$$

$$\leq k\exp(\frac{\|\rho\|_{L^1}}{\alpha'})\|f_n\|_{L^1(Q_T)}. \tag{4.23}$$

* For the (4.18), we take

$$\widetilde{T}_k(r)=\int_0^r\exp(G(s))T_k(s)^+ds,$$

then

$$\int_{Q_\tau}\frac{\partial u_n}{\partial t}\exp(G(u_n))T_k(u_n)^+dxdt=\int_\Omega\widetilde{T}_k(u_n(\tau))dx-\int_\Omega\widetilde{T}_k(u_n(0))dx. \tag{4.24}$$

By definition we have $\int_\Omega\widetilde{T}_k(u_n(\tau))dx\geq 0$ and $\int_\Omega\widetilde{T}_k(u_n(0))dx\leq k\exp(\frac{\|\rho\|_{L^1}}{\alpha'})\|u_0\|_{L^1(\Omega)}$.

* For the (4.20) we use (4.5) and Young inequality, we get

$\int_{Q_\tau}\Phi_n(x,t,u_n)\nabla(\exp(G(u_n))T_k(u_n)^+)dxdt$

$$\leq\frac{\|c(.,.)\|_{L^\infty(Q_T)}}{\alpha'}\Big[\alpha_0\int_{Q_\tau}M(x,u_n)\rho(u_n)\exp(G(u_n))T_k(u_n)^+dxdt$$

$$+\int_{Q_\tau}M(x,\nabla u_n)\rho(u_n)\exp(G(u_n))T_k(u_n)^+dxdt\Big]$$

$$+\|c(.,.)\|_{L^\infty(Q_T)}\alpha_0\int_{Q_\tau}M(x,u_n)\exp(G(u_n))dxdt$$

$$+\|c(.,.)\|_{L^\infty(Q_T)}\int_{Q_\tau}M(x,|\nabla T_k(u_n)^+|)\exp(G(u_n))dxdt.$$

* For the (4.21) we have,

$$\int_{Q_\tau}K_n(x,t,u_n,\nabla u_n)\exp(G(u_n))T_k(u_n)^+dxdt\leq k\exp(\frac{\|\rho\|_{L^1}}{\alpha'})\int_{Q_T}|h(x,t)|dxdt$$

$$+\int_{Q_\tau}\rho(u_n)\exp(G(u_n))M(x,\nabla u_n)T_k(u_n)^+dxdt.$$

Finally using the previous inequalities and (4.4), we obtain

$$\left\{\begin{array}{l}
\frac{1}{\alpha'}\int_{Q_\tau} M(x,u_n)\rho(u_n)\exp(G(u_n))T_k(u_n)^+dxdt+\frac{\alpha}{\alpha'}\int_{Q_T} M(x,\nabla u_n)\rho(u_n)\exp(G(u_n))T_k(u_n)^+dxdt\\
+\int_{Q_\tau} a(x,t,u_n,\nabla u_n)\exp(G(u_n))\nabla T_k(u_n)^+dxdt+\int_{Q_t} nT_n(u_n-\zeta)^- sg_{\frac{1}{n}}(u_n)\exp(G(u_n))T_k(u_n)^+dxdt\\
\leq \frac{\|c(.,.)\|_{L^\infty(Q_T)}}{\alpha'}\Big[\alpha_0\int_{Q_\tau} M(x,u_n)\rho(u_n)\exp(G(u_n))T_k(u_n)^+dxdt\\
+\int_{Q_\tau} M(x,\nabla u_n)\rho(u_n)\exp(G(u_n))T_k(u_n)^+dxdt\Big]\\
+\alpha_0\|c(.,.)\|_{L^\infty(Q_T)}\int_{\{0\leq u_n\leq k\}} M(x,u_n)\exp(G(u_n))dxdt\\
+\|c(.,.)\|_{L^\infty(Q_T)}\int_{Q_\tau} M(x,\nabla T_k(u_n)^+)\exp(G(u_n))dxdt\\
+\int_{Q_\tau} M(x,\nabla u_n)\rho(u_n)\exp(G(u_n))T_k(u_n)^+dxdt\\
+k\Big[\exp(\frac{\|\rho\|_{L^1}}{\alpha'}(\|f\|_{L^1(Q_T)}+\|u_0\|_{L^1(\Omega)}+\int_{Q_T}|h(x,t)|dxdt\Big],
\end{array}\right. \tag{4.25}$$

Using again (4.4) in (4.25) we get

$$\begin{aligned}
&\Big[\frac{1-\alpha_0\|c(.,.)\|_{L^\infty(Q_T)}}{\alpha'}\Big]\int_{Q_\tau} M(x,u_n)\rho(u_n)\exp(G(u_n))T_k(u_n)^+dxdt\\
&\qquad+\Big[\frac{\alpha-\|c(.,.)\|_{L^\infty(Q_T)}-\alpha'}{\alpha'}\Big]\int_{Q_\tau} M(x,\nabla u_n)\rho(u_n)\exp(G(u_n))T_k(u_n)^+dxdt\\
&+\int_{Q_\tau} a(x,t,u_n,\nabla u_n)\exp(G(u_n))\nabla T_k(u_n)^+dxdt+\int_{Q_t} nT_n(u_n-\zeta)^- sg_{\frac{1}{n}}(u_n)\exp(G(u_n))T_k(u_n)^+dxdt\\
&\leq\frac{\|c(.,.)\|_{L^\infty(Q_T)}}{\alpha}\Big[\alpha_0\alpha\int_{\{0\leq u_n\leq k\}} M(x,u_n)\exp(G(u_n))dxdt+\alpha M(x,\nabla T_k(u_n)^+)\exp(G(u_n))dxdt\Big]\\
&\qquad+kc_1.
\end{aligned}$$

If we choose α' such that $\alpha'<\alpha-\|c(.,.)\|_{L^\infty(Q_T)}$ and using again (4.4) we get

$$\begin{aligned}
&\Big[1-\frac{\|c(.,.)\|_{L^\infty(Q_T)}}{\alpha}\Big]\int_{Q_\tau} a(x,t,u_n,\nabla u_n)\exp(G(u_n))\nabla T_k(u_n)^+dxdt\\
&\qquad+\int_{Q_t} nT_n(u_n-\zeta)^- sg_{\frac{1}{n}}(u_n)\exp(G(u_n))T_k(u_n)^+dxdt\leq kc_1.
\end{aligned}\tag{4.26}$$

Taking $\frac{1}{c_2}=\Big[1-\frac{\|c(.,.)\|_{L^\infty(Q_T)}}{\alpha}\Big]$. Thus,

$$\begin{aligned}
&\int_{Q_\tau} a(x,t,u_n,\nabla u_n)\exp(G(u_n))\nabla T_k(u_n)^+dxdt\\
&\qquad+c_2\int_{Q_\tau} nT_n(u_n-\zeta)^- sg_{\frac{1}{n}}(u_n)\exp(G(u_n))T_k(u_n)^+dxdt\leq kc_1c_2.
\end{aligned}$$

It follow that

$$0 \leq \int_{Q_\tau} nT_n(u-\zeta)^- sg_{\frac{1}{n}}(u_n)\exp(G(u_n))\frac{T_k(u_n)^+}{k}dxdt \leq c_1,$$

by Fatou's lemma as $k \to 0$ we have

$$0 \leq \int_{\{u_n \geq 0\}} nT_n(u_n-\zeta)^- sg_{\frac{1}{n}}(u_n)\exp(G(u_n))dxdt \leq c_1.$$

Return to (4.26), we deduce easily

$$\int_{\{0\leq u_n \leq k\}} a(x,t,u_n,\nabla u_n)\exp(G(u_n))\nabla T_k(u_n)dxdt \leq kc_1c_2.$$

And as one has $\exp(G(u_n)) \geq 1$ for $0 \leq u_n \leq k$, then

$$\int_{\{0\leq u_n \leq k\}} a(x,t,u_n,\nabla u_n)\nabla T_k(u_n)dxdt \leq kc_1c_2, \tag{4.27}$$

by ((4.4))

$$\int_{Q_\tau} M(x,|\nabla T_k(u_n)^+|)dxdt \leq \frac{kc_1c_2}{\alpha}, \tag{4.28}$$

and

$$0 \leq \int_{\{u_n \geq 0\}} nT_n(u_n-\zeta)^- sg_{\frac{1}{n}}(u_n)dxdt \leq c_1. \tag{4.29}$$

Similarly, taking $\exp(-G(u_n))T_k(u_n)^-\chi_{(0,\tau)}$ as a test function in problem ((4.15)), we get

$$\int_{Q_\tau} \frac{\partial u_n}{\partial t}\exp(-G(u_n))T_k(u_n)^- dxdt \tag{4.30}$$

$$+\int_{Q_\tau} a_n(x,t,u_n,\nabla u_n)\nabla(\exp(-G(u_n))\nabla T_k(u_n)^-)dxdt \tag{4.31}$$

$$+\int_{Q_\tau} \Phi_n(x,t,u_n)\nabla(\exp(-G(u_n))\nabla T_k(u_n)^-)dxdt \tag{4.32}$$

$$+\int_{Q_\tau} K_n(x,t,u_n,\nabla u_n)\exp(-G(u_n))T_k(u_n)^- dxdt \tag{4.33}$$

$$+\int_{Q_\tau} nT_n(u_n-\zeta)^- sg_{\frac{1}{n}}(u_n)\exp(-G(u_n))T_k(u_n)^- dxdt \tag{4.34}$$

$$\geq \int_{Q_\tau} f_n\exp(-G(u_n))T_k(u_n)^- dxdt. \tag{4.35}$$

We take also

$$\widetilde{T}_k(r) = \int_0^r \exp(-G(s))T_k(s)^- ds,$$

then

$$\int_{Q_\tau} \frac{\partial u_n}{\partial s}\exp(-G(u_n))T_k(u_n)^- dxdt = \int_\Omega \widetilde{T}_k(u_n(\tau))dx - \int_\Omega \widetilde{T}_k(u_n(0))dx, \tag{4.36}$$

and using same techniques, we obtain also
$\int_{Q_\tau} a(x,t,u_n,\nabla u_n)\exp(-G(u_n))\nabla T_k(u_n)dxdt$

$$+c_2\int_{Q_\tau} nT_n(u-\zeta)^- sg_{\frac{1}{n}}(u_n)\exp(-G(u_n))T_k(u_n)^- dxdt \le kc_1c_2. \tag{4.37}$$

It follow that

$$0 \le -\int_{Q_\tau} nT_n(u_n-\zeta)^- sg_{\frac{1}{n}}(u_n)\exp(-G(u_n))\frac{T_k(u_n)^-}{k}dxdt \le c_1,$$

we deduce by Fatou's lemma as $k \to 0$ that

$$0 \le -\int_{\{u_n\le 0\}} nT_n(u_n-\zeta)^- sg_{\frac{1}{n}}(u_n)\exp(-G(u_n))dxdt \le c_1,$$

And as one has $\exp(-G(u_n)) \ge 1$ since $-k \le u_n \le 0$, then

$$\int_{\{-k\le u_n\le 0\}} a(x,t,u_n,\nabla u_n)\nabla T_k(u_n)dxdt \le kc_1c_2, \tag{4.38}$$

$$\int_{Q_\tau} M(x,|\nabla T_k(u_n)^-|)dxdt \le \frac{kc_1c_2}{\alpha}, \tag{4.39}$$

and

$$0 \le -\int_{\{u_n\le 0\}} nT_n(u_n-\zeta)^- sg_{\frac{1}{n}}(u_n)dxdt \le c_1. \tag{4.40}$$

Combining now (4.27) and (4.38) we get,

$$\int_{Q_T} a(x,t,u_n,\nabla u_n)\nabla T_k(u_n)dxdt \le kC_1. \tag{4.41}$$

Of the same with (4.28) and (4.39) we get,

$$\int_{Q_T} M(x,|\nabla T_k(u_n)|)dxdt \le kC_2. \tag{4.42}$$

we conclude that $T_k((u_n))$ is bounded in $W_0^{1,x}L_M(Q_T)$ independently of n and for any $k>0$, so there exists a subsequence still denoted by u_n such that

$$T_k(u_n) \rightharpoonup \xi_k \quad \text{weakly in} \quad W_0^{1,x}L_M(Q_T). \tag{4.43}$$

On the other hand, using (4.42), we have

$$\inf_{x\in\Omega} M(x,\frac{k}{\delta})meas\{|u_n|>k\} \le \int_{|u_n|>k} M(x,\frac{|T_k(u_n)|}{\delta})dxdt$$

$$\le \int_{Q_T} M(x,|\nabla T_k(u_n)|)dxdt \le kC_2.$$

Then

$$meas\{|u_n| > k\} \leq \frac{kC_2}{\inf_{x\in\Omega} M(x, \frac{k}{\delta})},$$

for all n and for all k.
Assuming that there exists a positive function M such that $\lim_{t\to\infty} \frac{M(t)}{t} = +\infty$ and $M(t) \leq ess\inf_{x\in\Omega} M(x,t)$, $\forall t \geq 0$. Thus, we get

$$\lim_{k\to\infty} meas\{|u_n| > k\} = 0. \tag{4.44}$$

□

Step 3:

Now we turn to prove the almost every convergence of u_n and convergence of $a_n(x,t,T_k(u_n),\nabla T_k(u_n))$.

Proposition 4.2.1. *Let u_n be a solution of the approximate problem, then*

$$u_n \to u \quad a.e\ in \quad Q_T, \tag{4.45}$$

$$a_n(x,t,T_k(u_n),\nabla T_k(u_n)) \rightharpoonup \varpi_k \quad in \quad (L_{\overline{M}}(Q))^N, \quad for \quad \sigma(\Pi L_{\overline{M}}, \Pi E_M), \tag{4.46}$$

for some $\varpi_k \in (L_{\overline{M}}(Q))^N$.

Proof.
Proof of (4.45) **:** Let $\lambda > 0$ then

$$meas\{|u_m - u_n| > \lambda\} \leq meas\{|u_m| > k\} + meas\{|u_n| > k\} + meas\{|T_k(u_m) - T_k(u_n)| > \lambda\}.$$

By (4.43), we can assume that $T_k(u_n)$ is a Cauchy sequence in measure in Q_T and using (4.44) we deduce that for any $\varepsilon > 0$ there exists some $k(\varepsilon) > 0$ such that

$$meas\{|u_m - u_n| > \lambda\} \leq \varepsilon \quad \text{for all} \quad n,m > N_{k(\varepsilon),\lambda}.$$

Which means that u_n is a Cauchy sequence in measure in Q_T, thus converge almost every where to some measurable function u.
Proof of (4.46) **:** We shall prove that $\{a(x,t,T_k(u_n),\nabla T_k(u_n))\}_n$ is bounded in $\left(L_{\overline{M}}(\Omega)\right)^N$ for all $k > 0$.

Let $w \in (E_M(\Omega))^N$ be arbitrary. By condition (4.3) we have,

$$(a(x,t,u_n,\nabla u_n) - a(x,t,u_n,w))(\nabla u_n - w) > 0,$$

then

$$\int_{\{|u_n|\leq k\}} a(x,t,u_n,\nabla u_n) w dxdt \leq \int_{\{|u_n|\leq k\}} a(x,t,u_n,\nabla u_n)\nabla u_n dxdt + \int_{\{|u_n|\leq k\}} a(x,t,u_n,w)(w - \nabla u_n) dxdt,$$

by (4.2) we have for $\nu > \beta$

$$\int_{\{|u_n|\leq k\}} \overline{M}_x(x, \frac{a(x,t,u_n,\frac{w}{k_2})}{3\nu}) dxdt \leq \frac{\beta}{3\nu} \int_{Q_T} \left[\overline{M}(x,a_0(x,t)) + \gamma(x,k_1|T_k(u_n)|) + M(x,|w|)\right] dxdt$$

$$\leq \frac{\beta}{3\nu}\int_{Q_T}\Big[\overline{M}(x,a_0(x,t))+\gamma(x,k_1k)+M(x,|w|)\Big]dxdt. \tag{4.47}$$

Thus $\{a(x,t,T_k(u_n),\frac{w}{k_2})\}$ is bounded in $(L_{\overline{M}}(\Omega))^N$.

By (4.47),(4.16) and by the theorem of Banach–Steinhaus, the sequence $\{a(x,t,T_k(u_n),\nabla T_k(u_n))\}$ remains bounded in $(L_{\overline{M}}(\Omega))^N$ and we conclude (4.46). □

Lemma 4.2.2. *If the subsequence u_n satisfies* (4.15)*, then*

$$\lim_{m\to+\infty}\limsup_{n\to+\infty}\int_{\{m\leq|u_n|\leq m+1\}}a(x,t,u_n,\nabla u_n)\nabla u_n dxdt=0. \tag{4.48}$$

Proof. Taking the function $Z_m(u_n)=T_1(u_n-T_m(u_n))^-$ and multiplying the approximating equation (4.15) by the test function $\exp(-G(u_n))Z_m(u_n)$ we get

$$\begin{cases}\displaystyle\int_\Omega \widetilde{T}_m(x,u_n(T))dx+\int_{Q_T}a_n(x,t,u_n,\nabla u_n)\nabla(\exp(-G(u_n))Z_m(u_n))dxdt\\ \displaystyle+\int_{Q_T}\Phi_n(x,t,u_n)\nabla(\exp(-G(u_n))Z_m(u_n))dxdt+\int_{Q_T}K_n(x,t,u_n,\nabla u_n)\exp(-G(u_n))Z_m(u_n)dxdt\\ \displaystyle+\int_{Q_T}nT_n(u_n-\zeta)^-sg_{\frac{1}{n}}(u_n)\exp(-G(u_n))Z_m(u_n)\,dxdt\\ \displaystyle=\int_{Q_T}f_n\exp(-G(u_n))Z_m(u_n)\,dxdt+\int_\Omega\widetilde{T}_m(u_{0n})dx,\end{cases} \tag{4.49}$$

where $\widetilde{T}_m(r)=\displaystyle\int_0^r\exp(-G(s))Z_m(s)ds$.

we know that $\displaystyle\int_\Omega\widetilde{T}_m(x,u_n(T))dx\geq 0$,

and $\displaystyle\int_{Q_T}nT_n(u_n-\zeta)^-sg_{\frac{1}{n}}(u_n)\exp(-G(u_n))Z_m(u_n)\,dxdt\geq 0$.

Then, using the same argument in step 2 to remove the term $\displaystyle\int_{Q_T}\rho(u_n)\exp(-G(u_n))\nabla u_nZ_m(u_n)dxdt$, we obtain,

$$\begin{cases}\displaystyle\int_{\{-(m+1)\leq u_n\leq -m\}}a_n(x,t,u_n,\nabla u_n)\nabla u_ndxdt+\int_{Q_T}\Phi_n(x,t,u_n)\exp(-G(u_n))\nabla Z_m(u_n)dxdt\\ \displaystyle\leq\exp(\frac{\|\rho\|_{L^1}}{\alpha'})\Big[\int_{Q_T}|f_n|Z_m(u_n)\,dxdt\int_{Q_T}|h(x,t)|Z_m(u_n)dxdt+\int_{|u_{0n}|>m}|u_{0n}|dx\Big].\end{cases} \tag{4.50}$$

Thanks to the (4.5) and (4.50) we obtain,

$$\int_{Q_T}M(x,|\nabla Z_m(u_n)|)\exp(-G(u_n))dxdt\leq\frac{c_2}{\alpha}\exp(\frac{\|\rho\|_{L^1}}{\alpha'})\Big(\int_{Q_T}f_nZ_m(u_n)dxdt$$
$$+\int_{Q_T}|h(x,t)|Z_m(u_n)dxdt+\int_{\{|u_{0n}|>m\}}|u_{0n}|dx\Big),$$

Passing to limit as $n\to+\infty$, since the pointwise convergence of u_n and strongly convergence in $L^1(Q_T)$ of f_n and u_{0n} we get

$$\lim_{n\to+\infty}\int_{Q_T}M(x,|\nabla Z_m(u_n)|)\exp(-G(u_n))dxdt\leq C\Big(\int_{Q_T}fZ_m(u)dxdt$$
$$+\int_{Q_T}|h(x,t)|Z_m(u)dxdt+\int_{\{|u_0|>m\}}|u_0|dx\Big).$$

By using Lebesgue's theorem and passing to limit as $m \to +\infty$, in the all term of the right-hand side, we get

$$\lim_{m\to+\infty}\lim_{n\to+\infty}\int_{Q_T} M(x,|\nabla Z_m(u_n)|)\exp(-G(u_n))dxdt = 0. \tag{4.51}$$

On the other hand, we have

$$\lim_{m\to+\infty}\lim_{n\to+\infty}\int_{Q_T} \Phi_n(x,t,u_n)\exp(-G(u_n))\nabla Z_m(u_n)dxdt$$

$$\leq \lim_{m\to+\infty}\lim_{n\to+\infty}\int_{Q_T} M(x,|\nabla Z_m(u_n)|)\exp(-G(u_n))dxdt$$

$$+ \lim_{m\to+\infty}\lim_{n\to+\infty}\int_{m\leq|u_n|\leq m+1} \overline{M}(x,|\Phi_n(x,t,u_n)|)\exp(-G(u_n))dxdt.$$

Using the pointwise convergence of u_n and Lebegue's theorem in the second term of the right side, we get

$$\lim_{n\to+\infty}\int_{m\leq|u_n|\leq m+1} \overline{M}(x,|\Phi_n(x,t,u_n)|)\exp(-G(u_n))dxdt = \int_{m\leq|u|\leq m+1} \overline{M}(x,|\Phi(x,t,u)|)\exp(-G(u))dxdt,$$

and also, by Lebesgue's theorem

$$\lim_{m\to+\infty}\int_{m\leq|u|\leq m+1} \overline{M}(x,|\Phi(x,t,u)|)dxdt = 0. \tag{4.52}$$

Then, we deduce

$$\lim_{m\to+\infty}\lim_{n\to+\infty}\int_{Q_T} \Phi_n(x,t,u_n)\exp(-G(u_n))\nabla Z_m(u_n)dx\,dt = 0.$$

Finally passing to the limit in (4.50), we get

$$\lim_{m\to+\infty}\lim_{n\to+\infty}\int_{\{-(m+1)\leq u_n\leq -m\}} a_n(x,t,u_n,\nabla u_n)\nabla u_n dx\,dt = 0.$$

In the same way we take $Z_m(u_n) = T_1(u_n - T_m(u_n))^+$ and multiplying the approximating equation (4.15) by the test function $\exp(G(u_n))Z_m(u_n)$ and we also obtain

$$\lim_{m\to+\infty}\lim_{n\to+\infty}\int_{\{m\leq u_n\leq m+1\}} a_n(x,t,u_n,\nabla u_n)\nabla u_n dx\,dt = 0.$$

On the above we get (4.48). □

Step 4: Convergence of the gradients ∇u_n.

This step is devoted to introduce a time regularization of the $T_k(u)$ for $k > 0$ in order to perform the monotonicity method.

Lemma 4.2.3. *Under assumptions* (4.2)-(4.8)*, and let* (z_n) *be a sequence in* $W_0^{1,x}L_M(Q_T)$ *such that:*

$$z_n \rightharpoonup z \quad for \quad \sigma(\Pi L_M(Q_T), \Pi E_{\overline{M}}(Q_T)), \tag{4.53}$$

$$(a(x,t,z_n,\nabla z_n)) \quad is\ bounded\ in\ (L_{\overline{M}}(Q_T)^N, \tag{4.54}$$

$$\int_{Q_T}[a(x,t,z_n,\nabla z_n)-a(x,t,z_n,\nabla z\chi_s)][\nabla z_n-\nabla z\chi_s]dx\,dt\to 0, \tag{4.55}$$

as n and s tend to $+\infty$, and where χ_s is the characteristic function of $Q_s=\{(x,t)\in Q_T;|\nabla z|\le s\}$ then,

$$\nabla z_n\to\nabla z \quad a.e.\ in\ Q_T, \tag{4.56}$$

$$\lim_{n\to+\infty}\int_{Q_T}a(x,t,z_n,\nabla z_n)\nabla z_n dx\,dt=\int_{Q_T}a(x,t,z,\nabla z)\nabla z dx\,dt, \tag{4.57}$$

$$M(x,|\nabla z_n|)\to M(x,|\nabla z|) \quad in\ L^1(Q_T). \tag{4.58}$$

Proof. It is easily adapted from that given in [29]. □

Let $v_j\in\mathscr{D}(Q_T)$ be a sequence such that $v_j\to u$ in $W_0^{1,x}L_M(Q_T)$ for the modular convergence. This specific time regularization of $T_k(v_j)$ (for fixed $k\ge 0$) is defined as follows.
Let $(\alpha_0^\mu)_\mu$ be a sequence of functions defined on Ω such that

$$\alpha_0^\mu\in L^\infty(\Omega)\cap W_0^1L_M(\Omega) \quad \text{for all} \quad \mu>0, \tag{4.59}$$

$$\|\alpha_0^\mu\|_{L^\infty(\Omega)}\le k, \text{ for all} \quad \mu>0,$$

and

$$\alpha_0^\mu \text{ converges to } \quad T_k(u_0) \text{ a.e. in } \quad \Omega \quad \text{ and } \quad \frac{1}{\mu}\|\alpha_0^\mu\|_{M,\Omega} \text{ converges to } \quad 0 \quad \mu\to+\infty$$

For $k\ge 0$ and $\mu>0$, let us consider the unique solution $(T_k(v_j))_\mu\in L^\infty(Q)\cap W_0^{1,x}L_M(Q)$ of the monotone problem:

$$\frac{\partial(T_k(v_j))_\mu}{\partial t}+\mu((T_k(v_j))_\mu-T_k(v_j))=0 \text{ in } D'(\Omega),$$

$$(T_k(v_j))_\mu(t=0)=\alpha_0^\mu \text{ in } \Omega.$$

Remark that due to

$$\frac{\partial(T_k(v_j))_\mu}{\partial t}\in W_0^{1,x}L_M(Q_T).$$

We recall that,

$$(T_k(v_j))_\mu\to T_k(u) \quad \text{a.e. in} \quad Q_T, \quad \text{weakly}-* \quad \text{in} \quad L^\infty(Q_T)$$

$(T_k(v_j))_\mu\to(T_k(u))_\mu$ in $W_0^{1,x}L_M(Q_T)$ for the modular convergence as $j\to+\infty$.
$(T_k(u))_\mu\to T_k(u)$ in $W_0^{1,x}L_M(Q_T)$ for the modular convergence as $\mu\to+\infty$.

$$||(T_k(v_j))_\mu||_{L^\infty(Q_T)}\le max(||(T_k(u))||_{L^\infty(Q_T)}, \quad ||\alpha_0^\mu||_{L^\infty(\Omega)})\le k, \ \forall\ \mu>0, \quad \forall\ k>0.$$

We introduce a sequence of increasing $\mathbf{C}^1(\mathbb{R})$-functions S_m such that

$$S_m(r)=1 \text{ for } |r|\le m,\ S_m(r)=m+1-|r|, \text{ for } m\le|r|\le m+1,\ S_m(r)=0 \text{ for } |r|\ge m+1$$

for any $m \geq 1$.

For fixed $k \geq 0$, let $W_{\mu,\eta}^{n,j} = T_\eta(T_k(u_n) - T_k(\upsilon_j)_\mu)^+$ and $W_{\mu,\eta}^{j} = T_\eta(T_k(u) - T_k(\upsilon_j)_\mu)^+$.

Multiplying the approximating equation by $\exp(G(u_n)))W_{\mu,\eta}^{n,j} S_m(u_n)$ and using the same technique in step 2 we obtain:

$$\begin{cases} \displaystyle\int_{Q_T} < \frac{\partial u_n}{\partial t} \exp(G(u_n)) W_{\mu,\eta}^{n,j} S_m(u_n)\, dx\, dt \\ \displaystyle + \int_{Q_T} a_n(x,t,u_n,\nabla u_n) \exp(G(u_n)) \nabla(W_{\mu,\eta}^{n,j}) S_m(u_n)\, dx\, dt \\ \displaystyle + \int_{Q_T} a_n(x,t,u_n,\nabla u_n) \nabla u_n \exp(G(u_n)) W_{\mu,\eta}^{n,j} S_m'(u_n)\, dx\, dt \\ \displaystyle - \int_{Q_T} \Phi_n(x,t,u_n) \exp(G(u_n)) \nabla(W_{\mu,\eta}^{n,j}) S_m(u_n)\, dx\, dt \\ \displaystyle - \int_{Q_T} \Phi_n(x,t,u_n) \nabla u_n \exp(G(u_n)) W_{\mu,\eta}^{n,j} S_m'(u_n)\, dx\, dt \\ \displaystyle \leq \int_{Q_T} f_n \exp(G(u_n)) W_{\mu,\eta}^{n,j} S_m(u_n)\, dx\, dt + \int_{Q_T} h(x,t) \exp(G(u_n)) W_{\mu,\eta}^{n,j} S_m(u_n)\, dx\, dt. \end{cases} \tag{4.60}$$

Now we pass to the limit in (4.60) for k real number fixed.

In order to perform this task we prove below the following results for any fixed $k \geq 0$:

$$\int_{Q_T} \frac{\partial u_n}{\partial t} \exp(G(u_n)) W_{\mu,\eta}^{n,j} S_m(u_n)\, dx\, dt \geq \varepsilon(n,\mu,\eta,j) \qquad \text{for any } m \geq 1, \tag{4.61}$$

$$\int_{Q_T} \Phi_n(x,t,u_n) S_m(u_n) \exp(G(u_n)) \nabla(W_{\mu,\eta}^{n,j})\, dx\, dt = \varepsilon(n,j,\mu) \quad \text{for any } m \geq 1, \tag{4.62}$$

$$\int_{Q_T} \Phi_n(x,t,u_n) \nabla u_n S_m'(u_n) \exp(G(u_n)) W_{\mu,\eta}^{n,j}\, dx\, dt = \varepsilon(n,j,\mu) \qquad \text{for any } m \geq 1, \tag{4.63}$$

$$\int_{Q_T} a_n(x,t,u_n,\nabla u_n) \nabla u_n S_m'(u_n) \exp(G(u_n)) W_{\mu,\eta}^{n,j}\, dx\, dt \leq \varepsilon(n,m), \tag{4.64}$$

$$\int_{Q_T} a_n(x,t,u_n,\nabla u_n) S_m(u_n) \exp(G(u_n)) \nabla(W_{\mu,\eta}^{n,j})\, dx\, dt \leq C\eta + \varepsilon(n,j,\mu,m), \tag{4.65}$$

$$\int_{Q_T} f_n S_m(u_n) \exp(G(u_n)) W_{\mu,\eta}^{n,j}\, dx\, dt + \int_{Q_T} h(x,t) \exp(G(u_n)) W_{\mu,\eta}^{n,j} S_m(u_n)\, dx\, dt \leq C\eta + \varepsilon(n,\eta), \tag{4.66}$$

$$\int_{Q_T} \Big[a(x,t,T_k(u_n),\nabla T_k(u_n)) - a(x,t,T_k(u_n),\nabla T_k(u)) \Big] \Big[\nabla T_k(u_n) - \nabla T_k(u) \Big] dx\, dt \to 0. \tag{4.67}$$

Proof of (4.61):

Lemma 4.2.4.

$$\int_{Q_T} \frac{\partial u_n}{\partial t} \exp(G(u_n)) W_{\mu,\eta}^{n,j} S_m(u_n) dx dt \geq \varepsilon(n,\mu,\eta,\eta,j) \quad m \geq 1. \tag{4.68}$$

Proof. Is a particular case of the proof in [20], with $b(x,u) = u$. □

Proof of (4.62): If we take $n > m+1$, we get

$$\Phi_n(x,t,u_n)\exp(G(u_n))S_m(u_n) = \Phi(x,t,T_{m+1}(u_n))\exp(G(T_{m+1}(u_n)))S_m(T_{m+1}(u_n)),$$

then $\Phi_n(x,t,u_n)\exp(G(u_n))S_m(u_n)$ is bounded in $L_{\overline{M}}(Q)$, thus, by using the pointwise convergence of u_n and Lebesgue's theorem we obtain

$$\Phi_n(x,t,u_n)\exp(G(u_n))S_m(u_n) \to \Phi(x,t,u)\exp(G(u))S_m(u),$$

with the modular convergence as $n \to +\infty$,
then $\Phi_n(x,t,u_n)\exp(G(u_n))S_m(u_n) \to \Phi(x,t,u)\exp(G(u))S_m(u)$ for $\sigma(\prod L_{\overline{M}}, \prod L_M)$.
In the other hand $\nabla W_{\mu,\eta}^{n,j} = \nabla T_k(u_n) - \nabla(T_k(\upsilon_j))_\mu$ for $|T_k(u_n) - (T_k(\upsilon_j))_\mu| \le \eta$ converge to $\nabla T_k(u) - \nabla(T_k(\upsilon_j))_\mu$ weakly in $(L_M(Q_T))^N$, then

$$\int_{Q_T} \Phi_n(x,t,u_n)\exp(G(u_n))S_m(u_n)\nabla W_{\mu,\eta}^{n,j}\,dx\,dt \to \int_{Q_T} \Phi(x,t,u)S_m(u)\exp(G(u))\nabla W_{\mu,\eta}^{j}\,dx\,dt$$

as $n \to +\infty$.

By using the modular convergence of $W_{\mu,\eta}^{j}$ as $j \to +\infty$ and letting μ tends to infinity, we get (4.62).

Proof of (4.63):

For $n > m+1 > k$, we have $\nabla u_n S_m'(u_n) = \nabla T_{m+1}(u_n)$ a.e. in Q_T. By the almost every where convergence of u_n we have $\exp(G(u_n))W_{\mu,\eta}^{n,j} \to \exp(G(u))W_{\mu,\eta}^{j}$ in $L^\infty(Q_T)$ weak-* and since the sequence $(\Phi_n(x,t,T_{m+1}(u_n)))_n$ converge strongly in $E_{\overline{M}}(Q_T)$ then

$$\Phi_n(x,t,T_{m+1}(u_n))\exp(G(u_n))\ W_{\mu,\eta}^{n,j} \to \Phi(x,t,T_{m+1}(u))\ \exp(G(u))W_{\mu,\eta}^{j},$$

converge strongly in $E_{\overline{M}}(Q_T)$ as $n \to +\infty$. By virtue of $\nabla T_{m+1}(u_n) \to \nabla T_{m+1}(u)$ weakly in $(L_M(Q_T))^N$ as $n \to +\infty$ we have

$$\int_{m\le|u_n|\le m+1} \Phi_n(x,t,T_{m+1}(u_n))\nabla u_n S_m'(u_n)\exp(G(u_n))W_{\mu,\eta}^{n,j}\,dx\,dt$$

$$\to \int_{m\le|u|\le m+1} \Phi(x,t,u))\nabla u\exp(G(u))W_{\mu,\eta}^{j}\,dx\,dt$$

as $n \to +\infty$ with the modular convergence of $W_{\mu,\eta}^{j}$ as $j \to +\infty$ and letting $\mu \to +\infty$ we get (4.63).

Proof of (4.64):

For (4.64), we have

$$\int_{Q_T} a_n(x,t,u_n,\nabla u_n)S_m'(u_n)\nabla u_n\exp(G(u_n))\exp(G(u_n))W_{\mu,\eta}^{n,j}\,dx\,dt$$

$$= \int_{m\le|u_n|\le m+1} a_n(x,t,u_n,\nabla u_n)S_m'(u_n)\nabla u_n\exp(G(u_n))W_{\mu,\eta}^{n,j}\,dx\,dt$$

$$\le \eta C\int_{m\le|u_n|\le m+1} a_n(x,t,u_n,\nabla u_n)\nabla u_n\,dx\,dt.$$

Using (4.48), we get

$$\int_{Q_T} a_n(x,t,u_n,\nabla u_n)S_m'(u_n)\nabla u_n\exp(G(u_n))W_{\mu,\eta}^{n,j}\,dx\,ds \le \varepsilon(n,\mu,m).$$

Proof of (4.66):

Since $S_m(r) \le 1$ and $W_{\mu,\eta}^{n,j} \le \eta$ we get

$$\int_{Q_T} f_n S_m(u_n) \exp(G(u_n)) W_{\mu,\eta}^{n,j} \quad dx\,dt \le \varepsilon(n,\eta),$$

$$\int_{Q_T} h(x,t) \exp(G(u_n)) W_{\mu,\eta}^{n,j} S_m(u_n)\, dx\,dt \le C\eta.$$

Proof of (4.65):

$$\int_{Q_T} a_n(x,t,u_n,\nabla u_n) S_m(u_n) \exp(G(u_n)) \nabla W_{\mu,\eta}^{n,j}\, dx\,dt$$

$$= \int_{\{|u_n|\le k\}\cap\{0\le T_k(u_n)-T_k(\upsilon_j)_\mu)\le\eta\}} a_n(x,t,T_k(u_n),\nabla T_k(u_n)) S_m(u_n) \exp(G(u_n)) (\nabla T_k(u_n) - \nabla T_k(\upsilon_j)_\mu)\, dx\,dt$$

$$- \int_{\{|u_n|>k\}\cap\{0\le T_k(u_n)-T_k(\upsilon_j)_\mu)\le\eta\}} a_n(x,t,u_n,\nabla u_n) S_m(u_n) \exp(G(u_n)) \nabla T_k(\upsilon_j)_\mu\, dx\,dt. \qquad (4.69)$$

Since $a_n(x,t,T_{k+\eta}(u_n),\nabla T_{k+\eta}(u_n))$ is bounded in $(L_{\overline{M}}(Q_T))^N$, there exist some $\varpi_{k+\eta} \in (L_{\overline{M}}(Q_T))^N$ such that $a_n(x,t,T_{k+\eta}(u_n),\nabla T_{k+\eta}(u_n)) \to \varpi_{k+\eta}$ weakly in $(L_{\overline{M}}(Q_T))^N$. Consequently,

$$\int_{\{|u_n|>k\}\cap\{0\le T_k(u_n)-T_k(\upsilon_j)_\mu)\le\eta\}} a_n(x,t,u_n,\nabla u_n) S_m(u_n) \exp(G(u_n)) \nabla T_k(\upsilon_j)_\mu\, dx\,dt$$

$$= \int_{\{|u|>k\}\cap\{0\le T_k(u)-T_k(\upsilon_j)_\mu)\le\eta\}} S_m(u) \exp(G(u)) \nabla T_k(\upsilon_j)_\mu \varpi_{k+\eta}\, dx\,dt + \varepsilon(n), \qquad (4.70)$$

where we have used the fact that

$$S_m(u_n) \exp(G(u_n)) \nabla T_k(\upsilon_j)_\mu) \chi_{\{|u_n|>k\}\cap\{0\le T_k(u_n)-T_k(\upsilon_j)_\mu)\le\eta\}}$$

$$\to S_m(u) \exp(G(u)) \nabla T_k(\upsilon_j)_\mu) \chi_{\{|u|>k\}\cap\{0\le T_k(u)-T_k(\upsilon_j)_\mu)\le\eta\}},$$

strongly in $(E_M(Q_T))^N$.

Letting $j \to +\infty$, we obtain

$$\int_{\{|u|>k\}\cap\{0\le T_k(u)-T_k(\upsilon_j)_\mu)\le\eta\}} S_m(u) \exp(G(u)) \nabla T_k(\upsilon_j)_\mu \varpi_{k+\eta}\, dx\,dt$$

$$= \int_{\{|u|>k\}\cap\{0\le T_k(u)-T_k(u)_\mu)\le\eta\}} S_m(u) \exp(G(u)) \nabla T_k(u)_\mu \varpi_{k+\eta}\, dx\,dt + \varepsilon(n,j).$$

One easily has,

$$\int_{\{|u|>k\}\cap\{0\le T_k(u)-T_k(u)_\mu)\le\eta\}} S_m(u) \exp(G(u)) \nabla T_k(u)_\mu \varpi_{k+\eta}\, dx\,dt = \varepsilon(n,j,\mu).$$

By (4.60)-(4.66), (4.69) and (4.70) we obtain

$$\int_{\{|u_n|\le k\}\cap\{0\le T_k(u_n)-T_k(\upsilon_j)_\mu)|\le\eta\}} a_n(x,t,T_k(u_n),\nabla T_k(u_n)) S_m(u_n) \exp(G(u_n)) (\nabla T_k(u_n) - \nabla T_k(\upsilon_j)_\mu)\, dx\,dt$$

$$\le C\eta + \varepsilon(n,j,\mu,m),$$

we know that $\exp(G(u_n)) \geq 1$ and $S_m(u_n) = 1$ for $|u_n| \leq k$ then

$$\int_{\{|u_n|\leq k\}\cap\{0\leq T_k(u_n)-T_k(\upsilon_j)_\mu)|\leq\eta\}} a_n(x,t,T_k(u_n),\nabla T_k(u_n))(\nabla T_k(u_n)-\nabla T_k(\upsilon_j)_\mu)\,dxdt$$

$$\leq C\eta+\varepsilon(n,j,\mu,m). \tag{4.71}$$

Proof of (4.67):

Setting for $s>0,\ Q^s=\{(x,t)\in Q:|\nabla T_k(u)|\leq s\}$ and $Q_j^s=\{(x,t)\in Q:|\nabla T_k(\upsilon_j)|\leq s\}$ and denoting by χ^s and χ_j^s the characteristic functions of Q^s and Q_j^s respectively, we deduce that letting $0<\delta<1$, define

$$\Theta_{n,k}=(a(x,t,T_k(u_n),\nabla T_k(u_n))-a(x,t,T_k(u_n),\nabla T_k(u)))(\nabla T_k(u_n)-\nabla T_k(u)).$$

For $s>0,$ we have

$$0\leq\int_{Q^s}\Theta_{n,k}^\delta\,dxdt=\int_{Q^s}\Theta_{n,k}^\delta\chi_{|T_k(u_n)-T_k(\upsilon_j)_\mu|\leq\eta)}\,dxdt+\int_{Q^s}\Theta_{n,k}^\delta\chi_{|T_k(u_n)-T_k(\upsilon_j)_\mu|>\eta)}\,dxdt.$$

The first term of the right-side hand, with the Hölder inequality,

$$\int_{Q^s}\Theta_{n,k}^\delta\chi_{|T_k(u_n)-T_k(\upsilon_j)_\mu|\leq\eta)}\,dxdt\leq(\int_{Q^s}\Theta_{n,k}\chi_{|T_k(u_n)-T_k(\upsilon_j)_\mu|\leq\eta)}\,dxdt)^\delta(\int_{Q^s}dxdt)^{1-\delta}$$

$$\leq C_1(\int_{Q^s}\Theta_{n,k}\chi_{|T_k(u_n)-T_k(\upsilon_j)_\mu|\leq\eta)}\,dxdt)^\delta.$$

Also using the Hölder inequality, the second term of the right-side hand is

$$\int_{Q^s}\Theta_{n,k}^\delta\chi_{|T_k(u_n)-T_k(\upsilon_j)_\mu|>\eta)}\,dxdt\leq(\int_{Q^s}\Theta_{n,k}\,dxdt)^\delta(\int_{|T_k(u_n)-T_k(\upsilon_j)_\mu|>\eta)}dxdt)^{1-\delta},$$

since $a(x,t,T_k(u_n),\nabla T_k(u_n))$ is bounded in $(L_{\overline{M}}(Q_T))^N,$ while $\nabla T_k(u_n)$ is bounded in $(L_M(Q_T))^N$ then

$$\int_{Q^s}\Theta_{n,k}^\delta\chi_{|T_k(u_n)-T_k(\upsilon_j)_\mu|>\eta)}\,dxdt\leq C_2meas\{(x,t)\in Q_T:|T_k(u_n)-T_k(\upsilon_j)_\mu|>\eta\}^{1-\delta}.$$

We obtain,

$$\int_{Q^s}\Theta_{n,k}^\delta\,dxdt\leq C_1(\int_{Q^s}\Theta_{n,k}\chi_{|T_k(u_n)-T_k(\upsilon_j)_\mu|\leq\eta)}\,dxdt)^\delta$$

$$+C_2meas\{(x,t)\in Q_T:|T_k(u_n)-T_k(\upsilon_j)_\mu|>\eta\}^{1-\delta}.$$

On the other hand,

$$\int_{Q^s}\Theta_{n,k}\chi_{|T_k(u_n)-T_k(\upsilon_j)_\mu|\leq\eta)}\,dxdt$$

$$\leq\int_{|T_k(u_n)-T_k(\upsilon_j)_\mu|\leq\eta)}(a(x,t,T_k(u_n),\nabla T_k(u_n))-a(x,t,T_k(u_n),\nabla T_k(u)\chi_s))\times(\nabla T_k(u_n)-\nabla T_k(u)\chi_s)\,dxdt.$$

For each $s > r, r > 0$, one has

$$
\begin{aligned}
0 &\leq \int_{Q^r \cap \{|T_k(u_n) - T_k(\upsilon_j)_\mu| \leq \eta)\}} (a(x,t,T_k(u_n),\nabla T_k(u_n)) - a(x,t,T_k(u_n),\nabla T_k(u))) \times (\nabla T_k(u_n) - \nabla T_k(u))\, dx\, dt \\
&\leq \int_{Q^s \cap \{|T_k(u_n) - T_k(\upsilon_j)_\mu| \leq \eta)\}} (a(x,t,T_k(u_n),\nabla T_k(u_n)) - a(x,t,T_k(u_n),\nabla T_k(u))) \times (\nabla T_k(u_n) - \nabla T_k(u))\, dx\, dt \\
&= \int_{Q^s \cap \{|T_k(u_n) - T_k(\upsilon_j)_\mu| \leq \eta)\}} (a(x,t,T_k(u_n),\nabla T_k(u_n)) - a(x,t,T_k(u_n),\nabla T_k(u)\chi_s)) \times (\nabla T_k(u_n) - \nabla T_k(u)\chi_s)\, dx\, dt \\
&\leq \int_{Q \cap \{|T_k(u_n) - T_k(\upsilon_j)_\mu| \leq \eta)\}} (a(x,t,T_k(u_n),\nabla T_k(u_n)) - a(x,t,T_k(u_n),\nabla T_k(u)\chi^s)) \times (\nabla T_k(u_n) - \nabla T_k(u)\chi^s)\, dx\, dt \\
&= \int_{|T_k(u_n) - T_k(\upsilon_j)_\mu| \leq \eta} (a(x,t,T_k(u_n),\nabla T_k(u_n)) - a(x,t,T_k(u_n),\nabla T_k(\upsilon_j)\chi_j^s)) \times (\nabla T_k(u_n) - \nabla T_k(\upsilon_j)\chi_j^s)\, dx\, dt \\
&+ \int_{|T_k(u_n) - T_k(\upsilon_j)_\mu| \leq \eta} a(x,t,T_k(u_n),\nabla T_k(u_n))(\nabla T_k(\upsilon_j)\chi_j^s - \nabla T_k(u)\chi^s)\, dx\, dt \\
&+ \int_{|T_k(u_n) - T_k(\upsilon_j)_\mu| \leq \eta} (a(x,t,T_k(u_n),\nabla T_k(\upsilon_j)\chi_j^s) - a(x,t,T_k(u_n),\nabla T_k(u)\chi^s))\nabla T_k(u_n)\, dx\, dt \\
&- \int_{|T_k(u_n) - T_k(\upsilon_j)_\mu| \leq \eta} a(x,t,T_k(u_n),\nabla T_k(\upsilon_j)\chi_j^s)\nabla T_k(\upsilon_j)\chi_j^s)\, dx\, dt \\
&+ \int_{|T_k(u_n) - T_k(\upsilon_j)_\mu| \leq \eta} a(x,t,T_k(u_n),\nabla T_k(u)\chi^s)\nabla T_k(u)\chi^s)\, dx\, dt \\
&= I_1(n,j,s) + I_2(n,j) + I_3(n,j) + I_4(n,j,\mu) + I_5(n,\mu).
\end{aligned}
$$

We go to the limit as n, j, μ, and $s \to +\infty$

$$
\begin{aligned}
I_1 = &\int_{|T_k(u_n) - T_k(\upsilon_j)_\mu| \leq \eta} a(x,t,T_k(u_n),\nabla T_k(u_n))(\nabla T_k(u_n) - \nabla T_k(\upsilon_j)_\mu)\, dx\, dt \\
&- \int_{|T_k(u_n) - T_k(\upsilon_j)_\mu| \leq \eta} a(x,t,T_k(u_n),\nabla T_k(u_n))(\nabla T_k(\upsilon_j)\chi_j^s - \nabla T_k(\upsilon_j)_\mu)\, dx\, dt \\
&- \int_{|T_k(u_n) - T_k(\upsilon_j)_\mu| \leq \eta} a(x,t,T_k(u_n),\nabla T_k(\upsilon_j)\chi_j^s))(\nabla T_k(u_n) - \nabla T_k(\upsilon_j)\chi_j^s))\, dx\, dt.
\end{aligned}
$$

Using (4.71), the first term of the right-hand side, we get

$$
\begin{aligned}
&\int_{|T_k(u_n) - T_k(\upsilon_j)_\mu| \leq \eta} a(x,t,T_k(u_n),\nabla T_k(u_n))(\nabla T_k(u_n) - \nabla T_k(\upsilon_j)_\mu)\, dx\, dt \\
&\leq C\eta + \varepsilon(n,m,j,s) - \int_{|u|>k \cap |T_k(u) - T_k(\upsilon_j)_\mu| \leq \eta} a(x,t,T_k(u),0)\nabla T_k(\upsilon_j)_\mu\, dx\, dt \\
&\leq C\eta + \varepsilon(n,m,j,\mu).
\end{aligned}
$$

The second term of the right-hand side tends to

$$
\int_{|T_k(u) - T_k(\upsilon_j)_\mu| \leq \eta} \varpi_k(\nabla T_k(\upsilon_j)\chi_j^s - \nabla T_k(\upsilon_j)_\mu)\, dx\, dt,
$$

since $a(x,t,T_k(u_n),\nabla T_k(u_n))$ is bounded in $(L_{\overline{M}}(Q_T))^N$, there exist some $\varpi_k \in (L_{\overline{M}}(Q_T))^N$ such that (for a subsequence still denoted by u_n

$$a(x,t,T_k(u_n),\nabla T_k(u_n)) \to \varpi_k \quad \text{in} \quad (L_M(Q_T))^N \quad \text{for} \quad \sigma(\Pi L_{\overline{M}}, \Pi E_M).$$

In view of the fact that

$$(\nabla T_k(\upsilon_j)\chi_j^s - \nabla T_k(\upsilon_j)_\mu)\chi_{|T_k(u_n)-T_k(\upsilon_j)_\mu|\le\eta} \to (\nabla T_k(\upsilon_j)\chi_j^s - \nabla T_k(\upsilon_j)_\mu)\chi_{|T_k(u)-T_k(\upsilon_j)_\mu|\le\eta},$$

strongly in $(E_M(Q_T))^N$ as $n \to +\infty$.

The third term of the right-hand side tends to

$$\int_{|T_k(u)-T_k(\upsilon_j)_\mu|\le\eta} a(x,t,T_k(u),\nabla T_k(\upsilon_j)\chi_j^s))(\nabla T_k(u) - \nabla T_k(\upsilon_j)\chi_j^s))\,dx\,dt.$$

Since

$$a(x,t,T_k(u_n),\nabla T_k(\upsilon_j)\chi_j^s))\chi_{|T_k(u_n)-T_k(\upsilon_j)_\mu|\le\eta} \to a(x,t,T_k(u),\nabla T_k(\upsilon_j)\chi_j^s))\chi_{|T_k(u)-T_k(\upsilon_j)_\mu|\le\eta},$$

in $(E_{\overline{M}}(Q_T))^N$ while

$$(\nabla T_k(u_n) - \nabla T_k(\upsilon_j)\chi_j^s)) \to (\nabla T_k(u) - \nabla T_k(\upsilon_j)\chi_j^s)),$$

in $(L_M(Q_T))^N$ for $\sigma(\Pi L_{\overline{M}}, \Pi E_M)$.

Passing to limit as $j \to +\infty$ and $\mu \to +\infty$ and using Lebesgue's theorem, we have

$$I_1 \le C\eta + \varepsilon(n,j,s,\mu).$$

For what concerns I_2, by letting $n \to +\infty$, we have

$$I_2 \to \int_{|T_k(u)-T_k(\upsilon_j)_\mu|\le\eta)} \varpi_k(\nabla T_k(\upsilon_j)\chi_j^s - \nabla T_k(u)\chi^s)\,dx\,dt.$$

Since $a(x,t,T_k(u_n),\nabla T_k(u_n)) \rightharpoonup \varpi_k$ in $(L_{\overline{M}}(Q_T))^N$, for $\sigma(\Pi L_{\overline{M}}, \Pi E_M)$, while

$$\begin{aligned}&(\nabla T_k(\upsilon_j)\chi_j^s - \nabla T_k(u)\chi^s)\chi_{|T_k(u_n)-T_k(\upsilon_j)_\mu|\le\eta}\\ &\to (\nabla T_k(\upsilon_j)\chi_j^s - \nabla T_k(u)\chi^s)\chi_{|T_k(u)-T_k(\upsilon_j)_\mu|\le\eta},\end{aligned}$$

strongly in $(E_M(Q_T))^N$.

Passing to limit $j \to +\infty$, and using Lebesgue's theorem, we have

$$I_2 = \varepsilon(n,j).$$

Similar ways as above give

$$\begin{aligned}I_3 &= \varepsilon(n,j).\\ I_4 &= \int_{|T_k(u)-T_k(u)_\mu|\le\eta)} a(x,t,T_k(u),\nabla T_k(u))\nabla T_k(u)\,dx\,dt + \varepsilon(n,j,\mu,s,m).\\ I_5 &= \int_{|T_k(u)-T_k(u)_\mu|\le\eta)} a(x,t,T_k(u),\nabla T_k(u))\nabla T_k(u)\,dx\,dt + \varepsilon(n,j,\mu,s,m).\end{aligned}$$

Finally, we obtain,

$$\int_{Q^s} \Theta_{n,k}\,dx\,dt \leq C_1(C\eta + \varepsilon(n,\mu,\eta,m))^{\delta} + C_2(\varepsilon(n,\mu,))^{1-\delta}.$$

Which yields, by passing to the limit sup over n, j, μ , s and η

$$\int_{Q^r} \Big[(a(x,t,T_k(u_n),\nabla T_k(u_n)) - a(x,t,T_k(u_n),\nabla T_k(u)))(\nabla T_k(u_n) - \nabla T_k(u))\Big]^{\delta} dx\,dt = \varepsilon(n).$$

Thus, passing to a subsequence if necessary, $\nabla u_n \to \nabla u$ a.e. in Q^r, and since r is arbitrary,

$$\nabla u_n \to \nabla u\ , \quad \text{a.e. in} \quad Q_T.$$

Step 5: Equi-integrability of the nonlinearity sequence

We shall prove that $K_n(x,t,u_n,\nabla u_n) \to K(x,t,u,\nabla u)$ strongly in $L^1(\Omega)$.
Consider $g_0(u_n) = \int_0^{u_n} \rho(s)\chi_{\{s>h\}}ds$ and multiply (4.15) by $\exp(G(u_n))g_0(u_n)$, we get

$$\int_\Omega \widetilde{T}_h(u_n)(T)dx + \int_{Q_T} a(x,u_n,\nabla u_n)\nabla(\exp(G(u_n))g_0(u_n))dxdt$$

$$+ \int_{Q_T} \Phi_n(x,u_n,\nabla u_n)\nabla(\exp(G(u_n))g_0(u_n))dxdt + \int_{Q_T} K_n(x,t,u_n,\nabla u_n)\exp(G(u_n))g_0(u_n))dxdt$$

$$\leq (\int_h^{+\infty} \rho(s)dx)\exp\big(\frac{\|\rho\|_{L^1(\mathbb{R})}}{\alpha'}\big)\Big[\|f\|_{L^1(Q_T)} + \|u_0\|_{L^1(\Omega)} + \|h(.,.)\|_{L^1(Q_T)}\Big].$$

where $\widetilde{T}_h(r) = \int_0^r g_0(s)\exp(G(s))ds \geq 0$,
then using same technique in step 2 we can have

$$\int_{\{u_n>h\}} \rho(u_n)M(x,\nabla u_n)dxdt \leq C(\int_h^{+\infty} \rho(s)dx).$$

Since $\rho \in L^1(\mathbb{R})$, we get

$$\limsup_{h\to\infty} \sup_{n\in\mathbb{N}} \int_{\{u_n>h\}} \rho(u_n)M(x,\nabla u_n)dxdt = 0.$$

Similarly, let $g_0(u_n) = \int_{u_n}^0 \rho(s)\chi_{\{s<-h\}}dx$ in (4.15), we have also

$$\limsup_{h\to\infty} \sup_{n\in\mathbb{N}} \int_{\{u_n<-h\}} \rho(u_n)M(x,\nabla u_n)dxdt = 0.$$

We conclude that

$$\limsup_{h\to\infty} \sup_{n\in\mathbb{N}} \int_{\{|u_n|>h\}} \rho(u_n)M(x,\nabla u_n)dxdt = 0. \tag{4.72}$$

Let $D \subset \Omega$ then

$$\int_D \rho(u_n)M(x,\nabla u_n)dxdt \leq \max_{\{|u_n|\leq h\}}(\rho(x)) \int_{D\cap\{|u_n|\leq h\}} M(x,\nabla u_n)dxdt + \int_{D\cap\{|u_n|>h\}} \rho(u_n)M(x,\nabla u_n)dxdt.$$

Consequently $\rho(u_n)M(x,\nabla u_n)$ is equi-integrable. Then $\rho(u_n)M(x,\nabla u_n)$ converge to $\rho(u)M(x,\nabla u)$ strongly in $L^1(I\!R)$. By ((4.6)), we get our result.

Step 6: We show that u satisfies (4.9)

Firstly show that $u \geq \zeta$ a.e. in Q_T.

In fact, from (4.29) and (4.40) we get $0 \leq \int_{Q_T} T_n(u_n-\zeta)^- dxdt \leq \frac{c_1}{n}$.

Let n tends to $+\infty$ we obtain $\int_{Q_T}(u-\zeta)^- dxdt = 0$, then $(u-\zeta)^- = 0$ a.e. in Q_T;

thus $u \geq \zeta$ a.e. in Q_T.

Secondly passing Now to the limit in (4.73) to show that u satisfies the equation (4.9).
Let $v \in W_0^1 L_M(Q_T) \cap L^\infty(Q_T)$ such that $\frac{\partial v}{\partial t} \in W^{-1,x}L_{\overline{M}}(Q_T) + L^1(Q_T)$, then by theorem (2.5.3) we take $\overline{v} = v$ on Q_T, $\overline{v} \in W^{1,x}L_M(\Omega \times I\!R) \cap L^1(\Omega \times I\!R) \cap L^\infty(\Omega \times I\!R)$, $\frac{\partial \overline{v}}{\partial t} \in W^{-1,x}L_{\overline{M}}(Q_T) + L^1(Q_T)$, and there exists $v_j \in \mathscr{D}(\Omega \times I\!R)$ such that

$$v_j \to \overline{v} \quad \text{in} \quad W_0^{1,x}L_M(\Omega \times I\!R) \quad \text{and} \quad \frac{\partial v_j}{\partial t} \to \frac{\partial \overline{v}}{\partial t} \in W^{-1,x}L_{\overline{M}}(Q_T) + L^1(Q_T),$$

for the modular convergence in $W_0^1 L_M(Q_T)$, with $\|v_j\|_{L^\infty(Q_T)} \leq (N+2)\|v\|_{L^\infty(Q_T)}$.
Pointwise multiplication of the approximate equation (4.15) by $T_k(u_n - v_j)$, we get

$$\begin{cases} \int_0^T < \frac{\partial u_n}{\partial s}, T_k(u_n - v_j) > ds + \int_{Q_T} a_n(x,s,u_n,\nabla u_n))\nabla T_k(u_n - v_j)dxds \\ + \int_{Q_T} \Phi_n(x,s,u_n)\nabla T_k(u_n - v_j)dxds + \int_Q T_n(u_n-\zeta)^- sg_{\frac{1}{n}}(u_n)T_k(u_n - v_j)dxds \\ + \int_{Q_T} K_n(x,s,u_n,\nabla u_n)\nabla T_k(u_n - v_j)dxds = \int_{Q_T} f_n T_k(u_n - v_j)dxds. \end{cases} \tag{4.73}$$

We pass to the limit as in (4.73), n tend to $+\infty$ and j tend to $+\infty$.

- Limit of the first term of (4.73):
  The first term can be written

$$\int_0^T < \frac{\partial u_n}{\partial s}, T_k(u_n - v_j) > ds = \int_0^T < \frac{\partial (u_n - v_j)}{\partial s}, T_k(u_n - v_j) > ds + \int_0^T < \frac{\partial v_j}{\partial s}, T_k(u_n - v_j) > ds$$

$$= S_k(u_n(T) - v_j(T)) - S_k(u_n(0) - v_j(0)) + \int_0^T < \frac{\partial v_j}{\partial s}, T_k(u_n - v_j) > ds.$$

  We pass to the limit as $n \to +\infty$ and $j \to +\infty$ we can easily deduce

$$\int_0^T < \frac{\partial u_n}{\partial s}, T_k(u_n - v_j) > ds \to \int_\Omega S_k(u_n(T) - v(T))dx - \int_\Omega S_k(u_n(0) - v(0))dx + \int_0^T < \frac{\partial v}{\partial s}, T_k(u_n - v) > ds.$$

- We can follow same way in [31] to prove that

$$\liminf_{j\to\infty} \liminf_{n\to\infty} \int_{Q_T} a(x,s,u_n,\nabla u_n)\nabla T_k(u_n - v_j)dxds \geq \int_{Q_T} a(x,s,u,\nabla u)\nabla T_k(u - v)dxds.$$

- For $n \geq k+(N+2)\|v\|_{L^\infty(Q_T)}$

$$\Phi_n(x,s,u_n)\nabla T_k(u_n-v_j) = \Phi(x,s,T_{k+(N+2)\|v\|_{L^\infty(Q_T)}}(u_n))\nabla T_k(u_n-v_j).$$

The pointwise convergence of u_n to u as n tends to $+\infty$ and (4.5), then

$$\Phi(x,s,T_{k+(N+2)\|v\|_{L^\infty(Q_T)}}(u_n))\nabla T_k(u_n-v_j) \rightharpoonup \Phi(x,s,T_{k+(N+2)\|v\|_{L^\infty(Q_T)}}(u))\nabla T_k(u-v_j),$$

weakly for $\sigma(\Pi L_v, \Pi L_{\overline{M}})$.
In a similar way, we obtain
$\lim\limits_{j\to\infty}\int_{Q_T} \Phi(x,s,T_{k+(N+2)\|v\|_{L^\infty(Q_T)}}(u))\nabla T_k(u-v_j)dxds$

$$\begin{aligned}&= \int_{Q_T} \Phi(x,s,T_{k+(N+2)\|v\|_{L^\infty(Q_T)}}(u))\nabla T_k(u-v)dxds\\ &= \int_{Q_T} \Phi(x,s,u)\nabla T_k(u-v)dxds.\end{aligned}$$

- Limit of $K_n(x,s,u_n,\nabla u_n)T_k(u_n-v_j)$:
Since $K_n(x,s,u_n,\nabla u_n)$ converge strongly to $K(x,t,u,\nabla u)$ in $L^1(Q_T)$ and and the pointwise convergence of u_n to u as $n\to+\infty$, it is possible to prove that $K_n(x,s,u_n,\nabla u_n)T_k(u_n-v_j)$ converge to $K(x,s,u,\nabla u)T_k(u-v_j)$ in $L^1(Q_T)$ and

$$\lim_{j\to\infty}\int_{Q_T} K(x,s,u,\nabla u)T_k(u-v_j)dxds = \int_{Q_T} K(x,s,u,\nabla u)T_k(u-v)dxds.$$

- Since f_n converge strongly to f in $L^1(Q_T)$, and $T_k(u_n-v_j)\to T_k(u-v_j)$ weakly* in $L^\infty(Q_T)$, we have $\int_{Q_T} f_nT_k(u_n-v_j)dxds \to \int_{Q_T} fT_k(u-v_j)dxds$ as $n\to\infty$
and also we have $\int_{Q_T} fT_k(u-v_j)dxds \to \int_{Q_T} fT_k(u-v)dxds$ as $j\to\infty$.
Finally we know that $\int_Q T_n(u_n-\zeta)^- sg_{\frac{1}{n}}(u_n)T_k(u_n-v_j)dxds \geq 0$.

As a conclusion, the proof of Theorem 4.2.1 is complete. □

Remark 4.2.3. *we generated this result by replacing $\frac{\partial u}{\partial t}$ by $\frac{\partial b(x,u)}{\partial t}$ in [63] and thus we achieved the same result.*

Part III

Applications

5

Existence and uniqueness of discrete solution of parabolic equations with lower term in Orlicz Spaces

In this chapter, we consider the weak formulation of

$$\begin{cases} \frac{\partial b(u)}{\partial t} - \operatorname{div}(a(\nabla u) + \Phi(u)) = f, \\ u = 0 \quad \text{on} \quad \partial\Omega \times (0,T), \\ b(u(.,0)) = b(u_0(x)) \quad \text{in} \quad \partial\Omega. \end{cases} \tag{5.1}$$

We propose a convergent full discretization combining a piecewise constant finite element approximation with the backward Euler scheme.Then we construct an approximation solution sequence for problem (5.1) and establish a priori estimates. Our study is done on the isotropic case and generalizes [70] and [97] where the authors studied only the case $b(u) = u$ and $\Phi = 0$.

5.0.1 Statement of the problem

Beginning with the nonlinear Dirichlet equation (5.1) where $A(u) = -div(a(\nabla u))$ is a Leray-Lions operator defined on the inhomogeneous Orlicz-Sobolev space $W^{1,x}L_M(Q)$, M is a N-function related to the growth of $A(u)$ (see assumptions (5.3)-(5.5)), and to the growth of the lower order Carathéodory function $\Phi(u)$ (see assumptions (5.6)-(5.7)), b a strictly increasing C^1-function, the data f and $b(u_0)$ in $L^1([0,T];L^2(\Omega))$ and $L^2(\Omega)$ respectively.

In classical spaces, starting with [54], the authors proved an existence result of a weak solutions for the non coercive problem (5.1) in the stationary case $b(u) = 0$ using the symmetrization method. More later R. Di Nardo et al. [55] have shown the existence of renormalized solution for the parabolic version, more precisely in the linear case $b(u) = u$, and the uniqueness for such solution in [39], A. Aberqi et al. [4] have proved the existence of a renormalized solutions for (5.1) with more general parabolic terms $b(x,s)$.

In the Orlicz spaces we refer to [9] where L. Aharouch, J. Bennouna have proved the existence and uniqueness of entropy solutions in the framework of Orlicz Sobolev spaces $W_0^1 L_M(\Omega)$ assuming the $\triangle_2$-condition on the $N - function\ M$. Recently, F.KH. Mukminov in [90] proved the uniqueness of renormalized solutions to the Cauchy problem for anisotrpic parabolic equation using Kruzhkovis method of doubling the variable.

In our case, the diffculty encountered during the proof of the existence and uniqueness of discrete solution, is that the fact the Orlicz spaces $L_M(Q)$ is not isometrically isomorphic to the Orlicz space $L_M(0,T;L_M(\Omega))$, the term Φ does not satisfy the coercivity condition and the nonlinearities are characterized by N-function M, for which $\triangle_2$-condition not imposed, will lose the reflexivity of the space $L_M(Q)$ and $W_0^1 L_M(Q)$.

5.1 Technical Lemmas

Let define the Orlicz class $\mathscr{L}_M(Q)$ as the set of real-valued measurable functions u on Q such that $\int_Q M(u(x,t))dxdt < \infty$

Lemma 5.1.1. *[72]*
For some $R > 0$, let $h : \overline{B}(0,R) \to \mathbb{R}^m$ be continuous, where $\overline{B}(0,R) \subset \mathbb{R}^m$ denotes the closed ball with respect to some norm $\|.\|_{\mathbb{R}^m}$ on $\mathbb{R}^m$. If

$$h(v).v \geq 0 \quad \textit{for all} \quad v \subset \mathbb{R}^m \quad \textit{with} \quad \|v\|_{\mathbb{R}^m} = R,$$

then there exists $\widetilde{v} \in \overline{B}(0,R)$ such that $h(\widetilde{v}) = 0$.

Lemma 5.1.2. *[70]*
Let $\{\xi_l\} \in \mathscr{L}_M(Q)$ and there exists $C > 0$ such that $\int_Q M(|\xi_l|)dxdt \leq C$ for all $l \in \mathbb{N}$. Then there exists $\xi \in \mathscr{L}_M(Q)$ and a subsequence, denoted by l', such that $\xi_{l'} \rightharpoonup \xi$ weakly in $L^1(Q)$ and

$$\int_Q M(|\xi|)dxdt \leq \liminf_{l' \to \infty} \int_Q M(|\xi_{l'}|)dxdt.$$

Lemma 5.1.3. *[70]*
Let $w \in \mathscr{W} = \{w \in W^{1,1}(0,T;L^2(\Omega)) : \nabla w \in (\mathscr{L}_M(Q))^d, \gamma_0(w(.,t)) = 0$ for almost $t \in (0,T)\}$.
For any $\varepsilon > 0$ there is then a smooth function w_ε witch vanishes in $\partial\Omega \times (0,T)$ such that

$$\|w_\varepsilon - w\|_{W^{1,1}(0,T;L^2(\Omega))} < \frac{\varepsilon}{2},$$

and for all $\eta \in L_{\overline{M}}(Q)$

$$|\int_Q (\nabla(w_\varepsilon - w)\eta dxdt| < \frac{\varepsilon}{2}.$$

Notation:
Let X be a Banach space, by $\mathscr{C}([0,T];X)$, we denote the usual space of continuous functions $u : [0,T] \to X$, and $\mathscr{C}_w([0,T];X)$ denotes the space of demicontinuous functions (i.e., continuous with respect to the weak topology in X).

5.2 Assumptions and full discretization

Let Throughout this chapter, we assume that the following assumptions hold true.
Let $\Omega \subset \mathbb{R}^N$ is a bounded domain with Lipschitz boundary. Let M is a $\mathscr{N} - function$ and both M and $\overline{M}$ are locally integrable.
$b : \mathbb{R} \to \mathbb{R}$ is a strictly increasing $\mathscr{C}^1(\mathbb{R})$-function, $b(0) = 0$,

$$0 < b_0 < b'(s) < b_1 = 2b_0 \quad \forall s \in \mathbb{R}, \tag{5.2}$$

$a : \mathbb{R}^N \to \mathbb{R}^N$ is a continuous function such that for all $\xi, \xi^* \in \mathbb{R}^N$

$$|a(\xi)| \leq \overline{M}^{-1} M(|\xi|), \tag{5.3}$$

$$(a(\xi) - a(\xi^*))(\xi - \xi^*) \geq 0, \tag{5.4}$$

$$a(\xi)\xi \geq \nu M(\xi)) \quad \text{such that} \quad \nu > 0, \tag{5.5}$$

$\Phi : \mathbb{R} \to \mathbb{R}^N$ is a continuous function such that

$$|\Phi(s)| \leq \nu_0 \overline{M}^{-1} M(\frac{s}{\lambda}) \quad \text{for all} \quad s \quad \text{in} \quad \mathbb{R}, \tag{5.6}$$

$$|\Phi(s) - \Phi(s')| \leq L(x)|s - s'| \quad \text{for all} \quad s \in \mathbb{R}, \quad x \in \Omega, \tag{5.7}$$

where $L(.) \in E_{\overline{M}}(\Omega)$.
and

$$f \in L^1([0,T]; L^2(\Omega)). \tag{5.8}$$

A full discretization:

1) For $N \in \mathbb{N}$ $(N \geq 1)$, let $\tau = \frac{T}{N}$ and $t_n = n\tau$ $(n = 0, 1, ..., N)$. Moreover, we consider a generalized internal approximation $(V_m)_{m \in \mathbb{N}}$ of space

$$V = \{v \in L^2(\Omega) : \nabla v \in (E_M(\Omega))^d, \gamma_0 v = 0\}, \quad \|v\|_V = \|v\|_{2,\Omega} + \|\nabla v\|_{M,\Omega},$$

and restriction operators $R_m : V \to V_m$ such that for any sequence $\{m_l\}_{l \in \mathbb{N}}$ with $m_l \to \infty$ as $l \to \infty$ there holds

$$R_{m_l} v \to v \quad \text{in} \quad V \quad \text{as} \quad l \to +\infty \quad v \in V. \tag{5.9}$$

Since V is separable Banach space, there exits a Galerkin basis and an internal approximation scheme for V.

2) the Galerkin scheme :
According to Taylor-Young's formula

$$b(u(x,t_n)) - b(u(x,t_{n-1})) = \partial_t b(u(x,t_{n-1}))\tau + \tau\varepsilon(\tau) \quad \text{such that} \quad \lim_{\tau \to 0} \varepsilon(\tau) = 0,$$

then we seek to find a $\{u^n\}_{n=1}^N \subset V_m$ such that for $n = 1, 2, ..., N$

$$\int_\Omega \Big[(\frac{b(u^n) - b(u^{n-1})}{\tau}) v + a(\nabla u^n)\nabla v + \Phi(u^n)\nabla v \Big] dx = \int_\Omega f(.,t_n) v dx \quad \forall v \in V_m. \tag{5.10}$$

Here, $u^0 \in V_m$ denotes a suitable approximation of the initial $u_0 \in L^2(\Omega)$ and we can also take $f^n = \frac{1}{\tau} \int_{t_{n-1}}^{t_n} f(.,t)dt$, $(n = 1, 2, ..., N)$.

Our main results consist to prove that there is a unique solution to the problem (5.10), then show the convergence of the numerical solution and finally obtain the estimation of the error for the temporal semidiscretization.

5.3 Existence and stability result

5.3.1 Solvability

Theorem 5.3.1. *Let $u^0 \in V_m$ and $f \in L^1([0,T];L^2(\Omega))$. Then there exists a unique solution $\{u^n\}_{n=1}^N \subset V_m$ to (5.10).*

Proof. Assume that $u^{n-1} \in L^2(\Omega)$ is given and let is show the existence of a unique $u^n \in V_m$ satisfying (5.10).
In fact, since V_m is finite dimensional, we can have $V_m = span\{\varphi_1, \varphi_2, ..., \varphi_m\}$.We construct a one-to-one mapping between V_m and $\mathbb{R}^m$ as

$$v = [v_1, v_2, ...v_m] \in \mathbb{R}^m \Leftrightarrow V_m \ni v = \sum_{j=1}^{m} v_j \varphi_j,$$

and $\|v\|_{\mathbb{R}^m} := \|v\|_{2,\Omega}$ defines a norm on $\mathbb{R}^m$. We define the mapping h by

$$h_i(v) := \int_\Omega \Big[(\frac{b(v) - b(u^{n-1})}{\tau}) \varphi_i + a(\nabla v)\nabla\varphi_i + \Phi(v)\nabla\varphi_i \Big] dx - \int_\Omega f(.,t_n)\varphi_i dx.$$

Remark that, any solution $u^n \in V_m$ corresponds to a zero u^n of h and vice versa and the function $h : \mathbb{R}^m \to \mathbb{R}^m$ is continuous since b, a and Φ are continuous.

$$h(v).v = \int_\Omega \Big[(\frac{b(v) - b(u^{n-1})}{\tau}) v + a(\nabla v)\nabla v + \Phi(v)\nabla v \Big] dx - \int_\Omega f(.,t_n) v dx$$

Firstly taking $\widetilde{\Phi}(t) = \int_0^t \Phi(s)ds$ then $\widetilde{\Phi}(0) = 0$ and $\widetilde{\Phi} \in (\mathscr{C}^1(\mathbb{R}))^N$.
By divergence theorem we obtain

$$\int_\Omega \Phi(u)\nabla u dx = \int_\Omega div(\widetilde{\Phi}(u))dx = \int_{\partial\Omega} \widetilde{\Phi}(0)\overrightarrow{n}\, dx = 0, \tag{5.11}$$

since $u = 0$ on $\partial\Omega$ with $\widetilde{\Phi} = (\widetilde{\Phi}_1, \widetilde{\Phi}_2, ..., \widetilde{\Phi}_d)$ such that $\overrightarrow{n}$ is the normal vector on $\partial\Omega$.
Secondly the Cauchy-Schwartz inequality, the coercivity assumption (5.5) and (5.2) imply that

$$h(v).v \geq \frac{1}{\tau}\|v\|_{2,\Omega}\Big(b_0\|v\|_{2,\Omega} - b_1\|u^{n-1}\|_{2,\Omega} - \tau\|f\|_{L^1([0,T];L^2(\Omega))} \Big)$$

$$\geq \frac{1}{\tau}\|v\|_{2,\Omega} b_0 (\|v\|_{2,\Omega} - R) \geq 0$$

where $R > \frac{1}{b_0}(b_1\|u^{n-1}\|_{2,\Omega} + \tau\|f\|_{L^1([0,T];L^2(\Omega))})$.
By Lemma 5.1.1, there exists u^n such that $h(u^n) = 0$ and u^n solves (5.10) at level n.
Return to uniqueness and let v and w are two solutions of (5.10) and taking $\frac{1}{k}T_k(v-w)$ as a test function, thus

$$\frac{1}{k}\int_\Omega (\frac{(b(v) - b(w)}{\tau}) T_k(v-w)dx = -\frac{1}{k}\int_\Omega (a(\nabla v) - a(\nabla w))\nabla T_k(v-w)dx$$

$$-\frac{1}{k}\int_\Omega(\Phi(v)-\Phi(w))\nabla T_k(v-w)dx.$$

We have

$$\lim_{k\to 0}\frac{1}{k}\int_\Omega(\frac{b(v)-b(w)}{\tau})T_k(v-w)dx=\int_\Omega\frac{1}{\tau}(\frac{b(v)-b(w)}{\tau})sgn(v-w)dx\geq\frac{b_0}{\tau}\int_\Omega|v-w|dx.$$

By (5.4) we have $-\frac{1}{k}\int_\Omega(a(\nabla v)-a(\nabla w))\nabla T_k(v-w)dx\leq 0$
Using (5.7) we get

$$\frac{1}{k}\int_\Omega(\Phi(v)-\Phi(w))\nabla T_k(v-w)dx\leq\frac{1}{k}\int_{|v-w|\leq k}L(x)|v-w||\nabla T_k(v-w)|dx$$

$$\leq\int_{|v-w|\leq k}L(x)|\nabla(v-w)|dx.$$

Since $L(x)|\nabla(v-w)|\in L^1(\Omega)$ then $\lim_{k\to 0}\frac{1}{k}\int_\Omega(\Phi(v)-\Phi(w))\nabla T_k(v-w)dx=0.$
Thus

$$\frac{b_0}{\tau}\int_\Omega|v-w|dx\leq 0.$$

This implies that $v=w$. □

5.3.2 A priori estimates

Theorem 5.3.2. *Let $u^0\in V_m$ and $f\in L^1([0,T];L^2(\Omega))$. Let $\{u^n\}_{n=1}^N\subset V_m$ be the solution to (5.10). Let $\tau\leq\tau_0<1$. Then there exits a positif constants c_1, c_2, c_3 and c_4 depends on b_0, b_1, ν, T and τ_0 such that for all $(n=1,2,...,N)$*

$$\|b(u^n)\|^2_{2,\Omega}+c_1\sum_{j=1}^n\|b(u^j)-b(u^{j-1})\|^2_{2,\Omega}+c_2\sum_{j=1}^n\int_\Omega M(\nabla u^j)dx$$

$$\leq c_3\|f\|^2_{L^1([0,T];L^2(\Omega))}+c_1\|b(u^0)\|^2_{2,\Omega},\tag{5.12}$$

and

$$\sum_{j=1}^n\|u^n-u^{n-1}\|_{2,\Omega}\leq c_4.\tag{5.13}$$

Proof. Taking $v=b(u^n)$ in (5.10) for the discrete time derivative, we have

$$\|b(u^n)\|^2_{2,\Omega}-\|b(u^{n-1})\|^2_{2,\Omega}+\|b(u^n)-b(u^{n-1})\|^2_{2,\Omega}$$

$$+2\tau\int_\Omega[b'(u^n)a(\nabla u^n)\nabla u^n+b'(u^n)\Phi(\nabla u^n)\nabla u^n]dx\leq 2\tau\int_\Omega|f_n(x)||b(u^n)|dx$$

Using the assumptions (5.5), (5.6), (9) and using the Young inequality, we obtain

$$\|b(u^n)\|^2_{2,\Omega}-\|b(u^{n-1})\|^2_{2,\Omega}+\|b(u^n)-b(u^{n-1})\|^2_{2,\Omega}+2\nu\tau b_0\int_\Omega M(\nabla u^n)dx$$

$$\leq 2\tau \int_\Omega |f_n(x)||b(u^n)|dx + 2\tau b_1 \int_\Omega [M(\frac{u^n}{\lambda}) + M(\nabla u^n)]dx$$
$$\leq 2\tau \int_\Omega |f_n(x)||b(u^n)|dx + 4\tau b_1 \nu_0 \int_\Omega M(\nabla u^n)dx$$

Choosing ν_0 such that $\nu > \frac{2b_1}{b_0}\nu_0 = 4\nu_0$, and summation for all $(n = 1,2,...,N)$ implies

$$\|b(u^n)\|^2_{2,\Omega} + \sum_{j=1}^{n} \|b(u^j) - b(u^{j-1})\|^2_{2,\Omega} + 2\tau(\nu b_0 - 2b_1\nu_0) \sum_{j=1}^{n} \int_\Omega M(\nabla u^j)dx$$

$$\leq 2\tau \sum_{j=1}^{n} \int_\Omega |f_n(x)||b(u^n)|dx + \|b(u^0)\|^2_{2,\Omega}.$$

Taking $\|b(u^n)\|_{2,\Omega} = \max_{j=1,...,N} \|b(u^j)\|_{2,\Omega}$, $\tau \sum_{j=1}^{n} \|f_n\|^2_{L^1([0,T];L^2(\Omega))} \leq \|f\|^2_{L^1([0,T];L^2(\Omega))}$ and using Holder inequality, we obtain

$$\|b(u^n)\|^2_{2,\Omega} + \sum_{j=1}^{n} \|b(u^j) - b(u^{j-1})\|^2_{2,\Omega} + 2\tau(\nu b_0 - 2b_1\nu_0) \sum_{j=1}^{n} \int_\Omega M(\nabla u^j)dx$$

$$\leq \frac{1}{\varepsilon^2}\|f\|^2_{L^1([0,T];L^2(\Omega))} + \varepsilon^2\|b(u^n)\|^2_{2,\Omega} + \|b(u^0)\|^2_{2,\Omega},$$

for any $0 < \varepsilon < 1$. From where we have (5.12).

For the second inequality we use only (5.2) to get $b_0|u^n - u^{n-1}| \leq |b(u^n) - b(u^{n-1})|$ and by (5.12) we deduce (5.13). □

5.4 Convergence of the numerical solution

Let consider a sequences $\{m_l, N_l\}_{l\in\mathbb{N}}$ such that $m_l \to \infty$, $N_l \to \infty$ as $l \to \infty$ and suppose that $\tau_l \leq \tau_0 < 1$ for all $l \in \mathbb{N}$.
We consider a sequence $\{u_l^0\}_{l\in\mathbb{N}}$ of approximation of the initial value such that $u_l^0 \in V_m$ and

$$u_l^0 \to u_0 \quad \text{in} \quad L^2(\Omega) \quad \text{as} \quad l \to \infty. \tag{5.14}$$

From the discrete solution $\{u^n\}_{n=1}^N$ corresponding to the time step size $\tau_l = \frac{T}{N_l}$, we construct approximate solutions on the whole time interval as follows: Let u_l denote the piecewise constant function such that

$$u_l(.,t) = u^n \quad \text{if} \quad t \in (t_{n-1}, t_n](n = 1,2,...,N_l), \quad u_l(.,0) = u^1$$

and let $\widehat{u}_l$ denote the linear spline interpolating $(t_0, b(u^0)), (t_1, b(u^1)), ..., (t_{N_l}, b(u^{N_l}))$, i.e.

$$\widehat{u}_l(t) = b(u^{n-1}) + \frac{b(u^n) - b(u^{n-1})}{\tau_l}(t - t_{n-1}) \quad \text{for} \quad t \in [t_{n-1}, t_n](n = 1,2,...,N_l).$$

We also use the piecewise constant in time approximation f_l defined by

$$f_l(.,t) = f(.,t_n) \quad \text{if} \quad t \in (t_{n-1}, t_n](n = 1,2,...,N_l), \quad f_l(.,0) = f(.,t_1).$$

Theorem 5.4.1. *Let $u_0 \in L^2(\Omega)$ and $f \in L^1([0,T];L^2(\Omega))$. Consider the numerical solution of (5.1) by the scheme (5.10) on a sequence of finite dimensional subspaces such that (5.9) is satisfied, and time step sizes which tend to zero and are bounded away from one. For the approximation of the initial value, assume (5.14).*
Then there is a subsequences, denoted by $\{u_{l'}\}$ and $\{\widehat{u}_{l'}\}$ of piecewise constant in time and piecewise linear in time prolongations, respectively, of the numerical solutions converges weakly* in $L^\infty(0,T;L^2(\Omega))$ to an exact solution $u \in \mathscr{C}_w([0,T];L^2(\Omega))$ to (5.1) and to $b(u)$ respectively. Moreover and

$$b(u_{l'}(.,T)) = \widehat{u}_{l'}(.,T) \rightharpoonup b(u(.,T)) \quad \text{weakly in} \quad L^2(\Omega), \tag{5.15}$$

$$\nabla u_{l'} \rightharpoonup \nabla u \quad \text{weakly* in} \quad (L_M(Q))^d \quad \text{and} \quad \nabla u \in (\mathscr{L}_M(Q))^d, \tag{5.16}$$

$$a(\nabla u_{l'}) \rightharpoonup a(\nabla u) \quad \text{weakly* in} \quad (L_{\overline{M}}(Q))^d \quad \text{and} \quad a(\nabla u) \in (\mathscr{L}_{\overline{M}}(Q))^d, \tag{5.17}$$

Lemma 5.4.1. *Let the assumptions of theorem 5.4.1, there is a subsequence, denoted by l', and elements $u \in L^\infty(0,T;L^2(\Omega))$ with $\nabla u \in (\mathscr{L}_M(Q))^d$ and $\gamma_0 u(.,t) = 0$ for almost all $t \in (0,T)$, $z \in L^2(\Omega)$, $\alpha \in (\mathscr{L}_{\overline{M}}(Q))^d$, such that, as $l \to \infty$,*

$$b(u_l) - \widehat{u}_l \to 0 \quad in \quad L^2(Q); \quad b(u_{l'}), \widehat{u}_{l'} \rightharpoonup b(u) \quad in \quad L^\infty(0,T;L^2(\Omega)), \tag{5.18}$$

$$\widehat{u}_{l'}(.,T) = b(u_{l'}(.,T)) \rightharpoonup b(z) \quad in \quad L^2(\Omega); \quad \nabla u_{l'} \rightharpoonup \nabla u \quad in \quad (L_M(Q))^d, \tag{5.19}$$

$$\Phi(u_{l'}) \rightharpoonup \Phi(u) \quad \text{weakly* in} \quad (L_{\overline{M}}(Q))^d; \quad a(\nabla u_{l'}) \rightharpoonup \alpha \quad \text{weakly* in} \quad (L_{\overline{M}}(Q))^d. \tag{5.20}$$

Proof of Lemma 5.4.1

1. By (5.14), the sequence $\{u_l^0\}$ is bounded in $L^2(\Omega)$.Using the definition of u_l and $\widehat{u}$ we obtain

$$\|b(u_l) - \widehat{u}_l\|_{2,Q}^2 = \frac{\tau}{3} \sum_{n=1}^{N_l} \|b(u^n) - b(u^{n-1})\|_{2,Q}^2$$

   and by (5.12), we have $\lim_{l\to\infty} \|b(u_l) - \widehat{u}_l\|_{2,Q}^2 = 0$.
   Also the definition of the approximate solutions allows to get

$$\|b(u_l)\|_{L^\infty(0,T;L^2(\Omega))} = \max_{n=1,2,\ldots,N_l} \|b(u^n)\|_{2,\Omega}, \quad \|\widehat{u}_l\|_{L^\infty(0,T;L^2(\Omega))} = \max_{n=1,2,\ldots,N_l} \|b(u^n)\|_{2,\Omega},$$

   and the inequality (5.12) shows the boundedness of $\{b(u_l)\}$ and $\{\widehat{u}_l\}$ in $L^\infty(0,T;L^2(\Omega))$.We thus have weak* convergence of a subsequence in $L^\infty(0,T;L^2(\Omega))$.
   Since the difference of both the sequences tends to zero in $L^2(Q)$ Their limits must coincide and denoted ϖ. The condition (5.2) allows you to write ϖ in the form $\varpi = b(u)$. That is $b(u_{l'}), b(\widehat{u}_{l'}) \rightharpoonup b(u) \quad \text{in} \quad L^\infty(0,T;L^2(\Omega))$.

2. Since $\|\widehat{u}_l(.,T)\|_{2,\Omega} = \|b(u_l)(.,T)\|_{2,\Omega} = \|b(u^{N_l})\|_{2,\Omega}$, the a priori estimate in (5.12) also proves the asserted weak convergence of a subsequence of $\{\widehat{u}_l(.,T)\}$ in $L^2(\Omega)$ and also its limit can be written as $b(z)$ where $z \in L^2(\Omega)$.
   In view of the discrete a priori estimate of gradients of $\{u_l\}$, we observe that

$$\int_Q M(\nabla u_l) dxdt = \tau \sum_{n=1}^{N_l} \int_\Omega M(\nabla u^n) dx$$

is uniformly bounded (see (5.12)). However, from the boundedness of the modular boundedness of the Luxemburg norm follows. Therefore, $\{\nabla u_l\} \subset (\mathscr{L}_M(Q))^d \subseteq (L_M(Q))^d$ is bounded with respect to $\|\|_{M,Q}$. Since $(L_M(Q))^* = E_{\overline{M}}(Q)$ is separable Banach space, we obtain weak* convergence of a subsequence in $(L_M(Q))^d$ to an element $\xi \in (L_M(Q))^d$ such that $\nabla u_{l'} \rightharpoonup^* \xi$. On the other hand we know that $\mathscr{C}_c^\infty(\Omega) \otimes \mathscr{C}_c^\infty(0;T) \subset E_{\overline{M}}(Q)$, then for all $g \in \mathscr{C}_c^\infty(\Omega)$ and all $\psi \in \mathscr{C}_c^\infty(0;T)$

$$\int_Q \xi g \psi dxdt = \lim_{l' \to \infty} \int_Q \nabla u_{l'} g \psi dxdt = \lim_{l' \to \infty} \int_Q u_{l'} \nabla g \psi dxdt = \int_Q u \nabla g \psi dxdt$$

since $u_{l'} \rightharpoonup u$ weakly in $L^\infty(0,T;L^2(\Omega))$, thus $\xi = \nabla u$.
In view of Lemma 5.1.2, we finally get $\nabla u \in (\mathscr{L}_M(\Omega))^d$.
Since the trace of $u_{l'}$ is zero and since, in particular,

$$\int_Q \nabla u_{l'} z dxdt \to \int_Q \nabla u z dxdt, \quad \int_Q u_{l'} \nabla z dxdt \to \int_Q u \nabla z dxdt$$

for all $z \in L^\infty(0,T;W^{1,q}(\Omega))$ with $q \geq 2$ as $l \to \infty$, also the limit u must have vanishing trace for all $t \in (0,T)$.

3. Finally in view of (5.12) and (5.3) we have

$$\int_Q \overline{M}(a(\nabla u_l)) dxdt \leq \int_Q M(\nabla u_l) dxdt = \tau \sum_{n=1}^{N_l} \int_\Omega M(\nabla u^n) dx$$

and by (5.6) and (9) we have

$$\int_Q \overline{M}(|\Phi(u_l)|) dxdt \leq \int_Q M(\frac{u_l}{\lambda}) dxdt \leq \tau C \sum_{n=1}^{N_l} \int_\Omega M(\nabla u^n)) dx$$

are uniformly bounded. We deduce that there exists α and β in $(L_M(Q))^d$ such that $a(\nabla u_{l'}) \rightharpoonup \alpha$ and $\Phi(u_{l'}) \rightharpoonup \beta$ weakly in $(L_M(Q))^d$ for a subsequence. And by Lemma 5.1.2, $\alpha \in (\mathscr{L}_M(Q))^d$ and $\beta \in (\mathscr{L}_M(Q))^d$.
On the other hand using (5.7) we get $|\Phi(u_l) - \Phi(u)| \leq L(x)|u_l - u|$ and as we have $u_l \to u$ a.e. in Q, we obtain $\beta = \Phi(u)$.

Remark 5.4.1.
We omit writing l' for the sequence from Lemma 5.4.1.

Proof of theorem 5.4.1:
Step 1:
Remark that by definition of $\widehat{u}_l$, the numerical scheme (5.10) can be written as

$$\int_\Omega \Big[\partial_t \widehat{u}_l v + (a(\nabla u_l) + \Phi(u_l))\nabla v\Big] dx = \int_\Omega f_l v dx \quad \text{for all} \quad v \in V_{m_l}. \tag{5.21}$$

This equation holds almost everywhere in $(0,T)$ as well as in the weak sense. This implies

$$-\int_Q \widehat{u}_l R_{m_l} v \psi' dxdt + \int_\Omega \widehat{u}_l(.,T) R_{m_l} v \psi(T) dx - \int_\Omega \widehat{u}_l(.,0) R_{m_l} v \psi(0) dx$$

$$+\int_Q (a(\nabla u_l)+\Phi(u_l))\nabla R_{m_l} v\psi dxdt = \int_Q f_l R_{m_l} v\psi dxdt \quad \text{for all} \quad v\in V, \psi\in \mathscr{C}^1([0,T]).$$

where $\widehat{u}_l(.,T) = b(u^{N_l})$ and $\widehat{u}_l(.,0) = b(u_l^0)$ and by (5.2) we have

$$|b(u_l^0) - b(u_0)| \le b_1 |u_l^0 - u_0|$$

then using Lemma 5.4.1 and (5.14) we obtain respectively $\widehat{u}_l(.,T) \to b(z)$ and $\widehat{u}_l(.,0) \to b(u_0)$ strongly in $L^2(\Omega)$ and also $f_l \to f$ strongly $L^1(0,T;L^2(\Omega))$.

Let now tends l to ∞ for the others terms, we get

$$\left\{\begin{array}{c} R_{m_l} v\psi' \to v\psi' \quad \text{in} \quad L^1(0,T;L^2(\Omega)), \\ R_{m_l} v \to v \quad \text{in} \quad L^2(\Omega), \\ \nabla R_{m_l} v\psi \to \nabla v\psi \quad \text{in} \quad (E_M(Q))^d, \\ R_{m_l} v\psi \to v\psi \quad \text{in} \quad L^1(0,T;L^{2'}(\Omega)). \end{array}\right. \tag{5.22}$$

Apply Lemma 5.4.1 and (5.22), we obtain as $l\to\infty$

$$-\int_Q b(u) v\psi' dxdt + \int_\Omega b(z) v\psi(T) dx - \int_\Omega b(u_0) v\psi(0) dx$$

$$+\int_Q (\alpha+\Phi(u))\nabla v\psi dxdt = \int_Q f v\psi dxdt \quad \text{for all} \quad v\in V, \psi\in\mathscr{C}^1([0,T]). \tag{5.23}$$

This follows from (5.9) and the definition of the norm in V. Observe that
$\|\nabla R_{m_l} v\psi - \nabla v\psi\|_{M,Q} \le \|\psi\|_{\infty,[0,T]} \max(1,T) \|\nabla R_{m_l} v - \nabla v\|_{M,Q}$ and $V \hookrightarrow W^{1,1}(\Omega)\cap L^2(\Omega)$.

Relation (5.23) implies, by density arguments,

$$-\int_Q b(u)\partial_t w dxdt + \int_\Omega b(z) w(.,T) dx - \int_\Omega b(u_0) w(.,0) dx$$

$$+\int_Q (\alpha+\Phi(u))\nabla w dxdt = \int_Q f w dxdt \quad \text{for all} \quad w\in\mathscr{W}. \tag{5.24}$$

Now, remark that the tensor product $V\otimes\mathscr{C}^1([0,T]) \subset \mathscr{W}$, which shows that (5.23) is a particular case of (5.24). The function w_ε that exists in view of Lemma 5.1.3 for any $w\in\mathscr{W}$ can be approximated, with respect to the strong convergence in $\mathscr{C}^1(\overline{Q})$, by a polynomial vanishing at $\partial\Omega\times[0,T]$, which possesses a tensor structure and thus belongs to $V\otimes\mathscr{C}^1([0,T])$. For any $u\in L^\infty(0,T;L^2(\Omega))$, $z,u_0\in L^2(\Omega)$, $\alpha\in(\mathscr{L}_{\overline{M}}(Q))^d$, $f\in\mathscr{C}(0,T;L^2(\Omega))$, any $\varepsilon>0$ $w\in\mathscr{W}$, there is hence (recalling also the continuous embedding of $W^{1,1}(0,T;L^2(\Omega))$ into $\mathscr{C}(0,T;L^2(\Omega))$ an element $w_\varepsilon\in V\otimes\mathscr{C}^1([0,T])$ such that

$$|\int_Q b(u)\partial_t(w_\varepsilon - w)dxdt| + |\int_\Omega b(z)(w_\varepsilon(.,T) - w(.,T))dx| + |\int_\Omega b(u_0)(w_\varepsilon(.,0) - w(.,0))dx|$$

$$+|\int_Q (\alpha+\Phi(u))\nabla(w_\varepsilon - w)dxdt| + |\int_Q f(w_\varepsilon - w)dxdt| < \varepsilon,$$

On the other hand, since $u\in L^\infty(0,T;L^2(\Omega))$, with $\nabla u\in(\mathscr{L}_M(Q))^d \subset L^1(0,T;L_M(\Omega))$, $\alpha\in(\mathscr{L}_{\overline{M}}(Q))^d \subset L^1(0,T;L_{\overline{M}}(\Omega))$, and $f\in\mathscr{C}(0,T;L^2(\Omega))$, we see that for any $v\in V$ the functions

$$t\mapsto\int_\Omega b(u(.,t))v dx, \quad t\mapsto\int_\Omega \alpha(.,t)\nabla v dx, \quad t\mapsto\int_\Omega \Phi(u)(.,t)\nabla v dx, \quad t\mapsto\int_\Omega f(.,t)v dx,$$

are in $L^1(0,T)$ and with (5.23), we get

$$\frac{d}{dt}\int_\Omega b(u(.,t))vdx = \int_\Omega (f(.,t)vdx - \int_\Omega (\alpha(.,t)+\Phi(u)(.,t))\nabla v)dx \tag{5.25}$$

holds true in the weak sense.

Thus, the function $t \mapsto \int_\Omega b(u(.,t))vdx$ is absolutely continuous and since V is dense in $L^2(\Omega)$ with respect to the strong convergence in $L^2(\Omega)$, we obtain $b(u) \in \mathscr{C}_w(0,T;L^2(\Omega))$.

Step 2: Initial and final values

- We now prove $b(u(.,0)) = b(u_0) \in L^2(\Omega)$.
  For any $v \in V$, we have with (5.21)

$$\int_\Omega \widehat{u}_l^0 R_{m_l} vdx = \Big[\int_\Omega \widehat{u}_l(.,t)R_{m_l}vdx\frac{t-T}{T}\Big]_{t=0}^T$$

$$= \int_0^T \Big(\int_\Omega \partial_t \widehat{u}_l R_{m_l}vdx\frac{t-T}{T} + \int_\Omega \widehat{u}_l R_{m_l}vdx\frac{1}{T}\Big)dt$$

$$= \int_0^T \Big([\int_\Omega f_l R_{m_l}vdx - \int_\Omega (a(\nabla u_l)+\Phi(u_l))\nabla R_{m_l}vdx]\frac{t-T}{T}$$

$$+ \int_\Omega \widehat{u}_l R_{m_l}vdx\frac{1}{T}\Big)dt$$

  Pass to the limit, we obtain with integration by parts and using (5.25)

$$\int_\Omega b(u_0)vdx = \int_0^T \Big([\int_\Omega fvdx - \int_\Omega (\alpha+\Phi(u))\nabla vdx]\frac{t-T}{T} + \int_\Omega b(u)vdx\frac{1}{T}\Big)dt$$

$$= \Big[\int_\Omega b(u)vdx\frac{t-T}{T}\Big]_{t=0}^T = \int_\Omega b(u(.,0))vdx.$$

- Changing now the function $t \to \frac{t-T}{T}$ by $t \to \frac{t}{T}$ and using the same argumentation as above provides that the limit $b(z)$ of $\widehat{u}_l(.,T) = b(u_l(.,T))$ weakly in $L^2(\Omega)$, is exactly $b(u(.,T))$,

$$\widehat{u}_l(.,T) \to b(z) = b(u(.,T)) \in L^2(\Omega) \quad \text{as} \quad l \to \infty.$$

Step 3:

- For the convergence of the term $\int_Q \Phi(u_l)\nabla b(u_l)dxdt$ we use (5.7) and Lemma 5.4.1, to deduce that $\Phi(u_l)$ converges to $\Phi(u)$ in $(L_{\overline{M}}(Q))^d$ and $\int_Q \Phi(u_l)\nabla b(u_l)dxdt \to \int_Q \Phi(u)\nabla b(u)dxdt$.

- For the source term, we know that

$$\int_Q f_l b(u_l)dxdt \to \int_Q fb(u)dxdt \quad \text{as} \quad l \to \infty.$$

- We now show that $\alpha = a(\nabla u)$.
  For proving this, we employ a variant of Minty's monotony trick.
  Using $(a-b)a \geq \frac{1}{2}(a^2-b^2)$, we find

$$\int_Q \partial_t \widehat{u} b(u_l) dxdt = \sum_{n=1}^{N_l} \int_\Omega (b(u^n) - b(u^{n-1})) b(u^n) dx \geq \frac{1}{2}(\|b(u^{N_l})\|_{2,\Omega}^2 - \|b(u_l^0)\|_{2,\Omega}^2)$$

$$= \frac{1}{2}(\|b(u_l(.,T))\|_{2,\Omega}^2 - \|b(u_l^0)\|_{2,\Omega}^2),$$

  which implies, because of the weak lower semicontinuity of the norm, the weak convergence of $u_l(.,T)$ to $z = u(T)$ in $L^2(\Omega)$ and the strong convergence (5.14),

$$\frac{1}{2}(\|b(u(.,T))\|_{2,\Omega}^2 - \|b(u^0)\|_{2,\Omega}^2) \leq \liminf_{l\to\infty} \int_Q \partial_t \widehat{u} b(u_l) dxdt. \tag{5.26}$$

  For all $\eta \in (L^\infty(Q))^d$ with (5.5) and (5.2),

$$\int_Q a(\nabla u_l) \nabla b(u_l) dxdt \geq b_0 \int_Q a(\nabla u_l)\eta dxdt + b_0 \int_Q a(\eta)(\nabla u_l - \eta) dxdt.$$

  Remark that $a(\eta) \in (E_{\overline{M}}(Q))^d$ since $\eta \in (L^\infty(Q))^d$ and a is continuous. In the limit, we thus obtain (see again Lemma 5.4.1)

$$\liminf_{l\to\infty} \int_Q a(\nabla u_l)\nabla u_l dxdt \geq b_0 \int_Q \alpha\eta dxdt + b_0 \int_Q a(\eta)(\nabla u - \eta) dxdt. \tag{5.27}$$

  Now taking $v = u_l(.,t) \in V_{m_l}$ in (5.21) and using (5.26) and (5.27), we thus come up with

$$\frac{1}{2}(\|b(u(T))\|_{2,\Omega} - \|b(u^0)\|_{2,\Omega}) + b_0 \int_Q \alpha\eta dxdt + b_0 \int_Q a(\eta)(\nabla u_l - \eta) dxdt$$

$$+ \int_Q \Phi(u)\nabla b(u) dxdt \leq \int_Q f b(u) dxdt \tag{5.28}$$

  We therefore consider the centered Steklov average of u, given by

$$(S_h u)(.,t) = \frac{1}{2h} \int_{t-h}^{t+h} b(u(.,s)) ds, \quad t \in [0,T],$$

  where $h > 0$ and where u is extended by zero outside $[0,T]$. The properties of b and u imply that $S_h u \in \mathscr{W}$. It is know that

$$\lim_{h\to 0} \int_Q f S_h u dxdt = \int_Q f b(u) dxdt.$$

  On the other hand, we find with (5.24)

$$\int_Q f S_h u dxdt = -\int_Q b(u)\partial_t S_h u dxdt + \int_\Omega b(u(.,T)) S_h u(.,T) dx$$

$$-\int_\Omega b(u(.,0))S_h u(.,0)dx+\int_Q(\alpha+\Phi(u))\nabla S_h u dxdt,$$

where $\partial_t S_h u(.,t)=\frac{b(u(.,t+h))-b(u(.,t-h))}{2h}$ and thus

$$\int_Q b(u)\partial_t S_h u dxdt=\frac{1}{2h}\int_0^T\int_\Omega b(u(.,t))(b(u(.,t+h))-b(u(.,t-h)))dxdt$$

$$=\frac{1}{2h}\int_0^{T-h}\int_\Omega b(u(.,t))b(u(.,t+h))dxdt-\frac{1}{2h}\int_h^T\int_\Omega b(u(.,t))b(u(.,t-h))dxdt=0 \quad (5.29)$$

Moreover, we have

$$\int_\Omega b(u(.,T))S_h u(.,T)dx=\frac{1}{2h}\int_{T-h}^T\int_\Omega b(u(.,T))b(u(.,s))dxds$$

$$\to\frac{1}{2}\int_\Omega b(u(.,T))^2dx=\frac{1}{2}\|b(u(.,T))\|^2_{2,\Omega}\quad\text{as}\quad h\to 0. \quad (5.30)$$

Recall here that $u\in\mathscr{C}_w([0,T];L^2(\Omega))$ and thus $s=T$ is a Lebesgue point of the mapping $[0,T]\ni s\mapsto\int_\Omega b(u(.,T))b(u(.,s))dx$.
Similarly, we have

$$\int_\Omega b(u(.,0))S_h u(.,0)dx\to\frac{1}{2}\|b(u_0)\|^2_{2,\Omega}\quad\text{as}\quad h\to 0.$$

Finally, we observe that

$$\int_Q\alpha\nabla S_h u dxdt-\int_Q\alpha\nabla b(u)dxdt$$

$$=\frac{1}{2}\int_0^T\int_{t-h}^{t+h}\int_\Omega\alpha(.,t)\nabla(b(u(.,s))-b(u(.,t)))dxdsdt$$

$$=\frac{1}{2}\int_{-1}^1\int_0^T\int_\Omega\alpha(.,t)\nabla(b(u(.,t+rh))-b(u(.,t)))dxdrdt\to 0\quad\text{as}\quad h\to 0. \quad (5.31)$$

Similarly we have

$$\int_Q\Phi(u)\nabla S_h u dxdt-\int_Q\Phi(u)\nabla b(u)dxdt\to 0\quad\text{as}\quad h\to 0. \quad (5.32)$$

since the translation of a function in the Orlicz space $L_M(Q)$ is continuous with respect to the weak convergence in $E_M(Q)$ (see [77]).
Finally we obtain as $h\to 0$

$$\frac{1}{2}(\|b(u(T))\|_{2,\Omega}-\|b(u^0)\|_{2,\Omega})+b_0\int_Q\alpha\nabla b(u)dxdt+\int_Q\Phi(u)\nabla b(u)dxdt=\int_Q fb(u)dxdt \quad (5.33)$$

and from (5.28), (5.33) and (5.2), we get for all $\eta\in(L^\infty(Q))^d$

$$0\le\int_Q[b_0a(\eta)-(b_1-b_0)\alpha](\eta-\nabla u)dxdt=b_0\int_Q(a(\eta)-\alpha)(\eta-\nabla u)dxdt.$$

Following the modification of Minty's trick in [79], we set $Q_k = \{(x,t) : |\nabla u(x,t)| > k\}$ for any $k \in I\!N$. For arbitrary $i, j \in I\!N$ with $j < i$, arbitrary $\lambda > 0$, and arbitrary $\xi \in (L^\infty(Q)^d$, we take

$$\eta = \nabla u \chi_{Q\setminus Q_i} + \lambda\zeta\chi_{Q\setminus Q_j} = \begin{cases} 0 & \text{in} \quad Q_i, \\ \nabla u & \text{in} \quad Q_j\setminus Q_i, \\ \nabla u + \lambda\zeta & \text{in} \quad Q\setminus Q_j. \end{cases}$$

Thus

$$0 \le -\int_{Q_i} (a(0)-\alpha)\nabla u dxdt + \lambda \int_{Q\setminus Q_j} (a(\nabla u + \lambda\zeta) - \alpha)\zeta dxdt.$$

Since $(a(0)-\alpha)\nabla u \in L^1(Q)$, we have

$$\int_{Q_i} (a(0)-\alpha)\nabla u dxdt \to 0 \quad \text{as} \quad i \to \infty$$

Then

$$0 \le \lambda \int_{Q\setminus Q_j} (a(\nabla u + \lambda\zeta) - \alpha)\zeta dxdt.$$

On the other hand a is monotone, such that for $\lambda \in [0,1]$

$$a(\nabla u + \lambda\zeta)\zeta \le a(\nabla u + \zeta)\zeta \in L^1(Q),$$

a is continuous, then by dominated convergence theorem we get

$$\int_{Q\setminus Q_j} (a(\nabla u + \lambda\zeta) - \alpha)\zeta dxdt \to \int_{Q\setminus Q_j} (a(\nabla u) - \alpha)\zeta dxdt \quad \text{as} \quad \lambda \to 0$$

and thus

$$0 \le \int_{Q\setminus Q_j} (a(\nabla u) - \alpha)\zeta dxdt$$

for any $j \in I\!N$ and any $\xi \in (L^\infty(Q))^d$.

The choice $\begin{cases} \zeta = -\frac{a(\nabla u)-\alpha}{|a(\nabla u)-\alpha|} & \text{if} \quad a(\nabla u) \ne 0, \\ \zeta = 0, & \text{otherwise.} \end{cases}$

allows us to get

$$\int_{Q\setminus Q_j} |a(\nabla u) - \alpha| dxdt \le 0$$

and thus $\alpha = a(\nabla u)$ almost everywhere in $Q\setminus Q_j$. Since j was arbitrary, this proves the equality almost everywhere in Q, which complete the proof.

Step 4:Uniqueness

Let u and v be two solutions to the problem with the same data (u_0, f). From the proof above, we have

$$\int_Q (b(u)-b(v))\partial_t w dxdt + \int_\Omega (b(u(.,T)) - b(v(.,T)))w(.,T)dx + \int_Q (a(\nabla u) - a(\nabla v))\nabla w dxdt$$

$$+ \int_Q (\Phi(u) - \Phi(v))\nabla w dxdt = 0 \quad \text{for all} \quad w \in \mathscr{W}. \tag{5.34}$$

Thus

$$\int_Q \partial_t(b(u)-b(v))w dxdt \leq -\int_Q (a(\nabla u)-a(\nabla v))\nabla w dxdt$$

$$-\int_Q (\Phi(u)-\Phi(v))\nabla w dxdt = 0 \quad \text{for all} \quad w \in \mathscr{W}. \tag{5.35}$$

For all $\bar{t} \in [0;T]$, taking $w_k = \frac{1}{k} T_k(b(u)-b(v))w_{\varepsilon,\bar{t}}$ where

$$w_{\varepsilon,\bar{t}}(t) = \begin{cases} 1 & \text{if} \quad 0 \leq t \leq \bar{t}-\varepsilon, \\ \frac{\bar{t}-t}{\varepsilon} & \text{if} \quad \bar{t}-\varepsilon < t \leq \bar{t}, \\ 0 & \text{otherwise.} \end{cases}$$

In the same way as the above

$$lim_{k\to 0} \int_Q (\Phi(u)-\Phi(v))\nabla w_k dxdt = 0$$

and by the monotonicity of a we obtain

$$\int_Q \partial_t(b(u)-b(v))sgn(b(u)-b(v))w_{\varepsilon,\bar{t}} dxdt = lim_{k\to 0} \int_Q \partial_t(b(u)-b(v))w_k dxdt \leq 0.$$

On the other hand, we have

$$\int_Q \partial_t(b(u)-b(v))sgn(b(u)-b(v))w_{\varepsilon,\bar{t}} dxdt = \int_0^{\bar{t}-\varepsilon} \int_\Omega \partial_t(b(u)-b(v))sgn(b(u)-b(v))dxdt$$

$$+\int_{\bar{t}-\varepsilon}^{\bar{t}} \frac{\bar{t}-t}{\varepsilon} \int_\Omega \partial_t(b(u)-b(v))sgn(b(u)-b(v))dxdt.$$

Employing dominated convergence theorem, as $\varepsilon \to 0$ we get

$$0 \geq \int_0^{\bar{t}} \int_\Omega \partial_t(b(u)-b(v))sgn(b(u)-b(v))dxdt = \int_0^{\bar{t}} \frac{d}{dt}\|b(u)-b(v)\|_{L^1(\Omega)} dt,$$

and thus

$$\|b(u(.,\bar{t}))-b(v(.,\bar{t}))\|_{L^1(\Omega)} = 0 \quad \text{for all} \quad \bar{t} \in (0;T],$$

and thus the uniqueness.

5.5 Error estimate for the temporal semidiscretization

We just give the error estimate in the temporal semidiscretization case since in the complete discretization is far from being easy to make. Let $e^n = u(.,t_n) - u^n$ be the error between the exact solution and numerical solution.

Theorem 5.5.1. *Let $u_0, u^0 \in L^2(\Omega)$, $f \in L^1([0,T];L^2(\Omega))$, $u, \partial_t u, \partial_{tt}^2 u \in L^1([0,T];L^2(\Omega))$ with $u(.,t) \in \mathscr{V} = \{v \in L^2(\Omega) : \nabla v \in (\mathscr{L}_M(\Omega))^d, \gamma_0 u = 0\}$, $a(\nabla u(.,t)), \Phi(u(.,t)) \in (\mathscr{L}_{\overline{M}}(\Omega))^d$ for all*

$t \in [0,T]$. *Let* $u^n \in \mathscr{V}$ *with* $a(\nabla u^n), \Phi(u^n) \in (\mathscr{L}_{\overline{M}}(\Omega))^d$ *be an approximation of* $u(.,t_n)$ *such that* $n = 1,2,...,N$,

$$\int_\Omega \Big(\frac{b(u^n) - b(u^{n-1})}{\tau} v + a(\nabla u^n)\nabla v + \Phi(u^n)\nabla v \Big) dx = \int_\Omega f(.,t_n) v dx \quad \textit{for all} \quad v \in \mathscr{V}.$$

Then for $n = 1,2,...,N$,

$$\|e^n\|_{L^1(\Omega)} \leq C(\|u_0 - u^0\|_{L^1(\Omega)} + \tau \|\partial_{tt}^2 u\|_{L^1([0,T];L^2(\Omega))} + \|\overline{f} - f\|_{L^1(\Omega)}) \tag{5.36}$$

where $\overline{f}$ *denotes the piecewise constant in time interpolation of* f *with respect* $(t_n)_{n=1}^N$.

Proof. Let $R^n = b(u(.,t_n)) - b(u^n)$ then

$$\int_\Omega \frac{R^n - R^{n-1}}{\tau} v dx = \int_\Omega \frac{b(u(.,t_n)) - b(u(.,t_{n-1}))}{\tau} v dx - \int_\Omega \frac{b(u^n) - b(u^{n-1})}{\tau} v dx.$$

By integrate par parts and (5.10) we have

$$\begin{aligned}
\int_\Omega \frac{R^n - R^{n-1}}{\tau} v dx &= \int_\Omega \partial_t b(u(.,t_n)) v dx - \frac{1}{\tau} \int_{t_{n-1}}^{t_n} \int_\Omega (t - t_{n-1}) \partial_{tt} b(u(.,t)) v dx dt \\
&\quad - \int_\Omega \frac{b(u^n) - b(u^{n-1})}{\tau} v dx \\
&= \int_\Omega (f(.,t_n) - f_n) v dx - \int_\Omega (a(\nabla u(.,t_n)) - a(\nabla u^n)) v dx - \int_\Omega (\Phi(u(.,t_n)) - \Phi(u^n)) v dx \\
&\quad - \frac{1}{\tau} \int_{t_{n-1}}^{t_n} \int_\Omega (t - t_{n-1}) \partial_{tt} b(u(.,t)) v dx dt
\end{aligned} \tag{5.37}$$

Taking $v_k = \frac{1}{k} T_k(R^n)$ and using (5.4) we have

$$\Big| \int_\Omega (f(.,t_n) - f_n) v dx \Big| \leq \|f(.,t_n) - f_n\|_{L^1(\Omega)}$$

$$\begin{aligned}
-\int_\Omega (a(\nabla u(.,t_n)) - a(\nabla u^n)) \nabla v dx &\leq -b_0 \int_{\{|R^n| \leq k\}} (a(\nabla u(.,t_n)) - a(\nabla u^n)) \\
&\quad \times \nabla (u(.,t_n) - u^n) dx \leq 0
\end{aligned}$$

$$\frac{1}{\tau} \int_{t_{n-1}}^{t_n} \int_\Omega (t - t_{n-1}) \partial_{tt}^2 b(u(.,t)) v dx dt \leq \frac{1}{\tau} \int_{t_{n-1}}^{t_n} \int_\Omega (t - t_{n-1}) |\partial_{tt}^2 b(u(.,t))| dx dt$$

and

$$\begin{aligned}
\Big| \int_\Omega (\Phi(u(.,t_n)) - \Phi(u^n)) v dx \Big| &\leq \frac{b_0}{k} \int_{\{|R^n| \leq k\}} L(x) |u(.,t_n)) - u^n| |\nabla u(.,t_n)) - \nabla u^n| dx \\
&\leq \frac{b_0}{k} \int_{\{|R^n| \leq k\}} L(x) |\nabla u(.,t_n) - \nabla u^n| dx
\end{aligned}$$

Since $L(x)|\nabla u(.,t_n)) - \nabla u^n| \in L^1(\Omega)$ we pass to the limit as $k \to +\infty$, and we have

$$\lim_{k\to+\infty} \int_\Omega (\Phi(u(.,t_n)) - \Phi(u^n))v dx = 0$$

and also

$$\lim_{k\to+\infty} \int_\Omega \frac{R^n - R^{n-1}}{\tau} v_k dx = \int_\Omega \frac{R^n - R^{n-1}}{\tau} sgn(R^n) dx$$

Recalling (5.37) we obtain

$$\int_\Omega \frac{|R^n|}{\tau} dx \leq \|f(.,t_n) - f_n\|_{L^1(\Omega)} + \frac{1}{\tau} \int_{t_{n-1}}^{t_n} \int_\Omega (t - t_{n-1}) |\partial_{tt}^2 b(u(.,t))| dx dt + \int_\Omega \frac{|R^{n-1}|}{\tau} dx$$

Thus summation $n = 1, ..., N$ and using (5.2)

$$b_0 \|e^n\|_{L^1(\Omega)} \leq \tau \sum_{n=1}^{N} \|f(.,t_n) - f_n\|_{L^1(\Omega)} + 2\tau \int_0^T |\partial_{tt}^2 b(u(.,t))|_{L^1(\Omega)} dt + b_1 \|e^0\|_{L^1(\Omega)}$$

Thus, together with the estimate $\tau \sum_{n=1}^{N} \|f(.,t_n) - f_n\|_{L^1(\Omega)} \leq c\|\overline{f} - f\|_{L^1(\Omega)}$, we deduce (5.36). Remark that $L^1(0,T;L^2(\Omega)) \subset L^1(0,T;L^1(\Omega))$ □

5.6 Numerical illustration

we will apply these results on this following simple example.
To simplify and without losing any generality we take

- $\Omega = [0;1] \times [0;1], Q = \Omega \times [0;1]$
- $M(t) = |t|^{p_1} + |t|^{p_2}, \quad p_1 > 2, p_2 > 2$ and $\overline{M}(t) = \mu_1 |t|^{q_1} + \mu_2 |t|^{q_2}$ such that $\frac{1}{p_1} + \frac{1}{q_1} = 1$ and $\frac{1}{p_2} + \frac{1}{q_2} = 1$.
- b(s)=s.
- $a(\xi) = (\mu_1 |\xi|^{p_1 - 2} + \mu_2 |\xi|^{p_2 - 2})\xi$, where $\mu_1 = \frac{2p_1 - 1}{(2p_1)^{q_1}}$ and $\mu_2 = \frac{2p_2 - 1}{(2p_2)^{q_2}}$.
- $\Phi(u) = \Big((\mu_1 |u|^{p_1 - 1} + \mu_2 |u|^{p_2 - 1})u, \mu_1 |u|^{p_1 - 1} + \mu_2 |u|^{p_2 - 1})u \Big)$.
- we choose f such that $u(x,y,t) = e^{-t}(x - x^2)(y - y^2)$ is the solution exact of the equation (5.1).

It is easy to show that all the terms b, a, Φ and f verify the conditions (5.4)-(5.7).

We preferred the finite element method rather than the finite difference method or the finite volume method, because we are looking for a weak solution that requires an integration of the problem (5.10).

We use a particular conforming finite element approximation in the double-dimensional case where $p_1 = 4$, $p_2 = 5$ and the time step size is taken as the square of the spatial mesh size. Each time step solves a nonlinear problem (5.10) using the Backward Euler method. We give The table of convergence errors measured (**Table 1**), also we give the exact solution (**Fig. 1**), the numerical solution (**Fig. 2**) and the difference between the exact solution and the numerical solution (**Fig. 3**) as 3D plots . Finally we take $u^0(x,y) = e^{-t}(x - x^2)(y - y^2)(1 + \frac{1}{N})$.

h	τ	$\|u^n - u_l\|_{L^\infty(0,T,(L^2(\Omega))}$
1/2	1/4	0.04
1/3	1/9	0.025
1/5	1/25	0.0011
1/9	1/81	3.4358E-04
1/10	1/100	2.7966E-04
1/15	1/225	1.2573E-04
1/20	1/400	7.1008E-05
1/25	1/900	4.5529E-05

Table 5.1 – The table of convergence errors measured as $\|u^n - u_l\|_{L^\infty(0,T,(L^2(\Omega))}$ where h is the mesh size and τ is the time step.

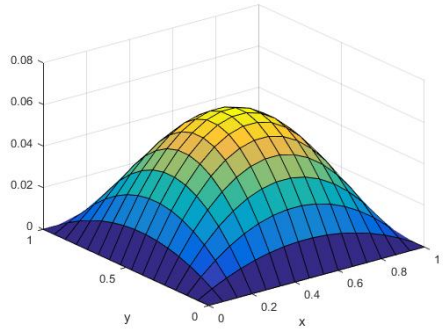


Figure 5.1 – *Exact solution* $t_n = 5\tau$ *where* $h = 1/15$ *and* $\tau = 1/225$.

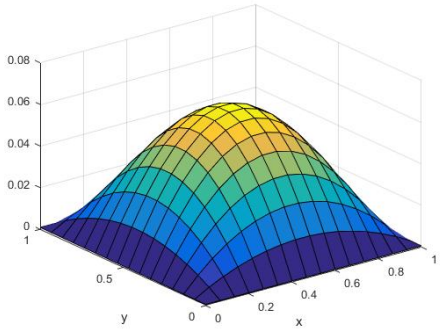


Figure 5.2 – *Numerical solution at* $t_n = 5\tau$ *where* $h = 1/15$ *and* $\tau = 1/225$.

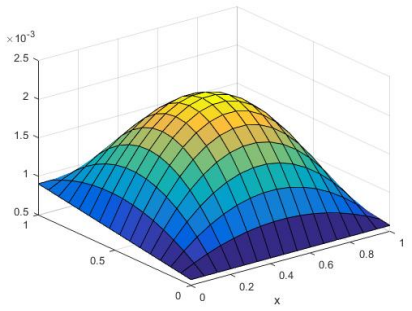


Figure 5.3 – *The difference* $|u^n(x,y) - u(t_n,x,y)|$ *at* $t_n = 5\tau$ *where* $h = 1/15$ *and* $\tau = 1/225$.

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