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Study of some nonlinear elliptic systems with variable exponents

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Résumé

Dans cette thèse, nous étudions quelques systèmes aux dérivées partielles non linéaires de type elliptiques faisant intervenir des opérateurs non homogènes. L'une des difficultés majeures qui apparaissent lors de l'étude de ces systèmes est la perte de compacité dans les inclusions de Sobolev car ou le systéme est défini sur l'espace $\mathbb{R}^N$ tout entier ou bien il contient des exposants critiques. Une seconde difficulté concerne le choix des nonlinéarités, qui doivent satisfaire des conditions de croissance. Ces conditions garantissent la différentiabilité, la coercivité et la semicontinuité la fonctionnelle d'énergie associée au système. Enfin, la derniére difficulté est liée à la construction des espaces fonctionnels à poids pour assurer la structure d'espaces de Banach reflexifs ainsi que les différentes inclusions de Sobolev.

Mots-clefs: Solutions faibles, unicité, solutions multiples, espaces à exposants variables , exposants critiques de Sobolev, $p(x)$-Laplacien, problèmes de type Kirchhoff, principe de criticité symétrique, principe de concentration-compacité, condition de Palais-Smale, théorème du col de la montagne, théorie des points critiques.

Abstract

We study some nonlinear partial differential elliptic systems involving nonhomogeneous operators. One of the major difficulties which appear while we study these systems is the loss of compactness in the Sobolev inclusions because the system is either defined on the entire space $\mathbb{R}^N$ or contains critical exponents. A second difficulty concerns the choice of nonlinearities, which must satisfy growth conditions. These conditions guarantee the differentiability, coercivity and semicontinuity of the energy functional associated with the system. Finally, the last difficulty is linked to the construction of functional spaces with weight to ensure the structure of reflexive Banach spaces as well as the different Sobolev inclusions.

keywords: Weak solutions, uniqueness, multiple solutions, Variable exponents spaces,critical Sobolev exponents, $p(x)$-Laplacian, Kirchhoff-type problems, principle of symmetric criticality, concentration-compactness principle, Palais-Smale condition, Mountain Pass theorem, critical points theory.

Résumé de thèse

Dans ce travail, nous étudions des problèmes elliptiques non-linéaires faisant intervenir l'opérateur p(x)-Laplacien. La non homogénéité de l'opérateur rend l'étude délicate et fait appel à des espaces fonctionnels non classiques. Ces espaces sont des cas particuliers des espaces d'Orlicz dits espaces de Sobolev généralisés et notés W, dont la topologie est induite par une norme de manipulation ardue, appelée norme du Luxemburg.

Dans **Le chapitre I**, nous commençons par donner un bref exposé des définitions et résultats nécessaires à la suite de ce travail. Nous introduisons les espaces de Lebesgue à poids généralisés et de Sobolev à poids généralisés, en faisant remarquer que la quasi-totalité des propriétés classiques surtout les immersions, demeurent vraies. Enfin nous nous sommes efforcés de reprendre les résultats relatifs à la théorie des points critiques pour rendre la lecture plus aisée.

Le chapitre II, nous traitons les questions d'existence et d'unicité des solutions de certains (p(x),q(x))-systèmes elliptiques de type potentiel de la forme

$$\begin{cases} -\Delta_{p(x)}u = \frac{\partial F}{\partial u}(x,u,v) & \text{dans } \Omega, \\ -\Delta_{q(x)}v = \frac{\partial F}{\partial v}(x,u,v) & \text{dans } \Omega, \\ u = v = 0 & \text{sur } \partial\Omega; \end{cases}$$

où Ω est un domaine borné de $\mathbb{R}^N$, $\Delta_{p(x)}u = \operatorname{div}(|\nabla u|^{p(x)-2}\nabla u)$ est l'opérateur $p(x)$-Laplacian, $p(x)$ et $q(x)$ sont des fonctions réelles continues telles que $1 < p(x), q(x) < N$ $(N \geq 2) \forall x \in \mathbb{R}^N$ et la fonction F appartient à $C^1(\Omega \times \mathbb{R}^2)$. Le résultat d'existence est obtenu dans le cas Ω quelconque mais le résultat d'unicité conccerne uniquement le cas des solutons radiales lorsque Ω est une boule.

Dans **Le chapitre 3**, pour λ appartenant à un certain intervalle défini au préalable, nous montrons que le système suivant admet trois solutions faibles :

$$-\Delta_{p_i(x)}u_i + a_i(x)|u_i|^{p_i(x)-2}u_i = \lambda F_{u_i}(x, u_1, u_2, ..., u_n) \quad \text{in } \mathbb{R}^N.$$

Pour cela nous utilisons un recent résultat abstrait en théorie des points critiques établi par G. Bonanno et S. A. Marano.

Dans **Le chapitre 4**, nous étudions les questions d'existence et de multiplicité des solutions pour le système elliptique quasilinéaire nonlocal dde type potontiel suivant :

$$\begin{cases} -M_1\left(L_p(u)\right)(\Delta_{p(x)}u - a(x)|u|^{p(x)-2}u) = \lambda F_u(x,u,v) & \text{dans } \mathbb{R}^N, \\ -M_2\left(L_q(v)\right)(\Delta_{q(x)}v - b(x)|v|^{q(x)-2}v) = \lambda F_v(x,u,v) & \text{dans } \mathbb{R}^N; \end{cases}$$

avec $N \geq 2$, les exposants p and $q \in C_*(\mathbb{R}^N) := \{r \in C(\mathbb{R}^N) : 1 < r^- = \inf_{x\in\mathbb{R}^N} r(x) \leq r(x) \leq r^+ = \sup_{x\in\mathbb{R}^N} r(x) < N, \forall x \in \mathbb{R}^N\}$, λ un paramétre positif réel et a, $b \in L^\infty(\mathbb{R}^N)$ vérifient $a := ess\inf_{x\in\mathbb{R}^N} a(x) > 0$ et $b := ess\inf_{x\in\mathbb{R}^N} b(x) > 0$.

M_1 et M_2 sont des fonctions continues et bornées, $F \in C^1(\mathbb{R}^N \times \mathbb{R}^2)$ vérifie des conditions de croissance convenables, $\Delta_{p(x)}u$ et $L_r(u)$ sont donnés par

$$\Delta_{p(x)}u := \operatorname{div}(|\nabla u|^{p(x)-2}\nabla u), \qquad L_r(u) = \int_{\mathbb{R}^N} \frac{1}{r(x)}\left(|\nabla u|^{r(x)} + a(x)|u|^{r(x)}\right)dx.$$

Les resultats obtenus au chapitre 4 généralisent ceux obtenus au chapitre 3 au cas d'operateurs nonlocaux de type kirchhoff et utilisent les mêmes techniques de démonstration.

Dans **le cinquième chapitre**, nous montrons l'existence de solutions non trivials $u = (u_1, ..., u_n)$ pour la classe suivante de systèmes elliptiques quasilinéaires nonlocaux à exposants variables critiques :

$$\begin{cases} -M_i\left(\int_\Omega \frac{1}{p_i(x)}|\nabla u_i|^{p_i(x)}\right)\Delta_{p_i(x)}u_i = |u_i|^{q_i(x)-2}u_i + \lambda F_{u_i}(x, u_1, u_2, ..., u_n) & \text{dans } \Omega, \\ u_i = 0 & \text{sur } \partial\Omega; \end{cases}$$

où Ω est un domaine borné de $\mathbb{R}^N (N \geq 2)$, à frontière $\partial\Omega$ suffisamment régulière , $\Delta_{p_i(x)}u_i := \operatorname{div}(|\nabla u_i|^{p_i(x)-2}\nabla u_i)$ pour $1 \leq i \leq n$, λ est un paramètre réel positif et $p_i(x)$ and $q_i(x)$ sont des fonctions réelles de Lipschitz telles que

$$1 < p_i^- := \inf_{x\in\overline{\Omega}} p_i(x) \leq p_i(x) \leq p_i^+ = \sup_{x\in\overline{\Omega}} p_i(x) < N, \quad 1 \leq q_i(x) \leq p_i^\star(x) = \frac{Np_i(x)}{N - p_i(x)},$$
$$\mathcal{A}_{p_i} = \{x \in \Omega, q_i(x) = p_i^\star(x)\} \neq \emptyset.$$

La fonction F appartient à $C^1(\Omega \times \mathbb{R}^n)$ et $M_i : \mathbb{R}_0^+ \to \mathbb{R}^+$ est une fonction continue croissante, vérifiant certaines condition de croissance.

La démonstration des résultats de ce chapitre est essentiellement basée sur le principe de concentration-compacité dû initialement à P. L. Lions.

Finalement, dans **le chapitre 6** , nous étendons les résultats du chapitre 5 au cas de systèmes elliptiques quasilinéaires nonlocaux à exposants variables critiques du type :

$$\begin{cases} -M_i\left(\mathcal{A}_i(u_i)\right)\operatorname{div}\Big(\mathcal{B}_i(\nabla u_i)\Big) = |u_i|^{s_i(x)-2}u_i + \lambda F_{u_i}(x,u) & \text{in } \Omega, \\ u_i = 0 & \text{on } \partial\Omega; \end{cases}$$

pour $1 \leq i \leq n$ ($n \in \mathbb{N}$), avec Ω est un domaine borné de $\mathbb{R}^N (N \geq 2)$, à frontière $\partial\Omega$ suffisamment réguliére et λ est un paramétre positif.

Les opérateurs $\mathcal{B}_i : X_i \to \mathbb{R}^n$ et $\mathcal{A}_i : X_i \to \mathbb{R}$, étant respectivement définies par :

$$\mathcal{B}_i(u_i) = a_i(|\nabla u_i|^{p_i(x)})|\nabla u_i|^{p_i(x)-2}\nabla u_i, \text{ and } \mathcal{A}_i(u_i) = \int_\Omega \frac{1}{p_i(x)} A_i(|\nabla u_i|^{p_i(x)})dx,$$

où X_i est un espace de Banach

$$X_i := W_0^{1,p_i(x)}(\Omega) \cap W_0^{1,\gamma_i(x)}(\Omega),$$

$A_i(.)$ est la fonction $A_i(t) = \displaystyle\int_0^t a_i(k)dk$, avec $a_i(.)$ une fonction de classe $\mathcal{C}^1$ encadrée par deux fonctions convenables. Comme au chapitre 5, nous montrons l'existence de solutions non triviales dans le cas où les exposants variables sont critiques en utilisant les mêmes techniques de démonstration appliquées au chapitre 5.

L'une des difficultés majeures qui apparaissent lors de l'étude de ces systèmes est la perte de compacité dans les inclusions de Sobolev car le système est défini sur l'espace $\mathbb{R}^N$ tout entier. En guise de compensation, la pondération par des fonctions poids s'avère incontournable. Une seconde difficulté concerne le choix des nonlinéarités, qui doivent satisfaire des conditions de croissance. Ces conditons garantissent la différentiabilité, la coercivité et la semicontinuité la fonctionnelle d'énergie associée au système. Enfin la dernière difficulté est liée à la construction des espaces fonctionnels à poids pour assurer la structure d'espaces de Banach reflexif ainsi que les différentes inclusions de Sobolev.

Contents

Introduction

Partial differential equations are a precise, elegant, rich, and captivating subject, which is quite old, and its history is broad and deep. Elliptic partial differential equations are startling due to their elegance and clarity. One progresses very rapidly from the basics of the linear Laplace, Poisson, and Helmholtz equations to profound results concerning nonlinear elliptic problems and the qualitative analysis of their solutions. The study of physical models has remained up to the present one of the fundamental concerns of the development of partial differential equations. The idea of using analytic methods to study partial differential equations has its foundations in the ideas of Poincaré. This modern approach provides a new way to view the analysis of linear or nonlinear partial differential equations. As stated by Brezis and Browder [15], ”Poincaré emphasized that a wide variety of physically significant problems arising in very different areas (such as electricity, hydrodynamics, heat flow, magnetism, optics, elasticity, etc.) have a family resemblance-un air de famille in Poincaré's words-and should be treated by common methods”.

Historically, variable exponent Lebesgue spaces appeared in the literature in 1931 in the paper by Orlicz [68]. He was interested in the study of function spaces that contain all measurable functions $u : \Omega \longrightarrow \mathbb{R}$ such that

$$\rho(\lambda u) \int_{\Omega} \varphi(\lambda \left|u(x)\right|)dx,$$

for some $\lambda > 0$ and φ satisfying some natural assumptions, where Ω is an open set in $\mathbb{R}^N$. This space is denoted by L^{φ} and it is now called Orlicz space. However, we point out that in [68] the case $\left|u(x)\right|^{p(x)}$ corresponding to variable exponents was not included. In the 1950's these problems were systematically studied by Nakano [67], who developed the theory of modular function spaces. Nakano explicitly mentioned variable exponent Lebesgue spaces as an example of more general spaces he considered, see Nakano ([67], p. 284). Later, Polish mathematicians investigated the modular function spaces, see Musielak [66]. Variable exponent Lebesgue spaces on the real line have been independently developed by Russian researchers. In that context, we refer to the work of Tsenov [81] and Sharapudinov [79]. They were interested in the minimization of functionals like

$$\int_a^b |u(x) - v(x)|^{p(x)}\, dx,$$

where u is a fixed function and v varies over a finite dimensional subspace of $L^{p(x)}([a, b])$. Zhikov [86] started a new direction of investigation, which created the relationship between spaces with variable exponent and variational integrals with nonstandard growth conditions. We also point out the contributions of Marcellini [62], who studied minimization problems with (p, q)-growth, namely

$$\inf \int_\Omega F(x, |\nabla u|)dx,$$

where $t^p \leq F(x, t) \leq t^q + 1$ for all $t \geq 0$. The case corresponding to the variable exponent corresponds to $F(x, t) = t^{p(x)}$, where $p : \Omega \to (1, \infty)$ is a bounded function.

In 1991, Kovacik and Rakosnik [53] established several basic properties of spaces $L^{p(x)}$ and $W^{1,p(x)}$ with variable exponents. Their results were extended by Fan and Zhao [38] in the framework of Sobolev spaces $W^{m,p(x)}$. Pioneering regularity results for functionals with nonstandard growth are due to Acerbi and Mingione [2]. Density of smooth functions in $W^{m,p(x)}(\Omega)$ and related Sobolev embedding properties are due to Edmunds and Rakosnik [36, 37]. We also point out the important contributions of the Finnish research group on variable exponent spaces and image processing, whose main goal was to study nonlinear potential theory in variable exponents Sobolev spaces. The abstract theory of Lebesgue and Sobolev spaces with variable exponents was developed in the monograph by Diening, Harjulehto, Hästö and Ružicka; [31].

The study of differential equations and variational problems involving $p(x)$-growth conditions is a consequence of their applications. In 1920 Bingham was surprised to discover that some paints do not run like honey. He studied such a behavior and described a strange phenomenon. There are fluids that first flow, then stop spontaneously (Bingham fluids). Inside them, the forces that create the flows reach a threshold. As this threshold is not reached, the fluid flow deforms as a solid. Invented in the 17th century, the "Flemish medium" makes painting oil thixotropic: it flows under pressure of the brush, but freezes as soon as you leave it to rest. While the exact composition of the Flemish medium remains unknown, it is known that the bonds form gradually between its components, which is why the picture freezes in a few minutes. Thanks to this wonderful medium, Rubens was able to paint La Kermesse in only 24 hours.

Recent systematic study of partial differential equations with variable exponents was motivated by the description of several relevant models in electrorheological and thermorheological fluids, image processing, or robotics. In what follows, we give two relevant examples that justify the mathematical study of models involving variable exponents. The first example is due to Chen, Levine, Rao [17] and it concerns applications to image restoration. Let us consider an input I that corresponds to shades of gray in a domain $\Omega \subset \mathbb{R}^2$. We assume that I is made up of the true image corrupted by the noise. Suppose that the noise is additive, that is, $I = T + \eta$ where T is the true image and η is a random variable with zero mean. Thus, the effect of the noise can be eliminated by smoothing the input, since this will cause the effect of the zeromean random variables at nearby locations to cancel. Smoothing corresponds to minimizing the energy

$$\mathcal{E}_1(u) = \int_\Omega (|\nabla u(x)|^2 + |u(x) - I(x)|^2)dx.$$

Unfortunately, smoothing destroys the small details of the image, so this procedure is not useful. A better approach is the total variation smoothing. Since an edge in the image gives rise to a very large gradient, the level sets around the edge are very distinct, so this method does a good job of preserving edges. Total variation smoothing corresponds to minimizing the energy

$$\mathcal{E}_2(u) = \int_\Omega (|\nabla u(x)| + |u(x) - I(x)|^2)dx.$$

Unfortunately, total variation smoothing not only preserves edges, but it also creates edges where there were none in the original image. This is called the staircase effect.

Looking at $\mathcal{E}_1$ and $\mathcal{E}_2$, Chen, Levine and Rao suggested that an appropriate energy is

$$\mathcal{E}(u) = \int_\Omega (|\nabla u(x)|^{p(x)} + |u(x) - I(x)|^2)dx,$$

where $1 \leq p(x) \leq 2$.

This function should be close to 1 where there are likely to be no edges, and close to 2 where there are likely to be no edges. The approximate location of the edges can be determined by just smoothing the input data and looking where the gradient is large.

The diffusion resulting from the model proposed by Chen, Levine and Rao is a combination of the Gaussian smoothing and regularization based on the total variation. More exactly, the following adaptive model was proposed

$$\min_{I=u+v, u \in BV \cap L^2(\Omega)} \int_\Omega \varphi(x, \nabla u)dx + \lambda. \|u\|^2_{L^2(\Omega)}, \tag{0.0.1}$$

where $\Omega \subset \mathbb{R}^2$ is an open domain,

$$\varphi(x,r) = \begin{cases} \frac{1}{p(x)}|r|^{p(x)}, & \text{for} \quad |r| \leq \beta \\ |r| - \frac{\beta.p(x) - \beta^{p(x)}}{p(x)}, & \text{for} \quad |r| > \beta, \end{cases}$$

where $\beta > 0$ is fixed and $1 < \alpha \leq p(x) \leq 2$. the function $p(x)$ nvolved here depends on the location of x in the model. For instance, it can be

$$p(x) = 1 + \frac{1}{1 + k|\nabla G_\sigma * I|^2},$$

where $\nabla G_\sigma(x) = \frac{1}{\sigma}\exp(-|x|^2/(4\sigma^2))$ is the Gaussian filter and $k > 0$ and $\sigma > 0$ are fixed parameters. For problem (0.0.1), Chen, Levine and Rao established the existence and uniqueness of the solution and the long-time behavior of the associated flow of the proposed model. The effectiveness of the model in image restoration is illustrated by some experimental results included in their paper.

The next example is related to electrorheological fluids. The constitutive equation for the motion of an electrorheological fluid is

$$u_t + divS(u) + (u.\nabla)u + \nabla\pi = F, \tag{0.0.2}$$

where $u : \mathbb{R}^{3,1} \to \mathbb{R}^3$ is the velocity of the fluid at a point in space-time, $\pi : \mathbb{R}^{3,1} \to \mathbb{R}$ represents external forces, and the stress tensor $S : W^{1,1}_{loc} \to \mathbb{R}^{3,1}$ is of the form

$$S(u)(x) = \mu(x)\left[1 + |Du(x|^2)\right]^{\frac{p(x)-2}{p(x)}} Du(x),$$

where $Du(\nabla u + \nabla u^T)/2$ is the symmetric part of the gradient of u.

We observe that the highest order differential term in (0.0.2) is

$$div\Big(\big(1 + |Du(x|^2)\big)^{\frac{p(x)-2}{p(x)}} Du(x)\Big).$$

In what follows, we describe a model that takes into account the delicate interaction between the electromagnetic fields and the moving fluids. In particular, in the context of continuum mechanics, these fluids are treated as non-Newtonian fluids. The system modeling the phenomenon arising from this study is

$$\begin{cases} \text{div } E = 0, \text{curl } E = 0 \\ \frac{\partial v}{\partial t} - \text{div} S(x, E, \mathcal{E}(v)) + |\nabla v|v + \nabla\pi = g(x, E) \\ \text{div } v = 0, \end{cases}$$

where $E(x)$ is the electromagnetic field, $v : \Omega \subset \mathbb{R}^3 \to \mathbb{R}^3$is the velocity of the field, $\mathcal{E}(v)$ is the symmetric part of the gradient, S is the extra stress tensor and π the pressure.

The constitutive relation for the extra stress tensor S is

$$S(x, E, z) = v(E)(1 + |z|^2)^{\frac{p-2}{2}} z + \text{terms of the same growth}$$

for all symmetric 3×3 matrices z and where $p = p(|E|^2)$. The structure of the system allows to determine E so that it depends on x and thus $p = p(x)$.

The extra stress tensor S is chosen so that it is a monotone vector field satisfying the ellipticity condition

$$D_z S(x, E, z)\lambda \otimes \lambda \geq v(E)(1 + |z|^2)^{\frac{p-2}{2}} |\lambda|^2$$

where $v(E) \geq \bar{v} > 0$ for any 3×3 symmetric matrices z, λ with null trace.

For the system described above, Rajagopal and Ruzicka [90] established an existence theory, which works particularly in the stationary case

$$-\text{div} S(x, \mathcal{E}(v)) + |\nabla v| \, v + \nabla \pi = g(x).$$

The elliptic systems involving $p(x)$-Kirchhoff are a generalizations of the following Kirchhoff equation, introduced by Kirchhof in [51]

$$\rho \frac{\partial^2 u}{\partial t^2} - \left(\frac{\rho_0}{h} + \frac{E}{2L} \int_0^L \left| \frac{\partial u}{\partial x} \right|^2 dx \right) \frac{\partial^2 u}{\partial x^2} = 0, \tag{0.0.3}$$

where ρ, ρ_0, E and L are constants. This equation extends the classical D'Alembert's wave equation by considering the effects of the changes in the length of the strings during the vibrations. A distinguishing feature of equation (0.0.3) is that the equation contains a nonlocal coefficient $\frac{\rho_0}{h} + \frac{E}{2L} \int_0^L \left| \frac{\partial u}{\partial x} \right|^2 dx$ which depends on the average $\frac{1}{2L} \int_0^L \left| \frac{\partial u}{\partial x} \right|^2 dx$, and hence the equation is no longer a pointwise equation. The parameters in equation (0.0.3) have the following meanings: E is the Young modulus of the material (also referred to as the elastic modulus, it measures the strings resistance to being deformed elastically), ρ is the mass density, L is the length of the string, h is the area of cross-section, and ρ_0 is the initial tension. Almost one century later, Jacques-Louis Lions [58] returned to the equation and proposed a general Kirchhoff equation in arbitrary dimension with external force term which was written as

$$\begin{cases} \frac{\partial^2 u}{\partial t^2} - \left(a + b \int_\Omega |\nabla u|^2 \, dx \right) \Delta u = f(x, u) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega; \end{cases} \tag{0.0.4}$$

this problem is often called a nonlocal problem because it contains an integral over Ω. This causes some mathematical difficulties which make the study of such a problem particularly interesting. The nonlocal problem models several physical and biological systems, where u describes a process which depends on the average of itself, such as the population density,see [23] and its references therein. For a more detailed reference on this subject we refer the interested reader to [5, 16, 24, 25, 26, 48, 64, 71].

For more applications, we refer to the pioneering work by Halsey [45] and the monograph by Ruzicka [78]. A survey of recent contributions to the study of partial differential equations with variable exponents is contained in the paper by Rădulescu [73].

The objective of this work is the study of elliptic systems of partial differential equations involving an operator in divergent form of the type $p(x)-$Laplacian defined of $W^{1,p(x)}(\mathbb{R}^N)$ in $W^{-1,p(x)}(\mathbb{R}^N)$. These problems are generally not integrable (i.e. we cannot determine the explicit form). For that we are satisfied to show the existence of weak solutions (i.e. in the sense of distributions), which are precisely the critical points of the Euler-Lagrange functional.

In certain works authors show that this solution is unique [30, 33, 37]. Other works ensure the existence of two solutions [52], three solutions [57, 75, 76], multiple solutions [9, 83], and even an infinity of solutions [65]; based respectively on recent results from Bonnano and Marano [10]; and Ricceri in [76].

This thesis is structured as follows, The first chapter is naturally devoted to notations, to the definitions and properties of functional spaces and to the fundamental results on which we rely in the following. All these results are given without proof and we refer to the bibliography for details.

In the second chapter, we establish two results on the existence and uniqueness of solutions for $(p(x), q(x))$-Laplacian potential-type system such as

$$\begin{cases} -\Delta_{p(x)}u = \frac{\partial F}{\partial u}(x,u,v) & \text{in } \Omega, \\ -\Delta_{q(x)}v = \frac{\partial F}{\partial v}(x,u,v) & \text{in } \Omega, \\ u = v = 0 & \text{on } \partial\Omega, \end{cases} \tag{0.0.5}$$

the first result concerns the case where $\frac{\partial F}{\partial u}(x,u,v)$ and $\frac{\partial F}{\partial v}(x,u,v)$ are decreasing in both u and v and the proof is made using the Browder-Minty Theorem. The second deals with the radial case and assumes that $\frac{\partial F}{\partial u}(r,u,v)$ and $\frac{\partial F}{\partial v}(r,u,v)$ are nondecreasing in both u and v and the method of proof is based on a comparison principle and the use of critical points theory.

In the third chapter, we deal with the multiplicity of weak solutions to a class of a parametric problem for doubly eigenvalue elliptic systems involving the $(p_1(x), ..., p_n(x))$-Laplacian operator of the form:

$$-\Delta_{p_i(x)}u_i + a_i(x)|u_i|^{p_i(x)-2}u_i = \lambda F_{u_i}(x, u_1, u_2, ..., u_n) \quad \text{in } \mathbb{R}^N, \tag{0.0.6}$$

for $1 \le i \le n$. Our technical approach is based on variational methods and recent three critical points theorem obtained by Bonanno and Marano.

In the fourth chapter, we extend the result of the previous chapter for a parametric doubly eigenvalue quasilinear elliptic $(p(x), q(x))$-Kirchhoff type system of the form :

$$\begin{cases} -M_1\left(L_p(u)\right)\left(\Delta_{p(x)}u - a(x)|u|^{p(x)-2}u\right) = \lambda F_u(x,u,v) & \text{in } \mathbb{R}^N, \\ -M_2\left(L_q(v)\right)\left(\Delta_{q(x)}v - b(x)|v|^{q(x)-2}v\right) = \lambda F_v(x,u,v) & \text{in } \mathbb{R}^N. \end{cases} \tag{0.0.7}$$

Our approch is based on a variational method and the three critical points theorem obtained by Bonano and Marano.

In the fifth chapter, by using the concentration-compactness principle of Lions for variable exponents and variational arguments, we obtain the existence of solutions for a class of $(p_1(x), \dots, p_n(x))$-Kirchhoff-Type Potential Systems type with critical exponents.

In sixth and final chapter, by using the concentration-compactness principle of P. L. Lions for variable exponents spaces found in [11] and the Mountain Pass Theorem without the Palais-Smale condition given in [72], we obtain the existence and multiplicity of solutions $u = (u_1, u_2, \dots . u_n)$, for a class of Kirchhoff-type potential systems with critical exponent, namely

$$\begin{cases} -M_i\Big(\mathcal{A}_i(u_i)\Big)\operatorname{div}\Big(\mathcal{B}_i(\nabla u_i)\Big) = |u_i|^{s_i(x)-2}u_i + \lambda F_{u_i}(x,u) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega; \end{cases}$$

where Ω is a bounded smooth domain in $\mathbb{R}^N (N \geq 2)$, and

$$\mathcal{B}_i(\nabla u_i) = a_i\Big(|\nabla u_i|^{p_i(x)}\Big)|\nabla u_i|^{p_i(x)-2}\nabla u_i.$$

The functions M_i, $\mathcal{A}_i$, a_i and a_i, $(1 \leq i \leq n)$, are given functions, whose properties will be introduced hereafter, λ is positive parameter, and the real function F belongs to $C^1(\Omega \times \mathbb{R}^n)$, F_{u_i} denotes the partial derivative of F with respect to u_i. Our result extend, complement and complete in several ways some of many works in particular those in [21]. We want to emphasize that a difference by respect to some previous research is that the conditions on $a_i(.)$ are general enough to incorporate some differential operators of great interest. In particular, we can cover a general class of nonlocal operators for $p_i(x) > 1$ for all $x \in \overline{\Omega}$.

Chapter 1

Functional framework

1.1 The generalized Lebesgue and Sobolev spaces

We recall in this part some definitions and properties of Lebesgue-Sobolev spaces with variable exponents commonly known as generalized Sobolev spaces. We refer the book [31], and the papers by O. Kováčik and J. Rákosní [53],and by X. Fan and D. Zhao [38], for more detailed properties.

Let's pose

$$C_+(\mathbb{R}^N) = \{h \in C(\mathbb{R}^N) : \inf_{x\in\mathbb{R}^N} h(x) > 1\}.$$

and

$$L^\infty_+(\mathbb{R}^N) = \{h \in L^\infty(\mathbb{R}^N) : \inf_{x\in\mathbb{R}^N} h(x) > 1\}.$$

For any $h \in C_+(\mathbb{R}^N)$, define

$$h^+ = \sup_{x\in\mathbb{R}^N} h(x) \quad \text{and} \quad h^- = \inf_{x\in\mathbb{R}^N} h(x).$$

For any $p \in C_+(\mathbb{R}^N)$, define the variable exponent Lebesgue space as

$$L^{p(x)}(\mathbb{R}^N) = \{u : \quad \text{u is a measurable real-valued function and } \int_{\mathbb{R}^N} |u(x)|^{p(x)}dx < \infty\},$$

endowed with the Luxemburg norm

$$|u|_{p(x)} := |u|_{L^{p(x)}} = \inf\left\{\mu > 0; \int_{\mathbb{R}^N} \left|\frac{u(x)}{\mu}\right|^{p(x)} dx \le 1\right\}.$$

On the other hand, the variable exponent Sobolev space $W^{1,p(x)}(\mathbb{R}^N)$ is defined by

$$W^{1,p(x)}(\mathbb{R}^N) = \{u \in L^{p(x)}(\mathbb{R}^N) : |\nabla u| \in L^{p(x)}(\mathbb{R}^N)\},$$

and is endoweded with the norm

$$\|u\|_{1,p(x)} := \|u\|_{W^{1,p(x)}(\mathbb{R}^N)} = |u|_{p(x)} + |\nabla u|_{p(x)}, \qquad \forall u \in W^{1,p(x)}(\mathbb{R}^N).$$

The space $W_0^{1,p(x)}(\mathbb{R}^N)$ is defined as the closure of $C_0^\infty(\mathbb{R}^N)$ in $W^{1,p(x)}(\mathbb{R}^N)$ and is equipped with the norm

$$\|u\|_{p(x)} = |\nabla u|_{p(x)}, \quad \forall u \in W_0^{1,p(x)}(\mathbb{R}^N).$$

It is well known that the spaces $L^{p(x)}(\mathbb{R}^N), W^{1,p(x)}(\mathbb{R}^N)$ and $W_0^{1,p(x)}(\mathbb{R}^N)$ are separable and reflexive Banach spaces.

Proposition 1.1.1 (see [31, 38]). *The conjugate space of $L^{p(x)}(\mathbb{R}^N)$ is $L^{p'(x)}(\mathbb{R}^N)$, where*

$$\frac{1}{p(x)} + \frac{1}{p'(x)} = 1.$$

Moreover, for any $(u, v) \in L^{p(x)}(\mathbb{R}^N) \times L^{p'(x)}(\mathbb{R}^N)$ we have

$$|\int_{\mathbb{R}^N} uvdx| \leq (\frac{1}{p^-} + \frac{1}{(p')^-})|u|_{p(x)}|v|_{p'(x)} \leq 2|u|_{p(x)}|v|_{p'(x)}.$$

Definition 1.1.2. *We call modulus of u the quantity*

$$\rho(u) = \int_{\mathbb{R}^N} |u|^{p(x)}dx, \text{ for all } u \in L^{p(x)}(\mathbb{R}^N).$$

Proposition 1.1.3 (see [31, 38]). *Let u and v be two measurable functions on Ω, and let α and β be two positive constants such that $\alpha + \beta = 1$ then :*

$$\rho(\alpha u + \beta v) \leq \alpha\rho(u) + \beta\rho(v).$$

The relation between the Luxembourg norm and the modulus of u can be summed up in the following properties:

$$\min\{|u|_{p(x)}^{p^-}, |u|_{p(x)}^{p^+}\} \leq \rho(u) \leq \max\{|u|_{p(x)}^{p^-}, |u|_{p(x)}^{p^+}\}$$

(i) $|u|_{p(x)} < 1$ *(resp.* $= 1, > 1$*)* $\Leftrightarrow \rho(u) < 1$ *(resp.* $= 1, > 1$*);*

(ii) $|u|_{p(x)} > 1 \Rightarrow |u|_{p(x)}^{p^-} \leq \rho(u) \leq |u|_{p(x)}^{p^+}$*;*

(iii) $|u|_{p(x)} < 1 \Rightarrow |u|_{p(x)}^{p^+} \leq \rho(u) \leq |u|_{p(x)}^{p^-}$*;*

(iii) $\rho\left(\frac{u}{|u|_{p(x)}}\right) = 1.$

Proposition 1.1.4 (see [37]). *Let $p(x)$ and $q(x)$ be measurable functions such that $p \in L^{\infty}(\mathbb{R}^N)$ and $1 \leq p(x), q(x) \leq \infty$ almost everywhere in Ω. If $u \in L^{q(x)}(\mathbb{R}^N)$, $u \not\equiv 0$. Then, we have*

$$|u|_{p(x)q(x)} \leq 1 \Rightarrow |u|^{p^-}_{p(x)q(x)} \leq \left||u|^{p(x)}\right|_{q(x)} \leq |u|^{p^+}_{p(x)q(x)},$$
$$|u|_{p(x)q(x)} \geq 1 \Rightarrow |u|^{p^+}_{p(x)q(x)} \leq \left||u|^{p(x)}\right|_{q(x)} \leq |u|^{p^-}_{p(x)q(x)}.$$

In particular, if $p(x) = p$ is constant, then

$$||u|^p|_{q(x)} = |u|^p_{pq(x)}.$$

Proposition 1.1.5 (see [31, 38]). *If $u, u_n \in L^{p(x)}(\mathbb{R}^N)$, $n = 1, 2, \dots,$ then the following statements are equivalent to each other:*

(1) $\lim_{n\to\infty} |u_n - u|_{p(x)} = 0,$

(2) $\lim_{n\to\infty} \rho_p(u_n - u) = 0,$

(3) $u_n \to u$ *in measure in* Ω *and* $\lim_{n\to\infty} \rho(u_n) = \rho(u).$

Definition 1.1.6. *We define the critical Sobolev exponent of $p(x)$ by*

$$p^*(x) = \begin{cases} \frac{Np(x)}{N-p(x)} & \text{for } p(x) < N \\ +\infty & \text{for } p(x) \geq N \end{cases}.$$

Proposition 1.1.7 (see [31, 37]). *Let $p \in C^{0,1}_{+}(\mathbb{R}^N)$, the space of Lipschitz-continuous functions defined on $\mathbb{R}^N$. We have continuous inclusion*

$$W^{1,p(x)}(\mathbb{R}^N) \hookrightarrow L^{p^\star(x)}(\mathbb{R}^N).$$

That is to say, there exists a positive constant c such that

$$|u|_{p^*(x)} \leq c\|u\|_{p(x)}, \quad \forall u \in W^{1,p(x)}(\mathbb{R}^N).$$

Proposition 1.1.8 (see [31, 37]). *Let us denote $q(x) \ll p^*(x)$ when* $\inf_{x\in\Omega}(p^*(x) - q(x)) > 0.$
(1) If $q \in L^{\infty}_{+}(\mathbb{R}^N)$ and $p(x) \leq q(x) \ll p^*(x)$, for all $x \in \mathbb{R}^N$, then the embedding $W^{1,p(x)}(\mathbb{R}^N) \hookrightarrow L^{q(x)}(\mathbb{R}^N)$ is continuous but not compact.

(2) if we replace $\mathbb{R}^N$ by a bounded domain Ω and if p is continuous on $\overline{\Omega}$ and q is a measurable function on Ω, with $p(x) < q(x) < p^*(x)$ for all $x \in \Omega$, then the embedding $W^{1,p(x)}(\Omega) \hookrightarrow L^{q(x)}(\Omega)$ is compact.

Proposition 1.1.9 (see [31, 37]). *There is a constant $C > 0$ such that*

$$\|u\|_{p(x)} \leq c\|\nabla u\|_{p(x)}, \quad \forall u \in W^{1,p(x)}_0(\Omega).$$

$\|u\|_{p(x)} = \|\nabla u\|_{p(x)}$ *and* $\|u\|_{1,p(x)}$ *are equivalent norms on* $W^{1,p(x)}_0(\Omega).$

1.2 The generalized weighted Lebesgue-Sobolev spaces

There is a rich literature on weighted Lebesgue spaces. In a similar way, we define the generalized weighted Lebesgue spaces (see [6]) by

$$L^{p(x)}_{a(x)}(\mathbb{R}^N) = \left\{ u : u \text{ is a measurable real-valued function and } \int_{\mathbb{R}^N} a(x)|u(x)|^{p(x)}dx < \infty \right\},$$

a being a locally integrable positive function. In other words $a \in L^1_{loc}(\mathbb{R}^N)$. for example

$$a(x) = (1 + |x|^2)^{\frac{1}{2}}.$$

We endow $L^{p(x)}_a(\mathbb{R}^N)$ with the norm

$$|u|_{p(x),a(x)} := |u|_{L^{p(x)}_{a(x)}(\mathbb{R}^N)} = \inf \left\{ \mu > 0 : \int_{\mathbb{R}^N} a(x) \left| \frac{u(x)}{\mu} \right|^{p(x)} dx \leq 1 \right\}.$$

This space has the same properties as the Lebesgue space with variable exponent. In a similar way we define the weight modulus

$$\rho_{a(x)} = \int_{\mathbb{R}^N} a(x) \left|u(x)\right|^{p(x)} dx, \ \forall u \in L^{p(x)}_a(\mathbb{R}^N).$$

The link between weighted Lebesgue space with variable exponent $L^{p(x)}_a(\mathbb{R}^N)$ and Lebesgue space with variable exponent $L^{p(x)}(\mathbb{R}^N)$ is summarized in the following proposition

Proposition 1.2.1. *When there is $C > 0$ such that $a(x) \geq C$, then we have the continuous inclusion $L^{p(x)}_a(\mathbb{R}^N) \hookrightarrow L^{p(x)}(\mathbb{R}^N)$. In other words we have*

$$C \int_{\mathbb{R}^N} \left|u(x)\right|^{p(x)} dx \leq \int_{\mathbb{R}^N} a(x) \left|u(x)\right|^{p(x)} dx,$$

and

$$C \left\|u\right\|_{L^{p(x)}(\mathbb{R}^N)} \leq \left\|u\right\|_{L^{p(x)}_{a(x)}(\mathbb{R}^N)}.$$

In the same way, we define the generalized weighted Sobolev space, when the conditions $1 < p^- \leq p(x) \leq p^+ < \infty$ et $a^{-\frac{1}{p(x)-1}} \in L^1_{loc}(\mathbb{R}^N)$ are verified, by

$$W^{1,p(x)}_a(\mathbb{R}^N) = \{u \in L^{p(x)}_{a(x)}(\mathbb{R}^N) : |\nabla u| \in L^{p(x)}_{a(x)}(\mathbb{R}^N)\},$$

endowed with the norm

$$\|u\|_{p(x),a(x)} := |u|_{p(x),a(x)} + |\nabla u|_{p(x)}.$$

From Proposition (1.1.3), we have:

$$\|u\|_a^{p^-} \leq \int_{\mathbb{R}^N} \left(|\nabla u(x)|^{p(x)} + a(x)\, |u(x)|^{p(x)} \right) dx \leq \|u\|_a^{p^+} \quad \text{if} \quad \|u\|_a \geq 1. \tag{1.2.1}$$

$$\|u\|_a^{p^+} \leq \int_{\mathbb{R}^N} \left(|\nabla u(x)|^{p(x)} + a(x)\, |u(x)|^{p(x)} \right) dx \leq \|u\|_a^{p^-} \quad \text{if} \quad \|u\|_a \leq 1. \tag{1.2.2}$$

We also note that $C_0^\infty(\mathbb{R}^N)$ is dense in $W_{a(x)}^{1,p(x)}(\mathbb{R}^N)$ (see [6]).

1.3 A few reminders on the theory of critical points

The main purpose of this section is to present a few reminders on the theory of critical points of C^1-functionals on a Banach space.

A number of problems in the theory of differential equations can be expressed in the form of an equation

$$Au = 0, \tag{1.3.1}$$

where $A : X \to Y$ is a mapping between Banach spaces X and Y. The interesting case is the situation where this equation has a variational structure, that is, there exists a functional $\phi : X \to \mathbb{R}$ such that

$$\langle A(u), v \rangle = \lim_{t \to 0} \frac{\phi(u + tv) - \phi(u)}{t},$$

where $Y = X^\star$, $\langle .,. \rangle$ is a duality pairing between X and its dual $X^\star$. In this case we can write $A = \phi'$ and equation (1.3.1) becomes

$$\langle \phi'(u), v \rangle = 0 \quad \text{for each } v \in X. \tag{1.3.2}$$

Equation (1.3.2) says that solutions of (1.3.1) are critical points of the functional ϕ. By writing equation (1.3.2) we have expressed equation (1.3.1) in a weak (distributional)form. The problem that we have to solve is to find critical points of ϕ.

Definition 1.3.1. *We say that $\varphi : X \to \bar{\mathbb{R}}$ is lower semi-continuous (in abbreviated L.S.C) at point $u \in X$, si $\varphi(u) = -\infty$ or if at all $\alpha < \varphi(u)$ we can associate a neighborhood V_α of u such that $\alpha < \varphi(u)$ for $u \in V_\alpha$ (or again, $\varphi^{-1}(]\alpha, +\infty[)$is a neighborhood of u). In terms of sequence, if $u_n \to u$ then*

$$\varphi(u) \leq \liminf_{n \to \infty} \varphi(u_n).$$

The function φ is said to be L.S.C on X if it is so at any point of X.

Definition 1.3.2. *Let X and Y be real Banach spaces and let $A : X \to X^\star$ be an operator. Then*

i) A is called coercive, iff

$$\lim_{\|u\| \to \infty} \frac{|\langle Au, u\rangle|}{\|u\|} = +\infty.$$

ii) A is called monotone iff

$$\langle Au - Av, u - v\rangle \geq 0 \textit{ for all } (u, v) \in X^2.$$

iii) A is called strictly monotone, iff

$$\langle Au - Av, u - v\rangle > 0 \textit{ for all } (u, v) \in X^2 \textit{ with } u \neq v.$$

iv) A is called uniformly monotone iff

$$\langle Au - Av, u - v\rangle \geq a(\|u - v\|) \|u - v\| . \textit{ for all } u, v \in X$$

where the continuous function $a : \mathbb{R}_+ \to \mathbb{R}_+$ is strictly monotone increasing with $a(0) = 0$ and $a(t) \to +\infty$ as $t \to +\infty$.

For example, we may choose $a(t) = c\,|t|^{p-1}$, with $c > 0$ and $p > 1$. In this case, We obtain

$$\langle Au - Av, u - v\rangle \geq c\,\|u - v\|^p \textit{ for all } u, v \in .X$$

Moreover, when $p = 2$, the operator A is said to be strongly monotone, and we have the following implications (see [25.3 of [84]]).

A is strongly monotone $\Rightarrow$ A is uniformly monotone $\Rightarrow$
A is strictly monotone $\Rightarrow$ A is monotone.

Finally, if A is uniformly monotone then is A is coercive.

iv) We say that A is semi-continuous, if the function

$$\begin{array}{ccc} [0,1] & \to & \mathbb{R} \\ t & \mapsto & \langle A(u + tv), w\rangle \end{array}$$

is continuous on $[0, 1]$ for all $u, v, w \in X$.

Theorem 1.3.3. *(see [84]) Let E be a functional on X a reflexive Banach space. If E is weakly lower semi-continuous and coercive then it admits a global minimum.*

let us recall the Browder-Minty Theorem.

Theorem 1.3.4 (cf. [56]). *Let X be a reflexive Banach real space. Moreover, let $T : X \to X^\star$ be a bounded, semi-continuous, coercive and monotone operator on the space X. Then, the equation $T(u) = f$ has at least one solution $u \in X$ for each $f \in X^\star$. If moreover, T is strictly monotone, then for every $f \in X^\star$, the equation $T(u) = f$ has exactly one solution $u \in X$.*

Theorem 1.3.5. *(see [26: A of [84]]) (Browder (1963); Minty (1963))*
If the operator A is uniformly monotone, coercive and semi-continuous on a real reflexive Banach space X, then A admits a continuous inverse on X.

Now we state Theorem A due to G. Bonanno and S. A. Marano [10]; which ensures the existence of the three critical points, which is a more precise version of Theorem 3.2 of [8].

Theorem 1.3.6 (see [10, Theorem 3.6]). *Let X a reflexive real Banach space; $\Phi : X \to \mathbb{R}$ be a coercive, continuously Gâteaux differentiable and sequentially weakly lower semicontinuous functional whose Gâteaux derivative admits a continuous inverse on X^*; $\Psi : X \to \mathbb{R}$ a continuously Gâteaux differentiable functional whose Gâteaux derivative is compact such that*

$$\Phi(0) = \Psi(0) = 0.$$

Assume that there exist $e > 0$ and $\overline{x} \in X$, with $e < \Phi(\overline{x})$, such that

(a$_1$) $\frac{\sup_{\Phi(x)\leq e} \Psi(x)}{e} < \frac{\Psi(\overline{x})}{\Phi(\overline{x})}$;

(a$_2$) *for each $\lambda \in \Lambda_e :=]\frac{\Phi(\overline{x})}{\Psi(\overline{x})}, \frac{r}{\sup_{\Phi(x)\leq e} \Psi(x)}[$, the functional $\Phi - \lambda\Psi$ is coercive.*

Then, for each $\lambda \in \Lambda_e$ the functional $\Phi - \lambda\Psi$ has at least three distinct critical points in X.

1.4 Mountain pass theorem

The mountain pass theorem was established by Ambrosetti and Rabinowitz [4]. Their original proof relies on some deep deformation techniques developed by Palais and Smale [69], [70], who put the main ideas of the Morse theory into the framework of differential topology on infinite dimensional manifolds. In this way, Palais and Smale replaced the finite dimensionality assumption with an appropriate compactness hypothesisl. As pointed out by Brezis and Browder [15], the mountain pass theorem "extends ideas already presented by Poincaré and Birkhoff".

Definition 1.4.1. *(Palais-Smale sequences). Let X be a Banach space and*

$$E : X \to \mathbb{R}$$

a $\mathcal{C}^1-$functional. We call a sequence $u_n \in X$ a Palais-Smale sequence (PS-sequence) on X if $E(u_n)$ is bounded and $E'(u_n) \to 0$. If it happens that $E(u_n) \to c$ for some $c \in \mathbb{R}$ the PS-sequence will be called a PSc-sequence.

We can now with the help of the above definition define what is meant by the Palais-Smale condition.

Definition 1.4.2. *(The Palais-Smale condition). Let E and X be as above (definition* (1.4.1)*), We say that the functional E satisfies the Palais-Smale condition on X denoted by (PS), if every PS-sequence has a converging subsequence.*

Definition 1.4.3. *((PS)c). For X, E and c as in definition* (1.4.1)*, we say that E satisfies the (local) Palais-Smale condition at the level c denoted by (PS)c if every PSc-sequence has a converging subsequence.*

Remark 1.4.4. *The condition (PS) is stronger than (PS)c in the sense that if (PS) holds, then (PS)c is satisfied for all $c \in \mathbb{R}$ while the converse is not necessarily true.*

Observe that if $E \in \mathcal{C}^1(X, R)$ satisfies the condition $(PS)c$, any accumulation point $\bar{u}$ of a sequence u_n of $(PS)c$ is a critical point of E. We have implicitly $E'(\bar{u}) = 0$ and $E(\bar{u}) = c$.

For a functional E which is not bounded, looking for its critical points amounts to looking for saddle points. These points are determined by a min-max argument, which brings us back to the use of the mountain pass theorem.

Theorem 1.4.5 (see [4])**.** *Let X be a real infinite dimensional Banach space and $E \in C^1(X, \mathbb{R})$ such that $E(0_X) = 0$ and satisfying the (PS) condition. Suppose that*

- *($\mathcal{I}_1$) There are $\mathcal{R}, \rho > 0$ such that $E(u) \geq \mathcal{R}$ and for all $u \in \partial B_\rho \cap X$;*
- *($\mathcal{I}_2$) There exists $e \in X$ with $\|e\| > \rho$ such that $E(e) < 0$.*

Then E possesses a critical value $c \geq \mathcal{R}$, which can be characterized as

$$c := \inf_{\xi \in \Gamma} \sup_{t \in [0,1]} E_\lambda(\xi(t)),$$

where

$$\Gamma = \{\xi : [0,1] \to X, \textit{continuous and } \xi(0) = 0_X, E(\xi(1)) < 0\}.$$

We state the $\mathbb{Z}_2$-symmetric mountain pass version (that is, for even functionals).

Theorem 1.4.6 (see [72]). *Let X be a real infinite dimensional Banach space and $E \in C^1(X, \mathbb{R})$ be even, satisfying the Palais-Smale condition and $E(0_X) = 0$. Suppose that condition ($\mathcal{I}_1$) holds in addition to the following:*

($\mathcal{I}_2'$) *For each finite dimensional subspace $X_1 \subset X$, the set $S_1 := \{u \in X_1 : E(u) \geq 0\}$ is bounded in X.*

Then, E has has an unbounded sequence of critical values.

1.5 The concentration-compactness principle

This method introduced by P.L. Lions in [59] and [60] is one of the most powerful methods to deal with variational problems posed in unbounded domains, for example $\mathbb{R}^N$ or under the existence of critical exponents or non-compact injections.

Now, we recall an important version of concentration compactness principle of Lions for variable exponents found in [11], which will be used in the proof of our main results in chapters 5 and 6.

Theorem 1.5.1 (see [11]). *Let $q(x)$ and $p(x)$ be two continuous functions such that*

$$1 < \inf_{x\in\Omega} p(x) \leq \sup_{x\in\Omega} p(x) < N \quad \textit{and} \quad 1 \leq q(x) \leq p^*(x) \quad \textit{in } \Omega.$$

Let $\{u_n\}_{n\in\mathbb{N}}$ be a weakly convergent sequence in $W_0^{1,p(x)}(\Omega)$ with weak limit u, and such that:

- $|\nabla u_n|^{p(x)} \rightharpoonup \mu$ *weakly-* in the sense of measures.*
- $|u_n|^{q(x)} \longrightarrow \nu$ *weakly-* in the sense of measures.*

Also assume that $\mathcal{A} = \{x \in \Omega\colon q(x) = p^*(x)\}$ is nonempty. Then, for some countable index set J, we have:

$$\nu = |u|^{q(x)} + \sum_{j\in I} \nu_J \delta_{x_j} \quad \nu_j > 0 \tag{1.5.1}$$

$$\mu \geq |\nabla u|^{p(x)} + \sum_{j\in I} \mu_j \delta_{x_j} \quad \mu_j > 0 \tag{1.5.2}$$

$$S \nu_i^{1/p^*(x_j)} \leq \mu_i^{1/p(x_j)} \quad \forall j \in J. \tag{1.5.3}$$

where $\{x_j\}_{j\in J} \subset \mathcal{A}$, δ_{x_j} is the Dirac mass at $x_j \in \overline{\Omega}$ and S is the best constant in the Gagliardo-Nirenberg-Sobolev inequality for variable exponents, namely

$$S = S_q(\Omega) := \inf_{\phi\in C_0^\infty(\Omega)} \frac{\||\nabla\phi|\|_{L^{p(x)}(\Omega)}}{\|\phi\|_{L^{q(x)}(\Omega)}}. \tag{1.5.4}$$

Chapter 2

Existence and Uniqueness Results for the $(p(x), q(x))$-Laplacian potential-type system

2.1 Introduction :

The aim of this chapter is to investigate existence and uniqueness questions for solutions of the following boundary value problem

$$\begin{cases} -\Delta_{p(x)} u = \frac{\partial F}{\partial u}(x,u,v) & \text{in } \Omega, \\ -\Delta_{q(x)} v = \frac{\partial F}{\partial v}(x,u,v) & \text{in } \Omega, \\ u = v = 0 & \text{on } \partial\Omega; \end{cases} \tag{2.1.1}$$

where Ω is a bounded domain in $\mathbb{R}^N$, $\Delta_{p(x)} u = \operatorname{div}(|\nabla u|^{p(x)-2}\nabla u)$ is the so-called $p(x)$-Laplacian operator, $p(x)$ and $q(x)$ are continuous real-valued functions such that $1 < p(x), q(x) < N$ $(N \geq 2)$ for all $x \in \mathbb{R}^N$ and the real-valued function F belongs to $C^1(\Omega \times \mathbb{R}^2)$.

The existence and multiplicty of the solutions for (2.1.1) have been studied by many authors [50, 80, 83], where the nonlinear potential F has different mixed growth conditions.

When $\Omega = \mathbb{R}^N$, and p and q are constants, the authors show in [33] the existence of nontrivial solutions for the following (p, q)-Laplacian system

$$\begin{cases} -\Delta_p u = \frac{\partial F}{\partial u}(x,u,v) & \text{in } \mathbb{R}^N \\ \\ -\Delta_q v = \frac{\partial F}{\partial v}(x,u,v) & \text{in } \mathbb{R}^N \end{cases} \tag{2.1.2}$$

where the potential function F satisfies mixed and subcritical growth conditions and, in addition, is supposed to be intimately connected with the first eigenvalue of the (Δ_p, Δ_q) operator. In that work, the authors apply the Mountain Pass theorem to get existence of nontrivial solutions for system (2.1.1).

On the other hand, existence of nontrivial solutions for system (2.1.2), when $(p,q) \in [C(\mathbb{R}^N)]^2$ $(N \geq 2)$ and $F \in C^1(\mathbb{R}^N \times \mathbb{R}^2, \mathbb{R})$ verifies some mixed growth conditions, has been shown by the authors in [32], also by using critical points theory.

For general bounded domain Ω of $\mathbb{R}^N$ with sufficiently smooth boundary $\partial\Omega$, $N \geq 2$, $(p,q) \in [C(\overline{\Omega})]^2$ and $F \in C^1(\mathbb{R}^N \times \mathbb{R}^2, \mathbb{R})$, the authors obtain in [35] existence and multiplicity of the solutions for system (2.1.1). By introducing natural growth hypotheses on the right-hand side of the system to ensure the mountain pass geometry and Palais-Smale condition for the corresponding Euler-Lagrange functional of the system, the authors focuses on subcritical case for function F to obtain existence and multiplicity results for the solutions of (2.1.1).

Our study is a continuation of these works. Actually, under certain growth conditions on the potential, we show the existence and uniqueness of solutions for problem (2.1.1) by using Browder-Minty Theorem in case where $\frac{\partial F}{\partial u}(x,u,v)$ and $\frac{\partial F}{\partial v}(x,u,v)$ are nonincreasing in both u and v, and by using a comparison principle and critical points theory tools in case $\frac{\partial F}{\partial u}(x,u,v)$ and $\frac{\partial F}{\partial v}(x,u,v)$ are nondecreasing in both u and v.

In comparison with the study of uniqueness of the solutions of problem (2.1.1) in general form, the study of the radial solutions of problem (2.1.1) in the radially symmetric case is easier since the corresponding differential system becomes ordinary.

A positive radial solution for (2.1.1) is of the form $(u(|x|), v(|x|))$ and satisfies

$$\begin{cases} -(r^{N-1}|u'(r)|^{p(r)-2}u'(r))' = r^{N-1}\frac{\partial F}{\partial u}(r,u,v) & \text{in }]0,R[, \\ -(r^{N-1}|v'(r)|^{q(r)-2}v'(r))' = r^{N-1}\frac{\partial F}{\partial v}(r,u,v) & \text{in }]0,R[, \\ u'(0) = u(R) = 0 \quad , v'(0) = v(R) = 0; \end{cases} \tag{2.1.3}$$

with $r = |x|$ and R is the radius of the ball. We shall therefore be concerned with existence and uniqueness of the solutions for problem (2.1.3).

Consider the product space

$$W_{p(x),q(x)}(\Omega) := W_0^{1,p(x)}(\Omega) \times W_0^{1,q(x)}(\Omega),$$

equipped with the norm

$$\|(u,v)\| := \max\left\{\|u\|_{p(x)}, \|v\|_{q(x)}\right\}, \forall (u,v) \in W_{p(x),q(x)}(\Omega).$$

Denote by $W'_{p(x),q(x)}(\Omega)$ the topological dual of $W_{p(x),q(x)}(\Omega)$ equipped with the usual dual norm.

Definition 2.1.1. *(u,v) is called a weak solution of the system* (2.1.1) *if*

$$\int_\Omega |\nabla u|^{p(x)-2}\nabla u \nabla \omega \, dx + \int_\Omega |\nabla v|^{q(x)-2}\nabla v \nabla z \, dx = \int_\Omega \frac{\partial F}{\partial u}(x,u,v)(\omega)\, dx + \int_\Omega \frac{\partial F}{\partial v}(x,u,v)(z)\, dx,$$

for all $(\omega, z) \in W_{p(x),q(x)}$.

For every (u,v) in $W_{p(x),q(x)}$, let us define the functional Ψ by

$$\Psi(u,v) = \int_\Omega F(x,u(x),v(x))dx,$$

Then, $\Psi \in C^1(W_{p(x),q(x)}, \mathbb{R})$ and

$$\Psi'(u,v)(\omega,z) = D_1\Psi(u,v)(\omega) + D_2\Psi(u,v)(z),$$

where

$$D_1\Psi(u,v)(\omega) = \int_\Omega \frac{\partial F}{\partial u}(x,u,v)(\omega)\, dx,$$

and

$$D_2\Psi(u,v)(z) = \int_\Omega \frac{\partial F}{\partial v}(x,u,v)(z)\, dx.$$

Also, define Φ by

$$\Phi(u,v) = \int_\Omega \frac{1}{p(x)}|\nabla u|^{p(x)}dx + \int_\Omega \frac{1}{q(x)}|\nabla v|^{q(x)}dx,$$

We recall that Φ is a C^1-functional, weakly lower semi-continuous and its derivative is given by

$$\Phi'(u,v)(\omega,z) = D_1\Phi(u,v)(\omega) + D_2\Phi(u,v)(z)$$

where

$$D_1\Phi(u,v)(\omega) = \int_\Omega |\nabla u|^{p(x)-2}\nabla u \nabla \omega \, dx$$

and

$$D_2\Phi(u,v)(\omega) = \int_\Omega |\nabla v|^{q(x)-2}\nabla v \nabla z \, dx$$

The Euler-Lagrange functional associated to the system (2.1.1) is defined by

$$E(u,v) = \Phi(u,v) - \Psi(u,v).$$

According to [32], we have $E \in C^1(W_{p(x),q(x)}, \mathbb{R})$ and for all $(\omega,z) \in W_{p(x),q(x)}$

$$E'(u,v)(\omega,z) = \Phi'(u,v)(\omega,z) - \Psi'(u,v)(\omega,z).$$

Consequently, (u,v) in $W_{p(x),q(x)}(\Omega)$ is a weak solution of (2.1.1) if and only if (u,v) is a critical point of E.

2.2 Main results

In this section we state the main results. Theorem (2.2.1) shows that the system (2.1.1) has a unique weak solution in the case that $\frac{\partial F}{\partial u}(x,u,v)$ and $\frac{\partial F}{\partial v}(x,u,v)$ are decreasing in both u and v. Theorem (2.2.2) shows that the system (2.1.1) has at most one positive radial solution in the case that $\frac{\partial F}{\partial u}(x,u,v)$ and $\frac{\partial F}{\partial v}(x,u,v)$ are nondecreasing in both u and v, and finally, Theorem (2.2.3) yields the existence of at least a positive radial soluton to the system (2.1.1)

Theorem 2.2.1. *Suppose that F satisfies the following conditions*

(F_1) $F \in C^1(\Omega \times \mathbb{R} \times \mathbb{R}, \mathbb{R})$,

(F_2) *There exist positive functions $a_i, b_i (i = 1, 2)$, such that*

$$|\frac{\partial F}{\partial u}(x,u,v)| \leq a_1(x)|u|^{\mu_1 - 1} + a_2(x)|v|^{\mu_2 - 1},$$
$$|\frac{\partial F}{\partial v}(x,u,v)| \leq b_1(x)|u|^{\nu_1 - 1} + b_2(x)|v|^{\nu_2 - 1},$$

where $1 < \mu_1, \mu_2, \nu_1, \nu_2 < \inf(p(x), q(x))$ for all $x \in \Omega$ and the weight-functions a_1, b_2 (resp a_2, b_1), belong to the generalized Lebesgue spaces $L^{\alpha_i}(\Omega)$ (resp $L^{\beta}(\Omega)$), with

$$\alpha_1(x) = \frac{p(x)}{p(x)-1}, \alpha_2(x) = \frac{q(x)}{q(x)-1}, \quad \beta(x) = \frac{p^*(x)q^*(x)}{p^*(x)q^*(x) - p^*(x) - q^*(x)}.$$

(F_3) *$\frac{\partial F}{\partial u}(x,u,v)$ and $\frac{\partial F}{\partial v}(x,u,v)$ are decreasing with respect to both u and v.*
$(F_4)\frac{\partial F}{\partial u}(x,0,0) \not\equiv 0$ *or* $\frac{\partial F}{\partial u}(x,0,0) \not\equiv 0$.
Then, (2.1.1) *has a unique weak solution.*

Theorem 2.2.2. *Let $\Omega = B_R$ be the ball in $\mathbb{R}^N$ centered at the origin and of radius R. Suppose that F satisfies the following conditions*

(F_5) $F \in C^1(\Omega \times \mathbb{R}^+ \times \mathbb{R}^+, \mathbb{R})$

(F_6) *$\forall r \in [0, R]$ $F(r, 0, 0) = 0$, and $\frac{\partial F}{\partial u}(r,u,v) > 0$, $\frac{\partial F}{\partial v}(r,u,v) > 0$, for any $r \in [0, R]$, $u > 0$ and $v > 0$.*

(F_7) *$\frac{\partial F}{\partial u}(r,u,v)$ and $\frac{\partial F}{\partial v}(r,u,v)$ are nondecreasing with respect to both u and v.*

(F_8) *$\frac{1}{u^{p_- - 1}}\frac{\partial F}{\partial u}(r,u,v)$ is strictly decreasing in $u \in]0, +\infty[$ and $\frac{1}{v^{q_- - 1}}\frac{\partial F}{\partial v}(r,u,v)$ is strictly decreasing in $v \in]0, +\infty[$.*

Then, (2.1.3) *has at most one positive solution.*

Theorem 2.2.3. *Let $\Omega = B_R$. Suppose that F satisfies $(F_1) - (F_2), (F_5) - (F_8)$ and*

(F_9) *There exist constants $M > 0, 0 < \theta < 1$ and a positive function $H : \Omega \times \mathbb{R}^+ \times \mathbb{R}^+ \to \mathbb{R}$ such that for $x \in \Omega$, $0 \leq u, v \leq M$ and sufficiently small $t > 0$, we have*

$$F(x, t^{1/p^+}u, t^{1/q^+}v) \geq t^{\theta} H(x,u,v).$$

Then, (2.1.1) *has a unique positive radial solution.*

Remark

Assumption (F_9) implies that the potential function F is positive in a neighborhood of zero.

2.3 Proofs of main results

Proof. (of Theorem (2.2.1)) By definition (u,v) is a weak solution of (2.1.1) if and only if $E'(u,v)=0$ in $W'_{p(x),q(x)}(\Omega)$. In order to apply Browder-Minty theorem, we split the proof in several steps.

step 1 We prove that E' is bounded and continuous on $W_{p(x),q(x)}(\Omega)$.

Since $E\in C^1(W_{p(x),q(x)}(\Omega),\mathbb{R})$. Then E' are bounded and continuous.

step 2 We prove that E' is strictly monotone. We recall the elementary inequality for any $\alpha,\beta\in\mathbb{R}^N$

$$\begin{cases} |\alpha-\beta|^\gamma \le 2^\gamma \left(|\alpha|^{\gamma-2}\alpha-|\beta|^{\gamma-2}\beta\right)\cdot(\alpha-\beta) & \text{if } \gamma\ge 2\\ |\alpha-\beta|^2 \le \frac{1}{\gamma-1}(|\alpha|+|\beta|)^{2-\gamma}\left(|\alpha|^{\gamma-2}\alpha-|\beta|^{\gamma-2}\beta\right)\cdot(\alpha-\beta) & \text{if } 1<\gamma<2 \end{cases} \tag{2.3.1}$$

where $\cdot$ denotes the standard inner product in $\mathbb{R}^N$. from which we deduce that Φ' is strictly monotone. Also, we get

$$\left\langle D_1\Psi(u_1,v_1)-D_1\Psi(u_2,v_2),(u_1-u_2)^+\right\rangle=\int_\Omega\Big(\frac{\partial F}{\partial u}(x,u_1,v_1)-\frac{\partial F}{\partial u}(x,u_2,v_2)\Big)(u_1-u_2)^+\,dx\le 0.$$

Similarly, we obtain

$$\left\langle D_2\Psi(u_1,v_1)-D_2\Psi(u_2,v_2),(v_1-v_2)^+\right\rangle=\int_\Omega\Big(\frac{\partial F}{\partial u}(x,u_1,v_1)-\frac{\partial F}{\partial u}(x,u_2,v_2)\Big)(v_1-v_2)^+\,dx\le 0.$$

Thus we get,

$$\begin{aligned}\langle\Psi'(u_1,v_1)-\Psi'(u_2,v_2),(u_1,v_1)-(u_2,v_2)\rangle &= \langle D_1\Psi(u_1,v_1)-D_1\Psi(u_2,v_2),u_1-u_2\rangle\\ &\quad+\langle D_2\Psi(u_1,v_1)-D_2\Psi(u_2,v_2),v_1-v_2\rangle\\ &\le 0.\end{aligned}$$

Consequently, E' is strictly monotone.

step 3 We prove that E' is is coercive,
for all $(u_1,v_1)\neq(0,0)\in W_{p(x),q(x)}$ we have,

$$\begin{aligned}\frac{1}{\|(u,v)\|}\langle E'(u,v),(u,v)\rangle &= \frac{1}{\|(u,v)\|}\Big(\int_\Omega |\nabla u|^{p(x)}\,dx + \int_\Omega |\nabla v|^{q(x)}\,dx\,dx \\ &\quad - \int_\Omega \frac{\partial F}{\partial u}(x,u,v)u\,dx - \int_\Omega \frac{\partial F}{\partial v}(x,u,v)v\,dx\Big) \\ &\geq \frac{1}{\|(u,v)\|}\Big(\rho(\nabla u) + \rho(\nabla v) \\ &\quad - \int_\Omega (a_1(x)|u|^{\mu_1-1} + a_2(x)|v|^{\mu_2-1})u\,dx \\ &\quad - \int_\Omega (b_1(x)|u|^{\nu_1-1} + b_2(x)|v|^{\nu_2-1})u\,dx\Big).\end{aligned}$$

By applying Propositions 1.1.3, 1.1.4, 1.1.7 and the Young inequality, we obtain

$$\begin{aligned}\frac{1}{\|(u,v)\|}\langle E'(u,v),(u,v)\rangle &\geq \frac{1}{\|(u,v)\|}\Big(\min\Big\{\|u\|_{p(x)}^{p^-}, \|u\|_{p(x)}^{p^+}\Big\} + \min\Big\{\|v\|_{q(x)}^{q^-}, \|v\|_{q(x)}^{q^+}\Big\} \\ &\quad - c_1\Big(|a_1|_{\alpha_1(x)}||u|^{\mu_1}|_{p(x)} + |a_2|_{\beta(x)}||v|^{\mu_2-1}|_{q^*(x)}|u|_{p^*(x)} \\ &\quad + |b_1|_{\beta(x)}||u|^{\nu_1-1}|_{p^*(x)}|v|_{q^*(x)} + |b_2|_{\alpha_2(x)}||v|^{\nu_2}|_{q(x)}\Big) \\ &\geq \frac{1}{\|(u,v)\|}\Big(\min\Big\{\|u\|_{p(x)}^{p^-}, \|u\|_{p(x)}^{p^+}\Big\} + \min\Big\{\|v\|_{q(x)}^{q^-}, \|v\|_{q(x)}^{q^+}\Big\} \\ &\quad - c_1\Big(|a_1|_{\alpha_1(x)}||u|^{\mu_1}|_{p(x)} + |a_2|_{\beta(x)}(\frac{\mu_2-1}{\mu_2}\|v\|_{q(x)}^{\mu_2} + \frac{1}{\mu_2}\|u\|_{p(x)}^{\mu_2}) \\ &\quad + |b_1|_{\beta(x)}(\frac{\nu_1-1}{\nu_1}\|u\|_{p(x)}^{\nu_1} + \frac{1}{\nu_1}\|v\|_{q(x)}^{\nu_1}) + |b_2|_{\alpha_2(x)}||v|^{\nu_2}|_{q(x)}\Big) \\ &\geq \frac{1}{\|(u,v)\|}\Big(\min\Big\{\|u\|_{p(x)}^{p^-}, \|u\|_{p(x)}^{p^+}\Big\} + \min\Big\{\|v\|_{q(x)}^{q^-}, \|v\|_{q(x)}^{q^+}\Big\} \\ &\quad - c\Big(|a_1|_{\alpha_1(x)}\|u\|_{p(x)}^{\mu_1} + |a_2|_{\beta(x)}\|v\|_{q(x)}^{\mu_2} + |a_2|_{\beta(x)}\|u\|_{p(x)}^{\mu_2} + \\ &\quad + |b_1|_{\beta(x)}\|u\|_{p(x)}^{\nu_1} + |b_1|_{\beta(x)}\|v\|_{q(x)}^{\nu_1} + |b_2|_{\alpha_2(x)}\|v\|_{q(x)}^{\nu_2}\Big),\end{aligned}$$

which yields the coercivity of E', because $1 < \mu_1, \mu_2, \nu_1, \nu_2 < \inf(p(x), q(x))$.

The previous steps imply the existence of solutions for the problem since, by the Browder-Minty theorem (Proposition (1.3.4)), the operator E from X to X' is surjective; while the uniqueness of the solution is guarranted by the strict monotonicity of E.

Note that the unique solution cannot be trivial according to hypothesis $(F4)$. □

Before proving theorem (2.2.3) we shall give some preliminaries.

First, We recall the Principle of Symmetric Criticality of Krawcewicz and Marzantowicz [[54], p.1045], which will be crucial in the proof of Theorem (2.2.3). Let G be a compact Lie group which acts linear isometrically on the real Banach space $(X, \|.\|)$, i.e., the action $G \times X \to X : [g, u] \mapsto gu$ is continuous, $1.u = u$, $(g_1 g_2)u = g_1(g_2 u)$ for every $g_1, g_2 \in G$, and the map $u \mapsto gu$ is linear such thath $\|gu\| = \|u\|$ for every $g \in G$ and $u \in X$

A function $h : X \to \mathbb{R}$ is G-invariant if $h(gu) = h(u)$ for all $g \in G$. Let

$$X_G = \{u \in X : gu = u \quad \text{for all} \quad g \in G\}.$$

Lemma 2.3.1 (cf. [54]). *Assume that a compact Lie group G acts linearly isometrically on a Banach space X. If $h : X \to \mathbb{R}$ is G-invariant, locally lipshitz function and if $u \in X_G$ is a critical point of h restricted to X_G, then u is a critical point of h.*

Let $W_{p(x),q(x)}(\Omega)$ be the Cartesian product $W_0^{1,p(x)}(\Omega) \times W_0^{1,q(x)}(\Omega)$ with the norm $\|(u, v)\| := \max\left\{\|u\|_{p(x)}, \|v\|_{q(x)}\right\} \forall (u, v) \in W_{p(x),q(x)}(\Omega)$. Then G acts linearly isometrically on $W_{p(x),q(x)}(\Omega)$, where the action $G \times W_{p(x),q(x)}(\Omega) \to W_{p(x),q(x)}(\Omega)$ is defined by

$$g(u, v) = (gu, gv)$$

for all $g \in G$ and $u, v \in W_{p(x),q(x)}(\Omega)$. Moreover,

$$\begin{aligned} W^G_{p(x),q(x)}(\Omega) &= (W_0^{1,p(x)}(\Omega) \times W_0^{1,q(x)}(\Omega))^G \\ &= \left\{(u, v) \in W^G_{p(x),q(x)}(\Omega) : g(u, v) = (gu, gv) \quad \text{for all } g \in G\right\}, \\ &= (W_0^{1,p(x)}(\Omega)^G \times W_0^{1,q(x)}(\Omega)^G. \end{aligned}$$

In the following, we assume $G = O(N)$ be the group of orthogonal linear transformations in $\mathbb{R}^N$ and take $W^G_{p(x),q(x)}(\Omega) = W^r_{p(x),q(x)}(\Omega)$, where

$$W^r_{p(x),q(x)}(\Omega) = \left\{(u, v) \in W_{p(x),q(x)}(\Omega) : \ u \text{ and } v \text{ are radially symmetric } \right\}.$$

It is immediate that E is $O(N)$-invariant. Then, by the principle of symmetric criticality of Krawcewicz and Marzantowicz (2.3.1), we know that (u_0, v_0) is a critical point of E if and only if (u_0, v_0) is a critical point of $E_r = E|_{W^r_{p(x),q(x)}(\Omega)}$

The next result we need, in order to prove theorem (2.2.2), concerns a compara- ison principle for some unidimensional operators associated with the equations in problem (2.1.3).

Denote $\varphi_p(r, x) = |x|^{p(r)-2}x$ for $x \in \mathbb{R}$ and $r \in [0, R]$. Then for each $r \in [0, R]$, the function $\varphi_p(r, .)$ is a strictly increasing homeomorphism from $\mathbb{R}$ to $\mathbb{R}$. The inverse function of $\varphi_p(r, .)$ is denoted by $\varphi^{-1}_{p,r}(.)$. It is easy to see that $\varphi_p(r, .)$ and $\varphi^{-1}_{p,r}(.)$ are odd, and that $\varphi^{-1}_{p,r}(y) = \frac{1}{y^{p(r)-1}}$ for $y \geq 0$.

Let $C([0,R])$ be the Banach space of all continuous real functions defined on $[0,R]$ with the maximum norm. Set

$$C^+([0,R]) = \{u \in C([0,R]) : u(r) \geq 0 \quad \text{for } r \in [0,R]\}.$$

Denote $w_1(r) = \varphi_p(r,u') = |u'|^{p(r)-2}u'$, $w_2(r) = \varphi_q(r,v') = |v'|^{q(r)-2}v'$, $h_1(r) = -\frac{\partial F}{\partial u}(r,u(r),v(r))$ and $h_2(r) = -\frac{\partial F}{\partial v}(r,u(r),v(r))$; then the system

$$\begin{cases} (r^{N-1}w_1(r))' = r^{N-1}h_1(r) \\ (r^{N-1}w_2(r))' = r^{N-1}h_2(r) \end{cases} \tag{2.3.2}$$

is equivalent to

$$\begin{cases} w_1(r) = \frac{1}{r^{N-1}}\int_0^r t^{N-1}h_1(t)dt, & 0 < r < R, \\ w_2(r) = \frac{1}{r^{N-1}}\int_0^r t^{N-1}h_2(t)dt, & 0 < r < R. \\ w_1(0) = w_2(0) = 0. \end{cases} \tag{2.3.3}$$

For every $h \in C([0,R])$, define $T_p(h)$ by

$$\begin{aligned} T_p(h)(s) &= \int_s^R \varphi_{p,r}^{-1}(w(r))dr \\ &= \int_s^R \varphi_{p,r}^{-1}\left(\frac{1}{r^{N-1}}\int_0^r t^{N-1}h(t)dt\right)dr \quad \text{for } s \in [0,R]. \end{aligned} \tag{2.3.4}$$

It is easy to see that $(T_p(h_1),T_q(h_2)) \in C^1([0,R]) \times C^1([0,R])$ and satisfies :

$$\begin{cases} -\left(r^{N-1}|(T_p(h_1))'(r)|^{p(r)-2}(T_p(h_1))'(r)\right)' = r^{N-1}h_1(r) & \text{in }]0,R[, \\ -\left(r^{N-1}|(T_q(h_2))'(r)|^{q(r)-2}(T_q(h))'(r)\right)' = r^{N-1}h_2(r) & \text{in }]0,R[, \\ T_p(h_1)(0) = T_q(h_2)(0) = 0 \quad , T_p(h_1)(R) = T_q(h_2)(R) = 0. \end{cases} \tag{2.3.5}$$

Note that when $h \in C^+([0,R])$, then

$$T_p(h)(s) = \int_s^R \left(\frac{1}{r^{N-1}}\int_0^r t^{N-1}h(t)dt\right)^{\frac{1}{p(r)-1}} dr \quad \text{for } s \in [0,R], \tag{2.3.6}$$

and for any $K > 0$, we have

$$K^{\frac{1}{p_+-1}}T_p(h) \leq T_p(Kh) \leq K^{\frac{1}{p_--1}}T_p(h) \quad \text{if} \quad K > 1, \tag{2.3.7}$$

and

$$K^{\frac{1}{p_--1}}T_p(h) \leq T_p(Kh) \leq K^{\frac{1}{p_+-1}}T_p(h) \quad \text{if} \quad K < 1. \tag{2.3.8}$$

It is also clear that, (u,v) is a solution of system (2.1.3) if and only if $(u,v) = (T_p(\frac{\partial F}{\partial u}(r,u(r),v(r))), T_q(\frac{\partial F}{\partial v}(r,u(r),v(r))))$; that is

$$\begin{cases} u(s) = \int_s^R \left(\frac{1}{r^{N-1}}\int_0^r t^{N-1}h_1(t)dt\right)^{\frac{1}{p(r)-1}} dr & \text{for } s \in [0,R], \\ v(s) = \int_s^R \left(\frac{1}{r^{N-1}}\int_0^r t^{N-1}h_2(t)dt\right)^{\frac{1}{q(r)-1}} dr & \text{for } s \in [0,R]. \end{cases} \tag{2.3.9}$$

Lemma 2.3.2. *(see [[39]]) We have the following properties:*

1. *The mapping T_p is continuous and bounded from $C([0,R])$ to $C^1([0,R])$, and compact from $C([0,R])$ into itself.*
2. *If $h_1 \leq h_2$ then $T_p(h_1) \leq T_p(h_2)$ and if $h_1(s) < h_2(s)$ for some $s \in]0,R[$, then $T_p(h_1)(s) \leq T_p(h_2)(s)$.*
3. *If $h \in C^+([0,R])$, then $T_p(h) \in C^+([0,R])$ and $(T_p(h))'(s) \leq 0$ for any $s \in]0,R]$. Moreover, if $h(s) \neq 0$, then $(T_p(h))(s) > 0$ for any $s \in [0,R[$ and $(T_p(h))'(R) < 0$.*

Proof. (of Theorem (2.2.2))

Let (F_1) hold. Suppose that (u_1,v_1) and (u_2,v_2) are two positive solutions of (2.1.2). In order to show that $(u_1,v_1)=(u_2,v_2)$, we shall only prove that $u_1 \geq u_2$ and $v_1 \geq v_2$, since the proof of $u_1 \leq u_2$ and $v_1 \leq v_2$ is similar.

Define

$$D = \{\lambda > 0 \quad ; \ u_1(s) \geq \lambda u_2(s) \text{ and } \ v_1(s) \geq \lambda v_2(s) \quad \text{for } s \in [0,R]\}.$$

From Lemma 2.3.2, we know that $u_1, u_2 \in C^1([0,R])$; $u_1(s) > 0$ and $u_2(s) > 0$ for $s \in [0,R[$; $u_1(R)=0$, and $u_2(R)=0$; and ${u_1}'(R) < 0$ and ${u_2}'(R) < 0$. Hence, it follows that there exist positive constants η_1, A_1, A_2, B_1 and B_2 with $0 < \eta_1 < R$, $A_1 < A_2$ and $B_1 < B_2$ such that

$$-A_2 \leq u_1'(s) \leq -A_1 \quad ; \ -A_2 \leq u_2'(s) \leq -A_1 \quad , \quad \forall s \in [R-\eta_1, R]$$

and

$$B_1 \leq u_1(s) \leq B_2 \quad ; B_1 \leq u_2(s) \leq B_2 \quad , \quad \forall s \in [0, R-\eta_1].$$

For any $\lambda \in]0, \frac{A_1}{A_2}]$ and $s \in [R-\eta_1, R]$, we have

$$u_1'(s) \leq -A_1 \leq -\lambda A_2 \leq \lambda u_2'(s)$$

and, consequently

$$u_1(s) = -\int_s^R u_1'(t)dt \geq -\int_s^R \lambda u_2'(t)dt = \lambda u_2(s).$$

For any $\lambda \in]0, \frac{B_1}{B_2}]$ and $s \in [0, R-\eta_1]$, we have

$$u_1(s) \geq B_1 \geq \lambda B_2 \geq \lambda u_2(s).$$

Thus, for any λ with $0 < \lambda \leq \lambda_1 = \min\{\frac{A_1}{A_2}, \frac{B_1}{B_2}\}$, we have $u_1(s) \geq \lambda u_2(s)$ for all $s \in [0,R]$, and similarly, for any λ with $0 < \lambda \leq \lambda_2 = \min\{\frac{A_1'}{A_2'}, \frac{B_1'}{B_2'}\}$, we have $v_1(s) \geq \lambda v_2(s)$ for all $s \in [0,R]$, then for any $0 < \lambda \leq \min\{\lambda_1, \lambda_2\}$, we have

$u_1(s) \geq \lambda u_2(s)$ and $v_1(s) \geq \lambda v_2(s)$, and this shows that $D \neq \emptyset$. On the other hand, it is obvious that the set $D \subset \mathbb{R}$ is bounded from above.
Setting $\lambda_\star = \sup_{\lambda \in D} \lambda$, we have $\lambda_s \in]0, +\infty[$; and from the continuity of u_1,u_2,v_2 and v_2 we can see that $\lambda_\star \in D$; that is λ_s is the maximum of all $\lambda \in D$ (that is $D =]0, \lambda_\star]$).

Now, we have $u_1 \geq \lambda_\star u_2$ and $v_1 \geq \lambda_\star v_2$. To prove $u_1 \geq u_2$ and $v_1 \geq \lambda_\star v_2$, it is sufficient to prove that $\lambda_\star \geq 1$. To this end, arguing by contradiction, we assume $\lambda_\star < 1$. Then, it follows from (F_8) that

$$\frac{\partial F}{\partial u}(t, u_1(t), v_1(t) \geq \frac{\partial F}{\partial u}(t, \lambda_\star u_2(t), v_2(t)) > \lambda_\star^{p_- -1}\frac{\partial F}{\partial u}(t, u_2(t), v_2(t)) \quad , \text{for } t \in]0, R[. \tag{2.3.10}$$

Denote $z = T_p\left(\lambda_\star^{p_- -1}\frac{\partial F}{\partial u}((t, u_2(t), v_2(t))\right)$. Note that $u_1 = T_p(\frac{\partial F}{\partial u}(t, u_1(t), v_1(t))$ and $u_2 = T_p(\frac{\partial F}{\partial u}(t, u_2(t), v_2(t)))$. By (2.3.10) and Lemma 2.3.2, we have :

$$u_1(s) > z(s) \quad , \text{for } s \in]0, R[.$$

Moreover, it follows from (2.3.10) and (2.3.6) that

$$-u_1'(R) > -z'(R) > 0.$$

By the continuity of u_1' and z' at R, there exists $\eta_2 \in]0, R[$ such that

$$-u'(s) > -z'(s) > 0 \quad , \forall s \in [R - \eta_2, R].$$

By the compactness of $[R-\eta_2, R]$, there exists σ_1 such that for any $\sigma \in]0, \sigma_1]$, we have

$$-u_1'(s) > -\sigma z'(s) > 0 \quad , \forall s \in [R - \eta_2, R]$$

and consequently

$$u_1(s) = -\int_s^R u_1'(t)dt \geq -\int_s^R \sigma z'(t)dt = \sigma z(s) \quad , \forall s \in [R - \eta_2, R].$$

Since $u_1(s) > z(s)$ for $s \in [0, R - \eta_2]$ and $[0, R - \eta_2] \subset \mathbb{R}$ is a compact set, then there exists $\sigma_2 > 1$ such that for any $\sigma \in]0, \sigma_2]$,

$$u_1(s) \geq \sigma z(s) \quad , \forall s \in [0, R - \eta_2].$$

Thus, by setting $\sigma = \min\{\sigma_1, \sigma_2\}$, we have $\sigma > 1$ and $u_1(s) \geq \sigma u_2(s)$, for all $s \in [0, R]$. On the other hand, by (2.3.10) and (2.3.8), we have

$$z \geq \left(\lambda_\star^{p_- -1}\right)^{\frac{1}{p_- -1}} T_p(\frac{\partial F}{\partial u}(t, u_1(t), v_1(t))) = \lambda_\star u_2.$$

Hence, we get $u_1 \geq \sigma\lambda_\star u_2$ which contradicts with the definition of $\lambda_\star$. Therfore, we obtain $u_1 \geq u_2$ and the proof of $v_1 \geq v_2$ is also similar. The proof is now complete.

□

Proof. (of Theorem (2.2.3))

Define $F^\star : [0,R] \times \mathbb{R} \times \mathbb{R} \longrightarrow \mathbb{R}$ by

$$F^\star(r,u,v) = \begin{cases} F(r,u,v) & \text{if } \ u,v \geq 0, \quad , r \in [0,R] \\ 0 & \text{if } \ u < 0 \ \text{ or } \ v < 0 \ , r \in [0,R] \end{cases}$$

It is well-known that the functional $\Psi_\star$ defined on $W_{p(x),q(x)}(\Omega)$ by

$$\Psi_\star(u,v) = \int_\Omega F^\star(x,u,v)dx$$

belongs to $C^1(W_{p(x),q(x)}(\Omega))$, in virtue of (F_1) and (F_2). In addition, the operator $\Psi'_\star$ defined from $W_{p(x),q(x)}(\Omega))$ to $(W_{p(x),q(x)}(\Omega))'$ by

$$\Psi'_\star(u,v)(\omega,z) = \int_\Omega \Big(\frac{\partial F^\star}{\partial u}(x,u,v)\omega + \frac{\partial F^\star}{\partial v}(x,u,v)z\Big)dx,$$

is compact (see [40]).

Define

$$E(u,v) = \Phi(u,v) - \Psi_\star(u,v),$$

for any $(u,v) \in W_{p(x),q(x)}(\Omega)$.
Hereafter, we write $p(x) = p(|x|) = p(r)$, $F^\star(|x|,u,v) = F^\star(r,u,v)$.
From (F_1) and (F_2) we can see that $E \in C^1(W_{p(x),q(x)}(\Omega),\mathbb{R})$. Define

$$W^r_{p(x),q(x)}(\Omega) = \big\{(u,v) \in W_{p(x),q(x)}(\Omega) : \ u \text{ and } v \text{ are radially symmetric } \big\}$$

and $E_r = E_{|W^r_{p(x),q(x)}(\Omega)}$. Then $W^r_{p(x),q(x)}(\Omega)$ is a closed linear subspace of $W_{p(x),q(x)}(\Omega)$, $W^r_{p(x),q(x)}(\Omega)$ is also reflexive and $E_r \in C^1(W_{p(x),q(x)}(\Omega),\mathbb{R})$. Let us prove that E_r has a nontrivial critical point.

The functional $\Phi(u,v)$ is continuous and convex on $W_{p(x),q(x)}(\Omega)$, and hence it is sequentially weakly lower semicontinuous on $W_{p(x),q(x)}(\Omega)$. Once again, from (F_1) and (F_2), we have the following Sobolev compact imbedding

$$W_{p(x),q(x)}(\Omega) \hookrightarrow L^{n(x)}(\Omega) \times L^{m(x)}(\Omega),$$

for all $(n,m) \in [p(x),p^*(x)[\times[q(x),q^*(x)[$. Moreover, one can obviously see that the functional $\Psi_\star$ is weakly lower semi-continuous on $W_{p(x),q(x)}(\Omega)$.

The functional $E_r : W^r_{p(x),q(x)}(\Omega) \longrightarrow \mathbb{R}$ is also sequentially weakly lower semicontinuous . Let us show that it is coercive.
For $(u,v) \in W^r_{p(x),q(x)}(\Omega)$ with $\|u\|_{p(x)}, \|v\|_{q(x)} \geq 1$, we have

$$\begin{aligned}
E_r(u,v) &= \int_\Omega \left(\frac{1}{p(x)} |\nabla u|^{p(x)} + \frac{1}{q(x)} |\nabla v|^{q(x)} - F^\star(x,u,v) \right) dx \\
&= \int_\Omega \left(\frac{1}{p(x)} |\nabla u|^{p(x)} + \frac{1}{q(x)} |\nabla v|^{q(x)} \right) dx \\
&\quad - \int_\Omega \left(\int_0^u \frac{\partial F^\star}{\partial s}(x,s,v)ds + \int_0^v \frac{\partial F^\star}{\partial s}(x,0,s)ds + F^\star(x,0,0) \right) dx \\
&\geq \int_\Omega \left(\frac{1}{p^+} |\nabla u|^{p(x)} + \frac{1}{q^+} |\nabla v|^{q(x)} \right) dx \\
&\quad - \int_\Omega (a_1(x)|u|^{\mu_1} + a_2(x)|v|^{\mu_2 - 1}|u| + b_2(x)|v|^{\nu_2})dx \\
&\geq \frac{1}{p^+}\rho(\nabla u) + \frac{1}{q^+}\rho(\nabla v) - \Big(|a_1|_{\alpha_1(x)} ||u|^{\mu_1}|_{p(x)} \\
&\quad + |a_2|_{\beta(x)} ||v|^{\mu_2 - 1}|_{q^*(x)} |u|_{p^*(x)} + |b_2|_{\alpha_2(x)} ||v|^{\nu_2}|_{q(x)} \Big) \\
&\geq \frac{1}{p^+} \|u\|_{p(x)}^{p^-} + \frac{1}{q^+} \|v\|_{q(x)}^{q^-} - \Big(|a_1|_{\alpha_1(x)} \|u\|_{p(x)}^{\mu_1} \\
&\quad + |a_2|_{\beta(x)} \big(\frac{\mu_2 - 1}{\mu_2} \|v\|_{q(x)}^{\mu_2} + \frac{1}{\mu_2} \|u\|_{p(x)}^{\mu_2} \big) + |b_2|_{\alpha_2(x)} \|v\|_{q(x)}^{\nu_2} \Big) \\
&\geq \frac{1}{p^+} \|u\|_{p(x)}^{p^-} + \frac{1}{q^+} \|v\|_{q(x)}^{q^-} - c\Big(|a_1|_{\alpha_1(x)} \|u\|_{p(x)}^{\mu_1} \\
&\quad + |a_2|_{\beta(x)} \|v\|_{q(x)}^{\mu_2} + |a_2|_{\beta(x)} \|u\|_{p(x)}^{\mu_2} + |b_2|_{\alpha_2(x)} \|v\|_{q(x)}^{\nu_2} \Big).
\end{aligned}$$

This shows that $E_r(u,v) \to +\infty$ as $\|(u,v)\| \to \infty$, since $1 < \mu_1, \mu_2, \nu_1, \nu_2 < \inf(p(x), q(x))$; that is , E_r is coercive on $W^r_{p(x),q(x)}(\Omega)$. Consequently, E_r has a global minimizer $(\overline{u}, \overline{v}) \in W^r_{p(x),q(x)}(\Omega)$. On the other hand, E_r is $C^1(W^r_{p(x),q(x)}(\Omega), \mathbb{R})$. Hence this minimum is necessarily characterized by a critical point of E_r, which is a weak solution of (2.1.1). Furthermore, this solution is nontrivial. Indeed, as $E(0,0) = 0$, it is sufficient to show that there exists $(u,v) \in W^r_{p(x),q(x)}(\Omega)$ such that $E_r(u,v) < 0$. Let $M > 0$, $0 < \theta < 1$ and

$(\varphi, \psi) \in C_0^\infty(\Omega) \times C_0^\infty(\Omega)$ with $0 < \varphi, \psi < M$. According to $(F9)$, one has

$$\begin{aligned}
&E_r(t^{1/p^+}\varphi, t^{1/q^+}\psi)\\
&= \Phi(t^{1/p^+}\varphi, t^{1/q^+}\psi) - \Psi(t^{1/p^+}\varphi, t^{1/q^+}\psi)\\
&\le t\int_\Omega \left(\frac{1}{p^-}|\nabla\varphi|^{p(x)} + \frac{1}{q^-}|\nabla\psi|^{q(x)}\right)dx - \int_\Omega F^\star(x, t^{1/p^+}\varphi, t^{1/q^+}\psi)dx\\
&\le t\left(\frac{1}{p^-}\rho(\nabla\varphi) + \frac{1}{q^-}\rho(\nabla\psi)\right) - t^\theta\int_\Omega H(x,\varphi,\psi)dx\\
&\le t\left(\frac{1}{p^-}\max\{|\nabla\varphi|_{p(x)}^{p^-}, |\nabla\varphi|_{p(x)}^{p^+}\} + \frac{1}{q^-}\max\{|\nabla\psi|_{q(x)}^{q^-}, |\nabla\psi|_{q(x)}^{q^+}\}\right)\\
&\quad - t^\theta\int_\Omega H(x,\varphi,\psi)dx\\
&\le t\left(\frac{1}{p^-}\max\{\|\nabla\varphi\|_{p(x)}^{p^-}, \|\nabla\varphi\|_{p(x)}^{p^+}\} + \frac{1}{q^-}\max\{\|\nabla\psi\|_{q(x)}^{q^-}, \|\nabla\psi\|_{q(x)}^{q^+}\}\right)\\
&\quad - t^\theta\int_\Omega H(x,\varphi,\psi)\,dx < 0,
\end{aligned}$$

for $t > 0$ sufficiently small.

This shows that there exists $(\overline{u}, \overline{v}) \in W^r_{p(x),q(x)}(\Omega)$ such that $E_r(\overline{u}, \overline{v}) < 0$ and hence $(\overline{u}, \overline{v}) \neq (0,0)$, because $E(0,0) = 0$. Thus, $(\overline{u}, \overline{v})$ is a nontrivial critical point of E_r.

By the principle of symmetric criticality due to Krawcewicz and Marzantowicz [54] (see lemma (2.3.1)), every critical point of E_r is also a critical point of E. Thus $(\overline{u}, \overline{v})$ is also a critical point of E_r and hence $(\overline{u}, \overline{v})$ is a weak solution of the following problem

$$\begin{cases} -\Delta_{p(x)}\overline{u} = \frac{\partial F^\star}{\partial u}(x,u,v) & \text{in } \Omega,\\ -\Delta_{q(x)}\overline{v} = \frac{\partial F^\star}{\partial v}(x,u,v) & \text{in } \Omega,\\ \overline{u} = \overline{v} = 0 & \text{on } \partial\Omega; \end{cases} \tag{2.3.11}$$

From the definition of $F^\star$, we can see that $(\overline{u}, \overline{v})$ is positive, thus $F^\star(x, \overline{u}, \overline{v}) = F(x, \overline{u}, \overline{v})$ and consequently, $(\overline{u}, \overline{v})$ is a weak solution of problem (2.1.1). Since $(\overline{u}, \overline{v}) \in W^r_{p(x),q(x)}(\Omega)$, $(\overline{u}, \overline{v})$ is radially symmetric, and so $(\overline{u}, \overline{v})$ is a radial solution of (2.1.1), i.e., a solution of (2.1.3). By item (3) of Lemma (2.3.2), $(\overline{u}(t), \overline{v}(t))$ is strictly positive for $t \in [0, R)$. So $(\overline{u}, \overline{v})$ is a positive solution of (2.1.3). By Theorem (2.2.2), the positive solution of (2.1.3) is unique. The proof is now complete.

□

Chapter 3

Multiple Solutions to a $(p_1(x), \ldots, p_n(x))$-Laplacian-type Systems in Unbounded Domain

3.1 Introduction

In this work, we deal with the multiplicity of weak solutions for nonlinear elliptic system:

$$-\Delta_{p_i(x)}u_i + a_i(x)|u_i|^{p_i(x)-2}u_i = \lambda F_{u_i}(x, u_1, u_2, ..., u_n) \quad \text{in } \mathbb{R}^N, \tag{3.1.1}$$

for $1 \leq i \leq n$, where $\Delta_{p_i(x)}u_i := \operatorname{div}(|\nabla u_i|^{p_i(x)-2}\nabla u_i)$ is the $p_i(x)$-Laplacian operator for all $1 \leq i \leq n$, $p_i(x)$ are continuous real-valued functions such that $1 < p_i^- = \inf_{x\in\mathbb{R}^N} p_i(x) \leq p_i(x) \leq p_i^+ = \sup_{x\in\mathbb{R}^N} p_i(x) < N\,(N \geq 2)$ for all $x \in \mathbb{R}^N$, λ is a positive parameter, $a_i \in L^\infty(\mathbb{R}^N)$ such that $a_i := ess\inf_{x\in\mathbb{R}^N} a_i(x) > 0$, the real function F belongs to $C^1(\mathbb{R}^N \times \mathbb{R}^n)$, F_{u_i} denotes the partial derivative of F with respect to u_i.

The goal of this work is to establish the existence of some interval which includes λ, where the system (3.1.1) admits at least three weak solutions, by means of a very recent abstract critical points result of G. Bonanno and S.A. Marano [10], which is a more precise version of Theorem 3.2 of [8]. For other basic notations and definitions we refer to [84].

In the following discussions, we will use the product space

$$X := \prod_{i=1}^{n} W_{a_i}^{1,p_i(x)}(\mathbb{R}^N),$$

which is equipped with the norm

$$\|u\| := \sum_{i=1}^{n} \|u\|_{a_i}, \qquad \forall u = (u_1, u_2, ..., u_n) \in X,$$

where $\|u\|_{a_i}$ is the norm of $W_{a_i}^{1,p_i(x)}(\mathbb{R}^N)$. The space $X^\star$ denotes the dual space of X equipped with the usual dual norm.

Definition 3.1.1. *$u = (u_1, u_2, ..., u_n) \in X$ is called a weak solution of the system* (3.1.1) *if*

$$\sum_{i=1}^{n} \int_{\mathbb{R}^N} \Big(|\nabla u_i(x)|^{p_i(x)-2} \nabla u_i \nabla v_i + a_i(x) \left| u_i \right|^{p_i(x)-2} u_i v_i \Big)\, dx -$$

$$-\lambda \sum_{i=1}^{n} F_{u_i}(x, u_1, ... u_n) v_i \, dx = 0$$

for all $v = (v_1, v_2, ..., v_n) \in X$.

We denote by E_λ the energy functional associated with the problem (3.1.1)

$$E_\lambda(.) := \Phi(.) - \lambda \Psi(.),$$

where $\Phi, \Psi : X \longrightarrow \mathbb{R}$ are defined as follows:

$$\Phi(u) = \sum_{i=1}^{n} \int_{\mathbb{R}^N} \frac{1}{p_i(x)} \left(|\nabla u_i(x)|^{p_i(x)} dx + a_i(x) |u_i(x)|^{p_i(x)} \right) dx,$$

$$\Psi(u) = \int_{\mathbb{R}^N} F(x, u_1(x), ..., u_n(x)) dx.$$

for any $u = (u_1, ..., u_n)$ in X.

It is well known that $E_\lambda \in C^1(X, \mathbb{R})$ and that a critical point of E_λ corresponds to a weak solution of problem (3.1.1).

3.2 Hypotheses

We assume the following conditions:

(H1) $F \in C^1(\mathbb{R}^N \times \mathbb{R}^n, \mathbb{R})$ and $F(x, 0, ..., 0) = 0$.

(H2) There exist positive functions b_{ij} $(1 \leq i, j \leq n)$, such that

$$\left| \frac{\partial F}{\partial u_i}(x, u_1, ..., u_n) \right| \leq \sum_{j=1}^{n} b_{ij}(x) |u_j|^{\mu_{ij}-1},$$

where $1 < \mu_{ij} < \inf_{x \in \mathbb{R}^N} p_i(x)$, and $p_i(x) > \frac{N}{2}$, for all $x \in \mathbb{R}^N$ and for all $i \in \{1, 2, ..., n\}$. The weight-functions b_{ii} (resp b_{ij} if $i \neq j$) belong to the generalized Lebesgue spaces $L^{\alpha_i}(\mathbb{R}^N)$ (resp $L^{\alpha_{ij}}(\mathbb{R}^N)$), with

$$\alpha_i(x) = \frac{p_i(x)}{p_i(x) - 1}, \quad \alpha_{ij}(x) = \frac{p_i^*(x) p_j^*(x)}{p_i^*(x) p_j^*(x) - p_i^*(x) - p_j^*(x)}.$$

(H3) Assume that there exist $r > 0$ and $w = (w_1, ..., w_n) \in X$ such that the following conditions are satisfied:

(c1) $\sum_{i=1}^{n} \frac{\min\left\{\|w_i\|_{a_i}^{p^-}, \|w_i\|_{a_i}^{p^+}\right\}}{p_i^+} > r$

(c2) $\frac{\int_{\mathbb{R}^N} \sup_{(\xi_1,...\xi_n)\in K(sr)} F(x, \xi_1, ...\xi_n)dx}{r} < \frac{\int_{\mathbb{R}^N} F(x, w_1, ..., w_n)dx}{\sum_{i=1}^{n} \max\left\{\|w_i\|_{a_i}^{p^-}, \|w_i\|_{a_i}^{p^+}\right\}}$

where $K(t) := \left\{(\xi_1, ...\xi_n) \in \mathbb{R}^N : \sum_{i=1}^{n} \min\left\{|\xi_i|_{p^\star(x)}^{(p^\star)^-}, |\xi_i|_{p^\star(x)}^{(p^\star)^-}\right\} \leq t\right\}$ with $t > 0$
and $s = \min\left\{p_i^+ \min\left\{c_{p_i(x)}^{(p_i^\star)^-}, c_{p_i(x)}^{(p_i^\star)^+}\right\}\right\}$, such that $c_{p_i(x)}$ represent the constants associated with Proposition (1.1.7).

3.3 Main results

We will use the three critical points theorem obtained by Bonanno and Marano together with the following lemmas, to get our main results.

Lemma 3.3.1. *The functional Φ is continuously Gâteaux differentiable and sequentially weakly lower semicontinuous, coercive, whose Gâteaux derivative admits a continuous inverse on $X^\star$.*

Proof. It is well-known that the functional Φ is well defined and is continuously Gâteaux differentiable functional whose derivative at the point $u = (u_1, ..., u_n) \in X$ is the functional $\Phi'(u)$ given by

$$\Phi'(u)(v) = \int_\Omega \sum_{i=1}^{n} \left(|\nabla u_i(x)|^{p_i(x)-2}\nabla u_i(x)\nabla v_i(x) + a_i(x)|u_i(x)|^{p_i(x)}u_i(x)v_i(x)\right) dx,$$

for every $v = (v_1, ..., v_n) \in X$.

Let us show that it is coercive. By using (1.2.1) and (1.2.2), we have for all $u = (u_1, ..., u_n) \in X$

$$\begin{aligned}\Phi(u) &= \int_{\mathbb{R}^N} \sum_{i=1}^{n} \frac{1}{p_i(x)} \left(|\nabla u_i(x)|^{p_i(x)}dx + a_i(x)|u_i(x)|^{p_i(x)}\right) dx,\\ &\geq \sum_{i=1}^{n} \frac{1}{p_i^+} \int_{\mathbb{R}^N} \left(|\nabla u_i(x)|^{p_i(x)}dx + a_i(x)|u_i(x)|^{p_i(x)}\right) dx,\\ &\geq \sum_{i=1}^{n} \frac{1}{p_i^+} \min\left\{\|u_i\|_{a_i}^{p_i^-}, \|u_i\|_{a_i}^{p_i^+}\right\}.\end{aligned}$$

This shows that $\Phi(u) \to +\infty$ as $\|u\| \to \infty$; that is , Φ is coercive on X.

Now let us show that the operator $\Phi' : X \longrightarrow X^\star$ is strictly monotone, it suffices to prove that Φ is strictly convex

Consider the functional $\Phi_{p_i} : W_{a_i}^{1,p_i(x)}(\mathbb{R}^N) \longrightarrow \mathbb{R}$ defined by

$$\Phi_{p_i}(u_i) = \int_{\mathbb{R}^N} \frac{1}{p_i(x)} \left(|\nabla u_i|^{p_i(x)} + a_i(x)|u_i|^{p_i(x)}\right) dx \text{ for all } u_i \in W_{a_i}^{1,p_i(x)}(\mathbb{R}^N)$$

,
whose Gâteaux derivative at point $u_i \in W_{a_i}^{1,p_i(x)}(\mathbb{R}^N)$ is given by

$$\left\langle \Phi'_{p_i}(u_i), \varphi_i \right\rangle = \int_{\mathbb{R}^N} \left(|\nabla u_i(x)|^{p_i(x)-2} \nabla u \nabla \varphi_i + a_i(x)\, |u_i|^{p_i(x)-2}\, u_i \varphi_i \right) dx \text{ for all } \varphi_i \in W_{a_i}^{1,p_i(x)}(\mathbb{R}^N).$$

Taking into account the inequality (see, e.g., Chapter I in [29]) for $\gamma > 1$ there exists a positive constant C_γ such that

$$\langle |\alpha|^{\gamma-2}\alpha - |\beta|^{\gamma-2}\beta, \alpha - \beta\rangle \quad \geq \begin{cases} C_\gamma |\alpha-\beta|^\gamma & \text{if } \gamma \geq 2 \\ C_\gamma \frac{|\alpha-\beta|^2}{(|\alpha|+|\beta|)^{2-\gamma}}, (\alpha,\beta) \neq (0,0) & \text{if } 1 < \gamma < 2 \end{cases} \tag{3.3.1}$$

for any $\alpha, \beta \in \mathbb{R}^N$. Therefore,

$$\left\langle \Phi'_{p_i}(u_{i1}) - \Phi'_{p_i}(u_{i2}), u_{i1} - u_{i2} \right\rangle > 0,$$

for all $u_{i1} \neq u_{i2} \in W_{a_i}^{1,p_i(x)}(\mathbb{R}^N)$ which means that Φ'_{p_i} is strictly monotone. So by [[84], Prop.25.10], Φ_{p_i} is strictly convex. Hence Φ is a strictly convex in X, and so $\Phi' = \sum_{i=1}^n \Phi'_{p_i}$ is strictly monotone.

It is clear that Φ' is an injection since Φ' is a strictly monotone operator in X. Since

$$\lim_{\|u=(u_1,u_2,\ldots,u_n)\| \to +\infty} \frac{\langle \Phi'(u), u\rangle}{\|u\|} = \lim_{\|u\| \to +\infty} \frac{\sum_{i=1}^n \left(\int_{\mathbb{R}^N} \left(|\nabla u_i|^{p_i(x)} + a_i(x)|u_i|^{p_i(x)}\right) dx\right)}{\|u\|} = +\infty,$$

Φ' is coercive (see inequality (1.2.1) chapter 1), thus Φ' is a surjection. Now, since Φ' is hemicontinuous in X, then by applying Minty-Browder theorem (Theorem 26.A of [84]) we deduce that Φ' admits a continuous inverse on $X^\star$. Moreover, the monotonicity of Φ' on $X^\star$ ensures that Φ is sequentially lower-semicontinuousl on X (see [84], proposition 25. 20).

◀ □

Lemma 3.3.2. *Under the assumptions* $(H1)$ *and* $(H2)$*, the functional* Ψ *is well defined, and it is of class* C^1 *on* X*. Moreover, its derivative is*

$$\Psi'(u)h = \sum_{i=1}^n \int_{\mathbb{R}^N} \frac{\partial F}{\partial u_i}(x, u_1(x), \ldots, u_n(x)h_i(x)\, dx$$

$$\forall u = (u_1, \ldots, u_n), h = (h_1, \ldots, h_n) \in X.$$

Proof. For all $u = (u_1, ..., u_n) \in X$, under the assumptions $(H1)$ and $(H2)$, we can write

$$F(x, u_1, ..., u_n) = \sum_{i=1}^{n} \int_0^{u_i} \frac{\partial F}{\partial s}(x, u_1, ..., s, ..., u_n)\, ds + F(x, 0, ..., 0),$$

$$F(x, u_1, ..., u_n) \leq c_1 \left[\sum_{i=1}^{n} \left(\sum_{j=1}^{n} b_{ij}(x) |u_j(x)|^{\mu_{ij}-1} |u_i(x)| \right) \right]. \tag{3.3.2}$$

Then,

$$\int_{\mathbb{R}^N} F(x, u_1, ..., u_n)\, dx \leq c_2 \left[\sum_{i=1}^{n} \left(\int_{\mathbb{R}^N} \sum_{j=1}^{n} b_{ij}(x) |u_j(x)|^{\mu_{ij}-1} |u_i(x)|\, dx \right) \right]. \tag{3.3.3}$$

If we consider the fact that $W^{1,p(x)}(\mathbb{R}^N) \hookrightarrow L^{\mu(x)}(\mathbb{R}^N)$, for $\mu(x) > 1$, then there exists $c > 0$ such that

$$||u|^{\mu}|_{p(x)} = |u|^{\mu}_{\mu p(x)} \leq c \|u\|^{\mu}_{p(x)},$$

and if we apply Propositions 1.1.1, 1.1.4 and 1.1.5 and take $b_{ii} \in L^{\alpha_i(x)}$, $b_{ij} \in L^{\alpha_{ij}(x)}$ if $i \neq j$, then we have

$$\begin{aligned}
\int_{\mathbb{R}^N} F(x, u_1, ..., u_n) &\leq c_3 \Big[\sum_{i=1}^{n} \Big(\sum_{j=1}^{n} |b_{ij}|_{\alpha_{ij}(x)} ||u_j|^{\mu_{ij}-1}|_{p_j^\star(x)} |u_i|_{p_i^\star(x)} \Big) \Big] && (3.3.4)\\
&\leq c_3 \Big[\sum_{i=1}^{n} \Big(\sum_{j=1}^{n} |b_{ij}|_{\alpha_{ij}(x)} |u_j|^{\mu_{ij}-1}_{(\mu_{ij}-1)p_j^\star(x)} |u_i|_{p_i^\star(x)} \Big) \Big] && (3.3.5)\\
&\leq c_3 \Big[\sum_{i=1}^{n} \Big(\sum_{j=1}^{n} |b_{ij}|_{\alpha_{ij}(x)} \|u_j\|^{\mu_{ij}-1}_{p_j(x)} \|u_i\|_{p_i(x)} \Big) \Big] < \infty. && (3.3.6)
\end{aligned}$$

Hence, Ψ is well defined. Moreover, one can easily see that Ψ' is also well defined on X. Indeed, using $(F2)$ for all $h = (h_1, ..., h_n) \in X$, we have

$$\begin{aligned}
\Psi'(u)h &= \sum_{i=1}^{n} \int_{\mathbb{R}^N} \frac{\partial F}{\partial u_i}(x, u_1, ..., u_n) h_i\, dx \\
&\leq \sum_{i=1}^{n} \Big(\sum_{j=1}^{n} b_{ij}(x) |u_j(x)|^{\mu_{ij}-1} \Big) |h_i(x)| dx.
\end{aligned}$$

Following Hölder inequality, we obtain

$$\Psi'(u)h \leq c_4 \Big[\sum_{i=1}^{n} \Big(\sum_{j=1}^{n} |b_{ij}|_{\alpha_{ij}(x)} ||u_j|^{\mu_{ij}-1}|_{p_j^\star(x)} |h_i|_{p_i^\star(x)} \Big) \Big].$$

The above propositions yield

$$\Psi'(u)h \leq c\Big[\sum_{i=1}^{n}\Big(\sum_{j=1}^{n}|b_{ij}|_{\beta_{ij}(x)}\|u_j\|_{p_j(x)}^{\mu_{ij}-1}\|h_i\|_{p_i(x)}\Big)\Big] < \infty.$$

Now let us show that Ψ is differentiable in the sense of Frechet, that is, for fixed $u=(u_1,...,u_n)\in X$ and given $\varepsilon>0$, there must be a $\delta=\delta_{\varepsilon,u_1,...,u_n}>0$ such that

$$|\Psi(u_1+h_1,...,u_n+h_n)-\Psi(u_1,...,u_n)-\Psi'(u_1,...,u_n)(h_1,...,h_n)| \leq \varepsilon\sum_{i=1}^{n}(\|h_i\|_{p_i(x)})$$

for all $h=(h_1,...,h_n)\in X$ with $\sum_{i=1}^{n}(\|h_i\|_{p_i(x)})\leq\delta$.

Let B_R be the ball of radius R which is centered at the origin of $\mathbb{R}^N$ and denote $B_R'=\mathbb{R}^N-B_R$. Moreover, let us define the functional Ψ_R on $\prod_{i=1}^{n}W_{a_i}^{1,p_i(x)}(B_R)$ as follows:

$$\Psi_R(u)=\int_{B_R}F(x,u_1(x),...,u_n(x))\,dx.$$

If we consider $(H1)$ and $(H2)$, it is easy to see that $\Psi_R\in C^1(\prod_{i=1}^{n}W_{a_i}^{1,p_i(x)}(B_R))$, and in addition for all $h=(h_1,...,h_n)\in\prod_{i=1}^{n}W_{a_i}^{1,p_i(x)}(B_R)$, we have

$$\Psi_R'(u)h=\sum_{i=1}^{n}\int_{B_R}\frac{\partial F}{\partial u_i}(x,u_1(x),...,u_n(x))h_i(x)\,dx.$$

Also as we know, the operator $\Psi_R':X\to X^\star$ is compact [40]. Then, for all $u=(u_1,...,u_n), h=(h_1,...,h_n)\in X$, we can write

$$\begin{aligned}&|\Psi(u_1+h_1,...,u_n+h_n)-\Psi(u_1,...,u_n)-\Psi'(u_1,...,u_n)(h_1,...,h_n)|\\ &\leq|\Psi_R(u_1+h_1,...,u_n+h_n)-\Psi_R(u_1,...,u_n)-\Psi_R'(u_1,...,u_n)(h_1,...,h_n)|\\ &+\Bigg|\int_{B_R'}\Big(F(x,u_1+h_1,...,u_n+h_n)-F(x,u_1,...,u_n)-\\ &-\sum_{i=1}^{n}\int_{B_R}\frac{\partial F}{\partial u_i}(x,u_1,...,u_n)h_i\Big)\,dx\Bigg|.\end{aligned}$$

According to a classical theorem, there exist $\xi_1,...,\xi_n\in]0,1[$ such that

$$\Bigg|\int_{B_R'}\Big(F(x,u_1+h_1,...,u_n+h_n)-F(x,u_1,...,u_n)\Big)-$$

$$-\sum_{i=1}^{n}\int_{B_R}\frac{\partial F}{\partial u_i}(x,u_1,...,u_n)h_i\,dx\Bigg|$$

$$= \left| \int_{B'_R} \left(\sum_{i=1}^{n} \frac{\partial F}{\partial u_i}(x, u_1, ..., u_i + \xi_i h_i, ..., u_n) h_i - \sum_{i=1}^{n} \frac{\partial F}{\partial u_i}(x, u_1, ..., u_n) h_i \right) \, dx \right|.$$

Using the condition $(H2)$, we have

$$\Big| \int_{B'_R} \Big(F(x, u_1 + h_1, ..., u_n + h_n) - F(x, u_1, ..., u_n) \Big) - \sum_{i=1}^{n} \int_{B_R} \frac{\partial F}{\partial u_i}(x, u_1, ..., u_n) h_i \, dx \Big|$$

$$\leq \Big| \sum_{i=1}^{n} \Big(\sum_{j=1}^{n} \int_{B'_R} b_{ij}(x) (|u_j + \xi_j h_j|^{\mu_{ij}-1} - |u_j|^{\mu_{ij}-1}) h_i dx \Big) \Big|.$$

Using the elementary inequality $|a+b|^s \leq 2^{s-1}(|a|^s + |b|^s)$ for $a, b \in \mathbb{R}^N$, we can write

$$\leq \sum_{i=1}^{n} \Big(\sum_{j=1}^{n} \Big((2^{\mu_{ij}-1} - 1) \int_{B'_R} b_{ij}(x) |u_j|^{\mu_{ij}-1} |h_i| dx +$$

$$+ (\xi_j 2)^{\mu_{ij}-1} \int_{B'_R} b_{ij}(x) |h_j|^{\mu_{ij}-1} |h_i| dx \Big) \Big).$$

Then, applying Propositions 1.1.1, 1.1.4, and 1.1.5, we have

$$\leq \sum_{i=1}^{n} c \Big(\sum_{j=1}^{n} \Big(|b_{ij}(x)|_{\alpha_{ij}} \|u_j\|_{p_1^{\star}(x)}^{\mu_{ij}-1} + |b_{ij}(x)|_{\alpha_{ij}} \|h_j\|_{p_j^{\star}(x)}^{\mu_{ij}-1} \Big) \Big) \|h_i\|_{p_i(x)},$$

and by the fact that

$$|b_{ii}(x)|_{L^{\alpha_i}(B'_R)} \longrightarrow 0,$$

$$|b_{ij}(x)|_{L^{\alpha_{ij}}(B'_R)} \longrightarrow 0$$

for $1 \leq i, j \leq n$, as $R \to \infty$, and for R sufficiently large, we obtain the estimate

$$\left| \int_{B'_R} \Big(F(x, u_1 + h_1, ..., u_n + h_n) - F(x, u_1, ..., u_n) - \right.$$

$$\left. - \sum_{i=1}^{n} \frac{\partial F}{\partial u_i}(x, u_1, ..., u_n) h_i \Big) \, dx \right| \leq \varepsilon \sum_{i=1}^{n} (\|h_i\|_{p_i(x)}).$$

It remains only to show that Ψ' is continuous on X. Let $u^m = (u_1^m, ..., u_n^m)$ be such that $u^m \to u$ as $m \to \infty$. Then, for $h = (h_1, ..., h_n) \in X$, we have

$$|\Psi'(u^m)h - \Psi'(u)h| \leq |\Psi'_R(u^m)h - \Psi'_R(u)h|$$

$$+ \sum_{i=1}^{n} \int_{B'_R} \Big| (\frac{\partial F}{\partial u_i}(x, u_1^m, ..., u_n^m) h_i - \frac{\partial F}{\partial u_i}(x, u_1, ..., u_n) h_i) \, dx \Big|.$$

Since Ψ'_R is continuous on $\prod_{i=1}^{n} W_{a_i}^{1,p_i(x)}(B_R)$(see [40]), we have

$$|\Psi^{'}_R(u^m)h - \Psi^{'}_R(u)h| \longrightarrow 0,$$

as $m \to \infty$. Now, using $(H2)$ once again and taking into account that the other terms on the right-hand side of the above inequality tend to zero, we conclude that Ψ' is continuous on X. ◄ □

Lemma 3.3.3. *Under the assumptions* $(H1)$ *and* $(H2)$*,* $\Psi^{'}$ *is compact from* X *to* $X^\star$*.*

Proof. Let $u^m = (u_1^m, ..., u_n^m)$ be a bounded sequence in X. Then, there exists a subsequence (we denote it also as $u^m = (u_1^m, ..., u_n^m)$) which converges weakly in X to $u = (u_1, ..., u_n) \in X$. Then, if we use the same arguments as above, we have

$$\begin{aligned}|\Psi^{'}(u^m)h - \Psi^{'}(u)h| &\leq |\Psi^{'}_R(u^m)h - \Psi^{'}_R(u)h| \\ &\quad + \sum_{i=1}^{n} \int_{B^{'}_R} |(\frac{\partial F}{\partial u_i}(x, u_1^m, ..., u_n^m)h_i - \frac{\partial F}{\partial u_i}(x, u_1, ..., u_n)h_i)\, dx|\end{aligned}$$

Since the restriction operator is continuous, we have $u^m \rightharpoonup u$ in $\prod_{i=1}^{n} W_{a_i}^{1,p_i(x)}(B_R)$. Because of the compactness of $\Psi^{'}$, the first expression on the right-hand side of the inequality tends to 0, as $m \longrightarrow \infty$, and, as above, for sufficiently large R we obtain

$$\sum_{i=1}^{n} \int_{B^{'}_R} \Big|\Big(\frac{\partial F}{\partial u_i}(x, u_1^m, ..., u_n^m)h_i - \frac{\partial F}{\partial u_i}(x, u_1, ..., u_n)h_i\Big)\, dx\Big| \longrightarrow 0.$$

This implies $\Psi^{'}$ is compact from X to $X^\star$. ◄ □

Theorem 3.3.4. *Under the assumptions* $(H1) - (H3)$*, for each*

$$\lambda \in \left] \frac{\sum_{i=1}^{n} \max\left\{\|w_i\|_{a_i}^{p_i^-}, \|w_i\|_{a_i}^{p_i^+}\right\}}{\int_{\mathbb{R}^N} F(x, w_1(x), ..., w_n(x))dx}, \frac{r}{\int_{\mathbb{R}^N} \sup_{(\xi_1,...\xi_n)\in K(sr)} F(x, \xi_1, ..., \xi_n)dx} \right[,$$

the system (4.1.1) *admits at least three distinct weak solutions in* X*.*

Proof. By Lemma (5.3.1), Φ is coercive and $\Phi(0) = \Psi(0) = 0$. Also, we see that the required hypothesis $\Phi(\overline{x}) > r$ follows from (c_1) and the definition of Φ by choosing $\overline{x} = w = (w_1, ..., w_n)$. Moreover, by applying Proposition 4, for each $u_i \in W_{a_i}^{1,p_i(x)}$

$$|u_i|_{p_i^\star} \leq c\|u_i\|_{a_i},$$

with $1 \leq i \leq n$, we have for each $u = (u_1, ..., u_n) \in X$

$$\frac{1}{s}\sum_{i=1}^{n} \min\left\{|u_i|_{p_i^\star}^{p^-}, |u_i|_{p_i^\star}^{p^+}\right\} \leq \sum_{i=1}^{n} \frac{1}{p_i^+} \min\left\{\|u_i\|_{a_i}^{p^-}, \|u_i\|_{a_i}^{p^+}\right\} \tag{3.3.7}$$

with $s = \min\left\{p_i^+ \min\left\{c_{p_i(x)}^{(p_i^\star)^-}, c_{p_i(x)}^{(p_i^\star)^+}\right\}\right\}$. From (4.3.2), for each $r > 0$ we obtain

$$\begin{aligned}\Phi^{-1}(]-\infty, r[) &= \{u = (u_1, ..., u_n) \in X : \Phi(u) \leq r\} \\ &= \left\{u = (u_1, ..., u_n) \in X : \sum_{i=1}^{n} \frac{1}{p_i^+} \min\left\{\|u_i\|_{a_i}^{p_i^-}, \|u_i\|_{a_i}^{p_i^+}\right\} \leq r\right\} \\ &\subseteq \left\{u = (u_1, ..., u_n) \in X : \sum_{i=1}^{n} \min\left\{|u_i|_{p_i^\star}^{p_i^-}, |u_i|_{p_i^\star}^{p_i^+}\right\} \leq sr\right\}.\end{aligned}$$

Then,

$$\begin{aligned}\sup_{u\in\Phi^{-1}(]-\infty,r[)} \Psi(u) &= \sup_{u\in\Phi^{-1}(]-\infty,r[)} \int_{\mathbb{R}^N} F(x, u_1, ..., u_n)dx, \\ &\leq \int_{\mathbb{R}^N} \sup_{(\xi_1,...\xi_n)\in K(sr)} F(x, \xi_1, ..., \xi_n)dx.\end{aligned}$$

Therefore, from the condition (c_2), we have

$$\begin{aligned}\sup_{u\in\Phi^{-1}(]-\infty,r[)} \Psi(u) &\leq r\frac{\int_{\mathbb{R}^N} F(x, w_1(x), ..., w_n(x))dx}{\sum_{i=1}^{n} \max\left\{\|w_i\|_{a_i}^{p_i^-}, \|w_i\|_{a_i}^{p_i^+}\right\}}, \\ &\leq r\frac{\Psi(w)}{\Phi(w)}.\end{aligned}$$

from which (a_1) of Lemma 1 follows.

To show that the functional $\Phi - \lambda\Psi$ is coercive, we use the inequallity (9). Then for all $u \in X$, we have by virtue of $(H1)$ and $(H2)$

$$\begin{aligned}\Phi(u) - \lambda\Psi(u) &= \sum_{i=1}^{n} \int_{\mathbb{R}^N} \frac{1}{p_i(x)} \left(|\nabla u_i(x)|^{p_i(x)}dx + a_i(x)|u_i(x)|^{p_i(x)}\right) dx \\ &\quad - \lambda \int_{\mathbb{R}^N} F(x, u_1(x), ..., u_n(x))dx, \\ &\geq \sum_{i=1}^{n} \frac{1}{p_i^+} \int_{\mathbb{R}^N} \left(|\nabla u_i(x)|^{p_i(x)}dx + a_i(x)|u_i(x)|^{p_i(x)}\right) dx \\ &\quad - c_3\Big[\sum_{i=1}^{n}\Big(\sum_{i=j}^{n} |b_{ij}|_{\alpha_{ij}(x)}\|u_j\|_{p_j(x)}^{\mu_{ij}-1}\|u_i\|_{p_i(x)}\Big)\Big].\end{aligned}$$

Using Young's inequality, we obtain

$$\begin{aligned}\Phi(u) - \lambda\Psi(u) &\geq \sum_{i=1}^{n} \frac{1}{p_i^+}\|u_i\|_{a_i}^{p_i^-} - c_3\Big[\sum_{i=1}^{n}\Big(\sum_{j=1}^{n}|b_{ij}|_{\alpha_{ij}(x)}\Big(\frac{\mu_{ij}-1}{\mu_{ij}}\|u_j\|_{p_j(x)}^{\mu_{ij}} \\ &\quad + \frac{1}{\mu_{ij}}\|u_i\|_{p_i(x)}^{\mu_{ij}}\Big)\Big)\Big], \\ &\geq \sum_{i=1}^{n} \frac{1}{p_i^+}\|u_i\|_{a_i}^{p_i^-} - c_4\Big[\sum_{i=1}^{n}\Big(\sum_{j=1}^{n}\Big(|b_{ij}|_{\alpha_{ij}(x)}\|u_j\|_{p_j(x)}^{\mu_{ij}} \\ &\quad + |b_{ij}|_{\alpha_{ij}(x)}\|u_i\|_{p_i(x)}^{\mu_{ij}}\Big)\Big)\Big].\end{aligned}$$

This shows that $\Phi - \lambda\Psi \to +\infty$ as $\|u\|_X \to +\infty$, since $1 < \mu_{ij} < \inf_{x\in\mathbb{R}^N} p_i(x)$; that is, $\Phi - \lambda\Psi$ is coercive on X for every parameter λ, in particular, for every $\lambda \in \Lambda_r := \big]\frac{\Phi(w)}{\Psi(w)}, \frac{r}{\sup_{\Phi(w)\leq r}\Psi((w))}\big[$. Then, condition (a_2) also holds. Now all the hypotheses of Lemma 1 are satisfied. Also note that the solutions of the equation $\Phi'(u) - \lambda\Psi'(u) = 0$ are exactly the weak solutions of(4.1.1). Thus for each

$$\lambda \in \left]\frac{\sum_{i=1}^{n}\max\left\{\|w_i\|_{a_i}^{p_i^-}, \|w_i\|_{a_i}^{p_i^+}\right\}}{\int_{\mathbb{R}^N}F(x, w_1(x), ..., w_n(x))dx}, \frac{r}{\int_{\mathbb{R}^N}\sup_{(\xi_1,...\xi_n)\in K(sr)}F(x, \xi_1, ..., \xi_n)dx}\right[$$

the system (4.1.1) admits at least three weak solutions in X. ◀ □

3.4 Example

Let

$$F(x, u, v) = a(x)|u|^{\beta(x)}|v|^{\gamma(x)},$$

where $\frac{\beta(x)}{p(x)} + \frac{\gamma(x)}{q(x)} < 1$ and a is a positive function in $L^{s(x)}(\mathbb{R}^N)$ such that

$$s(x) = \frac{p^{\star(x)}q^{\star(x)}}{p^{\star(x)}q^{\star(x)} - \beta(x)q^{\star(x)} - \gamma(x)p^{\star(x)}} \text{ for each } x \in \mathbb{R}^N.$$

We can easily verify that $F(x, u, v)$ satisfies the conditions (H_1) and (H_3). Moreover, by using Young inequality we easily check that the condition (H_2) is also satisfied, and then the conclusion of Theorem 1 holds true.

Chapter 4

Multiple Solutions for elliptic $(p(x), q(x))$-Kirchhoff type potential Systems in Unbounded Domains

4.1 Introduction

The aim of this chapter is to show the existence of at least three weak solutions for the following class of nonlocal quasilinear elliptic systems in $\mathbb{R}^N$

$$\begin{cases} -M_1\left(L_p(u)\right)\left(\Delta_{p(x)}u - a(x)|u|^{p(x)-2}u\right) = \lambda F_u(x,u,v) & \text{in } \mathbb{R}^N, \\ -M_2\left(L_q(v)\right)\left(\Delta_{q(x)}v - b(x)|v|^{q(x)-2}v\right) = \lambda F_v(x,u,v) & \text{in } \mathbb{R}^N; \end{cases} \tag{4.1.1}$$

where $N \geq 2$, p and $q \in C_*(\mathbb{R}^N) := \{r \in C(\mathbb{R}^N) : 1 < r^- = \inf_{x\in\mathbb{R}^N} r(x) \leq r(x) \leq r^+ = \sup_{x\in\mathbb{R}^N} r(x) < N, \forall x \in \mathbb{R}^N\}$, λ is a positive real parameter and a, $b \in L^\infty(\mathbb{R}^N)$ are such that $a := ess\inf_{x\in\mathbb{R}^N} a(x) > 0$ and $b := ess\inf_{x\in\mathbb{R}^N} b(x) > 0$. M_1 and M_2 are bounded continuous functions, F belongs to $C^1(\mathbb{R}^N \times \mathbb{R}^2)$ and satisfies adequate growth assumptions and $F_u(resp. F_v)$ denotes the partial derivative of F with respect to u (resp. v). Here, we denote by $\Delta_{p(x)}u := \operatorname{div}\left(|\nabla u|^{p(x)-2}\nabla u\right)$ the so-called $p(x)$-Laplacian operator and $L_r(u) = \displaystyle\int_{\mathbb{R}^N} \frac{1}{r(x)}\left(|\nabla u|^{r(x)} + a(x)|u|^{r(x)}\right)dx$.

The existence and multiplicty of solutions for the elliptic systems involving $p(x)$-Kirchhoff model have been studied by many authors, where the nonlinear source F has different mixed growth conditions. We refer the reader to [28, 49, 63] and the references therein for an overview on this subject. In connexion to our context, the author obtained in [27] the existence and multiplicity of solutions for the vector valued elliptic system

$$\begin{cases} -M_1\left(\int_{\mathbb{R}^N} \dfrac{1}{p(x)}|u|^{p(x)}\right) \operatorname{div}\left(|\nabla u|^{p(x)-2}\nabla u\right) = \dfrac{\partial F}{\partial u}(x,u,v) & \text{in } \Omega, \\ -M_2\left(\int_{\mathbb{R}^N} \dfrac{1}{q(x)}|v|^{q(x)}\right) \operatorname{div}\left(|\nabla v|^{q(x)-2}\nabla v\right) = \dfrac{\partial F}{\partial v}(x,u,v) & \text{in } \Omega, \\ u = v = 0 & \text{in } \partial\Omega; \end{cases} \tag{4.1.2}$$

where Ω is a bounded domain in $\mathbb{R}^N$, with smooth boundary $\partial\Omega$, p and $q \in C_*(\Omega) = \{r \in C(\Omega) : 1 < r^- = \inf_{x\in\Omega} r(x) < r(x) < r^+ = \sup_{x\in\Omega} r(x) < N, \forall x \in \Omega\}$,, $M_1(t)$ and $M_2(t)$ are continuous functions such that $M_1(t) = M_2(t)$. The author apply a direct variational approach and the theory of variable exponents Sobolev spaces.

On the contrary, by using the Mountain Pass theorem the authors in [1], showed the existence of nontrivial solutions for the system (4.1.1), when $(p,q) \in [C(\mathbb{R}^N)]^2$ $(N \geq 2)$, $M_1(t)$ and $M_2(t)$ are continuous functions such that $M_1(t) = M_2(t)$, $a(x) = b(x) = 0$, $\lambda = 1$ and $F \in C^1(\mathbb{R}^N \times \mathbb{R}^2, \mathbb{R})$ verifies some mixed growth conditions.

The goal of this work is to establish the existence of a definite interval in which λ lies such that the system (4.1.1) admits at least three weak solutions, by applying the following very recent abstract critical points result of G. Bonanno and S.A. Marano [10], which is a more precise version of Theorem 3.2 of [8].

In the following, we shall use the product space

$$X := W_a^{1,p(x)}(\mathbb{R}^N) \times W_b^{1,q(x)}(\mathbb{R}^N),$$

which is equipped with the norm

$$\|(u,v)\| := \max\left\{\|u\|_a, \|v\|_b\right\} \qquad ,\forall (u,v) \in X;$$

where $\|.\|_a$ (resp., $\|.\|_b$) is the norm in $W_a^{1,p(x)}(\mathbb{R}^N)$ (resp., $W_b^{1,q(x)}(\mathbb{R}^N)$), defined above. We denote by $X^\star$ the dual space of X equipped with the usual dual norm.

Definition 4.1.1. *$(u,v) \in X$ is called a weak solution of the system* (4.1.1) *if*

$$\begin{aligned} &M_1\left(\int_{\mathbb{R}^N} \frac{|\nabla u|^{p(x)} + a(x)|u|^{p(x)}}{p(x)} dx\right) \int_{\mathbb{R}^N} \left(|\nabla u(x)|^{p(x)-2}\nabla u \nabla\varphi + a(x)\,|u|^{p(x)-2}\,u\varphi\right) dx \\ &+M_2\left(\int_{\mathbb{R}^N} \frac{|\nabla v|^{q(x)} + b(x)|v|^{q(x)}}{q(x)} dx\right) \int_{\mathbb{R}^N} \left(|\nabla v(x)|^{q(x)-2}\nabla v \nabla\psi + b(x)\,|v|^{q(x)-2}\,v\psi\right) dx \\ &\qquad -\lambda \int_{\mathbb{R}^N} F_u(x,u,v)\phi dx - \lambda \int_{\mathbb{R}^N} F_v(x,u,v)\psi dx = 0, \end{aligned}$$

for all $(\varphi, \psi) \in X$.

We denote by E_λ the energy functional associated with the problem (4.1.1)

$$E_\lambda(.) := \Phi(.) - \lambda\Psi(.),$$

where $\Phi, \Psi : X \longrightarrow \mathbb{R}$ are defined as follows

$$\begin{aligned}
\Phi(u,v) &= \widehat{M_1}\left(\int_{\mathbb{R}^N} \frac{|\nabla u|^{p(x)} + a(x)|u|^{p(x)}}{p(x)}dx\right) + \widehat{M_2}\left(\int_{\mathbb{R}^N} \frac{|\nabla v|^{q(x)} + b(x)|v|^{q(x)}}{q(x)}dx\right), \\
&= \widehat{M_1}(L_p(u)) + \widehat{M_2}(L_q(v)) \\
&= \Phi_1(L_p(u)) + \Phi_2(L_q(v)), \\
\Psi(u,v) &= \int_{\mathbb{R}^N} F(x,u,v)dx.
\end{aligned}$$

for any $w = (u,v)$ in X, with

$$\widehat{M_i}\,(t) := \int_0^t M_i(s)ds, \qquad \text{for all } t \geq 0, (i = 1,2).$$

Note that we have following formula

$$F(x,u,v) = \int_0^u \frac{\partial F}{\partial s}(x,s,v)ds + \int_0^v \frac{\partial F}{\partial s}(x,0,s)ds + F(x,0,0).$$

It is well know that $E_\lambda \in C^1(X,\mathbb{R})$ and that critical points of E_λ correspond to weak solutions of problem (4.1.1).

4.2 Hypotheses

In this chapter, we use the following assumptions

(H1) $F \in C^1(\mathbb{R}^N \times \mathbb{R}^2, \mathbb{R})$ and $F(x,0,0) = 0$.

(H2) There exist positive functions a_i and b_i $(i = 1,2)$ such that

$$\begin{aligned}
\left|\frac{\partial F}{\partial u}(x,u,v)\right| &\leq a_1(x)|u|^{\mu_1 - 1} + a_2(x)|v|^{\mu_2 - 1}, \\
\frac{\partial F}{\partial v}(x,u,v) &\leq b_1(x)|u|^{\nu_1 - 1} + b_2(x)|v|^{\nu_2 - 1},
\end{aligned}$$

where $1 < \mu_1, \mu_2, \nu_1, \nu_2 < \inf(p(x), q(x))$ and $p(x)$, $q(x) > \frac{N}{2}$, for all $x \in \mathbb{R}^N$, and the weight-functions a_1, b_2 (resp a_2, b_1), belong to the generalized Lebesgue spaces $L^{\alpha_i}(\mathbb{R}^N)$ (resp $L^\beta(\mathbb{R}^N)$), with

$$\alpha_1(x) = \frac{p(x)}{p(x)-1}, \alpha_2(x) = \frac{q(x)}{q(x)-1}, \quad \beta(x) = \frac{p^*(x)q^*(x)}{p^*(x)q^*(x) - p^*(x) - q^*(x)}.$$

(H3) $M_i : \mathbb{R}^+ \to \mathbb{R}$ are continuous and increasing functions such that $0 < m_0 \leq M_i(t) \leq m_1$, for all $t \geq 0$, $(i = 1, 2)$.

(H4) There exist $r > 0$ and $(w_1, w_2) \in X$ such that the following conditions are satisfied

$$\text{(C1)}\ \frac{m_0}{p^+} \min\left\{\|w_1\|_a^{p^-}, \|w_1\|_a^{p^+}\right\} + \frac{m_0}{q^+} \min\left\{\|w_2\|_b^{q^-}, \|w_2\|_b^{q^+}\right\} > r,$$

$$\text{(C2)}\ \frac{1}{r} \int_{\mathbb{R}^N} \sup_{(\xi_1,\xi_2)\in K(\frac{sr}{m_0})} F(x, \xi_1, \xi_2) dx$$
$$< \frac{1}{m_1\Big(\max\left\{\|w_1\|_a^{p^-}, \|w_1\|_a^{p^+}\right\} + \max\left\{\|w_2\|_b^{q^-}, \|w_2\|_b^{q^+}\right\}\Big)} \int_{\mathbb{R}^N} F(x, w_1, w_2) dx,$$

where

$$K(t) := \left\{(\xi_1, \xi_2) \in \mathbb{R}^2 : \min\left\{|\xi_1|_{p^\star(x)}^{(p^\star)^-}, |\xi_1|_{p^\star(x)}^{(p^\star)^+}\right\} + \min\left\{|\xi_2|_{q^\star(x)}^{(q^\star)^-}, |\xi_2|_{q^\star(x)}^{(q^\star)^+}\right\} \leq t\right\},$$

$$s = \min\left\{p^+ \min\left\{c_{p(x)}^{(p^\star)^-}, c_{p(x)}^{(p^\star)^+}\right\}, q^+ \min\left\{c_{q(x)}^{(q^\star)^-}, c_{q(x)}^{(q^\star)^+}\right\}\right\},$$

with $t > 0$ and $c_{p(x)}$ and $c_{q(x)}$ representing the constants defined in proposition (1.1.7).

4.3 Main results

We will use the three critical points theorem obtained by Bonano and Marano together with the following lemmas to get our main results.

Lemma 4.3.1. *The functional Φ is continuously Gâteaux differentiable and sequentially weakly lower semicontinuous, coercive whose Gâteaux derivative admits a continuous inverse on $X^\star$.*

Proof. It is well-known that the functional Φ is well defined and is continuously Gâteaux differentiable functionals whose derivative at the point $(u, v) \in X$ is the functional $\Phi^{'}(u, v)$ given by

$$\begin{aligned}\langle \Phi'(u, v), (\varphi, \psi)\rangle &= M_1\left(L_p(u)\right) \int_{\mathbb{R}^N} \left(|\nabla u(x)|^{p(x)-2}\nabla u \nabla\varphi + a(x) |u|^{p(x)-2} u\varphi\right) dx \\ &\quad + M_2\left(L_q(u)\right) \int_{\mathbb{R}^N} \left(|\nabla v(x)|^{q(x)-2}\nabla v \nabla\psi + b(x) |v|^{q(x)-2} v\psi\right) dx, \\ &= \langle \Phi_1'(u), \varphi\rangle + \langle \Phi_2'(v), \psi\rangle .\end{aligned}$$

where

$$\langle \Phi_1'(u), \varphi\rangle = M_1\left(L_p(u)\right)\int_{\mathbb{R}^N}\Big(|\nabla u(x)|^{p(x)-2}\nabla u\nabla\varphi + a(x)\,|u|^{p(x)-2}\,u\varphi\Big)\,dx,$$

$$\langle \Phi_2'(v), \psi\rangle = M_2\left(L_q(u)\right)\int_{\mathbb{R}^N}\Big(|\nabla v(x)|^{q(x)-2}\nabla v\nabla\psi + b(x)\,|v|^{q(x)-2}\,v\psi\Big)\,dx,$$

for every $(\varphi,\psi) \in X$ and $L_r(u) = \displaystyle\int_{\mathbb{R}^N}\frac{1}{r(x)}\left(|\nabla u|^{r(x)} + a(x)|u|^{r(x)}\right)dx$, for $r \in C_*(\mathbb{R}^N)$.

Let us show that Φ is coercive. By using (1.2.1) and (1.2.2), we have for all $(u,v) \in X$

$$\begin{aligned}\Phi(u,v) &= \widehat{M_1}\left(\int_{\mathbb{R}^N}\frac{|\nabla u|^{p(x)} + a(x)|u|^{p(x)}}{p(x)}dx\right) + \widehat{M_2}\left(\int_{\mathbb{R}^N}\frac{|\nabla v|^{q(x)} + b(x)|v|^{q(x)}}{q(x)}dx\right)\\ &\geq \frac{m_0}{p^+}\int_{\mathbb{R}^N}\left(|\nabla u(x)|^{p(x)}dx + a(x)|u(x)|^{p(x)}\right)\\ &\quad + \frac{m_0}{q^+}\int_{\mathbb{R}^N}\left(|\nabla v(x)|^{q(x)}dx + b(x)|v(x)|^{q(x)}\right)dx,\\ &\geq \frac{m_0}{p^+}\min\left\{\|u\|_a^{p^-}, \|u\|_a^{p^+}\right\} + \frac{m_0}{q^+}\min\left\{\|v\|_b^{q^-}, \|v\|_b^{q^+}\right\}.\end{aligned}$$

This shows that $\Phi(u,v) \to +\infty$ as $\|(u,v)\| \to +\infty$; that is , Φ is coercive on X.

Now let us show that the operator $\Phi' : X \longrightarrow X^\star$ is strictly monotone, it suffices to prove that Φ is strictly convex

Consider the functional $L_p : W_a^{1,p(x)}(\mathbb{R}^N) \longrightarrow \mathbb{R}$ defined by

$$L_p(u) = \int_{\mathbb{R}^N}\frac{1}{p(x)}\left(|\nabla u|^{p(x)} + a(x)|u|^{p(x)}\right)dx \text{ for all } u \in W_a^{1,p(x)}(\mathbb{R}^N)$$

,
whose Gâteaux derivative at point $u \in W_a^{1,p(x)}(\mathbb{R}^N)$ is given by

$$\left\langle L_p'(u), \varphi\right\rangle = \int_{\mathbb{R}^N}\Big(|\nabla u(x)|^{p(x)-2}\nabla u\nabla\varphi + a(x)\,|u|^{p(x)-2}\,u\varphi\Big)\,dx \text{ for all } \varphi \in W_a^{1,p(x)}(\mathbb{R}^N).$$

Taking into account the inequality (see, e.g., Chapter I in [29]) for $\gamma > 1$ there exists a positive constant C_γ such that

$$\langle |\alpha|^{\gamma-2}\alpha - |\beta|^{\gamma-2}\beta, \alpha-\beta\rangle \;\geq \begin{cases} C_\gamma|\alpha-\beta|^\gamma & \text{if } \gamma \geq 2\\ C_\gamma\frac{|\alpha-\beta|^2}{(|\alpha|+|\beta|)^{2-\gamma}}, (\alpha,\beta) \neq (0,0) & \text{if } 1 < \gamma < 2\end{cases} \tag{4.3.1}$$

for any $\alpha, \beta \in \mathbb{R}^N$. Therefore,

$$\left\langle L_p'(u_1) - L_p'(u_2), u_1 - u_2 \right\rangle > 0,$$

for all $u_1 \neq u_2 \in W_a^{1,p(x)}(\mathbb{R}^N)$ which means that L_p' is strictly monotone. So by [[84], Prop.25.10], L_p is strictly convex. Moreover, since the Kirchhoff function M_1 is nondecreasing, $\widehat{M_1}$ is convex in $[0, +\infty[$ Thus, for every $u_1, u_2 \in W_a^{1,p(x)}(\mathbb{R}^N)$ with $u_1 \neq u_2$, and every $s, t \in]0, 1[$ with $s + t = 1$, on has

$$\widehat{M_1}(L(su_1 + tu_2)) < \widehat{M_1}((sL(u_1) + tL(u_2))) \leq s\widehat{M_1}(L(u_1)) + t\widehat{M_1}(L(u_2)).$$

This shows that Φ_1 is strictly convex in $W_a^{1,p(x)}(\mathbb{R}^N)$, Similarly, we have that Φ_2 is strictly convex in $W_b^{1,p(x)}(\mathbb{R}^N)$. Hence Φ is a strictly convex in X, and so $\Phi' = \Phi_1' + \Phi_2'$ is strictly monotone.

It is clear that Φ' is an injection since Φ' is a strictly monotone operator in X. Since

$$\lim_{\|(u,v)\| \to +\infty} \frac{\langle \Phi'(u,v), (u,v) \rangle}{\|(u,v)\|} \geq$$

$$\lim_{\|(u,v)\| \to +\infty} \frac{m_0 \left(\int_{\mathbb{R}^N} \left(|\nabla u|^{p(x)} + a(x)|u|^{p(x)} \right) dx + \int_{\mathbb{R}^N} \left(|\nabla v|^{q(x)} + b(x)|v|^{q(x)} \right) dx \right)}{\|(u,v)\|} = +\infty,$$

Φ' is coercive (see (1.2.1)), thus Φ' is a surjection. Now, since Φ' is hemicontinuous in X, then by applying Minty-Browder theorem (Theorem 26.A of [84]) we deduce that Φ' admits a continuous inverse on $X^\star$. Moreover, the monotonicity of Φ' on $X^\star$ ensures that Φ is sequentially lower-semicontinuousl on X (see [84], proposition 25. 20). ☐

Lemma 4.3.2 (see [34]). *Under the assumptions* $(H1)$ *and* $(H2)$, *the functional* Ψ *is well defined and is of class* C^1 *on* X. *Moreover, its derivative is given by*

$$\Psi'(u,v)(\varphi,\psi) = \int_{\mathbb{R}^N} \frac{\partial F}{\partial u}(x,u,v)\varphi + \frac{\partial F}{\partial v}(x,u,v)\psi \, dx, \qquad \forall (u,v),(\varphi,\psi) \in X.$$

Moreover, Ψ' *is compact from* X *to* $X^\star$.

Theorem 4.3.3. *Under the assumptions* $(H1) - (H4)$, *the system* (4.1.1) *admits at least three distinct weak solutions in* X *for each*

$$\lambda \in \left] \frac{m_1 \left(\max\left\{ \|w_1\|_a^{p^-}, \|w_1\|_a^{p^+} \right\} + \max\left\{ \|w_2\|_b^{q^-}, \|w_2\|_b^{q^+} \right\} \right)}{\int_{\mathbb{R}^N} F(x, w_1(x), w_1(x)) dx}, \frac{r}{\int_{\mathbb{R}^N} \sup_{(\xi_1,\xi_2) \in K(sr/m_0)} F(x, \xi_1, \xi_2) dx} \right[.$$

Proof. by Lemma (4.3.1), Φ is coercive and by the definitions of Φ and Ψ and from hypothesis (H_1), we have $\Phi(0,0) = \Psi(0,0) = 0$. Moreover, the required hypothesis $\Phi(\overline{x}) > r$ follows form condition $(C1)$ and the defnition of Φ by choosing $\overline{x} = (w_1, w_2)$. On the other hand, by applying Proposition (1.1.5) for $(u, v) \in X$, we have

$$\frac{1}{s}\Big(\min\Big\{|u|_{p^\star}^{p^-}, |u|_{p^\star}^{p^+}\Big\} + \min\Big\{|v|_{s^\star}^{q^-}, |v|_{q^\star}^{q^+}\Big\}\Big) \leq \frac{1}{p^+}\min\Big\{\|u\|_a^{p^-}, \|u\|_a^{p^+}\Big\} + \frac{1}{q^+}\min\Big\{\|v\|_b^{q^-}, \|v\|_b^{q^+}\Big\}, \quad (4.3.2)$$

with $s = \min\Big\{p^+ \min\Big\{c_{p(x)}^{(p^\star)^-}, c_{p(x)}^{(p^\star)^+}\Big\}, q^+ \min\Big\{c_{q(x)}^{(q^\star)^-}, c_{q(x)}^{(q^\star)^+}\Big\}\Big\}$, where $c_{p(x)}$ and $c_{q(x)}$ represent the constants defined in proposition (1.1.7). Now, from (4.3.2) we obtain for each $r > 0$

$$\begin{aligned}\Phi^{-1}(]-\infty, r[) &= \{x = (u, v) \in X : \Phi(u, v) \leq r\}\\ &\subseteq \Big\{(u, v) \in X : \frac{m_0}{p^+}\min\Big\{\|u\|_a^{p^-}, \|u\|_a^{p^+}\Big\} + \frac{m_0}{q^+}\min\Big\{\|v\|_b^{q^-}, \|v\|_b^{q^+}\Big\} \leq r\Big\}\\ &\subseteq \Big\{(u, v) \in X : \min\Big\{|u|_{p^\star}^{p^-}, |u|_{p^\star}^{p^+}\Big\} + \min\Big\{|v|_{s^\star}^{q^-}, |v|_{q^\star}^{q^+}\Big\} \leq \frac{sr}{m_0}\Big\} = K\Big(\frac{sr}{m_0}\Big).\end{aligned}$$

Then,

$$\begin{aligned}\sup_{(u,v)\in\Phi^{-1}(]-\infty,r[)} \Psi(u) &= \sup_{(u,v)\in\Phi^{-1}(]-\infty,r[)} \int_{\mathbb{R}^N} F(x, u, v)dx\\ &\leq \int_{\mathbb{R}^N} \sup_{(\xi_1,\xi_2)\in K(\frac{sr}{m_0})} F(x, \xi_1, \xi_2)dx.\end{aligned}$$

Therefore, from the condition $(C2)$ we have

$$\begin{aligned}\sup_{(u,v)\in\Phi^{-1}(]-\infty,r[)} \Psi(u) &\leq r\frac{\displaystyle\int_{\mathbb{R}^N} F(x, w_1(x), w_1(x))dx}{m_1\Big(\max\Big\{\|w_1\|_a^{p^-}, \|w_1\|_a^{p^+}\Big\} + \max\Big\{\|w_2\|_b^{q^-}, \|w_2\|_b^{q^+}\Big\}\Big)},\\ &\leq r\frac{\Psi(w_1, w_2)}{\Phi(w_1, w_2)};\end{aligned}$$

from which condition (a_1) of Lemma (1.3.6) follows.

To show that the functional $E_\lambda = \Phi - \lambda\Psi$ is coercive, we use inequality (3.3.6).

For all $(u, v) \in X$, we have in virtue of $(H1)$ and $(H2)$

$$
\begin{aligned}
E_\lambda(u,v) &= \widehat{M_1}\left(\int_{\mathbb{R}^N} \frac{|\nabla u|^{p(x)} + a(x)|u|^{p(x)}}{p(x)} dx\right) + \widehat{M_2}\left(\int_{\mathbb{R}^N} \frac{|\nabla v|^{q(x)} + b(x)|v|^{q(x)}}{q(x)} dx\right) \\
&\quad - \lambda \int_{\mathbb{R}^N} F(x, u(x), v(x)) dx \\
&\geq \frac{m_0}{p^+} \int_{\mathbb{R}^N} \left(|\nabla u(x)|^{p(x)} dx + a(x)|u(x)|^{p(x)}\right) \\
&\quad + \frac{m_0}{q^+} \int_{\mathbb{R}^N} \left(|\nabla v(x)|^{q(x)} dx + b(x)|v(x)|^{q(x)}\right) dx, \\
&\quad - \lambda \int_{\mathbb{R}^N} \left(\int_0^u \frac{\partial F}{\partial s}(x,s,v) ds + \int_0^v \frac{\partial F}{\partial s}(x,0,s) ds + F(x,0,0)\right) dx \\
&\geq \frac{m_0}{p^+}\rho_a(u) + \frac{m_0}{q^+}\rho_b(v) - \lambda \int_{\mathbb{R}^N} \Big(a_1(x)|u|^{\mu_1} + a_2(x)|v|^{\mu_2 - 1}|u| + b_2(x)|v|^{\nu_2}\Big) dx \\
&\geq \frac{m_0}{p^+}\rho_a(u) + \frac{m_0}{q^+}\rho_b(v) - \lambda\Big(|a_1|_{\alpha_1(x)} ||u|^{\mu_1}|_{p(x)} + |a_2|_{\beta(x)} ||v|^{\mu_2-1}|_{q^*(x)} |u|_{p^*(x)} \\
&\quad + |b_2|_{\alpha_2(x)} ||v|^{\nu_2}|_{q(x)}\Big).
\end{aligned}
$$

Using Young's inequality, we obtain

$$
\begin{aligned}
E_\lambda(u,v) &\geq \frac{m_0}{p^+}\|u\|_a^{p^-} + \frac{m_0}{q^+}\|v\|_b^{q^-} - \lambda\Big(|a_1|_{\alpha_1(x)}\|u\|_{p(x)}^{\mu_1} \\
&\quad + |a_2|_{\beta(x)}\big(\frac{\mu_2 - 1}{\mu_2}\|v\|_{q(x)}^{\mu_2} + \frac{1}{\mu_2}\|u\|_{p(x)}^{\mu_2}\big) + |b_2|_{\alpha_2(x)}\|v\|_{q(x)}^{\nu_2}\Big) \\
&\geq \frac{m_0}{p^+}\|u\|_a^{p^-} + \frac{m_0}{q^+}\|v\|_b^{q^-} - c\Big(|a_1|_{\alpha_1(x)}\|u\|_{p(x)}^{\mu_1} \\
&\quad + |a_2|_{\beta(x)}\|v\|_{q(x)}^{\mu_2} + |a_2|_{\beta(x)}\|u\|_{p(x)}^{\mu_2} + |b_2|_{\alpha_2(x)}\|v\|_{q(x)}^{\nu_2}\Big).
\end{aligned}
$$

This shows that $\Phi - \lambda\Psi \to +\infty$ as $\|(u,v)\|_X \to \infty$, since we have $1 < \mu_1, \mu_2, \nu_1, \nu_2 < \inf(p(x), q(x))$; that is $\Phi - \lambda\Psi$ is coercive on X, for every parameter λ; in particular, for every $\lambda \in \Lambda_r := \left]\frac{\Phi(w_1,w_2)}{\Psi(w_1,w_2)}, \frac{r}{\sup_{\Phi(u,v)\leq r}\Psi((u,v))}\right[$. Then, condition (a_2) in lemma (1.3.6) also holds . Now, all the hypotheses of Lemma(1.3.6) are satisfied. Note that the solutions of the equation: $\Phi'(u,v) - \lambda\Psi'(u,v) = 0$, are exactly the weak solutions of(4.1.1). Thus, for each

$$
\lambda \in \left] \frac{m_1\Big(\max\Big\{\|w_1\|_a^{p^-}, \|w_1\|_a^{p^+}\Big\} + \max\Big\{\|w_2\|_b^{q^-}, \|w_2\|_b^{q^+}\Big\}\Big)}{\displaystyle\int_{\mathbb{R}^N} F(x, w_1(x), w_1(x)) dx}, \frac{r}{\displaystyle\int_{\mathbb{R}^N} \sup_{(\xi_1,\xi_2)\in K(\frac{sr}{m_0})} F(x,\xi_1,\xi_2) dx} \right[,
$$

system (4.1.1) admits at least three weak solutions in X. □

Chapter 5

Existence of Solutions for a critical $(p_1(x), ..., p_n(x))$-KIRCHHOFF TYPE POTENTIAL SYSTEMS

5.1 Introduction

The aim of this chapter is to show the existence of non-trivial solutions for the following class of nonlocal quasilinear elliptic systems

$$\begin{cases} -M_i\left(\displaystyle\int_\Omega \frac{1}{p_i(x)}|\nabla u_i|^{p_i(x)}\right)\Delta_{p_i(x)}u_i = |u_i|^{q_i(x)-2}u_i + \lambda F_{u_i}(x,u_1,u_2,...,u_n) & \text{in } \Omega, \\ u_i = 0 & \text{on } \partial\Omega; \end{cases} \tag{5.1.1}$$

for $1 \leq i \leq n$, where Ω is a bounded domain in $\mathbb{R}^N (N \geq 2)$, with smooth boundary $\partial\Omega$, $\Delta_{p_i(x)}u_i := \operatorname{div}\left(|\nabla u_i|^{p_i(x)-2}\nabla u_i\right)$ is the $p_i(x)$-Laplacian operator for all $1 \leq i \leq n$, λ is positive parameter, $p_i(x)$ and $q_i(x)$ are Lipschitz continuous real-valued functions such that

$$1 < p_i^- := \inf_{x\in\overline{\Omega}} p_i(x) \leq p_i(x) \leq p_i^+ = \sup_{x\in\overline{\Omega}} p_i(x) < N, \quad 1 \leq q_i(x) \leq p_i^\star(x) = \frac{Np_i(x)}{N-p_i(x)},$$

$$\mathcal{A}_{p_i} = \{x \in \Omega, q_i(x) = p_i^\star(x)\} \neq \emptyset.$$

The real function F belongs to $C^1(\Omega \times \mathbb{R}^n)$, F_{u_i} denotes the partial derivative of F with respect to u_i.

$M_i : \mathbb{R}_0^+ \to \mathbb{R}^+$ is a nondecreasing and continuous function, Let us assume throughout this paper that

$(\mathcal{M}_1)$ There exists $\mathfrak{M}_i^0 > 0$, such that

$$M_i(t) \geq \mathfrak{M}_i^0, \quad \forall t \in \mathbb{R}_0^+, (i = 1, 2, ..., n).$$

$(\mathcal{M}_2)$ There exists $\sigma_i \in (\frac{p_i^+}{q_i^-}, 1]$ such that

$$\widehat{M_i}\,(t) \geq \sigma_i M_i(t)t, \quad \forall t \in \mathbb{R}_0^+;$$

where $\widehat{M_i}\,(t) := \int_0^t M_i(s)ds$.

We can see that there are many functions satisfying conditions $(\mathcal{M}_1) - (\mathcal{M}_2)$, for example $M(t) = \mathfrak{M}^0 + bt^{\frac{1}{\sigma}}$ with $\sigma \leq 1$, $\mathfrak{M}^0 > 0$ and $b \geq 0$.

The system (5.1.1) is a generalization of the following Kirchhoff equation, introduced by Kirchhof in [51]

$$\rho \frac{\partial^2 u}{\partial t^2} - \left(\frac{\rho_0}{h} + \frac{E}{2L} \int_0^L \left| \frac{\partial u}{\partial x} \right|^2 dx \right) \frac{\partial^2 u}{\partial x^2} = 0, \tag{5.1.2}$$

where ρ, ρ_0, E and L are constants. This equation extends the classical D'Alembert's wave equation by considering the effects of the changes in the length of the strings during the vibrations. A distinguishing feature of equation (5.1.2) is that the equation contains a nonlocal coefficient $\frac{\rho_0}{h} + \frac{E}{2L} \int_0^L \left| \frac{\partial u}{\partial x} \right|^2 dx$ which depends on the average $\frac{1}{2L} \int_0^L \left| \frac{\partial u}{\partial x} \right|^2 dx$, and hence the equation is no longer a pointwise equation. The parameters in equation (0.0.3) have the following meanings: E is the Young modulus of the material (also referred to as the elastic modulus, it measures the strings resistance to being deformed elastically), ρ is the mass density, L is the length of the string, h is the area of cross-section, and ρ_0 is the initial tension. Almost one century later, Jacques-Louis Lions [58] returned to the equation and proposed a general Kirchhoff equation in arbitrary dimension with external force term which was written as

$$\begin{cases} \frac{\partial^2 u}{\partial t^2} - \left(a + b \int_\Omega |\nabla u|^2 \, dx\right) \Delta u = f(x, u) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega; \end{cases} \tag{5.1.3}$$

this problem is often called a nonlocal problem because it contains an integral over Ω. This causes some mathematical difficulties which make the study of such a problem particularly interesting. The nonlocal problem models several physical and biological systems, where u describes a process which depends on the average of itself, such as the population density,see [23] and its references therein. For a more detailed reference on this subject we refer the interested reader to [5, 16, 24, 25, 26, 48, 64, 71].

Moreover, elliptic problems involving critical nonlinearities have received great attention since the seminal work of Brezis and Nirenberg [13], in the case $p = 2$. The difficulty in problems involving critical nonlinearities is due to the lack of compactness of the embedding $W_0^{1,p(.)}(\Omega) \hookrightarrow L^{p^\star(.)}(\Omega)$. To overcome this difficulty, we use a Concentration-Compactness Principle for variable exponent space proved by Bonder and Silva in [11], which is slightly more general than the result proved in [41].

In the present work, we will show the existence of solutions for the nonlocal problem 5.1.1. The main theorems extend in several directions previous results recently appeared in the literature, see for example [3, 46, 55, 82, 85],and references therein. The difficulty in this case, is due to the lack of compactness of the embedding $W_0^{1,p(.)}(\Omega) \hookrightarrow L^{p^\star(.)}(\Omega)$ and the Palais-Smale condition for the corresponding energy functional could not be checked directly. To deal with this difficulty, we use a version of the concentration-compactness lemma due to Lions for variable exponents [11].

5.2 Hypotheses and main results

In the following discussions, we will use the product space

$$X := \prod_{i=1}^{n} W_0^{1,p_i(x)}(\Omega),$$

which is equipped with the norm

$$\|u\| := \max\left\{\|u_i\|_{p_i(x)}\right\}, \qquad \forall u = (u_1, u_2, ..., u_n) \in X,$$

where $\|u_i\|_{p_i(x)}$ is the norm of $W_0^{1,p_i(x)}(\Omega)$. The space $X^\star$ denotes the dual space of X and equipped with the usual dual norm.

Definition 5.2.1. *$u = (u_1, u_2, ..., u_n) \in X$ is called a weak solution of the system* (5.1.1) *if*

$$\sum_{i=1}^{n} M_i\left(\int_\Omega \frac{1}{p_i(x)}|\nabla u_i|^{p_i(x)}\right)\int_\Omega |\nabla u_i(x)|^{p_i(x)-2}\nabla u_i \nabla v_i\, dx - \sum_{i=1}^{n}\int_\Omega |u_i|^{q_i(x)-2}u_i v_i\, dx$$
$$-\lambda \sum_{i=1}^{n} F_{u_i}(x, u_1, ...u_n)v_i\, dx = 0,$$

for all $v = (v_1, v_2, ..., v_n) \in X$.

We denote by E_λ the energy functional associated with the problem (5.1.1)

$$E_\lambda(.) := \Phi(.) - \Theta(.) - \lambda\Psi(.),$$

where $\Phi, \Theta, \Psi : X \longrightarrow \mathbb{R}$ are defined as follows

$$\Phi(u) = \sum_{i=1}^{n} \widehat{M_i}\left(\int_{\Omega} \frac{1}{p_i(x)} |\nabla u_i|^{p_i(x)} dx\right),$$
$$\Theta(u) = \sum_{i=1}^{n} \int_{\Omega} \frac{1}{q_i(x)} |u_i|^{q_i(x)} dx,$$
$$\Psi(u) = \int_{\Omega} F(x, u_1(x), ..., u_n(x)) dx,$$

for any $u = (u_1, ..., u_n)$ in X.

In order to ensure that the function F satisfies the topological conditions and the geometric conditions of the mountain pass theorem (see [42]), we assume some growth conditions.

Hypotheses.

$(\mathcal{H}1)$ $F \in C^1(\Omega \times \mathbb{R}^n, \mathbb{R})$ and $F(x, 0, ..., 0) = 0$

$(\mathcal{H}2)$ There exist positive functions b_{ij} $(1 \leq i, j \leq n)$, such that

$$\left|\frac{\partial F}{\partial u_i}(x, u_1, ..., u_n)\right| \leq \sum_{j=1}^{n} b_{ij}(x) |u_j|^{\mu_{ij} - 1},$$

where $1 < \mu_{ij} < \inf_{x \in \mathbb{R}^N} p_i(x)$ for all $x \in \Omega$ and for all $i \in \{1, 2, ..., n\}$. The weight-functions b_{ii} (resp b_{ij} if $i \neq j$) belong to the generalized Lebesgue spaces $L^{\alpha_i}(\Omega)$ (resp $L^{\alpha_{ij}}(\Omega)$), with

$$\alpha_i(x) = \frac{p_i(x)}{p_i(x) - 1}, \quad \alpha_{ij}(x) = \frac{p_i^*(x) p_j^*(x)}{p_i^*(x) p_j^*(x) - p_i^*(x) - p_j^*(x)}.$$

$(\mathcal{H}3)$ Assume that there exist $K > 0$ and $\exists \theta_i \in (p_i^+, q_i^-)$ for all $(x, u_1, ..., u_n) \in \Omega \times \mathbb{R}^n$ where $|u_i|^{\theta_i} \geq K$

$$0 < F(x, u_1, ..., u_n) < \sum_{i=1}^{n} \frac{u_i}{\theta_i} \frac{\partial F}{\partial u_i}(x, u_1, ..., u_n).$$

$(\mathcal{H}4)$ $\exists c > 0$ such that

$$|F(x, u_1, ..., u_n)| \leq c\Big(\sum_{i=1}^{n} |u_i|^{r_i(x)}\Big), \forall (x, u_1, ..., u_n) \in \Omega \times \mathbb{R}^n,$$

where $r_i \in C_+(\overline{\Omega})$ and $p_i^+ < r_i^- \leq r_i^+ << q^- \leq q_i^+ \quad \forall 1 \leq i \leq n$.

Note that according to the above hypothesis, we have $E_\lambda \in C^1(X, \mathbb{R})$ and for all $v = (v_1, v_2, ..., v_n) \in X$

$$
\begin{aligned}
E'_\lambda(u)v = &\sum_{i=1}^{n} M_i \left(\int_\Omega \frac{1}{p_i(x)} |\nabla u_i|^{p_i(x)} \right) \int_\Omega |\nabla u_i(x)|^{p_i(x)-2} \nabla u_i \nabla v_i \, dx \\
&- \sum_{i=1}^{n} \int_\Omega |u_i|^{q_i(x)-2} u_i v_i \, dx - \lambda \sum_{i=1}^{n} F_{u_i}(x, u_1, ...u_n) v_i \, dx.
\end{aligned}
$$

Consequently, $u = (u_1, u_2, ..., u_n)$ in X is a weak solution of (5.1.1) if and only if u is a critical point of E_λ.

We end this section by stating the following existence result.

Theorem 5.2.2. *suppose that* $(\mathcal{M}_1) - (\mathcal{M}_2)$ *and* $(\mathcal{H}1) - (\mathcal{H}4)$ *hold. Then, there exists* $\lambda_\star > 0$*, such that problem* (4.1.1) *has at least one nontrivial solution in* X *for all* $\lambda \geq \lambda_\star$*.*

5.3 Main results

To prove the main result of this paper which is given in form of Theorem 2.2.1, we need to first prove few lemmas related to the mountain pass theorem and Palais-Smale condition.

Lemma 5.3.1. *Under the assumptions* $(\mathcal{H}1)$ *and* $(\mathcal{H}2)$*, the functional* Ψ *is well defined, lower weakly semicontinuous and it is of class* C^1 *on* X*. Moreover, the operator* Ψ' *is compact from* X *to* $X^\star$*.*

The proof of the above Lemma follows the very same arguments as in [22].

Lemma 5.3.2. *Let* $\{u_m = (u_{1m}, u_{2m}, ..., u_{nm})\}$ *be a Palais-Smale sequence for the Euler-Lagrange functional* E_λ*. If* $(\mathcal{H}3)$ *is satisfied, then* $\{u_m\}$ *is bounded.*

Proof. Let $\{u_m = (u_{1m}, u_{2m}, ..., u_{nm})\}$ be a Palais-Smale sequence for the Euler-Lagrange functional E_λ, we have

$$
\begin{aligned}
E_\lambda(u_m) = &\sum_{i=1}^{n} \widehat{M_i} \left(\int_\Omega \frac{1}{p_i(x)} |\nabla u_{im}(x)|^{p_i(x)} dx \right) - \sum_{i=1}^{n} \int_\Omega \frac{1}{q_i(x)} |u_{im}(x)|^{q_i(x)} dx \\
&- \lambda \int_\Omega F(x, u_{1m}(x), u_{2m}(x)..., u_{nm}(x)) dx \\
= &\, C + o_m(1).
\end{aligned}
$$

On the other hand for all $v=(v_1,v_2,...,v_n)\in X$, we have

$$E'_\lambda(u_m)v=\sum_{i=1}^{n}M_i\left(\int_\Omega\frac{1}{p_i(x)}|\nabla u_{im}|^{p_i(x)}dx\right)\int_\Omega|\nabla u_{im}|^{p_i(x)-2}\nabla u_{im}\nabla v_i\,dx$$
$$-\sum_{i=1}^{n}\int_\Omega|u_{im}|^{q_i(x)-2}u_{im}v_i\,dx-\lambda\sum_{i=1}^{n}F_{u_i}(x,u_{1m},u_{2m},...,u_{nm})v_i\,dx=o_m(1). \tag{5.3.1}$$

Then

$$\begin{aligned}E_\lambda(u_m)-E'_\lambda(u_m)\Big(\frac{u_m}{\theta}\Big)\geq&\sum_{i=1}^{n}\Big(\frac{\sigma_i\mathfrak{M}_i^0}{p_i^+}-\frac{\mathfrak{M}_i^0}{\theta_i}\Big)\int_\Omega\frac{1}{p_i(x)}|\nabla u_{im}|^{p_i(x)}dx\\&+\sum_{i=1}^{n}\Big(\frac{1}{\theta_i}-\frac{1}{q_i^-}\Big)\int_\Omega\frac{1}{q_i(x)}|u_{im}|^{q_i(x)}dx\\&+\lambda\int_\Omega\left[\sum_{i=1}^{n}\frac{u_{im}}{\theta_i}\frac{\partial F}{\partial u_i}(x,u_{1m},u_{2m},...,u_{nm})-F(x,u_{1m},u_{2m},...,u_{nm})\right]dx.\end{aligned}$$

Using $(\mathcal{H}3)$, we obtain

$$E_\lambda(u_m)-E'_\lambda(u_m)\Big(\frac{u_m}{\theta}\Big)\geq\sum_{i=1}^{n}\Big(\frac{\sigma_i\mathfrak{M}_i^0}{p_i^+}-\frac{\mathfrak{M}_i^0}{\theta_i}\Big)\int_\Omega\frac{1}{p_i(x)}|\nabla u_{im}|^{p_i(x)}dx.$$

Proposition 1.1.3 gives us

$$E_\lambda(u_m)-E'_\lambda(u_m)\Big(\frac{u_m}{\theta}\Big)\geq\sum_{i=1}^{n}\Big(\frac{\sigma_i\mathfrak{M}_i^0}{p_i^+}-\frac{\mathfrak{M}_i^0}{\theta_i}\Big)\min\left\{\|u_{im}\|_{p_i(x)}^{p_i^-},\|u_{im}\|_{p_i(x)}^{p_i^+}\right\},$$

thus

$$C+o_m(1)\geq\sum_{i=1}^{n}\Big(\frac{\sigma_i\mathfrak{M}_i^0}{p_i^+}-\frac{\mathfrak{M}_i^0}{\theta_i}\Big)\min\left\{\|u_{im}\|_{p_i(x)}^{p_i^-},\|u_{im}\|_{p_i(x)}^{p_i^+}\right\}.$$

Now, without loss of generality, we have may $\|u_{im}\|_{p_i(x)}\geq\|u_{nm}\|_{p_n(x)},\forall i\neq n$. Therefore, for m large enough, we get

$$C+o_m(1)\geq\sum_{i=1}^{n}\Big(\frac{\sigma_i\mathfrak{M}_i^0}{p_i^+}-\frac{\mathfrak{M}_i^0}{\theta_i}\Big)\min\left\{\|u_{nm}\|_{p_n(x)}^{p_i^-},\|u_{nm}\|_{p_n(x)}^{p_i^+}\right\},$$

hence, $\{u_m\}$ is bounded in X. □

Lemma 5.3.3. *Let $\{u_m = (u_{1m}, u_{2m}, ..., u_{nm})\}_{m\in\mathbb{N}} \subset X$ be a Palais-Smale sequence with energy level C_λ, if*

$$C_\lambda < \inf_{1\leq i\leq n}\left\{\Big(\frac{1}{\theta_i} - \frac{1}{q_i^-}\Big)S_{p_i}^N(\mathfrak{M}_i^0)^{N/p_i(x_j)}\right\},$$

then there exists a subsequence strongly convergent in X. S_{p_i} are the best Sobolev constants corresponding to the embedding $W^{1,p_i(x)}(\Omega) \hookrightarrow L^{q_i(x)}$.

Proof. Let $\{u_m\}_{m\in\mathbb{N}}$ be a bounded Palais-Smale sequence for the functional E_λ. Then there is a subsequence still denoted by $\{u_m\}_{m\in\mathbb{N}}$ which converges weakly in X. So there exists positive and bounded measures μ_i, $\nu_i \in \Omega$ such that

$$|\nabla u_{im}|^{p_i(x)} \rightharpoonup \mu_i, \quad |u_{im}|^{q_i(x)} \rightharpoonup \nu_i.$$

Hence by Proposition 1.5.1, if $J_i = \emptyset$ then $u_{im} \rightharpoonup u_i$ in $L^{q_i(x)}(\Omega)$ with $i = 1, 2, ..., n$. Let us show that if $C_\lambda < \inf_{1\leq i\leq n}\left\{\Big(\frac{1}{\theta_i} - \frac{1}{q_i^-}\Big)S_{p_i}^N(\mathfrak{M}_i^0)^{N/p_i(x_j)}\right\}$ and $\{u_m\}_{m\in\mathbb{N}}$ is a Palais-Smale sequence with energy level C_λ then $J_i = \emptyset$. Suppose that J_i is nonempty, let us consider $\phi \in C_0^\infty(\mathbb{R}^N, [0,1])$ with $|\nabla\phi|_\infty \leq 2$ and

$$\phi(x) = \begin{cases} 1, & \text{if } |x| < 1, \\ 0, & \text{if } |x| \geq 2. \end{cases}$$

We define, for any $\varepsilon > 0$ and $j \in J_i$, the function

$$\phi_{j,\varepsilon} := \phi\Big(\frac{x - x_j}{\varepsilon}\Big), \quad \forall x \in \mathbb{R}^N.$$

Note that $\phi_{j,\varepsilon} \in C_0^\infty(\mathbb{R}^N, [0,1])$, $|\nabla\phi_{j,\epsilon}|_\infty \leq \frac{2}{\varepsilon}$ and

$$\phi_{j,\varepsilon}(x) = \begin{cases} 1, & x \in B(x_j, \varepsilon), \\ 0, & x \in \mathbb{R}^N \setminus B(x_j, 2\varepsilon); \end{cases}$$

where $x_j \in \overline{\Omega}$ belongs to the support of ν_i. Since $\{u_{im}\phi_{j,\varepsilon}\}$ is bounded in the space $W^{1,p_i(x)}$, it then follows from **??** that $E'_\lambda(u_{1m}, ..., u_{im}, ..., u_{nm})(0, ..., u_{im}\phi_{j,\varepsilon}, ..., 0) \to 0$ as $m \to +\infty$, that is, we obtain

$$\begin{aligned} E'_\lambda(u_m)(0, ...u_{im}\phi_{j,\varepsilon}, ..., 0) = {} & M_i\left(\int_\Omega \frac{1}{p_i(x)}|\nabla u_{im}|^{p_i(x)}dx\right)\int_\Omega |\nabla u_{im}(x)|^{p_i(x)-2}\nabla u_{im}\nabla(u_{im}\phi_{j,\varepsilon})\,dx \\ & - \int_\Omega |u_{im}|^{q_i(x)-2}u_{im}(u_{im}\phi_{j,\epsilon})\,dx \\ & - \lambda\int_\Omega F_{u_i}(x, u_{1m}, ..., u_{im}, ..., u_{nm})u_{im}\phi_{j,\varepsilon}\,dx \to 0 \text{ as } m \to +\infty. \end{aligned}$$

Because of the compactness of $F_{u_i}$and Proposition 1.5.1, we obtain

$$M_i\left(\int_\Omega \frac{1}{p_i(x)}|\nabla u_{im}|^{p_i(x)}dx\right)\int_\Omega |\nabla u_{im}|^{p_i(x)-2}\nabla u_{im}\nabla\phi_{j,\epsilon}u_{im}\,dx = -M_i\left(\int_\Omega \frac{1}{p_i(x)}|\nabla u_i^m|^{p_i(x)}dx\right)$$
$$\times\int_\Omega \phi_{j,\epsilon}\,d\mu_i + \int_\Omega \phi_{j,\epsilon}\,d\nu_i + \lambda\int_\Omega F_{u_i}(x,u_{1m},...,u_{im},...,u_{nm})u_{im}\phi_{j,\varepsilon}\,dx + o_m(1). \tag{5.3.2}$$

Now, we will prove that

$$\lim_{\varepsilon\to 0}\left\{\limsup_{m\to+\infty} M_i\left(\int_\Omega \frac{1}{p_i(x)}|\nabla u_{im}|^{p_i(x)}dx\right)\int_\Omega |\nabla u_{im}|^{p_i(x)-2}\nabla u_{im}\nabla\phi_{j,\varepsilon}u_{im}\,dx\right\} = 0.$$

First, using the Hölder inequality, we obtain

$$\left|\int_\Omega |\nabla u_{im}|^{p_i(x)-2}\nabla u_{im}\nabla\phi_{j,\varepsilon}u_{im}\,dx\right| \le 2\left||\nabla u_{im}|^{p_i(x)-1}\right|_{\frac{p_i(x)}{p_i(x)-1}}|\nabla\phi_{j,\varepsilon}u_{im}|_{p_i(x)},$$

since $\{u_{im}\}$ is bounded, the real-valued sequence $\left||\nabla u_{im}|^{p_i(x)-1}\right|_{\frac{p_i(x)}{p_i(x)-1}}$ is also bounded, then there is a positive constant C, such that

$$\left|\int_\Omega |\nabla u_{im}|^{p_i(x)-2}\nabla u_{im}\nabla\phi_{j,\varepsilon}u_{im}\,dx\right| \le C|\nabla\phi_{j,\varepsilon}u_{im}|_{p_i(x)}.$$

Moreover $\{u_{im}\}$ is bounded in $W^{1,p_i(x)}(B(x_j,2\varepsilon))$, then there exists a subsequence denoted again $\{u_{im}\}$ weakly convergente to u_i in $L^{p_i(x)}(B(x_j,2\varepsilon))$. Hence

$$\begin{aligned}\limsup_{m\to+\infty}\left|\int_\Omega |\nabla u_{im}|^{p_i(x)-2}\nabla u_{im}\nabla\phi_{j,\varepsilon}u_{im}\,dx\right| &\le C|\nabla\phi_{j,\varepsilon}u_i|_{p_i(x)}\\ &\le 2C\limsup_{\varepsilon\to 0}||\nabla\phi_{j,\varepsilon}|^{p_i(x)}|_{(\frac{p_i^\star(x)}{p_i(x)})',B(x_j,2\varepsilon)}||u_i|^{p_i(x)}|_{\frac{p_i^\star(x)}{p_i(x)},B(x_j,2\varepsilon)}\\ &\le 2C\limsup_{\varepsilon\to 0}||\nabla\phi_{j,\varepsilon}|^{p_i(x)}|_{\frac{N}{p_i(x)},B(x_j,2\varepsilon)}||u_i|^{p_i(x)}|_{\frac{N}{N-p_i(x)},B(x_j,2\varepsilon)}.\end{aligned}$$

Note that

$$\int_{B(x_j,2\varepsilon)}(|\nabla\phi_{j,\varepsilon}|^{p_i(x)})^{(\frac{p_i^\star(x)}{p_i(x)})'}dx = \int_{B(x_j,2\varepsilon)}|\nabla\phi_{j,\varepsilon}|^N dx \le \left(\frac{2}{\varepsilon}\right)^N meas(B(x_j,2\varepsilon)) = \frac{4^N}{N}\omega_N,$$

where ω_N is the surface area of an N-dimensional unit sphere. As $\int_{B(x_j,2\varepsilon)}(|u_i|^{p_i(x)})^{\frac{p_i^\star(x)}{p_i(x)}}dx \to 0$ when $\varepsilon\to 0$, we obtain that $|\nabla\phi_{j,\varepsilon}u_i|_{p_i(x)}\to 0$, which implies

$$\lim_{\varepsilon\to 0}\left\{\limsup_{n\to+\infty}\left|\int_\Omega |\nabla u_{im}|^{p_i(x)-2}\nabla u_{im}\nabla\phi_{j,\epsilon}u_{im}\,dx\right|\right\} = 0. \tag{5.3.3}$$

Since $\{u_{im}\}$ is bounded in $W^{1,p_i(x)}(\Omega)$, we may assume that $\displaystyle\int_\Omega \frac{1}{p_i(x)}|\nabla u_{im}|^{p_i(x)}dx \to t_i \geq 0$ as $m \to +\infty$. Observing that $M_i(t)$ is is continuous, we then have

$$M_i\Big(\int_\Omega \frac{1}{p_i(x)}|\nabla u_{im}|^{p_i(x)}dx\Big) \to M_i(t_i) \geq \mathfrak{M}_i^0 > 0, \quad \text{as } m \to +\infty.$$

Hence, by 6.3.7, we obtain

$$\lim_{\varepsilon\to 0}\left\{\limsup_{m\to+\infty} M_i\left(\int_\Omega \frac{1}{p_i(x)}|\nabla u_{im}|^{p_i(x)}dx\right)\int_\Omega |\nabla u_{im}|^{p_i(x)-2}\nabla u_{im}\nabla\phi_{j,\varepsilon}u_{im}\,dx\right\} = 0. \tag{5.3.4}$$

Similarly, we can also get

$$\lim_{\varepsilon\to 0}\int_\Omega \frac{\partial F}{\partial u_i}(x,u_{1m},...,u_{im},...,u_{nm})\phi_{j,\epsilon}u_{im}dx = 0, \text{ as } m \to +\infty. \tag{5.3.5}$$

Indeed, using Hölder's inequality with $(\mathcal{H}2)$ and since $0 \leq \phi_{j,\varepsilon} \leq 1$ we obtain

$$\begin{aligned}\lim_{\varepsilon\to 0}\int_\Omega \frac{\partial F}{\partial u_i}(x,u_{1m},...,u_{im},...,u_{nm})\phi_{j,\varepsilon}u_{im}dx &\leq \lim_{\varepsilon\to 0}\int_\Omega\left(\sum_{j=1}^n b_{ij}(x)|u_jm|^{\mu_{ij}-1}\right)\phi_{j,\varepsilon}u_{im}\\ &\leq \lim_{\varepsilon\to 0}\int_\Omega\left(\sum_{j=1}^n b_{ij}(x)|u_j|^{\mu_{ij}-1}\right)|\phi_{j,\varepsilon}u_{im}|dx\\ &\leq \lim_{\varepsilon\to 0}c_1\Big(\sum_{j=1}^n |b_{ij}|_{\alpha_{ij}(x)}||u_{jm}|^{\mu_{ij}-1}|_{p_j^\star(x)}|\phi_{j,\epsilon}u_{im}|_{p_i^\star(x)}\Big).\end{aligned}$$

The above propositions yield

$$\lim_{\varepsilon\to 0}\int_\Omega \frac{\partial F}{\partial u_i}(x,u_{1m},...,u_{im},...,u_{nm})\phi_{j,\varepsilon}u_{im}dx \leq \lim_{\varepsilon\to 0}c_1\Big(\sum_{j=1}^n |b_{ij}|_{\beta_{ij}(x)}\|u_{jm}\|_{p_j(x)}^{\mu_{ij}-1}\Big)\|u_{im}\|_{p_i(x),B(x_j,2\varepsilon)}.$$

and this last goes to zero because of

$$\sum_{j=1}^n |b_{ij}|_{\beta_{ij}(x)}\|u_j\|_{p_j(x)}^{\mu_{ij}-1} < \infty.$$

Since $\phi_{j,\varepsilon}$ has compact support, going to the limit $m \to +\infty$ in 5.3.2, from 5.3.4 and 5.3.5,we obtain

$$M_i(t_i)\int_\Omega \phi_{j,\varepsilon}d\mu_i + o_\varepsilon(1) \leq \int_\Omega \phi_{j,\varepsilon}d\nu_i.$$

Letting $\varepsilon \to 0$, and using the standard theory of Radon measures, we conclude that $\mathfrak{M}_i^0 \mu_{ij} \le M_i(t_i)\mu_{ij} \le \nu_{ij}$. Using (1.5.3) we have

$$S\nu_{ij}^{\frac{1}{p_i^\star(x_j)}} \le \mu_{ij}^{\frac{1}{p_i(x_j)}} \le \left(\frac{\nu_{ij}}{\mathfrak{M}_i^0}\right)^{\frac{1}{p_i(x_j)}}.$$

which implies that $\nu_{ij} = 0$ or $\nu_{ij} \ge S_{p_i}^N (\mathfrak{M}_i^0)^{N/p_i(x_j)}$ for all $j \in J$.

On the other hand, from the conditions $(\mathcal{M}_1)$, $(\mathcal{M}_2)$ and $(\mathcal{H}3)$, we get

$$\begin{aligned}
C_\lambda &= E_\lambda(u_{1m}, ..., u_{im}, ..., u_{nm}) - E'_\lambda(u_{1m}, ..., u_{im}, ..., u_{nm})\Big(\frac{u_{1m}}{\theta_1}, .., \frac{u_{im}}{\theta_i}, ..., \frac{u_{nm}}{\theta_n}\Big) \\
&= \sum_{i=1}^n \widehat{M_i}\left(\int_\Omega \frac{1}{p_i(x)}|\nabla u_{im}|^{p_i(x)}dx\right) - \sum_{i=1}^n \int_\Omega \frac{1}{q_i(x)}|u_{im}|^{q_i(x)}dx \\
&\quad - \lambda \int_\Omega F(x, u_{1m}(x), ..., u_{nm}(x))dx - \sum_{i=1}^n M_i\left(\int_\Omega \frac{1}{p_i(x)}|\nabla u_{im}|^{p_i(x)}dx\right)\int_\Omega \frac{|\nabla u_{im}(x)|^{p_i(x)}}{\theta_i}\,dx \\
&\quad + \sum_{i=1}^n \int_\Omega \frac{|u_{im}(x)|^{q_i(x)}}{\theta_i}\,dx + \sum_{i=1}^n \frac{\lambda}{\theta_i}\int_\Omega F_{u_i}(x, u_{1m}(x), ..., u_{nm}(x))u_{im}\,dx + o_m(1) \\
&\ge \sum_{i=1}^n \frac{\sigma_i \mathfrak{M}_i^0}{p_i^+}\int_\Omega |\nabla u_{im}(x)|^{p_i(x)} - \sum_{i=1}^n \frac{1}{q_i^-}\int_\Omega |u_{im}(x)|^{q_i(x)} - \lambda\int_\Omega F(x, u_{1m}(x), ..., u_{nm}(x))dx \\
&\quad - \sum_{i=1}^n \frac{\mathfrak{M}_i^0}{\theta_i}\int_\Omega |\nabla u_{im}(x)|^{p_i(x)} + \sum_{i=1}^n \frac{1}{\theta_i}\int_\Omega |u_{im}(x)|^{q_i(x)} \\
&\quad + \sum_{i=1}^n \frac{\lambda}{\theta_i}\int_\Omega F_{u_i}(x, u_{1m}, ..., u_{nm})u_{im}\,dx + o_m(1) \\
&\ge \sum_{i=1}^n \mathfrak{M}_i^0\Big(\frac{\sigma_i}{p_i^+} - \frac{1}{\theta_i}\Big)\int_\Omega |u_{im}|^{p_i(x)}dx + \sum_{i=1}^n \Big(\frac{1}{\theta_i} - \frac{1}{q_i^-}\Big)\int_\Omega |u_{im}|^{q_i(x)}dx \\
&\quad + \lambda\int_\Omega \left[\sum_{i=1}^n \frac{u_{im}}{\theta_i}\frac{\partial F}{\partial u_i}(x, u_{1m}, ..., u_{nm}) - F(x, u_{1m}, ..., u_{nm})\right]dx + o_m(1),
\end{aligned}$$

hence

$$C_\lambda \ge \sum_{i=1}^n \Big(\frac{1}{\theta_i} - \frac{1}{q_i^-}\Big)\int_\Omega |u_{im}|^{q_i(x)}dx + o_m(1)$$

When $m \to +\infty$ we obtain

$$
\begin{aligned}
C_\lambda &\geq \sum_{i=1}^{n}\Big(\frac{1}{\theta_i}-\frac{1}{q_i^-}\Big)\Big(\int_\Omega |u_i|^{q_i(x)}dx+\sum_{j\in J_i}\nu_{ij}\delta_{x_j}\Big)\\
&\geq \sum_{i=1}^{n}\Big(\frac{1}{\theta_i}-\frac{1}{q_i^-}\Big)\Big(\int_\Omega |u_i|^{q_i(x)}dx+S_{p_i}^N(\mathfrak{M}_i^0)^{N/p_i(x_j)}Card J_i\Big)
\end{aligned}
$$

suppose that $\cup_{i=1}^n J_i \neq \emptyset$ and thus

$$C_\lambda \geq \inf_{1\leq i\leq n}\left\{\Big(\frac{1}{\theta_i}-\frac{1}{q_i^-}\Big)S_{p_i}^N(\mathfrak{M}_i^0)^{N/p_i(x_j)}\right\}.$$

Therefore, if $C_\lambda < \inf_{1\leq i\leq n}\left\{\Big(\frac{1}{\theta_i}-\frac{1}{q_i^-}\Big)S_{p_i}^N(\mathfrak{M}_i^0)^{N/p_i(x_j)}\right\}$, the set $\cup_{i=1}^n J_i$ is embty, which means that $|u_{im}|_{q_i(x)} \to |u_i|_{q_i(x)}$ for all $i=1,2,...,n$. Taking this together with the fact that $(u_{1m},...,u_{nm}) \rightharpoonup (u_1,...,u_n)$ in X, we have $u_{im}\to u_i$ strongly in $L^{q_i(x)}(\Omega)$ for all $i\in\{1,2,...,n\}$. On the other hand

$$
\begin{aligned}
&\langle E'_\lambda(u_{1m},...,u_{nm})-E'_\lambda(u_{1k},...,u_{nk}),(u_{1m}-u_{1k},0,...,0)\rangle =\\
&\qquad\langle \Phi'(u_{1m},...,u_{nm})-\Phi'(u_{1k},...,u_{nk}),(u_{1m}-u_{1k},0,...,0)\rangle\\
&\qquad-\langle \Theta'(u_{1m},...,u_{nm})-\Theta'(u_{1k},...,u_{nk}),(u_{1m}-u_{1k},0,...,0)\rangle\\
&\qquad\qquad-\lambda\langle \Psi'(u_{1m},...,u_{nm})-\Psi'(u_{1k},...,u_{nk}),(u_{1m}-u_{1k},0,...,0)\rangle,
\end{aligned}
$$

thus $\Phi'(u_{1m},...,u_{nm})\to 0$, i.e $\Phi'(u_{1m},...,u_{nm})$ is a Cauchy sequence in $X^\star$. Moreover, by Hölder's inequality

$$
\begin{aligned}
&\langle \Phi'(u_{1m},...,u_{nm})-\Phi'(u_{1k},...,u_{nk}),(u_{1m}-u_{1k},0,...,0)\rangle\\
&=\int_\Omega\Big(|u_{1m}|^{q_1(x)-2}u_{1m}-|u_{1k}|^{q_1(x)-2}u_{1k}\Big)(u_{1m}-u_{1k})dx\\
&\leq \||u_{1m}|^{q_1(x)-2}u_{1m}-|u_{1k}|^{q_1(x)-2}u_{1k}\|_{L^{\frac{q_1(x)}{q_1(x)-1}}(\Omega)}\|u_{1m}-u_{1k}\|_{L^{q_1(x)}(\Omega)}.
\end{aligned}
$$

Since $\{u_{1m}\}$ is a Cauchy sequence in $L^{q_1(x)}(\Omega)$, $\Psi'(u_{1m},...,u_{nm})$ is a Cauchy sequence in $X^\star$. The compactness of θ' gives

$$(u_{1m},...,u_{nm})\rightharpoonup(u_1,...,u_n)\Rightarrow \theta'(u_{1m},...,u_{nm})\to\theta'(u_1,...,u_n).$$

Therefore, according to the elementary inequalities (see, e.g., Chapter I in [29]) for any $\alpha,\beta\in\mathbb{R}^N$

$$|\alpha-\beta|^\gamma \leq \begin{cases} 2^\gamma\left(|\alpha|^{\gamma-2}\alpha-|\beta|^{\gamma-2}\beta\right)\cdot(\alpha-\beta) & \text{if } \gamma\geq 2\\ (|\alpha|-|\beta|)^{\frac{\gamma(2-\gamma)}{2}}\left(|\alpha|^{\gamma-2}\alpha-|\beta|^{\gamma-2}\beta\right)\cdot(\alpha-\beta)^{\frac{\gamma}{2}} & \text{if } 1<\gamma<2\end{cases} \tag{5.3.6}$$

where $\cdot$ denotes the standard inner product in $\mathbb{R}^N$. Replacing α and β by ∇u_{1m} and ∇u_{1k} respectively and integrating over Ω, we obtain

$$\|u_{1m}-u_{1k}\|^{p_1^-} \leq \Big| \langle \Phi'(u_{1m},...,u_{nm}) - \Phi'(u_{1k},...,u_{nk}), (u_{1m}-u_{1k},0,...,0)\rangle, \quad \text{if } p_1(x) \geq 2$$

and if $1 < p_1(x) < 2$, we get

$$\begin{aligned}\|u_{1m}-u_{1k}\|^2 \leq \Big| \langle \Phi'(u_{1m},...,u_{nm}) - \Phi'(u_{1k},...,u_{nk}), (u_{1m}-u_{1k},0,...,0)\rangle \Big| \\ \times \left(\|u_{1m}|^{p_1^-}_{1,p_1(x)} - \|u_{1k}|^{p_1^-}_{1,p_1(x)} \right)^{\frac{p_1^-(2-p_1^+)}{2}}\end{aligned}$$

Taking into account the fact that $\{u_{1m}\}$ is bounded in $W^{1,p_1(x)}(\Omega)$

$$\langle \Phi'(u_{1m},...,u_{nm}) - \Phi'(u_{1k},...,u_{nk}), ((u_{1m}-u_{1k},0,...,0)\rangle \to 0 \quad \text{as} \quad m,k\to\infty,$$

we find that $\{u_{1m}\}$ is a Cauchy sequence in $W^{1,p_1(x)}(\Omega)$. We proceed similarly for $\{u_{im}\}$ with $\langle \Phi'(u_{1m},...,u_{im},...,u_{nm}) - \Phi'(u_{1k},...,u_{ik},...,u_{nk}), (0,...,u_{im}-u_{ik},0,...,0)\rangle$ for all $i \in \{2,3,...,n\}$. $\square$

Now we are in position to prove Theorem 1.

Proof. The proof is an immediate consequence of the mountain pass theorem, Lemma 5.3.2 and Lemma 5.3.3. Precisely, it suffices to verify that E_λ has the mountain pass geometry and that $E_\lambda(tu_1,...,tu_n) < 0$ for some $t > 0$.

From $(\mathcal{M}_2)$, we can obtain for $t > t_0$

$$\widehat{M_i}(t) \leq \frac{\widehat{M_i}(t_0)}{t_0^{\frac{1}{\sigma_i}}} t^{\frac{1}{\sigma_i}} \leq c_i t_0^{\frac{1}{\sigma_i}}.$$

About the latter condition, we have

$$\begin{aligned}E_\lambda(u) = &\sum_{i=1}^n \widehat{M_i}\left(\int_\Omega \frac{1}{p_i(x)}|\nabla u_i|^{p_i(x)}dx\right) - \sum_{i=1}^n \int_\Omega \frac{1}{q_i(x)}|u_i|^{q_i(x)}dx, \\ &-\lambda \int_\Omega F(x,u_1(x),...,u_n(x))dx.\end{aligned}$$

Then, because of $\int_\Omega F(x,u_1(x),...,u_n(x))dx > 0$, it's clear that for $(z_1,...,z_n) \in X/\{(0,...,0)\}$ and any $t > 1$

$$\begin{aligned}E_\lambda(tz_1,...,tz_n) &\leq \sum_{i=1}^n c_i\Big(\int_\Omega \frac{1}{p_i(x)}|t\nabla z_i|^{p_i(x)}dx\Big)^{\frac{1}{\sigma_i}} - \sum_{i=1}^n \int_\Omega \frac{1}{q_i(x)}|tz_i|^{q_i(x)}dx, \\ &\leq \sum_{i=1}^n c_i t^{\frac{p_i^+}{\sigma_i}}\Big(\int_\Omega \frac{1}{p_i(x)}|\nabla z_i|^{p_i(x)}dx\Big)^{\frac{1}{\sigma_i}} - \sum_{i=1}^n t^{q_i^-} \int_\Omega \frac{1}{q_i(x)}|z_i|^{q_i(x)}dx,\end{aligned}$$

which tends to $-\infty$ as $t \to +\infty$ since $\sigma_i \geq \frac{p_i^+}{q_i^-}$.

On the other hand, for $\|(u_1, ..., u_n)\| = R$ is small enough and from from $(\mathcal{H}4)$, we get

$$\begin{aligned}
E_\lambda(u_1, ..., u_n) &\geq \sum_{i=1}^{n} \frac{\sigma_i \mathfrak{M}_i^0}{p_i^+} \int_\Omega |\nabla u_i|^{p_i(x)} dx - \sum_{i=1}^{n} \frac{1}{q_i^-} \int_\Omega |u_i|^{q_i(x)} dx - \lambda c \int_\Omega \Big(\sum_{i=1}^{n} |u_i|^{r_i(x)} \Big) \\
&\geq \sum_{i=1}^{n} \frac{\sigma_i \mathfrak{M}_i^0}{p_i^+} |\nabla u_i|_{p_i(x)}^{p_i^+} - \sum_{i=1}^{n} \frac{1}{q_i^-} |u_i|_{q_i(x)}^{q_i^+} - \sum_{i=1}^{n} \lambda c |u_i|_{r_i(x)}^{r_i(x)} \\
&\geq \sum_{i=1}^{n} \frac{\sigma_i \mathfrak{M}_i^0}{p_i^+} \|\nabla u_i\|_{p_i(x)}^{p_i^+} - \sum_{i=1}^{n} \frac{1}{q_i^-} \|u_i\|_{q_i(x)}^{q_i^-} - \sum_{i=1}^{n} \lambda c \|u_i\|_{p_i(x)}^{r_i^-} \\
&\geq \sum_{i=1}^{n} \Big(\frac{\sigma_i \mathfrak{M}_i^0}{p_i^+} \|\nabla u_i\|_{p_i(x)}^{p_i^+} - \frac{1}{q_i^-} \|u_i\|_{q_i(x)}^{q_i^-} - \lambda c \|u_i\|_{p_i(x)}^{r_i^-} \Big).
\end{aligned}$$

So, it easy to check that $f_i(R) > a > 0, i = 1, 2, ..., n$, where

$$f_i(t) = \frac{\sigma_i \mathfrak{M}_i^0}{p_i^+} \|\nabla u_i\|_{p_i(x)}^{p_i^+} - \frac{1}{q_i^-} \|u_i\|_{q_i(x)}^{q_i^-} - \lambda c \|u_i\|_{p_i(x)}^{r_i^-},$$

since $r_i^-, q_i^- > p_i^+$. That means the existence of an element $(u_1^0, ..., u_n^0)$ of X such that $E_\lambda(u_1^0, ..., u_n^0) < 0$. Consequentely, the critical value is

$$C_\lambda := \inf_{\xi \in \Gamma} \sup_{t \in [0,1]} E_\lambda(\xi(t)),$$

where

$$\Gamma = \big\{ \xi : [0, 1] \to X, \text{continuous and } \xi(0) = (0, ..., 0), \xi(1) = (u_1^0, ..., u_n^0) \big\}$$

That concludes the proof. □

Chapter 6

Existence and Multiplicity of Solutions for Kirchhoff-type potential systems With Variable Critical Growth Exponent

6.1 Introduction

The aim of this chapter is to show the existence and multiplicity of solutions for the following class of nonlocal quasilinear elliptic systems

$$\begin{cases} -M_i\left(\mathcal{A}_i(u_i)\right)\operatorname{div}\Big(\mathcal{B}_i(\nabla u_i)\Big) = |u_i|^{s_i(x)-2}u_i + \lambda F_{u_i}(x,u) & \text{in } \Omega, \\ u_i = 0 & \text{on } \partial\Omega; \end{cases} \tag{6.1.1}$$

for $1 \leq i \leq n$ ($n \in \mathbb{N}$), where Ω is a bounded domain in $\mathbb{R}^N (N \geq 2)$, with smooth boundary $\partial\Omega$, λ is a positive parameter, $p_i(x), q_i(x), r_i(x)$ and $s_i(x)$ are Lipschitz continuous real-valued functions such that

$$1 < p_i^- \leq p_i(x) \leq p_i^+ < q_i^- \leq q_i(x) \leq q_i^+ < N, \tag{6.1.2}$$

and

$$\gamma_i^- \leq \gamma_i(x) \leq \gamma_i^+ \leq r_i^- \leq r_i(x) \leq r_i^+ \leq s_i^- \leq s_i(x) \leq s_i^+ \leq \gamma_i^{\star}(x) < \infty, \tag{6.1.3}$$

for all $x \in \overline{\Omega}$, where $p_i^- := \inf_{x\in\overline{\Omega}} p_i(x)$, $p_i^+ := \sup_{x\in\overline{\Omega}} p_i(x)$, and analogously to $r_i^-, r_i^+, q_i^-, q_i^+, \gamma_i^-, \gamma_i^+, s_i^-$ and s_i^+, with $\gamma_i(x) = (1-\mathcal{H}(k_i^3))p_i(x) + \mathcal{H}(k_i^3)q_i(x)$ where k_i^3 is given in ($\boldsymbol{H}_2$) and

$$\gamma_i^*(x) = \begin{cases} \frac{N\gamma_i(x)}{N-\gamma_i(x)} & \text{for } \gamma_i(x) < N, \\ +\infty & \text{for } \gamma_i(x) \geq N, \end{cases}$$

for all $x \in \overline{\Omega}$, where $\mathcal{H} : \mathbb{R}_0^+ \to \{0,1\}$ is given by

$$\mathcal{H}(k_i) = \begin{cases} 1 & \text{if } k_i > 0, \\ 0 & \text{if } k_i < 0. \end{cases}$$

Moreover, we consider the set $\mathbf{K}_{\gamma_i} := \{x \in \Omega, s_i(x) = \gamma_i^\star(x)\} \neq \emptyset$.

The operator $\mathcal{B}_i : X_i \to \mathbb{R}^n$, and the operator $\mathcal{A}_i : X_i \to \mathbb{R}$, are respectively defined by

$$\mathcal{B}_i(u_i) = a_i(|\nabla u_i|^{p_i(x)})|\nabla u_i|^{p_i(x)-2}\nabla u_i, \text{ and } \mathcal{A}_i(u_i) = \int_\Omega \frac{1}{p_i(x)} A_i(|\nabla u_i|^{p_i(x)})dx,$$

where X_i is the Banach space

$$X_i := W_0^{1,p_i(x)}(\Omega) \cap W_0^{1,\gamma_i(x)}(\Omega),$$

$A_i(.)$ is the function $A_i(t) = \displaystyle\int_0^t a_i(k)dk$, and the function $a_i(.)$ is described in the hypothesis $(\boldsymbol{H}_1)$.

In this article, we consider the function $a_i : \mathbb{R}^+ \to \mathbb{R}^+$ satisfying the following hypotheses for all $1 \le i \le n$:

$(\boldsymbol{H}_1)$ The function $a_i(.)$ is of class C^1.

$(\boldsymbol{H}_2)$ There exist positive constants k_i^0, k_i^1, k_i^2 and k_i^3 for all $1 \le i \le n$, such that

$$k_i^0 + \mathcal{H}(k_i^3)k_i^2 \tau^{\frac{q_i(x)-p_i(x)}{p_i(x)}} \le a_i(\tau) \le k_i^1 + k_i^3 \tau^{\frac{q_i(x)-p_i(x)}{p_i(x)}},$$

for all $\tau \ge 0$ and for almost every $x \in \overline{\Omega}$.

$(\boldsymbol{H}_3)$ There exists $c > 0$ such that

$$\min\left\{ a_i(\tau^{p_i(x)})\tau^{p_i(x)-2}, a_i(\tau^{p_i(x)})\tau^{p_i(x)-2} + \tau \frac{\partial(a_i(\tau^{p_i(x)})\tau^{p_i(x)-2})}{\partial \tau} \right\} \ge c\tau^{p_i(x)-2},$$

for almost every $x \in \Omega$ and for all $\tau > 0$.

$(\boldsymbol{H}_4)$ There exists positive constants β_i, θ_i and σ_i for all $i \in \{1, 2, ..., n\}$ such that

$$A_i(\tau) \ge \frac{1}{\beta_i} a_i(\tau)\tau \text{ with } \gamma_i^+ < \theta_i < s_i^- \text{ and } \frac{q_i^+}{p_i^+} \le \frac{\beta_i}{\sigma_i} < \frac{\theta_i}{p_i^+},$$

for all $\tau \ge 0$ and σ_i satisfy $(\mathcal{M}_2)$.

The real function F belongs to $C^1(\Omega \times \mathbb{R}^n)$ and F_{u_i} denotes the partial derivative of F with respect to u_i.

$M_i : \mathbb{R}_0^+ \to \mathbb{R}^+$ is a nondecreasing and continuous function, satisfying

$(\mathcal{M}_1)$ There exists $\mathfrak{M}_i^0 > 0$, such that

$$M_i(t) \geq \mathfrak{M}_i^0, \quad \forall t \in \mathbb{R}_0^+, (i = 1, 2, ..., n).$$

$(\mathcal{M}_2)$ There exists $\sigma_i \in (\frac{q_i^+}{s_i^-}, 1]$ such that

$$\widehat{M_i}(t) \geq \sigma_i M_i(t) t, \quad \forall t \in \mathbb{R}_0^+;$$

where $\widehat{M_i}(t) := \int_0^t M_i(s) ds$.

We can see that there are many functions satisfying conditions $(\mathcal{M}_1) - (\mathcal{M}_2)$, for example $M(t) = \mathfrak{M}^0 + bt^{\frac{1}{\sigma}}$ with $\sigma \leq 1$, $\mathfrak{M}^0 > 0$ and $b \geq 0$.

The system (6.1.1) is related (in the case of a single equation) to the following Kirchhoff equation, introduced by Kirchhof in [51]

$$\rho \frac{\partial^2 u}{\partial t^2} - \left(\frac{\rho_0}{h} + \frac{E}{2L} \int_0^L \left| \frac{\partial u}{\partial x} \right|^2 dx \right) \frac{\partial^2 u}{\partial x^2} = 0, \tag{6.1.4}$$

where ρ, ρ_0, E and L are constants. This equation extends the classical D'Alembert's wave equation by considering the effects of the changes in the length of the strings during the vibrations. A distinguishing feature of equation (6.1.4) is that the equation contains a nonlocal coefficient $\frac{\rho_0}{h} + \frac{E}{2L} \int_0^L \left| \frac{\partial u}{\partial x} \right|^2 dx$ which depends on the average $\frac{1}{2L} \int_0^L \left| \frac{\partial u}{\partial x} \right|^2 dx$, and hence the equation is no longer a pointwise equation. The parameters in equation (6.1.4) have the following meanings: E is the Young modulus of the material (also referred to as the elastic modulus, it measures the strings resistance to being deformed elastically), ρ is the mass density, L is the length of the string, h is the area of cross-section, and ρ_0 is the initial tension. Almost one century later, Jacques-Louis Lions [58] returned to the equation and proposed a general Kirchhoff equation in arbitrary dimension with external force term which was written as

$$\begin{cases} \frac{\partial^2 u}{\partial t^2} - \left(a + b \int_\Omega |\nabla u|^2 dx\right) \Delta u = f(x, u) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega; \end{cases} \tag{6.1.5}$$

this problem is often called a nonlocal problem because it contains an integral over Ω. This causes some mathematical difficulties which make the study of such a problem particularly interesting. The nonlocal problem models several physical and biological systems, where u describes a process which depends on the average of itself, such as the population density,see [23] and its references therein. For a more detailed reference on this subject we refer the interested reader to [5, 16, 24, 25, 26, 48, 64, 71].

Moreover, elliptic problems involving critical nonlinearities have received great attention since the seminal work of Brezis and Nirenberg [13], in the case $p = 2$. The difficulty in problems involving critical nonlinearities is due to the lack of compactness of the embedding $W_0^{1,p(x)}(\Omega) \hookrightarrow L^{p^\star(x)}(\Omega)$. To overcome this difficulty, we use a Concentration-Compactness Principle for variable exponent space proved by Bonder and Silva in [11], which is slightly more general than the result proved in [41].

The motivation to study these types of problems comes from the fact that they arise in many applications in different fields of research. We note that the $p_i(x)$-Laplacian operator is a special case of the divergence form operator

$$\operatorname{div}\Big(a_i(|\nabla u_i|^{p_i(x)})|\nabla u_i|^{p_i(x)-2}\nabla u\Big),$$

for which the natural functional framework is described by the Sobolev space with variable exponent $W^{1,p_i(x)}$. We recall that, in the last two decades, particular attention has been given to variable exponent Lebesgue and Sobolev spaces, $L^{p(x)}$ and $W^{1,p(x)}$, where $p(x)$ is a real function. With the apparition of nonlinear problems in applied sciences and engineering, Lebesgue spaces L^p and Sobolev spaces $W^{1,p}$ with p constant, has shown its limitations in applications. The variable exponent Lebesgue spaces and Sobolev spaces has been used in the last decades to model phenomena concerning nonhomogeneous materials, this is, a new field research and reflects a new type of physical phenomena, for example electrorheological fluids (sometimes referred to as "smart fluid"). In these fields, the exponent p must be allowed to vary. In fact, electrorehological fluids are fluids that dramatically change their mechanical properties at the presence of an electromagnetic field, which have been used in robotics and space technology. Another field of application of these spaces is in image restoration and image processing. Moreover, other applications have emerged in thermorheological fluids, mathematical biology, flow in porous media, polycrystal plasticity, the growth of heterogeneous sand piles, and fluid dynamics, for more details see [7, **?**, **?**, 43, 44, 47, 74, 77, 78] and the references therein.

Now, we will illustrate the class of problems that the general divergence operator wraps. Let's set beforehand some examples which are interesting from the mathematical point of view, and, also, of great physical interest due to applications in different areas. The following operators satisfy the hypotheses $(\boldsymbol{H}_1) - (\boldsymbol{H}_3)$:

Example I. If $a_i \equiv 1$, we have the $p(x)$-Laplacian:

$$-\operatorname{div}\Big(a_i(|\nabla u|^{p(x)})|\nabla u|^{p(x)-2}\nabla u\Big) = \operatorname{div}\,(|\nabla u|^{p(x)-2}\nabla u),$$

which coincides with the usual p-Laplacian when $p(x) = p$, and with the Laplacian when $p(x) = 2$.

Example II. If $a_i(t) = 1 + t^{\frac{q(x)-p(x)}{p(x)}}$, we have the $p\&q$-Laplacian:

$$-\text{div}\left(a_i(|\nabla u|^{p(x)})|\nabla u|^{p(x)-2}\nabla u\right) = -\text{div}\,(|\nabla u|^{p(x)-2}\nabla u) - \text{div}\,(|\nabla u|^{q(x)-2}\nabla u),$$

Example III. If $a_i(t) = 1 + \frac{t}{\sqrt{1+t^2}}$, we obtain the operator $p(x)$-Laplacian like:

$$-\text{div}\left(a_i(|\nabla u|^{p(x)})|\nabla u|^{p(x)-2}\nabla u\right) = -\text{div}\left(\left(1 + \frac{|\nabla u|^{p(x)}}{\sqrt{1+|\nabla u|^{2p(x)}}}\right)|\nabla u|^{p(x)-2}\nabla u\right).$$

In the present work, we will show the existence and multiplicity of solutions for the nonlocal problem 6.1.1. The main theorems extend in several directions previous results recently appeared in the literature, see for example [3, 21, 46, 55, 82, 85],and references therein. The difficulty in this case, is due to the lack of compactness of the embedding $W_0^{1,\gamma(.)}(\Omega) \hookrightarrow L^{\gamma^\star(.)}(\Omega)$ and the Palais-Smale condition for the corresponding energy functional could not be checked directly. To deal with this difficulty, we use a version of the concentration-compactness lemma due to Lions for variable exponents [11].

The rest of the chapter is organized as follows: in section 2 we state the main results; while section 3 is dedicated to prove the main results.

6.2 Hypotheses and main results

In the following discussions, we will use the product space

$$X := \prod_{i=1}^{n}\left(W_0^{1,p_i(x)}(\Omega) \cap W_0^{1,\gamma_i(x)}(\Omega)\right),$$

which is equipped with the norm

$$\|u\| := \max\{\|u_i\|_i\}, \qquad \forall u = (u_1, u_2, ..., u_n) \in X,$$

where $\|u_i\|_i := \|\nabla u_i\|_{p_i(x)} + \mathcal{H}(k_i^3)\|\nabla u_i\|_{q_i(x)}$ is the norm of $W_0^{1,p_i(x)}(\Omega) \cap W_0^{1,\gamma_i(x)}(\Omega)$. The space $X^\star$ denotes the dual space of X and equipped with the usual dual norm.

Definition 6.2.1. *Let X be a Banach space, an element $u = (u_1, u_2, ..., u_n) \in X$ is called a weak solution of the system* (6.1.1) *if*

$$\sum_{i=1}^{n} M_i\left(\mathcal{A}_i(u_i)\right)\int_\Omega a_i(|\nabla u_i|^{p_i(x)})|\nabla u_i|^{p_i(x)-2}\nabla u_i\nabla v_i\,dx - \sum_{i=1}^{n}\int_\Omega |u_i|^{s_i(x)-2}u_i v_i\,dx$$
$$-\lambda\sum_{i=1}^{n}\int_\Omega F_{u_i}(x, u_1, ...u_n)v_i\,dx = 0,$$

for all $v = (v_1, v_2, ..., v_n) \in X = \prod_{i=1}^{n}(W_0^{1,p_i(x)}(\Omega) \cap W_0^{1,\gamma_i(x)}(\Omega))$.

We denote by E_λ the energy functional associated with the problem (6.1.1)

$$E_\lambda(.) := \Phi(.) - \Theta(.) - \lambda\Psi(.),$$

where Φ, Θ and $\Psi : X \longrightarrow \mathbb{R}$ are defined as follows

$$\begin{aligned} \Phi(u) &= \sum_{i=1}^{n} \widehat{M_i}\left(\mathcal{A}_i(u_i(x))\right), \\ \Theta(u) &= \sum_{i=1}^{n} \int_\Omega \frac{1}{s_i(x)} |u_i|^{s_i(x)} dx, \\ \Psi(u) &= \int_\Omega F(x, u_1(x), ..., u_n(x)) dx, \end{aligned}$$

for any $u = (u_1, ..., u_n)$ in X.

In order to ensure that the function F satisfies the topological conditions and the geometric conditions of the mountain pass theorem (see [72]), we assume some growth conditions.

Hypotheses.

($\mathcal{F}1$) $F \in C^1(\Omega \times \mathbb{R}^n, \mathbb{R})$ and $F(x, 0, ..., 0) = 0$

($\mathcal{F}2$) There exist positive functions b_{ij} $(1 \leq i, j \leq n)$, such that

$$\Big|\frac{\partial F}{\partial u_i}(x, u_1, ..., u_n)\Big| \leq \sum_{j=1}^{n} b_{ij}(x)|u_j|^{\mu_{ij}-1},$$

where $1 < \mu_{ij} < \inf_{x\in\Omega} q_i(x)$ for all $x \in \Omega$ and for all $(i, j) \in \{1, 2, ..., n\}^2$. The weight-functions b_{ii} (resp b_{ij} if $i \neq j$) belong to the generalized Lebesgue spaces $L^{\alpha_i}(\Omega)$ (resp $L^{\alpha_{ij}}(\Omega)$), with

$$\alpha_i(x) = \frac{q_i(x)}{q_i(x) - 1}, \quad \alpha_{ij}(x) = \frac{q_i^*(x)q_j^*(x)}{q_i^*(x)q_j^*(x) - q_i^*(x) - q_j^*(x)}.$$

($\mathcal{F}3$) There exist $K > 0$ and $\exists\theta_i \in (\gamma_i^+, s_i^-)$ for all $(x, u_1, ..., u_n) \in \Omega \times \mathbb{R}^n$ where $|u_i|^{\theta_i} \geq K$

$$0 < F(x, u_1, ..., u_n) < \sum_{i=1}^{n} \frac{u_i}{\theta_i} \frac{\partial F}{\partial u_i}(x, u_1, ..., u_n).$$

($\mathcal{F}4$) There exists $c > 0$ such that

$$|F(x, u_1, ..., u_n)| \leq c\Big(\sum_{i=1}^{n} |u_i|^{r_i(x)}\Big), \forall (x, u_1, ..., u_n) \in \Omega \times \mathbb{R}^n,$$

where $r_i \in C_+(\overline{\Omega})$ and $q_i^+ < r_i^- \leq r_i^+ << s^- \leq s_i^+ \quad \forall 1 \leq i \leq n$.

Note that according to the above hypothesis, we have $E_\lambda \in C^1(X, \mathbb{R})$ and for all $v = (v_1, v_2, ..., v_n) \in X$

$$\begin{aligned} E'_\lambda(u)v = & \sum_{i=1}^{n} M_i\left(\mathcal{A}_i(u_i)\right) \int_\Omega a_i(|\nabla u_i|^{p_i(x)})|\nabla u_i|^{p_i(x)-2}\nabla u_i \nabla v_i\, dx \\ & - \sum_{i=1}^{n} \int_\Omega |u_i|^{s_i(x)-2} u_i v_i\, dx - \lambda \sum_{i=1}^{n} \int_\Omega F_{u_i}(x, u_1, ...u_n) v_i\, dx. \end{aligned}$$

Consequently, $u = (u_1, u_2, ..., u_n)$ in X is a weak solution of (6.1.1) if and only if u is a critical point of E_λ.

We end this section by stating the following existence and multiplicity results.

Theorem 6.2.2. *suppose that* $(\mathcal{M}_1) - (\mathcal{M}_2)$ *and* $(\mathcal{H}1) - (\mathcal{H}4)$ *hold. Then, there exists* $\lambda_\star > 0$*, such that problem* (6.1.1) *has at least one nontrivial solution in* X *for all* $\lambda \geq \lambda_\star$.

Theorem 6.2.3. *Assume* $(\mathcal{M}_1) - (\mathcal{M}_2)$, $(\mathcal{H}1) - (\mathcal{H}4)$, *and* $F(u_1, ..., u_n)$ *is even in* u_i *for all* $i \in \{1, 2, ..., n\}$. *Then, there exists* $\lambda_\star > 0$*, such that problem* (6.1.1) *has infinitely many weak solutions for for all* $\lambda \geq \lambda_\star$.

6.3 Proof of main results

To prove the main result of this paper which is given in Theorem 2.2.1, we need to first prove few lemmas related to the mountain pass theorem and Palais-Smale condition.

Lemma 6.3.1. *Under the assumptions* $(\mathcal{H}1)$ *and* $(\mathcal{H}2)$*, the functional* Ψ *is well defined, lower weakly semicontinuous and it is of class* C^1 *on* X*. Moreover, the operator* Ψ' *is compact from* X *to* $X^\star$.

The proof of the above Lemma follows the very same arguments as in [22].

Lemma 6.3.2. *Let* $\{u_m = (u_{1m}, u_{2m}, ..., u_{nm})\}$ *be a Palais-Smale sequence for the Euler-Lagrange functional* E_λ*. If* $(\mathcal{H}3)$ *is satisfied, then* $\{u_m\}$ *is bounded.*

Proof. Let $\{u_m = (u_{1m}, u_{2m}, ..., u_{nm})\}$ be a Palais-Smale sequence for the Euler-Lagrange functional E_λ, we have

$$\begin{aligned} E_\lambda(u_m) &= \sum_{i=1}^{n} \widehat{M}_i\left(\mathcal{A}_i(u_{im}(x))\right) - \sum_{i=1}^{n} \int_\Omega \frac{1}{s_i(x)} |u_{im}(x)|^{s_i(x)} dx \\ &\quad - \lambda \int_\Omega F(x, u_{1m}(x), u_{2m}(x)..., u_{nm}(x)) dx \\ &= C + o_m(1). \end{aligned}$$

On the other hand for all $v=(v_1,v_2,...,v_n)\in X$, we have

$$E'_\lambda(u_m)v=\sum_{i=1}^n M_i\left(\mathcal{A}_i(u_{im})\right)\int_\Omega a_i(|\nabla u_{im}|^{p_i(x)})|\nabla u_{im}|^{p_i(x)-2}\nabla u_{im}\nabla v_i\,dx$$
$$-\sum_{i=1}^n\int_\Omega |u_{im}|^{s_i(x)-2}u_{im}v_i\,dx-\lambda\sum_{i=1}^n\int_\Omega F_{u_i}(x,u_{1m},u_{2m},...,u_{nm})v_i\,dx=o_m(1). \tag{6.3.1}$$

Then

$$\begin{aligned}E_\lambda(u_m)-E'_\lambda(u_m)\Big(\frac{u_m}{\theta}\Big)&\geq\sum_{i=1}^n\Big(\widehat{M_i}\left(\mathcal{A}_i(u_{im})\right)-\frac{1}{\theta_i}M_i\left(\mathcal{A}_i(u_{im})\right)\int_\Omega a_i(|\nabla u_{im}|^{p_i(x)})|\nabla u_{im}|^{p_i(x)}\Big)\\&+\sum_{i=1}^n\Big(\frac{1}{\theta_i}-\frac{1}{s_i^-}\Big)\int_\Omega|u_{im}|^{s_i(x)}dx\\&+\lambda\int_\Omega\left[\sum_{i=1}^n\frac{u_{im}}{\theta_i}\frac{\partial F}{\partial u_i}(x,u_{1m},u_{2m},...,u_{nm})-F(x,u_{1m},u_{2m},...,u_{nm})\right]dx.\end{aligned}$$

Using $(\boldsymbol{H}_4),(\mathcal{M}_1)-(\mathcal{M}_2)$ and $(\mathcal{F}3)$, we obtain

$$\begin{aligned}E_\lambda(u_m)-E'_\lambda(u_m)\Big(\frac{u_m}{\theta}\Big)&\geq\sum_{i=1}^n\mathfrak{M}_i^0\Big(\frac{\sigma_i}{p_i^+\beta}\int_\Omega A_i(|\nabla u_{im}|^{p_i(x)})dx-\frac{1}{\theta_i}\int_\Omega a_i(|\nabla u_{im}|^{p_i(x)})|\nabla u_{im}|^{p_i(x)}dx\Big)\\&\geq\sum_{i=1}^n\Big(\frac{\sigma_i\mathfrak{M}_i^0}{p_i^+\beta}-\frac{\mathfrak{M}_i^0}{\theta_i}\Big)\int_\Omega a_i(|\nabla u_{im}|^{p_i(x)})|\nabla u_{im}|^{p_i(x)}dx.\end{aligned}$$

Therefore, by using $(\boldsymbol{H}_2)$, there are positive constants C_{i1} and C_{i2} such that

$$E_\lambda(u_m)-E'_\lambda(u_m)\Big(\frac{u_m}{\theta}\Big)\geq\sum_{i=1}^n\Big[C_{i1}\Big(\int_\Omega|\nabla u_{im}|^{p_i(x)}\Big)+C_{i2}\mathcal{H}(k_i^3)\Big(\int_\Omega|\nabla u_{im}|^{q_i(x)}\Big)\Big]. \tag{6.3.2}$$

Suppose, by contradiction, that there exists a subsequence, still denoted by $\{u_{im}\}$, such that $\|u_{im}\|_i\to+\infty$.

If $k_i^3=0$, proposition 1.1.3 gives us

$$E_\lambda(u_m)-E'_\lambda(u_m)\Big(\frac{u_m}{\theta}\Big)\geq\sum_{i=1}^n C_i\|u_{im}\|_i^{p_i^-},$$

thus

$$C+o_m(1)\geq\sum_{i=1}^n C_i\|u_{im}\|_i^{p_i^-},$$

which is a contradiction because $p_i^- > 1$. Thus, we conclude that $\{u_m\}$ is bounded in X.

On the other hand, if $k_i^3 > 0$, we will need to analyze the following cases :

(i) $\|u_{im}\|_{p_i(x)} \to +\infty$ and $\|u_{im}\|_{q_i(x)} \to +\infty$ as $m \to +\infty$;
(ii) $\|u_{im}\|_{p_i(x)} \to +\infty$ and $\|u_{im}\|_{q_i(x)}$ is bounded;
(iii) $\|u_{im}\|_{p_i(x)}$ is bounded and $\|u_{im}\|_{q_i(x)} \to +\infty$.

In the case (i), for m large enough, $\|u_{im}\|^{q^-}_{q_i(x)} \geq \|u_{im}\|^{p^-}_{q_i(x)}$ Hence, by (6.3.2), we get

$$\begin{aligned} C + o_m(1) &\geq \sum_{i=1}^{n} \Big[C_i \|u_{im}\|^{p_i^-}_{p_i(x)} + C_i \mathcal{H}(k_i^3)\|u_{im}\|^{q_i^-}_{q_i(x)}\Big], \\ &\geq \sum_{i=1}^{n} \Big[C_i \|u_{im}\|^{p_i^-}_{p_i(x)} + C_i \mathcal{H}(k_i^3)\|u_{im}\|^{p_i^-}_{q_i(x)}\Big], \\ &\geq \sum_{i=1}^{n} C_i \|u_{im}\|^{p_i^-} \end{aligned}$$

which is absurd.

In the case (ii), by (6.3.2), we have

$$C + o_m(1) \geq \sum_{i=1}^{n} C_i \|u_{im}\|^{p_i^-}_{p_i(x)},$$

Thence, since $p_i^- > 1$, taking limit as $m \to +\infty$, we obtain a contradiction.

The case (iii) is similar to case (ii).

Therefore, we conclude that $\{u_m\}$ is bounded in X.

□

Lemma 6.3.3. *Let $\{u_m = (u_{1m}, u_{2m}, ..., u_{nm})\}_{m\in\mathbb{N}} \subset X$ be a Palais-Smale sequence with energy level C_λ, if*

$$C_\lambda < \inf_{1\leq i\leq n} \left\{ \Big(\frac{1}{\theta_i} - \frac{1}{s_i^-}\Big) S_i^N \Big(\mathfrak{M}_i^0(k_i^0(1 - \mathcal{H}(k_i^3) + \mathcal{H}(k_i^3)k_i^2))\Big)^{N/\gamma_i(x_j)} \right\},$$

then there exists a subsequence strongly convergent in X. S_i are the best positive constants of the Gagliardo-Nirenberg-Sobolev embedding, see 1.5.4.

Proof. Let $\{u_m\}_{m\in\mathbb{N}}$ be a bounded Palais-Smale sequence for the functional E_λ. Then there is a subsequence still denoted by $\{u_m\}_{m\in\mathbb{N}}$ which converges weakly in X. So there exists positive and bounded measures μ_i, $\nu_i \in \Omega$ such that

$$|\nabla u_{im}|^{\gamma_i(x)} \rightharpoonup \mu_i, \quad |u_{im}|^{s_i(x)} \rightharpoonup \nu_i.$$

Hence by Proposition 1.5.1, if $J_i = \emptyset$ then $u_{im} \rightharpoonup u_i$ in $L^{s_i(x)}(\Omega)$ with $i = 1, 2, ..., n$. Let us show that if $C_\lambda < \inf_{1\leq i\leq n}\left\{\left(\frac{1}{\theta_i} - \frac{1}{s_i^-}\right)S_i^N\left(\mathfrak{M}_i^0(k_i^0(1-\mathcal{H}(k_i^3)+\mathcal{H}(k_i^3)k_i^2))\right)^{N/\gamma_i(x_j)}\right\}$ and $\{u_m\}_{m\in\mathbb{N}}$ is a Palais-Smale sequence with energy level C_λ then $J_i = \emptyset$. Suppose that J_i is nonempty, let us consider $\phi \in C_0^\infty(\mathbb{R}^N, [0,1])$ with $|\nabla\phi|_\infty \leq 2$ and

$$\phi(x) = \begin{cases} 1, & \text{if } |x| < 1,\\ 0, & \text{if } |x| \geq 2. \end{cases}$$

We define, for any $\varepsilon > 0$ and $j \in J_i$, the function

$$\phi_{j,\varepsilon} := \phi\Big(\frac{x - x_j}{\varepsilon}\Big), \quad \forall x \in \mathbb{R}^N.$$

Note that $\phi_{j,\varepsilon} \in C_0^\infty(\mathbb{R}^N, [0,1])$, $|\nabla\phi_{j,\epsilon}|_\infty \leq \frac{2}{\varepsilon}$ and

$$\phi_{j,\varepsilon}(x) = \begin{cases} 1, & x \in B(x_j, \varepsilon),\\ 0, & x \in \mathbb{R}^N \setminus B(x_j, 2\varepsilon); \end{cases}$$

where $x_j \in \overline{\Omega}$ belongs to the support of ν_i. Since $\{u_{im}\phi_{j,\varepsilon}\}$ is bounded in the space $W_0^{1,p_i(x)}(\Omega) \cap W_0^{1,\gamma_i(x)}(\Omega)$, it then follows from 6.3.1 that $E'_\lambda(u_{1m}, ..., u_{im}, ..., u_{nm})(0, ..., u_{im}\phi_{j,\varepsilon}, ..., 0) \to 0$ as $m \to +\infty$, that is, we obtain

$$\begin{aligned} E'_\lambda(u_m)(0, ...u_{im}\phi_{j,\varepsilon}, ..., 0) = M_i\left(\mathcal{A}_i(u_{im})\right)\int_\Omega a_i(|\nabla u_{im}|^{p_i(x)})|\nabla u_{im}|^{p_i(x)-2}\nabla u_{im}\nabla(u_{im}\phi_{j,\varepsilon})\,dx\\ - \int_\Omega |u_{im}|^{s_i(x)-2}u_{im}(u_{im}\phi_{j,\epsilon})\,dx\\ - \lambda\int_\Omega F_{u_i}(x, u_{1m}, ..., u_{im}, ..., u_{nm})u_{im}\phi_{j,\varepsilon}\,dx \to 0 \text{ as } m \to +\infty. \end{aligned}$$

that is,

$$\begin{aligned} M_i\left(\mathcal{A}_i(u_{im})\right)\int_\Omega a_i(|\nabla u_{im}|^{p_i(x)})|\nabla u_{im}|^{p_i(x)-2}\nabla u_{im}\nabla\phi_{j,\varepsilon}u_{im}\,dx = \int_\Omega |u_{im}|^{s_i(x)}\phi_{j,\epsilon}\,dx\\ -M_i\left(\mathcal{A}_i(u_{im})\right)\int_\Omega a_i(|\nabla u_{im}|^{p_i(x)})|\nabla u_{im}|^{p_i(x)}\phi_{j,\varepsilon}\,dx + \lambda\int_\Omega F_{u_i}(x, u_{1m}, ..., u_{im}, ..., u_{nm})u_{im}\phi_{j,\varepsilon}\,dx + o_m(1). \end{aligned}$$

(6.3.3

Now, we will prove that

$$\lim_{\varepsilon\to 0}\left\{\limsup_{m\to+\infty} M_i\left(\mathcal{A}_i(u_{im})\right)\int_\Omega a_i(|\nabla u_{im}|^{p_i(x)})|\nabla u_{im}|^{p_i(x)-2}\nabla u_{im}\nabla\phi_{j,\varepsilon}u_{im}\,dx\right\} = 0. \tag{6.3.4}$$

We remark that, due to the hypotheses $(\boldsymbol{H}_2)$ enough to show that

$$\lim_{\varepsilon\to 0}\left\{\limsup_{m\to+\infty} M_i\left(\mathcal{A}_i(u_{im})\right)\int_\Omega |\nabla u_{im}|^{p_i(x)-2}\nabla u_{im}\nabla\phi_{j,\varepsilon}u_{im}\,dx\right\}=0. \tag{6.3.5}$$

and

$$\lim_{\varepsilon\to 0}\left\{\limsup_{m\to+\infty} M_i\left(\mathcal{A}_i(u_{im})\right)\int_\Omega |\nabla u_{im}|^{q_i(x)-2}\nabla u_{im}\nabla\phi_{j,\varepsilon}u_{im}\,dx\right\}=0. \tag{6.3.6}$$

First, using the Hölder inequality, we obtain

$$\left|\int_\Omega |\nabla u_{im}|^{p_i(x)-2}\nabla u_{im}\nabla\phi_{j,\varepsilon}u_{im}\,dx\right|\le 2\left||\nabla u_{im}|^{p_i(x)-1}\right|_{\frac{p_i(x)}{p_i(x)-1}}\left|\nabla\phi_{j,\varepsilon}u_{im}\right|_{p_i(x)},$$

since $\{u_{im}\}$ is bounded, the real-valued sequence $\left||\nabla u_{im}|^{p_i(x)-1}\right|_{\frac{p_i(x)}{p_i(x)-1}}$ is also bounded, then there is a positive constant C, such that

$$\left|\int_\Omega |\nabla u_{im}|^{p_i(x)-2}\nabla u_{im}\nabla\phi_{j,\varepsilon}u_{im}\,dx\right|\le C|\nabla\phi_{j,\varepsilon}u_{im}|_{p_i(x)}.$$

Moreover $\{u_{im}\}$ is bounded in $W^{1,p_i(x)}(B(x_j,2\varepsilon))$, then there exists a subsequence denoted again $\{u_{im}\}$ weakly convergente to u_i in $L^{p_i(x)}(B(x_j,2\varepsilon))$. Hence

$$\begin{aligned}\limsup_{m\to+\infty}\left|\int_\Omega |\nabla u_{im}|^{p_i(x)-2}\nabla u_{im}\nabla\phi_{j,\varepsilon}u_{im}\,dx\right| &\le C|\nabla\phi_{j,\varepsilon}u_i|_{p_i(x)}\\ \le 2C\limsup_{\varepsilon\to 0}\left||\nabla\phi_{j,\varepsilon}|^{p_i(x)}\right|_{(\frac{p_i^\star(x)}{p_i(x)})',B(x_j,2\varepsilon)}&\left||u_i|^{p_i(x)}\right|_{\frac{p_i^\star(x)}{p_i(x)},B(x_j,2\varepsilon)}\\ \le 2C\limsup_{\varepsilon\to 0}\left||\nabla\phi_{j,\varepsilon}|^{p_i(x)}\right|_{\frac{N}{p_i(x)},B(x_j,2\varepsilon)}&\left||u_i|^{p_i(x)}\right|_{\frac{N}{N-p_i(x)},B(x_j,2\varepsilon)}.\end{aligned}$$

Note that

$$\int_{B(x_j,2\varepsilon)}(|\nabla\phi_{j,\varepsilon}|^{p_i(x)})^{(\frac{p_i^\star(x)}{p_i(x)})'}dx=\int_{B(x_j,2\varepsilon)}|\nabla\phi_{j,\varepsilon}|^N dx\le\left(\frac{2}{\varepsilon}\right)^N meas(B(x_j,2\varepsilon))=\frac{4^N}{N}\omega_N,$$

where ω_N is the surface area of an N-dimensional unit sphere. As

$$\int_{B(x_j,2\varepsilon)}(|u_i|^{p_i(x)})^{\frac{p_i^\star(x)}{p_i(x)}}dx\to 0,$$

when $\varepsilon \to 0$, we obtain that $|\nabla \phi_{j,\varepsilon} u_i|_{p_i(x)} \to 0$, which implies

$$\lim_{\varepsilon \to 0} \left\{ \limsup_{n \to +\infty} \left| \int_\Omega |\nabla u_{im}|^{p_i(x)-2} \nabla u_{im} \nabla \phi_{j,\epsilon} u_{im} \, dx \right| \right\} = 0. \tag{6.3.7}$$

Since $\{u_{im}\}$ is bounded in $W_0^{1,p_i(x)}(\Omega)$, we may assume that $\mathcal{A}_i(u_{im}) \to t_i \geq 0$ as $m \to +\infty$. Observing that $M_i(t_i)$ is is continuous, we then have

$$M_i\Big(\mathcal{A}_i(u_{im})\Big) \to M_i(t_i) \geq \mathfrak{M}_i^0 > 0, \quad \text{as } m \to +\infty.$$

Hence, by 6.3.7, we obtain

$$\lim_{\varepsilon \to 0} \left\{ \limsup_{m \to +\infty} M_i\left(\mathcal{A}_i(u_{im})\right) \int_\Omega |\nabla u_{im}|^{p_i(x)-2} \nabla u_{im} \nabla \phi_{j,\varepsilon} u_{im} \, dx \right\} = 0. \tag{6.3.8}$$

Analogously, we verify 6.3.6. Therefore, we conclude the proof of (6.3.4). Similarly, we can also get

$$\lim_{\varepsilon \to 0} \int_\Omega \frac{\partial F}{\partial u_i}(x, u_{1m}, ..., u_{im}, ..., u_{nm}) \phi_{j,\epsilon} u_{im} dx = 0, \text{ as } m \to +\infty. \tag{6.3.9}$$

Indeed, using Hölder's inequality with $(\mathcal{F}2)$ and since $0 \leq \phi_{j,\varepsilon} \leq 1$ we obtain

$$\begin{aligned}
\lim_{\varepsilon \to 0} \int_\Omega \frac{\partial F}{\partial u_i}(x, u_{1m}, ..., u_{im}, ..., u_{nm}) \phi_{j,\varepsilon} u_{im} dx &\leq \lim_{\varepsilon \to 0} \int_\Omega \left(\sum_{j=1}^n b_{ij}(x) |u_j m|^{\mu_{ij}-1} \right) \phi_{j,\varepsilon} u_{im} \\
&\leq \lim_{\varepsilon \to 0} \int_\Omega \left(\sum_{j=1}^n b_{ij}(x) |u_j|^{\mu_{ij}-1} \right) |\phi_{j,\varepsilon} u_{im}| dx \\
&\leq \lim_{\varepsilon \to 0} c_1 \Big(\sum_{j=1}^n |b_{ij}|_{\alpha_{ij}(x)} \big| |u_{jm}|^{\mu_{ij}-1} \big|_{q_j^\star(x)} |\phi_{j,\epsilon} u_{im}|_{q_i^\star(x)} \Big).
\end{aligned}$$

The above propositions yield

$$\lim_{\varepsilon \to 0} \int_\Omega \frac{\partial F}{\partial u_i}(x, u_{1m}, ..., u_{im}, ..., u_{nm}) \phi_{j,\varepsilon} u_{im} dx \leq \lim_{\varepsilon \to 0} c_1 \Big(\sum_{j=1}^n |b_{ij}|_{\beta_{ij}(x)} \|u_{jm}\|_{q_j(x)}^{\mu_{ij}-1} \Big) \|u_{im}\|_{q_i(x), B(x_j, 2\varepsilon)}.$$

and this last goes to zero because of

$$\sum_{j=1}^n |b_{ij}|_{\beta_{ij}(x)} \|u_j\|_{q_j(x)}^{\mu_{ij}-1} < \infty.$$

Since $\phi_{j,\varepsilon}$ has compact support, going to the limit $m \to +\infty$ and letting $\varepsilon \to 0$ in 6.3.3, from 6.3.4 and 6.3.9,we obtain

$$0 = -\lim_{\varepsilon\to 0}\left[\limsup_{m\to+\infty}\Big(M_i\left(\mathcal{A}_i(u_{im})\right)\int_\Omega a_i(|\nabla u_{im}|^{p_i(x)})|\nabla u_{im}|^{p_i(x)}\phi_{j,\varepsilon}\,dx\Big)\right] + \nu_{ij} \tag{6.3.10}$$

$$\leq -\mathfrak{M}_i^0\lim_{\varepsilon\to 0}\left[\limsup_{m\to+\infty}\Big(\int_\Omega a_i(|\nabla u_{im}|^{p_i(x)})|\nabla u_{im}|^{p_i(x)-2}\phi_{j,\varepsilon}\,dx\Big)\right] + \nu_{ij} \tag{6.3.11}$$

$$\leq -\mathfrak{M}_i^0\lim_{\varepsilon\to 0}\left[\limsup_{m\to+\infty}\Big(\int_\Omega (k_i^0|\nabla u_{im}|^{p_i(x)} + \mathcal{H}(k_i^3)k_i^2|\nabla u_{im}|^{q_i(x)})\phi_{j,\varepsilon}\,dx\Big)\right] + \nu_{ij}. \tag{6.3.12}$$

Note that, when $k_i^3 = 0$, we have $\gamma_i(x) = p_i(x)$. Hence, by using (1.5.3) we have

$$\begin{aligned} 0 &\leq \nu_{ij} - \mathfrak{M}_i^0 k_i^0 \lim_{\varepsilon\to 0}\int_\Omega \phi_{j,\varepsilon}\,d\mu_i \\ &\leq \nu_{ij} - \mathfrak{M}_i^0 k_i^0\mu_{ij} - \mathfrak{M}_i^0 k_i^0 \lim_{\varepsilon\to 0}\int_\Omega |\nabla u_i|^{p_i(x)}\phi_{j,\varepsilon}\,dx \end{aligned}$$

By using Lebesgue Dominated Convergence Theorem, we have

$$\lim_{\varepsilon\to 0}\int_\Omega |\nabla u_i|^{p_i(x)}\phi_{j,\varepsilon}\,dx = 0$$

Then, we get

$$\mathfrak{M}_i^0 k_i^0 \mu_{ij} \leq \nu_{ij}. \tag{6.3.13}$$

On the other hand, if $k_i^3 > 0$, then $\gamma_i(x) = q_i(x)$ Therefore, follows from (1.5.4) and (6.3.12) that

$$\begin{aligned} 0 &\leq \nu_{ij} - \mathfrak{M}_i^0\lim_{\varepsilon\to 0}\left[\limsup_{m\to 0}\Big(\int_\Omega \mathcal{H}(k_i^3)k_i^2|\nabla u_{im}|^{q_i(x)}\phi_{j,\varepsilon}\,dx\Big)\right] \\ &\leq \nu_{ij} - \mathfrak{M}_i^0\mathcal{H}(k_i^3)k_i^2\lim_{\varepsilon\to 0}\int_\Omega \phi_{j,\varepsilon}\,d\mu_i \\ &\leq \nu_{ij} - \mathfrak{M}_i^0\mathcal{H}(k_i^3)k_i^2\mu_{ij} - \mathfrak{M}_i^0\mathcal{H}(k_i^3)k_i^2\lim_{\varepsilon\to 0}\int_\Omega |\nabla u_i|^{p_i(x)}\phi_{j,\varepsilon}\,dx \end{aligned}$$

and by using Lebesgue Dominated Convergence Theorem, we have

$$\lim_{\varepsilon\to 0}\int_\Omega |\nabla u_i|^{q_i(x)}\phi_{j,\varepsilon}\,dx = 0$$

Then, we get

$$\mathfrak{M}_i^0\mathcal{H}(k_i^3)k_i^2\mu_{ij} \leq \nu_{ij}. \tag{6.3.14}$$

Then, by combining 6.3.13 and 6.3.14, we get $\mathfrak{M}_i^0((1-\mathcal{H}(k_i^3))k_i^0 + \mathcal{H}(k_i^3)k_i^2)\mu_{ij} \leq \nu_{ij}$. Using (1.5.4), we have

$$S\nu_{ij}^{\frac{1}{\gamma_i^\star(x_j)}} \leq \mu_{ij}^{\frac{1}{\gamma_i(x_j)}} \leq \left(\frac{\nu_{ij}}{\mathfrak{M}_i^0((1-\mathcal{H}(k_i^3))k_i^0+\mathcal{H}(k_i^3)k_i^2)}\right)^{\frac{1}{\gamma_i(x_j)}}.$$

which implies that $\nu_{ij}=0$ or $\nu_{ij} \geq S_{p_i}^N\Big(\mathfrak{M}_i^0(k_i^3(1-\mathcal{H}(k_i^3)+\mathcal{H}(k_i^3)k_i^2))\Big)^{N/\gamma_i(x_j)}$ for all $j \in J$.

On the other hand, from the conditions $(\mathcal{M}_1)$, $(\mathcal{M}_2)$ and $(\mathcal{H}3)$, we get

$$\begin{aligned}
C_\lambda &= E_\lambda(u_{1m},...,u_{im},...,u_{nm}) - E'_\lambda(u_{1m},...,u_{im},...,u_{nm})\Big(\frac{u_{1m}}{\theta_1},..,\frac{u_{im}}{\theta_i},...,\frac{u_{nm}}{\theta_n}\Big)\\
&= \sum_{i=1}^n \widehat{M_i}\left(\mathcal{A}_i(u_{im})\right) - \sum_{i=1}^n\int_\Omega \frac{1}{s_i(x)}|u_{im}|^{s_i(x)}dx\\
&\quad -\lambda\int_\Omega F(x,u_{1m}(x),...,u_{nm}(x))dx - \sum_{i=1}^n M_i\left(\mathcal{A}_i(u_{im})\right)\int_\Omega a_i(|\nabla u_{im}(x)|^{p_i(x)})\frac{|\nabla u_{im}(x)|^{p_i(x)}}{\theta_i}\,dx\\
&\quad +\sum_{i=1}^n\int_\Omega\frac{|u_{im}(x)|^{s_i(x)}}{\theta_i}\,dx + \sum_{i=1}^n\frac{\lambda}{\theta_i}\int_\Omega F_{u_i}(x,u_{1m}(x),...,u_{nm}(x))u_{im}\,dx + o_m(1)\\
&\geq \sum_{i=1}^n\frac{\sigma_i\mathfrak{M}_i^0}{p_i^+\beta_i}\int_\Omega a_i(|\nabla u_{im}(x)|^{p_i(x)})|\nabla u_{im}(x)|^{p_i(x)} - \sum_{i=1}^n\frac{1}{s_i^-}\int_\Omega|u_{im}(x)|^{s_i(x)}\\
&\quad -\lambda\int_\Omega F(x,u_{1m}(x),...,u_{nm}(x))dx - \sum_{i=1}^n\frac{\mathfrak{M}_i^0}{\theta_i}\int_\Omega a_i(|\nabla u_{im}(x)|^{p_i(x)})|\nabla u_{im}(x)|^{p_i(x)}\\
&\quad +\sum_{i=1}^n\frac{1}{\theta_i}\int_\Omega|u_{im}(x)|^{s_i(x)} + \sum_{i=1}^n\frac{\lambda}{\theta_i}\int_\Omega F_{u_i}(x,u_{1m},...,u_{nm})u_{im}\,dx + o_m(1)\\
&\geq \sum_{i=1}^n\mathfrak{M}_i^0\Big(\frac{\sigma_i}{p_i^+\beta_i}-\frac{1}{\theta_i}\Big)\int_\Omega a_i(|\nabla u_{im}(x)|^{p_i(x)})|u_{im}|^{p_i(x)}dx + \sum_{i=1}^n\Big(\frac{1}{\theta_i}-\frac{1}{s_i^-}\Big)\int_\Omega|u_{im}|^{s_i(x)}dx\\
&\quad +\lambda\int_\Omega\left[\sum_{i=1}^n\frac{u_{im}}{\theta_i}\frac{\partial F}{\partial u_i}(x,u_{1m},...,u_{nm}) - F(x,u_{1m},...,u_{nm})\right]dx + o_m(1),
\end{aligned}$$

hence

$$C_\lambda \geq \sum_{i=1}^n\Big(\frac{1}{\theta_i}-\frac{1}{s_i^-}\Big)\int_\Omega|u_{im}|^{s_i(x)}dx + o_m(1)$$

When $m \to +\infty$ we obtain

$$\begin{aligned}
C_\lambda &\geq \sum_{i=1}^{n} \Big(\frac{1}{\theta_i} - \frac{1}{s_i^-}\Big)\Big(\int_\Omega |u_i|^{s_i(x)}dx + \sum_{j\in J_i} \nu_{ij}\delta_{x_j}\Big)\\
&\geq \sum_{i=1}^{n} \Big(\frac{1}{\theta_i} - \frac{1}{s_i^-}\Big)\Big(\int_\Omega |u_i|^{s_i(x)}dx + S_i^N\Big(\mathfrak{M}_i^0(k_i^3(1-\mathcal{H}(k_i^3)+\mathcal{H}(k_i^3)k_i^2))\Big)^{N/\gamma_i(x_j)} Card J_i\Big)
\end{aligned}$$

suppose that $\cup_{i=1}^n J_i \neq \emptyset$ and thus

$$C_\lambda \geq \inf_{1\leq i\leq n}\Big\{\Big(\frac{1}{\theta_i} - \frac{1}{s_i^-}\Big)S_i^N\Big(\mathfrak{M}_i^0(k_i^3(1-\mathcal{H}(k_i^3)+\mathcal{H}(k_i^3)k_i^2))\Big)^{N/\gamma_i(x_j)}\Big\}.$$

Therefore, if $C_\lambda < \inf_{1\leq i\leq n}\Big\{\Big(\frac{1}{\theta_i} - \frac{1}{s_i^-}\Big)S_i^N(\mathfrak{M}_i^0)^{N/p_i(x_j)}\Big\}$, the set $\cup_{i=1}^n J_i$ is embty, which means that $|u_{im}|_{s_i(x)} \to |u_i|_{s_i(x)}$ for all $i = 1, 2, ..., n$. Taking this together with the fact that $(u_{1m}, ..., u_{nm}) \rightharpoonup (u_1, ..., u_n)$ in X, we have $u_{im} \to u_i$ strongly in $L^{s_i(x)}(\Omega)$ for all $i \in \{1, 2, ..., n\}$. On the other hand

$$\begin{aligned}
\langle E'_\lambda(u_{1m}, ..., u_{nm}) &- E'_\lambda(u_{1k}, ..., u_{nk}), (u_{1m} - u_{1k}, 0, ..., 0)\rangle =\\
&\langle \Phi'(u_{1m}, ..., u_{nm}) - \Phi'(u_{1k}, ..., u_{nk}), (u_{1m} - u_{1k}, 0, ..., 0)\rangle\\
&- \langle \Theta'(u_{1m}, ..., u_{nm}) - \Theta'(u_{1k}, ..., u_{nk}), (u_{1m} - u_{1k}, 0, ..., 0)\rangle\\
&\quad - \lambda\,\langle \Psi'(u_{1m}, ..., u_{nm}) - \Psi'(u_{1k}, ..., u_{nk}), (u_{1m} - u_{1k}, 0, ..., 0)\rangle\,,
\end{aligned}$$

thus $E'_\lambda(u_{1m}, ..., u_{nm}) \to 0$, i.e $E'_\lambda(u_{1m}, ..., u_{nm})$ is a Cauchy sequence in $X^\star$. Moreover, again by Hölder's inequality, we obtain

$$\begin{aligned}
&\langle \Theta'(u_{1m}, ..., u_{nm}) - \Theta'(u_{1k}, ..., u_{nk}), (u_{1m} - u_{1k}, 0, ..., 0)\rangle\\
&= \int_\Omega \Big(|u_{1m}|^{s_1(x)-2}u_{1m} - |u_{1k}|^{s_1(x)-2}u_{1k}\Big)(u_{1m} - u_{1k})dx\\
&\leq \big\||u_{1m}|^{s_1(x)-2}u_{1m} - |u_{1k}|^{s_1(x)-2}u_{1k}\big\|_{L^{\frac{s_1(x)}{s_1(x)-1}}(\Omega)} \|u_{1m} - u_{1k}\|_{L^{s_1(x)}(\Omega)}.
\end{aligned}$$

Since $\{u_{1m}\}$ is a Cauchy sequence in $L^{s_1(x)}(\Omega)$, $\Theta'(u_{1m}, ..., u_{nm})$ is a Cauchy sequence in $X^\star$. The compactness of Ψ' gives

$$(u_{1m}, ..., u_{nm}) \rightharpoonup (u_1, ..., u_n) \Rightarrow \Psi'(u_{1m}, ..., u_{nm}) \to \Psi'(u_1, ..., u_n),$$

i.e. $\Psi'(u_{1m}, ..., u_{nm})$ is a Cauchy sequence in $X^\star$.

Therefore, according to the elementary inequalities (see, e.g., Auxiliary Results in [46]) for any $\varrho, \zeta \in \mathbb{R}^N$

$$\begin{cases}
|\varrho - \zeta|^{p_i(x)} \leq c_{p_i}\Big(\mathcal{B}_i(\varrho) - \mathcal{B}_i(\zeta)\Big)\cdot(\varrho - \zeta) & \text{if } p_i(x) \geq 2\\
|\varrho - \zeta|^2 \leq c(|\varrho| + |\zeta|)^{2-p_i(x)}\Big(\mathcal{B}_i(\varrho) - \mathcal{B}_i(\zeta)\Big)\cdot(\varrho - \zeta) & \text{if } 1 < p_i(x) < 2
\end{cases} \tag{6.3.15}$$

where $\cdot$ denotes the standard inner product in $\mathbb{R}^N$. Replacing ϱ and ζ by ∇u_{1m} and ∇u_{1k} respectively and integrating over Ω, we obtain

$$c\int_\Omega |u_{1m}-u_{1k}|^{p_1(x)}dx \leq \left\langle \Phi'(u_{1m},...,u_{nm}) - \Phi'(u_{1k},...,u_{nk}), (u_{1m}-u_{1k},0,...,0)\right\rangle,$$

if $p_1(x)\geq 2$, and if $1<p_i(x)<2$, we get

$$\int_\Omega \sigma_1(x)^{p_i(x)-2}|u_{1m}-u_{1k}|^2dx \leq \left\langle \Phi'(u_{1m},...,u_{nm}) - \Phi'(u_{1k},...,u_{nk}), (u_{1m}-u_{1k},0,...,0)\right\rangle$$

where $\sigma_1(x) = C(|\nabla u_{1m}|+|\nabla u_{1k}|)$. Hence by Hölder's inequality and by lemma , we get

$$\begin{aligned}
\int_\Omega |u_{1m}-u_{1k}|^{p_1(x)}dx &= \int_\Omega \sigma_1^{\frac{p_1(x)(p_1(x)-2)}{2}}\Big(\sigma_1^{\frac{p_1(x)(p_1(x)-2)}{2}}|u_{1m}-u_{1k}|^{p_1(x)}\Big)dx\\
&\leq C\|\sigma_1^{\frac{p_1(x)(2-p_1(x))}{2}}\|_{L^{\frac{2}{2-p_1(x)}}(\Omega)}\|\sigma_1^{\frac{p_1(x)(p_1(x)-2)}{2}}|u_{1m}-u_{1k}|^{p_1(x)}\|_{L^{\frac{2}{p_1(x)}}(\Omega)}\\
&\leq C\max\left\{\|\sigma_1\|_{L^{p_1(x)}(\Omega)}^{[\frac{p_1(x)(p_1(x)-2)}{2}]^-}, \|\sigma_1\|_{L^{p_1(x)}(\Omega)}^{[\frac{p_1(x)(p_1(x)-2)}{2}]^+}\right\}\times\\
&\max\left\{\Big(\int_\Omega \sigma_1^{p_1(x)-2}|u_{1m}-u_{1k}|^2dx\Big)^{\frac{p_1^-}{2}}, \Big(\int_\Omega \sigma_1^{p_1(x)-2}|u_{1m}-u_{1k}|^2dx\Big)^{\frac{p_1^+}{2}}\right\}
\end{aligned}$$

Taking into account the fact that $\{u_{1m}\}$ is bounded in $W_0^{1,p_1(x)}\cap W_0^{1,\gamma_1(x)}(\Omega)$

$$\left\langle \Phi'(u_{1m},...,u_{nm}) - \Phi'(u_{1k},...,u_{nk}), ((u_{1m}-u_{1k},0,...,0)\right\rangle \to 0 \quad \text{as} \;\; m,k\to\infty,$$

we find that $\{u_{1m}\}$ is a Cauchy sequence in $W_0^{1,p_1(x)}\cap W_0^{1,\gamma_1(x)}(\Omega)$. We proceed similarly for $\{u_{im}\}$ with $\left\langle \Phi'(u_{1m},...,u_{im},...,u_{nm}) - \Phi'(u_{1k},...,u_{ik},...,u_{nk}), (0,...,u_{im}-u_{ik},0,...,0)\right\rangle$ for all $i\in\{2,3,...,n\}$. $\square$

Now we are in position to prove Theorem 1.

Proof. (of Theorem (2.2.1)) The proof is an immediate consequence of the mountain pass theorem, Lemma 6.3.2 and Lemma 6.3.3. Precisely, it suffices to verify that E_λ has the mountain pass geometry and that $E_\lambda(tu_1,...,tu_n)<0$ for some $t>0$.

From $(\mathcal{M}_2)$, we can obtain for $t>t_0$

$$\widehat{M_i}(t) \leq \frac{\widehat{M_i}(t_0)}{t_0^{\frac{1}{\sigma_i}}}t^{\frac{1}{\sigma_i}} \leq c_i t^{\frac{1}{\sigma_i}}. \tag{6.3.16}$$

About the latter condition, we have

$$E_\lambda(u)=\sum_{i=1}^{n}\widehat{M}_i\left(\mathcal{A}_i(u_i)\right)-\sum_{i=1}^{n}\int_\Omega\frac{1}{s_i(x)}|u_i|^{s_i(x)}dx-\lambda\int_\Omega F(x,u_1(x),...,u_n(x))dx.$$

Then, because of $\int_\Omega F(x,u_1(x),...,u_n(x))dx>0$ and 6.3.16, we obtain for $(z_1,...,z_n)\in X/\{(0,...,0)\}$ and any $t>1$

$$\begin{aligned}E_\lambda(tz_1,...,tz_n)&\leq\sum_{i=1}^{n}c_i\left(\mathcal{A}_i(tz_i)\right)^{\frac{1}{\sigma_i}}-\sum_{i=1}^{n}\int_\Omega\frac{1}{s_i(x)}|tz_i|^{s_i(x)}dx,\\&\leq\sum_{i=1}^{n}c_i\left(\int_\Omega\frac{1}{p_i(x)}A_i(|\nabla(tz_i)|^{p_i(x)})dx\right)^{\frac{1}{\sigma_i}}-\sum_{i=1}^{n}\int_\Omega\frac{1}{s_i(x)}|tz_i|^{s_i(x)}dx,\\&\leq\sum_{i=1}^{n}c_i\Big(\int_\Omega\Big(\frac{k_i^1}{p_i(x)}|t\nabla z_i|^{p_i(x)}+\frac{k_i^3}{q_i(x)}|t\nabla z_i|^{q_i(x)}\Big)dx\Big)^{\frac{1}{\sigma_i}}-\sum_{i=1}^{n}\int_\Omega\frac{1}{s_i(x)}|tz_i|^{s_i(x)}dx,\\&\leq\sum_{i=1}^{n}c_i t^{\frac{q_i^+}{\sigma_i}}\Big(\int_\Omega\Big(\frac{k_i^1}{p_i(x)}|\nabla z_i|^{p_i(x)}+\frac{k_i^3}{q_i(x)}|\nabla z_i|^{q_i(x)}\Big)dx\Big)^{\frac{1}{\sigma_i}}-\sum_{i=1}^{n}t^{s_i^-}\int_\Omega\frac{1}{s_i(x)}|z_i|^{s_i(x)}dx.\end{aligned}$$

which tends to $-\infty$ as $t\to+\infty$ since $\sigma_i\geq\frac{q_i^+}{s_i^-}$.

On the other hand, let us fix $\lambda>0$. For all $u=(u_1,...,u_n)\in X$, under the assumptions $(\mathcal{M}_1)-(\mathcal{M}_2),(\boldsymbol{H}_2)$ and $(\mathcal{F}_4)$, we obtain

$$\begin{aligned}E(u)\geq&\sum_{i=1}^{n}\mathfrak{M}_i\left[\int_\Omega\frac{k_i^0}{p_i(x)}\left|\nabla u_i\right|^{p_i(x)}dx+\mathcal{H}(k_i^3)k_i^2\int_\Omega\frac{1}{q_i(x)}\left|\nabla u_i\right|^{q_i(x)}dx\right]\\&-\sum_{i=1}^{n}\int_\Omega\frac{1}{s_i(x)}\left|\nabla u_i\right|^{s_i(x)}dx-\lambda c\int_\Omega\Big(\sum_{i=1}^{n}|u_i|^{r_i(x)}\Big).\end{aligned}$$

Consider $0<\|u\|=\sum_{i=1}^{n}\|u_i\|_i=\rho<1$ with $\|u_i\|_i=\|\nabla u_i\|_{p_i(x)}+\mathcal{H}(k_i^3)\|\nabla u_i\|_{q_i(x)}$. By Propositons (1.1.1), (1.1.4) and (1.1.5), we have

$$\begin{aligned}E(u)&\geq\sum_{i=1}^{n}c\Big(\left\|\nabla u_i\right\|_{p_i(x)}^{q^+}+\mathcal{H}(k_i^3)\left\|\nabla u_i\right\|_{q_i(x)}^{q^+}\Big)-\sum_{i=1}^{n}\frac{c}{s_i^-}\left\|u_i\right\|_i^{s_i^-}-\sum_{i=1}^{n}\lambda c\|u_i\|_i^{r_i^-}\\&\geq\sum_{i=1}^{n}\Big(c\left\|u_i\right\|_i^{q_i^+}-\frac{c}{s_i^-}\left\|u_i\right\|_i^{s_i^-}-\lambda c\|u_i\|_i^{r_i^-}\Big)\end{aligned}$$

Hence, since $q_i^+<r_i^-<s_i^-$, follows that there are $0<\rho<1$ small enough and $\mathcal{R}>0$ such that

$$E_\lambda(u) \geq \mathcal{R} > 0 \quad \text{as } \|u\| = \rho.$$

That means the existence of an element $(u_1^0, ..., u_n^0)$ of X such that $E_\lambda(u_1^0, ..., u_n^0) < 0$. Consequentely, the critical value is

$$C_\lambda := \inf_{\xi \in \Gamma} \sup_{t \in [0,1]} E_\lambda(\xi(t)),$$

where

$$\Gamma = \left\{\xi : [0,1] \to X, \text{continuous and } \xi(0) = (0, ..., 0), \xi(1) = (u_1^0, ..., u_n^0)\right\}.$$

That concludes the proof. □

Next we will prove under some symmetry condition on the function F that 6.1.1 possesses infinitely many nontrivial solutions.

Proof. (of Theorem (2.2.2)) We will use a $\mathbb{Z}_2$-symmetric version of the Mountain Pass theorem 1.4.6, to accomplish the proof of theorem 2.2.2. By assumption the function F is even, the functional E_λ is even too. Considering the proof of theorem 2.2.1, we need only check the condition $(\mathcal{I}_2')$. In fact by using 6.3.16 and $\int_\Omega F(x, u_1(x), ..., u_n(x))dx > 0$, we obtain

$$\begin{aligned}
E_\lambda(u_1, ..., u_n) &\leq \sum_{i=1}^n c_i \left(\mathcal{A}_i(u_i)\right)^{\frac{1}{\sigma_i}} - \sum_{i=1}^n \int_\Omega \frac{1}{s_i(x)} |u_i|^{s_i(x)} dx, \\
&\leq \sum_{i=1}^n c_i \Big(\int_\Omega \Big(\frac{k_i^1}{p_i^-} |\nabla u_i|^{p_i(x)} + \frac{k_i^3}{q_i^-} |\nabla u_i|^{q_i(x)} \Big) dx \Big)^{\frac{1}{\sigma_i}} - \sum_{i=1}^n \int_\Omega \frac{1}{s_i^+} |u_i|^{s_i(x)} dx.
\end{aligned}$$

Let $u = (u_1, ..., u_n) \in X$ be arbitrary but fixed. we define

$$\Omega_< = \{x \in \Omega : |u_i(x)| < 1\}, \text{ and } \Omega_\geq = \Omega \backslash \Omega_<.$$

Then we have

$$\begin{aligned}
E_\lambda(u_1, ..., u_n) &\leq \sum_{i=1}^n c_i \Big(\int_\Omega \Big(\frac{k_i^1}{p_i^-} |\nabla u_i|^{p_i(x)} + \frac{k_i^3}{q_i^-} |\nabla u_i|^{q_i(x)} \Big) dx \Big)^{\frac{1}{\sigma_i}} - \sum_{i=1}^n \int_\Omega \frac{1}{s_i^+} |u_i|^{s_i(x)} dx, \\
&\leq \sum_{i=1}^n c_i \Big(\int_\Omega \Big(\frac{k_i^1}{p_i^-} |\nabla u_i|^{p_i(x)} + \frac{k_i^3}{q_i^-} |\nabla u_i|^{q_i(x)} \Big) dx \Big)^{\frac{1}{\sigma_i}} - \sum_{i=1}^n \int_{\Omega_\geq} \frac{1}{s_i^+} |u_i|^{s_i^-} dx, \\
&\leq \sum_{i=1}^n c_i \Big(\int_\Omega \Big(\frac{k_i^1}{p_i^-} |\nabla u_i|^{p_i(x)} + \frac{k_i^3}{q_i^-} |\nabla u_i|^{q_i(x)} \Big) dx \Big)^{\frac{1}{\sigma_i}} - \sum_{i=1}^n \int_\Omega \frac{1}{s_i^+} |u_i|^{s_i^-} dx \\
&\quad + \sum_{i=1}^n \int_{\Omega_<} \frac{1}{s_i^+} |u_i|^{s_i^-} dx, \\
&\leq \sum_{i=1}^n c_i \Big(\int_\Omega \Big(\frac{k_i^1}{p_i^-} |\nabla u_i|^{p_i(x)} + \frac{k_i^3}{q_i^-} |\nabla u_i|^{q_i(x)} \Big) dx \Big)^{\frac{1}{\sigma_i}} - \sum_{i=1}^n \int_\Omega \frac{1}{s_i^+} |u_i|^{s_i^-} dx.
\end{aligned}$$

The functional $|.|_{s_i^-} : X_i \to \mathbb{R}$ defined by

$$|u_i|_{s_i^-} = \Big(\int_\Omega |u_i|^{s_i^-} \, dx \Big)^{1/s_i^-}$$

is a norm in X_i. LetX_i^1 be a fixed finite dimensional subspace of X_i. Then $|.|_{s_i^-}$ and $\|.\|_i$ are equivalent norms on X_i, so there exists a positive constant $c = c(X_i^1)$ such that

$$\|u_i\|_i^{s_i^-} \le c \, |u_i|_{s_i^-}^{s_i^-}, \text{ for all } u_i \in X_i^1.$$

Assume $\|u_i\|_i > 1$ for convenience. According to proposition 1.1.3 and proposition 1.1.8, for any $u \in S_1$ we obtain

$$0 \le E_\lambda(u) \le \sum_{i=1}^{n} \Big(c \, \|u_i\|_i^{q_i^+/\sigma_i} - c \, \|u_i\|_i^{s_i^-} \Big)$$

since $s_i^- > q_i^+/\sigma_i$ we conclude thatS_1 is bounded in X. $\square$

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Abstract:

In this thesis, we study some nonlinear partial differential elliptic systems involving nonhomogeneous operators. One of the major difficulties which appear while we study these systems is the loss of compactness in the Sobolev inclusions because the system is defined on unbounded domains or contains critical exponents. A second difficulty concerns the choice of nonlinearities, which must satisfy growth conditions. These conditions guarantee the differentiability, coercivity and semicontinuity of the energy functional associated with the system. Finally, the last difficulty is linked to the construction of functional spaces with weight to ensure the structure of reflexive Banach spaces as well as the different Sobolev inclusions.

Keywords: *Weak solutions, uniqueness, multiple solutions, Variable exponents spaces, critical Sobolev exponents, p(x)-Laplacian, Kirchhof-type problems, principle of symmetric criticality, Palais-Smale condition, Mountain Pass theorem, critical points theory, concentration-compactness principle.*

Résumé :

Dans cette thèse, nous étudions quelques systèmes aux dérivées partielles non linéaires de type elliptiques faisant intervenir des opérateurs non homogènes. L'une des difficultés majeures qui apparaissent lors de l'étude de ces systèmes est la perte de compacité dans les inclusions de Sobolev car le système est défini sur des domaines non bornés ou bien il contient des exposants critiques. Une seconde difficulté concerne le choix des non linéarités, qui doivent satisfaire des conditions de croissance. Ces conditions garantissent la différentiabilité, la coercivité et la semicontinuité de la fonctionnelle d'énergie associée au système. Enfin, la dernière difficulté est liée à la construction des espaces fonctionnels à pois pour assurer la structure d'espaces de Banach réflexifs ainsi que les différentes inclusions de Sobolev.

Mots-clés: *Solutions faibles, unicité, solutions multiples, espaces a exposants variables, exposants critiques de Sobolev, p(x)-Laplacien, problèmes de type Kirchhoff , principe de criticité symétrique, principe de concentration-compacité, condition de Palais-Smale, théorème du col de la montagne, théorie des points critiques.*

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