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par

**Brahim EL ALAOUI**

## Une étude de certaines propriétés des graphes dans le contexte des graphes des diviseurs de zéro des anneaux commutatifs

**JURY**

| | | | |
|---|---|---|---|
| Abdelhak ZOGLAT | PES | Faculté des Sciences, Université Mohammed V, Rabat | Président |
| Zine El Abidine ABDELALI | PES | Faculté des Sciences, Université Mohammed V, Rabat | Rapporteur/Examinateur |
| Driss KARIM | PES | Faculté des Sciences et Techniques, Université Hassan II, Casablanca. | Rapporteur/Examinateur |
| Brahim FAHID | PH | Ecole Supérieure de Technologie, Université Ibn Tofail, Kenitra | Rapporteur/Examinateur |
| Khalid OUARGHI | MC | Ecole Normale Supérieure, Université Mouly Ismail, Meknès. | Invité |
| Driss BENNIS | PES | Faculté des Sciences, Université Mohammed V, Rabat | Directeur de thése |

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✉ Faculté des Sciences, avenue Ibn Battouta, BP. 1014 RP, Rabat –Maroc
☎ 00212(0)5 37 77 18 76 00212(0)5 37 77 42 61 ; http://fsr.um5.ac.ma/

# A study of certain graph properties in the context of zero-divisor graphs of commutative rings

Thesis Dissertation
Under supervision of Professor Driss BENNIS

Mohammed V University in Rabat, Faculty of Sciences.

Brahim EL ALAOUI

# Dedication

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# Abstract

This thesis addresses concepts and problems related to graph theory and the theory of commutative rings. Namely, we study certain graph properties in the context of some graphs associated with commutative rings. Specifically, we study the notions of a global defensive $k$-alliance, partitioning into global defensive alliances, complementedness and uniquely complementedness in the context of zero-divisor graphs and extended zero-divisor graphs of rings.

The global defensive $k$-alliance is a very well-studied notion in graph theory, it provides a method of classification of graphs based on relations between members of a particular set of vertices. In this thesis, we explore this notion in zero-divisor graphs of finite commutative rings. The established results generalize and improve recent work by Muthana and Mamouni [62], in which they treated a particular case for $k = -1$ known by the global defensive alliance.

Another well-studied problem in graph theory is the partitioning of the vertex set of a graph. It involves dividing the set of vertices of a graph into disjoint subsets or partitions, based on specific criteria or constraints. Then, we are also interested in partitioning the zero-divisor graph of a commutative ring into global defensive alliances. This problem has been well investigated in graph theory. Here, we connected it with the ring theoretical context. We characterize various finite commutative rings for which the zero-divisor graph is partitionable into global defensive alliances. We also provide several examples to illustrate and delimit the scope of the established results.

Also, we deal with complementedness and uniquely complementedness notions of graphs. These notions were initially introduced for a general graph in [9, 58] and subsequently investigated within the framework of zero-divisor graphs of commutative rings. We continue the study started in [24] concerning when the extended zero-divisor graph of a commutative ring is complemented or uniquely complemented. We give a complete characterization of when the extended zero-divisor graph of a finite commutative ring is complemented. Various examples are given using the direct product of rings and the idealization of modules.

At the end of this thesis, we present some proposals for future studies of certain graph-theoretic conjectures with a view to the zero-divisor graph theory.

[1]**Key Words:** Zero-divisor graph; extended zero-divisor graph; commutative ring; idealization; zero-dimensional ring; complemented; global defensive alliance; partitioning into global defensive alliances.

# Résumé

Cette thèse traite des concepts et des problèmes en relation avec la théorie des graphes et la théorie des anneaux commutatifs. Plus précisément, nous étudions certaines propriétés graphiques dans le contexte de certains graphes associés aux anneaux commutatifs. En particulier, nous étudions les notions de $k$-alliance défensive globale, de partitionnement en alliance défensive globale, de complémentarité et de complémentarité unique dans le contexte des graphes des diviseurs de zéro et des graphes des diviseurs de zéro étendus des anneaux commutatifs.
La $k$-alliance défensive globale est une notion très étudiée en théorie des graphes, elle offre une méthode de classification des graphes basée sur les relations entre les membres d'un ensemble particulier de sommets. Dans cette thèse, nous explorons cette notion dans les graphes des diviseurs de zéro des anneaux commutatifs finis. Les résultats établis généralisent et améliorent le travail récent de Muthana et Mamouni, dans [62], qui ont traité un cas particulier pour $k = -1$, connu sous le nom d'alliance défensive globale.
Un autre problème bien étudié en théorie des graphes est le partitionnement de l'ensemble des sommets d'un graphe. Il implique de diviser l'ensemble des sommets d'un graphe en sous-ensembles ou partitions disjointes, en fonction de certains critères ou contraintes spécifiques. Nous sommes également intéressés par le partitionnement du graphe des diviseurs de zéro d'un anneau commutatif en alliances défensives globales. Ce problème a été bien étudié en théorie des graphes. Ici, nous le relions au contexte théorique des anneaux. Nous caractérisons divers anneaux commutatifs finis pour lesquels le graphe des diviseurs de zéro est partitionnable en alliances défensives globales. Plusieurs exemples sont également fournis pour illustrer et délimiter la portée des résultats établis.
Nous abordons également les notions de complémentarité et de complémentarité unique des graphes. Ces notions ont été initialement introduites pour un graphe quelconque dans [9, 58] et ils ont ensuite été étudiées dans le cadre des graphes des diviseurs de zéro d'anneaux commutatifs. Nous poursuivons l'étude commencée dans [24] concernant le caractère complémenté ou uniquement complémenté du graphe des diviseurs de zéro étendu d'un anneau commutatif. Nous donnons une caractérisation complète du caractère complémenté du graphe des diviseurs de zéro étendu d'un anneau commutatif fini. Divers exemples sont donnés en utilisant le produit direct d'anneaux et l'idéalisation des modules.
A la fin de cette thèse, nous présentons quelques propositions pour des études futures de certaines conjectures de la théorie des graphes avec une perspective dérivée de la théorie des graphes des diviseurs de zéro.

[1]**Mots-clés:** Graphe des diviseurs de zéro; graphe des diviseurs de zéro étendu; anneau commutatif; idéalisation; anneau de dimension zéro; complémenté; alliance défensive globale; partitionnement en alliances défensives globales.

# Résumé étendu

La théorie des graphes est l'étude des graphes en tant que concepts mathématiques utilisés pour représenter les connexions entre des entités. Sa portée s'étend bien au-delà du domaine des mathématiques théoriques, offrant une utilité pratique pour résoudre des problèmes du monde réel. En effet, en cartographiant et en explorant visuellement les relations complexes entre les composants, la théorie des graphes fournit des informations inestimables et des solutions efficaces dans toute une série de domaines, des transports et de la planification urbaine aux réseaux sociaux et à l'informatique (voir [36, 43, 60, 63, 65, 75, 83], et les préliminaires du Chapitre 1).

Les racines de la théorie des graphes remontent au 18ème siècle, et sa naissance est attribuée au travail du mathématicien suisse Leonhard Euler, souvent considéré comme le précurseur du domaine. L'exploit monumental d'Euler en 1736, lorsqu'il s'est attaqué au célèbre problème des sept ponts de Königsberg, a servi de démonstration puissante de l'efficacité de la théorie des graphes dans la résolution d'énigmes concrètes. Au fil des années, divers mathématiciens et scientifiques ont continué à développer la théorie, en affinant ses principes et en étendant ses applications.

La théorie des anneaux, une branche importante de l'algèbre abstraite, a une riche histoire qui remonte à la fin du 19ème siècle et au début du 20ème siècle. Ses origines sont liées à l'étude des structures algébriques et à la théorie des nombres. Le concept d'anneau a été introduit implicitement pour la première fois par Richard Dedekind dans le contexte des champs de nombres algébriques dans les années 1870 [34]. Les travaux de Dedekind sur les idéaux dans les anneaux d'entiers ont jeté les bases de la définition formelle des anneaux. Le terme "anneau" lui-même a été inventé par David Hilbert dans les années 1890 pour décrire certains ensembles de fonctions polynomiales. En 1921 [64], Emmy Noether a révolutionné le domaine en introduisant la théorie des idéaux dans les anneaux commutatifs, fournissant un cadre plus systématique et plus abstrait pour la théorie des anneaux. Ses travaux ont contribué à façonner l'algèbre moderne. Tout au long du 20ème siècle, la théorie des anneaux s'est rapidement développée, incorporant à la fois les anneaux commutatifs et non commutatifs et trouvant des applications dans divers domaines des mathématiques, y compris la géométrie, l'analyse et la topologie.

Cette thèse traite des concepts et des problèmes en relation avec la théorie des graphes et la théorie des anneaux commutatifs. Plus précisément, dans le chapitre 2, nous traitons la notion de $k$-alliance défensive globale dans les graphes des diviseurs de zéro des anneaux commutatifs. Nous commençons par étendre [62, Proposition 2.2] qui donne une borne supérieure sur la cardinalité de $Z(R)$ en termes du nombre d'alliances défensives globales de $\Gamma(R)$. À savoir, nous donnerons une borne supérieure sur la cardinalité de $Z(R)$ en termes du nombre $k$-alliance défensive globale (voir Proposition 2.2). La borne supérieure établie est plus optimale que celle donnée dans [62, Proposition 2.2] comme le montrent les Exemples 2.1 et les Exemples 2.3. Ensuite, nous nous intéressons à l'étude du nombre

$k$-alliance défensive globale du graphe des diviseurs de zéro d'un anneau local fini. Il semble difficile de le déterminer pour tout anneau. Cependant, en tant que deuxième résultat principal, nous réussissons à le calculer pour $\mathbb{Z}_{p^n}$ pour un nombre premier $p$ et un entier positif $n$ (voir Théorème 2.1). Le nombre $k$-alliance défensive globale du graphe des diviseurs de zéro d'un anneau local fini avec un idéal maximal nilpotent d'indice 2 est également donné (voir Proposition 2.3). Dans la Section 2.2, nous calculons le nombre $k$-alliance défensive globale du graphe des diviseurs de zéro pour certains types de produits directs de corps finis. Nous commençons par déterminer le nombre $k$-alliance défensive globale du graphe des diviseurs de zéro d'un produit direct de deux corps finis (voir Théorème 2.2) et, comme cas particulier, nous retrouvons [62, Proposition 2.3]. De plus, nous obtenons des résultats pour l'alliance défensive globale forte qui n'est rien d'autre que la 0-alliance défensive globale (voir Section 1.1.2). Déterminer la $k$-alliance défensive globale pour le produit direct de corps finis $\prod_{i=1}^{n} F_i$ avec $n \geq 3$ un entier positif et $F_i$ un corps fini pour chaque $i \in 1, \ldots, n$ est encore une question ouverte. Cependant, comme principaux résultats, nous la déterminons pour $\mathbb{Z}_2 \times \mathbb{Z}_2 \times F$ avec $|F| \geq 2$, et pour $\mathbb{Z}_2 \times F \times K$ avec $|K| \geq |F| \geq 3$ (voir Théorèmes 2.3 et 2.4). La Section 2.3 est consacrée à l'étude du nombre $k$-alliance défensive globale du graphe des diviseurs de zéro pour un produit direct de $\mathbb{Z}_2$ et un anneau fini. Nous commençons par donner une borne supérieure et inférieure pour le nombre $k$-alliance défensive globale de $\Gamma(\mathbb{Z}_2 \times R)$ où $R$ est un anneau fini (voir Théorème 2.5). Dans [62, Proposition 2.4], Muthana et Mamouni ont établi l'égalité $\gamma_a(\Gamma(\mathbb{Z}_2 \times R)) = \left\lceil \frac{|R|}{2} \right\rceil$ pour un anneau local $R$. Ici, nous donnons des égalités pour certains entiers $k \in [\![1 - |R|; 1]\!]$ (voir Théorèmes 2.6 et 2.7). Pour un anneau local $R$ avec un idéal maximal nilpotent d'indice 2, nous améliorons l'inégalité du Théorème 2.5 et donnons l'égalité pour les cas restants autres que ceux étudiés dans les Théorèmes 2.6 et 2.7.

Dans le chapitre 3, nous nous concentrons sur la tâche de partitionner l'ensemble des sommets des graphes des diviseurs de zéro des anneaux commutatifs en alliances défensives globales. Ce problème a été précédemment étudié, pour les graphes généraux, par Eroh et Gera [39, 40] et par Haynes et Lachniet [50]. Ici, nous le connectons au contexte théorique des anneaux. Nous caractérisons divers anneaux finis commutatifs pour lesquels le graphe des diviseurs de zéro est partitionnable en alliances défensives globales. À savoir, dans la Section 3.1, nous étudions quand les graphes des diviseurs de zéro de certains types d'anneaux locaux finis sont partitionnables en alliances défensives globales et calculons leurs nombres de partition d'alliances défensives globales. Nous commençons par donner une borne inférieure sur la cardinalité de l'ensemble des diviseurs de zéro $Z(R)$, pour un anneau commutatif $R$, en termes du nombre de partition d'alliances défensives globales de $\Gamma(R)$ (voir Proposition 3.1). Prouver que le graphe des diviseurs de zéro peut être partitionnable en alliances défensives globales pour tout anneau semble difficile. Cependant, comme deuxième résultat principal, nous réussissons à calculer le nombre de partition d'alliances défensives globales pour $\mathbb{Z}_{p^n}$ pour un nombre premier $p$ et un entier positif $n$ (voir Théorème 3.1). Le nombre de partition d'alliances défensives globales du graphe des diviseurs de zéro d'un anneau local fini avec un idéal maximal nilpotent d'indice 2 est également donné (voir Proposition 3.2). Dans la Section 3.2, nous étudions quand le graphe des diviseurs de zéro de certains types de produits directs de corps finis est partitionnable en alliances défensives globales. Nous commençons par caractériser quand un graphe des diviseurs de zéro de $\mathbb{Z}_2 \times F$, pour un corps fini $F$, est partitionnable en alliances défensives globales (voir Théorème 3.2). Ensuite, nous étudions quand les graphes des diviseurs de zéro d'un produit direct de deux corps finis sont partitionnables

en alliances défensives globales (voir Théorème 3.3). Ensuite, nous présentons quand $\Gamma(\mathbb{Z}_2 \times \mathbb{Z}_2 \times F)$, pour un corps fini $F$, est partitionnable en alliances défensives globales. Nous terminons cette section en montrant que $\Gamma(\mathbb{Z}_2 \times F \times K)$, pour des corps finis $F$ et $K$ avec $|F|, |K| \geq 3$, n'est pas partitionnable en alliances défensives globales. Dans la section 3.3, nous étudions quand le graphe des diviseurs de zéro du produit direct d'un corps fini $F$ et d'un anneau local fini $R$, qui n'est pas un corps et tel que son idéal maximal est nilpotent d'indice 2, est partitionnable en alliances défensives globales (voir Théorèmes 3.6 et 3.7). Dans la Section 3.4, nous donnons des caractérisations complètes pour la partition des graphes des diviseurs de zéro des anneaux finis avec $\gamma_a(\Gamma(R)) = 1, 2$. À savoir, nous prouvons qu'un graphe des diviseurs de zéro, $\Gamma(R)$, d'un anneau fini $R$ avec $\gamma_a(\Gamma(R)) = 1$ est partitionnable en alliances défensives globales si et seulement si $R$ est isomorphe à l'un des anneaux $\mathbb{Z}_2 \times \mathbb{Z}_2$, $\mathbb{Z}_9$, $\mathbb{Z}_3[X]/(X^2)$ (voir Théorème 3.8), et nous prouvons que pour un graphe des diviseurs de zéro d'un anneau fini avec $\gamma_a(\Gamma(R)) = 2$, $\Gamma(R)$ est partitionnable en alliances défensives globales si et seulement si $R$ est isomorphe à l'un des anneaux $\mathbb{Z}_2 \times \mathbb{Z}_4$, $\mathbb{Z}_2 \times \mathbb{Z}_2[X]/(X^2)$, $\mathbb{Z}_3 \times \mathbb{Z}_3$, $\mathbb{Z}_3 \times \mathbb{F}_4$, $\mathbb{Z}_{25}$, $\mathbb{Z}_5[X]/(X^2)$ et $\mathbb{F}_4 \times \mathbb{F}_4$ (voir Théorème 3.9 et Corollaire 3.9).
Dans le chapitre 4, nous traitons des notions de complémentarité et de complémentarité unique des graphes. Ces notions ont été initialement introduites pour un graphe général dans [9, 58] et ensuite étudiées dans le cadre des graphes des diviseurs de zéro des anneaux commutatifs. Dans cette thèse, nous explorons ces notions dans le contexte des graphes étendus des diviseurs de zéro des anneaux commutatifs. Nous poursuivons l'investigation commencée dans [24] pour étudier plus en profondeur quand le graphe étendu des diviseurs de zéro d'un anneau commutatif est complémenté et quand il est uniquement complémenté. À savoir, dans la Section 4.1, nous étudions quand le graphe étendu des diviseurs de zéro d'un anneau commutatif est complémenté. Nous commençons par montrer que, si $\overline{\Gamma}(R)$ est complémenté et que $|Z(R)| \geq 4$, alors l'anneau $R$ a au plus un élément nilpotent non nul (voir Théorème 4.1 et Exemples 4.2, 4.3 et 4.4). Lorsque $R$ est fini, nous obtenons le contraire du Théorème 4.1 (voir Corollaire 4.2). En fait, cela découle de la caractérisation des anneaux finis avec des graphes étendus des diviseurs de zéro complémentés (voir Théorème 4.2). Dans la Section 4.2, nous montrons comme résultat principal que, lorsque $\Gamma(R) \neq \overline{\Gamma}(R)$, les notions de complémentarité et de complémentarité unique coïncident (voir Théorème 4.3). Dans la Section 4.3, nous montrons que, lorsque $\Gamma(R) \neq \overline{\Gamma}(R)$, l'anneau quotient total $T(R)$ de $R$ est zéro-dimensionnel une fois que $\Gamma(R)$ est complémenté (voir Théorème 4.4). La preuve de ce résultat nous amène à montrer que lorsque $\overline{\Gamma}(R)$ est complémenté, chaque élément non nilpotent a un orthogonal qui n'est pas nilpotent (voir Théorème 4.5). De plus, si $\overline{\Gamma}(R)$ est complémenté et que $\overline{\Gamma}(R) \neq \Gamma(R)$, alors chaque orthogonal à l'élément nilpotent non nul unique ne peut pas être une extrémité (voir Corollaire 4.4). À la fin de cette section, nous prouvons que, pour tout anneau $R$ tel que $|\mathrm{Nil}(R)| = 2$, $R$ n'est pas local ou $\overline{\Gamma}(R)$ n'est pas complémenté (voir Proposition 4.1). Enfin, la Section 4.4 est consacrée à l'étude de quand le graphe étendu des diviseurs de zéro d'un produit direct fini d'anneaux ainsi que celui d'une idéalisation d'un $R$-module sont complémentés (voir Théorèmes 4.6, 4.7 et 4.8, et Proposition 4.2).
A la fin de cette thèse, nous présentons quelques conjectures intéressantes en théorie des graphes qui restent non résolues. À savoir, la conjecture de double couverture par cycles, la conjecture de Hadwiger, la conjecture de Hedetniemi et la conjecture de Vizing. Ainsi, comme perspective est d'étudier ces conjectures dans le cadre de la théorie des graphes des diviseurs zéro.

# Articles involved in this thesis

1. Bennis D., El Alaoui B., Ouarghi K. (2023). On global defensive k-alliances in zero-divisor graphs of finite commutative rings. *J. Algebra Appl.* **22(06)**: 2350127.
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3. Bennis D., El Alaoui B., L'Hamri R. (2024). Rings whose associated extended zero-divisor graphs are complemented. *Bull. Korean Math. Soc.* **61(03)**: 763–777.

## Other papers:

1. Bennis D., El Alaoui B., Fahid B., Farnik M., L'hamri R. (2021). The i-extended zero-divisor graphs of commutative rings. *Comm Algebra* **49**: 4661–4678.
2. Bennis D., El Alaoui B., L'hamri R. (2024). The i-extended zero-divisor graphs of idealizations. *J Algebra Appl* **23(6)**: 2450127.
3. El Alaoui B., L'Hamri R. (2024). The extended zero-divisor graphs of the amalgamated duplication of a ring along an ideal. Submitted for publication.
4. Bennis D., El Alaoui B. Automorphisms of the zero-divisor graph of the incidence ring. In preparation.

# Contents

# Notation

- Throughout this thesis, $R$ is a commutative ring with identity $1 \neq 0$.
- The set of zero-divisors and the set of units of $R$ are denoted respectively by $Z(R)$ and $U(R)$.
- Let $x$ be an element of $R$. The annihilator of $x$ is defined as $\mathrm{Ann}(x) := \{y \in R \mid xy = 0\}$.
- Let $I$ be an ideal of $R$, we denote by $\sqrt{I}$, the radical of $I$.
- An element $x$ of $R$ is called nilpotent if $x^n = 0$ for some positive integer $n$.
- $\mathrm{Nil}(R) := \sqrt{0}$ is the set of all nilpotent elements of $R$.
- A ring $R$ is considered reduced if $\mathrm{Nil}(R) = \{0\}$.
- $\mathbb{Z}_n$ represents the ring of residues modulo an integer $n$, which is equivalent to the ring $\mathbb{Z}/n\mathbb{Z}$.
- For a subset $S$ of $R$, $S^* := S \setminus \{0\}$.
- Let $r$ be a real number, then $\lceil r \rceil$ (resp., $\lfloor r \rfloor$) denotes the ceiling of $r$, i.e., the least integer greater than or equals $r$ (resp., the floor of $r$, i.e., the greatest integer less than or equals $r$).
- Let $n$ be a positive integer, the interval of integers from 1 up to $n$ is denoted by $[\![1; n]\!] := \{1, \ldots, n\}$.

# Introduction

Graph Theory is the study of graphs as mathematical concepts used to represent connections between entities. Its reach extends far beyond the realm of theoretical mathematics, offering practical utility in solving real-world problems. Indeed, by visually mapping and exploring complex relationships between components, graph theory provides invaluable insights and effective solutions in a range of domains, from transportation and urban planning to social networks and computational sciences (see [36, 43, 60, 63, 65, 75, 83], and the preliminaries in Chapter 1).

The roots of graph theory date back to the 18th century, with its inception attributed to the pioneering work of the swiss mathematician Leonhard Euler, often hailed as the progenitor of the field. Euler's monumental achievement in 1736 [26], when he famously tackled the Seven Bridges of Königsberg problem, served as a powerful demonstration of graph theory's effectiveness in solving concrete puzzles. Over time, various mathematicians and scientists have continued to develop the theory, refining its principles and expanding its applications.

Ring theory, an important branch of abstract algebra, has a rich history dating back to the end of the 19th century and the beginning of the 20th century. Its origins are intertwined with the study of algebraic structures and number theory. The concept of a ring was first implicitly introduced by Richard Dedekind in the context of algebraic number fields in the 1870s [34]. Dedekind's work on ideals in rings of integers laid the groundwork for the formal definition of rings. The term "ring" itself was coined by David Hilbert in the 1890s to describe certain sets of polynomial functions. In 1921 [64], Emmy Noether revolutionized the field by introducing the theory of ideals in commutative rings, providing a more systematic and abstract framework for ring theory. Her work was instrumental in shaping modern algebra. Throughout the 20th century, ring theory expanded rapidly, incorporating both commutative and non-commutative rings and finding applications in various areas of mathematics, including geometry, analysis, and topology.

This thesis combines Graph Theory and Ring Theory. Namely, we study certain graph properties in the context of some graphs associated with commutative rings. Specifically, we study the notions of global defensive $k$-alliance, partitioning into global defensive alliance, complementedness and uniquely complementedness in the context of zero-divisor graphs and extended zero-divisor graphs of rings (see Chapter 1 for the definitions). At the end of this thesis, we will present some proposals for future studies of certain graph-theoretic conjectures with a view to the zero-divisor graph theory.

The concept of the zero-divisor graph of a commutative ring was first introduced by Beck [17], to present the idea of coloring of a commutative ring, in order to establish a connection between graph theory and commutative ring theory, which turns out to be mutually beneficial for these two branches of mathematics. For a given commutative ring $R$, Beck's zero-divisor graph of $R$ is a simple graph with vertex set all elements of $R$, such that two distinct vertices $x$ and $y$ are adjacent if and only if $xy = 0$. In 1999, Anderson and Livingston defined a simplified version $\Gamma(R)$ of Beck's zero-divisor graph by including only nonzero zero-divisors of $R$ in the vertex set and leaving the definition of edges the same [8]. The reason for this simplification was to better capture the essence of the structure of the zero-divisors of the ring. Several properties of $\Gamma(R)$ have been investigated, such as connectedness, diameter, girth, chromatic number, domination number, etc. [3, 8, 66]. In addition, the isomorphism problem for such graphs has been solved for finite reduced rings [7]. Several authors have also investigated rings $R$ whose zero-divisor graph $\Gamma(R)$ belongs to a certain family of graphs, such as star graphs [3], complete graphs [8], complete $r$-partite graphs, and planar graphs [4, 84].

To understand well the concept of zero-divisor graphs, several authors have been interested in studying zero-divisor graphs of certain ring constructions. In [14], Axtell, Coykendall and Stickles studied the preservation of the diameter and girth of the zero-divisor graph of a ring under extensions to polynomial and power series rings. In [15], Axtell and Stickles investigated the preservation of the diameter and girth under idealizations. Specifically, they characterized the girth of the zero-divisor graph of an idealization and when it is complete. They also gave conditions for the zero-divisor graph with diameter 2. In [59], Maimani and Yassemi studied the diameter and girth of the zero-divisor graph of the amalgamated duplication of a ring along an ideal, and this work was extended by Kabbaj and Mimouni in [53] for amalgamations.

However, sometimes, the zero-divisor graph does not represent enough information to characterize its associated ring. This may be because it shows only the relationship between nonzero zero-divisors and not, for instance, the relationship between the power of nonzero zero-divisors. This was the purpose of introducing the extended zero-divisor graph, in 2016 by Bennis, Mikram, and Taraza [24, 25], which is a simple graph, denoted by $\overline{\Gamma}(R)$, with the same set of vertices as in the zero-divisor graph and two distinct vertices $x$ and $y$ are adjacent if and only if $x^n y^m = 0$ with $x^n \neq 0$ and $y^m \neq 0$ for some $n, m \in \mathbb{N}^*$. In this thesis, we continue the investigation begun in [24].

We refer the reader to [3, 5, 6, 7, 8, 10, 11, 14, 17, 21], for general background on the zero-divisor graph theory.

This thesis is organized as follows:

In Chapter 1, we recall some basic definitions and some interesting results on graph theory and zero-divisor graph theory that we use throughout this thesis.

In Chapter 2, we explore the notion of global defensive $k$-alliance in zero-divisor graphs of commutative rings. "Alliance" refers to a relationship or connection between individuals, families, states, or parties that share similar goals or characteristics. The graph-theoretic definition of alliances was first introduced by Kristiansen, Hedetniemi, and Hedetniemi in [57] to study the properties of real-world alliances. While they initially defined the notion by examining alliances between various nations, to defend each other or to attack a common enemy, the concept can be applied to other scenarios where a collection of similar elements is relevant. They define different types of alliances that have been extensively studied in the last decade. These types of alliances are defensive alliances [49], offensive alliances [72] and powerful alliances (or dual alliances)[30]. In this thesis, we are interested in the defensive alliances. Moreover, a generalization of the defensive alliance called the defensive $k$-alliance introduced by Shafique and Dutton [80, 81] has received special attention in recent years. It provides a method of classification of graphs based on relations between members of a particular set of vertices. In this thesis, we explore also this notion in zero-divisor graphs of commutative rings. Then, in Section 2.1,

we investigate the global defensive $k$-alliance of zero-divisor graphs of local rings. We start by extending [62, Proposition 2.2] which gives an upper bound on the cardinality of $Z(R)$ in terms of the global defensive alliance number of $\Gamma(R)$. Namely, we will give an upper bound on the cardinality of $Z(R)$ in terms of the global defensive $k$-alliance number (see Proposition 2.2). The established upper bound is more optimal than the one given in [62, Proposition 2.2] as shown by Examples 2.1 and Examples 2.3. Then, we are interested in studying the global defensive $k$-alliance number of the zero-divisor graph of a finite local ring. It seems not an easy task to determine it for any ring. However, as a second main result, we succeed in computing it for $\mathbb{Z}_{p^n}$ for a prime number $p$ and a positive integer $n$ (see Theorem 2.1). The global defensive $k$-alliance number of the zero-divisor graph of a finite local ring with a nilpotent maximal ideal of index 2 is also given (see Proposition 2.3). In Section 2.2, we compute the global defensive $k$-alliance number of the zero-divisor graph for some kind of direct products of finite fields. We start by determining the global defensive $k$-alliance number of the zero-divisor graph of a direct product of two finite fields (see Theorem 2.2) and as a particular case, we find again [62, Proposition 2.3]. Moreover, we get results for the global strong defensive alliance which is nothing but the global defensive 0-alliance (see Section 1.1.2). Determining the global defensive $k$-alliance for the direct product of finite fields $\prod_{i=1}^{n} F_i$ with $n \geq 3$ a positive integer and $F_i$ a finite field for every $i \in \{1, \ldots, n\}$ is still an open question. However, as main results we determine it for $\mathbb{Z}_2 \times \mathbb{Z}_2 \times F$ with $|F| \geq 2$, and for $\mathbb{Z}_2 \times F \times K$ with $|K| \geq |F| \geq 3$ (see Theorems 2.3 and 2.4). Section 2.3 is devoted to the study of the global defensive $k$-alliance number of the zero-divisor graph for a direct product of $\mathbb{Z}_2$ and a finite ring. We start by giving an upper and lower bound for the global defensive $k$-alliance number of $\Gamma(\mathbb{Z}_2 \times R)$ where $R$ is a finite ring (see Theorem 2.5). In [62, Proposition 2.4], Muthana and Mamouni established the equality $\gamma_a(\Gamma(\mathbb{Z}_2 \times R)) = \left\lceil \frac{|R|}{2} \right\rceil$ for a local ring $R$. Here, we give equalities for some integers $k \in [\![1 - |R|; 1]\!]$ (see Theorems 2.6 and 2.7). For a local ring $R$ with a nilpotent maximal ideal of index 2, we improve the inequality of Theorem 2.5 and give equality for the remaining cases other than the ones studied in Theorems 2.6 and 2.7.
Another well-studied problem in graph theory is the partitioning of the vertex set of a graph. It involves dividing the set of vertices of a graph into disjoint subsets or partitions, based on specific criteria or constraints. Diverse forms of vertex partitioning problems exist, each with unique objectives and applications. In Chapter 3, we focus on the task of partitioning the vertex set of the zero-divisor graphs of commutative rings into global defensive alliances. This problem has been previously studied, for general graphs, by Eroh and Gera [39, 40] and by Haynes and Lachniet [50]. Here, we connected it with the ring theoretical context. We characterize various commutative finite rings for which the zero divisor graph is partitionable into global defensive alliances. Namely, in Section 3.1, we study when zero-divisor graphs of some kind of finite local rings are partitionable into global defensive alliances and calculate their global defensive alliance partition numbers. We start by giving a lower bound on the cardinality of the set of zero-divisors $Z(R)$, for a commutative ring $R$, in terms of the global defensive alliance partition number of $\Gamma(R)$ (see Proposition 3.1). Proving that the zero-divisor graph can be partitionable into global defensive alliances for any ring appears difficult. However, as a second main result, we succeed in computing the global defensive alliance partition number for $\mathbb{Z}_{p^n}$ for a prime number $p$ and a positive integer $n$ (see Theorem 3.1). The global defensive alliance partition number of the zero-divisor graph of a finite local ring with a nilpotent maximal ideal of index 2 is also given (see Proposition 3.2). In Section 3.2, we study when the zero-divisor graph of some kind of a direct product of finite fields is partitionable into

global defensive alliances. We start by characterizing when a zero-divisor graph of $\mathbb{Z}_2 \times F$, for a finite field $F$, is partitionable into global defensive alliances(see Theorem 3.2). Next, we study when zero-divisor graphs of a direct product of two finite fields are partitionable into global defensive alliances (see Theorem 3.3). Next, we present when $\Gamma(\mathbb{Z}_2 \times \mathbb{Z}_2 \times F)$, for a finite field $F$, is partitionable into global defensive alliances. We end this section by showing that $\Gamma(\mathbb{Z}_2 \times F \times K)$, for finite fields $F$ and $K$ with $|F|, |K| \geq 3$, is not partitionable into global defensive alliances. In section 3.3, we study when the zero-divisor graph of the direct product of a finite field $F$ and a finite local ring $R$, which is not a field and such that its maximal ideal is nilpotent of index 2, is partitionable into global defensive alliances (see Theorems 3.6 et 3.7 ). In Section 3.4, we give complete characterizations for partitioning zero-divisor graphs of finite rings with $\gamma_a(\Gamma(R)) = 1, 2$. Namely, we prove that a zero-divisor graph, $\Gamma(R)$, of a finite ring $R$ with $\gamma_a(\Gamma(R)) = 1$ is partitionable into global defensive alliances if and only if $R$ is isomorphic to one of the rings $\mathbb{Z}_2 \times \mathbb{Z}_2$, $\mathbb{Z}_9$, $\mathbb{Z}_3[X]/(X^2)$ (see Theorem 3.8), and we prove that for a zero-divisor graph of a finite ring with $\gamma_a(\Gamma(R)) = 2$, $\Gamma(R)$ is partitionable into global defensive alliances if and only if $R$ is isomorphic to one of the rings $\mathbb{Z}_2 \times \mathbb{Z}_4$, $\mathbb{Z}_2 \times \mathbb{Z}_2[X]/(X^2)$, $\mathbb{Z}_3 \times \mathbb{Z}_3$, $\mathbb{Z}_3 \times \mathbb{F}_4$, $\mathbb{Z}_{25}$, $\mathbb{Z}_5[X]/(X^2)$ and $\mathbb{F}_4 \times \mathbb{F}_4$ (see Theorem 3.9 and Corollary 3.9).

In Chapter 4, we deal with complementedness and uniquely complementedness notions of graphs. These notions were initially introduced for a general graph in [9, 58] and subsequently investigated within the framework of zero-divisor graphs of commutative rings. In this thesis, we explore these notions in the context of the extended zero-divisor graphs of commutative rings. We continue the investigation begun in [24] to further study when the extended zero-divisor graph of a commutative ring is complemented and when it is uniquely complemented. Namely, in Section 4.1, we study when the extended zero-divisor graph of a commutative ring is complemented. We start by showing that, if $\overline{\Gamma}(R)$ is complemented such that $|Z(R)| \geq 4$, then the ring $R$ has at most one nonzero nilpotent element (see Theorem 4.1 and Examples 4.2, 4.3 and 4.4). When $R$ is finite, we get the converse of Theorem 4.1 (see Corollary 4.2). In fact, this is a consequence of the characterization of finite rings with complemented extended zero-divisor graphs (see Theorem 4.2). In Section 4.2, we show as a main result that, when $\Gamma(R) \neq \overline{\Gamma}(R)$, the complementedness and the uniquely complementedness notions coincide (see Theorem 4.3). In Section 4.3, we show that, when $\Gamma(R) \neq \overline{\Gamma}(R)$, the total quotient ring $T(R)$ of $R$ is zero-dimensional once $\Gamma(R)$ is complemented. (see Theorem 4.4). The proof of this result leads us to show that when $\overline{\Gamma}(R)$ is complemented, every non-nilpotent element has an orthogonal which is not nilpotent (see Theorem 4.5). Also, if $\overline{\Gamma}(R)$ is complemented such that $\overline{\Gamma}(R) \neq \Gamma(R)$, then every orthogonal to the unique nonzero nilpotent element cannot be an end (see Corollary 4.4). At the end of this section we prove that, for any ring $R$ such that $|\mathrm{Nil}(R)| = 2$, $R$ is not local or $\overline{\Gamma}(R)$ is not complemented (see Proposition 4.1). Finally, Section 4.4 is devoted to the study of when the extended zero-divisor graph of a finite direct product of rings as well as the one of an idealization of an $R$-module are complemented (see Theorems 4.6, 4.7 and 4.8, and Proposition 4.2).

In Appendix A, we present some interesting graph theory conjectures that remain unsolved. Namely, the cycle double cover conjecture, Hadwiger's conjecture, Hedetniemi's conjecture and Vizing's conjecture. Thus, one perspective is to investigate these conjectures within zero-divisor graph theory.

# Chapter 1

# Preliminaries

In this chapter, we recall some basic definitions and classical results from both graph and zero-divisor graph theories.

## 1.1 Graph Theory

Graph Theory is the study of graphs as mathematical concepts used to represent connections between entities. Its reach extends far beyond the realm of theoretical mathematics, offering practical utility in solving real-world problems. Indeed, by visually mapping and exploring complex relationships between components, graph theory provides invaluable insights and effective solutions in a range of domains, from transportation and urban planning to social networks and computational sciences [36, 43, 60, 63, 65, 75, 83].

The roots of graph theory date back to the 18th century, with its inception attributed to the pioneering work of the Swiss mathematician Leonhard Euler, often hailed as the progenitor of the field. Euler's monumental achievement in 1736 [26], when he famously tackled the Seven Bridges of Königsberg problem, served as a powerful demonstration of graph theory's effectiveness in solving concrete puzzles. Over time, a wide variety of mathematicians and scientists have continued to develop the theory, refining its principles and expanding its applications.

Graph theory revolves around the study of graphs, which are made up of vertices (also known as nodes) and edges (connections between nodes). These nodes and edges can represent a wide range of entities and their relationships. The power of graph theory lies in its ability to capture complex interactions and represent them in a simplified but meaningful way.

Graph theory finds myriad essential applications across various domains, including:
**Transportation:** Graph theory has many applications in the modeling of transportation systems, whether road networks, air routes or public transport systems. By representing these systems as graphs, analysts can optimize traffic flows, identify bottlenecks and plan new transport routes. Graph theory algorithms can determine the shortest paths between two points, facilitating navigation systems and optimizing logistics operations.

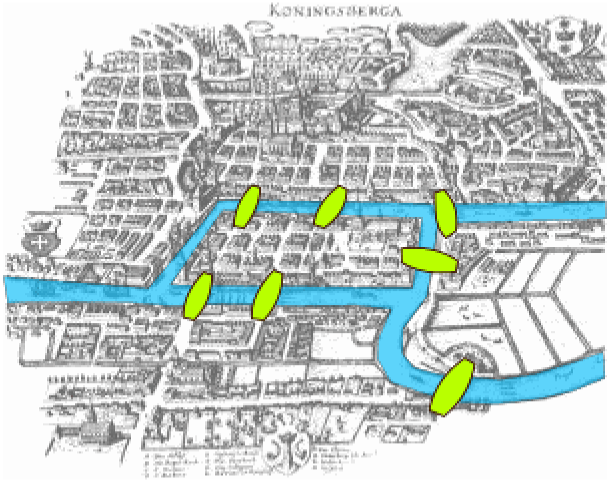



Figure 1.1: Map of Königsberg at the time of Euler showing the current layout of the seven bridges, with the Pregel and the bridges marked(https://en.wikipedia.org/wiki/Seven_Bridges_of_Königsberg).

**Scheduling:** Scheduling problems, such as task allocation or meeting planning, can be handled efficiently using graph theory. By modeling these scenarios as graphs, optimal solutions can be derived, guaranteeing efficient resource allocation and time management. Graph theory algorithms help to find the most efficient way of assigning tasks to workers, minimizing downtime and maximizing productivity.

**Social Networks:** With the advent of social media platforms, graph theory has become increasingly important for modeling social networks. By representing individuals as nodes and their connections as edges, graph theory enables the study of information diffusion, the identification of influential individuals and recommendation systems. Platforms such as Facebook and Twitter rely on graph theory to suggest friends, promote engagement and analyze social dynamics.

**Computer Science:** Graph theory forms the backbone of many fundamental algorithms in computer science. Shortest path algorithms, such as Dijkstra's algorithm, help determine the most efficient routes in network optimization. Minimum spanning tree algorithms find applications in network design and clustering problems. Maximum flow algorithms help with resource allocation and capacity planning. The versatility of graph theory makes it an indispensable tool for solving complex computing problems.

**Other Applications:** Beyond its influence on transportation, planning, social networks and computer science, graph theory has applications in many other fields. In biology, graphs are used to model genetic interactions and protein networks. In chemistry, they represent molecular structures and chemical reactions. In economics, graphs are used to analyze supply and demand networks and market dynamics. In physics, graphs model complex systems and their interconnections, enabling the study of complex phenomena.

In the following sections of this chapter, we recall some basic notions and some classical results on graph theory, which are used throughout this thesis. We refer the reader to [26, 27, 28, 31, 42], for general background on graph theory.

### 1.1.1 Simple graphs

In this section, we recall some basic definitions and notions in graph theory. First, let us fix some notations:

- $\mathbb{N} := \{0, 1, 2, 3, \dots\}$ is the set of non-negative integer.
- $|S|$ means the size (or the cardinal) of a set $S$.
- $\mathcal{P}(S)$, the **powerset** of $S$, which is the set of all subsets of $S$.
- Let $k \in \mathbb{N}$, we denote $\mathcal{P}_k(S)$, the $k$-**element subsets** of $S$. For instance,

$$\mathcal{P}_2(\{v_1, v_2, v_3\}) = \{\{v_1, v_2\}, \{v_1, v_3\}, \{v_2, v_3\}\}.$$

#### Definitions

The graphs we consider in this thesis are "simple graphs".

**Definition 1.1** ([42]). *A **simple graph** is an ordered pair $(V, E)$, where $V$ is a (finite) set of elements and $E$ is a set of 2-subsets of $V$ (i.e., $E \subseteq \mathcal{P}_2(V)$).*

**Example 1.1.** *The ordered pair*

$$(V, E) = (\{v_1, v_2, v_3, v_4, v_5\}, \{\{v_1, v_3\}, \{v_1, v_4\}, \{v_3, v_4\}\}, \{v_3, v_5\}\})$$

*is a simple graph. Indeed,*

$$\mathcal{P}_2(V) = \{\{v_1, v_2\}, \{v_1, v_3\}, \{v_1, v_4\}, \{v_1, v_5\}, \{v_2, v_3\}, \{v_2, v_4\}, \{v_2, v_5\}, \{v_3, v_4\}, \{v_4, v_5\}\}.$$

*and $E \subset \mathcal{P}_2(V)$.*

**Example 1.2.** *Let $n \in \mathbb{N}^*$. The ordered pair $(V, E)$, where $V := \{1, 2, \dots, n\}$ and $E := \{\{u, v\} \in \mathcal{P}_2(V) | gcd(u, v) = 1\}$, is a simple graph denoted by $Cop_n$, which is called the $n$-**th coprimality graph**.*

In the following we will use "graph" as shorthand for "simple graph".

**Definition 1.2** ([42]). *Let $G = (V, E)$ be a graph.*

- *$V$ is called the **vertex set** of $G$ and it is denoted by $V(G)$. Its elements are called **vertices** (or the **nodes**) of $G$.*
- *$E$ is called the **edge set** of $G$ and it is denoted by $E(G)$. Its elements are called **edges** of $G$. For $u, v \in V(G)$, we shall often use the notation $u - v$ for $\{u, v\}$.*
- *Let $u, v \in V(G)$. Then, $u$ and $v$ are said to be **adjacent** if $u - v \in E(G)$. In other words, the edge $u - v$ is said to **join** $u$ and $v$, and $u$ and $v$ are called the **endpoints** of the edge $u - v$. $u$ and $v$ are said to be **non-adjacent** if $u - v \notin E(G)$.*
- *Let $v \in V(G)$. Then, the **open-neighborhood** of $v$ in $G$, denoted by $N_G(v)$ (or simply by $N(v)$), is the set $\{u \in V(G) | \, u - v \in E(G)\}$ and the **closed-neighborhood** of $v$, denoted by $N_G[v]$ (or simply by $N[v]$), is the open-neighborhood including the vertex $v$ itself. The elements of $N(v)$ are called the **neighbors** of $v$. More generally, if $S$ is a nonempty subset of $V$, then, its open-neighborhood $S$ is $N(S) := \cup_{x \in S} N(x)$ and its closed-neighborhood is $N[S] := N(S) \cup S$.*

**Example 1.3.** *Let $G$ be the graph, from Example 1.1,*

$$(\{v_1, v_2, v_3, v_4, v_5\}, \{\{v_1, v_3\}, \{v_1, v_4\}, \{v_3, v_4\}, \{v_3, v_5\}\}).$$

- *The vertex set:* $V(G) = \{v_1, v_2, v_3, v_4, v_5\}$.
- *The edge set:* $E(G) = \{\{v_1, v_3\}, \{v_1, v_4\}, \{v_3, v_4\}, \{v_3, v_5\}\}$.
- *The vertices $v_1$ and $v_3$ are adjacent since $v_1 - v_3 \in E(G)$.*
- *The vertices $v_1$ and $v_2$ are non-adjacent since $v_1 - v_2 \notin E(G)$.*
- *The endpoints of the edge $v_3 - v_5$ are $v_3$ and $v_5$.*
- $N(v_3) = \{v_1, v_4, v_5\}$ *and* $N[v_3] = \{v_1, v_3, v_4, v_5\}$.
- *The neighbors of $v_3$ are $v_1$, $v_4$ and $v_5$.*

### Drawing graphs

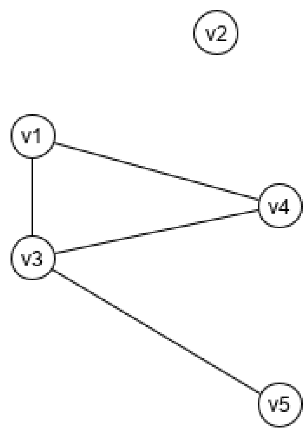
A simple graph can be represented by a collection of points in the plane together with a set of curves connecting specific pairs of these points. To clarify, we have the following definition:

**Definition 1.3** ([42]). *A simple graph $G$ can be illustrated by* ***drawing*** *it on the plane. Then, every vertex of $G$ can be represented as a point (at which we put the name of the vertex), and each edge $u - v$ of $G$ can be drawn as a curve connecting the point that represents $u$ with the point that represents $v$. The arrangement of the points and the shapes of curves can be selected freely, as long as they allow the reader to unambiguously reconstruct the graph $G$ from the picture.*

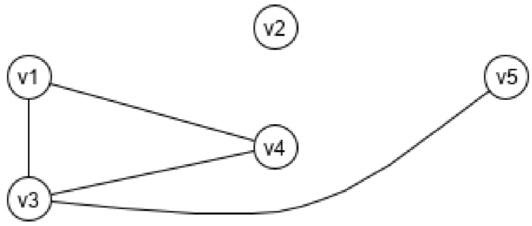
**Example 1.4.** *Let $G$ be the simple graph, from Example 1.1,*

$$(\{v_1, v_2, v_3, v_4, v_5\}, \{\{v_1, v_3\}, \{v_1, v_4\}, \{v_3, v_4\}\}, \{v_3, v_5\}\}).$$

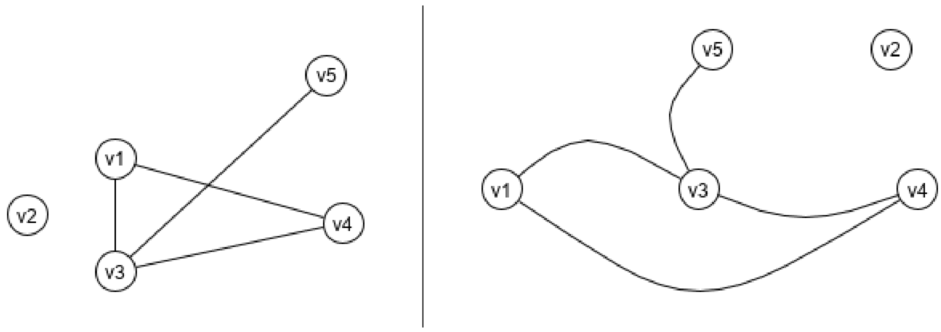
*This graph can be drawn in the following way:*




*The edges, in this representation, are represented by straight lines (it is the simplest way to draw this simple graph). Also, we can draw it in several other ways, as follows:*




*Here, we have arranged the points that represent the vertices $v_1, v_2, v_3, v_4, v_5$ in a different way. We cannot draw the edge $v_3 - v_5$ as a straight line because it will overlap the vertex $v_4$, making the graph ambiguous (the edge $v_3 - v_5$ can be mistaken for two edges $v_3 - v_4$ and $v_4 - v_5$).*
*Below are two more drawings for this simple graph:*




### Degrees

**Definition 1.4** ([42]). *Let $G$ be a graph and $v \in V(G)$. Then, the* ***degree*** *of $v$ in $G$, denoted by $deg_G(v)$ (or simply by $deg(v)$ if there is no ambiguity), is the number of neighbors of $v$. Namely,*

$$deg_G(v) := |N_G(v)|.$$

*More generally, for every nonempty subset $S \subseteq V(G)$ and every vertex $v \in V(G)$, the degree of $v$ over $S$, denoted by $deg_S(v)$, is defined as $deg_S(v) := |S \cap N(v)|$. So, in particular, $deg_{V(G)}(v) = deg_G(v) = deg(v)$.*

In the graph of Example 1.1, the vertices have degrees:

$$deg(v_1) = 2, \quad deg(v_3) = 3, \quad deg(v_4) = 2, \quad deg(v_2) = 0, \quad deg(v_5) = 1.$$

In the following, we recall some basic results concerning degrees in simple graphs.

**Proposition 1.1** ([42]). *Let $G$ be a graph with $n$ vertices and $v \in V(G)$. Then,*

$$deg(v) \in \{0, 1, \ldots, n-1\}.$$

The following result shows that the sum of the degrees of all vertices of a graph is twice the number of its edges.

**Proposition 1.2** (Euler 1736). *Let $G$ be a graph. Then,*

$$\sum_{v \in V(G)} deg(v) = 2|E(G)|.$$

**Corollary 1.1** (handshake lemma)**.** *Let $G$ be a graph. Then, the number of vertices of $G$ whose degree is odd is even.*

This consequence (i.e., Corollary 1.1) is often expressed in the following way: The number of people with an odd number of friends is even in a group of people.

**Proposition 1.3** ([42])**.** *Let $G$ be a graph such that $|V(G)| \geq 2$. Then, there exist $v, u \in V(G)$ such that*

$$deg(u) = deg(v).$$

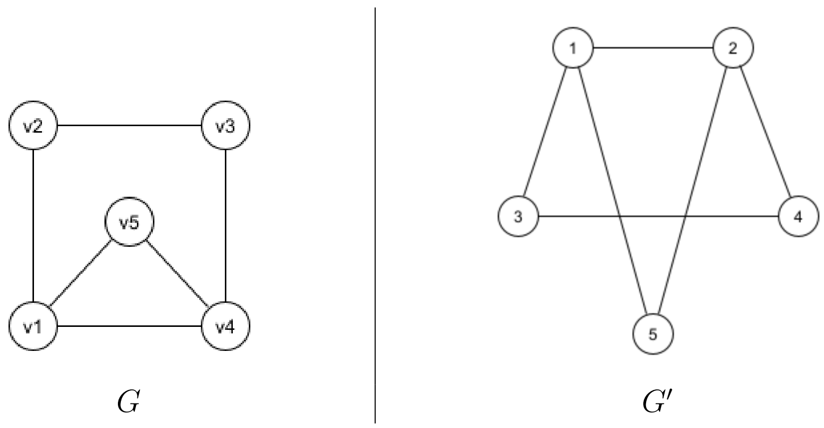
### Graph isomorphism

**Definition 1.5** ([42])**.** *Let $G$ and $G'$ be two graphs. A* ***graph isomorphism*** *from $G$ to $G'$ is a bijection $\theta : V(G) \to V(G')$ that preserves adjacency and non-adjacency, i.e., for every $u, v \in V(G)$,*

$$u - v \in E(G) \Leftrightarrow \theta(u) - \theta(v) \in E(G').$$

*$G$ and $G'$ are said to be* ***isomorphic****, and write $G \cong G'$, if there exists a graph isomorphism from $G$ to $G'$.*

**Example 1.5.** *Consider the following illustrated graphs:*




*Let define a map $\theta : V(G) \to V(G')$ as follows:*

$$\theta : \begin{pmatrix} v_1 & v_2 & v_3 & v_4 & v_5 \\ 1 & 3 & 4 & 2 & 5 \end{pmatrix}$$

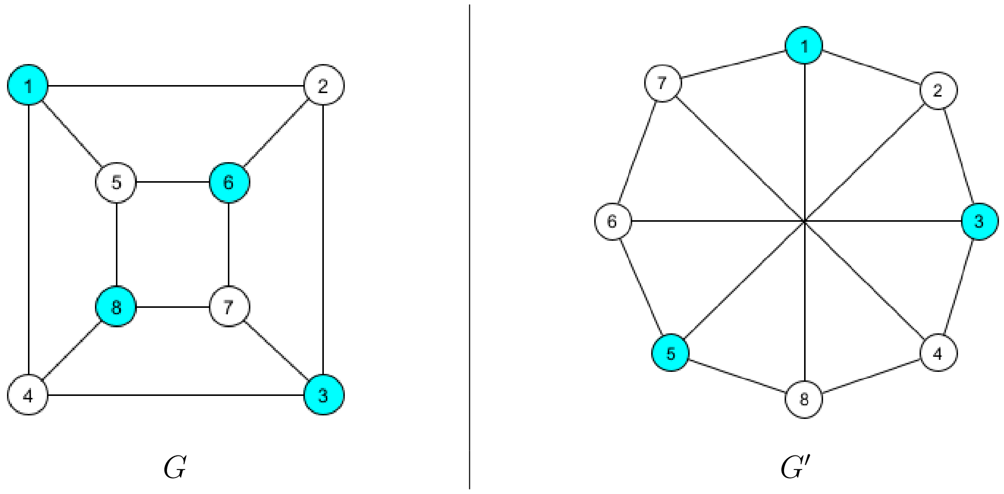
*Then, $\theta$ is onto since each vertex in $G'$ gets mapped to, and it is also one-to-one because no two vertices of $G$ map to the same vertex of $G'$. Thus, $\theta$ is a bijection. On the other hand, $\theta$ preserves adjacency and non-adjacency as follows:*

$$\begin{gathered} v_1 - v_2 \in E(G) \to \theta(v_1) - \theta(v_2) = 1 - 3 \in E(G') \\ v_2 - v_3 \in E(G) \to \theta(v_2) - \theta(v_3) = 3 - 4 \in E(G') \\ \dots \\ v_4 - v_5 \in E(G) \to \theta(v_4) - \theta(v_5) = 2 - 5 \in E(G') \\ v_1 - v_3 \notin E(G) \to \theta(v_1) - \theta(v_3) = 1 - 4 \notin E(G') \\ \dots \\ v_3 - v_5 \notin E(G) \to \theta(v_3) - \theta(v_5) = 4 - 5 \notin E(G'). \end{gathered}$$

*Then, $\theta$ is a graph isomorphism from $G$ to $G'$. Therefore, $G \cong G'$.*

An example of graphs that are not isomorphic to each other is shown below.

**Example 1.6.** *Consider the following two simple graphs:*




*These two graphs are not isomorphic (i.e., $G \ncong G'$) because the graph $G$ has four mutually non-adjacent vertices but the graph $G'$ does not have any set of four mutually adjacent vertices.*

In the following, we give some basic properties of graph isomorphisms.

**Proposition 1.4** ([42]). *Consider two simple graphs, $G$ and $G'$. If $\theta$ is a graph isomorphism from $G$ to $G'$, then the inverse of $\theta$ is also a graph isomorphism, but this time from $G'$ to $G$*

**Proposition 1.5** ([42]). *Consider three simple graphs, $G$, $G'$ and $G''$. If $\theta$ is a graph isomorphism from $G$ to $G'$ and $\sigma$ is a graph isomorphism from $G'$ to $G''$, then $\sigma \circ \theta$ is a graph isomorphism from $G$ to $G''$.*

From Proposition 1.4 and Proposition 1.5, we can easily show that the relation "$\cong$", on the class of all simple graphs, is an equivalence relation.

Graph isomorphisms preserve all characteristics of a graph. For instance:

**Proposition 1.6** ([42]). *Consider two graphs $G$ and $G'$. Let $\theta$ be a graph isomorphism from $G$ to $G'$. We have the following characteristics:*

- $deg_G(v) = deg_{G'}(\theta(v))$ *for every* $v \in V(G)$.
- $|E(G')| = |E(G)|$ *and* $|V(G')| = |V(G)|$.

Graph isomorphisms can be employed to rename the nodes of a graph. As an illustration, we can rename the nodes of an $n$-vertex graph to be $1, 2, \ldots, n$, or any set of $n$ distinct entities.

**Proposition 1.7** ([42]). *If $G$ is a graph and $S$ is a (finite) set such that $|S| = |V(G)|$, then there exists a graph $G'$ which is isomorphic to $G$ and whose vertex set $V(G') = S$*

### Some families of graphs

In this subsection, we set some particularly significant families of simple graphs. We start with the definitions of complete graphs and empty graphs.

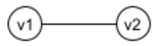
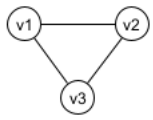
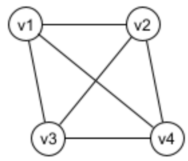
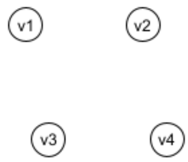
#### *Complete graph and empty graph*

The complete graphs and the empty graphs are the simplest families of graphs:

**Definition 1.6** ([42]). *1. A* ***complete graph****, denoted by $K_n$, on $n$ vertices is a graph with an edge between any pair of vertices.*

*2. An* ***empty graph*** *is a graph without edges (i.e., the edge set is empty).*

**Example 1.7.** *This example shows the complete graphs $K_0$, $K_1$, $K_2$, $K_3$ and $K_4$, and the empty graphs with 0, 1, 2, 3 and 4 (respectively) vertices.*

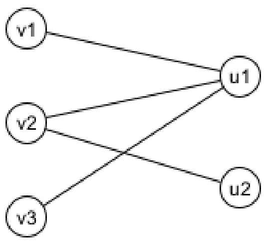
| | $K_0$ | $K_1$ | $K_2$ | $K_3$ | $K_4$ |
|---|---|---|---|---|---|
| *Complete graphs* | |  |  |  |  |
| *Empty graphs* | $(\emptyset, \emptyset)$ | $(\{v_1\}, \emptyset)$  | $(\{v_1, v_2\}, \emptyset)$  | $(\{v_1, v_2, v_3\}, \emptyset)$  | $(\{v_1, v_2, v_3, v_4\}, \emptyset)$  |

**Proposition 1.8** ([42]). *Let $G$ be a graph. Then, $G$ is a complete graph with $n$ vertices (i.e., $|V(G)| = n$ and $E(G) = \mathcal{P}_2(V(G))$) if and only if $G$ is isomorphic to $K_n$.*

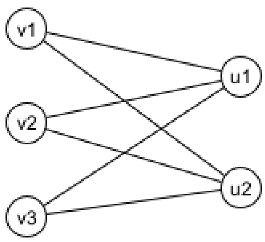
#### *Bipartite graph*

**Definition 1.7** ([28]). *A* ***bipartite graph*** *is a graph whose vertex set can be partitioned into two sets $X$ and $Y$ such that every edge $u - v \in E$ has $u \in X$ and $v \in Y$. If it has every possible edge between the two sets of vertices, then it is called a* ***complete bipartite graph****, and it is denoted by $K_{n,m}$ where $|X| = n$ and $|Y| = m$. A complete bipartite graph is called a* ***star graph*** *if either $|X| = 1$ or $|Y| = 1$.*

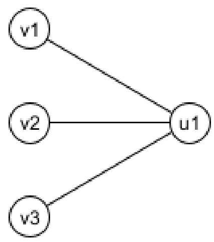
**Example 1.8.** *Here, we give an example of a bipartite graph, a complete bipartite graph, and a star graph.*




*Bipartite graph*




*Complete bipartite graph*




*Star graph*

***Path graph and cycle graph***

**Definition 1.8** ([42])**.** *Let $n \in \mathbb{N}^*$.*

- *The $n$-**th path graph**, denoted by $P_n$, is the simple graph defined as follows:*

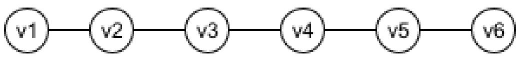
$$(\{v_1, v_2, \ldots, v_n\}, \{v_i - v_{i+1} | 1 \leq i \leq n-1\}).$$

- *The $n$-**th cycle graph**, denoted by $C_n$, is the simple graph defined as follows:*

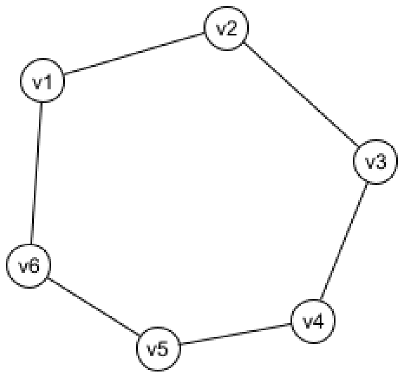
$$(\{v_1, v_2, \ldots, v_n\}, \{v_i - v_{i+1} | 1 \leq i \leq n-1\} \cup \{v_n - v_1\}).$$

Note that, the $n$-th path graph has $n$ vertices and $n-1$ edges, and the $n$-th cycle graph has $n$ vertices and $n$ edges (unless the case $n = 2$, it has just one edge).

**Example 1.9.** *The 6-th path graph ($P_6$) and the 6-th cycle graph ($C_6$) are illustrated as follows:*




*Path graph ($P_6$)*




*Cycle graph ($C_6$)*

***Connected graph and regular graph***

Before we define a connected graph let us recall two of the most fundamental features that graphs can have which are Walks and Paths.

**Definition 1.9** ([42])**.** *Let $G$ be a graph.*

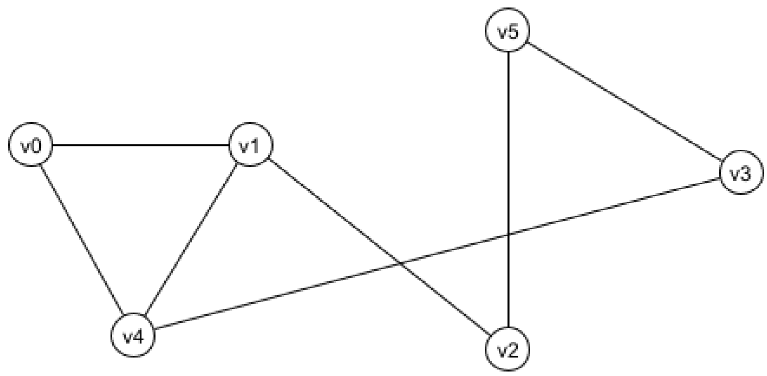
- *A **walk** of $G$ is a (finite) sequence $(v_0, v_1, \ldots, v_n)$ of vertices of $G$ where $v_0 - v_1$, $v_1 - v_2$, ..., $v_{n-1} - v_n \in E(G)$.*
- *Let $\boldsymbol{w} := (v_0, v_1, \ldots, v_n)$ be a walk of $G$, then*
  - *$v_0, v_1, \ldots, v_n$ are the vertices of $\boldsymbol{w}$.*
  - *$v_0 - v_1, v_1 - v_2, \ldots, v_{n-1} - v_n$ are the edges of $\boldsymbol{w}$. The number of edges of $\boldsymbol{w}$, counted with multiplicity, is $n$.*
  - *$n$ is called the **length** of $\boldsymbol{w}$. The number of all vertices of $\boldsymbol{w}$, counted with multiplicity, is one greater than $n$.*
  - *$v_0$ is the **starting point** of $\boldsymbol{w}$(we say, $\boldsymbol{w}$ **starts** (or **begins**) at $v_0$).*
  - *$v_n$ is the **ending point** of $\boldsymbol{w}$(we say, $\boldsymbol{w}$ **ends** at $v_n$).*

- *A **path** of $G$ is a walk of $G$ with distinct vertices. Namely, it is a walk $(v_0, v_1, \ldots, v_n)$ where $v_0, v_1, \ldots, v_n$ are distinct.*
- *Let $v_i, v_j \in V(G)$. A **walk from** $v_i$ **to** $v_j$ means a walk that starts at $v_i$ and ends at $v_j$.*

**Example 1.10.** *Let $G$ be the graph*

$$(\{v_0, v_1, v_2, v_3, v_4, v_5\}, \{\{v_0, v_1\}, \{v_1, v_2\}, \{v_1, v_4\}, \{v_2, v_5\}, \{v_4, v_3\}, \{v_3, v_5\}\}),$$

*illustrated as follows:*

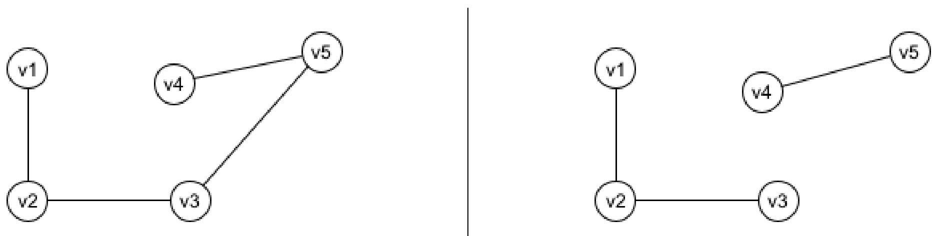



Then,

- The sequence $(v_0, v_1, v_4, v_0, v_1, v_2, v_5, v_3)$ is a walk in $G$, from $v_0$ to $v_3$, of length 7. It is not a path.
- The sequence $(v_1, v_2, v_0, v_4)$ is not a walk because $v_2 - v_0 \notin E(G)$.
- The sequence $(v_1, v_4, v_3, v_5, v_2, v_1)$ is a walk, from $v_1$ to $v_1$, of length 5. But, it is not a path (the starting point and the ending point are the same).
- The sequence $(v_3)$ is a walk, from $v_3$ to $v_3$, of length 0, and it is also a path. In general, each vertex $v$ of a simple graph gives rise to a path of length 0, $(v)$.
- The sequence $(v_3, v_4)$ is a walk, from $v_3$ to $v_4$, of length 1, and it is also a path. In general, each edge $u - v$ of a simple graph gives rise to a path of length 1, $(u, v)$.

**Definition 1.10** ([28])**.** *A graph $G$ is **connected** if, for every pair of vertices $u$ and $v$ in $V(G)$, there exists a path from $u$ to $v$. $G$ is **disconnected** if it is not connected.*

**Example 1.11.** *This example illustrates a connected graph (left) and a disconnected graph (right).*

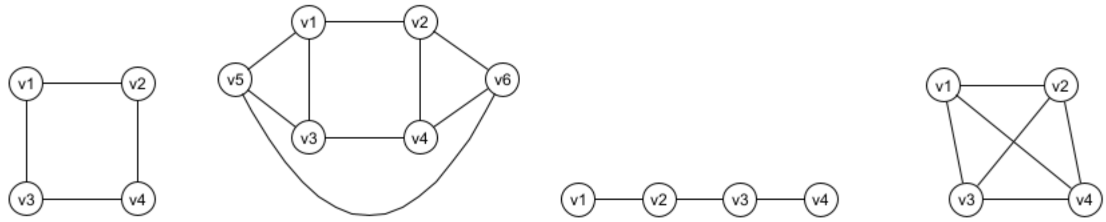



*Connected graph* | *Disconnected graph*

Now, we give the definition of a regular graph.

**Definition 1.11** ([28]). *Let $r \in \mathbb{N}$. A graph $G$ is said to be $r$-**regular** if $deg_G(v) = r$ for every $v \in V(G)$.*

**Example 1.12.**




*$C_4$ is 2-regular* *3-regular (cubic)* *$P_4$ is not regular* *$K_4$ is 3-regular(cubic)*

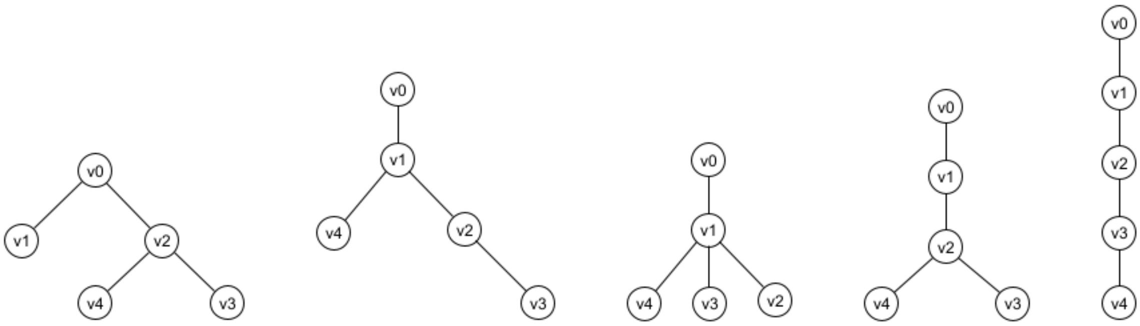
Note that cycles are 2-regular graphs, the complete graph $K_n$ is $(n-1)$-regular graph, and the path graphs, with lengths greater than 2, are not regular.

***Forests and Trees***

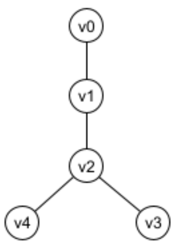
**Definition 1.12** ([28]). *A **tree** is a connected **acyclic** graph, while an acyclic graph is a graph without cycles.*
*A **leaf** (or **terminal vertex**) is a vertex with degree 1 in a tree.*
*An **internal vertex** (or **inner vertex**) is a vertex with degree at least 2 in a tree.*
*A **branch vertex** is a vertex with degree at least 3 in a tree.*
*A **trivial tree** is a graph with just one vertex.*

According to these definitions, every component of an acyclic graph is a tree. For this reason acyclic graphs are commonly referred to as **forests** (i.e., a forest is a collection of trees).

**Example 1.13.** *Here, we give some examples of trees on five vertices:*




If we consider the following tree:



Then,

- The three vertices $v_0$, $v_3$ and $v_4$ are leaves.
- The two vertices $v_1$ and $v_2$ are internal vertices (or inner vertices).
- The vertex $v_2$ is a branch vertex.

Now, we recall some interesting results related to trees.

**Proposition 1.9** ([28]). *Each pair of vertices of a tree $T$ has a unique path that connects them.*

**Proposition 1.10** ([28]). *There are at least two leaves in every non-trivial tree.*

**Proposition 1.11** ([28]). *If $T$ is a tree, then*

$$|E(T)| = |V(T)| - 1.$$

### The diameter and the girth

**Definition 1.13** ([28]). *Let $u, v \in V(G)$. The **distance** between $u$ and $v$, denoted $d(u, v)$, is the length of the shortest path from $u$ to $v$.*
*If there is no path between $u$ and $v$, then the distance between them is defined to be $\infty$ (i.e., $d(u, v) = \infty$).*

**Definition 1.14** ([28]). *The **diameter** of a graph $G$ is denoted by $diam(G)$ and it is defined as follows:*

$$diam(G) := \max\{d(u, v)|\ u, v \in V(G)\}.$$

**Example 1.14.** *In the graph $G$ in Example 1.10 we have:*

- $d(v_i, v_i) = 0$ *for* $i = 0, \ldots, 5$,
- $d(v_0, v_1) = d(v_0, v_4) = d(v_1, v_4) = d(v_1, v_2) = d(v_2, v_5) = d(v_3, v_5) = d(v_3, v_4) = 1$,
- $d(v_0, v_2) = d(v_0, v_3) = d(v_1, v_3) = d(v_1, v_5) = d(v_2, v_4) = d(v_4, v_5) = 2$,
- $d(v_0, v_5) = 3$.

*Thus, $diam(G) = 3$.*

We need to recall the following definition before defining the girth of a graph.

**Definition 1.15.** *Let $G$ be a graph. Then,*

1. *A **closed walk** of $G$ is a walk $(v_0, v_1, \ldots, v_{n-1}, v_n)$, where $n$ is a non-negative integer, such that the vertex $v_0$ is identical to the vertex $v_n$.*
2. *A **cycle** of $G$ is a closed walk $(v_0, v_1, \ldots, v_{n-1}, v_n)$ such that $n \geq 3$ and the vertices $v_0, v_1, \ldots, v_{n-1}$ are distinct.*

**Example 1.15.** *In the graph of Example 1.10.*

- $(v_1, v_4, v_3, v_5, v_2, v_1)$, $(v_1, v_4, v_3, v_5, v_2, v_1, v_0, v_4, v_1)$ *and* $(v_0, v_4, v_1, v_0)$ *are closed walks.*

- $(v_1, v_4, v_3, v_5, v_2, v_1)$, $(v_0, v_4, v_1, v_0)$ *and* $(v_4, v_3, v_5, v_2, v_1, v_0, v_4)$ *are cycles.*

**Definition 1.16** ([28]). *Let* $G$ *be a graph. Then, The* ***girth*** *of* $G$*, denoted by* $girth(G)$*, is the length of the shortest cycle of* $G$*. If there are no cycles in the graph* $G$*, then the girth is defined to be equal to* $\infty$ *(i.e.,* $girth(G) = \infty$*).*

**Example 1.16.**
- *In the graph of Example 1.10,* $(v_0, v_4, v_1, v_0)$ *is a shortest cycle. Then, the girth of this graph is* $3$*.*
- *The girth of star graphs and path graphs is* $\infty$*.*
- *The girth of* $K_{n,m}$*, where* $n, m \geq 2$*, is* $4$*.*
- *The girth of* $C_n$ *is* $n$*.*
- *The girth of* $K_n$*, where* $n \geq 3$*, is* $3$*.*

### Subgraphs

**Definition 1.17** ([42]). *Let* $G$ *be a graph.*

- *A graph* $G'$ *is said to be a* ***subgraph*** *of* $G$ *if* $V(G') \subseteq V(G)$ *and* $E(G') \subseteq E(G)$*.*
- *A graph* $G'$ *is an* ***induced subgraph*** *of* $G$ *if* $G'$ *is a subgraph* $G$ *such that* $E(G') = E(G) \cap \mathcal{P}_2(V(G'))$*.*

The induced subgraphs can be characterized as follows:

**Proposition 1.12** ([42]). *Let* $G$ *be a graph and* $G'$ *be a subgraph of* $G$*. Then,* $G'$ *is an induced subgraph of* $G$ *if and only if every edge* $u - v$ *of* $G$*, where* $u, v \in V(G')$*, is an edge of* $G'$*.*

**Example 1.17.** *Let* $n \geq 3$ *be an integer. Then,*

- $P_n$ *is a subgraph of* $C_n$*.*
- $P_n$ *is not an induced subgraph of* $C_n$ *since it contains the two vertices* $v_1$ *and* $v_n$ *of* $C_n$ *but* $v_1 - v_n \notin E(P_n)$*.*
- $P_{n-1}$ *is an induced subgraph of* $P_n$*.*
- $C_{n-1}$ *is not a subgraph of* $C_n$ *because the edge* $v_{n-1} - v_1$ *belongs to* $C_{n-1}$ *but not to* $C_n$*.*

## 1.1.2 Global defensive alliances in graphs

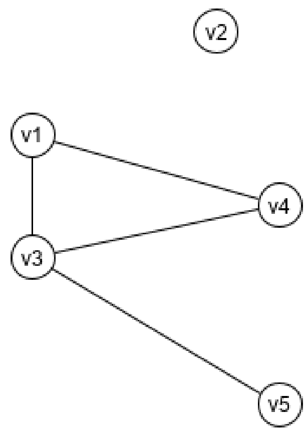
The graph-theoretic definition of alliances was first introduced by Hedetniemi, et. al. [57] to study the properties of real-world alliances. They defined different types of alliances that have been extensively studied in the last decade. This section presents one type of these alliances. Namely, the global defensive alliances and their variants as well as the parameters associated with them.

### The global defensive $k$-alliances in graphs

We start this subsection with the definition of a dominating set.

**Definition 1.18** ([28]). *Let $G$ be a graph and $S$ be a subset of $V(G)$. Then, $S$ is a* ***dominating set*** *if its closed neighborhood equals $V(G)$. The* ***domination number*** *of $G$ is the minimal cardinality among all dominating sets of $G$. It is denoted by $\gamma(G)$.*

**Example 1.18.** *Consider the graph $G$ in Example 1.1:*




- $\{v_1, v_4, v_2\}$, $\{v_1, v_5, v_4\}$ *and* $\{v_2, v_5\}$ *are not dominating sets.*
- $\{v_1, v_2, v_3, v_4, v_5\}, \{v_1, v_2, v_3, v_5\}, \{v_1, v_2, v_5\}$ *and* $\{v_2, v_3\}$ *are dominating sets.*
- $\gamma(G) = 2$.

Now, we give the definition of a global defensive alliance in a graph.

**Definition 1.19** ([57, 49]). *Let $G$ be a graph and $S$ be a non-empty subset of $V(G)$. Then,*

- *$S$ is said to be a* ***defensive alliance*** *if for every $x \in S$, $|N[x] \cap S| \geq |N(x) \cap \bar{S}|$, or equivalently, if $deg_S(x) + 1 \geq deg_{\bar{S}}(x)$ for every $x \in S$, where $\bar{S} = V \setminus S$.*
- *$S$ is said to be* ***strong defensive alliance*** *(it is also known as a* ***cohesive set*** *(see [79])) if for every vertex $x \in S$, $|N[x] \cap S| > |N(x) \cap \bar{S}|$, or equivalently, if $deg_S(x) \geq deg_{\bar{S}}(x)$ for every $x \in S$. In this case, we say that every vertex in $S$ is strongly defended.*
- *$S$ is said to be* ***global defensive alliance*** *if it is a defensive alliance and it forms a dominating set.*
- *The minimum cardinality among all global defensive alliances in $G$ is called the* ***global defensive alliance number*** *of $G$, and it is denoted by $\gamma_a(G)$.*
- *The minimum cardinality among all global strong defensive alliances in $G$ is called the* ***global strong defensive alliance number*** *of $G$, and it is denoted by $\gamma_{\hat{a}}(G)$.*

**Example 1.19.** *Consider the same simple graph $G$ as in Example 1.1. Then,*

- $\{v_2, v_4, v_5\}$ *is not a defensive alliance (since* $deg_{\{v_2,v_4,v_5\}}(v_4)+1 = 1 < deg_{\{v_1,v_3\}}(v_4)$*).*

- $\{v_1, v_3\}$ *is a defensive alliance, but it is not a strong defensive alliance (since* $deg_{\{v_1,v_3\}}(v_3) = 1 < deg_{\{v_2,v_4,v_5\}}(v_3) = 2$*).*

- $\{v_1, v_2, v_4\}$ *is a strong defensive alliance.*

- $\{v_1, v_2, v_3\}$ *is a global defensive alliance.*

- $\gamma_a(G) = 3$.

For what follows we adopt the following notations: For a simple graph $G$, we denote by $\Delta(G)$ (resp., $\delta(G)$) (or simply $\Delta$ (resp., $\delta$) if there is no ambiguity) the maximum degree (resp., the minimum degree) among all degrees of vertices of $G$.
The notion of the defensive alliance is parameterized in the following sense (see [57, 74]):

**Definition 1.20** ([74])**.** *Let $G$ be a graph, $S$ be a non-empty subset of $V(G)$ and $k$ be an integer in $[\![-\Delta; \Delta]\!]$. Then,*

- *$S$ is said to be a **defensive $k$-alliance**, if for every $x \in S$, $deg_S(x) \geq deg_{\bar{S}}(x) + k$, or equivalently, if $deg(x) \geq 2deg_{\bar{S}}(x) + k$ for every $x \in S$.*

- *$S$ is said to be a **global defensive $k$-alliance** if it is a defensive $k$-alliance and it forms a dominating set.*

- *The minimum cardinality among all global defensive $k$-alliances in $G$ is called the **global defensive $k$-alliance number** of $G$, and it is denoted by $\gamma_k^d(G)$.*

Note that, a defensive alliance is nothing but a defensive $(-1)$-alliance and a strong defensive alliance is nothing but a defensive 0-alliance. Then, the global defensive alliance number is the global defensive $(-1)$-alliance number and the global strong alliance number is the global defensive 0-alliance number.

**Example 1.20.** *Consider the graph $G$ in Example 1.1. Then, $[\![-\Delta; \Delta]\!] = [\![-3; 3]\!]$, and so*

- *$\{v_2, v_3\}$ is a global defensive $(-3)$-alliance. But it is not a global defensive $k$-alliance for $k \in [\![-2; 3]\!]$.*

- *$\{v_1, v_2, v_3\}$ is a global defensive $k$-alliance for every $k \in [\![-3; 0]\!]$. But, it is not a global defensive $k$-alliance for every $k \in [\![1; 3]\!]$.*

- *There is no global defensive $k$-alliance, where $k \in [\![1; 3]\!]$, in $G$ because if $S$ is a global defensive $k$-alliance, then $v_2 \in S$ (since $S$ is a dominating set) and so $deg_S(v_2) \geq deg_{\bar{S}}(v_2) + k$ implies that $0 \geq k$, which is not possible.*

In general, for some graphs, defensive $k$-alliances do not exist for some values of $k$ in $[\![-\Delta; \Delta]\!]$. For instance, in the case of the star graph, defensive $k$-alliances do not exist for $k \geq 2$. From the definition of the defensive $k$-alliance we conclude that, in any graph, there are defensive $k$-alliances for $k \in [\![-\Delta; \delta]\!]$. For instance, a defensive $\delta$-alliance in $G$ is $V$. Moreover, if $x \in V$ is a vertex of a minimum degree, $deg(x) = \delta$, then $S = \{x\}$ is a defensive $k$-alliance for every $k \leq -\delta$.
Clearly, $\gamma_{k+1}^d(G) \geq \gamma_k^d(G) \geq \gamma(G)$.

Now, let us recall some interesting results that we will need throughout this thesis. We start with the following propositions which present the global defensive alliance and global strong defensive alliance numbers for complete graphs and complete bipartite graphs.

**Proposition 1.13** ([49], Proposition 2). *Let $n \in \mathbb{N}^*$. Then,*

- $\gamma_a(K_n) = \left\lfloor \frac{n+1}{2} \right\rfloor$, *and*
- $\gamma_{\hat{a}}(K_n) = \left\lceil \frac{n+1}{2} \right\rceil$.

**Proposition 1.14** ([49], Proposition 3). *Let $n, m \in \mathbb{N}^*$. Then,*

- $\gamma_a(K_{1,m}) = \left\lfloor \frac{m}{2} \right\rfloor + 1$.
- $\gamma_a(K_{n,m}) = \left\lfloor \frac{n}{2} \right\rfloor + \left\lfloor \frac{m}{2} \right\rfloor$, *if $n, m \geq 2$.*
- $\gamma_{\hat{a}}(K_{n,m}) = \left\lceil \frac{n}{2} \right\rceil + \left\lceil \frac{m}{2} \right\rceil$.

The following results give some lower and upper bounds of the global defensive alliance and the global strong defensive alliance numbers of a graph in terms of its order, maximum degree and minimum degree.

**Theorem 1.1** ([49], Theorem 11 and Theorem 14). *Let $G$ be a graph such that $|V(G)| = n$. Then,*

- $\gamma_a(G) \geq \frac{\sqrt{4n+1}-1}{2}$.
- $\gamma_{\hat{a}}(G) \geq \sqrt{n}$.

*and these bounds are sharp.*

**Theorem 1.2** ([49], Theorem 15 and Theorem 16). *Let $G$ be a bipartite graph such that $|V(G)| = n$. Then,*

- $\gamma_a(G) \geq \frac{2n}{\Delta+3}$,
- $\gamma_{\hat{a}}(G) \geq \frac{2n}{\Delta+2}$,

*and these bounds are sharp.*

**Theorem 1.3** ([49], Theorem 17 and Theorem 18). *Let $T$ be a tree such that $|V(T)| = n$. Then,*

- $\gamma_a(T) \geq \frac{n+2}{4}$,
- $\gamma_{\hat{a}}(T) \geq \frac{n+2}{3}$,

*and these bounds are sharp.*

**Proposition 1.15** ([49], Proposition 19). *Let $G$ be a graph with no isolated vertices. Then,*

- $\gamma_a(G) \leq n - \left\lceil \frac{\delta}{2} \right\rceil$,
- $\gamma_{\hat{a}}(G) \leq n - \left\lfloor \frac{\delta}{2} \right\rfloor$,

*and these bounds are sharp.*

**Theorem 1.4** ([49], Theorem 21). *Let $T$ be a tree such that $|V(T)| = n \geq 4$. Then,*

$$\gamma_a(T) \leq \frac{3n}{5},$$

*and this bound is sharp.*

**Theorem 1.5** ([49], Theorem 22). *Let $T$ be a tree such that $|V(T)| = n \geq 3$. Then,*

$$\gamma_{\hat{a}}(T) \leq \frac{3n}{4},$$

*and this bound is sharp.*

Now, we recall some interesting results that generalized the previous results. These results showed by supposing that the graph contains defensive $k$-alliances. Then, we start with the following one which gives the global defensive $k$-alliance number of complete graphs, $K_n$ for some $n \in \mathbb{N}^*$.

**Proposition 1.16** ([73]). *Let $n \in \mathbb{N}^*$ and $k \in [\![1-n; n-1]\!]$. Then,*

$$\gamma_k^d(K_n) = \left\lceil \frac{n+k+1}{2} \right\rceil.$$

**Theorem 1.6** ([73], Theorem 2). *Let $G$ be a graph such that $|V(G)| = n$. Then,*

$$\frac{\sqrt{4n+k^2}+k}{2} \leq \gamma_k^d(G) \leq n - \left\lfloor \frac{\delta - k}{2} \right\rfloor.$$

**Theorem 1.7** ([73], Theorem 3). *Let $G$ be a graph such that $|V(G)| = n$. Then,*

$$\gamma_k^d(G) \geq \left\lceil \frac{n}{\left\lfloor \frac{\Delta - k}{2} \right\rfloor + 1} \right\rceil.$$

**Proposition 1.17** ([73], Corollary 10). *Let $T$ be a tree such that $|V(T)| = n$. Then,*

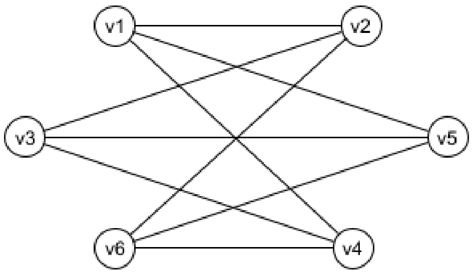
$$\gamma_k(T) \geq \left\lceil \frac{n+2}{3-k} \right\rceil.$$

### Partitioning graphs into global defensive $k$-alliances

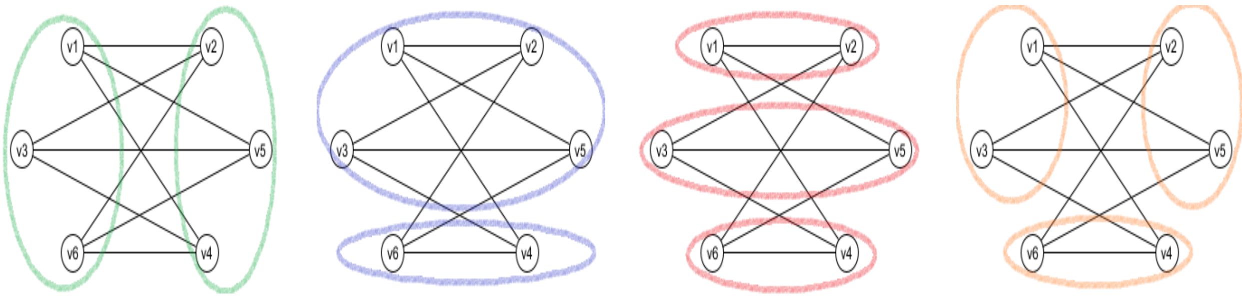
The problem of partitioning graphs into global defensive $k$-alliances has been the subject of much research for arbitrary graphs (see for example [78, 79, 90]). The problem is to determine whether or not a given graph is partitionable into global defensive $k$-alliances. First, recall the definition of a partition.

**Definition 1.21.** *Let $r \geq 2$ be a positive integer and $S$ be a set of elements. Let $S_1, S_2, \ldots, S_r$ be subsets of $S$. Then, $\{S_1, S_2, \ldots, S_r\}$ is a* ***partition*** *of $S$ if $\bigcup_{i=1}^{r} S_i = S$ and $S_i \cap S_j = \emptyset$ for every $i \neq j \in [\![1; r]\!]$.*

**Example 1.21.** *Consider the following illustrated simple graph:*




*This graph can be partitioned in several ways, for instance:*



*(a)* *(b)* *(c)* *(d)*

**Definition 1.22.** *Let $G$ be a graph.*

- *$G$ is said to be* ***partitionable into global defensive alliances*** *if there exists a partition $\{S_1, S_2, \ldots, S_r\}$ of $V(G)$ such that, for every $i \in [\![1; r]\!]$, $S_i$ is a global defensive alliance.*

- *The maximum cardinality among all partitions of $V(G)$ into global defensive alliances is called* ***the global defensive alliance partition number*** *of $G$, and it is denoted by $\psi_g(G)$.*

Notice that, a graph $G$ is partitionable into global defensive alliances if and only if $\psi_g(G) \geq 2$.
The graph in Example 1.1 is not partitionable into global defensive alliances since $v_2$ must be in any dominating set. Here, we give a simple example of a partitionable graph.

**Example 1.22.** *In the previous example, (b) and (c) form two partitions for this simple graph into global defensive alliances. But, (a) and (d) are not partitions of this graph into global defensive alliances.*
*The global defensive alliance partition number of this graph is equal to 3 since there is not a global defensive alliance of cardinality smaller than 2.*

The problem of partitioning a graph into global defensive alliances is parameterized in the following way (see [90]).

**Definition 1.23** ([90])**.** *Let $G$ be a simple graph and $k \in [\![-\Delta; \delta]\!]$. Then,*

- *$G$ is said to be* ***partitionable into global defensive $k$-alliances*** *if there exists a partition $\{S_1, S_2, \ldots, S_r\}$ of $V(G)$ such that, for every $i \in [\![1; r]\!]$, $S_i$ is a global defensive $k$-alliance.*

- *The maximum cardinality among all partitions of $V(G)$ into global defensive $k$-alliances is called* ***the global defensive $k$-alliance partition number*** *of $G$, and it is denoted by $\psi_k^d(G)$.*

Notice that the global defensive alliance partition number is the global defensive $k$-alliance partition number when $k$ equals $-1$. Also, a graph $G$ is partitionable into global defensive $k$-alliances if and only if $\psi_k^{gd}(G) \geq 2$.

Now, we set some interesting results about the global defensive $k$-alliance partition number of a graph.

**Proposition 1.18** ([90])**.** *Let $G$ be a graph such that $|V(G)| = n$. Then,*

$$\gamma_k^d(G)\psi_k^{gd}(G) \leq n.$$

**Theorem 1.8** ([90], Theorem 2.1)**.** *Let $G$ be a graph. If $G$ is partitionable into global defensive $k$-alliances, then*

1. $\psi_k^{gd}(G) \leq \left\lfloor \frac{\sqrt{k^2+4n}-k}{2} \right\rfloor$.
2. $\psi_k^{gd}(G) \leq \left\lfloor \frac{\delta-k+2}{2} \right\rfloor$.

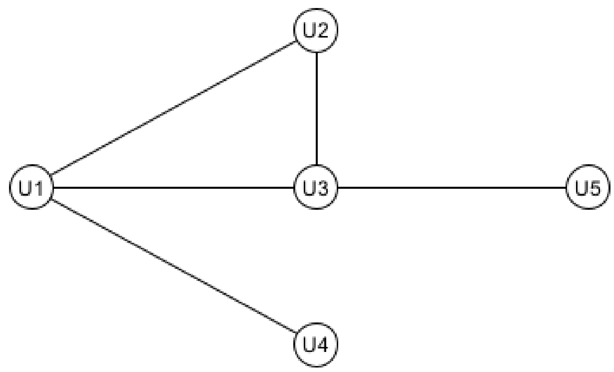
### 1.1.3 Complementedness and uniquely complementedness of graphs

The complementedness and uniquely complementedness notions of graphs were initially introduced for a general graph in [9, 58] and subsequently investigated within the framework of zero-divisor graphs of commutative rings. In this section, we recall the definitions of complemented graph and uniquely complemented graph. These definitions are very useful in Chapter 4.

Like in [58], for two distinct vertices $u, v \in V(G)$ of $G$, we define $u \leq v$ if $u$ and $v$ are not adjacent and $N(u) \subseteq N(v)$, and define $u \sim v$ if $u \leq v$ and $v \leq u$. Thus, $u \sim v$ if and only if $N(u) = N(v)$. $\sim$ is an equivalence relation on $G$.

**Definition 1.24** ([9, 58])**.** *Let $G$ be a graph and $u, v \in V(G)$. Then, $u$ and $v$ are said to be* ***orthogonal****, and we write $u \perp v$, if the edge $u - v$ is not part of any triangle in $G$ (i.e., $u$ and $v$ are adjacent and there is no other vertex $w$ adjacent to both $u$ and $v$).*

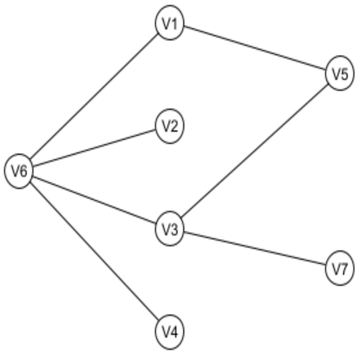
**Example 1.23.** *Consider the following illustrated graph.*



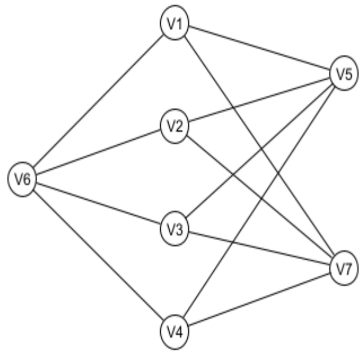

*Then, $u_1$ and $u_4$ are orthogonal, $u_3$ and $u_5$ are orthogonal, $u_1$ and $u_2$ are not orthogonal, $u_1$ and $u_3$ are not orthogonal, and $u_2$ and $u_3$ are not orthogonal.*

**Definition 1.25** ([9, 58])**.** *Let $G$ be a graph. Then, $G$ is said to be **complemented** if for every vertex $u \in V(G)$, there is a vertex $v \in V(G)$, called a **complement** (or an **orthogonal**) of $u$, such that $u \perp v$. This graph is said to be **uniquely complemented** if it is complemented and whenever $u \perp v$ and $u \perp w$, then $v \sim w$.*

**Example 1.24.** *The graph in Example 1.23 is not complemented since $u_2$ has no complement. Here, we give two examples of complemented graphs (one is uniquely complemented (the graph $H$) and one is not uniquely complemented (the graph $G$)).*




$G$




$H$

The graph $G$ is complemented since every two adjacent vertices are orthogonal, but it is not uniquely complemented since $v_5 \perp v_3$ and $v_3 \perp v_7$ but $N(v_5) = \{v_3, v_1\} \neq N(v_7) = \{v_3, v_4\}$. The graph $H$ is uniquely complemented.

## 1.2 Zero-divisor graph theory

Beck first introduced the concept of the zero-divisor graph of a commutative ring [17], to present the idea of coloring of a commutative ring, in order to establish a connection between graph theory and commutative ring theory, which turns out to be mutually beneficial for these two branches of mathematics. For a given commutative ring $R$, Beck's zero-divisor graph is a simple graph with the set of vertices is the set of all elements of $R$, and two distinct vertices $x$ and $y$ are adjacent if $xy = 0$. In 1999, Anderson and Livingston defined a simplified version $\Gamma(R)$ of Beck's zero-divisor graph by including only nonzero zero-divisors of $R$ in the vertex set and leaving the definition of edges the same [8]. The reason for this simplification was to capture better the essence of the zero-divisor structure of the ring. Several properties of $\Gamma(R)$ have been investigated, such as connectedness, diameter, girth, chromatic number, etc. [3, 8]. In addition, the isomorphism problem for such graphs has been solved for finite reduced rings [7]. Several authors have also investigated rings $R$ whose graph $\Gamma(R)$ belongs to a certain family of graphs, such as star graphs [3], complete graphs [8], complete $r$-partite graphs and planar graphs [4, 84]. However, sometimes, the zero-divisor graph does not represent enough information to characterize its associated ring. This may be because it shows only the relationship between nonzero zero-divisors and not, for instance, the relationship between the power of nonzero zero-divisors. This was the purpose of introducing the extended zero-divisor graph, in 2016 by Bennis, Mikram, and Taraza, which is a simple graph, denoted by $\overline{\Gamma}(R)$, with the same set of vertices as in the zero-divisor graph and two distinct vertices $x$ and $y$ are adjacent if and only if $x^n y^m = 0$ with $x^n \neq 0$ and $y^m \neq 0$ for some $n, m \in \mathbb{N}^*$ [24].

This chapter is devoted to recalling some definitions and classical results related to both the zero-divisor graph and the extended zero-divisor graph of a commutative ring, which are used throughout this thesis.

### 1.2.1 The zero-divisor graph of a commutative ring

We start this section with the definition of the zero-divisor graph of a commutative ring and then we set some classical results related to it which are used in this thesis.

**Definition 1.26** ([8]). ***The zero-divisor graph*** *of $R$, denoted $\Gamma(R)$, is a simple graph with vertex set is the set of nonzero zero-divisors of $R$ and two distinct vertices $x$ and $y$ are adjacent if $xy = 0$.*

In the following, we give some illustrated examples of zero-divisor graphs of some kind of commutative rings (see Figure 1.2).

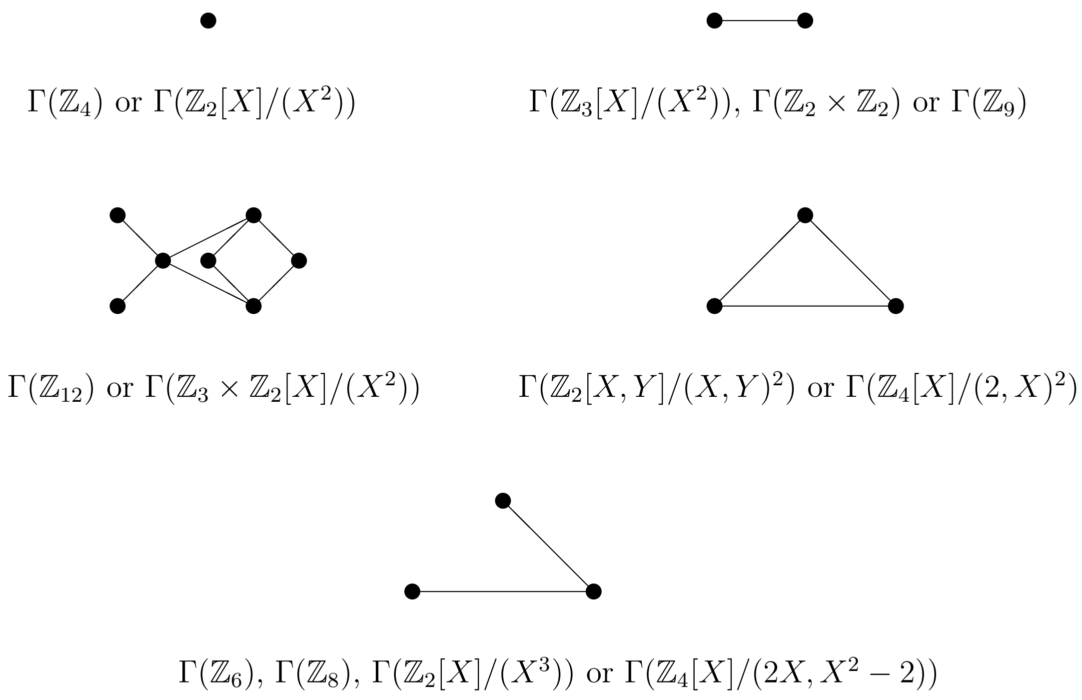



Figure 1.2: The zero-divisor graphs of some rings

Now, let us recall some interesting results on the zero-divisor graph of a commutative ring. We start by characterizing when $\Gamma(R)$ is complete, for a commutative ring $R$. But first, we give the following theorem.

**Theorem 1.9** ([8], Theorem 2.5). *The existence of a vertex in $\Gamma(R)$ that is adjacent to every other vertex is equivalent to either $R$ is isomorphic to $\mathbb{Z}_2 \times D$ where $D$ is an integral domain, or $Z(R) = \mathrm{Ann}(x)$ for some $x \in R$.*

**Theorem 1.10** ([8], Theorem 2.8). *$\Gamma(R)$ is a complete graph is equivalent to either $R$ is isomorphic to $\mathbb{Z}_2 \times \mathbb{Z}_2$ or $xy = 0$ for all $x, y \in Z(R)$.*

The following theorem characterizes when $\Gamma(R)$ is finite, for a commutative ring $R$.

**Theorem 1.11** ([8], Theorem 2.2). *$\Gamma(R)$ is finite is equivalent to either $R$ is finite or $R$ is an integral domain.*

The following result shows an upper bound of the cardinality of $R$ in terms of the cardinality of the set of zero-divisors.

**Theorem 1.12** ([41], Theorem 1). *If $R$ is a non-integral domain such that $|Z(R)| < \infty$. Then, $|R| \leq |Z(R)|^2$.*

We also set the following interesting results which will be very useful in this thesis.

**Theorem 1.13** ([67], Theorem 2). *If $R$ is a finite ring, whose zero-divisors form an additive group $I$. Then,*

1. *$I$ is the Jacobson radical of $R$.*
2. *there exist a prime number $p$ and two positive integers $n$ and $r$ such that $|R| = p^{nr}$ and $|I| = p^{(n-1)r}$.*
3. *there exists a positive integer $n$ such that $I^n = (0)$.*

**Theorem 1.14** ([54], Theorem 82). *If $R$ is a Noetherian ring, $M$ is a finitely generated non-zero $R$-module and $S$ is a subring contained in $Z(M)$. Then, there exists a single non-zero element $x$ in $M$ such that $Sx = 0$.*

**Corollary 1.2.** *If $(R, M)$ is a finite local ring that is not a field, then we have the following statements:*

1. *$M = Z(R) = \mathrm{Ann}(x)$, for some $x \in Z(R)^*$.*
2. *$|R| = p^{nr}$ and $|M| = p^{(n-1)r}$ for some prime $p$ and some positive integers $n, r$.*
3. *$M^n = (0)$.*

**Corollary 1.3.** *If $(R, M)$ is a finite local ring, then $\Gamma(R)$ is complete if and only if $Z(R) = M$ and $M$ is nilpotent of index 2.*

**Proposition 1.19** ([18],Corollary 1). *Let $R$ be a ring with $|Z(R)| = p_1^{k_1} p_2^{k_2} \dots p_n^{k_n}$, where $n \geq 1$, $1 \leq k_i \leq 4$ and $p_i's$ are distinct prime numbers. Then there exist $0 \leq s \leq \sum_{i=1}^{n} k_i$ and $t \geq 0$ such that*

$$R \cong R_1 \times \cdots \times R_s \times F_{q_1} \times \cdots \times F_{q_t}$$

*where $F'_{q_i}s$ are finite fields and each $R_i$ is a local ring with $|Z(R_i)| = p_j^{t_j}$ for some $p_j$ $(1 \leq j \leq n)$ and $1 \leq t_j \leq k_j$.*

In [68] and [69], all graphs on $n = 6, 7, ..., 14$ vertices which can be realized as zero-divisor graphs of commutative rings with identity $1 \neq 0$, and the list of all rings (up to isomorphism) which produce these graphs, were given. Below we list the established ring and graphs for $n = 3$ to 6. The tables for $n = 1, 2, 3$ and 4 can be found in [7]. The results for $n = 5$ can be found in [70].

| vertices | $R$ | $\|R\|$ | Graph | Type |
| --- | --- | --- | --- | --- |
| 3 | $\mathbb{Z}_6$ | 6 | $K^{1,2}$ | reduced |
| | $\mathbb{Z}_8$ | 8 | $K^{1,2}$ | local |
| | $\mathbb{Z}_2[X]/(X^3)$ | 8 | $K^{1,2}$ | local |
| | $\mathbb{Z}_4[X]/(2X, X^2-2)$ | 8 | $K^{1,2}$ | local |
| | $\mathbb{Z}_2[X,Y]/(X,Y)^2$ | 8 | $K^3$ | local |
| | $\mathbb{Z}_4[X]/(2,X)^2$ | 8 | $K^3$ | local |
| | $\mathbb{F}_4[X]/(X^2)$ | 16 | $K^3$ | local |
| | $\mathbb{Z}_4[X]/(X^2+X+1)$ | 16 | $K^3$ | local |

Table 1.1

| vertices | $R$ | $\|R\|$ | Graph | Type |
| --- | --- | --- | --- | --- |
| 4 | $\mathbb{Z}_2 \times \mathbb{F}_4$ | 8 | $K^{1,3}$ | reduced |
| | $\mathbb{Z}_3 \times \mathbb{Z}_3$ | 9 | $K^{2,2}$ | reduced |
| | $\mathbb{Z}_{25}$ | 25 | $K^4$ | local |
| | $\mathbb{Z}_5[X]/(X^2)$ | 25 | $K^4$ | local |

Table 1.2

| vertices | $R$ | $\|R\|$ | Graph | Type |
| --- | --- | --- | --- | --- |
| 5 | $\mathbb{Z}_2 \times \mathbb{Z}_5$ | 10 | $K^{1,4}$ | reduced |
| | $\mathbb{Z}_3 \times \mathbb{F}_4$ | 12 | $K^{2,3}$ | reduced |
| | $\mathbb{Z}_2 \times \mathbb{Z}_4$ | 8 | Figure 1.3 | mixed |
| | $\mathbb{Z}_2 \times \mathbb{Z}_2[X]/(X^2)$ | 8 | Figure 1.3 | mixed |

Table 1.3

| vertices | $R$ | $\|R\|$ | Graph | Type |
| --- | --- | --- | --- | --- |
| 6 | $\mathbb{Z}_3 \times \mathbb{Z}_5$ | 15 | $K^{2,4}$ | reduced |
| | $\mathbb{F}_4 \times \mathbb{F}_4$ | 16 | $K^{3,3}$ | reduced |
| | $\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$ | 8 | Figure 1.4 | reduced |
| | $\mathbb{Z}_{49}$ | 49 | $K^6$ | local |
| | $\mathbb{Z}_7[X]/(X^2)$ | 49 | $K^6$ | local |

Table 1.4

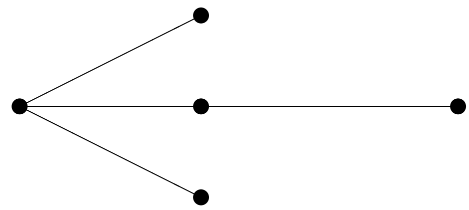


Figure 1.3: The zero-divisor graph of $\mathbb{Z}_2 \times \mathbb{Z}_4$ or $\mathbb{Z}_2 \times \mathbb{Z}_2[X]/(X^2)$

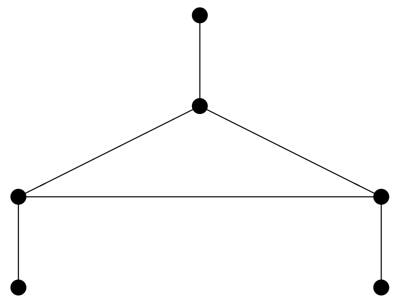


Figure 1.4: The zero-divisor graph of $\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$

### 1.2.2 The extended zero-divisor graph of a commutative ring

We start this section with the definition of the extended zero-divisor graph of a commutative ring and then we set some interesting results related to it which are used in this thesis.

**Definition 1.27.** ***The extended zero-divisor graph*** *of $R$, denoted by $\overline{\Gamma}(R)$, is a simple graph with vertex set is the set of nonzero zero-divisors of $R$ and two distinct vertices $x$ and $y$ are adjacent if $x^n y^m = 0$ with $x^n \neq 0$ and $y^m \neq 0$ for some $n, m \in \mathbb{N}^*$.*

In the following, we give illustrated examples of extended zero-divisor graphs of some commutative rings and compare them with their associated zero-divisor graphs (see Figure 1.5).

One can see that the zero-divisor graph is a subgraph of the extended zero-divisor graph. So, there is a natural question that arises in this context that is, when the zero-divisor graph and the extended zero-divisor graph of a commutative ring coincide? The answer to this question is in Theorem 1.15. But first, we set the following lemma which is an interesting result and can help to prove Theorem 1.15.

**Lemma 1.1** ([24]). *Let $x$ be a nonzero element of $R$. Then,*

1. *If $x \in \mathrm{Nil}(R)$, then $\mathrm{Ann}(x) \subsetneq \mathrm{Ann}(x^n)$ for every integer $n \geq 2$.*

2. *If $x \notin \mathrm{Nil}(R)$, then $\mathrm{Ann}(x^2) = \mathrm{Ann}(x)$ if and only if $\mathrm{Ann}(x^n) = \mathrm{Ann}(x)$ for every integer $n \geq 2$.*

**Theorem 1.15** ([24], Theorem 2.1). *The two following statements are equivalent:*

1. *$\overline{\Gamma}(R) = \Gamma(R)$.*

2. *$R$ satisfies the two following conditions:*

*(a)* *If* $\mathrm{Nil}(R) \neq \{0\}$*, then every nonzero nilpotent element has index 2, and*

*(b)* *For every* $x \in Z(R) \setminus \mathrm{Nil}(R)$*,* $\mathrm{Ann}(x^2) = \mathrm{Ann}(x)$*.*

**Corollary 1.4** ([24], Corollary 2.6 and Corollary 2.7)**.** *We have the following statements:*

1. *If* $R$ *contains a nilpotent element of index* $3$*, then* $\overline{\Gamma}(R) \neq \Gamma(R)$*.*

2. *If* $R$ *is reduced, then* $\overline{\Gamma}(R) = \Gamma(R)$*.*

The following result characterizes when the classical zero-divisor graph and the extended zero-divisor graph of a finite direct product of commutative rings coincide.

**Proposition 1.20** ([24], Proposition 2.8)**.** *Let* $n \geq 2$ *be an integer and* $(R_i)_{1 \leq i \leq n}$ *be a finite family of rings. Then,* $\overline{\Gamma}(\prod_{i=1}^{n} R_i) = \Gamma(\prod_{i=1}^{n} R_i)$ *if and only if* $R_i$ *is reduced for every* $1 \leq i \leq n$*.*

**Corollary 1.5** ([24], Corollary 2.9)**.** *Let* $n = \prod_{i=1}^{k} p_i^{\alpha_i}$ *be the prime factorization of an integer* $n$*. Let* $m := \sup\{\alpha_i |\ 1 \leq i \leq k\}$*. Then,* $\overline{\Gamma}(\mathbb{Z}_n) \neq \Gamma(\mathbb{Z}_n)$ *if and only if* $m \geq 3$ *or* $(m = 2$ *and* $k \geq 2)$*.*

To describe the conditions under which the extended zero-divisor graph is complete, we present the following theorems.

**Theorem 1.16** ([24], Theorem 3.2)**.** *Let* $R$ *be a ring. Then, there is a vertex* $x$ *of* $\overline{\Gamma}(R)$ *that is adjacent to every other vertex if and only if either* $R \cong \mathbb{Z}_2 \times D$*, where* $D$ *is an integral domain, or* $Z(R) = \sqrt{\mathrm{Ann}(x^{n_x - 1})}$*.*

**Theorem 1.17** ([24], Theorem 3.3)**.** *Let* $R$ *be a ring. Then,* $\overline{\Gamma}(R)$ *is complete if and only if either* $R \cong \mathbb{Z}_2 \times \mathbb{Z}_2$ *or* $Z(R) = Nil(R)$ *and for every* $x, y \in Z(R)^*$*,* $x^{n_x - 1} y^{n_y - 1} = 0$*.*

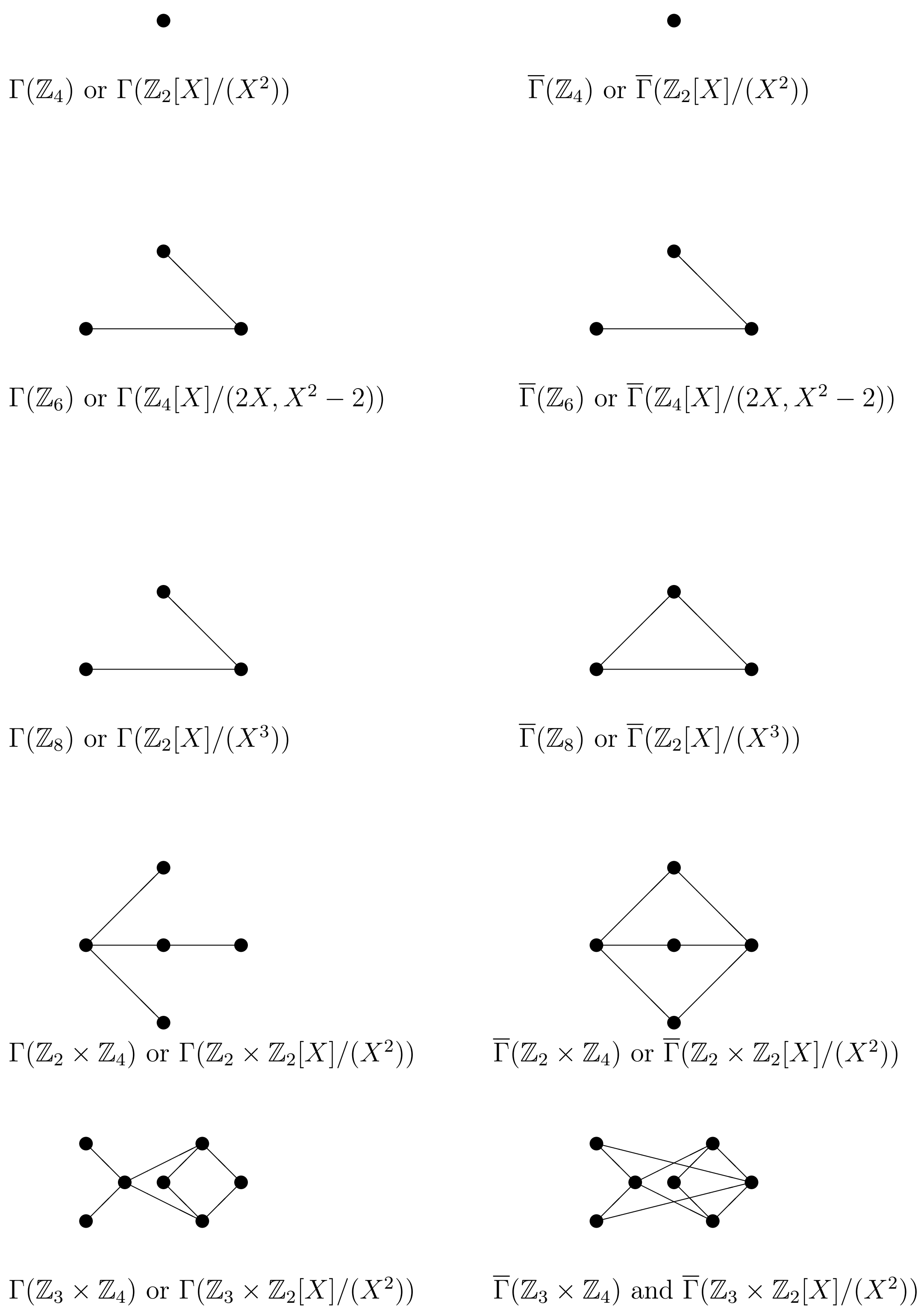



Figure 1.5: The zero-divisor graphs and the extended zero-divisor graphs of some rings

# Chapter 2

# The global defensive $k$-alliances in zero-divisor graphs of rings

This chapter focuses on the global defensive $k$-alliance notion introduced by Rodríguez-Velázquez and Sigarreta in [73]. It is a very well-studied notion in graph theory, it provides a method of classification of graphs based on relations between members of a particular set of vertices. Here, we explore this notion in the context of zero-divisor graphs of commutative rings. In 2021, Muthana and Mamouni started to study the global defensive alliances of zero-divisor graphs of commutative rings which is a particular case for $k = -1$ [62]. Thus, in this chapter, several established results are generalized and improved. We give various examples to illustrate and delimit the scope of the established results. See the introduction for a complete overview of this chapter.

## 2.1 The global defensive $k$-alliances of the zero-divisor graphs of local rings

In this section, we study the global defensive $k$-alliances of the zero-divisor graphs of local rings. We start by extending and improving the following result.

**Proposition 2.1** ([62], Proposition 2.2). *We have the following inequality for any finite ring $R$ which is not a field.*

$$|Z(R)| \leq \gamma_a(\Gamma(R))^2 + \gamma_a(\Gamma(R)) + 1.$$

*In addition, if $R$ is a local ring, then $|Z(R)| \leq \gamma_a(\Gamma(R))^2 + 2$.*

Before giving results for the case of local rings, we start with an extension of the first inequality of Proposition 2.1.

Here, we extend and ameliorate this inequality to the other global defensive $k$-alliances which provide a more optimal upper bound (see Examples 2.1 and 2.3).

**Lemma 2.1.** *For any finite ring $R$, we have*

$$|Z(R)| \leq \min_{k \in [\![-\Delta;\delta]\!]} \{1 + \gamma_k^d(\Gamma(R))^2 - k\gamma_k^d(\Gamma(R))\}.$$

*Moreover, if there is a global defensive $k$-alliance set $S = \{x_1, ..., x_r\}$ in $\Gamma(R)$ with $r = \gamma_k^d(\Gamma(R))$ and a subset $\Lambda$ of $Z(R)^*$ such that $\Lambda \subseteq N(x_i)$ for every $x_i \in S$, then*

$$|Z(R)| \leq 1 + |\Lambda| + r^2 - r(k + |\Lambda|).$$

*Proof.* Let $k \in [\![-\Delta; \delta]\!]$, and set $r = \gamma_k^d(\Gamma(R))$ and $S = \{x_1, ..., x_r\}$ be a global defensive $k$-alliance set. We have $N[S] = Z(R)^*$, since $S$ is a dominating set. Thus, $\bar{S} \subset \bigcup_{i=1}^r \mathrm{Ann}_R(x_i)$. Hence,

$$\begin{aligned}
|\bar{S}| &= |\bar{S} \cap \bigcup_{i=1}^{r} \mathrm{Ann}_R(x_i)| \\
&\leq \sum_{i-1}^{r} |\bar{S} \cap \mathrm{Ann}_R(x_i)| \\
&\leq \sum_{i=1}^{r} deg_{\bar{S}}(x_i) \\
&\leq \sum_{i=1}^{r} deg_S(x_i) - k \\
&\leq \sum_{i=1}^{r} (r-1) - k = r^2 - (1+k)r.
\end{aligned}$$

Then, for every $k \in [\![-\Delta; \delta]\!]$, $|Z(R)| = 1 + |S| + |\bar{S}| \leq 1 + r^2 - kr$. Hence, $|Z(R)| \leq \min_{k \in [\![-\Delta;\delta]\!]}\{1 + \gamma_k^d(\Gamma(R))^2 - k\gamma_k^d(\Gamma(R))\}$. Now, if there exists a subset $\Lambda$ of $Z(R)^*$ such that $\Lambda \subseteq N(x_i)$ for every $x_i \in S$. Then,

$$\begin{aligned}
|\bar{S}| &= |\Lambda| + |\bigcup_{i=1}^{r} (\bar{S} \cap \mathrm{Ann}_R(x_i)) \setminus (\mathrm{Ann}_R(x_i) \cap \Lambda)| \\
&\leq |\Lambda| + \sum_{i=1}^{r} (|\bar{S} \cap \mathrm{Ann}_R(x_i)| - |\mathrm{Ann}_R(x_i) \cap \Lambda|) \\
&\leq |\Lambda| + \sum_{i=1}^{r} (deg_{\bar{S}}(x_i) - deg_{\Lambda}(x_i)) \\
&\leq |\Lambda| + \sum_{i=1}^{r} (r - 1 - k - |\Lambda|) \\
&= |\Lambda| + r^2 - r - rk - r|\Lambda|
\end{aligned}$$

and so $|Z(R)| = 1 + |S| + |\bar{S}| \leq 1 + |\Lambda| + r^2 - r(k + |\Lambda|)$. $\square$

The following examples show that the bounds in Lemma 2.1 are optimal.
For what follows, we adopt the following notation: $A_k = 1 + \gamma_k^d(\Gamma(R))^2 - k\gamma_k^d(\Gamma(R))$.

**Example 2.1.** *Let $R = \mathbb{Z}_{12}$. The zero-divisor graph of this ring is illustrated in Figure 2.1. Different values of the global defensive $k$-alliance number and $A_k$ are presented in Table* 2.1.

*This example gives an upper bound of the cardinality of $Z(R)$ which is smaller than the upper bound given in Proposition 2.1. Namely, $|Z(R)| < \min_{k \in [\![-4;1]\!]}\{A_k\} = 9 < \gamma_a(\Gamma(R))^2 + \gamma_a(\Gamma(R)) + 1 = 13$.*

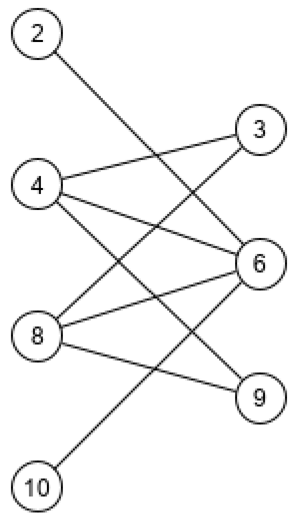



Figure 2.1: The zero-divisor graph of $\mathbb{Z}_{12}$

| $k$ | $\gamma_k^d(\Gamma(R))$ | $A_k$ |
|---|---|---|
| $-4$ | 2 | 13 |
| $-3$ | 2 | 11 |
| $-2$ | 2 | 9 |
| $-1$ | 3 | 13 |
| 0 | 4 | 17 |
| 1 | 5 | 21 |

Table 2.1: Values of $\gamma_k^d(\Gamma(R))$ and $A_k$

**Example 2.2.** *Consider the ring $R = \mathbb{Z}_2 \times \mathbb{Z}_4$. The zero-divisor graph of this ring is illustrated in Figure 2.2. Different values of the global defensive $k$-alliance number and $A_k$ are presented in Table* 2.2.

*This ring provides an example satisfying the second inequality for $k = -1$. Namely, we have the global defensive $(-1)$-alliance $\{(1,0),(1,2)\}$ and the set $\Lambda = \{(0,2)\}$ satisfying the condition of Lemma 2.1, that is, $\Lambda \subseteq N((1,0)) \cap N((1,2))$. Then, $|Z(R)| \leq 1 + |\Lambda| + \gamma_{-1}^d(\Gamma(R))^2 - \gamma_{-1}^d(\Gamma(R))(-1+|\Lambda|) = 6$.*
*Notice that $|Z(R)| < \min_{k \in [\![-3;1]\!]}\{A_k\} = \gamma_a(\Gamma(R))^2 + \gamma_a(\Gamma(R)) + 1 = 7$.*

Now, we give our first desired result.

| $k$ | $\gamma_k^d(\Gamma(\mathbb{Z}_2 \times \mathbb{Z}_4))$ | $A_k$ |
|---|---|---|
| $-3$ | 2 | 11 |
| $-2$ | 2 | 9 |
| $-1$ | 2 | 7 |
| 0 | 3 | 10 |
| 1 | 4 | 7 |

Table 2.2: Values of $\gamma_k^d(\Gamma(\mathbb{Z}_2 \times \mathbb{Z}_4))$ and $A_k$

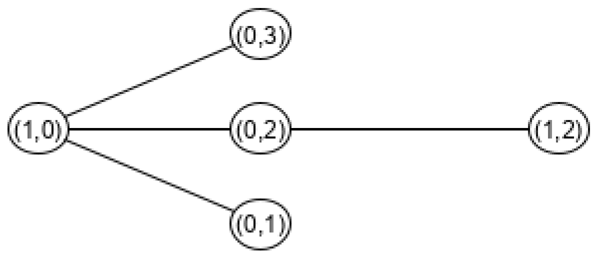



Figure 2.2: The zero-divisor graph of $\mathbb{Z}_2 \times \mathbb{Z}_4$

| $k$ | $\gamma_k^d(\Gamma(R))$ | $B_k$ | $C_k$ |
|---|---|---|---|
| $-1$ | 1 | 3 | 3 |
| 0 | 2 | 4 | 4 |
| 1 | 2 | 3 | 2 |

Table 2.3: Values of $\gamma_k^d(\Gamma(R))$, $B_k$ and $C_k$

**Proposition 2.2.** *Let $R$ be a finite local ring. Then,* $|Z(R)| \leq \max\{ \min_{k\in[\![-\Delta;\delta]\!]}\{2\gamma_k^d(\Gamma(R)) - k\},\ \min_{k\in[\![-\Delta;\delta]\!]}\{2 + \gamma_k^d(\Gamma(R))^2 - (k+1)\gamma_k^d(\Gamma(R))\}\}$.

*Proof.* Let $k \in [\![-\Delta;\delta]\!]$ and set $r = \gamma_k^d(\Gamma(R))$ and $S = \{x_1, ..., x_r\}$ be a global defensive $k$-alliance. We have $N[S] = Z(R)^*$, since $S$ is a dominating set. Thus, $\bar{S} \subset \bigcup_{i=1}^{r} \mathrm{Ann}_R(x_i)$ and since $R$ is a finite local ring, there exists $x \in Z(R)^*$ such that its maximal ideal $M$ is $Z(R) = \mathrm{Ann}_R(x)$. If $x \notin S$, then by taking $\Lambda = \{x\}$ and using Lemma 2.1, we get the following inequality for every $k \in [\![-\Delta;\delta]\!]$:

$$|Z(R)| = 1 + |S| + |\bar{S}| \leq 2 + r^2 - (k+1)r.$$

If $x \in S$, then $deg_S(x) \geq deg_{\bar{S}}(x) + k$. That is $|S| - 1 \geq |\bar{S}| + k$. Then, for every $k \in [\![-\Delta;\delta]\!]$, $|Z(R)| = 1 + |S| + |\bar{S}| \leq 2r - k$. Hence, $|Z(R)| \leq max\{\min_{k\in[\![-\Delta;\delta]\!]}\{2r - k\}, \min_{k\in[\![-\Delta;\delta]\!]}\{2 + r^2 - (k+1)r\}\}$. □

We give examples proving that the bound in Proposition 2.2 are sharp.
For what follows, we adopt the notations: $B_k = 2\gamma_k^d(\Gamma(R)) - k$ and $C_k = 2 + \gamma_k^d(\Gamma(R))^2 - (k+1)\gamma_k^d(\Gamma(R))$.

**Example 2.3.** *Let $R = \mathbb{Z}_9$. The different values of the global defensive $k$-alliance number and $A_k$ are presented in Table 2.3. Then,*

$$\begin{cases} \min_{k\in[\![-1;1]\!]}\{B_k\} = 3, \\ \min_{k\in[\![-1;1]\!]}\{C_k\} = 2, \end{cases}$$

*and so* $|Z(R)| = \max\{\min_{k\in\{-1,0,1\}}\{B_k\},\ \min_{k\in\{-1,0,1\}}\{C_k\}\} = 3$.

**Example 2.4.** *Consider the ring $R = \mathbb{Z}_8$. The zero-divisor graph of $R$ is illustrated in Figure 2.3. Different values of the global defensive $k$-alliance number and $A_k$ are presented in Table 2.4. Then,*

$$\begin{cases} \min_{k\in[\![-2;1]\!]}\{B_k\} = 4, \\ \min_{k\in[\![-2;1]\!]}\{C_k\} = 4, \end{cases}$$

*and so* $|Z(R)| = \max\{\min_{[\![-2;1]\!]}\{B_k\},\ \min_{[\![-2;1]\!]}\{C_k\}\} = 4$.

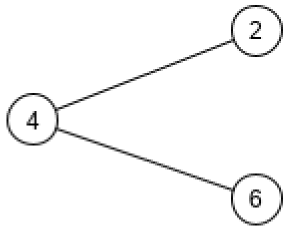



Figure 2.3: $\Gamma(\mathbb{Z}_8)$

| $k$ | $\gamma_k^d(\Gamma(R))$ | $B_k$ | $C_k$ |
|---|---|---|---|
| $-2$ | 1 | 4 | 4 |
| $-1$ | 2 | 5 | 6 |
| 0 | 2 | 4 | 4 |
| 1 | 3 | 5 | 5 |

Table 2.4: Values of $\gamma_k^d(\Gamma(R))$, $B_k$ and $C_k$

These examples show also that $\min_{[\![-\Delta;\delta]\!]}\{B_k\}$ and $\min_{[\![-\Delta;\delta]\!]}\{C_k\}$ are not comparable in the sens that $\min_{[\![-\Delta;\delta]\!]}\{C_k\} < \min_{[\![-\Delta;\delta]\!]}\{B_k\}$ for certain rings but they are not for others.

**Remark 2.1.** *Notice that for a finite local ring $R$, $\gamma_k^d(\Gamma(R)) \geq 2+k$ if and only if $2\gamma_k^d(\Gamma(R)) - k \leq 2 + (\gamma_k^d(\Gamma(R)))^2 - (k+1)\gamma_k^d(\Gamma(R))$. So, For every $k \in [\![-\Delta;\delta]\!]$,*

$$|Z(R)| \leq \begin{cases} 2\gamma_k^d(\Gamma(R)) - k & \text{if } \gamma_k^d(\Gamma(R)) \leq 2+k, \\ 2 + (\gamma_k^d(\Gamma(R)))^2 - (k+1)\gamma_k^d(\Gamma(R)) & \text{otherwise.} \end{cases}$$

It is not clear how to determine the global defensive $k-$alliances of any finite local ring. But we can determine them for $\mathbb{Z}_{p^n}$ for a prime number $p$ and an integer $n$. However, we know that for a finite local ring $(R, M)$, $\Gamma(R)$ is complete if and only if $Z(R) = M$ with $M^2 = 0$, (see Theorem 1.10). And we know from Proposition 1.16 that the global defensive $k$-alliance number of a complete graph is determined as follows: for every $k \in [\![1-n; n-1]\!]$, $\gamma_k^d(K_n) = \lceil \frac{n+k+1}{2} \rceil$. So, we have the following consequence for this simple case.

**Proposition 2.3.** *Let $R$ be a finite local ring such that its maximal ideal $M$ is nilpotent of index 2. Then, for every $k \in [\![2-|M|; |M|-2]\!]$, $\gamma_k^d(\Gamma(R)) = \left\lceil \frac{|M|+k}{2} \right\rceil$.*

Idealization can be used to give a family of examples of rings whose zero-divisor graph is complete. Recall that the idealization of an $R-$module $M$ called also the trivial extension of $R$ by $M$, denoted by $R(+)M$, is the commutative ring $R \times M$ with the following addition and multiplication: $(a,n)+(b,m) = (a+b, n+m)$ and $(a,n)(b,m) = (ab, am+bn)$ for every $(a,n),(b,m) \in R(+)M$, [52].

**Example 2.5.** *Let $n$ be a positive integer and $p$ be a prime number. Then, $\mathbb{Z}_p(+)(\mathbb{Z}_p)^n$ is a finite local ring of maximal ideal $0(+)(\mathbb{Z}_p)^n$. We have $(0(+)(\mathbb{Z}_p)^n)^2 = 0$ and so $\Gamma(\mathbb{Z}_p(+)(\mathbb{Z}_p)^n)$ is a complete graph. Then, $\gamma_k^d(\mathbb{Z}_p(+)(\mathbb{Z}_p)^n) = \left\lceil \frac{p^n+k}{2} \right\rceil$ for every $k \in [\![1-p^n; p^n-1]\!]$.*

Now, we give the second main result of this section. We determine the global defensive $k$-alliance number of the zero-divisor graph $\Gamma(\mathbb{Z}_{p^n})$. If $n=2$, then the maximal ideal $Z(\mathbb{Z}_{p^2}) =< p >$ is nilpotent of index 2 and so by Proposition 2.3 we have $\gamma_k^d(\Gamma(\mathbb{Z}_{p^2})) = \lceil \frac{p+k}{2} \rceil$ for every $k \in [\![2-p; p-2]\!]$. For $n \geq 3$, we have the following theorem.

**Theorem 2.1.** *Let $p$ be a prime number and $n \geq 3$ be an integer. Then, for every $k \in [\![2-p^{n-1}; p-1]\!]$,*

$$\gamma_k^d(\Gamma(\mathbb{Z}_{p^n})) = \left\lceil \frac{p^{n-1}+k}{2} \right\rceil.$$

*Proof.* We have $Z(\mathbb{Z}_{p^n}) = \{\overline{ap} \mid 0 \leq a < p^{n-1}\}$ and $|Z(\mathbb{Z}_{p^n})| = p^{n-1}$. For each $1 \leq r \leq n-1$, we define $A_r = \{\overline{ap^r} \mid p \text{ does not divide } a\}$ and so $Z(\mathbb{Z}_{p^n}) = \bigcup_{r=1}^{r=n-1} A_r$. There are two cases to discuss:

Case $p=2$: Let $k \in [\![2-2^{n-1}; 1]\!]$ and set $S = S_1 \cup \{2^{n-1}\}$ such that $S_1 \subseteq A_1$ with $|S_1| = \left\lceil \frac{2^{n-1}+k}{2} \right\rceil - 1$. We have $deg_S(\bar{2}) = 1$ and $deg_{\bar{S}}(\bar{2}) + k = 0 + k$. Then, $deg_{\bar{S}}(\bar{2}) + k \leq deg_S(\bar{2})$. And, $deg_S(2^{n-1}) = |S_1| = \left\lceil \frac{2^{n-1}+k}{2} \right\rceil - 1$ and $deg_{\bar{S}}(2^{n-1}) + k = |\bar{S}| + k = |Z(\mathbb{Z}_{p^n})| - 1 - |S| + k = 2^{n-1} + k - 1 - \left\lceil \frac{2^{n-1}+k}{2} \right\rceil \leq \left\lceil \frac{2^{n-1}+k}{2} \right\rceil - 1$. So, $deg_{\bar{S}}(2^{n-1}) + k \leq deg_S(2^{n-1})$. Hence, $S$ is a global defensive $k$-alliance of cardinality $\left\lceil \frac{2^{n-1}+k}{2} \right\rceil$.

Now, let $S$ be a global defensive $k$-alliance of minimal cardinality $\gamma_k^d(\Gamma(\mathbb{Z}_{2^n}))$. If $2^{n-1} \notin S$. Then, $A_1 \subseteq S$ and so $deg_S(\bar{2}) \geq deg_{\bar{S}}(\bar{2}) + k = 1 + k$. In cases $k=0$ and $k=1$, we get a contradiction. Hence, $2^{n-1} \in S$ and so $deg_S(2^{n-1}) \geq deg_{\bar{S}}(2^{n-1}) + k$. Then, $|S| - 1 \geq |\bar{S}| + k = |Z(\mathbb{Z}_{2^n})| - 1 - |S| + k$ and so $|S| \geq \frac{2^{n-1}+k}{2}$. Thus, $|S| \geq \left\lceil \frac{2^{n-1}+k}{2} \right\rceil$. Hence, $\gamma_k^d(\Gamma(\mathbb{Z}_{2^n})) = \left\lceil \frac{2^{n-1}+k}{2} \right\rceil$.

Case $p \geq 3$: Let $k \in [\![2-p^{n-1}; p-1]\!]$ and $S$ be a global defensive $k$-alliance of minimal cardinality $\gamma_k^d(\Gamma(\mathbb{Z}_{p^n}))$. If $A_{n-1} \cap S = \emptyset$. Then, $A_1 \subseteq S$ and so $deg_S(\bar{p}) \geq deg_{\bar{S}}(\bar{p}) + k = |A_{n-1}| + k = p - 1 + k$, a contradiction when $k \in [\![2-p; p-1]\!]$. Then, $A_{n-1} \cap S \neq \emptyset$ and so, there exists $z \in A_{n-1} \cap S$ such that $deg_S(z) \geq deg_{\bar{S}}(z) + k$ and so $|S| - 1 \geq |Z(\mathbb{Z}_{p^n})^*| - |S| + k$. Thus, $|S| \geq \frac{p^{n-1}+k}{2}$. Hence, $\gamma_k^d(\Gamma(\mathbb{Z}_{p^n})) \geq \left\lceil \frac{p^{n-1}+k}{2} \right\rceil$.

Now, to prove that the cardinality of $S$ is less than or equal to $\left\lceil \frac{p^{n-1}+k}{2} \right\rceil$, we have to prove that there exists a global defensive $k$-alliance of cardinality $\left\lceil \frac{p^{n-1}+k}{2} \right\rceil$ for every $k \in [\![2-p^{n-1}; p-1]\!]$. So we have the following sub-cases:

**Sub-case** $k \in [\![2-p^{n-1}; 2(p-1)-p^{n-1}]\!]$:

Let $S \subseteq A_{n-1}$ such that $|S| = \left\lceil \frac{p^{n-1}+k}{2} \right\rceil$. It is clear that $S$ is a dominating set. Let $x \in S$, then $deg_S(x) = |S| - 1$ and $deg_{\bar{S}}(x) + k = |Z(\mathbb{Z}_{p^n})| - 1 - |S| + k = p^{n-1} + k - 1 - \left\lceil \frac{p^{n-1}+k}{2} \right\rceil \leq \left\lceil \frac{p^{n-1}+k}{2} \right\rceil - 1$ and so $deg_S(x) \geq deg_{\bar{S}}(x) + k$. Hence, $S$ is a global defensive $k$-alliance of cardinality $|S| = \left\lceil \frac{p^{n-1}+k}{2} \right\rceil$.

**Sub-case** $k \in [\![2(p-1)-p^{n-1}+1; 2(p^2-1)-p^{n-1}]\!]$:

Let $S \subseteq A_{n-1} \cup A_{n-2}$ with $S \cap A_{n-1} \neq \emptyset$ and $S \cap A_{n-2} \neq \emptyset$ such that $|S| = \left\lceil \frac{p^{n-1}+k}{2} \right\rceil$. Clearly, $S$ is a dominating set. Let $x \in S \cap A_{n-1}$, we have $deg_S(x) = |S| - 1$ and $deg_{\bar{S}}(x) + k = |Z(\mathbb{Z}_{p^{n-1}})| - 1 - |S| + k = p^{n-1} - 1 - \left\lceil \frac{p^{n-1}+k}{2} \right\rceil + k \leq |S| - 1$ and so $deg_S(x) \geq deg_{\bar{S}}(x) + k$. Let $x \in S \cap A_{n-2}$, we have $deg_S(x) = |S| - 1$ and $deg_{\bar{S}}(x) + k =$

$|Z(\mathbb{Z}_{p^n})|-1-(|S|+|A_1|)+k = p^{n-1}-1-|S|-|A_1|+k = p^{n-1}+k-\left\lceil\frac{p^{n-1}+k}{2}\right\rceil-|A_1|-1 \leq \left\lceil\frac{p^{n-1}+k}{2}\right\rceil - 1$ and so $deg_S(x) \geq deg_{\bar{S}}(x)+k$. Hence, $S$ is a global defensive $k$-alliance of cardinality $|S| = \left\lceil\frac{p^{n-1}+k}{2}\right\rceil$.

**Sub-case** $k \in [\![2(p^2-1)-p^{n-1}+1; 2(p^3-1)-p^{n-1}]\!]$:

Let $S \subseteq A_{n-1} \cup A_{n-2} \cup A_{n-3}$ with $S \cap A_{n-1} \neq \emptyset$, $S \cap A_{n-2} \neq \emptyset$ and $S \cap A_{n-3} \neq \emptyset$ such that $|S| = \left\lceil\frac{p^{n-1}+k}{2}\right\rceil$. Clearly, $S$ is a dominating set. Let $x \in S \cap A_{n-1}$. We have $deg_S(x) = |S|-1$ and $deg_{\bar{S}}(x)+k = |Z(\mathbb{Z}_{p^n})|-1-|S|+k = p^{n-1}+k-\left\lceil\frac{p^{n-1}+k}{2}\right\rceil+k \leq |S|-1$ and so $deg_S(x) \geq deg_{\bar{S}}(x)+k$. Let $x \in S \cap A_{n-2}$. We have $deg_S(x) = |S|-1$ and $deg_{\bar{S}}(x)+k = |Z(\mathbb{Z}_{p^n})|-1-(|S|+|A_1|)+k = p^{n-1}-1-|S|-|A_1|+k = p^{n-1}+k-\left\lceil\frac{p^{n-1}+k}{2}\right\rceil-|A_1|-1 \leq \left\lceil\frac{p^{n-1}+k}{2}\right\rceil-1$ and so $deg_S(x) \geq deg_{\bar{S}}(x)+k$. Now, let $x \in S \cap A_{n-3}$, then $deg_S(x) = |S|-1$ and $deg_{\bar{S}}(x)+k = |Z(\mathbb{Z}_{p^n})|-1-(|S|+|A_1|+|A_2|)+k = p^{n-1}-1-|S|-|A_1|-|A_2|+k = p^{n-1}+k-\left\lceil\frac{p^{n-1}+k}{2}\right\rceil-|A_1|-|A_2|-1 \leq \left\lceil\frac{p^{n-1}+k}{2}\right\rceil-1$ and so $deg_S(x) \geq deg_{\bar{S}}(x)+k$. Hence, $S$ is a global defensive $k$-alliance of cardinality $|S| = \left\lceil\frac{p^{n-1}+k}{2}\right\rceil$.

**Sub-cases** $k \in [\![2(p^{\alpha-1}-1)-p^{n-1}+1; 2(p^{\alpha}-1)-p^{n-1}]\!]$ for $3 \leq \alpha \leq \frac{n}{2}$:

Let $S \subseteq A_{n-1} \cup A_{n-2} \cup \cdots \cup A_{n-\alpha}$ with $S \cap A_{n-1} \neq \emptyset$, $S \cap A_{n-2} \neq \emptyset$, $\ldots$, $S \cap A_{n-\alpha} \neq \emptyset$. such that $|S| = \left\lceil\frac{p^{n-1}+k}{2}\right\rceil$. Thus, similarly to the previous sub-case, we prove that $S$ is a global defensive $k$-alliance of cardinality $|S| = \left\lceil\frac{p^{n-1}+k}{2}\right\rceil$.

**Sub-case** $k \in [\![2(p^{\frac{n}{2}}-1)-p^{n-1}+1; 2(p^{\frac{n}{2}+1}-1)-p^{n-1}]\!]$:

Let $S \subseteq A_{n-1} \cup A_{n-2} \cup \cdots \cup A_{\frac{n}{2}-1}$ with $S \cap A_{n-1} \neq \emptyset$, $S \cap A_{n-2} \neq \emptyset$, $\ldots$, $S \cap A_{\frac{n}{2}-1} \neq \emptyset$ and

$$\begin{cases} |S \cap A_{\frac{n}{2}-1}| \leq \left\lfloor\frac{|A_1|+|A_2|+\cdots+|A_{\frac{n}{2}-1}|}{2}\right\rfloor, \\ \\ |S \cap A_{\frac{n}{2}-1}| + |S \cap A_{\frac{n}{2}}| \leq \left\lfloor\frac{|A_1|+|A_2|+\cdots+|A_{\frac{n}{2}}|}{2}\right\rfloor \end{cases} \tag{2.1}$$

such that $|S| = \left\lceil\frac{p^{n-1}+k}{2}\right\rceil$. It is clear that $S$ is a dominating set. Let $x \in S \cap A_\beta$ with $\frac{n}{2}+1 \leq \beta \leq n-1$. We have $deg_S(x) = |S|-1$ and $deg_{\bar{S}}(x)+k = |Z(\mathbb{Z}_{p^n})|-1-|S|-(|A_1|+\cdots+|A_{n-\beta-1}|)$. Then, $deg_S(x) \geq deg_{\bar{S}}(x)+k$. Let $x \in S \cap A_{\frac{n}{2}}$. We have $deg_S(x) = |S|-1-|S \cap A_{\frac{n}{2}-1}|$ and $deg_{\bar{S}}(x)+k = |Z(\mathbb{Z}_{p^n})|-1-|S|-(|A_1|+\cdots+|A_{\frac{n}{2}-2}|+|\bar{S} \cap A_{\frac{n}{2}-1}|)+k \leq |S|-1-(|A_1|+\cdots+|A_{\frac{n}{2}-2}|+|\bar{S} \cap A_{\frac{n}{2}-1}|)$. Then, by (2.1), $deg_S(x) \geq deg_{\bar{S}}(x)+k$. Let $x \in S \cap A_{\frac{n}{2}}$. We have $deg_S(x) = |S|-1-(|S \cap A_{\frac{n}{2}}|+|S \cap A_{\frac{n}{2}-1}|)$ and $deg_{\bar{S}}(x)+k = |Z(\mathbb{Z}_{p^n})|-1-|S|-(|A_1|+|A_2|+\cdots+|A_{\frac{n}{2}-2}|+|\bar{S} \cap A_{\frac{n}{2}-1}|+|\bar{S} \cap A_{\frac{n}{2}}|) \leq |S|-1-(|A_1|+|A_2|+\cdots+|A_{\frac{n}{2}-2}|+|\bar{S} \cap A_{\frac{n}{2}-1}|+|\bar{S} \cap A_{\frac{n}{2}}|)$. So, by (2.1), $deg_S(x) \geq deg_{\bar{S}}(x)+k$. Hence, $S$ is a global defensive $k-$alliance of cardinality $|S| = \left\lceil\frac{p^{n-1}+k}{2}\right\rceil$.

**Sub-cases** $k \in [\![2(p^{\frac{n}{2}+\alpha-1}-1)-p^{n-1}+1; 2(p^{\frac{n}{2}+\alpha}-1)-p^{n-1}]\!]$ for $2 \leq \alpha \leq \frac{n}{2}-2$:

Let $S \subseteq A_{n-1} \cup A_{n-2} \cup \cdots \cup A_{\frac{n}{2}-\alpha}$ with $S \cap A_{n-1} \neq \emptyset$, $S \cap A_{n-2} \neq \emptyset$, $\ldots$, $S \cap A_{\frac{n}{2}-\alpha} \neq \emptyset$ and

$$\begin{cases} |S\cap A_{\frac{n}{2}-\alpha}| \leq \left\lfloor \frac{|A_1|+\cdots+|A_{\frac{n}{2}-\alpha}|}{2} \right\rfloor, \\ |S\cap A_{\frac{n}{2}-\alpha}| + |S\cap A_{\frac{n}{2}-\alpha+1}| \leq \left\lfloor \frac{|A_1|+\cdots+|A_{\frac{n}{2}-\alpha+1}|}{2} \right\rfloor, \\ \dots\dots \quad \dots\dots \quad \dots\dots\dots \quad \dots\dots \\ |S\cap A_{\frac{n}{2}-\alpha}| + |S\cap A_{\frac{n}{2}-\alpha+1}| + \cdots + |S\cap A_{\frac{n}{2}}| \leq \left\lfloor \frac{|A_1|+|A_2|+\cdots+|A_{\frac{n}{2}}|}{2} \right\rfloor \end{cases}$$

such that $|S| = \left\lceil \frac{p^{n-1}+k}{2} \right\rceil$. Then, similarly to the previous sub-case, we prove that $S$ is a global defensive $k-$alliance.

**Sub-case $k \in [\![2(p^{n-2}-1) - p^{n-1} + 1; p-1]\!]$:**

Let $S \subseteq A_{n-1} \cup A_{n-2} \cup \cdots \cup A_1$ with $S \cap A_{n-1} \neq \emptyset$, $S \cap A_{n-2} \neq \emptyset$, $\ldots$, $S \cap A_1 \neq \emptyset$ and

$$\begin{cases} |S\cap A_1| \leq \left\lfloor \frac{|A_1|}{2} \right\rfloor, \\ |S\cap A_1| + |S\cap A_2| \leq \left\lfloor \frac{|A_1|+|A_2|}{2} \right\rfloor, \\ \dots\dots \quad \dots\dots \quad \dots\dots\dots \quad \dots\dots \\ |S\cap A_1| + |S\cap A_2| + \cdots + |S\cap A_{\frac{n}{2}}| \leq \left\lfloor \frac{|A_1|+|A_2|+\cdots+|A_{\frac{n}{2}}|}{2} \right\rfloor \end{cases}$$

such that $|S| = \left\lceil \frac{p^{n-1}+k}{2} \right\rceil$. So, it is not difficult to verify that $S$ is a global defensive $k$-alliance of cardinality $\left\lceil \frac{p^{n-1}+k}{2} \right\rceil$.

Hence, for every $k \in [\![2 - p^{n-1}; p-1]\!]$, $\gamma_k^d(\Gamma(\mathbb{Z}_{p^n})) = \left\lceil \frac{p^{n-1}+k}{2} \right\rceil$. □

## 2.2 The global defensive $k$-alliances of zero-divisor graphs of some direct products of finite fields

In this section, we study the global defensive $k$-alliances of zero-divisor graphs of some direct products of finite fields. So, we start with the following result which determines the global defensive $k$-alliance number of zero-divisor graphs of the direct product of two finite fields.

**Theorem 2.2.** *If $F$ and $K$ are two finite fields. Then,*

1. *If $|F| = 2$ ($F \cong \mathbb{Z}_2$) and $|K| \geq 2$. Then, for every $k \in [\![1 - |K|; 1]\!]$,*

$$\gamma_k^d(\Gamma(F \times K)) = \left\lceil \frac{|K| + k + 1}{2} \right\rceil$$

2. *If $|K| \geq |F| \geq 3$,*

$$\gamma_k^d(\Gamma(F \times K)) = \begin{cases} 2 & \text{if } k = 1 - |K|, \\ \left\lceil \frac{|K|+k+1}{2} \right\rceil & \text{if } k \in [\![2 - |K|; 3 - |F|]\!], \\ \left\lfloor \frac{|F|+k}{2} \right\rfloor + \left\lfloor \frac{|K|+k}{2} \right\rfloor & \text{if } k \in [\![4 - |F|; |F| - 1]\!]. \end{cases}$$

*Proof.* 1. Let $k \in [\![1-|K|;1]\!]$ and let $S$ be a global defensive $k-$alliance of cardinality $r = \gamma_k^d(\Gamma(F\times K))$. If $(1,0) \notin S$. Then, $(0,u) \in S$ for all $u \in K^*$. Thus, for $k=0$ (*resp.*, $k=1$), we have $deg_S((0,u)) = 0 \geq deg_{\bar{S}}((0,u)) = 1$ (*resp.*, $deg_S((0,u)) = 0 \geq deg_{\bar{S}}((0,u)) + 1 = 2$), a contradiction. Consequently, $(1,0) \in S$ and so $deg_S((1,0)) \geq deg_{\bar{S}}((1,0)) + k$. That is $|S\cap\{0\}\times K^*| \geq |\bar{S}\cap\{0\}\times K^*|+k$. Then, $|S\cap\{0\}\times K^*| \geq |\{0\}\times K^*|-|S\cap\{0\}\times K^*|+k$. Thus, $|S\cap\{0\}\times K^*| \geq \frac{|K|-1+k}{2}$. Hence, $r \geq |S\cap\{0\}\times K^*| + 1 \geq \frac{|K|+k+1}{2}$ and so $r \geq \left\lceil \frac{|K|+k+1}{2} \right\rceil$. Now, set $S = \{(1,0)\}\cup\{0\}\times S_1$, such that $S_1 \subseteq K^*$ and $|S_1| = \left\lceil \frac{|K|+k+1}{2} \right\rceil - 1$. Clearly, $S$ is a dominating set. We have $deg_S((1,0)) = |S_1| = \left\lceil \frac{|K|+k+1}{2} \right\rceil - 1$ and $deg_{\bar{S}}((1,0)) + k = |\{0\}\times K^*| - |S_1| + k = |K| + k - \left\lceil \frac{|K|+k+1}{2} \right\rceil$. Then, $deg_{\bar{S}}((1,0)) + k \leq deg_S((1,0))$. Let $u \in K^*$, then $deg_S((0,u)) = 1 \geq k = deg_{\bar{S}}(0,u) + k$. Hence, $S$ is a global defensive $k$-alliance and so $r \leq |S| = \left\lceil \frac{|K|+k+1}{2} \right\rceil$. Then, by the first part $r = \left\lceil \frac{|K|+k+1}{2} \right\rceil$.

2. **Case** $k = 1-|K|$:
Since every dominating set in $\Gamma(F\times K)$ has a cardinality greater than or equal 2. Then, every global defensive $k$-alliance $S$ has cardinality $|S| \geq 2$. Thus, $\gamma_k^d(\Gamma(F\times K)) \geq 2$.
Now, set $S = \{(0,v),(u,0)\}$ such that $v \in K^*$ and $u \in F^*$. It is clear that $S$ is a dominating set.
We have $deg_S((0,v)) = 1$ and $deg_{\bar{S}}((0,v))+1-|K| = |F|-2+1-|K| = |F|-|K|-1$. Then, $deg_S((0,v)) \geq deg_{\bar{S}}((0,v)) + k$. Moreover, $deg_S((u,0)) = 1$ and $deg_{\bar{S}}((u,0)) + 1 - |K| = |K| - 2 + 1 - |K| = -1$. Thus, $deg_S((u,0)) \geq deg_{\bar{S}}((u,0)) + k$. So, $S$ is a global defensive $k$-alliance of cardinality $|S| = 2$. Hence, $r = \gamma_k^d(\Gamma(F\times K)) = |S| = 2$.

**Case** $k \in [\![2-|K|;3-|F|]\!]$:
Set $S = \{(u,0)\}\cup\{0\}\times S_1$ such that $u \in F^*$, $S_1 \subseteq K^*$ and $|S_1| = \left\lceil \frac{|K|+k+1}{2} \right\rceil - 1$. Clearly, $S$ is a dominating set. We have $deg_S((u,0)) = |S_1| = \left\lceil \frac{|K|+k+1}{2} \right\rceil - 1$ and $deg_{\bar{S}}((u,0))+k = |\{0\}\times K^*| - |S_1| + k = |K| + k - \left\lceil \frac{|K|+k+1}{2} \right\rceil \leq \left\lceil \frac{|K|+k+1}{2} \right\rceil - 1$. Then, $deg_{\bar{S}}((u,0)) + k \leq deg_S((u,0))$.
Let $v \in S_1$. We have, $deg_S((0,v)) = 1$ and $deg_{\bar{S}}(0,v)+k = |\bar{S}\cap F^*\times\{0\}|+k = |F|-2+k$. Thus, $deg_S((0,v)) \geq deg_{\bar{S}}(0,v)+k$. Hence, $S$ is a global defensive $k$-alliance of cardinality $|S| = \left\lceil \frac{|K|+k+1}{2} \right\rceil$.
Now, let $S$ be a global defensive $k$-alliance of minimal cardinality $r = \gamma_k^d(\Gamma(F\times K))$. Since $S$ is a dominating set, it must contain elements of the form $(0,v)$ and $(u,0)$ with $v \in K^*$ and $u \in F^*$. Suppose that there exist $(u_1,0),(u_2,0) \in S$ with $u_1 \neq u_2$. Then, for every $(0,v) \in S$ we have $deg_S((0,v)) = |S\cap F^*\times\{0\}| \geq 2$. On the other hand, $deg_S((u_1,0)) \geq deg_{\bar{S}}((u_1,0)) + k$ and so $|S\cap\{0\}\times K^*| \geq \frac{|K|+k-1}{2}$. Hence, $\left\lceil \frac{|K|+k+1}{2} \right\rceil \geq |S| \geq \frac{|K|+k-1}{2} + 2 = \frac{|K|+k+1}{2} + 1$, a contradiction. Then, $u_1 = u_2$ and so $deg_S((0,v)) = 1$ for every $(0,v) \in S$. Hence, $\left\lceil \frac{|K|+k+1}{2} \right\rceil \geq |S| \geq \frac{|K|+k-1}{2} + 1$ and so $r = |S| = \left\lceil \frac{|K|+k+1}{2} \right\rceil$.

**Case** $k \in [\![4-|F|;|F|-1]\!]$:
Let $S = S_1\times\{0\}\cup\{0\}\times S_2$ such that $S_1 \subseteq F^*$ and $S_2 \subseteq K^*$ with $|S_1| = \left\lfloor \frac{|F|+k}{2} \right\rfloor$ and $|S_2| = \left\lfloor \frac{|K|+k}{2} \right\rfloor$. Clearly, $S$ is a dominating set. We have, $deg_S((u,0)) = |S_2| = \left\lfloor \frac{|K|+k}{2} \right\rfloor$ and $deg_{\bar{S}}((u,0))+k = |\bar{S}\cap\{0\}\times K^*|+k = |\{0\}\times K^*|-|S_2|+k = |K|+k-1-\left\lfloor \frac{|K|+k}{2} \right\rfloor \leq \frac{|K|+k}{2} - \frac{1}{2} \leq \left\lfloor \frac{|K|+k}{2} \right\rfloor$. Then, $deg_S((u,0)) \geq deg_{\bar{S}}((u,0))+k$. Let $v \in S_2$, then $deg_S((0,v)) =$

$|S_1| = \left\lfloor \frac{|F|+k}{2} \right\rfloor$ and $deg_S((0,v))+k = |\bar{S} \cap F^* \times \{0\}|+k = |F^* \times \{0\}| - |S \cap F^* \times \{0\}| + k = |F| + k - 1 - \left\lfloor \frac{|F|+k}{2} \right\rfloor$. Thus, $deg_{\bar{S}}((0,v)) + k \leq \frac{|F|+k}{2} - \frac{1}{2} \leq deg_S((0,v))$. Hence, $S$ is a global defensive $k$-alliance.
Now, let $S$ be a global defensive $k$-alliance of cardinality $r = \gamma_k^d(\Gamma(F \times K))$. Assume that for all $u \in F^*$, $(u,0) \notin S$. Then, $(0,v) \in S$ for all $v \in K^*$ since $S$ is a dominating set. Thus, for every $v \in K^*$, $deg_S((0,v)) \geq deg_{\bar{S}}((0,v)) + k$. Then, $0 \geq |\bar{S} \cap F^* \times \{0\}| + k = |F| - 1 + k \geq 1$, a contradiction. We also get a contradiction when we suppose that $(0,v)$ is not in $S$ for all $v$ in $K^*$. Hence, $(0,v), (u,0) \in S$ for some $u \in F^*$ and $v \in K^*$. Then, $deg_S((u,0)) \geq deg_{\bar{S}}((u,0)) + k$, that is $|S \cap \{0\} \times K^*| \geq |\bar{S} \cap \{0\} \times K^*| + k$. Thus, $|S \cap \{0\} \times K^*| \geq |\{0\} \times K^*| - |S \cap \{0\} \times K^*| + k$, then $|S \cap \{0\} \times K^*| \geq \frac{|K|-1+k}{2}$ and so $|S \cap \{0\} \times K^*| \geq \left\lceil \frac{|K|-1+k}{2} \right\rceil = \left\lfloor \frac{|K|+k}{2} \right\rfloor$. Similarly, we have $deg_S((0,v)) \geq deg_{\bar{S}}((0,v)) + k$ and so $|S \cap F^* \times \{0\}| \geq \left\lfloor \frac{|F|+k}{2} \right\rfloor$. Hence, $|S| = |S \cap \{0\} \times K^*| + |S \cap F^* \times \{0\}| \geq \left\lfloor \frac{|F|+k}{2} \right\rfloor + \left\lfloor \frac{|K|+k}{2} \right\rfloor$ and so $r = \left\lfloor \frac{|F|+k}{2} \right\rfloor + \left\lfloor \frac{|K|+k}{2} \right\rfloor$. □

As a particular case of Theorem 2.2, we fined again [62, Proposition 2.3]. Moreover, we get a result for the global strong defensive alliance which corresponds to the global defensive 0-alliance.

**Corollary 2.1.** *If $F$ and $K$ are two finite fields. Then,*

1. $\gamma_a(\Gamma(\mathbb{Z}_2 \times F)) = \left\lfloor \frac{|F|-1}{2} \right\rfloor + 1$, *[62, Proposition 2.3].*
2. $\gamma_{\hat{a}}(\Gamma(\mathbb{Z}_2 \times F)) = \left\lceil \frac{|F|}{2} \right\rceil + 1$.
3. $\gamma_a(\Gamma(F \times K)) = \left\lfloor \frac{|F|-1}{2} \right\rfloor + \left\lfloor \frac{|K|-1}{2} \right\rfloor$, *if* $|F|, |K| \geq 3$, *[62, Proposition 2.3].*
4. $\gamma_{\hat{a}}(\Gamma(F \times K)) = \left\lceil \frac{|F|-1}{2} \right\rceil + \left\lceil \frac{|K|-1}{2} \right\rceil$.

**Corollary 2.2.** *If $p$ and $q$ are two prime numbers. Then,*

1. ***Case*** $p = 2$ *and* $q \geq 2$ : *For every* $k \in [\![1-q; 1]\!]$, $\gamma_k^d(\Gamma(\mathbb{Z}_p \times \mathbb{Z}_q)) = \left\lceil \frac{q+k+1}{2} \right\rceil$.
2. ***Case*** $3 \leq p \leq q$ :

$$\gamma_k^d(\Gamma(\mathbb{Z}_p \times \mathbb{Z}_q)) = \begin{cases} 2 & \text{if } k = 1-q, \\ \left\lceil \frac{q+k+1}{2} \right\rceil & \text{if } k \in [\![2-q; 3-p]\!], \\ \left\lfloor \frac{p+k}{2} \right\rfloor + \left\lfloor \frac{q+k}{2} \right\rfloor & \text{if } k \in [\![4-p; p-1]\!]. \end{cases}$$

In [62, Theorem 2.6], Muthana and Mamouni established the equality $\gamma_a(\Gamma(\mathbb{Z}_2 \times \mathbb{Z}_2 \times F)) = |F|$ for a finite field $F$ of cardinality $|F| \geq 3$. Here, we give equalities for $|F| \geq 2$ and for all $k \in [\![1-2|F|; 1]\!]$.

**Theorem 2.3.** *If $F$ is a finite field. Then,*

$$\gamma_k^d(\Gamma(\mathbb{Z}_2 \times \mathbb{Z}_2 \times F)) = \begin{cases} 3 & \text{if } k \in [\![1-2|F|; 3-2|F|]\!], \\ |F| + \left\lceil \frac{1+k}{2} \right\rceil & \text{if } k \in [\![4-2|F|; 1]\!]. \end{cases}$$

*Proof.* **Case** $k \in [\![1-2|F|; 3-2|F|]\!]$: There is no dominating set in $\Gamma(\mathbb{Z}_2 \times \mathbb{Z}_2 \times F)$ of cardinality smaller than or equal to two. On the other hand, we consider the set $S = \{(0,0,z),(1,0,0),(0,1,0)\}$ with $z \in F^*$ and we can prove easily that $S$ is a global defensive $k$-alliance with $k \in [\![1-2|F|; 3-2|F|]\!]$. Hence, $\gamma_k^d(\Gamma(\mathbb{Z}_2 \times \mathbb{Z}_2 \times F)) = 3$ for every $k \in [\![1-2|F|; 3-2|F|]\!]$.
**Case** $k \in [\![4-2|F|; 1]\!]$: Let $S = \{0\} \times \{0\} \times S_1 \cup \{(1,0,0),(0,1,0)\}$ with $S_1$ is a subset of $F^*$ and $|S_1| = |F| + \lceil \frac{1+k}{2} \rceil - 2$. Clearly, S is a dominating set. We have, $deg_S((0,1,0)) = 1+|S_1| = |F| + \lceil \frac{1+k}{2} \rceil - 1$ and $deg_{\bar{S}}((0,1,0)) + k = |\mathbb{Z}_2 \times \{0\} \times F| - 1 - (|\{0\} \times \{0\} \times S_1| + 1) + k = |F| - \lceil \frac{1+k}{2} \rceil + k \le |F| + \lceil \frac{1+k}{2} \rceil - 1$. Then, $deg_S((0,1,0)) \ge deg_{\bar{S}}((0,1,0)) + k$. Similarly, $deg_S((1,0,0)) \ge deg_{\bar{S}}((1,0,0)) + k$. Let $z \in S_1$, then $deg_S((0,0,z)) = 2$ and $deg_{\bar{S}}((0,0,z)) + k = |\mathbb{Z}_2 \times \mathbb{Z}_2 \times \{0\}| - 1 - 2 + k = 1 + k$. Thus, $deg_S((0,0,z)) \ge deg_{\bar{S}}((0,0,z)) + k$. Hence, $S$ is a global defensive $k$-alliance of cardinality $|S| = |F| + \lceil \frac{1+k}{2} \rceil$. Now, let $S$ be a global defensive $k$-alliance of minimal cardinality $r = \gamma_k^d(\Gamma(\mathbb{Z}_2 \times \mathbb{Z}_2 \times F))$. From the first part, we have $r \le |F| + \lceil \frac{1+k}{2} \rceil$. Suppose that $(1,0,0) \notin S$, then $(0,1,z) \in S$ for all $z \in F^*$, thus $deg_S((0,1,z)) \ge deg_{\bar{S}}((0,1,z)) + k$ and so $|S \cap \mathbb{Z}_2 \times \{0\} \times \{0\}| = 0 \ge 1+k$ which is not true for $k = 0, 1$. Hence, $(1,0,0) \in S$ and so $deg_S((1,0,0)) \ge deg_{\bar{S}}((1,0,0)) + k$, then $|S \cap \{0\} \times \mathbb{Z}_2 \times F| \ge |F| + \frac{k-1}{2}$. Thus, $|F| + \frac{k+1}{2} \le |S| \le |F| + \lceil \frac{1+k}{2} \rceil$. Hence, $r = |F| + \lceil \frac{1+k}{2} \rceil$. □

**Corollary 2.3.** *If $p$ is a prime number. Then,*

$$\gamma_k^d(\Gamma(\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_p)) = \begin{cases} 3 & \text{if } k \in [\![1-2p; 3-2p]\!], \\ p + \lceil \frac{1+k}{2} \rceil & \text{if } k \in [\![4-2p; 1]\!]. \end{cases}$$

In the following theorem we extend the equality $\gamma_a(\Gamma(\mathbb{Z}_2 \times F \times K)) = \left\lceil \frac{|F||K|}{2} \right\rceil$ giving in [62, Theorem 2.8] to the other cases.

**Theorem 2.4.** *If $F$ and $K$ are two finite fields such that $|K| \ge |F| \ge 3$. Then,*

$$\gamma_k^d(\Gamma(\mathbb{Z}_2 \times F \times K)) = \begin{cases} 3 & \text{if } k \in [\![1-|K||F|; 5-|K||F|]\!], \\ \left\lceil \frac{|F||K|+k+1}{2} \right\rceil & \text{if } k \in [\![6-|F||K|; 1]\!]. \end{cases}$$

*Proof.* **Case** $k \in [\![1-|K||F|; 5-|K||F|]\!]$: Let $S$ be a global defensive $k$-alliance of minimal cardinality $r = \gamma_k^d(\Gamma(\mathbb{Z}_2 \times F \times K))$. Since $S$ is a dominating set, the cardinality of $S$ must be greater than or equal 3. On the other hand, set $S = \{(1,0,0),(0,u,0),(0,0,v)\}$ with $u \in F^*$ and $v \in K^*$. It is clear that $S$ is a dominating set. We have, $deg_S((1,0,0)) = 2$ and $deg_{\bar{S}}((1,0,0)) + k = (|K^*||F^*| + |K^*| - 1 + |F^*| - 1) + k = |F||K| - 3 + k \le |F||K| - 3 + 5 - |F||K| = 2$. Then, $deg_{\bar{S}}((1,0,0)) + k \le deg_S((1,0,0))$. We have, $deg_S((0,u,0)) = 2$ and $deg_{\bar{S}}((0,u,0)) + k = |K^*| + |K^*| - 1 + k = 2|K| - 3 + k \le 2|K| - 3 + 5 - |K||F| \le 2 + |K|(2 - |F|)$. Then, $deg_{\bar{S}}((0,u,0)) + k \le deg_S((0,u,0))$ (since $|F| > 2$). We have, $deg_S((0,0,v)) = 2$ and $deg_{\bar{S}}((0,0,v)) + k = |F^*| + |F^*| - 1 + k = 2|F| - 3 + k \le 2|F| - 3 + 5 - |K||F| \le 2 + |F|(2 - |K|)$. Then, $deg_{\bar{S}}((0,0,v)) + k \le deg_S((0,0,v))$. Thus, $S$ is a global defensive $k$-alliance of cardinality $|S| = 3$. Hence, $r = 3$.
**Case** $k \in [\![6-|K||F|; 5-2|K|]\!]$: Let $S = \{(1,0,0),(0,u,0),(0,0,v)\} \cup S_1$ with $S_1 \subseteq \{0\} \times F^* \times K^*$ such that $|S_1| = \left\lceil \frac{|F||K|+k+1}{2} \right\rceil - 3$. Clearly, $S$ is a dominating set. We have, $deg_S((1,0,0)) = |S_1| + 2 = \left\lceil \frac{|K||F|+k+1}{2} \right\rceil - 1$ and $deg_{\bar{S}}((1,0,0)) + k = |F||K| - 3 - |S_1| + k =$

$|F||K| - \left\lceil \frac{|K||F|+k+1}{2} \right\rceil + k \leq \left\lceil \frac{|K||F|+k+1}{2} \right\rceil - 1$. So, $deg_S((1,0,0)) \geq deg_{\bar{S}}((1,0,0)) + k$. We have, $deg_S((0,u,0)) = 2$ and $deg_{\bar{S}}((0,u,0)) + k = 2|K| - 3 + k$ and so $deg_S((0,u,0)) \geq deg_{\bar{S}}((0,u,0)) + k$. Also, $deg_S((0,0,v)) = 2$ and $deg_{\bar{S}}((0,0,v)) + k = 2|F| - 3 + k$. Then, $deg_S((0,0,v)) \geq deg_{\bar{S}}((0,0,v)) + k$. Let $(0,v_1,v_2) \in S_1$, we have $deg_S((0,v_1,v_2)) = 1 \geq k = deg_{\bar{S}}((0,v_1,v_2)) + k$. Hence, $S$ is a global defensive $k-$alliance of cardinality $|S| = \left\lceil \frac{|F||K|+k+1}{2} \right\rceil$.

Now, let $S$ be a global defensive $k-$alliance of minimal cardinality $\gamma_k^d(\Gamma(\mathbb{Z}_2 \times F \times K))$. If $(1,0,0) \notin S$, then $\{0\} \times F^* \times K^* \subseteq S$ and also $S$ contains at least two vertices from $\{0\} \times F^* \times \{0\} \cup \{0\} \times \{0\} \times K^* \cup \{1\} \times \{0\} \times K^* \cup \{1\} \times F^* \times \{0\}$. Then, $|S| \geq |F^*||K^*| + 2$ or from the first part, $|S| \leq \left\lceil \frac{|F||K|+k+1}{2} \right\rceil$ and so $|F^*||K^*| + 2 \leq \left\lceil \frac{|F||K|+k+1}{2} \right\rceil$ which is not true for every $k \in [\![6 - |K||F|; 5 - 2|K|]\!]$. Hence, $(1,0,0) \in S$ and so $deg_S((1,0,0)) \geq deg_{\bar{S}}((1,0,0)) + k$, then $|S \cap \{0\} \times F \times K| \geq |\{0\} \times F \times K| - |S \cap \{0\} \times F \times K| + k - 1$. Thus, $|S \cap \{0\} \times F \times K| \geq \frac{|F||K|+k-1}{2}$ and so $|S| \geq \frac{|F||K|+k-1}{2} + 1 = \frac{|F||K|+k+1}{2}$. Hence, $|S| = \left\lceil \frac{|F||K|+k+1}{2} \right\rceil$ (by the first part).

<u>**Case** $k \in [\![6 - 2|K|; 5 - 2|F|]\!]$</u>: Let $S = \{(1,0,0),(0,u,0)\} \cup \{0\} \times \{0\} \times S_1 \cup S_2$ with $S_1 \subseteq K^*$ and $S_2 \subseteq \{0\} \times F^* \times K^*$ such that $|S_1| = \left\lceil \frac{2|K|+k+1}{2} \right\rceil - 2$ and $|S_2| = \left\lceil \frac{|K||F|+k+1}{2} \right\rceil - \left\lceil \frac{2|K|+k+1}{2} \right\rceil$. Clearly, $S$ is a dominating set. We have, $deg_S((1,0,0)) = |S_1| + |S_2| + 1 = \left\lceil \frac{|K||F|+k+1}{2} \right\rceil - 1$ and $deg_{\bar{S}}((1,0,0)) + k = |\{0\} \times F^* \times K^*| - |S_2| + |K^*| - |S_1| + |F^*| - 1 + k = |F||K| + k - \left\lceil \frac{|K||F|+k+1}{2} \right\rceil \leq \left\lceil \frac{|K||F|+k+1}{2} \right\rceil - 1$. So, $deg_S((1,0,0)) \geq deg_{\bar{S}}((1,0,0)) + k$. We have, $deg_S((0,u,0)) = |S_1| + 1 = \left\lceil \frac{2|K|+k+1}{2} \right\rceil - 1$ and $deg_{\bar{S}}((0,u,0)) + k = |K^*| + |K^*| - |S_1| + k = 2|K| + k - \left\lceil \frac{2|K|+k+1}{2} \right\rceil \leq \left\lceil \frac{2|K|+k+1}{2} \right\rceil - 1$. Then, $deg_{\bar{S}}((0,u,0)) + k \leq deg_S((0,u,0))$. Let $v \in S_1$. We have $deg_S((0,0,v)) = 2$ and $deg_{\bar{S}}((0,0,v)) + k = |F^*| - 1 + |F^*| + k$. So, $deg_{\bar{S}}((0,0,v)) + k \leq deg_S((0,0,v))$. Let $(0,v_1,v_2) \in S_2$, then $deg_S((0,v_1,v_2)) = 1$ and $deg_{\bar{S}}((0,v_1,v_2)) + k = 0 + k = k$. So, $deg_{\bar{S}}((0,v_1,v_2)) + k \leq deg_S((0,v_1,v_2))$. Hence, $S$ is a global defensive $k-$alliance of cardinality $|S| = \left\lceil \frac{|F||K|+k+1}{2} \right\rceil$.

Now, let $S$ be a global defensive $k-$alliance of minimal cardinality $\gamma_k^d(\Gamma(\mathbb{Z}_2 \times F \times K))$. If $(1,0,0) \notin S$, then $\{0\} \times F^* \times K^* \subseteq S$ and also $S$ contains at least two vertices from $\{0\} \times F^* \times \{0\} \cup \{0\} \times \{0\} \times K^* \cup \{1\} \times \{0\} \times K^* \cup \{1\} \times F^* \times \{0\}$. Then, $|S| \geq |F^*||K^*| + 2$ and $|S| \leq \left\lceil \frac{|F||K|+k+1}{2} \right\rceil$ from the first part. So, $|F^*||K^*| + 2 \leq \left\lceil \frac{|F||K|+k+1}{2} \right\rceil$ which is not true since $k \in [\![6 - 2|K|; 5 - 2|F|]\!]$. Hence, $(1,0,0) \in S$ and so $deg_S((1,0,0)) \geq deg_{\bar{S}}((1,0,0)) + k$, then $|S \cap \{0\} \times F \times K| \geq |\{0\} \times F \times K| - |S \cap \{0\} \times F \times K| + k - 1$. Thus, $|S \cap \{0\} \times F \times K| \geq \frac{|F||K|+k-1}{2}$ and so $|S| \geq \frac{|F||K|+k-1}{2} + 1 = \frac{|F||K|+k+1}{2}$. Hence, $|S| = \left\lceil \frac{|F||K|+k+1}{2} \right\rceil$.

<u>**Case** $k \in [\![6 - 2|F|; 1]\!]$</u>: Set $S = \{(1,0,0)\} \cup \{0\} \times S_1 \times \{0\} \cup \{0\} \times \{0\} \times S_2 \cup S_3$ with $S_1 \subseteq F^*$, $S_2 \subseteq K^*$, and $S_3 \subseteq \{0\} \times F^* \times K^*$ such that $|S_1| = \left\lceil \frac{2|F|+k+1}{2} \right\rceil - 2$, $|S_2| = \left\lceil \frac{2|K|+k+1}{2} \right\rceil - 2$ and $|S_3| = \left\lceil \frac{|F||K|+k+1}{2} \right\rceil - \left\lceil \frac{2|F|+k+1}{2} \right\rceil - \left\lceil \frac{2|K|+k+1}{2} \right\rceil + 3$. Clearly, $S$ is a dominating set.

We have $deg_S((1,0,0)) = |S_1| + |S_2| + |S_3| = \left\lceil \frac{|F||K|+k+1}{2} \right\rceil - 1$ and $deg_{\bar{S}}((1,0,0)) + k = |F^*||K^*| - |S_3| + |F^*| - |S_1| + |K^*| - |S_2| + k = (|F^*||K^*| + |F^*| + |K^*|) - (|S_3| + |S_2| + |S_1|) + k =$

$|F||K| + k - \left\lceil \frac{|F||K|+k+1}{2} \right\rceil \leq \left\lceil \frac{|F||K|+k+1}{2} \right\rceil - 1$. Then, $deg_S((1,0,0)) \geq deg_{\bar{S}}((1,0,0)) + k$. Let $u \in S_1$, then $deg_S((0,u,0)) = 1 + |S_2| = \left\lceil \frac{2|K|+k+1}{2} \right\rceil - 1$ and $deg_{\bar{S}}((0,u,0)) + k = |K^*| - |S_2| + |K^*| + k = 2|K| + k - \left\lceil \frac{2|K|+k+1}{2} \right\rceil \leq \left\lceil \frac{2|K|+k+1}{2} \right\rceil - 1$. So, $deg_S((0,u,0)) \geq deg_{\bar{S}}((0,u,0)) + k$. Let $v \in S_2$, then $deg_S((0,0,v)) = 1 + |S_1| = \left\lceil \frac{2|F|+k+1}{2} \right\rceil - 1$ and $deg_{\bar{S}}((0,0,v)) + k = |F^*| - |S_1| + |F^*| + k = 2|F| + k - \left\lceil \frac{2|F|+k+1}{2} \right\rceil \leq \left\lceil \frac{2|F|+k+1}{2} \right\rceil - 1$. So, $deg_S((0,0,v)) \geq deg_{\bar{S}}((0,0,v)) + k$. Let $(0,u,v) \in S_3$, we have $deg_S((0,u,v)) = 1$ and $deg_{\bar{S}}((0,u,v)) + k = 0 + k$. Thus, $deg_S((0,u,v)) \geq deg_{\bar{S}}((0,u,v)) + k$. Hence, $S$ is a global defensive $k$-alliance of cardinality $|S| = \left\lceil \frac{|F||K|+k+1}{2} \right\rceil$.
Now, let $S$ be a global defensive $k$-alliance of minimal cardinality $r = \gamma_k^d(\Gamma(\mathbb{Z}_2 \times F \times K))$. If $(1,0,0) \notin S$, then $\{0\} \times F^* \times K^* \subseteq S$ and so $deg_S((0,u,v)) = 0 \geq deg_{\bar{S}}((0,u,v)) + k = 1 + k$ which is not true for $k \in \{0,1\}$. Hence, $(1,0,0) \in S$ and so $deg_S((1,0,0)) \geq deg_{\bar{S}}((1,0,0)) + k$, then $|S \cap \{0\} \times F \times K| \geq |\{0\} \times F \times K| - 1 - |S \cap \{0\} \times F \times K| + k$. Thus, $|S \cap \{0\} \times F \times K| \geq \frac{|F||K|+k-1}{2}$ and so $|S| \geq \frac{|F||K|+k-1}{2} + 1 = \frac{|F||K|+k+1}{2}$. Since, $|S| \leq \left\lceil \frac{|F||K|+k+1}{2} \right\rceil$ (by the first part), $r = |S| = \left\lceil \frac{|F||K|+k+1}{2} \right\rceil$. □

**Corollary 2.4.** *If $p \geq 3$ and $q \geq 3$ are two prime numbers. Then,*

$$\gamma_k^d(\Gamma(\mathbb{Z}_2 \times \mathbb{Z}_p \times \mathbb{Z}_q)) = \begin{cases} 3 & \text{if } k \in [\![1-pq; 5-pq]\!], \\ \left\lceil \frac{pq+k+1}{2} \right\rceil & \text{if } k \in [\![6-pq; 1]\!]. \end{cases}$$

## 2.3 The global defensive $k$-alliances of zero-divisor graphs of the direct products of $\mathbb{Z}_2$ and finite rings

The purpose of this section is to study the global defensive $k$-alliances of zero-divisor graphs of the direct product of $\mathbb{Z}_2$ and a finite ring. We start by giving an upper and lower bound of the global defensive $k$-alliance number of $\Gamma(\mathbb{Z}_2 \times R)$ for every $k \in [\![1-|R|; 1]\!]$ where $R$ is a finite ring.

**Theorem 2.5.** *Let $R$ be a finite ring. Then, for every $k \in [\![1-|R|; 1]\!]$,*

$$\left\lceil \frac{|R|+k+1}{2} \right\rceil \leq \gamma_k^d(\Gamma(\mathbb{Z}_2 \times R)) \leq \left\lceil \frac{|R|+2|Z(R)|+k-1}{2} \right\rceil.$$

*Proof.* If $R$ is a finite field, then, $\gamma_k^d(\Gamma(\mathbb{Z}_2 \times R)) = \left\lceil \frac{|R|+k+1}{2} \right\rceil$, by Theorem 2.2. Thus, we can assume that $R$ is not a finite field. Let $S$ be a global defensive $k$-alliance of minimal cardinality $r = \gamma_k^d(\Gamma(\mathbb{Z}_2 \times R))$. If $(1,0) \notin S$, then $\{0\} \times U(R) \subseteq S$. So, for every $y \in U(R)$ we have $deg_S((0,y)) \geq deg_{\bar{S}}((0,y)) + k$ and so $k \leq -1$ which is not true for $k = 0$ and $k = 1$. Hence, $(1,0)$ must be inside of $S$ and so $deg_S((1,0)) \geq deg_{\bar{S}}((1,0)) + k$. Then, $|S \cap \{0\} \times R^*| \geq |\bar{S} \cap \{0\} \times R^*| + k$ and so $|S \cap \{0\} \times R^*| \geq \frac{|R|+k-1}{2}$. Then, $|S| \geq \frac{|R|+k+1}{2}$. Hence, $r = |S| \geq \left\lceil \frac{|R|+k+1}{2} \right\rceil$. It is easy to check the other bound. Indeed, just add $|Z(R)^*|$ elements from $\mathbb{Z}_2 \times Z(R)^*$ to have the domination property, that is $\gamma_k^d(\Gamma(\mathbb{Z}_2 \times R)) \leq \left\lceil \frac{|R|+k+1}{2} \right\rceil + |Z(R)| - 1 = \left\lceil \frac{|R|+2|Z(R)|+k-1}{2} \right\rceil$. □

In [62, Proposition 2.4], Muthana and Mamouni established the equality $\gamma_a(\Gamma(\mathbb{Z}_2\times R)) = \left\lceil \frac{|R|}{2} \right\rceil$ for a local ring $R$. Here, we give equalities for some other integers $k \in [\![1-|R|;1]\!]$. The following result presents the cases $k=0$ and $k=1$.

**Theorem 2.6.** *Let $R$ be a finite local ring which is not a field. Then,*

$$\gamma_k^d(\Gamma(\mathbb{Z}_2\times R)) = \begin{cases} \left\lceil \frac{|R|+1}{2} \right\rceil & \text{if } k=0, \\ \left\lceil \frac{|R|}{2} \right\rceil + 2 & \text{if } k=1. \end{cases}$$

*Proof.* <u>Case $k=0$</u>: We have, $|Z(R)| \le \left\lfloor \frac{|R|}{2} \right\rfloor$. Then, we can consider a set $S_1 \subseteq R^*$ such that $Z(R)^* \subseteq S_1$ and $|S_1| = \left\lfloor \frac{|R|}{2} \right\rfloor$. Set $S = \{(1,0)\}\cup\{0\}\times S_1$ and let us prove that it is a global strong alliance set. Clearly, $S$ is a dominating set. We have, $deg_S((1,0)) = |\{0\}\times S_1| = |S_1| = \left\lfloor \frac{|R|}{2} \right\rfloor$ and $deg_{\bar{S}}((1,0)) = |\bar{S}\cap\{0\}\times R^*| = |\{0\}\times R^*| - |S_1| = |R| - \left\lfloor \frac{|R|}{2} \right\rfloor - 1 \le \left\lfloor \frac{|R|}{2} \right\rfloor$. Then, $deg_S((1,0)) \ge deg_{\bar{S}}((1,0))$. Let $z \in Z(R)^*$. If $z^2=0$, then $deg_S((0,z)) = |S\cap\mathbb{Z}_2\times \mathrm{Ann}_R(z)| = |\{(1,0)\}\cup\{0\}\times\mathrm{Ann}_R(z)| - 2 = |\mathrm{Ann}_R(z)| - 1$ and $deg_{\bar{S}}((0,z)) = |\bar{S}\cap\mathbb{Z}_2\times\mathrm{Ann}_R(z)| = |\{1\}\times\mathrm{Ann}_R(z)| - 1 = |\mathrm{Ann}_R(z)| - 1$. Thus, $deg_S((0,z) \ge deg_{\bar{S}}((0,z))$. If $z^2 \ne 0$. Then, $deg_S((0,z)) = |S\cap\mathbb{Z}_2\times\mathrm{Ann}_R(z)| = |\{(1,0)\}\cup\{0\}\times\mathrm{Ann}_R(z)| - 1 = |\mathrm{Ann}_R(z)|$ and $deg_{\bar{S}}((0,z)) = |\bar{S}\cap\mathbb{Z}_2\times\mathrm{Ann}_R(z)| = |\{1\}\times\mathrm{Ann}_R(z)| - 1 = |\mathrm{Ann}_R(z)| - 1$. Thus, $deg_S((0,z) \ge deg_{\bar{S}}((0,z))$. Let $u \in S_1 \setminus Z(R)$. Then, $deg_S((0,u)) = 1 \ge 0 = deg_{\bar{S}}((0,u))$. Hence, $S$ is a global strong alliance set and so $\gamma_0^d(\Gamma(\mathbb{Z}_2\times R)) \le \left\lfloor \frac{|R|}{2} \right\rfloor + 1$.
Now, let $S$ be a global strong alliance set of minimal cardinality $r = \gamma_0^d(\mathbb{Z}_2\times R)$. Suppose that $(1,0)\notin S$. Then, for every element $u \in U(R)$, $(0,u)\in S$, since $S$ is a dominating set. Or, $S$ is a strong alliance set, then $deg_S((0,u)) = 0 \ge deg_{\bar{S}}((0,u)) = 1$, a contradiction. Hence, $(1,0)\in S$ and so $deg_S((1,0)) \ge deg_{\bar{S}}((1,0))$ which implies that $|S\cap\{0\}\times R^*| \ge |\bar{S}\cap\{0\}\times R^*| = |\{0\}\times R^*| - |S\cap\{0\}\times R^*|$ and so $|S\cap\{0\}\times R^*| \ge \frac{|R|-1}{2}$. Thus, $|S| \ge \frac{|R|}{2} - \frac{1}{2} + 1$. Hence, $\frac{|R|}{2} + \frac{1}{2} \le r \le \left\lfloor \frac{|R|}{2} \right\rfloor + 1$ which implies that $\gamma_0^d(\Gamma(\mathbb{Z}_2\times R)) = \left\lceil \frac{|R|+1}{2} \right\rceil$.
<u>Case $k=1$</u>: Consider the set $S_1 \subseteq R^*$ such that $Z(R)^* \subseteq S_1$ and $|S_1| = \left\lceil \frac{|R|}{2} \right\rceil$. Let $x \in Z(R)^*$ such that $\mathrm{Ann}_R(x) = Z(R)$ ($R$ is a finite local ring) and set $S = \{(1,0),(1,x)\}\cup\{0\}\times S_1$. Clearly, $S$ is a dominating set. We have, $deg_S((1,0)) = |\{0\}\times S_1| = |S_1| = \left\lceil \frac{|R|}{2} \right\rceil$ and $deg_{\bar{S}}((1,0)) + 1 = |\bar{S}\cap\{0\}\times R^*| + 1 = |\{0\}\times R^*| - |S_1| + 1 = |R| - \left\lceil \frac{|R|}{2} \right\rceil \le \left\lceil \frac{|R|}{2} \right\rceil$. Then, $deg_S((1,0)) \ge deg_{\bar{S}}((1,0)) + 1$. We have, $deg_S((1,x)) = |S\cap\{0\}\times Z(R)| = |\{0\}\times Z(R)| - 1 = |Z(R)^*|$ and $deg_{\bar{S}}((1,x)) + 1 = |\bar{S}\cap\{0\}\times Z(R)| + 1 = 1$. Hence, $deg_S((1,x)) \ge deg_{\bar{S}}((1,x)) + 1$. Let $y \in Z(R)$. If $y^2=0$. Then, $deg_S((0,y)) = |S\cap\mathbb{Z}_2\times\mathrm{Ann}_R(y)| = |\{(1,0),(1,x)\}\cup\{0\}\times\mathrm{Ann}_R(y)| - 2 = |\mathrm{Ann}_R(y)|$ and $deg_{\bar{S}}((0,y)) + 1 = |\bar{S}\cap\mathbb{Z}_2\times\mathrm{Ann}_R(y)| = |\{1\}\times\mathrm{Ann}_R(y)| - 2 + 1 = |\mathrm{Ann}_R(y)| - 1$. Thus, $deg_S((0,y) \ge deg_{\bar{S}}((0,y))$. If $y^2 \ne 0$, then $deg_S((0,y)) = |S\cap\mathbb{Z}_2\times\mathrm{Ann}_R(y)| = |\{(1,0),(1,x)\}| + |\{0\}\times\mathrm{Ann}_R(y)| - 1 = |\mathrm{Ann}_R(y)| + 1$ and $deg_{\bar{S}}((0,y)) + 1 = |\bar{S}\cap\mathbb{Z}_2\times\mathrm{Ann}_R(y)| + 1 = |\{1\}\times\mathrm{Ann}_R(y)| - 2 + 1 = |\mathrm{Ann}_R(y)| - 1$. Then, $deg_S((0,y) \ge deg_{\bar{S}}((0,y)) + 1$. Let $u \in R\setminus Z(R)$. Then, $deg_S((0,u) = 1 \ge 0 + 1 = deg_{\bar{S}}((0,u)) + 1$. Hence, $S$ is a global defensive 1-alliance set.
Now, let $S$ be a global defensive 1-alliance set of minimal cardinality $r = \gamma_1^d(\Gamma(\mathbb{Z}_2\times R))$. If $(1,0)\notin S$. Then, $(0,u)\in S$, for every $u \in U(R)$ and so $deg_S((0,u)) = 0 \ge deg_{\bar{S}}((0,u)) + 1 = 2$, a contradiction. Hence, $(1,0)\in S$ and so $deg_S(1,0) \ge deg_{\bar{S}}((1,0)) + 1$, then

$|S\cap\{0\}\times R^*|\geq|\bar{S}\cap\{0\}\times R^*|+1$. Thus, $|S|\geq\frac{|R|}{2}+1$. Suppose that for all $y\in Z(R)^*$, $(1,y)\in\bar{S}$, then for all $x\in Z(R)^*$, $(0,x)\in S$. Otherwise, there exists $(0,y)\in\bar{S}$ which is adjacent to at least one element of the form $(0,x)$ ($\Gamma(R)$ is a connected graph). Thus, $deg_S((0,x))\geq deg_{\bar{S}}(0,x)+1$ and so if $x^2\neq 0$, we have $|S\cap\mathbb{Z}_2\times\mathrm{Ann}_R(x)|\geq|\bar{S}\cap\mathbb{Z}_2\times\mathrm{Ann}_R(x)|+1$, then $|S\cap\mathbb{Z}_2\times\mathrm{Ann}_R(x)|\geq|\bar{S}\cap\mathbb{Z}_2\times\mathrm{Ann}_R(x)|+1$ and so $|\mathrm{Ann}_R(x)|-1\geq|\mathrm{Ann}_R(x)|-2$, a contradiction. Thus, for all $x\in Z(R)^*$, $(0,x)\in S$. Or, $R$ is a finite local ring, then there exists $y\in Z(R)^*$ such that $\mathrm{Ann}_R(y)=Z(R)$. Thus, $deg_S((0,y))\geq deg_{\bar{S}}((0,y))+1$ and so $|S\cap\mathbb{Z}_2\times\mathrm{Ann}_R(x)|\geq|\bar{S}\cap\{0\}\times\mathrm{Ann}_R(x)|+1$. Then, $|Z(R)^*|\geq|Z(R)^*|+1$, a contradiction. Hence, there exists $z\in Z(R)^*$ such that $(1,z)\in S$ and so $\left\lceil\frac{|R|}{2}\right\rceil+2\geq|S|\geq\frac{|R|}{2}+2$. Thus, $\gamma_1^d(\Gamma(\mathbb{Z}_2\times R))=\left\lceil\frac{|R|}{2}\right\rceil+2$. □

Also, we have the following results.

**Theorem 2.7.** *Let $R$ be a finite local ring which is not a field. Then,*

$$\gamma_k^d(\Gamma(\mathbb{Z}_2\times R))=\begin{cases}2 & \text{if } k\in[\![1-|R|;3-|R|]\!],\\ \left\lceil\frac{|R|+k+1}{2}\right\rceil & \text{if } k\in[\![4-|R|;4-2|Z(R)|]\!].\end{cases}$$

*Proof.* Case $k\in[\![1-|R|;3-|R|]\!]$: Let $x\in Z(R)^*$ such that $Z(R)=\mathrm{Ann}_R(x)$ (since $R$ is a finite local ring) and set $S=\{(1,0),(0,x)\}$. Clearly, $S$ is a dominating set. We have, $deg_S((1,0))=1$ and $deg_{\bar{S}}((1,0))+k=|U(R)|+|Z(R)^*|-1+k=|R|-2+k$. Then, $deg_S((1,0))\geq deg_{\bar{S}}((1,0))+k$. We have, $deg_S((0,x))=1$ and $deg_{\bar{S}}((0,x))+k=|Z(R)^*|+|Z(R)^*|-1+k=2|Z(R)|-3+k\leq 2|Z(R)|-|R|$. So, $deg_S((0,x))\geq deg_{\bar{S}}((0,x))+k$. Hence, $S$ is a global defensive $k$-alliance of cardinality $|S|=2$.
Now, let $S$ be a global defensive $k$-alliance of minimal cardinality $r=\gamma_k^d(\Gamma(\mathbb{Z}_2\times R))$. Then, $r=|S|\leq 2$ and since $S$ is a dominating set, $|S|\geq 2$. Hence, $r=\gamma_k^d(\Gamma(\mathbb{Z}_2\times R))=2$.
Case $k\in[\![4-|R|;4-2|Z(R)|]\!]$: Let $S=\{(1,0),(0,x)\}\cup\{0\}\times S_1$ such that $x\in Z(R)^*$, $Z(R)=\mathrm{Ann}(x)$ and $S_1\subseteq U(R)$ with $|S_1|=\left\lceil\frac{|R|+k+1}{2}\right\rceil-2$. It is clear that $S$ is a dominating set. We have, $deg_S((1,0))=1+\left\lceil\frac{|R|+k+1}{2}\right\rceil-2=\left\lceil\frac{|R|+k+1}{2}\right\rceil-1$ and $deg_{\bar{S}}((1,0))+k=|\{0\}\times R^*|-|S_1|-1+k=|R|-2-\left\lceil\frac{|R|+k+1}{2}\right\rceil+k\leq\left\lceil\frac{|R|+k+1}{2}\right\rceil-3$. Thus, $deg_S((1,0))\geq deg_{\bar{S}}((1,0))+k$. Let $u\in S_1$, then $deg_S((0,u))=1$ and $deg_{\bar{S}}((0,u))+k=k$. Then, $deg_S((0,u))\geq deg_{\bar{S}}((0,u))+k$. We have, $deg_S((0,x))=1$ and $deg_{\bar{S}}((0,x))+k=2|Z(R)|-3+k\leq 1$. Thus, $deg_S((0,x))\geq deg_{\bar{S}}((0,x))+k$. Hence, $S$ is a global defensive $k$-alliance of cardinality $|S|=\left\lceil\frac{|R|+k+1}{2}\right\rceil$.
Now, let $S$ be a global defensive $k$-alliance of minimal cardinality $r=\gamma_k^d(\Gamma(\mathbb{Z}_2\times R))$.
If $(1,0)\notin S$. Then, $\{0\}\times U(R)\subseteq S$ and also $S$ contains at least one element of the form $(0,z)$ or $(1,z)$ with $z\in Z(R)^*$. Thus, $|S|\geq|U(R)|+1=|R|-|Z(R)|+1$ and so $\frac{|R|+k+1}{2}+1\leq|S|\leq\left\lceil\frac{|R|+k+1}{2}\right\rceil$, a contradiction. Then, $(1,0)$ must be in $S$ and so $deg_S((1,0))\geq deg_{\bar{S}}((1,0))+k$. Then, $|S\cap\{0\}\times R^*|\geq|\bar{S}\cap\{0\}\times R^*|+k$ and so $|S\cap\{0\}\times R^*|\geq\frac{|R|+k-1}{2}$. Then, $|S|\geq\frac{|R|+k+1}{2}$ and by the first part, $|S|\leq\left\lceil\frac{|R|+k+1}{2}\right\rceil$. Hence, $r=|S|=\left\lceil\frac{|R|+k+1}{2}\right\rceil$. □

Notice that for $R\cong\mathbb{Z}_4$ or $R\cong\mathbb{Z}_2[X]/(X^2)$ where $|Z(R)|=2$, $[\![4-|R|;4-2|Z(R)|]\!]=\{0\}$. In this case, the equalities of Theorems 2.6 and 2.7 are the same.

**Corollary 2.5.** *Let $p$ be a prime number and $n$ be a positive integer. Then,*

$$\gamma_k^d(\Gamma(\mathbb{Z}_2 \times \mathbb{Z}_{p^n})) = \begin{cases} 2 & \text{if } k \in [\![1-p^n; 3-p^n]\!], \\ \left\lceil \frac{p^n+k+1}{2} \right\rceil & \text{if } k \in [\![4-p^n; 4-2p^{n-1}]\!], \\ \left\lceil \frac{p^n}{2} \right\rceil & \text{if } k = -1, \\ \left\lceil \frac{p^n+1}{2} \right\rceil & \text{if } k = 0, \\ \left\lceil \frac{p^n}{2} \right\rceil + 2 & \text{if } k = 1. \end{cases}$$

*and* $\left\lceil \frac{p^n+k+1}{2} \right\rceil \leq \gamma_k^d(\Gamma(\mathbb{Z}_2 \times \mathbb{Z}_{p^n})) \leq \left\lceil \frac{p^{n-1}(p+2)+k-1}{2} \right\rceil$ *for every* $k \in [\![5-2p^{n-1}; -2]\!]$.

For a finite local ring $R$ with a nilpotent maximal ideal $M$ of index 2, we can improve the inequality of Theorem 2.5 and give equality for the remaining cases other than the ones studied in Theorems 2.6 and 2.7. We only deal with the case where $|M| \geq 4$. The other cases are rather simple. In fact, if $|M| = 2$, then $R \cong \mathbb{Z}_4$ or $\mathbb{Z}_2[X]/(X^2)$, by Proposition 1.19. So, $\gamma_{-3}^d(\Gamma(\mathbb{Z}_2 \times R)) = \gamma_{-2}^d(\Gamma(\mathbb{Z}_2 \times R)) = \gamma_{-1}^d(\Gamma(\mathbb{Z}_2 \times R)) = 2$, $\gamma_0^d(\Gamma(\mathbb{Z}_2 \times R)) = 3$ and $\gamma_1^d(\Gamma(\mathbb{Z}_2 \times R)) = 4$. If $|M| = 3$, then $R \cong \mathbb{Z}_9$ or $\mathbb{Z}_3[X]/(X^2)$, by Proposition 1.19. So, $\gamma_{-8}^d(\Gamma(\mathbb{Z}_2 \times R)) = \gamma_{-7}^d(\Gamma(\mathbb{Z}_2 \times R)) = \gamma_{-6}^d(\Gamma(\mathbb{Z}_2 \times R)) = 2$, $\gamma_{-5}^d(\Gamma(\mathbb{Z}_2 \times R)) = \gamma_{-4}^d(\Gamma(\mathbb{Z}_2 \times R)) = 3$, $\gamma_{-3}^d(\Gamma(\mathbb{Z}_2 \times R)) = \gamma_{-2}^d(\Gamma(\mathbb{Z}_2 \times R)) = 4$, $\gamma_{-1}^d(\Gamma(\mathbb{Z}_2 \times R)) = \gamma_0^d(\Gamma(\mathbb{Z}_2 \times R)) = 5$ and $\gamma_1^d(\Gamma(\mathbb{Z}_2 \times R)) = 7$.

**Theorem 2.8.** *Let $R$ be a finite local ring such that its maximal ideal $M$ is nilpotent of index 2. Then, for $k \in [\![5-2|M|; -2]\!]$ with $|M| \geq 4$, we have* $\gamma_k^d(\Gamma(\mathbb{Z}_2 \times R)) = \left\lceil \frac{|R|+k+1}{2} \right\rceil$.

*Proof.* Let $k \in [\![5-2|M|; -2]\!]$ and set $S = \{(1,0)\} \cup \{0\} \times S_1 \cup \{0\} \times S_2$ with $S_1 \subseteq M^*$ and $S_2 \subseteq U(R)$ such that $|S_1| = \left\lceil \frac{2|M|+k+1}{2} \right\rceil - 1$ and $|S_2| = \left\lceil \frac{|R|+k+1}{2} \right\rceil - \left\lceil \frac{2|M|+k+1}{2} \right\rceil$. We have $deg_S((1,0)) = |S_1| + |S_2| = \left\lceil \frac{|R|+k+1}{2} \right\rceil - 1$ and $deg_{\bar{S}}((1,0)) + k = |R| - 1 - (|S_1| + |S_2|) + k = |R| + k + 1 - \left\lceil \frac{|R|+k+1}{2} \right\rceil - 1 \leq \left\lceil \frac{|R|+k+1}{2} \right\rceil - 1$. Then, $deg_S((1,0)) \geq deg_{\bar{S}}((1,0)) + k$. Let $z \in S_1$, we have $deg_S((0,z)) = |S_1|$ and $deg_{\bar{S}}((0,z)) + k = |M| - 1 - |S_1| + |M| - 1 + k = 2|M| + k - \left\lceil \frac{2|M|+k+1}{2} \right\rceil \leq \left\lceil \frac{2|M|+k+1}{2} \right\rceil - 1$. Then, $deg_S((0,z)) \geq deg_{\bar{S}}((0,z)) + k$. Let $u \in S_2$, we have $deg_S((0,u)) = 1$ and $deg_{\bar{S}}((0,u)) + k = 0 + k$. Thus, $deg_S((0,u)) \geq deg_{\bar{S}}((0,u)) + k$. Hence, $S$ is a global defensive $k-$alliance of cardinality $|S| = \left\lceil \frac{|R|+k+1}{2} \right\rceil$.
Now, let $S$ be a global defensive $k-$alliance of minimal cardinality $\gamma_k^d((\mathbb{Z}_2 \times R))$. From the first part and Theorem 2.5, we get the equality $\gamma_k^d((\mathbb{Z}_2 \times R)) = \left\lceil \frac{|R|+k+1}{2} \right\rceil$. □

**Corollary 2.6.** *Let $p$ be a prime number. Then, for every $k \in [\![1-p^2; 1]\!]$,* $\gamma_k^d(\Gamma(\mathbb{Z}_2 \times \mathbb{Z}_{p^2})) = \left\lceil \frac{p^2+k+1}{2} \right\rceil$.

# Chapter 3

# Partitioning zero-divisor graphs of rings into global defensive alliances

The chapter focuses on the problem of partitioning the vertex set of the zero-divisor graph of a commutative ring into global defensive alliances. This problem has been previously studied, for general graphs, by Eroh and Gera [39, 40] and by Haynes and Lachniet [50]. Here, we connected it with the ring theoretical context. We characterize various commutative finite rings for which the zero divisor graph is partitionable into global defensive alliances. We also give several examples to illustrate the scopes and limits of our results. See the introduction for a complete overview of this chapter.

## 3.1 Partitioning zero-divisor graphs of local rings into global defensive alliances

In this section, we study when zero-divisor graphs of certain finite local commutative rings are partitionable into global defensive alliances and calculate their global defensive alliance partition numbers. First, we start by giving a lower bound on the cardinality of the set of zero-divisors $Z(R)$ in terms of the global defensive alliance partition number of $\Gamma(R)$.

**Proposition 3.1.** *Let $R$ be a ring. Then,*

$$|Z(R)| \geq \psi_g(\Gamma(R))^2 - \psi_g(\Gamma(R)) + 1.$$

*Proof.* Assume that $\Gamma(R)$ is partitionable into global defensive alliances. Let $r = \psi_g(\Gamma(R))$ and $\{S_1, \ldots, S_r\}$ be a partition of $\Gamma(R)$ into global defensive alliances. Then, for every $i \in [\![1; r]\!]$, every $x \in S_i$ is connected to at least one vertex in $S_j$ for every $i \neq j \in [\![1; r]\!]$ (since every set $S_j$ is a dominating set). So, as all these sets are degreewise disjoint, $deg_{\bar{S}_i}(x) \geq r - 1$ for every $x \in S_i$. On the other hand, for every $i \in [\![1; r]\!]$ and every $x \in S_i$, $|S_i| - 1 \geq \deg_{S_i}(x) \geq deg_{\bar{S}_i}(x) - 1$ (since $S_i$ is a defensive alliance). Thus, for every $i \in [\![1; r]\!]$, $|S_i| - 1 \geq r - 2$. Then, $|Z(R)| - 1 = \sum_{i=1}^{r} |S_i| \geq r^2 - r$. Therefore, $|Z(R)| \geq r^2 - r + 1$. If $\Gamma(R)$ is not partitionable into global defensive alliances, that is $\psi_g(\Gamma(R)) = 1$, we have $|Z(R)| \geq 1^2 - 1 + 1 = 1$. Hence, $|Z(R)| \geq \psi_g(\Gamma(R))^2 - \psi_g(\Gamma(R)) + 1$. □

The bound in Proposition 3.1 is sharp. To show this, we give the following example.

**Example 3.1.** *Let $R = \mathbb{Z}_9$. Then, the zero-divisor graph is just an edge joining $\bar{3}$ and $\bar{6}$. So, the pair $\{S_1, S_2\}$ where $S_1 = \{\bar{3}\}$ and $S_2 = \{\bar{6}\}$, forms a partition of $\Gamma(R)$ into two global defensive alliances. Then, $\psi_g(\Gamma(R)) = 2$ and so $|Z(R)| = \psi_g(\Gamma(R))^2 - \psi_g(\Gamma(R)) + 1 = 2^2 - 2 + 1 = 3$.*

For a finite local ring $(R, M)$ which is not a field, $M = Z(R) = \mathrm{Ann}(x)$ for some $x \in Z(R)^*$ and $|R| = p^{nr}$ and $|M| = p^{(n-1)r}$ for some prime number $p$ and positive integers $n$ and $r$. However, we know that for a finite local ring $(R, M)$, $\Gamma(R)$ is complete if and only if $Z(R) = M$ with $M^2 = 0$ [8, Theorem 2.8]. So, we have the following result for this simple case.

**Proposition 3.2.** *Let $(R, M)$ be a finite local ring such that its maximal ideal $M$ is nilpotent of index 2. Then,*

1. *if $|M|$ is odd, $\Gamma(R)$ is partitionable into global defensive alliances with $\psi_g(\Gamma(R)) = 2$.*
2. *if $|M|$ is even, $\Gamma(R)$ is not partitionable into global defensive alliances.*

*Proof.* (1) Let $S_1$ and $S_2$ be two distinct subsets of $M^*$ such that $|S_1| = |S_2| = \frac{|M|-1}{2}$. Then, $\{S_1, S_2\}$ is a partition of $\Gamma(R)$ into global defensive alliances and so $\psi_g(\Gamma(R)) \geq 2$. Since, $\gamma_a(\Gamma(R)) = \frac{|M|-1}{2}$ and $\gamma_a(\Gamma(R)) \times \psi_g(\Gamma(R)) \leq |M| - 1$, $\psi_g(\Gamma(R)) \leq 2$. Hence, $\psi_g(\Gamma(R)) = 2$.
(2) Suppose that $\Gamma(R)$ is partitionable into global defensive alliances, then $\psi_g(\Gamma(R)) \geq 2$ and so, by [20, Proposition 3.6], $\left\lceil \frac{|M|-1}{2} \right\rceil \times 2 \leq \gamma_a(\Gamma(R)) \times \psi_g(\Gamma(R)) \leq |M| - 1$. Thus, $|M| \leq |M| - 1$, a contradiction.

□

**Corollary 3.1.** *Let $p$ be a prime number. Then, we have two cases:*

1. *if $p = 2$, then $\Gamma(\mathbb{Z}_{p^2})$ has only one vertex and so it is not partitionable into global defensive alliances.*
2. *if $p \neq 2$, then $\Gamma(\mathbb{Z}_{p^2})$ is partitionable into global defensive alliances and $\psi_g(\Gamma(\mathbb{Z}_{p^2})) = 2$.*

In the following theorem, we study when $\Gamma(\mathbb{Z}_{p^n})$ is partitionable into global defensive alliances for a prime number $p$ and a positive integer $n \geq 3$.

**Theorem 3.1.** *Let $p$ be a prime number and $n \geq 3$ be a positive integer. Then,*

1. *If $p = 2$, then $\Gamma(\mathbb{Z}_{p^n})$ is not partitionable into global defensive alliances.*
2. *If $p \geq 3$, then $\Gamma(\mathbb{Z}_{p^n})$ is partitionable into global defensive alliances and $\psi_g(\Gamma(\mathbb{Z}_{p^n})) = 2$.*

*Proof.* We have $Z := Z(\mathbb{Z}_{p^n}) = \{\overline{mp} \mid 0 \leq m < p^{n-1}\}$ and $|Z(\mathbb{Z}_{p^n})| = p^{n-1}$. Then,
(1) Suppose that $\Gamma(\mathbb{Z}_{2^n})$ is partitionable into global defensive alliances. Then, $\psi_g(\Gamma(\mathbb{Z}_{2^n})) \geq 2$. Since $\gamma_a(\Gamma(\mathbb{Z}_{2^n})) = 2^{n-2}$, by [62, Theorem 2.9 ], then $2^{n-2} \times 2 \leq \gamma_a(\Gamma(\mathbb{Z}_{2^n}))\psi_g(\Gamma(\mathbb{Z}_{2^n})) \leq 2^{n-1} - 1$, a contradiction.
(2) For each $1 \leq k \leq n-1$, set $A_k = \{\overline{ap^k} \in Z \mid p \text{ does not divide } a\}$. The sets $A_k$ are disjoint, $|A_k| = p^{n-k} - p^{n-k-1}$ which is an even number, and $Z^* = \cup_{k=1}^{n-1} A_k$. Let $S_1 = \cup_{k=1}^{n-1} A'_k$ and $S_2 = \cup_{k=1}^{n-1} A''_k$ such that $A'_k$ has a half of elements of $A_k$ and $A''_k$

the other half. By the proof of [62, Theorem 2.9], $S_1$ and $S_2$ are two global defensive alliances and so $\{S_1, S_2\}$ is a partition of $\Gamma(\mathbb{Z}_{p^n})$ into global defensive alliances. Then, $\psi_g(\Gamma(\mathbb{Z}_{p^n})) \geq 2$. On the other hand $\gamma_a(\Gamma(\mathbb{Z}_{p^n})) = \left\lceil \frac{p^{n-1}-1}{2} \right\rceil$, by [62, Theorem 2.9 ], and since $\gamma_a(\Gamma(\mathbb{Z}_{p^n}))\psi_g(\Gamma(\mathbb{Z}_{p^n})) \leq p^{n-1} - 1$, $\psi_g(\Gamma(\mathbb{Z}_{p^n})) \leq 2$. Hence, $\psi_g(\Gamma(\mathbb{Z}_{p^n})) = 2$

□

## 3.2 Partitioning zero-divisor graphs of some kind of a direct product of finite fields into global defensive alliances

In this section, we study when the zero-divisor graph of some kind of a direct product of finite fields is partitionable into global defensive alliances. We start with the following theorem which characterizes when $\Gamma(\mathbb{Z}_2 \times F)$, for a finite field $F$, is partitionable into global defensive alliances.

**Theorem 3.2.** *Let $F$ be a finite field. Then, $\Gamma(\mathbb{Z}_2 \times F)$ is partitionable into global defensive alliances if and only if $F \cong \mathbb{Z}_2$.*

*Proof.* $\Leftarrow$) Let $S_1 = \{(1, 0)\}$ and $S_2 = \{(0, 1)\}$, then $S_1$ and $S_2$ are both global defensive alliances and so $\{S_1, S_2\}$ is the only partition into global defensive alliances of $\Gamma(\mathbb{Z}_2 \times \mathbb{Z}_2)$. Then, $\psi_g(\Gamma(\mathbb{Z}_2 \times \mathbb{Z}_2)) = 2$.
$\Rightarrow$) Assume that $|F| \geq 3$ and suppose that $\Gamma(\mathbb{Z}_2 \times F)$ is partitionable into global defensive alliances. Let $S_1$ be a global defensive alliance in a partition of $\Gamma(\mathbb{Z}_2 \times F)$ such that $(1, 0) \notin S_1$. Then, $\{0\} \times F^* \subseteq S_1$. Otherwise, there exists $(0, a) \notin S_1$ for some $a \in F^*$. Then, $(0, a)$ can not be adjacent to any vertex in $S_1$. This contradicts the fact that $S_1$ is a dominating set. Thus, $S_2 = \{(1, 0)\}$ is the other global defensive alliance such that $\{S_1, S_2\}$ is a partition of $\Gamma(\mathbb{Z}_2 \times F)$ into global defensive alliances. Then, $deg_{S_2}(1, 0) + 1 \geq deg_{\bar{S_2}}(1, 0)$ and so $1 \geq |F| - 1$, a contradiction.

□

**Corollary 3.2.** *Let $p$ be a prime number. Then,*

1. *If $p = 2$, then $\Gamma(\mathbb{Z}_2 \times \mathbb{Z}_p)$ is partitionable into global defensive alliances and $\psi_g(\Gamma(\mathbb{Z}_2 \times \mathbb{Z}_p)) = 2$.*
2. *If $p \neq 2$, then $\Gamma(\mathbb{Z}_2 \times \mathbb{Z}_p)$ is not partitionable into global defensive alliances.*

In the following result, we study when zero-divisor graphs of a direct product of two finite fields are partitionable into global defensive alliances.

**Theorem 3.3.** *Let $F$ and $K$ be two finite fields such that $|F|, |K| \geq 3$. Then, $\Gamma(F \times K)$ is partitionable into global defensive alliances and*

$$\psi_g(\Gamma(F \times K)) = \begin{cases} 3 & \text{if } |F| = |K| = 4, \\ 2 & \text{otherwise.} \end{cases}$$

*Proof.* We have the following cases:
**Case** $|F| = |K| = 4$: The zero-divisor graph of $\Gamma(F \times K)$ is illustrated in Figure 3.1.

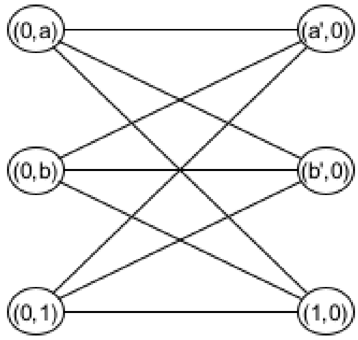



Figure 3.1: $\Gamma(F \times K)$

So, $\{\{(0,a),(a',0)\},\{(0,b),(b',0)\},\{(0,1),(1,0)\}\}$ is a partition of $\Gamma(F\times K)$ into global defensive alliances, then $\psi_g(\Gamma(F \times K)) \geq 3$. On the other hand, $\gamma_a(\Gamma(F \times K))\psi_g(\Gamma(F \times K)) \leq 6$. Thus, $\psi_g(\Gamma(F \times K)) = 3$.

**Case** $|F| \neq 4$ or $|K| \neq 4$: Let $S_1 = F_1 \times \{0\} \cup \{0\} \times K_1$ and $S_2 = F_2 \times \{0\} \cup \{0\} \times K_2$ such that $F_1, F_2 \subset F^*$ and $K_1, K_2 \subset K^*$ with

$$\begin{cases} F_1 \cap F_2 = K_1 \cap K_2 = \emptyset, \\ |F_1| = \left\lfloor \frac{|F|-1}{2} \right\rfloor, \\ |F_2| = |F| - 1 - \left\lfloor \frac{|F|-1}{2} \right\rfloor, \\ |K_1| = \left\lfloor \frac{|K|-1}{2} \right\rfloor, \\ |K_2| = |K| - 1 - \left\lfloor \frac{|K|-1}{2} \right\rfloor. \end{cases}$$

So, $S_1$ and $S_2$ are two global defensive alliances and $Z(F\times K)^* = S_1 \cup S_2$ and $S_1 \cap S_2 = \emptyset$. Thus, $\Gamma(F \times K)$ is partitionable into global defensive alliances and so $\psi_g(\Gamma(F \times K)) \geq 2$. Now, suppose that $\psi_g(\Gamma(F \times K)) \geq 3$, then $3 \times \gamma_a(\Gamma(F \times K)) \leq \gamma_a(\Gamma(F \times K))\psi_g(\Gamma(F \times K)) \leq |F| + |K| - 2$ and since $\gamma_a(\Gamma(F \times K)) = \left\lfloor \frac{|F|-1}{2} \right\rfloor + \left\lfloor \frac{|K|-1}{2} \right\rfloor$, by [62, Proposition 2.3], then $3 \times (\left\lfloor \frac{|F|-1}{2} \right\rfloor + \left\lfloor \frac{|K|-1}{2} \right\rfloor) \leq |F| + |K| - 2$. So, we have four sub-cases to discuss:

**sub-case 1**. 2 divides both $|F|$ and $|K|$: Then, $|F| + |K| \leq 8$, a contradiction since one of the $|F|$ and $|K|$ is different from 4.

**sub-case 2**. 2 divides $|F|$ and 2 does not divide $|K|$: Then, $|F|+|K| \leq 5$, a contradiction.

**sub-case 3**. 2 does not divide $|F|$ and 2 divide $|K|$: Similar to sub-case 2.

**sub-case 4**. 2 does not divide both $|F|$ and $|K|$: Then, $|F| + |K| \leq 2$, a contradiction.

Hence, $\psi_g(\Gamma(F \times K)) = 2$.

□

**Corollary 3.3.** *If $p, q \geq 3$ are two prime numbers. Then, $\Gamma(\mathbb{Z}_p \times \mathbb{Z}_q)$ is partitionable into global defensive alliances and $\psi_g(\Gamma(\mathbb{Z}_p \times \mathbb{Z}_q)) = 2$.*

The following theorem presents when $\Gamma(\mathbb{Z}_2 \times \mathbb{Z}_2 \times F)$, for a finite field $F$, is partitionable into global defensive alliances.

**Theorem 3.4.** *Let $F$ be a finite field. Then,*

1. *If $|F| \leq 4$, then $\Gamma(\mathbb{Z}_2 \times \mathbb{Z}_2 \times F)$ is partitionable into global defensive alliances with $\psi_g(\Gamma(\mathbb{Z}_2 \times \mathbb{Z}_2 \times F)) = 2$.*

2. *If $|F| > 4$, then $\Gamma(\mathbb{Z}_2 \times \mathbb{Z}_2 \times F)$ is not partitionable into global defensive alliances.*

*Proof.* (1) If $|F| = 2$ that is $F \cong \mathbb{Z}_2$, then we take the partition $\{S_1, S_2\}$ such that $S_1 = \{(\bar{1}, \bar{0}, \bar{0}), (\bar{0}, \bar{0}, \bar{1}), (\bar{0}, \bar{1}, \bar{0})\}$ and $S_2 = \{(\bar{1}, \bar{1}, \bar{0}), (\bar{0}, \bar{1}, \bar{1}), (\bar{1}, \bar{0}, \bar{1})\}$. It is easy to see that $S_1$ and $S_2$ are both global defensive alliances. Since $Z(\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2) = \{(\bar{0}, \bar{0}, \bar{0})\} \cup S_1 \cup S_2$, $\Gamma(\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2)$ is partitionable into global defensive alliances and $\psi_g(\Gamma(\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2)) \geq 2$. On the other hand, $\gamma_a(\Gamma(\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2))\psi_g(\Gamma(\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2)) \leq |Z(\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2)^*|$. Thus, $\psi_g(\Gamma(\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2)) \leq 2$, by [20, Theorem 4.5]. Then, $\psi_g(\Gamma(\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2)) = 2$. If $|F| = 3$ or $|F| = 4$, then we take the partition $\{S_1, S_2\}$ such that $S_1 = \{(\bar{1}, \bar{0}, 0), (\bar{0}, \bar{1}, 0)\} \cup \{(\bar{0}, \bar{0}, x) |\ x \in F - \{0, 1\}\}$ and $S_2 = \{(\bar{0}, \bar{0}, 1), (\bar{1}, \bar{1}, 0)\} \cup \{(\bar{0}, \bar{1}, x) |\ x \in F^*\} \cup \{(\bar{1}, \bar{0}, x) |\ x \in F^*\}$. Hence, $\Gamma(\mathbb{Z}_2 \times \mathbb{Z}_2 \times F)$ is partitionable into global defensive alliances and $\psi_g(\Gamma(\mathbb{Z}_2 \times \mathbb{Z}_2 \times F)) \geq 2$. On the other hand, since the minimal degree of $\Gamma(\mathbb{Z}_2 \times \mathbb{Z}_2 \times F)$ is $\delta = 1$, $\psi_g(\Gamma(\mathbb{Z}_2 \times \mathbb{Z}_2 \times F)) \leq 2$, by [90, Theorem 2.1]. Thus, $\psi_g(\Gamma(\mathbb{Z}_2 \times \mathbb{Z}_2 \times F)) = 2$.
(2) $\Gamma(\mathbb{Z}_2 \times \mathbb{Z}_2 \times F)$ is a simple graph of minimal degree $\delta = 1$, then by [90, Theorem 2.1], $\psi_g(\Gamma(\mathbb{Z}_2 \times \mathbb{Z}_2 \times F)) \leq 2$. Suppose that $\{S_1, S_2\}$ is a partition of $\Gamma(\mathbb{Z}_2 \times \mathbb{Z}_2 \times F)$ into global defensive alliances. Assume that $(\bar{0}, \bar{1}, 0) \in S_1$. Then, we have the following two cases:
<u>**Case 1** $(\bar{1}, \bar{0}, 0) \in S_1$</u>: If $\{\bar{0}\} \times \{\bar{0}\} \times F^* \subset S_1$, then $(\bar{1}, \bar{1}, 0) \in S_2$ (since $S_2$ is a dominating set) and so $deg_{S_2}((\bar{1}, \bar{1}, 0)) + 1 \geq deg_{\bar{S_2}}((\bar{1}, \bar{1}, 0))$, then $2 \geq |F|$, a contradiction. Otherwise, there exists a vertex $(\bar{0}, \bar{0}, u) \in S_2$, then $(\bar{1}, \bar{1}, 0) \in S_2$, otherwise $deg_{S_2}((\bar{0}, \bar{0}, u)) + 1 \geq deg_{\bar{S_2}}((\bar{0}, \bar{0}, u))$ implies $1 \geq 3$, a contradiction. On the other hand $\{\bar{0}\} \times \{\bar{1}\} \times F^* \subset S_2$ (since $S_2$ is a dominating set). Thus, if there exists $u \neq v \in F^*$ such that $(\bar{0}, \bar{0}, v) \in S_2$, then $deg_{S_1}((\bar{1}, \bar{0}, 0)) + 1 \geq deg_{\bar{S_1}}((\bar{1}, \bar{0}, 0))$ and so $-1 \geq 0$, a contradiction, otherwise $deg_{S_2}((\bar{1}, \bar{1}, 0)) + 1 \geq deg_{\bar{S_2}}((\bar{1}, \bar{1}, 0))$ implies that $4 \geq |F|$, a contradiction.
<u>**Case 2** $(\bar{1}, \bar{0}, 0) \in S_2$</u>: If $\{\bar{0}\} \times \{\bar{0}\} \times F^* \subset S_2$, then $(\bar{1}, \bar{1}, 0) \in S_1$ (since $S_1$ is a dominating set) and so $deg_{S_1}((\bar{1}, \bar{1}, 0)) + 1 \geq deg_{\bar{S_1}}((\bar{1}, \bar{1}, 0))$, thus $2 \geq |F|$, a contradiction. Then, there exists $u \in F^*$ such that $(\bar{0}, \bar{0}, u) \in S_1$ and so $deg_{S_2}((\bar{0}, \bar{1}, 0)) + 1 \geq deg_{\bar{S_2}}((\bar{0}, \bar{1}, 0))$ which implies that $0 \geq 2$, a contradiction.
Hence, $\Gamma(\mathbb{Z}_2 \times \mathbb{Z}_2 \times F)$ is not partitionable into global defensive alliances.

□

**Corollary 3.4.** *Let $p$ be a prime number. Then, $\Gamma(\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_p)$ is partitionable into global defensive alliances if and only if $p \in \{2, 3\}$. Namely,*

$$\psi_g(\Gamma(\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_p)) = \begin{cases} 2 & \textit{if } p \in \{2, 3\}, \\ \\ 1 & \textit{otherwise.} \end{cases}$$

We end this section with the following result which shows that $\Gamma(\mathbb{Z}_2 \times F \times K)$, for finite fields $F$ and $K$ with $|F|, |K| \geq 3$, is not partitionable into global defensive alliances.

**Theorem 3.5.** *If $F$ and $K$ are two finite fields such that $|F|, |K| \geq 3$. Then, $\Gamma(\mathbb{Z}_2 \times F \times K)$ is not partitionable into global defensive alliances.*

*Proof.* $\Gamma(\mathbb{Z}_2 \times F \times K)$ is a simple graph of minimal degree $\delta = 1$. Then, by [90, Theorem 2.1], $\psi_g(\Gamma(\mathbb{Z}_2 \times F \times K)) \leq 2$. Suppose that $\psi_g(\Gamma(\mathbb{Z}_2 \times F \times K)) = 2$ that is $\Gamma(\mathbb{Z}_2 \times F \times K)$ is partitionable into two global defensive alliance, say $S_1$ and $S_2$. Assume that $(\bar{1}, 0, 0) \in S_1$, then $\{\bar{0}\} \times F^* \times K^* \subseteq S_2$. Otherwise, there exists a vertex $(\bar{0}, a, b) \notin S_2$ for some $a \in F^*$ and $b \in K^*$. Then, there no vertex in $S_2$ that is adjacent to $(\bar{0}, a, b)$, a contradiction with the fact that $S_2$ is dominating. Thus, $deg_{S_1}((\bar{1}, 0, 0)) + 1 \geq deg_{\bar{S_1}}((\bar{1}, 0, 0))$ implies that $|F^*| + |K^*| + 1 \geq |F^*||K^*|$ (since $deg_{S_1}((\bar{1}, 0, 0)) \leq |F^*| + |K^*|$ and $deg_{\bar{S_1}}((\bar{1}, 0, 0)) \geq |F^*||K^*|$). So, we have the following cases:

**Case 1.** $|F| \geq 4$ and $|K| \geq 4$: In this case we get a contradiction.

**Case 2.** $|F| = |K| = 3$: The zero-divisor graph of $\mathbb{Z}_2 \times F \times K$ is illustrated in Figure 3.2. Then, $S_1$ contains at least three vertices from the set $\{(\bar{0}, 0, b), (\bar{0}, 0, b'), (\bar{0}, a, 0), (\bar{0}, a', 0)\}$,

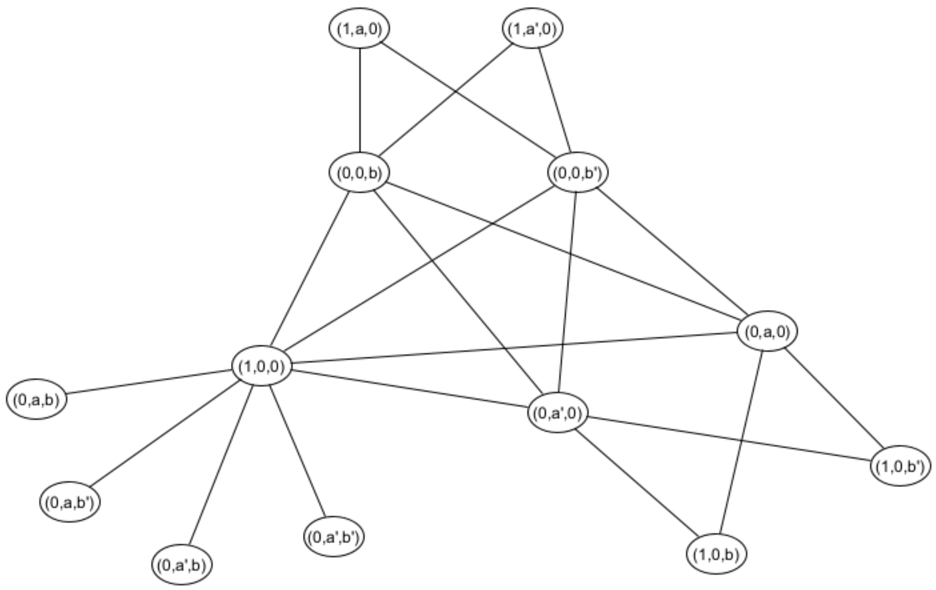



Figure 3.2: $\Gamma(\mathbb{Z}_2 \times F \times K)$

so we can assume that the $\{(\bar{0}, 0, b), (\bar{0}, 0, b'), (\bar{0}, a, 0)\} \subset S_1$. Then, $\{(\bar{1}, a, 0), (\bar{1}, a', 0)\} \subset S_2$ (since $S_2$ is a dominating set). Thus, $deg_{S_2}((\bar{1}, a, 0)) + 1 \geq deg_{\bar{S_2}}((\bar{1}, a, 0))$ and so $1 \geq 2$, a contradiction.

**Case 3.** $|F| = 4$ and $|K| = 3$: The zero-divisor graph of $\mathbb{Z}_2 \times F \times K$ is illustrated in Figure 3.3. Then, $\{(\bar{0}, 0, b), (\bar{0}, 0, b'), (\bar{0}, 0, b''), (\bar{0}, a, 0), (\bar{0}, a', 0)\} \subset S_1$ (since $deg_{S_1}((\bar{1}, 0, 0)) + 1 \geq deg_{\bar{S_1}}((\bar{1}, 0, 0))$ ). Thus, $deg_{S_2}((\bar{1}, a, 0)) + 1 \geq deg_{\bar{S_2}}((\bar{1}, a, 0))$ implies that $1 \geq 3$, a contradiction.

**Case 4.** $|F| \geq 5$ and $|K| = 3$: We have $|F^*| + |K^*| + 1 \geq |F^*||K^*|$, then $|F| \leq 4$, a contradiction.

Hence, $\psi_g(\Gamma(\mathbb{Z}_2 \times F \times K)) < 2$ and so $\Gamma(\mathbb{Z}_2 \times F \times K)$ is not partitionable into global defensive alliances.

□

**Corollary 3.5.** *If $p, q \geq 3$ are two prime numbers. Then, $\psi_g(\Gamma(\mathbb{Z}_2 \times \mathbb{Z}_p \times \mathbb{Z}_q)) = 1$.*

## 3.3 Partitioning zero-divisor graphs of the direct product of finite fields and finite local rings

In this section, we study when the zero-divisor graph of the direct product of finite fields and finite local rings is partitionable into global defensive alliances. So, we start with the

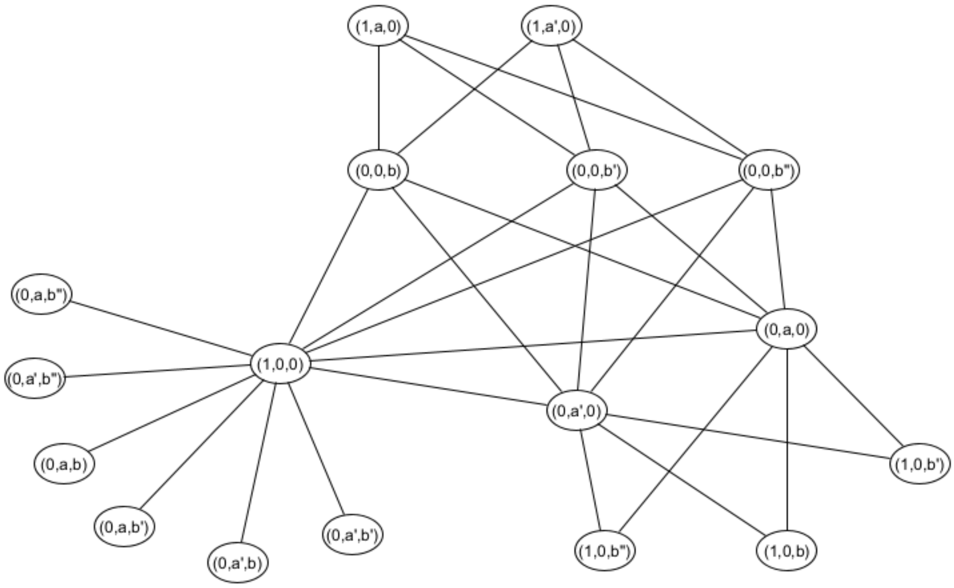



Figure 3.3: $\Gamma(\mathbb{Z}_2 \times F \times K)$

following result which characterizes when $\Gamma(\mathbb{Z}_2 \times R)$ is partitionable into global defensive alliances, for a finite local ring $R$ which is not a field and such that its maximal ideal is nilpotent of index 2.

**Theorem 3.6.** *Let $R$ be a finite local ring that is not a field and such that its maximal ideal is nilpotent of index 2. Then, $\Gamma(\mathbb{Z}_2 \times R)$ is partitionable into global defensive alliances if and only if $R \cong \mathbb{Z}_4$ or $R \cong \mathbb{Z}_2[X]/(X^2)$.*

*Moreover, if $\Gamma(\mathbb{Z}_2 \times R)$ is partitionable into global defensive alliances, then $\psi_g(\Gamma(\mathbb{Z}_2 \times R)) = 2$.*

*Proof.* $\Leftarrow$) Assume that $R \cong \mathbb{Z}_4$. The zero-divisor graph of $\mathbb{Z}_2 \times \mathbb{Z}_4$ is illustrated in Figure 3.4. Set $S_1 = \{(\bar{1}, \bar{0}), (\bar{0}, \bar{2})\}$ and $S_2 = \{(\bar{0}, \bar{1}), (\bar{0}, \bar{3}), (\bar{1}, \bar{2})\}$. So, $S_1$ and $S_2$ are two global

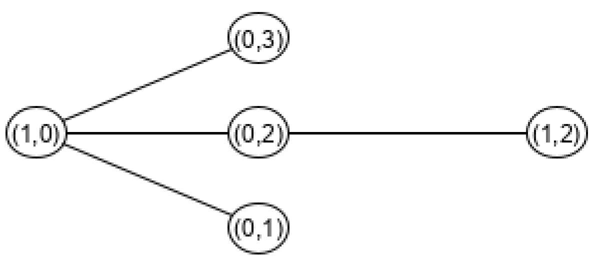



Figure 3.4: $\Gamma(\mathbb{Z}_2 \times \mathbb{Z}_4)$

defensive alliances since $S_1$ and $S_2$ are dominating sets and

$$\begin{cases} deg_{S_1}((\bar{1},\bar{0}))+1=2\geq deg_{\bar{S_1}}((\bar{1},\bar{0}))=2, \\ deg_{S_1}((\bar{0},\bar{2}))+1=2\geq deg_{\bar{S_1}}((\bar{0},\bar{2}))=1, \\ deg_{S_2}((\bar{0},\bar{1}))+1=1\geq deg_{\bar{S_2}}((\bar{0},\bar{1}))=1, \\ deg_{S_2}((\bar{0},\bar{3}))+1=1\geq deg_{\bar{S_2}}((\bar{0},\bar{3}))=1, \\ deg_{S_2}((\bar{1},\bar{2}))+1=1\geq deg_{\bar{S_2}}((\bar{1},\bar{2}))=1. \end{cases}$$

On the other hand $Z(\mathbb{Z}_2\times\mathbb{Z}_4)^* = S_1\cup S_2$ and $S_1\cap S_2=\emptyset$ and so $\{S_1,S_2\}$ forms a partition of $\Gamma(\mathbb{Z}_2\times R)$ into global defensive alliances. Then, $\psi_g(\Gamma(\mathbb{Z}_2\times R))\geq 2$. Thus, by [90, Theorem 2.1 ], $\psi_g(\Gamma(\mathbb{Z}_2\times R))\leq \lfloor\frac{\delta+1+2}{2}\rfloor=2$ and so $\psi_g(\Gamma(\mathbb{Z}_2\times R))=2$. Similarly, when $R\cong\mathbb{Z}_2[X]/(X^2)$, we take the partition $\{S_1,S_2\}$ with $S_1=\{(\bar{1},\bar{0}),(\bar{0},\bar{X})\}$ and $S_2=\{(\bar{0},\bar{1}),(\bar{0},\overline{1+X}),(\bar{1},\bar{X})\}$.

$\Rightarrow$) Assume that $R\ncong\mathbb{Z}_4$ and $R\ncong\mathbb{Z}_2[X]/(X^2)$. Suppose that $\Gamma(\mathbb{Z}_2\times R)$ is partitionable into two global defensive alliances $S_1$ and $S_2$. So, assume that $(\bar{1},0)\in S_1$. Then, we have two cases:

**Case 1.** $\{0\}\times U(R)\subset S_2$: If $\{\bar{0}\}\times Z(R)^*\subseteq S_1$, then $\{\bar{1}\}\times Z(R)^*\subseteq S_2$. Otherwise, there exists $z\in Z(R)^*$ such that $(\bar{1},z)\in S_1$ and so, $S_2$ is not a dominating set (since there is not a vertex in $S_2$ that is adjacent to $(\bar{1},z)$). Thus, for every $z$ in $Z(R)^*$, $deg_{S_2}(\bar{1},z)+1\geq deg_{S_1}(\bar{1},z)$ implies $2\geq|Z(R)|$ a contradiction (since $R$ is not a field, $R\ncong\mathbb{Z}_4$ and $R\ncong\mathbb{Z}_2[X]/(X^2)$). Then, there exists $z\in Z(R)^*$ such that $(\bar{0},z)\in S_2$. Thus, $|Z(R)^*|-1+1\geq deg_{S_1}(\bar{1},0)+1\geq deg_{S_2}(\bar{1},0)\geq|U(R)|+1$. Then, $|Z(R)|\geq\frac{|R|}{2}+1$, a contradiction (since, by Corollary 1.2, $|Z(R)|\leq\frac{|R|}{2}$).

**Case 2.** There exists $u\in U(R)$ such that $(\bar{0},u)\in S_1$: Then, $S_2$ is not a dominating set since there is no vertex adjacent to $(\bar{0},u)$ other than $(\bar{1},0)$, a contradiction.

Hence, $\Gamma(\mathbb{Z}_2\times R)$ is not partitionable into global defensive alliances.

□

**Corollary 3.6.** *Let $p$ be a prime number. Then, $\Gamma(\mathbb{Z}_2\times\mathbb{Z}_{p^2})$ is partitionable into global defensive alliances if and only if $p=2$.*

The following theorem characterizes when $\Gamma(F\times R)$ is partitionable into global defensive alliances, for a finite local ring $R$ which is not a field and such that its maximal ideal is nilpotent of index 2, and a finite field $F$ with $|F|\geq 3$.

**Theorem 3.7.** *Let $R$ be a finite local ring such that its maximal ideal is nilpotent of index 2 and $F$ be a finite field with $|F|\geq 3$. Then,*

1. *If both $|Z(R)|$ and $|F|$ are odd, then $\Gamma(F\times R)$ is partitionable into global defensive alliances and $\psi_g(\Gamma(F\times R))=2$.*

2. *If $|Z(R)|$ or $|F|$ is even, then $\Gamma(F\times R))$ is not partitionable into global defensive alliances.*

*Proof.* (1) Assume that both $|Z(R)|$ and $|F|$ are odd. We set $S_1=(A_1\times\{0\})\cup(\{0\}\times A_2)\cup(\{0\}\times A_3)\cup(A_1\times A_3)\cup(A_1\times B_3)$ and $S_2=(B_1\times\{0\})\cup(\{0\}\times B_2)\cup(\{0\}\times B_3)\cup$

$(B_1 \times B_3) \cup (B_1 \times A_3)$ such that

$$\begin{cases} A_1, B_1 \subset F^* \text{ with } |A_1| = |B_1| = \frac{|F|-1}{2}, \text{ and } A_1 \cap B_1 = \emptyset, \\ A_2, B_2 \subset U(R) \text{ with } |A_2| = \left\lceil \frac{|U(R)|}{2} \right\rceil, \ |B_2| = |U(R)| - \left\lceil \frac{|U(R)|}{2} \right\rceil, \text{ and } A_2 \cap B_2 = \emptyset, \\ A_3, B_3 \subset Z(R)^* \text{ with } |A_3| = |B_3| = \frac{|Z(R)|-1}{2}, \text{ and } A_3 \cap B_3 = \emptyset. \end{cases}$$

It is clear that $S_1$ is a dominating set. So, let us prove that $S_1$ is a defensive alliance. Let $(x,0) \in A_1 \times \{0\}$, then $deg_{S_1}((x,0)) + 1 = |A_2| + |A_3| + 1 = \left\lceil \frac{|U(R)|}{2} \right\rceil + \frac{|Z(R)|+1}{2}$ and $deg_{\bar{S_1}}((x,0)) = |B_2| + |B_3| = |U(R)| - \left\lceil \frac{|U(R)|}{2} \right\rceil + \frac{|Z(R)|-1}{2} \le \frac{|U(R)|}{2} + \frac{|Z(R)|-1}{2}$. Then, $deg_{S_1}((x,0)) + 1 \ge deg_{\bar{S_1}}((x,0))$. Next, let $(0,y) \in \{0\} \times A_2$, then $deg_{S_1}((0,y)) + 1 = |A_1| + 1 = \frac{|F|+1}{2}$ and $deg_{\bar{S_1}}((0,y)) = |B_1| = \frac{|F|-1}{2}$. Then, $deg_{S_1}((0,y)) + 1 \ge deg_{\bar{S_1}}((0,y))$. Next, let $(0,y) \in \{0\} \times A_3$, then $deg_{S_1}((0,y)) + 1 = |A_1| + |A_3| - 1 + |A_1||A_3| + |A_1||B_3| + 1 = \frac{|F||Z(R)|-1}{2}$ and $deg_{\bar{S_1}}((0,y)) = |B_1| + |B_3| + |B_1||B_3| + |B_1||A_3| = \frac{|F||Z(R)|-1}{2}$. Then, $deg_{S_1}((0,y)) + 1 \ge deg_{\bar{S_1}}((0,y))$. Finally, let $(x,y) \in (A_1 \times A_3) \cup (A_1 \times B_3)$, then $deg_{S_1}((x,y)) + 1 = |A_3| + 1 = \frac{|Z(R)|-1}{2} + 1 = \frac{|Z(R)|+1}{2}$ and $deg_{\bar{S_1}}((x,y)) = |B_3| = \frac{|Z(R)|-1}{2}$. Then, $deg_{S_1}((x,y)) + 1 \ge deg_{\bar{S_1}}((x,y))$. Thus, $S_1$ is a global defensive alliance. Similarly, we prove that $S_2$ is a global defensive alliance. Since $S_1 \cap S_2 = \emptyset$ and $S_1 \cup S_2 = Z(F \times R)^*$, $\{S_1, S_2\}$ is a partition of $\Gamma(F \times R)$ into global defensive alliances. Thus, $\psi_g(\Gamma(F \times R)) \ge 2$. Now, suppose that $\psi_g(\Gamma(F \times R)) \ge 3$. Then, $3 \times \gamma_a(\Gamma(F \times R)) \le \psi_g(\Gamma(F \times R)) \times \gamma_a(\Gamma(F \times R)) \le |Z(F \times R)| - 1$ and so by [62, Theorem 2.5 ], $\frac{|R|}{2} + |Z(R)|(\frac{|F|}{2} - 2) \le -1$, a contradiction. Hence, $\psi_g(\Gamma(F \times R)) = 2$.

(2) Assume that $|Z(R)|$ or $|F|$ is even. Suppose that $\Gamma(F \times R)$ is partitionable into two global defensive alliances, $S_1$ and $S_2$. Then, we discuss the following two cases:

**Case $|Z(R)| = 2$:** Then, $R \cong \mathbb{Z}_4$ or $R \cong \mathbb{Z}_2[X]/(X^2)$. Assume that $R \cong \mathbb{Z}_4$. Then, let us assume that $(0,\bar{2}) \in S_1$, then $F^* \times \{\bar{2}\} \subset S_2$ (since $S_2$ is a dominating set). Since $deg_{S_2}((0,\bar{2})) + 1 \ge deg_{\bar{S_2}}((0,\bar{2}))$, $S_1$ contains at least $|F^*| - 1$ elements from $F^* \times \{\bar{0}\}$. If $F^* \times \{\bar{0}\} \subset S_1$, then either $(0,\bar{1})$ and $(0,\bar{3})$ are both in $S_1$ or one of them is in $S_2$, if $\{(0,\bar{1}),(0,\bar{3})\} \subset S_1$, then $S_2$ is not a dominating set, a contradiction, if one of them (i.e., $(0,\bar{1})$ or $(0,\bar{3})$ ), say $(0,\bar{1})$, is in $S_2$, then $deg_{S_2}((0,\bar{1})) + 1 \ge deg_{\bar{S_2}}((0,\bar{1}))$ and so $|F| \le 2$, a contradiction. Hence, there exists $(x,\bar{0}) \in S_2$ for some $x \in F^*$ and so $deg_{S_1}((0,\bar{2})) + 1 \ge deg_{\bar{S_1}}((0,\bar{2}))$, which implies $|F^*| - 1 + 1 \ge |F^*| + 1$, a contradiction. Thus, $\Gamma(F \times R)$ is not partitionable into global defensive alliances. Similarly, when $R \cong \mathbb{Z}_2[X]/(X^2)$.

**Case $|Z(R)| > 2$:** Suppose that $\{0\} \times Z(R)^* \subset S_1$, then $F^* \times Z(R)^* \subset S_2$ (since $S_2$ is a dominating set). Thus, for every $(x,r) \in F^* \times Z(R)^* \cap S_2$, $deg_{S_2}((x,r)) + 1 \ge deg_{\bar{S_2}}((x,r))$ and so $|Z(R)| \le 2$, a contradiction. Then, $\{0\} \times Z(R)^* \cap S_2 \ne \emptyset$ (similarly, $\{0\} \times Z(R)^* \cap S_1 \ne \emptyset$). Thus, for every $(0,y) \in S_2 \cap \{0\} \times Z(R)^*$, $deg_{S_2}((0,y)) + 1 \ge deg_{\bar{S_2}}((0,y))$. Then, $|S_2 \cap F^* \times \{0\}| + |S_2 \cap \{0\} \times Z(R)^*| - 1 + |S_2 \cap F^* \times Z(R)^*| + 1 \ge |\bar{S_2} \cap F^* \times \{0\}| + |\bar{S_2} \cap \{0\} \times Z(R)^*| + |\bar{S_2} \cap F^* \times Z(R)^*|$. Then, $2(|S_2 \cap F^* \times \{0\}| + |S_2 \cap \{0\} \times Z(R)^*| + |S_2 \cap F^* \times Z(R)^*|) \ge |F^*| + |Z(R)^*| + |F^*||Z(R)^*|$. Thus, $|S_2 \cap F^* \times \{0\}| + |S_2 \cap \{0\} \times Z(R)^*| + |S_2 \cap F^* \times Z(R)^*| \ge \frac{|F||Z(R)|-1}{2}$. Since $|F|$ or $|Z(R)|$ is even, $|S_2 \cap F^* \times \{0\}| + |S_2 \cap \{0\} \times Z(R)^*| + |S_2 \cap F^* \times Z(R)^*| \ge \left\lceil \frac{|F||Z(R)|-1}{2} \right\rceil$. Similarly, we prove that $|S_1 \cap F^* \times \{0\}| + |S_1 \cap \{0\} \times Z(R)^*| + |S_1 \cap F^* \times Z(R)^*| \ge \left\lceil \frac{|F||Z(R)|-1}{2} \right\rceil$. Then, $|F||Z(R)| - 1 = |S_1 \cap F^* \times \{0\}| + |S_1 \cap \{0\} \times Z(R)^*| + |S_1 \cap F^* \times Z(R)^*| + |S_2 \cap$

$F^* \times \{0\}| + |S_2 \cap \{0\} \times Z(R)^*| + |S_2 \cap F^* \times Z(R)^*| \geq \left\lceil \frac{|F||Z(R)|-1}{2} \right\rceil + \left\lceil \frac{|F||Z(R)|-1}{2} \right\rceil$. Since $|F|$ or $|Z(R)|$ is even, $|F||Z(R)| - 1 \geq |F||Z(R)|$, a contradiction. Hence, $\Gamma(F \times R)$ is not partitionable into global defensive alliances. □

**Corollary 3.7.** *Let $p$ and $q$ be two prime numbers such that $p \neq 2$. Then, $\Gamma(\mathbb{Z}_p \times \mathbb{Z}_{q^2})$ is partitionable into global defensive alliances if and only if $q \neq 2$. Namely,*

$$\psi_g(\Gamma(\mathbb{Z}_p \times \mathbb{Z}_{q^2})) = \begin{cases} 2 & \text{if } q \neq 2, \\ 1 & \text{otherwise.} \end{cases}$$

To give a family of examples for Theorem 3.7, in the case $|Z(R)| > 2$, we use the idealization of modules.

**Example 3.2.** *Let $n \geq 2$ be a positive integer, and $p$ and $q$ be two prime numbers. Then, $\mathbb{Z}_q(+)(\mathbb{Z}_q)^n$ is a finite local ring of maximal ideal $0(+)(\mathbb{Z}_q)^n$. We have $(0(+)(\mathbb{Z}_q)^n)^2 = 0$ and so $\Gamma(\mathbb{Z}_p \times (\mathbb{Z}_q(+)(\mathbb{Z}_q)^n))$ is not partitionable into global defensive alliances if and only if $n \times q$ is even. Namely,*

$$\psi_g(\Gamma(\mathbb{Z}_p \times (\mathbb{Z}_q(+)(\mathbb{Z}_q)^n))) = \begin{cases} 2 & \text{if } q \neq 2 \text{ and } n \text{ is odd }, \\ 1 & \text{otherwise.} \end{cases}$$

## 3.4 Rings with small global defensive alliance numbers and small global defensive alliance partition numbers

In this section, we investigate rings with small global defensive alliance numbers and small global defensive alliance partition numbers. Namely, $\gamma_a(\Gamma(R)) = 1, 2$ and $\psi_g(\Gamma(R)) = 2, 3$.

**Theorem 3.8.** *Let $R$ be a finite ring such that $\gamma_a(\Gamma(R)) = 1$. Then, the following statements are equivalent:*

1. *$\Gamma(R)$ is partitionable into global defensive alliances.*
2. *$|Z(R)| = 3$.*
3. *$R$ is isomorphic to one of the rings $\mathbb{Z}_2 \times \mathbb{Z}_2$, $\mathbb{Z}_9$, $\mathbb{Z}_3[X]/(X^2)$.*

*Proof.* (1) ⇒ (2) Assume that $\Gamma(R)$ is partitionable into global defensive alliances, then $\psi_g(\Gamma(R)) \geq 2$ and so $|Z(R)| = 3$, by Proposition 3.1 and [62, Proposition 2.2].
(2) ⇒ (3) Follows from [18, Corollary 1].
(3) ⇒ (1) The zero-divisor graphs of these rings are isomorphic to a simple graph with two vertices and one edge. □

**Corollary 3.8.** *Let $R$ be a finite ring such that $\Gamma(R)$ is partitionable into global defensive alliances. If $\gamma_a(\Gamma(R)) = 1$, then $\psi_g(\Gamma(R)) = 2$.*

**Theorem 3.9.** *Let $R$ be a finite ring such that $\gamma_a(\Gamma(R)) = 2$. Then, $\Gamma(R)$ is partitionable into global defensive alliances if and only if either $\psi_g(\Gamma(R)) = 2$ or $R \cong \mathbb{F}_4 \times \mathbb{F}_4$.*

*Proof.* ⇒) Assume that $\Gamma(R)$ is partitionable into global defensive alliances and suppose that $R \ncong \mathbb{F}_4 \times \mathbb{F}_4$. Then $\psi_g(\Gamma(R)) \geq 2$. So, we need just to prove the other inequality. We have $\gamma_a(\Gamma(R)) = 2$, then by Proposition 3.1 and [62, Proposition 2.2], $\psi_g(\Gamma(R))^2 - \psi_g(\Gamma(R)) - 6 \leq 0$. Then, $\psi_g(\Gamma(R))$ is either 2 or 3. Suppose that $\psi_g(\Gamma(R)) = 3$, then by Proposition 3.1, $|Z(R)| \geq 7$. Thus, using [62, Proposition 3.3], $R \cong \mathbb{F}_4 \times \mathbb{F}_4$, a contradiction by the hypothesis. Thus, $\psi_g(\Gamma(R)) = 2$.
⇐) If $\psi_g(\Gamma(R)) = 2$, then $\Gamma(R)$ is partitionable into two global defensive alliances. If $R \cong \mathbb{F}_4 \times \mathbb{F}_4$, then $\Gamma(R)$ is isomorphic to the graph illustrated in Figure 3.1. Thus, $\Gamma(R)$ is partitionable into three global defensive alliances.

□

**Corollary 3.9.** *Let $R$ be a finite ring such that $\gamma_a(\Gamma(R)) = 2$. Then, we have the following equivalents:*

1. *$\psi_g(\Gamma(R)) = 2$ if and only if $R$ is isomorphic to one of the rings $\mathbb{Z}_2 \times \mathbb{Z}_4$, $\mathbb{Z}_2 \times \mathbb{Z}_2[X]/(X^2)$, $\mathbb{Z}_3 \times \mathbb{Z}_3$, $\mathbb{Z}_3 \times \mathbb{F}_4$, $\mathbb{Z}_{25}$ and $\mathbb{Z}_5[X]/(X^2)$.*
2. *$\psi_g(\Gamma(R)) = 3$ if and only if $R \cong \mathbb{F}_4 \times \mathbb{F}_4$.*

*Proof.* It follows from Theorem 3.9 and [62, Proposition 3.3]. □

# Chapter 4

# Rings whose associated extended zero-divisor graphs are complemented

This chapter deals with complementedness and uniquely complementedness notions of graphs (see Section 1.1.3 of Chapter 1 for the definitions). In [9, Theorem 3.5], these notions were used, for the zero-divisor graph, to characterize when the total quotient ring of a reduced ring $R$ is von Neumann regular. Also, [24, Proposition 4.8] gives a similar result. Namely, it was shown that when $\overline{\Gamma}(R) \neq \Gamma(R)$, $\overline{\Gamma}(R)$ is complemented is a sufficient condition so that the total quotient ring of $R$ is zero-dimensional. But, it seems that the proof holds true only when $girth(\overline{\Gamma}(R)) = 4$. In this chapter, using a new treatment, we prove that [24, Proposition 4.8] still holds true without any further assumption (see Theorem 4.4). Namely, in this chapter, we continue the investigation begun in [24] to further study when $\overline{\Gamma}(R)$ is complemented and when it is uniquely complemented. See the introduction for a complete overview of this chapter.

## 4.1 When the extended zero-divisor graph of a commutative ring is complemented?

In this section, we study when the extended zero-divisor graph of a commutative ring is complemented. We start by showing that the ring $R$ have at most one nonzero nilpotent element if $\overline{\Gamma}(R)$ is complemented and $|Z(R)| \geq 4$. But first, we need the following lemmas which will be very useful throughout this paper.

**Lemma 4.1.** *Let $R$ be a non-reduced ring. If $\overline{\Gamma}(R)$ is complemented, then every nonzero nilpotent element has index* 2.

*Proof.* Assume that $\mathrm{Nil}(R) \neq \{0\}$. Let $x \in \mathrm{Nil}(R)$ such that $n_x \geq 3$. Let $z \in Z(R)$ such that $z$ is adjacent to $x$. If $x^{n_x-1} \neq z$, then $x^{n_x-1}$ is adjacent to both $z$ and $x$. Otherwise, we can easily see that $x^{n_x-1} + x$ is adjacent to both $x^{n_x-1}$ and $x$. Then, $\overline{\Gamma}(R)$ is not complemented. □

Notice that the converse of this lemma does not hold in general since, for instance, the extended zero-divisor graph $\overline{\Gamma}(\mathbb{Z}_{18})$, illustrated in Figure 4.1, is not complemented (since, for example, $\overline{6}$ has not an orthogonal element) even if the index of nilpotency of every nilpotent element is 2.

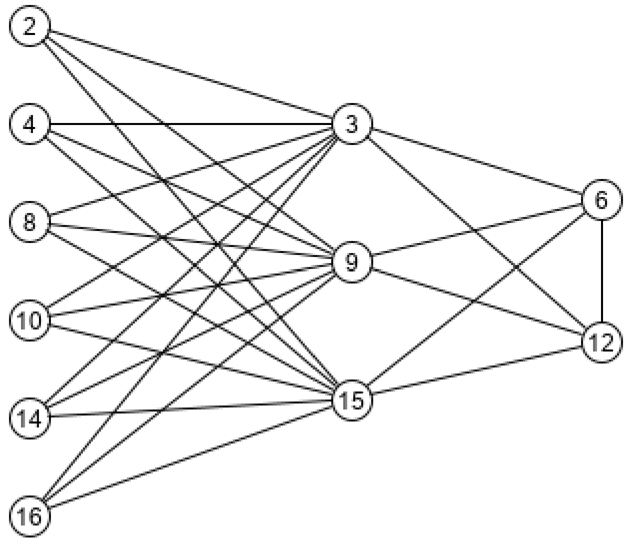



Figure 4.1: $\overline{\Gamma}(\mathbb{Z}_{18})$

**Example 4.1.** *1. Let $p$ be a prime number and $n$ be a positive integer. Then, $\overline{\Gamma}(\mathbb{Z}_{p^n})$ is complemented if and only if $n = 2$ and $p = 3$ (since $K_2$ is the only complete graph that is complemented).*

*2. Consider the ring $\mathbb{R}[X,Y]/(X^3, XY^3)$. The index of nilpotency of $\overline{X}$ is 3, and the graph $\overline{\Gamma}(\mathbb{R}[X,Y]/(X^3, XY^3))$ is not complemented since the vertex $\overline{Y}$ has not an orthogonal element.*

**Lemma 4.2.** *Let $R$ be a ring such that $|Z(R)^*| \geq 3$. If $\overline{\Gamma}(R)$ is complemented, then the following assertions hold:*

*1. For every $\alpha \in \mathrm{Nil}(R)^*$, $2\alpha = 0$.*

*2. For every $\alpha \in \mathrm{Nil}(R)^*$, if $\beta \in Z(R)^*$ such that $\beta \perp \alpha$, then $\beta \notin \mathrm{Nil}(R)$.*

*Proof.* 1. Assume that there exists $\alpha \in \mathrm{Nil}(R)^*$ such that $2\alpha \neq 0$. Then, $\alpha$ is adjacent to $(-\alpha)$. On the other hand, $|Z(R)^*| \geq 3$ and since $\overline{\Gamma}(R)$ is connected, there exists $z \in Z(R)^* \setminus \{\alpha, -\alpha\}$ which is adjacent to $\alpha$. But, such an element is adjacent to $(-\alpha)$. Namely, this means that $\alpha$ has not an orthogonal, which is a contradiction with the fact that $\overline{\Gamma}(R)$ is complemented.
2. Let $\alpha \in \mathrm{Nil}(R)^*$ and consider $\beta \in Z(R)^*$ such that $\alpha \perp \beta$. If $\beta \in \mathrm{Nil}(R)^*$, then $\alpha + \beta \neq 0$, otherwise $\alpha = -\beta$ and with the fact that $2\alpha = 0$, $\alpha = \beta$, a contradiction since $\alpha \perp \beta$. Thus, $\alpha + \beta$ is adjacent to both $\alpha$ and $\beta$ (since $\alpha$ and $\beta$ are adjacent, and by Lemma 4.1, $\beta^2 = \alpha^2 = 0$). So, $\alpha$ and $\beta$ are not orthogonal, a contradiction. □

Now, we are in position to show that when $\overline{\Gamma}(R)$ is complemented and $|Z(R)| \geq 4$, the ring $R$ has at most one nonzero nilpotent element.
Notice that, if $|Z(R)| = 2$, which means that $R$ is isomorphic to $\mathbb{Z}_4$ or $\mathbb{Z}_2[X]/(X^2)$, $\overline{\Gamma}(R)$ is not complemented. If $|Z(R)| = 3$, then $\overline{\Gamma}(R)$ is complemented. Explicitly, $R$ is either isomorphic to $\mathbb{Z}_9$ or $\mathbb{Z}_3[X]/(X^2)$ (and in this case $\mathrm{Nil}(R) = \{0, a, -a\} = Z(R)$ for some $0 \neq a \in R$), or $R$ is isomorphic to $\mathbb{Z}_2 \times \mathbb{Z}_2$ (and in this case $\mathrm{Nil}(R) = \{0\}$).

**Theorem 4.1.** *Let $R$ be a ring such that $|Z(R)| \geq 4$. If $\overline{\Gamma}(R)$ is complemented, then $|\mathrm{Nil}(R)| \leq 2$.*

*Proof.* Assume that there exist $a, b \in \mathrm{Nil}(R)^*$ such that $a \neq b$. Then, $a + b \in \mathrm{Nil}(R)^*$ by Lemma 4.2. Let $x, y, z \in Z(R) \setminus \mathrm{Nil}(R)$ such that $x \perp a$, $y \perp b$ and $z \perp a + b$. Let $n$ be a positive integer such that $z^n(a + b) = 0$. We have the two following cases:
**Case** $ab \neq 0$: Since $z^n(a + b) = 0$, $z^n ab = -z^n b^2 = 0$ by Lemma 4.1. Thus, $ab$ is adjacent to both $z$ and $a + b$ ($ab \neq z$ since $ab \in \mathrm{Nil}(R)^*$ and also $ab \neq a + b$), a contradiction.
**Case** $ab = 0$: If $z^n a = 0$, then $a$ is adjacent to both $z$ and $a + b$, a contradiction. Then, $z^n a \neq 0$. If $z^n a \neq a$, then $z^n a$ is adjacent to both $a$ and $x$, a contradiction. Otherwise, since $z^n(a + b) = 0$ and $b \in \mathrm{Nil}(R)^*$, $z^n a = -z^n b = z^n b$. Then, $z^n a = a = z^n b$ is adjacent to both $b$ and $y$, a contradiction.

□

**Example 4.2.** *Consider the ring $D \times \mathbb{Z}_2[X]/(X^2)$, where $D$ is an integral domain. Then, $\mathrm{Nil}(D \times \mathbb{Z}_2[X]/(X^2)) = \{(0, \bar{0}), (0, \bar{X})\}$ and its extended zero-divisor graph is illustrated in Figure 4.2. Namely, $\overline{\Gamma}(D \times \mathbb{Z}_2[X]/(X^2))$ is a complete bipartite graph and hence it is complemented.*

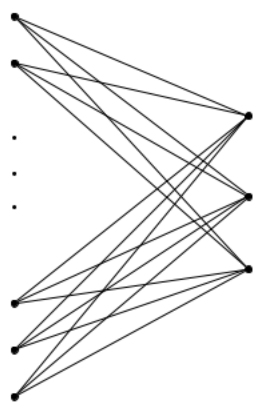


Figure 4.2: $\overline{\Gamma}(D \times \mathbb{Z}_2[X]/(X^2))$

**Example 4.3.** *For the ring $\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_4$, we have $\mathrm{Nil}(\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_4) = \{(\bar{0}, \bar{0}, \bar{0}), (\bar{0}, \bar{0}, \bar{2})\}$. The extended zero-divisor graph of this ring is illustrated in Figure 4.3. We can easily show*

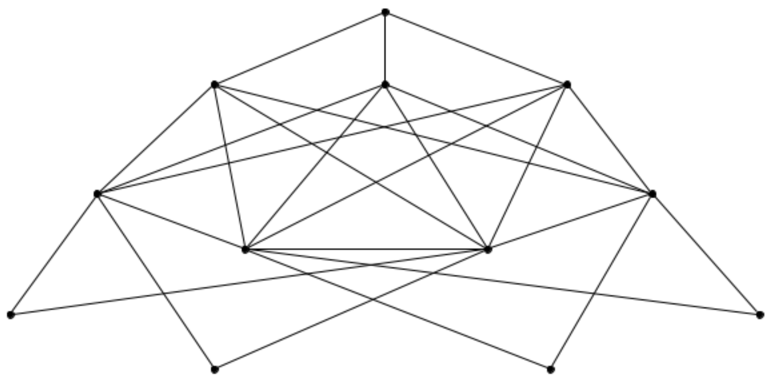


Figure 4.3: $\overline{\Gamma}(\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_4)$

*that $\overline{\Gamma}(\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_4)$ is complemented.*

**Example 4.4.** *For the ring* $\mathbb{Z}_2[X,Y]/(X^3,XY)$, *we have* $\mathrm{Nil}(\mathbb{Z}_2[X,Y]/(X^3,XY)) = \{\bar{0},\bar{X},\bar{X}^2,\bar{X}+\bar{X}^2\}$. *The extended zero-divisor of this ring is illustrated in Figure 4.4.*

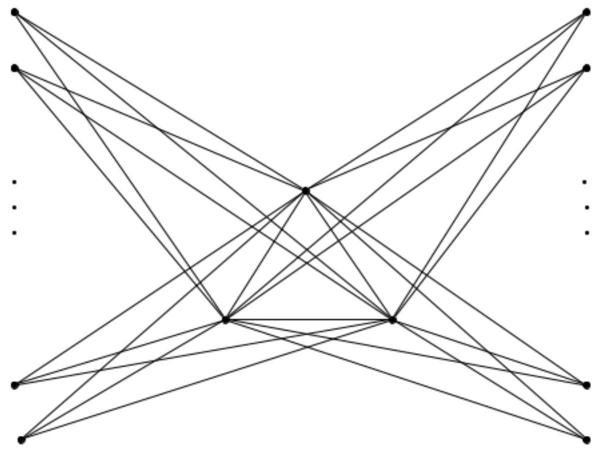


Figure 4.4: $\overline{\Gamma}(\mathbb{Z}_2[X,Y]/(X^3,XY))$

*Since* $\bar{X}+\bar{Y}$ *has not an orthogonal element,* $\overline{\Gamma}(\mathbb{Z}_2[X,Y]/(X^3,XY))$ *is not complemented.*

When $R$ is finite, the converse of Theorem 4.1 holds as shown in Corollary 4.2 which is a consequence of the following one.

**Theorem 4.2.** *Let* $R$ *be a finite ring such that* $\Gamma(R) \neq \overline{\Gamma}(R)$. *Then,* $\overline{\Gamma}(R)$ *is complemented if and only if* $R \cong B \times A_1 \times \cdots \times A_n$ *such that* $B \cong \mathbb{Z}_4$ *or* $\mathbb{Z}_2[X]/(X^2)$ *and* $A_1,\ldots,A_n$ *are finite fields.*

*Proof.* $\Leftarrow$) This follows by induction using Theorems 4.6 and 4.7 given in Section 5.
$\Rightarrow$) Since $R$ is a finite ring, $R \cong A_1 \times \cdots \times A_n$ such that $A_i$ is a finite local ring for all $i \in \{1,\ldots,n\}$, by [13, Theorem 87]. Then, for all $i \in \{1,\ldots,n\}$, $Z(A_i) = \mathrm{Nil}(A_i)$. By Theorem 4.1, $|\mathrm{Nil}(R)| \leq 2$, and since $\overline{\Gamma}(R) \neq \Gamma(R)$, $|\mathrm{Nil}(R)| = 2$. So, one of the $A_i's$ is isomorphic to $\mathbb{Z}_4$ or $\mathbb{Z}_2[X]/(X^2)$ and the other rings are finite fields. Notice that $\overline{\Gamma}(\mathbb{Z}_4)$ and $\overline{\Gamma}(\mathbb{Z}_2[X]/(X^2))$ are not complemented which guarantee the existence of the fields. □

**Corollary 4.1.** *Let* $n \in \mathbb{N}^*$ *such that* $\Gamma(\mathbb{Z}_n) \neq \overline{\Gamma}(\mathbb{Z}_n)$. *Then,* $\overline{\Gamma}(\mathbb{Z}_n)$ *is complemented if and only if* $n = 2^2 p_1 \ldots p_r$ *with* $p_1,\ldots,p_r$ *are distinct prime numbers and* $r \geq 1$ *is a positive integer.*

Now, let us prove the converse of Theorem 4.1 in the case of a finite ring.

**Corollary 4.2.** *Let* $R$ *be a finite ring such that* $\Gamma(R) \neq \overline{\Gamma}(R)$. *If* $\mathrm{Nil}(R) = \{0,a\}$ *for some* $a \in R^*$, *then* $\overline{\Gamma}(R)$ *is complemented.*

*Proof.* Since $R$ is a finite ring, by [13, Theorem 87], $R \cong A_1 \times \cdots \times A_n$ such that $A_i$ is a finite local ring for all $i \in \{1,\ldots,n\}$. If $R$ is indecomposable, then using the fact that $|\mathrm{Nil}(R)| = 2 = |Z(R)|$, $R \cong \mathbb{Z}_4$ or $R \cong \mathbb{Z}_2[X]/(X^2)$. Then, this contradicts the fact that $\Gamma(R) \neq \overline{\Gamma}(R)$. Thus, $R \cong A_1 \times \cdots \times A_n$ such that $Z(A_i) = \mathrm{Nil}(A_i)$ for every $i \in \{1,\ldots,n\}$ and $n \geq 2$. Since $|\mathrm{Nil}(R)| = 2$, one of the $A_i's$ is isomorphic to $\mathbb{Z}_4$ or $\mathbb{Z}_2[X]/(X^2)$ and the other rings are integral domains. Then, by Theorem 4.2, $\overline{\Gamma}(R)$ is complemented. □

The authors are not able to prove the equivalence of Theorem 4.2 for infinite rings. We let it then as an open important question.

## 4.2 Complementedness and uniquely complementedness notions coincide for the extended zero-divisor graphs

In [9, Theorem 3.5], it was shown that, when $R$ is reduced, $\Gamma(R)(=\overline{\Gamma}(R))$ is uniquely complemented if and only if $\Gamma(R)$ is complemented if and only if $T(R)$ is von Neumann regular. The main result of this section generalizes [9, Theorem 3.5]. Namely, it shows that, when $R$ is not reduced, the complementedness and the uniquely complementedness notions coincide. To show this, we first prove the following lemma.

**Lemma 4.3.** *Let $a, b, c \in Z(R) \setminus \mathrm{Nil}(R)$ such that $b \perp a$ and $a \perp c$ in $\overline{\Gamma}(R)$, then $b \sim c$.*

*Proof.* We have $a^{n_1}b^{m_1} = a^{n_2}c^{m_2} = 0$ for some $n_1, m_1, n_2, m_2 \in \mathbb{N}^*$. We first show that $c$ and $b$ are not adjacent; that is, $b^\alpha c^\beta \neq 0$ for every $\alpha, \beta \in \mathbb{N}^*$. If $b^\alpha c^\beta = 0$ for some $\alpha, \beta \in \mathbb{N}$. Then, $b = c$ or $a = c$ (since $b \perp a$ and $a \perp c$). Thus, $b \in \mathrm{Nil}(R)$ or $a \in \mathrm{Nil}(R)$, a contradiction. Then, $c$ and $b$ are not adjacent. Now, let us prove that $N(b) = N(c)$. Let $d \in N(b)$, then $d^n b^m = 0$ with $d^n \neq 0$ for some $n, m \in \mathbb{N}$. Thus, $(d^n c^{m_2})a^{n_2} = d^n(c^{m_2}a^{n_2}) = 0$ and $(d^n c^{m_2})b^m = (d^n b^m)c^{m_2} = 0$. Then, $d^n c^{m_2} = 0$, otherwise $d^n c^{m_2}$ is adjacent to both $a$ and $b$ (and $d^n c^{m_2} \neq a$ and $d^n c^{m_2} \neq b$ since $a, b \notin \mathrm{Nil}(R)$) which contradicts the fact that $b \perp a$. This shows that, $N(b) \subseteq N(c)$. Similarly, we show the other inclusion and then $b \sim c$.

□

Now, we are ready to prove the main result of this section.

**Theorem 4.3.** *Let $R$ be a ring such that $\overline{\Gamma}(R) \neq \Gamma(R)$. Then, $\overline{\Gamma}(R)$ is uniquely complemented if and only if $\overline{\Gamma}(R)$ is complemented.*

*Proof.* $\Rightarrow$) By definition of uniquely complemented.
$\Leftarrow$) Suppose that $\overline{\Gamma}(R)$ is complemented, then, by Theorem 4.1, $\mathrm{Nil}(R) = \{0, \alpha\}$ for some $0 \neq \alpha \in R$. So, by Lemma 4.3, we have just to prove that, for every $b, c \in Z(R)^*$, if $\alpha \perp b$ and $\alpha \perp c$, then $b \sim c$, and if $\alpha \perp c$ and $b \perp c$, then $\alpha \sim b$. Let us prove the first implication. So, suppose by contradiction that there exist $b, c \in Z(R)^*$ such that $\alpha \perp b$ and $\alpha \perp c$ but $b \nsim c$. Then, there exists $x \in N(c) \setminus N(b)$; that is, $x^{n_1}c^{m_1} = 0$ for some $n_1, m_1 \in \mathbb{N}^*$ and $x^n b^m \neq 0$ for every $n, m \in \mathbb{N}^*$. If $xb \neq c$. Then, $(xb)^{n_1}c^{m_1} = 0$ and so $xb$ and $c$ are adjacent. On the other hand, $\alpha$ and $b$ are adjacent. Then, $\alpha b^t = 0$ for some $t \in \mathbb{N}^*$. Thus, $\alpha(xb)^t = 0$ which shows that $xb$ is adjacent to both $\alpha$ and $c$, a contradiction since $\alpha \perp c$. Then, $xb = c$, and with $x^{n_1}c^{m_1} = 0$ we get $x^{n_1+m_1}b^{m_1} = x^{n_1}(xb)^{m_1} = 0$, a contradiction. Then, $N(c) \subseteq N(b)$. Similarly, we prove the other inclusion.
Now, we prove the second implication. Assume that $\alpha \perp c$ and $b \perp c$. Then, $\alpha c^{m_1} = b^{n_1}c^{m_2} = 0$ for some $n_1, m_1, m_2 \in \mathbb{N}^*$. Thus, $\alpha$ is not adjacent to $b$, otherwise $b$ is adjacent to both $\alpha$ and $c$, a contradiction since $\alpha \perp c$. Let $d \in N(\alpha)$, then $d^n\alpha = 0$ for some $n \in \mathbb{N}^*$ and so $(d^n b^{n_1})c^{m_2} = d^n(b^{n_1}c^{m_2}) = 0$ and $(d^n b^{n_1})\alpha = b^{n_1}d^n\alpha = 0$. If $d^n b^{n_1} \in Z(R) \setminus \mathrm{Nil}(R)$, then $d^n b^{n_1} \neq \alpha$ and $d^n b^{n_1} \neq c$. Thus, $d^n b^{n_1}$ is adjacent to both $c$ and $\alpha$, a contradiction (since $\alpha \perp c$). Then, $d^n b^{n_1} \in \mathrm{Nil}(R)$. If $d^n b^{n_1} = 0$, then, $d$ is adjacent to $b$ ($d \neq b$ since $d, b \in Z(R) \setminus \mathrm{Nil}(R)$). Thus, $d \in N(b)$. If $d^n b^{n_1} = \alpha$, then, $d^{2n}b^{2n_1} = \alpha^2 = 0$ ($d^{2n} \neq 0$, $b^{2n_1} \neq 0$ and $d \neq b$ since $b, d \in Z(R) \setminus \mathrm{Nil}(R)$). Thus, $d \in N(b)$. This shows that $\alpha \sim b$. Therefore, $\overline{\Gamma}(R)$ is uniquely complemented.

□

**Corollary 4.3.** *Let $R$ be a ring such that $\Gamma(R) \neq \overline{\Gamma}(R)$ and $\overline{\Gamma}(R)$ is complemented. Then, for every orthogonal $b \in Z(R)^*$ to the nonzero nilpotent element $\alpha$, we have $b \sim \alpha + b$.*

*Proof.* Assume that $\Gamma(R) \neq \overline{\Gamma}(R)$ and $\overline{\Gamma}(R)$ is complemented. Then, by Theorem 4.2, $\mathrm{Nil}(R) = \{0, \alpha\}$ for some $0 \neq \alpha \in R$. Let $b \in Z(R)^* \setminus \{\alpha\}$ such that $\alpha \perp b$; that is, $\alpha b^n = 0$ for some positive integer $n$ and there is no vertex adjacent to both $\alpha$ and $b$. Let us prove that $\alpha \perp (\alpha + b)$. We have $\alpha(\alpha+b)^n = \alpha(b^n + n\alpha b^{n-1} + \cdots + \alpha^n) = \alpha b^n = 0$. Since $\alpha + b \neq \alpha$ and $(\alpha+b)^n \neq 0$ (because $b \notin \mathrm{Nil}(R)$), $\alpha$ and $\alpha + b$ are adjacent. Now, assume that there exists $c$ which is adjacent to both $\alpha$ and $\alpha + b$. Then, $c^{n_1}\alpha = 0 = c^{n_1}(\alpha+b)^{m_1} = m_1 c^{n_1}\alpha b^{m_1-1} + c^{n_1}b^{m_1} = 0 + c^{n_1}b^{m_1}$. So, $c$ is adjacent to $b$, a contradiction since $\alpha \perp b$. Therefore, $\alpha \perp (\alpha + b)$, which shows using Theorem 4.3 that $b \sim \alpha + b$.

□

## 4.3 Complemented extended zero-divisor graphs and zero-dimensional rings

If $|Z(R)| = 2$, then $R$ is isomorphic to $\mathbb{Z}_4$ or $\mathbb{Z}_2[X]/(X^2)$ and so $T(R)$ is zero-dimensional. If $|Z(R)| = 3$, then $R$ is isomorphic to $\mathbb{Z}_2 \times \mathbb{Z}_2$, $\mathbb{Z}_9$ or $\mathbb{Z}_3[X]/(X^2)$ and so, $\overline{\Gamma}(R)$ is complemented and $T(R)$ is zero-dimensional. In this section, we show that, when $\Gamma(R) \neq \overline{\Gamma}(R)$ (in particular $|Z(R)| \geq 4$), $T(R)$ is zero-dimensional once $\overline{\Gamma}(R)$ is complemented. In fact, this result was already given in [24, Proposition 4.8]. But, in the third line of the proof, [24, Corollary 3.4] is used to show that an element $z_0$ is not nilpotent. This means that we have supposed that the girth of $\overline{\Gamma}(R)$ is not 3. But, there are some extended zero-divisor graphs, $\overline{\Gamma}(R)$, which are complemented with girth equal to 3. For this consider $\overline{\Gamma}(\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_4)$ (see Figure 4.3). Now, using a new way, we show that [24, Proposition 4.8] holds true. To show that, we need the following lemma.

**Lemma 4.4.** *Let $R$ be a ring such that $\Gamma(R) \neq \overline{\Gamma}(R)$. If $\Gamma(R)$ is uniquely complemented, then $\overline{\Gamma}(R)$ is not complemented.*

*Proof.* The result holds because once $\Gamma(R)$ is uniquely complemented it will be a star graph by [9, Theorem 3.9]. In this case $\overline{\Gamma}(R)$ is not complemented.

□

Using the previous lemma, we get the main result of this section.

**Theorem 4.4.** *Let $R$ be a ring such that $\Gamma(R) \neq \overline{\Gamma}(R)$. If $\overline{\Gamma}(R)$ is complemented, then $T(R)$ is zero-dimensional.*

*Proof.* There are two cases to discuss:
**<u>Case 1.</u>** <u>For every $x \in Z(R)^*$, $x^\perp \cap (Z(R) \setminus \mathrm{Nil}(R)) \neq \emptyset$:</u> In this case, we show that for every $\frac{x_1}{x_2}$ in $T(R)$, there exists $\frac{m_1}{m_2} \in T(R)$ such that $\frac{x_1}{x_2} + \frac{m_1}{m_2}$ is a unit and $\frac{x_1}{x_2}\frac{m_1}{m_2}$ is nilpotent. This shows that $T(R)$ is $\pi$-regular and so zero-dimensional (see [52, Theorems 3.1 and 3.2]). Then, let $\frac{x_1}{x_2}$ in $T(R)$. We distinguish three sub-cases:
**<u>Sub-case 1.</u>** Assume that $x_1 \in R \setminus Z(R)$. Since $\overline{\Gamma}(R)$ is complemented and $\overline{\Gamma}(R) \neq \Gamma(R)$, $|\mathrm{Nil}(R)| = 2$. We denote by $\alpha$ the nonzero nilpotent element of $R$. Using Lemma 4.2, we have $\alpha^2 = 2\alpha = 0$. It is clear that $\frac{x_1}{x_2}\frac{\alpha}{x_2}$ is nilpotent and also $\frac{x_1}{x_2} + \frac{\alpha}{x_2}$ is a unit since $(x_1 + \alpha)^2 = {x_1}^2 \notin Z(R)$.
**<u>Sub-case 2.</u>** Assume that $x_1 = \alpha$. We have $\frac{x_1}{x_2}\frac{1}{x_2}$ is nilpotent and also $\frac{x_1}{x_2} + \frac{1}{x_2}$ is a unit since $(x_1 + 1)^2 = 1 \notin Z(R)$.
**<u>Sub-case 3.</u>** Assume that $x_1 \in Z(R) \setminus \mathrm{Nil}(R)$. Then, there exists $m_1 \in {x_1}^\perp \cap (Z(R) \setminus \mathrm{Nil}(R))$. Since $x_1$ and $m_1$ are adjacent, $\frac{x_1}{x_2}\frac{m_1}{x_2}$ is nilpotent. So, it remains to show that

$\frac{x_1}{x_2} + \frac{m_1}{x_2}$ is a unit, which means to prove that $x_1 + m_1$ does not belong to $Z(R)$. Otherwise, there exists $z \in R^*$ such that $z(x_1 + m_1) = 0$. We have $x_1 m_1$ is nilpotent (since $x_1$ and $m_1$ are adjacent), then there are the two following sub-subcases to discuss:

**Sub-subcase 1.** Suppose that $x_1 m_1 = 0$. We have $z(x_1 + m_1) = 0$, then $zx_1 m_1 + zm_1{}^2 = 0$ and $zx_1{}^2 + zx_1 m_1 = 0$, so $zm_1{}^2 = 0$ and $zx_1{}^2 = 0$. Then, $z \neq x_1$ and $z \neq m_1$ since $x_1$ and $m_1$ are not nilpotent. Thus, $z$ is adjacent to both $x_1$ and $m_1$, a contradiction (since $x_1$ and $m_1$ are orthogonal).

**Sub-subcase 2.** Suppose that $x_1 m_1 = \alpha$. We have $zx_1{}^2 + zx_1 m_1 = 0$ and $zx_1 m_1 + zm_1{}^2 = 0$, then $zx_1{}^2 + z\alpha = 0$ and $z\alpha + zm_1{}^2 = 0$. Thus, $z\alpha x_1{}^2 = 0$ and $z\alpha m_1{}^2 = 0$. Then, $z\alpha \neq x_1$ and $z\alpha \neq m_1$ since $x_1$ and $m_1$ are not nilpotent. If $z\alpha \neq 0$, then it is adjacent to both $x_1$ and $m_1$, a contradiction (since $x_1$ and $m_1$ are orthogonal). Then, $z\alpha = 0$ which implies that $zx_1^2 = 0 = zm_1^2$. Thus, $z \neq x_1$ and $z \neq m_1$ since $x_1$ and $m_1$ are not nilpotent. Then, $z$ is adjacent to both $x_1$ and $m_1$, a contradiction (since $x_1$ and $m_1$ are orthogonal).

**Case 2.** There exists $x \in Z(R)^*$, $x^\perp = \{\alpha\} \subset \mathrm{Nil}(R) = \{0, \alpha\}$: In this case, one can show that $N_{\overline{\Gamma}(R)}(x) = \{\alpha\}$. Otherwise, there exist $s$ and $t$ in $N_{\overline{\Gamma}(R)}(x)$ such that $s$ and $t$ are adjacent. Since $x^\perp = \{\alpha\}$, $s\alpha = \alpha$. Also, since $s$ and $t$ are adjacent, $st = \alpha$ or $st = 0$. Then, $s\alpha t = 0$, which implies that $\alpha t = 0$. Thus, $t$ and $\alpha$ are adjacent, which is a contradiction with the fact that $x^\perp = \{\alpha\}$. Thus, $N_{\overline{\Gamma}(R)}(x) = \{\alpha\}$. On the other hand, $\Gamma(R)$ is not uniquely complemented (using Lemma 4.4). Then, in this case, there are two sub-cases to discuss:

**Sub-case 1.** $\Gamma(R)$ is complemented: Since $\Gamma(R)$ is not uniquely complemented, by [9, Theorem 3.14], $R \cong \mathbb{Z}_4 \times D$ or $R \cong \mathbb{Z}_2[X]/(X^2) \times D$ such that $D$ is an integral domain, but this contradicts the fact that $|x^\perp| = 1$ (since for every $y \in Z(R)^*$, $|y^\perp| > 1$).

**Sub-case 2.** $\Gamma(R)$ is not complemented: Then, there exists $b \in Z(R)^*$ which has an orthogonal in $\overline{\Gamma}(R)$ and not in $\Gamma(R)$ (one can see that $b \neq \alpha$ and $bx \notin \{0, \alpha\}$ since $\alpha$ has $x$ as an orthogonal in $\Gamma(R)$ and $N_{\overline{\Gamma}(R)}(x) = \{\alpha\}$). Then, there exists $t \in b^\perp$ such that $bt = \alpha$. One can show that $t \neq \alpha$. Otherwise, $b\alpha = \alpha$ and so for every $n \in \mathbb{N}^*$, $b^n \alpha = \alpha \neq 0$. Then, $b$ and $\alpha$ are not adjacent in $\overline{\Gamma}(R)$, which is a contradiction with the fact that $t = \alpha$ and $b$ are orthogonal. There are two sub-subcases to discuss:

**Sub-subcase 1.** $b\alpha \neq 0$: Then, $b\alpha = \alpha$. Since for every $z \in N_{\overline{\Gamma}(R)}(b)$, $zb = \alpha$ or $zb = 0$, $z(b\alpha) = 0$ for every $z \in N_{\overline{\Gamma}(R)}(b)$, then $z\alpha = 0$ for every $z \in N_{\overline{\Gamma}(R)}(b)$ (since $b\alpha = \alpha$). In this case, we show that $(bx)^\perp = \emptyset$, therefore, we determine firstly $N_{\overline{\Gamma}(R)}(bx)$. Let $h \in N_{\overline{\Gamma}(R)}(bx)$, then there exist $n, m$ in $\mathbb{N}^*$ such that $(bx)^n h^m = 0$ with $(bx)^n \neq 0$ and $h^m \neq 0$. Then, $(bh)^n x^n = 0$ (resp., $(bh)^m x^n = 0$), if $n \geq m$ (resp., $m \geq n$). If $(bh)^n \neq 0$, then $bh = \alpha$ since $N_{\overline{\Gamma}(R)}(x) = \{\alpha\}$. Then, $h = \alpha$ or $h \in N_{\overline{\Gamma}(R)}(b)$. If $(bh)^n = 0$, then $h = \alpha$ or $h \in N_{\overline{\Gamma}(R)}(b)$. Thus, $N_{\overline{\Gamma}(R)}(bx) = N_{\overline{\Gamma}(R)}(b) \cup \{\alpha\}$. Thus, $(bx)^\perp = \emptyset$ since $z\alpha = 0$ for every $z \in N_{\overline{\Gamma}(R)}(b)$, which is a contradiction with the fact that $\overline{\Gamma}(R)$ is complemented.

**Sub-subcase 2.** $b\alpha = 0$: Then, $t\alpha \neq 0$ since $t \in b^\perp$, which implies that $t\alpha = \alpha$. As in the previous case, $z\alpha = 0$ for every $z \in N_{\overline{\Gamma}(R)}(t)$. In this case, we show that $(tx)^\perp = \emptyset$. First, let us determine $N_{\overline{\Gamma}(R)}(tx)$. Let $h$ in $N_{\overline{\Gamma}(R)}(tx)$, then there exist $n, m$ in $\mathbb{N}^*$ such that $(tx)^n h^m = 0$ with $(tx)^n \neq 0$ and $h^m \neq 0$. Then, $(th)^n x^n = 0$ (resp. $(th)^m x^n = 0$), if $n \geq m$ (resp., $m \geq n$). If $(th)^n \neq 0$, then $th = \alpha$ since $N_{\overline{\Gamma}(R)}(x) = \{\alpha\}$. Then, $h = \alpha$ or $h \in N_{\overline{\Gamma}(R)}(t)$. If $(th)^n = 0$, then $h = \alpha$ or $h \in N_{\overline{\Gamma}(R)}(t)$. Thus, $N_{\overline{\Gamma}(R)}(tx) = N_{\overline{\Gamma}(R)}(t) \cup \{\alpha\}$. Then, $(tx)^\perp = \emptyset$ since for every $z \in N_{\overline{\Gamma}(R)}(t)$, $z\alpha = 0$, which is a contradiction with the fact that $\overline{\Gamma}(R)$ is complemented.

□

The following example shows that $T(R)$ is zero-dimensional does not imply that $\overline{\Gamma}(R)$ is complemented.

**Example 4.5.** *$T(\mathbb{Z}_{16})$ is zero dimensional and $\overline{\Gamma}(\mathbb{Z}_{16})$ is not complemented.*

*Proof.* Since $\mathbb{Z}_{16}$ is zero-dimensional, $T(\mathbb{Z}_{16}) \cong \mathbb{Z}_{16}$ is zero-dimensional. On the other hand, $\overline{\Gamma}(\mathbb{Z}_{16})$ is not complemented ($\overline{2}$ is a nilpotent element in $\mathbb{Z}_{16}$ of index of nilpotency 4). Then, by Lemma 4.1, $\overline{\Gamma}(\mathbb{Z}_{16})$ is not complemented.

□

An observation of the proof of Theorem 4.4 leads us to show that, if $\overline{\Gamma}(R)$ is complemented, then every non-nilpotent element has a non-nilpotent orthogonal, as shown in the following result.

**Theorem 4.5.** *Let $R$ be a ring such that $\overline{\Gamma}(R)$ is complemented and $\overline{\Gamma}(R) \neq \Gamma(R)$. Then, for all $x \in Z(R) \setminus \mathrm{Nil}(R)$, $x^{\perp} \cap (Z(R) \setminus \mathrm{Nil}(R)) \neq \emptyset$.*

*Proof.* Since $\overline{\Gamma}(R)$ is complemented and $\overline{\Gamma}(R) \neq \Gamma(R)$, $|\mathrm{Nil}(R)^*| = 1$ and $n_x \leq 2$ for every nilpotent element $x$. We denote by $\alpha$ the nonzero nilpotent element of $R$. We suppose that there exists $x_1 \in Z(R)\setminus\mathrm{Nil}(R)$ such that ${x_1}^{\perp} = \{\alpha\}$. On the other hand, by Theorem 4.4, $T(R)$ is zero-dimensional. Then, using [52, Theorems 3.1 and 3.2], there exists $m_1$ in $R$ such that $\frac{x_1}{x_2}\frac{m_1}{m_2}$ is nilpotent and $\frac{x_1}{x_2} + \frac{m_1}{m_2}$ is a unit for some $m_2, x_2$ in $R \setminus Z(R)$. Since $\frac{x_1}{x_2}\frac{m_1}{m_2}$ is nilpotent, $x_1m_1 \in \mathrm{Nil}(R)$. Then, there are two cases to discuss:
**<u>Case 1.</u>** Suppose that $x_1m_1 = 0$. Then, $x_1$ and $m_1$ are adjacent ($x_1 \neq m_1$ and $m_1 \neq 0$ since $x_1 \in Z(R)\setminus\mathrm{Nil}(R)$ and $\frac{x_1}{x_2}+\frac{m_1}{m_2}$ is a unit). If $m_1$ is nilpotent, then $m_1(x_1m_2+m_1x_2) = 0$. Thus, $\frac{x_1}{x_2} + \frac{m_1}{m_2}$ is not a unit, a contradiction. Otherwise, $m_1$ and $x_1$ are not orthogonal, then there exists $z \in Z(R)^*$ that is adjacent to both $x_1$ and $m_1$. Then, $z^2{x_1}^2 = z^2{m_1}^2 = 0$ (since $zx_1$ and $zm_1$ are nilpotent). If $z^2x_1 = 0$ and $z^2m_1 = 0$, then $z^2(x_1m_2 + x_2m_1) = 0$. Thus, $\frac{x_1}{x_2} + \frac{m_1}{m_2}$ is not a unit, a contradiction. Otherwise, $z^2x_1(x_1m_2 + x_2m_1) = 0$ with $z^2x_1 \neq 0$ or $z^2m_1(x_1m_2 + x_2m_1) = 0$ with $z^2x_1 \neq 0$. Then, $\frac{x_1}{x_2} + \frac{m_1}{m_2}$ is not a unit, a contradiction.
**<u>Case 2.</u>** Suppose that $x_1m_1 = \alpha$. If $m_1$ is nilpotent, then $x_1$ and $m_1$ are adjacent. Thus, there exists $n \in \mathbb{N}^*$ such that ${x_1}^n m_1 = 0$. Consider $\beta$ such that ${x_1}^{\beta}m_1 = 0$ and ${x_1}^{(\beta-1)}m_1 \neq 0$. We have ${x_1}^{(\beta-1)}m_1(x_1m_2 + m_1x_2) = 0$, then $\frac{x_1}{x_2} + \frac{m_1}{m_2}$ is not a unit, a contradiction. If $m_1$ is not nilpotent, then ${x_1}^2{m_1}^2 = 0$ with ${x_1}^2 \neq 0$ and ${m_1}^2 \neq 0$. Then, $x_1$ and $m_1$ are adjacent, but they are not orthogonal (since $m_1$ is not nilpotent). Then, there exists $z \in Z(R)^*$ that is adjacent to both $x_1$ and $m_1$. Thus, $z^2(x_1)^2 = 0$ and $z^2(m_1)^2 = 0$. If $z^2x_1 = 0$ and $z^2m_1 = 0$, then $z^2(x_1m_2 + m_1x_2) = 0$. Thus, $\frac{x_1}{x_2} + \frac{m_1}{m_2}$ is not a unit, a contradiction. Otherwise, $z^2x_1(x_1m_2 + m_1x_2) = 0$ if $z^2x_1 \neq 0$, or $z^2m_1(x_1m_2 + m_1x_2) = 0$ if $z^2m_1 \neq 0$ and $z^2x_1 = 0$. Then, $\frac{x_1}{x_2} + \frac{m_1}{m_2}$ is not a unit, a contradiction.

□

It was proven in [9, Lemma 3.7] that if $\Gamma(R)$ is uniquely complemented and $|R| > 9$, then there exists a unique nonzero nilpotent element in $R$ and any orthogonal to such an element is an end. This is not the case for $\overline{\Gamma}(R)$ as shown in the following corollary.

**Corollary 4.4.** *Let $R$ be a ring such that $\overline{\Gamma}(R) \neq \Gamma(R)$. If $\overline{\Gamma}(R)$ is complemented, then every orthogonal to the nonzero nilpotent element is not an end.*

In Corollary 4.2, we showed, for a finite ring $R$, that when $\Gamma(R) \neq \overline{\Gamma}(R)$ and $|\mathrm{Nil}(R)| = 2$, $\overline{\Gamma}(R)$ is complemented. For the infinite case, we get the following result.

**Proposition 4.1.** *Let $R$ be an infinite ring such that* $\mathrm{Nil}(R) = \{0, \alpha\}$ *for some* $\alpha \in R^*$. *Then, either $R$ is not local or* $\overline{\Gamma}(R)$ *is not complemented.*

*Proof.* Assume that $\mathrm{Nil}(R) = \{0, \alpha\}$ and suppose that $R$ is local with the maximal ideal $\mathrm{Ann}(\alpha)$ and that $\overline{\Gamma}(R)$ is complemented. Let $x \in Z(R) \setminus \{0, \alpha\}$. Then, there exists $y \in Z(R) \setminus \{0, \alpha\}$ such that $x \perp y$, by Theorem 4.5. But, since $\mathrm{Ann}(\alpha)$ is the maximal ideal of $R$, $x, y \in \mathrm{Ann}(\alpha)$. So, $x - y$ is a part of a triangle, a contradiction.

□

## 4.4 When the graphs $\overline{\Gamma}(R_1 \times R_2)$ and $\overline{\Gamma}(R(+)M)$ are complemented?

In the first part of this section, we investigate when the extended zero-divisor graph of the product of two rings, $R_1 \times R_2$, is complemented. Namely, we treat three cases following the cardinality of $Z(R_2)$: $|Z(R_2)| = 1$, $|Z(R_2)| = 2$ and $|Z(R_2)| \geq 3$.
For the case when $R_2$ is an integral domain, we give the following theorem.

**Theorem 4.6.** *Let $R_1$ and $R_2$ be two rings such that $R_2$ is an integral domain. Then,* $\overline{\Gamma}(R_1 \times R_2)$ *is complemented if and only if either* $|Z(R_1)| = 2$ *or (* $\overline{\Gamma}(R_1)$ *is complemented and* $|\mathrm{Nil}(R_1)| \leq 2$*).*

*Proof.* $\Rightarrow$) Assume that $\overline{\Gamma}(R_1 \times R_2)$ is complemented and $|Z(R_1)| \neq 2$. If $|\mathrm{Nil}(R_1)| \geq 3$, then $|\mathrm{Nil}(R_1 \times R_2)| \geq 3$, a contradiction (by Theorem 4.1). Now, suppose that $\overline{\Gamma}(R_1)$ is not complemented, then there exists $z \in Z(R_1)^*$ such that, $x$ is not an orthogonal to $z$ for every $x \in Z(R_1)^*$. We have $(z, 0) \in Z(R_1 \times R_2)$. Let $(a, b) \in Z(R_1 \times R_2)$ such that $(a, b)$ is adjacent to $(z, 0)$. So, $(a, b)^n (z, 0)^m = (0, 0)$ for some $n, m \in \mathbb{N}^*$ with $(a, b)^n \neq (0, 0)$ and $(z, 0)^m \neq (0, 0)$, then $a^n z^m = 0$ and so we have three cases to discuss:
<u>Case 1.</u> If $a^n = 0$ and $b \neq 0$, then for every vertex $y$ adjacent to $z$, $(y, 0)$ is adjacent to both $(a, b)$ and $(z, 0)$.
<u>Case 2.</u> If $a^n \neq 0$ and $b \neq 0$, then $a$ is adjacent to $z$ and so there exists $x$ adjacent to both $a$ and $z$ since $z$ is not orthogonal to $a$. Thus, $(x, 0)$ is adjacent to both $(z, 0)$ and $(a, b)$.
<u>Case 3.</u> If $a^n \neq 0$ and $b = 0$, then $(0, 1)$ is adjacent to both $(a, b)$ and $(z, 0)$.
In all cases, $(z, 0)$ has not an orthogonal in $\overline{\Gamma}(R_1 \times R_2)$, a contradiction.
$\Leftarrow$) If $|Z(R_1)| = 2$, then $R_1 \cong \mathbb{Z}_4$ or $\mathbb{Z}_2[X]/(X^2)$. Thus, $\overline{\Gamma}(R_1 \times R_2)$ is a complete bipartite graph. Then, $\overline{\Gamma}(R_1 \times R_2)$ is complemented.
Now, assume that $\overline{\Gamma}(R_1)$ is complemented and $|\mathrm{Nil}(R_1)| \leq 2$. If $|\mathrm{Nil}(R_1)| = 1$, then $Z(R_1 \times R_2) = (R_1 \setminus Z(R_1) \times \{0\}) \cup (Z(R_1) \times \{0\}) \cup (Z(R_1) \times R_2^*)$. If $(a, b) \in R_1 \setminus Z(R_1) \times \{0\}$, then $(a, 0) \perp (0, 1)$. If $(a, b) \in Z(R_1) \times \{0\}$, then $b = 0$ and $(a, 0) \perp (c, 1)$ with $c \in a^\perp$. If $(a, b) \in Z(R_1) \times R_2^*$, then $(a, b) \perp (c, 0)$ with $c \in a^\perp$.
If $|\mathrm{Nil}(R_1)| = 2$, then $\mathrm{Nil}(R_1) = \{0, \alpha\}$ for some $0 \neq \alpha \in R_1$ and $Z(R_1 \times R_2) = (R_1 \setminus Z(R_1) \times \{0\}) \cup (Z(R_1) \setminus \mathrm{Nil}(R_1) \times R_2) \cup (\mathrm{Nil}(R_1) \times R_2)$. Let $(a, b) \in Z(R_1 \times R_2)^*$. If $(a, b) \in R_1 \setminus Z(R_1) \times \{0\}$, then $(a, 0) \perp (0, 1)$. If $(a, b) \in Z(R_1) \setminus \mathrm{Nil}(R_1) \times R_2$, then if $b = 0$, $(a, 0) \perp (c, b')$ with $c \in a^\perp$, otherwise $(a, b) \perp (c, 0)$ with $c \in a^\perp$. For the case where $(a, b) \in \mathrm{Nil}(R_1) \times R_2$, we distinguish three cases:
If $a = \alpha$ and $b = 0$, then $(\alpha, 0) \perp (c, b')$ with $c \in \alpha^\perp$ and $b' \in R_2^*$.
If $a = \alpha$ and $b \neq 0$, then $(\alpha, b) \perp (c, 0)$ with $c \in R_1 \setminus Z(R_1)$.
If $a = 0$ and $b \neq 0$, then $(0, b) \perp (c, 0)$ with $c \in R_1 \setminus Z(R_1)$.
This completes the proof.

□

Now, for the case when $|Z(R_2)| = 2$, we give the following result.

**Theorem 4.7.** *Let $R_1$ and $R_2$ be two rings such that $|Z(R_2)| = 2$. Then, $\overline{\Gamma}(R_1 \times R_2)$ is complemented if and only if $\overline{\Gamma}(R_1)$ is complemented and $R_1$ is reduced.*

*Proof.* $\Rightarrow$) Assume that $\overline{\Gamma}(R_1 \times R_2)$ is complemented. We have $|\mathrm{Nil}(R_2)| = |Z(R_2)| = 2$, then if $R_1$ is not reduced, $|\mathrm{Nil}(R_1 \times R_2)| \geq 3$, a contradiction (by Theorem 4.1). Now, suppose that $\overline{\Gamma}(R_1)$ is not complemented, then there exists $z \in Z(R_1)^*$ which has not an orthogonal. We have $(z, 0) \in Z(R_1 \times R_2)$. Suppose that there exists $(a, b) \in Z(R_1 \times R_2)$ such that $(z, 0) \perp (a, b)$. Then, $(a, b)^n (z, 0)^m = (0, 0)$ for some $n, m \in \mathbb{N}^*$ and so $a^n z^m = 0$. Thus, we have two cases to discuss:
__**Case 1.**__ If $a^n = 0$, then $b^n \neq 0$. So, consider $y \in Z(R_1)^*$ which is adjacent to $z$. Then, $(y, 0)$ is adjacent to both $(a, b)$ and $(z, 0)$, a contradiction.
__**Case 2.**__ If $a^n \neq 0$, then $a$ is adjacent to $z$ and so there exists $x \in Z(R_1)^*$ such that $x$ is adjacent to both $z$ and $a$. Thus, $(x, 0)$ is adjacent to both $(a, b)$ and $(z, 0)$, a contradiction. Hence, $(z, 0)$ has not an orthogonal in $\overline{\Gamma}(R_1 \times R_2)$, a contradiction.
$\Leftarrow$) We have $Z(R_1 \times R_2) = (R_1 \setminus Z(R_1) \times Z(R_2)) \cup (Z(R_1) \times R_2 \setminus Z(R_2)) \cup (Z(R_1) \times Z(R_2))$. Let $(a, b) \in Z(R_1 \times R_2)$. If $(a, b) \in R_1 \setminus Z(R_1) \times Z(R_2)$, then $(a, b) \perp (0, 1)$. If $(a, b) \in Z(R_1) \times R_2 \setminus Z(R_2)$, then $(a, b) \perp (c, 0)$ with $c \in a^{\perp}$. If $(a, b) \in Z(R_1) \times Z(R_2)$, then $(a, b) \perp (c, 1)$ with $c \in a^{\perp}$.

□

For the case where $R_2$ is a non reduced ring such that $|Z(R_2)| \geq 3$, we give the following theorem.

**Theorem 4.8.** *Let $R_1$ be a ring and $R_2$ be a non reduced ring such that $|Z(R_2)| \geq 3$. Then, $\overline{\Gamma}(R_1 \times R_2)$ is complemented if and only if $\overline{\Gamma}(R_2)$ and $\overline{\Gamma}(R_1)$ are both complemented and $R_1$ is reduced.*

*Proof.* $\Rightarrow$) If $R_1$ is not reduced, then $|\mathrm{Nil}(R_1 \times R_2)| \geq 3$ since $R_2$ is not reduced. Then, $\overline{\Gamma}(R_1 \times R_2)$ is not complemented, by Theorem 4.1, since $|Z(R_1 \times R_2)| \geq 4$, a contradiction. Now, assume that $\overline{\Gamma}(R_2)$ is not complemented. Then, there exists $z \in Z(R_2)^*$ which has not an orthogonal. Let $(a, b) \in Z(R_1 \times R_2)^*$ such that $(a, b)$ is adjacent to $(0, z)$. Then, $(a, b)^n (0, z)^m = (0, 0)$ for some $n, m \in \mathbb{N}^*$ with $(a, b)^n \neq (0, 0)$ and $(0, z)^m \neq (0, 0)$. Thus, $b^n z^m = 0$. Then, we have two cases to discuss:
__**Case 1.**__ If $b^n \neq 0$, then $b$ is adjacent to $z$. So, there exists a vertex $x$ adjacent to both $z$ and $b$. Thus, $(0, x)$ is adjacent to both $(0, z)$ and $(a, b)$.
__**Case 2.**__ If $b^n = 0$, then $a^n \neq 0$. So, consider $y \in Z(R_2)^*$ which is adjacent to $z$. Then, $(0, y)$ is adjacent to both $(a, b)$ and $(0, z)$, a contradiction (since $\overline{\Gamma}(R_1 \times R_2)$ is complemented). Similarly, we can prove that $\overline{\Gamma}(R_1)$ is complemented (because, if $\overline{\Gamma}(R_1)$ is not complemented, then $|Z(R_1)| \geq 3$).
$\Leftarrow$) We have $Z(R_1 \times R_2) = (Z(R_1) \times Z(R_2)) \cup (R_1 \setminus Z(R_1) \times Z(R_2)) \cup (Z(R_1) \times R_2 \setminus Z(R_2))$. Let $(a, b) \in Z(R_1 \times R_2)^*$. If $(a, b) \in R_1 \setminus Z(R_1) \times Z(R_2) \setminus \mathrm{Nil}(R_2)$, then $(a, b) \perp (0, c)$ with $c \in b^{\perp}$. If $(a, b) \in Z(R_1)^* \times R_2 \setminus Z(R_2)$, then $(a, b) \perp (c, 0)$ with $c \in a^{\perp}$. If $(a, b) \in Z(R_1)^* \times Z(R_2) \setminus \mathrm{Nil}(R_2)$, then $(a, b) \perp (c_1, c_2)$ with $c_1 \in a^{\perp}$ and $c_2 \in b^{\perp}$. If $(a, b) \in R_1 \setminus Z(R_1) \times \mathrm{Nil}(R_1)^*$, then $(a, b) \perp (0, c)$ with $c \in R_2 \setminus Z(R_2)$. If $(a, b) \in Z(R_1)^* \times \mathrm{Nil}(R_2)^*$, then $(a, b) \perp (c_1, c_2)$ with $c_1 \in a^{\perp}$ and $c_2 \in R_2 \setminus Z(R_2)$. If $(a, b) \in \{0\} \times \mathrm{Nil}(R_2)^*$, then $(a, b) \perp (c_1, c_2)$ with $c_1 \in R_1 \setminus Z(R_1)$ and $c_2 \in b^{\perp}$.

□

In the following result, we study when $\overline{\Gamma}(R(+)M)$ is complemented. Notice that, if $|M| \geq 4$, then $|Z(R(+)M)| \geq 4$ and $|\mathrm{Nil}(R(+)M)| \geq 3$ and so $\overline{\Gamma}(R(+)M)$ is not complemented, by Theorem 4.1. If $M \cong \mathbb{Z}_3$, then $|\mathrm{Nil}(R(+)\mathbb{Z}_3)| \geq 3$ and so $\overline{\Gamma}(R(+)\mathbb{Z}_3)$ is complemented if and only if $R$ is an integral domain and $\mathbb{Z}_3$ is a torsion free $R$-module (in particular, $\overline{\Gamma}(R(+)\mathbb{Z}_3)$ is an edge). Then, only the case where $M \cong \mathbb{Z}_2$ is of interest.

**Proposition 4.2.** *Let $R$ be a non-integral domain such that, for every $x \in Z(R) \cap Z(\mathbb{Z}_2)$, $x^{\perp} \setminus Z(\mathbb{Z}_2) \neq \emptyset$. Then, $\overline{\Gamma}(R(+)\mathbb{Z}_2)$ is complemented if and only if $R$ is reduced and $\overline{\Gamma}(R)$ is complemented.*

*Proof.* $\Leftarrow$) We have $Z(R(+)\mathbb{Z}_2) = Z(R) \cup Z(\mathbb{Z}_2)(+)\mathbb{Z}_2 = \{(a, \bar{0}), (a, \bar{1}) |\ a \in Z(R) \cup Z(\mathbb{Z}_2)\}$. Let $a \in Z(R) \cup Z(\mathbb{Z}_2)$, then we have the following three cases:
<u>**Case 1.**</u> Suppose that $a \in Z(\mathbb{Z}_2) \setminus Z(R)$. Then, $(a, \bar{0}) \perp (0, \bar{1})$ and $(a, \bar{1}) \perp (0, \bar{1})$.
<u>**Case 2.**</u> Suppose that $a \in Z(R) \setminus Z(\mathbb{Z}_2)$. Then, since $\overline{\Gamma}(R)$ is complemented, $(a, \bar{0}) \perp (x, \bar{0})$ with $x \in a^{\perp}$, and either $(a, \bar{1}) \perp (y, \bar{0})$ with $y \in a^{\perp} \cap Z(\mathbb{Z}_2)$, or $(a, \bar{1}) \perp (y, \bar{1})$ with $y \in a^{\perp} \setminus Z(\mathbb{Z}_2)$.
<u>**Case 3.**</u> Suppose that $a \in Z(R) \cap Z(\mathbb{Z}_2)$. Then, there exists $x \in a^{\perp} \setminus Z(\mathbb{Z}_2)$ such that $(a, \bar{0}) \perp (x, \bar{0})$ and $(a, \bar{1}) \perp (x, \bar{0})$.
$\Rightarrow$) Assume that $\overline{\Gamma}(R(+)\mathbb{Z}_2)$ is complemented. Then, $|\mathrm{Nil}(R(+)\mathbb{Z}_2)| = 2$ (by Theorem 4.1). In particular $R$ is reduced. Now, let us prove that $\overline{\Gamma}(R)$ is complemented. Let $a \in Z(R)^*$. If $a \in Z(R) \cap Z(\mathbb{Z}_2)$, then $a$ has an orthogonal, by the hypotheses.
If $a \in Z(R) \setminus Z(\mathbb{Z}_2)$, then $(a, \bar{0}) \in Z(R(+)\mathbb{Z}_2)^*$ and so $(a, \bar{0})$ has an orthogonal in $\overline{\Gamma}(R(+)\mathbb{Z}_2)$. Since the vertices adjacent to $(a, \bar{0})$ are of the form $(b, \bar{1})$ or $(b, \bar{0})$ with $b \in Z(R)^*$, $(a, \bar{0}) \perp (c, \bar{0})$ or $(a, \bar{0}) \perp (c, \bar{1})$ for some $c \in a^{\perp}$. Hence, $a$ has $c$ as an orthogonal.

□

# Appendix A

# Exploring graph theory conjectures through algebraic structures: Future Directions.

Here, we present several interesting graph theory conjectures that remain unsolved, to investigate them in the framework of Zero-divisor graph theory.

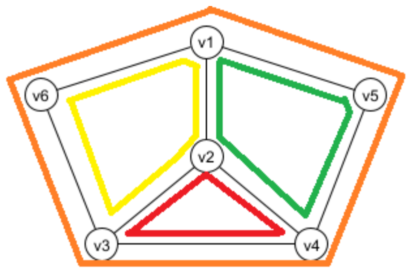
## A.1 Cycle Double Cover Conjecture

The "pencil-and-paper" problems, found in many books, involve the endeavor to trace a diagram without lifting the pen from the paper or retracing any portion of the figure. Euler, in his famous resolution to the Seven Bridges of Königsberg problem, essentially identified the conditions under which this feat can be accomplished. The problem can be restated as covering the diagram, represented by a graph, with a cycle. Such graphs are now called *Eulerian*. The connected Eulerian graphs are those in which every vertex has even degree. It's noteworthy that Euler himself didn't furnish a proof for the characterization of Eulerian graphs. The first proof goes to Hierholzer, as documented in [32]. The subsequent conjecture can be perceived as an extension of these kinds of problems, incorporating graphs with vertices of odd degrees.

The *Cycle Double Cover conjecture* states as follows:

«***Every bridgeless graph has a cycle double cover***».

A *bridgeless graph* is a graph without bridges. Recall that a *bridge* is an edge whose deletion disconnects the graph. For instance, every edge of a tree is a bridge.
A *cycle double cover (CDC)* of a graph $G$ is a list of cycles of $G$ such that every edge of $G$ is contained in exactly two cycles in the list. Notice that, this list may have repeated cycles, as in the case with the cycle graph $C_n$. As a simple example, we consider the following illustrated graph:




Then, this graph has a cycle double cover given by the cycles

$$(v_1, v_2, v_4, v_5, v_1), \quad (v_2, v_3, v_4, v_2), \quad (v_1, v_2, v_3, v_6, v_1), \quad (v_1, v_5, v_4, v_3, v_6, v_1).$$

The reason that the cycle double cover is so interesting is that it has connections with many other areas of graph theory, including the famous *Four Color Theorem*.

The *CDC conjecture* is one of the most famous conjectures in graph theory. It was, in fact, independently posed by George Szekeres in 1973 [86] and by Paul Seymour, in 1979 [77]. This conjecture has connections to embeddings of graphs on surfaces; that is, drawings of graphs on different surfaces so that no two edges cross. The simplest case is the family of planar graphs, which have an embedding in the plane. If each face in the embedding corresponds to a cycle in the graph, then the faces form a *CDC* as in the previous illustrated graph, as is true for all connected, bridgeless, planar graphs. That there is an embedding in some surfaces where each face corresponds to a cycle is the *Strong Embedding conjecture*, which is a stronger conjecture than the *CDC conjecture*.

A stronger conjecture than the *CDC* conjecture is the *Small Cycle Double Cover conjecture*:

«***Every bridgeless graph on*** $n$ ***vertices has a CDC of size at most*** $n-1$».

## A.2 Hadwiger's Conjecture

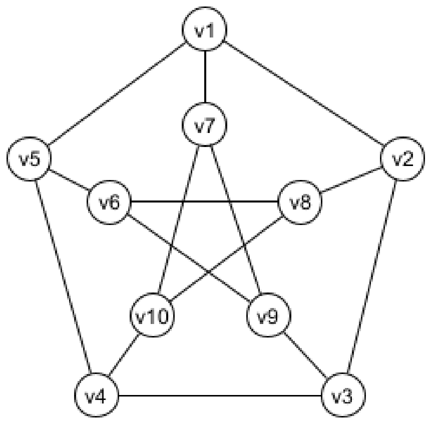
In graph theory, coloring refers to the act of assigning labels, traditionally referred to as "*colors*", to elements of a graph subject to certain constraints. In its simplest form, it is a way of coloring the vertices of a graph such that adjacent vertices receive distinct colors; this is called a *vertex coloring problem*. If coloring is done using at most $k$ colors, it is called *$k$-coloring*. Similarly, an *edge coloring problem* involves assigning a color to each edge so that two adjacent edges (i.e., edges that share a common vertex) receive distinct colors, and a *face coloring problem* of a planar graph involves assigning a color to each face or region so that two faces that share a boundary have distinct colors.

Coloring has attracted the attention of many researchers in graph theory from the early stages of the discipline. The chromatic number of $G$, written $\chi(G)$, is the minimum number of colors in a vertex coloring. In other words, it is the minimum non-negative integer $k$ such that $G$ is $k$-colorable. The most famous theorem proved so far in graph theory is the *Four-Color theorem* [12], which states that every planar graph is 4-colorable. All known proofs of this fact are computer-assisted.

One of the interesting results of Wagner [89] states that a graph is planar if and only if it does not have $K_5$ or $K_{3,3}$ as a minor.
Note that a graph is a *minor* of $G$ if it results by repeatedly performing one of the following operations:

- deleting a vertex,
- deleting an edge, or
- contracting an edge (i.e., shrinking an edge to a vertex and preserving adjacencies and non-adjacencies with vertices outside the edge).

As an example, the Petersen graph shown below has $K_5$ as a minor, and hence, it is not planar.




*Hadwiger's Conjecture*, dating back to 1943 [44], relates graph coloring to minors and it states as follows.

«***For*** $m \geq 2$***, a graph not having*** $K_m$ ***as a minor is*** $(m-1)$***-colorable***».

The cases $m = 2$ and 3 are simple since a graph without $K_2$ as a minor has no edges, and a graph not having $K_3$ as a minor is a forest. In the case, $m = 4$, Dirac [37] and Hadwiger [44] proved that graphs not having $K_4$ as a minor have a vertex of degree at most 2 and, hence, can be 3-colored using a greedy algorithm. Although the case $m = 7$ is open, in 2005 Kawarabayashi and Toft [55] proved that any 7-chromatic graph has $K_7$ or $K_{4,4}$ as a minor.

*Hadwiger's Conjecture* is open for $m \geq 7$. The startling case for small $m$ is $m = 5$, which Wagner showed [89] to reduce to the Four-Color theorem. Hence, *Hadwiger's Conjecture* may be viewed as a broad generalization of that theorem. Robertson, Seymour, and Thomas settled the case $m = 6$ [71] by showing that a minimal counterexample to the conjecture is planar after the removal of one vertex (so this also reduces to the Four-Color theorem).

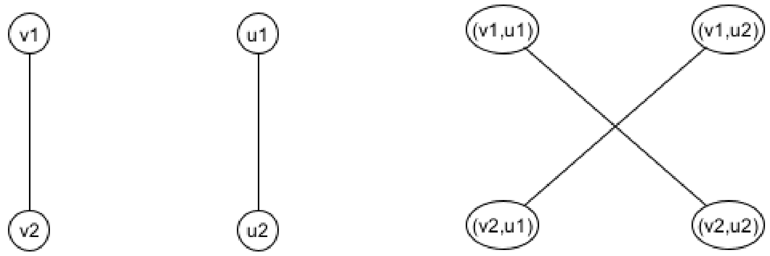
## A.3 Hedetniemi's Conjecture

Hedetniemi's conjecture concerns the connection between the chromatic number of a graph and the categorical product (also known as the tensor or the Kronecker product) of graphs. First, let's define the Categorical product of two graphs.

**Definition A.1.** ***The categorical product*** *of two graphs $G$ and $G'$, denoted by $G \times G'$, is a graph with the set of vertices $V(G) \times V(G')$ and $(x, y) - (u, v)$ is an edge of $G \times G'$ if and only if $x - u$ is an edge of $G$ and $y - v$ is an edge of $G'$.*

The following example illustrates the categorical product of two graphs, which motivates the notation for this product.

**Example A.1.** *Here, the categorical product of two graphs that are isomorphic to the complete graph $K_2$:*




$G$ $G'$ $G \times G'$

*Hedetniemi's Conjecture*, proposed by Stephen T. Hedetniemi in 1966 [51] while he was a graduate student. The conjecture deals with the chromatic number of the categorical product of graphs. It states the following.

«***For a given two graphs*** $G$ ***and*** $G'$**,** $\chi(G \times G') = \min\{\chi(G), \chi(G')\}$».

This conjecture was stated independently by Burr, Erdös, and Lovász in 1976. It received a lot of attention in the past half century (see [38, 56, 76, 87, 91, 92]).

The Categorical product graph $G \times G'$ can be visualized as replacing every vertex $v$ in $G$ with a set of vertices corresponding to the vertices of $G'$. These new vertices are labeled as $(v, h)$. Then, add the edges $(v, h) - (u, k)$ if $v - u$ is an edge of $G$ and $h - k$ is an edge of $G'$. Next, assume a proper coloring of graph $G$. For each vertex $v$ in $G$, assign the same color to all vertices $(v, h)$. Since $(v, h)$ and $(v, k)$ are not connected by an edge, this coloring is also valid for the product graph $G \times G'$. Consequently, we have $\chi(G \times G') \leq \chi(G)$. Similarly, we have $\chi(G \times G') \leq \chi(G')$.

*Hedetniemi's Conjecture* has a convenient restatement that is often used. Let $H(n)$, where $n$ is a positive integer, represent the following statement.

«***If*** $\chi(G \times G') = n$**, *then either*** $\chi(G) = n$ ***or*** $\chi(G') = n$».

Then, *Hedetniemi's Conjecture* is equivalent to $H(n)$ holds true for every $n \geq 1$, which permits an incremental approach. Demonstrating the truth of $H(1)$ and $H(2)$ is not overly challenging. El-Zahar and Sauer established the truth of $H(3)$ in 1985, but little is known about $H(n)$ for $n > 3$.

Burr, Erdös, and Lovász in their 1976 article demonstrated that for a graph $G$ where each vertex is part of a complete subgraph of order $n$, and if $G'$ is a connected graph with $\chi(G \times G') = n$, then it follows that $\min\{\chi(G), \chi(G')\} = n$. (In Figure A.1, we have that $n = 2$). The existence of $K_n$ in $G$ implies that $\chi(G) \geq n$. Thus, any attempt to prove or disprove the conjecture needs to focus on graphs characterized by high chromatic numbers and the existence of small complete subgraphs.

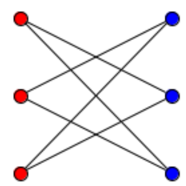


Figure A.1: The graph $P_2 \times K_3$

In 2019, Shitov refuted Hedetniemi's conjecture [82]. He proved that the conjecture fails for sufficiently large $n$. Let $t$ be the minimum number of vertices of a graph $G$ of odd girth 7 and fractional chromatic number greater than $3 + 4/(t-1)$. So, Shitov proved that the conjecture fails for some $n$ about $3^t t^3$. The current known upper bound for $t$ is 83. Thus Shitov's result shows that Hedetniemi's conjecture fails for some $n$ that is about 395. On the other hand, we do not know if Hedetniemi's conjecture holds for any integer $n \geq 4$. A natural question is whether Hedetniemi's conjecture fails for relatively small $n$.

In 2020, Zhu shows that the conjecture fails for some graphs $G$ and $G'$ with chromatic number $3\lceil (p+1)/2 \rceil$ and with $p\lceil (p-1)/2 \rceil$ and $3\lceil p+12 \rceil (p+1)-p$ vertices, respectively [93]. He concludes, using the upper bound $t \leq 83$, that the conjecture fails for $n = 125$ (and hence $G$ and $G'$ can be assumed to have chromatic number 126). The number of vertices in $G$ and $G'$ are $3,403$ and $10,501$, respectively.

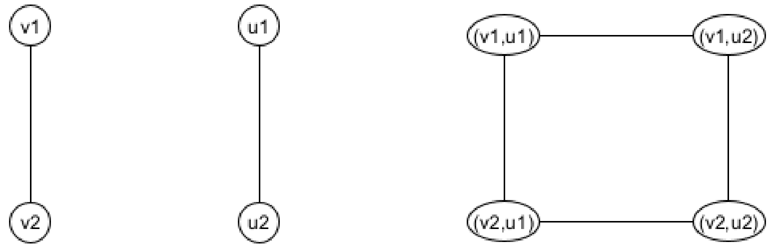
## A.4 Vizing's Conjecture

Vizing's conjecture concerns the connection between the domination number of graphs and the Cartesian product of graphs. First, let's define the Cartesian product of two graphs.

**Definition A.2.** ***The Cartesian product*** *of two graphs $G$ and $G'$, denoted by $G\Box G'$, is a graph with the set of vertices $V(G) \times V(G')$ and $(x,y)-(u,v)$ is an edge of $G\Box G'$ if and only if $x = u$ and $y - v$ is an edge of $G'$ or $y = v$ and $x - u$ is an edge of $G$.*

The following example illustrates the Cartesian product of two graphs, which motivates the notation for this product.

**Example A.2.** *Here, the Cartesian product of two isomorphic graphs $G$ and $G'$ to the complete graph $K_2$.*




$G$ $\quad$ $G'$ $\quad$ $G \Box G'$

*Vizing's Conjecture* was posed by him in 1968 [88]. This conjecture states as follows:

«***For a given two graphs*** $G$ ***and*** $G'$, $\gamma(G \Box G') \geq \gamma(G)\gamma(G')$».

where $\gamma$ is the domination number.
In the field of domination theory, this conjecture is probably the most important open problem. In what follows, we would like to present some interesting results which are related to this conjecture. One of the first results that shows the truth of Vizing's conjecture for a class of graphs is due to Barcalkin and German [16], and it stated as follows:

**Theorem A.1** ([16])**.** *Let* $G'$ *be a decomposable graph and* $G$ *be a spanning subgraph of* $G'$*. If* $\gamma(G) = \gamma(G')$*, then* $\gamma(G \Box H) \geq \gamma(G)\gamma(H)$ *for every graph* $H$*.*

Where a decomposable graph $G$, with domination number $\gamma(G) = k$, is a graph in which the vertex set can be covered by $k$ complete subgraphs.

The following result shows that the graphs of Type $\mathcal{X}$ satisfy Vizing's conjecture. Recall that a graph $G$ is of Type $\mathcal{X}$ if $\{S, SC, BC, C\}$ form a partition of $V(G)$, where,

$$\begin{cases} \gamma(G) = n = k + t + m + 1, \\ S = S_1 \cup \cdots \cup S_k, \\ BC = B_1 \cup \cdots \cup B_t, \\ C = C_1 \cup \cdots \cup C_m, \end{cases}$$

such that each of $SC, B_1, \ldots, B_t, C_1, \ldots, C_m$ induces a clique. $SC \subseteq N(\overline{SC})$, each $B_i$ in $\{B_1, \ldots, B_t\}$ has at least one vertex which has no neighbors outside $B_i$, and every $S_i$ in $\{S_1, \ldots, S_k\}$ is *starlike* in that it contains a star centered at a vertex $v_i$ which is adjacent to each vertex in $T_i = S_i - \{v_i\}$. The vertex $v_i$ has no neighbors besides those in $T_i$. Although other pairs of vertices in $T_i$ may be adjacent (i.e., $S_i$ does not necessarily induce a star), $S_i$ does not induce a clique nor can more edges be added in the subgraph induced by $S_i$ without lowering the domination number of $G$. Furthermore, there are no edges between vertices in $S$ and vertices in $C$.

**Theorem A.2** ([45])**.** *Let* $G'$ *be a spanning subgraph of a graph* $G$ *of Type* $\mathcal{X}$ *such that* $\gamma(G) = \gamma(G')$*. Then, for every graph* $H$*,* $\gamma(G' \Box H) \geq \gamma(G')\gamma(H)$*.*

For a graph $G$, a set $S \subseteq V(G)$ is called 2-packing if it satisfies the property that every two distinct vertices $v_1, v_2 \in S$, $N[v_1] \cap N[v_2] = \emptyset$. The 2-packing number of $G$, denoted by $\rho(G)$ is the maximum cardinality of a 2-packing in $G$. Then, as a consequence of Theorem A.2, we have the following corollary.

**Corollary A.1** ([45])**.** *Let $G$ be a graph such that $\gamma(G) = \rho(G) + 1$, then for every graph $G'$, $\gamma(G \Box G') \geq \gamma(G)\gamma(G')$.*

Recall that an independent set of a graph $G$ is a subset of vertices of $V(G)$ such that no two vertices are adjacent. It is denoted $I(G)$ or simply $I$. Let $\gamma^i(G)$ denote the maximum, over all independent sets $I$ in $G$, of the smallest cardinality of a set $D$ that dominates $I$ (i.e., such that $I \subseteq N[D]$). Then, we have the following result:

**Theorem A.3** ([2])**.** *If $G$ and $G'$ are two graphs. Then, $\gamma(G \Box G') \geq \gamma^i(G)\gamma(G')$.*

It was shown by Aharoni, Berger and Ziv in [1] that $\gamma^i$ and $\gamma$ agree on chordal graphs. So, as a consequence of the previous theorem, if $G$ is a chordal graph then it satisfies Vizing's conjecture. Also, we have the following theorem.

**Theorem A.4** ([33])**.** *If $G$ and $G'$ are two graphs. Then,*

$$\gamma(G \Box G') \geq \frac{1}{2}\gamma(G)\gamma(G').$$

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ROYAUME DU MAROC
Université Mohammed V - RABAT -
جامعة محمد الخامس – الرباط –
Faculté des sciences
كلية العلوم
*CENTRE D'ETUDES DOCTORALES - SCIENCES ET TECHNOLOGIES*

# Résumé

Cette thèse traite des concepts et des problèmes en relation avec la théorie des graphes et la théorie des anneaux commutatifs. Plus précisément, nous étudions certaines propriétés graphiques dans le contexte de certains graphes associés aux structures algébriques des anneaux. En particulier, nous étudions les notions de k-alliance défensive globale, de partitionnement en alliance défensive globale, de complémentarité et de complémentarité unique dans le contexte des graphes des diviseurs de zéro et des graphes des diviseurs de zéro étendus des anneaux commutatifs. La k-alliance défensive globale est une notion très étudiée en théorie des graphes, elle offre une méthode de classification des graphes basée sur les relations entre les membres d'un ensemble particulier de sommets. Dans cette thèse, nous explorons cette notion dans les graphes des diviseurs de zéro des anneaux commutatifs finis. Les résultats établis généralisent et améliorent le travail récent de Muthana et Mamouni, dans [62], qui ont traité un cas particulier pour $k = -1$, connu sous le nom d'alliance défensive globale. Un autre problème bien étudié en théorie des graphes est le partitionnement de l'ensemble des sommets d'un graphe. Il implique de diviser l'ensemble des sommets d'un graphe en sous-ensembles ou partitions disjointes, en fonction de certains critères ou contraintes spécifiques. Nous sommes également intéressés par le partitionnement du graphe des diviseurs de zéro d'un anneau commutatif en alliances défensives globales. Ce problème a été bien étudié en théorie des graphes. Ici, nous le relions au contexte théorique des anneaux. Nous caractérisons divers anneaux commutatifs finis pour lesquels le graphe des diviseurs de zéro est partitionnable en alliances défensives globales. Plusieurs exemples sont également fournis pour illustrer et délimiter la portée des résultats établis. Nous abordons également les notions de complémentarité et de complémentarité unique des graphes. Ces notions ont été initialement introduites pour un graphe quelconque dans [9, 58] et ils ont ensuite été étudiées dans le cadre des graphes des diviseurs de zéro d'anneaux commutatifs. Nous poursuivons l'étude commencée dans [24] concernant le caractère complémenté ou uniquement complémenté du graphe des diviseurs de zéro étendu d'un anneau commutatif. Nous donnons une caractérisation complète du caractère complémenté du graphe des diviseurs de zéro étendu d'un anneau commutatif fini. Divers exemples sont donnés en utilisant le produit direct d'anneaux et l'idéalisation des modules. A la fin de cette thèse, nous présentons quelques propositions pour des études futures de certaines conjectures de la théorie des graphes avec une perspective dérivée de la théorie des graphes des diviseurs de zéro.

**Mots-clés :** Graphe des diviseurs de zéro ; graphe des diviseurs de zéro étendu ; anneau commutatif ; idéalisation ; anneau de dimension zéro ; complémenté ; alliance défensive globale ; partitionnement en alliances défensives globales.

# Abstract

This thesis addresses concepts and problems related to graph theory and the theory of commutative rings. Namely, we study certain graph properties in the context of some graphs associated with algebraic structures of rings. Specifically, we study the notions of global defensive k-alliance, partitioning into global defensive k-alliance, complementedness and uniquely complementedness in the context of zero-divisor graphs and extended zero-divisor graphs of rings. The global defensive k-alliance is a very well-studied notion in graph theory, it provides a method of classification of graphs based on relations between members of a particular set of vertices. In this thesis, we explore this notion in zero-divisor graphs of finite commutative rings. The established results generalize and improve recent work by Muthana and Mamouni, in [60], who treated a particular case for $k = -1$ known by the global defensive alliance. Another well-studied problem in graph theory is the partitioning of the vertex set of a graph. It involves dividing the set of vertices of a graph into disjoint subsets or partitions, based on specific criteria or constraints. Then, we are also interested in partitioning the zero-divisor graph of a commutative ring into global defensive alliances. This problem has been well investigated in graph theory. Here we connected it with the ring theoretical context. We characterize various finite commutative rings for which the zero-divisor graph is partitionable into global defensive alliances. Several examples are also provided which illustrate and delimit the scope of the established results. Also, we deal with complementedness and uniquely complementedness notions of graphs. These notions were initially introduced for a general graph in [9, 56] and subsequently investigated within the framework of zero-divisor graphs of commutative rings. We continue the study started in [24] concerning when the extended zero-divisor graph of a commutative ring is complemented or uniquely complemented. We give a complete characterization of when the extended zero-divisor graph of a finite commutative ring is complemented. Various examples are given using the direct product of rings and idealizations of modules. At the end of this thesis, we will present some proposals for future studies of certain graph-theoretic conjectures with a view to the zero-divisor graph theory.

**Keywords**: Zero-divisor graph; extended zero-divisor graph; commutative ring; idealization; zero-dimensional ring; complemented; global defensive alliance; partitioning into global defensive alliances.

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✉ Faculté des Sciences, avenue Ibn Battouta, BP. 1014 RP, Rabat –Maroc
☎ 00212(0)5 37 77 18 76 🖷 00212(0)5 37 77 42 61 ; http://fsr.um5.ac.ma/