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Sujet:

TRANSFERT DE CERTAINES PROPRIÉTÉS ALGÉBRIQUES DANS DIFFÉRENTS CONTEXTES D'ANNEAUX COMMUTATIFS.

Présentée par

Abdelhaq El khalfi

Date prévue: le 23-01-2021, devant le jury composé de:

Pr. El Hassan El Kinani	**FS Meknès**	**Président**
Pr. Chahrazade Bakkari	**FS Meknès**	**Rapporteur**
Pr. Lahcen Oukhtite	**FST Fès**	**Rapporteur**
Pr. Mohammed Tamekkante	**FS Meknès**	**Rapporteur**
Pr. Mohamed Chhiti	**FSJES Fès**	**Examinateur**
Pr. Abdellah MAMOUNI	**FS Meknès**	**Examinateur**
Pr. Ali MOUHIB	**FP Taza**	**Examinateur**
Pr. Aziza Rahmouni Hassani	**FST Fès**	**Co-encadrante**
Pr. Najib Mahdou	**FST Fès**	**Directeur de Thèse**

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Dédicace

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"Le monde devrait remercier cette étonnante cohorte de gens qui font toujours preuve d'une insolente et illogique gentillesse". (Hélène Thomas)

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(3) A. El khalfi and N. Mahdou, *Almost Bézout rings and almost GCD-rings*, Asian-European J. Math., 13(6) (2020) 2050107 (10 pages).

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Summary (In French)

Les travaux de cette thèse sont consacrés à l'étude de certaines propriétés algébriques dans différentes extensions d'anneaux commutatifs. Cette thèse comporte sept chapitres rendant compte respectivement des travaux de sept articles.

Le premier axe qui se compose des trois premiers chapitres porte sur l'étude de quelques propriétés liées aux anneaux de valuation. Dans le premier chapitre, nous étudions le transfert de ces propriétés à l'extension triviale. Nos résultats nous permettent de construire des nouvelles classes d'anneaux commutatifs satisfaisant ces propriétés. Comme application de ces résultats, nous montrons l'échec du théorème de Badawi (resp. de Dobbs) concernant la clôture intégrale d'un domaine pseudo presque de valuation (resp. pseudo de valuation cohérent), lorsque l'anneau n'est plus intègre. Le deuxième chapitre concerne l'étude des idéaux fortement primaires dans le contexte d'anneaux commutatifs avec diviseurs de zéro. Notre étude conduit à généraliser des résultats prouvés dans le cas des anneaux intègres sur les idéaux fortement primaires. De plus, nous montrons via des exemples que certains résultats ne peuvent pas s'étendre aux anneaux avec diviseurs de zéro. Dans le troisième chapitre, nous élargissons la classe des domaines presque de GCD à des anneaux commutatifs contenant des diviseurs de zéro. Ensuite, nous étudions le transfert de cette propriété et de la propriété des anneaux presque de Bézout à l'extension triviale et à l'amalgamation d'anneaux.

Le deuxième axe de cette thèse est présenté dans le chapitre 4. Cet axe s'intéresse au transfert des propriétés (A)-anneaux et (A)-anneaux forts dans la bi-amalgamation algébrique, afin de construire de nouveaux exemples d'anneaux satisfaisants ces propriétés.

Le troisième axe de notre recherche, qui regroupe les chapitres 5, 6, et 7, consiste à poursuivre l'étude de la classe des anneaux dans lesquels le nilradical est un idéal premier divisé noté $\mathscr{H}$. Dans le cinquième chapitre, nous étudions le transfert des propriétés ϕ-anneaux, anneaux ϕ-enchainé et anneaux ϕ-pseudo de valuation à l'amalgamation algébrique. Les chapitres six et sept sont consacrés à l'étude de certaines propriétés dans le cadre des anneaux de la classe $\mathscr{H}$, à savoir: anneau pseudo de Krull, anneau Noethérien

par morceaux et anneau w-Noethérien par morceaux. Aussi, nous utilisons l'extension triviale et l'amalgamé algébrique pour construire des exemples d'anneaux satisfaisant ces propriétés et qui ne sont pas intègre.

Notre thèse se termine par quelques perspectives que nous souhaiterons aborder dans nos travaux futurs.

Mots clés: (A)-anneau, anneau pseudo de Krull, AV-anneau, APV-anneau, PV-anneau, PAV-anneau, (A)-anneau fort, idéal fortement primaire, extension triviale, amalgamation d'anneaux le long d'un idéal, bi-amalgamation algébrique, ϕ-anneau.

Laboratoire d'accueil:
Laboratoire Modélisation et Structures Mathématiques.
Faculté des Sciences et Techniques.
Université Sidi Mohamed Ben Abdellah Fès, Maroc.

Summary

This thesis has developed in the theory of commutative algebra and focuses on the transfer of some algebraic properties to certain ring extensions. It consists of seven chapters covering seven papers.

The first three chapters concern the study of certain properties related to the notion of valuation ring. The first chapter investigates the transfer of these notions to trivial ring extensions. Our aim is to provide new classes of commutative rings satisfying these properties. As an immediate consequence, we show that certain domain-theoretic results of Badawi and Dobbs on the integral closure of a pseudo-almost valuation domain and the integral closure of a coherent pseudo-valuation domain do not generalize to the context of trivial ring extensions of domains. The second chapter focuses on a generalization of strongly primary ideals to the context of arbitrary commutative rings. Moreover, we give some examples to show the failure of certain domain-theoretic results beyond the context of integral domains. In the third chapter, we study the possible transfer of the property of being an *AB*-ring to trivial ring extensions and amalgamated algebras along an ideal. Also, we extend the notion of an almost *GCD*-domain to the context of arbitrary rings, and we study the possible transfer of this notion to trivial ring extensions and amalgamated algebras along an ideal.

The second part is presented in chapter 4, deals with the transfer of the notions of (A)-rings and strong (A)-rings to the bi-amalgamated algebra. Our results generate new original examples of rings satisfying these properties.

The third part of the thesis, which includes chapters 5, 6 and 7, aims to study certain properties of the set $\mathscr{H}$ of rings in which the nilradical is a divided prime ideal. The fifth chapter investigates the possible transfer of the properties of being a ϕ-ring, a ϕ-chained ring, and a ϕ-pseudo-valuation ring to the amalgamated algebra $A \bowtie^f J$. In the sixth and the seventh chapter, we are interested in extending some properties of integral domains to the context of the class $\mathscr{H}$: the pseudo-Krull property, the piecewise Noetherian property and the piecewise w-Noetherian property. We also use trivial ring extention and amalgamation of rings to construct examples of rings satisfying these properties which are not

integral domains.

Keys words: (A)-ring, AV-ring, APV-ring, PAV-ring, strong (A)-ring, pseudo-Krull ring, strongly primary ideal, trivial ring extension, amalgamation of rings along an ideal, bi-amalgamated algebra, ϕ-ring.

Host laboratory:
Laboratory of Modeling and Mathematical structures.
Faculty of Sciences and Technologies.
University Sidi Mohamed Ben Abdellah Fez, Morocco.

Introduction (In French)

Tous les anneaux considérés dans cette thèse sont supposés commutatifs, unitaires et les modules sont unitaires.

Le premier axe de notre recherche poursuit l'étude de quelques propriétés liées à la notion d'anneau de valuation. En 1987, Hedstrom et Houston ont introduit la classe des domaines pseudo de valuation. Un domaine R est dit pseudo de valuation si tout idéal premier de R est fortement premier, dans le sens où pour tous x, y dans le corps des fractions de R tels que $xy \in P$ on a $x \in P$ ou $y \in P$. Dans la même année, Dobbs a démontré que la clôture intégrale d'un domaine pseudo de valuation de conducteur fini est un domaine de valuation. Des informations supplémentaires sur les domaines pseudo de valuation peuvent être trouvées dans l'article intéressant [19]. En 1997, Anderson, Badawi et Dobbs ont généralisé l'étude sur les domaines pseudo de valuation dans le contexte d'anneaux commutatifs avec diviseurs de zéro comme suit. Rappelons d'abord qu'un idéal premier P d'un anneau R est fortement premier si pour tous $a, b \in R$, les idéaux aP et bR sont comparable. Un anneau R est pseudo de valuation (PV-anneau) si tout idéal premier de R est fortement premier. De nombreux résultats concernant cette classe d'anneaux sont donnés dans [3]. Citons par exemple la caratérisation suivante : un anneau R est un anneau pseudo de valuation si et seulement si R est un anneau local dont l'idéal maximal est fortement premier [3, Theorem 2]. Dans [9], Anderson et Zaffrullah ont introduit et étudié la notion de domaine presque de valuation (AVD) comme étant un domaine R tel que pour tout élément x dans K, où K est le corps des fractions de R, il existe un entier strictement positif n tel que $x^n \in K$ ou $x^{-n} \in K$. Un peu plus tard, dans [84] et [73], les auteurs ont introduit une généralisation des domaines presque de valuation dans le contexte des anneaux avec des diviseurs de zéro. Un anneau R est dit presque de valuation (AV-anneau) si pour tous éléments a et b dans R, il existe un entier strictement positif n tel que a^n divise b^n ou b^n divise a^n. Une autre notion très importante qui a été introduite par Badawi en 2007 est la notion des domaines pseudo presque de valuation. D'après [20], Un idéal

premier P d'un domaine R est dit pseudo-fortement premier si pour tous éléments x et y dans le corps des fractions de R tels que $xyP \subseteq P$, il existe un entier strictement positif n tel que $x^n \in R$ ou $y^nP \subseteq P$. Un domaine est dit pseudo-presque de valuation (PAVD) si tout idéal premier est un idéal pseudo-fortement premier. En 2015, Jahani-Nezhad et Khoshayand ont généralisé l'étude sur les domaines pseudo-presque de valuation dans le contexte d'anneaux commutatifs avec diviseurs de zéro comme suit: Un idéal premier P d'un anneau R est dit pseudo-fortement premier si pour tous éléments a et b dans R, il existe un entier strictement positif n tel que $a^nR \subseteq b^nR$ ou $b^nP \subseteq a^nP$; un anneau est dit pseudo-presque de valuation (*PAV*-anneau) si tout idéal maximal est pseudo-fortement premier. D'après [23], un idéal primaire I d'un domaine R est dit fortement primaire, si pour tous $x, y \in K$ tels que $xy \in I$, il existe un certain entier strictement positif n tel que $x \in I$ ou $y^n \in I$, où K est le corps des fractions de R. Cette classe d'idéaux généralise la classe d'idéaux fortement premiers et donne lieu à une nouvelle classe des domaines. Un domaine R est dit presque pseudo de valuation si tout idéal premier de R est fortement primaire. D'autre part, dans [9], Anderson et Zaffrullah ont introduit et étudié la notion de domaine presque de Bézout (AB-domaine) comme étant un domaine R tel que pour tous a et b dans R, il existe un entier strictement positif n tel que l'idéal $(a^n, b^n)R$ est principal. La notion de domaines presque de Bézout suit des lignes assez similaires à celles des domaines de Bézout (ie. domaines pour lesquels tout idéal de type fini est principal). Un peu plus tard, les auteurs dans [84] ont introduit une généralisation des domaines presque de Bézout dans le contexte des anneaux avec des diviseurs de zéro. En 1985, une autre notion très importante a été introduite par Zaffrullah comme une génénralisation de domaines de GCD. Un domaine R est dit presque de *GCD* si pour tous a et b dans R, il existe un entier strictement positif n tel que l'idéal $a^nR \cap b^nR$ est principal.

Le deuxième axe de cette thèse consiste à poursuivre l'étude des (A)-anneaux et des (A)-anneaux forts. Noter que dans un anneau commutatif Noethérien, l'annulateur d'un idéal I entièrement engendré par des diviseurs de zéro est non nul. Cependant, ce résultat n'est pas vrai pour certains anneaux qui ne sont pas Noethériens, même si l'idéal I est de type fini. En 1979, Huckaba et Keller ont introduit la propriété suivante: un anneau commutatif R vérifié la propriété (A) si tout idéal de type fini de R entièrement engendré par des diviseurs de zéro possède un annulateur non nul. La propriété (A) a été initialement introduite par Quentel [92](C'est la condition(C) de Quentel). Un peu plus tard, Mahdou et Rahmouni ont introduit la notion d'(A)-anneau fort comme étant un anneau sur lequel tout idéal de type fini engendré par un nombre fini de diviseurs de zéro admet un annulateur non nul. Ils ont étudié le transfert de la propriété (A) et de la propriété (A) forte à l'extension triviale et à la duplication amalgamée d'un anneau le long d'un idéal et ils ont présenté également une classe des (A)-anneaux qui ne sont pas des (A)-anneaux forts.

D'après [18], un idéal premier d'un anneau R est divisé s'il est comparable à tout idéal de R et un anneau R est dit divisé si chaque idéal premier de R est divisé. Aussi, un idéal I de R est dit nonnil si $I \nsubseteq \mathrm{Nil}(R)$. Un peu plus tard, Badawi a introduit et

étudié la notion $\mathscr{H}$ comme étant l'ensemble de tous les anneaux tels que le nilradical est un idéal premier divisé. Soient R un anneau et $T(R)$ son anneau total des fractions tels que $\mathrm{Nil}(R)$ est un idéal premier divisé de R. Comme dans [14], nous définissons $\phi : T(R) \longrightarrow K := R_{\mathrm{Nil}(R)}$ tel que $\phi(\frac{a}{b}) = \frac{a}{b}$ pour chaque $a \in R$ et tout $b \in R \setminus Z(R)$. Alors ϕ est un homomorphisme d'anneau de $T(R)$ dans K et ϕ restrient à R est aussi un homomrphisme de R dans K donné par $\phi(x) = \frac{x}{1}$ pour chaque $x \in R$. Un anneau de l'ensemble $\mathscr{H}$ est dit un ϕ-anneau. D'autre part, un anneau $R \in \mathscr{H}$ est dit ϕ-pseudo de valuation si pour tous $a,b \in R \setminus \mathrm{Nil}(R)$, $aR \subseteq bR$ ou $bc \in aR$ pour tout élément non unité $c \in R$, ainsi R est dit ϕ-enchainé si pour tous $a,b \in R \setminus \mathrm{Nil}(R)$ les idéaux aR et bR sont comparables. En 1987, Oda a introduit la notion de domaine pseudo de Krull comme étant un domaine D tel qu' elle existe une famille $\mathscr{F} = \{D_i\}$ des sur-anneaux de D tel que $D = \cap D_i$, où chaque D_i est un sur-anneau Noethérien pseudo de valuation. Une autre notion qui nous intéresse dans cette partie est celle d'anneaux Noethériens par morceaux. Le concept de cette propriété a été introduit et étudié pour la premiére fois par Beachy et Weakley dans [26]. Un anneau commutatif R est dit Noethérien par morceaux si (*i*) l'ensemble des idéaux premiers de R satisfait la condition des chaines ascendantes (CCA), (*ii*) R satisfait la condition des chaines ascendantes sur les idéaux P-primaires pour chaque idéal premier P, et (*iii*) chaque idéal n'a qu'un nombre fini d'idéaux premiers minimaux sur lui. Rappelons qu'un anneau commutatif R est dit avoir un spectre Noethérien s'il satisfait la (CCA) sur les idéaux radicaux. On remarque d'après [27, Lemme 1] qu'un anneau commutatif R est Noethérien par morceaux si et seulement si R a un spectre Noethérien et il satisfait la (CCA) sur les idéaux P-primaires, pour chaque idéal premier P de R. Cette caractérisation montre clairement que chaque anneau Noethérien est Noethérien par morceaux et elle est utilisée pour définir la notion d'anneaux w-Noethérien par morceaux dans [51]. En se basant sur toutes ces notions, nous étudions, dans le cinquième chapitre, le transfert des propriétés des ϕ-anneaux, des anneaux ϕ-enchainé et des anneaux ϕ-pseudo de valuation dans l'amalgamation d'anneaux. De plus, noter que si D est un domaine alors D est dans $\mathscr{H}$. Cela nous motive, dans les chapitres six et sept, à étudier la propriété d'anneaux pseudo de Krull (resp. Noethérien par morceaux et w-Noethérien par morceaux) dans la classe $\mathscr{H}$.

Notre travail se base sur le transfert de certaines propriétés d'anneaux commutatifs dans différents contextes d'extension d'anneaux. Avant la description de la majorité des contributions de chaque chapitre, nous présentons d'abord quelques définitions, terminologies et notations.

En 1956, Nagata a introduit la notion d'extension triviale d'un anneau par un module comme suit:

Définition 0.0.1 *Soient A un anneau et E un A-module. L'extension triviale de A par E est l'anneau $A \times E$ où l'addition est définie de façon naturelle et la multiplication est*

donnée par:

$$(a,e)(b,f) = (ab, af+be).$$

Cette notion a fait l'objet de plusieurs travaux dans la littérature et constitue un champ de recherche fertile dans la théorie des anneaux [60, 69]. Les extensions triviales d'anneaux ont été largement étudiées; et un travail considérable, dont une partie résumée dans le livre de Glaz [60] et le livre de Huckaba [69], a été concerné par ces extensions. Principalement, ces extension ont été utiles pour la résolution de nombreux problèmes ouverts et pour les conjectures en théorie d'anneaux commutatifs et non-commutatifs [8, 25, 45, 69, 76].

En 2006, Marco D'Anna et Marco Fontana [35] ont introduit une nouvelle construction dite la duplication amalgamée d'un anneau A le long d'un sous A-module E de $Q(A)$ (l'anneau total des fractions de A) vérifiant $E^2 \subseteq E$. Quand $E^2 = \{0\}$ cette construction coïncide avec l'extension triviale de A par E.

Récemment, dans [33, 34, 35, 36, 37, 38], sont discutées les motivations ainsi que les applications de cette duplication amalgamée $A \bowtie E$ de A le long du sous A-module E de $Q(A)$, essentiellement dans le cas particulier où E est un idéal de A.
En 2010, une extension de la construction de la duplication amalgamée $A \bowtie I$ de A le long de l'idéal I de A a été définie dans [33] comme suit:

Définition 0.0.2 *Soient A et B deux anneaux, J un idéal de B et soit $f : A \to B$ un homomorphisme d'anneaux. On appelle l'amalgamation de A et B suivant J et respectant f, le sous-anneau de $A \times B$:*

$$A \bowtie^f J := \{(a, f(a)+j) \mid a \in A, j \in J\}$$

Cette construction généralise la duplication amalgamée d'un anneau le long d'un idéal (présenté et étudié par D'Anna et Fontana dans [35, 36, 37]).
Lorsque $A = B$, $J = I$ est un idéal de A et $f = id_A$. On obtient la définition suivante:

Définition 0.0.3 *Soient A un anneau et I un idéal de A. Le sous-anneau de $A \times A$:*

$$A \bowtie I = \{(a, a+i) \mid a \in A \text{ et } i \in I\}$$

est appelé la duplication amalgamée d'anneau le long d'un idéal.

Par ailleurs, d'autres constructions classiques (tels que $A + XB[X]$, $A + XB[[X]]$, et les constructions $D+M$) peuvent être étudiées comme des cas particuliers de l'amalgamation ([33, Exemples 2.5 et 2.6]). D'autres constructions classiques, telle que l'idéalisation du Nagata, également appelé l'anneau extension triviale (cf. [87, page 2]), et les extensions

CPI (dans le sens de Boisen et Sheldon [28]) sont strictement liées ([33, Exemple 2.7 et Remarque 2.8]). Voir, par exemple [33, 34, 37, 38].

En 2013, dans [74] Kabbaj, Tamekkante et Louartiti ont introduit une extension de la construction amalgamée algébrique $A \bowtie^f J$ de A le long de l'idéal J de B comme suit:

Définition 0.0.4 *Soient $f : A \longrightarrow B$ et $g : A \longrightarrow C$ deux homomorphismes d'anneaux et soit J (resp.,J') un idéal de B (resp.,C) tel que $f^{-1}(J) = g^{-1}(J')$. On considère le sous anneau de $B \times C$ suivant :*

$$A \bowtie^{f,g} (J,J') := \{(f(a)+j, g(a)+j') | a \in A, j \in J, j' \in J'\}$$

appelé la bi-amalgamation de A avec (B,C) le long de (J,J') par rapport à (f,g).

Cette construction est une généralisation de l'amalgamé algébrique le long d'un idéal (dans le sens où $A \bowtie^f J = A \bowtie^{Id,f} (f^{-1}(J),J)$). Dans [74], les auteurs ont étudiés les propriétés de bases de cette construction (e.g., la caractérisation de $A \bowtie^{f,g} (J,J')$ pour être un anneau Noethérien, un domaine, un anneau réduit ...).

Soient A un anneau, I un idéal de A, S un sous-ensemble de A et E un A-module. Nous utiliserons les notations suivantes:

- $Z(A) := \{a \in A / ax = 0 \text{ pour un certain } 0 \neq x \in A\}$, l'ensemble des diviseurs de zéro de A.
- $Ann(E) := \{a \in A / ax = 0 \text{ pour tout } 0 \neq x \in E\}$, l'annulateur de E.
- $T(A)$, l'anneau total des fractions de A.
- Si A est un anneau intègre, on note par $qf(A) = K$, le corps des fractions de A.
- Nil(A), l'ensemble des éléments nilpotents de A.
- J(R), le radial de Jacobson de A.
- $\sqrt{I}$, le radical d'un idéal I de A, dans le sens de [78, page 17].
- $Spec(A)$, l'ensemble des idéaux premiers de A.
- $Reg(A)$, l'ensemble des éléments réguliers de A.
- $E(S) := \{x \in qf(A) / x^n \notin S \text{ pour tout } n \geq 1\}$
- $(I : I) := \{x \in T(A) / xI \subseteq I\}$.
- A', la clôture intégrale de A.

Présentation des résultats

Dans **le premier chapitre**, nous étudions le transfert de certaines propriétés liée à la notion d'anneau de valuation à l'extension triviale. Nos résultats nous permettent de construire de nouvelles classes d'anneaux commutatifs satisfaisants ces propriétés. Comme application de ces résultats, nous montrons l'échec du théorème de Badawi (resp. de

Dobbs) concernant la clôture intégrale d'un domaine pseudo presque de valuation (resp. pseudo de valuation cohérent), lorsque l'anneau n'est plus intègre. Nous commençons par énoncer le transfert de la propriété d'*AV*-anneau dans les extensions triviales.

Théorème 0.0.5 *Soient A un anneau, E un A-module et $R := A \ltimes E$ l'extension triviale de A par E.*

1. *Si R est un AV-anneau, alors A l'est aussi.*
2. *On suppose que $Z(A) = \mathrm{Nil}(A)$ et E est un A-module divisible. Alors R est un AV-anneau si et seulement si A l'est aussi.*
3. *Supposons que $\mathbb{Q} \subseteq A$, $Z(A) = \mathrm{Nil}(A)$ et E est un A-module sans torsion. Alors R est un AV-anneau si et seulement si A est un AV-anneau et E est un A-module divisible.*
4. *On suppose que A est un anneau local d'idéal maximal M vérifiant $\sqrt{Ann(E)} = M$. Alors R est un AV-anneau si et seulement si A l'est aussi.*

Le second résultat principal de ce chapitre est le théorème suivant qui nous donne le transfert de la propriété de *PV*-anneau à l'extension triviale.

Théorème 0.0.6 *Soient A un anneau, E un A-module et $R := A \ltimes E$ l'extension triviale de A par E.*

1. *Si R est un PV-anneau, alors A l'est aussi.*
2. *On suppose que $Ann(E) \neq 0$. Alors, R est un PV-anneau si et seulement si A est un PV-anneau local d'idéal maximal M tel que $M^2 = 0$ et $ME = 0$.*
3. *Supposons que R est un PV-anneau. Alors, A est un anneau local d'idéal maximal M tel que soit A est un PVD ou $M^2 = 0$, E est un A-module divisible, et pour tous $e, f \in E$ on a, $Ae \subseteq Mf$ ou $Mf \subseteq Ae$.*
4. *Si E est un A-module non nul de type fini et R est un PV-anneau, alors A est un anneau local d'idéal maximal M vérifiant $M^2 = 0$.*

Dans [41, Proposition 4.2], Dobbs a prouvé que la clôture intégrale d'un domaine pseudo de valuation qui est de conducteur fini est un domaine de valuation. Le résultat suivant nous donne un exemple qui illustre l'échec du théorème de Dobbs lorsque l'anneau n'est pas un domaine. Plus précisement, nous montrons qu'un *PV*-anneau de conducteur fini et qui est de plus intégralement clos dans son anneau total de quotient n'est pas forcément un anneau de valuation.

Proposition 0.0.7 *Soit (A,M) un anneau local d'idéal maximal M qui n'est pas un corps tel que $M^2 = 0$. Soient E un A-module tel que $ME = 0$ et $R := A \ltimes E$ l'extension triviale de A par E.*

1. *R est un PV-anneau.*
2. *On suppose que M est un idéal de type fini et E est un A-module de type fini. Alors R est de conducteur fini.*
3. *Supposons que M est l'unique idéal premier de A et que E est un A-module non nul de type fini. Alors $R' = R$ n'est pas un anneau de valuation.*

Le théorème suivant s'intéresse au transfert de la *PAV*-propriété dans l'extension triviale.

Théorème 0.0.8 *Soient A un anneau, E un A-module et $R := A \ltimes E$ l'extension triviale de A par E.*

1. *Si R est un PAV-anneau, alors A l'est aussi.*
2. *On suppose que $Z(A) = \mathrm{Nil}(A)$ et E est un A-module divisible. Alors R est un PAV-anneau si et seulement si A l'est aussi.*
3. *On suppose que A est un anneau local d'idéal maximal M vérifiant $\sqrt{Ann(E)} = M$. Alors R est un PAV-anneau si et seulement si A l'est aussi.*
4. *Supposons que $A \subseteq E$ est une extension de domaines telle que $\mathbb{Q} \subseteq E$. Alors R est un PAV-anneau si et seulement si A l'est aussi et $qf(A) \subseteq E$.*

Dans [20, Proposition 4.11], Badawi a prouvé que si (R,M) est un domaine pseudo presque de valuation (PAVD) tel que M est de type fini, alors la clôture intégrale de R est un domaine de valuation. L'exemple suivant illustre l'échec du théorème de Badawi lorsque l'anneau n'est plus un domaine.

Exemple 0.0.9 *Soit (A,M) un anneau de valuation Noethérien qui n'est pas un corps et soit E un A-module non nul de type fini tel que $ME = 0$. Alors $A \ltimes E$ est un PAV-anneau Noethérien mais $R' = R$ n'est pas un anneau de valuation.*

La dernière partie de ce chapitre est consacrée à élargir la classe des domaines presque pseudo de valuation (APVD) à des anneaux contenant des diviseurs de zéro. Un anneau R est dit presque pseudo de valuation (APV-anneau) si pour tout idéal maximal M de R et pour tous $a,b \in R$, il existe un entier naturel $n \geq 1$ tel que $a^n R \subseteq b^n M$ ou $bM \subseteq aM$. On s'intèresse dans cette partie à l'étude de quelques propriétés de cette nouvelle classe d'anneaux. En effet, après avoir donner une caractérisation de cette notion, nous étudions la stabilité sous l'image homomorphe ainsi que le transfert de cette propriété à l'extension triviale. Notre résultat du transfert est donnée dans le théorème suivant:

Théorème 0.0.10 *Soient A un anneau, E un A-module et $R := A \ltimes E$ l'extension triviale de A par E.*

1. *Si R est un APV-anneau, alors A l'est aussi.*
2. *On suppose que $Z(A) = \mathrm{Nil}(A)$ et E est un A-module divisible. Alors R est un APV-anneau si et seulement si A l'est aussi.*
3. *On suppose que A est un anneau local d'idéal maximal M vérifiant $ME = 0$. Alors R est un APV-anneau si et seulement si A l'est aussi.*
4. *Soit $A \subseteq E$ une extension de domaines. Alors R est un APV-anneau si et seulement si A est un APV-anneau et $qf(A) \subseteq E$.*

Le deuxième chapitre consiste à introduire et étudier une généralisation des idéaux fortement primaires dans le contexte des anneaux avec des diviseurs de zéro. Un idéal primaire P d'un anneau R est dit fortement primaire si pour tous $a, b \in R$, il existe un entier naturel $n \geq 1$ tel que $aP \subseteq bR$ ou $b^n R \subseteq a^n P$. Dans la première section, nous nous somme intéressé à établir quelques résultats de base sur les idéaux fortement primaire. Plus précisément, nous montrons qu'un idéal fortement primaire qui est de plus un idéal premier est comparable avec tout idéal principal. Ensuite, Nous examinons la stabilité des idéaux fortement primaires sous l'image homomorphique et sous la localisation. De plus, nous abordons une classe d'anneaux dans lesquels la notion d'idéal fortement primaire et la notion d'idéal fortement premier coïncident.
La deuxième section de ce chapitre est consacrée à l'étude des idéaux fortement primaires dans les extensions triviales. Notre résultat du transfert est donné dans le théorème suivant:

Théorème 0.0.11 *Soient A un anneau, P un idéal primaire de A, et E un A-module. Soit $R := A \ltimes E$ l'extension triviale de A par E.*

1. *Si $P \ltimes E$ est un idéal fortement primaire de R, alors P est un idéal fortement primaire de A.*
2. *On suppose que A est un domaine et E est un A-module divisible. Alors, un idéal J de R est fortement primaire si et seulement si $J = P \ltimes E$ tel que P est un idéal fortement primaire de A, ou $J = 0 \ltimes F$ où F est un sous module 0-primaire de E.*
3. *Supposons que A est un anneau local idéal maximal M tel que $ME = 0$. Alors, $P \ltimes E$ est fortement primaire si et seulement si P est fortement primaire.*

Dans [23], Badawi et Houston ont montré que dans un anneau intègre, les idéaux premiers contenus strictement dans un idéal fortement primaire sont fortement premiers. L'exemple suivant montre que ce résultat ne peut pas s'étendre aux anneaux avec diviseurs de zéro.

Exemple 0.0.12 *Soit A un APVD d'idéal maximal M où la dimension de Krull de A et supérieur ou égale à 2. Soient P un idéal premier de A tel que $P \neq M$ et $R := A \ltimes E$ l'extension triviale de A par E où $E = A/M$. Alors $P \ltimes E$ n'est pas fortement premier.*

Le but de la dernière section est d'étudier le transfert de la propriété des idéaux fortement primaires dans la duplication amalgamé. Notre résultat du transfert permet la construction de nouvelles classes originales d'idéaux satisfaisants cette propriété.

Théorème 0.0.13 *Soient A un anneau, P un idéal primaire de A, et I un idéal propre de A.*

1. *Si $P \bowtie I$ est fortement primaire, alors P est fortement primaire, $I \subseteq \mathrm{Nil}(A)$, et $I = aI$ pour tout élément régulier $a \in P$.*

2. *Supposons que A est un anneau local d'idéal maximal M tel que $MI = 0$. Alors, $P \bowtie I$ est fortement primaire si et seulement si P est fortement primaire.*

3. *On suppose que $Z(A) = \mathrm{Nil}(A)$ et que $I = aI$ pour tout élément régulier a de A. Alors, $P \bowtie I$ est fortement primaire si et seulement si P est fortement primaire et $I \subseteq \mathrm{Nil}(A)$.*

Dans **le troisième chapitre**, nous examinons le transfert de la propriété d'anneaux presque de Bézout et de la propriété d'anneaux presque de *GCD* dans les extensions triviales et dans l'anneau amalgamé algébrique le long d'un idéal. Nos résultats nous permettent de construire des nouvelles classes originales d'anneaux satisfaisants ces propriétés.

La première section de ce chapitre est consacrée à l'étude de la propriété d'*AB*-anneau dans les extensions triviales et dans l'anneau amalgamé algébrique le long d'un idéal. Nous commencons par énoncer le transfert de cette propriété dans les extensions triviales.

Théorème 0.0.14 *Soient A un anneau, E un A-module et $R := A \ltimes E$ l'extension triviale de A par E.*

1. *Si R est un AB-anneau, alors A l'est aussi et pour tout $e \in E$ et tout élément régulier a de A il existe $f \in E$ tel que $na^{n-1}e = a^n f$ pour un certain entier positive $n \geq 1$.*

2. *On suppose que $Z(A) = \mathrm{Nil}(A)$ et E est un A-module divisible. Alors R est un AB-anneau si et seulement si A l'est aussi.*

3. *On suppose que A est un anneau local d'idéal maximal M vérifiant $\sqrt{Ann(E)} = M$. Alors R est un AB-anneau si et seulement si A l'est aussi.*

Le second résultat principal de cette section est le théoréme suivant qui nous donne le transfert de la propriété d'*AB*-anneau à l'amalgamation algébrique.

Théorème 0.0.15 *Soient A et B deux anneaux, $f : A \to B$ un homomorphisme d'anneaux et J un idéal propre régulier de B. Si $R := A \bowtie^f J$ est un AB-anneau, alors $f^{-1}(J) \subseteq Z(A)$. En particulier, si A est un domaine alors R est un AB-anneau si et seulement si $f^{-1}(J) = 0$ et $f(A) + J$ et un AB-anneau.*

La deuxième section de ce chapitre s'intéresse à élargir les domaines presque de *GCD* à des anneaux contenant des diviseurs de zéro. Un anneau R est dit presque de *GCD* (*AGCD*-anneau) si pour tous $a, b \in R$, il existe un entier naturel $n \geq 1$ tel que $a^n R \cap b^n R$ est principal. Ensuite, nous étudions le transfert de l'*AGCD*-propriété dans l'extension triviale et dans l'amalgamé algébrique d'un anneau le long d'un idéal.

Théorème 0.0.16 *Soient A un anneau, E un A-module et $R := A \ltimes E$ l'extension triviale de A par E.*

1. *Si R est un AGCD-anneau, alors A l'est aussi.*
2. *On suppose que $Z(A) = \mathrm{Nil}(A)$ et E est un A-module divisible. Alors R est un AGCD-anneau si et seulement si A l'est aussi.*
3. *On suppose que A est un anneau local d'idéal maximal M vérifiant $\sqrt{Ann(E)} = M$. Alors R est un AGCD-anneau si et seulement si A l'est aussi.*
4. *Supposons que $A \subseteq E$ est une extension de domaines telle que $\mathbb{Q} \subseteq E$. Alors R est un AGCD-anneau si et seulement si A l'est aussi et $qf(A) \subseteq E$.*

On termine cette section par donner un autre résulat principal qui examine le transfert de l'*AGCD*-propriété dans l'amalgamation algébrique.

Théorème 0.0.17 *Soient A et B deux anneaux, $f : A \to B$ un homomorphisme d'anneaux et J un idéal non-principal régulier de B. Si A est un domaine, alors $A \bowtie^f J$ est un AGCD-anneau si et seulement si $f^{-1}(J) = 0$ et $f(A) + J$ et un AGCD-anneau.*

Le quatrième chapitre a pour but de poursuivre l'étude d'(A)-anneaux et d'(A)-anneaux forts. Dans un premier temps, nous commençons ce chapitre par donner une caractérisation d'(A)-anneaux fort en terme d'(A)anneaux. Plus précisément, nous montrons qu'un anneau R est un (A)-anneau fort si et seulement si R est un (A)-anneau et $Z(R)$ est un idéal de R. De plus, lorsque R est un anneau réduit alors R est (A)-anneau fort si et seulement si pour tout idéal de type fini I de R qui est engendré par un nombre fini de diviseurs de zéro, I est contenu dans un idéal premier minimal. Ensuite, un autre résultat très important est donné. Nous prouvons que si $A \subseteq B$ est une extension d'anneaux telle que $mB \subseteq A$ pour un certain élément régulier m de B, alors A est un (A)-anneau fort (resp. (A)-anneau) si et seulement si B est un (A)-anneau fort (resp. (A)-anneau). Le reste de ce chapitre est consacré à l'étude de ces propriétés dans la bi-amalgamation d'anneaux. Notre principal résultat du transfert est donné dans le théorème suivant:

Théorème 0.0.18 *Soient $f : A \longrightarrow B$ et $g : A \longrightarrow C$ deux homomorphismes d'anneaux et soit J (resp., J') un idéal propre de B (resp., C) tel que $f^{-1}(J) = g^{-1}(J')$.*

1. *Si $A \bowtie^{f,f} (J,J)$ est un (A)-anneau fort (resp. (A)-anneau), alors $f(A)+J$ l'est aussi.*
2. *On suppose que $J \times J'$ est un idéal régulier de $A \times B$. Alors*
   a) *$A \bowtie^{f,g} (J,J')$ est un (A)-anneau si et seulement si B et C sont des (A)-anneaux.*
   b) *$A \bowtie^{f,g} (J,J')$ n'est jamais un (A)-anneau fort.*

Notre étude enrichit la littérature courante avec des exemples originaux des (A)-anneaux qui ne sont pas des (A)-anneaux fort.

Dans **le cinquième chapitre**, nous étudions le transfert de certaines propriétés liée à l'ensemble $\mathscr{H}$ dans l'amalgamation d'anneaux. Nos résultats permettent la construction de nouvelles classes originales d'anneaux commutatifs satisfaisants ces propriétés. Nous commençons par étudier le transfert de la propriété de ϕ-anneau à l'amalgamé algébrique le long d'un idéal.

Théorème 0.0.19 *Soient A et B deux anneaux, $f : A \to B$ un homomorphisme d'anneaux et J un idéal non nul de B. Posons $R := A \bowtie^f J$ et $N(J) := \mathrm{Nil}(B) \cap J$. Alors*

1. *Si J est un idéal nonnil de B, alors R est un ϕ-anneau si et seulement si $f^{-1}(J) = 0$, A est un domaine, et $N(J)$ est un idéal premier divisé de $f(A)+J$.*
2. *Si $J \subseteq \mathrm{Nil}(B)$, alors R est un ϕ-anneau si et seulement si A est un ϕ-anneau, et pour tous $i, j \in J$ et tout $a \in A \setminus \mathrm{Nil}(A)$, ils existent $x \in \mathrm{Nil}(A)$ et $k \in J$ tels que $xa = 0$ et $j = kf(a) + i(f(x)+k)$.*

Dans la deuxième section de ce chapitre, nous nous sommes intéressés particulièrement à l'étude du transfert des propriétés d'anneaux ϕ-enchainé et d'anneaux ϕ-pseudo de valuation dans l'amalgamation algébrique. Pour cela, nous pésentons le résultat principal suivant.

Théorème 0.0.20 *Soient A et B deux anneaux, $f : A \to B$ un homomorphisme d'anneaux et $J \subseteq \mathrm{Nil}(B)$ un idéal de B. Si $A \bowtie^f J$ est un ϕ-anneau, alors $A \bowtie^f J$ est un anneau ϕ-enchainé (resp. ϕ-pseudo de valuation) si et seulement si A est un anneau ϕ-enchainé (resp. ϕ-pseudo de valuation).*

Le sixième chapitre consiste à introduire et étudier une généralisation des domaines pseudo de Krull aux anneaux de la classe $\mathscr{H}$. Rappelons d'après Badawi qu'un anneau $R \in \mathscr{H}$ est dit nonnil Noethérien si tout idéal nonnil (contient un élément nonnilpotent) est de type fini. Un anneau $R \in \mathscr{H}$ est dit ϕ-pseudo de Krull si elle existe une famille

$\mathscr{F} = \{R_i\}$ des sur-anneaux de $\phi(R)$ tels que $\phi(R) = \cap R_i$ où chaque R_i est un sur-anneau nonnil Noethérien qui est de plus un anneau ϕ-pseudo de valuation. On s'intéresse dans la première section de ce chapitre à l'ètude de quelques propriétés de cette nouvelle classe d'anneaux. Le théorème suivant est l'un des résultats principaux dans ce chapitre.

Théorème 0.0.21 *Soit $R \in \mathscr{H}$. Alors R est un anneau ϕ-pseudo de Krull si et seulement si $R/\mathrm{Nil}(R)$ est un domaine pseudo de Krull.*

Ce théorème nous permet de généraliser des résultats prouvés dans le cas des domaines pseudo de Krull. Un anneau R dans $\mathscr{H}$ est dit satisfait la propriété $(*)$ si tout élément nonnilpotent de R est contenu dans un nombre fini des idéaux premier de hauteur un. Deux caractérisations des anneaux ϕ-pseudo de Krull sont données dans les deux résultats suivants.

Proposition 0.0.22 *Soit $R \in \mathscr{H}$. Alors les assertions suivantes sont équivalentes:*

1. *R est un anneau ϕ-pseudo de Krull.*
2. *R_P est un anneau ϕ-pseudo de Krull pour tout idéal premier P et R satisfait la propriété $(*)$.*
3. *R_M est un anneau ϕ-pseudo de Krull pour tout idéal maximal M et R satisfait la propriété $(*)$.*

Théorème 0.0.23 *Soit $R \in \mathscr{H}$. Alors R est un anneau ϕ-pseudo de Krull si et seulement $\phi(R)$ est isomorphe à un anneau A obtenu à partir du diagramme suivant :*

$$\begin{array}{ccc} A & \longrightarrow & S \\ \downarrow & & \downarrow \\ T & \longrightarrow & T/M \end{array}$$

*Où T est un anneau local de dimension zéro d'*id*éal maximal M, $S = A/M$ est un sous-anneau pseudo de Krull de T/M, les flèches verticales sont les inclusions naturelles et les flèches horizontales sont les surjections naturelles.*

La dernière section consiste à donner des exemples d'anneaux ϕ-pseudo de Krull qui ne sont pas intègre en utilisant les extensions triviales et l'amalgamation d'anneaux.

Le dernier chapitre a pour but de généraliser la notion de domaines Noethérien par morceaux (resp. w-Noethérien par morceaux) dans le contexte des anneaux de la classe $\mathscr{H}$. Dans la première section, notre étude se porte sur les anneaux ϕ-Noethérien par morceaux. Un anneau $R \in \mathscr{H}$ est dit ϕ-Noethérien par morceaux si les conditions suivantes sont vérifiées: (i) l'ensemble des idéaux premiers nonnil de R satisfait la condition

des chaines ascendantes (CCA), (*ii*) R satisfait la condition des chaines ascendantes sur les idéaux P-primaires nonnil pour chaque idéal premier nonnil P et (*iii*) chaque idéal nonnil n'a qu'un nombre fini d'idéaux premiers minimaux sur lui. De plus, nous nous sommes intéressés à établir quelques résultats préliminaires de la propriété d'anneaux ϕ-Noethérien par morceaux. Dans ce sens, nous montrons qu'un anneau $R \in \mathscr{H}$ est ϕ-Noethérien par morceaux si et seulement si $R/\mathrm{Nil}(R)$ est un domaine Noethérien par morceaux. Ce dernier résultat nous aide à donner des exemples d'anneaux ϕ-Noethérien par morceaux qui ne sont ni intègre ni Noethérien par morceaux, aussi il nous permet de généraliser beaucoup de résultats. Parmi ces résultats principaux, nous donnons les deux théorèmes suivants:

Théorème 0.0.24 *Soit $R \in \mathscr{H}$ un anneau ϕ-enchainé. Supposons que* $\mathrm{Nil}(R)$ *a une dimension de Krull comme un R-module. Alors, les conditions suivantes sont équivalentes:*

1. *R est un anneau ϕ-Noethérien par morceaux.*
2. *Si P est un idéal premier de R, alors $P^2 \neq P$ ou $P = \mathrm{Nil}(R)$.*
3. *R admet une dimension de Krull comme un R-module.*

Théorème 0.0.25 *Soit $R \in \mathscr{H}$ un anneau de ϕ-Prüfer. Alors R est un anneau de ϕ-Dedekind généralisé si et seulement si R est un anneau ϕ-Noethérien par morceaux.*

La deuxième section de ce chapitre consiste à introduire et étudier la notion d'anneaux ϕ-w-Noethérien par morceaux. Soient R un anneau, $T(R)$ son anneau total des fractions et $\mathbf{F}(R)$ l'ensemble des idéaux fractionnels non nul de R. Pour un idéal $I \in \mathbf{F}(R)$, on note $I^{-1} = \{x \in T(R) \mid xI \subseteq R\}$ et $I_w = \{x \in T \mid xJ \subseteq I$ pour un certain idéal de type fini J tel que $J^{-1} = R\}$. Aussi, un idéal $I \in \mathbf{F}(R)$ est dit w-ideal si $I_w = I$. D'après [80], un idéal nonnil I d'un anneau $R \in \mathscr{H}$ est dit ϕ-w-idéal si $\phi(I)$ est un w-idéal de $\phi(R)$. Ainsi, R est dit ϕ-w-Noethérien par morceaux si (*i*) l'ensemble des ϕ-w-idéaux premiers nonnil de R satisfait la condition des chaines ascendantes (CCA), (*ii*) R satisfait la condition des chaines ascendantes sur les idéaux P-primaires nonnil pour chaque ϕ-w-idéal premier P et (*iii*) chaque ϕ-w-idéal n'a qu'un nombre fini d'idéaux premiers minimaux sur lui. Dans cette section, nous montrons qu'un anneau R est ϕ-w-Noethérien par morceaux si et seulement si $R/\mathrm{Nil}(R)$ est un domaine w-Noethérien par morceaux. Ce théorème nous permet de généraliser des résultats prouvés dans le cas des anneaux intègres. Parmi ces résultats, nous prouvons que R est un anneau de ϕ-Krull généralisé si et seulement si R est un anneau ϕ-w-Noethérien par morceaux qui est un ϕ-PvM-anneau.

Introduction

The present thesis is devoted to the study of various properties of commutative rings and ideals. The first part of our research pursues the study of some properties related to the notion of valuation ring. If R is a domain, K denotes the quotient field of R. In [65], Hedstrom and Houston introduced the class of pseudo-valuation domains (in short, PVDs), which is closely related to the class of valuation domains. A domain R with quotient field K is called a PVD if, whenever P is a prime ideal of R and $xy \in P$ with $x \in K$ and $y \in K$, then either $x \in P$ or $y \in P$. In [41, Proposition 4.2], Dobbs proved that if R is a finite-conductor PVD, then R' is a valuation domain. Additional information about PVDs can be found in the interesting survey article [19]. In [3], D. F. Anderson, Badawi and Dobbs generalized the study of PVDs to the context of arbitrary rings (possibly with nontrivial zero-divisors) as follows. A prime ideal P of a ring R is said to be a *strongly prime ideal* (*of* R) if aP and bR are comparable (under inclusion) for all $a, b \in R$; a ring R is called a *pseudo-valuation ring* (in short, a PV-*ring*) if each prime ideal of R is a strongly prime ideal of R. In [9], D. D. Anderson and Zaffrullah introduced and studied the following notion. A domain R is called an *almost valuation domain* (in short, an AVD) if, for each nonzero $x \in K$, there exists a positive integer n such that either $x^n \in R$ or $x^{-n} \in R$. In [84] and [73], a generalization of AVDs to the context of arbitrary rings was considered as follows. A ring R is called an *almost valuation ring* (in short, an AV-ring) if, for any two elements a and b in R, there exists a positive integer n such that either a^n divides b^n (in R) or b^n divides a^n. In [20], Badawi introduced a class of domains as follows. A prime ideal P of a domain R is called a *pseudo-strongly prime ideal* (*of* R) if, whenever $xyP \subseteq P$ with $x \in K$ and $y \in K$, there is a positive integer n such that either $x^n \in R$ or $y^n P \subseteq P$; if each prime ideal of a domain R is a pseudo-strongly prime ideal of R, then R is called a *pseudo-almost valuation domain* (in short, a PAVD). In [72], a generalization of PAVDs to arbitrary rings was considered as follows. A prime ideal P of a ring R is said to be a *pseudo-strongly prime ideal* (*of* R) if, for all $a, b \in R$, there is a positive integer n such that either $a^n R \subseteq b^n R$ or $b^n P \subseteq a^n P$; a ring R is called a *pseudo-almost valuation ring* (in short, a PAV-ring) if each maximal ideal of R is a pseudo-strongly prime ideal of

R. Next recall from Badawi-Houston [23] that a domain R is said to be an *almost pseudo-valuation domain* (in short, an APVD) if each prime ideal P of R is a *strongly primary ideal* of R, in the sense that $xy \in P$ with $x \in K$ and $y \in K$ implies that either $x^n \in P$ for some $n \geq 1$ or $y \in P$. In [9], D. D. Anderson and Zaffrullah introduced and studied the notion of almost Bézout domain (AB-domain). An integral domain A is an AB-domain if for $a, b \in A - \{0\}$ there is n such that (a^n, b^n) is principal. In [84], the generalization of the almost Bézout domains to arbitrary commutative rings (with zero-divisors) is considered as follows: a ring A is called an almost Bézout ring (AB-ring for short) if, for any two elements a and b in A, there exists a positive integer n such that the ideal (a^n, b^n) is principal. An integral domain A is said to be an almost GCD (AGCD) domain if for every $a, b \in A$, there exists a positive integer n such that $Aa^n \cap Ab^n$ is a principal ideal. AGCD domains were introduced by Zafrullah in [95] as a generalization of GCD domains.

The second part of the thesis, which includes chapter 4, aims to study the notions of (A)-ring and strong (A)-ring. Huckaba and Keller [70] introduced the following: a commutative ring R has property (A) if every finitely generated ideal of R consisting entirely of zero divisors has a nonzero annihilator. Property (A) was originally studied by Quentel [92]. Quentel used the term condition (C) for property (A). The class of commutative rings with property (A) is quite large; for example, Noetherian rings [78, p. 56], rings whose prime ideals are maximal [66], the polynomial ring $R[X]$ for every ring R and rings whose total ring of quotients are Von Neumann regular [66]. In [64], Mahdou and Rahmouni introduced and defined a strong (A)-ring as a ring in which every finitely generated ideal of R which is generated by a finite number of zero-divisors elements of R, has a nonzero annihilator. They studied the transfer of property (A) and strong property (A) in trivial ring extensions and amalgamated duplication of a ring along an ideal.

Badawi defined the divided prime ideal of a ring R in [18], as a prime ideal which is comparable to every ideal of R and R is said to be divided if each prime ideal of R is divided. An ideal I of R is said to be a *nonnil ideal* if $I \nsubseteq \mathrm{Nil}(R)$. In [5], Anderson and Badawi introduced the notion $\mathscr{H}$ to be the set of all rings R such that $\mathrm{Nil}(R)$ is a divided prime ideal and named these rings as ϕ-rings. The object of the final part is to study certain properties of ϕ-rings. Recall from [14] that a ring R is a ϕ-PVR if $\mathrm{Nil}(R)$ is a divided prime ideal and for every $a, b \in R \setminus \mathrm{Nil}(R)$, either $a \mid b$ or $b \mid ac$ in R for each nonunit element $c \in R$. A ring R is called a *ϕ-chained ring* if $\mathrm{Nil}(R)$ is divided prime and for every $a, b \in R \setminus \mathrm{Nil}(R)$ either $a \mid b$ or $b \mid a$ [15]. In [88] Oda introduced the class of pseudo-Krull domains. An integral domain R is called a pseudo-Krull domain if there exists a family $\mathscr{F} = \{R_i\}$ of overrings of R such that $R = \cap R_i$, where each R_i is a Noetherian pseudo-valuation overring of R, and every nonzero element of R is not a unit in only a finite number of members of $\mathscr{F}$. Many characterizations and properties of pseudo-Krull domains are given in [79], [88], and [90]. Another very important notion that appears in [26] is the notion of piecewise Noetherian ring. A commutative ring R is piecewise Noetherian if it satisfies the ascending chain condition (ACC) on its prime ideals and has finite ideal lengths. In [26], the authors showed that being piecewise Noetherian is

inherited by factor rings and localizations and extends to integral extensions and polynomial rings, and they distinguished these rings from rings with Krull dimension. Recall that a commutative ring R is said to have Noetherian spectrum if it satisfies the ACC on radical ideals. It is remarked just before [27, Lemma 1] that a commutative ring R is piecewise Noetherian if and only if R has Noetherian spectrum and satisfies the ACC on P-primary ideals, for each prime ideal P of R. This characterization shows clearly that every Noetherian ring is piecewise Noetherian, and is used to define the notion of piecewise w-Noetherian rings by the authors in [51].

In 1956, Nagata introduced the notion of trivial ring extension of a ring A by a module E as follows :

Definition 0.0.0.1 *Let A be a ring and E an A-module. The trivial ring extension $R = A \ltimes E$ of A by E is the set of pairs (a,e) with $a \in A$ and $e \in E$ under coordinate-wise addition and adjusted multiplication defined by:*

$$(a,e)(b,f) = (ab, af + be).$$

This notion has been the subject of several works in the area, and it is a fertile field of research in the theory of rings (see [60, 69] and the reference therein). Trivial ring extensions have been studied extensively; and considerable work, part of is summarized in Glaz's book [60] and Huckaba's book [69], has been concerned with these extensions. Mainly, trivial ring extensions have been useful for solving many open problems and conjectures in both commutative and non-commutative ring theory. See for instance [8, 25, 45, 69, 76].

In 2006, M. D'Anna and M. Fontana [35] introduced a new construction, called amalgamated duplication of a ring A along an A-submodule E of $Q(A)$ (the total ring of fractions of A) such that $E^2 \subseteq E$. When $E^2 = \{0\}$, this construction coincides with the trivial ring extension of A by E. Motivations and more applications of the amalgamated duplication $A \bowtie E$ of A along an A-submodule E of $Q(A)$ are discussed in more details, especially in the particular case where E is an ideal of A, in recent papers, for instance, see [33, 34, 35, 36, 37, 38].
In 2010, D'Anna, Finocchiaro and Fontana [33] extended the notion of amalgamated duplication construction $A \bowtie I$ of a ring A along an ideal I of A to the general context of ring homomorphism extensions as follows:

Definition 0.0.0.2 *Let A and B be two rings with identity elements, J be an ideal of B and let $f : A \to B$ be a ring homomorphism. In this setting, we consider the following subring of $A \times B$; $A \bowtie^f J := \{(a, f(a) + j) \mid a \in A, j \in J\}$ called the amalgamation of A and B along J with respect to f.*

This construction is a generalization of the amalgamated duplication of a ring along an ideal (introduced and studied by D'Anna and Fontana in [35, 36, 37]). Moreover, other classical constructions (such as $A+XB[X]$, $A+XB[[X]]$, and the $D+M$ constructions) can be studied as particular cases of the amalgamation ([33, Examples 2.5 and 2.6]). Other classical constructions, such as the Nagata's idealization, also called trivial ring extension ([87, page 2]), and the CPI extensions (in the sense of Boisen and Sheldon [28]) are strictly related to it ([33, Example 2.7 and Remark 2.8]). See for instance [33, 34, 37, 38].

In 2013, Kabbaj, Tamekkante and Louartiti [74] extended the notion of amalgamated algebra construction $A \bowtie J$ of a rings A and B along ideal J of B to the general context of ring homomorphism extensions as follows:

Definition 0.0.0.3 *Let $f : A \longrightarrow B$ and $g : A \longrightarrow C$ be two ring homomorphisms and let J (resp.,J') be an ideal of B (resp.,C) such that $f^{-1}(J) = g^{-1}(J')$. In this setting, we can consider the following subring of $B \times C$:*

$$A \bowtie^{f,g} (J,J') := \{(f(a)+j, g(a)+j') | a \in A, j \in J, j' \in J'\}$$

called the bi-amalgamation of A with (B,C) along (J,J') with respect to (f,g).

This construction is a generalisation of the amalgamated algebra along an ideal (introduced and studied by D'Anna and Fontana in [33, 34, 37, 38]) as follows $A \bowtie^{f} J = A \bowtie^{Id,f} (f^{-1}(J), J)$. In [74], the authors studied the basic properties of this construction (e.g., characterized for $A \bowtie^{f,g} (J,J')$ to be a Noetherian ring, an integral domain, a reduced ring ...)

For a ring R, S a set of R, I an ideal of R, and M an R-module. We use the following notations:

- $\mathrm{Z}(R) := \{a \in R \mid ax = 0 \text{ for some } 0 \neq x \in R\}$, the set of zero divisors of R.
- $\mathrm{Ann}(M)$, the annihilator of M.
- $\mathrm{T}(R) = \mathrm{tq}(R) := R_{R \setminus \mathrm{Z}(R)}$, the total quotient ring of R.
- If R is an integral domain, then $\mathrm{qf}(R) = K$ denotes the quotient field of R.
- $\mathrm{Nil}(R)$, the set of nilpotent elements of R.
- $\mathrm{J}(R)$, the jacobson radical of R.
- $\sqrt{\mathrm{I}}$, denotes the radical of I.
- $\mathrm{Spec}(R)$, the set of prime ideals of R.
- $\mathrm{Reg}(R)$, the set of regular elements of R.
- $E(S) := \{x \in qf(R) \mid x^n \notin S \text{ for each } n \geq 1\}$.
- $(I : I) := \{x \in \mathrm{T}(R) \mid xI \subseteq I\}$.
- R', denotes the integral closure of R.

CHAPTER 1

TRIVIAL EXTENSIONS SATISFYING CERTAIN VALUATION-LIKE PROPERTIES

Abstract.

In this chapter [*], we study the possible transfer of the properties of being an almost valuation ring, a pseudo-valuation ring and a pseudo-almost valuation ring between a commutative ring A and $A \ltimes E$ (the trivial ring extension of A arising from an A-module E). Also, we extend the notion of an almost pseudo-valuation domain to the context of arbitrary commutative rings (possibly with nontrivial zero-divisors), and we also study the possible transfer of this notion between A and $A \ltimes E$. Our aim is to provide examples of new classes of commutative rings satisfying the above-mentioned properties.

1.1 Introduction

The next two paragraphs summarize most of the background on the underlying domain-theoretic properties and their ring-theoretic generalizations that will be relevant in this work. A reader may choose to skim the next long paragraph initially, returning to it later as needed.

In [65], Hedstrom and Houston introduced the class of pseudo-valuation domains (in short, PVDs), which is closely related to the class of valuation domains. A domain R with quotient field K is called a PVD if, whenever P is a prime ideal of R and $xy \in P$ with $x \in K$ and $y \in K$, then either $x \in P$ or $y \in P$. Any valuation domain is a PVD [65, Proposition 1.1]; and any PVD is local [65, Corollary 1.3]. In [41, Proposition 4.2], Dobbs

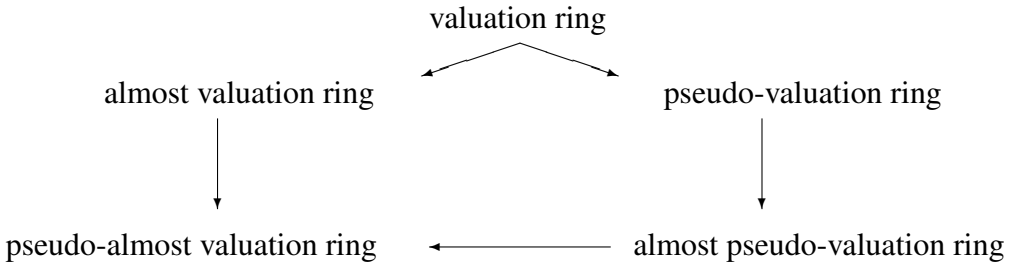
[*]Comm. Algebra, 47(5) (2019), 2060-2077 (in collaboration with D. E. Dobbs and N. Mahdou).

proved that if R is a finite-conductor PVD, then R' is a valuation domain. Additional information about PVDs can be found in the interesting survey article [19]. In [3], D. F. Anderson, Badawi and Dobbs generalized the study of PVDs to the context of arbitrary rings (possibly with nontrivial zero-divisors) as follows. A prime ideal P of a ring R is said to be a *strongly prime ideal* (*of* R) if aP and bR are comparable (under inclusion) for all $a,b \in R$; a ring R is called a *pseudo-valuation ring* (in short, a PV-*ring*) if each prime ideal of R is a strongly prime ideal of R. Any chained ring is a PV-ring [3, Corollary 4]; and the prime ideals of any PV-ring are linearly ordered by inclusion (whence any PV-ring must be local) [3, Lemma 1]. A ring R is a PV-ring if and only if it is local and its maximal ideal is a strongly prime ideal of R [3, Theorem 2]. If (R,M) is a PV-ring, then R' is also a PV-ring with (unique) maximal ideal M [3, Theorem 19]. As explained in [3, page 58], a domain is a PV-ring if and only if it is a PVD. In [9], D. D. Anderson and Zaffrullah introduced and studied the following notion. A domain R is called an *almost valuation domain* (in short, an AVD) if, for each nonzero $x \in \mathrm{qf}(R)$, there exists a positive integer n such that either $x^n \in R$ or $x^{-n} \in R$. It is clear that any valuation domain is an AVD; the proof of [9, Theorem 5.6] showed that if R is an AVD, then R' is a valuation domain. In [84] and [73], a generalization of AVDs to the context of arbitrary rings was considered as follows. A ring R is called an *almost valuation ring* (in short, an AV-ring) if, for any two elements a and b in R, there exists a positive integer n such that either a^n divides b^n (in R) or b^n divides a^n. Any almost valuation ring is local [73, Proposition 2.2]. In [20], Badawi introduced a class of domains as follows. A prime ideal P of a domain R is called a *pseudo-strongly prime ideal* (*of* R) if, whenever $xyP \subseteq P$ with $x \in \mathrm{qf}(R)$ and $y \in \mathrm{qf}(R)$, there is a positive integer n such that either $x^n \in R$ or $y^nP \subseteq P$; if each prime ideal of a domain R is a pseudo-strongly prime ideal of R, then R is called a *pseudo-almost valuation domain* (in short, a PAVD). Any AVD is a PAVD [20, Proposition 2.12]. A domain R is a PAVD if and only if, for each nonzero $x \in \mathrm{qf}(R)$, there is a positive integer n such that either $x^n \in R$ or $ax^{-n} \in R$ for each nonunit $a \in R$ [20, Theorem 2.8]. The proof of [20, Theorem 4.6] showed that if R is a PAVD, then R' is a PVD; one part of [20, Proposition 4.11] showed that the integral closure of a Noetherian PAVD must be a valuation domain. In [72], a generalization of PAVDs to arbitrary rings was considered as follows. A prime ideal P of a ring R is said to be a *pseudo-strongly prime ideal* (*of* R) if, for all $a,b \in R$, there is a positive integer n such that either $a^nR \subseteq b^nR$ or $b^nP \subseteq a^nP$; a ring R is called a *pseudo-almost valuation ring* (in short, a PAV-ring) if each maximal ideal of R is a pseudo-strongly prime ideal of R. Any PV-ring is a PAV-ring [72, Proposition 3.6]; any PAV-ring has a linearly ordered prime spectrum [72, Corollary 3.3] and so must be local. It is then immediate from the definitions that any AV-ring is a PAV-ring. Also, as explained in [72, page 819], a domain is a PAV-ring if and only if it is a PAVD. Next recall from Badawi-Houston [23] that a domain R is said to be an *almost pseudo-valuation domain* (in short, an APVD) if each prime ideal P of R is a *strongly primary ideal* of R, in the sense that $xy \in P$ with $x \in \mathrm{qf}(R)$ and $y \in \mathrm{qf}(R)$ implies that either $x^n \in P$ for some $n \geq 1$ or $y \in P$. It was shown in [23, Theorem 3.4] that a domain R is an APVD if and

only if R is local, say with maximal ideal M, such that, for each nonzero $x \in \mathrm{qf}(R)$, either $x^n \in M$ for some positive integer n or $ax^{-1} \in M$ for each nonunit $a \in R$. It is clear that any PVD is an APVD; any APVD is a PAVD [20, Proposition 2.11]; and [23, Example 3.9] (resp., [20, Example 3.5]) has given an example of an APVD that is not a PVD (resp., a PAVD that is neither an APVD nor an AVD).

We can now specify more precisely the main purposes of this chapter. Our results study the possible transfer of the properties of being a PV-ring, an AV-ring and a PAV-ring between a ring A and a trivial ring extension $A \ltimes E$, with an emphasis on finding naturally occurring kinds of A and E for which the property in question transfers from A to $A \ltimes E$. Applications include Examples 1.2.26 and 1.2.11, which show that certain domain-theoretic results of Badawi-Houston [23] and Dobbs [41] on the integral closure of an almost-pseudo valuation domain and the integral closure of a coherent pseudo-valuation domain do not generalize to the context of trivial ring extensions of domains. Also, we generalize the notion of an almost pseudo-valuation domain to the context of arbitrary rings and then study the possible transfer of this generalized property between A and $A \ltimes E$. For the main transfer results in this chapter, see Theorems 1.2.2, 1.2.8, 1.2.13 and 1.2.23.

As we proceed to study the above-mentioned classes of rings, the reader may find it helpful to keep in mind the implications noted in the following figure.



We will be using the following definition (which agrees with the classical one if A is a domain). An A-module E is said to be a *torsion A-module* if, for each $e \in E$, there exists $a \in A \setminus \{0\}$ such that $ae = 0$. We will also use the following three standard definitions. A *regular element* of a ring A is any element of $A \setminus \mathrm{Z}(A)$; an A-module E is *divisible* if, for each $u \in E$ and each regular element a of A, there exists $f \in E$ such that $e = af$; an A-module E is (a) *torsion-free* (A-*module*) if $a \in A$, $e \in E$, $ae = 0$ implies that either $a \in \mathrm{Z}(A)$ or $e = 0$.

1.2 Results

There are two reasons that we begin by recalling some results from [76]. First, Theorem 1.2.1 (2) will be used in a proof later in this chapter. Second, Theorem 1.2.1 shows that, for a couple of useful kinds of A-modules E, one can characterize when the trivial ring extension $A \ltimes E$ is a valuation ring. That theme will motivate the choice of the contexts studied in our later results, which, for the most part, seek to identify situations admitting positive transfer results for the other ring-theortic properties being considered here.

Theorem 1.2.1 (*Kabbour and Mahdou [76, Theorem 2.1]*) *Let A be a ring, E a nonzero A-module, and $R := A \ltimes E$. Then:*

(1) *Suppose that E is not a torsion A-module. Then R is a valuation ring if and only A is a valuation domain and $E \cong \mathrm{qf}(A)$ as an A-module.*

(2) *Suppose that E is a finitely generated A-module. Then R is a valuation ring if and only if A is a field and $E \cong A$ as an A-module.*

Our first new result studies the possible transfer of the AV-ring property between a ring A and a trivial ring extension $A \ltimes E$.

Theorem 1.2.2 *Let A be a ring, E a nonzero A-module, and $R := A \ltimes E$. Then:*

(1) *If R is an* AV*-ring, then so is A.*

(2) *Suppose that $\mathrm{Z}(A) = \mathrm{Nil}(A)$ and E is a divisible A-module. Then R is an* AV*-ring if and only if A is an* AV*-ring.*

(3) *Suppose that $\mathbb{Q} \subseteq A$, $\mathrm{Z}(A) = \mathrm{Nil}(A)$ and E is a torsion-free A-module. Then R is an* AV*-ring if and only if A is an* AV*-ring and E is a divisible A-module.*

(4) *Let (A, M) be a local ring (with maximal ideal M) and let E be an A-module such that $M = \sqrt{\mathrm{Ann}(E)}$. Then R is an* AV*-ring if and only if A is an* AV*-ring.*

Proof. (1) This was established in [84, Theorem 2.1 (1)].

(2) By (1), we need only prove that if A is an AV-ring (along with the hypotheses that $\mathrm{Z}(A) = \mathrm{Nil}(A)$ and E is a divisible A-module), then R is an AV-ring. We will show that if $\alpha := (a, e), \beta := (b, f) \in R$, then there exists a positive integer n such that either $\alpha^n \in \beta^n R$ or $\beta^n \in \alpha^n R$. There are two cases. In the first case, a and b are each regular elements of A. Then, since A is an AV-ring, there exists a positive integer n such that either $a^n A \subseteq b^n A$ or $b^n A \subseteq a^n A$. We may assume, without loss of generality, that $a^n A \subseteq b^n A$. Then $a^n = b^n c$ for some $c \in A$. Since E is a divisible A-module and b^n is a regular element

of A, there exists $d \in E$ such that $b^n d = na^{n-1}e - ncb^{n-1}f$. Therefore (with $a^0 := 1$), we have $\alpha^n = (a,e)^n = (a^n, na^{n-1}e) =$

$$(b^n c, b^n d + ncb^{n-1}f) = (b^n, nb^{n-1}f)(c,d) = \beta^n(c,d) \in \beta^n R,$$

as desired.

In the remaining case, either a or b is a zero-divisor in A. Without loss of generality, $a \in \mathrm{Z}(A) = \mathrm{Nil}(A)$. Then there exists a positive integer n such that $a^n = 0$. Hence $\alpha^{n+1} = (a^{n+1}, (n+1)a^n e) = (0,0) \in \beta^n R$, as desired.

(3) By (1) and (2), we need only prove that if R is an AV-ring (along with the hypotheses that $\mathbb{Q} \subseteq A$ and E is a torsion-free A-module), then E is a divisible A-module; that is, if $0 \neq e \in E$ and a is a regular element of A, then $e \in aE$. Since R is an AV-ring, there exists a positive integer n such that either $(a,0)^n \in (a,e)^n R$ or $(a,e)^n \in (a,0)^n R$. There are two cases. In the first case, $(a,0)^n = (a,e)^n(c,f)$ for some $(c,f) \in R$. Then

$$(a^n, 0) = (a,0)^n = (a,e)^n(c,f) = (a^n, na^{n-1}e)(c,f) = (a^n c, a^n f + na^{n-1}ce),$$

so that $a^n c = a^n$ and $a^n f + na^{n-1}ce = 0$. These facts can be rewritten as $a^n(c-1) = 0$ and $a^{n-1}(af + nce) = 0$. Since a^n and a^{n-1} are each regular elements of A, the hypothesis that E is a torsion-free A-module gives first that $c = 1$ and next that $af + ne = 0$. Multiplying by $n^{-1} \in \mathbb{Q} \subseteq A$, we get $e = n^{-1}(-af) = a(-n^{-1}f) \in aE$, as desired.

In the remaining case, $(a,e)^n = (a,0)^n(d,g)$ for some $(d,g) \in R$. Simplifying and equating second coordinates, we have $a^n g = na^{n-1}e$. As above, the hypotheses that E is a torsion-free A-module and a is a regular element give that $ag = ne$. It follows that $e = a(n^{-1}g) \in aE$, as desired.

(4) By (1), we need only prove that if A is an AV-ring, along with the hypothesis that $M = \sqrt{\mathrm{Ann}(E)}$, then R is an AV-ring. Let $\alpha = (a,e), \beta = (b,f) \in R$. If a is a unit in A, then α is a unit in R, so that $\beta^1 = \beta \in R = \alpha R = \alpha^1 R$. Similarly, if b is a unit in A, then $\alpha^1 \in \beta^1 R$. Thus, we may assume, without loss of generality, that $a \in M$ and $b \in M$. Then, since $M \subseteq \sqrt{\mathrm{Ann}(E)}$, there exist positive integers m and p such that $a^m \in \mathrm{Ann}(E)$ and $b^p \in \mathrm{Ann}(E)$. As A is an AV-ring, there exists a positive integer n such that either $a^n \in b^n A$ or $b^n \in a^n A$. We may assume, without loss of generality, that $a^n \in b^n A$. Taking $(mp+1)^{\text{th}}$ powers and rewriting, we get $a^{(mp+1)n} = b^{(mp+1)n} r$ for some $r \in A$. Then, since $(mp+1)n - 1 \geq \max(m,p)$,

$$\alpha^{(mp+1)n} = (a,e)^{(mp+1)n} = (a^{(mp+1)n}, (mp+1)na^{(mp+1)n-1}e) =$$

$$(a^{(mp+1)n}, 0) = (b^{(mp+1)n} r, 0) = (b^{(mp+1)n} r, (mp+1)nb^{(mp+1)n-1}f) =$$

$$(b^{(mp+1)n}, (mp+1)nb^{(mp+1)n-1}f)(r,0) = (b,f)^{(mp+1)n}(r,0) \in \beta^{(mp+1)n} R,$$

as desired. This completes the proof.

If A is a domain and E is a divisible A-module, Theorem 1.2.2 (2) specializes to the following result.

Corollary 1.2.3 *Let A be a domain and E a divisible A-module. Then $A \ltimes E$ is an* AV*-ring if and only if A is an* AV*-ring.*

If (A,M) is a local ring and E a suitable A-module, then Theorem 1.2.2 (4) specializes to recover one result in Corollary 3.2.4 (1) and to obtain a new application in Corollary 3.2.4 (2). Background on SPIRs (special principal ideal rings) can be found in, for instance, [96, page 245]. It follows easily from that background that each SPIR is a valuation ring.

Corollary 1.2.4 *Let (A,M) be a local ring, E an A-module, and $R := A \ltimes E$. Then:*

(1) *(Mahdou, Mimouni and Moutui [84, Theorem 2.1(3)]) Suppose that $ME = 0$. Then R is an* AV*-ring if and only if A is an* AV*-ring.*

(2) *Suppose that M is the only prime ideal of A (for instance, suppose that A is an* SPIR*). Then R is an* AV*-ring if and only if A is an* AV*-ring.*

Proof. If $E = 0$, then $R \cong A$ and so the assertions are clear in this case. Thus, we may henceforth assume, without loss of generality, that $E \neq 0$. Then, in the context of either (1) or (2), we can give the following unified argument. We have $\mathrm{Ann}(E) \neq A$, whence $\sqrt{\mathrm{Ann}(E)} = M$ (cf. [78, Theorem 26], [96, Note II, page 151]), and so an application of Theorem 1.2.2 (4) completes the proof.

Our next two examples illustrate parts (2) and (4) of Theorem 1.2.2.

Example 1.2.5 *Let F be a finite field and X an indeterminate over F. Put $H := F(X)$, the quotient field of $F[X]$, and let Y be an analytic indeterminate over H. Set $D := H + Y^3H[[Y]]$, $K := \mathrm{qf}(D)$, $A := D \ltimes K$ and $E := K \ltimes K$. As A is a subring of E, we can view E as an A-module. Then $A \ltimes E$ is an* AV*-ring which is neither a domain nor a valuation ring.*

Proof. By [20, Example 2.20], D is an AV-domain which is not a valuation domain. Therefore, by [84, Theorem 2.1 (2)], A is an AV-ring. However, since D is not a valuation domain, it follows from [76, Theorem 2.1 (1)] that A is not a valuation ring. It is straightforward to check that $\mathrm{Z}(A) = 0 \ltimes K = \mathrm{Nil}(A)$ and E is a divisible A-module. Hence, Theorem 1.2.2 (2) gives that $A \ltimes E$ is an AV-ring. As A is not a valuation ring, $A \ltimes E$ is not a valuation ring. Of course, $A \ltimes E$ is not a domain, since each element of $0 \ltimes E$ is nilpotent. This completes the proof.

Example 1.2.6 *Let (A,M) be a valuation domain, but not a field. Then $A \ltimes A/M^2$ is an* AV*-ring which is neither a domain nor a valuation ring.*

Proof. The first assertion follows from Theorem 1.2.2 (4); the second assertion follows as in the proof of Example 1.2.5; and, since A is not a field, the third assertion follows from [76, Theorem 2.1 (2)].

Remark 1.2.7 (a) As suggested by the final comment in the first paragraph of this section, the hypotheses of Theorem 1.2.1 (and the later results) have been chosen so as to be especially tractable. So, it should not be surprising that some of those assertions do not extend if one deletes certain hypotheses. In fact, the assertion in Theorem 1.2.1 (2) fails if one deletes the hypothesis that E is nonzero. (Perhaps the easiest way to see this is to take A to be any valuation ring that is not a domain and $E = 0$.) It seems noteworthy that each of the assertions in the statement of Theorem 1.2.2 (1)–(4) does extend to the case $E = 0$.

(b) Despite expectations that may have been aroused by the final comment in (a), one should note that the conclusions in Theorem 1.2.2 may fail if one deletes some of the attendant special hypotheses. In particular, the conclusion in Theorem 1.2.2 (2) does fail if one deletes the hypothesis that E is a divisible A-module. To see this, consider the following data. Let (A,M) be a local domain such that M is not a finitely generated ideal of A and suppose that $\mathbb{Q} \subseteq A$. (For instance, take A be a (necessarily non-Noetherian, Krull dimension 1) valuation domain whose associated valuation has a non-discrete rank 1 value group (such as $\mathbb{Q}$ or $\mathbb{R}$) and arrange also that $\mathbb{Q} \subseteq A$.) It is clear that A is a torsion-free A-module, A is an AV-ring and A is not a divisible A-module; and it is easy to check that $\mathrm{Z}(A) = 0$ $(= \mathrm{Nil}(A))$. Hence, by Theorem 1.2.2 (3), $R := A \ltimes A$ is not an AV-ring.

(c) While Corollary 1.2.4 (2) was proven above by using Theorem 1.2.2 (4), another proof of it could have been given by using Theorem 1.2.2 (2). The underlying point is that if A is a ring with a unique prime ideal P, then $\mathrm{Z}(A) = P = \mathrm{Nil}(A)$, so that any A-module must be divisible.

(d) It is interesting that the final assertion in the statement of Example 1.2.6 would fail if A were a field. Indeed, if F is any field, $F \ltimes F$ is a valuation ring: cf. [76, Theorem 2.1 (2)].

The following theorem studies the possible transfer of the PV-ring property between a ring A and a trivial ring extension $A \ltimes E$.

Theorem 1.2.8 *Let A be a ring, E a nonzero A-module, and $R := A \ltimes E$. Then:*

(1) *If R is a* PV*-ring, then so is A.*

(2) *Suppose that* $\mathrm{Ann}(E) \neq 0$. *Then R is a* PV*-ring if and only if A is a local ring with maximal ideal M such that $M^2 = 0$ and $ME = 0$.*

(3) *Suppose that R is a* PV*-ring. Then A is a local ring, say with maximal ideal M; E is a divisible A-module; either A is a* PVD *or $M^2 = 0$; and for each $e, f \in E$, either $Ae \subseteq Mf$ or $Mf \subseteq Ae$.*

(4) *If E is a nonzero finitely generated A-module and R is a* PV*-ring, then A is local and its maximal ideal, say M, satisfies $M^2 = 0$.*

Proof. (1) This was established in [22, Theorem 3.1 (a)].

(2) As the assignment $P \mapsto P \ltimes E$ determines an order-isomorphism $\mathrm{Spec}(A) \to \mathrm{Spec}(R)$, we may assume, without loss of generality, that A is local, with unique maximal ideal M. Then R is local, with unique maximal ideal $N := M \ltimes E$. By hypothesis, we can pick a nonzero element $a \in \mathrm{Ann}(E)$.

Suppose first that R is a PV-ring. To show that $ME = 0$, it suffices to prove that if $\mu \in M$ and $e \in E$, then $\mu e = 0$. In fact, since R is a PV-ring, either $(a,0)R \subseteq (0,e)N$ or $(0,e)N \subseteq (a,0)R$. The former possibility cannot hold since $(0,e)N \subseteq 0 \ltimes E$ and $a \neq 0$, and so $(0,e)N \subseteq (a,0)R$. Thus $(0,e)(\mu,0) = (a,0)(b,t)$ for some $(b,t) \in R$, whence $(0,\mu e) = (ab,at) = (ab,0)$, and so $\mu e = 0$, which proves that $ME = 0$.

Next, to show that $M^2 = 0$, it suffices to prove that $mn = 0$ for all $m,n \in M$. By hypothesis, we can pick a nonzero element $g \in E$. Since R is a PV-ring, either $(0,g)R \subseteq (m,0)N$ or $(m,0)N \subseteq (0,g)R$. The former possibility cannot hold since $(m,0)N \subseteq mM \ltimes mE \subseteq M^2 \ltimes 0$ and $g \neq 0$, and so $(m,0)N \subseteq (0,g)R$. Thus $(m,0)(n,0) = (0,g)(c,s)$ for some $(c,s) \in R$, whence $mn = 0$, which proves that $M^2 = 0$.

Conversely, suppose that $M^2 = 0$ and $ME = 0$. To show that R is a PV-ring, [3, Theorem 2] reduces our task to showing that if $\xi = (d,k), \eta = (h,f) \in R$, then either $\xi R \subseteq \eta N$ or $\eta N \subseteq \xi R$. We will examine three cases. If $d \notin M$, then ξ is a unit of R, so that $\eta N \subseteq R = \xi R$. Next, if $d \in M$ and $h \notin M$, then η is a unit of R, so that $\xi R \subseteq dA \ltimes E \subseteq M \ltimes E = N = \eta N$. Lastly, if both d and h are in M, then since $M^2 = 0$ and $ME = 0$ we have $\eta N \subseteq hM \ltimes (hE + Mf) \subseteq M^2 \ltimes ME = \{(0,0)\} \subseteq \xi R$.

(3) As in the proof of (2), A is a local ring, say with maximal ideal M, so that R is a local ring with maximal ideal $N = M \ltimes E$. Let $e,f \in E$. As R is a PV-ring, either $(0,e)R \subseteq (0,f)N$ or $(0,f)N \subseteq (0,e)R$. Since $(0,e)R = 0 \ltimes Ae$ and $(0,f)N = 0 \ltimes Mf$, it follows that either $Ae \subseteq Mf$ or $Mf \subseteq Ae$.

Next, by [21, Theorem 3.1 (a)], E is a divisible A-module. Also, by [21, Theorem 3.1 (a)] (or (1)), A is a PV-ring, and so if A is a domain, then A is a PVD. Thus, it remains only to prove that either A is a domain or $M^2 = 0$.

By (2), we may assume, without loss of generality, that $\mathrm{Ann}(E) = 0$. Suppose first that $A = \mathrm{tq}(A)$. We will prove that $M^2 = 0$ in this case. It suffices to show that $am = 0$ for all $a,m \in M \setminus \{0\}$. As A is a total quotient ring, each nonunit of A is in $\mathrm{Z}(A)$. Thus, there exists a nonzero element $b \in M$ such that $ba = 0$. Since $b \notin \mathrm{Ann}(E)$, there exists $g \in E$ such that $bg \neq 0$. Since R is a PV-ring, either $(0,g)R \subseteq (a,0)N$ or $(a,0)N \subseteq (0,g)R$. The former possibility cannot hold, for if $(0,g) = (a,0)(c,d)$ for some $(c,d) \in N$, then $g = ad$, whence $bg = (ba)d = 0 \cdot d = 0$, a contradiction. Thus $(a,0)N \subseteq (0,g)R$. In particular, $(a,0)(m,0) \in (0,g)R \subseteq 0 \ltimes E$, whence $am = 0$. This completes the proof that $M^2 = 0$ if $A = \mathrm{tq}(A)$.

In the remaining case, $A \neq \mathrm{tq}(A)$. We will give an indirect proof that A is a domain in

this case. In other words, we assume that A is not a domain and we seek a contradiction. Pick $a,b \in A \setminus \{0\}$ such that $ab = 0$. As A is not a total quotient ring, M must contain some regular element d of A. Hence, since R is a PV-ring, we get that, for each $0 \neq e \in E$, either $(a,0)N \subseteq (0,e)R$ or $(0,e) \in (a,0)N$. The former possibility cannot hold, for $(0,e)R \subseteq 0 \ltimes E$ and $a \neq 0$. Therefore, $(0,e) = (a,0)(c,f)$ for some $(c,f) \in N$. It follows that $e = af$, and so $be = baf = (ab)f = 0 \cdot f = 0$; that is, $b \in \mathrm{Ann}(E)$. However, since R is a PV-ring, we also have that either $(b,0)N \subseteq (0,e)R$ or $(0,e) \in (b,0)N$. We see as above that the former condition cannot hold (since $b \neq 0$). Thus, for each $0 \neq e \in E$, there exists $(d,h) \in N$ such that $(0,e) = (b,0)(d,h)$. Then $e = bh \in bE = 0$, the desired contradiction.

(4) As above, A is a local ring, say with maximal ideal M, and R is local with maximal ideal $N = M \ltimes E$. Suppose the assertion fails. Then, by (2) and the proof of (3), A is a PVD but not a field. Pick nonzero elements $d \in M$ and $e \in E$. Since R is a PV-ring, either $(d,0)N \subseteq (0,e)R$ or $(0,e) \in (d,0)N$. As above, the former condition cannot hold (since $d \neq 0$). Hence $(0,e) = (d,0)(c,f)$ for some $(c,f) \in N$. Therefore, $e = df \in ME$. Consequently $E = ME$. Then Nakayama's Lemma gives $E = 0$, the desired contradiction. This completes the proof.

Corollary 1.2.9 *Let A be a domain. Then:*

(1) *Let E be a nonzero finitely generated A-module. Then $R := A \ltimes E$ is a* PV*-ring if and only if A is a field.*

(2) *$A \ltimes A$ is a* PV*-ring if and only if A is a field.*

Proof. As (2) is a special case of (1), we need only prove (1). If R is a PV-ring, then Theorem 1.2.8 (4) gives that the domain A is local and its maximal ideal M satisfies $M^2 = 0$, whence $M = 0$ and A is a field. Conversely, if A is a field, then E is a divisible A-module, and so (regardless of whether E is finitely generated as an A-module) an application of [22, Theorem 3.1 (b)] yields that R is a PV-ring. This completes the proof.

In [41, Proposition 4.2], Dobbs proved that the integral closure of a finite-conductor pseudo-valuation domain is a valuation domain. As Example 1.2.11 illustrates, that part of the statement of [41, Proposition 4.2] cannot be extended beyond domains to the context of trivial ring extensions. We isolate the reasoning behind that example as the parts of the next result. To appreciate the hypotheses of Proposition 1.2.10, note that the assertion in Proposition 1.2.10 (3) would fail for $E = 0$ if the hypotheses were weakened to allow A to be a field.

Proposition 1.2.10 *Let (A,M) be a local ring, but not a field, such that $M^2 = 0$. Let E be an A-module such that $ME = 0$. Put $R := A \ltimes E$. Then:*

(1) *R is a* PV*-ring.*

(2) *Suppose that M is a finitely generated ideal of A and E is a finitely generated A-module. Then R is a finite-conductor ring.*

(3) *Suppose that M is the only prime ideal of A and E is a nonzero finitely generated A-module. Then $R' = R$ is not a valuation ring.*

Proof. The hypotheses ensure that $\mathrm{Ann}(E) \neq 0$. Thus, (1) follows from Theorem 1.2.8 (2).

(2) As (A,M) is a local ring, so is (R,N), where $N := M \ltimes E$. Moreover, N is a finitely generated ideal of R, for if $M = \sum Am_i$ and $E = \sum Ae_j$, then $N = \sum_i R(m_i,0) + \sum_j R(0,e_j)$. Also, since $M^2 = 0$ and $ME = 0$, it follows that $N^2 = 0$. Hence, by [75, Example 2.4], R is a finite-conductor ring.

(3) Since $M \ltimes E$ is the only prime ideal of R, it follows from [78, Theorem 84] that $\mathrm{tq}(R) = R$. It is then trivially the case that $R' = R$. On the other hand, Theorem 1.2.1 (2) yields that R is not a valuation ring. The proof is complete.

Example 1.2.11 *Let $A := \mathbb{Z}/4\mathbb{Z}$ and $E = A/M$ ($\cong \mathbb{Z}/2\mathbb{Z}$). Put $R := A \ltimes E$. Then R is a finite-conductor ring and a* PV*-ring, but $R = R'$ is not a valuation ring.*

Remark 1.2.12 (a) We pause to consider a particular kind of base ring that was encountered in the statement of Theorem 1.2.8 (3), namely, a local ring whose maximal ideal M satisfies $M^2 = 0$. This kind of ring has attracted attention (cf. [75, Example 2.4]), but we will focus here on the case where A takes the form of a trivial ring extension $A := K \ltimes V$, where K is a field and V is a K-vector space. Note that A is a field $\Leftrightarrow$ A is a domain $\Leftrightarrow$ $V = 0$. So, we will assume henceforth that $V \neq 0$. Then, apart from being quite accessible, such a ring A is among the simplest non-domain examples having the property of being both an AV-ring and a PV-ring. We verify that property in the next paragraph, which will also discuss some additional features of such rings A.

If $\dim_K(V) < \infty$, then $A := K \ltimes V$ is a finitely generated K-module and hence a Noetherian ring, in fact, an Artinian ring by [96, Theorem 2, page 203] since A has Krull dimension 0. On the other hand, if $\dim_K(V)$ is infinite, then A is not a Noetherian ring, since its maximal ideal $0 \ltimes V$ is not finitely generated in this case. Regardless of whether $\dim_K(V)$ is finite, the fact that K is a field ensures, thanks to [84, Theorem 2.1 (2)], that $A = K \ltimes V$ is an AV-ring. Moreover, since every vector space is a divisible module, [22, Theorem 3.1 (b)] ensures that A is also a PV-ring (regardless of whether $\dim_K(V)$ is finite). However, A is typically not a valuation ring. Indeed, [76, Theorem 2.1] shows that, although $K \ltimes K$ is a valuation ring, it is the case that $K \ltimes V$ is not a valuation ring whenever $2 \leq \dim_K(V) \leq \infty$. Another context where $\dim_K(V) = 1$ implies anomalous behavior will be discussed in the next paragraph.

The case where V is a nonzero finite-dimensional K-vector space merits additional attention, as the (Noetherian) ring $A = K \ltimes V$ then satisfies an additional finiteness condition. Indeed, when $1 \leq \dim_K(V) < \infty$, the ring extension $K \subset A$ satisfies the FC property (also known as the FCP): every chain of K-subalgebras of A is finite. This can be seen by applying [47, Theorem 4.2 (a)], as $K/(K:A) = K/0 \cong K$ is an Artinian ring and A is a finitely generated K-module. However, the ring extension $K \subset A$ typically does not satisfy the (stronger) FIP property; that is, there are typically infinitely many K-subalgebras of A. Surely, the easiest way to see that is first to note, using [43, Remark 2.9], that the K-subalgebras of A are the rings of the form $K \ltimes W$ as W runs through the K-vector subspaces of V and then to observe that there are infinitely many such W (and hence infinitely many such $K \ltimes W$) if K is an infinite field and $\dim_K(V) \geq 2$. For a more esoteric way to establish the failure of FIP to hold under these conditions, observe that A is the seminormalization of K in A and apply the characterization of FIP for integral extensions of local rings [47, Theorem 5.18]. In fact, that characterization shows that if $1 \leq \dim_K(V) < \infty$, then $K \subset A = K \ltimes V$ satisfies FIP if and only if either K is finite or (in case K is infinite) there exist $\alpha, \beta \in A$ such that $A = K[\alpha]$, $K[\alpha^2] = K[\beta]$ and $\alpha^3 = 0 = \beta^3$. The reader is invited to confirm directly that if K is infinite (and $V \neq 0$), the existence of such α and β is equivalent to $\dim_K(V) = 1$.

When $1 \leq \dim_K(V) < \infty$, one can use a method involving iterated idealizations (that is, iterated constructions of trivial ring extensions), as in [8, Proposition 2.3], to provide another interesting and fruitful way to view $A = K \ltimes V$ up to K-algebra isomorphism. That method can be applied more generally than in our context, in particular, for ring extensions of the form $\Lambda \subset \Lambda \ltimes E$ where Λ is any ring and E is any nonzero finitely generated free Λ-module. For finite-dimensional K-vector spaces V, (b) treats another, somewhat more elementary, way to view $K \ltimes V$ as a K-algebra.

(b) In seeking an example of a local ring whose maximal ideal M is nonzero and satisfies $M^2 = 0$, a reader may not have initially thought of using the above $K \ltimes V$ construction. It would be quite natural to instead proceed as follows. Let K be a field, let I be a nonempty index set, and let $\{X_i \mid i \in I\}$ be a set of commuting, algebraically independent indeterminates over K. Consider the polynomial ring $S = K[\{X_i \mid i \in I\}]$, let J be the ideal of S generated by the set $\{X_iX_j \mid i, j \in I\}$, and put $B := S/J$. For each $i \in I$, put $x_i := X_i + J$. It is clear that $x_ix_j = 0$ for all $i, j \in I$, and so $B = K[\{x_i \mid i \in I\}] = K \oplus \oplus_{i \in I} Kx_i$. In addition, B is local with unique maximal ideal $N := \oplus_{i \in I} Kx_i$ and $N^2 = 0$. Of course, B is not a domain, since $0 \neq x_i \in \mathrm{Nil}(B)$ for each $i \in I$. One could document, point-by-point, that B has all the ring-theoretic properties of A that were noted in (a). But suffice it to say that if $\{v_i \mid i \in I\}$ is a (nonempty) basis of V as a K-vector space, then B is K-algebra isomorphic to A via the function $B \to A$ given by

$$\xi + \sum \xi_i x_i \mapsto (\xi, 0) + \sum_i (0, \xi_i v_i) = (\xi, \sum \xi_i v_i),$$

where all $\xi, \xi_i \in K$. (This isomorphism was noted, for the more general context $\Lambda \subset \Lambda \ltimes E$

where E is a nonzero finitely generated free module over a ring Λ, in [8, Proposition 2.2].) Perhaps it is worthwhile to draw the following moral explicitly. A ring that is not presented (pun intended) as a trivial ring extension may, in fact, be (isomorphic to) a trivial ring extension.

(c) Viewing A as B via the above isomorphism permits certain calculations that may seem more natural to the reader than their counterparts when one works with A. For instance, one could give the following *ad hoc* proof that B is a PV-ring. Suppose that $a,b \in B$ with $aR \not\subseteq bR$. Then b is not a unit of R; that is, $b \in N$. Hence $bN \subseteq N^2 = \{0\} \subseteq aN$, which completes the proof.

(d) We next comment on the possibility of some natural PV-ring-theoretic generalizations of some of the results in (a). We saw there that if K is a field, then the K-algebra $K \ltimes K$ ($\cong K[X]/X^2$) exhibits singular behavior. Consider the ring extension $K \subset K \ltimes K$ (viewed as an inclusion via the injection ring homomorphism given by $\xi \mapsto (\xi,0)$ for all $\xi \in K$). Not only does $K \subset K \ltimes K$ satisfy FIP (and FCP), the fact that $\dim_K(K \ltimes K) = 2$ implies that $K \subset K \ltimes K$ is a minimal ring extension, in the sense of [59].

Suppose now that A is a ring and E is a A-module. View $A \subseteq A \ltimes E$ as above (via $a \mapsto (a,0)$). Then, by [43, Theorem 2.4], $A \subset A \ltimes E$ is a minimal ring extension if and only if there exists a maximal ideal M of A such that $E \cong A/M$ as an A-module. We will apply this next in case A is a domain.

Now, suppose that D is a domain, but not a field, and set $L := \mathrm{qf}(D)$. We claim that there cannot exist a D-module E such that $D \ltimes E$ is a PV-ring and $D \subset D \ltimes E$ is a minimal ring extension. Suppose the claim fails. Then, by [22, Theorem 3.1], E must be a nonzero L-vector space, and so E is a divisible D-module. However, as recalled above from [43], there must exist a maximal ideal M of D such that $E \cong D/M$ as a D-module. Since D is not a field, $M \neq 0$. As $ME = 0$, it follows that E is not a divisible D-module, the desired contradiction, thus proving the above claim.

In fact, more is true: for any domain D which is not a field and any E that is a nonzero vector space over $L := \mathrm{qf}(D)$, it follows from [47, Theorem 4.2 (a)] that $D \subset D \ltimes E$ does not satisfy FCP. To see this, we need only prove that such a $D \ltimes E$ cannot be a finitely generated D-module. Suppose the assertion fails. Then it follows easily that $E = \sum_{i=1}^{n} De_i$ for some finite nonempty set $\{e_i\}$. It also follows that $\dim_L(E) < \infty$. Without loss of generality, we can identify $E = \oplus_{j=1}^{m} L$ for some positive integer m. View each e_i as an m-tuple with entries from L. Then, by clearing denominators, we see that for each i, $1 \leq i \leq n$, there exists a nonzero element $d_i \in D$ such that $d_i e_i \in \oplus_{j=1}^{m} D$. Hence, $d := \prod_{i=1}^{n} d_i \in D \setminus \{0\}$ satisfies

$$\oplus_{j=1}^{m} dL = d \oplus_{j=1}^{m} L = dE = d \sum_{i=1}^{n} De_i = \sum_{i=1}^{n} Dde_i \subseteq \oplus_{j=1}^{m} D,$$

so that $dL \subseteq D$; that is, $L \subseteq d^{-1}D$. Thus $d^{-2} = d^{-1}\delta$ for some $\delta \in D$. In fact, $d^{-1} = d^{-2}d = \delta \in D$, whence $L \subseteq d^{-1}D \subseteq D$. Therefore, $D = L$ is a field, the desired contradiction, which completes the proof.

The following theorem studies the possible transfer of the "pseudo-almost valuation ring" property between a ring A and a trivial ring extension $A \ltimes E$.

Theorem 1.2.13 *Let A be a ring, E an A-module, and $R := A \ltimes E$. Then:*

(1) *If R is a* PAV*-ring, then so is A.*

(2) *Suppose that $Z(A) = \mathrm{Nil}(A)$ and E is a divisible A-module. Then R is a* PAV*-ring if and only if A is a* PAV*-ring.*

(3) *Suppose that A is a local ring with maximal ideal M such that $M = \sqrt{\mathrm{Ann}(E)}$. Then R is a* PAV*-ring if and only if A is a* PAV*-ring.*

(4) *Suppose that $A \subseteq E$ is an extension of domains and $\mathbb{Q} \subseteq E$. Then R is a* PAV*-ring if and only if A is a* PAV*-ring and $\mathrm{qf}(A) \subseteq E$.*

Proof. (1) If R is a PAV-ring, then R is a local ring with maximal ideal $N = M \ltimes E$ such that M is the maximal ideal of A. To show that A is a PAV-ring, we must prove that if $a, b \in A$, then there exists a positive integer n such that either $a^n \in b^n A$ or $b^n M \subseteq a^n M$. This, in turn, follows easily from the fact that R being a PAV-ring supplies a positive integer n such that either $(a,0)^n \in (b,0)^n R$ or $(b,0)^n N \subseteq (a,0)^n N$, the underlying point being that $(c,0)^n = (c^n,0)$ for all $c \in A$.

(2) By (1), it suffices to prove the "if" assertion. Suppose that A is a PAV-ring, $Z(A) = \mathrm{Nil}(A)$ and E is a divisible A-module. Then A is local, say with maximal ideal M. Hence R is local, with maximal ideal $N = M \ltimes E$. We must show that if (a,e) and (b,f) are in R, then there exists a positive integer n such that either $(a,e)^n \in (b,f)^n R$ or $(b,f)^n N \subseteq (a,e)^n N$. We will consider two cases. The first case uses the fact that if $(c,g) \in R$ and $k \geq 2$, then $(c,g)^k = (c^k, kc^{k-1}g)$. In the first case, either a or b is in $Z(A)$. Then either $a^p = 0$ or $b^q = 0$ for some positive integers p and q, so that either $(a,e)^{p+1} = (0,0) \in (b,f)^{p+1} R$ or $(b,f)^{q+1} N = \{(0,0)\} \subseteq (a,e)^{q+1} N$. In the second case, a and b are each regular elements of A. Then, since A is a PAV-ring and E is a divisible A-module, reasoning as in the proof of [72, Proposition 3.15 (2)] produces n with the desired behavior.

(3) By (1), it suffices to prove the "if" assertion. As (A,M) is a (nonzero) local ring such that $M = \sqrt{\mathrm{Ann}(E)}$, we have that ($E \neq 0$ and) R is a local ring with maximal ideal $N = M \ltimes E$. Let $\xi = (a,e), \eta = (b,f) \in R$. If b (resp., a) is a unit of A, then η (resp., ξ) is a unit of R, and so $\xi^1 \in R = \eta^1 R$ (resp., $\eta^1 N \subseteq N = \xi^1 N$). Thus, we may assume, without loss of generality, that both a and b are in M. As $M = \sqrt{\mathrm{Ann}(E)}$, there exist positive integers m and p such that $a^m \in \mathrm{Ann}(E)$ and $b^p \in \mathrm{Ann}(E)$. Since A is a PAV-ring, there exists a positive integer n such that either $a^{(mp+1)n} \in b^{(mp+1)n} A$ or $b^{(mp+1)n} M \subseteq a^{(mp+1)n} M$. If $a^{(mp+1)n} = b^{(mp+1)n} r$ for some $r \in A$, then $(a,e)^{(mp+1)n} = (a^{mp+1}, (mp+1)a^{mp} e)^n = (a^{mp+1},0)^n = (a^{(mp+1)n},0) = (b^{(mp+1)n} r, 0) = (b^{(mp+1)n},0)(r,0) = (b,f)^{(mp+1)n}(r,0) \in$

$(b,f)^{(mp+1)n}R$. In the remaining case, $b^{(mp+1)n}M \subseteq a^{(mp+1)n}M$. Then, for each $m \in M$, we get $b^{(mp+1)n}m = a^{(mp+1)n}s$ for some $s \in M$. Hence, for each $(m,t) \in N$, we have

$$(b,f)^{(mp+1)n}(m,t) = (b^{(mp+1)n},0)(m,t) = (b^{(mp+1)n}m, b^{(mp+1)n}t) = (b^{(mp+1)n}m,0)$$

$$= (a^{(mp+1)n}s,0) = (a^{(mp+1)n},0)(s,0) = (a,e)^{(mp+1)n}(s,0) \in (a,e)^{(mp+1)n}N.$$

This completes the proof that R is a PAV-ring.

(4) Without loss of generality, A is local, say with maximal ideal M, so that R is local with maximal ideal $N = M \ltimes E$. The "if" assertion follows from (2) because the hypothesis that $\mathrm{qf}(A) \subseteq E$ ensures that E is a divisible A-module.

We turn to the "only if" assertion. As R is now assumed to be a PAV-ring, (1) reduces our task to proving that $\mathrm{qf}(A) \subseteq E$. To that end, it suffices to show that if $0 \neq a \in M$, then a is a unit of E, that is, that $a^{-1} \in E$. Since R is a PAV-ring, there exists a positive integer n such that either $(a^2,0)^nR \subseteq (a^2,1)^nR$ or $(a^2,1)^nN \subseteq (a^2,0)^nN$. If $(a^2,0)^n = (a^2,1)^n(c,t)$ for some $(c,t) \in R$, then

$$(a^{2n},0) = (a^2,1)^n(c,t) = (a^{2n}, na^{2n-2})(c,t) = (a^{2n}c, a^{2n}t + na^{2n-2}c),$$

so that $a^{2n} = a^{2n}c$ and $a^{2n}t + na^{2n-2}c = 0$. Since $A \subseteq E$ are domains, we get $c = 1$ and $a^2t + n = 0$, whence $a(-atn^{-1}) = 1$, so that $a^{-1} = -atn^{-1} \in E$, as desired. In the remaining case, $(a^2,1)^nN \subseteq (a^2,0)^nN$. As $a \in M$, we then have $(a^2,1)^n(a,0) = (a^2,0)^n(d,e)$ for some $(d,e) \in N$. Hence

$$(a^{2n}, na^{2n-2})(a,0) = (a^{2n},0)(d,e),$$

so that $a^{2n+1} = a^{2n}d$ and $na^{2n-1} = a^{2n}e$. Consequently, $a = d$ and $n = ae$, whence $a^{-1} = n^{-1}e \in E$, as desired. The proof is complete.

The next two corollaries are immediate applications of parts (2) and (3) of Theorem 1.2.13.

Corollary 1.2.14 (*Cf. Jahani-Nezhad and Khoshayand [72, Proposition 3.15]*) *Let A be a domain and E a divisible A-module. Then $A \ltimes E$ is a* PAV*-ring if and only if A is a* PAV*-ring* (*that is, if and only if A is a* PAVD).

Corollary 1.2.15 *Let (A,M) be a local ring and E an A-module such that $ME = 0$. Then $A \ltimes E$ is a* PAV*-ring if and only if A is a* PAV*-ring.*

In [20, Proposition 4.11], Badawi proved that if (R,M) is a PAVD such that M is a finitely generated ideal of R (for instance, such that R is Noetherian), then R' is a valuation domain. The next example shows that [20, Proposition 4.11] does not extend from domains to the context of trivial ring extensions.

Example 1.2.16 *Let (A,M) be a Noetherian valuation domain, but not a field, and let E be a nonzero finitely generated A-module such that $ME = 0$. Then $R := A \ltimes E$ is a Noetherian* PAV*-ring, but $R' = R$ is not a valuation ring.*

Proof. It follows from Corollary 1.2.15 that R is a PAV-ring. To show that R is a Noetherian ring, one can apply Cohen's criterion (cf. [78, Theorem 8]) as follows: if $\{a_i\}$ generates a prime ideal P of A and $\{e_j\}$ generates E as an A-module, then $\{(a_i,0)\} \cup \{(0,e_j)\}$ generates $P \ltimes E$ as an ideal of R. Next, [76, Theorem 2.1 (2)] ensures that R is not a valuation ring, since A is not a field. It remains only to show that $R = R'$. Thus, it suffices to prove that $\mathrm{tq}(R) = R$. This, in turn, will be shown to follow from the description of the total quotient ring of an idealization in [69, Lemma 25.4 and Theorem 25.5 (1)]. To see this, let S denote the complement in A of the set of zero-divisors on E. Then $S = A \setminus M$, the set of units of A. Thus, we can identify $A_S = A$ as A-algebras and $E_S = E$ as A-modules. However, since A is a domain, the cited results from [69] identify $\mathrm{tq}(R)$ (as an A-algebra) with $A_{\mathscr{S}} \ltimes E_{\mathscr{S}}$, that is, with $A \ltimes E = R$. This completes the proof.

Theorem 1.2.13 enriches the literature with new examples of PAV-rings, as is illustrated next by Examples 1.2.17 and 1.2.18.

Example 1.2.17 *Let F be a finite field and X an indeterminate over F. Put $H := F(X)$, the quotient field of $F[X]$, and let Y be an analytic indeterminate over H. Set $R := F + HY^2 + Y^4H[[Y]]$, $K := \mathrm{qf}(R)$, $A := R \ltimes K$ and $E := K \ltimes K$. As A is a subring of E, we can view E as an A-module. Then $A \ltimes E$ is a* PAV*-ring which is neither a domain nor an* AV*-ring.*

Proof. By [73, Example 2.6], A is a PAV-ring which is not an AV-ring. Since $\mathrm{Z}(A) = 0 \ltimes K = \mathrm{Nil}(A)$ and E is a divisible A-module, Theorem 1.2.13 (2) gives that $A \ltimes E$ is a PAV-ring. Also, as in the proof of Example 1.2.5, $A \ltimes E$ is not a domain. Finally, as A is not an AV-ring, [84, Theorem 2.1 (1)] gives that $A \ltimes E$ is not an AV-ring. This completes the proof.

Example 1.2.18 *Let (A,M) be an* SPIR *(for instance, $\mathbb{Z}/8\mathbb{Z}$, with $M = 2A$) and let E be an A-module. Then $A \ltimes E$ is a* PAV*-ring.*

Proof. As in Remark 1.2.7 (c), we have $\mathrm{Z}(A) = \mathrm{Nil}(A)$ and every A-module is divisible, since M is the only prime ideal of A. Moreover, since A is an SPIR, it is a valuation ring, hence a PAV-ring. Therefore, an application of Theorem 1.2.13 (2) completes the proof.

Following [72, Definition 3.7], we will say that a (commutative) ring A is *root closed* if, whenever $a,b \in A$ and $a^nA \subseteq b^nA$ for some positive integer n, then $aA \subseteq bA$. The reader is advised that another, older definition of the term "root closed ring" appears widely in

the literature and that the definitions agree if A is a domain. Any root closed PAVD (resp., any root closed PAV-ring) must be a PVD (resp., a PV-ring) [20, Theorem 2.13] (resp., [72, Theorem 3.8]). Another assertion in the same spirit will be given in Proposition 1.2.19. Besides having accommodated [72, Theorem 3.8], another advantage of adopting the definition of "root closed ring" from [72] is that the proof of Proposition 1.2.19 follows immediately from the definitions.

Proposition 1.2.19 *Any root closed* AV*-ring is a valuation ring.*

Next, we extend the notion of "almost pseudo-valuation domain" to the setting of arbitrary rings. A ring R is said to be an *almost pseudo-valuation ring* (in short, an APV-ring) if, for every maximal ideal M of R and any two elements $a,b \in R$, either $a^nR \subseteq b^nM$ for some positive integer n or $bM \subseteq aM$. It is clear from the definitions that any APV-ring is a PAV-ring; and that a domain is an APV-ring if and only if it is an APVD. The next result shows that any APV-ring is local and gives some easy characterizations of such rings.

Proposition 1.2.20 *For any ring R, the following three conditions are equivalent:*

(1) R is an APV*-ring;*

(2) R is a local ring with maximal ideal M such that, for any two elements $a,b \in R$, either $a^nR \subseteq b^nM$ for some positive integer n or $bM \subseteq aM$;

(3) R is a local ring with maximal ideal M such that, for any two elements $a,b \in R$, either $a^nR \subseteq b^nM$ for some positive integer n or $bM \subseteq aR$.

Proof. It is clear that (2) $\Rightarrow$ (1) and that (2) $\Rightarrow$ (3).

(1) $\Rightarrow$ (2): It suffices to prove that R is local. Suppose the assertion fails. Pick distinct maximal ideals, say M and M', of R. Pick any $a \in M' \setminus M$, and set $b := 1 \in R$. As R is an APV-ring, either $a^nR \subseteq b^nM$ $(= M)$ for some positive integer n or $(M =)$ $bM \subseteq aM$ $(\subseteq M')$. As $M \not\subseteq M'$, we get $a^nR \subseteq M$, whence $a \in M$, the desired contradiction.

(3) $\Rightarrow$ (2): Suppose the assertion fails, with (R,M) local. Then there exist $a,b \in R$ such that $a^nR \not\subseteq b^nM$ for each positive integer n, $bM \subseteq aR$, and $bM \not\subseteq aM$. It follows that there exist $m \in M$ and $r \in R \setminus M$ such that $bm = ar$. As M is the unique maximal ideal of R, r must be a unit of R. Thus $a = br^{-1}m \in bM$, whence $a^1R \subseteq b^1M$, the desired contradiction. The proof is complete.

The next result identifies an important class of APV-rings and then shows that, within the universe of root closed rings, the APV-rings are the same as the PV-rings. Note that Proposition 1.2.21 (2) also follows from the result of Jahani-Nezhad and Khoshayand [72, Theorem 3.8] that any root closed PAV-ring is a PV-ring.

Proposition 1.2.21 *(1) Any* APV*-ring is an* PAV*-ring.*
(2) Let R be a root closed ring. Then R is a APV*-ring if and only if R is a* PV*-ring.*

Proof. We noted (1) above. Its statement is included here for the sake of completeness and because the "if" assertion of (2) is a special case of (1). We next prove the "only if" assertion of (2). Assume that R is a root closed APV-ring. By Proposition 1.2.20, R is a local ring, say with maximal ideal M. To prove that R is a PV-ring, we will show that if $a,b \in R$, then either $aR \subseteq bM$ or $bM \subseteq aR$. As R is a APV-ring, there exists a positive integer n such that either $a^nR \subseteq b^nM$ or $bM \subseteq aM$. As $bM \subseteq aR$, we may suppose, without loss of generality, that $a^nR \subseteq b^nM$ ($\subseteq b^nR$). Then the hypothesis that R is root closed yields $r \in R$ such that $a = br$. Without loss of generality, $r \in R \setminus M$; that is, r is a unit of R. Hence $bM = (ar^{-1})M = a(r^{-1}M) = aM \subseteq aR$, as desired. The proof is complete.

Next, we give an easy (but useful) result that is the APV-theoretic analogue of a result of Jahani-Nezhad and Khoshayand [72, Proposition 3.10] on PAVRs.

Proposition 1.2.22 *Let R be an* APV*-ring and I an ideal of R. Then R/I is an* APV*-ring.*

Proof. By Proposition 1.2.20, R is a local ring, say with maximal ideal M. Then $N := M/I$ is the unique maximal ideal of $\overline{R} := R/I$. Our task is to prove that if $\xi, \eta \in \overline{R}$, then either $\xi^n\overline{R} \subseteq \eta^n N$ for some positive integer n or $\eta N \subseteq \xi N$. Pick $a,b \in R$ such that $a+I = \xi$ and $b+I = \eta$. Since R is an APV-ring, either $a^nR \subseteq b^nM$ for some positive integer n or $bM \subseteq aM$. Observe that

$$a^nR \subseteq b^nM \Rightarrow \xi^n\overline{R} = (a^nR+I)/I \subseteq (b^nM+I)/I = \eta^nN;$$

and $bM \subseteq aM \Rightarrow \eta N = (bM+I)/I \subseteq (aM+I)/I = \xi N$. The proof is complete.

The next result identifies some contexts for which one can characterize when a trivial ring extension is an almost pseudo-valuation ring.

Theorem 1.2.23 *Let A be a ring, E an A-module and $R := A \ltimes E$. Then:*

(1) If R is an APV*-ring, then so is A.*

(2) Suppose that $\mathrm{Z}(A) = \mathrm{Nil}(A)$ and E is a divisible A-module. Then R is an APV*-ring if and only if A is an* APV*-ring.*

(3) Suppose that A is a local ring whose maximal ideal M satisfies $ME = 0$. Then R is an APV*-ring if and only if A is an* APV*-ring.*

(4) Let $A \subseteq E$ be an extension of domains. Then R is an APV*-ring if and only if A is an* APV*-ring and $\mathrm{qf}(A) \subseteq E$.*

Proof. (1) Since $R/(0 \ltimes E) \cong A$, the assertion follows from Proposition 1.2.22.

(2) By (1), it suffices to prove the "if" assertion. Suppose that A is a APV-ring, $\mathrm{Z}(A) = \mathrm{Nil}(A)$ and E is a divisible A-module. By Proposition 1.2.20, A is local, say with maximal ideal M. Hence R is local, with maximal ideal $N = M \ltimes E$. According to Proposition 1.2.20, it will suffice to show that if $\xi = (a,e)$ and $\eta = (b,f)$ are in R, then either $\xi^n \in \eta^n N$ for some positive integer n or $\eta N \subseteq \xi R$.

Suppose first that $a \in \mathrm{Z}(A)$ $(= \mathrm{Nil}(A))$. Then $a^p = 0$ for some positive integer p, and so $\xi^{p+1} = (a^{p+1}, (p+1)a^p e) = (0,0) \in \eta^{p+1}N$, as desired. Thus, we may assume henceforth that a is a regular element of A.

Suppose next that $bM \subseteq aM$. We claim that $\eta N \subseteq \xi R$, that is, that $\eta(m,g) \in \xi R$ for each $(m,g) \in N$. As $m \in M$, there exists $s \in M$ such that $bm = as$. Also, the regularity of a in A and the divisibility of E as an A-module combine to give $h \in E$ such that $bg + mf - se = ah$. Hence $\eta(m,g) = (b,f)(m,g) = (a,e)(s,h) = \xi(s,h) \in \xi R$, which proves the above claim.

Since A is a APV-ring, there is only one remaining case, namely, where $a^n \in b^n M$ for some positive integer n (and a is a regular element of A). Without loss of generality, $n \geq 2$. As a is a regular element of A, so is a^n; hence so is b^n. Also, $a^n = b^n d$ for some $d \in M$. We claim that $\xi^n \in \eta^n N$. The regularity of b^n in A and the divisibility of E as an A-module combine to give $k \in E$ such that

$$na^{n-1}e - nb^{n-1}df = b^n k,$$

and so $\xi^n = (a^n, na^{n-1}e) = (b^n, nb^{n-1}f)(d,k) \in \eta^n N$. This proves the above claim and thus completes the proof of (2).

(3) As (A,M) is local, it follows that R is local with maximal ideal $N = M \ltimes E$. By (1), it suffices to prove the "if" assertion. It will be useful to note that the hypothesis that $ME = 0$ entails that if $(c,k) \in N$, then $(c,k)N = cM \ltimes 0$ and so $(c,k)^n N = c^n M \ltimes 0 \subseteq cM \ltimes 0$ for all positive integers n. According to Proposition 1.2.20, it will suffice to show that if $\xi = (a,e)$ and $\eta = (b,f)$ are in R, then either $\xi^n \in \eta^n N$ for some positive integer n or $\eta N \subseteq \xi R$.

Suppose first that $a \notin M$ and $b \in M$. Then $\xi \notin N$, ξ is a unit of R, $\xi R = R$, and so $\eta N \subseteq R = \xi R$, as desired.

Next, suppose that $b \notin M$ and $a \in M$. Then η is a unit of R, whence $\eta^1 N = \eta N = N$, and so $\xi^1 = \xi \in N = \eta^1 N$, as desired.

By the preceding two paragraphs, we can assume, without loss of generality, that $a \in M$ and $b \in M$; that is, $\xi \in N$ and $\eta \in N$. As A is an APV-ring, there are two cases: either $a^n \in b^n M$ for some positive integer n or $bM \subseteq aM$.

Suppose that $a^n = b^n s$ for some positive integer n and some $s \in M$. It suffices to prove that $\xi^n \in \eta^n N$. Without loss of generality, $n \geq 2$. Then, since $ME = 0$, we get

$$\xi^n = (a^n, na^{n-1}e) = (a^n, 0) = (b^n s, 0) = (b^n, 0)(s,0) = \eta^n(s,0) \in \eta^n N,$$

as desired.

In the final case, $bM \subseteq aM$. It suffices to prove that $\eta N \subseteq \xi R$. As η and ξ are in N (and $ME = 0$), it follows from the above remarks that

$$\eta N = (b,f)N = bM \ltimes 0 \subseteq aM \ltimes 0 = (a,e)N = \xi N \subseteq \xi R.$$

This completes the proof of (3).

(4) Without loss of generality, A is local, say with maximal ideal M, so that R is local with maximal ideal $N = M \ltimes E$. The "if" assertion follows from (2) because the hypothesis that $\mathrm{qf}(A) \subseteq E$ ensures that E is a divisible A-module.

We turn to the "only if" assertion. As R is now assumed to be an APV-ring, (1) reduces our task to proving that $\mathrm{qf}(A) \subseteq E$. To that end, it suffices to show that if $0 \neq a \in M$, then a is a unit of E, that is, that $a^{-1} \in E$. Since R is a APV-ring, either $(a^2,0)^n \in (a^2,1)^n N$ for some positive integer n or $(a^2,1)N \subseteq (a^2,0)N$. The former (apparent) possibility implies that some $(c,g) \in N$ satisfies $(a^2,0)^n = (a^2,1)^n(c,g)$, whence $a^{2n} = a^{2n}c$, so that $1 = c \in M$, a contradiction. Therefore $(a^2,1)N \subseteq (a^2,0)N$. In particular, there exists $(d,h) \in N$ such that $(a^2,1)(a,0) = (a^2,0)(d,h)$. It follows that $a^3 = a^2 d$ and $a = a^2 h$. We conclude first that $a = d$ and next that $1 = ah$, so that $a^{-1} = h \in E$, as desired. The proof is complete.

Parts (2) and (3) of Theorem 1.2.23 each have the following result as an immediate consequence.

Corollary 1.2.24 *Let A be a domain with quotient field K and let E be a K-vector space. Then $A \ltimes E$ is an* APV*-ring if and only if A is an* APV*-ring.*

As another consequence of Theorem 1.2.23, we next give a non-domain example of a PAV-ring that is not an APV-ring.

Example 1.2.25 *Let F,H,R,K,A and E be as in Example 1.2.17. Then $A \ltimes E$ is a* PAV*-ring but neither a domain nor a* APV*-ring.*

Proof. It was shown in Example 1.2.17 that $A \ltimes E$ is a PAV-ring which is not a domain. However, R is not an APVD, by [20, Example 3.5]. Thus, by either part (2) or part (3) of Theorem 1.2.23, A is not an APV-ring. Hence, Theorem 1.2.23 (1) ensures that $A \ltimes E$ is not an APV-ring. The proof is complete.

In [23, Proposition 3.7], Badawi and Houston proved that the integral closure of a almost pseudo-valuation domain is a pseudo-valuation domain. The next example illustrates the failure of [23, Proposition 3.7] to extend beyond domains to the context of trivial ring extensions.

Example 1.2.26 *Let $\mathbb{Q}[[t]]$ be the formal power series ring over the field $\mathbb{Q}$ of rational numbers (in an analytic indeterminate t) and let $D := \mathbb{Q}[[t^2,t^3]]$. Consider the maximal ideal $M := (t^2,t^3)$ of D. Let E be a nonzero vector space over D/M, and put $R := D \ltimes E$. Then R is an* APV*-ring and R is integrally closed (that is, $R' = R$), but R is not a* PV*-ring.*

Proof. It was shown by Chang, Nam and Park [32, Example 2.1] that D is an APVD with (unique) maximal ideal M and that D is not a PVD. As $ME = 0$, it follows from Theorem 1.2.23 (3) that R is an APV-ring. Next, since $E \neq 0$, $M \neq 0$ and $ME = 0$, it follows that E is not divisible as a D-module. Therefore, [22, Theorem 3.1 (1)] ensures that R is not a PV-ring. It remains only to show that $R' = R$. Thus, it suffices to prove that $\mathrm{tq}(R) = R$. This, in turn, can be done as in the proof of Example 1.2.16. The proof is complete.

Much of the motivation for the classes of rings studied in this chapter (and in several other papers) comes from earlier work on going-down domains (as introduced by Dobbs [39] and characterized by Dobbs and Papick [46, Theorem 1]) and on locally divided domains (as introduced by Dobbs [40, Remark 2.7 (b)]), as well as subsequent work on going-down rings (as introduced by Dobbs [42]). By definition, a ring R is a *going-down ring* if R/P is a going-down domain for all prime ideals P of R. A domain is a going-down ring if and only if it is a going down domain [42, Remark (a), page 4]. There is a clear connection between some of the themes in this chapter (and in some earlier papers) and work on going-down domains. One such connection is the result of Badawi and Dobbs [21, Proposition 3.8] that a trivial ring extension $A \ltimes E$ is a going-down ring if and only if A is a going-down ring. Another such connection is the fact that [42] argued strongly for the condition "$\mathrm{Z}(A) = \mathrm{Nil}(A)$" (often in the equivalent form "0 is a primary ideal of R" [42, Proposition 2.3 (a)]) as a ring-theoretic context in which one could move beyond domain-theoretic work. Specifically, [42, Corollary 2.6] established that if $\mathrm{Z}(A) = \mathrm{Nil}(A)$, then A is a going-down ring if and only if $A \subset B$ satisfies GD for each overring B of A. (As usual, by an *overring* of a ring A, we mean any A-subalgebra of $\mathrm{tq}(A)$; and, following [78, page 28], GD denotes the going-down property of ring extensions.) Subsequently, Section 2 of [21] went on to exhibit additional uses of the condition "$\mathrm{Z}(A) = \mathrm{Nil}(A)$" (and of certain related conditions that went on to enjoy extensive scrutiny in later works), while [21, Remark 3.11] expressed the need (which still exists) for parallel treatments in the absence of the "$\mathrm{Z}(A) = \mathrm{Nil}(A)$" condition. Accordingly, it seems fitting to close this chapter with results showing that each type of ring studied here (and in earlier related papers) is a going-down ring (Proposition 1.2.27) and that when such rings A satisfy "$\mathrm{Z}(A) = \mathrm{Nil}(A)$", all their overring extensions must satisfy GD (Corollary 1.2.28).

Proposition 1.2.27 *Let R be a* PAV*-ring (for instance, an* AV*-ring, a* PV*-ring, an* APV*-ring). Then R is a going-down ring.*

Proof. As for the parenthetical comment, recall that AV-ring $\Rightarrow$ PAV-ring; and PV-ring $\Rightarrow$ APV-ring $\Rightarrow$ PAV-ring. Now, let R be a PAV-ring. We must prove that if P is any

prime ideal of R, then R/P is a going-down domain. By [72, Proposition 3.10], the domain R/P is a PAV-ring and, hence, a PAVD. As it now suffices to show that every PAVD is a going-down domain, an appeal to [44, Theorem 1] completes the proof.

Corollary 1.2.28 *Let R be a* PAV*-ring* (*for instance, an* AV*-ring, a* PV*-ring, an* APV*-ring*). *Assume also that* $\mathrm{Z}(R) = \mathrm{Nil}(R)$. *Then $R \subseteq T$ satisfies* GD *for each overring T of R.*

Proof. By Proposition 1.2.27, R is a going-down ring. Therefore, since $\mathrm{Z}(R) = \mathrm{Nil}(R)$, an appeal to [42, Theorem 2.4 (b)] completes the proof.

CHAPTER 2

STRONGLY PRIMARY IDEALS IN RINGS WITH ZERO-DIVISORS

Abstract.

In this chapter*, we consider A an integral domain with quotient field K. A. Badawi and E. Houston called a strongly primary ideal I of A if whenever $x, y \in K$ and $xy \in I$, we have $x \in I$ or $y^n \in I$ for some $n \geq 1$. We study the generalization of a strongly primary ideal to the context of arbitrary commutative rings. We define a primary ideal P of a ring A to be strongly primary if for each $a, b \in A$, we have $aP \subseteq bA$ or $b^n A \subseteq a^n P$ for some $n \geq 1$.

2.1 Introduction

Recall from [65] that a prime ideal P of an integral domain A, with quotient field K is said to be strongly prime if, whenever $xy \in P$ for elements $x, y \in K$, we have $x \in P$ or $y \in P$. In [3], D. F. Anderson, A. Badawi and D.E. Dobbs generalized the study of strongly prime ideals to the context of arbitrary rings (possibly with nontrivial zero-divisors) as follows. A prime ideal P of a ring A is called strongly prime if aP and bA are comparable for all $a, b \in A$. A ring A is called a *pseudo-valuation ring* (in short, a PV-*ring*) if each prime ideal of A is a strongly prime ideal of A; equivalently, A is local and its maximal ideal is a strongly prime ideal of A. For an extensive study of strongly prime see Jahani-Nezhad's paper [71]. As a generalization of strongly prime ideal A. Badawi and E. Houston introduced the notion of strongly primary ideal in [23]. An ideal I of integral domain A, with quotient field K, is called strongly primary, if whenever $xy \in I$ with $x, y \in K$, we have $x \in I$ or $y^n \in I$ for some $n \geq 1$; equivalently, $x^{-1}I \subseteq I$ for each $x \in E(I)$. They showed that an ideal I of A is strongly primary if and only if I is a primary ideal in some valuation overring of A. Also, they showed that a strongly primary ideal

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of A is comparable to every radical ideal of A. Moreover, an integral domain A is said to be an almost pseudo-valuation domain (in short, an APVD) in case each prime ideal P of A is a strongly primary ideal; equivalently, A is local with maximal ideal M such that M is strongly primary ideal. Also, recall from [45] that an arbitrary ring A is said to be an *almost pseudo-valuation ring* (in short, an APV-ring) if, A is local with maximal ideal M such that, for any two elements $a,b \in A$, either $b^nA \subseteq a^nM$ for some positive integer n or $aM \subseteq bA$, that is, M is a strongly primary ideal of A.

In this chapter, we define a primary ideal P of a ring A to be a strongly primary ideal, if for every $a,b \in A$, $aP \subseteq bA$ or $b^nA \subseteq a^nP$ for some positive integer $n \geq 1$. Clearly every strongly prime is strongly primary. Simple example show that "strongly prime" and "strongly primary" are different notions. More precisely, we take a strongly primary ideal which is not prime. This chapter is organized as follows. The second section is dedicate to a number of results concerning strongly primary ideals. Among many results of this part, we show that if P is prime which is strongly primary ideal of A then P is comparable to $Z(A)$ and the prime ideals contained in P are linearely ordred. Also, we show that for a prime ideal P of root closed ring, P is strongly primary if and only if P is strongly prime. The third and fourth section deals with the transfer of strongly primary ideal to the pre-mentioned ring extensions in particular cases. At this point, we construct a non-trivial examples of strongly primary ideals. For the main results of this chapter, see Propositions 2.2.5, 2.2.6, 2.2.7 and Theorems 2.3.1, 2.4.1.

2.2 Basic results

In the following Proposition, we give several characterizations of strongly primary ideals in integral domains.

Proposition 2.2.1 *Let A be an integral domain and P be a primary ideal of A. The following statements are equivalent:*

1. *P is a strongly primary ideal of A.*
2. *$x^{-1}P \subseteq P$ for each $x \in E(P)$.*
3. *For every $a,b \in A$, $aP \subseteq bA$ or $b^nA \subseteq a^nP$ for some $n \geq 1$.*
4. *$P \subset xA$ for every $x \in E(P)$.*

Proof. $1 \Leftrightarrow 2)$ By [23, Lemma 2.3].

$2 \Rightarrow 3)$. Let $a,b \in A$. Suppose that $b^nA \nsubseteq a^nP$. We may assume that $a \neq 0$. Set $x = \frac{b}{a}$. Assume that there exists a positive integer n such that $x^n \in P$. Then $b^n \in a^nP$, a contradiction. Hence $x^n \notin P$ for each positive integer n. By (2), $x^{-1}P \subseteq P$ and hence $aP \subseteq bP \subseteq bA$.

$3 \Rightarrow 4$). Let $x \in E(P)$. Set $x = \frac{b}{a}$. Then $aP \subseteq bA$ since $b^nA \nsubseteq a^nP$. Hence $P \subseteq \frac{b}{a}A = xA$. Clearly, $P \neq xA$.
$4 \Rightarrow 3$). Let $a, b \in A$. If $b^nA \nsubseteq a^nP$ for each $n \geq 1$, then $x = \frac{b}{a} \in E(P)$ and so $P \subset xA$ by (4). Then we have $aP \subseteq bA$.
$3 \Rightarrow 2$). Let $x = \frac{b}{a} \in E(P)$. Then $b^nA \nsubseteq a^nP$ for each $n \geq 1$. By (3), $aP \subseteq bA$ and so $x^{-1}P \subseteq P$.

Our interest in the above proposition is to extend the notion of strongly primary to an arbitrary ring.

Definition 2.2.2 *Let A be a ring and P be a primary ideal of A. P is called strongly primary ideal if for every $a, b \in A$, $aP \subseteq bA$ or $b^nA \subseteq a^nP$ for some integer $n \geq 1$.*

Our first result extends [23, Proposition 2.1].

Proposition 2.2.3 *A primary ideal of a valuation ring is strongly primary.*

Proof. It follows immediately from the fact that in a valuation ring, every two ideals are comparable.

The following Proposition gives another class of rings in which every primary ideal is strongly primary. Recall from [29], that a ring R is said to be an UN-ring if every nonunit element a of R is a product of a unit and a nilpotent elements, or equivalently every element of R is either nilpotent or unit.

Proposition 2.2.4 *Let A be an UN-ring. Then every primary ideal is strongly primary.*

Proof. Straightforward.

Proposition 2.2.5 *Let P be a strongly primary ideal and I be a radical ideal of A. Then one of the following statements hold:*

(1) $P \subseteq I$.

(2) $I \subseteq P$.

(3) $P \backslash I \subseteq Z(A)$.

In particular, if A is an integral domain, then P and I are comparable.

Proof. Assume that $P \nsubseteq I$. Let $p \in P \backslash I$ and $x \in I$. Since P is strongly primary, we get $xP \subseteq p^2A$ or $p^{2n}A \subseteq x^nP$ for some positive integer n. If $p^{2n}A \subseteq x^nP$, then $p^{2n} \in I$ and hence $p \in I$, a contradiction. Thus, since $p \in P$ there is $c \in A$ such that $xp = p^2c$. If p is a regular element, then $x \in P$ and hence $I \subseteq P$. Now, assume that $I \nsubseteq P$. Then p is zero-divisor and so $P \backslash I \subseteq Z(A)$. This completes the proof of Proposition.

Proposition 2.2.6 *Let P be a prime ideal of a ring A which is strongly primary. Then :*

1. *P is comparable to each principal ideal.*
2. *If P_1 and P_2 are two prime ideals of A contained in P, then P_1 and P_2 are comparable.*

Proof. (1) Let $x \in A \setminus P$. Since P is strongly primary, $P \subseteq xA$ or $x^n \in P$ for some integer $n \geq 1$. As P is prime, we get $P \subseteq xA$.
(2) Suppose that $P_1 \nsubseteq P_2$ and let $a \in P_1 \setminus P_2$. Let $b \in P_2$. Since P is strongly primary, $aP \subseteq bA$ or $b^n \in a^nP$ for some positive integer n. If $aP \subseteq bA \subseteq P_2$, then $a \in P_2$, which is a contradiction. Hence, $b^n \in a^nP \subseteq P_1$, and so $b \in P_1$. Therefore, $P_2 \subseteq P_1$.

Following [72, Definition 3.7], we will say that a ring A is *root closed* if, whenever $a, b \in A$ and $a^nA \subseteq b^nA$ for some positive integer n, then $aA \subseteq bA$. The next result shows that for prime ideals, within the universe of root closed rings, the strongly primary ideals are the same as the strongly prime ideals.

Proposition 2.2.7 *Let P be a prime ideal of a root closed ring. Then P is strongly prime if and only if P is strongly primary.*

Proof. Clearly if P is strongly prime, then P is strongly primary. Now, let P be a strongly primary ideal of a root closed ring A. To prove that P is strongly prime, we will show that if $a, b \in A$, then either $bA \subseteq aP$ or $aP \subseteq bA$. As P is strongly primary, there exists a positive integer $n \geq 1$ such that either $b^nA \subseteq a^nP$ or $aP \subseteq bA$. If $b^nA \subseteq a^nP$ ($\subseteq a^nA$), then there exists $s \in A$ such that $b = as$ since A is root closed . Without loss of generality, assume that $s \notin P$. By Proposition 2.2.6, $P \subseteq sA$. Hence, $aP \subseteq asA = bA$. Therefore $bA \subseteq aP$ or $aP \subseteq bA$, as desired.

Proposition 2.2.8 *Let P be a strongly primary ideal of a ring A. Then*

1. *P is comparable to $Z(A)$.*
2. *$S^{-1}P$ is a strongly primary ideal of $S^{-1}A$ for any multiplicatively closed subset S of A such that $S \cap P = \emptyset$. Moreover, if P is prime and contains a non zero-divisor of A, then $P = P_P$ and A_P is an APV-ring.*
3. *Let $I \subseteq P$ be a proper ideals. Then P/I is a strongly primary ideal of A/I.*

Proof. (1) Assume that $P \nsubseteq Z(A)$. We show that $Z(A) \subseteq P$. Let $a \in Z(A)$. Since $Z(A)$ is a union of prime ideals of A [78, Page 3], there exists a prime ideal Q such that $a \in Q \subseteq Z(A)$. By Proposition 2.2.5, $P \subseteq Q$ or either $Q \subseteq P$ or $P \setminus Q \subseteq Z(A)$. Since $P \nsubseteq Z(A)$, we get $P \setminus Q \nsubseteq Z(A)$. Then $P \subseteq Q$ or $Q \subseteq P$. If $P \subseteq Q$, we get $P \subseteq Z(A)$, a contradiction. Then $a \in P$ and hence $Z(A) \subseteq P$.

(2) Suppose that $x,y \in S^{-1}A$ and $y^nS^{-1}A \nsubseteq x^nS^{-1}P$. Set $x = \frac{a}{s}$ and $y = \frac{b}{t}$ for some $a,b \in A$ and $s,t \in S$. Since P is strongly primary, we get $aP \subseteq bA$ or $b^nA \subseteq a^nP$. Our task is to show that $xS^{-1}P \subseteq yS^{-1}A$. Consider first the case that $aP \subseteq bA$. Let $\frac{p}{u} \in S^{-1}P$. Hence, there exists $r \in A$ such that $\frac{ap}{su} = \frac{brt}{sut} = \frac{b}{t}.\frac{rt}{su} \in \frac{b}{t}S^{-1}A$. Then $xS^{-1}P \subseteq yS^{-1}A$. In the remaining case, there exists $p \in P$ such that $y^n = \frac{b^n}{t^n} = \frac{pa^ns^n}{t^ns^n} = \frac{ps^n}{t^n}.\frac{a^n}{s^n} \in x^nS^{-1}P$, a contradiction. For the "moreover" statement, by (1), we have $Z(A) \subseteq P$. Let $\frac{p}{s} \in P_P$ with $p \in P$ and $s \in A \setminus P$. By Proposition 2.2.6, $P \subseteq sA$ and hence $\frac{p}{s} \in P_P \cap A = P$. Therefore $P_P = P$. Clearly P_P is the only maximal ideal of A_P which is strongly primary. By [45, Proposition 2.20], A_P is an APV-ring.

(3) Let $\bar{a},\bar{b} \in \bar{A} := A/I$ and $\bar{b}^n\bar{A} \nsubseteq \bar{a}^n\bar{P}$. Since P is a strongly primary ideal, we get $aP \subseteq bA$ or $b^nA \subseteq a^nP$. If $b^nA \subseteq a^nP$, we get easily $\bar{b}^n\bar{A} \subseteq \bar{a}^n\bar{P}$, a contradiction. Let $\bar{p} \in \bar{P}$. Since $ap = bc$ for some $c \in A$, we conclude that $\bar{a}\bar{p} = \bar{b}\bar{c} \in \bar{b}\bar{A}$, as desired.

Proposition 2.2.9 *Let P be a prime ideal of A which is strongly primary. Then for each $a \in P$ and $b \in A\setminus P$, $a^2 \in b^2P$.*

Proof. Let $a \in P$ and $b \in A\setminus P$. Since P is strongly primary, we get $aP \subseteq b^2A$ or $b^{2n}A \subseteq a^nP$ for some positive integer n. If $b^{2n} \in a^nP \subseteq P$ then $b \in P$, a contradiction. Then $aP \subseteq b^2A$ and hence $a^2 \in b^2A$ since $a \in P$, as desired.

Proposition 2.2.10 *Let P be a strongly primary ideal of A. Suppose that P contain a prime ideal Q of A. Then for each $q \in Q$ and $p \in P\setminus Q$ there is an $m \geq 1$ such that $q^m \in p^mQ$.*

Proof. Let $q \in Q$ and $p \in P\setminus Q$. Since P is strongly primary, we get $pP \subset qA$ or $q^m \in p^mP$. Assume that $pP \subseteq qA \subseteq Q$. Since P is prime, we get $p \in Q$ or $P \subseteq Q$, a contradiction. Then $q^m \in p^mP$ for some positive integer m. Hence $q^m = p^ma$ for some $a \in P$. Since $p^m \notin Q$, we have $a \in Q$. Therefore, $q^m \in p^mQ$, as desired.

Proposition 2.2.11 *Let P be a prime ideal of A such that $Z(A) \subseteq P$. If P is strongly primary, then for every $a,b \in A$, either $aP \subseteq bP$ or $b^nA \subseteq a^nA$ for some positive integer n.*

Proof. Let $a,b \in A$. Since P is strongly primary, we get that $aP \subseteq bA$ or $b^nA \subseteq a^nP$. If $b^nA \subseteq a^nP$, then $b^nA \subseteq a^nA$. Now, assume that $aP \subseteq bA$ and let $p \in P$. If $aP \nsubseteq bP$ then $ap = bc$ for some $c \in A\setminus P$. By Proposition 2.2.6, $P \subseteq cA$. Hence $p = cd$ for some $d \in A$. Since c is a regular element, we get that $b \in aA$ and so $b^1A \subseteq a^1A$, a contradiction.

We say that an ideal I of a ring A satisfies property $(*)$ if for every $x,y \in T(A)$ such that $xy \in I$, we have $x \in I$ or $y^n \in I$ for some integer $n \geq 1$. The ideal I is necessarily primary, and in the domain case I is strongly primary. If A is any ring with $T(A) = A$, then all primary ideals of A satisfy property $(*)$. Next, we do show that a maximal ideal which is strongly primary satisfies property $(*)$.

Theorem 2.2.12 *Let A be an APV-ring with maximal ideal M. Then M satisfies property $(*)$.*

We start by the following lemma.

Lemma 2.2.13 *Let A be an APV-ring with maximal ideal M. If $y^n \in T(A) \setminus M$ for each integer $n \geq 1$, then $y^{-1} \in (M : M)$.*

Proof. Put $y = \frac{a}{b} \in T(A)$ such that $a \in A$ and b is a regular element of A. Then $bM \subseteq aA$ or $a^n \in b^n M$ for some integer $n \geq 1$ since M is strongly primary. As $y^n \notin M$ for each $n \geq 1$, we get $bM \subseteq aA$. Assume that $bM \nsubseteq aM$, then for each $t \in M$ we have $bt = as$ for some $s \notin M$. Thus, s is unit and so $a \in bM$ which is a contradiction. Therefore $bM \subseteq aM$. Since $T(A) \neq A$, M contains a regular element m. Hence $bm = at$ for some $t \in A$. Note that a and t are regular elements since b and m are regular elements. Then $y^{-1} = \frac{b}{a} \in T(A)$, and $\frac{b}{a}M \subseteq M$. Hence $y^{-1} \in (M : M)$.

Proof of Theorem 2.2.12. Let $x, y \in T(A)$ such that $xy \in M$. Suppose that $y^n \notin M$ for each integer $n \geq 1$. By Lemma 2.2.13, $y^{-1}M \subseteq M$. Then $x = xyy^{-1} \in M$ and this complete the proof of the Theorem.

We next determine necessary and sufficient conditions for a maximal ideal M which satisfies property $(*)$ to be strongly primary.

Theorem 2.2.14 *Let A be a local ring with maximal ideal M satisfies property $(*)$. Then M is strongly primary if and only if the following statements hold,*

1. *$Z(A)$ is a prime ideal of A and is comparable to each principal ideal of A.*
2. *For all $a, b \in Z(A)$, we have $aM \subseteq bA$ or $b^n \in a^n M$ for some integer $n \geq 1$.*

Proof. Assume that M is strongly primary. Clearly, $Z(A)$ is prime since $Z(A)$ is a union of prime ideals of A [78, Page 3], and the set of all prime ideals of A is linearly ordered by Proposition 2.2.6. Now let $a \in Z(A)$, and $b \in A \setminus Z(A)$. We wish to show that $a \in bA$ (we may assume that b is non-unit). As M is strongly primary, we get $aM \subseteq b^2A$ or $b^{2n} \in a^n M$ for some integer $n \geq 1$. If $b^{2n} \in a^n M$, then $b^{2n} = a^n m$ for some $m \in M$, which is a contradiction since b is regular. Hence $aM \subseteq b^2A$, and so $ab = b^2 r$ for some $r \in A$. As b is regular, we get $a = br$, which implies that $a \in bA$. Since M is strongly primary, (2) holds. Conversely, suppose that (1) and (2) hold and let $a, b \in M$. By (2) we may assume that $a \notin Z(A)$ or $b \notin Z(A)$. If $a \in Z(A)$ and $b \notin Z(A)$, then by (1) $a \in bA$. Hence $aM \subseteq bA$. Now, suppose that $a \notin Z(A)$ and $b \in Z(A)$. By (1) we have $b \in aA$, which implies that $b = ar$ for some $r \in A$. If $r \in A \setminus M$, then r is unit, and so $a \in bA$, which is absurd since a is regular. Therefore, $bA \subseteq aM$. In the remaining case, $a \notin Z(A)$ and $b \notin Z(A)$. Let $m \in M$. As M satisfies property $(*)$ and $\frac{am}{b}\frac{b}{a} = m \in M$, we get $\frac{am}{b} \in M \subseteq A$ or $\frac{b^n}{a^n} \in M$.

Hence, $am \in bA$ or $b^n \in a^nM$. Therefore M is strongly primary, as desired.

2.3 Trivial extensions

In this section, we investigate the transfer of strongly primary ideals to trivial ring extension in some particular cases.

Theorem 2.3.1 *Let A be a ring, P a primary ideal of A and let E be an A-module. Put $R = A \ltimes E$. Then*

1. *If $P \ltimes E$ is a strongly primary ideal of R, then P is strongly primary.*
2. *Assume that A is an integral domain and E is a divisible A-module. An ideal J of R is strongly primary if and only if $J = P \ltimes E$, where P is a strongly primary ideal of A or $J = 0 \ltimes F$, where F is a 0-primary submodule of E.*
3. *Suppose that (A,M) be a local ring with maximal ideal M and E an A-module such that $ME = 0$. Then $P \ltimes E$ is strongly primary if and only if P is strongly primary.*

Proof. (1) Let $a,b \in A$. Then $(a,0),(b,0) \in A \ltimes E$. So, $(a,0)P \ltimes E \subseteq (b,0)A \ltimes E$ or $(b,0)^nA \ltimes E \subseteq (a,0)^nP \ltimes E$. Then $aP \subseteq bA$ or $b^nA \subseteq a^nP$, as desired.
(2) Assume that J is a strongly primary ideal of R. By [8, Corollary 3.4], J has the form $P \ltimes E$ or $J = 0 \ltimes F$. So, if $J = P \ltimes E$, then by (1) P is a strongly primary ideal of A. Otherwise, $J = 0 \ltimes F$ and F is 0-primary submodule of E by [8, Theorem 3.6]. Conversely, assume that $J = P \ltimes E$ and P is a strongly primary ideal of A. Let (a,e), $(b,f) \in A \ltimes E$ and $(b,f)^nA \ltimes E \nsubseteq (a,e)^nP \ltimes E$. Since P is strongly primary, we get $aP \subseteq bA$ or $b^nA \subseteq a^nP$. Assume that $aP \subseteq bA$. Thus if $p \in P$, there exists $c \in A$ such that $ap = bc$. Let $h \in E$. As $b \neq 0$ and E is a divisible A-module, there exists $g \in E$ such that $ah + pe = bg + cf$. Then $(a,e)P \ltimes E \subseteq (b,f)A \ltimes E$. If $b^nA \subseteq a^nP$ for some positive integer n, then by similar reasoning we get a contradiction. Now, suppose that $J = 0 \ltimes F$ where F is 0-primary. By [8, Theorem 3.6(b)] $0 \ltimes F$ is a primary ideal of $A \ltimes E$. Let $(a,e),(b,f) \in A \ltimes E$. Two cases are then possible:
Case 1: $a = 0$ or $b = 0$. Without loss of generality, we may assume that $a = 0$. Then $(0,0) = (0,e)0 \ltimes F \subseteq (b,f)A \ltimes F$, as desired.
Case 2: $a \neq 0$ and $b \neq 0$. Let $g \in F$. As $b \neq 0$ and E is a divisible A-module, there exists $k \in E$ such that $ag = bk$. Therefore, $(a,e)(0,g) = (0,ag) = (0,bk) = (b,f)(0,k) \in (b,f)A \ltimes E$, as desired.

(3) If $P \ltimes E$ is a strongly primary ideal of $A \ltimes E$, then P is strongly primary by (1). Conversely, let $(a,e),(b,f) \in A \ltimes E$. Assume that (a,e) or (b,f) is a unit. Without loss of generality, we may assume that (b,f) is unit. Then $(a,e)P \ltimes E \subseteq A \ltimes E = (b,f)A \ltimes E$,

as desired. Now, assume that $a \in M$ and $b \in M$ and $(b,f)^n A \ltimes E \nsubseteq (a,e)^n P \ltimes E$. Since P is strongly primary, we get $b^n A \subseteq a^n P$ for some positive integer n or $aP \subseteq bA$. First, if $b^n A \subseteq a^n P$, then $b^n = a^n p$ for some $p \in P$. Without loss of generality, we suppose that $n \geq 2$. Since $ME = 0$, we get $(b,f)^n = (b^n, nb^{n-1}f) = (a^n p, 0) = (a^n, 0)(p,0) = (a,e)^n (p,0) \in (a,e)^n P \ltimes E$, a contradiction. In the remaining case, $ap = bc$ for some $c \in A$. It follows from $b^n A \nsubseteq a^n P$ that $c \in M$. Let $h \in E$. Then $(a,e)(p,h) = (ap,0) = (bc,0) = (b,f)(c,0) \in (b,f)A \ltimes E$. Therefore $(a,e)P \ltimes E \subseteq (b,f)A \ltimes E$, as desired.

In [23, Theorem 2.8], the authors proved that the prime ideals which are properly contained in a strongly primary ideal are strongly prime. The following example shows that [23, Theorem 2.8] does not extend from domains to the context of rings with zero-divisors.

Example 2.3.2 *Let A be an APVD with maximal ideal M of krull dimension greater than or equal to 2 and $R = A \ltimes E$, where $E = A/M$. Let P be a nonzero prime ideal properly contained in M. Then $P \ltimes E$ is not strongly prime.*

Proof. Since M is strongly primary, we get $M \ltimes E$ is strongly primary by Theorem 2.3.1(3). But $P \ltimes E$ is not strongly prime. Deny, let $x = (0,e)$ and $y = (p,0)$ where $0 \neq e \in E$ and $p \in P$. Then $(0,e)A \ltimes E \subseteq (p,0)P \ltimes E$ or $(p,0)P \ltimes E \subseteq (0,e)A \ltimes E$. If $(0,e)A \ltimes E \subseteq (p,0)P \ltimes E$, then $(0,e) = (p,0)(q,f)$ for some $(q,f) \in P \ltimes E$ and hence $e = pf = 0$, a contradiction. Now, let $q \in P$. Since $(p,0)P \ltimes E \subseteq (0,e)A \ltimes E$, we get $(p,0)(q,0) = (0,e)(c,f)$ and hence $P^2 = 0$. Therefore $P = 0$, a desired contradiction.

As another consequence of Theorem 2.3.1, we give an example of a strongly primary ideal.

Example 2.3.3 *Let (V,M) be a local domain with maximal ideal M (for instance, see [23, Example 1.10]) and E be a V/M-vector space. Let P be strongly primary ideal of V. Then $P \ltimes E$ is a strongly primary ideal of $V \ltimes E$.*

2.4 Amalgamated duplication

Our next result develops a result on the transfer of strongly primary ideals to duplication amalgamated. It is easy to see that if $I = A$, then $A \bowtie I \simeq A \times A$ and so every ideal of $A \bowtie I$ is not strongly primary. In the sequel, we suppose that I is a proper ideal of A.

Theorem 2.4.1 *Let A be a ring, I and P be proper ideals of A such that P is primary.*

1. *If $P \bowtie I$ is strongly primary then P is strongly primary, $I \subseteq \mathrm{Nil}(A)$ and $I = pI$ for each regular p of P.*

2. *Assume that (A,M) is a local ring such that $MI = 0$. Then $P \bowtie I$ is strongly primary if and only if P is strongly primary.*

3. *Assume that $Z(A) = \mathrm{Nil}(A)$ and $I = aI$ for each $a \in Reg(A)$. Then $P \bowtie I$ is strongly primary if and only if P is strongly primary and $I \subseteq \mathrm{Nil}(A)$.*

Proof. (1) Let $a,b \in A$. Then $(a,a),(b,b) \in A \bowtie I$. Since $P \bowtie I$ is strongly primary, we get $(a,a)P \bowtie I \subseteq (b,b)A \bowtie I$ or $(b,b)^n A \bowtie I \subseteq (a,a)^n P \bowtie I$ for some positive integer n. Then $aP \subseteq bA$ or $b^n A \subseteq a^n$P. Now, let i be a nonzero element of I. Thus $(0,i)P \bowtie I \subseteq (i,0)A \bowtie I$ or $(i,0)^n A \bowtie I \subseteq (0,i)^n P \bowtie I$ for some positive integer n. Then $i^2 = 0$ or $i^n = 0$ for some positive integer n. Therefore $i \in \mathrm{Nil}(A)$ and hence $I \subseteq \mathrm{Nil}(A)$. Finally, let p be a regular element of P and $i \in I$. Since $P \bowtie I$ is strongly primary, we get $(0,i)P \bowtie I \subseteq (p^2,p^2)A \bowtie I$ or $(p^{2n},p^{2n}) \in (0,i)^n P \bowtie I$ for some positive integer n. If $(p^{2n},p^{2n}) \in (0,i)^n P \bowtie I$ then $p^{2n} = 0$ and hence $p = 0$, a contradiction. Then $(0,i)(p,p) = (p^2,p^2)(c,c+k)$ for some $c \in A$ and $k \in I$. Thus $p^2 c = 0$ implies $c = 0$ and consequently $ip = p^2 k$. Therefore $i = pk$, as desired.

(2) By (1) it suffices to prove that if P is a strongly primary then $P \bowtie I$ is strongly primary. Let $(a,a+i),(b,b+j) \in A \bowtie I$ such that $(b,b+j)^n A \bowtie I \nsubseteq (a,a+i)^n P \bowtie I$. Assume that a or b is unit. Without loss of generality, we may assume that b is unit. Then $(b,b+j)$ is unit since $I \subseteq \mathrm{Nil}(A)$. Therefore $(a,a+i)P \bowtie I \subseteq (b,b+j)A \bowtie I$ and so we are done. Now, we may assume that $a,b \in M$. Since P is strongly primary, we get $aP \in bA$ or $b^n A \in a^n P$ for some integer $n \geq 1$. It is clear that $b^n A \nsubseteq a^n P$ for each integer $n \geq 1$. Then $ap = bc$ for some $c \in A$. So, $(a,a+i)(p,p+k) = (ap,ap) = (bc,bc) = (b,b+j)(c,c)$ since $c \in M$. Therefore $(a,a+i)P \bowtie I \subseteq (b,b+j)A \bowtie I$, as desired.

(3) Assume that $Z(A) = \mathrm{Nil}(A)$ and $I = aI$ for each $a \in Reg(A)$. If $P \bowtie I$ is strongly primary then P is strongly primary and $I \subseteq \mathrm{Nil}(A)$ by (1). Conversely, let $(a,a+i),(b,b+j) \in A \bowtie I$. Then two cases are then possible.
Case 1: a or $b \in Z(A) = \mathrm{Nil}(A)$. Without loss of generality, we may assume that $a \in \mathrm{Nil}(A)$. Then $(a,a+i)^n P \bowtie I = (0,0) \subseteq (b,b+j)^n A \bowtie I$ for some positive integer n.
Case 2: a and b are regular elements of A. Since P is a strongly primary ideal of A, we get $aP \subseteq bA$ or $b^n A \subseteq a^n P$. First, assume that $aP \subseteq bA$. Then $ap = bc$ for some $c \in A$. Clearly $b+j$ is a regular element of A. Then by our assumption, there exists $t \in I$ such that $ak + pi + ik - jc = (b+j)t$ for each $p \in P$ and $k \in I$. Therefore $(a,a+i)P \bowtie I \subseteq (b,b+j)A \bowtie I$ and so we are done. Thus, we may assume that there exists a positive integer n such that $b^n \in a^n P$. Then $b^n = a^n p$ for some $p \in P$. By the same argument as in the first part, there exists $k \in I$ such that $(b,b+j)^n = (a,a+i)^n (p,p+k)$, as desired.

Corollary 2.4.2 *Let A be a reduced ring, I an ideal of A and P be a primary ideal of A. Then $P \bowtie I$ is strongly primary if and only if P is strongly primary and $I = 0$.*

Proof. Follows immediately from Theorem 2.4.1(1).

We close with several examples. In particular, Example 2.4.5 show that [23, Proposition 3.3] does not extend to the context of rings with zero-divisors.

Example 2.4.3 *Let (A,M) be a valuation domain, E a nonzero A-module such that $ME = 0$ (for instance, let $\mathbb{Z}$ be the ring of integers, p a positive prime integer, X an indeterminate over $\mathbb{Z}$, and set $A = \mathbb{Z}/p\mathbb{Z}$ and $E = A/pA + XA/pA$) and $R = A \ltimes E$ be the trivial extension of A by E. Let $K = J \ltimes E$, where J is a primary ideal of A. Set $I = 0 \ltimes E$ be an ideal of R. Then $K \bowtie I$ is strongly primary.*

Proof. Follows from [23, Proposition 2.1], Theorem 2.3.1(3) and Theorem 2.4.1(2).

Example 2.4.4 *Let A be APVD with quotient field K and $R = A \ltimes K$ the trivial extension of A by K. Let P be a prime ideal of A. Set $J = P \ltimes K$ and $I = 0 \ltimes K$. Then $J \bowtie I$ is a strongly primary ideal of $R \bowtie I$.*

Proof. By Theorem 2.3.1(2), J is a strongly primary ideal of A. Then the result follows immediately from Theorem 2.4.1(3).

Example 2.4.5 *Let $A = \mathbb{Z}/8\mathbb{Z}$ and $I = 4\mathbb{Z}/8\mathbb{Z}$ be an ideal of A. Put $R = A \bowtie I$, then:*

1. *Every primary ideal of R is strongly primary.*
2. *R is not a PV-ring (in particular, R is not a valuation ring).*

Proof. (1) Follows immediately from Proposition 2.2.4.
(2) Assume that R is a PV-ring. Then $(\bar{2},\bar{2})2\mathbb{Z}/8\mathbb{Z} \bowtie 4\mathbb{Z}/8\mathbb{Z}$ and $(0,\bar{4})R$ are comparable, which is a contradiction.

CHAPTER 3

ON ALMOST BÉZOUT RINGS AND ALMOST GCD-RINGS

Abstract.

In this chapter *, we study the possible transfer of the property of being an *AB*-ring to trivial ring extensions and amalgamated algebras along an ideal. Also, we extend the notion of an almost GCD-domain to the context of arbitrary rings, and we study the possible transfer of this notion to trivial ring extensions and amalgamated algebras along an ideal. Our aim is to provide examples of new classes of commutative rings satisfying the above-mentioned properties.

3.1 Introduction

In [9], Anderson and Zaffrullah introduced and studied the notion of almost Bézout domain (AB-domain). An integral domain A is an AB-domain if for $a,b \in A-\{0\}$ there is n such that (a^n,b^n) is principal. The notion of almost Bézout domains runs along lines somewhat similar to those of Bézout domains (i.e., every two generated, equivalently, every finitely generated, ideal is principal). In [10], Anderson, Knopp, and Lewin continued the study of almost Bézout domains, and after observing that each almost Bézout domain is nearly Bézout, they used the construction $K+XL[X]$ to disprove the converse. The same example shows that a Noetherian almost Bézout domain need not be an almost principal ideal domain (*API*-domain), even though each Noetherian Bézout domain is a principal ideal domain (*PID*). In[7], Anderson and Zaffrullah continued their study of almost Bézout domains and gave a new characterization of Cohen-Kaplansky domains. In [84], the generalization of the almost Bézout domains to arbitrary commutative rings (with zero-divisors) is considered as follows: a ring A is called an almost Bézout ring

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(AB-ring for short) if, for any two elements a and b in A, there exists a positive integer n such that the ideal (a^n, b^n) is principal. An integral domain A is said to be an almost GCD (AGCD) domain if for every $a, b \in A$, there exists a positive integer n such that $Aa^n \cap Ab^n$ is a principal ideal. AGCD domains were introduced by Zafrullah in [95] as a generalization of GCD domains (also see [9] and [49]).

The purpose of this chapter is to study the possible transfer of the property of being an *AB*-ring to trivial ring extensions and amalgamated algebras along an ideal. Also, we generalize the notion of an almost *GCD* domain to the context of arbitrary rings and then we study the possible transfer of this generalized property to trivial ring extensions and amalgamated algebras along an ideal.

3.2 AB-rings

Our first result studies the possible transfer of the AB-ring property between a ring A and a trivial ring extension $A \ltimes E$.

Theorem 3.2.1 *Let A be a ring, E a nonzero A-module, and $R := A \ltimes E$. Then:*

(1) *If R is an* AB*-ring, then A is an* AB*-ring and for all $e \in E$ and a regular element a of A there are $f \in E$ and a positive integer $n \geq 1$ such that $na^{n-1}e = a^n f$.*

(2) *Suppose that $\mathrm{Z}(A) = \mathrm{Nil}(A)$ and E is a divisible A-module. Then R is an* AB*-ring if and only if A is an* AB*-ring.*

(3) *Let (A, M) be a local ring (with maximal ideal M) and let E be an A-module such that $M = \sqrt{\mathrm{Ann}(E)}$. Then R is an* AB*-ring if and only if A is an* AB*-ring.*

Proof. (1) The fact that A is an AB-ring was established in [84, Theorem 3.1 (1)]. Now let a be a regular element of A and $e \in E$. Since R is an AB-ring, there is a positive integer n such that $R(a,0)^n + R(a,e)^n = R(c,d)$ for some $c \in A$ and $d \in E$. Then there are $(\alpha_1, \beta_1), (\alpha_2, \beta_2), (c_1, d_1)$, and (c_2, d_2) in R such that:

$$(a,0)^n = (a^n, 0) = (\alpha_1, \beta_1)(c,d)$$

$$(a,e)^n = (a^n, na^{n-1}e) = (\alpha_2, \beta_2)(c,d)$$

$$(c,d) = (c_1, d_1)(a^n, 0) + (c_2, d_2)(a^n, na^{n-1}e)$$

Then we obtain the following equations:

(1) $a^n = \alpha_1 c$

(2) $a^n = \alpha_2 c$

(3) $0 = \alpha_1 d + c\beta_1$

(4) $na^{n-1}e = \alpha_2 d + c\beta_2$

(5) $c = (c_1 + c_2)a^n$

As $a^n = \alpha_1 c$, we get that c is a regular element of A since a is regular. From the first and the second equation, we have $\alpha_1 = \alpha_2 = \alpha$. Finally, from the third, the fourth, and the fifth equation, we have $na^{n-1}e = c(\beta_2 - \beta_1) = a^n(c_1 + c_2)(\beta_2 - \beta_1) = a^n f$.
(2) By (1), we need only prove that if A is an AB-ring (along with the hypotheses that $Z(A) = \mathrm{Nil}(A)$ and E is a divisible A-module), then R is an AB-ring. We will show that if $\alpha := (a,e), \beta := (b,f) \in R$, then there exists a positive integer n such that $R\alpha^n + R\beta^n$ is principal. There are two cases. In the first case, a and b are each regular elements of A. Then, since A is an AB-ring, there exists a positive integer n such that $Aa^n + Ab^n$ is principal. Then $Aa^n + Ab^n = Ac$ for some $c \in A$. Let y be an element of $R\alpha^n + R\beta^n$. Hence $y = (ra^n + tb^n, d)$ for some $r,t \in A$ and $d \in E$, and $ra^n + tb^n = \lambda c$ for some $\lambda \in A$. Since E is a divisible A-module and c is a regular element of A (since $a^n \in Ac$ and a is regular), there exists $z \in E$ such that $d = cz$. Therefore, $y = (ra^n + tb^n, d) = (\lambda c, cz) = (c,0)(\lambda, z)$. So $R\alpha^n + R\beta^n \subseteq R(c,0)$. Conversely, we have $c = ra^n + tb^n$ for some $r,t \in A$. Since E is a divisible A-module and a^n is a regular element of A, there exists $d \in E$ such that $a^n d + nra^{n-1}e + ntb^{n-1}f = 0$. Hence $(c,0) = (a,e)^n(r,d) + (b,f)^n(t,0) \in R\alpha^n + R\beta^n$. Therefore, $R\alpha^n + R\beta^n = R(c,0)$, as desired.

In the remaining case, either a or b is a zero-divisor in A. Without loss of generality, we may assume that $a \in Z(A) = \mathrm{Nil}(A)$. Then there exists a positive integer n such that $a^n = 0$. Hence $\alpha^{n+1} = (a^{n+1}, (n+1)a^n e) = (0,0)$ and so $R(a,e)^{n+1} + R(b,f)^{n+1} = R(b,f)^{n+1}$ is principal, as desired.

(3) By (1), we need only prove that if A is an AB-ring, along with the hypothesis that $M = \sqrt{\mathrm{Ann}(E)}$, then R is an AB-ring. Let $\alpha = (a,e), \beta = (b,f) \in R$. If a is a unit in A, then α is a unit in R, so that $R\beta + R\alpha = R\alpha$. Similarly, if b is a unit in A, then $R\alpha + R\beta = R\beta$. Thus, we may assume, without loss of generality, that $a \in M$ and $b \in M$. Then, since $M \subseteq \sqrt{\mathrm{Ann}(E)}$, there exist positive integers m and p such that $a^m \in \mathrm{Ann}(E)$ and $b^p \in \mathrm{Ann}(E)$. Therefore, $\alpha^{mp+1} = (a,e)^{mp+1} = (a^{(mp+1)}, (mp+1)a^{mp}e) = (a^{mp+1}, 0)$ and $\beta^{mp+1} = (b,f)^{mp+1} = (b^{mp+1}, (mp+1)a^{mp}f) = (b^{mp+1}, 0)$. As A is an AB-ring, there exists a positive integer n such that $Aa^{(mp+1)n} + Ab^{(mp+1)n}$ is principal. Therefore $R(a^{mp+1}, 0)^n + R(b^{mp+1}, 0)^n$ is principal, which implies that $R\alpha^{(mp+1)n} + R\beta^{(mp+1)n}$ is principal. This completes the proof.

If A is a domain and E is a divisible A-module, Theorem 3.2.1(2) specializes to the following result.

Corollary 3.2.2 *Let A be a domain and E a divisible A-module. Then $A \ltimes E$ is an* AB*-ring if and only if A is an* AB*-domain.*

Corollary 3.2.3 *Let A be a domain and E a torsion-free A-module such that $\mathbb{Q}.E \subseteq E$ then, $A \ltimes E$ is an* AB*-ring if and only if A is an* AB*-domain and E is a divisible A-module.*

Proof. By Theorem 3.2.1 (1) and (2), we need only prove that if R is an AB-ring, then E is a divisible A-module; that is, if $0 \neq e \in E$ and $0 \neq a$ is a element of A, then $e \in aE$. Since R is an AB-ring, by Theorem 3.2.1(1), there is $f \in E$ and a positive integer $n \geq 1$ such that $na^{n-1}e = a^n f$. As E is a torsion-free A-module, we get $ne = af$. Therefore $e = an^{-1}f \in aE$ since $\mathbb{Q}.E \subseteq E$.

If (A,M) is a local ring and E an A-module such that $ME = 0$, then Theorem 3.2.1 (3) specializes to recover one result in Corollary 3.2.4 (1) and to obtain a new application in Corollary 3.2.4 (2).

Corollary 3.2.4 *Let (A,M) be a local ring, E an A-module, and $R := A \ltimes E$. Then:*

(*1*) (*Mahdou, Mimouni and Moutui [84, Theorem 3.1(3)]*) *Suppose that $ME = 0$. Then R is an* AB*-ring if and only if A is an* AB*-ring.*

(*2*) *Suppose that M is the only prime ideal of A (for instance, suppose that A is an* SPIR*). Then R is an* AB*-ring if and only if A is an* AB*-ring.*

Proof. We may assume without loss of generality, that $E \neq 0$. Then, in the context of either (1) or (2), we can give the following unified argument. We have $\mathrm{Ann}(E) \neq A$, whence $\sqrt{\mathrm{Ann}(E)} = M$, and so an application of Theorem 3.2.1 (3) completes the proof.

Our next example illustrate parts (2) of Theorem 3.2.1.

Example 3.2.5 *Let A be a Bézout domain, $K = qf(A)$ the quotient field of A, and F be a K-vector space such that $dim_K(F) \geq 2$. Set $B = A \ltimes F$ and $E = K \ltimes F$. Then, $R = B \ltimes E$ is an* AB*-ring wihch is not a Bézout ring.*

Proof. By [84, Example 3.4], B is an AB-ring wihch is not a Bézout ring. We have $Z(B) = 0 \ltimes F = \mathrm{Nil}(B)$ and E is a divisible B-module. By Thoerem 3.2.1(2), R is an AB-ring wihch is not a Bézout ring.

Our following theorem studies the transfer of AB-ring property to amalgamation of rings:

Theorem 3.2.6 *Let A and B be a pair of rings, $f : A \longrightarrow B$ be a ring homomorphism, and J be a regular proper ideal of B. If $R = A \bowtie^f J$ is an* AB*-ring, then $f^{-1}(J) \subseteq Z(A)$.*
In particular, if A is an integral domain then R is an AB*-ring if and only if $f^{-1}(J) = 0$ and $f(A) + J$ is an* AB*-ring.*

Proof. Assume that $f^{-1}(J)$ contains a regular element i and let j be a regular element of J. Then, there is a positive integer n such that $R(i,0)^n + R(0,j)^n$ is principal. Hence there is $(r, f(r)+k) \in R$ such that $R(i,0)^n + R(0,j)^n = R(r, f(r)+k)$. Therefore, we have :

$$(i,0)^n = (i^n,0) = (\alpha_1, f(\alpha_1)+k_1)(r, f(r)+k)$$

$$(0,j)^n = (0,j^n) = (\alpha_2, f(\alpha_2)+k_2)(r, f(r)+k)$$

$$(r, f(r)+k) = (c_1, f(c_1)+d_1)(i^n,0) + (c_2, f(c_2)+d_2)(0,j^n)$$

for some $(\alpha_1, f(\alpha_1)+k_1), (\alpha_2, f(\alpha_2)+k_2), (c_1, f(c_1)+d_1)$ and $(c_2, f(c_2)+d_2)$ in R.

Then, we obtain the following equations:

(1) $i^n = \alpha_1 r$.

(2) $\alpha_2 r = 0$.

(3) $j^n = (f(\alpha_2)+k_2)(f(r)+k)$.

(4) $f(r)+k = (f(c_2)+d_2)j^n$.

From the first equation we get that r is a regular element of A since i is regular, and so $\alpha_2 = 0$ by the second equation. Also, $f(r)+k$ is a regular element of B, since j is regular. Hence, from the third and the fourth equation, we obtain $f(r)+k = k_2(f(c_2)+d_2)(f(r)+k)$. Therefore, $1 = k_2(f(c_2)+d_2) \in J$, a contradiction.
In particular, if A is an integral domain, then $f^{-1}(J) \subseteq Z(A) = 0$ and so $f^{-1}(J) = 0$. Moreover $f(A)+J$ is an AB-ring by [84, Lemma 3.7]. Conversely, since $f^{-1}(J) = 0$, we have from [34, Proposition 2.1(3)] $A \bowtie^f J \cong f(A)+J$. Therefore R is an AB-ring.

If $f = id_A$, $B = A$ and $J = I$ be an ideal of A, we get the following Corollary.

Corollary 3.2.7 *Let A be an integral domain and I a proper ideal of A. Then $A \bowtie I$ is an* AB*-ring if and only if so is A and $I = 0$.*

If A is a local ring, then we have the following result :

Proposition 3.2.8 *Let A and B be a pair of rings and $f : A \longrightarrow B$ be a ring homomorphism. Suppose that A is a local ring with maximal ideal M, and J is a proper ideal of B such that $f(M)J = 0$ and $J \subseteq \mathrm{Nil}(B)$. Set $R := A \bowtie^f J$. Then R is an* AB*-ring if and only if A is an* AB*-ring.*

Proof. Assume that R is an AB-ring. By [84, Lemma 3.7] A is an AB-ring. Conversely, let $\alpha = (a, f(a)+i)$ and $\beta = (b, f(b)+j)$ be two elements of R. Without loss of generality, we may assume that $a, b \in M$. Since $i, j \in J$, there are positives integers p and m such

that $i^p = 0$ and $j^m = 0$. As A is an AB-ring, there exists a positive integer n such that $Aa^{npm} + Ab^{npm} = Ac$ for some $c \in A$. By using the fact that $f(M)J = 0$, we conclude easily that $R\alpha^{npm} + R\beta^{npm} = R(a, f(a))^{npm} + R(b, f(b))^{npm} = R(c, f(c))$. Therefore R is an AB-ring.

3.3 AGCD-rings

We start this section by extend Zafrullah's definition of AGCD-domain to an arbitrary commutative ring in the following way.

Definition 3.3.1 *Let A be a ring. A is called an almost GCD-ring (AGCD-ring for short) if, for any two elements a and b in A, there exists a positive integer n such that the ideal $Aa^n \cap Ab^n$ is principal.*

The following theorem studies the possible transfer of the ACGD-ring property between a ring A and a trivial ring extension $A \ltimes E$.

Theorem 3.3.2 *Let A be a ring, E be a nonzero A-module, and $R := A \ltimes E$. Then:*

(1) *If R is an* AGCD*-ring, then A is an* AGCD*-ring.*

(2) *Suppose that $\mathrm{Z}(A) = \mathrm{Nil}(A)$ and E is a divisible A-module. Then R is an* AGCD*-ring if and only if A is an* AGCD*-ring.*

(3) *Let (A, M) be a local ring (with maximal ideal M) and let E be an A-module such that $M = \sqrt{\mathrm{Ann}(E)}$. Then R is an* AGCD*-ring if and only if A is an* AGCD*-ring.*

(4) *Suppose that $A \subseteq E$ is an extension of domains and $\mathbb{Q} \subseteq E$. Then R is an* AGCD*-ring if and only if A is an* AGCD*-domain and $\mathrm{qf}(A) \subseteq E$.*

Proof. (1) Assume that R is an AGCD-ring, and let $a, b \in A$. Then there is a positive integer n such that $R(a,0)^n \cap R(b,0)^n = R(c,d)$ for some $(c,d) \in R$. Hence $Aa^n \cap Ab^n = Ac$, as desired.
(2) By (1), it suffices to prove the "if" assertion. Suppose that A is an AGCD-ring, $\mathrm{Z}(A) = \mathrm{Nil}(A)$ and E is a divisible A-module. We must show that if (a,e) and (b,f) are in R, then there exists a positive integer n such that $R(a,e)^n \cap R(b,f)^n$ is principal. We will consider two cases. In the first case, either a or b is in $\mathrm{Z}(A)$. We may assume that $a \in \mathrm{Z}(A) = \mathrm{Nil}(A)$. Then $a^p = 0$ for some positive integer p, so that $R(a,e)^{p+1} \cap R(b,f)^{p+1} = R(a,e)^{p+1} = (0,0)$ is principal (We use the fact that if $(a,e) \in R$ and $k \geq 2$, then $(a,e)^k = (a^k, ka^{k-1}e)$). In the second case, a and b are each regular elements of A. As A is an AGCD-ring, there is a positive integer n such that $Aa^n \cap Ab^n = Ac$ for some $c \in A$. Let $y \in R(a,e)^n \cap R(b,f)^n$, then $y = (\alpha c, z)$ for some $\alpha \in A$ and $z \in E$. Since E is a divisible A-module and c is a regular element of A (since $a^n b^n \in Ac$), there exists $\beta \in E$ such

that $z = c\beta$. Hence $y = (\alpha,\beta)(c,0) \in R(c,0)$. Therefore $R(a,e)^n \cap R(b,f)^n \subseteq R(c,0)$. On the other hand, we have $c \in Aa^n \cap Ab^n$, so that $c = \alpha a^n = \beta b^n$ for some α, β in A. It follows from divisibility of E that $0 = \alpha n a^{n-1} e + a^n k$ and $0 = \beta n b^{n-1} f + b^n t$ for some $k,t \in E$. Hence $(c,0) = (\alpha,k)(a,e)^n = (\beta,t)(b,f)^n \in R(a,e)^n \cap R(b,f)^n$. Therefore $R(a,e)^n \cap R(b,f)^n = R(c,0)$, and hence R is an AGCD-ring.
3) By (1), it suffices to prove the "if" assertion. Let $\alpha = (a,e), \beta = (b,f) \in R$. If a (resp., b) is a unit of A, then α (resp., β) is a unit of R, and so $R\alpha \cap R\beta = R\beta$ (resp. $= R\alpha$). Thus, we may assume, without loss of generality, that both a and b are in M. As $M = \sqrt{\mathrm{Ann}(E)}$, there exist positive integers m and p such that $a^m \in \mathrm{Ann}(E)$ and $b^p \in \mathrm{Ann}(E)$. Then we get,

$$\alpha^{mp+1} = (a,e)^{mp+1} = (a^{(mp+1)}, (mp+1)a^{mp}e) = (a^{mp+1}, 0)$$
$$\beta^{mp+1} = (b,f)^{mp+1} = (b^{mp+1}, (mp+1)a^{mp}f) = (b^{mp+1}, 0).$$

Since A is a AGCD-ring, there exists a positive integer n such that $Aa^{(mp+1)n} \cap Ab^{(mp+1)n}$ is principal. Therefore, $R\alpha^{(mp+1)n} \cap R\beta^{(mp+1)n}$ is principal, as desired.
4) The "if" assertion follows from (2) because the hypothesis that $\mathrm{qf}(A) \subseteq E$ ensures that E is a divisible A-module.

We turn to the "only if" assertion. As R is now assumed to be a AGCD-ring, (1) reduces our task to proving that $\mathrm{qf}(A) \subseteq E$. To that end, it suffices to show that if $0 \neq a \in M$, then a is a unit of E, that is, $a^{-1} \in E$. Since R is a AGCD-ring, there exists a positive integer n such that $R(a,0)^n \cap R(a,1)^n = R(c,d)$ for some $(c,d) \in R$. Hence there exist $\alpha_1, \alpha_2, \xi \in A$ and $\beta_1, \beta_2, \eta \in E$ such that :

$$(c,d) = (\alpha_1,\beta_1)(a,0)^n = (\alpha_1 a^n, \beta_1 a^n),$$
$$(c,d) = (\alpha_2,\beta_2)(a,1)^n = (\alpha_2 a^n, \beta_2 a^n + n\alpha_2 a^{n-1}),$$
$$\textit{and},\ \ (0,a^n) = (\xi,\eta)(c,d).$$

Then, we obtain the following equations :

1) $c = \alpha_1 a^n = \alpha_2 a^n$.

2) $d = \beta_1 a^n = \beta_2 a^n + n\alpha_2 a^{n-1}$

3) $\xi c = 0$.

4) $\xi d + \eta c = a^n$.

Hence, from the first equation we have $\alpha_1 = \alpha_2 = \alpha$. Also, from the second equation we obtain $n\alpha = a(\beta_1 - \beta_2)$. Moreover, from the third and the fourth equation we have $\xi = 0$ and so $a^n = \eta c$. Finally, from the first equation we obtain, $a^n = \eta c = \eta\alpha a^n$. Then $\eta\alpha = 1$ and so $a\eta(\beta_1 - \beta_2)n^{-1} = 1$. Therefore a is a unit of E. This completes the proof.

The next corollaries are immediate applications of parts (2) and (3) of Theorem 3.3.2.

Corollary 3.3.3 *Let A be a domain and E a divisible A-module. Then $A \ltimes E$ is an* AGCD-*ring if and only if A is a* AGCD-*domain.*

Corollary 3.3.4 *Let A be a domain such that $\mathbb{Q} \subseteq A$, then $A \ltimes A$ is an* AGCD-*ring if and only if A is a field.*

Corollary 3.3.5 *Let (A,M) be a local ring and E an A-module such that $ME = 0$. Then $A \ltimes E$ is an* AGCD-*ring if and only if A is an* AGCD-*ring.*

Example 3.3.6 *Let A be a GCD-domain, $K = qf(A)$, and $R = A \ltimes K$ be the trivial extension of A by K. Then :*

1) R is an AGCD-*ring by Corollary 3.3.3.*

2) R is not a coherent ring by [75, Theorem 3.1(1)].

Example 3.3.7 *Let A be a domain containing the field of rational numbers, but not a field (For instance, let $A = Q[X]$). By Theorem 3.3.2 (4) $A \ltimes A$ is not an* AGCD-*ring..*

The following result studies the transfer of AGCD-ring property to amalgamation of rings:

Theorem 3.3.8 *Let A and B be a pair of rings, $f : A \longrightarrow B$ a ring homomorphism, and J be a non-principal regular ideal of B. If A is an integral domain then $R = A \bowtie^f J$ is an* AGCD-*ring if and only if $f^{-1}(J) = 0$ and $f(A) + J$ is an* AGCD-*ring.*

Before proving the theorem, we need the following lemma :

Lemma 3.3.9 *Let A and B be a pair of rings, $f : A \longrightarrow B$ a ring homomorphism, and J be a nonzero proper ideal of B. If $A \bowtie^f J$ is an* AGCD-*ring, then so are A and $f(A) + J$.*

Proof. This follows immediately from [34, Proposition 2.1 (3)] and the fact that if A is an AGCD-ring and I is an ideal of A, then A/I is an AGCD-ring.

Proof of Thoerem 3.3.8. Assume that R is an AGCD-ring (along with the hypotheses that A is a domain and J is a non-principal regular ideal of B), then $f^{-1}(J) = 0$. Deny, let $0 \neq i \in f^{-1}(J)$ and $0 \neq j$ be a regular element of J. Since R is an AGCD-ring, there is a positive integer n such that $R(i,j)^n \cap R(0,j)^n$ is principal. Hence there exists $(c, f(c) + d) \in R$ such that $R(i,j)^n \cap R(0,j)^n = R(c, f(c) + d)$. Then, there exist $(\alpha_1, f(\alpha_1) + \beta_1), (\alpha_2, f(\alpha_2) + \beta_2), (\alpha_3, f(\alpha_3) + \beta_3) \in R$ such that :

$$(c, f(c) + d) = (\alpha_1, f(\alpha_1) + \beta_1)(i,j)^n = (\alpha_2, f(\alpha_2) + \beta_2)(0,j)^n$$

$$and\ for\ each\ x \in J,\ (0, xj^n) = (\alpha_3, f(\alpha_3) + \beta_3)(c, f(c) + d)$$

Therefore we have $c = \alpha_1 i^n = 0$, so that $\alpha_1 = 0$. Also, we obtain $d = (f(\alpha_1) + \beta_1) j^n = \beta_1 j^n$. On the other hand, for each $x \in J$ we have $x j^n = (f(\alpha_3) + \beta_3) d = (f(\alpha_3) + \beta_3) \beta_1 j^n$. As j is regular, we get $x = (f(\alpha_3) + \beta_3)\beta_1 \subseteq B\beta_1$. Therefore $J \subseteq B\beta_1 \subseteq J$ and thus J is a principal ideal of B, a contradiction. By Lemma 3.3.9 $f(A) + J$ is an AGCD-ring.
Conversely, if $f^{-1}(J) = 0$, by [34, Proposition 2.1(3)] we have $A \bowtie^f J \cong f(A) + J$ and so the conclusion is straightforward.

If A is a local, then we get the following result :

Proposition 3.3.10 *Let A and B be a pair of rings and $f : A \longrightarrow B$ be a ring homomorphism. Suppose that A is a local ring with maximal ideal M, and J is a proper ideal of B such that $f(M)J = 0$ and $J \subseteq \mathrm{Nil}(B)$. Then R is an* AGCD*-ring if and only if A is an* ACGD*-ring.*

Proof. If R is an AB-ring then so is A by Lemma 3.3.9. Conversely, let $\alpha = (a, f(a) + i)$ and $\beta = (b, f(b) + j)$ be two elements of R. Without loss of generality, we may assume that $a, b \in M$. Since $i, j \in J$, there are positives integers p and m such that $i^p = 0$ and $j^m = 0$. As A is an AGCD-ring, there is a positive integer n such that $Aa^{npm} \cap Ab^{npm} = Ac$ for some $c \in A$. Since $f(M)J = 0$, we conclude that $R\alpha^{npm} \cap R\beta^{npm} = R(a, f(a))^{npm} \cap R(b, f(b))^{npm} = R(c, f(c))$. Therefore R is an AGCD-ring. This completes the proof.

CHAPTER 4

BI-AMALGAMATION OF RINGS DEFINED BY CERTAIN ANNIHILATOR CONDITIONS

Abstract.

Let $f : A \to B$ and $g : A \to C$ be two ring homomorphisms and let J (resp. J') be an ideal of B (resp., C) such that $f^{-1}(J) = g^{-1}(J')$. In this chapter*, we investigate the transfer of the notions of (A)-rings and strong (A)-rings to the bi-amalgamation of A with (B,C) along (J,J') with respect to (f,g) (denoted by $A \bowtie^{f,g} (J,J')$), introduced and studied by Kabbaj, Louartiti and Tamekkante in 2013. Our results generate new original examples of rings satisfying these properties.

4.1 Introduction

One of important properties of commutative Noetherian rings is that the annihilator of an ideal I consisting entirely of zero-divisors is nonzero [78, p. 56]. However, this result fails for some non-Noetherian rings, even if the ideal I is finitely generated [78, p. 63]. Huckaba and Keller [70] introduced the following: a commutative ring R has property (A) if every finitely generated ideal of R consisting entirely of zero divisors has a nonzero annihilator. Property (A) was originally studied by Quentel [92]. Quentel used the term condition (C) for property (A). The class of commutative rings with property (A) is quite large; for example, Noetherian rings [78, p. 56], rings whose prime ideals are maximal [66], the polynomial ring $R[X]$ for every ring R and rings whose total ring of quotients are Von Neumann regular [66]. Using this property, Hinkle and Huckaba [67] extend the concept Kronecker function rings from integral domains to rings with zero

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divisors. Many authors have studied commutative ring R with property (A), and have obtained several results which are useful studying commutative rings with zero-divisors. For instance, see [11, 66, 69, 70, 82, 83, 92]. In [64], Mahdou and Rahmouni introduced and defined a strong (A)-ring as a ring in which every finitely generated ideal of R which is generated by a finite number of zero-divisors elements of R, has a nonzero annihilator. They studied the transfer of property (A) and strong property (A) in trivial ring extensions and amalgamated duplication of a ring along an ideal. They also exhibit a class of rings which satisfy property (A) and do not satisfy strong property (A). In [48], Dobbs and Shapiro studied a (commutative unital) ring R with only finitely many minimal prime ideals (for instance, a Noetherian ring) and they established that R is reduced and a strong (A)-ring if and only if R is an integral domain. Thus, the smallest reduced ring which has Property (A) but is not a strong (A)-ring is $\mathbb{Z}_2 \times \mathbb{Z}_2$. A Noetherian ring R is a strong (A)-ring if and only if Ass(R) has a unique maximal element [48, Proposition 2.3].

In this chapter, we investigate the transfer of property (A) and strong property (A) in bi-amalgamation of rings, introduced and studied by Kabbaj, Louartiti and Tamekkante in [74].

4.2 Results

We start this section by the following result which gives an important characterization of strong (A)-rings.

Theorem 4.2.1 *Let R be a ring. Then R is a strong (A)-ring if and only if R is an (A)-ring and Z(R) is an ideal of R.*

Proof. Assume that R is a strong (A)-ring. Clearly, R is an (A)-ring. Now, let $a, b \in Z(R)$ then $a+b \in Z(R)$. Deny, assume that $a+b$ is a regular element of R. As $a+b \in Ra+Rb$, we get that $\mathrm{Ann}(Ra+Rb) = 0$ which is a contradiction since R is a strong (A)-ring. Conversely, let $I = \sum_{i=1}^{n} Ra_i$ be a finitely generated ideal of R such that $a_i \in Z(R)$ for each $i = 1, 2, ..., n$. Clearly, as $Z(R)$ is an ideal of R we have $I \subseteq Z(R)$. Therefore $\mathrm{Ann}(I) \neq 0$ since R is an (A)-ring, as desired.

The next corollary is an immediate application of Theorem 4.2.1 and [70, Theorem 1].

Corollary 4.2.2 *If R is nontrivial graded ring, then R is a strong (A)-ring if and only if Z(R) is an ideal of R.*

We say that a ring R is a total ring if every element of R is either a unit or a zero divisor. By Theorem 4.2.1 we get the next corollary.

Corollary 4.2.3 *Let R be a total ring. Then R is a strong (A)-ring if and only if R is a local (A)-ring.*

A ring R is called a reduced ring if it has no nonzero nilpotent elements. The following result is an analogue of [70, Theorem 2].

Proposition 4.2.4 *Let R be a reduced ring. Then the following two statements are equivalent:*

(1) R is a strong (A)-ring.

(2) If I is a finitely generated ideal of R which is generated by a finite number of zero-divisors elements of R, then I is contained in a minimal prime ideal of R.

Proof. $(1) \Rightarrow (2)$. Assume that R is a strong (A)-ring. Let I be a finitely generated ideal of R which is generated by a finite number of zero-divisors elements of R not contained in a minimal prime ideal of R. Then Ann$(I) \neq 0$ and so there is $a \neq 0$ such that $aI = 0$. Therefore a is in every minimal prime ideal of R which implies that $a = 0$, a contradiction.
$(2) \Rightarrow (1)$. Let $I = \sum_{i=1}^{n} Ra_i$ be a finitely generated ideal of R such that $a_i \in Z(R)$ and $a_i \in P$ for some P minimal prime of R. As R is a reduced ring and by [66, p.111] there is $b_i \in R \setminus P$ such that $b_i a_i = 0$ for each $i = 1, 2, ..., n$. Thus, $b = b_1 b_2 ... b_n \neq 0$ and $b \in$ Ann(I), as desired.

The following result gives an example of a non strong (A)-ring.

Proposition 4.2.5 *Let A and B be a pair of rings. Then $A \times B$ is never a strong (A)-ring.*

Proof. $A \times B$ is never a strong (A)-ring since $Z(A \times B)$ is not an ideal of $A \times B$. Indeed, we have $(1,0), (0,1) \in Z(A \times B)$ but $(1,0) + (0,1) = (1,1) \notin Z(A \times B)$.

If $A \subseteq B$ is a ring extension, then we have the next theorem.

Theorem 4.2.6 *Let $A \subseteq B$ be a ring extension such that $mB \subseteq A$, for some regular element $m \in B$. Then B is a strong (A)-ring (resp. an (A)-ring) if and only if A is a strong(A)-ring (resp. an (A)-ring).*

Proof. Assume that B is a strong (A)-ring, and let $I = \sum_{i=1}^{n} Aa_i$ be a finitely generated ideal of A such that $a_i \in Z(A) \subseteq Z(B)$. Set $J = \sum_{i=1}^{n} Ba_i$, then J has a nonzero annihilator. Hence there exists $0 \neq b \in B$ such that $ba_i = 0$ for each $i = 1, 2, ..., n$. Therefore, $mba_i = 0$ with $0 \neq mb \in A$. It follows that I has a nonzero annihilator and so A is a strong (A)-ring. Conversely, assume that A is a strong (A)-ring, and let $J = \sum_{i=1}^{n} Bb_i$ be a finitely generated ideal of B such that $b_i \in Z(B)$. Set $I = \sum_{i=1}^{n} Amb_i$ be an ideal of A. As $b_i \in Z(B)$, we get

$mb_i \in Z(A)$ for each $i = 1...n$. Hence I has a nonzero annihilator since A is a strong (A)-ring. Therefore there exists $0 \neq a \in A$ such that $amb_i = 0$ for each $i = i = 1, 2, ..., n$, and so $aJ = 0$. Then B is a strong (A)-ring.

Similarly suppose that B is an (A)-ring, and let $I = \sum_{i=1}^{n} Aa_i$ be a finitely generated ideal of A such that $I \subseteq Z(A)$. Set $J = \sum_{i=1}^{n} Ba_i$. We wish to show that $J \subseteq Z(B)$. Let $a \in J$, hence $a = \sum_{i=1}^{n} \alpha_i a_i$ for some $\alpha_i \in B$. Clearly, we have $am \in I \subseteq Z(A)$ which implies that $a \in Z(B)$. Then J has a nonzero annihilator since B is an (A)-ring . Hence there exists $0 \neq b \in B$ such that $ba_i = 0$ for each $i = i = 1, 2, ..., n$. Therefore, $mba_i = 0$ with $0 \neq mb \in A$. It follows that I has a nonzero annihilator and so A is an (A)-ring. Conversely, assume that A is an (A)-ring, and let $J = \sum_{i=1}^{n} Bb_i$ be a finitely generated ideal of B such $J \subseteq Z(B)$. Set $I = \sum_{i=1}^{n} Amb_i$ be an ideal of A. Clearly, $I \subseteq J \subseteq Z(B)$. Thus, for each $a \in I$, there is $0 \neq b \in B$ such that $ba = 0$. So $mba = 0$ with $0 \neq mb \in A$ which implies that $I \subseteq Z(A)$. Hence I has a nonzero annihilator since A is an (A)-ring. Then there exists $0 \neq a \in A$ such that $amb_i = 0$ for each $i = 1, 2, ..., n$, and so $aJ = 0$. Therefore B is an (A)-ring.

An immediate consequence of Theorem 4.2.6 is the next corollary.

Corollary 4.2.7 *Let $A \subseteq B$ be a ring extension and $B[X]$ the polynomial ring over B. Then $A + XB[X]$ is an (A)-ring.*

Proof. The proof is clear by Theorem 4.2.6 and [70, Corollary 1] since $A + XB[X] \subseteq B[X]$, $XB[X] \subseteq A + XB[X]$ and $B[X]$ is an (A)-ring.

A pair of rings $A \subseteq B$ is called a strong (A)-ring (resp. an (A)-ring) pair if C is a strong (A)-ring (resp. an (A)-ring) for each ring C such that $A \subseteq C \subseteq B$. In the context of Theorem 4.2.6, we obtain:

Corollary 4.2.8 *Let A, B, and m be as in Theorem 4.2.6. Then (A,B) is a strong (A)-ring (resp. an (A)-ring) pair if and only if B is a strong (A)-ring (resp. an (A)-ring).*

Proof. Let C be a ring such that $A \subseteq C \subseteq B$. Hence, $mB \subseteq A \subseteq C$. Therefore, by Theorem 4.2.6 C is a strong (A)-ring (resp. an (A)-ring) if and only if B is a strong (A)-ring (resp. an (A)-ring).

Recall that a regular ideal of a ring R is an ideal which contains a regular element of R. Our next result studies the transfer of (A)-property and strong (A)-property in Bi-amalgamated algebras along ideals.

Theorem 4.2.9 *Let $f : A \to B$ and $g : A \to C$ be two ring homomorphisms and let J and J' be two ideals of B and C, respectively, such that $f^{-1}(J) = g^{-1}(J')$. Then :*

(1) If $A \bowtie^{f,f} (J, J)$ is a strong (A)-ring (resp. an (A)-ring), then so is $f(A) + J$.

(2) Assume that $J \times J'$ is a regular ideal of $B \times C$. Then :
(a) $A \bowtie^{f,g} (J,J')$ is an (A)-ring if and only if B and C are (A)-rings.
(b) $A \bowtie^{f,g} (J,J')$ is not a strong (A)-ring.

Lemma 4.2.10 *Let $f : A \to B$ and $g : A \to C$ be two ring homomorphisms and let J and J' be two ideals of B and C, respectively, such that $f^{-1}(J) = g^{-1}(J')$. Assume that J and J' are regular ideals of B and C, respectively. Then $Q(A \bowtie^{f,g} (J,J'))$ is canonically isomorphic to $Q(B) \times Q(C)$.*

Proof. As $J \times J'$ is the conductor of $A \bowtie^{f,g} (J,J')$ in $B \times C$, reasoning as in the proof of [34, Proposition 3.1], the result becomes clear.

Proof of Thoerem 4.2.9. (1) Assume that $A \bowtie^{f,f} (J,J)$ is a strong (A)-ring, and let $K = \sum_{i=1}^{n}(f(A)+J)(f(a_i)+j_i)$ be a finitely generated ideal of $f(A)+J$ such that $f(a_i)+j_i \in Z(f(A)+J)$ for each $i = i = 1,2,...,n$. Hence, there is $(0,0) \neq f(b_i)+t_i \in f(A)+J$ such that $(f(b_i)+t_i)(f(a_i)+j_i) = 0$. Now, set $H = \sum_{i=1}^{n} A \bowtie^{f,f} (J,J)(f(a_i)+j_i, f(a_i)+j_i)$ be a finitely generated ideal of $A \bowtie^{f,f} (J,J)$. Clearly, $(f(a_i)+j_i, f(a_i)+j_i) \in Z(A \bowtie^{f,f} (J,J))$ since $(f(a_i)+j_i, f(a_i)+j_i)(f(b_i)+t_i, f(b_i)+t_i) = (0,0)$. As $A \bowtie^{f,f} (J,J)$ is a strong (A)-ring, there is $(0,0) \neq (f(a)+i, f(a)+j) \in A \bowtie^{f,f} (J,J)$ such that $(f(a)+i, f(a)+j)H = 0$. If $f(a)+i \neq 0$, then $(f(a)+i)K = 0$. Now, assume that $f(a)+i = 0$, then $f(a)+j \neq 0$ and so $(f(a)+j)K = 0$. In all cases, K has a nonzero annihilator. So $f(A)+J$ is a strong (A)-ring.
Now assume that $A \bowtie^{f,f} (J,J)$ is an (A)-ring, and let $K = \sum_{i=1}^{n}(f(A)+J)(f(a_i)+j_i)$ be a finitely generated ideal of $f(A)+J$ such that $K \subseteq Z(f(A)+J)$. Set $H = \sum_{i=1}^{n} A \bowtie^{f,f} (J,J)(f(a_i)+j_i, f(a_i)+j_i)$ be a finitely generated ideal of $A \bowtie^{f,f} (J,J)$. Let $(f(a)+i, f(a)+j) \in H$, then $(f(a)+i, f(a)+j) = \sum_{i=1}^{n}(f(\alpha_i)+\beta_i, f(\alpha_i)+\lambda_i)(f(a_i)+j_i, f(a_i)+j_i)$. Clearly, $f(a)+i \in K$ and so there is $0 \neq f(b)+k \in f(A)+J$ such that $(f(a)+i)(f(b)+k) = 0$. Hence, if $(f(b)+k)J = 0$, then $(f(b)+k, f(b)+k)(f(a)+i, f(a)+j) = (f(b)+k, f(b)+k)(f(a)+i, f(a)+i-i+j) = (0,0)$. Now, assume that $(f(b)+k)t \neq 0$ for some $t \in J$, then $((f(b)+k)t, 0)(f(a)+i, f(a)+j) = (0,0)$. In all cases we get $H \subseteq Z(A \bowtie^{f,f} (J,J))$. As $A \bowtie^{f,f} (J,J)$ is an (A)-ring, there is $(0,0) \neq (f(a)+i, f(a)+j) \in A \bowtie^{f,f} (J,J)$ such that $(f(a)+i, f(a)+j)H = 0$. If $f(a)+i \neq 0$ then $(f(a)+i)K = 0$. Now, assume that $f(a)+i = 0$, then $f(a)+j \neq 0$ and $(f(a)+j)K = 0$. The conclusion is clear.
(2-a) As $J \times J'$ is a regular ideal of $B \times C$, by Lemma 4.2.10, we get $Q(A \bowtie^{f,g} (J,J')) \cong Q(B) \times Q(C)$. By [68, Proposition 2.14], $A \bowtie^{f,g} (J,J')$ is an (A)-ring if and only if so is $Q(B) \times Q(C)$. So, by using [68, Proposition 1.3], $Q(B)$ (resp. $Q(C)$) is an (A)- ring, if and only if B (resp. C) is an (A)-ring. Hence, $A \bowtie^{f,g} (J,J')$ is an (A)-ring if and only if so are B and C.
(2-b) By using Lemma 4.2.10, we get $Q(A \bowtie^{f,g} (J,J')) \cong Q(B) \times Q(C)$. The result follows immediately from [64, Lemma 2.3] and Proposition 4.2.5.

The following corollaries are immediate applications of part (2) of Theorem 4.2.9.

Corollary 4.2.11 *Let $(A;B;C)$ be integral domains, $f : A \to B$ and $g : A \to C$ be two ring homomorphisms and let J and J' be two nonzero ideals of B and C, respectively, such that $f^{-1}(J) = g^{-1}(J')$. Then $A \bowtie^{f,g} (J,J')$ is an (A)-ring which is not a strong (A)-ring.*

Recall that the amalgamation of A with B along an ideal J of B with respect the ring homomorphism $f : A \to B$, is given by

$$A \bowtie^f J := \big\{(a, f(a)+j) \mid a \in A, j \in J\big\}.$$

Clearly, every amalgamation can be viewed as a special bi-amalgamation, since $A \bowtie^f J = A \bowtie^{id_A,f} (f^{-1}(J),J)$. The following result recovers [85, Theorem 2.1].

Corollary 4.2.12 *Under the above notation, assume that $f^{-1}(J) \times J$ is a regular ideal of $A \times B$. Then $A \bowtie^f J$ is an (A)-ring if and only if A and B are (A)-rings.*

Let I be a proper ideal of A. The (amalgamated) duplication of A along I is a special amalgamation given by

$$A \bowtie I := A \bowtie^{id_A} I = \big\{(a, a+i) \mid a \in A, i \in I\big\}.$$

The next corollary is an immediate consequence of Corollary 4.2.12 on the transfer of property (A) and strong property (A) to duplications.

Corollary 4.2.13 *Let A be a ring and I be a regular ideal of A. Then $A \bowtie I$ is an (A)-ring if and only if A is an (A)-ring.*

The next lemma study the set of zero-divisors of the bi-amalgamation of ring.

Lemma 4.2.14 *Let $f : A \to B$ and $g : A \to C$ be two ring homomorphisms and let J and J' be two ideals of B and C, respectively, such that $f^{-1}(J) = g^{-1}(J')$. Set $L = \{(f(a)+i, g(a)+j) \in A \bowtie^{f,g} (J,J') \mid f(a)+i \in Z(f(A)+J)\}$ and $K = \{(f(a)+i, g(a)+j) \in A \bowtie^{f,g} (J,J') \mid (g(a)+j)k = 0$ for some $0 \neq k \in J'\}$. Then $Z(A \bowtie^{f,g} (J,J')) \subseteq L \cup K$, with equality if at least one of the following conditions holds:*

(1) *$Ann(J) = 0$.*

(2) *f is injective, $J \subseteq f(A)$ and $J' \subseteq Nil(C)$.*

(3) *$Z(f(A)+J) \subseteq J$ and $J \neq 0$.*

Proof. Let $(f(a)+i, g(a)+j) \in \mathrm{Z}(A \bowtie^{f,g} (J,J'))$. Then $(f(a)+i, g(a)+j)(f(b)+i', g(b)+j') = (0,0)$ for some $(0,0) \neq (f(b)+i', g(b)+j') \in A \bowtie^{f,g} (J,J')$. Hence, $(f(a)+i)(f(b)+i') = 0$ and $(g(a)+j)(g(b)+j') = 0$. If $f(b)+i' \neq 0$, then $(f(a)+i, g(a)+j) \in L$. Now, assume that $f(b)+i' = 0$, hence $0 \neq g(b)+j' \in J'$ and so $(f(a)+i, g(a)+j) \in K$. This proves the inclusion.
To complete the proof, it is enough for us to show that $L \cup K \subseteq \mathrm{Z}(A \bowtie^{f,g} (J,J'))$ under the validity of any of condition 1)-3). Let $(f(a)+i, g(a)+j) \in K$. Hence, there is $0 \neq k \in J'$ such that $(g(a)+j)k = 0$. Then, $(f(a)+i, g(a)+j)(0,k) = (0,0)$ and so $(f(a)+i, g(a)+j) \in \mathrm{Z}(A \bowtie^{f,g} (J,J'))$. It remains to show that $L \subseteq \mathrm{Z}(A \bowtie^{f,g} (J,J'))$. Let $(f(a)+i, g(a)+j) \in L$ such that $(f(a)+i)(f(b)+k) = 0$ for some $0 \neq f(b)+k \in f(A)+J$.
(1) Assume that Ann(J)$= 0$, then there exists $t \in J$ such that $(f(b)+k)t \neq 0$. Therefore $(f(b)+k)t, 0)(f(a)+i, g(a)+j) = (0,0)$ and so $(f(a)+i, g(a)+j) \in \mathrm{Z}(A \bowtie^{f,g} (J,J'))$.
(2) As $J \subseteq f(A)$, we have $f(A)+J = f(A)$. Then $(f(a)+i)(f(b)+k) = f(a')f(b') = 0$ and so $a'b' = 0$ since f is injective. Therefore, $(f(a'), g(a'))(f(b'), g(b')) = (0,0)$. On the other hand, we have $(f(a)+i, g(a)+j) = (f(a'), g(a')+j') = (f(a'), g(a')) + (0, j')$. Since $J' \subseteq \mathrm{Nil}(C)$ and $(f(a'), g(a')) \in \mathrm{Z}(A \bowtie^{f,g} (J,J'))$, we get $(f(a)+i, g(a)+j) \in \mathrm{Z}(A \bowtie^{f,g} (J,J'))$.
(3) Assume that $f(a)+i \neq 0$. Since $\mathrm{Z}(f(A)+J) \subseteq J$ we get $f(b)+k \in J$. Hence $(f(a)+i, g(a)+j)(f(b)+k, 0) = (0,0)$. If $f(a)+i = 0$, then $(0, g(a)+j)(t,0) = (0,0)$ for some $0 \neq t \in J$. This proves the equality.

The following result is a partial result when a bi-amalgamation is a strong (A)-ring.

Proposition 4.2.15 *Let $f : A \to B$ and $g : A \to C$ be two ring homomorphisms and let J and J' be two ideals of B and C, respectively, such that $f^{-1}(J) = g^{-1}(J')$. Then :*

(1) Assume that $f(A)+J$ and $g(A)+J'$ are Noetherian rings, $f(A)+J$ is reduced and $J' \cap$ Nil(C)$= 0$. Then $A \bowtie^{f,g} (J,J')$ is a strong (A)-ring if and only if $J = 0$ and $g(A)+J'$ is a domain, or $J' = 0$ and $f(A)+J$ is a domain.

(2) Assume that J' is a nonzero ideal of C such that $J'^2 = 0$. Then $A \bowtie^{f,g} (J,J')$ is a strong (A)-ring if at least one of the following conditions holds:

(a) A and $f(A)+J$ are domains and J' is a torsion free A-module (with the structure induced by g).

(b) A is local with maximal ideal M, $J \times J' \subseteq Jac(B \times C)$ and $g(M)J' = 0$.

Proof. (1) Assume that $f(A)+J$ and $g(A)+J'$ are Noetherian rings, $f(A)+J$ is reduced and $J' \cap$ Nil(C)$= 0$. It follows from [74, Proposition 4.2 and 4.7] that $A \bowtie^{f,g} (J,J')$ is a Noetherian reduced ring. By [48, Proposition 2.1], $A \bowtie^{f,g} (J,J')$ is a strong (A)-ring if and only if $A \bowtie^{f,g} (J,J')$ is a domain. By [74, Proposition 4.5], $A \bowtie^{f,g} (J,J')$ is a domain

if and only if $J = 0$ and $g(A) + J'$ is a domain, or $J' = 0$ and $f(A) + J$ is a domain. This complete the proof of (1).
(2-a) Assume that A and $f(A) + J$ are domains, J' is a nonzero ideal of C such that $J'^2 = 0$, and J' is a torsion free A-module. Let $H = \sum_{i=1}^{n} A \bowtie^{f,g} (J,J')(f(a_i) + j_i, g(a_i) + j_i')$ be a finitely generated ideal of $A \bowtie^{f,g} (J,J')$ such that $(f(a_i) + j_i, g(a_i) + j_i') \in \mathrm{Z}(A \bowtie^{f,g} (J,J'))$. Our aim is to show that $\mathrm{Z}(A \bowtie^{f,g} (J,J')) \subseteq J \times J'$. By Lemma 4.2.14, we have $\mathrm{Z}(A \bowtie^{f,g} (J,J')) \subseteq L \cup K$ and as $f(A) + J$ is a domain, we get $L = \{0\} \times J' \subseteq J \times J'$. Now let $(f(a) + i, g(a) + j) \in K$. Then there is $0 \neq k \in J'$ such that $(g(a) + j)k = 0$ and so $ak = 0$ since $J'^2 = 0$. As J' is a torsion free A-module, we have $a = 0$. This implies that $(f(a) + i, g(a) + j) = (i, j) \in J \times J'$. Therefore $\mathrm{Z}(A \bowtie^{f,g} (J,J')) \subseteq J \times J'$. But since $J'^2 = 0$, we get $(\{0\} \times J')J \times J' = (0,0)$. Then there is $0 \neq k \in J'$ such that $(0,k)(f(a_i) + j_i, g(a_i) + j_i') = (0,0)$ for each $i = 1, 2, ..., n$, that implies $(0,k)H = 0$. Hence $A \bowtie^{f,g} (J,J')$ is a strong (A)-ring, as desired.
(2-b) Since A is local with maximal ideal M and $J \times J' \subseteq Jac(B \times C)$, we get by [74, Proposition 5.4(2)] $A \bowtie^{f,g} (J,J')$ is local with maximal ideal $M \bowtie^{f,g} (J,J')$. As $J'^2 = 0$ and $g(M)J' = 0$, we get $(0, j)M \bowtie^{f,g} (J,J') = 0$ for each $0 \neq j \in J'$. Hence $\mathrm{Z}(A \bowtie^{f,g} (J,J'))$= $M \bowtie^{f,g} (J,J')$ and so for every finitely generated ideal H of $A \bowtie^{f,g} (J,J')$ which is generated by a finite number of zero-divisors elements of $M \bowtie^{f,g} (J,J')$ we have $(0, j)H = 0$ for each $0 \neq j \in J'$. Therefore $A \bowtie^{f,g} (J,J')$ is a strong (A)-ring .This complete the proof.

Proposition 4.2.15 enriches the literature with new original examples of strong (A)-rings. Recall that for a ring A and an A-module E, the trivial ring extension of A by E (also called idealization of E over A) is the ring $R := A \ltimes E$ whose underlying group is $A \times E$ with multiplication given by $(a,e)(a',e') = (aa', ae' + a'e)$.

Example 4.2.16 *Let $(A,m) := (A_1 \ltimes E_1, m_1 \ltimes E_1)$ be the trivial ring extension of A_1 by E_1, (for instance $(A_1,m_1) := (\mathbb{Z}/4\mathbb{Z}, 2\mathbb{Z}/4\mathbb{Z})$) and E_1 is a nonzero $\frac{A_1}{m_1}$-vector space (for instance $E_1 = (\mathbb{Z}/4\mathbb{Z})/(2\mathbb{Z}/4\mathbb{Z})$). Let $B := A_1$. Consider*

$$\begin{array}{rccl} f: & A & \to & B \\ & (a,e) & \to & f((a,e)) = a; \end{array}$$

Set $J = m_1$ the maximal ideal of B. Let $C := A \ltimes E$ be the trivial ring extension of A by a nonzero A/m-vector space E and let

$$\begin{array}{rccl} g: & A & \hookrightarrow & C \\ & (a,e) & \hookrightarrow & g((a,e)) = ((a,e),0); \end{array}$$

Set $J' := m \ltimes E = (m_1 \ltimes E_1) \ltimes E$ the maximal ideal of C. Clearly, $f^{-1}(J) = g^{-1}(J') = m_1 \ltimes E_1$. Then :
(1) By Proposition 4.2.15 (2-b) $A \bowtie^{f,g} (J,J')$ is a strong (A)-ring.
(2) Clearly, $Q(A \bowtie^{f,g} (J,J')) = A \bowtie^{f,g} (J,J')$ is not a Von Neumann Regular ring since it is not reduced by [74, Proposition 4.7].

The next result is another partial result when a Bi-amalgamation is an (A)-ring (resp. a strong (A)-ring) in case J and J' are not regular ideals of $f(A)+J$ and $g(A)+J'$ respectively.

Theorem 4.2.17 *Let $f : A \to B$ and $g : A \to C$ be two ring homomorphisms and let J and J' be two ideals of B and C, respectively, such that $f^{-1}(J) = g^{-1}(J')$. Then*

(1) Assume that $f(Reg(A))=Reg(B)$, and $J^2 = 0$. If A is an (A)-ring (resp. a strong (A)-ring), then so is $A \bowtie^f J$.

(2) Suppose that $J = Z(f(A)+J)$ and $Z(g(A)+J') \subseteq J'$. If $f(A)+J$ is an (A)-ring (resp. a strong (A)-ring), then so is $A \bowtie^{f,g} (J,J')$.

Proof. (1) Assume that A is an (A)-ring (along with the hypotheses that $f(\mathrm{Reg}(A))=\mathrm{Reg}(B)$, and $J^2 = 0$). Let $K = \sum_{i=1}^{n} A \bowtie^f J(a_i, f(a_i)+j_i) \subseteq Z(A \bowtie^f J)$ be a finitely generated ideal of $A \bowtie^f J$. Consider $H = \sum_{i=1}^{n} Aa_i$. Our aim is to show that $H \subseteq \mathrm{Z}(A)$. Let $a \in H$, so there is $j \in J$ such that $(a, f(a)+j) \in K$. Hence $(a, f(a)+j)(b, f(b)+k) = (0,0)$ for some $(b, f(b)+k) \neq (0,0)$. If $b \neq 0$, clearly $a \in \mathrm{Z}(A)$. Suppose that $b = 0$, then $k \neq 0$ and $(f(a)+j)k = 0$. As $J^2 = 0$ we get $f(a)k = 0$ and $f(a) \in \mathrm{Z}(B)$. Using the fact $f(\mathrm{Reg}(A))=\mathrm{Reg}(B)$, $a \in \mathrm{Z}(A)$. Therefore there exists a nonzero element $b \in A$ such that $bH = 0$ since A is an (A)-ring. If $f(b) \in \mathrm{Ann}(J)$, then $(b, f(b))(a_i, f(a_i)+j_i) = (0,0)$ for all $i = 1,2,...,n$. If $f(b) \notin \mathrm{Ann}(J)$, then there is a nonzero element $t \in J$ such that $f(b)t \neq 0$. Hence $(0, f(b)t)(a_i, f(a_i)+j_i) = (0,0)$ for each $i = 1,2,...,n$. It follows that K has a nonzero annihilator in all cases. Therefore $A \bowtie^f J$ is an (A)-ring.
Now, similarly assume that A is a strong (A)-ring . Let $K = \sum_{i=1}^{n} A \bowtie^f J(a_i, f(a_i)+j_i)$ be a finitely generated ideal of $A \bowtie^f J$ such that $(a_i, f(a_i)+j_i) \in Z(A \bowtie^f J)$. Consider $H = \sum_{i=1}^{n} Aa_i$. We wish to show that $a_i \subseteq \mathrm{Z}(A)$ for each $i = 1,2,...,n$. Suppose that there is $i \in \{1,2,...,n\}$ such that a_i is regular. As $(a_i, f(a_i)+j_i)$ is zero-divisor, there exists $(b, f(b)+k) \neq (0,0)$ such that $(a_i, f(a_i)+j_i)(b, f(b)+k) = (0,0)$. Since a_i is regular, we have $b = 0$ and $f(a_i)+j_i \in \mathrm{Z}(B)$. As $J^2 = 0$ we get $f(a_i) \in \mathrm{Z}(B)$. Using the fact $f(\mathrm{Reg}(A))=\mathrm{Reg}(B)$, we have $a_i \in \mathrm{Z}(A)$, a contradiction. Therefore there exists a nonzero element $b \in A$ such that $bH = 0$ since A is a strong (A)-ring. If $f(b) \in \mathrm{Ann}(J)$, then $(b, f(b))(a_i, f(a_i)+j_i) = (0,0)$ for all $i = 1,2,...,n$. If $f(b) \notin \mathrm{Ann}(J)$, then there is a nonzero element $t \in J$ such that $f(b)t \neq 0$. Hence $(0, f(b)t)(a_i, f(a_i)+j_i) = (0,0)$ for each $i = 1,2,...,n$. It follows that K has a nonzero annihilator in all cases. Therefore $A \bowtie^f J$ is a strong (A)-ring.
(2) Assume that $f(A)+J$ is an (A)-ring. Let $K = \sum_{i=1}^{n} A \bowtie^{f,g} (J,J')(f(a_i)+j_i, g(a_i)+j'_i) \subseteq Z(A \bowtie^{f,g} (J,J'))$ be a finitely generated ideal of $A \bowtie^{f,g} (J,J')$. Set $H = \sum_{i=1}^{n} (f(A)+J)(f(a_i)+j_i)$ and let $f(a)+j \in H$. Hence $(f(a)+j, g(a)+j') \in K$ for some $j' \in J'$. Then, there exists $(f(b)+i, g(b)+k) \neq (0,0)$ such that $(f(a)+j, g(a)+j')(f(b)+i, g(b)+k) = (0,0)$. If $f(b)+i \neq 0$, then $f(a)+j \in Z(f(A)+J)$. Assume that $f(b)+i = 0$. Hence $g(b)+k \neq 0$ and $g(a)+j' \in \mathrm{Z}(g(A)+J') \subseteq J'$, which implies that $g(a) \in J'$. Therefore

$f(a) \in J$ and so $f(a)+j \in J = Z(f(A)+J)$. Hence H has a nonzero annihilator since $f(A)+J$ is an (A)-ring. Then there exists a nonzero element $f(b)+j \in f(A)+J$ such that $(f(b)+j)H = 0$. It follows that $f(b)+j \in J$ and so $(f(b)+j,0)K = (0,0)$. Therefore K has a nonzero annihilator, as desired.
Assume that $f(A)+J$ is a strong (A)-ring. Let $K = \sum_{i=1}^{n} A \bowtie^{f,g} (J,J')(f(a_i)+j_i, g(a_i)+j'_i)$ be a finitely generated ideal of $A \bowtie^{f,g} (J,J')$ such that $(f(a_i)+j_i, g(a_i)+j'_i) \in Z(A \bowtie^{f,g} (J,J'))$ for each $i = 1,2,...,n$. Set $H = \sum_{i=1}^{n} (f(A)+J)(f(a_i)+j_i)$. Let $i \in \{1,2,...,n\}$. Then, there exists $(f(b)+i, g(b)+k) \neq (0,0)$ such that $(f(a_i)+j_i, g(a_i)+j'_i)(f(b)+i, g(b)+k) = (0,0)$. If $f(b)+i \neq 0$, then $f(a_i)+j_i \in Z(f(A)+J)$. Suppose that $f(b)+i = 0$. Hence $g(b)+k \neq 0$ and $g(a_i)+j'_i \in Z(g(A)+J') \subseteq J'$, which implies that $g(a_i) \in J'$. Therefore $f(a_i) \in J$ and so $f(a_i)+j_i \in J = Z(f(A)+J)$. Hence H has a nonzero annihilator since $f(A)+J$ is a strong (A)-ring. Then there is $0 \neq f(b)+j \in f(A)+J$ such that $(f(b)+j)H = 0$. It follows that $f(b)+j \in J$ and so $(f(b)+j,0)K = (0,0)$. Therefore K has a nonzero annihilator and so $A \bowtie^{f,g} (J,J')$ is a strong (A)-ring. This completes the proof.

The next corollary is an immediate application of Theorem 4.2.17 and [64, Theorem 3.1 (1) and (2-a)].

Corollary 4.2.18 *Let A be a ring and I be an ideal of A such that $I^2 = 0$ or $I = Z(A)$. Then $A \bowtie I$ is an (A)-ring (resp. a strong (A)-ring) if and only if A is an (A)-ring (resp. a strong (A)-ring).*

Example 4.2.19 *Let $f : \mathbb{Z} \longrightarrow \frac{\mathbb{Z}}{4\mathbb{Z}}$ and $g : \mathbb{Z} \longrightarrow \frac{\mathbb{Z}}{8\mathbb{Z}}$ be the canonical surjections. Set $J = \frac{2\mathbb{Z}}{4\mathbb{Z}}$ and $J' = \frac{2\mathbb{Z}}{8\mathbb{Z}}$. We have $f^{-1}(J) = g^{-1}(J') = 2\mathbb{Z}$. By Theorem 4.2.17(2) $A \bowtie^{f,g} (J,J')$ is a strong (A)-ring (resp. an (A)-ring).*

CHAPTER 5

AMALGAMATED ALGEBRAS ISSUED FROM ϕ-CHAINED RINGS AND ϕ-PVRS

Abstract.

Let $f : A \longrightarrow B$ be a ring homomorphism and let J be an ideal of B. In this chapter [*], we study the possible transfer of the properties of being a ϕ-ring, a ϕ-chained ring, and a ϕ-pseudo-valuation ring to the amalgamated algebra $A \bowtie^f J$. Our aim is to provide examples of new classes of commutative rings satisfying the aforementioned properties.

5.1 Introduction

In [65], Hedstrom and Houston introduced the class of pseudo-valuation domains, which is closely related to the class of valuation domains. A domain R with quotient field K is called a pseudo-valuation domain (in short, PVD) if, whenever P is a prime ideal of R and $xy \in P$ with $x \in K$ and $y \in K$, then either $x \in P$ or $y \in P$. Any valuation domain is a PVD [65, Proposition 1.1]; any PVD is local, in the sense that it has a unique maximal ideal [65, Corollary 1.3]. Additional information about PVDs can be found in the interesting survey article [19]. In [3], D. F. Anderson, Badawi and Dobbs generalized the study of PVDs to the context of arbitrary rings (possibly with nontrivial zero-divisors) as follows. A prime ideal P of a ring R is called a *strongly prime ideal* of R if aP and bR are comparable (under inclusion) for all $a, b \in R$; a ring R is called a *pseudo-valuation ring* (in short, a *PVR*) if each prime ideal of R is strongly prime. Any chained ring is a PVR [3, Corollary 4]; the prime ideals of any PVR are linearly ordered by inclusion (whence any PVR must be local) [3, Lemma 1]. A ring R is a PVR if and only if it is local and its maximal ideal is a strongly prime ideal of R [3, Theorem 2]. Recall from [18, 40] that a

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prime ideal P of R is said to be *divided* if it is comparable to every ideal of R, equivalently if $P \subseteq (x)$ for any $x \in R \setminus P$. A ring R is called a *divided ring* if every prime ideal of R is divided. Recently A. Badawi, in [13, 14, 15, 16, 17], has studied the following class of rings: $\mathscr{H} = \{R \mid R$ is a commutative ring and $\mathrm{Nil}(R)$ is a divided prime ideal of $R\}$. If $R \in \mathscr{H}$, then R is called a *ϕ-ring*. It is obvious by the definition that every integral domain is a ϕ-ring. An ideal I of R is said to be a *nonnil ideal* if $I \nsubseteq \mathrm{Nil}(R)$. If I is a nonnil ideal of a ϕ-ring R, then $\mathrm{Nil}(R) \subseteq I$. In [14], the author gave another generalization of pseudo valuation domains to the context of arbitrary rings (possibly with nonzero zerodivisors). Let R be a ring with total quotient ring T such that $\mathrm{Nil}(R)$ is a divided prime ideal of R. As in [14], we define $\phi : T \longrightarrow K := R_{\mathrm{Nil}(R)}$ such that $\phi(\frac{a}{b}) = \frac{a}{b}$ for every $a \in R$ and every $b \in R \setminus Z(R)$. Then ϕ is a ring homomorphism from T into K, and ϕ restricted to R is also a ring homomorphism from R into K given by $\phi(x) = \frac{x}{1}$ for every $x \in R$. A prime ideal Q of $\phi(R)$ is said to be *K-strongly prime* if $xy \in Q$, $x \in K$, $y \in K$ implies that either $x \in Q$ or $y \in Q$. A prime ideal P of R is said to be *ϕ-strongly prime* if $\phi(P)$ is a K-strongly prime ideal of $\phi(R)$. If each prime ideal of R is ϕ-strongly prime, then R is called a *ϕ-pseudo-valuation ring* (in short *ϕ-PVR*). It is shown in [14, Corollary 7(2)] that a ring R is a ϕ-PVR if and only if $\mathrm{Nil}(R)$ is a divided prime ideal and for every $a, b \in R \setminus \mathrm{Nil}(R)$, either $a \mid b$ or $b \mid ac$ in R for each nonunit element $c \in R$. A ring R is called a *ϕ-chained ring* if $\mathrm{Nil}(R)$ is divided prime and for each $x \in R_{\mathrm{Nil}(R)} \setminus \phi(R)$ we have $x^{-1} \in \phi(R)$ [15]. Then it is easy to see that R is a ϕ-chained ring if and only if $\mathrm{Nil}(R)$ is divided prime and for every $a, b \in R \setminus \mathrm{Nil}(R)$ either $a \mid b$ or $b \mid a$.

This chapter investigates properties of ϕ-chained rings and ϕ-PVRs in the amalgamated algebra along an ideal. Our aim is to provide new classes of commutative rings satisfying these properties.

5.2 On ϕ-rings

Our first result characterizes when the amalgamation of rings is a ϕ-ring.

Theorem 5.2.1 *Let A and B be two rings, J a nonzero ideal of B, and $f : A \longrightarrow B$ be a ring homomorphism. Set $R := A \bowtie^f J$ and $N(J) := \mathrm{Nil}(B) \cap J$. Then*

1. *If J is a nonnil ideal of B, then R is a ϕ-ring if and only if $f^{-1}(J) = 0$, A is an integral domain, and $N(J)$ is a divided prime ideal of $f(A) + J$.*

2. *If $J \subseteq \mathrm{Nil}(B)$, then R is a ϕ-ring if and only if A is a ϕ-ring, and for each $i, j \in J$ and each $a \in A \setminus \mathrm{Nil}(A)$, there exist $x \in \mathrm{Nil}(A)$ and $k \in J$ such that $xa = 0$ and $j = kf(a) + i(f(x) + k)$.*

We need the following two lemmas.

Lemma 5.2.2 *Let A and B be two rings, J a nonzero ideal of B, and $f : A \longrightarrow B$ be a ring homomorphism. Let $0 \neq a \in A$ and $i \in J$. Then the following two conditions are equivalent for a principal ideal $R(a, f(a)+i)$ of $R := A \bowtie^f J$.*

1. *$R(a, f(a)+i)$ has the form $Aa \bowtie^f J$.*
2. *For each $j \in J$, there exist $x \in A$ and $k \in J$ such that $ax = 0$ and $j = kf(a) + i(f(x) + k)$.*

Proof. $(1) \Rightarrow (2)$ Assume that $R(a, f(a)+i)$ has the form $I \bowtie^f J$ with $I = Aa$. Then $\{0\} \times J \subseteq H$. Hence, for each $j \in J$ we have $(0, j) = (a, f(a)+i)(x, f(x)+k)$ for some $(x, f(x)+k) \in R$. Therefore $ax = 0$ and $j = kf(a) + i(f(x)+k)$.

$(2) \Rightarrow (1)$ It suffices to prove that $\{0\} \times J \subseteq R(a, f(a)+i)$. Let $j \in J$. Then there exist $x \in A$ and $k \in J$ such that $ax = 0$ and $j = kf(a) + i(f(a)+k)$. Hence $(0, j) = (a, f(a)+i)(x, f(x)+k)$, which implies that $\{0\} \times J \subseteq R(a, f(a)+i)$, as desired.

Lemma 5.2.3 *Let A and B be two rings, J a nonzero ideal of B, and $f : A \longrightarrow B$ be a ring homomorphism. If $A \bowtie^f J$ is a ϕ-ring, then the following conditions hold:*

1. *$f^{-1}(J) \subseteq \mathrm{Nil}(A)$, and if $f^{-1}(J) \neq 0$, then $J \subseteq \mathrm{Nil}(B)$*
2. *A and $f(A) + J$ are ϕ-rings.*

Proof. Clearly $\mathrm{Nil}(A \bowtie^f J) = \{(a, f(a) + j) \mid a \in \mathrm{Nil}(A), j \in N(J)\}$.

(1) If $f^{-1}(J) = 0$, then $f^{-1}(J) \subseteq \mathrm{Nil}(A)$. Hence we may assume that $f^{-1}(J) \neq 0$. Assume on the contrary that $f^{-1}(J)$ contains a non-nilpotent element, say i. Let $j \in J$. Then we have $(i,0)(0, j) = (0,0) \in \mathrm{Nil}(A \bowtie^f J)$. Since $A \bowtie^f J$ is a ϕ-ring, we get that $\mathrm{Nil}(A \bowtie^f J)$ is prime. Since $(i,0) \notin \mathrm{Nil}(A \bowtie^f J)$, we have $(0, j) \in \mathrm{Nil}(A \bowtie^f J)$. Thus $J \subseteq \mathrm{Nil}(B)$. Hence $\mathrm{Nil}(A) \times J \subseteq f^{-1}(J) \times \{0\}$ since $\mathrm{Nil}(A \bowtie^f J)$ is divided. So we get $J = 0$, which is a contradiction. Therefore $f^{-1}(J) \subseteq \mathrm{Nil}(A)$.

Assume that $f^{-1}(J) \neq 0$ and suppose on the contrary that $J \nsubseteq \mathrm{Nil}(B)$. By similar reasoning as above, we have $f^{-1}(J) \subseteq \mathrm{Nil}(A)$, and so $f^{-1}(J) \times \{0\} \subseteq \mathrm{Nil}(A) \times N(J) \subseteq \{0\} \times J$. Therefore we have $f^{-1}(J) = 0$, which is absurd.

(2) Since $A \bowtie^f J$ is a ϕ-ring, it follows by (1) that $f^{-1}(J) \subseteq \mathrm{Nil}(A)$. Now the result follows from [93, Theorem 2.1].

Proof of Theorem 5.2.1 (1) Since $J \nsubseteq \mathrm{Nil}(B)$, by Lemma 5.2.3 we get $f^{-1}(J) = 0$. Now let j be a non-nilpotent element of J. As R is a ϕ-ring, we have $\mathrm{Nil}(R) \subseteq R(0, j)$. Hence $\mathrm{Nil}(R) = \{0\} \times N(J)$. Let $a, b \in A$ such that $ab = 0$. Then $(a, f(a))(b, f(b)) = (0,0) \in \mathrm{Nil}(R)$. Since $\mathrm{Nil}(R)$ is prime, we have either $(a, f(a)) \in \mathrm{Nil}(R)$ or $(b, f(b)) \in \mathrm{Nil}(R)$, and so either $a = 0$ or $b = 0$. Thus A is an integral domain. As $\{0\} \times N(J)$ is a divided prime ideal of R, it is clear that $N(J)$ is a divided prime ideal of $f(A) + J$.

For the converse, we first prove that $\mathrm{Nil}(R) = \{0\} \times N(J)$. Clearly $\{0\} \times N(J) \subseteq \mathrm{Nil}(R)$. Let $x := (a, f(a) + j) \in \mathrm{Nil}(R)$. Then $a \in \mathrm{Nil}(A)$ and $j \in N(J)$. Since A is an integral domain, we have $a = 0$, and hence $x = (0, j) \in \{0\} \times N(J)$. Therefore, $\mathrm{Nil}(R) = \{0\} \times N(J)$. Now since $f^{-1}(J) = 0$ and $N(J)$ is a divided prime ideal of $f(A) + J$, we get that $\mathrm{Nil}(R)$ is a divided prime ideal of R. Hence R is a ϕ-ring.

(2) Assume that R is a ϕ-ring and $J \subseteq \mathrm{Nil}(B)$. Then A is a ϕ-ring by Lemma 5.2.3(2). Let $a \in A \setminus \mathrm{Nil}(A)$, it is clear that $(a, f(a)+i) \in R \setminus \mathrm{Nil}(R)$ for each $i \in J$. Hence $\{0\} \times J \subseteq R(a, f(a)+i)$, and so $R(a, f(a)+i)$ has the form $I \bowtie^f J$ for some ideal I of R. By Lemma 5.2.2, there exist $x \in A$ and $k \in J$ such that $ax = 0$ and $j = kf(a) + i(f(x) + k)$ for each $j \in J$. As $\mathrm{Nil}(A)$ is prime and $ax = 0 \in \mathrm{Nil}(A)$, we get that $x \in \mathrm{Nil}(A)$.

Conversely, since $J \subseteq \mathrm{Nil}(B)$, we have $\mathrm{Nil}(R) = \mathrm{Nil}(A) \bowtie^f J$. Hence $\mathrm{Nil}(R)$ is prime since $\mathrm{Nil}(A)$ is prime. Let $j \in J$ and $(a, f(a)+i) \in R \setminus \mathrm{Nil}(R)$. Since $a \in A \setminus \mathrm{Nil}(A)$, there exist $x \in A$ and $k \in J$ such that $ax = 0$ and $j = kf(a) + i(f(x) + k)$. By Lemma 5.2.3 $R(a, f(a)+i) = I \bowtie^f J$ for some ideal I of R. As $I \nsubseteq \mathrm{Nil}(A)$ and A is a ϕ-ring, we get that $\mathrm{Nil}(A) \subseteq I$. Therefore $\mathrm{Nil}(R) = \mathrm{Nil}(A) \bowtie^f J \subseteq I \bowtie^f J = R(a, f(a)+i)$. □

Let A be a ring and E be an A-module. Then $A \ltimes E$, the *trivial* (*ring*) *extension of A by E*, is the ring whose additive structure is that of the external direct sum $A \oplus E$ and whose multiplication is defined by $(a,e)(b,f) := (ab, af + be)$ for all $a, b \in A$ and all $e, f \in E$. (This construction is also known by other terminology and other notation, such as the *idealization* $A(+)E$.) The basic properties of trivial ring extensions are summarized in the books [60, 69]. Trivial ring extensions have been studied or generalized extensively, often because of their usefulness in constructing new classes of examples of rings satisfying various properties (cf., [8, 25, 75]). The next corollary studies when the trivial ring extension is a ϕ-ring.

Corollary 5.2.4 *Let A be a ring and E a nonzero A-module. Then $A \ltimes E$ is a ϕ-ring if and only if A is a ϕ-ring and $E = aE$ for each $a \in A \setminus \mathrm{Nil}(A)$. In particular, If A is an integral domain, then $A \ltimes E$ is a ϕ-ring if and only if E is a divisible A-module.*

Proof. Consider a ring homomorphism

$$\begin{array}{rccc} f & : & A & \hookrightarrow & A \ltimes E \\ & & a & \mapsto & f(a) = (a, 0) \end{array}$$

and a nonzero ideal $J := 0 \ltimes E$ of $A \ltimes E$. Then we have $A \bowtie^f J \cong A \ltimes E$ and $J \subseteq \mathrm{Nil}(A \ltimes E)$ since $J^2 = 0$.

Assume that $A \ltimes E$ is a ϕ-ring. Then so is $A \bowtie^f J$. By Theorem 5.2.1(2), A is a ϕ-ring and for each $a \in A \setminus \mathrm{Nil}(A)$ and $e, f \in E$, there exist $x \in \mathrm{Nil}(A)$ and $d \in E$ such that $e = ad + xf$. As $x \in \mathrm{Nil}(A)$ and $\mathrm{Nil}(A)$ is a divided prime ideal of A, we get that $x = at$ for some $t \in A$. Hence $e = ad + atf = a(d + tf) \in aE$. Therefore $E = aE$.

Conversely, let $a \in A \setminus \mathrm{Nil}(A)$ and $e, f \in E$. Then there exist $x = 0 \in \mathrm{Nil}(A)$ and $d \in E$ such that $xa = 0$ and $e = ad + xf$. Since A is a ϕ-ring, it follows from Theorem 5.2.1(2) that $A \ltimes E$ is a ϕ-ring.

Moreover, we give the following corollaries as consequences of the previous theorem.

Corollary 5.2.5 *Let A and B be two rings, J a nonzero ideal of B, and $f : A \longrightarrow B$ be a ring homomorphism. Assume that $J = \mathrm{Nil}(B)$ and $f^{-1}(J) \subseteq \mathrm{Nil}(A)$. If A and $f(A) + J$ are ϕ-rings, then so is $A \bowtie^f J$.*

Proof. Let $a \in A \setminus \mathrm{Nil}(A)$ and $i \in J$. Since $f(A) + J$ is a ϕ-ring and $J = \mathrm{Nil}(B)$, we have $\mathrm{Nil}(f(A)+J) = J$, which is a divided prime ideal of $f(A)+J$. Then $J \subseteq (f(A)+J)(f(a)+i)$. Hence for each $j \in J$ we have $j = (f(a)+i)(f(b)+t)$ for some $f(b)+t \in f(A)+J$. As J is prime and $f(a)+i \notin J$, we get $f(b)+t \in J$. Set $k := f(b)+t$ and $x := 0$. Then $ax = 0$ and $j = kf(a) + i(f(x)+k)$. Since A is a ϕ-ring, it follows by Theorem 5.2.1(2) that $A \bowtie^f J$ is a ϕ-ring.

Corollary 5.2.6 *Let A be a ring. Then the polynomial ring $A[X]$ is a ϕ-ring if and only if A is an integral domain.*

Proof. Consider the canonical injection $f : A \hookrightarrow B = A[X]$ and set $J := XA[X]$. Then we have $A \bowtie^f J \cong A[X]$. Assume that $A[X]$ is a ϕ-ring. As $J \nsubseteq \mathrm{Nil}(B)$, it follows by Theorem 5.2.1(1) that A is an integral domain. Conversely, if A is an integral domain, then so is $A[X]$, and hence $A[X]$ is a ϕ-ring.

Corollary 5.2.7 *Let A be a reduced ring and I be an ideal of A. Then $A \bowtie I$ is a ϕ-ring if and only if A is a ϕ-ring and $I = 0$.*

Proof. Set $f := Id_A$, the identity map on A. Then we have $A \bowtie I = A \bowtie^f I$. Assume that $I \neq 0$. Then I contains a non-nilpotent element since A is reduced. Hence by Theorem 5.2.1(1), we get that $f^{-1}(I) = I = 0$, which is a contradiction. The converse is clear by the isomorphism $A \bowtie 0 \cong A$.

Corollary 5.2.8 *Let A and B be two rings such that A is a local ring with maximal ideal $M = \mathrm{Nil}(A)$. Let $J \subseteq \mathrm{Nil}(B)$ be an ideal of B. Then $A \bowtie^f J$ is a ϕ-ring.*

Proof. It is clear that A is a ϕ-ring. Let $a \in A \setminus \mathrm{Nil}(A) = M$. Then a is a unit element of A. Hence, for each $i, j \in J$, there exist $x := 0 \in \mathrm{Nil}(A)$ and $k := j(f(a)+i)^{-1} \in J$ such that $xa = 0$ and $j = kf(a) + i(f(x)+k)$. By Theorem 5.2.1(2), $A \bowtie^f J$ is a ϕ-ring.

Example 5.2.9 *In general, $A \bowtie^f J$ need not be a ϕ-ring. Indeed, let $f : A \longrightarrow B$ be a surjective homomorphism of integral domains and J a nonzero ideal of B . By Theorem 5.2.1(1), $A \bowtie^f J$ is not a ϕ-ring since $f^{-1}(J) \neq 0$.*

Example 5.2.10 *Let $A = \mathbb{Z}/8\mathbb{Z}$, $I = 4\mathbb{Z}/8\mathbb{Z}$, and $B = A/I \cong \mathbb{Z}/4\mathbb{Z}$. Let $f : A \longrightarrow B$ be the canonical surjection and $J = 2\mathbb{Z}/4\mathbb{Z}$ an ideal of B. By using Corollary 5.2.5, we conclude that $A \bowtie^f J$ is a ϕ-ring.*

5.3 On ϕ-chained rings and ϕ-PVRs

Now, we study when $A \bowtie^f J$ is a ϕ-chained ring.

Theorem 5.3.1 *Let A and B be two rings, J a nonnil ideal of B, and $f : A \longrightarrow B$ be a ring homomorphism. If $A \bowtie^f J$ is a ϕ-ring, then $A \bowtie^f J$ is a ϕ-chained ring if and only if A is a valuation domain, $J = (f(a) + j)J$ for each $j \in J$ and $0 \neq a \in A$ and for each two non-nilpotent elements $i, j \in J$, $(f(A) + J)i$ and $(f(A) + J)j$ are comparable.*

We need the following lemma.

Lemma 5.3.2 *Let A and B be two rings, J a nonzero ideal of B, and $f : A \longrightarrow B$ be a ring homomorphism. If $A \bowtie^f J$ is a ϕ-chained ring, then so are A and $f(A) + J$. Moreover we have $J \subseteq \mathrm{J}(B)$.*

Proof. Assume that $A \bowtie^f J$ is a ϕ-chained ring, then A is a ϕ-ring by Lemma 5.2.3. Now, let $a, b \in A \setminus \mathrm{Nil}(A)$. Hence $(a, f(a)), (b, f(b)) \in A \bowtie^f J \setminus \mathrm{Nil}(A \bowtie^f J)$. Therefore, $A \bowtie^f J(a, f(a)) \subseteq A \bowtie^f J(b, f(b))$ or $A \bowtie^f J(b, f(b)) \subseteq A \bowtie^f J(a, f(a))$. So $Aa \subseteq Ab$ or $Ab \subseteq Aa$. Then A is a ϕ-chained ring.
Since $A \bowtie^f J$ is ϕ-ring, so is $f(A) + J$, by Lemma 5.2.3(2). Let $f(a) + i$ and $f(b) + j \in f(A) + J \setminus \mathrm{Nil}(f(A) + J)$. Hence $(a, f(a) + i), (b, f(b) + j) \in A \bowtie^f J \setminus \mathrm{Nil}(A \bowtie^f J)$. Therefore, the two ideals $A \bowtie^f J(a, f(a) + i)$ and $A \bowtie^f J(b, f(b) + j)$ are comparable. So $f(a) + i \in (f(A) + J)(f(b) + j)$ or $f(b) + j \in (f(A) + J)(f(a) + i)$, and so, $f(A) + J$ is a ϕ-chained ring. Moreover since $A \bowtie^f J$ is local, we have $J \subseteq \mathrm{J}(B)$ [38, Corollary 2.7(3)].

Proof of Theorem 5.3.1 For simplicity set $R := A \bowtie^f J$. Suppose that R is a ϕ-chained ring. Then by Theorem 5.2.1(1), A is an integral domain, and so A is a valuation domain by Lemma 5.3.2. Let $i \in J$ and $f(a) + j \in f(A) + J$ such that $0 \neq a \in A$. If i is a nilpotent element, then $(0, i) \in R(a, f(a) + j)$ since $A \bowtie^f J$ is a ϕ-ring. Now, assume that i is a non-nilpotent element of B. As R is a ϕ-chained ring, we conclude that $R(a, f(a) + j) \subseteq R(0, i)$ or $R(0, i) \subseteq R(a, f(a) + j)$. If $(a, f(a) + j) \in R(0, i)$, then $(a, f(a) + j) = (0, i)(b, f(b) + k)$ for some $(b, f(b) + k) \in R$. Therefore $a = 0$, which is a contradiction. So, in all cases, we get that $R(0, i) \subseteq R(a, f(a) + j)$. Hence there is $(d, f(d) + k) \in R$ such that $(0, i) = (a, f(a) + j)(d, f(d) + k)$. Then $ad = 0$ and $i = (f(a) + j)(f(d) + k)$. Hence $d = 0$ and $i = (f(a) + j)k \in (f(a) + j)J$. Therefore $J = (f(a) + j)J$ for each $0 \neq a \in A$. Now, let $i, j \in J$ be two non-nilpotent elements. It follows from the fact that R is a ϕ-chained ring that $R(0, i) \subseteq R(0, j)$ or $R(0, j) \subseteq R(0, i)$. Therefore $(f(A) + J)i \subseteq (f(A) + J)j$ or $(f(A) + J)j \subseteq (f(A) + J)i$.

Conversely, let $(a, f(a)+i), (b, f(b)+j) \in R \setminus \mathrm{Nil}(R)$. Since A is a valuation domain, we get that $Aa \subseteq Ab$ or $Ab \subseteq Aa$. Without loss of generality, we may suppose that $Aa \subseteq Ab$. If $b = 0$, whence $a = 0$, and thus, i and j are non-nilpotent elements. By hypothesis, we get $R(0, j) \subseteq R(0, i)$ or $R(0, i) \subseteq R(0, j)$. So we may assume that $b \neq 0$. Hence $a = br$ for some $r \in A$. Since $b \neq 0$, we get that $i - jf(r) = (f(b) + j)k$ for some $k \in J$. Therefore, $(a, f(a)+i) = (b, f(b)+j)(r, f(r)+k) \in R(b, f(b)+j)$, as desired. □

In the case when $J \subseteq \mathrm{Nil}(B)$, we give the following result.

Theorem 5.3.3 *Let A and B be two rings, J an ideal of B such that $J \subseteq \mathrm{Nil}(B)$, and $f : A \longrightarrow B$ be a ring homomorphism. If $A \bowtie^f J$ is a ϕ-ring, then $A \bowtie^f J$ is a ϕ-chained ring if and only if A is a ϕ-chained ring.*

Proof. Assume that $A \bowtie^f J$ is a ϕ-chained ring. Then so is A by Lemma 5.3.2. Conversely, let $(a, f(a)+i), (b, f(b)+j) \in A \bowtie^f J \setminus \mathrm{Nil}(A \bowtie^f J)$. Then $a, b \notin \mathrm{Nil}(A)$. Since $A \bowtie^f J$ is a ϕ-ring, it follows by Theorem 5.2.1(2) that for each $t \in J$, there exist x (resp., x') $\in A$ and k (resp., k') $\in J$ such that $xa = 0$ (resp., $x'b = 0$) and $t = kf(a) + i(f(x) + k)$ (resp., $t = k'f(b) + j(f(x') + k')$). By Lemma 5.2.2, the two principal ideals $(A \bowtie^f J)(a, f(a)+i)$ and $(A \bowtie^f J)(b, f(b)+j)$ have respectively the form $Aa \bowtie^f J$ and $Ab \bowtie^f J$. As A is a ϕ-chained ring, we get that $Aa \subseteq Ab$ or $Ab \subseteq Aa$. Therefore $Aa \bowtie^f J \subseteq Ab \bowtie^f J$ or $Ab \bowtie^f J \subseteq Aa \bowtie^f J$, as desired.

The following result studies when the amalgamation of rings is a ϕ-PVR.

Theorem 5.3.4 *Let A and B be two rings, J a nonnil ideal of B, and $f : A \longrightarrow B$ be a ring homomorphism. If $R := A \bowtie^f J$ is a ϕ-ring, then R is a ϕ-PVR if and only if A is a PVD, $J = (f(a) + j)J$ for each $j \in J$ and $0 \neq a \in A$, $J \subseteq \mathrm{J}(B)$ and for each two non-nilpotent elements $i, j \in J$, $(f(A)+J)i \subseteq (f(A)+J)j$ or $j(f(c)+t) \in (f(A)+J)i$ for every nonunit element $f(c)+t \in f(A)+J$.*

To prove this theorem, we need the following lemma.

Lemma 5.3.5 *Let A and B be two rings, J a nonzero ideal of B, and $f : A \longrightarrow B$ be a ring homomorphism. If $R := A \bowtie^f J$ is a ϕ-PVR, then A and $f(A)+J$ are ϕ-PVRs, and $J \subseteq \mathrm{J}(B)$.*

Proof. As R is a local ring, we get $J \subseteq \mathrm{J}(B)$ [38, Corollary 2.7(3)].

Assume that R is a ϕ-PVR. Then A is a ϕ-ring by Lemma 5.2.3. Let $a, b \in A \setminus \mathrm{Nil}(A)$. Hence $(a, f(a)), (b, f(b)) \in R \setminus \mathrm{Nil}(R)$. Therefore either $R(a, f(a)) \subseteq R(b, f(b))$ or $(b, f(b))(c, f(c)+t) \in R(a, f(a))$ for each nonunit $(c, f(c)+t) \in R$. So either $Aa \subseteq Ab$ or $bc \in Aa$ for each nonunit $c \in A$. Then A is a ϕ-PVR.

Since R is a ϕ-ring, then so is $f(A)+J$ by Lemma 5.2.3(2). Let $f(a)+i, f(b)+j \in f(A)+J \setminus \mathrm{Nil}(f(A)+J)$. Then $(a, f(a)+i), (b, f(b)+j) \in R \setminus \mathrm{Nil}(R)$. Thus $R(a, f(a)+$

$i) \subseteq R(b, f(b)+i)$ or $(b, f(b)+j)(c, f(c)+t) \in R(a, f(a)+i)$ for each nonunit $(c, f(c)+t) \in R$. Hence $(f(A)+J)(f(a)+i) \subseteq (f(A)+J)(f(b)+j)$ or $(f(b)+j)(f(c)+t) \in (f(A)+J)(f(a)+i)$ for each nonunit $f(c)+t \in f(A)+J$, and so $f(A)+J$ is a ϕ-PVR.

Proof of Theorem 5.3.4 Assume that R is a ϕ-PVR. Then by Theorem 5.2.1(1), A is an integral domain, and so A is a PVD and $J \subseteq \mathrm{J}(B)$ by Lemma 5.3.5. Let $i \in J$ and $f(a)+j \in f(A)+J$ such that $0 \neq a \in A$. If i is a nilpotent element, then $(0,i) \in R(a, f(a)+j)$ since R is a ϕ-ring. Now, suppose that i is a non-nilpotent element of B. Then $R(0,i) \subseteq R(a, f(a)+j)$ or $(a, f(a)+j)(c, f(c)+t) \in R(0,i)$ for each nonunit element $(c, f(c)+t) \in R$. Without loss of generality, we may assume that a is a nonunit element of A. If $(a, f(a)+j)(a, f(a)) = (0,i)(b, f(b)+k)$ for some $(b, f(b)+k) \in R$. Then $a = 0$, which is a contradiction. So we have $R(0,i) \subseteq R(a, f(a)+j)$. Hence there is $(d, f(d)+k) \in R$ such that $(0,i) = (a, f(a)+j)(d, f(d)+k)$. Then $ad = 0$ and $i = (f(a)+j)(f(d)+k)$. Hence $i = (f(a)+j)k \in (f(a)+j)J$ since $d = 0$. Therefore $J = (f(a)+j)J$ for each $0 \neq a \in A$ and $j \in J$. Now, as R is a ϕ-PVR we get that, for each non-nilpotent two element $i, j \in J$ and every nonunit element $f(c)+t \in f(A)+J$ we have $(0, j)(c, f(c)+t) \in R(0,i)$ or $R(0,i) \subseteq R(0, j)$. That implies $j(f(c)+t) \in (f(A)+J)i$ or $(f(A)+J)i \subseteq (f(A)+J)j$.

Conversely, let $(a, f(a)+i), (b, f(b)+j) \in R \setminus \mathrm{Nil}(R)$. As A is a PVD, we get that $Aa \subseteq Ab$ or $bc \in Aa$ for each nonunit $c \in A$. Suppose that $Aa \subseteq Ab$. Hence $a = br$ for some $r \in A$. If $b = 0$, whence $a = 0$, and thus, i and j are non-nilpotent elements. By using the hypothesis, we get that $(0, j)(c, f(c)+t) \in R(0,i)$ or $R(0,i) \subseteq R(0, j)$ for every nonunit element $(c, f(c)+t) \in R$. So we may assume that $b \neq 0$. Thus, we get $i - jf(r) = (f(b)+j)k$ for some $k \in J$. Therefore, $(a, f(a)+i) = (b, f(b)+j)(r, f(r)+k) \in R(b, f(b)+j)$. Now assume that for each nonunit $c \in A$ we have $bc \in Aa$. Let $(c, f(c)+t)$ be a nonunit element of R. Then c is a nonunit of A since $J \subseteq \mathrm{J}(B)$. We may assume that $a \neq 0$. Hence $bc = ar$ for some $r \in A$, and $f(b)t + j(f(c)+t) = (f(a)+i)k + f(r)i$ for some $k \in J$ (since $J = (f(a)+i)J$). Therefore $(b, f(b)+j)(c, f(c)+t) = (a, f(a)+i)(r, f(r)+k) \in R(a, f(a)+i)$, as desired. □

If $J \subseteq \mathrm{Nil}(B)$, then we obtain the next result.

Theorem 5.3.6 *Let A and B be two rings, J a nonzero ideal of B such that $J \subseteq \mathrm{Nil}(B)$, and $f : A \longrightarrow B$ be a ring homomorphism. If $R := A \bowtie^f J$ is a ϕ-ring, then R is a ϕ-PVR if and only if A is a ϕ-PVR.*

Proof. Assume that R is a ϕ-PVR. Then so is A by Lemma 5.3.5. Conversely, let $(a, f(a)+i), (b, f(b)+j) \in R \setminus \mathrm{Nil}(R)$. Then $a, b \in A \setminus \mathrm{Nil}(A)$. Since R is a ϕ-ring, it follows by Theorem 5.2.1(2) and Lemma 5.2.2 that the two principal ideals $R(a, f(a)+i)$ and $R(b, f(b)+j)$ have respectively the form $Aa \bowtie^f J$ and $Ab \bowtie^f J$. As A is a ϕ-PVR, we get that $Aa \subseteq Ab$ or $bc \in Aa$ for each nonunit $c \in A$. Therefore $Aa \bowtie^f J \subseteq Ab \bowtie^f J$ or $(b, f(b)+j)(c, f(c)+k) \in Aa \bowtie^f J$ for each nonunit $(c, f(c)+k) \in R$, as desired.

The next two corollaries are immediate applications of Theorem 5.3.3 and Theorem 5.3.6.

Corollary 5.3.7 *Let A be a ring and E a nonzero A-module. Then $A \ltimes E$ is a ϕ-chained ring (resp., ϕ-PVR) if and only if A is a ϕ-chained ring (resp., ϕ-PVR) and $E = aE$ for each $a \in A \setminus \mathrm{Nil}(A)$.*

Corollary 5.3.8 *Let A be not a reduced ring and I a nonzero ideal of A such that $I = \mathrm{Nil}(A)$. Then $A \bowtie I$ is a ϕ-chained ring (resp., ϕ-PVR) if and only if A is a ϕ-chained ring (resp., ϕ-PVR).*

Proof. Suppose that $A \bowtie I$ is a ϕ-chained ring. By Lemma 5.3.2(1), we have A a ϕ-chained ring. Conversely, assume that A is a ϕ-chained ring, then A is a ϕ-ring. As $I = \mathrm{Nil}(A)$, it follows by Corollary 5.2.5 that $A \bowtie I$ a ϕ-ring. Therefore $A \bowtie I$ is a ϕ-chained ring by Theorem 5.3.3. Similarly, by using Corollary 5.2.5 and Theorem 5.3.6, we can prove the ϕ-PVR case.

The following example illustrates Theorem 5.3.1 and Theorem 5.3.4.

Example 5.3.9 *Let K be a field, $B := K[[X]] = K + XK[[X]]$ the ring of formal power series in an indeterminate X, with coefficient in K, and Q be the quotient field of B. It is well-known that B is a valuation domain. Also, if F is a proper subfield of K, then $D = F + XK[[X]]$ is a PVD which is not a valuation domain by [65, Example 2.1]. Consider $f : K \longrightarrow B \ltimes Q$ and $g : F \longrightarrow B \ltimes Q$ be two injective ring homomorphisms such that $f(a) = (a,0)$ and $g(b) = (b,0)$ for $a \in K$ and $b \in F$. Let $J := XK[[X]] \ltimes Q$ be the maximal ideal of $B \ltimes Q$. Then:*

1. *$K \bowtie^f J$ is a ϕ-chained ring by Theorem 5.3.1.*
2. *$F \bowtie^g J$ is a ϕ-PVR by Theorem 5.3.4.*

Proof. (1). Clearly $f^{-1}(J) = 0$ and $\mathrm{Nil}(B \ltimes Q) \cap J = 0 \ltimes Q$ is a divided prime ideal of $B \ltimes Q$. Thus $K \bowtie^f J$ is a ϕ-ring by Theorem 5.2.1(1). Also, for every nonzero element $a \in K$ and $j \in J$, $f(a) + j$ is a unit element of $B \ltimes Q$ which implies that $J = (f(a) + j)J$. Moreover since $B \ltimes Q$ is a ϕ-chained ring, by Corollary 5.3.7, the last condition of Theorem 5.3.1 holds and this completes the proof of (1).
(2). Similarly, this follows form Theorem 5.3.4 and by using the fact that $D \ltimes Q$ is a ϕ-PVR by Corollary 5.3.7 to see that for each two non-nilpotent elements $i, j \in J$, $(g(F)+J)i \subseteq (g(F)+J)j$ or $j(g(c)+t) \in (g(F)+J)i$ for every nonunit element $g(c)+t \in g(F)+J$.

Our next two examples illustrate Theorem 5.3.3.

Example 5.3.10 *Let $A = \mathbb{Z}/4\mathbb{Z}$. Then $A \ltimes A$ is a ϕ-chained ring, but not a chained ring.*

Proof. We have A is a local ring with maximal ideal $M = 2\mathbb{Z}/4\mathbb{Z} = \mathrm{Nil}(A)$. Clearly A is a ϕ-chained ring and for each $a \in A \setminus \mathrm{Nil}(A)$, a is unit. Hence $A \ltimes A$ is a ϕ-chained ring by Corollary 5.3.8. As A is not an integral domain, it follows by [8, Theorem 4.16(3)] that $A \ltimes A$ is not a chained ring.

Example 5.3.11 *Let A be a valuation domain (which is not a field) with quotient field K. Set $B = A \ltimes K$ and $I = 0 \ltimes K$. Hence $B \bowtie I$ is a ϕ-chained (resp., ϕ-PVR) ring which is not a PVR.*

Proof. By [8, Theorem 4.16(3)] B is a valuation ring, and we have $\mathrm{Nil}(B) = 0 \ltimes K = I$. Clearly, by Corollary 5.3.8 $B \bowtie I$ is a ϕ-chained ring (rep., ϕ-PVR). Moreover $B \bowtie I$ is not a PVR. Indeed. Let x, y be two nonzero elements of I and $0 \neq a$ be a nonunit element of A. Then we have $(x,0) = (0,y)(r, r+t)$ or $(0,y)(a,a) = (0,x)(r, r+t)$ for some $(r, r+t) \in B \bowtie I$. If $(x,0) = (0,y)(r, r+t)$, then $x = 0$, which is a contradiction. Hence, $(0,y)(a,a) = (x,0)(r, r+t)$, and so $ya = 0$, which is absurd. Therefore $B \bowtie I$ is not a PVR (and hence it is not a chained ring).

Theorem 5.3.6 enriches the literature with new examples of ϕ-PVR, as is illustrated next by Example 5.3.12.

Example 5.3.12 *Let D be a PVD with quotient field K and $A = D \ltimes K$. Let $f : A \longrightarrow K \ltimes K$ be the natural injection, and set $J = 0 \ltimes K$. By using Corollary 5.3.7, Corollary 5.2.5 and Theorem 5.3.6, we conclude that $A \bowtie^f J$ is a ϕ-PVR.*

CHAPTER 6

ON ϕ-PSEUDO-KRULL RINGS

Abstract.

The purpose of this chapter*is to introduce a new classe of rings that is closely related to the classe of pseudo-Krull domains. Let $\mathscr{H} = \{R \mid R$ is a commutative ring and $\mathrm{Nil}(R)$ is a divided prime ideal of $R\}$. Let $R \in \mathscr{H}$ be a ring with total quotient ring $T(R)$, and set $\phi : T(R) \longrightarrow R_{\mathrm{Nil}(R)}$ such that $\phi(\frac{a}{b}) = \frac{a}{b}$ for every $a \in R$ and every $b \in R \setminus Z(R)$. Then ϕ is a ring homomorphism from $T(R)$ into $R_{\mathrm{Nil}(R)}$, and ϕ restricted to R is also a ring homomorphism from R into $R_{\mathrm{Nil}(R)}$ given by $\phi(x) = \frac{x}{1}$ for every $x \in R$. We say that R is a ϕ-pseudo-Krull ring if $\phi(R) = \cap R_i$, where each R_i is a nonnil-Noetherian ϕ-pseudo valuation overring of $\phi(R)$, and for every nonnilpotent element $x \in R$, $\phi(x)$ is a unit in all but finitely many R_i. We show that the theories of ϕ-pseudo Krull rings resemble those of pseudo-Krull domains.

6.1 Introduction

In [65], Hedstrom and Houston introduced the class of pseudo-valuation domains, which is closely related to the class of valuation domains. A domain R with quotient field K is called a pseudo-valuation domain (in short, *PVD*) if, whenever P is a prime ideal of R and $xy \in P$ with $x \in K$ and $y \in K$, then either $x \in P$ or $y \in P$. Any valuation domain is a PVD [65, Proposition 1.1]; any PVD is quasilocal, in the sense that it has a unique maximal ideal [65, Corollary 1.3]. Additional information about PVDs can be found in the interesting survey article [19]. In [3], D. F. Anderson, Badawi and Dobbs generalized the study of PVDs to the context of arbitrary rings (possibly with nontrivial zero-divisors). Recall from [18, 40] that a prime ideal P of R is said to be *divided* if it is comparable to every ideal of R, equivalently if $P \subseteq (x)$ for any $x \in R \setminus P$. A ring R is called a *divided ring* if every prime ideal of R is divided. Recently A. Badawi, in

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[13, 14, 15, 16, 17], has studied the following class of rings: $\mathscr{H} = \{R \mid R$ is a commutative ring and $\mathrm{Nil}(R)$ is a divided prime ideal of $R\}$. If $R \in \mathscr{H}$, then R is called a *ϕ-ring*. It is easy to see that every integral domain is a ϕ-ring. An ideal I of R is said to be a *nonnil ideal* if $I \nsubseteq \mathrm{Nil}(R)$. If I is a nonnil ideal of a ϕ-ring R, then $\mathrm{Nil}(R) \subseteq I$. Let R be a ring with total quotient ring T such that $\mathrm{Nil}(R)$ is a divided prime ideal of R. As in [14], we define $\phi : T(R) \longrightarrow K := R_{\mathrm{Nil}(R)}$ such that $\phi(\frac{a}{b}) = \frac{a}{b}$ for every $a \in R$ and every $b \in R \setminus Z(R)$. Then ϕ is a ring homomorphism from $T(R)$ into K, and ϕ restricted to R is also a ring homomorphism from R into K given by $\phi(x) = \frac{x}{1}$ for every $x \in R$. Observe that if $R \in \mathscr{H}$, then $\phi(R) \in \mathscr{H}$, $Ker(\phi) \subseteq \mathrm{Nil}(R)$, $\mathrm{Nil}(T(R)) = \mathrm{Nil}(R)$, $\mathrm{Nil}(R_{\mathrm{Nil}(R)}) = \phi(\mathrm{Nil}(R)) = \mathrm{Nil}(\phi(R)) = Z(\phi(R))$, $T(\phi(R)) = R_{\mathrm{Nil}(R)}$ is local with maximal ideal $\mathrm{Nil}(\phi(R))$, and $R_{\mathrm{Nil}(R)}/\mathrm{Nil}(\phi(R)) = T(\phi(R))/\mathrm{Nil}(\phi(R))$ is the quotient field of $\phi(R)/\mathrm{Nil}(\phi(R))$. In [14], the author gave another generalization of pseudo valuation domains to the context of arbitrary rings (possibly with nonzero zerodivisors). A prime ideal Q of $\phi(R)$ is said to be *K-strongly prime* if $xy \in Q$, $x \in K$, $y \in K$ implies that either $x \in Q$ or $y \in Q$. A prime ideal P of R is said to be *ϕ-strongly prime* if $\phi(P)$ is a K-strongly prime ideal of $\phi(R)$. If each prime ideal of R is ϕ-strongly prime, then R is called a *ϕ-pseudo-valuation ring* (in short *ϕ-PVR*). It is shown in [14, Corollary 7(2)] that a ring R is a ϕ-PVR if and only if $\mathrm{Nil}(R)$ is a divided prime ideal and for every $a, b \in R \setminus \mathrm{Nil}(R)$, either $a \mid b$ or $b \mid ac$ in R for each nonunit element $c \in R$. A ring R is called a *ϕ-chained ring* if $Nil(R)$ is divided prime and for each $x \in R_{\mathrm{Nil}(R)} \setminus \phi(R)$ we have $x^{-1} \in \phi(R)$ [15]. Then it is easy to se that R is a ϕ-chained ring if and only if $\mathrm{Nil}(R)$ is divided prime and for every $a, b \in R \setminus \mathrm{Nil}(R)$ either $a \mid b$ or $b \mid a$. Also, recall from [17] that a ring $R \in \mathscr{H}$ is called a nonnil-Noetherian ring if every nonnil ideal of R is finitely generated. It was shown in [17] that a ring $R \in \mathscr{H}$ is a nonnil-Noetherian ring if and only if $R/\mathrm{Nil}(R)$ is a Noetherian domain. Recall that a ring R is called a t-closed ring if for every element $(a, r, c) \in R^3$, such that $a^3 + arc - c^2 = 0$, there exists an element $b \in R$ such that $a = b^2 - rb$ and $c = b^3 - rb^2$. Also, from [86] an integral domain R is called an infra-Krull domain if $R = \cap[R_P \mid P$ is a height-one prime ideal of $R]$, where this representation is of finite character and each R_P is a Noetherian ring for every height-one prime ideal P of R.

Recall from [88] that an integral domain R is called a pseudo-Krull domain if there exists a family $\mathscr{F} = \{R_i\}$ of overrings of R such that $R = \cap R_i$, where each R_i is a Noetherian pseudo-valuation overring of R, and every nonzero element of R is not a unit in only a finite number of members of $\mathscr{F}$. Many characterizations and properties of pseudo-Krull domains are given in [79], [88], and [90]. In this chapter, we are interested in extending the pseudo-Krull property to ϕ-rings. Let $R \in \mathscr{H}$, we say that R is a ϕ-pseudo-Krull ring if $\phi(R) = \cap R_i$, where each R_i is a nonnil-Noetherian ϕ-pseudo valuation overring of $\phi(R)$, and for every nonnilpotent element $x \in R$, $\phi(x)$ is a unit in all but finitely many R_i. Among many results in this chapter, we introduce the notions of ϕ-t-closed rings and ϕ-infra-Krull rings and we show that a ϕ-ring R is a ϕ-pseudo-Krull ring if and only if R is a ϕ-t-closed ϕ-infra-Krull ring. We also use trivial ring extention and amalgamation of rings to construct examples of ϕ-pseudo-Krull rings which are not integral domains.

6.2 Basic results

We start this section by one of the main results on ϕ-pseudo Krull rings. Before we proceed, we recall the following Lemma from [5].

Lemma 6.2.1 *([5, Lemma 2.5]) Let $R \in \mathscr{H}$ and let P be a prime ideal of R. Then R/P is ring-isomorphic to $\phi(R)/\phi(P)$. In particular, $R/\mathrm{Nil}(R)$ is ring-isomorphic to $\phi(R)/\mathrm{Nil}(\phi(R))$.*

Lemma 6.2.2 *Let $R \in \mathscr{H}$. Then R is a nonnil-Noetherian ϕ-pseudo valuation ring if and only if $R/\mathrm{Nil}(R)$ is a Noetherian pseudo-valuation domain.*

Proof. It follows from [13, Proposition 2.9] and [17, Theorem 1.2].

Theorem 6.2.3 *Let $R \in \mathscr{H}$. Then R is a ϕ-pseudo Krull ring if and only if $R/Nil(R)$ is a pseudo-Krull domain.*

Proof. Assume that R is a ϕ-pseudo Krull ring. Hence, there exists a family $\mathscr{F} = \{R_i\}$ of overring of $\phi(R)$ such that $\phi(R) = \cap R_i$ where each R_i is a nonnil-Noetherian ϕ-pseudo valuation overring of $\phi(R)$, and for every nonnilpotent element $x \in R$, $\phi(x)$ is a unit in all but finitely many R_i. Since $T(\phi(R)/\mathrm{Nil}(\phi(R)) = T(\phi(R))/\mathrm{Nil}(\phi(R)) = R_{\mathrm{Nil}(R)}/\mathrm{Nil}(\phi(R))$, Lemma 6.2.2 implies that each $R_i/\mathrm{Nil}(\phi(R))$ is a Noetherian pseudo-valuation overring of $\phi(R)/\mathrm{Nil}(\phi(R))$. Then $\phi(R)/\mathrm{Nil}(\phi(R)) = \cap R_i/\mathrm{Nil}(\phi(R))$ and every nonzero element of $\phi(R)/\mathrm{Nil}(\phi(R))$ is a unit in all but finitely many $R_i/\mathrm{Nil}(\phi(R))$. Thus $\phi(R)/\mathrm{Nil}(\phi(R))$ is a pseudo-Krull domain. As $R/\mathrm{Nil}(R)$ is ring-isomorphic to $\phi(R)/\mathrm{Nil}(\phi(R))$ by Lemma 6.2.1, we get that $R/\mathrm{Nil}(R)$ is a pseudo-Krull domain. Conversely, assume that $R/\mathrm{Nil}(R)$ is a pseudo-Krull domain. Since $R/\mathrm{Nil}(R)$ is ring-isomorphic to $\phi(R)/\mathrm{Nil}(\phi(R))$ by Lemma 6.2.1, we can conclude that $\phi(R)/\mathrm{Nil}(\phi(R))$ is a pseudo-Krull domain. By Lemma 6.2.2 and the fact that $T(\phi(R)/\mathrm{Nil}(\phi(R)) = T(\phi(R))/\mathrm{Nil}(\phi(R)) = R_{\mathrm{Nil}(R)}/\mathrm{Nil}(\phi(R))$, we conclude that $\phi(R)/\mathrm{Nil}(\phi(R)) = \cap R_i/\mathrm{Nil}(\phi(R))$, where each R_i is a nonnil-Noetherian ϕ-pseudo valuation overring of $\phi(R)$. Thus $\phi(R) = \cap R_i$. Since for every nonnilpotent element $x \in R$, $\phi(x) + \mathrm{Nil}(\phi(R))$ is a unit in all but finitely many $R_i/\mathrm{Nil}(\phi(R))$, we get that $\phi(x)$ is a unit in all but finitely many R_i. Therefore R is a ϕ-pseudo Krull ring.

For a ring $R \in \mathscr{H}$, let ϕ_R denotes the ring-homomorphism $\phi : T(R) \longrightarrow R_{\mathrm{Nil}(R)}$. We have the following result which is an analog of [88, Theorem 1.6]

Theorem 6.2.4 *Let $R \in \mathscr{H}$ be a ϕ-pseudo Krull ring such that $dim(R) \geq 1$. Then :*

(1) R_P is a nonnil-Noetherian ϕ-pseudo valuation ring for every height-one prime ideal P of R.

(2) $\phi_R(R) = \cap \phi_{R_P}(R_P)$ for all height-one prime ideals Pof R.

(3) Any nonnil proper principal ideal of R may be expressed as a finite intersection of (nonnil) primary ideals of height-one.

Proof. Assume that R is a ϕ-pseudo Krull ring. Set $D = R/\mathrm{Nil}(R)$ and let P be a height-one prime ideal of R.
(1) By Theorem 6.2.3, D is a pseudo-Krull domain which implies that $D_{P/\mathrm{Nil}(R)}$ is a Noetherian pseudo-valuation domain by [88, Theorem 1.6]. As $D_{P/\mathrm{Nil}(R)}$ is ring-isomorphic to $R_P/\mathrm{Nil}(R_P)$, we get that R_P is a nonnil-Noetherian ϕ-pseudo valuation ring by Lemma 6.2.2.
(2) First observe that $\mathrm{Nil}(\phi_{R_P}(R_P)) = \mathrm{Nil}(\phi_R(R))$. Since $R_P/\mathrm{Nil}(R_P)$ is ring-isomorphic to $\phi_{R_P}(R_P)/\mathrm{Nil}(\phi_{R_P}(R_P))$, Lemma 6.2.2 implies that $\phi_{R_P}(R_P)$ is a nonnil-Noetherian ϕ-pseudo valuation ring. Since D is a pseudo-Krull domain by Theorem 6.2.3 and D is is ring-isomorphic to $\phi(R)/\mathrm{Nil}(\phi(R))$ by Lemma 6.2.1 and [6, Lemma 3.8], we conclude that $\phi(R)/\mathrm{Nil}(\phi(R))$ is a pseudo-Krull domain. Thus,

$$\phi_R(R)/\mathrm{Nil}(\phi_R(R)) = \cap \phi_R(R)_{\phi_R(P)}/\mathrm{Nil}(\phi_R(R) = \cap \phi_{R_P}(R_P)/\mathrm{Nil}(\phi_R(R))$$

for each height-one prime ideal P of R. Hence one can easily see that $\phi_R(R) = \cap \phi_{R_P}(R_P)$ for each height-one prime ideal P of R.
(3) Since D is a pseudo Krull domain by Thoerem 6.2.3, for any nonnil proper principal ideal I of R, $I/\mathrm{Nil}(R)$ may be expressed as a finite intersection of primary ideals of height-one by [88, Theorem 1.6]. Therefore, I may be expressed as a finite intersection of (nonnil) primary ideals of height-one.

Let $R \in \mathscr{H}$. We say that R satisfies property (*) if each nonnilpotent element of R is contained in only finitely many prime ideal of height-one. Recall from [88, Proposition 1.15] that an integral domain R is a pseudo Krull domain if and only if R_P is a pseudo-Kull domain for every prime ideal P of R and R satisfies (*) if and only if R_M is a pseudo-Krull domain for every maximal ideal M of R and R satisfies (*). The following is the analogous characterization of ϕ-pseudo-Krull ring.

Proposition 6.2.5 *Let $R \in \mathscr{H}$. Then the following statements are equivalent:*

(1) R is ϕ-pseudo-Krull ring.

(2) R_P is a ϕ-pseudo-Kull ring for every prime ideal P of R and R satisfies (*).

(3) R_M is a ϕ-pseudo-Krull ring for every maximal ideal M of R and R satisfies (*).

Proof. Set $D = R/\mathrm{Nil}(R)$. $(1) \Rightarrow (2)$. Since D is a pseudo-Krull domain by Theorem 6.2.3, we conclude that $D_{P/\mathrm{Nil}(R)}$ is a pseudo-Kull domain for every prime ideal P of R

by [88, Proposition 1.15]. As $D_{P/\mathrm{Nil}(R)}$ is ring-isomorphic to $R_P/\mathrm{Nil}(R)R_P = R_P/\mathrm{Nil}(R_P)$ and $R_P \in \mathscr{H}$, we get that R_P is a a ϕ-pseudo-Kull ring by Theorem 6.2.3. Also, since D is a pseudo-Krull domain, [88, Proposition 1.15] implies that each nonzero element of D is contained in only finitely many prime ideal of height-one. Thus each nonnilpotent element of R is contained in only finitely many prime ideal of height-one.
$(2) \Rightarrow (3)$. Clear.
$(3) \Rightarrow (1)$ As $R_M \in \mathscr{H}$ for each maximal ideal M of R, we conclude that $R_M/\mathrm{Nil}(R_M)$ is a pseudo-Krull domain for each maximal ideal M of R by Theorem 6.2.3, and D satisfies (*). Hence $D_{M/Nil(R)}$ is a pseudo-Krull domain for each maximal ideal M of R. Thus D is a pseudo-Krull domain by [88, Proposition 1.15]. Therefore, R is a ϕ-pseudo-Krull ring by Theorem 6.2.3.

As a generalization of infra-Krull domains and t-closed domains, we give the following definitions of ϕ-t-closed rings and ϕ-infra-Krull rings.

Definitions 6.2.6 *Let $R \in \mathscr{H}$. Then:*

(1) We say that R is a ϕ-infra-Krull ring if $\phi_R(R) = \cap[\phi_{R_P}(R_P) \mid P$ is a height-one prime ideal of $R]$, where this representation is of finite character and each R_P is a nonnil-Noetherian ring for every height-one prime ideal P of R.

(2) We say that R is a ϕ-t-closed ring if for every nonnilpotent elements a, c of R and each $r \in R$, such that $a^3 + arc - c^2 = 0$, there exist an element $b \in R$ and $w_1, w_2 \in \mathrm{Nil}(R)$ such that $a = b^2 - (r - w_1)b$ and $c = b^3 - (r - w_2)b^2$.

We need the next lemma.

Lemma 6.2.7 *Let $R \in \mathscr{H}$. Then:*

(1) R is a ϕ-infra-Krull ring if and only if $R/Nil(R)$ is a infra-Krull domain.

(2) R is a ϕ-t-closed ring if and only if $R/Nil(R)$ is a t-closed domain.

Proof. (1) Suppose that R is a ϕ-infra-Krull ring. Then, for each height-one prime ideal of R, $\phi_R(R) = \cap\phi_{R_P}(R_P)$ where this representation is of finite character and each R_P is a nonnil-Noetherian ring. Since $\mathrm{Nil}(\phi_{R_P}(R_P)) = \mathrm{Nil}(\phi_R(R))$, then each $\phi_{R_P}(R_P)/\mathrm{Nil}(\phi_R(R))$ is a Noetherian domain by [17, Theorem 1.4]. Then $\phi_R(R)/\mathrm{Nil}(\phi_R(R)) = \cap\phi_{R_P}(R_P)/\mathrm{Nil}(\phi_R(R))$ for each height-one prime ideal P of R and this representation is of finite character. Thus $\phi_R(R)/\mathrm{Nil}(\phi_R(R))$ is a infra-Krull domain. As $\phi_R(R)/\mathrm{Nil}(\phi_R(R))$ is ring-isomorphic to $R/\mathrm{Nil}(R)$ by Lemma 6.2.1, we get that $R/\mathrm{Nil}(R)$ is a infra-Krull domain. Conversely, assume that $R/\mathrm{Nil}(R)$ is a infra-Krull domain. Since $R/\mathrm{Nil}(R)$ is ring-isomorphic to $\phi_R(R)/\mathrm{Nil}(\phi_R(R))$ by Lemma 6.2.1, we conclude that $\phi_R(R)/\mathrm{Nil}(\phi_R(R))$ is a infra-Krull domain. By [17, Theorem 1.4] and [6, Lemma 3.8], we conclude that $\phi_R(R)/\mathrm{Nil}(\phi_R(R)) =$

$\cap\phi_{R_P}(R_P)/\mathrm{Nil}(\phi(R))$ for all height-one prime ideal of R, where each R_P is a nonnil-Noetherian ring. Thus $\phi_R(R) = \cap\phi_{R_P}(R_P)$ where this representation is of finite character. Therefore R is a ϕ-infra Krull ring, as desired.
(2) Set $D = R/\mathrm{Nil}(R)$. Assume that R is a ϕ-t-closed ring and let $(\bar{a},\bar{r},\bar{c}) \in D^3$ such that $\bar{a}^3 + \bar{a}\bar{r}\bar{c} - \bar{c}^2 = 0$. We may assume that $\bar{a} \neq 0$ and $\bar{c} \neq 0$ and so a and c are non-nilpotent elements of R. Since $a^3 + arc - c^2 \in \mathrm{Nil}(R)$ and $\mathrm{Nil}(R)$ is a divided prime ideal of R, $a^3 + (r-z)ac - c^2 = 0$ for some $z \in \mathrm{Nil}(R)$. As R is a ϕ-t-closed ring, there exist an element $b \in R$ and $w_1, w_2 \in \mathrm{Nil}(R)$ such that $a = b^2 - (r - z - w_1)b$ and $c = b^3 - (r - z - w_2)b^2$. That implies $\bar{a} = \bar{b}^2 - \bar{r}\bar{b}$ and $\bar{c} = \bar{b}^3 - \bar{r}\bar{b}^2$. Hence D is a t-closed domain. Conversely, let $(a,r,c) \in R^3$ such that a and c are nonnilpotents and $a^3 + arc - c^2 = 0$. Since D is a t-closed domain and $\bar{a}^3 + \bar{a}\bar{r}\bar{c} - \bar{c}^2 = 0$, there exists an element $(0 \neq)\bar{b} \in D$ such that $\bar{a} = \bar{b}^2 - \bar{r}\bar{b}$ and $\bar{c} = \bar{b}^3 - \bar{r}\bar{b}^2$. Hence $a - b^2 + rb \in \mathrm{Nil}(R)$ and $c - b^3 + rb^2 \in \mathrm{Nil}(R)$. That implies $a - b^2 + rb = w_1 b$ and $c - b^3 + rb^2 = w_2 b^2$ for some $w_1, w_2 \in \mathrm{Nil}(R)$ since $\mathrm{Nil}(R)$ is divided and $b \notin \mathrm{Nil}(R)$. Thus $a = b^2 - (r - w_1)b$ and $c = b^3 - (r - w_2)b^2$. Therefore R is a ϕ-t-closed ring and this completes the proof.

It is well-known [90, Proposition 3.6] that an integral domain R is a pseudo-Krull domain if and only if R is a t-closed infra-Krull domain. We have a similar characterization for ϕ-pseudo-Krull rings.

Theorem 6.2.8 *Let $R \in \mathscr{H}$. Then R is a ϕ-pseudo-Krull ring if and only if R is a ϕ-t-closed ϕ-infra-Krull ring.*

Proof. Set $D = R/\mathrm{Nil}(R)$. Assume that R is a ϕ-pseudo-Krull ring. Since D is a pseudo-Krull domain by Theorem 6.2.3, we conclude that D is a t-closed infra-Krull domain by [90, Proposition 3.6]. Hence Lemma 6.2.7 implies that R is a ϕ-t-closed ϕ-infra-Krull ring.
Conversely, since R is a ϕ-t-closed ϕ-infra-Krull ring, by Lemma 6.2.7 D is a t-closed infra-Krull domain. Thus, D is a pseudo-Krull domain by [90, Proposition 3.6] and so R is a ϕ-pseudo Krull ring by Theorem 6.2.3.

We have the following pullback characterization of ϕ-pseudo-Krull rings.

Theorem 6.2.9 *Let $R \in \mathscr{H}$. Then R is a ϕ-pseudo-Krull ring if and only if $\phi(R)$ is ring-isomorphic to a ring A obtained from the following pullback diagram:*

$$\begin{array}{ccc} A & \longrightarrow & S \\ \downarrow & & \downarrow \\ T & \longrightarrow & T/M \end{array}$$

where T is a zero-dimensional quasilocal ring with maximal ideal M, $S := A/M$ is a

pseudo-Krull subring of T/M, the vertical arrows are the usual inclusion maps, and the horizontal arrows are the usual surjective maps.

Proof. Suppose that $\phi(R)$ is ring-isomorphic to a ring A obtained from the given diagram. Then $A \in \mathscr{H}$ and $\mathrm{Nil}(A) = Z(A) = M$. Since A/M is a pseudo-Krull domain, A is a ϕ-pseudo-Krull ring by Theorem 6.2.3, and thus R is a ϕ-pseudo-Krull ring. Conversely, suppose that R is a ϕ-pseudo-Krull ring. Then, set $T = R_{\mathrm{Nil}(R)}$, $M = \mathrm{Nil}(R_{\mathrm{Nil}(R)})$, and $A = \phi(R)$ yields the desired pullback diagram.

Reacall that an ideal J of a commutative ring R is called a Glaz-Vasconcelos ideal or a GV-ideal, denoted by $J \in GV(R)$, if J is finitely generated and the natural homomorphism $\alpha : R \longrightarrow Hom_R(J,R)$, defined by $\alpha(r)(a) = ra$, $\forall r \in R$, $\forall a \in J$, is an isomorphism. An R-module M is called a GV-torsion-free module if whenever $Jx = 0$, for some $J \in GV(R)$ and $x \in M$, then $x = 0$. A GV-torsion-free R-module M is said to be a w-module if $\mathrm{Ext}^1_R(R/J,M) = 0$ for any $J \in GV(R)$. Let $R \in \mathscr{H}$. Recall from [80] that that a nonnil ideal J is a ϕ-GV-ideal of R if $\phi(J)$ is a GV-ideal of $\phi(R)$. Also, a nonnil ideal I is a ϕ-w-ideal if $\phi(I)$ is a w-ideal of $\phi(R)$, and R is called a ϕ-Stongly-Mori ring if it satisfies the ascending chain condition (a.c.c.) on ϕ-w-ideals.
It is well-known [79, Proposition 3.3] that if R is a one-dimensional integral domain. Then R is a pseudo-Krull domain if and only if R is a t-closed Strongly-Mori domain. We have the following analogous result for ϕ-pseudo-Krull rings.

Proposition 6.2.10 *Let $R \in \mathscr{H}$ be a one-dimensional ring. Then R is a ϕ-pseudo-Krull ring if and only if R is a ϕ-t-closed ϕ-Strongly-Mori ring.*

Proof. Set $D = R/\mathrm{Nil}(R)$. Clearly D is a one-dimensional integral domain. Now, Assume that R is a ϕ-pseudo-Krull ring. Hence D is a pseudo-Krull domain by Theorem 6.2.3 and so D is a t-closed Strongly-Mori domain by [79, Proposition 3.3]. Thus R is a ϕ-t-closed ϕ-Strongly-Mori ring by Lemma 6.2.7(2) and [80, Theorem 2.4]. Conversely, D is a t-closed Strongly-Mori domain by Lemma 6.2.7(2) and [80, Theorem 2.4]. That implies D is a pseudo-Krull domain by [79, Proposition 3.3] and hence R is a ϕ-pseudo-Krull ring by Theorem 6.2.3.

For a ring R, let R' denote the integral closure of R in $T(R)$ and let R^* denote the complete integral closure of R in $T(R)$. Recall that a ring $R \in \mathscr{H}$ is called ϕ-(completely) integrally closed if $\phi(R)$ is (completely) integrally closed in $T(\phi(R)) = R_{Nil(R)}$.

Lemma 6.2.11 *[6, Lemma 2.8] Let $R \in \mathscr{H}$ and set $D = \phi(R)/\mathrm{Nil}(\phi(R))$. Then one has that $D' = \phi(R)'/\mathrm{Nil}(\phi(R))$ and $D^* = \phi(R)^*/\mathrm{Nil}(\phi(R))$. In particular, R is ϕ-(completely) integrally closed if and only if D is (completely) integrally closed, if and only if $R/\mathrm{Nil}(R)$ is (completely) integrally closed.*

Recall from [6] that a ring $R \in \mathscr{H}$ is called a ϕ-Krull ring if $\phi(R) = \cap V_i$, where each V_i is a discrete ϕ-chained overring of $\phi(R)$, and for every nonnilpotent element $x \in R$, $\phi(x)$ is a unit in all but finitely many V_i.

It is well-known that if R is a pseudo-Krull domain, then the complete integral closure of R is a Krull domain. Also, an integral domain R is a Krull domain if and only if R is an integrally closed pseudo-Krull domain. We have the following analogous result.

Theorem 6.2.12 *Let $R \in \mathscr{H}$. Then:*

(1) If R is a ϕ-pseudo-Krull ring, then $\phi(R)^*$ is a ϕ-Krull ring.

(2) R is a ϕ-Krull ring if and only if R is a ϕ-integrally closed ϕ-pseudo-Krull ring.

Proof. (1) Set $D = \phi(R)/\mathrm{Nil}(\phi(R))$. Since D is ring-isomorphic to $R/\mathrm{Nil}(R)$ by Lemma 6.2.1, we conclude that D is a pseudo-Krull domain by Theorem 6.2.3. Since $D^* = \phi(R)^*/Nil(\phi(R))$ by Lemma 6.2.11 and D^* is a Krull domain, we get that $\phi(R)^*$ is a ϕ-Krull ring by [6, Theorem 3.1].
(2) Set $D = R/\mathrm{Nil}(R)$. By using [6, Theorem 3.1] R is a ϕ-Krull ring if and only if D is a Krull domain if and only if D is an integrally closed pseudo-Krull domain if and only if R is a ϕ-integrally closed ϕ-pseudo-Krull ring by Lemma 6.2.11 and Theorem 6.2.3.

Recall that a ring S is called an overring of a ring R if $R \subseteq S \subseteq T(R)$. It is well-known [88, Proposition 5.5] that a flat overring of a pseudo-Krull domain is a pseudo-Krull domain. We have the next result.

Proposition 6.2.13 *Let $R \in \mathscr{H}$ be a ϕ-pseudo-Krull ring. Then every flat overring of R is a ϕ-pseudo-Krull ring.*

Proof. Let T be a flat overring of R. Then $T \in \mathscr{H}$, $\mathrm{Nil}(T) = \mathrm{Nil}(R)$, and $T/\mathrm{Nil}(R)$ is a flat overring of $R/\mathrm{Nil}(R)$. Since $R/\mathrm{Nil}(R)$ is a pseudo-Krull domain and $T/\mathrm{Nil}(R)$ is a flat overring of $R/\mathrm{Nil}(R)$, we conclude that $T/\mathrm{Nil}(R)$ is a pseudo-Krull domain by [88, Proposition 5.5]. Thus T is a ϕ-pseudo-Krull ring by Theorem 6.2.3.

6.3 Examples of ϕ-pseudo-Krull rings

In this section we use trivial ring extention and amalgamation of rings to construct examples of ϕ-pseudo-Krull rings which are not integral domains.

Proposition 6.3.1 *Let $A \in \mathscr{H}$ be a ring and $E = A_{\mathrm{Nil}(A)}$ an A-module. Then $R := A \ltimes E$ is a ϕ-pseudo-Krull ring if and only if A is a ϕ-pseudo-Krull ring.*

Proof. First we show that R is a ϕ-ring. It is easy to see that $\mathrm{Nil}(R) = \mathrm{Nil}(A) \ltimes E$. Since $\mathrm{Nil}(A)$ is prime, $\mathrm{Nil}(R)$ is a prime ideal of R. Now, we show that $\mathrm{Nil}(R)$ is divided. Let $(a,e) \in R \setminus \mathrm{Nil}(R)$ and let $(b,f) \in \mathrm{Nil}(R)$. Hence $a \in A \setminus \mathrm{Nil}(A)$ and so $b = ac$ for some $c \in \mathrm{Nil}(A)$ since $A \in \mathscr{H}$. Then, $(b,f) = (ac, ad + ce)$ for some $d \in E$. Thus, $(b,f) = (a,e)(c,d) \in (a,e)R$ and hence $\mathrm{Nil}(R)$ is a divided prime ideal of R. Now, observe that $R/\mathrm{Nil}(R) \cong A/\mathrm{Nil}(A)$. Then, by using Theorem 6.2.3, R is a ϕ-pseudo-Krull ring if and only if $R/\mathrm{Nil}(R)$ is a pseudo-Krull domain if and only if so is $A/\mathrm{Nil}(A)$ if and only if A is a ϕ-pseudo-Krull ring. This completes the proof.

We combine this proposition with [69, Theorem 25.1] to get the following corollary

Corollary 6.3.2 *Let A be a pseudo-Krull domain with quotient field K and Krull dimension n. Then $A \ltimes K \in \mathscr{H}$ is a ϕ-pseudo-Krull ring with Krull dimension n.*

The following is an example of a ϕ-pseudo-Krull ring which is not a ϕ-Krull ring.

Example 6.3.3 *Let A be a pseudo-Krull domain which is not a Krull-domain with quotient field K. Then $R := A \ltimes K$ is a ϕ-pseudo-Krull ring by Proposition 6.3.1 which is not a ϕ-Krull ring by [6, Theorem 3.1] since $R/\mathrm{Nil}(R)$ is ring-isomorphic to A.*

Example 6.3.4 *Let $L \subseteq K$ be a proper finite algebraic field extension, and let $T = K[[X_1, ..., X_n]]$ be a ring of power series in $n \geq 2$ variables over K. Then $T = K + M$, where $M = X_1T + + X_nT$ is the unique maximal ideal of T. Note that T is an n-dimensional Noetherian domain. Let $A = L + M$ with quotient field E and let $R := A \ltimes E$. Then:*

1. *R is a ϕ-t-closed ϕ-Strongly-Mori ring with Krull dimension n.*
2. *R is not a ϕ-pseudo-Krull ring.*

Proof. (1) By [79, Remark 3.5] A is a t-closed strongly-Mori domain. Since $\mathrm{Nil}(R) = 0 \ltimes E$ and $R/\mathrm{Nil}(R)$ is ring-isomorphic to A, R is a ϕ-t-closed ϕ-Strongly-Mori ring with Krull dimension n by Lemma 6.2.7(2), [80, Theorem 2.4] and [69, Theorem 25.1].
(2) A is not a pseudo-Krull domain by [88, Example 3.6]. Then R is not a ϕ-pseudo-Krull ring by Proposition 6.3.1.

In [52, 93] the authors studied when the amalgamation of rings is a ϕ-ring. Now, we study when $A \bowtie^f J$ is a ϕ-pseudo-Krull ring.

Proposition 6.3.5 *Let A and B be two rings, J an ideal of B such that $J \subseteq \mathrm{Nil}(B)$, and $f : A \longrightarrow B$ be a ring homomorphism. Set $R := A \bowtie^f J$ and assume that $R \in \mathscr{H}$. Then R is a ϕ-pseudo-Krull ring if and only if so is A.*

Proof. First Observe that $\mathrm{Nil}(R) = \{(a, f(a)+j) \mid a \in \mathrm{Nil}(A), j \in J \cap \mathrm{Nil}(B)\}$. It is easy to see that if $J \subseteq \mathrm{Nil}(B)$, then $\mathrm{Nil}(R) = \mathrm{Nil}(A) \bowtie^f J$. Now, assume that R is a ϕ-pseudo-Krull ring. Hence $R/\mathrm{Nil}(R)$ is a pseudo-Krull domain by Theorem 6.2.3. Since $A/\mathrm{Nil}(A)$ is ring-isomorphic to $R/\mathrm{Nil}(R)$, we conclude that $A/\mathrm{Nil}(A)$ is a pseudo-Krull domain. Thus A is a ϕ-pseudo-Krull ring. The converse follows by similar reasoning.

We combine [93, Corollary 2.3] with Proposition 6.3.5 to get the following examples.

Example 6.3.6 *Let B be a pseudo-Krull domain with quotient field K. Set $A := B \ltimes K$ and $I = 0 \ltimes K$. Then, the amalgamated duplication $R := A \bowtie I$ is a ϕ-pseudo-Krull ring.*

Proof. By [93, Corollary 2.3] R is a ϕ-ring since $I = \mathrm{Nil}(A)$. Thus, Proposition 6.3.5 completes the proof.

Example 6.3.7 *Let C be a pseudo-Krull domain with quotient field K. Let $f : A := C \ltimes K \longrightarrow B := K \ltimes K$ be the natural injection, and set $J = 0 \ltimes K$. Then $A \bowtie^f J$ is a ϕ-pseudo-Krull ring.*

CHAPTER 7

ON ϕ-PIECEWISE NOETHERIAN RINGS

Abstract.

The purpose of this chapter * is to introduce two new classes of rings that is closely related to the classes of piecewise Noetherian domains and piecewise w-Noetherian domains. Let $\mathscr{H} = \{R \mid R \text{ is a commutative ring and } \mathrm{Nil}(R) \text{ is a divided prime ideal of } R\}$. Let $R \in \mathscr{H}$ be a ring with total quotient ring $T(R)$, and set $\phi : T(R) \longrightarrow R_{\mathrm{Nil}(R)}$ such that $\phi(\frac{a}{b}) = \frac{a}{b}$ for every $a \in R$ and every $b \in R \setminus Z(R)$. Then ϕ is a ring homomorphism from $T(R)$ into $R_{\mathrm{Nil}(R)}$, and ϕ restricted to R is also a ring homomorphism from R into $R_{\mathrm{Nil}(R)}$ given by $\phi(x) = \frac{x}{1}$ for every $x \in R$. At this setting, we introduce the notions of ϕ-piecewise Noetherian ring and ϕ-piecewise w-Noetherian ring and we show that the theories of ϕ-piecewise Noetherian (resp. ϕ-piecewise w-Noetherian) rings resemble those of piecewise Noetherian (resp. piecewise w-Noetherian) domains.

7.1 Introduction

Recall from [18, 40] that a prime ideal P of R is said to be *divided* if it is comparable to every ideal of R, equivalently if $P \subseteq (x)$ for any $x \in R \setminus P$. A ring R is called a *divided ring* if every prime ideal of R is divided. Recently A. Badawi, in [13, 14, 15, 16, 17], has studied the following class of rings: $\mathscr{H} = \{R \mid R \text{ is a commutative ring and } \mathrm{Nil}(R)$ is a divided prime ideal of $R\}$. If $R \in \mathscr{H}$, then R is called a *ϕ-ring*. It is easy to see that every integral domain is a ϕ-ring. An ideal I of R is said to be a *nonnil ideal* if $I \nsubseteq \mathrm{Nil}(R)$. If I is a nonnil ideal of a ϕ-ring R, then $\mathrm{Nil}(R) \subseteq I$. Let R be a ring with total quotient ring $T(R)$ such that $\mathrm{Nil}(R)$ is a divided prime ideal of R. As in [14], we define $\phi : T(R) \longrightarrow K := R_{\mathrm{Nil}(R)}$ such that $\phi(\frac{a}{b}) = \frac{a}{b}$ for every $a \in R$ and every $b \in R \setminus Z(R)$. Then ϕ is a ring homomorphism from $T(R)$ into K, and ϕ restricted to R is also a ring homomorphism from R into K given by $\phi(x) = \frac{x}{1}$ for every $x \in$

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R. Observe that if $R \in \mathscr{H}$, then $\phi(R) \in \mathscr{H}$, $Ker(\phi) \subseteq \mathrm{Nil}(R)$, $\mathrm{Nil}(T(R)) = \mathrm{Nil}(R)$, $\mathrm{Nil}(R_{\mathrm{Nil}(R)}) = \phi(\mathrm{Nil}(R)) = \mathrm{Nil}(\phi(R)) = Z(\phi(R))$, $T(\phi(R)) = R_{\mathrm{Nil}(R)}$ is local with maximal ideal $\mathrm{Nil}(\phi(R))$, and $R_{\mathrm{Nil}(R)}/\mathrm{Nil}(\phi(R)) = T(\phi(R))/\mathrm{Nil}(\phi(R))$ is the quotient field of $\phi(R)/\mathrm{Nil}(\phi(R))$. A ring R is called a *ϕ-chained ring* if $\mathrm{Nil}(R)$ is divided prime and for each $x \in R_{\mathrm{Nil}(R)} \setminus \phi(R)$ we have $x^{-1} \in \phi(R)$ [15]. Then it is easy to see that R is a ϕ-chained ring if and only if $\mathrm{Nil}(R)$ is divided prime and for every $a, b \in R \setminus \mathrm{Nil}(R)$ either $a \mid b$ or $b \mid a$.

The concept of piecewise Noetherian rings was first introduced and studied by Beachy and Weakley in [26]. They originally defined piecewise Noetherian rings as follows: A commutative ring R is piecewise Noetherian if it satisfies the ascending chain condition (ACC) on its prime ideals and has finite ideal lengths (a concept that goes back to the work of Krull). As pointed out in [89], this notion of finite ideal length was used before localization techniques were developed. In [26], the authors showed that being piecewise Noetherian is inherited by factor rings and localizations and extends to integral extensions and polynomial rings, and they distinguished these rings from rings with Krull dimension. In a later paper [27], Beachy and Weakley proved the existence of an associated prime ideal for any module over a commutative piecewise Noetherian ring. They also showed that for a piecewise Noetherian ring the smallest test set (for injectivity) of prime ideals is the set of essential prime ideals. In [89], Pearson extended and studied the notion of piecewise Noetherian rings to the noncommutative setting.

Recall that a commutative ring R is said to have Noetherian spectrum if it satisfies the ACC on radical ideals. This is equivalent to the condition that R satisfies the ACC on prime ideals and each ideal has only finitely many prime ideals minimal over it. It is remarked just before [27, Lemma 1] that a commutative ring R is piecewise Noetherian if and only if R has Noetherian spectrum and satisfies the ACC on P-primary ideals, for each prime ideal P of R. This characterization shows clearly that every Noetherian ring is piecewise Noetherian, and is used to define the notion of piecewise w-Noetherian rings. In [51], the authors introduced the notion of piecewise w-Noetherian domains and showed that an integral domain R is a generalized Krull domain if and only if R is a piecewise w-Noetherian PvMD.

Let R be a ring with total quotient ring $T(R)$ and $\mathbf{F}(R)$ the set of nonzero fractional ideals of R. For a nonzero ideal $I \in \mathbf{F}(R)$, we let $I^{-1} = \{x \in T(R) \mid xI \subseteq R\}$, $I_v = (I^{-1})^{-1}$, $I_t = \bigcup\{J_v \mid J \subseteq I$ is a nonzero finitely generated fractional ideal of $R\}$, and $I_w = \{x \in T(R) \mid xJ \subseteq I$ for some finitely generated ideal J with $J^{-1} = R\}$. An ideal $I \in \mathbf{F}(R)$ is called a v-ideal (resp. t-ideal, w-ideal) if $I = I_v$ (resp. $I = I_t, I = I_w$). For $* = v, w$, an ideal I is said to be of $*$-finite type (or $*$-finite) if $I_* = J_*$ for some finitely generated ideal J of R. A finitely generated ideal J of R is called a GV-ideal if $J^{-1} = R$. Denote by $GV(R)$ the set of GV-ideals of R. Recall from [80] that a nonnil ideal I is a ϕ-w-ideal if $\phi(I)$ is a w-ideal of $\phi(R)$, and R is called a ϕ-strong Mori (ϕ-SM) ring if it satisfies the ascending chain condition (ACC) on ϕ-w-ideals.

In this chapter, we are interested in extending the piecewise Noetherian (resp. w-

Noetherian) property to ϕ-rings. A ring $R \in \mathscr{H}$ is said to be ϕ-piecewise Noetherian if (1) the set of nonnil prime ideals of R satisfies the ACC; (2) R has the ACC on nonnil P-primary ideals for each nonnil prime ideal P; and (3) each nonnil ideal has only finitely many prime ideals minimal over it. Among many results in this chapter, we show that many of the properties of piecewise Noetherian (resp. w-Noetherian) domains are also true for ϕ-piecewise Noetherian (resp. w-Noetherian) rings. We also use trivial ring extension and amalgamation of rings to construct examples of ϕ-piecewise Noetherian (resp. w-Noetherian) rings which are not integral domains.

7.2 On ϕ-piecewise Noetherian rings

Recall that a non-zerodivisor of a ring R is called a regular element and an ideal of R is said to be regular if it contains a regular element. We say that a ring R is a r-piecewise Noetherian ring if (1) the set of regular prime ideals of R satisfies the ACC, (2) R has the ACC on regular P-primary ideals for each prime ideal P, and (3) each regular ideal has only finitely many prime ideals minimal over it. We have the following characterization of ϕ-piecewise Noetherian rings in terms of r-piecewise Noetherian rings.

Theorem 7.2.1 *Let $R \in \mathscr{H}$. Then R is a ϕ-piecewise Noetherian ring if and only if $\phi(R)$ is an r-piecewise Noetherian ring.*

Proof. First observe that each regular ideal of $\phi(R)$ is the image of a unique nonnil ideal of R and $\phi(I)$ is a regular ideal of $\phi(R)$ for each nonnil ideal I of R. Thus for a prime ideal P of R, an ideal $\phi(I)$ of $\phi(R)$ is regular $\phi(P)$-primary if and only if I is a nonnil P-primary ideal of R. Hence a chain of nonnil P-primary ideals (resp. prime ideals) of R stabilizes if and only if the corresponding chain of regular $\phi(P)$-primary ideals (resp. prime ideals) of $\phi(R)$ stabilizes. Clearly a regular ideal $\phi(I)$ of $\phi(R)$ has only finitely many prime ideals minimal over it if and only if the nonnil ideal I has only finitely many prime ideals minimal over it. Therefore R is a ϕ-piecewise Noetherian ring if and only if $\phi(R)$ is an r-piecewise Noetherian ring.

We state our first main result of this chapter.

Theorem 7.2.2 *Let $R \in \mathscr{H}$. Then R is a ϕ-piecewise Noetherian ring if and only if $R/\mathrm{Nil}(R)$ is a piecewise Noetherian domain.*

Proof. Suppose that R is a ϕ-piecewise Noetherian ring. Let $\{I_i/\mathrm{Nil}(R)\}$ be an ascending chain of nonzero $P/\mathrm{Nil}(R)$-primary ideal of $R/\mathrm{Nil}(R)$ for a prime ideal P of R where each I_i is a nonnil ideal of R. Hence $\{I_i\}$ is an ascending chain of nonnil P-primary ideal of R. Thus there exists an integer $n \geq 1$ such that $I_n = I_m$ for each $m \geq n$. It follows that $I_n/\mathrm{Nil}(R) = I_m/\mathrm{Nil}(R)$ for each $m \geq n$. Similarly, the set of prime ideals of $R/\mathrm{Nil}(R)$

satisfies the ACC. Moreover for a nonzero ideal $J = I/\mathrm{Nil}(R)$ of $R/\mathrm{Nil}(R)$, $P/\mathrm{Nil}(R)$ is minimal over J if and only if P is minimal over I. Since I is a nonnil ideal of R which has only finitely many prime ideals minimal over it, it follows that J has only finitely many prime ideals minimal over it.

Conversely, suppose that $R/\mathrm{Nil}(R)$ is a piecewise Noetherian domain. For a prime ideal P of R, let $\{I_i\}$ be an ascending chain of nonnil P-primary ideals of R. Hence $\{I_i/\mathrm{Nil}(R)\}$ is an ascending chain of nonzero $P/\mathrm{Nil}(R)$-primary ideals of $R/\mathrm{Nil}(R)$. Thus there exists an integer $n \geq 1$ such that $I_n/\mathrm{Nil}(R) = I_m/\mathrm{Nil}(R)$ for each $m \geq n$. Therefore we get $I_n = I_m$ for each $m \geq n$. The other two assertions follow by similar reasoning.

Using the following proposition, we state an example of a ϕ-piecewise Noetherian ring which is neither an integral domain nor a piecewise Noetherian ring.

Proposition 7.2.3 *Let A be an integral domain and E be an A-module. If $A \ltimes E$ is a piecewise Noetherian ring, then E is a Noetherian A-module.*

Proof. Set $P = 0 \ltimes E$ and $R = A \ltimes E$. Let $\{N_i\}$ be an ascending chain of submodules of E. Then $\{0 \ltimes N_i\}$ is an ascending chain of P-primary ideals of R by [69, Theorem 25.2]. Since R is a piecewise Noetherian ring, there exists an integer $n \geq 1$ such that $0 \ltimes N_n = 0 \ltimes N_m$ for each $m \geq n$. Hence $N_n = N_m$ for each $m \geq n$, and thus E is a Noetherian A-module.

Example 7.2.4 Let A be a piecewise Noetherian domain with quotient field K. Set $R = A \ltimes K$. Then $R \in \mathscr{H}$ and R is a ϕ-piecewise Noetherian ring which is neither an integral domain nor a piecewise Noetherian ring.

Proof. Clearly $\mathrm{Nil}(R) = \{0\} \ltimes K$ is a divided prime ideal of R. Thus $R \in \mathscr{H}$. Since $R/\mathrm{Nil}(R)$ is ring-isomorphic to A, we conclude that R is a ϕ-piecewise Noetherian ring by Theorem 7.2.2. But R is not a piecewise Noetherian ring by Proposition 7.2.3.

Example 7.2.5 *Let A be a ϕ-piecewise Noetherian ring which is not a piecewise Noetherian ring. Then $R = A \bowtie \mathrm{Nil}(A)$ is a ϕ-piecewise Noetherian ring which is not a piecewise Noetherian ring.*

Proof. It is clear by [93, Corollary 2.3] that R is a ϕ-ring and $\mathrm{Nil}(R) = \mathrm{Nil}(A) \bowtie \mathrm{Nil}(A)$. As $R/\mathrm{Nil}(R)$ is ring-isomorphic to $A/\mathrm{Nil}(A)$, we conclude that R is a ϕ-piecewise Noetherian ring by Theorem 7.2.2. But R is not a piecewise Noetherian ring by [31, Corollary 1.16].

Example 7.2.6 *Let A be a ϕ-piecewise Noetherian ring, $B = A \bowtie \mathrm{Nil}(A)$, and consider the injective ring homomorphism $f : A \longrightarrow B$ given by $f(a) = (a,a)$. Set $J = \mathrm{Nil}(B)$. Hence $R = A \bowtie^f J$ is a ϕ-piecewise Noetherian ring.*

Proof. It follows from [93, Corollary 2.3] that $\mathrm{Nil}(R) = \mathrm{Nil}(A) \bowtie^f J$ is a divided prime ideal of R since A is a ϕ-ring and $f(A)+J = A \bowtie \mathrm{Nil}(A)$ is a ϕ-ring by the above example. Thus $R \in \mathscr{H}$. Since $R/\mathrm{Nil}(R)$ is ring-isomorphic to $A/\mathrm{Nil}(A)$, we conclude that R is a ϕ-piecewise Noetherian ring by Theorem 7.2.2.

Recall from [26, Proposition 2.2] that each maximal ideal of a piecewise Noetherian ring is finitely generated. We have the following result.

Proposition 7.2.7 *Let $R \in \mathscr{H}$ be a ϕ-piecewise Noetherian ring. Then each nonnil maximal ideal of R is finitely generated.*

Proof. Since R is a ϕ-piecewise Noetherian ring, by Theorem 7.2.2 $R/\mathrm{Nil}(R)$ is a piecewise Noetherian domain. Hence, [26, Proposition 2.2] implies that $M/\mathrm{Nil}(R)$ is finitely generated for each nonnil maximal ideal M of R. That implies $M/\mathrm{Nil}(R) = (a_1 + \mathrm{Nil}(R), \ldots, a_m + \mathrm{Nil}(R))$ for some $a_1, \ldots, a_m$ of M. Let a be a nonnilpotent element of M. Then $a + \mathrm{Nil}(R) = b_1a_1 + \cdots + b_ma_m + \mathrm{Nil}(R)$ in $R/\mathrm{Nil}(R)$ for some b_n's in R. Thus there is $t \in \mathrm{Nil}(R)$ such that $a + t = b_1a_1 + \cdots + b_ma_m$ in R. Since a is nonnilpotent and $\mathrm{Nil}(R)$ is divided, $t = ad$ for some $d \in \mathrm{Nil}(R)$. Then $a + t = a + ad = a(1+d) = b_1a_1 + \cdots + b_ma_m$. Since $d \in \mathrm{Nil}(R)$, $1+d$ is a unit of R. Hence $a \in (a_1, \ldots, a_m)$, and thus M is a finitely generated ideal of R.

The Krull dimension of an R-module M will be denoted by K-dim(M), and is defined as follows: If M is Artinian, then K-dim$(M) = 0$; if α is an ordinal and K-dim$(M) \not< \alpha$, then K-dim$(M) = \alpha$ if there is no infinite descending chain $\{M_i\}$ such that K-dim$(M_{i-1}/M_i) \not< \alpha$ for $i = 1, 2, \ldots$. The Krull dimension of a ring R is its Krull dimension as an R-module. Recall the following lemmas from [61].

Lemma 7.2.8 *[61, Lemma 1.1] If M is a module and N is a submodule of M, then K-dim$(M) = \sup\{K\text{-dim}(M/N), K\text{-dim}(N)\}$ if either side exists.*

Lemma 7.2.9 *[61, Proposition 1.2] Every homomorphic image of a ring R with Krull dimension has Krull dimension less than or equal to K-dim(R).*

Now, we have the next result.

Proposition 7.2.10 *Let $R \in \mathscr{H}$ and set $D = R/Nil(R)$. If D has Krull dimension as a D-module, then R is a ϕ-piecewise Noetherian ring.*

Proof. As D has Krull dimension as a D-module, by [26, Proposition 2.3] we get that D is a piecewise Noetherian domain. Hence R is a ϕ-piecewise Noetherian ring by Theorem 7.2.2.

Let $R \in \mathscr{H}$. Set $D := R/\mathrm{Nil}(R)$. Then we observe that D has Krull dimension as a D-module if and only if D has Krull dimension as an R-module since the submodules of

D as an R-module are exactly the ideals of D.

In view of Proposition 7.2.10, Lemma 7.2.9, and the remark above, we have the following corollary.

Corollary 7.2.11 *If $R \in \mathscr{H}$ has Krull dimension as an R-module, then R is ϕ-piecewise Noetherian.*

It is well-known [26, Theorem 2.9] that a valuation domain R is piecewise Noetherian if and only if $P^2 \neq P$ or $P = 0$ for each prime ideal P of R if and only if R has Krull dimension as a module. For a ring $R \in \mathscr{H}$, we have the following analog.

Theorem 7.2.12 *Let $R \in \mathscr{H}$ be a ϕ-chained ring. Assume that* $\mathrm{Nil}(R)$ *has Krull dimension as an R-module. Then the following conditions are equivalent:*

(1) R is a ϕ-piecewise Noetherian ring.

(2) If P is a prime ideal of R, then $P^2 \neq P$ or $P = \mathrm{Nil}(R)$.

(3) R has Krull dimension as an R-module.

Proof. Let $D = R/\mathrm{Nil}(R)$. As R is a ϕ-chained ring, by [5, Theorem 2.7] we get that D is a valuation domain.

$(1) \Rightarrow (2)$ By Theorem 7.2.2, we conclude that D is a piecewise Noetherian domain. Hence [26, Theorem 2.9] implies that if P is a nonnil prime ideal of R, then $(P/\mathrm{Nil}(R))^2 \neq P/\mathrm{Nil}(R)$. Thus, $P^2 \neq P$ for each nonnil prime ideal P of R.

$(2) \Rightarrow (3)$ Clearly for every prime ideal P of R, we have $(P/\mathrm{Nil}(R))^2 \neq P/\mathrm{Nil}(R)$ or $P/\mathrm{Nil}(R) = 0$. Hence by [26, Theorem 2.9], D has Krull dimension as a D-module, and hence as an R-module. Since $\mathrm{Nil}(R)$ has Krull dimension as an R-module, Lemma 7.2.8 implies that R has Krull dimension as an R-module.

$(3) \Rightarrow (1)$ As R has Krull dimension as an R-module, we get that D has Krull dimension as an R-module by Lemma 7.2.9, and so as a D-module. Thus D is a piecewise Noetherian domain by [26, Theorem 2.9]. Therefore R is a ϕ-piecewise Noetherian ring by Theorem 7.2.2.

A ring R is called a Prüfer ring if every finitely generated regular ideal of R is invertible, i.e., if I is a finitely generated regular ideal of R and $I^{-1} = \{x \in T(R) \mid xI \subseteq R\}$, then $II^{-1} = R$. From [5], a nonnil ideal I of R is said to be ϕ-invertible if $\phi(I)$ is an invertible ideal of $\phi(R)$. If every finitely generated nonnil ideal of R is ϕ-invertible, then we say that R is a ϕ-Prüfer ring. The next result is an analog of [26, Proposition 2.10].

Theorem 7.2.13 *Let $R \in \mathscr{H}$ be a ϕ-Prüfer ring in which every nonnilpotent element is contained in only finitely maximal ideals. Assume that* $\mathrm{Nil}(R)$ *has Krull dimension as an*

R-module. Then R is a ϕ-piecewise Noetherian ring if and only if R has Krull dimension as an R-module.

Proof. Set $D = R/\mathrm{Nil}(R)$. By [5, Theorem 2.6], D is a Prüfer domain since R is a ϕ-Prüfer ring. Moreover, every nonzero element of D is contained in only finitely maximal ideals.

Suppose that R is a ϕ-piecewise Noetherian ring. Then D is a piecewise Noetherian domain by Theorem 7.2.2, and thus D has Krull dimension as a D-module by [26, Proposition 2.10], and hence as an R-module. That implies R has Krull dimension as an R-module by Lemma 7.2.8 since $\mathrm{Nil}(R)$ has Krull dimension as an R-module.

Conversely, since R has Krull dimension as an R-module, we obtain that D has Krull dimension as an R-module by Lemma 7.2.9, and so as a D-module. Hence [26, Proposition 2.10] implies that D is a piecewise Noetherian domain. Therefore R is ϕ-piecewise Noetherian by Theorem 7.2.2.

Recall that a ring R is said to be Laskerian if every ideal of R can be expressed as a finite intersection of primary ideals. Also, recall from [17] that a ring $R \in \mathscr{H}$ is called a nonnil-Noetherian ring if every nonnil ideal of R is finitely generated. In light of [26, Theorem 1.8] we get the following result for a ring $R \in \mathscr{H}$.

Theorem 7.2.14 *Let $R \in \mathscr{H}$ be a Laskerian ring. Then R is a nonnil-Noetherian ring if and only if R is a ϕ-piecewise Noetherian ring.*

Proof. Set $D = R/\mathrm{Nil}(R)$. Let J be a nonzero ideal of D. Hence $J = I/\mathrm{Nil}(R)$ for some nonnil ideal I of R. Since R is Laskerian, I is a finite intersection of nonnil primary ideals of R. Thus J can be expressed as a finite intersection of nonzero primary ideals of D. Therefore D is a Laskerian domain. Now, since R is a nonnil-Noetherian ring, by [17, Theorem 1.2] we conclude that D is a Noetherian domain. Hence D is a piecewise Noetherian domain by [26, Theorem 1.8], which implies that R is a ϕ-piecewise Noetherian ring by Theorem 7.2.2.

Conversely, assume that R is a ϕ-piecewise Noetherian ring. Thus it follows from Theorem 7.2.2 that D is a piecewise Noetherian domain. Hence D is a Noetherian domain by [26, Theorem 1.8]. Therefore R is a nonnil-Noetherian ring by [17, Theorem 1.2].

For a ring R, let R' denote the integral closure of R in $T(R)$.

Lemma 7.2.15 *[6, Lemma 2.8] Let $R \in \mathscr{H}$ and set $D = \phi(R)/\mathrm{Nil}(\phi(R))$. Then $D' = \phi(R)'/\mathrm{Nil}(\phi(R))$.*

Lemma 7.2.16 *[5, Lemma 2.5] Let $R \in \mathscr{H}$ and let P be a prime ideal of R. Then R/P is ring-isomorphic to $\phi(R)/\phi(P)$. In particular, $R/\mathrm{Nil}(R)$ is ring-isomorphic to $\phi(R)/\mathrm{Nil}(\phi(R))$.*

It is well-known [26, Proposition 3.7(1)] that the integral closure of any Noetherian domain is piecewise Noetherian. We have a similar characterization for ϕ-piecewise Noetherian rings.

Theorem 7.2.17 *Let $R \in \mathscr{H}$ be a nonnil-Noetherian ring. Then $\phi(R)'$ is a ϕ-piecewise Noetherian ring.*

Proof. Set $D = \phi(R)/\mathrm{Nil}(\phi(R))$. Since D is ring-isomorphic to $R/\mathrm{Nil}(R)$ by Lemma 7.2.16, we conclude that D is a Noetherian domain by [17, Theorem 2.1]. Since $D' = \phi(R)'/\mathrm{Nil}(\phi(R))$ by Lemma 7.2.15 and D' is a piecewise Noetherian domain by [26, Proposition 3.7(1)], we get that $\phi(R)'$ is a ϕ-piecewise Noetherian ring by Theorem 7.2.2.

Note that if $\mathrm{Nil}(R) = Z(R)$, then $\phi(R) = R$. Thus we may state the following corollary of Theorem 7.2.17.

Corollary 7.2.18 *Let $R \in \mathscr{H}$ be a nonnil-Noetherian ring such that $Nil(R) = Z(R)$. Then R' is a ϕ-piecewise Noetherian ring.*

Theorem 7.2.19 *Let $R \in \mathscr{H}$. Then R is a ϕ-piecewise Noetherian ring if and only if $\phi(R)$ is ring-isomorphic to a ring A obtained from the following pullback diagram:*

$$\begin{array}{ccc} A & \longrightarrow & S \\ \downarrow & & \downarrow \\ T & \longrightarrow & T/M \end{array}$$

where T is a zero-dimensional local ring with maximal ideal M, $S := A/M$ is a piecewise Noetherian subring of T/M, the vertical arrows are the usual inclusion maps, and the horizontal arrows are the usual surjective maps.

Proof. Suppose that $\phi(R)$ is ring-isomorphic to a ring A obtained from the given diagram. Then $A \in \mathscr{H}$ and $\mathrm{Nil}(A) = Z(A) = M$. Since A/M is a piecewise Noetherian domain, A is a ϕ-piecewise Noetherian ring by Theorem 7.2.2, and thus R is a ϕ-piecewise Noetherian ring. Conversely, suppose that R is a ϕ-piecewise Noetherian ring. Then setting $T := R_{\mathrm{Nil}(R)}$, $M := \mathrm{Nil}(R_{\mathrm{Nil}(R)})$, and $A := \phi(R)$ yields the desired pullback diagram.

Recall that a ring S is called an overring of a ring R if $R \subseteq S \subseteq T(R)$. In this case we have $\mathrm{Nil}(S) = \mathrm{Nil}(R)$. Let $R \in \mathscr{H}$. Then we say that an overring T of R is ϕ-flat over R if $T_M = R_{M \cap R}$ for all nonnil maximal ideals M of T.

Lemma 7.2.20 *Let $R \in \mathscr{H}$. Then an overring T of R is ϕ-flat if and only if $T/\mathrm{Nil}(T)$ is flat over $R/\mathrm{Nil}(R)$.*

Proof. Assume that T is ϕ-flat over R and let $M/\mathrm{Nil}(R)$ be a maximal ideal of $T/\mathrm{Nil}(T)$. Hence M is a nonnil maximal ideal of T. Thus $T_M = R_{M\cap R}$, and so $T_M/\mathrm{Nil}(T)T_M = R_{M\cap R}/\mathrm{Nil}(T)R_{M\cap R}$. That implies

$$(T/\mathrm{Nil}(T))_{M/\mathrm{Nil}(T)} = (R/\mathrm{Nil}(R))_{(M/\mathrm{Nil}(T))\cap(R/\mathrm{Nil}(R))}$$

since $\mathrm{Nil}(T) = \mathrm{Nil}(R)$. Therefore $T/\mathrm{Nil}(T)$ is flat over $R/\mathrm{Nil}(R)$. The converse follows by similar reasoning.

It is well-known [31, Theorem 1.4] that a flat overring of a piecewise Noetherian ring is a piecewise Noetherian ring. We have the next analog.

Proposition 7.2.21 *Let $R \in \mathscr{H}$ be a ϕ-piecewise Noetherian ring. Then every ϕ-flat overring of R is a ϕ-piecewise Noetherian ring.*

Proof. Let T be a ϕ-flat overring of R. Then $T \in \mathscr{H}$, $\mathrm{Nil}(T) = \mathrm{Nil}(R)$, and $T/\mathrm{Nil}(R)$ is a flat overring of $R/\mathrm{Nil}(R)$. Since $R/\mathrm{Nil}(R)$ is a piecewise Noetherian domain and $T/\mathrm{Nil}(R)$ is a flat overring of $R/\mathrm{Nil}(R)$, we conclude that $T/\mathrm{Nil}(R)$ is a piecewise Noetherian domain by [31, Theorem 1.4]. Thus T is a ϕ-piecewise Noetherian ring by Theorem 7.2.2.

Let R be a ϕ-ring. Then R is said to be of ϕ-finite character if each nonnilpotent nonunit element of R is contained in only finitely many maximal ideal. It is clear that R has ϕ-finite character if and only if $R/\mathrm{Nil}(R)$ has finite character. The following is an analog of [31, Corollary 1.8].

Theorem 7.2.22 *Let $R \in \mathscr{H}$ be a ring of ϕ-finite character. Then R is ϕ-piecewise Noetherian if and only if R_M is ϕ-piecewise Noetherian for all maximal ideals M of R.*

Proof. Set $D = R/\mathrm{Nil}(R)$. Clearly D is of finite character since R is of ϕ-finite character. Since D is a piecewise Noetherian domain by Theorem 7.2.2, we conclude that $D_{M/\mathrm{Nil}(R)}$ is a piecewise Noetherian domain for every maximal ideal M of R by [31, Corollary 1.8]. As $D_{M/\mathrm{Nil}(R)}$ is ring-isomorphic to $R_M/\mathrm{Nil}(R)R_M = R_M/\mathrm{Nil}(R_M)$ and $R_M \in \mathscr{H}$, we get that R_M is a ϕ-piecewise Noetherian ring by Theorem 7.2.2.

Conversely, as $R_M \in \mathscr{H}$ for each maximal ideal M of R, we conclude that $R_M/\mathrm{Nil}(R_M)$ is a piecewise Noetherian domain for each maximal ideal M of R by Theorem 7.2.2. Hence $D_{M/\mathrm{Nil}(R)}$ is a piecewise Noetherian domain for each maximal ideal M of R. Thus D is a piecewise Noetherian domain by [31, Corollary 1.8]. Therefore R is a ϕ-piecewise Noetherian ring by Theorem 7.2.2.

Recall from [91] that an integral domain R is a generalized Dedekind domain if and only if it is a Prüfer domain, $P \neq P^2$ for every nonzero prime ideal P of R, and every

prime ideal of R is the radical of a finitely generated ideal. Now, we extend the notion of generalized Dedekind domains to the setting of ϕ-rings.

Definition 7.2.23 Let $R \in \mathscr{H}$. We say that R is a ϕ-generalized Dedekind ring if R is a ϕ-Prüfer ring, $P \neq P^2$ for every nonnil prime ideal P of R, and every nonnil prime ideal of R is the radical of a nonnil finitely generated ideal.

We have the next result.

Theorem 7.2.24 *Let $R \in \mathscr{H}$. Then R is a ϕ-generalized Dedekind ring if and only if $R/\mathrm{Nil}(R)$ is a generalized Dedekind domain.*

Proof. Since R is a ϕ-Prüfer ring, $R/\mathrm{Nil}(R)$ is a Prüfer domain by [5, Theorem 2.6]. Now let $P/\mathrm{Nil}(R)$ be a nonzero prime ideal of $R/\mathrm{Nil}(R)$. Hence P is a nonnil prime ideal of R such that $P \neq P^2$ since R is a ϕ-generalized Dedekind ring. Thus $P/\mathrm{Nil}(R) \neq (P/\mathrm{Nil}(R))^2$. Also, for a nonzero prime ideal Q of $R/\mathrm{Nil}(R)$, we have $Q = P/\mathrm{Nil}(R)$ for a nonnil prime ideal P of R such that $P = \sqrt{I}$ where I is a nonnil finitely generated ideal of R. Hence one can easily see that $Q = \sqrt{I/\mathrm{Nil}(R)}$ such that $I/\mathrm{Nil}(R)$ is a finitely generated ideal.

Conversely, by [5, Theorem 2.6] R is a ϕ-Prüfer ring since $R/\mathrm{Nil}(R)$ is a Prüfer domain. Moreover, if P is a nonnil prime ideal of R, then $P/\mathrm{Nil}(R)$ is a nonzero prime ideal of $R/\mathrm{Nil}(R)$ such that $P/\mathrm{Nil}(R) \neq (P/\mathrm{Nil}(R))^2$. That implies $P \neq P^2$. Now let P be a nonnil prime ideal of R. Then $P/\mathrm{Nil}(R) = \sqrt{I/\mathrm{Nil}(R)}$ where $I/\mathrm{Nil}(R)$ is a finitely generated ideal. Thus by similar reasoning as in the proof of Proposition 6.3.5, we get that I is a finitely generated ideal of R, and so $P = \sqrt{I}$. This completes the proof.

It is well-known [58, Proposition 3.1] that generalized Dedekind domains can be characterized as piecewise Noetherian Prüfer domains. We have the next similar result.

Theorem 7.2.25 *Let $R \in \mathscr{H}$ be a ϕ-Prüfer ring. Then R is a ϕ-generalized Dedekind ring if and only if R is a ϕ-piecewise Noetherian ring.*

Proof. Set $D = R/\mathrm{Nil}(R)$. Then it follows from [5, Theorem 2.6] that D is a Prüfer domain. Assume that R is a ϕ-generalized Dedekind ring. Hence D is a generalized Dedekind domain by Theorem 7.2.24, and so D is a piecewise Noetherian domain by [58, Proposition 3.1]. Thus R is ϕ-piecewise Noetherian ring by Theorem 7.2.2.

Conversely, since R is a ϕ-piecewise Noetherian ring, by Theorem 7.2.2 we conclude that D is a piecewise Noetherian domain. Thus D is a generalized Dedekind domain by [58, Proposition 3.1]. Then by using Theorem 7.2.24, we conclude that R is a ϕ-generalized Dedekind ring.

7.3 On ϕ-piecewise w-Noetherian rings

In this section, we introduce the ϕ-piecewise w-Noetherian rings and prove some properties of these rings.

Definition 7.3.1 Let $R \in \mathscr{H}$ be a ring. We say that R is a ϕ-piecewise w-Noetherian ring if (1) R satisfies the ACC on prime ϕ-w-ideals, (2) R has the ACC on nonnil P-primary ideals for each prime ϕ-w-ideal P, and (3) each ϕ-w-ideal has only finitely many prime ideals minimal over it.

Recall the next lemma from [80].

Lemma 7.3.2 *[80, Lemma 2.3(b)] Let $R \in \mathscr{H}$. Then:*

(1) For each nonnil ideal I of R, I is a ϕ-w-ideal of R if and only if $I/\mathrm{Nil}(R)$ is a w-ideal of $R/\mathrm{Nil}(R)$.

(2) For each nonnil ideal I of R, I is a maximal ϕ-w-ideal of R if and only if $I/\mathrm{Nil}(R)$ is a maximal w-ideal of $R/\mathrm{Nil}(R)$.

(3) For any nonnil ideals I and J of R, $(\phi(I) : \phi(J))/\mathrm{Nil}(\phi(R))$ is isomorphic to $(I/\mathrm{Nil}(R) : J/\mathrm{Nil}(R))$.

Theorem 7.3.3 *Let $R \in \mathscr{H}$. Then R is a ϕ-piecewise w-Noetherian ring if and only if $R/Nil(R)$ is a piecewise w-Noetherian domain.*

Proof. Assume that R is a ϕ-piecewise w-Noetherian ring. Let $\{I_i/\mathrm{Nil}(R)\}$ be an ascending chain of nonzero $P/\mathrm{Nil}(R)$-primary ideals for a prime w-ideal $P/\mathrm{Nil}(R)$ of $R/\mathrm{Nil}(R)$. By Lemma 7.3.2(1), P is a prime ϕ-w-ideal of R and $\{I_i\}$ is an ascending chain of nonnil P-primary ideals of R. Thus there exists an integer $n \geq 1$ such that $I_n = I_m$ for each $m \geq n$. It follows that $I_n/\mathrm{Nil}(R) = I_m/\mathrm{Nil}(R)$ for each $m \geq n$. Similarly, the set of prime ϕ-w-ideals of $R/\mathrm{Nil}(R)$ satisfies the ACC. Now let $J = I/\mathrm{Nil}(R)$ be a nonzero w-ideal of $R/\mathrm{Nil}(R)$. Then by Lemma 7.3.2(1), I is a ϕ-w-ideal of R. Since I has only finitely many prime ideals minimal over it, so is J. Therefore $R/\mathrm{Nil}(R)$ is a piecewise w-Noetherian domain.

Conversely, suppose that $R/\mathrm{Nil}(R)$ is a piecewise Noetherian domain. Let $\{I_i\}$ be an ascending chain of nonnil P-primary ideals of R for a prime ϕ-w-ideal P of R. Hence $P/\mathrm{Nil}(R)$ is a w-ideal of $R/\mathrm{Nil}(R)$ by Lemma 7.3.2(1) and $\{I_i/\mathrm{Nil}(R)\}$ is an ascending chain of nonzero $P/\mathrm{Nil}(R)$-primary ideals of $R/Nil(R)$. Thus there exists an integer $n \geq 1$ such that $I_n/\mathrm{Nil}(R) = I_m/\mathrm{Nil}(R)$ for each $m \geq n$. Therefore we get $I_n = I_m$ for each $m \geq n$. By similar reasoning we prove that R satisfies the ACC on prime ϕ-w-ideals. Moreover, let I be a ϕ-w-ideal of R. Then $I/\mathrm{Nil}(R)$ is a w-ideal of $R/\mathrm{Nil}(R)$ by Lemma 7.3.2(1), and hence $I/\mathrm{Nil}(R)$ has only finitely many prime ideals minimal over it since $R/\mathrm{Nil}(R)$ is a

piecewise w-Noetherian domain. Therefore I has only finitely many prime ideals minimal over it. This completes the proof.

Using the following proposition, we state an example of a ϕ-piecewise w-Noetherian ring which is neither an integral domain nor a piecewise w-Noetherian ring.

Proposition 7.3.4 *Let A be an integral domain with quotient field K and E be a vector space over K. If $A \ltimes E$ is a piecewise w-Noetherian ring, then E is a w-Noetherian A-module.*

Proof. Set $P = 0 \ltimes E$ and $R = A \ltimes E$. Let $\{N_i\}$ be an ascending chain of w-submodules over A of E. Then $\{0 \ltimes N_i\}$ is an ascending chain of P-primary w-ideals of R by [69, Theorem 25.2] and [30, Proposition 2.2]. Since R is a piecewise w-Noetherian ring, there exists an integer $n \geq 1$ such that $0 \ltimes N_n = 0 \ltimes N_m$ for each $m \geq n$. Hence $N_n = N_m$ for each $m \geq n$, and thus E is a w-Noetherian A-module.

Example 7.3.5 Let A be a piecewise w-Noetherian domain with quotient field K. Set $R = A \ltimes K$. Then $R \in \mathscr{H}$ and R is a ϕ-piecewise w-Noetherian ring which is neither an integral domain nor a piecewise w-Noetherian ring.

Proof. Since $\mathrm{Nil}(R) = \{0\} \ltimes K$ is a divided prime ideal of R, we have $R \in \mathscr{H}$. As $R/\mathrm{Nil}(R)$ is ring-isomorphic to A, we conclude that R is a ϕ-piecewise w-Noetherian ring by Theorem 7.3.3. Moreover R is not an integral domain. Now assume on the contrary that R is a piecewise w-Noetherian ring. Then K is a w-Noetherian A-module by Proposition 7.3.4. In particular, K is of w-finite type, and so $K = 0$ by [94, Exercise 7.16], a contradiction.

Example 7.3.6 *Let D be a piecewise w-Noetherian domain with quotient field K and $A = D \ltimes K$. Let $f : A \longrightarrow K \ltimes K$ be the natural injection, and set $J = 0 \ltimes K$. By using [93, Corollary 2.3] and Theorem 7.3.3, we conclude that $A \bowtie^f J$ is a ϕ-piecewise w-Noetherian ring.*

Let $R \in \mathscr{H}$. We say that an overring T of R is ϕ-t-flat over R if $T_M = R_{M \cap R}$ for all maximal ϕ-w-ideals M of T. R is said to be of ϕ-w-finite character if each nonnilpotent nonunit of R is contained in only finitely many maximal ϕ-w-ideals. Also, recall that a nonnil ideal I of R is said to be of ϕ-w-finite type if there exists a nonnil finitely generated ideal $J \subseteq I$ such that $\phi(I)_w = \phi(J)_w$.

Lemma 7.3.7 *The following holds for each $R \in \mathscr{H}$.*

(1) An overring T of R is ϕ-t-flat if and only if $T/\mathrm{Nil}(T)$ is t-flat over $R/\mathrm{Nil}(R)$.

(2) R is of ϕ-w-finite character if and only if $R/\mathrm{Nil}(R)$ is of w-finite character.

(3) A nonnil ideal I of R is of ϕ-w-finite type if and only if $I/\mathrm{Nil}(R)$ is an ideal of $R/Nil(R)$ of w-finite type.

Proof. (1) Assume that T is ϕ-t-flat over R and let $M/\mathrm{Nil}(R)$ be a maximal w-ideal of $T/\mathrm{Nil}(T)$. Hence M is a maximal ϕ-w-ideal of T by Lemma 7.3.2(2). Thus $T_M = R_{M\cap R}$, and so $T_M/\mathrm{Nil}(T)T_M = R_{M\cap R}/\mathrm{Nil}(T)R_{M\cap R}$. That implies

$$(T/\mathrm{Nil}(T))_{M/\mathrm{Nil}(R)} = (R/\mathrm{Nil}(R))_{M/\mathrm{Nil}(R)\cap R/\mathrm{Nil}(R)}$$

since $\mathrm{Nil}(T) = \mathrm{Nil}(R)$. Therefore $T/\mathrm{Nil}(T)$ is t-flat over $R/\mathrm{Nil}(R)$. The converse follows by similar reasoning.

(2) Suppose that R is of ϕ-w-finite character and let $r+\mathrm{Nil}(R)$ be a nonzeo nonunit element of $R/\mathrm{Nil}(R)$. Thus r is a nonnilpotent nonnuit element of R, and so r is contained in only finitely many maximal ϕ-w-ideals. By using Lemma 7.3.2(2), we can immediately conclude that $r+\mathrm{Nil}(R)$ is contained in only finitely many maximal w-ideals. By similar reasoning, we prove that if $R/\mathrm{Nil}(R)$ is of w-finite character, then R is of ϕ-w-finite character.

(3) The assertion follows by Lemma 7.3.2(3), the fact that

$$\phi(I)_w = \bigcup_{\phi(J)\in GV(\phi(R))} (\phi(I) : \phi(J))$$

and since a nonnil ideal K of R is finitely generated if and only if $K/\mathrm{Nil}(R)$ is a finitely generated ideal of $R/\mathrm{Nil}(R)$.

Next we have the following result.

Theorem 7.3.8 *Let $R \in \mathscr{H}$ be a ϕ-piecewise w-Noetherian ring. If T is a ϕ-t-flat overring of R, then T is a ϕ-piecewise w-Noetherian ring.*

Proof. Let T be a ϕ-t-flat overring of R. Then $T \in \mathscr{H}$, $\mathrm{Nil}(T) = \mathrm{Nil}(R)$, and by Lemma 7.3.7(1) $T/\mathrm{Nil}(T)$ is a t-flat overring of $R/\mathrm{Nil}(R)$. Since $R/\mathrm{Nil}(R)$ is a piecewise w-Noetherian domain, by Theorem 7.3.3 and the fact that $T/\mathrm{Nil}(T)$ is a t-flat overring of $R/\mathrm{Nil}(R)$, we conclude that $T/\mathrm{Nil}(T)$ is a piecewise w-Noetherian domain by [31, Theorem 2.3]. Thus T is a ϕ-piecewise w-Noetherian ring by Theorem 7.3.3.

Theorem 7.3.9 *Let $R \in \mathscr{H}$ be a ϕ-piecewise w-Noetherian ring of ϕ-w-finite character and let M be a maximal ϕ-w-ideal of R. Then M is of ϕ-w-finite type.*

Proof. By using Theorem 7.3.3 and Lemma 7.3.7(2), we conclude that $R/\mathrm{Nil}(R)$ is a piecewise w-Noetherian domain of w-finite character. Also, $M/\mathrm{Nil}(R)$ is a maximal w-ideal of $R/\mathrm{Nil}(R)$ by Lemma 7.3.2(2). Hence $M/\mathrm{Nil}(R)$ is of w-finite type by [31, Theorem 2.5]. It follows from Lemma 7.3.7(3) that M is of ϕ-w-finite type.

The following result is an analog of [31, Theorem 2.7].

Theorem 7.3.10 *Let $R \in \mathscr{H}$ be a ring of ϕ-w-finite character. Then R is ϕ-piecewise w-Noetherian if and only if R_M is ϕ-piecewise w-Noetherian for each maximal ϕ-w-ideal M of R.*

Proof. Set $D = R/\mathrm{Nil}(R)$. Clearly, D is of w-finite character by Lemma 7.3.7(2). Since D is a piecewise w-Noetherian domain by Theorem 7.3.3, we conclude that $D_{M/\mathrm{Nil}(R)}$ is a piecewise w-Noetherian domain for every maximal ϕ-w-ideal M of R by Lemma 7.3.2(2) and [31, Theorem 2.7]. Since $D_{M/\mathrm{Nil}(R)}$ is ring-isomorphic to $R_M/\mathrm{Nil}(R)R_M = R_M/\mathrm{Nil}(R_M)$ and $R_M \in \mathscr{H}$, we get that R_M is a ϕ-piecewise w-Noetherian ring by Theorem 7.3.3.

Conversely, as $R_M \in \mathscr{H}$ for each maximal ϕ-w-ideal M of R, we conclude that $R_M/\mathrm{Nil}(R_M)$ is a piecewise w-Noetherian domain by Theorem 7.3.3. Hence $D_{M/\mathrm{Nil}(R)}$ is a piecewise w-Noetherian domain for each maximal ϕ-w-ideal M of R. Thus D is a piecewise w-Noetherian domain by [31, Theorem 2.7]. Therefore R is a ϕ-piecewise w-Noetherian ring by Theorem 7.3.3.

Theorem 7.3.11 *Let $R \in \mathscr{H}$. Then R is a ϕ-piecewise w-Noetherian ring if and only if $\phi(R)$ is ring-isomorphic to a ring A obtained from the following pullback diagram:*

$$\begin{array}{ccc} A & \longrightarrow & S \\ \downarrow & & \downarrow \\ T & \longrightarrow & T/M \end{array}$$

where T is a zero-dimensional quasilocal ring with maximal ideal M, $S := A/M$ is a piecewise w-Noetherian subring of T/M, the vertical arrows are the usual inclusion maps, and the horizontal arrows are the usual surjective maps.

Proof. Suppose that $\phi(R)$ is ring-isomorphic to a ring A obtained from the given diagram. Then $A \in \mathscr{H}$ and $\mathrm{Nil}(A) = Z(A) = M$. Since A/M is a piecewise w-Noetherian domain, A is a ϕ-piecewise w-Noetherian ring by Theorem 7.3.3, and thus R is a ϕ-piecewise w-Noetherian ring.

Conversely, suppose that R is a ϕ-piecewise w-Noetherian ring. Then setting $T := R_{\mathrm{Nil}(R)}$, $M := \mathrm{Nil}(R_{\mathrm{Nil}(R)})$, and $A := \phi(R)$ yields the desired pullback diagram.

Following [81], we say that a commutative ring R is w-Laskerian if each proper w-ideal of R may be expressed as a finite intersection of primary w-ideals of R. For a ring $R \in \mathscr{H}$, we say that R is ϕ-w-Laskerian if each proper ϕ-w-ideal of R may be expressed as a finite intersection of primary ϕ-w-ideals of R. Clearly, by using Lemma 7.3.2(1), R is ϕ-w-Laskerian if and only if $R/\mathrm{Nil}(R)$ is a w-Laskerian domain. It is shown in [31, Theorem 2.18] that a piecewise w-Noetherian domain R is an SM domain if and only if R is a w-Laskerian domain. Our next result generalizes this result to ϕ-rings.

Theorem 7.3.12 *Let $R \in \mathscr{H}$ be a ϕ-piecewise w-Noetherian ring. Then R is a ϕ-SM ring if and only if R is a ϕ-w-Laskerian ring.*

Proof. It follows from Theorem 7.3.3 that $R/\mathrm{Nil}(R)$ is a piecewise w-Noetherian domain. As R is a ϕ-SM ring, we get that $R/\mathrm{Nil}(R)$ is an SM domain by [80, Theorem 2.4]. Thus $R/\mathrm{Nil}(R)$ is a w-Laskerian domain by [31, Theorem 2.18], and so R is a ϕ-w-Laskerian ring. The converse follows by similar reasoning.

The following is an example of a ϕ-piecewise w-Noetherian ring that is not a ϕ-*SM* ring.

Example 7.3.13 Let D be an SM domain with quotient field T such that $D \neq T$. Set $R = D + XT[X]$ and let K be the quotient field of R. By [31, Example 2.16] R is a piecewise w-Noetherian domain which is not *SM*. Clearly $S := R \ltimes K \in \mathscr{H}$, and thus S is a ϕ-piecewise w-Noetherian ring that is not a ϕ-*SM* ring by Theorem 7.3.3 and [80, Theorem 2.4] since $S/\mathrm{Nil}(S)$ is ring-isomorphic to R.

We recall the following lemma from [24].

Lemma 7.3.14 *[24, Lemma 2.4(b)] Let $R \in \mathscr{H}$. Then under the derived isomorphism $\beta : R_{\mathrm{Nil}(R)}/\mathrm{Nil}(\phi(R)) \to T(R/\mathrm{Nil}(R))$, $(\phi(R) : \phi(I))/\mathrm{Nil}(\phi(R))$ is isomorphic to $(R/\mathrm{Nil}(R) : I/\mathrm{Nil}(R))$ for each nonnil ideal I.*

Let $R \in \mathscr{H}$. We say that a nonnil ideal I is a ϕ-t-ideal of R if $\phi(I)$ is a t-ideal of $\phi(R)$. Recall that a nonnil ideal I of R is said to be of ϕ-v-finite type (or ϕ-v-finite) if there exists a nonnil finitely generated ideal J such that $\phi(I)_v = \phi(J)_v$.

The next lemma follows from Lemma 7.3.14 and the above definitions. We leave the proof to the reader.

Lemma 7.3.15 *The following hold for each $R \in \mathscr{H}$.*

(1) A nonnil ideal I of R is a ϕ-t-ideal if and only if $I/\mathrm{Nil}(R)$ is a t-ideal of $R/\mathrm{Nil}(R)$.

(2) A nonnil ideal I of R is ϕ-v-finite if and only if $I/\mathrm{Nil}(R)$ is a v-finite ideal of $R/\mathrm{Nil}(R)$.

Recall form [62] that a Prüfer v-multiplication domain (for short, a PvMD) is a domain whose localizations at maximal t-ideals are valuation domains. Following [50], a PvMD D is strongly discrete if D_P is a strongly discrete valuation domain for each prime t-ideal P of D; equivalently, if $(P^2)_t = P$ for each prime t-ideal P of D. A generalized Krull domain (for short *GK*-domain) is a strongly discrete PvMD in which every prime t-ideal is the radical of a v-finite ideal. For more results on GK-domains see [50, 51]. Hereafter we use the term "*GK*-domain" instead of "generalized Krull domain" in our sense in order

to avoid confusion with the classical generalized Krull domains.
By a seminal book on ideal systems [63, Chapter 22], a domain is a generalized Krull domain if and only if its multiplicative monoid of nonzero elements is a generalized Krull monoid. That is, the concept of generalized Krull domains is purely multiplicative. In the following, we will see that the concept of GK-domains is also purely multiplicative.

Remark 7.3.16 Let D be an integral domain. In [77], Kang and Park defined a nonzero ideal A of D to be a t-SFT-ideal if there exist a finitely generated ideal $B \subseteq A$ and a positive integer k such that $a^k \in B_v$ for each $a \in A_t$. The ring D is said to be a t-SFT-ring if each nonzero ideal of D is a t-SFT-ideal. Then they proved that a domain D is a GK-domain if and only if D is a t-SFT and PvMD. Now we will see that GK-domains have purely multiplicative structure. It is known that D is a PvMD if and only if $D \setminus 0$ is t-Prüfer ([63, p. 187]). On the other hand, similarly in the domain case, we define a nonzero ideal A of a cancellative monoid H to be a t-SFT-ideal if there exist a finite set $B \subseteq A$ and a positive integer k such that $a^k \in B_v$ for each $a \in A_t$. A cancellative monoid H is said to be a t-SFT-monoid if each nonzero ideal of H is a t-SFT-ideal. Then it is easily seen that a domain D is a t-SFT-ring if and only if its multiplicative monoid of nonzero elements is a t-SFT-monoid. Thus the concept of GK-domains is purely multiplicative.

Now, let $R \in \mathscr{H}$. We say that R is a ϕ-PvMR if every localization at a maximal ϕ-t-ideal is a ϕ-chained ring. Also, R is said to be a ϕ-GK-ring if R is a ϕ-PvMR such that $(\phi(P)^2)_t = \phi(P)$ for each prime ϕ-t-ideal P of R and every prime ϕ-t-ideal is the radical of a ϕ-v-finite ideal. Then we have the following lemma.

Lemma 7.3.17 *Let $R \in \mathscr{H}$. Then:*

(1) R is a ϕ-PvMR if and only if $R/\mathrm{Nil}(R)$ is a PvMD.

(2) R is a ϕ-GK-ring if and only if a $R/\mathrm{Nil}(R)$ is a GK-domain.

Proof. (1) Set $D = R/\mathrm{Nil}(R)$. Assume that R is a ϕ-PvMR and let $M/\mathrm{Nil}(R)$ be a nonzero maximal t-ideal of $R/\mathrm{Nil}(R)$. Then M is a maximal ϕ-t-ideal of R by Lemma 7.3.15(1), and so R_M is a ϕ-chained ring. By using [5, Theorem 2.7], we conclude that $R_M/\mathrm{Nil}(R_M)$ is a valuation domain. Thus $D_{M/\mathrm{Nil}(R)}$ is a valuation domain, which implies that D is a PvMD.

Conversely, let M be a maximal ϕ-t-ideal of R. Since $M/\mathrm{Nil}(R)$ is a maximal t-ideal of D, by Lemma 7.3.15(1) $D_{M/\mathrm{Nil}(R)}$ is a valuation domain. Thus $R_M/\mathrm{Nil}(R_M)$ is a valuation domain, and so R_M is a ϕ-chained ring by [5, Theorem 2.7]. Therefore R is a ϕ-PvMR.

(2) The assertion follows immediately by (1), Lemma 7.3.14, and Lemma 7.3.15.

It is well-known [51, Theorem 2.6] that an integral domain R is a GK-domain if and only if R is a piecewise w-Noetherian PvMD. We have a similar characterization for ϕ-GK-rings.

Theorem 7.3.18 *Let $R \in \mathscr{H}$. Then R is a ϕ-GK-ring if and only if R is a ϕ-piecewise w-Noetherian ϕ-PvMR.*

Proof. Set $D = R/\mathrm{Nil}(R)$. Since R is a ϕ-*GK*-ring, by Lemma 7.3.17(2) we get that D is a *GK*-domain. Thus [51, Theorem 2.6] implies that D is piecewise w-Noetherian PvMD. It follows by Theorem 7.3.3 and Lemma 7.3.17(1) that R is a ϕ-piecewise w-Noetherian ϕ-PvMR.

Conversely, by Theorem 7.3.3 and Lemma 7.3.17(1) D is a piecewise w-Noetherian PvMD. That implies D is a *GK*-domain by [51, Theorem 2.6], and thus R is a ϕ-*GK*-ring by Lemma 7.3.17(1).

Perspectives

To end this thesis, we introduce the following perspectives:

1. Generalize the piecewise Noetherian property to the context of modules.

2. Study the graph whose vertices are all non-trivial ideals of a ring R and two distinct vertices I and J are adjacent if they are comparable.

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