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Sous la direction du Pr. **YOUSSEF AKDIM**

Intitulée

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Etude de Problèmes de Stefan dans le cadre des Espaces de Sobolev avec poids

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Le temps met tout
en lumière.
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Etude de Problèmes de Stefan dans le cadre des Espaces de Sobolev avec poids

Résumé. L'objectif de ce travail est l'étude de divers problèmes d'équations aux dérivées partielles non linéaires du type elliptique-parabolique faisant intervenir un opérateur en forme divergentielle du type Leray-Lions avec des données peu régulières. Ces équations sont d'une façon générale mal posées dans le cadre de solutions faibles (i.e. au sens des distributions), car en général on n'a pas l'unicité. Des formulations plus appropriées ont alors vu le jour : les solutions appelées SOLA, les solutions entropiques et les solutions renormalisées. Cette thèse est divisée en deux parties principales, présente des résultats d'existence et d'unicité de solutions renormalisées pour trois problèmes non linéaires du type mentionnés ci-dessous. Ces types regroupent des problèmes dont les solutions se trouvent dans des espaces dépendants de la variable d'espace : il s'agit des espaces de Sobolev avec poids et Sobolev avec poids à exposant variable.

Dans la premiere partie, après un bref exposé de définitions et résultats nécessaires à la suite du travail, nous prouvons au chapitre 2 l'existence de solutions renormalisées pour un problème elliptique du type diffusion-convection avec des conditions non linéaires sur le bord $\beta(u) - div(a(x, Du) + F(u)) \ni f$ avec une donnée $L^\infty(\Omega)$. Dans le même axe, au chapitre 3, nous allons prouver l'existence et l'unicité de solutions renormalisées de même problème précédant dans le cas $L^1(\Omega)$.

Le chapitre 4 a pour but de présenter un résultat d'existence de solutions renormalisées pour un problème parabolique de Stefan non linéaire dans l'espace de Sobolev avec poids $\beta(u)_t - div(a(x, Du) + F(u)) \ni f$.

La seconde partie de cette thèse est constituée de deux chapitres. Dans Le premier chapitre, nous traitons le même problème elliptique de chapitre 2 dans les espaces de Sobolev à exposants variables avec poids $W_0^{1,p(x)}(\Omega, \omega)$, après avoir élaboré un cadre fonctionnel adéquat pour l'étude de ce dernier, nous allons prouver l'existence et l'unicité de solutions renormalisées.

Le dernier résultat présenté au deuxième chapitre, est l'existence et l'unicité de solutions renormalisées dans l'espace de Sobolev avec poids à exposant variable pour le problème parabolique associé à l'équation : $\frac{\partial b(x,u)}{\partial t} - div(a(x, t, u, \nabla u)) + H(x, t, u, \nabla u) = f$.

Mots clés : Espaces de Sobolev á exposant variable avec poids, Espaces de Sobolev avec poids , Inégalité de Young, Probémes Elliptiques, Probémes Paraboliques, semigroupes non linéaires, opérateur accrétif, integration par parties, Solutions Renormalisées.

AMS Classification : 35J15, 35J70, 35J85,35K45, 35K61, 35K65,A7A15, A6A32, 47D20.

Study the Stefan Problems in the framework of weighted Sobolev spaces

Abstract. The objective of this study was the study of various problems of nonlinear partial differential equations elliptic-parabolic type involving a divergentielle form operator of Leray-Lions type with regular data. These equations are generally are badly posed within the framework of weak solutions (i.e. the sense of distributions) because in general there is no uniqueness. More appropriate formulations were then born :the solutions called SOLA, entropic solutions and renormalized solutions. This thesis is divided into two main parts, presents the results of existence and uniqueness of renormalized solutions for three nonlinear problems of the type mentioned above, these types regroup the problems whose solutions are in spaces depending on the variable of space : it's about the Weighted Sobolev spaces and weighted Sobolev exponent variable.

In the first part, after a brief presentation of definitions and results needed further work, we prove in Chapter 2 the existence of renormalized solution for an elliptic problem type diffusion-convection with nonlinear terms on the board.

In the same axis, in Chapter 3, we will prove the existence and uniqueness of renormalized solution of the same problem preceding in the case $L^1(\Omega)$.

The objective of Chapter 4 is to have a result of the existence of renormalized solutions to a parabolic problem of Nonlinear Stefan in the weighted Sobolev space. $\beta(u)_t - div(a(x, Du) + F(u)) \ni f$.

The second part of this thesis consists of two chapters. In the first chapter we discuss the same elliptic problem of Chapter 2 in weighted variable exponent Sobolev space $W_0^{1,p(x)}(\Omega, \omega)$ after having drawn up a adequate functional framework for the study of this last, we will prove the existence and uniqueness of renormalized solution.

The last result presented to the second chapter, is the existence and uniqueness of renormalized solutions in the weighted variable exponent Sobolev space of for the parabolic problem associated with the equation : $\frac{\partial b(x,u)}{\partial t} - div(a(x, t, u, \nabla u)) + H(x, t, u, \nabla u) = f$.

Key Words : Weighted variable exponent Sobolev space, Weighted Sobolev spaces , Young's Inequality, elliptic problems, Parabolic problems,Nonlinear semigroup, Integration by parties, Renomalized Solution,

AMS Classification 35J15, 35J70, 35J85,35K45, 35K61, 35K65,A7A15, A6A32, 47D20.

NOTATION

Ω : Ouvert de $\mathbb{R}^N$ avec $(N \geq 1)$.
$\partial\Omega$: Frontiére topologique de Ω.
$x = (x_1, x_2, ...x_N)$: Point génriéque de $\mathbb{R}^N$.
$dx = dx_1 dx_2 ....dx_N$:Mesure de Lebesgue sur Ω.
∇u : Gradient de u.
Q : $\Omega \times (0, T), t$ variable du temps, $T > 0$.
$\text{supp}(f)$: Support d'une fonction f.
f^+, f^- : $\max(f, 0), \min(f, 0)$.
T_k : Fonction troncature de niveau $k > 0$.
$D(\Omega), D(Q)...$: Espace des fonctions différentiables et á support compact dans $\Omega, Q...$
$D_+(\Omega), D_+(Q)$:Espace des fonctions positives de $D(\Omega), D(Q)$.
$C_0(\Omega), C_0(Q)$:Espace des fonctions continues nulles au bord de Ω, Q.
$C^k(\Omega), C^k(Q)$: Espace des fonctions k-fois continuement différentiables dans Ω, Q.
$L^p(\Omega)$:Espace des fonctions de puissance p éme intégrables sur Ω pour la mesure dx.
$|||f||| = \Big(\sum_{i=1}^N \int_\Omega |\frac{\partial f(x)}{\partial x_i}|^p \omega_i(x)\, dx \Big)^{1/p}$.
$W_0^{1,p}(\Omega, \omega)$: Adhérence de $C_0^\infty(\Omega)$ dans $W^{1,p}(\Omega, \omega)$.
$W^{-1,p'}(\Omega, \omega^*)$: Espace dual de $W_0^{1,p}(\Omega, \omega)$.
Si X est un espace de Banach
$L^p(0, T; X)$=$\{f : (0, T) \to X mesurable : \int_0^T \|f(t)\|_X^p dt < \infty\}$.
$L^\infty(0, T; X) = \Big\{u : (0, T) \to X\ measurable,\ \exists\ C > 0\ /\|u(t)\|_X \leq\ C\ a.e.\Big\}$.
$C([0, T]; X)$: Espace des fonctions continuement différentiables de $[0, T] \to X$.
$D([0, T]; X)$: Espace des fonctions continuement différentiablesá support compact dans $[0, T]$.
Soit $p(\cdot) : \Omega \to (1, \infty)$ est une fonction mesurable.
$L^{p(\cdot)}(\Omega)$=$\{u \in P(\Omega) \int_\Omega |u(x)|^{p(x)} < \infty\}$.
$||u||_{L^{p(\cdot)}(\Omega,\omega)}$=$\inf\{\lambda > 0 : \int_\Omega |\frac{u(x)}{\lambda}|^{p(x)} \omega(x) dx \leq 1\}$.
$W^{1,p(\cdot)}(\Omega, \omega)$=$\{u \in L^{p(\cdot)}(\Omega)$ and $|\nabla u| \in L^{p(\cdot)}(\Omega, \omega)\}$.
$||u||_{W^{1,p(\cdot)}(\Omega,\omega)}$=$||u||_{L^{p(\cdot)}} + ||\nabla u||_{L^{p(\cdot)}(\Omega,\omega)}, \quad \forall u \in W^{1,p(\cdot)}(\Omega)$.
$W_0^{1,p(\cdot)}(\Omega, \omega)$: Fermeture de $C_0^\infty(\Omega)$dans$W^{1,p(\cdot)}(\Omega, \omega)$.
$\rho(u)$=$\int_\Omega \omega(x)|u(x)|^{p(x)} dx\ \ \forall u \in L^{p(\cdot)}(\Omega, \omega)$: Module convexe.

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Introduction Générale

Les équations aux dérivées partielles sont un outil essentiel de modélisation, et leur étude occupe les mathématiciens depuis le 18 ème siècle, avec les travaux d'Euler, de d'Alembert, de Lagrange et de Laplace etc (voir [50]). Elles permettent d'aborder d'un point de vue mathématique des phénomènes observés, par exemple dans les domaines de la physique et de la chimie. Les situations dépendant du temps se traduisent plus particulièrement par des équations d'évolution tenant compte d'éventuelles interactions entre objets et événements. La préoccupation première du mathématicien confronté à une équation aux dérivées partielles est de lui donner un sens dans des espaces fonctionnels appropriés et d'y démontrer l'existence et l'unicité de la solution. Ces équations modélisent de nombreux phénomènes physiques comme l'élasticité non linéaire, les fluides electrorhéologiques (l'interaction entre les fluides et les champs électromagnétiques) et termorhéologiques, le traitement d'image, la propagation à travers les milieux poreux et le calcul de variations avec des conditions de croissance non standard [28, 57, 65].
L'objectif de ce travail est l'étude de divers problèmes d'équations aux dérivées partielles non linéaires de type elliptiques et de type paraboliques faisant intervenir un opérateur du type Leray- Lions avec des données peu régulières. Nous avons étudié trois types de problèmes. Ces types regroupent des problèmes dont les solutions se trouvent dans des espaces dépendant de la variable d'espace : il s'agit des espaces de Lebesgue et de Sobolev avec poids et les espaces de Sobolev à exposants variables, avec poids.
Notre étude est portée sur le cas où les données sont dans $L^\infty(\Omega)$, $L^1(\Omega)$ ou $L^1(Q)$. Cette thèse, représente des résultats d'existence et d'unicité de solutions renormalisées pour trois problèmes non linéaires du type mentionnés ci-dessous. Nous allons présenter ici d'une manière détaillée le contenu de chaque partie.

Notre thèse est composée de deux parties principales :

- La premiere comporte quatre chapitres.

Le premier chapitre est naturellement dévolu aux notations, aux définitions et propriètès des espaces fonctionnels et aux rèsultats fondamentaux sur lesquels nous nous appuyons dans la suite. Des propriétés sur les opérateurs multivoques et les bonnes solutions sont également rappelées en ce chapitre. On verra régulièrement, dans cette thèse, des liens entre la théorie des opérateurs, accrétifs en particulier, l'approche par les bonnes solutions et l'approche par les solutions renormalisées. Tous ces rèsultats sont donnès sans démonstration et nous renvoyons a la bibliographie pour les détails. De plus, nous allons introduire quelques notations et les fonctions qui seront utilisées fréquemment.

Nous établirons dans le chapitre 2 des résultats d'existence de solutions renormalisée de problèmes :

$$\begin{gathered} \beta(u) - div(a(x,Du) + F(u)) \ni f \quad \text{in} \quad \Omega, \\ u = 0 \quad \text{on} \quad \partial\Omega, \end{gathered} \tag{0.0.1}$$

dans le cadre des espaces de Sobolev avec poids, avec $f \in L^\infty$ et F est localement lipschitzienne et β est un graphe maximal monotone.

On a supposé que l'operateur de Carathéodory $a(.,.)$ vérifiant les hypothèses du type Leray-Lions suivantes :

(A_1) Il existe une constante positive λ tels que

$$a(x,\xi)\cdot\xi \geq \lambda \sum_{i=1}^{N} \omega_i |\xi_i|^p$$

pour tout $\xi \in \mathbb{R}^N$ et presque tout $x \in \Omega$.

(A_2) $|a_i(x,\xi)| \leq \alpha \omega_i^{\frac{1}{p}}(x)[k(x) + \sum_{j=1}^{N} \omega_j^{\frac{1}{p'}}(x)|\xi_j|^{p-1}]$ pour presque tout $x \in \Omega$, $\forall i = 1, ..., N$, $\forall \xi \in \mathbb{R}^N$,où $k(x)$ est une fonction non négative dans $L^{p'}(\Omega)$, $p' = p/p-1$, et $\alpha > 0$.

(A_3) $(a(x,\xi) - a(x,\eta)) \cdot (\xi - \eta) \geq 0$ pour presque tout $x \in \Omega$ et pour tout $\xi, \eta \in \mathbb{R}^N$.

Le concept de solutions renormalisées a été introduit par Diperna et P.-L. Lions [56] dans leur étude de l'équation de Boltzmann. Cette notion a ensuite été adapté à une version elliptique (0.0.1) par L. Boccardo et al. [38] lorsque le côté droit est en $W^{-1,p'}(\Omega)$, par J.-M. Rakotoson [83] lorsque le côté droit est en $L^1(\Omega)$, et enfin par G. Dal Maso, F. Murat, L.Orsina et A. Prignet [54] pour le cas du côté droit sont des données générales de mesure. La notion équivalente de solution d'entropie a été introduite par Bénilan et al. dans [38]. Pour obtenir des résultats sur l'existence de solutions renormalisées de problèmes elliptiques de type (0.0.1) avec un $a(.,.)$ satisfaisant une condition de croissance variable, nous renvoyons à [89, 49, 62] et [25] (pour les résultats associées voir aussi [33, 34, 91]).

Dans le chapitre 3, nous étudierons le même problème (0.0.1) avec $f \in L^1(\Omega)$ en utilisant des techniques de troncature et la méthode de monotonicité généralisée dans l'espace de Sobolev avec poids, en aboutissant à des résultats d'existence de solutions renormalisés. Sous une hypothèse supplémentaire de monotonicité stricte l'unicité de la solution renormalisée est établie . On ajoutant le tereme "arctan" dans le probleme aproché et on utilise la Remarque 3.3.3 pour bien montrer l'unicité de la solution à travers un principe de comparaison qui sera jouer un rôle important dans le rapprochement des solutions renormalisées.
Notez que dans le cas avec des exposants variables et des espaces Orlicz les problème a été étudié par Wittbold et al. [62, 89]. D'autres travaux dans ce sens peut être trouvée dans [5, 33, 35]. Effets anisotropes ont été considérés comme des problèmes de type (0.0.1) avec des exposants constants dans [31, 48] et avec des exposants variables dans [78] (voir aussi [71]), où l'existence d'une solution renormalisée a été fourni avec $\beta = 0, F = 0$.

Le chapitre 4 présente un résultat d'existence de solutions renormalisées dans l'espace de Sobolev avec poids pour le problème parabolique de Stefan :

$$\beta(u)_t - div(a(x, Du) + F(u)) \ni f \quad \text{in} \quad Q_T \tag{0.0.2}$$

où $a(,)$ vérifiant les hypothèses $(A_1)-(A_3)$ et u est assujettie à vérifier la condition initiale $\beta(u(0,.)) \ni b_0$ sur Ω et la condition de Dirichlet : $u = 0$ sur Σ_T. Ici β est un graphe maximal monotone avec $0 \in \beta(0), f \in L^1(Q)$ et $b_0 \in L^1(\Omega)$. Nombreux travaux ont été dédiés aux questions d'existence et d'unicité de solutions faibles ou renormalisées pour ce type de problèmes, citons par exemple les travaux de de H.W. Alt et S. Luckhaus [90], X. Xu [93], D. Blanchard et G. Francfort [45], D. Berda [40], N. Igbida et J.M. Urbano [67], K. Ammar et P. Wittbold [23], D. Blanchard et A.

Porretta [46] et N. Igbida et P. Wittbold [68], etc... On notera au passage le travail de F. Andreu et al [25, 27] sur le problème de Stefan avec des conditions dynamiques au bord. Dans le troisième chapitre, nous allons étudier l'existence et l'unicité des solutions renormalisées au problème elliptique (0.0.1), ces résultats nous servira de base pour l'étude du problème de l'évolution associée à le même opérateur convection-diffusion : Plus précisément, à partir de ces résultats on en déduit qu'il existe une bonne solution du problème de Cauchy abstrait correspondant à (0.0.2) dans le sens de la théorie des semi-groupe non linéaire. La théorie des semi-groupe non linéaire donne une idée générale de la solution, appelée " bonne" solution pour les problèmes de Cauchy abstraits de la forme :

$$\frac{du}{dt} + Au \ni f$$

correspondant au problème (0.0.2), où A est un opérateur non linéaire multivoque dans un espace de Banach X et $f \in L^1(0, T, X)$.

On commence en un premier temps par prouver, grâce à la théorie des semi-groupes non linéaires, l'existence de solutions faibles du problème

$$(P, \psi, f, b_0) \begin{cases} \beta(u)_t + \psi(u) - div(a(x, Du) + F(u)) \ni f & \text{in} \quad Q_T, \\ u = 0 & \text{on} \quad \Sigma_T, \\ \beta(u(0, .)) \ni b_0 & \text{in} \quad \Omega, \end{cases}$$

où $\psi : \mathbb{R} \to \mathbb{R}, \psi(0) = 0$ est une fonction continu strictement monotone. La méthode de dédoublement de variables de S.N. Kruzhkov [74] combinée avec les techniques de D. Blanchard et A. Porretta [41] permet d'obtenir des estimations a priori sur la suite de solutions $(u_k)_k$ du problème précédent et de passer à la limite avec k vers l'infini. En un deuxième temps, nous choisissons une suite de perturbations $(\psi_{m,n})_{m,n}$ croissante en m et décroissante en n, nous approchons également les données f et b_0 par des approximations bi-monotones et nous montrons que les solutions faibles du problème $(P, \psi_{m,n}, f_{m,n}, b_0^{m,n})$ convergent lorsque $m, n \to \infty$ vers des solutions de (P, f, b_0) que l'on appellera renormalisées. Un lemme de comparaison de solutions (cf. Lemme 4.3.13) jouera un rôle clef lors de ce passage à la limite.

L'équation (0.0.2) peut être considérée comme une généralisation de l'équation parabolique de la $p(x)$ -Laplacian

$$(L,f,u_0)=\begin{cases} u_t - div(|Du|^{p(x)-2}Du) = f \quad in\ Q_T \\ u = 0 on \quad \Sigma_T \\ u(0,.) = u_0(.) \quad in\ \Omega. \end{cases}$$

Dans [33] l'existence et l'unicité des solutions renormalisées à (L,f,u_0) a été montré pour le données L^1.

- La deuxième partie de cette thèse concerne l'étude de deux problèmes elliptique et parabolique dans les espaces de Soblev à exposants variables et avec poids, elle est constituée de deux chapitres.

Dans le chapitre 5 nous étudierons le même problème (0.0.1) où $\Omega \subset \mathbb{R}^N$ est un domaine borné, f appartient à $L^\infty(\Omega)$ ou $L^1(\Omega)$, et $a(x,\xi)$ est l'opérateur de type $p(x)$ -Laplacian pondérée. Sous certaines hypothèses sur F et β, l'existence de solutions renormalisées a été prouvé. La preuve suit plutôt une procédure normale (mais non trivial) : Première f est estimée par la fonction L^∞ pour lesquels l'existence faible est connu. Ensuite, en utilisant la définition d'une solution faible, on montre que la suite de solutions formant une suite de Cauchy dans une norme appropriée (qui passe à une sous-séquence si nécessaire). Toutefois, cela ne suffit pas pour passer à une limite sous le signe de l'intégrale de solutions renormalisées mais plus d'information est nécessaire sur les gradients de troncatures : la prochaine étape est de montrer par un argument de compacité que les gradients de troncatures convergent fortement. La dernière étape est de valider que la limite est une solution renormalisée. Lorsque p (.) est une fonction constante, l'existence et l'unicité de solutions renormalisées de (0.0.1) a été obtenu. Ce chapitre peut être considéré comme une extension naturelle de [10] pour le cas de l'exposant variable. Compte tenu de certaines nouvelles propriétés pour l'exposant variables espaces de Lebesgue-Sobolev avec poids développés dans [70] et les techniques de troncature, nous avons obtenu des résultats d'existence similaires.
Un exemple de modèle de ce type est le problème homogène de valeur de Dirichlet pour le p-Laplacian $\Delta_p(u) = -div(|\nabla u|^{p-2}\nabla u)$, i.e. the equation $\beta(x,u) - \Delta_p(u) - divF_x(u) \ni f$.

L'équation (0.0.1) peut être considérée comme généralisation de la p (x) -Laplacian équation

$$(L, f) = \begin{cases} -div(|Du|^{p(x)-2}Du) = f \quad in\ \Omega \\ u = 0 \quad on\ \partial\Omega \end{cases}$$

Dans [34], l'existence et l'unicité des solutions renormalisées à (L, f) a été montré pour les fonctions continues $p : \bar{\Omega} \to (1, \infty)$ telle que $2 - \frac{1}{N} < \min_{x \in \Omega} p(x)$. Il est bien connu, même dans ce cas particulier, que pour les L^1-données une solution faible peut ne pas exister en général ou peut ne pas être unique.
Une des motivations pour l'étude (0.0.1) et (0.0.2) provient d'applications fluides électro-rhéologiques (voir [85], [64] pour plus de détails) comme classe importante de fluides non-newtoniens. D'autres applications importantes sont liés au traitement et à l'élasticité image (voir [53], [99]).

Dans le chapitre 6, nous étudierons le problème non linéaire $p(x)$- parabolique suivant :

$$\begin{aligned} \frac{\partial b(x,u)}{\partial t} - div(a(x,t,u,\nabla u)) + H(x,t,u,\nabla u) = f \text{ in } Q = \Omega \times (0,T), \\ b(x,u) \mid_{t=0} = b(x,u_0) \quad \text{in } \Omega \\ u = 0 \quad \text{ on } \partial\Omega \times (0,T), \end{aligned} \tag{0.0.3}$$

avec second membre f est dans $L^1(Q)$ et $b(x, u_0) \in L^1(\Omega)$. Nous montrons un résultat d'existence de solution renormalisée dans les espaces de Sobolev avec poids à exposent variable pour ce probléme sans aucune condition de signe supposée sur le terme fortement non linéaires H mais vérifié la condition de croissance :

$$|H(x,t,s,\xi)| \leq \gamma(x,t) + g(s)\omega|\xi|^{p(x)} \tag{0.0.4}$$

avec $g : \mathbb{R} \to \mathbb{R}^+$ est une fonction bornée continue et positive tel que $g \in L^1(\mathbb{R})$ et $\gamma \in L^1(Q)$. Lorsque le second membre f est dans le dual $L^{p'}\big(0, T, W^{-1,p'}(\Omega, \omega^*)\big)$ ce problème à été traité dans [3].
Pour montrer que $T_k(u_n)$ est bornée par une constante indépendante de n, on montre d'abord :

$$\int_Q g(u_n)|\nabla u_n|^{p(x)}\omega(x) \exp(G(u_n))dxdt \leq C$$

par les choix de fonctions tests suivant : $\displaystyle\int_0^{u_n} g(s)\chi_{\{s>0\}}ds \quad et \quad \int_{u_n}^0 g(s)\chi_{\{s<0\}}ds.$
On peut consulter aussi les travaux de Porretta [79] ou la condition de signe a été supprimée. On peut se référer aux travaux de A. El-Mahi et D. Meskine [60, 61] dans le cadre des espace d'Orlicz-Sobolev et avec $b(x,u) = u$. Dans le cas dégénéré (c.a.d. dans les espaces de Sobolev avec poids) et $b(x,u) = u$, L. Aharouch, E. Azroul et M. Rhoudaf [2, 3, 4] ont montré l'existence des solutions faibles dans le cas variationel et ont supposé la condition de signe et la condition de croissance, sur H. Pour le cas ou $f \in L^1$ ils ont montrés l'existence de solutions entropiques . Le cas d'une donnée L^1 et H vérifée la condition (0.0.4) (sans condition de signe) est traité par Y. Akdim, J. Bennouna, M. Mekkour [7] et Y. Akdim, C.Allaou, N. elgourch and M. Mekkour [16] avec $b(x,u) = u$ en utilisant des solutions renormalisées. Ce même problème a ètè étudier par Y. Akdim, N. Elgorch and M. Mekkour [18] dans les espaces de Sobolev à exposent variable avec seconde membre $f - divF$, avec $f \in L^1(Q)$ et $F \in (L^{P'(x)}(Q))^N$.
Notons que le contenu de cette thèse est l'objet de quelques travaux [10, 11, 12, 13, 14] publiés.

Publications

1) Y.Akdim, C.Allalou, Existence and Uniqueness of Renormalized Solution of Nonlinear Degenerated Elliptic Problems, Anal.Theory Appl., 30(2014),318-343.

2)Y.Akdim, C.Allalou, Existence Of a Renormalized Solution of Nonlinear Degenerate Elliptic Problems, Applicationes Mathematicae 41,2-3 (2014), pp. 131-140.

3)Youssef Akdim, Chakir Allalou, Existence of Renormalized Solutions of Nonlinear Elliptic Problems in Weighted Variable-Exponent Space, J. Math.Study, Vol. 48, No. 4 (2015), pp. 375-397.

4)Youssef Akdim, Chakir Allalou and Nezha El gorch, Solvability of Degenerated $p(x)$-Parabolic Equations with three Unbounded Nonlinearities, Electronic Journal of Mathematical Analysis and Applications, Vol. 4(1) Jan. 2016, pp. 42-62.

5)Youssef Akdim, Chakir Allalou, Renormalized solutions of Stefan degenerat parabolic nonlinear problems, International Journal of Evolution Equations Volume 9, Number 3, pp. 247–275.

6)Youssef Akdim, Chakir Allalou, Renormalized solutions of Stefan degenerat elliptic nonlinear problems with variable exponent, Journal of Nonlinear Evolution Equations and Applications,Volume 2015, Number 7, pp. 105–119.

This part is constituted of the following chapters :

CHAPTER I

Preliminaries.

CHAPTER II

Existence of Renormalized solution of nonlinear degenerate problems elliptic.

CHAPTER III

Existence and uniqueness of Renormalized Solution of Nonlinear Degenerated Elliptic Problems.

CHAPTER IV

Renormalized Solutions of Stefan Degenerated Parabolic Nonlinear Problems.

Chapitre 1

Preliminaries

This chapter will be devoted to introduce too the notion of Weighted Sobolev Spaces (see Subsection 1) and weighted variable exponent Lebesgue–Sobolev spaces (see Subsection 2), and some interesting definitions and properties, which are essential to prove some results of existence for solutions of the Elliptic-Parabolic nonlinear problems studied in this thesis.

1.1 Weighted Sobolev spaces

Let Ω be a bounded open set of $\mathbb{R}^N$ $(N \geq 1)$ p be a real number such that $1 < p < \infty$ and $\omega = \{\omega_i(x),\ 0 \leq i \leq N\}$ be a vector of weight functions, i.e., every component $\omega_i(x)$ is a measurable function which is positive a.e. in Ω. Further, we suppose in all our considerations that

$$\omega_i \in L^1_{\mathrm{loc}}(\Omega), \tag{1.1.1}$$

$$\omega_i^{\frac{-1}{p-1}} \in L^1_{\mathrm{loc}}(\Omega), \tag{1.1.2}$$

for any $0 \leq i \leq N$. We denote by $W^{1,p}(\Omega,\omega)$ the space of all real-valued functions $u \in L^p(\Omega,\omega_0)$ such that the derivatives in the sense of distributions fulfill

$$\frac{\partial u}{\partial x_i} \in L^p(\Omega,\omega_i) \quad \text{for } i = 1,\ldots,N.$$

Which is a Banach space under the norm

$$\|u\|_{1,p,\omega} = \Big[\int_\Omega |u(x)|^p \omega_0(x)\,dx + \sum_{i=1}^N \int_\Omega |\frac{\partial u(x)}{\partial x_i}|^p \omega_i(x)\,dx\Big]^{1/p}. \tag{1.1.3}$$

The condition (1.1.1) implies that $C_0^\infty(\Omega)$ is a subspace of $W^{1,p}(\Omega,\omega)$ and consequently, we can introduce the subspace $X = W_0^{1,p}(\Omega,\omega)$ of $W^{1,p}(\Omega,\omega)$ as the closure

of $C_0^\infty(\Omega)$ with respect to the norm (1.1.3). Moreover, condition (1.1.2) implies that $W^{1,p}(\Omega,\omega)$ as well as $W_0^{1,p}(\Omega,\omega)$ are reflexive Banach spaces.
We recall that the dual space of weighted Sobolev spaces $W_0^{1,p}(\Omega,\omega)$ is equivalent to $W^{-1,p'}(\Omega,\omega^*)$, where $\omega^* = \{\omega_i^* = \omega_i^{1-p'},\, i = 0,\dots,N\}$ and where p' is the conjugate of p ; i.e., $p' = \frac{p}{p-1}$.(for more details we refer to [72]).

1.1.1 Basic assumptions in weighted Sobolev spaces

Assumption (H1)

The expression

$$|||u|||_X = \Big(\sum_{i=1}^N \int_\Omega |\frac{\partial u(x)}{\partial x_i}|^p \omega_i(x)\,dx\Big)^{1/p} \tag{1.1.4}$$

is a norm defined on X and is equivalent to the norm (1.1.3).
There exist a weight function σ on Ω and a parameter q, $1 < q < \infty$, such that

$$\sigma^{1-q'} \in L^1(\Omega), \tag{1.1.5}$$

with $q' = q/(q-1)$. The Hardy inequality,

$$\Big(\int_\Omega |u(x)|^q \sigma\,dx\Big)^{1/q} \leq c\Big(\sum_{i=1}^N \int_\Omega |\frac{\partial u(x)}{\partial x_i}|^p \omega_i(x)\,dx\Big)^{1/p}, \tag{1.1.6}$$

holds for every $u \in X$ with a constant $c > 0$ independent of u, and moreover, the imbedding

$$X \hookrightarrow\hookrightarrow L^q(\Omega,\sigma), \tag{1.1.7}$$

expressed by the inequality (1.1.6) is compact. Note that $(X, |||.|||_X)$ is a uniformly convex (and thus reflexive) Banach space.

Remark 1.1.1 If we assume that $\omega_0(x) \equiv 1$ and in addition the integrability condition : There exists $\nu \in]\frac{N}{p}, +\infty[\cap]\frac{1}{p-1}, +\infty[$ such that

$$w_i^{-\nu} \in L^1(\Omega) \text{ and } \omega_i^{\frac{N}{N-1}} \in L^1_{\text{loc}}(\Omega), \text{ for all } i = 1.....N. \tag{1.1.8}$$

Note that the assumptions *(1.1.1)* and *(1.1.8)* imply that,

$$|||u||| = \Big(\sum_{i=1}^N \int_\Omega |\frac{\partial u}{\partial x_i}|^p \omega_i(x)\,dx\Big)^{1/p} \tag{1.1.9}$$

which is a norm defined on $W_0^{1,p}(\Omega,\omega)$ and its equivalent to *(1.1.3)* and that, the imbedding

$$W_0^{1,p}(\Omega,\omega) \hookrightarrow L^q(\Omega), \tag{1.1.10}$$

is compact for all $1 \leq q \leq p_1^*$ if $p\nu < N(N+1)$ and for all $q \geq 1$ if $p\nu \geq N(\nu+1)$ where $p_1 = p\nu/\nu+1$ *and* p_1^* is the Sobolev conjugate of p_1 *[*see *[58] ,pp.30-31].*

1.1.2 Some technical results in weighted Sobolev spaces

Some weighted embedding and compactness results

In this subsection we establish some imbedding and compactness results in weighted Sobolev Spaces which allow in particular to extend in the settings of weighted Sobolev spaces.

Let $X = W_0^{1,p}(\Omega,\omega)$, $H = L^2(\Omega,\sigma)$ and let $X^* = W^{-1,p'}(\Omega,\omega^*)$,with $(2 \leq p < \infty)$. Let $V = L^p(0,T,X)$ the dual space of V is $V^* = L^{p'}(0,T,X^*)$ where $\frac{1}{p}+\frac{1}{p'} = 1$ and denoting the space $W_p^1(0,T,X,H) = \{v \in V : \ v' \in V^*\}$ endowed with the norm

$$\|u\|_{W_p^1} = \|u\|_V + \|u'\|_{V^*}, \tag{1.1.11}$$

is a Banach space. Here u' stands for the generalized derivative of u ; i.e.,

$$\int_0^T u'(t)\varphi(t)dt = -\int_0^T u(t)\varphi'(t)dt \quad \text{for all } \varphi \ \in C_0^\infty(0,T).$$

Lemma 1.1.2 *([95])*
1) The evolution triple $X \subseteq H \subseteq X^*$ is verified.
2) The imbedding $W_p^1(0,T,X,H) \subseteq C(0,T,H)$ is continuous.
3) The imbedding $W_p^1(0,T,X,H) \subseteq L^p(Q,\sigma)$ is compact.

Characterization of the time mollification of a function u

In order to deal with time derivative, we introduce a time mollification of a function u belonging to a some weighted Lebesgue space. For all $\mu > 0$ and all $(x,t) \in Q_T$, we denote this time regularized function by $(T_k(u))_\mu : Q_T \to \mathbb{R}$ defined by

$$(T_k(u))_\mu(t,x) := \mu \int_{-\infty}^t \exp(\mu(s-t))T_k(u(s,x))ds \tag{1.1.12}$$

where, for $s \leq 0$ and $\mu > 0$ we extend $u(s,x)$ by a function $\nu_0^\mu \in W_0^{1,p}(\Omega,\omega) \cap L^\infty(\Omega)$ such that $(\nu_0^\mu)_\mu$ is a sequence of functions with $\|\nu_0^\mu\|_{L^\infty(\Omega)} \leq k$ for all $\mu > 0$, $\frac{1}{\mu}\|\nu_0^\mu\|_{W_0^{1,p}(\Omega,\omega)} \to 0$ as $\mu \to \infty$, $\nu_0^\mu \to T_k(u_0)$ almost everywhere in Ω as $\mu \to \infty$

and $u_0 : \Omega \to \overline{\mathbb{R}}$ is a measurable function such that $b_0 \in \beta(u_0)$ almost everywhere in Ω. We can easily check that $(T_k(u))_\mu \in L^p(0,T,W_0^{1,p}(\Omega,\omega)) \cap L^\infty(Q_T)$ is the unique solution of the equation

$$\partial_t(T_k(u))_\mu + \mu\Big((T_k(u)) - T_k(u)_\mu\Big) = 0$$

in $D'(Q_T)$ satisfying the initial condition $(T_k(u))_\mu(0,x) = \nu_0^\mu$ almost everywhere in Ω. In particular, $(T_k(u))_\mu$ is differentiable for almost every $t \in (0,T)$ with $\partial_t(T_k(u))_\mu = \mu(T_k(u) - (T_k(u)_\mu) \in L^p(0,T,W_0^{1,p}(\Omega,w)) \cap L^\infty(Q_T)$. Moreover, we have $D(T_k(u))_\mu) = (DT_k(u))_\mu$ in $D'([0,T)\times\Omega)$, $\|(T_k(u))_\mu\|_{L^\infty(Q_T)} \leq k$ for all $\mu > 0$, $(T_k(u))_\mu \to T_k(u)$ almost everywhere in Q_T , weak-$*$ in $L^\infty(Q_T)$ and strongly in $L^p(0,T,W_0^{1,p}(\Omega,\omega))$ as $\mu \to \infty$ (for more details we refer to [3]).

1.2 Weighted variable exponent Sobolev spaces

In this section, we state some elementary properties for the Weighted Variable Exponent Lebesgue–Sobolev spaces $L^{p(x)}(\Omega,\omega)$ which will be used in the next sections. The basic properties of the variable exponent Lebesgue–Sobolev spaces $W^{1,p(x)}(\Omega,\omega)$, that is, when $\omega(x) \equiv 1$ can be found from [59, 75].
Let Ω be a bounded open subset of $\mathbb{R}^N (N \geq 2)$.
Set

$$C_+(\overline{\Omega}) = \{p \in C(\overline{\Omega}) : \min_{x\in\overline{\Omega}} p(x) > 1\}.$$

For any $p \in C_+(\overline{\Omega})$, we define

$$p^+ = \max_{x\in\overline{\Omega}} p(x), \quad p^- = \min_{x\in\overline{\Omega}} p(x).$$

For any $p \in C_+(\overline{\Omega})$, we introduce the weighted variable exponent Lebesgue space $L^{p(x)}(\Omega,\omega)$ that consists of all measurable real-valued functions u such that

$$L^{p(x)}(\Omega,\omega) = \{u : \Omega \to \mathbb{R}, \text{measurable}, \int_\Omega |u(x)|^{p(x)}\omega(x)dx < \infty\}.$$

Then, $L^{p(x)}(\Omega,\omega)$ endowed with the Luxemburg norm

$$|u|_{L^{p(x)}(\Omega,\omega)} = \inf\{\lambda > 0 : \int_\Omega |\frac{u(x)}{\lambda}|^{p(x)}\omega(x)dx \leq 1\}$$

becomes a normed space. When $\omega(x) \equiv 1$, we have $L^{p(x)}(\Omega,\omega) \equiv L^{p(x)}(\Omega)$ and we use the notation $|u|_{L^{p(x)}(\Omega)}$ instead of $|u|_{L^{p(x)}(\Omega,\omega)}$. The following Hölder type inequality is useful for the next sections. The weighted variable exponent Sobolev space $W^{1,p(x)}(\Omega,\omega)$ is defined by

$$W^{1,p(x)}(\Omega,\omega)=\{u\in L^{p(x)}(\Omega);|\nabla u|\in L^{p(x)}(\Omega,\omega)\},$$

where the norm is given by

$$\|u\|_{W^{1,p(x)}(\Omega,\omega)}=|u|_{L^{p(x)}(\Omega)}+|\nabla u|_{L^{p(x)}(\Omega,\omega)}, \tag{1.2.13}$$

or, equivalently by

$$\|u\|_{W^{1,p(x)}(\Omega,\omega)}=\inf\{\lambda>0:\int_\Omega|\frac{u(x)}{\lambda}|^{p(x)}+\omega(x)|\frac{\nabla u(x)}{\lambda}|^{p(x)}dx\leq 1\}$$

for all $u\in W^{1,p(x)}(\Omega,\omega)$.
It should be emphasised that smooth functions are not dense in $W^{1,p(x)}(\Omega)$ without additional assumptions on the exponent $p(x)$. This feature was observed by Zhikov [98] in connection with the Lavrentiev phenomenon. However, if the exponent $p(x)$ is log-Hölder continuous, i.e., there is a constant C such that

$$|p(x)-p(y)|\leq\frac{C}{-\log|x-y|} \tag{1.2.14}$$

for every x,y with $|x-y|\leq\frac{1}{2}$, then smooth functions are dense in variable exponent Sobolev spaces and there is no confusion in defining the Sobolev space with zero boundary values $W_0^{1,p(x)}(\Omega)$, as the completion of $C_0^\infty(\Omega)$ with respect to the norm $\|u\|_{W^{1,p(x)}(\Omega)}$ (see[66]).
The space $W_0^{1,p(x)}(\Omega,\omega)$ is defined as the completion of $C_0^\infty(\Omega)$ with respect to the norm $\|u\|_{W^{1,p(x)}(\Omega,\omega)}$.

Lemma 1.2.1 *(Generalised Hölder inequality,See [59, 75]).*
i) *For any functions $u\in L^{p(x)}(\Omega)$ and $v\in L^{p'(x)}(\Omega)$, we have*
$|\int_\Omega uvdx|\leq(\frac{1}{p^-}+\frac{1}{p'^-})||u||_{p(\cdot)}||v||_{p'(\cdot)}\leq 2||u||_{p(x)}||v||_{p'(x)}$.
ii) *For all $p,\ q\in C_+(\bar\Omega)$ such that $p(x)\leq q(x)$ a.e. in Ω, we have $L^{q(x)}\hookrightarrow L^{p(x)}$ and the embedding is continuous.*

Lemma 1.2.2 *(See [59].) Denote $\rho(u)=\int_\Omega\omega(x)|u(x)|^{p(x)}dx$ for all $u\in L^{p(x)}(\Omega,\omega)$.*

Then,

$$|u|_{L^{p(x)}(\Omega,\omega)} < 1 (= 1; > 1) \text{ if and only if } \rho(u) < 1 (= 1; > 1), \tag{1.2.15}$$

$$\text{if } |u|_{L^{p(x)}(\Omega,\omega)} > 1 \text{ then } |u|^{p^-}_{L^{p(x)}(\Omega,\omega)} \leq \rho(u) \leq |u|^{p^+}_{L^{p(x)}(\Omega,\omega)}, \tag{1.2.16}$$

$$\text{if } |u|_{L^{p(x)}(\Omega,\omega)} < 1 \text{ then } |u|^{p^+}_{L^{p(x)}(\Omega,\omega)} \leq \rho(u) \leq |u|^{p^-}_{L^{p(x)}(\Omega,\omega)}. \tag{1.2.17}$$

Remark 1.2.3 *(See [70].) If we set*

$$I(u) = \int_\Omega |u(x)|^{p(x)} + \omega(x)|\nabla u(x)|^{p(x)} dx,$$

then, following the same argument, we have

$$\min\{\|u\|^{p^-}_{W^{1,p(x)}(\Omega,\omega)}, \|u\|^{p^+}_{W^{1,p(x)}(\Omega,\omega)}\} \leq I(u) \leq \max\{\|u\|^{p^-}_{W^{1,p(x)}(\Omega,\omega)}, \|u\|^{p^+}_{W^{1,p(x)}(\Omega,\omega)}\}. \tag{1.2.18}$$

Let ω be a weight function satisfying the conditions :
($\mathbf{W}_1$) $\omega \in L^1_{loc}(\Omega)$ and $\omega^{-\frac{1}{(p(x)-1)}} \in L^1_{loc}(\Omega)$;
($\mathbf{W}_2$) $\omega^{-s(x)} \in L^1(\Omega,\omega)$ with $s(x) \in (\frac{N}{p(x)}, \infty) \cap [\frac{1}{p(x)-1}, \infty)$.
The reasons that we assume (W_1) and (W_2) can be found in [70].

Remark 1.2.4 *([70].)*
(i) *If ω is a positive measurable and finite function, then $L^{p(x)}(\Omega,\omega)$ is a reflexive Banach space.*
(ii) *Moreover, if (W_1) holds, then $W^{1,p(x)}(\Omega,\omega)$ is a reflexive Banach space.*

For $p, s \in C_+(\overline{\Omega})$, denote $p_s(x) := \frac{p(x)s(x)}{s(x)+1} < p(x)$, where $s(x)$ is given in (W_2). Assume that we fix the variable exponent restrictions

$$\begin{cases} p_s^*(x) := \frac{p(x)s(x)N}{(s(x)+1)N - p(x)s(x)} & \text{if } N > p_s(x), \\ p_s^*(x) \text{ arbitrary} & \text{if } N \leq p_s(x) \end{cases}$$

for almost all $x \in \Omega$. These definitions play a key role in our thesis. We shall frequently make use of the following (compact) imbedding theorem for the weighted variable exponent Lebesgue–Sobolev space in the next sections.

Lemma 1.2.5 *([70].) Let $p, s \in C_+(\overline{\Omega})$ satisfy the log-Hölder continuity condition* (1.2.14) *and let (W_1) and (W_2) be satisfied. If $r \in C_+(\overline{\Omega})$ and $1 < r(x) \leq p_s^*(x)$ then, we obtain the continuous imbedding*

$$W^{1,p(x)}(\Omega,\omega) \hookrightarrow L^{r(x)}(\Omega).$$

Moreover, we have the compact imbedding

$$W^{1,p(x)}(\Omega,\omega) \hookrightarrow L^{r(x)}(\Omega),$$

provided that $1 < r(x) < p_s^*(x)$ for all $x \in \overline{\Omega}$.

From Lemma 1.2.5, we have the Poincaré–type inequality immediately.

Corollary 1.2.6 *([70].) Let $p \in C_+(\overline{\Omega})$ satisfy the log–Hölder continuity condition (1.2.14). If (W_1) and (W_2) hold, then the estimate*

$$\|u\|_{L^{p(x)}(\Omega)} \leq C\|\nabla u\|_{L^{p(x)}(\Omega,\omega)}$$

holds, for every $u \in C_0^\infty(\Omega)$ with a positive constant C independent of u.

Throughout this the thesis, let $p \in C_+(\overline{\Omega})$ satisfy the log-Hölder continuity condition (1.2.14) and $X := W_0^{1,p(x)}(\Omega, \omega)$ be the weighted variable exponent Sobolev space that consists of all real valued functions u from $W^{1,p(x)}(\Omega, \omega)$ which vanish on the boundary $\partial\Omega$, endowed with the norm

$$\|u\|_X = \inf\{\lambda > 0 : \int_\Omega \Big|\frac{\nabla u(x)}{\lambda}\Big|^{p(x)} \omega(x)dx \leq 1\},$$

which is equivalent to the norm (1.2.13) due to Corollary 1.2.6. The following proposition gives the characterization of the dual space $(W_0^{k,p(x)}(\Omega, \omega))^*$, which is analogous to [[75], Theorem 3.16]. We recall that the dual space of the weighted Sobolev spaces $W_0^{1,p(x)}(\Omega, \omega)$ is equivalent to $W^{-1,p'(x)}(\Omega, \omega^*)$, where $\omega^* = \omega^{1-p'(x)}$.

Proposition 1.2.7 Let $p \in C_+(\overline{\Omega})$ and α be a multi-index with $|\alpha| \leq k$. Further, let ω be a given family $\{\omega_\alpha(x) : |\alpha| \leq k\}$ of weight functions $\omega_\alpha(x)$, $x \in \Omega$ satisfying $(\mathbf{W}_1)$. Then for every $G \in (W_0^{k,p(x)}(\Omega, \omega))^*$, there exists a unique system of functions $\{g_\alpha \in L^{p'(x)}(\Omega, \omega_\alpha^{1-p'(x)}) : |\alpha| \leq k\}$ such that
$G(f) = \Sigma_{|\alpha|\leq k} \int_\Omega D^\alpha f(x) g_\alpha(x)dx$, for all $f \in W_0^{1,p(x)}(\Omega, \omega)$.

Lemma 1.2.8 *([34].) Let $g \in L^{p(x)}(Q, \omega)$ and let $g_n \in L^{p(x)}(Q, \omega)$, with $\|g_n\|_{L^{p(x)}(Q,\omega)} \leq c$, $1 < r(x) < \infty$. If $g_n(x) \to g(x)$ a.e. in Q, then $g_n \rightharpoonup g$ in $L^{p(x)}(Q, \omega)$, where $\rightharpoonup$ denotes weak convergence and ω is a weight function on Q.*

We will also use the standard notation for Bochner spaces, i.e, if $q \geq 1$ and X is a Banach space then $L^q(0, T; X)$ denotes the space of strongly measurable function $u : (0, T) \to X$ for which $t \to \|u(t)\|_X \in L^q(0, T)$ Morever, $C([0; T]; X)$ denotes the space of continuous function $u : [0; T] \to X$ endowed with the norm $\|u\|_{C([0;T];X)} = max_{t\in[0;T]} \|u\|_X$,

$$L^{p^-}(0, T; W_0^{1,p(x)}(\Omega, \omega)) = \{u : (0, T) \to W_0^{1,p(x)}(\Omega, \omega) \text{ measurable;}$$
$$\Big(\int_0^T \|u(t)\|^{p^-}_{W_0^{1,p(x)}(\Omega,\omega)}\Big)^{1/p^-} < \infty\}$$

and we define the space

$$L^\infty(0,T;X)=\{u:(0,T)\to X \text{ measurable}; \exists C>0/\|u(t)\|_X\le C \text{ a.e.}\}$$

where the norm is defined by :

$$\|u\|_{L^\infty(0,T;X)}=\inf\{C>0; \|u(t)\|_X\le C \text{ a.e.}\}.$$

We introduce the functional space see [34]

$$V=\{f\in L^{p^-}(0,T;W_0^{1,p(x)}(\Omega,\omega)); |\nabla f|\in L^{p(x)}(Q,\omega)\} \tag{1.2.19}$$

which endowed with the norm

$$\|f\|_V=\|\nabla f\|_{L^{p(x)}(Q,\omega)}$$

or, the equivalent norm

$$|||f|||_V=\|f\|_{L^{p^-}(0,T;W_0^{1,p(x)}(\Omega,\omega))}+\|\nabla f\|_{L^{p(x)}(Q,\omega)},$$

is a separable and reflexive Banach space. The equivalence of the two norms is an easy consequence of the continuous embedding $L^{p(x)}(Q)\hookrightarrow L^{p^-}(0,T;L^{p(x)}(\Omega))$ and the Poincaré inequality. We state some further properties of V in the following lemma.

Lemma 1.2.9 *Let V be defined as in (1.2.19) and its dual space be denote by V^*. Then,*
(i) *We have the following continuous dense embeddings :*

$$L^{p^+}(0,T;W_0^{1,p(x)}(\Omega,\omega))\hookrightarrow V\hookrightarrow L^{p^-}(0,T;W_0^{1,p(x)}(\Omega,\omega)).$$

In particular, since $D(Q)$ is dense in $L^{p^+}(0,T;W_0^{1,p(x)}(\Omega,\omega)$; it is dense in V and for the corresponding dual spaces, we have

$$L^{(p^-)'}(0,T;(W_0^{1,p(x)}(\Omega,\omega))^*)\hookrightarrow V^*\hookrightarrow L^{(p^+)'}(0,T;(W_0^{1,p(x)}(\Omega,\omega))^*).$$

Note that, we have the following continuous dense embeddings

$$L^{p^+}(0,T;L^{p(\cdot)}(\Omega,\omega))\hookrightarrow L^{p(x)}(Q,\omega)\hookrightarrow L^{p^-}(0,T;L^{p(x)}(\Omega,\omega)).$$

(ii) *One can represent the elements of V^* as follows : if $T\in V^*$; then there exists $F=(f_1,....,f_N)\in(L^{p'(x)}(Q))^N$ such that $T=div_X F$ and*

$$\langle T,\xi\rangle_{V^*,V}=\int_0^T\int_\Omega F\cdot\nabla\xi dxdt.$$

for any $\xi\in V$. Moreover, we have

$$\|T\|_{V^*}=\max\{\|f_i\|_{L^{p(\cdot)}(Q,\omega)},\ i=1,....,n\}.$$

Remark 1.2.10 *The space $V \cap L^\infty(Q)$, is endowed with the norm definie by the formula :*

$$\|v\|_{V\cap L^\infty(Q)} := \max\left\{\|v\|_V, \|v\|_{L^\infty(Q)}\right\},\ v \in V \cap L^\infty(Q).$$

Is a Banach space. In fact, it is the dual space of the Banach space $V^* + L^1(Q)$ endowed with the norm :

$$\|v\|_{V^*+L^1(Q)} := \inf\left\{\|v_1\|_{V^*} + \|v_2\|_{L^1(Q)}\right\};\ v = v_1 + v_2, v_1 \in V^*, v_2 \in L^1(Q).$$

1.3 Multivalued operators and theory of mild solutions

An mapping $A : X \to P(X)$ is called multivalued operator, its domain effective is given by

$$D(A) = \{x \in X; A(x) \neq \emptyset\}$$

and its range is defined by

$$R(A) = \{y \in X; \exists x \in X, y \in A(x)\}.$$

If for all $x \in D(A)$, $A(x)$ is reduced to an element, we talk about the single valued operator. By convention, identifies the operator A with its graph in $X \times X$, note $G(A) = \{(x, y) \in X \times X; y \in A(x)\}$.
Given $2^{\mathbb{R}} = \{X; X \subseteq \mathbb{R}\}$ and a mapping $A : \mathbb{R} \to 2^{\mathbb{R}}$ where $A(x) = [0, |x|]$ for every $x \in \mathbb{R}$. We know that for every $x \in \mathbb{R}$, $A(x)$ is a closed interval. Hence, range of A is a set of closed intervals. It is one example of multivalued mappings.
Let us particular classes of non-linear operators.

Definition 1.3.1 *Let A be a multivalued operator to X.*
♦ *A is accretive in X if*

$$(\forall \alpha > 0), \forall (x, y), (\tilde{x}, \tilde{y}) \in A \quad \|x - \tilde{x}\| \leq \|x - \tilde{x} + \alpha(y - \tilde{y})\|.$$

In other words, A is accretive if the $(I + \alpha A)^{-1} : R(I + \alpha A) \to X$ is a contraction.
♦ *A is m- accretive in X if*

$$A \text{ is accretive and verifies } R(I + \alpha A) = X\ \forall \alpha > 0.$$

♦ *A is T-accretive in X if*

$$\forall \alpha > 0; \forall\ (x, \tilde{x}),\ (y, \tilde{y}) \in A;\ \|(x - \tilde{x})^+\|_X \leq \|x - \tilde{x} + \alpha(y - \tilde{y})^+\|_X.$$

♦ *A is $m - T$-accretive in X if*

$$A \text{ is } T- \text{ accretive and verifies } R(I + \alpha A) = X \quad \forall \alpha > 0.$$

Proposition 1.3.2 *Let A be an operator in $L^1(\Omega)$ such that*
- *A is T-accretive*
- *$R(I+\alpha A)$ is dense in $L^1(\Omega)$ $\forall \alpha > 0$.*

Then the closure $\overline{A}$ of A in $L^1(\Omega)$ is m-t-accretive operator in $L^1(\Omega)$.

Mild solutions

Consider the following Cauchy problem :

$$(CP)(u_0,f)\begin{cases} u_t + Au \ni f \\ u(0) = u_0, \end{cases}$$

where A is a multivalued nonlinear operator in X, u_0 and $f \in L^1(0,T,X)$.
A notion of solutions for the problem $(CP)(u_0,f)$ is the one introduced by Ph.Bénilan [36].
Let $\varepsilon > 0$,one chooses a subdivision $(t_i)_i; i = 1, ..., l$ of $(0,T)$ satisfying

$$\begin{cases} 0 = t_0 < t_1 ... < t_l < T \\ t_i - t_{i-1} < \varepsilon, \forall i = 1, ..., l \\ T - t_l < \varepsilon. \end{cases}$$

On approach the function f by the function f_ε piecewise constant, defined by $f^\varepsilon(t) = f_i^\varepsilon$ on $[t_{i-1}, t_i[$ for $i = 1, ..., l$, or $f_i^\varepsilon \in X$ and verify

$$\sum_{i=1}^{l} \int_{t_{i-1}}^{t_i} \|f(t) - f_i^\varepsilon\|_X < \varepsilon.$$

This set of data is called a ε-discrétisation the problem $(CP)(u_0,f)$, and notes $D_A^\varepsilon(t_0, ..., t_l; f_1, ..., f_l; u_0)$. We consider the Cauchy problem discretized in time

$$(P_\varepsilon)\begin{cases} \frac{u_i^\varepsilon - u_{i-1}^\varepsilon}{t_i - t_{i-1}} + Au_i^\varepsilon \ni f_i^\varepsilon \\ u(0) = u_0 \end{cases}$$

Definition 1.3.3 *We call mild solution of the Cauchy problem $(CP)(u_0,f)$ any function $u \in C([0,T];X)$ with $u(0) = u_0$ and satisfying for all $\varepsilon > 0$,there exists a ε-discrétisation $D_A^\varepsilon(t_0, ..., t_l; f_1, ..., f_l; u_0)$ and a solution ε-approchée u^ε the problem (P_ε) such that $\sup_{0 \le t \le t_l} \|u^\varepsilon(t) - u(t)\|_X < \varepsilon$.*

Theorem 1.3.4 *Let A be a m-accretive operator in X,then for all $u_0 \in \overline{D(A)}^{\|\cdot\|_X}$ and any function $f \in L^1(0,T;X)$, there exists a unique mild solution u of* *the problem $(CP)(u_0,f)$. Specifically, for every $\varepsilon>0$ and any ε-discrétisation $D_A^{\varepsilon}(t_0,...,t_l;f_1,...,f_l;u_0)$, the solution ε-approchée u^{ε} of the problem (P^{ε}) exists and we $\sup\limits_{0\le t\le t_l}\|u^{\varepsilon}(t)-u(t)\|_X<\varepsilon$ when $\varepsilon\to 0$.*

Remark 1.3.5 *Let A be an accretive operator in X such that $\overline{R(I+\alpha A)}=X$. Then for all $u_0\in\overline{D(A)}^{\|.\|_X}$ and any function $f\in L^1(0,T;X)$, there is a unique mild solution of the Cauchy problem $(CP)(u_0,f)$.*

For a exposed of the mild solutions, the reader may consult the work of Ph. Bénilan [36, 37].

The following notations will be used throughout the paper : for $k\ge 0$, the truncation at heigth k is defined by

$$T_k(r):=\begin{cases}-k, & if\ r\le -k,\\ r, & if\ |r|<k,\\ k, & if\ r\ge k,\end{cases}$$

and let $h_l:\mathbb{R}\to\mathbb{R}$ be defined by

$$h_l(r):=\min\Big((l+1-|r|)^+,1\Big)\quad \text{for each } r\in\mathbb{R}.$$

For $r\in\mathbb{R}$: Let $r\to r^+:=\max(r,0)$, $r\to Sign_0(r)$ the usual sign function which is equal to -1 on $]-\infty,0[$, to 1 on $]0,+\infty[$ and to 0 for $r=0$,

$$r\to Sign_0^+(r):=\begin{cases}1 & if\ r>0\\ 0 & if\ r\le 0\end{cases}.$$

For $\delta>0$, we define $H_\delta^+:\mathbb{R}\to\mathbb{R}$ by

$$H_\delta^+(r):=\begin{cases}0 & if\ r<0\\ \frac{1}{\delta}r & if\ 0\le r\le\delta\\ 1 & if\ r>\delta\end{cases}$$

and $H_\delta:\mathbb{R}\to\mathbb{R}$ by

$$H_\delta(r)=\begin{cases}-1 & if\ r<-\delta\\ \frac{1}{\delta}r & if\ -\delta\le r\le\delta\\ 1 & if\ r>\delta\end{cases}.$$

Chapitre 2

Existence of Renormalized solution of nonlinear degenerate elliptic problems

[1] In this chapter, we study a general class of nonlinear elliptic problems associated with the differential inclusion

$$\beta(u) - div(a(x, Du) + F(u)) \ni f \quad \text{in} \quad \Omega,$$
$$u = 0 \quad \text{on} \quad \partial\Omega$$

where $f \in L^\infty(\Omega)$. The vector field a(., .) is a Carathéodory function.

2.1 Introduction

Let Ω be a bounded open set of $\mathbb{R}^N$ $(N \geq 1)$ with Lipschitz boundary if $N \geq 2$, p be a real number such that $1 < p < \infty$ and $\omega = \{\omega_i(x), 0 \leq i \leq N\}$ be a vector of weight functions on Ω,(i.e, every component $\omega_i(x)$ is a measurable function which is positive a.e. in Ω). Let $W_0^{1,p}(\Omega, \omega)$ be the weighted Sobolev space associated with the vector ω. Our aim is to show existence of renormalized solutions to the following nonlinear elliptic equation

$$(E, f) \begin{cases} \beta(u) - div(a(x, Du) + F(u)) \ni f & \text{in} \quad \Omega \\ u = 0 & \text{on} \quad \partial\Omega. \end{cases}$$

with a right-hand side $f \in L^\infty(\Omega)$ for (E, f). Furthermore, F and β are two functions satisfying the following assumption :
$(\mathbf{A}_0)$ $F : \mathbb{R} \to \mathbb{R}^N$ is locally lipschitz continuous and $\beta : \mathbb{R} \to 2^{\mathbb{R}}$ a set

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valued, maximal monotone mapping such that $0 \in \beta(0)$, Moreover, we assume that

$$\beta^0(l) \in L^1(\Omega). \tag{2.1.1}$$

for each $l \in \mathbb{R}$, where β^0 denotes the minimal selection of the graph of β. Namely $\beta_0(l)$ is the minimal in the norm element of $\beta(l)$, that is, $\beta_0(l) = \inf\{|r|/r \in \mathbb{R} \text{ and } r \in \beta(l)\}$.

Moreover, $a: \Omega \times \mathbb{R}^N \to \mathbb{R}^N$ is a Carathéodory function satisfying the following assumptions :

$(\mathbf{A}_1)$ There existe a positive constant λ such that

$$a(x,\xi)\cdot\xi \geq \lambda \sum_{i=1}^{N} \omega_i |\xi_i|^p$$

for all $\xi \in \mathbb{R}^N$ and almost every $x \in \Omega$.

$(\mathbf{A}_2)$ $|a_i(x,\xi)| \leq \alpha \omega_i^{\frac{1}{p}}(x)[k(x) + \sum_{j=1}^{N} \omega_j^{\frac{1}{p'}}(x)|\xi_j|^{p-1}]$ for almost every $x \in \Omega$, all $i = 1, ..., N$, and every $\xi \in \mathbb{R}^N$, where $k(.)$ is a nonnegative function in $L^{p'}(\Omega)$, $p' = p/p-1$, for a.e. and $\alpha > 0$.

$(\mathbf{A}_3)$ $(a(x,\xi) - a(x,\eta))\cdot(\xi - \eta) \geq 0$ for almost every $x \in \Omega$ and every $\xi, \eta \in \mathbb{R}^N$.

Basic assumption

Assumption (H1)

The expression $|||u|||_X = \left(\sum_{i=1}^{N} \int_\Omega |\frac{\partial u(x)}{\partial x_i}|^p \omega_i(x)\, dx\right)^{1/p}$ is a norm defined on X and is equivalent to the norm (1.1.3).There exist a weight function σ on Ω and a parameter q, $1 < q < \infty$, such that

$$\sigma^{1-q'} \in L^1(\Omega), \tag{2.1.2}$$

with $q' = \frac{q}{q-1}$. The Hardy inequality,

$$\Big(\int_\Omega |u(x)|^q \sigma\, dx\Big)^{1/q} \leq c\Big(\sum_{i=1}^{N} \int_\Omega |\frac{\partial u(x)}{\partial x_i}|^p \omega_i(x)\, dx\Big)^{1/p}, \tag{2.1.3}$$

holds for every $u \in X$ with a constant $c > 0$ independent of u, and moreover, the imbedding

$$X \hookrightarrow\hookrightarrow L^q(\Omega,\sigma), \tag{2.1.4}$$

expressed by the inequality (2.1.3) is compact. Note that $(X, |||.|||_X)$ is a uniformly convex (and thus reflexive) Banach space.

2.2 Notion of solutions and Existence results

2.2.1 Renormalized solutions

Definition 2.2.1 A renormalized solution to (E, f) is a pair of functions (u, b) satisfying the following conditions :

(R1) $u : \Omega \to \mathbb{R}$ *is measurable,* $b \in L^1(\Omega)$, $u(x) \in D(\beta(x))$ and $b(x) \in \beta(u(x))$ for a.e. $x \in \Omega$.

(R2) *For each* $k > 0$, $T_k(u) \in W_0^{1,p}(\Omega, \omega)$ *and*

$$\int_\Omega b \cdot h(u)\varphi + \int_\Omega (a(x, Du) + F(u)) \cdot D(h(u)\varphi) = \int_\Omega f h(u)\varphi \tag{2.2.5}$$

holds for all $h \in \mathcal{C}_c^1(\mathbb{R})$ *and all* $\varphi \in W_0^{1,p}(\Omega, w) \cap L^\infty(\Omega)$, *where* $T_k(.)$ *is a truncation at heigth* k.

(R3) $\int_{\{k \leq |u| \leq k+1\}} a(x, Du) \cdot Du \longrightarrow 0 \quad \text{as} \quad k \longrightarrow \infty.$

2.2.2 Existence results

Theorem 2.2.2 *Under assumptions* (H_1), $(A_0)-(A_3)$ *and* $f \in L^\infty(\Omega)$ *there exists at least one renormalized solution* (u, b) *to* (E, f).

2.3 Proof of Theorem 2.2.2

The proof of Theorem 2.2.2 will be divided into several steps.

2.3.1 Approximate Problem :

At first we approximate (E, f) for $f \in L^\infty(\Omega)$ by problems for which existence can be proved by standard variational arguments. For $0 < \varepsilon \leq 1$, let $\beta_\varepsilon : \mathbb{R} \longrightarrow \mathbb{R}$ be the Yosida approximation of β (see [51]). We introduce the operators

$$\begin{array}{llcl} A_{1,\varepsilon} & : W_0^{1,p}(\Omega,\omega) & \longrightarrow & W^{-1,p'}(\Omega,\omega^*) \\ & \quad u & \longrightarrow & \beta_\varepsilon(T_{\frac{1}{\varepsilon}}(u)) - div\ a(x,Du) \end{array} \quad \text{and}$$

$$\begin{array}{llcl} A_{2,\varepsilon} & : W_0^{1,p}(\Omega,\omega) & \longrightarrow & W^{-1,p'}(\Omega,\omega^*) \\ & \quad u & \longrightarrow & -div\ F(T_{\frac{1}{\varepsilon}}(u)) \end{array}$$

Because of (A2) and (A3), $A_{1,\varepsilon}$, is well-defined and monotone (see [76], p.157). Since $\beta_\varepsilon \circ T_{\frac{1}{\varepsilon}}$ are bounded and continuous and thanks to the growth condition (A2) on a, it follows that $A_{1,\varepsilon}$, is hemicontinuous (see [76], p.157). From the continuity and boundedness of $F \circ T_{\frac{1}{\varepsilon}}$ it follows that $A_{2,\varepsilon}$ is strongly continuous. Therefore the operator $A_\varepsilon := A_{1,\varepsilon} + A_{2,\varepsilon}$ is pseudomonotone. Using the monotonicity of β_ε, the Gauss-Green Theorem for Sobolev functions and the boundary condition on the convection term $\int_\Omega F(T_{\frac{1}{\varepsilon}}(u)) \cdot Du$, we show with similar arguments as in [33] that A_ε is coercive and bounded. Then it follows from [76], Theorem 2.7, that A_ε is surjective, i.e., for each $0 < \varepsilon \leq 1$ and $f \in W^{-1,p'}(\Omega,\omega^*)$ there exists at least one solution $u_\varepsilon \in W_0^{1,p}(\Omega,\omega)$ to the problem

$$(E_\varepsilon, f) \begin{cases} \beta_\varepsilon(T_{\frac{1}{\varepsilon}}(u_\varepsilon)) - div(a(x,Du_\varepsilon) + F(T_{\frac{1}{\varepsilon}}(u_\varepsilon))) = f & \text{in} \quad \Omega, \\ u_\varepsilon = 0 & \text{on} \quad \partial\Omega, \end{cases}$$

such that

$$\int_\Omega \beta_\varepsilon(T_{\frac{1}{\varepsilon}}(u_\varepsilon))\varphi + \int_\Omega \big(a(x,Du_\varepsilon) + F(T_{\frac{1}{\varepsilon}}(u_\varepsilon))\big) \cdot D\varphi =< f, \varphi > \tag{2.3.6}$$

for all $\varphi \in W_0^{1,p}(\Omega,\omega)$.

2.3.2 A priori estimates

Lemma 2.3.1 *For $0 < \varepsilon \leq 1$ and $f \in L^\infty(\Omega)$ let $u_\varepsilon \in W_0^{1,p}(\Omega,\omega)$ be a solution of (E_ε, f). Then,*

i) there exists a constant $C_1 = C_1(\|f\|_\infty, \lambda, p, N) > 0$, not depending on ε, such that

$$|||u_\varepsilon||| \leq C_1. \tag{2.3.7}$$

ii)

$$\|\beta_\varepsilon(T_{\frac{1}{\varepsilon}}(u_\varepsilon))\|_\infty \le \|f\|_\infty. \tag{2.3.8}$$

iii) For all $l, k > 0$ *we have*

$$\int_{\{l \le |u_\varepsilon| \le l+k\}} a(x, Du_\varepsilon)\cdot Du_\varepsilon \le k \int_{\{|u_\varepsilon|>l\}} |f|. \tag{2.3.9}$$

Proof : i) Taking u_ε as a test function in (2.3.6) we obtain

$$\int_\Omega \beta_\varepsilon(T_{\frac{1}{\varepsilon}}(u_\varepsilon))u_\varepsilon dx + \int_\Omega a(x, Du_\varepsilon)\cdot Du_\varepsilon dx + \int_\Omega F(T_{\frac{1}{\varepsilon}}(u_\varepsilon)))\cdot Du_\varepsilon dx = \int_\Omega f u_\varepsilon dx$$

As the first term on the left-hand side is nonnegative and the integral over the convection term vanishes by $(A1)$ we have :

$$\lambda \sum_{i=1}^N \int_\Omega |\frac{\partial u_\varepsilon}{\partial x_i}|^p \omega_i(x)dx \le \sum_{i=1}^N \int_\Omega a_i(x, Du_\varepsilon)\cdot \frac{\partial u_\varepsilon}{\partial x_i}dx \le \int_\Omega f u_\varepsilon dx$$

$$\le C\|f\|_\infty \Big(\sum_{i=1}^N \int_\Omega |\frac{\partial u_\varepsilon(x)}{\partial x_i}|^p \omega_i(x)dx\Big)^{\frac{1}{p}} \Big(\int_\Omega \sigma^{1-q'}dx\Big)^{\frac{1}{q'}} (\text{due to}(1.1.6))$$

Then $|\|u_\varepsilon|\|^p \le C_2 |\|u_\varepsilon|\|$ where C_2 a positive constant. Hence we can deduce that u_ε remains bounded in $W_0^{1,p}(\Omega,\omega)$i.e.; $|\|u_\varepsilon|\| \le C_1$.
ii) Taking $\frac{1}{\delta}[T_{k+\delta}(\beta_\varepsilon(T_{\frac{1}{\varepsilon}}(u_\varepsilon))) - T_k(\beta_\varepsilon(T_{\frac{1}{\varepsilon}}(u_\varepsilon)))]$ as a test function in (2.3.6), letting $\delta \to 0$ and choosing $k > \|f\|_\infty$ we obtain (ii).
iii) For $k, l > 0$ fixed we take $T_k(u_\varepsilon - T_l(u_\varepsilon))$ as a test function in (2.3.6). Using $\int_\Omega a(x, Du_\varepsilon)\cdot DT_k(u_\varepsilon - T_l(u_\varepsilon))dx = \int_{\{l<|u_\varepsilon|<l+k\}} a(x, Du_\varepsilon)\cdot Du_\varepsilon dx$, and as the first term on the left-hand side is nonnegative and the convection term vanishes, we get

$$\int_{\{l<|u_\varepsilon|<l+k\}} a(x, Du_\varepsilon)\cdot Du_\varepsilon dx \le \int_\Omega f T_k(u_\varepsilon - T_l(u_\varepsilon))dx \quad \le k\int_{\{|u_\varepsilon|>l\}} |f|dx.$$

Remark 2.3.2 *For* $k > 0$, *since*

$$|\{|u_\varepsilon| > l\}| \le \frac{C_2}{l^{1-\frac{1}{p}}}, \tag{2.3.10}$$

from Lemma 2.3.1 (iii) we deduce that

$$\int_{\{l \le |u_\varepsilon| \le l+k\}} a(x, Du_\varepsilon)\cdot Du_\varepsilon \le k\|f\|_\infty |\{|u_\varepsilon| > l\}| \le \frac{C_2(k)}{l^{1-\frac{1}{p}}}. \tag{2.3.11}$$

2.3.3 Basic convergence results

Lemma 2.3.3 For $0 < \varepsilon \leq 1$ and $f \in L^\infty(\Omega)$ let $u_\varepsilon \in W_0^{1,p}(\Omega, \omega)$ be the solution of (E_ε, f). There exist $u \in W_0^{1,p}(\Omega, \omega)$ and $b \in L^\infty(\Omega)$ such that for a not relabeled subsequence of $(u_\varepsilon)_{0<\varepsilon\leq 1}$ as $\varepsilon \downarrow 0$:

$$u_\varepsilon \rightharpoonup u \qquad \text{in } W_0^{1,p}(\Omega, \omega) \quad \text{and a.e. in} \quad \Omega, \tag{2.3.12}$$

$$T_k(u_\varepsilon) \rightharpoonup T_k(u) \qquad \text{in } W_0^{1,p}(\Omega, \omega) \text{ and strongly in } L^q(\Omega, \sigma), \tag{2.3.13}$$

$$\beta_\varepsilon(T_{\frac{1}{\varepsilon}}(u_\varepsilon)) \rightharpoonup b \qquad \text{in } L^\infty(\Omega), \tag{2.3.14}$$

Moreover, for any $k > 0$

$$DT_k(u_\varepsilon) \rightharpoonup DT_k(u) \qquad \text{in} \quad \Pi_{i=1}^N L^p(\Omega, \omega_i), \tag{2.3.15}$$

$$a(x, DT_k(u_\varepsilon)) \rightharpoonup a(x, DT_k(u)) \quad \text{in} \quad \Pi_{i=1}^N L^{p'}(\Omega, \omega_i^*). \tag{2.3.16}$$

Proof : (2.3.14) follow directly from Lemma 2.3.1 and Remark 2.3.2. From (2.3.10), (2.3.7) and (1.1.7) we deduce with classical argument (see,e.g.,[1]) that for a subsequence still indexed by ε, (2.3.12)-(2.3.13) and (2.3.15) hold as ε tend to 0, where u is a measurable function defined on Ω. It is left to prove (2.3.16). For this, by $(A2)$ and (2.3.7) it follows that given any subsequence of $(a(x, DT_k(u_\varepsilon))_\varepsilon)$, there exists a subsequence, still denoted by $(a(x, DT_k(u_\varepsilon))_\varepsilon)$ such that $a(x, DT_k(u_\varepsilon)) \rightharpoonup \Phi_k$ in $\Pi_{i=1}^N L^{p'}(\Omega, \omega_i^*)$. We will prove that $\Phi_k = a(x, DT_k(u))$ a.e. of Ω. The proof consists in three steps.

Step i : For every function $h \in W^{1,\infty}(\mathbb{R})$, $h \geq 0$ with $supp(h)$ compact, we will prove that,

$$\limsup_{\varepsilon\to 0} \int_\Omega a(x, DT_k(u_\varepsilon)) \cdot D[h(u_\varepsilon)(T_k(u_\varepsilon) - T_k(u))]dx \leq 0. \tag{2.3.17}$$

Taking $h(u_\varepsilon)(T_k(u_\varepsilon) - T_k(u))$ as test function in (2.3.6) we have

$$\int_\Omega \beta_\varepsilon(T_{\frac{1}{\varepsilon}}(u_\varepsilon))h(u_\varepsilon)(T_k(u_\varepsilon) - T_k(u)) + \int_\Omega a(x, Du_\varepsilon) \cdot D[h(u_\varepsilon)(T_k(u_\varepsilon) - T_k(u))]$$
$$+ \int_\Omega F(T_{\frac{1}{\varepsilon}}(u_\varepsilon)) \cdot D[h(u_\varepsilon)(T_k(u_\varepsilon) - T_k(u))] = \int_\Omega fh(u_\varepsilon)(T_k(u_\varepsilon) - T_k(u)) \tag{2.3.18}$$

Using $|h(u_\varepsilon)(T_k(u_\varepsilon) - T_k(u))| \leq 2k\|h\|_\infty$, by Lebesgue's Dominated Convergence Theorem we find that $\lim_{\varepsilon\to 0}\int_\Omega fh(u_\varepsilon)(T_k(u_\varepsilon) - T_k(u)) = 0$ and $\lim_{\varepsilon\to 0}\int_\Omega F(T_{\frac{1}{\varepsilon}}(u_\varepsilon)) \cdot D[h(u_\varepsilon)(T_k(u_\varepsilon) - T_k(u))] = 0$. By using the same arguments in [25] we can prove that

$$\limsup_{\varepsilon\to 0}\int_\Omega \beta_\varepsilon(T_{\frac{1}{\varepsilon}}(u_\varepsilon)) \cdot [h(u_\varepsilon)(T_k(u_\varepsilon) - T_k(u))]dx \geq 0.$$

Passing to the limit in (2.3.18) and using the above results we obtain the (2.3.17).

Step ii : We now prove that for every $k > 0$,

$$\limsup_{\varepsilon\to 0}\int_\Omega a(x, DT_k(u_\varepsilon)) \cdot [DT_k(u_\varepsilon) - DT_k(u)]dx \leq 0 \tag{2.3.19}$$

Indeed, for $k > l$, take $h_l(u_\varepsilon)(T_k(u_\varepsilon) - T_k(u))$ as a test function in (2.3.6). Letting $\varepsilon \downarrow 0$ and then $l \to \infty$, we obtain

$$\begin{aligned}
&\int_\Omega a(x, DT_k(u_\varepsilon)) \cdot D[h_l(u_\varepsilon)(T_k(u_\varepsilon) - T_k(u))]dx \\
&\qquad = \int_{[|u_\varepsilon|\leq k]} h_l(u_\varepsilon)a(x, DT_k(u_\varepsilon)) \cdot [DT_k(u_\varepsilon) - DT_k(u)]dx \\
&\qquad + \int_{[|u_\varepsilon|>k]} h_l(u_\varepsilon)a(x, DT_k(u_\varepsilon)) \cdot (-DT_k(u))dx \\
&\qquad + \int_\Omega h_l'(u_\varepsilon)(T_k(u_\varepsilon) - T_k(u))a(x, DT_k(u_\varepsilon)) \cdot Du_\varepsilon dx \\
&\qquad = E_1 + E_2 + E_3.
\end{aligned}$$

Since $l > k$, on the set $[|u_\varepsilon| \leq k]$ we have $h_l(u_\varepsilon) = 1$ so that we can write

$$\limsup_{\varepsilon\to 0} E_1 = \limsup_{\varepsilon\to 0}\int_\Omega a(x, DT_k(u_\varepsilon)) \cdot (DT_k(u_\varepsilon) - DT_k(u))dx.$$

For E_2, using Lebesgue's dominated convergence theorem we get

$$\lim_{\varepsilon\to 0} E_2 = -\int_{[|u|>k]} h_l(u)\Phi_{l+1} \cdot DT_k(u)dx = 0.$$

For E_3 , we have
$-(\int_\Omega h_l'(u_\varepsilon)(T_k(u_\varepsilon) - T_k(u))a(x, DT_k(u_\varepsilon))Du_\varepsilon dx) \leq 2k\int_{[l<|u_\varepsilon|\leq l+1]} a(x, Du_\varepsilon)Du_\varepsilon dx.$
Using (2.3.11), we deduce that

$$\limsup_{l\to+\infty}\limsup_{\varepsilon\to 0}(-\int_\Omega h_l'(u_\varepsilon)(T_k(u_\varepsilon) - T_k(u))a(x, DT_k(u_\varepsilon)) \cdot Du_\varepsilon dx \leq 0$$

Applying (2.3.17) whit h replaced by h_l , $l > k$ we get

$$\limsup_{\varepsilon\to 0}\int_\Omega a(x,DT_k(u_\varepsilon))\cdot(DT_k(u_\varepsilon)-DT_k(u))dx$$
$$\leq \limsup_{\varepsilon\to 0}(-\int_\Omega h_l^{'}(u_\varepsilon)(T_k(u_\varepsilon)-T_k(u))a(x,DT_k(u_\varepsilon))\cdot Du_\varepsilon dx.$$

Now letting $l\to+\infty$ yields (2.3.19).

Step iii : In this step we prove by monotonicity arguments that for $k>0$, $\Phi_k = a(x,DT_k(u))$ for almost every $x\in\Omega$. Let $\varphi\in D(\Omega)$ and $\tilde{\alpha}\in\mathbb{R}$. Using (2.3.19), we have

$$\tilde{\alpha}\lim_{\varepsilon\to 0}\int_\Omega a(x,DT_k(u_\varepsilon))\cdot D\varphi dx \geq \tilde{\alpha}\int_\Omega a(x,D(T_k(u)-\tilde{\alpha}\varphi))\cdot D\varphi dx.$$

Dividing by $\tilde{\alpha}>0$ and by $\tilde{\alpha}<0$, and letting $\tilde{\alpha}\to 0$ we obtain

$$\lim_{\varepsilon\to 0}\int_\Omega a(x,DT_k(u_\varepsilon))\cdot D\varphi dx = \int_\Omega a(x,DT_k(u))\cdot D\varphi dx.$$

This means that for all $k>0$, $\int_\Omega \Phi_k\cdot D\varphi dx = \int_\Omega a(x,DT_k(u))\cdot D\varphi dx$ and so $\Phi_k = a(x,DT_k(u))$ in $D^{'}(\Omega)$ for all $k>0$. Hence $\Phi_k = a(x,DT_k(u))$ a.e. in Ω and so $a(x,DT_k(u_\varepsilon))\rightharpoonup a(x,DT_k(u))$ weakly in $\Pi_{i=1}^N L^{p'}(\Omega,\omega_i^*)$.

2.3.4 Proof of Existence.

Let $h\in\mathcal{C}_c^1(\mathbb{R})$ and $\phi\in W_0^{1,p}(\Omega,\omega)\cap L^\infty(\Omega)$. Taking $h_l(u_\varepsilon)h(u)\phi$ as a test function in (2.3.6), we obtain

$$I_{\varepsilon,l}^1 + I_{\varepsilon,l}^2 + I_{\varepsilon,l}^3 = I_{\varepsilon,l}^4 \qquad (2.3.20)$$

where

$$I_{\varepsilon,l}^1 = \int_\Omega \beta_\varepsilon(T_{\frac{1}{\varepsilon}}(u_\varepsilon))h_l(u_\varepsilon)h(u)\phi,$$
$$I_{\varepsilon,l}^2 = \int_\Omega a(x,Du_\varepsilon)\cdot D(h_l(u_\varepsilon)h(u)\phi),$$
$$I_{\varepsilon,l}^3 = \int_\Omega F(T_{\frac{1}{\varepsilon}}(u_\varepsilon))\cdot D(h_l(u_\varepsilon)h(u)\phi),$$
$$I_{\varepsilon,l}^4 = \int_\Omega f h_l(u_\varepsilon)h(u)\phi.$$

Step i : Letting $\varepsilon \downarrow 0$ using the convergence results (2.3.12), (2.3.14) from Lemma 2.3.3 we can immediately calculate the following limits :

$$\lim_{\varepsilon \to 0} I^1_{\varepsilon,l} = \int_\Omega b h_l(u) h(u)\phi, \tag{2.3.21}$$

$$\lim_{\varepsilon \to 0} I^4_{\varepsilon,l} = \int_\Omega f h_l(u) h(u)\phi. \tag{2.3.22}$$

We $I^2_{\varepsilon,l} = I^{2,1}_{\varepsilon,l} + I^{2,2}_{\varepsilon,l}$ where
$I^{2,1}_{\varepsilon,l} = \int_\Omega h'_l(u_\varepsilon) a(x, Du_\varepsilon) \cdot Du_\varepsilon h(u)\phi$, $\quad I^{2,2}_{\varepsilon,l} = \int_\Omega h_l(u_\varepsilon) a(x, Du_\varepsilon) \cdot D(h(u)\phi)$.
Using (2.3.11) we get the estimate

$$|\lim_{\varepsilon \to 0} I^{2,1}_{\varepsilon,l}| \le \|h\|_\infty \|\phi\|_\infty \cdot C_2 l^{-(1-\frac{1}{p})}. \tag{2.3.23}$$

By Lebesgue's Dominated Convergence Theorem it follows that for any $i \in \{1, ..., N\}$ we have

$$h_l(u_\varepsilon) \frac{\partial}{\partial x_i}(h(u)\phi) \to h_l(u) \frac{\partial}{\partial x_i}(h(u)\phi) \quad \text{in } L^p(\Omega, \sigma) \text{ as } \varepsilon \downarrow 0.$$

Keeping in mind that $I^{2,2}_{\varepsilon,l} = \int_\Omega h_l(u_\varepsilon) a(x, DT_{l+1}(u_\varepsilon)) \cdot D(h(u)\phi)$, by (2.3.16), we get

$$\lim_{\varepsilon \to 0} I^{2,2}_{\varepsilon,l} = \int_\Omega h_l(u) a(x, DT_{l+1}(u)) \cdot D(h(u)\phi). \tag{2.3.24}$$

Let us write $I^3_{\varepsilon,l} = I^{3,1}_{\varepsilon,l} + I^{3,2}_{\varepsilon,l}$, where

$$I^{3,1}_{\varepsilon,l} = \int_\Omega h'_l(u_\varepsilon) F(T_{\frac{1}{\varepsilon}}(u_\varepsilon)) \cdot Du_\varepsilon h(u)\phi$$

$$I^{3,2}_{\varepsilon,l} = \int_\Omega h_l(u_\varepsilon) F(T_{\frac{1}{\varepsilon}}(u_\varepsilon)) \cdot D(h(u)\phi).$$

For any $l \in \mathbb{N}$, there exists $\varepsilon_0(l)$ such that for all $\varepsilon < \varepsilon_0(l)$,

$$I^{3,1}_{\varepsilon,l} = \int_\Omega h'_l(T_{l+1}(u_\varepsilon)) F(T_{l+1}(u_\varepsilon)) \cdot DT_{l+1}(u_\varepsilon) h(u)\phi. \tag{2.3.25}$$

Using Gauss-Green Theorem for Sobolev functions in (2.3.25) we get

$$I^{3,1}_{\varepsilon,l} = -\int_\Omega \int_0^{T_{l+1}(u_\varepsilon)} h'_l(r) F(r) dr \cdot D(h(u)\phi). \tag{2.3.26}$$

Now, using (2.3.12) and the Gauss-Green Theorem, after letting $\varepsilon \downarrow 0$ we get

$$\lim_{\varepsilon \to 0} I^{3,1}_{\varepsilon,l} = \int_\Omega h'_l(u) F(u) \cdot Du \; h(u)\phi. \tag{2.3.27}$$

Choosing ε small enough, we can write

$$I_{\varepsilon,l}^{3,2} = \int_\Omega h_l(u_\varepsilon) F(T_{l+1}(u_\varepsilon)) \cdot D(h(u)\phi) \tag{2.3.28}$$

and conclude

$$\lim_{\varepsilon\to 0} I_{\varepsilon,l}^{3,2} = \int_\Omega h_l(u) F(u) \cdot D(h(u)\phi). \tag{2.3.29}$$

Step ii : We let $l \to \infty$. Combining (2.3.20) and (3.3.33)- (2.3.29) we find

$$I_l^1 + I_l^2 + I_l^3 + I_l^4 + I_l^5 = I_l^6 \tag{2.3.30}$$

where $I_l^1 = \int_\Omega b h_l(u) h(u)\phi$, $\quad I_l^2 = \int_\Omega h_l(u) a(x, DT_{l+1}(u)) \cdot D(h(u)\phi)$,
$|I_l^3| \le C_2 l^{-(1-\frac{1}{p})} \|h\|_\infty \|\phi\|_\infty$, $\quad I_l^4 = \int_\Omega h_l(u) F(u) \cdot D(h(u)\phi)$,
$I_l^5 = \int_\Omega h_l'(u) F(u) \cdot Du\, h(u)\phi$, $\quad I_l^6 = \int_\Omega f\, h_l(u) h(u)\phi$.
Obviously, we have

$$\lim_{l\to\infty} I_l^3 = 0. \tag{2.3.31}$$

Choosing $m > 0$ such that *supp* $h \subset [-m, m]$, we can replace u by $T_m(u)$ in $I_l^1, I_l^2, ..., I_l^6$, and
$h_l'(u) = h_l'(T_m(u)) = 0$ if $l + 1 > m$, $\ h_l(u) = h_l(T_m(u)) = 1$ if $l > m$.
Therefore, letting $l \to \infty$ and combining (2.3.30) with (2.3.31) we obtain

$$\int_\Omega b h(u)\phi + \int_\Omega (a(x, Du) + F(u)) \cdot D(h(u)\phi) = \int_\Omega f h(u)\phi, \tag{2.3.32}$$

for all $h \in C_c^1(\mathbb{R})$ and all $\phi \in W_0^{1,p}(\Omega, \omega) \cap L^\infty(\Omega)$.
Step iii : Subdifferential argument : It is left to prove that $u(x) \in D(\beta(x))$ and $b(x) \in \beta(u(x))$ for almost all $x \in \Omega$. Since β a is maximal monotone graph, there exists a convex, l.s.c., and proper function $j : \mathbb{R} \to [0, \infty]$ such that $\beta(r) = \partial j(r)$ for all $r \in \mathbb{R}$. According to [51], for $0 < \varepsilon \le 1$, $j_\varepsilon : \mathbb{R} \to \mathbb{R}$ defined by $j_\varepsilon(r) = \int_0^r \beta_\varepsilon(s) ds$ has the following properties : [see [89]]. Using the same argument in [89] we can prove that for all $r \in \mathbb{R}$ and almost every $x \in \Omega$, $u \in D(\beta)$ and $b \in \beta(u)$ almost everywhere in Ω. With this last step the proof of Theorem 2.2.2 is completed.

Chapitre 3

Existence and uniqueness of Renormalized solution of nonlinear degenerated elliptic problems

[1] In this chapter, we study a general class of nonlinear degenerated elliptic problems associated with the differential inclusion $\beta(u)-div(a(x,Du)+F(u))\ni f$ in Ω where $f\in L^1(\Omega)$. A vector field $a(.,.)$ is a Carathéodory function. Using truncation techniques and the generalized monotonicity method in the functional spaces we prove existence of renormalized solutions for general L^1-data. Under an additional strict monotonicity assumption uniqueness of the renormalized solution is established.

3.1 Introduction

Let Ω be a bounded open set of $\mathbb{R}^N$ $(N\geq 1)$ with Lipschitz boundary if $N\geq 2$, p be a real number such that $1<p<\infty$ and $\omega=\{\omega_i(x),\ 0\leq i\leq N\}$ be a vector of weight functions on Ω, i.e. each $\omega_i(x)$ is a measurable a.e. positive on Ω. Let $W_0^{1,p}(\Omega,\omega)$ be the weighted Sobolev space associated with the vector ω. Our aim is to show existence and uniqueness of renormalized solutions to the following nonlinear elliptic equation

$$(\mathrm{Eq},f)\begin{cases}\beta(u)-div(a(x,Du)+F(u))\ni f & \text{in}\quad\Omega\\ u=0 & \text{on}\quad\partial\Omega.\end{cases}$$

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with a right-hand side f is assumed to belong either to $L^{\infty}(\Omega)$ or to $L^{1}(\Omega)$, for (Eq, f) Furthermore, $F: \mathbb{R} \rightarrow \mathbb{R}^{N}$ is locally lipschitz continuous and $\beta: \mathbb{R} \rightarrow 2^{\mathbb{R}}$ a set valued, maximal monotone mapping such that $0 \in \beta(0)$, Moreover, we assume that

$$\beta^{0}(l) \in L^{1}(\Omega), \tag{3.1.1}$$

for each $l \in \mathbb{R}$, where β^{0} denotes the minimal selection of the graph of β. Namely $\beta_{0}(l)$ is the minimal in the norm element of $\beta(l)$

$$\beta_{0}(l)=\inf\{|r| \ / r \in \mathbb{R} \text{ and } r \in \beta(l)\}$$

$a: \Omega \times \mathbb{R}^{N} \rightarrow \mathbb{R}^{N}$ is a Carathéodory function satisfying to following assumptions :(A_{1}), (A_{2}), (A_{3}) set out General Introduction

We use in this chapter the the framework of renormalized solutions. This notion was introduced by Diperna and P.-L. Lions [56] in their study of the Boltzmann equation. This notion was then adapted to an elliptic version of (Eq, f) by L. Boccardo et al. [38] when the right hand side is in $W^{-1,p'}(\Omega)$, by J.-M. Rakotoson [83] when the right hand side is in $L^{1}(\Omega)$, and finally by G. Dal Maso, F. Murat, L. Orsina and A. Prignet [54] for the case of right hand side is general measure data. The equivalent notion of entropy solution has been introduced by Bénilan et al. in [38]. For results on existence of renormalized solutions of elliptic problems of type (Eq, f) with $a(,)$ satisfying a variable growth condition we refer to [[89],[62]] and [25]. One of the motivations for studying (Eq, f) comes from applications to electrorheological fluids (see [85] for more details) as an important class of non-Newtonian fluids.

The plan of the chapter is as follows. In Section 2 we make precise all the assumptions on a, β, F, f and we introduce the notions of weak and also renormalized solution for problem (Eq, f). Our first main result, existence of a renormalized solution to (Eq, f) for any L^{∞} - data f, are collected in Section 3. Our second main result, existence of a renormalized solution to (Eq, f) for any L^{1} - data f , and the results on uniqueness of renormalized solutions are collected in Section 4. Section 5 is devoted to an example for illustrating our abstract result.

3.2 Basic assumptions and notion of solutions

3.2.1 Basic assumptions

Assumption (H1)

The expression

$$|||u|||_X = \Big(\sum_{i=1}^{N}\int_\Omega |\frac{\partial u(x)}{\partial x_i}|^p \omega_i(x)\,dx\Big)^{1/p} \tag{3.2.2}$$

is a norm defined on X and is equivalent to the norm (1.1.3).
There exist a weight function σ on Ω and a parameter q, $1 < q < \infty$, such that

$$\sigma^{1-q'} \in L^1(\Omega), \tag{3.2.3}$$

with $q' = \frac{q}{q-1}$. The Hardy inequality,

$$\Big(\int_\Omega |u(x)|^q \sigma\,dx\Big)^{1/q} \leq c\Big(\sum_{i=1}^{N}\int_\Omega |\frac{\partial u(x)}{\partial x_i}|^p \omega_i(x)\,dx\Big)^{1/p}, \tag{3.2.4}$$

holds for every $u \in X$ with a constant $c > 0$ independent of u, and moreover, the imbedding

$$X \hookrightarrow\hookrightarrow L^q(\Omega,\sigma), \tag{3.2.5}$$

expressed by the inequality (3.2.4) is compact. Note that $(X, |||.|||_X)$ is a uniformly convex (and thus reflexive) Banach space.

Assumption (H2)

$$|a_i(x,\xi)| \leq \alpha\omega_i^{1/p}(x)[k(x) + \sum_{j=1}^{N}\omega_j^{1/p'}(x)|\xi_j|^{p-1}], \tag{3.2.6}$$

$$[a(x,\xi) - a(x,\eta)](\xi - \eta) \geq 0 \quad \text{for all } (\xi,\eta) \in \mathbb{R}^N \times \mathbb{R}^N, \tag{3.2.7}$$

$$a(x,\xi).\xi \geq \lambda\sum_{i=1}^{N}\omega_i|\xi_i|^p, \tag{3.2.8}$$

where $k(x)$ is a positive function in $L^{p'}(\Omega)$ and α, λ are positive constants.

Assumption (H3)

Let $\beta,\ F$ such that

$$\beta : \mathbb{R} \to 2^{\mathbb{R}} \ \textit{set valued, maximal monotone mapping} \tag{3.2.9}$$

such that $0 \in \beta(0)$ and satisfying (3.1.1),

$$F : \mathbb{R} \to \mathbb{R}^N \ \textit{is locally lipschitz continuous.} \tag{3.2.10}$$

Remark 3.2.1 The lipschitz character of F and Stokes formula together with the boundary condition ($u = 0$ on $\partial\Omega$) of problem give $\int_\Omega F(u)DT_k(u)dx = 0$*(see[89])*.

3.2.2 Notions of solutions

Definition 3.2.2 *A weak solution to* (Eq, f) *is a pair of functions* $(u, b) \in W_0^{1,p}(\Omega, \omega) \times L^1(\Omega)$ *satisfying* $F(u) \in (L^1_{\text{loc}}(\Omega))^N$, $b \in \beta(u)$ *almost everywhere in* Ω *and*

$$b - \operatorname{div}(a(x, Du) + F(u)) = f \quad \text{in } D^{'}(\Omega). \tag{3.2.11}$$

Definition 3.2.3 *A renormalized solution to* (Eq, f) *is a pair of functions* (u, b) *satisfying the following conditions :*

(R1) $u : \Omega \to \mathbb{R}$ *is measurable,* $b \in L^1(\Omega)$, $u(x) \in D(\beta(x))$ *and* $b(x) \in \beta(u(x))$ *for a.e.* $x \in \Omega$.

(R2) *For each* $k > 0$, $T_k(u) \in W_0^{1,p}(\Omega, \omega)$ *and*

$$\int_\Omega b \cdot h(u)\varphi + \int_\Omega (a(x, Du) + F(u)) \cdot D(h(u)\varphi) = \int_\Omega f h(u)\varphi, \tag{3.2.12}$$

holds for all $h \in \mathcal{C}_c^1(\mathbb{R})$ *and all* $\varphi \in W_0^{1,p}(\Omega, \omega) \cap L^\infty(\Omega)$.

(R3) $\int_{\{k<|u|<k+1\}} a(x, Du) \cdot Du \longrightarrow 0 \quad as \quad k \longrightarrow \infty.$

Remark 3.2.4 *For* $p \in (1, \infty)$, $\tau_0^{1,p}(\Omega, \omega)$ *is defined as the set of measurable functions* $u : \Omega \to \mathbb{R}$ *such that for* $k > 0$ *the truncated functions* $T_k(u) \in W_0^{1,p}(\Omega, \omega)$ *and for every* $u \in \tau_0^{1,p}(\Omega, \omega)$ *there exists a unique measurable function* $v : \Omega \to \mathbb{R}^{\mathbb{N}}$ *such that* $\nabla T_K(u) = v\chi_{\{|u|<k\}}$ *(see[1],[38]) for more details.*

Remark 3.2.5 *Note that if* (u, b) *is a renormalized solution to* (Eq, f) *such that* $u \in W_0^{1,p}(\Omega, \omega)$ *, then* (u, b) *in general is not a weak solution in the sense of Definition 3.2.2, since we did not assume a growth condition on* F *and therefore* $F(u)$ *in general may fail to be locally integrable. If* (u, b) *is a renormalized solution of* (Eq, f) *such that* $u \in L^\infty(\Omega)$*, it is a direct consequence of Definition 3.2.2 that* u *is in* $W_0^{1,p}(\Omega, \omega)$ *and since* (3.2.12) *holds with the formal choice* $h \equiv 1$, (u, b) *is a weak solution. Indeed, let us choose* $\varphi \in D(\Omega)$ *and plug* $h_l(u)\varphi$ *as a test function in* (3.2.12)*). Since* $u \in L^\infty(\Omega)$*, we can pass to the limit with* $l \to \infty$ *and find that* u *solves* (Eq, f) *in the sense of distributions.*

3.3 Case where $f \in L^\infty(\Omega)$ -data

3.3.1 Resultat d'existence

In this subsection we will state existence of renormalized solutions to (Eq, f) in Theorem 3.3.1. In the next subsections we will present the proof.

Theorem 3.3.1 *Under assumptions* $(H_1) - (H_3)$ *and* $f \in L^\infty(\Omega)$ *there exists at least one renormalized solution* (u, b) *to* (Eq, f)*.*

Remark 3.3.2 *The problem on the renormalized solution to the degenerated problem* (Eq, f) *for right-hand side* $f \in L^\infty(\Omega)$ *to be a weak solution is still open.*

3.3.2 Proof of Theorem 3.3.1

The proof of Theorem 3.3.1 will be divided into several steps.

Approximate problem

At first we approximate (Eq, f) for $f \in L^\infty(\Omega)$ by problems for which existence can be proved by standard variational arguments. For $0 < \varepsilon \leq 1$, let $\beta_\varepsilon : \mathbb{R} \longrightarrow \mathbb{R}$ be the Yosida approximation (see [51]) of β . We introduce the operators

$$\begin{array}{rlcl} A_{1,\varepsilon} & : W_0^{1,p}(\Omega,\omega) & \longrightarrow & W^{-1,p'}(\Omega,\omega^*) \\ & u & \longrightarrow & \beta_\varepsilon(T_{\frac{1}{\varepsilon}}(u)) + \varepsilon \arctan(u) - div\ a(x,Du), \end{array}$$

and

$$\begin{array}{rlcl} A_{2,\varepsilon} & : W_0^{1,p}(\Omega,\omega) & \longrightarrow & W^{-1,p'}(\Omega,\omega^*), \\ & u & \longrightarrow & -div\ F(T_{\frac{1}{\varepsilon}}(u)). \end{array}$$

Because of (A2) and (A3), $A_{1,\varepsilon}$, is well-defined and monotone (see [76], p.157). Since $\beta_\varepsilon \circ T_{\frac{1}{\varepsilon}}$ and arctan are bounded and continuous and thanks to the growth condition (A2) on a, it follows that $A_{1,\varepsilon}$, is hemicontinuous (see [76], p.157). From the continuity and boundedness of $F \circ T_{\frac{1}{\varepsilon}}$ it follows that $A_{2,\varepsilon}$ is strongly continuous. Therefore the operator $A_\varepsilon := A_{1,\varepsilon} + A_{2,\varepsilon}$ is pseudomonotone. Using the monotonicity of β_ε, the Gauss-Green Theorem for Sobolev functions and the boundary condition on the convection term $\int_\Omega F(T_{\frac{1}{\varepsilon}}(u)) \cdot Du$, we show with similar arguments as in [33] that A_ε is coercive and bounded. Then it follows from [76], Theorem 2.7, that A_ε is surjective, i.e., for each $0 < \varepsilon \leq 1$ and $f \in W^{-1,p'}(\Omega,\omega^*)$ there exists at least one solution $u_\varepsilon \in W_0^{1,p}(\Omega,\omega)$ to the problem

$$(\mathrm{Eq}_\varepsilon, f) \begin{cases} \beta_\varepsilon(T_{\frac{1}{\varepsilon}}(u_\varepsilon)) + \varepsilon \arctan(u_\varepsilon) - div(a(x,Du_\varepsilon) + F(T_{\frac{1}{\varepsilon}}(u_\varepsilon))) = f & \text{in} \quad \Omega, \\ \\ u_\varepsilon = 0 & \text{on} \quad \partial\Omega, \end{cases}$$

such that

$$\int_\Omega \Big(\beta_\varepsilon(T_{\frac{1}{\varepsilon}}(u_\varepsilon)) + \varepsilon \arctan(u_\varepsilon)\Big)\varphi + \int_\Omega \Big(a(x,Du_\varepsilon) + F(T_{\frac{1}{\varepsilon}}(u_\varepsilon))\Big) \cdot D\varphi =< f, \varphi > \tag{3.3.13}$$

holds for all $\varphi \in W_0^{1,p}(\Omega,\omega)$, where $< .,. >$ denotes the duality pairing between $W_0^{1,p}(\Omega,\omega)$ and $W^{-1,p'}(\Omega,\omega^*)$.

In the next remark, we establish uniqueness of solutions u_ε of $(\mathrm{Eq}_\varepsilon, f)$ with right-hand sides $f \in L^\infty(\Omega)$ through a comparison principle that will play an important role in the approximation of renormalized solutions to (Eq, f) with $f \in L^1(\Omega)$.

Remark 3.3.3 For $0 < \varepsilon \leq 1$ fixed and $f, \tilde{f} \in L^\infty(\Omega)$ let $u_\varepsilon, \tilde{u}_\varepsilon \in W_0^{1,p}(\Omega, \omega)$ be solutions of $(\mathrm{Eq}_\varepsilon, f)$ and $(\mathrm{Eq}_\varepsilon, \tilde{f})$ respectively. Then, the following comparison principle holds :

$$\varepsilon \int_\Omega (\arctan(u_\varepsilon) - \arctan(\tilde{u}_\varepsilon))^+ \leq \int_\Omega (f - \tilde{f}) sign_0^+(u_\varepsilon - \tilde{u}_\varepsilon). \tag{3.3.14}$$

Proof : We use the test function $\varphi = H_\delta^+(u_\varepsilon - \tilde{u}_\varepsilon)$ in the weak formulation (3.3.13) for u_ε and $\tilde{u}_\varepsilon$. Subtracting the resulting inequalities, we obtain

$$I^1_{\varepsilon,\delta} + I^2_{\varepsilon,\delta} + I^3_{\varepsilon,\delta} + I^4_{\varepsilon,\delta} = I^5_{\varepsilon,\delta},$$

where

$$\begin{aligned}
I^1_{\varepsilon,\delta} &= \int_\Omega (\beta_\varepsilon(T_{\frac{1}{\varepsilon}}(u_\varepsilon)) - \beta_\varepsilon(T_{\frac{1}{\varepsilon}}(\tilde{u}_\varepsilon)) H_\delta^+(u_\varepsilon - \tilde{u}_\varepsilon),\\
I^2_{\varepsilon,\delta} &= \int_\Omega (\varepsilon \arctan(u_\varepsilon) - \varepsilon \arctan(\tilde{u}_\varepsilon)) H_\delta^+(u_\varepsilon - \tilde{u}_\varepsilon),\\
I^3_{\varepsilon,\delta} &= \int_\Omega (a(x, Du_\varepsilon) - a(x, D\tilde{u}_\varepsilon)) \cdot DH_\delta^+(u_\varepsilon - \tilde{u}_\varepsilon),\\
I^4_{\varepsilon,\delta} &= \int_\Omega (F(T_{\frac{1}{\varepsilon}}(u_\varepsilon)) - F(T_{\frac{1}{\varepsilon}}(\tilde{u}_\varepsilon))) \cdot DH_\delta^+(u_\varepsilon - \tilde{u}_\varepsilon),\\
I^5_{\varepsilon,\delta} &= \int_\Omega (f - \tilde{f}) \cdot H_\delta^+(u_\varepsilon - \tilde{u}_\varepsilon).
\end{aligned}$$

Passing to the limit with $\delta \downarrow 0$, (3.3.14) follows.

Remark 3.3.4 Let $f, \tilde{f} \in L^\infty(\Omega)$ be such that $f \leq \tilde{f}$ almost everywhere in Ω, $\varepsilon > 0$ and $u_\varepsilon, \tilde{u}_\varepsilon \in W_0^{1,p}(\Omega, \omega)$ solutions to $(\mathrm{Eq}_\varepsilon, f)$ and $(\mathrm{Eq}_\varepsilon, \tilde{f})$ respectively. Then it is an immediate consequence of Remark 3.3.3 that $u_\varepsilon \leq \tilde{u}_\varepsilon$ almost everywhere in Ω. Furthermore, from the monotonicity of $\beta_\varepsilon \circ T_{\frac{1}{\varepsilon}}$ it follows that also $\beta_\varepsilon(T_{\frac{1}{\varepsilon}}(u_\varepsilon)) \leq \beta_\varepsilon(T_{\frac{1}{\varepsilon}}(\tilde{u}_\varepsilon))$ almost everywhere in Ω.

Lemma 3.3.5 *For $0 < \varepsilon \leq 1$ and $f \in L^\infty(\Omega)$ let $u_\varepsilon \in W_0^{1,p}(\Omega,\omega)$ be a solution of (Eq_ε, f). Then,*

i) There exists a constant $C_1 = C_1(\|f\|_\infty, \lambda, p, N) > 0$, not depending on ε, such that

$$\||u_\varepsilon\|| \leq C_1. \tag{3.3.15}$$

ii) The expression

$$\|\beta_\varepsilon(T_{\frac{1}{\varepsilon}}(u_\varepsilon))\|_\infty \leq \|f\|_\infty. \tag{3.3.16}$$

holds.

iii) For all $l, k > 0$, we have

$$\int_{\{l<|u_\varepsilon|<l+k\}} a(x, Du_\varepsilon) \cdot Du_\varepsilon \leq k \int_{\{|u_\varepsilon|>l\}} |f|. \tag{3.3.17}$$

Proof : i) Taking u_ε as a test function in (3.3.13), we obtain

$$\int_\Omega \Big(\beta_\varepsilon(T_{\frac{1}{\varepsilon}}(u_\varepsilon)) + \varepsilon \arctan(u_\varepsilon)\Big) u_\varepsilon dx + \int_\Omega a(x, Du_\varepsilon) \cdot Du_\varepsilon dx$$
$$+ \int_\Omega F(T_{\frac{1}{\varepsilon}}(u_\varepsilon))) \cdot Du_\varepsilon dx = \int_\Omega f u_\varepsilon dx.$$

As the first term on the left-hand side is nonnegative and the integral over the convection term vanishes by (3.2.8), we have

$$\begin{aligned}
\lambda \sum_{i=1}^N \int_\Omega |\frac{\partial u_\varepsilon}{\partial x_i}|^p \omega_i(x) dx &\leq \sum_{i=1}^N \int_\Omega a_i(x, Du_\varepsilon) \cdot \frac{\partial u_\varepsilon}{\partial x_i} dx \\
&\leq \int_\Omega f u_\varepsilon dx \\
&\leq \|f\|_\infty \Big(\int_\Omega |u_\varepsilon|^q \sigma dx\Big)^{\frac{1}{q}} \cdot \Big(\int_\Omega \sigma^{1-q'} dx\Big)^{\frac{1}{q'}} \\
&\leq C\|f\|_\infty \Big(\sum_{i=1}^N \int_\Omega |\frac{\partial u_\varepsilon(x)}{\partial x_i}|^p \omega_i(x) dx\Big)^{\frac{1}{p}} \Big(\int_\Omega \sigma^{1-q'} dx\Big)^{\frac{1}{q'}} \text{(due to(3.2.4))}.
\end{aligned}$$

Then $|||u_\varepsilon|||^p \leq C_2 |||u_\varepsilon|||$ where C_2 a positive constant. Then, we can deduce that u_ε remains bounded in $W_0^{1,p}(\Omega,\omega)$i.e.,

$$|||u_\varepsilon||| \leq C_1.$$

ii) Taking $\frac{1}{\delta}[T_{k+\delta}(\beta_\varepsilon(T_{\frac{1}{\varepsilon}}(u_\varepsilon))) - T_k(\beta_\varepsilon(T_{\frac{1}{\varepsilon}}(u_\varepsilon)))]$ as a test function in (3.3.13), passing to the limit with $\delta \to 0$ and choosing $k > \|f\|_\infty$ we obtain (ii).
iii) For $k, l > 0$ fixed we take $T_k(u_\varepsilon - T_l(u_\varepsilon))$ as a test function in (3.3.13), we obtain

$$\int_\Omega (\beta_\varepsilon(T_{\frac{1}{\varepsilon}}(u_\varepsilon)))T_k(u_\varepsilon - T_l(u_\varepsilon))dx + \varepsilon \int_\Omega \arctan(u_\varepsilon) T_k(u_\varepsilon - T_l(u_\varepsilon))dx$$
$$+ \int_\Omega (a(x, Du_\varepsilon) \cdot DT_k(u_\varepsilon - T_l(u_\varepsilon))dx + \int_\Omega F(T_{\frac{1}{\varepsilon}}(u_\varepsilon))) \cdot D(T_k(u_\varepsilon - T_l(u_\varepsilon)))dx$$
$$= \int_\Omega f T_k(u_\varepsilon - T_l(u_\varepsilon))dx,$$

using $\displaystyle\int_\Omega a(x, Du_\varepsilon) \cdot DT_k(u_\varepsilon - T_l(u_\varepsilon))dx = \int_{\{l<|u_\varepsilon|<l+k\}} a(x, Du_\varepsilon) \cdot Du_\varepsilon dx.$

As the first term and the second on the left-hand side is nonnegative and the convection term vanishes we get

$$\int_{\{l<|u_\varepsilon|<l+k\}} a(x, Du_\varepsilon) \cdot Du_\varepsilon dx \leq \int_\Omega f T_k(u_\varepsilon - T_l(u_\varepsilon))dx \leq k \int_{\{|u_\varepsilon|>l\}} |f|dx.$$

Remark 3.3.6 For $k > 0$, from Lemma 3.3.5 iii) we deduce

$$|\{|u_\varepsilon| \geq l\}| \leq \frac{C_2}{l^{1-\frac{1}{p}}}. \tag{3.3.18}$$

$$\int_{\{l<|u_\varepsilon|<l+k\}} a(x, Du_\varepsilon) \cdot Du_\varepsilon \leq k\|f\|_\infty |\{|u_\varepsilon| > l\}| \leq \frac{C_2(k)}{l^{1-\frac{1}{p}}}. \tag{3.3.19}$$

Indeed, let $l > 0$ large enough we have :

$$l|\{|u_\varepsilon| \geq l\}| = \int_{\{|u_\varepsilon| \geq l\}} |T_l(u_\varepsilon)| \sigma^{\frac{1}{q}} \sigma^{-\frac{1}{q}} dx$$
$$\leq \Big(\int_\Omega |T_l(u_\varepsilon)|^q \sigma dx\Big)^{\frac{1}{q}} \Big(\int_\Omega \sigma^{1-q'} dx\Big)^{\frac{1}{q'}}$$
$$\leq C\Big(\sum_{i=1}^N \int_\Omega |\frac{\partial T_l(u_\varepsilon)}{\partial x_i}|^p \omega_i dx\Big)^{\frac{1}{p}} \leq C_2 l^{\frac{1}{p}} \ (\textit{Due to } (3.2.8))$$

which implies $|\{|u_\varepsilon| \geq l\}| \leq C_2 l^{\frac{1}{p}-1}$. So we have

$$\lim_{l \to +\infty} |\{|u_\varepsilon| \geq l\}| = 0.$$

Hence (2.3.10) provides (3.3.19).

Basic convergence results

In the following it is always understood that ε takes values in a sequence in $(0,1)$ tending to zero. The a priori estimates in Lemma 3.3.5 and Remark 3.3.6 imply the following basic convergences :

Lemma 3.3.7 *For $0 < \varepsilon \leq 1$ and $f \in L^\infty(\Omega)$, let $u_\varepsilon \in W_0^{1,p}(\Omega, \omega)$ be the solution of (Eq_ε, f). There exist $u \in W_0^{1,p}(\Omega, \omega)$, $b \in L^\infty(\Omega)$ such that for a not relabeled subsequence of $(u_\varepsilon)_{0<\varepsilon\leq 1}$ as $\varepsilon \downarrow 0$:*

$$u_\varepsilon \rightharpoonup u \quad in \quad W_0^{1,p}(\Omega, \omega) \quad and\ a.e.\ in \quad \Omega, \tag{3.3.20}$$

$$T_k(u_\varepsilon) \rightharpoonup T_k(u) \quad in \quad W_0^{1,p}(\Omega, \omega)\ and\ strongly\ in\ L^q(\Omega, \sigma), \tag{3.3.21}$$

$$\beta_\varepsilon(T_{\frac{1}{\varepsilon}}(u_\varepsilon)) \rightharpoonup b \quad in \quad L^\infty(\Omega). \tag{3.3.22}$$

$Moreover, for\ any \quad k > 0,$

$$DT_k(u_\varepsilon) \rightharpoonup DT_k(u) \quad in \quad \Pi_{i=1}^N L^p(\Omega, \omega_i), \tag{3.3.23}$$

$$a(x, DT_k(u_\varepsilon)) \rightharpoonup a(x, DT_k(u)) \quad in \quad \Pi_{i=1}^N L^{p'}(\Omega, \omega_i^*). \tag{3.3.24}$$

Proof : Since (3.3.22) follow directly from Lemma 3.3.5 and Remark 3.3.6.
From (3.3.18),(3.3.15) and (3.2.5), we deduce with classical argument (see,e.g.,[1]) that for a subsequence still indexed by ε (3.3.20)-(3.3.21) and (3.3.23) as ε tend to 0, where u is a measurable function defined on Ω which is function a.e. in Ω, (3.3.24) is left to prove .
To this end By (A2) and (3.3.15) it follows that given any subsequence of $a(x, DT_k(u_\varepsilon))_\varepsilon$, there exists a subsequence, still denoted by $a(x, DT_k(u_\varepsilon))_\varepsilon$ such that

$$a(x, DT_k(u_\varepsilon)) \rightharpoonup \Phi_k \quad \text{in} \ \Pi_{i=1}^N L^{p'}(\Omega, \omega_i^*).$$

Let us prove that $\Phi_k = a(x, DT_k(u))$ a.e. of Ω. The proof consists in three steps.
Step1 : For every function $h \in W^{1,\infty}(\mathbb{R})$, $h \geq 0$ with $supp(h)$ compact, we prove that,

$$\limsup_{\varepsilon \to 0} \int_\Omega a(x, DT_k(u_\varepsilon)) \cdot D[h(u_\varepsilon)(T_k(u_\varepsilon) - T_k(u))]dx \leq 0. \tag{3.3.25}$$

Taking $h(u_\varepsilon)(T_k(u_\varepsilon) - T_k(u))$ as test function in (3.3.13), we have

$$\begin{aligned} &\int_\Omega \Big(\beta_\varepsilon(T_{\frac{1}{\varepsilon}}(u_\varepsilon) + \varepsilon \arctan(u_\varepsilon)\Big) h(u_\varepsilon)(T_k(u_\varepsilon) - T_k(u)) \\ &+ \int_\Omega \Big(a(x, Du_\varepsilon) + F(T_{\frac{1}{\varepsilon}}(u_\varepsilon))\Big) \cdot D[h(u_\varepsilon)(T_k(u_\varepsilon) - T_k(u))] \\ &\qquad = \int_\Omega f h(u_\varepsilon)(T_k(u_\varepsilon) - T_k(u)). \end{aligned} \tag{3.3.26}$$

Using

$$|h(u_\varepsilon)(T_k(u_\varepsilon) - T_k(u))| \leq 2k\|h\|_\infty.$$

Since by the generalized dominated convergence theorem we have

$$\lim_{\varepsilon \to 0} \int_\Omega f h(u_\varepsilon)(T_k(u_\varepsilon) - T_k(u)) = 0.$$

In addition we see that $h(u_\varepsilon)(T_k(u_\varepsilon) - T_k(u)) \rightharpoonup 0$ weakly in $W_0^{1,p}(\Omega, \omega)$. Indeed the sequence $\big(h(u_\varepsilon)(T_k(u_\varepsilon) - T_k(u))\big)_{\varepsilon > 0}$ is bounded in $W_0^{1,p}(\Omega, \omega)$ and converge to zero almost everywhere on Ω. This implie that

$$\lim_{\varepsilon \to 0} \int_\Omega F(T_{\frac{1}{\varepsilon}}(u_\varepsilon)) \cdot D[h(u_\varepsilon)(T_k(u_\varepsilon) - T_k(u))] = 0.$$

By using the same arguments in [25] we can prove that

$$\limsup_{\varepsilon \to 0} \int_\Omega \beta_\varepsilon(T_{\frac{1}{\varepsilon}}(u_\varepsilon)) \cdot [h(u_\varepsilon)(T_k(u_\varepsilon) - T_k(u))]dx \geq 0.$$

Passing the limit in (3.3.26) and using the above results, we obtain the inequality (3.3.25).
Step2 : We prove that for every $k > 0$,

$$\limsup_{\varepsilon \to 0} \int_\Omega a(x, DT_k(u_\varepsilon)) \cdot [DT_k(u_\varepsilon) - DT_k(u)]dx \leq 0. \tag{3.3.27}$$

Indeed : For $k > l$, and take $h_l(u_\varepsilon)(T_k(u_\varepsilon) - T_k(u))$ as a test function in (3.3.13) and passing to the limit whit $\varepsilon \downarrow 0$ and then with $l \to \infty$, we obtain

$$\begin{aligned}
&\int_\Omega a(x, DT_k(u_\varepsilon)) \cdot D[h_l(u_\varepsilon)(T_k(u_\varepsilon) - T_k(u))]dx \\
&= \int_{[|u_\varepsilon| \le k]} h_l(u_\varepsilon) a(x, DT_k(u_\varepsilon)) \cdot [DT_k(u_\varepsilon) - DT_k(u)]dx \\
&+ \int_{[|u_\varepsilon| > k]} h_l(u_\varepsilon) a(x, DT_k(u_\varepsilon)) \cdot (-DT_k(u))dx \\
&+ \int_\Omega h_l^{'}(u_\varepsilon)(T_k(u_\varepsilon) - T_k(u)) a(x, DT_k(u_\varepsilon)) \cdot Du_\varepsilon dx \\
&= E_1 + E_2 + E_3.
\end{aligned}$$

Since $l > k$, on the set $[|u_\varepsilon| \le k]$ we have $h_l(u_\varepsilon) = 1$ so that we can write E_1 as

$$\begin{aligned}
E_1 &= \int_{[|u_\varepsilon| \le k]} h_l(u_\varepsilon) a(x, DT_k(u_\varepsilon)) \cdot [DT_k(u_\varepsilon) - DT_k(u)]dx \\
&= \int_\Omega a(x, DT_k(u_\varepsilon)) \cdot (DT_k(u_\varepsilon) - DT_k(u))dx,
\end{aligned}$$

hence we obtain

$$\limsup_{\varepsilon \to 0} E_1 = \limsup_{\varepsilon \to 0} \int_\Omega a(x, DT_k(u_\varepsilon)) \cdot (DT_k(u_\varepsilon) - DT_k(u))dx.$$

Let us write the term E_2 as

$$E_2 = -\int_{[|u_\varepsilon| > k]} h_l(u_\varepsilon) a(x, DT_{l+1}(u_\varepsilon)) \cdot DT_k(u)dx$$

Using Lebesgue dominated convergence theorem we get

$$\lim_{\varepsilon \to 0} E_2 = -\int_{[|u| > k]} h_l(u) \Phi_{l+1} \cdot DT_k(u)dx = 0.$$

For the term E_3, we have

$$\begin{aligned}
&-(\int_\Omega h_l^{'}(u_\varepsilon)(T_k(u_\varepsilon) - T_k(u)) a(x, DT_k(u_\varepsilon)) \cdot Du_\varepsilon dx) \\
&\le |\int_\Omega h_l^{'}(u_\varepsilon)(T_k(u_\varepsilon) - T_k(u)) a(x, Du_\varepsilon) \cdot Du_\varepsilon dx| \\
&\le 2k \int_{[l < |u_\varepsilon| < l+1]} a(x, Du_\varepsilon) \cdot Du_\varepsilon dx.
\end{aligned}$$

Using (3.3.19), we deduce that

$$\limsup_{l\to+\infty}\limsup_{\varepsilon\to 0}(-\int_\Omega h_l^{'}(u_\varepsilon)(T_k(u_\varepsilon)-T_k(u))a(x,DT_k(u_\varepsilon))\cdot Du_\varepsilon dx \leq 0.$$

Applying (3.3.25) whit h replaced by h_l, $l>k$ we get

$$\begin{aligned}&\limsup_{\varepsilon\to 0}\int_\Omega a(x,DT_k(u_\varepsilon))\cdot(DT_k(u_\varepsilon)-DT_k(u))dx\\ &\leq \limsup_{\varepsilon\to 0}(-\int_\Omega h_l^{'}(u_\varepsilon)(T_k(u_\varepsilon)-T_k(u))a(x,DT_k(u_\varepsilon))\cdot Du_\varepsilon dx.\end{aligned}$$

So that $l\to+\infty$ yields the inequality (3.3.27).

Step3 : In this step we prove by monotonicity arguments that for $k>0$, $\Phi_k = a(x,DT_k(u))$ for almost every $x\in\Omega$. Let $\varphi\in D(\Omega)$ and $\tilde{\alpha}\in\mathbb{R}$. Using (3.3.27), we have

$$\begin{aligned}&\tilde{\alpha}\lim_{\varepsilon\to 0}\int_\Omega a(x,DT_k(u_\varepsilon))\cdot D\varphi dx\\ &\geq \limsup_{\varepsilon\to 0}\int_\Omega a(x,DT_k(u_\varepsilon))\cdot[DT_k(u_\varepsilon)-DT_k(u)+D(\tilde{\alpha}\varphi)]dx\\ &\geq \limsup_{\varepsilon\to 0}\int_\Omega a(x,D(T_k(u)-\tilde{\alpha}\varphi))\cdot[DT_k(u_\varepsilon)-DT_k(u)+D(\tilde{\alpha}\varphi)]dx\\ &\geq \limsup_{\varepsilon\to 0}\int_\Omega a(x,D(T_k(u)-\tilde{\alpha}\varphi))\cdot D(\tilde{\alpha}\varphi)]dx\\ &\geq \tilde{\alpha}\int_\Omega a(x,D(T_k(u)-\tilde{\alpha}\varphi))\cdot D\varphi dx.\end{aligned}$$

Dividing by $\tilde{\alpha}>0$ and by $\tilde{\alpha}<0$, pasing the limit with $\tilde{\alpha}\to 0$ we obtain

$$\lim_{\varepsilon\to 0}\int_\Omega a(x,DT_k(u_\varepsilon))\cdot D\varphi dx = \int_\Omega a(x,DT_k(u))\cdot D\varphi dx.$$

This means that $\forall k>0$, $\int_\Omega \Phi_k\cdot D\varphi dx = \int_\Omega a(x,DT_k(u))\cdot D\varphi dx$ and then

$$\Phi_k = a(x,DT_k(u)) \quad \text{in} \quad D^{'}(\Omega)$$

for all $k>0$. Hence $\Phi_k = a(x,DT_k(u))$ a.e. in Ω and so $a(x,DT_k(u_\varepsilon)) \rightharpoonup a(x,DT_k(u))$ weakly in $\Pi_{i=1}^N L^{p^{'}}(\Omega,\omega_i^*)$.

Remark 3.3.8 As immediate consequence of (3.3.27) and (A3) we obtain

$$\lim_{\varepsilon\to 0}\int_\Omega a(x, DT_k(u_\varepsilon)) - a(x, DT_k(u))\cdot(DT_k(u_\varepsilon) - DT_k(u)) = 0. \tag{3.3.28}$$

Lemma 3.3.9 The limit u of the approximate solution u_ε of $(\mathrm{Eq}_\varepsilon, f)$ satisfies

$$\lim_{l\to\infty}\int_{\{l<|u|<l+1\}} a(x, Du)\cdot Du dx = 0. \tag{3.3.29}$$

Proof : To this end, observe that for any fixed $l \geq 0$, one has

$$\begin{aligned}\int_{\{l<|u_\varepsilon|<l+1\}} a(x, Du_\varepsilon)\cdot Du_\varepsilon dx &= \int_\Omega a(x, Du_\varepsilon)\cdot(DT_{l+1}(u_\varepsilon) - DT_l(u_\varepsilon))dx \\ = \int_\Omega a(x, DT_{l+1}(u_\varepsilon))\cdot DT_{l+1}(u_\varepsilon)dx &- \int_\Omega a(x, DT_l(u_\varepsilon))\cdot DT_l(u_\varepsilon))dx.\end{aligned}$$

According (3.3.28) is at liberty passing the limit as $\varepsilon \to 0$ for fixed $l \geq 0$ and to obtain

$$\begin{aligned}&\lim_{\varepsilon\to 0}\int_{\{l<|u_\varepsilon|<l+1\}} a(x, Du_\varepsilon)\cdot Du_\varepsilon dx \\ &= \int_\Omega a(x, DT_{l+1}(u))\cdot DT_{l+1}(u)dx - \int_\Omega a(x, DT_l(u))\cdot DT_l(u))dx \\ &= \int_{\{l<|u|<l+1\}} a(x, Du)\cdot Du dx.\end{aligned} \tag{3.3.30}$$

Taking the limit as $l \to +\infty$ in (3.3.30) and using the estimate (3.3.19) show that u satisfies (R3) and the proof of the lemma is complete.

Conclusion of the proof of Theorem 3.3.1

Now, we are able to conclude the proof of Theorem 3.3.1

Proof : Let $h \in \mathcal{C}_c^1(\mathbb{R})$ and $\phi \in W_0^{1,p}(\Omega, \omega) \cap L^\infty(\Omega)$ be arbitrary. Taking $h_l(u_\varepsilon)h(u)\phi$ as a test function in (3.3.13), we obtain

$$I^1_{\varepsilon,l} + I^2_{\varepsilon,l} + I^3_{\varepsilon,l} + I^4_{\varepsilon,l} = I^5_{\varepsilon,l}, \tag{3.3.31}$$

where

$$I^1_{\varepsilon,l} = \int_\Omega \beta_\varepsilon(T_{\frac{1}{\varepsilon}}(u_\varepsilon))h_l(u_\varepsilon)h(u)\phi,$$
$$I^2_{\varepsilon,l} = \varepsilon \int_\Omega \arctan(u_\varepsilon)h_l(u_\varepsilon)h(u)\phi,$$
$$I^3_{\varepsilon,l} = \int_\Omega a(x, Du_\varepsilon) \cdot D(h_l(u_\varepsilon)h(u)\phi),$$
$$I^4_{\varepsilon,l} = \int_\Omega F(T_{\frac{1}{\varepsilon}}(u_\varepsilon)) \cdot D(h_l(u_\varepsilon)h(u)\phi),$$
$$I^5_{\varepsilon,l} = \int_\Omega f h_l(u_\varepsilon)h(u)\phi.$$

Step 1 : Passing to the limit as $\varepsilon \downarrow 0$ obviously,

$$\lim_{\varepsilon\to 0} I^2_{\varepsilon,l} = 0. \tag{3.3.32}$$

Using the convergence results (3.3.20), (3.3.22) from Lemma 3.3.7 we can immediately calculate the following limits :

$$\lim_{\varepsilon\to 0} I^1_{\varepsilon,l} = \int_\Omega b h_l(u)h(u)\phi, \tag{3.3.33}$$
$$\lim_{\varepsilon\to 0} I^5_{\varepsilon,l} = \int_\Omega f h_l(u)h(u)\phi. \tag{3.3.34}$$

We write

$$I^3_{\varepsilon,l} = I^{3,1}_{\varepsilon,l} + I^{3,2}_{\varepsilon,l},$$

where

$$I^{3,1}_{\varepsilon,l} = \int_\Omega h'_l(u_\varepsilon)a(x, Du_\varepsilon) \cdot Du_\varepsilon h(u)\phi,$$
$$I^{3,2}_{\varepsilon,l} = \int_\Omega h_l(u_\varepsilon)a(x, Du_\varepsilon) \cdot D(h(u)\phi).$$

Using (3.3.19) we get the estimate

$$|\lim_{\varepsilon\to 0} I^{3,1}_{\varepsilon,l}| \le \|h\|_\infty \|\phi\|_\infty \cdot C_2 l^{-(1-\frac{1}{p})}. \tag{3.3.35}$$

By Lebesgue's Dominated Convergence Theorem it follows that for any $i \in \{1, ..., N\}$, we have

$$h_l(u_\varepsilon)\frac{\partial}{\partial x_i}(h(u)\phi) \to h_l(u)\frac{\partial}{\partial x_i}(h(u)\phi) \textit{ in } L^p(\Omega, \sigma) \textit{ as } \varepsilon \downarrow 0.$$

Keeping in mind that

$$I^{3,2}_{\varepsilon,l} = \int_\Omega h_l(u_\varepsilon) a(x, DT_{l+1}(u_\varepsilon)) \cdot D(h(u)\phi),$$

by (3.3.24), we get

$$\lim_{\varepsilon\to 0} I^{3,2}_{\varepsilon,l} = \int_\Omega h_l(u) a(x, DT_{l+1}(u)) \cdot D(h(u)\phi). \tag{3.3.36}$$

Let us write

$$I^{4}_{\varepsilon,l} = I^{4,1}_{\varepsilon,l} + I^{4,2}_{\varepsilon,l},$$

where

$$\begin{aligned} I^{4,1}_{\varepsilon,l} &= \int_\Omega h'_l(u_\varepsilon) F(T_{\frac{1}{\varepsilon}}(u_\varepsilon)) \cdot Du_\varepsilon h(u)\phi, \\ I^{4,2}_{\varepsilon,l} &= \int_\Omega h_l(u_\varepsilon) F(T_{\frac{1}{\varepsilon}}(u_\varepsilon)) \cdot D(h(u)\phi). \end{aligned}$$

For any $l \in \mathbb{N}$, there exists $\varepsilon_0(l)$; such that for all $\varepsilon < \varepsilon_0(l)$,

$$I^{4,1}_{\varepsilon,l} = \int_\Omega h'_l(T_{l+1}(u_\varepsilon)) F(T_{l+1}(u_\varepsilon)) \cdot DT_{l+1}(u_\varepsilon) h(u)\phi. \tag{3.3.37}$$

Using Gauss-Green Theorem for Sobolev functions in (3.3.37) we get

$$I^{4,1}_{\varepsilon,l} = -\int_\Omega \int_0^{T_{l+1}(u_\varepsilon)} h'_l(r) F(r) dr \cdot D(h(u)\phi). \tag{3.3.38}$$

Now, using (3.3.20) and the Gauss-Green Theorem, after the passing the limit as $\varepsilon \downarrow 0$ we get

$$\lim_{\varepsilon\to 0} I^{4,1}_{\varepsilon,l} = \int_\Omega h'_l(u) F(u) \cdot Du\; h(u)\phi. \tag{3.3.39}$$

Choosing ε small enough, we can write

$$I^{4,2}_{\varepsilon,l} = \int_\Omega h_l(u_\varepsilon) F(T_{l+1}(u_\varepsilon)) \cdot D(h(u)\phi) \tag{3.3.40}$$

and conclude

$$\lim_{\varepsilon\to 0} I^{4,2}_{\varepsilon,l} = \int_\Omega h_l(u) F(u) \cdot D(h(u)\phi). \tag{3.3.41}$$

Step 2 : Passing to the limit as $l \to \infty$. Combining (3.3.31) and (3.3.32)- (3.3.41) we find

$$I_l^1 + I_l^2 + I_l^3 + I_l^4 + I_l^5 = I_l^6, \tag{3.3.42}$$

where

$$\begin{aligned}
I_l^1 &= \int_\Omega b h_l(u) h(u)\phi, \\
I_l^2 &= \int_\Omega h_l(u) a(x, DT_{l+1}(u)) \cdot D(h(u)\phi), \\
|I_l^3| &\leq C_2 l^{-(1-\frac{1}{p})} \|h\|_\infty \|\phi\|_\infty, \\
I_l^4 &= \int_\Omega h_l(u) F(u) \cdot D(h(u)\phi), \\
I_l^5 &= \int_\Omega h_l'(u) F(u) \cdot Du\ h(u)\phi, \\
I_l^6 &= \int_\Omega f\ h_l(u) h(u)\phi.
\end{aligned}$$

Obviously, we have

$$\lim_{l\to\infty} I_l^3 = 0. \tag{3.3.43}$$

Choosing $m > 0$ such that *supp* $h \subset [-m, m]$, we can replace u by $T_m(u)$ in $I_l^1, I_l^2, ..., I_l^6$ and $h_l'(u) = h_l'(T_m(u)) = 0$ if $l + 1 > m$ and $h_l(u) = h_l(T_m(u)) = 1$,if $l > m$. Therefore, it follows that

$$\lim_{l\to\infty} I_l^1 = \int_\Omega b h(u)\phi, \tag{3.3.44}$$

$$\lim_{l\to\infty} I_l^2 = \int_\Omega a(x, Du) \cdot D(h(u)\phi), \tag{3.3.45}$$

$$\lim_{l\to\infty} I_l^4 = \int_\Omega F(u) \cdot D(h(u)\phi), \tag{3.3.46}$$

$$\lim_{l\to\infty} I_l^5 = 0, \tag{3.3.47}$$

$$\lim_{l\to\infty} I_l^6 = \int_\Omega f h(u)\phi. \tag{3.3.48}$$

Combining (3.3.42)) with (3.3.43)) - (3.3.48) we obtain

$$\int_\Omega b h(u)\phi + \int_\Omega (a(x, Du) + F(u)) \cdot D(h(u)\phi) = \int_\Omega f h(u)\phi, \tag{3.3.49}$$

for all $h \in C_c^1(\mathbb{R})$ and all $\phi \in W_0^{1,p}(\Omega, w) \cap L^\infty(\Omega)$.

Step3 : Subdifferential argument

It remains to prove that $u(x) \in D(\beta(x))$ and $b(x) \in \beta(u(x))$ for almost all $x \in \Omega$. Since β a is maximal monotone graph, there exists a convex; l.s.c. and proper function $j : \mathbb{R} \to [0, \infty]$, such that

$$\beta(r) = \partial j(r) \quad \text{for all } r \in \mathbb{R}.$$

According to [51], for $0 < \varepsilon \leq 1$, $j_\varepsilon : \mathbb{R} \to \mathbb{R}$ defined by $j_\varepsilon(r) = \int_0^r \beta_\varepsilon(s)ds$ has the following properties :

i) For any $0 < \varepsilon \leq 1$, j_ε is convex and differentiable for all $r \in \mathbb{R}$, such that

$$j'_\varepsilon(r) = \beta_\varepsilon(r) \quad \text{for all } r \in \mathbb{R} \quad \text{and any } 0 < \varepsilon \leq 1.$$

ii) $j_\varepsilon(r) \uparrow j(r)$ pointwise in $\mathbb{R}$ as $\varepsilon \to 0$

from i) it follows that for any $0 < \varepsilon \leq 1$

$$j_\varepsilon(r) \geq j_\varepsilon(T_{\frac{1}{\varepsilon}}(u_\varepsilon)) + (r - T_{\frac{1}{\varepsilon}}(u_\varepsilon))\beta_\varepsilon(T_{\frac{1}{\varepsilon}}(u_\varepsilon)) \tag{3.3.50}$$

holds for all $r \in \mathbb{R}$ and almost everywhere in Ω. Let $E \subset \Omega$ be an arbitrary measurable set and χ_E its characteristic function. We fix $\varepsilon_0 > 0$. Multiplying (3.3.50) by $h_l(u_\varepsilon)\chi_E$, integrating over Ω and using ii), we obtain

$$j(r)\int_E h_l(u_\varepsilon)dx \geq \int_E j_{\varepsilon_0}((T_{l+1}(u_\varepsilon))h_l(u_\varepsilon) + (r - T_{l+1}(u_\varepsilon))h_l(u_\varepsilon)\beta_\varepsilon(T_{\frac{1}{\varepsilon}}(u_\varepsilon))dx \tag{3.3.51}$$

for all $r \in \mathbb{R}$ and all $0 < \varepsilon < \min(\varepsilon_0, \frac{1}{l})$. As $\varepsilon \to 0$, taking into account that E is arbitrary we obtain from (3.3.51)

$$j(r)h_l(u) \geq j_{\varepsilon_0}(T_{l+1}(u))h_l(u) + bh_l(u)(r - T_{l+1}(u)) \tag{3.3.52}$$

for all $r \in \mathbb{R}$ almost everywhere in Ω. Passing to the limit with $l \to \infty$ and then with $\varepsilon_0 \to 0$ in (3.3.52) finally yields

$$j(r) \geq j(u(x)) + b(x)(r - u(x)) \tag{3.3.53}$$

for all $r \in \mathbb{R}$ and almost every $x \in \Omega$, hence $u \in D(\beta)$ and $b \in \beta(u)$ almost everywhere in Ω.With this last step the prof of Thoerem 3.3.1 is concluded.

Next, we plug $\frac{1}{k}T_k(u_\varepsilon)h_l(u_\varepsilon)$ as a test function into $(\mathrm{Eq}_\varepsilon, f_\varepsilon)$. Applying Remark 3.3.6

on the diffusion term, the Stokes Theorem on the convection term and neglecting nonnegative terms we can pass the limit as $\varepsilon \to 0$ and obtain :

$$\int_\Omega b\frac{1}{k}T_k(u)h_l dx \leq \int_\Omega |f|dx. \tag{3.3.54}$$

Thanks to (3.3.53) a.e. in Ω for $l \to \infty$ therefore, passing the limit as $k \downarrow 0$ in (3.3.54) and using the Fatou Lemma we find

$$\int_\Omega |b|dx \leq \|f\|_{L^1(\Omega)} \tag{3.3.55}$$

and $b \in L^1(\Omega)$.

3.4 Case where $f \in L^1(\Omega)$ -data

In this section we establish the existence and uniqueness of renormalized solution to the degenerated problem (Eq, f) with $f \in L^1(\Omega)$.

3.4.1 Results of existence and uniqueness

Theorem 3.4.1 *Under the assumptions* $(H_1)-\ (H_3)$ *and* $f \in L^1(\Omega)$ *there exists at least one renormalized solution* (u, b) *to the degenerated problem* (Eq, f).

Theorem 3.4.2 *Let the assumptions of Theorem 3.4.1 be satisfied. Moreover, let* $\beta : \mathbb{R} \to 2^{\mathbb{R}}$ *be such that is strictly monotone for almost every* $x \in \Omega$, *for* $f \in L^1(\Omega)$ *let* (u, b), $(\tilde{u}, \tilde{b})$ *be renormalized solution to* (Eq, f), *then* $u = \tilde{u}$ *and* $b = \tilde{b}$.

3.4.2 Proof of theorem 3.4.1

Approximate problem and a priori estimates

To prove Theorem 3.4.1, we will introduce and solve approximating problems. To this end, for $f \in L^1(\Omega)$, and $n, m \in \mathbb{N}$ we define $f_{m,n} : \Omega \to \mathbb{R}$ by

$$f_{m,n} = \max(\min(f(x), m), -n)$$

for almost every $x \in \Omega$, clearly $f_{m,n} \in L^{\infty}(\Omega)$ for each $m, n \in \mathbb{N}$, $|f_{m,n}(x)| \leq |f(x)|$ a.e. in Ω hence

$$\lim_{n\to\infty}\lim_{m\to\infty} f_{m,n} = f \text{ in } L^1(\Omega) \text{ for almost everywhere in } \Omega.$$

The comparison principle from Remark 3.3.3 will be the main tool in the second approximation procedure. For $f \in L^1(\Omega)$ and $m, n \in \mathbb{N}$, let $f_{m,n} \in L^{\infty}(\Omega)$ be defined as above. From Theorem 3.3.1 it follows that for any $m, n \in \mathbb{N}$ there exists $u_{m,n} \in W_0^{1,p}(\Omega, \omega)$, $b_{m,n} \in L^{\infty}(\Omega)$ such that $(u_{m,n}, b_{m,n})$ is a renormalized solution of $(\mathrm{Eq}, f_{m,n})$. Therefore

$$\int_\Omega b_{m,n} h(u_{m,n})\phi + \int_\Omega \big(a(x, Du_{m,n}) + F(u_{m,n})\big)\cdot D(h(u_{m,n})\phi) = \int_\Omega f_{m,n} h(u_{m,n})\phi \tag{3.4.56}$$

holds for all $m, n \in \mathbb{N}, h \in C_c^1(\mathbb{R})$, $\phi \in W_0^{1,p}(\Omega, \omega) \cap L^{\infty}(\Omega)$.
In the next Lemma, we give a priori estimates that will be important in the following :

Lemma 3.4.3 *For $m, n \in \mathbb{N}$ let $(u_{m,n}, b_{m,n})$ be a renormalized solution of $(Eq, f_{m,n})$. Then,*

i) For any $k > 0$ we have

$$\sum_{i=1}^{N} \int_\Omega |\frac{\partial T_k(u_{m,n})}{\partial x_i}|^p \omega_i dx \leq \frac{k}{\lambda}\|f\|_1. \tag{3.4.57}$$

ii) For $k > 0$, there exists a constant $C_3(k) > 0$, not depending on $m, n \in \mathbb{N}$, such that

$$\sum_{i=1}^{N} \int_\Omega |\frac{\partial T_k(u_{m,n})}{\partial x_i}|^p \omega_i dx \leq C_3(k). \tag{3.4.58}$$

iii) For all $m, n \in \mathbb{N}$, we have

$$\|b_{m,n}\|_1 \leq \|f\|_1. \tag{3.4.59}$$

Proof : For $l, k > 0$, we plug $h_l(u_{m,n})T_k(u_{m,n})$ as a test function in (3.4.56). Then i) and ii) follow with similar arguments as used in the proof of Lemma3.3.5. To prove iii), we neglect the positive term $\int_\Omega a(x, DT_k(u_{m,n}))DT_k(u_{m,n})$ and keep

$$\int_\Omega b_{m,n} T_k(u_{m,n}) \leq \int_\Omega f_{m,n} u_{m,n}. \tag{3.4.60}$$

Since $b_{m,n} \in \beta(u_{m,n})$ a.e. in Ω, from (3.4.60) it follows that

$$\int_{\{|u_{m,n}|>k\}} |b_{m,n}| \leq \int_\Omega |f| \tag{3.4.61}$$

and we find iii) by passing the limit as $k \downarrow 0$.
By definition we have

$$f_{m,n} \leq f_{m+1,n} \text{ and } f_{m,n+1} \leq f_{m,n}. \tag{3.4.62}$$

From Remark 3.3.3 it follows that

$$u^\varepsilon_{m,n} \leq u^\varepsilon_{m+1,n} \text{ and } u^\varepsilon_{m,n+1} \leq u^\varepsilon_{m,n} \tag{3.4.63}$$

almost everywhere in Ω for any $m, n \in \mathbb{N}$ and all $\varepsilon > 0$, hence passing the limit as $\varepsilon \downarrow 0$ in (3.4.63) yields

$$u_{m,n} \leq u_{m+1,n} \text{ and } u_{m,n+1} \leq u_{m,n} \tag{3.4.64}$$

almost everywhere in Ω for any $m, n \in \mathbb{N}$. Setting $b_\varepsilon = \beta_\varepsilon(T_{\frac{1}{\varepsilon}}(u_\varepsilon))$ using (3.4.63), Remark 3.3.4 and the fact that $b^\varepsilon_{m,n} \rightharpoonup b_{m,n}$ in $L^\infty(\Omega)$ and this convergence preserves order we get

$$b_{m,n} \leq b_{m+1,n} \text{ and } b_{m,n+1} \leq b_{m,n} \tag{3.4.65}$$

almost everywhere in Ω for any $m, n \in \mathbb{N}$. By (3.4.65) and (3.4.59), for any $n \in \mathbb{N}$ there exists $b^n \in L^1(\Omega)$ such that $b_{m,n} \to b^n$ as $m \to \infty$ in $L^1(\Omega)$ and almost everywhere in Ω and $b \in L^1(\Omega)$, such that $b^n \to b$ as $n \to \infty$ in $L^1(\Omega)$ and almost everywhere in Ω. By (3.4.64), the sequence $(u_{m,n})_m$ is monotone increasing, hence, for any $n \in \mathbb{N}$,$u_{m,n} \to u^n$ almost everywhere in Ω, where $u^n : \Omega \to \overline{\mathbb{R}}$ is a measurable function. Using (3.4.64) again, we conclude that the sequence $(u^n)_n$ is monotone decreasing, hence $u^n \to u$ almost everywhere in Ω, where $u : \Omega \to \overline{\mathbb{R}}$ is a measurable function.
In order to show that u is finite almost everywhere we will give an estimate on the level sets of $u_{m,n}$ in the next lemma :

Lemma 3.4.4 *For* $m, n \in \mathbb{N}$ *let* $(u_{m,n}, b_{m,n})$ *be a renormalized solution of* $(Eq, f_{m,n})$. *Then, there exist a constant* $C_4 > 0$, *not depending on* $m, n \in \mathbb{N}$, *such that*

$$|\{|u_{m,n}| \geq l\}| \leq C_4 l^{\frac{1}{p}-1} \tag{3.4.66}$$

for all $l \geq 1$.

Proof : With the same arguments as in Remark 3.3.6 we obtain

$$l|\{|u_{m,n}| \geq l\}| \leq C\Big(\sum_{i=1}^{N}\int_{\Omega}|\frac{\partial T_l(u_{m,n})}{\partial x_i}|^p \omega_i dx\Big)^{\frac{1}{p}}\Big(\int_{\Omega}\sigma^{1-q'}\Big)^{\frac{1}{q'}} \tag{3.4.67}$$

for all $m, n \in \mathbb{N}$ where C is the constant from embedding (1.1.7). Now, we plug (3.4.57) into (3.4.67) to obtain (3.4.66). Note that, as $(u_{m,n})_m$ is pointwise increasing with respect to m,

$$\lim_{m\to\infty}|\{|u_{m,n}| \geq l\}| = |\{|u^n| \geq l\}|, \tag{3.4.68}$$

and

$$\lim_{m\to\infty}|\{|u_{m,n}| \geq -l\}| = |\{|u^n| \geq -l\}|. \tag{3.4.69}$$

Combining (3.4.66) with (3.4.68) and (3.4.69), we get

$$|\{|u^n| \geq l\}| + |\{|u^n| \geq -l\}| \leq C_4 l^{\frac{1}{p}-1} \tag{3.4.70}$$

for any $l \geq 1$, hence u^n is finite almost everywhere for any $n \in \mathbb{N}$. By the same arguments we get

$$|\{|u| \geq l\}| + |\{|u| \geq -l\}| \leq C_4 l^{\frac{1}{p}-1} \tag{3.4.71}$$

from (3.4.70), hence u is finite almost everywhere. Now, since $b_{m,n} \in \beta(u_{m,n})$ almost everywhere in Ω it follows by a subdifferential argument that $b^n \in \beta(u^n)$ and $b \in \beta(u)$ almost everywhere in Ω.

Remark 3.4.5 *The expression*

$$\int_{\{l<|u_{m,n}|<l+k\}} a(x, Du_{m,n}) \cdot Du_{m,n} \leq k\Big(\int_{\{|u_{m,n}|>l\}\cap\{|f|<\delta\}}|f| + \int_{\{|f|>\delta\}}|f|\Big) \tag{3.4.72}$$

for any $k, l, \delta > 0$. Now, applying (3.4.66) to (3.4.72) , we find that

$$\int_{\{l<|u_{m,n}|<l+k\}} a(x, Du_{m,n}) \cdot Du_{m,n} \leq k\delta C_4 l^{\frac{1}{p}-1} + k \int_{\{|f|>\delta\}} |f|) \qquad (3.4.73)$$

holds for any $k, \delta > 0$, $l \geq 1$ uniformly in $m, n \in \mathbb{N}$.

Basic convergence

Lemma 3.4.6 *For $m, n \in \mathbb{N}$ let $(u_{m,n}, b_{m,n})$ be a renormalized solution of $(Eq, f_{m,n})$. There exists a subsequence $(m(n))_n$ such that setting $f_n := f_{m(n),n}$, $b_n := b_{m(n),n}$, $u_n := u_{m(n),n}$, we have*

$$u_n \to u \quad \textit{almost everywhere in}\, \Omega. \qquad (3.4.74)$$

Moreover, for any $k > 0$,

$$T_k(u_n) \to T_k(u) \quad \textit{in} \quad W_0^{1,p}(\Omega, \omega), \qquad (3.4.75)$$

$$DT_k(u_n) \rightharpoonup DT_k(u) \quad \textit{in} \quad \prod_{i=1}^{N} L^p(\Omega, \omega_i), \qquad (3.4.76)$$

$$a(x, DT_k(u_n)) \rightharpoonup a(x, DT_k(u)) \quad \textit{in} \quad \prod_{i=1}^{N} L^{p'}(\Omega, \omega_i^*). \qquad (3.4.77)$$

as $n \to \infty$.

Proof : We construct a subsequence $(m(n))_n$, such that

$$\arctan(u_{m(n),n}) \to \arctan(u),\ b_n := b_{m(n),n} \to b,\ f_n := f_{m(n),n} \to f$$

as $n \to \infty$ in $L^1(\Omega)$ and almost everywhere in Ω. It follows that (3.4.74) and (3.4.75) hold. Combining (3.4.75) with (3.4.58), we get $T_k(u) \in W_0^{1,p}(\Omega, \omega)$, $T_k(u_n) \to T_k(u)$ in $W_0^{1,p}(\Omega, \omega)$ and (3.4.76)) holds for any $k > 0$. From (3.4.57) and (A2) it follows that, for fixed $k > 0$, given any subsequence of $(a(x, DT_k(u_n)))_n$, there exists a subsequence, still denoted by $a(x, DT_k(u_n))_n$, such that

$$a(x, DT_k(u_n)))_n \rightharpoonup \Phi_k \textit{ in } \quad \prod_{i=1}^{N} L^{p'}(\Omega, \omega_i^*).$$

as $n \to \infty$. Since $h_l(u_n)(T_k(u_n) - T_k(u))$ is an admissible test function in (3.4.56), we obtain

$$\limsup_{n\to\infty} \int_\Omega a(x, DT_k(u_n)) \cdot D(T_k(u_n) - T_k(u)) \le 0 \tag{3.4.78}$$

holds. Then, (3.4.77) follows with the same arguments as in the proof of Lemma 3.3.7 .

Remark 3.4.7 *With the same arguments as in Remark 3.3.8 and Lemma 3.3.9, we have*

$$\lim_{n\to\infty} \int_\Omega \Big(a(x, DT_k(u_n)) - a(x, DT_k(u))\Big) \cdot D(T_k(u_n) - T_k(u)) = 0. \tag{3.4.79}$$

$$\lim_{l\to\infty} \int_{\{l<|u|<l+1\}} a(x, Du) \cdot Du = 0. \tag{3.4.80}$$

Conclusion of the proof of Theorem 3.4.1

It is left to prove that (u, b) satisfies

$$\int_\Omega bh(u)\phi + \int_\Omega \big(a(x, Du) + F(u)\big) \cdot D(h(u)\phi) = \int_\Omega fh(u)\phi, \tag{3.4.81}$$

for all $h \in C_c^1(\mathbb{R})$, $\phi \in W_0^{1,p}(\Omega, \omega) \cap L^\infty(\Omega)$. To this end, we take $h \in C_c^1(\mathbb{R})$ $\phi \in W_0^{1,p}(\Omega, \omega) \cap L^\infty(\Omega)$ arbitrary and plug $h_l(u_n)h(u)\phi$ into (3.4.56) to obtain

$$I_{n,l}^1 + I_{n,l}^2 + I_{n,l}^3 = I_{n,l}^4, \tag{3.4.82}$$

where

$$\begin{aligned}
I_{n,l}^1 &= \int_\Omega b_n h_l(u_n)h(u)\phi, \\
I_{n,l}^2 &= \int_\Omega a(x, Du_n) \cdot D(h_l(u_n)h(u)\phi), \\
I_{n,l}^3 &= \int_\Omega F(u_n) \cdot D(h_l(u_n)h(u)\phi), \\
I_{n,l}^4 &= \int_\Omega fh_l(u_n)h(u)\phi.
\end{aligned}$$

Step 1 : Passing the limit as $n \to \infty$, applying the convergence results from Lemma 3.4.6, we get,

$$\lim_{n\to\infty} I_{n,l}^1 = \int_\Omega b h_l(u)h(u)\phi, \tag{3.4.83}$$
$$\lim_{n\to\infty} I_{n,l}^4 = \int_\Omega f h_l(u)h(u)\phi. \tag{3.4.84}$$

Let us write

$$I_{n,l}^2 = I_{n,l}^{2,1} + I_{n,l}^{2,2},$$

where

$$I_{n,l}^{2,1} = \int_\Omega h_l(u_n)a(x, Du_n) \cdot D(h(u)\phi),$$
$$I_{n,l}^{2,2} = \int_\Omega h_l^{'}(u_n)a(x, Du_n) \cdot Du_n h(u)\phi.$$

With similar arguments as in the proof of (3.3.36) it follows that

$$\lim_{n\to\infty} I_{n,l}^{2,1} = \int_\Omega h_l(u)a(x, Du) \cdot D(h(u)\phi). \tag{3.4.85}$$

By (3.4.73), we get the estimate

$$|\lim_{n\to\infty} I_{n,l}^{2,2}| \leq \|h\|_\infty\|\phi\|_\infty\Big(\delta C_4 l^{\frac{1}{p}-1} + \int_{\{|f|>\delta\}} |f|\Big), \tag{3.4.86}$$

for all $n \in \mathbb{N}$ and all $l \geq 1$, $\delta > 0$.
Next, we write

$$I_{n,l}^3 = I_{n,l}^{3,1} + I_{n,l}^{3,2},$$

where

$$\lim_{n\to\infty} I_{n,l}^{3,1} = \int_\Omega h_l(u)F(u) \cdot D(h(u)\phi), \tag{3.4.87}$$
$$\lim_{n\to\infty} I_{n,l}^{3,2} = \int_\Omega h_l^{'}(u)F(u) \cdot Du\ h(u)\phi, \tag{3.4.88}$$

follows with the same arguments as in (3.3.37)-(3.3.41).
Step 2 : Passing to the limit as $l \to \infty$. Combining (3.4.82) with (3.4.83)-(3.4.88), we get for all $\delta > 0$ and all $l \geq 1$

$$I_l^1 + I_l^2 + I_l^3 + I_l^4 + I_l^5 = I_l^6, \tag{3.4.89}$$

and

$$
\begin{aligned}
I_l^1 &= \int_\Omega b h_l(u) h(u) \phi, \\
I_l^2 &= \int_\Omega h_l(u) a(x, DT_{l+1}(u)) \cdot D(h(u)\phi), \\
|I_l^3| &\leq \|h\|_\infty \|\phi\|_\infty \Big(\delta C_4 l^{\frac{1}{p}-1} + \int_{\{|f|>\delta\}} |f| \Big),
\end{aligned}
$$

for any $\delta > 0$ and

$$
\begin{aligned}
I_l^4 &= \int_\Omega h_l^{'}(u) F(u) h(u) \phi Du \\
I_l^5 &= \int_\Omega h_l(u) F(u) \cdot D(h(u)\phi) \\
|I_l^6| &= \int_\Omega f h_l(u) h(u) \phi.
\end{aligned}
$$

Choosing $m > 0$ such that *supp* $h \subset [-m, m]$, we can replace u by $T_m(u)$ in I_l^1,... ,I_l^6, hence

$$\lim_{l \to \infty} I_l^1 = \int_\Omega b h(u) \phi, \tag{3.4.90}$$

$$\lim_{l \to \infty} I_l^2 = \int_\Omega a(x, Du) \cdot D(h(u)\phi), \tag{3.4.91}$$

$$\lim_{l \to \infty} |I_l^3| \leq \|h\|_\infty \|\phi\|_\infty \int_{\{|f|>\sigma\}} |f|, \tag{3.4.92}$$

$$\lim_{l \to \infty} I_l^4 = 0, \tag{3.4.93}$$

$$\lim_{l \to \infty} I_l^5 = \int_\Omega F(u) \cdot D(h(u)\phi), \tag{3.4.94}$$

$$\lim_{l \to \infty} |I_l^6| = \int_\Omega f h(u) \phi, \tag{3.4.95}$$

for all $\delta > 0$. Combining (3.4.89) with (3.4.90) - (3.4.95), we finally obtain that (3.4.56) holds for all $h \in C_c^1(\mathbb{R})$ and all $\phi \in W_0^{1,p}(\Omega, \omega) \cap L^\infty(\Omega)$.
Hence (u, b) satisfies $(R1), (R2)$ and $(R3)$ and proof of theorem is completed.

3.4.3 Proof of Theorem 3.4.2 (Uniqueness)

Lemma 3.4.8 *For $f, \tilde{f} \in L^1(\Omega)$ let $(u,b), (\tilde{u}, \tilde{b})$ be the renormalized solutions to (Eq, f) and $(Eq, \tilde{f})$ respectively, then*

$$\int_\Omega (b - \tilde{b}) Sign_0^+(u - \tilde{u}) dx \leq \int_\Omega (f - \tilde{f}) Sign_0^+(u - \tilde{u}) dx. \tag{3.4.96}$$

Proof : For $\delta > 0$ let H_δ^+ be a Lipschitz approximation of the $sign_0^+$ -function. Since $(u,b), (\tilde{u}, \tilde{b})$ are renormalized solutions, it follows that $T_{l+1}(u), T_{l+1}(\tilde{u}) \in X \cap L^\infty(\Omega)$ for all $l > 0$. Hence $H_\delta^+(T_{l+1}(u) - T_{l+1}(\tilde{u}))$ is in $X \cap L^\infty(\Omega)$ for $l, \delta > 0$. Now, we choose $H_\delta^+(T_{l+1}(u) - T_{l+1}(\tilde{u}))$ as a test function in the renormalized formulation with $h = h_l$ for (u,b) and for $(\tilde{u}, \tilde{b})$ respectively. Subtracting the resulting equalities, we obtain

$$I_{l,\delta}^1 + I_{l,\delta}^2 + I_{l,\delta}^3 + I_{l,\delta}^4 + I_{l,\delta}^5 = I_{l,\delta}^6, \tag{3.4.97}$$

where $K = \{0 < T_{l+1}(u) - T_{l+1} < \delta\}$ and

$$\begin{aligned}
I_{l,\delta}^1 &= \int_\Omega \Big(b h_l(u) - \tilde{b} h_l(\tilde{u}) \Big) H_\delta^+(T_{l+1}(u) - T_{l+1}(\tilde{u})) dx, \\
I_{l,\delta}^2 &= \int_\Omega \Big(h_l^{'}(u) a(x, Du) \cdot Du - h_l^{'}(\tilde{u}) a(x, D\tilde{u}) \cdot D\tilde{u} \Big) H_\delta^+(T_{l+1}(u) - T_{l+1}(\tilde{u})) dx, \\
I_{l,\delta}^3 &= \frac{1}{\delta} \int_K \Big(h_l(u) a(x, Du) - h_l(\tilde{u}) a(x, D\tilde{u}) \Big) \cdot D(T_{l+1}(u) - T_{l+1}(\tilde{u})) dx, \\
I_{l,\delta}^4 &= \int_\Omega \Big(h_l^{'}(u) F(u) \cdot Du - h_l^{'}(\tilde{u}) F(\tilde{u}) \cdot D\tilde{u} \Big) H_\delta^+(T_{l+1}(u) - T_{l+1}(\tilde{u})) dx, \\
I_{l,\delta}^5 &= \frac{1}{\delta} \int_K \Big(h_l(u) F(u) - h_l(\tilde{u}) F(\tilde{u}) \Big) \cdot D(T_{l+1}(u) - T_{l+1}(\tilde{u})) dx, \\
I_{l,\delta}^6 &= \int_\Omega \Big(f h_l(u) - \tilde{f} h_l(\tilde{u}) \Big) H_\delta^+(T_{l+1}(u) - T_{l+1}(\tilde{u})) dx.
\end{aligned}$$

Using the same arguments as in [89] i.e., neglecting the nonnegative part of $I_{l,\delta}^3$ and using that F is locally Lipschitz continuous, we can pass to the limit as $\delta \to 0$. Using the energy dissipation condition $(R3)$ we can pass the limit as $l \to \infty$ and obtain (3.4.96).

Now, we are in the position to give the proof of Theorem 3.4.2.

Assuming $f = \tilde{f}$, from Lemma 3.4.8 we get

$$\int_\Omega (b - \tilde{b}) Sign_0^+(u - \tilde{u}) dx \leq 0, \tag{3.4.98}$$

hence $(b-\tilde{b})\ Sign_0^+(u-\tilde{u}) = 0$ almost everywhere in Ω. Now, let us write $\Omega = \Omega_1 \cup \Omega_2$, where $\Omega_1 = \{Sign_0^+(u(x)-\tilde{u}(x)) = 0\}$, $\Omega_2 = \{(b(x)-\tilde{b}(x)) = 0\}$. Since $r \to \beta(r)$ is strictly increasing for $x \in \Omega$; we can define the function $\beta^{-1} : \mathbb{R} \to \mathbb{R}$ such that $\beta^{-1}(r) = s$. For all $(s,r) \in \mathbb{R}^2$ satisfying $r \in \beta(s)$ for a.e. $x \in \Omega_2$, we have $b(x) = \tilde{b}(x)$, hence $u(x) = \beta^{-1}(b(x)) = \beta^{-1}(\tilde{b}(x)) = \tilde{u}(x)$. Therefore, $u(x) = \tilde{u}(x)$ a.e. in Ω_2 and $\mathrm{sign}_0^+(u-\tilde{u}) = 0$ a.e in Ω . Interchanging the roles of u and $\tilde{u}$ and repeating the arguments, we get $\mathrm{sign}_0^+(\tilde{u}-u) = 0$ a.e in Ω and we finally arrive at $u = \tilde{u}$ a.e. in Ω. Now, we write the renormalized formulation for (u,b) and $(\tilde{u},\tilde{b})$ respectively. Subtracting the resulting equalities, we obtain,

$$\int_\Omega (b-\tilde{b})h(u)\varphi dx = 0$$

for all $h \in C_c^1(\mathbb{R})$ and all $\varphi \in D(\Omega)$ choosing $h(u) = h_l(u)$ and passing to the limit as $l \to \infty$, we obtain $b = \tilde{b}$ a.e. in Ω.

3.5 Example

Let us consider the special case :

$$\beta(r) = (r-1)^+ - (r-1)^-, F : \mathbb{R} \to (F_i)_{i=1,\dots,N} \in \mathbb{R}^{\mathrm{N}},$$

where F is locally lipschitz continuous function, and

$$a_i(x,\xi) = \omega_i(x) \cdot |\xi_i|^{p-1}\mathrm{sgn}(\xi_i) \quad i = 1, \dots N,$$

with $\omega_i(x)$ is a weight function $(i = 1, ..., N)$. For simplicity, we shall suppose that

$$w_0 \equiv 1,\ \omega_i(x) = \omega(x),\ \text{for } i = 1, ..., N-1,\ \text{and } \omega_N(x) \equiv 0$$

it is easy to show that the $a_i(x,\xi)$ are Caracthéodory function satisfying the growth condition (3.2.6) and the coercivity (3.2.8). On the other hand the monotonicity condition is verified. In fact

$$\sum_{i=1}^{N} \Big(a_i(x,\xi) - a_i(x,\hat{\xi})\Big)(\xi_i - \hat{\xi}_i)$$

$$= \omega(x) \sum_{i=1}^{N-1} \Big(|\xi_i|^{p-1}\mathrm{sgn}(\xi_i) - |\hat{\xi}_i|^{p-1}\mathrm{sgn}(\hat{\xi}_i)\Big)(\xi_i - \hat{\xi}_i) \geq 0$$

for almost all $x \in \Omega$ and for all $\xi, \hat{\xi} \in \mathbb{R}^{\mathbb{N}}$. This last inequality can not be strict, since for $\xi \neq \hat{\xi}$ with $\xi_N \neq \hat{\xi}_N$ and $\xi = \hat{\xi}, i = 1, ..., N-1$. The corresponding expression is zero. In particular, let us use special weight functions w expressed in terms of the distance to the bounded $\partial\Omega$. Denote $d(x) = dist(x, \partial\Omega)$ and set $\omega(x) = d^{\delta}(x)$, such that,

$$\delta < \min\big(\frac{p}{N}, p-1\big). \tag{3.5.99}$$

Remark 3.5.1 *The condition* (3.5.99) *is sufficient for* (1.1.8)*. Finally, the hypotheses of Theorem 3.4.1 are satisfied. Therefore, for all $f \in L^1(\Omega)$, the following problem :*

$$\begin{cases} T_k(u) \in W_0^{1,p}(\Omega, \omega), for(k > 0) \,; b \in L^1(\Omega) \; and \; b(x) \in \beta(u(x)), \\ \lim_{l\to\infty} \int_{\{l<|u|<l+1\}} a(x, Du) \cdot Du dx = 0, \\ \int_\Omega bh(u)\varphi dx + \int_\Omega h(u) \sum_{i=1}^N \omega_i |\frac{\partial u}{\partial x_i}|^{p-1} sgn(\frac{\partial u}{\partial x_i}) \cdot \frac{\partial \varphi}{\partial x_i} dx \\ + \int_\Omega h'(u) \sum_{i=1}^N \omega_i |\frac{\partial u}{\partial x_i}|^{p-1} sgn(\frac{\partial u}{\partial x_i}) \cdot \frac{\partial u}{\partial x_i} \varphi dx + \int_\Omega F(u) \cdot D(h(u)\varphi) dx, \\ = \int_\Omega f \cdot D(h(u)\varphi), \\ \forall \varphi \in W_0^{1,p}(\Omega, \omega) \cap L^\infty(\Omega) \; and \; h \in \mathcal{C}_c^1(\mathbb{R}), \end{cases}$$

has at least one renormalized solution.

Chapitre 4

Renormalized Solutions of Stefan Degenerat Parabolic Nonlinear problems

[1] In this chapter, We study a general class of nonlinear parabolic problems associated with the differential inclusion $\beta(u)_t - div(a(x, Du) + F(u)) \ni f$ in Ω, where $f \in L^1(Q_T)$. A vector field $a(.,.)$ is a Carathéodory function. Using truncation techniques and a generalized Minty method in the functional spaces we prove existence of weak solution and existence of renormalized solutions for general L^1-data.

4.1 Introduction

Let Ω be a bounded open set of $\mathbb{R}^N$ $(N \geq 1)$ with Lipschitz boundary if $N \geq 2$, p be a real number such that $1 < p < \infty$ and $\omega = \{\omega_i(x),\, 0 \leq i \leq N\}$ be a vector of weight functions on Ω, i.e. each $\omega_i(x)$ is a measurable a.e. positive on Ω. Let $W_0^{1,p}(\Omega, \omega)$ be the weighted Sobolev space associated with the vector ω. Our aim is to show existence of renormalized solutions to the following nonlinear parabolic equation

$$(P, f, b_0) \begin{cases} \beta(u)_t - div(a(x, Du) + F(u)) \ni f & \text{in} \quad Q_T \\ u = 0 & \text{on} \quad \Sigma_T, \\ \beta(u(0,.)) \ni b_0 & \text{in} \quad \Omega, \end{cases}$$

with a right-hand side $f \in L^1(Q_T)$, for (P, f, b_0) Furthermore, $F : \mathbb{R} \to \mathbb{R}^N$ is locally lipschitz continuous and $\beta : \mathbb{R} \to 2^{\mathbb{R}}$ a set-valued, maximal monotone

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mapping such that $0 \in \beta(0)$, Moreover, we assume that

$$\beta^0(l) \in L^1(\Omega). \tag{4.1.1}$$

for each $l \in \mathbb{R}$, where β^0 denotes the minimal selection of the graph of β. Namely $\beta_0(l)$ is the minimal in the norm element of $\beta(l)$

$$\beta_0(l) = \inf\{|r| \ /r \in \mathbb{R} \text{ and } r \in \beta(l)\}$$

$a: \ \Omega \times \mathbb{R}^N \ \rightarrow \ \mathbb{R}^N$ is a Carathéodory function satisfying to following assumptions :(A_1), (A_2), (A_3) set out General Introduction.

We use in this chapter the the framework of renormalized solutions. This notion was introduced by Diperna and P.-L. Lions [56] in their study of the Boltzmann equation. This notion was then adapted to an elliptic version of (Eq, f) by L. Boccardo et al. [47] when the right hand side is in $W^{-1,p'}(\Omega)$, by J.-M. Rakotoson [83] when the right hand side is in $L^1(\Omega)$, and finally by G. Dal Maso, F. Murat, L. Orsina and A. Prignet [54] for the case of right hand side is general measure data. Note that in the case with variable exponent the problem is studied by Wittbold and al. in [89]. Let us point out that other work in this direction can be found in [52, 86].

In chapter III, existence and uniqueness of renormalized solutions to (Eq, f) has been shown for the weighted Sobolev space, from these results we deduce that there exists a mild solution of the abstract Cauchy problem corresponding to (P, f, b_0) in the sense of nonlinear semigroup theory. The nonlinear semigroup theory gives a general notion of solution, called 'mild' solution for abstract Cauchy problems of the form

$$\frac{du}{dt} + Au \ni f$$

where A is (a possibly multivalued) operator in a Banach space X and $f \in L^1(0, T; X)$. The mild solution is, roughly speaking, the uniform limit of piecewise constant approximate solutions of time-discretized equations given by an implicit Euler scheme. This result will lead us to the appropriate energy space for weak and renormalized solutions of (P, f, b_0) in Weighted Sobolev spaces.

One of the motivations for studying (P, f, b_0) comes from applications to electrorheological fluids (see [85] for more details) as an important class of non-Newtonian fluids.

The plan of the chapter is as follows. In Section 2 we make precise all the assumptions on $a, F,\ f$ and u_0, and we give some technical results, we give the definition of a renormalized solution and weak solution of (P, f, b_0), and main results. Section 3 Proof of theorem 4.2.9. In Section 4 Proof of theorem 4.2.10.

4.2 Basic assumptions and Main results

4.2.1 Basic assumptions

Assumption (H1)

The expression

$$|||u|||_X = \Big(\sum_{i=1}^{N} \int_\Omega |\frac{\partial u(x)}{\partial x_i}|^p \omega_i(x)\,dx\Big)^{1/p} \tag{4.2.2}$$

is a norm defined on X and is equivalent to the norm (1.1.3).
There exist a weight function σ on Ω and a parameter q, $1 < q < \infty$, such that

$$\sigma^{1-q'} \in L^1(\Omega), \tag{4.2.3}$$

with $q' = q/(q-1)$. The Hardy inequality,

$$\Big(\int_\Omega |u(x)|^q \sigma\,dx\Big)^{1/q} \leq c\Big(\sum_{i=1}^{N} \int_\Omega |\frac{\partial u(x)}{\partial x_i}|^p \omega_i(x)\,dx\Big)^{1/p}, \tag{4.2.4}$$

holds for every $u \in X$ with a constant $c > 0$ independent of u, and moreover, the imbedding

$$X \hookrightarrow\hookrightarrow L^q(\Omega, \sigma), \tag{4.2.5}$$

expressed by the inequality (4.2.4) is compact. Note that $(X, |||.|||_X)$ is a uniformly convex (and thus reflexive) Banach space.

Assumption (H2)

We assume that

$$|a_i(x,\xi)| \leq \alpha\omega_i^{1/p}(x)[k(x) + \sum_{j=1}^{N}\omega_j^{1/p'}(x)|\xi_j|^{p-1}], \tag{4.2.6}$$

$$[a(x,\xi) - a(x,\eta)](\xi - \eta) \geq 0 \quad \text{for all } (\xi,\eta) \in \mathbb{R}^N \times \mathbb{R}^N, \tag{4.2.7}$$

$$a(x,\xi).\xi \geq \lambda\sum_{i=1}^{N}\omega_i|\xi_i|^p, \tag{4.2.8}$$

where $k(x)$ is a positive function in $L^{p'}(\Omega)$ and α, λ are positive constants.

Assumption (H3)

Let $\beta,\ F$ such that

$$\beta : \mathbb{R} \to 2^{\mathbb{R}} \text{ set valued, maximal monotone mapping, such that}$$
$$0 \in \beta(0) \text{ and satisfying (4.1.1),}$$
$$F : \mathbb{R} \to \mathbb{R}^N \text{ is locally lipschitz continuous.}$$

4.2.2 Some technical results

a) Mild solution

It follows from Theorem 3.4.1 of the previous chapter that for all $f \in L^1(\Omega)$, there exists a renormalized solution (u, b) to (Eq, f). For $f, \tilde{f} \in L^1(\Omega)$ let (u, b), and $(\tilde{u}, \tilde{b})$ be renormalized solutions of (Eq, f), $(\mathrm{Eq}, \tilde{f})$ respectively. Writing $|b - \tilde{b}| = (b - \tilde{b})^+ + (\tilde{b} - b)^+$ and applying the comparison principle from Lemma 3.4.8, we find that

$$\|b - \tilde{b}\|_{L^1(\Omega)} \leq \|f - \tilde{f}\|_{L^1(\Omega)}. \tag{4.2.9}$$

In terms of nonlinear operators the preceeding results read as follows : if A_β is the nonlinear operator defined in $L^1(\Omega)$ by :

Corollary 4.2.1 *Assume that $(H1) - (H3)$ are satisfied. Then, the operator*

$$A_\beta := \{(b, w) \in L^1(\Omega) \times L^1(\Omega) : \exists u : \Omega \to \mathbb{R} \text{ measurable, } u \in W_0^{1,p}(\Omega, w) \cap L^\infty(\Omega),$$
$$b \in \beta(u) \text{ a.e in } \Omega, u \text{ is a renormalized solution of } - div(a(x, Du) + F(u)) = w\}$$

verifies the following properties :

i) A_β is m-accretive in $L^1(\Omega)$,

ii) $R(I + \lambda A_\beta) = L^1(\Omega),\ \lambda > 0$,

iii) $\overline{D(A_\beta)}^{\|.\|_{L^1(\Omega)}} = \{b \in L^1(\Omega) :\ b \in \overline{R(\beta)} \text{ a.e in } \Omega\}$.

Proof : A_β is m-accretive in $L^1(\Omega)$, i.e., the resolvent mapping $f \in L^1(\Omega) \to (I + \lambda A_\beta)^{-1} f := J^\lambda_{A_\beta}(f)$ is a contraction in the L^1-norm (because of (4.2.9) and the range condition

$$R(I + \lambda A_\beta) = L^1(\Omega). \tag{4.2.10}$$

Indeed, for any $f \in L^1(\Omega)$, $\lambda > 0$ there exists $(b, w) \in A_\beta$ such that

$$b + \lambda w = f \tag{4.2.11}$$

almost everywhere in Ω : If (u, b) is the renormalized solution to (Eq, f), then we have $b \in \beta(u)$ almost everywhere in Ω and u is the renormalized solution to

$$-\lambda div(a(x, Du) + F(u)) = f - b.$$

Therefore $(b, \frac{f-b}{\lambda}) \in A_\beta$ and (4.2.11) holds with $w = \frac{f-b}{\lambda}$.
For iii) see Proposition 4.1.1 [89].

Remark 4.2.2 *By the general theory of nonlinear semigroups (see [30], [37]), we conclude that the abstract Cauchy problem corresponding to* (P, f, b_0)

$$(ACP)(f, b_0) \begin{cases} \frac{db}{dt} + A_\beta b \ni f \quad in \quad (0, T) \\ \\ b(0) = b_0 \end{cases}$$

admits a unique mild solution $b \in C([0, T]; L^1(\Omega))$ *for any initial datum* $b_0 \in \overline{D(A_\beta)}^{\|.\|_{L^1(\Omega)}}$ *and any right-hand side* $f \in L^1(0, T; L^1(\Omega)) \simeq L^1(Q_T)$.
Roughly speaking, a mild solution is a continuous abstract function $b \in C([0, T]; L^1(\Omega))$ *which is the uniform limit of piecewise constant functions.*

Lemma 4.2.3 *For* $f \in L^1(Q_T)$, $b_0 \in \overline{D(A_\beta)}^{\|\|_{L^1(\Omega)}}$, $0 < \varepsilon \leq 1$,$N(\varepsilon) \in \mathbb{N}$ *and*

$$(D_\varepsilon) \begin{cases} t_0^\varepsilon = 0 < t_1^\varepsilon < .... < t_{N(\varepsilon)}^\varepsilon = T, \\ \\ t_j^\varepsilon - t_{j-1}^\varepsilon = \varepsilon, T - t_{N(\varepsilon)}^\varepsilon \leq \varepsilon \; \forall j = 1, ..., N(\varepsilon), \\ \\ f_j^\varepsilon \in L^\infty(\Omega), \; j = 1, ..., N(\varepsilon) : \; \sum_{j=1}^{N(\varepsilon)} \int_{t_{j-1}^\varepsilon}^{t_j^\varepsilon} \|f(t) - f_j^\varepsilon\|_{L^1(\Omega)} dt \leq \varepsilon \\ \\ b_0^\varepsilon \in L^\infty(\Omega) \; : \|b_0^\varepsilon - b_0\|_{L^1(\Omega)} \leq \varepsilon. \end{cases}$$

Let $(b_j^\varepsilon, u_j^\varepsilon)_{j=1}^{N(\varepsilon)}$ *be a solution of the discretized problem*

$$(DP_\varepsilon)\begin{cases} b_j^\varepsilon \in L^1(\Omega),\ u_j^\varepsilon : \Omega \to \mathbb{R}\ \textit{measurable},\ T_k(u_j^\varepsilon) \in W_0^{1,p}(\Omega,\omega), \forall k > 0, \\ \int_{\{n<|u_j^\varepsilon|<n+1\}} a(x, Du_j^\varepsilon)\cdot Du_j^\varepsilon \to 0\ \textit{as}\ n \to \infty, \\ \int_\Omega \frac{b_j^\varepsilon - b_{j-1}^\varepsilon}{t_j^\varepsilon - t_{j-1}^\varepsilon} h(u_j^\varepsilon)\varphi + \int_\Omega (a(x, Du_j^\varepsilon) + F(u_j^\varepsilon))\cdot D(h(u_j^\varepsilon)\varphi) = \int_\Omega f_j^\varepsilon h(u_j^\varepsilon)\varphi \\ \forall \varphi \in W_0^{1,p}(\Omega,\omega)\cap L^\infty(\Omega),\ h \in C_c^1(\mathbb{R}), \\ b_j^\varepsilon \in \beta(u_j^\varepsilon) \quad \textit{a.e. in}\ \Omega,\ \textit{for all}\ j = 1, ..., N(\varepsilon). \end{cases}$$

For $k > 0$, *we define the piecewise constant functions* $f_\varepsilon : (0, T] \to L^1(\Omega)$,$b_\varepsilon : [0, T] \to L^1(\Omega)$, *and* $T_k(u_\varepsilon) : (0, T] \to W_0^{1,p}(\Omega, \omega)$ *as follows :* $f_\varepsilon(t) = f_j^\varepsilon$, $b_\varepsilon(0) = b_0^\varepsilon$, $b_\varepsilon(t) = b_j^\varepsilon$, *and* $T_k(u_\varepsilon(t)) = T_k(u_j^\varepsilon)$ *for* $t \in (t_j^\varepsilon, t_{j-1}^\varepsilon)$ *and* $j = 1, ..., N(\varepsilon)$. *If* $t_{N(\varepsilon)}^\varepsilon < T$, f_ε, b_ε *and* u_ε *are extended by setting* $f_\varepsilon(t) = f_j^{N(\varepsilon)}$, $T_k(u_\varepsilon(t)) = T_k(u_{N(\varepsilon)}^\varepsilon)$ *and* $b_\varepsilon(t) = b_j^{N(\varepsilon)}$ *for all* $t \in (t_\varepsilon^{N(\varepsilon)}, T]$.

Then the following estimates hold true for all $k > 0$ *and* $0 < \varepsilon \le 1$ *:*

i)There exists a constant $C_1(, \lambda, \|f\|_{L^1(Q_T)}, \|b_0\|_{L^1(\Omega)}, k, T) > 0$ *not depending on* $\varepsilon > 0$, *such that*

$$\int_0^T \int_\Omega \sum_{i=1}^N |\frac{\partial u_\varepsilon}{\partial x_i}|^p \omega_i dxdt \le C_1(\lambda, \|f\|_{L^1(Q_T)}, \|b_0\|_{L^1(\Omega)}, k). \tag{4.2.12}$$

ii) There exists a constant $C_2(\lambda, \|f\|_{L^1(Q_T)}, \|b_0\|_{L^1(\Omega)}, k,) > 0$ *not depending on* $\varepsilon > 0$, *such that*

$$\|T_k(u_\varepsilon)\|_{L^p(0,T,W_0^{1,p}(\Omega,\omega))} \le C_2(\lambda, \|f\|_{L^1(Q_T)}, \|b_0\|_{L^1(\Omega)}, k). \tag{4.2.13}$$

iii)There exists a constant $C_3(\lambda, \|f\|_{L^1(Q_T)}, \|b_0\|_{L^1(\Omega)}) > 0$ *not depending on* $\varepsilon > 0$, *such that*

$$\sum_{i=1}^{N}\int_0^T\int_\Omega |a_i(x,DT_k(u_\varepsilon))|^{p'}\omega_i^{1-p'}dxdt \le C_3(\lambda,\|f\|_{L^1(Q_T)},\|b_0\|_{L^1(\Omega)}). \tag{4.2.14}$$

Proof : For $j\in\{1,...,N(\varepsilon)\}$ we take $T_k(u_j^\varepsilon)h_l(u_j^\varepsilon)$, $k,l>0$, as a test function in (DP_ε) to obtain $I_1+I_2+I_3=I_4$ where

$$I_1=\int_\Omega \frac{b_j^\varepsilon-b_{j-1}^\varepsilon}{t_j^\varepsilon-t_{j-1}^\varepsilon}h_l(u_j^\varepsilon)T_k(u_j^\varepsilon),\;\; I_2=\int_\Omega a(x,Du_j^\varepsilon)\cdot D(h_l(u_j^\varepsilon)T_k(u_j^\varepsilon)),$$
$$I_3=\int_\Omega F(u_j^\varepsilon))\cdot D(h_l(u_j^\varepsilon)T_k(u_j^\varepsilon)),\;\; I_4=\int_\Omega f_j^\varepsilon h_l(u_j^\varepsilon)T_k(u_j^\varepsilon).$$

By Gauss-Green Theorem, it follows that $I_3=0$ for all $l>k$. Applying (4.2.8) in I_2, we can pass to the limit with $l\to\infty$ and find

$$\int_\Omega \frac{b_j^\varepsilon-b_{j-1}^\varepsilon}{t_j^\varepsilon-t_{j-1}^\varepsilon}T_k(u_j^\varepsilon)+\lambda\int_\Omega\sum_{i=1}^N\Big|\frac{\partial T_k(u_j^\varepsilon)}{\partial x_i}\Big|^p\omega_i(x)\le k\int_\Omega|f_j^\varepsilon|. \tag{4.2.15}$$

If we define the convex, l.s.c., proper function $\phi_{T_k}:\mathbb{R}\to\mathbb{R}\cup\{+\infty\}$ by

$$\phi_{T_k}(r)=\begin{cases}\int_0^r T_k((\beta^{-1})^0(\sigma))d\sigma, \; if\ r\in\overline{R(\beta)}\\ +\infty \quad otherwise\end{cases}$$

then, $T_k(u_j^\varepsilon)\subset\partial\phi_{T_k}(b_j^\varepsilon)$ for all $j=1,...,N(\varepsilon)$ and

$$\phi_{T_k}(b_j^\varepsilon)-\phi_{T_k}(b_{j-1}^\varepsilon)\le(b_j^\varepsilon-b_{j-1}^\varepsilon)T_k(u_j^\varepsilon) \tag{4.2.16}$$

holds almost everywhere in Ω. Therefore from (4.2.15) and (4.2.16) it follows that

$$\int_\Omega\frac{\phi_{T_k}(b_j^\varepsilon)-\phi_{T_k}(b_{j-1}^\varepsilon)}{t_j^\varepsilon-t_{j-1}^\varepsilon}+\lambda\sum_{i=1}^N\int_\Omega|\frac{\partial T_k(u_j^\varepsilon)}{\partial x_i}|^p\omega_i(x)dx\le k\int_\Omega|f_j^\varepsilon|. \tag{4.2.17}$$

Integrating (4.2.17) over $(t_{j-1},t_j]$, and taking the sum over $j=1,...,N(\varepsilon)$ yields

$$\int_\Omega\phi_{T_k}(b_\varepsilon(T))dx+\lambda\int_0^T\sum_{i=1}^N\int_\Omega|\frac{\partial T_k(u_\varepsilon)}{\partial x_i}|^p\omega_i(x)dxdt\le\int_\Omega\phi_k(b_\varepsilon(0))+k\int_0^T\int_\Omega|f_\varepsilon|dxdt. \tag{4.2.18}$$

According to (D_ε), $b_\varepsilon(0)=b_0^\varepsilon$ converges to b_0 and f_ε converges to f in $L^1(0,T,L^1(\Omega))$ as $\varepsilon\downarrow 0$. Therefore, the right-hand side of (4.2.18) is bounded by a constant $C_1(\|f\|L^1(Q_T),\|b_0\|_L^1(\Omega),k)>0$ that does not depend on ε. Now, i)

and ii) follows from (4.2.18) if we neglect the positive term and use (4.2.8).
To prove iii :), we use (4.2.6) and the same arguments as above.

b) Integration-by-parts-formula

In the next Lemma, we prove an integration-by-parts-formula that will be crucial in the following. The idea of the proof is the same as in [21] and the generalisations considered in [77].

Lemma 4.2.4 *Let $\beta \subset \mathbb{R} \times \mathbb{R}$ be a maximal monotone graph, $u \in V$, $b \in L^1(Q_T)$ such that $b \in \beta(u)$ almost everywhere in Q_T, $b_0 \in L^1(\Omega)$ with $b(0,x) = b_0$ almost everywhere in Ω and $u_0 : \Omega \to \overline{\mathbb{R}}$ be a measurable function such that $b_0 \in \beta(u_0)$ almost everywhere in Ω. Furthermore, we assume that there exists $G \in V + L^1(Q_T)$ satisfying*

$$\int_{Q_T} (b - b_0)\xi_t = \langle G, \xi \rangle \tag{4.2.19}$$

for all $\xi \in D([0,T) \times \Omega)$ where $< .,. >$ denotes the duality pairing between $V' + L^1(Q_T)$ and $V \cap L^\infty(Q_T)$. Then,

$$\int_{Q_T} \xi_t \int_{b_0}^{b(t,x)} h((\beta^{-1})(\sigma))d\sigma = \langle G, h(u)\xi \rangle \tag{4.2.20}$$

for all $h \in C_c^1(\mathbb{R})$ and $\xi \in D([0,T) \times \overline{\Omega})$.

Proof : The proof of following Lemma 4.2.4 will be omitted since it is very similar to that Lemma 4.2.11, p.63-67 in [89]. The a priori estimates in Lemma 4.2.3 naturally lead to an appropriate notion of a renormalized solution to (P, f, b_0).

4.2.3 Notions of solutions

Definition 4.2.5 *For $f \in L^1(Q_T), b_0 \in L^1(\Omega)$ a weak solution to (P, f, b_0) is a pair of function $(u, b) \in L^p(0, T, W_0^{1,p}(\Omega, \omega)) \times L^1(Q_T)$ satisfying $F(u) \in (L^{p'}(Q_T))^N$,*

$b \in \beta(u)$ *almost everywhere in* Q_T, $b(0,x) = b_0$ *almost everywhere in* Ω *such that*

$$-\int_{Q_T}(b-b_0)\xi_t + \int_{Q_T}\Big(a(x,Du)+F(u)\Big)\cdot D(\xi)dxdt = \int_{Q_T} f\xi dxdt, \qquad (4.2.21)$$

holds for all $\xi \in D([0,T)\times\Omega)$.

Definition 4.2.6 *For* $f \in L^1(Q_T)$, $b_0 \in L^1(\Omega)$ *a renormalized solution to* (P,f,b_0) *is a pair of function* (u,b) *satisfying the following conditions :*

(P1) $u : Q_T \to \mathbb{R}$ *is measurable,* $b \in L^1(Q_T), u(t,x) \in D(\beta(t,x))$ *and* $b(t,x) \in \beta(u(t,x))$ *for a.e.* $(t,x) \in Q_T$.

(P2) $b(0,x) = b_0(x)$ *a.e. in* Ω,

(P3) *For each* $k > 0, T_k(u) \in L^p(0,T,W_0^{1,p}(\Omega,\omega))$ *and* $DT_k(u) \in \prod_{i=1}^N L^p(Q_T,\omega_i)$,

(P4) $-\int_{Q_T}\xi_t\int_{b_0}^{b(t,x)} ho(\beta^{-1})^0(r)drdxdt + \int_{Q_T}\Big(a(x,Du)+F(u)\Big)\cdot D(h(u)\xi)dxdt$

$$= \int_{Q_T} fh(u)\xi dxdt,$$

holds for all $h \in \mathcal{C}_c^1(\mathbb{R})$ *and all* $\xi \in D([0,T)\times\Omega)$

(P5) $\int_{Q_T\cap\{k\le|u|\le k+1\}} a(x,Du)\cdot Dudxdt \to 0$ *as* $k\to\infty$.

Remark 4.2.7 *Note that if* (u,b) *is a renormalized solution to* (P,f,b_0) *such that* $u \in L^\infty(Q_T)$ *then* (u,b) *is a weak solution to* (P,f,b_0).

Indeed*, as an immediate consequence from* $(P1)$, *and* $(P3)$ *we get* $u \in L^p(0,T,W_0^{1,p}(\Omega,\omega))$. *Now we fix* $\xi \in D([0,T)\times\Omega)$ *and choose* $h_l(u)\xi$ *as a test function in* $(P4)$. *As usual, we apply the Gauss-Green Theorem and the boundary condition on the convection term* $\int_{Q_T} h_l'(u)\xi F(u)\cdot Du$ *and* $(P5)$ *to estimate* $\int_{Q_T} h_l'(u)\xi a(x,Du)\cdot Du$. *Passing to the limit with* $l\to\infty$, *we find* $(4.2.21)$. *The remaining conditions for being a weak solution follow from* $(P1)$ *and* $(P2)$.

Proposition 4.2.8 *For $f \in L^1(Q_T)$, $b_0 \in L^1(\Omega)$ such that there exists a measurable function $u_0 : \Omega \to \overline{\mathbb{R}}$ with $b_0 \in \beta(u_0)$ almost everywhere in Ω let (u, b) be a weak solution to (P, f, b_0). Then (u, b) is a renormalized solution to (P, f, b_0).*

Proof : Clearly, (u, b) satisfies $(P1), (P2)$ and $(P5)$. By assumption we have $u \in L^p(0, T, W_0^{1,p}(\Omega, \omega))$, hence u is finite almost everywhere in Q_T and it follows that $|\{k < |u| < k+1\}| \to 0$ as $k \to \infty$. In particular, $|Du|^p \in L^1(Q_T, \sigma)$ and therefore $(P3)$ holds. From (4.2.13), we get that $(b - b_0)_t \in L^{p'}(0, T, W^{-1,p'}(\Omega, \omega_i^*)) + L^1(Q_T)$. Now, $(P4)$ follows from the integration-by parts- formula in Lemma 4.2.4.
The main result of this section is the following theorem :

4.2.4 Main results.

Theorem 4.2.9 *For $f \in L^\infty(Q_T)$ and $b_0 \in \overline{D(A_\beta)}^{\|\cdot\|_{L^1(\Omega)}} \cap L^\infty(\Omega)$. Assume that $(H1) - (H3)$, there exists a weak solution (u, b) to (P, f, b_0). In particular, b is the mild solution of $(ACP)(f, b_0)$.*

Theorem 4.2.10 *For each $f \in L^1(Q_T)$ and $b_0 \in \overline{D(A_\beta)}^{\|\cdot\|_{L^1(\Omega)}}$. Assume that $(H1)-(H3)$, there exists a renormalized solution to (P, f, b_0).*

To prove Theorem 4.2.10, we will use several approximation procedures. First, we prove existence of weak solutions for L^∞-data in Theorem 4.2.9.

4.3 Proof of Theorem 4.2.9

4.3.1 Auxiliary problem

Step1 : Approximate problems.
In a first step, for bounded data $f \in L^\infty(Q_T)$, $b_0 \in \overline{D(A_\beta)}^{\|\cdot\|_{L^1(\Omega)}} \cap L^\infty(\Omega)$, we prove existence of a weak solution to our elliptic-parabolic problem with an additional

strictly monotone and continuous perturbation $\psi : \mathbb{R} \to \mathbb{R},\ \psi(0) = 0$, i.e.,

$$(P, \psi, f, b_0) \begin{cases} \beta(u)_t + \psi(u) - div(a(x, Du) + F(u)) \ni f & \text{in} \quad Q_T, \\ u = 0 & \text{on} \quad \Sigma_T, \\ \beta(u(0, .)) \ni b_0 & \text{in} \quad \Omega, \end{cases}$$

To this end, we define the nonlinear operator

$$\begin{aligned} A_{\beta,\psi} := \{ & (b, w) \in L^1(\Omega) \times L^1(\Omega) : \exists u : \Omega \to \mathbb{R} \textit{ measurable, } b \in \beta(u) \textit{ a.e in } \Omega, \\ & u \textit{ is a renormalized solution of } - div(a(x, Du) + F(u)) + \psi(u) = w \} \end{aligned}$$

where a definition of a renormalized solution to the above problem is obtained from Definition 3.2.3 upon setting $f = w - \psi(u) - b_0$. Using the same arguments as in Corollary 4.2.1 it follows that $A_{\beta,\psi}$ is m-accretive in $L^1(\Omega)$ and $\overline{D(A_{\beta,\psi})}^{\|.\|_{L^1(\Omega)}} = \overline{D(A_{\beta})}^{\|.\|_{L^1(\Omega)}}$, i.e., to each $(f, b_0) \in L^1(Q_T) \times \overline{D(A_{\beta,\psi})}^{\|.\|_{L^1(\Omega)}}$ there exists a unique mild solution $b \in \mathcal{C}([0, T]; L^1(\Omega))$ of the abstract Cauchy problem

$$(ACP)(f, \psi, b_0) \begin{cases} \frac{db}{dt} + A_{\beta,\psi} b \ni f & \text{in} \quad (0, T) \\ b(0) = b_0 \end{cases}$$

corresponding to (P, ψ, f, b_0). Moreover, for $f \in L^\infty(Q_T),\ b_0 \in \overline{D(A_\beta)}^{\|.\|_{L^1(\Omega)}} \cap L^\infty(\Omega)$, b is the uniform limit of piecewise constant functions $b_\varepsilon : (0, T) \to L^1(\Omega)$ define by $b_\varepsilon = b_j^\varepsilon$ on $]t_{j-1}^\varepsilon, t_j^\varepsilon[,\ j = 1, ..., N(\varepsilon),\ b_\varepsilon(0) = b_0^\varepsilon$ where $(u_j^\varepsilon, b_j^\varepsilon)_{j=1}^{N(\varepsilon)}$ is a solution of the discretized problem (see [10])

$$(DP_{\varepsilon,\psi}) \begin{cases} b_j^\varepsilon \in L^1(\Omega), \quad u_j^\varepsilon \in W_0^{1,p}(\Omega, \omega) \cap L^\infty(\Omega), \\ \int_{\{n<|u_j^\varepsilon|<n+1\}} a(x, Du_j^\varepsilon) \cdot Du_j^\varepsilon \to 0 \text{ as } n \to \infty, \\ \int_\Omega \frac{b_j^\varepsilon - b_{j-1}^\varepsilon}{\varepsilon} \varphi + \int_\Omega (a(x, Du_j^\varepsilon) + F(u_j^\varepsilon)) \cdot D\varphi + \int_\Omega \psi(u_j^\varepsilon)\varphi = \int_\Omega f_j^\varepsilon \varphi, \\ \forall \varphi \in W_0^{1,p}(\Omega, \omega), \\ b_j^\varepsilon \in \beta(u_j^\varepsilon) \quad \text{a.e. in } \Omega, \\ j = 1, ..., N(\varepsilon), \end{cases}$$

given by an equidistant time discretisation of the form

$$(D_\varepsilon) \begin{cases} t_0^\varepsilon = 0 < t_1^\varepsilon < .... < t_{N(\varepsilon)}^\varepsilon = T, \\ t_j^\varepsilon - t_{j-1}^\varepsilon = \varepsilon, \forall j = 1, ..., N(\varepsilon), \\ f_j^\varepsilon \in L^\infty(\Omega),\ j = 1, ..., N(\varepsilon) :\ \sum_{j=1}^{N(\varepsilon)} \int_{t_{j-1}^\varepsilon}^{t_j^\varepsilon} \|f(t) - f_j^\varepsilon\|_{L^1(\Omega)} dt \le \varepsilon \\ b_0^\varepsilon \in L^\infty(\Omega) \ : \|b_0^\varepsilon - b_0\|_{L^1(\Omega)} \le \varepsilon. \end{cases}$$

If we define the piecewise constant function $u_\varepsilon : (0,T) \to W_0^{1,p}(\Omega,\omega)$ by $u_\varepsilon(t) = u_j^\varepsilon$ for $t \in (t_{j-1}^\varepsilon, t_j^\varepsilon]$ and $j = 1, ..., N(\varepsilon)$, the following a priori estimates hold :

Step2 : A priori estimates.

Lemma 4.3.1 *Let u_ε be defined as above. Then, the following results hold for all $\varepsilon > 0$:*

i)There exists a constant $C_1(\|f\|_{L^\infty(Q_T)}, \|b_0\|_{L^\infty(\Omega)}) > 0$ not depending on $\varepsilon > 0$, such that

$$\|\psi(u_\varepsilon)\|_{L^\infty(Q_T)} \leq C_1(\|f\|_{L^\infty(Q_T)}, \|b_0\|_{L^\infty(\Omega)}). \tag{4.3.22}$$

ii) There exists a constant $C_2(\|f\|_{L^\infty(Q_T)}, \|b_0\|_{L^\infty(\Omega)}, \psi) > 0$ not depending on $\varepsilon > 0$, such that

$$\|u_\varepsilon\|_{L^\infty(Q_T)} \leq C_2(\|f\|_{L^\infty(Q_T)}, \|b_0\|_{L^\infty(\Omega)}, \psi). \tag{4.3.23}$$

iii)There exists a constant $C_3(\lambda, \|f\|_{L^\infty(Q_T)}, \|b_0\|_{L^\infty(\Omega)}) > 0$ not depending on $\varepsilon > 0$, such that

$$\int_0^T \int_\Omega \sum_{i=1}^N |\frac{\partial u_\varepsilon}{\partial x_i}|^p \omega_i dxdt \leq C_3(\lambda, \|f\|_{L^\infty(Q_T)}, \|b_0\|_{L^\infty(\Omega)}). \tag{4.3.24}$$

Proof : For $j = 1, ..., N(\varepsilon)$ we choose $p(u_j^\varepsilon)$ as a test function in $(DP_{\varepsilon,\psi})$ where $p \in \mathcal{P}_0 = \{p \in \mathcal{C}^\infty(\mathbb{R}); 0 \leq p' < 1, \text{ supp } p' \text{ compact}, 0 \notin \text{supp } p\}$. Upon integrating over (t_{j-1}, t_j) and summing over $j = 1, ..., N(\varepsilon)$ as in [22],[39], we obtain i) and from i) we deduce ii) since ψ is strictly increasing and continuous.
To prove iii), for $j = 1, ..., N(\varepsilon)$ we plug u_j^ε a test function in $(DP_{\varepsilon,\psi})$ to obtain

$$I_1 + I_2 + I_3 + I_4 = I_5$$

where $I_1 = \int_\Omega \frac{b_j^\varepsilon - b_{j-1}^\varepsilon}{\varepsilon} u_j^\varepsilon$, $I_2 = \int_\Omega a(x, Du_j^\varepsilon) \cdot Du_j^\varepsilon$, $I_3 = \int_\Omega F(u_j^\varepsilon)) \cdot Du_j^\varepsilon$,
$I_4 = \int_\Omega \psi(u_j^\varepsilon) u_j^\varepsilon$, $I_5 = \int_\Omega f_j^\varepsilon u_j^\varepsilon$.
Applying (4.2.8) in I_2. The Lipschitz character of F and stokes formula together

with the boundary condition 2 of problem give we find

$$\int_\Omega \frac{b_j^\varepsilon - b_{j-1}^\varepsilon}{\varepsilon} u_j^\varepsilon + \lambda \int_\Omega \sum_{i=1}^N |\frac{\partial u_j^\varepsilon}{\partial x_i}|^p \omega_i(x)dx + \int_\Omega \psi(u_j^\varepsilon)u_j^\varepsilon \le \int_\Omega f_j^\varepsilon u_j^\varepsilon. \qquad (4.3.25)$$

If we define the convex, l.s.c., proper function $\phi_{id} : \mathbb{R} \to \mathbb{R} \cup \{+\infty\}$ by

$$\phi_{id}(r) = \begin{cases} \int_0^r (\beta^{-1})^0(\sigma)d\sigma, \ if \ r \in \overline{R(\beta)} \\ +\infty \quad otherwise \end{cases}$$

then $u_j^\varepsilon \in \partial\phi_{id}(b_j^\varepsilon)$ for all $j = 1, ..., N(\varepsilon)$ and

$$\phi_{id}(b_j^\varepsilon) - \phi_{id}(b_{j-1}^\varepsilon) \le (b_j^\varepsilon - b_{j-1}^\varepsilon)u_j^\varepsilon \qquad (4.3.26)$$

holds almost everywhere in Ω. Therefore from (4.3.25) and (4.3.26) it follows that

$$\int_\Omega \frac{\phi_{id}(b_j^\varepsilon) - \phi_{id}(b_{j-1}^\varepsilon)}{\varepsilon} + \lambda \int_\Omega \sum_{i=1}^N \left|\frac{\partial u_j^\varepsilon}{\partial x_i}\right|^p \omega_i(x)dx \le \int_\Omega f_j^\varepsilon u_j^\varepsilon. \qquad (4.3.27)$$

Integrating (4.3.27) over $(t_{j-1}, t_j]$, and taking the sum over $j = 1, ..., N(\varepsilon)$ yields

$$\int_\Omega \phi_{id}(b_\varepsilon(T)) + \lambda \int_0^T \int_\Omega \sum_{i=1}^N |\frac{\partial u_\varepsilon}{\partial x_i}|^p \omega_i(x)dxdt \le \int_\Omega \phi_{id}(b_\varepsilon(0)) \qquad (4.3.28)$$

$$+C\|f_\varepsilon\|_{L^\infty(Q)}\Big(\int_0^T \int_\Omega \sum_{i=1}^N |\frac{\partial u_\varepsilon(x)}{\partial x_i}|^p \omega_i(x)dxdt\Big)^{\frac{1}{p}}\Big(\int_{Q_T} \sigma^{1-q'}dx\Big)^{\frac{1}{q'}} \text{(due to(1.1.6))}.$$

According to (D_ε), $b_\varepsilon(0) = b_0^\varepsilon$ converges to b_0 and f_ε converges to f in $L^1(0, T, L^1(\Omega))$ as $\varepsilon \downarrow 0$ Therefore, the right-hand side of (4.3.28) is bounded by a constant $C_1(\|f\|_{L^\infty(Q_T)}, \|b_0\|_{L^1(\Omega)}) > 0$ that does not depend on ε. Now, (iii) follows from (4.3.28) if we neglect the positive term .

4.3.2 Cas where β continuous, non-decreasing

Lemma 4.3.2 *Let β be a continuous and non-decreasing function, $f \in L^\infty(Q_T)$, $b_0 \in \overline{D(A_\beta)}^{\|.\|_{L^1(\Omega)}} \cap L^\infty(\Omega)$. For ε, $\delta > 0$ let $(D_\varepsilon), (D_\delta)$ be equidistant time discretisations and $(u_i^\varepsilon, b_i^\varepsilon)_{i=1}^{N(\varepsilon)}$, $(u_j^\delta, b_j^\delta)_{j=1}^{M(\delta)}$ solutions of the corresponding discretised problems $(DP_{\varepsilon,\psi})$ and $(DP_{\delta,\psi})$. Assume that the piecewise constant functions $b_\varepsilon, b_\delta : [0, T] \to L^1(\Omega)$ defined by $b_\varepsilon(0) = b_0^\varepsilon$, $b_\delta(0) = b_0^\delta$, $b_\varepsilon(t) = b_i^\varepsilon$, $b_\delta(t) = b_j^\delta$ for*

$t \in (t_{i-1}^{\varepsilon}, t_i^{\varepsilon}]$ and $t \in (t_{j-1}^{\delta}, t_j^{\delta}]$ respectively, $i = 1, ..., N(\varepsilon), j = 1, ..., M(\delta)$ converge to a function $b \in \mathcal{C}([0,T]\,;\,L^1(\Omega))$ as $\varepsilon, \delta \downarrow 0$ in $L^\infty(0,T; L^1(\Omega))$.

$$\textit{Then,} \qquad \lim_{\varepsilon,\delta\downarrow 0} \int_0^T \int_\Omega |\psi(u_\varepsilon) - \psi(u_\delta)| = 0 \tag{4.3.29}$$

holds for the piecewise constant functions $u_\varepsilon, u_\delta : (0,T) \to W_0^{1,p}(\Omega,\omega)$ defined by $u_\varepsilon(t) = u_i^\varepsilon$ for $t \in (t_{i-1}^{\varepsilon}, t_i^{\varepsilon}]$ and $i = 1, ..., N(\varepsilon)$, $u_\delta(t) = u_j^\delta$ for $(t_{j-1}^{\delta}, t_j^{\delta}]$ and $j = 1, ..., M(\delta)$.

Proof : Use analogous arguments (see [89]).
The a priori estimates of Lemma 4.3.1 and Lemma 4.3.2 imply the following convergence results for the approximate solutions of $(DP_{\varepsilon,\psi})$ for $\varepsilon \downarrow 0$:

Lemma 4.3.3 *Let $\varepsilon > 0$ take values in a sequence in (0, 1) tending to 0. For $f \in L^\infty(Q_T)$, $b_0 \in \overline{D(A_\beta)}^{\|.\|_{L^1(\Omega)}} \cap L^\infty(\Omega)$ let u_ε, b_ε be the piecewise constant functions defined by $(DP_{\varepsilon,\psi})$. Then there exist functions $b \in C([0,T]; L^1(\Omega))$, $u \in L^p(0,T; W_0^{1,p}(\Omega,\omega)) \cap L^\infty(Q_T)$ and $\Phi \in \prod_{i=1}^N L^{p'}(Q_T, \omega_i^*)$ such that for a not relabeled subsequence of $(u_\varepsilon)_\varepsilon$ we have the following convergence results for $\varepsilon \downarrow 0$:*

i) *$u_\varepsilon \to u$ almost everywhere in Q_T, weak-$*$ in $L^\infty(Q_T)$ and weak in $L^p(0,T; W_0^{1,p}(\Omega,\omega))$,*

ii) *$b_\varepsilon \to b$ in $L^\infty(0,T,L^1(\Omega))$ and $b = \beta(u)$ almost everywhere in Q_T,*

iii) *$Du_\varepsilon \rightharpoonup Du$ in $\prod_{i=1}^N L^p(Q_T, \omega_i)$,*

iv) *$a(x, Du_\varepsilon) \rightharpoonup \Phi$ in $\prod_{i=1}^N L^{p'}(Q_T, \omega_i^*)$.*

Proof : If $(\varepsilon_n)_n \subset (0,1)$ is a sequence tending to 0 as $n \to \infty$, applying Lemma 4.3.2 with $u_\varepsilon = u_{\varepsilon_n}$, $u_\delta = u_{\varepsilon_m}$ for $m, n \in \mathbb{N}$ yields that (passing to a subsequence if necessary)

$$|\psi(u_{\varepsilon_n}) - \psi(u_{\varepsilon_m})| \to 0$$

almost everywhere in Q_T as $m, n \to \infty$. Hence, $(u_{\varepsilon_n})_n$ is a Cauchy sequence almost everywhere in Q_T and there exists a measurable function $u : Q_T \to \mathbb{R}$ such that $u_{\varepsilon_n} \to u$ almost everywhere in Q_T as $n \to \infty$. By (4.3.23) and (4.3.24) it follows that $u \in V \cap L^\infty(Q_T)$ and i), iii) hold. By definition of the operator $A_{\beta,\psi}$, assuming β to be a continuous, non-decreasing function it follows that $b_\varepsilon(t) = \beta(u_\varepsilon(t))$ a.e. in $(0, T)$ for all $\varepsilon > 0$. Keeping in mind that by nonlinear semigroup theory, $(b_\varepsilon)_\varepsilon$ converges to the mild solution $b \in \mathcal{C}([0, T]; L^1(\Omega))$ of $(ACP)(f, \psi, b_0)$ as $\varepsilon \downarrow 0$ and using i) and the continuity of β, ii) holds. Applying (4.3.24) and (4.2.6) (and passing to a subsequence if necessary), we find that $a(x, Du_\varepsilon) \rightharpoonup \Phi$ in $\prod_{i=1}^N L^{p'}(Q_T, \omega_i^*)$ for some $\Phi \in \prod_{i=1}^N L^{p'}(Q_T, \omega_i^*)$.
Using the convergence results of the preceeding Lemma, we have the following existence result.

Proposition 4.3.4 *If $\beta : \mathbb{R} \to \mathbb{R}$ is a continuous and non-decreasing function, then there exists a weak solution (in the sense of Definition 4.2.5)* $(u, b = \beta(u))$ *to* (P, ψ, f, b_0) *for any* $f \in L^\infty(Q_T)$ *and* $b_0 \in \overline{D(A_\beta)}^{\|\cdot\|_{L^1(\Omega)}} \cap L^\infty(\Omega)$.

Proof : Let $\tilde{b}_\varepsilon : [0, T] \to L^1(\Omega)$ be the piecewise linear function defined by $\tilde{b}_\varepsilon(t) = b^\varepsilon_{j-1} + \frac{t - t^\varepsilon_{j-1}}{t^\varepsilon_j - t^\varepsilon_{j-1}}(b^\varepsilon_j - b^\varepsilon_{j-1})$ for $t \in [t^\varepsilon_{j-1}, t^\varepsilon_j]$, $j = 1, ..., N(\varepsilon)$. For arbitrary $\xi \in D([0, T) \times \Omega)$ and $t \in [0, T)$ the function $\Omega \ni x \to \xi(t, x)$ is in $D(\Omega)$, hence we can use it as a test function each equation of $(DP_{\varepsilon,\psi})$. Integrating over $(t^\varepsilon_{j-1}, t^\varepsilon_j)$ and summing over $j = 1, ..., N(\varepsilon)$ we find

$$-\int_0^T \int_\Omega \tilde{b}_\varepsilon \xi_t + \Big(a(x, Du_\varepsilon) + F(u_\varepsilon)\Big) \cdot D\xi + \psi(u_\varepsilon)\xi - \int_\Omega \tilde{b}_\varepsilon(0)\xi(0, .) = \int_0^T \int_\Omega f_\varepsilon \xi. \tag{4.3.30}$$

Since $\tilde{b}_\varepsilon \to b$ in $\mathcal{C}([0, T]; L^1(\Omega))$ as $\varepsilon \downarrow 0$ using the convergence results of Lemma 4.3.3 we can pass to the limit in (4.3.30) to obtain

$$-\int_0^T \int_\Omega (b - b_0)\xi_t + (\Phi + F(u)) \cdot D\xi + \psi(u)\xi = \int_0^T \int_\Omega f\xi, \tag{4.3.31}$$

for all $\xi \in D([0, T) \times \Omega)$, where $b = \beta(u)$. It is left to prove that $\Phi = a(x, Du)$. To this end let κ be a non-negative function in $\mathcal{C}_c^\infty([0, T))$. We discretise κ with respect to $(DP_{\varepsilon,\psi})$ by setting $\kappa_\varepsilon(0) = \kappa(0)$ and $\kappa_\varepsilon(t) = \kappa(t^\varepsilon_j)$ for $t \in (t^\varepsilon_{j-1}, t^\varepsilon_j]$ and

$j=1,......,N(\varepsilon)$. Taking $\kappa(t_j^\varepsilon)u_j^\varepsilon$ as a test function in $(DP_{\varepsilon,\psi})$ yields :

$$\kappa(t_j^\varepsilon)\int_\Omega \frac{b_j^\varepsilon - b_{j-1}^\varepsilon}{\varepsilon}u_j^\varepsilon + (a(x,Du_j^\varepsilon)+F(u_j^\varepsilon))\cdot Du_j^\varepsilon + \psi(u_j^\varepsilon)u_j^\varepsilon = \int_\Omega f_j^\varepsilon \kappa(t_j^\varepsilon)u_j^\varepsilon \tag{4.3.32}$$

for all $j=1,...,N(\varepsilon)$. If we define $\phi_{id}:\mathbb{R}\to\mathbb{R}\cup\{+\infty\}$ by

$$\phi_{id}(r)=\begin{cases}\int_0^r(\beta^{-1})^0(\sigma)d\sigma, & if\ r\in\overline{R(\beta)}\\ +\infty & otherwise,\end{cases} \tag{4.3.33}$$

since $b_j^\varepsilon=\beta(u_j^\varepsilon)$ for all $j=1,...,N(\varepsilon)$ it follows that

$$\frac{b_j^\varepsilon - b_{j-1}^\varepsilon}{\varepsilon}u_j^\varepsilon \geq \frac{1}{\varepsilon}\int_{b_{j-1}^\varepsilon}^{b_j^\varepsilon}(\beta^{-1})^0(\sigma)d\sigma = \frac{1}{\varepsilon}(\phi_{id}(b_j^\varepsilon)-\phi_{id}(b_{j-1}^\varepsilon)) \tag{4.3.34}$$

a.e. in Ω. Now, integration over $(t_{j-1}^\varepsilon,t_j^\varepsilon)$ in (4.3.32) and summation over $j=1,...,N(\varepsilon)$ yields :

$$\begin{aligned}&\sum_{j=1}^{N(\varepsilon)}\int_\Omega\Big(\phi_{id}(b_j^\varepsilon)-\phi_{id}(b_{j-1}^\varepsilon)\Big)\kappa(t_j^\varepsilon)\\ &+\int_{Q_T}\kappa_\varepsilon((a(x,Du_\varepsilon)+F(u_\varepsilon))\cdot Du_\varepsilon+\psi(u_\varepsilon)u_\varepsilon)\leq\int_{Q_T}f_\varepsilon u_\varepsilon\kappa_\varepsilon,\end{aligned} \tag{4.3.35}$$

where $u_\varepsilon:(0,T)\to W_0^{1,p}(\Omega,\omega)$ is the piecewise constant function defined by $u_\varepsilon(t)=u_j^\varepsilon$, for $t\in(t_{j-1}^\varepsilon,t_j^\varepsilon]$, $j=1,...,N(\varepsilon)$ and $f_\varepsilon:(0,T)\to L^\infty(\Omega)$ is the piecewise constant function defined by $f_\varepsilon(t)=f_j^\varepsilon$ for $t\in(t_{j-1}^\varepsilon,t_j^\varepsilon]$, $j=1,...,N(\varepsilon)$. Using summation by parts in the first term of (4.3.35) and setting $b_\varepsilon(t)=b_j^\varepsilon$ for $t\in(t_{j-1}^\varepsilon,t_j^\varepsilon]$, $j=1,...,N(\varepsilon)$, $b_\varepsilon(t)=b_0^\varepsilon$ for $t\in(-\varepsilon,0]$ it follows that

$$\begin{aligned}\int_{Q_T}\kappa_\varepsilon a(x,Du_\varepsilon)\cdot Du_\varepsilon\leq\int_{-\varepsilon}^{T-\varepsilon}\int_\Omega\kappa_t(t+\varepsilon)\phi_{id}(b_\varepsilon)+\int_\Omega\phi_{id}(b_0^\varepsilon)\kappa_\varepsilon(0)\\ -\int_{Q_T}\kappa_\varepsilon(F(u_\varepsilon)\cdot Du_\varepsilon+(\psi(u_\varepsilon)-f_\varepsilon)u_\varepsilon).\end{aligned} \tag{4.3.36}$$

Using the convergence results of Lemma 4.3.3, there is no problem to pass to the limit with $\varepsilon\downarrow 0$ on the right-hand side of (4.3.36). Moreover,

$$\begin{aligned}&\limsup_{\varepsilon\downarrow 0}\int_{Q_T}\kappa_\varepsilon a(x,Du_\varepsilon)\cdot Du_\varepsilon\\ &\geq\limsup_{\varepsilon\downarrow 0}\int_{Q_T}\kappa a(x,Du_\varepsilon)\cdot Du_\varepsilon+\liminf_{\varepsilon\downarrow 0}\int_{Q_T}(\kappa_\varepsilon-\kappa)a(x,Du_\varepsilon)\cdot Du_\varepsilon\end{aligned} \tag{4.3.37}$$

where the second term on the right hand side of (4.3.37) is 0 by (A2), (4.3.24) and since $\|\kappa_\varepsilon - \kappa\|_{L^\infty(0,T)} \to 0$. Combining (4.3.36) with (4.3.37) and passing to the limit with $\varepsilon \downarrow 0$, we find

$$\limsup_{\varepsilon\downarrow 0} \int_{Q_T} \kappa a(x, Du_\varepsilon) \cdot Du_\varepsilon$$
$$\leq \int_{Q_T} \kappa_t(t)\phi_{id}(b) + \int_\Omega \phi_{id}(b_0)\kappa(0) - \int_{Q_T} \kappa(F(u) \cdot Du + \psi(u)u - fu). \quad (4.3.38)$$

Since (4.3.31) holds, we can apply the integration-by-parts formula of Lemma 4.2.4 with $h(u) = u$ and $\xi = \kappa\chi_\Omega$ to obtain

$$\int_{Q_T} \kappa_t \int_{b_0}^{b(t,x)} (\beta^{-1})^0(\sigma)d\sigma = \int_{Q_T} \kappa(F(u) \cdot Du + \Phi \cdot Du + \psi(u)u - fu) \quad (4.3.39)$$

for all $\kappa \in \mathcal{C}_c^\infty([0,T))$, $\kappa \geq 0$. Combining (4.3.38) and (4.3.39), we finally get

$$\limsup_{\varepsilon\downarrow 0} \int_{Q_T} \kappa a(x, Du_\varepsilon) \cdot Du_\varepsilon \leq \int_{Q_T} \kappa\Phi \cdot Du. \quad (4.3.40)$$

Therefore it follows that

$$\limsup_{\varepsilon\downarrow 0} \int_{Q_T} \kappa(a(x, Du_\varepsilon) - a(x, Du)) \cdot (Du_\varepsilon - Du) \leq 0 \quad (4.3.41)$$

for all $\kappa \in \mathcal{C}_c^\infty([0,T))$, $\kappa \geq 0$. Using (4.2.7) and Minty's monotonicity argument we get $\Phi = a(x, Du)$ from (4.3.40) and (4.3.41). Moreover, choosing $\kappa = \chi_{(0,\tau)}$ for $0 < \tau < T$, from (4.3.41) we obtain $a(x, Du_\varepsilon) \cdot Du_\varepsilon \rightharpoonup a(x, Du) \cdot Du$ weak in $L^1((0,\tau) \times \Omega)$.

4.3.3 The general case of multivalued β

Now, let $\beta \subset \mathbb{R} \times \mathbb{R}$ be a maximal monotone graph. To continue the proof of Theorem 4.2.10, we proceed as in [86] (see also [87]) in the case of a constant exponent and combine the techniques developed in [41] with the approach from [22] so that we do not need the additional assumption that β^{-1} is continuous and defined on $\mathbb{R}$ if we accept one more approximation procedure. Once we have found the appriopriate energy space $L^p(0, T; W_0^{1,p}(\Omega, \omega))$. For the first approximation procedure let us regularize β by $\beta_k := \beta + \frac{1}{k}I$, $k > 0$. Clearly the results of Subsection

4.2.4 still apply to the nonlinear operator

$$A_{\beta_k,\psi} := \{(b_k, w_k) \in L^1(\Omega) \times L^1(\Omega) : \ \exists u_k : \Omega \to \mathbb{R} \text{ measurable, } b_k \in \beta(u_k)$$
$$a.e. \text{ in } \Omega \text{ and } u_k \text{ is a renormalized solution of}$$
$$-div(a(x, Du_k) + F(u_k)) + \psi(u_k) = w_k\}$$

and therefore there exists a unique mild solution $b^k \in \mathcal{C}([0,T]; L^1(\Omega))$ of the abstract Cauchy problem

$$(ACP)_k(, \psi, f, b_0^k) \begin{cases} \frac{db^k}{dt} + A_{\beta_k,\psi} b^k \ni f \text{ in } (0,T) \\ b^k(0) = b_0^k \end{cases}$$

corresponding to (P_k, ψ, f, b_0^k) for any given $f \in L^1(Q_T)$, $b_0^k \in \overline{D(A_{\beta_k,\psi})}^{\|.\|_{L^1(\Omega)}}$ and $k > 0$. Moreover, it follows from the results for the elliptic case (see Theorem 3.3.1) that for any $f \in L^1(\Omega)$

$$\lim_{k\to\infty} \|(I + A_{\beta_k,\psi})^{-1} f - (I + A_{\beta,\psi})^{-1} f\|_{L^1(\Omega)} = 0. \tag{4.3.42}$$

Applying the a priori estimates of Lemma 4.3.1, we get the following convergence results for the solutions of the discretized problems $(DP^k_{\varepsilon,\psi})$:

Lemma 4.3.5 *For $f \in L^\infty(Q_T)$, $b_0^k \in \overline{D(A_{\beta_k,\psi})}^{\|.\|_{L^1(\Omega)}} \cap L^\infty(\Omega)$ and $\varepsilon, k > 0$, let $(b_j^{\varepsilon,k}, u_j^{\varepsilon,k})_{j=1}^{N(\varepsilon)}$ be a solution of the discretized problem $(DP^k_{\varepsilon,\psi})$. For $k > 0$, let $b^k \in \mathcal{C}([0,T]; L^1(\Omega))$ be the $L^\infty(0,T; L^1(\Omega))$ -limit of the sequence of piecewise constant functions $(b^k_\varepsilon)_\varepsilon$ defined by $b^k_\varepsilon : (0,T) \to L^1(\Omega)$, $b^k_\varepsilon(0) = b_0^{\varepsilon,k}$, $b^k_\varepsilon(t) = b_j^{\varepsilon,k}$ for $t \in (t^\varepsilon_{j-1}, t^\varepsilon_j]$ and $j = 1, ..., N(\varepsilon)$. If we define $u^k_\varepsilon : (0,T) \to W_0^{1,p}(\Omega,\omega) \cap {}^\infty(\Omega)$ by $u^k_\varepsilon(t) = u_j^{\varepsilon,k}$ for $t \in (t^\varepsilon_{j-1}, t^\varepsilon_j]$, $j = 1, ..., N(\varepsilon)$, there exists $u^k \in L^p(0,T; W_0^{1,p}(\Omega,\omega)) \cap L^\infty(Q_T)$ and a subsequence of $(u^k_\varepsilon)_\varepsilon$ such that, as $\varepsilon \downarrow 0$,*

i) *$u^k_\varepsilon \to u^k$ almost everywhere in Q_T, weak-$*$ in $L^\infty(Q_T)$ and weak in $L^p(0,T, W_0^{1,p}(\Omega,\omega))$,*

ii) *$b^k_\varepsilon \to b^k$ in $L^\infty(0,T; L^1(\Omega))$, in $L^1(Q_T)$ and almost everywhere in Q_T . Moreover, $b^k \in \beta_k(u^k)$ almost everywhere in Q_T,*

iii) $Du^k_\varepsilon \rightharpoonup Du^k$ *in* $\prod_{i=1}^N L^p(Q_T,\omega_i)$,

iv) $a(x,Du^k_\varepsilon) \rightharpoonup a(x,Du^k)$ *in* $\prod_{i=1}^N L^{p'}(Q_T,\omega_i^*)$.

Proof : Using the a priori estimates (4.3.22), (4.3.23) and (4.3.24), it follows immediately that there exists $u^k \in L^p(0,T;W_0^{1,p}(\Omega,\omega)) \cap L^\infty(Q_T)$ such that, passing to a subsequence if necessary, iii) holds and $u^k_\varepsilon \rightharpoonup u^k$ weak-$*$ in $L^\infty(Q_T)$ and weak in $L^p(0,T;W_0^{1,p}(\Omega,\omega))$. The convergence of b^k_ε to b^k in $L^\infty(0,T;L^1(\Omega))$ follows immediately from nonlinear semigroup theory and this implies the other convergence results for a subsequence of $(b^k_\varepsilon)_\varepsilon$. By $(DP^k_{\varepsilon,\psi})$, we have $b^k_\varepsilon \in \beta_k(u^k_\varepsilon)$ almost everywhere in Q_T .
Since β is a maximal monotone graph, $(\beta+\frac{1}{k}I)^{-1} = k(k\beta+I)^{-1}$ is single-valued and Lipschitz continuous in $\mathbb{R}$, hence $u^k_\varepsilon := (\beta+\frac{1}{k}I)^{-1}b^k_\varepsilon$ converges to $u^k = (\beta+\frac{1}{k}I)^{-1}b^k$ almost everywhere in Q_T. Therefore i) and ii) hold. Finally, iv) follows with the same arguments as in the proof of Lemma 4.3.3 and Proposition 4.3.4. Using the convergence results of Lemma 4.3.5 we can prove the following result :

Proposition 4.3.6 *For any* $k>0$, $f \in L^\infty(Q_T)$ *and* $b_0^k \in \overline{D(A_{\beta_k,\psi})}^{\|\cdot\|_{L^1(\Omega)}} \cap L^\infty(\Omega)$ *there exists a weak solution* (u^k,b^k) *to* (P_k,ψ,f,b_0^k). *In particular,* b^k *is the mild solution of* $(ACP)_k(\psi,f,b_0^k)$.

Proof : The assertion follows according to the convergence results of Lemma 4.3.5 and by similar arguments as in the proof of Proposition 4.3.4. Next we want to obtain a weak solution (u,b) of (P,ψ,f,b_0) for $f \in L^\infty(Q_T)$ and $b_0 \in \overline{D(A_{\beta_k,\psi})}^{\|\cdot\|_{L^1(\Omega)}} \cap L^\infty(\Omega)$ by passing to the limit with $k\to\infty$ in the approximate equations (P_k,ψ,f,b_0). The convergence of the sequence $(b^k)_k$ is an immediate consequence of nonlinear semigroup theory.

Lemma 4.3.7 *If* b_0 *is in* $\overline{D(A_{\beta,\psi})}^{\|\cdot\|_{L^1(\Omega)}}$ *such that there exists* $(b_0^k)_k \subset L^1(\Omega)$ *with* $b_0^k \in \overline{D(A_{\beta_k,\psi})}^{\|\cdot\|_{L^1(\Omega)}}$ *for all* $k>0$ *and* $b_0^k \to b_0$ *in* $L^1(\Omega)$ *as* $k\to\infty$, *then* b^k *converges in* $\mathcal{C}([0,T];L^1(\Omega))$ *to the mild solution* b *of* $(ACP)(\psi,f,b_0)$ *as* $k\to\infty$.

Proof : From (4.3.42) it follows that $A_{\beta,\psi} \subset \liminf_{k\to\infty} A_{\beta_k,\psi}$ and therefore the assertion follows according to nonlinear semigroup theory (see, e.g. [37]). The following comparison principle is a corresponding result for multivalued β and was proved in [41] and [86] in the constant exponent case.

Lemma 4.3.8 *For $f \in L^1(Q_T)$, $l, k > 0, b_0^k, b_0^l \in L^1(\Omega)$ such that*

$$\lim_{k,l\to\infty} \|b_0^k - b_0^l\|_{L^1(\Omega)} = 0$$

let $(u^k, b^k), (u^l, b^l)$ be the weak solution of $(P_k, \psi, f, b_0^k), (P_l, \psi, f, b_0^l)$ respectively. Then,

$$\lim_{k,l\to\infty} \int_\tau^\theta \int_\Omega |\psi(u_k) - \psi(u_l)| = 0 \tag{4.3.43}$$

holds for all $0 < \tau < \theta < T$.

Proof : The proof of this lemma follows the same lines as the proof of the corresponding result in the case of a constant exponent p as stated in [86], Proposition 4.2.2., and (see also [41], Proposition 4.2) and is omitted here in detail. It is based on Kruzhkov's doubling of variable technique (see e.g. [73]) that has been adapted by other authors (see [86, 41]) to prove uniqueness results and comparison principles for elliptic-parabolic problems. In our particular case, we only have to double the time variables : Let t, s denote two variables in $[0, T]$. We write the t variable in the weak formulation of (P_k, ψ, f, b_0^k) and the s variable in the weak formulation of (P_l, ψ, f, b_0^l). For $\delta > 0$, and $r \in \mathbb{R}$ we define the function $r \to \eta_\delta(r)$ by $\eta_\delta(r) := \frac{1}{\delta} T_\delta(r)$. According to the integration-by-parts formula of Lemma 4.2.4 we choose $\eta_\delta\Big(u_\delta(t,x) - u_l(s,x) + \delta\pi(x)\Big)\phi(t)\rho_n(t-s)$ as a test function in (P_k, ψ, f, b_0^k) and (P_l, ψ, f, b_0^l) where $\pi \in D(\Omega)$ such that $0 \le \pi \le 1$, $\phi \in D([0,T))$ such that $\phi \ge 0$ and $(\rho_n)_n$ is a sequence of mollifiers in $\mathbb{R}$. There is no problem to pass to the limit with $\delta \downarrow 0$ in the diffusion and the convection term because F is assumed to be locally Lipschitz continuous.

Using this result, we can prove the following :

Proposition 4.3.9 *For any $f \in L^\infty(Q_T)$ and $b_0 \in \overline{D(A_{\beta_k,\psi})}^{\|\cdot\|_{L^1(\Omega)}} \cap L^\infty(\Omega)$ there exists $u \in L^p(0,T; W_0^{1,p}(\Omega,\omega)) \cap L^\infty(Q_T)$ and $b \in \mathcal{C}([0,T]; L^1(\Omega))$ such that (u, b) is a weak solution to (P, ψ, f, b_0). In particular, b is the mild solution of $(ACP)(\psi, f, b_0)$.*

Proof : According to Corollary4.2.1 (iii), we have

$$\overline{D(A_{\beta,\Psi})}^{\|\cdot\|_{L^1(\Omega)}} \subset \overline{D(A_{\beta_k,\psi})}^{\|\cdot\|_{L^1(\Omega)}}$$

for all $k > 0$. Therefore, if b_0 is in $\overline{D(A_{\beta,\psi})}^{\|\cdot\|_{L^1(\Omega)}} \cap L^\infty(\Omega)$ it is in $\overline{D(A_{\beta_k,\psi})}^{\|\cdot\|_{L^1(\Omega)}} \cap L^\infty(\Omega)$ for all $k > 0$ and by Proposition 4.3.6, (P_k, ψ, b_0, f) has a weak solution $(u^k, b^k) \subset \Big(L^p(0,T; W_0^{1,p}(\Omega,\omega)) \cap L^\infty(Q_T)\Big) \times \mathcal{C}([0,T]; L^1(\Omega))$ for all $k > 0$. In particular,

$$\int_{Q_T} -(b^k - b_0)\xi_t + (a(x, Du^k) + F(u^k)) \cdot D\xi + \psi(u^k)\xi = \int_{Q_T} f\xi \qquad (4.3.44)$$

holds for any $\xi \in D([0,T) \times \Omega)$. By Lemma 4.3.7, $b^k \to b$ as $k \to \infty$ in $\mathcal{C}([0,T]; L^1(\Omega))$, where $b \in \mathcal{C}([0,T]; L^1(\Omega))$ is the mild solution of $(ACP)(\psi, f, b_0)$ and therefore $b(0) = b_0$ almost everywhere in Ω. From Lemma 4.3.8 it follows with the same arguments as in the proof of Lemma 4.3.3 that there exists a measurable function $u : Q_T \to \mathbb{R}$ and a (not relabeled) subsequence of $(u^k)_k$ such that $u^k \to u$ almost everywhere as $k \to \infty$. Since the a priori estimates of Lemma 4.3.1 still hold for u^k, independently of $k > 0$, it follows that, as $k \to \infty$ and up to a non-relabeled subsequence, u^k converges to u weak-$*$ in $L^\infty(Q_T)$ and weak in $L^p(0,T; W_0^{1,p}(\Omega,\omega))$, $Du^k \rightharpoonup Du$ in $\prod_{i=1}^N L^p(Q_T, \omega_i)$, hence u is in $L^p(0,T; W_0^{1,p}(\Omega,\omega)) \cap L^\infty(Q_T)$. Moreover, there exists $\Phi \in \prod_{i=1}^N L^{p'}(Q_T, \omega_i^*)$ such that $a(x, Du^k) \rightharpoonup \Phi$ weak in $\prod_{i=1}^N L^{p'}(Q_T, \omega_i^*)$ as $k \to \infty$. Using these convergence results we can pass to the limit in (4.3.44)) and find that

$$-\int_{Q_T} (b - b_0)\xi_t + \int_{Q_T} (\Phi + F(u)) \cdot D\xi + \int_{Q_T} \psi(u)\xi = \int_{Q_T} f\xi \qquad (4.3.45)$$

holds for all $\xi \in D([0,T) \times \Omega)$. Next, we prove $a(x, Du) = \Phi$. To this end, we fix $\sigma \in D([0,T))$, $\sigma \geq 0$ and $l > 0$. Since (4.3.44) holds, by Lemma 4.2.4 we can use $\sigma T_l(u^k)$ as a test function and obtain

$$\begin{aligned} -\int_{Q_T} \sigma_t \int_{b_0}^{b^k} T_l \circ (\beta + \frac{1}{k}I)^{-1}(s)ds + \int_{Q_T} \sigma a(x, Du^k) \cdot DT_l(u^k) \\ = -\int_{Q_T} \sigma\big(F(u^k) \cdot DT_l(u^k) + (\psi(u^k) - f)T_l(u^k)\big). \qquad (4.3.46) \end{aligned}$$

There is no problem to pass to the limit with $k \to \infty$ on the right-hand side of (4.3.46). To pass to the limit in the first term on the left-hand side, we write

$$\int_{b_0}^{b^k} T_l \circ (\beta + \frac{1}{k}I)^{-1}(s)ds = I_1 + I_2 \qquad (4.3.47)$$

where, since $b^k \in (\beta + \frac{1}{k}I)u^k$ almost everywhere in Q_T ,

$$I_1 = \int_0^{b^k} T_l \circ (\beta + \frac{1}{k}I)^{-1}(s)ds = b^k T_l(u^k) - \int_0^{T_l(u^k)} (\beta^0 + \frac{1}{k}I)(s)ds \qquad (4.3.48)$$

(see [26, 77]) almost everywhere in Q_T and

$$I_2 = -\int_0^{b_0} T_l \circ (\beta + \frac{1}{k}I)^{-1}(s)ds \qquad (4.3.49)$$

Now, setting $u_0 := (\beta^{-1})^0(b_0)$ we have $(b_0 + \frac{1}{k}u_0) \in (\beta + \frac{1}{k}I)u_0$, hence

$$I_2 = -b_0 T_l(u_0) + \int_0^{T_l(u_0)} (\beta^0 + \frac{1}{k}I)(s)ds - \int_{b_0}^{b_0+\frac{1}{k}u_0} T_l \circ (\beta + \frac{1}{k}I)^{-1}(s)ds \quad (4.3.50)$$

almost everywhere in Ω. Passing to the limit with $k \to \infty$ we find :

$$\lim_{k\to\infty} I_1 = -bT_l(u) + \int_0^{T_l(u)} (\beta^0)(s)ds = -\int_0^b T_l \circ (\beta^{-1})^0(s)ds \qquad (4.3.51)$$

almost everywhere in Q_T and

$$\lim_{k\to\infty} I_2 = -b_0 T_l(u_0) + \int_0^{T_l(u_0)} (\beta^0)(s)ds = -\int_0^{b_0} T_l \circ (\beta^{-1})^0(s)ds \qquad (4.3.52)$$

almost everywhere in Ω. Now, thanks to Lebesgue Dominated Convergence Theorem it follows that

$$\lim_{k\to\infty} -\int_{Q_T} \sigma_t \int_{b_0^k}^{b^k} T_l \circ (\beta + \frac{1}{k}I)^{-1}(s)ds = -\int_{Q_T} \sigma_t \int_{b_0}^{b} T_l \circ (\beta^{-1})^0(s)ds \quad (4.3.53)$$

and therefore

$$\begin{aligned} &\limsup_{k\to\infty} \int_{Q_T} \sigma a(x, Du^k) \cdot DT_l(u^k) = \\ &\int_{Q_T} \sigma_t \int_{b_0}^{b} T_l \circ (\beta^{-1})^0(s)ds - \int_{Q_T} \sigma(F(u) \cdot DT_l(u) + (\psi(u) - f)T_l(u)). \end{aligned} \quad (4.3.54)$$

Now, we use $\sigma T_l(u)$ as a test function in (4.3.45). By Lemma 4.2.4 we get

$$\begin{aligned} \int_{Q_T} \sigma\Phi \cdot DT_l(u) = &\int_{Q_T} \sigma_t \int_{b_0}^{b} T_l \circ (\beta^{-1})^0(s)ds \\ &- \int_{Q_T} \sigma(F(u) \cdot DT_l(u) + (\psi(u) - f)T_l(u)). \end{aligned} \qquad (4.3.55)$$

Subtracting (4.3.55) from (4.3.54) and choosing $l = \|u\|_{L^\infty(Q_T)}$ it follows that

$$\limsup_{k\to\infty} \int_{Q_T} \sigma a(x, Du^k) \cdot Du^k \leq \int_{Q_T} \sigma \Phi \cdot Du \tag{4.3.56}$$

for all $\sigma \in D([0,T))$, $\sigma \geq 0$. Furthermore, using (4.3.56) we have

$$\lim_{k\to\infty} \int_{Q_T} \sigma(a(x, Du^k) - a(x, Du)) \cdot (Du^k - Du) = 0 \tag{4.3.57}$$

for all $\sigma \in D([0,T))$, $\sigma \geq 0$. Now, $\Phi = a(x, Du)$ follows from (4.3.57) by the Minty monotonicity argument. It is left to prove that $b \in \beta(u)$ almost everywhere in Q_T . Since we have $b^k \in \beta_k(u^k)$ almost everywhere in Q_T , for any $k > 0$ there exists $B^k \in \beta(u^k)$ such that $b^k = B^k + \frac{1}{k}u^k$ and since $B^k \to b$ for $k \to \infty$ almost everywhere in Q_T . If we define $j : \mathbb{R} \to \mathbb{R} \sqcup \{+\infty\}$ by $j(r) = \int_0^r \beta^0(\sigma) d\sigma$ if $r \in \overline{D(\beta)}$ and $j(r) = +\infty$ otherwise, it is easy to see that j is a convex, l.s.c, proper function such that $\beta = \partial j$. Therefore,

$$j(r) \geq j(u^k) + B^k(r - u^k) \tag{4.3.58}$$

holds for any $r \in \mathbb{R}$ and almost everywhere in Q_T . Now, by the almost everywhere convergence of B^k to b and u^k to u, from (4.3.58) it follows that $b \in \beta(u)$ almost everywhere in Q_T.

4.3.4 L^1 -contraction and uniqueness of renormalized solutions

Proposition 4.3.10 *For $f_1,\ f_2 \in L^1(Q_T),\ b_0^1,\ b_0^2 \in L^1(\Omega)$, let $(u_1, b_1), (u_2, b_2)$ be renormalized solutions of (P, f_1, b_0^1), (P, f_2, b_0^2) respectively. Then*

$$\int_\Omega (b_1(t) - b_2(t))^+ \leq \int_0^t \int_\Omega (f_1 - f_2)^+ + \int_\Omega (b_0^1 - b_0^2)^+ \tag{4.3.59}$$

holds for almost all $t \in (0,T)$.

Proof : We can copy the proof in [41], Theorem 4.1 for the case of a constant exponent with slight modifications such as exchanging the space $L^p(0,T; W_0^{1,p(\cdot)}(\Omega))$ by $L^p(0,T; W_0^{1,p}(\Omega, \omega))$.

Remark 4.3.11 *The result of Proposition 4.3.10 still holds if we replace (P, f_i, b_0^i) by (P, ψ_i, f_i, b_0^i), where $\psi_i : \mathbb{R} \to \mathbb{R}$ is a continuous, non-decreasing function for $i = 1, 2$.*

Remark 4.3.12 *Uniqueness of renormalized solutions is a direct consequence of Proposition 4.3.10 : If (u, b) is a renormalized solution to (P, f, b_0) for $f \in L^1(Q_T)$ and $b_0 \in \overline{D(A_\beta)}^{\|\cdot\|_{L^1(\Omega)}}$, then b is unique. We cannot expect uniqueness of the function u without additional assumptions on β .*

4.3.5 A comparison principle and weak solutions for L^1-data

Lemma 4.3.13 *Let $b_0, \tilde{b}_0$ be in $L^1(\Omega)$ $f,\ \tilde{f} \in L^1(Q_T)$, $\psi, \tilde{\psi} : \mathbb{R} \to \mathbb{R}$ be strictly increasing, continuous functions and (u, b), $(\tilde{u}, \tilde{b})$ be weak solutions to (P, ψ, f, b_0) and $(P, \tilde{\psi}, \tilde{f}, \tilde{b}_0)$ respectively. If we have $b_0 \leq \tilde{b}_0$ almost everywhere in Ω, $f \leq \tilde{f}$ almost everywhere in Q_T and $\tilde{\psi}(r) \leq \psi(r)$ for all $r \in \mathbb{R}$, then $u \leq \tilde{u}$ holds almost everywhere in Q_T.*

Proof : As in the proof of the corresponding result in the case of a constant exponent ([86], Lemma 4.3.1., p. 120 and [41], Proposition 4.2.).

4.3.6 Conclusion of the proof of Theorem 4.2.9

For $n \in \mathbb{N}$, we define the continuous, strictly increasing function $\psi_n : \mathbb{R} \to \mathbb{R}$ by

$$\psi_n(r) := \frac{1}{n}(\arctan(r) + \frac{\pi}{2}),\ r \in \mathbb{R}.$$

Then, by Proposition 4.3.9, there exists a weak solution $(u_n, b_n) \in \big(L^p(0, T; W_0^{1,p}(\Omega, \omega)) \cap L^\infty(Q_T)\big) \times \mathcal{C}([0, T]; L^1(Q_T))$ to (P, ψ_n, f, b_0) for any $n \in \mathbb{N}$. Since $A_\beta \subset \liminf_{n\to\infty} A_{\beta,\psi_n}$ and b_n is the mild solution of $(ACP)(\psi_n, f, b_0)$, it follows that b_n converges in $\mathcal{C}([0, T]; L^1(\Omega))$ to the mild solution b of $(ACP)(f, b_0)$ as $n \to \infty$. Since $\psi_n \geq \psi_{n+1}$ in $\mathbb{R}$ for all $n \in \mathbb{N}$, from Lemma 4.3.13 it follows that $u_n \leq u_{n+1}$ almost everywhere in Q_T for all $n \in \mathbb{N}$, hence there exists a measurable function $u : Q_T \to \overline{\mathbb{R}}$ such that $u_n \uparrow u$ almost everywhere in Q_T. Moreover, $\arctan(r) - \frac{\pi}{2} \leq \psi_n \leq \arctan(r) + \frac{\pi}{2}$ for all $r \in \mathbb{R}$ and all $n \in \mathbb{N}$ and from Lemma 4.3.13 it follows that $u_{\frac{\pi}{2}} \leq u_n \leq u_{-\frac{\pi}{2}}$ almost everywhere in Q_T for all $n \in \mathbb{N}$ where $(u_{\frac{\pi}{2}}, b_{\frac{\pi}{2}}), (u_{-\frac{\pi}{2}}, b_{-\frac{\pi}{2}}) \in \big(L^p(0, T; W_0^{1,p}(\Omega, \omega)) \cap L^\infty(Q_T)\big) \times \mathcal{C}([0, T], L^1(\Omega))$ are the weak solutions to $(P, \arctan + \frac{\pi}{2}, f, b_0)$ and $(P, \arctan - \frac{\pi}{2}, f, b_0)$ respectively. Therefore $u \in L^\infty(Q_T)$ and $u_n \to u$ weak-$*$ in $L^\infty(Q_T)$ for a not relabeled subsequence of $(u_n)_n$. For $\delta > 0$ we define $\phi_\delta : [0, T] \to \mathbb{R}$ by $\phi_\delta(t) := \min(\frac{1}{\delta}\max(T - \delta - t, 0), 1)$. Thanks to the integration-by-parts formula of Lemma 4.2.4, can use $\phi_\delta T_k(u_n)$ as a test function and obtain for $k = \|u\|_{L^\infty(Q_T)}$:

$$\begin{aligned} &\frac{1}{\delta}\int_{T-2\delta}^{T+2\delta}\int_\Omega\int_0^{b_n(t,x)} T_{\|u\|_{L^\infty(Q_T)}} \circ (\beta^{-1})^0(\sigma)d\sigma - \int_\Omega\int_0^{b_0} T_{\|u\|_{L^\infty(Q_T)}} \circ (\beta^{-1})^0(\sigma)d\sigma \\ &+ \int_{Q_T}\phi_\delta\Big(((a(x, Du_n) + F(u_n)) \cdot Du_n) + \frac{1}{n}\arctan(u_n) \cdot u_n\Big) = \int_{Q_T}\phi_\delta u_n(f - \frac{\pi}{2}). \end{aligned} \tag{4.3.60}$$

We neglect positive terms and use (4.2.8), pass to the limit with $\delta \downarrow 0$ and obtain

$$\int_{Q_T}\sum_{i=1}^N |\frac{\partial u_n}{\partial x_i}|^p \omega_i dxdt \le C\|u\|_{L^\infty(Q_T)}\Big(\||f| + \frac{\pi}{2}\|_{L^1(Q_T)} + \|b_0\|_{L^1(\Omega)}\Big), \tag{4.3.61}$$

where $C > 0$ is a positive constant not depending on $n \in N$. From (4.3.61) we get $u \in L^p(0, T; W_0^{1,p}(\Omega, \omega))$ and there exists a (not relabeled) subsequence of $(u_n)_n$, such that $Du_n \rightharpoonup Du$ weak in $\prod_{i=1}^N L^p(Q_T, \omega_i)$ and $a(x, Du_n) \rightharpoonup \Phi$ weak in $\prod_{i=1}^N L^{p'}(Q_T, \omega_i^*)$ for a function $\Phi \in \prod_{i=1}^N L^{p'}(Q_T, \omega_i^*)$. Now we can pass to the limit with $n \to \infty$ in the weak formulation for (P, f, ψ_n, b_0) and obtain

$$-\int_{Q_T}(b - b_0)\xi_t + (\Phi + F(u)) \cdot D\xi = \int_{Q_T} f \cdot \xi \tag{4.3.62}$$

for all $\xi \in D([0, T) \times \Omega)$. With the same arguments as in the proof of Proposition 4.3.9 it follows that $\Phi = a(x, Du)$ (by Minty monotonicity argument) and $b \in \beta(u)$ (by a subdifferential argument).

To prove Theorem 4.2.10, we will use several approximation procedures.

4.4 Proof of Theorem 4.2.10

As in the proof of Theorem 3.3.1 for the elliptic case, we will construct monotone sequences of weak solutions for L^∞-data and show convergence (up to a subsequence) to a renormalized solution. The comparison principles from Lemma 4.3.13 and Lemma 4.3.10 will be a main tool in this approximation procedure.

Step1 : Approximate solutions and a priori estimates

Let f be in $L^1(Q_T)$ and $b_0 \in \overline{D(A_\beta)}^{\|\cdot\|_{L^1(\Omega)}}$. For $m,n \in \mathbb{N}$, let $f_{m,n} := \max(\min(f,m),-n)$, $b_0^{m,n} := \max(\min(b_0,m),-n)$. Furthermore, we define $\psi_{m,n} : \mathbb{R} \to \mathbb{R}$ by $\quad \psi_{m,n}(r) := \frac{1}{m}\max(r,0) - \frac{1}{n}\max(-r,0) \quad for\ r \in \mathbb{R}$.
By Proposition 4.3.9 , $(P, \psi_{m,n}, f_{m,n}, b_0^{m,n})$ has a weak solution $(u_{m,n}, b_{m,n})$ for all $m,n \in \mathbb{N}$. We have $\psi_{m,n} \geq \psi_{m+1,n}$ for all $n \in \mathbb{N}$ and $\psi_{m,n} \leq \psi_{m,n+1}$ on $\mathbb{R}$. By Lemma 4.3.13 it follows that
$u_{m,n} \leq u_{m+1,n}$ and $u_{m,n+1} \leq u_{m,n}$ almost everywhere in Q_T for any $m,n \in \mathbb{N}$.
Hence, there exist measurable functions $u^n : Q_T \to \mathbb{R} \cup \{+\infty\}$, $u : Q_T \to \overline{\mathbb{R}}$ such that $u_{m,n} \uparrow u^n$ *as* $m \to \infty$ and $u^n \downarrow u$ as $n \to \infty$ almost everywhere in Q_T. By Lemma 4.3.10, it follows that $b_{m,n} \leq b_{m+1,n}$ and $b_{m,n+1} \leq b_{m,n}$ almost everywhere in Q_T for any $n,m \in \mathbb{N}$. Note that we have also $\psi_{m,n}(r) \downarrow \psi^n(r) := -\frac{1}{n}\max(-r,0)$ as $m \to \infty$ and $\psi^n(r) \uparrow 0$ as $n \to \infty$ for all $r \in \mathbb{R}$. Therefore, $A_{\psi_n,\beta} \subset \liminf_{m\to 0} A_{\psi_{m,n},\beta}$ and $A_\beta \subset \liminf_{n \to 0} A_{\psi_n,\beta}$. By nonlinear semigroupe theory it follows that $b_{m,n} \uparrow b^n$ as $m \to \infty$ in $\mathcal{C}([0,T]; L^1(\Omega))$, where b^n is the mild solution of (ACP) (ψ^n, f^n, b_0^n) with $\psi^n(r) := -\frac{1}{n}\max(-r,0)$, $f^n := \max(f,-n)$, and $b_0^n := \max(b_0,-n)$ for $n \in \mathbb{N}$. Moreover, $b^n \downarrow b$ as $n \to \infty$ in $\mathcal{C}([0,T]; L^1(\Omega))$ where b is the mild solution of (ACP)(f, b_0). In the next steps, we will prove that (u,b) is a renormalized solution to (P, f, b_0). Therefore we need the following a priori estimate :

Lemma 4.4.1 *For $m,n \in \mathbb{N}$, $f \in L^1(Q_T)$ and $b_0 \in \overline{D(A_\beta)}^{\|\cdot\|_{L^1(\Omega)}}$ let $(u_{m,n}, b_{m,n})$ be the weak solution to $(P, \psi_{m,n}, f_{m,n}, b_0^{m,n})$. Then there exists a constant $C > 0$ not depending on $m,n \in \mathbb{N}$ such that*

$$\int_0^T \int_\Omega \sum_{i=1}^N |\frac{\partial T_k(u_{m,n})}{\partial x_i}|^p \omega_i dxdt \leq Ck(\|f\|_{L^1(Q_T)} + \|b_0\|_{L^1(\Omega)}) \tag{4.4.63}$$

holds for any $k > 0$ and all $m,n \in \mathbb{N}$.

Proof : We fix $k > 0$. For $\delta > 0$ we define $\phi_\delta : [0,T] \to \mathbb{R}$
by $\phi_\delta(t) = \min(\frac{1}{\delta}\max(T-\delta-t,0),1)$. Using the integration -by parts formula of Lemma 4.2.4 and density arguments, we plug $\phi_\delta T_k(u_{m,n})$ as a test function into $((P, \psi_{m,n}, f_{m,n}, b_0^{m,n}))$. Then, for $\delta > 0$ small enough, we find

$$I_1 + I_2 + I_3 + I_4 = I_5, \tag{4.4.64}$$

where

$$I_1 = -\int_{T-2\delta}^{T-\delta}\int_\Omega (\phi_\delta)_t \int_{b_0^{m,n}}^{b_{m,n}(t,x)} T_k \circ (\beta^{-1})^0(\sigma)d\sigma$$
$$= \frac{1}{\delta}\int_{T-2\delta}^{T-\delta}\int_\Omega \int_0^{b_{m,n}(t,x)} T_k \circ (\beta^{-1})^0(\sigma)d\sigma - \int_\Omega \int_0^{b_0^{m,n}} T_k \circ (\beta^{-1})^0(\sigma)d\sigma. \quad (4.4.65)$$

By (4.2.8) we get

$$I_2 = \int_{Q_T} \phi_\delta a(x, DT_k(u_{m,n})) \cdot DT_k(u_{m,n}) \geq \lambda \int_0^{T-2\delta} \int_\Omega \sum_{i=1}^N |\frac{\partial T_k(u_{m,n})}{\partial x_i}|^p \omega_i dxdt. \quad (4.4.66)$$

Note that applying Gauss-Green Theorem and the boundary condition

$$\int_\Omega F(T_k(u_{m,n}(t))) \cdot DT_k(u_{m,n}(t)) = 0$$

for almost all $t \in (0,T)$. Hence,

$$I_3 = \int_0^T \phi_\delta \int_\Omega F(T_k(u_{m,n})) \cdot DT_k(u_{m,n}) = 0. \quad (4.4.67)$$

Moreover, by monotonicity of $\psi_{m,n}$ we have

$$I_4 = \int_{Q_T} \psi_{m,n}(u_{m,n})T_k(u_{m,n})\phi_\delta \geq 0,$$

$$I_5 = \int_{Q_T} f_{m,n}T_k(u_{m,n})\phi_\delta \leq \|f\|_{L^1(Q_T)}k. \quad (4.4.68)$$

Now, plugging (4.4.65)-(4.4.68) into (4.4.64) and neglecting non-negative terms we arrive at

$$\lambda \int_0^{T-2\delta} \int_\Omega \sum_{i=1}^N |\frac{\partial T_k(u_{m,n})}{\partial x_i}|^p \omega_i \leq k\|f\|_{L^1(Q_T)} + \int_\Omega \int_0^{b_0^{m,n}} T_k \circ (\beta^{-1})^0(\sigma)d\sigma$$
$$\leq k\big(\|f\|_{L^1(Q_T)} + \|b_0\|_{L^1(\Omega)}\big), \quad (4.4.69)$$

for all $k >$ and all $m, n \in \mathbb{N}$. For $\delta \downarrow 0$, the assertion follows.

Remark 4.4.2 *There exists a constant $C > 0$, not depending on n, $l \in \mathbb{R}$ such that*

$$|\{|u_{m,n}| \geq l\}| \leq Cl^{-(1-\frac{1}{p})}, \quad (4.4.70)$$

and from (4.4.70) *it follows that*

$$\lim_{l\to\infty} |\{|u| \geq l\}| = 0. \tag{4.4.71}$$

Proof : Let $k > 0$ large enough

$$\begin{aligned} k|\{|u_{m,n}| > k\} \times [0,T]| &= \int_0^T \int_{\{|u_{m,n}|>k\}} |T_k(u_{m,n}|dxdt \\ &\leq \Big(\int_0^T \int_\Omega |T_k(u_{m,n}|^p \sigma dxdt\Big)^{\frac{1}{p}} \Big(\int_0^T \int_\Omega \sigma^{1-p'} dxdt\Big)^{\frac{1}{p'}} \\ &\leq TC\Big(\int_Q \sum_{i=1}^N |\frac{\partial T_k(u_{m,n})}{\partial x_i}|^p \omega_i\Big)^{\frac{1}{p}} \leq Ck^{\frac{1}{p}} \end{aligned}$$

which implies that,

$$|\{|u_{m,n}| > k\} \times [0,T]| \leq Ck^{\frac{1}{p}-1}, \;\; \forall k \geq 1.$$

So, we have

$$\lim_{k\to+\infty} |\{|u_{m,n}| > k\} \times [0,T]| = 0.$$

Since $b_{m,n} \in \beta(u_{m,n})$ almost everywhere in Q_T, it follows with subdifferential arguments as in the proof of Proposition 4.3.9 that $b^n \in \beta(u^n)$ and $b \in \beta(u)$ almost everywhere in Q_T.

Step2 : Convergence results

Now applying the diagonal principle and lemma we get the following convergence results :

Lemma 4.4.3 *For* $m, n \in \mathbb{N}$, $f \in L^1(Q_T)$ *and* $b_0 \in \overline{D(A_\beta)}^{\|.\|_{L^1(\Omega)}}$ *let* $(u_{m,n}, b_{m,n})$ *be the weak solution to* $(P, \psi_{m,n}, f_{m,n}, b_0^{m,n})$. *Then, there exists a subsequence* $(m(n))_n$ *such that setting* $\psi^n := \psi_{m(n),n}$, $f_n := f_{m(n),n}$, $b_{0,n} := b_0^{m(n),n}$, $b_n := b_{m(n),n}$, $u_n := u_{m(n),n}$ *we have the following convergence results for* $n \to \infty$:

i) $f_n \to f$ *in* $L^1(Q_T)$,

ii) $u_n \to u$ *almost everywhere in* Q_T, *iii)* $b_{0,n} \to b_0$ *in* $L^1(\Omega)$, $b_n \to b$ *in* $\mathcal{C}([0,T], L^1(\Omega))$, *and* $b \in \beta(u)$ *almost everywhere in* Q_T, *and the uniform renormalized condition*

$$\limsup_{l\to\infty}\sup_{n}\int_{\{l\leq|u_n|\leq l+1\}} a(x,Du_n)\cdot Du_n = 0 \tag{4.4.72}$$

holds true. Moreover, for any $k > 0$, *we have*

iv) $T_k(u_n) \rightharpoonup T_k(u)$ *in* $L^p(0,T,W_0^{1,p}(\Omega,\omega))$,

v) $DT_k(u_n) \rightharpoonup DT_k(u)$ *in* $\prod_{i=1}^N L^p(Q_T,\omega_i)$,

vi) $a(x,DT_k(u_n)) \rightharpoonup a(x,DT_k(u))$ *in* $\prod_{i=1}^N L^{p'}(Q_T,\omega_i^*)$,

vii) $a(x,DT_k(u_n))\cdot DT_k(u_n) \rightharpoonup a(x,DT_k(u))\cdot DT_k(u)$ *weak in* $L^1((0,\tau)\times\Omega)$ *for any* $0<\tau<T$.

$i) - v)$ are direct consequences of the approximation procedure, Lemma 4.4.1 and Remark 4.4.2 . To prove the uniform renormalized condition, we take $T_k(u_n)\phi_\delta$, $\phi_\delta(t) = \min\left(\frac{1}{\delta}\max(T-\delta-t,0),1\right)$ as a test function and apply Lemma 4.2.4 in the weak formulation for (P,ψ^n,f_n,b_0^n). By Gauss-Green Theorem for sobolev function and the boundary condition, we have

$$\int_\Omega F(T_k(u_n(t)))\cdot DT_k(u_n(t)) = 0$$

for almost all $t\in(0,T)$, Hence the convection terms vanishes Then we set $k=l+1$ and after $k=l$. Subtracting the corresponding equalities and neglecting positive terms we obtain

$$\int_{\{l<|u_n|<l+1\}} \phi_\delta a(x,Du_n)\cdot Du_n \leq \int_{\{|u_n|>l\}} |f| + \int_\Omega\int_0^{|b_0|} G_l((\beta^{-1})^0)(s)ds \tag{4.4.73}$$

where $G_l = T_{l+1} - T_l$ and the uniform renormalized condition follows applying (4.4.70) in (4.4.73). It is left to show that vi) and vii) hold.

From Lemma 4.4.1 and (4.2.6) it follows that for any $k>0$ there exists $\Phi_k \in \prod_{i=1}^N L^{p'}(Q_T,\omega_i^*)$ such that $a(x,DT_k(u_n)) \rightharpoonup \Phi_k$ weak in $\prod_{i=1}^N L^{p'}(Q_T,\omega_i^*)$ as $n\to\infty$. To prove that $\Phi_k = a(x,DT_k(u))$, we proceed as in as in [86] for the case of a constant exponent (see also [33] for variable exponent and [41], Theorem 3.6, [22],

[26],[43] for constant exponent) and using a special time regularisation of $T_k(u)$ by the regularisation method of Landes ,see subsection 4.2.2. Now, for $\kappa \in D^+([0,T))$ we show that

$$\limsup_{\mu\to\infty}\limsup_{n\to\infty}\int_{Q_T}\kappa a(x,DT_k(u_n))\cdot\Big(T_k(u_n)-(T_k(u))_\mu\Big)\leq 0. \tag{4.4.74}$$

We use the sequence $(T_k(u))_\mu$ of approximations of $T_k(u)$, and plug the test function $\kappa h_l(u_n)(T_k(u_n)-(T_k(u))_\mu)$ (for $\mu>0$) in (P,ψ^n,f_n,b_0^n) and obtain

$$I_1+I_2+I_3+I_4+I_5=I_6 \tag{4.4.75}$$

where

$$I_1=\langle (b_n-b_0^n)_t,\kappa h_l(u_n)(T_k(u_n)-(T_k(u))_\mu)\rangle \tag{4.4.76}$$

and $\langle .,.\rangle$ denotes the duality pairing between $L^{p'}(0,T,W^{-1,p'}(Q_T,\omega^*))+L^1(Q_T)$ and $L^p(0,T,W_0^{1,p}(\Omega,\omega))\cap L^\infty(Q_T)$,

$$\begin{aligned}
I_2&=\int_{Q_T}\kappa h_l(u_n)a(x,Du_n)\cdot D(T_k(u_n)-(T_k(u))_\mu),\\
I_3&=\int_{Q_T}\kappa h_l'(u_n)(T_k(u_n)-(T_k(u))_\mu)a(x,Du_n)\cdot Du_n,\\
I_4&=\int_{Q_T}\kappa F(u_n)\cdot D(h_l(u_n)(T_k(u_n)-(T_k(u))_\mu)),\\
I_5&=\int_{Q_T}\kappa h_l(u_n)\psi^n(u_n)(T_k(u_n)-(T_k(u))_\mu),\\
I_6&=\int_{Q_T}\kappa f_n h_l(u_n)(T_k(u_n)-(T_k(u))_\mu).
\end{aligned}$$

Now we want to pass to the limit with $n\to\infty$ and then with $\mu\to\infty$. To handle I_2, we choose $l>k$ and apply the uniform renormalized condition. It is easy to see that I_6, I_5 and I_4 tend to 0 as $n\to\infty$, $\mu\to\infty$. Using the uniform renormalized condition (4.4.72), it follows that $I_3\leq w(l,k)$ and $w(l,k)\to 0$ as $l\to\infty$. To handle the parabolic term I_1, we need the following :

Lemma 4.4.4

$$\liminf_{n\to\infty}\liminf_{\mu\to\infty}\langle (b_n-b_0^n)_t,\kappa h_l(u_n)(T_k(u_n)-(T_k(u))_\mu)\rangle\geq 0 \tag{4.4.77}$$

The proof is similar to the proof of the corresponding result in the case of a constant exponent (see [33] for the case of a variable exponent and [86, 41, 26, 43],for the case of a constant exponent). If (4.4.74) holds, then, by (4.2.7) it follows that

$$\limsup_{n\to\infty}\int_{Q_T}\kappa a(x,DT_k(u_n))\cdot D(T_k(u_n)-(T_k(u)))\leq 0 \tag{4.4.78}$$

and vi) follows from (4.4.78) by the standard Minty-Browder argument. vii) follows from (4.4.78) by choosing κ to be a smooth approximation of $\chi_{(0,T)}$ for $0<\tau<T$.

Step3 : Conclusion of the proof of Theorem 4.2.10

Now, we are able to conclude the proof of Theorem 4.2.10 : By Remark 4.4.2 and Lemma 4.4.3 it follows immediately that $(P1)$, $(P2)$ and $(P3)$ hold for all $k>0$. For $h\in C_c^1(\mathbb{R})$ and $\xi\in D([0,T)\times\Omega)$ we can plug $h(u_n)\xi$ into (P,ψ^n,f_n,b_0^n) by integration-by parts formula of Lemma 4.2.4 and obtain

$$I_1+I_2+I_3+I_4=I_5, \tag{4.4.79}$$

where
$I_1=\int_{Q_T}\xi_t\int_{b_0^n}^{b_n}h\circ(\beta^{-1})^0(s)ds,\ I_2=\int_{Q_T}a(x,Du_n)\cdot D(h(u_n)\xi),$
$I_3=\int_{Q_T}F(u_n)\cdot D(h(u_n)\xi),\quad I_4=\int_{Q_T}\psi^n(u_n)h(u_n)\xi,\quad I_5=\int_{Q_T}f_nh(u_n)\xi.$ Thanks to the convergence results of Lemma 4.4.3 , we can pass to limit with $n\to\infty$ in $I_1,...,I_5$: It follows immediately that

$$\lim_{n\to\infty}I_1=\int_{Q_T}\xi_t\int_{b_0}^{b}h\circ(\beta^{-1})^0(s)ds. \tag{4.4.80}$$

Now we choose $m>0$ such that $supp\,h\subset[-m,m]$. Next, we write

$$I_2=I_{2,1}+I_{2,2}, \tag{4.4.81}$$

where

$$I_{2,1}=\int_{(0,\tau)\times\Omega}h'(T_m(u_n))\xi a(x,DT_m(u_n))\cdot DT_m(u_n),$$

for $0<\tau<T$ is such that $supp\xi\subset[0,T)\times\Omega)$. By Lemma 4.4.3, $vii)$, $a(x,DT_m(u_n))\cdot DT_m(u_n)\rightharpoonup a(x,DT_m(u))\cdot DT_m(u)$ weak in $L^1([0,T)\times\Omega)$. Since $h'(u_n)\xi\to h'(u)\xi$ almost everywhere in Q_T and $\|h(u_n)\xi\|_{L^\infty((0,T)\times\Omega)}\leq\|h\|_{L^\infty(Q_T)}\|\xi\|_{L^\infty(Q_T)}$, we may pass to the limit in $I_{2,1}$ and obtain

$$\lim_{n\to\infty}I_{2,1}=\int_{Q_T}h'(u)\xi a(x,Du)\cdot Du. \tag{4.4.82}$$

By Lebesgue Dominated Convergence Theorem it follows that $h(T_m(u_n)) \to h(T_m(u))$ as $n \to \infty$ in $L^p(Q_T, \sigma)$. Using Lemma 4.4.3 $vi)$, we can pass to the limit in

$$I_{2,2} = \int_{Q_T} h(T_m(u_n))a(x, DT_m(u_n)) \cdot D\xi, \tag{4.4.83}$$

$$\text{and find} \quad \lim_{n\to\infty} I_{2,2} = \int_{Q_T} h(u)a(x, Du) \cdot D\xi. \tag{4.4.84}$$

Next we write $I_3 = I_{3,1} + I_{3,2}$ where
$I_{3,1} = \int_{Q_T} h'(T_m(u_n))\xi F(T_m(u_n)) \cdot DT_m(u_n)$,
$I_{3,2} = \int_{Q_T} h(T_m(u_n))F(T_m(u_n)) \cdot D\xi$.
Since $h'(T_m(u_n))F(T_m(u_n))$ converge to $h'(T_m(u))F(T_m(u))$ in $\prod_{i=1}^N L^{p'}(Q_T, w_i^*)$ as $n \to \infty$ using Lemma 4.4.3 $v)$ we have

$$\lim_{n\to\infty} I_{3,1} = \int_{Q_T} h'(u)\xi F(u) \cdot Du, \tag{4.4.85}$$

$$\text{and moreover} \quad \lim_{n\to\infty} I_{3,2} = \int_{Q_T} h(u)F(u) \cdot D\xi. \tag{4.4.86}$$

$$\text{Note that} \quad |I_4| \leq \frac{m}{n}\|h\|_{L^\infty(Q_T)}\|\xi\|_{L^\infty(Q_T)} \to 0 \tag{4.4.87}$$

for $n \to \infty$. Finally, we have

$$\lim_{n\to\infty} I_5 = \int_{Q_T} fh(u)\xi \tag{4.4.88}$$

and from (4.4.79)-4.4.88 it follows that (u, b) satisfies the renormalized formulation $(P4)$. $(P5)$ follows from the uniform renormalized condition (4.4.72) and Lemma 4.4.3, $vii)$.

This part is constituted of the following chapters :

CHAPTER V

Existence of Renormalized Solutions of Nonlinear Elliptic Problems in Weighted Variable -Exponent Space.

CHAPTER VI

Renormalized solutions of nonlinear parabolic problems in Weighted variable exponent Sobolev spaces.

Chapitre 5

Existence of Renormalized Solutions of Nonlinear Elliptic Problems in Weighted Variable -Exponent Space

[1] In this chapter, we study a general class of nonlinear degenerated elliptic problems associated with the differential inclusion $\beta(u)-div(a(x,Du)+F(u))\ni f$ in Ω where $f\in L^1(\Omega)$. A vector field $a(.,.)$ is a Carathéodory function. Using truncation techniques and the generalized monotonicity method in the framework of weighted variable exponent Sobolev spaces, we prove existence of renormalized solutions for general L^1-data.

5.1 Introduction

Let Ω be a bounded open set of $\mathbb{R}^N$ $(N\geq 1)$ with Lipschitz boundary if $N\geq 2$, where the variable exponent $p:\overline{\Omega}\to(1,\infty)$ is a continuous function, and ω be a weight functions on Ω, i.e. each ω is a measurable a.e. positive on Ω. Let $W_0^{1,p(.)}(\Omega,\omega)$ be the weighted variable exponent Sobolev space associated with the vector ω. We are intersted in existence of renormalized solutions to the following nonlinear elliptic equation

$$(E,f)\begin{cases}\beta(u)-div(a(x,Du)+F(u))\ni f & \text{in} \quad \Omega\\ u=0 & \text{on} \quad \partial\Omega,\end{cases}$$

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with a right-hand side f which is assumed to belong either $L^\infty(\Omega)$ or to $L^1(\Omega)$ for Eq. (E,f). Furthermore, F and β are two functions satisfying the following assumption :

$(\mathbf{C}_0)$ $F:\ \mathbb{R}\ \rightarrow\ \mathbb{R}^N$ is locally lipschitz continuous and $\beta:\mathbb{R}\ \rightarrow\ 2^{\mathbb{R}}$ a set valued, maximal monotone mapping such that $0\in\beta(0)$, Moreover, we assume that

$$\beta^0(l)\in L^1(\Omega), \tag{5.1.1}$$

for each $l\in\mathbb{R}$, where β^0 denotes the minimal selection of the graph of β. Namely $\beta_0(l)$ is the minimal in the norm element of $\beta(l)$

$$\beta_0(l)=\inf\{|r|\ \ /r\in\mathbb{R} \text{ and } r\in\beta(l)\}.$$

Moreover, $a:\ \Omega\times\mathbb{R}^N\ \rightarrow\ \mathbb{R}^N$ is a Carathéodory function satisfying the following assumptions :

$(\mathbf{C}_1)$ There existe a positive constant λ such that $a(x,\xi)\cdot\xi\geq\lambda\omega(x)|\xi|^{p(x)}$ holds for all $\xi\in\mathbb{R}^N$ and almost every $x\in\Omega$.

$(\mathbf{C}_2)$ $|a_i(x,\xi)|\leq\alpha\omega^{1/p(x)}(x)[k(x)+\omega^{1/p'(x)}(x)|\xi|^{p(x)-1}]$ for almost every $x\in\Omega$, all $i=1,....,N$, every $\xi\in\mathbb{R}^N$, where $k(x)$ is a nonnegative function in $L^{p'(\cdot)}(\Omega)$, $p'(x):=p(x)/(p(x)-1)$, and $\alpha>0$.

$(\mathbf{C}_3)$ $(a(x,\xi)-a(x,\eta))\cdot(\xi-\eta)\geq 0$ for almost every $x\in\Omega$ and every $\xi,\eta\in\mathbb{R}^N$.

We use in this chapter the framework of renormalized solutions. This notion was introduced by Diperna and P.-L. Lions [56] in their study of the Boltzmann equation. This notion was then adapted to an elliptic version of (E,f) by L. Boccardo et al. [38] when the right hand side is in $W^{-1,p'}(\Omega)$, by J.-M. Rakotoson [83] when the right hand side is in $L^1(\Omega)$, and finally by G. Dal Maso, F. Murat, L. Orsina and A. Prignet [54] for the case of right hand side is general measure data. The equivalent notion of entropy solution has been introduced by Bénilan et al. in [38]. For results on existence of renormalized solutions of elliptic problems of type (E,f) with $a(,)$ satisfying a variable growth condition, we refer to [89],[62], [25] and [10]. One of the motivations for studying (E,f) comes from applications to electrorheological fluids (see [85] for more details) as an important class of non-Newtonian fluids.
For the convenience of the readers, we recall some definitions and basic properties of the weighted variable exponent Lebesgue spaces $L^{p(x)}(\Omega,\omega)$ and the weighted variable exponent Sobolev spaces $W^{1,p(x)}(\Omega,\omega)$. Set

$$C_+(\overline{\Omega})=\{p\in C(\overline{\Omega}):\min_{x\in\overline{\Omega}}p(x)>1\}.$$

For any $p \in C_+(\overline{\Omega})$, we define

$$p^+ = \max_{x \in \overline{\Omega}} p(x), \quad p^- = \min_{x \in \overline{\Omega}} p(x).$$

For any $p \in C_+(\overline{\Omega})$, we introduce the weighted variable exponent Lebesgue space $L^{p(x)}(\Omega, \omega)$ that consists of all measurable real-valued functions u such that

$$L^{p(x)}(\Omega, \omega) = \{u : \Omega \to \mathbb{R}, measurable, \int_\Omega |u(x)|^{p(x)} \omega(x) dx < \infty\}.$$

Then, $L^{p(x)}(\Omega, \omega)$ endowed with the Luxemburg norm

$$|u|_{L^{p(x)}(\Omega,\omega)} = \inf\{\lambda > 0 : \int_\Omega |\frac{u(x)}{\lambda}|^{p(x)} \omega(x) dx \leq 1\}$$

becomes a normed space. When $\omega(x) \equiv 1$, we have $L^{p(x)}(\Omega, \omega) \equiv L^{p(x)}(\Omega)$ and we use the notation $|u|_{L^{p(x)}(\Omega)}$ instead of $|u|_{L^{p(x)}(\Omega,\omega)}$. The weighted variable exponent Sobolev space $W^{1,p(x)}(\Omega, \omega)$ is defined by

$$W^{1,p(x)}(\Omega, \omega) = \{u \in L^{p(x)}(\Omega); |\nabla u| \in L^{p(x)}(\Omega, \omega)\},$$

where the norm is

$$\|u\|_{W^{1,p(x)}(\Omega,\omega)} = |u|_{L^{p(x)}(\Omega)} + |\nabla u|_{L^{p(x)}(\Omega,\omega)} \tag{5.1.2}$$

or equivalently

$$\|u\|_{W^{1,p(x)}(\Omega,\omega)} = \inf\{\lambda > 0 : \int_\Omega |\frac{u(x)}{\lambda}|^{p(x)} + \omega(x)|\frac{\nabla u(x)}{\lambda}|^{p(x)} dx \leq 1\}$$

for all $u \in W^{1,p(x)}(\Omega, \omega)$.
It should be emphasised that smooth functions are not dense in $W^{1,p(x)}(\Omega)$ without additional assumptions on the exponent $p(x)$. This feature was observed by Zhikov [98] in connection with the Lavrentiev phenomenon. However, if the exponent $p(x)$ is log-Hölder continuous, i.e., there is a constant C such that

$$|p(x) - p(y)| \leq \frac{C}{-\log |x - y|} \tag{5.1.3}$$

for every x, y with $|x - y| \leq \frac{1}{2}$, then smooth functions are dense in variable exponent Sobolev spaces and there is no confusion in defining the Sobolev space with zero boundary values,$W_0^{1,p(x)}(\Omega)$, as the completion of $C_0^\infty(\Omega)$ in $W^{1,p(x)}(\Omega, \omega)$ with respect to the norm $\|u\|_{W^{1,p(x)}(\Omega)}$ (see [66]).

The space $W_0^{1,p(x)}(\Omega,\omega)$ is defined as the completion of $C_0^\infty(\Omega)$ with respect to the norm $\|u\|_{W^{1,p(x)}(\Omega,\omega)}$.
Throughout the chapter, we assume that $p \in C_+(\overline{\Omega})$ and ω is a measurable positive and a.e. finite function in Ω.
The plan of the chapter is as follows. In section 2, we give some preliminaries of the weighted variable exponent Lebesgue-Sobolev spaces which is given in [70] and we introduce the notions of weak and also renormalized solution for problem (E,f). Our first main result, existence of a renormalized solution to (E,f) for any L^∞ - data f, are collected in section 3. Our second main result, existence of a renormalized solution to (E,f) for any L^1 - data f is collected in section 4.

5.1.1 Notions of solutions

Definition 5.1.1 *A weak solution to (E,f) is a pair of functions $(u,b) \in W_0^{1,p(.)}(\Omega,\omega) \times L^1(\Omega)$ satisfying $F(u) \in (L^1_{\text{loc}}(\Omega))^N$, $b \in \beta(u)$ almost everywhere in Ω and*

$$b - \text{div}(a(x,Du) + F(u)) = f \quad \text{in } D^{'}(\Omega). \tag{5.1.4}$$

Definition 5.1.2 A renormalized solution to (E,f) is a pair of functions (u,b) satisfying the following conditions :

$(R1)$ $u : \Omega \to \mathbb{R}$ is measurable, $b \in L^1(\Omega)$, $u(x) \in D(\beta(x))$ and $b(x) \in \beta(u(x))$

for a.e. $x \in \Omega$.

$(R2)$ For each $k > 0$, $T_k(u) \in W_0^{1,p(.)}(\Omega,\omega)$ and

$$\int_\Omega b \cdot h(u)\varphi + \int_\Omega (a(x,Du) + F(u)) \cdot D(h(u)\varphi) = \int_\Omega f h(u)\varphi \tag{5.1.5}$$

holds for all $h \in \mathcal{C}_c^1(\mathbb{R})$ and all $\varphi \in W_0^{1,p(.)}(\Omega,\omega) \cap L^\infty(\Omega)$.

$(R3)$ $\int_{\{k<|u|<k+1\}} a(x,Du) \cdot Du \longrightarrow 0 \quad \text{as} \quad k \longrightarrow \infty.$

Remark 5.1.3 For $p \in (1,\infty)$, $\tau_0^{p(.)}(\Omega)$ is defined as the set of measurable functions $u : \Omega \to \mathbb{R}$ such that for $k > 0$ the truncated functions $T_k(u) \in W_0^{1,p(.)}(\Omega, \omega)$ and for every $u \in \tau_0^{p(.)}(\Omega)$ there exists a unique measurable function $v : \Omega \to \mathbb{R}^{\mathbb{N}}$ such that $\nabla T_K(u) = v\chi_{\{|u|<k\}}$ for a.e. $x \in \Omega$, [see [96], [38] for more details].

Remark 5.1.4 Note that if (u, b) is a renormalized solution to (E, f) such that $u \in W_0^{1,p(.)}(\Omega, \omega)$, then (u, b) in general is not a weak solution in the sense of Definition (5.1.1), since we did not assume a growth condition on F and therefore $F(u)$ in general may fail to be locally integrable. If (u, b) is a renormalized solution of (E, f) such that $u \in L^\infty(\Omega)$, it is a direct consequence of
Definition (5.1.1) that u is in $W_0^{1,p(.)}(\Omega, \omega)$ and since (5.1.5) holds with the formal choice $h \equiv 1$, (u, b) is a weak solution.
Indeed, let us choose $\varphi \in D(\Omega)$ and plug $h_l(u)\varphi$ as a test function in (5.1.5). Since $u \in L^\infty(\Omega)$, we can pass to the limit with $l \to \infty$ and find that u solves (E, f) in the sense of distributions.

5.2 Case where $f \in L^\infty(\Omega)$ -data

5.2.1 Resultat d'existence

In this subsection we will state existence of renormalized solutions to (E, f) in Theorem 5.2.1. In the next subsections we will present the proof.

Theorem 5.2.1 Under assuptions $(W_1)-(W_2)$, $(C_0)-(C_3)$ and $f \in L^\infty(\Omega)$. There, exists at least one renormalized solution (u, b) to (E, f).

5.2.2 Proof of Theorem 5.2.1

Approximate problem

First we approximate (E,f) for $f \in L^{\infty}(\Omega)$ by problems for which existence can be proved by standard variational arguments. For $0<\varepsilon\leq 1$, let $\beta_\varepsilon:\mathbb{R}\longrightarrow\mathbb{R}$ be the Yosida approximation (see [51]) of β . We introduce the operators

$$A_{1,\varepsilon}: W_0^{1,p(.)}(\Omega,\omega) \longrightarrow W^{-1,p'(.)}(\Omega,\omega^*)$$
$$u \longrightarrow \beta_\varepsilon(T_{\frac{1}{\varepsilon}}(u))+\varepsilon\arctan(u)-div\ a(x,Du),$$
$$A_{2,\varepsilon}\ : W_0^{1,p(.)}(\Omega,\omega)\longrightarrow\ W^{-1,p'(.)}(\Omega,\omega^*),\ u\longrightarrow -div\ F(T_{\frac{1}{\varepsilon}}(u)).$$

Because of (C_2) and (C_3), $A_{1,\varepsilon}$, is well-defined and monotone (see [76], p.157). Since $\beta_\varepsilon\circ T_{\frac{1}{\varepsilon}}$ and arctan are bounded and continuous and thanks to the growth condition (C_2) on a, it follows that $A_{1,\varepsilon}$, is hemicontinuous (see [76], p.157). From the continuity and boundedness of $F\circ T_{\frac{1}{\varepsilon}}$ it follows that $A_{2,\varepsilon}$ is strongly continuous. Therefore the operator $A_\varepsilon := A_{1,\varepsilon}+A_{2,\varepsilon}$ is pseudomonotone. Using the monotonicity of β_ε, the Gauss-Green Theorem for Sobolev functions and the boundary condition on the convection term $\int_\Omega F(T_{\frac{1}{\varepsilon}}(u))\cdot Du$, we show by similar arguments as in [70] that A_ε is coercive and bounded. Then it follows from [[76], Theorem 2.7], that A_ε is surjective, i.e., for each $0<\varepsilon\leq 1$ and $f\in W^{-1,p'(.)}(\Omega,\omega^*)$ there exists at least one solution $u_\varepsilon\in W_0^{1,p(.)}(\Omega,\omega)$ to the problem

$$(E_\varepsilon,f)\begin{cases}\beta_\varepsilon(T_{\frac{1}{\varepsilon}}(u_\varepsilon))+\varepsilon\arctan(u_\varepsilon)-div(a(x,Du_\varepsilon)+F(T_{\frac{1}{\varepsilon}}(u_\varepsilon)))=f & \text{in} \quad \Omega\\ u_\varepsilon=0 & \text{on} \quad \partial\Omega.\end{cases}$$

such that

$$\int_\Omega\Big(\beta_\varepsilon(T_{\frac{1}{\varepsilon}}(u_\varepsilon))+\varepsilon\arctan(u_\varepsilon)\Big)\varphi+\int_\Omega\Big(a(x,Du_\varepsilon)+F(T_{\frac{1}{\varepsilon}}(u_\varepsilon))\Big)\cdot D\varphi=<f,\varphi> \tag{5.2.6}$$

holds for all $\varphi\in W_0^{1,p(.)}(\Omega,\omega)$, where $<.,.>$ denotes the duality pairing between $W_0^{1,p(.)}(\Omega,\omega)$ and $W^{-1,p'(.)}(\Omega,\omega^*)$.

In the next remark, we establish uniqueness of solutions u_ε of (E_ε,f) with right-hand sides $f\in L^\infty(\Omega)$ through a comparison principle that will play an important role in the approximation of renormalized solutions to (E,f) with $f\in L^1(\Omega)$.

Remark 5.2.2 For $0<\varepsilon\leq 1$ fixed and $f,\tilde{f}\in L^\infty(\Omega)$ let $u_\varepsilon,\tilde{u}_\varepsilon\in W_0^{1,p(.)}(\Omega,\omega)$ be solutions of (E_ε,f) and $(E_\varepsilon,\tilde{f})$ respectively. Then, the following comparison principle holds :

$$\varepsilon \int_\Omega (\arctan(u_\varepsilon) - \arctan(\tilde{u}_\varepsilon))^+ \le \int_\Omega (f - \tilde{f}) sign_0^+(u_\varepsilon - \tilde{u}_\varepsilon). \tag{5.2.7}$$

Proof 5.2.3 *We use the test function* $\varphi = H_\delta^+(u_\varepsilon - \tilde{u}_\varepsilon)$ *in the weak formulation* (5.2.6) *for* u_ε *and* $\tilde{u}_\varepsilon$. *Subtracting the resulting inequalities, we obtain*

$$I^1_{\varepsilon,\delta} + I^2_{\varepsilon,\delta} + I^3_{\varepsilon,\delta} + I^4_{\varepsilon,\delta} = I^5_{\varepsilon,\delta}.$$

Where

$$\begin{aligned}
I^1_{\varepsilon,\delta} &= \int_\Omega (\beta_\varepsilon(T_{\frac{1}{\varepsilon}}(u_\varepsilon)) - \beta_\varepsilon(T_{\frac{1}{\varepsilon}}(\tilde{u}_\varepsilon)) H_\delta^+(u_\varepsilon - \tilde{u}_\varepsilon),\\
I^2_{\varepsilon,\delta} &= \int_\Omega (\varepsilon \arctan(u_\varepsilon) - \varepsilon \arctan(\tilde{u}_\varepsilon)) H_\delta^+(u_\varepsilon - \tilde{u}_\varepsilon),\\
I^3_{\varepsilon,\delta} &= \int_\Omega (a(x, Du_\varepsilon) - a(x, D\tilde{u}_\varepsilon)) \cdot DH_\delta^+(u_\varepsilon - \tilde{u}_\varepsilon),\\
I^4_{\varepsilon,\delta} &= \int_\Omega (F(T_{\frac{1}{\varepsilon}}(u_\varepsilon)) - F(T_{\frac{1}{\varepsilon}}(\tilde{u}_\varepsilon))) \cdot DH_\delta^+(u_\varepsilon - \tilde{u}_\varepsilon),\\
I^5_{\varepsilon,\delta} &= \int_\Omega (f - \tilde{f}) \cdot H_\delta^+(u_\varepsilon - \tilde{u}_\varepsilon).
\end{aligned}$$

Passing to the limit with $\delta \downarrow 0$, (5.2.7) *follows.*

Remark 5.2.4 Let $f, \tilde{f} \in L^\infty(\Omega)$ be such that $f \le \tilde{f}$ almost everywhere in Ω, $\varepsilon > 0$ and $u_\varepsilon, \tilde{u}_\varepsilon \in W_0^{1,p(.)}(\Omega, \omega)$ solutions to $(E_\varepsilon, f), (E_\varepsilon, \tilde{f})$ respectively. Then it is an immediate consequence of Remark 5.2.2 that $u_\varepsilon \le \tilde{u}_\varepsilon$ almost everywhere in Ω. Furthermore, from the monotonicity of $\beta_\varepsilon \circ T_{\frac{1}{\varepsilon}}$ it follows that also $\beta_\varepsilon(T_{\frac{1}{\varepsilon}}(u_\varepsilon)) \le \beta_\varepsilon(T_{\frac{1}{\varepsilon}}(\tilde{u}_\varepsilon))$ almost everywhere in Ω.

A priori estimates

Lemma 5.2.5 For $0 < \varepsilon \le 1$ and $f \in L^\infty(\Omega)$ let $u_\varepsilon \in W_0^{1,p(.)}(\Omega, \omega)$ be a solution of (E_ε, f). Then,

i) There exists a constant $C_1 = C_1(\|f\|_\infty, \lambda, p(.), N) > 0$, not depending on ε, such that

$$\|Du_\varepsilon\|_{L^{p(.)}(\Omega,\omega)} \leq C_1. \tag{5.2.8}$$

ii)

$$\|\beta_\varepsilon(T_{\frac{1}{\varepsilon}}(u_\varepsilon))\|_\infty \leq \|f\|_\infty. \tag{5.2.9}$$

iii) For all $l, k > 0$, *we* have

$$\int_{\{l<|u_\varepsilon|<l+k\}} a(x, Du_\varepsilon) \cdot Du_\varepsilon \leq k \int_{\{|u_\varepsilon|>l\}} |f|. \tag{5.2.10}$$

Proof 5.2.6 *i) Taking* u_ε *as a test function in 5.2.6, we obtain*

$$\int_\Omega \Big(\beta_\varepsilon(T_{\frac{1}{\varepsilon}}(u_\varepsilon)) + \varepsilon \arctan(u_\varepsilon)\Big) u_\varepsilon dx + \int_\Omega a(x, Du_\varepsilon) \cdot Du_\varepsilon dx$$
$$+ \int_\Omega F(T_{\frac{1}{\varepsilon}}(u_\varepsilon)) \cdot Du_\varepsilon dx = \int_\Omega f u_\varepsilon dx.$$

As the first term on the left-hand side is nonnegative and the integral over the convection term vanishes by (C_1), we have :

$$\begin{aligned}
\lambda \int_\Omega |Du_\varepsilon|^{p(.)} \omega(x) dx &\leq \int_\Omega a(x, Du_\varepsilon) \cdot Du_\varepsilon dx \\
&\leq \int_\Omega f u_\varepsilon dx = \int_\Omega f u_\varepsilon \omega^{1/p(x)} \omega^{-1/p(x)} dx \\
&\leq C(p(.), N) \|f\|_\infty \|Du_\varepsilon\|_{L^{p(.)}(\Omega,\omega)},
\end{aligned} \tag{5.2.11}$$

where $C(p(.), N) > 0$ *is a constant coming from the Hölder and Poincaré inequalities. From (1.2.18) and* (5.2.11) *it follows that either*

$$\|Du_\varepsilon\|_{L^{p(.)}(\Omega,\omega)} \leq \Big(\frac{1}{\lambda} C(p(.), N) \|f\|_\infty\Big)^{\frac{1}{p^- - 1}}$$

or

$$\|Du_\varepsilon\|_{L^{p(.)}(\Omega,\omega)} \leq \Big(\frac{1}{\lambda} C(p(.), N) \|f\|_\infty\Big)^{\frac{1}{p^+ - 1}}$$

Setting $C_1 := \max\Big(\Big(\frac{1}{\lambda}C(p(.),N)\|f\|_\infty\Big)^{\frac{1}{p^+-1}}, \Big(\frac{1}{\lambda}C(p(.),N)\|f\|_\infty\Big)^{\frac{1}{p^--1}}\Big)$*, we get i).*

ii) Taking $\frac{1}{\delta}[T_{k+\delta}(\beta_\varepsilon(T_{\frac{1}{\varepsilon}}(u_\varepsilon)))-T_k(\beta_\varepsilon(T_{\frac{1}{\varepsilon}}(u_\varepsilon)))]$ *as a test function in* (5.2.6)*, passing to the limit with* $\delta \to 0$ *and choosing* $k > \|f\|_\infty$*, we obtain (ii).*

iii) For $k, l > 0$ *fixed, we take* $T_k(u_\varepsilon - T_l(u_\varepsilon))$ *as a test function in* (5.2.6)*. Using* $\int_\Omega a(x, Du_\varepsilon)\cdot DT_k(u_\varepsilon - T_l(u_\varepsilon))dx = \int_{\{l<|u_\varepsilon|<l+k\}} a(x, Du_\varepsilon)\cdot Du_\varepsilon dx$*, and as the first term and the second on the left-hand side is nonnegative and the convection term vanishes, we get*

$$\begin{aligned}\int_{\{l<|u_\varepsilon|<l+k\}} a(x, Du_\varepsilon)\cdot Du_\varepsilon dx &\leq \int_\Omega f T_k(u_\varepsilon - T_l(u_\varepsilon))dx\\ &\leq k\int_{\{|u_\varepsilon|>l\}} |f|dx.\end{aligned}$$

Remark 5.2.7 For $k > 0$, from Lemma 5.2.5 (iii), we deduce

$$|\{|u_\varepsilon| \geq l\}| \leq l^{-(p_s^*)^-}C(p(.), p^-, \lambda, C_1) \tag{5.2.12}$$

$$\int_{\{l<|u_\varepsilon|<l+k\}} a(x, Du_\varepsilon)\cdot Du_\varepsilon \leq k\|f\|_\infty |\{|u_\varepsilon| > l\}| \leq C_2(k)l^{-(p_s^*)^-}. \tag{5.2.13}$$

Indeed, we have the following continuous embeddings

$$W_0^{1,p(x)}(\Omega,\omega) \hookrightarrow L^{p_s^*(x)}(\Omega) \hookrightarrow L^{(p_s^*)^-}(\Omega),$$

where $(p_s^*(x))^- := \frac{p^-s-N}{(s^-+1)N-p^-s^-}$.

It follows from Remark 1.2.3 that for every $l > 1$,

$$\|T_l(u)\|_{L^{(p_s^*)^-}(\Omega)} \leq C\|DT_l(u)\|_{L^{p(x)}(\Omega,\omega)} \leq C\Big(\int_\Omega \omega(x)|DT_l(u)|^{p(x)}dx\Big)^\nu$$

where

$$\nu = \begin{cases} \frac{1}{p^-} & if \quad \|DT_l(u)\|_{L^{p(x)}(\Omega,\omega)} \geq 1\\ \frac{1}{p^+} & if \quad \|DT_k(u)\|_{L^{p(x)}(\Omega,\omega)} \leq 1.\end{cases}$$

Noting that $\{|u_\varepsilon| \geq l\} = \{|T_l(u_\varepsilon)| \geq l\}$, we have

$$\begin{aligned}|\{|u_\varepsilon| \geq l\}| &\leq \Big(\frac{\|T_l(u)\|_{L^{(p_s^*)^-}(\Omega)}}{l}\Big)^{(p_s^*)^-}\\ &\leq l^{-(p_s^*)^-}\Big(C\Big(\int_\Omega \omega(x)|DT_l(u)|^{p(x)}dx\Big)^\nu\Big)^{(p_s^*)^-}.\end{aligned} \tag{5.2.14}$$

Combining (5.2.11), (5.2.8) and (5.2.14), setting

$$C(p(.),(p_s^*)^-,\lambda,C_1)=C^{(p_s^*)^-}\Big(\frac{C(p(.),N)\|f\|_\infty}{\lambda}C_1\Big)^{\nu(p_s^*)^-}>0,$$

we obtain

$$|\{|u_\varepsilon|\geq l\}|\leq C(p(.),(p_s^*)^-,\lambda,C_1)l^{-(p_s^*)^-}. \tag{5.2.15}$$

So we have

$$\lim_{l\to+\infty}|\{|u_\varepsilon|\geq l\}|=0.$$

Hence (5.2.15) provides (5.2.13)) with $C_2(k):=C(p(.),(p_s^*)^-,\lambda,C_1)k\|f\|_\infty$.

Basic convergence results

Lemma 5.2.8 For $0<\varepsilon\leq 1$ and $f\in L^\infty(\Omega)$, let $u_\varepsilon\in W_0^{1,p(.)}(\Omega,\omega)$ be the solution of (E_ε,f). There exist $u\in W_0^{1,p(.)}(\Omega,\omega)$ and $b\in L^\infty(\Omega)$ such that for a not relabeled subsequence of $(u_\varepsilon)_{0<\varepsilon\leq 1}$ as $\varepsilon\downarrow 0$:

$$u_\varepsilon\rightharpoonup u \quad \text{in} \quad L^{p(.)}(\Omega,\omega) \quad \text{and a.e. in} \quad \Omega \tag{5.2.16}$$

$$Du_\varepsilon\rightharpoonup Du \quad \text{in} \ (L^{p(.)}(\Omega,\omega))^N \tag{5.2.17}$$

$$\text{and} \quad \beta_\varepsilon(T_{\frac{1}{\varepsilon}}(u_\varepsilon))\rightharpoonup b \quad \text{in} \quad L^\infty(\Omega). \tag{5.2.18}$$

$Moreover,\ for\ any \quad k>0,$

$$DT_k(u_\varepsilon)\rightharpoonup DT_k(u) \quad \text{in} \quad (L^{p(.)}(\Omega,\omega))^N \tag{5.2.19}$$

$$a(x,DT_k(u_\varepsilon))\rightharpoonup a(x,DT_k(u)) \quad \text{in} \quad (L^{p'(.)}(\Omega,\omega^*))^N. \tag{5.2.20}$$

Proof 5.2.9 *Since* (5.2.16)-(5.2.19) *follow directly from Lemma 5.2.5 and Remark 5.2.7.*

It is left to prove (5.2.20). *For this end by* (C_2) *and* (5.2.8) *it follows that given any subsequence of* $(a(x,DT_k(u_\varepsilon))_\varepsilon)$, *there exists a subsequence, still denoted by*

$(a(x,DT_k(u_\varepsilon))_\varepsilon)$, *such that* $a(x,DT_k(u_\varepsilon)) \rightharpoonup \Phi_k \quad in \ \ (L^{p'(.)}(\Omega,\omega^*))^N$.

We will we prove that $\Phi_k = a(x,DT_k(u))$ *a.e. of* Ω. *The proof consists in three assertions.*

Assertion i : *For every function* $h \in W^{1,\infty}(\mathbb{R})$, $h \geq 0$ *with supp*(h) *compact, we will prove that,*

$$\limsup_{\varepsilon\to 0} \int_\Omega a(x,DT_k(u_\varepsilon)) \cdot D[h(u_\varepsilon)(T_k(u_\varepsilon)-T_k(u))]dx \leq 0. \tag{5.2.21}$$

Taking $h(u_\varepsilon)(T_k(u_\varepsilon)-T_k(u))$ *as test function in* (5.2.6), *we have*

$$\begin{aligned}\int_\Omega \Big(\beta_\varepsilon(T_{\frac{1}{\varepsilon}}(u_\varepsilon)+\varepsilon\arctan(u_\varepsilon)\Big)h(u_\varepsilon)(T_k(u_\varepsilon)-T_k(u))\\ +\int_\Omega\Big(a(x,Du_\varepsilon)+F(T_{\frac{1}{\varepsilon}}(u_\varepsilon))\Big)\cdot D[h(u_\varepsilon)(T_k(u_\varepsilon)-T_k(u))]\\ =\int_\Omega fh(u_\varepsilon)(T_k(u_\varepsilon)-T_k(u)).\end{aligned} \tag{5.2.22}$$

Using $\quad |h(u_\varepsilon)(T_k(u_\varepsilon)-T_k(u))| \leq 2k\|h\|_\infty$,

by Lebesgue's dominated convergence theorem, we find that

$$\lim_{\varepsilon\to 0}\int_\Omega fh(u_\varepsilon)(T_k(u_\varepsilon)-T_k(u)) = 0.$$

and $\quad \lim_{\varepsilon\to 0}\int_\Omega F(T_{\frac{1}{\varepsilon}}(u_\varepsilon))\cdot D[h(u_\varepsilon)(T_k(u_\varepsilon)-T_k(u))] = 0.$

By using the same arguments in [25], we can prove that

$$\limsup_{\varepsilon\to 0}\int_\Omega \beta_\varepsilon(T_{\frac{1}{\varepsilon}}(u_\varepsilon))\cdot[h(u_\varepsilon)(T_k(u_\varepsilon)-T_k(u))]dx \geq 0.$$

Passage to limit in (5.2.22) *and using the above results, we obtain* (5.2.21).

assertion ii : *We prove that for every* $k > 0$,

$$\limsup_{\varepsilon\to 0}\int_\Omega a(x,DT_k(u_\varepsilon))\cdot[DT_k(u_\varepsilon)-DT_k(u)]dx \leq 0. \tag{5.2.23}$$

Indeed : *See [10].*

Assertion iii : *In this step, we prove by monotonicity arguments that for* $k > 0$,

$\Phi_k = a(x, DT_k(u))$ *for almost every* $x \in \Omega$. *Let* $\varphi \in D(\Omega)$ *and* $\tilde{\alpha} \in \mathbb{R}$. *Using* (5.2.23), *we have*

$$\begin{aligned}
&\tilde{\alpha} \lim_{\varepsilon \to 0} \int_\Omega a(x, DT_k(u_\varepsilon)) \cdot D\varphi dx \\
&\geq \limsup_{\varepsilon \to 0} \int_\Omega a(x, DT_k(u_\varepsilon)) \cdot [DT_k(u_\varepsilon) - DT_k(u) + D(\tilde{\alpha}\varphi)]dx \\
&\geq \limsup_{\varepsilon \to 0} \int_\Omega a(x, D(T_k(u) - \tilde{\alpha}\varphi)) \cdot [DT_k(u_\varepsilon) - DT_k(u) + D(\tilde{\alpha}\varphi)]dx \\
&\geq \limsup_{\varepsilon \to 0} \int_\Omega a(x, D(T_k(u) - \tilde{\alpha}\varphi)) \cdot D(\tilde{\alpha}\varphi)]dx \\
&\geq \tilde{\alpha} \int_\Omega a(x, D(T_k(u) - \tilde{\alpha}\varphi)) \cdot D\varphi dx.
\end{aligned}$$

Dividing by $\tilde{\alpha} > 0$ *and by* $\tilde{\alpha} < 0$, *passing the limit with* $\tilde{\alpha} \to 0$, *we obtain*

$$\lim_{\varepsilon \to 0} \int_\Omega a(x, DT_k(u_\varepsilon)) \cdot D\varphi dx = \int_\Omega a(x, DT_k(u)) \cdot D\varphi dx.$$

This means that $\forall k > 0$, $\int_\Omega \Phi_k \cdot D\varphi dx = \int_\Omega a(x, DT_k(u)) \cdot D\varphi dx$ *and so*

$$\Phi_k = a(x, DT_k(u)) \quad in \quad D^{'}(\Omega)$$

for all $k > 0$. *Hence* $\Phi_k = a(x, DT_k(u))$ *a.e. in* Ω *and so* $a(x, DT_k(u_\varepsilon)) \rightharpoonup a(x, DT_k(u))$ *weakly in* $(L^{p'(\cdot)}(\Omega, \omega^*))^N$.

Remark 5.2.10 As immediate consequence of (5.2.23) and (C_3), we obtain

$$\lim_{\varepsilon \to 0} \int_\Omega a(x, DT_k(u_\varepsilon)) - a(x, DT_k(u)) \cdot (DT_k(u_\varepsilon) - DT_k(u)) = 0. \tag{5.2.24}$$

Lemma 5.2.11 The limit u of the approximate solution u_ε of (E_ε, f) satisfies

$$\lim_{l \to \infty} \int_{\{l < |u| < l+1\}} a(x, Du) \cdot Dudx = 0. \tag{5.2.25}$$

Proof 5.2.12 *To this end, observe that for any fixed $l \geq 0$, one has*

$$\int_{\{l<|u_\varepsilon|<l+1\}} a(x, Du_\varepsilon) \cdot Du_\varepsilon dx = \int_\Omega a(x, Du_\varepsilon) \cdot (DT_{l+1}(u_\varepsilon) - DT_l(u_\varepsilon))dx$$
$$= \int_\Omega a(x, DT_{l+1}(u_\varepsilon)) \cdot DT_{l+1}(u_\varepsilon)dx - \int_\Omega a(x, DT_l(u_\varepsilon)) \cdot DT_l(u_\varepsilon))dx.$$

According (5.2.24) *is at liberty to passe to the limit as $\varepsilon \to 0$ for fixed $l \geq 0$ and to obtain*

$$\lim_{\varepsilon\to 0} \int_{\{l<|u_\varepsilon|<l+1\}} a(x, Du_\varepsilon) \cdot Du_\varepsilon dx$$
$$= \int_\Omega a(x, DT_{l+1}(u)) \cdot DT_{l+1}(u)dx - \int_\Omega a(x, DT_l(u)) \cdot DT_l(u))dx$$
$$= \int_{\{l<|u|<l+1\}} a(x, Du) \cdot Dudx. \tag{5.2.26}$$

Taking the limit as $l \to +\infty$ in (5.2.26) *and using the estimate* (5.2.13) *show that u satisfies (R3) and the proof of the lemma is complete.*

5.2.3 Concluding the proof of Theorem 5.2.1 .

Let $h \in \mathcal{C}_c^1(\mathbb{R})$ and $\phi \in W_0^{1,p(.)}(\Omega, \omega) \cap L^\infty(\Omega)$ be arbitrary. Taking $h_l(u_\varepsilon)h(u)\phi$ as a test function in (5.2.6), we obtain

$$I^1_{\varepsilon,l} + I^2_{\varepsilon,l} + I^3_{\varepsilon,l} + I^4_{\varepsilon,l} = I^5_{\varepsilon,l} \tag{5.2.27}$$

where

$$I^1_{\varepsilon,l} = \int_\Omega \beta_\varepsilon(T_{\frac{1}{\varepsilon}}(u_\varepsilon))h_l(u_\varepsilon)h(u)\phi$$
$$I^2_{\varepsilon,l} = \varepsilon \int_\Omega \arctan(u_\varepsilon)h_l(u_\varepsilon)h(u)\phi$$
$$I^3_{\varepsilon,l} = \int_\Omega a(x, Du_\varepsilon) \cdot D(h_l(u_\varepsilon)h(u)\phi)$$
$$I^4_{\varepsilon,l} = \int_\Omega F(T_{\frac{1}{\varepsilon}}(u_\varepsilon)) \cdot D(h_l(u_\varepsilon)h(u)\phi)$$
$$I^5_{\varepsilon,l} = \int_\Omega fh_l(u_\varepsilon)h(u)\phi.$$

Step i : Passing to the limit with $\varepsilon \downarrow 0$ obviously,

$$\lim_{\varepsilon\to 0} I^2_{\varepsilon,l} = 0. \tag{5.2.28}$$

Using the convergence results (5.2.16), (5.2.18) from Lemma 5.2.8, we can immediately calculate the following limits :

$$\lim_{\varepsilon\to 0} I^1_{\varepsilon,l} = \int_\Omega b h_l(u)h(u)\phi, \tag{5.2.29}$$

$$\lim_{\varepsilon\to 0} I^5_{\varepsilon,l} = \int_\Omega f h_l(u)h(u)\phi. \tag{5.2.30}$$

We write $\quad I^3_{\varepsilon,l} = I^{3,1}_{\varepsilon,l} + I^{3,2}_{\varepsilon,l}$ where
$I^{3,1}_{\varepsilon,l} = \int_\Omega h_l'(u_\varepsilon)a(x,Du_\varepsilon)\cdot Du_\varepsilon h(u)\phi$ and $I^{3,2}_{\varepsilon,l} = \int_\Omega h_l(u_\varepsilon)a(x,Du_\varepsilon)\cdot D(h(u)\phi)$.
Using (5.2.13), we get the estimate

$$|\lim_{\varepsilon\to 0} I^{3,1}_{\varepsilon,l}| \le \|h\|_\infty\|\phi\|_\infty \cdot C_2(1) l^{-(p^*_s)^-}. \tag{5.2.31}$$

Since modular convergence is equivalent to norm convergence in $L^{p(.)}(\Omega,\omega)$, by Lebesgue Dominated Convergence Theorem it follows that for any $i \in \{1,...,N\}$, we have

$$h_l(u_\varepsilon)\frac{\partial}{\partial x_i}(h(u)\phi) \to h_l(u)\frac{\partial}{\partial x_i}(h(u)\phi) \ in\ L^{p(.)}(\Omega,\omega)\ as\ \varepsilon\downarrow 0.$$

Keeping in mind that $I^{3,2}_{\varepsilon,l} = \int_\Omega h_l(u_\varepsilon)a(x,DT_{l+1}(u_\varepsilon))\cdot D(h(u)\phi)$.
By (5.2.20), we get

$$\lim_{\varepsilon\to 0} I^{3,2}_{\varepsilon,l} = \int_\Omega h_l(u)a(x,DT_{l+1}(u))\cdot D(h(u)\phi). \tag{5.2.32}$$

Let us write $I^4_{\varepsilon,l} = I^{4,1}_{\varepsilon,l} + I^{4,2}_{\varepsilon,l}$, where

$$I^{4,1}_{\varepsilon,l} = \int_\Omega h_l'(u_\varepsilon)F(T_{\frac{1}{\varepsilon}}(u_\varepsilon))\cdot Du_\varepsilon h(u)\phi,$$
$$I^{4,2}_{\varepsilon,l} = \int_\Omega h_l(u_\varepsilon)F(T_{\frac{1}{\varepsilon}}(u_\varepsilon))\cdot D(h(u)\phi).$$

For any $l\in\mathbb{N}$, there exists $\varepsilon_0(l)$; such that for all $\varepsilon < \varepsilon_0(l)$,

$$I^{4,1}_{\varepsilon,l} = \int_\Omega h_l'(T_{l+1}(u_\varepsilon))F(T_{l+1}(u_\varepsilon))\cdot DT_{l+1}(u_\varepsilon)h(u)\phi. \tag{5.2.33}$$

Using Gauss-Green Theorem for Sobolev functions in (5.2.33), we get

$$I^{4,1}_{\varepsilon,l} = -\int_\Omega \int_0^{T_{l+1}(u_\varepsilon)} h'_l(r)F(r)dr \cdot D(h(u)\phi). \tag{5.2.34}$$

Now, using (5.2.16) and the Gauss-Green Theorem, after letting $\varepsilon \downarrow 0$, we get

$$\lim_{\varepsilon\to 0} I^{4,1}_{\varepsilon,l} = \int_\Omega h'_l(u)F(u) \cdot Du\; h(u)\phi. \tag{5.2.35}$$

Choosing ε small enough, we can write

$$I^{4,2}_{\varepsilon,l} = \int_\Omega h_l(u_\varepsilon)F(T_{l+1}(u_\varepsilon)) \cdot D(h(u)\phi) \tag{5.2.36}$$

and conclude

$$\lim_{\varepsilon\to 0} I^{4,2}_{\varepsilon,l} = \int_\Omega h_l(u)F(u) \cdot D(h(u)\phi). \tag{5.2.37}$$

Step ii : Passing to the limit with $l \to \infty$. Combining (5.2.27) and (5.2.28)-(5.2.37)), we find

$$I^1_l + I^2_l + I^3_l + I^4_l + I^5_l = I^6_l \tag{5.2.38}$$

where

$I^1_l = \int_\Omega bh_l(u)h(u)\phi,\qquad I^2_l = \int_\Omega h_l(u)a(x, DT_{l+1}(u)) \cdot D(h(u)\phi),$

$|I^3_l| \leq C_2(1)l^{-(p^*_s)^-}\|h\|_\infty\|\phi\|_\infty,\qquad I^4_l = \int_\Omega h_l(u)F(u) \cdot D(h(u)\phi),$

$I^5_l = \int_\Omega h'_l(u)F(u) \cdot Du\; h(u)\phi,\qquad I^6_l = \int_\Omega f\; h_l(u)h(u)\phi.$

Obviously, we have

$$\lim_{l\to\infty} I^3_l = 0. \tag{5.2.39}$$

Choosing $m > 0$ such that *supp* $h \subset [-m, m]$, we can replace u by $T_m(u)$ in $I^1_l, I^2_l, ..., I^6_l$, and $h'_l(u) = h'_l(T_m(u)) = 0$ if $l + 1 > m$, $h_l(u) = h_l(T_m(u)) = 1$, if $l > m$.

Therefore, letting $l \to \infty$ and combining (5.2.38) with (5.2.39), we obtain

$$\int_\Omega bh(u)\phi + \int_\Omega (a(x, Du) + F(u)) \cdot D(h(u)\phi) = \int_\Omega fh(u)\phi, \tag{5.2.40}$$

for all $h \in C^1_c(\mathbb{R})$ and all $\phi \in W^{1,p(.)}_0(\Omega, \omega) \cap L^\infty(\Omega)$.

Step iii : Subdifferential argument. It is left to prove that $u(x) \in D(\beta(x))$

and $b(x) \in \beta(u(x))$ for almost all $x \in \Omega$. Since β a is maximal monotone graph, exists a convex; l.s.c. and proper function $j : \mathbb{R} \to [0, \infty]$, such that $\beta(r) = \partial j(r)$ for all $r \in \mathbb{R}$. According to [51], for $0 < \varepsilon \leq 1$, $j_\varepsilon : \mathbb{R} \to \mathbb{R}$ defined by $j_\varepsilon(r) = \int_0^r \beta_\varepsilon(s)ds$ has the properties see chapter 3 page 56. Using the same argument in [89], we can prove that for all $r \in \mathbb{R}$ and almost every $x \in \Omega$, $u \in D(\beta)$ and $b \in \beta(u)$ almost everywhere in Ω. With this last step the proof of Theorem 5.2.1 is completed.

5.3 Case where $f \in L^1(\Omega)$ -data

In this section we establish the existence and uniqueness of renormalized solution to the degenerated problem (E, f) with $f \in L^1(\Omega)$.

5.3.1 Results of existence

Theorem 5.3.1 Under assuptions $(H_1) - (H_2)$, $(C_0) - (C_3)$ and $f \in L^1(\Omega)$, there exists at least one renormalized solution (u, b) to the degenerated problem (E, f).

5.3.2 Proof of theorem 5.3.1

Approximate problem and a priori estimates

To prove Theorem 5.3.1, we will introduce and solve approximating problems. To this end, for $f \in L^1(\Omega)$, and $n, m \in \mathbb{N}$, we define $f_{m,n} : \Omega \to \mathbb{R}$ by

$$f_{m,n} = \max(\min(f(x), m), -n)$$

for almost every $x \in \Omega$, clearly $f_{m,n} \in L^\infty(\Omega)$ for each $m, n \in \mathbb{N}$,
$|f_{m,n}(x)| \leq |f(x)|$ a.e in Ω hence

$$\lim_{n\to\infty} \lim_{m\to\infty} f_{m,n} = f \text{ in } L^1(\Omega) \text{ for almost everywhere in } \Omega.$$

The comparison principle from Remark 5.2.2 will be the main tool in the second approximation procedure. For $f \in L^1(\Omega)$ and $m, n \in \mathbb{N}$, let $f_{m,n} \in L^\infty(\Omega)$ be defined as above. From Theorem 5.2.1 it follows that for any $m, n \in \mathbb{N}$, there exists $u_{m,n} \in W_0^{1,p(.)}(\Omega, \omega)$, $b_{m,n} \in L^\infty(\Omega)$ such that $(u_{m,n}, b_{m,n})$ is a renormalized solution of $(E, f_{m,n})$. Therefore

$$\int_\Omega b_{m,n} h(u_{m,n})\phi + \int_\Omega \big(a(x, Du_{m,n}) + F(u_{m,n})\big) \cdot D(h(u_{m,n})\phi) = \int_\Omega f_{m,n} h(u_{m,n})\phi \tag{5.3.41}$$

holds for all $m, n \in \mathbb{N}, h \in C_c^1(\mathbb{R})$, $\phi \in W_0^{1,p(.)}(\Omega, \omega) \cap L^\infty(\Omega)$.
In the next Lemma, we give a priori estimates that will be important in the following :

Lemma 5.3.2 For $m, n \in \mathbb{N}$ let $(u_{m,n}, b_{m,n})$ be a renormalized solution of $(E, f_{m,n})$.

Then,

i) For any $k > 0$, we have

$$\int_\Omega |DT_k(u_{m,n})|^{p(x)} \omega(x) dx \leq \frac{k}{\lambda} \|f\|_1. \tag{5.3.42}$$

ii) For $k > 0$, there exists a constant $C_3(k) > 0$, not depending on $m, n \in \mathbb{N}$, such that

$$\|DT_k(u_{m,n})\|_{W_0^{1,p(.)}(\Omega,\omega)} \leq C_3(k). \tag{5.3.43}$$

iii) For all $m, n \in \mathbb{N}$, we have :

$$\|b_{m,n}\|_1 \leq \|f\|_1. \tag{5.3.44}$$

Proof 5.3.3 *For $l, k > 0$, we plug $h_l(u_{m,n})T_k(u_{m,n})$ as a test function in* (5.3.41). *Then i) and ii) follow with similar arguments as used in the proof of Lemma 5.2.5. To prove iii), we neglect the positive term $\int_\Omega a(x, DT_k(u_{m,n}))DT_k(u_{m,n})$ and keep*

$$\int_\Omega b_{m,n} T_k(u_{m,n}) \leq \int_\Omega f_{m,n} u_{m,n}. \tag{5.3.45}$$

Since $b_{m,n} \in \beta(u_{m,n})$ a.e. in Ω, from (5.3.45) *it follows that*

$$\int_{\{|u_{m,n}|>k\}} |b_{m,n}| \leq \int_\Omega |f| \tag{5.3.46}$$

and we find iii) by passing to the limit with $k \downarrow 0$.

By definition we have

$$f_{m,n} \leq f_{m+1,n} \text{ and } f_{m,n+1} \leq f_{m,n}. \tag{5.3.47}$$

From Remark 5.2.2 it follows that

$$u^{\varepsilon}_{m,n} \leq u^{\varepsilon}_{m+1,n} \text{ and } u^{\varepsilon}_{m,n+1} \leq u^{\varepsilon}_{m,n} \tag{5.3.48}$$

almost everywhere in Ω for any $m, n \in \mathbb{N}$ and all $\varepsilon > 0$, hence passing to the limit with $\varepsilon \downarrow 0$ in (5.3.48) yields

$$u_{m,n} \leq u_{m+1,n} \text{ and } u_{m,n+1} \leq u_{m,n} \tag{5.3.49}$$

almost everywhere in Ω for any $m, n \in \mathbb{N}$. Setting $b_\varepsilon = \beta_\varepsilon(T_{\frac{1}{\varepsilon}}(u_\varepsilon))$ using (5.3.48), Remark 5.2.4 and the fact that $b^{\varepsilon}_{m,n} \rightharpoonup b_{m,n}$ in $L^\infty(\Omega)$ and this convergence preserves order we get

$$b_{m,n} \leq b_{m+1,n} \text{ and } b_{m,n+1} \leq b_{m,n} \tag{5.3.50}$$

almost everywhere in Ω for any $m, n \in \mathbb{N}$. By (5.3.50) and (5.3.44), for any $n \in \mathbb{N}$ there exists $b^n \in L^1(\Omega)$ such that $b_{m,n} \to b^n$ as $m \to \infty$ in $L^1(\Omega)$ and almost everywhere in Ω and $b \in L^1(\Omega)$, such that $b^n \to b$ as $n \to \infty$ in $L^1(\Omega)$ and almost everywhere in Ω. By (5.3.49), the sequence $(u_{m,n})_m$ is monotone increasing, hence, for any $n \in \mathbb{N}$,$u_{m,n} \to u^n$ almost everywhere in Ω, where $u^n : \Omega \to \overline{\mathbb{R}}$ is a measurable function. Using (5.3.49) again, we conclude that the sequence $(u^n)_n$ is monotone decreasing, hence $u^n \to u$ almost everywhere in Ω, where $u : \Omega \to \overline{\mathbb{R}}$ is a measurable function. In order to show that u is finite almost everywhere, we will give an estimate on the level sets of $u_{m,n}$ in the next lemma.

Lemma 5.3.4 For $m,n \in \mathbb{N}$ let $(u_{m,n}, b_{m,n})$ be a renormalized solution of $(E, f_{m,n})$. Then, there exist a constant $C_4 > 0$, not depending on $m,n \in \mathbb{N}$, such that

$$|\{|u_{m,n}| \geq l\}| \leq C_4 l^{\frac{1}{\kappa}-1} \tag{5.3.51}$$

for all $l \geq 1$.

Proof 5.3.5 *With the same arguments as in Remark 5.2.7, we obtain*

$$\begin{aligned}
l|\{|u_{m,n}| \geq l\}| &= \int_{\{u_{m,n} \geq l\}} |T_l(u_{m,n})| dx \\
&\leq C\|DT_l(u_{m,n})\|_{L^{p(x)}(\Omega,\omega)} \\
&\leq C\Big(\int_\Omega |DT_l(u_{m,n})|^{p(x)}\omega(x)dx\Big)^\kappa \\
&\leq Cl^{\frac{1}{\kappa}}.
\end{aligned}$$

Where

$$\kappa = \begin{cases} p^- & if \quad \|DT_l(u_{m,n})\|_{L^{p(x)}(\Omega,\omega)} \leq 1 \\ \\ p^+ & if \quad \|DT_l(u_{m,n})\|_{L^{p(x)}(\Omega,\omega)} > 1 \end{cases}$$

which implies that,

$$|\{|u_{m,n}| \geq l\}| \leq \frac{C_4}{l^{1-\frac{1}{\kappa}}} \quad \forall\, l > 1.$$

Note that, as $(u_{m,n})_m$ is pointwise increasing with respect to m,

$$\lim_{m\to\infty} |\{|u_{m,n}| \geq l\}| = |\{|u^n| \geq l\}|, \tag{5.3.52}$$

and

$$\lim_{m\to\infty} |\{|u_{m,n}| \geq -l\}| = |\{|u^n| \geq -l\}|. \tag{5.3.53}$$

Combining (5.3.51) *with* (5.3.52) *and* (5.3.53), *we get*

$$|\{|u^n| \geq l\}| + |\{|u^n| \geq -l\}| \leq C_4 l^{\frac{1}{\kappa}-1}, \tag{5.3.54}$$

for any $l \geq 1$, *hence* u^n *is finite almost everywhere for any* $n \in \mathbb{N}$. *By the same arguments we get*

$$|\{|u| \geq l\}| + |\{|u| \geq -l\}| \leq C_4 l^{\frac{1}{\kappa}-1} \tag{5.3.55}$$

from (5.3.54), *hence* u *is finite almost everywhere. Now, since* $b_{m,n} \in \beta(u_{m,n})$ *almost everywhere in* Ω *it follows by a subdifferential argument that* $b^n \in \beta(u^n)$ *and* $b \in \beta(u)$ *almost everywhere in* Ω.

Remark 5.3.6

$$\int_{\{l<|u_{m,n}|<l+k\}} a(x, Du_{m,n}) \cdot Du_{m,n} \leq k\Big(\int_{\{|u_{m,n}|>l\}\cap\{|f|<\delta\}} |f| + \int_{\{|f|>\delta\}} |f|\Big) \tag{5.3.56}$$

for any $k, l, \delta > 0$. Now, applying (5.3.51) to (5.3.56), we find that

$$\int_{\{l<|u_{m,n}|<l+k\}} a(x, Du_{m,n}) \cdot Du_{m,n} \leq k\delta C_4 l^{\frac{1}{\kappa}-1} + k \int_{\{|f|>\delta\}} |f| \tag{5.3.57}$$

holds for any $k, \delta > 0$, $l \geq 1$ uniformly in $m, n \in \mathbb{N}$.

Basic convergence

Lemma 5.3.7 For $m, n \in \mathbb{N}$ let $(u_{m,n}, b_{m,n})$ be a renormalized solution of $(E, f_{m,n})$. There exists a subsequence $(m(n))_n$ such that setting $f_n := f_{m(n),n}$, $b_n := b_{m(n),n}$, $u_n := u_{m(n),n}$, we have

$$u_n \to u \quad \text{almost everywhere in } \Omega. \tag{5.3.58}$$

Moreover, for any $k > 0$,

$$T_k(u_n) \to T_k(u) \quad \text{in} \quad W_0^{1,p(.)}(\Omega, \omega) \tag{5.3.59}$$

$$DT_k(u_n) \rightharpoonup DT_k(u) \quad \text{in} \quad (L^{p(.)}(\Omega, \omega))^N \tag{5.3.60}$$

$$a(x, DT_k(u_n)) \rightharpoonup a(x, DT_k(u)) \quad \text{in} \quad (L^{p'(.)}(\Omega, \omega^*))^N. \tag{5.3.61}$$

as $n \to \infty$.

Proof 5.3.8 *We construct a subsequence $(m(n))_n$, such that*

$$\arctan(u_{m(n),n}) \to \arctan(u),\ b_n := b_{m(n),n} \to b,\ f_n := f_{m(n),n} \to f$$

as $n \to \infty$ in $L^1(\Omega)$ and almost everywhere in Ω. It follows that (5.3.58) and (5.3.59) hold. Combining (5.3.59) with (5.3.43), we get $T_k(u) \in W_0^{1,p(.)}(\Omega, \omega)$, $T_k(u_n) \to T_k(u)$ in $W_0^{1,p(.)}(\Omega, \omega)$ and (5.3.60) holds for any $k > 0$. From (5.3.42) and (C_2) it follows that, for fixed $k > 0$, given any subsequence of $(a(x, DT_k(u_n)))_n$ there exists a subsequence, still denoted by $a(x, DT_k(u_n))_n$, such that

$$a(x, DT_k(u_n)))_n \rightharpoonup \Phi_k \ in \quad (L^{p'(.)}(\Omega, \omega^*))^N.$$

as $n \to \infty$. Since $h_l(u_n)(T_k(u_n) - T_k(u))$ is an admissible test function in (5.3.41)), we obtain

$$\limsup_{n\to\infty} \int_\Omega a(x, DT_k(u_n)) \cdot D(T_k(u_n) - T_k(u)) \leq 0 \tag{5.3.62}$$

holds. Then, (5.3.61) follows with the same arguments as in the proof of Lemma 5.2.8 .

Remark 5.3.9 *With the same arguments as in Remark 5.2.10 and Lemma 5.2.11, we have*

$$\lim_{n\to\infty} \int_\Omega \Big(a(x, DT_k(u_n)) - a(x, DT_k(u))\Big) \cdot D(T_k(u_n) - T_k(u)) = 0. \tag{5.3.63}$$

$$\lim_{l\to\infty} \int_{\{l<|u|<l+1\}} a(x, Du) \cdot Du = 0. \tag{5.3.64}$$

Conclusion of the proof of Theorem 5.3.1

It is left to prove that (u, b) satisfies

$$\int_\Omega bh(u)\phi + \int_\Omega \big(a(x, Du) + F(u)\big) \cdot D(h(u)\phi) = \int_\Omega fh(u)\phi, \tag{5.3.65}$$

for all $h \in C^1_c(\mathbb{R})$, $\phi \in W_0^{1,p(.)}(\Omega, \omega) \cap L^\infty(\Omega)$. To this end, we take $h \in C^1_c(\mathbb{R})$ $\phi \in W_0^{1,p(.)}(\Omega, \omega) \cap L^\infty(\Omega)$ arbitrary and plug $h_l(u_n)h(u)\phi$ into (5.3.41) to obtain

$$I^1_{n,l} + I^2_{n,l} + I^3_{n,l} = I^4_{n,l}, \tag{5.3.66}$$

where

$$I^1_{n,l} = \int_\Omega b_n h_l(u_n)h(u)\phi, \quad I^2_{n,l} = \int_\Omega a(x, Du_n) \cdot D(h_l(u_n)h(u)\phi)$$
$$I^3_{n,l} = \int_\Omega F(u_n) \cdot D(h_l(u_n)h(u)\phi), \quad I^4_{n,l} = \int_\Omega fh_l(u_n)h(u)\phi.$$

Step 1 : Passing to the limit with $n \to \infty$, applying the convergence results from Lemma 5.3.7, we get

$$\lim_{n\to\infty} I^1_{n,l} = \int_\Omega bh_l(u)h(u)\phi \tag{5.3.67}$$

$$\lim_{n\to\infty} I^4_{n,l} = \int_\Omega fh_l(u)h(u)\phi. \tag{5.3.68}$$

Let us write

$$I^2_{n,l} = I^{2,1}_{n,l} + I^{2,2}_{n,l},$$

where

$$I^{2,1}_{n,l} = \int_\Omega h_l(u_n)a(x, Du_n) \cdot D(h(u)\phi)$$
$$I^{2,2}_{n,l} = \int_\Omega h_l'(u_n)a(x, Du_n) \cdot Du_n h(u)\phi.$$

With similar arguments as in the proof of (5.2.32) it follows that

$$\lim_{n\to\infty} I^{2,1}_{n,l} = \int_\Omega h_l(u)a(x, Du) \cdot D(h(u)\phi). \tag{5.3.69}$$

By (5.3.57), we get the estimate

$$|\lim_{n\to\infty} I^{2,2}_{n,l}| \leq \|h\|_\infty \|\phi\|_\infty \Big(\delta C_4 l^{\frac{1}{\kappa}-1} + \int_{\{|f|>\delta\}} |f|\Big), \tag{5.3.70}$$

for all $n \in \mathbb{N}$ and all $l \geq 1$, $\delta > 0$.
Next, we write

$$I^3_{n,l} = I^{3,1}_{n,l} + I^{3,2}_{n,l},$$

where

$$\lim_{n\to\infty} I^{3,1}_{n,l} = \int_\Omega h_l(u)F(u)\cdot D(h(u)\phi) \tag{5.3.71}$$

$$\lim_{n\to\infty} I^{3,2}_{n,l} = \int_\Omega h_l^{'}(u)F(u)\cdot Du\; h(u)\phi \tag{5.3.72}$$

follows with the same arguments as in (5.2.33)-(5.2.37).
Step 2 : Passing to the limit with $l \to \infty$. Combining (5.3.66) with (5.3.67)-(5.3.72), we get for all $\delta > 0$ and all $l \geq 1$

$$I^1_l + I^2_l + I^3_l + I^4_l + I^5_l = I^6_l \tag{5.3.73}$$

where,

$$I^1_l = \int_\Omega b h_l(u)h(u)\phi$$

$$I^2_l = \int_\Omega h_l(u)a(x, DT_{l+1}(u))\cdot D(h(u)\phi)$$

$$|I^3_l| \leq \|h\|_\infty\|\phi\|_\infty\Big(\delta C_4 l^{\frac{1}{\kappa}-1} + \int_{\{|f|>\delta\}}|f|\Big)$$

for any $\delta > 0$ and

$$I^4_l = \int_\Omega h_l^{'}(u)F(u)h(u)\phi Du$$

$$I^5_l = \int_\Omega h_l(u)F(u)\cdot D(h(u)\phi)$$

$$|I^6_l| = \int_\Omega f h_l(u)h(u)\phi.$$

Choosing $m > 0$ such that *supp* $h \subset [-m, m]$, we can replace u by $T_m(u)$ in I_l^1,... ,I_l^6, hence

$$\lim_{l\to\infty} I_l^1 = \int_\Omega bh(u)\phi \tag{5.3.74}$$

$$\lim_{l\to\infty} I_l^2 = \int_\Omega a(x, Du) \cdot D(h(u)\phi) \tag{5.3.75}$$

$$\lim_{l\to\infty} |I_l^3| \leq \|h\|_\infty \|\phi\|_\infty \int_{\{|f|>\sigma\}} |f| \tag{5.3.76}$$

$$\lim_{l\to\infty} I_l^4 = 0 \tag{5.3.77}$$

$$\lim_{l\to\infty} I_l^5 = \int_\Omega F(u) \cdot D(h(u)\phi) \tag{5.3.78}$$

$$\lim_{l\to\infty} |I_l^6| = \int_\Omega fh(u)\phi \tag{5.3.79}$$

for all $\delta > 0$. Combining (5.3.73) with (5.3.74) - (5.3.79), we finally obtain that (5.3.65) holds for all $h \in C_c^1(\mathbb{R})$ and all $\phi \in W_0^{1,p(.)}(\Omega, \omega) \cap L^\infty(\Omega)$.
Hence (u, b) satisfies $(R1), (R2)$ and $(R3)$ and proof of theorem is completed.

Chapitre 6

Renormalized solutions of nonlinear parabolic problems in Weighted variable exponent Sobolev spaces

[1] In this chapter, we study the existence of renormalized solutions for the nonlinear $p(x)$-parabolic problem with $f \in L^1(Q)$ and $b(x, u_0) \in L^1(\Omega)$. The main contribution of our work is to prove the existence of renormalized solutions of the weighted variable exponent Soblev spaces and we suppose that $H(x, t, u, \nabla u)$ is the non linear term satisfying some growth condition but no sign condition or the coercivity condition.

6.1 Introduction

Let Ω be a bounded domain in $\mathbb{R}^N (N \geq 1), T$ is a positive real number, and $Q = \Omega \times (0, T)$. We are interested in existence of renormalized solutions to the following nonlinear parabolic problem

$$(\mathcal{P}) \begin{cases} \frac{\partial b(x,u)}{\partial t} - div(a(x, t, u, \nabla u)) + H(x, t, u, \nabla u) = f \text{ in } Q = \Omega \times (0, T) \\ b(x, u) \mid_{t=0} = b(x, u_0) \quad \text{in } \Omega \\ u = 0 \quad \text{ on } \partial\Omega \times (0, T), \end{cases}$$

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where $f \in L^1(Q)$, $b(x, u_0) \in L^1(\Omega)$. The operator $-div(a(x,t,u,\nabla u)$ is a Leray-Lions operator defined on $L^{p^-}(0,T;W_0^{1,p(\cdot)}(\Omega,\omega))$ (see assumption (6.3.6)-(6.3.8) of section 3) which is coercive $b(x,u)$ is an unbounded function of u, H is a non linear lower order term. The notion of renormalized solutions was introduced by R. J. Diperna and P. L. Lions [56] for the study of the Boltzmann equation. It was then used by L. Boccardo and al [47] when the right hand side is in $W^{-1,p'}(\Omega)$ and by J. M Rakoston [84] when the right hand side is in $L^1(\Omega)$.
It is our purpose to prove the existence of renormalized solution of weighted variable exponent Sobolev spaces for the problem $(\mathcal{P})$ setting without the sign condition and without the coercivity condition, the critical growth condition on H is only with respect to ∇u and not with respect to u (see assumption H2). Where the right hand side is assumed to satisfy : f belongs to $L^1(Q)$. Other work in this direction can be found in [[3],[9],[81],[82]].
For the convenience of the readers, we recall some definitions and basic properties of the weighted variable exponent Lebesgue spaces $L^{p(x)}(\Omega,\omega)$ and the weighted variable exponent Sobolev spaces $W^{1,p(x)}(\Omega,\omega)$. Set

$$C_+(\overline{\Omega}) = \{p \in C(\overline{\Omega}) : \min_{x\in\overline{\Omega}} p(x) > 1\}.$$

For any $p \in C_+(\overline{\Omega})$, we define $\quad p^+ = \max_{x\in\overline{\Omega}} p(x), \quad p^- = \min_{x\in\overline{\Omega}} p(x)$.
For any $p \in C_+(\overline{\Omega})$, we introduce the weighted variable exponent Lebesgue space $L^{p(x)}(\Omega,\omega)$ that consists of all measurable real-valued functions u such that

$$L^{p(x)}(\Omega,\omega) = \{u : \Omega \to \mathbb{R}, measurable,\ \int_\Omega |u(x)|^{p(x)}\omega(x)dx < \infty\}.$$

Then, $L^{p(x)}(\Omega,\omega)$ endowed with the Luxemburg norm

$$|u|_{L^{p(x)}(\Omega,\omega)} = \inf\{\lambda > 0 : \int_\Omega |\frac{u(x)}{\lambda}|^{p(x)}\omega(x)dx \leq 1\}$$

becomes a normed space. When $\omega(x) \equiv 1$, we have $L^{p(x)}(\Omega,\omega) \equiv L^{p(x)}(\Omega)$ and we use the notation $|u|_{L^{p(x)}(\Omega)}$ instead of $|u|_{L^{p(x)}(\Omega,\omega)}$. The following Hölder type inequality is useful for the next sections. The weighted variable exponent Sobolev space $W^{1,p(x)}(\Omega,\omega)$ is defined by

$$W^{1,p(x)}(\Omega,\omega) = \{u \in L^{p(x)}(\Omega); |\nabla u| \in L^{p(x)}(\Omega,\omega)\},$$

where the norm is

$$\|u\|_{W^{1,p(x)}(\Omega,\omega)} = |u|_{L^{p(x)}(\Omega)} + |\nabla u|_{L^{p(x)}(\Omega,\omega)} \tag{6.1.1}$$

or, equivalently

$$\|u\|_{W^{1,p(x)}(\Omega,\omega)} = \inf\{\lambda > 0 : \int_\Omega |\frac{u(x)}{\lambda}|^{p(x)} + \omega(x)|\frac{\nabla u(x)}{\lambda}|^{p(x)} dx \leq 1\}$$

for all $u \in W^{1,p(x)}(\Omega, \omega)$.
It is significant that smooth functions are not dense in $W^{1,p(x)}(\Omega)$ without additional assumptions on the exponent $p(x)$. This feature was observed by Zhikov [98] in connection with the Lavrentiev phenomenon. However, if the exponent $p(x)$ is log-Hölder continuous, i.e., there is a constant C such that

$$|p(x) - p(y)| \leq \frac{C}{-\log|x-y|} \tag{6.1.2}$$

for every x, y with $|x-y| \leq \frac{1}{2}$, then smooth functions are dense in variable exponent Sobolev spaces and there is no confusion in defining the Sobolev space with zero boundary values, $W_0^{1,p(x)}(\Omega)$, as the completion of $C_0^\infty(\Omega)$ with respect to the norm $\|u\|_{W^{1,p(x)}(\Omega)}$ (see [66]).
The space $W_0^{1,p(x)}(\Omega, \omega)$ is defined as the completion of $C_0^\infty(\Omega)$ in $W^{1,p(x)}(\Omega, \omega)$ with respect to the norm $\|u\|_{W^{1,p(x)}(\Omega,\omega)}$.
Throughout the chapter, we assume that $p \in C_+(\overline{\Omega})$ and ω is a measurable positive and a.e. finite function in Ω.
This chapter is organized as follows. In section 2, we state some basic results for the weighted variable exponent Lebesgue-Sobolev spaces which is given in [70]. In section 3, we make precise all the assumption on b, a, H, f and $b(x, u_0)$ and give the definition of a renormalized solution of the problem $(\mathcal{P})$ and main results, which is proved in section 4.

6.2 Some Technical Results

Lemma 6.2.1 *Assume* (6.3.6) -(6.3.8) *and let* $(u_n)_n$ *be a sequence in* $L^{p^-}(0, T, W_0^{1,p(\cdot)}(\Omega, \omega))$ *such that* $u_n \rightharpoonup u$ *weakly in* $L^{p^-}(0, T, W_0^{1,p(\cdot)}(\Omega, \omega))$ *and*

$$\int_Q \Big(a(x, t, u_n, \nabla u_n) - a(x, t, u_n, \nabla u)\Big) \cdot \nabla(u_n - u) dxdt \to 0. \tag{6.2.3}$$

Then, $u_n \to u$ *strongly in* $L^{p^-}(0, T, W_0^{1,p(\cdot)}(\Omega, \omega))$.

Proof.
Let $D_n = [a(x,t,u_n,\nabla u_n) - a(x,t,u_n,\nabla u)]\nabla(u_n - u)$, thanks to (6.3.7), we have D_n is a positive function, and by (6.2.3) , $D_n \to 0$ in $L^1(Q)$ as $n \to \infty$.
Extracting a subsequence, still denoted by u_n, we can write $u_n \rightharpoonup u$ a.e. in Q and since $D_n \to 0$ a.e. in Q. There exists a subset B in Q with measure zero such that for all $(t,x) \in Q \backslash B$,

$$|u(x,t)| < \infty, \quad |\nabla u(x,t)| < \infty, \quad K(x,t) < \infty, \quad u_n \to u, \quad D_n \to 0.$$

Taking $\xi_n = \nabla u_n$ and $\xi = \nabla u$, we have

$$\begin{aligned}
D_n(x,t) &= [a(x,t,u_n,\xi_n) - a(x,t,u_n,\xi)] \cdot (\xi_n - \xi) \\
&= a(x,t,u_n,\xi_n)\xi_n + a(x,t,u_n,\xi)\xi - a(x,t,u_n,\xi_n)\xi - a(x,t,u_n,\xi)\xi_n \\
&\geq \alpha\omega(x)|\xi_n|^{p(x)} + \alpha\omega(x)|\xi|^{p(x)} \\
&- \beta\omega^{1/p(x)}(x)\Big(k(x,t) + \omega^{1/p'(x)}(x)|u_n|^{p(x)-1} + \omega^{1/p'(x)}(x)|\xi_n|^{p(x)-1}\Big)|\xi| \\
&- \beta\omega^{1/p(x)}(x)\Big(k(x,t) + \omega^{1/p'(x)}(x)|u_n|^{p(x)-1} + \omega^{1/p'(x)}(x)|\xi|^{p(x)-1}\Big)|\xi_n| \\
&\geq \alpha\omega(x)|\xi_n|^{p(x)} - C_{x,t}[1 + \omega^{1/p'(x)}(x)|\xi_n|^{p(x)-1} + \omega^{1/p(x)}(x)|\xi_n|],
\end{aligned}$$

where $C_{x,t}$ depending on x, but does not depend on n. (Since $u_n(x,t) \to u(x,t)$ then, $(u_n)_n$ is bounded), we obtain

$$D_n(x,t) \geq |\xi_n|^{p(x)}\Big(\alpha\omega(x) - \frac{C_{x,t}}{|\xi_n|^{p(x)}} - \frac{C_{x,t}\omega^{\frac{1}{p'(x)}}}{|\xi_n|} - \frac{C_{x,t}\omega^{\frac{1}{p(x)}}}{|\xi_n|^{p(x)-1}}\Big),$$

by the standard argument $(\xi_n)_n$ is bounded almost everywhere in Q. Indeed, if $|\xi_n| \to \infty$ in a measurable subset $E \in Q$ then,

$$\lim_{n\to\infty}\int_Q D_n(x,t)dx \geq \lim_{n\to\infty}\int_E |\xi_n|^{p(x)}\Big(\alpha\omega(x) - \frac{C_{x,t}}{|\xi_n|^{p(x)}} - \frac{C_{x,t}\omega^{\frac{1}{p'(x)}}}{|\xi_n|} - \frac{C_{x,t}\omega^{\frac{1}{p(x)}}}{|\xi_n|^{p(x)-1}}\Big) = \infty,$$

which is absurd since $D_n(x,t) \to 0$ in $L^1(Q)$). Let ξ^* an accumulation point of $(\xi_n)_n$, we have $|\xi^*| < \infty$ and by continuity of $a(.,.,.,.)$, we obtain

$$a(x,t,u(x,t),\xi^*) - a(x,t,u(x,t),\xi)] \cdot (\xi_n - \xi) = 0,$$

thanks to (6.3.7), we have $\xi^* = \xi$, the uniqueness of the accumulation point implies that $\nabla u_n(x,t) \to \nabla u(x,t)$ a.e. in Q. Since the sequence $a(x,t,u,\nabla u_n)$ is bounded in $(L^{p'(x)}(Q,\omega^*))^N$ and $a(x,t,u,\nabla u_n) \to a(x,t,u,\nabla u)$ a.e. in Q, Lemma 1.2.8 implies

$$a(x,t,u_n,\nabla u_n) \rightharpoonup a(x,t,u,\nabla u) \quad \text{in } (L^{p'(x)}(Q,\omega^*))^N.$$

Let us taking $\bar{y}_n = a(x,t,u_n,\nabla u_n)\nabla u_n$ and $\bar{y} = a(x,t,u,\nabla u)\nabla u$, then $\bar{y}_n \to \bar{y}$ in $L^1(Q)$, according to the condition (6.3.8), we have

$$\alpha\omega(x)|\nabla u_n|^{p(x)} \leq a(x,t,u_n,\nabla u_n)\nabla u_n.$$

Let $z_n = |\nabla u_n|^{p(x)}\omega,\ z = |\nabla u|^{p(x)}\omega$ and $y_n = \frac{\bar{y}_n}{\alpha},\ y = \frac{\bar{y}}{\alpha}$. Then, by Fatou's Lemma, we obtain

$$\int_Q 2y dxdt \leq \liminf_{n\to\infty} \int_Q (y_n + y - |z_n - z|)dxdt,$$

i.e., $0 \leq \limsup_{n\to\infty} \int_Q |z_n - z|dxdt$, hence

$$0 \leq \liminf_{n\to\infty} \int_Q |z_n - z|dx \leq\leq \limsup_{n\to\infty} \int_Q |z_n - z|dx \leq 0,$$

this implies

$$\nabla u_n \to \nabla u \quad \text{in } (L^{p(x)}(Q,\omega))^N,$$

we deduce that

$$u_n \to u \quad \text{in } L^{p^-}(0,T,W_0^{1,p(\cdot)}(\Omega,\omega)),$$

which completes our proof.

Let $X = L^{p^-}(0,T;W_0^{1,p(x)}(\Omega,\omega))$, the dual space of X is $X^* = L^{p^-}(0,T;(W_0^{1,p(x)}(\Omega,\omega))^*)$.

Lemma 6.2.2 *(See[80].)*

$W := \Big\{u \in V; u_t \in V^* + L^1(Q)\Big\} \hookrightarrow C([0,T];L^1(\Omega))$

and

$$W \cap L^\infty(Q) \hookrightarrow C([0,T];L^2(\Omega)).$$

Definition 6.2.3 *A monotone map $T : D(T) \to X^*$ is called maximal monotone if its graph*

$$G(T) = \Big\{(u,T(u)) \in X \times X^* \ \mathit{for\ \ all}\ u \in D(T)\Big\},$$

is not a proper subset of any monotone set in $X \times X^*$.

Let us consider the operator $\frac{\partial}{\partial t}$ which induces a linear map L from the subset

$$D(L) = \Big\{v \in X : v' \in X^*, v(0) = 0\Big\} \ \mathit{of}\ X\ \mathit{in\ to}\ X^*\ \mathit{by}$$

$$\left\langle Lu, v \right\rangle_X = \int_0^T \langle u'(t), v(t)\rangle_V dt \quad u \in D(L),\ v \in X.$$

Definition 6.2.4 *A mapping S is called pseudo-monotone with $u_n \rightharpoonup u$ and $Lu_n \rightharpoonup Lu$ and $\lim_{n\to\infty} \sup\langle S(u_n), u_n - u\rangle \leq 0$, that we have*

$\lim_{n\to\infty} \sup \left\langle S(u_n), u_n - u \right\rangle = 0$ *and $S(u_n) \rightharpoonup S(u)$ as $n \to \infty$.*

6.3 Assumption and Main Results

Throughout the paper, we assume that the following assumption hold true.

Assumption (H1)

Let Ω be a bounded open subset of $\mathbb{R}^N (N \geq 1)$, $p \in C_+(\overline{\Omega})$ and $b : \Omega \times \mathbb{R} \to \mathbb{R}$ is a Carathéodory function such that for every $x \in \Omega$, $b(x, .)$ is a strictly increasing C^1 function with

$$b(x, 0) = 0. \tag{6.3.4}$$

Next, for any $k > 0$, there exist $\lambda_k > 0$ and functions $A_k \in L^\infty(\Omega)$ and $B_k \in L^{p(\cdot)}(\Omega)$ such that

$$\lambda_k \leq \frac{\partial b(x, s)}{\partial s} \leq A_k(x) \text{ and } \quad \left| D_x\Big(\frac{\partial b(x, s)}{\partial s}\Big)\right| \leq B_k(x), \tag{6.3.5}$$

for almost every $x \in \Omega$ and every s such that $|s| \leq k$, we denote by $D_x(\partial b(x, s) \setminus \partial s)$ the gradient of $\partial b(x, s) \setminus \partial s$ defined in the sense of distributions.

Assumption (H2)

We consider a Leray -Lions operator defined by the formula :

$$Au = -div\ a(x, t, u, \nabla u),$$

where $a : \Omega \times [0, T] \times \mathbb{R} \times \mathbb{R}^N \to \mathbb{R}$ is a Caratheodory function i.e., (measurable with respect to x in Ω for every (s, ξ) in $\mathbb{R} \times \mathbb{R}^N$ and continuous with respect to (s, ξ) in $\mathbb{R} \times \mathbb{R}^N$, for almost every x in Ω) which satisfies the following conditions there exist $k \in L^{p'(\cdot)}(Q)$ and $\alpha > 0$, $\beta > 0$ such that for almost every $(x, t) \in Q$ all $(s, \xi) \in \mathbb{R} \times \mathbb{R}^N$.

$$|a(x, t, s, \xi)| \leq \beta\omega^{1/p(x)}(x)[k(x, t) + \omega^{1/p'(x)}|s|^{p(x)-1} + \omega^{1/p'(x)}(x)|\xi|^{p(x)-1}], \tag{6.3.6}$$

$$[a(x, t, s, \xi) - a(x, t, s, \eta)] \cdot (\xi - \eta) > 0 \ \ \forall\, \xi \neq \eta \in \mathbb{R}^N, \tag{6.3.7}$$

$$a(x, t, s, \xi) \cdot \xi \geq \alpha\omega|\xi|^{p(x)}. \tag{6.3.8}$$

Assumption (H3)

Let $H : \Omega \times [0,T] \times \mathbb{R} \times \mathbb{R}^N \to \mathbb{R}$ is a Carathéodory function such that for a.e. $(x,t) \in Q$ and for all $s \in \mathbb{R}$, $\xi \in \mathbb{R}^N$, the growth condition

$$|H(x,t,s,\xi)| \leq \gamma(x,t) + g(s)\omega|\xi|^{p(x)} \tag{6.3.9}$$

is satisfied, where $g : \mathbb{R} \to \mathbb{R}^+$ is a bounded continuous positive function that belongs to $L^1(\mathbb{R})$, while $\gamma \in L^1(Q)$.

We recall that, for $k > 0$ and $s \in \mathbb{R}$, the truncation function $T_k(.)$ defined by

$$T_k(s) = \begin{cases} s & if \quad |s| \leq k \\ k\frac{s}{|s|} & if \quad |s| > k. \end{cases}$$

Definition 6.3.1 *Let $f \in L^1(Q)$ and $b(., u_0) \in L^1(\Omega)$. A real-valued function u defined on Q is renormalized solutions of problem $(\mathcal{P})$ if :*

$$T_k(u) \in L^{p^-}(0,T; W_0^{1,p(\cdot)}(\Omega,\omega)) \text{ for all } k \geq 0,\ b(x,u) \in L^\infty(0,T; L^1(\Omega)), \tag{6.3.10}$$

$$\int_{\{m\leq|u|\leq m+1\}} a(x,t,u,\nabla u)\nabla u dxdt \to 0 \ as \ \ m \to \infty, \tag{6.3.11}$$

$$\begin{aligned} \frac{\partial B_S(x,u)}{\partial t} - div\Big(S'(u)a(x,t,u,\nabla u)\Big) + S''(u)a(x,t,u,\nabla u)\nabla u \\ +H(x,t,u,\nabla u)S'(u) = fS'(u) \ in \ D'(Q), \end{aligned} \tag{6.3.12}$$

for all $S \in W^{2,\infty}(\mathbb{R})$, which are piecewise C^1 and such that S' has a compact support in $\mathbb{R}$, where $B_S(x,z) = \int_0^z \frac{\partial b(x,r)}{\partial r} S'(r)dr$ and

$$B_S(x,u)\mid_{t=0} = B_S(x,u_0) \quad in \quad \Omega. \tag{6.3.13}$$

Remark 6.3.2 *Equation* (6.3.12) *is formally obtained through pointwise multiplication of problem $(\mathcal{P})$ by $S'(u)$. However, while $a(x,t,u,\nabla u)$ and $H(x,t,u,\nabla u)$ do not in general make sense in $(\mathcal{P})$, all the terms in* (6.3.12) *have a meaning in $D'(Q)$. Indeed, if M is such that supp $S \subset [-M, M]$, the following identifications are made in (6.3.12) :*

– *$S(u)$ belongs to $V \cap L^\infty(Q)$. Since S is a bounded function.*

– $S'(u)\ a(x,t,u,\nabla u)$ *identifies with* $S'(u)\ a(x,t,T_M(u),\nabla T_M(u))$ *a.e. in* Q, *for any* $\varphi \in D(Q)$, *using Hölder inequality*

$$\int_Q S'(u)a(x,t,u,\nabla u)\nabla\varphi dxdt = \int_Q S'(u)a(x,t,T_M(u),\nabla T_M(u))\nabla\varphi dxdt$$
$$\leq C_M\|S'\|_{L^\infty(Q)} \max\Big\{\Big(\int_Q |\nabla T_M(u)|^{p(x)}\omega\Big)^{\frac{1}{p'^-}}, \Big(\int_Q |\nabla T_M(u)|^{p(x)}\omega\Big)^{\frac{1}{p'^+}}\Big\}\|\nabla\varphi\|_{L^{p(\cdot)}(Q)},$$

where $M > 0$ *is that supp* $S' \subset [-M, M]$. *As* $D(Q)$ *is dense in* V, *we deduce that*

$$div(S'(u)a(x,t,u,\nabla u)) \in V^*.$$

– $S''(u)\ a(x,t,u,\nabla u)\nabla u$ *identifies with* $S''(u)\ a(x,u,T_M(u),\nabla T_M(u))\nabla T_M(u)$ *and*

$$S''(u)a(x,u,T_M(u),\nabla T_M(u))\nabla T_M(u) \in L^1(Q).$$

– $S'(u)H(x,t,u,\nabla u)$ *identifies with* $S'(u)H(x,t,T_M(u),\nabla T_M(u))$ *a.e. in* Q. *Since* $|T_M(u)| \leq M$ *a.e. in* Q *and* $S'(u) \in L^\infty(Q)$, *we see from (6.3.9) and (6.3.10) that* $S'(u)H(x,t,T_M(u),\nabla T_M(u)) \in L^1(Q)$.

– $S'(u)\ f$ *belongs to* $L^1(Q)$.

The above considerations show that equation (6.3.12)*) hold in* $D'(Q)$ *and that*

$$\frac{\partial B_S(x,u)}{\partial t} \in V^* + L^1(Q).$$

Due to the properties of S *and* (6.3.12), $\frac{\partial S(u)}{\partial t} \in V^* + L^1(Q)$, *using Lemma 1.2.9 which implies that* $S(u) \in C^0([0,T]; L^1(\Omega))$. *So that the initial condition* (6.3.13) *makes sense since, due to the properties of* S *(increasing) and* (6.3.5), *we have*

$$\Big|(B_S(x,r) - B_S(x,r')\Big| \leq A_k(x)\Big|S(r) - S(r')\Big| \text{ for all } r, r' \in \mathbb{R}. \tag{6.3.14}$$

Theorem 6.3.3 *Let* $f \in L^1(Q)$, $p(\cdot) \in C_+(\bar\Omega)$ *and assume that* u_0 *is a measurable function such that* $b(., u_0) \in L^1(\Omega)$. *Assume that* $(H1) - (H3)$ *hold true. Then* *there, exists a renormalized solution u of problem $(\mathcal{P})$ in the sense of Definition 6.3.1.*

6.4 Proof of Main Results.

6.4.1 Approximate problem

For $n > 0$, we define approximations of b, H, f and u_0. First set

$$b_n(x,r) = b(x, T_n(r)) + \frac{1}{n}r. \tag{6.4.15}$$

b_n is a Carathéodory function and satisfies (6.3.5). There exist $\lambda_n > 0$ and functions $A_n \in L^\infty(\Omega)$ and $B_n \in L^{p(\cdot)}(\Omega)$ such that

$$\lambda_n \leq \frac{\partial b_n(x,s)}{\partial s} \leq A_n(x) \text{ and } \quad \left| D_x\Big(\frac{\partial b_n(x,s)}{\partial s}\Big)\right| \leq B_n(x) \; a.e. \text{ in } \; \Omega, \; s \in \mathbb{R}.$$

Next, set

$$\begin{aligned} H_n(x,t,s,\xi) &= \frac{H(x,t,s,\xi)}{1+\frac{1}{n}|H(x,t,s,\xi)|}. \\ \text{Note that} \quad |H_n(x,t,s,\xi)| &\leq |H(x,t,s,\xi)| \\ \text{and} \quad |H_n(x,t,s,\xi)| &\leq n \text{ for all } (s,\xi) \in \mathbb{R}\times\mathbb{R}^N \end{aligned}$$

and select f_n, u_{0n} and b_n. So that

$$f_n \in L^{p'(\cdot)}(Q) \textit{ and } f_n \to f \textit{ a.e. in } Q, \text{ strongly in } L^1(Q) \text{ as } n \to \infty, \tag{6.4.16}$$

$$u_{0n} \in D(\Omega), \qquad \|b_n(x,u_{0n})\|_{L^1(\Omega)} \leq \| b_n(x,u_0)\|_{L^1(\Omega)}, \tag{6.4.17}$$

$$b_n(x,u_{0n}) \to b(x,u_0) \text{ a.e. in } \Omega \text{ and strongly in } L^1(\Omega). \tag{6.4.18}$$

Let us now consider the approximate problem

$$(\mathcal{P}_n)\begin{cases} \frac{\partial b_n(x,u_n)}{\partial t} - div(a(x,t,u_n,\nabla u_n)) + H_n(x,t,u_n,\nabla u_n) = f_n \quad \text{in } D'(Q), \\ b_n(x,u_n)\mid_{t=0} = b_n(x,u_{0n}) \quad \text{in } \Omega, \\ u_n = 0 \quad \text{on} \quad \partial\Omega\times(0,T). \end{cases}$$

Theorem 6.4.1 *Let $f_n \in L^{p'^-}(0,T;W^{-1,p'(\cdot)}(\Omega,\omega^*))$, $p(\cdot) \in C_+(\overline{\Omega})$ for fixed n, the approximate problem $(\mathcal{P}_n)$ has at least one weak solution $u_n \in L^{p^-}(0,T;W_0^{1,p(\cdot)}(\Omega,\omega))$.*

Proof.
We define the operator $L_n : L^{p^-}(0,T;W_0^{1,p(x)}(\Omega,\omega)) \to L^{p'^-}(0,T;W^{-1,p'(\cdot)}(\Omega,\omega^*))$ by
$\Big\langle L_n u, v\Big\rangle = \int_Q \frac{\partial b_n(x,u)}{\partial t} v dxdt = \int_Q \frac{\partial b_n(x,u)}{\partial u}\frac{\partial u}{\partial t} v dxdt \quad \forall u,v \in L^{p^-}(0,T;W_0^{1,p(\cdot)}(\Omega,\omega))$, then,

$$\begin{aligned}
\Big|\Big\langle L_n u, v\Big\rangle\Big| &\le \Big|\int_0^T\int_\Omega A_n(x)\frac{\partial u}{\partial t} v dxdt\Big| = \Big|\int_0^T\int_\Omega A_n(x)\frac{\partial u}{\partial t}\omega^{-\frac{1}{p(x)}} v\omega^{\frac{1}{p(x)}} dxdt\Big| \\
&\le \Big(\frac{1}{p^-}+\frac{1}{p'^-}\Big)\|A_n\|_{L^\infty}\int_0^T \|\frac{\partial u}{\partial t}\|_{L^{p'(x)}(\Omega,\omega^*)}\|v\|_{L^{p(x)}(\Omega,\omega)} dt \\
&\le C\Big(\frac{1}{p^-}+\frac{1}{p'^-}\Big)\|A_n\|_{L^\infty}\int_0^T \|\frac{\partial u}{\partial t}\|_{W^{-1,p'(\cdot)}(\Omega,\omega^*)}\|v\|_{W_0^{1,p(x)}(\Omega,\omega)} dt \qquad (6.4.19)\\
&\le C\Big(\frac{1}{p^-}+\frac{1}{p'^-}\Big)\|A_n\|_{L^\infty}\|\frac{\partial u}{\partial t}\|_{L^{p'^-}(0,T,W^{-1,p'(\cdot)}(\Omega,\omega^*))}\|v\|_{L^{p^-}(0,T,W_0^{1,p(x)}(\Omega,\omega))} \\
&\le C_1\|v\|_{L^{p^-}(0,T,W_0^{1,p(x)}(\Omega,\omega))}.
\end{aligned}$$

We define the operator $G_n : L^{p^-}(0,T;W_0^{1,p(\cdot)}(\Omega,\omega)) \to L^{p^-}(0,T,W^{-1,p'(\cdot)}(\Omega,\omega^*))$

$$by, \quad \Big\langle G_n u, v\Big\rangle = \int_Q H_n(x,t,u,\nabla u) v dxdt \quad \forall u,v \in L^{p^-}(0,T;W_0^{1,p(\cdot)}(\Omega,\omega)).$$

Thanks to the Hölder inequality, we have that for $u,v \in L^{p^-}(0,T;W_0^{1,p(\cdot)}(\Omega,\omega))$

$$\begin{aligned}
&\int_Q H_n(x,t,u,\nabla u) v dxdt \le \Big|\int_0^T\int_\Omega H_n(x,t,u,\nabla u) v dxdt\Big| \\
&\le \Big|\int_0^T\int_\Omega H_n(x,t,u,\nabla u)\omega^{-\frac{1}{p(x)}} v\omega^{\frac{1}{p(x)}} dxdt\Big| \\
&\le \Big(\frac{1}{p^-}+\frac{1}{p'^-}\Big)\int_0^T \Big(\int_\Omega \Big|H_n(x,t,u,\nabla u)\Big|^{p'(x)}\omega^{-\frac{p'(x)}{p(x)}} dx\Big)^\theta \|v\|_{L^{p(x)}(\Omega,\omega)} dt \\
&\le C\Big(\frac{1}{p^-}+\frac{1}{p'^-}\Big)\int_0^T n^{\theta p'^+}\Big(\int_\Omega \omega^{-\frac{p'(x)}{p(x)}} dx\Big)^\theta \|v\|_{W_0^{1,p(x)}(\Omega,\omega)} dt \\
&\le C_2\|v\|_{L^{p^-}(0,T;W_0^{1,p(\cdot)}(\Omega,\omega))}. \qquad (6.4.20)
\end{aligned}$$

with $\theta = \begin{cases} 1/p'^- & if \quad \|H_n(x,t,u,\nabla u)\|_{L^1(Q)} > 1 \\ 1/p'^+ & if \quad \|H_n(x,t,u,\nabla u)\|_{L^1(Q)} \le 1. \end{cases}$

Lemma 6.4.2 *Let* $B_n : L^{p^-}(0,T;W_0^{1,p(\cdot)}(\Omega,\omega)) \to L^{p'^-}(0,T,W^{-1,p'(\cdot)}(\Omega,\omega^*))$.

The operator $B_n = A + G_n$ *is*

a)coercive

b) pseudo-monotone

c) bounded and demi continuous.

Proof. **a)** For the coercivity, we have for any $u \in L^{p^-}(0,T;W_0^{1,p(\cdot)}(\Omega,\omega))$

$$\begin{aligned}
&\Big\langle B_n u, u\Big\rangle = \Big\langle G_n u, u\Big\rangle + \Big\langle Au, u\Big\rangle \\
&\Rightarrow \Big\langle B_n u, u\Big\rangle - \Big\langle G_n u, u\Big\rangle = \Big\langle Au, u\Big\rangle \\
\textit{then,}\ \ &\Big\langle B_n u, u\Big\rangle - \Big\langle G_n u, u\Big\rangle = \int_Q a(x,t,u,\nabla u)\nabla u dxdt \\
&= \int_0^T\int_\Omega a(x,t,u,\nabla u)\nabla u dxdt \\
&\geq \int_0^T \alpha(\int_\Omega |\nabla u|^{p(x)}\omega(x)dx)dt \quad \text{(using (6.3.8))} \\
&\geq \alpha\|\nabla u\|^{\delta}_{L^{p^-}(0,T;W_0^{1,p(\cdot)}(\Omega,\omega))} \geq \beta\|u\|^{\delta}_{L^{p^-}(0,T;W_0^{1,p(\cdot)}(\Omega,\omega))},
\end{aligned}$$

which is due to Poincaré inequality with
$\delta = \begin{cases} p^- & if\ \|\nabla u\|_{L^{p^-}(0,T;W_0^{1,p(\cdot)}(\Omega,\omega))} > 1 \\ p^+ & if\ \|\nabla u\|_{L^{p^-}(0,T;W_0^{1,p(\cdot)}(\Omega,\omega))} \leq 1, \end{cases}$

$$\textit{hence,}\ \ \Big\langle B_n u, u\Big\rangle - \Big\langle G_n u, u\Big\rangle \geq \beta\|u\|^{\delta}_{L^{p^-}(0,T;W_0^{1,p(\cdot)}(\Omega,\omega))}$$
$$\textit{then,}\ \Big\langle B_n u, u\Big\rangle \geq \beta\|\ u\|^{\delta}_{L^{p^-}(0,T;W_0^{1,p(\cdot)}(\Omega,\omega))} - C_2\|u\|_{L^{p^-}(0,T;W_0^{1,p(\cdot)}(\Omega,\omega))}$$

then, we have

$$\frac{\Big\langle B_n u, u\Big\rangle}{\|u\|_{L^{p^-}(0,T;W_0^{1,p(\cdot)}(\Omega,\omega))}} \geq \beta\|u\|^{\delta-1}_{L^{p^-}(0,T;W_0^{1,p(\cdot)}(\Omega,w))} - C_2 \to +\infty$$
$$\Rightarrow \frac{\Big\langle B_n u, u\Big\rangle}{\|u\|_{L^{p^-}(0,T;W_0^{1,p(x)}(\Omega,\omega))}} \to +\infty \quad as\ \|u\|_{L^{p^-}(0,T;W_0^{1,p(\cdot)}(\Omega,\omega))} \to +\infty$$

then, B_n is coercive.
b)It remains to show that B_n is pseudo-monotone.

Let $(u_k)_k$ a sequence in $L^{p^-}(0,T;W_0^{1,p(\cdot)}(\Omega,\omega))$ such that

$$\begin{aligned}
&u_k \rightharpoonup u \;\text{ in }\; L^{p^-}(0,T;W_0^{1,p(\cdot)}(\Omega,\omega))\\
&L_n u_k \rightharpoonup L_n u \;\text{ in }\; L^{p'^-}(0,T;W^{-1,p'(\cdot)}(\Omega,\omega^*))\\
&\lim_{k\to\infty}\sup\Big\langle B_n u_k, u_k - u\Big\rangle \leq 0
\end{aligned}\tag{6.4.21}$$

that, we have prove that

$$B_n u_k \rightharpoonup B_n u \quad in \quad L^{p'^-}(0,T;W_0^{1,p(\cdot)}(\Omega,\omega)) \; and \; \langle B_n u_k, u_k\rangle \to \langle B_n u, u\rangle.$$

By the definition of the operator L_n defined in definition 6.2.3, we obtain that u_k is bounded in $W_0^{1,p(\cdot)}(\Omega,\omega)$ and since $W_0^{1,p(\cdot)}(\Omega,\omega) \hookrightarrow L^{p'(\cdot)}(\Omega)$ then $u_k \to u$ in $L^{p^-}(0,T;W_0^{1,p(\cdot)}(\Omega,\omega))$, then the growth condition (6.3.6) $(a(x,t,u_k,\nabla u_k))_k$ is bounded in $(L^{p'(\cdot)}(Q,\omega^*))^N$ therefore, there exists a function $\varphi \in (L^{p'(\cdot)}(Q,\omega^*))^N$ such that

$$a(x,t,u_k,\nabla u_k) \rightharpoonup \varphi \; as \; k \to +\infty. \tag{6.4.22}$$

Similarly, using condition (6.3.9), $\Big(H_n(x,t,u_k,\nabla u_k)\Big)_k$ is bounded in $L^1(Q)$, then there exists a function $\psi_n \in L^1(Q)$ such that :

$$H_n(x,t,u_k,\nabla u_k) \to \psi_n \;\text{ in } L^1(Q) \text{ as } k \to +\infty. \tag{6.4.23}$$

$$\begin{aligned}
\lim_{k\to\infty}\Big\langle B_n u_k, u_k\Big\rangle &= lim_{k\to\infty}\Big[\Big\langle G_n u_k, u_k\Big\rangle + \Big\langle A u_k, u_k\Big\rangle\Big]\\
&= \lim_{k\to\infty}\Big[\int_Q a(x,t,u_k,\nabla u_k)\nabla u_k dxdt + \int_Q H(x,t,u_k,\nabla u_k)u_k dxdt\Big]\\
&= \int_Q \varphi\nabla u_k dxdt + \int_Q \psi_n u_k dxdt
\end{aligned}\tag{6.4.24}$$

using (6.4.21) and (6.4.24), we obtain

$$\begin{aligned}
\lim_{k\to\infty}\sup\Big\langle B_n u_k, u_k\Big\rangle &= \lim_{k\to\infty}\sup\Big\{\int_Q a(x,t,u_k,\nabla u_k)\nabla u_k dxdt\\
&+ \int_Q H(x,t,u_k,\nabla u_k)u_k dxdt\Big\}\\
&\leq \int_Q \varphi\nabla u dxdt + \int_Q \psi_n u dxdt
\end{aligned}\tag{6.4.25}$$

thanks to (6.4.23), we have :

$$\int_Q H_n(x,t,u_k,\nabla u_k)dxdt \to \int_Q \psi_n dxdt. \tag{6.4.26}$$

Therefore,

$$\lim_{k\to\infty}\sup\int_Q a(x,t,u_k,\nabla u_k)\nabla u_k \le \int_Q \varphi\nabla u dxdt \qquad (6.4.27)$$

on the other hand, using (6.3.7), we have

$$\int_Q \Big[a(x,t,u_k,\nabla u_k)-a(x,t,u_k,\nabla u)\Big](\nabla u_k-\nabla u)dxdt \ge 0. \qquad (6.4.28)$$

Then,

$$\begin{aligned}\int_Q a(x,t,u_k,\nabla u_k)\nabla u_k dxdt \ \ &\ge \ -\int_Q a(x,t,u_k,\nabla u)\nabla u dxdt\\ &+\int_Q a(x,t,u_k,\nabla u_k)\nabla u dxdt\\ &+\int_Q a(x,t,u_k,\nabla u)\nabla u_k dxdt\end{aligned}$$

and by (6.4.22), we get

$$\lim_{k\to\infty}\inf\int_Q a(x,t,u_k,\nabla u_k)\nabla u_k dxdt \ge \int_Q \varphi\nabla u dxdt,$$

this implies, thanks to (6.4.27) that

$$\lim_{k\to\infty}\int_Q a(x,t,u_k,\nabla u_k)\nabla u_k dxdt = \int_Q \varphi\nabla u dxdt. \qquad (6.4.29)$$

Now, by (6.4.29)), we can obtain

$$\lim_{k\to\infty}\int_Q a(x,t,u_k,\nabla u_k)-a(x,t,u_k,\nabla u))(\nabla u_k-\nabla u)dxdt = 0.$$

In view of the Lemma 6.2.1, we obtain

$$\begin{aligned} u_k \ &\to \ u \quad in \quad L^{p^-}(0,T;W_0^{1,p(\cdot)}(\Omega,\omega)),\\ \nabla u_k \ &\to \ \nabla u \quad a.e.\ in \quad Q.\end{aligned}$$

Then,

$$a(x,t,u_k,\nabla u_k)\rightharpoonup a(x,t,u,\nabla u) \quad in \quad (L^{p'(\cdot)}(Q,\omega^*))^N,$$

$$H_n(x,t,u_k,\nabla u_k)\rightharpoonup H_n(x,t,u,\nabla u) \quad in \quad L^1(Q),$$

we deduce that

$$Au_k \rightharpoonup Au \quad in \quad (L^{p'^-}(Q,\omega^*))^N$$

and

$$G_n u_k \rightharpoonup G_n u \quad in\ L^1(Q),$$

which implies

$$B_n u_k \rightharpoonup B_n u \quad in\ L^{p'^-}(0,T;W_0^{1,p(\cdot)}(\Omega,\omega))$$

and

$$\Big\langle B_n u_k, u_k \Big\rangle \to \Big\langle B_n u, u \Big\rangle$$

completing the proof of assertion(b).

c) Using $H\ddot{o}lder's$ inequality and the growth condition (6.3.6), we can show that the operator A is bounded, and by using (6.4.20), we conclude that B_n is bounded. For to show that B_n is demicontinuous.

Let $u_k \to u$ in $L^{p^-}(0,T;W_0^{1,p(\cdot)}(\Omega,\omega))$ and prove that :

$$\Big\langle B_n u_k, \psi \Big\rangle \to \langle B_n u, \psi\rangle \quad for\ all\ \psi \in\ L^{p^-}(0,T;W_0^{1,p(\cdot)}(\Omega,\omega)).$$

Since $a(x,t,u_k,\nabla u_k) \to a(x,t,u,\nabla u)$ as $k \to \infty$ a.e. in Q. Then, by the growth condition (6.3.6) and Lemma 1.2.8

$$a(x,t,u_k,\nabla u_k) \rightharpoonup a(x,t,u,\nabla u)\ in\ (L^{p'(\cdot)}(Q,\omega^*))^N$$

and for all $\varphi \in L^{p^-}(0,T;W_0^{1,p(\cdot)}(\Omega,\omega))$, $\langle Au_k,\varphi\rangle \to \langle Au,\varphi\rangle$ as $k \to \infty$
similarly, $G_n u_k \to G_n u$ as $k \to \infty$ a.e. in Q, then by the (6.3.9) and Lemma 1.2.8 $G_n u_k \rightharpoonup G_n u$ in $L^{p'(\cdot)}(Q,\omega^*)$ and for all $\phi \in L^{p^-}(0,T;W_0^{1,p(\cdot)}(\Omega,\omega))$,
$\Big\langle G_n u_k, \phi \Big\rangle \to \Big\langle G_n u, \phi \Big\rangle$ as $k \to \infty$ which implies B_n is demi continuous.
In view of Theorem 6.4.1, there exists at least one weak solution $u_n \in L^{p^-}(0;T;W_0^{1,p(\cdot)}(\Omega,\omega))$ of the problem (P_n).(See [76].)

6.4.2 A Priori Estimates

Proposition 6.4.3 *Let u_n a solution of the approximate problem (P_n). Then, there exists a constant C(which does not depend on the n and k) such that*

$$\|T_k(u_n)\|_{L^{p^-}(0,T;W_0^{1,p(\cdot)}(\Omega,\omega))} \le kC \quad \forall k > 0.$$

Proof.

Let $\varphi \in L^{p^-}(0,T;W_0^{1,p(\cdot)}(\Omega,\omega)) \cap\ L^\infty(Q)$, with $\varphi > 0$. Choosing $v = \exp(G(u_n))\varphi$

as a test function in $(\mathcal{P}_n)$, where

$$G(s)=\int_0^s(\frac{g(r)}{\alpha})dr,$$

(the function g appears in (6.3.9)), we have

$$\int_Q\frac{\partial b_n(x,u_n)}{\partial t}\exp(G(u_n))\varphi dxdt+\int_Q a(x,t,u_n,\nabla u_n)\nabla(\exp(G(u_n))\varphi)dxdt$$
$$+\int_Q H_n(x,t,u_n,\nabla u_n)\exp(G(u_n))\varphi dxdt=\int_Q f_n\exp(G(u_n))\varphi dxdt.$$

In view of (6.3.9), we obtain

$$\int_Q\frac{\partial b_n(x,u_n)}{\partial t}\exp(G(u_n))\varphi dxdt+\int_Q a(x,t,u_n,\nabla u_n)\nabla u_n\frac{g(u_n)}{\alpha}\exp(G(u_n))\varphi dxdt$$
$$+\int_Q a(x,t,u_n,\nabla u_n)\exp(G(u_n))\nabla\varphi dxdt\le\int_Q\gamma(x,t)\exp(G(u_n))\varphi dxdt$$
$$+\int_Q f_n\exp(G(u_n))\varphi dxdt+\int_Q g(u_n)|\nabla u_n|^{p(x)}\omega(x)\exp(G(u_n))\varphi dxdt.$$

By using (6.3.8), we obtain

$$\int_Q\frac{\partial b_n(x,u_n)}{\partial t}\exp(G(u_n))\varphi dxdt+\int_Q a(x,t,u_n,\nabla u_n)\exp(G(u_n))\nabla\varphi dxdt$$
$$\le\int_Q\gamma(x,t)\exp(G(u_n))\varphi dxdt+\int_Q f_n\exp(G(u_n))\varphi dxdt \tag{6.4.30}$$

for all $\varphi\in L^{p^-}(0,T;W_0^{1,p(\cdot)}(\Omega,\omega))\cap L^\infty(Q)$, with $\varphi>0$.
On the other hand, taking $v=\exp(-G(u_n))\varphi$ as a test function in $(\mathcal{P}_n)$, we deduce as in (6.4.30) that

$$\int_Q\frac{\partial b_n(x,u_n)}{\partial t}\exp(-G(u_n))\varphi dxdt+\int_Q a(x,t,u_n,\nabla u_n)\exp(-G(u_n))\nabla\varphi dxdt$$
$$+\int_Q\gamma(x,t)\exp(-G(u_n))\varphi dxdt\ge\int_Q f_n\exp(-G(u_n))\varphi dxdt \tag{6.4.31}$$

for all $\varphi \in L^{p^-}(0,T;W_0^{1,p(\cdot)}(\Omega,\omega)) \cap L^\infty(Q)$, with $\varphi > 0$.
Letting $\varphi = T_k(u_n)^+\chi_{(0,\tau)}$ for every $\tau \in [0,T]$, in (6.4.30), we have

$$\begin{aligned}\int_\Omega B_{k,G}^n(x,u_n(\tau))dx &+ \int_{Q^\tau} a(x,t,u_n,\nabla u_n)\exp(G(u_n))\nabla T_k(u_n)^+dxdt \\ \le \int_{Q^\tau}\gamma(x,t)\exp(G(u_n))T_k(u_n)^+dxdt &+ \int_{Q^\tau} f_n\exp(G(u_n))T_k(u_n)^+dxdt \\ &+ \int_\Omega B_{k,G}^n(x,u_{0n})dx, \end{aligned} \tag{6.4.32}$$

where,

$$B_{k,G}^n(x,r) = \int_0^r \frac{\partial b_n(x,s)}{\partial s}T_k(s)^+\exp(G(s))ds.$$

Due to the definition of $B_{k,G}^n$ and $|G(u_n)| \le \exp(\frac{\|g\|_{L^1(\mathbb{R})}}{\alpha})$, we have

$$0 \le \int_\Omega B_{k,G}^n(x,u_{0n})dx \le k\exp\Big(\frac{\|g\|_{L^1(\mathbb{R})}}{\alpha}\Big)\|b(.,u_0\|_{L^1(\Omega)}. \tag{6.4.33}$$

Using (6.4.33), $B_{k,G}^n(x,u_n) \ge 0$, we obtain

$$\begin{aligned}&\int_{Q^\tau} a(x,t,u_n,\nabla T_k(u_n)^+)\exp(G(u_n))\nabla T_k(u_n)^+dxdt \\ &\le k\exp\Big(\frac{\|g\|_{L^1(\mathbb{R})}}{\alpha}\Big)\Big[\|f_n\|_{L^1(Q)} + \|\gamma\|_{L^1(Q)} + \|b_n(x,u_{0n}\|_{L^1(\Omega)}\Big].\end{aligned}$$

Thanks to (6.3.8), we have

$$\begin{aligned}\alpha\int_{Q^\tau}|\nabla T_k(u_n)^+|^{p(x)}\omega(x)\exp(G(u_n))dxdt &\le k\exp\Big(\frac{\|g\|_{L^1(\mathbb{R})}}{\alpha}\Big)\Big[\|f_n\|_{L^1(Q)} \\ &+ \|\gamma\|_{L^1(Q)} + \|b_n(x,u_{0n}\|_{L^1(\Omega)}\Big].\end{aligned} \tag{6.4.34}$$

Let us observe that if we take : $\varphi = \rho(u_n) = \int_0^{u_n} g(s)\chi_{\{s>0\}}ds$ in (6.4.30) and use (6.3.8), we obtain

$$\begin{aligned}\int_\Omega\Big[B_g^n(x,u_n)\Big]_0^T dx &+ \alpha\int_Q|\nabla u_n|^{p(x)}\omega(x)g(u_n)\chi_{\{u_n>0\}}\exp(G(u_n))dxdt \\ &\le \Big(\int_0^\infty g(s)ds\Big)\exp\Big(\frac{\|g\|_{L^1(\mathbb{R})}}{\alpha}\Big)\Big[\|f_n\|_{L^1(Q)} + \|\gamma\|_{L^1(Q)}\Big],\end{aligned}$$

where

$$B_g^n(x,r) = \int_0^r \frac{\partial b_n(x,s)}{\partial s}\rho(s)\exp(G(s))ds,$$

which implies, using $B_g^n(x,r) \geq 0$, we obtain

$$\alpha \int_{\{u_n>0\}} |\nabla u_n|^{p(x)} \omega(x) g(u_n) \exp(G(u_n)) dxdt$$
$$\leq \|g\|_\infty \exp\Big(\frac{\|g\|_{L^1(\mathbb{R})}}{\alpha}\Big) \Big[\|\gamma\|_{L^1(Q)} + \|f_n\|_{L^1(Q)} + \|b_n(x, u_{0n}\|_{L^1(\Omega)}\Big]$$
$$then, \quad \int_{\{u_n>0\}} g(u_n) |\nabla u_n|^{p(x)} \omega(x) \exp(G(u_n)) dxdt \leq C_3.$$

Similarly, taking $\varphi = \int_{u_n}^0 g(s)\chi_{\{s<0\}} ds$ as a test function in (6.4.31),
we conclude that

$$\int_{\{u_n<0\}} g(u_n) |\nabla u_n|^{p(x)} \omega(x) \exp(G(u_n)) dxdt \leq C_4.$$

Consequently,

$$\int_Q g(u_n) |\nabla u_n|^{p(x)} \omega(x) \exp(G(u_n)) dxdt \leq C_5. \tag{6.4.35}$$

Above, $C_1, ..., C_5$ are constants independent of n, we deduce that

$$\int_Q |\nabla T_k(u_n)^+|^{p(x)} \omega(x) dxdt \leq k\ C_6. \tag{6.4.36}$$

Similarly to (6.4.36), we take $\varphi = T_k(u_n)^- \chi(0,\tau)$ in (6.4.31) to deduce that

$$\int_Q |\nabla T_k(u_n)^-|^{p(x)} \omega(x) dxdt \leq k\ C_7. \tag{6.4.37}$$

Combining (6.4.36), (6.4.37) and Remark 1.2.3, we conclude that

$$\int_0^T \min\Big\{\|T_k(u_n)\|^{p^+}_{W_0^{1,p(\cdot)}(\Omega,\omega)}, \|T_k(u_n)\|^{p^-}_{W_0^{1,p(\cdot)}(\Omega,\omega)}\Big\} dt \leq \rho(\nabla T_k(u_n)) \leq kC_8.$$
$$\|T_k(u_n)\|_{L^{p^-}(0,T;W_0^{1,p(\cdot)}(\Omega,\omega))} \leq\ k\ C_8. \tag{6.4.38}$$

Where C_6, C_7, C_8 are constants independent of n. Thus, $T_k(u_n)$ is bounded in $L^{p^-}(0,T; W_0^{1,p(\cdot)}(\Omega,\omega))$ independently of n for any $k > 0$. Then,
we deduce from (6.4.32)), (6.4.33) and (6.4.38) that

$$\int_\Omega B_{k,G}^n(x, u_n(\tau)) dx \leq\ kC. \tag{6.4.39}$$

6.4.3 Almost everywhere convergence of the gradients

Now, we turn to proving the almost everywhere convergence of u_n and $b_n(x,u_n)$. Consider a non decreasing function $g_k \in C^2(\mathbb{R})$ such that : $g_k(s) = \begin{cases} s & if\ |s| \leq \frac{k}{2} \\ k & if\ |s| \geq k. \end{cases}$
Multiplying the approximate equation by $g'_k(u_n)$, we get

$$\frac{\partial B_k^n(x,u_n)}{\partial t} - div(a(x,t,u_n,\nabla u_n)g'_k(u_n)) + a(x,t,u_n,\nabla u_n)g''_k(u_n)\nabla u_n$$
$$+H_n(x,t,u_n,\nabla u_n)g'_k(u_n) = f_n g'_k(u_n), \tag{6.4.40}$$

where

$$B_k^n(x,z) = \int_0^z \frac{\partial b_n(x,s))}{\partial s} g'_k(s)ds.$$

As a consequence of (6.4.38), we deduce that $g_k(u_n)$ is bounded in $L^{p^-}(0,T;W_0^{1,p(\cdot)}(\Omega,\omega))$ and $\frac{\partial B_k^n(x,u_n)}{\partial t}$ is bounded in $L^1(Q)+V^*$. Due to the properties of g_k and (6.3.5)), we conclude that $\frac{\partial g_k(u_n)}{\partial t}$ is bounded in $L^1(Q)+V^*$, which implies that $g_k(u_n)$ is compact in $L^1(Q)$.
Due to the choice of g_k, we conclude that for each k, the sequence $T_k(u_n)$ converges almost everywhere in Q, which implies that u_n converges almost everywhere to some measurable function v in Q. Thus by using the same argument as in [42], [43], [44], we can show the following lemma.

Lemma 6.4.4 *Let u_n be a solution of the approximate problem $(\mathcal{P}_n)$ then,*

$$u_n \rightarrow u \quad a.e. \quad in \quad Q.$$

$$b_n(x,u_n) \rightarrow b(x,u) \quad a.e. \quad in \quad Q.$$

We can deduce from (6.4.38)*) that*

$$T_k(u_n) \rightharpoonup T_k(u) \quad in \quad L^{p^-}(0,T;W_0^{1,p(\cdot)}(\Omega,\omega))$$

which implies, by using (6.3.6)*, that for all $k>0$ there exists $\varphi_k \in (L^{p'(\cdot)}(Q,\omega^*))^N$, such that*

$$a(x,t,T_k(u_n),\nabla T_k(u_n)) \rightharpoonup \varphi_k \quad in \quad \left(L^{p'(\cdot)}(Q,\omega^*)\right)^N.$$

Remark 6.4.5 *$b(.,u)$ it belongs to $L^\infty(0,T;L^1(\Omega))$.*

Proof.

Let u_n be a solution of the approximate problem $(\mathcal{P}_n)$ passing to $\liminf$ in (6.4.39) as $n \to \infty$, we obtain

$$\frac{1}{k}\int_\Omega B_{k,G}(x,u(\tau))dx \leq C, \text{ for a.e. } \quad \tau \text{ in } \quad [0,\tau].$$

Due to the definition of $B_{k,G}(x,s)$ and the fact that $\frac{1}{k}B_{k,G}(x,s)$ converge pointwise to $\int_0^u sgn(s)\frac{\partial b(x,s)}{\partial s}\exp(G(s))ds \geq |b(x,u)|$ as $k \to \infty$, it follows that $b(.,u)$ belongs to $L^\infty(0,T;L^1(\Omega))$.

Lemma 6.4.6 *Let u_n be a solution of the approximate problem $(\mathcal{P}_n)$. Then,*

$$\lim_{m\to\infty}\limsup_{n\to\infty}\int_{\{m\leq|u_n|\leq m+1\}} a(x,t,u_n,\nabla u_n)\nabla u_n dxdt = 0. \tag{6.4.41}$$

Proof.

Set $\varphi = T_1(u_n - T_m(u_n))^+ = \alpha_m(u_n)$ in (6.4.30), this function is admissible since $\varphi \in L^{p^-}(0,T;W_0^{1,p(\cdot)}(\Omega,\omega))$ and $\varphi \geq 0$. Then, we have

$$\begin{aligned}
&\int_Q \frac{\partial b_n(x,u_n)}{\partial t}\exp(G(u_n))\alpha_m(u_n)dxdt \\
&\qquad + \int_{\{m\leq u_n\leq m+1\}} a(x,t,u_n,\nabla u_n)\exp(G(u_n))\nabla u_n dxdt \\
&\leq \int_Q |\gamma(x,t)|\exp(G(u_n))\alpha_m(u_n)dxdt + \int_Q |f_n|\exp(G(u_n))\alpha_m(u_n)dxdt.
\end{aligned}$$

This gives, by setting

$$B^m_{n,G}(x,r) = \int_0^r \frac{\partial b_n(x,s)}{\partial s}\exp(G(s))\alpha_m(s)ds,$$

and by Young's Inequality,

$$\begin{aligned}
&\int_\Omega B^m_{n,G}(x,u_n)(T)dx + \int_{\{m\leq u_n\leq m+1\}} a(x,t,u_n,\nabla u_n)\exp(G(u_n))\nabla u_n dxdt \\
&\qquad \leq \exp\Big(\frac{\|g\|_{L^1(\mathbb{R})}}{\alpha}\Big)\Big[\int_{\{|u_n|>m\}} [|\gamma| + |f_n| + \|b_n(x,u_{0n}\|_{L^1(\Omega)}\Big]dxdt.
\end{aligned}$$

Since $B^m_{n,G}(x,u_n)(T) > 0$ and use (6.3.8), we obtain

$$\alpha \int_{\{m\le u_n\le m+1\}} |\nabla u_n)|^{p(x)} \exp(G(u_n))\nabla u_n dxdt$$
$$\le \exp\Big(\frac{\|g\|_{L^1(\mathbb{R})}}{\alpha}\Big)\Big[\int_{\{|u_n|>m\}} |\gamma| + |f_n| dxdt + \|b_n(x,u_{0n}\|_{L^1(\Omega)}\Big]. \tag{6.4.42}$$

Taking $\varphi = \rho_m(u_n) = \int_0^T g(s)\chi_{\{s>m\}}ds$ as a test function in (6.4.30), we obtain

$$\Big[\int_\Omega B^m_{m,n}(x,u_n)dx\Big]_0^T + \int_Q a(x,t,u_n,\nabla u_n)\exp(G(u_n))g(u_n)\nabla u_n \chi_{\{u_n>m\}} dxdt$$
$$\le \Big(\int_m^\infty g(s)\chi_{\{u_n>m\}}ds\Big)\exp\Big(\frac{\|g\|_{L^1(\mathbb{R})}}{\alpha}\Big)\Big[\|f_n\|_{L^1(Q)} + \|\gamma\|_{L^1(Q)}\Big],$$

where $B^m_{m,n}(x,r) = \int_0^r \frac{\partial b_n(x,s)}{\partial s}\rho_m(s)\exp(G(s))ds$ which implies,
since $B^m_{m,n}(x,r) \ge 0$, by (6.3.8) and Young's Inequality

$$\alpha \int_{\{u_n>m\}} |\nabla u_n|^{p(x)}\omega(x)g(u_n)\exp(G(u_n))dxdt \le$$
$$\Big(\int_m^\infty g(s)ds\Big)\exp\Big(\frac{\|g\|_{L^1(\mathbb{R})}}{\alpha}\Big)\Big[\|f_n\|_{L^1(Q)} + \|\gamma\|_{L^1(Q)} + \|b_n(x,u_{0n}\|_{L^1(\Omega)}\Big]. \tag{6.4.43}$$

Using (6.4.43) and the strong convergence of f_n in $L^1(\Omega)$ and $b_n(x,u_{0n})$ in $L^1(\Omega)$, $\gamma \in L^1(\Omega)$, $g \in L^1(\mathbb{R})$, by Lebesgue's theorem, passing to limit in (6.4.42), we conclude that

$$\lim_{m\to\infty}\limsup_{n\to\infty} \int_{\{m\le u_n\le m+1\}} a(x,t,u_n,\nabla u_n)\nabla u_n dxdt = 0. \tag{6.4.44}$$

On the other hand, taking $\varphi = T_1(u_n - T_m(u_n))^-$ as a test function
in (6.4.31) and reasoning as in the proof (6.4.44), we deduce that

$$\lim_{m\to\infty}\limsup_{n\to\infty} \int_{\{-(m+1)\le u_n\le -m\}} a(x,t,u_n,\nabla u_n)\nabla u_n dxdt = 0. \tag{6.4.45}$$

By using (6.4.44) and (6.4.45), we have

$$\lim_{m\to\infty}\limsup_{n\to\infty} \int_{\{m\le |u_n|\le m+1\}} a(x,t,u_n,\nabla u_n)\nabla u_n dxdt = 0. \tag{6.4.46}$$

To this end, we prove the strong convergence of truncation of $T_k(u_n)$ that we will use the following function of one real variable s, which is define as where $m > k$,

$$h_m(s) = \begin{cases} 1 & if \quad |s| \le m \\ 0 & if \quad |s| > m+1 \\ m+1+|s| & if \quad m \le |s| \le m+1. \end{cases}$$

Let $\psi_i \in D(\Omega)$ be a sequence which converges strongly to u_0 in $L^1(\Omega)$.
Set $w^i_\mu = (T_k(u))_\mu + e^{-\mu t}T_k(\psi_i)$ where $(T_k(u))_\mu$ is the mollification of $T_k(u)$ with respect to time. Note that w^i_μ is a smooth function having the following properties :

$$\frac{\partial w^i_\mu}{\partial t} = \mu(T_k(u) - w^i_\mu), \quad w^i_\mu(0) = T_k(\psi_i), \quad |w^i_\mu| \leq k, \tag{6.4.47}$$

$$w^i_\mu \to T_k(u) \quad \text{in } L^{p^-}(0,T; W_0^{1,p(\cdot)}(\Omega,\omega)) \quad \text{as } \mu \to \infty. \tag{6.4.48}$$

The very definition of the sequence w^i_μ makes it possible to establish the following lemma.

Lemma 6.4.7 *(See[44, 6].) For $k \geq 0$, we have*

$$\int_{\{T_k(u_n)-w^i_\mu \geq 0\}} \frac{\partial b_n(x,u_n)}{\partial t} \exp(G(u_n))(T_k(u_n) - w^i_\mu)h_m(u_n)dxdt \geq \varepsilon(n,m,\mu,i).$$

Proposition 6.4.8 *The subsequence of u_n solution of problem $(\mathcal{P}_n)$ satisfies for any $k \geq 0$ following assertion :*

$$\lim_{n\to\infty}\int_Q \Big[a(T_k(u_n),\nabla T_k(u_n)) - a(T_k(u_n),\nabla T_k(u))\Big]\cdot\Big[\nabla T_k(u_n) - \nabla T_k(u)\Big]dxdt = 0.$$

Proof.
For $m > k$, let $\varphi = (T_k(u_n) - w^i_\mu)^+ h_m(u_n) \in L^{p^-}(0,T; W_0^{1,p(\cdot)}(\Omega,\omega)) \cap L^\infty(Q)$ and $\varphi \geq 0$. If we take this function in (6.4.30), we obtain

$$\begin{aligned}
&\int_{\{T_k(u_n)-w^i_\mu \geq 0\}} \frac{\partial b_n(x,u_n)}{\partial t} \exp(G(u_n))(T_k(u_n) - w^i_\mu)h_m(u_n)dxdt \\
&+ \int_{\{T_k(u_n)-w^i_\mu \geq 0\}} a(x,t,u_n,\nabla u_n)\nabla(T_k(u_n) - w^i_\mu)h_m(u_n)dxdt \\
&- \int_{\{m\leq u_n\leq m+1\}} \exp(G(u_n))a(x,t,u_n,\nabla u_n)\nabla u_n(T_k(u_n) - w^i_\mu)^+dxdt \\
&\leq \int_Q (f_n+\gamma)\exp(G(u_n))(T_k(u_n) - w^i_\mu)^+h_m(u_n)dxdt
\end{aligned} \tag{6.4.49}$$

Observe that,

$$\begin{aligned}
&\Big|\int_{\{m\leq u_n\leq m+1\}} \exp(G(u_n))a(x,t,u_n,\nabla u_n)\nabla u_n(T_k(u_n) - w^i_\mu)^+dxdt\Big| \\
&\qquad \leq 2k\exp\Big(\frac{\|g\|_{L^1(\mathbb{R})}}{\alpha}\Big)\int_{\{m\leq u_n\leq m+1\}} a(x,t,u_n,\nabla u_n)\nabla u_n dxdt.
\end{aligned}$$

Tanks to (6.4.41) the third and fourth integrals on the right hand side tend to zero as n and m tend to infinity and by Lebesgue's theorem, we deduce that the right hand side converges to zero as $n,\ m$ and μ tend to infinity . Since $\Big(T_k(u_n) - w_\mu^i\Big)^+ h_m(u_n) \rightharpoonup \Big(T_k(u) - w_\mu^i\Big)^+ h_m(u)$ in $L^\infty(Q)$ as $n \to \infty$ and strongly in $L^{p^-}(0,T;W_0^{1,p(\cdot)}(\Omega,\omega))$ and $(T_k(u_n) - w_\mu^i)^+ h_m(u_n) \rightharpoonup 0$ in $L^\infty(Q)$ and strongly in $L^{p^-}(0,T;W_0^{1,p(\cdot)}(\Omega,\omega))$ as $\mu \to \infty$, it follows that the first and second integrals on the right-hand side of (6.4.49)) converge to zeros as $n,\ m,\ \mu \to \infty$, using [8] Lemma 6.4.7 and Lemma 6.2.1, the proof of Proposition 6.4.8 is complete. Thanks to the Lemma 6.2.1, we have

$$T_k(u_n) \to T_k(u) \text{ strongly } \text{ in } L^{p^-}(0,T;W_0^{1,p(\cdot)}(\Omega,\omega)), \quad \forall k \tag{6.4.50}$$

and $\nabla u_n \to \nabla u$ a.e. in Q, which implies that

$$a(x,t,T_k(u_n),\nabla T_k(u_n)) \rightharpoonup a(x,t,T_k(u),\nabla T_k(u)) \ \ in\ (L^{p'(\cdot)}(Q,\omega^*))^N. \tag{6.4.51}$$

6.4.4 Equi-Integrability of the non Linearity Sequence

Proposition 6.4.9 *Let u_n be a solution of problem $(\mathcal{P}_n)$. Then $H_n(x,t,u_n,\nabla u_n) \to H(x,t,u,\nabla u)$ strongly in $L^1(Q)$.*

Proof. By using Vitali's theorem. Since $H_n(x,t,u_n,\nabla u_n) \to H(x,t,u,\nabla u)$ a.e. in Q, considering now, $\varphi = \rho_h(u_n) = \int_0^{u_n} g(s)\chi_{\{s>h\}}ds$ as a test function in (6.4.30), we obtain

$$\begin{aligned}&\Big[\int_\Omega B_h^n(x,u_n)dx\Big]_0^T + \int_Q a(x,t,u_n,\nabla u_n)\nabla u_n g(u_n)\chi_{\{u_n>h\}}\exp(G(u_n))dxdt\\ &\quad\le \Big(\int_h^\infty g(s)\chi_{\{s>h\}}ds\Big)\exp\Big(\frac{\|g\|_{L^1(\mathbb{R})}}{\alpha}\Big)\Big[\|f_n\|_{L^1(Q)} + \|\gamma\|_{L^1(Q)}\Big],\end{aligned}$$

where $B_h^n(x,r) = \int_0^r \frac{\partial b_n(x,s)}{\partial s}\rho_h(s)\exp(G(s))ds$,
which implies, in view of $B_h^n(x,r) \ge 0$ and (6.3.8)

$$\begin{aligned}&\alpha\int_{\{u_n>h\}} |\nabla u_n|^{p(x)}\omega(x)g(u_n)\exp(G(u_n))dxdt\\ &\qquad\le \Big(\int_h^\infty g(s)ds\Big)\exp\Big(\frac{\|g\|_{L^1(\mathbb{R})}}{\alpha}\Big)\Big[\|f_n\|_{L^1(Q)} + \|\gamma\|_{L^1(Q)} + \|b_n(x,u_{0n}\|_{L^1(\Omega)}\Big]\end{aligned}$$

and since $g \in L^1(\mathbb{R})$, we deduce that

$$\lim_{h\to\infty} \sup_{n\in\mathbb{N}} \int_{\{u_n>h\}} |\nabla u_n|^{p(x)}\omega(x)g(u_n)dxdt = 0.$$

Similarly, taking $\varphi = \rho_h(u_n) = \int_{u_n}^0 g(s)\chi_{\{s<-h\}}ds$ as a test function in (6.4.31), we conclude that : $\lim_{h\to\infty} \sup_{n\in\mathbb{N}} \int_{\{u_n<-h\}} |\nabla u_n|^{p(x)}\omega(x)g(u_n)dxdt = 0$.
Consequently, $\lim_{h\to\infty} \sup_{n\in\mathbb{N}} \int_{\{|u_n|>h\}} |\nabla u_n|^{p(x)}\omega(x)g(u_n)dxdt = 0$.
Which implies, for h large enough and for a subset E of Q,

$$\begin{aligned}\lim_{measE\to 0} \int_E |\nabla u_n|^{p(x)}\omega(x)g(u_n)dxdt \leq \|g\|_\infty \lim_{measE\to 0} \int_E |\nabla T_h u_n|^{p(x)}\omega(x)dxdt \\ + \int_{\{|u_n|>h\}} |\nabla u_n|^{p(x)}\omega(x)g(u_n)dxdt,\end{aligned}$$

so $g(u_n)|\nabla u_n|^{p(x)}\omega(x)$ is equi-integrable. Thus we have shown that

$$g(u_n)|\nabla u_n|^{p(x)}(x)\omega(x) \to g(u)|\nabla u|^{p(x)}(x)\omega(x) \text{ stongly in } L^1(Q).$$

Consequently, by using (6.3.9), we conclude that

$$H_n(x,t,u_n,\nabla u_n) \to H(x,t,u,\nabla u) \;\text{ strongly }\; \text{in } L^1(Q). \tag{6.4.52}$$

6.4.5 Concluding the proof of Theorem 6.3.3

a) Proof that u satisfies (6.3.11). For any fixed $m \geq 0$, we have

$$\begin{aligned}&\int_{\{m\leq|u_n|\leq m+1\}} a(x,t,u_n,\nabla u_n)\nabla u_n dxdt \\ &\qquad = \int_Q a(x,t,u_n,\nabla u_n)\Big[\nabla T_{m+1}(u_n) - \nabla T_m(u_n)\Big]dxdt \\ &\qquad = \int_Q a(x,t,T_{m+1}(u_n),\nabla T_{m+1}(u_n))\nabla T_{m+1}(u_n) \\ &\qquad\quad - \int_Q a(x,t,T_m(u_n),\nabla T_m(u_n))\nabla T_m(u_n)dxdt.\end{aligned}$$

According to (6.4.50) and (6.4.51), one can pass to the limit as $n \to \infty$ for fixed $m \geq 0$ to obtain

$$\lim_{n\to\infty} \int_{\{m\leq |u_n|\leq m+1\}} a(x,t,u_n,\nabla u_n)\nabla u_n dxdt$$
$$= \int_Q a(x,t,T_{m+1}(u),\nabla T_{m+1}(u))\nabla T_{m+1}(u)$$
$$- \int_Q a(x,t,T_m(u),\nabla T_m(u))\nabla T_m(u)dxdt$$
$$= \int_{\{m\leq |u|\leq m+1\}} a(x,t,u,\nabla u)\nabla u dxdt. \qquad (6.4.53)$$

Taking the limit as $m \to \infty$ in (6.4.53) and using the estimate (6.4.41), shows that u satisfies (6.3.11).

b) Proof that u satisfies (6.3.12)

Let $S \in W^{2,\infty}(\mathbb{R})$ be such that S' has a compact support. Let $M > 0$ such that supp$(S')\subset [-M, M]$. Pointwise multiplication of the approximate problem $(\mathcal{P}_n)$ by $S'(u_n)$, leads to

$$\frac{\partial B_S^n(x,u_n)}{\partial t} - div\Big[S'(u_n)a(x,t,u_n,\nabla u_n)\Big] + S''(u_n)a(x,t,u_n,\nabla u_n)\nabla u_n$$
$$+ H_n(x,t,u_n,\nabla u_n)S'(u_n) = f_n S'(u_n) \text{ in } D'(Q). \qquad (6.4.54)$$

In what follows, we pass to the limit in (6.4.54) as n tends to ∞.

• Limit of $\frac{\partial B_S^n(x,u_n)}{\partial t}$.

Since S is bounded and continuous, $u_n \to u$ a.e. in Q implies that $B_S^n(x,u_n)$ converge to $B_S(x,u)$ a.e. in Q and L^∞ weakly

$$Then, \quad \frac{\partial B_S^n(x,u_n)}{\partial t} \to \frac{\partial B_S(x,u)}{\partial t} \quad \text{in } D'(Q), \text{ as } n\to\infty.$$

• Limit of $-div\Big[S'(u_n)a(x,t,u_n,\nabla u_n)\Big]$.

Since supp$(S')\subset [-M, M]$, we have, for $n \geq M$

$$S'(u_n)a(x,t,u_n,\nabla u_n) = S'(u_n)a(x,t,T_M(u_n),\nabla T_M(u_n)) \ a.e. \text{ in } Q.$$

The pointwise convergence of u_n to u and (6.4.51) and the boundedness of S' yied, as $n \to \infty$,

$$S'(u_n)a(x,t,u_n,\nabla u_n) \rightharpoonup S'(u)a(x,t,T_M(u),\nabla T_M(u)) \text{ in } (L^{p'(\cdot)}(Q,\omega^*))^N \quad (6.4.55)$$

as $n \to \infty$,
$S'(u)a(x,t,T_M(u),\nabla T_M(u))$ has been denoted by $S'(u)a(x,t,u,\nabla u)$ in equation (6.3.12).

• Limit of $S''(u_n)a(x,t,u_n,\nabla u_n)\nabla u_n$.
Consider the "energy" term
$S''(u_n)a(x,t,u_n,\nabla u_n)\nabla u_n = S''(u_n)a(x,t,T_M(u_n),\nabla T_M(u_n))\nabla T_M(u_n)$ a.e. in Q.
The pointwise convergence of $S'(u_n)$ to $S'(u)$ and (6.4.51) as $n \to \infty$ and the boundedness of S'' yield

$$S''(u_n)a(x,t,u_n,\nabla u_n)\nabla u_n \rightharpoonup S''(u)a(x,t,T_M(u),\nabla T_M(u))\nabla T_M(u) \text{ in } L^1(Q). \tag{6.4.56}$$

Recall that $S''(u)a(x,t,T_M(u),\nabla T_M(u))\nabla T_M((u)) = S''(u)a(x,t,u,\nabla u)\nabla u$ a.e. in Q.

• Limit of $S'(u_n)H_n(x,t,u_n,\nabla u_n)$. From supp($S'$)$\subset [-M,M]$ and (6.4.52),
we have

$$S'(u_n)H_n(x,t,u_n,\nabla u_n) \to S'(u)H(x,t,u,\nabla u) \textit{ strongly in } L^1(Q) \textit{ as } \ n \to \infty. \tag{6.4.57}$$

• Limit of $S'(u_n)f_n$. Since $u_n \to u$ a.e. in Q,
we have $S'(u_n)f_n \to S'(u)f$ strongly in $L^1(Q)$, as $n \to \infty$.
As a consequence of the above convergence result, we are in a position to pass to the limit as $n \to \infty$ in equation (6.4.54) and to conclude that u satisfies (6.3.12).

c) Proof that u satisfies (6.3.13)

S is bounded and $B_S^n(x,u_n)$ is bounded in $L^\infty(Q)$. Secondly by (6.4.54), we have $\frac{\partial B_S^n(x,u_n)}{\partial t}$ is bounded in $L^1(Q)+V^*$.
As a consequence, an Aubin type Lemma (see, e.g, [88] implies that $B_S^n(x,u_n)$ lies in a compact set in $C^0([0,T],L^1(\Omega))$.
It follows that on the hand, $B_S^n(x,u_n)\ |_{t=0} = B_S^n(x,u_0^n)$ converge to $B_S(x,u)\ |_{t=0}$ strongly in $L^1(\Omega)$ implies that : $B_S(x,u)\ |_{t=0} = B_S(x,u_0)$ in Ω.
As a conclusion, the proof of Theorem 6.3.3 is complete.

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