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Contributions to r-monotone operators, r-convex functions and sequential vector optimization

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Résumé

La thèse s'intéresse à deux thèmes de l'optimisation et inéquations variationnelles. La première partie est consacrée aux opérateurs r-monotones et fonctions r-convexes. Nous y introduisons et étudions les nouveaux concepts de cyclicité et maximalité pour les opérateurs r-monotones. Nous établissons aussi leurs caractérisations du premier ordre. Nous donnons également plusieurs critères du premier et second ordre pour les fonctions r-convexes. La deuxième partie est dédiée à l'optimisation vectorielle séquentielle. Nous y prouvons des formules séquentielles pour les sous-différentiels de Pareto de l'opérateur composé et l'opérateur somme de plusieurs fonctions vectorielles. Comme application, nous dérivons des conditions nécessaires et suffisantes d'optimalité séquentielles pour le problème général d'optimisation vectorielle et un problème multi-objectif fractionnaire avec contraintes géométriques et coniques. Les résultats obtenus sont valables sans conditions de qualification.

Mots-clefs : Opérateur r-monotone, r-monotonie cyclique et maximale, fonction r-convexe, optimisation vectorielle séquentielle, analyse non lisse

Abstract

The thesis is interested in two themes of optimization and variational inequalities. The first part is devoted to r-monotone operators and r-convex functions. We there introduce and study new concepts of cyclicity and maximality for r-monotone operators. We also establish their first-order characterizations. Moreover, we give several first and second order criteria for r-convex functions. The second part is dedicated to sequential vector optimization. We there prove sequential formulas for Pareto subdifferentials of the composed operator and the operator sums of several m vector mappings. As application, we derive necessary and sufficient sequential optimality conditions for the general problem of vector optimization and a multi-objective fractional problem with geometric and cone constraints. The obtained results hold without conditions of qualification.

Key Words : r-monotone operator, cyclic and maximal r-monotonicity, r-convex function, sequential vector optimization, nonsmooth analysis

Résumé détaillé

Dans cette thèse, nous nous intéressons à deux thèmes de l'optimisation et inéquations variationnelles. La première partie concerne la convexité et monotonie généralisée précisément les opérateurs r-monotones et les fonctions r-convexes. La deuxième partie est dédiée à l'optimisation vectorielle séquentielle.

Partie I : Opérateurs r-monotones et fonctions r-convexes

Cette partie de thèse s'intéresse aux opérateurs r-monotones. C'est une classe d'opérateurs monotones généralisés définie la première fois par Crouzeix et Hassouni [25]. Soit X un espace de Banach réel et X^* son dual topologique. On dit qu'un opérateur $T : X \rightrightarrows X^*$ est r-monotone ($r \neq 0$) si l'opérateur $T_r : X \times \mathbb{R} \rightrightarrows X^* \times \mathbb{R}$ défini par :

$$T_r(x,t) := \begin{cases} T(x) \times \{-\frac{1}{rt}\} & \text{si } (x,t) \in X \times]0,+\infty[. \\ \\ \emptyset & \text{sinon.} \end{cases}$$

est quasimonotone. La classe de la r-monotonie ($r > 0$) présente un intérêt. En effet, elle est intermédiaire entre la monotonie et la pseudomonotonie. De plus, elle intervient dans la théorie des consommateurs et de la demande en économie mathématique : voir le problème de la monotonie généralisée de l'opérateur demande [25].

Nous nous intéressons également aux fonctions r-convexes introduite par Avriel [5]. On dit qu'une fonction $f : X \longrightarrow \mathbb{R} \cup \{+\infty\}$ est r-convexe ($r > 0$) si la fonction $f_r := \exp(rf)$ est convexe. Notons que la r-convexité s'applique à l'économie mathématique et à l'optimisation convexe généralisée.

Dans le chapitre 1, nous introduisons une nouvelle notion de cyclicité pour les opérateurs r-monotones puis nous l'étudions. Sa définition est reliée à celle de la cyclique quasi-monotonie de Daniilidis et Hadjisavvas [27, 28]. On dit qu'un opérateur $T : X \rightrightarrows X^*$ est cyclique r-monotone ($r \neq 0$) si l'opérateur T_r est cyclique quasimonotone. Nous démontrons également plusieurs caractérisations du premier ordre de la r-convexité via différents sous-différentiels pour des fonctions semi-continues inférieurement. En particulier, nous établissons les liens entre r-monotonie ou cyclique r-monotonie et r-convexité. De ce fait, nous complétons les relations déjà établies entre convexité et monotonie [21, 22, 52, 68], pseudoconvexité et pseudomonotonie, quasiconvexité et quasimonotonie [3, 28, 34, 38, 51, 65].

Dans le chapitre 2, nous poursuivons l'étude des opérateurs r-monotones. Nous introduisons la maximalité dans un sens naturel. Un opérateur r-monotone T sera dit r-monotone maximal s'il n'y a aucun opérateur r-monotone dont le graphe contient strictement le graphe de T. Puis, nous démontrons les propriétés de base. Nous étendons ainsi des résultats classiques de la monotonie et la monotonie maximale. Nous introduisons aussi une polarité adaptée appelée r-polarité et présentons des caractérisations des opérateurs r-monotones et r-monotones maximaux en terme du r-polaire. Dans le cas classique monotone, plusieurs recherches ont été menées : voir à ce sujet la revue de la littérature de Borwein [8] et ses références. En revanche, il n'y a eu que peu de travaux sur la maximalité des opérateurs pseudomonotones et quasimonotones [4, 11, 12, 37, 31, 23]. Nous nous intéressons également aux caractérisations du premier ordre en dimension finie. En premier lieu, nous démontrons des critères du premier ordre du type [24, 47, 54] pour les opérateurs r-monotones dans le cas lisse et le cas non lisse localement Lipschitzien. L'approche se base sur un lemme de séparabilité de la Jacobienne généralisée de Clarke. En second lieu, nous prouvons les analogues r-monotone d'un résultat connu qui dit : Un opérateur différentiable est monotone si et seulement si sa matrice Jacobienne est semi-définie positive. Dans le cas non lisse, les résultats obtenus sont exprimés en terme de la Jacobienne de Clarke généralisée ou la codérivée de Mordukhovich. Pour les caractérisations avec codérivée dans le cas monotone, voir [16, 15, 62, 59]. Comme applications, nous donnons de nouvelles caractérisations de second-ordre pour les fonctions r-convexes. Ceci généralise au cas non lisse ou étend certains résultats de la littérature [5, 15, 14, 16, 43].

Partie II : Optimisation vectorielle séquentielle

Notre but dans cette deuxième partie est de contribuer au calcul séquentiel sans condition de qualification en optimisation vectorielle.

Les travaux de cette partie sont une suite naturelle du travail de Thibault [72] sur le calcul séquentiel sans condition de qualification en optimisation scalaire et de l'article [33] sur le calcul avec condition de qualification en optimisation vectorielle. Dans la suite, nous prenons X, Y et Z des espaces de Banach réels. Nous noterons par $+\infty_Y$ et $+\infty_Z$ des éléments abstraits jouant le rôle de l'infinie dans Y et Z respectivement.

Dans le chapitre 3 de cette thèse, nous établissons une formule séquentielle, sans condition de qualification, pour les sous-différentiels de Pareto (faible, propre et fort) de l'opérateur vectoriel $f + g \circ h$ où $f : X \longrightarrow Y \cup \{+\infty_Y\}$ est propre, Y_+-convexe et semi-continue inférieurement, $h : X \longrightarrow Z \cup \{+\infty_Z\}$ est propre, Z_+-convexe et à épigraphe fermé, et $g : Z \longrightarrow Y \cup \{+\infty_Y\}$ est propre, Y_+-convexe, semi-continue inférieurement et (Z_+, Y_+)-croissante. La technique de scalarisation [33] est utilisée. Au chapitre 3, X et Z sont supposés réflexifs.

Dans le chapitre 4, nous démontrons trois formules pour le sous-différentiel de Pareto (faible, propre et fort) de la somme de m fonctions vectorielles $f_1, ..., f_m : X \longrightarrow Z \cup \{+\infty_Z\}$ propres, Z_+-convexes et semi-continues inférieurement où $m \geq 2$. La méthode est basée sur un outil intéressant appelé outil épigraphique. La troisième formule généralise la formule de la somme de deux fonctions vectorielles que nous avons donnée dans [49]. Elle généralise aussi, au cadre vectoriel, la formule correspondante de Jules, Lassonde [44]. Comme première application, nous dérivons des conditions nécessaires et suffisantes d'optimalité, séquentielles sans condition de qualification, au sens faible, propre et fort du problème général d'optimisation vectorielle :

$$(VOP): \quad \inf_{\substack{x \in C \\ h(x) \in -Y_+}} f(x)$$

où : $f : X \longrightarrow Z \cup \{+\infty_Z\}$ est propre, Z_+-convexe et semi-continue inférieurement, $h : X \longrightarrow Y \cup \{+\infty_Y\}$ est propre, Y_+-convexe et à épigraphe fermé, C est un convexe fermé non vide de X, et Y_+ est un cône convexe fermé non vide de Y. Comme deuxième application, nous donnons des conditions d'optimalité séquentielles sans contraintes de qualification au sens faible et fort pour le problème multi-objectif fractionnaire avec contrainte géométrique et conique suivant :

$$(MFP): \quad \inf_{\substack{x \in C \\ h(x) \in -Y_+}} \{\frac{f_1(x)}{g_1(x)}, \cdots, \frac{f_q(x)}{g_q(x)}\}$$

où : $f_1, ..., f_q : X \longrightarrow [0, +\infty[$ sont convexes et semi-continues inférieurement, $g_1, ..., g_q : X \longrightarrow]0, +\infty[$ sont concaves et semi-continues supérieurement, et $h : X \longrightarrow Y \cup \{+\infty_Y\}$ est propre, Y_+-convexe et à épigraphe fermé. Au chapitre 4, X et Y sont supposés réflexifs.

Contents

General Introduction

Optimization is a mathematical specialty that studies problems that minimize or maximize an objective function on a set of constraints. It is extremely useful for many other mathematical disciplines (Analysis, Statistics etc.) and for various applications areas (Engineering, Industry, Finance, Economics etc.). In a schematic way, the optimization comprises three steps : a modelization step, a theoretical study step using convex analysis [69, 71] and a numerical-algorithmic step. It contains several subdomains according to the type of the objective function and the constraint of the optimization problem, for example : linear optimization, convex optimization, robust optimization, stochastic optimization, nonsmooth optimization, vector optimization, set-valued optimization, generalized convexity etc.

Optimization problems are part of the more general framework of equilibrium problems that also contains variational inequality problems. These latter are somehow in duality with optimization problems. The aim is to find points $x_0 \in K$ that verify for all $x \in K : T(x_0)(x - x_0) \geq 0$ where $T : K \subseteq X \rightrightarrows X^*$ is an operator between a Banach space X and its topological dual X^*. The theoretical part of variational inequalities makes use of the theory of monotone and generalized monotone operators.

In this thesis, we are interested by two themes of optimization and variational inequalities. The first part is about generalized convexity and monotonicity, more precisely r-monotone operators and r-convex functions. The second part is dedicated to sequential vector optimization.

Part I : r-monotone operators and r-convex functions

The theme of generalized monotonicity has attracted a lot of attention from the community of mathematical optimization. Its main fields of application are : mathematical economics, variational inequalities and equilibrium problems.

Researchers in this theme study generalized monotone operators, that is operators defined by a property generalizing monotonicity. The two most important notions are the pseudomonotonicity introduced by Karamardian [45] and the quasimonotonicity introduced by Hassouni [38] (See also Karamardian and Schaible [46]). The essential objective is to extend and generalize concepts and results from the classical theory of monotone operators.

On the other hand, the theory of generalized monotone operators is closely related to that of generalized convexity through gradient or generalized gradients. The latter considers the classes of functions that generalize convexity by retaining some remarkable properties. Its two central examples are the pseudoconvexity and the quasiconvexity. For more details about generalized convexity and monotonicity, see the book of Cambini and Martein [13].

This part of the thesis is interested by r-monotone operators. It is a class of generalized monotone operators first defined by Crouzeix and Hassouni [25]. Roughly, an r-monotone operator is an operator such that its hyperbolic perturbation is quasimonotone. Precisely, let X be a real Banach space and X^* be its topological dual. An operator $T : X \rightrightarrows X^*$ is said to be r-monotone ($r \neq 0$) if the operator $T_r : X \times \mathbb{R} \rightrightarrows X^* \times \mathbb{R}$ defined by :

$$T_r(x,t) := \begin{cases} T(x) \times \{-\frac{1}{rt}\} & \text{if } (x,t) \in X \times]0,+\infty[. \\ \\ \emptyset & \text{otherwise.} \end{cases}$$

is quasimonotone. The class of r-monotonicity ($r > 0$) is of interest. Indeed, it is intermediate between monotonicity and pseudomonotonicity. Moreover, it appears in the consumer and demand theory in mathematical economics : see the problem of the generalized monotonicity of the operator demand [25].

We are also interested by r-convex functions introduced by Avriel [5]. See also Martos [56] for a similar contribution and the paper of Balogh and Ewerhart [6] for a historical approach. A function $f : X \longrightarrow \mathbb{R} \cup \{+\infty\}$ is called r-convex ($r > 0$) if the function $f_r := \exp(rf)$ is convex. Note that the r-convexity applies to mathematical economics and generalized convex optimization.

In the chapter 1, we introduce a new notion of cyclicity for r-monotone operators then we study it. Its definition is connected to that of cyclic quasimonotonicity of Daniilidis and Hadjisavvas [27, 28]. We say that an operator $T : X \rightrightarrows X^*$ is cyclically r-monotone ($r \neq 0$) if the operator T_r is cyclically quasimonotone. Let us mention the cyclic pseudomonotonicity [27, 28] that is adapted to pseudomonotone operators. We also prove several first-order characterizations of r-convexity through various subdifferentials for lower semi-continuous functions. In particular, we establish the links between r-monotonicity or cyclic monotonicity and r-convexity. As a result, we complete the already established relations between convexity and monotonicity [21, 22, 52, 68], pseudoconvexity and pseudomonotonicity, quasiconvexity and quasimonotonicity [3, 28, 34, 38, 51, 65].

In the chapter 2, we continue the investigation of r-monotone operators. We introduce maximality in a natural sense. A r-monotone operator T is called maximal r-monotone if there does not exist a r-monotone operator whose graph strictly contains the graph of T. Then, we prove the basic properties. We thus extend some classical results of monotonicity and maximal monotonicity. We also introduce an adapted polarity called r-polarity and present characterizations of r-monotone and maximal r-monotone operators in terms of the r-polar. In the classical monotone case, several researches have been conducted : see the survey of Borwein [8] and references therein. In contrast, there has been little work on maximality of pseudomonotone and quasimonotone operators [4, 11, 12, 37, 31, 23]. We are also interested by first-order characterizations in finite dimension. First, we prove first-order criteria of type [24, 47, 54] for r-monotone operators in the smooth and nonsmooth locally Lipschitz cases. The approach is based on a lemma about separability of the generalized Jacobian of Clarke. Second, we show the r-monotone counterparts of the next well known result : A differentiable operator is monotone if and only if its Jacobian matrix is positive semidefinite. In the nonsmooth case, the presented results are expressed in terms of the generalized Jacobian of Clarke or the Mordukhovich coderivative. For the characterizations using the coderivative in the monotone case, see [16, 15, 62, 59]. As applications, we give new second-order characterizations for r-convex functions. This generalizes to the nonsmooth case or extends some results from the literature [5, 15, 14, 16, 43].

Part II : Sequential vector optimization

Vector optimization studies optimization problems where objective function and constraints are vectorial. It thus generalizes convex analysis and classical optimization where functions are real-valued. The reader is invited to consult the book of Jahn [41] and the one of Luc [53] for more information about this theory.

Our aim in this second part is to contribute to sequential calculus without qualification condition in vector optimization. Historically, the first formula without condition of qualification is due to Hiriart-Urruty and Phelps [39]. It expresses the subdifferential of the sum of two scalar proper, convex and lower semicontinuous functions in terms of the approximate subdifferential. A bit later, Thibault [72] continued the contribution [39] and provided sequential formulas without qualification condition of the subdifferential of the operations sum and composition of (scalar) convex functions. These results involve the subdifferential of convex analysis. To do this, he makes use of Ekeland variational principle [32] and a new version of the Brondsted-Rockafellar theorem. The adjective sequential means that the points of calculus of the subdifferential are elements of sequences converging to the initial point. Thibault also obtained sequential Lagrange multipliers.

The sequential calculus is still an active topic [20, 36]. Its interest and necessity come from the fact that the constraints of qualification are rarely satisfied, even in finite dimension. Recall that unlike the classical differential calculus, the subdifferential calculus on convex functions needs an additional condition called constraint of qualification or condition of qualification. Let us give an example. Let X be a real Banach space. Let $f, g : X \longrightarrow \mathbb{R} \cup \{+\infty\}$ be convex functions and $x_0 \in X$. We denote the effective domains of f and g by $\operatorname{dom} f$ and $\operatorname{dom} g$ respectively. As usual, the Fenchel-Moreau subdifferential will be denoted by ∂. Then, we do not always have the right to write the following "exact" formula:

$$\partial(f+g)(x_0) = \partial f(x_0) + \partial g(x_0). \tag{1}$$

However, if one of these functions is continuous at a point $\bar{x} \in \operatorname{dom} f \cap \operatorname{dom} g$ (constraint of qualification of Moreau-Rockafellar [64]), the relation (1) is valid at any point. Other conditions ensuring (1) exist in the literature, among which the constraint of qualification Attouch-Brézis [1] :

$$\cup_{\lambda \geq 0} \lambda[\operatorname{dom} f - \operatorname{dom} g]$$

is a closed subspace of X with f, g proper, convex and lower semicontinuous. For more details about the constraints of qualification of the operations composition and sum/composition, see [19, 18] and references therein.

The works of this part are natural continuations of Thibault's work and the article [33] where the authors have developed the vector subdifferential calculus with condition of qualification. Next, let X, Y and Z be real Banach spaces. We will denote by $+\infty_Y$ and $+\infty_Z$ abstract elements playing the role of infinity in Y and Z respectively.

In chapter 3 of this thesis, we establish a sequential formula, free of qualification condition, for the Pareto subdifferentials (weak, proper, strong) of the vectorial operator $f + g \circ h$ where $f : X \longrightarrow Y \cup \{+\infty_Y\}$ is proper, Y_+-convex and lower semicontinuous, $h : X \longrightarrow Z \cup \{+\infty_Z\}$ is proper, Z_+-convex and epi-closed, and $g : Z \longrightarrow Y \cup \{+\infty_Y\}$ is proper, Y_+-convex, lower semicontinuous and (Z_+, Y_+)-nondecreasing. The technique of scalarization [33] is used. In chapter 3, X and Z are assumed reflexive.

In the chapter 4, we prove three formulas for the (weak, proper, strong) Pareto subdifferential of the sums of m vector mappings $f_1, ..., f_m : X \longrightarrow Z \cup \{+\infty_Z\}$ proper, Z_+-convex and lower semicontinuous where $m \geq 2$. The method is based on the interesting epigraphical tool. The third formula generalizes the formula of the sum of two vector mappings that we have given in [49]. It also generalizes to the vector framework the corresponding formula of Jules, Lassonde [44]. As a first application, we derive sequential necessary and sufficient optimality conditions, free of condition of qualification, in the weak, proper and strong senses for the general vector optimization problem :

$$(VOP): \quad \inf_{\substack{x \in C \\ h(x) \in -Y_+}} f(x)$$

where : $f : X \longrightarrow Z \cup \{+\infty_Z\}$ is proper, Z_+-convex and lower semicontinuous, $h : X \longrightarrow Y \cup \{+\infty_Y\}$ is proper, Y_+-convex and epi-closed, C is a nonempty closed convex subset of X, and Y_+ is a nonempty closed convex cone of Y. As a second application, we give sequential optimality conditions without constraints of qualification, in the weak and strong senses for the following multi-objective fractional optimization problem with geometric and cone constraints :

$$(MFP): \quad \inf_{\substack{x \in C \\ h(x) \in -Y_+}} \{\frac{f_1(x)}{g_1(x)}, \cdots, \frac{f_q(x)}{g_q(x)}\}$$

where : $f_1, ..., f_q : X \longrightarrow [0, +\infty[$ are convex and lower semicontinuous, $g_1, ..., g_q : X \longrightarrow]0, +\infty[$ are concave and upper semicontinuous, and $h : X \longrightarrow Y \cup \{+\infty_Y\}$ is proper, Y_+-convex and epi-closed. In chapter 4, X and Y are assumed reflexive.

Part I

r-monotone operators and r-convex functions

Chapter 1

Characterizations of r-convexity through subdifferentials and cyclic *r*-monotonicity

1.1 Introduction

r-convexity [5] is of interest and appears in the practical areas of generalized convex optimization and Economics.

This chapter proves the important question of characterizing *r*-convex functions through Clarke-Rockafellar subdifferential. Unlike quasiconvexity and pseudoconvexity [3, 28, 34, 38, 51, 65], no interest has been accorded to this issue. We employ *r*-monotonicity, a generalized monotonicity concept due to Crouzeix and Hassouni [25]. The approach is based on a separable property of the subdifferential of Clarke-Rockafellar coupled with a characterization of *r*-convex functions by means of quasiconvexity of their logarithmic pertubations.

In section 1.4, we strengthen r-monotonicity by introducing the new notion of cyclically r-monotone operator. Examples of such operators are subdifferentials of r-convex functions. In the section 1.5, we study the validity of the above characterizations for other subdifferentials by using an appropriate abstract subdifferential.

Throughout the chapter, the focus is on the interesting nonconvex case where the parameter r is positive.

1.2 Preliminaries

For convenience of the readers, we recall some necessary notions and results. Let X be a real Banach space and X^* its topological dual paired in duality by $\langle x^*, x\rangle := x^*(x)$ where $(x, x^*) \in X \times X^*$.

Let $f : X \longrightarrow \mathbb{R} \cup \{+\infty\}$ be an extended real-valued function whose effective domain is $\operatorname{dom} f := \{x \in X : f(x) \in \mathbb{R}\}$. f is said to be lower semicontinuous shortly lsc (resp. convex) if its epigraph $\operatorname{epi} f := \{(x,t) \in X \times \mathbb{R} : f(x) \leq t\}$ is closed (resp. convex). f is called :

- r-convex ($r > 0$) if e^{rf} is convex ($e^{r(+\infty)} := +\infty$).
- quasiconvex if for all $\alpha \in \mathbb{R}$, $L(f,\alpha) := \{x \in X : f(x) \leq \alpha\}$ is convex.
- pseudoconvex (f lsc, w.r.t. a subdifferential ∂) if for all $x, y \in X$:

$$\exists x^* \in \partial f(x) : \langle x^*, y - x\rangle \geq 0 \Longrightarrow f(x) \leq f(y).$$

It is easy to see that : f is convex $\Longrightarrow$ f is r-convex $\Longrightarrow$ f is quasiconvex.

Let $F : X \rightrightarrows X^*$ be a multivalued operator on X. We denote by $G(F) := \{(x, x^*) \in X \times X^* : x^* \in F(x)\}$ its graph. F is called :

- quasimonotone if $\forall (x, x^*), (y, y^*) \in G(F) : \langle x^*, y - x\rangle > 0 \Longrightarrow \langle y^*, y - x\rangle \geq 0$.

- r-monotone $(r > 0)$ if $F_r : X \times \mathbb{R} \rightrightarrows X^* \times \mathbb{R}$

$$F_r(x,t) := \begin{cases} F(x) \times \{-\frac{1}{rt}\} & \text{if } (x,t) \in X \times]0, +\infty[. \\ \\ \emptyset & \text{otherwise.} \end{cases}$$

  is quasimonotone.

- pseudomonotone if $\forall (x, x^*), (y, y^*) \in G(F)$:

$$\langle x^*, y - x\rangle > 0 \Longrightarrow \langle y^*, y - x\rangle > 0.$$

Remark 1.2.1. In [25], r-monotonicity corresponds to $(-r)$-monotonicity in the sense of the above definition.

Cyclic counterparts of quasimonotonicity and pseudomonotonicity are as follows.

F is cyclically quasimonotone if for all $\{(x_k, x_k^*)\}_{k\in\{1,2,\ldots,p\}} \subseteq G(F)$:

$$\Big(\forall k \in \{1, \ldots, p-1\}, \langle x_k^*, x_{k+1} - x_k\rangle > 0\Big) \Longrightarrow \langle x_p^*, x_1 - x_p\rangle \leq 0.$$

F is cyclically pseudomonotone if for all $\{(x_k, x_k^*)\}_{k\in\{1,2,\ldots,p\}} \subseteq G(F)$:

$$\Big(\forall k \in \{1, \ldots, p-1\}, \langle x_k^*, x_{k+1} - x_k\rangle \geq 0\Big) \Longrightarrow \langle x_p^*, x_1 - x_p\rangle \leq 0.$$

Recall the geometric definition of Clarke-Rockafellar subdifferential [17, 70].

Definition 1.2.2. Let Y be a real normed space. The tangent and normal cones of $C \subseteq Y$ at $\bar{y} \in C$ are respectively defined by :
$T_C(\bar{y}) = \{d \in Y : \forall \{t_n\}_n \to 0^+, \forall \{y_n\}_n \subseteq C \to \bar{y}, \exists \{d_n\}_n \to d, \{y_n + t_n d_n\}_n \subseteq C\}$,
$N_C(\bar{y}) = \{y^* \in Y^* : \forall d \in T_C(\bar{y}), \langle y^*, d\rangle \leq 0\}$.

Definition 1.2.3. The Clarke-Rockafellar subdifferential of f (lsc) at $\bar{x} \in \operatorname{dom} f$ is :

$$\partial f(\bar{x}) := \{x^* \in X^* : (x^*, -1) \in N_{\operatorname{epi} f}((\bar{x}, f(\bar{x})))\}.$$

For $\bar{x} \notin \operatorname{dom} f$, $\partial f(\bar{x}) := \emptyset$.

The next characterizations will be needed.

Theorem 1.2.4. *[3, 27, 28] Let $f : X \longrightarrow \mathbb{R} \cup \{+\infty\}$ be a lsc function. The following assertions are equivalent.*
(i) f is quasiconvex.
(ii) ∂f is quasimonotone.
(iii) ∂f is cyclically quasimonotone.
(iv) For every $x, y \in X$:
$\Big(\exists x^* \in \partial f(x) : \langle x^*, y - x\rangle > 0\Big) \Longrightarrow \forall \lambda \in [0,1],\ f(\lambda x + (1-\lambda)y) \leq f(y)$.

Theorem 1.2.5. *[65] Let $f : X \longrightarrow \mathbb{R} \cup \{+\infty\}$ be a lsc and radially continuous function. Then, f is pseudoconvex if and only if ∂f is pseudomonotone.*

Theorem 1.2.6. *[25] Let $F : X \rightrightarrows X^*$ be a multivalued operator. Consider the following assertions.*
(i) F is r-monotone.

(ii) F is pseudomonotone.

(iii) $\forall (x,x^*),(y,y^*) \in G(F) : \langle x^*, y-x\rangle > 0 \Longrightarrow \frac{1}{\langle x^*,y-x\rangle} - \frac{1}{\langle y^*,y-x\rangle} \geq -r.$

Then, (i) is equivalent to ((ii) and (iii)).

1.3 Characterizations via Clarke-Rockafellar subdifferential

Let $f : X \longrightarrow \mathbb{R}\cup\{+\infty\}$ be a lsc function. This section establishes that f is r-convex if and only if ∂f is r-monotone. In addition, a mixed characterization involving f and ∂f as well as a link between the classes of r-convex and pseudoconvex functions are provided. To this end, the following lemmas are essential.

Lemma 1.3.1. *Let $(X,\|.\|)$ and $(Y,\|.\|)$ be two real Banach spaces. Let $f : X \longrightarrow \mathbb{R}\cup\{+\infty\}$ and $g : Y \longrightarrow \mathbb{R}\cup\{+\infty\}$ be two lsc functions. Define h on $X\times Y$ by $h(x,y) := f(x)+g(y)$. If g is continuously differentiable at $\bar{y} \in dom\ g$, then*

$$\partial h(\bar{x},\bar{y}) = \partial f(\bar{x}) \times \partial g(\bar{y})$$

for every $\bar{x} \in X$.

Proof. ($\subseteq$) is well-known. ($\supseteq$) Let $\bar{x} \in \text{dom } f$, $x^* \in \partial f(\bar{x})$ and $y^* \in \partial g(\bar{y}) = \{d_{\bar{y}} g\}$, where $d_{\bar{y}} g$ is the differential of g at $\bar{y}$. To show that $(x^*, d_{\bar{y}} g) \in \partial h(\bar{x},\bar{y})$ or equivalently

$$\langle x^*, u\rangle + d_{\bar{y}} g(v) - s \leq 0$$

for any $(u,v,s) \in T_{\text{epi }h}((\bar{x},\bar{y},h(\bar{x},\bar{y})))$, it suffices to prove

$$(u, s - d_{\bar{y}}\, g(v)) \in T_{\text{epi }f}((\bar{x}, f(\bar{x})))$$

for fixed $(u,v,s) \in T_{\text{epi }h}((\bar{x},\bar{y},h(\bar{x},\bar{y})))$.

For this, consider two sequences $\{(x_n, r_n)\}_n \subseteq \text{epi } f$ and $\{t_n\}_n \subseteq \mathbb{R}$ with

$$(x_n, r_n) \xrightarrow[n\to\infty]{} (\bar{x}, f(\bar{x})) \text{ and } t_n \xrightarrow[n\to\infty]{} 0^+.$$

By applying

$$(u,v,s) \in T_{\text{epi }h}((\bar{x},\bar{y},h(\bar{x},\bar{y})))$$

with $\{t_n\}_n$ and $\{(x_n, \bar{y}, r_n + g(\bar{y}))\}_n$, we obtain a sequence $\{(u_n, v_n, \hat{s}_n)\}_n$ with

$$(u_n, v_n, \hat{s}_n) \xrightarrow[n\to\infty]{} (u,v,s)$$

such that for every n, one has

$$f(x_n + t_n u_n) + g(\bar{y} + t_n v_n) \leq r_n + t_n \hat{s}_n + g(\bar{y}). \tag{1.1}$$

Since g is continuously differentiable at $\bar{y}$, we can write

$$g(\bar{y} + t_n v_n) = g(\bar{y}) + t_n d_{\bar{y}}\, g(v_n) + t_n \|v_n\| \epsilon(t_n v_n),$$

where $\epsilon(x) \xrightarrow[x\to 0]{} 0$ and $n \geq n_0$. Now define $s_n := \hat{s}_n - d_{\bar{y}}\, g(v_n) - \|v_n\|\epsilon(t_n v_n)$ for

$n \geq n_0$. Then after replacing $\hat{s}_n$ by $s_n + d_{\bar{y}}\, g(v_n) + \|v_n\|\epsilon(t_n v_n)$ in (1.1), we get

$$f(x_n + t_n u_n) \leq r_n + t_n s_n$$

that is

$$(x_n, r_n) + t_n(u_n, s_n) \in \text{epi } f$$

for $n \geq n_0$. We complete the proof by setting $(u_n, s_n) := (0,0)$ for $n < n_0$ and observing that $(u_n, s_n) \xrightarrow[n\to\infty]{} (u, s - d_{\bar{y}}\, g(v))$. $\square$

In the sequel the function g_r is defined as

$$\begin{array}{rl} g_r : & \mathbb{R} \longrightarrow \mathbb{R} \cup \{+\infty\} \\ & t \longrightarrow \begin{cases} -\frac{\ln(t)}{r} & \text{if } t \in]0, +\infty[\\ +\infty & \text{otherwise} \end{cases} . \end{array}$$

Lemma 1.3.2. *A function $f : X \longrightarrow \mathbb{R} \cup \{+\infty\}$ is r-convex if and only if its logarithmic perturbation $f_r : (x,t) \rightarrow f(x) + g_r(t)$ is quasiconvex on $X \times \mathbb{R}$.*

Proof. ($\Longrightarrow$) Fix $\alpha \in \mathbb{R}$ and consider the function defined on $X \times \mathbb{R}$ by

$$g_{r,\alpha}(x,t) := e^{rf(x)} - e^{r\alpha} \cdot t.$$

It is easy to see that $g_{r,\alpha}$ is convex since f is assumed to be r-convex. Then the quasiconvexity of f_r follows from the fact that

$$\mathrm{L}(f_r, \alpha) = \mathrm{L}(g_{r,\alpha}, 0) \cap (X \times]0, +\infty[)$$

for every $\alpha \in \mathbb{R}$.

($\Longleftarrow$) The converse is deduced from the relation $\mathrm{epi}(e^{rf}) = \mathrm{L}(f_r, 0)$. □

We can state our main results.

Theorem 1.3.3. *Let $f : X \longrightarrow \mathbb{R} \cup \{+\infty\}$ be a lsc function. Then, the following assertions are equivalent.*

(i) f is r-convex.

(ii) ∂f is r-monotone.

(iii) For every $x, y \in X$ and $t, s \in]0, +\infty[$:

$\left(\exists x^* \in \partial f(x) : \langle x^*, y - x\rangle > \frac{s-t}{rt}\right) \Longrightarrow \forall \lambda \in [0,1], f(\lambda x + (1-\lambda)y) \leq f(y) + \frac{\ln(1+\lambda\frac{t-s}{s})}{r}.$

Proof. (i) $\Longleftrightarrow$ (ii) By Lemma 1.3.2, f is r-convex if and only if f_r is quasiconvex on $X \times \mathbb{R}$. The function g_r is lower semicontinuous since its limit at 0^+ is $+\infty$. Hence, the separable sum f_r is also lower semicontinuous. Therefore, by Theorem 1.2.4, f is r-convex is equivalent to ∂f_r is quasimonotone. Now according to Lemma 1.3.1,

$$\partial f_r(x,t) = \begin{cases} \partial f(x) \times \{-\frac{1}{rt}\} & \text{if } (x,t) \in X \times]0, +\infty[. \\ \emptyset & \text{otherwise.} \end{cases}.$$

Thus, by definition, ∂f is r-monotone. The equivalence (i) $\Longleftrightarrow$ (iii) is proved similarly.

□

Corollary 1.3.4. *Let $f : X \longrightarrow \mathbb{R} \cup \{+\infty\}$ be a lsc radially continuous function. Consider the following assertions.*

(i) f is r-convex.

(ii) f is pseudoconvex.

(iii) $\forall (x,x^*),(y,y^*) \in G(\partial f) : \langle x^*, y-x\rangle > 0 \Longrightarrow \frac{1}{\langle x^*, y-x\rangle} - \frac{1}{\langle y^*, y-x\rangle} \geq -r.$

Then, (i) is equivalent to ((ii) and (iii)).

Proof. Follows from Theorem 1.3.3, Theorem 1.2.6, Theorem 1.2.5. □

1.4 Cyclic r-monotonicity

In this section, we define the new notion of cyclic r-monotonicity in order to strengthen Theorem 1.3.3. Moreover, we study its relationship with cyclic pseudomonotonicity and cyclic monotonicity.

Definition 1.4.1. A multivalued operator $F : X \rightrightarrows X^*$ is said to be cyclically r-monotone if the operator $F_r : X \times \mathbb{R} \rightrightarrows X^* \times \mathbb{R}$

$$F_r(x,t) = \begin{cases} F(x) \times \{-\frac{1}{rt}\} & \text{if } (x,t) \in X \times]0,+\infty[. \\ \\ \emptyset & \text{otherwise.} \end{cases}$$

is cyclically quasimonotone.

Remark 1.4.2. Consider the following assertion.

$\forall \{(x_k, x_k^*)\}_{k\in\{1,2,\ldots,p\}} \subseteq G(F)$, $\forall \{t_k\}_{k\in\{1,2,\ldots,p\}} \subseteq]0,+\infty[$:

$$\Big(\forall k \in \{1,\ldots,p-1\}, \langle x_k^*, x_{k+1} - x_k\rangle > \frac{t_{k+1} - t_k}{rt_k}\Big) \Longrightarrow \langle x_p^*, x_1 - x_p\rangle \leq \frac{t_1 - t_p}{rt_p}. \quad (1.2)$$

Then, F is cyclically r-monotone if and only if (1.2) holds.

We also have the following formulations of cyclic r-monotonicity.

Lemma 1.4.3. *Let $F : X \rightrightarrows X^*$ be a multivalued operator. Consider the following assertion.*

$$\forall\{(x_k, x_k^*)\}_{k\in\{1,2,\ldots,p\}} \subseteq G(F),\ \forall\{\mu_k\}_{k\in\{1,\ldots,p-1\}} \subseteq \mathbb{R} \text{ with } 1 + r\mu_k > 0 : \tag{1.3}$$

$$\Big(\forall k \in \{1,\ldots,p-1\}, \langle x_k^*, x_{k+1} - x_k\rangle > \mu_k\Big) \Longrightarrow \langle x_p^*, x_1 - x_p\rangle \le \frac{1}{r}\Big(\prod_{k=1}^{p-1}(1 + r\mu_k)^{-1} - 1\Big).$$

Then, F is cyclically r-monotone if and only if (1.3) holds.

Proof. ($\Longrightarrow$) Let $\{(x_k, x_k^*)\}_{k\in\{1,2,\ldots,p\}} \subseteq G(F)$, $\{\mu_k\}_{k\in\{1,\ldots,p-1\}} \subseteq \mathbb{R}$ with $1 + r\mu_k > 0$. Assume that $\langle x_k^*, x_{k+1} - x_k\rangle > \mu_k$ for all $k \in \{1,\ldots,p-1\}$. Then, by choosing

$$t_k := \begin{cases} \prod_{i=1}^{k-1}(1 + r\mu_i) & \text{if } k \in \{2,\ldots,p\}. \\ \\ 1 & \text{if } k = 1. \end{cases},$$

it is easy to see that every t_k is positive and

$$\langle x_k^*, x_{k+1} - x_k\rangle > \mu_k = \frac{t_{k+1} - t_k}{rt_k}$$

for $k \in \{1,\ldots,p-1\}$.

Therefore by (1.2),

$$\langle x_p^*, x_1 - x_p\rangle \le \frac{t_1 - t_p}{rt_p} = \frac{1}{r}\Big(\prod_{k=1}^{p-1}(1 + r\mu_k)^{-1} - 1\Big).$$

($\Longleftarrow$) Let $\{(x_k, x_k^*)\}_{k\in\{1,2,\ldots,p\}} \subseteq G(F)$ and $\{t_k\}_{k\in\{1,2,\ldots,p\}} \subseteq]0,+\infty[$. Suppose that $\langle x_k^*, x_{k+1} - x_k\rangle > \frac{t_{k+1}-t_k}{rt_k}$ for each $k \in \{1,\ldots,p-1\}$. Let $k \in \{1,\ldots,p-1\}$. Then by picking

$$\mu_k := \frac{t_{k+1} - t_k}{rt_k},$$

we have

$$1 + r\mu_k = \frac{t_{k+1}}{t_k} > 0$$

and

$$\langle x_k^*, x_{k+1} - x_k\rangle > \mu_k.$$

So it follows from (1.3),

$$\langle x_p^*, x_1 - x_p\rangle \le \frac{1}{r}\Big(\prod_{k=1}^{p-1}(1 + r\mu_k)^{-1} - 1\Big) = \frac{t_1 - t_p}{rt_p}.$$

□

Lemma 1.4.4. *Let $F : X \rightrightarrows X^*$ be a multivalued operator. Consider the following assertion.*

$\forall\{(x_k, x_k^*)\}_{k\in\{1,2,\ldots,p\}} \subseteq G(F)$ *:*

$$\Big(\forall k \in \{1,\ldots,p-1\}, 1 + r\langle x_k^*, x_{k+1} - x_k\rangle > 0\Big) \Longrightarrow \prod_{k=1}^{p}(1 + r\langle x_k^*, x_{k+1} - x_k\rangle) \le 1 \quad (1.4)$$

with the convention $x_{p+1} := x_1$.

Then, F is cyclically r-monotone if and only if (1.4) holds.

Proof. By Lemma 1.4.3, it is equivalent to prove (1.3) $\Longleftrightarrow$ (1.4).

(1.3) $\Longrightarrow$ (1.4) Let $\{(x_k, x_k^*)\}_{k\in\{1,2,\dots,p\}} \subseteq G(F)$ that satisfies

$$1 + r\langle x_k^*, x_{k+1} - x_k\rangle > 0$$

for every $k \in \{1, \dots, p-1\}$. One can consider $p-1$ sequences $\{\mu_k^n\}_{n\in\mathbb{N}} \subseteq \mathbb{R}$ such that :

$$1 + r\mu_k^n > 0,$$

$$\langle x_k^*, x_{k+1} - x_k\rangle > \mu_k^n,$$

$$\mu_k^n \xrightarrow[n\to\infty]{} \langle x_k^*, x_{k+1} - x_k\rangle$$

for $k \in \{1, \dots, p-1\}$ and $n \in \mathbb{N}$.

Then by (1.3), for every $n \in \mathbb{N}$,

$$(1 + r\langle x_p^*, x_1 - x_p\rangle)\prod_{k=1}^{p-1}(1 + r\mu_k^n) \leq 1.$$

Thus, by letting $n \to \infty$, we obtain

$$\prod_{k=1}^{p}(1 + r\langle x_k^*, x_{k+1} - x_k\rangle) \leq 1.$$

(1.4) $\Longrightarrow$ (1.3) Let $\{(x_k, x_k^*)\}_{k\in\{1,2,\dots,p\}} \subseteq G(F)$ and $\{\mu_k\}_{k\in\{1,\dots,p-1\}} \subseteq \mathbb{R}$ with

$$1 + r\mu_k > 0, \tag{1.5}$$

$$\langle x_k^*, x_{k+1} - x_k\rangle > \mu_k. \tag{1.6}$$

Clearly, we have

$$1 + r\langle x_k^*, x_{k+1} - x_k\rangle > 0, \tag{1.7}$$

for $k \in \{1, ..., p-1\}$.

Let $\mathbb{R}_+^{p-1}$ denote the nonnegative orthant of $\mathbb{R}^{p-1}$. It is easy to verify that the function

$$\begin{array}{lll} \psi: & \prod_{k=1}^{p-1}]-\frac{1}{r}, +\infty[& \longrightarrow \mathbb{R} \\ & (y_1, ..., y_{p-1}) & \longrightarrow \frac{1}{r}\Big(\prod_{k=1}^{p-1}(1+ry_k)^{-1} - 1\Big) \end{array}$$

is $(\mathbb{R}_+^{p-1}, \mathbb{R}_+)$-nonincreasing.

Thus, from (1.4),

$$\langle x_p^*, x_1 - x_p\rangle \le \psi\Big(\langle x_1^*, x_2 - x_1\rangle, ..., \langle x_{p-1}^*, x_p - x_{p-1}\rangle\Big).$$

Now according to (1.5) and (1.7), μ_k and $\langle x_k^*, x_{k+1} - x_k\rangle$ belong to $]-\frac{1}{r}, +\infty[$ for $k \in \{1, ..., p-1\}$. Therefore, by (1.6) and the monotonicity of ψ, we get

$$\langle x_p^*, x_1 - x_p\rangle \le \psi\Big(\mu_1, ..., \mu_{p-1}\Big) = \frac{1}{r}\Big(\prod_{k=1}^{p-1}(1 + r\mu_k)^{-1} - 1\Big).$$

□

Remark 1.4.5. Since cyclic quasimonotonicity implies quasimonotonicity [28], one obtains that every cyclically r-monotone operator is r-monotone.

Proposition 1.4.6. *Let $F : X \rightrightarrows X^*$ be a multivalued operator. If F is cyclically r-monotone then F is cyclically pseudomonotone.*

Proof. Let $\{(x_k, x_k^*)\}_{k\in\{1,2,...,p\}} \subseteq G(F)$ and suppose $\langle x_k^*, x_{k+1} - x_k\rangle \ge 0$ for all $k \in \{1, ..., p-1\}$. Now choose any $p-1$ real numbers $\mu_k \in]-\frac{1}{r}, 0[$. Obviously $\langle x_k^*, x_{k+1} -$

$x_k\rangle > \mu_k$ and $1 + r\mu_k > 0$ for $k \in \{1, ..., p-1\}$. Therefore by (1.3),

$$\langle x_p^*, x_1 - x_p\rangle \le \frac{1}{r}\Big(\prod_{k=1}^{p-1}(1 + r\mu_k)^{-1} - 1\Big).$$

Hence, by letting $\mu_k \to 0^-$ for all $k \in \{1, ..., p-1\}$, we obtain $\langle x_p^*, x_1 - x_p\rangle \le 0$.

□

In the same line as the proof of Theorem 1.3.3, we have :

Theorem 1.4.7. *Let $f : X \longrightarrow \mathbb{R} \cup \{+\infty\}$ be a lsc function. Then, f is r-convex if and only if ∂f is cyclically r-monotone.*

Corollary 1.4.8. *Let $F : X \rightrightarrows X^*$ be a multivalued operator and $f : X \longrightarrow \mathbb{R} \cup \{+\infty\}$ a lsc r-convex function such that $F \subseteq \partial f$. Then, F is cyclically r-monotone.*

Recall that $F : X \rightrightarrows X^*$ is cyclically monotone if for all $\{(x_k, x_k^*)\}_{k\in\{1,2,...,p\}} \subseteq G(F)$:

$$\sum_{i=1}^{p}\langle x_k^*, x_{k+1} - x_k\rangle \le 0$$

with the convention $x_{p+1} := x_1$.

Corollary 1.4.9. *Let $F : X \rightrightarrows X^*$ be a multivalued operator. If F is cyclically monotone then F is cyclically r-monotone for all $r > 0$.*

Proof. From the cyclic monotonicity of F and by the existence of Rockafellar's integration, there exists a proper lower semicontinuous convex function $f_F : X \longrightarrow \mathbb{R} \cup \{+\infty\}$ that satisfies

$$F \subseteq \partial^{FM} f_F = \partial f_F$$

where ∂^{FM} is the subdifferential of convex analysis. Since f_F is convex, it is r-convex for every r positive. Therefore according to Corollary 1.4.8, F is cyclically r-monotone for every r positive. □

1.5 The case of other subdifferentials

The aim of this section is to extend the characterizations established before to a larger class of subdifferentials. The approach is inspired by the abstract unified subdifferentials [2, 27].

Definition 1.5.1. A separable subdifferential ∂ is a map that associates to any real Banach space X, any lsc function $h : X \longrightarrow \mathbb{R} \cup \{+\infty\}$ and any $\bar{x} \in X$, a subset $\partial h(\bar{x})$ of X^* such that :

(i) If $\bar{x} \notin \operatorname{dom} h$, then $\partial h(\bar{x}) = \emptyset$.

(ii) If h is convex, then $\partial h(\bar{x}) = \{x^* \in X^* : h(x) \geq h(\bar{x}) + \langle x^*, x - \bar{x}\rangle, \forall x \in X\}$.

(iii) If $\bar{x} \in \operatorname{dom} h$ is a local minimum of h, then $0 \in \partial h(\bar{x})$.

(iv) If $f, g : X \longrightarrow \mathbb{R} \cup \{+\infty\}$ are lsc with g real-valued convex continuous and ∂-differentiable at $\bar{x}$, then $\partial(f + g)(\bar{x}) \subseteq \partial f(\bar{x}) + \partial g(\bar{x})$. (Here g is ∂-differentiable at $\bar{x}$ means that $\partial g(\bar{x}) \neq \emptyset$ and $\partial(-g)(\bar{x}) \neq \emptyset$)

(v) If $v^* \in X^*$, then $\partial(h + v^*)(\bar{x}) = \partial h(\bar{x}) + v^*$.

(vi) If $h(x,t) := f(x) + g(t)$, then $\partial h(\bar{x}, \bar{t}) = \partial f(\bar{x}) \times \{d_{\bar{t}} g\}$. Here, $f : X \longrightarrow \mathbb{R} \cup \{+\infty\}$ and $g : \mathbb{R} \longrightarrow \mathbb{R} \cup \{+\infty\}$ are lsc with g convex continuously differentiable at $\bar{t} \in \operatorname{dom} g$.

Remark 1.5.2. It is well-known that axioms (i), (ii), (iii) and (iv) are verified by the following subdifferentials :

- the Fréchet subdifferential ∂^F,
- the Lipschitz smooth subdifferential ∂^{LS},
- the upper Dini subdifferential ∂^D,
- the lower Dini subdifferential ∂_D,
- the Clarke-Rockafellar subdifferential ∂^{CR}.

Also, it is easy to see that these subdifferentials satisfy (v).

Concerning (vi), it is fulfilled using the directional derivatives for $\partial \in \{\partial^D, \partial_D\}$ and the very definition for $\partial \in \{\partial^F, \partial^{LS}\}$. For the sake of completeness, we recall the proofs. The following lemmas give separability results for the separable sum $h(x,y) := f(x) + g(y)$ with $f : X \longrightarrow \mathbb{R} \cup \{+\infty\}$ and $g : Y \longrightarrow \mathbb{R} \cup \{+\infty\}$ lsc functions (X and Y are real Banach spaces).

Lemma 1.5.3. *Let* $(\bar{x}, \bar{y}) \in X \times Y$. *Then,*

$$\partial^{LS} h(\bar{x}, \bar{y}) = \partial^{LS} f(\bar{x}) \times \partial^{LS} g(\bar{y}).$$

$$\partial^F h(\bar{x}, \bar{y}) = \partial^F f(\bar{x}) \times \partial^F g(\bar{y}).$$

Proof. We treat the case ∂^{LS}. Similarly, one can prove the Fréchet subdifferential case. Let $(\bar{x}, \bar{y}) \in \operatorname{dom} h = \operatorname{dom} f \times \operatorname{dom} g$. ($\subseteq$) Let $(x^*, y^*) \in \partial^{LS} h(\bar{x}, \bar{y})$. Then, by definition, there exist $c > 0$ and $\alpha > 0$ such that :

$$f(x) + g(y) \geq f(\bar{x}) + g(\bar{y}) + \langle x^*, x - \bar{x} \rangle + \langle y^*, y - \bar{y} \rangle - c(\max(\|x - \bar{x}\|, \|y - \bar{y}\|))^2 \quad (1.8)$$

for every $(x,y) \in \mathbb{B}(\bar{x},\alpha) \times \mathbb{B}(\bar{y},\alpha)$ ($\mathbb{B}(a,r)$ denotes the open ball of center a and radius r (in a normed space)). By setting $(x,y) := (x,\bar{y})$ and $(x,y) := (\bar{x},y)$ respectively in (1.8), we get respectively, $x^* \in \partial^{LS} f(\bar{x})$ and $y^* \in \partial^{LS} g(\bar{y})$.

Conversely ($\supseteq$), consider $x^* \in \partial^{LS} f(\bar{x})$, $y^* \in \partial^{LS} g(\bar{y})$ and $\epsilon > 0$. Then, there exist $c_f, c_g, \alpha_f, \alpha_g > 0$ such that :

$$f(x) \geq f(\bar{x}) + \langle x^*, x - \bar{x} \rangle - c_f \|x - \bar{x}\|^2, \quad \forall x \in \mathbb{B}(\bar{x}, \alpha_f) \tag{1.9}$$

$$g(y) \geq g(\bar{y}) + \langle y^*, y - \bar{y} \rangle - c_g \|y - \bar{y}\|^2, \quad \forall y \in \mathbb{B}(\bar{y}, \alpha_g) \tag{1.10}$$

By taking $\alpha := \min(\alpha_f, \alpha_g)$, $c := c_f + c_g$ and summing (1.9) and (1.10), we easily obtain $(x^*, y^*) \in \partial^{LS} h(\bar{x}, \bar{y})$. □

Let us recall that the definitions of the lower and upper Dini directional derivatives of f at $x \in \operatorname{dom} f$ in the direction $d \in X$ are respectively :

$$f_D(x,d) := \liminf_{t \to 0_+} \frac{f(x+td) - f(x)}{t},$$

$$f^D(x,d) := \limsup_{t \to 0_+} \frac{f(x+td) - f(x)}{t}.$$

Lemma 1.5.4. *Let $(x,y,d_1,d_2) \in dom\ f \times dom\ g \times X \times Y$. Then,*

(i) $h_D((x,y),(d_1,d_2)) \geq f_D(x,d_1) + g_D(y,d_2)$.

(ii) $h^D((x,y),(d_1,d_2)) \leq f^D(x,d_1) + g^D(y,d_2)$.

(iii) $\partial_D f(x) \times \partial_D g(y) \subset \partial_D h(x,y)$.

(iv) $\partial^D f(x) \times \partial^D g(y) \supset \partial^D h(x,y)$.

If $\lim_{t \to 0_+} \frac{g(y+td_2)-g(y)}{t}$ exists then equalities hold in (i) and (ii).

If $\lim_{t \to 0_+} \frac{g(y+td)-g(y)}{t}$ exists for all $d \in Y$ then equalities hold in (iii) and (vi).

Proof. We only treat the lower case (the upper case is similar). We first prove the part of lower Dini directional derivative. By definition,

$$\begin{aligned} h_D((x,y),(d_1,d_2)) &= \liminf_{t\to 0_+} \frac{h((x,y)+t(d_1,d_2))-h(x,y)}{t} \\ &= \liminf_{t\to 0_+} \frac{f(x+td_1)-f(x)+g(y+td_2)-g(y)}{t}. \end{aligned}$$

Using usual properties of inferior limits, we have :

$$h_D((x,y),(d_1,d_2)) \geq \liminf_{t\to 0_+} \frac{f(x+td_1)-f(x)}{t} + \liminf_{t\to 0_+} \frac{g(y+td_2)-g(y)}{t}.$$

Therefore, $h_D((x,y),(d_1,d_2)) \geq f_D(x,d_1)+g_D(y,d_2)$ with equality if :

$$\lim_{t\to 0_+} \frac{g(y+td_2)-g(y)}{t}$$

exists. Now, let $x^* \in \partial_D f(x)$ and $y^* \in \partial_D g(y)$. By the lower Dini subdifferential's definition, we have $\langle x^*, v_1\rangle \leq f_D(x,v_1)$ for all $v_1 \in X$ and $\langle y^*, v_2\rangle \leq g_D(y,v_2)$ for all $v_2 \in Y$. Therefore, by adding these inequalities, we obtain :

$$\langle (x^*,y^*),(v_1,v_2)\rangle \leq f_D(x,v_1)+g_D(y,v_2) \leq h_D((x,y),(v_1,v_2)).$$

Hence, $(x^*,y^*) \in \partial_D h(x,y)$. Thus, $\partial_D f(x)\times\partial_D g(y) \subset \partial_D h(x,y)$. It remains to prove the equality in case of the existence of $\lim_{t\to 0_+} \frac{g(y+td)-g(y)}{t}$ for all $d \in Y$. So let $(x^*,y^*) \in \partial_D h(x,y)$. Then, for all $(v_1,v_2) \in X\times Y$, we have :

$$\langle x^*, v_1\rangle + \langle y^*, v_2\rangle \leq h_D((x,y),(v_1,v_2)) = f_D(x,v_1)+g_D(y,v_2).$$

By setting successively $v_2 = 0$ and $v_1 = 0$, we are led to $x^* \in \partial_D f(x)$ and $y^* \in \partial_D g(y)$. □

Remark 1.5.5. If g is continuously differentiable at y then it is well known that $\lim_{t\to 0_+} \frac{g(y+td)-g(y)}{t}$ exists for all $d \in Y$ and is equal to $d_y g(d_2)$.

Definition 1.5.6. Let X be a real Banach space and ∂ a separable subdifferential. Let $Z \in \{X, X\times\mathbb{R}\}$. ∂ is said to be a Z-MVT subdifferential if for every $h : Z \longrightarrow \mathbb{R}\cup\{+\infty\}$ lsc and $\bar{z}, z \in Z$ with $h(z) > h(\bar{z})$, there exists $\lambda \in [0,1[$ and sequences $\{z_n\} \subseteq \operatorname{dom} h$, $\{z_n^*\} \subseteq Z^*$ with $z_n \xrightarrow[n\to\infty]{} \lambda z + (1-\lambda)\bar{z}$, $z_n^* \in \partial h(z_n)$, and for all $t > 0$,

$$\langle z_n^*, \bar{z} + (t+\lambda)(z-\bar{z}) - z_n\rangle > 0.$$

Remark 1.5.7. We recall from [2] that the mean value property in Definition 1.5.6 is satisfied under the additional regularity assumption that Z admits a ∂-smooth renorm [2].

Using [27, Proposition 2] and [27, Theorem 9] instead of Theorem 1.2.4 in the proof of Theorem 1.3.3, we get :

Theorem 1.5.8. *Let $f : X \longrightarrow \mathbb{R}\cup\{+\infty\}$ be a lower semicontinuous function and ∂ a separable subdifferential with $\partial \subseteq \partial^{CR}$ or $\partial \subseteq \partial^D$. Assume that $X\times\mathbb{R}$ admits a ∂-smooth renorm. Then the following assertions are equivalent.*
(i) f is r-convex.
(ii) ∂f is r-monotone.
(iii) ∂f is cyclically r-monotone.
(iv) For every $x, y \in X$ and $t, s \in]0,+\infty[$:
$\left(\exists x^* \in \partial f(x) : \langle x^*, y-x\rangle > \frac{s-t}{rt}\right) \Longrightarrow \forall\lambda \in [0,1],\ f(\lambda x + (1-\lambda)y) \le f(y) + \frac{\ln(1+\lambda\frac{t-s}{s})}{r}.$

By applying Theorem 1.5.8, Theorem 1.2.6 and [2, Theorem 4.2], we obtain :

Corollary 1.5.9. *Let $f : X \longrightarrow \mathbb{R} \cup \{+\infty\}$ be a lower semicontinuous function and ∂ a separable subdifferential with $\partial \subseteq \partial^{CR}$ or $\partial \subseteq \partial^{D}$. Assume that X and $X \times \mathbb{R}$ admit a ∂-smooth renorm. Consider the following assertions.*

(i) f is r-convex.

(ii) f is pseudoconvex.

(iii) $\forall (x,x^*),(y,y^*) \in G(\partial f) : \langle x^*, y-x\rangle > 0 \Longrightarrow \frac{1}{\langle x^*, y-x\rangle} - \frac{1}{\langle y^*, y-x\rangle} \geq -r.$

Then (i) is equivalent to ((ii) and (iii)) if f is radially continuous.

Chapter 2

Maximality and first-order criteria of r-monotone operators

2.1 Introduction

r-monotone operators were first introduced by Crouzeix and Hassouni [25]. They constitute a class of generalized monotonicity intermediate between monotonicity and pseudomonotonicity. They occur in the consumer and demand theory. See the problem of generalized monotonicity of the demand map [25]. Recently, Fajri and Hassouni [35] have contributed to r-monotone operators by showing important links with r-convex functions and by introducing their cyclic versions called cyclic r-monotone operators (see also Chapter 1).

It is known that the essence of the subject of generalized monotonicity is to extend notions and results from the theory of monotone operators [67, 69]. In this direction and as a sequel of [35], the present chapter investigates maximality and first-order criteria for r-monotone operators. Let us recall that these aspects are prominent in the classical monotone setting. Notice also the recent important coderivative characterizations of maximal monotonicity by Mordukhovich and co-authors in [15, 62] and in the recent monograph of Mordukhovich [59]. For pseudomonotone and quasimonotone operators, some interests have been accorded : see [4, 37, 11, 12] and references therein (for the different concepts of maximality and their study) and [24, 47, 54] (for first-order conditions in smooth and locally Lipschitz cases).

Section 2.2 of this chapter is preliminary with some new results. We present a new reformulation of r-monotone operators that prove to be crucial in the study of their maximality. We also prove that r-monotonicity is a local notion and establish some of its stability operations. In section 2.3, we introduce maximal r-monotone operators and show the basic properties. Note that we consider maximality in the natural sense, that is, according to graph inclusion. Furthermore, we give characterizations of r-monotone and maximal r-monotone operators using newly introduced concept of polarity that we call r-polar. For further details about polars, see [55, 12, 11]. Section 2.4 is mainly devoted to first-order characterizations. First, we derive first-order criteria of type [24, 47, 54] for r-monotone maps that are differentiable, continuously differentiable and locally Lipschitz. The approach is based on a new separability lemma of Clarke generalized Jacobians of a separable product of operators. Second, we prove the r-monotone counterpart in differentiable and locally Lipschitz situations of a classical result saying that a differentiable operator is monotone if and only if its Jacobian matrix is positive semidefinite. In the locally Lipschitz setting, the obtained results are formulated in terms of Clarke generalized Jacobians or Mordukhovich coderivatives. As application, new second-order characterizations of r-convex functions that are smooth or nonsmooth are obtained. In particular, this generalizes Avriel's characterization of r-convexity [5] valid for twice-continuously differentiable functions to nonsmooth framework by employing generalized Hessians in the sense of Hiriart-Urruty et al. or Mordukhovich second-order subdifferentials. In addition, the first-order conditions in the case of affine r-monotone maps are stated.

In what follows, we treat the interesting non-monotone case where $r > 0$.

2.2 Preliminaries and r-monotone operators

In this section, we recall concepts from generalized monotonicity. Then, we focus on r-monotone operators by recalling known results and establishing some new ones.

Let $(X, \|.\|)$ be a real Banach space and $(X^*, \|.\|_*)$ be its topological dual with duality pairing defined for $(x,x^*) \in X \times X^*$ by $\langle x^*, x\rangle := x^*(x)$. Let C be a nonempty subset of X and $T : C \rightrightarrows X^*$ a set-valued operator. As usual, we shall denote by $\mathrm{dom}(T)$ its domain and by $\mathrm{G}(T)$ its graph. The operator T is said to be pseudomonotone if for all $(x,x^*),(y,y^*) \in \mathrm{G}(T)$ one has : $\langle x^*, y-x\rangle \geq 0 \implies \langle y^*, y-x\rangle \geq 0$. It is called quasimonotone if we have $\langle x^*, y-x\rangle > 0 \implies \langle y^*, y-x\rangle \geq 0$ that holds for every $(x,x^*),(y,y^*) \in \mathrm{G}(T)$. We also recall that T is monotone if for every $(x,x^*),(y,y^*) \in \mathrm{G}(T) : \langle y^* - x^*, y-x\rangle \geq 0$.

The next generalized monotonicty notion is central in this chapter.

Definition 2.2.1. [25] Let $T : C \rightrightarrows X^*$ be a set-valued operator. T is said to be r-monotone ($r > 0$) if $T_r : C \times (0,+\infty) \rightrightarrows X^* \times \mathbb{R}$ defined for every $(x,t) \in C \times (0,+\infty)$ by $T_r(x,t) := T(x) \times \{-\frac{1}{rt}\}$ is pseudomonotone.

Remark 2.2.2. Next, we will take C equals the whole space X or assume C nonempty, open, convex and X of finite dimension. We will also interest by the nontrivial case where T is non null. Therefore, by Theorem 5.2 [25], T is r-monotone is equivalent to T_r is quasimonotone. Note that this latter coincide with $(-r)$-monotonicity in the sense of [25].

Remark 2.2.3. By identification with their graphs, operators can be manipulated as subsets of $X \times X^*$. A subset E of $X \times X^*$ is said to be r-monotone if the associated operators is also.

Remark 2.2.4. [25] r-monotonicity is a radial, that is, $T : C \subseteq X \rightrightarrows X^*$ is r-monotone if and only if $\psi_{x,d}$ is r-monotone for every $x \in C$ and $d \in X$, where $\psi_{x,d}(t) := \{\langle x^*, d\rangle : x^* \in T(x+td)\}$ and $t \in I_{x,d} := \{t \in \mathbb{R} : x + td \in C\}$.

r-monotone operators are intermediate between monotone and pseudomonotone operators. More precisely, we have the following relationships.

Remark 2.2.5. [25] Let $T : X \rightrightarrows X^*$ be a set-valued operator. Then, T is monotone if and only if T is r-monotone for all $r > 0$.

Lemma 2.2.6. *[25] Let $T : X \rightrightarrows X^*$ be a set-valued operator. Then, T is r-monotone if and only if T is pseudomonotone and for all $(x, x^*), (y, y^*) \in G(T)$, one has :*

$$\langle x^*, y - x\rangle > 0 \Longrightarrow \frac{1}{\langle x^*, y - x\rangle} - \frac{1}{\langle y^*, y - x\rangle} \geq -r.$$

The forthcoming result is a new interesting formulation of r-monotone operators. It does not involve pseudomonotonicity and is expressed better than Lemma 2.2.6. It will be important for the investigation of maximal r-monotone operators in the next section and also useful for proving stability operations for r-monotonicity.

Lemma 2.2.7. *Let $T : X \rightrightarrows X^*$ be a set-valued operator. Then, T is r-monotone if and only if for all $(x,x^*),(y,y^*) \in \mathrm{G}(T)$, one has :*

$$\Big(1 + r\langle x^*, y-x\rangle > 0 \text{ or } 1 + r\langle y^*, x-y\rangle > 0\Big) \Longrightarrow r\langle x^*, y-x\rangle\langle y^*, y-x\rangle + \langle y^*-x^*, y-x\rangle \geq 0.$$

Proof. By the very definition, T is r-monotone if and only if for all $x, y \in X$, $t, t' > 0$, $x^* \in T(x)$, $y^* \in T(y)$, one has : $\langle x^*, y-x\rangle - \frac{t'-t}{rt} \geq 0 \Longrightarrow \langle y^*, y-x\rangle - \frac{t'-t}{rt'} \geq 0$. This is equivalent to for all $(x,x^*),(y,y^*) \in \mathrm{G}(T)$:

$$1 + \inf_{\rho>0}[\,\rho(r\langle y^*, y-x\rangle - 1) : \rho \leq 1 + r\langle x^*, y-x\rangle\,] \geq 0. \tag{2.1}$$

($\Longrightarrow$) Let $(x,x^*),(y,y^*) \in \mathrm{G}(T)$. If $1 + r\langle x^*, y-x\rangle > 0$, by taking $\rho := 1 + r\langle x^*, y-x\rangle$ in (2.1), we get : $r\langle x^*, y-x\rangle\langle y^*, y-x\rangle + \langle y^*-x^*, y-x\rangle \geq 0$. If $1 + r\langle y^*, x-y\rangle > 0$, then proceed as the precedent case for the permuted couples (y,y^*) and (x,x^*). ($\Longleftarrow$) Let $(x,x^*),(y,y^*) \in \mathrm{G}(T)$. Assume that $1 + r\langle x^*, y-x\rangle > 0$. If $r\langle y^*, y-x\rangle - 1 \geq 0$, then obviously (2.1) holds. If $r\langle y^*, y-x\rangle - 1 < 0$, then $\rho \mapsto \rho(r\langle y^*, y-x\rangle - 1)$ is decreasing. Hence its infimum on $]0, 1 + r\langle x^*, y-x\rangle]$ is atteined for $\rho = 1 + r\langle x^*, y-x\rangle$. From the assumption, we deduce (2.1). Therefore T is r-monotone. □

Corollary 2.2.8. *Let $S \subseteq \mathbb{R}^n \times \mathbb{R}^n$. Let $T : \mathbb{R}^n \rightrightarrows \mathbb{R}^n$, $A \in M_{n,p}(\mathbb{R})$ and $a \in \mathbb{R}^n$. Then, we have :*

(i) S is r-monotone $\Longrightarrow$ the topological closure $\bar{S}$ is r-monotone.

(ii) T is r-monotone $\Longrightarrow$ the operator $x \in \mathbb{R}^p \mapsto A^t T(Ax + a)$ is r-monotone.

(iii) T is r-monotone and $\lambda > 0 \Longrightarrow \lambda T$ is $\frac{r}{\lambda}$-monotone.

Proof. (i) Let $(x,x^*),(y,y^*) \in \bar{S}$. Then there exist sequences $\{(x_m,x_m^*)\}$, $\{(y_m,y_m^*)\} \subseteq S$ such that : $(x_m,x_m^*) \xrightarrow[m\to\infty]{} (x,x^*)$ and $(y_m,y_m^*) \xrightarrow[m\to\infty]{} (y,y^*)$. By Lemma 2.2.7, for all $m,p \in \mathbb{N}$, we have :

$$\Big(1 + r\langle x_m^*, y_p - x_m\rangle \le 0 \text{ and } 1 + r\langle y_p^*, x_m - y_p\rangle \le 0\Big)$$
$$\text{or } \Big(r\langle x_m^*, y_p - x_m\rangle\langle y_p^*, y_p - x_m\rangle + \langle y_p^* - x_m^*, y_p - x_m\rangle \ge 0\Big).$$

For fixed p, we have three cases. Case 1 : There exists $m_0 \in \mathbb{N}$ such that for all $m \ge m_0$: $1 + r\langle x_m^*, y_p - x_m\rangle \le 0$ and $1 + r\langle y_p^*, x_m - y_p\rangle \le 0$. Case 2 : There exists $m_0 \in \mathbb{N}$ such that for all $m \ge m_0$: $r\langle x_m^*, y_p - x_m\rangle\langle y_p^*, y_p - x_m\rangle + \langle y_p^* - x_m^*, y_p - x_m\rangle \ge 0$. Case 3 : There exist subsequences $(x_{\alpha(m)}, x_{\alpha(m)}^*)$ and $(x_{\beta(m)}, x_{\beta(m)}^*)$ such that : $1 + r\langle x_{\alpha(m)}^*, y_p - x_{\alpha(m)}\rangle \le 0$ and $1 + r\langle y_p^*, x_{\alpha(m)} - y_p\rangle \le 0$ and $r\langle x_{\beta(m)}^*, y_p - x_{\beta(m)}\rangle\langle y_p^*, y_p - x_{\beta(m)}\rangle + \langle y_p^* - x_{\beta(m)}^*, y_p - x_{\beta(m)}\rangle \ge 0$ for all m.
In all cases, by letting m to $+\infty$, we get :

$$\Big(1 + r\langle x^*, y_p - x\rangle \le 0 \text{ and } 1 + r\langle y_p^*, x - y_p\rangle \le 0\Big)$$
$$\text{or } \Big(r\langle x^*, y_p - x\rangle\langle y_p^*, y_p - x\rangle + \langle y_p^* - x^*, y_p - x\rangle \ge 0\Big).$$

By repeating the precedent argument, we get :

$$\Big(1 + r\langle x^*, y - x\rangle \le 0 \text{ and } 1 + r\langle y^*, x - y\rangle \le 0\Big)$$
$$\text{or } \Big(r\langle x^*, y - x\rangle\langle y^*, y - x\rangle + \langle y^* - x^*, y - x\rangle \ge 0\Big).$$

So, by Lemma 2.2.7, $\bar{S}$ is r-monotone.

(ii) and (iii) : Follow by using Lemma 2.2.7.

□

It is known that monotonicity is local. Next, we will see that pseudomonotonicity and thus r-monotonicity are also. The approach is similar to the monotone case.

Proposition 2.2.9. *Let $T : \mathbb{R}^n \rightrightarrows \mathbb{R}^n$ with nonempty, convex open domain. Then, T is pseudomonotone if and only if T is locally pseudomonotone on* $\mathrm{dom}(T)$*, that is, for all $\bar{x} \in \mathrm{dom}(T)$, there exists a neighborhood U of $\bar{x}$ where T is pseudomonotone.*

Proof. ($\Longrightarrow$) is straightforward. ($\Longleftarrow$) Let $x, y \in \mathrm{dom}(T)$, $x^* \in T(x)$, $y^* \in T(y)$ such that $\langle x^*, y - x\rangle \geq 0$. By assumption, there exists a subdivision of $[x, y]$, denoted by $([x + \alpha_i(y - x), x + \alpha_{i+1}(y - x)])_{i=0...m}$ with $\alpha_0 := 0$, $\alpha_{m+1} := 1$ and $(\alpha_i)_{i=0...m+1}$ an increasing sequence, and T pseudomonotone on each $[x + \alpha_i(y - x), x + \alpha_{i+1}(y - x)]$. Thus, for all $i = 0...m$ and $z_i^* \in T(x + \alpha_i(y - x))$, we have : $\langle z_i^*, y - x\rangle \geq 0 \Longrightarrow \langle z_{i+1}^*, y - x\rangle \geq 0$. Hence, since $\langle x^*, y - x\rangle \geq 0$, we deduce that : $\langle y^*, y - x\rangle \geq 0$. □

Corollary 2.2.10. *Let $T : \mathbb{R}^n \rightrightarrows \mathbb{R}^n$ with nonempty, convex open domain. Then, T is r-monotone if and only if T is locally r-monotone on* $\mathrm{dom}(T)$*, that is, for all $\bar{x} \in \mathrm{dom}(T)$, there exists a neighborhood U of $\bar{x}$ where T is r-monotone.*

2.3 Maximal r-monotone operators

In this section, we define the new concept of maximal r-monotone operator. Then, we establish the basic properties and characterize r-monotone and maximal r-monotone operators using a new notion of polarity suitable for r-monotonicity.

Definition 2.3.1. Let $T : X \rightrightarrows X^*$ be a r-monotone operator. T is said to be maximal r-monotone if its graph is not properly contained in the graph of any other r-monotone operator.

Remark 2.3.2. Recall that an extension of a given operator $T : X \rightrightarrows X^*$ is any operator $S : X \rightrightarrows X^*$ whose graph contains the graph of T. Then, every r-monotone operator admits a maximal r-monotone extension. The proof is analogous to its monotone counterpart, it makes use of Zorn's lemma (and Lemma 2.2.7).

Lemma 2.3.3. *Let $T : X \rightrightarrows X^*$ be a set-valued operator. T is maximal r-monotone if and only if for all $x \in X$ and $x^* \in X^*$ one has :*

$$\Big[\forall (y,y^*) \in \mathrm{G}(T), \Big(1 + r\langle x^*, y-x\rangle \le 0 \text{ and } 1 + r\langle y^*, x-y\rangle \le 0\Big) \\ \text{or } \Big(r\langle x^*, y-x\rangle\langle y^*, y-x\rangle + \langle y^* - x^*, y-x\rangle \ge 0\Big)\Big] \\ \iff (x,x^*) \in \mathrm{G}(T).$$

Proof. ($\Longleftarrow$) r-monotonicity of T follows directly from Lemma 2.2.7. Let $S : X \rightrightarrows X^*$ be a r-monotone extension of T and $(x,x^*) \in \mathrm{G}(S)$. Hence, by r-monotonicity of S and Lemma 2.2.7, one has :

$$\Big(1 + r\langle x^*, y-x\rangle \le 0 \text{ and } 1 + r\langle y^*, x-y\rangle \le 0\Big) \\ \text{or } \Big(r\langle x^*, y-x\rangle\langle y^*, y-x\rangle + \langle y^* - x^*, y-x\rangle \ge 0\Big)$$

for all $(y,y^*) \in \mathrm{G}(T) \subseteq \mathrm{G}(S)$. Therefore, $(x,x^*) \in \mathrm{G}(T)$. So, $S = T$.

($\Longrightarrow$) Let $x \in X$ and $x^* \in X^*$. Since T is r-monotone and by Lemma 2.2.7, it remains to show the implication :

$$\Big[\forall (y,y^*)\in \mathrm{G}(T), \Big(1+r\langle x^*, y-x\rangle \leq 0 \text{ and } 1+r\langle y^*, x-y\rangle \leq 0\Big)$$
$$\text{or } \Big(r\langle x^*, y-x\rangle\langle y^*, y-x\rangle + \langle y^*-x^*, y-x\rangle \geq 0\Big)\Big]$$
$$\Longrightarrow (x,x^*)\in \mathrm{G}(T).$$

For this, define $\bar{T}: X \rightrightarrows X^*$ by $\bar{T}(u) := T(u)$ for $u \neq x$ and $\bar{T}(x) := T(x)\cup\{x^*\}$. By maximality of T, we have : $\bar{T} = T$. Hence, $(x,x^*)\in \mathrm{G}(T)$. □

As maximal monotone operators, maximal r-monotone operators have interesting properties of values set, graph, zeros set and continuity. In the sequel, the weak-star topology is denoted by w^*.

Theorem 2.3.4. *Let $T: X \rightrightarrows X^*$ be a maximal r-monotone operator. Then,*
(i) $T(x)$ is w^*-closed and convex for all $x \in X$.
(ii) $\mathrm{G}(T)$ is sequentially $\|.\| \times w^*$-closed.
(iii) $\mathrm{G}(T)$ is $\|.\| \times \|.\|_*$-closed.
(iv) The zeros set $Z(T) := \{x \in X : 0 \in T(x)\}$ is convex.

Proof. (i) Let $x \in X$. By Lemma 2.3.3, we have : $T(x) = \bigcap_{(y,y^*)\in \mathrm{G}(T)} S(y,y^*,x)$. Here, if $1+r\langle y^*, x-y\rangle \leq 0$, we set :

$$S(y,y^*,x) := \{x^*\in X^* : r\langle x^*, y-x\rangle\langle y^*, y-x\rangle + \langle y^*-x^*, y-x\rangle \geq 0\}$$
$$\cup\, \{x^*\in X^* : 1+r\langle x^*, y-x\rangle \leq 0\}.$$

If not, $S(y,y^*,x) := \{x^*\in X^* : r\langle x^*, y-x\rangle\langle y^*, y-x\rangle + \langle y^*-x^*, y-x\rangle \geq 0\}$. Then the w^*-closedness of $T(x)$ follows from w^*-continuity of the maps $1+r\langle ., y-x\rangle$ and

$\phi(.) := r\langle ., y-x\rangle\langle y^*, y-x\rangle + \langle y^* - ., y-x\rangle$. Since ϕ is convex, it suffices to show the convexity of $T(x)$ by observing that $S(y,y^*,x)$ is convex in the case where $1 + r\langle y^*, x-y\rangle \leq 0$. Indeed, if $1 + r\langle y^*, x-y\rangle < 0$, we rewrite as follows :

$$S(y,y^*,x) = \{x^* \in X^* : \langle x^*, x-y\rangle \leq \frac{\langle y^*, x-y\rangle}{1 + r\langle y^*, x-y\rangle}\} \cup \{x^* \in X^* : \langle x^*, x-y\rangle \geq \frac{1}{r}\}.$$

Since $\frac{\langle y^*,x-y\rangle}{1+r\langle y^*,x-y\rangle} \geq \frac{1}{r}$, we have : $S(y,y^*,x) = X^*$. The remaining case where $1 + r\langle y^*, x-y\rangle = 0$ leads straightforwardly to $S(y,y^*,x) = X^*$.

(ii) Let two convergent sequences $x_m \xrightarrow[m\to\infty]{} x$ (w.r.t. $\|.\|$) and $x_m^* \xrightarrow[m\to\infty]{} x^*$ (w.r.t. w^*) such that $x_m^* \in T(x_m)$. Let $(y,y^*) \in \mathrm{G}(T)$. By Lemma 2.2.7, for all $m \in \mathbb{N}$:

$$\Big(1 + r\langle x_m^*, y - x_m\rangle \leq 0 \text{ and } 1 + r\langle y^*, x_m - y\rangle \leq 0\Big)$$
$$\text{or } \Big(r\langle x_m^*, y - x_m\rangle\langle y^*, y - x_m\rangle + \langle y^* - x_m^*, y - x_m\rangle \geq 0\Big).$$

As in the proof of Corollary 2.2.8 (i), we deduce :

$$\Big(1 + r\langle x^*, y - x\rangle \leq 0 \text{ and } 1 + r\langle y^*, x - y\rangle \leq 0\Big)$$
$$\text{or } \Big(r\langle x^*, y - x\rangle\langle y^*, y - x\rangle + \langle y^* - x^*, y - x\rangle \geq 0\Big).$$

Therefore, by Lemma 2.3.3, $(x,x^*) \in \mathrm{G}(T)$.

(iii) is proved as (ii) and by using that $\|.\| \times \|.\|_*$ is normed.

(iv) Let $x_0, x_1 \in Z(T)$, $\lambda \in (0,1)$, $x_\lambda := (1-\lambda)x_0 + \lambda x_1$ and $(y,y^*) \in \mathrm{G}(T)$. Clearly, by Lemma 2.2.7, $\langle y^*, y - x_i\rangle \geq 0$ $(i = 0,1)$. Thus, $\langle y^*, y - x_\lambda\rangle = (1-\lambda)(y-x_0) + \lambda(y-x_1) \geq 0$. Hence, the condition of Lemma 2.3.3 holds for fixed $(x_\lambda, 0)$ and every $(y,y^*) \in \mathrm{G}(T)$. So, by Lemma 2.3.3, $x_\lambda \in Z(T)$. □

Theorem 2.3.5. *Let $T : X \to X^*$ be a single-valued r-monotone map. Assume that the restriction of T to any finite dimensional affine subspace of X is $\|.\| \times w^*$-continuous. Then, T is maximal r-monotone. In particular, if T is continuous then T is maximal r-monotone.*

Proof. Let $x \in X$. Let $x^* \in X^*$ be such that, for every $y \in X$, one has :

$$\Big(1 + r\langle x^*, y - x\rangle \leq 0 \text{ and } 1 + r\langle y^*, x - y\rangle \leq 0\Big)$$
$$\text{or } \Big(r\langle x^*, y - x\rangle\langle T(y), y - x\rangle + \langle T(y) - x^*, y - x\rangle \geq 0\Big).$$

Let $h \in X$ and $m \in \mathbb{N}^*$ arbitrary. Then, for $y := x + \frac{1}{m}h$, we have :

$$\Big(1 + \frac{1}{m}r\langle x^*, h\rangle \leq 0 \text{ and } 1 - \frac{1}{m}r\langle y^*, h\rangle \leq 0\Big)$$
$$\text{or } \Big(\frac{1}{m^2}r\langle x^*, h\rangle\langle T(x + \frac{1}{m}h), h\rangle + \frac{1}{m}\langle T(x + \frac{1}{m}h) - x^*, h\rangle \geq 0\Big).$$

There exists m_0 such that for all $m \geq m_0$:

$$\frac{1}{m^2}r\langle x^*, h\rangle\langle T(x + \frac{1}{m}h), h\rangle + \frac{1}{m}\langle T(x + \frac{1}{m}h) - x^*, h\rangle \geq 0.$$

After multiplying by m and letting $m \longrightarrow +\infty$, we get : $\langle T(x) - x^*, h\rangle \geq 0$. Thus, $x^* = T(x)$. Whence T is maximal r-monotone by Lemma 2.3.3. □

Next, we define the novel notion of r-polar and present characterizations of r-monotone and maximal r-monotone operators using this polarity. The proofs are ommited since they are similar to those of [55].

Definition 2.3.6. Let $T : X \rightrightarrows X^*$ be a set-valued operator. The r-polar of T is the set-valued operator defined for every $x \in X$ and $x^* \in X^*$ by :

$$x^* \in T^r(x) \Longleftrightarrow \forall (y,y^*) \in \mathrm{G}(T), \Big(1 + r\langle x^*, y-x\rangle \leq 0 \text{ and } 1 + r\langle y^*, x-y\rangle \leq 0\Big) \\ \text{or } \Big(r\langle x^*, y-x\rangle\langle y^*, y-x\rangle + \langle y^*-x^*, y-x\rangle \geq 0\Big).$$

Proposition 2.3.7. *Let $T : X \rightrightarrows X^*$ be a set-valued operator. We have :*

(i) $T \mapsto T^r$ is a polarity [7].

(ii) $T \subseteq (T^r)^r$.

(iii) $((T^r)^r)^r = T^r$.

(iv) $T \subseteq S \Longrightarrow S^r \subseteq T^r$.

Proposition 2.3.8. *Let $T : X \rightrightarrows X^*$ be a set-valued operator. Then,*

(i) $T^r(x)$ is w^*-closed and convex for all $x \in X$.

(ii) $\mathrm{G}(T^r)$ is sequentially $\|.\| \times w^*$-closed.

(iii) $\mathrm{G}(T^r)$ is $\|.\| \times \|.\|_*$-closed.

(iv) $Z(T^r)$ is convex.

Proof. Very similar to the proof of Theorem 2.3.4. □

Proposition 2.3.9. *Let $T : X \rightrightarrows X^*$ be a r-monotone operator and $(x,x^*) \in X \times X^*$. Then, $(x,x^*) \in G(T^r) \Longleftrightarrow G(T) \cup \{(x,x^*)\}$ is r-monotone.*

Theorem 2.3.10. *Let $T : X \rightrightarrows X^*$ be a set-valued operator. The following assertions are equivalent :*

(i) T is r-monotone.

(ii) $T \subseteq T^r$.

(iii) $(T^r)^r \subseteq T^r$.

(iv) $(T^r)^r$ is r-monotone.

Theorem 2.3.11. *Let $T : X \rightrightarrows X^*$ be a set-valued operator. Then, T is maximal r-monotone if and only if $T = T^r$.*

This section ends up by a counter-example which shows that maximal r-monotone operators do not have in general the local boundedness property in the interior of the domain and compact values.

Example 2.3.12. Take $X := \mathbb{R}$, $T(t) := \frac{1}{rt}$ $(t \neq 0)$, $T(0) := \mathbb{R}$. At 0, any maximal r-monotone extension of T is not locally bounded and does not have compact value.

2.4 First-order criteria of r-monotone maps

In this section, we establish first-order characterizations of smooth and nonsmooth locally Lipschitz r-monotone maps. As application, we obtain new second-order characterizations of smooth and nonsmooth r-convex functions.

Throughout the section, $X := \mathbb{R}^n$ unless otherwise stated. Let $T : S \subseteq \mathbb{R}^n \to \mathbb{R}^n$ be a single-valued map with S nonempty, open and convex. When T is differentiable at $x \in S$, its Jacobian matrix at x is denoted by $T'(x)$. When T is nonsmooth locally Lipschitz at x, it is known [17, 54] that its Clarke generalized Jacobian at x is the convex hull of matrices given by :

$$\partial T(x) := \operatorname{conv}\left\{\lim_{i\to\infty} T'(x_i) : x_i \xrightarrow[i\to\infty]{} x \text{ and } T \text{ is differentiable at } x_i\right\}.$$

By abuse of notation, if T is differentiable at x, we set $\partial T(x) := T'(x)$.

In the sequel, as in [54], we put : $D_-T(x;v) := \inf\{\langle v, Av\rangle : A \in \partial T(x)\}$ and $D_+T(x;v) := \sup\{\langle v, Av\rangle : A \in \partial T(x)\}$ $((x,v) \in S \times \mathbb{R}^n)$. We will also consider the next assumptions where $(s,x,v,r,t) \in \mathbb{R} \times S \times \mathbb{R}^n \times (0,+\infty) \times (0,+\infty)$:

$$c(s,x,v,r,t) : \langle T(x), v\rangle = \tfrac{s}{tr} \Longrightarrow D_{-}T(x;v) \geq -\tfrac{s^2}{t^2 r}$$

$$\bar{c}(s,x,v,r,t) : \langle T(x), v\rangle = \tfrac{s}{tr} \Longrightarrow D_{+}T(x;v) \geq -\tfrac{s^2}{t^2 r}$$

$$a(s,x,v,r,t) : \Big(\langle T(x), v\rangle = \tfrac{s}{tr} \text{ and } -\tfrac{s^2}{t^2 r} \in \{\langle v, Av\rangle : A \in \partial T(x)\} \text{ and } \exists \hat{t} < 0 : \langle T(x+\hat{t}v), v\rangle > \tfrac{s}{(t+\hat{t}s)r}\Big) \Longrightarrow \exists \tilde{t} > 0, \forall \rho \in [0,\tilde{t}], \langle T(x+\rho v), v\rangle \geq \tfrac{s}{(t+\rho s)r}$$

$$\bar{a}(s,x,v,r,t) : \Big(\langle T(x), v\rangle = \tfrac{s}{tr} \text{ and } -\tfrac{s^2}{t^2 r} \in \{\langle v, Av\rangle : A \in \partial T(x)\}\Big) \Longrightarrow \exists \tilde{t} > 0, \forall \rho \in [0,\tilde{t}], \langle T(x+\rho v), v\rangle \geq \tfrac{s}{(t+\rho s)r}$$

$$h(s,x,v,r,t) : \Big(\langle T(x), v\rangle = \tfrac{s}{tr} \text{ and } -\tfrac{s^2}{t^2 r} \in \{\langle v, Av\rangle : A \in \partial T(x)\}\Big) \Longrightarrow \exists \tilde{t} > 0, \forall \rho \in [0,\tilde{t}], D_{-}T(x+\rho v; v) \geq -\tfrac{s^2}{t+\rho s^2 r}.$$

For $s = 0$, the parameters r and t are superfluous therefore will be ommitted in the notations.

For convenience, we recall the first-order characterizations of pseudomonotonicity and quasimonotonicity [47, 24, 54] in case of maps that never vanish. We express the statements using our notations.

Lemma 2.4.1 ([47, 24]). *Let $T : S \to \mathbb{R}^n$ be differentiable. Then, T is quasimonotone (resp. pseudomonotone) if and only if $a(0,x,v)$ (resp. $\bar{a}(0,x,v)$) and $c(0,x,v)$ hold for all $(x,v) \in S \times \mathbb{R}^n$. In addition, if T is continuously differentiable with $T(x) \neq 0$ for all $x \in S$, then T is pseudomonotone (resp. quasimonotone) if and only if $c(0,x,v)$ holds for all $(x,v) \in S \times \mathbb{R}^n$.*

Lemma 2.4.2 ([54]). *Let $T : S \to \mathbb{R}^n$ be locally Lipschitz with $T(x) \neq 0$ for all $x \in S$. Then,*

(i) T is quasimonotone (resp. pseudomonotone) if and only if $a(0,x,v)$ (resp. $\bar{a}(0,x,v)$) and $\bar{c}(0,x,v)$ hold for all $(x,v) \in S \times \mathbb{R}^n$.

(ii) If $c(0,x,v)$ or ($\bar{c}(0,x,v)$ and $h(0,x,v)$) hold for all $(x,v) \in S \times \mathbb{R}^n$ then T is pseudo-monotone.

The next lemma shows the separability of the Clarke generalized jacobian of the separable map defined by the separable cartesian product of locally Lipschitz maps $F : S \to \mathbb{R}^n$ and $G : U \to \mathbb{R}^p$, where $S \subseteq \mathbb{R}^n$ and $U \subseteq \mathbb{R}^p$ are nonempty open and convex sets. We make use a notation of set of block matrices :

$$\begin{pmatrix} \partial F(x) & 0 \\ 0 & \partial G(y) \end{pmatrix} := \left\{ \begin{pmatrix} A_F & 0 \\ 0 & A_G \end{pmatrix} : A_F \in \partial F(x), A_G \in \partial G(y) \right\}.$$

Lemma 2.4.3. *Let $F : S \to \mathbb{R}^n$ and $G : U \to \mathbb{R}^p$ be locally Lipschitz maps. Let $H(x,y) := (F(x), G(y))$ defined on $S \times U$. If G is continuously differentiable then the Clarke generalized Jacobian of H at $(x,y) \in S \times U$ is given by :*

$$\partial H(x,y) = \begin{pmatrix} \partial F(x) & 0 \\ 0 & \partial G(y) \end{pmatrix}. \tag{2.2}$$

Proof. Step 1 : Let $x \in S$ and $y \in U$. We have : F is differentiable at x and G is differentiable at y if and only if H is differentiable at (x,y). Furthermore, the Jacobian of H is formulated as :

$$H'(x,y) = \begin{pmatrix} F'(x) & 0 \\ 0 & G'(y) \end{pmatrix}. \tag{2.3}$$

Indeed, for the direct sense, take $(s,t) \in \mathbb{R}^n \times \mathbb{R}^p$. By definition of differentiability, one has : $H(x+s, y+t) = H(x,y) + (d_x F(s), d_y G(t)) + (\|s\|\alpha(s), \|t\|\beta(t))$. Here $d_a f(.)$ is the differential, $\alpha(s) \xrightarrow[s \to 0]{} 0$, $\beta(s) \xrightarrow[s \to 0]{} 0$ and $\|v\|$ is the supremum of coordinates of v. Thus, H is differentiable at (x,y) with $d_{(x,y)}H(s,t) = (d_xF(s), d_yG(t))$. Hence, we infer the relation (2.3). The converse follows by symmetry and by taking the first n components of the following equality : $(F(x+s), G(y)) = H(x,y) + d_{(x,y)}H(s,0) + \|s\|\epsilon(s,0)$ where $\epsilon(s,t) \xrightarrow[(s,t)\to 0]{} 0$.

Step 2 : By the very definition, F and G are locally Lipschitz if and only if H is also. Assume F and G are locally Lipschitz. Let us now show the equality (2.2). ($\subseteq$) Let $A \in \partial H(x,y)$. Thus, there exists coefficients $\{\lambda_j\}_{j=1,\ldots,m} \subseteq [0,1]$ with $\sum_{j=1}^m \lambda_j = 1$ and sequences $\{x_i^j, y_i^j\}_i \subseteq S \times U$ such that : $(x_i^j, y_i^j) \xrightarrow[i\to\infty]{} (x,y)$, H is differentiable at the points (x_i^j, y_i^j), $A = \sum_{j=1}^m \lambda_j A_j$ and $A_j = \lim_{i\to\infty} H'(x_i^j, y_i^j)$. By the Step1, F and G are differentiable at x_i^j and y_i^j respectively. From (2.3), we get :

$$H'(x_i^j, y_i^j) = \begin{pmatrix} F'(x_i^j) & 0 \\ 0 & G'(y_i^j) \end{pmatrix}.$$

Whence,

$$A = \begin{pmatrix} \sum_{j=1}^m \lambda_j \lim_{i\to\infty} F'(x_i^j) & 0 \\ 0 & \sum_{j=1}^m \lambda_j \lim_{i\to\infty} G'(y_i^j) \end{pmatrix}.$$

Then, $A \in \begin{pmatrix} \partial F(x) & 0 \\ 0 & \partial G(y) \end{pmatrix}$.

($\supseteq$) Recall that G is assummed continuously differentiable. Let

$$A \in \begin{pmatrix} \partial F(x) & 0 \\ 0 & \partial G(y) = \{G'(y)\} \end{pmatrix}$$

Therefore, one finds $\{\alpha_j\}_{j=1,\ldots l} \subseteq [0,1]$ with $\sum_{j=1}^l \alpha_j = 1$ and sequences $\{z_i^j\}_i$ such that : $z_i^j \xrightarrow[i\to\infty]{} x$, F is differentiable at z_i^j and $A = \begin{pmatrix} \sum_{j=1}^{l} \alpha_j A_j & 0 \\ 0 & G'(y) \end{pmatrix}$ with $A_j = \lim_{i\to\infty} F'(z_i^j)$. Thus, $A = \sum_{j=1}^{l} \lambda_j \lim_{i\to\infty} \begin{pmatrix} F'(z_i^j) & 0 \\ 0 & G'(y) \end{pmatrix}$. Using (2.3), we deduce that : $A \in \partial H(x,y)$.

□

The following results provide first-order characterizations of type of [47, 24, 54] for smooth and nonsmooth locally Lipschitz r-monotone maps in terms of Jacobian or Clarke generalized Jacobian. The particular case of affine maps is also treated.

Theorem 2.4.4. *Let $T : S \to \mathbb{R}^n$ be a differentiable map. Then, T is r-monotone if and only if $c(s,x,v,r,t)$ and $(a(s,x,v,r,t)$ or $\bar{a}(s,x,v,r,t))$ hold for all $(s,x,v,t) \in \mathbb{R}\times S\times\mathbb{R}^n\times(0,+\infty)$. In addition, if $T : S \to \mathbb{R}^n$ is continuously differentiable, then T is r-monotone if and only if $c(s,x,v,r,t)$ holds for all $(s,x,v,t) \in \mathbb{R}\times S\times\mathbb{R}^n\times(0,+\infty)$.*

Proof. By Definition 2.2.1 (resp. Remark 2.2.2), T is r-monotone if and only if T_r is pseudomonotone (resp. quasimonotone). On the other hand, by the proof of Lemma 2.4.3 and formula (2.3), T_r is differentiable on $S\times U$ with Jacobian given by :

$$T_r'(x,t) = \begin{pmatrix} T'(x) & 0 \\ 0 & \frac{1}{t^2 r} \end{pmatrix}.$$

The result follows by applying Lemma 2.4.1 to T_r and observing : $\langle T_r(x,t),(v,s)\rangle = \langle T(x),v\rangle - \frac{s}{tr}$ and $\langle T_r'(x,t)(v,s),(v,s)\rangle = \langle T'(x)v,v\rangle + \frac{s^2}{t^2 r}$.

The continuously differentiable case is obtained similarly after seeing that $T_r(x,t) \neq 0$ for all $(x,t) \in S\times(0,+\infty)$.

□

Corollary 2.4.5. *Let $T(x) := Ax+b$ be a non null affine map with $x \in S$, $A \in M_{n,n}(\mathbb{R})$ and $b \in \mathbb{R}^n$. Then, T is r-monotone if and only if for all $(s,x,v,t) \in \mathbb{R}\times S\times\mathbb{R}^n\times(0,+\infty)$ one has : $\langle Ax+b,v\rangle = \frac{s}{tr} \Longrightarrow \langle Av,v\rangle \geq -\frac{s^2}{t^2 r}$.*

Theorem 2.4.6. *Let $T : S \to \mathbb{R}^n$ be a locally Lipschitz map. Then,*
(i) T is r-monotone if and only if $\bar{c}(s,x,v,r,t)$ and ($a(s,x,v,r,t)$ or $\bar{a}(s,x,v,r,t)$) hold for all $(s,x,v,t) \in \mathbb{R}\times S\times\mathbb{R}^n\times(0,+\infty)$.
(ii) If $c(s,x,v,r,t)$ or ($\bar{c}(s,x,v,r,t)$ and $h(s,x,v,r,t)$) hold for all $(s,x,v,t) \in \mathbb{R}\times S\times\mathbb{R}^n\times(0,+\infty)$ then T is r-monotone.

Proof. According to Lemma 2.4.3, T_r is locally Lipschitz and its Clarke generalized Jacobian is expressed by : $\partial T_r(x,t) = \begin{pmatrix} \partial T(x) & 0 \\ 0 & \frac{1}{t^2 r} \end{pmatrix}$. To conclude, it remains to use Lemma 2.4.2 with T_r and remark that :

$$\{\langle (v,s), A(v,s)\rangle : A \in \partial T_r(x,t)\} = \{\langle v, Av\rangle + \frac{s^2}{t^2 r} : A \in \partial T(x)\},$$

$$D_+ T_r((x,t);(v,s)) = D_+ T(x;v) + \frac{s^2}{t^2 r},$$

$$D_- T_r((x,t);(v,s)) = D_- T(x;v) + \frac{s^2}{t^2 r}.$$

□

The next lemma characterizes r-convexity by pseudoconvexity of its logarithmic pertubation. Note that its direct implication is implicitly included in the proof of Theorem 5.3 [25]. The authors used rather complicated concept of convexity index [26, 29]. The proof that we give here is elementary and simpler.

Let us recall some definitions. Let X be a real Banach space and $f : X \to \mathbb{R} \cup \{+\infty\}$ be a function with domain $\mathrm{dom}(f)$. According to [5], f is r-convex if $f_r := \exp(rf)$ is convex. Let $I \subsetneq (0,+\infty)$ be a nonempty open interval. The logarithmic perturbation of f is the function defined by $g_r(x,t) := f(x) - \frac{\ln(t)}{r}$ for $(x,t) \in \mathrm{dom}(f) \times I$ and $g_r(x,t) := +\infty$ elsewhere. Given a function $g : X \to \mathbb{R} \cup \{+\infty\}$ with nonempty open convex domain and that admits one-sided directional derivatives $g'(.,.)$, we say that it is pseudoconvex if for all $x, y \in \mathrm{dom}(g)$, one has : $g(y) < g(x) \Longrightarrow g'(x, y-x) < 0$. If $f : S \to \mathbb{R}$ is real-valued with $S \subseteq X$ convex, f is called r-convex (resp. pseudoconvex) if and only if its extension $\bar{f}$ defined by $\bar{f}(x) := f(x)$ if $x \in S$ and $\bar{f}(x) := +\infty$ if $x \in X \setminus S$ is also.

Lemma 2.4.7. *Let X be a real Banach space. Let $f : X \to \mathbb{R} \cup \{+\infty\}$ be a function with nonempty, open and convex domain. We have :*
(i) f is r-convex $\Longrightarrow g_r$ is pseudoconvex.
(ii) Assume f is Gâteaux-differentiable and $I := (0,+\infty)$.
Then, g_r is pseudoconvex $\Longrightarrow f$ is r-convex.

Proof. (i) Let $(x,t),(y,s) \in \mathrm{dom}(g_r) = \mathrm{dom}(f) \times I$ with $g_r(y,s) < g_r(x,t)$. Since f_r is convex, f admits directional derivatives on $\mathrm{dom}(f)$. Moreover, we have : $f'(x,y-x) = \frac{f_r'(x,y-x)}{rf_r(x)} \leq \frac{f_r(y)-f_r(x)}{rf_r(x)}$. Also g_r admits directional derivatives given by : $g_r'((x,t),(y,s)-(x,t)) = \frac{f_r'(x,y-x)}{rf_r(x)} - \frac{s-t}{rt}$. Thus, $g_r'((x,t),(y,s)-(x,t)) \leq \frac{1}{r}\left(\frac{f_r(y)}{f_r(x)} - \frac{s}{t}\right)$. Then, by using the fact $g_r(y,s) < g_r(x,t)$, we get : $g_r'((x,t),(y,s)-(x,t)) < 0$.
(ii) Now the directional derivatives are considered two-sided. Since f is Gâteaux-differentiable, f_r and g_r are also. By contradiction suppose the existence of $x, y \in \mathrm{dom}(f)$ such that $rf_r(x)f'(x,y-x) = f_r'(x,y-x) > f_r(y) - f_r(x)$. Then, we deduce : $g_r'((x,f_r(x)),(y,f_r(y)) - (x,f_r(x))) = f'(x,y-x) - \frac{f_r(y)-f_r(x)}{rf_r(x)} > 0$. Now, since

g_r is pseudoconvex, it is quasiconvex. Then, we have :

$$0 = g_r(y, f_r(y)) > g_r(x, f_r(x)) = 0,$$

which is a contradiction. □

Remark 2.4.8. One can derive second-order characterizations of type of [47, 24, 54] for r-convex functions that are smooth or nonsmooth. Indeed, let $f : S \to \mathbb{R}$ be twice differentiable, twice continuously differentiable or Gâteaux-differentiable with locally Lipschitz gradient. By applying Lemma 2.4.7 to $\bar{f}$, we have : f is r-convex if and only if g_r is pseudoconvex. Since g_r is Gâteaux-differentiable on its domain $S \times \mathbb{R}$, the latter is equivalent to pseudomonotonicity of the gradient map $\nabla g_r(x,t) = (\nabla f(x), -\frac{1}{rt})((x,t) \in S \times (0,+\infty))$ [45]. In other words, $\nabla f(.)$ is r-monotone. We conclude by applying Theorem 2.4.4 or Theorem 2.4.6 to the operator $\nabla f(.)$.

It is known from the monotone operator theory that a differentiable map $T : S \to \mathbb{R}^n$ is monotone if and only if $T'(x)$ is positive semidefinite for every $x \in S$. In [54], Luc and Schaible proved the nonsmooth version : If T is locally Lipschitz then T is monotone if and only if A is positive semidefinite for every $x \in S$ and $A \in \partial T(x)$.
The forthcoming theorems provide the counterparts of these results for r-monotone maps. Before that, we will denote by u^t the transpose of vector $u \in \mathbb{R}^n$. We also recall the following characterization of r-convex functions [5].

Lemma 2.4.9 ([5])**.** *Let I be an open interval. Let $f : I \to \mathbb{R}$ be twice differentiable. Then, f is r-convex if and only if $f'' + rf'^2 \geq 0$.*

Theorem 2.4.10. *Let $T : S \to \mathbb{R}^n$ be a differentiable map. Then, T is r-monotone if and only if the matrix $T'(x) + rT(x)T(x)^t$ is positive semidefinite for every $x \in S$.*

Proof. First, we will prove the case $n = 1$, that is, $T : S \to \mathbb{R}$. Consider the function defined on S by : $f(t) := \int_{t_0}^{t} T(x)dx$ ($t_0 \in S$). Then, f is twice differentiable on S and satisfies $f' = T$. Thus, T is r-monotone if and only if f' is r-monotone. As in Remark 2.4.8, this is equivalent to f is r-convex. Then, by Lemma 2.4.9, $T'+rT^2 = f''+rf'^2 \geq 0$. Now since r-monotonicity is radial (Remark 2.2.4), the extension to $n \geq 2$ is possible. In this setting $\psi_{x,d}(t) = \langle T(x+td), d\rangle$ for all $x \in S$, $d \in \mathbb{R}^n$ and $t \in I_{x,d}$. Elementary computations show that $\psi_{x,d}$ is differentiable, $\psi'_{x,d}(t) = \langle T'(x+td)d, d\rangle$ and $r\psi^2_{x,d}(t) = \langle rT(x+td)(T(x+td))^t d, d\rangle$ ($t \in I_{x,d}$). By Remark 2.2.4, T is r-monotone if and only if $\psi_{x,d}$ is r-monotone for every $x \in S$ and $d \in \mathbb{R}^n$. This is equivalent to : $\langle [T'(x+td) + rT(x+td)(T(x+td))^t]d, d\rangle \geq 0$ for every $x \in S$, $d \in \mathbb{R}^n$and $t \in I_{x,d}$. That is, for all $x \in S$, $T'(x) + rT(x)T(x)^t$ is positive semidefinite. □

Remark 2.4.11. If T is r-monotone, differentiable at $\bar{x}$ and continuous on S then $T'(\bar{x}) + rT(\bar{x})T(\bar{x})^t$ is positive semidefinite. This follows from the precedent proof and the fact that a differentiable convex function $f : S \to \mathbb{R}$ that admits a second derivative at a point $t_0 \in S$ satisfies : $f''(t_0) \geq 0$.

Corollary 2.4.12. *Let $T(x) := Ax+b$ be a non null affine map with $x \in S$, $A \in M_{n,n}(\mathbb{R})$ and $b \in \mathbb{R}^n$. Then, T is r-monotone if and only if $A + r(Ax+b)(Ax+b)^t$ is positive semidefinite for every $x \in S$.*

Theorem 2.4.13. *Let $T : S \to \mathbb{R}^n$ be a locally Lipschitz map. Then, T is r-monotone if and only if $A + rT(x)T(x)^t$ is positive semidefinite for all $x \in S$ and $A \in \partial T(x)$.*

Proof. ($\Longrightarrow$) Let $x \in S$ and $A \in \partial T(x)$. One can write : $A = \sum_{j=1}^{m} \lambda_j A_j$ with $\{\lambda_j\}_{j=1,\dots m} \subseteq [0,1]$ such that $\sum_{j=1}^{m} \lambda_j = 1$ and $A_j = \lim_{i\to\infty} T'(x_i^j)$. Where, T is differentiable at x_i^j and $x_i^j \xrightarrow[i\to\infty]{} x$. Since T is continuous, $T(x_i^j)T(x_i^j)^t \xrightarrow[i\to\infty]{} T(x)T(x)^t$. Then, for every $d \in \mathbb{R}^n$, we have :

$$\langle [A + rT(x)T(x)^t]d, d\rangle = \lim_{i\to\infty} \sum_{j=1}^{m} \lambda_j \langle (T'(x_i^j) + rT(x_i^j)T(x_i^j)^t)d, d\rangle.$$

This is nonnegative by Remark 2.4.11.

($\Longleftarrow$) As the proof of Theorem 2.4.10, we begin by the case $n = 1$. Recall that $f(.) := \int_{t_0}^{.} T(x)dx$ ($t_0 \in S$). Then f is continuously differentiable with locally Lipschitz derivative with $f' = T$. By Remark 2.4.8, T is r-monotone if and only if f is r-convex, that is, f_r is convex. According to Corollary 3.6 in [43], this is equivalent to for all $t \in S$ and $z \in \partial^2 f_r(t)$, $z \geq 0$. Here $\partial_H^2 f_r := \partial f_r'$ is the generalized Hessian in the sense of Hiriart-Urruty et al. [40]. By the product rule, $\partial f_r'(t) = \partial(rf'f_r)(t) \subseteq rf_r(t)\partial f'(t) + rf'(t)\partial f_r(t) = rf_r(t)(\partial T(t) + rT^2(t))$. Thus, for all $t \in S$, every element of $\partial f_r'(t)$ is nonnegative. So, T is r-monotone. For $n \geq 2$, let $x \in S$ and $d \in \mathbb{R}^n$. Let $t \in I_{x,d}$. Using the chain rules of the Clarke generalized Jacobians, we get :

$$\partial \psi_{x,d}(t) = \langle \partial T(x + .d)(t), d\rangle (:= \{\langle B, d\rangle \,:\, B \in \partial T(x + .d)(t)\}).$$

and

$$\partial T(x + .d)(t) \subseteq \mathrm{conv}\,\{\partial T(x + td)d\}\,(:= \mathrm{conv}\,\{Cd : C \in \partial T(x + td)\}).$$

Let $A \in \partial\psi_{x,d}(t)$. Then,

$$A + r\psi_{x,d}^2(t) = \sum_{j=1}^{l} \alpha_j \langle (C_j + rT(x+td)T(x+td)^t)d, d\rangle.$$

with $C_j \in \partial T(x+td)$ and $\{\alpha_j\}_{j=1,\ldots l} \subseteq [0,1]$ are such that $\sum_{j=1}^{l} \alpha_j = 1$. Then, $A + r\psi_{x,d}^2(t) \geq 0$. Therefore, by the precedent step where $n = 1$, $\psi_{x,d}$ is r-monotone. So, T is r-monotone. □

In [43], Jeyakumar, Luc and Schaible have shown that a Gâteaux-differentiable function with locally Lipschitz gradient is convex if and only if all elements of its generalized Hessians (in the sense of Hiriart-Urruty et al.) are positive semidefinite. Next, we give a corresponding result for r-convex functions. The obtained second-order characterization extends a result of Avriel [5] (see Lemma 2.4.9 where $n = 1$) from continuously twice differentiable functions to nonsmooth functions.

Corollary 2.4.14. *Let $f : S \to \mathbb{R}$ be a Gâteaux-differentiable function with locally Lipschitz gradient. Then, f is r-convex if and only if $A + r\nabla f(x)\nabla f(x)^t$ is positive semidefinite for all $x \in S$ and $A \in \partial_H^2 f(x)$.*

Proof. From Remark 2.4.8, we have f is r-convex if and only if $\nabla f(.)$ is r-monotone on S. Therefore, by using Theorem 2.4.13 with $F := \nabla f(.)$, we obtain the announced result. □

We finish by stating first-order criteria of r-monotonicity involving Mordukhovich coderivatives for locally Lipschitz maps. As application, we provide a second-order characterization in terms of Mordukhovich second-order subdifferential of r-convex functions that are Fréchet-differentiable with locally

Lipschitz gradient. For the definitions and details on generalized differentiation notions, we refer to the books of Mordukhovich [60, 61, 59] and the articles [58, 63].

Let $T : \mathbb{R}^n \to \mathbb{R}^n$ be a locally Lipschitz map and $x \in \mathbb{R}^n$. The Mordukhovich coderivative of T at $(x, T(x))$ will be denoted by $D^*T(x) : \mathbb{R}^n \rightrightarrows \mathbb{R}^n$. We have [57] the following relationship with Clarke generalized Jacobian :

$$\forall v \in \mathbb{R}^n : (\partial T(x))^t v = co(D^*T(x)(v)) \tag{2.4}$$

where co denotes the convex hull and $(\partial T(x))^t := \{A^t : A \in \partial T(x)\}$.

Remark 2.4.15. Using (2.4), $\{\langle v, Av\rangle : A \in \partial T(x)\} = \{\langle u, v\rangle : u \in co(D^*T(x)(v))\}$. Then, $D_-T(x;v) = \inf\{\langle u, v\rangle : u \in D^*T(x)(v)\}$, $D_+T(x;v) = \sup\{\langle u, v\rangle : u \in D^*T(x)(v)\}$. Thus, Theorem 2.4.6 can be expressed using Mordukhovich coderivative instead of Clarke generalized Jacobian for $T : \mathbb{R}^n \to \mathbb{R}^n$ locally Lipschitz.

The next result is the r-monotone counterpart for locally Lipschitz maps of the coderivative characterizations of maximal monotonicity : see [16] (for $T : \mathbb{R}^n \to \mathbb{R}^n$ continuous) and [15, 62, 59] (for general set-valued operators).

Theorem 2.4.16. *Let $T : \mathbb{R}^n \to \mathbb{R}^n$ be a locally Lipschitz map. Then, T is r-monotone if and only if $v^t u + r(v^t T(x))^2 \geq 0$ for all $x, v \in \mathbb{R}^n$ and for all $u \in D^*T(x)(v)$.*

Proof. Follows from Theorem 2.4.13 and by using (2.4). □

Corollary 2.4.17. *Let $T : \mathbb{R}^n \to \mathbb{R}^n$ be a locally Lipschitz map. Then, T is monotone if and only if $v^t u \geq 0$ for all $x, v \in \mathbb{R}^n$ and for all $u \in D^*T(x)(v)$.*

Proof. Follows from Remark 2.2.5 and Theorem 2.4.16 □

Let $f : \mathbb{R}^n \to \mathbb{R}$ be Fréchet-differentiable with locally Lipschitz gradient and $x \in \mathbb{R}^n$. Let $\partial^2_M f(x) = D^*\nabla f(x)$ be the Mordukhovich second-order subdifferential (called also Mordukhovich generalized Hessian) of f at x relatively to $\nabla f(x)$. We now derive nonsmooth second-order characterization of r-convexity of f by means of Mordukhovich second-order subdifferential. See [15, 14, 16] for results in the setting of convex functions.

Corollary 2.4.18. *Let $f : \mathbb{R}^n \to \mathbb{R}$ be a Fréchet-differentiable function with locally Lipschitz gradient. Then, f is r-convex if and only if $v^t u + r(v^t \nabla f(x))^2 \geq 0$ for all $x, v \in \mathbb{R}^n$ and for all $u \in \partial^2_M f(x)(v)$.*

Proof. Follows from Theorem 2.4.16 applied for ∇f. □

Remark 2.4.19. Theorem 2.4.16, Corollary 2.4.17 and Corollary 2.4.18 are still valid if one replace $D^*T(x)(v)$ by its convex hull $co(D^*T(x)(v))$ or $\partial^2_M f(x)(v)$ by its convex hull $co(\partial^2_M f(x)(v))$.

Remark 2.4.20. From Remark 2.4.15 and Remark 2.4.8, one can deduce another kind of second-order characterization of Fréchet-differentiable with locally Lipschitz gradient r-convex functions $f : \mathbb{R}^n \to \mathbb{R}$ involving second-order subdifferential (in the sense of Mordukhovich).

Part II

Sequential vector optimization

Chapter 3

Sequential Pareto subdifferential composition rules and an application to sequential vector efficiency

3.1 Introduction

This chapter has been motivated by the contribution of Thibault [72] where the author formulated, in the absence of constraint qualifications, formulas for the subdifferential of the sum and the composition of convex functions in terms of the subdifferential of the data functions at nearby points.

In this chapter, we establish sequential formulas for the weak, proper and strong Pareto subdifferential of the convex composed operator $f + g \circ h$ where f, g and h are vector-valued convex mappings and g is nondecreasing. The approach that we will use is to reduce the calculus of Pareto subdifferential of the composed operator $f + g \circ h$ to that of the sum of convex functions via a scalarization theorem [33]. As an application of this result, sequential optimality conditions are obtained for a (weak, proper and strong) Pareto minimal point of a vector optimization problem with cone constraints. These sequential results are obtained without imposing qualification condition.

The organization is as follows. In section 3.2, we recall some notions and give some preliminary results. In section 3.3, we show the sequential formulas for the Pareto subdifferential of the convex composed operator $f + g \circ h$. In section 3.4, we derive from the obtained formulas the sequential efficiency optimality conditions for a vector cone-constrained optimization problem. The main results are presented in the setting of reflexive Banach spaces in order to avoid the use of nets.

The present chapter is the object of a publication [48].

3.2 Preliminaries

In this section, we give some basic definitions and results. Let X, Y and Z be real topological vector spaces and their respective continuous dual spaces be X^*, Y^* and Z^* with duality pairing denoted by $\langle .,. \rangle$. Let $Y_+ \subset Y$ and $Z_+ \subset Z$ be two nontrivial convex cones with $\text{int}_Y Y_+ \neq \emptyset$. Let $l(Y_+) = Y_+ \cap -Y_+$ be the lineality of Y_+, when it is nul, the cone Y_+ is said to be pointed. On Y, we define the following ordering relations

$$\begin{aligned} y_1 \quad \leq_{Y_+} \quad y_2 &\iff y_2 - y_1 \in Y_+, \\ y_1 \quad <_{Y_+} \quad y_2 &\iff y_2 - y_1 \in \text{int } Y_+, \\ y_1 \quad \lneq_{Y_+} \quad y_2 &\iff y_2 - y_1 \in Y_+ \backslash \text{l}(Y_+). \end{aligned}$$

To Y, we attach an abstract maximal element with respect to " $\leq_{Y_+}$ ", denoted by $+\infty_Y$. Then for every $y \in Y$ one has $y \leq_{Y_+} +\infty_Y$, and we consider the following operations

$$y + (+\infty_Y) = (+\infty_Y) + y = +\infty_Y, \quad \forall y \in Y \cup \{+\infty_Y\},$$

$$\alpha \cdot (+\infty_Y) = +\infty_Y, \quad \forall \alpha \geq 0.$$

The dual cone Y_+^* of Y_+ and the strict polar cone $(Y_+^*)^\circ$ of Y_+ are defined respectively as

$$Y_+^* := \{y^* \in Y^* : y^*(Y_+) \subseteq \mathbb{R}_+\}$$

and

$$(Y_+^*)^\circ := \{y^* \in Y^* : y^*(Y_+ \setminus l(Y_+)) \subseteq \mathbb{R}_+ \setminus \{0\}\}.$$

A mapping $f : X \longrightarrow Y \cup \{+\infty_Y\}$ is said to be

- Y_+-convex if for every $\lambda \in [0,1]$ and $x_1, x_2 \in X$

$$f(\lambda x_1 + (1-\lambda)x_2) \leq_{Y_+} \lambda f(x_1) + (1-\lambda) f(x_2).$$

- proper if its effective domain

$$\text{dom} f := \{x \in X : f(x) \in Y\} \neq \emptyset.$$

- Y_+-epi-closed if its epigraph

$$\text{Epi} f := \{(x,y) \in X \times Y : f(x) \leq_{Y_+} y\} \text{ is closed.}$$

- lower semicontinuous [50, 66] at $\bar{x} \in \text{dom f}$ if for any neighborhood V of $f(\bar{x})$ in Y, there exists a neighborhood U of $\bar{x}$ such that

$$f(U) \subseteq (V + Y_+) \cup \{+\infty_Y\}. \tag{3.1}$$

  When $f(\bar{x}) = +\infty_Y$, f is said to be lower semicontinuous at $\bar{x}$ if for any $y \in Y$, any neighborhood V of y, there exists a neighborhood U of $\bar{x}$ such that (3.1) is satisfied. f is said to be lower semicontinuous if it is lower semicontinuous at every point of X.

Let " $\leq_{Z_+}$ " be the partial order on Z induced by a nonempty convex cone $Z_+ \subset Z$. We say that a mapping $g : Z \longrightarrow Y \cup \{+\infty_Y\}$ is (Z_+, Y_+)-nondecreasing if for each $z_1, z_2 \in Z$ we have

$$z_1 \leq_{Z_+} z_2 \Longrightarrow g(z_1) \leq_{Y_+} g(z_2).$$

If $h : X \longrightarrow Z \cup \{+\infty_Z\}$, then the composed mapping $g \circ h : X \longrightarrow Y \cup \{+\infty_Y\}$ is defined by

$$(g \circ h)(x) := \begin{cases} g(h(x)) & \text{if } x \in \text{dom h} \\ \\ +\infty_Y & \text{otherwise} \end{cases}.$$

It is easy to see that if g is (Z_+, Y_+)-nondecreasing and Y_+-convex and h is Z_+-convex then $g \circ h$ is Y_+-convex.

Let $f : X \longrightarrow Y \cup \{+\infty_Y\}$ be a lower semicontinuous mapping. In the sequel, we shall need the lower semicontinuity of the function $y^* \circ f$ for any $y^* \in Y_+^* \setminus \{0\}$. For this, one can apply Proposition 3.1 in [50].

Now, we consider the vector minimization problem

$$\text{(P)} \quad \min_{x \in C} f(x)$$

where $f : X \longrightarrow Y \cup \{+\infty_Y\}$ is a mapping and C is a nonempty subset of X. A point $\bar{x} \in \text{dom} f \cap C$ is said to be

- a weak efficient solution (w-efficient) of (P) if

$$\nexists x \in C, \; f(x) <_{Y_+} f(\bar{x}).$$

- a proper efficient solution (p-efficient) of (P) if

$$\exists \hat{Y}_+ \subsetneq Y \text{ convex cone such that } Y_+ \setminus l(Y_+) \subseteq \text{int}\hat{Y}_+, \nexists x \in C, \; f(x) \lneq_{\hat{Y}_+} f(\bar{x}).$$

- a strong efficient solution (s-efficient) of (P) if $\forall x \in C, \; f(\bar{x}) \leq_{Y_+} f(x)$.

The sets of weak, proper and strong efficient points will be denoted respectively by $\text{E}_w(f, C)$, $\text{E}_p(f, C)$ and $\text{E}_s(f, C)$.

These notions enable us to define the weak, proper and strong subdifferential of a mapping $f : X \longrightarrow Y \cup \{+\infty_Y\}$ at $\bar{x} \in$ dom f as follows

- weak subdifferential

$$\partial^w f(\bar{x}) := \{A \in \text{L}(X, Y) : \nexists x \in X, \, f(x) - f(\bar{x}) <_{Y_+} A(x - \bar{x})\},$$

- proper subdifferential

$$\partial^p f(\bar{x}) := \{A \in \mathrm{L}(X,Y) : \exists \hat{Y}_+ \subsetneq Y \text{ convex cone such that}$$
$$Y_+ \setminus l(Y_+) \subseteq \operatorname{int} \hat{Y}_+,\ \nexists x \in X,\ f(x) - f(\bar{x}) \lneq_{\hat{Y}_+} A(x - \bar{x})\},$$

- strong subdifferential

$$\partial^s f(\bar{x}) := \{A \in \mathrm{L}(X,Y)\ :\ \forall x \in X,\ A(x - \bar{x}) \leq_{Y_+} f(x) - f(\bar{x})\},$$

where $\mathrm{L}(X,Y)$ stands for the space of linear continuous operators from X to Y. In the sequel, for simplicity, we shall regroup in one notation $\mathrm{E}_\sigma(f,C)$, σ-efficient solution, $\partial^\sigma f(\bar{x})$ for $\sigma \in \{w,p,s\}$ and

$$Y_+^\sigma := \begin{cases} Y_+^* \setminus \{0\} & \text{if } \sigma \in \{w,s\} \\ (Y_+^*)^\circ & \text{if } \sigma = p \end{cases}.$$

An important property follows immediately from the above definitions

$$\bar{x} \in E_\sigma(f,X) \Longleftrightarrow 0 \in \partial^\sigma f(\bar{x}) \quad (\sigma \in \{w,p,s\}).$$

When f is a real-valued function, the subdifferential of f at $\bar{x} \in \mathrm{dom\ f}$ is given by

$$\partial f(\bar{x}) := \{x^* \in X^* : f(x) - f(\bar{x}) \geq \langle x^*, x - \bar{x} \rangle,\ \forall x \in X\}.$$

The vector indicator mapping of a nonempty subset $C \subset X$, denoted by δ_C^v, is defined as $\delta^v_C : X \longrightarrow Y \cup \{+\infty_Y\}$

$$\delta^v_C(x) = \begin{cases} 0 & \text{if } x \in C \\ +\infty_Y & \text{otherwise} \end{cases}.$$

The vector indicator mapping appears to possess properties like the scalar one. The normal cone of C is defined as

$$N_C(\bar{x}) := \{x^* \in X^* : \langle x^*, x - \bar{x}\rangle \leq 0, \quad \forall x \in C\}.$$

The following scalarization result will be needed.

Theorem 3.2.1. *([33]) Let X, Y be two Hausdorff topological vector spaces and $f : X \to Y \cup \{+\infty_Y\}$ be an Y_+-convex mapping. Let $\sigma \in \{w, p, s\}$ and $\bar{x} \in X$, then Case $\sigma \in \{w, p\}$ with Y_+ pointed as $\sigma = p$:*

$$\partial^\sigma f(\bar{x}) = \bigcup_{y^* \in Y^\sigma_+} \{A \in L(X, Y) : y^* \circ A \in \partial(y^* \circ f)(\bar{x})\}.$$

Case $\sigma = s$ and Y_+ is closed :

$$\partial^s f(\bar{x}) = \bigcap_{y^* \in Y^s_+} \{A \in L(X, Y) : y^* \circ A \in \partial(y^* \circ f)(\bar{x})\}.$$

Let us recall a result due to Thibault [72], establishing in the absence of any constraint qualification, a sequential general formula for the subdifferential of the sum of two proper lower semicontinuous convex functions defined on a reflexive Banach space. Let $(X, \|.\|_X)$ be a real reflexive Banach space and $(X^*, \|.\|_{X^*})$ its topological dual space. Let $(x_n)_{n\in\mathbb{N}}$ be a sequence in X (resp. $(x^*_n)_{n\in\mathbb{N}}$

be a sequence in X^*) and $x \in X$ (resp. $x^* \in X^*$). We write $x_n \xrightarrow{\|\cdot\|_X} x$ (resp. $x_n^* \xrightarrow{\|\cdot\|_{X^*}} x^*$) if $\|x_n - x\|_X \xrightarrow[n\to\infty]{} 0$ (resp. $\|x_n^* - x^*\|_{X^*} \xrightarrow[n\to\infty]{} 0$) .

Theorem 3.2.2. *([72]) Let X be a real reflexive Banach space and $f_1, f_2 : X \longrightarrow \mathbb{R} \cup \{+\infty\}$ be two proper convex and lower semicontinuous functions on X. Let $\bar{x} \in \mathrm{dom\ f_1} \cap \mathrm{dom\ f_2}$. Then, $x^* \in \partial(f_1 + f_2)(\bar{x})$ if and only if there exist $(x_{1,n}, x_{2,n}) \in \mathrm{dom\ f_1} \times \mathrm{dom\ f_2}$, $x_{1,n}^* \in \partial f_1(x_{1,n})$ and $x_{2,n}^* \in \partial f_2(x_{2,n})$ such that*

$$\begin{cases} x_{1,n}^* + x_{2,n}^* \xrightarrow{\|\cdot\|_{X^*}} x^*, \ x_{1,n} \xrightarrow{\|\cdot\|_X} \bar{x}, \ x_{2,n} \xrightarrow{\|\cdot\|_X} \bar{x}, \\ \\ f_1(x_{1,n}) - \langle x_{1,n}^*, x_{1,n} - \bar{x} \rangle - f_1(\bar{x}) \xrightarrow[n\to\infty]{} 0, \\ \\ f_2(x_{2,n}) - \langle x_{2,n}^*, x_{2,n} - \bar{x} \rangle - f_2(\bar{x}) \xrightarrow[n\to\infty]{} 0. \end{cases}$$

Remark 3.2.3. The above theorem holds if we take the convergence of the sequence $x_{1,n}^* + x_{2,n}^* \xrightarrow{\|\cdot\|_{X^*}} x^*$ with respect to the weak star convergence $\sigma(X^*, X)$ instead of the norm convergence $\|.\|_{X^*}$.

3.3 Sequential Pareto subdifferential composition rules

In this section we state the sequential composition rules for (weak, proper and strong) Pareto subdifferential. The approach that we will use for computing the sequential Pareto subdifferential of the composed convex mapping is to transform it as the sequential Pareto subdifferential of the sum of two convex functions. In what follows $(X,\|.\|_X)$ and $(Z,\|.\|_Z)$ stand for two real reflexive Banach spaces, $(X^*,\|.\|_{X^*})$, $(Z^*,\|.\|_{Z^*})$ are their respective topological dual spaces and $(Y,\|.\|_Y)$ is a real normed vector space. On $X\times Z$ we use the norm $\|(x,z)\| = \sqrt{\|x\|^2+\|z\|^2}$ for $(x,z)\in X\times Z$. Similarly we define the norm on $X^*\times Z^*$. Let $f:X\longrightarrow Y\cup\{+\infty_Y\}$ be a proper and Y_+-convex mapping, $h:X\longrightarrow Z\cup\{+\infty_Z\}$ be a proper and Z_+-convex mapping, and $g:Z\longrightarrow Y\cup\{+\infty_Y\}$ be a proper and Y_+-convex mapping. Let us consider the following auxiliary mappings

$$\begin{aligned} F_1: \quad X\times Z &\longrightarrow Y\cup\{+\infty_Y\}\\ (x,z) &\longrightarrow F_1(x,z) := f(x)+g(z), \end{aligned}$$

$$\begin{aligned} F_2: \quad X\times Z &\longrightarrow Y\cup\{+\infty_Y\}\\ (x,z) &\longrightarrow F_2(x,z) := \delta^v_{\mathrm{Epi}h}(x,z). \end{aligned}$$

In order to compute $\partial(y^*\circ F_i)$ $(i=1,2)$ by means of data mappings f,g and h, we will need the following lemma.

Lemma 3.3.1. *Let* $(\bar{x},\bar{z})\in(\mathrm{dom}f\times\mathrm{dom}g)\cap\mathrm{Epi}h$ *and* $y^*\in Y^*_+$. *We have*

1. $(x^*,z^*)\in\partial(y^*\circ F_1)(\bar{x},\bar{z}) \Longleftrightarrow x^*\in\partial(y^*\circ f)(\bar{x})$ *and* $z^*\in\partial(y^*\circ g)(\bar{z})$
2. $(x^*,z^*)\in\partial(y^*\circ F_2)(\bar{x},\bar{z}) \Longleftrightarrow \begin{cases} -z^*\in Z^*_+\\ x^*\in\partial(-z^*\circ h)(\bar{x})\\ \langle z^*,\bar{z}-h(\bar{x})\rangle = 0\end{cases}$

Proof. (1) ($\Longrightarrow$) By using the variational form of $(x^*, z^*) \in \partial(y^* \circ F_1)(\bar{x}, \bar{z})$ i.e.

$$(y^* \circ f)(x) + (y^* \circ g)(z) \geq (y^* \circ f)(\bar{x}) + (y^* \circ g)(\bar{z}) + \langle x^*, x - \bar{x}\rangle + \langle z^*, z - \bar{z}\rangle, \quad \forall (x,z) \in X \times Z$$

and by letting respectively $x := \bar{x}$ and $z := \bar{z}$ we obtain $x^* \in \partial(y^* \circ f)(\bar{x})$ and $z^* \in \partial(y^* \circ g)(\bar{z})$. The converse ($\Longleftarrow$) is easily obtained by adding the variational inequalities of the two subdifferentials $\partial(y^* \circ g)(\bar{z})$ and $\partial(y^* \circ f)(\bar{x})$.
(2) ($\Longrightarrow$) By using the fact that $y^* \circ \delta^v_{\mathrm{Epi}h} = \delta_{\mathrm{Epi}h}$, we have

$$(x^*, z^*) \in \partial(y^* \circ F_2)(\bar{x}, \bar{z}) \Longleftrightarrow \delta_{\mathrm{Epi}h}(x,z) \geq \langle x^*, x - \bar{x}\rangle + \langle z^*, z - \bar{z}\rangle,\ \forall (x,z) \in X \times Z. \quad (3.2)$$

Let $v \in Z_+$ and put $z := v + \bar{z}$. It is clear that $(\bar{x}, z) \in \mathrm{Epi}h$ and hence by taking $x := \bar{x}$, we get from (3.2) that $-z^* \in Z^*_+$. We check that $x^* \in \partial(-z^* \circ h)(\bar{x})$. For this, let $x \in \mathrm{dom}h$ and set $z := h(x)$ in (3.2), then we obtain $0 \geq \langle x^*, x - \bar{x}\rangle + \langle z^*, h(x) - \bar{z}\rangle$. Since $-z^* \in Z^*_+$ and $(\bar{x}, \bar{z}) \in \mathrm{Epi}h$, it follows that $(-z^* \circ h)(\bar{x}) \leq \langle -z^*, \bar{z}\rangle$ and therefore we get

$$(-z^* \circ h)(x) - (-z^* \circ h)(\bar{x}) \geq \langle x^*, x - \bar{x}\rangle, \quad \forall x \in \mathrm{dom}h$$

and by using the convention $(-z^*)(+\infty_Z) = +\infty$, we obtain $x^* \in \partial(-z^* \circ h)(\bar{x})$. Now, it remains to prove that $\langle z^*, \bar{z} - h(\bar{x})\rangle = 0$. Let $w \in Z_+$. Since $(\bar{x}, w + h(\bar{x})) \in \mathrm{Epi}h$ and by letting $x := \bar{x}$ and $z := w + h(\bar{x})$ in (3.2), we get $\langle z^*, w - \bar{z} + h(\bar{x})\rangle \leq 0$, for any $w \in Z_+$ which yields that $z^* \in N_{Z_+}(\bar{z} - h(\bar{x}))$. From the fact that Z_+ is a convex cone, we get easily

$$z^* \in N_{Z_+}(\bar{z} - h(\bar{x})) \Longleftrightarrow \begin{cases} -z^* \in Z_+^* \\ \langle z^*, \bar{z} - h(\bar{x}) \rangle = 0 \end{cases}.$$

Conversely ($\Longleftarrow$). Let $x^* \in \partial(-z^* \circ h)(\bar{x})$ i.e.

$$\langle x^*, x - \bar{x} \rangle \leq (-z^* \circ h)(x) + (z^* \circ h)(\bar{x}), \quad \forall x \in X. \tag{3.3}$$

Let $z^* \in -Z_+^*$ such that $\langle z^*, \bar{z} - h(\bar{x}) \rangle = 0$, since Z_+ is a convex cone, it follows that

$$\langle z^*, z - \bar{z} + h(\bar{x}) \rangle \leq 0, \quad \forall z \in Z_+. \tag{3.4}$$

Let $(x, w) \in \text{Epi}h$. Since $w - h(x) \in Z_+$ and by letting $z := w - h(x)$ in (3.4), we obtain

$$\langle z^*, w - \bar{z} \rangle \leq (z^* \circ h)(x) - (z^* \circ h)(\bar{x}), \quad \forall (x, w) \in \text{Epi}h. \tag{3.5}$$

By summing the two inequalities (3.3) and (3.5), we obtain

$$\langle x^*, x - \bar{x} \rangle + \langle z^*, w - \bar{z} \rangle \leq 0, \qquad \forall (x, w) \in \text{Epi}h$$

and since $(\bar{x}, \bar{z}) \in \text{Epi}h$, it results that

$$\langle x^*, x - \bar{x} \rangle + \langle z^*, w - \bar{z} \rangle \leq \delta_{\text{Epi}h}(x, w) - \delta_{\text{Epi}h}(\bar{x}, \bar{z}), \quad \forall (x, w) \in X \times Z$$

i.e. $(x^*, z^*) \in \partial \delta_{\text{Epi}h}(\bar{x}, \bar{z})$. By using the fact that $y^* \circ \delta^v_{\text{Epi}h} = \delta_{\text{Epi}h}$, we obtain the desired result $(x^*, z^*) \in \partial(y^* \circ \delta^v_{\text{Epi}h})(\bar{x}, \bar{z})$. □

Now, we can state our main result.

Theorem 3.3.2. *Let* $f : X \longrightarrow Y \cup \{+\infty_Y\}$ *be proper,* Y_+*-convex and lower semicontinuous mapping,* $h : X \longrightarrow Z \cup \{+\infty_Z\}$ *be proper,* Z_+*-convex and* Z_+*-epi-closed mapping, and* $g : Z \longrightarrow Y \cup \{+\infty_Y\}$ *be proper,* Y_+*-convex, lower semicontinuous and* (Z_+, Y_+)*-nondecreasing mapping. Let* $\bar{x} \in \mathrm{dom} f \cap \mathrm{dom} h \cap h^{-1}(\mathrm{dom} g)$ *and* $\sigma \in \{w, p, s\}$*. Assume* Y_+ *is pointed as* $\sigma = p$ *(resp.* Y_+ *closed as* $\sigma = s$*). Then,* $A \in \partial^\sigma (f + g \circ h)(\bar{x})$ *if and only if there exists (resp. for all as* $\sigma = s$*)* $y^* \in Y_+^\sigma$*, there exist* $(x_{1,n}, z_{1,n}) \in \text{dom f} \times \text{dom g}$*,* $(x_{2,n}, z_{2,n}) \in X \times Z$*,* $(x^*_{i,n}, z^*_{i,n}) \in X^* \times Z^*$ $(i = 1, 2)$ *satisfying*

$$\begin{cases} x_{i,n} \xrightarrow{\|\cdot\|_X} \bar{x},\ z_{i,n} \xrightarrow{\|\cdot\|_Z} h(\bar{x}),\ h(x_{2,n}) \leq_{Z_+} z_{2,n}\ (i = 1, 2), \\ x^*_{1,n} \in \partial(y^* \circ f)(x_{1,n}),\ z^*_{1,n} \in \partial(y^* \circ g)(z_{1,n}),\ \ x^*_{2,n} \in \partial(-z^*_{2,n} \circ h)(x_{2,n}), \\ z^*_{2,n} \in -Z^*_+, \quad \langle z^*_{2,n}, z_{2,n} - h(x_{2,n}) \rangle = 0, \end{cases}$$

such that

$$\begin{cases} x^*_{1,n} + x^*_{2,n} \xrightarrow{\|\cdot\|_{X^*}} y^* \circ A \quad \textit{and} \quad z^*_{1,n} + z^*_{2,n} \xrightarrow{\|\cdot\|_{Z^*}} 0, \\ \\ (y^* \circ f)(x_{1,n}) - \langle x^*_{1,n}, x_{1,n} - \bar{x} \rangle \xrightarrow[n\to\infty]{} (y^* \circ f)(\bar{x}), \\ \\ (y^* \circ g)(z_{1,n}) - \langle z^*_{1,n}, z_{1,n} - h(\bar{x}) \rangle \xrightarrow[n\to\infty]{} (y^* \circ g)(h(\bar{x})), \\ \\ \langle x^*_{2,n}, x_{2,n} - \bar{x} \rangle + \langle z^*_{2,n}, z_{2,n} - h(\bar{x}) \rangle \xrightarrow[n\to\infty]{} 0. \end{cases}$$

Proof. Let $A \in \partial^\sigma (f + g \circ h)(\bar{x})$. By applying the scalarization Theorem 3.2.1, there exists (resp. for all as $\sigma = s$) $y^* \in Y_+^\sigma$:

$$y^* \circ A \in \partial (y^* \circ f + y^* \circ g \circ h)(\bar{x}). \tag{3.6}$$

By introducing the scalar indicator function $\delta_{\mathrm{Epi}h}$ and using the monotonicity of the mapping g and the fact that $y^* \circ \delta^v_{\mathrm{Epi}h} = \delta_{\mathrm{Epi}h}$, it follows that (3.6) becomes equivalent to

$$(y^* \circ A, 0) \in \partial (y^* \circ F_1 + y^* \circ F_2)(\bar{x}, h(\bar{x})).$$

It is easy to check that F_1 and F_2 are lower semicontinuous, Y_+-convex and proper on $X \times Z$ and as y^* is Y_+–nondecreasing, it follows that the scalar functions $y^* \circ F_1$ and $y^* \circ F_2$ are proper, convex and lower semicontinuous. Let us note that $\mathrm{dom}\ \mathrm{F_i} = \mathrm{dom}\ (\mathrm{y^*} \circ \mathrm{F_i}) \quad (i = 1, 2)$ and the condition $\bar{x} \in \mathrm{dom} f \cap \mathrm{dom} h \cap h^{-1}(\mathrm{dom} g)$ is equivalent to $(\bar{x}, h(\bar{x})) \in \mathrm{dom}\ (\mathrm{y^*} \circ \mathrm{F_1}) \cap \mathrm{dom}\ (\mathrm{y^*} \circ \mathrm{F_2})$. The functions $y^* \circ F_1$ and $y^* \circ F_2$ satisfy all the assumptions of Theorem 3.2.2 and therefore there exist $(x_{1,n}, z_{1,n}) \in \mathrm{dom}\ \mathrm{F_1} = \mathrm{dom\ f} \times \mathrm{dom\ g}$, $(x_{2,n}, z_{2,n}) \in \mathrm{dom}\ \mathrm{F_2} = \mathrm{Epi}h$, $(x^*_{i,n}, z^*_{i,n}) \in$

$X^* \times Z^*$ $(i = 1,2)$ satisfying

$$(x_{i,n}, z_{i,n}) \xrightarrow{\|\cdot\|_{X\times Z}} (\bar{x}, h(\bar{x}))\ (i = 1,2), \tag{3.7}$$

$$(x^*_{i,n}, z^*_{i,n}) \in \partial(y^* \circ F_i)(x_{i,n}, z_{i,n})\ (i = 1,2)\ (n \in \mathbb{N}), \tag{3.8}$$

$$(y^* \circ F_i)(x_{i,n}, z_{i,n}) - \langle (x^*_{i,n}, z^*_{i,n}), (x_{i,n}, z_{i,n}) - (\bar{x}, h(\bar{x}))\rangle \xrightarrow[n\to\infty]{} (y^* \circ F_i)(\bar{x}, h(\bar{x}))(i = 1,2), \tag{3.9}$$

$$(x^*_{1,n}, z^*_{1,n}) + (x^*_{2,n}, z^*_{2,n}) \xrightarrow{\|\cdot\|_{X^*\times Z^*}} (y^* \circ A, 0). \tag{3.10}$$

It is obvious that (3.7) is equivalent to $x_{i,n} \xrightarrow{\|\cdot\|_X} \bar{x}$ and $z_{i,n} \xrightarrow{\|\cdot\|_Z} h(\bar{x})$. By using Lemma 3.3.1, (3.8) may be rewritten as

$$\begin{cases} (x^*_{1,n}, z^*_{1,n}) \in \partial(y^* \circ F_1)(x_{1,n}, z_{1,n}) \Longleftrightarrow \begin{cases} x^*_{1,n} \in \partial(y^* \circ f)(x_{1,n}) \\ z^*_{1,n} \in \partial(y^* \circ g)(z_{1,n}) \end{cases} \\ (x^*_{2,n}, z^*_{2,n}) \in \partial(y^* \circ F_2)(x_{2,n}, z_{2,n}) \Longleftrightarrow \begin{cases} -z^*_{2,n} \in Z^*_+ \\ x^*_{2,n} \in \partial(-z^*_{2,n} \circ h)(x_{2,n}) \\ \langle z^*_{2,n},\ z_{2,n} - h(x_{2,n})\rangle = 0 \end{cases} \end{cases}.$$

By means of the data mappings, the expression

$$(y^* \circ F_1)(x_{1,n}, z_{1,n}) - \langle (x^*_{1,n}, z^*_{1,n}), (x_{1,n}, z_{1,n}) - (\bar{x}, h(\bar{x}))\rangle \xrightarrow[n\to\infty]{} (y^* \circ F_1)(\bar{x}, h(\bar{x}))$$

can be expressed as

$$[(y^* \circ f)(x_{1,n}) - \langle x^*_{1,n}, x_{1,n} - \bar{x}\rangle - (y^* \circ f)(\bar{x})] \\ + [(y^* \circ g)(z_{1,n}) - \langle z^*_{1,n}, z_{1,n} - h(\bar{x})\rangle - (y^* \circ g)(h(\bar{x}))] \xrightarrow[n\to\infty]{} 0.$$

Since $x^*_{1,n} \in \partial(y^* \circ f)(x_{1,n})$ and $z^*_{1,n} \in \partial(y^* \circ g)(z_{1,n})$, hence it follows that the two terms $[(y^* \circ f)(x_{1,n}) - \langle x^*_{1,n}, x_{1,n} - \bar{x}\rangle - (y^* \circ f)(\bar{x})]$ and $[(y^* \circ g)(z_{1,n}) - \langle z^*_{1,n}, z_{1,n} - h(\bar{x})\rangle - (y^* \circ g)(h(\bar{x}))]$ are both nonpositives and therefore the condition (3.9) for $i = 1$ becomes equivalent to

$$\begin{cases} (y^* \circ f)(x_{1,n}) - \langle x^*_{1,n}, x_{1,n} - \bar{x}\rangle \xrightarrow[n\to\infty]{} (y^* \circ f)(\bar{x}). \\ (y^* \circ g)(z_{1,n}) - \langle z^*_{1,n}, z_{1,n} - h(\bar{x})\rangle \xrightarrow[n\to\infty]{} (y^* \circ g)(h(\bar{x})). \end{cases}$$

Since $(x_{2,n}, z_{2,n}) \in \text{Epi}h$ and $(\bar{x}, \bar{z}) \in \text{Epi}h$, the following expression

$$(y^* \circ F_2)(x_{2,n}, z_{2,n}) - \langle (x^*_{2,n}, z^*_{2,n}), (x_{2,n}, z_{2,n}) - (\bar{x}, h(\bar{x}))\rangle - (y^* \circ F_2)(\bar{x}, h(\bar{x})) \xrightarrow[n\to\infty]{} 0$$

becomes equivalent to $\langle x^*_{2,n}, x_{2,n} - \bar{x}\rangle + \langle z^*_{2,n}, z_{2,n} - h(\bar{x})\rangle \xrightarrow[n\to\infty]{} 0$. It is immediate that (3.10) is equivalent to

$$x^*_{1,n} + x^*_{2,n} \xrightarrow{\|\cdot\|_{X^*}} y^* \circ A \quad \text{and} \quad z^*_{1,n} + z^*_{2,n} \xrightarrow{\|\cdot\|_{Z^*}} 0,$$

which completes the proof. □

Consider now the case of composition with a continuous linear operator $L : X \longrightarrow Z$ and let $g : Z \longrightarrow Y \cup \{+\infty_Y\}$ be a proper and Y_+-convex mapping. Put

$Z_+ = \{0_Z\}$, obviously the mapping g is (Z_+, Y_+)-nondecreasing. So, by applying Theorem 3.3.2, one gets the following result.

Corollary 3.3.3. *Let $f : X \longrightarrow Y \cup \{+\infty_Y\}$ be proper, Y_+-convex and lower semicontinuous mapping, $L : X \longrightarrow Z$ be a continuous linear operator and $g : Z \longrightarrow Y \cup \{+\infty_Y\}$ be proper, Y_+-convex and lower semicontinuous mapping. Let $\bar{x} \in \mathrm{dom} f \cap L^{-1}(\mathrm{dom} g)$ and $\sigma \in \{w, p, s\}$. Assume Y_+ is pointed as $\sigma = p$ (resp. Y_+ closed as $\sigma = s$). Then, $A \in \partial^\sigma (f + g \circ L)(\bar{x})$ if and only if there exists (resp. for all as $\sigma = s$) $y^* \in Y_+^\sigma$, there exist $(x_{1,n}, z_{1,n}) \in \mathrm{dom\ f} \times \mathrm{dom\ g}$, $x_{2,n} \in X$, $(x^*_{i,n}, z^*_{i,n}) \in X^* \times Z^*$ $(i = 1, 2)$ satisfying*

$$\begin{cases} x_{i,n} \xrightarrow{\|\cdot\|_X} \bar{x},\ z_{1,n} \xrightarrow{\|\cdot\|_Z} L(\bar{x})\ (i = 1, 2), \\ \\ x^*_{1,n} \in \partial(y^* \circ f)(x_{1,n}),\ z^*_{1,n} \in \partial(y^* \circ g)(z_{1,n}), \end{cases}$$

such that

$$\begin{cases} x^*_{1,n} - L^*(z^*_{2,n}) \xrightarrow{\|\cdot\|_{X^*}} y^* \circ A \quad \text{and} \quad z^*_{1,n} + z^*_{2,n} \xrightarrow{\|\cdot\|_{Z^*}} 0, \\ \\ (y^* \circ f)(x_{1,n}) - \langle x^*_{1,n}, x_{1,n} - \bar{x} \rangle \xrightarrow[n\to\infty]{} (y^* \circ f)(\bar{x}), \\ \\ (y^* \circ g)(z_{1,n}) - \langle z^*_{1,n}, z_{1,n} - L(\bar{x}) \rangle \xrightarrow[n\to\infty]{} (y^* \circ g)(L(\bar{x})), \end{cases}$$

where $L^* : Z^* \longrightarrow X^*$ is the adjoint operator of L.

Proof. By applying Theorem 3.3.2, it suffices to observe that $L(x_{2,n}) = z_{2,n}$ and the condition $x^*_{2,n} \in \partial(-z^*_{2,n} \circ L)(x_{2,n})$ is equivalent to $-L^*(z^*_{2,n}) = x^*_{2,n}$. Also, let us notice that the expression $\langle x^*_{2,n}, x_{2,n} - \bar{x} \rangle + \langle z^*_{2,n}, z_{2,n} - L(\bar{x}) \rangle \xrightarrow[n\to\infty]{} 0$ in Theorem

3.3.2 becomes superfluous since

$$\langle x_{2,n}^*, x_{2,n} - \bar{x}\rangle + \langle z_{2,n}^*, z_{2,n} - L(\bar{x})\rangle = \langle -L^*(z_{2,n}^*), x_{2,n} - \bar{x}\rangle + \langle L^*(z_{2,n}^*), x_{2,n} - \bar{x}\rangle = 0.$$

□

By taking $Z = X$ and $L = id_X$ (the identity of X), we obtain the sequential Pareto subdifferential sum rules.

Corollary 3.3.4. *Let* $f : X \longrightarrow Y \cup \{+\infty_Y\}$ *be proper,* Y_+*-convex and lower semicontinuous mapping and* $g : X \longrightarrow Y \cup \{+\infty_Y\}$ *be proper,* Y_+*-convex and lower semicontinuous mapping. Let* $\bar{x} \in \mathrm{dom} f \cap dom g$ *and* $\sigma \in \{w, p, s\}$*. Assume* Y_+ *is pointed as* $\sigma = p$ *(resp.* Y_+ *closed as* $\sigma = s$*). Then,* $A \in \partial^\sigma(f+g)(\bar{x})$ *if and only if there exists (resp. for all as* $\sigma = s$*)* $y^* \in Y_+^\sigma$*, there exist* $(x_n, z_n) \in \mathrm{dom\ f} \times \mathrm{dom\ g}$*,* $x_n^*, z_n^* \in X^*$ *satisfying*

$$\begin{cases} x_n \xrightarrow{\|\cdot\|_X} \bar{x},\ z_n \xrightarrow{\|\cdot\|_X} \bar{x}, \\ \\ x_n^* \in \partial(y^* \circ f)(x_n),\ z_n^* \in \partial(y^* \circ g)(z_n), \end{cases}$$

such that

$$\begin{cases} x_n^* + z_n^* \xrightarrow{\|\cdot\|_{X^*}} y^* \circ A, \\ \\ (y^* \circ f)(x_n) - \langle x_n^*, x_n - \bar{x}\rangle \xrightarrow[n\to\infty]{} (y^* \circ f)(\bar{x}), \\ \\ (y^* \circ g)(z_n) - \langle z_n^*, z_n - \bar{x}\rangle \xrightarrow[n\to\infty]{} (y^* \circ g)(\bar{x}). \end{cases}$$

3.4 An application to sequential vector efficiency

Let us consider the following vector optimization problem with cone constraints

$$(P) \quad \min_{h(x)\in -Z_+} f(x)$$

where $f : X \longrightarrow Y \cup \{+\infty_Y\}$ is proper and Y_+-convex, $h : X \longrightarrow Z \cup \{+\infty_Z\}$ is proper and Z_+-convex, and Z_+ is closed. This model problem is thoroughly studied due to its utility in various practical areas. When characterizing σ-efficient solutions of this large class of problems, usually constraint qualifications have to be satisfied. It is known that, in the absence of a regularity condition (such as the generalized Slater constraint qualification or the closed-cone condition), the existence of a σ-efficient solution of problem (P) does not hold. Our main objective, in this section, is to establish sequential σ-efficient optimality conditions for the convex programming problem (P). In fact, by introducing the vector indicator mapping $\delta^v_{-Z_+}$, the problem (P) becomes equivalent to the unconstrained vector minimization problem

$$\min_{x\in X}(f + \delta^v_{-Z_+} \circ h)(x)$$

in the following sense

Lemma 3.4.1. *i) For $\sigma \in \{w, p, s\}$, we have*

$$\mathrm{E}_\sigma(f, h^{-1}(-Z_+)) = \mathrm{E}_\sigma(f + \delta^v_{-Z_+} \circ h, X).$$

ii) If C is closed then δ^v_C is lower semicontinuous.

Proof. i) Let $\bar{x} \in \mathrm{E}_p(f, h^{-1}(-Z_+))$. If $\bar{x} \notin \mathrm{E}_p(f + \delta^v_{-Z_+} \circ h, X)$, then for every convex cone $\hat{Y}_+ \subsetneq Y$ such that $Y_+ \setminus l(Y_+) \subseteq \text{int}\hat{Y}_+$ there exists $x_0 \in \text{dom f} \cap h^{-1}(-Z_+)$ verifying $f(x_0) \lneq_{\hat{Y}_+} f(\bar{x})$ which contradicts the fact that $\bar{x} \in \text{E}_p(f, h^{-1}(-Z_+))$. Conversely, let $\bar{x} \in \text{E}_p(f + \delta^v_{-Z_+} \circ h, X)$, then $\bar{x} \in \text{dom f} \cap \text{h}^{-1}(-\text{Z}_+)$. If we suppose that $\bar{x} \notin \text{E}_p(f, h^{-1}(-Z_+))$, then for every convex cone $\hat{Y}_+ \subsetneq Y$ such that $Y_+ \setminus l(Y_+) \subseteq \text{int}\hat{Y}_+$ there exists $x_0 \in \text{dom f} \cap \text{h}^{-1}(-\text{Z}_+)$ verifying $f(x_0) \lneq_{\hat{Y}_+} f(\bar{x})$ i.e. $f(x_0) + (\delta^v_{-Z_+} \circ h)(x_0) \lneq_{\hat{Y}_+} f(\bar{x}) + (\delta^v_{-Z_+} \circ h)(\bar{x})$ contradicting $\bar{x} \in \text{E}_p(f + \delta^v_{-Z_+} \circ h, X)$. The case $\sigma = w$ can be proved similarly. The case $\sigma = s$ is easy.

ii) If $\bar{x} \in C$, let V be any neighborhood of $\delta^v_C(\bar{x}) = 0_Y$ in Y. By choosing $U := X$ as a neighborhood of $\bar{x}$ in X and by distinguishing the cases $u \in C$, $u \in X \setminus C$ ($u \in U$), the inclusion $\delta^v_C(U) \subset (V + Y_+) \cup \{+\infty_Y\}$ follows immediately. Now if $\bar{x} \in X \setminus C$, let $y \in Y$ and V any neighborhood of y. By putting $U := X \setminus C$ which is an open neighborhood of $\bar{x}$ (since C is closed), one can easily check $\delta^v_C(U) \subset (V + Y_+) \cup \{+\infty_Y\}$. □

Theorem 3.4.2. *Let $\bar{x} \in \text{dom} f \cap \text{dom} h \cap h^{-1}(-Z_+)$ and $\sigma \in \{w, p, s\}$. Assume Y_+ is pointed as $\sigma = p$ (resp. Y_+ closed as $\sigma = s$).. Then, $\bar{x}$ is a σ-efficient solution of (P) if and only if there exists (resp. for all as $\sigma = s$) $y^* \in Y_+^\sigma$, there exist $(x_{1,n}, z_{1,n}) \in$* dom f $\times (-Z_+)$, $(x_{2,n}, z_{2,n}) \in X \times Z$, $(x^*_{i,n}, z^*_{i,n}) \in X^* \times Z^*$ $(i = 1, 2)$ *satisfying*

$$\begin{cases} x_{i,n} \xrightarrow{\|\cdot\|_X} \bar{x}, \quad z_{i,n} \xrightarrow{\|\cdot\|_Z} h(\bar{x}), \quad h(x_{2,n}) \leq_{Z_+} z_{2,n} \ (i = 1, 2), \\ x^*_{1,n} \in \partial(y^* \circ f)(x_{1,n}), \quad z^*_{1,n} \in Z^*_+, \quad \langle z^*_{1,n}, z_{1,n} \rangle = 0, \\ x^*_{2,n} \in \partial(-z^*_{2,n} \circ h)(x_{2,n}), \quad z^*_{2,n} \in -Z^*_+, \quad \langle z^*_{2,n}, z_{2,n} - h(x_{2,n}) \rangle = 0, \end{cases}$$

such that

$$\begin{cases} x^*_{1,n} + x^*_{2,n} \xrightarrow{\|\cdot\|_{X^*}} 0 \quad \textit{and} \quad z^*_{1,n} + z^*_{2,n} \xrightarrow{\|\cdot\|_{Z^*}} 0, \\ (y^* \circ f)(x_{1,n}) - \langle x^*_{1,n}, x_{1,n} - \bar{x} \rangle \xrightarrow[n\to\infty]{} (y^* \circ f)(\bar{x}), \\ \langle z^*_{1,n}, z_{1,n} - h(\bar{x}) \rangle \xrightarrow[n\to\infty]{} 0, \\ \langle x^*_{2,n}, x_{2,n} - \bar{x} \rangle + \langle z^*_{2,n}, z_{2,n} - h(\bar{x}) \rangle \xrightarrow[n\to\infty]{} 0. \end{cases}$$

Proof. According to Lemma 3.4.1, $\bar{x}$ is a σ-efficient solution of problem (P) if and only if $\bar{x} \in \mathrm{E}_\sigma(f, h^{-1}(-Z_+)) = \mathrm{E}_\sigma(f + \delta^v_{-Z_+} \circ h, X)$ i.e. $0 \in \partial^\sigma(f + \delta^v_{-Z_+} \circ h)(\bar{x})$. The properness and the convexity of $\delta^v_{-Z_+}$ are immediate since dom $\delta^v_{-Z_+} = -Z_+ \neq \emptyset$ and Z_+ is convex. By virtue of Lemma 3.4.1, $\delta^v_{-Z_+}$ is lower semicontinuous and let us recall that $\delta^v_{-Z_+}$ is (Z_+, Y_+)-nondecreasing (see [33]) and since the mappings f, $g := \delta^v_{-Z_+}$ and h satisfy together all the assumptions of Theorem 3.3.2, it then follows that there exists (resp. for all as $\sigma = s$) $y^* \in Y^\sigma_+$, there exist $(x_{1,n}, z_{1,n}) \in \mathrm{dom\ f} \times -\mathrm{Z}_+$, $(x_{2,n}, z_{2,n}) \in X \times Z$, $(x^*_{i,n}, z^*_{i,n}) \in X^* \times Z^*$ $(i = 1, 2)$ satisfying

$$\begin{cases} x_{i,n} \xrightarrow{\|\cdot\|_X} \bar{x}, \quad z_{i,n} \xrightarrow{\|\cdot\|_Z} h(\bar{x}), \quad h(x_{2,n}) \leq_{Z_+} z_{2,n} \ (i = 1, 2), \\ x^*_{1,n} \in \partial(y^* \circ f)(x_{1,n}), z^*_{1,n} \in N_{-Z_+}(z_{1,n}), \\ x^*_{2,n} \in \partial(-z^*_{2,n} \circ h)(x_{2,n}), \quad z^*_{2,n} \in -Z^*_+, \quad \langle z^*_{2,n}, z_{2,n} - h(x_{2,n}) \rangle = 0, \end{cases}$$

such that

$$\begin{cases} x^*_{1,n} + x^*_{2,n} \xrightarrow{\|\cdot\|_{X^*}} 0 \quad \text{and} \quad z^*_{1,n} + z^*_{2,n} \xrightarrow{\|\cdot\|_{Z^*}} 0, \\ (y^* \circ f)(x_{1,n}) - \langle x^*_{1,n}, x_{1,n} - \bar{x}\rangle \xrightarrow[n\to\infty]{} (y^* \circ f)(\bar{x}), \\ \langle z^*_{1,n}, z_{1,n} - h(\bar{x})\rangle \xrightarrow[n\to\infty]{} 0, \\ \langle x^*_{2,n}, x_{2,n} - \bar{x}\rangle + \langle z^*_{2,n}, z_{2,n} - h(\bar{x})\rangle \xrightarrow[n\to\infty]{} 0. \end{cases}$$

We end up the proof by observing that

$$z^*_{1,n} \in N_{-Z_+}(z_{1,n}) \Longleftrightarrow z^*_{1,n} \in Z^*_+ \text{ and } \langle z^*_{1,n}, z_{1,n}\rangle = 0.$$

□

Chapter 4

Sequential formulas for Pareto subdifferential of the sums of m vector mappings and applications to sequential efficiency

4.1 Introduction

The sequential subdifferential calculus and the sequential efficiency are important and active topics of Mathematical Optimization [20, 36]. Recently, Laghdir et al. [49] have shown a sequential formula for the weak and proper Pareto subdifferential of the sum of two proper convex lsc vector valued mappings. As a corollary, they derived sequential efficiency optimality conditions for vector optimization problems with geometric constraint. The contribution [49] motivates the present chapter.

In this chapter, by applying interesting results of Boţ, Wanka [9] and Jeyakumar [42], we obtain three new sequential formulas without conditions of qualification for the weak, proper and strong Pareto subdifferential of the sums of $m \geq 2$ proper, cone-convex and Penot-Théra lower semicontinuous vector valued mappings. The first formula is expressed in terms of the epigraphs of the conjugate of the data vector valued mappings. The second involves the approximate subdifferential. The third one is by means of the scalar subdifferential and extends to m vector mappings the sum rule formula of [49]. It is worth noting that in the latter situation the induction principle is useless. As an application, we provide sequential without a constraint qualification necessary and sufficient optimality conditions for weak, proper and strong efficient solution of general vector optimization problem with geometric and cone constraints generalizing the corresponding result of [49]. We also give sequential weak and strong efficiency for general multi-objective fractional programming problem with geometric and cone constraints. To the best of our knowledge it is the first time sequential Pareto subdifferential calculus is used in the context of multi-objective fractional optimization to derive sequential efficiency optimality conditions.

The outline is as follows. The next section presents preliminary facts. In section 4.3, we establish the sequential Pareto subdifferentials sums rule formulas for vector mappings. In sections 4.4 and 4.5, we derive the sequential efficiency optimality conditions for vector and multi-objective fractional optimization respectively.

4.2 Preliminaries

Throughout this chapter, let X and Y be two real reflexive Banach spaces paired in duality by $\langle .,. \rangle$ with their topological duals X^* and Y^*. For simplicity, the norms and the dual norms as well as the associated topologies are denoted by $\|.\|$ and $\|.\|_*$ respectively. We will use the symbol w^* for the weak-star topology on the dual spaces and $\tau_{\mathbb{R}}$ for the euclidean topology on the real line $\mathbb{R}$. The product space $X \times Y$ will be endowed with the norm $\|(x,y)\| := \sqrt{\|x\|^2 + \|y\|^2}$ and similarly the norm on $X^* \times Y^*$ will be choosen.

Let Y_+ be a nontrivial convex cone of Y with nonempty topological interior int Y_+. The associated dual and strict polar cones are :

$$Y_+^* := \{y^* \in Y^* \ : \ \langle y^*, y \rangle \geq 0, \ \forall y \ \in Y_+\},$$

$$(Y_+^*)^\circ := \{y^* \in Y^* \ : \ \langle y^*, y \rangle > 0, \ \forall y \ \in Y_+ \setminus l(Y_+)\}.$$

When the lineality $l(Y_+) := Y_+ \cap -Y_+$ reduces to $\{0\}$, Y_+ is said to be pointed. The cone Y_+ induces the following binary relations :

$$\begin{aligned}
y_1 \quad &\leq_{Y_+} \quad y_2 :\Longleftrightarrow y_2 - y_1 \in Y_+, \\
y_1 \quad &<_{Y_+} \quad y_2 :\Longleftrightarrow y_2 - y_1 \in \text{int } Y_+, \\
y_1 \quad &\lneq_{Y_+} \quad y_2 :\Longleftrightarrow y_2 - y_1 \in Y_+ \setminus l(Y_+),
\end{aligned}$$

for $y_1, y_2 \in Y$. With respect to " $\leq_{Y_+}$ " the augmented set $Y \cup \{+\infty_Y\}$ is considered where $+\infty_Y$ is an abstract element verifying natural relations :

$$y \leq_{Y_+} +\infty_Y,$$

$$y + (+\infty_Y) = (+\infty_Y) + y = +\infty_Y,$$

$$\alpha \cdot (+\infty_Y) = +\infty_Y,$$

for every $y \in Y \cup \{+\infty_Y\}$ and $\alpha \geq 0$.

Definition 4.2.1. Let $f : X \longrightarrow Y \cup \{+\infty_Y\}$ be a vector valued mapping.

- f is said to be proper if its effective domain

$$\text{dom} f := \{x \in X \ : \ f(x) \in Y\} \neq \emptyset.$$

- f is said to be Y_+-convex if for every $\lambda \in [0,1]$ and $x_1, x_2 \in X$

$$f(\lambda x_1 + (1-\lambda)x_2) \leq_{Y_+} \lambda f(x_1) + (1-\lambda) f(x_2).$$

- f is said to be Y_+-epi-closed if its epigraph

$$\text{epi} f := \{(x,y) \in X \times Y \ : \ f(x) \leq_{Y_+} y\} \text{ is closed.}$$

- f is said to be lower semicontinuous [50, 66] at $\bar{x} \in \text{dom} f$ if for every neighborhood V of $f(\bar{x})$ in Y, there exists a neighborhood U of $\bar{x}$ such that

$$f(U) \subseteq (V + Y_+) \cup \{+\infty_Y\}.$$

When $f(\bar{x}) = +\infty_Y$, f is said to be lower semicontinuous at $\bar{x}$ if for any $y \in Y$, any neighborhood V of y, there exists a neighborhood U of $\bar{x}$ such that the above inclusion is satisfied.

f is said to be lower semicontinuous if it is lower semicontinuous at every point of X.

- The weak subdifferential of f at $\bar{x} \in \mathrm{dom} f$ is

$$\partial^w f(\bar{x}) := \{A \in \mathrm{L}(X,Y) \,:\, \nexists x \in X, f(x) - f(\bar{x}) <_{Y_+} A(x - \bar{x})\}.$$

- The proper subdifferential of f at $\bar{x} \in \mathrm{dom} f$ is

$$\begin{aligned}\partial^p f(\bar{x}) :=\{A \in \mathrm{L}(X,Y) : \exists \hat{Y}_+ \subsetneq Y \text{ convex cone such that}\\ Y_+ \setminus l(Y_+) \subseteq \mathrm{int}\hat{Y}_+, \nexists x \in X, f(x) - f(\bar{x}) \lneq_{\hat{Y}_+} A(x - \bar{x})\}.\end{aligned}$$

- The strong subdifferential of f at $\bar{x} \in \mathrm{dom} f$ is

$$\partial^s f(\bar{x}) := \{A \in \mathrm{L}(X,Y) \,:\, \forall x \in X, A(x - \bar{x}) \leq_{Y_+} f(x) - f(\bar{x})\}.$$

Here $\mathrm{L}(X,Y)$ is the space of linear continuous operators from X to Y.

Remark 4.2.2. By applying Proposition 1.1 in [50], we obtain that if $f : X \longrightarrow Y \cup \{+\infty_Y\}$ is lower semicontinuous in the sense of Penot-Théra then the scalar function $y^* \circ f$ for any $y^* \in Y_+^* \setminus \{0\}$ is lower semicontinuous. This fact is needed.

Definition 4.2.3. Let S be a nonempty subset of X.

- The vector indicator mapping of S is

$$\delta_S^v : \quad X \longrightarrow Y \cup \{+\infty_Y\}$$
$$x \longrightarrow \begin{cases} 0 & \text{if } x \in S \\ +\infty_Y & \text{otherwise} \end{cases}.$$

- The vector normal cone $N_S^v(\bar{x})$ of S at $\bar{x} \in S$ is the strong subdifferential of δ_S^v at $\bar{x}$.

Remark 4.2.4. If $Y = \mathbb{R}$, $Y_+ = [0, +\infty[$, δ_S^v reduces to the scalar indicator function denoted by δ_S and $N_S^v(\bar{x})$ becomes, for S convex, the classical normal cone defined by :

$$N_S(\bar{x}) := \{x^* \in X^* \ : \ \forall x \in S, \langle x^*, x - \bar{x}\rangle \leq 0\}.$$

Let $f : X \longrightarrow Y \cup \{+\infty_Y\}$ be a vector valued mapping and S a nonempty subset of X. The vector minimization problem

$$(\text{P}) \quad \inf_{x \in S} f(x)$$

is considered in the following senses

Definition 4.2.5. $\bar{x} \in \operatorname{dom} f \cap S$ is called :

- a weak efficient solution of (P) if $\nexists x \in S,\ f(x) <_{Y_+} f(\bar{x})$.

- a proper efficient solution of (P) if

$$\exists \hat{Y}_+ \subsetneq Y \text{ convex cone such that } Y_+ \setminus l(Y_+) \subseteq \operatorname{int} \hat{Y}_+, \nexists x \in S,\ f(x) \lneq_{\hat{Y}_+} f(\bar{x}).$$

- a strong efficient solution of (P) if $\forall x \in S,\ f(\bar{x}) \leq_{Y_+} f(x)$.

The set of weak, proper and strong efficient solutions of (P) are respectively denoted by $\mathrm{E}_w(f,S)$, $\mathrm{E}_p(f,S)$ and $\mathrm{E}_s(f,S)$.

We deduce two useful relations ($\sigma \in \{w,p,s\}$) (see also [33]).

$$\bar{x} \in E_\sigma(f,X) \Longleftrightarrow 0 \in \partial^\sigma f(\bar{x}). \tag{4.1}$$

$$\bar{x} \in E_\sigma(f,S) \Longleftrightarrow \bar{x} \in E_\sigma(f+\delta_S^v, X). \tag{4.2}$$

Remark 4.2.6. In the particular case where $Y = \mathbb{R}^q$ is the q dimensional Euclidian space, $Y_+ = \mathbb{R}^q_+$ is the nonnegative orthant and (P) is of the form

$$\inf_{x\in S} \{r_1(x), \cdots, r_q(x)\},$$

with $r_1, ..., r_q : X \longrightarrow \mathbb{R}$, the definition of weak efficient solution becomes : $\bar{x} \in S$ is a weak efficient solution of (P) if there does not exist $x \in S$ such that

$$r_i(x) < r_i(\bar{x})$$

for all $i \in \{1, ..., q\}$.

The next convention is adopted. Let $g : Y \longrightarrow \mathbb{R} \cup \{+\infty\}$ be a scalar function and $h : X \longrightarrow Y \cup \{+\infty_Y\}$ a vector valued mapping then the composed function

$g \circ h : X \longrightarrow \mathbb{R} \cup \{+\infty\}$ is given by

$$(g \circ h)(x) := \begin{cases} g(h(x)) & \text{if } x \in \text{dom} h, \\ \\ +\infty & \text{otherwise.} \end{cases}$$

Analogously the composed mapping is defined when g is vector valued.

For convenience we also recall well known concepts from scalar convex analysis.

Definition 4.2.7. Let $f : X \to \mathbb{R} \cup \{+\infty\}$ be a convex scalar function.

- The ϵ-approximate subdifferential of f at $\bar{x} \in \text{dom} f$ with $\epsilon \geq 0$ is

$$\partial_\epsilon f(\bar{x}) := \{x^* \in X^* \ : \ \forall x \in X, f(x) \geq f(\bar{x}) + \langle x^*, x - \bar{x} \rangle - \epsilon\}.$$

- The subdifferential of f at $\bar{x} \in \text{dom} f$ is :

$$\partial f(\bar{x}) := \{x^* \in X^* \ : \ \forall x \in X, f(x) \geq f(\bar{x}) + \langle x^*, x - \bar{x} \rangle\}.$$

- The conjugate of f is the function given by :

$$\begin{array}{cccc} f^* & : & X^* & \longrightarrow & \mathbb{R} \cup \{-\infty, +\infty\} \\ & & x^* & \longmapsto & \sup_{x \in X}\{\langle x^*, x \rangle - f(x)\} \end{array}.$$

We point out the relation between the subdifferential and the conjugate :

$$\partial f(\bar{x}) := \{x^* \in X^* \ : \ f(\bar{x}) + f^*(x^*) = \langle x^*, \bar{x} \rangle\}.$$

The following results will be useful.

Lemma 4.2.8. (Boţ and Wanka [9]) Let $f_1, f_2 : X \longrightarrow \mathbb{R} \cup \{+\infty\}$ be two proper, convex and lower semicontinuous scalar functions verifying $\operatorname{dom} f_1 \cap \operatorname{dom} f_2 \neq \emptyset$. Then,

$$\operatorname{epi}(f_1 + f_2)^* = \operatorname{cl}_{w^* \times \tau_{\mathbb{R}}}(\operatorname{epi} f_1^* + \operatorname{epi} f_2^*),$$

where cl denotes the topological closure.

Using induction, the fact that the conjugate is convex lower semicontinuous and the relation $cl(clA + clB) = cl(A + B)$ for subsets in a topological vector space, the precedent lemma is easily reformulated for m scalar functions with $m \geq 2$ and for the dual norm.

Lemma 4.2.9. *Let $f_1, ..., f_m : X \longrightarrow \mathbb{R} \cup \{+\infty\}$ be m proper, convex and lower semicontinuous scalar functions satisfying $\cap_{i=1}^m \operatorname{dom} f_i \neq \emptyset$. Then,*

$$\operatorname{epi}(\sum_{i=1}^m f_i)^* = \operatorname{cl}_{\|.\|_* \times \tau_{\mathbb{R}}}(\sum_{i=1}^m \operatorname{epi} f_i^*).$$

Lemma 4.2.10. (Jeyakumar [42]) Let $f : X \longrightarrow \mathbb{R} \cup \{+\infty\}$ be a proper, convex and lower semicontinuous scalar function. Let $a \in \operatorname{dom} f$, then

$$\operatorname{epi} f^* = \bigcup_{\epsilon \geq 0} \{(s, \epsilon + \langle s, a \rangle - f(a)) : s \in \partial_\epsilon f(a)\}.$$

Lemma 4.2.11. (Thibault [72], [10]) Let $f : X \longrightarrow \mathbb{R} \cup \{+\infty\}$ be a proper, convex and lower semicontinuous scalar function. Let $\bar{x} \in \operatorname{dom} f$, then for any real number $\epsilon > 0$ and any $x^* \in \partial_\epsilon f(\bar{x})$, there exists $(x_\epsilon, x_\epsilon^*) \in X \times X^*$ such that

$$x_\epsilon^* \in \partial f(x_\epsilon),$$

$$\|x_\epsilon - \bar{x}\| \leq \sqrt{\epsilon},$$

$$\|x_\epsilon^* - x^*\|_* \leq \sqrt{\epsilon},$$

$$|f(x_\epsilon) - \langle x_\epsilon^*, x_\epsilon - \bar{x}\rangle - f(\bar{x})| \leq 2\epsilon.$$

Lemma 4.2.12. (El Maghri and Laghdir [33]) Let $f : X \longrightarrow Y \cup \{+\infty_Y\}$ be Y_+-convex vector valued mapping and $\bar{x} \in X$.
Case $\sigma \in \{w, p\}$ with Y_+ pointed as $\sigma = p$:

$$\partial^\sigma f(\bar{x}) = \bigcup_{y^* \in Y_+^\sigma} \{A \in L(X, Y) : y^* \circ A \in \partial(y^* \circ f)(\bar{x})\}.$$

Case $\sigma = s$ and Y_+ is closed :

$$\partial^s f(\bar{x}) = \bigcap_{y^* \in Y_+^s} \{A \in L(X, Y) : y^* \circ A \in \partial(y^* \circ f)(\bar{x})\}.$$

Where :

$$Y_+^\sigma := \begin{cases} Y_+^* \setminus \{0\} & \text{if } \sigma \in \{w, s\} \\ (Y_+^*)^\circ & \text{if } \sigma = p \end{cases}.$$

Remark 4.2.13. In the sequel Z is a normed space. All the above notations, concepts and results stated with Y remain true for Z. Sometimes the departure space will be taken equal to $X \times Y$.

4.3 Sequential weak, proper and strong Pareto subdifferential sums rules

Using Lemma 4.2.9 and Lemma 4.2.10, that describe the epigraph of the conjugate of the sums of functions and the relationship of the epigraph of the conjugate with the approximate subdifferential respectively, and a refined version of the well known Brøndsted-Rockafellar theorem (Lemma 4.2.11), we show three new sequential without constraint qualification weak, proper and strong Pareto subgradient characterization formulas for the sums of m proper, cone-convex and lower semicontinuous vector valued mappings with $m \geq 2$.

The first sequential formula is by means of the epigraphs of the conjugate of data vector valued mappings.

Theorem 4.3.1. *Let $f_1, ..., f_m : X \longrightarrow Z \cup \{+\infty_Z\}$ be m proper, Z_+-convex and lower semicontinuous vector valued mappings. Let $\bar{x} \in \cap_{i=1}^m \operatorname{dom} f_i$ and suppose that Z_+ is pointed as $\sigma = p$ (resp. closed as $\sigma = s$). Then, $A \in \partial^\sigma(\sum_{i=1}^m f_i)(\bar{x})$ if and only if there exists $z^* \in Z_+^\sigma$ (resp. for every $z^* \in Z_+^s$ as $\sigma = s$) and there exist sequences $\{(u_{i,n}^*, r_{i,n})\}_n \subseteq \operatorname{epi}(z^* \circ f_i)^*$ with $i \in \{1, \cdots, m\}$ and $\{t_n\}_n \subseteq \mathbb{R}_+$ satisfying*

$$\sum_{i=1}^m u_{i,n}^* \xrightarrow{\|\cdot\|_*} z^* \circ A,$$

$$t_n + \sum_{i=1}^m r_{i,n} \xrightarrow[n \to \infty]{} \langle z^* \circ A, \bar{x} \rangle - \sum_{i=1}^m (z^* \circ f_i)(\bar{x}).$$

Proof. Let $\sigma \in \{w, p\}$ and $A \in \partial^\sigma(\sum_{i=1}^m f_i)(\bar{x})$. By applying Lemma 4.2.12, there exists some $z^* \in Z_+^\sigma$ such that $z^* \circ A \in \partial(\sum_{i=1}^m (z^* \circ f_i))(\bar{x})$. For each $x^* \in X^*$, we introduce the function φ_{x^*} by

$$\begin{array}{rcccl} \varphi_{x^*} & : & X & \longrightarrow & \mathbb{R}\cup\{+\infty\} \\ & & x & \longmapsto & [\sum_{i=1}^m (z^*\circ f_i)(x)] - \langle x^*, x-\bar{x}\rangle \end{array}.$$

Then, one can check that

$$z^*\circ A \in \partial(\sum_{i=1}^m (z^*\circ f_i))(\bar{x}) \Longleftrightarrow (0, -\varphi_{z^*\circ A}(\bar{x})) \in \mathrm{epi}(\varphi_{z^*\circ A})^* \tag{4.3}$$

and also

$$\mathrm{epi}(-\langle z^*\circ A, . - \bar{x}\rangle)^* = \{-z^*\circ A\}\times[-\langle z^*\circ A, \bar{x}\rangle, +\infty[. \tag{4.4}$$

Using successively (4.3), Lemma 4.2.9 and (4.4), we obtain that

$$A \in \partial^\sigma(\sum_{i=1}^m f_i)(\bar{x})$$

if and only if

$$(0, -\varphi_{z^*\circ A}(\bar{x})) \in \mathrm{cl}_{\|.\|_*\times\tau_{\mathbb{R}}}((\sum_{i=1}^m \mathrm{epi}\,(z^*\circ f_i)^*) + \{-z^*\circ A\}\times[-\langle z^*\circ A, \bar{x}\rangle, +\infty[)$$

or equivalently $\exists\{(u^*_{i,n}, r_{i,n})\}_n \subseteq \mathrm{epi}\,(z^*\circ f_i)^*$, $\exists\{s_n\}_n \subseteq [-\langle z^*\circ A, \bar{x}\rangle, +\infty[$ $(i \in \{1\cdots m\})$:

$$(\sum_{i=1}^m u^*_{i,n}) - z^*\circ A \xrightarrow{\|.\|_*} 0,$$

$$(\sum_{i=1}^m r_{i,n}) + s_n \xrightarrow[n\to\infty]{} -\varphi_{z^*\circ A}(\bar{x}).$$

By putting $t_n := s_n + \langle z^*\circ A, \bar{x}\rangle$ for $n \in \mathbb{N}$, the announced result follows. The case

$\sigma = s$ is analogous. □

The second formula is in terms of the approximate subdifferentials.

Theorem 4.3.2. *Let $f_1,...,f_m : X \longrightarrow Z \cup \{+\infty_Z\}$ be m proper, Z_+-convex and lower semicontinuous vector valued mappings. Let $\bar{x} \in \cap_{i=1}^m \operatorname{dom} f_i$ and suppose that Z_+ is pointed as $\sigma = p$ (resp. closed as $\sigma = s$). Then, $A \in \partial^\sigma(\sum_{i=1}^m f_i)(\bar{x})$ if and only if there exists $z^* \in Z_+^\sigma$ (resp. for every $z^* \in Z_+^s$ as $\sigma = s$), there exist sequences $\{\epsilon_n\}_n \subseteq \mathbb{R}_+$ and $\{u_{i,n}^*\}_n \subseteq X^*$ with $i \in \{1,\cdots,m\}$ satisfying*

$$\epsilon_n \xrightarrow[n\to\infty]{} 0,$$

$$\sum_{i=1}^m u_{i,n}^* \xrightarrow{\|\cdot\|_*} z^* \circ A,$$

$$u_{i,n}^* \in \partial_{\epsilon_n}(z^* \circ f_i)(\bar{x}),$$

with $i \in \{1,\cdots,m\}$ and $n \in \mathbb{N}$.

Proof. The focus is on the case $\sigma \in \{w,p\}$.
($\Longrightarrow$) Let $A \in \partial^\sigma(\sum_{i=1}^m f_i)(\bar{x})$. By applying Theorem 4.3.1, there exist $z^* \in Z_+^\sigma$ and sequences $\{(u_{i,n}^*, r_{i,n})\}_n \subseteq \operatorname{epi}(z^* \circ f_i)^*$ with $i \in \{1,\cdots,m\}$ and $\{t_n\}_n \subseteq \mathbb{R}_+$ satisfying

$$\sum_{i=1}^m u_{i,n}^* \xrightarrow{\|\cdot\|_*} z^* \circ A \tag{4.5}$$

$$t_n + \sum_{i=1}^m r_{i,n} \xrightarrow[n\to\infty]{} \langle z^* \circ A, \bar{x}\rangle - \sum_{i=1}^m (z^* \circ f_i)(\bar{x}) \tag{4.6}$$

and according to Lemma 4.2.10, there exist another sequences $\{\epsilon_{i,n}\}_n \subseteq \mathbb{R}_+$ that satisfy

$$u^*_{i,n} \in \partial_{\epsilon_{i,n}}(z^* \circ f_i)(\bar{x}) \tag{4.7}$$

$$r_{i,n} = \epsilon_{i,n} + \langle u^*_{i,n}, \bar{x}\rangle - (z^* \circ f_i)(\bar{x}) \tag{4.8}$$

with $i \in \{1, \cdots, m\}$ and $n \in \mathbb{N}$.

From (4.7) and by setting $\epsilon_n := \max_{i\in\{1,\cdots,m\}}(\epsilon_{i,n})$ for each $n \in \mathbb{N}$, we have

$$u^*_{i,n} \in \partial_{\epsilon_n}(z^* \circ f_i)(\bar{x})$$

with $i \in \{1, \cdots, m\}$ and $n \in \mathbb{N}$.

Now in view of (4.8), it is easy to see that

$$0 \leq \epsilon_n \leq t_n + \sum_{i=1}^m r_{i,n} + [\sum_{i=1}^m (z^* \circ f_i)(\bar{x})] - \langle \sum_{i=1}^m u^*_{i,n}, \bar{x}\rangle.$$

Then, by (4.5) and (4.6), we obtain that $\epsilon_n \xrightarrow[n\to\infty]{} 0$.

To prove the converse ($\Longleftarrow$), it suffices to sum over i and let $n \to +\infty$ in accordance to the variational inequalities associated to $u^*_{i,n} \in \partial_{\epsilon_n}(z^* \circ f_i)(\bar{x})$, that is, $(z^* \circ f_i)(x) - (z^* \circ f_i)(\bar{x}) \geq \langle u^*_{i,n}, x - \bar{x}\rangle - \epsilon_n$ for all $x \in X$, with $i \in \{1, \cdots, m\}$ and $n \in \mathbb{N}$. Then $z^* \circ A \in \partial(\sum_{i=1}^m (z^* \circ f_i))(\bar{x})$. We conclude by Lemma 4.2.12. □

The third formula involves the scalar subdifferentials.

Theorem 4.3.3. *Let $f_1, ..., f_m : X \longrightarrow Z \cup \{+\infty_Z\}$ be m proper, Z_+-convex and lower semicontinuous vector valued mappings. Let $\bar{x} \in \cap_{i=1}^m \operatorname{dom} f_i$ and suppose that Z_+ is pointed as $\sigma = p$ (resp. closed as $\sigma = s$). Then, $A \in \partial^\sigma(\sum_{i=1}^m f_i)(\bar{x})$ if and only if there exists $z^* \in Z_+^\sigma$ (resp. for every $z^* \in Z_+^s$ as $\sigma = s$), there exist sequences $\{x_{i,n}\}_n \subseteq \operatorname{dom} f_i$*

and $\{x^*_{i,n}\}_n \subseteq X^*$ *with* $i \in \{1, \cdots, m\}$ *satisfying*

$$x^*_{i,n} \in \partial(z^* \circ f_i)(x_{i,n}),$$

$$x_{i,n} \xrightarrow{\|\cdot\|} \bar{x},$$

$$\sum_{i=1}^{m} x^*_{i,n} \xrightarrow{\|\cdot\|_*} z^* \circ A,$$

$$(z^* \circ f_i)(x_{i,n}) - \langle x^*_{i,n}, x_{i,n} - \bar{x}\rangle \xrightarrow[n\to\infty]{} (z^* \circ f_i)(\bar{x}),$$

with $i \in \{1, \cdots, m\}$ *and* $n \in \mathbb{N}$.

Proof. We treat the situation with $\sigma \in \{w, p\}$, the case $\sigma = s$ is similar.

($\Longrightarrow$) By Theorem 4.3.2, $A \in \partial^\sigma(\sum_{i=1}^m f_i)(\bar{x})$ if and only if there exist $z^* \in Z^\sigma_+$, sequences $\{\epsilon_n\}_n \subseteq \mathbb{R}_+$ and $\{u^*_{i,n}\}_n \subseteq X^*$ with $i \in \{1, \cdots, m\}$ satisfying

$$\epsilon_n \xrightarrow[n\to\infty]{} 0, \tag{4.9}$$

$$\sum_{i=1}^{m} u^*_{i,n} \xrightarrow{\|\cdot\|_*} z^* \circ A, \tag{4.10}$$

$$u^*_{i,n} \in \partial_{\epsilon_n}(z^* \circ f_i)(\bar{x}),$$

with $i \in \{1, \cdots, m\}$ and $n \in \mathbb{N}$.

Therefore from Lemma 4.2.11 with $u^*_{i,n} \in \partial_{\epsilon_n}(z^* \circ f_i)(\bar{x})$, we obtain sequences $\{x_{i,n}\}_n \subseteq \operatorname{dom} f_i$ and $\{x^*_{i,n}\}_n \subseteq X^*$ with $i \in \{1, \cdots, m\}$ such that

$$x^*_{i,n} \in \partial(z^* \circ f_i)(x_{i,n}),$$

$$\|x_{i,n} - \bar{x}\| \leq \sqrt{\epsilon_n}, \tag{4.11}$$

$$\|x^*_{i,n} - u^*_{i,n}\|_* \leq \sqrt{\epsilon_n}, \tag{4.12}$$

$$|(z^* \circ f_i)(x_{i,n}) - \langle x^*_{i,n}, x_{i,n} - \bar{x}\rangle - (z^* \circ f_i)(\bar{x})| \leq 2\epsilon_n, \tag{4.13}$$

with $i \in \{1, \cdots, m\}$ and $n \in \mathbb{N}$.

Hence by letting $n \to +\infty$ in (4.11) and (4.13), we get

$$x_{i,n} \xrightarrow{\|\cdot\|} \bar{x},$$

$$(z^* \circ f_i)(x_{i,n}) - \langle x^*_{i,n}, x_{i,n} - \bar{x}\rangle \xrightarrow[n\to\infty]{} (z^* \circ f_i)(\bar{x}),$$

with $i \in \{1, \cdots, m\}$.

It remains to prove that $\sum_{i=1}^m x^*_{i,n} \xrightarrow{\|\cdot\|_*} z^* \circ A$ and this easily follows by using (4.10), (4.12) and (4.9).

($\Longleftarrow$) The variational inequalities associated to $x^*_{i,n} \in \partial(z^* \circ f_i)(x_{i,n})$ lead us to

$$(z^* \circ f_i)(x) - ((z^* \circ f_i)(x_{i,n}) - \langle x^*_{i,n}, x_{i,n} - \bar{x}\rangle) \geq \langle x^*_{i,n}, x - \bar{x}\rangle, \quad \forall x \in X$$

with $i \in \{1, \cdots, m\}$ and $n \in \mathbb{N}$. Thus by summing over i and letting $n \to +\infty$, we obtain $z^* \circ A \in \partial(\sum_{i=1}^m (z^* \circ f_i))(\bar{x})$. □

Remark 4.3.4. The above formulas are also valid when all data vector valued mappings f_i satisfy that for any $z^* \in Z^*_+ \setminus \{0\}$, the scalar function $z^* \circ f_i$ is lower semicontinuous.

4.4 Application to sequential weak, proper and strong efficiency of general vector optimization problem

In this section we use the sequential Pareto subdifferential calculus to obtain sequential without any constraint qualification necessary and sufficient weak, proper and strong efficient optimality conditions of the following general vector optimization problem with geometric and cone constraints

$$(VOP): \quad \inf_{\substack{x\in C \\ h(x)\in -Y_+}} f(x)$$

where :

- $f : X \longrightarrow Z \cup \{+\infty_Z\}$ is proper, Z_+-convex and lower semicontinuous vector valued mapping.
- $h : X \longrightarrow Y \cup \{+\infty_Y\}$ is proper, Y_+-convex and Y_+-epi-closed vector valued mapping.
- C is a nonempty closed convex subset of X.
- Y_+ is a nonempty closed convex cone of Y.

For convenience efficient solutions are shortned as σ-efficient solution with $\sigma \in \{w, p, s\}$.

Theorem 4.4.1. *Let $\bar{x} \in \mathrm{dom} f \cap C \cap h^{-1}(-Y_+)$, $\sigma \in \{w, p, s\}$ and assume Z_+ pointed as $\sigma = p$ (resp. closed as $\sigma = s$). Then, $\bar{x}$ is a σ-efficient solution of (VOP) if and only if there exists $z^* \in Z_+^{\sigma}$ (resp. for every $z^* \in Z_+^{s}$ as $\sigma = s$), there exist sequences*

$\{x_n\}_n \subseteq \operatorname{dom} f$, $\{c_n\}_n \subseteq C$, $\{y_n\}_n \subseteq -Y_+$, $\{(u_n, v_n)\}_n \subseteq \operatorname{epi} h$ *and* $\{x_n^*\}_n, \{c_n^*\}_n, \{u_n^*\}_n \subseteq X^*$, $\{y_n^*\}_n, \{v_n^*\}_n \subseteq Y^*$ *satisfying*

$$x_n^* \in \partial(z^* \circ f)(x_n),\ c_n^* \in N_C(c_n),$$

$$y_n^* \in Y_+^*,\ \langle y_n^*, y_n \rangle = 0,$$

$$u_n^* \in \partial((-v_n^*) \circ h)(u_n),\ v_n^* \in -Y_+^*,\ \langle v_n^*, v_n - h(u_n) \rangle = 0,$$

$$x_n \xrightarrow{\|\cdot\|} \bar{x}, c_n \xrightarrow{\|\cdot\|} \bar{x},\ u_n \xrightarrow{\|\cdot\|} \bar{x},\ y_n \xrightarrow{\|\cdot\|} h(\bar{x}),\ v_n \xrightarrow{\|\cdot\|} h(\bar{x}),$$

$$x_n^* + c_n^* + u_n^* \xrightarrow{\|\cdot\|_*} 0, \quad y_n^* + v_n^* \xrightarrow{\|\cdot\|_*} 0,$$

$$(z^* \circ f)(x_n) - \langle x_n^*, x_n - \bar{x} \rangle \xrightarrow[n\to\infty]{} (z^* \circ f)(\bar{x}),$$

$$\langle c_n^*, c_n - \bar{x} \rangle \xrightarrow[n\to\infty]{} 0,$$

$$\langle y_n^*, y_n - h(\bar{x}) \rangle \xrightarrow[n\to\infty]{} 0,$$

$$\langle u_n^*, u_n - \bar{x} \rangle + \langle v_n^*, v_n - h(\bar{x}) \rangle \xrightarrow[n\to\infty]{} 0,$$

with $n \in \mathbb{N}$.

Proof. By (4.2) and (4.1), $\bar{x}$ is a σ-efficient solution of (VOP) if and only if

$$0 \in \partial^\sigma (f + \delta_C^v + \delta_{-Y_+}^v \circ h)(\bar{x}). \tag{4.14}$$

Introduce the vector valued mappings with arrival set $Z \cup \{+\infty_Z\}$ defined by : $f_1(x,y) := f(x)$, $f_2(x,y) := \delta_C^v(x)$, $f_3(x,y) := \delta_{-Y_+}^v(y)$, $f_4(x,y) := \delta_{\operatorname{epi} h}^v(x,y)$, where $(x,y) \in X \times Y$. By using the definition of Pareto subdifferentials it is not difficult to see that (4.14) is equivalent to $(0,0) \in \partial^\sigma(f_1 + f_2 + f_3 + f_4)(\bar{x}, h(\bar{x}))$. Taking into account Remark 4.3.4 and by Theorem 4.3.3, there exists $z^* \in Z_+^\sigma$ (resp. for every $z^* \in Z_+^s$ as $\sigma = s$), there exist sequences $\{(x_n, \bar{x}_n)\}_n \subseteq \operatorname{dom} f \times Y$, $\{(c_n, \bar{c}_n)\}_n \subseteq C \times Y$, $\{(\bar{y}_n, y_n)\}_n \subseteq X \times -Y_+$, $\{(u_n, v_n)\}_n \subseteq \operatorname{epi} h$ and $\{x_n^*\}_n, \{c_n^*\}_n, \{\bar{y}_n^*\}_n, \{u_n^*\}_n \subseteq X^*$, $\{\bar{x}_n^*\}_n, \{\bar{c}_n^*\}_n, \{y_n^*\}_n, \{v_n^*\}_n \subseteq Y^*$ satisfying for every $n \in \mathbb{N}$:

$$
\begin{aligned}
&(x_n^*, \bar{x}_n^*) \in \partial(z^* \circ f_1)(x_n, \bar{x}_n) = \partial(z^* \circ f)(x_n) \times \{0\},\\
&(c_n^*, \bar{c}_n^*) \in \partial(z^* \circ f_2)(c_n, \bar{c}_n) = N_C(c_n) \times \{0\},\\
&(\bar{y}_n^*, y_n^*) \in \partial(z^* \circ f_3)(\bar{y}_n, y_n) = \{0\} \times N_{-Y_+}(y_n),\\
&(u_n^*, v_n^*) \in \partial \delta_{\operatorname{epi} h}(u_n, v_n),\\
&x_n \xrightarrow{\|\cdot\|} \bar{x}, c_n \xrightarrow{\|\cdot\|} \bar{x}, \bar{y}_n \xrightarrow{\|\cdot\|} \bar{x}, u_n \xrightarrow{\|\cdot\|} \bar{x},\\
&\bar{x}_n \xrightarrow{\|\cdot\|} h(\bar{x}), \bar{c}_n \xrightarrow{\|\cdot\|} h(\bar{x}), y_n \xrightarrow{\|\cdot\|} h(\bar{x}), v_n \xrightarrow{\|\cdot\|} h(\bar{x}),\\
&x_n^* + c_n^* + \bar{y}_n^* + u_n^* \xrightarrow{\|\cdot\|_*} 0,\\
&\bar{x}_n^* + \bar{c}_n^* + y_n^* + v_n^* \xrightarrow{\|\cdot\|_*} 0,\\
&(z^* \circ f)(x_n) - \langle x_n^*, x_n - \bar{x}\rangle - \langle \bar{x}_n^*, \bar{x}_n - h(\bar{x})\rangle \xrightarrow[n\to\infty]{} (z^* \circ f)(\bar{x}),\\
&\langle c_n^*, c_n - \bar{x}\rangle + \langle \bar{c}_n^*, \bar{c}_n - h(\bar{x})\rangle \xrightarrow[n\to\infty]{} 0,\\
&\langle \bar{y}_n^*, \bar{y}_n - \bar{x}\rangle + \langle y_n^*, y_n - h(\bar{x})\rangle \xrightarrow[n\to\infty]{} 0,\\
&\langle u_n^*, u_n - \bar{x}\rangle + \langle v_n^*, v_n - h(\bar{x})\rangle \xrightarrow[n\to\infty]{} 0.
\end{aligned}
\tag{4.15}
$$

Now, for $(x^*, y^*) \in X^* \times Y^*$, simple computations show that

$$\delta^*_{\mathrm{epi}\,h}(x^*, y^*) = ((-y^*) \circ h)^*(x^*) + \delta^*_{Y_+}(y^*).$$

Then, for $n \in \mathbb{N}$, (4.15) is equivalent to

$$((-v_n^*) \circ h)^*(u_n^*) + \delta^*_{Y_+}(v_n^*) - \langle u_n^*, u_n \rangle - \langle v_n^*, v_n \rangle = 0. \tag{4.16}$$

Since $(u_n, v_n) \in \mathrm{epi}\, h$, we define $z_n := v_n - h(u_n) \in Y_+$ for $n \in \mathbb{N}$. Consequently (4.16) is reformulated as follows

$$[((-v_n^*) \circ h)^*(u_n^*) + ((-v_n^*) \circ h)(u_n) - \langle u_n^*, u_n \rangle] + [\delta^*_{Y_+}(v_n^*) + \delta_{Y_+}(z_n) - \langle v_n^*, z_n \rangle] = 0.$$

According to Fenchel-Young inequality, this is equivalent to

$$u_n^* \in \partial((-v_n^*) \circ h)(u_n),\ v_n^* \in N_{Y_+}(v_n - h(u_n))$$

and from the fact that Y_+ is a convex cone, we obtain

$$v_n^* \in N_{Y_+}(v_n - h(u_n)) \Longleftrightarrow \begin{cases} v_n^* \in -Y_+^* \\ \langle v_n^*, v_n - h(u_n) \rangle = 0 \end{cases}.$$

Therefore the result follows straightforwardly after observing that $\{\bar{x}_n^*\}_n, \{\bar{c}_n^*\}_n, \{\bar{y}_n^*\}_n$ are null and $\{\bar{x}_n\}_n, \{\bar{c}_n\}_n, \{\bar{y}_n\}_n$ are superfluous.

□

4.5 Application to sequential weak and strong efficiency in multi-objective fractional programming

Multi-objective fractional optimization problems appear in many practical areas such as Economics and Management science. In this section we are interested in establishing sequential without any constraint qualification necessary and sufficient weak and strong efficient optimality conditions for the following multi-objective fractional optimization problem.

$$(MFP): \quad \inf_{\substack{x\in C\\ h(x)\in -Y_+}} \{\frac{f_1(x)}{g_1(x)}, \cdots, \frac{f_q(x)}{g_q(x)}\}$$

where :

- $f_1,...,f_q : X \longrightarrow [0,+\infty[$ are convex and lower semicontinuous scalar functions.
- $g_1,...,g_q : X \longrightarrow]0,+\infty[$ are concave and upper semicontinuous scalar functions.
- $h : X \longrightarrow Y \cup \{+\infty_Y\}$ is a proper, Y_+-convex and Y_+-epi-closed vector valued mapping.
- C is a nonempty closed convex subset of X.
- Y_+ is a nonempty closed convex cone of Y.

The approach is based on sequential Pareto subdifferential calculus.

Theorem 4.5.1. *Let* $\bar{x} \in C \cap h^{-1}(-Y_+)$, $\Omega := \{k \in \{1, \cdots, q\} \ : \ f_k(\bar{x}) > 0\}$. *Let* $\sigma \in \{w, s\}$. *Then,* $\bar{x}$ *is a* σ*-efficient solution of* (MFP) *if and only if there exist index set* $\Delta \subseteq \{1, \cdots, q\}$ *nonempty,* $\{\lambda_i\}_{i\in\Delta} \subseteq]0, +\infty[$ *(resp. for every* Δ *and every* $\{\lambda_i\}_{i\in\Delta}$ *as* $\sigma = s$*) and there exist sequences* $\{x_{i,n}\}_n, \{w_{j,n}\}_n \subseteq X$, $\{c_n\}_n \subseteq C$, $\{y_n\}_n \subseteq -Y_+$, $\{(u_n, v_n)\}_n \subseteq \operatorname{epi} h$, $\{x^*_{i,n}\}_n, \{w^*_{j,n}\}_n, \{c^*_n\}_n, \{u^*_n\}_n \subseteq X^*$, $\{y^*_n\}_n, \{v^*_n\}_n \subseteq Y^*$ *with* $i \in \Delta$, $j \in \Delta \cap \Omega$ *such that :*

$$x^*_{i,n} \in \partial f_i(x_{i,n}),\ w^*_{j,n} \in \partial(-g_j)(w_{j,n}),\ c^*_n \in N_C(c_n),$$

$$y^*_n \in Y^*_+,\ \langle y^*_n, y_n\rangle = 0,$$

$$u^*_n \in \partial((-v^*_n)\circ h)(u_n),\ v^*_n \in -Y^*_+,\ \langle v^*_n, v_n - h(u_n)\rangle = 0,$$

$$x_{i,n} \xrightarrow{\|\cdot\|} \bar{x},\ w_{j,n} \xrightarrow{\|\cdot\|} \bar{x},\ c_n \xrightarrow{\|\cdot\|} \bar{x},\ u_n \xrightarrow{\|\cdot\|} \bar{x},\ y_n \xrightarrow{\|\cdot\|} h(\bar{x}),\ v_n \xrightarrow{\|\cdot\|} h(\bar{x}),$$

$$[\sum_{i\in\Delta} \lambda_i x^*_{i,n}] + [\sum_{j\in\Delta\cap\Omega} (\lambda_j \frac{f_j(\bar{x})}{g_j(\bar{x})}) w^*_{j,n}] + c^*_n + u^*_n \xrightarrow{\|\cdot\|_*} 0,\ y^*_n + v^*_n \xrightarrow{\|\cdot\|_*} 0,$$

$$f_i(x_{i,n}) - \langle x^*_{i,n}, x_{i,n} - \bar{x}\rangle \xrightarrow[n\to\infty]{} f_i(\bar{x}),$$

$$g_j(w_{j,n}) + \langle w^*_{j,n}, w_{j,n} - \bar{x}\rangle \xrightarrow[n\to\infty]{} g_j(\bar{x}),$$

$$\langle c^*_n, c_n - \bar{x}\rangle \xrightarrow[n\to\infty]{} 0,$$

$$\langle y^*_n, y_n - h(\bar{x})\rangle \xrightarrow[n\to\infty]{} 0,$$

$$\langle u^*_n, u_n - \bar{x}\rangle + \langle v^*_n, v_n - h(\bar{x})\rangle \xrightarrow[n\to\infty]{} 0,$$

with $i \in \Delta$, $j \in \Delta \cap \Omega$ *and* $n \in \mathbb{N}$.

Proof. We study the situation $\sigma = w$.

First we proceed using the parametric approach [30] by considering

$$(MFP_{\bar{x}}): \quad \inf_{\substack{x\in C \\ h(x)\in -Y_+}} \{f_1(x) - \frac{f_1(\bar{x})}{g_1(\bar{x})}g_1(x), \cdots, f_q(x) - \frac{f_q(\bar{x})}{g_q(\bar{x})}g_q(x)\}.$$

Directly from Remark 4.2.6, we deduce that $\bar{x}$ is a weak efficient solution of (MFP) if and only if $\bar{x}$ is a weak efficient solution of $(MFP_{\bar{x}})$. By (4.2) and (4.1), this is equivalent to

$$0 \in \partial^w((\sum_{i=1}^{q} F^i) + (\sum_{i=1}^{q} G^i_{\bar{x}}) + \delta^v_C + \delta^v_{-Y_+} \circ h)(\bar{x}). \tag{4.17}$$

where

$$\begin{array}{rccc} F^i: & X & \longrightarrow & \mathbb{R}^q \\ & x & \longrightarrow & \left(0, ..., f_i(x), ..., 0\right) \end{array}, \qquad \begin{array}{rccc} G^i_{\bar{x}}: & X & \longrightarrow & \mathbb{R}^q \\ & x & \longrightarrow & \left(0, ..., -\frac{f_i(\bar{x})}{g_i(\bar{x})}g_i(x), ..., 0\right) \end{array}.$$

Now consider the following auxiliary vector valued mappings with values in $\mathbb{R}^q$:

$$\phi_i(x,y) := F^i(x),$$

$$\varphi_i(x,y) := G^i_{\bar{x}}(x),$$

$$\gamma(x,y) := \delta^v_C(x),$$

$$\psi(x,y) := \delta^v_{-Y_+}(y),$$

$$H(x,y) := \delta^v_{\mathrm{epi}h}(x,y),$$

where $(x,y) \in X \times Y$ and $i \in \{1, \cdots, q\}$. Similarly to the precedent section, we have that (4.17) is equivalent to :

$$(0,0) \in \partial^w((\sum_{i=1}^{q} \phi_i) + (\sum_{i=1}^{q} \varphi_i) + \gamma + \psi + H)(\bar{x}, h(\bar{x})).$$

According to Theorem 4.3.3, there exist $\lambda^* := (\lambda_1, \cdots, \lambda_q) \in \mathbb{R}_+^q \setminus \{0\}$ and sequences $\{(x_{i,n}, \bar{x}_{i,n})\}_n, \{(w_{i,n}, \bar{w}_{i,n})\}_n \subseteq X \times Y$, $\{(c_n, \bar{c}_n)\}_n \subseteq C \times Y$, $\{(\bar{y}_n, y_n)\}_n \subseteq X \times -Y_+$, $\{(u_n, v_n)\}_n \subseteq \operatorname{epi} h$, $\{(\hat{x}_{i,n}^*, \bar{x}_{i,n}^*)\}_n$, $\{(\hat{w}_{i,n}^*, \bar{w}_{i,n}^*)\}_n$, $\{(c_n^*, \bar{c}_n^*)\}_n$, $\{(\bar{y}_n^*, y_n^*)\}_n$, $\{(u_n^*, v_n^*)\}_n \subseteq X^* \times Y^*$ with $i \in \{1, \cdots, q\}$ satisfying

$$
\begin{gathered}
(\hat{x}_{i,n}^*, \bar{x}_{i,n}^*) \in \partial(\lambda^* \circ \phi_i)(x_{i,n}, \bar{x}_{i,n}) = \partial(\lambda_i f_i)(x_{i,n}) \times \{0\}, \\
(\hat{w}_{i,n}^*, \bar{w}_{i,n}^*) \in \partial(\lambda^* \circ \varphi_i)(w_{i,n}, \bar{w}_{i,n}) = \partial(\lambda_i \frac{f_i(\bar{x})}{g_i(\bar{x})}(-g_i))(w_{i,n}) \times \{0\}, \\
(c_n^*, \bar{c}_n^*) \in \partial(\lambda^* \circ \gamma)(c_n, \bar{c}_n) = N_C(c_n) \times \{0\}, \\
(\bar{y}_n^*, y_n^*) \in \partial(\lambda^* \circ \psi)(\bar{y}_n, y_n) = \{0\} \times N_{-Y_+}(y_n), \\
(u_n^*, v_n^*) \in \partial(\lambda^* \circ H)(u_n, v_n) = \partial \delta_{\operatorname{epi} h}(u_n, v_n), \qquad (4.18) \\
x_{i,n} \xrightarrow{\|\cdot\|} \bar{x},\ w_{i,n} \xrightarrow{\|\cdot\|} \bar{x},\ c_n \xrightarrow{\|\cdot\|} \bar{x},\ \bar{y}_n \xrightarrow{\|\cdot\|} \bar{x},\ u_n \xrightarrow{\|\cdot\|} \bar{x}, \\
\bar{x}_{i,n} \xrightarrow{\|\cdot\|} h(\bar{x}),\ \bar{w}_{i,n} \xrightarrow{\|\cdot\|} h(\bar{x}),\ \bar{c}_n \xrightarrow{\|\cdot\|} h(\bar{x}),\ y_n \xrightarrow{\|\cdot\|} h(\bar{x}),\ v_n \xrightarrow{\|\cdot\|} h(\bar{x}), \\
(\sum_{i=1}^{q} \hat{x}_{i,n}^*) + (\sum_{i=1}^{q} \hat{w}_{i,n}^*) + c_n^* + \bar{y}_n^* + u_n^* \xrightarrow{\|\cdot\|_*} 0,\ (\sum_{i=1}^{q} \bar{x}_{i,n}^*) + (\sum_{i=1}^{q} \bar{w}_{i,n}^*) + \bar{c}_n^* + y_n^* + v_n^* \xrightarrow{\|\cdot\|_*} 0, \\
\lambda_i f_i(x_{i,n}) - \langle \hat{x}_{i,n}^*, x_{i,n} - \bar{x} \rangle - \langle \bar{x}_{i,n}^*, \bar{x}_{i,n} - h(\bar{x}) \rangle \xrightarrow[n \to \infty]{} \lambda_i f_i(\bar{x}), \\
\lambda_i \frac{f_i(\bar{x})}{g_i(\bar{x})}(-g_i)(w_{i,n}) - \langle \hat{w}_{i,n}^*, w_{i,n} - \bar{x} \rangle - \langle \bar{w}_{i,n}^*, \bar{w}_{i,n} - h(\bar{x}) \rangle \xrightarrow[n \to \infty]{} \lambda_i \frac{f_i(\bar{x})}{g_i(\bar{x})}(-g_i)(\bar{x}), \\
\langle c_n^*, c_n - \bar{x} \rangle + \langle \bar{c}_n^*, \bar{c}_n - h(\bar{x}) \rangle \xrightarrow[n \to \infty]{} 0,
\end{gathered}
$$

$$\langle \bar{y}_n^*, \bar{y}_n - \bar{x} \rangle + \langle y_n^*, y_n - h(\bar{x}) \rangle \xrightarrow[n\to\infty]{} 0,$$

$$\langle u_n^*, u_n - \bar{x} \rangle + \langle v_n^*, v_n - h(\bar{x}) \rangle \xrightarrow[n\to\infty]{} 0,$$

with $i \in \{1, \cdots, q\}$ and $n \in \mathbb{N}$.

As in the section before, (4.18) is equivalent to :

$$u_n^* \in \partial((-v_n^*) \circ h)(u_n),\ v_n^* \in -Y_+^*,\ \langle v_n^*, v_n - h(u_n) \rangle = 0.$$

For $i \in \{1, \cdots, q\}$, $\{\bar{x}_{i,n}^*\}_n$, $\{\bar{w}_{i,n}^*\}_n$, $\{\bar{c}_n^*\}_n$, $\{\bar{y}_n^*\}_n$ are null and $\{\bar{x}_{i,n}\}_n$, $\{\bar{w}_{i,n}\}_n$, $\{\bar{c}_n\}_n$, $\{\bar{y}_n\}_n$ are superfluous. Thus the announced result follows by setting $\Delta := \{k \in \{1, \cdots, q\} : \lambda_k > 0\}$. Similarly we have the result of strong efficient solutions. □

Remark 4.5.2. Using the precedent approach of sequential Pareto subdifferential calculus, the case of sequential proper efficiency of (MFP) is more delicate. Also, it is interessant to provide the nonsmooth counterpart of Theorem 4.5.1 or even develop second order approach. These may be the objects of future researchs.

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