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Contact problems with Coulomb friction in elasto-plasticity and electro-elasto-plasticity for Hencky type materials: Mathematical study and numerical resolution

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Abstract

This thesis work concerns the variational and numerical analysis of some non-linear contact problems with or without friction resulting from contact mechanics. This analysis is treated in static processes and under the hypothesis of small deformations for elasto-plastic and electro-elasto-plastic Hencky-type materials. The general context of the study presented in this manuscript is that of non-linear partial differential equations intervening in the field of contact mechanics. This work is composed of four parts which are successively articulated around the mechanical modeling, variational and numerical analysis of some non-linear problems arising from contact mechanics.

The first part is devoted to the mathematical and mechanical tools necessary for a better understanding of the problems studied in this thesis. We present the various mechanical contact models considered, then we recall the functional spaces and the main notations used in this manuscript.

The second part is devoted to the mathematical and numerical study of a unilateral contact problem with non-local Coulomb friction. After having established a theorem of existence and uniqueness of the weak solution under certain hypotheses, we present a successive iterative scheme of the fixed point that transforms the elasto-plastic contact problem with Coulomb's law of friction into a sequence of contact problems with Tresca's law in linear elasticity. The contact boundary conditions are treated numerically, using the Augmented Lagrangian method combined with Uzawa block relaxation on the one hand, and the penalization approach on the other hand.

The third part concerns the theoretical and numerical study of two mathematical models that describe the contact between a piezoelectric body and a rigid conductive foundation. The non-linear constitutive law of the Hencky-type material is assumed to be electro-elasto-plastic in static processes. The contact is modeled using Signorini's conditions or contact conditions with normal compliance, the non-local Coulomb friction law and a regularized electrical contact condition. For each mathematical model, we give a theorem of existence and uniqueness. Then, we study an interesting result from a physical and numerical point of view, where we can observe that the solution of the problem which describes the nonlocal frictional contact between a nonlinear piezoelectric body and an electrically conductive foundation, approaches so closely that of the frictionless contact problem between a nonlinear piezoelectric body and non-conductive foundation, as the friction and electrical conductivity coefficients approach so closely to zero. Finally, we obtain numerical solutions to the problems using a successive iterative fixed-point method; we also establish their convergence, and realize the numerical treatment of the contact conditions using an Augmented Lagrangian type formulation combined with Uzawa type algorithms.

The fourth part consists of implementing the numerical methods through a number of numerical experiments on two-dimensional test problems aimed to illustrate, among other things, the convergence and the performance of the algorithms presented in the previous chapters using $\mathbb{P}_1$ triangular finite element method.

Keywords: *Elasto-plastic materials; Piezoelectric materials; Nonlinear elastic constitutive Hencky's law; Contact laws; Friction laws; Variational inequalities; Numerical methods; Numerical simulation*

Résumé

Ce travail de thèse concerne l'analyse variationnelle et numérique de quelques problèmes non linéaires de contact avec ou sans frottement issus de la mécanique du contact. Cette analyse est traitée dans des processus statiques et sous l'hypothèse des petites déformations pour des matériaux de type Hencky élasto-plastique et électro-élasto-plastique. Le contexte général dans lequel se situe l'étude présentée dans ce mémoire est celui des équations aux dérivées partielles non linéaires intervenant dans le domaine de la mécanique du contact. Ce travail est composé de quatre parties qui s'articulent successivement autour de la modélisation mécanique, analyse variationnelle et numérique de quelques problèmes non linéaires issus de la mécanique du contact.

La première partie est consacrée aux outils mathématiques et mécaniques nécessaires à une meilleure compréhension des problèmes étudiés dans ce mémoire. Nous présentons les divers modèles mécaniques de contact considérés, puis nous rappelons les espaces fonctionnels et les principales notations utilisées dans ce manuscrit.

La deuxième partie est consacrée à l'étude mathématique et numérique d'un problème de contact unilatéral avec frottement non-locale de Coulomb. Après avoir établi un théorème d'existence et d'unicité de la solution faible sous certaines hypothèses, nous présentons un schéma itératif successive du point fixe qui transforme le problème de contact élasto-plastique avec la loi de frottement de Coulomb en une suite de problèmes de contact avec la loi de Tresca en élasticité linéaire. Les conditions aux limites de contact sont traités numériquement, en utilisant la méthode de Lagrangienne Augmentée combinée avec la relaxation par bloc d'Uzawa, d'une part, et l'approche par pénalisation, d'autre part.

La troisième partie concerne l'étude théorique et numérique de deux modèles mathématiques qui décrivent le contact entre un corps piézoélectrique et une fondation rigide conductrice. La loi constitutive non linéaire du matériau de type Hencky est supposée électro-élasto-plastique dans des processus statiques. Le contact est modélisé en utilisant les conditions de Signorini ou conditions de contact avec compliance normale, la loi de frottement de Coulomb non local et une condition de contact électrique régularisée. Pour chaque modèle mathématique, nous donnons un théorème d'existence et d'unicité. Puis, nous étudions un résultat intéressant d'un point de vue physique et numérique, où nous pouvons observer que la solution du problème qui décrit le contact avec frottement non-local entre un corps piézoélectrique et une fondation électriquement conductrice, s'approche si près de celle du problème du contact sans frottement entre un corps piézoélectrique et une fondation non-conductrice, quand les coefficients de frottement et de conductivité électrique se rapprochent si près de zéro. Enfin, nous obtenons des solutions numériques des problèmes à l'aide d'une méthode itérative successive du point fixe; nous établissons également leur convergence, et nous réalisons le traitement numérique des conditions de contact par une formulation de type Lagrangien Augmentée combinée avec des algorithmes de type Uzawa.

La quatrième partie consiste à mettre en œuvre les méthodes numériques à travers une série d'expériences numériques sur des problèmes test bidimensionnels visant à illustrer, entre autres, la convergence et la performance des algorithmes présentés dans les chapitres précédents par la méthode des éléments finis triangulaire $\mathbb{P}_1$.

Mots clés : *Matériaux élasto-plastiques; Matériaux piézoélectriques; Lois de contact; Lois de frottement; Inégalités variationnelles; Méthodes numériques; Simulations numériques.*

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Notations and Symbols

Ω	a domain of $\mathbb{R}^d$ $(d = 1, 2, 3)$.
$\overline{\Omega}$	the adherence of Ω.
Γ	the boundary of Ω.
$(\Gamma_i)_{i=1,2,3}$	a measurable partition of Γ.
$(\Gamma_i)_{i=a,b}$	another measurable partition of Γ.
$meas(\Gamma_i)$	Lebesgue's measure of Γ_i.
$\mathbb{S}^d$	the space of symmetric tensors of second order on $\mathbb{R}$.
ν	the unit outward normal vector on Γ.
v_ν, v_τ	the normal and tangential components of the vector v.
$v_{i,j}$	the partial derivation $\dfrac{\partial v_i}{\partial x_j}$ of the component v_i respect to the variable x_j.
ϱ	the normalized gap.
Div σ	the divergence operator for tensors, *i.e.*, Div $\sigma = (\sigma_{ij,j})$.
$\mathcal{D}(\Omega)$	the space of test functions on Ω.
$\mathcal{D}'(\Omega)$	the dual space of $\mathcal{D}(\Omega)$.
$\mathcal{C}^m(\Omega)$	the space of real functions m times continuously differentiable on Ω.
$L^p(\Omega)$	the space of measurable functions v defined on Ω with the p-power absolutely integrable
$W^{m,p}(\Omega)$	the Sobolev space $\{\, v \;;\; D^\alpha v \in L^p(\Omega) \quad \text{for} \quad \lvert\alpha\rvert \leq m\}$.
H	the product space $L^2(\Omega)^d$.
H_1	the product space $H^1(\Omega)^d$.
$\mathcal{H}$	the space $\{\, \sigma = (\sigma_{ij})\,;\; \sigma_{ij} = \sigma_{ji} \in L^2(\Omega)\}$.
$\mathcal{H}_1$	the space $\{\, \sigma \in \mathcal{H}\,;\; \mathrm{Div}\,\sigma \in H\}$.
$H^{\frac{1}{2}}(\Gamma)$	the Sobolev space of order $\frac{1}{2}$ on Γ.

H_Γ	the product space $H^{\frac{1}{2}}(\Gamma)^d$.
$H^{-\frac{1}{2}}(\Gamma)$	the dual space of $H^{\frac{1}{2}}(\Gamma)$.
H'_Γ	the dual space of H_Γ.
$\gamma : H_1 \to H_\Gamma$	the trace map for vectorial functions.
V	the space $\{ v \in H_1\, ;\ v = 0 \text{ on } \Gamma_1\}$.
W	the space $\{ \varphi \in H^1(\Omega)\, ;\ \varphi = 0 \text{ on } \Gamma_a\}$.
K	the space of admissible displacements $\{ v \in V\, ;\ v_\nu - \varrho \leq 0 \text{ on } \Gamma_1\}$.
X	the product space $V \times W$.
U	the product space $K \times W$.
$\langle \cdot, \cdot \rangle_\Gamma$	the duality pairing between H'_Γ and H_Γ.
$(\cdot, \cdot)_X$	the scalar product of the real Hilbert space X.
$\| \cdot \|_X$	the associated norm to the real Hilbert space X.
$\mathscr{L}(X, Y)$	the space of linear continuous operators from X to Y.
$\| . \|_{\mathscr{L}(X,Y)}$	the associated norm to space $\mathscr{L}(X, Y)$.
$x_n \to x$	the strong convergence of the sequence x_n towards the element x of X.
$x_n \rightharpoonup x$	the weak convergence of the sequence x_n towards the element x of X.
$dom\, f$	the domain of a function f, *i.e.*, $\{ x\, ;\ f(x) < +\infty\}$.
$supp\, f$	the support of a function f.
∇f	the gradient of the function f.
$rot\, B$	the rotational of a field B.
$\mathrm{div}\ f$	the divergence of the function f.
φ	the electric potential.
u	the displacement field.
D	the electric displacement field.
σ	the stress tensor.
$\overline{\sigma}$	the deviatoric part of σ.
$\varepsilon(v)$	the linearized tensor of deformations.
$\overline{\varepsilon}(v)$	the deviatoric part of $\varepsilon(v)$.
$\mathfrak{F}$	the elasticity operator.
$\mathcal{E}$	the piezoelectric tensor.
$\mathcal{E}^*$	the transposed of $\mathcal{E}$.
β	the electric tensor.
I	the identity tensor.
tr	the trace operator.
f_0	the volume force of density.

f_2	the surface traction of density.
q_0	the density of volume electric charges.
q_2	the surface electric charges density.
μ	the friction coefficient.
k_e	the electrical conductivity coefficient.
k_0	the bulk modulo (a material coefficient).
h	the discretization parameter.
ϵ	the penalization parameter.
$\liminf$	the lower limit.
$\limsup$	the upper limit.
a.e.	almost every where.

General introduction

The phenomena of contact between bodies, deformable or not, remain until today omnipresent in daily life. It might be considered as one of the foundations of mechanical engineering, with a particularly wide field of application, for example, we can mention the railway sector, the automotive industry, civil engineering and the aeronautics field. The mathematical modeling of such phenomena is not without difficulties. Indeed, their representation uses partial differential equation systems with boundary conditions that may be relatively difficult to establish, depending on the complexity of the considered problem. The scope of this manuscript lies at the junction between two fields, namely applied mathematics and mechanics. The transversality of such a combination has several aspects which are manifested initially by the mathematical and numerical modeling of nonlinear mechanical problems, in this case, the frictional contact associated with the behavioral laws of certain materials. Mathematical considerations then take precedence over mechanics through approximation and variational analysis, the study of the existence and uniqueness of a solution to the problem, numerical analysis and scientific calculation more broadly. In addition, although the interpretation of results obtained using a numerical method is mainly mechanical, it is possible to validate some theoretical results using numerical simulations.

It should be noted that although the mathematical theory of contact problems is based, in particular, on principles of mechanics of continuous media, as well as the variational and numerical analysis of models, this is in fact a relatively recent topic of study.

Significant breakthroughs were made by Coulomb in 1785 [44], to confirm and extend the results of the Amontons formulated in 1699 on friction laws. Then, in the XIXth century, Hertz was able to solve analytically a problem involving a frictionless contact between two elastic bodies for specific geometries [65]. It should also be noted that its results are still frequently used in the literature as a reference solution to assess the accuracy of a solution obtained using a numerical method. It would, however, be necessary to wait for Signorini's work [139] in 1933, to set the basis for a contact problem between a deformable body and a rigid foundation, and his student Fichera, who solve the problem [56] via the minimization of the total energy on a convex. Moreover, the use of variational inequalities to formulate a contact problem is due to him. From this basis, Duvaut and Lions established a mathematical theory of the mechanics of contact in [51]; they have introduced some variational formulations as well as results of existence and uniqueness. Of course, this does not stop there; many works on the numerical resolution and analysis of variational problems resulting from contact mechanics followed. In this regard, we should mention the contributions of Hlaváček et al. [71], those of Moreau [104] or those of Kikuchi and Oden [79].

Although there are still a significant number of problems to be explored to this day, such a development necessarily requires the elaboration of new contact and friction models. We propose to review some of the contact laws discussed in this manuscript. In the literature, Signorini's law, which describes the contact between a deformable solid and a perfectly rigid foundation, remains one of the most widely used laws nowadays. Nevertheless, it is an idealization of reality insofar as it does not take into account irregularities inherent in contact surfaces such as micro-asperities, regardless of applied forces. For this reason, the conditions that characterize Signorini's law can be replaced by an approximation involving a penalty called normal compliance, introduced for the first time in [113]. The foundation, or at least its surface, is then considered deformable, thereby making the obtained problem easier to study. Once the existence and uniqueness of the solution to the problem have been established using abstract results from variational inequalities theory or a fixed point method, among others, the difficulty resides in the passage to the limit. Indeed, to proceed in this way implies that the rigidity of the foundation tends towards infinity, thus, considering a perfect foundation from a mechanical point of view, permitting to come closer to the solution of the original problem (cf. [14, 36]). Nevertheless, here again, the normal compliance law has its limits; inherent in the case where there is no penetration. The existence of a foundation in which penetration would not be limited remains quite questionable from a physical point of view. To overcome this, one possible idea is to combine the two laws mentioned above to obtain the law of normal compliance with unilateral constraint, introduced in [76], for which the foundation can be seen as a base made of a rigid material where a layer of deformable asperities rests.

This goes without saying that it is difficult to conceive of a relevant model of the contact phenomenon without taking into account the friction phenomenon, with the disadvantage of making the problem even more complex, from a mechanical, mathematical and numerical point of view. Even though, the frictionless case has been fully addressed in [56], it will take a few more years to consider a problem involving a friction law (cf. [47]). Let us recall for all intents and purposes that Coulomb's friction remains, to this day, one of the most widely used laws in the literature. Such a law has two notable characteristics: the notion of a friction limit and a coupling with normal stress, as opposed to the Tresca law for which the limit is assumed to be constant. The Signorini problem with friction was solved mathematically in [110, 118], under the assumption that the friction coefficient is small enough. This is a recurrent difficulty in the mathematical study of contact problems; the question of whether this is a limitation inherent in the mathematical tools used or an intrinsic characteristic of the mechanics of the problem, remains open. Nevertheless, it has been shown that a too high coefficient of friction can lead not only to the loss of the uniqueness of the solution (cf. [9, 67, 68]) but also to the loss of the existence of the solution (cf. [84, 94]). Moreover, although such a law is commonly used, it remains insufficient for taking into account the influence of the microstructure of the contact surface insofar as the friction coefficient is considered constant. Studies have shown that physical quantities such as temperature (cf. [123, 124]), or the norm of the tangential velocity (cf. [136]) can have a significant influence on the friction coefficient; such dependence may also be a source of non-uniqueness (cf. [131]). In addition, there are two characteristic values for friction: a static friction coefficient, representing the ratio between the normal and tangential stresses below which there is no slip, and a dynamic friction coefficient representing the value of this ratio when the slip occurs during the actual movement. Typically, the dynamic friction coefficient is lower than the static friction coefficient, reflecting the idea that it is easier to maintain an object in a sliding motion status than to slide an object at rest. Such a phenomenon has been studied in [122] within the geophysical context of earthquakes. In this respect, we can refer to [14, 23, 24, 31, 98, 131, 138], in which varying friction coefficients intervene. It should also be noted that such problems cannot be addressed with standard convex analysis and quadratic programming tools. Indeed, a non-monotonous friction law has the consequence of making problems non-convex, forcing the use of new tools resulting from the nonsmooth and nonlinear analysis. One possible numerical treatment for this type of problem consists in bringing it down to a series of convex problems, each of which is solved with standard tools. For more details on the subject, the reader may refer to [101, 102, 130]. At this stage, we draw attention to the fact that we have only mentioned here a sample of the existing contact laws, which were later used in this manuscript; other laws modeling contact at the interface, as well as materials behavior, are included in [4, 88, 127, 144, 155].

As we have already mentioned so far, problems arising from contact mechanics are, in most cases, too complex to be treated analytically in dimensions greater than one. This is why it is necessary to employ numerical approaches. Among the possible methods, and due to its proximity to solid mechanics, Finite element method (FEM) is, and still, one of the most widely used to discretize the geometry of the body into a set of elements of which the size is denied. The pioneers in this field initially worked on frictionless elastic problems, see [7, 33, 43, 74, 86, 154], followed a few years later by friction problems, as in [105, 106, 125, 135]. However, the nonlinear and nonregular nature of a mathematical problem with contact and friction is not without some difficulties, from a numerical point of view, which is why it is necessary to develop methods that are sufficiently robust, accurate and stable to overcome them.

Without being exhaustive, we propose to review some of the major families of methods used in contact mechanics. The two most common methods used to treat contact and friction conditions are the Augmented Lagrangian method [57, 58, 60, 86, 155] and the penalty method [21, 24, 36, 49, 155], the last-mentioned can also be seen as a special case of the Augmented Lagrangian and is, in particular, directly related to the normal compliance condition. Penalty method provides two main advantages: First, it appears very simple to implement. The second is that this technique does not require the introduction of additional variables in the problem. This simplicity makes it one of the most used methods by finite element calculation codes. Indeed, this approach does not require the modification of the code structure. The nonlinear systems that result from these techniques are, in most cases, solved by the Uzawa algorithm [87] or a Newtonian method [3, 6, 132]. Indeed, to this day, the generalized Newton method remains widely used in contact mechanics, in particular for its ability to treat nonlinearities in a single iteration (plasticity, contact, friction) as well as for its quadratic convergence speed in the neighborhood of the solution. Alart and Curnier presented [7] a formulation based on Augmented Lagrangian method combined with a generalized Newton method to solve continuous but nondifferentiable equations resulting from frictional contact problems. Considering this same Lagrangian, Simo and Laursen proposed a method [141] combining the Uzawa algorithm, for contact constraint, and the generalized Newton method for displacement, thus, the penalty parameter allows to accelerate the convergence of the Uzawa algorithm and the multipliers allow to check the contact and friction conditions in a strict way when approaching the solution. Despite the advantages of the Newtonian method, the sequence of problems resulting from this method has no mechanical sens, for this reason, in this manuscript we propose an iterative solution scheme based on the Kačanov method [23, 24, 61, 63, 64, 109] that deals with the nonlinearity that arises from the contact conditions and from the constitutive law; it transforms the constitutive equations into an incremental recursive form, thus, we obtain a linear behavior in each iteration. It should be noted that linear systems resulting from contact problems are known to be, on one hand, particularly ill-conditioned and, on the other hand, augmented in part because of Lagrange multipliers. These are the two main reasons why some authors have sought to implement methods that do not use them. As such, we can cite Nitsche's method [35, 37–39] as well as the active set strategy [69, 70, 72, 73], for which it is then necessary to determine all the nodes in contact with the foundation by directly imposing relatively simple boundary conditions. There are many other methods in the literature: the method of successive relaxations with projection [128], a robust method but which may require a high number of iterations, the Gauss-Seidel non-linear method [77], also referred to in this context as the successive equilibrium algorithm, the projected and preconditioned conjugated gradient method, used in particular in [133] in the context of granular media, relaxation procedures [26], to solve unilateral contact problems with friction, or the Lemke method [89], a direct type method. It is also possible to mention the stabilized Lagrangian techniques which offer interesting properties in terms of reliability and robustness [83]. We can also refer to [126, 129], in which a presentation of several numerical resolution methods for contact problems is given, some of which have been mentioned. Note that in the case of generalized Newtonian methods, the question of the choice of linear solver and preconditioner arises, due to the nonregular nature of the contact problems and the conditioning of the linear systems resulting from them, after discretization. Such considerations are addressed in [5, 8].

This thesis is divided into fourth parts.

The first part is devoted to the mathematical and mechanical tools necessary for a better understanding of the problems studied in this thesis. More specifically, Chapter 1 aims to set the physical framework, by presenting the fundamental contact problem we are dealing with. The intention is to present the various laws that govern the behavior of the materials used and then to model the different types of contact treated. In Chapter 2, we recall some elements of functional analysis and discuss real-valued function spaces.

In the second part, which is composed of two chapters, we study and analyze a problem of contact with friction under the hypothesis of small deformations involving nonlinear Hencky-type materials. Chapter 3 concerns a static elasto-plasticity problem with Signorini conditions and Coulomb's nonlocal friction law. After having established the strong formulation and made the necessary assumptions on data to obtain a weak formulation, it is necessary to show the existence and uniqueness of a weak solution, and that is carried out in several steps and based on arguments of variational inequalities, pseudomonotone operators and a fixed point theorem. Afterward, an iterative scheme of the fixed point that transforms the elasto-plastic contact problem with Coulomb's friction law into a sequence of contact problems with Tresca's friction law in linear elasticity is pointed out. Finally, a variational formulation equivalent to an incremental step is transformed into quadratic programming which is suitable for numerical analysis, and by using the Augmented Lagrangian method we have enforced the contact constraints where the normal contact forces are treated as independent variables, and we have reached a solution by using Uzawa block relaxation method. Chapter 4 deals with the same model presented in the precedent one. A numerical model based on the penalty method has been formulated for this model, and an existence and uniqueness result in addition to a convergence result are stated. Here, we are interested in the numerical analysis of the problem in the form of a numerical approximation using finite element method. This chapter ends with an iterative solution scheme as well as its convergence.

In the third part, we deal with two chapters on the theoretical and numerical study of two static problems which describe the contact between a piezoelectric body and an obstacle, the so-called foundation. The constitutive relation of the material is assumed to be electro-elastic and involves the nonlinear elastic constitutive Hencky's law. The contact conditions used are Coulomb's friction law, and the regularized electrical conductivity condition with a unilateral contact condition in Chapter 5, and normal compliance with unilateral constraint in Chapter 6. For each type of problem, we give a physical formulation as well as a weak formulation. The existence and uniqueness of the weak solution are studied. In addition, in Chapter 6, we have studied an interesting result from a physical and numerical point of view, where we can observe that the solution of the problem which describes the nonlocal frictional contact between a nonlinear piezoelectric body and an electrically conductive foundation, approached so closely to that of the frictionless contact problem between a nonlinear piezoelectric body and non-conductive foundation, as the friction and electrical conductivity coefficients have approached so close to zero. Finally, at the end of each chapter the numerical solutions of the problems are achieved by using a successive iterative fixed-point method; their convergence are also established, and the numerical treatment of the contact conditions is realized using an Augmented Lagrangian type formulation combined with Uzawa type algorithms.

The fourth part consists of two chapters in which we are interested in implementing the numerical methods through a number of numerical experiments on two-dimensional test problems aimed to illustrate, among other things, the convergence and the performances of the algorithms presented in the previous chapters using $\mathbb{P}_1$ triangular finite element method.

The manuscript concludes with the results obtained and an outline of possible mechanical, mathematical and numerical perspectives that are in line with the continuity of this research work.

Part I

Preliminary and pre-requirements

Chapter 1

Physical modeling of some mechanical structures in contact

In order to facilitate the comprehension of this manuscript, we thought it would be useful to present in this first chapter; the physical and functional framework in which we will work: we start by specifying the physical framework, briefly recalling some notions of the mechanics of continuous media, the behavior laws for the materials studied, as well as the conditions at the contact limits with or without friction. All the variables in this chapter are assumed to have a sufficient degree of smoothness so that all the necessary mathematical manipulations are justified.

1.1 Description of elastic structures in a contact process

1.1.1 Mechanical framework

A body occupying, in its reference configuration, an open and bounded domain $\Omega \subset \mathbb{R}^d$, $d = 2, 3$ is considered, with a sufficiently regular boundary Γ, partitioned into three disjoint measurable parts Γ_1, Γ_2 and Γ_3, such that $meas(\Gamma_1) > 0$. A volume force of density f_0 acts in Ω. The body is supposed to be clamped on Γ_1, and thus the displacement field vanishes there. A surface traction of density f_2 acts on Γ_2. In the initial configuration, the body may come in contact over Γ_3 with a rigid foundation.

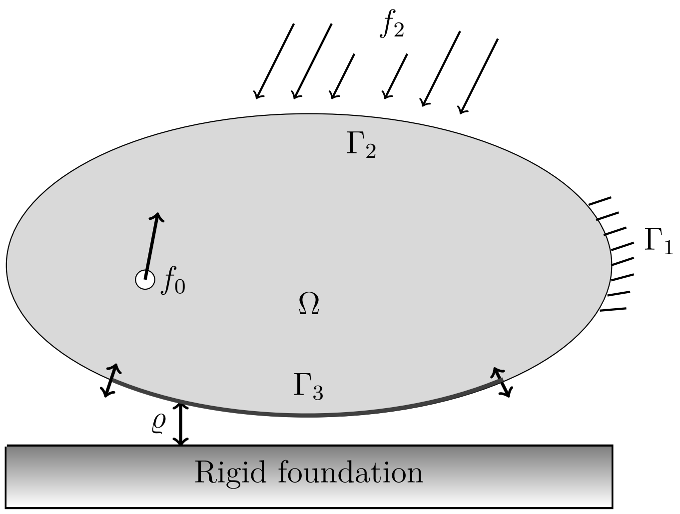


Figure 1.1 – Mechanical framework.

Here and through this manuscript, we do not indicate explicitly the dependence of various functions on the spatial variable $x \in \overline{\Omega}$ in order to simplify the notation. Moreover, in the sequel, the indices i, j run between 1 and d, the summation convention over repeated index is used, and the index that follows a comma indicates the partial derivative with respect to the corresponding component of the independent variable, e.g., $u_{i,j} = \dfrac{\partial u_i}{\partial x_j}$.

We denote by $\mathbb{S}^d$ the space of second order symmetric tensor on $\mathbb{R}^d$. We define the inner products and the corresponding norms on $\mathbb{R}^d$ and $\mathbb{S}^d$ by

$$u \,.\, v = u_i\, v_i \ ; \quad \|v\| = (v \,.\, v)^{\frac{1}{2}} \quad \forall\, u,\, v \in \mathbb{R}^d,$$

$$\sigma \,:\, \tau = \sigma_{ij}\, \tau_{ij} \ ; \quad \|\tau\| = (\tau \,:\, \tau)^{\frac{1}{2}} \quad \forall\, \sigma,\, \tau \in \mathbb{S}^d.$$

Throughout this manuscript, we adopt the following notation: $u \,:\, \Omega \to \mathbb{R}^d$ for the displacement field and $\sigma \,:\, \Omega \to \mathbb{S}^d$, $\sigma = (\sigma_{ij})$ for the stress tensor. Moreover, let $\varepsilon(u) = (\varepsilon_{ij}(u))$ denote the linearized strain tensor giving by $\varepsilon_{ij}(u) = 1/2(u_{i,j} + u_{j,i})$ and " Div " denote the divergence operator for tensors, *i.e.*, $\mathrm{Div}\,\sigma = (\sigma_{ij,j})$. Let ν be the unit outward normal vector on Γ, we denote the normal and tangential components of the displacement vector and the stress tensor by

$$v_\nu = v \,.\, \nu \ ; \quad v_\tau = v - v_\nu; \quad \sigma_\nu = \sigma\nu \,.\, \nu \ ; \quad \sigma_\tau = \sigma\nu - \sigma_\nu\, \nu.$$

The governing equations consist of the equilibrium equation and constitutive relation.

Since here the process is assumed to be static the equilibrium equation is given by

$$\operatorname{Div} \sigma + f_0 = 0 \quad \text{in } \Omega. \tag{1.1}$$

To complete the mechanical model, we prescribe the boundary conditions

$$u = 0 \quad \text{on } \Gamma_1, \tag{1.2}$$

$$\sigma\nu = f_2 \quad \text{on } \Gamma_2, \tag{1.3}$$

It should be noted that the considerations that follow are only valid in the case of small deformations. A behavioral law, or constitutive law, establishes a relationship between the stress tensor σ and the strain tensor ε.

In the following, we will recall the laws of behavior for elastic and elasto-plastic materials that we will use in the following chapters.

1.1.2 Elastic behavior law

A homogeneous media is said to be elastic if there is a reference configuration without stress and if the stress tensor depends only on the tensor of infinitesimal deformations ε calculated from this reference configuration, *i.e.*,

$$\sigma = \mathfrak{F}\varepsilon, \tag{1.4}$$

with $\mathfrak{F}$, a function that can be non-linear. In the case where $\mathfrak{F}$ explicitly depends on $x \in \Omega$ and the tensor of the infinitesimal deformations, the material is said to be non-homogeneous elastic.

If the material is linear elastic, then there is a linear relationship between the stress tensor σ and the infinitesimal strain tensor ε whose behavior can be characterized by Hooke's law, which in the three-dimensional case is of the shape

$$\sigma = \mathcal{A}\varepsilon, \tag{1.5}$$

where $\mathcal{A}$ is the fourth-order elasticity tensor, which comprises 81 components called elasticity constants of the material that are independent of the tensor of the deformations in the homogeneous case. However, due to the different material symmetries inherent in the assumed homogeneity and isotropy of the material (*i.e.*, all directions around a point are equivalent), we have

$$\mathcal{A}_{ijkl} = \mathcal{A}_{jikl} = \mathcal{A}_{ijlk}, \tag{1.6}$$

therefore, only 21 of them are independent. The components of such a tensor verify that

$$\mathcal{A}_{ijkh} = \lambda\delta_{ij}\delta_{kh} + \theta(\delta_{ik}\delta_{jh} + \delta_{ih}\delta_{jk}), \tag{1.7}$$

where the constants λ and θ are Lamé's coefficients, and δ_{ij} is Kronecker's symbol. Therefore, by combining (1.5) and (1.7), we obtain

$$\sigma_{ij} = \lambda\varepsilon_{kk}\delta_{ij} + 2\theta\varepsilon_{ij}. \tag{1.8}$$

Note that if the material is elastic, homogeneous, linear and isotropic, as considered in the above equation, it corresponds to the so-called classical elasticity scheme.

It should be noted that λ and θ can also be expressed as a function of E, Young's modulus, and κ, Poisson's coefficient,

$$\lambda = \frac{\kappa E}{(1+\kappa)(1-2\kappa)}, \quad \theta = \frac{E}{2(1+\kappa)}$$

which can be physically interpreted. In a simple tensile test of a specimen made of a material obeying the classical elasticity scheme, Young's modulus is the ratio between tension and longitudinal expansion, while Poisson's coefficient is the ratio between transverse contraction and longitudinal expansion.

1.1.3 Elasto-plastic behavior law

Let us denote by $\overline{\sigma}$ and $\overline{\varepsilon}$ the deviatoric part, respectively, of the stress and strain tensors which are defined by

$$\overline{\sigma} = \sigma - \frac{1}{d}\,\mathrm{tr}(\sigma)\,\mathrm{I}, \tag{1.9}$$

$$\overline{\varepsilon} = \varepsilon - \frac{1}{d}\,\mathrm{tr}(\varepsilon)\,\mathrm{I}\,. \tag{1.10}$$

where $\mathrm{tr}(\sigma) = \sigma_{ii}$, $\mathrm{tr}(\varepsilon) = \varepsilon_{ii}$ is the trace operator, and I is the identity tensor of second order. For a linearly isotropic, homogeneous material, the stored energy function is

$$M_0(\varepsilon) = \frac{1}{2}\,\lambda\,|\,\mathrm{tr}(\varepsilon)|^2 + \theta\,\|\varepsilon\|^2,$$

it can also be expressed as follows

$$M_0(\varepsilon) = \frac{1}{2}\, k_0 \,|\, \mathrm{tr}(\varepsilon)|^2 + \theta \,\|\overline{\varepsilon}\|^2,$$

where $k_0 = \lambda + 2\,\dfrac{\theta}{d}$ is the bulk modulus, λ and θ are Lamé coefficients. A nonlinear Hencky material is characterized by a stored energy function of the form

$$M(\varepsilon) = \frac{1}{2}\, k_0 \,|\, \mathrm{tr}(\varepsilon)|^2 + \theta\, \phi(\|\overline{\varepsilon}\|^2), \tag{1.11}$$

for a non-negative, continuously differentiable function ϕ satisfying

$$\phi(0) = 0, \quad \phi'(\xi) \geq c > 0. \tag{1.12}$$

The constitutive law $\dfrac{\partial M}{\partial \varepsilon}$ describes the behavior of the Hencky's materials (see for example [61, 63, 64, 109]), that is

$$\sigma = k_0 \,\mathrm{tr}(\varepsilon)\, \mathrm{I} + 2\, g(\|\overline{\varepsilon}\|^2)\overline{\varepsilon}, \tag{1.13}$$

with $g(\xi) = \theta\, \phi'(\|\xi\|^2)$, and we have

$$\overline{\sigma} = 2\theta\, \phi'(\|\overline{\varepsilon}(u)\|^2)\overline{\varepsilon}(u).$$

For the special case when $\phi(\xi) = \xi$, we obtain a linearly elastic material, and the constitutive law becomes

$$\sigma = k_0 \,\mathrm{tr}(\varepsilon)\, \mathrm{I} + 2\theta\, \overline{\varepsilon},$$

1.2 Description of piezoelectric structures in a contact process

1.2.1 Physical framework

A piezoelectric body occupies, in its reference configuration, an open and bounded domain $\Omega \subset \mathbb{R}^d$, $d = 2, 3$ with a sufficiently regular boundary Γ, partitioned into three disjoint measurable parts Γ_1, Γ_2 and Γ_3, such that $meas(\Gamma_1) > 0$, on one hand, and a partition of $\Gamma_1 \cup \Gamma_2$ into two open parts Γ_a and Γ_b, such that $meas(\Gamma_a) > 0$, on the other hand.

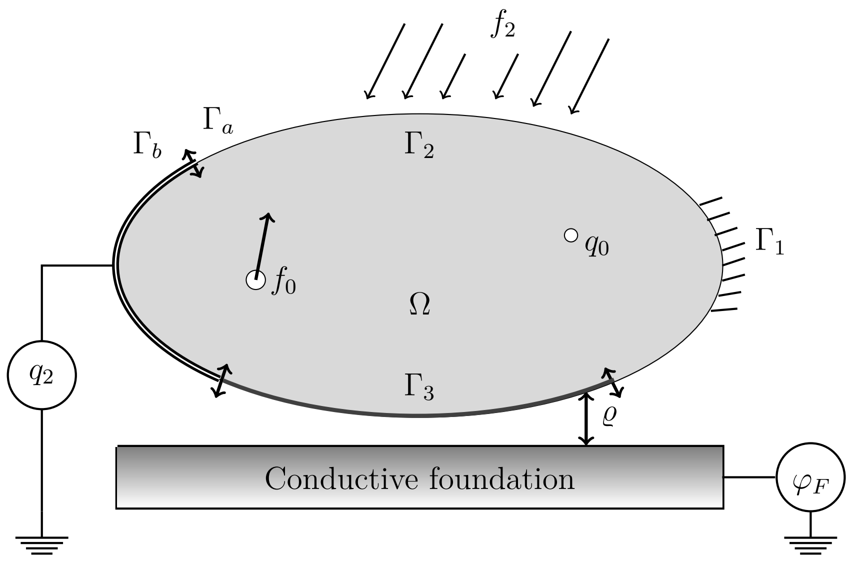


Figure 1.2 – Physical framework.

In the case of a piezoelectric material, the behavioral law contains a new unknown, the electric field E, hence the need to introduce another equation of equilibrium to manage it. This is the Maxwell-Gauss equation or the charge conservation equation

$$\operatorname{div} \mathrm{D} = q_0 \quad \text{in} \ \ \Omega, \tag{1.14}$$

where $\operatorname{div} \mathrm{D} = (\mathrm{D}_{j,j})$, and q_0 the density of volume electric charges. This equation is valid in a non-magnetic environment. This hypothesis has been confirmed experimentally by H.F. Tiersten [149].

We recall that the tensor ε of order 2 depends only on the vector u, in the same way, the electric field vector E derives other quantities, in particular from the electric scalar potential. More precisely, we know that an electric field varying with time induces a magnetic field B and vice versa; this phenomenon is reflected in the Maxwell-Ampére equations

$$rot\, \mathrm{B} = \mu_1 \frac{\partial\, \mathrm{D}}{\partial t}, \tag{1.15}$$

and of Maxwell-Faraday

$$rot\, \mathrm{E} = -\frac{\partial\, \mathrm{B}}{\partial t}, \tag{1.16}$$

where μ_1 denotes the magnetic permeability of the material. The fourth Maxwell equation is an add-on to the equations already mentioned (Maxwell-Gauss, Maxwell-Ampére, Maxwell-

Faraday) and reflects the law of magnetic flux conservation

$$\operatorname{div} \mathrm{B} = 0. \tag{1.17}$$

For the construction and study of these equations, we refer to [47] and [120]. The conservation law (1.17) implies the existence of a vector A called magnetic potential vector such as

$$\mathrm{B} = rot\, \mathrm{A}\,. \tag{1.18}$$

The last equation combined with the Maxwell-Farraday equation implies that the sum of vectors $\mathrm{E} + \dfrac{\partial\, \mathrm{A}}{\partial t}$ has a nil rotational value, thus it is deriving from a scalar potential φ, *i.e.*

$$\mathrm{E} = -\nabla\varphi - \frac{\partial\, \mathrm{A}}{\partial t}. \tag{1.19}$$

The unknowns of the piezoelectricity system as written are now the displacement u, the scalar potential φ and the vector potential A.

In the dynamic case, to manage the unknown A, we use the Maxwell-Ampére equation (1.15). In general, the piezoelectric material is a continuous and electrically neutral media; consequently, it is referred to as an insulator. In many applications, we can restrict ourselves to the quasi-electrostatic approximation, which means assuming that the transformation is thermodynamically adiabatic (no heat exchange and therefore no Joule effect). It is also assumed that only the effect of the electro-mechanical interaction is important, so the magnetic part in the Maxwell-Faraday equation (1.16) is neglected ($\mathrm{A} = 0$), then, using (1.19), this leads us to

$$\mathrm{E} = -\nabla\varphi. \tag{1.20}$$

We consider the electrical potential as the electrical unknown rather than its gradient (see Banks al. [13]), this is justified by the boundary conditions that will later be imposed on the electrical potential.

Since the process is assumed to be static, the governing equations consist of the equilibrium equations and constitutive relation, and they are given by

$$\operatorname{Div} \sigma + f_0 = 0, \quad \operatorname{div} \mathrm{D} = q_0 \quad \text{in } \Omega. \tag{1.21}$$

To complete the model, we have to prescribe the mechanic and electric boundary conditions.

According to the physical setting, we use

$$u = 0 \quad \text{on } \Gamma_1, \quad \sigma\nu = f_2 \quad \text{on } \Gamma_2, \quad \varphi = 0 \quad \text{on } \Gamma_a, \quad \mathrm{D}\,.\,\nu = q_2 \quad \text{on } \Gamma_b, \tag{1.22}$$

where f_2 and q_2 the density of tractions and surface electric charges.

1.2.2 Piezo-elasto-plastic behavior law

The systematic characterization of the electro-mechanical properties of piezoelectric media is based on a tensor representation of the coupling between the electrical and mechanical systems [75]. This approach is particularly necessary due to the anisotropy inherent in the existence of piezoelectricity itself.

Macroscopic local quantities, generally chosen as mechanical and electrical variables in continuous media, are, respectively, the strain tensor ε, the stress tensor σ, the electric displacements field D and the electric field E. The representation of the piezoelectric properties of the material therefore leads, depending on the system of independent variables (ε, D), (σ, E), (σ, D) or (ε, E) chosen, to the definition of four pairs of fundamental relationships reflecting the direct and inverse effects of energy conservation. Within the framework of the classical hypotheses of elasticity theory, assuming in particular, that the amplitudes of the deformations remain small, the symmetry properties of the strain and stress tensors make it possible to reduce the initial tensor relations to matrix relations that are easier to handle. Thus, by choosing for example the intensive variables (σ, E) as a pair of independent variables, the piezoelectric properties of the material are reflected in the following relationships:

$$\begin{cases} \sigma = \mathfrak{F}\varepsilon(u) - \mathcal{E}^*\,\mathrm{E}(\varphi), \\ \mathrm{D} = \mathcal{E}\varepsilon(u) + \beta\,\mathrm{E}(\varphi), \end{cases} \tag{1.23}$$

where $\mathfrak{F} : \Omega \times \mathbb{S}^d \to \mathbb{S}^d$ is the elasticity operator not necessarily linear with a nil electric field (short-circuited piezoelectric material), for example the nonlinear elasticity operator that describes the behavior of the Hencky's materials, defined in (1.13). $\mathcal{E} : \Omega \times \mathbb{S}^d \to \mathbb{R}^d$ is the linear piezoelectric operator which translates the proportionality between the charge and the deformation with constant or nil field, and $\beta : \Omega \times \mathbb{R}^d \to \mathbb{R}^d$ is the linear electric permittivity operator with nil deformation that constitutes a symmetric definite positive tensor. In addition, $\mathcal{E}^*$ denotes the transposed of tensor $\mathcal{E}$, such as

$$\mathcal{E}\sigma\, v = \sigma\mathcal{E}^* v \quad \forall\sigma \in \mathbb{S}^d, \quad \forall v \in \mathbb{R}^d, \tag{1.24}$$

1.3 Contact conditions

These conditions concern the contact boundary Γ_3, and relate the normal components of the displacement field u_ν and the normal components of the constraint field σ_ν. These laws can be found in [4, 88, 127, 144, 155].

1.3.1 Unilateral contact

Proposed in 1933 by Antonio Signorini [139], this contact law remains one of the most popular laws in literature nowadays. It is described by the following three conditions:

$$\begin{cases} \text{(a) Non-penetration condition} & u_\nu \leq 0, \\ \text{(b) Compression condition} & \sigma_\nu \leq 0, \\ \text{(c) Complementarity condition} & \sigma_\nu \, u_\nu = 0. \end{cases} \tag{1.25}$$

Physically, this is like considering a perfectly rigid foundation; no matter what compressive force is applied, there will be no penetration. The condition of complementarity simply reflects that:

- If $u_\nu = 0$, there is contact and the normal contact force is negative ($\sigma_\nu \leq 0$).
- If $u_\nu < 0$, there is no contact and in this case the contact force is zero ($\sigma_\nu = 0$).

The multivoque graph of this law is shown in Figure 1.3.

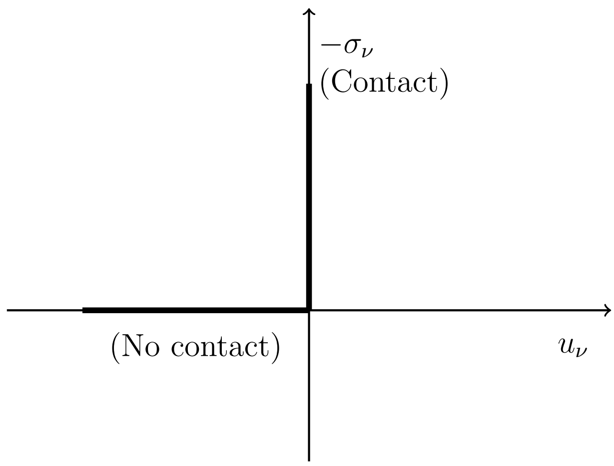


Figure 1.3 – Signorini contact law.

If we assume that there is a gap ϱ between the foundation and the body, then Signorini's conditions have the form:

$$\begin{cases} \text{(a) Non-penetration condition} & (u_\nu - \varrho) \leq 0, \\ \text{(b) Compression condition} & \sigma_\nu \leq 0, \\ \text{(c) Complementarity condition} & \sigma_\nu (u_\nu - \varrho) = 0. \end{cases} \tag{1.26}$$

The multivoque graph of this law is shown in Figure 1.4.

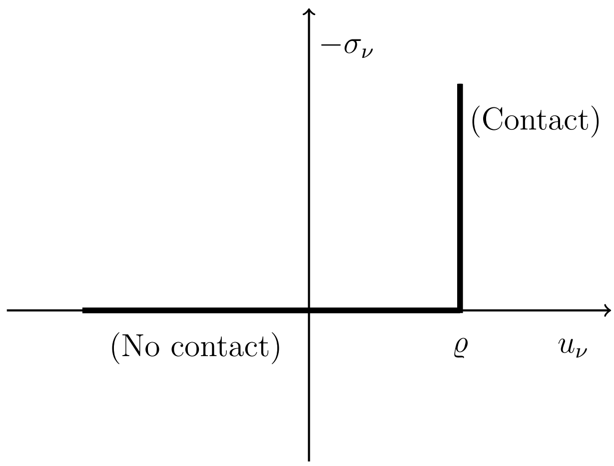


Figure 1.4 – Signorini contact law with a gap.

1.3.2 Normal compliance contact

Let us now consider another type of law, as popular as the previous one, in which the foundation is supposed to be deformable. It was first used in [95] in a visco-plastic dynamic problem. The normal constraint σ_ν is supposed to satisfy the following condition:

$$-\sigma_\nu = p(u_\nu), \tag{1.27}$$

where p refers to a function given a value in $\mathbb{R}^+$, due to the compression of the body, vanishing each other out for any negative argument. In particular, this means that such a law allows penetration, of which the normal constraint σ_ν is a function. The representation of the normal complication law depends on the compliance function p chosen; in this case, Figure 1.5. corresponds to

$$p(r) = c[r]^+. \tag{1.28}$$

where c can be assimilated to the stiffness coefficient of the foundation.

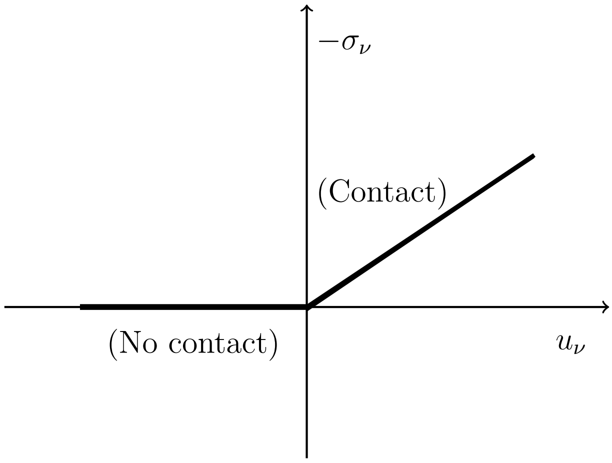


Figure 1.5 – Normal compliance condition.

It should also be noted that this choice of p gives rise to another interpretation: by tending the c rigidity coefficient towards infinity, we find a perfectly rigid foundation, namely Signorini's law mentioned above. We can, therefore, consider that the normal compliance law represents a penalization of Signorini's law. Beyond the mechanical interest that this type of law can offer, it is frequently used to regularize Signorini's law for the analysis of contact problems.

1.3.3 Contact with normal compliance and unilateral constraint

This contact condition was first introduced in [76] in the study of a dynamic problem for elasto-visco-plastic materials and aims to overcome the limits of the two laws from which it is derived. Indeed, neither Signorini's conditions or the normal compliance condition are entirely admissible from a physical point of view; in the first one, it is assumed that the foundation is perfectly rigid while the second one allows uncontrollable penetration. We will, therefore, consider an intermediate case where the foundation is made of a perfectly rigid material on which rests a fine layer of deformable asperities with a depth of ϱ, thus limiting penetration. Such a law is described by the following relationships:

$$\begin{cases} (u_\nu - \varrho) \leq 0, \quad \sigma_\nu + p(u_\nu - \varrho) \leq 0, \\ (\sigma_\nu + p(u_\nu - \varrho))\ (u_\nu - \varrho) = 0. \end{cases} \tag{1.29}$$

a graphical representation of which is provided below.

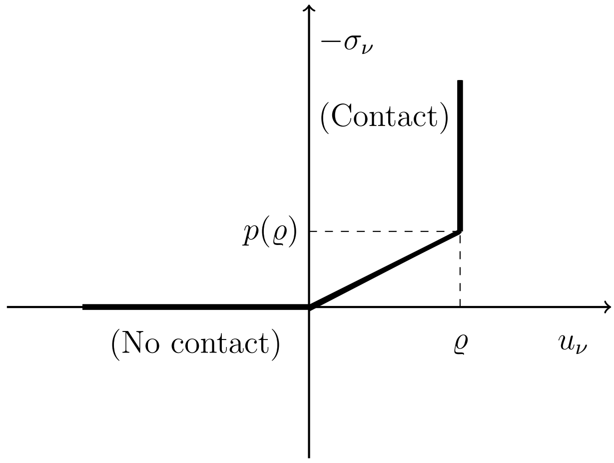


Figure 1.6 – Normal compliance condition with unilateral constraint.

1.3.4 Electrical contact condition

In Chapters 5 and 6, we will study a contact problem between a piezoelectric body and an electrically conductive foundation. It, therefore, seems natural to introduce a law to model electrical contact. We use here, a condition similar to the one used in [20, 90], defined as follows

$$\mathrm{D}\ .\ \nu = \psi(u_\nu - \varrho)\phi_L(\varphi - \varphi_F) \quad \text{on} \quad \Gamma_3, \tag{1.30}$$

Equation (1.30), represents the regularized condition of electrical contact, which can be obtained as follows. In the case of detachment (*i.e.*, $(u_\nu - \varrho) < 0$), we assume that the gap is insulating (for example, it is filled with air), there are no free electrical charges at the surface and the normal component of the electric displacement field is nil. So,

$$(u_\nu - \varrho) < 0 \Rightarrow \mathrm{D} \,.\, \nu = 0. \tag{1.31}$$

During the contact process (*i.e.*, when $(u_\nu - \varrho) \geq 0$), we assume that the normal component of the electric displacement field is proportional to the difference between the potential of the foundation φ_F and the potential of the body surface, then

$$(u_\nu - \varrho) \geq 0 \Rightarrow \mathrm{D} \,.\, \nu = k_e(\varphi - \varphi_F), \tag{1.32}$$

with k_e is a positive or nil constant, called the electrical conductivity coefficient. By combining (1.31) and (1.32), we obtain

$$\mathrm{D}\,.\,\nu = k_e\ \chi_{[0,\infty)}(u_\nu - \varrho)(\varphi - \varphi_F), \tag{1.33}$$

where $\chi_{[0,\infty)}(\,.\,)$ is the characteristic function of the interval $[0,\infty)$, defined by

$$\chi_{[0,\infty)}(s) = \begin{cases} 0 & \text{if} \quad s < 0, \\ 1 & \text{if} \quad s \geq 0. \end{cases}$$

Condition (1.32) describes a perfect electrical contact, it is somewhat similar to Signorini's contact conditions. We can rewrite it in the regularized form given in (1.30), where the functions ψ and ϕ_L are given as follows (for more details see [20, 90]):

$$\psi(s) = \begin{cases} 0, & \text{if } s < 0, \\ k_e\ \delta\ s, & \text{if } 0 \leq s \leq 1/\delta, \\ k_e, & \text{if } s > 1/\delta, \end{cases} \qquad \phi_L(s) = \begin{cases} -L, & \text{if } s < -L, \\ s, & \text{if } -L \leq s \leq L, \\ L, & \text{if } s > L. \end{cases} \tag{1.34}$$

in which, $\delta > 0$ is a small parameter, and L is a large positive constant.

This regularization is introduced here for mathematical reasons. Indeed, the truncation function ϕ_L is used to control the boundedness of $(\varphi - \varphi_F)$. Note that the presence of the ϕ_L function does not pose a practical problem, since L can be chosen arbitrarily large, in this case we can take $\phi_L(\varphi - \varphi_F) = (\varphi - \varphi_F)$. Also, to avoid discontinuity of in the free electric charge when contact is established and therefore we regularize the indicator function of the interval $[0,\infty)$ with the Lipschitz continuous function ψ. The choice of the function ψ means that during the process of contact the electrical conductivity increases as the contact among the surface asperities improves, and stabilizes when the penetration $(u_\nu - \varrho)$ reaches the value $1/\delta$. Finally, let us observe that when $k_e = 0$, then $\psi \equiv 0$ in (1.30). This, therefore, leads to

$$\mathrm{D}\,.\,\nu = 0 \quad \text{on} \quad \Gamma_3, \tag{1.35}$$

the condition (1.35) models the case where the foundation is perfectly insulating.

Note that contact law mentioned in subsection 1.3.3, can be modeled in the case of electro-mechanical system as follows

$$\left.\begin{array}{l}(u_\nu - \varrho) \le 0, \quad (\sigma_\nu(u,\varphi) + h_\nu(\varphi - \varphi_F)p_\nu(u_\nu - \varrho)) \le 0,\\ (\sigma_\nu(u,\varphi) + h_\nu(\varphi - \varphi_F)p_\nu(u_\nu - \varrho))\ (u_\nu - \varrho) = 0,\end{array}\right\} \quad \text{on} \quad \Gamma_3,$$

In this condition, p_ν is a prescribed nonnegative function which vanishes when its argument is negative and h_ν is a positive function, which depends on the difference between the potential of the foundation and the body's surface. This shows that when there is no contact (*i.e.*, $(u_\nu - \varrho) < 0$) the reaction of the foundation vanishes and, therefore, $\sigma_\nu = 0$. When $(u_\nu - \varrho) > 0$, we have $-\sigma_\nu = h_\nu(\varphi - \varphi_F)p_\nu(u_\nu - \varrho)$, which means that the reaction of the foundation depends on the normal displacement and the difference between the potential of the foundation and that of the body surface. When $u_\nu - \varrho = 0$, then $\sigma_\nu \le 0$. We note that if $p_\nu = 0$, in the above condition becomes the classical Signorini's conditions with a gap ϱ, defined in (1.26).

1.4 Friction laws

We now consider the conditions relating to the contact boundary Γ_3 linking the tangential constraint σ_τ and the normal constraint σ_ν. The purpose here is to characterize a stress limit below which no slippage is possible.

1.4.1 Absence of friction

This is the simplest condition in which an idealized surface with no friction is considered, *i.e.*,

$$\sigma_\tau = 0. \tag{1.36}$$

It is important to note that despite everything, such an approximation of reality remains reasonable in some situations, thus making its use in many problems completely legal.

1.4.2 Tresca's friction law

Named in this way by analogy with the Tresca criterion in plasticity [152], we assume here that the stress limit $S > 0$ is fixed. If the tangential stress is below the limit in norm, then there is adherence between the body and the foundation. If, on the other hand, this limit is reached, then the body slips on the foundation while the tangential stress opposes the movement.

$$\|\sigma_\tau\| \leq S \begin{cases} \text{if} \quad \|\sigma_\tau\| < S \quad \text{(adherence)}, \\ \text{if} \quad \|\sigma_\tau\| = S \quad \text{(slipping)}. \end{cases} \tag{1.37}$$

1.4.3 Coulomb's friction law

This is undoubtedly the most well-known friction law and one of the most widely used in the literature. First formulated by Amontons in 1699 in his thesis, Charles-Augustin Coulomb was able to confirm it and extend its scope of validity to the case of very high charges in 1785.

The Coulomb device can be described as follows: we consider a horizontal platform finished by two stops on which a sled can slide, which can be loaded, pulled by a rope, whose tension is adjustable. The objective is to establish the value of the tension of the rope sufficient to overcome the friction force; the sled then begins to slide. Among Coulomb's conclusions, it can be noted that:

- The friction limit is proportional to the weight of the solid placed on the horizontal plate.

- For a given load, this limit depends only on the nature of the surfaces in contact.

In the case of the Coulomb's friction law, unlike the Tresca friction law, the threshold limit is no longer fixed and now depends on the friction coefficient $\mu(\,.\,)$ and the normal contact stress σ_ν. The Coulomb's friction law will then put in competition, via the friction coefficient $\mu(\,.\,)$, the tangential friction stress σ_τ and the normal contact stress σ_ν.

$$\|\sigma_\tau\| \leq \mu(\|u_\tau\|)|\sigma_\nu| \begin{cases} \text{if} \quad \|\sigma_\tau\| < \mu(\|u_\tau\|)|\sigma_\nu| \Rightarrow \quad u_\tau = 0, \\ \text{if} \quad \|\sigma_\tau\| = \mu(\|u_\tau\|)|\sigma_\nu| \Rightarrow \quad \exists \lambda \in \mathbb{R}^+ \text{ such that } \sigma_\tau = -\lambda u_\tau. \end{cases} \tag{1.38}$$

where $|\,.\,|$ refers to the absolute value if it applies to a scalar or, the Euclidean norm, if it applies to an element of $\mathbb{R}^d$.
This law implies that σ_τ is a multivalent application of u_τ, it is represented in Figure 1.7. Also as in the case of unilateral contact of Signorini, this law can be regularized by a normal compliance law (see [11, 80, 81]).

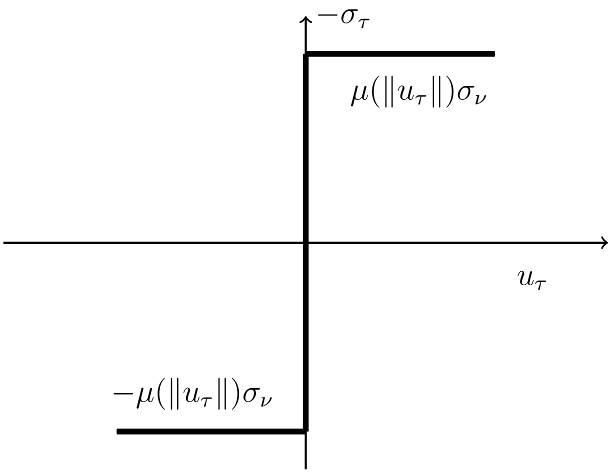


Figure 1.7 – Coulomb's friction law.

However, the punctual application of Coulomb's law produces great mathematical difficulties. Since generally the stress σ is only square integrable, it is not necessarily continuous, and it does not have a well-defined trace on Γ_3. Indeed, the variational formulation of this problem, as we will see later, is a quasi-variational inequality that contains the term $\int_{\Gamma_3} \mu(\|u_\tau\|)|\sigma_\nu(u)|\|v_\tau\| da$. So, if $u \in V = \{ v \in H^1(\Omega)^d ;\ v = 0 \quad \text{on } \Gamma_1\}$, then $\sigma_\nu(u)$ is not defined on Γ_3. Moreover, even if $u \in H^1(\Omega)^d$ verifies the equation (1.1) for $f_0 \in L^2(\Omega)^d$ then $|\sigma_\nu(u)|$ does not have a mathematical meaning. Indeed, it is known (see for example [50]) that

$$\sigma_\nu(u) \in H^{-\frac{1}{2}}(\Gamma), \quad \forall u \in H^1_{\text{Div}}(\Omega) = \{ v \in H^1(\Omega)^d ;\ \text{Div}\, \sigma(v) \in L^2(\Omega)^d \}.$$

$\sigma_{ij}\nu_j$ being defined by the generalized Green formula

$$\langle \sigma_{ij}\nu_j, \gamma v\rangle_\Gamma = \int_\Omega \sigma_{ij}(u)\, \varepsilon_{ij}(v) dx + \int_\Omega \sigma_{ij,j}(u)\, v_i\, dx \quad \forall u \in H^1_{\text{Div}}(\Omega), \quad \forall v \in H^1(\Omega)^d, \qquad (1.39)$$

where $\langle \,.\, , \,.\, \rangle_\Gamma$ denotes the duality pairing between $H^{-\frac{1}{2}}(\Gamma)$ and $H^{\frac{1}{2}}(\Gamma)$, $H^{-\frac{1}{2}}(\Gamma)$ being the dual of $H^{\frac{1}{2}}(\Gamma)$.

Therefore, if $u \in H^1(\Omega)^d$ verifies (1.1) for $f_0 \in L^2(\Omega)^d$, then $-\sigma_{ij,j} = f_{0i} \in L^2(\Omega)$ hence $\sigma_\nu(u) \in H^{-\frac{1}{2}}(\Gamma)$ and $|\sigma_\nu(u)|$ does not have a mathematical meaning.

This difficulty can be overcome by using a non-local variant of Coulomb's law, proposed by Duvaut [50]. This law stipulates that motion in a point of contact between two elastic bodies happens when the modulus of stress in that point reaches a value proportional to an average of the normal stress in a neighborhood of that point. The character of the effective local neighborhood and how neighboring stresses contribute to the slip condition depends on the microstructure of the materials in contact. The non-local character of the law is given by the regularization of the normal constraint σ_ν which can be defined in any point as a convolution around this point, so the local interactions between asperities are taken into account (see [48, 50, 114, 115]). Indeed, let ρ be a positive number and let $(\omega_\rho), 0 < \rho \leq \rho_0$, be a sequence of test functions in the class $\mathcal{D}(-l, l)$ of infinitely differentiable functions with support in the interval $(-l, l)$; *i.e.*, if ϕ is an arbitrary test function in $\mathcal{C}_0^\infty(-l, l)$, then (ω_ρ) has the property that

$$\lim_{\rho \to 0} \int_{-l}^{l} \omega_\rho \, \phi \, dx = \phi(0) \quad \forall \phi \in \mathcal{D}(-l, l),$$

Then the limit of the sequence (ω_ρ) in the space $\mathcal{D}'(-l, l)$ of distributions defined on $(-l, l)$ is the Dirac delta:

$$\langle \delta, \phi \rangle = \lim_{\rho \to 0} \langle \omega_\rho, \phi \rangle = \phi(0) \quad \forall \phi \in \mathcal{D}(-l, l),$$

here $\langle \cdot, \cdot \rangle$ denotes duality pairing on $\mathcal{D}'(-l, l) \times \mathcal{D}(-l, l)$. For illustration purposes we can choose here the sequence of smooth test functions which assumes non-zero values only within an interval of radius ρ of the origin given by

$$\omega_\rho(x) = \begin{cases} C \exp\Big(\dfrac{\rho^2}{x^2 - \rho^2}\Big) & 0 \leq |x| \leq \rho, \\ 0 & |x| > \rho. \end{cases}$$

with C a constant such that $\displaystyle\int_{-l}^{l} \omega_\rho(x) dx = 1$.
Thus, the regularization of the normal stress vector σ_ν can be given by

$$\sigma_\nu^*(u) = \int_{\Gamma_3} \omega_\rho(|x - y|) \, (-\sigma_\nu(u(y))) \, dy.$$

Chapter 2

Functional spaces

After a brief reminder of the mechanics of continuous media, we introduce the spaces that represent the mathematical framework for the upcoming demonstrations. Then, we briefly recall below some classical definitions and theorems (for the demonstrations we refer to the bibliography), and discuss real-valued function spaces, such as the $\mathcal{C}^m$, L^p or Sobolev function space that will be used in the following chapters. Here and throughout this manuscript, all the functions considered are of real value.

For a point $x = (x_1, \ldots, x_d) \in \mathbb{R}^d$, we will note the differential operator $\dfrac{\partial}{\partial x_i}$ $(1 \leq i \leq d)$ by D_i.

If $\alpha = (\alpha_1, \ldots, \alpha_d)$ is a multi-index, then, we define the differential operator D^α of order α, with $|\alpha| = \sum_{i=1}^{d} \alpha_i$, by

$$D^\alpha = D_1^{\alpha_1} \ldots D_d^{\alpha_d} = \frac{\partial^{|\alpha|}}{\partial x_1^{\alpha_1} \ldots \partial x_d^{\alpha_d}}.$$

Obviously, D_i^0 represents the identity.

If $A \subset \mathbb{R}^d$, we will note by $\mathcal{C}(A)$ the space of the real continuous functions on A.
Let Ω be an open subset of $\mathbb{R}^d$ and K a subset of Ω. We will note $K \subset\subset \Omega$ if K is relatively compact in Ω *i.e.*, the adherence of K, noted $\overline{K}$, is a compact (*i.e.*, closed and bounded) included in Ω.

For any positive integer m, we can consider the spaces $\mathcal{C}^m(\Omega)$, respectively $\mathcal{C}^m(\overline{\Omega})$, real functions m times continuously differentiable on Ω, respectively $\overline{\Omega}$, that is

$$\mathcal{C}^m(\Omega) = \{\, v \in \Omega \ ; \ D^\alpha v \in C(\Omega) \quad \text{for} \quad \alpha \leq m\}. \tag{2.1}$$

When $m = 0$, we often abbreviate $\mathcal{C}(\Omega) \equiv \mathcal{C}^0(\Omega)$ and $\mathcal{C}(\overline{\Omega}) \equiv \mathcal{C}^0(\overline{\Omega})$.

We call the support of a function $v : \Omega \mapsto \mathbb{R}$ is defined as closed set

$$supp\, v = \overline{\{ x \in \Omega \; ; \; v(x) \neq 0\}}. \tag{2.2}$$

We say that the function v has a compact support in Ω if there exists a compact K in Ω such that $v(x) = 0 \quad \forall x \in \Omega \setminus K$ or, equivalently, $supp\, v \subset\subset \Omega$.

Then, we shall denote by $\mathcal{C}_0^m(\Omega)$, respectively $\mathcal{C}_0^m(\overline{\Omega})$, the subspace of $\mathcal{C}^m(\Omega)$, respectively $\mathcal{C}^m(\overline{\Omega})$, formed by compact support functions in Ω. Obviously, for m finite integer and Ω bounded, $\mathcal{C}_0^m(\overline{\Omega})$ is a Banach space for the norm

$$\|v\|_{\mathcal{C}^m(\Omega)} = \sum_{|\alpha| \le m} \sup_{x \in \overline{\Omega}} |D^\alpha v(x)|. \tag{2.3}$$

Let

$$\mathcal{C}^\infty(\Omega) = \bigcap_{m=0}^{\infty} \mathcal{C}^m(\Omega).$$

the space of functions indefinitely differentiable in Ω.

To understand more clearly the meaning of the differential operator $D^\alpha v$ for functions v whose derivatives do not exist in the classical sense, we will briefly recall the definition of the distributions on Ω ([2, 116, 137, 156]).

We denote by $\mathcal{D}(\Omega)$ called the space of test functions, the space $\mathcal{C}^\infty(\Omega)$ equipped with the inductive limit topology as in the Schwartz theory of distributions [137].

Definition 2.1 *A sequence $(\varphi_k)_k \subset \mathcal{C}_0^\infty(\Omega)$ is said to converge to a function $\varphi \in \mathcal{C}_0^\infty(\Omega)$ in (the sense of the space) $\mathcal{D}(\Omega)$ provided that the following conditions are fulfilled:*

i) There exists a compact subset K of Ω such that $supp\, (\varphi_k - \varphi) \subset K$ for all k.

ii) $D^\alpha \varphi_k \to D^\alpha \varphi$ uniformly on K for all multi-index α.

The dual space $\mathcal{D}'(\Omega)$ of $\mathcal{D}(\Omega)$ is called the space of (Schwartz) distributions (or, generalized functions). Hence, any distribution T is a linear and continuous functional on $\mathcal{D}(\Omega)$, *i.e.*, $\mathrm{T}(\varphi_k) \to \mathrm{T}(\varphi)$ in $\mathbb{R}$ whenever $\varphi_k \to \varphi$ in $\mathcal{D}(\Omega)$. As dual of $\mathcal{D}(\Omega)$, the space $\mathcal{D}'(\Omega)$ is equipped with the weak-star topology: $\mathrm{T}_k \to \mathrm{T}$ in $\mathcal{D}'(\Omega)$ if and only if $\mathrm{T}_k(\varphi) \to \mathrm{T}(\varphi)$ in $\mathbb{R}$, for every $\varphi \in \mathcal{D}(\Omega)$.

Every distribution is infinitely differentiable in the following sense: $\mathrm{T} \in \mathcal{D}'(\Omega)$, then, for all multi-index α, the function $D^\alpha\,\mathrm{T}$ defined on $\mathcal{D}(\Omega)$ by

$$D^\alpha\,\mathrm{T}(\varphi) = (-1)^{|\alpha|}\,\mathrm{T}(D^\alpha\varphi). \tag{2.4}$$

is a distribution. In addition, the operator D^α from $\mathcal{D}'(\Omega)$ into $\mathcal{D}'(\Omega)$ is continuous.

Any function $u \in L^1_{loc}(\Omega)$ generates a distribution $\mathrm{T}_u \in \mathcal{D}'(\Omega)$ defined by

$$\mathrm{T}_u(\varphi) = \int_\Omega u(x)\,\varphi(x)\,dx \quad \forall\varphi \in \mathcal{D}(\Omega). \tag{2.5}$$

Therefore, for any multi-index α, there exists the α-th derivative of T_u, namely the distribution $D^\alpha\,\mathrm{T}_u \in \mathcal{D}'(\Omega)$ defined by (2.4), *i.e.*,

$$D^\alpha\,\mathrm{T}_u(\varphi) = (-1)^{|\alpha|}\,\mathrm{T}_u(D^\alpha\varphi). \tag{2.6}$$

But not any distribution is generated by a locally integrable function.

Definition 2.2 *We shall say that the function $u \in L^1_{loc}(\Omega)$ possesses the distributional (or generalized or weak) partial derivative of order α on Ω, denoted by $D^\alpha u$, if there exists a function $v_\alpha \in L^1_{loc}(\Omega)$ which generates the distribution $D^\alpha\,\mathrm{T}_u \in \mathcal{D}'(\Omega)$, i.e.,*

$$D^\alpha\,\mathrm{T}_u = \mathrm{T}_{v_\alpha}\,.$$

Thus, from the last three relations, it follows that $D^\alpha u = v_\alpha$, is the distributional partial derivative of u if $v_\alpha \in L^1_{loc}(\Omega)$ satisfies

$$\int_\Omega u(x)\,D^\alpha\varphi(x)\,dx = (-1)^{|\alpha|}\int_\Omega v_\alpha\,\varphi(x)\,dx \quad \forall\varphi \in \mathcal{D}(\Omega). \tag{2.7}$$

Obviously, the distributional derivative is uniquely defined up to a set of measure zero.

In fact, this formula generalizes the (classical) partial derivative of order α, obtained, for a function $u \in \mathcal{C}^{|\alpha|}(\Omega)$, by integrating by parts $|\alpha|$ times

$$\int_\Omega D^\alpha u(x)\,\varphi(x)\,dx = (-1)^{|\alpha|}\int_\Omega u(x)\,D^\alpha\varphi(x)\,dx \quad \forall\varphi \in \mathcal{D}(\Omega). \tag{2.8}$$

Of course, in this case, $D^\alpha u$ is also a distributional partial derivative of u. However, it should be noted that the derivative in the sense of distributions of a function, even sufficiently smooth, may exist, even if it does not exist in the classical sense.

We denote by $L^p(\Omega)$ for $1 \le p < +\infty$, the space of (equivalence classes of) real functions v

defined on Ω with the p-power absolutely integrable, *i.e.*,

$$\int_\Omega |v(x)|^p\, dx < \infty,$$

where $dx = dx_1,\, dx_2,\, \ldots,\, dx_d$ is the Lebesque measure. The elements of $L^p(\Omega)$, being equivalence classes of measurable functions, are identical if they are equal almost everywhere (*a.e.*) on Ω. Thus, we write $v = 0$ in $L^p(\Omega)$ if $v(x) = 0$ *a.e.* $x \in \Omega$.

We also denote by $L^\infty(\Omega)$ the space consisting of all (equivalence classes of) measurable real functions v that are essentially bounded on Ω, *i.e.*, there exists a constant C such that $|v(x)| \leq C$ *a.e.* on Ω.

The space $L^p(\Omega)$ endowed with the norm

$$\|v\|_{L^p(\Omega)} = \begin{cases} \left(\displaystyle\int_\Omega |v(x)|^p\, dx\right)^{\frac{1}{p}} & \text{if} \quad 1 \leq p < +\infty, \\ \operatorname*{ess\,sup}_{x\in\Omega} |v(x)| = \inf\{\, C\ ;\ |v(x)| \leq C\ a.e. \text{ on } \Omega\} & \text{if} \quad p = +\infty. \end{cases} \tag{2.9}$$

is a Banach space. In addition, the space $L^p(\Omega)$ is separable if $1 \leq p < +\infty$ and reflexive if $1 < p < +\infty$.

If $p \in [1,\, +\infty)$, then, the exponent conjugate to p is the number denoted by p' defined by the relation

$$\frac{1}{p} + \frac{1}{p'} = 1,$$

and we use the convention

$$p' = \begin{cases} \infty & \text{if} \quad p = 1, \\ 1 & \text{if} \quad p = +\infty. \end{cases}$$

From Riesz representation Theorem A.6 for Hilbert spaces it follows that, for $p \in [1,\, +\infty)$, the dual space of $L^p(\Omega)$ is the space $(L^p(\Omega))' = L^{p'}(\Omega)$ where p' is the exponent conjugate to p. The dual space of $L^\infty(\Omega)$ is a space larger than $L^1(\Omega)$ (for more details, see [156], p. 118).

In the case $p = 2$, the space $L^2(\Omega)$ is a Hilbert space with respect to the inner product

$$(u,\, v)_{L^2(\Omega)} = \int_\Omega u(x)\; v(x)\, dx. \tag{2.10}$$

Definition 2.3 *We say that a measurable function v defined a.e. on Ω is locally p-integrable on Ω if $v \in L^p(A)$ for every measurable set $A \subset\subset \Omega$.*

We shall denote by $L^p_{loc}(\Omega)$ the space of all locally p-integrable functions on Ω.

Theorem 2.1 *Let $\Omega \subset \mathbb{R}^d$ be an open set. The following assertions hold.*

1. *Let $1 < p,\, q < +\infty$.*
   - *If $u \in L^p(\Omega)$ and $v \in L^q(\Omega)$, then, $u\,v \in L^r(\Omega)$ with $\dfrac{1}{r} = \dfrac{1}{p} + \dfrac{1}{q}$.*
   - *If $u_n \to u \in L^p(\Omega)$ and $v_n \to v \in L^q(\Omega)$, then, $u_n\,v_n \to u\,v \in L^r(\Omega)$ with $\dfrac{1}{r} = \dfrac{1}{p} + \dfrac{1}{q}$.*
   - *If $u \in L^p(\Omega)$ and $v \in L^{p'}(\Omega)$, where p' is the exponent conjugate to p, then, $u\,v \in L^1(\Omega)$ and Hölder's inequality holds:*

   $$|\int_\Omega u(x)\; v(x)\,dx| \leq \|u\|_{L^p(\Omega)}\|v\|_{L^{p'}(\Omega)}. \tag{2.11}$$

   When $p = p' = 2$, we get the Cauchy-Schwarz inequality.

2. *For $1 \leq p \leq +\infty$, every Cauchy sequence in $L^p(\Omega)$ has a subsequence converging pointwise a.e. on Ω.*

3. *$L^p(\Omega) \subset L^1_{loc}(\Omega)$ for all p with $1 \leq p < +\infty$.*

4. *Let $v \in L^1_{loc}(\Omega)$, such that $\displaystyle\int_\Omega v(x)\,\varphi(x)\,dx = 0$ for all $\varphi \in \mathcal{D}(\Omega)$. Then, $v(x) = 0$ a.e. on Ω.*

5. *$\mathcal{C}^\infty_0(\Omega)$ is dense in $L^p(\Omega)$ for all p with $1 \leq p < +\infty$.*

In the sequel, for $(X,\, \|\,.\,\|_X)$, $(Y,\, \|\,.\,\|_Y)$ two normed spaces with $X \subset Y$, we shall write $X \hookrightarrow Y$ to designate the continuously embedding of X in Y provided the identity operator $\mathrm{I} : X \mapsto Y$ is continuous. This is equivalent, since I is linear, to the existence of a constant C such that

$$\|u\|_Y \leq C\|u\|_X \quad \forall u \in X.$$

We also say that the normed space X is compactly embedded in the normed space Y and write $X \hookrightarrow_c Y$ if the identity operator I is compact, *i.e.*, every bounded sequence in X has a subsequence converging in Y, or, equivalently, if $(u_k)_k$ is a sequence which converges weakly to u in X, and we write $u_k \rightharpoonup u$, then, $(u_k)_k$ converges strongly to u in Y, and we write $u_k \to u$.

The following theorem gives an embedding result for the spaces $L^p(\Omega)$ and some of its consequences.

Theorem 2.2 *Let $\Omega \subset \mathbb{R}^d$ be an open set with $vol(\Omega) = \displaystyle\int_\Omega dx < \infty$. Then, the following statements are valid*

1. *For all p, q such that $1 \leq p, q \leq +\infty$ we have $L^q(\Omega) \hookrightarrow L^p(\Omega)$ and*

$$\|v\|_{L^p(\Omega)} \leq vol(\Omega)^{\frac{1}{p}-\frac{1}{q}} \|v\|_{L^q(\Omega)} \quad \forall v \in L^q(\Omega).$$

2. $\lim\limits_{p\to\infty} \|v\|_{L^p(\Omega)} = \|v\|_{L^\infty(\Omega)} \quad \forall v \in L^\infty(\Omega).$

3. *Suppose that for all $1 \leq p < +\infty$, we have $v \in L^p(\Omega)$ and that there exists a constant C such that $\|v\|_{L^p(\Omega)} \leq C$. Then, $v \in L^\infty(\Omega)$.*

In particular, the relation (2.5) brings out a linear and continuous mapping $u \mapsto \mathrm{T}_u$ from $L^p(\Omega)$ into $\mathcal{D}'(\Omega)$ and so, we may identify the distribution T_u with the integrable function u. The same identification may be made for $\mathcal{D}(\Omega)$.Thus, we have

$$\mathcal{D}(\Omega) \hookrightarrow L^p(\Omega) \hookrightarrow \mathcal{D}'(\Omega).$$

Using the last result and the definition (2.6), Sobolev [142] expanded in a natural way the space $L^p(\Omega)$ by considering those functions which, for some nonnegative integer m, possess distributional partial derivatives of all orders $|\alpha| \leq m$ in $L^p(\Omega)$. This is the definition of the Sobolev space

$$W^{m,p}(\Omega) = \{ v \; ; \; D^\alpha v \in L^p(\Omega) \quad \text{for} \quad |\alpha| \leq m\}.$$

The space $W^{m,p}(\Omega)$ is a Banach space with the norm

$$\|v\|_{W^{m,p}(\Omega)} = \begin{cases} \left(\sum\limits_{|\alpha|\leq m} \|D^\alpha v\|^p_{L^p(\Omega)} \right)^{\frac{1}{p}} & \text{if} \quad p \in [1, +\infty), \\ \max\limits_{|\alpha|\leq m} \|D^\alpha v\|_{L^\infty(\Omega)} & \text{if} \quad p = +\infty. \end{cases} \tag{2.12}$$

Obviously, $W^{0,p}(\Omega) = L^p(\Omega)$ for $p \in [1, +\infty)$. The semi-norm over $W^{m,p}(\Omega)$ is defined by

$$|v|_{W^{m,p}(\Omega)} = \begin{cases} \left(\sum\limits_{|\alpha|= m} \|D^\alpha v\|^p_{L^p(\Omega)} \right)^{\frac{1}{p}} & \text{if} \quad p \in [1, +\infty), \\ \max\limits_{|\alpha|= m} \|D^\alpha v\|_{L^\infty(\Omega)} & \text{if} \quad p = +\infty. \end{cases} \tag{2.13}$$

We denote by $W_0^{m,p}(\Omega)$ the closure of $\mathcal{C}_0^\infty(\Omega)$ in the space $W^{m,p}(\Omega)$ for the norm $\| \, . \, \|_{W_0^{m,p}(\Omega)}$. For $p \in [1, +\infty)$, we have the following chain of embeddings

$$W_0^{m,p}(\Omega) \hookrightarrow W^{m,p}(\Omega) \hookrightarrow L^p(\Omega).$$

and, since $\mathcal{C}_0^\infty(\Omega)$ is dense in $L^p(\Omega)$, it is clear that $W_0^{0,\,p}(\Omega) = L^p(\Omega)$.

It is easy to see that, if the open set Ω is bounded, the semi-norm $|\,.\,|_{W^{m,\,p}(\Omega)}$ is a norm over $W_0^{m,\,p}(\Omega)$ equivalent to the norm $\|\,.\,\|_{W^{m,\,p}(\Omega)}$.

In the case $p = 2$, we use the notation

$$H^m(\Omega) = W^{m,\,2}(\Omega).$$

Endowed with the scalar product

$$(u,\, v)_{H^m(\Omega)} = \sum_{|\alpha|\leq m} (D^\alpha u,\, D^\alpha v)_{L^p(\Omega)}, \tag{2.14}$$

the Sobolev space $H^m(\Omega)$ is a Hilbert space. Also we denote $H_0^m(\Omega) = W_0^{m,\,2}(\Omega)$.

If Ω is bounded, then, without any hypothesis on the regularity of Ω, we have

$$H_0^1(\Omega) \hookrightarrow_c L^2(\Omega).$$

Many different symbols are being used to denote these norms, when no confusion may occur: $\|\,.\,\|_{m,\,p,\,\Omega}$ or $\|\,.\,\|_{m,\,p}$ instead of $\|\,.\,\|_{W^{m,\,p}(\Omega)}$, $\|\,.\,\|_{m,\,\Omega}$ or $\|\,.\,\|_m$ instead of $\|\,.\,\|_{H^m(\Omega)}$, and $\|\,.\,\|_{0,\,\Omega}$ or $\|\,.\,\|_0$ instead of $\|\,.\,\|_{L^2(\Omega)}$.

If $m \geq 1$ and $1 \leq p < +\infty$, we denote by $W^{-m,\,p'}(\Omega)$ the dual space of $W_0^{m,\,p}(\Omega)$, p' being the exponent conjugate to p (in fact, $W_0^{-m,\,p'}(\Omega)$ is the notation for a space of some distributions on Ω which is isometrically isomorphic to the dual space $(W_0^{m,\,p}(\Omega))'$; for details, see [2]). Endowed with the norm

$$\|f\|_{W^{-m,\,p'}(\Omega)} = \sup_{\substack{u\in W^{m,\,p}(\Omega),\\ u\neq 0}} \frac{\langle f,\, u\rangle}{\|u\|_{W^{m,\,p}(\Omega)}}, \tag{2.15}$$

the space $W^{-m,\,p'}(\Omega)$ is a Banach space which is separable and reflexive if $1 < p < +\infty$. Here $\langle\,.\,,\,.\,\rangle$ is the duality pairing between $W^{-m,\,p'}(\Omega)$ and $W^{m,\,p}(\Omega)$.

We note that if X, Y are two Hilbert spaces such that $X \hookrightarrow Y$ dense, then, (see, for instance, [12] p. 51]) $Y^* \hookrightarrow X^*$ dense, where Y^* and X^* denote their dual spaces.

If Ω is bounded, then, $\mathcal{D}(\Omega)$ is dense in $H_0^m(\Omega)$, and so, we can identify the dual space $H^{-m}(\Omega)$ of $H_0^m(\Omega)$ with a subspace of $\mathcal{D}'(\Omega)$:

$$\mathcal{D}(\Omega) \hookrightarrow H_0^m(\Omega) \hookrightarrow L^2(\Omega) \hookrightarrow H^{-m}(\Omega) \hookrightarrow \mathcal{D}'(\Omega).$$

Now, it should be noted that most results in Sobolev's spaces are obtained by considering first regular functions, and then extending these results by reasoning by density. It is the embedding and density theorems that show how and whether smooth functions can approximate an element of a Sobolev's space. Since these theorems require additional regularity properties for the open set Ω, we recall some definitions of them. Later, we will use some of these assumptions on for getting regularity properties of the solutions of some concrete variational inequalities.

Definition 2.4 *We say that the open subset Ω of $\mathbb{R}^d$ has the cone property if there exists a finite open bounded cover $(O_j)_{j\in J}$ of the boundary Γ of Ω and, for any j, there exists a cone C_j with the vertex at 0, such that, for all $x \in O_j \cap \Omega$, $x + C_j$ do not intersect $O_j \cap \Gamma$.*

Definition 2.5 *We say that the open set $\Omega \subset \mathbb{R}^d$ has the segment property if there exists a locally finite open cover $(U_j)_{j\in J}$ of the boundary Γ of Ω, and a corresponding sequence (y_j) of nonzero vectors such that if $x \in \widetilde{\Omega} \cap U_j$ for some j, then, $x + ty_j \in \Omega$ for $0 < t < 1$. In this case, Ω must have $(d-1)$-dimensional boundary and cannot simultaneously lie on both sides of its boundary.*

Definition 2.6 *Let $r > 1$ an integer. An open bounded set $\Omega \in \mathbb{R}^d$ is said to be $\mathscr{C}^r$-smooth (or, of class $\mathscr{C}^r$) if there exists a covering of the boundary Γ of Ω by a finite number of bounded open subsets $(U_j)_{j\in J}$ and, for any $j \in J$, there exists a $\mathscr{C}^r$-homeomorphisms j such that:*

i) $\theta(U_j) = S = \{\, \mathbf{y} = (y', y_d) \in \mathbb{R}^d \,;\ |y'| < 1,\ |y_d| < 1\}$,

ii) $\theta(U_j \cap \Omega) = S_+ = \{\, \mathbf{y} \in S \,;\ y_d > 0\}$,

iii) $\theta(U_j \cap \Gamma) = S_0 = \{\, \mathbf{y} \in S \,;\ y_d = 0\}$.

Concerning the approximation by smooth functions, we have the following results (see, for instance, [[51], p. 40], [[92], p. 44], or [[137], p. 11]).

Theorem 2.3 *Let $\Omega \subset \mathbb{R}^d$ be an open bounded set. Then, the following approximation results are true.*

1. *$\mathcal{C}_0^\infty(\Omega)$ is dense in $W_0^{m,p}(\Omega)$.*
2. *If Ω has the cone property, then, $\mathcal{C}^\infty(\overline{\Omega})$ is dense in $W^{m,p}(\Omega)$.*
3. *If Ω is $\mathscr{C}^\infty$-smooth, then, $\mathcal{D}(\overline{\Omega})$ is dense in $H^m(\Omega)$.*

We now recall the following Sobolev embedding theorem (see [2, 92, 147] for more details and proofs) which will be used frequently in this manuscript.

Theorem 2.4 (Sobolev Embedding Theorem) *Suppose that the open bounded set Ω has the cone property and $1 \leq p < +\infty$. Then, the following assertions hold.*

1. *If $mp < d$, then,*

   i) $W^{m,p}(\Omega) \hookrightarrow L^{p^*}(\Omega)$ *where* $p^* = \dfrac{dp}{d-mp}$.

   ii) $W^{m,p}(\Omega) \hookrightarrow_c L^q(\Omega)$ *for any q with $1 \leq q < p^*$.*

2. *If $mp = d$, then, $W^{m,p}(\Omega) \hookrightarrow_c L^q(\Omega)$ for any q with $1 \leq q < +\infty$.*

3. *If $mp > d$, then,*

   i) $W^{m,p}(\Omega) \hookrightarrow_c L^q(\Omega)$ *for any q with $1 \leq q < +\infty$.*

   ii) $W^{m,p}(\Omega) \hookrightarrow_c \mathcal{C}^k(\overline{\Omega})$ *for any integer k with* $\dfrac{mp-d}{p} - 1 \leq k < \dfrac{mp-d}{p}$

As a consequence of this theorem, we have the following particular cases that we shall often use:

$$\begin{aligned} &H^1(\Omega) \hookrightarrow_c \mathcal{C}(\overline{\Omega}) \quad \text{if} \quad d = 1, \\ &H^1(\Omega) \hookrightarrow_c L^q(\Omega) \quad \text{where} \quad \begin{cases} q \in [1, +\infty) & \text{if} \quad d = 1, \\ q = 6 & \text{if} \quad d = 3. \end{cases}, \\ &H^2(\Omega) \hookrightarrow_c \mathcal{C}(\overline{\Omega}) \quad \text{if} \quad d \in \{1, 2\}. \end{aligned}$$

We note that a function $v \in H^1(\Omega)$ is not necessary continuous on Ω, neither on $\overline{\Omega}$, and so, we may not define, in the classical sense, the values of v on the boundary Γ of Ω. The trace theorems show that it is not necessary for a function to be continuous in order to define, in the trace sense, its restriction on the boundary Γ of Ω. Their purpose is to determine the space of functions defined on the boundary Γ of Ω containing the traces of functions in $W^{m,p}(\Omega)$.

The next theorem (see [[51], p. 40] or [[147], p. 9]) allows to define every function $v \in H^1(\Omega)$ almost everywhere on Γ.

Theorem 2.5 (Trace Theorem for $H^1(\Omega)$) *Let Ω be an open bounded set in $\mathbb{R}^d$ of class $\mathscr{C}^1$ with its boundary. Then, we can uniquely define the trace $\gamma_0 v$ of $v \in H^1(\Omega)$ on Γ such that $\gamma_0 v$ coincides with the usual definition*

$$\gamma_0 v(x) = v(x) \quad x \in \Gamma, \tag{2.16}$$

if $v \in \mathcal{C}^1(\overline{\Omega})$. Moreover, the mapping $\gamma_0 : H^1(\Omega) \to L^2(\Gamma)$ is linear continuous and the range of $\gamma_0(H^1(\Omega))$ is a space smaller than $L^2(\Gamma)$ denoted by $H^{\frac{1}{2}}(\Gamma)$.

Theorem 2.6 *Suppose that Ω is sufficiently smooth. Then,*

$$W^{m,p}(\Omega) \hookrightarrow L^q(\Gamma),$$

where $q = \dfrac{dp-p}{d-mp}$ *if* $mp < d$, *and* $1 \leq q < \infty$ *if* $mp = d$.

In particular, we have the following frequently useful results.

$$H^1(\Omega) \hookrightarrow L^q(\Gamma) \quad \text{where} \quad \begin{cases} q \in [1, +\infty) & \text{if} \quad d = 2, \\ q = \dfrac{2(n-1)}{n-2} & \text{if} \quad d \geq 3. \end{cases}$$

2.1 Functional frameworks

In the study of physical problems, we frequently use specific spaces. In this section, we give the function spaces as well as some fundamental results that will be used in the study of the problems we have considered.

We introduce the following functional spaces

$$H = \{\, u = (u_i)\,;\; u_i \in L^2(\Omega)\}, \quad \mathcal{H} = \{\, \sigma = (\sigma_{ij})\,;\; \sigma_{ij} = \sigma_{ji} \in L^2(\Omega)\}, \tag{2.17}$$

$$H_1 = \{\, u = (u_i)\,;\; u_i \in H^1(\Omega)\}, \quad \mathcal{H}_1 = \{\, \sigma \in \mathcal{H}\,;\; \sigma_{ij,j} \in H\}, \tag{2.18}$$

$$\mathcal{W} = \{\mathrm{D} = (\mathrm{D}_i) \in L^2(\Omega)^d\,;\; div\ \mathrm{D} \in L^2(\Omega)\}. \tag{2.19}$$

These are real Hilbert spaces endowed with the inner products

$$(u,\, v)_H = \int_\Omega u_i\, v_i\, dx, \quad (\sigma,\, \tau)_{\mathcal{H}} = \int_\Omega \sigma_{ij}\, \tau_{ij}\, dx, \quad (u,\, v)_{H_1} = (u,\, v)_H + (\varepsilon(u),\, \varepsilon(v))_{\mathcal{H}},$$

$$(\sigma,\, \tau)_{\mathcal{H}_1} = (\sigma,\, \tau)_{\mathcal{H}} + (Div\, \sigma,\, Div\, \tau)_H, \quad (\mathrm{D}, \mathrm{E})_{\mathcal{W}} = (\mathrm{D}, \mathrm{E})_H + (div\ \mathrm{D}, div\ \mathrm{E})_{L^2(\Omega)},$$

with the associated norms $\|\,.\,\|_H$, $\|\,.\,\|_{\mathcal{H}}$, $\|\,.\,\|_{H_1}$, $\|\,.\,\|_{\mathcal{H}_1}$ and $\|\,.\,\|_{\mathcal{W}}$, respectively.

Let $H_\Gamma = H^{\frac{1}{2}}(\Gamma)^d$ and let $\gamma : H_1 \to H_\Gamma$ be the trace map. For every element $v \in H_1$, we also use the notation v to note the trace γv of v on Γ.

Let H'_Γ be the dual of H_Γ and let $\langle\, .\,,\, .\,\rangle_\Gamma$ denote the duality pairing between H'_Γ and H_Γ. For every $\sigma \in \mathcal{H}_1$, $\sigma\nu$ can be defined as the element in the H'_Γ which satisfies Green's formula (1.39):

$$\langle\, \sigma\nu\,,\, \gamma v\,\rangle_\Gamma = (\sigma,\, \varepsilon(v))_{\mathcal{H}} + (Div\, \sigma,\, v)_H \quad \forall v \in H_1.$$

Moreover, if σ is continuously differentiable on $\overline{\Omega}$, then

$$\langle\, \sigma\nu \,,\, \gamma v \,\rangle_\Gamma = \int_\Gamma \sigma\nu \,.\, v \, da,$$

for all $v \in H_1$, where da is the surface measure element.

Let us also introduce $H_{\Gamma_3}^{\frac{1}{2}} \subset L^2(\Gamma_3)$, the space of normal traces on Γ_3:

$$H_{\Gamma_3}^{\frac{1}{2}} = \{\, v_\nu \in L^2(\Gamma_3)\,;\; \exists v \in H_1,\, v_\nu = \gamma v \,.\, \nu \},$$

and its dual $H_{\Gamma_3}^{-\frac{1}{2}}$, with norms correspondingly

$$\|v_\nu\|_{H_{\Gamma_3}^{\frac{1}{2}}} = \inf_{v \,\in\, H_1} \{\, \|v\|_{H_1}\,;\; v_\nu = \gamma v \,.\, \nu \,\} \quad \forall v_\nu \,\in\, H_{\Gamma_3}^{\frac{1}{2}},$$

$$\|\sigma_\nu\|_{H_{\Gamma_3}^{-\frac{1}{2}}} = \sup_{\substack{v_\nu \in H_{\Gamma_3}^{\frac{1}{2}} \\ v_\nu \neq 0_{H_{\Gamma_3}^{\frac{1}{2}}}}} \frac{\langle\, \sigma_\nu \,,\, v_\nu \,\rangle_{\Gamma_3}}{\|v_\nu\|_{H_{\Gamma_3}^{\frac{1}{2}}}} \quad \forall \sigma_\nu \,\in\, H_{\Gamma_3}^{-\frac{1}{2}},$$

where $\langle\,.\,,\,.\,\rangle_{\Gamma_3}$ denotes the duality pairing between $H_{\Gamma_3}^{-\frac{1}{2}}$ and $H_{\Gamma_3}^{\frac{1}{2}}$. Keeping in mind the boundary condition (1.22), we introduce the spaces of the displacements and the electric potential

$$V = \{\, v \in H_1\,;\; v = 0 \quad \text{on} \;\; \Gamma_1 \}, \tag{2.20}$$

$$W = \{ \xi \in H^1(\Omega)\,;\; \xi = 0 \text{ on } \Gamma_a \}, \tag{2.21}$$

and K be the set of admissible displacements

$$K = \{\, v \in V\,;\; (v_\nu - \varrho) \le 0 \quad \text{on} \;\; \Gamma_3 \}. \tag{2.22}$$

Since $meas(\Gamma_1) > 0$ and $meas(\Gamma_a) > 0$, Korn's (A.50) and the Friedrichs-Poincaré inequalities hold: There exists $c_K > 0$ and $c_F > 0$ which depends only on Ω, $\Gamma_1$and Γ_a such that

$$\|\varepsilon(v)\|_{\mathcal{H}} \ge c_K \|v\|_{H_1} \qquad \forall v \in V, \tag{2.23}$$

$$\|\nabla\xi\|_H \ge c_F \|\xi\|_{H^1(\Omega)} \quad \forall \xi \in W. \tag{2.24}$$

Therefore, the space V endowed with the inner product $(u,\, v)_V = (\varepsilon(u),\, \varepsilon(v))_{\mathcal{H}}$ is a real Hilbert space, and its associated norm $\|v\|_V = \|\varepsilon(v)\|_{\mathcal{H}}$ is equivalent on V to the usual norm $\|\,.\,\|_{H_1}$.

On W, we consider the inner product given by $(\varphi, \xi)_W = (\nabla\varphi, \nabla\xi)_H$. It follows from (2.24) that $\|\,.\,\|_{H^1(\Omega)}$ and $\|\,.\,\|_W$ are equivalent norms on W and thus $(W, \|\,.\,\|_W)$ is a real Hilbert space. By Sobolev's trace theorem, there exist two positive constants c_0 and $\widetilde{c_0}$ which depends only on Ω, Γ_3, Γ_1 and Γ_a such that

$$\|v\|_{L^2(\Gamma)^d} \leq c_0\|v\|_V \quad \forall v \in V, \tag{2.25}$$

$$\|\xi\|_{L^2(\Gamma_3)} \leq \widetilde{c_0}\|\xi\|_W \quad \forall \xi \in W. \tag{2.26}$$

Part II

Nonlinear elastic materials

Chapter 3

Unilateral contact with non-local Coulomb's friction law in elasto-plasticity

In this chapter, we study a static contact problem modeling the interaction between an elasto-plastic body, behavior already presented in subsection 1.1.3, and a rigid foundation. The contact is described by Signorini's conditions, as detailed in (1.25), the friction follows Coulomb's non-local law. This chapter is divided into six sections. In Section 3.1, we propose a strong formulation of the problem. In Section 3.2, we introduce some preliminary, list assumptions on data and we give a weak formulation of the model. In Section 3.3, we state an existence and uniqueness result, Theorem 3.1, and we provide its proof where it is carried out in several steps and based on arguments of variational inequalities, pseudomonotone operators and a fixed point theorem. In Section 3.4, we propose an iterative solution scheme, and we prove its convergence. In Section 3.5, we use an appropriate Augmented Lagrangian formulation to improve the conditioning of the iterative problem which will lead us to use Uzawa type algorithms. The results presented in this chapter have been discussed in the paper [23].

3.1 The mechanical problem

An elasto-plastic body occupying, in its reference configuration, an open and bounded domain $\Omega \subset \mathbb{R}^d$, $d = 2, 3$ is considered, with a sufficiently regular boundary Γ, partitioned into three disjoint measurable parts Γ_1, Γ_2 and Γ_3, such that $meas(\Gamma_1) > 0$. A volume force of density f_0 acts in Ω. The body is supposed to be clamped on Γ_1, and thus the displacement field vanishes there. A surface traction of density f_2 acts on Γ_2. During the process, the body may come in contact over Γ_3 with a rigid foundation (Figure 1.1.). The strong formulation of the problem can then be summarized as follows

Problem (P). *Find a displacement field $u : \Omega \to \mathbb{R}^d$ and a stress filed $\sigma : \Omega \to \mathbb{S}^d$ such that*

$$\sigma = \mathfrak{F}\varepsilon(u) \quad \text{in} \quad \Omega, \tag{3.1}$$

$$\operatorname{Div} \sigma + f_0 = 0 \quad \text{in} \quad \Omega, \tag{3.2}$$

$$u = 0 \quad \text{on} \quad \Gamma_1, \tag{3.3}$$

$$\sigma\nu = f_2 \quad \text{on} \quad \Gamma_2, \tag{3.4}$$

$$(u_\nu - \varrho) \leq 0, \quad \sigma_\nu(u) \leq 0, \quad \sigma_\nu(u)\ (u_\nu - \varrho) = 0 \quad \text{on} \quad \Gamma_3, \tag{3.5}$$

$$\left.\begin{array}{l} \|\sigma_\tau\| \leq \mu(\|u_\tau\|)|\mathrm{R}\sigma_\nu(u)|, \\ \|\sigma_\tau\| < \mu(\|u_\tau\|)|\mathrm{R}\sigma_\nu(u)| \;\Rightarrow\; u_\tau = 0, \\ \|\sigma_\tau\| = \mu(\|u_\tau\|)|\mathrm{R}\sigma_\nu(u)| \Rightarrow \exists\lambda \in \mathbb{R}^+ \text{ such that } \sigma_\tau = -\lambda\, u_\tau, \end{array}\right\} \quad \text{on} \quad \Gamma_3. \tag{3.6}$$

Here, (3.1) represents the constitutive law, in which $\mathfrak{F} : \Omega \times \mathbb{S}^d \to \mathbb{S}^d$ is the nonlinear elasticity operator that describes the behavior of the Hencky's materials ($\mathfrak{F}\varepsilon = k_0 \operatorname{tr}(\varepsilon) \mathrm{I} + 2\, g(\|\bar{\varepsilon}\|^2)\bar{\varepsilon}$), already defined in (1.13). Then, (3.2) is the equilibrium equation in the case of a static problem. Then, (3.3) and (3.4) represent the conditions at the limits of the displacement and the traction, respectively. Conditions (3.5), represent the classical Signorini's contact conditions with the gap ϱ between Γ_3 and the rigid foundation, measured along the outward normal ν. Relation (3.6) represent the Coulomb's friction law in which μ is the coefficient of friction, and R is a regularization operator. Note that in (3.6), the coefficient of friction is assumed to depend on the slip $\|u_\tau\|$ which leads to a nonstandard frictional contact problem.

3.2 Variational formulation

To derive a variational formulation for problem (P), we shall use the functional frameworks introduced in Chapter 2. Then, we make the following assumptions on data

(h_1) The function g is continuously differentiable in $[0, +\infty)$ and satisfies

(a) $0 < g_0 \leq g(\xi) \leq \dfrac{d}{2}k_0,$

(b) $0 < \alpha \leq g(\xi) + 2g'(\xi)\,\xi \leq \beta,$

where g_0, α and β are given positive constants.

(h_2) The coefficient of friction $\mu : \Gamma_3 \times \mathbb{R}_+ \to \mathbb{R}_+$ satisfies

(a) The function $x \mapsto \mu(x, u)$ is measurable on Γ_3, for all $u \in \mathbb{R}_+$.

(b) There exists $\mu^* > 0$ such that $0 \le \mu(x,u) \le \mu^*$, for all $u \in \mathbb{R}_+$, a.e. $x \in \Gamma_3$.

(c) There exists $L_\mu > 0$ such that

$$|\mu(.,u) - \mu(.,v)| \le L_\mu |u - v|, \text{ for all } u,\, v \in \mathbb{R}_+, \text{ a.e. on } \Gamma_3.$$

(h_3) The mapping $\mathrm{R} : H_{\Gamma_3}^{-\frac{1}{2}} \to L^\infty(\Gamma_3)$ is linear and continuous with $\|\mathrm{R}\| = c_{\mathrm{R}}$.

(h_4) The densities of the body forces and surface tractions have the regularity

$$f_0 \in L^2(\Omega)^d, \quad f_2 \in L^2(\Gamma_2)^d.$$

Next, we define the functional $j : V \times V \to \mathbb{R}$ by

$$j(u,v) = \int_{\Gamma_3} \mu(\|u_\tau\|)\, |\mathrm{R}\sigma_\nu(u)|\, \|v_\tau\|\, da. \tag{3.7}$$

Using Riesz's representation theorem A.6, we define $f \in V$ as follows,

$$(f,v)_V = \int_\Omega f_0 \,.\, v\, dx + \int_{\Gamma_2} f_2 \,.\, v\, da \quad \forall v \in V. \tag{3.8}$$

Keeping in mind the assumptions $(h_2)(a) - (h_4)$ it follows that the integrals in (3.7) and (3.8) are well-defined.
Thus, we are led to the following weak formulation.

Problem (PV). *Find a displacement field $u \in K$ such that:*

$$(Au, v - u)_V + j(u,v) - j(u,u) \ge (f, v - u)_V \quad \forall v \in K, \tag{3.9}$$

where the operator $A : V \to V$ is defined by

$$(Au,v)_V = \int_\Omega k_0 \,\mathrm{tr}(\varepsilon(u))\, tr(\varepsilon(v)) + 2g(\|\overline{\varepsilon}(u)\|^2)(\overline{\varepsilon}(u) : \varepsilon(v))dx \quad \forall v \in V. \tag{3.10}$$

Now, we are able to state the following result of existence and uniqueness

3.3 Existence and uniqueness result

Theorem 3.1 *Assume that (h_1), $(h_2)(a) - (b)$ and $(h_3) - (h_4)$ hold. Then,*

1. *The problem (PV) has at least one solution* $u \in K$.

2. *Under the assumption* $(h_2)(c)$, *there exists* $L^* > 0$ *such that if* $\mu^* + L_\mu < L^*$, *then the problem (PV) has a unique solution.*

The proof of Theorem 3.1 will be carried out in several steps, and it is based on arguments of variational inequalities and Schauder's fixed point theorem A.4. To this end, we assume in the following that (h_1), $(h_2)(a) - (b)$ and $(h_3) - (h_4)$ hold. In the sequel, we define a closed convex set of $L^2(\Gamma_3)$ as follows:

$$\mathcal{K} = \{\eta \in L^2(\Gamma_3)\,;\ \eta \geq 0 \text{ and } \|\eta\|_{L^2(\Gamma_3)} \leq \kappa\},$$

with κ to be specified later. For every $\eta \in L^2(\Gamma_3)$ we define the function

$$j_\eta(v) = \int_{\Gamma_3} \eta \|v_\tau\|\, da \quad \forall v \in V. \tag{3.11}$$

In the first step, we consider the following intermediate problem where Coulomb's friction law is replaced by Tresca's friction law.

Problem (PV$^\eta$). *Find a displacement field* $u_\eta \in K$ *such that:*

$$(Au_\eta, v - u_\eta)_V + j_\eta(v) - j_\eta(u_\eta) \geq (f, v - u_\eta)_V \quad \forall v \in K. \tag{3.12}$$

We have the following existence and uniqueness result.

Lemma 3.1 *For any* $\eta \in \mathcal{K}$, *the problem (PV$^\eta$) has a unique solution* $u_\eta \in K$. *Then, there exists a constant* $c_1 > 0$ *such that the solution of the problem (PV$^\eta$) satisfies*

$$\|u_\eta\|_V \leq c_1 \|f\|_V. \tag{3.13}$$

Proof. First, we show that the operator $A : V \to V$ given by (3.10) is strongly monotone and Lipschitz continuous.

Consider two elements u_1, $u_2 \in V$. Using (3.10) we get

$$\begin{aligned}(Au_1 - Au_2, u_1 - u_2)_V = \Big(k_0 \operatorname{tr}(\varepsilon(u_1 - u_2)), \operatorname{tr}(\varepsilon(u_1 - u_2))\Big)_{\mathcal{H}} \\ + 2\Big(g(\|\overline{\varepsilon}(u_1)\|^2)\overline{\varepsilon}(u_1) - g(\|\overline{\varepsilon}(u_2)\|^2)\overline{\varepsilon}(u_2), \overline{\varepsilon}(u_1 - u_2)\Big)_{\mathcal{H}}.\end{aligned}$$

Using the equality

$$g(\|\overline{\varepsilon}(u_1)\|^2)\overline{\varepsilon}(u_1) - g(\|\overline{\varepsilon}(u_2)\|^2)\overline{\varepsilon}(u_2) = \int_0^1 \frac{d}{dt}[g(\|\delta\|^2)\delta]dt,$$

with $\delta = \overline{\varepsilon}(u_2) + t\overline{\varepsilon}(u_1 - u_2)$, it follows that

$$(Au_1 - Au_2, u_1 - u_2)_V = \int_0^1 \mathcal{J} dt, \tag{3.14}$$

where

$$\mathcal{J} = \int_\Omega k_0 |\operatorname{tr}(\varepsilon(u_1 - u_2))|^2 + 2\Big[g(\|\delta\|^2)\|\overline{\varepsilon}(u_1 - u_2)\|^2 + 2g'(\|\delta\|^2)(\delta : \overline{\varepsilon}(u_1 - u_2))^2\Big]dx.$$

Let $g'(t,x) \geq 0$ for all x in Ω, thus

$$\mathcal{J} \geq \int_\Omega \Big[k_0 |\operatorname{tr}(\varepsilon(u_1 - u_2))|^2 + 2g(\|\delta\|^2)\|\overline{\varepsilon}(u_1 - u_2)\|^2\Big]dx,$$

and, keeping in mind (1.10), (h_1), we find

$$\begin{aligned} \mathcal{J} &\geq \int_\Omega \Big[(k_0 - \frac{2}{d}g_0)|\operatorname{tr}(\varepsilon(u_1 - u_2))|^2 + 2g_0\|\varepsilon(u_1 - u_2)\|^2\Big]dx \\ &\geq 2g_0 \int_\Omega \|\varepsilon(u_1 - u_2)\|^2 dx = 2g_0\|u_1 - u_2\|_V^2. \end{aligned} \tag{3.15}$$

Let us now assume that $g'(t,x) \leq 0$ for all x in Ω. We use the Cauchy-Schwarz inequality $(\delta : (\overline{\varepsilon}(u_1 - u_2)))^2 \leq \|\delta\|^2 \, \|\overline{\varepsilon}(u_1 - u_2)\|^2$, the relation (1.10) and the assumption (h_1), to find

$$\begin{aligned} \mathcal{J} &\geq \int_\Omega k_0 |\operatorname{tr}(\varepsilon(u_1 - u_2))|^2 + 2\Big[g(||\delta||^2) + 2g'(||\delta||^2)||\delta||^2\Big]||\overline{\varepsilon}(u_1 - u_2)||^2 dx, \\ &\geq \int_\Omega \Big[k_0 |\operatorname{tr}(\varepsilon(u_1 - u_2))|^2 + 2\alpha||\overline{\varepsilon}(u_1 - u_2)||^2\Big]dx, \\ &\geq \int_\Omega \Big[(k_0 - \frac{2}{d}\alpha)|\operatorname{tr}(\varepsilon(u_1 - u_2))|^2 + 2\alpha||\varepsilon(u_1 - u_2)||^2\Big]dx. \end{aligned}$$

Since we may assume that $g_0 \geq \alpha$, we obtain

$$\mathcal{J} \geq 2\alpha \int_\Omega \|\varepsilon(u_1 - u_2)\|^2 dx = 2\alpha\|u_1 - u_2\|_V^2. \tag{3.16}$$

Combining now (3.14), (3.15) and (3.16), we deduce that

$$(Au_1 - Au_2, u_1 - u_2)_V \geq m_A \|u_1 - u_2\|_V^2 \quad \forall u_1,\, u_2 \in V, \tag{3.17}$$

with $m_A = 2\alpha$.
On the other hand, using again (3.10) and Cauchy-Schwarz inequality, we have

$$|(Au_1 - Au_2, v)_V| \leq \int_0^1 \mathcal{I} dt \quad \forall v \in V, \tag{3.18}$$

where

$$\mathcal{I} = \int\limits_\Omega k_0 |\operatorname{tr}(\varepsilon(u_1 - u_2))| \, |\operatorname{tr}(\varepsilon(v)| + 2\Big[g(\|\delta\|^2) + 2|g'(\|\delta\|^2)| \, \|\delta\|^2\Big] \, \|\overline{\varepsilon}(u_1 - u_2)\| \, \|\overline{\varepsilon}(v)\| dx.$$

Let $g'(t, x) \geq 0$ for all x in Ω, hence using (h_1), Cauchy-Schwarz inequality and we may assume that $\beta \leq \dfrac{d}{2} k_0$, thus we obtain the following result, we obtain

$$\begin{aligned}
&|(Au_1 - Au_2, v)_V| \\
&\leq k_0 \Big(\int\limits_\Omega |\operatorname{tr}(\varepsilon(u_1 - u_2))|^2 dx \Big)^{\frac{1}{2}} \Big(\int\limits_\Omega |\operatorname{tr}(\varepsilon(v)|^2 \, dx \Big)^{\frac{1}{2}} \\
&\qquad\qquad + 2\beta \Big(\int\limits_\Omega \|\overline{\varepsilon}(u_1 - u_2)\|^2 dx \Big)^{\frac{1}{2}} \Big(\int\limits_\Omega \|\overline{\varepsilon}(v)\|^2 dx \Big)^{\frac{1}{2}} \\
&\leq dk_0 \Big(\int\limits_\Omega |\operatorname{tr}(\varepsilon(u_1 - u_2))|^2 + \|\overline{\varepsilon}(u_1 - u_2)\|^2 dx \Big)^{\frac{1}{2}} \Big(\int\limits_\Omega |\operatorname{tr}(\varepsilon(v))|^2 + \|\overline{\varepsilon}(v)\|^2 dx \Big)^{\frac{1}{2}}.
\end{aligned}$$

Thus we obtain following result using the relation (1.10), we get

$$\begin{aligned}
|(Au_1 - Au_2, v)_V| &\leq dk_0 \Big(\int\limits_\Omega d\|\varepsilon(u_1 - u_2)\|^2 dx \Big)^{\frac{1}{2}} \Big(\int\limits_\Omega d\|\varepsilon(v)\|^2 dx \Big)^{\frac{1}{2}} \\
&\leq d^2 k_0 \|\varepsilon(u_1 - u_2)\|_{\mathcal{H}} \|\varepsilon(v)\|_{\mathcal{H}} = d^2 k_0 \|u_1 - u_2\|_V \|v\|_V.
\end{aligned} \tag{3.19}$$

Let now assume that $g'(t, x) \leq 0$ for all x in Ω. Moreover, with the inequality $(g(\|\delta\|^2) +$

$2g'(\|\delta\|^2)\,\|\delta\|^2) > 0$ it follows that

$$|(Au_1 - Au_2, v)_V| \leq k_0 \int\limits_\Omega |\operatorname{tr}(\varepsilon(u_1-u_2))|\,|\operatorname{tr}(\varepsilon(v)|\,dx + \int\limits_\Omega 4g(\|\delta\|^2)\,\|\overline{\varepsilon}(u_1-u_2)\|\,\|\overline{\varepsilon}(v)\|dx.$$

By using (h_1), Cauchy-Schwarz inequality, we find from (1.10) that

$$\begin{aligned} |(Au_1 - Au_2, v)_V| &\leq 2dk_0\Big(\int\limits_\Omega d\|\varepsilon(u_1-u_2)\|^2dx\Big)^{\frac{1}{2}}\Big(\int\limits_\Omega d\|\varepsilon(v)\|^2dx\Big)^{\frac{1}{2}} \\ &\leq 2d^2k_0\|\varepsilon(u_1-u_2)\|_{\mathcal{H}}\|\varepsilon(v)\|_{\mathcal{H}} = 2d^2k_0\|u_1-u_2\|_V\|v\|_V. \end{aligned} \tag{3.20}$$

Now, we take $v = Au_1 - Au_2$ in the inequalities (3.18), (3.19) and (3.20), we deduce that

$$\|Au_1 - Au_2\|_V \leq\; M_A\|u_1-u_2\|_V \quad \forall u_1,\, u_2 \in V, \tag{3.21}$$

with $M_A = 2d^2k_0$.

We employ now (3.17) and (3.21) to see that the operator A is a strongly monotone Lipschitz continuous operator on V. It can be easily verified that j_η given by (3.11) is a proper convex lower semi-continuous function for every $\eta \in \mathcal{K}$. Note also (3.8) implies that $f \in V$. Since K is non-empty closed convex set of V, then, the existence of a unique solution u_η for the problem (PV^η) is guaranteed by Theorem B.3.

Let $\eta \in \mathcal{K}$, we denote by u_η the corresponding solution of the problem (PV^η). We use (3.12) with the choice $v = 0$ and notice that $j_\eta(u_\eta) \geq 0$, it follows from (3.17) that

$$\|u_\eta\|_V \leq\; c_1\|f\|_V,$$

where $c_1 = \dfrac{1}{m_A}$, which concludes the proof of Lemma 3.1. □

In the second step, we consider the operator $\Lambda\,:\,L^2(\Gamma_3) \to L^2(\Gamma_3)$ defined by

$$\Lambda\eta = \mu(\|u_{\eta\tau}\|)|\,\mathrm{R}\,\sigma_\nu(u_\eta)|. \tag{3.22}$$

Clearly, by assumptions $(h_2)(a)$ and (h_3), we see that the operator Λ is well defined on $\mathcal{K}$. In this step we show that the operator Λ has a fixed point. To this end, we need the following result

Lemma 3.2 *The mapping $\eta \mapsto u_\eta$, where u_η is the solution of the problem (PV^η), is weakly continuous from $L^2(\Gamma_3)$ to V.*

Proof. Let (η_n) be subsequence of $L^2(\Gamma_3)$ such that $\eta_n \geq 0$ for all $n \in \mathbb{N}$, converging weakly to η, we denote by $u_{\eta_n} \in K$ the solution of the problem (PV$^\eta$) corresponding to η_n. Then, we have

$$(Au_{\eta_n}, v - u_{\eta_n})_V + j_{\eta_n}(v) - j_{\eta_n}(u_{\eta_n}) \geq (f, v - u_{\eta_n})_V \quad \forall v \in K, \tag{3.23}$$

taking $v = 0$ in (3.23) and using (3.17), we deduce

$$\|u_{\eta_n}\|_V \leq c\left(\|f\|_V + \|\underbrace{\eta_n}_{\text{bounded}}\|_{L^2(\Gamma_3)}\right).$$

This shows that the sequence (u_{η_n}) is bounded in V, then, there exists $\tilde{u} \in V$ and a subsequence, denoted again (u_{η_n}), such that (u_{η_n}) converges weakly to $\tilde{u} \in V$. Since, K is a closed convex set in a real Hilbert space V, therefore, K is weakly closed, then $\tilde{u} \in K$. As the trace map $\gamma : V \to L^2(\Gamma_3)^d$ is a compact operator, from the weak convergence $u_{\eta_n} \rightharpoonup \tilde{u}$ in V, we obtain the convergence $u_{\eta_n} \to \tilde{u}$ strongly in $L^2(\Gamma_3)^d$.

Next, let us prove that $\tilde{u}$ is the solution of Problem (PV$^\eta$). From (3.23) we have

$$(Au_{\eta_n}, u_{\eta_n} - v)_V \leq (f, u_{\eta_n} - v)_V + j_{\eta_n}(v) - j_{\eta_n}(u_{\eta_n}) \quad \forall v \in K. \tag{3.24}$$

Furthermore, we have

$$|j_{\eta_n}(u_{\eta_n}) - j_{\eta_n}(\tilde{u})| \leq \|\underbrace{\eta_n}_{\text{bounded}}\|_{L^2(\Gamma_3)}\|u_{\eta_n} - \tilde{u}\|_{L^2(\Gamma_3)^d}.$$

Since $u_{\eta_n} \to \tilde{u}$ strongly in $L^2(\Gamma_3)^d$, we obtain that $j_{\eta_n}(u_{\eta_n})$ converge to $j_\eta(\tilde{u})$ as $n \to +\infty$. We deduce then

$$\limsup_{n\to+\infty}(Au_{\eta_n}, u_{\eta_n} - v)_V \leq (f, \tilde{u} - v)_V + j_\eta(v) - j_\eta(\tilde{u}) \quad \forall v \in K.$$

On the other hand, we have for all $v \in K$

$$\begin{aligned}
\limsup_{n\to+\infty}(Au_{\eta_n}, u_{\eta_n} - \tilde{u})_V &= \limsup_{n\to+\infty}\left[(Au_{\eta_n}, u_{\eta_n} - v)_V + (Au_{\eta_n}, v - \tilde{u})_V\right] \\
&\leq \limsup_{n\to+\infty}\left[(Au_{\eta_n}, u_{\eta_n} - v)_V + \|Au_{\eta_n}\|_V\|v - \tilde{u}\|_V\right] \\
&\leq (f, \tilde{u} - v)_V + j_\eta(v) - j_\eta(\tilde{u}) + \limsup_{n\to+\infty}\|Au_{\eta_n}\|_V\|v - \tilde{u}\|_V.
\end{aligned}$$

Since $\|Au_{\eta_n}\|_V$ is bounded on V, if we take $v = \tilde{u}$, the last inequality becomes

$$\limsup_{n\to+\infty}(Au_{\eta_n}, u_{\eta_n} - \tilde{u})_V \leq 0.$$

By pseudo-monotonicity of A, we get

$$(A\tilde{u}, \tilde{u} - v)_V \leq \liminf_{n\to+\infty}(Au_{\eta_n}, u_{\eta_n} - v)_V. \tag{3.25}$$

Combining (3.24) and (3.25), we deduce

$$(A\tilde{u}, v - \tilde{u})_V + j_\eta(v) - j_\eta(\tilde{u}) \geq (f, v - \tilde{u})_V.$$

Which means that $\tilde{u} \in K$ is a solution of Problem (PV$^\eta$), and from the uniqueness of the solution for this variational inequality, we obtain $\tilde{u} = u_\eta$. Since u_η is the unique weak limit of any subsequence of (u_{η_n}), we deduce that the whole sequence (u_{η_n}) is weakly convergent in V to u_η, ensures that the mapping $\eta \mapsto u_\eta$ is weakly continuous from $L^2(\Gamma_3)$ to V. □

Lemma 3.3 *If $\kappa = \mu^* meas(\Gamma_3)^{\frac{1}{2}} c_{\mathrm{R}} c_0 c_1 \|f\|_V$, Λ is an operator of $\mathcal{K}$ into itself, and has at least one fixed point.*

Proof. Let us consider η an element of $\mathcal{K}$, we have

$$\|\eta\|_{L^2(\Gamma_3)} \leq \kappa.$$

From (3.22), $(h_2)(b)$, (h_3), (2.25) and (3.13) it follows that

$$\begin{aligned}\|\Lambda\eta\|_{L^2(\Gamma_3)} &= \|\mu(\|u_{\eta\tau}\|)\,|\,\mathrm{R}\,\sigma_\nu(u_\eta)|\|_{L^2(\Gamma_3)}\\ &\leq \mu^* meas(\Gamma_3)^{\frac{1}{2}} c_{\mathrm{R}} c_0 c_1 \|f\|_V.\end{aligned}$$

Hence, by choosing $\kappa = \mu^* meas(\Gamma_3)^{\frac{1}{2}} c_{\mathrm{R}} c_0 c_1 \|f\|_V$. We conclude that Λ is an operator from $\mathcal{K}$ into itself. Note that $\mathcal{K}$ is a nonempty, convex, closed and bounded subset of the reflexive space $L^2(\Gamma_3)$, which implies that $\mathcal{K}$ is weakly compact. Using the properties of μ and R, we deduce that Λ is weakly continuous and hence, by the Schauder's fixed point theorem the operator Λ has at least one fixed point. □

Proof of Theorem 3.1 Let η^* be the fixed point of operator Λ. We denote by u^* the solution of the variational problem (PV$^\eta$) for $\eta = \eta^*$. Using (3.12) and (3.22), it is easy to see that u^*

is a solution of (PV). This proves the existence part of Theorem 3.1. Next, we show that if $\mu^* + L_\mu < L^*$ the solution is unique.
Let u_1, $u_2 \in K$ denote the solutions of Problem (PV). From (3.9), we have

$$(Au_1, v - u_1)_V + j(u_1, v) - j(u_1, u_1) \geq (f, v - u_1)_V \quad \forall v \in K,$$

$$(Au_2, v - u_2)_V + j(u_2, v) - j(u_2, u_2) \geq (f, v - u_2)_V \quad \forall v \in K.$$

Taking $v = u_2$ in the first inequality, $v = u_1$ in the second one and we add the two resulting inequalities, we get

$$(Au_1 - Au_2, u_1 - u_2)_V \leq j(u_1, u_2) - j(u_1, u_1) + j(u_2, u_1) - j(u_2, u_2) = G. \tag{3.26}$$

From (3.7), it is straightforward to show that

$$G = \int_{\Gamma_3} \mu(\|u_{1\tau}\|) \left(|\,\mathrm{R}\,\sigma_\nu(u_1)| - |\,\mathrm{R}\,\sigma_\nu(u_2)|\right) \left(\|u_{1\tau}\| - \|u_{2\tau}\|\right) da$$
$$+ \int_{\Gamma_3} |\,\mathrm{R}\,\sigma_\nu(u_2)| \left(\mu(\|u_{1\tau}\|) - \mu(\|u_{2\tau}\|)\right) \left(\|u_{1\tau}\| - \|u_{2\tau}\|\right) da.$$

$$G \leq \Bigg(\Big\| \Big(\mu(\|u_{1\tau}\|) - \mu(\|u_{1\tau}\|)\Big) |\,\mathrm{R}\,\sigma_\nu(u_1)|$$
$$+ \mu(\|u_{2\tau}\|)\Big(|\,\mathrm{R}\,\sigma_\nu(u_1)| - |\,\mathrm{R}\,\sigma_\nu(u_2)|\Big)\Big\|_{L^2(\Gamma_3)}\Bigg) \|u_1 - u_2\|_{L^2(\Gamma_3)^d}$$

Using the property of μ, the continuity of R and (2.25), we deduce

$$G \leq c_0^2 \left(\mu^* meas(\Gamma_3)^{\frac{1}{2}} c_{\mathrm{R}} + \|\,\mathrm{R}\,\|_{L^\infty(\Gamma_3)} L_\mu\right) \|u_1 - u_2\|_V^2. \tag{3.27}$$

So, we combine (3.17), (3.26) and (3.27) to deduce that there exists a positive constant c^* such that

$$\|u_1 - u_2\|_V^2 \leq \frac{c^*}{m_A}\Big(\mu^* + L_\mu\Big)\|u_1 - u_2\|_V^2.$$

Let $L^* = \dfrac{m_A}{c^*}$, then if we have $\mu^* + L_\mu < L^*$, we obtain $u_1 = u_2$ and it implies the uniqueness of the solution.

3.4 Iteration method

The main purpose of this section is to propose an iterative solution scheme for the quasi-variational problem (PV), and to prove its convergence. This iteration technique allows the numerical treatment of the nonlinearity due to the friction's term and that of the material's behavior. Moreover, it is interpreted mechanically as a sequence of linear unilateral contact problems with Tresca's law of friction. Whenever we discuss the proposed iterative method, no summation is implied over the repeated index n.

The iteration technique consists of the following natural procedure. Let u_n be the n-th approximation of the solution to the problem (PV). We seek for the weak solution u_{n+1} of the linear problem

$$\begin{cases} \text{Given an initial guess } u_0 \in K, \text{ find } u_{n+1} \in K \text{ such that} \\ B(u_n; u_{n+1}, v - u_{n+1}) + j(u_n, v) - j(u_n, u_{n+1}) \geq (f, v - u_{n+1})_V \quad \forall v \in K, \end{cases} \tag{3.28}$$

where $B : K \times V \times V \to \mathbb{R}$ is an operator defined by

$$B(u; v, w) = \Big(k_0 \operatorname{tr}(\varepsilon(v)) \, \mathrm{I} + 2g(\|\overline{\varepsilon}(u)\|^2)\overline{\varepsilon}(v), \varepsilon(w)\Big)_{\mathcal{H}}.$$

For a fixed $u \in K$, it's clear that $(v, w) \mapsto B(u; v, w)$ is a bilinear, symmetric form, and by arguments similar to those used in the proof of Theorem 3.1, we can show that

$$\begin{cases} (a) \quad |B(u; v, w)| \leq M_A \|v\|_V \, \|w\|_V \quad \forall\, v,\, w \in V, \\ (b) \quad B(u; v, v) \geq m_A \|v\|_V^2 \qquad\qquad \forall\, v \in V. \end{cases}$$

We have the following convergence result.

Theorem 3.2 *Under assumptions of Theorem 3.1, the iteration method* (3.28) *converges, i.e.,*

$$\|u_n - u\|_V \to 0 \quad \textit{as } n \to +\infty,$$

where u is the unique solution to Problem (PV).

Proof. Let $u_n \in K$ the solution of the problem (3.28), thus we have for all $v \in K$,

$$B(u_{n-1}; u_n, v - u_n) + j(u_{n-1}, v) - j(u_{n-1}, u_n) \geq (f, v - u_n)_V. \tag{3.29}$$

Taking $v = 0$ in (3.29), by using the properties of B and the fact that $j(u_{n-1}, u_n) > 0$, we have

$$\|u_n\|_V \leq c_1 \|f\|_V.$$

Afterward, since the sequence $(u_n)_{n\geq 1}$ is bounded in V, then, there exists $\omega \in V$ and a sub-sequence, denoted again $(u_n)_{n\geq 1}$, such that $(u_n)_{n\geq 1}$ converges weakly to $\omega \in V$. Since, K is a closed convex set in a real Hilbert space V, therefore, K is weakly closed, then $\omega \in K$. Moreover, using the compactness of the trace map $\gamma : V \to L^2(\Gamma_3)^d$, it follows from the weak convergence of $(u_n)_{n\geq 1}$ that $u_n \to \omega$ strongly in $L^2(\Gamma_3)^d$ as $n \to +\infty$.

Next, let us prove that ω is the solution of Problem (PV). Using (3.7) keeping in mind the properties of μ and R, we get

$$\begin{aligned}|j(u_{n-1}, u_n) - j(\omega, \omega)| \leq \mu^* meas(\Gamma_3)^{\frac{1}{2}} \| \mathrm{R} \|_{L^\infty(\Gamma_3)} \|u_n - \omega\|_{L^2(\Gamma_3)^d} \\ + \left(\mu^* meas(\Gamma_3)^{\frac{1}{2}} c_{\mathrm{R}} + L_\mu \| \mathrm{R} \|_{L^\infty(\Gamma_3)}\right) \|u_{n-1} - \omega\|_{L^2(\Gamma_3)^d} \|\omega\|_{L^2(\Gamma_3)^d}.\end{aligned}$$

Since $u_n \to \omega$ strongly in $L^2(\Gamma_3)^d$, we obtain

$$j(u_{n-1}, u_n) \to j(\omega, \omega) \quad \text{as } n \to +\infty.$$

We deduce from (3.29) that for all $v \in K$

$$\limsup_{n\to+\infty} B(u_{n-1}; u_n, u_n - v) \leq (f, \omega - v)_V + j(\omega, v) - j(\omega, \omega).$$

On the other hand, we have for all $v \in K$,

$$\begin{aligned}\limsup_{n\to+\infty} B(u_{n-1}; u_n, u_n - \omega) &= \limsup_{n\to+\infty} \Big[B(u_{n-1}; u_n, u_n - v) + B(u_{n-1}; u_n, v - \omega)\Big] \\ &\leq \limsup_{n\to+\infty} \Big[B(u_{n-1}; u_n, u_n - v) + M_A \|u_n\|_V \|v - \omega\|_V\Big] \\ &\leq (f, \omega - v)_V + j(\omega, v) - j(\omega, \omega) + \limsup_{n\to+\infty} M_A \|u_n\|_V \|v - \omega\|_V.\end{aligned}$$

Note that $\|u_n\|_V$ is bounded on V, we may then substitute $v = \omega$ into the last inequality to obtain

$$\limsup_{n\to+\infty} B(u_{n-1}; u_n, u_n - \omega) \leq 0.$$

Therefore, by pseudo-monotonicity of B, we get

$$B(\omega;\omega,\omega-v) \leq \liminf_{n\to+\infty} B(u_{n-1};u_n,u_n-v). \tag{3.30}$$

Combining (3.29) and (3.30) we deduce

$$B(\omega;\omega,v-\omega) + j(\omega,v) - j(\omega,\omega) \geq (f,\omega-v)_V.$$

Which means that $\omega \in K$ is a solution of Problem (PV), and from the uniqueness of the solution for this quasi-variational inequality, we get $\omega = u$. Since u is the unique weak limit of any subsequence of $(u_n)_{n\geq 1}$, we deduce that the whole sequence $(u_n)_{n\geq 1}$ is weakly convergent in V to u.

Let us now prove that

$$\|u_n - u\|_V \to 0 \quad \text{as } n \to +\infty.$$

To this end, let $u_n \in K$ be a solution of (3.28) and $u \in K$ a solution of the problem (PV). By using the strong monotonicity of A, we get

$$m_A\|u_n-u\|_V^2 \leq (Au_n - Au, u_n-u)_V = (Au_n, u_n-u)_V - (Au, u_n-u)_V.$$

Using (3.9) with $v = u_n$ and the fact that $(Au,v)_V = B(u;u,v)$ for all u, v in V, we obtain

$$m_A\|u_n-u\|_V^2 \leq B(u_n;u_n,u_n-u) - (f,u_n-u)_V + j(u,u_n) - j(u,u).$$

We conclude by using the fact that $(u_n)_{n\geq 1}$ is bounded, weakly convergent to u in V and the continuity properties of B, j and $(f,\,.\,)_V$, to get

$$u_n \to u \quad \text{strongly in } V \quad \text{as } n \to +\infty.$$

□

3.5 Augmented Lagrangian for the iterative problem

In this section, we present an Augmented Lagrangian formulation of the iterative problem introduced in Section 3.4. It is a method that combines both the penalty and Lagrangian method, and it has the advantage of not depending on the penalty parameter and moreover it offers an excellent convergence. In our context, we will use the Augmented Lagrangian method to uncouple the displacement components, normal and tangential, over the contact zone.

Thus, an equivalent constrained minimization problem of (3.28) is formulated, which is suitable for numerical analysis. The proposed minimization problem is as follows

$$\begin{cases} \textit{Find } u \in K \textit{ such that} \\ J_n(u) + j_n(u) \leq J_n(v) + j_n(v) \quad \forall v \in K, \end{cases} \tag{3.31}$$

where

$$j_n(v) = j(u_n, v) \quad \forall v \in V,$$

and J_n is the energy functional due to non-frictional effects given by

$$J_n(v) = \frac{1}{2}B(u_n; v, v) - (f, v)_V \quad \forall v \in V.$$

The quadratic functional J_n is strictly convex and Gateaux differentiable on V. Moreover, the friction functional j_n is convex, and lower semi-continuous on V, thus, the existence of a unique solution to problem (3.31) is guaranteed by Theorem A.10.

We can now define an Augmented Lagrangian to (3.31), but this will lead us to Uzawa block relaxation method in every iteration. To achieve a solution, we need additional steps. Following Glowinski and Le Tallec [60], we introduce the set

$$C = \{\varphi \in L^2(\Gamma_3)\, ;\; \varphi - \varrho \leq 0 \text{ on } \Gamma_3\},$$

and the characteristic functional $I_C : L^2(\Gamma_3) \to \mathbb{R} \cup \{+\infty\}$ of the set C, is defined by

$$I_C(\varphi) = \begin{cases} 0, & \text{if } \varphi \in C, \\ +\infty, & \text{if } \varphi \notin C. \end{cases}$$

Let $\varphi = (\varphi_f, \varphi_c)$, where φ_f (friction) and φ_c (contact) are auxiliary variables. It is easy to see that the problem giving in (3.31) is equivalent to the following constrained minimization problem

Find $(u,\phi) \in V \times L^2(\Gamma_3)^2$ *such that*

$$J_n(u) + j_n(\phi_f) + I_C(\phi_c) \le J_n(v) + j_n(\varphi_f) + I_C(\varphi_c)\, \forall (v,\varphi) \in V \times L^2(\Gamma_3)^2, \tag{3.32}$$

$$\left.\begin{aligned} u_\nu - \phi_c = 0, \\ u_\tau - \phi_f = 0, \end{aligned}\right\} \quad \text{on } \Gamma_3. \tag{3.33}$$

From (3.32)-(3.33) the Augmented Lagrangian functional $\mathcal{L}_r$ is defined over $V \times L^2(\Gamma_3)^2 \times L^2(\Gamma_3)^2$ by

$$\begin{aligned} \mathcal{L}_r(v,\varphi;\theta) =& J_n(v) + j_n(\varphi_f) + I_C(\varphi_c) + (\theta_c, v_\nu - \varphi_c)_{L^2(\Gamma_3)} \\ &+ (\theta_f, v_\tau - \varphi_f)_{L^2(\Gamma_3)} + \frac{r}{2}\|v_\nu - \varphi_c\|^2_{L^2(\Gamma_3)} + \frac{r}{2}\|v_\tau - \varphi_f\|^2_{L^2(\Gamma_3)}, \end{aligned}$$

where the constant $r > 0$ is the penalty parameter and $\theta = (\theta_f, \theta_c)$. Since the functional $J_n + j_n$ is strictly convex, and the constraints (3.33) are linear, a saddle point of $\mathcal{L}_r$ exists, and it is the solution of the saddle-point problem

$$\begin{cases} \textit{Find } \big((u,\phi);\lambda\big) \in V \times L^2(\Gamma_3)^2 \times L^2(\Gamma_3)^2 \textit{ such that} \\ \mathcal{L}_r(u,\phi;\theta) \le \mathcal{L}_r(u,\phi;\lambda) \le \mathcal{L}_r(v,\varphi;\lambda) \quad \forall \big((v,\varphi);\theta\big) \in V \times L^2(\Gamma_3)^2 \times L^2(\Gamma_3)^2, \end{cases}$$

where we have set $\lambda = (\lambda_f, \lambda_c)$. Equivalently, $\big((u,\phi);\lambda\big)$ is the solution of the min-max problem

$$\max_{\theta} \min_{(v,\varphi)} \mathcal{L}_r(v,\varphi;\theta) = \min_{(v,\varphi)} \max_{\theta} \mathcal{L}_r(v,\varphi;\theta),$$

Uzawa block relaxation method is obtained by minimizing $\mathcal{L}_r$, successively, over u and ϕ, as follows (starting with ϕ^0 and λ^0)

$$\mathcal{L}_r(u^{k+1},\phi^k;\lambda^k) = \min_{v} \mathcal{L}_r(v,\phi^k;\lambda^k), \tag{3.34}$$

$$\mathcal{L}_r(u^{k+1},\phi^{k+1};\lambda^k) = \min_{\varphi} \mathcal{L}_r(u^{k+1},\varphi;\lambda^k), \tag{3.35}$$

$$\lambda^{k+1} = \lambda^k + r(u^{k+1} - \phi^{k+1}). \tag{3.36}$$

The solution of (3.34) can be characterized by the Euler-Lagrange equation [54, 57, 82], since $v \mapsto \mathcal{L}_r(v, \varphi; \theta)$ is convex and differentiable

$$B(u_n; u^{k+1}, v) + r(u_\nu^{k+1}, v_\nu)_{L^2(\Gamma_3)} + r(u_\tau^{k+1}, v_\tau)_{L^2(\Gamma_3)}$$
$$= (f, v)_V + (r\phi_c^k - \lambda_c^k, v_\nu)_{L^2(\Gamma_3)} + (r\phi_f^k - \lambda_f^k, v_\tau)_{L^2(\Gamma_3)}.$$

In (3.35) the subproblems in φ_f and φ_c are uncoupled. Consequently, we can minimize the functional $\varphi \to \mathcal{L}_r(u^{k+1}, \varphi; \lambda^k)$ separately in φ_c and φ_f.

3.5.1 Subproblem in φ_c: Frictionless contact subproblem

Over the constraints set C the functional $\varphi \mapsto \mathcal{L}_r(u^{k+1}, \varphi; \lambda^k)$ can be simplified

$$\mathrm{F}(\varphi) = \mathcal{L}_r(u^{k+1}, \varphi; \lambda_c^k) = \frac{r}{2}\|\varphi\|_{L^2(\Gamma_3)} - (\lambda_c^k + r u_\nu^{k+1}, \varphi)_{L^2(\Gamma_3)} + \zeta,$$

where

$$\zeta = J_n(u^{k+1}) + (\lambda_c^k, u_\nu^{k+1})_{L^2(\Gamma_3)} + \frac{r}{2}\|u_\nu^{k+1}\|_{L^2(\Gamma_3)},$$

is a constant which does not count in the minimization.
The minimization problem (3.35) becomes

Find $\phi_c^{k+1} \in L^2(\Gamma_3)$ such that

$$\mathrm{F}(\phi_c^{k+1}) \leq \mathrm{F}(\varphi), \quad \forall \varphi \in L^2(\Gamma_3), \tag{3.37}$$

$$\phi_c^{k+1} - \varrho \leq 0, \qquad \text{on } \Gamma_3. \tag{3.38}$$

The functional F is convex and coercive over the convex set C. We can associate to (3.37)-(3.38) the Lagrangian functional $\mathfrak{L}$ defined over $L^2(\Gamma_3) \times L^2(\Gamma_3)$

$$\mathfrak{L}(\varphi, \gamma) = \mathrm{F}(\varphi) + (\gamma^k, \varphi - \varrho)_{L^2(\Gamma_3)}, \tag{3.39}$$

where $\gamma^k \geq 0$ is referred as a Lagrange (Kuhn-Tucker) multiplier for the constraints set C (see [54, 82]).

The minimum ϕ_c^{k+1} of the functional F must satisfied the Kuhn-Tucker conditions

$$\frac{d}{dt}\Big[\mathfrak{L}(\phi_c^{k+1}+t\varphi,\gamma^k)\Big]_{|t=0} = 0, \quad \forall\varphi\in\ L^2(\Gamma_3),$$

$$(\gamma^k,\phi_c^{k+1}-\varrho)_{L^2(\Gamma_3)} = 0,$$

thus,

$$r(\phi_c^{k+1},\varphi)_{L^2(\Gamma_3)} - (\lambda_c^k + ru_\nu^{k+1},\varphi)_{L^2(\Gamma_3)} + (\gamma_c^k,\varphi)_{L^2(\Gamma_3)} = 0 \quad \forall\varphi\in L^2(\Gamma_3), \tag{3.40}$$

$$(\gamma^k,\phi_c^{k+1}-\varrho)_{L^2(\Gamma_3)} = 0. \tag{3.41}$$

From (3.40) we deduce that

$$\phi_c^{k+1} = \frac{1}{r}(\lambda_c^k + ru_\nu^{k+1} - \gamma^k). \tag{3.42}$$

Substituting this result into (3.41), we obtain

$$\Big(\gamma^k,\frac{1}{r}(\lambda_c^k + ru_\nu^{k+1}-\gamma^k)-\varrho\Big)_{L^2(\Gamma_3)} = 0,$$

if $\gamma^k>0$, we must have

$$\lambda_c^k + ru_\nu^{k+1} - \gamma^k - r\varrho = 0.$$

We deduce that the Lagrange multiplier is

$$\gamma^k = \max(0,\lambda_c^k + r(u_\nu^{k+1}-\varrho)) = \Big(\lambda_c^k + r(u_\nu^{k+1}-\varrho)\Big)^+. \tag{3.43}$$

Substituting (3.43) into (3.42) we get the solution of the non-penetration minimization sub-problem

$$\phi_c^{k+1} = u_\nu^{k+1} + \frac{1}{r}\Big[\lambda_c^k - \Big(\lambda_c^k + r(u_\nu^{k+1}-\varrho)\Big)^+\Big]. \tag{3.44}$$

Note that

- if $\gamma^k>0$ then, the contact constraint is active. *i.e.*, $\phi_c^{k+1}-\varrho=0$,
- if $\gamma^k=0$ then, from (3.43)-(3.44) we have $\lambda_c^k + ru_\nu^{k+1}\le r\varrho$ and $\phi_c^{k+1} = \frac{1}{r}(\lambda_c^k+ru_\nu^{k+1})$. We then deduce that

$$\phi_c^{k+1}-\varrho = \frac{1}{r}(\lambda_c^k + ru_\nu^{k+1}) - \varrho \le 0.$$

3.5.2 Subproblem in φ_f: Frictional contact subproblem

For the friction subproblem, using the Fenchel duality theory (see [54, 82]), we get

$$\phi_f^{k+1} = \begin{cases} \dfrac{|\lambda_f^k + r u_\tau^{k+1}| - s_n}{r|\lambda_f^k + r u_\tau^{k+1}|}(\lambda_f^k + r u_\tau^{k+1}), & if \ \ |\lambda_f^k + r u_\tau^{k+1}| > s_n, \\ \\ 0, & if \ \ |\lambda_f^k + r u_\tau^{k+1}| \le s_n, \end{cases}$$

where

$$s_n = \mu(\|u_{n,\tau}\|)|\,\mathrm{R}\,\sigma_\nu(u_n)|.$$

3.5.3 Uzawa block relaxation method

With the results of the previous subsections, we can now present our Uzawa block relaxation method **Algorithm 2**.

Algorithm 2. Uzawa block relaxation method for (3.31)

Initialization. $r > 0$, $\phi^0 = (\phi_f^0, \phi_c^0)$ and $\lambda^0 = (\lambda_f^0, \lambda_c^0)$ are given.

Iteration $k > 0$. Compute successively u^{k+1}, $\phi^{k+1} = (\phi_f^{k+1}, \phi_c^{k+1})$ and $\lambda^{k+1} = (\lambda_f^{k+1}, \lambda_c^{k+1})$ as follows

Step 1. Find $u^{k+1} \in V$ such that: for all $v \in V$

$$\begin{aligned} &B(u_n; u^{k+1}, v) + r(u_\nu^{k+1}, v_\nu)_{L^2(\Gamma_3)} + r(u_\tau^{k+1}, v_\tau)_{L^2(\Gamma_3)} \\ &\qquad = (f, v)_V + (r\phi_c^k - \lambda_c^k, v_\nu)_{L^2(\Gamma_3)} + (r\phi_f^k - \lambda_f^k, v_\tau)_{L^2(\Gamma_3)}. \end{aligned}$$

Step 2. Compute the auxiliary contact and friction variables

$$\phi_c^{k+1} = u_\nu^{k+1} + \frac{1}{r}\Big[\lambda_c^k - \Big(\lambda_c^k + r(u_\nu^{k+1} - \varrho)\Big)^+\Big],$$

$$\phi_f^{k+1} = \begin{cases} \dfrac{|\lambda_f^k + r u_\tau^{k+1}| - s_n}{r|\lambda_f^k + r u_\tau^{k+1}|}(\lambda_f^k + r u_\tau^{k+1}), & if \ \ |\lambda_f^k + r u_\tau^{k+1}| > s_n, \\ \\ 0, & if \ \ |\lambda_f^k + r u_\tau^{k+1}| \le s_n. \end{cases}$$

Step 3. Update the Lagrange multipliers

$$\lambda_c^{k+1} = \lambda_c^k + r(u_\nu^{k+1} - \phi_c^{k+1}),$$
$$\lambda_f^{k+1} = \lambda_f^k + r(u_\tau^{k+1} - \phi_f^{k+1}).$$

We iterate until the relative error on u^k, ϕ_f^k and ϕ_c^k is sufficiently "small", *i.e.*,

$$\frac{\|u^{k+1} - u^k\|^2_{L^2(\Omega)} + \|\phi_c^{k+1} - \phi_c^k\|^2_{L^2(\Gamma_3)} + \|\phi_f^{k+1} - \phi_f^k\|^2_{L^2(\Gamma_3)}}{\|u^{k+1}\|^2_{L^2(\Omega)} + \|\phi_c^{k+1}\|^2_{L^2(\Gamma_3)} + \|\phi_f^{k+1}\|^2_{L^2(\Gamma_3)}} < \epsilon^2, \tag{3.45}$$

and with the above results, the solution method for (3.28) is presented in **Algorithm 1**.

Algorithm 1. Solution for (3.28)

Initialization. s_0 and u_0 are given.

Iteration $n \geq 0$. Compute u_{n+1} and s_{n+1} successively as follows

- Compute u_{n+1} using **Algorithm 2**.
- Update the friction function $s_{n+1} = \mu(\|u_{n+1,\tau}\|)|\,\mathrm{R}\,\sigma_\nu(u_{n+1})|$.

The fixed-point iteration terminates if the relative error on s_n becomes sufficiently "small", *i.e.*,

$$\frac{\|s_{n+1} - s_n\|^2_{L^2(\Gamma_3)}}{\|s_{n+1}\|^2_{L^2(\Gamma_3)}} < \epsilon_{fp}^2. \tag{3.46}$$

Chapter 4

Penalty approach of a frictional contact problem in elasto-plasticity

This chapter deals with the same model presented in the precedent one; it is about a unilateral frictional contact problem involving nonlinear Hencky-type materials, where a slip-dependence describes the friction, and unlike the model in the precedent chapter, we assume here that there is no gap between the foundation and the body ($\varrho = 0$). This chapter is divided into tree sections. In Section 4.1, we propose a penalty formulation of the elasto-plastic frictional contact problem, state an existence and uniqueness result as well as a convergence result, Theorem 4.1 and Theorem 4.2, and we provide their proofs. In Section 4.2, we introduce and study the finite element approximation of the penalized problem. Finally, in Section 4.3, we propose an iterative solution scheme, and prove its convergence. The results presented in this chapter have been discussed in the paper [24].

4.1 Penalty formulation of the contact problem

The resolution of contact problems by the penalization method is very often used in practice. The main idea of penalization is to approach by a variational equation depending on a positive parameter ϵ a variational inequality sought, this solution will be the limit when ϵ tends to zero.

Therefore, to give the numerical approach of the frictional elasto-plastic contact problem by the penalty method, we replace the contact conditions on Γ_3 by $\sigma_\nu(u) = -\dfrac{1}{\epsilon}[u_\nu]^+$, where $[r]^+ = \max(r, 0)$ and $\epsilon > 0$, and we approximate the non-differentiable term $j(u, \,.\,)$ by a family of differentiable ones $j_\epsilon : V \times V \to \mathbb{R}$ given by

$$j_\epsilon(u, v) = \int_{\Gamma_3} \mu(\|u_\tau\|)|\, \mathrm{R}\,\sigma_\nu(u)|\, \Psi_\epsilon(u_\tau)\; v_\tau da \quad \forall u, v \in V, \tag{4.1}$$

where the family of differentiable functions $\Psi_\epsilon : \mathbb{R}^d \to \mathbb{R}^d$ has the form

$$\Psi_\epsilon(v) = \frac{v}{\|v\| + \epsilon} \quad \forall v \in \mathbb{R}^d \quad \forall \epsilon > 0.$$

and $\epsilon > 0$ is a small penalization parameter. Let $\Phi : V \times V \to \mathbb{R}$

$$\Phi(u, v) = \int_{\Gamma_3} [u_\nu]^+ \, v_\nu da. \tag{4.2}$$

We can introduce now the following variational problem.

Problem (PV_ϵ) *Find a displacement field $u_\epsilon \in V$ such that*

$$(Au_\epsilon, v)_V + \frac{1}{\epsilon}\Phi(u_\epsilon, v) + j_\epsilon(u_\epsilon, v) = (f, v)_V \quad \forall v \in V. \tag{4.3}$$

We have the following results.

Theorem 4.1 *Under the assumptions of Theorem 3.1 with the same value of L^*, the problem (PV_ϵ) has a unique solution u_ϵ in V.*

Remark 4.1 *Assume that (h_1), $(h_2)(a) - (b)$ and $(h_3) - (h_4)$ hold. Then, the problem (PV_ϵ) has at least one solution $u_\epsilon \in V$.*

We have the following convergence result.

Theorem 4.2 *Under the assumptions of Theorem 4.1 the solution u_ϵ of the penalized problem (PV_ϵ) converges to the solution u of the problem (PV). i.e.*

$$u_\epsilon \to u \quad \textit{strongly in } V \quad \textit{as } \epsilon \to 0.$$

Proof of Theorem 4.1

Before giving the proof of the Theorem 4.1, we introduce some intermediate results. To this end, we assume in the following that (h_1), $(h_2)(a) - (b)$ and $(h_3) - (h_4)$ hold.
For every $\eta \in L^2(\Gamma_3)$ we define the function

$$j_{\epsilon\eta}(u, v) = \int_{\Gamma_3} \eta \, \Psi_\epsilon(u_\tau) \, v_\tau da \quad \forall v \in V.$$

Let $A_\epsilon : V \to V$ be an operator defined by

$$(A_\epsilon u, v)_V = (Au, v)_V + \frac{1}{\epsilon}\Phi(u, v) + j_{\epsilon\eta}(u, v) \quad \forall u,\, v \in V, \tag{4.4}$$

where the operator A is giving in (3.9).
In the first step, we consider the following intermediate problem.

Problem $(\mathrm{PV}_\epsilon^\eta)$. *Find a displacement field* $u_{\epsilon\eta} \in V$ *such that*

$$(A_\epsilon u_{\epsilon\eta}, v)_V = (f, v)_V \quad \forall v \in V, \tag{4.5}$$

The unique solvability of Problem $(\mathrm{PV}_\epsilon^\eta)$ follows from the following result.

Lemma 4.1 *For any* $\eta \in \mathcal{K}$*, the problem* $(\mathrm{PV}_\epsilon^\eta)$ *has a unique solution* $u_{\epsilon\eta} \in V$*. Moreover, there exists a constant* $c_1 > 0$ *such that*

$$\|u_{\epsilon\eta}\| \le c_1 \|f\|_V. \tag{4.6}$$

Proof. To establish the proof of Lemma 4.1 we will need the following result

Lemma 4.2 Ψ_ϵ *is a Lipschitz continuous function, strictly monotone and bounded.*

Proof. let $u,\, v \in \mathbb{R}^d$, we have

$$\begin{aligned}\Psi_\epsilon(u) - \Psi_\epsilon(v) &= \frac{u - v}{\|u\| + \epsilon} + \frac{\|v\| - \|u\|}{(\|u\| + \epsilon)\,(\|v\| + \epsilon)}\, v \\ &\le \frac{\|u - v\|}{\|u\| + \epsilon} + \frac{\|v - u\|}{\|u\| + \epsilon}\,\frac{\|v\|}{\|v\| + \epsilon},\end{aligned}$$

using the inequalities

$$\frac{1}{\|u\| + \epsilon} \le \frac{1}{\epsilon}, \qquad \frac{\|v\|}{\|v\| + \epsilon} \le 1,$$

we deduce that

$$\|\Psi_\epsilon(u) - \Psi_\epsilon(v)\| \le L_{\Psi_\epsilon}\|u - v\|, \quad \text{with} \quad L_{\Psi_\epsilon} = \frac{2}{\epsilon},$$

which proves that Ψ_ϵ is Lipschitz continuous.

It comes from the following inequality

$$\begin{aligned} u_\tau\, v_\tau \leq\ & \|u_\tau\|\,\|v_\tau\|, \\ & \leq (\|u_\tau\| + \epsilon)\,(\|v_\tau\| + \epsilon). \end{aligned}$$

that,

$$\begin{aligned} u_\tau\, v_\tau - \|u_\tau\|^2 \leq\ & (\|u_\tau\| + \epsilon)\,(\|v_\tau\| + \epsilon) - \|u_\tau\|^2, \\ & \leq\ (\|u_\tau\| + \epsilon)\,(\|v_\tau\| + \epsilon) - (\|u_\tau\|^2 - \epsilon^2), \end{aligned}$$

by dividing the last inequality by $(\|u_\tau\| + \epsilon) > 0$, we find that

$$\frac{u_\tau\,(v_\tau - u_\tau)}{\|u_\tau\| + \epsilon} \leq\ (\|v_\tau\| + \epsilon) - (\|u_\tau\| - \epsilon),$$

thus,

$$\Psi(u_\tau)(v_\tau - u_\tau) \leq\ \|v_\tau\| - \|u_\tau\|. \tag{4.7}$$

A similar argument shows that

$$\Psi(v_\tau)(u_\tau - v_\tau) \leq\ \|u_\tau\| - \|v_\tau\|, \tag{4.8}$$

and, adding (4.7) and (4.8), we find that

$$\Big(\Psi_\epsilon(u_\tau) - \Psi_\epsilon(v_\tau)\Big)(u_\tau - v_\tau) \geq 0 \quad u,\, v \in V. \tag{4.9}$$

Thus, Ψ_ϵ is strictly monotone. Moreover, for all $v \in \mathbb{R}^d$

$$\|\Psi_\epsilon(v)\| = \frac{\|v\|}{\|v\| + \epsilon} \leq\ 1.$$

□

Now, let u_1 and u_2 be two elements of V, it follows from (4.4) that for $\eta \in \mathcal{K}$

$$\begin{aligned} (A_\epsilon u_1 - A_\epsilon u_2, u_1 - u_2)_V = (Au_1 - Au_2, u_1 - u_2)_V + \frac{1}{\epsilon}\Big(\Phi(u_1, u_1 - u_2) - \Phi(u_2, u_1 - u_2)\Big) \\ + j_{\epsilon\eta}(u_1, u_1 - u_2) - j_{\epsilon\eta}(u_2, u_1 - u_2), \end{aligned}$$

from (4.2), we have

$$\Phi(u_1, u_1 - u_2) - \Phi(u_2, u_1 - u_2) \geq 0 \quad \forall u_1,\, u_2 \in V.$$

Combining the last result with (4.9) and the fact that the operator A is strong monotone , we get

$$(A_\epsilon u_1 - A_\epsilon u_2, u_1 - u_2)_V \geq m_A \|u_1 - u_2\|_V^2 \quad \forall u_1,\, u_2 \in V. \tag{4.10}$$

Furthermore, using Cauchy-Schwarz inequality, (2.25) and the fact that the operator A and the function Ψ_ϵ are Lipschitz continuous, it follows for $\eta \in \mathcal{K}$ that

$$\begin{aligned}(A_\epsilon u_1 - A_\epsilon u_2, v)_V &= (Au_1 - Au_2, v)_V + \frac{1}{\epsilon}\Big(\Phi(u_1, v) - \Phi(u_2, v)\Big) + j_{\epsilon\eta}(u_1, v) - j_{\epsilon\eta}(u_2, v) \\ &\leq M_A \|u_1 - u_2\|_V \|v\|_V + \frac{1}{\epsilon} c_0^2 \|u_1 - u_2\|_V \|v\|_V \\ &\quad + \|\eta\Big(\Psi_\epsilon(u_{1,\tau}) - \Psi_\epsilon(u_{2,\tau})\Big)\|_{L^2(\Gamma_3)} \|v\|_{L^2(\Gamma_3)^d}.\end{aligned}$$

Since $\eta \in L^2(\Gamma_3)$ and $\|\eta\|_{L^2(\Gamma_3)} \leq \kappa$, then, Theorem 2.2 leads us to: $\eta \in L^\infty(\Gamma_3)$, *i.e.*, there exist $c_\eta > 0$ such that $|\eta| \leq c_\eta$ a.e. on Γ_3. Hence

$$(A_\epsilon u_1 - A_\epsilon u_2, v)_V \leq \Big(M_A + \frac{1}{\epsilon} c_0^2 + c_0^2 c_\eta L_{\Psi_\epsilon}\Big) \|u_1 - u_2\|_V \|v\|_V.$$

Taking $v = A_\epsilon u_1 - A_\epsilon u_2$, we get

$$\|A_\epsilon u_1 - A_\epsilon u_2\|_V \leq \Big(M_A + \frac{1}{\epsilon} c_0^2 + c_0^2 c_\eta L_{\Psi_\epsilon}\Big) \|u_1 - u_2\|_V \quad \forall u_1,\, u_2 \in V.$$

We conclude by using Theorem A.9, that for all $\epsilon > 0$ fixed, Problem ($\mathrm{PV}_\epsilon^\eta$) has a unique solution $u_{\epsilon\eta}$.

Using a mathematical manipulation similar to that in the proof of Theorem 3.1, we find that

$$\|u_{\epsilon\eta}\|_V \leq c_1 \|f\|_V \quad \forall u_{\epsilon\eta} \in V.$$

□

Next, we consider the operator $\Lambda : L^2(\Gamma_3) \to L^2(\Gamma_3)$ introduced in (3.22)

$$\Lambda\eta = \mu(\|u_{\epsilon\eta,\tau}\|) |\, \mathrm{R}\, \sigma_\nu(u_{\epsilon\eta})|.$$

Using Lemma 3.3, we conclude that Λ is an operator weakly continuous from a weakly compact subset $\mathcal{K}$ into it self. Hence, by the Schauder's fixed point theorem the operator Λ has at least one fixed point.

Proof of Theorem 4.1. Let η^* be the fixed point of operator Λ. We denote by u^* the solution of the variational problem $(\mathrm{PV}_\epsilon^\eta)$ for $\eta = \eta^*$. Using (4.5) and (3.22), it is easy to see that u^* is a solution of (PV_ϵ). This proves the existence part of Theorem 4.1. Next, we show that if $\mu^* + L_\mu < L^*$ the solution is unique.

Let $u_{\epsilon,1},\ u_{\epsilon,2} \in V$ denote the solutions of Problem (PV_ϵ). From (4.4), we have

$$(A_\epsilon u_{\epsilon,1}, v - u_{\epsilon,1})_V = (f, v - u_{\epsilon,1})_V \quad \forall v \in V,$$

$$(A_\epsilon u_{\epsilon,2}, v - u_{\epsilon,2})_V = (f, v - u_{\epsilon,2})_V \quad \forall v \in V.$$

Taking $v = u_{\epsilon,2}$ in the first equality and $v = u_{\epsilon,1}$ in the second, adding the two inequalities obtained, we get

$$(A_\epsilon u_{\epsilon,1} - A_\epsilon u_{\epsilon,2}, u_{\epsilon,1} - u_{\epsilon,2})_V = 0.$$

Then, from (4.4), we have

$$(Au_{\epsilon,1} - Au_{\epsilon,2}, u_{\epsilon,1} - u_{\epsilon,2})_V + \frac{1}{\epsilon}\Big(\Phi(u_{\epsilon,1}, u_{\epsilon,1} - u_{\epsilon,2}) - \Phi(u_{\epsilon,2}, u_{\epsilon,1} - u_{\epsilon,2})\Big) = G, \tag{4.11}$$

where

$$G = j_\epsilon(u_{\epsilon,2}, u_{\epsilon,1} - u_{\epsilon,2}) - j_\epsilon(u_{\epsilon,1}, u_{\epsilon,1} - u_{\epsilon,2}).$$

Now, from (4.1), we have

$$G = \int_{\Gamma_3} \Big[\mu(\|u_{\epsilon,2,\tau}\|)|\,\mathrm{R}\,\sigma_\nu(u_{\epsilon,2})|\Psi_\epsilon(u_{\epsilon,2,\tau}) - \mu(\|u_{\epsilon,1,\tau}\|)|\,\mathrm{R}\,\sigma_\nu(u_{\epsilon,1})|\Psi_\epsilon(u_{\epsilon,1,\tau})\Big](u_{\epsilon,1,\tau} - u_{\epsilon,2,\tau})\,da,$$

thus,

$$\begin{aligned}
|G| \le &\int_{\Gamma_3} \mu(\|u_{\epsilon,2,\tau}\|)\ |\,\mathrm{R}\,\sigma_\nu(u_{\epsilon,2})|\ \|\Psi_\epsilon(u_{\epsilon,2,\tau}) - \Psi_\epsilon(u_{\epsilon,1,\tau})\|\ \|u_{\epsilon,1,\tau} - u_{\epsilon,2,\tau}\|da \\
&+ \int_{\Gamma_3} \mu(\|u_{\epsilon,2,\tau}\|)\ \Big||\,\mathrm{R}\,\sigma_\nu(u_{\epsilon,2})| - |\,\mathrm{R}\,\sigma_\nu(u_{\epsilon,1})|\Big|\ \|\Psi_\epsilon(u_{\epsilon,1,\tau})\|\ \|u_{\epsilon,1,\tau} - u_{\epsilon,2,\tau}\|da \\
&+ \int_{\Gamma_3} \Big|\mu(\|u_{\epsilon,2,\tau}\|) - \mu(\|u_{\epsilon,1,\tau}\|)\Big|\ |\,\mathrm{R}\,\sigma_\nu(u_{\epsilon,1})|\ \|\Psi_\epsilon(u_{\epsilon,1,\tau})\|\ \|u_{\epsilon,1,\tau} - u_{\epsilon,2,\tau}\|da.
\end{aligned}$$

Using $(h_2)(b)-(c)$, (h_3), (3.6) and Lemma 4.2,the properties of μ, the continuity of R and (3.6) it is straightforward to show that

$$|G| \leq c_0^2\Big(\mu^* \| \mathrm{R} \|_{L^\infty(\Gamma_3)} L_{\Psi_\epsilon} + \mu^* meas(\Gamma_3)^{\frac{1}{2}} c_{\mathrm{R}} + \| \mathrm{R} \|_{L^\infty(\Gamma_3)} L_\mu\Big) \|u_{\epsilon,1} - u_{\epsilon,2}\|_V^2. \tag{4.12}$$

So, we combine (4.10), (4.11) and (4.12) to deduce that there exists a positive constant c^* such that

$$\|u_{\epsilon,1} - u_{\epsilon,2}\|_V^2 \leq \frac{c^*}{m_A}(\mu^* + L_\mu)\|u_{\epsilon,1} - u_{\epsilon,2}\|_V^2.$$

Let $L^* = \dfrac{m_A}{c^*}$, then if we have $\mu^* + L_\mu < L^*$, we obtain $u_{\epsilon,1} = u_{\epsilon,2}$ and it implies the uniqueness of the solution.

Proof of Theorem 4.2

Taking $v = u_\epsilon$ in (PV_ϵ), we obtain

$$(Au_\epsilon, u_\epsilon)_V + \frac{1}{\epsilon}\Phi(u_\epsilon, u_\epsilon) + j_\epsilon(u_\epsilon, u_\epsilon) = (f, u_\epsilon)_V.$$

Since, $j_\epsilon(u_\epsilon, u_\epsilon) \geq 0$ and $\Phi(u_\epsilon, u_\epsilon) \geq 0$, then

$$(Au_\epsilon, u_\epsilon)_V \leq (f, u_\epsilon)_V.$$

Using Cauchy-Schwarz inequality and the strong monotonicity of the operator A, we get

$$\|u_\epsilon\|_V \leq c_1\|f\|_V. \tag{4.13}$$

We also have

$$\begin{aligned} \frac{1}{\epsilon}\int_{\Gamma_3} [u_{\epsilon,\nu}]^+ \, u_{\epsilon,\nu} \, da &= \frac{1}{\epsilon}\int_{\Gamma_3} [u_{\epsilon,\nu}]^{+2} \, da \\ &= \frac{1}{\epsilon}\|[u_{\epsilon,\nu}]^+\|_{L^2(\Gamma_3)}^2 \leq c_0^2 c_1^2 \, \|f\|_V^2. \end{aligned}$$

Thus,

$$\|[u_{\epsilon,\nu}]^+\|_{L^2(\Gamma_3)} \leq \sqrt{\epsilon}\, c_0 c_1 \, \|f\|_V. \tag{4.14}$$

From (4.13), we deduce that there exists $\tilde{u} \in V$ and a subsequence of (u_ϵ), denoted again by (u_ϵ), such that

$$u_\epsilon \rightharpoonup \tilde{u} \quad \text{weakly in } V \quad \text{as } \epsilon \to 0. \tag{4.15}$$

Since the trace map $\gamma : V \to L^2(\Gamma_3)^d$ is a compact operator, we deduce that

$$u_\epsilon \to \widetilde{u} \text{ strongly in } L^2(\Gamma_3)^d \quad \text{as } \epsilon \to 0, \tag{4.16}$$

and we have

$$\lim_{\epsilon \to 0} \|[u_{\epsilon,\nu}]^+\|_{L^2(\Gamma_3)} = \|[\widetilde{u}_\nu]^+\|_{L^2(\Gamma_3)},$$

also, from (4.14), we deduce that

$$\lim_{\epsilon \to 0} \|[u_{\epsilon,\nu}]^+\|_{L^2(\Gamma_3)} = 0,$$

we find that $[\widetilde{u}_\nu]^+ = 0,$ *a.e.* on Γ_3, it follows that $\widetilde{u}_\nu \leq 0$ *a.e.* on Γ_3, which shows that $\widetilde{u} \in K$. It may be easily verified by using (4.7) that

$$j_\epsilon(u, v - u) \leq j(u, v) - j(u, u) \quad \forall u,\, v \in V, \tag{4.17}$$

Since $\Phi(u_\epsilon, v - u_\epsilon) \leq 0$ for all $v \in K$ and by using (4.3), (4.4) and (4.17), we obtain

$$(Au_\epsilon, v - u_\epsilon)_V + j(u_\epsilon, v) - j(u_\epsilon, u_\epsilon) \geq (f, v - u_\epsilon)_V \quad \forall v \in K. \tag{4.18}$$

Then, from (4.16) and the properties of μ, R and σ, we have

$$j(u_\epsilon, v) - j(u_\epsilon, u_\epsilon) \to j(u, v) - j(u, u) \quad \text{as } \epsilon \to 0.$$

Using (4.15), (4.17), (4.18) and a lower semi-continuity argument, we find that

$$(A\widetilde{u}, v - \widetilde{u})_V + j(\widetilde{u}, v) - j(\widetilde{u}, \widetilde{u}) \geq (f, v - \widetilde{u})_V \quad \text{as } \epsilon \to 0,$$

for any $v \in K$, and by the uniqueness of the solution to Problem (PV), we deduce that $\widetilde{u} = u$. We conclude that u is the unique weak limit in V of any subsequence of (u_ϵ) and therefore, we find that the whole sequence (u_ϵ) converges weakly to the element $u \in K$ as $\epsilon \to 0$.

Finally, let us prove that u_ϵ is strongly convergent in V to u.
From the strong monotonicity of the operator A, (4.18) with $v = u$, we have

$$\begin{aligned} m_A \|u_\epsilon - u\|_V^2 &\leq (Au_\epsilon - Au, u_\epsilon - u)_V \\ &\leq (Au, u - u_\epsilon)_V - (f, u - u_\epsilon)_V + j(u_\epsilon, u) - j(u_\epsilon, u_\epsilon). \end{aligned}$$

The weak convergence of u_ϵ to u in V and (4.16) lead us to

$$\|u_\epsilon - u\|_V \to 0 \quad \text{as } \epsilon \to 0.$$

4.2 Finite element setting and discrete penalty problem

In this section, we will see how to approximate the variational formulation (PV$_\epsilon$). Concerning the finite element approximation, we will follow the classical techniques mentioned in [40, 58, 159]. The finite element method consists of a spatial discretization of the body into simple geometric elements connected by a finite number of nodes. For this purpose, let us assume that Ω is a polygonal domain, and let $\mathcal{T}^h$ be a regular family of triangular finite element partitions of $\overline{\Omega}$ that are compatible with the partition of the boundary decompositions $\Gamma = \Gamma_1 \cup \Gamma_2 \cup \Gamma_3$, that is, any point when the boundary condition type changes is a vertex of the partition, then the side lies entirely in $\overline{\Gamma} = \overline{\Gamma}_1 \cup \overline{\Gamma}_2 \cup \overline{\Gamma}_3$. Corresponding to each partition $\mathcal{T}^h$. We denote by $\mathbb{P}_1(\Omega^e)$ the space of polynomials of global degree less or equal to one in Ω^e. Let us consider a finite-dimensional space $V^h \subset V$, approximating the space V that is

$$V^h = \{\, v^h \in C(\overline{\Omega})^d ;\; v^h_{|\Omega^e} \in \mathbb{P}_1(\Omega^e)^d,\; \Omega^e \in \mathcal{T}^h,\; v^h = 0 \text{ on } \Gamma_1 \,\},$$

where $h > 0$ is a discretization parameter. We also introduce $W^h(\Gamma_3)$, the space of normal traces on Γ_3 for discrete functions in V^h

$$W^h(\Gamma_3) = \{\xi^h \in C(\overline{\Gamma}_3) ;\; \exists v^h \in V^h,\, v^h.\nu = \xi^h\}.$$

We consider the following discrete approximation of Problem (PV$_\epsilon$)

Problem (PV$_\epsilon^h$). *Find a discrete displacement field $u_\epsilon^h \in V^h$ such that*

$$(Au_\epsilon^h, v^h)_V + \frac{1}{\epsilon}\Phi(u_\epsilon^h, v^h) + j_\epsilon(u_\epsilon^h, v^h) = (f, v^h)_V \quad \forall v^h \in V^h. \tag{4.19}$$

Applying Theorem 4.1, for the case when V is replaced by V^h, we find that Problem (PV$_\epsilon^h$) has a unique solution $u_\epsilon^h \in V^h$. We have the following convergence result.

Theorem 4.3 *Let us denote by u_ϵ and u_ϵ^h the respective solutions to Problem (PV_ϵ) and (PV_ϵ^h).*

Under the assumptions of Theorem 4.1, we have

$$\|u_\epsilon^h - u_\epsilon\|_V \to 0 \quad \textit{as } h \to 0.$$

Proof. We consider the following space

$$\mathfrak{U} = \{\, v \in C^\infty(\overline{\Omega})^d\,;\ v_i = 0 \ \text{ in a neighbourhood of } \Gamma_1\}.$$

Then $\overline{\mathfrak{U}} = V$ (see [58], Section 6.2.1, pp. 12). We define $r^h : \mathfrak{U} \to V^h$ by

$$\begin{cases} r^h v \in V^h \quad \forall v \in \mathfrak{U}, \\ (r^h v)(P) = v(P), \quad P \text{ is a vertex of triangulation.} \end{cases}$$

Then, since $r^h v$ is the "linear" interpolate of v on τ^h, under the assumptions made on τ^h, we have (see [40], Theorem 3.1.6, pp. 124)

$$\|r^h v - v\|_V \leq c\, h\, |v|_{H^2(\Omega)^d} \quad \forall v \in \mathfrak{U},$$

with c independent of h and v. This implies

$$r^h v \ \to\ v \quad \text{strongly in } V \quad \text{as } h \to 0. \tag{4.20}$$

We now prove the boundedness of the sequence (u_ϵ^h) in V.
Taking $v^h = u_\epsilon^h$ in (4.19), we have

$$(A_\epsilon u_\epsilon^h, u_\epsilon^h)_V + \frac{1}{\epsilon}\Phi(u_\epsilon^h, u_\epsilon^h) + j_\epsilon(u_\epsilon^h, u_\epsilon^h) = (f, u_\epsilon^h)_V,$$

as $\Phi(u_\epsilon^h, u_\epsilon^h) \geq 0$ and $j_\epsilon(u_\epsilon^h, u\epsilon^h) \geq 0$, then

$$(Au_\epsilon^h, u_\epsilon^h)_V \leq (f, u_\epsilon^h)_V.$$

Using Cauchy-Schwarz inequality and the strong monotonicity of the operator A, we get

$$\|u_\epsilon^h\|_V \leq c_1 \|f\|_V.$$

Thus, there exist $u^* \in V$ and subsequences of the sequence (u_ϵ^h), denoted again by (u_ϵ^h), such that

$$u_\epsilon^h \rightharpoonup u^* \quad \text{weakly in } V \quad \text{as } h \to 0. \tag{4.21}$$

Since the trace mapping $\gamma : V \to L^2(\Gamma_3)^d$ is compact operator, it follows from (4.21) that

$$u_\epsilon^h \to u^* \quad \text{strongly in } L^2(\Gamma_3)^d \quad \text{as } h \to 0. \tag{4.22}$$

Next, we prove that u^* is a solution of (PV_ϵ). Since u_ϵ^h is a solution to Problem (PV_ϵ^h) and $r^h v \in V^h$, for all h, $v \in \mathfrak{U}$, we have

$$(Au_\epsilon^h, u_\epsilon^h - r^h v)_V + \frac{1}{\epsilon}\Phi(u_\epsilon^h, u_\epsilon^h - r^h v) + j_\epsilon(u_\epsilon^h, u_\epsilon^h - r^h v) = (f, u_\epsilon^h - r^h v)_V. \tag{4.23}$$

From, (4.20), (4.22) and the properties of μ, R and σ, we have

$$\lim_{h\to 0} j_\epsilon(u_\epsilon^h, u_\epsilon^h - r^h v) = j_\epsilon(u^*, u^* - v) \quad \forall v \in \mathfrak{U}, \tag{4.24}$$

$$\lim_{h\to 0} \Phi(u_\epsilon^h, u_\epsilon^h - r^h v) = \Phi(u^*, u^* - v) \quad \forall v \in \mathfrak{U}. \tag{4.25}$$

We also have

$$\begin{aligned}(Au_\epsilon^h, u_\epsilon^h - u^*)_V &= (Au_\epsilon^h, u_\epsilon^h - r^h v)_V + (Au_\epsilon^h, r^h v - u^*)_V \\ &\leq (f, u_\epsilon^h - r^h v)_V + \frac{1}{\epsilon}\Phi(u_\epsilon^h, r^h v - u_\epsilon^h) + j_\epsilon(u_\epsilon^h, r^h v - u_\epsilon^h) \\ &\quad + \|Au_\epsilon^h\|_V \|r^h v - u^*\|_V.\end{aligned}$$

Therefore, by (4.24) and (4.25)

$$\begin{aligned}\limsup_{h\to 0}(Au_\epsilon^h, u_\epsilon^h - u^*)_V \leq (f, u^* - v)_V &+ \frac{1}{\epsilon}\Phi(u^*, v - u^*) + j_\epsilon(u^*, v - u^*) \\ &+ \limsup_{h\to 0} \|Au_\epsilon^h\|_V \|v - u^*\|_V,\end{aligned}$$

for all $v \in \mathfrak{U}$. Since A is Lipschitz continuous and u_ϵ^h is bounded, $\|Au_\epsilon^h\|_V$ is bounded. We may then substitute $v = u^*$ into the previous inequality to obtain

$$\limsup_{h\to 0}(Au_\epsilon^h, u_\epsilon^h - u^*)_V \leq 0.$$

By pseudo-monotonicity of A, we get

$$(Au^*, u^* - v)_V \leq \liminf_{h\to 0}(Au^h_\epsilon, u^h_\epsilon - r^h v)_V. \tag{4.26}$$

Combining (4.23), (4.24), (4.25) and (4.26), we get

$$(Au^*, u^* - v)_V + \frac{1}{\epsilon}\Phi(u^*, u^* - v) + j_\epsilon(u^*, u^* - v) \leq (f, u^* - v)_V \quad \forall v \in \mathfrak{U}, \tag{4.27}$$

Since $\mathfrak{U}$ is dense in V, from (4.27) we obtain

$$(Au^*, u^* - v)_V + \frac{1}{\epsilon}\Phi(u^*, u^* - v) + j_\epsilon(u^*, u^* - v) \leq (f, u^* - v)_V \quad \forall v \in V. \tag{4.28}$$

By setting $v = u^* \pm v^*$ in (4.28) with v^* is an arbitrary element of V, we find

$$(Au^*, v^*)_V + \frac{1}{\epsilon}\Phi(u^*, v^*) + j_\epsilon(u^*, v^*) = (f, v^*)_V \quad \forall v^* \in V.$$

Which means that u^* is a solution to Problem (PV_ϵ). By Theorem 4.1, the solution of (PV_ϵ) is unique and hence $u^* = u_\epsilon$. Then, u_ϵ is the only cluster point of (u^h_ϵ) in the weak topology of V. Hence, The hole sequence (u^h_ϵ) converge to u_ϵ weakly in V as $h \to 0$.
From the strong monotonicity of the operator A and (4.23), we get

$$\begin{aligned}
m_A\|u^h_\epsilon - u_\epsilon\|^2_V &\leq (Au^h_\epsilon - Au_\epsilon, u^h_\epsilon - u_\epsilon)_V \\
&= (Au^h_\epsilon, r^h v - u_\epsilon)_V + (Au^h_\epsilon, u^h_\epsilon - r^h v)_V - (Au_\epsilon, u^h_\epsilon - u_\epsilon)_V \\
&= (Au^h_\epsilon, r^h v - u_\epsilon)_V - \frac{1}{\epsilon}\Phi(u^h_\epsilon, u^h_\epsilon - r^h v) - j_\epsilon(u^h_\epsilon, u^h_\epsilon - r^h v) \\
&\qquad + (f, u^h_\epsilon - r^h v)_V - (Au_\epsilon, u^h_\epsilon - u_\epsilon)_V \\
&\leq \|Au^h_\epsilon\|_V \|r^h v - u_\epsilon\|_V - \frac{1}{\epsilon}\Phi(u^h_\epsilon, u^h_\epsilon - r^h v) \\
&\qquad - j_\epsilon(u^h_\epsilon, u^h_\epsilon - r^h v) + (f, u^h_\epsilon - r^h v)_V - (Au_\epsilon, u^h_\epsilon - u_\epsilon)_V.
\end{aligned}$$

Taking account the boundedness of $\|Au^h_\epsilon\|_V$, (4.20) and the weak convergence of (u^h_ϵ) to u_ϵ in V, we obtain from the previous inequality that

$$\lim_{h\to 0}\left(m_A\|u^h_\epsilon - u_\epsilon\|^2_V\right) \leq c\,\|v - u_\epsilon\|_V - \frac{1}{\epsilon}\Phi(u_\epsilon, u_\epsilon - v) - j_\epsilon(u_\epsilon, u_\epsilon - v) + (f, u_\epsilon - v)_V, \tag{4.29}$$

for all $v \in \mathfrak{U}$. By the density of $\mathfrak{U}$ in V, (4.29) holds, for all $v \in V$.
With replacing v by u_ϵ in (4.29), we obtain

$$\lim_{h\to 0}\left(\|u_\epsilon^h - u_\epsilon\|_V^2\right) = 0.$$

□

Theorem 4.4 *Let us denote by u^h and u_ϵ^h the respective solutions to Problem (PV^h) and (PV_ϵ^h), we have the following error estimates*

$$\|\sigma_\nu(u^h) + \frac{1}{\epsilon}[u_\nu^h]^+\|_{H_{\Gamma_3}^{-\frac{1}{2}}} \le CM_A\|u^h - u_\epsilon^h\|_V, \tag{4.30}$$

where $C > 0$ independent of ϵ and h.

Proof. Let $\mathcal{P}^h : L^2(\Gamma_3) \to W^h(\Gamma_3)$ denote the $L^2(\Gamma_3)$-projection operator onto $W^h(\Gamma_3)$. We suppose that the mesh associated to $W^h(\Gamma_3)$ and the mesh contact boundary are quasi-uniform. We recall now some results (see [36, 49]): there exists a constant $C > 0$ such that

- $\forall s \in [0,1], \quad \forall v \in H^s(\Gamma_3)$

$$\|\mathcal{P}^h v\|_{H^s(\Gamma_3)} \le C\|v\|_{H^s(\Gamma_3)}, \quad \|v - \mathcal{P}^h v\|_{L^2(\Gamma_3)} \le Ch^s\|v\|_{H^s(\Gamma_3)}.$$

- $\exists \mathcal{R}^h : W^h(\Gamma_3) \to V^h, \quad \forall v^h \in W^h(\Gamma_3)$

$$\mathcal{R}^h(v^h)_{|\Gamma_3}.\nu = v^h, \quad \|\mathcal{R}^h(v^h)\|_{H^1(\Omega)} \le C\|v^h\|_{H^{\frac{1}{2}}(\Omega)}.$$

Using Green's formula, we get

$$(Au, v)_V = \langle\, \sigma_\nu(u)\,,\, v_\nu \,\rangle_{\Gamma_3} + (f, v)_V \quad \forall v \in V,$$

$$(Au_\epsilon, v)_V = \langle\, -\frac{1}{\epsilon}[u_{\epsilon,\nu}]^+\,,\, v_\nu \,\rangle_{\Gamma_3} + (f, v)_V \quad \forall v \in V,$$

When we replace V by V^h and by subtracting the second equality from the first one, we obtain

$$\langle\, \sigma_\nu(u^h) + \frac{1}{\epsilon}[u_{\epsilon,\nu}^h]^+\,,\, v_\nu^h \,\rangle_{\Gamma_3} = (Au^h - Au_\epsilon^h, v^h)_V \quad \forall v^h \in V^h, \tag{4.31}$$

and it results from the continuity of the operator A that

$$\sup_{v\in H^{\frac{1}{2}}_{\Gamma_3}} \frac{\langle\, \sigma_\nu(u^h) + \frac{1}{\epsilon}[u^h_{\epsilon,\nu}]^+\,,\, \mathcal{P}^h v\,\rangle_{\Gamma_3}}{\|\mathcal{P}^h v\|_{H^{\frac{1}{2}}_{\Gamma_3}}} = \sup_{v\in H^{\frac{1}{2}}_{\Gamma_3}} \frac{\langle\, \sigma_\nu(u^h) + \frac{1}{\epsilon}[u^h_{\epsilon,\nu}]^+\,,\, \mathcal{R}^h(\mathcal{P}^h v)_{|\Gamma_3}.\nu\,\rangle_{\Gamma_3}}{\|\mathcal{P}^h v\|_{H^{\frac{1}{2}}_{\Gamma_3}}}$$

$$= \sup_{v\in H^{\frac{1}{2}}_{\Gamma_3}} \frac{(Au^h - Au^h_\epsilon, \mathcal{R}^h(\mathcal{P}^h v))_V}{\|\mathcal{P}^h v\|_{H^{\frac{1}{2}}_{\Gamma_3}}}$$

Thus, using the Lipschitz continuity of the operator A, we obtain

$$\sup_{v\in H^{\frac{1}{2}}_{\Gamma_3}} \frac{\langle\, \sigma_\nu(u^h) + \frac{1}{\epsilon}[u^h_{\epsilon,\nu}]^+\,,\, \mathcal{P}^h v\,\rangle_{\Gamma_3}}{\|\mathcal{P}^h v\|_{H^{\frac{1}{2}}_{\Gamma_3}}} \le M_A\|u^h - u^h_\epsilon\|_V \sup_{v\in H^{\frac{1}{2}}_{\Gamma_3}} \frac{\|\mathcal{R}^h(\mathcal{P}^h v)\|_{H^1(\Omega)}}{\|\mathcal{P}^h v\|_{H^{\frac{1}{2}}_{\Gamma_3}}},$$

which lead us to

$$\|\sigma_\nu(u^h) + \frac{1}{\epsilon}[u^h_{\epsilon,\nu}]^+\|_{H^{-\frac{1}{2}}_{\Gamma_3}} \le CM_A\|u^h - u^h_\epsilon\|_V \sup_{v\in H^{\frac{1}{2}}_{\Gamma_3}} \frac{\|\mathcal{P}^h v\|_{H^{\frac{1}{2}}_{\Gamma_3}}}{\|\mathcal{P}^h v\|_{H^{\frac{1}{2}}_{\Gamma_3}}},$$

□

4.3 Iteration method

We consider the bilinear, symmetric operator $B : V \times V \times V \to \mathbb{R}$, defined in Section 3.4. Then, we can write the successive iteration technique for the problem (PV^h_ϵ) as follows

$$\begin{cases} \text{Given an initial guess } u^h_{\epsilon,0} \in V^h, \text{ find } u^h_{\epsilon,(n+1)} \in V^h \text{ such that} \\ B(u^h_{\epsilon,n}; u^h_{\epsilon,(n+1)}, v^h) + \dfrac{1}{\epsilon}\Phi(u^h_{\epsilon,(n+1)}, v^h) + J_\epsilon(u^h_{\epsilon,n}; u^h_{\epsilon,(n+1)}, v^h) = (f, v^h)_V \;\; \forall v^h \in V^h, \end{cases} \tag{4.32}$$

where the functional $J_\epsilon : V \times V \times V \to \mathbb{R}$ is given by

$$J_\epsilon(u; w, v) = \int_{\Gamma_3} \mu(\|u_\tau\|)|\,\mathrm{R}\,\sigma_\nu(u)|\frac{w_\tau}{\|u_\tau\| + \epsilon} v_\tau da \quad \forall u,\, v,\, w \in V. \tag{4.33}$$

It may be easily verified that

$$J_\epsilon(u; u, v) = j_\epsilon(u, v) \quad \forall u,\, v \in V. \tag{4.34}$$

We have the following convergence result.

Theorem 4.5 *Under assumptions of Theorem 4.1, the iteration method (4.32) converges:*

$$\|u^h_{\epsilon,n} - u^h_\epsilon\|_V \to 0 \quad \text{as } n \to +\infty \quad \forall \epsilon > 0 \text{ and } \quad h > 0,$$

where u^h_ϵ is the unique solution to Problem $(\mathrm{PV}^{\mathrm{h}}_\epsilon)$.

Proof. Let $u^h_{\epsilon,n} \in V^h$ be the solution of the problem (4.32), thus we have for all $v^h \in V^h$

$$B(u^h_{\epsilon,(n-1)}; u^h_{\epsilon,n}, u^h_{\epsilon,n} - v^h) + \frac{1}{\epsilon}\Phi(u^h_{\epsilon,n}, u^h_{\epsilon,n} - v^h) + J_\epsilon(u^h_{\epsilon,(n-1)}; u^h_{\epsilon,n}, u^h_{\epsilon,n} - v^h) = (f, u^h_{\epsilon,n} - v^h)_V. \quad (4.35)$$

Taking $v = 0$ in (4.35), by using the properties of B and the fact that $\Phi(u^h_{\epsilon,n}, u^h_{\epsilon,n}) \geq 0$, $J_\epsilon(u^h_{\epsilon,(n-1)}, u^h_{\epsilon,n}, u^h_{\epsilon,n}) \geq 0$, we obtain

$$\|u^h_{\epsilon,n}\|_V \leq c_1 \|f\|_V.$$

Afterward, since the sequence $(u^h_{\epsilon,n})_{n\geq 1}$ is bounded in V, then, there exists $\omega \in V$ and a subsequence, denoted again $(u^h_{\epsilon,n})_{n\geq 1}$, such that $(u^h_{\epsilon,n})_{n\geq 1}$ converges weakly to $\omega \in V$. Moreover, using the compactness of the trace map $\gamma : V \to L^2(\Gamma_3)^d$, it follows from the weak convergence of $(u^h_{\epsilon,n})_{n\geq 1}$ that, for all $\epsilon > 0$ and $h > 0$, $u^h_{\epsilon,n} \to \omega$ strongly in $L^2(\Gamma_3)^d$.

Next, let us prove that ω is the solution of Problem (PV^h_ϵ). Using (4.1), (4.33), and keeping in mind the properties of μ, R and Lemma 4.2, we get

$$\begin{aligned}
&\left| J_\epsilon(u^h_{\epsilon,(n-1)}; u^h_{\epsilon,n}, u^h_{\epsilon,n} - v^h) - j_\epsilon(\omega, \omega - v^h) \right| \\
&\leq \mu^* \| \mathrm{R} \|_{L^\infty(\Gamma_3)} \Big(\|u^h_{\epsilon,n} - u^h_{\epsilon,(n-1)}\|_{L^2(\Gamma_3)^d} + L_{\Psi_\epsilon} \|u^h_{\epsilon,(n-1)} - \omega\|_{L^2(\Gamma_3)^d} \Big) \|u^h_{\epsilon,n} - v^h\|_{L^2(\Gamma_3)^d} \\
&\quad + \Big(\mu^* meas(\Gamma_3)^{\frac{1}{2}} c_{\mathrm{R}} + L_\mu \| \mathrm{R} \|_{L^\infty(\Gamma_3)} \Big) \|u^h_{\epsilon,(n-1)} - \omega\|_{L^2(\Gamma_3)^d} \|u^h_{\epsilon,n} - v^h\|_{L^2(\Gamma_3)^d} \\
&\qquad + \mu^* meas(\Gamma_3)^{\frac{1}{2}} \| \mathrm{R} \|_{L^\infty(\Gamma_3)} \|u^h_{\epsilon,n} - \omega\|_{L^2(\Gamma_3)^d}.
\end{aligned}$$

Since, $(u^h_{\epsilon,n})_{n\geq 1}$ is a Cauchy sequence and $u^h_{\epsilon,n} \to \omega$ strongly in $L^2(\Gamma_3)^d$, we obtain

$$J_\epsilon(u^h_{\epsilon,(n-1)}; u^h_{\epsilon,n}, u^h_{\epsilon,n} - v^h) \to j_\epsilon(\omega, \omega - v^h) \quad \text{as } n \to +\infty \quad \forall v^h \in V^h.$$

Then, from (4.35), we deduce that

$$\begin{aligned}&\limsup_{n\to+\infty} B(u^h_{\epsilon,(n-1)}; u^h_{\epsilon,n}, u^h_{\epsilon,n} - v^h)\\ &= (f, \omega - v^h)_V + j_\epsilon(\omega, v^h - \omega) + \frac{1}{\epsilon}\Phi(\omega, v^h - \omega) \quad \forall v^h \in V^h.\end{aligned}$$

On the other hand, we have for all for all $v^h \in V^h$

$$\begin{aligned}&\limsup_{n\to+\infty} B(u^h_{\epsilon,(n-1)}; u^h_{\epsilon,n}, u^h_{\epsilon,n} - \omega)\\ &= \limsup_{n\to+\infty} \left[B(u^h_{\epsilon,(n-1)}; u^h_{\epsilon,n}, u^h_{\epsilon,n} - v^h) + B(u^h_{\epsilon,(n-1)}; u^h_{\epsilon,n}, v^h - \omega)\right]\\ &\leq (f, \omega - v^h)_V + j_\epsilon(\omega, v^h - \omega) + \frac{1}{\epsilon}\Phi(\omega, v^h - \omega) + \limsup_{n\to+\infty}\left[M_A \|u^h_{\epsilon,n}\|_V \|v^h - \omega\|_V\right]\end{aligned}$$

Since, $(u^h_{\epsilon,n})_{n\geq 1}$ is bounded on V, taking $v^h = \omega$, we get

$$\limsup_{n\to+\infty} B(u^h_{\epsilon,(n-1)}; u^h_{\epsilon,n}, u^h_{\epsilon,n} - \omega) \leq 0.$$

By pseudo-monotonicity of B, we get for all $v^h \in V^h$

$$B(\omega; \omega, \omega - v^h) \leq \liminf_{n\to+\infty} B(u^h_{\epsilon,(n-1)}; u^h_{\epsilon,n}, u^h_{\epsilon,n} - v^h). \tag{4.36}$$

Combining (4.35) and (4.36), we get

$$B(\omega; \omega, \omega - v^h) + \frac{1}{\epsilon}\Phi(\omega, \omega - v^h) + j_\epsilon(\omega, \omega - v^h) \leq (f, \omega - v^h)_V.$$

By setting $v^h = \omega \pm w^h$ in last inequality with w^h is an arbitrary element of V^h, we find

$$B(\omega; \omega, w^h) + \frac{1}{\epsilon}\Phi(\omega, w^h) + j_\epsilon(\omega, w^h) = (f, w^h)_V \quad \forall w^h \in V^h.$$

Which means that ω is a solution to Problem (PV^h_ϵ). By Theorem 4.1, the solution of (PV^h_ϵ) is unique and hence $u^h_\epsilon = \omega$, then, u^h_ϵ is the only cluster point of $(u^h_{\epsilon,n})_{n\geq 1}$ in the weak topology of V. Hence, The hole sequence $(u^h_{\epsilon,n})_{n\geq 1}$ converge to u^h_ϵ weakly in V, for all $\epsilon > 0$ and $h > 0$.

Finally, let us prove that $u^h_{\epsilon,n}$ is strongly convergent in V to u^h_ϵ, for all $\epsilon > 0$ and $h > 0$. Let $u^h_{\epsilon,n} \in V^h$ be a solution of (4.32) and $u^h_\epsilon \in V^h$ a solution of the problem (PV^h_ϵ). Then, we have from (4.19)

$$(Au_\epsilon^h, v^h - u_\epsilon^h)_V + \frac{1}{\epsilon}\Phi(u_\epsilon^h, v^h - u_\epsilon^h) + j_\epsilon(u_\epsilon^h, v^h - u_\epsilon^h) = (f, v^h - u_\epsilon^h)_V \quad \forall v^h \in V^h. \tag{4.37}$$

From the strong monotonicity of A, we get

$$\begin{aligned} m_A\|u_{\epsilon,n}^h - u_\epsilon^h\|_V^2 &\leq (Au_{\epsilon,n}^h - Au_\epsilon^h, u_{\epsilon,n}^h - u_\epsilon^h)_V \\ &= (Au_{\epsilon,n}^h, u_{\epsilon,n}^h - u_\epsilon^h)_V - (Au_\epsilon^h, u_{\epsilon,n}^h - u_\epsilon^h)_V \end{aligned}$$

Using (4.37) with $v^h = u_{\epsilon,n}^h$ and the fact that $(Au, v)_V = B(u; u, v)$ for all u, $v \in V$, we obtain

$$\begin{aligned} &m_A\|u_{\epsilon,n}^h - u_\epsilon^h\|_V^2 \\ &\leq B(u_{\epsilon,n}^h; u_{\epsilon,n}^h, u_{\epsilon,n}^h - u_\epsilon^h) - (f, u_{\epsilon,n}^h - u_\epsilon^h)_V + j_\epsilon(u_\epsilon^h, u_{\epsilon,n}^h - u_\epsilon^h) + \frac{1}{\epsilon}\Phi(u_\epsilon^h, u_{\epsilon,n}^h - u_\epsilon^h). \end{aligned}$$

We conclude by using fact that $(u_{\epsilon,n}^h)_{n\geq 1}$ is bounded, weakly convergent to u_ϵ^h in V and the continuity properties of B, j_ϵ and $(f, \,.\,)_V$, we get $u_{\epsilon,n}^h \to u_\epsilon^h$ strongly in $V \quad \forall \epsilon > 0$ and $h > 0$.
□

With the above results, the solution method for (4.32) is presented in **Algorithm 3**.

Algorithm 3. The solution method for (4.32).

Initialization. $\epsilon > 0$ and $u_{\epsilon,0}^h$ are given.

Iteration $n \geq 0$. Compute $u_{\epsilon,(n+1)}^h$ as follows

$$\begin{cases} \text{Find } u_{\epsilon,(n+1)}^h \in V^h \text{ such that: for all } v^h \in V^h \\ B(u_{\epsilon,n}^h; u_{\epsilon,(n+1)}^h, v^h) + \dfrac{1}{\epsilon}\Phi(u_{\epsilon,(n+1)}^h, v^h) + J_\epsilon(u_{\epsilon,n}^h; u_{\epsilon,(n+1)}^h, v^h) = (f, v^h)_V. \end{cases}$$

We iterate until the relative error on $u_{\varepsilon,n}$ becomes "sufficiently" small. *i.e.*,

$$\frac{\|u_{\epsilon,(n+1)}^h - u_{\epsilon,n}^h\|_{L^2(\Omega)}^2}{\|u_{\epsilon,(n+1)}^h\|_{L^2(\Omega)}^2} < \varepsilon_{fp}^2. \tag{4.38}$$

Part III

Nonlinear electro-elastic materials

Chapter 5

Unilateral contact with non-local Coulomb's friction law in electro-elasto-plasticity

In this chapter, we study a static contact problem modeling the interaction between an electro-elasto-plastic body, behavior already presented in subsection 1.2.2, and a conductive foundation. The contact is described by Signorini's conditions and electrical contact condition, as detailed in (1.24) and (1.30), respectively, and the friction follows Coulomb's non-local law. This chapter is divided into five sections. In Section 5.1, we propose a strong formulation of the problem. In Section 5.2, we introduce some preliminary, list assumptions on the problem data and we give a weak formulation of the model as a coupled system involving elastic displacement and electric potential. In Section 5.3, we state our existence and uniqueness result, Theorem 5.2, then, we provide its proof. In Section 5.4, we propose an iterative solution scheme, and we prove its convergence. In Section 5.5, we suggested an appropriate Augmented Lagrangian formulation to treat the unilateral contact numerically and to improve the conditioning of the iterative problem. The results presented in this chapter have been discussed in the paper [22].

5.1 The physical problem

The physical setting is as follows. An electro-elasto-plastic body occupies, in its reference configuration, an open and bounded domain $\Omega \subset \mathbb{R}^d$, $d = 2, 3$ with a sufficiently regular boundary Γ, partitioned into three disjoint measurable parts Γ_1, Γ_2 and Γ_3, such that $meas(\Gamma_1) > 0$, on one hand, and a partition of $\Gamma_1 \cup \Gamma_2$ into two open parts Γ_a and Γ_b, such that $meas(\Gamma_a) > 0$, on the other hand. During the process, the body may come in contact over Γ_3 with a rigid foundation (Figure 1.2.). The strong formulation of the problem can then be summarized as follows.

Problem $(\widetilde{\mathrm{P}})$**.** *Find a displacement field* $u : \Omega \to \mathbb{R}^d$*, a stress field* $\sigma : \Omega \to \mathbb{S}^d$*, an electric* *potential* $\varphi : \Omega \to \mathbb{R}$ *and an electric displacement field* $\mathrm{D} : \Omega \to \mathbb{R}^d$ *such that*

$$\sigma = \mathfrak{F}\varepsilon(u) - \mathcal{E}^* \mathrm{E}(\varphi) \quad \text{in} \quad \Omega, \tag{5.1}$$

$$\mathrm{D} = \mathcal{E}\varepsilon(u) + \beta\, \mathrm{E}(\varphi) \quad \text{in} \quad \Omega, \tag{5.2}$$

$$\mathrm{Div}\, \sigma + f_0 = 0 \quad \text{in} \quad \Omega, \tag{5.3}$$

$$\mathrm{div}\, \mathrm{D} = q_0 \quad \text{in} \quad \Omega, \tag{5.4}$$

$$u = 0 \quad \text{on} \quad \Gamma_1, \tag{5.5}$$

$$\sigma\nu = f_2 \quad \text{on} \quad \Gamma_2, \tag{5.6}$$

$$\varphi = 0 \quad \text{on} \quad \Gamma_a, \tag{5.7}$$

$$\mathrm{D} \,.\, \nu = q_2 \quad \text{on} \quad \Gamma_b, \tag{5.8}$$

$$\left.\begin{array}{l} (u_\nu - \varrho) \le 0, \;\; \sigma_\nu(u,\varphi) \le 0, \;\; \sigma_\nu(u,\varphi)(u_\nu - \varrho) = 0, \\ \left\{\begin{array}{l} \|\sigma_\tau\| \le \mu(\|u_\tau\|)|\,\mathrm{R}\,\sigma_\nu(u,\varphi)|, \\ \|\sigma_\tau\| < \mu(\|u_\tau\|)|\,\mathrm{R}\,\sigma_\nu(u,\varphi)| \Rightarrow u_\tau = 0, \\ \|\sigma_\tau\| = \mu(\|u_\tau\|)|\mathrm{R}\sigma_\nu(u)| \Rightarrow \exists \lambda \in \mathbb{R}^+ \text{ such that } \sigma_\tau = -\lambda\, u_\tau, \end{array}\right. \\ \mathrm{D} \,.\, \nu = \psi(u_\nu - \varrho)\phi_L(\varphi - \varphi_F), \end{array}\right\} \text{on } \Gamma_3. \tag{5.9}$$

Here, (5.1) represents the constitutive law, in which $\mathfrak{F} : \Omega \times \mathbb{S}^d \to \mathbb{S}^d$ is the nonlinear elasticity operator that describes the behavior of the Hencky's materials ($\mathfrak{F}\varepsilon = k_0 \,\mathrm{tr}(\varepsilon)\,\mathrm{I} + 2\, g(\|\bar{\varepsilon}\|^2)\bar{\varepsilon}$), already defined in (1.13). Then, (5.3) and (5.4) are the equilibrium equations in the case of a static problem. Then, (5.5)-(5.8) represent the mechanic and electric boundary conditions. Conditions (5.9), represent the Signorini conditions, the regularized Coulomb law and the regularized electrical conductivity condition.

5.2 Variational formulation

To derive a variational formulation for problem $(\widetilde{\mathrm{P}})$, we shall use the functional frameworks introduced in Chapter 2. Then, we make the following assumptions on data

$(\widetilde{h}_1)$ The function g is continuously differentiable in $[0, +\infty)$ and satisfies

(a) $0 < g_0 \le g(t) \le \dfrac{d}{2} k_0.$

(b) $0 < \alpha_1 \le g(t) + 2g'(t)\, t \le \alpha_2.$

where g_0, α_1 and α_2 are given positive constants.

$(\widetilde{h}_2)$ The coefficient of friction $\mu : \Gamma_3 \times \mathbb{R}_+ \to \mathbb{R}_+$ satisfies

(a) The function $x \mapsto \mu(x,u)$ is measurable on Γ_3, for all $u \in \mathbb{R}_+$.

(b) There exists $\mu^* > 0$ such that $0 \le \mu(x,u) \le \mu^*$, for all $u \in \mathbb{R}_+$, a.e. $x \in \Gamma_3$.

(c) There exists $L_\mu > 0$ such that

$$|\mu(.,u) - \mu(.,v)| \le L_\mu |u - v|, \text{ for all } u,\, v \in \mathbb{R}_+, \text{ a.e. on } \Gamma_3.$$

$(\widetilde{h}_3)$ The piezoelectric tensor $\mathcal{E} = (e_{ijk})$ satisfies $e_{ijk} = e_{ikj} \in L^\infty(\Omega)$.

$(\widetilde{h}_4)$ The electric permittivity tensor

(a) $\beta_{ij} = \beta_{ji} \in L^\infty(\Omega)$.

(b) There exists $m_\beta > 0$ such that $\beta_{ij}\xi_i\xi_j \ge m_\beta \|\xi\|^2$, for all $\xi \in \mathbb{R}^d$, a.e. $x \in \Omega$.

$(\widetilde{h}_5)$ The surface electrical conductivity function satisfies

(a) $\psi : \Gamma_3 \times \mathbb{R} \to \mathbb{R}_+$.

(b) There exists $M_\psi > 0$ such that $|\psi(x,u)| \le M_\psi$, for all $u \in \mathbb{R}$, a.e. $x \in \Gamma_3$.

(c) $x \mapsto \psi(x,u)$ is measurable on Γ_3, for all $u \in \mathbb{R}$.

(d) $x \mapsto \psi(x,u) = 0$ for all $u \le 0$.

(e) There exists $L_\psi > 0$ such that $|\psi(x,u_1) - \psi(x,u_2)| \le L_\psi |u_1 - u_2|$, for all $u_1,\, u_2 \in \mathbb{R}$, a.e. $x \in \Gamma_3$.

$(\widetilde{h}_6)$ The mapping $\mathrm{R} : H_{\Gamma_3}^{-\frac{1}{2}} \to L^\infty(\Gamma_3)$ is linear and continuous with $\|\,\mathrm{R}\,\| = c_{\mathrm{R}}$.

$(\widetilde{h}_7)$ The densities of the body force, surface traction, volume electric charge, surface electric charge and the potential of the foundation have the regularity

$$f_0 \in L^2(\Omega)^d, \quad f_2 \in L^2(\Gamma_2)^d, \quad q_0 \in L^2(\Omega), \quad q_2 \in L^2(\Gamma_b) \text{ and } \varphi_F \in L^2(\Gamma_3).$$

Next, using Riesz's representation theorem, we define $f \in V$ and $q \in W$ for all $v \in V$ and $\xi \in W$ by

$$(f,v)_V = \int_\Omega f_0 \,.\, v\, dx + \int_{\Gamma_2} f_2 \,.\, v\, da, \quad (q,\xi)_W = \int_\Omega q_0\, \xi\, dx - \int_{\Gamma_2} q_2\, \xi\, da, \tag{5.10}$$

and, we define the mappings $\ell : V \times W \times W \to \mathbb{R}$ and $j : V \times V \to \mathbb{R}$, respectively, by

$$\ell(u, \varphi, \xi) = \int_{\Gamma_3} \psi(u_\nu - \varrho)\phi_L(\varphi - \varphi_F)\,\xi\,da, \tag{5.11}$$

$$j(u, v) = \int_{\Gamma_3} \mu(\|u_\tau\|)\,|\,\mathrm{R}\,\sigma_\nu(u, \varphi)|\,\|v_\tau\|\,da. \tag{5.12}$$

Keeping in mind the assumptions $(\widetilde{h}_2)(a)$, $(\widetilde{h}_5)(c)$ and $(\widetilde{h}_6)$-$(\widetilde{h}_7)$ it follows that the integrals in (5.10), (5.11) and (5.12) are well-defined. Thus, with these notations and a standard procedure based on Green's formula we can derive the following variational formulation of Problem $(\widetilde{\mathrm{P}})$

Problem $(\widetilde{\mathrm{PV}})$. *Find a displacement field $u \in K$ and an electric potential $\varphi \in W$ such that:* for all $v \in K$ and $\xi \in W$, we have

$$(\mathfrak{F}\varepsilon(u), \varepsilon(v) - \varepsilon(u))_{\mathcal{H}} + (\mathcal{E}^*\nabla\varphi, \varepsilon(v) - \varepsilon(u))_H + j(u, v) - j(u, u) \geq (f, v - u)_V, \tag{5.13}$$

$$(\beta\nabla\varphi, \nabla\xi)_H - (\mathcal{E}\varepsilon(u), \nabla\xi)_H + \ell(u, \varphi, \xi) = (q, \xi)_W. \tag{5.14}$$

Now, we are able to state the following result of existence and uniqueness.

5.3 Existence and uniqueness result

The unique solvability of problem $(\widetilde{\mathrm{PV}})$ follows from the followings results.

Theorem 5.1 *Assume that $(\widetilde{h}_1)$, $(\widetilde{h}_2)(a) - (b)$, $(\widetilde{h}_3)$, $(\widetilde{h}_4)$, $(\widetilde{h}_5)(a) - (d)$, $(\widetilde{h}_6)$ and $(\widetilde{h}_7)$ hold. Then,*

1. *Problem $(\widetilde{\mathrm{PV}})$ has at least one solution.*

2. *Under the assumptions $(\widetilde{h}_2)(c)$ and $(\widetilde{h}_5)(e)$, there exists $L^* > 0$, such that if $L_\mu + \mu^* + L_\psi L + M_\psi < L^*$ the solution is unique.*

Let us consider the product space $X = V \times W$ endowed with the inner product

$$(x, y)_X = (u, v)_V + (\varphi, \xi)_W \quad \forall\, x = (u, \varphi),\; y = (v, \xi) \in X, \tag{5.15}$$

and the associated norm $\|\,.\,\|_X$. Let $U = K \times W$ be non-empty closed convex subset of X. We define the operator $A : X \to X$, the functions $\widetilde{j}$ and $\widetilde{\ell}$ on $X \times X$ and the element $f_3 \in X$ by

$$(Ax, y)_X = (\mathfrak{F}\varepsilon(u), \varepsilon(v))_{\mathcal{H}} + (\mathcal{E}^*\nabla\varphi, \varepsilon(v))_H + (\beta\nabla\varphi, \nabla\xi)_H - (\mathcal{E}\varepsilon(u), \nabla\xi)_H, \tag{5.16}$$

$$\widetilde{\ell}(x, y) = \ell(u, \varphi, \xi), \quad \widetilde{j}(x, y) = j(u, v) \quad \text{and} \quad f_3 = (f, q) \in X, \tag{5.17}$$

for all $x = (u, \varphi)$ and $y = (v, \xi)$ in X. With the above notations, we get the following equivalent problem

Problem $(\widetilde{\widetilde{\mathrm{PV}}})$. *Find $x = (u, \varphi) \in U$ such that:*

$$(Ax, y - x)_X + \widetilde{j}(x, y) - \widetilde{j}(x, x) + \widetilde{\ell}(x, y - x) \geq (f_3, y - x)_X \quad \forall y = (v, \xi) \in U. \tag{5.18}$$

Lemma 5.1 *The couple $x = (u, \varphi) \in U$ is a solution to Problem $(\widetilde{\mathrm{PV}})$ if and only if it is a solution to Problem $(\widetilde{\widetilde{\mathrm{PV}}})$.*

Proof. Let $x = (u, \varphi) \in U$ be a solution to Problem $(\widetilde{\mathrm{PV}})$ and let $y = (v, \xi) \in U$. We use the test function $\xi - \varphi$ in (5.14), add the corresponding inequality to (5.13) and use (5.15)-(5.17) to obtain (5.18). Conversely, let $x = (u, \varphi) \in U$ be a solution to the elliptic variational inequalities (5.18). We take $y = (v, \varphi)$ in (5.18), where v is an arbitrary element of K and obtain (5.13). Then for any $\xi \in W$, we take successively $y = (v, \varphi + \xi)$, and $y = (v, \varphi - \xi)$ in (5.18) to obtain (5.14), which concludes the proof of Lemma 5.1. □

The proof of Theorem 5.1 is obtained by using Schauder's fixed point theorem A.4 combined with arguments of abstract variational inequalities, in a similar way to that in the proof of Theorem 3.1. All we have to do is make sure that the operator A is strongly monotone and Lipschitz continuous. For this, we have already shown that the nonlinear operator of elasticity $\mathfrak{F}$ defined in (1.13) is strongly monotone and Lipschitz continuous.

Let $x_1 = (u_1, \varphi_1) \in X$, $x_1 = (u_2, \varphi_2) \in X$, using (5.16), we have

$$\begin{aligned}
&(Ax_1 - Ax_2, x_1 - x_2)_X \\
&= (\mathfrak{F}\varepsilon(u_1) - \mathfrak{F}\varepsilon(u_2), \varepsilon(u_1) - \varepsilon(u_2))_{\mathcal{H}} + (\mathcal{E}^*\nabla\varphi_1 - \mathcal{E}^*\nabla\varphi_2, \varepsilon(u_1) - \varepsilon(u_2))_H \\
&\quad + (\beta\nabla\varphi_1 - \beta\nabla\varphi_2, \nabla\varphi_1 - \nabla\varphi_2)_H - (\mathcal{E}\varepsilon(u_1) - \mathcal{E}\varepsilon(u_2), \nabla\varphi_1 - \nabla\varphi_2)_H,
\end{aligned}$$

using (1.24), we get $(\mathcal{E}\varepsilon(u), \nabla\varphi)_H$ for all $x = (u, \varphi) \in X$, thus

$$
\begin{aligned}
&(Ax_1 - Ax_2, x_1 - x_2)_X \\
&= (\mathfrak{F}\varepsilon(u_1) - \mathfrak{F}\varepsilon(u_2), \varepsilon(u_1) - \varepsilon(u_2))_{\mathcal{H}} + (\beta\nabla\varphi_1 - \beta\nabla\varphi_2, \nabla\varphi_1 - \nabla\varphi_2)_H.
\end{aligned}
$$

We use now (3.17), $(\widetilde{h}_4)$ and (2.26), there exist $m_A > 0$ depend only on $\mathfrak{F}$, β, Ω, Γ_a such that

$$
(Ax_1 - Ax_2, x_1 - x_2)_X \geq m_A\Big(\|u_1 - u_2\|_V^2 + \|\varphi_1 - \varphi_2\|_W^2\Big) = m_A\|x_1 - x_2\|_X^2 \quad \forall x_1,\, x_2 \in X,
$$

and keeping in mind (5.15), we obtain

$$
(Ax_1 - Ax_2, x_1 - x_2)_X \geq m_A\|x_1 - x_2\|_X^2 \quad \forall x_1,\, x_2 \in X. \tag{5.19}
$$

In the same way, using (3.21) and $(\widetilde{h}_4)$, after an algebraic manipulations, it follows that there exist $c_2 > 0$ depend only on $\mathfrak{F}$, β and $\mathcal{E}$ such that

$$
(Ax_1 - Ax_2, y)_X \leq c_2\Big(\|u_1 - u_2\|_V\|v\|_V + \|\varphi_1 - \varphi_2\|_W\|v\|_V + \|\varphi_1 - \varphi_2\|_W\|\xi\|_W + \|u_1 - u_2\|_V\|\xi\|_W\Big),
$$

for all $y = (v, \xi) \in X$. We use (5.15) and the previous inequality to obtain

$$
(Ax_1 - Ax_2, y)_X \leq 4c_2\|x_1 - x_2\|_X\|y\|_X \quad \forall y \in X,
$$

and taking $y = Ax_1 - Ax_2 \in X$, we find

$$
\|Ax_1 - Ax_2\|_X \leq M_A\,\|x_1 - x_2\|_X \quad \forall x_1,\, x_2 \in X. \tag{5.20}
$$

with $M_A = 4c_2$.

5.4 Iteration method

Let $x_n = (u_n, \varphi_n)$ be the n-th approximation of the solution to the problem $(\widetilde{\mathrm{PV}})$. We seek for the weak solution $x_{n+1} = (u_{n+1}, \varphi_{n+1})$ of the linear problem.

$$
\begin{cases}
\text{Given an initial guess } x_0 = (u_0, \varphi_0) \in U, \text{ find } x_{n+1} = (u_{n+1}, \varphi_{n+1}) \in U \text{ such that} \\
B(x_n; x_{n+1}, y - x_{n+1}) + \widetilde{j}(x_n, y) - \widetilde{j}(x_n, x_{n+1}) + \widetilde{\ell}(x_n, y - x_{n+1}) \geq (f_3, y - x_{n+1})_X,
\end{cases} \tag{5.21}
$$

for all $y = (v, \xi)$ in U, where the operator $B : U \times X \times X \to \mathbb{R}$ is the operator defined by

$$B(x; y, z) = \Big(k_0 tr(\varepsilon(v))I + 2g(\|\overline{\varepsilon}(u)\|^2)\overline{\varepsilon}(v), \varepsilon(w)\Big)_{\mathcal{H}} \\ + (\mathcal{E}^*\nabla\eta, \varepsilon(w))_H + (\beta\nabla\eta, \nabla\xi)_H - (\mathcal{E}\varepsilon(v), \nabla\xi)_H,$$

for all $x = (u, \varphi)$, $y = (v, \eta)$ and $z = (w, \xi) \in X$. For a fixed $x = (u, \varphi) \in X$, it's clear that $(y, z) \mapsto B(x; y, z)$ is a bilinear form and by arguments similar to those used in the precedent section show that

$$\begin{cases} \text{(a)} \quad B(x; y, y) \geq m_A\|y\|_X^2, & \text{for all } y = (v, \eta) \in X. \\ \text{(b)} \quad |B(x; y, z)| \leq M_A\|y\|_X\|z\|_X, & \text{for all } y = (v, \eta),\ z = (w, \xi) \in X. \end{cases}$$

We have the following convergence result.

Theorem 5.2 *Under assumptions of Theorem 5.1, the iteration method (5.21) converges:*

$$\|x_n - x\|_X \to 0 \quad \text{as } n \to +\infty.$$

where x is the solution to Problem $(\widetilde{\widetilde{\widetilde{\mathrm{PV}}}})$.

Proof. Let $x_n = (u_n, \varphi_n) \in U$ the solution of the iterative problem (5.21), thus we have for all $y = (v, \xi) \in U$,

$$B(x_{n-1}; x_n, y - x_n) + \widetilde{j}(x_{n-1}, y) - \widetilde{j}(x_{n-1}, x_n) + \widetilde{\ell}(x_{n-1}, y - x_n) \geq (f_3, y - x_n)_X. \tag{5.22}$$

Taking $y = (0, 0)$ in (5.22), by using the fact that $\widetilde{j}(x_{n-1}, x_n) > 0$, we have

$$B(x_{n-1}; x_n, x_n) \leq (f_3, x_n)_X - \widetilde{\ell}(x_{n-1}, x_n).$$

It follows from the properties of B, the boundedness of ψ and ϕ_L and (2.26), that

$$\|x_n\|_X \leq c_1(\|f_3\|_X + M_\psi L\, \widetilde{c}_0\, meas(\Gamma_3)^{\frac{1}{2}}), \quad \text{with } c_1 = \frac{1}{m_A}.$$

Afterward, since the sequence $(x_n)_{n\geq 1}$ is bounded in X, then, there exists $\widetilde{x} = (\widetilde{u}, \widetilde{\varphi}) \in X$ and a subsequence, denoted again $(x_n)_{n\geq 1}$, such that $(x_n)_{n\geq 1}$ converges weakly to $\widetilde{x} \in X$. Since, U is a closed convex set in a real Hilbert space X, therefore, U is weakly closed, then $\widetilde{x} \in U$. Moreover, using the compactness of the trace map $\gamma : X \to L^2(\Gamma_3)^d \times L^2(\Gamma_3)$, it follows from the weak convergence of $(x_n)_{n\geq 1}$ that $x_n \to \widetilde{x}$ strongly in $L^2(\Gamma_3)^d \times L^2(\Gamma_3)$ as $n \to +\infty$.

Next, let us prove that $\widetilde{x}$ is the solution of Problem $(\widetilde{\widetilde{\mathrm{PV}}})$. Using (5.17) and by keeping in mind the proprieties of μ, R, ψ and ϕ_L, we get

$$|\widetilde{j}(x_{n-1}, x_n) - \widetilde{j}(\widetilde{x}, \widetilde{x})| \leq \mu^* meas(\Gamma_3)^{\frac{1}{2}} \|\,\mathrm{R}\,\|_{L^\infty(\Gamma_3)} \|u_n - \widetilde{u}\|_{L^2(\Gamma_3)^d} + \left(\mu^* meas(\Gamma_3)^{\frac{1}{2}} c_R + L_\mu \|\,\mathrm{R}\,\|_{L^\infty(\Gamma_3)}\right) \|u_{n-1} - \widetilde{u}\|_{L^2(\Gamma_3)^d} \|\widetilde{u}\|_{L^2(\Gamma_3)^d}. \tag{5.23}$$

Moreover,

$$|\widetilde{\ell}(x_{n-1}, y - x_n) - \widetilde{\ell}(\widetilde{x}, y - \widetilde{x})| \leq M_\psi \|\varphi_{n-1} - \widetilde{\varphi}\|_{L^2(\Gamma_3)} \|\xi - \varphi_n\|_{L^2(\Gamma_3)} + L_\psi \ L \ \|u_{n-1} - \widetilde{u}\|_{L^2(\Gamma_3)^d} \|\xi - \varphi_n\|_{L^2(\Gamma_3)} + M_\psi \ L \ meas(\Gamma_3)^{\frac{1}{2}} \ \|\varphi_n - \widetilde{\varphi}\|_{L^2(\Gamma_3)}. \tag{5.24}$$

Since $x_n \to \widetilde{x}$ strongly in $L^2(\Gamma_3)^d \times L^2(\Gamma_3)$, it follows from (5.23) and (5.24), that

$$\left.\begin{array}{l} \widetilde{j}(x_{n-1}, x_n) \ \to \ \widetilde{j}(\widetilde{x}, \widetilde{x}), \\ \widetilde{\ell}(x_{n-1}, y - x_n) \ \to \ \widetilde{\ell}(\widetilde{x}, y - \widetilde{x}), \end{array}\right\} \quad \text{as } n \to +\infty.$$

We deduce from (5.22) that for all $y = (v, \xi) \in U$

$$\limsup_{n\to+\infty} B(x_{n-1}; x_n, x_n - y) \leq \ (f_3, x - y)_X + \widetilde{j}(\widetilde{x}, y) - \widetilde{j}(\widetilde{x}, \widetilde{x}) + \widetilde{\ell}(\widetilde{x}, y - \widetilde{x}).$$

On the other hand, we have for all $y = (v, \xi) \in U$,

$$\begin{aligned} &\limsup_{n\to+\infty} B(x_{n-1}; x_n, x_n - \widetilde{x}) \\ &= \limsup_{n\to+\infty} \left[B(x_{n-1}; x_n, x_n - y) + B(x_{n-1}; x_n, y - \widetilde{x})\right] \\ &\leq \ \limsup_{n\to+\infty} \left[B(x_{n-1}; x_n, x_n - y) + M_A \|x_n\|_X \|y - \widetilde{x}\|_X\right] \\ &\leq \ (f_3, \widetilde{x} - y)_X + \widetilde{j}(\widetilde{x}, y) - \widetilde{j}(\widetilde{x}, \widetilde{x}) + \widetilde{\ell}(\widetilde{x}, y - \widetilde{x}) + \limsup_{n\to+\infty} M_A \|x_n\|_X \|y - \widetilde{x}\|_X. \end{aligned}$$

Note that $\|x_n\|_X$ is bounded on X, we may then substitute $y = \widetilde{x}$ into the last inequality to obtain

$$\limsup_{n\to+\infty} B(x_{n-1}; x_n, x_n - \widetilde{x}) \leq 0.$$

Therefore, by pseudo-monotonicity of B, we get

$$B(\widetilde{x};\widetilde{x},\widetilde{x}-y) \leq \liminf_{n\to+\infty} B(x_{n-1};x_n,x_n-y). \tag{5.25}$$

Combining (5.22) and (5.25) we deduce

$$B(\widetilde{x};\widetilde{x},y-\widetilde{x}) + \widetilde{j}(\widetilde{x},y) - \widetilde{j}(\widetilde{x},\widetilde{x}) + \widetilde{\ell}(\widetilde{x},y-\widetilde{x}) \geq (f_3,y-\widetilde{x})_X.$$

Which means that $\widetilde{x} \in U$ is a solution of Problem $(\widetilde{\widetilde{\widetilde{\mathrm{PV}}}})$, and from the uniqueness of the solution for this variational inequality, we obtain $\widetilde{x} = x$. Since x is the unique weak limit of any subsequence of $(x_n)_{n\geq 1}$, we deduce that the whole sequence $(x_n)_{n\geq 1}$ is weakly convergent in X to x as $n \to +\infty$.

Let us now prove that

$$\|x_n - x\|_X \to 0 \quad \text{as } n \to +\infty.$$

To this end, let $x_n \in U$ be a solution of (5.21) and $x \in U$ a solution of the problem $(\widetilde{\widetilde{\widetilde{\mathrm{PV}}}})$. By using the strong monotonicity of A, we get

$$m_A\|x_n - x\|_X^2 \leq (Ax_n - Ax, x_n - x)_X = (Ax_n, x_n - x)_X - (Ax, x_n - x)_X.$$

Using (5.18) with $y = x_n$ and the fact that $(Ax,y)_X = B(x;x,y)$ for all x, y in X, we obtain

$$m_A\|x_n - x\|_X^2 \leq B(x_n;x_n,x_n-x) - (f_3,x_n-x)_X + \widetilde{j}(x,x_n) - \widetilde{j}(x,x) + \widetilde{\ell}(x,x_n-x).$$

We conclude by using the boundedness of ψ and ϕ_L, the fact that $(x_n)_{n\geq 1}$ is bounded, weakly convergent to x in X and the continuity properties of B, $\widetilde{j}$, $\widetilde{\ell}$ and $(f_3,\,.\,)_X$, to get

$$x_n \to x \quad \text{strongly in } X \quad \text{as } n \to +\infty.$$

□

5.5 Augmented Lagrangian for the iterative problem

The bilinear form B is positive definite, but not symmetric since the global matrix of piezoelectricity is antisymmetric. However, it is possible to use a second equivalent variational formulation characterized by a symmetric bilinear form $\check{B}$. To this end, we will need the following additional step. By subtracting the equation (5.14) from the inequality (5.13), we obtain

$$(\check{A}x, y-x)_X + \widetilde{j}(x,y) - \widetilde{j}(x,x) + \check{\ell}(x, y-x) \geq (\check{f}_3, y-x)_X \quad \forall y=(v,\xi) \in U, \tag{5.26}$$

with

$$(\check{A}x, y)_X = (\mathfrak{F}\varepsilon(u), \varepsilon(v))_{\mathcal{H}} + (\mathcal{E}^*\nabla\varphi, \varepsilon(v))_H - (\beta\nabla\varphi, \nabla\xi)_H + (\mathcal{E}\varepsilon(u), \nabla\xi)_H, \tag{5.27}$$

$$\check{\ell}(x,y) = -\int_{\Gamma_3} \psi(u_\nu - \varrho)\phi_L(\varphi - \varphi_F)\,\xi\, da, \tag{5.28}$$

$$(\check{f}_3, y)_X = \int_\Omega f_0 \,.\, v\, dx + \int_{\Gamma_2} f_2 \,.\, v\, da - \int_\Omega q_0\,\xi\, dx + \int_{\Gamma_2} q_2\,\xi\, da, \tag{5.29}$$

for all $y=(v,\xi) \in X$.

Lemma 5.2 *The variational formulations* (5.26) *and* (5.18) *are equivalent.*

Proof. Let $x=(u,\varphi) \in U$ the solution to the variational inequality (5.18). Since for all $\xi \in W$, we have $-\xi \in W$, thus

$$(Ax, (v,-\xi)-x)_X + \widetilde{j}(x,(v,-\xi)) - \widetilde{j}(x,x) + \widetilde{\ell}(x,(v,-\xi)-x) \geq (f_3, (v,-\xi)-x)_X \quad \forall (v,\xi) \in U.$$

Since for all $(v,\xi) \in X$, we have

$$(\check{A}x, y)_X = (Ax, (v,-\xi))_X,\ \check{\ell}(x,y) = \widetilde{\ell}(x,(v,-\xi)) \text{ and } (\check{f}_3, y)_X = (f_3, (v,-\xi))_X. \tag{5.30}$$

So $x=(u,\varphi)$ is also a solution of (5.26). Similarly, we show that the solution of (5.18) is also a solution of (5.26). □

Next, by applying the iteration method used in the precedent section to the variational inequality (5.26), we get the following iterative problem.

$$\begin{cases} \text{Given an initial guess } x_0=(u_0,\varphi_0) \in U, \text{ find } x_{n+1}=(u_{n+1},\varphi_{n+1}) \in U \text{ such that} \\ \check{B}(x_n; x_{n+1}, y-x_{n+1}) + \widetilde{j}(x_n,y) - \widetilde{j}(x_n,x_{n+1}) + \check{\ell}(x_n, y-x_{n+1}) \geq (\check{f}_3, y-x_{n+1})_X, \end{cases} \tag{5.31}$$

for all $y=(v,\xi)$ in U, where the bilinear symmetric form $\check{B} : U \times X \times X \to \mathbb{R}$ is given by

$$\check{B}(x; y, z) = B(x; y, (w,-\xi)) \quad \forall x=(u,\varphi),\ y=(v,\eta) \text{ and } z=(w,\xi) \in X.$$

Hence, an equivalent constrained minimization problem of (5.31) can be formulated. The proposed minimization problem is as follows

$$\begin{cases} Find\ x = (u, \varphi) \in U\ such\ that \\ J_n(x) + \widetilde{j}_n(x) \leq\ J_n(y) + \widetilde{j}_n(y) \quad \forall y = (v, \xi) \in U, \end{cases} \tag{5.32}$$

where

$$\widetilde{j}_n(y) = \widetilde{j}(x_n, y) \quad \forall y = (v, \xi) \in X.$$

J_n is the piezoelectric deformation energy functional due to non-frictional effects given by

$$J_n(y) = \frac{1}{2}\check{B}(x_n; y, y) - (f_n, y)_X \quad \forall y = (v, \xi) \in X,$$

and

$$(f_n, y)_X = (\check{f}_3, y)_X - \check{\ell}(x_n, y) \quad \forall y = (v, \xi) \in X.$$

The quadratic functional J_n is strictly convex and Gateaux differentiable on X. Moreover, the friction functional $\widetilde{j}_n$ is convex and lower semi-continuous on X, thus, the existence of a unique solution to problem (5.32) is guaranteed by Theorem A.10.

Let $\mathfrak{p} = (\mathfrak{p}_f, \mathfrak{p}_c)$, where $\mathfrak{p}_f$ (friction) and $\mathfrak{p}_f$ (contact) are auxiliary variables. we introduce the set

$$C = \{\mathfrak{p}_c \in L^2(\Gamma_3)\,;\ (\mathfrak{p}_c - \varrho) \leq 0 \text{ on } \ \Gamma_3\},$$

and the characteristic functional $I_C : L^2(\Gamma_3) \to \mathbb{R} \cup \{+\infty\}$ of the set C, is defined by

$$I_C(\mathfrak{p}_c) = \begin{cases} 0, & \text{if } \mathfrak{p}_c \in C, \\ +\infty, & \text{if } \mathfrak{p}_c \notin\ C. \end{cases}$$

It is easy to see that the problem giving in (5.32) is equivalent to the following constrained minimization problem

Find $x = (u, \varphi) \in X$ *and* $\mathfrak{p} = (\mathfrak{p}_f, \mathfrak{p}_c) \in L^2(\Gamma_3)^2$ *such that, for all* $y = (v, \xi) \in X$ *and* $\mathfrak{q} = (\mathfrak{q}_f, \mathfrak{q}_c) \in L^2(\Gamma_3)^2$

$$J_n(x) + \widetilde{j}_n(\mathfrak{p}_f) + I_C(\mathfrak{p}_c) \leq J_n(y) + \widetilde{j}_n(\mathfrak{q}_f) + I_C(\mathfrak{q}_c), \tag{5.33}$$

$$\left.\begin{aligned} u_\nu - \mathfrak{p}_c = 0, \\ u_\tau - \mathfrak{p}_f = 0, \end{aligned}\right\} \quad \text{on } \Gamma_3. \tag{5.34}$$

From (5.33)-(5.34) the Augmented Lagrangian functional $\mathcal{L}_r$ is defined over $X \times L^2(\Gamma_3)^2 \times$

$L^2(\Gamma_3)^2$ by

$$\mathcal{L}_r(y, \mathfrak{q}; \theta) = J_n(y) + \widetilde{j}_n(\mathfrak{q}_f) + I_C(\mathfrak{q}_c) + (\theta_c, v_\nu - \mathfrak{q}_c)_{L^2(\Gamma_3)} + (\theta_f, v_\tau - \mathfrak{q}_f)_{L^2(\Gamma_3)} + \frac{r}{2}\|v_\nu - \mathfrak{q}_c\|^2_{L^2(\Gamma_3)} + \frac{r}{2}\|v_\tau - \mathfrak{q}_f\|^2_{L^2(\Gamma_3)},$$

where the constant $r > 0$ is the penalty parameter and $\theta = (\theta_f, \theta_c)$. Since the functional $J_n + \widetilde{j}_n$ is strictly convex, and the constraints (5.34) are linear, a saddle point of $\mathcal{L}_r$ exists, and it is the solution of the saddle-point problem

$$\begin{cases} \text{Find } \big((x, \mathfrak{p}); \lambda\big) \in X \times L^2(\Gamma_3)^2 \times L^2(\Gamma_3)^2 \text{ such that} \\ \mathcal{L}_r(x, \mathfrak{p}; \theta) \le \mathcal{L}_r(x, \mathfrak{p}; \lambda) \le \mathcal{L}_r(y, \mathfrak{q}; \lambda) \quad \forall\big((y, \mathfrak{q}); \theta\big) \in X \times L^2(\Gamma_3)^2 \times L^2(\Gamma_3)^2, \end{cases}$$

where we have set $\lambda = (\lambda_f, \lambda_c)$. Equivalently, $\big((x, \mathfrak{p}); \lambda\big)$ is the solution of the min-max problem

$$\max_{\theta} \min_{(y,\mathfrak{q})} \mathcal{L}_r(y, \mathfrak{q}; \theta) = \min_{(y,\mathfrak{q})} \max_{\theta} \mathcal{L}_r(y, \mathfrak{q}; \theta),$$

Uzawa block relaxation method is obtained by minimizing $\mathcal{L}_r$, successively, over x and $\mathfrak{p}$, as follows, starting with $\mathfrak{p}^0$ and λ^0

$$\mathcal{L}_r(x^{k+1}, \mathfrak{p}^k; \lambda^k) = \min_y \mathcal{L}_r(y, \mathfrak{p}^k; \lambda^k), \tag{5.35}$$

$$\mathcal{L}_r(x^{k+1}, \mathfrak{p}^{k+1}; \lambda^k) = \min_{\mathfrak{p}} \mathcal{L}_r(x^{k+1}, \mathfrak{q}; \lambda^k), \tag{5.36}$$

$$\lambda^{k+1} = \lambda^k + r(u^{k+1} - \mathfrak{p}^{k+1}). \tag{5.37}$$

The solution of (5.35) can be characterized by the Euler-Lagrange equation [54], since $y \mapsto \mathcal{L}_r(y, \mathfrak{p}; \theta)$ is convex and differentiable

$$\check{B}(x_n; x^{k+1}, y) + r(u_\nu^{k+1}, v_\nu)_{L^2(\Gamma_3)} + r(u_\tau^{k+1}, v_\tau)_{L^2(\Gamma_3)} = (f_n, y)_X + (r\mathfrak{p}_c^k - \lambda_c^k, v_\nu)_{L^2(\Gamma_3)} + (r\mathfrak{p}_f^k - \lambda_f^k, v_\tau)_{L^2(\Gamma_3)}.$$

In (5.36) the subproblems in $\mathfrak{p}_c$ and $\mathfrak{p}_f$ are uncoupled. Consequently, we can minimize the functional $\mathfrak{p} \to \mathcal{L}_r(x^{k+1}, \mathfrak{p}; \lambda^k)$ separately in $\mathfrak{p}_c$ and $\mathfrak{p}_f$. For the contact subproblem, straightforward calculations using Karush-Kuhn-Tucker optimality conditions yield to (see [54])

$$\mathfrak{p}_c^{k+1} = u_\nu^{k+1} + \frac{1}{r}\Big[\lambda_c^k - \Big(\lambda_c^k + r(u_\nu^{k+1} - \varrho)\Big)^+\Big].$$

For the friction subproblem, using the Fenchel duality theory(see [54, 82]), we get

$$\mathfrak{p}_f^{k+1} = \begin{cases} \dfrac{|\lambda_f^k + r u_\tau^{k+1}| - s_n}{r|\lambda_f^k + r u_\tau^{k+1}|}(\lambda_f^k + r u_\tau^{k+1}), & if \ \ |\lambda_f^k + r u_\tau^{k+1}| > s_n, \\ 0, & if \ \ |\lambda_f^k + r u_\tau^{k+1}| \leq s_n, \end{cases}$$

where

$$s_n = \mu(\|u_{n,\tau}\|)|\,\mathrm{R}\,\sigma_\nu(u_n, \varphi_n)|.$$

With the previous results, we can now present our Uzawa block relaxation method **Algorithm 5**.

Algorithm 5. Uzawa block relaxation for (5.32)

Initialization. $r > 0$, $\mathfrak{p}^0 = (\mathfrak{p}_f^0, \mathfrak{p}_c^0)$ and $\lambda^0 = (\lambda_f^0, \lambda_c^0)$ are given.

Iteration $k > 0$. Compute successively $x^{k+1} = (u^{k+1}, \varphi^{k+1})$, $\mathfrak{p}^{k+1} = (\mathfrak{p}_f^{k+1}, \mathfrak{p}_c^{k+1})$ and $\lambda^{k+1} = (\lambda_f^{k+1}, \lambda_c^{k+1})$ as follows

Step 5. Find $x^{k+1} = (u^{k+1}, \varphi^{k+1}) \in X$ such that

$$\begin{aligned} &\check{B}(x_n; x^{k+1}, y) + r(u_\nu^{k+1}, v_\nu)_{L^2(\Gamma_3)} + r(u_\tau^{k+1}, v_\tau)_{L^2(\Gamma_3)} \\ &\qquad = (f_n, y)_X + (r\mathfrak{p}_c^k - \lambda_c^k, v_\nu)_{L^2(\Gamma_3)} + (r\mathfrak{p}_f^k - \lambda_f^k, v_\tau)_{L^2(\Gamma_3)}. \end{aligned}$$

Step 2. Compute the auxiliary contact and friction variables

$$\mathfrak{p}_c^{k+1} = u_\nu^{k+1} + \frac{1}{r}\Big[\lambda_c^k - \Big(\lambda_c^k + r(u_\nu^{k+1} - \varrho)\Big)^+\Big],$$

$$\mathfrak{p}_f^{k+1} = \begin{cases} \dfrac{|\lambda_f^k + r u_\tau^{k+1}| - s_n}{r|\lambda_f^k + r u_\tau^{k+1}|}(\lambda_f^k + r u_\tau^{k+1}), & if \ \ |\lambda_f^k + r u_\tau^{k+1}| > s_n, \\ 0, & if \ \ |\lambda_f^k + r u_\tau^{k+1}| \leq s_n. \end{cases}$$

Step 3. Update the Lagrange multipliers

$$\lambda_c^{k+1} = \lambda_c^k + r(u_\nu^{k+1} - \mathfrak{p}_c^{k+1}),$$
$$\lambda_f^{k+1} = \lambda_f^k + r(u_\tau^{k+1} - \mathfrak{p}_f^{k+1}).$$

We iterate until the relative error on x^k, $\mathfrak{p}_f^k$ and $\mathfrak{p}_c^k$ is sufficiently "small", *i.e.*,

$$\frac{\|x^{k+1}-x^k\|^2_{L^2(\Omega)}+\|\mathfrak{p}_c^{k+1}-\mathfrak{p}_c^k\|^2_{L^2(\Gamma_3)}+\|\mathfrak{p}_f^{k+1}-\mathfrak{p}_f^k\|^2_{L^2(\Gamma_3)}}{\|x^{k+1}\|_{L^2(\Omega)}+\|\mathfrak{p}_c^{k+1}\|^2_{L^2(\Gamma_3)}+\|\mathfrak{p}_f^{k+1}\|^2_{L^2(\Gamma_3)}}<\epsilon^2, \tag{5.38}$$

and with the above results, the solution method for (5.21) is presented in **Algorithm 4**.

Algorithm 4. Solution for (5.21)

Initialization. s_0 and $x_0=(u_0,\varphi_0)\in X$ are given.

Iteration $n\geq 0$. Compute x_{n+1} and s_{n+1} successively as follows

- Compute $x_{n+1}=(u_{n+1},\varphi_{n+1})\in X$ using **Algorithm 5**.
- Update $s_{n+1}=\mu(\|u_{n+1,\tau}\|)|\,\mathrm{R}\,\sigma_\nu(u_{n+1},\varphi_{n+1})|$ and $(f_{n+1},\,.\,)_X=(\check{f}_3,\,.\,)_X-\check{\ell}(x_{n+1},\,.\,)$.

The fixed-point iteration terminates if the relative error (3.46) on s_n becomes sufficiently "small".

Chapter 6

Frictional contact with normal compliance and unilateral constraint in electro-elasto-plasticity

In this chapter, we consider two static problems which describe the contact between a piezoelectric body and an obstacle, the so-called foundation. The constitutive relation of the material is assumed to be electro-elastic and involves the nonlinear elastic constitutive Hencky's law. In the first problem, the contact is assumed to be frictionless, and the foundation is non-conductive, while in the second it is supposed to be frictional, and the foundation is electrically conductive. The contact is modeled with the normal compliance condition with finite penetration, the regularized Coulomb law, and the regularized electrical conductivity condition, as detailed in 1.3.3, 1.4.3 and 1.2.2, respectively. This chapter is divided into five sections. In Section 6.1, we propose strong formulations of the problems. In Section 6.2, we introduce some preliminary, list assumptions on the problem data and we give weak formulations of the models. In Section 5.3, we state our existence and uniqueness result, Theorem 6.2, then, we provide their proofs using the theory of variational inequalities involving nonlinear strongly monotone Lipschitz continuous operators and Schauder's fixed-point theorem, we also prove that the solution of the second problem converges towards that of the first one as the friction and electrical conductivity coefficients converge towards zero. In Section 6.4, the numerical solutions of the problems are achieved by using a successive iteration techniques; their convergence are also established. In Section 6.5, The numerical treatment of the contact condition is realized using an Augmented Lagrangian type formulation that will lead us to use Uzawa type algorithms. The results presented in this chapter have been discussed in the paper [25].

6.1 The physical problems

The physical setting is the same one presented in Chapter 5-Section 5.1. The strong formulation of one of these two problems can be summarized as follows.

Problem (P_2)**.** *Find a displacement field* $u : \Omega \to \mathbb{R}^d$*, a stress field* $\sigma : \Omega \to \mathbb{S}^d$*, an electric potential* $\varphi : \Omega \to \mathbb{R}$ *and an electric displacement field* $\mathrm{D} : \Omega \to \mathbb{R}^d$ *such that*

$$\sigma = \mathfrak{F}\varepsilon(u) - \mathcal{E}^* \,\mathrm{E}(\varphi) \quad \text{in} \quad \Omega, \tag{6.1}$$

$$\mathrm{D} = \mathcal{E}\varepsilon(u) + \beta\,\mathrm{E}(\varphi) \quad \text{in} \quad \Omega, \tag{6.2}$$

$$\mathrm{Div}\,\sigma + f_0 = 0 \quad \text{in} \quad \Omega, \tag{6.3}$$

$$\mathrm{div}\,\mathrm{D} = q_0 \quad \text{in} \quad \Omega, \tag{6.4}$$

$$u = 0 \quad \text{on} \quad \Gamma_1, \tag{6.5}$$

$$\sigma\nu = f_2 \quad \text{on} \quad \Gamma_2, \tag{6.6}$$

$$\varphi = 0 \quad \text{on} \quad \Gamma_a, \tag{6.7}$$

$$\mathrm{D}\,.\nu = q_2 \quad \text{on} \quad \Gamma_b, \tag{6.8}$$

$$\left.\begin{array}{l} (u_\nu - \varrho) \le 0, \quad (\sigma_\nu(u,\varphi) + h_\nu(\varphi - \varphi_F) p_\nu(u_\nu - \varrho)) \le 0, \\ (\sigma_\nu(u,\varphi) + h_\nu(\varphi - \varphi_F) p_\nu(u_\nu - \varrho))\;(u_\nu - \varrho) = 0, \end{array}\right\} \quad \text{on} \quad \Gamma_3, \tag{6.9}$$

$$\left.\begin{array}{l} \left\{\begin{array}{l} \|\sigma_\tau\| \le \mu |\,\mathrm{R}\,\sigma_\nu(u,\varphi)|, \\ \|\sigma_\tau\| < \mu |\,\mathrm{R}\,\sigma_\nu(u,\varphi)| \Rightarrow u_\tau = 0, \\ \|\sigma_\tau\| = \mu\,\mathrm{R}\,\sigma_\nu(u,\varphi)| \Rightarrow \exists \lambda \in \mathbb{R}^+ \text{ such that } \sigma_\tau = -\lambda\, u_\tau, \end{array}\right. \\ \mathrm{D}\,.\nu = \psi(u_\nu - \varrho)\phi_L(\varphi - \varphi_F), \end{array}\right\} \quad \text{on} \Gamma_3. \tag{6.10}$$

Here, (6.1) represents the constitutive law, in which $\mathfrak{F} : \Omega \times \mathbb{S}^d \to \mathbb{S}^d$ is the nonlinear elasticity operator that describes the behavior of the Hencky's materials ($\mathfrak{F}\varepsilon = k_0\,\mathrm{tr}(\varepsilon)\,\mathrm{I} + 2\,g(\|\bar{\varepsilon}\|^2)\bar{\varepsilon}$), already defined in (1.13). Then, (6.3) and (6.4) are the equilibrium equations in the case of a static problem. Then, (6.5)-(6.8) represent the mechanic and electric boundary conditions. Conditions (6.9), represent the contact with normal compliance and unilateral constraint condition. Conditions (6.10)the regularized Coulomb law and the regularized electrical conductivity condition.

When the tangential stresses on Γ_3 are supposed to be nil, and the foundation is non-conductive, *i.e.*,

$$\sigma_\tau = 0, \quad \mathrm{D}\,.\,\nu = 0 \quad \text{on}\;\Gamma_3, \tag{6.11}$$

the resulting physical problem is a frictionless contact problem of a nonlinear electro-elastic body with a non-conductive foundation, and it may be formulated classically as follows.

Problem (P_1)**.** *Find a displacement field* $u:\Omega\to\mathbb{R}^d$*, a stress field* $\sigma:\Omega\to\mathbb{S}^d$*, an electric potential* $\varphi:\Omega\to\mathbb{R}$ *and an electric displacement field* $\mathrm{D}:\Omega\to\mathbb{R}^d$ *satisfies* (6.1)-(6.9) *and* (6.11)*.*

6.2 Variational formulation

To study the problems (P_1) and (P_2), we shall use the functional frameworks introduced in Chapter 2. Then, we make the same assumptions made on the data of Problem $(\widetilde{\mathrm{P}})$ with the following changes

$(\widetilde{h}_2)$ The coefficient of friction μ satisfies

$$\mu\in L^\infty(\Gamma_3),\quad \mu\ge 0\quad\text{and}\quad \|\mu\|_{L^\infty(\Gamma_3)}\le\mu^*.$$

$(\widetilde{h}_6)$ The functions $p_\nu:\Gamma_3\times\mathbb{R}\to\mathbb{R}_+$ satisfies

(a) There exists $M_{p_\nu}>0$ such that $|p_\nu(x,u)|\le M_{p_\nu}$, for all $u\in\mathbb{R}$, a.e. $x\in\Gamma_3$.

(b) The mapping $x\mapsto p_\nu(x,u)$ is measurable on Γ_3, for all $u\in\mathbb{R}$.

(c) The mapping $x\mapsto p_\nu(x,u)=0$ for all $u\le 0$.

(d) There exists $L_{p_\nu}>0$ such that $|p_\nu(x,u_1)-p_\nu(x,u_2)|\le L_{p_\nu}|u_1-u_2|$, for all u_1, $u_2\in\mathbb{R}$, a.e. $x\in\Gamma_3$.

$(\widetilde{h}_7)$ The functions $h_\nu:\Gamma_3\times\mathbb{R}\to\mathbb{R}_+$ satisfies

(a) There exists $M_{h_\nu}>0$ such that $|h_\nu(x,u)|\le M_{h_\nu}$, for all $u\in\mathbb{R}$, a.e. $x\in\Gamma_3$.

(b) The mapping $x\mapsto h_\nu(x,u)$ is measurable on Γ_3, for all $u\in\mathbb{R}$.

(c) There exists $L_{h_\nu}>0$ such that $|h_\nu(x,u_1)-h_\nu(x,u_2)|\le L_{h_\nu}|u_1-u_2|$, for all u_1, $u_2\in\mathbb{R}$, a.e. $x\in\Gamma_3$.

$(\widetilde{h}_8)$ The mapping $R:H_{\Gamma_3}^{-\frac{1}{2}}\to L^2(\Gamma_3)$ is linear and continuous with $\|R\|=c_R$.

$(\widetilde{h}_9)$ The densities of the body force, surface traction, volume electric charge, surface electric charge and the potential of the foundation have the regularity

$$f_0\in L^2(\Omega)^d,\quad f_2\in L^2(\Gamma_2)^d,\quad q_0\in L^2(\Omega),\quad q_2\in L^2(\Gamma_b)\text{ and }\varphi_F\in L^2(\Gamma_3).$$

Next, we define the mappings $j_1 : V \times W \times V \to \mathbb{R}$, $j_2 : V \times W \times V \to \mathbb{R}$ and $j : V \times W \times V \to \mathbb{R}$, respectively, by

$$j_1(u, \varphi, v) = \int_{\Gamma_3} h_\nu(\varphi - \varphi_F) p_\nu(u_\nu - \varrho)\, v_\nu \, da, \tag{6.12}$$

$$j_2(u, \varphi, v) = \int_{\Gamma_3} \mu\, |\mathrm{R}\, \sigma_\nu(u, \varphi)|\, \|v_\tau\| \, da, \tag{6.13}$$

$$j(u, \varphi, v) = j_1(u, \varphi, v) + j_2(u, \varphi, v). \tag{6.14}$$

Keeping in mind the assumptions $(\widetilde{h}_2)$, $(\widetilde{h}_6)(b)$ and $(\widetilde{h}_7)(b)$ and $(\widetilde{h}_8)$ it follows that the integrals in (6.12)-(6.13) are well-defined. Thus, with these notations and a standard procedure based on Green's formula, we can derive the following variational formulation of the physical problems (P_1) and (P_2).

Problem (PV_1). *Find a displacement field $u \in K$ and an electric potential $\varphi \in W$ such that:* for all $v \in K$ and $\xi \in W$ we have

$$(\mathfrak{F}\varepsilon(u), \varepsilon(v) - \varepsilon(u))_{\mathcal{H}} + (\mathcal{E}^*\nabla\varphi, \varepsilon(v) - \varepsilon(u))_H + j_1(u, \varphi, v) - j_1(u, \varphi, u) \geq (f, v - u)_V, \tag{6.15}$$

$$(\beta\nabla\varphi, \nabla\xi)_H - (\mathcal{E}\varepsilon(u), \nabla\xi)_H = (q, \xi)_W. \tag{6.16}$$

Problem (PV_2). *Find a displacement field $u \in K$ and an electric potential $\varphi \in W$ such that:* for all $v \in K$ and $\xi \in W$, we have

$$(\mathfrak{F}\varepsilon(u), \varepsilon(v) - \varepsilon(u))_{\mathcal{H}} + (\mathcal{E}^*\nabla\varphi, \varepsilon(v) - \varepsilon(u))_H + j(u, \varphi, v) - j(u, \varphi, u) \geq (f, v - u)_V, \tag{6.17}$$

$$(\beta\nabla\varphi, \nabla\xi)_H - (\mathcal{E}\varepsilon(u), \nabla\xi)_H + \ell(u, \varphi, \xi) = (q, \xi)_W. \tag{6.18}$$

6.3 Existence and uniqueness results

The unique solvability of problems (PV_1) and (PV_2) follows from the followings results.

Theorem 6.1 *Assume that $(\widetilde{h}_1)$, $(\widetilde{h}_3) - (\widetilde{h}_4)$, $(\widetilde{h}_6)(a) - (c)$, $(\widetilde{h}_7)(a) - (b)$ and $(\widetilde{h}_9)$ hold. Then,*

1. *Problem (PV_1) has at least one solution.*

2. *Under the assumption $(\widetilde{h}_6)(d)$ and $(\widetilde{h}_7)(c)$ there exists $L_1^* > 0$, such that if $M_{h_\nu} L_{p_\nu} + L_{h_\nu} M_{p_\nu} < L_1^*$ the solution is unique.*

Theorem 6.2 *Assume that* $(\widetilde{h}_1) - (\widetilde{h}_5)(a) - (c)$, $(\widetilde{h}_6)(a) - (c)$, $(\widetilde{h}_7)(a) - (b)$ *and* $(\widetilde{h}_8) - (\widetilde{h}_9)$ *hold. Then,*

1. *Problem* (PV_2) *has at least one solution.*

2. *Under the assumption* $(h_5)(d)$, $(h_6)(d)$ *and* $(h_7)(c)$ *there exists* $L_2^* > 0$, *such that if* $M_{h_\nu}L_{p_\nu} + L_{h_\nu}M_{p_\nu} + \mu^* + L_\psi L + M_\psi < L_2^*$ *the solution is unique.*

The proofs of Theorem6.1 and Theorem6.2 can be obtained in a similar way the proof of Theorem 5.1 is obtained.

Let us consider the product space $X = V \times W$ endowed with the inner product (5.15). We define the functions $\widetilde{j}$ and $\widetilde{\ell}$ on $X \times X$ by

$$\widetilde{j}_1(x,y) = j_1(u,\varphi,v), \qquad \widetilde{j}_2(x,y) = j_2(u,\varphi,v), \qquad \widetilde{j}(x,y) = j(u,\varphi,v), \tag{6.19}$$

$$\widetilde{\ell}(x,y) = \ell(u,\varphi,\xi), \qquad f_3 = (f,q) \in X, \tag{6.20}$$

for all $x = (u,\varphi)$ and $y = (v,\xi)$ in X, where th operator A is defined in (5.14). With the above notations, we get the following equivalent problems.

Problem $(\widetilde{\mathrm{PV}}_1)$. *Find* $x = (u,\varphi) \in U$ *such that:*

$$(Ax, y-x)_X + \widetilde{j}_1(x,y) - \widetilde{j}_1(x,x) \geq (f_3, y-x)_X \quad \forall y = (v,\xi) \in U. \tag{6.21}$$

Problem $(\widetilde{\mathrm{PV}}_2)$. *Find* $x = (u,\varphi) \in U$ *such that:*

$$(Ax, y-x)_X + \widetilde{j}(x,y) - \widetilde{j}(x,x) + \widetilde{\ell}(x,y-x) \geq (f_3, y-x)_X \quad \forall y = (v,\xi) \in U. \tag{6.22}$$

Lemma 6.1 *The couple* $x = (u,\varphi) \in U$ *is a solution to Problem* (PV_1) *if and only if it is a solution to Problem* $(\widetilde{\mathrm{PV}}_1)$.

Lemma 6.2 *The couple* $x = (u,\varphi) \in U$ *is a solution to Problem* (PV_2) *if and only if it is a solution to Problem* $(\widetilde{\mathrm{PV}}_2)$.

The proofs of Lemma 6.1 and Lemma 6.2 can be obtained in a similar way the proof of Lemma 5.1 is obtained. Now, we are able to state the following existence and uniqueness results.

6.4 A convergence result

We are now interested in the problem at the limit verified by $x_{(\mu,k_e)} = (u_{(\mu,k_e)}, \varphi_{(\mu,k_e)})$ (solution to Problem $(\widetilde{\mathrm{PV}}_2)$ with μ and k_e are, respectively, the friction and electrical conductivity coefficients). When we take $\mu = 0$ and $k_e = 0$ in the conditions at the limits given by (6.10), we obtain $\sigma_\tau = 0$ and $\mathrm{D}\,.\nu = 0$ on Γ_3. We are then in the presence of a frictionless contact problem with a non-conductive foundation. The following theorem shows that $x_{(\mu,k_e)} = (u_{(\mu,k_e)}, \varphi_{(\mu,k_e)})$ converges towards the solution to Problem $(\widetilde{\mathrm{PV}}_1)$.

Theorem 6.3 *Assume that the assumptions of Theorem 6.1 and Theorem 6.2 hold. Let us denote by $x = (u, \varphi)$ and $x_{(\mu,k_e)} = (u_{(\mu,k_e)}, \varphi_{(\mu,k_e)})$ the respective solutions to Problem $(\widetilde{\mathrm{PV}}_1)$ and Problem $(\widetilde{\mathrm{PV}}_2)$. Then, we have*

$$x_{(\mu,k_e)} = (u_{(\mu,k_e)}, \varphi_{(\mu,k_e)}) \to x = (u, \varphi) \quad \textit{as } (\|\mu\|_{L^\infty(\Gamma_3)}, k_e) \to (0, 0).$$

Proof. Let $\psi_0 : \Gamma_3 \times \mathbb{R} \to \mathbb{R}_+$ the application given by

$$\psi_0(s) = \begin{cases} 0, & \text{if } s < 0, \\ \delta\ s, & \text{if } 0 \le s \le 1/\delta, \\ 1, & \text{if } s > 1/\delta, \end{cases}$$

with $\delta > 0$ is a small parameter. It's may easy to verify that ψ_0 satisfies $(\widetilde{h}_5)$ and

$$\psi(s) = k_e\ \psi_0(s),$$

where ψ is given in (1.34).
Taking $y = (0, 0)$ in (6.22), using the fact that

$$\begin{cases} \widetilde{j}_2(x_{(\mu,k_e)}, x_{(\mu,k_e)}) > 0, \\ \phi_L(\varphi_{(\mu,k_e)} - \varphi_F)(\varphi_{(\mu,k_e)} - \varphi_F) > 0, \end{cases}$$

we have

$$(Ax_{(\mu,k_e)}, x_{(\mu,k_e)})_X \le (f_3, x_{(\mu,k_e)})_X - \widetilde{j}_1(x_{(\mu,k_e)}, x_{(\mu,k_e)}) - \ell(u_{(\mu,k_e)}, \varphi_{(\mu,k_e)}, \varphi_F).$$

Taking into account the boundedness of h_ν, p_ν and ϕ_L, $(\widetilde{h}_1)$, $(\widetilde{h}_4)$ and $(\widetilde{h}_5)$, we get

$$\begin{aligned}
&m_{\mathfrak{F}}\|u_{(\mu,k_e)}\|_V^2 + m_\beta\|\varphi_{(\mu,k_e)}\|_W^2\\
&\le \|f\|_V\|u_{(\mu,k_e)}\|_V + \|q\|_W\|\varphi_{(\mu,k_e)}\|_W + M_{h_\nu}M_{p_\nu}meas(\Gamma_3)^{\frac{1}{2}}c_0\|u_{(\mu,k_e)}\|_V\\
&\qquad + k_e c_0 L_{\psi_0} L\|\varphi_F\|_{L^2(\Gamma_3)}\|u_{(\mu,k_e)}\|_V.
\end{aligned}$$

Thus,

$$\|x_{(\mu,k_e)}\|_X \le c_1 c\Big(2\|f_3\|_X + M_{h_\nu}M_{p_\nu}meas(\Gamma_3)^{\frac{1}{2}}c_0 + \underbrace{k_e c_0 M L_{\psi_0} L\|\varphi_F\|_{L^2(\Gamma_3)}}_{\longrightarrow 0 \text{ as } k_e\to 0}\Big).$$

with $c_1 = \dfrac{1}{m_A}$ and c is a constant independent of μ and k_e. This shows that the sequence $(x_{(\mu,k_e)}) = (u_{(\mu,k_e)}, \varphi_{(\mu,k_e)})$ is bounded in X, then, there exists $\widetilde{x} = (\widetilde{u}, \widetilde{\varphi}) \in X$ and a subsequence, denoted again $(x_{(\mu,k_e)}) = (u_{(\mu,k_e)}, \varphi_{(\mu,k_e)})$, such that $(x_{(\mu,k_e)})$ converge weakly to $\widetilde{x} \in X$ as $(\|\mu\|_{L^\infty(\Gamma_3)}, k_e) \to (0,0)$. Since, U is a closed convex set in a real Hilbert space X, therefore, U is weakly closed, then $\widetilde{x} \in U$. Moreover, using the compactness of the trace map $\gamma : X \to L^2(\Gamma_3)^d \times L^2(\Gamma_3)$, it follows from the weak convergence of $(x_{(\mu,k_e)})$ that $x_{(\mu,k_e)} \to \widetilde{x}$ strongly in $L^2(\Gamma_3)^d \times L^2(\Gamma_3)$ as $(\|\mu\|_{L^\infty(\Gamma_3)}, k_e) \to (0,0)$.

Next, let us prove that $\widetilde{x} = (\widetilde{u}, \widetilde{\varphi})$ is the solution Problem $(\widetilde{\mathrm{PV}}_1)$. Using (6.19) keeping in mind the proprieties of μ, R, ψ, ϕ_L, h_ν and p_ν, we get

$$\begin{aligned}
|\widetilde{j}_1(x_{(\mu,k_e)}, x_{(\mu,k_e)}) - \widetilde{j}_1(\widetilde{x},\widetilde{x})| &\le c_0\Big(L_{h_\nu}M_{p_\nu} + M_{h_\nu}L_{p_\nu}\Big)\|x_{(\mu,k_e)}\|_X\|u_{(\mu,k_e)} - \widetilde{u}\|_{L^2(\Gamma_3)^d}\\
&\quad + M_{h_\nu}M_{p_\nu}meas(\Gamma_3)^{\frac{1}{2}}\|u_{(\mu,k_e)} - \widetilde{u}\|_{L^2(\Gamma_3)}. \qquad (6.23)
\end{aligned}$$

$$|\widetilde{j}_2(x_{(\mu,k_e)}, x_{(\mu,k_e)})| \le \|\mu\|_{L^\infty(\Gamma_3)} c_{\mathrm{R}} c_0^2 \|x_{(\mu,k_e)}\|_X^2. \qquad (6.24)$$

Moreover,

$$|\widetilde{\ell}(x_{(\mu,k_e)}, y - x_{(\mu,k_e)})| \le k_e M_{\psi_0} L\ meas(\Gamma_3)^{\frac{1}{2}}\ \|\xi - \varphi_{(\mu,k_e)}\|_{L^2(\Gamma_3)}. \qquad (6.25)$$

Since $x_{(\mu,k_e)} \to \widetilde{x}$ strongly in $L^2(\Gamma_3)^d \times L^2(\Gamma_3)$, it follows from the boundedness of $(x_{(\mu,k_e)})$ in X, (6.23)-(6.25), that

$$\left.\begin{aligned}
\widetilde{j}_1(x_{(\mu,k_e)}, x_{(\mu,k_e)}) &\to \widetilde{j}_1(\widetilde{x},\widetilde{x}),\\
\widetilde{j}_2(x_{(\mu,k_e)}, x_{(\mu,k_e)}) &\to 0,\\
\widetilde{\ell}(x_{(\mu,k_e)}, y - x_{(\mu,k_e)}) &\to 0,
\end{aligned}\right\} \quad \text{as } (\|\mu\|_{L^\infty(\Gamma_3)}, k_e) \to (0,0).$$

We deduce from (6.22) that for all $y = (v, \xi) \in U$

$$\limsup_{(\|\mu\|_{L^\infty(\Gamma_3)}, k_e) \to (0,0)} (Ax_{(\mu,k_e)}, x_{(\mu,k_e)} - y)_X \leq (f_3, \widetilde{x} - y)_X + \widetilde{j}_1(\widetilde{x}, y) - \widetilde{j}_1(\widetilde{x}, \widetilde{x}).$$

On the other hand, we have for all $y = (v, \xi) \in U$,

$$\begin{aligned}
&\limsup_{(\|\mu\|_{L^\infty(\Gamma_3)}, k_e) \to (0,0)} (Ax_{(\mu,k_e)}, x_{(\mu,k_e)} - \widetilde{x})_X \\
&= \limsup_{(\|\mu\|_{L^\infty(\Gamma_3)}, k_e) \to (0,0)} \Big[(Ax_{(\mu,k_e)}, x_{(\mu,k_e)} - y)_X + (Ax_{(\mu,k_e)}, y - \widetilde{x})_X\Big] \\
&\leq \limsup_{(\|\mu\|_{L^\infty(\Gamma_3)}, k) \to (0,0)} \Big[(Ax_{(\mu,k_e)}, x_{(\mu,k_e)} - y)_X + \|Ax_{(\mu,k_e)}\|_X \|y - \widetilde{x}\|_X\Big] \\
&\leq (f_3, \widetilde{x} - y)_X + \widetilde{j}_1(\widetilde{x}, y) - \widetilde{j}_1(\widetilde{x}, \widetilde{x}) + \limsup_{(\|\mu\|_{L^\infty(\Gamma_3)}, k_e) \to (0,0)} \|Ax_{(\mu,k_e)}\|_X \|y - \widetilde{x}\|_X.
\end{aligned}$$

Note that $\|Ax_{(\mu,k_e)}\|_X$ is bounded on X, we may then substitute $y = \widetilde{x}$ into the last inequality to obtain

$$\limsup_{(\|\mu\|_{L^\infty(\Gamma_3)}, k_e) \to (0,0)} (Ax_{(\mu,k_e)}, x_{(\mu,k_e)} - \widetilde{x})_X \leq 0.$$

Therefore, by pseudo-monotonicity of A, we get

$$(A\widetilde{x}, \widetilde{x} - y)_X \leq \liminf_{(\|\mu\|_{L^\infty(\Gamma_3)}, k_e) \to (0,0)} (Ax_{(\mu,k_e)}, x_{(\mu,k_e)} - y)_X. \tag{6.26}$$

Combining (6.22) and (6.26), we deduce

$$(A\widetilde{x}, y - \widetilde{x})_X + \widetilde{j}_1(\widetilde{x}, y) - \widetilde{j}_1(\widetilde{x}, \widetilde{x}) \geq (f_3, y - \widetilde{x})_X.$$

Which means that $\widetilde{x} \in U$ is a solution to Problem $(\widetilde{\mathrm{PV}}_1)$, and from the uniqueness of the solution for this variational inequality we obtain $\widetilde{x} = x$. Since x is the unique weak limit of any subsequence of $(x_{(\mu,k_e)})$, we deduce that the whole sequence $(x_{(\mu,k_e)})$ is weakly convergent in X towards x as $(\|\mu\|_{L^\infty(\Gamma_3)}, k_e) \to (0,0)$.

Let us now prove that

$$\|x_{(\mu,k_e)} - x\|_X \to 0 \quad \text{as } (\|\mu\|_{L^\infty(\Gamma_3)}, k_e) \to (0,0).$$

To this end, let $x_{(\mu,k_e)} = (u_{(\mu,k_e)}, \varphi_{(\mu,k_e)}) \in U$ be a solution Problem $(\widetilde{\mathrm{PV}}_2)$ and $x = (u, \varphi) \in U$

a solution to Problem $(\widetilde{\mathrm{PV}}_1)$, thus we have

$$\begin{aligned}
&(Ax_{(\mu,k_e)}, x_{(\mu,k_e)} - y)_X \\
&\leq (f_3, x_{(\mu,k_e)} - y)_X + \widetilde{j}(x_{(\mu,k_e)}, y) - \widetilde{j}(x_{(\mu,k_e)}, x_{(\mu,k_e)}) + \widetilde{\ell}(x_{(\mu,k_e)}, y - x_{(\mu,k_e)}), \\
&(Ax, x - y)_X \leq (f_3, x - y)_X + \widetilde{j}_1(x, y) - \widetilde{j}_1(x, x).
\end{aligned}$$

Taking $y = x$ in the first inequality, $y = x_{(\mu,k_e)}$ in the second one and we add the two resulting inequalities, we get

$$(Ax_{(\mu,k_e)} - Ax, x_{(\mu,k_e)} - x)_X \leq G + \widetilde{\ell}(x_{(\mu,k_e)}, x - x_{(\mu,k_e)}), \tag{6.27}$$

with,

$$G = \widetilde{j}(x_{(\mu,k_e)}, x) - \widetilde{j}(x_{(\mu,k_e)}, x_{(\mu,k_e)}) + \widetilde{j}_1(x, x_{(\mu,k_e)}) - \widetilde{j}_1(x, x).$$

From (6.12)-(6.14), it is straightforward to show that

$$\begin{aligned}
G &= \widetilde{j}_1(x_{(\mu,k_e)}, x) - \widetilde{j}_1(x_{(\mu,k_e)}, x_{(\mu,k_e)}) + \widetilde{j}_1(x, x_{(\mu,k_e)}) - \widetilde{j}_1(x, x) + \widetilde{j}_2(x_{(\mu,k_e)}, x) - \widetilde{j}_2(x_{(\mu,k_e)}, x_{(\mu,k_e)}) \\
&\leq M_{h_\nu} L_{p_\nu} c_0^2 \|u_{(\mu,k_e)} - u\|_V^2 + L_{h_\nu} M_{p_\nu} \widetilde{c}_0 c_0 \|\varphi_{(\mu,k_e)} - \varphi\|_W \|u_{(\mu,k_e)} - u\|_V \\
&\quad + \|\mu\|_{L^\infty(\Gamma_3)} c_{\mathrm{R}} c_0^2 \|u_{(\mu,k_e)}\|_V \|u_{(\mu,k_e)} - u\|_V \\
&\leq \Big(M_{h_\nu} L_{p_\nu} c_0^2 + L_{h_\nu} M_{p_\nu} \widetilde{c}_0 c_0\Big) \|x_{(\mu,k_e)} - x\|_X^2 + \|\mu\|_{L^\infty(\Gamma_3)} c_{\mathrm{R}} c_0^2 \|x_{(\mu,k_e)}\|_X \|x_{(\mu,k_e)} - x\|_X.
\end{aligned} \tag{6.28}$$

So, we combine (6.27), (6.28) and the strong monotonicity of the operator A to deduce that

$$\begin{aligned}
&m_A \|x_{(\mu,k_e)} - x\|_X^2 \\
&\leq \Big(M_{h_\nu} L_{p_\nu} c_0^2 + L_{h_\nu} M_{p_\nu} \widetilde{c}_0 c_0\Big) \|x_{(\mu,k_e)} - x\|_X^2 + \|\mu\|_{L^\infty(\Gamma_3)} c_{\mathrm{R}} c_0^2 \|x_{(\mu,k_e)}\|_X \|x_{(\mu,k_e)} - x\|_X \\
&\quad + k_e M_{\psi_0} L \widetilde{c}_0 \; meas(\Gamma_3)^{\frac{1}{2}} \|x_{(\mu,k_e)} - x\|_X.
\end{aligned}$$

Thus,

$$\|x_{(\mu,k_e)} - x\|_X \leq c\Big(\|\mu\|_{L^\infty(\Gamma_3)} + k_e\Big).$$

with c is a constant independent of μ and k_e. This proves that $(x_{(\mu,k_e)})$ converges strongly to x in X as μ and k_e converge towards zero. □

6.5 Iteration method

The iteration method for problems $(\widetilde{\text{PV}}_1)$ and $(\widetilde{\text{PV}}_2)$ consists of the following procedures, respectively

$$\begin{cases} \text{Given an initial guess } x_0 = (u_0, \varphi_0) \in U, \text{ find } x_{n+1} = (u_{n+1}, \varphi_{n+1}) \in U \text{ such that} \\ B(x_n; x_{n+1}, y - x_{n+1}) + \tilde{j}_1(x_n, y) - \tilde{j}_1(x_n, x_{n+1}) \geq (f_3, y - x_{n+1})_X, \end{cases} \tag{6.29}$$

$$\begin{cases} \text{Given an initial guess } x_0 = (u_0, \varphi_0) \in U, \text{ find } x_{n+1} = (u_{n+1}, \varphi_{n+1}) \in U \text{ such that} \\ B(x_n; x_{n+1}, y - x_{n+1}) + \tilde{j}(x_n, y) - \tilde{j}(x_n, x_{n+1}) + \tilde{\ell}(x_n, y - x_{n+1}) \geq (f_3, y - x_{n+1})_X, \end{cases} \tag{6.30}$$

for all $y = (v, \xi)$ in U, where the operator $B : U \times X \times X \to \mathbb{R}$ is the bilinear operator defined in Section 5.4.

We have the following convergence results.

Theorem 6.4 *Under assumptions of Theorem 6.1, the iteration method (6.29) converges, i.e.,*

$$\|x_n - x\|_X \to 0 \quad \text{as } n \to +\infty,$$

where x is the unique solution to Problem $(\widetilde{\text{PV}}_1)$.

Theorem 6.5 *Under assumptions of Theorem 6.2, the iteration method (6.30) converges, i.e.,*

$$\|x_n - x\|_X \to 0 \quad \text{as } n \to +\infty,$$

where x is the unique solution to Problem $(\widetilde{\text{PV}}_2)$.

The proofs of Theorem 6.4 and Theorem 6.5 can be obtained in a similar way the proof of Theorem 5.2 is obtained.

6.6 Augmented Lagrangian for the iterative problems

By subtracting the equation (6.18) from the inequality (6.17), we obtain

$$(\check{A}x, y - x)_X + \tilde{j}(x, y) - \tilde{j}(x, x) + \check{\ell}(x, y - x) \geq (\check{f}_3, y - x)_X \quad \forall y = (v, \xi) \in U, \tag{6.31}$$

where $\check{A}$, $\check{f}_3$ and $\check{\ell}$ are defined in (5.27)-(5.29).

Lemma 6.3 *The variational formulations* (6.31) *and* (6.22) *are equivalent.*

The proof of Lemma 6.3 can be obtained in a similar way the proof of Lemma 5.2 is obtained.

Next, by applying the iteration method (6.30) presented in the precedent section to the variational inequality (6.31), we get the following iterative problem

$$\begin{cases} \text{Given an initial guess } x_0 = (u_0, \varphi_0) \in U, \text{ find } x_{n+1} = (u_{n+1}, \varphi_{n+1}) \in U \text{ such that} \\ \check{B}(x_n; x_{n+1}, y - x_{n+1}) + \tilde{j}(x_n, y) - \tilde{j}(x_n, x_{n+1}) + \check{\ell}(x_n, y - x_{n+1}) \geq (\check{f}_3, y - x_{n+1})_X, \end{cases} \tag{6.32}$$

Hence, an equivalent constrained minimization problems of (6.32) can be formulated. The proposed minimization problem is as follows

$$\begin{cases} \textit{Find } x = (u, \varphi) \in U \textit{ such that} \\ J_n(x) + \tilde{j}_{2,n}(x) \leq J_n(y) + \tilde{j}_{2,n}(y) \quad \forall y = (v, \xi) \in U, \end{cases} \tag{6.33}$$

J_n is the piezoelectric deformation energy functional due to non-frictional effects given by

$$J_n(y) = \frac{1}{2}\check{B}(x_n; y, y) - (f_{1,n}, y)_X \quad \forall y = (v, \xi) \in X,$$

where

$$(f_{1,n}, y)_X = (\check{f}_3, y)_X - \tilde{j}_1(x_n, y) - \check{\ell}(x_n, y) \quad \forall y = (v, \xi) \in X.$$

$$\tilde{j}_{1,n}(y) = \tilde{j}_1(x_n, y), \quad \tilde{j}_{2,n}(y) = \tilde{j}_2(x_n, y) \quad \forall y = (v, \xi) \in X.$$

The quadratic functional J_n is strictly convex and Gateaux differentiable on X. Moreover, the friction functional $\tilde{j}_{2,n}$ is convex, and lower semi-continuous on X, thus, the existence of a unique solution to problem (6.33) is guaranteed by Theorem A.10.

Let $\mathfrak{p} = (\mathfrak{p}_f, \mathfrak{p}_c)$, where $\mathfrak{p}_f$ (friction) and $\mathfrak{p}_c$ (contact) are auxiliary variables. we introduce the set

$$C = \{\mathfrak{p}_c \in L^2(\Gamma_3)\,;\ (\mathfrak{p}_c - \varrho) \leq 0 \text{ on } \ \Gamma_3\},$$

and the characteristic functional $I_C : L^2(\Gamma_3) \to \mathbb{R} \cup \{+\infty\}$ of the set C, is defined by

$$I_C(\mathfrak{p}_c) = \begin{cases} 0, & \text{if } \mathfrak{p}_c \in C, \\ +\infty, & \text{if } \mathfrak{p}_c \notin C. \end{cases}$$

It is easy to see that the problem giving in (6.32) is equivalent to the following constrained minimization problem

Find $x = (u, \varphi) \in X$ *and* $\mathfrak{p} = (\mathfrak{p}_f, \mathfrak{p}_c) \in L^2(\Gamma_3)^2$ *such that, for all* $y = (v, \xi) \in X$ *and* $\mathfrak{q} = (\mathfrak{q}_f, \mathfrak{q}_c) \in L^2(\Gamma_3)^2$

$$J_n(x) + \widetilde{j}_{2,n}(\mathfrak{p}_f) + I_C(\mathfrak{p}_c) \leq J_n(y) + \widetilde{j}_{2,n}(\mathfrak{q}_f) + I_C(\mathfrak{q}_c), \tag{6.34}$$

$$\left.\begin{aligned} u_\nu - \mathfrak{p}_c = 0, \\ u_\tau - \mathfrak{p}_f = 0, \end{aligned}\right\} \quad \text{on } \Gamma_3. \tag{6.35}$$

From (6.34)-(6.35) the Augmented Lagrangian functional $\mathcal{L}_{2,r}$ is defined over $X \times L^2(\Gamma_3)^2 \times L^2(\Gamma_3)^2$ by

$$\begin{aligned} \mathcal{L}_{2,r}(y, \mathfrak{q}; \theta) = J_n(y) + \widetilde{j}_{2,n}(\mathfrak{q}_f) + I_C(\mathfrak{q}_c) + (\theta_c, v_\nu - \mathfrak{q}_c)_{L^2(\Gamma_3)} \\ + (\theta_f, v_\tau - \mathfrak{q}_f)_{L^2(\Gamma_3)} + \frac{r}{2}\|v_\nu - \mathfrak{q}_c\|^2_{L^2(\Gamma_3)} + \frac{r}{2}\|v_\tau - \mathfrak{q}_f\|^2_{L^2(\Gamma_3)}, \end{aligned}$$

where the constant $r > 0$ is the penalty parameter and $\theta = (\theta_f, \theta_c)$. Uzawa block relaxation method is obtained as follows, starting with $\mathfrak{p}^0$ and λ^0

$$\mathcal{L}_{2,r}(x^{k+1}, \mathfrak{p}^k; \lambda^k) = \min_y \mathcal{L}_{2,r}(y, \mathfrak{p}^k; \lambda^k), \tag{6.36}$$

$$\mathcal{L}_{2,r}(x^{k+1}, \mathfrak{p}^{k+1}; \lambda^k) = \min_{\mathfrak{p}} \mathcal{L}_{2,r}(x^{k+1}, \mathfrak{q}; \lambda^k), \tag{6.37}$$

$$\lambda^{k+1} = \lambda^k + r(u^{k+1} - \mathfrak{p}^{k+1}). \tag{6.38}$$

The solution of (6.36) can be characterized by the Euler-Lagrange equation [54], since $y \mapsto \mathcal{L}_{2,r}(y, \mathfrak{p}; \theta)$ is convex and differentiable

$$\begin{aligned} \check{B}(x_n; x^{k+1}, y) + r(u_\nu^{k+1}, v_\nu)_{L^2(\Gamma_3)} + r(u_\tau^{k+1}, v_\tau)_{L^2(\Gamma_3)} \\ = (f_{1,n}, y)_X + (r\mathfrak{p}_c^k - \lambda_c^k, v_\nu)_{L^2(\Gamma_3)} + (r\mathfrak{p}_f^k - \lambda_f^k, v_\tau)_{L^2(\Gamma_3)}. \end{aligned}$$

In (6.37) the subproblems in $\mathfrak{p}_f$ and $\mathfrak{p}_c$ are uncoupled. Consequently, we can minimize the functional $\mathfrak{p} \to \mathcal{L}_{2,r}(x^{k+1}, \mathfrak{p}; \lambda^k)$ separately in $\mathfrak{p}_c$ and $\mathfrak{p}_f$. For the contact subproblem, straightforward calculations using Karush-Kuhn-Tucker optimality conditions yield to (see [54])

$$\mathfrak{p}_c^{k+1} = u_\nu^{k+1} + \frac{1}{r}\Big[\lambda_c^k - \Big(\lambda_c^k + r(u_\nu^{k+1} - \varrho)\Big)^+\Big].$$

For the friction subproblem, using the Fenchel duality theory (see [54, 82]), we get

$$\mathfrak{p}_f^{k+1} = \begin{cases} \dfrac{|\lambda_f^k + r u_\tau^{k+1}| - s_n}{r|\lambda_f^k + r u_\tau^{k+1}|}(\lambda_f^k + r u_\tau^{k+1}), & if \;\; |\lambda_f^k + r u_\tau^{k+1}| > s_n, \\ 0, & if \;\; |\lambda_f^k + r u_\tau^{k+1}| \leq s_n, \end{cases}$$

where

$$s_n = \mu |\, \mathrm{R}\, \sigma_\nu(u_n, \varphi_n)|.$$

With the previous results, we can now present our Uzawa block relaxation method **Algorithm 7**. We iterate until the relative error (5.38) on x^k, $\mathfrak{p}_f^k$ and $\mathfrak{p}_c^k$ is sufficiently "small".

Algorithm 7. Uzawa block relaxation for (6.33).

Initialization. $r > 0$, $\mathfrak{p}^0 = (\mathfrak{p}_f^0, \mathfrak{p}_c^0)$ and $\lambda^0 = (\lambda_f^0, \lambda_c^0)$ are given.

Iteration $k > 0$. Compute successively $x^{k+1} = (u^{k+1}, \varphi^{k+1})$, $\mathfrak{p}^{k+1} = (\mathfrak{p}_f^{k+1}, \mathfrak{p}_c^{k+1})$ and $\lambda^{k+1} = (\lambda_f^{k+1}, \lambda_c^{k+1})$ as follows

Step 1. Find $x^{k+1} = (u^{k+1}, \varphi^{k+1}) \in X$ such that

$$\begin{aligned} &\check{B}(x_n; x^{k+1}, y) + r(u_\nu^{k+1}, v_\nu)_{L^2(\Gamma_3)} + r(u_\tau^{k+1}, v_\tau)_{L^2(\Gamma_3)} \\ &\qquad = (f_{1,n}, y)_X + (r\mathfrak{p}_c^k - \lambda_c^k, v_\nu)_{L^2(\Gamma_3)} + (r\mathfrak{p}_f^k - \lambda_f^k, v_\tau)_{L^2(\Gamma_3)}, \end{aligned}$$

Step 2. Compute the auxiliary contact and friction variables

$$\mathfrak{p}_c^{k+1} = u_\nu^{k+1} + \frac{1}{r}\Big[\lambda_c^k - \Big(\lambda_c^k + r(u_\nu^{k+1} - \varrho)\Big)^+\Big],$$

$$\mathfrak{p}_f^{k+1} = \begin{cases} \dfrac{|\lambda_f^k + r u_\tau^{k+1}| - s_n}{r|\lambda_f^k + r u_\tau^{k+1}|}(\lambda_f^k + r u_\tau^{k+1}), & if \;\; |\lambda_f^k + r u_\tau^{k+1}| > s_n, \\ 0, & if \;\; |\lambda_f^k + r u_\tau^{k+1}| \leq s_n. \end{cases}$$

Step 3. Update the Lagrange multipliers

$$\lambda_c^{k+1} = \lambda_c^k + r(u_\nu^{k+1} - \mathfrak{p}_c^{k+1}),$$

$$\lambda_f^{k+1} = \lambda_f^k + r(u_\tau^{k+1} - \mathfrak{p}_f^{k+1}).$$

With the above results, the solution method for (6.31) is presented in **Algorithm 6**.

Algorithm 6. Solution for (6.31).

Initialization. s_0 and $x_0 = (u_0, \varphi_0) \in X$ are given.

Iteration $n \geq 0$. Compute x_{n+1} and s_{n+1} successively as follows

- Compute $x_{n+1} = (u_{n+1}, \varphi_{n+1}) \in X$ using **Algorithm 7**.
- Update $s_{n+1} = \mu |\, \mathrm{R}\, \sigma_\nu(u_{n+1}, \varphi_{n+1})|$ and $(f_{1,n+1}, \,.\,)_X = (\check{f}_3, \,.\,)_X - \tilde{j}_1(x_{n+1}, .) - \check{\ell}(x_{n+1}, \,.\,)$.

The fixed-point iteration terminates if the relative error (3.46) on s_n becomes sufficiently "small".

An equivalent variational formulation to (6.21) is given by

$$(\check{A}x, y - x)_X + \tilde{j}_1(x, y) - \tilde{j}_1(x, x) \geq (\check{f}_3, y - x)_X \quad \forall y = (v, \xi) \in U, \tag{6.39}$$

In order to define the solution method for (6.39), we follow the same steps used to define that of (6.31). The resulting algorithm is given by

Algorithm 8. Solution for (6.39).

Initialization. $x_0 = (u_0, \varphi_0) \in X$ is given.

Iteration $n \geq 0$. Compute $x_{n+1} \in X$ using **Algorithm 9**. Then, update

$$(f_{2,n+1}, \,.\,)_X = (\check{f}_3, \,.\,)_X - \tilde{j}_1(x_{n+1}, .).$$

The fixed-point iteration terminates if the relative error on x_n becomes sufficiently "small", *i.e.*,

$$\frac{\|x_{n+1} - x_n\|^2_{L^2(\Omega}}{\|x_{n+1}\|^2_{L^2(\Omega)}} < \epsilon^2_{fp}. \tag{6.40}$$

where

Algorithm 9. Uzawa block relaxation.

Initialization. $r > 0$, $\mathfrak{p}_c^0$ and λ_c^0 are given.

Iteration $k > 0$. Compute successively $x^{k+1} = (u^{k+1}, \varphi^{k+1})$, $\mathfrak{p}_c^{k+1}$ and λ_c^{k+1} as follows

Step 1. Find $x^{k+1} = (u^{k+1}, \varphi^{k+1}) \in X$ such that

$$\check{B}(x_n; x^{k+1}, y) + r(u_\nu^{k+1}, v_\nu)_{L^2(\Gamma_3)} = (f_{2,n}, y)_X + (r\mathfrak{p}_c^k - \lambda_c^k, v_\nu)_{L^2(\Gamma_3)}.$$

Step 2. Compute the auxiliary contact variable

$$\mathfrak{p}_c^{k+1} = u_\nu^{k+1} + \frac{1}{r}\Big[\lambda_c^k - \Big(\lambda_c^k + r(u_\nu^{k+1} - \varrho)\Big)^+\Big].$$

Step 3. Update the Lagrange multipliers

$$\lambda_c^{k+1} = \lambda_c^k + r(u_\nu^{k+1} - \mathfrak{p}_c^{k+1}).$$

We iterate until the relative error on x^k and $\mathfrak{p}_c^k$ is sufficiently "small", *i.e.*,

$$\frac{\|x^{k+1} - x^k\|^2_{L^2(\Omega)} + \|\mathfrak{p}_c^{k+1} - \mathfrak{p}_c^k\|^2_{L^2(\Gamma_3)}}{\|x^{k+1}\|_{L^2(\Omega)} + \|\mathfrak{p}_c^{k+1}\|^2_{L^2(\Gamma_3)}} < \epsilon^2, \tag{6.41}$$

Part IV

Numerical simulations

Chapter 7

Numerical experiments for elasto-plastic materials

In this chapter, the objective is to present a number of numerical experiments on two-dimensional test problems aimed to illustrate, among other things, the convergence and the performance of the algorithms presented in subsection 3.5.3 and Section 4.3 using $\mathbb{P}_1$ triangular finite element method. These simulations concern the static contact problem with Coulomb's friction law of an elasto-plastic body with an obstacle perfectly rigid, which was analyzed in Part II. We also compare the results obtained by the numerical methods "Augmented Lagrangian" and "penalty approach" used, respectively, in Sections 3.5 and 4.1.

7.1 Resolution method

The program code proposed here employs $\mathbb{P}_1$ triangular finite element method to calculate a numerical solution U approximating the solution u of Problem (PV). Let $\Omega \subset \mathbb{R}^2$ be a bounded domain with a polygonal edge Γ. Let $\mathcal{T}$ be a regular triangularization of Ω, *i.e.*, $\overline{\Omega} = \bigcup_{T\in\mathcal{T}} \mathrm{T}$ and $\mathcal{N}$ is the set of nodes in $\mathcal{T}$. We set $(\Phi_1, \Phi_2, \ldots, \Phi_{2\times \mathrm{N}_t}) = (\psi_1 e_1, \psi_1 e_2, \ldots, \psi_{\mathrm{N}_t} e_1, \psi_{\mathrm{N}_t} e_2)$ the nodal base of the finite dimension space $\mathcal{S}$, N_t is the total number of nodes in the mesh and $\psi_i,\ \ i = 1, 2, \ldots, 2 \times \mathrm{N}_t$ is the scalar hat function of the i^{th} node in the triangularization $\mathcal{T}$, *i.e.*, $\psi_i(i) = 1$ and $\psi_i(j) = 0$ for any $j \in \mathcal{N}$ with $j \neq i$. Taking into account the conditions at the edges

$$\int_\Omega \varepsilon(\Phi_i) : \sigma(u^h)\, dx = \int_\Omega f.\Phi_i\, dx, \quad i = 1, 2, \ldots, 2 \times \mathrm{N}_t \tag{7.1}$$

For the vector of discrete displacements, we have $u^h = \sum_{j=1}^{2\times \mathrm{N}_t} \mathrm{U}_j\, \Phi_j$ and by (7.1) we obtain the following system of equations

$$\sum_{j=1}^{2\times \mathrm{N}_t} \left(\int_\Omega \varepsilon(\Phi_i) : \sigma(\Phi_j)\, dx \right) \mathrm{U}_j = \int_\Omega f.\Phi_i\, dx, \quad i = 1, 2, \dots, 2 \times \mathrm{N}_t \tag{7.2}$$

The coefficient matrix (global matrix) $\mathrm{A} = (\mathrm{A}_{ij}) \in \mathbb{R}^{2\times \mathrm{N}_t \times 2\times \mathrm{N}_t}$ and the second member $b = (b_i) \in \mathbb{R}^{2\times \mathrm{N}_t}$ are defined as follows

$$\mathrm{A}_{ij} = \int_\Omega \varepsilon(\Phi_i) : \sigma(\Phi_j)\, dx, \quad \text{and} \quad b_i = \int_\Omega f.\Phi_i\, dx. \tag{7.3}$$

Using the space $\mathcal{S}$ and its nodal base, the integrals in (7.3) can be calculated as a sum across all the elements, *i.e.*, for $i, j = 1, 2, \dots, 2 \times \mathrm{N}_t$

$$\mathrm{A}_{ij} = \sum_{T\in\mathcal{T}} \int_T \varepsilon(\Phi_i) : \sigma(\Phi_j)\, dx, \tag{7.4}$$

$$b_i = \sum_{T\in\mathcal{T}} \int_T f.\Phi_i\, dx. \tag{7.5}$$

We consider the representation $\eta : V \to L^2(\Omega)$ of the linear tensor of the deformations

$$\eta(u) = \begin{pmatrix} \dfrac{\partial u_1}{\partial x_1} \\ \dfrac{\partial u_2}{\partial x_2} \\ \dfrac{\partial u_2}{\partial x_1} + \dfrac{\partial u_1}{\partial x_2} \end{pmatrix} = \begin{pmatrix} \varepsilon_{11}(u) \\ \varepsilon_{22}(u) \\ 2\varepsilon_{12}(u) \end{pmatrix}. \tag{7.6}$$

For

$$\sigma = \left(k_0 - \frac{2}{3} g(\|\overline{\varepsilon}(u)\|^2)\right) \mathrm{tr}(\varepsilon(u))\, \mathrm{I} + 2g(\|\overline{\varepsilon}(u)\|^2)\varepsilon(u),$$

we have the following relationship

$$\begin{pmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{12} \end{pmatrix} = \begin{pmatrix} (k_0 + \frac{4}{3} g(\|\overline{\varepsilon}(u)\|^2)) & (k_0 - \frac{2}{3} g(\|\overline{\varepsilon}(u)\|^2)) & 0 \\ (k_0 - \frac{2}{3} g(\|\overline{\varepsilon}(u)\|^2)) & (k_0 + \frac{4}{3} g(\|\overline{\varepsilon}(u)\|^2)) & 0 \\ 0 & 0 & g(\|\overline{\varepsilon}(u)\|^2) \end{pmatrix} \begin{pmatrix} \varepsilon_{11}(u) \\ \varepsilon_{22}(u) \\ 2\varepsilon_{12}(u) \end{pmatrix} = \mathbb{C}_1 \eta(u), \tag{7.7}$$

where $g(\,.\,)$ is the nonlinear function (plasticity function), witch has the form

$$g(\xi) = \begin{cases} \mu_1, & \text{if } \xi \leq \xi_0, \\ \mu_1 \dfrac{\xi_0}{\xi}(\ln(\dfrac{\xi}{\xi_0}) + 1), & \text{if } \xi > \xi_0. \end{cases} \tag{7.8}$$

i.e., The material behaves linearly for sufficiently small strains (see [64]), with

$$\mu_1 = \frac{\mathrm{E}}{(2+2\nu)}, \quad k_0 = \frac{\mathrm{E}}{(3-6\nu)}, \tag{7.9}$$

where E, ν and ξ_0 are, respectively, Young's modulus, Poisson's coefficient and the elasticity limit.

Remark 7.1 *We assume that we have a semi-infinite body along the X_1 axis, in an orthogonal reference frame (O, X_1, X_2, X_3). The mechanical loads applied to the body are supposed to be constant along the X_1 direction. As a consequence, the tensors ε and σ are nil along X_1. In other words, the material is in a state of plane deformation with respect to the plane (O, X_2, X_3), we get then*

$$\varepsilon_{11} = \varepsilon_{12} = \varepsilon_{13} = 0. \tag{7.10}$$

Thus, we obtain

$$\varepsilon(v) : \mathbb{C}_1\eta(u) = \sigma_{22}\varepsilon_{22}(v) + \sigma_{33}\varepsilon_{33}(v) + \underbrace{\sigma_{23}\varepsilon_{23}(v) + \sigma_{32}\varepsilon_{32}(v)}_{=2\sigma_{23}\varepsilon_{23}(v)} = \eta^\top(v)\mathbb{C}_1\eta(u). \tag{7.11}$$

To carry out specific calculations, we have

$$\begin{aligned} \|\overline{\varepsilon}(u)\|^2 &= \|\varepsilon(u)\|^2 - \frac{1}{3}|\operatorname{tr}(\varepsilon(u))|^2 \\ &= \varepsilon(u) : \Big(\varepsilon(u) - \frac{1}{3}\operatorname{tr}(\varepsilon(u))\,\mathrm{I}\Big) \\ &= \frac{1}{3}\eta^\top(u)\mathbb{C}_2\eta(u), \end{aligned} \tag{7.12}$$

with

$$\mathbb{C}_2 = \begin{pmatrix} 2 & -1 & 0 \\ -1 & 2 & 0 \\ 0 & 0 & \frac{3}{2} \end{pmatrix}.$$

Let $s_1, \ldots, s_N$ be the vertices of an element T. For two-dimensional problems, we have $2 \times \mathrm{N}$ basic functions with support in T, named

$$\Phi_1 = \psi_{s_1} e_1, \quad \Phi_2 = \psi_{s_1} e_2,$$
$$\vdots$$
$$\Phi_{2\times\mathrm{N}-1} = \psi_{s_\mathrm{N}} e_1, \quad \Phi_{2\times\mathrm{N}} = \psi_{s_\mathrm{N}} e_2,$$

we note $\mathrm{K}(\mathrm{T}) = (\mathrm{K}(\mathrm{T})_{kl}),\ k,\, l = 1, \ldots, 2\times\mathrm{N}$ the local elementary matrix on each element T,

$$\mathrm{K}(\mathrm{T})_{kl} = \int_\mathrm{T} \eta^\top(\Phi_i)\mathbb{C}_1\eta(\Phi_j)\, dx = |\,\mathrm{T}\,|\eta^\top(\Phi_i)\mathbb{C}_1\eta(\Phi_j). \tag{7.13}$$

If we assume that the element T admits three vertices s_1, s_2 and s_3, we have

$$\eta(\sum_{j=1}^{6} u_j\Phi_j)_{|\mathrm{T}} = \begin{pmatrix} \frac{\partial\varphi_{s_1}}{\partial x_1} & 0 & \frac{\partial\varphi_{s_2}}{\partial x_1} & 0 & \frac{\partial\varphi_{s_3}}{\partial x_1} & 0 \\ 0 & \frac{\partial\varphi_{s_1}}{\partial x_2} & 0 & \frac{\partial\varphi_{s_2}}{\partial x_2} & 0 & \frac{\partial\varphi_{s_3}}{\partial x_2} \\ \frac{\partial\varphi_{s_1}}{\partial x_2} & \frac{\partial\varphi_{s_1}}{\partial x_1} & \frac{\partial\varphi_{s_2}}{\partial x_2} & \frac{\partial\varphi_{s_2}}{\partial x_1} & \frac{\partial\varphi_{s_3}}{\partial x_2} & \frac{\partial\varphi_{s_3}}{\partial x_1} \end{pmatrix} \begin{pmatrix} u_1 \\ \vdots \\ u_6 \end{pmatrix} = R \begin{pmatrix} u_1 \\ \vdots \\ u_6 \end{pmatrix}. \tag{7.14}$$

Thus, according to (7.13)-(7.14), we get

$$\mathrm{K}(\mathrm{T}) = |\,\mathrm{T}\,|\, R^\top\, \mathbb{C}_1\, R. \tag{7.15}$$

7.2 Presentation of numerical results

Through this section, the coefficient of friction is chosen in way to satisfy the assumption (h_2), whose form is as follows

$$\mu(\|u_{n,\tau}\|) = 0.5 + \frac{0.2}{\rho \times \|u_{n,\tau}\| + 2}, \tag{7.16}$$

where ρ is a positive parameter. The tolerances in the stopping criteria (3.45) and (3.46) are

$$\epsilon = 10^{-5}, \;\; \epsilon_{fp} = 10^{-5}.$$

7.2.1 Augmented Lagrangian method

In this subsection, we present two numerical examples, the first one treats a problem of frictional contact through Hertz problem, and the second deals with an academic example of a parallelepiped bar resting on a rigid foundation.

Example 7.1 *In general, it is particularly difficult to find an analytical solution for contact problems due to their complexity. However, some of the problems studied by Hertz in [65] deviate from this rule.*

Let us consider an infinitely long cylinder with radius $R = 8$ resting in a rigid foundation and subjected to a uniform load along its top of intensity $f_2 = (0, -1600)$ (force unit per area unit). The body force f_0 is assumed to be zero. The cylinder is supposed made from a homogeneous, isotropic, elastic material and its behavior is linear with Young's modulus $\mathrm{E} = 2000$ *and Poisson's ratio $\nu = 0.3$. The Hertz solution yields a contact pressure*

$$p(x) = \frac{2\|f_2\|}{\pi b^2}\sqrt{b^2 - x_1^2},$$

(see [54]) where b is the half-width of the contact surface defined by

$$b = 2\sqrt{\|f_2\| R \frac{(1-\nu^2)}{\pi \mathrm{E}}}.$$

Only quarter of the cylinder section is consider in the finite element discretization, for symmetry reasons, Figure 7.1. The contact surface is $\Gamma_3 = \{(x_1, x_2) \in (0, 8)^2\,;\; x_1^2 + x_2^2 = 64\}$ and we prescribe $u_\tau = 0$ on $\Gamma_1 = \{0\} \times (0, 8)$.

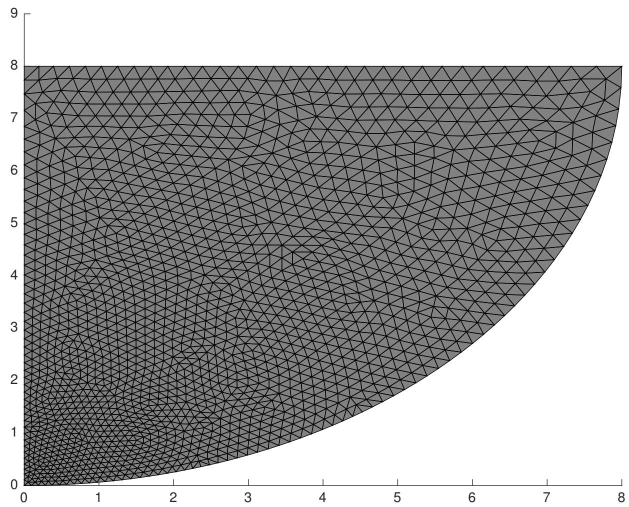
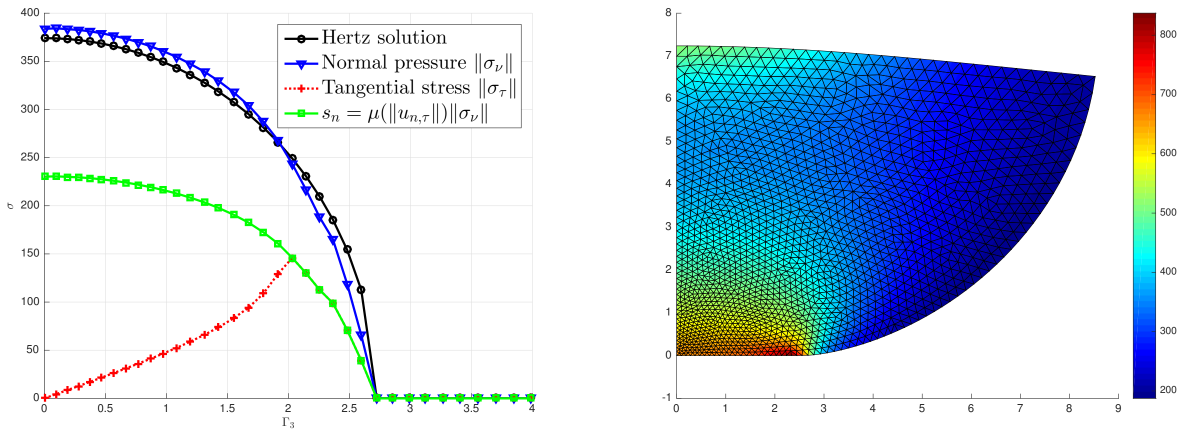


Figure 7.1 – Initial configuration of Hertz problem.

Figure 7.2 (a) compares the contact pressure and the tangential stress distributions on Γ_3

with the classical Hertz solution for $\rho = 10^4$ *and* $r = 0.13 \times \mathrm{E}$ *while Figure 7.2 (b) shows the deformed configuration with Von Mises effective stress distribution.*



(a) (b)

Figure 7.2 – (a) Normal and tangential stress distributions on Γ_3 with Coulomb friction. (b) Deformed configuration with Von Mises effective stress distribution.

Example 7.2 *In this example, we will study the case of a bar resting on a rigid foundation. We have chosen the academic example of a parallelepiped bar, Figure 7.3 which has the following dimensions: the height* $h = 1\,\mathrm{mm}$ *and the length is equal to four times the height, with* $\Gamma_1 = \{0\} \times [0, 1]$, $\Gamma_2 = [2, 4] \times \{1\}$ *and* $\Gamma_3 = [0, 4] \times \{0\}$.

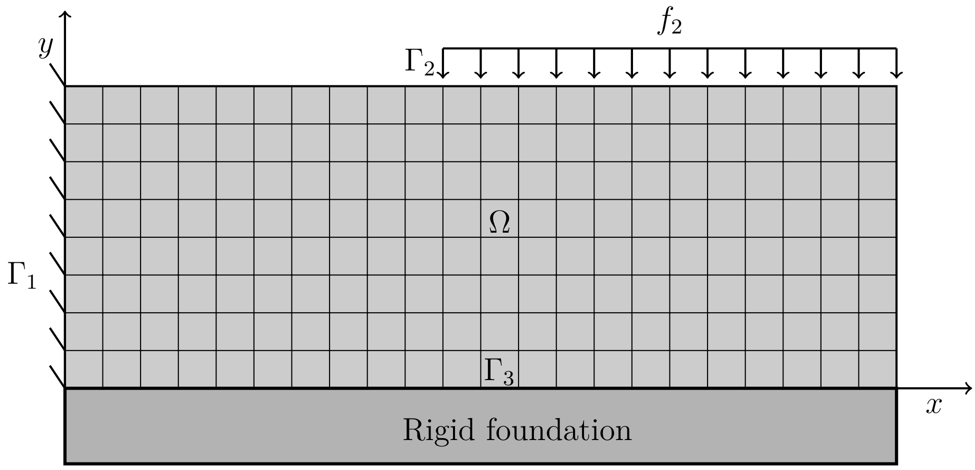


Figure 7.3 – Initial configuration of the academic example of a parallelepiped bar.

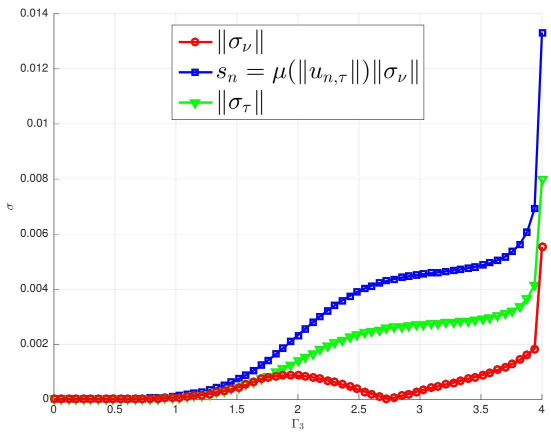
The body force f_0 *is assumed to be zero. The body is clamped on* Γ_1 *and thus* $u = 0$ *there. A surface traction of density* $f_2 = (0, -0.005)\,\mathrm{daN/mm^2}$ *acts on* Γ_2*. The gap between* Γ_3

and the foundation is $\varrho = 0$. *The characteristics of the material are: the Young's modulus* $\mathrm{E} = 1\,\mathrm{daN/mm^2}$ *and Poisson's ratio* $\nu = 0.3$.

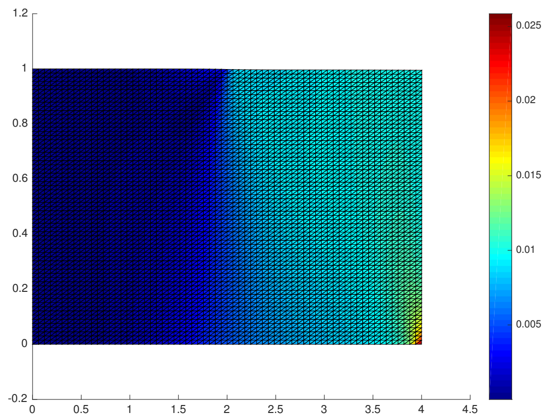
We assume that the nonlinear function $g(\,.\,)$ *defined in* (1.13), *has the form* (7.8) *with* (7.9) *and an elasticity limit* $\xi_0 = 0.05$. *The penalty parameter is* $r = 2.5 \times 10^4 \times \mathrm{E}$ *for all mesh sizes.*

To determine the contact pressure s_n, *we first solve a pure contact problem in the initialization step. Then we set* $s_n = \mu(\|u_{n,\tau}\|)\|\sigma_\nu\|$. *Note that in* ***Algorithm 2***, $\lambda_c \approx -\sigma_\nu$ *and* $\lambda_f \approx -\sigma_\tau$, *see e.g., [54, 82].*

The deformed configuration with Von Mises effective stress distribution is shown in Figure 7.4 (b) while Figure 7.4 (a) shows the normal and tangential stress distributions on Γ_3. *The sticking zone* $\|\sigma_\tau\| < s_n$ *and sliding zone* $\|\sigma_\tau\| = s_n$ *are clearly identified.*



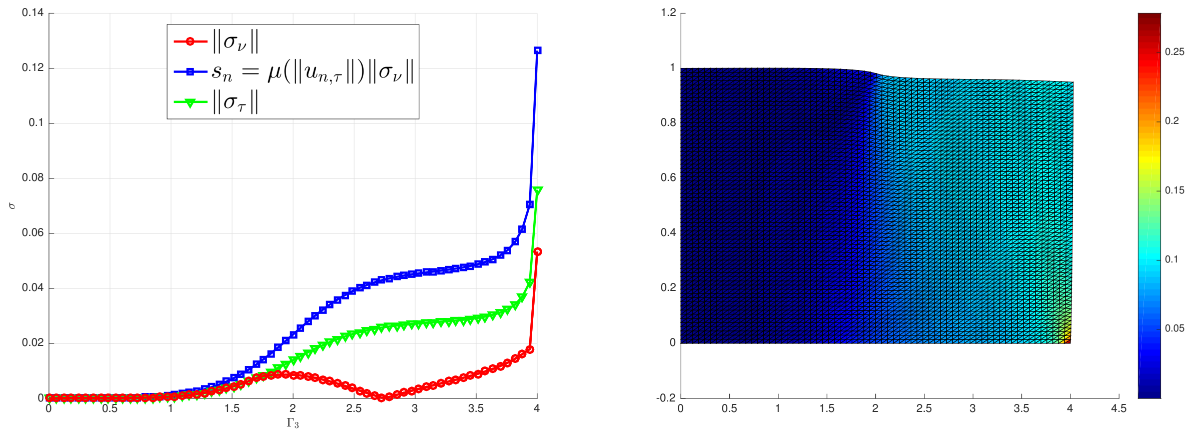
(a)



(b)

Figure 7.4 – (a) Normal and tangential stress distributions on Γ_3 with Coulomb friction. (b) Deformed configuration with Von Mises effective stress distribution.

In order to study the influence of the magnitude of the deformations on iteration process, we increase the value of the surface traction applied on Γ_2: $f_2 = (0, -0.05)\,\mathrm{daN/mm^2}$.



(a) (b)

Figure 7.5 – (a) Normal and tangential stress distributions on Γ_3 with Coulomb friction. (b) Deformed configuration with Von Mises effective stress distribution.

In Table 7.1, we give the number of Kačanov iterations and the computation time (CPU-time) necessary for the numerical resolution of the mechanical problem considered, for different values of the parameter ρ for $f_2 = (0, -0.005)\,\mathrm{daN/mm^2}$ and $f_2 = (0, -0.05)\,\mathrm{daN/mm^2}$.

	$f_2 = (0, -0.005)\,\mathrm{daN/mm^2}$		$f_2 = (0, -0.05)\,\mathrm{daN/mm^2}$	
ρ	Kačanov iterations	CPU-time (s)	Kačanov iterations	CPU-time (s)
10^5	3	5.0900	5	8.4000
10^6	3	5.5000	5	8.6700
10^7	4	6.6700	6	10.0500
6×10^7	6	10.0100	8	13.63000

Tableau 7.1 – Number of iterations and the computation time for different values of ρ and surface traction density.

Our numerical tests consist of measuring the error $\|u - u^h\|_V$ using $L^2(\Omega)$ norm, for a uniform mesh size of step $h \in \{\frac{1}{8}, \frac{1}{16}, \frac{1}{32}, \frac{1}{128}, \frac{1}{256}\}$. It is not possible to refine the size of the elements uniformly as much as necessary, for obvious reasons of calculation costs. As we do not have an analytical solution for our problem, the error $\|u - u^h\|_V$ is evaluated numerically by $\|u_{ref} - u^h\|_V$. The reference solution u_{ref} is calculated on a reference mesh comprising 65536 element and 33153 degrees of liberty (corresponding to $h = \frac{1}{256}$). In Figure 7.6, the order of convergence of the method for different discretization steps h is shown.

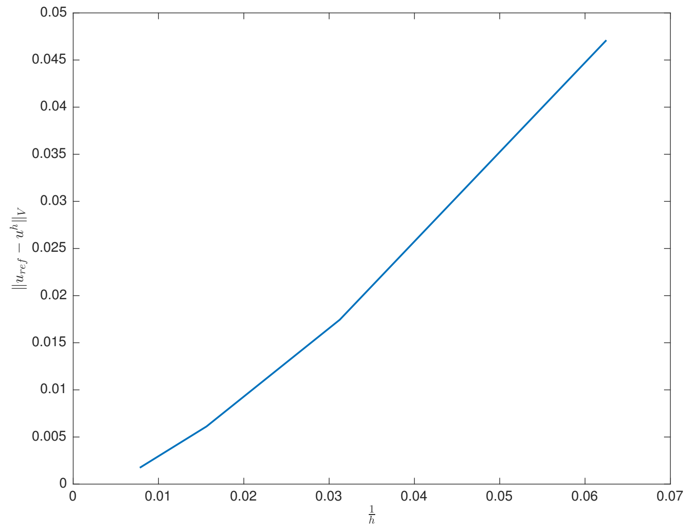
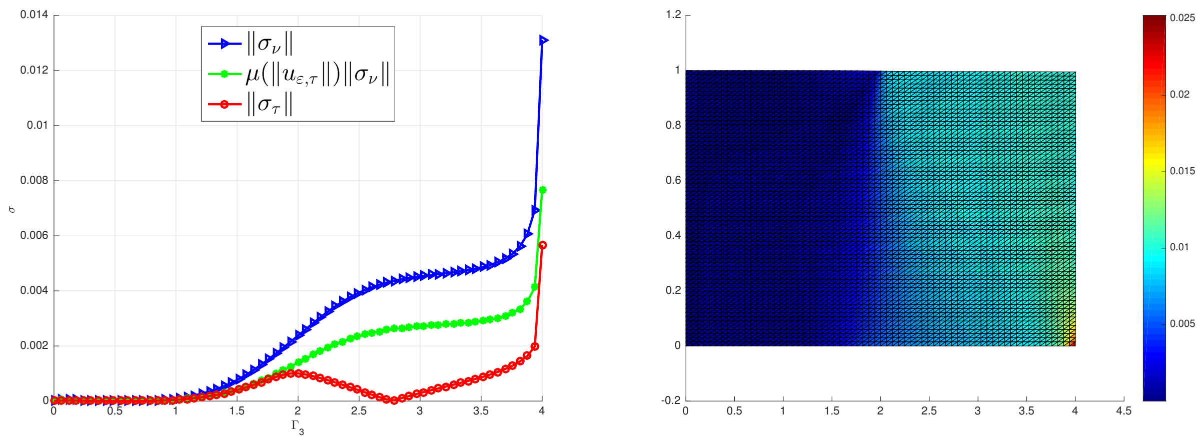


Figure 7.6 – Error estimation.

7.2.2 Penalty method

In this section, we take up again the academic example 7.2 presented in section 7.2.1, Figure 7.3, with the same characteristics.



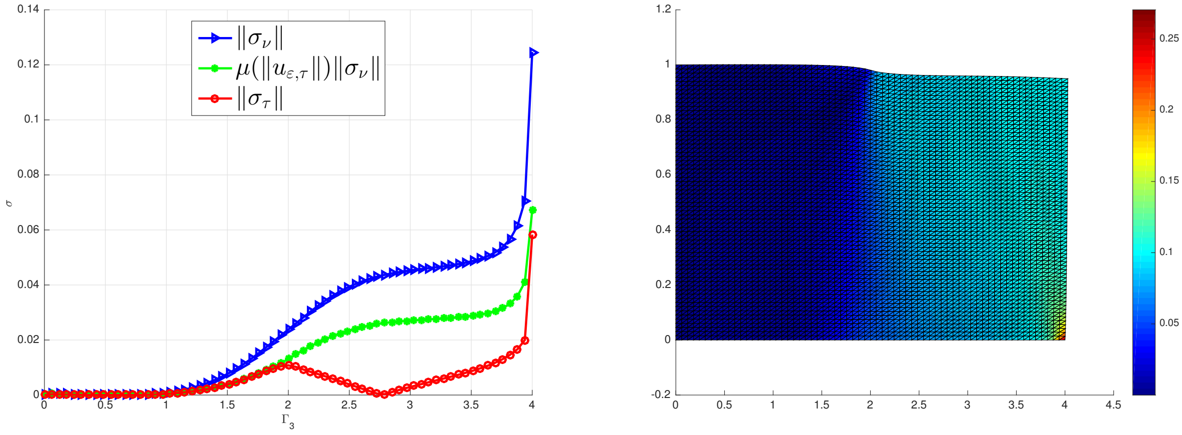
(a) (b)

Figure 7.7 – (a) Normal and tangential stress distributions on Γ_3 with Coulomb friction. (b) Deformed configuration with Von Mises effective stress distribution.

In Figure 7.7 (a) the normal and tangential stress distributions on Γ_3 with Coulomb friction are shown for $\rho = 10^4$, $\epsilon = 1.05 \times 10^{-5}$ and $f_2 = (0, -0.005)\,\mathrm{daN/mm^2}$, while Figure 7.7

(b) shows the deformed configuration with Von Mises effective stress distribution for the same parameters.

In Figure 7.8 (a) the normal and tangential stress distributions on Γ_3 with Coulomb friction are shown for $\rho = 10^4$, $\epsilon = 1.05 \times 10^{-5}$ and $f_2 = (0, -0.05)\,\text{daN/mm}^2$, while Figure 7.8 (b) shows the deformed configuration with Von Mises effective stress distribution for the same parameters.



(a) (b)

Figure 7.8 – (a) Normal and tangential stress distributions on Γ_3 with Coulomb friction. (b) Deformed configuration with Von Mises effective stress distribution.

These tests concern the influence of the penalization parameter onto the iteration method. In Table 7.2, we give the number of iterations carried out by **Algorithm 3** for the numerical resolution of Problem (PV), for different values of the penalization parameter ϵ for $f_2 = (0, -0.005)\,\text{daN/mm}^2$ and $f_2 = (0, -0.05)\,\text{daN/mm}^2$.

	$f_2 = (0, -0.005)\,\text{daN/mm}^2$	$f_2 = (0, -0.05)\,\text{daN/mm}^2$
ϵ	Number of iterations	
10^{-2}	6	10
10^{-3}	9	14
10^{-4}	13	15
10^{-5}	14	16

Tableau 7.2 – Number of iterations for different values of the penalization parameter and surface traction density.

These different tests allowed us to observe that the numerical solutions of the test problem by both methods (Augmented Lagrangian and penalty approach) are the same.

Chapter 8

Numerical experiments for electro-elasto-plastic materials

In this chapter, we take up again the static contact problem with Coulomb's friction law of an electro-elasto-plastic body with a conductive foundation, which was analyzed in Part III, with a perspective of implementing the corresponding simulations using $\mathbb{P}_1$ triangular finite element method. A number of numerical experiments through two-dimensional test problems have been carried out to illustrate the performances of the algorithms presented in Sections 5.4 and 6.5.

8.1 Presentation of numerical results

We assume that we have a semi-infinite body along the X_1 axis, in an orthogonal reference frame (O, X_1, X_2, X_3) of the crystallographs: there is no possible variation of the calculated degrees of freedom (the displacement u and the electric potential φ) in this direction. As a consequence, the tensors E, D, ε and σ are nil along X_1. In other words, the material is in a state of plane deformation with respect to the plane (O, X_2, X_3), and with nil electric displacement along X_1, we get then

$$\varepsilon_{11} = \varepsilon_{12} = \varepsilon_{13} = 0, \quad \mathrm{D}_1 = 0. \tag{8.1}$$

For numerical tests, a transverse anisotropic hexagonal electro-elastic material of class 6mm is considered (*i.e.*, having an axis of symmetry X_1 which represents the direction of polarization P, and the six planes of symmetry). The material PZT-5A is used as an example (see [75]). The main consequence of this is that the law of behavior is expressed,in the two equations (6.1) and (6.2), only in the plane (O, X_2, X_3); this makes it possible to work in two dimensions only. The mechanical stress tensor and the vector of electrical displacements characterizing the PZT-5A are expressed in the following matrix form:

$$\begin{pmatrix} \sigma_{22} \\ \sigma_{33} \\ \sigma_{23} \\ \mathrm{D}_2 \\ \mathrm{D}_3 \end{pmatrix} = \begin{pmatrix} (k_0 + \frac{4}{3}g(\|\bar{\varepsilon}(u)\|^2)) & (k_0 - \frac{2}{3}g(\|\bar{\varepsilon}(u)\|^2)) & 0 & 0 & e_{32} \\ (k_0 - \frac{2}{3}g(\|\bar{\varepsilon}(u)\|^2)) & (k_0 + \frac{4}{3}g(\|\bar{\varepsilon}(u)\|^2)) & 0 & 0 & e_{33} \\ 0 & 0 & g(\|\bar{\varepsilon}(u)\|^2) & e_{24} & 0 \\ 0 & 0 & e_{24} & -\beta_{22} & 0 \\ e_{32} & e_{33} & 0 & 0 & -\beta_{33} \end{pmatrix} \begin{pmatrix} \varepsilon_{22}(u) \\ \varepsilon_{33}(u) \\ 2\varepsilon_{23}(u) \\ -\mathrm{E}_2 \\ -\mathrm{E}_3 \end{pmatrix}, \tag{8.2}$$

In equation (8.2), using the symmetry properties of the piezoelectric tensor, the change from a tensor of order 3 to a tensor of order 2, is given by the identification set which, to any pair of indices or exponents which can be interchanged, associates an integer according to the following scheme

$$e_{ijk} \equiv e_{pq} = \begin{pmatrix} 0 & 0 & e_{24} \\ e_{32} & e_{33} & 0 \end{pmatrix},$$

with

$$jk = 22 \to q = 2,$$
$$jk = 33 \to q = 3,$$
$$jk = 23 \text{ or } 32 \to q = 4.$$

For numerical tests, we consider a piezoelectric material PZT-5A characterized by the coefficients given in Table 8.1, established by Feng and Wu in [55]. The permittivity constant of the vacuum is $\epsilon_0 = 8.885\mathrm{e}^{-12}\ \mathrm{C}^2/\mathrm{Nm}^2$.

Elasticity (GPa)				
Young's modulo (GPa)			Poisson's ratio	
58.7102			0.3912	
Piezoelectricity ($\mathrm{C/m^2}$)			Permittivity ($\mathrm{C^2/Nm^2}$)	
e_{32}	e_{33}	e_{24}	β_{22}/ϵ_0	β_{33}/ϵ_0
-5.4	15.8	12.3	916	830

Tableau 8.1 – The material PZT-5A coefficients values with $\epsilon_0 = 8.885\mathrm{e}^{-12}\ \mathrm{C}^2/\mathrm{Nm}^2$.

8.1.1 Numerical tests for Problem $(\widetilde{\mathrm{PV}})$

Example 8.1 *In this example, we will study the academic example of a parallelepiped bar which has the following dimensions: the height $h = 4$ m and the length is equal to two times the height,* *with* $\Gamma_a = \Gamma_1 = (\{0\} \times [0,4])$*,* $\Gamma_2 = \Gamma_b = ([0,8] \times \{4\})$ *and* $\Gamma_3 = ([0,8] \times \{0\})$*. The body force* f_0 *and the volume electric charge* q_0 *are assumed to be zero. The body is clamped on* Γ_1 *and thus* $u = 0$ *there. A surface traction and electric charge of densities* $f_2(x) = (0, -10x)\,\mathrm{N/m^2}$*,* $q_2(x) = 0\,\mathrm{C/m^2}$*, respectively, act on* Γ_2 *(Figure 8.1). We suppose here that the behavior of the material is linear. The penalty parameter is* $r = 0.25 \times \mathrm{E}$ *for all mesh sizes. The characteristics of the material are given in Table 8.1.*

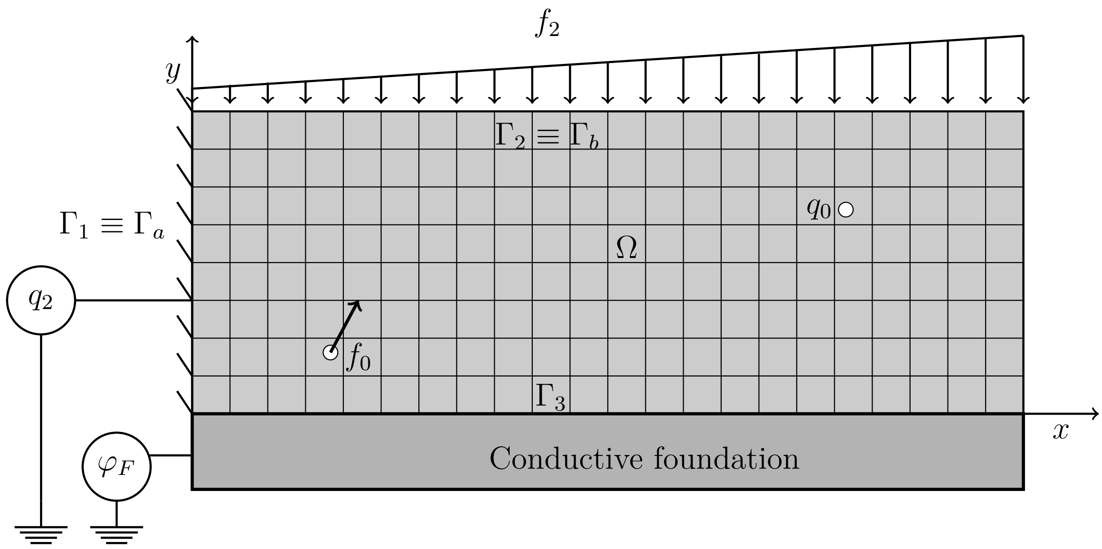
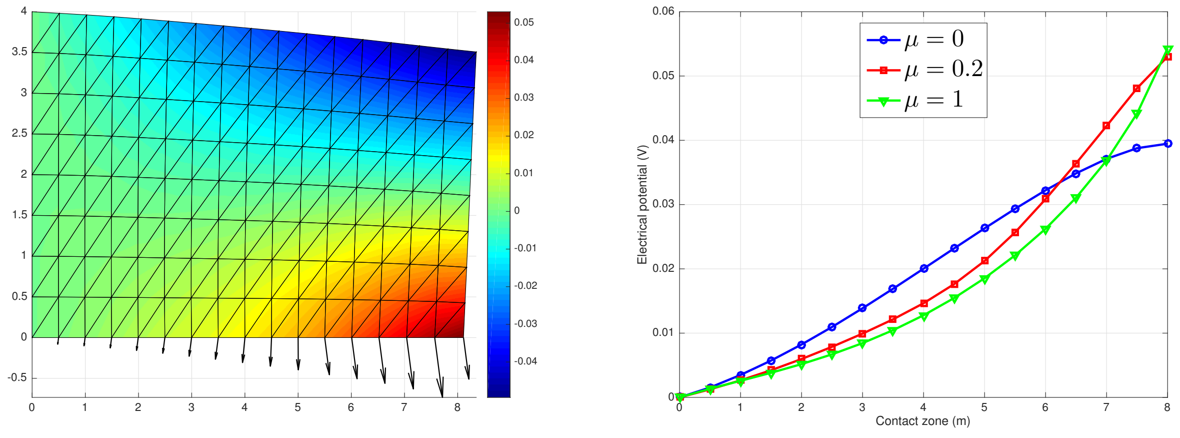


Figure 8.1 – Initial configuration of a piezoelectric body resting on a conductive foundation.

In the case where the rigid foundation is non-conductive ($k_e = 0$) *and the coefficient of friction is independent of the slip* ($\mu = 0.2$)*, the same results were obtained as in [117]. In order to have a visible deformation, we have increased the pressure on the* Γ_2 *part of the body, the material will deform so that the barycenter of the positive and negative loads will diverge. An electric dipole has thus been created which, by reaction, will cause charges of opposite signs to appear on the surface; Figure 8.2(a) shows the electrical potential distribution on the body. It is normal that the electrical potential values are higher where the stresses are higher, i.e., where there is a strong reaction between the body and the foundation. This is an example of how a piezoelectric sensor works: electrodes can be placed at any point on the surface of the material and a change in stress can be detected.*

At the contact interface, there is frictional contact; the arrows show the frictional contact forces. This contact force is increasingly directed to the right, proportionally to the intensity of the applied force. To evaluate the effects of the friction coefficient at the contact interface, three values were tested, $\mu = 0$*,* $\mu = 0.2$ *and* $\mu = 1$*; Figure 8.2 (b) shows the electrical potential distribution on* Γ_3 *for those different values of the friction coefficient.*

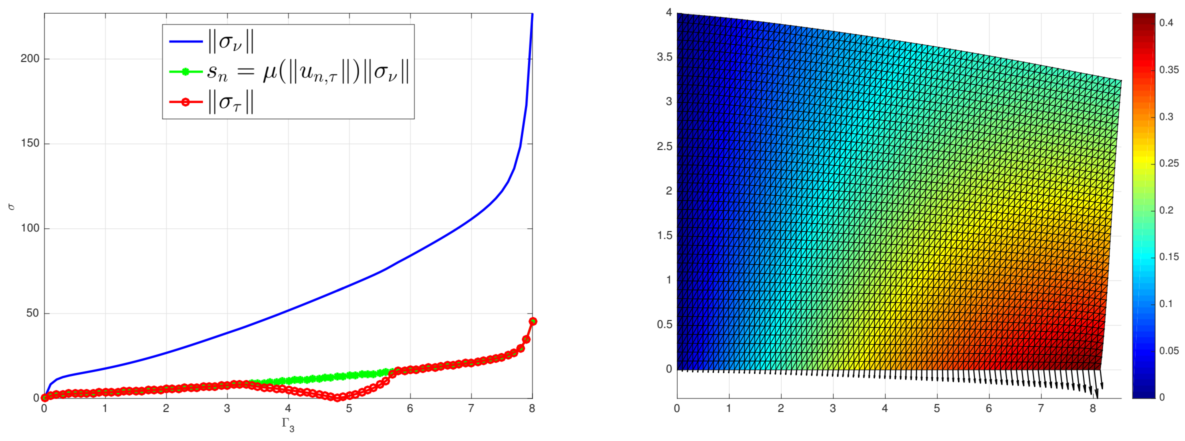


(a) (b)

Figure 8.2 – For $k_e = 0$ (a) Deformed configuration and electrical potential distribution with contact forces (arrows). (b) Electrical potential distribution on Γ_3 for different values of the friction coefficient μ.

Next, we refine the mesh, we assume that the rigid foundation is conductive, we increase the value of the surface traction applied on Γ_2 *up to:* $f_2(x) = (0, -15x)\,\mathrm{N/m^2}$, *and we use the coefficient of friction defined in* (7.16).

The normal and tangential stress distributions on Γ_3 *are shown on Figure* 8.3 *(a) while Figure* 8.3 *(b) shows the deformed configuration with electrical potential distribution. The sticking zone* $\|\sigma_\tau\| < s_n$ *and sliding zone* $\|\sigma_\tau\| = s_n$ *are clearly identified.*



(a) (b)

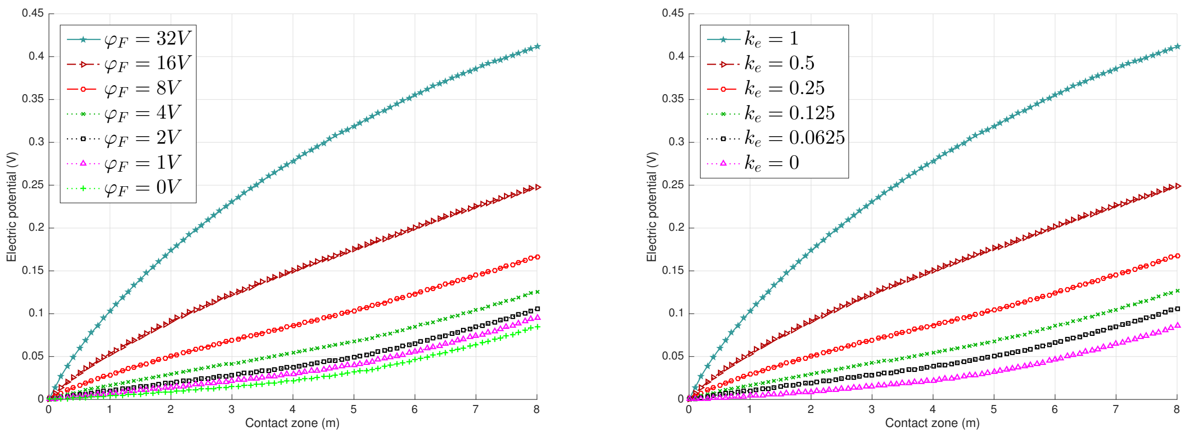
Figure 8.3 – For $k_e = 1$ and $\varphi_F = 32$ V (a) Normal and tangential stress distributions on Γ_3 with Coulomb friction. (b) Deformed configuration and electrical potential distribution with contact forces (arrows).

In Table 8.2, we give the number of iterations carried out by ***Algorithm 4*** *for the numerical resolution of the physical problem* $(\widetilde{\mathrm{P}})$ *different values of the foundation's electrical potential* φ_F.

	$f_2(x) = (0, -15x)\,\mathrm{N/m^2}$						
$\varphi_F\ (V)$	0	1	2	4	8	16	32
Number of iterations	4	5	5	5	5	5	5

Tableau 8.2 – Number of iterations for different values of "φ_F" and $k_e = 1$.

Figure 8.4 (a) (resp. (b)) shows the electrical potential distribution on Γ_3 *when the electrical conductivity coefficient is equal to one* $(k_e = 1)$ *with different values of the foundation's electrical potential* φ_F *(resp. for* $\varphi_F = 32\,V$ *and different values of* k_e*). We can clearly see the influence of the foundation's electrical potential (resp. the electrical conductivity coefficient) on that of the contact zone* Γ_3.

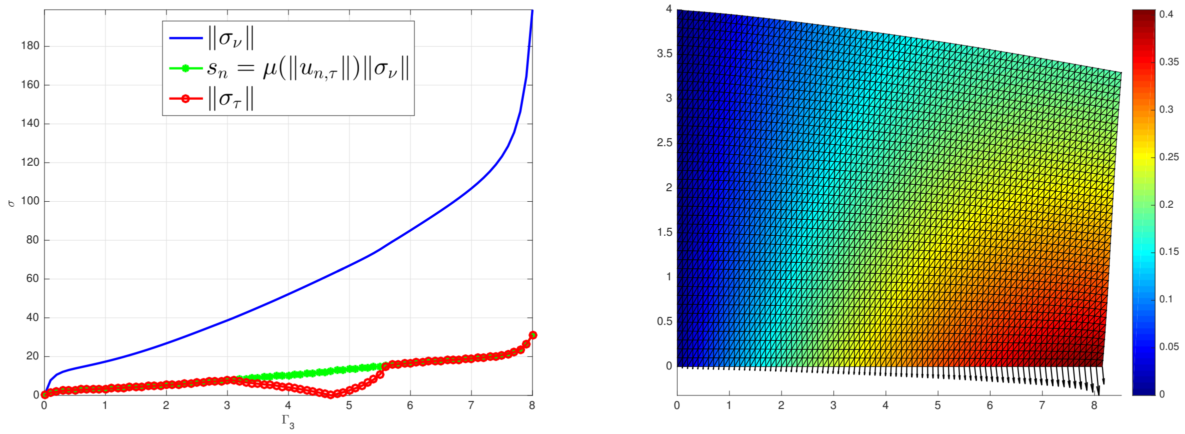


(a) (b)

Figure 8.4 – The electrical potential distribution on Γ_3 (a) For $k_e = 1$ and different values of φ_F. (b) For $\varphi_F = 32\,V$ and different values of k_e.

Example 8.2 *We consider here the same data from the previous example, and we assume that the nonlinear function* $g(\,.\,)$ *defined in (1.13), has the form (7.8) with (7.9) and an elasticity limit* $\xi_0 = 1.8$.

The normal and tangential stress distributions on Γ_3 *are shown on Figure 8.5 (a) while Figure 8.5 (b) shows the deformed configuration with electrical potential distribution.*



(a) (b)

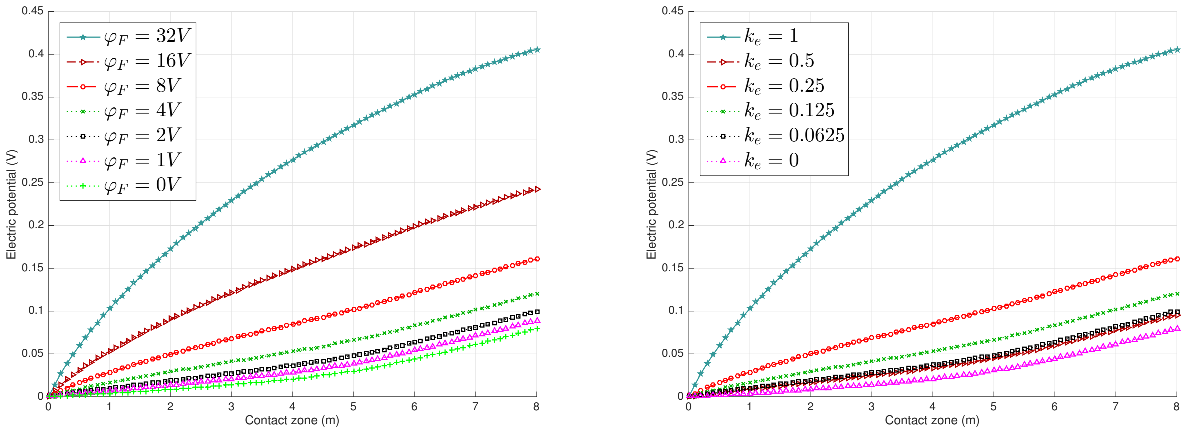
Figure 8.5 – For $k_e = 1$ and $\varphi_F = 32$ V (a) Normal and tangential stress distributions on Γ_3 with Coulomb friction. (b) Deformed configuration and electrical potential distribution with contact forces (arrows).

*In Table 8.3, we give the number of iterations carried out by the fixed point algorithm (**Algorithm 4**) for different values of the foundation's electrical potential φ_F where we can clearly see that this number increased by 4 iterations than Table 8.2.*

	$f_2(x) = (0, -15x)\,\mathrm{N/m^2}$						
φ_F (V)	0	1	2	4	8	16	32
Number of iterations	8	9	9	9	9	9	9

Tableau 8.3 – Number of iterations for different values of "φ_F" and $k_e = 1$.

Figure 8.6 (a) (resp. (b)) shows the electrical potential distribution on Γ_3 when the electrical conductivity coefficient is equal to one ($k_e = 1$) with different values of the foundation's electrical potential φ_F (resp. for $\varphi_F = 32\,V$ and different values of k_e). We can clearly see the influence of the foundation's electrical potential (resp. the electrical conductivity coefficient) on that of the contact zone Γ_3.



(a) (b)

Figure 8.6 – The electrical potential distribution on Γ_3 (a) For $k_e = 1$ and different values of φ_F. (b) For $\varphi_F = 32\,V$ and different values of k_e.

It can be seen that the electrical potential varies in the same way as the electrical conductivity coefficient k_e and the foundation's electrical potential φ_F. In other words, the pattern of the electrical potential distributions increases as the coefficient of electrical conductivity k_e or the foundation's electrical potential φ_F increases. These results are consistent with the electrical contact boundary condition we use on the contact interface so they show the effect of foundation conductivity on the process.

To measure errors respectively $\|u - u^h\|_V$ and $\|\varphi - \varphi^h\|_W$ with the norms of $H^1(\Omega)$ and $L^2(\Omega)$, we have used a process analogous to the one developed in the previous chapter, keeping the same geometric properties of the model (see Figure 8.1). We have considered uniform mesh size of step $h \in \{\frac{1}{8}, \frac{1}{16}, \frac{1}{32}, \frac{1}{128}, \frac{1}{256}\}$. As we do not have an analytical solution for our problem, the error $\|u - u^h\|_V + \|\varphi - \varphi^h\|_W$ is evaluated numerically by $\|u_{ref} - u^h\|_V + \|\varphi_{ref} - \varphi^h\|_W$. The reference solution (u_{ref}, φ_{ref}) is calculated on a reference mesh comprising 65536 element and 33153 degrees of liberty (corresponding to $h = \frac{1}{256}$). In Figure 8.7, the order of convergence of the method for different discretization steps h is shown.

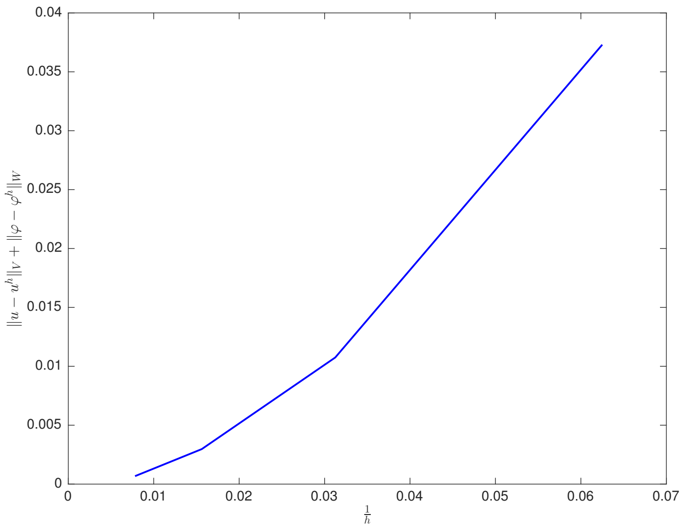


Figure 8.7 – Error estimation.

8.1.2 Numerical tests for Problems (PV_1) and (PV_2)

In this section, we describe numerical results for problems (PV_1) and (PV_2) in two dimensions to verify the performance of the iterative schemes presented in Section 6.5.

A possible choice of the functions h_ν and p_ν is (see [15])

$$h_\nu(s) = c_\nu \times \begin{cases} \alpha_\nu, & \text{if } |s| > 128, \\ 1 + (\alpha_\nu - 1) \times \dfrac{|s|}{128}, & \text{if } 0 \leq |s| \leq 128, \end{cases} \qquad p_\nu(s) = \begin{cases} 0, & \text{if } s < 0, \\ s, & \text{if } 0 \leq s \leq n_\nu, \\ n_\nu, & \text{if } s > n_\nu, \end{cases}$$

where c_ν, α_ν and n_ν are positive constants, $\alpha_\nu > 1$. The tolerances in the stopping criteria (6.40) and (6.41) are those given in Section 7.2.

Example 8.3 *In this example, we will study the academic example of a parallelepiped bar which has the following dimensions:* $\Omega = ([0,12] \times [0,2])$*, with* $\Gamma_1 = \Gamma_b = (\{0\} \times [0,2] \cup \{12\} \times [0,2])$, $\Gamma_2 = \Gamma_a = ([0,12] \times \{2\} \cup [0,2] \times \{0\} \cup [10,12] \times \{0\})$ *and* $\Gamma_3 = ([2,10] \times \{0\})$*. The body force* f_0 *and the volume electric charge* q_0 *are assumed to be zero. The body is clamped on* Γ_1 *and thus* $u = 0$ *there. A surface traction and electric charge of densities* $f_2(x) = (0,-5)\,\mathrm{N/m^2}$, $q_2(x) = 0\,\mathrm{C/m^2}$ *act, respectively, on* Γ_2, Γ_b*. The gap between the body and the conductive foundation is* $\varrho = 0.5\,m$ *(Figure 8.8).*

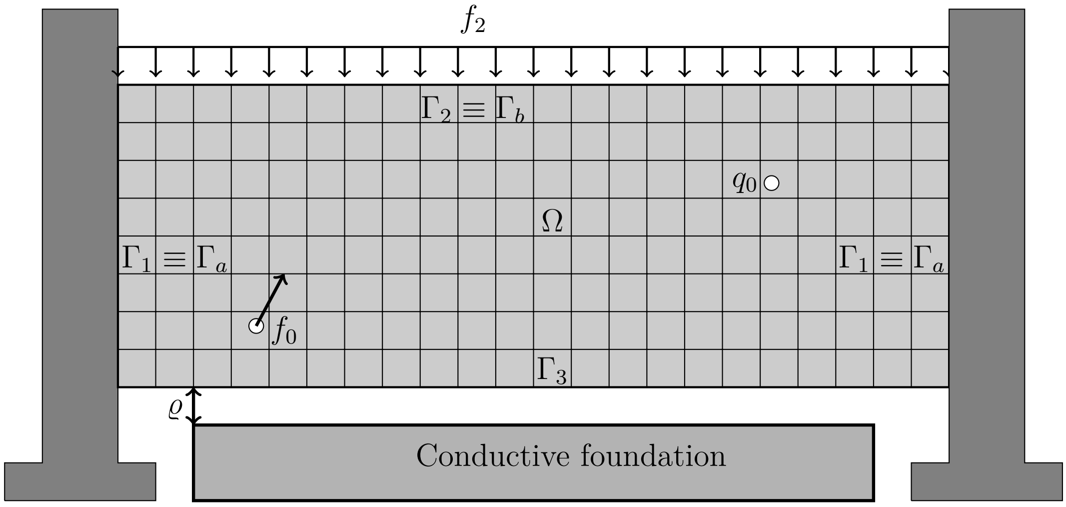
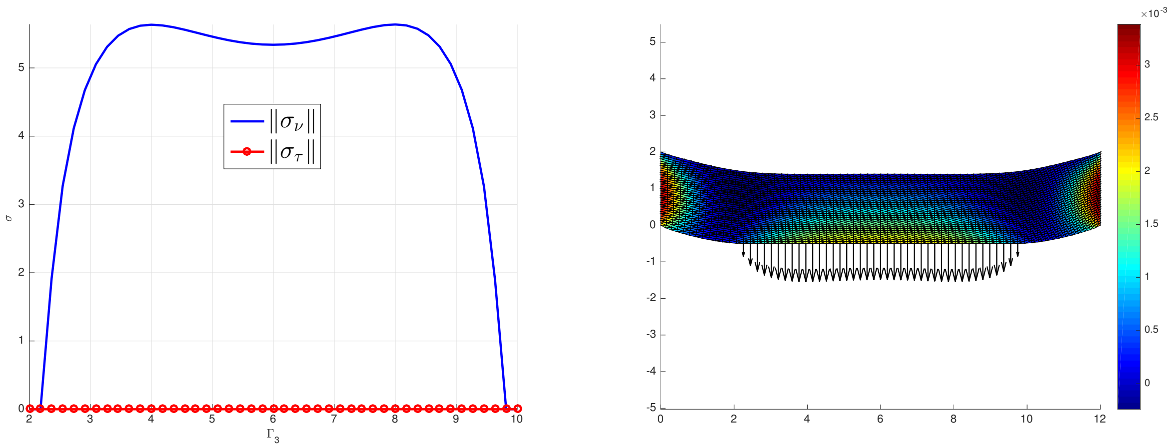


Figure 8.8 – Physical framework.

The penalty parameter is $r = 0.25 \times \mathrm{E}$ *for all mesh sizes. The characteristics of the material are given in Table 8.1. We assume that the nonlinear function* $g(\,.\,)$ *in* (1.13), *has the form* (7.8), *with* (7.9) *and an elasticity limit* $\xi_0 = 1.8$.

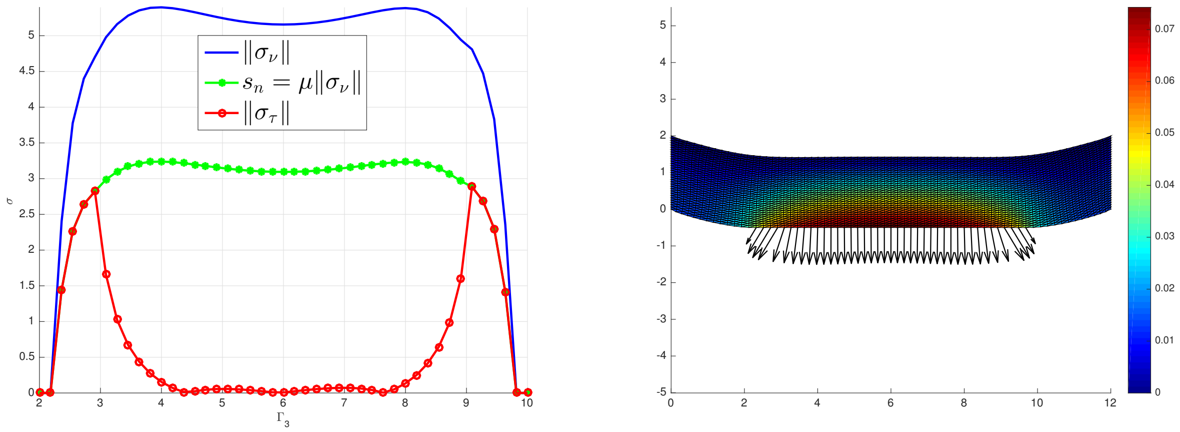


(a) (b)

Figure 8.9 – (a) Normal and tangential stress distributions on Γ_3. (b) Deformed configuration and electrical potential distribution with contact forces (arrows).

Algorithm 9 *stops after 2 iterations. The normal and tangential stress distributions on* Γ_3 *are shown on Figure 8.9 (a) while Figure 8.9 (b) shows the deformed configuration with electrical potential distribution.*

Example 8.4 *We consider here the same data from the previous example. For $\mu = 0.6$, $k_e = 1$ and $\varphi_F = 32\,V$, the normal and tangential stress distributions on Γ_3 are shown on Figure 8.10 (a) while Figure 8.10 (b) shows the deformed configuration with electrical potential distribution. The sticking zone $\|\sigma_\tau\| < s_n$ and sliding zone $\|\sigma_\tau\| = s_n$ are clearly identified.*



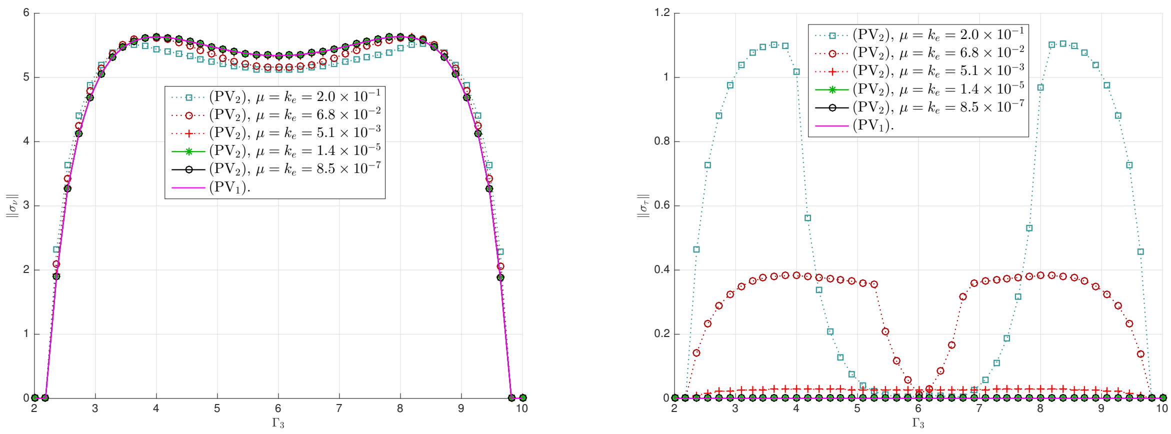
(a) (b)

Figure 8.10 – For $\mu = 0.6$, $k_e = 1$ and $\varphi_F = 32\,V$ (a) Normal and tangential stress distributions on Γ_3. (b) Deformed configuration and electrical potential distribution with contact forces (arrows).

To show that the numerical results are in good agreement with the theoretical analysis given in Section 6.4, we will study the evolution of the number of iterations, the distribution of the electrical potential, normal and tangential stresses on Γ_3 as a function of the friction coefficient μ and the electrical conduction coefficient k_e. The results are given in Table 8.4, Figure 8.11(a)-(b) and Figure 8.12.

$\mu = k_e$	Number of iterations
6.0×10^{-1}	7
2.0×10^{-1}	5
6.8×10^{-2}	4
5.1×10^{-3}	3
6.2×10^{-4}	2
1.4×10^{-5}	2
3.5×10^{-6}	2
8.5×10^{-7}	2

Tableau 8.4 – Number of iterations for different values of (μ, k_e).



(a) (b)

Figure 8.11 – For $\varphi_F = 32V$ and different values of $\mu = k_e$ (a) Normal stress distribution on Γ_3. (b) Tangential stress distribution on Γ_3.

For $\varphi_F = 32\,V$ and different values of $\mu = k_e$, Figure 3(a)-(b) show the normal and tangential stress distribution on Γ_3, while Figure 8.12 shows the electrical potential distribution on Γ_3, where we can clearly see that the contact zone's electrical potential, normal and tangential stress of Problem (PV_2) *approaches those of Problem* (PV_1) *when $\mu = k_e$ approaches zero.*

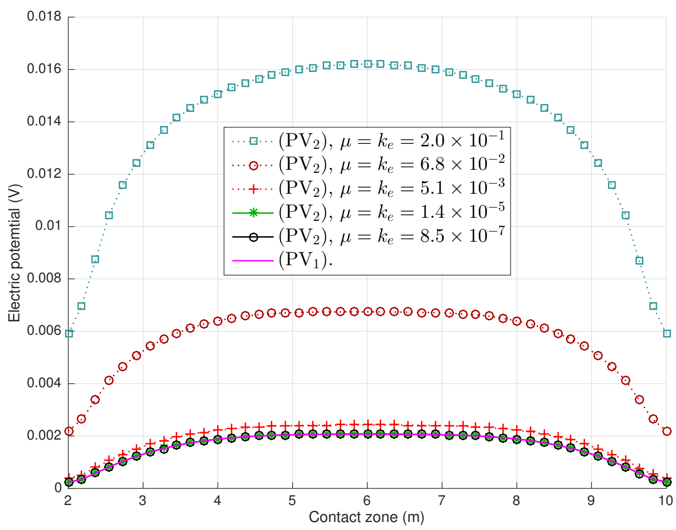


Figure 8.12 – Electrical potential distribution on Γ_3 for $\varphi_F = 32V$ and different values of $\mu = k_e$.

General conclusion and perspectives

Through this thesis, we have studied some static problems of contact between a deformable body and a foundation by treating various laws of contact and friction in elasto-plasticity and piezo-elasto-plasticity. In small deformations, we have proposed strong and weak formulations, and gave results of existence and uniqueness.

We also studied the numerical approach to problems, using a finite element scheme. The existence and uniqueness of the discretized problems considered as well as the results of error estimation under regularity assumptions were shown.

The Augmented Lagrangian and the penalty methods were used to solve problems resulting from modeling the contact with friction of an elasto-plastic or piezo-elasto-plastic body with a foundation. The resulting nonlinear vector equations are linearized by using iterative methods.

The theoretical models and numerical algorithms were tested on simple cases to verify the reliability, the efficiency and also the asymptotic behavior of convergence.

Several perspectives can be envisaged in this research work. We can first extend the static cases developed in this manuscript to quasistatic and dynamic cases. It could also be interesting to consider other conditions aimed to model more complex problems. In this regard, it is reasonable to assume that the contact phenomena is accompanied by that of wear and tear of the materials or that the dissipation of energy due to friction could cause the heating of the material, thus leading to the thermal expansion of the body at the contact boundary. Another way is to formulate theoretical models and numerical algorithms, at the contact surface, between two bodies. As mentioned in the introduction, the mathematical theory of contact mechanics is relatively a young field of study. Consequently, it is still possible to describe, model, study and analyze a considerable number of physical phenomena that intervene in everyday life. Numerical methods are not to be neglected either. In our work we only consider two-dimensional test problems, to take into account all the properties of piezoelectric materials, it would be desirable to extend our work to dimension three. Furthermore, it would be necessary to compare the solutions obtained during the simulations with real measurements.

In this manuscript, we have considered some examples of boundary problems, but their study has led to other new problems. These succinct lines above only scratch some of these multiple potential openings, the list is far from exhaustive. Mechanics is a privileged field of mathematical applications, and at the same time, its open problems are inexhaustible sources of inspiration that lead us to forge new mathematical tools. We therefore hope to make the reader aware of the richness of the problems raised, and to encourage him to go even further, in order to explore the depths of this immense field of research in contact mechanics.

Annex

Annex A

This annex presents preliminary material from functional analysis which we have used in the previous chapters. Some of the results are stated without proofs, since they are standard and can be found in many references. Nevertheless, we pay particular attention to the results which are repeatedly used in the previous chapters of this manuscript. They include Banach's and Schauder's fixed point theorems, the projection lemma, the Riesz representation theorem and the Weierstrass theorem, among others. All the linear spaces considered in this manuscript including abstract normed spaces, Banach spaces, Hilbert spaces and various function spaces are assumed to be real linear spaces.

A.1 Normed spaces

We start this section with basic definitions, notations and results concerning the normed spaces. Then we recall two main fixed point results: Banach's and Schauder's fixed point theorems.

A.1.1 Basic definitions

Given a linear space X, we recall that a norm $\| \, . \, \|_X$ is a function from X to $\mathbb{R}$ with the following properties.

1. $\|u\|_X \geq 0 \quad \forall u \in X$, and $\|u\|_X = 0$ if and only if $u = 0_X$.
2. $\|\alpha u\|_X = |\alpha| \|u\|_X \quad \forall u \in X, \, \alpha \in \mathbb{R}$.
3. $\|u + v\|_X \leq \|u\|_X + \|v\|_X \quad u, \, v \in X$.

The pair $(X, \| \, . \, \|_X)$ is called a normed space. Here and everywhere in this manuscript 0_X will denote the zero element of X. Also, we will simply say X is a normed space when the definition of the norm is understood from the context.

On a linear space various norms can be defined. Sometimes it is desirable to know if two norms are related. Let $\| \,.\, \|^{(1)}$ and $\| \,.\, \|^{(2)}$ be two norms over a linear space X. The two norms are said to be equivalent if there exist two constants c_1, $c_2 > 0$ such that

$$c_1 \|u\|^{(1)} \leq \|u\|^{(2)} \leq c_2 \|u\|^{(1)} \quad \forall u \in X. \tag{A.1}$$

We recall that a sequence $(u_n) \subset X$ is said to converge (strongly) to $u \in X$ if

$$\|u_n - u\|_X \to 0 \quad \text{as} \quad n \to +\infty. \tag{A.2}$$

In this case u is called the (strong) limit of the sequence (u_n) and we write

$$\lim_{n \to +\infty} u_n = u \quad \text{or} \quad u_n \to u \quad \text{in} \quad X.$$

It is straightforward to verify that the limit of a sequence, if it exists, is unique. The adjective "strong" is introduced in the previous definition to distinguish this convergence from the weak convergence which will be introduced later.

A sequence $(u_n) \subset X$ is said to be bounded if there exists $M > 0$ such that

$$\|u_n\|_X \leq M \quad \forall n \in \mathbb{N}, \tag{A.3}$$

or, equivalently, if

$$\sup_n \|u_n\|_X \leq +\infty.$$

To test the convergence of a sequence without knowing its limit, it is usually convenient to refer to the notion of a Cauchy sequence. Let X be a normed space. A sequence $(u_n) \subset X$ is called a Cauchy sequence if

$$\|u_m - u_n\|_X \to 0 \quad \text{as} \quad m, n \to +\infty.$$

Obviously, a convergent sequence is a Cauchy sequence, but in a general infinite dimensional space, a Cauchy sequence may fail to converge. This justifies the following definition.

Definition A.1 *A normed space is said to be complete if every Cauchy sequence from the space converges to an element in the space. A complete normed space is called a Banach space.*

From this definition it follows that $(X, \| \,.\, \|_X)$ is a Banach space if and only if for every

Cauchy sequence $(u_n) \subset X$ there exists an element $u \in X$ such that $u_n \to u$ in X. Moreover, using (A.1) and (A.2) it is easy to see that, for two equivalent norms, convergence in one norm implies convergence in the other norm. As a consequence, if $\| \,.\, \|^{(1)}$ and $\| \,.\, \|^{(2)}$ are equivalent norms on the linear space X then $(X, \| \,.\, \|^{(1)})$ is a Banach space if and only if $(X, \| \,.\, \|^{(2)})$ is a Banach space.

We introduce in what follows a particular type of normed space, in which the norm is defined in a special way. Given a linear space X we recall that an inner product $(X, \| \,.\, \|_X)$ is a function from $X \times X$ to $\mathbb{R}$ with the following properties.

1. $(u, u)_X \geq 0 \quad \forall u \in X$, and $(u, u)_X = 0$ if and only if $u = 0_X$.

2. $(u, v)_X = (v, u)_X \quad \forall u, v \in X$.

3. $(\alpha u + \beta v, w)_X = \alpha(u, w)_X + \beta(v, w)_X \quad \forall u, v, w \in X, \forall \alpha, \beta \in \mathbb{R}$.

The pair $(X, \| \,.\, \|_X)$ is called an inner product space. When the definition of the inner product $(\,.\,,\,.\,)_X$ is clear from the context, we simply say X is an inner product space.

Next, it is well known that an inner product $(\,.\,,\,.\,)_X$ induces a norm through the formula

$$\|u\|_X = \sqrt{(u, u)_X} \quad \forall u \in X, \tag{A.4}$$

and we note that everywhere in this manuscript the norm in an inner product space is the one induced by the inner product through the above formula. For an inner product space, we have the Cauchy-Schwarz inequality:

$$|(u, v)_X| \leq \|u\|_X \|v\|_X \quad \forall u, v \in X, \tag{A.5}$$

with the equality holding if and only if u and v are linearly dependent. Moreover, the following identity holds:

$$\|u + v\|_X^2 + \|u - v\|_X^2 = 2\Big(\|u\|_X^2 + \|v\|_X^2\Big) \quad \forall u, v \in X. \tag{A.6}$$

Identity (A.6) is called the parallelogram identity or the parallelogram law.

Among the inner product spaces, of particular importance are the Hilbert spaces.

Definition A.2 *A complete inner product space is called a Hilbert space.*

From the definition, we see that an inner product space X is a Hilbert space if X is a Banach space under the norm induced by the inner product.

A.1.2 Linear continuous operators

Let $(X, \|\,.\,\|_X)$ and $(Y, \|\,.\,\|_Y)$ be two normed spaces and let $\mathrm{L} : X \to Y$ be an operator. We recall that $\mathrm{L} : X \to Y$ is linear if

$$\mathrm{L}(\alpha_1 v_1 + \alpha_2 v_2) = \alpha_1 \mathrm{L}(v_1) + \alpha_2 \mathrm{L}(v_2) \quad \forall v_1,\, v_2 \in X, \quad \forall \alpha_1,\, \alpha_2 \in \mathbb{R}.$$

The operator L is said to be continuous if

$$u_n \to u \quad \text{in} \quad X \Rightarrow \mathrm{L}(u_n) \to \mathrm{L}(u) \quad \text{in} \quad Y.$$

It can be proved that, if L is linear, then L is continuous if and only if it is bounded, *i.e.*, there exists $M > 0$ such that

$$\|\mathrm{L}(v)\|_Y \leq M \|v\|_X \quad \forall v \in X.$$

We will use the notation $\mathscr{L}(X, Y)$ for the set of all linear continuous operators from X to Y. For $\mathrm{L} \in \mathscr{L}(X, Y)$, the quantity

$$\|\,\mathrm{L}\,\|_{\mathscr{L}(X,Y)} = \sup_{\substack{v \in X \\ v \neq 0_X}} \frac{\|\mathrm{L}(v)\|_Y}{\|v\|_X}, \tag{A.7}$$

is called the operator norm of L, and $\mathrm{L} \mapsto \|\,\mathrm{L}\,\|_{\mathscr{L}(X,Y)}$ defines a norm on the space $\mathscr{L}(X, Y)$. Moreover, if Y is a Banach space then $\mathscr{L}(X, Y)$ is also a Banach space. For a linear operator L, we usually write $\mathrm{L}(v)$ as $\mathrm{L}\, v$, but sometimes we also write $\mathrm{L}\, v$ even when L is not linear.

For a normed space X, the space $\mathscr{L}(X, \mathbb{R})$ is called the dual space of X and is denoted by X'. The elements of X' are linear continuous functionals on X. Recall that a linear functional $\ell : X \to \mathbb{R}$ belongs to X' if and only if it is continuous, *i.e.*,

$$u_n \to u \quad \text{in} \quad X \Rightarrow \ell(u_n) \to \ell(u) \quad \text{in} \quad \mathbb{R},$$

or, equivalently, if it is bounded, *i.e.*, there exists $M > 0$ such that

$$|\ell(v)| \leq M \|v\|_X \quad \forall v \in X.$$

The duality pairing between X' and X is usually denoted by $\ell(v)$ or $\langle v', v \rangle$ or $\langle v', v \rangle_{X' \times X}$

for ℓ, $v \in X'$ and $v \in X$. It follows from (A.7) that a norm on X' is

$$\|\ell\|_{X'} = \sup_{\substack{v \in X \\ v \neq 0_X}} \frac{|\ell(v)|}{\|v\|_X}. \tag{A.8}$$

Moreover, $(X', \|\,.\,\|_{X'})$ is always a Banach space.

We can now introduce another kind of convergence in a normed space. A sequence $(u_n) \subset X$ is said to converge weakly to $u \in X$ if for each $\ell \in X'$,

$$\ell(u_n) \to \ell(u) \quad \text{as} \quad n \to +\infty.$$

In this case u is called the weak limit of (u_n) and we write

$$u_n \rightharpoonup u \quad \text{in} \quad X.$$

It follows from the Hahn-Banach theorem that the weak limit of a sequence, if it exists, is unique. Moreover, it is easy to see that strong convergence implies weak convergence, *i.e.*, if $u_n \to u$ in X, then $u_n \rightharpoonup u$ in X. The converse of this property is not true in general.

Assume now that $(X, (\,.\,,\,.\,)_X)$ is an inner product space and note that in this case $u \mapsto (u, v)_X$ is a linear continuous functional on X, for all $v \in X$.
Therefore, it follows from the definition of the weak convergence that

$$u_n \rightharpoonup u \quad \text{in} \quad X \Rightarrow (u_n, v)_X \to (u, v)_X \quad \text{as} \quad n \to +\infty \quad \forall v \in X. \tag{A.9}$$

We shall see later that, in the case when $(X, (\,.\,,\,.\,)_X)$ is a Hilbert space, the converse of (A.9) is true.

The convergence of sequences is used to define closed subsets in a normed space.

Definition A.3 *Let X be a normed space. A subset $K \subset X$ is called:*

i) (strongly) closed if the limit of each convergent sequence of elements of K belongs to K, that is

$$(u_n) \subset K, \quad u_n \to u \quad \textit{in} \quad X \Rightarrow u \in K.$$

ii) weakly closed if the limit of each weakly convergent sequence of elements of K belongs to K, that is

$$(u_n) \subset K, \quad u_n \rightharpoonup u \quad \textit{in} \quad X \Rightarrow u \in K.$$

Evidently, every weakly closed subset of X is (strongly) closed, but the converse is not true, in general. An exception is provided by the class of convex subsets of a Banach space, as shown in the following result.

Theorem A.1 (Mazur's theorem) *A convex subset of a Banach space is (strongly) closed if and only if it is weakly closed.*

Here, for the convenience of the reader, we recall that a subset K of a linear space is said to be convex if it has the property

$$u,\, v \in K \Rightarrow (1-t)u + tv \in K \quad \forall t \in [0,1].$$

Let X and Y be linear spaces. A mapping $a : X \times Y \to \mathbb{R}$ is called a bilinear form if it is linear in each argument, that is for any $u_1,\, u_2,\, u \in X,\, v_1,\, v_2,\, v \in Y$, and $\alpha_1,\, \alpha_2 \in \mathbb{R}$,

$$a(\alpha_1 u_1 + \alpha_2 u_2, v) = \alpha_1 a(u_1, v) + \alpha_2 a(u_2, v),$$

$$a(u, \alpha_1 v_1 + \alpha_2 v_2) = \alpha_1 a(u, v_1) + \alpha_2 a(u, v_2).$$

For the case $X = Y$ we say that a bilinear form is symmetric if,

$$a(u, v) = a(v, u) \quad \forall u,\, v \in X.$$

Let now $(X, \|\,.\,\|_X)$ and $(Y, \|\,.\,\|_Y)$ be two normed spaces. A bilinear form $a : X \times Y \to \mathbb{R}$ is said to be continuous if there exists a constant $M > 0$ such that

$$|a(u, v)| \leq M \|u\|_X \|v\|_Y \quad \forall u \in X, \quad v \in Y.$$

For the case $X = Y$ we say that a bilinear form is positive if

$$a(u, u) \geq 0 \quad \forall u \in X,$$

and X-elliptic if there exists a constant $m > 0$ such that

$$a(u, u) \geq m \|u\|_X^2 \quad \forall u \in X.$$

It is easy to see that if the bilinear form $a(\,.\,,\,.\,)$ is X-elliptic then it is positive. The converse of this property is not true, in general.

A.2 Fixed point theorems

Let X be a Banach space with the norm $\|\,.\,\|_X$, and K a subset of X. Also, let $\Lambda : K \to X$ be an operator defined on K. We are interested in the existence of a solution $u \in K$ of the operator equation

$$\Lambda u = u. \tag{A.10}$$

An element $u \in K$ which satisfies (A.10) is called a fixed point of the operator Λ.

We introduce in what follows two main theorems which state the existence of the fixed points of nonlinear operators: Banach's and Schauder's fixed point theorems. Both of them represent major results of functional analysis.

Theorem A.2 (Banach's fixed point theorem) *Let K be a nonempty closed subset of a Banach space $(X, \|\,.\,\|_X)$. Assume that $\Lambda : K \to K$ is a contraction, i.e., there exists a constant $\alpha \in [0, 1)$ such that*

$$\|\Lambda u - \Lambda v\|_X \leq \alpha \|u - v\|_X \quad \forall u,\, v \in K. \tag{A.11}$$

Then there exists a unique $u \in K$ such that $\Lambda u = u$.

We also recall in what follows an other version of Banach's fixed point theorem. To this end, for an operator Λ, we define its powers inductively by the formula $\Lambda^m = \Lambda(\Lambda^{m-1})$ for $m \geq 2$.

Theorem A.3 *Assume that K is a nonempty closed subset of a Banach space X and let $\Lambda : K \to K$. Assume also that $\Lambda^m : K \to K$ is a contraction for some positive integer m. Then Λ has a unique fixed point.*

We turn now to Schauder's fixed point theorem. To introduce it we need the following preliminaries.

Definition A.4 *Let X be a normed space. A subset $K \subset X$ is called:*

i) Bounded if there exists $M > 0$ such that

$$\|u\|_X \leq M \quad \forall u \in K.$$

ii) Relatively sequentially compact if each sequence in K has a convergent subsequence in X.

It follows from Definition A.4 i) that if $K \subset X$ is bounded, then, every sequence $(u_n) \subset K$ satisfies (A.3) and, therefore, is bounded. Moreover, Definition A.4 ii) shows that a subset $K \subset X$ is relatively sequentially compact if for each $(u_n) \subset K$ there exists a subsequence (u_{n_k})

and an element $u \in X$ such that $u_{n_k} \to u$ in X. Note also that, for simplicity, everywhere below we shall use the terminology *relatively compact* instead of *relatively sequentially compact*.

Definition A.5 *Let X and Y be normed spaces. The operator $\Lambda : K \subset X \to Y$ is called:*

i) Continuous at the point $u \in K$ if for each sequence $(u_n) \subset K$ which converges in X to u, the sequence $(\Lambda u_n) \subset Y$ converges to Λu in Y, that is

$$(u_n) \subset K, \quad u_n \to u \quad in \quad X \Rightarrow \Lambda u_n \to \Lambda u \quad in \quad Y.$$

ii) Continuous if it is continuous at each point $u \in K$.

iii) Compact if it is continuous and maps bounded sets into relatively compact sets.

It follows from Definitions A.5 iii) and A.4 ii) that a continuous operator $\Lambda : K \subset X \to Y$ is compact if and only if for each bounded sequence $(u_n) \subset K$ there exists a subsequence $(u_{n_k}) \subset K$ such that the sequence (Λu_{n_k}) is convergent in Y.

We proceed with the following result.

Theorem A.4 (Schauder's fixed point theorem) *Let K be a nonempty closed convex bounded subset of a Banach space X, and let $\Lambda : K \to X$ be a compact operator such that $\Lambda(K) \subset K$. Then Λ has at least one fixed point*

The proof of Theorem A.4 requires a number of preliminary results. Nevertheless, we indicate that such a proof can be found in [[158], p. 56]. Also, note that, unlike Banach's fixed point theorem, Schauder's fixed point theorem does not provide the uniqueness of the fixed point of the operator Λ.

A.3 Hilbert spaces

We introduce in what follows some useful results which are valid in Hilbert spaces. This concerns the projection operators, some properties related to the Riesz representation theorem, together with its consequences.

A.3.1 Projection operators

The projection operators represent an important class of nonlinear operators defined in a Hilbert space, to introduce them we need the following existence and uniqueness result.

Theorem A.5 (The projection lemma) *Let K be a nonempty closed convex subset of a Hilbert space X. Then, for each $f \in X$ there exists a unique element $u \in K$ such that*

$$\|u - f\|_X = \min_{v \in K} \|v - f\|_X. \tag{A.12}$$

Theorem A.5 allows us to introduce the following definition.

Definition A.6 *Let K be a nonempty closed convex subset of a Hilbert space X. Then, for each $f \in X$ the element u which satisfies (A.12) is called the projection of f on K and is usually denoted $\mathcal{P}_K f$. Moreover, the operator $\mathcal{P}_K : X \to K$ is called the projection operator on K.*

It follows from Definition A.6 that

$$f = \mathcal{P}_K f \Leftrightarrow f \in K. \tag{A.13}$$

We conclude from (A.13) that the element $f \in X$ is a fixed point of the projection operator $\mathcal{P}_K$ if and only if $f \in K$.

Next, we present the following characterization of the projection.

Proposition A.1 *Let K be a nonempty closed convex subset of a Hilbert space X and let $f \in X$. Then $u = \mathcal{P}_K f$ if and only if*

$$u \in K, \quad (u, v - u)_X \geq (f, v - u)_X \quad \forall v \in K. \tag{A.14}$$

Note that, besides the characterization of the projection in terms of inequalities, Proposition A.1 provides, implicitly, the existence of a unique solution to the inequality (A.14). Moreover, using this proposition it is easy to prove the following results.

Proposition A.2 *Let K be a nonempty closed convex subset of a Hilbert space X. Then the projection operator $\mathcal{P}_K$ satisfies the following inequalities:*

$$(\mathcal{P}_K u - \mathcal{P}_K v, u - v)_X \geq 0 \quad \forall u,\, v \in X, \tag{A.15}$$

$$\|\mathcal{P}_K u - \mathcal{P}_K v\|_X \leq \|u - v\|_X \quad \forall u,\, v \in X. \tag{A.16}$$

Proposition A.3 *Let K be a nonempty closed convex subset of a Hilbert space X and let $G_K : X \to X$ be the operator defined by*

$$G_K u = u - \mathcal{P}_K u \quad \forall u \in X. \tag{A.17}$$

Then, the following properties hold:

$$(G_K u - G_K v, u - v)_X \geq 0 \quad \forall u,\, v \in X, \tag{A.18}$$

$$\|G_K u - G_K v\|_X \leq 2\|u - v\|_X \quad \forall u,\, v \in X, \tag{A.19}$$

$$(G_K u, v - u)_X \leq 0 \quad \forall u,\, v \in X, \tag{A.20}$$

$$G_K u = 0 \quad \textit{if and only if} \quad u \in K. \tag{A.21}$$

A.3.2 Duality and weak convergence

On Hilbert spaces, linear continuous functionals are limited in the forms they can take. The following theorem makes this more precise.

Theorem A.6 (The Riesz representation theorem) *Let $(X, (\,.\,,\,.\,)_X)$ be a Hilbert space and let $\ell \in X'$. Then there exists a unique $u \in X$ such that*

$$\ell(v) = (u, v)_X \quad \forall v \in X. \tag{A.22}$$

Moreover,

$$\|\ell\|_{X'} = \|u\|_X. \tag{A.23}$$

The proof of Theorem A.6 can be found in [[144], p. 17].

We proceed with some important consequences of the Riesz representation theorem concerning the weak convergence in a Hilbert space. First, we recall that if $(X, (\,.\,,\,.\,)_X)$ is an inner product space then, (A.9) holds.

Assume now that $(X, (\,.\,,\,.\,)_X)$ is a Hilbert space. Then, it follows from Riesz's representation theorem that the converse of (A.9) is true, *i.e.*,

$$(u_n, v)_X \to (u, v)_X \quad \text{as} \quad n \to +\infty, \quad \forall v \in X \Rightarrow u_n \rightharpoonup u \quad \text{in} \quad X.$$

We conclude from this that a sequence $(u_n) \subset X$ converges weakly to $u \in X$ if and only if

$$(u_n, v)_X \to (u, v)_X \quad \text{as} \quad n \to +\infty, \quad \forall v \in X.$$

The Riesz representation theorem also allows to identify a Hilbert space with its dual and, therefore, with its bidual which, roughly speaking, shows that each Hilbert space is reflexive.

Based on this result we have the following important property which represents a particular case of the well-known Eberlein-Smulyan theorem.

Theorem A.7 *If X is a Hilbert space, then any bounded sequence in X has a weakly convergent subsequence.*

It follows that if X is a Hilbert space and the sequence $(u_n) \subset X$ is bounded, that is, $\sup_n \|u_n\|_X < +\infty$, then there exists a subsequence $(u_{n_k}) \subset (u_n)$ and an element $u \in X$ such that $u_{n_k} \rightharpoonup u$ in X. Furthermore, if the limit u is independent of the subsequence, then the whole sequence (u_n) converges weakly to u, as stated in the following result.

Theorem A.8 *Let X be a Hilbert space and let (u_n) be a bounded sequence of elements in X such that each weakly convergent subsequence of (u_n) converges weakly to the same limit $u \in X$. Then $u_n \rightharpoonup u$ in X.*

A.4 Elements of nonlinear analysis

In the study of variational inequalities presented in Chapters 3, 4, 5 and 6, we need several results on nonlinear operators and convex functions that we introduce in this section

A.4.1 Monotone operators

The projection operator on a convex subset K of a Hilbert space is, in general, a nonlinear operator on X. Its properties (A.15) and (A.16) can be extended as follows

Definition A.7 *Let X be a space with inner product $(\,.\,,\,.\,)_X$ and norm $\|\,.\,\|_X$ and let $A : X \to X$ be an operator. The operator A is said to be monotone if*

$$(Au - Av, u - v)_X \geq 0 \quad \forall u,\, v \in X.$$

The operator A is strictly monotone if

$$(Au - Av, u - v)_X > 0 \quad \forall u,\, v \in X, \quad u \neq v,$$

and strongly monotone if there exists a constant $m_A > 0$ such that

$$(Au - Av, u - v)_X \geq m_A \|u - v\|_X^2 \quad \forall u,\, v \in X. \tag{A.24}$$

The operator A is nonexpansive if

$$\|Au - Av\|_X \leq \|u - v\|_X \quad \forall u,\, v \in X,$$

and Lipschitz continuous if there exists $M_A > 0$ such that

$$\|Au - Av\|_X \leq M_A \|u - v\|_X \quad \forall u,\, v \in X. \tag{A.25}$$

Finally, the operator A is hemicontinuous if the real-valued function

$$\theta \mapsto (A(u + \theta v), w)_X \quad \textit{is continuous on} \quad \mathbb{R}, \quad \forall u,\, v,\, w \in X,$$

and, recalling Definition A.5, A is continuous if

$$u_n \to u \quad \textit{in} \quad X \Rightarrow Au_n \to Au \in X.$$

It follows from the definition above that each strongly monotone operator is strictly monotone and a nonexpansive operator is Lipschitz continuous, with Lipschitz constant $M_A = 1$. Also, it is easy to check that a Lipschitz continuous operator is continuous and a continuous operator is hemicontinuous. Moreover, it follows from Proposition A.2 that the projection operators are monotone and nonexpansive.

In many applications it is not necessary to define nonlinear operators on the entire space X. Indeed, in the study of the variational inequalities presented in Chapters 3, 4, 5 and 6, we shall consider strongly monotone Lipschitz continuous operators defined on a subset $K \subset X$. For this reason, we complete Definition A.7 with the following one.

Definition A.8 *Let X be a space with inner product $(\,.\,,\,.\,)_X$ and norm $\|\,.\,\|_X$ and let $K \subset X$. An operator $A : K \to X$ is said to be strongly monotone if there exists a constant $m_A > 0$ such that*

$$(Au - Av, u - v)_X \geq m_A \|u - v\|_X^2 \quad \forall u,\, v \in K. \tag{A.26}$$

The operator A is Lipschitz continuous if there exists $M_A > 0$ such that

$$\|Au - Av\|_X \leq M_A \|u - v\|_X \quad \forall u,\, v \in K. \tag{A.27}$$

The following result involving monotone operators is used in Chapters 3, 4, 5 and 6, in the analysis of elliptic variational inequalities.

Proposition A.4 *Let $(X, (\,.\,,\,.\,)_X)$ be an inner product space and let $A : X \to X$ be a monotone hemicontinuous operator. Assume that (u_n) is a sequence of elements in X which converges weakly to the element $u \in X$, i.e.,*

$$u_n \rightharpoonup u \quad as \quad n \to +\infty. \tag{A.28}$$

Moreover, assume that

$$\limsup_{n\to+\infty} (Au_n, u_n - u)_X \le 0. \tag{A.29}$$

Then, for all $v \in X$, the following inequality holds:

$$\liminf_{n\to+\infty} (Au_n, u_n - v)_X \ge (Au, u - v)_X. \tag{A.30}$$

An operator $A : X \to X$ for which (A.28) and (A.29) imply (A.30) for all $v \in X$ is called a pseudomonotone operator. We conclude from Proposition A.4 that every monotone hemicontinuous operator on a Hilbert space is a pseudomonotone operator.

We proceed with the following existence and uniqueness result in the study of nonlinear equations involving monotone operators.

Theorem A.9 *Let X be a Hilbert space and let $A : X \to X$ be a strongly monotone Lipschitz continuous operator. Then, for each $f \in X$ there exists a unique element $u \in X$ such that $Au = f$.*

Theorem A.9 shows that if $A : X \to X$ is a strongly monotone Lipschitz continuous operator defined on a Hilbert space X, then A is invertible. The properties of its inverse, denoted A^{-1}, are given by the following result.

Proposition A.5 *Let X be a Hilbert space and let $A : X \to X$ be a strongly monotone Lipschitz continuous operator. Then, $A^{-1} : X \to X$ is a strongly monotone Lipschitz continuous operator.*

A.4.2 Convex lower semicontinuous functions

Convex lower semicontinuous functions represent a crucial ingredient in the study of variational inequalities. To introduce them, we start with the following definitions.

Definition A.9 *Let X be a linear space and let K be a nonempty convex subset of X. A function $\varphi : K \to \mathbb{R}$ is said to be convex if*

$$\varphi((1-t)u + tv) \le (1-t)\varphi(u) + t\varphi(v) \quad \forall u,\, v \in K, \quad and \quad t \in [0,1]. \tag{A.31}$$

The function φ is strictly convex if the inequality in (A.31) is strict for $u \neq v$ and $t \in (0,1)$.

We note that if φ, $\psi : K \to \mathbb{R}$ are convex and $\lambda \geq 0$, then the functions $\varphi + \psi$ and $\lambda\varphi$ are also convex.

Definition A.10 *Let $(X, \| \,.\, \|_X)$ be a normed space and let K be a nonempty closed convex subset of X. A function $\varphi : K \to \mathbb{R}$ is said to be lower semicontinuous (l.s.c.) at $u \in K$ if*

$$\liminf_{n \to +\infty} \varphi(u_n) \geq \varphi(u), \tag{A.32}$$

for each sequence $(u_n) \subset K$ converging to u in X. The function φ is l.s.c. if it is l.s.c. at every point $u \in K$. When inequality (A.32) holds for each sequence $(u_n) \subset K$ that converges weakly to u, the function φ is said to be weakly lower semicontinuous at u. The function φ is weakly l.s.c. if it is weakly l.s.c. at every point $u \in K$.

We note that if φ, $\psi : K \to \mathbb{R}$ are *l.s.c.* functions and $\lambda \geq 0$, then the functions $\varphi + \psi$ and $\lambda\varphi$ are also lower semicontinuous. Moreover, if $\varphi : K \to \mathbb{R}$ is a continuous function then it is also lower semicontinuous. The converse is not true and a lower semicontinuous function can be discontinuous. Since strong convergence in X implies weak convergence, it follows that a weakly lower semicontinuous function is lower semicontinuous. Moreover, the following results hold.

Proposition A.6 *Let $(X, \| \,.\, \|_X)$ be a Banach space, K a nonempty closed convex subset of X and $\varphi : K \to \mathbb{R}$ a convex function. Then φ is lower semicontinuous if and only if it is weakly lower semicontinuous.*

Proposition A.7 *Let $(X, \| \,.\, \|_X)$ be a normed space, K a nonempty closed convex subset of X and $\varphi : K \to \mathbb{R}$ a convex lower semicontinuous function. Then φ is bounded from below by an affine function, i.e., there exist $\ell \in X'$ and $\alpha \in \mathbb{R}$ such that $\varphi(v) \geq \ell(v) + \alpha$ for all $v \in K$.*

The proof of Proposition A.6 is a consequence of Mazur's theorem. It follows from this proposition that a convex continuous function $\varphi : K \to \mathbb{R}$ defined on Banach space X is weakly lower semicontinuous. In particular, the norm function $v \mapsto \|v\|_X$ is weakly lower semicontinuous. A second example of a lower semicontinuous function is provided by the following result.

Proposition A.8 *Let $(X, \| \,.\, \|_X)$ be a normed space and let $a : X \times X \to \mathbb{R}$ be a bilinear symmetric continuous and positive form. Then the function $v \mapsto a(v, v)$ is strictly convex and lower semicontinuous.*

In particular, it follows from Proposition A.8 that, if $(X, (\,.\,,\,.\,)_X)$ is an inner product space then the function $v \longmapsto \|v\|_X^2 = (v, v)_X$ is strictly convex and lower semicontinuous.

We now recall the definition of Gâteaux differentiable functions.

Definition A.11 *Let $(X, (\,.\,,\,.\,)_X)$ be an inner product space, $\varphi : X \to \mathbb{R}$ and $u \in X$. Then φ is Gâteaux differentiable at u if there exists an element $\nabla\varphi(u) \in X$ such that*

$$\lim_{t \to 0} \frac{\varphi(u + tv) - \varphi(u)}{t} = (\nabla\varphi(u), v)_X \quad \forall v \in X. \tag{A.33}$$

The element $\nabla\varphi(u)$ which satisfies (A.33) is unique and is called the gradient of φ at u. The function $\varphi : X \to \mathbb{R}$ is said to be Gâteaux differentiable if it is Gâteaux differentiable at every point of X. In this case the operator $\nabla\varphi : X \to X$ which maps every element $u \in X$ into the element $\nabla\varphi(u)$ is called the gradient operator of φ.

The convexity of Gâteaux differentiable functions can be characterized as follows.

Proposition A.9 *Let $(X, (\,.\,,\,.\,)_X)$ be an inner product space and let $\varphi : X \to \mathbb{R}$ be a Gâteaux differentiable function. Then the following statements are equivalent:*

i) φ is a convex function.

ii) φ satisfies the inequality

$$\varphi(v) - \varphi(u) \geq (\nabla\varphi(u), v - u)_X \quad \forall u,\, v \in X. \tag{A.34}$$

iii) the gradient of φ is a monotone operator, that is

$$(\nabla\varphi(u) - \nabla\varphi(v), u - v)_X \geq 0 \quad \forall u,\, v \in X. \tag{A.35}$$

From the previous proposition, we easily deduce the following result.

Corollary A.1 *Let $(X, (\,.\,,\,.\,)_X)$ be an inner product space and let $\varphi : X \to \mathbb{R}$ be a convex Gâteaux differentiable function. Then, φ is lower semicontinuous.*

A large number of boundary value problems in Contact Mechanics lead to variational formulations in which the frictional term is associated with a continuous seminorm. For this reason, we collect at the end of this subsection some results on continuous seminorms defined on a normed space, which we shall need in this manuscript.

Given a linear space X, we recall that a seminorm j is a function from X to $\mathbb{R}$ satisfying the following properties:

1. $j(u) \geq 0 \quad \forall u \in X$.

2. $j(\alpha u) = |\alpha| j(u) \quad \forall u \in X, \quad \forall \alpha \in \mathbb{R}$.

3. $j(u+v) \leq j(u) + j(v) \quad \forall u, v \in X$.

It follows from the above that a seminorm satisfies the properties of a norm except that $j(u) = 0$ does not necessarily imply $u = 0_X$. In particular, note that $j(0_X) = 0$. The continuity of a seminorm defined on a normed space is characterized by the following result.

Proposition A.10 *Let j be a seminorm on the normed space $(X, \|\,.\,\|_X)$. Then j is continuous if and only if there exists $m > 0$ such that*

$$j(v) \leq m\|v\|_X \quad \forall v \in X. \tag{A.36}$$

It is easy to see that a seminorm defined on a linear space is a convex function. Therefore, a direct consequence of Proposition A.10 is given by the following result.

Corollary A.2 *Let $(X, \|\,.\,\|_X)$ be a normed space and let j be a seminorm on X which satisfies (A.36) with some $m > 0$. Then j is a convex lower semicontinuous function.*

Note that in convex analysis it is usual to consider functions φ defined on a normed space X with values on $(-\infty, +\infty]$. For such functions, concepts such as convexity and lower semicontinuity are introduced in Definitions A.9 and A.10, respectively, by using the convention that $+\infty + \infty = +\infty$ while an expression of the form $+\infty - \infty$ is undefined. Nevertheless, in many applications in mechanics the domain of definition of various functions (like the energy function) is subjected to constraints. For this reason, above, we restricted ourselves to considering only functions $\varphi : K \to \mathbb{R}$ defined on a subset K of a normed or inner product space X. This choice is not restrictive. Indeed, assume that K is a nonempty closed convex subset of X, $\varphi : K \to \mathbb{R}$ is a given function and let $\widetilde{\varphi} : X \to (-\infty, +\infty]$ be the function given by

$$\widetilde{\varphi}(v) = \begin{cases} \varphi(v) = 0 & \text{if} \quad v \in K, \\ +\infty & \text{if} \quad n \notin K. \end{cases}$$

Then, it is easy to see that $\widetilde{\varphi}$ is convex and $l.s.c.$ if and only if φ is convex and $l.s.c.$

A.4.3 Minimization problems

An important property of convex lower semicontinuous functions is given by the following well-known theorem

Theorem A.10 (The Weierstrass theorem) *Let X be a Hilbert space and K a nonempty closed convex subset of X. Let $J : K \to \mathbb{R}$ be a convex lower semicontinuous function. Then J is bounded from below and attains its infimum on K whenever one of the following two conditions hold:*

i) *K is bounded.*

ii) *J is coercive, i.e., $J(u) \to +\infty$ as $\|u\|_X \to +\infty$.*

Moreover, if J is a strictly convex function, then, J attains its infimum on K at only one point.

We can use Theorem A.10 to prove the following result.

Theorem A.11 *Let X be a Hilbert space, K a nonempty closed convex subset of X, $a : X \times X \to \mathbb{R}$ a bilinear continuous symmetric and X-elliptic form, $j : K \to \mathbb{R}$ a convex lower semicontinuous function and $f \in X$. Denote by $J : K \to \mathbb{R}$ the function given by*

$$J(v) = \frac{1}{2}a(v, v) + j(v) - (f, v)_X \quad \forall v \in K, \tag{A.37}$$

and consider the minimization problem

$$u \in K, \quad J(v) \geq J(u) \quad \forall v \in K. \tag{A.38}$$

Then:

1. *an element u is a solution to problem (A.38) if and only if*

$$u \in K, \quad a(u, v - u) + j(v) - j(u) \geq (f, v - u)_X \quad \forall v \in K, \tag{A.39}$$

2. *there exists a unique solution to problem (A.38).*

We conclude from the first part of Theorem A.11 that the minimization problem (A.38) and inequality (A.39) are equivalent. Since the existence of a unique minimizer for J is guaranteed by the second part of this theorem, we deduce the following existence and uniqueness result.

Corollary A.3 *Let X be a Hilbert space, K a nonempty closed convex subset of X, $a : X \times X \to \mathbb{R}$ a bilinear continuous symmetric and X-elliptic form and $j : K \to \mathbb{R}$ a convex lower semicontinuous function. Then, for each $f \in X$ there exists a unique element u such that (A.39) holds.*

A.5 Decomposition-coordination methods by Augmented Lagrangian

A.5.1 Principle of the methods

The decomposition-coordination methods are based on the following obvious equivalence result.

Theorem A.12 *The following problem of minimization*

$$\min_{v\in V}\{F(Bv)+G(v)\}, \tag{A.40}$$

where

- V, Y *are topological vector spaces,*
- $B\in\mathscr{L}(V,Y)$,
- $F:Y\to\overline{\mathbb{R}}=\mathbb{R}\cup\{+\infty\}$, $G:V\to\overline{\mathbb{R}}$ *are convex proper l.s.c. functional.*

is equivalent to

$$\min_{(v,q)\in W}\{F(q)+G(v)\}, \tag{A.41}$$

where

$$W=\{(v,q)\in V\times Y\,;\ Bv-q=0\}.$$

In the sequel, we shall assume that V and Y are Hilbert spaces with inner products and norms denoted by $(V,(\,.\,,\,.\,)_V)$, $\|\,.\|_V$ and $(Y,(\,.\,,\,.\,)_Y)$, $\|\,.\|_Y$, respectively. We then define a Lagrangian functional $\mathfrak{L}$ associated with problem (A.41), by

$$\mathfrak{L}(v,q,\mu)=F(q)+G(v)+(\mu,Bv-q)_Y, \tag{A.42}$$

and for $r>0$ an Augmented Lagrangian $\mathfrak{L}_r$ is defined by

$$\mathfrak{L}_r(v,q,\mu)=\mathfrak{L}(v,q,\mu)+\frac{r}{2}\|Bv-q\|_Y^2. \tag{A.43}$$

Remark A.1 *Augmented Lagrangian methods for solving general optimization problems have been introduced by Hestenes [66] and Powell [121] . Augmented Lagrangian methods for solving problems like* (A.42) *via* (A.43) *have been introduced by Glowinski and Marrocco [59].*

A.6 Properties of problem (A.40) and of the Saddle Points of $\mathfrak{L}$ and $\mathfrak{L}_r$

A.6.1 Existence and uniqueness properties for problem (A.40)

Let us define $J : V \to \overline{\mathbb{R}}$ by

$$J(v) = F(Bv) + G(v).$$

Then, problem (A.40) can also be written

$$J(u) < J(v), \quad \forall v \in V, \ u \in V. \tag{A.44}$$

Let $j : X \to \overline{\mathbb{R}}$, we define the so-called domain of $j(\,.\,)$ by

$$dom(j) = \{x \in X\,;\ j(x) \in \mathbb{R}\}.$$

Then, if

$$dom(F \circ B) \cap dom(G) = \emptyset, \tag{A.45}$$

J is convex, proper, and $l.s.c.$. Therefore, sufficient conditions for problem (A.42) to have a unique solution are in Theorem A.10 (Ekeland and Temam [53]).

Remark A.2 *If B is an injection from V to Y, with $R(B)$ (= range of B) closed in Y, then $\|Bv\|_Y$ is a norm on V equivalent to $\|v\|_V$.*

A.6.2 Properties of the saddle points of $\mathfrak{L}$ and $\mathfrak{L}_r$

We have the following result.

Theorem A.13 *Let (u, p, λ) be a saddle point of $\mathfrak{L}$ on $V \times Y \times Y$, then (u, p, λ) is also a saddle point of $\mathfrak{L}_r$, $\forall r > 0$ and conversely. Moreover, u is a solution of problem (A.40) and $p = Bu$.*

The proof of Theorem A.13 can be found in [58].

A.7 Korn's inequality and its consequence

Before we enunciate Korn's inequality, let's start with two simple inequalities:

Lemma A.1 *Let Ω be an open subset of $\mathbb{R}^d$, for all v in $H^1(\Omega)^d$, we have*

$$\|\varepsilon(v)\|_{L^2(\Omega)^d} \leq \|\nabla v\|_{L^2(\Omega)^d}, \tag{A.46}$$

$$\|\operatorname{div}(v)\|_{L^2(\Omega)^d} \leq d\|\nabla v\|_{L^2(\Omega)^d}. \tag{A.47}$$

The inequality inverse of (A.46) is false. However, in the case of functions in $H_0^1(\Omega)^d$, the following result holds.

Lemma A.2 *Let Ω be a regular open subset of $\mathbb{R}^d$, for all function v in $H_0^1(\Omega)^d$, we have*

$$\|\nabla v\|_{L^2(\Omega)^d} \leq \sqrt{2}\|\varepsilon(v)\|_{L^2(\Omega)^d}. \tag{A.48}$$

In the previous lemma, the function v is nil at the boundary. We now state Korn's inequality, which generalizes the previous lemma to non-nil functions at the boundary. The cost is that the open subset Ω must be bounded.

Proposition A.11 (Korn's inequality) *Let Ω be a bounded, regular open subset of $\mathbb{R}^d$. There exists a constant C_Ω such that, for all function v in $H^1(\Omega)^d$, we have*

$$\|v\|^2_{H^1(\Omega)} \leq C_\Omega\Big(\|v\|^2_{L^2(\Omega)^d} + \|\varepsilon(v)\|^2_{L^2(\Omega)^d}\Big). \tag{A.49}$$

An important consequence of Korn's inequality, which we need in the study of elasticity problems, is the following proposition.

Proposition A.12 *Let Ω be a connected, bounded and regular open subset of $\mathbb{R}^d$, with $d = 2, 3$. Let $\Gamma_1 \subset \partial\Omega$ be a subset of the Ω boundary with a non-nil surface measure, i.e., meas$(\Gamma_1) > 0$, and let*

$$V = \{\, v \in H^1(\Omega)^d\,;\ v = 0 \text{ on } \Gamma_1\},$$

there exists a constant C_Ω such that, for all function v in V, we have

$$\|v\|^2_{H^1(\Omega)^d} \leq C_\Omega\|\varepsilon(v)\|^2_{L^2(\Omega)^d}. \tag{A.50}$$

There have been several nice proofs of Korn's inequality in the literature, see for instance [41, 52] and the references therein.

Annex B

In this annex, we present theorems on the unique solvability of elliptic variational inequalities with strongly monotone Lipschitz continuous operators. We start with an existence and uniqueness result for elliptic variational inequalities of the first kind, then, we extend it to elliptic variational inequalities of the second kind. We also present various convergence results. Besides their own interest, the results presented in this chapter represent crucial tools in the analysis of static frictionless and frictional contact problems with elastic materials. Even if most of the results we present in this chapter still remain valid for more general cases, we restrict ourselves to the framework of strongly monotone Lipschitz continuous operators in Hilbert spaces. Everywhere in this chapter, X denotes a real Hilbert space with inner product $(\,.\,,\,.\,)_X$ and norm $\|\,.\,\|_X$.

B.1 Variational inequalities of the first kind

In this section, we provide an extension of the existence and uniqueness result of Theorem A.9. Thus, given an operator $A : X \to X$, a subset $K \subset X$ and an element $f \in X$, we consider the problem of finding an element u such that

$$u \in K, \quad (Au, v-u)_X \geq (f, v-u)_X \quad \forall v \in K. \tag{B.1}$$

An inequality of the form (B.1) is called an elliptic variational inequality of the first kind. An example is provided by inequality (A.14) which characterizes the projection on the nonempty closed convex subset $K \subset X$. Moreover, Theorem A.5, Definition A.6 and Proposition A.1 provide, implicitly, a first existence result in the study of such inequalities.

B.1.1 Existence and uniqueness

The first result we present in this section is the following.

Theorem B.1 *Let X be a Hilbert space and let $K \subset X$ be a nonempty closed convex subset. Assume that $A : K \to X$ is a strongly monotone Lipschitz continuous operator, i.e., it satisfies conditions* (A.26) *and* (A.27)*. Then, for each $f \in X$ the variational inequality* (B.1) *has a unique solution.*

Assume now that $K = X$. Then, taking $v = u \pm w$ it is easy to see that the variational inequality (B.1) is equivalent to the variational equation

$$(Au, w)_X = (f, w)_X \quad \forall w \in X,$$

which, in turn, is equivalent to the nonlinear equation $Au = f$. We conclude from above that Theorem B.1 represents an extension of Theorem A.9, as claimed at the beginning of this section.

B.1.2 Penalization

We now investigate the unique solvability of the variational inequality (B.1) by using a penalization method. To this end, we assume in what follows that $A : X \to X$ and we consider an operator G which satisfies the following conditions:

$$\left.\begin{array}{ll} (a) & G : X \to X \quad \text{is a monotone Lipschitz continuous operator,} \\ (b) & (Gu, v - u)_X \leq 0 \quad \forall u \in X,\ v \in K, \\ (c) & Gu = 0_X \quad \text{if and only if} \quad u \in K. \end{array}\right\} \tag{B.2}$$

Note that such an operator G always exists, as shown in Proposition A.3. For each $\epsilon > 0$, we consider the problem of finding an element u_ϵ such that

$$u_\epsilon \in X, \quad Au_\epsilon + \frac{1}{\epsilon} Gu_\epsilon = f. \tag{B.3}$$

We have the following existence, uniqueness and convergence result.

Theorem B.2 *Let X be a Hilbert space and let $K \subset X$ be a nonempty closed convex subset. Assume that $A : X \to X$ is a strongly monotone Lipschitz continuous operator, G is an operator which satisfies* (B.2) *and $f \in X$. Then:*

1. *For each $\epsilon > 0$ there exists a unique element u_ϵ which solves the nonlinear equation* (B.3).

2. *There exists a unique element u which solves the variational inequality* (B.1).

3. *The solution u_ϵ of* (B.3) *converges strongly to the solution u of* (B.1)*, i.e.,*

$$u_\epsilon \to u \quad in\ X \quad as \quad \epsilon \to 0. \tag{B.4}$$

Note that the convergence (B.4) above is understood in the following sense: for every sequence $(\epsilon_n) \subset \mathbb{R}_+$ converging to 0 as $n \to +\infty$ one has $u_{\epsilon_n} \to u$ in X as $n \to +\infty$. We shall use such notation to indicate various convergences in the rest of the manuscript.

The proof of Theorem B.2 will be carried out in several steps, based on arguments of compactness and monotonicity. We suppose in what follows that the assumptions of Theorem B.2 hold. We use c to denote a positive constant which does not depend on ϵ, and whose value may change from line to line. We start with the unique solvability of the nonlinear equation (B.3).

Lemma B.1 *For each $\epsilon > 0$ there exists a unique element which solves the nonlinear equation* (B.3).

Next, we perform a priori estimates on the solution of equation (B.3), which imply the following convergence result.

Lemma B.2 *There exists an element $u \in X$ and a subsequence of the sequence (u_ϵ), again denoted (u_ϵ), which converges weakly to u, i.e.,*

$$u_\epsilon \rightharpoonup u \quad in\ X \quad as \quad \epsilon \to 0. \tag{B.5}$$

Next, we investigate the properties of the element u defined in Lemma B.2.

Lemma B.3 *The element u satisfies the variational inequality* (B.1) *and, moreover, it is the unique solution of this inequality.*

We proceed our analysis with the following weak convergence result.

Lemma B.4 *The whole sequence (u_ϵ) converges weakly in X to the element u.*

The last step is provided by the following strong convergence result.

Lemma B.5 *The sequence* (u_ϵ) *converges strongly in* X *to the element* u*, that is*

$$u_\epsilon \to u \quad in\ X \quad as \quad \epsilon \to 0. \tag{B.6}$$

We can now easily provide the proof of Theorem B.2.

Proof. The points 1., 2. and 3. of Theorem B.2 are direct consequences of Lemmas B.1, B.3 and B.5, respectively. □

The interest in Theorem B.2 is twofold; first, it provides the existence and uniqueness of the solution to the variational inequality (B.1); second, it shows that the solution of (B.1) represents the strong limit of the sequence of solutions u_ϵ of the problem (B.3), as $\epsilon \to 0$.

B.2 Variational inequalities of the second kind

Given a set $K \subset X$, an operator $A : K \to X$, a function $j : K \to \mathbb{R}$ and an element $f \in X$, in this section we consider the problem of finding an element u such that

$$u \in K, \quad (Au, v-u)_X + j(v) - j(u) \geq (f, v-u)_X \quad \forall v \in K. \tag{B.7}$$

A variational inequality of the form (B.7) is called an elliptic variational inequality of the second kind. An example is provided by inequality (A.39) which characterizes the minimizers of the functional (A.37) on K. Moreover, Corollary A.3 provides a first existence and uniqueness result in the study of this kind of inequality. Finally, note that in the particular case when $j \equiv 0$, the variational inequality (B.7) represents an elliptic variational inequality of the form (B.1), *i.e.*, an elliptic variational inequality of the first kind.

B.2.1 Existence and uniqueness

In the study of (B.7) we consider the following assumptions:

$$K \text{ is a nonempty closed convex subset of } X. \tag{B.8}$$

$$\begin{aligned}&A : K \to X \text{ is a stronglymonotone Lipschitz continuous operator, } i.e.,\\ &\quad \text{it satisfies conditions (A.26) and (A.27).}\end{aligned} \tag{B.9}$$

$$j : K \to \mathbb{R} \quad \text{is a convex } l.s.c. \text{ function.} \tag{B.10}$$

The main result of this section is the following.

Theorem B.3 *Let X be a Hilbert space and assume that* (B.8)-(B.10) *hold. Then, for each $f \in X$ the variational inequality* (B.7) *has a unique solution.*

The proof of Theorem B.3 will be carried out in two steps. The first step consists of solving (B.7) in the case when A is the identity operator and obtaining an estimated result.

Lemma B.6 *Assume* (B.8) *and* (B.10). *Then, for each $f \in X$ there exists a unique element u such that*

$$u \in K, \quad (u, v-u)_X + j(v) - j(u) \geq (f, v-u)_X \quad \forall v \in K. \tag{B.11}$$

Moreover, if u_1 and u_2 denote the solutions of the inequality (B.11) *for $f = f_1 \in X$ and $f = f_2 \in X$, respectively, then*

$$\|u_1 - u_2\|_X \leq \|f_1 - f_2\|_X. \tag{B.12}$$

Proof. The first part of the lemma is a direct consequence of Corollary A.3. Let now $f_1,\ f_2 \in X$ and let $u_1,\ u_2 \in K$ be such that

$$(u_1, v-u_1)_X + j(v) - j(u_1) \geq (f_1, v-u_1)_X \quad \forall v \in K, \tag{B.13}$$

$$(u_2, v-u_2)_X + j(v) - j(u_2) \geq (f_2, v-u_2)_X \quad \forall v \in K. \tag{B.14}$$

Taking $v = u_2$ in (B.13), $v = u_1$ in (B.14) and adding the resulting inequalities, we find that

$$\|u_1 - u_2\|_X^2 \leq (f_1 - f_2, u_1 - u_2)_X, \tag{B.15}$$

which implies (B.12). □

Lemma B.6 allows us to introduce the following definition.

Definition B.1 *Let X be a Hilbert space, $K \subset X$ a nonempty closed convex subset and $j : K \to \mathbb{R}$ a convex lower semicontinuous function. Then, for each $f \in X$, the solution u of the variational inequality* (B.11) *is called the proximal element of f with respect to the function j and it is usually denoted $\mathrm{prox}_j f$. The operator $\mathrm{prox}_j : X \to K$ defined by $f \mapsto \mathrm{prox}_j f$ is called the proximity operator of the function j.*

The proximity operators were first introduced in [103] Note that Lemma B.6 states that prox_j is a nonexpansive operator, *i.e.*,

$$\|\mathrm{prox}_j f_1 - \mathrm{prox}_j f_2\|_X \leq \|f_1 - f_2\|_X \quad \forall f_1,\ f_2 \in X. \tag{B.16}$$

Moreover, it follows from (B.15) that prox_j is a monotone operator, since

$$(\text{prox}_j f_1 - \text{prox}_j f_2, f_1 - f_2)_X \geq 0 \quad \forall f_1,\, f_2 \in X.$$

Finally, we remark that if K is a nonempty closed convex subset of X, we may consider the proximity operator of the zero function on K, denoted prox_K. It follows from Definition B.1 that $u = \text{prox}_K f$ if and only if

$$u \in K, \quad (u, v - u)_X \geq (f, v - u)_X \quad \forall v \in K.$$

Therefore, using Proposition A.1 we obtain that $\text{prox}_K = \mathcal{P}_K$, where $\mathcal{P}_K$ denotes the projection operator on K. We conclude from the above that the projection operators represent a particular type of proximity operator.

We have now all the ingredients to provide the proof of Theorem B.3.

Recall that for $j \equiv 0$ the variational inequality (B.7) reduces to the variational inequality (B.1) and, moreover, for $K = X$ and $j \equiv 0$ it reduces to the nonlinear equation $Au = f$. We conclude that Theorems A.9 and B.1 can be recovered by Theorem B.3.

B.2.2 A convergence result

We proceed with a result concerning the dependence of the solution of the variational inequality (B.7) with respect to the function j. To this end, we assume in what follows that (B.8)-(B.10) hold, $f \in X$ and, for each $\epsilon > 0$, let j_ϵ be a perturbation of j which satisfies (B.10). We consider the problem of finding an element u_ϵ, such that

$$u_\epsilon \in K, \quad (Au_\epsilon, v - u_\epsilon)_X + j_\epsilon(v) - j_\epsilon(u_\epsilon) \geq (f, v - u_\epsilon)_X \quad \forall v \in K, \tag{B.17}$$

We deduce from Theorem B.3 that inequality B.7 has a unique solution $u \in K$ and, for each $\epsilon > 0$, inequality (B.17) has a unique solution $u_\epsilon \in K$.

Consider now the following assumption.

$$\left.\begin{array}{l} \text{There exists } G : \mathbb{R}_+ \to \mathbb{R}_+ \text{ such that:} \\ (a) \quad j_\epsilon(u_1) - j_\epsilon(u_2) + j(u_2) - j(u_1) \leq G(\epsilon)\|u_1 - u_2\|_X \\ \qquad \forall u_1,\, u_2 \in K, \quad \text{for each} \quad \epsilon > 0, \\ (b) \quad \lim\limits_{\epsilon \to 0} G(\epsilon) = 0. \end{array}\right\} \tag{B.18}$$

Then, the behavior of the solution u_ϵ as ϵ converges to zero is given in the following theorem.

Theorem B.4 *Under the assumption* (B.18), *the solution* u_ϵ *of problem* (B.17) *converges to the solution* u *of problem* (B.7), *i.e.,*

$$u_\epsilon \to u \quad \textit{in } X \quad \textit{as} \quad \epsilon \to 0. \tag{B.19}$$

Note that, as in the case of (B.4), the convergence (B.19) is understood in the following sense: for every sequence $(\epsilon_n) \subset \mathbb{R}_+$ converging to 0 as $n \to +\infty$ one has $u_{\epsilon_n} \to u$ in X as $n \to +\infty$.

B.2.3 Regularization

We consider now the variational inequality (B.7) in the case when $K = X$ and we investigate its unique solvability by using a regularization method. Thus, we assume in what follows that $A : X \to X$, $j : X \to \mathbb{R}$ and $f \in X$ are given and we consider the problem of finding an element u such that

$$u \in X, \quad (Au, v - u)_X + j(v) - j(u) \geq (f, v - u)_X \quad \forall v \in X. \tag{B.20}$$

Let $\epsilon > 0$ be a parameter. We also assume that there exists a family of functionals (j_ϵ) which satisfies:

$$\left.\begin{array}{ll} (a) & j_\epsilon : X \to \mathbb{R} \quad \text{is a convex Gâteaux differentiable function, for each } \epsilon > 0. \\ (b) & \nabla j_\epsilon : X \to X \quad \text{is a Lipschitz continuous operator, for each } \epsilon > 0. \end{array}\right\} \tag{B.21}$$

$$\left.\begin{array}{ll} \multicolumn{2}{l}{\text{There exists } F : \mathbb{R}_+ \to \mathbb{R}_+ \text{ such that}} \\ (a) & |j_\epsilon(v) - j(v)| \leq F(\epsilon) \quad \forall v \in X, \quad \text{for each } \epsilon > 0. \\ (b) & \lim\limits_{\epsilon \to 0} F(\epsilon) = 0. \end{array}\right\} \tag{B.22}$$

For each $\epsilon > 0$ we consider the problem of finding an element u_ϵ such that

$$u_\epsilon \in X, \quad Au_\epsilon + \nabla j_\epsilon(u_\epsilon) = f. \tag{B.23}$$

We have the following existence, uniqueness and convergence result.

Theorem B.5 *Let* X *be a Hilbert space. Assume that* $A : X \to X$ *is a strongly monotone Lipschitz continuous operator,* $j : X \to \mathbb{R}$, (j_ϵ) *is a family of functionals which satisfies* (B.21), (B.22) *and* $f \in X$. *Then:*

1. *For each* $\epsilon > 0$ *there exists a unique element* u_ϵ *which solves the nonlinear equation* (B.23).

2. *There exists a unique element* u *which solves the variational inequality* (B.20).

3. *The solution* u_ϵ *of* (B.23) *converges strongly to the solution* u *of* (B.20), *i.e.,*

$$u_\epsilon \to u \quad \textit{in } X \quad \textit{as} \quad \epsilon \to 0.$$

The proof of Theorem B.5 will be carried out in several steps, based on arguments of compactness and monotonicity. We suppose in what follows that the assumptions of Theorem B.5 hold, moreover, we use c to denote a positive constant which may depend on A and f, but is independent of ϵ, and whose value may change from line to line. We start with the unique solvability of the nonlinear equation (B.23).

Lemma B.7 *For each* $\epsilon > 0$ *there exists a unique element which solves the nonlinear equation* (B.23).

Next, we use the uniform convergence assumption (B.22) to derive the following result on the functional j.

Lemma B.8 *The functional* j *is convex and lower semicontinuous.*

Next, we perform a priori estimates on the solution of the equation (B.23), which imply the following convergence result.

Lemma B.9 *There exists an element* $u \in X$ *and a subsequence of the sequence* (u_ϵ), *again denoted* (u_ϵ), *which converges weakly to* u, *i.e.,*

$$u_\epsilon \rightharpoonup u \quad \textit{in } X \quad \textit{as} \quad \epsilon \to 0. \tag{B.24}$$

Next, we investigate the properties of the element u defined in Lemma B.9.

Lemma B.10 *The element* u *satisfies the variational inequality B.20 and, moreover, it is the unique solution of this inequality.*

We proceed through our analysis with the following weak convergence result.

Lemma B.11 *The whole sequence* (u_ϵ) *converges weakly in* X *to the element* u.

The last step is provided by the following strong convergence result.

Lemma B.12 *The sequence* (u_ϵ) *converges strongly in* X *to the element* u*, that is*

$$u_\epsilon \to u \quad in \quad X \quad as \quad \epsilon \to 0. \tag{B.25}$$

We can now easily provide the proof of Theorem B.5.

Proof. The points 1., 2. and 3. of Theorem B.5 are direct consequences of Lemmas B.7, B.10 and B.12, respectively. □

The interest in Theorem B.5 is twofold; first, it provides the existence and uniqueness of the solution to the variational inequality (B.20); second, it shows that the solution of (B.20) represents the strong limit of the sequence of solutions u_ϵ of the problems (B.23), as $\epsilon \to 0$.

B.3 Quasivariational inequalities

For the variational inequalities studied in this section we allow the function j to depend on the solution. Therefore, given a subset $K \subset X$, an operator $A : K \to X$, a function $j : K \times K \to \mathbb{R}$ and an element $f \in X$, in this section, we consider the problem of finding an element u such that

$$u \in K, \quad (Au, v-u)_X + j(u,v) - j(u,u) \geq (f, v-u)_X \quad \forall v \in K. \tag{B.26}$$

A problem of the form (B.26) is called an elliptic quasivariational inequality. Its solvability is obtained, usually, by using fixed point arguments

B.3.1 The Banach fixed point argument

In the study of (B.26), besides (B.8) and (B.9) we consider the following assumption on the function j.

$$\left.\begin{array}{l} (a) \quad \text{For all } \eta \in K,\ j(\eta, .\,) : K \to \mathbb{R} \text{ is convex and } l.s.c. \\ (b) \quad \text{There exists } \alpha \geq 0 \text{ such that} \\ \quad j(\eta_1, v_2) - j(\eta_1, v_1) + j(\eta_2, v_1) - j(\eta_2, v_2) \\ \quad\quad \leq \alpha \|\eta_1 - \eta_2\|_X \|v_1 - v_2\|_X \quad \forall \eta_1,\, \eta_2,\, v_1,\, v_2 \in K. \end{array}\right\} \tag{B.27}$$

Our first result in this section is the following.

Theorem B.6 *Let* X *be a Hilbert space and assume that* (B.8), (B.9), *and* (B.27) *hold. Moreover, assume that* $m_A > \alpha$ *where* $m_A > 0$ *is the constant defined in* (A.26). *Then, for each* $f \in X$ *the quasivariational inequality* (B.26) *has a unique solution.*

The proof of Theorem B.6 will be carried out in three steps. We assume in what follows that (B.8), (B.9) and (B.27) hold and let $f \in X$. In the first step, for each $\eta \in K$, we consider the auxiliary problem of finding u_η which solves the elliptic variational inequality

$$u \in K, \quad (Au_\eta, v - u_\eta)_X + j(\eta, v) - j(\eta, u_\eta) \geq (f, v - u_\eta)_X \quad \forall v \in K. \tag{B.28}$$

We use Theorem B.3 to obtain the following existence and uniqueness result.

Lemma B.13 *For each $\eta \in K$, there exists a unique solution u_η of inequality (B.28).*

Next, we consider the operator $\Lambda : K \to K$ defined by

$$\Lambda = u_\eta \quad \forall \eta \in K, \tag{B.29}$$

where $u_\eta \in K$ is the unique solution of (B.28), guaranteed by Lemma B.13. We have the following fixed point result.

Lemma B.14 *If $m_A > \alpha$, then, Λ has a unique fixed point $\eta^* \in K$.*

Proof. Let η_1, $\eta_2 \in K$ and let u_i denote the solution of (B.28) for $\eta = \eta_i$, *i.e.*, $u_i = u_{\eta_i}$, $i = 1, 2$. We have

$$(Au_1, v - u_1)_X + j(\eta_1, v) - j(\eta_1, u_1) \geq (f, v - u_1)_X \quad \forall v \in K,$$

$$(Au_2, v - u_2)_X + j(\eta_2, v) - j(\eta_2, u_2) \geq (f, v - u_2)_X \quad \forall v \in K.$$

We take $v = u_2$ in the first inequality, $v = u_1$ in the second one and add the resulting inequalities to obtain

$$(Au_1 - Au_2, u_1 - u_2)_X \leq j(\eta_1, u_2) - j(\eta_1, u_1) + j(\eta_2, u_1) - j(\eta_2, u_2).$$

We now use the properties (A.26) and (B.27)(b) of A and j, respectively, to see that

$$\|u_1 - u_2\|_X \leq \frac{\alpha}{m_A} \|\eta_1 - \eta_2\|_X. \tag{B.30}$$

Since $m_A > \alpha$ the inequality (B.30) shows that the operator Λ given by (B.29) is a contraction on K and, therefore, Lemma B.14 follows from Theorem A.2. □

We can now proceed with the proof of Theorem B.6.

Proof. Let η^* be the fixed point of the operator Λ obtained in Lemma B.14. Since $\eta^* = \Lambda\eta^* = u_{\eta^*}$it follows from (B.28) that η^* is a solution of the quasivariational inequality (B.26), which concludes the proof of the existence part.

Next, let u be a solution of (B.26). It follows that u is a solution of the variational inequality (B.28) with $\eta = u$ and, since by Lemma B.13 this inequality has a unique solution denoted u_η, we have $u = u_\eta$. This equality shows that $\eta = u_\eta$ and, keeping in mind the definition (B.29) of the operator Λ, we deduce that $\eta = \Lambda\eta$. Since Lemma B.14 guarantees that the operator Λ has a unique fixed point, denoted η^*, we find that $\eta = \eta^*$. Therefore, $u = \eta^*$, which concludes the proof of the uniqueness part. □

B.3.2 The Schauder fixed point argument

Next, we investigate the quasivariational inequality (B.26) by using the Schauder fixed point argument. To this end we assume in what follows that (B.8) and (B.9) hold and, in addition,

$$0_X \in K. \tag{B.31}$$

Moreover, we assume that $j : K \times K \to \mathbb{R}$ satisfies the following conditions.

$$\left.\begin{array}{l} \text{For all } \eta \in K, \text{ the function } v \mapsto j(\eta, v) : K \to \mathbb{R} \\ \text{is convex, } j(\eta, v) \geq 0 \text{ for all } v \in K \text{ and } j(\eta, 0_X) = 0. \end{array}\right\} \tag{B.32}$$

$$\left.\begin{array}{l} \text{For all sequences } (\eta_n) \subset K \text{ and } (u_n) \subset K \text{ such that:} \\ \eta_n \rightharpoonup \eta \text{ in } X, \ u_n \rightharpoonup u \text{ in } X \text{ and for all } v \in K, \\ \text{the inequality below holds:} \\ \limsup_{n\to+\infty} \Big[j(\eta_n, v) - j(\eta_n, u_n)\Big] \leq j(\eta, v) - j(\eta, u). \end{array}\right\} \tag{B.33}$$

Our second result in this section is the following.

Theorem B.7 *Let X be a Hilbert space and assume that (B.8), (B.9), (B.31), (B.32) and (B.33) hold. Then, for each $f \in X$ there exists at least one solution to the quasivariational inequality (B.26).*

The proof of Theorem B.7 will be carried out in five steps. We assume in what follows that (B.8), (B.9), (B.31), (B.32) and (B.33) hold, and let $f \in X$. As in the proof of Theorem B.6, we start with the study of the elliptic variational inequality (B.28), defined for each $\eta \in K$.

Lemma B.15 *For each $\eta \in K$, the inequality (B.28) has a unique solution $u_\eta \in K$. Moreover, the solution satisfies*

$$\|u_\eta\|_X \leq \frac{1}{m_A}(\|f\|_X + \|A0_X\|_X). \tag{B.34}$$

Lemma B.15 allows us, again, to consider the operator Λ defined by (B.29). The properties of this operator are investigated in the next three steps.

Lemma B.16 *The operator $\Lambda : K \to K$ is weakly continuous, i.e., $(\eta_n) \subset K$, $\eta_n \rightharpoonup \eta$ in X imply $\Lambda\eta_n \rightharpoonup \Lambda\eta$ in X.*

Lemma B.17 *The operator $\Lambda : K \to K$ maps weak convergent sequences into strong convergent sequences, i.e., $(\eta_n) \subset K$, $\eta_n \rightharpoonup \eta$ in X imply $\Lambda\eta_n \to \Lambda\eta$ in X.*

Lemma B.18 *The operator Λ is compact, i.e., it is continuous and maps bounded sets into relatively compact sets.*

We have now all the ingredients needed to prove Theorem B.7.

Proof. Let $f \in X$ and let $\widetilde{K}$ denote the set given by

$$\widetilde{K} = \{\, v \in K\,;\ \|v\|_X \leq \frac{1}{m_A}(\|f\|_X + \|A0_X\|_X)\}$$

Clearly, $\widetilde{K}$ is a nonempty closed convex bounded subset of X. Consider Λ, the operator given by (B.29). It follows from Lemma B.15 that $\Lambda\eta \in \widetilde{K}$ for all $\eta \in K$ and, therefore, considering the restriction of Λ to $\widetilde{K}$, we have $\Lambda(\widetilde{K}) \subset \widetilde{K}$. Recall also that, as proved in Lemma B.18, the operator Λ is compact. Therefore, we can use Theorem A.4 to deduce that there exists an element $\eta^* \in \widetilde{K}$ such that $\Lambda\eta^* = \eta^*$. By (B.29) it follows now that $u_{\eta^*} = \eta^*$ and, writing (B.28) for $\eta = \eta^*$, it is easy to see that η^* is a solution of the quasivariational inequality (B.26). □

Theorems B.6 and B.7 provide existence results in the study of the elliptic quasivariational inequality (B.26). However, besides the fact that the arguments used in their proofs are different, we note that the statements of these theorems are different too, since the assumptions on the functional j in the two theorems are different. Moreover, in contrast with Theorem B.6, Theorem B.7 does not provide the uniqueness of the solution of the quasivariational inequality (B.26).

B.3.3 A convergence result

We proceed with a result concerning the dependence of the solution of the quasivariational inequality (B.26) with respect to the function j. To this end, we assume in what follows that (B.8), (B.9) and (B.27) hold and, moreover, $m_A > \alpha$. Let $f \in X$ and, for each $\epsilon > 0$, let j_ϵ be a perturbation of j which satisfies (B.27) with $0 < \alpha_\epsilon < m_A$. We consider the problem of finding an element u_ϵ such that

$$u_\epsilon \in K, \quad (Au_\epsilon, v - u_\epsilon)_X + j_\epsilon(u_\epsilon, v) - j_\epsilon(u_\epsilon, u_\epsilon) \geq (f, v - u_\epsilon)_X \quad \forall v \in K. \tag{B.35}$$

We deduce from Theorem B.6 that inequality (B.26) has a unique solution $u \in K$ and, for each $\epsilon > 0$, inequality (B.35) has a unique solution $u_\epsilon \in K$.

Consider now the following assumption:

$$\left.\begin{aligned}
&\text{There exists } G : \mathbb{R}_+ \to \mathbb{R}_+ \text{ such that:}\\
&(a) \quad j_\epsilon(u_1, u_2) - j_\epsilon(u_1, u_1) + j(u_2, u_1) - j(u_2, u_2)\\
&\qquad\quad \leq G(\epsilon)\|u_1 - u_2\|_X + \alpha\|u_1 - u_2\|_X^2 \quad \forall u_1,\, u_2 \in K,\ \text{for each } \epsilon > 0.\\
&(b) \quad \lim_{\epsilon \to 0} G(\epsilon) = 0.
\end{aligned}\right\} \tag{B.36}$$

Then, the behavior of the solution u_ϵ as $\epsilon \to 0$ is given in the following theorem.

Theorem B.8 *Under the assumption (B.36), the solution u_ϵ of problem (B.35) converges to the solution u of problem (B.26), i.e.,*

$$u_\epsilon \to u \quad \textit{in } X \quad \textit{as} \quad \epsilon \to 0. \tag{B.37}$$

We end this section with the remark that the results presented in this section represent extensions of the results presented in Section B.2, in the study of elliptic variational inequalities. Indeed, assume the particular case when j does not depend on the first variable. Then it is easy to see that (B.27)(a) implies (B.10) and (B.27)(b) holds with $\alpha = 0$. It follows from this that the smallness assumption $\alpha < m_A$ is satisfied and, therefore, Theorem B.5 can be used in this case and allows us to recover the existence and uniqueness result in Theorem B.3. Moreover, assumption (B.36) implies assumption (B.18) and, therefore, Theorem B.8 implies Theorem B.4.

Annex C

C.1 The deviatoric of a matrix

Definition C.1 (Deviatoric) *The deviatoric* $\overline{\mathrm{A}}$ *of a matrix* $\mathrm{A} \in \mathbb{R}^{d\times d}_{sym}$ *is defined as*

$$\overline{\mathrm{A}} = \mathrm{A} - \frac{1}{d}\,\mathrm{tr}(\mathrm{A})\,\mathrm{I}\,. \tag{C.1}$$

where $\mathrm{tr}(\mathrm{A}) = \mathrm{A}_{ii}$ *is the trace operator, and* I *is the identity tensor of second order.*

The mapping $\mathrm{A} \mapsto \overline{\mathrm{A}}$ is linear with kernel

$$\ker(\overline{\mathrm{A}}) = \{\,\lambda\,\mathrm{A}\,\mathrm{I}\,;\;\; \lambda \in \mathbb{R}\}.$$

The image of the deviatoric mapping is the subspace of all matrices with trace zero.

$$\begin{aligned}
\mathrm{tr}(\overline{\mathrm{A}}) &= \mathrm{tr}\left(\mathrm{A} - \frac{1}{d}\,\mathrm{tr}(\mathrm{A})\,\mathrm{I}\right) \\
&= \mathrm{tr}(\mathrm{A}) - \frac{1}{d}\,\mathrm{tr}(\mathrm{A})\;\mathrm{tr}(\mathrm{I}) \\
&= \mathrm{tr}(\mathrm{A}) - \mathrm{tr}(\mathrm{A}) = 0.
\end{aligned}$$

Theorem C.1 *For* A, $\mathrm{B} \in \mathbb{R}^{d\times d}_{sym}$*, we have*

$$\overline{\mathrm{A}} \,:\, \mathrm{B} = \overline{\mathrm{A}} \,:\, \overline{\mathrm{B}}. \tag{C.2}$$

Proof. Consider the identities $\mathrm{tr}(\mathrm{I}) = d$ and $\mathrm{I} : \mathrm{A} = \mathrm{tr}(\mathrm{A})$, we have

$$\begin{aligned}\overline{\mathrm{A}} : \mathrm{B} &= \left(\mathrm{A} - \frac{1}{d}\,\mathrm{tr}(\mathrm{A})\,\mathrm{I}\right) : \mathrm{B} \\ &= \mathrm{A} : \mathrm{B} - \frac{1}{d}\,\mathrm{tr}(\mathrm{A})\,\mathrm{tr}(\mathrm{B}).\end{aligned}$$

$$\begin{aligned}\overline{\mathrm{A}} : \overline{\mathrm{B}} &= \left(\mathrm{A} - \frac{1}{d}\,\mathrm{tr}(\mathrm{A})\,\mathrm{I}\right) : \left(\mathrm{B} - \frac{1}{d}\,\mathrm{tr}(\mathrm{B})\,\mathrm{I}\right) \\ &= \mathrm{A} : \mathrm{B} - \frac{2}{d}\,\mathrm{tr}(\mathrm{A})\,\mathrm{tr}(\mathrm{B}) - \frac{1}{d^2}\,\mathrm{tr}(\mathrm{A})\,\mathrm{tr}(\mathrm{B})\,d \\ &= \mathrm{A} : \mathrm{B} - \frac{1}{d}\,\mathrm{tr}(\mathrm{A})\,\mathrm{tr}(\mathrm{B}).\end{aligned}$$

□

Remark C.3 *It follows from Theorem C.1 that*

$$\overline{\mathrm{A}} : \mathrm{A} = |\overline{\mathrm{A}}|^2, \tag{C.3}$$

and the proof shows that

$$|\overline{\mathrm{A}}|^2 \leq |\,\mathrm{A}\,|^2. \tag{C.4}$$

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