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Recursiveness approach to multi-dimensional truncated moment problems and subnormal completion.

JURY

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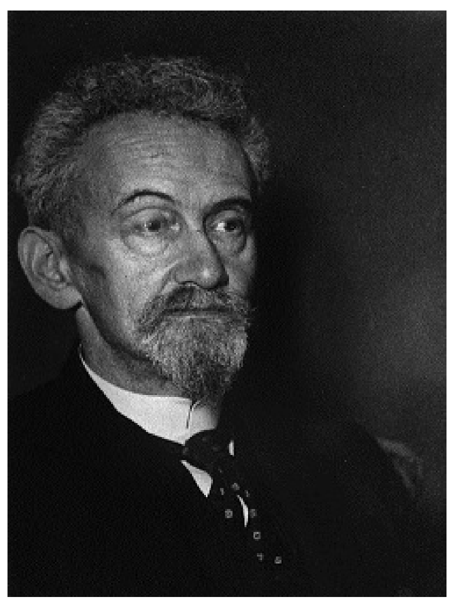
To my Parents.
To my wife ,
my daughters Tasnim
and my son Yahya.

Thomas Jean Stieltjes (1856 – 1894)

Hans Ludwig Hamburger
(1889-1956)



Felix Hausdorff
(1868 - 1942)

"The moment problems is a classical question in analysis, remarkable not only for its own elegance, but also for its extraordinary range of subjects theoretical and applied, which it has illuminated. From it flow developments in function theory, in functional analysis, in spectral representation of operators, in probability and statics, in Fourier analysis and the prediction of stochastic process, in approximation and numerical methods, in inverse problems and the design of algorithms for simulating physical systems."

H. Landau [40]

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Résumé

Dans cette dissertation, nous considérons le K-problème des moments n-dimensionnel tronqué. Tout d'abord, nous étudions la notion de récursivité bidimensionnelle. Les résultats obtenus permettent d'établir une nouvelle condition nécessaire et suffisante de l'existence d'une solution du problème des moments tronqué dans le cas complexe. En conséquence, nous montrons que le problème des moments complexe tronqué généralisé à toujours une solution (la construction explicite de la solution sera donnée). Puis nous résoudrons le problème des moments complexe tronqué dont les colonnes de la matrice des moments, notées ici par $(\overline{Z}^i Z^j)_{i,j\in\mathbb{N}}$, vérifient $Z^{k+1} = \sum_{0\leq n+m\leq k} a_{nm}\overline{Z}^n Z^m$ ($a_{nm} \in \mathbb{C}$). De plus, nous résoudrons le cas quintic du problème des moments complexe tronqué. Ensuite, nous présentons une généralisation (des résultats obtenus) au K-problème des moments n-dimensionnel tronqué.

D'autre part, l'étude des shifts à poids récurrents en deux dimensions, nous mènera à une nouvelle méthode pour résoudre le problème de complétion sous-normale bidimensionnel. Nous donnerons en particulier une solution concrète du problème de complétion sous-normale bidimensionnel avec une mesure de Berger minimale.

Mot-clés: K-problème des moments n-dimensionnel tronqué, suite récurrente doublement indexée, problème des moments complexe tronqué généralisé, problème des moments complexe quintic, shifts à poids récurrent en deux dimensions, problème de complétion sous-normale bidimensionnel.

Abstract

In this dissertation we are dealing with the topic of the truncated multivariable K-moment problems. First, we investigate the notion of recursiveness in two variables. We obtain several useful results that we will use to deduce new necessary and sufficient conditions for the truncated complex moment problem to own a solution. As a consequence, we show that the general truncated complex moment problem always have a solution (concrete construction of such solution is given). Furthermore, we solve the truncated complex moment problem with column dependence relations $Z^{k+1} = \sum\limits_{0 \le n+m \le k} a_{nm} \overline{Z}^n Z^m$ ($a_{nm} \in \mathbb{C}$). Also, we solve the quintic complex moment problem. Next, we present generalizations to multivariable K-moment problems. Finally, we extend the notion of recursively generated weighted shifts to 2-variable case, and provided a new methodology for solving the 2-variable subnormal completion problem. As a consequence, we furnish a concrete solution to the 2-variable subnormal completion problem with minimal Berger measure.

key words: Truncated multivariable K-moment problems, bi-recursive sequences, general truncated complex moment problem, quintic moment problem, 2-variable recursively generated weighted shifts, 2-variable subnormal completion problem.

CONTENTS

Résumé de la thèse

Le problème des moments (ou devrait-on dire les problèmes de moment, au pluriel, car il y a plusieurs variantes du problème des moments) est une question classique en analyse, qui a donné lieu à une vaste théorie avec des ramifications multiples.

Dans cette dissertation, nous considérons le K-problème des moments d-dimensionnel: étant donnés une suite $\beta^{(m)} \equiv (\beta_{(i_1,\ldots,i_d)})_{i_1+\ldots+i_d \leq m}$ (ici $m \in \mathbb{N} \cup \{+\infty\}$) et un fermé K de $\mathbb{R}^d$, le K-problème des moments n-dimensionnel consiste à trouver une mesure positive μ supportée sur K telle que

$$\beta_{(i_1,\ldots,i_d)} = \int x_1^{i_1} \ldots x_d^{i_d} \, d\mu \qquad \text{pour tout } (i_1,\ldots,i_d) \in \mathbb{N}^d. \tag{1}$$

Dans ce cas, on dit que $\beta^{(m)}$ est une K-suite des moments de mesure de représentation μ.

Si $m < +\infty$, le Problème (1) est connu dans la littérature sous le nom de K-problème des moments tronqué. Tandis que si $m = +\infty$, le Problème (1) est appelé K-problème des moments complet.

J. Stochel [55] a montré que le K-problème des moments tronqué est plus général que le K-problème des moments complet: "$\beta^{(\infty)}$ a une K-mesure de représentation si et seulement si $\beta^{(m)}$ admet une K-mesure de représentation pour tout $m \in \mathbb{N}$".

Notons que (1) englobe tous les problèmes des moments unidimensionnels classiques du 20ème siècle:

- Le problème des moments de Hamburger; $K = \mathbb{R}$ et $(\beta_i)_{i\in\mathbb{N}} \subset \mathbb{R}$.
- Le problème des moments de Stieltjes; $K = \mathbb{R}_+$ et $(\beta_i)_{i\in\mathbb{N}} \subset \mathbb{R}_+$.
- Le problème des moments de Hausdorff; $K = [a,b]$ et $(\beta_i)_{i\in\mathbb{N}} \subset \mathbb{R}$.
- Le problème des moments de Toeplitz; $K = \pi$, le cercle unité de $\mathbb{C}$, et $(\beta_i)_{i\in\mathbb{N}} \subset \mathbb{C}$.

M. Riesz a démontré que $\{\beta_{\mathbf{i}}\}_{\mathbf{i}\in\mathbb{N}^d}$ admet une mesure de représentation supportée sur $K \subset \mathbb{R}^d$ si et seulement si le fonctionnel $\Lambda : p(\mathbf{x}) \equiv \sum_{\mathbf{i}} a_{\mathbf{i}} \mathbf{x}^{\mathbf{i}} \longmapsto \Lambda(p) := \sum_{\mathbf{i}} a_{\mathbf{i}} \beta_{\mathbf{i}}$ est K-positive (i.e., $p_{|K} \geq 0 \Rightarrow \Lambda(p) \geq 0$).

Dans le cas unidimensionnel, la description des polynômes positifs sur $K = \mathbb{R}$ (resp. $K = \mathbb{R}_+$, $K = [a,b]$ et $K = \pi$) en termes de somme de carrés conduit à une solution concrète des problèmes des moments mentionnés ci-dessus. Par exemple, Hilbert (1880) a montré que tout polynôme positif de $\mathbb{R}[X]$ s'écrit comme somme de carrés de polynômes. Donc (via le résultat de Riesz) le problème des moments de Hamburger a une solution si et seulement si $\Lambda(p^2) \geq 0$ pour tout $p \in \mathbb{R}[X]$, ce qui est équivalent à

$$\begin{pmatrix} \beta_0 & \beta_1 & \cdots & \beta_n \\ \beta_1 & \beta_2 & \cdots & \beta_{n+1} \\ \vdots & \vdots & \ddots & \vdots \\ \beta_n & \beta_{n+1} & \cdots & \beta_{2n} \end{pmatrix} \geq 0, \qquad \text{pour tout } n \in \mathbb{N}.$$

Dans le cas multidimensionnel, le K-problème des moments est un problème ouvert, sauf dans le cas où K est un compacts semi-algébrique. La difficulté de résoudre ce problème vient, principalement, du fait que l'on ne peut pas représenter tout polynôme positif sur $K \subseteq \mathbb{R}^d$ comme somme de carrés de polynômes.

Dans le but de résoudre le problème (1) dans le cas où $K = \mathbb{R}^2 = \mathbb{C}$, R. Curto et L. Fialkow ont introduit le problème des moments complexe tronqué: étant donnés un entier $r \in \mathbb{N}$ et une suite tronquée de nombres complexes $\gamma \equiv \gamma^{(r)} \equiv \{\gamma_{ij}\}_{i,j\in\mathbb{N}; i+j\leq r}$ tels que $\gamma_{00} > 0$ et $\gamma_{ji} = \overline{\gamma}_{ij}$, le problème des moments complexe tronqué consiste à trouver une mesure positive μ, supportée sur $\mathbb{C}$, telle que

$$\gamma_{ij} = \int \overline{z}^i z^j \, d\mu \qquad \text{pour tous } i,j \in \mathbb{N}, \text{ tels que } i+j \leq r. \tag{2}$$

Une condition nécessaire évidente est

$$\Lambda(p\overline{p}) = \int p\overline{p}d\mu = \sum a_{ij}\overline{a}_{nm}\gamma_{i+m,j+n} \geq 0, \quad \text{pour tout } p \equiv \sum_{i+j\leq[\frac{r}{2}]} a_{ij}\overline{z}^i z^j \in \mathbb{C}[z,\overline{z}].$$

Cette dernière correspond à la positivité de la matrice $M(n) \equiv M(\gamma^{(2n)}) := (\gamma_{i+k,j+h})_{\substack{0\leq i+j\leq n,\\ 0\leq h+k\leq n}}$, connue dans la littérature sous le nom de matrice des moments complexe. Les colonnes et les lignes de $M(n)$ sont notées par ordre lexicographique suivant: $1, Z, \overline{Z}, Z^2, \overline{Z}Z, \overline{Z}^2, \ldots, Z^n, \overline{Z}Z^{n-1}, \ldots, \overline{Z}^{n-1}Z, \overline{Z}^n$.

Par exemple,

$$M(2) = \begin{array}{c c} & \begin{array}{cccccc} 1 & Z & \overline{Z} & Z^2 & \overline{Z}Z & \overline{Z}^2 \end{array} \\ \begin{array}{c} 1 \\ Z \\ \overline{Z} \\ Z^2 \\ \overline{Z}Z \\ \overline{Z}^2 \end{array} & \begin{pmatrix} \gamma_{00} & \gamma_{01} & \gamma_{10} & \gamma_{02} & \gamma_{11} & \gamma_{20} \\ \gamma_{10} & \gamma_{11} & \gamma_{20} & \gamma_{12} & \gamma_{21} & \gamma_{30} \\ \gamma_{01} & \gamma_{02} & \gamma_{11} & \gamma_{03} & \gamma_{12} & \gamma_{21} \\ \gamma_{20} & \gamma_{21} & \gamma_{30} & \gamma_{22} & \gamma_{31} & \gamma_{40} \\ \gamma_{11} & \gamma_{21} & \gamma_{21} & \gamma_{13} & \gamma_{22} & \gamma_{31} \\ \gamma_{02} & \gamma_{03} & \gamma_{12} & \gamma_{04} & \gamma_{13} & \gamma_{22} \end{pmatrix} \end{array}. \tag{3}$$

Malgré les nombreuses tentatives pour résoudre le problème des moments complexe tronqué, seul le cas quadratique ($r = 2$), le cas cubique ($r = 3$) et le cas quartique ($r = 4$) qui sont complétement résolus; les autre cas ($r = 5, 6, \ldots$) sont ouverts.

Dans le Chapitre 4, on va résoudre le problème des moment complex tronqué dans le cas quintique ($r = 5$).

R. Curto et S Yoo, dans leur article [20], ont résolu le cas sixtique sous les hypothéses: $M(3) \geq 0, M(2) > 0$ et les colonnes de la matrice $M(3)$ vérifient la relation de dépendance linéaire $Z^3 = itZ - u\overline{Z}$ où t et u sont des paramètres réels. Le problème du moment sixtique (reste ouvert) est seulement partiellement résolu.

Nous donnerons dans le Chapitre 5 une généralisation du résultat de Curto-Yoo mentionné ci-dessus. Pour ce faire, nous montrerons que pour chaque suite des moments tronquée $\gamma^{(r)}$ elle existe une suite infinie $\gamma^{(\infty)}$, telle que $\gamma^{(r)} \subseteq \gamma^{(\infty)}$, défini par les relations suivantes:

$$\begin{aligned} &i) \text{ Pour tous } i, j \geq 0, && \gamma_{ji} = \overline{\gamma}_{ij}; \\ &ii) \text{ pour tous } n, m \geq 0, && \gamma_{n,m+s+1} = \sum_{i+j=0}^{s} a_{ij}\gamma_{n+i,m+j} \qquad (a_{ij} \in \mathbb{C}). \end{aligned} \tag{4}$$

Nous appellerons suite récurrente doublement indexée toute suite vérifiant (4). Nous caractériserons les suites récurrentes doublement indexées admettant une mesure représentative. Une application directe de ce résultat donne une solution concrète au problème des moments complexe tronqué pour toute matrice des moments complexes $M(n+1)$ dont les colonnes vérifient la relation de dépendance linéaire $Z^{k+1} = \sum_{0\leq n+m\leq k} a_{nm}\overline{Z}^n Z^m$ ($a_{nm} \in \mathbb{C}$).

Il est naturel de se demander quelles sont les conditions nécéssaire et suffisantes pour qu'une suite donnée admette une mesure représentative signée. Autrement dit, étant donnée une suite $\{\gamma_{ij}\}_{(i,j)\in I}$ où $I \subseteq \mathbb{Z}_+ \times \mathbb{Z}_+$, résoudre le problème des moments complexe généralisé tronqué consiste à trouver une mesure signée ν telle que

$$\int \overline{z}^i z^j d\nu = \gamma_{ij}, \qquad \text{pour tout } (i,j) \in I. \tag{5}$$

Le cas unidimensionnels (du problème (5)) a été résolu par C.E. Chidume, M. Rachidi and E. H. Zerouali [5].

Comme pour le problème des moments tronqué classique, nous avons réalisé qu'il y a des difficultés avec la généralisation, en particulier pour le cas bidimensionnel (complexe). Notre approche pour résoudre le problème des moments complexe tronqué (donné au Chapitre 5) nous permet de surmonter cette difficulté et fournir la généralisation souhaitée. En effet, dans le Chapitre 6 nous allons voir que le problème des moments complexe généralisé tronqué admet toujours une solution.

Dans le Chapitre 7, nous allons étendre notre méthode pour résoudre le problème des moments complexe tronqué au K-problème des moments d-dimensionnel tronqué. Ce qui nous permettra de généraliser des résultats du chapitre 5.

Notons que le $\mathbb{R}_+^d$-problème des moments est étroitement lié au problème de completion sous-normal d-dimensionnel (voir [37]).

Dans le dernière chapitre, nous allons élargir la notion des shifts à poids récurrent (introduit par R. Curto et L. Fialkow pour résoudre le problème de completion sous-normal [7, 8]) au deux-dimention. L'étude des shifts à poids récurrent bidimensionnelle, nous mènera à une nouvelle méthode pour résoudre le problème de complétion sous-normale bidimensionnel. De plus, nous donnerons une solution concrète du problème de complétion sous-normale bidimensionnel avec une mesure de Berger est minimale.

CHAPTER 1

INTRODUCTION

The moment problem (or one should say moment problems, plural, because there are several different moment problems) is a classical question in analysis, which has given rise to rich theory in many branches of science and mathematics. Its application can be found in the operator theory, invariant subspace problem, polynomial optimization, positive polynomials, control theory, Nevanlinna-Pick problems and signal processing, among others. Some of these applications are described in several recent surveys and monographs (cf. [32] [28] [43] [45] [49]).

In this dissertation, we will consider the K-moment problem: given a d-dimensional real sequence $\beta^{(m)} \equiv \{\beta_{\mathbf{i}}\}_{\mathbf{i}\in\mathbb{Z}_+^d, |\mathbf{i}|\leq m}$ (here $m \in \mathbb{Z}_+ \cup \{+\infty\}$) and K is a closed subset of $\mathbb{R}^d$, *The K-moment problem* for $\beta^{(m)}$ asks when does there exist a positive Borel measure μ, supported in K, such that

$$\beta_{\mathbf{i}} = \int \mathbf{x}^{\mathbf{i}} d\mu, \qquad \text{for all } \mathbf{i} \in \mathbb{Z}_+^d \text{ with } \mid \mathbf{i} \mid \leq 2n. \tag{1.1}$$

Where, for $\mathbf{x} \equiv (x_1,\dots,x_d) \in \mathbb{R}^d$ and $\mathbf{i} \equiv (i_1,\dots,i_n) \in \mathbb{Z}_+^n$, we set $\mid \mathbf{i} \mid = i_1 + \ldots + i_d$ and $\mathbf{x}^{\mathbf{i}} = x_1^{i_1} \ldots x_d^{i_d}$. A sequence $\beta^{(m)}$ satisfying (1.1) will be called a *K-moment sequence* and μ is said to be a *K-representing measure* for $\beta^{(m)}$.

For $m = +\infty$; the Problem (1.1) is called *the full K-moment problem*. When $m < +\infty$; the Problem (1.1) is known as *the truncated K-moment problem*.

J. Stochel [55] has shown that the truncated K-moment problem is more general than the full K-moment problem, in the following sense: $\beta^{(\infty)}$ has a K-representing measure if and only if $\beta^{(m)}$ has a K-representing measure for every $m \geq 1$.

A theorem of Riesz-Haviland shows that the K-moment problem for a sequence $\{\beta_{\mathbf{i}}\}_{\mathbf{i}\in\mathbb{Z}_+^d}$ is solvable if and only if the functional $\Lambda : p(\mathbf{x}) \equiv \sum\limits_{\mathbf{i}} a_{\mathbf{i}}\mathbf{x}^{\mathbf{i}} \longmapsto \Lambda(p) := \sum\limits_{\mathbf{i}} a_{\mathbf{i}}\beta_{\mathbf{i}}$ is K-positive, i.e., $p_{|K} \geq 0 \Rightarrow \Lambda(p) \geq 0$. To apply this result representations of positive polynomials in terms of sums of squares (sometimes abbreviated as SOS) are needed. When the latter characterization is available, it will translate into conditions on the sequence.

In one dimension, characterization of polynomials that are nonnegative on a given subset $K \subseteq \mathbb{R}$ is fairly complete. That leads to explicit solution to all classical one-dimensional moment problems (Problem (1.1) with $d = 1$) of the 20th century: Stieltjes problem ($K = [0,+\infty[$), Hamburger problem ($K = \mathbb{R}$) and Hausdorff problem ($K = [0,1]$).

In multi-dimensional case, polynomials that are nonnegative on a given set $K \subseteq \mathbb{R}^d$ (with $d > 1$) have no simple characterization, except for compact basic semi-algebraic sets. Indeed, as already observed by Hilbert, there are positive polynomials in $d \geq 2$ variables that are not SOS, see (2.4). As a consequence, there exists a linear functional Λ on $\mathbb{R}[\mathbf{x}]$ which is positive for any SOS polynomial (i.e. $\Lambda(p^2) \geq 0$ for all $p \in \mathbb{R}[\mathbf{x}]$), but Λ

is not a moment functional (that is, there is no positive measure μ for which $\Lambda(p) = \int p d\mu$ for all $p \in \mathbb{R}[\mathbf{x}]$). Therefore, the multi-dimensional K-moment problem is still unsolved for general sets $K \subseteq \mathbb{R}^d$, including $K = \mathbb{R}^2 = \mathbb{C}$.

In the aim to solve the $\mathbb{R}^2$-moment problem, R. Curto and L. Fialkow [9] introduce *the truncated complex moment problem* (which is equivalent to the truncated $\mathbb{R}^2$-moment problem): given a truncated sequence of complex numbers $\gamma \equiv \gamma^{(r)} \equiv \{\gamma_{ij}\}_{i,j\in\mathbb{Z}_+;i+j\leq r}$ (here $r \in \mathbb{Z}_+$) with $\gamma_{00} > 0$ and $\gamma_{ji} = \overline{\gamma}_{ij}$. The *truncated complex moment problem*, associated with $\gamma^{(r)}$, entails finding a positive Borel measure μ, supported on $\mathbb{C}$, such that

$$\gamma_{ij} = \int \overline{z}^i z^j d\mu \qquad (i,j \in \mathbb{Z}_+; i+j \leq r). \tag{1.2}$$

An obvious necessary condition is then

$$\Lambda(p\overline{p}) = \int p\overline{p} d\mu = \sum a_{ij}\overline{a}_{nm}\gamma_{i+m,j+n} \geq 0, \quad \text{for any } p \equiv \sum_{i+j\leq[\frac{r}{2}]} a_{ij}\overline{z}^i z^j \in \mathbb{C}[z,\overline{z}].$$

This last corresponds to the positivity of the $l \times l$-matrix (where $l \equiv l(n) := \frac{(n+1)(n+2)}{2}$)

$$M(n)(\gamma^{(2n)}) \equiv M(\gamma^{(2n)}) \equiv M(n) := (\gamma_{i+k,j+h})_{\substack{0\leq i+j\leq n,\\ 0\leq h+k\leq n}}$$

known as *the complex moment matrix*. The columns and rows, of $M(n)$, are labeled by the following lexicographic order

$$1, Z, \overline{Z}, Z^2, \overline{Z}Z, \overline{Z}^2, \ldots, Z^n, \overline{Z}Z^{n-1}, \ldots, \overline{Z}^{n-1}Z, \overline{Z}^n.$$

Hence, the entry of $M(n)$ in row $\overline{Z}^h Z^k$ and column $\overline{Z}^i Z^j$ is $< \overline{Z}^i Z^j, \overline{Z}^h Z^k >_{M(n)} := \gamma_{i+k,j+h}$.

For instance, given $\gamma^{(4)} = \{\gamma_{ij}\}_{0\leq i+j\leq 2n}$, then

$$M(2) = \begin{array}{c} \\ 1 \\ Z \\ \overline{Z} \\ Z^2 \\ \overline{Z}Z \\ \overline{Z}^2 \end{array} \begin{array}{c} \begin{array}{cccccc} 1 & Z & \overline{Z} & Z^2 & \overline{Z}Z & \overline{Z}^2 \end{array} \\ \begin{pmatrix} \gamma_{00} & \gamma_{01} & \gamma_{10} & \gamma_{02} & \gamma_{11} & \gamma_{20} \\ \gamma_{10} & \gamma_{11} & \gamma_{20} & \gamma_{12} & \gamma_{21} & \gamma_{30} \\ \gamma_{01} & \gamma_{02} & \gamma_{11} & \gamma_{03} & \gamma_{12} & \gamma_{21} \\ \gamma_{20} & \gamma_{21} & \gamma_{30} & \gamma_{22} & \gamma_{31} & \gamma_{40} \\ \gamma_{11} & \gamma_{21} & \gamma_{21} & \gamma_{13} & \gamma_{22} & \gamma_{31} \\ \gamma_{02} & \gamma_{03} & \gamma_{12} & \gamma_{04} & \gamma_{13} & \gamma_{22} \end{pmatrix} \end{array}. \tag{1.3}$$

The truncated complex moment problem has interested many authors, such as, for instance, R. Curto [22], L. Fialkow [27], F. vasilescu [58], and others. Although interesting results were discovered, various basic situations are considered as open problems. For example, in the truncated complex moment problem associated with $\gamma^{(r)} \equiv \{\gamma_{ij}\}_{0\leq i+j\leq r}$, only the cases $r = 1,2,3,4$ (the quadratic [27], the cubic [38] and the quartic [21] moment problem) have been (recently) complectly achieved. All the other cases are open: quintic, sixtic, ...; as indicated in many recent papers (see, for instance, [22, 23]).

We will provide, in Chapter 4, a concrete solution to the quintic truncated complex moment problem (i.e., $r = 5$), when one desires a minimal representing measure. Our method intended to be useful for all odd-degree moment problems.

In [20], R. Curto and S. Yoo consider a subproblem of the sextic moment problem (i.e., $r = 6$). Indeed, [20] focuses on the case when $M(3) \geq 0$ and $M(2) > 0$, and there is a column dependence relation $Z^3 = itZ - u\overline{Z}$, with real parameters t and u. The sextic moment problem remains open and is only partially solved.

We will give, in Chapter 5, a generalisation of the above mentioned Curto-Yoo's result. To this aim, we will show that every truncated *moment* sequence $\gamma^{(r)}$ is a subsequence of an infinite bi-sequence $\gamma^{(\infty)}$ defined by the next relations,

$$\begin{aligned} &i)\text{ For all } i,j \geq 0, && \gamma_{ji} = \overline{\gamma}_{ij}; \\ &ii)\text{ For all } n,m \geq 0, && \gamma_{n,m+s+1} = \sum_{i+j=0}^{s} a_{ij}\gamma_{n+i,m+j} \qquad \text{where } a_{ij} \in \mathbb{C}. \end{aligned} \tag{1.4}$$

We refer to sequences verifying (1.4) as *recursive doubly indexed sequences* (RDIS for short). We will characterise RDIS admitting a representing measure. A simple application gives a concrete solution to the complex moment problem for any complex moment matrix $M(n+1)$ with column dependence relation $Z^{k+1} = \sum\limits_{0\leq n+m\leq k} a_{nm}\overline{Z}^n Z^m$ ($a_{nm} \in \mathbb{C}$).

It seem be natural to ask for the existence of *signed* representing measure μ associated with a given sequence $\gamma^{(m)}$. In other words, for $\{\gamma_{ij}\}_{(i,j)\in I}$, with $I \subseteq \mathbb{Z}_+ \times \mathbb{Z}_+$, solving the general complex moment problem consists in finding a signed measure μ, such that

$$\int \overline{z}^i z^j d\mu = \gamma_{ij}, \qquad \text{for all } (i,j) \in I.$$

Our interest for the general truncated complex moment problem is not just by curiosity, but is motivated by a useful applications in physics; for instance, the general truncated (one and two dimensional) moment problem arises in many-body quantum physics (cf. [29] [30] [60] [61] [62]).

In [5], C.E. Chidume, M. Rachidi and E. H. Zerouali give an explicit solution to the one-real variable version of the truncated general moment problem.

In the Chapter 6, we realized that (like the, classical, truncated moment problem) there are difficulties with the generalization, in particularly for the two-dimensional (complex) case. Our approach for solving the TCMP (given in Chapter 3) allow us to overcome this difficulty and provide the desired generalization.

We devote Chapter 7 to generalize the notion of RDIS to multi-variable version, that leads to a new method for solving the Problem (1.1) (with $d > 1$). As application, we shortcut two (celebrate) theorems of R. Curto and L. Fialkow on multi-dimensional moment problem.

In the last chapter we involve the recursiveness approach for solving the multivariable K-moment problem (developed in the Chapters 5 and 7) to give an abstract solution to an other open problem, known as the 2-variable subnormal completion problem: Given $m \geq 0$ and a finite collection of pairs of positive numbers $\Omega_m = \{(\alpha_{ij};\beta_{ij})\}_{i,j\in\mathbb{Z}_+;i+j\leq m}$, the 2-variable subnormal completion problem entails finding a necessary and sufficient conditions to guarantee the existence of a subnormal 2-variable weighted shift whose initial weights are given by Ω_m.

The complete solution of the subnormal completion problem in one variable was given by R. curto and L. Fialkow [7, 8], the explicit calculation requires recursively generated weighted shifts (we will extend this notion to 2-variable case).

We will show that if a given, finite, collection of weights Ω_m admits a subnormal completion in 2-variable, then there exists a 2-variable subnormal weighted shift (solution of the subnormal completion problem for Ω_m) with moment sequence obey to some recurrence relations, we shall refer to such shifts as 2-variable recursively generated weighted shifts. We will provide necessary and sufficient conditions for the existence of 2-variable recursively generated weighted shift completion, and thus a solution to the 2-variable subnormal completion problem. Moreover, we will give a concrete necessary and sufficient conditions for the existence of a subnormal completion with minimal Berger measure, and also a short alternative proof to the propagation phenomena, for the subnormal weighted shifts in 2-variable.

CHAPTER 2

PRELIMINARIES

This chapter is devoted to introduce some basic terminology, foundational results, examples, concepts and ideas which will enable us to build the proofs of our main results.

2.1 Polynomials

It is well known that the one variable polynomials are involved for solving the one-dimensional truncated moment problems. In our approach to multi-dimensional, real and complex, truncated moment problems the multi-variable polynomials will play a crucial role [1, 5, 6]. We thus need to discuss polynomials and the polynomial space in the multi-variable versions.

2.1.1 Multi-variable Real Polynomials

Let $\mathbf{i}$ be a multi-index $(i_1, i_2, \ldots, i_d)$ of positive integers. We will write $|\,\mathbf{i}\,| = i_1 + \ldots + i_d$ and $\mathbf{x}^{\mathbf{i}} = x_1^{i_1} \ldots x_d^{i_d}$ whenever $\mathbf{x} \equiv (x_1, \ldots, x_d)$ is a d-tuple of real numbers. Let us denote the vector space of d-variables real polynomials as follows

$$\mathbb{R}[x_1, \ldots, x_d] \equiv \mathbb{R}[\mathbf{x}] := \{ \sum_{\mathbf{i} \in \mathbb{Z}_+^d} a_{\mathbf{i}} \mathbf{x}^{\mathbf{i}} \text{ with } a_{\mathbf{i}} \in \mathbb{R} \}.$$

Similarly, we denote the set of polynomials with nonnegative coefficients such that

$$\mathbb{R}^+[x_1, \ldots, x_d] \equiv \mathbb{R}^+[\mathbf{x}] := \{ \sum_{\mathbf{i} \in \mathbb{Z}_+^d} a_{\mathbf{i}} \mathbf{x}^{\mathbf{i}} \text{ with } a_{\mathbf{i}} \in \mathbb{R}^+ \}.$$

This last well be used in the multi-variable subnormal completion problem. We need also to define the ring of d-variables real polynomials of total degree less than or equal to n:

$$\mathbb{R}_n[x_1, \ldots, x_d] \equiv \mathbb{R}_n[\mathbf{x}] := \{ \sum_{|\mathbf{i}| \le n} a_{\mathbf{i}} \mathbf{x}^{\mathbf{i}} \text{ with } a_{\mathbf{i}} \in \mathbb{R} \}.$$

We will denote the cone of all nonnegative d-variables real polynomials of total degree less than or equal to n by

$$\mathbb{R}_n^+[x_1, \ldots, x_d] \equiv \mathbb{R}_n^+[\mathbf{x}] := \mathbb{R}_n[\mathbf{x}] \cap \mathbb{R}^+[\mathbf{x}].$$

2.1.2 Polynomials in z and $\bar{z}$

Let us define the vector space of polynomials in z and $\bar{z}$, with complex coefficients, as follows

$$\mathbb{C}[z,\bar{z}] := \{ \sum_{i,j\in\mathbb{Z}_+} a_{ij}\bar{z}^i z^j \text{ with } a_{ij} \in \mathbb{C}\}.$$

The ring of polynomials in z and $\bar{z}$ with complex coefficients and total degree less than or equal to n is define as follows

$$\mathbb{C}_n[z,\bar{z}] := \{ \sum_{i+j\leq n} a_{ij}\bar{z}^i z^j \text{ with } a_{ij} \in \mathbb{C}\}.$$

2.1.3 Lexicographic Order on Monomials

A monomial order (sometimes called a term order or an admissible order) is a total order on the set of all (monic) monomials in a given polynomial ring, satisfying the property of respecting multiplication, that is, if $u \leq v$ and w is any other monomial, then $uw \leq vw$.

The ordering that we will adopt in this thesis is the, well-known, lexicographic order (lex for short).

Definition 2.1.1 — lexicographic orde. Let $\alpha,\beta \in \mathbb{Z}_+^n$, we say $\mathbf{x}^\alpha >_{lex} \mathbf{x}^\beta$ if $\alpha \neq \beta$ and the left-most nonzero entry of $\alpha - \beta$ is positive.

Example 2.1.1 (a) The lexicographic order on monomials of $\mathbb{R}_n[x,y]$:

$$1,x,y,x^2,xy,y^2,\ldots,x^n,x^{n-1}y,\ldots,xy^{n-1},y^n. \tag{2.1}$$

(b) The lexicographic order on monomials of $\mathbb{C}_n[z,\bar{z}]$:

$$1,z,\bar{z},z^2,\bar{z}z,\bar{z}^2,\ldots,z^n,\bar{z}z^{n-1},\ldots,\bar{z}^{n-1}z,\bar{z}^n. \tag{2.2}$$

2.1.4 Sums of squares and positive polynomials

A polynomial $p \in \mathbb{R}[x_1,\ldots,x_d]$ is said to be a *sum of squares of polynomials* (SOS in short), if p can be written as $p = \sum_{j=1}^m u_j^2$ for some $u_1,\ldots,u_m \in \mathbb{R}[x_1,\ldots,x_d]$. Denote two sets of polynomials

$$\Sigma_d := \{p \in \mathbb{R}[x_1,\ldots,x_d] \mid p \text{ is a } SOS\},$$
$$\Sigma_{d,n} := \Sigma_d \cap \mathbb{R}_n[x_1,\ldots,x_d].$$

Obviously any polynomial which is SOS is nonnegative on $\mathbb{R}^d$; that is,

$$\Sigma_d \subseteq \mathbb{R}^+[x_1,\ldots,x_d] \text{ and } \Sigma_{d,n} \subseteq \mathbb{R}_n^+[x_1,\ldots,x_d]. \tag{2.3}$$

Equality holds in (2.3) for $d = 1$ (i.e. for univariate polynomials), but the inclusion $\sum_d \subseteq \mathbb{R}^+[x_1,\ldots,x_d]$ is strict for $d \geq 2$. Indeed, The polynomial

$$M(x,y,z) = x^4y^2 + x^2y^4 - 3x^2y^2 + 1. \tag{2.4}$$

introduced by Motzkin for other purposes, was recognised by Taussky–Todd to be the first concrete example of a positive semi-definite polynomial that is not a sum of squares of polynomials.

The following celebrated result of Hilbert classifies the pairs (d,n) for which equality $\Sigma_{d,n} = \mathbb{R}_n^+[x_1,\ldots,x_d]$ holds.

Theorem 2.1.1 — Hilbert's theorem. The equality $\Sigma_{d,n} = \mathbb{R}_n^+[x_1,\ldots,x_d]$ holds if and only if
- $d = 1$, or
- $d = 2$, or
- $(d,n) = (2,4)$.

Let us denote by $Pos(M)$ (resp. $Pos(M)_n$) the set of $p \in \mathbb{R}[x]$ (resp. $p \in \mathbb{R}_n[x]$) that are nonnegative on $M \subseteq \mathbb{R}$. The following three propositions contain all results on positive polynomials needed for solving the moment problem on intervals. The formulas containing polynomial degrees will be used for the truncated moment problems.

Proposition 2.1.2 $Pos(\mathbb{R}) = \Sigma_1 = \{f^2+g^2 : f,g \in \mathbb{R}[x]\}$ and $Pos(\mathbb{R})_{2n} = \Sigma_{1,n} = \{f^2+g^2 : f,g \in \mathbb{R}_n[x]\}$.

Proposition 2.1.3 $Pos([0,+\infty)) = \{f + xg : f,g \in \Sigma_1\}$.

Proposition 2.1.4 Suppose that $a,b \in \mathbb{R}, a < b$. Then
$Pos([a,b]))_{2n+1} = \{(b-x)f + (x-a)g : f,g \in \Sigma_1\}$ and $Pos([a,b])_{2n} = \{f + (b-x)(x-a)g : f \in \Sigma_{1,n}, g \in \Sigma_{1,n-1}\}$.

An immediate extension is to consider characterizations of polynomials that are nonnegative on a basic semi-algebraic set (see definition below).

Definition 2.1.2 — Basic semi-algebraic set. A set K is said to be a basic semi-algebraic, defined by polynomials $f_1,\ldots,f_m \in \mathbb{R}[x_1,\ldots,x_d]$, if

$$K = \{\mathbf{x} \in \mathbb{R}^d \mid f_i(\mathbf{x}) \geq 0 \text{ for all } i = 1,\ldots,m\}.$$

The first representation (via sums of squares) result, known as Schmüdgen's Positivstellensatz, was an important breakthrough as it was the first to provide a simple characterization of polynomials positive on a compact basic semi-algebraic set K, and with no additional assumption on K (or on its description). In order to state the theorem, we need the following definition.

Definition 2.1.3 For $F = \{f_1,\ldots,f_m\} \subset \mathbb{R}[\mathbf{x}]$ and a set $J \subseteq \{1,\ldots,m\}$, we denote by $f_J \in \mathbb{R}[\mathbf{x}]$ the polynomial $\mathbf{x} \mapsto f_J(\mathbf{x}) := \prod_{j\in J} f_j(\mathbf{x})$, with convention that $f_\emptyset = 1$. The set

$$P(f_1,\ldots,f_m) = \{\sum_{J\subseteq\{1,\ldots,m\}} q_J f_J : q_J \in \Sigma_d\}$$

is called the *preordering* generated by $f_1,\ldots,f_m$.

Now we are able to state the celebrated Schmüdgen's Positivstellensatz.

Theorem 2.1.5 — Schmüdgen's Positivstellensatz. Let $(g_j)_{j=1}^m \subset \mathbb{R}[\mathbf{x}]$ be such that the basic semi-algebraic set

$$K = \{\mathbf{x} \in \mathbb{R}^d \;\;:\;\; g_j(\mathbf{x}) \geq 0, j = 1,\ldots,m\} \tag{2.5}$$

is compact. If $f \in \mathbb{R}[\mathbf{x}]$ is strictly positive on K, then $f \in P(g_1,\ldots,g_m)$.

In the next we provide an other interesting characterization of polynomials positive on a compact basic semi-algebraic set K called *Putinar's Positivstellensatz* in literature. To this aim, we associated with the finite family $(g_j)_{j=1}^m \subset \mathbb{R}[\mathbf{x}]$, the set

$$Q(g) = \{q_0 + \sum_{j=1}^{m} q_j g_j : (q_j)_{j=0}^m \subset \Sigma_d\}.$$

Theorem 2.1.6 — Putinar's Positivstellensatz. Let $K \in \mathbb{R}^d$ be as in (2.5) and assume that there exists $u \in Q(g)$ such that $\{\mathbf{x} \in \mathbb{R}^d : u(\mathbf{x}) \geq 0\}$ is compact. If $f \in \mathbb{R}[\mathbf{x}]$ is strictly positive on K, then $f \in Q(g)$.

2.2 Ideals and Varieties

This section is devoted to define the basic geometric objects used in this thesis.

2.2.1 Affine Varieties

In the next, we give a definition to the affine varieties.

Definition 2.2.1 Let K be a field and let $S \subseteq K[x_1,\dots,x_d]$. The *affine variety* (or simply variety) given by S is defined as

$$V(S) = \{(a_1,\dots,a_d) \in K^n \mid f(a_1,\dots,a_n) = 0 \text{ for all } f \in S\}.$$

Thus, an affine variety $V(f_1,\dots,f_s) \subseteq K^n$ is the set of all solutions of the system of equations $f_1(x_1,\dots,x_d) = \dots = f_s(x_1,\dots,x_d) = 0$.

2.2.2 Ideals in $\mathbb{K}[x_1,\dots,x_d]$

In the next we discuss a special subset of $K[x_1,\dots,x_d]$, where K is a field (we will often consider $K \equiv \mathbb{R}$ or $\mathbb{C}$).

Definition 2.2.2 A subset $I \subseteq K[x_1,\dots,x_d]$ is an *ideal* if it satisfies:
- (i) $0 \in I$.
- (ii) If $f, g \in I$, then $f + g \in I$.
- (iii) If $f \in I$ and $h \in K[x_1,\dots,x_d]$, then $hf \in I$.

The first natural example of an ideal is the ideal generated by a finite number of polynomials.

Lemma 2.2.1 Let $f_1,\dots,f_s$ be polynomials in $K[x_1,\dots,x_d]$, then

$$< f_1,\dots,f_s > := \{\sum_{i=1}^{s} h_i f_i \mid h_1,\dots,h_s \in K[x_1,\dots,x_d]\}$$

is an ideal of $K[x_1,\dots,x_d]$. We will call $< f_1,\dots,f_s >$ the *ideal generated by* $f_1,\dots,f_s$.

Let us provide an other example of an ideal which we will need to introduce the, celebrate, Hilbert's Nullstellensatz, see Section 2.2.3.

Lemma 2.2.2 If I is an ideal $K[x_1,\dots,x_n]$, then

$$\sqrt{I} := \{f \mid f^m \in I \text{ for some integer } m \geq 1\}$$

is an ideal in $K[x_1,\dots,x_n]$ containing I. We will call $\sqrt{I}$ the radical of I.

We will next see how affine varieties give rise to an interesting class of ideals

Lemma 2.2.3 Let $V \subseteq K^d$ be an affine variety. Then

$$I(V) := \{f \in K[x_1,\ldots,x_n] \mid f(a_1,\ldots,a_n) = 0 \text{ for all } (a_1,\ldots,a_n) \in V\}$$

is an ideal. We will call $I(V)$ the *ideal of* V.

2.2.3 Hilbert's Nullstellensatz

Hilbert's Nullstellensatz is a celebrated theorem which identifies exactly which ideals correspond to varieties. This will allow us to construct a "bridge" between geometry and algebra, whereby any statement about varieties can be translated into a statement about ideals (and conversely). Before we state it, let us recall that a field K is said to be *algebraically closed* if every nonconstant polynomial in $K[x]$ has a root in K.

Theorem 2.2.4 — Hilbert's Nullstellensatz. Let K be an algebraically closed field. If $\mathfrak{I}$ is an ideal in $K[x_1,\ldots,x_n]$, then

$$I(V(\mathfrak{I})) = \sqrt{\mathfrak{I}}.$$

2.3 Riesz Functional

This section is devoted primarily to introduce the notion of Riesz Functional, which is a fundamental tool for studying the moment problem.

Definition 2.3.1 Let V be a linear space over a field $\mathbb{F}$. A *linear functional* is a map $T : V \to \mathbb{F}$ which satisfies the following properties:

a) $T(x+y) = T(x) + T(y)$ for all $x, y \in V$,

b) $T(\alpha x) = \alpha T(x)$ for all $\alpha \in \mathbb{F}$ and for all $x \in V$.

The set of all linear functionals on V is denote by V^*.

In the next we define an important class of linear functionals, called Riesz-functional in literature. There are two versions: real and complex.

Definition 2.3.2 — Riesz functional–complex case. Let $m \in \mathbb{Z}_+ \cup \{\infty\}$ and let $\gamma^{(m)} \equiv \{\gamma_{ij}\}_{i,j \in \mathbb{Z}_+, i+j \leq m}$ be a given complex bi-sequence, the Riesz functional corresponding to $\gamma^{(m)}$ is the linear functional defined as follows

$$\begin{array}{rcc} & \mathbb{C}_m[z,\bar{z}] \quad \longrightarrow & \mathbb{C} \\ \Lambda_{\gamma^{(m)}} : & p(\mathbf{x}) = \sum\limits_{i+j \leq m} a_{ij} \bar{z}^i z^j \quad \longrightarrow & \sum\limits_{i+j \leq m} a_{ij} \gamma_{ij}. \end{array}$$

Definition 2.3.3 — Riesz functional–real case. Let $\beta^{(m)} \equiv \{\beta_{\mathbf{i}}\}_{\mathbf{i} \in \mathbb{Z}_+^d, |\mathbf{i}| \leq m}$, where $m \in \mathbb{Z}_+ \cup \{\infty\}$. The Riesz functional corresponding to $\beta^{(m)}$ is the linear functional defined as follows

$$\begin{array}{rcc} & \mathbb{R}_m[x_1,\ldots,x_d] \quad \longrightarrow & \mathbb{R} \\ \Lambda_{\beta^{(m)}} : & p(\mathbf{x}) = \sum\limits_{|\mathbf{i}| \leq m} a_{\mathbf{i}} \mathbf{x}^{\mathbf{i}} \quad \longrightarrow & \sum\limits_{|\mathbf{i}| \leq m} a_{\mathbf{i}} \beta_{\mathbf{i}}. \end{array}$$

In fact, we are not interested by the Riesz functional itself, instead it is the positivity of this functional that we need. Indeed, according to Riesz-Haviland Theorem 3.3.1, the K-positivity of the Riesz functional (see definition below) is intimately related to the moment problems.

Definition 2.3.4 Let $m \in \mathbb{Z}_+ \cup \{+\infty\}$ and let K be a closed subset of $\mathbb{C}$ (resp. $\mathbb{R}^d$). A Riesz functional $\Lambda_{\beta^{(m)}}$ is said to be K-positive if whenever $p \in \mathbb{C}_m[z,\bar{z}]$ (resp. $p \in \mathbb{R}_m[x_1,\dots,x_d]$) and $p_{|K} \geq 0$, then $L(p) \geq 0$. For $K = \mathbb{C}$ or $\mathbb{R}^d$, we say simply that $\Lambda_{\beta^{(m)}}$ is positive.

In the next section we will present, the notion of *moment matrices*, an interesting tool to detect the positivity of Riesz functional. Furthermore, in the aim to detect the K-positivity, of the Riesz functional, with K is a closed semi-algebraic set, we introduce the notion of *localizing matrices*.

2.4 Matrices

Matrices are a key tool in linear algebra; they have been used in a variety of areas (in particularly, they are crucial in the study of truncated moment problems and the subnormal completion problem). We devote this section to presenting some results and definitions (in related with the matrix theory) which will be used in this Thesis.

2.4.1 Smul'jan's Theorem

The matrix extension problems play a fundamental role in many areas such as electronic engineering, system sciences, mathematics, etc. For example, positive matrix extension is closely related to the truncated moment problems.

Let $k,l \in \mathbb{Z}_+$. We say that $\tilde{A} \in M_{k+l}(\mathbb{C})$ is a matrix extension of $A \in M_k(\mathbb{C})$ if there exist $B \in M_{k,l}(\mathbb{C})$ and $C \in M_l(\mathbb{C})$ such that

$$\tilde{A} = \begin{pmatrix} A & B \\ B^* & C \end{pmatrix}.$$

Theorem 2.4.1 — Smul'jan's Theorem. Let $\tilde{A}, A, B$ and C be as above. Then $\tilde{A} \geq 0$ if and only if

(i) $A \geq 0$

(ii) there exists $W \in M_{k,l}(\mathbb{C})$ such that $B = AW$ (equivalently, $RanB \subseteq RanA$),

(iii) $C \leq W^*AW$ (Since $A = A^*$, then W^*AW is independent of W provided $B = AW$).

Moreover, $rank\ \ \tilde{A} = rank\ \ A$ if and only if $C = W^*AW$.

Conversely, if $A \geq 0$, any extension $\tilde{A}$ satisfying $rank\ \ \tilde{A} = rank\ \ A$ is necessarily positive.

Notice also that from the expression

$$\begin{pmatrix} I & 0 \\ -W^* & I' \end{pmatrix} \tilde{A} \begin{pmatrix} I & -W \\ 0 & I' \end{pmatrix} = \begin{pmatrix} A & 0 \\ 0 & C - W^*AW \end{pmatrix},$$

where I and I' denote the unit matrices of size l and k, respectively; we deduce that

$$rank\ \ \tilde{A} = rank\ \ A + rank\ \ (C - W^*AW). \tag{2.6}$$

2.4.2 Vandermonde Matrices

A Vandermonde matrix, named after Alexandre Théophile Vandermonde (1735–1796), is a matrix with rows (or column) given by increasing powers and such matrices appear in many different circumstances, both in abstract mathematics and various applications. We next give a formal definition of this matrix.

Definition 2.4.1 A Vandermonde matrix is an $m \times n$-matrix of the form

$$V_{mn}(x_1,\dots,x_n) = \begin{pmatrix} 1 & l & \cdots & 1 \\ x_1 & x_2 & \cdots & x_n \\ \vdots & \vdots & \ddots & \vdots \\ x_1^{m-1} & x_2^{m-1} & \cdots & x_n^{m-1} \end{pmatrix}$$

where $x_i \in \mathbb{C}$, $i = 1,\dots,n$. If the matrix is square, $n = m$, the notation $V_n = V_{nm}$ will be used.

Often it is not the Vandermonde matrix itself that is useful, instead it is the multivariate polynomial given by its determinant that is examined and used. The determinant of the Vandermonde matrix is usually called the Vandermonde determinant and can be written using an exceptionally simple formula.

Theorem 2.4.2 The vandermonde determinant is given by

$$\det V_n(x_1,\dots,x_n) = \prod_{1 \le i < j \le n} (x_i - x_j).$$

Observe that if all the numbers x_i are distinct, then the Vandermonde determinant is non-zero.

2.4.3 Complex Moment Matrices

The notion of complex moment matrix, introduced by R. Curto and L. Fialkow [9], play a central role in the truncated complex moment problem. Before giving the structure of this matrix let us fixed some notations.

Given $n \geq 1$, let $m \equiv m(n) := \frac{(n+1)(n+2)}{2}$ and let $\gamma^{(2n)} \equiv \{\gamma_{ij}\}_{i+j \le 2n}$ be a truncated bi-sequence of complex numbers verifying that $\overline{\gamma}_{ij} = \gamma_{ji}$. We define the $(i+1)\times(j+1)$-matrix $M[i,j]$, with $0 \le i,j \le n$, as follows:

$$M[i,j] := \begin{pmatrix} \gamma_{i,j} & \gamma_{i+1,j-1} & \cdots & \gamma_{i+j,0} \\ \gamma_{i-1,j+1} & \gamma_{i,j} & \cdots & \gamma_{i+j-1,1} \\ \vdots & \vdots & \ddots & \vdots \\ \gamma_{0,i+j} & \gamma_{1,i+j-1} & \cdots & \gamma_{j,i} \end{pmatrix}.$$

Observe in passing that $M[i,j]$ has a Toeplitz form (that is, is each of its diagonals contains constant entries) and all its entries are moments of order $i+j$.

We now define the *Complex moment matrix* $M(n)(\gamma^{(2n)}) \equiv M(n) \equiv M(\gamma^{(2n)}) \in M_m(\mathbb{C})$ via the block decomposition

$$M(n) := (M[i,j])_{0 \le i,j \le n} = \begin{pmatrix} M[0,0] & M[0,1] & \dots & M[0,n] \\ M[1,0] & M[1,1] & \dots & M[1,n] \\ \vdots & \vdots & \ddots & \vdots \\ M[n,0] & M[n,1] & \dots & M[n,n] \end{pmatrix}. \tag{2.7}$$

Considering the complex lexicographic order (see (2.2)) to denote rows and columns of the moment matrix

$M(n)$. For example, The $M(3)$ matrix is

$$M(3) := \begin{array}{c} \\ 1 \\ Z \\ \overline{Z} \\ Z^2 \\ Z\overline{Z} \\ \overline{Z}^2 \\ Z^3 \\ Z^2\overline{Z} \\ Z\overline{Z}^2 \\ \overline{Z}^3 \end{array} \begin{array}{c} \begin{array}{cccccccccc} 1 & Z & \overline{Z} & Z^2 & Z\overline{Z} & \overline{Z}^2 & Z^3 & Z^2\overline{Z} & Z\overline{Z}^2 & \overline{Z}^3 \end{array} \\ \left(\begin{array}{c|cc|ccc|cccc} \gamma_{00} & \gamma_{01} & \gamma_{10} & \gamma_{02} & \gamma_{11} & \gamma_{20} & \gamma_{03} & \gamma_{12} & \gamma_{21} & \gamma_{30} \\ \hline \gamma_{10} & \gamma_{11} & \gamma_{20} & \gamma_{12} & \gamma_{21} & \gamma_{30} & \gamma_{13} & \gamma_{22} & \gamma_{31} & \gamma_{40} \\ \gamma_{01} & \gamma_{02} & \gamma_{11} & \gamma_{03} & \gamma_{12} & \gamma_{21} & \gamma_{04} & \gamma_{13} & \gamma_{22} & \gamma_{31} \\ \hline \gamma_{20} & \gamma_{21} & \gamma_{30} & \gamma_{22} & \gamma_{31} & \gamma_{40} & \gamma_{23} & \gamma_{32} & \gamma_{41} & \gamma_{50} \\ \gamma_{11} & \gamma_{12} & \gamma_{21} & \gamma_{13} & \gamma_{22} & \gamma_{31} & \gamma_{14} & \gamma_{23} & \gamma_{32} & \gamma_{41} \\ \gamma_{02} & \gamma_{03} & \gamma_{12} & \gamma_{04} & \gamma_{13} & \gamma_{22} & \gamma_{05} & \gamma_{14} & \gamma_{23} & \gamma_{32} \\ \hline \gamma_{30} & \gamma_{31} & \gamma_{40} & \gamma_{32} & \gamma_{41} & \gamma_{50} & \gamma_{33} & \gamma_{42} & \gamma_{51} & \gamma_{60} \\ \gamma_{21} & \gamma_{22} & \gamma_{31} & \gamma_{23} & \gamma_{32} & \gamma_{41} & \gamma_{24} & \gamma_{33} & \gamma_{42} & \gamma_{51} \\ \gamma_{12} & \gamma_{13} & \gamma_{22} & \gamma_{14} & \gamma_{23} & \gamma_{32} & \gamma_{15} & \gamma_{24} & \gamma_{33} & \gamma_{42} \\ \gamma_{03} & \gamma_{04} & \gamma_{13} & \gamma_{05} & \gamma_{14} & \gamma_{23} & \gamma_{06} & \gamma_{15} & \gamma_{24} & \gamma_{33} \end{array}\right) \end{array}.$$

Associate to the sequence $\gamma^{(2n)}$ the Riesz functional $\Lambda_{\gamma^{(2n)}}$ given by

$$\Lambda_{\gamma^{(2n)}} : p(\bar{z},z) \equiv \sum_{0 \le i+j \le 2n} a_{ij}\bar{z}^i z^j \longrightarrow \sum_{0 \le i+j \le 2n} a_{ij}\gamma_{ij}.$$

It is an immediate observation that the rows $\overline{Z}^k Z^l$, columns $\overline{Z}^i Z^j$ entry of the matrix $M(n)$ is equal to $\Lambda_{\gamma^{(2n)}}(\bar{z}^{i+l}z^{j+k}) = \gamma_{i+l,j+k}$. For reason of simplicity, we identify a polynomial $p(\bar{z},z) \equiv \sum a_{ij}\bar{z}^i z^j$ with its coefficient vector $p = (a_{ij})$ with respect to the basis of monomials of $\mathbb{C}_n[\bar{z},z]$ in degree-lexicographic order. Clearly, $M(n)$ acts on these coefficient vectors as follows:

$$\langle M(n)p,q\rangle := q^* M(n)p = \Lambda_{\gamma^{(2n)}}(p\bar{q}). \tag{2.8}$$

This last, Equalitie (2.8), yields that the matrix $M(n)$ detects the positivity of the Riesz functional $\Lambda_{\gamma^{(2n)}}$ on the cone generated by the collection $\{p\bar{p} : p \in \mathbb{C}_n[\bar{z},z]\}$.

2.4.4 Real Moment and Localizing Matrix

We next define the important notions of, multi-dimensional, real moment matrix and localizing matrix.

Real Moment Matrices: Let $\beta^{(2n)} \equiv \{\beta_{\mathbf{i}}\}_{\mathbf{i}\in\mathbb{Z}^d_+,|\mathbf{i}|\le 2n}$ be a given truncated multisequence. The matrix $M(n) \equiv M_d(n) \equiv M_d(\beta^{(2n)}) \equiv M(\beta^{(2n)}) := (\beta_{\mathbf{i}+\mathbf{j}})_{|\mathbf{i}|,|\mathbf{j}|\in\mathbb{Z}^d_+ : |\mathbf{i}|,|\mathbf{j}|\le n}$ is known in Curto-Fialkow's terminology as the *multi-dimensional real moment matrix*. The columns and rows, of $M(n)$, are labeled by the lexicographic order on monomials of $\mathbb{R}_n[x_1,\dots,x_d]$, see Section 2.1.3. Clearly, the entry of $M(n)$ in row $\mathbf{X}^{\mathbf{i}}$ and column $\mathbf{X}^{\mathbf{j}}$ is $M(n)_{\mathbf{ij}} = \beta_{\mathbf{i}+\mathbf{j}}$.

Let

$$\Lambda_{\beta^{(2n)}} : p(x_1,\dots,x_d) \equiv \sum_{|\mathbf{i}|\le 2n} a_{\mathbf{i}}\mathbf{x}^{\mathbf{i}} \longrightarrow \sum_{|\mathbf{i}|\le 2n} a_{\mathbf{i}}\beta_{\mathbf{i}}$$

be the Riesz functional associated with $\beta^{(2n)}$. Similarly to the complex case, we will identify a polynomial $p(\mathbf{x}) \equiv \sum a_{\mathbf{i}}\mathbf{x}^{\mathbf{i}}$ with its coefficient vector $p = (a_{\mathbf{i}})$ with respect to the basis of monomials of $\mathbb{R}_n[\mathbf{x}]$ in degree-lexicographic order. Clearly $M(n)$ acts on these coefficient vectors as follows:

$$< M(n)p,q > = q^T M(n)p = \Lambda_{\beta^{(2n)}}(pq), \qquad p,q \in \mathbb{R}_n[\mathbf{x}]. \tag{2.9}$$

Therefore, in view of (2.9), the matrix $M(n)$ detects the positivity of the Riesz functional $\Lambda_{\beta^{(2n)}}$ on the cone generated by $\{p^2 : p \in \mathbb{R}_n[x_1,\dots,x_d]\}$, the sum of squares of polynomials (sometimes abbreviated as SOS).

In particular, with a given bi-sequence $\beta^{(2n)} \equiv \{\beta_{\mathbf{i}}\}_{\mathbf{i}\in\mathbb{Z}_+^2, |\mathbf{i}|\le 2n} \equiv \{\beta_{ij}\}_{i+j\le 2n}$, we associate the moment matrix $M(n) \equiv M(n)(\beta^{(2n)})$ (in this case, i.e. $d=2$, $M(n)$ is called the real moment matrix) build as follows.

$$M(n) := \begin{pmatrix} N[0,0] & N[0,1] & \dots & N[0,n] \\ N[1,0] & N[1,1] & \dots & N[1,n] \\ \vdots & \vdots & \ddots & \vdots \\ N[n,0] & N[n,1] & \dots & N[n,n] \end{pmatrix}, \tag{2.10}$$

where

$$N[i,j] := \begin{pmatrix} \beta_{i+j,0} & \beta_{i+j-1,1} & \dots & \beta_{i,j} \\ \beta_{i+j-1,1} & \beta_{i+j-2,2} & \dots & \beta_{i-1,j-1} \\ \vdots & \vdots & \ddots & \vdots \\ \beta_{i,j} & \beta_{i-1,j+1} & \dots & \beta_{0,i+j} \end{pmatrix}. \tag{2.11}$$

Remark that every $N[i,j]$ has the Hankel-like property, i.e. its entries are constant on each diagonal to the left-lower corner. Considering the lexicographic order on monomials of $\mathbb{R}_n[X,Y]$ (see (2.1)) to denote rows and columns of the moment matrix $M(n)$.

For example, if $n=d=2$, then

$$M(2) = \begin{array}{c} \\ 1 \\ X \\ Y \\ X^2 \\ XY \\ Y^2 \end{array} \begin{array}{c} \begin{array}{cccccc} 1 & X & Y & X^2 & XY & Y^2 \end{array} \\ \begin{pmatrix} \beta_{00} & \beta_{10} & \beta_{01} & \beta_{20} & \beta_{11} & \beta_{02} \\ \beta_{10} & \beta_{20} & \beta_{11} & \beta_{30} & \beta_{21} & \beta_{12} \\ \beta_{01} & \beta_{11} & \beta_{02} & \beta_{21} & \beta_{12} & \beta_{03} \\ \beta_{20} & \beta_{30} & \beta_{21} & \beta_{40} & \beta_{31} & \beta_{22} \\ \beta_{11} & \beta_{21} & \beta_{12} & \beta_{31} & \beta_{22} & \beta_{13} \\ \beta_{02} & \beta_{12} & \beta_{03} & \beta_{22} & \beta_{13} & \beta_{04} \end{pmatrix} \end{array}.$$

Localizing Moment Matrices: Given $P \in \mathbb{R}[x_1,\dots,x_d]$, with coefficient vector $\mathbf{P} \equiv \{P_\zeta\}$, and $\beta^{(2t)} \equiv \{\beta_{\mathbf{i}}\}_{\mathbf{i}\in\mathbb{Z}_+^d, |\mathbf{i}|\le 2t}$, where $t \in \mathbb{Z}_+ \cup \{+\infty\}$. Define the new sequence

$$P * \beta^{(2t)} := M(\beta^{(2t)})\mathbf{P} = M(2t)\mathbf{P},$$

called *shifted vector*, with α-th entry is $(P*\beta^{(2t)})_\alpha := \sum_\zeta P_\zeta \beta_{\zeta+\alpha}$.

The moment matrix of $P*\beta^{(2t)}$, i.e. $M(n)(P*\beta) = M_P(n + [\frac{1+\deg P}{2}])$ is called the *localizing matrix*, since they can be used to "localize" the support of a representing measure for $\beta^{(2t)}$.

For example, given $\beta^{(3)} \equiv \{\beta_{ij}\}_{i+j\le 3}$, then

$$M_x(2) = \begin{pmatrix} \beta_{10} & \beta_{20} & \beta_{11} \\ \beta_{20} & \beta_{30} & \beta_{21} \\ \beta_{11} & \beta_{21} & \beta_{12} \end{pmatrix} \text{ and } M_y(2) = \begin{pmatrix} \beta_{01} & \beta_{11} & \beta_{02} \\ \beta_{11} & \beta_{21} & \beta_{12} \\ \beta_{02} & \beta_{12} & \beta_{03} \end{pmatrix}.$$

Remark that

$$< M_P(n)f, g > = < M(n)Pf, g > = (Pf)^T M(n) g = \Lambda_{\beta^{(2n)}}(Pfg), \qquad f,g \in \mathbb{R}_n[\mathbf{x}]. \tag{2.12}$$

Hence the localizing matrix $M_P(n)$ detect the positivity of the Riesz functional $\Lambda_{\beta^{(2n)}}$ on the based semi-algebraic set $K = \{\mathbf{x} \in \mathbb{R}^d \mid P(\mathbf{x}) \ge 0\}$. In a similar way one obtain the following proposition.

Proposition 2.4.3 Let $\Lambda_{\beta^{(2n)}}$ be the Riesz functional associated with the multi-sequence $\beta^{(2n)} \equiv \{\beta_{\mathbf{i}}\}_{\mathbf{i}\in\mathbb{Z}_d^+, |\mathbf{i}|\le 2n}$ (here $n \in \mathbb{Z}^+ \cup \{+\infty\}$) and let K be a basic semi-algebraic set defined by polynomials $P_1, P_2, \dots, P_m \in \mathbb{R}[\mathbf{x}]$. Then

$$\Lambda_{\beta^{(2n)}} \text{ is } K\text{-positive} \iff M_{P_j}(n) \ge 0 \text{ for any } j = 1,\dots,m.$$

We conclude this section with a proposition, due to R. curto and L. Fialkow, which establish the equivalence between truncated real moment problems and truncated complex moment problems through a unitary transformation.

Before stating the proposition, let us first fixed some notation. Setting $M(n)(\gamma^{(2n)}) \equiv M_\gamma, M(\beta^{(2n)}) \equiv M_\beta, \psi(x,y) := z \equiv x+iy, \Psi(x,y) := (z,\bar{z})$ and let $\hat{p}$ and $\widetilde{p \circ \Psi}$ be the coefficient vectors associated with p and $p \circ \Psi$, respectively.

Proposition 2.4.4 — (13, Theorem 1.5). Let M_γ and M_β be the moment matrices associated with γ and β, and define the matrix L by $L\hat{p} := \widetilde{p \circ \Psi} \qquad (p \in \mathbb{C}_n[z,\bar{z}])$.

(i) $M_\gamma = L^* M_\beta L$.
(ii) L is inversible.
(iii) $M_\gamma \geq 0 \Leftrightarrow M_\beta \geq 0$.
(iv) $rank \;\; M_\gamma = rank \;\; M_\beta$.
(v) The formula $\mu_{\mathbb{R}} = \mu \circ \Psi$ establishes a one-to-one correspondence between the sets of representing measures for β and γ, which preserves measure class and cardinality of the support; moreover, $\psi(supp \;\; \mu_{\mathbb{R}}) = supp \;\; \mu$.
(vi) M_γ admits a flat extension if and only if M_β admits a flat extension.
(vii) For $p \in \mathbb{C}_n[z,\bar{z}], \;\; p(Z,\overline{Z}) = L^*((p \circ \Psi)(X,Y))$.

2.5 Linear Recurrence Relations

This section is devoted to introduce the generalized Fibonacci sequences of finite order and present some of its fundamental properties related in particular to the explicit representation of the general term of these sequences. We are interested primarily in the analytic expression (Binet's formula). This basic formulation will be widely used in this thesis.

2.5.1 Weighted r-Generalized Fibonacci Sequences

Let $a_0, a_1, \ldots, a_{r-1}$ (here $r \geq 2$) be some fixed complex numbers with $a_{r-1} \neq 0$ and consider the sequence $S = (s_k)_{k \in \mathbb{Z}_+}$ defined by the following linear recurrence relation of order r,

$$s_{k+1} = a_0 s_k + a_1 s_{k-1} + \ldots + a_{r-1} s_{k-r+1}, \qquad \text{for all } k \geq r-1, \tag{2.13}$$

where $s_0, s_1, \ldots, s_{r-1}$ are specified by the initial conditions.

The sequence (2.13) is known in the literature as *r-generalized Fibonacci sequences*.

The polynomial $P(x) = x^r - a_0 x^{r-1} - \ldots - a_{r-2}x - a_{r-1}$ is called a *characteristic polynomial* associated to (2.13), and any solution λ of the characteristic equation $P(x) = 0$ is called a *characteristic root* for (2.13).

An r-generalized Fibonacci sequence can be defined in various ways using different characteristic polynomials as is shown in the following example.

Let $S = (s_k)_{k \in \mathbb{Z}_+}$ with $s_k = 2^k$, then S may be defined by the following recursive relations,

- $s_{k+1} = 2s_k$ with $s_0 = 1$.
- $s_{k+2} = s_{k+1} + 2s_k$ with $s_0 = 1$ and $s_1 = 2$.

Therefore, $P_1(x) = x - 2$ and $P_2(x) = x^2 - x - 2$ are two characteristic polynomials associated with S.

Let $\mathscr{P}_S$ be the set of characteristic polynomials associated with the sequence $S = (s_k)_{k \in \mathbb{Z}_+}$.

Proposition 2.5.1 The subset $\mathscr{P}_S$ is an ideal of $\mathbb{C}[x]$.

Proof. Let $p(x) = \sum_i a_i x^i, q(x) = \sum_i b_i x^i \in \mathscr{P}_S$, then

$$\sum_i a_i s_{k+i} = \sum_i b_i s_{k+i} = 0, \qquad \text{for all } k \in \mathbb{Z}_+,$$

hence

$$\sum_i (a_i + b_i) s_{k+i} = 0$$

and thus $p + q \in \mathscr{P}_S$.

Given an integer $n \in \mathbb{Z}_+$, one have

$$\sum_i a_i s_{k+i} = 0, \qquad \text{for all } k \in \mathbb{Z}_+,$$

that implies

$$\sum_i a_i s_{k+n+i} = 0, \qquad \text{for all } k \in \mathbb{Z}_+,$$

that is, $x^n p \in \mathscr{P}_S$. Thus, for any polynomial $H(x) \in \mathbb{C}[x]$, one have $H(x)p(x) \in \mathscr{P}_S$. Therefore, $\mathscr{P}_S$ is an ideal of $\mathbb{C}[x]$. ∎

Since $\mathbb{C}[x]$ is a principal ideal domain and according to Proposition 2.5.1, the following proposition holds.

Proposition 2.5.2 For every sequence $S = (s_k)_{k\in\mathbb{Z}_+}$ given by (2.13), there exists a unique, monic, characteristic polynomial $P_S \in \mathscr{P}_S$ with minimal degree. Moreover, every $P \in \mathscr{P}_S$ is a multiple of P_S.

The polynomial P_S is called the *minimal polynomial* associated with $S = (s_k)_{k\in\mathbb{Z}_+}$.

2.5.2 Binet's Formula

Binet's formula is an explicit formula used to find the terms of the Fibonacci sequence, given by (2.13), without using recursiveness.

Theorem 2.5.3 [24, Theorem 1](Binet formula) Let $S \equiv \{s_k\}_{k\in\mathbb{Z}_+}$ be a generalized Fibonacci sequence, associated with the characteristic polynomial $p(x) = \prod_{i=0}^{n-1}(x-\lambda_i)^{k_i}$, then

$$s_k = \sum_{i=0}^{n-1}\sum_{j=0}^{k_i-1} c_{i,j} k^j \lambda_i^k \qquad (c_{i,k_i} \neq 0), \tag{2.14}$$

where the $c_{i,j}$ are determined by the initial condition $s_k = 0, 1, \ldots, \deg p - 1$.

Let us observe that if the characteristic polynomial of $S \equiv \{s_k\}_{k\in\mathbb{Z}_+}$ has distinct roots, say $p(x) = \prod_{i=0}^{n-1}(x-\lambda_i)$, then (2.14) can be written as follows:

$$s_k = \sum_{i=0}^{n-1} c_i \lambda_i^k \qquad , \tag{2.15}$$

where the c_i are only determined by the initial condition $\{s_k\}_{0\le k\le n-1}$.

2.6 Bounded Operators on Hilbert space

In this section will be presented definitions, properties and results regarding (single) bounded operators on Hilbert space, n-tuples of commuting operators and (one-variable and multi-variable) weighted shifts.

2.6.1 Single Operators

Let $\mathscr{H}$ be a complex Hilbert space and let $\mathscr{L}(\mathscr{H})$ be the algebra of all bounded operators on $\mathscr{H}$. We denote by T^* the adjoint operator of T and by $[T,S] := TS - ST$ the commutator of S and T in $\mathscr{L}(\mathscr{H})$. An operator $T \in \mathscr{L}(\mathscr{H})$ is said to be

- normal if $[T^*,T] = 0$,
- hyponormal if $[T^*,T] \geq 0$,
- subnormal if $T = N_{|\mathscr{H}}$, where N is a normal operator on some Hilbert space $\mathscr{K} \supseteq \mathscr{H}$.

The concepts of subnormal and hyponormal operators were introduced by Paul R. Halmos in [31]. The first notion, hyponormal, reflects the geometric nature of normality with the corresponding implications in terms of positive matrices; while subnormal is intimately related to the notion of analyticity for complex functions, through the restriction of the functional calculus to invariant subspaces.

In order to establish a bridge between operator theory and matrix theory, we recall the Bram-Halmos criterion for subnormality, which says that an operator T is subnormal if and only if

$$\sum_{i,j}(T^i x_j, T^j x_i) \geq 0 \tag{2.16}$$

for any $x_0, x_1, \ldots, x_k \in \mathscr{H}$ ([4]). An application of the Choleski algorithm for operator matrices shows that (2.16) is equivalent to the positivity test

$$M_k(T) := \begin{pmatrix} [T^*,T] & [T^{*2},T] & \ldots & [T^{*k},T] \\ [T^*,T^2] & [T^{*2},T^2] & \ldots & [T^{*k},T^2] \\ \vdots & \vdots & \ddots & \vdots \\ [T^*,T^k] & [T^{*2},T^k] & \ldots & [T^{*k},T^k] \end{pmatrix} \geq 0 \qquad (\text{for all } k \geq 0). \tag{2.17}$$

To illustrate and to study the gap between subnormal and hyponormal operators A. Athavale ([2]) introduced the classes of k-hyponormal operators as follows: an operator $T \in \mathscr{L}(\mathscr{H})$ is k-hyponormal if $M_k(T) \geq 0$. Clearly,

T is subnormal $\Leftrightarrow$ T is k-hyponormal (for all $k \in \mathbb{N}$)

and

$(k+1)$-hyponormal $\Rightarrow$ k-hyponormal $\Rightarrow$ 1-hyponormal $\Leftrightarrow$ hyponormal.

2.6.2 One-Variable Weighted Shifts

Weighted shifts provide several examples and counter examples in operator theory and hence they are an important motivation in the analysis of operators.

Let $l^2(\mathbb{Z}_+)$ be the Hilbert space of square-summable sequences and let $\{e_n\}_{n=0}^{\infty}$ be the associated canonical orthonormal basis.

Given a sequence of positive numbers $\alpha \equiv \{\alpha_i\}_{i\geq 0} \in l^2(\mathbb{Z}_+)$ (*called weights*), the *unilateral weighted shift* W_α associated with α is the operator on $l^2(\mathbb{Z}_+)$ defined by

$$W_\alpha e_n := \alpha_n e_{n+1} \qquad \text{for all } n \geq 0.$$

The moments of α are given by

$$\gamma_0 := 1 \text{ and } \gamma_{n+1} := \alpha_n^2 \gamma_n \qquad (n \geq 0).$$

It is straightforward to check that W_α can never be normal, and that W_α is hyponormal if and only if $\alpha_n \leq \alpha_{n+1}$ for all $n \geq 0$.

The Stieltjes moment problem(see Section 3.2) associated with a given sequence $\{\gamma_n\}_{n\geq 0}$ entails finding a positive Borel measure μ supported in $\mathbb{R}^+$ such that

$$\gamma_n = \int t^n d\mu \quad \text{for every } n \geq 0. \tag{2.18}$$

When the moment problem owns a solution μ, then μ is said to be a *representing measure* of the moment sequence $\{\gamma_n\}_{n\geq 0}$. The well known Berger theorem says that a weighted shift W_α is subnormal precisely when the sequence of its moments is a moment sequence of a positive measure supported in $[0, \|W_\alpha\|]$.

A description of subnormality for an abstract operator T in terms of weighted shifts can be found in [39]. Namely, an operator T is subnormal if and only if, for each $h \neq 0$ in $\mathscr{H}$, the weighted shift associated with the weight sequence $\{\|T^{n+1}h\|/\|T^n h\|\}$ is subnormal.

J. Stampfli [52] showed that for subnormal weighted shifts W_α, a propagation phenomenon occurs which forces the flatness of W_α whenever two equal weights are present. That is, if $\alpha_k = \alpha_{k+1}$ for some $k \geq 0$, then $\alpha_n = \alpha_{n+1}$ for every $n \geq 1$ (see also [26]). Later, R. Curto proved, in [7], that the above result remains valid for 2-hyponormal weighted shifts.

2.6.3 Commuting Tuple Operators

We start by defining the n-tuples of commuting operators.

Definition 2.6.1 An operator $\mathbf{T} = (T_1, \ldots, T_n) \in \mathscr{L}(\mathscr{H})^n$ is called a commuting n-tuple if $T_iT_j = T_jT_i$ for every $1 \leq i, j \leq n$.

We say that an n-tuple $\mathbf{T} = (T_1, \ldots, T_n)$ of operators on $\mathscr{H}$ is (jointly) hyponormal if the operator matrix

$$[\mathbf{T}^*; \mathbf{T}] := \begin{pmatrix} [T_1^*, T_1] & [T_2^*, T_1] & \ldots & [T_n^*, T_1] \\ [T_1^*, T_2] & [T_2^*, T^2] & \ldots & [T_n^*, T_2] \\ \vdots & \vdots & \ddots & \vdots \\ [T_1^*, T_n] & [T_2^*, T^n] & \ldots & [T_n^*, T_n] \end{pmatrix}$$

is positive on the direct sum of n copies of $\mathscr{H}$.

For $k \geq 1$, T is k-hyponormal if $(I, T, \ldots, T^k)$ is (jointly) hyponormal. The Bram-Halmos characterization of subnormality can be paraphrased as follows: T is subnormal if and only if T is k-hyponormal for every $k \geq 1$.

The n-tuple $\mathbf{T} \equiv (T_1, \ldots, T_n)$ is said to be *normal* if $\mathbf{T}$ is commuting and each T_i is normal, and $\mathbf{T}$ is *subnormal* if $\mathbf{T}$ is the restriction of a normal n-tuple to a common invariant subspace.

2.6.4 Two-Variable Weighted Shifts

Consider double-indexed positive bounded sequences $\alpha \equiv \{\alpha_{\mathbf{k}}\}_{\mathbf{k}\in\mathbb{Z}_+^2}$ and $\beta \equiv \{\beta_{\mathbf{k}}\}_{\mathbf{k}\in\mathbb{Z}_+^2}$. We define the 2-variable weighted shift $\mathbf{T} \equiv (T_1, T_2)$ acting on the Hilbert space $l^2(\mathbb{Z}_+^2)$, associated with $\alpha \equiv \{\alpha_{\mathbf{k}}\}_{\mathbf{k}\in\mathbb{Z}_+^2}$ and $\beta \equiv \{\beta_{\mathbf{k}}\}_{\mathbf{k}\in\mathbb{Z}_+^2}$, by

$$T_1 e_{\mathbf{k}} := \alpha_{\mathbf{k}} e_{\mathbf{k}+\varepsilon_1} \text{ and } T_2 e_{\mathbf{k}} := \beta_{\mathbf{k}} e_{\mathbf{k}+\varepsilon_2},$$

where $\varepsilon_1 := (1,0)$ and $\varepsilon_2 := (0,1)$. We recall here that, the vector space $l^2(\mathbb{Z}_+^2)$ is canonically isometrically isomorphic to $l^2(\mathbb{Z}_+) \otimes l^2(\mathbb{Z}_+)$, equipped with its canonical orthonormal basis $\{e_{\mathbf{k}}\}_{\mathbf{k}\in\mathbb{Z}_+^2}$.

Clearly,

$$T_1T_2 = T_2T_1 \Leftrightarrow \beta_{\mathbf{k}+\varepsilon_1}\alpha_{\mathbf{k}} = \alpha_{\mathbf{k}+\varepsilon_2}\beta_{\mathbf{k}} \qquad (\text{ for all } \mathbf{k} \in \mathbb{Z}_+^2). \tag{2.19}$$

Given $\mathbf{k} \equiv (k_1, k_2) \in \mathbb{Z}_+^2$, the *moment* of (α, β) of order $\mathbf{k}$ is

$$\gamma_{\mathbf{k}} := \|T_1^{k_1} T_2^{k_2} e_{\mathbf{0}}\|^2 = \begin{cases} 1 & \text{if } k = 0, \\ \alpha_{(0,0)}^2 \cdots \alpha_{(k_1-1,0)}^2 & \text{if } k_1 \geq 1 \text{ and } k_2 = 0, \\ \beta_{(0,0)}^2 \cdots \beta_{(0,k_2-1)}^2 & \text{if } k_1 = 0 \text{ and } k_2 \geq 1, \\ \alpha_{(0,0)}^2 \cdots \alpha_{(k_1-1,0)}^2 \cdot \beta_{(k_1,0)}^2 \cdots \beta_{(k_1,k_2-1)}^2 & \text{if } k_1 \geq 1 \text{ and } k_2 \geq 1. \end{cases} \tag{2.20}$$

Conversely, one can recover the weights from the moments, by using the following relations:

$$\alpha_{\mathbf{k}} = \sqrt{\frac{\gamma_{\mathbf{k}+\varepsilon_1}}{\gamma_{\mathbf{k}}}} \text{ and } \beta_{\mathbf{k}} = \sqrt{\frac{\gamma_{\mathbf{k}+\varepsilon_2}}{\gamma_{\mathbf{k}}}}. \tag{2.21}$$

We remark that, due to the commutativity condition (2.19), $\gamma_{\mathbf{k}}$ can be computed using any nondecreasing path from $(0,0)$ to (k_1, k_2).

The presence of consecutive equal weights, of a 2-variable subnormal weighted shift, leads to horizontal or vertical flatness. Explicitly, if, for some $k_1, k_2 \geq 1$, $\alpha_{(k_1,k_2)} = \alpha_{(k_1+1,k_2)}$ (resp. $\beta_{(k_1,k_2)} = \beta_{(k_1,k_2+1)}$), then $\alpha_{(k_1,k_2)} = \alpha_{(1,1)}$ (resp. $\beta_{(k_1,k_2)} = \beta_{(1,1)}$) for all $k_1, k_2 \geq 1$. This result is known as propagation phenomena.

A characterization of subnormality for multi-variable weighted shifts is given in [37]. More precisely, $\mathbf{T}$ admits a commuting normal extension if and only if there is a probability measure μ, which we call the Berger measure of $\mathbf{T}$, defined on the 2-dimensional rectangle $R = [0, \|T_1\|^2] \times [0, \|T_2\|^2]$ such that

$$\gamma_{\mathbf{k}} = \int_R x^{k_1} y^{k_2} d\mu(x,y), \qquad \text{for all } \mathbf{k} \equiv (k_1, k_2) \in \mathbb{Z}_+^2. \tag{2.22}$$

A measure μ satisfies (2.22) is also known as representing measure for $\{\gamma_{\mathbf{k}}\}_{\mathbf{k}\in\mathbb{Z}_+^2}$.

CHAPTER 3

A GENERAL THEORY OF THE K-MOMENT PROBLEMS

This chapter is designed to serve as a brief introduction to K-moment problems, to describe a number of interrelated techniques for establishing the existence of K-representing measures, and to the basic results of moment problems. The first section presents a historical overview of the moment problem. Section 3.2 is devoted to define truncated and full K-moment problems in one-dimensional and multi-dimensional versions. In Section 3.3 we discuss representing measures arising from K-positivity of the Riesz functional. Section 3.4 about measures arising from flat extensions of positive moment matrices. In the last section we discuss the link between the truncated K-Moment Problem and the full multi-variable K-moment problem, also we provide the celebrated Tchakaloff's theorem and its generalizations and applications to truncated K-moment problem.

3.1 Brief Historical Review

The moment problem arose in $1894-1895$ in the hands of Thomas Jan Stieltjes $(1856-1894)$ as a means of studying the analytic behavior of continued fractions. Indeed at the end of Chapter 4 of his memoir "*Recherches sur les fractions continues*" Stieltjes writes:

> " *nous appellerons problème des moments le problème suivant: Trouver une distribution de masse positive sur une droite* $(0,\infty)$ *les moments d'ordre* k $(k=0,1,2,3,\dots)$ *étant donnés.*"
>
> T.J. Stieltjes [53]

Even though it was Stieltjes who introduced the concept of the moment problem, he was not the first one who dealt with moments. Already in 1874 Chebyshev(1821-1894) considered the problem of how an upper (resp. lower) bound for the value of the integral

$$\int_a^b f(x)dx \tag{3.1}$$

over an interval $[a,b]$ can be determined given the values of the integrals

$$\int_A^B f(x)dx, \int_A^B xf(x)dx, \dots, \int_A^B x^m f(x)dx$$

over a larger interval $[A,B]$. Here $f(x)$ is an unknown function, which remains positive between the endpoints of integration.

The Chebyshev's method of finding an upper and lower bound, for the integral (3.1), is related to the moment problem and can be viewed as a precursor for it, but Chebyshev only worked with a finite number of moments and he primarily saw the method as a tool to prove some important limit theorems in probability theory.

Stieltjes' approach to continued fractions has its root in his interest in divergent series of the form

$$\frac{c_0}{x}+\frac{c_1}{x^2}+\frac{c_2}{x^3}+\dots.$$

After Cauchy (1789-1857) in 1821 stressed that one cannot talk of a sum of a divergent series, mathematicians became more careful in their work with such series. However, in many branches of physics and in particular in celestial mechanics, divergent series were widely used. Stieltjes' career as a scientist began in the observatory of Leiden in Holland where he worked as an astronomer from 1877 to 1883, so he must have been familiar with the use of divergent series. But first of all Stieltjes was a mathematician. So from a mathematical point of view it must have been a challenge for Stieltjes to figure out why the use of divergent series in so many cases gave the right results. In his doctoral thesis from 1886 "*Recherches sur quelques séries semiconvergentes*" he gaves one of the first rigorous approaches to divergent series.

Stieltjes introduced the moment problem on $(0,\infty)$ and solved the problems about existence and uniqueness in his famous memoir [53], the first part of which came out just before he died in December 1894. It was the first general investigation of the analytic theory of continued fractions as part of complex function theory. Besides that, it introduced several new ideas and concepts such as the Stieltjes integral and the moment problem, and it became famous for its rigorous style.

Stieltjes devoted his memoir to a complete investigation of the question of convergence of continued fractions of the form

$$\cfrac{1}{a_1 z + \cfrac{1}{a_2 + \cfrac{1}{a_3 z + \cfrac{1}{a_4 + \dots}}}} \tag{3.2}$$

where $a_n > 0$ and $z \in \mathbb{C}$. Let us denote by $T_n(z)$ and $U_n(z)$ the numerators and denominators of the consecutive finite nth convergent, to the continued fraction, so that

$$\frac{T_1(z)}{U_1(z)} = \frac{1}{a_1 z}, \frac{T_2(z)}{U_2(z)} = \cfrac{1}{a_1 z + \cfrac{1}{a_2}}, \frac{T_3(z)}{U_3(z)} = \cfrac{1}{a_1 z + \cfrac{1}{a_2 + \cfrac{1}{a_3 z}}}, \dots .$$

Consider the sequence $(c_n)_{n\geq 0}$ uniquely determined via the asymptotic expansion

$$\frac{T_n(z)}{U_n(z)} = \frac{c_0}{z} - \frac{c_1}{z^2} + \frac{c_3}{z^3} - \dots + (-1)^{n-1}\frac{c_{n-1}}{z^n} + O(\frac{1}{z^{n+1}}), \mid z \mid \to \infty.$$

Stieltjes showed that if the sequence $(c_n)_{n\geq 0}$ arises as described above then the determinants

$$A_n = \begin{vmatrix} c_0 & c_1 & \dots & c_{n-1} \\ c_1 & c_2 & \dots & c_n \\ \vdots & \vdots & \ddots & \vdots \\ c_{n-1} & c_n & \dots & c_{2n-2} \end{vmatrix} \text{ and } B_n = \begin{vmatrix} c_1 & c_2 & \dots & c_n \\ c_2 & c_3 & \dots & c_{n+1} \\ \vdots & \vdots & \ddots & \vdots \\ c_n & c_{n+1} & \dots & c_{2n} \end{vmatrix} \tag{3.3}$$

are positive for all $n \in \mathbb{Z}_+$. By converting these arguments Stieltjes could show that

> *" The moment problem corresponding to a given sequence $c_0, c_1, c_2, \dots$ is solvable if and only if the determinants $A_n > 0$ and $B_n > 0$ for all natural numbers n."*

Stieltjes pointed out that one has to distinguish between two cases:

$$\sum_{n\geq 0} a_n < \infty \text{ and } \sum_{n\geq 0} a_n = \infty.$$

In the first case (called the indeterminate case) the continued fraction diverges for all $z \in \mathbb{C}$. However, the even convergents and the odd convergents each have a limit as $n \to \infty$ for $z \in \mathbb{C} \setminus (-\infty, 0]$. The limits are different and of the form

$$\lim_{n\to\infty} \frac{T_{2n}(z)}{U_{2n}(z)} = \int_0^\infty \frac{d\nu_1(t)}{z+t} \text{ and } \lim_{n\to\infty} \frac{T_{2n-1}(z)}{U_{2n-1}(z)} = \int_0^\infty \frac{d\nu_2(t)}{z+t},$$

where ν_1 and ν_2 are different positive (and discrete) measures on $[0,\infty)$ with moments $(c_n)_{n\geq 0}$.

In the second case – the determinate case – the continued fraction converges uniformly on compact subsets of $\mathbb{C}\setminus(-\infty,0]$ even though the polynomials $T_n(z)$ and $U_n(z)$ diverge as $n\to\infty$. The limit of the nth convergent has the form

$$\lim_{n\to\infty}\frac{T_n(z)}{U_n(z)}=\int_0^\infty\frac{d\nu(t)}{z+t},$$

where ν is a positive measure on $[0,\infty)$ with moments $(c_n)_{n\geq 0}$. In fact, ν is the only positive measure on $[0,\infty)$ with moments $(c_n)_{n\geq 0}$.

After Stieltjes there were attempts to extend his theory to other types of continued fractions. At this early stage the moment problem was closely connected to continued fractions, and the successors' work on Stieltjes' theory also tells something about the extension of the Stieltjes moment problem to the whole real axis and the difficulties mathematicians had to deal with in order to solve that problem.

The German mathematician Hans Ludwig Hamburger (1889-1956) introduced the extension of the moment problem to the whole real axis in 1919. He also introduced the modern definition of determinateness and indeterminateness, which does not depend on the corresponding continued fractions.

The next year, in 1920, Hamburger published the first part of the extensive work "*Über eine Erweiterung des Stieltjesschen Momentproblems*". He was the first profound and complete treatment of the moment problem as a theory of its own and considered more general continued fractions than the one in (3.2). The role of $[0,\infty)$ in Stieltjes' work is taken over by the real line in Hamburger's work. A key result, often referred to as Hamburger's theorem, says that

> *" The moment problem corresponding to a given sequence $(c_n)_{n\geq 0}$ has a solution if and only if the determinants $A_n>0$ (3.3) for all $n\in\mathbb{Z}_+$".*

In 1920 (published 1921) the German mathematician Felix Hausdorff (1868-1942) solved the moment problem on a bounded interval. This is called the Hausdorff moment problem and consists of finding an increasing function $\chi(u)$ defined on $[0,1]$ such that

$$c_n=\int_0^1 u^n d\chi(u)\qquad n=0,1,2,\ldots,$$

where $(c_n)_{n\geq 0}$ is a given sequence of real numbers.

Hausdorff's work was the first in which the moment problem was completely disconnected from its roots–the continued fractions–and simultaneously the same happened for the moment problem on the whole real axis.

With the work of Nevanlinna and Riesz the moment problem freed itself from continued fractions and moved into other fields of mathematics–complex function theory and functional analysis–and today it no longer bears the marks of its origin.

3.2 Statement of The K-Moment Problem

Given a bi-sequence of complex numbers $\gamma\equiv\gamma^{(m)}\equiv\{\gamma_{ij}\}_{i,j\in\mathbb{Z}_+;i+j\leq m}$, with $\gamma_{00}>0$ and $\gamma_{ji}=\overline{\gamma}_{ij}$, and a non empty closed subset $K\subseteq\mathbb{C}$. The *complex K-moment problem*, associated with $\gamma^{(m)}$, entails finding a positive Borel measure μ supported on K and such that

$$\gamma_{ij}=\int\bar{z}^i z^j d\mu\qquad (i,j\in\mathbb{Z}_+;i+j\leq m).\tag{3.4}$$

A sequence $\gamma\equiv\{\gamma_{ij}\}_{i,j\in\mathbb{Z}_+;i+j\leq m}$ satisfying (3.4) will be called a *moment sequence* and μ is said to be a *K-representing measure* for γ.

For $m=+\infty$; the Problem (3.4) is called *the full complex K-moment problem*. When $m<+\infty$; the Problem (3.4) is known as *the truncated K-complex moment problem*.

For $K=\mathbb{C}$, we refer to the Problem (3.4) simply as *the – truncated or full– complex moment problem* (TCMP or FCMP in short) and to μ as *representing measure*.

The truncated complex moment problem serves as a prototype for several other moment problems; for instance, it is equivalent to the 2-dimensional real moment problems (see definition below).

We notice in passing that when $K:=supp\ \ \mu\subset\mathbb{R}$, we get $\bar{z}=z=t$ and then

$$\gamma_{ij}=\int t^{i+j}d\mu=:\gamma_{j+i}.$$

Therefore, the moment problem for $K \subset \mathbb{R}$ is reduced to singly indexed sequence,

$$\gamma_j = \int z^j d\mu, \qquad \text{for all } \ j \geq 0. \tag{3.5}$$

Similarly, if K is the unit circle, then from $\bar{z} = \frac{1}{z} = e^{-it}$ it follows that

$$\gamma_{kl} = \int e^{i(k-l)t} dt = \gamma_{k-l}.$$

Singly indexed moment problems have been widely studied, and arise in various fields of mathematics and applied science. The most known moment problems are

- the Hamburger moment problem, $K = \mathbb{R}$,
- the Stieltjes moment problem, $K = \mathbb{R}^+$,
- The Hausdorff moment problem, $K = [0,1]$,
- the Toeplitz moment problem $K = \mathbb{T} := \{t \in \mathbb{C} : | t | = 1\}$.

Now let us define the truncated multi-dimensional moment problem in the real case; to this aim, let $\{\beta_{\mathbf{i}}\}_{\mathbf{i} \in \mathbb{Z}^d_+, |\mathbf{i}| \leq 2n}$ be a given a real multisequence and let K be a nonempty subset of $\mathbb{R}^d$. *The truncated K-moment problem* for $\{\beta_{\mathbf{i}}\}_{\mathbf{i} \in \mathbb{Z}^d_+, |\mathbf{i}| \leq 2n}$ entails finding a positive Borel measure μ, supported in K such that

$$\beta_{\mathbf{i}} = \int \mathbf{x}^{\mathbf{i}} d\mu \qquad (\mathbf{i} \in \mathbb{Z}^d_+, | \mathbf{i} | \leq 2n). \tag{3.6}$$

Let $\mathbf{i}$ be a multi-index $(i_1, i_2, \ldots, i_d)$ of positive integers. We will write $| \mathbf{i} | = i_1 + \ldots + i_d$ and $\mathbf{x}^{\mathbf{i}} = x_1^{i_1} \ldots x_d^{i_d}$ whenever $\mathbf{x} \equiv (x_1, \ldots, x_d)$ is a d-tuple of real numbers.

A solution μ of (3.6) is called a *K-representing measure* and $\{\beta_{\mathbf{i}}\}_{\mathbf{i} \in \mathbb{Z}^d_+, |\mathbf{i}| \leq 2n}$ is known as a *truncated K-moment sequence*. The *full K- moment problem* prescribes moments of all orders. More precisely, an infinite multisequence $\{\beta_{\mathbf{i}}\}_{\mathbf{i} \in \mathbb{Z}^d_+}$ is given and we aim to find a positive Borel measure μ supported in K such that

$$\beta_{\mathbf{i}} = \int \mathbf{x}^{\mathbf{i}} d\mu \qquad \text{for all } \mathbf{i} \in \mathbb{Z}^d_+; \tag{3.7}$$

If $K = \mathbb{R}^d$; (3.7) is often, and we would, called *the moment problem* for $\{\beta_{\mathbf{i}}\}_{\mathbf{i} \in \mathbb{Z}^d_+}$.

3.3 Moment Problem Via K-Positivity

Haviland's Theorem shows that there is a close link between positive polynomials and the moment problem. However, in order to apply this result reasonable desriptions of positive polynomials are needed. In this section, we solve the K-moment problem by combining Haviland's theorem with the descriptions of positive polynomial given in Section 2.1.4.

3.3.1 K-Positivity Approach to the Full Moment Problem

We begin by stating the celebrated Riesz-Haviland theorem.

Theorem 3.3.1 — Riesz-Haviland theorem. Let K be a closed subset in $\mathbb{R}^d$ and Λ be a linear functional on $\mathbb{R}[\mathbf{x}]^*$. The following asserting are equivalent:

(i) $\Lambda(p) \geq 0$ for any polynomial $p \in \mathbb{R}[\mathbf{x}]$ such that $p \geq 0$ on K.

(ii) Λ is a moment functional, that is, there exists a measure μ on K such that $\Lambda(p) = \int_K p d\mu$ for all $p \in \mathbb{R}[\mathbf{x}]$.

For a first application of Riesz-Haviland, consider the classical theorems of Hamburger for $K = \mathbb{R}$. Recall that we denote by $H_n(y)$ the $(n+1) \times (n+1)$-Hankel matrix, defined by $H_n(\mathbf{y})(i,j) = y_{i+j}$.

Theorem 3.3.2 — Hamburger. Let $\mathbf{y} = (y_i)_{i \in \mathbb{Z}_+} \subset \mathbb{R}$, then $\mathbf{y}$ has a representing Borel measure μ on $\mathbb{R}$ if and only if the quadratic form

$$\mathbb{R}^{n+1} \ni \mathbf{x} \longrightarrow \Lambda_n(\mathbf{x}) = \sum_{i,j=0}^{n} y_{i+j}x_i x_j \tag{3.8}$$

is positive semi-definite for all $n \in \mathbb{Z}_+$. Equivalently, $H_n(y) \geq 0$ for all $n \in \mathbb{Z}_+$.

Proof. If $y_n = \int_{\mathbb{R}} t^n d\mu$, then

$$\Lambda_n(\mathbf{x}) = \sum_{i,j=0}^{n} x_i x_j \int t^{i+j} d\mu = \int (\sum_{i=0}^{n} x_i t^i)^2 d\mu \geq 0.$$

Conversely, we assume that (3.8) holds, or, equivalently $H_n(y) \geq 0$ for all $n \in \mathbb{Z}_+$. Therefore, for every $\mathbf{q} \in \mathbb{R}^{n+1}$ we have $\mathbf{q}^T H_n(y)\mathbf{q} \geq 0$. Now, let $p \in \mathbb{R}_+[x]$, according to the Proposition 2.1.2, one can write $p = f^2 + g^2$ for some polynomials $f, g \in \mathbb{R}[x]$. Then

$$\Lambda_{\mathbf{y}}(p) = \Lambda_{\mathbf{y}}(f^2 + g^2) = \mathbf{f}^T H_n(y)\mathbf{f} + \mathbf{g}^T H_n(y)\mathbf{g} \geq 0,$$

where $\mathbf{f}$ and $\mathbf{g}$ are vectors of coefficients of polynomial f and g, respectively. As $p \geq 0$ was arbitrary, then, in the light of Theorem 3.3.1 with $K = \mathbb{R}$, the sequence $\mathbf{y}$ has a representing measure. ■

We next consider the theorem of Stieltjes and Hausdorff for $K = [0, +\infty)$ and $K = [a, b]$, respectively; but for this we need additional notation. Let us denote by $S\mathbf{y} = (y_1, y_2, \dots, y_m)$ (here $m \in \mathbb{Z}_+ \cup \{+\infty\}$) the shifted sequence of $\mathbf{y} = (y_0, y_1, \dots, y_m)$; obviously $S^2\mathbf{y} = (y_2, y_3, \dots, y_m)$.

Theorem 3.3.3 — Stieltjes. Let $\mathbf{y} = (y_i)_{i \in \mathbb{Z}_+} \subset \mathbb{R}$, then $\mathbf{y}$ has a representing Borel measure μ supported on $[0, +\infty)$ if and only if the quadratic forms (3.8) and

$$\mathbb{R}^{n+1} \ni \mathbf{x} \longrightarrow u_n(\mathbf{x}) = \sum_{i,j=0}^{n} y_{i+j+1}x_i x_j \tag{3.9}$$

are positive semi-definite for all $n \in \mathbb{Z}_+$. Equivalently, $H_n(y) \geq 0$ and $H_n(Sy) \geq 0$ for all $n \in \mathbb{Z}_+$.

Proof. The proof is almost the same as the proof of Theorem 3.3.2; instead of Proposition 2.1.2 we apply Proposition 2.1.3. ■

Combining Haviland's theorem with Proposition 2.1.4 the same reasoning used in the proofs of Theorems 3.3.2 and 3.3.3 yields the following result.

Theorem 3.3.4 — Hausdorff. Let $\mathbf{y} = (y_i)_{i \in \mathbb{Z}_+} \subset \mathbb{R}$, then $\mathbf{y}$ has a representing Borel measure μ supported on $[a, b]$ if and only if the quadratic forms (3.8) and

$$\mathbb{R}^{n+1} \ni \mathbf{x} \longrightarrow v_n(\mathbf{x}) = \sum_{i,j=0}^{n} (-y_{i+j+2} + (b+a)y_{i+j+1} - ab y_{i+j})x_i x_j \tag{3.10}$$

are positive semi-definite for all $n \in \mathbb{Z}_+$. Equivalently, $H_n(y) \geq 0$ and $-H_n(S^2y) + (a+b)H_n(Sy) - abH_n(Sy) \geq 0$ for all $n \in \mathbb{Z}_+$.

The passage from one-dimensional to multidimensional moment problems leads to fundamental new difficulties. As already observed by Hilbert (Theorem 2.1.1) there are positive polynomials in $d \geq 2$ variables that are not sums of squares, see (2.4). As a consequence, there exists a linear functional Λ on $\mathbb{R}[\mathbf{x}]$ which is positive for any SOS polynomial (i.e. $\Lambda(p^2) \geq 0$ for all $p \in \mathbb{R}[\mathbf{x}]$), but Λ is not a moment functional (that is, there is no positive measure μ for which $\Lambda(p) = \int p d\mu$ for all $p \in \mathbb{R}[\mathbf{x}]$). Therefore, when $K = \mathbb{R}^d$ $(d > 1)$, the most direct analogue of Hamburger's Theorem 3.3.2 is false, and the $K = \mathbb{R}^d$ full moment problem (including the full complex moment problem for $K = \mathbb{C}$) remains unsolved. For this reason the K-moment problem is invented. It asks for representing measures with support contained in a given closed subset K of $\mathbb{R}^d$.

Previously we derived solvability criteria of the moment problem on intervals in terms of positivity conditions. It seems to be natural to look for similar characterizations in higher dimensions as well. This leads us immediately into the realm of real algebraic geometry and the descriptions of positive polynomials on basic semi-algebraic sets (see Definition 2.1.2). In the next we treat this approach for basic closed compact semi-algebraic subsets of $\mathbb{R}^d$. It turns out that for such sets there is a close interaction between the moment problem and Positivstellensätze for strictly positive polynomials, see Theorems 2.1.5 and 2.1.6.

Theorem 3.3.5 — Schmüdgen (48). Let polynomials $g_1,\ldots,g_m$ be such that the associated basic semi-algebraic set K is compact and let Λ_β be the Riesz functional associated with $\beta \equiv \{\beta_{\mathbf{i}}\}_{\mathbf{i}\in\mathbb{Z}_+^d}$. Then β has a K-representing measure if and only if

$$\Lambda_\beta(h^2 g_J) \geq 0, \qquad \text{for every } J \subseteq \{1,\ldots,m\} \text{ and } h \in \mathbb{R}[\mathbf{x}], \tag{3.11}$$

or, equivalently, if and only if

$$\text{the localizing matrix } M_{g_J}(\infty) \geq 0 \text{ for each } J \subseteq \{1,\ldots,m\}. \tag{3.12}$$

Proof. For every $J \subseteq \{1,\ldots,m\}$ and $f \in \mathbb{R}[\mathbf{x}]$, the polynomial $f^2 g_J$ is nonnegative on K. Therefore, if β has a K-representing measure μ, then $\int f^2 g_J d\mu \geq 0$. Equivalently $\Lambda_\beta(f^2 g_J) \geq 0$ or, in view of Proposition 2.4.3, $M_{g_J}(\infty) \geq 0$.

Conversely, let $f \in \mathbb{R}[\mathbf{x}]$ such that $f_{|K} \geq 0$, then for arbitrary $\varepsilon > 0$ one have $f + \varepsilon > 0$ on K; as K is a compact basic semi-algebraic set, so that, by Theorem 2.1.5,

$$f + \varepsilon = \sum_{J\subseteq\{1,\ldots,m\}} f_J g_J$$

for some SOS $f_J \in \Sigma_d$. Hence, from (3.11) and from the linearity of Λ_β, we have $\Lambda_\beta(f+\varepsilon) = \Lambda_\beta(f) + \varepsilon\beta_0 \geq 0$. As $\varepsilon > 0$ was arbitrary, $\Lambda_\beta(f) \geq 0$ follows. Therefore, Theorem 3.3.1 implies that β has a K-representing measure. ∎

Theorem 3.3.6 — Putinar (47). Let polynomials $g_1,\ldots,g_m$ be such that the associated basic semi-algebraic set K is compact and let Λ_β be the Riesz functional associated with $\beta \equiv \{\beta_{\mathbf{i}}\}_{\mathbf{i}\in\mathbb{Z}_+^d}$. Assume that there exists $u \in \mathbb{R}[\mathbf{x}]$ of the form

$$u = u_0 + \sum_{j=0}^{m} u_i g_i, \qquad \text{where } u_i \in \Sigma_d, i = 0,1,\ldots,m,$$

and such that the set $\{\mathbf{x} \in \mathbb{R}^d : u(\mathbf{x}) \geq 0\}$ is compact. Then β has a representing measure with support contained in K if and only if

$$\Lambda_\beta(h^2 g_j) \geq 0, \qquad \text{for every } j = 1,\ldots,m \text{ and } h \in \mathbb{R}[\mathbf{x}], \tag{3.13}$$

or, equivalently, if and only if

$$\text{the localizing matrix } M_{g_j}(\infty) \geq 0 \text{ for each } j = 1,\ldots,m. \tag{3.14}$$

Proof. The proof is similar to that of Theorem 3.3.5; instead of Theorem 2.1.5 we apply Theorem 2.1.6. ∎

3.3.2 K-Positivity Approach to the Truncated Moment Problem

Given a truncated K*-moment sequence* $\beta^{(2n)} \equiv \{\beta_{\mathbf{i}}\}_{\mathbf{i}\in\mathbb{Z}_+,|\mathbf{i}|\leq 2n}$, that is, there exists a positive measure, say μ, supported on K such that $\Lambda_{\beta^{(2n)}}(p) = \int p d\mu$, for every $p \in \mathbb{R}[\mathbf{x}]$; where $\Lambda_{\beta^{(2n)}}$ denote the Riesz functional associated with $\beta^{(2n)}$. Obviously, if $p_{|K} \geq 0$, then $\Lambda_{\beta^{(2n)}}(p) \geq 0$.

It follows from above that if $\beta^{(2n)}$ is a truncated K-moment sequence, then $\Lambda_{\beta^{(2n)}}$ is K-positive. Tchakaloff [57] proved that the converse is true when K is compact. However, we will show in the next example (taken from [17]) that in general it is not true that $\beta^{(2n)}$ has a K-representing measure whenever $\Lambda_{\beta^{(2n)}}$ is K-positive, so the most direct analogue of Riesz-Haviland Theorem 3.3.1 for the truncated moment problem is false.

Example 3.3.1 Define $\beta^{(4)}$ by $\beta_0 = \beta_1 = \beta_2 = \beta_3 = 1$ and $\beta_4 = 2$, so that

$$M(2) = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 2 \end{pmatrix} \geq 0.$$

Let us show that the Riesz functional $\Lambda_{\beta^{(4)}}$, associated with the truncated sequence $\beta^{(4)}$, is $\mathbb{R}$-positive. According to Proposition 2.1.2, if the polynomial $p \in \mathbb{R}_4[x]$ is nonnegative, then there exist $f, g \in \mathbb{R}_2[x]$ such that $p^2 = f^2 + g^2$. Hence $\Lambda_{\beta^{(4)}}(p) = \Lambda_{\beta^4}(f^2+g^2) =< M(2)f, f > + < M(2)g, g > \geq 0$ and thus $\Lambda_{\beta^{(4)}}$ is $\mathbb{R}$-positive.

Assume that $\beta^{(4)}$ has a representing measure, say μ, then

$$\int (x-1)^2 d\mu = L((x-1)^2) = v^T M(2) v = 0,$$

where $v \equiv (-1, 1, 0)$ denote the coefficient vector, associated to the polynomial $-1+x$, with respect to the basis of monomials of $\mathbb{R}_2[x]$ in degree-lexicographic order; hence $supp\mu = \{1\}$. Since $\int d\mu = \beta_0 = 1$, then $\mu \equiv \delta_{\{1\}}$. Therefore $\beta_4 = \int x^4 d\mu = 1$, a contradiction. Thus $\Lambda_{\beta^{(4)}}$ is K-positive, but $\beta^{(4)}$ has no representing measure.

Despite the preceding difficulties in adapting Riesz-Haviland to TKMP, R. Curto and L. Fialkow give in [17] an appropriate analogue.

Theorem 3.3.7 — Truncated Riesz-Haviland Theorem (17). $\beta \equiv \beta^{(2n)}$ or $\beta \equiv \beta^{(2n+1)}$ has a K-representing measure if and only if Λ_β admits an extension to a sequence $\widetilde{\beta} \equiv \widetilde{\beta}^{(2n+2)}$ such that $\Lambda_{\widetilde{\beta}}$ is K-positive.

Note that Theorem 3.3.7 does not provide a concrete solution to truncated K-moment problem, as discussed above, we have no general structure theorem for K-positive polynomial, so we have trouble in checking K-positivity of the Riesz functional. Several authors have addressed this issue from a variety of viewpoints; for instance, F.H. Vasilescu [59] has studied truncated moment problem using techniques from function spaces and $\mathbb{C}^*$-algebras (in another direction see [45, 49]).

3.4 Moment Problem Via Flat Extensions of Positive Moment Matrices

R. Curto and L. Fialkow developed in a series of papers [6, 9, 10, 12] an approach for truncated moment problems based on positivity and flat extensions of the moment matrix. The key result about flat extensions of moment matrices states that a sequence $\beta^{(2n)} \equiv \{\beta_{\mathbf{i}}\}_{\mathbf{i} \in \mathbb{Z}_+^d, |\mathbf{i}| \leq 2n}$ has a representing measure if, and only if, their associated moment matrix $M(n)$ admits a positive finite rank moment matrix extension $M(n+1)$. In general the existence of such extension is difficult to determine, but in various special cases concrete results are known.

In this section we group some of the main results about the truncated moment problem, mostly from R. Curto and L. Fialkow.

3.4.1 The Truncated Univariate Moment Problem

In [6] R. Curto and L. Fialkow provided some of the main results about the truncated moment problem in the univariate case. Namely they give characterizations for the truncated sequences $\{\beta_i\}_{0 \leq i \leq m}$ having a representing measure on an interval $K = \mathbb{R}, K = [0, +\infty), K = [a, b]$ and on the unit circle $K = \mathbb{T}$.

Begin by stating a concrete solution to the so-called Hamburger truncated moment problem. We need a definition.

For a sequence $\beta^{(m)} \equiv \{\beta_i\}_{0 \leq i \leq m}$ and $0 \leq l \leq k \leq m$, we set $\beta[l,k]^T = (\beta_l, \beta_{l+1}, \ldots, \beta_k)$. Consider the moment matrix $M(n) \equiv M(\beta^{(2n)})$, indexed by $\{1, X, \ldots, X^n\}$. Then, $rank(\beta^{(2n)})$ is defined as the smallest integer r for which the column of $M(n)$ indexed by X^r lies in the linear span of the columns indexed by $\{1, X, \ldots, X^{r-1}\}$ if such integer exists, and $rank(\beta^{(2n)}) := n+1$ otherwise.

Theorem 3.4.1 — Hamburger truncated moment problem (6).

(i) **(odd case)** The truncated sequence $\beta^{(2n+1)}$ has a representing measure on $\mathbb{R}$ if and only if $M(n) \geq 0$ and $\beta[n+1; 2n+1] \in RanM(n)$.

(ii) **(even case)** The truncated sequence $\beta^{(2n)}$ has a representing measure on $\mathbb{R}$ if and only if $M(n) \geq 0$ and $rankM(n) = rank\beta^{(2n)}$.

To illustrate the above theorem we consider the following example.

Example 3.4.1 consider the sequence $\beta^{(4)} = (1,1,1,1,2)$ from Example 3.3.1. Then $M(2) \geq 0$ but $\beta^{(4)}$ has no representing measure, which is consist with the fact that $rank(\beta^{(4)}) = 1 < rankM(2) = 2$.

The next theorem provides a solution of the Stieltjes truncated moment problems.

Theorem 3.4.2 — Stieltjes truncated moment problem [6].

(i) **(odd case)** The truncated sequence $\beta^{(2n+1)}$ has a representing measure on $[0,+\infty)$ if and only if $M(n) \geq 0$, $M_x(n) \geq 0$ and $\beta[n+1;2n+1] \in RanM(n)$.

(ii) **(even case)** The truncated sequence $\beta^{(2t)}$ has a representing measure on $[0,+\infty)$ if and only if $M(n) \geq 0$, $M_x(n-1) \geq 0$ and $\beta[n+1;2n] \in RanM_x(n-1)$.

In the next we characterize moment sequence having a representing measure on an interval $[a,b]$.

Theorem 3.4.3 — Hausdorff truncated moment problem [6]. Let $a < b$ be real numbers.

(i) **(odd case)** The truncated sequence $\beta^{(2n+1)}$ has a representing measure on $[a,b]$ if and only if $bM(n) - M_x(n) \geq 0$, $M_x(n) - aM(n) \geq 0$.

(ii) **(even case)** The truncated sequence $\beta^{(2n)}$ has a representing measure on $[a,b]$ if and only if $M(n) \geq 0$, $-M_{x^2}(n-1) + (a+b)M_x(n-1) - abM(n-1) \geq 0$.

We now present characterizations for moment sequence having a representing measure on the unit circle $\mathbb{T}$. We need to recall that if $s = (s_j)_{j=0}^m$ is a truncated (complex) sequence and $n \leq m$, then $T_n(s) = (t_{jk})_{j,k=0}^n$ denotes the $(n+1) \times (n+1)$-Toeplitz matrix with entries $t_{jk} = s_{k-j}$.

Theorem 3.4.4 — Toeplitz truncated moment problem [49, Theorem 11.5]. Let $n \in \mathbb{Z}_+$. For a sequence $(s_j)_{j=0}^n$ the following are equivalent:

(i) There is a positive measure μ on the unit circle $\mathbb{T}$ such that

$$s_j = \int z^{-j} d\mu(z) \qquad \text{for } j = 0,1,\ldots,n. \tag{3.15}$$

(ii) There is a k-atomic measure μ on $\mathbb{T}$, $k \leq 2n+1$, such that (3.15) holds.

(iii) The Toeplitz matrix $T_n(s)$ is positive semi-definite.

(iv) $\sum_{j,k=0}^n s_{j-k} c_k \overline{c_j} \geq 0$ for all $(c_0,\ldots,c_n)^T \in \mathbb{C}^{n+1}$.

3.4.2 The Truncated Complex Moment Problem

In [9, 10] R. Curto and L. Fialkow introduced an approach to study the existence and uniqueness of solutions of the truncated complex moment problem (Problem (3.4) with $K = \mathbb{C}$ and $m < +\infty$) based on positivity and extensions of the moment matrix. The key result of this approach is given in the following theorem.

Theorem 3.4.5 — [9, Theorem 5.13]. Let $\gamma^{(2n)} \equiv \{\gamma_{ij}\}_{i+j \leq 2n}$ be a given complex bi-sequence such that $\gamma_{00} > 0$ and $\overline{\gamma_{ij}} = \gamma_{ji}$. Then $\gamma^{(2n)}$ has a *rank* $M(n)$-atomic representing measure if and only if $M(n) \geq 0$ and $M(n)$ admits a flat extension $M(n+1)$, i.e., $M(n)$ can be extended to a positive moment matrix $M(n+1)$ satisfying $rank\ M(n+1) = rank\ M(n)$.

Moreover, an extended version of the flat extension theorem says if $M(n)$ admits a positive extension $M(n+k)$ for some $k \in \mathbb{Z}_+$ that has a flat extension $M(n+k+1)$, then $\gamma^{(2n)}$ has a *rank* $M(n+k)$-atomic measure μ.

In general the existence of such extension is difficult to determine, but in various special cases concrete results are known. Indeed, only the cases $m = 1,2,3,4$ (the quadratic, the cubic and the quartic moment problem) have been completely achieved.

We first present existence uniqueness criteria for the quadratic moment problem (i. e., the Problem (3.4) with $m = 2n = 2$) .

Theorem 3.4.6 — The quadratic moment problem (9, Theorem 6.1). Given the data $\gamma^{(2)} := \{\gamma_{ij}\}_{i+j\leq 2}$, let $M(1)$ be the associated moment matrix and let $r := rank\ \ M(1)$. The following are equivalent:

(i) $\gamma^{(2)}$ has a representing measure;

(ii) $\gamma^{(2)}$ has an r-atomic representing measure;

(iii) $M(1) \geq 0$.

In this case, if $r = 1$ there exists a unique representing measure; if $r = 2$ the 2-atomic representing measures are parameterized by a line; if $r = 3$ the 3-atomic representing measures contain a sub-parametrization by a circle.

A complete solution to the cubic complex moment problem (when $m = 3$) was given in [38], the solution is based on commutativity conditions of matrices determined by $\{\gamma_{ij}\}_{0\leq i+j\leq 3}$. Recently, an alternative solution to nonsingular cubic moment problems, using positive flat extension techniques, was provided by R. Curto and S. Yoo [23]. We need to fix some notation.

Let $\gamma^{(3)} \equiv \{\gamma_{ij}\}_{i+j\leq 3}$ be a given complex-valued sequence, put

$$\Phi = \begin{pmatrix} \gamma_{00} & \gamma_{01} & \gamma_{10} \\ \gamma_{10} & \gamma_{11} & \gamma_{20} \\ \gamma_{01} & \gamma_{02} & \gamma_{11} \end{pmatrix}, \Phi_z = \begin{pmatrix} \gamma_{01} & \gamma_{02} & \gamma_{11} \\ \gamma_{11} & \gamma_{12} & \gamma_{21} \\ \gamma_{02} & \gamma_{03} & \gamma_{12} \end{pmatrix} \text{ and } \Phi_{\bar{z}} = (\Phi_z)^*.$$

Theorem 3.4.7 — The cubic moment problem (38, Theorems 3.2 and 3.4).
Let Φ, Φ_z and $\Phi_{\bar{z}}$ be as above.

$A)$ Suppose $r := rank\ \ \Phi = 1, 2$, then the following are equivalent:

(i) $\Phi \geq 0$ and there exist matrices Θ_z and $\Theta_{\bar{z}}$ such that $\Phi\Theta_z = \Phi_z$, $\Phi\Theta_{\bar{z}} = \Phi_{\bar{z}}$;

(ii) $\gamma^{(3)}$ has a r-atomic representing measure;

(iii) $\gamma^{(3)}$ has a atomic representing measure.

$B)$ Suppose $\Phi > 0$ and put $\Theta_z = \Phi^{-1}\Phi_z$ and $\Theta_{\bar{z}} = \Phi^{-1}\Phi_{\bar{z}}$. Then

(i) $\gamma^{(3)}$ has a 3-atomic representing measure if and only if the matrices Θ_z and $\Theta_{\bar{z}}$ commute.

(ii) $\gamma^{(3)}$ has a 4-atomic representing measure if and only if the matrices Θ_z and $\Theta_{\bar{z}}$ do not commute.

In [21] concrete solution to the *nonsingular* quartic moment problem (i.e., when $m = 2n = 4$) is given.

Theorem 3.4.8 — (21, Theorem 2.1). Assume $M(2)$ is positive and invertible. Then $M(2)$ admits a representing measure, with exactly 6 atoms; that is, $M(2)$ actually admits a flat extension $M(3)$.

The next theorem give a general solution of the *singular* quartic moment problem and for $M(n)$ whenever the submatrix $M(2)$ is singular. We need definition.

The moment matrix $M(n)$ is said to be recursively generated if for any column relation in $M(n)$ of the form $p(Z,\overline{Z}) = 0$, one automatically has $(pq)(Z,\overline{Z}) = 0$, for each polynomial q with $\deg(pq) \leq n$. Recall in passing the following basic property of positive moment matrices:

$$M(n+1) \geq 0 \Longrightarrow M(n) \textit{ is recursively generated } [9, \text{Theorem } 3.14]. \tag{3.16}$$

Let us denote by $\nu \equiv \nu_{\gamma^{(m)}}$ the variety associated with $\gamma^{(m)}$ as the intersection of the zero-sets of the polynomials $p(z;\bar{z})$ such that $M(n)p = 0$.

Theorem 3.4.9 — Cf. (11, 12, 14, 15, 27). Let $p \in \mathbb{C}[z,\bar{z}]$, with $\deg p \leq 2$. For $n \geq \deg p$, the following are equivalent:

(i) $\gamma^{(2n)}$ has a representing measure supported in the curve $p(z,\bar{z}) = 0$;

(ii) $M(n)$ has a column dependence relation $p(Z,\overline{Z}) = 0$ and admits a positive, recursively generated extension $M(n+1)$;

(iii) (concrete condition) $M(n)$ has a column dependence relation $p(Z,\overline{Z}) = 0$, $M(n) \geq 0$, $M(n)$ is recursively generated, and $rank\ M(n) \leq card\ \nu$.

In general, if $\gamma^{(2n)}$ has a representing measure, then $M(n)$ is positive and rank $M(n) \leq$ card $\nu_{\gamma^{(2n)}}$. In Curto-Fialkow-Moller [18], TCMP is solved for the "extremal" case when rank $M(n) =$ card $\nu_{\gamma^{(2n)}}$. In this case, $\gamma^{(2n)}$ has a representing measure if and only if $M(n) \geq 0$ and $\gamma^{(2n)}$ is consistent; that is, if $p \in \mathbb{C}_{2n}[z;\bar{z}]$ and $p_{|\mathcal{V}_{\gamma^{(2n)}}} \equiv 0$, then $\Lambda_{\gamma^{(2n)}}(p) = 0$. The proof of this fact does not require results on moment matrix extensions; it is based on elementary tools from vector space duality, convex analysis, and interpolation by polynomials.

As noted above, only the cases $m = 1,2,3,4$ are completely solved. In a joint work with H. El azhar, A. Harrat and E. H. Zerouali [25] we are solved the quintic moment problem (when $m = 5$). Therefore, the first open case of truncated complex moment problem concerns $m = 2n = 6$ with M(2) positive definite. Within this problem, [20] identifies a subproblem which is "extremal", with rank $M(3) = \text{card } \mathcal{V}_{\gamma^{(6)}} = 7$. Indeed, [20] focuses on the case when $M(3) \geq 0$ and $M(2) > 0$, and there is a column dependence relation $p(Z;\overline{Z}) = 0$ (and automatically, $\overline{p}(Z;\overline{Z}) = 0$), where p is a harmonic polynomial of the form

$$p(z,\bar{z}) = z^3 - itz - u\bar{z} \qquad (t,u \in \mathbb{R}).$$

Lemma 2.3 in [20] states that if $0 <| u |< t < 2 | u |$, then $p \equiv z^3 - itz - u\bar{z}$ has exactly 7 zeros in the complex plane. This later can be disproved by the next two examples.

Example 3.4.2 The equality $z^3 = 2iz - \frac{5}{4}\bar{z}$ admits only 3 zeros, 0 and $\pm\frac{\sqrt{13}}{2}e^{i\frac{\pi}{4}}$, although $t = 2$ and $u = -\frac{5}{4}$ verify the condition $0 < |u| < t < 2|u|$.

Example 3.4.3 The equality $z^3 = -2iz + \frac{5}{4}\bar{z}$ admits 7 zeros, 0 ; $\pm\frac{\sqrt{3}}{2}e^{-i\frac{\pi}{4}}$ and $(\pm\frac{3\sqrt{2}}{4} \pm i\frac{\sqrt{2}}{4})e^{-i\frac{\pi}{4}}$, although $t = -2$ and $u = \frac{5}{4}$ does not verify the condition $0 < |u| < t < 2|u|$.

This "inattention" led to the next incorrect version of the main theorem in [20].

Theorem 3.4.10 [20, Theorem 1.5] Let $M(3) \geq 0$, with $M(2) > 0$ and $z^3 - itz - u\bar{z} = 0$. For $u,t \in \mathbb{R}$, assume that $0 < |u| < t < 2|u|$. The following statements are equivalent.
i) There exists a representing measure for $M(3)$.
ii) $\Lambda_{\gamma^{(6)}}(q_{LC}) = \Lambda_{\gamma^{(6)}}(zq_{LC}) = 0$.
iii) $\mathfrak{Re}\gamma_{12} - \mathfrak{Im}\gamma_{12} = u(\mathfrak{Re}\gamma_{01} - \mathfrak{Im}\gamma_{01})$ and $\gamma_{22} = (t+u)\gamma_{11} - 2u\mathfrak{Im}\gamma_{02}$.
iv) $q_{LC} := i(z - i\bar{z})(z\bar{z} - u) = 0$.

Example 3.4.3 shows that for $u < 0$ the last theorem is not valid, because $6 \leq \text{rang } M(3)$ and card $\mathcal{V}_{\gamma^{(6)}} \leq 3$ (recall that, if $\gamma^{(6)}$ is a moment sequence then rang $M(3) \leq$ card $\mathcal{V}_{\gamma^{(6)}}$). Theorem 3.4.10 implies Corollary 4.3 in the same paper (dealing with the TCMP for cubic column relations in $M(3)$ of the forms $Z^3 = aZ + b\overline{Z}$ where $a,b \in \mathbb{R}$) which has inherited the same mistakes.

From the previous discussion, it appears natural to give a new solution to the TCMP for cubic column relations in $M(3)$ of the forms $Z^3 = itZ + u\overline{Z}$ and $Z^3 = aZ + b\overline{Z}$, where a,b,t and u are real numbers. This is the main goal of Section 5.4, by using our methodology on recursive sequences.

We conclude the section with a theorem.

Theorem 3.4.11 — (10, Theorem 3.1). Suppose $M(n)$ is positive and recursively generated. If $1 \leq k \leq [\frac{n}{2}] + 1$ and the columns in $M(n)$ satisfying $Z^k = p(Z,\overline{Z})$ for some $p \in \mathbb{R}_{k-1}[z,\bar{z}]$, then $M(n)$ admits a unique flat extension $M(n+1)$, which corresponds to the unique minimal representing measure for $\gamma^{(n)}$.

3.4.3 The Truncated Multi-Dimensional Moment Problem

We have considering in the precedent section the truncated complex moment problem which is equivalent, via the Proposition 2.4.4, to the two-dimensional real moment problem. For instance, we may paraphrase Theorem 3.4.9 as follows.

Theorem 3.4.12 Let $p \in \mathbb{R}_2[x,y]$. For $n \geq \deg p$ and $\beta^{(2n)} \equiv \{\beta_{\mathbf{i}}\}_{\mathbf{i}\in\mathbb{Z}^2_+,|\mathbf{i}|\leq 2n}$, the following are equivalent:
(i) $\beta^{(2n)}$ has a representing measure supported in the curve $p(x,y) = 0$;
(ii) $M(n)$ has a column dependence relation $p(X,Y) = 0$ and admits a positive, recursively generated extension $M(n+1)$;
(iii) (concrete condition) $M(n)$ has a column dependence relation $p(X,Y) = 0$, $M(n) \geq 0$, $M(n)$ is recursively generated, and $r \leq v$.

In this section we give the most important results for higher-dimensional truncated moment problem.

Similarly to the previous section we begin by stating the key result of the positive flat extension approach.

Theorem 3.4.13 — (12, 16). Let $\beta \equiv \{\beta_{\mathbf{i}}\}_{\mathbf{i}\in\mathbb{Z}_+^d}$ be a real multisequence and let $M(\beta)$ be its associated moment matrix. If $M(\beta) \geq 0$ and $M(\beta)$ have finite *rank*, then β has a unique $rankM(\beta)$-atomic representing measure.

The next theorem characterizes the existence of a finitely atomic K-representing measure, in the case when K is a semi-algebraic set, that is, $K = K_{\mathscr{Q}} := \{(t_1,\dots,t_d) \in \mathbb{R}^d : q_i(t_1,\dots,t_d) \geq 0, 1 \leq i \leq m\}$ where $\mathscr{Q} \equiv \{q_i\}_{i=1}^m \subset \mathbb{R}[t_1,\dots,t_d]$.

Theorem 3.4.14 — (16, Theorem 2.9). An d-dimensional real sequence $\beta^{(2n)} \equiv \{\beta_{\mathbf{i}}\}_{\mathbf{i}\in\mathbb{Z}_+^d,|\mathbf{i}|\leq 2n}$ admits a $rankM(n)$-atomic representing measure supported in $K_{\mathscr{Q}}$ if and only if $M(n) \geq 0$ and $M(n)$ admits a flat extension $M(n+1)$ such that $M_{q_i}(n+[\frac{\deg q_i+1}{2}]) \geq 0$ $(1 \leq i \leq m)$. In this case, $M(n+1)$ admits a unique representing measure μ, which is a $rankM(n)$-atomic (minimal) $K_{\mathscr{Q}}$-representing measure for $\beta^{(2n)}$; moreover, μ has precisely $rankM(n) - rankM_{q_i}(n+[\frac{\deg q_i+1}{2}])$ atoms in $\mathscr{Z}(q_i) \equiv \{t \in \mathbb{R}^d : q_i(t) = 0\}$, $1 \leq i \leq m$.

An alternative proof, of Theorems 3.4.13 and 3.4.14, was provided by M. Laurent in [44], based on Hilbert's Nullstellensatz instead of the functional analytic tools used in the original proof of Curto and Fialkow. Also, F.H. Vasilescu was introduced in [58, 59] an approach, based on the dimensional stability, equivalent to the concept of flatness.

3.5 Two Important Theorems on Moment Problem

This section is devoted to present two fundamental results, which play a crucial role in our approach for moment problem.

3.5.1 Link Between the Truncated K-Moment Problem and the Full K-Moment Problem.

A result of J. Stochel [55] shows that a solution to Truncated K-moment problem actually implies a solution to full K-Moment Problem.

Theorem 3.5.1 — (55, Theorem 4). Let K be a nonempty closed subset of $\mathbb{R}^d$ and let $\beta^{(\infty)} \equiv \{\beta_{\mathbf{i}}\}_{\mathbf{i}\in\mathbb{Z}_+^d}$ be a multisequence. Then $\beta^{(\infty)}$ has a K-representing measure if and only if $\beta^{(m)} \equiv \{\beta_{\mathbf{i}}\}_{\mathbf{i}\in\mathbb{Z}_+^d,|\mathbf{i}|\leq m}$ has a K-representing measure for each $m \geq 1$.

The above theorem furnishes a tool for solving the full K-moment problem without explicitly using Riesz-Haviland. This section is devoted to illustrate this approach by provided an alternate proof of Theorems 3.3.2, 3.3.3, 3.3.4 and a solution to the full Toeplitz moment problem. Also, we illustrate this approach in the proof of Theorem 3.5.3, which present the full version of Theorem 3.4.9.

In the next theorem we adopt notations used in Theorems 3.4.3 and 3.4.4.

Theorem 3.5.2 Let $\mathbf{y} \equiv \{y_i\}_{i\in\mathbb{Z}_+} \subset \mathbb{Z}_+$. Then:

(a) (Hamburger) $\mathbf{y}$ has a representing measure μ on $\mathbb{R}$ if and only if $H_n(\mathbf{y}) \geq 0$ for all $n \in \mathbb{Z}_+$.

(b) (Stieltjes) $\mathbf{y}$ has a representing measure μ on $\mathbb{R}_+$ if and only if $H_n(\mathbf{y}) \geq 0$ and $H_n(S\mathbf{y}) \geq 0$ for all $n \in \mathbb{Z}_+$.

(c) (Hausdorff) $\mathbf{y}$ has a representing measure μ on $[a,b]$ if and only if $H_n(\mathbf{y}) \geq 0$ $-H_n(S^2\mathbf{y}) + (a+b)H_n(S\mathbf{y}) - abH_n(\mathbf{y}) \geq 0$ for all $n \in \mathbb{Z}_+$.

(d) (Toeplitz) $\mathbf{y}$ has a representing measure μ supported on the unit circle $\mathbb{T}$ if and only if $T_n(\mathbf{y}) \geq 0$ for all $n \in \mathbb{Z}_+$.

Proof. (a) Let $K = \mathbb{R}$ and suppose μ is a representing measure for $\mathbf{y}$, it follows as in the preceding proof that $H_n(\mathbf{y}) \geq 0$ for all $n \in \mathbb{Z}_+$.

Conversely, let $l \in \mathbb{Z}_+$, since $H_{l+1}(\mathbf{y}) \geq 0$, then Theorem 2.4.1 implies that $H_l(\mathbf{y}) \geq 0$ and $\mathbf{y}[l+1;2l+1] := (y_{l+1},\dots,y_{2l+1})^T \in RanH_l(\mathbf{y})$. According to Theorem 3.4.1 the truncated sequence $\mathbf{y}^{(2l+1)} \equiv \{\mathbf{y}_i\}_{i=0}^{2l+1}$, and in force $\mathbf{y}^{(2l)}$, has a representing measure supported in $\mathbb{R}$. Since l is an arbitrary integer, Theorem 3.5.1 yields $\mathbf{y} \equiv \{y_i\}_{i\in\mathbb{Z}_+} \subset \mathbb{Z}_+$ has a representing measure supported in $\mathbb{R}$.

(b) The proof is similar to the proof of Hamburger's Theorem (a); instead of Theorem 3.4.1 we apply Theorem 3.4.2.

(c) The proof is almost the same as the proof of (a); instead of Theorem 3.4.1 we apply Theorem 3.4.3.

(d) Assume that $\mathbf{y}$ has a representing measure μ on the unit circle $\mathbb{T}$. Let $p(z) = \sum_{i=0}^{n} c_i z^n \in \mathbb{C}[z]$, then

$$0 \leq \int_{\mathbb{T}} |\, p\,|^2 \, d\mu = \int_{\mathbb{T}} \sum_{i,j=0}^{n} c_i \overline{c_j} z^{i-j} d\mu = \sum_{i,j=0}^{n} c_i \overline{c_j} t_{i-j} = \mathbf{p}^T T_n(\mathbf{y})\mathbf{p},$$

where $\mathbf{p} \in \mathbb{C}^{n+1}$ denote the coefficient vector associated to the polynomial p with respect to the degree-lexicographic order. Since p is arbitrary, then $T_n(\mathbf{y}) \geq 0$ for all $n \in \mathbb{Z}_+$.
Conversely, let $n \in \mathbb{Z}_+$ and let $T_n(\mathbf{y}) \geq 0$, then Theorem 3.4.4 implies that $\{y_i\}_{i=0}^{n}$ has a representing measure μ_n supported on the unit circle $\mathbb{T}$. Since n is arbitrary, then Theorem 3.5.1 yields that $\mathbf{y} \equiv \{y_i\}_{i \in \mathbb{Z}_+}$ has a representing measure μ supported on the unit circle $\mathbb{T}$. ∎

The moment problem on curves was first studied in [54]. Indeed, for $p \in \mathbb{R}_2[x,y]$ and $K \equiv \mathscr{Z}_p := \{(x,y) \in \mathbb{R}^2 \mid p(x,y) = 0\}$ the solution of the Full K-Moment Problem is due to J. Stochel [54]; we may paraphrase Stochel's result as follows.

Theorem 3.5.3 — Stochel (54). Let $p \in \mathbb{R}_2[x,y]$ and let $M(\infty)$ be the moment matrix associated with the bi-indexed sequence of real numbers $\beta^{(\infty)} \equiv \{\beta_{\mathbf{i}}\}_{\mathbf{i} \in \mathbb{Z}_+^2}$. Then $\beta^{(\infty)}$ has a representing measure supported in $\mathscr{Z}_p$ if and only if $M(\infty) \geq 0$ and $p(X,Y) = 0$ in $Col \;\; M(\infty)$, the column space of $M(\infty)$.

Proof. Assume that $\beta^{(\infty)}$ has a representing measure μ supported in $\mathscr{Z}_p$. Clearly, $M(\infty) \geq 0$. Since $supp \;\; \mu \subseteq \mathscr{Z}_p$, then

$$< p(X,Y), \mathbf{q} >= \Lambda_{\beta^{(\infty)}}(pq) = \int pq d\mu = 0, \text{ for all } q \in \mathbb{R}[x,y].$$

It follows that $p(X,Y) = 0$ in $Col \;\; M(\infty)$.

Conversely, suppose $M(\infty) \geq 0$ and $p(X,Y) = 0$. Let $l \geq 2$, since $M(l+2) \geq 0$, then (3.16) yields $M(l+1)$ is a positive and recursively generated moment matrix extension of $M(l)$; also, we have $p(X,Y) = 0$ in $ColM(l)$, therefore Theorem 3.4.12 implies that $\beta^{(2l)}$ admits a representing measure supported in $\mathscr{Z}_p$. Since $l \geq 2$ is arbitrary, it now follows from Theorem 3.5.1 that $\beta^{(\infty)}$ has a representing measure supported in $\mathscr{Z}_p$. ∎

3.5.2 Tchakaloff's Theorem

In this section, we will draw from the generalization of Tchakaloff's Theorem a very useful result, for our approach to truncated moment problem.

Begin by recall that *a cubature formula* of degree n for a Borel measure μ on $\mathbb{R}^d$, is given by an integer $m \geq 1$, points $\{\mathbf{x}_i\}_{i=1}^{m}$ in the support of μ and positive scalars $\{\lambda_i\}_{i=1}^{m}$ such that

$$\int_{\mathbb{R}^d} p(\mathbf{x}) d\mu = \sum_{i=1}^{m} \lambda_i p(\mathbf{x}_i),$$

for all polynomials $p \in \mathbb{R}_n[\mathbf{x}]$. The next result is the basis of existence of cubature formulas for numerical integration and is an extension of the celebrated Tchakaloff's theorem.

Theorem 3.5.4 — (3, Theorem 2). Let μ be a finite Borel measure with support $\mathbb{K} \subset \mathbb{R}^d$ and let $n \geq 1$ be a fixed positive integer. Then there exist m points $\{\mathbf{x}_i\}_{i=1}^{m} \subset \mathbb{K}$ (with $m \leq \binom{d+n}{d}$) and positive weights λ_i, $i = 1, \ldots, m$, such that

$$\int_{\mathbb{K}} p d\mu = \sum_{i=1}^{m} \lambda_i p(\mathbf{x}_i),$$

for every polynomials $p \in \mathbb{R}_n[\mathbf{x}]$.

We derive from the above result the following useful theorem.

Theorem 3.5.5 If a truncated sequence $\{y_{\mathbf{i}}\}_{\mathbf{i} \in \mathbb{Z}_+^d, |\mathbf{i}| \leq n}$ has a representing measure μ, then it has another representing measure ν which is finitely atomic with $supp(\nu) \subseteq supp(\mu)$ and $card \;\; supp(\nu) \leq \binom{d+n}{d}$.

CHAPTER 4

THE QUINTIC COMPLEX MOMENT PROBLEM

The truncated complex moment problem (Problem (3.4) with $K = \mathbb{C}$ and $m < +\infty$) have been completely achieved only for the cases $m = 1; 2; 3$ and 4 (the quadratic, the cubic and the quartic moment problem), all the other cases (quintic, sixtic, ...) are open, as noted at the Subsection 3.4.2.

In this chapter, we provide a concrete solution to the, almost all, quintic moment problem (i.e. $m = 5$) when one desires a minimal representing measure. To this aim, we investigate the structure of recursive complex-valued bi-indexed sequences and we combine the obtained observations with some results due to R.Curto and L. Fialkow, to provide a new technique for solving the odd-degree TCMP. We notice that our techniques furnish a short solution to the cubic moment problem (we omit the proof because the cubic moment problem is already solved, see Theorem 3.4.7) and expected to be useful for higher odd-degree truncated moment problems.

Let $\gamma^{(5)} = \{\gamma_{ij}\}_{0\le i+j\le 5}$ be a given complex valued bi-sequence. We associate with $\gamma^{(5)}$ the next two matrices that will play a crucial role in our approach.

$$M(2) := \begin{pmatrix} \gamma_{00} & \gamma_{01} & \gamma_{10} & \gamma_{02} & \gamma_{11} & \gamma_{20} \\ \gamma_{10} & \gamma_{11} & \gamma_{20} & \gamma_{12} & \gamma_{21} & \gamma_{30} \\ \gamma_{01} & \gamma_{02} & \gamma_{11} & \gamma_{03} & \gamma_{12} & \gamma_{21} \\ \gamma_{20} & \gamma_{21} & \gamma_{30} & \gamma_{22} & \gamma_{31} & \gamma_{40} \\ \gamma_{11} & \gamma_{12} & \gamma_{21} & \gamma_{13} & \gamma_{22} & \gamma_{31} \\ \gamma_{02} & \gamma_{03} & \gamma_{12} & \gamma_{04} & \gamma_{13} & \gamma_{22} \end{pmatrix} \text{ and } B := \begin{pmatrix} \gamma_{03} & \gamma_{12} & \gamma_{21} & \gamma_{30} \\ \gamma_{13} & \gamma_{22} & \gamma_{31} & \gamma_{40} \\ \gamma_{04} & \gamma_{13} & \gamma_{22} & \gamma_{31} \\ \gamma_{23} & \gamma_{32} & \gamma_{41} & \gamma_{50} \\ \gamma_{14} & \gamma_{23} & \gamma_{32} & \gamma_{41} \\ \gamma_{05} & \gamma_{14} & \gamma_{23} & \gamma_{32} \end{pmatrix}. \tag{4.1}$$

Let us recall that thanks to Douglas factorization theorem, we have $Rang\ \ B \subseteq Rang\ \ M(2)$ if, and only if, there exists a matrix W such that $B = M(2)W$. We will show, in Section 4.1, that the Hermitian matrix $W^*M(2)W$ is symmetric with respect to the second diagonal, then one can set

$$W^*M(2)W = \begin{pmatrix} a & b & c & d \\ \overline{b} & e & f & c \\ \overline{c} & \overline{f} & e & b \\ \overline{d} & \overline{c} & \overline{b} & a \end{pmatrix} \tag{4.2}$$

As we will see in the Section 4.3, the entries a, b, e and f in the matrix $W^*M(2)W$ encodes the complete information on the cardinal of the support of the minimal representing measure.

Theorem 4.0.1 Let $\gamma^{(5)} \equiv \{\gamma_{ij}\}_{i+j\le 5}$ be a given finite sequence, such that

$M(2) \ge 0$, $Rang\ \ B \subseteq Rang\ \ M(2)$ and $a \neq e$ or $b = f$.

Then the quintic moment problem, associated with $\gamma^{(5)}$, admits a solution μ. Moreover, The smallest cardinality of $supp\ \ \mu$ is

- $card\ \ supp\ \ \mu = r \iff a = e$ and $b = f$,
- $card\ \ supp\ \ \mu = r+1 \iff \quad a \neq e$ and $\frac{a-e}{2} <| b - f |$,

- *card supp* $\mu = r+2 \iff \quad a > e$ and $\frac{a-e}{2} \geq | b-f |$;

where $r := \;$ *card* $M(2)$ and the numbers a, b, e and f are given by (4.2).

We will show in Section 4.1 that $M(2) \geq 0$ and *Rang* $B \subseteq$ *Rang* $M(2)$ are two necessary conditions for the quintic TCMP, associated with $\gamma^{(5)}$, then Theorem 4.0.1 provides a concrete solution to the quintic complex moment problem, except for the case $a = e$ and $b \neq f$. The difficulty that we encountered in solving the remaining case ($a = e$ and $b \neq f$) is technical, not a failure in the method, see Section 4.3.

This chapter is organized as follows. In Section 4.1, we will give useful tools and results usually used in the treatment of the truncated complex moment problems. We will investigate in Section 4.2 the complex-valued recursive bi-sequences and we will exhibit important results for quintic TCMP. Finally, in Section 4.3, we solve the quintic complex moment problem together with the minimal support problem, an illustrate example is given.

4.1 Preliminaries

First, we recall a fundamental necessary condition. To this end, let us assume that $\gamma^{(2n)} \equiv \{\gamma_{ij}\}_{i+j\leq 2n}$ is a given moment sequence and let μ be the associated representing measure, then, for every $p \equiv \sum\limits_{h,k} a_{hk}\bar{z}^h z^k \in \mathbb{C}[\bar{z}, z]$,

$$0 \leq \int | p |^2 \, d\mu = \sum_{h,k,h',k'} a_{hk}\overline{a_{h'k'}} \int \bar{z}^{h+k'} z^{k+h'} = \sum_{h,k,h',k'} a_{hk}\overline{a_{h'k'}} \gamma_{h+k',k+h'},$$

or, equivalently, the moment matrix $M(n) \equiv M(n)(\gamma^{(2n)})$, defined in Subsection 2.4.3, is semi-definite positive.

In the light of Smul'jan's Theorem 2.4.1, $M(n) \geq 0$ admits a (necessarily positive) flat extension

$$M(n+1) = \begin{pmatrix} M(n) & B \\ B^* & C \end{pmatrix} \tag{4.3}$$

in the form of a moment matrix $M(n+1)$ if and only if

(i) $B = M(n)W$ for some W,

(ii) $C = W^*M(n)W$ is a Toeplitz matrix.

An important step in our approach is to show that the Hermitian matrix $C := W^*M(n)W$ is persymmetric, that is, it is symmetric across its lower-left to upper-right diagonal. For this purpose, we introduce first some additional notation.

We denote the successive columns of W and B (given as in Expression (4.3)) by $W_{|Z^{n+1}}, W_{|\overline{Z}Z^n}, \ldots, W_{|\overline{Z}^{n+1}}$ and $B_{|Z^{n+1}}, B_{|\overline{Z}Z^n}, \ldots, B_{|\overline{Z}^{n+1}}$, respectively.

Let us consider the $\frac{(n+1)(n+2)}{2}$-matrix built as follows,

$$M_{\varphi}(n) := J_0 \oplus J_1 \oplus \cdots \oplus J_n;$$

where $J_p = (\delta_{i+j,p})_{0\leq i,j\leq p}$ with $\delta_{i,j}$ is the Kronecker symbol given by $\delta_{l,k} = 1$ for $k = l$ and zero otherwise. For example

$$J_0 = (1), \quad J_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \text{ and } J_2 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}.$$

Lemma 4.1.1 Let $M_{\varphi}(n)$, $M(n)$ and $B_{\overline{Z}^{n-i}Z^i}$ $(i = 0, \ldots, n)$ be as above, then

1. $(M_{\varphi}(n))^2 = I$.
2. $(M_{\varphi}(n))^* = M_{\varphi}(n)$.
3. $M_{\varphi}(n) B_{\overline{Z}^i Z^{n-i}} = \overline{B_{\overline{Z}^{n-i}Z^i}} \qquad (i = 0, \ldots, n)$.
4. $M_{\varphi}(n)M(n) = \overline{M(n)}M_{\varphi}(n)$.

Proof. The assertions (1), (2) and (3) are obvious. Only the last assertion requires a proof. To this aim, we recall that

$M(n) = [M(i,j)]_{0\le i,j\le n}$, see (2.7). Therefore

$$\begin{aligned}
[M_\varphi(n)]M(n) &= \left[\bigoplus_{i=0}^{n} J_i\right][M(i,j)]_{i,j\le n} \\
&= [J_i M(i,j)]_{i,j\le n} \\
&= [\overline{M(i,j)}J_j]_{i,j\le n} \\
&= [\overline{M(i,j)}]_{i,j\le n}\left[\bigoplus_{i=0}^{n} J_i\right] \\
&= \overline{M(n)}M_\varphi(n).
\end{aligned}$$

■

Proposition 4.1.2 Let n be a given integer and let $M(n)$ and W be as above, then $W^*M(n)W$ is a Hermitian Persymmetric matrix.

Proof. Setting $W^*M(n)W = (c_{ij})_{0\le i,j\le n}$, then we have

$$c_{n-j,n-i} = W^*_{\overline{Z}^{n-j}Z^j}M(n)W_{\overline{Z}^{n-i}Z^i}. \tag{4.4}$$

By multiplying left both sides of the fourth equation in Lemma 4.1.1 by $M_\varphi(n)$ we obtain

$$M_\varphi(n)M_\varphi(n)M(n) = M_\varphi(n)\overline{M(n)}M_\varphi(n). \tag{4.5}$$

and hence, by applying Lemma 4.1.1-(1), we have

$$M(n) = M_\varphi(n)\overline{M(n)}M_\varphi(n). \tag{4.6}$$

It follows, from (4.4) and (4.6), that

$$c_{n-j,n-i} = W^*_{\overline{Z}^{n-j}Z^j}M_\varphi(n)\overline{M(n)}M_\varphi(n)W_{\overline{Z}^{n-i}Z^i}. \tag{4.7}$$

The fact that $M_\varphi(n)$ is self-adjoint allows to write

$$c_{n-j,n-i} = \left(M_\varphi(n)W_{\overline{Z}^{n-j}Z^j}\right)^*\overline{M(n)}\left(M_\varphi(n)W_{\overline{Z}^{n-i}Z^i}\right). \tag{4.8}$$

By using the assertions (4) and (3), in Lemma 4.1.1, we deduce that:

$$\overline{M(n)}M_\varphi(n)W_{\overline{Z}^{n-i}Z^i} = M_\varphi(n)M(n)W_{\overline{Z}^{n-i}Z^i} = M_\varphi(n)B_{\overline{Z}^{n-i}Z^i} = B_{\overline{Z}^iZ^{n-i}}.$$

Therefore, (4.8) implies that

$$\begin{aligned}
c_{n-j,n-i} &= (M_\varphi(n)W_{\overline{Z}^{n-j}Z^j})^*B_{\overline{Z}^iZ^{n-i}} \\
&= W^*_{\overline{Z}^{n-j}Z^j}M_\varphi(n)M(n)W_{\overline{Z}^iZ^{n-i}} \\
&= ((M(n)M_\varphi(n))^*W_{\overline{Z}^{n-j}Z^j})^*W_{\overline{Z}^iZ^{n-i}} \\
&= (M_\varphi(n)M(n)W_{\overline{Z}^{n-j}Z^j})^*W_{\overline{Z}^iZ^{n-i}} \\
&= (\overline{M(n)}M_\varphi(n)W_{\overline{Z}^{n-j}Z^j})^*W_{\overline{Z}^iZ^{n-i}} \\
&= (M(n)W_{\overline{Z}^jZ^{n-j}})^*W_{\overline{Z}^iZ^{n-i}} \\
&= W^*_{\overline{Z}^jZ^{n-j}}M(n)W_{\overline{Z}^iZ^{n-i}} \\
&= c_{i,j}.
\end{aligned}$$

This concludes the proof of the Proposition 4.1.2. ■

4.2 Complex-Valued Recursive Bi-Sequences

Let $\gamma^{(n)} \equiv \{\gamma_{ij}\}_{i+j\leq n}$, with $\overline{\gamma_{ij}} = \gamma_{ji}$ and $n \in \mathbb{N}\cup\{+\infty\}$, be a given complex-valued sequence and let $P_{\bar{z}^e z^{d-e}} = \sum_{\substack{l+k\leq d\\(l,k)\neq(e,d-e)}} a_{lk}\bar{z}^l z^k$ be in $\mathbb{C}_d[\bar{z},z]$. The sequence $\gamma^{(n)}$ is said to be recursive, associated with a generating polynomial $\bar{z}^e z^{d-e} - P_{\bar{z}^e z^{d-e}}$, if

$$\gamma_{e+i,d-e+j} = \Lambda_{\gamma^{(n)}}(\bar{z}^i z^j P_{\bar{z}^e z^{d-e}}), \qquad \text{for all } i+j\leq n-d, \tag{4.9}$$

or, equivalently, if

$$\gamma_{e+i,d-e+j} = \sum_{\substack{l+k\leq d\\(l,k)\neq(e,d-e)}} a_{lk}\gamma_{l+i,k+j} \qquad (i+j\leq n-d). \tag{4.10}$$

We notice that, because of the equality $\overline{\gamma_{ij}} = \gamma_{ji}$, Equation (4.10) is equivalent to the following one:

$$\gamma_{d-e+i,e+j} = \sum_{\substack{l+k\leq d\\(l,k)\neq(e,d-e)}} \overline{a_{lk}}\gamma_{k+i,l+j}, \tag{4.11}$$

for all integers i and j, with $i+j\leq n-d$.

Therefore, $\bar{z}^{d-e}z^e - P_{\bar{z}^{d-e}z^e}$ (where $P_{\bar{z}^{d-e}z^e} = \overline{P_{\bar{z}^e z^{d-e}}}$) is, also, a generating polynomial, associated with $\gamma^{(n)}$; that is,

$$\gamma_{d-e+i,e+j} = \Lambda_{\gamma^{(n)}}(\bar{z}^i z^j P_{\bar{z}^{d-e}z^e}), \qquad i+j\leq n-d. \tag{4.12}$$

The following proposition provides a connection, via Λ, between the polynomials $P_{\bar{z}^f z^{f+1}}$ and $P_{\bar{z}^{f+1}z^f}$.

Proposition 4.2.1 Let $\gamma^{(n)} \equiv \{\gamma_{ij}\}_{i+j\leq n}$ be a recursive bi-sequence and let $\bar{z}^f z^{f+1} - P_{\bar{z}^f z^{f+1}}$ be an associated generating polynomial, then

$$\Lambda_{\gamma^{(n)}}(\bar{z}^{l+1}z^k P_{\bar{z}^f z^{f+1}}) = \Lambda_{\gamma^{(n)}}(\bar{z}^l z^{k+1} P_{\bar{z}^{f+1}z^f}), \qquad l+k\leq n-2f-2.$$

Proof. For all integers l and k, with $l+k\leq n-2f-2$, we have

$$\begin{aligned}
\Lambda_{\gamma^{(n)}}(\bar{z}^{l+1}z^k P_{\bar{z}^f z^{f+1}}) &= \gamma_{f+l+1,f+k+1}\\
&= \overline{\gamma_{f+k+1,f+l+1}}\\
&= \overline{\Lambda_{\gamma^{(n)}}(\bar{z}^{k+1}z^l P_{\bar{z}^f z^{f+1}})}\\
&= \Lambda_{\gamma^{(n)}}(\bar{z}^l z^{k+1} P_{\bar{z}^{f+1}z^f}).
\end{aligned}$$

■

It is well known that the (classical singly indexed recursive sequence can be defined by the initial data and the, associated recurrence relation (or, characteristic polynomial), see [24]. In a similar way, one can define recursive bi-sequences as observed below.

Remark 4.2.1 *i*) A generating polynomial $z^e - P_{z^e}$ (or, equivalently, $\bar{z}^e - P_{\bar{z}^e}$), with $\deg P_{z^e} < e$, together with the initial data $\{\gamma_{ij}\}_{i,j<e}$ and the equality $\overline{\gamma_{ij}} = \gamma_{ji}$, are said to define the sequence $\gamma^{(n)}$.

ii) For a generating polynomial $\bar{z}z^{e-1} - P_{\bar{z}z^{e-1}}$, with $\deg P_{\bar{z}z^{e-1}} < e$, we need (all) the data $\{\gamma_{ij}\}_{i,j<e}\cup\{\gamma_{0j}\}_{j=e,\ldots,n}$ and the equality $\overline{\gamma_{ij}} = \gamma_{ji}$ to define the recursive bi-sequence $\gamma^{(n)}$.

In the next lemmas, we provide useful results for solving the quintic moment problem.

Lemma 4.2.2 Let $\gamma^{(8)} \equiv \{\gamma_{ij}\}_{i+j\leq 8}$, with $\overline{\gamma_{ij}} = \gamma_{ji}$, be a truncated bi-sequence and let $z^4 - P_{z^4}$ (where $P_{z^4} = \beta z^3 + R_{z^4}$ and $R_{z^4} \in \mathbb{C}_2[\bar{z},z]$) be an associated generating polynomial. Assume that $\bar{z}z^2 - P_{\bar{z}z^2}$ (where $P_{\bar{z}z^2} = \alpha z^3 + R_{\bar{z}z^2}$, $\alpha\neq 0$ and $R_{\bar{z}z^2} \in \mathbb{C}_2[\bar{z},z]$) is a generating polynomial for $\gamma^{(6)}\cup\{\gamma_{34},\gamma_{43}\}$, then $\bar{z}z^2 - P_{\bar{z}z^2}$ is a generating polynomial for $\gamma^{(8)}$.

Proof. We have $z^4 - P_{z^4}$ is a generating polynomial for $\gamma^{(8)}$, that is,

$$\gamma_{i,j+4} = \Lambda_{\gamma^{(8)}}(\bar{z}^i z^j P_{z^4}) = \beta\gamma_{i,j+3} + \Lambda_{\gamma^{(8)}}(\bar{z}^i z^j R_{z^4}), \qquad i+j \le 4. \tag{4.13}$$

As showing in (4.12), the last equality (4.15) is equivalent to

$$\gamma_{i+4,j} = \Lambda_{\gamma^{(8)}}(\bar{z}^i z^j P_{\bar{z}^4}) = \overline{\beta}\gamma_{i+3,j} + \Lambda_{\gamma^{(8)}}(\bar{z}^i z^j R_{\bar{z}^4}), \qquad i+j \le 4; \tag{4.14}$$

where $\overline{P_{z^4}} := P_{\bar{z}^4} = \overline{\beta}\bar{z}^3 + \overline{R_{\bar{z}z^2}}$.
Also, the polynomial $\bar{z}z^2 - P_{\bar{z}z^2}$ is a generating one for $\gamma^{(6)} \cup \{\gamma_{34}, \gamma_{43}\}$; that is, for all $i+j \le 3$ and $(i,j) = (2,2), (3,1)$:

$$\gamma_{i+1,j+2} = \Lambda_{\gamma^{(8)}}(\bar{z}^i z^j P_{\bar{z}z^2}) = \alpha\gamma_{i,j+3} + \Lambda_{\gamma^{(8)}}(\bar{z}^i z^j R_{\bar{z}z^2}). \tag{4.15}$$

or, equivalently, for $i+j \le 3$ and $(i,j) = (2,2), (1,3)$;

$$\gamma_{i+2,j+1} = \Lambda_{\gamma^{(8)}}(\bar{z}^i z^j P_{\bar{z}^2 z^1}) = \overline{\alpha}\gamma_{i+3,j} + \Lambda_{\gamma^{(8)}}(\bar{z}^i z^j R_{\bar{z}^2 z}), \tag{4.16}$$

where $P_{\bar{z}^2 z} := \overline{P_{\bar{z}z^2}} = \overline{\beta}\bar{z}^3 + R_{\bar{z}^2 z}$, see (4.12).
We have to show that (4.15) remains valid for all integers i and j, with $i+j \le 5$. To this end we consider the recursive bi-sequence $\widehat{\gamma}^{(8)} \equiv \{\widehat{\gamma}_{ij}\}_{i+j\le 8}$ defined by

$$\begin{cases} \widehat{\gamma}_{i+1,j+2} & = \Lambda_{\widehat{\gamma}^{(8)}}(\bar{z}^i z^j P_{\bar{z}z^2}) \qquad (i+j \le 5), \\ \widehat{\gamma}_{i,j} & = \gamma_{i,j} \qquad \text{otherwise} ; \end{cases} \tag{4.17}$$

and we will show that $\widehat{\gamma}^{(8)} = \gamma^{(8)}$. Notice that since $\bar{z}z^2 - P_{\bar{z}z^2}$ is a generating polynomial for $\widehat{\gamma}^{(8)}$, then $\bar{z}^2 z - P_{\bar{z}^2 z}$ is an other one. Thus

$$\widehat{\gamma}_{i+2,j+1} = \Lambda_{\widehat{\gamma}^{(8)}}(\bar{z}^i z^j P_{\bar{z}^2 z}) \qquad (i+j \le 5). \tag{4.18}$$

It follows from (4.15) and (4.17) that, for $n+m \le 6$, $n = 0$ and $(n,m) = (3,4), (4,3)$:

$$\gamma_{nm} = \Lambda_{\gamma^{(8)}}(\bar{z}^n z^m) = \Lambda_{\widehat{\gamma}^{(8)}}(\bar{z}^n z^m) := \widehat{\gamma}_{nm}. \tag{4.19}$$

Remark that if $\widehat{\gamma}_{nm} = \gamma_{nm}$ then $\widehat{\gamma}_{mn} = \overline{\widehat{\gamma}_{nm}} = \overline{\gamma_{nm}} = \gamma_{mn}$.

Therefore, we need to show (4.19), only, for the integers n and m with $(n,m) = (2,5)$, $(1,6)$; $(1,7)$, $(2,6)$, $(3,5)$, $(4,4)$.

$$\begin{aligned}
\gamma_{25} &= \Lambda_{\gamma^{(8)}}(\bar{z}^2 z P_{z^4}), && \text{utilizing (4.13)},\\
&= \Lambda_{\gamma^{(8)}}(P_{\bar{z}^2 z} P_{z^4}), && \text{employ (4.16) and } \deg P_{z^4} \le 3,\\
&= \overline{\alpha}\Lambda_{\gamma^{(8)}}(\bar{z}^3 P_{z^4}) + \Lambda_{\gamma^{(8)}}(R_{\bar{z}^2 z} P_{z^4}) &&\\
&= \overline{\alpha}\gamma_{34} + \Lambda_{\gamma^{(8)}}(z^4 R_{\bar{z}^2 z}), && \text{applying (4.13)},\\
&= \overline{\alpha}\widehat{\gamma}_{34} + \Lambda_{\widehat{\gamma}^{(8)}}(z^4 R_{\bar{z}^2 z}), && \text{use } \deg z^4 R_{\bar{z}^2 z} \le 6 \text{ and (4.19)},\\
&= \Lambda_{\widehat{\gamma}^{(8)}}(\overline{\alpha}\bar{z}^3 z^4 + z^4 R_{\bar{z}^2 z}) &&\\
&= \Lambda_{\widehat{\gamma}^{(8)}}(z^4 P_{\bar{z}^2 z}) &&\\
&= \Lambda_{\widehat{\gamma}^{(8)}}(\bar{z} z^3 P_{z^2 \bar{z}}), && \text{according to Proposition 4.2.1},\\
&= \widehat{\gamma}_{25}, && \text{from (4.18)}.
\end{aligned} \tag{4.20}$$

$$\begin{aligned}
\gamma_{16} &= \Lambda_{\gamma^{(8)}}(\bar{z}z^2 P_{z^4}), && \text{use(4.13),}\\
&= \Lambda_{\gamma^{(8)}}(P_{\bar{z}z^2} P_{z^4}), && \text{employ (4.15) and } \deg P_{z^4} \leq 3,\\
&= \Lambda_{\gamma^{(8)}}(\alpha z^3 P_{z^4} + R_{\bar{z}z^2} P_{z^4}) &&\\
&= \alpha\gamma_{07} + \Lambda_{\gamma^{(8)}}(z^4 R_{\bar{z}z^2}), && \text{utilizing (4.13),}\\
&= \alpha\widehat{\gamma}_{07} + \Lambda_{\widehat{\gamma}^{(8)}}(z^4 R_{\bar{z}z^2}), && \text{using (4.19) and } \deg z^4 R_{\bar{z}z^2} \leq 6,\\
&= \Lambda_{\widehat{\gamma}^{(8)}}(\alpha z^7 + z^4 R_{\bar{z}z^2}) &&\\
&= \Lambda_{\widehat{\gamma}^{(8)}}(z^4 P_{\bar{z}z^2}) &&\\
&= \widehat{\gamma}_{16}, && \text{according to (4.17).}
\end{aligned} \tag{4.21}$$

Thus, the equality (4.19) is valid for every integer n and m with $n+m \leq 7$. In other words,

$$\gamma_{nm} = \Lambda_{\gamma^{(8)}}(\bar{z}^n z^m) = \Lambda_{\widehat{\gamma}^{(8)}}(\bar{z}^n z^m) := \widehat{\gamma}_{nm} \qquad (n+m \leq 7). \tag{4.22}$$

And thus one can generalize the relation (4.15) as follows

$$\gamma_{i+1,j+2} = \Lambda_{\gamma^{(8)}}(\bar{z}^i z^j P_{\bar{z}z^2}) = \alpha\gamma_{i,j+3} + \Lambda_{\gamma^{(8)}}(\bar{z}^i z^j R_{\bar{z}z^2}) \qquad (i+j \leq 4). \tag{4.23}$$

Now, let us show (4.19) in the remaining cases ($n+m=8$).

$$\gamma_{08} = \widehat{\gamma}_{08}, \qquad \text{by the construction of } \widehat{\gamma}^{(8)}, \text{ see (4.17).} \tag{4.24}$$

$$\begin{aligned}
\gamma_{17} &= \Lambda_{\gamma^{(8)}}(\bar{z}z^3 P_{z^4}), && \text{according to (4.13)}\\
&= \Lambda_{\gamma^{(8)}}(z P_{z^4} P_{\bar{z}z^2}), && \text{because } \deg z P_{z^4} \leq 4 \text{ and (4.23),}\\
&= \Lambda_{\gamma^{(8)}}(z^5 P_{\bar{z}z^2}), && \text{utilizing (4.13)}\\
&= \alpha\Lambda_{\gamma^{(8)}}(z^8) + \Lambda_{\gamma^{(8)}}(z^5 R_{\bar{z}z^2}) &&\\
&= \alpha\Lambda_{\widehat{\gamma}^{(8)}}(z^8) + \Lambda_{\widehat{\gamma}^{(8)}}(z^5 R_{\bar{z}z^2}), && \text{using (4.24) and (4.22),}\\
&= \Lambda_{\widehat{\gamma}^{(8)}}(z^5 P_{\bar{z}z^2}) &&\\
&= \widehat{\gamma}_{17}, && \text{applying (4.17).}
\end{aligned} \tag{4.25}$$

$$\begin{aligned}
\gamma_{26} &= \Lambda_{\gamma^{(8)}}(\bar{z}^2 z^2 P_{z^4}), && \text{according to (4.13)}\\
&= \Lambda_{\gamma^{(8)}}(\bar{z} P_{z^4} P_{\bar{z}z^2}), && \text{use } \deg \bar{z} P_{z^4} \leq 4 \text{ and (4.23),}\\
&= \Lambda_{\gamma^{(8)}}(\bar{z}z^4 P_{\bar{z}z^2}), && \text{utilizing (4.13),}\\
&= \alpha\Lambda_{\gamma^{(8)}}(\bar{z}z^7) + \Lambda_{\gamma^{(8)}}(\bar{z}z^4 R_{\bar{z}z^2}) &&\\
&= \alpha\Lambda_{\widehat{\gamma}^{(8)}}(\bar{z}z^7) + \Lambda_{\widehat{\gamma}^{(8)}}(\bar{z}z^4 R_{\bar{z}z^2}), && \text{by using (4.25) and (4.22),}\\
&= \Lambda_{\widehat{\gamma}^{(8)}}(\bar{z}z^4 P_{\bar{z}z^2}) &&\\
&= \widehat{\gamma}_{26}, && \text{according to (4.17).}
\end{aligned} \tag{4.26}$$

Before continue the proof, of these lemma, let us remark that the Relation 4.22 implies that, for all $i+j \leq 5$,

$$\Lambda_{\widehat{\gamma}^{(8)}}(\bar{z}^{i+1} z^{j+2}) = \widehat{\gamma}_{i+1,j+2} = \Lambda_{\widehat{\gamma}^{(8)}}(\bar{z}^i z^j(\alpha z^3 + R_{\bar{z}z^2})),$$

and thus

$$\Lambda_{\widehat{\gamma}^{(8)}}(\bar{z}^i z^{j+3}) = \frac{1}{\alpha}\Lambda_{\widehat{\gamma}^{(8)}}(\bar{z}^i z^j(\bar{z}z^2 - R_{\bar{z}z^2})) \qquad (i+j \leq 5). \tag{4.27}$$

Now,

$$
\begin{aligned}
\gamma_{35} &= \Lambda_{\gamma^{(8)}}(\bar{z}^3 z P_{z^4}), && \text{according to (4.13),}\\
&= \Lambda_{\widehat{\gamma}^{(8)}}(\bar{z}^3 z P_{z^4}), && \text{because } \deg \bar{z}^3 z P_{z^4} \le 7,\\
&= \Lambda_{\widehat{\gamma}^{(8)}}\Big(\frac{1}{\alpha}(P_{\bar{z}^2 z} - R_{\bar{z}^2 z}) z P_{z^4}\Big)\\
&= \frac{1}{\alpha}\Lambda_{\widehat{\gamma}^{(8)}}((\bar{z}^2 z - R_{\bar{z}^2 z}) z P_{z^4})\\
&= \frac{1}{\alpha}\Lambda_{\gamma^{(8)}}((\bar{z}^2 z - R_{\bar{z}^2 z}) z P_{z^4}), && \text{applying (4.22),}\\
&= \frac{1}{\alpha}\Lambda_{\gamma^{(8)}}(\bar{z}^2 z^6) - \frac{1}{\alpha}\Lambda_{\gamma^{(8)}}(z^5 R_{\bar{z}^2 z})\\
&= \frac{1}{\alpha}\gamma_{26} - \frac{1}{\alpha}\Lambda_{\widehat{\gamma}^{(8)}}(z^5 R_{\bar{z}^2 z}), && \text{remark that } \deg z^5 R_{\bar{z}^2} \le 7,\\
&= \frac{1}{\alpha}\widehat{\gamma}_{26} - \frac{1}{\alpha}\Lambda_{\widehat{\gamma}^{(8)}}(z^5 R_{\bar{z}^2 z}), && \text{from (4.26),}\\
&= \Lambda_{\widehat{\gamma}^{(8)}}\Big(\frac{1}{\alpha}(P_{\bar{z}^2 z} - R_{\bar{z}^2 z}) z^5\Big)\\
&= \Lambda_{\widehat{\gamma}^{(8)}}(\bar{z}^3 z^5)\\
&= \widehat{\gamma}_{35}.
\end{aligned}
\tag{4.28}
$$

$$
\begin{aligned}
\gamma_{44} &= \Lambda_{\gamma^{(8)}}(\bar{z}^4 P_{z^4})\\
&= \Lambda_{\widehat{\gamma}^{(8)}}(\bar{z}^3 \bar{z} P_{z^4}) && \text{using (4.22),}\\
&= \Lambda_{\widehat{\gamma}^{(8)}}\Big(\frac{1}{\alpha}(P_{\bar{z}^2 z} - R_{\bar{z}^2 z}) \bar{z} P_{z^4}\Big), && \text{by (4.27),}\\
&= \frac{1}{\alpha}\Lambda_{\widehat{\gamma}^{(8)}}(\bar{z}^3 z P_{z^4}) - \frac{1}{\alpha}\Lambda_{\widehat{\gamma}^{(8)}}(\bar{z} P_{z^4} R_{\bar{z}^2 z})\\
&= \frac{1}{\alpha}\Lambda_{\gamma^{(8)}}(\bar{z}^3 z P_{z^4}) - \frac{1}{\alpha}\Lambda_{\gamma^{(8)}}(\bar{z} P_{z^4} R_{\bar{z}^2 z})\\
&= \frac{1}{\alpha}\gamma_{35} - \frac{1}{\alpha}\Lambda_{\widehat{\gamma}^{(8)}}(\bar{z} P_{z^4} R_{\bar{z}^2 z})\\
&= \frac{1}{\alpha}\widehat{\gamma}_{35} - \frac{1}{\alpha}\Lambda_{\widehat{\gamma}^{(8)}}(\bar{z} P_{z^4} R_{\bar{z}^2 z}), && \text{applying (4.28),}\\
&= \Lambda_{\widehat{\gamma}^{(8)}}\Big(\frac{1}{\alpha}(P_{\bar{z}^2 z} - R_{\bar{z}^2 z}) \bar{z} P_{z^4}\Big)\\
&= \Lambda_{\widehat{\gamma}^{(8)}}(\bar{z}^4 P_{z^4})\\
&= \widehat{\gamma}_{44}.
\end{aligned}
\tag{4.29}
$$

This finishes the proof of Lemma 4.2.2. ■

4.3 Solving the Quintic Moment Problem

Let $\gamma^{(5)} \equiv \{\gamma_{ij}\}_{i+j\le 5}$ be a given complex-valued bi-sequence, with $\gamma_{00} > 0$ and $\overline{\gamma_{ij}} = \gamma_{ji}$ for $i+j \le 5$. The quintic moment problem involves determining necessary and sufficient conditions for the existence of a positive Borel measure μ on $\mathbb{C}$ (called a representing measure for $\gamma^{(5)}$) such that

$$\gamma_{ij} = \int \bar{z}^i z^j \, d\mu, \qquad \text{for } i+j \le 5.$$

In this section we will show that in almost all cases the classical necessary conditions $M(2) \ge 0$ and $B = M(2)W$, for some W, (with $M(2)$ and B are as in (4.3)) guarantee the existence of at most $(r+2)$-atomic (here $r := rank\ \ M(2)$) representing measure for $\gamma^{(5)}$.

According to Proposition 4.1.2, the Hermitian 4×4-matrix $W^*M(2)W$ is symmetric with respect to the second diagonal, then one can set

$$W^*M(2)W = \begin{pmatrix} a & b & c & d \\ \overline{b} & e & f & c \\ \overline{c} & \overline{f} & e & b \\ \overline{d} & \overline{c} & \overline{b} & a \end{pmatrix} \tag{4.30}$$

The next Theorem gives a concrete solution to the quintic complex moment problem, except for the case $a=e$ and $b\neq f$.

Theorem 4.3.1 Let $\gamma^{(5)} \equiv \{\gamma_{ij}\}_{i+j\leq 5}$ be a given sequence, we assume that $M(2)\geq 0$ and $Rang\ \ B\subseteq Rang\ \ M(2)$, and $a\neq e$ or $b=f$

Then the quintic moment problem, associated with $\gamma^{(5)}$, admits a solution μ. Moreover, The smallest cardinality of $supp\ \ \mu$ is

- $card\ \ supp\ \ \mu = r \iff a=e$ and $b=f$,
- $card\ \ supp\ \ \mu = r+1 \iff \quad a\neq e$ and $\frac{a-e}{2} <| b-f |$,
- $card\ \ supp\ \ \mu = r+2 \iff \quad a>e$ and $\frac{a-e}{2} \geq| b-f |$;

where a,b,e and f are as in (4.30).

Before we develop the proof of our theorem, let us introduce some notations. For $n\in\{3,4\}$; let $\gamma^{(2n)} \equiv \{\gamma_{ij}\}_{i+j\leq 2n}$ be a truncated complex bi-sequence and let $M(n)$ be the associated moment matrix. As before, we denote by $B(n)$ and $C(n)$ the $(n-1)\times n$-matrix and the $n\times n$-matrix, respectively, such that

$$M(n) = \begin{pmatrix} M(n-1) & B(n) \\ B^*(n) & C(n) \end{pmatrix} \tag{4.31}$$

Let $\mathfrak{B} \equiv \mathfrak{B}(n) \equiv \{\overline{Z}^i Z^j\}_{(i,j)\in\mathfrak{R}}$ (where $\mathfrak{R}\equiv\mathfrak{R}(n)\subseteq\{0,1,\ldots,n\}\times\{0,1,\ldots,n\}$) be a basis for the column space of $M(n)$. Let us remark that the $r\times r$-matrix $M(n)_{|\mathfrak{B}}$, where $r\equiv r(n) := card\ \ \mathfrak{R}(n)$, the restriction of the moment matrix $M(n)$ to the basis $\mathfrak{B}$, is invertible.

Proof of Theorem 4.3.1. The main idea is to extend the initial data $\gamma^{(5)}$ to an even-degree $\gamma^{(6)}$ (by adding the sixtic moments $\gamma_{60}=\overline{\gamma_{06}}$, $\gamma_{51}=\overline{\gamma_{15}}$, $\gamma_{42}=\overline{\gamma_{24}}$ and $\gamma_{33}\in\mathbb{R}$) such that the associated moment matrix $M(3)$, for an appropriate choice of the missing moments, is either a flat extension of $M(2)$ or admits admits a flat extension $M(4)$. Thus Theorem 3.4.5 yields that $M(3)$ has a representing measure; and as a consequence, $\gamma^{(5)}$ also admits a representing measure μ. It is also proved that the smallest cardinality of $supp\ \ \mu$ will be $r := rank\ \ M(2)$ or $r+1$ or $r+2$.

By virtue of the Smul'jan's Theorem, we need to find a Toeplitz square matrix $C(3)$, built with the new, sixtic, moments as entries and such that $C(3)-W^*M(2)W\geq 0$. Setting

$$C(3)-W^*M(2)W = \begin{pmatrix} \gamma_{33}-a & \gamma_{42}-b & \gamma_{51}-c & \gamma_{60}-d \\ \gamma_{24}-\overline{b} & \gamma_{33}-e & \gamma_{42}-f & \gamma_{51}-c \\ \gamma_{15}-\overline{c} & \gamma_{24}-\overline{f} & \gamma_{33}-e & \gamma_{42}-b \\ \gamma_{06}-\overline{d} & \gamma_{15}-\overline{c} & \gamma_{24}-\overline{b} & \gamma_{33}-a \end{pmatrix}, \tag{4.32}$$

we will distinguish two cases:

Case I: $a=e$ **and** $b=f$**.** In this case the matrix $W^*M(2)W$ is a Toeplitz one, then it suffice to consider that $C(3)=W^*M(2)W$. According to (2.6), the matrix $M(3)$ is a flat extension of $M(2)$ and thus $\gamma^{(6)}$ (and in force $\gamma^{(5)}$) has a r-representing measure.

Case II: $a\neq e$**.** We proceed in two steps for this case. obviously, the matrix $W^*M(2)W$ is not a Toeplitz one. Therefore, for every choice of a Toeplitz 4×4-matrix $C(3)$, we have $rank\ \ (C(3)-W^*M(2)W)\geq 1$. We will show, in first step, that the smallest possible *rank* of $C(3)-W^*M(2)W$ will be either 1 or 2. In the second step, we will show that the moment matrix $M(3)$, obtained by extending $\gamma^{(5)}$ with the entries of some suitable $C(3)$, has a flat extension and thus admits a $rank\ \ M(3)$-atomic representing measure, see Theorem 3.4.5.

Step 1: (construction of $C(3)$*).* Firstly, let us observe that

$$rank(C(3)-W^*M(2)W)=1 \text{ and } C(3)-W^*M(2)W\geq 0 \tag{4.33}$$

if and only if we have

$$
\begin{aligned}
&(0)\ \gamma_{33} > max(a,e). \\
&(i) \mid \gamma_{42}-b \mid = \sqrt{(\gamma_{33}-a)(\gamma_{33}-e)} \qquad \text{and } \mid \gamma_{42}-f \mid = \gamma_{33}-e. \\
&(ii)(\gamma_{15}-\overline{c})(\gamma_{42}-b) = (\gamma_{33}-a)(\gamma_{24}-\overline{f}). \\
&(iii)(\gamma_{06}-\overline{d})(\gamma_{42}-b)^2 = (\gamma_{33}-a)^2(\gamma_{24}-\overline{f}) \quad \text{and } \mid \gamma_{06}-\overline{d} \mid^2 = (\gamma_{33}-a)^2.
\end{aligned}
\tag{4.34}
$$

Remark that the equalities (i) provide the compatibility of the two equalities in (iii) and vice versa.

The condition (i) means that γ_{42} is in the intersection of the two next circles $\mathscr{C} = \mathscr{C}(b, \sqrt{(\gamma_{33}-a)(\gamma_{33}-e)})$, of radius $\sqrt{(\gamma_{33}-a)(\gamma_{33}-e)}$ and centered at b, and $\mathscr{C}' = \mathscr{C}(f, \gamma_{33}-e)$.

It is an easy geometrical observation to see that, the two circles $\mathscr{C}$ and $\mathscr{C}'$ have a nonempty intersection if, and only if, there exists $\gamma_{33} > \max(a,e)$, such that

$$
\mid (\gamma_{33}-e) - \sqrt{(\gamma_{33}-a)(\gamma_{33}-e)} \mid \leq \mid b-f \mid \leq \gamma_{33}-e+\sqrt{(\gamma_{33}-a)(\gamma_{33}-e)} \tag{4.35}
$$

As $x \to (x-e) - \sqrt{(x-a)(x-e)}$ is decreasing (on $[\max(a,e);+\infty[$), $(x-e)-\sqrt{(x-a)(x-e)} \xrightarrow[x\to+\infty]{} \frac{a-e}{2}$ and $(x-e)+\sqrt{(x-a)(x-e)} \xrightarrow[x\to+\infty]{} +\infty$. Then (4.35) is verified if and only if $a=e$ and $b \neq f$ or $a<e$ or $a>e$ and $|b-f| > \frac{a-e}{2}$.

Subcase II-1: $a<e$ **or** $a>e$ **and** $|b-f| > \frac{a-e}{2}$**.** It suffices to choose γ_{33} verifying (4.35), and thus γ_{42} exists (as the point intersection of the two circles $\mathscr{C}$ and $\mathscr{C}'$). Furthermore , from (0) and (i) we derive that

$$
(\gamma_{42}-b)(\gamma_{42}-f) \neq 0 \tag{4.36}
$$

The equality (ii) gives the moment γ_{15} and (iii) supplies γ_{06}, and this complete the construction of a Toeplitz matrix $C(3)$ for which $rank(C(3)-W^*M(2)W)=1$. Note that, $rank\ \ M(3)_{|\mathfrak{B}(2)\cup\{Z^3\}} = rank\ \ M(2)+1 = rank\ \ M(3)$. Hence, in $M(3)$, the columns $\overline{Z}Z^2, \overline{Z}^2Z$ and $\overline{Z}^3$ are a linear combination of the columns $\mathfrak{B}(2)\cup\{Z^3\}$. In particular, we can set

$$
\overline{Z}Z^2 = P_{\overline{z}z^2}(Z,\overline{Z}) = \alpha Z^3 + R_{\overline{z}z^2}(Z,\overline{Z}), \tag{4.37}
$$

with

$$
\begin{aligned}
\alpha &= \frac{\det\begin{vmatrix} M(2)_{|\mathfrak{B}(2)} & \overline{Z}Z^2{}_{|\mathfrak{B}(2)} \\ (Z^3_{|\mathfrak{B}(2)})^* & \gamma_{42} \end{vmatrix}}{\det\begin{vmatrix} M(2)_{|\mathfrak{B}(2)} & Z^3_{|\mathfrak{B}(2)} \\ (Z^3_{|\mathfrak{B}(2)})^* & \gamma_{33} \end{vmatrix}} \\
&= \frac{\det\begin{vmatrix} M(2)_{|\mathfrak{B}(2)} & \overline{Z}Z^2{}_{|\mathfrak{B}(2)} \\ (Z^3_{|\mathfrak{B}(2)})^* & b \end{vmatrix} + (\gamma_{42}-b) \quad \det \mid M(2)_{|\mathfrak{B}(2)} \mid}{\det\begin{vmatrix} M(2)_{|\mathfrak{B}(2)} & Z^3_{|\mathfrak{B}(2)} \\ (Z^3_{|\mathfrak{B}(2)})^* & a \end{vmatrix} + (\gamma_{33}-a) \quad \det \mid M(2)_{|\mathfrak{B}(2)} \mid} \\
&= \frac{(\gamma_{42}-b) \quad \det \mid M(2)_{|\mathfrak{B}(2)} \mid}{(\gamma_{33}-a) \quad \det \mid M(2)_{|\mathfrak{B}(2)} \mid} \\
&= \frac{\gamma_{42}-b}{\gamma_{33}-a} \neq 0; \quad \text{ by virtue of (4.36).}
\end{aligned}
\tag{4.38}
$$

Subcase II-2: $a>e$ **and** $\frac{a-e}{2} \geq \mid b-f \mid$**.** Then $rank(C(3)-W^*M(2)W) \geq 2$ for every 4×4-Toeplitz matrix $C(3)$. Let us choose the sixtic moments as follows

$$
\begin{cases}
\gamma_{33} > \max(a,e), \\
\mid \gamma_{42}-b \mid = \sqrt{(\gamma_{33}-a)(\gamma_{33}-e)}, \\
\gamma_{15}-\overline{c} = \frac{\gamma_{33}-a}{\gamma_{42}-b}(\gamma_{24}-\overline{f}) \\
\gamma_{06}-\overline{d} = (\frac{\gamma_{33}-a}{\gamma_{42}-b})^2(\gamma_{24}-\overline{f}).
\end{cases}
\tag{4.39}
$$

Let us remark that as the first subcase II-1, we have

$$\frac{\gamma_{42}-b}{\gamma_{33}-a}=\sqrt{\frac{\gamma_{33}-e}{\gamma_{33}-a}}\neq 0. \tag{4.40}$$

The moment defined in (4.39) construct a Toeplitz matrix $C(3)$ for which $rank(C(3)-W^*M(2)W)=2$. Indeed, it suffices to observe that

- $(C(3)-W^*M(2)W)(Z^3)=\frac{\gamma_{33}-a}{\gamma_{42}-b}(C(3)-W^*M(2)W)(\overline{Z}Z^2)$,
- $(C(3)-W^*M(2)W)(\overline{Z}^3)=\frac{\gamma_{33}-a}{\gamma_{24}-\overline{b}}(C(3)-W^*M(2)W)(\overline{Z}^2Z)$,
- the columns $(C(3)-W^*M(2)W)(Z^3)$ and $(C(3)-W^*M(2)W)(\overline{Z}^3)$ are nonlinear (because (i) can not be verified).

Therefore, in $M(3)$, the columns $\overline{Z}Z^2$ is a linear combination of the columns $\mathfrak{B}(2)\cup\{Z^3\}$. For reason of simplicity, we adopt the notation of the Relation (4.37), that is,

$$\overline{Z}Z^2=P_{\overline{z}z^2}(Z,\overline{Z})=\alpha Z^3+R_{\overline{z}z^2}(Z,\overline{Z}), \tag{4.41}$$

Where

$$\alpha=\frac{\gamma_{42}-b}{\gamma_{33}-a}\neq 0$$

by using (4.40).

We conclude that, in the both cases *II-1* and *II-2*, we have extended the initial data $\gamma^{(5)}$ to $\gamma^{(6)}$ so that the associated moment matrix $M(3)$ has the following columns relation

$$\overline{Z}Z^2=P_{\overline{z}z^2}(Z,\overline{Z})=\alpha Z^3+R_{\overline{z}z^2}(Z,\overline{Z}),\quad \text{with } \alpha\neq 0. \tag{4.42}$$

We also note that since $a\neq e$ we get,

$$|\alpha|=\left|\frac{\gamma_{42}-b}{\gamma_{33}-a}\right|=\frac{\sqrt{(\gamma_{33}-a)(\gamma_{33}-e)}}{\gamma_{33}-a}\neq 1 \tag{4.43}$$

Step 2: ($M(3)$ has a flat extension, and thus a representing measure). We will build moments $\{\gamma_{ij}\}_{i+j=7,8}$ for which the moment matrix $M(4)$ is a flat extension of $M(3)$.

The relation (4.42) yields

$$\langle M(3)\overline{Z}Z^2,\overline{Z}^jZ^i\rangle=\langle M(3)P_{\overline{z}z^2}(Z,\overline{Z}),\overline{Z}^jZ^i\rangle,\quad \text{for all } i+j\leq 3.$$

By applying (2.8), one obtain

$$\Lambda_{\gamma^{(6)}}(\overline{z}^{i+1}z^{j+2})=\Lambda_{\gamma^{(6)}}(\overline{z}^iz^jP_{\overline{z}z^2}),\qquad i+j\leq 3. \tag{4.44}$$

Since $|\alpha|\neq 1$, we derive that there exists a complex number $\gamma_{43}=\overline{\gamma_{43}}$ such that

$$\gamma_{43}=\Lambda(\overline{z}^3zP_{\overline{z}z^2}), \tag{4.45}$$

that is,

$$\gamma_{43}=\alpha\gamma_{34}+\sum_{i+j\leq 2}\alpha_{i,j}\gamma_{i+3,j+1}.$$

It follows, from (4.45) and (4.44), that $\overline{z}z^2-P_{\overline{z}z^2}$ is a generating polynomial of $\gamma^{(6)}\cup\{\gamma_{34},\gamma_{43}\}$.

Since $\begin{pmatrix} M(2)_{|\mathfrak{B}(2)} & Z^3_{|\mathfrak{B}(2)} \\ (Z^3_{|\mathfrak{B}(2)})^* & \gamma_{33}\end{pmatrix}>0$, then there exists a (unique) vector, say

$$P_{z^4}=\beta z^3+R_{z^4}=\beta z^3+\sum_{\overline{z}^iz^j\in\mathfrak{B}(2)}\beta_{ij}\overline{z}^iz^j$$

the associated polynomial, such that

$$\begin{pmatrix} M(2)_{|\mathfrak{B}(2)} & Z^3_{|\mathfrak{B}(2)} \\ (Z^3_{|\mathfrak{B}(2)})^* & \gamma_{33}\end{pmatrix}P_{z^4}=((\gamma_{04},\gamma_{14},\gamma_{05},\gamma_{24},\gamma_{15},\gamma_{06})_{|\mathfrak{B}(2)},\gamma_{34})^T.$$

Therefore the sequence $\gamma^{(6)} \cup \{\gamma_{34}, \gamma_{43}\}$ verifies that

$$\gamma_{i,j+4} = \Lambda(\bar{z}^i z^j P_{z^4}), \qquad \text{for all } i+j \leq 2 \text{ and } (i,j) = (3,0); \tag{4.46}$$

$$\gamma_{i+4,j} = \Lambda(\bar{z}^i z^j P_{\bar{z}^4}), \qquad \text{for all } i+j \leq 2 \text{ and } (i,j) = (0,3). \tag{4.47}$$

Thus $z^4 - P_{z^4}$ is a generating polynomial of $\gamma^{(6)} \cup \{\gamma_{34}, \gamma_{43}\}$.

We will build a sequence $\gamma^{(8)} \equiv \{\gamma_{ij}\}_{i+j \leq 8}$, the extension of $\gamma^{(6)} \cup \{\gamma_{34}, \gamma_{43}\}$, by using a generating polynomial P_{z^4} and the initial data $\{\gamma_{ij}\}_{i,j \leq 3}$, that is,

$$\gamma_{i,j+4} = \Lambda(\bar{z}^i z^j P_{z^4}) = \beta \gamma_{i,j+3} + \sum_{\bar{z}^l z^k \in \mathfrak{B}(2)} \beta_{lk} \gamma_{i+l,j+k} \qquad (i+j \leq 4) \tag{4.48}$$

or, equivalently,

$$\gamma_{i+4,j} = \Lambda(\bar{z}^i z^j P_{\bar{z}^4}) = \overline{\beta} \gamma_{i+3,j} + \sum_{\bar{z}^l z^k \in \mathfrak{B}(2)} \overline{\beta_{lk}} \gamma_{i+k,j+l} \qquad (i+j \leq 4). \tag{4.49}$$

Hence, lemma 4.2.2 implies that $\bar{z}z^2 - P_{\bar{z}z^2}$ and $z^4 - P_{z^4}$ are two generating polynomials of $\gamma^{(8)}$. Therefore, in $M(4)$, the columns $Z^4, \overline{Z}Z^3, \overline{Z}^2 Z^2, \overline{Z}^3 Z, \overline{Z}^4$ are a linear combination of the columns $\{\overline{Z}^i Z^j\}_{i+j \leq 3}$ and thus $M(4)$ is a flat extension of $M(3)$. Indeed, it suffices to observe that $P_{z^4}, P_{\bar{z}^4}, P_{\bar{z}z^2}, P_{\bar{z}^2 z} \in V \equiv Vect(Z^3, \overline{Z}^3, Z^2, \overline{Z}Z, \overline{Z}^2, \overline{Z}, Z, 1)$ and thus $zP_{\bar{z}z^2}, \bar{z}P_{\bar{z}^2 z}, \bar{z}P_{\bar{z}z^2} \in V$; also one have, for all $i+j \leq 4$,

$$\langle M(4)Z^4, \overline{Z}^i Z^j \rangle = \langle M(4)P_{z^4}, \overline{Z}^i Z^j \rangle;$$
$$\langle M(4)\overline{Z}^4, \overline{Z}^i Z^j \rangle = \langle M(4)P_{\bar{z}^4}, \overline{Z}^i Z^j \rangle;$$
$$\langle M(4)\overline{Z}Z^3, \overline{Z}^i Z^j \rangle = \langle M(4)zP_{\bar{z}z^2}, \overline{Z}^i Z^j \rangle;$$
$$\langle M(4)\overline{Z}^2 Z^2, \overline{Z}^i Z^j \rangle = \langle M(4)\bar{z}P_{\bar{z}z^2}, Z^i Z^j \rangle;$$
$$\text{and } \langle M(4)\overline{Z}^3 Z, \overline{Z}^i Z^j \rangle = \langle M(4)\bar{z}P_{\bar{z}^2 z}, \overline{Z}^i Z^j \rangle.$$

This finishes the proof of the theorem. ■

We close this chapter with an illustrate example.

Example 4.3.1

$$M(2) = \begin{pmatrix} 6 & 1+i & 1-i & 2i & 6 & -2i \\ 1-i & 6 & -2i & 2+2i & 2-2i & -2-2i \\ 1+i & 2i & 6 & -2+2i & 2+2i & 2-2i \\ -2i & 2-2i & -2-2i & 8 & -4i & 0 \\ 6 & 2+2i & 2-2i & 4i & 8 & -4i \\ 2i & -2+2i & 2+2i & 0 & 4i & 8 \end{pmatrix} > 0 \text{ and } B = \begin{pmatrix} -2+2i & 2+2i & 2-2i & -2-2i \\ 4i & 8 & -4i & 0 \\ 0 & 4i & 8 & -4i \\ 4+4i & 4-4i & -4-4i & -4+4i \\ -4+4i & 4+4i & 4-4i & -4-4i \\ -4-4i & -4+4i & 4+4i & 4-4i \end{pmatrix}.$$

The fact that $M(2)$ is positive definite implies,

$$W = (M(2))^{-1}B = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ \frac{3}{4}+\frac{3i}{4} & \frac{1}{4}-\frac{i}{4} & -\frac{1}{4}-\frac{i}{4} & -\frac{3}{4}+\frac{3i}{4} \\ 0 & 0 & 0 & 0 \\ -\frac{3}{4}-\frac{3i}{4} & -\frac{1}{4}+\frac{i}{4} & \frac{1}{4}+\frac{i}{4} & \frac{3}{4}-\frac{3i}{4} \end{pmatrix} \text{ and } W^* M(2) W = \begin{pmatrix} 12 & -8i & -4 & 8i \\ 8i & 12 & -8i & -4 \\ -4 & 8i & 12 & -8i \\ -8i & -4 & 8i & 12 \end{pmatrix}$$

Since $a = e = 12$ and $b = f = -8i$, according to theorem 4.3.1, our sequence is a moment matrix for a 6 atoms measure. In fact, from W, we can see that $Z^3 + \frac{3(1+i)}{4}(\bar{Z}^2 - Z^2) - \bar{Z}$ and $Z^2\bar{Z} + \frac{(1-i)}{4}(\bar{Z}^2 - Z^2) - Z$ are two characteristic polynomials for the moment sequence. The comment roots of the two polynomials are

$$\{\pm 1, \pm i, 0, 1+i\}$$

Finally, we get that $\mu = \delta_1 + \delta_{-1} + \delta_i + \delta_{-1} + \delta_0 + \delta_{1+i}$.

CHAPTER 5

THE TRUNCATED COMPLEX MOMENT PROBLEM AND RECURSIVE RELATIONS

We notice below that the recursiveness in the truncated complex moment problem (TCMP in short) is a natural concept and is totally inherent.

Let $\omega \equiv \{\gamma_{ij}\}_{0\leq i,j\leq n}$ be a given doubly indexed truncated moment sequence, i.e., ω admits a representing measure. A result of C. Bayer and J. Teichmann (Theorem 3.5.5) states that if a finite double sequence of complex numbers has a representing measure, then it has a finitely atomic representing measure. It follows that ω has a finite support representing measure, say μ.

Obviously, μ is a representing measure of the infinite complex-valued bi-sequence $\gamma \equiv \{\gamma_{ij}\}_{i,j\in\mathbb{Z}_+}$, where

$$\gamma_{ij} = \int \bar{z}^i z^j d\mu \qquad\qquad (n,m \in \mathbb{Z}_+).$$

The bi-sequence γ satisfying a recurrence relation. Indeed, Let $p \equiv z^{s+1} - \sum\limits_{i+j=0}^{s} a_{ij}\bar{z}^i z^j$ be a polynomial vanishing on $supp\ \ (\mu)$, then

$$\gamma_{n,m+s+1} - \sum_{i+j=0}^{s} a_{ij}\gamma_{n+i,m+j} = \int \bar{z}^n z^m p(z,\bar{z})d\mu = 0 \qquad\qquad \text{for all } n,m \in \mathbb{Z}_+.$$

Therefore the representing measure μ give rise to an infinite recursively generated sequence $\gamma \equiv \{\gamma_{ij}\}_{i,j\geq 0}$ $(\omega \subset \gamma)$. This last can be defined by the next relations,

$i)$ For all $i,j \geq 0$,

$$\gamma_{ji} = \overline{\gamma}_{ij}. \tag{5.1}$$

$ii)$ For all $n,m \geq 0$,

$$\gamma_{n,m+s+1} = \sum_{i+j=0}^{s} a_{ij}\gamma_{n+i,m+j}. \tag{5.2}$$

We shall refer to sequence verifying (5.1) and (5.2) as recursive double indexed sequence (RDIS for short) associated with the characteristic polynomial $p(z,\bar{z}) = z^{s+1} - \sum\limits_{i+j=0}^{s} a_{ij}\bar{z}^i z^j$. The sequence γ can be associated with several characteristic polynomials, see Section 5.1.

Thus every truncated complex moment sequence is a subsequence of a recursively moment sequence. We deduce that solving the TCMP is actually equivalent to solve the recursive full complex moment problem (FCMP in short). The main goal in this chapter is to investigate use the recursive double sequences (verifying (5.1) and (5.2)) to get an approach, based on the localization of zeros of the characteristic polynomials, for solving the TCMP (see also [35]).

We define in Section 5.1 recursive doubly indexed sequences. We show that for such sequences the TCMP and FCMP are equivalent. We devote Section 5.2 to the study of RDIS associated with analytic characteristic polynomial and we show that the family of analytic characteristic polynomials is a principal ideal of $\mathbb{C}[X]$. Section 5.3 is devoted to give a characterization of recursive doubly indexed moment sequences. In Section 5.4 we apply our results to give an explicit solution for the TCMP with cubic column relations in $M(3)$ of the form $Z^3 = aZ + b\overline{Z}$ $(a, b \in \mathbb{R})$ and also to regain the correct form of Theorem 3.4.10. In the last section, we involve the RDIS to give a necessary and sufficient condition for the existence of a solution to the TCMP with column dependence relations of the form $Z^{k+1} = \sum\limits_{0 \le n+m \le k} a_{nm}\overline{Z}^n Z^m$ $(a_{nm} \in \mathbb{C})$.

5.1 Recursive Double indexed sequences

Let $\{a_{lk}\}_{0 \le l,k \le r}$ be some fixed complex numbers and let $\gamma \equiv \{\gamma_{ij}\}_{i,j \ge 0}$ be a doubly indexed sequence, with $\gamma_{00} > 0$, defined by the following relations:

i) For all $i, j \ge 0$,

$$\gamma_{ji} = \overline{\gamma}_{ij}. \tag{5.3}$$

ii) For all i and n with $0 \le i$ and $r \le n$,

$$\gamma_{i,n+1} = \sum_{0 \le l+k \le r} a_{lk}\gamma_{l+i,n+k-r}. \tag{5.4}$$

Where $\omega \equiv \{\gamma_{ij}\}_{0 \le i \le j \le r}$ are given initial conditions.

In the sequel we shall refer to such sequence as Recursive Double Indexed Sequence, RDIS for short. The polynomial $P(z,\bar{z}) = z^{r+1} - \sum\limits_{0 \le l+k \le r} a_{lk}\bar{z}^l z^k$ is called a characteristic polynomial associated with γ, given by (5.3) and (5.4). This last polynomial has a finite roots set. More precisely P has at most $(r+1)^2$ roots (see Proposition 4.4 in [21]).

A RDIS can be defined in various ways using different characteristic polynomials as is shown in the following example.

Example 5.1.1 Let $\gamma = \{\gamma_{ij}\}_{i,j \ge 0}$ with $\gamma_{ij} = \frac{(-1)^{i+j}}{2} + \frac{1}{2}\mathfrak{Re}((1-2i)^i(1+2i)^j)$. Then γ may be defined by the following recursive relations,

- $\gamma_{n+2,m} = -2\gamma_{n,m+1} - \gamma_{n,m}$, with $\gamma_{n,m} = \overline{\gamma}_{m,n}$, for $\gamma_{00} = 1, \gamma_{01} = \gamma_{10} = 0, \gamma_{11} = 3$.
- $\gamma_{n+3,m} = \gamma_{n+2,m} - 3\gamma_{n+1,m} - 5\gamma_{n,m}$, with $\gamma_{n,m} = \overline{\gamma}_{m,n}$, for $\gamma_{00} = 1, \gamma_{01} = \gamma_{10} = 0, \gamma_{11} = 3, \gamma_{02} = \gamma_{20} = -1, \gamma_{21} = \gamma_{12} = 2, \gamma_{22} = 13$.

Therefore, $P_1(z,\bar{z}) = z^2 + 2\bar{z} + 1$ and $P_2(z,\bar{z}) = Q(z) = z^3 - z^2 + 3z + 5$ are two characteristic polynomials associated with γ.

Let us denote by $\mathscr{P}_\gamma$ the set of characteristic polynomials associated with the sequence $\gamma \equiv \{\gamma_{ij}\}_{i,j \ge 0}$.

Remark 5.1.1 *i*) The subset $\mathscr{P}_\gamma$ is an ideal of $\mathbb{C}[z,\bar{z}]$.

ii) The characteristic polynomial P, together with the initial conditions and the relations (5.3) and (5.4), are said to define the sequence γ.

iii) We notice that because of condition (5.3) , Equation (5.4) is equivalent to : For all n and j with $0 \le j$ and $r \le n$,

$$\gamma_{n+1,j} = \sum_{0 \le l+k \le r} \bar{a}_{lk}\gamma_{n-r+k,l+j}. \tag{5.5}$$

The polynomial $Q(z,\bar{z}) = \bar{z}^{r+1} - \sum\limits_{0 \le l+k \le r} \bar{a}_{lk}\bar{z}^k z^l$ is a characteristic polynomial associated with $\{\gamma_{i,j}\}_{i,j \ge 0}$ given by (5.5), where $\{\gamma_{ij}\}_{0 \le j \le i \le r}$ are given initial conditions.

The following result is an immediate consequence of (5.4).

Lemma 5.1.1 Let $M(\gamma)$ be the infinite complex moment matrix associated with the doubly indexed sequence $\gamma \equiv \{\gamma_{ij}\}_{i,j \ge 0}$ and let $p(z,\bar{z}) \in \mathbb{C}[z,\bar{z}]$. Then $p(z,\bar{z})$ is a characteristic of γ if and only if $M(\gamma)p = 0$.

We use a structural properties of moment matrices to get the following interesting result.

Lemma 5.1.2 Let $\gamma \equiv \{\gamma_{ij}\}_{i,j\geq 0}$ be a doubly indexed sequence and let $M(\gamma)$ be the associated infinite complex moment matrix. Then, for every $f,g,h \in \mathbb{R}[x_1,\ldots,x_d]$, we have

$$f^T M(\gamma)(gh) = (fg)^T M(\gamma)h. \tag{5.6}$$

Proof. Let $f,g,h \in \mathbb{R}[x_1,\ldots,x_d]$ be polynomials. We write $f = \sum_{\mathbf{i}} f_{\mathbf{i}}\mathbf{x}^{\mathbf{i}}$, $g = \sum_{\mathbf{j}} g_{\mathbf{j}}\mathbf{x}^{\mathbf{j}}$ and $h = \sum_{\mathbf{k}} h_{\mathbf{k}}\mathbf{x}^{\mathbf{k}}$. As the entry of the moment matrix corresponding to the column $\mathbf{x}^{\mathbf{i}}$ and the line $\mathbf{x}^{\mathbf{j}}$ is $\gamma_{\mathbf{i}+\mathbf{j}}$, we obtain

$$\begin{aligned} f^T M(\gamma)(gh) &= (\sum_{\mathbf{i}} f_{\mathbf{i}}\mathbf{x}^{\mathbf{i}})^T M(\gamma)(\sum_{\mathbf{j},\mathbf{k}} g_{\mathbf{j}}h_{\mathbf{k}}\mathbf{x}^{\mathbf{j}+\mathbf{k}}) \\ &= \sum_{\mathbf{i},\mathbf{j},\mathbf{k}} f_{\mathbf{i}}g_{\mathbf{j}}h_{\mathbf{k}}\gamma_{\mathbf{i}+\mathbf{j}+\mathbf{k}} \\ &= (fg)^T M(\gamma)h. \end{aligned}$$

■

It follows that

Proposition 5.1.3 Let $\gamma \equiv \{\gamma_{ij}\}_{i,j\geq 0}$ be a doubly indexed sequence such that its associated moment matrix $M(\gamma)$ is semidefinite positive. Then, for any polynomial $p \in \mathbb{C}[z,\bar{z}]$ and any integer $n \geq 1$,

$$M(\gamma)p^n = 0 \Longrightarrow M(\gamma)p = 0. \tag{5.7}$$

Proof. If $M(\gamma)p^2 = 0$, then $0 = M(\gamma)p^2 = 1^T M(\gamma)p^2 = p^T M(\gamma)p$, from (5.6); since $M(\gamma) \geq 0$, we obtain $M(\gamma)p = 0$ and hence (5.7) holds for $n = 2$. By induction, (5.7) remains valid for any power of 2. Now , if $M(\gamma)p^n = 0$ we choose r in such a way that $r+k$ is a power of 2 to ensure that

$$M(\gamma)p^{n+r} = (p^r)^T M(\gamma)p^n = 0.$$

Which gives $M(\gamma)p = 0$. ■

In the next, we involve the celebrated Hilbert's Nullstellensatz (Theorem 2.2.4) to obtain a very useful result. Let $I_p \equiv (p)$ be the ideal of $\mathbb{C}[z,\bar{z}]$ generated by p. The set $V(I_p) := \{z \in \mathbb{C} \mid f(z) = 0 \text{ for every } f \in I_p\}$ is the (complex) variety associated with I_p. The sets $I(V(I_p)) := \{f(z) \in \mathbb{C}[z,\bar{z}] \mid f(x) = 0 \text{ for every } z \in V(I_p)\}$ and $\sqrt{I_p} := \{f \in \mathbb{C}[z,\bar{z}] \mid f^k \in I_p \text{ for some integer } k \geq 1\}$ are again ideals in $\mathbb{C}[z,\bar{z}]$, that obviously contain the ideal I_p. The Hilbert's Nullstellensatz states that

$$I(V(I_p)) = \sqrt{I_p}.$$

Now let us consider a polynomial $q \in \mathbb{C}[z,\bar{z}]$ satisfying that $Z(p) := \{z \in \mathbb{C} \text{ such that } p(z,\bar{z}) = 0\} \subseteq Z(q)$. We have $q \in I(V(I_p))$ hence $q \in \sqrt{I_p}$, that is, there exists some integer $k \geq 1$ such that $q^k \in I_p$. Thus, from Remark 5.1.1-*i*) and Lemma 5.1.1, we have $M(\gamma)q^k = 0$. This implies, by Proposition 5.1.3, that $M(\gamma)q = 0$ and therefore, from Lemma 5.1.1, q is a characteristic polynomial of γ.

Proposition 5.1.4 Let $\gamma \equiv \{\gamma_{ij}\}_{i,j\geq 0}$ be a RDIS, associated with the characteristic polynomial $p(z,\bar{z})$. Then every polynomial vanishing at all points of $Z(p) := \{z \in \mathbb{C} \text{ such that } p(z,\bar{z}) = 0\}$ is a characteristic polynomial of γ.

In the following proposition, we show that for a given RDIS the TCMP and the FCMP are equivalent.

Proposition 5.1.5 Let $\gamma \equiv \{\gamma_{ij}\}_{i,j\geq 0}$ be a RDIS whose initial conditions and characteristic polynomial are $\{\gamma_{ij}\}_{0\leq i\leq j\leq r}$ and $P(z,\bar{z}) = z^{r+1} - \sum_{0\leq l+k\leq r} a_{lk}\bar{z}^l z^k$, respectively. The following are equivalent.

i) There exists a positive Borel measure μ, solution of the FCMP for $\gamma \equiv \{\gamma_{ij}\}_{i,j\geq 0}$.

ii) There exists a positive Borel measure μ, solution of the TCMP for $\omega \equiv \{\gamma_{ij}\}_{0\leq i\leq j\leq r}$ with

$$\text{supp}(\mu) \subseteq Z(P) := \{z \in \mathbb{C} \text{ such that } P(z,\bar{z}) = 0\}.$$

Proof.

$i) \Rightarrow ii)$ It suffices to prove that $supp(\mu) \subseteq Z(P)$.
Write $\gamma_{ij} = \int \bar{z}^i z^j d\mu$, for $i, j \geq 0$. According to (5.4), one has

$$\gamma_{i,n+1} - \sum_{0 \leq l+k \leq r} a_{lk} \gamma_{l+i,n+k-r} = 0, \qquad \text{for every } i \geq 0 \text{ and } n \geq r.$$

We get

$$\int \bar{z}^i z^{n+1} - \sum_{0 \leq l+k \leq r} a_{lk} \bar{z}^{l+i} z^{n+k-r} d\mu = 0, \qquad \text{for every } i \geq 0 \text{ and } n \geq r.$$

Hence

$$\int \bar{z}^i z^{n-r} P(z,\bar{z}) d\mu = 0 \qquad \text{for every } i \geq 0 \text{ and } r \leq n.$$

Taking an adequate combination, we manage to obtain

$$\int \overline{P}(z,\bar{z}) P(z,\bar{z}) d\mu = \int \mid P(z,\bar{z}) \mid^2 d\mu = 0.$$

It follows that $P.\mu = 0$, and thus $supp(\mu) \subset Z(P)$.

$ii) \Rightarrow i)$ Suppose that $\gamma_{ij} = \int \bar{z}^i z^j d\mu$, for all integers i and j such that $0 \leq j \leq i \leq r$, and that $supp\ \ (\mu) \subseteq Z(P)$. Since $\gamma_{ij} = \overline{\gamma}_{ji}$, we also have $\gamma_{ij} = \int \bar{z}^i z^j d\mu$, for every $i, j = 0, \ldots, r$.
Now, for every $i = 0, \ldots, r$, we have

$$\begin{aligned} \gamma_{i,r+1} &= \sum_{0 \leq l+k \leq r} a_{lk} \gamma_{l+i,k}, && \text{from (5.4)}, \\ &= \int \bar{z}^i \Big(\sum_{0 \leq l+k \leq r} a_{lk} \bar{z}^l z^k \Big) d\mu \\ &= \int \bar{z}^i z^{r+1} d\mu && \text{because } supp\ \ (\mu) \subseteq Z\Big(z^{r+1} - \sum_{0 \leq l+k \leq r} a_{lk} \bar{z}^l z^k\Big). \end{aligned}$$

It yields from (5.3) that $\gamma_{r+1,j} = \int \bar{z}^{r+1} z^j d\mu$, for all $j = 0, \ldots, r$. By reapplying (5.4) one obtain

$$\gamma_{ij} = \int \bar{z}^i z^j \qquad \text{for all } i, j = 0, \ldots, r+1.$$

By induction we obtain $\gamma_{ij} = \int \bar{z}^i z^j d\mu$ (for all $i, j \in \mathbb{Z}_+$) and consequently, μ is a solution of the FCMP for $\{\gamma_{ij}\}_{i,j \geq 0}$.

■

5.2 Recursive Sequences of Fibonacci Type

In this section, we focus ourself on a particular case of RDIS, which will play a crucial role in the sequel.

We shall refer to a RDIS with analytic characteristic polynomial as Recursive Sequences of Fibonacci Type (RSFT for short). In other words, a sequence $\gamma \equiv \{\gamma_{ij}\}_{i,j \geq 0}$, with $\gamma_{00} > 0$, is said to be RSFT, associated with the characteristic polynomial $P(x) = x^r - a_0 x^{r-1} - \ldots - a_{r-2} x - a_{r-1}$, if it verifies the following relations:

$i)$ For all $i, j \geq 0$,

$$\gamma_{ji} = \overline{\gamma}_{ij}. \tag{5.8}$$

$ii)$ For all i and n such that $0 \leq i \leq r-1 \leq n$,

$$\gamma_{i,n+1} = a_0 \gamma_{i,n} + a_1 \gamma_{i,n-1} + \ldots + a_{r-1} \gamma_{i,n-r+1}. \tag{5.9}$$

We denote by $\mathscr{A}_\gamma$ the family of analytic characteristic, monic, polynomial associated with γ.

Remark 5.2.1

- $\mathscr{A}_\gamma \subseteq \mathscr{P}_\gamma$.
- A sequence $\gamma \equiv \{\gamma_{ij}\}_{i,j \geq 0}$ is a RSFT if and only if $\mathscr{A}_\gamma \neq \emptyset$.
- Because of condition (5.8), Equation (5.9) is equivalent to : For all integers n and j, with $0 \leq j$ and $r \leq n$,

$$\gamma_{n+1,j} = \overline{a_0} \gamma_{n,j} + \overline{a_1} \gamma_{n-1,j} + \ldots + \overline{a_{r-1}} \gamma_{n-r+1,j}. \tag{5.10}$$

Proposition 5.2.1 Let $\gamma \equiv \{\gamma_{ij}\}_{i,j\geq 0}$ be a RSFT, satisfying the relations (5.8) and (5.9). Then $\mathscr{A}_\gamma$ is a principal ideal of $\mathbb{C}[X]$.

Proof. As $\mathbb{C}[X]$ is a principal ideal domain, it suffices to prove that $\mathscr{P}_\gamma$ is an ideal of $\mathbb{C}[X]$. To this aim let us consider the polynomials $H(x) = b_m x^m + \ldots + b_1 x + b_0, P(x) = x^r - a_0 x^{r-1} - \ldots - a_{r-2}x - a_{r-1} \in \mathscr{P}_\gamma$, with $m \leq r$. It follows, from (5.9), that, for all $j \in \mathbb{Z}_+$,

$$\gamma_{i,j+r} - a_0\gamma_{i,j+r-1} - a_1\gamma_{i,j+r-2} - \ldots - a_{r-1}\gamma_{i,j} = 0$$
$$\gamma_{i,j+m} - b_0\gamma_{i,j+m-1} - b_1\gamma_{i,j+m-2} - \ldots - b_{m-1}\gamma_{i,j} = 0.$$

Hence, for all $j \in \mathbb{Z}_+$, we have

$$(\gamma_{i,j+r} - a_0\gamma_{i,j+r-1} - a_1\gamma_{i,j+r-2} - \ldots - a_{r-1}\gamma_{i,j}) + (\gamma_{i,j+m} - b_0\gamma_{i,j+m-1} - b_1\gamma_{i,j+m-2} - \ldots - b_{m-1}\gamma_{i,j}) = 0$$

and thus $H(x) + P(x) \in \mathscr{P}_\gamma$.

Now, let $Q(x) = c_k x^k + \ldots + c_1 x + c_0 \in \mathbb{C}[x]$ and let $P(x) = x^r - a_0 x^{r-1} - \ldots - a_{r-2}x - a_{r-1} \in \mathscr{P}_\gamma$. Equality (5.9) yields

$$\gamma_{i,n+j+1} - a_0\gamma_{i,n+j} - a_1\gamma_{i,n+j-1} - \ldots - a_{r-1}\gamma_{i,n+j-r+1} = 0 \qquad (j = 0,\ldots,k),$$

hence

$$\sum_{j=0}^{k} c_j(\gamma_{i,n+j+1} - a_0\gamma_{i,n+j} - a_1\gamma_{i,n+j-1} - \ldots - a_{r-1}\gamma_{i,n+j-r+1}) = 0,$$

therefore

$$Q(x)P(x) = \sum_{j=0}^{k} c_j(x^{j+r} - a_0 x^{j+r-1} - a_1 x^{j+r-2} - \ldots - a_{r-1}x^j) \in \mathscr{P}_\gamma,$$

as desired. ■

Let $\gamma \equiv \{\gamma_{ij}\}_{i,j\geq 0}$ be a RSFT, we call P_γ the minimal polynomial associated with γ. Below, we associate with every RSFT its minimal polynomial.

Corollary 5.2.2 Let $\gamma \equiv \{\gamma_{ij}\}_{i,j\geq 0}$ be a RSFT, associated with the minimal analytic characteristic polynomial $P_\gamma \in \mathbb{C}[z]$. If $M(\gamma) \geq 0$, then P_γ has distinct roots.

Proof. Setting $P_\gamma(z) = \prod_{i=1}^{r}(z-\lambda_i)^{n_i}$ for the characteristic polynomial of γ. By applying Lemma 5.1.1, we get

$$M(\gamma)\prod_{i=1}^{r}(z-\lambda_i)^{n_i} = 0.$$

It follows that

$$M(\gamma)\prod_{i=1}^{r}(z-\lambda_i)^{m} = (\prod_{i=1}^{r}(z-\lambda_i)^{m-n_i})^T M(\gamma)\prod_{i=1}^{r}(z-\lambda_i)^{n_i} = 0, \qquad \text{where } m = \max_{i=1}^{r} n_i,$$

and thus, by using Proposition 5.1.3, one obtain

$$M(\gamma)\prod_{i=1}^{r}(z-\lambda_i) = 0.$$

Therefore, Lemma 5.1.1 yields that the polynomial $\prod_{i=1}^{r}(z-\lambda_i)$ is a characteristic polynomial of γ. Since P_γ is minimal and $\prod_{i=1}^{r}(z-\lambda_i)$ divides P_γ, then $P_\gamma = \prod_{i=1}^{r}(z-\lambda_i)$. ■

The next proposition establishes a link between the positivity of the moment matrix $M(\infty)$ and that of $M(n)$, for a RSFT.

Proposition 5.2.3 Let γ be a RSFT such that $\deg P_\gamma = r$ and let $n \geq 2r-2$. Then $M(n)$ is positive semi-definite if and only if $M(\infty)$ is positive semi-definite. Moreover, $rank\ \ M(n) = rank\ \ M(\infty)$.

Proof. We only need to show the direct implication. To this aim, we construct a matrix $W \in M_{m,n+2}(\mathbb{C})$ (where $n \geq 2r-2$) such that the successive rows are defined by:

$$\overline{Z}^i Z^{n+1-i} = \sum_{j=0}^{r-1} a_j e_{(i,n-i-j)}, \qquad \text{for all } \ 0 \leq i \leq r-1,$$

and

$$\overline{Z}^i Z^{n+1-i} = \sum_{j=0}^{r-1} \overline{a_j} e_{(i-j-1,n+1-i)}, \qquad \text{for all } \ r \leq i \leq 2r-1;$$

where $\{e_{ij}\}_{0\leq i+j\leq n}$ denote the canonical basis of $\mathbb{C}^m$, that is, e_{ij} is the vector with 1 in the $\bar{z}^i z^j$ entry and 0 all other position, according to the lexicographic order (see Section 2.1.3).

Clearly, (5.9) yields

$$M(n+1)(\gamma) = \begin{pmatrix} M(n)(\gamma) & B \\ B^* & C \end{pmatrix},$$

with $B = M(n)(\gamma)W$ and $C = B^*W$.

Therefore, Smul'jan's Theorem 2.4.1 implies that

$$M(n)(\gamma) \geq 0 \Longrightarrow M(n+1)(\gamma) \geq 0 \text{ and } rank\, M(n)(\gamma) = rank\, M(n+1)(\gamma).$$

Since $M(2r-2) \geq 0$, then, by induction, we obtain

$$M(\infty) \geq 0 \text{ and } rank\ \ M(n) = rank\ \ M(\infty), \qquad \text{for any integer } n \geq 2r-2.$$

■

We are able now to give a necessary and sufficient condition for a RSFT to be a moment sequence.

Theorem 5.2.4 Let $\gamma \equiv \{\gamma_{ij}\}_{i,j\geq 0}$ be a RSFT and let P_γ, of degree r, be its minimal analytic characteristic polynomial. Then γ admits a representing measure μ if and only if $M(2r-2)$ is positive semi-definite. Moreover, $supp(\mu) = Z(P_\gamma)$.

Proof. If γ admits a representing measure μ, then

$$p^T M(2r-2)p = \int |\, p\,|^2\, d\mu \geq 0 \qquad \text{for any } p \in \mathbb{C}_{r-1}[z,\bar{z}]$$

and thus $M(2r-2)$ is positive semi-definite. Conversely, if $M(2r-2) \geq 0$, then Proposition 5.2.3 yields $M(\infty) \geq 0$.

Now let $P_\gamma(z) = \prod_{i=1}^{r}(z-\lambda_i)$ and let $L_{\lambda_j} = \prod_{\substack{1\leq i\leq r\\ i\neq j}} \frac{z-\lambda_i}{\lambda_j-\lambda_i} \in \mathbb{C}[z,\bar{z}]$ $(i = 1\cdots,r)$ be the interpolation polynomials at the points of $Z(P_\gamma)$. Since the polynomials $Q(z,\bar{z}) = z^n - \sum_{i=1}^{r} \lambda_i^n L_{\lambda_i}$ and $H(z,\bar{z}) = \bar{z}^m - \sum_{j=1}^{r} \overline{\lambda_j^m L_{\lambda_j}}$ vanish at all points of $Z(P_\gamma)$, then we derive (from the Proposition 5.1.4) that $Q(z,\bar{z})$ and $H(z,\bar{z})$ are characteristic polynomials of γ. Hence Lemma 5.1.1 ensures that

$$M(\infty)Q(z,\bar{z}) = M(\infty)H(z,\bar{z}) = 0$$

and thus γ_{mn} $(m,n \in \mathbb{Z}_+)$ can be expressed as follows:

$$\begin{aligned}
\gamma_{mn} &= (z^m)^T M(\infty) z^n \\
&= 1^T M(\infty)(z^n \bar{z}^m), && \text{by applying (5.6),} \\
&= 1^T M(\infty)(\sum_{i=1}^{r} \lambda_i^n L_{\lambda_i} \bar{z}^m), && \text{from Remark 5.1.1-i) and Lemma 5.1.1,} \\
&= 1^T M(\infty)(\sum_{i=1}^{r} \lambda_i^n L_{\lambda_i} \sum_{j=1}^{r} \overline{\lambda}_j^m \overline{L_{\lambda_j}}) \\
&= 1^T M(\infty)(\sum_{i,j=1}^{r} \lambda_i^n \overline{\lambda}_j^m L_{\lambda_i} \overline{L_{\lambda_j}}) \\
&= \sum_{i,j=1}^{r} \lambda_i^n \overline{\lambda}_j^m 1^T M(\infty)(L_{\lambda_i} \overline{L_{\lambda_j}}).
\end{aligned}$$

If $i \neq j$, then $L_{\lambda_i} L_{\lambda_j}$ vanishing at all points of $Z(P_\gamma)$, hence (in this case and from Lemma 5.1.4) $L_{\lambda_i} L_{\lambda_j}$ is a characteristic polynomial of γ, that is, $M(\infty)(L_{\lambda_i} \overline{L_{\lambda_j}}) = 0$. Hence

$$\gamma_{mn} = \sum_{i=1}^{r} \overline{\lambda}_i^m \lambda_i^n 1^T M(\infty)(L_{\lambda_i} \overline{L_{\lambda_i}}) = \sum_{i=1}^{r} \lambda_i^n \overline{\lambda}_i^m L_{\lambda_i}^T M(\infty) L_{\lambda_i}.$$

Since $M(\infty) \geq 0$, then $c_i = L_{\lambda_i}^T M(\infty) L_{\lambda_i} \geq 0$ for $i = 0, \ldots, r$. Therefore, the measure $\mu = \sum_{i=1}^{r} c_i \delta_{\lambda_i}$ is a representing measure for γ.

It remains to show that $supp(\mu) = Z(P_\gamma)$, that is, $c_i \neq 0$ for all $i = 0, \ldots, r$. To this end, let us assume that $c_i = 0$ for some $i \in \{1, \ldots, r\}$, then $L_{\lambda_i}^T M(\infty) L_{\lambda_i} = 0$. Since $M(\infty) \geq 0$, then $M(\infty) L_{\lambda_i} = 0$. It follows from Lemma 5.1.1 that L_{λ_i} is a analytic characteristic polynomial of γ, and this is a contradiction, because the degree of the polynomial L_{λ_i} is less strictly than the degree of the minimal polynomial P_γ. Therefore, $c_i \neq 0$ for all $i = 0, \ldots, r$ and thus $supp(\mu) = Z(P_\gamma)$. ∎

5.3 Solving the complex moment problem for RDIS

Let $\gamma = \{\gamma_{ij}\}_{i,j \geq 0}$ be a double indexed recursive *moment* sequence, associated with the characteristic polynomial $P(z,\bar{z}) = z^r - \sum_{0 \leq l+k \leq r-1} a_{lk} \bar{z}^l z^k$. Proposition 5.1.5 ensures the existence of a representing measure μ associated with γ such that $supp(\mu) \subseteq Z(P) := \{\lambda_0, \ldots, \lambda_n\}$. It follows that $Q(z) = \prod_{i=1}^{n} (z - z_i) = z^n - a_1 z^{n-1} - \cdots - a_n$ is also a characteristic polynomial associated with γ; that is, for every $i \geq 0$ and $n \geq r$, we have

$$\gamma_{i,n} - a_1 \gamma_{i,n-1} - a_2 \gamma_{i,n-2} - \ldots - a_r \gamma_{i,n-r} = \int \bar{z}^i z^{n-r} Q(z) d\mu = 0.$$

We conclude that every double indexed recursive *moment* sequence is a RSFT.

In order to obtain necessary and sufficient condition for a RDIS to be moment sequence, we need to find the smallest n which verifies the following equivalence

$$M(n) \geq 0 \Longleftrightarrow M(\gamma) \equiv M(\infty) \geq 0.$$

Since γ admits a characteristic polynomial of the form $z^r - P_{r-1}(z,\bar{z})$ (with $\deg P_{r-1} \leq r-1$) the best known result in this direction is the one given by Curto-Fialkow (Theorem 3.4.11) which guarantees the equivalence for every n satisfying the inequality $r \leq [\frac{n}{2}] + 1$. In the next theorem we involve the characteristic polynomials $P(z,\bar{z})$ and $Q(z)$ (associated with γ) to refine this result.

Theorem 5.3.1 Let γ be a RDIS associated with the characteristic polynomials $P(z,\bar{z}) = z^r - \sum_{0 \leq l+k \leq r-1} a_{lk} \bar{z}^l z^k$ and $Q(z) = \prod_{i=1}^{n} (z - \lambda_i)$, where $Z(P) = \{\lambda_1, \ldots, \lambda_n\}$, then:

i) There exists a characteristic polynomial, $h \in \mathbb{C}[z,\bar{z}]$, associated with γ such that $Q(z) = h(z,\bar{z}) + f_1(z,\bar{z}) P(z,\bar{z}) + f_2(z,\bar{z}) \overline{P}(z,\bar{z})$, where $f_1, f_2 \in \mathbb{C}[z,\bar{z}]$, with $d_z(h) < r$ and $d_{\bar{z}}(h) < r$.

ii) Let A_h be the set of monomials $\bar{z}^i z^j$ in h such that $i+j=d_h$, and denote $c_1 := \max\{k \mid \bar{z}^l z^k \in A_h\}$ and $c_1' := \max\{l \mid \bar{z}^l z^k \in A_h\}$. We also define $c_2 := \max\{k \mid \bar{z}^l z^k \in A_h \setminus \{\bar{z}^{d_h-c_1} z^{c_1}\}\}$ and $c_2' := \max\{l \mid \bar{z}^l z^k \in A_h \setminus \{\bar{z}^{c_1'} z^{d_h-c'-1}\}\}$, if card $A_h \geq 2$. We put for convenience $c_2 = c_2' = -\infty$, in the case where $card(A_h) = 1$. We finally denote $c = sup(c_1, c_1')$, $\alpha_{c_1} = inf(r-c_1, c_1-c_2)$ and $\alpha_{c_1'} = inf(r-c_1', c_1'-c_2')$. Then

$$M(\infty) \geq 0 \Longleftrightarrow M(2r-2-\alpha_c) \geq 0.$$

Proof. *i*) It obvious to show that there exists $h(z,\bar{z})$ such that $Q(z) = h(z,\bar{z}) + f_1(z,\bar{z})P(z,\bar{z}) + f_2(z,\bar{z})\overline{P}(z,\bar{z})$, for some $f_1, f_2 \in \mathbb{C}[z,\bar{z}]$, with $d_z(h) < r$ and $d_{\bar{z}}(h) < r$. Since $P, \overline{P}$ and Q are characteristic polynomials then $h(z,\bar{z})$ is also a characteristic polynomia associated with (γ).

ii) Since the two conditions are symmetric, we only need to prove the case $c_1 \geq c_1'$. Recall first that for $m \geq 0$, $M_m(\mathbb{C})$ denotes the algebra of $m \times m$ complex matrices and for $n \geq 0$, let $m \equiv m(n) := (n+1)(n+2)/2$ and $M(n)(\gamma) \in M_m(\mathbb{C})$, as in the introduction. We define a basis $\{e_{ij}\}_{0 \leq i+j \leq n}$ of $\mathbb{C}^m$ as follows: e_{ij} is the vector with 1 in the $\overline{Z}^i Z^j$ entry and 0 all other positions.

The main idea is to write, from the characteristic polynomials associated with γ, monomials of order $2r-1-\alpha_c+e-1$ ($e \in \mathbb{N}$) as linear combination of monomials of order strictly less than $2r-1-\alpha_c+e$ modulo $P, \overline{P}, h$ and $\overline{h}$, that is,

$$\bar{z}^i z^{2r-1-\alpha_c+e-i} = H_{(i,2r-1-\alpha_c+e-i)}(z,\bar{z}) \bmod \{P, \overline{P}, h, \overline{h}\},$$

with $0 \leq i \leq 2r-1-\alpha_c+e$ and $\deg H_{(i,2r-1-\alpha_c+e-i)}(z,\bar{z}) \leq 2r-2-\alpha_c$.

We construct for every $e \in \mathbb{Z}_+$ a matrix $W_e \in M_{m(2r-2-\alpha_c+e),2r-1-\alpha_c+e}(\mathbb{C})$ such that the coefficients of the column $\overline{Z}^i Z^{2r-1-\alpha_c+e-i}$, $0 \leq i \leq 2r-1-\alpha_c+e$, are that of the polynomial $H_{(i,2r-1-\alpha_c+e-i)}$ (considering the lexicographic order cited in the introduction).

Since $P, \overline{P}, h$ and $\overline{h}$ are characteristic polynomials associated with γ, the above discussion leads, in view of Lemma 5.1.1, to the following equality:

$$M(2r-1-\alpha_c+e)(\gamma) = \begin{pmatrix} M(2r-2-\alpha_c+e)(\gamma) & B \\ B^* & C \end{pmatrix},$$

such that $B = M(2r-2-\alpha_c+e)(\gamma)W_e$ and $C = B^* W_e$, for all $e \in \mathbb{N}$. According to Smul'jan's theorem we have

$$M(2r-2-\alpha_c+e)(\gamma) \geq 0 \Longleftrightarrow M(2r-1-\alpha_c+e)(\gamma) \geq 0,$$

and hence it follows by induction that

$$M(2r-2-\alpha_c)(\gamma) \geq 0 \Longleftrightarrow M(\infty)(\gamma) \geq 0.$$

We distinguish 3 cases,

- $r \leq i \leq 2r-1-\alpha_c+e$. Since $\bar{z}^r = \overline{P} + \sum_{0 \leq l+k \leq r-1} \bar{a}_{lk} \bar{z}^k z^l$, we obtain

$$\bar{z}^i z^{2r-1-\alpha_c+e-i} = \bar{z}^{i-r} z^{2r-1-\alpha_c+e-i} \overline{P} + \sum_{0 \leq l+k \leq r-1} \bar{a}_{lk} \bar{z}^{k+i-r} z^{l+2r-1-\alpha_c+e-i}.$$

  Hence

$$\overline{Z}^i Z^{2r-1-\alpha_c+e-i} = \sum_{0 \leq l+k \leq r-1} \bar{a}_{lk} e_{(k+i-r,l+2r-1-\alpha_c+e-i)}.$$

- $0 \leq i \leq r-1-\alpha_c$. As $z^r = P + \sum_{0 \leq l+k \leq r-1} a_{lk} \bar{z}^l z^k$, we get

$$\bar{z}^i z^{2r-1-\alpha_c+e-i} = \bar{z}^i z^{r-1-\alpha_c+e-i} P + \sum_{0 \leq l+k \leq r-1} a_{lk} \bar{z}^{l+i} z^{k+r-1-\alpha_c+e-i},$$

  and thus,

$$\overline{Z}^i Z^{2r-1-\alpha_c+e-i} = \sum_{0 \leq l+k \leq r-1} a_{lk} e_{(l+i,k+r-1-\alpha_c+e-i)}.$$

- $r-\alpha_c \le i \le r-1$. This third case requires more work. We will distinguish two subcases.

$a)$ Card $A=1$. (In this case we have $\alpha_c = r-c_1$). Let

$$h(z,\bar{z}) = \bar{z}^{d_h-c_1} z^{c_1} - \sum_{0\le l+k\le d_h-1} a_{lk}\bar{z}^l z^k,$$

we have

$$\bar{z}^{d_h-c_1} z^{c_1} = h(z,\bar{z}) + \sum_{0\le l+k\le d_h-1} a_{lk}\bar{z}^l z^k.$$

and since $d_h \le c_1 + c_1' \le 2c_1$, we get

$$\begin{array}{rcl} \bar{z}^{r+c_1+e-1-i} z^i & = & \bar{z}^{r+2c_1+e-1-i-d_h} z^{i-c_1}\left(\bar{z}^{d_h-c_1} z^{c_1}\right), \\ & = & \bar{z}^{r+2c_1+e-1-i-d_h} z^{i-c_1} h(z,\bar{z}) \\ & & + \sum\limits_{0\le l+k\le d_h-1} a_{lk}\bar{z}^{l+r+2c_1+e-1-i-d_h} z^{k+i-c_1}. \end{array}$$

Hence,

$$\overline{Z}^{r+c_1+e-1-i} Z^i = \sum_{0\le l+k\le d_h-1} a_{lk} e_{(l+r+2c_1+e-1-i-d_h,\,k+i-c_1)},$$

and it follows that,

$$\overline{Z}^{2r-1-\alpha_c+e-i} Z^i = \sum_{0\le l+k\le d_h-1} a_{lk} e_{(l+3r-2\alpha_c+e-1-i-d_h,\,k+i-r+\alpha_c)}.$$

$b)$ Card $A\ge 2$. Let

$$\bar{z}^{d_h-c_1} z^{c_1} = h(z,\bar{z}) + \sum_{l+k=d_h} \alpha_{lk}\bar{z}^l z^k + \sum_{0\le l+k\le d_h-1} a_{lk}\bar{z}^l z^k,$$

we deduce that,

$$\begin{array}{rl} \bar{z}^{c_2-c_1+r} z^{c_1} & = \left(\bar{z}^{r-d_h+c_2}\right)\left(\bar{z}^{d_h-c_1} z^{c_1}\right) \\ & = \bar{z}^{r-d_h+c_2} h + \sum\limits_{l+k=d_h} \alpha_{lk}\bar{z}^{l+r-d_h+c_2} z^k \\ & + \sum\limits_{0\le l+k\le d_h-1} a_{lk}\bar{z}^{l+r-d_h+c_2} z^k. \end{array}$$

Since in the above equality, all monomials in the sum $\sum\limits_{l+k=d_h} \alpha_{lk}\bar{z}^{r+l-d_h+c_2} z^k$ satisfy $d_{\bar{z}}(\bar{z}^{l+r-d_h+c_2} z^k) \ge r$ (since $k\le c_2$ and $l+k=d_h$, then $l-d_h+c_2\ge 0$), we get

$$\sum_{l+k=d_h} \alpha_{lk}\bar{z}^{l+r-d_h+c_2} z^k = \overline{P}\sum_{l+k=d_h} \bar{z}^{l-d_h+c_2} z^k - \sum_{0\le l'+k'\le r+c_2-1} a_{l'k'}\bar{z}^{l'} z^{k'},$$

where $\{a_{l'k'}\}_{0\le l'+k'\le r+c_2-1}$ are complex numbers. Thus, there exists $\alpha_{l''k''}\in\mathbb{C}$ such that

$$\bar{z}^{c_2-c_1+r} z^{c_1} = \sum_{0\le l''+k''\le r+c_2-1} \alpha_{l''k''}\bar{z}^{l''} z^{k''} \, mod(\overline{P},h). \tag{5.11}$$

Here again we discuss two situations,
*) If $r-c_1 \le c_1-c_2$, (that is, $\alpha_c = r-c_1$).
Then $0\le c_1-r+c_1-c_2 \le i-r+c_1-c_2$ and $0\le c_2-c_1+r$ (since $c_2\le c_1\le i\le r$), hence (5.11) yields

$$\begin{array}{rl} \bar{z}^i z^{r+c_1+e-1-i} & = \left(\bar{z}^{i-c_2+c_1-r} z^{r+e-1-i}\right)\left(\bar{z}^{c_2-c_1+r} z^{c_1}\right) \\ & = \sum\limits_{0\le l''+k''\le r+c_2-1} \alpha_{l''k''}\bar{z}^{l''+i-c_2+c_1-r} z^{k''+r+e-1-i} mod(\overline{P},h). \end{array}$$

Then

$$\overline{Z}^i Z^{r+c_1+e-1-i} = \sum_{0\le l''+k''\le r+c_2-1} \alpha_{l''k''} e_{(l''+i-c_2+c_1-r,\,k''+r+e-1-i)},$$

that is,

$$\overline{Z}^i Z^{2r-1-\alpha_c+e-i} = \sum_{0\le l''+k''\le r+c_2-1} \alpha_{l''k''} e_{(l''+i-c_2+c_1-r, k''+r+e-1-i)}.$$

**) If $r-c_1 > c_1-c_2$, (that is, $\alpha_c = c_1-c_2$).
By (5.11), we get

$$\begin{aligned}\overline{z}^{2r-c_1+c_2+e-1-i} z^i &= (\overline{z}^{r+e-1-i} z^{i-c_1})(\overline{z}^{r-c_1+c_2} z^{c_1})\\ &= \sum_{0\le l''+k''\le r+c_2-1} \alpha_{l''k''} \overline{z}^{l''+r+e-1-i} z^{k''+i-c_1} mod(\overline{P},h),\end{aligned}$$

and then

$$\overline{z}^i z^{2r-1-\alpha_c+e-i} = \sum_{0\le l''+k''\le r+c_2-1} \overline{\alpha}_{l''k''} \overline{z}^{k''+i-c_1} z^{l''+r+e-1-i} mod(P,\overline{h}).$$

This finally gives

$$\overline{Z}^i Z^{2r-1-\alpha_c+e-i} = \sum_{0\le l''+k''\le r+c_2-1} \overline{\alpha}_{l''k''} e_{(k''+i-c_1, l''+r+e-1-i)},$$

as required.

■

Corollary 5.3.2 Let $\gamma \equiv \{\gamma_{ij}\}_{i,j\ge 0}$ be a RDIS associated with the characteristic polynomial $P = z^r - \sum\limits_{0\le l+k\le r-1} a_{lk}\overline{z}^l z^k$, then:

$$M(\infty)(\gamma) \ge 0 \Longleftrightarrow M(2r-2)(\gamma) \ge 0.$$

Proof. It suffices to rewrite the first and second cases, in the above theorem's proof, with $\alpha_c = 0$. ■

In the next theorem and in the sequel, we set $\xi \equiv \xi_\gamma = 2r-2-\alpha_c$.

Theorem 5.3.3 Let $\gamma \equiv \{\gamma_{ij}\}_{i,j\ge 0}$ be a RDIS associated with the characteristic polynomial $P = z^r - \sum\limits_{0\le l+k\le r-1} a_{lk}\overline{z}^l z^k$ and let $Z(P) = \{\lambda_1, \lambda_2, \ldots, \lambda_n\}$. The following are equivalent:

- γ is a moment sequence;
- $Q(z) = \Pi_{i=1}^n (z-\lambda_i) \in \mathscr{P}_\gamma$ and $M(\xi) \ge 0$.

Proof. Suppose that γ admits a representing measure μ, then it follows, from Proposition 5.1.5, that

$$\mathrm{supp}(\mu) \subset Z(P) = \{\lambda_1, \lambda_2, \ldots, \lambda_n\}.$$

Setting $Q(z) = \Pi_{i=1}^n (z-\lambda_i) = z^n - \alpha_1 z^{n-1} - \ldots - \alpha_n$, we have

$$\begin{aligned}\gamma_{im} &= \int \overline{z}^i z^m d\mu, && \text{where} \quad 0 \le i \text{ and } n \le m,\\ &= \int \overline{z}^i (\alpha_1 z^{m-1} + \alpha_2 z^{m-2} + \ldots + \alpha_n z^{m-n}) d\mu, && \text{because } \mathrm{supp}(\mu) \subset Z(Q),\\ &= \alpha_1 \gamma_{i,m-1} + \alpha_2 \gamma_{i,m-2} + \ldots + \alpha_n \gamma_{i,m-n}.\end{aligned}$$

Hence $Q(z)$ is a characteristic polynomial associated with γ. The condition $M(\xi) \ge 0$ is obvious. Conversely, since $M(\gamma)(\xi) \ge 0$ then, from Theorem 5.3.1, $M(\gamma)(\infty) \ge 0$ and thus Theorem 5.2.4 yields that the sequence γ owns a representing measure. ■

Corollary 5.3.4 Let $\gamma \equiv \{\gamma_{ij}\}_{i,j\ge 0}$ be a RDIS associated with the characteristic polynomial $P = z^r - \sum\limits_{0\le l+k\le r-1} a_{lk}\overline{z}^l z^k$, then γ is a moment sequence if and only if $M(2r-2)(\gamma) \ge 0$.

Proof. We only need to show the converse implication. As noted in the introduction the polynomial $P(z,\bar{z})$ has finite number of roots, say $Z(P) := \{\lambda_1, \lambda_2, \ldots, \lambda_n\}$. Since $M(2r-2)(\gamma) \geq 0$, Corollary 5.3.2 yields $M(\infty)(\gamma) \geq 0$. Let $Q(z) = \Pi_{i=1}^{n}(z-\lambda_i)$, applying Proposition 5.1.4, we obtain $Q(z,\bar{z}) \in \mathscr{P}_\gamma$. Therefore, Theorem 5.3.3 implies that γ is a moment sequence. ■

5.4 Cubic column relations in truncated moment problems

5.4.1 The TCMP with cubic relation of the form $z^3 + az + b\bar{z} = 0$

Whenever we have the zeros of a characteristic polynomial associated with a RDIS, Theorem 5.3.3 allows us to give a concrete, computable, necessary and sufficient conditions for the existence of a representing measure associated with this sequence. In this section we apply the above mentioned theorem to solve the complex moment problem for a RDIS, associated with a harmonic characteristic polynomial of the form $z^3 + az + b\bar{z}$, where $a, b \in \mathbb{R}$. First, we start by giving the number of zeros of the harmonic polynomial $P(z,\bar{z}) = z^3 + az + b\bar{z}$. We solve the equation $z^3 + az + b\bar{z} = 0$, for completeness. Writing $z = x+iy$, we get $(x+iy)^3 + a(x+iy) + b(x-iy) = 0$, and then $x^3 - 3xy^2 + (a+b)x - i(y^3 - (3x^2 + a + d)y) = 0$. It follows that

$$\begin{cases} x(x^2 - 3y^2 + a + b) & = 0, \\ y(y^2 - (3x^2 + a - b)) & = 0. \end{cases}$$

- If $y = 0$; then $x = 0$ or $x^2 = -a - b$.
- If $x = 0$; then $y = 0$ or $y^2 = a - b$.
- If $xy \neq 0$; then $x^2 = \frac{-a+2b}{4}$ and $y^2 = \frac{a+2b}{4}$.

We deduce from the above cases that $z^3 + az + b\bar{z}$ has at most 7 roots and it has exactly 7 roots if and only if $b < |a| < 2b$.

Let $\lambda_1, \lambda_2, \ldots, \lambda_7$ be the roots of the polynomial $P(z,\bar{z}) = z^3 + az + b\bar{z}$, with $b < -a < 2b$. Direct computations lead to the expression

$$\begin{aligned} Q(z) &= \Pi_{i=1}^{7}(z-\lambda_i) \\ &= z^7 + (2a+b)z^5 + (a^2+b^2+ab)z^3 + (b^3+ab^2)z \\ &= (z^4 + (a+b)z^2 - b\bar{z}z + b^2)P(z,\bar{z}) - b^2(\bar{z}z^2 - \bar{z}^2 z - bz + b\bar{z}) \\ &= (z^4 + (a+b)z^2 - b\bar{z}z + b^2)P(z,\bar{z}) - b^2 h(z,\bar{z}). \end{aligned}$$

Now we are able to solve the moment problem of this section, in the case $b < -a < 2b$. Recall again that, $1, Z, \overline{Z}, \ldots, Z^n, \ldots, \overline{Z}^n$ denote the successive columns of $M(\omega)(n)$.

Theorem 5.4.1 Let $\omega \equiv \{\gamma_{ij}\}_{0 \leq i+j \leq 6}$, with $\overline{\gamma}_{ij} = \gamma_{ji}$ and $\gamma_{00} > 0$, be a truncated complex sequence, and let $M(\omega)(3)$ be its associated moment matrix. If $M(\omega)(3) \geq 0$ and has cubic column relation of the form $Z^3 = -aZ - b\overline{Z}$, with $a, b \in \mathbb{R}$ and $b < -a < 2b$. Then the following statements are equivalent:

i) There exists a representing measure for ω.

ii) There exists a representing measure for the RDIS $\gamma \equiv \{\gamma_{ij}\}_{i,j \geq 0}$, whose initial conditions and characteristic polynomial are $\{\gamma_{ij}\}_{0 \leq i \leq j \leq 2}$ and $P(z,\bar{z}) = z^3 + az + b\bar{z}$, respectively .

iii)

$$\begin{cases} \Lambda_\omega(h) & = 0, \\ \Lambda_\omega(zh) & = 0. \end{cases}$$

iv)

$$\begin{cases} \mathfrak{Im}\gamma_{12} & = b\mathfrak{Im}\gamma_{01}, \\ \gamma_{22} + 2b\mathfrak{Re}\gamma_{20} + (a-b)\gamma_{11} & = 0. \end{cases}$$

v) $h(z,\bar{z}) = \bar{z}z^2 - \bar{z}^2 z - bz + b\bar{z} \in \mathscr{P}_\gamma$, where γ is the RDIS defined in *ii)*.

vi) $\overline{Z}Z^2-\overline{Z}^2Z-bZ+b\overline{Z}=0.$

Proof. It is straightforward to see that *iii)* $\Leftrightarrow$ *iv)*, *ii)* $\Rightarrow$ *i)* and *v)* $\Rightarrow$ *vi)* $\Rightarrow$ *iii)*. Thus, it is enough to show *i)* $\Rightarrow$ *ii)*, *iii)* $\Rightarrow$ *v)* and *v)* $\Leftrightarrow$ *ii)*.

i) $\Longrightarrow$ *ii)*. Suppose that ω admits a representing measure μ. Because the cubic column in $M(\omega)(3)$ verifies $Z^3=-aZ-b\overline{Z}$, we get the following relations

$$\begin{aligned}\int \overline{z}^3P(z,\overline{z})d\mu &= \int \overline{z}^3(z^3+az+b\overline{z})d\mu &&= \gamma_{3,3}+a\gamma_{3,1}+b\gamma_{4,0}=0,\\ \int zP(z,\overline{z})d\mu &= \int z(z^3+az+b\overline{z})d\mu &&= \gamma_{0,4}+a\gamma_{0,2}+b\gamma_{1,1}=0,\\ \int \overline{z}P(z,\overline{z})d\mu &= \int \overline{z}(z^3+az+b\overline{z})d\mu &&= \gamma_{1,3}+a\gamma_{1,1}+b\gamma_{2,0}=0,\end{aligned}$$

thus $\int \mid P\mid^2 d\mu=\int(\overline{z}^3+a\overline{z}+bz)Pd\mu=0$ hence $supp(\mu)\subseteq Z(P)$. It follows, from Proposition 5.1.5, that μ is a representing measure for γ.
iii) $\Rightarrow$ *v)*. It suffices to show that $\Lambda_\gamma(\overline{z}^iz^jh(z,\overline{z}))=0$, for all $i,j\geq 0$.
Remark that, whenever $z^3=-az-b\overline{z}$, we have:

$$\begin{aligned} z^2h &= (b-a)h,\\ \overline{z}zh &= (a-b)h,\\ \overline{h} &= -h.\end{aligned}$$

Then

$$\begin{aligned}&\Lambda_\gamma(z^2h)=(b-a)\Lambda_\omega(h)=0,\\ &\Lambda_\gamma(\overline{z}zh)=(a-b)\Lambda_\omega(h)=0,\\ &\Lambda_\gamma(\overline{h})=-\Lambda_\omega(h)=0,\\ &\Lambda_\gamma(\overline{z}h)=\overline{\Lambda_\omega(z\overline{h})}=-\overline{\Lambda_\omega(zh)}=0,\\ &\Lambda_\gamma(\overline{z}z^2h)=(b-a)\Lambda_\omega(\overline{z}h)=-(b-a)\overline{\Lambda_\omega(zh)}=0.\end{aligned}$$

Since γ is a RDIS associated with the characteristic polynomial $z^3+az+b\overline{z}$, then, for every $i,j\in\mathbb{Z}_+$, we have

$$\Lambda_\gamma(\overline{z}^iz^jh(z,\overline{z}))=\sum_{0\leq l,k\leq 2}a_{lk}\Lambda_\omega(\overline{z}^lz^kh(z,\overline{z}))=0,$$

where $\{a_{lk}\}_{0\leq l,k\leq 2}$ are real numbers.
v) $\Rightarrow$ *ii)*. We know that $Q(z)=(z^4+(a+b)z^2-b\overline{z}z+b^2)P(z,\overline{z})-b^2h(z,\overline{z})$. Since $P(z,\overline{z})$ and $h(z,\overline{z})$ are characteristic polynomials, $Q(z)$ is also a characteristic polynomial associated with γ. As $M(\xi)(\gamma)\equiv M(3)\geq 0$ (observe that $\xi=2\times 3-2+1=3$) and $Q(z)\in\mathscr{P}_\gamma$, then Theorem 5.3.3 yields γ admits a representing measure.
ii) $\Rightarrow$ *v)*. We have

$$h(z,\overline{z})=\frac{1}{b^2}(z^4+(a+b)z^2-b\overline{z}z+b^2)P(z,\overline{z})-\frac{1}{b^2}Q(z),$$

since (by Theorem 5.3.3) $Q(z)$ is a characteristic polynomial of γ, as well as the polynomial $P(z,\overline{z})$, then $h(z,\overline{z})$ is a characteristic polynomial associated with γ (by Remark 5.1.1-*i)*). ■

Let us now suppose that $b<a<2b$, then

$$\begin{aligned}Q(z)&=\Pi_{i=1}^7(z-\lambda_i)\\ &=z^7+(2a-b)z^5+(a^2+b^2-ab)z^3+(ab^2-b^3)z\\ &=(z^4+(a-b)z^2-b\overline{z}z+b^2)P(z,\overline{z})+b^2(z+\overline{z})(\overline{z}z-b).\end{aligned}$$

Let $h(z,\overline{z})=(z+\overline{z})(\overline{z}z-b)$, we have

$$\begin{aligned}z^2h(\overline{z},z)&=(\overline{z}z+\overline{z}^2-b)P(z,\overline{z})-b\overline{P}(\overline{z},z)-(a+b)h(\overline{z},z),\\ z\overline{z}h(\overline{z},z)&=(z^2-b)\overline{P}(\overline{z},z)+(\overline{z}^2-b)P(\overline{z},z)-(a+b)h(\overline{z},z),\\ \overline{h}&=h.\end{aligned}$$

If $\Lambda_\omega(h) = \Lambda_\omega(zh) = 0$, we get

$$\begin{aligned}
\Lambda_\gamma(\bar{h}) &= \Lambda_\omega(h) &&= 0,\\
\Lambda_\gamma(\bar{z}h) &= \overline{\Lambda_\gamma(z\bar{h})} = \overline{\Lambda_\omega(zh)} &&= 0,\\
\Lambda_\gamma(z\bar{z}h) &= -(a+b)\Lambda_\omega(h) &&= 0,\\
\Lambda_\gamma(z^2h) &= -(a+b)\Lambda_\omega(h) &&= 0,\\
\Lambda_\gamma(\bar{z}z^2h) &= -(a+b)\overline{\Lambda_\omega(zh)} &&= 0,\\
\Lambda_\gamma(\bar{z}^2zh) &= -(a+b)\Lambda_\omega(zh) &&= 0.
\end{aligned}$$

The above equalities and cubic relation allow us to prove that $\Lambda_\gamma(z^i\bar{z}^jh) = 0$, for all $i, j \in \mathbb{Z}_+$; that is, $h(\bar{z}, z) \in \mathscr{P}_\gamma$. From this observation and similarly to the above theorem's proof, we are able to state the following theorem.

Theorem 5.4.2 Let $\omega \equiv \{\gamma_{ij}\}_{0\le i+j\le 6}$, with $\overline{\gamma}_{ij} = \gamma_{ji}$ and $\gamma_{00} > 0$, be a given truncated complex sequence, and let $M(\omega)(3)$ be its associated moment matrix. If $M(\omega)(3) \ge 0$ and has cubic column relation of the form $Z^3 = -aZ - b\overline{Z}$, with $a, b \in \mathbb{R}$ and $b < a < 2b$. Then the following statements are equivalent:

i) There exists a representing measure for ω.

ii) There exists a representing measure for the RDIS $\gamma \equiv \{\gamma_{ij}\}_{i,j\ge 0}$, whose initial conditions and characteristic polynomial are $\{\gamma_{ij}\}_{0\le i\le j\le 2}$ and $P(z,\bar{z}) = z^3 + az + b\bar{z}$, respectively .

iii) $\Lambda_\omega(h) = \Lambda_\omega(zh) = 0$.

iv)

$$\begin{cases} \mathfrak{Re}\gamma_{12} &= b\mathfrak{Re}\gamma_{01},\\ \gamma_{22} &= 2b\mathfrak{Re}\gamma_{20} + (a+b)\gamma_{11}. \end{cases}$$

v) $h(z,\bar{z}) = (z+\bar{z})(z\bar{z}-b) \in \mathscr{P}_\gamma$, where γ is the RDIS defined in *ii)*.

vi) $(Z+\overline{Z})(Z\overline{Z}-b) = 0$.

By the same technique we treat the other cases, we find the following results.

$(b<0)$	N	Zeros	$h(z,\bar{z})$	Necessary and sufficient conditions	
$-b \le a$	3	$\pm i\sqrt{a-b}$ 0	$z+\bar{z}$	$M(2)\ge 0$ $\Lambda_\gamma(h)=0$ $\Lambda_\gamma(zh)=0$ $\Lambda_\gamma(z^2h)=0$	$M(2)\ge 0$ $\mathfrak{Im}\gamma_{01}=0$ $\gamma_{11}+\gamma_{02}=0$ $\gamma_{12}=a\gamma_{01}+b\gamma_{10}$
$\lvert a\rvert < -b$	5	$\pm i\sqrt{a-b}$ $\pm\sqrt{-a-b}$ 0	$z^2\bar{z}+a\bar{z}+bz$	$M(3)\ge 0$ $\Lambda_\gamma(h)=0$ $\Lambda_\gamma(zh)=0$ $\Lambda_\gamma(\bar{z}h)=0$	$M(3)\ge 0$ $\gamma_{21}+a\gamma_{01}+b\gamma_{10}=0$ $\gamma_{20}=\gamma_{02}$ $\gamma_{22}+a\gamma_{01}+b\gamma_{10}=0$
$a \le b$	3	$\pm\sqrt{-a-b}$ 0	$z-\bar{z}$	$M(2)\ge 0$ $\Lambda_\gamma(h)=0$ $\Lambda_\gamma(zh)=0$ $\Lambda_\gamma(z^2h)=0$	$M(2)\ge 0$ $\gamma_{01}=\gamma_{10}$ $\gamma_{02}=\gamma_{11}$ $a\gamma_{01}+b\gamma_{10}+\gamma_{12}=0$

Table 5.1: b negative

$(b>0)$	N	Zeros	$h(z,\bar{z})$	Necessary and sufficient conditions	
$2b \leq a$	3	$\pm i\sqrt{a-b}$ 0	$z+\bar{z}$	$M(2) \geq 0$ $\Lambda_\gamma(h)=0$ $\Lambda_\gamma(zh)=0$ $\Lambda_\gamma(z^2h)=0$	$M(2) \geq 0$ $\mathfrak{Im}\gamma_{01}=0$ $\gamma_{11}+\gamma_{02}=0$ $\gamma_{12}=a\gamma_{01}+b\gamma_{10}$
$b<a<2b$	7	$\pm i\sqrt{a-b}$; $\pm\frac{\sqrt{-a+2b}}{2}$ $\pm i\frac{\sqrt{a+2b}}{2}$; 0.	$(z+\bar{z})(z\bar{z}-u)$	$M(3) \geq 0$ $\Lambda_\gamma(h)=0$ $\Lambda_\gamma(zh)=0$	$M(3) \geq 0$ $\mathfrak{Re}\gamma_{12}=b\mathfrak{Re}\gamma_{01}$ $\gamma_{22}=2b\mathfrak{Re}\gamma_{20}+(a+b)\gamma_{11}$
$-b \leq a \leq b$	1	0			$\gamma_{00}>0$ and $\gamma_{ij}=0$ for all $0 \leq i \leq j \leq 2$
$b<-a<2b$	7	$\pm\sqrt{-a-b}$; $\pm\frac{\sqrt{-a+2b}}{2}$ $\pm i\frac{\sqrt{a+2b}}{2}$; 0.	$(z-\bar{z})(z\bar{z}-u)$	$M(3) \geq 0$ $\Lambda_\gamma(h)=0$ $\Lambda_\gamma(zh)=0$	$M(3) \geq 0$ $\mathfrak{Im}\gamma_{12}=b\mathfrak{Im}\gamma_{01}$ $\gamma_{22}+2b\mathfrak{Re}\gamma_{20}=(b-a)\gamma_{11}$
$a \leq -2b$	3	$\pm\sqrt{-a-b}$ 0	$z-\bar{z}$	$M(2) \geq 0$ $\Lambda_\gamma(h)=0$ $\Lambda_\gamma(zh)=0$ $\Lambda_\gamma(z^2h)=0$	$M(2) \geq 0$ $\gamma_{01}=\gamma_{10}$ $\gamma_{02}=\gamma_{11}$ $a\gamma_{01}+b\gamma_{10}+\gamma_{12}=0$

Table 5.2: b positive

5.4.2 The TCMP with cubic relation in $M(3)$ of the form $Z^3 = itZ + u\overline{Z}$

We end this section by considering another class of cubic column relations in truncated moment problems.

Set $w = e^{-i\frac{\pi}{4}}z$, the form $w^3 = itw + u\overline{w}$ became $z^3 + tz + u\bar{z} = 0$.

We take $t = a$ and $u = b$, then $w^3 = itw + u\overline{w}$ owns exactly 7 roots if and only if $u < |t| < 2u$, see Table (5.2). Hence if $u \leq 0$ then rank $M(3) < 7$ (more precisely rank $M(3) \leq 5$); as noted in the introduction this case is not interesting.

In view of Theorems 5.4.1 and 5.4.2 we deduce the solution of the TCMP for cubic column relations in $M(3)$ of the form $Z^3 = itZ + u\overline{Z}$, where $u < |t| < 2u$.

If $0 < u < t < 2u$; then

$$\begin{aligned}
&h(z,\bar{z}) = 0,\\
&(z+\bar{z})(z\bar{z}-u) = 0,\\
&(e^{\frac{\pi}{4}i}w + e^{-\frac{\pi}{4}i}\overline{w})(\overline{w}w-u) = 0,\\
&i(w-i\overline{w})(\overline{w}w-u) = 0.
\end{aligned}$$

If $0 < u < -t < 2u$; then

$$\begin{aligned}
&h(z,\bar{z}) = 0,\\
&(z-\bar{z})(z\bar{z}-u) = 0,\\
&(e^{\frac{\pi}{4}i}w - e^{-\frac{\pi}{4}i}\overline{w})(\overline{w}w-u) = 0,\\
&i(w+i\overline{w})(\overline{w}w-u) = 0.
\end{aligned}$$

Now we are able to state the main theorem in [20].

Theorem 5.4.3 Let $\omega \equiv \{\gamma_{ij}\}_{0\leq i+j\leq 6}$, with $\overline{\gamma}_{ij} = \gamma_{ji}$ and $\gamma_{00} > 0$, be a truncated complex sequence, and let $M(\omega)(3)$ be its associated moment matrix. If $M(\omega)(3) \geq 0$ and has cubic column relation of the form $Z^3 = itZ - u\overline{Z}$, with $t,u \in \mathbb{R}$ and $u < t < 2u$. Then the following statements are equivalent:

i) There exists a representing measure for ω.

ii) There exists a representing measure for the RDIS $\gamma \equiv \{\gamma_{ij}\}_{i,j\geq 0}$, whose initial conditions and characteristic polynomial are $\{\gamma_{ij}\}_{0\leq i\leq j\leq 2}$ and $P(z,\bar{z}) = z^3 - itz - u\bar{z}$, respectively .

iii)

$$\begin{cases} \Lambda_\omega(h) &= 0, \\ \Lambda_\omega(zh) &= 0. \end{cases}$$

iv)

$$\begin{cases} \mathfrak{Re}\gamma_{12} - \mathfrak{Im}\gamma_{12} &= u(\mathfrak{Re}\gamma_{01} - \mathfrak{Im}\gamma_{01}), \\ \gamma_{22} &= (t+u)\gamma_{11} - 2u\mathfrak{Im}\gamma_{02}. \end{cases}$$

v) $h(z,\bar{z}) = i(z - i\bar{z})(z\bar{z} - u) \in \mathscr{P}_\gamma$, where γ is the RDIS defined in *ii*).

vi) $Z^2\overline{Z} - iZ\overline{Z}^2 - uZ + iu\overline{Z} = 0$.

Theorem 5.4.4 Let $\omega \equiv \{\gamma_{ij}\}_{0\leq i+j\leq 6}$, with $\overline{\gamma}_{ij} = \gamma_{ji}$ and $\gamma_{00} > 0$, be a truncated complex sequence, and let $M(\omega)(3)$ be its associated moment matrix. If $M(\omega)(3) \geq 0$ and has cubic column relation of the form $Z^3 = itZ - u\overline{Z}$, with $t,u \in \mathbb{R}$ and $u < -t < 2u$. Then the following statements are equivalent:

i) There exists a representing measure for ω.

ii) There exists a representing measure for the RDIS $\gamma \equiv \{\gamma_{ij}\}_{i,j\geq 0}$, whose initial conditions and characteristic polynomial are $\{\gamma_{ij}\}_{0\leq i\leq j\leq 2}$ and $P(z,\bar{z}) = z^3 - itz - u\bar{z}$, respectively .

iii)

$$\begin{cases} \Lambda_\omega(h) &= 0, \\ \Lambda_\omega(zh) &= 0. \end{cases}$$

iv)

$$\begin{cases} \mathfrak{Re}\gamma_{12} + \mathfrak{Im}\gamma_{12} &= u(\mathfrak{Re}\gamma_{01} + \mathfrak{Im}\gamma_{01}), \\ \gamma_{22} &= (u-t)\gamma_{11} + 2u\mathfrak{Im}\gamma_{02}. \end{cases}$$

v) $h(z,\bar{z}) = i(z + i\bar{z})(z\bar{z} - u) \in \mathscr{P}_\gamma$, where γ is the RDIS defined in *ii*).

vi) $Z^2\overline{Z} + iZ\overline{Z}^2 - uZ - iu\overline{Z} = 0$.

5.5 Column Relations in Truncated Complex Moment Problems

In this section, we involve the RDIS to solve the TCMP associated with the truncated sequence $\gamma \equiv \{\gamma_{ij}\}_{0\leq i+j\leq 2k+2}$, with $\gamma_{ij} = \gamma_{ji}$ and $M(k+1)(\gamma)$ has a column relation of the form

$$Z^{k+1} = \sum_{0\leq n+m\leq k} a_{nm}\overline{Z}^n Z^m \qquad (a_{nm} \in \mathbb{C}, \text{ for all } n,m \in \mathbb{Z}_+ \text{ and } n+m \leq k). \tag{5.12}$$

According to (5.12), we have $\gamma_{i+k+1,j} = \sum_{0\leq n+m\leq k} a_{nm}\gamma_{n+i,m+j}$, for all $i,j \in \mathbb{Z}_+$ such that $i+j \leq k+1$. Hence γ is a subsequence of the RDIS $\widetilde{\gamma}$, defined by the initial conditions $\{\gamma_{ij}\}_{0\leq i\leq j\leq k}$ and by the characteristic polynomial $p(z,\bar{z}) = z^{k+1} - \sum_{0\leq n+m\leq k} a_{nm}\bar{z}^n z^m$.

We give now the main result of this section.

Theorem 5.5.1 Let $M(k+1)(\gamma)$ has a column dependence relations of the form $Z^{k+1} = \sum_{0\leq n+m\leq k} a_{nm}\overline{Z}^n Z^m$ and let $\widetilde{\gamma}$ be a RDIS defined as above. Then $M(k+1)(\gamma)$ admits a representing measure if and only if $M(2k)(\widetilde{\gamma})$ is positive semidefinite.

Proof. Suppose that $\gamma \equiv \{\gamma_{ij}\}_{0\leq i+j\leq 2k+2}$ is a moment sequence, then there exists a positive Borel measure μ verifies the relation

$$\gamma_{ij} = \int \bar{z}^i z^j d\mu \qquad (i+j\leq 2k+2).$$

Set $p(\bar{z},z) = z^{k+1} - \sum_{0\leq n+m\leq k} a_{nm}\bar{z}^n z^m$. Since $M(k+1)(\gamma)$ has a column dependence relations of the form $p(\overline{Z},Z)=0$, then

$$\int \bar{z}^i z^j p(\bar{z},z)d\mu = \gamma_{i+k+1,j} - \sum_{0\leq n+m\leq k} a_{nm}\gamma_{i+n,j+m} = 0,$$

for every $i,j\in\mathbb{Z}_+$ such that $i+j\leq k+1$. Hence

$$\begin{aligned}\int \mid p(\bar{z},z)\mid^2 d\mu &= \int \overline{p(\bar{z},z)}p(\bar{z},z)d\mu \\ &= \int \bar{z}^{k+1}p(\bar{z},z)d\mu - \sum_{0\leq n+m\leq k}\overline{a}_{nm}\int \bar{z}^n z^m p(\bar{z},z)d\mu \\ &= 0,\end{aligned}$$

thus $supp\mu \subset Z(p)$. It follows, from Proposition (5.1.5), that $\widetilde{\gamma}$ is a moment sequence, and obviously $M(2k)(\widetilde{\gamma})$ is positive semidefinite. Conversely, if $M(2k)(\widetilde{\gamma})\geq 0$, then Corollary (5.3.4) yields $\widetilde{\gamma}$ has a representing measure, and thus $\gamma \equiv \{\gamma_{ij}\}_{0\leq i+j\leq 2k+2}$ is a moment sequence. ■

On account of Theorem (5.5.1) we can formulate the following corollary, which proved a complete solution to the truncated moment problems with cubic column relations.

Corollary 5.5.2 Let $\gamma \equiv \{\gamma_{ij}\}_{0\leq i+j\leq 6}$ (with $\overline{\gamma_{ij}} = \gamma_{ji}$) be a complex numbers, let $M(3)(\gamma)$ admits a cubic column relations of the form $Z^3 = \sum_{0\leq i+j\leq 2} a_{ij}\overline{Z}^i Z^j$ and let $\widetilde{\gamma}$ be the RDIS defined by the initial condition $\{\gamma_{ij}\}_{0\leq i,j\leq 2}$ and by the characteristic polynomial $z^3 - \sum_{0\leq i+j\leq 2} a_{ij}\bar{z}^i z^j$. Then γ admits a representing measure if and only if $M(4)(\widetilde{\gamma})\geq 0$.

CHAPTER 6

THE GENERAL TRUNCATED COMPLEX MOMENT PROBLEM

Let $\gamma \equiv \{\gamma_{ij}\}_{(i,j)\in I}$, with $I \subseteq \mathbb{Z}_+ \times \mathbb{Z}_+$, be a given finite complex-valued sequence. The truncated complex moment problem, TCMP in short, (resp., the general TCMP) consists in determining necessary and sufficient conditions for the existence of a positive Borel measure (resp., a signed measure) μ on $\mathbb{C}$ such that $\gamma_{ij} = \int_{\mathbb{C}} \bar{z}^i z^j d\mu$ for $(i,j) \in I$ (see [34]).

In this chapter, we involve the necessary and sufficient conditions for the TCMP, obtained in the preceding chapter, to show that the general TCMP always have a solution. A concrete construction of the solution and an illustrating example are also given.

6.1 Introduction

The general truncated (one and two dimensional) moment problem arises in many-body quantum physics (see, for instance, [5, 30, 29, 60, 61, 62]). In [5], the (one-real variable) general truncated moment problem for a weighted r-generalized Fibonacci sequence, see 2.5.1, is solved.

Theorem 6.1.1 — (5, Theorem 4.1). Let $S = (s_k)_{k\geq 0}$ be an r-Generalized Fibonacci Sequences. Then S admits a generating measure μ if and only if P_S has distinct real roots. Moreover, $Supp(\mu) = Z(P_S)$.

As a consequence of Theorem 6.1.1, an explicit solution to the one-real variable version can formulate as follows.

Corollary 6.1.2 For any given truncated sequence $(s_k)_{0\leq k\leq r-1}$, there exists a solution of the associated truncated general momet problem.

We realized that (like the, classical, truncated moment problem) there are difficulties with the generalization, in particularly for the two-dimensional case. Our approach for solving the TCMP, given in the precedent chapter, allowed us to overcome this difficulty and provide the desired generalization, see Theorem 6.2.3.

6.2 Solving the General Truncated Complex Moment Problem

The main purpose of the general moment problem is to drop the requirement of positivity of the measure solution of the moment problem. In other words, for $\{\gamma_{ij}\}_{(i,j)\in I}$, with $I \subseteq \mathbb{Z}_+ \times \mathbb{Z}_+$, solving the general moment problem consists in finding a signed measure μ, such that

$$\int \bar{z}^i z^j d\mu = \gamma_{ij}, \qquad \text{for all } (i,j) \in I.$$

To this end, let us consider a RSFT $\gamma \equiv \{\gamma_{ij}\}_{i,j\in\mathbb{Z}_+}$, see Section 5.2, and let $M(\gamma)$ be the associated moment matrix. We need the following proposition.

Proposition 6.2.1 Let $\gamma \equiv \{\gamma_{ij}\}_{i,j\in\mathbb{Z}}$ be a bi-sequence of complex number such that $\overline{\gamma_{ij}} = \gamma_{ji}$. Then γ is a DRSF if and only if the associated (infinite) moment matrix $M(\gamma)$ has a finite rank.

Proof. Let $\gamma \equiv \{\gamma_{ij}\}_{i,j\in\mathbb{Z}}$ be a sequence of complex numbers obeying (5.9), then every column $\overline{Z}^i Z^j$, with i or $j \geq r+1$, in $M(\gamma)$ is a linear combination of columns indexed by monomial of total degree less strictly than $\deg \overline{Z}^i Z^j$. Indeed, (5.9) yields

$$< \overline{z}^l z^k, M(\gamma)\overline{z}^i z^{n+1} >= \sum_{e=0}^{r} a_e < \overline{z}^l z^k, M(\gamma)\overline{z}^i z^{n-e} > \qquad (l,k\in\mathbb{Z}_+).$$

Also, by applying (5.10), we obtain

$$< \overline{z}^l z^k, M(\gamma)\overline{z}^{n+1} z^i >= \sum_{e=0}^{r} \overline{a}_e < \overline{z}^l z^k, M(\gamma)\overline{z}^{n-e} z^i > \qquad (l,k\in\mathbb{Z}_+).$$

Therefore, any column of $M(\gamma)$ can be expressed as a linear combination of columns indexed by $\{\overline{Z}^i Z^j$ with $i,j\leq r\}$, and thus $M(\gamma)$ has a finite rank.

Conversely, if $M(\gamma)$ has a finite rank, then there exists an integer r such that the column, of $M(\gamma)$, indexed by Z^{r+1} is a linear combination of columns indexed by $1, Z, \ldots, Z^r$. Then

$$< \overline{z}^i z^{n-r}, M(\gamma)\overline{z}^{r+1} >= \sum_{e=0}^{r} a_e < \overline{z}^i z^{n-r}, M(\gamma)\overline{z}^{r-e} >, \qquad i, n-r\in\mathbb{Z}_+,$$

for some $a_0,\ldots,a_r\in\mathbb{C}$. That is,

$$\gamma_{i,n+1} = a_0\gamma_{i,n} + a_1\gamma_{i,n-1} + \ldots + a_r\gamma_{i,n-r}, \qquad \text{for all } i,n\in\mathbb{Z}_+ \text{ with } n\geq r,$$

and this concludes the proof. ■

According to Proposition 6.2.1, the matrix $M(\gamma)$ has a finite rank. Therefore $M(\gamma)$ has a finite orthonormal set of eigenvectors $v_1,\ldots,v_r$ corresponding to the eigenvalues $\lambda_1,\ldots,\lambda_r$.

Setting

$$\begin{aligned} \gamma^+_{i+l,j+k} &:= \sum_{all\ \lambda_n>0} < e_{ij} \mid v_n > \lambda_n < v_n \mid e_{k,l} >, \\ \gamma^-_{i+l,j+k} &:= \sum_{all\ \lambda_n<0} < e_{ij} \mid v_n > |\lambda_n| < v_n \mid e_{k,l} >; \end{aligned} \tag{6.1}$$

where $\{e_{ij}\}_{i,j\in\mathbb{Z}_+}$ denotes the (infinite) canonical basis associated with the columns of $M(\gamma)$, with respect to the lexicographic ordering of $\mathbb{C}[\overline{Z},Z]$ (in the same way as the truncated version, see Subsection 2.1.3), that is, e_{ij} is the vector with 1 in the $\overline{Z}^i Z^j$ entry and is 0 all other positions.

Let $M(\gamma^+)$ and $M(\gamma^-)$ denote the moment matrices associated with $\{\gamma^+_{i,j}\}_{i,j\in\mathbb{Z}_+}$ and $\{\gamma^-_{i,j}\}_{i,j\in\mathbb{Z}_+}$, respectively. It is clear that

(i) $\overline{\gamma}^+_{j,i} = \gamma^+_{i,j}$ and $\overline{\gamma}^-_{j,i} = \gamma^-_{i,j}$ $\quad (i,j\in\mathbb{Z}_+)$,
(ii) $\gamma_{i,j} = \gamma^+_{i,j} - \gamma^-_{i,j}$ $\quad (i,j\in\mathbb{Z}_+)$,
(iii) $M(\gamma) = M(\gamma^+) - M(\gamma^-)$,
(iv) $M(\gamma^+)$ and $M(\gamma^-)$ are positives,
(v) $rankM(\gamma) = rankM(\gamma^+) + rankM(\gamma^-)$.

Since γ is a RSFT, then, by virtue of Proposition 6.2.1, the matrix $M(\gamma)$ has a finite rank. Thus, the equality (v) yields that the matrices $M(\gamma^+)$ and $M(\gamma^-)$ have also finite rank. Hence, from Proposition 6.2.1 and (i), the sequences γ^+ and γ^- are RSFT. Therefore, by applying Theorem 5.2.4 and taking into account (iv) , the sequences $\{\gamma^+_{i,j}\}_{i,j\in\mathbb{Z}_+}$ and $\{\gamma^-_{i,j}\}_{i,j\in\mathbb{Z}_+}$ owns a representing measure μ^+ and μ^-, respectively.

It follows that the measure $\mu := \mu^+ - \mu^-$ give a positive answer to the general moment problem associated with γ. Indeed,

$$\gamma_{ij} = \gamma^+_{i,j} - \gamma^-_{i,j} = \int \overline{z}^i z^j d\mu^+ - \int \overline{z}^i z^j d\mu^+ = \int \overline{z}^i z^j d\mu.$$

Theorem 6.2.2 For every given RSFT $\gamma \equiv \{\gamma_{ij}\}_{i,j\in\mathbb{Z}_+}$, there exists a solution of the associated general complex moment problem.

Since every finite data $\{\gamma_{ij}\}_{(i,j)\in I}$ can be regarded as a subset of some RSFT $\gamma \equiv \{\gamma_{ij}\}_{i,j\in\mathbb{Z}_+}$ (i. e., $\{\gamma_{ij}\}_{(i,j)\in I} \subset \gamma$), then, in the light of Theorem 6.2.2, one can solve the truncated general complex moment problem.

Theorem 6.2.3 For any given finite data $\{\gamma_{ij}\}_{(i,j)\in I}$, with $\overline{\gamma_{ij}} = \gamma_{ji}$, there exists a solution of the associated truncated complex moment problem.

Example 6.2.1 Let $\gamma_{00} = \gamma_{11} = 1$, $\gamma_{01} = \overline{\gamma}_{10} = i$ and $\gamma_{02} = \overline{\gamma}_{20} = -2i$. Hence the moment matrix associated with the sequence $\{\gamma_{ij}\}_{i+j\leq 2}$ is

$$M(\gamma)(1) = \begin{pmatrix} 1 & i & -i \\ -i & 1 & 2i \\ i & -2i & 1 \end{pmatrix}, \text{ remark that } rankM(\gamma)(1) = 3.$$

Thus

$$M(\gamma^-)(1) = \begin{pmatrix} \frac{\sqrt{6}-1}{6} & \frac{1-\sqrt{6}}{6} + i\frac{\sqrt{6}-6}{12} & \frac{1-\sqrt{6}}{6} - i\frac{\sqrt{6}-6}{12} \\ \frac{1-\sqrt{6}}{6} - i\frac{\sqrt{6}-6}{12} & \frac{5\sqrt{6}-5}{12} & \frac{1-\sqrt{6}}{12} - i\frac{6-\sqrt{6}}{6} \\ \frac{1-\sqrt{6}}{6} + i\frac{\sqrt{6}-6}{12} & \frac{1-\sqrt{6}}{12} + i\frac{6-\sqrt{6}}{6} & \frac{5\sqrt{6}-5}{12} \end{pmatrix}$$

and

$$M(\gamma^+)(1) = \begin{pmatrix} \frac{5+\sqrt{6}}{6} & \frac{1-\sqrt{6}}{6} + i\frac{6+\sqrt{6}}{12} & \frac{1-\sqrt{6}}{6} - i\frac{6+\sqrt{6}}{12} \\ \frac{1-\sqrt{6}}{6} - i\frac{6+\sqrt{6}}{12} & \frac{7+5\sqrt{6}}{12} & \frac{1-\sqrt{6}}{12} + i\frac{6+\sqrt{6}}{6} \\ \frac{1-\sqrt{6}}{6} + i\frac{6+\sqrt{6}}{12} & \frac{1-\sqrt{6}}{12} - i\frac{6+\sqrt{6}}{6} & \frac{7+5\sqrt{6}}{12} \end{pmatrix}.$$

Since $M(\gamma^-)(1) \geq 0$ and $rankM(\gamma^-)(1) = 1$, then γ^- has a unique representing measure $\mu^- = \frac{5+\sqrt{6}}{6} d\delta_{-1-i\frac{\sqrt{6}}{2}}$, see Proposition 6.2 in [9].

Since $M(\gamma^+)(1) \geq 0$ and $rankM(\gamma^+)(1) = 2$, then $\mu^+ = \rho_0\delta_{z_0} + \rho_1\delta_{z_1}$ (where $z_0 = \frac{1}{2}$, $z_1 = \frac{\gamma^+_{02} - z_0\gamma^+_{01}}{\gamma^+_{01} - z_0\gamma^+_{00}}$, $\rho_1 = \frac{\gamma^+_{00}z_0 - \gamma^+_{01}}{z_0 - z_1}$ and $\rho_0 = \gamma_{00} - \rho_1$) is a representing measure associated with γ^+ , see Proposition 6.3 in [9].

Hence the signed measure $\mu = \mu^+ - \mu^-$ is a solution to the general truncated complex moment problem associated with $\gamma \equiv \{\gamma_{ij}\}_{0\leq i+j\leq 2}$.

CHAPTER 7

MULTIVARIABLE MOMENT PROBLEMS AND RECURSIVE RELATIONS

In view of its fundamental importance in various field of mathematics and applied science, the results of K-moment problem have interesting application; for instant, the Curto-fialkow's results ([9, Theorems 4.7] and [12, Theorems 1.6]) have been crucial in the Lasserre's method for minimizing a polynomial over a semialgebraic set (which is NP-hard in general), see for instance [42, 41, 43]. This wide area of applications motivated Curto an Fialkow to give a generalization in several variable (see Theorems 3.4.13 and 3.4.14).

An alternative proof, of Theorems 3.4.13 and 3.4.14, was provided by M. Laurent in [44], based on Hilbert's Nullstellensatz (see the Subsection 2.2.3) instead of the functional analytic tools used in the original proof of Curto and Fialkow. Also, F.-H. Vasilescu was introduced in [58, 59] an approach, based on the dimensional stability, equivalent to the concept of flatness introduced by R. Curto and L. Fialkow.

In the present chapter we introduce an approach for the multivariable K-truncated moment problem, based on the recursiveness, not only for simplify and shortcut Theorems 3.4.13 and 3.4.14, but also to provide a new methodology for solving the moment problem in several variable (see also [33]).

This paper is organized as follows. In Section 7.1, we introduce the recursively generated multisequence and we exhibit useful results. We devote Section 7.2 to prove Theorems 3.4.13 and 3.4.14. In the last section, we provide an application to the subnormal completion problem in several variable.

7.1 On the Recursively Generated Multisequences and Moment Matrix

In this section, we state two necessary conditions for the existence of a finitely atomic K-representing measure. We will prove that these conditions are, also, sufficient for the existence of a, unique minimal, representing measure.

7.1.1 Two Necessary Conditions

The first necessary condition is established by studying the matrix positivity. To this end, let us assume that $\beta \equiv \{\beta_{\mathbf{i}}\}_{\mathbf{i}\in\mathbb{Z}_+^d, |\mathbf{i}|\leq 2n}$ is a moment sequence, that is, there exists a positive Borel measure μ supported on $\mathbb{R}^d$ such that

$$\beta_{\mathbf{i}} = \int \mathbf{x}^{\mathbf{i}} d\mu, \qquad \mathbf{i}\in\mathbb{Z}_+^d.$$

Then given $p(\mathbf{x}) = \sum\limits_{|\mathbf{i}|\leq n} a_{\mathbf{i}}\mathbf{x}^{\mathbf{i}} \in \mathbb{R}_n[x_1,\ldots,x_d]$, we have

$$0 \leq \int p^2(\mathbf{x})d\mu = \int \sum_{\mathbf{i},\mathbf{j}} a_{\mathbf{i}}a_{\mathbf{j}}\mathbf{x}^{\mathbf{i}+\mathbf{j}}d\mu = \sum_{\mathbf{i},\mathbf{j}} a_{\mathbf{i}}a_{\mathbf{j}}\int \mathbf{x}^{\mathbf{i}+\mathbf{j}}d\mu = \sum_{\mathbf{i},\mathbf{j}} a_{\mathbf{i}}a_{\mathbf{j}}\beta_{\mathbf{i}+\mathbf{j}},$$

and thus the moment matrix $M(n)(\beta)$ is positive semi-definite.

Moreover, if μ is supported on the closed semialgebraic set $F = \{\mathbf{x}\in\mathbb{R}^d \mid h_1(\mathbf{x})\geq 0,\ldots,h_m(\mathbf{x})\geq 0\}$, where $h_j\in\mathbb{R}[x_1,\ldots,x_d]$, then, for all $l=0,\ldots,m$,

$$
\begin{aligned}
0 \leq \int p^2(\mathbf{x}) h_l(\mathbf{x}) d\mu &= \sum_{\mathbf{i},\mathbf{j}} a_{\mathbf{i}} a_{\mathbf{j}} \int \mathbf{x}^{\mathbf{i}+\mathbf{j}} h_l(\mathbf{x}) d\mu \\
&= \sum_{\mathbf{i},\mathbf{j}} a_{\mathbf{i}} a_{\mathbf{j}} \int \sum_{\alpha} (h_l)_\alpha \mathbf{x}^{\mathbf{i}+\mathbf{j}+\alpha} d\mu \\
&= \sum_{\mathbf{i},\mathbf{j}} a_{\mathbf{i}} a_{\mathbf{j}} (\sum_{\alpha} (h_l)_\alpha \beta_{\mathbf{i}+\mathbf{j}+\alpha}) \\
&= \sum_{\mathbf{i},\mathbf{j}} a_{\mathbf{i}} a_{\mathbf{j}} (h_l * \beta)_{\mathbf{i}+\mathbf{j}}.
\end{aligned}
$$

It follows that, if $\beta \equiv \{\beta_{\mathbf{i}}\}_{\mathbf{i}\in\mathbb{Z}_+^d, |\mathbf{i}|\leq 2n}$ admits a representing measure supported on F, then the localizing matrices $M_{h_1}(n+[\frac{\deg h_1+1}{2}]), \ldots, M_{h_m}(n+[\frac{\deg h_m+1}{2}])$ and the moment matrix $M(n)$ are semidefinite positive.

The second necessary condition gives rise to recurrence relations. Indeed, suppose that $\beta \equiv \{\beta_{\mathbf{i}}\}_{\mathbf{i}\in\mathbb{Z}_+^d, |\mathbf{i}|\leq 2n}$ is a moment sequence, a result of C. Bayer and J. Teichmann (Theorem 3.5.5) states that every finite moment sequence admits a finite atomic representing measure, then β has a finite representing measure, say μ. Hence, for every polynomial $Q(\mathbf{x}) = \sum\limits_{\mathbf{n}} a_{\mathbf{n}} \mathbf{x}^{\mathbf{n}} \in \mathbb{R}[x_1, \ldots, x_d]$ with $Q_{|supp\mu} = 0$, we have

$$
0 = \int \mathbf{x}^{\mathbf{i}} Q(\mathbf{x}) d\mu = \sum_{\mathbf{n}} a_{\mathbf{n}} \beta_{\mathbf{x}^{\mathbf{n}+\mathbf{i}}}, \qquad \text{for all } \mathbf{i} \in \mathbb{Z}_+^d.
$$

Assume that $supp\ \ \mu \subseteq \times_{l=1}^d \{\lambda_{l,1}, \ldots, \lambda_{l,m_l+1}\}$, then the polynomials

$$
p_l(x) = \prod_{i=1}^{m_l+1} (x_l - \lambda_{l,i}) = x_l^{m_l+1} - a_0^{(l)} x_l^{m_l} - a_1^{(l)} x_l^{m_l-1} - \ldots - a_{m_l}^{(l)} \qquad (l = 0, \ldots, d)
$$

vanish on $supp\mu$, thus

$$
\beta_{(m_l+1)\varepsilon_l+\mathbf{i}} = a_0^{(l)} \beta_{m_l\varepsilon_l+\mathbf{i}} + a_1^{(l)} \beta_{(m_l-1)\varepsilon_l+\mathbf{i}} + \ldots + a_{m_l}^{(l)} \beta_{\mathbf{i}}, \quad \text{for all } \mathbf{i} \in \mathbb{Z}_+^d, \tag{7.1}
$$

where ε_l is the d-tuple with 1 in the l-th place and zero elsewhere. It follows that every truncated moment sequence can be regarded as initial data of an infinite moment sequence defined by a d recurrence relations, as in (7.1).

It results that the recursiveness is inherent in the truncated moment problem. This is our main motivation in the study of the recursive multisequences in connection with the moment problem.

7.1.2 Recursively Generated Multisequences

Let $\{a_j^{(l)}\}_{1\leq l\leq d, 0\leq j\leq m_l}$ be some fixed real numbers and let $\beta \equiv \{\beta_{\mathbf{i}}\}_{\mathbf{i}\in\mathbb{Z}_d^+}$ be a real multisequence defined by the following recurrence relations:

$$
\beta_{(m_l+1)\varepsilon_l+\mathbf{i}} = a_0^{(l)} \beta_{m_l\varepsilon_l+\mathbf{i}} + a_1^{(l)} \beta_{(m_l-1)\varepsilon_l+\mathbf{i}} + \ldots + a_{m_l}^{(l)} \beta_{\mathbf{i}} \qquad (1 \leq l \leq d), \tag{7.2}
$$

where $\omega = \{\beta_{\mathbf{i}}\}_{\mathbf{i}\in\times_{l=1}^d\{0,\ldots,m_l\}}$ are given initial conditions. In the sequel, we shall refer to such sequence as recursively generated multisequence associated with the d-tuple of characteristic polynomials $\equiv (p_1, \ldots, p_d)$, where

$$
p_l(x) = x^{m_l+1} - a_0^{(l)} x^{m_l} - a_1^{(l)} x^{m_l-1} - \ldots - a_l^{(l)} \in \mathbb{R}[x], \quad l = 0, \ldots, d.
$$

Observe that the characteristic polynomials $(p_1, \ldots, p_d)$, together with the initial conditions ω, are said to define the sequence β.

A recursively generated multisequence can be defined in various ways using different characteristic polynomials as is shown in the following example. Let $\{\beta_{(n,m,v)}\}_{(n,m,v)\in\mathbb{Z}_+^3}$ with $\beta_{(n,m,v)} = 5^m a^n (2^v - 1)$, where a is a nonzero real number. Then $p_\beta = (x-a, x^2+ax-5x-5a, x^2-x-2)$ and $p'_\beta = (x^2-a^2, x-5, x^3-2x^2-x+2)$ are both characteristic polynomials of β.

Let $\mathscr{P}_\beta$ denote the set of characteristic polynomials associated with β.

Remark 7.1.1 For every $p_\beta \equiv (p_1,\dots,p_d) \in \mathscr{P}_\beta$ and for every $Q_1(x),\dots,Q_d(x) \in \mathbb{R}[X]$, we have $(p_1Q_1,\dots,p_dQ_d) \in \mathscr{P}_\beta$.

For reason of simplicity, we identify a polynomial $p \equiv \sum_{|\mathbf{i}|\le n} a_{\mathbf{i}}\mathbf{x}^{\mathbf{i}}$ with its coefficient vector $p=(a_{\mathbf{i}})$ with respect to the basis of monomials of $\mathbb{R}_n[x_1,\dots,x_n]$ in degree-lexicographic order. Clearly, for every polynomials $p \equiv \sum_{\mathbf{i}} a_{\mathbf{i}}\mathbf{x}^{\mathbf{i}}, q \equiv \sum_{\mathbf{j}} b_{\mathbf{j}}\mathbf{x}^{\mathbf{j}} \in \mathbb{R}_n[x_1,\dots,x_n]$, we have $p^TM(\beta)q = \sum_{\mathbf{i},\mathbf{j}} a_{\mathbf{i}}b_{\mathbf{j}}\beta_{\mathbf{i}+\mathbf{j}}$.

The next lemma is an immediate consequence of (7.2).

Lemma 7.1.1 Let $\beta \equiv \{\beta_{\mathbf{i}}\}_{\mathbf{i}\in\mathbb{Z}_d^+}$ be a recursively generated multisequence and let $M(\beta)$ be the associated moment matrix. Then $(p_1,\dots,p_d)$ is a characteristic polynomials of β if and only if $M(\beta)p_l = 0$ (for all $l = 0,1,\dots,d$).

As observed in the Proposition 2.5.2 the singly indexed case, among all characteristic polynomials defining S, there exists a unique monic characteristic polynomial p_S of minimal degree, called the minimal characteristic polynomial, and which divides every characteristic polynomial. The next proposition gives a generalization of this result.

Proposition 7.1.2 For every recursively generated multisequence $\beta \equiv \{\beta_{\mathbf{i}}\}_{\mathbf{i}\in\mathbb{Z}_d^+}$ given by (7.2), there exists unique monic characteristic polynomials $p_\beta = (p_1^{(\beta)},\dots,p_d^{(\beta)}) \in \mathscr{P}_\beta$ with minimal degree. Moreover, for all $(Q_1,\dots,Q_d) \in \mathscr{P}_\beta$, Q_l is a multiple of p_l whenever $l \in \{1,\dots,d\}$.

Proof. Given $l \in \{1,\dots,d\}$ and let $\mathbf{I}_l = (i_1,\dots,i_{l-1},i_{l+1},\dots,i_d) \in \mathbb{Z}_+^{d-1}$ be a fixed $(d-1)$-tuple, we have

$$\beta_{(i_1,\dots,i_{l-1},m_l+1+i_l,i_{l+1},\dots,i_d)} = a_0^{(l)}\beta_{(i_1,\dots,i_{l-1},m_l+i_l,i_{l+1},\dots,i_d)} + \dots + a_{m_l}^{(l)}\beta_{(i_1,\dots,i_l,\dots,i_d)},$$

for every $i_l \in \mathbb{Z}_+^d$.

Hence $p_l(x) = x^{m_l+1} - a_0^{(l)}x^{m_l} - \dots - a_{m_l}^{(l)}$ is a characteristic polynomial associated with the general Fibonacci sequence $\beta^{(l)} : i_l \to \beta(i_1,\dots,i_l,\dots,i_d)$. Thus there exists a minimal characteristic polynomial $p_{\beta,\mathbf{I}_l}$, associated with $\beta^{(l)}$, divides p_l. As $\mathbf{I}_l$ is arbitrarily, then the polynomial $p_l^{(\beta)} = \bigwedge_{\mathbf{I}_l\in\mathbb{Z}_+^{d-1}} p_{\beta,\mathbf{I}_l}$, the smallest common multiple of all $p_{\beta,\mathbf{I}_l}$, provides a positive answer to the proposition. ■

In the remainder of this paper, we will associate with every recursively generated multisequence β its d-tuple of minimal characteristic polynomials, that we denote p_β.

Proposition 7.1.3 Let $\beta \equiv \{\beta_{\mathbf{i}}\}_{\mathbf{i}\in\mathbb{Z}_d^+}$ be a recursively generated multisequence, associated with the characteristic polynomials $(p_1,\dots,p_d)$. If $M(\beta) \ge 0$, then, for every $l \in \{1,\dots,d\}$, the polynomial $p_l(x)$ has distinct roots.

To prove Proposition 7.1.3, we need the following two lemmas of independent interest

Lemma 7.1.4 Under the notations above, for every $f,g,h \in \mathbb{R}[x_1,\dots,x_d]$, we have

$$f^TM(\beta)(gh) = (fg)^TM(\beta)h. \tag{7.3}$$

Proof. Let $f,g,h \in \mathbb{R}[x_1,\dots,x_d]$ be polynomials. We write $f = \sum_{\mathbf{i}} f_{\mathbf{i}}\mathbf{x}^{\mathbf{i}}$, $g = \sum_{\mathbf{j}} g_{\mathbf{j}}\mathbf{x}^{\mathbf{j}}$ and $h = \sum_{\mathbf{k}} h_{\mathbf{k}}\mathbf{x}^{\mathbf{k}}$. As the entry of the moment matrix corresponding to the column $\mathbf{x}^{\mathbf{i}}$ and the line $\mathbf{x}^{\mathbf{j}}$ is $\beta_{\mathbf{i}+\mathbf{j}}$, we obtain

$$\begin{aligned} f^TM(\beta)(gh) &= (\sum_{\mathbf{i}} f_{\mathbf{i}}\mathbf{x}^{\mathbf{i}})^TM(\beta)(\sum_{\mathbf{j},\mathbf{k}} g_{\mathbf{j}}h_{\mathbf{k}}\mathbf{x}^{\mathbf{j}+\mathbf{k}}) \\ &= \sum_{\mathbf{i},\mathbf{j},\mathbf{k}} f_{\mathbf{i}}g_{\mathbf{j}}h_{\mathbf{k}}\beta_{\mathbf{i}+\mathbf{j}+\mathbf{k}} \\ &= (fg)^TM(\beta)h. \end{aligned}$$

■

Lemma 7.1.5 For every polynomial $p \in \mathbb{R}[x_1,\ldots,x_d]$ and any integer $n \geq 1$, we have

$$M(\beta)p^n = 0 \Longrightarrow M(\beta)p = 0. \tag{7.4}$$

Proof. If $M(\beta)p^2 = 0$, then $0 = 1^T M(\beta)p^2 = p^T M(\beta)p$, from Lemma 7.1.4; since $M(\beta) \geq 0$, we obtain $M(\beta)p = 0$ and hence (7.4) holds for $n = 2$. By induction, (7.4) remains valid for any power of 2. Now , if $M(\beta)p^n = 0$ we choose r in such a way that $r+k$ is a power of 2, hence

$$M(\beta)p^{n+r} = 0.$$

Which gives $M(\beta)p = 0$. ■

Proof of Proposition 7.1.3. Let $p_l(x) = \prod_{i=0}^{m_l-1}(x-\lambda_{l,i})^{n_{l,i}}$ and let $M_l = \max_{i=0}^{m_l-1} n_{l,i}$. Applying Lemma 7.1.1 we obtain $M(\beta)\prod_{i=0}^{m_l-1}(x-\lambda_{l,i})^{n_{l,i}} = 0$, we derive that $M(\beta)(\prod_{i=0}^{m_l-1}(x-\lambda_{l,i}))^{M_l} = 0$, and hence Lemma 7.1.5 yields that $M(\beta)\prod_{i=0}^{m_l-1}(x-\lambda_{l,i}) = 0$. It follows, by Lemma 7.1.1, that $\prod_{i=0}^{m_l-1}(x-\lambda_{l,i})$ is a characteristic polynomial, of β, dividing $p_l(x)$. Since $p_l(x)$ is minimal, then $p_l(x) = \prod_{i=0}^{m_l-1}(x-\lambda_{l,i})$, as desired. ■

7.2 Main Results

We present a characterization of moment sequences involving the recursively generated multisequence and the moment matrix, that leads to new proofs of Theorem 3.4.13 and Theorem 3.4.14.

Theorem 7.2.1 Let $\beta \equiv \{\beta_{\mathbf{i}}\}_{\mathbf{i}\in\mathbb{Z}_+^d}$ be a multisequence of real numbers and let $M(\beta)$ and $M(n)$ be the moment matrices associated with β and $\{\beta_{\mathbf{i}}\}_{|\mathbf{i}|\leq 2n,\mathbf{i}\in\mathbb{Z}_+^d}$, respectively. The following are equivalent:

1. $M(\beta)$ is a finite-rank positive semidefinite matrix.
2. β is recursively generated, associated with the minimal characteristic polynomials $(p_1,\ldots,p_d)$ and $M(\tau) \geq 0$, with $\tau = \sum_{i=1}^{d}(\deg p_i - 1)$.
3. β has a unique representing measure, which is *rank*$M(\beta)$-atomic.

For the proof of Theorem 7.2.1 we will need the following auxiliary result which can be regarded as the Binet Formula (see Subsection 2.5.2) for the recursively generated multisequences.

Lemma 7.2.2 Let $\beta \equiv \{\beta_{(i_1,\ldots,i_d)}\}_{(i_1,\ldots,i_d)\in\mathbb{Z}_+^d}$ be a real recursively generated multisequence associated with the characteristic polynomials $(p_1,\ldots,p_d)$, where $p_j(x) = \prod_{l=0}^{m_j-1}(x-\lambda_{j,l})$. Then, there exists $c_{(l_1,\ldots,l_d)}$ real numbers, determined by the initial conditions $\{\beta_{(i_1,\ldots,i_d)}\}_{i_j\in\{0,\ldots,m_j-1\}}$ such that

$$\beta_{(i_1,\ldots,i_d)} = \sum_{l_1=0}^{m_1-1}\cdots\sum_{l_d=0}^{m_d-1} c_{(l_1,\ldots,l_d)}\lambda_{1,l_1}^{i_1}\ldots\lambda_{d,l_d}^{i_d}. \tag{7.5}$$

Proof. For $(i_2,\ldots,i_d) \in \mathbb{Z}_+^{d-1}$ given, the singly sequence $i_1 \to \beta_{(i_1,\ldots,i_d)}$ is a general Fibonacci sequence associated with the characteristic polynomial $p_1(x) = \prod_{l=0}^{m_1-1}(x-\lambda_{1,l})$. Then the Binet formula implies that

$$\beta_{(i_1,\ldots,i_d)} = \sum_{l_1=0}^{m_1-1} c_{l_1}^{(i_2,\ldots,i_d)}\lambda_{1,l_1}^{i_1}, \tag{7.6}$$

where the $c_{l_1}^{(i_2,\ldots,i_d)}$ are determined by the initial condition $\{\beta_{(i_1,\ldots,i_d)}\}_{0\leq i_1\leq m_1-1}$. We claim that, for every integer $l_1 \geq 0$, the single sequence $i_2 \to c_{l_1}^{(i_1,\ldots,i_d)}$ is a general Fibonacci sequence associated with the characteristic polynomial $p_2(x)$.

Indeed, since $i_2 \to \beta_{(i_1,i_2,\ldots,i_d)}$ is a Fibonacci sequence associated with $p_2(x)$, we obtain

$$\beta_{(i_1,i_2+m_2,i_3,\ldots,i_d)} - a_1^{(2)}\beta_{(i_1,i_2+m_2-1,i_3,\ldots,i_d)} - \ldots - a_{m_2}^{(2)}\beta_{(i_1,i_2,\ldots,i_d)} = 0.$$

It follows from (7.6),

$$\sum_{l_1=0}^{m_1-1} p_2(c_{l_1}^{(i_2,\ldots,i_d)})\lambda_{1,l_1}^{i_1} = 0$$

where $p_2(c_{l_1}^{(i_2,\ldots,i_d)}) = c_{l_1}^{(i_2+m_2,i_3,\ldots,i_d)} - a_1^{(2)}c_{l_1}^{(i_2+m_2-1,i_3,\ldots,i_d)} - \ldots - a_{m_2}^{(2)}c_{l_1}^{(i_2,i_3,\ldots,i_d)}$. By induction of i_1 over $0,1,\ldots,m_1-2$ and m_1-1, we derive the following Vandermonde system (see the Subsection 2.4.2)

$$\begin{cases} p_2(c_0^{(i_2,\ldots,i_d)})\lambda_{1,0}^{0} & +\ldots+p_2(c_{m_1-1}^{(i_2,\ldots,i_d)})\lambda_{1,m_1-1}^{0} & =0\\ \vdots & \vdots & \vdots\\ p_2(c_0^{(i_2,\ldots,i_d)})\lambda_{1,0}^{m_1-1} & +\ldots+p_2(c_{m_1-1}^{(i_2,\ldots,i_d)})\lambda_{1,m_1-1}^{m_1-1} & =0. \end{cases}$$

Since $\{\lambda_{1,l_1}\}_{l_1=0}^{m_1-1}$ are distinct, then the unique solution is zero,

$$p_2(c_{l_1}^{(i_2,\ldots,i_d)}) = 0, \qquad l_1 = 0,\ldots,m_2-1.$$

As the integer i_2 is arbitrary, then , for any $l_1 = 0,\ldots,m_1-1$, the sequence $i_2 \to c_{l_1}^{(i_2,\ldots,i_d)}$ is a general Fibonacci sequence associated with $p_2(x)$. Similarly, one can show that the singly indexed sequence $i_j \to c_{l_1}^{(i_2,\ldots,i_j,\ldots,i_d)}$ is a general Fibonacci sequence associated with the characteristic polynomial $p_j(x)$. By applying the Binet formula to the sequence $i_2 \to c_{l_1}^{(i_2,\ldots,i_d)}$, we get $c_{l_1}^{(i_2,\ldots,i_j,\ldots,i_d)} = \sum_{l_2=0}^{m_2-1} c_{(l_1,l_2)}^{(i_3,\ldots,i_d)}\lambda_{2,l_2}^{i_2}$, where $c_{(l_1,l_2)}^{(i_3,\ldots,i_d)}$ are determined by the initial condition $\{c_{l_1}^{(i_2,\ldots,i_d)}\}_{0\leq i_2\leq m_2-1}$. Hence

$$\beta_{(i_1,\ldots,i_d)} = \sum_{l_1=0}^{m_1-1}\lambda_{1,l_1}^{i_1}\sum_{l_2=0}^{m_2-1}\lambda_{2,l_2}^{i_2}c_{(l_1,l_2)}^{(i_3,\ldots,i_d)} = \sum_{l_1=0}^{m_1-1}\sum_{l_2=0}^{m_2-1}\lambda_{1,l_1}^{i_1}\lambda_{2,l_2}^{i_2}c_{(l_1,l_2)}^{(i_3,\ldots,i_d)}.$$

Now we will show that, for every $0 \leq l_1 \leq m_1-1$ and $0 \leq l_2 \leq m_2-1$, the sequence $i_3 \to c_{(l_1,l_2)}^{(i_3,\ldots,i_d)}$ is a general Fibonacci sequence associated with $p_3(x)$. To this aim it suffices to remark that $i_3 \to c_{l_1}^{(i_2,\ldots,i_j,\ldots,i_d)}$ is a general Fibonacci sequence associated with $p_3(x)$ and replace, in the above proof, the sequence $i_2 \to \beta_{(i_1,i_2,\ldots,i_d)}$ by $i_3 \to c_{l_1}^{(i_2,\ldots,i_j,\ldots,i_d)}$. Therefore, we obtain

$$\beta_{(i_1,\ldots,i_d)} = \sum_{l_1=0}^{m_1-1}\sum_{l_2=0}^{m_2-1}\sum_{l_3=0}^{m_3-1}\lambda_{1,l_1}^{i_1}\lambda_{2,l_2}^{i_2}\lambda_{3,l_3}^{i_3}c_{(l_1,l_2,l_3)}^{(i_4,\ldots,i_d)},$$

where $c_{(l_1,l_2,l_3)}^{(i_4,\ldots,i_d)}$ are determined by $\{c_{(l_1,l_2)}^{(i_3,\ldots,i_d)}\}_{0\leq i_3\leq m_3-1}$. By induction we get

$$\beta_{(i_1,\ldots,i_d)} = \sum_{l_1=0}^{m_1-1}\cdots\sum_{l_d=0}^{m_d-1}c_{(l_1,\ldots,l_d)}\lambda_{1,l_1}^{i_1}\ldots\lambda_{d,l_d}^{i_d},$$

where $c_{(l_1,\ldots,l_d)}$ are real numbers. ■

Using the multi index notations $\mathbf{i} = (i_1,\cdots,i_p)$, $\mathbf{l} = (l_1,\cdots,l_p)$ and $\lambda_{\mathbf{l}} = (\lambda_{1,\mathbf{l}},\cdots,\lambda_{p,\mathbf{l}})$, The expression (7.5) becomes

$$\beta_{\mathbf{i}} = \beta_{(i_1,\ldots,i_d)} = \sum_{\mathbf{l}\in\mathfrak{I}}c_{\mathbf{l}}\lambda_{1,\mathbf{l}}^{i_1}\ldots\lambda_{d,\mathbf{l}}^{i_d} = \sum_{\mathbf{l}\in\mathfrak{I}}c_{\mathbf{l}}\lambda_{\mathbf{l}}^{\mathbf{i}}, \tag{7.7}$$

where $\mathfrak{I} \equiv \mathfrak{I}(\beta) := \{\mathbf{l} \in \times_{k=1}^{d}\{0,\ldots,m_k-1\} \mid c_{\mathbf{l}} \neq 0\}$.

Proof of Theorem 7.2.1. Firstly, let's show the equivalence between the assertions (1) and (2). Let $j \in \{1,\dots,d\}$ and let $s_j = \inf\{n \in \mathbb{N} \mid X_j^{n+1} = a_0^{(j)} X_j^n - a_1^{(j)} X_j^{n-1} - \dots - a_n^{(j)} 1$ for some $a_0^{(j)},\dots,a_n^{(j)} \in \mathbb{R}\}$, where X_j^i denotes the index of columns (or rows) of $M(\beta)$ as in the above section. For every $\mathbf{i} \in \mathbb{Z}_+^d$, we will have

$$\beta_{\mathbf{i}+(s_j+1)\varepsilon_j} = a_0^{(j)} \beta_{\mathbf{i}+s_j\varepsilon_j} + \dots + a_{s_j}^{(j)} \beta_{\mathbf{i}}.$$

Hence β is recursively generated associated with the minimal characteristic polynomials $(p_1,\dots,p_d)$, where $p_j(x) = x^{s_j+1} - a_0^{(j)} x^{s_j} - \dots - a_{s_j}^{(j)}$ $(j \in \{1,\dots,d\})$.

Conversely, since β is recursively generated multisequence, every column $\mathbf{X}^{\mathbf{i}}$ in $M(\beta) \equiv M(\infty)(\beta)$, such that $i_l \geq \deg p_l$ for some $l \in \{1,\dots,d\}$, is a linear combination of lower (power index) columns; more precisely,

$$\mathbf{X}^{(l_1,\dots,l_j,\dots,_d)} = \sum_{i=1}^{m_l} a_i^{(j)} \mathbf{X}^{(l_1,\dots,l_j-i,\dots,_d)}.$$

and then $M(\beta)$ has a finite rank. It remains to show that $M(\beta) \geq 0$. To this end, let us construct a matrix $W_s \in M_{m(\tau+s),\tau+s+1}$, the algebra of $m(\tau+s)\times(\tau+s+1)$ real matrices, where $m(\tau+s)$ denote the number of columns (or rows) of $M(\tau+s)(\beta)$, such that the successive columns of W_s are defined by

$$\mathbf{X}^{\sum\limits_{k=1}^{d} l_k\varepsilon_k} = \sum_{i=1}^{\deg p_j} a_i^{(j)} e_{((l_j-i)\varepsilon_j + \sum\limits_{k\neq j} l_k\varepsilon_k)},$$

with $l_1+\dots+l_d = \tau+s+1$, $l_j \geq \deg p_j$ and $\{e_{\mathbf{i}}\}_{|\mathbf{i}|\leq\tau+s+1}$ denote the canonical basis of $\mathbb{R}^{m(\tau+s+1)}$, that is, $e_{\mathbf{i}}$ is the vector with 1 in the $\mathbf{X}^{\mathbf{i}}$ entry and 0 all other positions. Remark that if $\sum\limits_{k=1}^{d} n_k \geq \tau+1$, then there exists $j \in \{1,\dots,d\}$ such that $n_j \geq p_j$. Thus it follows, from (7.2), that

$$M(\tau+s+1) = \begin{pmatrix} M(\tau+s) & B \\ B^* & C \end{pmatrix},$$

with $B = M(\tau+s)W_{\tau+s}$ and $C = B^*W$. Therefore, if $M(\tau+s)\geq 0$, then (by Smul'jan's Theorem) we get $M(\tau+s+1)\geq 0$. As $M(\tau)\geq 0$ then, by induction over $s\geq 0$, we conclude that $M(\beta)\geq 0$.

We show now the equivalence between (2) and (3). From (7.1.3), the polynomial $p_l(x)$ has distinct roots, for all $l \in \{1,\dots,d\}$. We put $p_l(x) = \prod\limits_{i=0}^{m_l-1}(x-\lambda_{l,i})$ where $\lambda_{l,i}\in\mathbb{C}$. According to the relation (7.7), $\beta_{\mathbf{i}} \equiv \beta_{(i_1,\dots,i_d)}$ can be expressed as follows

$$\beta_{\mathbf{i}} = \beta_{(i_1,\dots,i_d)} = \sum_{\mathbf{l}\in\mathfrak{I}} c_{\mathbf{l}} \lambda_{1,\mathbf{l}}^{i_1}\dots\lambda_{d,\mathbf{l}}^{i_d} = \sum_{\mathbf{l}\in\mathfrak{I}} c_{\mathbf{l}} \lambda_{\mathbf{l}}^{\mathbf{i}}.$$

Thus the measure

$$\mu = \sum_{\mathbf{l}\in\mathfrak{I}} c_{\mathbf{l}}\, d\delta_{\lambda_{1,\mathbf{l}}}\dots d\delta_{\lambda_{d,\mathbf{l}}} = \sum_{l_1=0}^{m_1-1}\dots\sum_{l_d=0}^{m_d-1} c_{(l_1,\dots,l_d)}\, d\delta_{\lambda_{1,l_1}}\dots d\delta_{\lambda_{d,l_d}},$$

satisfies

$$\beta_{\mathbf{i}} = \int \mathbf{x}^{\mathbf{i}} d\mu.$$

Let us show that μ is a representing measure of β, that is, $c_{\mathbf{l}} > 0$ and $\lambda_{1,\mathbf{l}},\dots,\lambda_{d,\mathbf{l}} \in \mathbb{R}$, whenever $\mathbf{l}\in\mathfrak{I}$. We consider the the following family of interpolation polynomials at the atoms of the representing measure μ, say $supp\mu := \{\lambda_{\mathbf{l}_1},\dots,\lambda_{\mathbf{l}_m}\} \subset \mathbb{R}^d$,

$$\begin{aligned} L_{\lambda_{\mathbf{l}_s}}(x_1,\dots,x_d) &= L_{(\lambda_{1,\mathbf{l}_s},\dots,\lambda_{d,\mathbf{l}_s})}(x_1,\dots,x_d) \\ &= \prod_{i=0}^{d}\Big(\prod_{\substack{0\leq j\leq m \\ j\neq s}} \frac{x_i-\lambda_{i,\mathbf{l}_j}}{\lambda_{i,\mathbf{l}_s}-\lambda_{i,\mathbf{l}_j}}\Big), \quad (s\in\{1,\dots,m\}). \end{aligned}$$

Clearly

$$L_{\lambda_{\mathbf{l}_s}}(x_1,\dots,x_d)=\begin{cases}1 \text{ if } (x_1,\dots,x_d)=(\lambda_{1,\mathbf{l}_s},\dots,\lambda_{d,\mathbf{l}_s}),\\ 0 \text{ elsewhere.}\end{cases}$$

It follows that, for any $\mathbf{l}\in\mathfrak{I}$,

$$\begin{aligned} c_{\mathbf{l}} &= \int |L_{\lambda_{\mathbf{l}}}|^2\, d\mu \\ &= L_{\lambda_{\mathbf{l}}}^T M(\beta) L_{\lambda_{\mathbf{l}}} \geq 0, \end{aligned}$$

and also, for any $j\in\{0,\dots,d\}$,

$$\begin{aligned} \lambda_{j,\mathbf{l}} c_{\mathbf{l}} &= \int x_j\, |L_{\lambda_{\mathbf{l}}}|^2\, d\mu \\ &= L_{\lambda_{\mathbf{l}}}^T M_{x_j}(\beta) L_{\lambda_{\mathbf{l}}} \in \mathbb{R}, \end{aligned}$$

since the localizing matrix $M_{x_j}(\beta)$ (defined above) is a symmetric real matrix. As $c_{\mathbf{l}}\neq 0$, because $\mathbf{l}\in\mathfrak{I}$, then $c_{\mathbf{l}}>0$, and hence $\lambda_{j,\mathbf{l}}\in\mathbb{R}$, as desired.

It remains to show that μ is a $rankM(\beta)$-atomic and is the unique representing measure of β. To this aim, assume that $M(\beta)\sum\limits_{\mathbf{l}\in\mathfrak{I}} a_{\mathbf{l}}\frac{1}{\sqrt{c_{\mathbf{l}}}}L_{\lambda_{\mathbf{l}}}=0$, where $\{a_{\mathbf{l}}\}_{\mathbf{l}\in\mathfrak{I}}$ are real numbers (not all zero) and $\lambda_{\mathbf{l}}=(\lambda_{1,\mathbf{l}},\dots,\lambda_{d,\mathbf{l}})$. Since $\frac{1}{\sqrt{c_{\mathbf{i}}}}L_{\lambda_{\mathbf{i}}}^T M(\beta)\frac{1}{\sqrt{c_{\mathbf{j}}}}L_{\lambda_{\mathbf{j}}}=\delta_{\mathbf{i},\mathbf{j}}$, the Kronecker delta, we obtain

$$0=\Big(\sum_{\mathbf{i}\in\mathfrak{I}} a_{\mathbf{i}}\frac{1}{\sqrt{c_{\mathbf{i}}}}L_{\lambda_{\mathbf{i}}}\Big)^T M(\beta)\sum_{\mathbf{i}\in\mathfrak{I}} a_{\mathbf{i}}\frac{1}{\sqrt{c_{\mathbf{i}}}}L_{\lambda_{\mathbf{i}}}=\sum_{\mathbf{i}\in\mathfrak{I}} a_{\mathbf{i}}^2,$$

a contradiction. Thus card $supp\mu\leq rankM(\beta)$.

On the other hands, from (7.7), $M(\beta)=\sum\limits_{\mathbf{l}\in\mathfrak{I}} c_{\mathbf{l}}\zeta_{\lambda_{\mathbf{i}}}^T\zeta_{\lambda_{\mathbf{i}}}$, where $\zeta_{\lambda_{\mathbf{l}}}:=(\lambda_{\mathbf{l}}{}^{\alpha})_{\alpha\in\mathbb{Z}^d}\in\mathbb{R}^{\mathbb{Z}_+^d}$, hence

$$rankM(\beta)\leq \text{card }\mathfrak{I}=\text{card } supp\mu.$$

Therefore $rankM(\beta)=$ card $supp\mu$. To get uniqueness, let us suppose now that $\mu':=\sum\limits_{\mathbf{i}\in\mathfrak{I}'} c'_{\mathbf{i}}\delta_{\lambda_{\mathbf{i}}}$ is another representing measure for β; that is, $\int p\,d\mu=\int p\,d\mu'$, for every $p\in\mathbb{R}[x_1,\dots,x_d]$. Let $\{L_{\lambda_{\mathbf{i}}}\}_{\mathbf{i}\in\mathfrak{I}\cup\mathfrak{I}'}\subset\mathbb{R}[x_1,\dots,x_d]$ be the interpolating polynomials at the points of $\mathfrak{I}\cup\mathfrak{I}'$. If $supp\mu\neq supp\mu'$, then there exists $\mathbf{j}\in\mathfrak{I}'\backslash\mathfrak{I}$. Thus

$$0\neq c'_{\mathbf{j}}=\int L_{\lambda_{\mathbf{j}}}d\mu'=\int L_{\lambda_{\mathbf{j}}}d\mu=0,$$

a contradiction, hence $supp\mu=supp\mu'$. Also, we have

$$c'_{\mathbf{i}}=\int L_{\lambda_{\mathbf{i}}}d\mu'=\int L_{\lambda_{\mathbf{i}}}d\mu=c_{\mathbf{i}},\ \text{whenever } \mathbf{i}\in\mathfrak{I}\cup\mathfrak{I}'.$$

Therefore $\mu=\mu'$, as desired. The reverse implication follows directly from the Subsection 7.1.1. ■

Let us recall [9, Theorem 7.8]: If $M(n)$ is positive semidefinite and admits a flat extension $M(n+1)$, then $M(n+1)$ admits unique successive flat moment extensions $M(n+2),M(n+3),\dots,M(\infty)\equiv M(\beta)$. In addition with Theorem 7.2.1, we obtain the following corollary.

Corollary 7.2.3 Let $\beta^{(2n)}\equiv\{\beta_{\mathbf{i}}\}_{\mathbf{i}\in\mathbb{Z}_+^d,|\mathbf{i}|\leq 2n}$ be a truncated multisequence and let $M(n)$ be its associated moment matrix. If $M(n)$ is positive semidefinite and admits a flat extension $M(n+1)$, then $M(n+1)$ has a unique representing measure μ; such that card $supp\mu=rankM(n)$.

We give now a short proof of Theorem 3.4.14.

Proof of Theorem 3.4.14. We have shown in Section 2 that the positive semidefiniteness of $M(n)$ and $M_{q_i}(n+[\frac{\deg q_i+1}{2}])$, for all $q_i \in K_{\mathscr{Q}}$, are necessary conditions for the existence of a representing measure μ, for $\beta^{(2n)}$, supported in $K_{\mathscr{Q}}$. Also, μ is a representing measure for some recursively generated moment sequence $\beta \equiv \{\beta_{\mathbf{i}}\}_{\mathbf{i}\in\mathbb{Z}_+^d}$ (notice in passing that, $M(\beta) \geq 0$ and $rankM(\beta) < \infty$). Therefore, from Theorem 7.2.1, $rankM(\beta) = supp\mu$ $(= rankM(n))$ and hence $M(n)$ admits a rank-preserving extension.

We prove the reverse inclusion. As $M(n) \geq 0$ and $M(n)$ admits a flat extension $M(n+1)$, from Corollary 7.2.3, $M(n+1)$ admits a unique $rankM(n)$-atomic representing measure, write

$$\mu = \sum_{\mathbf{l}\in\mathfrak{I}} c_{\mathbf{l}} d\delta_{\lambda_{1,\mathbf{l}}} \ldots d\delta_{\lambda_{d,\mathbf{l}}}, \tag{7.8}$$

where $r = rankM(n)$, $c_1, \ldots, c_r$ are positive numbers and $\lambda_{\mathbf{l}} \in \mathbb{R}^d$, whenever $\mathbf{l} \in \mathfrak{I}$. By virtue of [9, Theorem 7.8], $M(n+1)$ admits a unique (positive) flat extension $M(\infty) \equiv M(\beta)$, that is, $\beta^{(2n)}$ is a subsequence of some recursively generated moment sequence $\beta \equiv \{\beta_{\mathbf{i}}\}_{\mathbf{i}\in\mathbb{Z}_+^d}$.

For $k \in \{1,\ldots,m\}$, denote $q_k(t_1,\ldots,t_d) = \sum_{\alpha} q_{k,\alpha} t_1^{\alpha_1} \ldots t_d^{\alpha_d}$. From (7.7), we have $\beta_{(i_1,\ldots,i_d)} = \sum_{\mathbf{l}\in\mathfrak{I}} c_{\mathbf{l}} \lambda_{1,\mathbf{l}}^{i_1} \ldots \lambda_{d,\mathbf{l}}^{i_d}$; then

$$\begin{aligned}(q_k * \beta)_{(i_1,\ldots,i_d)} &= \sum_{\alpha} q_{k,\alpha} \sum_{\mathbf{l}\in\mathfrak{I}} c_{\mathbf{l}} \lambda_{1,\mathbf{l}}^{i_1+\alpha_1} \ldots \lambda_{d,\mathbf{l}}^{i_d+\alpha_d} \\ &= \sum_{\mathbf{l}\in\mathfrak{I}} c_{\mathbf{l}} q_k(\lambda_{1,\mathbf{l}},\ldots,\lambda_{d,\mathbf{l}}) \lambda_{1,\mathbf{l}}^{i_1} \ldots \lambda_{d,\mathbf{l}}^{i_d} \\ &= \sum_{\mathbf{l}\in\mathfrak{I}} c_{\mathbf{l}} q_k(\lambda_{\mathbf{l}}) \lambda_{\mathbf{l}}^{\mathbf{i}}.\end{aligned} \tag{7.9}$$

As

$$\begin{aligned} c_{\mathbf{l}} q_k(\lambda_{1,\mathbf{l}},\ldots,\lambda_{d,\mathbf{l}}) &= \int \mid L_{(\lambda_{1,\mathbf{l}},\ldots,\lambda_{d,\mathbf{l}})} \mid^2 q_k d\mu \\ &= L^T_{(\lambda_{1,\mathbf{l}},\ldots,\lambda_{d,\mathbf{l}})} M_{q_k}(\beta) L_{(\lambda_{1,\mathbf{l}},\ldots,\lambda_{d,\mathbf{l}})},\end{aligned}$$

and $M_{q_k}(\beta) \equiv M(q_k * \beta) \geq 0$, then $c_{\mathbf{l}} q_k(\lambda_{1,\mathbf{l}},\ldots,\lambda_{d,\mathbf{l}}) \geq 0$. Thus, for every $\mathbf{l} = (l_1,\ldots,l_d) \in \mathfrak{I}$, we obtain $q_k(\lambda_{\mathbf{l}}) = q_k(\lambda_{1,\mathbf{l}},\ldots,\lambda_{d,\mathbf{l}}) \geq 0$ and this implies that $supp\mu \subseteq K_{\mathscr{Q}}$, as desired.

Since β is recursively generated, $M(\beta) \geq 0$ and $rankM(\beta) = rankM(n)$, we derive from Relation (7.7) and Theorem 7.2.1 that $\mu = \sum_{\mathbf{l}\in\mathfrak{I}} c_{\mathbf{l}} d\delta_{\lambda_{1,\mathbf{l}}} \ldots d\delta_{\lambda_{d,\mathbf{l}}}$ is the unique representing measure of β. Similarly, since for every $i = 1,\ldots,m$, $\{(q_i * \beta)_\alpha\}_{\alpha\in\mathbb{Z}_+^d}$ is a recursively generated sequence, with $rankM(q_i * \beta) = rankM_{q_i}(\infty) = rankM_{q_i}(n+[\frac{\deg q_i+1}{2}])$ and $M_{q_i}(n+[\frac{\deg q_i+1}{2}]) \geq 0$. We get, by applying Theorem 7.2.1 and Relation (7.9), that $\{(q_i * \beta)_\alpha\}_{\alpha\in\mathbb{Z}_+^d}$ admits a unique representing measure

$$\mu_i = \sum_{\mathbf{l}\in\mathfrak{I}} c_{\mathbf{l}} q_i(\lambda_{1,\mathbf{l}},\ldots,\lambda_{d,\mathbf{l}}) d\delta_{\lambda_{1,\mathbf{l}}} \ldots d\delta_{\lambda_{d,\mathbf{l}}}, \tag{7.10}$$

which is $M_{q_i}(n+[\frac{\deg q_i+1}{2}])$-atomic. Thus, from (7.8) and (7.10), μ has precisely $rankM(n) - rankM_{q_i}(n+[\frac{\deg q_i+1}{2}])$ atoms in $\mathscr{Z}(q_i) := \{t \in \mathbb{R}^d : q_i(t) = 0\}$, for every $1 \leq i \leq m$. ■

7.3 Multivariable Subnormal Completion Problem

In this section, we will involve the results of this paper to solve the subnormal completion problem in several variable. Firstly, let's recall some related basic definitions and properties.

The d-tuple $\mathbf{T} \equiv (T_1,\ldots,T_d)$ is said to be normal if $\mathbf{T}$ is commuting and each T_i, $i = 1,\ldots,d$, is normal, and $\mathbf{T}$ is subnormal if $\mathbf{T}$ is the restriction of a normal d-tuple to a common invariant subspace.

Let $\{e_{\mathbf{n}}\}_{\mathbf{n}\in\mathbb{Z}_+^d}$ denote the canonical orthonormal basis of $l^2(\mathbb{Z}_+^d)$, the Hilbert space of square-summable d-sequences, and $\varepsilon_j = (0,\ldots,0,1,0,\ldots,0) \in \mathbb{Z}_+^d$, where 1 is in the j-th position. Consider the bounded multisequences $\alpha^{(j)} = \{\alpha_{\mathbf{n}}^{(j)}\}_{\mathbf{n}\in\mathbb{Z}_+^d}$ for $j = 1,\ldots,d$, called weights. We define the multivariable weighted shift $\mathbf{T} \equiv (T_1,\ldots,T_d)$ acting on $l^2(\mathbb{Z}_+^d)$ by

$$T_j e_{\mathbf{n}} := \alpha_{\mathbf{n}}^{(j)} e_{\mathbf{n}+\varepsilon_j} \qquad \text{for } j = 1,\ldots,d. \tag{7.11}$$

It easy to check that $T_jT_k = T_kT_j$ if and only if

$$\alpha^{(k)}_{\mathbf{n}+\varepsilon_j}\alpha^{(j)}_{\mathbf{n}} = \alpha^{(j)}_{\mathbf{n}+\varepsilon_k}\alpha^{(k)}_{\mathbf{n}} \qquad \text{for } j,k = 1,\dots,d. \tag{7.12}$$

Given $\mathbf{m} \in \mathbb{Z}^d_+$, the moment of order $\mathbf{m}$ of $\mathscr{C}^\infty \equiv \{(\alpha^{(1)},\dots,\alpha^{(d)})\}$ is given by

$$\beta_{\mathbf{m}} := \begin{cases} 1 & \text{if } \mathbf{m} = 0_d, \\ (\alpha^{(j)}_{\mathbf{m}-\varepsilon_j})^2\beta_{\mathbf{m}-\varepsilon_j} & \text{if } m_j \neq 0. \end{cases} \tag{7.13}$$

Problem 7.1 (Subnormal completion problem in d-variables) Given a finite collection of positive real numbers $\Omega_m = \{(\alpha^{(1)}_{\mathbf{k}},\dots,\alpha^{(d)}_{\mathbf{k}})\}_{|\mathbf{k}|\leq m}$ which satisfies (7.12) for all $|\,\mathbf{k}\,|\leq m$ (where $|\,\mathbf{k}\,| := k_1+\ldots+k_d$), find necessary and sufficient conditions on Ω_m to guarantee the existence of a subnormal weighted shift whose initial weights are given by Ω_m.

In the aim to solve this problem we will generalized the notion of recursively generated weighted shifts, introduced by R. Curto and L. Fialkow in [7, 8], to multivariable.

Definition 7.3.1 A multivariable weighted shift is said to be recursively generated if the associated moment sequence, given by (7.13), is recursively generated.

We need also to recall that a commuting multivariable weighted shift (associated with the moment sequence $\{\beta_{\mathbf{i}}\}_{\mathbf{i}\in\mathbb{Z}^d_+}$) is subnormal if, and only if, there is a probability measure μ, often called Berger's measure, see [37], supported on the d-dimensional rectangle $R = [0,\|T_1\|]\times\ldots\times[0,\|T_d\|]$ such that

$$\beta_{\mathbf{i}} = \int \mathbf{x}^{\mathbf{i}}d\mu \qquad \text{for all } \mathbf{i}\in\mathbb{Z}^d_+. \tag{7.14}$$

Therefore the collection of weights Ω_m has a subnormal completion if, and only if, the associated moment sequence $\beta^{(m+1)} \equiv \{\beta_{\mathbf{k}}\}_{|\mathbf{k}|\leq m+1}$, given by (7.13), admits a representing measure supported in $\mathbb{R}^d_+$. From this and by using theorems 7.2.1 and 3.4.14 one can provide the following theorem.

Theorem 7.3.1 Let $\Omega_n := \{(\alpha_{\mathbf{k}},\beta_{\mathbf{k}});|\,\mathbf{k}\,|\leq n\}$ be a given collection of weights obeying (7.12) and $\gamma^{(n+1)}$ the associated moment sequence, given by (7.13). The following statements are equivalent.

i) Ω_n admits a 2-variable subnormal completion,

ii) Ω_n admits a recursively generated 2-variable subnormal completion,

iii) There exists a RDIS $\gamma \equiv \{\gamma_{\mathbf{i}}\}_{\mathbf{i}\in\mathbb{Z}^d_+}$ such that $\gamma^{(n+1)}\subset\gamma$ and the matrices $M(\tau)(\gamma)$ and $M_{x_k}(\tau+1)(\gamma)$, $k = 1,\dots,d$, are positive, where $(p_1,\dots,p_d)\in\mathscr{P}_\gamma$ and $\tau = \sum\limits_{k=1}^{d}(\deg p_k - 1)$.

CHAPTER 8

THE TWO-VARIABLE SUBNORMAL COMPLETION PROBLEM

In this chapter, we give a new approach to solving the 2-variable subnormal completion problem (Problem 8.2). To this aim, we extend the notion of recursively generated weighted shifts, introduced by R. Curto and L. Fialkow [7, 8], to 2-variable case. We next provide "concrete" necessary and sufficient conditions for the existence of solutions to the 2-variable subnormal completion problem with minimal Berger measure, defined in Subsection 2.6.4. Furthermore, a short alternative proof to the propagation phenomena (see the Subsection 2.6.4) for the subnormal weighted shifts in 2-variable, is given (see also [36]).

8.1 Introduction and Results

The notion of recursively generated weighted shifts, largely studied in the literature, is employed to solve various questions in operator theory. R. Curto and L. Fialkow [7, 8] have used this concept to solve the Subnormal Completion Problem (SCP for short) in one variable. In this chapter, we extend this notion to 2-variables, and we use it to provide a new approach to solve the open problem of 2-variable SCP. A concrete solution to the minimal 2-variable SCP (see Section 8.4) is given as well as an alternative proof for the propagation phenomena for subnormal 2-variable weighted shifts.

In the one variable case, the SCP was stated and solved abstractly by J. Stampfli in [51].

Problem 8.1 (One-Variable Subnormal Completion Problem) Given $m \geq 0$ and a finite collection of positive numbers $\Omega_m = \{\alpha_k\}_{k=0}^m$, find necessary and sufficient conditions on Ω_m to guarantee the existence of a subnormal weighted shift whose initial weights are given by Ω_m.

When $m = 0$ or $m = 1$ the solution can be given by the canonical completion $\alpha_0, \alpha_0, \alpha_0, \ldots$ and $\alpha_0 < \alpha_1, \alpha_1, \ldots$, with Berger measure $\mu := \delta_{\alpha_0^2}$ and $\mu := \frac{\alpha_0^2}{\alpha_1^2}\delta_{\alpha_1^2}$, respectively. In [51], J. Stampfli showed that given $\alpha : \sqrt{a}, \sqrt{b}, \sqrt{c}$ with $a < b < c$, there always exists a subnormal completion of α (this solves the case $m = 2$), but for $\alpha : \sqrt{a}, \sqrt{b}, \sqrt{c}, \sqrt{d}$, with $a < b < c < d$, such a subnormal completion may not exist. The complete solution of the SCP in one variable was given by R. curto and L. Fialkow [7], the explicit calculation requires recursively generated weighted shifts (such shifts have finite atomic Berger measures).

We now state the 2-variable SCP:

Problem 8.2 (2-Variable Subnormal Completion Problem) Given $m \geq 0$ and a finite collection of pairs of positive numbers $\Omega_m = \{(\alpha_{\mathbf{k}}; \beta_{\mathbf{k}})\}_{|\mathbf{k}| \leq m}$ satisfying (2.19) for all $|\mathbf{k}| \leq m$ (where $|\mathbf{k}| := k_1 + k_2$), find necessary and sufficient conditions to guarantee the existence of a subnormal 2-variable weighted shift whose initial weights are given by Ω_m.

In [19], R. Curto, S. H. Lee and J. Yoon have introduced an approach to the 2-variable SCP based on positivity and rank-preserving extension of the moment matrix, developed in [9, 10, 12]. Although this lead to an explicit criterion for SCP with quadratic moment data (i.e., $m = 1$), the 2-Variable Subnormal Completion Problem remains open. Recently, S. H. Lee and J. Yoon [46] found, by using the recursiveness, a necessary and sufficient conditions for the existence of 2-Variable subnormal completion with minimal Berger measure for the case $m = 2$ (i.e., rank $M(1)$-atomic Berger measure). We will provide , in Theorem 8.4.2, a complete (and concrete) solution for the 2-Variable SCP with minimal Berger measure.

In the present chapter, we will show that if a given, finite, collection of weights Ω_m admits a subnormal completion in 2-variable, then there exists a 2-variable subnormal weighted shift (solution of the SCP for Ω_m) with moment sequence obey to some recurrence relations, we shall refer to such shifts as 2-variable recursively generated weighted shifts (see Definition 8.2.3). In Theorem 8.3.1, we will provide necessary and sufficient conditions for the existence of 2-variable recursively generated weighted shift completion, and thus a solution to the SCP in 2-variable. As application, we provide a generalization of a recent result of S. H. Lee and J. Yoon [46, Theorem 2.2]. More precisely, in Theorem 8.4.2 we give a concrete necessary and sufficient conditions for the existence of a subnormal completion with minimal Berger measure.

This chapter is organized as follows. In Section 8.2, we introduce the recursively generated weighted shifts and we exhibit some useful results. We devote Section 8.3 to prove Theorem 8.3.1 and the phenomena propagation for subnormal weighted shifts. In section 8.4, we solve the minimal SCP in 2-variable.

8.2 The Two-Variable Recursively Weighted Shifts

We introduce below the notion of 2-variable weighted shifts and we give some useful properties.

Definition 8.2.1 Let $\mathbf{T} \equiv (T_1, T_2)$ be a 2-variable weighted shift and let $\gamma \equiv \{\gamma_{ij}\}_{i,j\in\mathbb{Z}_+}$ be its associated moment sequence. A polynomial $p(x,y) = \sum\limits_{i,j} p_{ij}x^iy^j \in \mathbb{R}[x,y]$ is said to be characteristic polynomial associated with $\mathbf{T}$, or with γ, if

$$\sum_{i,j} p_{i,j}\gamma_{i+n,j+m} = 0, \text{ for all } n,m \in \mathbb{Z}_+. \tag{8.1}$$

Remark 8.2.1 If p is a characteristic polynomial associated with γ, then, for every $q \in \mathbb{R}[x,y]$, the polynomial pq is also a characteristic polynomial. In particular, the set of all characteristic polynomials associated with γ is an ideal in $\mathbb{R}[x,y]$.

The following proposition is an immediate consequence of relations (2.8) and (8.1).

Proposition 8.2.1 Let $\gamma \equiv \{\gamma_{ij}\}_{i,j\in\mathbb{Z}_+}$ be a bi-indexed sequence and let $p(x,y) \in \mathbb{R}[x,y]$. Then p is a characteristic polynomial of γ if and only if $M(\infty)(\gamma)p = 0$.

Definition 8.2.2 A sequence $\gamma \equiv \{\gamma_{ij}\}_{i,j\in\mathbb{Z}_+}$ is said to be recursive double indexed sequence (*RDIS* in short) if it has two characteristic polynomials $p_1, p_2 \in \mathbb{R}[x,y]$, with

$$\left\{\begin{array}{ll} p_1(x,y) = & x^{r+1} - \sum\limits_{i+j\leq r} a_{ij}x^iy^j \\ p_2(x,y) = & y^{s+1} - \sum\limits_{i+j\leq s+1, j\neq s+1} b_{i,j}x^iy^j, \end{array}\right. \tag{8.2}$$

or in the symetric form

$$\left\{\begin{array}{ll} p_1(x,y) = & x^{r+1} - \sum\limits_{i+j\leq r+1, i\neq r+1} a_{ij}x^iy^j \\ p_2(x,y) = & y^{s+1} - \sum\limits_{i+j\leq s} b_{i,j}x^iy^j. \end{array}\right. \tag{8.3}$$

Without loss of generality, throughout this paper, we assume that the pair of characteristic polynomials of the *RDIS* $\gamma \equiv \{\gamma_{ij}\}_{i,j\in\mathbb{Z}_+}$ is given in the form (8.2).

Definition 8.2.3 A 2-variable weighted shift is said to be recursively generated if its associated moment sequence is a *RDIS*.

Let us show that *RDIS* are well defined. Consider a *RDIS* $\gamma \equiv \{\gamma_{ij}\}_{i,j\geq 0}$ associated with a pair of characteristic polynomials as in (8.2), that is, for all $n,m,e \in \mathbb{Z}_+$ with $n \geq r$ and $m \geq s$,

$$\gamma_{n+1,e} = \sum_{i+j\leq r} a_{ij}\gamma_{n-r+i,e+j} \text{ and } \gamma_{e,m+1} = \sum_{l+k\leq s} b_{lk}\gamma_{e+l,n-s+k}. \tag{8.4}$$

A direct computation shows that

$$\begin{aligned}\gamma_{n+1,m+1} &= \sum_{i+j\leq r} a_{ij}\gamma_{n-r+i,m+1+j} \quad \Big(= \sum_{l+k\leq s} b_{lk}\gamma_{n+1+l,m-s+k}\Big)\\ &= \sum_{i+j\leq r, l+k\leq s} a_{ij}b_{lk}\gamma_{n-r+i+l,m-s+j+k}.\end{aligned} \tag{8.5}$$

Equation (8.5) gives the compatibility condition of the two relations in (8.4). Hence the sequence $\gamma \equiv \{\gamma_{ij}\}_{i,j\geq 0}$ is well defined. The other case (i.e., when the characteristic polynomial are defined as in (8.3)) is treated similarly.

Different pair of characteristic polynomials can be associated with the same *RDIS*, as shown in the following example.

Example 8.2.1 Let us consider the bi-sequence $\gamma \equiv \{\gamma_{ij}\}_{i,j\in\mathbb{Z}_+}$ defined by $\gamma_{n,m} = a2^n + b3^m$, where a and b are real numbers. The polynomials $x^2 - 4x - \frac{1}{2}y + \frac{9}{2}$ and $y + 2x - 5$ are two characteristic polynomials of γ. Indeed,

$$\gamma_{n+2,m} - 4\gamma_{n+1,m} - \frac{1}{2}\gamma_{n,m+1} + \frac{9}{2}\gamma_{n,m} = a2^{n+2} + b3^m - 4a2^{n+1} - 4b3^m \quad -\frac{1}{2}a2^n - \frac{1}{2}b3^{m+1} + \frac{9}{2}a2^n + \frac{9}{2}b3^m = 0,$$

and

$$\gamma_{n,m+1} + 2\gamma_{n+1,m} - 5\gamma_{m,n} = a2^n + b3^{m+1} + 2a2^{n+1} + 2b3^m - 5a2^n - 5b3^m = 0.$$

On the other hand, we have

$$\begin{aligned}\gamma_{n+2,m} - 3\gamma_{n+1,m} + 2\gamma_{n,m} &= a2^{n+2} + b3^m - 3a2^{n+1} - 3b3^m + 2a2^n + 2b3^m\\ &= 0,\\ \gamma_{n,m+2} - 4\gamma_{n,m+1} + 3\gamma_{n,m} &= a2^n + b3^{m+2} - 4a2^n - 4b3^{m+1} + 3a2^n + 3b3^m\\ &= 0.\end{aligned}$$

Hence $(x^2 - 3x + 2; y^2 - 4y + 3)$ is, also, a pair of characteristic polynomials associated with γ. (The last characteristic polynomials are analytic, i.e., $(x^2 - 3x + 2; y^2 - 4y + 3) \in \mathbb{R}[X] \times \mathbb{R}[Y]$).

Let $\gamma \equiv \{\gamma_{ij}\}_{i,j\in\mathbb{Z}_+}$ be a *RDIS*, we denote by $\mathscr{P}_\gamma$ $(\subseteq \mathbb{R}[x,y] \times \mathbb{R}[x,y])$ the set of pair of characteristic polynomials associated with γ and we denote by $\mathscr{A}_\gamma$ $(\subseteq \mathbb{R}[X] \times \mathbb{R}[Y])$ the family of analytic, monic, characteristic polynomials associated with γ.

Remark 8.2.2 The pair of characteristic polynomials $(p_1, p_2) \in \mathscr{P}_\gamma$, together with the initial conditions $\{\gamma_{ij}\}_{\substack{0\leq i\leq \deg p_1 - 1,\\ 0\leq j\leq \deg p_2 - 1.}}$, are said to define the sequence γ.

We use structural properties of moment matrices to get the following interesting lemma.

Lemma 8.2.2 Let $\gamma \equiv \{\gamma_{ij}\}_{i,j\in\mathbb{Z}_+}$ be a bi-indexed sequence and let $f, g, h \in \mathbb{R}[x,y]$, then

$$f^T M(\infty)(\gamma)(gh) = (fg)^T M(\infty)(\gamma)h. \tag{8.6}$$

Proof. We write $f = \sum_{i_1,i_2} f_{(i_1,i_2)}x^{i_1}y^{i_2}$, $g = \sum_{j_1,j_2} g_{(j_1,j_2)}x^{j_1}y^{j_2}$ and $h = \sum_{k_1,k_2} h_{(k_1,k_2)}\mathbf{x}^{k_1}y^{k_2}$. As the entry, of the infinite moment matrix $M(\infty)(\gamma)$, corresponding to the column X^nY^m and the line X^lY^k is $\gamma_{n+l,m+k}$, we obtain

$$\begin{aligned}f^T M(\gamma)(gh) &= \Big(\sum_{i_1,i_2} f_{(i_1,i_2)}x^{i_1}y^{i_2}\Big)^T M(\infty)(\gamma)\Big(\sum_{j_1,j_2,k_1,k_2} g_{(j_1,j_2)}h_{(k_1,k_2)}x^{j_1+k_1}y^{j_2+k_2}\Big)\\ &= \sum_{i_1,i_2,j_1,j_2,k_1,k_2} f_{(i_1,i_2)}g_{(j_1,j_2)}h_{(k_1,k_2)}\gamma_{i_1+j_1+k_1,i_2+j_2+k_2}\\ &= (fg)^T M(\gamma)h,\end{aligned}$$

which is the required result. ∎

It follows that

Proposition 8.2.3 Let $\gamma \equiv \{\gamma_{ij}\}_{i,j\geq 0}$ be a bi-indexed sequence and let $M(\infty)(\gamma)$ be the associated infinite moment matrix. If $M(\infty)(\gamma) \geq 0$, Then, for any polynomial $p \in \mathbb{R}[x,y]$ and any integer $n \geq 1$, we have

$$M(\infty)(\gamma)p^n = 0 \Longrightarrow M(\infty)(\gamma)p = 0. \tag{8.7}$$

Proof. If $M(\infty)(\gamma)p^2 = 0$, then $0 = M(\infty)(\gamma)p^2 = 1^T M(\infty)(\gamma)p^2 = p^T M(\infty)(\gamma)p$, from (8.6); since $M(\infty)(\gamma) \geq 0$, we obtain $M(\infty)(\gamma)p = 0$ and hence (8.7) holds for $n = 2$. By induction, (8.7) remains valid for any power of 2. Now , if $M(\infty)(\gamma)p^n = 0$ we choose r in such a way that $r+k$ is a power of 2 to ensure that

$$M(\infty)(\gamma)p^{n+r} = (p^r)^T M(\infty)(\gamma)p^n = 0.$$

Which gives $M(\infty)(\gamma)p = 0$. ■

Before continuing our investigations on *RDIS*, we recall the next result on weighted r-generalized Fibonacci sequences $\{y_A(n)\}_{n=0}^{+\infty}$ where the initial conditions $A \equiv \{\alpha_n\}_{n=0}^{r-1} \subset \mathbb{C}$ are given and the sequence is associated with the characteristic polynomial $p(x) = x^r - a_1x^{r-1} - \ldots - a_r \in \mathbb{C}[x]$. It is also is the sequence generated by the difference equation with initial values:

$$\begin{cases} y_A(n) & = \alpha_n; \quad n = 0,1,\ldots,r-1, \\ y_A(n+r) & = a_1y_A(n+r-1) + \ldots + a_ry_A(n); \quad n = r, r+1, \ldots. \end{cases} \tag{8.8}$$

We have

Theorem 8.2.4 [24, Theorem 1] Let $\{y_A(n)\}_{n=0}^{+\infty}$ be a *RDIS* and $p(x)$ be the associated polynomial as above, with $p(x) = \prod_{i=1}^{k}(x-\lambda_i)^{m_i}$, $(m_1 + \ldots + m_k = r)$. Then the difference equation (8.8) has r independent solution $n^j\lambda_l^n$ $(j = 0,\ldots,m_l - 1; l = 1,\ldots,k)$. Moreover, any solution of (8.8) is of the form

$$y_A(n) = \sum_{l=1}^{k}\sum_{j=0}^{m_l-1} e_{l,j}n^j\lambda_l^n,$$

where $e_{l,j}$ are solutions of the following system of r-equations

$$\sum_{l=1}^{k}\sum_{j=0}^{m_l-1} e_{l,j}n^j\lambda_l^n = y_A(n), \qquad n = 0,\ldots,r-1.$$

As observed in [56, Proposition 2.1], among all characteristic polynomials defining S, there exists a unique monic characteristic polynomial p_0 of minimal degree, called the minimal characteristic polynomial, and which, moreover, divides every characteristic polynomial. The next proposition gives a generalization of this result to the 2-variable case.

Proposition 8.2.5 For every *RDIS* $\gamma \equiv \{\gamma_{ij}\}_{i,j\geq 0}$, with $\mathscr{A}_\gamma \neq \emptyset$, there exists a unique pair of characteristic polynomials $(p_1^\gamma, p_2^\gamma) \in \mathscr{A}_\gamma$ with minimal degree. Moreover, for every $(Q_1,Q_2) \in \mathscr{A}_\gamma$, the polynomials p_1^γ and p_2^γ divide Q_1 and Q_2, respectively.

Proof. Let $(Q_1,Q_2) \in \mathscr{A}_\gamma$, with $Q_1(x) = x^{r+1} - a_0x^r - \ldots - a_r$ and $Q_2(y) = y^{s+1} - b_0y^s - \ldots - b_s$.
Given an integer $j \in \mathbb{Z}_+$. Since, for all $i \in \mathbb{Z}_+$,

$$\gamma_{i+r+1,j} = a_0\gamma_{i+r,j} + \ldots + a_r\gamma_{i,j},$$

then Q_1 is a characteristic polynomial of the Fibonacci sequence $\gamma_j \equiv \{\gamma_{ij}\}_{i\in\mathbb{Z}_+} : i \to \gamma_{ij}$, hence there exists a minimal characteristic polynomial $Q_1^{(j)}$, which divides Q_1. Thus $Q_1^\gamma = \bigwedge_{j\geq 0} Q_1^{(j)}$, the smallest common multiple of $\{Q_1^{(j)}; j \in \mathbb{Z}_+\}$, divides Q_1 and is a characteristic polynomial of γ.

Similarly, Given an integer $i \geq 0$. Q_2 is a characteristic polynomial for the Fibonacci sequence $\gamma_i \equiv \{\gamma_{ij}\}_{j \in \mathbb{Z}_+} : j \to \gamma_{ij}$, then there exists a minimal characteristic polynomial $Q_2^{(i)}$ of γ_i, and hence $Q_2^\gamma = \bigwedge_{i \geq 0} Q_2^{(i)}$ is a characteristic polynomial of γ, which divides Q_2. We conclude that the pair of analytic characteristic polynomials (Q_1^γ, Q_2^γ) provides a positive answer to the proposition. ■

In the following proposition, as well as in the remainder of this paper, we associate every *RDIS* γ (with $\mathscr{A}_\gamma \neq \emptyset$) with its pair of minimal polynomials.

The next theorem, of independent interest, is a crucial point in our approach.

Theorem 8.2.6 Let $\gamma \equiv \{\gamma_{ij}\}_{i,j \in \mathbb{Z}_+}$ be a *RDIS* and let $(p_1, p_2) = (p_1^\gamma, p_2^\gamma) \in \mathscr{A}_\gamma$. If $M(\infty)(\gamma) \geq 0$, then the polynomials p_1 and p_2 have distinct roots.

Proof. Let $l = 1,2$ and let $p_l(x) = \prod_{i=1}^{m} (x - \lambda_i)^{d_i}$, where $\lambda_i \in \mathbb{C}$. We notice that since $p_l(x) \in \mathbb{R}[x,y]$, then if $\lambda_i \notin \mathbb{R}$, for some i, hence there exists $j \neq i$ such that $\lambda_i = \overline{\lambda_j}$ and $d_i = d_j$. Setting $M = \max_{i=1}^{m} d_i$. Since p_l is a characteristic polynomial of γ, then, from Proposition 8.2.1, $M(\infty)(\gamma) \prod_{i=1}^{m} (x - \lambda_i)^{d_i} = 0$ and hence $M(\infty)(\gamma) \prod_{i=1}^{m} (x - \lambda_i)^{M} = 0$. By using Lemma 8.2.3, it follows that $M(\infty)(\gamma) \prod_{i=1}^{m} (x - \lambda_i) = 0$. Hence, via Proposition 8.2.1, the polynomial $\prod_{i=1}^{m} (x - \lambda_i)$ is a characteristic polynomial of γ, which divides p_l. As p_l is minimal, we deduce that $p_l(x) = \prod_{i=1}^{m} (x - \lambda_i)$, as desired. ■

In the next, we give an extension of Theorem 8.2.4 to 2-variables.

Lemma 8.2.7 Let $\gamma \equiv \{\gamma_{ij}\}_{i,j \in \mathbb{Z}_+}$ be a *RDIS* and let $(P_1, P_2) \in \mathscr{A}_\gamma$, with $P_1(x) = \prod_{l=1}^{k_1} (x - \lambda_l)^{m_l}$ and $P_2(y) = \prod_{l=1}^{k_2} (y - \beta_l)^{n_l}$, $(\lambda_l, \beta_l \in \mathbb{C})$. Then

$$\gamma_{ij} = \sum_{a=1}^{k_1} \sum_{b=0}^{m_a - 1} \sum_{c=1}^{k_2} \sum_{d=0}^{n_c - 1} e_{a,b,c,d}\, i^b j^d \lambda_a^i \beta_c^j, \tag{8.9}$$

where $e_{a,b,c,d}$ are determined by using the initial condition $\{\gamma_{ij}\}_{0 \leq i \leq r-1, 0 \leq j \leq s-1}$.

Proof. Given an integer $j \in \mathbb{Z}_+$, the single sequence $\gamma_j : i \to \gamma_{ij}$ is a general Fibonacci sequence associated with P_1, setting $r := \deg P_1$. This implies, by virtue of Theorem 8.2.4, that

$$\gamma_{ij} = \sum_{a=1}^{k_1} \sum_{b=0}^{m_a - 1} e_{a,b}(j)\, i^b \lambda_a^i, \tag{8.10}$$

where $e_{a,b}$ are determined uniquely by the initial conditions $\{\gamma_{ij}\}_{0 \leq i \leq r-1}$.

Since, for every i, the sequence $j \to \gamma_{ij}$ is a general Fibonacci sequence associated with $P_2(x) = x^s - p_1 x^{s-1} - \ldots - p_s$, then

$$\gamma_{i,j+s} - p_1 \gamma_{i,j+s-1} - \ldots - p_s \gamma_{i,j} = 0.$$

Hence, using (8.10), we obtain

$$\sum_{a=1}^{k_1} \sum_{b=0}^{m_a - 1} (e_{a,b}(j+s) - p_1 e_{a,b}(j+s-1) - \ldots - p_s e_{a,b}(j))\, i^b \lambda_a^i = 0. \tag{8.11}$$

As $i \to i^b \lambda_a^i$ (with $b = 0, \ldots, m_a - 1; a = 1, \ldots, k_1$) are linearly independent, see Theorem 8.2.4, then

$$e_{a,b}(j+s) - p_1 e_{a,b}(j+s-1) - \ldots - p_s e_{a,b}(j) = 0.$$

Since j is arbitrary, the sequences $j \to e_{a,b}(j)$ $(b = 0,\dots,m_a - 1; a = 1,\dots,k_1)$ are general Fibonacci sequences, associated with the characteristic polynomial P_2. Hence, for all $b = 0,\dots,m_a - 1$ and $a = 1,\dots,k_1$, we obtain

$$e_{a,b}(j) = \sum_{c=1}^{k_2} \sum_{d=0}^{n_c-1} e_{a,b,c,d}\, j^d \beta_c^j, \tag{8.12}$$

where $e_{a,b,c,d}$ are determined by the initial conditions $\{e_{a,b}(j)\}_{0\le j\le s-1}$. We conclude, from (8.11) and (8.12), that

$$\gamma_{ij} = \sum_{a=1}^{k_1} \sum_{b=0}^{m_a-1} \sum_{c=1}^{k_2} \sum_{d=0}^{n_c-1} e_{a,b,c,d}\, i^b j^d \lambda_a^i \beta_c^j.$$

■

By virtue of Theorem 8.2.6 and Lemma 8.2.7, we have the next corollary.

Corollary 8.2.8 Let $\gamma \equiv \{\gamma_{ij}\}_{i,j\in\mathbb{Z}_+}$ be a *RDIS*, with $M(\infty)(\gamma) \ge 0$, and let $(P_1, P_2) \in \mathscr{A}_\gamma$. Then

$$\gamma_{ij} = \sum_{a=1}^{r+1} \sum_{c=1}^{s+1} e_{a,c} \lambda_a^i \beta_c^j, \tag{8.13}$$

where $Z(P_1) := \{z \in \mathbb{C} \text{ such that } P(z,\bar{z}) = 0\} = \{\lambda_1,\dots,\lambda_{r+1}\}$ and $Z(P_2) = \{\beta_1,\dots,\beta_{s+1}\}$.

The next lemma establishes, for a *RDIS* γ, a link between the positivity of infinite moment matrix and that of the finite one.

Lemma 8.2.9 Let $\gamma \equiv \{\gamma_{ij}\}_{i,j\in\mathbb{Z}_+}$ be a *RDIS* and let $(p_1, p_2) \in \mathscr{P}_\gamma$. Then

$$M(\infty)(\gamma) \ge 0 \Longleftrightarrow M(\deg p_1 + \deg p_2 - 2)(\gamma) \ge 0.$$

Moreover, $rank M(\infty)(\gamma) = rank M(\deg p_1 + \deg p_2 - 2)(\gamma)$.

Proof. Let (p_1, p_2) be as in (8.2) and let $H \in \mathbb{R}_{v+1}[x,y]$, with $v = r + s$. First, we will show that, for every $e = 0,\dots,v+1$, there exist some real numbers $\{\alpha_{lk}^{(e)}; l,k \in \mathbb{Z}_+ \text{ with } l + k \le v\}$ such that

$$\Lambda_\gamma(x^e y^{v+1-e} H) = \sum_{l+k\le v} \alpha_{lk}^{(e)} \Lambda_\gamma(x^l y^k H).$$

To this end we distinguish two cases.

(i) When $e \ge r + 1$. We have, from Remark 8.2.1, $\Lambda_\gamma(p_1 x^{e-r-1} y^{v+1-e} H) = 0$, then $\Lambda_\gamma(x^e y^{v+1-e} H - \sum_{i+j\le r} a_{ij} x^{i+e-r-1} y^{j+v+1-e} H) = 0$.

Hence

$$\begin{aligned}\Lambda_\gamma(x^e y^{v+1-e} H) &= \sum_{i+j\le r} a_{ij} \Lambda_\gamma(x^{i+e-r-1} y^{j+v+1-e} H)\\ &= \sum_{l+k\le v} \alpha_{lk}^{(e)} \Lambda_\gamma(x^l y^k H),\end{aligned}$$

with $l = i + e - r - 1$, $k = j + v + 1 - e$ and $a_{l-e+r+1,k-v-1+e} = \alpha_{lk}^{(e)}$.

(ii) When $e \le r$. It follows, from Remark 8.2.1, that $\Lambda(p_2 x^e y^{v-s-e} H) = 0$. Since $p_2 = y^{s+1} - \sum_{f=1}^{s+1} b_{f,s+1-f} x^f y^{s+1-f} - \sum_{i+j\le s} b_{ij} x^i y^j$, then

$$\Lambda_\gamma(x^e y^{v+1-e} H - \sum_{f=1}^{s+1} b_{f,s+1-f} x^{f+e} y^{v+1-e-f} H - \sum_{i+j\le s} b_{ij} x^{i+e} y^{j+v+1-e} H) = 0.$$

Hence,

$$\begin{aligned}\Lambda_\gamma(x^e y^{v+1-e}H) = \quad & \Lambda_\gamma(\sum_{f=1}^{s+1} b_{f,s+1-f} x^{f+e} y^{v+1-e-f} H) \\ & +\Lambda_\gamma(\sum_{i+j\le s} b_{ij} x^{i+e} y^{j+v+1-e} H).\end{aligned} \tag{8.14}$$

In the first term of the right-hand expression of equation (8.14), the Riesz functional Λ_γ is applied to a sum of monomials of total degree $v+1$ times H. With the aim of lowering the degree of the associated polynomial, we employ the method used in the case (i), for the monomials with power of x greater than $r+1$ (i.e., $f+e \ge r+1$). When $f+e \le r+1$, we reapply the technic of the case (ii) in order to increase strictly the power of x. It follows that each time when applying case (ii) the minimum power of x, of all monomials, increases strictly. Since we can decrease the total degree of each monomial with power in x greater than or equal $r+1$, by applying the case (i), we write

$$\Lambda_\gamma(x^e y^{v+1-e}H) = \sum_{l+k\le v} \alpha_{lk}^{(e)} \Lambda_\gamma(x^l y^k H).$$

Now we construct a $m(v)\times v$-matrix W_v with successive rows defined by the relation,

$$P_{x^e y^{v+1-e}} = \sum_{l+k\le v} \alpha_{lk}^{(e)} e_{(l,k)},$$

where $e = 0,\dots,v+1$ and $\{e_{(l,k)}; k+l \le v\}$ the canonical basis of $\mathbb{R}^{m(n)}$.

Therefore, it is easy to show that:

$$M(v+1)(\gamma) = \begin{pmatrix} M(v)(\gamma) & B \\ B^* & C \end{pmatrix}$$

with $B = M(v)(\gamma)W_v$ and $C = B^* W_v$. Since $M(v)(\gamma) \ge 0$, then, by using Smul'jan's Theorem 2.4.1, $M(v+1)(\gamma) \ge 0$ and $rankM(v) = rankM(v+1)$. In the same way one can show that $M(v+2)(\gamma) \ge 0$ and $rank(v+2) = rankM(v+1) = rankM(v)$. And thus, by induction, we conclude that $M(\infty)(\gamma) \ge 0$ and $rankM(v)(\gamma) = rankM(\infty)(\gamma)$, as desired. ■

8.3 The Two-Variable SCP and the Propagation Phenomena

In this section we involve the 2-variable recursively generated weighted shifts to obtain a solution to the SCP in 2-variable. Also, a simple and new proof to the propagation phenomena, for the 2-variable subnormal weighted shift, is given.

Theorem 8.3.1 Let $\Omega_n := \{(\alpha_{(k_1,k_2)}, \beta_{(k_1,k_2)}); k_1+k_2 \le 2n\}$ be a given collection of weights obeying (2.19) and $\gamma^{(2n+1)}$ be the associated moment sequence, given by (2.20). The following statements are equivalent.

i) Ω_n admits a 2-variable subnormal completion,

ii) Ω_n admits a recursively generated 2-variable subnormal completion,

iii) There exists a *RDIS* $\gamma \equiv \{\gamma_{ij}\}_{i,j\ge 0}$ such that $\gamma^{(2n+1)} \subset \gamma$ and the matrices $M(\deg p_1 + \deg p_2 - 2)(\gamma)$, $M_x(\deg p_1 + \deg p_2 - 1)(\gamma)$ and $M_y(\deg p_1 + \deg p_2 - 1)(\gamma)$ are positive, where $(p_1, p_2) \in \mathscr{P}_\gamma$.

Proof. First, let us show that $i) \Rightarrow ii)$. If Ω_n admits a 2-variable subnormal completion, then there is a Berger's measure ν, supported in $\mathbb{R}_+^2$, such that

$$\gamma_{ij} = \int x^i y^j d\nu, \qquad i+j \le 2n+1. \tag{8.15}$$

A result of C. Bayer and J. Teichmann in [3] states that if a finite bi-sequence of positive numbers $\{\gamma_{ij}\}_{0\le i+j\le 2n}$ has a probability measure verifies (8.15), supported in $\mathbb{R}_+^2$, then it has a finitely atomic positive measure μ verify the same relation and supported, also, in $\mathbb{R}_+^2$. Write supp$\mu \subset \{\lambda_1,\lambda_2,\dots,\lambda_r\} \times \{\beta_1,\beta_2,\dots,\beta_s\}$, where $\lambda_1,\lambda_2,\dots,\lambda_r$ and $\beta_1,\beta_2,\dots,\beta_s$ are real numbers. It is easy to see that $\int x^{i+1-r} y^j \prod_{k=1}^{r}(x-\lambda_k)d\mu = \int x^i y^{j+1-s} \prod_{k=1}^{s}(y-\beta_k)d\mu = 0$, for all $i \ge r-1$ and $j \ge s-1$. Then μ is a representing measure (that is, μ satisfies the relation (2.22)) for the *RDIS* $\gamma \equiv \{\gamma_{ij}\}_{i,j\ge 0}$, defined by the initial conditions $\{\gamma_{ij}\}_{\substack{0\le i\le r-1,\\ 0\le j\le s-1.}}$ and by the following linear recurrence relations:

$$\gamma_{i+1,j} = \sum_{n=0}^{r-1} a_n \gamma_{i-n,j} \text{ and } \gamma_{i,j+1} = \sum_{m=0}^{s-1} b_m \gamma_{i,j-m}, \tag{8.16}$$

for all $i \geq r-1$ and $j \geq s-1$, where

$$\begin{cases} a_{k-1} = (-1)^{k-1} \sum_{1 \leq i_1 < i_2 < \ldots < i_k \leq r} \lambda_{i_1}\lambda_{i_2}\ldots\lambda_{i_k}, \\ b_{k-1} = (-1)^{k-1} \sum_{1 \leq i_1 < i_2 < \ldots < i_k \leq s} \beta_{i_1}\beta_{i_2}\ldots\beta_{i_k}. \end{cases} \tag{8.17}$$

Remark that $x^r - \sum_{n=0}^{r-1} a_n x^{r-1-n} = \prod_{k=1}^{r}(x-\lambda_k)$ and $x^s - \sum_{n=0}^{s-1} b_n x^{s-1-n} = \prod_{k=1}^{s}(x-\beta_k)$. Hence γ is a *RDIS* admitting a Berger's measure, and thus the recursively generated 2-variable subnormal completion associated with γ gives a positive answer to the SCP associated with Ω_n.

To show that $ii) \Rightarrow iii)$, it suffices to remark that if Ω_n admits a recursively generated 2-variable subnormal completion, then there exists a finite Berger's measure μ and a *RDIS* $\gamma \equiv \{\gamma_{ij}\}$ such that $\gamma^{(2n)} \subset \gamma$ and $\int x^i y^j d\mu = \gamma_{ij}$, for all $i, j \in \mathbb{Z}_+$. Hence, for every $Q \in \mathbb{R}[x,y]$ with $\deg Q \leq \deg p_1 + \deg p_2 - 2$, we have

$$Q^T M(p_1 + \deg p_2 - 2)(\gamma) Q = \int Q^2 d\mu \geq 0.$$

Therefore $M(p_1 + \deg p_2 - 2)(\gamma) \geq 0$. Similarly, let $H \in \mathbb{R}[x,y]$ be with $\deg H \leq \deg p_1 + \deg p_2 - 2$, since μ is a positive measure supported in $\mathbb{R}^2_+$, we get

$$H^T M_x(p_1 + \deg p_2 - 1)(\gamma) H = \int x H^2 d\mu \geq 0$$

and

$$H^T M_y(p_1 + \deg p_2 - 1)(\gamma) H = \int y H^2 d\mu \geq 0.$$

Hence $M_x(p_1 + \deg p_2 - 1)(\gamma)$ and $M_y(p_1 + \deg p_2 - 1)(\gamma)$ are positive.

It remains to prove the implication $iii) \Rightarrow i)$. From Lemma 8.2.9 it follows that $M(\infty)(\gamma)$ has finite rank, then there exists an integer n (resp., an integer m) such that, in the matrix $M(\infty)(\gamma)$, the column X^{n+1} (resp., the column Y^{m+1}) is a linearly combination of the columns $1, X, \ldots, X^n$ (resp., the columns $1, Y, \ldots, Y^m$). So we write

$$X^{n+1} = \alpha_0 X^n + \ldots + a_n 1 \text{ and } Y^{m+1} = \beta_0 Y^m + \ldots + b_m.$$

Hence, for every $i, j \in \mathbb{Z}_+$, we have

$$\begin{cases} (X^iY^j)^T M(\infty)(\gamma) X^{n+1} & = (X^iY^j) M(\infty)(\gamma)(a_0 X^n + \ldots + a_n 1) \\ (X^iY^j)^T M(\infty)(\gamma) Y^{m+1} & = (X^iY^j) M(\infty)(\gamma)(b_0 Y^m + \ldots + b_m 1), \end{cases}$$

that is,

$$\begin{cases} \gamma_{i+n+1,j} & = a_0 \gamma_{i+n,j} + \ldots + a_n \gamma_{i,j} \\ \gamma_{i,j+m+1} & = b_0 \gamma_{i,j+m} + \ldots + b_m \gamma_{i,j}. \end{cases}$$

Setting $Q_1(x) = a_0 x^n + \ldots + a_n$ and $Q_2(y) = b_0 y^m + \ldots + b_m$, we have $(Q_1, Q_2) \in \mathscr{A}_\gamma$. Since $M(\deg p_1 + \deg p_2 - 2)(\gamma) \geq 0$, then, via Lemma 8.2.9, $M(\infty)(\gamma) \geq 0$ and thus there exits, by using Theorem 8.2.6, a pair of minimal analytic characteristic polynomials $(H_1, H_2) \in \mathscr{A}_\gamma$ with distinct roots, writing $H_1(x) = \prod_{l=1}^{r+1}(x - \lambda_l)$ and $H_2(x) = \prod_{l=1}^{s+1}(y - \beta_l)$, where $\lambda_l, \beta_l \in \mathbb{C}$.

Hence, via Corollary 8.2.8,

$$\gamma_{ij} = \sum_{a=1}^{r+1}\sum_{c=1}^{s+1} e_{a,c} \lambda_a^i \beta_c^j, \text{ for all } i, j \in \mathbb{Z}_+,$$

and thus the measure $\mu = \sum_{a=1}^{r+1}\sum_{c=1}^{s+1} e_{a,c} \delta_{(\lambda_a, \beta_c)}$ (Here, $\delta_{(\lambda_a, \beta_c)}$ is the Dirac measure at $(\lambda_a, \beta_c) \in \mathbb{R}^2$, having mass (λ_a, β_c) at x and mass 0 elsewhere) verifies

$$\gamma_{ij} = \int x^i y^j d\mu, \text{ for all } i, j \in \mathbb{Z}_+.$$

We will show that μ is a Berger measure, that is, μ is a positive measure supported in $\mathbb{R}_+^2$, which implies that γ is a moment sequence of some 2-variable subnormal weighted shifts, so that Ω_n admits a 2-variable subnormal completion.

Let $I_\mu = \{(a,b)$ such that $e_{a,c} \neq 0\}$ and let

$$L_{(\lambda_i,\beta_j)}(x,y) = \prod_{\substack{0 \le l \le r+1 \\ l \neq i}} (\frac{x-\lambda_i}{\lambda_l - \lambda_i}) \prod_{\substack{0 \le k \le s+1 \\ k \neq j}} (\frac{y-\beta_j}{\lambda_k - \lambda_j}) \tag{8.18}$$

Clearly

$$L_{(\lambda_i,\beta_j)}(x,y) = \begin{cases} 1 & \text{if } (i,j) = (l,k), \\ 0 & \text{if } (i,j) \neq (l,k). \end{cases} \tag{8.19}$$

Thus, for every $(a,b) \in I_\mu$, we have

$$e_{a,c} = \int | L_{(\lambda_a,\beta_c)} |^2 \, d\mu = L^T_{(\lambda_a,\beta_c)} M(\infty)(\gamma) L_{(\lambda_a,\beta_c)} \ge 0.$$

and then $e_{a,c} > 0$, because $e_{a,c} \in I_\mu$. Therefore, μ is a positive measure.
It remains to show that $(\lambda_a, \beta_b) \in \mathbb{R}_+^2$, for all $(a,b) \in I_\mu$. To this aim, let $(a,b) \in I_\mu$, we have

$$e_{a,c}\lambda_a = \int | L_{(\lambda_a,\beta_c)} |^2 \, x d\mu = L^T_{(\lambda_a,\beta_c)} M_x(\infty)(\gamma) L_{(\lambda_a,\beta_c)} \ge 0$$

and

$$e_{a,c}\beta_c = \int | L_{(\lambda_a,\beta_c)} |^2 \, y d\mu = L^T_{(\lambda_a,\beta_c)} M_y(\infty)(\gamma) L_{(\lambda_a,\beta_c)} \ge 0.$$

Since $e_{a,c} > 0$, we deduce that $(\lambda_a, \beta_c) \in \mathbb{R}_+^2$, as desired. ■

The following corollary , of Theorem 8.3.1, characterizes subnormal recursively generated 2-variables weighted shifts.

Corollary 8.3.2 Let $T \equiv (T_1, T_2)$ be a recursively generated 2-variable weighted shift and let $\gamma \equiv \{\gamma_{ij}\}_{i,j \ge 0}$ be the associated moment sequence, with $(P_1, P_2) \in \mathscr{A}_\gamma$. Then T is subnormal if and only if $M(deg(P_1) + deg(P_2) - 2)(\gamma) \ge 0$.

In particular, we have

Corollary 8.3.3 Every collection of positive numbers $\alpha_{(0,0)}, \beta_{(0,0)}, \alpha_{(0,1)}, \beta_{(1,0)}$, such that $\beta_{(1,0)}\alpha_{(0,0)} = \alpha_{(0,1)}\beta_{(0,0)}$, admits a subnormal completion.

Proof. Let $\{\gamma_{ij}\}_{0 \le i,j \le 1}$ be the collection of moments associated with $\omega \equiv \{\alpha_{(0,0)}, \beta_{(0,0)}, \alpha_{(0,1)}, \beta_{(1,0)}\}$, given by (2.20), and let $\gamma \equiv \{\gamma_{ij}\}_{i,j \in \mathbb{Z}_+}$ be the *RDIS* defined by the initial conditions $\{\gamma_{ij}\}_{0 \le i,j \le 1}$ and by the pair of, minimal, analytic characteristic polynomials (P_1, P_2). We set $P_1(X) = (X - \lambda_0)(X - \lambda_1)$ and $P_2(X) = (X - \beta_0)(X - \beta_1)$, with $0 \le \lambda_0 \le \lambda_1$ and $0 \le \beta_0 \le \beta_1$.

The measure $\mu = C_{00}\delta_{(\lambda_0,\beta_0)} + C_{10}\delta_{(\lambda_1,\beta_0)} + C_{01}\delta_{(\lambda_0,\beta_1)} + C_{11}\delta_{(\lambda_1,\beta_1)}$ is a representing measure for γ if and only if $(C_{ij})_{0 \le i,j \le 1}$ are nonnegative and satisfy the following linear system of 4 equations

$$\begin{cases} C_{00} + C_{10} + C_{01} + C_{11} & = \gamma_{00}, \\ C_{00}\beta_0 + C_{10}\beta_0 + C_{01}\beta_1 + C_{11}\beta_1 & = \gamma_{01}, \\ C_{00}\lambda_0 + C_{10}\lambda_1 + C_{01}\lambda_0 + C_{11}\lambda_1 & = \gamma_{10}, \\ C_{00}\beta_0\lambda_0 + C_{10}\beta_0\lambda_1 + C_{01}\beta_1\lambda_0 + C_{11}\beta_1\lambda_1 & = \gamma_{11}. \end{cases}$$

Since the determinant of the preceding system is

$$\begin{vmatrix} 1 & 1 & 1 & 1 \\ \beta_0 & \beta_0 & \beta_1 & \beta_1 \\ \lambda_0 & \lambda_1 & \lambda_0 & \lambda_1 \\ \beta_0\lambda_0 & \beta_0\lambda_1 & \beta_1\lambda_0 & \beta_1\lambda_1 \end{vmatrix} = -((\lambda_1 - \lambda_0)(\beta_1 - \beta_0))^2 \neq 0,$$

we obtain the existence of $\{C_{00}, C_{10}, C_{01}, C_{11}\}$. Thus for the existence of $\mathbf{T} \equiv (T_1, T_2)$ a subnormal completion of $\{\gamma_{ij}\}_{0\le i,j\le 1}$ it suffices show that $\{C_{ij}\}_{0\le i,j\le 1}$ can be positive.

The numbers C_{00}, C_{10}, C_{01} and C_{11} are given by

$$\begin{cases} C_{00} = \frac{\gamma_{11}-\lambda_1\gamma_{01}-\beta_1\gamma_{10}+\lambda_1\beta_1}{(\lambda_1-\lambda_0)(\beta_1-\beta_0)}, \\ C_{01} = \frac{-\gamma_{11}+\lambda_1\gamma_{01}+\beta_0\gamma_{10}-\lambda_1\beta_0}{(\lambda_1-\lambda_0)(\beta_1-\beta_0)}, \\ C_{10} = \frac{-\gamma_{11}+\lambda_0\gamma_{01}+\beta_1\gamma_{10}-\lambda_0\beta_1}{(\lambda_1-\lambda_0)(\beta_1-\beta_0)}, \\ C_{11} = \frac{\gamma_{11}-\lambda_0\gamma_{01}-\beta_0\gamma_{10}+\lambda_0\beta_0}{(\lambda_1-\lambda_0)(\beta_1-\beta_0)}. \end{cases}$$

Direct computations show that C_{00}, C_{10}, C_{01} and C_{11} are positive numbers precisely when,

$$\begin{cases} \max\{2\gamma_{10}, \frac{2\gamma_{11}}{\gamma_{01}}\} \le \lambda_1, \\ \max\{2\gamma_{01}, \frac{2\gamma_{11}}{\gamma_{10}}\} \le \beta_1, \\ 0 \le \lambda_0 \le \min\{\frac{\gamma_{11}}{2\gamma_{01}}, \frac{\gamma_{10}}{2}\}, \\ 0 \le \beta_0 \le \min\{\frac{\gamma_{01}}{2}, \frac{\gamma_{11}}{2\gamma_{10}}\}. \end{cases} \tag{8.20}$$

Therefore it suffices to choose the roots of the polynomials p_1 and p_2 obeying (8.20). ■

We employ Berger's Theorem and Equality (2.21) to give a simple proof to the propagation phenomena for 2-variable subnormal weighted shifts.

Definition 8.3.1 A 2-variable weighted shift $\mathbf{T} \equiv (T_1, T_2)$ is horizontally flat (resp. vertically flat) if $\alpha_{(k_1,k_2)} = \alpha_{(1,1)}$, for all $k_1, k_2 \ge 1$ (resp. $\beta_{(k_1,k_2)} = \beta_{(1,1)}$).

Theorem 8.3.4 Let $\mathbf{T} \equiv (T_1, T_2)$ be a 2-variable subnormal weighted shift associated with the weight sequences $\{\alpha_{\mathbf{k}}\}_{\mathbf{k}\in\mathbb{Z}_+^2}$ and $\{\beta_{\mathbf{k}}\}_{\mathbf{k}\in\mathbb{Z}_+^2}$. If $\alpha_{(k_1,k_2)} = \alpha_{(k_1+1,k_2)}$ for some $k_1, k_2 \ge 1$ (resp. $\beta_{(k_1,k_2)} = \beta_{(k_1,k_2+1)}$) then $\mathbf{T}$ is horizontally flat (resp. vertically flat).

Proof. Let $\gamma \equiv \{\gamma_{ij}\}_{i,j\ge 0}$ be the moment sequence of $\mathbf{T}$, see (2.20). Given an arbitrary number $n_0 \ge \max(k_1+1, k_2)$, let $\gamma^{(n_0)} \equiv \{\gamma_{ij}\}_{i,j\le n_0}$ be a truncated subsequence of γ. Since γ admits a Berger measure, then, via Tchakaloff's Theorem, there exists a finite supported Berger measure μ of $\gamma^{(n_0)}$, write $\mu = \sum\limits_{\substack{0\le i\le p,\\ 0\le j\le q.}} \rho_{i,j}\delta_{(\lambda_i,\vartheta_j)}$. Notice that $\{\vartheta_j; 0 \le j \le q\} \ne \{0\}$ because the moments of the subnormal weighted shift are strictly positive.

Since $\alpha_{(k_1,k_2)} = \alpha_{(k_1+1,k_2)}$, then it follows from (2.21) that

$$\frac{\gamma_{(k_1+1,k_2)}}{\gamma_{(k_1,k_2)}} = \frac{\gamma_{(k_1+2,k_2)}}{\gamma_{(k_1+1,k_2)}},$$

hence

$$\Big(\sum_{\substack{0\le i\le p,\\ 0\le j\le q.}} \rho_{ij}\lambda_i^{k_1+1}\vartheta_j^{k_2}\Big)^2 = \Big(\sum_{\substack{0\le i\le p,\\ 0\le j\le q.}} \rho_{ij}\lambda_i^{k_1}\vartheta_j^{k_2}\Big)\Big(\sum_{\substack{0\le i\le p,\\ 0\le j\le q.}} \rho_{ij}\lambda_i^{k_1+2}\vartheta_j^{k_2}\Big),$$

that is,

$$\sum_{\substack{0\le i<k\le p,\\ 0\le j,h\le q.}} \rho_{ij}\rho_{hk}(\lambda_i-\lambda_k)^2\lambda_i^{k_1}\lambda_k^{k_1}\vartheta_j^{k_2}\vartheta_h^{k_2} = 0.$$

We deduce that $\lambda_i = \lambda_k$ whenever $\lambda_i.\lambda_k \ne 0$, and thus

$$\gamma_{nl} = \lambda_1^n\Big(\sum_{0\le j\le q} \rho_{jl}\vartheta_j^l\Big), \text{ for all } n, l \le n_0.$$

Since n_0 is arbitrary, then

$$\alpha_{(n,m)} = \sqrt{\frac{\gamma_{(n+1,m)}}{\gamma_{(n,m)}}} = \sqrt{\frac{\gamma_{(2,1)}}{\gamma_{(1,1)}}} = \alpha_{(1,1)}, \text{ for all } n,m \in \mathbb{N}^*.$$

The vertically flatness can be proved analogously.

■

8.4 The Minimal Two-Variable Subnormal Completion

Given an integer $n \geq 0$, let $\Omega_n := \{(\alpha_{(k_1,k_2)}, \beta_{(k_1,k_2)}); k_1+k_2 \leq 2n\}$ be a given collection of weight obeying (2.19) and let $\gamma^{(2n+1)} \equiv \{\gamma_{ij}\}_{i+j \leq 2n+1}$ be the corresponding moment sequence given by (2.20). We say that Ω_n admits a minimal 2-variable subnormal completion if it has a subnormal completion with $rankM(n)$-atomic Berger measure.

For $n = 1$, a characterization of weights admitting minimal 2-variable subnormal completion is given by S. H. Lee and J. Yoon [46]. In this section we give the complete solution of this problem.

Recall that, an extension of $M(n)(\gamma^{(2n)})$ is a block matrix of the form

$$M = \begin{pmatrix} M(n)(\gamma^{(2n)}) & B \\ B^* & C \end{pmatrix}. \tag{8.21}$$

A theorem of Smul'jan [50] shows that $M(n)(\gamma^{(2n)})$ admits a positive extension in the form of moment matrix if, and only if,

i) $M(n) \geq 0$,
ii) there exists a matrix W such that $B = AW$,
iii) $C \geq B^*W$,
iv) C is a Hankel $(n+2) \times (n+2)$ matrix.

Moreover, $M(n)(\gamma^{(2n)})$ admits a flat extension, or M is flat, if, and only if, $C = B^*W$. Notice that the condition *iv*) serves only to ensure that M is a moment matrix, see (2.10) and (2.11). If the conditions *i*)-*iv*) are satisfied, we set $M \equiv M(n+1)(\widehat{\gamma}^{(2n+2)})$, where $\gamma^{(2n)} \subseteq \widehat{\gamma}^{(2n+2)}$, i.e., $\gamma_{ij} = \widehat{\gamma}_{ij}$ for all $i+j \leq 2n$. To simplify our notations, we set $\gamma^{(2n+2)} = \widehat{\gamma}^{(2n+2)}$.

Observe that if $M(n+1)(\gamma^{(2n+2)}) \equiv M(n+1)$ is flat (that is, $rankM(n+1) = rankM(n)$), then every column of $M(n+1)$, indexed by a monomial of degree $n+1$, is a linear combination of columns indexed by monomials of degree less than or equal to n. Explicitly, the columns in $M(n+1)$ satisfies

$$X^{n+1-e}Y^e = \sum_{i+j \leq n} a_{ij}^{(e)} X^i Y^j \qquad \text{with } a_{ij}^{(e)} \in \mathbb{R} \text{ and } e = 0, \ldots, n+1. \tag{8.22}$$

Hence

$$(X^l Y^k)^T M(n+1) X^{n+1-e} Y^e = \sum_{i+j \leq n} a_{ij}^{(e)} (X^l Y^k)^T M(n+1) X^i Y^j,$$

for $(l+k \leq n+1)$, that is

$$\gamma_{n+1-e+l,e+k} = \sum_{i+j \leq n} a_{ij}^{(e)} \gamma_{i+l,j+k} \qquad (e = 0, \ldots, n+1 \text{ and } l+k \leq n+1). \tag{8.23}$$

Setting $P_{X^{n+1-e}Y^e}(x,y) = \sum_{i+j \leq n} a_{ij}^{(e)} x^i y^j$.

Lemma 8.4.1 Let $\gamma^{(2n+2)} \equiv \{\gamma_{ij}\}_{i+j \leq 2n+2}$ be a finite collection of non negative numbers such that $rankM(n+1) = rankM(n)$ and let $P_{X^{n+1-e}Y^e}$ be as above. Then, for all $e \in \{0, \ldots, n+1\}$, the polynomial $x^{n+1-e}y^e - P_{X^{n+1-e}Y^e}$ is a characteristic one of the *RDIS* defined by the initial conditions $\{\gamma_{ij}\}_{i,j \leq n}$ and by the pair of characteristic polynomials $(x^{n+1} - P_{X^{n+1}}, y^{n+1} - P_{Y^{n+1}})$.

Proof. Let $\gamma \equiv \{\gamma_{ij}\}_{i,j\in\mathbb{Z}_+}$ be a *RDIS* defined by the initial conditions $\{\gamma_{ij}\}_{i,j\le n}$ and by the pair of characteristic polynomials $(x^{n+1}-P_{X^{n+1}}, y^{n+1}-P_{Y^{n+1}})$. Corresponding to the sequence γ, the Riesz functional $\Lambda : \mathbb{R}[x,y] \to \mathbb{R}$ defined by $\Lambda(\sum\limits_{i,j} p_{ij}x^iy^j) = \sum\limits_{i,j} p_{ij}\gamma_{ij}$. Then , for all $a,b\in\mathbb{Z}_+$, we have

$$\Lambda(x^ay^b(x^{n+1}-P_{X^{n+1}})) = \Lambda(x^ay^b(y^{n+1}-P_{Y^{n+1}})) = 0. \tag{8.24}$$

Let $e \in \{0,\ldots,n+1\}$ be a fixed integer, we will show that $x^{n+1-e}y^e - P_{X^{n+1-e}Y^e}$ is a characteristic polynomial of γ, that is, for all $a,b\in\mathbb{Z}_+$,

$$\Lambda(x^ay^b(x^{n+1-e}y^e - P_{X^{n+1-e}Y^e})) = 0. \tag{8.25}$$

We distinguish two cases;
1) $a+b\le n+1$, then (8.23) implies that $\Lambda(x^{a+n+1-e}y^{b+e}) = \Lambda(x^ay^bP_{X^{n+1-e}Y^e})$ and hence (8.25) is verified.
2) $a+b=n+2$, we show first that

$$\Lambda(x^fy^gQ_k) = 0 \text{ for all } f+g\le n+1, \tag{8.26}$$

where $Q_k = yP_{X^{n+1-k}Y^k} - xP_{X^{n-k}Y^{k+1}}$ and $k\in\{0,\ldots,n+1\}$.

For $f+g\le n$, we have

$$\begin{aligned}\Lambda(x^fy^gQ_k) &= \Lambda(x^fy^g(yP_{X^{n+1-k}Y^k} - xP_{X^{n-k}Y^{k+1}}))\\ &= \Lambda(x^fy^{g+1}P_{X^{n+1-k}Y^k} - x^{f+1}y^gP_{X^{n-k}Y^{k+1}}))\\ &= \Lambda(x^{f+n+1-k}y^{g+1+k} - x^{f+1+n-k}y^{g+k+1}),\ \text{due to (8.23)},\\ &= 0.\end{aligned} \tag{8.27}$$

For $f+g=n+1$, since $\deg Q_k\le n+1$, and according to (8.23), we derive that $\Lambda(x^fy^gQ_k) = \Lambda(P_{X^fY^g}Q_k)$. As $\deg P_{X^fY^g}\le n$, then (8.27) implies that $\Lambda(P_{X^fY^g}Q_k)=0$, and hence $\Lambda(x^fy^gQ_k)=0$ (for all $f+g\le n+1$).

Now we consider the case $a+b=n+2$, which we split in two subcases.
i) If $a\ge e$: according to (8.26), we have

$$\begin{aligned}\Lambda(x^{a-e}y^{b+e}P_{X^{n+1}}) &= \Lambda(x^{a-e+1}y^{b+e-1}P_{X^nY})\\ &= \ldots = \Lambda(x^ay^bP_{X^{n+1-e}Y^e})\end{aligned}$$

Thus

$$\begin{aligned}0 &= \Lambda(x^{a-e}y^{b+e}P_{X^{n+1}}) - \Lambda(x^ay^bP_{X^{n+1-e}Y^e})\\ &= \Lambda(x^{a-e+n+1}y^{b+e}) - \Lambda(x^ay^bP_{X^{n+1-e}Y^e}),\ \text{by applying (8.24)},\\ &= \Lambda(x^{a+n+1-e}y^{b+e} - x^ay^bP_{X^{n+1-e}Y^e})\\ &= \Lambda(x^ay^b(x^{n+1-e}y^e - P_{X^{n+1-e}Y^e})).\end{aligned} \tag{8.28}$$

ii) If $a\le e$: since $a+b=n+2$, then $b+e\ge n+1$ and $b+e\ge n+1$. According to (8.27), we have

$$\begin{aligned}\Lambda(x^ay^bP_{X^{n+1-e}Y^e}) &= \Lambda(x^{a+1}y^{b-1}P_{X^{n-e}Y^{e+1}})\\ &= \ldots = \Lambda(x^{a+n+1-e}y^{b-n-1+e}P_{Y^{n+1}})\end{aligned}.$$

Hence

$$\begin{aligned}0 &= \Lambda(x^{a+n+1-e}y^{b-n-1+e}P_{Y^{n+1}}) - \Lambda(x^ay^bP_{X^{n+1-e}Y^e})\\ &= \Lambda(x^{a+n+1-e}y^{b+e}) - \Lambda(x^ay^bP_{X^{n+1-e}Y^e}),\ \text{due to (8.24)},\\ &= \Lambda(x^{a+n+1-e}y^{b+e} - x^ay^bP_{X^{n+1-e}Y^e})\\ &= \Lambda(x^ay^b(x^{n+1-e}y^e - P_{X^{n+1-e}Y^e})).\end{aligned} \tag{8.29}$$

Therefore, we conclude that

$$\Lambda(x^ay^b(x^{n+1-e}y^e - P_{X^{n+1-e}Y^e})) = 0, \qquad \text{for all } a+b\le n+2. \tag{8.30}$$

By induction, we obtain $\Lambda(x^ay^b(x^{n+1-e}y^e - P_{X^{n+1-e}Y^e})) = 0,$ for all $a,b\in\mathbb{Z}_+$. ■

Now we are able to give a concrete solution to the minimal 2-variable SCP.

Theorem 8.4.2 Let $\Omega_n := \{(\alpha_{(k_1,k_2)}, \beta_{(k_1,k_2)}); k_1+k_2 \le 2n\}$ be a given collection of weights obeying (2.19) and let $\gamma^{(2n+1)}$, $M(n)$, $M_x(n)$, $M_y(n)$, B and $P_{x^{n+1-i}y^i}$ $(i=0,\dots,n+1)$ be as above, with $RangB \subseteq RangM(n)$ and $M(n)$, $M_x(n)$ and $M_y(n)$ are positive. Then Ω_n admits a minimal subnormal completion if and only if

$$P^T_{x^{n+1-j}y^j} M(n) P_{x^{n+1-i}y^i} = P^T_{x^{n-j}y^{j+1}} M(n) P_{x^{n+2-i}y^{i-1}} \tag{8.31}$$

for all integers i and j with $0 \le j \le n-1$ and $2+j \le i \le n+1$.

Proof. Let $m \equiv m(n)$ denote the number of rows (or columns) in $M(n)$. Adopting the notation in (8.21), with B is a $m \times (n+1)$ matrix. Observe that, the sequence $\gamma^{(2n+1)}$ fills only the matrices $M(n)$ and B. We are looking for new numbers (entries) for the matrix C in order that M be a flat moment matrix.

Let the rows $X^{n+1-j}Y^j$, columns $X^{n+1-i}Y^i$ $(i,j=0,\dots,n+1)$ entry of the matrix M be equal to $P^T_{x^{n+1-j}y^j} M(n) P_{x^{n+1-i}y^i}$ (i.e., $C=(M(n)W)^*W$). With this completion, M becomes a flat completion of $M(n)$.

Let us show that M is a moment matrix, that is, C is Hankel. The relation (8.31) implies that the upper left triangular part of the matrix C is Hankel type. Since $M(n)$ is symmetric, then C is also symmetric, and thus $C=(M(n)W)^*W$ is a Hankel matrix. Setting $M=M(n+1)(\gamma^{(2n+2)})$. It follows, from Lemma 8.4.1, that there exists a $DIRS$ γ such that $rankM(n) = rankM(\infty)(\gamma)$ (where $M(n) = M(n)(\gamma^{(2n)}) = M(n)(\gamma)$).

We show now that $rankM_x(n+1)(\gamma) = rankM_x(\infty)(\gamma)$ and $rankM_y(n+1)(\gamma) = rankM_y(\infty)(\gamma)$. We prove first that the columns in $M_x(n+2)$ verify the relation $X^eY^{n+1-e} = \sum_{i+j\le n} a^{(e)}_{ij} X^iY^j (= P_{x^ey^{n+1-e}}(X,Y))$. We have, for all $a+b \le n+1$,

$$\begin{aligned}
&(X^aY^b)^T M_x(n+2)(\gamma)(X^eY^{n+1-e} - P_{x^ey^{n+1-e}}(X,Y))\\
&\quad = < M_x(n+2)(\gamma)(x^ey^{n+1-e} - P_{x^ey^{n+1-e}}), x^ay^b >, \text{ see (2.8)},\\
&\quad = < M(n+1)(\gamma)(x^ey^{n+1-e} - P_{x^ey^{n+1-e}}), x^{a+1}y^b >\\
&\quad =0, \text{ since } x^ey^{n+1-e} - P_{x^ey^{n+1-e}} \text{ is a characteristic polynomial of } \gamma.
\end{aligned}$$

Hence $M_x(n+2)(\gamma)(X^eY^{n+1-e}) = M_x(n+2)(\gamma)(P_{x^ey^{n+1-e}}(X,Y))$. Since $\deg P_{x^ey^{n+1-e}} \le n$, then $rankM_x(n+1)(\gamma) = rankM_x(n+2)(\gamma)$, and thus we obtain, by induction, $rankM_x(n+1)(\gamma) = rankM_x(\infty)(\gamma)$. Similarly, one shows that $rankM_y(n+1)(\gamma) = rankM_y(\infty)(\gamma)$. Since $M_x(n+1)(\gamma)$ and $M_y(n+1)(\gamma)$ are positive, then, via Schmul'jan's theorem, $M_x(\infty)(\gamma)$ and $M_y(\infty)(\gamma)$ are positive.

Let $(p_1,p_2) \in \mathscr{A}_\gamma$. We conclude, from above, that $M(\deg p_1 + \deg p_2 - 2)(\gamma)$, $M_y(\deg p_1 + \deg p_2 - 1)(\gamma)$ and $M_y(\deg p_1 + \deg p_2 - 1)(\gamma)$ are positive. Now, by applying Theorem 8.3.1, Ω_n admits a subnormal completion with finite Berger measure, say $\mu = \sum_{(a,b)\in I_\mu} e_{(a,b)} d\delta_{(\lambda_a,\beta_b)}$ (with $e_{(a,b)} \neq 0$, for all $(a,b) \in I_\mu$).

It remain to show that $cardsupp\mu = rankM(n)(\gamma)$. Let $\zeta_{(\lambda_a,\beta_c)} := (1, \lambda_a, \beta_c, \lambda_a^2, \lambda_a\beta_c, \beta_c^2, \dots) = (\lambda_a^i\beta_c^j)_{(i,j)\in\mathbb{Z}_+^2} \in \mathbb{R}^{\mathbb{Z}_+^2}$. By using Corollary 8.2.8, the infinite moment matrix can be formulated as follows $M(\infty)(\gamma) = \sum_{(a,c)\in I_\mu} e_{(a,c)} \zeta_{(\lambda_a,\beta_c)} \zeta^T_{(\lambda_a,\beta_c)}$, then

$$rankM(\infty)(\gamma) \le cardI_\mu = cardsupp\mu.$$

On the other hand, Let $L_{(\lambda_a,\beta_c)}$ be as in (8.19) and let

$$M(\infty)(\gamma) \sum_{(a,c)\in I_\mu} \kappa_{(a,c)} \frac{1}{\sqrt{e_{(a,c)}}} L_{(\lambda_a,\beta_c)} = 0,$$

where $\{\kappa_{(a,c)}\}_{(a,c)\in I_\mu}$ are real numbers (not all zero).
Since $\frac{1}{\sqrt{e_{(i,j)}}} L^T_{(\lambda_i,\beta_j)} M(\infty)(\gamma) \frac{1}{\sqrt{e_{(n,m)}}} L_{(\lambda_n,\beta_m)}$ is 1 if $(i,j)=(n,m)$ and equal to 0 if $(i,j) \neq (n,m)$, then

$$\begin{aligned}
0 &= (\sum_{(a,c)\in I_\mu} \kappa_{(a,c)} \frac{1}{\sqrt{e_{(a,c)}}} L_{(\lambda_a,\beta_c)})^T M(\infty)(\gamma) \sum_{(a,c)\in I_\mu} \kappa_{(a,c)} \frac{1}{\sqrt{e_{(a,c)}}} L_{(\lambda_a,\beta_c)}\\
&= \sum_{(a,c)\in I_\mu} \kappa^2_{(a,c)},
\end{aligned}$$

a contradiction. Hence $cardsupp\mu \le rankM(\infty)(\gamma)$, and thus $cardsupp\mu = rankM(n)(\gamma)$, as desired. ■

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Abstract:

In this dissertation we are dealing with the topic of the truncated multivariable K-moment problems. First, we investigate the notion of recursiveness in two variables. We obtain several useful results that we will use to deduce new necessary and sufficient conditions for the truncated complex moment problem to own a solution. As a consequence, we show that the general truncated complex moment problem always have a solution (concrete construction of such solution is given). Furthermore, we solve the truncated complex moment problem with column dependence relations $Z^{k+1} = \sum_{0 \leq n+m \leq k} a_{nm} \bar{Z}^n Z^m$ $(a_{nm} \in \mathbb{C})$. Also, we solve the quintic complex moment problem. Next, we present generalizations to multivariable K-moment problems. Finally, we extend the notion of recursively generated weighted shifts to 2-variable case, and provided a new methodology for solving the 2-variable subnormal completion problem. As a consequence, we furnish a concrete solution to the 2-variable subnormal completion problem with minimal Berger measure.

Keywords: *Truncated multivariable K-moment problems; bi-recursive sequences; general truncated complex moment problem; quintic moment problem; 2-variable recursively generated weighted shifts; 2-variable subnormal completion problem.*

Résumé :

Dans cette dissertation, nous considérons le K-problème des moments n-dimensionnel tronqué. Tout d'abord, nous étudions la notion de récursivité bidimensionnelle. Les résultats obtenus permettent d'établir une nouvelle condition nécessaire et suffisante de l'existence d'une solution du problème des moments tronqué dans le cas complexe. En conséquence, nous montrons que le problème des moments complexe tronqué généralisé à toujours une solution (la construction explicite de la solution sera donnée). Puis nous résoudrons le problème des moments complexe tronqué dont les colonnes de la matrice des moments, notées ici par $(\bar{Z}^i Z^j)_{i,j \in \mathbb{N}}$, vérifient $Z^{k+1} = \sum_{0 \leq n+m \leq k} a_{nm} \bar{Z}^n Z^m$ $(a_{nm} \in \mathbb{C})$. De plus, nous résoudrons le cas quintic du problème des moments complexe tronqué. Ensuite, nous présentons une généralisation (des résultats obtenus) au K-problème des moments n-dimensionnel tronqué.

D'autre part, l'étude des shifts à poids récurrents en deux dimensions, nous mènera à une nouvelle méthode pour résoudre le problème de complétion sous-normale bidimensionnel. Nous donnerons en particulier une solution concrète du problème de complétion sous-normale bidimensionnel avec une mesure de Berger minimale.

Mots-clés : *K-problème des moments n-dimensionnel tronqué; suite récurrente doublement indexée; problème des moments complexe tronqué généralisé; problème des moments complexe quintic; shifts à poids récurrent en deux dimensions; problème de complétion sous-normale bidimensionnel.*

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