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Hankel and Volterra operators on Bergman spaces

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ABSTRACT

In this thesis, we study the behavior of the singular values of Hankel operators on weighted Bergman spaces $A_{\omega_\varphi}^2$, where $\omega_\varphi = e^{-\varphi}$ and φ is a subharmonic function. We consider compact Hankel operators $H_{\bar{\phi}}$, with anti-analytic symbols $\bar{\phi}$, and give estimates of the trace of $h(|H_{\bar{\phi}}|)$ for any convex function h . This allows us to give asymptotic estimates of the singular values $(s_n(H_{\bar{\phi}}))_n$ in terms of a nonincreasing rearrangement of $|\phi'|/\sqrt{\Delta\varphi}$ with respect to an adequate measure. For radial weights, we first prove that the critical decay of $(s_n(H_{\bar{\phi}}))_n$ is achieved by $(s_n(H_{\bar{z}}))_n$. Namely, we establish that if $s_n(H_{\bar{\phi}}) = o(s_n(H_{\bar{z}}))$, then $H_{\bar{\phi}} = 0$. Hence, we show that if $\Delta\varphi(z) \asymp \frac{1}{(1-|z|^2)^{2+\beta}}$ with $\beta \geq 0$, then $s_n(H_{\bar{\phi}}) = O(s_n(H_{\bar{z}}))$ if and only if ϕ' belongs to the Hardy space H^p , where $p = \frac{2(1+\beta)}{2+\beta}$. Next, we compute the asymptotics of $s_n(H_{\bar{\phi}})$ whenever $\phi' \in H^p$.

In the second part, we study weighted Bergman spaces that satisfy Hardy-Littlewood Theorem. That is

$$\|f\|_{A_\omega^p}^p \asymp \sum_{k=0}^{n-1} |f^{(k)}(0)|^p + \int_{\mathbb{D}} |f^{(n)}(z)|^p (1-|z|^2)^{np} \omega(z) dA(z), \quad f \in A_\omega^p, \quad n \geq 1.$$

We improve and extend results obtained in [2, 10] to a large class of weights. We prove that polynomials are dense in A_ω^p for weights ω belonging to this class. Then we prove several properties on the behavior of the kernel of A_ω^2 which allows us to study Toeplitz and Integration operators on these spaces.

keywords— Bergman spaces, Hankel operator, Toeplitz operator, Volterra operator, singular values, $\bar{\partial}$ - L^2 minimal solution, Hardy–Littlewood theorem.

RÉSUMÉ

Dans cette thèse, on étudie le comportement des valeurs singulières de l'opérateur de Hankel sur des espaces de Bergman à poids. On considère des opérateurs de Hankel compact à symbole anti-analytique et on donne une estimation de la trace pour toute fonction convexe h . Ceci nous permet de donner une estimation du comportement asymptotique des valeurs singulières. Pour les poids radiaux, on montre à quel moment la décroissance critique est achevée. Ensuite, on donne une condition nécessaire et suffisante pour que les valeurs singulières soient un grand O de la décroissance critique. Quand cette condition est vérifiée, on donne les asymptotiques des valeurs singulières. On étudie aussi des espaces de Bergman à poids qui vérifient le théorème de Hardy-Littlewood. On étend des résultats existants à une large classe de poids. On montre que les polynômes sont denses dans les espaces de Bergman à poids pour des poids dans cette classe. Ensuite on démontre plusieurs propriétés sur le comportement des noyaux reproduisant des espaces de Bergman à poids, ce qui nous permet d'étudier les opérateurs de Toeplitz et d'intégration sur ces espaces.

Dans cette thèse, on s'intéresse à deux sujets. Le premier concerne les valeurs singulières de l'opérateur de Hankel sur des espaces de Bergman et le second concerne les espaces de Bergman vérifiant le théorème de Hardy-Littlewood.

Dans le premier sujet, on étudie le comportement des valeurs singulières des opérateurs de Hankel sur des espaces de Bergman à poids $A_{\omega_\varphi}^2$ où $\omega_\varphi = e^{-\varphi}$ et φ est une fonction sous-harmonique. On considère des opérateurs de Hankel $H_{\bar{\phi}}$ à symboles anti-analytiques $\bar{\phi}$ et on donne des estimés de la trace de $h(|H_{\bar{\phi}}|)$ pour toutes fonction convexe h . C'est l'objet du théorème suivant :

Théorème .1. *Soit $\omega \in \mathcal{W}^*$ et soit $\phi \in \mathcal{B}_0^\omega$. Soit $h : [0, +\infty[\rightarrow [0, +\infty[$ une fonction convexe croissante telle que $h(0) = 0$. Alors il exist $B > 0$ ne dépendent que de ω , tel que*

$$\int_{\mathbb{D}} h \left(\frac{1}{B} \tau_\omega(z) |\phi'(z)| \right) d\lambda_\omega(z) \leq \text{Tr} (h(|H_{\bar{\phi}}|)) \leq \int_{\mathbb{D}} h (B \tau_\omega(z) |\phi'(z)|) d\lambda_\omega(z).$$

Ce résultat permet de donner des estimations asymptotiques des valeurs singulières $(s_n(H_{\bar{\phi}}))_n$ en termes du réarrangement décroissant de $|\phi'|/\sqrt{\Delta\varphi}$ par rapport à une mesure adéquate.

Pour les poids radiaux, on montre que la décroissance critique de $(s_n(H_{\bar{\phi}}))_n$ est obtenue par $(s_n(H_{\bar{z}}))_n$ dans le théorème suivant :

Théorème .2. *Soit $\omega \in \mathcal{W}^*$ tel que τ_ω soit équivalent à une fonction radiale et soit $\phi \in \mathcal{B}_0^\omega$. Alors*

1. $s_n(H_{\bar{\phi}}) = o(s_n(H_{\bar{z}})) \implies H_{\bar{\phi}} = 0$.
2. $s_n(H_{\bar{\phi}}) = o(\mathcal{R}_{z,\omega}^+(n)) \implies H_{\bar{\phi}} = 0$.

Pour les différentes notations, on réfère le lecteur au reste de la thèse.

On montre ensuite que si $\Delta\varphi(z) \asymp \frac{1}{(1-|z|^2)^{2+\beta}}$ avec $\beta \geq 0$, alors $s_n(H_{\bar{\phi}}) = O(s_n(H_{\bar{z}}))$ si et seulement si ϕ' appartient à l'espace de Hardy H^p , où $p = \frac{2(1+\beta)}{2+\beta}$. Lorsque $\phi' \in H^p$, on calcul les asymptotes de $s_n(H_{\bar{\phi}})$. C'est ce que donne le théorème suivant avec lequel on conclut la première partie de la thèse :

Théorème .3. *Soit $\omega \in \mathcal{W}^*$ tel que $\tau_\omega^2(z) \asymp (1-|z|)^{2+\beta}$ avec $\beta \geq 0$. Soit $p = \frac{2(1+\beta)}{2+\beta}$ et soit $\phi \in \mathcal{B}_0^\omega$. On a*

$$s_n(H_{\bar{\phi}}) = O(1/n^{1/p}) \iff \phi' \in H^p.$$

De plus, si ω est radiale et $s_n(H_{\bar{z}}) \sim \gamma/n^{\frac{1}{p}}$ pour un certain $\gamma \in (0, \infty)$, alors

$$s_n(H_{\bar{\phi}}) \sim \|\phi'\|_{H^p} \frac{\gamma}{n^{\frac{1}{p}}}, \quad \phi' \in H^p.$$

Dans le deuxième sujet de cette thèse, on étudie les espaces de Bergman vérifiant le théorème de Hardy-Littlewood. On dit que ω vérifie le théorème de Hardy-Littlewood et on note $\omega \in \mathcal{HL}$, si pour tout $p > 0$, $f \in \text{Hol}(\mathbb{D})$ et $n \in \mathbb{N}^*$, on a

$$\|f\|_{A_{\omega}^p}^p \asymp \sum_{k=0}^{n-1} |f^{(k)}(0)|^p + \int_{\mathbb{D}} |f^{(n)}(z)|^p (1-|z|^2)^{np} \omega(z) dA(z).$$

On commence par introduire certains poids vérifiant le théorème de Hardy-Littlewood et on montre le théorème suivant qui donne une caractérisation des poids vérifiant le théorème de Hardy-Littlewood, lorsque ces derniers sont dans une certaine classe \mathcal{C}_p .

Théorème .4. *On suppose que $\omega \in \mathcal{C}_{p_0}$ pour un certain $p_0 > 1$. Les assertions suivantes sont équivalentes.*

- (1) $\omega \in \mathcal{HL}$.
- (2) $\tilde{\omega} \in \mathcal{HL}$.
- (3) $\omega_{0,t} \lesssim \tilde{\omega}$ for some $t > 0$.

On remarque que si $\frac{\omega}{(1-|z|^2)^\eta} \in B_{p_0}(\eta)$ pour un certain $p_0 > 1$ et un $\eta > -1$, alors $\omega \in \mathcal{C}_{p_0}$ et $\omega_{0,t} \lesssim \tilde{\omega}$ pour tout $t \geq p_0(\eta + 2) - 2$. Cette remarque nous permet de définir une nouvelle classe de poids.

Définition .1. *Soit $p > 1$ et soit $t \geq 0$. On dit qu'un poids ω appartient à la classe $\mathcal{C}_{p,t}$ si $\omega \in \mathcal{C}_p$ et $\omega_{0,t} \lesssim \tilde{\omega}$.*

On donne une preuve simple pour la densité des polynômes dans A_ω^p pour tout poids ω dans la classe $\mathcal{C}_{p,t}$ avec $p > 1$ et $t \geq 0$.

Ensuite on montre que le noyau reproduisant de A_ω^2 vérifie certaines conditions lorsque le poids est dans la classe $\mathcal{C}_{p,t}$ avec $p > 1$ et $t \geq 0$. Ceci nous permet d'étudier l'opérateur de Toeplitz et l'opérateur d'intégration sur A_ω^2 .

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INTRODUCTION

In this thesis we are interested in two major subjects. The first one concerns the singular values of Hankel operators with anti-analytic symbol acting on some weighted Bergman spaces and the second one concerns Bergman spaces satisfying a Hardy-Littlewood theorem. These subjects lie in the intersection of functional analysis, operator theory and complex analysis.

Hankel operators are one of the most important classes of bounded linear operators acting on spaces of analytic functions. They have many connections with function theory, harmonic analysis, approximation theory, moment problems, spectral theory, orthogonal polynomials, stationary Gaussian processes, $\bar{\partial}$ -operator . . . etc. The book of V. Peller [42] is an acknowledged reference in the classical theory of Hankel operators and their various applications. We are interested in the behavior of the singular values of Hankel operators with anti-analytic symbols acting on Bergman spaces. The first work in this subject is due to Axler [9] who described boundedness and compactness of such operators in the classical Bergman space. Right after, Arazy, Fischer and Peetre [7] studied the membership for such operators in Schatten classes. They highlighted the existence of a cut-off by proving that these operators can not be of finite trace. In [23], Engliš and Rochberg described, using the Boutet de Monvel-Guillemin theory, all Hankel operators with anti-analytic symbols that belongs to the Dixmier class and gave an explicit formula for Dixmier Trace in this case. Several authors considered the same problems in other spaces of analytic functions [13, 28, 33, 36, 37, 46, 52]. In this work, we give the asymptotic behavior of the singular values of compact Hankel operators, we describe the class of symbols for which the associated Hankel operators have the critical decay and compute the asymptotics of singular values of Hankel operators associated with this class of symbols.

Throughout this thesis, ω will denote a function from \mathbb{D} into $]0, \infty[$ which is integrable with

respect to the Lebesgue measure. The associated Bergman space will be denoted by A_ω^2 and is given by

$$A_\omega^2 = \left\{ f \in \text{Hol}(\mathbb{D}) : \|f\|_\omega := \left(\int_{\mathbb{D}} |f(z)|^2 dA_\omega(z) \right)^{1/2} < \infty \right\}, \quad dA_\omega := \omega dA.$$

We will also assume that ω is bounded below by a positive constant on each compact set of \mathbb{D} . This implies that A_ω^2 is a reproducing kernel Hilbert space.

Let Ω be any set on the complex plane \mathbb{C} and let $(\mathcal{H}, \|\cdot\|)$ be a Hilbert space of complex-valued functions on Ω endowed with an inner product $\langle \cdot, \cdot \rangle$ such that the point evaluation

$$\begin{aligned} e_w &: \mathcal{H} \rightarrow \mathbb{C} \\ f &\mapsto f(w) \end{aligned}$$

is bounded at each point w in Ω . Then, by Riesz representation theorem, for each $w \in \Omega$, there exists a unique function $K_w = K(\cdot, w)$ in \mathcal{H} such that

$$f(w) = \langle f, K_w \rangle, \quad f \in \mathcal{H}. \quad (1)$$

Definition 0.1. Suppose Ω is any set on \mathbb{C} and \mathcal{H} is a Hilbert space of complex-valued functions on Ω . We say that \mathcal{H} is a *Reproducing kernel Hilbert space* if, for each $z \in \Omega$, there exists a function $K_z(\cdot)$ in \mathcal{H} satisfying (1).

The function $K_w(z) = K(z, w)$, with $(z, w) \in \Omega \times \Omega$, is called **the reproducing kernel** of \mathcal{H} .

The reproducing kernel of A_ω^2 will be denoted by K .

The orthogonal projection from $L_\omega^2 := L^2(\mathbb{D}, dA_\omega)$ onto A_ω^2 will be designated by P_ω . It can be represented as follows

$$P_\omega(f)(z) = \int_{\mathbb{D}} f(\zeta) K(z, \zeta) dA_\omega(\zeta), \quad f \in L_\omega^2.$$

Therefore, the domain of P_ω can be extended to all functions f such that $fK_z \in L_\omega^1 := L^1(\mathbb{D}, dA_\omega)$ for all $z \in \mathbb{D}$. Let $g \in L_\omega^2$ such that $gK_z \in L_\omega^2$ for all $z \in \mathbb{D}$. The (big) Hankel operator H_g with symbol g is the densely defined operator on A_ω^2 given by

$$H_g f = gf - P_\omega(gf),$$

where $f = \sum_{1 \leq i \leq n} c_i K_{z_i}$, with $c_i \in \mathbb{C}$ and $z_i \in \mathbb{D}$.

The explicit formula for H_g is given by

$$H_g f(z) = \int_{\mathbb{D}} (g(z) - g(w)) f(w) K(z, w) dA_\omega(w), \quad z \in \mathbb{D}.$$

We are interested in anti-analytic symbols on \mathbb{D} , $g = \bar{\phi}$ where ϕ is an analytic function.

We refer for the notations to the rest of the thesis.

We impose some conditions on the weight ω that allow us to characterize many proprieties satisfied by the Hankel operator, such as the boundedness, the compactness and the membership to Schatten classes. We also use in the proofs tools such as subharmonicity and Cauchy formula. We will need estimates of solutions of the $\bar{\partial}$ equation. Then we use a result on Toeplitz operators to prove one of the main results of our thesis which is the following trace estimates :

Theorem 0.1. *Let $\omega \in \mathcal{W}^*$ and let $\phi \in \mathcal{B}_0^\omega$. Let $h : [0, +\infty[\rightarrow [0, +\infty[$ be an increasing convex function such that $h(0) = 0$. Then there exists $B > 0$, which depends only on ω , such that*

$$\int_{\mathbb{D}} h \left(\frac{1}{B} \tau_\omega(z) |\phi'(z)| \right) d\lambda_\omega(z) \leq \text{Tr} (h(|H_{\bar{\phi}}|)) \leq \int_{\mathbb{D}} h (B \tau_\omega(z) |\phi'(z)|) d\lambda_\omega(z).$$

The previous theorem allows us to show the following results :

Theorem 0.2. *Let $\omega \in \mathcal{W}^*$ be such that τ_ω is equivalent to a radial function and let $\phi \in \mathcal{B}_0^\omega$. Then*

1. $s_n(H_{\bar{\phi}}) = o(s_n(H_{\bar{z}})) \implies H_{\bar{\phi}} = 0$.
2. $s_n(H_{\bar{\phi}}) = o(\mathcal{R}_{z,\omega}^+(n)) \implies H_{\bar{\phi}} = 0$.

This theorem gives a cut-off for the singular values of Hankel operator, which means that when the decay of the singular values of Hankel operator is faster than a certain rate then the Hankel operator is null.

The following result is also proven by using the trace estimates and the non-tangential maximal function :

Theorem 0.3. Let $\omega \in \mathcal{W}^*$ be a weight such that $\tau_\omega^2(z) \asymp (1 - |z|^2)^{2+\beta} \nu^2(\log(\frac{1}{1-|z|^2}))$, where $\beta > 0$ and ν is a monotone function such that $\nu(2t) \asymp \nu(t)$. Let $\phi \in \mathcal{B}_0^\omega$ and let $p = \frac{2(1+\beta)}{2+\beta}$. Then

$$s_n(H_{\overline{\phi}}) = O\left(\frac{1}{n^{\frac{1}{p}} \nu^{\frac{1}{1+\beta}}(\log n)}\right) \iff \phi' \in H^p.$$

In this case we have

$$s_n(H_{\overline{\phi}}) \lesssim \frac{\|\phi'\|_p}{n^{\frac{1}{p}} \nu^{\frac{1}{1+\beta}}(\log n)},$$

where the involved constant doesn't depend on ϕ .

Now we introduce some tools that will be useful in the proof of the theorem on asymptotics.

Let T be a compact operator between two complex Hilbert spaces. The decreasing sequence of singular values of T will be denoted by $(s_n(T))_n$. The counting function of the singular values of T is denoted by

$$n(s, T) = \#\{n : s_n(T) \geq s\}, \quad s > 0.$$

We denote \mathcal{C}_p the class of increasing continuous function $\psi : (0, +\infty) \rightarrow (0, +\infty)$ satisfying

$$\psi(\alpha t) \sim \alpha^p \psi(t), \quad (t \rightarrow 0^+),$$

The quantities $D_\psi(T)$ and $d_\psi(T)$ are given by

$$D_\psi(T) := \limsup_{s \rightarrow 0^+} \psi(s) n(s, T) \quad \text{and} \quad d_\psi(T) := \liminf_{s \rightarrow 0^+} \psi(s) n(s, T).$$

Let us denote

$$\Sigma_\psi := \{T \in S_\infty : \psi(s) n(s, T) = O(1)\}$$

and

$$\Sigma_\psi^0 := \{T \in S_\infty : \psi(s) n(s, T) = o(1)\}.$$

Now we state the Theorem of A. Pushnitski [[45], Theorem 2.2] in the following form

Theorem 0.4. Let T_1, T_2, \dots, T_n be a compact operators such that

$$T_i^* T_j \in \Sigma_{\psi \circ \sqrt{\cdot}}^0 \quad \text{and} \quad T_i T_j^* \in \Sigma_{\psi \circ \sqrt{\cdot}}^0, \quad \forall i \neq j.$$

Then

$$D_\psi \left(\sum_{k=1}^n T_k \right) = \limsup_{s \rightarrow 0^+} \psi(s) \sum_{k=1}^n n(s, T_k) \quad \text{and} \quad d_\psi \left(\sum_{k=1}^n T_k \right) = \liminf_{s \rightarrow 0^+} \psi(s) \sum_{k=1}^n n(s, T_k).$$

In particular, if for all $s > 0$, we have $n(s, T_k) = n(s, T_1)$, $k = 1, 2, \dots, n$, then

$$D_\psi \left(\sum_{k=1}^n T_k \right) = n D_\psi(T_1) \quad \text{and} \quad d_\psi \left(\sum_{k=1}^n T_k \right) = n d_\psi(T_1).$$

Now we state our main result which is a theorem on the asymptotics of the singular values of Hankel operator.

Theorem 0.5. Let $\omega \in \mathcal{W}^*$ be a radial weight such that $\tau_\omega^2(z) \asymp (1 - |z|^2)^{2+\beta} \nu^2(\log(\frac{1}{1-|z|^2}))$, where $\beta > 0$ and ν is a monotone function such that $\nu(2t) \asymp \nu(t)$. Suppose that

$$s_n(H_{\bar{z}}) \sim \frac{\gamma}{n^{\frac{1}{p}} \nu^{\frac{1}{1+\beta}}(\log n)},$$

where $\gamma > 0$ and $p = \frac{2(1+\beta)}{2+\beta}$. Then, for $\phi' \in H^p$, we have

$$s_n(H_{\bar{\phi}}) \sim \frac{\gamma}{n^{\frac{1}{p}} \nu^{\frac{1}{1+\beta}}(\log n)} \|\phi'\|_p.$$

Next, we give an example. For a radial weight ω , the sequence of the singular values of the Hankel operator $H_{\bar{z}}$ on A_ω^2 is the nonincreasing rearrangement of the sequence

$$\left(\left(\frac{\|z^{n+1}\|^2}{\|z^n\|^2} - \frac{\|z^n\|^2}{\|z^{n-1}\|^2} \right)^{1/2} \right)_n.$$

We consider the weight

$$\omega(z) = \exp \left(-\frac{\alpha}{(\log \frac{1}{|z|^2})^\beta} \right), \quad \alpha, \beta > 0.$$

We obtain after computation

$$s_n(H_{\bar{z}}) \sim \frac{\gamma}{n^{\frac{\beta+2}{2(\beta+1)}}}.$$

where $\gamma = \sqrt{\frac{(\alpha\beta)^{1/1+\beta}}{1+\beta}}$.

Now we get to the second part of the thesis in which we are interested in Bergman spaces satisfying a Hardy-Littlewood theorem. The weighted Bergman space A_ω^p , $p > 0$, associated with a weight ω is defined by

$$A_\omega^p := \{f \in \text{Hol}(\mathbb{D}) : \|f\|_{A_\omega^p}^p := \int_{\mathbb{D}} |f(z)|^p \omega(z) dA(z) < \infty\},$$

We say that ω satisfies Hardy-Littlewood theorem, and we write $\omega \in \mathcal{HL}$, if for all $p > 0$, $f \in \text{Hol}(\mathbb{D})$, and $n \in \mathbb{N}^*$, we have

$$\|f\|_{A_\omega^p}^p \asymp \sum_{k=0}^{n-1} |f^{(k)}(0)|^p + \int_{\mathbb{D}} |f^{(n)}(z)|^p (1 - |z|^2)^{np} \omega(z) dA(z).$$

A weight ω is said to be in the class of Bekollé-Bonami weights $B_p(\eta)$, for $p \in (1, \infty)$ and $\eta > -1$, if

$$\left(\int_{S(\theta, h)} \omega dA_\eta \right) \left(\int_{S(\theta, h)} \omega^{-p'/p} dA_\eta \right)^{p/p'} \lesssim (A_\eta(S(\theta, h)))^p$$

for any Carleson square

$$S(\theta, h) = \{re^{i\alpha} : 1 - h < r < 1, |\theta - \alpha| < h/2\}, \quad \theta \in [0, 2\pi], h \in (0, 1).$$

Where $1/p + 1/p' = 1$ and $dA_\eta(z) = (\eta + 1)(1 - |z|^2)^\eta dA(z)$.

In [2], Aleman and Constantin proved that if $\frac{\omega}{(1-|z|^2)^\eta} \in B_{p_0}(\eta)$, for some $p_0 > 1$ and $\eta > -1$, then $\omega \in \mathcal{HL}$. Conversely, if $\omega \in \mathcal{HL}$ is such that

$$\omega(z) \asymp \omega(w), \quad |z - w| < \delta(1 - |z|^2), \quad (2)$$

for some $\delta \in (0, 1)$ then $\frac{\omega}{(1-|z|^2)^\eta} \in B_{p_0}(\eta)$, for some $p_0 > 1$ and $\eta > -1$ [3].

Recently, G. Bao, Wulan and K. Zhu have introduced in [10] a new class of weights which satisfy (2.1). For $s > -1$ and $t \geq 0$, we say that $\omega \in \mathcal{W}(s, t)$ if

$$\omega_{s,t}(z) := \int_{\mathbb{D}} \frac{\omega(w)(1 - |w|^2)^s (1 - |z|^2)^t}{|1 - \bar{w}z|^{2+s+t}} dA(w) \lesssim \omega(z), \quad z \in \mathbb{D}.$$

It is proved in [10] that if there exist $s_0 \in (-1, 0)$ and $t_0 > 0$ such that $\omega \in \mathcal{W}(s, t)$ for all $s > s_0$ and all $t > t_0$ and there exists $\delta \in (0, 1)$ such that ω satisfies (2.1) then $\omega \in \mathcal{HL}$.

We have the following result:

Proposition 0.1. *Let ω be a weight such that $\omega \in \mathcal{W}(s, t)$ for some $s \in (-1, 0]$ and some $t \geq 0$. Then $\omega \in \mathcal{HL}$.*

We will consider the class \mathcal{C}_p ($p > 1$) of weights ω such that

$$\left(\int_{\Delta(z, \alpha)} \omega dA \right) \left(\int_{\Delta(z, \alpha)} \omega^{-p'/p} dA \right)^{p/p'} \lesssim A^p(\Delta(z, \alpha)),$$

for all $\alpha \in (0, 1)$. Where

$$\Delta(z, r) := \{w \in \mathbb{D} : |z - w| < r(1 - |z|)\}; \quad z \in \mathbb{D} \text{ and } 0 < r < 1.$$

Let

$$\tilde{\omega}(z) := \frac{1}{(1 - |z|^2)^2} \int_{\Delta(z, 1/2)} \omega(\zeta) dA(\zeta), \quad z \in \mathbb{D}.$$

We prove the following theorem which is a variant of the results in [2, 3] in this subject.

Theorem 0.6. *Suppose that $\omega \in \mathcal{C}_{p_0}$ for some $p_0 > 1$. The following assertions are equivalent.*

1. $\omega \in \mathcal{HL}$.
2. $\tilde{\omega} \in \mathcal{HL}$.
3. $\omega_{0,t} \lesssim \tilde{\omega}$ for some $t \geq 0$.

For a radial weights let

$$\tilde{\omega}(r) = \frac{1}{1 - r} \int_r^1 \omega(s) ds.$$

Then we get the following results:

Theorem 0.7. *Let ω be a radial weight. The following assertions are equivalent.*

1. $\omega \in \mathcal{HL}$.
2. $\tilde{\omega}$ is a weight and $\omega_{s,t}(r) \lesssim \tilde{\omega}(r)$, $r \in [0, 1)$, for some $s \in]-1, 0[$ and $t \geq 0$.

In this case we have $A_\omega^p = A_{\omega_{s,t}}^p$ with $\|f\|_{A_\omega^p} \asymp \|f\|_{A_{\omega_{s,t}}^p}$ for all $f \in \text{Hol}(\mathbb{D})$ and $p > 0$.

We get also the following result:

Theorem 0.8. *Let ω be a radial weight such that $\tilde{\omega}(r) \lesssim \omega(r)$ for all $r \in [0, 1)$. Then, the following are equivalent.*

1. $\omega \in \mathcal{HL}$ (or equivalently $\omega \in \mathcal{D}$).
2. $\omega \in \hat{D}$.
3. $\omega \in \mathcal{W}(s, t)$ for some $s \in (-1, 0)$ and $t \geq 0$.
4. $\omega/(1 - |z|^2)^\eta \in B_p(\eta)$ for some $\eta \geq 0$ and for all $p > 1$.
5. $\omega/(1 - |z|^2)^\eta \in B_p(\eta)$ for some $\eta \geq 0$ and some $p > 1$.

We consider the following class of weights.

Definition 0.2. *Let $p > 1$ and $t \geq 0$. We say that a weight ω (not necessarily radial) belongs to the class $\mathcal{C}_{p,t}$ if*

$$\omega \in \mathcal{C}_p \text{ and } \omega_{0,t} \lesssim \tilde{\omega}.$$

Next we consider Toeplitz and Integration operators on A_ω^2 for ω belonging to $\mathcal{C}_{p_0,t}$ for some $p_0 > 1$ and $t \geq 0$.

Recall that the Toeplitz operator on A_ω^2 associated with a positive Borel measure μ on \mathbb{D} is the transformation

$$T_\mu f(z) := \int_{\mathbb{D}} f(\zeta) K^\omega(z, \zeta) \omega(\zeta) d\mu(\zeta), \quad (z \in \mathbb{D}).$$

We denote by μ_ω the measure $d\mu_\omega(z) := \omega(z) d\mu(z)$, $z \in \mathbb{D}$. We have the following theorem :

Theorem 0.9. *Let μ be a positive finite Borel measure on \mathbb{D} . The following assertions are equivalent*

1. The Toeplitz operator T_μ is bounded (resp. compact) on A_ω^2 .
2. $\mu_\omega(\Delta_n) = O(A_\omega(\Delta_n))$ (resp. $o(A_\omega(\Delta_n))$), $n \rightarrow \infty$.

We also have the following theorem which improves and extends the result of Constantin in [16].

Theorem 0.10. *Suppose that $\omega \in \mathcal{C}_{p_0,t}$ for some $p_0 > 1$ and $t \geq 0$. Let μ be a positive finite Borel measure on \mathbb{D} such that T_μ is compact on A_ω^2 and let $p > 0$. Then T_μ belongs to $\mathcal{S}_p(A_\omega^2)$ if and only if*

$$\sum_{n=0}^{\infty} \left(\frac{\mu_\omega(\Delta_n)}{A_\omega(\Delta_n)} \right)^p < \infty.$$

The previous theorem ends our discussion on Toeplitz operators. Now we are going to put our interest in the generalized Volterra integration operator. For analytic functions f and g on \mathbb{D} , the generalized Volterra integration operator V_g is defined by

$$V_g f(z) := \int_0^z f(\zeta) g'(\zeta) d\zeta.$$

The boundedness, compactness and membership to Schatten classes of V_g are the goals of several papers on various spaces of analytic functions. For instance, see [1, 4, 5, 51]. A. Aleman and A. Siskakis characterized in [5] the boundedness and compactness of V_g on the standard Bergman space A_α^2 . They proved that V_g is bounded (resp. compact) on A_α^2 if and only if g belongs to the Bloch space \mathcal{B} (resp. to the little Bloch space \mathcal{B}_0). Recall that

$$\mathcal{B} := \{g \in \text{Hol}(\mathbb{D}), \sup_{z \in \mathbb{D}} (1 - |z|^2) |g'(z)| < \infty\},$$

and

$$\mathcal{B}_0 := \{g \in \text{Hol}(\mathbb{D}), \limsup_{|z| \rightarrow 1} (1 - |z|^2) |g'(z)| = 0\}.$$

The following result describes the boundedness of V_g on A_ω^2 , improving that obtained in [2, 16].

Theorem 0.11. *The Volterra operator V_g is bounded (resp. compact) on A_ω^2 if and only if $g \in \mathcal{B}$ (resp. \mathcal{B}_0).*

Also, they characterized the membership of V_g to $\mathcal{S}_p(A_\alpha^2)$: $V_g \in \mathcal{S}_p(A_\alpha^2)$, for $p > 1$ if and only if g belong to the Besov space

$$\mathcal{B}_p := \{g \in \text{Hol}(\mathbb{D}), \int_{\mathbb{D}} |g'(z)|^p (1 - |z|^2)^{p-2} dA(z) < \infty\}.$$

And $V_g \in \mathcal{S}_1(A_\alpha^2)$ if and only if g is constant. If $\frac{\omega}{(1-|z|^2)^\eta} \in B_p(\eta)$, Constantin prove in [16] that, for $p \geq 2$, the operator V_g belongs to $\mathcal{S}_p(A_\omega^2)$ if and only if g belongs to \mathcal{B}_p . We improve and extend that in the following theorem.

Theorem 0.12. *Suppose that $\omega \in \mathcal{C}_{p_0, t}$ for some $p_0 > 1$ and $t \geq 0$. Let g be an analytic function on \mathbb{D} such that that V_g is a compact operator on A_ω^2 . Then we have*

1. *If $p > 1$ then V_g belongs to $\mathcal{S}_p(A_\omega^2)$ if and only if g belongs to \mathcal{B}_p .*
2. *If $p \leq 1$ then V_g belongs to $\mathcal{S}_p(A_\omega^2)$ if and only if $g' = 0$ on \mathbb{D} .*

It is known that if ω is any radial weight on \mathbb{D} then the polynomials are dense on the Bergman space A_ω^2 . for the non-radial case, it is also known that the fact that polynomials are dense on A_ω^2 is not always true. It is so interesting to find the weights for which the polynomials are dense on weighted Bergman spaces. We have the following result.

Theorem 0.13. *Suppose that $\omega \in \mathcal{C}_{p_0, t}$ for some $p_0 > 1$ and $t \geq 0$. Then*

$$\|f_r\|_{A_\omega^p} \leq C(\omega) \|f\|_{A_\omega^p}, \quad f \in A_\omega^p.$$

And, the polynomials are dense in A_ω^p for all $p > 0$.

CHAPTER 1

ON SINGULAR VALUES OF HANKEL OPERATORS ON BERGMAN SPACES

1.1 Introduction

The space of all holomorphic functions on the unit disc \mathbb{D} in the complex plane \mathbb{C} will be denoted by $\text{Hol}(\mathbb{D})$. The Lebesgue measure on \mathbb{C} is denoted by dA . The standard Bergman space A_α^2 with $\alpha > -1$, consists of holomorphic functions f on \mathbb{D} such that

$$\|f\|_\alpha^2 := \int_{\mathbb{D}} |f(z)|^2 dA_\alpha(z) < \infty,$$

where $dA_\alpha(z) := \frac{(\alpha+1)}{\pi}(1 - |z|^2)^\alpha dA(z)$.

Recall that A_α^2 is a reproducing kernel Hilbert space with the kernel

$$K(z, w) = \frac{1}{(1 - \bar{w}z)^{\alpha+2}}, \quad z, w \in \mathbb{D}.$$

The orthogonal projection of $L_\alpha^2 := L^2(dA_\alpha)$ onto A_α^2 will be denoted by P_α . Given $g \in L^1(dA_\alpha)$, the linear transformation

$$H_g f = gf - P_\alpha(gf)$$

is a densely defined operator of A_α^2 into $L_\alpha^2 \ominus A_\alpha^2$. The operator H_g is called the (big) Hankel operator with symbol g . For general facts concerning Hankel operators on Bergman spaces we refer to [7, 52]. In this chapter we are interested in Hankel operator $H_{\bar{\phi}}$ with anti-analytic symbol

$\bar{\phi}$. First, recall that the Bloch space \mathcal{B} and the little Bloch space \mathcal{B}_0 are given by

$$\mathcal{B} = \{\phi \in \text{Hol}(\mathbb{D}), \quad \sup(1 - |z|^2)|\phi'(z)| < \infty\}$$

$$\mathcal{B}_0 = \{\phi \in \text{Hol}(\mathbb{D}), \quad \lim_{|z| \rightarrow 1^-} (1 - |z|^2)|\phi'(z)| < \infty\}$$

In [7], J. Arazy, S. Fisher and J. Peetre proved that $H_{\bar{\phi}}$ is bounded (*resp.* compact) on A_α^2 if and only if ϕ belongs to the Bloch space \mathcal{B} (*resp.* ϕ belongs to the little Bloch space \mathcal{B}_0). This result was first proved by S. Axler [9] in the case $\alpha = 0$. The membership in Schatten classes of Hankel operators $H_{\bar{\phi}}$ was also studied by J. Arazy, S. Fisher and J. Peetre in [7]. They proved that $H_{\bar{\phi}} \in \mathcal{S}_p(A_\alpha^2)$, for $p > 1$, if and only if ϕ belongs to the Besov space \mathcal{B}_p defined by

$$\mathcal{B}_p = \{\phi \in \text{Hol}(\mathbb{D}), \quad \int_{\mathbb{D}} |\phi'(z)|^p (1 - |z|^2)^{p-2} dA(z) < \infty\}.$$

For $p \leq 1$, they proved that $H_{\bar{\phi}} \in \mathcal{S}_p(A_\alpha^2)$ if and only if $H_{\bar{\phi}} = 0$ (which means that ϕ is a constant). These results were extended by Galanopoulos and Pau in [26] to large Bergman spaces associated with radial weights.

Let $\omega =: e^{-\varphi}$ be a weight on \mathbb{D} such that φ is a regular subharmonic function and let A_ω^2 be the weighted Bergman space given by

$$A_\omega^2 = \{f \in \text{Hol}(\mathbb{D}) : \|f\|_\omega := \left(\int_{\mathbb{D}} |f(z)|^2 dA_\omega(z) \right)^{1/2} < \infty\}, \quad dA_\omega := \omega dA.$$

As before, the Hankel operator with anti-analytic symbol $\bar{\phi}$ is the operator $H_{\bar{\phi}} : L^2(dA_\omega) \rightarrow L^2(dA_\omega) \ominus A_\omega^2$ given by

$$H_{\bar{\phi}}f = \bar{\phi}f - P_\omega(\bar{\phi}f),$$

where P_ω is the orthogonal projection from $L^2(dA_\omega)$ onto A_ω^2 .

For a class of radial rapidly decreasing weights $\omega(z) = e^{-\varphi(|z|)}$, Galanopoulos and Pau proved in [26] that $H_{\bar{\phi}}$ is bounded (*resp.* compact) if and only if $\frac{|\phi'|^2}{\Delta\varphi}$ is bounded (*resp.* $\lim_{|z| \rightarrow 1^-} \frac{|\phi'(z)|^2}{\Delta\varphi(z)} = 0$). They also prove, for $p > 1$, that $H_{\bar{\phi}} \in \mathcal{S}_p$ if and only if $\frac{|\phi'|^2}{\Delta\varphi} \in L^{p/2}(\Delta\varphi dA)$.

Note that for $\omega_\alpha(z) =: e^{-\varphi_\alpha(z)} = (1 - |z|^2)^\alpha$, $\alpha > 0$, we have $\Delta\varphi_\alpha \asymp \frac{1}{(1-|z|^2)^2}$ and thus the conditions given by Galanopoulos and Pau are the same as those given by Arazy, Fisher and Peetre in this case.

Our goal in this paper is to study the asymptotic behavior of the singular values of $H_{\bar{\phi}}$. We will consider the class of weights \mathcal{W}^* which includes weights considered in [7, 26] (see the first and

the third examples given in Section 1.2.1). In order to state our main results, we introduce some notations. The reproducing kernel of A_ω^2 is denoted by K ,

$$\tau_\omega(z) = \frac{1}{\omega^{1/2}(z)\|K_z\|} \quad \text{and} \quad d\lambda_\omega = \frac{dA}{\tau_\omega^2}.$$

It should be noted that in several, but not all, situations τ_ω^2 is comparable to $1/\Delta\varphi$. For more information, see the examples given in Section 1.2.1.

By analogy with the standard case, we write \mathcal{B}^ω (resp. \mathcal{B}_0^ω) for the space of analytic functions ϕ on \mathbb{D} such that $\sup_{z \in \mathbb{D}} \tau_\omega(z)|\phi'(z)| < \infty$ (resp. $\lim_{|z| \rightarrow 1^-} \tau_\omega(z)|\phi'(z)| = 0$).

The following theorem will play an important role in the sequel.

Theorem 1.1. *Let $\omega \in \mathcal{W}^*$ and let $\phi \in \mathcal{B}_0^\omega$. Let $h : [0, +\infty[\rightarrow [0, +\infty[$ be an increasing convex function such that $h(0) = 0$. Then there exists $B > 0$, which depends only on ω , such that*

$$\int_{\mathbb{D}} h\left(\frac{1}{B}\tau_\omega(z)|\phi'(z)|\right) d\lambda_\omega(z) \leq \text{Tr}(h(|H_\phi^-|)) \leq \int_{\mathbb{D}} h(B\tau_\omega(z)|\phi'(z)|) d\lambda_\omega(z).$$

It should be noted that theorem 2.4 is new even in the A_α^2 case.

Let us denote by $\mathcal{R}_{\phi,\omega}^+$ the nonincreasing rearrangement of the function $\tau_\omega|\phi'|$ with respect to $d\lambda_\omega$. Namely,

$$\mathcal{R}_{\phi,\omega}^+(x) := \sup\{t \in (0, \|\tau\phi'\|_\infty] : \mathcal{R}_{\phi,\omega}(t) \geq x\},$$

where $\mathcal{R}_{\phi,\omega}$ is the distribution function given by

$$\mathcal{R}_{\phi,\omega}(t) := \lambda_\omega(\{z \in \mathbb{D} : \tau_\omega(z)|\phi'(z)| > t\}).$$

As a first consequence of Theorem 2.4, we prove that if ρ is an increasing function such that $\rho(x)/x^\gamma$ is decreasing for some $\gamma < 1$, then

$$s_n(H_\phi^-) \asymp 1/\rho(n) \iff \mathcal{R}_{\phi,\omega}^+(n) \asymp 1/\rho(n).$$

The next result is motivated by a problem raised in [6, 7]. We prove in the following theorem that the critical decay of $(s_n(H_\phi^-))_n$ is achieved by the symbol $\phi = z$.

Theorem 1.2. *Let $\omega \in \mathcal{W}^*$ be such that τ_ω is equivalent to a radial function and let $\phi \in \mathcal{B}_0^\omega$. Then*

1. $s_n(H_{\bar{\phi}}) = o(s_n(H_{\bar{z}})) \implies H_{\bar{\phi}} = 0.$
2. $s_n(H_{\bar{\phi}}) = o(\mathcal{R}_{z,\omega}^+(n)) \implies H_{\bar{\phi}} = 0.$

Note that, in general, it is not difficult to estimate $\mathcal{R}_{z,\omega}^+(n)$. For example, if $\tau_\omega^2(z) \asymp (1 - |z|^2)^{2+\beta}$ with $\beta \geq 0$, then $\mathcal{R}_{z,\omega}^+(n) \asymp n^{-\frac{2+\beta}{2(1+\beta)}}$. Thus, Theorem 1.2 implies that

$$s_n(H_{\bar{\phi}}) = o\left(n^{-\frac{2+\beta}{2(1+\beta)}}\right) \implies H_{\bar{\phi}} = 0.$$

Now our goal is to describe the class of functions $\phi \in \mathcal{B}_0^\omega$ such that $(s_n(H_{\bar{\phi}}))_n$ has the critical decay. For simplicity, we state our result only in the case $\tau_\omega^2(z) \asymp (1 - |z|^2)^{2+\beta}$.

As usual, the Hardy space H^p , $p \geq 1$, consists of analytic functions f on \mathbb{D} such that

$$\|f\|_{H^p}^p := \sup_{0 \leq r < 1} \frac{1}{2\pi} \int_0^{2\pi} |f(re^{i\theta})|^p d\theta < \infty.$$

Before stating our result, recall that in the setting of the classical Bergman space $A^2 = A_0^2$ M. Engliš and R. Rochberg proved in [23] that $\sum_{j=1}^n s_j(H_{\bar{\phi}}) = O(1/n)$ if and only if $\phi' \in H^1$. Next, R. Tytgat [48, 49] extended this result to A_α^2 , $\alpha > -1$. We would also like to point out that M. Dostanic [17] has shown for the classical Bergman space A^2 that if ϕ is analytic in some neighborhood of $\bar{\mathbb{D}}$ then $s_n(H_{\bar{\phi}}) \sim \frac{\|\phi'\|_{H^1}}{n}$. The following theorem improves and extends all these results.

Theorem 1.3. *Let $\omega \in \mathcal{W}^*$ be such that $\tau_\omega^2(z) \asymp (1 - |z|^2)^{2+\beta}$ with $\beta \geq 0$. Set $p = \frac{2(1+\beta)}{2+\beta}$ and let $\phi \in \mathcal{B}_0^\omega$. We have*

$$s_n(H_{\bar{\phi}}) = O(1/n^{1/p}) \iff \phi' \in H^p.$$

Furthermore, if ω is radial and $s_n(H_{\bar{z}}) \sim \gamma/n^{\frac{1}{p}}$ for some $\gamma \in (0, \infty)$, then

$$s_n(H_{\bar{\phi}}) \sim \|\phi'\|_{H^p} \frac{\gamma}{n^{\frac{1}{p}}}, \quad \phi' \in H^p.$$

In what follows we study this problem in more general situations. It should be noted that if $\tau_\omega^2(z) \asymp (1 - |z|)^{2+\beta} \log^{-\alpha}(2/(1 - |z|^2))$ with $\alpha \geq 0$, then $\mathcal{R}_{z,\omega}^+(n) \asymp \frac{\log^{\alpha/1+\beta}(n)}{n^{1/p}}$. We prove in Theorem 1.9 that if $\beta > 0$, then

$$s_n(H_{\bar{\phi}}) = O\left(\frac{\log^{\alpha/1+\beta}(n)}{n^{1/p}}\right) \iff \phi' \in H^p.$$

However, if $\beta = 0$ and $\alpha > 0$, curiously, the previous result is not valid. This is the subject of Proposition 1.3.

For the standard Bergman spaces A_α^2 we have $\tau_{\omega_\alpha}^2(z) \asymp (1 - |z|)^2$. It is known, and can be easily seen, that $s_n(H_{\bar{z}}) \sim \frac{\sqrt{1+\alpha}}{n+1}$. Hence, Theorem 1.3 says that if $\phi \in \mathcal{B}_0$ then

$$s_n(H_{\bar{\phi}}) = O(1/n) \iff \phi' \in H^1.$$

And in this case $s_n(H_{\bar{\phi}}) \sim \frac{\sqrt{1+\alpha}}{n+1} \|\phi'\|_{H^1}$. Another example is given in Section 1.8.

In Section 2 we introduce definitions and notations that are used in the rest of the thesis. In section 3, we give a description of boundedness and compactness of Hankel operators $H_{\bar{\phi}}$ on A_ω^2 . We establish, in Section 4, an upper and a lower estimates of the Trace of $h(|H_{\bar{\phi}}|)$. The upper estimate is obtained from Hörmander type L_ω^2 estimates for $\bar{\partial}$ - equation and from recent estimates obtained by El-Fallah and El Ibboui for Toeplitz operators [19, 20]. The lower estimate is obtained through a sort of local Berezin transform of $H_{\bar{\phi}}$. Two direct consequences of trace estimates are obtained by a suitable choice of the convex function h . The first one gives a sharp asymptotic estimates of the singular values of compact operators of $H_{\bar{\phi}}$. The second, presented in Section 5, shows that the critical decay of the sequence $(s_n(H_{\bar{\phi}}))_n$ is achieved by the symbol $\phi = z$. In Section 6, we prove the first assertion of Theorem 1.3. The proof is based on Theorem 2.4 and on estimates of some maximal non-tangential functions. The second part of Theorem 1.3 is given in Section 7. The proof of this result is based on the first part of Theorem 1.3 and on a result on asymptotic orthogonality due to A. Pushnitski (see Appendix). Section 8 is devoted to an explicit example.

Throughout this thesis, the notation $A \lesssim B$ means that there is a constant c independent of the relevant variables such that $A \leq cB$. We write $A \asymp B$ if both $A \lesssim B$ and $B \lesssim A$. As usual, the notation $u_n \sim v_n$ means that $\lim_{n \rightarrow \infty} u_n/v_n = 1$. The letter C will denote an absolute constant whose value may change at different occurrences.

1.2 Preliminaries

1.2.1 The class of weights \mathcal{W}^*

Throughout this paper, ω will denote a function from \mathbb{D} into $]0, \infty[$ which is integrable with respect to the Lebesgue measure. The associated Bergman space will be denoted by A_ω^2 . We will also assume that ω is bounded below by a positive constant on each compact set of \mathbb{D} . This implies that A_ω^2 is a reproducing kernel Hilbert space. The kernel of A_ω^2 will be denoted by K . The orthogonal projection from $L_\omega^2 := L^2(\mathbb{D}, dA_\omega)$ onto A_ω^2 will be designated by P_ω . It can be represented as follows

$$P_\omega(f)(z) = \int_{\mathbb{D}} f(\zeta)K(z, \zeta)dA_\omega(\zeta), \quad f \in L_\omega^2.$$

So, the domain of P_ω can be extended to all functions f such that $fK_z \in L_\omega^1 := L^1(\mathbb{D}, dA_\omega)$, for all $z \in \mathbb{D}$. Let $g \in L_\omega^2$ such that $gK_z \in L_\omega^2$, for all $z \in \mathbb{D}$. The (big) Hankel operator H_g with symbol g is the densely defined operator on A_ω^2 defined by

$$H_g f = gf - P_\omega(gf),$$

where $f = \sum_{1 \leq i \leq n} c_i K_{z_i}$, with $c_i \in \mathbb{C}$ and $z_i \in \mathbb{D}$.

The explicit formula for H_g is

$$H_g f(z) = \int_{\mathbb{D}} (g(z) - g(w))f(w)K(z, w)dA_\omega(w), \quad z \in \mathbb{D}.$$

We are interested in this paper in anti-analytic symbols on \mathbb{D} , $g = \bar{\phi}$. In this case a direct computation gives the following useful formula

$$(H_{\bar{\phi}}K_a)(z) = (\bar{\phi}(z) - \bar{\phi}(a))K_a(z), \quad z, a \in \mathbb{D}. \quad (1.1)$$

First, we recall the definition of the class of weights \mathcal{W} introduced in [21]. Let

$$\tau_\omega(z) = \frac{1}{\omega^{1/2}(z) \|K_z\|_\omega}, \quad z \in \mathbb{D}.$$

Suppose that the reproducing kernel K satisfies the following conditions

$$\lim_{|z| \rightarrow 1^-} \|K_z\| = \infty \text{ and for every } \zeta \in \mathbb{D}, \quad |K(\zeta, z)| = o(\|K_z\|), \quad |z| \rightarrow 1^-. \quad (1.2)$$

We will suppose that τ_ω is such that

$$\tau_\omega(z) = O(1 - |z|), \quad z \in \mathbb{D}, \quad (1.3)$$

and that there exists constant $\eta > 0$ such that for $z, \zeta \in \mathbb{D}$ satisfying $|z - \zeta| \leq \eta\tau_\omega(z)$, we have

$$\tau_\omega(z) \asymp \tau_\omega(\zeta) \text{ and } \|K_z\|_\omega \|K_\zeta\|_\omega \lesssim |K(\zeta, z)|. \quad (1.4)$$

If the weight ω satisfies all the previous conditions, we shall say that the weight ω belongs to the class \mathcal{W} . Note that \mathcal{W} contains all standard weights. For more information, see the examples listed in [21].

The Laplace operator Δ is given by $\Delta = \partial\bar{\partial}$, with

$$\partial := \frac{1}{2}\left(\frac{\partial}{\partial x} - i\frac{\partial}{\partial y}\right), \quad \bar{\partial} := \frac{1}{2}\left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\right).$$

Let $\omega = e^{-\varphi} \in \mathcal{W}$ such that $\varphi \in \mathcal{C}^2(\mathbb{D})$. We shall suppose that

$$\tau_\omega^2(z)\Delta\varphi(z) \gtrsim 1, \quad z \in \mathbb{D}, \quad (1.5)$$

or, there exist a subharmonic function $\psi : \mathbb{D} \rightarrow \mathbb{R}^+$ and constants $\delta > 0$ and $t \in (-1, 0)$ such that for all $z \in \mathbb{D}$ we have

$$\tau_\omega^2(z)\Delta\psi(z) \geq \delta, \quad \Delta\varphi(z) \geq t\Delta\psi(z) \text{ and } |\partial\psi(z)|^2 \leq \Delta\psi(z). \quad (1.6)$$

Definition 1.1. We say that $\omega \in \mathcal{W}^*$ if $\omega \in \mathcal{W}$ and satisfies (1.5) or (1.6).

Examples

In this subsection, we give three examples which will be our references throughout this paper.

- **Standard Bergman spaces A_α^2 :** These spaces are associated with $\omega_\alpha(z) = \frac{(\alpha+1)}{\pi}(1 - |z|^2)^\alpha$, with $\alpha > -1$. The reproducing kernel of A_α^2 is $K(z, w) = \frac{1}{(1-\bar{w}z)^{\alpha+2}}$. Then

$$\tau_{\omega_\alpha}(z) = \sqrt{\frac{\pi}{\alpha+1}}(1 - |z|^2).$$

Clearly, $\omega_\alpha \in \mathcal{W}$. Note that if $\alpha > 0$, then $\tau_{\omega_\alpha}(z) \asymp 1/\sqrt{\Delta\varphi}$, where $\varphi = \log 1/\omega_\alpha$. Then ω_α satisfies (1.5) and $\omega_\alpha \in \mathcal{W}^*$. It is also clear that if $\alpha \in (-1, 0]$, then ω_α satisfies (1.6) with $t = \alpha$ and $\psi(z) = \log(1/(1 - |z|^2))$. Then $\omega_\alpha \in \mathcal{W}^*$.

- **Harmonically weighted Bergman spaces:** In this case we suppose that ω is a positive harmonic weight. It is proved in [22], that $\tau_\omega(z) \asymp 1 - |z|^2$ and $\omega \in \mathcal{W}$. One can easily see that $\varphi = \log(1/\omega)$ is subharmonic and that ω satisfies (1.6), with $t \in (-1, 0)$ and $\psi(z) = \log(1/(1 - |z|^2))$. It should be noted that in general τ_ω^2 and $1/\Delta\varphi$ are not comparable.

- **Large Bergman spaces:** The following class of weights was introduced by Hu, Lv and Schuster in [27]. It includes the classes considered in [14, 30, 34]. Let \mathcal{L}_0 be the class of functions $\tau \in \text{lip}(\mathbb{D}, \mathbb{R})$ such that $\lim_{|z| \rightarrow 1^-} \tau(z) = 0$. We denote \mathcal{W}_0 the set of weights $\omega = e^{-\varphi}$, where $\varphi \in \mathcal{C}^2$ is strictly subharmonic and for which there exists $\tau \in \mathcal{L}_0$ such that $\tau \asymp 1/\sqrt{\Delta\varphi}$. One can directly see, from [27], that if $\omega = e^{-\varphi} \in \mathcal{W}_0$, then $\omega \in \mathcal{W}^*$ and $\tau_\omega \asymp 1/\sqrt{\Delta\varphi}$.

1.2.2 Some inequalities involving convex functions.

The following elementary lemma is proved in [20].

Lemma 1.1. *Let $(a_n)_{n \geq 1}, (b_n)_{n \geq 1}$ be two decreasing sequences such that $\lim_{n \rightarrow \infty} a_n = \lim_{n \rightarrow \infty} b_n = 0$ and suppose that there exists $\gamma \in (0, 1)$ such that $(n^\gamma b_n)$ is increasing. Suppose that there exists $B > 0$ such that*

$$\sum_{n \geq 1} h(b_n/B) \leq \sum_{n \geq 1} h(a_n) \leq \sum_{n \geq 1} h(Bb_n),$$

for all increasing convex function h . Then $a_n \asymp b_n$.

We also need the following elementary result. For completeness, we include the proof.

Lemma 1.2. *Let A, p be two real numbers such that $0 < A < p$. Let ρ be an increasing function such that $\rho(x)/x^A$ is decreasing and let $(a_n)_n$ be a decreasing sequence such that,*

$$\sum_{n \geq 1} h(a_n) \leq \sum_{n \geq 1} h(1/\rho(n)),$$

for all increasing function h such that $h(t^p)$ is convex. Then

$$a_n \leq C(p, A)/\rho(n).$$

Proof. Let $\delta > 0$ and consider $h(t) = (t^{1/p} - \delta^{1/p})^+$. By hypothesis,

$$\sum_{n \geq 1} (a_n^{1/p} - \delta^{1/p})^+ \leq \sum_{n \geq 1} (1/\rho^{1/p}(n) - \delta^{1/p})^+.$$

Then we have

$$c(p) \sum_{a_n \geq 2\delta} a_n^{1/p} \leq \sum_{\rho(n) \leq 1/\delta} 1/\rho^{1/p}(n).$$

Using the fact that $n^{A/p}/\rho^{1/p}(n)$ is increasing and the fact that $A/p < 1$, we obtain

$$\delta^{1/p} \text{Card}\{n : a_n \geq 2\delta\} \leq C(p, A) \delta^{1/p} \text{Card}\{n : \rho(n) \leq 1/\delta\}.$$

This implies the result. □

The following elementary lemma will be useful in the proof of Theorem 2.4.

Lemma 1.3. *Let H, K be two Hilbert spaces and let $T : H \rightarrow K$ be a compact operator. Suppose there exist $(u_n)_{n \geq 1} \subset H, (v_n)_{n \geq 1} \subset K$ such that $\|u_n\| = \|v_n\| = 1$ and*

$$\sum_n |\langle u_n, u \rangle|^2 \leq C \|u\|^2 \text{ and } \sum_n |\langle v_n, v \rangle|^2 \leq C \|v\|^2 \quad (u \in H, v \in K),$$

for some $C > 0$. Then for any increasing convex function h such that $h(0) = 0$, we have

$$\sum_n h(|\langle Tu_n, v_n \rangle|) \leq C \sum_n h(s_n(T)).$$

Proof. Let $(e_n)_{n \geq 1}$ and $(f_n)_{n \geq 1}$ be the two orthonormal systems in H and K respectively such that $Tf = \sum_{k=1}^{+\infty} s_k(T) \langle f, e_k \rangle f_k$, for all $f \in H$. By Hölder's inequality we have

$$\sum_{k \geq 1} |\langle u_n, e_k \rangle \langle f_k, v_n \rangle| \leq \left(\sum_{k \geq 1} |\langle u_n, e_k \rangle|^2 \right)^{1/2} \left(\sum_{k \geq 1} |\langle f_k, v_n \rangle|^2 \right)^{1/2} \leq \|u_n\| \|v_n\| = 1, \quad \forall n \geq 1.$$

And

$$\sum_{n \geq 1} |\langle u_n, e_k \rangle \langle f_k, v_n \rangle| \leq \left(\sum_{n \geq 1} |\langle u_n, e_k \rangle|^2 \right)^{1/2} \left(\sum_{n \geq 1} |\langle f_k, v_n \rangle|^2 \right)^{1/2} \leq C \|e_k\| \|f_k\| = C, \quad \forall k \geq 1.$$

Since h is convex and $h(0) = 0$ we obtain

$$\begin{aligned} \sum_{n \geq 1} h(|\langle Tu_n, v_n \rangle|) &\leq \sum_{n \geq 1} h \left(\sum_{k \geq 1} s_k(T) |\langle u_n, e_k \rangle \langle f_k, v_n \rangle| \right) \\ &\leq \sum_{n \geq 1} \sum_{k \geq 1} h(s_k(T)) |\langle u_n, e_k \rangle \langle f_k, v_n \rangle| \\ &= \sum_{k \geq 1} h(s_k(T)) \sum_{n \geq 1} |\langle u_n, e_k \rangle \langle f_k, v_n \rangle| \\ &\leq C \sum_{k \geq 1} h(s_k(T)). \end{aligned}$$

The proof is complete. □

1.3 Boundedness and compactness of Hankel operators on A_ω^2

We need the following L^2 -estimates of solutions of the $\bar{\partial}$ -equation due to B. Berndtsson ([11], Theorem 3.1).

Theorem 1.4 (B. Berndtsson). *Let Ω be a domain in \mathbb{C} and let φ and ψ be subharmonic functions in Ω . Assume that ψ satisfies*

$$|\partial\psi(z)|^2 \leq \Delta\psi(z), \quad z \in \Omega. \quad (1.7)$$

Let $s \in (0, 1)$. Then, for any function g on Ω , there exists a solution u to the equation $\bar{\partial}u = g$ such that

$$\int_{\Omega} |u|^2 e^{-\varphi+s\psi} dA \lesssim \int_{\Omega} \frac{|g|^2}{\Delta\psi} e^{-\varphi+s\psi} dA.$$

Lemma 1.4. *Let $\omega \in \mathcal{W}^*$. There exists $C > 0$ such that for any function g on \mathbb{D} , there exists a solution u to the equation $\bar{\partial}u = g$ such that*

$$\int_{\mathbb{D}} |u(z)|^2 \omega(z) dA(z) \leq C \int_{\mathbb{D}} \tau^2(z) |g(z)|^2 \omega(z) dA(z).$$

Proof. Note that if ω satisfies (1.5), then the result comes directly from $\bar{\partial}$ -Hörmander's theorem. On the other hand, suppose that ω satisfies condition (1.6), that is there exist a subharmonic function $\psi : \mathbb{D} \rightarrow \mathbb{R}^+$ and constant $t \in (-1, 0)$ such that for all $z \in \mathbb{D}$ we have

$$\frac{1}{\Delta\psi(z)} \lesssim \tau_{\omega}^2(z), \quad t\Delta\psi(z) \leq \Delta\varphi(z) \quad \text{and} \quad |\partial\psi(z)|^2 \leq \Delta\psi(z).$$

Then by Theorem 1.4, applied to the couple of subharmonic functions $(\varphi - t\psi, \psi)$ with $s = -t$, there exists a solution u to the equation $\bar{\partial}u = g$ such that

$$\int_{\mathbb{D}} |u|^2 e^{-\varphi} dA \lesssim \int_{\mathbb{D}} \frac{|g|^2}{\Delta\psi} e^{-\varphi} dA.$$

Then, we get

$$\int_{\mathbb{D}} |u(z)|^2 \omega(z) dA(z) \lesssim \int_{\mathbb{D}} |g(z)|^2 \tau_{\omega}^2(z) \omega(z) dA(z).$$

□

In the sequel $k_z = \frac{K_z}{\|K_z\|}$ will denote the normalized reproducing kernel of A_{ω}^2 . The following result describes the boundedness of $H_{\bar{\varphi}}$ on A_{ω}^2 when $\omega \in \mathcal{W}^*$.

Theorem 1.5. *Let $\omega \in \mathcal{W}^*$. Then, the Hankel operator $H_{\bar{\phi}}$ is bounded on A_{ω}^2 if and only if $\phi \in \mathcal{B}^{\omega}$. In this case $\|H_{\bar{\phi}}\| \asymp \sup_{z \in \mathbb{D}} \tau_{\omega}(z) |\phi'(z)|$, where the implied constants depend only on ω .*

Proof. Suppose that $H_{\bar{\phi}}$ is bounded on A_{ω}^2 . Fix a in \mathbb{D} and let $\delta \leq \eta$ (η is the constant which appears in (1.4)). By the formula (1.1), we have

$$\begin{aligned} \int_{\mathbb{D}} |H_{\bar{\phi}} k_a(z)|^2 \omega(z) dA(z) &= \int_{\mathbb{D}} |\phi(z) - \phi(a)|^2 |k_a(z)|^2 \omega(z) dA(z) \\ &\geq \int_{D(a, \delta \tau_{\omega}(a))} |\phi(z) - \phi(a)|^2 |k_a(z)|^2 \omega(z) dA(z). \end{aligned}$$

Using the fact that $|k_a(z)|^2 \asymp K(z, z)$ and the fact that $\tau_{\omega}(z) \asymp \tau_{\omega}(a)$ when $z \in D(a, \delta \tau_{\omega}(a))$, we have

$$\int_{D(a, \delta \tau_{\omega}(a))} |\phi(z) - \phi(a)|^2 \|k_z\|^2 \omega(z) dA(z) \asymp \frac{1}{\tau_{\omega}^2(a)} \int_{D(a, \delta \tau_{\omega}(a))} |\phi(z) - \phi(a)|^2 dA(z).$$

Then we obtain

$$\int_{\mathbb{D}} |H_{\bar{\phi}} k_a(z)|^2 \omega(z) dA(z) \gtrsim \frac{1}{\tau_{\omega}^2(a)} \int_{D(a, \delta \tau_{\omega}(a))} |\phi(z) - \phi(a)|^2 dA(z).$$

By Cauchy's representation formula, we get

$$\tau_{\omega}(a) |\phi'(a)| \lesssim \frac{1}{\tau_{\omega}^2(a)} \int_{D(a, \delta \tau_{\omega}(a))} |\phi(z) - \phi(a)| dA(z).$$

It follows that

$$\begin{aligned} \tau_{\omega}^2(a) |\phi'(a)|^2 &\lesssim \frac{1}{\tau_{\omega}^2(a)} \int_{D(a, \delta \tau_{\omega}(a))} |\phi(z) - \phi(a)|^2 dA(z) \\ &\lesssim \int_{\mathbb{D}} |H_{\bar{\phi}} k_a(z)|^2 \omega(z) dA(z) \\ &= \|H_{\bar{\phi}} k_a\|^2 \\ &\leq \|H_{\bar{\phi}}\|^2. \end{aligned}$$

Hence, $\phi \in \mathcal{B}^{\omega}$.

Suppose now that $\sup_{z \in \mathbb{D}} \tau_{\omega}(z) |\phi'(z)| < \infty$ and let $f \in A_{\omega}^2$. By Lemma 1.4, there exists a solution u to the equation

$$\bar{\partial} u = \bar{\phi}' f \tag{1.8}$$

satisfying

$$\int_{\mathbb{D}} |u(z)|^2 \omega(z) dA(z) \lesssim \int_{\mathbb{D}} \tau_{\omega}^2(z) |\phi'(z)|^2 |f(z)|^2 \omega(z) dA(z).$$

Therefore, since $H_{\bar{\phi}} f$ is the L_{ω}^2 -minimal solution to the equation (1.8), we have

$$\int_{\mathbb{D}} |H_{\bar{\phi}} f(z)|^2 \omega(z) dA(z) \lesssim \int_{\mathbb{D}} \tau_{\omega}^2(z) |\phi'(z)|^2 |f(z)|^2 \omega(z) dA(z). \quad (1.9)$$

Hence

$$\|H_{\bar{\phi}}\| \lesssim \sup_{z \in \Omega} \tau_{\omega}(z) |\phi'(z)| < \infty.$$

□

Note that from equation (1.9), if $(f_n)_n$ is an orthonormal basis of A_{ω}^2 then

$$\begin{aligned} \sum_n \|H_{\bar{\phi}} f_n\|^2 &\lesssim \sum_n \int_{\mathbb{D}} |f_n(z)|^2 \tau_{\omega}^2(z) |\phi'(z)|^2 \omega(z) dA(z) \\ &\asymp \int_{\mathbb{D}} \|K_z\|^2 \tau_{\omega}^2(z) |\phi'(z)|^2 \omega(z) dA(z) \\ &= \int_{\mathbb{D}} |\phi'(z)|^2 dA(z). \end{aligned}$$

Consequently, if $\phi \in \mathcal{B}_2$ then $H_{\bar{\phi}} \in \mathcal{S}_2$. We will see in the next paragraph that the converse is also true.

We have the following description of the compactness of Hankel operators.

Theorem 1.6. *Let $\omega \in \mathcal{W}^*$. Then, the Hankel operator $H_{\bar{\phi}}$ is compact on A_{ω}^2 if and only if $\phi \in \mathcal{B}_0^{\omega}$.*

Proof. Suppose that $H_{\bar{\phi}}$ is compact. Since $\omega \in \mathcal{W}^*$, $k_z = \frac{K_z}{\|K_z\|}$ converges weakly to 0 when $|z| \rightarrow 1^-$. Then $\lim_{|z| \rightarrow 1^-} \|H_{\bar{\phi}} k_z\| = 0$. By the proof of Theorem 2.8, we have

$$\tau(z) |\phi'(z)| \lesssim \|H_{\bar{\phi}} k_z\|.$$

This implies that $\phi \in \mathcal{B}_0^{\omega}$.

Conversely, let $\phi \in \mathcal{B}_0^{\omega}$ and let $\phi_r(z) = \phi(rz)$. Clearly, ϕ_r converges to ϕ in \mathcal{B}^{ω} as $r \rightarrow 1^-$. Then by Theorem 2.8 $H_{\bar{\phi}_r}$ converges to $H_{\bar{\phi}}$. Now since $\phi_r \in \mathcal{B}_2$, $H_{\bar{\phi}_r}$ is compact. Hence, $H_{\bar{\phi}}$ is compact. □

1.4 Trace estimates for Hankel operators.

This section is devoted to the proof of Theorem 2.4. Before starting the proof, we recall the following covering lemma.

Lemma 1.5. ([21]) *Let X be a subset of \mathbb{C} and let $\tau : X \rightarrow (0, \infty)$ be a bounded function. Suppose that there are two constants $\gamma, C > 0$ such that, for every $z, w \in X$ with $|z - w| < \gamma\tau(z)$, we have $\frac{\tau(z)}{C} \leq \tau(w) \leq C\tau(z)$. Set $B = C + 1$ and let $\delta \leq \gamma/B$. There exists a sequence $(z_n)_{n \geq 1} \subset X$ such that*

1. $X \subset \cup_{n \geq 1} D(z_n, \delta\tau(z_n))$.
2. $D(z_n, \delta\tau(z_n)/2C) \cap D(z_m, \delta\tau(z_m)/2C) = \emptyset$ for $n \neq m$.
3. For $z \in D(z_n, \delta\tau(z_n))$ we have $D(z, \delta\tau(z)) \subset D(z_n, B\delta\tau(z_n))$.
4. $(D(z_n, B\delta\tau(z_n)))_n$ is a covering of X of finite multiplicity.

Such sequences will be called a (τ, δ) -lattice of X .

We say that $(R_n)_n \in \mathcal{L}_\omega$ if $(R_n)_n = (D(z_n, \delta\tau_\omega(z_n)))_n$ and satisfies the conditions (1) – (4) of the above lemma. In the sequel we fix $(R_n)_n = (D(z_n, \delta\tau_\omega(z_n)))_n \in \mathcal{L}_\omega$ and $b > 1$ such that $(bR_n)_n = (D(z_n, b\delta\tau_\omega(z_n)))_n$ is a covering of \mathbb{D} of finite multiplicity.

For a positive Borel measure μ on \mathbb{D} , the Toeplitz operator associated with μ and defined on A_ω^2 is given by

$$T_\mu f(z) = \int_{\mathbb{D}} f(w) K(z, w) \omega(w) d\mu(w), \quad z \in \mathbb{D}.$$

It is easy to verify that T_μ satisfies the following remarkable formula

$$\langle T_\mu f, f \rangle = \int_{\mathbb{D}} |f(z)|^2 \omega(z) d\mu(z).$$

For more properties of Toeplitz operators we refer to [19–21, 32].

The following lemma will be used in the proof of the lower estimate of $\text{Tr } h(|H_{\bar{\phi}}|)$.

Lemma 1.6. *Let $\omega \in \mathcal{W}^*$ and let $\phi \in \mathcal{B}^\omega$. For any increasing convex function h such that $h(0) = 0$, we have*

$$\sum_n h(\|\chi_{bR_n} H_{\bar{\phi}} k_{z_n}\|) \lesssim \sum_n h(s_n(H_{\bar{\phi}})),$$

where the implied constant depends only on $(bR_n)_n$ and ω .

Proof. Let $u_n = k_{z_n}$, $v_n = \frac{\chi_{bR_n} H_{\bar{\phi}} k_{z_n}}{\|\chi_{bR_n} H_{\bar{\phi}} k_{z_n}\|}$ and remark that $\|\chi_{bR_n} H_{\bar{\phi}} k_{z_n}\| = \langle H_{\bar{\phi}} u_n, v_n \rangle$. So, it suffices to prove that the conditions of Lemma 1.3 are satisfied. Indeed, let $f \in A_{\omega}^2$ and let $d\mu = \sum_n \frac{1}{\|K_{z_n}\|^2 \omega(z_n)} d\delta_{z_n}$. Since $\omega \in \mathcal{W}$, T_{μ} is bounded [21]. Then we have

$$\sum_n |\langle u_n, f \rangle|^2 = \sum_n \frac{|f(z_n)|^2}{\|K_{z_n}\|^2} = \langle T_{\mu} f, f \rangle \leq \|T_{\mu}\| \|f\|^2.$$

Now let $g \in L^2(\omega dA)$. By Hölder inequality, we have

$$\sum_n |\langle v_n, g \rangle|^2 = \sum_n \left| \left\langle \frac{\chi_{bR_n} H_{\bar{\phi}} k_{z_n}}{\|\chi_{bR_n} H_{\bar{\phi}} k_{z_n}\|}, \chi_{bR_n} g \right\rangle \right|^2 \leq \sum_n \|\chi_{bR_n} g\|^2 \lesssim \|g\|^2.$$

This completes the proof. \square

Proof of Theorem 2.4. First, we prove that $\sum_n h(s_n(H_{\bar{\phi}})) \leq \int_{\mathbb{D}} h(C|\phi'(z)|\tau_{\omega}(z)) d\lambda_{\omega}(z)$. The equation (1.9) implies that

$$H_{\bar{\phi}}^* H_{\bar{\phi}} \lesssim T_{\mu_{\phi}}. \quad (1.10)$$

Then, by the monotonicity Weyl's lemma, we have

$$s_n^2(H_{\bar{\phi}}) = \lambda_n(H_{\bar{\phi}}^* H_{\bar{\phi}}) \lesssim \lambda_n(T_{\mu_{\phi}}). \quad (1.11)$$

Let $\tilde{h}(t) = h(\sqrt{t})$. By Theorem 4.5 of [20], we have

$$\begin{aligned} \sum h(s_n(H_{\bar{\phi}})) &= \sum \tilde{h}(s_n^2(H_{\bar{\phi}})) \\ &\leq \sum \tilde{h}(C\lambda_n(T_{\mu_{\phi}})) \\ &\leq \sum \tilde{h}\left(C \frac{\mu_{\phi}(R_n)}{A(R_n)}\right) \\ &= \sum \tilde{h}\left(C \frac{1}{A(R_n)} \int_{R_n} \tau_{\omega}^2(z) |\phi'(z)|^2 dA(z)\right) \\ &\leq \sum \tilde{h}\left(C \int_{R_n} |\phi'(z)|^2 dA(z)\right). \end{aligned}$$

By subharmonicity, we have for all $z \in R_n$

$$|\phi'(z)| \lesssim \frac{1}{A(R_n)} \int_{bR_n} |\phi'(\zeta)| dA(\zeta).$$

Then,

$$\int_{R_n} |\phi'(z)|^2 dA(z) \lesssim \frac{1}{A(R_n)} \left(\int_{bR_n} |\phi'(\zeta)| dA(\zeta) \right)^2 \asymp \left(\int_{bR_n} |\phi'(\zeta)| \tau_{\omega}(\zeta) d\lambda_{\omega}(\zeta) \right)^2.$$

Since h is convex we obtain

$$\begin{aligned} \tilde{h} \left(C \int_{R_n} |\phi'(z)|^2 dA(z) \right) &\leq \tilde{h} \left(C \left(\int_{bR_n} |\phi'(\zeta)| \tau_\omega(\zeta) d\lambda_\omega(\zeta) \right)^2 \right) \\ &\leq h \left(C \int_{bR_n} |\phi'(\zeta)| \tau_\omega(\zeta) d\lambda_\omega(\zeta) \right) \\ &\leq \int_{bR_n} h(C|\phi'(\zeta)|\tau_\omega(\zeta)) d\lambda_\omega(\zeta). \end{aligned}$$

Combining these inequalities and the fact that $(bR_n)_n$ is of finite multiplicity, we get

$$\sum h(s_n(H_{\bar{\phi}})) \leq \int_{\mathbb{D}} h(C|\phi'(z)|\tau_\omega(z)) d\lambda_\omega(z).$$

Now we prove the lower inequality by using Lemma 2. We have

$$\begin{aligned} \|\chi_{bR_n} H_{\bar{\phi}} k_{z_n}\| &\asymp \left(\int_{bR_n} |\phi(z) - \phi(z_n)|^2 \|K_z\|^2 \omega(z) dA(z) \right)^{1/2} \\ &\asymp \left(\int_{bR_n} |\phi(z) - \phi(z_n)|^2 d\lambda_\omega(z) \right)^{1/2}. \end{aligned}$$

By Cauchy's formula we have

$$\tau_\omega(z)|\phi'(z)| \lesssim \left(\int_{bR_n} |\phi(\zeta) - \phi(z_n)|^2 d\lambda_\omega(\zeta) \right)^{1/2} \asymp \|\chi_{bR_n} H_{\bar{\phi}} k_{z_n}\|, \quad z \in R_n.$$

Then

$$\int_{\mathbb{D}} h(\tau_\omega(z)|\phi'(z)|) d\lambda_\omega(z) \asymp \sum_n \int_{R_n} h(\tau_\omega(z)|\phi'(z)|) d\lambda_\omega(z) \leq \sum_n h(C\|\chi_{bR_n} H_{\bar{\phi}} k_{z_n}\|).$$

By Lemma 2, we obtain the desired result. \square

Let us denote by

$$\mathcal{B}^{\omega,p} = \left\{ \phi \in \mathcal{B}^\omega : \int_{\mathbb{D}} (|\phi'(z)|\tau_\omega(z))^p d\lambda_\omega(z) < \infty \right\} \quad (p > 0).$$

Note that $\mathcal{B}^{\omega,2} = \mathcal{B}_2$ is the classical Dirichlet space and it doesn't depend on the weight ω . Note that since $\tau_\omega(z) = O((1 - |z|^2))$, $\mathcal{B}^{\omega,p} = \{0\}$ whenever $p \leq 1$. Using standard arguments, one can easily prove that an analytic function f on \mathbb{D} belongs to $\mathcal{B}^{\omega,p}$ if and only if

$$\sum_n \left(\frac{\mu(R_n)}{A(R_n)} \right)^p < \infty, \quad \text{where } d\mu(z) = \tau_\omega(z)|f'(z)|dA(z) \text{ and } (R_n)_n \in \mathcal{L}_\omega.$$

In particular, this implies that the family $(\mathcal{B}^{\omega,p})_p$ is increasing.

The following result, which extends the main results in [7, 26], is a direct consequence of Theorem 2.4.

Corollary 1.1. *Let $p \geq 1$ and let $\omega \in \mathcal{W}^*$. Let $\phi \in \mathcal{B}_0^\omega$. Then*

$$H_{\bar{\phi}} \in \mathcal{S}_p(A_\omega^2) \iff \phi \in \mathcal{B}^{\omega,p}.$$

As a consequence of Theorem 2.4, we give some estimates of the singular values of compact Hankel operators. To this end, let us recall that $\mathcal{R}_{\phi,\omega}(t) := \lambda_\omega(\{z \in \mathbb{D} : \tau_\omega(z)|\phi'(z)| > t\})$ and let $\mathcal{R}_{\phi,\omega}^+$ be the nonincreasing rearrangement of the function $\tau_\omega|\phi'|$ with respect to λ_ω . For any increasing function h , by a standard computation, we have

$$\int_{\mathbb{D}} h(\tau_\omega(z)|\phi'(z)|) d\lambda_\omega(z) = \int_0^\infty \mathcal{R}_{\phi,\omega}(t) dh(t).$$

Then, there exists $B > 0$ which depends only on ω such that

$$\sum_{n \geq 0} h\left(\frac{1}{B} \mathcal{R}_{\phi,\omega}^+(n)\right) \leq \int_{\mathbb{D}} h(\tau_\omega(z)|\phi'(z)|) d\lambda_\omega(z) \leq \sum_{n \geq 0} h(B \mathcal{R}_{\phi,\omega}^+(n)). \quad (1.12)$$

As a consequence of Theorem 2.4, we obtain the following result.

Theorem 1.7. *Let $\omega \in \mathcal{W}^*$ and let $\phi \in \mathcal{B}_0^\omega$. Let ρ be an increasing function such $\rho(x)/x^\gamma$ is decreasing for some $\gamma \in (0, 1)$. Then*

$$s_n(H_{\bar{\phi}}) \asymp 1/\rho(n) \iff \mathcal{R}_{\phi,\omega}^+(n) \asymp 1/\rho(n).$$

Proof. By Theorem 2.4 and inequalities (1.12), there exists $B > 0$ such that for every increasing convex function h we have

$$\sum_{n \geq 0} h\left(\frac{1}{B} \mathcal{R}_{\phi,\omega}^+(n)\right) \leq \sum_{n \geq 0} h(s_n(H_{\bar{\phi}})) \leq \sum_{n \geq 0} h(B \mathcal{R}_{\phi,\omega}^+(n)).$$

By Lemma 1.1, we obtain the desired result. \square

Theorem 1.8. *Let $\omega \in \mathcal{W}^*$ and let $\phi \in \mathcal{B}_0^\omega$. Let ρ be an increasing function such that $\rho(x)/x^A$ is decreasing for some $A > 0$. Suppose that $\mathcal{R}_{\phi,\omega}^+(x) = O(1/\rho(x))$, as $x \rightarrow \infty$. Then*

$$s_n(H_{\bar{\phi}}) = O(1/\rho(n)), \quad n \rightarrow \infty.$$

Proof. First, recall that $d\mu_\phi(z) = \tau_\omega^2(z)|\phi'(z)|^2 dA(z)$ and fix $(R_n)_n \in \mathcal{L}_\omega$. By Inequality (1.11) it suffices to prove that $\lambda_n(T_{\mu_\phi}) = O(1/\rho^2(n))$. By [20], for any $p > 0$ there exists $B > 0$ such that, for any increasing function h such that $h(t^p)$ is convex and $h(0) = 0$, we have

$$\sum h(\lambda_n(T_{\mu_\phi})) \leq \sum h(B\mu_\phi(R_n)/A(R_n)).$$

Let $b > 1$ such that $((bR_n)_n) \in \mathcal{L}_\omega$. By subharmonicity of $|\phi'|^{2/p}$, we have

$$\tau_\omega^2(z)|\phi'(z)|^2 \leq \left(\frac{C(b, p, \omega)}{A(R_n)} \int_{bR_n} \tau_\omega^{2/p}(\zeta)|\phi'(\zeta)|^{2/p} dA(\zeta) \right)^p, \quad \text{for all } z \in R_n.$$

We obtain from the convexity of $h_p(t) =: h(t^p)$ that

$$\begin{aligned} \sum h(B\mu_\phi(R_n)/A(R_n)) &\leq \sum h_p \left(\frac{C(b, p, \omega)}{A(R_n)} \int_{bR_n} \tau_\omega^{2/p}(\zeta)|\phi'(\zeta)|^{2/p} dA(\zeta) \right) \\ &\leq \sum \frac{1}{A(R_n)} \int_{bR_n} h(C(b, p, \omega)\tau_\omega^2(\zeta)|\phi'(\zeta)|^2) dA(\zeta) \\ &\lesssim \int_{\mathbb{D}} h(C(b, p, \omega)\tau_\omega^2(\zeta)|\phi'(\zeta)|^2) d\lambda_\omega(\zeta). \end{aligned}$$

Combining these inequalities with (1.12) and the hypothesis that $\mathcal{R}_{\phi, \omega}^+(n) \lesssim 1/\rho(n)$ we obtain

$$\sum h(\lambda_n(T_{\mu_\phi})) \leq \sum_{n \geq 0} h(B(\mathcal{R}_{\phi, \omega}^+(n))^2) \leq \sum_{n \geq 0} h(B/\rho^2(n)),$$

where B depends on ω, b, p and A . The result comes from Lemma 1.2. \square

1.5 The cut-off

In this section we consider weights $\omega \in \mathcal{W}^*$ such that τ_ω is equivalent to a radial function. We cite as examples, radial weights $\omega \in \mathcal{W}^*$, positive harmonic weights and weights $\omega = e^{-\varphi} \in \mathcal{W}_0$ such that $\Delta\varphi$ is equivalent to a radial function.

Proof of Theorem 1.2. Suppose that $s_n(H_{\bar{\phi}}) = o(s_n(H_{\bar{z}}))$. Let $\delta \in (0, 1)$ and let $h_\delta(t) = (t - \delta)^+$. By Theorem 2.4 we have

$$\int_{\mathbb{D}} h_\delta \left(\frac{1}{B} \tau_\omega(z)|\phi'(z)| \right) d\lambda_\omega(z) \leq \sum_{n \geq 1} h_\delta(s_n(H_{\bar{\phi}})).$$

Let $\rho \in (1/2, 1)$ and put $K = \int_0^{2\pi} |\phi'(\rho e^{it})| \frac{dt}{2\pi}$. By Jensen's inequality we have

$$\begin{aligned} \int_{\mathbb{D}} h_\delta \left(\frac{1}{B} \tau_\omega(z) |\phi'(z)| \right) d\lambda_\omega(z) &\geq 2 \int_0^1 h_\delta \left(\frac{1}{B} \tau_\omega(r) \int_0^{2\pi} |\phi'(r e^{it})| \frac{dt}{2\pi} \right) \frac{r dr}{\tau_\omega^2(r)} \\ &\geq \int_\rho^1 h_\delta \left(\frac{1}{B} \tau_\omega(r) \int_0^{2\pi} |\phi'(\rho e^{it})| \frac{dt}{2\pi} \right) \frac{dr}{\tau_\omega^2(r)} \\ &= \int_\rho^1 \left(\frac{K}{B} \tau_\omega(r) - \delta \right)^+ \frac{dr}{\tau_\omega^2(r)}. \end{aligned}$$

Suppose that $\phi' \neq 0$, then $K > 0$. For $\tau_\omega(r) \geq \frac{2B}{K} \delta$, we have $\left(\frac{K}{B} \tau_\omega(r) - \delta \right)^+ \geq \frac{K}{2B} \tau_\omega(r)$. Then we obtain

$$\int_{\mathbb{D}} h_\delta \left(\frac{1}{B} \tau_\omega(z) |\phi'(z)| \right) d\lambda_\omega(z) \geq \frac{K}{2B} \int_{\{r \in (\rho, 1): \tau_\omega(r) \geq \frac{2B}{K} \delta\}} \frac{dr}{\tau_\omega(r)}. \quad (1.13)$$

Let $\varepsilon \in (0, K/4B^2)$ and let N be such that for $n \geq N$ we have $s_n(H_{\bar{\phi}}) \leq \varepsilon s_n(H_{\bar{z}})$. Using Theorem 2.4, we have

$$\begin{aligned} \sum_{n \geq 1} h_\delta(s_n(H_{\bar{\phi}})) &\leq \sum_{n < N} h_\delta(s_n(H_{\bar{\phi}})) + \sum_{n \geq 1} h_\delta(\varepsilon s_n(H_{\bar{z}})) \\ &\leq N \|H_{\bar{\phi}}\| + \int_{\mathbb{D}} h_\delta(\varepsilon B \tau_\omega(z)) d\lambda_\omega(z) \\ &\leq N \|H_{\bar{\phi}}\| + 2\varepsilon B \int_{\{r \in (0, 1): \tau_\omega(r) \geq \frac{\delta}{\varepsilon B}\}} \frac{dr}{\tau_\omega(r)}. \end{aligned}$$

Since $1/\varepsilon B > 2B/K$, we obtain

$$\sum_{n \geq 1} h_\delta(s_n(H_{\bar{\phi}})) \leq N \|H_{\bar{\phi}}\| + 2\varepsilon B \int_{\{r \in (0, 1): \tau_\omega(r) \geq \frac{2B}{K} \delta\}} \frac{dr}{\tau_\omega(r)}. \quad (1.14)$$

Combining inequalities (1.13) and (1.14), we obtain

$$\frac{K}{2B} \int_{\{r \in (\rho, 1): \tau_\omega(r) \geq \frac{2B}{K} \delta\}} \frac{dr}{\tau_\omega(r)} \leq N \|H_{\bar{\phi}}\| + 2\varepsilon B \int_{\{r: \tau_\omega(r) \geq \frac{2B}{K} \delta\}} \frac{dr}{\tau_\omega(r)}.$$

Since $\int_0^1 \frac{dr}{\tau_\omega(r)} = \infty$, when δ goes to 0, we obtain $\frac{K}{2B} \leq 2\varepsilon B$. This gives a contradiction.

The second assertion is obtained by using the same argument. \square

Corollary 1.2. *Let $\omega \in \mathcal{W}^*$ be such that $\tau_\omega^2(z) \asymp (1 - |z|^2)^{2+\beta} \nu^2(\log(\frac{1}{1-|z|^2}))$, where $\beta \geq 0$ and ν is a monotone function such that $\nu(2t) \asymp \nu(t)$. Let $\phi \in \mathcal{B}_0^\omega$ and let $p = \frac{2(1+\beta)}{2+\beta}$. If*

$$s_n(H_{\bar{\phi}}) = o\left(\frac{1}{n^{\frac{1}{p}} \nu^{\frac{1}{1+\beta}}(\log n)}\right),$$

then $\phi' = 0$.

Proof. It is not difficult to verify that

$$\int_{\delta}^1 \frac{dt}{t^{2+\beta}\nu^2(\log 1/t)} \asymp \frac{1}{\delta^{1+\beta}\nu^2(\log 1/\delta)}.$$

Then, since $\tau_{\omega}^2(z) \asymp (1 - |z|^2)^{2+\beta}\nu^2(\log(\frac{1}{1-|z|^2}))$, we have

$$\begin{aligned} \mathcal{R}_{z,\omega}(t) &= \int_{\{z \in \mathbb{D}: \tau_{\omega}(z) \geq t\}} \frac{dA(z)}{(1 - |z|^2)^{2+\beta}\nu^2(\log 1/(1 - |z|^2))} \\ &\asymp \frac{1}{t^{\frac{2(1+\beta)}{2+\beta}} \nu^{\frac{2}{2+\beta}}(\log 1/t)}. \end{aligned}$$

This implies that $\mathcal{R}_{z,\omega}^+(n) \asymp \frac{1}{n^{\frac{1}{p}} \nu^{\frac{1}{1+\beta}}(\log n)}$. By the second assertion of Theorem 1.2, we get $\phi' = 0$. \square

Note that if ω is radial then

$$H_{\bar{z}}^* H_{\bar{z}} \left(\frac{z^n}{\|z^n\|} \right) = \left(\frac{\|z^{n+1}\|^2}{\|z^n\|^2} - \frac{\|z^n\|^2}{\|z^{n-1}\|^2} \right) \frac{z^n}{\|z^n\|} =: m_{\omega}^2(n) \frac{z^n}{\|z^n\|}, \quad n \geq 1.$$

So, the sequence of the singular values of $H_{\bar{z}}$ is the nonincreasing rearrangement of the sequence $(m_{\omega}(n))_{n \geq 1}$.

For the standard Bergman spaces A_{α}^2 , it is easy to see that

$$\|z^n\|^2 = \frac{\Gamma(n+1)\Gamma(\alpha+2)}{\Gamma(n+\alpha+2)} \text{ and } m_{\omega_{\alpha}}^2(n) = \frac{\alpha+1}{(n+\alpha+1)(n+\alpha+2)}.$$

where $\Gamma(x) = \int_0^{\infty} t^{x-1} e^{-t} dt$ denotes the Gamma function. Then

$$s_n(H_{\bar{z}}) \sim \frac{\sqrt{\alpha+1}}{n+1}.$$

For larger Bergman spaces we have the following result.

Proposition 1.1. *Let $\omega \in \mathcal{W}^*$ and let $\beta > 0$. Suppose that $\tau_{\omega}^2(z) \asymp (1 - |z|^2)^{2+\beta}\nu^2(\log(\frac{1}{1-|z|^2}))$, where ν is a monotone function which satisfies $\nu(2t) \asymp \nu(t)$. Then*

$$s_n(H_{\bar{z}}) \asymp \frac{1}{n^{\frac{1}{p}} \nu^{\frac{1}{1+\beta}}(\log n)}, \quad \text{where } p = \frac{2(1+\beta)}{2+\beta}.$$

Proof. From the hypothesis, we have $\mathcal{R}_{z,\omega}^+(n) \asymp \frac{1}{n^{\frac{1}{p}} \nu^{\frac{1}{1+\beta}}(\log n)}$. By Theorem 1.7, we obtain the result. \square

1.6 Critical decay

In this section we describe the class of functions $\phi \in \mathcal{B}_0^\omega$ such that $s_n(H_{\bar{\phi}}) = O(s_n(H_{\bar{z}}))$. This kind of problem was first examined by Arazy, Fisher and Peetre for standard Bergman spaces [7]. They proved that if ϕ belongs to the Besov space \mathcal{B}_1 given by

$$\mathcal{B}_1 := \left\{ f \in \text{Hol}(\mathbb{D}), \int_{\mathbb{D}} |f''(z)| dA(z) < \infty \right\},$$

then

$$\sigma_n(H_{\bar{\phi}}) = O(\log(n+1)), \text{ where } \sigma_n(H_{\bar{\phi}}) := \sum_{j=1}^n s_j(H_{\bar{\phi}}).$$

They also proved, in the same paper, that the converse is false. In [23], M. Engliš and R. Rochberg gave a complete answer to this problem for the classical Bergman space. They proved, by using Boutet de Monvel-Guillemin theory, that $\sigma_n(H_{\bar{\phi}}) = O(1/n)$ if and only if $\phi' \in H^1$. This result was extended by R. Tytgat [48, 49] to standard Bergman spaces A_α^2 . In what follows we study this problem in more general situations. Our approach is based on Theorem 2.4.

Proposition 1.2. *Let $\omega \in \mathcal{W}^*$ be a weight such that $\tau_\omega^2(z) \asymp (1 - |z|^2)^{2+\beta} \nu^2(\log(\frac{1}{1-|z|^2}))$ where $\beta \geq 0$ and ν is a monotone function such that $\nu(2t) \asymp \nu(t)$. Let $\phi \in \mathcal{B}_0^\omega$ and let $p = \frac{2(1+\beta)}{2+\beta}$. Then*

$$s_n(H_{\bar{\phi}}) = O\left(\frac{1}{n^{\frac{1}{p}} \nu^{\frac{1}{1+\beta}}(\log n)}\right) \implies \phi' \in H^p.$$

Proof. Suppose that $s_n(H_{\bar{\phi}}) \leq \frac{C}{n^{\frac{1}{p}} \nu^{\frac{1}{1+\beta}}(\log n)}$. Since $\tau_\omega^2(z) \asymp (1 - |z|^2)^{2+\beta} \nu^2(\log(\frac{1}{1-|z|^2}))$, we have

$$\mathcal{R}_{z,\omega}(t) \asymp \frac{1}{t^p \nu^{\frac{2}{2+\beta}}(\log(1/t))} \quad \text{and} \quad \mathcal{R}_{z,\omega}^+(n) \asymp \frac{1}{n^{\frac{1}{p}} \nu^{\frac{1}{1+\beta}}(\log(n))}.$$

Then for every increasing function h , we have

$$\sum_{n=1}^{\infty} h\left(\frac{1}{B_1 n^{\frac{1}{p}} \nu^{\frac{1}{1+\beta}}(\log(n))}\right) \leq \int_{\mathbb{D}} h(\tau_\omega(z)) d\lambda_\omega(z) \leq \sum_{n=1}^{\infty} h\left(\frac{B_1}{n^{\frac{1}{p}} \nu^{\frac{1}{1+\beta}}(\log(n))}\right).$$

where B_1 doesn't depend on h . If in addition h is convex and $h(0) = 0$, then by Theorem 2.4 we obtain

$$\begin{aligned} \int_{\mathbb{D}} h(\tau_\omega(z)|\phi'(z)|) d\lambda_\omega(z) &\lesssim \sum_{n=1}^{\infty} h(Bs_n(H_{\bar{\phi}})) \\ &\lesssim \sum_{n=1}^{\infty} h(BC\mathcal{R}_{z,\omega}^+(n)) \\ &\lesssim \int_{\mathbb{D}} h(B_1BC\tau_\omega(z)) d\lambda_\omega(z). \end{aligned}$$

Let $\varepsilon \in (0, 1)$ and put $p_\varepsilon = (1 - \varepsilon)p + \varepsilon$. Note that if $\beta = 0$ (i.e. $p = 1$) then $p_\varepsilon = 1$ and if $\beta > 0$ then $1 < p_\varepsilon < p$. Note that in the two cases we have

$$\int_0^1 \frac{dr}{\tau_\omega^{2-p_\varepsilon}(r)} = \infty.$$

The last inequality, with $h(t) = h_\delta(t^{p_\varepsilon})$, becomes

$$\int_{\mathbb{D}} h_\delta(\tau_\omega^{p_\varepsilon}(z)|\phi'(z)|^{p_\varepsilon}) d\lambda_\omega(z) \lesssim \int_{\mathbb{D}} h_\delta(K^{p_\varepsilon}\tau_\omega^{p_\varepsilon}(z)) d\lambda_\omega(z), \quad \text{where } K = CBB_1.$$

Using the convexity of h_δ , we get

$$\int_0^1 h_\delta\left(\tau_\omega^{p_\varepsilon}(r) \int_0^{2\pi} |\phi'(re^{it})|^{p_\varepsilon} \frac{dt}{2\pi}\right) \frac{dr}{\tau_\omega^2(r)} \leq \int_{\mathbb{D}} h_\delta(\tau_\omega^{p_\varepsilon}(z)|\phi'(z)|^{p_\varepsilon}) d\lambda_\omega(z) \lesssim \int_{\mathbb{D}} h_\delta(K^{p_\varepsilon}\tau_\omega^{p_\varepsilon}(z)) d\lambda_\omega(z).$$

Suppose that there exists $\rho \in (0, 1)$ such that $\|\phi'_\rho\|_{p_\varepsilon} \geq 2K$. We have

$$\int_\rho^1 h_\delta(\tau_\omega^{p_\varepsilon}(r)\|\phi'_\rho\|_{p_\varepsilon}^{p_\varepsilon}) \frac{dr}{\tau_\omega^2(r)} \lesssim \int_{\mathbb{D}} h_\delta(K^{p_\varepsilon}\tau_\omega^{p_\varepsilon}(z)) d\lambda_\omega(z).$$

Now, using the fact that $h_\delta(t) \geq t/2$ if $t \geq 2\delta$, we get

$$\|\phi'_\rho\|_{p_\varepsilon}^{p_\varepsilon} \int_{\{r \in (\rho, 1) : \tau_\omega^{p_\varepsilon}(r)\|\phi'_\rho\|_{p_\varepsilon}^{p_\varepsilon} \geq 2\delta\}} \frac{dr}{\tau_\omega^{2-p_\varepsilon}(r)} \lesssim K^{p_\varepsilon} \int_{\{r \in (0, 1) : \tau_\omega^{p_\varepsilon}(r)K^{p_\varepsilon} \geq \delta\}} \frac{dr}{\tau_\omega^{2-p_\varepsilon}(r)}.$$

Since $\|\phi'_\rho\|_{p_\varepsilon} \geq 2K$, $\{r \in (\rho, 1) : \tau_\omega^{p_\varepsilon}(r)K^{p_\varepsilon} \geq \delta\} \subset \{r \in (\rho, 1) : \tau_\omega^{p_\varepsilon}(r)\|\phi'_\rho\|_{p_\varepsilon}^{p_\varepsilon} \geq 2\delta\}$. Then

$$\|\phi'_\rho\|_{p_\varepsilon}^{p_\varepsilon} \int_{\{r \in (\rho, 1) : \tau_\omega^{p_\varepsilon}(r)K^{p_\varepsilon} \geq \delta\}} \frac{dr}{\tau_\omega^{2-p_\varepsilon}(r)} \lesssim K^{p_\varepsilon} \int_{\{r \in (0, 1) : \tau_\omega^{p_\varepsilon}(r)K^{p_\varepsilon} \geq \delta\}} \frac{dr}{\tau_\omega^{2-p_\varepsilon}(r)}.$$

Recall that $\int_0^1 \frac{dr}{\tau_\omega^{2-p_\varepsilon}(r)} = \infty$. We obtain, when δ goes to 0^+ , that $\|\phi'_\rho\|_{p_\varepsilon} \lesssim K$. When ε goes to 0, we get $\|\phi'_\rho\|_p \lesssim K$. This proves that $\phi' \in H^p$. \square

Proof of the first assertion of Theorem 1.3. Let $p = p_\beta (:= \frac{2(1+\beta)}{2+\beta})$. Suppose that $s_n(H_{\bar{\phi}}) = O(1/n^{1/p})$. By Proposition 1.2, with $\nu = 1$, we have $\phi' \in H^p$.

For the converse, by Theorem 1.8, it suffices to prove that $\mathcal{R}_{\phi, \omega}^+(x) = O(1/x^{1/p})$, whenever $\phi' \in H^p$. To this end, let U be the non-tangential maximal function of ϕ' . Since $\phi' \in H^p$ we have $U \in L^p$ and $\|U\|_p \lesssim \|\phi'\|_p$ (even if $p = 1$). We have

$$\begin{aligned} \mathcal{R}_{\phi, \omega}(t) &= \lambda_\omega(\{z \in \mathbb{D} : \tau_\omega(z)|\phi'(z)| \geq t\}) \\ &\leq \lambda_\omega(\{re^{i\theta} \in \mathbb{D} : \tau_\omega(r)U(e^{i\theta}) \geq t\}) \\ &\asymp \int_{\{re^{i\theta} \in \mathbb{D} : \tau_\omega(r)U(e^{i\theta}) \geq t\}} \frac{drd\theta}{\pi(1-r^2)^{2+\beta}} \\ &\lesssim \frac{1}{t^p} \int_0^{2\pi} U^p(e^{i\theta}) \frac{d\theta}{2\pi} \\ &\lesssim \frac{1}{t^p} \|\phi'\|_p^p. \end{aligned}$$

This is equivalent to $\mathcal{R}_{\phi, \omega}^+(x) \lesssim \frac{\|\phi'\|_p}{x^{1/p}}$. The proof is complete. \square

Now, we study the converse of Proposition 1.2, when $\tau_\omega^2(z) \asymp (1 - |z|^2)^{2+\beta} \nu^2(\log(\frac{1}{1-|z|^2}))$. In the following result we consider the case $\beta > 0$. The case $\beta = 0$, will be discussed right after.

Theorem 1.9. *Let $\omega \in \mathcal{W}^*$ be a weight such that $\tau_\omega^2(z) \asymp (1 - |z|^2)^{2+\beta} \nu^2(\log(\frac{1}{1-|z|^2}))$, where $\beta > 0$ and ν is a monotone function such that $\nu(2t) \asymp \nu(t)$. Let $\phi \in \mathcal{B}_0^\omega$ and let $p = \frac{2(1+\beta)}{2+\beta}$. Then*

$$s_n(H_{\bar{\phi}}) = O\left(\frac{1}{n^{\frac{1}{p}} \nu^{\frac{1}{1+\beta}}(\log n)}\right) \iff \phi' \in H^p.$$

In this case we have

$$s_n(H_{\bar{\phi}}) \lesssim \frac{\|\phi'\|_p}{n^{\frac{1}{p}} \nu^{\frac{1}{1+\beta}}(\log n)},$$

where the involved constant doesn't depend on ϕ .

Proof. By Proposition 1.2, It remains to prove that if $\phi' \in H^p$ then $s_n(H_{\bar{\phi}}) = O\left(\frac{1}{n^{\frac{1}{p}} \nu^{\frac{1}{1+\beta}}(\log n)}\right)$. We will proceed as in the proof of Theorem 1.3. Without loss of generality, suppose that $\|\phi'\|_p =$

1. Our goal is to show that

$$\mathcal{R}_{\phi,\omega}(t) = O\left(\frac{1}{t^p \nu^{\frac{2}{2+\beta}} (\log(1/t))}\right),$$

where the implied constant doesn't depend on ϕ .

We have

$$|\phi'(r\zeta)| \leq \frac{1}{(1-r)^{\frac{1}{p}}} = \frac{1}{(1-r)^{\frac{2+\beta}{2(1+\beta)}}}, \quad r \in (0, 1) \text{ and } \zeta \in \mathbb{T}. \quad (1.15)$$

This implies that

$$|\phi'(r\zeta)|\tau_\omega(r) \lesssim (1-r)^{\frac{(2+\beta)\beta}{2(1+\beta)}} \nu(\log(1/(1-r))).$$

Then there exists $r_0 \in (0, 1)$, which depends only on ω , such that

$$|\phi'(r\zeta)|\tau_\omega(r) \leq (1-r)^{\frac{(2+\beta)\beta}{4(1+\beta)}}, \quad r \in (r_0, 1).$$

So, if $|\phi'(r\zeta)|\tau_\omega(r) \geq t$ then $r \leq r_t$ where r_t is given by $(1-r_t)^{\frac{(2+\beta)\beta}{4(1+\beta)}} = t$.

Let U_t be the non-tangential maximal function associated with $z \in \mathbb{D} \rightarrow \phi'(r_t z)$. By inequality (1.15), we have

$$|\phi'(r\zeta)| \leq U_t(\zeta) \leq \frac{1}{(1-r_t)^{1/p}} = \frac{1}{t^{2/\beta}}, \quad r \leq r_t. \quad (1.16)$$

For t small enough, we have

$$\begin{aligned} \mathcal{R}_{\phi,\omega}(t) &= \lambda_\omega(\{re^{i\theta} \in \mathbb{D} : \tau_\omega(r)|\phi'(re^{i\theta})| \geq t\}) \\ &\lesssim \int_{\{re^{i\theta} : r \in (0, r_t), \tau_\omega(r)U_t(e^{i\theta}) \geq t\}} \frac{dr}{\tau_\omega^2(r)} d\theta \\ &\lesssim \int_0^{2\pi} \int_{\{r \in (0, r_t) : \tau_\omega(r) \geq \frac{t}{U_t(e^{i\theta})}\}} \frac{dr}{\tau_\omega^2(r)} d\theta \\ &\lesssim \int_0^{2\pi} \mathcal{R}_{z,\omega}\left(\frac{t}{U_t(e^{i\theta})}\right) d\theta. \end{aligned}$$

Since $\tau_\omega^2 \asymp (1-r^2)^{2+\beta} \nu^2(\log(1/(1-r^2)))$, we have $\mathcal{R}_{z,\omega}(t) \asymp \frac{1}{t^p \nu^{\frac{2}{2+\beta}} (\log 1/t)}$. First, note that if

$U_t(e^{i\theta}) \leq t^{1/2}$ then $\mathcal{R}_{z,\omega}\left(\frac{t}{U_t(e^{i\theta})}\right) \leq \mathcal{R}_{z,\omega}(t^{1/2}) \leq \mathcal{R}_{z,\omega}(t)$. Otherwise, from equation (1.16), we get

$$\mathcal{R}_{z,\omega}\left(\frac{t}{U_t(e^{i\theta})}\right) \asymp \frac{U_t^p(e^{i\theta})}{t^p \nu^{\frac{2}{2+\beta}} (\log U_t(e^{i\theta})/t)} \lesssim \frac{U_t^p(e^{i\theta})}{t^p \nu^{\frac{2}{2+\beta}} (\log 1/t)}.$$

Combining these inequalities we obtain

$$\mathcal{R}_{\phi,\omega}(t) \lesssim \frac{\|U_t\|_p^p}{t^p \nu^{\frac{2}{2+\beta}} (\log 1/t)} \lesssim \frac{1}{t^p \nu^{\frac{2}{2+\beta}} (\log 1/t)},$$

where the involved constant doesn't depend on ϕ' . By Theorem 1.8, we obtain the desired result. \square

Now, we will prove that the previous result is not true when $\beta = 0$ which is somewhat unexpected.

Proposition 1.3. *Let $\omega \in \mathcal{W}^*$ such that $\tau_\omega(z) \asymp \frac{(1-|z|^2)}{\log^\alpha(1/(1-|z|^2))}$ with $\alpha > 0$. Then for $\nu \in]\frac{\alpha+1}{2\alpha+1}, 1[$, there exists $\phi \in \mathcal{B}_0^\omega$ such that $\phi' \in H^1$ and*

$$s_n(H_{\bar{\phi}}) \asymp \frac{1}{n^\nu}.$$

Proof. Let $\gamma \in (1, \alpha + 1)$ be such that $\nu = \frac{\alpha+\gamma}{2\alpha+1}$. Let ϕ be such that

$$\phi'(z) = \frac{1}{(1-z) \log^\gamma\left(\frac{e}{1-z}\right)}, \quad z \in \mathbb{D}.$$

Since $\gamma > 1$, $\phi' \in H^1$. Write $\mathcal{R}_{\phi,\omega} = \mathcal{R}_1 + \mathcal{R}_2$, where

$$\mathcal{R}_1(t) = \lambda_\omega(\{re^{i\theta} \in \mathbb{D} : 1-r > |\theta| \text{ and } \tau_\omega(r)|\phi'(re^{i\theta})| \geq t\})$$

and

$$\mathcal{R}_2(t) = \lambda_\omega(\{re^{i\theta} \in \mathbb{D} : 1-r \leq |\theta| \text{ and } \tau_\omega(r)|\phi'(re^{i\theta})| \geq t\}).$$

Clearly, we have

$$\mathcal{R}_1(t) \asymp \lambda_\omega(\{re^{i\theta} \in \mathbb{D} : 1-r > |\theta| \text{ and } \frac{1}{\log^{\alpha+\gamma}(e/(1-r))} \geq t\}) \asymp \frac{1}{t^{\frac{2\alpha+1}{\alpha+\gamma}}}.$$

Similarly,

$$\mathcal{R}_2(t) \asymp \lambda_\omega(\{re^{i\theta} \in \mathbb{D} : 1-r \leq |\theta| \text{ and } \frac{1-r}{\log^\alpha(1/(1-r))} \frac{1}{|\theta| \log^\gamma(e/|\theta|)} \geq t\}) \asymp \frac{1}{t^{\frac{2\alpha+1}{\alpha+\gamma}}}.$$

Then, $\mathcal{R}_{\phi,\omega}(t) \asymp \frac{1}{t^{\frac{2\alpha+1}{\alpha+\gamma}}}$. By Theorem 1.7, we obtain that $s_n(H_{\bar{\phi}}) \asymp \frac{1}{n^\nu}$. The proof is complete. \square

1.7 Asymptotics

Now, we will precise the results obtained in Section 1.6 when $s_n(H_{\bar{z}})$ is regular. The main result of this section is the following theorem.

Theorem 1.10. *Let $\omega \in \mathcal{W}^*$ be a radial weight such that $\tau_{\omega}^2(z) \asymp (1 - |z|^2)^{2+\beta} \nu^2(\log(\frac{1}{1-|z|^2}))$, where $\beta > 0$ and ν is a monotone function such that $\nu(2t) \asymp \nu(t)$. Suppose that*

$$s_n(H_{\bar{z}}) \sim \frac{\gamma}{n^{\frac{1}{p}} \nu^{\frac{1}{1+\beta}}(\log n)},$$

where $\gamma > 0$ and $p = \frac{2(1+\beta)}{2+\beta}$. Then, for $\phi' \in H^p$, we have

$$s_n(H_{\bar{\phi}}) \sim \frac{\gamma}{n^{\frac{1}{p}} \nu^{\frac{1}{1+\beta}}(\log n)} \|\phi'\|_p.$$

To prove this result, we will introduce the following functionals (see [12, 45]). Let T be a compact operator between two Hilbert spaces. Let $n(s, T)$ be the singular values counting function given by

$$n(s, T) = \#\{n : s_n(T) \geq s\}, \quad s > 0.$$

The class of strictly increasing continuous functions $\psi : (0, +\infty) \rightarrow (0, +\infty)$ such that $\psi(0) = 0$ and such that

$$\psi(\alpha t) \sim \alpha^p \psi(t), \quad (t \rightarrow 0^+),$$

for some $p > 0$, will be denoted by \mathcal{C}_p . Let

$$D_{\psi}(T) := \limsup_{s \rightarrow 0^+} \psi(s) n(s, T) \quad \text{and} \quad d_{\psi}(T) := \liminf_{s \rightarrow 0^+} \psi(s) n(s, T).$$

From these definitions it is easy to see that if $\psi \in \mathcal{C}_p$, then

$$d_{\psi}(T) = D_{\psi}(T) \in (0, \infty) \implies s_n(T) \sim D_{\psi}(T)^{1/p} \psi^{-1}(1/n).$$

For more definitions and properties of these functionals see Section 1.8.

First, we need some results on the operator $H_{\bar{z}} P_{\omega} M_g$, where M_g denotes the multiplication operator defined on $L^2(dA_{\omega})$.

For an arc $\delta = \{e^{i\theta} : \theta_1 \leq \theta < \theta_2\} \subset \mathbb{T}$, R_N^δ will denote

$$R_N^\delta = \{z = re^{i\theta} : 0 < 1 - r \leq 2\pi/N, \theta_1 \leq \theta < \theta_2\}, \quad N = 1, 2, \dots$$

The proof of the following results is similar to that one given by A. Pushnitski in [45].

Lemma 1.7. *Let $\omega \in \mathcal{W}^*$ be a radial weight. Let $\psi \in \mathcal{C}_p$ such that $D_\psi(H_{\bar{z}})$ is finite. The following are true*

1. *Let g be a bounded function on \mathbb{D} . We have*

$$D_\psi(H_{\bar{z}}P_\omega M_g) \leq \|g\|_\infty^p D_\psi(H_{\bar{z}}).$$

2. *Let δ_1, δ_2 be two arcs of \mathbb{T} such that $\overline{\delta_1} \cap \overline{\delta_2} = \emptyset$. Then*

$$\left(H_{\bar{z}}P_\omega M_{\chi_{R_N^{\delta_1}}}\right)^* \left(H_{\bar{z}}P_\omega M_{\chi_{R_N^{\delta_2}}}\right) \in \cap_{p>0} \mathcal{S}_p.$$

3. *Let $\delta \subset \partial\mathbb{D}$ be an arc such that $|\delta| < 2\pi$. Then*

$$D_\psi(H_{\bar{z}}P_\omega \chi_{R_N^\delta}) \leq |\delta| D_\psi(H_{\bar{z}}).$$

4. *Let δ_1, δ_2 be two arcs of \mathbb{T} such that $\overline{\delta_1} \cap \overline{\delta_2}$ is reduced to one point. Then*

$$\left(H_{\bar{z}}P_\omega M_{\chi_{R_N^{\delta_1}}}\right)^* \left(H_{\bar{z}}P_\omega M_{\chi_{R_N^{\delta_2}}}\right) \in \Sigma_{\psi \circ \sqrt{\cdot}}^0.$$

Proof. Since g is bounded, M_g is bounded on L_ω^2 and $\|M_g\| = \|g\|_\infty$. Then $n(s, H_{\bar{z}}P_\omega M_g) \leq n(s, \|M_g\|H_{\bar{z}}) = n(s, \|g\|_\infty H_{\bar{z}})$, for all $s > 0$. Hence, by Proposition 1.5, we get

$$D_\psi(H_{\bar{z}}P_\omega M_g) \leq D_\psi(\|g\|_\infty H_{\bar{z}}) = \|g\|_\infty^p D_\psi(H_{\bar{z}}).$$

This proves the first assertion.

The same proof as that proposed by Pushnitski in [45], gives the second assertion.

To prove the third assertion, let $N \in \mathbb{N}^*$ such that $\frac{2\pi}{N+1} \leq |\delta| < \frac{2\pi}{N}$ and let

$$\delta_n := e^{2\pi in/N} \delta, \quad n = 1, 2, \dots, N.$$

Let $R_N^n := R_N^{\delta_n}$. Since $g := \sum_{n=1}^N \chi_{R_N^n} \leq 1$, we have

$$D_\psi(H_{\bar{z}}) \geq D_\psi(H_{\bar{z}}P_\omega M_g) = D_\psi\left(\sum_{n=1}^N H_{\bar{z}}P_\omega M_{\chi_{R_N^n}}\right).$$

For $i \neq j$ we have $\left(H_{\bar{z}}P_{\omega}M_{\chi_{R_N^i}}\right)\left(H_{\bar{z}}P_{\omega}M_{\chi_{R_N^j}}\right)^* = 0$. Since the closures of R_N^i and R_N^j are disjoint, we have

$$\left(H_{\bar{z}}P_{\omega}M_{\chi_{R_N^i}}\right)^*\left(H_{\bar{z}}P_{\omega}M_{\chi_{R_N^j}}\right) \in \Sigma_{\psi \circ \sqrt{\cdot}}^0.$$

Further, since ω is radial, the operators $H_{\bar{z}}P_{\omega}M_{\chi_{R_N^j}}$ are unitarily equivalent. It follows from Theorem 1.12 that

$$D_{\psi}(H_{\bar{z}}) \geq D_{\psi}\left(\sum_{n=1}^N H_{\bar{z}}P_{\omega}M_{\chi_{R_N^n}}\right) = ND_{\psi}(H_{\bar{z}}P_{\omega}M_{\chi_{R_N^{\delta}}}).$$

Therefore, $D_{\psi}(H_{\bar{z}}P_{\omega}M_{\chi_{R_N^{\delta}}}) \leq 1/ND_{\psi}(H_{\bar{z}}) \leq |\delta|D_{\psi}(H_{\bar{z}})$.

The last assertion is a consequence of the second and the third assertion. Indeed, let $\varepsilon > 0$ small enough. Suppose that $\delta_1 = \{e^{i\theta} : \theta_1 \leq \theta < \theta_2\}$ and $\delta_2 = \{e^{i\theta} : \theta_2 \leq \theta < \theta_3\}$ and let $\delta_1(\varepsilon) = \{e^{i\theta} : \theta_1 \leq \theta < \theta_2 - \varepsilon\}$. By (2), $\left(H_{\bar{z}}P_{\omega}M_{\chi_{R_N^{\delta_1(\varepsilon)}}}\right)^*\left(H_{\bar{z}}P_{\omega}M_{\chi_{R_N^{\delta_2}}}\right) \in \cap_{p>0}\mathcal{S}_p$. By Corollary 1.3, we get

$$D_{\psi \circ \sqrt{\cdot}}\left(\left(H_{\bar{z}}P_{\omega}M_{\chi_{R_N^{\delta_1(\varepsilon)}}}\right)^*\left(H_{\bar{z}}P_{\omega}M_{\chi_{R_N^{\delta_2}}}\right)\right) = D_{\psi \circ \sqrt{\cdot}}\left(\left(H_{\bar{z}}P_{\omega}M_{\chi_{R_N^{\delta_1} \setminus R_N^{\delta_1(\varepsilon)}}}\right)^*\left(H_{\bar{z}}P_{\omega}M_{\chi_{R_N^{\delta_2}}}\right)\right).$$

Applying Proposition 1.5, we have

$$\begin{aligned} D_{\psi \circ \sqrt{\cdot}}\left(\left(H_{\bar{z}}P_{\omega}M_{\chi_{R_N^{\delta_1} \setminus R_N^{\delta_1(\varepsilon)}}}\right)^*\left(H_{\bar{z}}P_{\omega}M_{\chi_{R_N^{\delta_2}}}\right)\right) &\leq 2D_{\psi}(H_{\bar{z}}P_{\omega}M_{\chi_{R_N^{\delta_1} \setminus R_N^{\delta_1(\varepsilon)}}})D_{\psi}(H_{\bar{z}}P_{\omega}M_{\chi_{R_N^{\delta_2}}}) \\ &\lesssim \varepsilon D_{\psi}^2(H_{\bar{z}}). \end{aligned}$$

Letting ε to 0, we obtain $D_{\psi \circ \sqrt{\cdot}}\left(\left(H_{\bar{z}}P_{\omega}M_{\chi_{R_N^{\delta_1}}}\right)^*\left(H_{\bar{z}}P_{\omega}M_{\chi_{R_N^{\delta_2}}}\right)\right) = 0$, as required. \square

Proposition 1.4. *Let $\omega \in \mathcal{W}^*$ be a radial weight. Let $\psi \in \mathcal{C}_p$ such that $0 < d_{\psi}(H_{\bar{z}}) \leq D_{\psi}(H_{\bar{z}}) < \infty$.*

- *Let $\delta \subset \partial\mathbb{D}$ be an arc such that $|\delta| = \frac{2\pi}{N}$, where $N \in \mathbb{N}^*$. Then*

$$D_{\psi}(H_{\bar{z}}P_{\omega}M_{\chi_{R_N^{\delta}}}) = \frac{1}{N}D_{\psi}(H_{\bar{z}}) \quad \text{and} \quad d_{\psi}(H_{\bar{z}}P_{\omega}M_{\chi_{R_N^{\delta}}}) = \frac{1}{N}d_{\psi}(H_{\bar{z}}).$$

- *Let g be a continuous function on $\overline{\mathbb{D}}$. We have*

$$\|g\|_p^p d_{\psi}(H_{\bar{z}}) \leq d_{\psi}(H_{\bar{z}}P_{\omega}M_g) \leq D_{\psi}(H_{\bar{z}}P_{\omega}M_g) \leq \|g\|_p^p D_{\psi}(H_{\bar{z}}).$$

Proof. Let

$$R_N = \{z = re^{i\theta} \in \bar{\mathbb{D}} : r > 1 - 1/N\},$$

$$R_N^k = \{z = re^{i\theta} \in R_N : \frac{2\pi k}{N} \leq \theta < \frac{2\pi(k+1)}{N}\}, \quad k = 0, 1, \dots, N-1.$$

And let

$$h := \sum_{k=0}^{N-1} \chi_{R_N^k} = 1 - \chi_{\mathbb{D} \setminus R_N}.$$

Hence, by Corollary 1.3, Lemma 1.7 and Theorem 1.12, we have

$$\begin{aligned} D_\psi(H_{\bar{z}}) &= D_\psi(H_{\bar{z}}P_\omega M_h) \\ &= D_\psi\left(\sum_{k=0}^{N-1} H_{\bar{z}}P_\omega M_{\chi_{R_N^k}}\right) \\ &= \sum_{k=0}^{N-1} D_\psi(H_{\bar{z}}P_\omega M_{\chi_{R_N^k}}) \\ &= ND_\psi(H_{\bar{z}}P_\omega M_{\chi_{R_N^\delta}}). \end{aligned}$$

Then $D_\psi(H_{\bar{z}}P_\omega M_{\chi_{R_N^\delta}}) = \frac{1}{N}D_\psi(H_{\bar{z}})$. Similarly, we obtain $d_\psi(H_{\bar{z}}P_\omega M_{\chi_{R_N^\delta}}) = \frac{1}{N}d_\psi(H_{\bar{z}})$.

Let ξ_k be the center of the arc $R_N^k \cap \partial\mathbb{D}$ and let

$$g_N = \sum_{k=0}^{N-1} g(\xi_k)\chi_{R_N^k}.$$

By Proposition 1.5, we have

$$D_\psi^{\frac{1}{p+1}}(H_{\bar{z}}P_\omega M_g) \leq D_\psi^{\frac{1}{p+1}}(H_{\bar{z}}P_\omega M_{g_N}) + D_\psi^{\frac{1}{p+1}}(H_{\bar{z}}P_\omega M_{g-g_N})$$

On one hand, by Lemma 1.7, we have

$$D_\psi(H_{\bar{z}}P_\omega M_{g-g_N}) \leq \|g - g_N\|_\infty^p D_\psi(H_{\bar{z}}).$$

On the other hand, by Lemma 1.7, Theorem 1.12 and the first assertion of this proposition, we obtain

$$\begin{aligned} D_\psi(H_{\bar{z}}P_\omega M_{g_N}) &\leq \sum_{k=0}^{N-1} D_\psi(g(\xi_k)H_{\bar{z}}P_\omega M_{\chi_{R_N^k}}) \\ &= \sum_{k=0}^{N-1} |g(\xi_k)|^p D_\psi(H_{\bar{z}}P_\omega M_{\chi_{R_N^k}}) \\ &= \frac{1}{N} \sum_{k=0}^{N-1} |g(\xi_k)|^p D_\psi(H_{\bar{z}}). \end{aligned}$$

Combining these inequalities and letting N going to ∞ , we obtain

$$D_\psi(H_{\bar{z}}P_\omega M_g) \leq \|g\|_p^p D_\psi(H_{\bar{z}}).$$

The lower estimates can be obtained by the same arguments. \square

Let $\Phi : R\mathbb{D} \times R\mathbb{D} \rightarrow \mathbb{C}$ be an analytic function, where $R > 1$. Let $A_\Phi : A_\omega^2 \rightarrow L_\omega^2$ be the operator defined by

$$A_\Phi f(z) = \int_{\mathbb{D}} (\bar{z} - \bar{\xi})^2 K(z, \xi) \overline{\Phi(z, \xi)} f(\xi) \omega(\xi) dA(\xi), \quad z \in \mathbb{D}. \quad (1.17)$$

And let A be the operator given by

$$Af(z) = \int_{\mathbb{D}} (\bar{z} - \bar{\xi})^2 K(z, \xi) f(\xi) \omega(\xi) dA(\xi), \quad z \in \mathbb{D}. \quad (1.18)$$

Now, we introduce notations which will be used in the proof of the next lemma. Let $r \in (0, 1)$ and let J_r be the embedding operator from A_ω^2 to $L^2(\omega \chi_{r\mathbb{D}} dA)$. Note that $T_r := J_r^* J_r$ is the Toeplitz operator with symbol $\chi_{r\mathbb{D}}$. Namely,

$$T_r f = \int_{r\mathbb{D}} f(\zeta) K(\cdot, \zeta) \omega(\zeta) dA(\zeta).$$

By [20], there exists $C = C(\omega)$ such that

$$\sum_{n \geq 1} \lambda_n^p(T_r) \leq \frac{C}{p} \int_0^r \frac{d\rho}{\tau_\omega^2(\rho)}, \quad p \in (0, 1).$$

This implies that

$$\lambda_n(T_r) \leq \inf_{p \in (0, 1)} \left(\frac{C}{pn} \int_0^r \frac{d\rho}{\tau_\omega^2(\rho)} \right)^{\frac{1}{p}} = \exp \left(- \frac{e^{-1}}{C \int_0^r \frac{d\rho}{\tau_\omega^2(\rho)}} n \right).$$

If we suppose that $\tau_\omega^2(z) \asymp (1 - |z|^2)^{2+\beta} \nu^2(\log(1/(1 - |z|^2)))$, then

$$\lambda_n(T_r) \leq \exp \left(-C(\omega)(1 - r)^{1+\beta} \nu^2(\log(1/(1 - r)))n \right).$$

In particular we have

$$\lambda_n(T_r) = O \left(\exp \left(-C(\omega, \varepsilon)(1 - r)^{1+\beta-\varepsilon} n \right) \right), \quad \forall \varepsilon > 0. \quad (1.19)$$

Lemma 1.8. *Let $\omega \in \mathcal{W}^*$ be a radial weight such that $\tau_\omega^2(z) \asymp (1 - |z|^2)^{2+\beta} \nu^2(\log(\frac{1}{1-|z|^2}))$, where $\beta \geq 0$ and ν is a monotone function such that $\nu(2t) \asymp \nu(t)$. Then, for all $\varepsilon > 0$, we have*

$$s_n(A_\Phi) = O\left(\frac{1}{n^{2/p-\varepsilon}}\right), \quad p = \frac{2(1+\beta)}{2+\beta}.$$

Proof. First, we prove the result for A which corresponds to $\Phi = 1$. Remark that

$$Af = H_{\bar{z}^2}f - 2H_{\bar{z}}P_\omega \bar{z}f \in A_\omega^{2\perp}.$$

Then Af is the L_ω^2 -minimal solution of $\bar{\partial}u = 2H_{\bar{z}}f$. Applying Lemma 1.4 twice, we get

$$\begin{aligned} \|Af\|^2 &\lesssim \int_{\mathbb{D}} |2H_{\bar{z}}f(z)|^2 \tau_\omega^2(z) \omega(z) dA(z) \\ &\lesssim \int_{r\mathbb{D}} |H_{\bar{z}}f(z)|^2 \omega(z) dA(z) + \tau_\omega^2(r) \int_{\mathbb{D} \setminus r\mathbb{D}} |H_{\bar{z}}f(z)|^2 \omega(z) dA(z) \\ &\lesssim \int_{r\mathbb{D}} |f(z)|^2 \omega(z) dA(z) + \int_{r\mathbb{D}} |P_\omega \bar{z}f(z)|^2 \omega(z) dA(z) + \tau_\omega^2(r) \int_{\mathbb{D}} |f(z)|^2 \tau_\omega^2(z) \omega(z) dA(z) \\ &\lesssim \|J_r f\|^2 + \|J_r P_\omega \bar{z}f\|^2 + \tau_\omega^2(r) \langle T_{\tau_\omega^2} f, f \rangle. \end{aligned}$$

Then we obtain

$$A^*A \lesssim T_r + (P_\omega \bar{z})^* T_r P_\omega \bar{z} + \tau_\omega^2(r) T_{\tau_\omega^2}$$

This implies that

$$s_{3n}^2(A) \lesssim \lambda_n(T_r) + \tau_\omega^2(r) \lambda_n(T_{\tau_\omega^2}). \quad (1.20)$$

Since $\int^1 \frac{dr}{\tau_\omega^{2-p-\varepsilon}(r)} < \infty$, by [20], $T_{\tau_\omega^2} \in S_{\frac{p}{2}+\varepsilon}$ for every $\varepsilon > 0$. Then,

$$\lambda_n(T_{\tau_\omega^2}) = O\left(\frac{1}{n^{\frac{p}{2}-\varepsilon}}\right), \quad \text{for all } \varepsilon > 0. \quad (1.21)$$

Combining inequalities (1.19), (1.20) and (1.21), we obtain

$$s_{3n}^2(A) \lesssim \inf_{r \in (0,1)} \left(\exp(-C(\omega, \varepsilon)(1-r)^{1+\beta-\varepsilon} n) + \frac{\tau_\omega^2(r)}{n^{\frac{p}{2}-\varepsilon}} \right), \quad \text{for all } \varepsilon > 0.$$

For a suitable choice of r , we obtain the result for A .

To complete the proof of the Lemma we use the argument given by M. Dostanic in [17]. Suppose that Φ is analytic and bounded by M on $R\mathbb{D} \times R\mathbb{D}$. Clearly, we have

$$\Phi(z, \zeta) = \sum_{k \geq 0} \frac{\Phi^{(k)}(z, 0)}{k!} \zeta^k \quad \text{and} \quad \frac{|\Phi^{(k)}(z, 0)|}{k!} \leq \frac{M}{R^k}.$$

And since $A_\Phi f = \sum_{k \geq 0} \frac{\overline{\Phi^{(k)}(z, 0)}}{k!} AP_\omega \zeta^k f$, we obtain

$$s_{(N+2)m}(A_\Phi) \leq \sum_{k \geq 0}^N \frac{|\Phi^{(k)}(z, 0)|}{k!} s_m(A) + \sum_{k > N} \frac{|\Phi^{(k)}(z, 0)|}{k!} \|A\| \leq \frac{M}{R-1} (s_m(A) + \|A\|).$$

Now for $Nm = n$ and $m \sim n^{1-\varepsilon}$, we get the result. □

Proof of Theorem 1.10. Let $\psi(t) = \frac{1}{\gamma^p} t^p \nu^{p/1+\beta} (\log 1/t)$. Since $s_n(H_{\bar{z}}) \sim \frac{\gamma}{n^{\frac{1}{p}} \nu^{\frac{1}{1+\beta}} (\log n)}$, $D_\psi(H_{\bar{z}}) = d_\psi(H_{\bar{z}}) = 1$. Note that

$$s_n(H_{\bar{\phi}}) \sim s_n(H_{\bar{z}}) \|\phi'\|_p \iff D_\psi(H_{\bar{\phi}}) = d_\psi(H_{\bar{\phi}}) = \|\phi'\|_p^p.$$

First, suppose that ϕ is analytic in a neighborhood of $\overline{\mathbb{D}}$. Then there exist $R > 1$ and an analytic bounded function Φ on $R\mathbb{D} \times R\mathbb{D}$ such that

$$\phi(z) - \phi(w) = (z-w)\phi'(w) + (z-w)^2\Phi(z, w), \quad z, w \in R\mathbb{D}.$$

We have

$$H_{\bar{\phi}} P_\omega = H_{\bar{z}} P_\omega \bar{\phi}' + A_\Phi P_\omega.$$

By Lemma 1.8

$$s_n(A_\Phi P_\omega) = o(s_n(H_{\bar{z}} P_\omega)), \quad n \rightarrow \infty.$$

So,

$$D_\psi(A_\Phi P_\omega) = 0.$$

By Corollary 1.3, we deduce that

$$D_\psi(H_{\bar{\phi}}) = D_\psi(H_{\bar{z}} P_\omega \bar{\phi}') \quad \text{and} \quad d_\psi(H_{\bar{\phi}}) = d_\psi(H_{\bar{z}} P_\omega \bar{\phi}').$$

We obtain, by Proposition 1.4, that

$$d_\psi(H_{\bar{z}}) \|\phi'\|_p^p \leq d_\psi(H_{\bar{\phi}}) \leq D_\psi(H_{\bar{\phi}}) \leq D_\psi(H_{\bar{z}}) \|\phi'\|_p^p.$$

Since $D_\psi(H_{\bar{z}}) = d_\psi(H_{\bar{z}}) = 1$, we obtain

$$d_\psi(H_{\bar{\phi}}) = D_\psi(H_{\bar{\phi}}) = \|\phi'\|_p^p.$$

Now, suppose that $\phi' \in H^p$. By Theorem 1.9, we have

$$s_n(H_{\bar{\phi}-\bar{\phi}_r}) \leq C \|\phi' - \phi'_r\|_p s_n(H_{\bar{z}}), \quad n \geq 1.$$

This implies that $D_\psi(H_{\bar{\phi}-\bar{\phi}_r}) \leq C^p \|\phi' - \phi'_r\|_p^p D_\psi(H_{\bar{z}})$. Then we have

$$\begin{aligned} |D_\psi(H_{\bar{\phi}})^{\frac{1}{p+1}} - (D_\psi(H_{\bar{z}})\|\phi'\|_p^p)^{\frac{1}{p+1}}| &= |D_\psi(H_{\bar{\phi}})^{\frac{1}{p+1}} - D_\psi(H_{\bar{\phi}_r})^{\frac{1}{p+1}}| \\ &\leq D_\psi(H_{\bar{\phi}-\bar{\phi}_r})^{\frac{1}{p+1}} \\ &\leq (C^p \|\phi' - \phi'_r\|_p^p D_\psi(H_{\bar{z}}))^{\frac{1}{p+1}}. \end{aligned}$$

When $r \rightarrow 1^-$, we obtain $D_\psi(H_{\bar{\phi}}) = D_\psi(H_{\bar{z}})\|\phi'\|_p^p = \|\phi'\|_p^p$. With the same arguments we have $d_\psi(H_{\bar{\phi}}) = d_\psi(H_{\bar{z}})\|\phi'\|_p^p$. The proof is complete. \square

Similarly, one can prove the following result which corresponds to the situation $\beta = 0$ and $\nu \asymp 1$ in Theorem 1.10.

Theorem 1.11. *Let $\omega \in \mathcal{W}^*$ be a radial weight such that $\tau_\omega^2(z) \asymp (1 - |z|^2)^2$. Suppose that*

$$s_n(H_{\bar{z}}) \sim \frac{\gamma}{n},$$

where $\gamma > 0$. Then, for $\phi' \in H^1$, we have

$$s_n(H_{\bar{\phi}}) \sim \frac{\gamma}{n} \|\phi'\|_1.$$

Proof of the second assertion of Theorem 1.3. The case $\beta = 0$ comes from Theorem 1.11 while the case $\beta > 0$ is a consequence of Theorem 7.1 with $\nu = 1$. \square

1.8 An example

For a radial weight ω , the sequence of the singular values of the Hankel operator $H_{\bar{z}}$ on A_ω^2 is the nonincreasing rearrangement of the sequence

$$\left(\left(\frac{\|z^{n+1}\|^2}{\|z^n\|^2} - \frac{\|z^n\|^2}{\|z^{n-1}\|^2} \right)^{1/2} \right)_n.$$

Recall that for the standard case $\omega_\alpha(z) = (\alpha + 1)(1 - |z|^2)^\alpha$, we have $s_n(H_{\bar{z}}) \sim \frac{\sqrt{\alpha+1}}{n}$. In this section, we consider the weight

$$\omega(z) = \exp\left(-\frac{\alpha}{(\log \frac{1}{|z|^2})^\beta}\right), \quad \alpha, \beta > 0.$$

We have

$$\begin{aligned}
\|z^n\|^2 &= \int_{\mathbb{D}} |z|^{2n} \exp\left(-\frac{\alpha}{(\log \frac{1}{|z|^2})^\beta}\right) dA(z) \\
&= \int_0^1 r^{2n} \exp\left(-\frac{\alpha}{(\log \frac{1}{r^2})^\beta}\right) 2r dr \\
&= \int_0^{+\infty} \exp\left(-(n+1)x - \frac{\alpha}{x^\beta}\right) dx.
\end{aligned}$$

Let $x_n := \left(\frac{\alpha\beta}{n+1}\right)^{1/1+\beta}$ be the minimum of the function $(n+1)x + \frac{\alpha}{x^\beta}$. After the change of variable $u = \frac{x-x_n}{x_n}$, we get

$$\|z^n\|^2 = x_n \exp\left(-(n+1)x_n - \frac{\alpha}{x_n^\beta}\right) \int_{-1}^{+\infty} \exp\left(-\frac{\alpha}{x_n^\beta} h(u)\right) du,$$

where $h(u) = \beta u + \frac{1}{(1+u)^\beta} - 1$.

In the sequel, we will use Laplace method [24] to get an expansion of $\|z^n\|^2$. By Laplace Theorem, we have

$$H(t) = \sqrt{\frac{th''(0)}{2\pi}} \int_{-1}^{+\infty} \exp(-th(u)) du \sim 1, \quad t \rightarrow +\infty.$$

Let $\eta > 0$. We have

$$\sqrt{\frac{th''(0)}{2\pi}} \int_{[-1, -\eta] \cup [\eta, +\infty[} \exp(-th(u)) du = O(e^{-c(\eta)t}), \quad t \rightarrow +\infty.$$

Let

$$\begin{aligned}
H_\eta^+(t) &= \sqrt{\frac{th''(0)}{2\pi}} \int_{0 < u < \eta} \exp(-th(u)) du, \\
H_\eta^-(t) &= \sqrt{\frac{th''(0)}{2\pi}} \int_{-\eta < u < 0} \exp(-th(u)) du.
\end{aligned}$$

Using the change of variable $v = h(u)$, when η is small enough, one can write

$$H_\eta^+(t) = \sum_{j=0}^{2N} \frac{c_j}{t^{j/2}} + o\left(\frac{1}{t^N}\right),$$

where $c_0, \dots, c_{2N} \in \mathbb{R}$. The same is also true for H_η^- . Then

$$H(t) = \sum_{j=0}^{2N} \frac{d_j}{t^{j/2}} + o\left(\frac{1}{t^N}\right),$$

where $d_0 = 1$ and $d_1, \dots, d_{2N} \in \mathbb{R}$. Let $A_n = \sqrt{\frac{2\pi}{h''(0)}} x_n^{1+\frac{\beta}{2}} \exp\left(-(n+1)x_n - \frac{\alpha}{x_n^\beta}\right)$. We have

$$\|z^n\|^2 = A_n H\left(\frac{\alpha}{x_n^\beta}\right).$$

By a direct calculation, we get

$$\frac{\|z^n\|^2}{\|z^{n-1}\|^2} = 1 - \frac{a}{n^{1/1+\beta}} + \frac{b}{n} + \frac{c}{n^{1+1/1+\beta}} + o(1/n^{1+\frac{1}{\beta+1}}),$$

where $a = (\alpha\beta)^{1/1+\beta}$. Then, we have

$$\frac{\|z^{n+1}\|^2}{\|z^n\|^2} - \frac{\|z^n\|^2}{\|z^{n-1}\|^2} \sim \frac{\gamma^2}{n^{\frac{\beta+2}{\beta+1}}},$$

where $\gamma = \sqrt{\frac{(\alpha\beta)^{1/1+\beta}}{1+\beta}}$. Finally, we obtain

$$s_n(H_{\bar{z}}) \sim \frac{\gamma}{n^{\frac{\beta+2}{2(\beta+1)}}}.$$

Appendix: Asymptotic Orthogonality

Let T be a compact operator between two complex Hilbert spaces. The decreasing sequence of singular values of T will be denoted by $(s_n(T))_n$. The counting function of the singular values of T is denoted by

$$n(s, T) = \#\{n : s_n(T) \geq s\}, \quad s > 0.$$

Recall that \mathcal{C}_p denotes the class of increasing continuous function $\psi : (0, +\infty) \rightarrow (0, +\infty)$ satisfying

$$\psi(\alpha t) \sim \alpha^p \psi(t), \quad (t \rightarrow 0^+),$$

As before, $D_\psi(T), d_\psi(T)$ are given by

$$D_\psi(T) := \limsup_{s \rightarrow 0^+} \psi(s)n(s, T) \quad \text{and} \quad d_\psi(T) := \liminf_{s \rightarrow 0^+} \psi(s)n(s, T).$$

The goal of this appendix is to extend the results obtained for $\psi(t) = t^p$, [12, 45, 50], to the class \mathcal{C}_p . We give here the proof for completeness.

Proposition 1.5. *Let T and V be compact operators and $\lambda \in \mathbb{C}$. We have*

1. $D_\psi(\lambda T) = |\lambda|^p D_\psi(T)$ and $d_\psi(\lambda T) = |\lambda|^p d_\psi(T)$.
2. $D_\psi(T + V)^{\frac{1}{p+1}} \leq D_\psi(T)^{\frac{1}{p+1}} + D_\psi(V)^{\frac{1}{p+1}}$.
3. $d_\psi(T + V)^{\frac{1}{p+1}} \leq d_\psi(T)^{\frac{1}{p+1}} + d_\psi(V)^{\frac{1}{p+1}}$.
4. $D_\psi(TV) \leq 2D_{\psi \circ t^2}(T)D_{\psi \circ t^2}(V)$.

Proof. 1. For $\lambda \neq 0$, we have

$$\begin{aligned}
D_\psi(\lambda T) &= \limsup_{s \rightarrow 0^+} \psi(s)n(s, \lambda T) = \limsup_{s \rightarrow 0^+} \psi(s)n(s/|\lambda|, T) \\
&= \limsup_{s \rightarrow 0^+} \psi(|\lambda|s)n(s, T) \\
&= |\lambda|^p \limsup_{s \rightarrow 0^+} \psi(s)n(s, T) \\
&= |\lambda|^p D_\psi(T).
\end{aligned}$$

Similarly, we have $d_\psi(\lambda T) = |\lambda|^p d_\psi(T)$, $\forall \lambda \in \mathbb{C}$.

2. Let $x \in (0, 1)$. For all $s > 0$, we have

$$\begin{aligned}
n(s, T + V) &= n(xs + (1-x)s, T + V) \leq n(xs, T) + n((1-x)s, V) \\
&= n\left(s, \frac{1}{x}T\right) + n\left(s, \frac{1}{1-x}V\right).
\end{aligned}$$

Hence, $D_\psi(T + V) \leq D_\psi\left(\frac{1}{x}T\right) + D_\psi\left(\frac{1}{1-x}V\right)$. By the first assertion we obtain

$$D_\psi(T + V) \leq x^{-p}D_\psi(T) + (1-x)^{-p}D_\psi(V).$$

It follows that

$$D_\psi(T + V) \leq \min_{x \in (0,1)} \left\{ x^{-p}D_\psi(T) + (1-x)^{-p}D_\psi(V) \right\},$$

and 2. is obtained.

By the same way we obtain also the third assertion.

4. For all $\alpha, s > 0$, we have

$$\begin{aligned}
n(s, TV) &= n(\alpha\sqrt{s} \alpha^{-1}\sqrt{s}, TV) \leq n(\alpha\sqrt{s}, T) + n(\alpha^{-1}\sqrt{s}, V) \\
&= n(\sqrt{s}, \frac{1}{\alpha}T) + n(\sqrt{s}, \alpha V).
\end{aligned}$$

Hence

$$\begin{aligned}\psi(s)n(s, TV) &\leq \psi(s)n(\sqrt{s}, \frac{1}{\alpha}T) + \psi(s)n(\sqrt{s}, \alpha V) \\ &= \tilde{\psi}(\sqrt{s})n(\sqrt{s}, \frac{1}{\alpha}T) + \tilde{\psi}(\sqrt{s})n(\sqrt{s}, \alpha V),\end{aligned}$$

with $\tilde{\psi} = \psi(t^2)$. Then

$$\begin{aligned}D_\psi(TV) &\leq D_{\tilde{\psi}}\left(\frac{1}{\alpha}T\right) + D_{\tilde{\psi}}(\alpha V) \\ &= \alpha^{-2p}D_{\tilde{\psi}}(T) + \alpha^{2p}D_{\tilde{\psi}}(V).\end{aligned}$$

It follows that

$$\begin{aligned}D_\psi(TV) &\leq \min_{\alpha>0} \left\{ \alpha^{-2p}D_{\tilde{\psi}}(T) + \alpha^{2p}D_{\tilde{\psi}}(V) \right\} \\ &\leq 2D_{\tilde{\psi}}(T)D_{\tilde{\psi}}(V).\end{aligned}$$

□

Let us denote

$$\Sigma_\psi := \{T \in S_\infty : \psi(s)n(s, T) = O(1)\}$$

and

$$\text{and } \Sigma_\psi^0 := \{T \in S_\infty : \psi(s)n(s, T) = o(1)\}.$$

The second and the third assertions of the Proposition 1.5 imply the following corollary.

Corollary 1.3. *Let T and V be compact operators. Suppose that $V \in \Sigma_\psi^0$, then*

$$D_\psi(T + V) = D_\psi(T) \quad \text{and} \quad d_\psi(T + V) = d_\psi(T).$$

Once we have that, we can state the Theorem of A. Pushnitski [[45], Theorem 2.2] in the following form

Theorem 1.12. Let T_1, T_2, \dots, T_n be compact operators such that

$$T_i^*T_j \in \Sigma_{\psi \circ \sqrt{\cdot}}^0 \quad \text{and} \quad T_iT_j^* \in \Sigma_{\psi \circ \sqrt{\cdot}}^0, \quad \forall i \neq j.$$

Then

$$D_\psi \left(\sum_{k=1}^n T_k \right) = \limsup_{s \rightarrow 0^+} \psi(s) \sum_{k=1}^n n(s, T_k) \quad \text{and} \quad d_\psi \left(\sum_{k=1}^n T_k \right) = \liminf_{s \rightarrow 0^+} \psi(s) \sum_{k=1}^n n(s, T_k).$$

In particular, if for all $s > 0$, we have $n(s, T_k) = n(s, T_1)$, $k = 1, 2, \dots, n$, then

$$D_\psi \left(\sum_{k=1}^n T_k \right) = nD_\psi(T_1) \quad \text{and} \quad d_\psi \left(\sum_{k=1}^n T_k \right) = nd_\psi(T_1).$$

Proof. Let $T = \sum_{k=1}^n T_k$. Since

$$TT^* = \sum_{i=1}^n T_iT_i^* + \sum_{i,j=1, i \neq j}^n T_iT_j^*,$$

and $T_iT_j^* \in \Sigma_{\psi \circ \sqrt{\cdot}}^0$, for $i \neq j$, we have

$$\begin{aligned} D_\psi(T) &= D_{\psi \circ \sqrt{\cdot}}(TT^*) = D_{\psi \circ \sqrt{\cdot}} \left(\sum_{i=1}^n T_iT_i^* \right) \\ &= D_{\psi \circ \sqrt{\cdot}}((JA)(JA)^*) \\ &= D_{\psi \circ \sqrt{\cdot}}((JA)^*(JA)). \end{aligned}$$

Where

$$J : \begin{array}{ccc} H^n & \longrightarrow & H \\ (f_1, \dots, f_n) & \longrightarrow & \sum_{i=1}^n f_i \end{array} \quad \text{and} \quad A : \begin{array}{ccc} H^n & \longrightarrow & H^n \\ (f_1, \dots, f_n) & \longrightarrow & (T_1f_1, T_2f_2, \dots, T_nf_n) \end{array}$$

The matrix of $(JA)^*(JA)$ is

$$\begin{pmatrix} T_1^*T_1 & T_1^*T_2 & \dots & T_1^*T_n \\ T_2^*T_1 & T_2^*T_2 & \dots & T_2^*T_n \\ \vdots & \vdots & \ddots & \vdots \\ T_n^*T_1 & T_n^*T_2 & \dots & T_n^*T_n \end{pmatrix} = \begin{pmatrix} T_1^*T_1 & & & 0 \\ & T_2^*T_2 & & \\ & & \ddots & \\ 0 & & & T_n^*T_n \end{pmatrix} + \begin{pmatrix} O & T_1^*T_2 & \dots & T_1^*T_n \\ T_2^*T_1 & 0 & \dots & T_2^*T_n \\ \vdots & \vdots & \ddots & \vdots \\ T_n^*T_1 & T_n^*T_2 & \dots & 0 \end{pmatrix} =: T' + T''.$$

Since $T_i^*T_j \in \Sigma_{\psi \circ \sqrt{\cdot}}^0$, for $i \neq j$, we deduce that

$$\begin{aligned} D_\psi(T) &= D_{\psi \circ \sqrt{\cdot}}(T' + T'') = D_{\psi \circ \sqrt{\cdot}}(T') \\ &= D_{\psi \circ \sqrt{\cdot}}(T_1^*T_1, T_2^*T_2, \dots, T_n^*T_n). \end{aligned}$$

Let now T_0 the operator

$$T_0 := \begin{pmatrix} T_1 & & & 0 \\ & T_2 & & \\ & & \ddots & \\ 0 & & & T_n \end{pmatrix}$$

We have

$$\begin{aligned} D_\psi(T) &= D_{\psi \circ \sqrt{\cdot}}(T_1^*T_1, T_2^*T_2, \dots, T_n^*T_n) = D_{\psi \circ \sqrt{\cdot}}(T_0^*T_0) \\ &= D_\psi(T_0) \\ &= \limsup_{s \rightarrow 0^+} \psi(s) \sum_{k=1}^n n(s, T_k). \end{aligned}$$

In the same way we prove that

$$d_\psi \left(\sum_{k=1}^n T_k \right) = \liminf_{s \rightarrow 0^+} \psi(s) \sum_{k=1}^n n(s, T_k).$$

□

CHAPTER 2

BERGMAN SPACES SATISFYING HARDY-LITTLEWOOD THEOREM

2.1 Introduction

A weight on the open unit disk \mathbb{D} of the complex plane \mathbb{C} is a function $\omega : \mathbb{D} \rightarrow]0, +\infty[$ which is integrable with respect to Lebesgue measure $dA(z) := dxdy/\pi$ on \mathbb{D} . The weighted Bergman space A_ω^p , $p > 0$, associated with a weight ω is defined by

$$A_\omega^p := \{f \in \text{Hol}(\mathbb{D}) : \|f\|_{A_\omega^p}^p := \int_{\mathbb{D}} |f(z)|^p \omega(z) dA(z) < \infty\},$$

where $\text{Hol}(\mathbb{D})$ is the set of holomorphic functions on \mathbb{D} . For $\alpha > -1$, the standard Bergman space $A_{\omega_\alpha}^p$, where $\omega_\alpha(z) = (\alpha + 1)(1 - |z|^2)^\alpha$, will be denoted by A_α^p . We say that ω satisfies Hardy-Littlewood theorem, and we write $\omega \in \mathcal{HL}$, if for all $p > 0$, $f \in \text{Hol}(\mathbb{D})$, and $n \in \mathbb{N}^*$, we have

$$\|f\|_{A_\omega^p}^p \asymp \sum_{k=0}^{n-1} |f^{(k)}(0)|^p + \int_{\mathbb{D}} |f^{(n)}(z)|^p (1 - |z|^2)^{np} \omega(z) dA(z).$$

Throughout this thesis, we write $A \lesssim B$ if there exists a positive constant C such that $A \leq CB$. Moreover, if $A \lesssim B$ and $B \lesssim A$, then we write $A \asymp B$.

It is well-known that $\omega_\alpha \in \mathcal{HL}$ for all $\alpha > -1$. A weight ω is said to be in the class of Bekollé-Bonami weights $B_p(\eta)$, for $p \in (1, \infty)$ and $\eta > -1$, if

$$\left(\int_{S(\theta, h)} \omega dA_\eta \right) \left(\int_{S(\theta, h)} \omega^{-p'/p} dA_\eta \right)^{p/p'} \lesssim (A_\eta(S(\theta, h)))^p,$$

for any Carleson square

$$S(\theta, h) = \{re^{i\alpha} : 1 - h < r < 1, |\theta - \alpha| < h/2\}, \theta \in [0, 2\pi], h \in (0, 1),$$

where $1/p + 1/p' = 1$ and $dA_\eta = \omega_\eta dA$.

Aleman and Constantin [2] proved that if $\frac{\omega}{(1-|z|^2)^\eta} \in B_{p_0}(\eta)$, for some $p_0 > 1$ and $\eta > -1$, then $\omega \in \mathcal{HL}$. Conversely, if ω is in \mathcal{HL} and satisfies

$$\omega(z) \asymp \omega(w), |z - w| < \delta(1 - |z|^2), \quad (2.1)$$

for some $\delta \in (0, 1)$, then $\frac{\omega}{(1-|z|^2)^\eta} \in B_{p_0}(\eta)$ for some $p_0 > 1$ and $\eta > -1$, see [3]. Recently, Bao, Wulan, and Zhu [10] have introduced a new class of weights which satisfy Hardy-Littlewood theorem. For $s > -1$ and $t \geq 0$, the class $\mathcal{W}(s, t)$ is the set of weights ω satisfying

$$\omega_{s,t}(z) := \int_{\mathbb{D}} \frac{\omega(w)(1-|w|^2)^s(1-|z|^2)^t}{|1-\bar{w}z|^{2+s+t}} dA(w) \lesssim \omega(z), z \in \mathbb{D}.$$

It is proved in [10] that if there exist $s_0 \in [-1, 0)$ and $t_0 \geq 0$ such that $\omega \in \mathcal{W}(s, t)$ for all $s > s_0$ and for all $t > t_0$, and if there exists $\delta \in (0, 1)$ such that ω satisfies (2.1), then $\omega \in \mathcal{HL}$. In Section 2.2, we prove that the existence of $s \in (-1, 0]$ and $t \geq 0$ such that $\omega \in \mathcal{W}(s, t)$ is enough to get $\omega \in \mathcal{HL}$. Next, we consider the following class appeared in the literature before, usually denoted by \mathcal{C}_p , see for instance [2, 16, 31]. For $p > 1$, a weight ω belongs to the class \mathcal{C}_p if

$$\left(\int_{\Delta(z,\alpha)} \omega dA \right) \left(\int_{\Delta(z,\alpha)} \omega^{-p'/p} dA \right)^{p/p'} \lesssim A^p(\Delta(z,\alpha)), z \in \mathbb{D},$$

for some $\alpha \in (0, 1)$ (or equivalently for all $\alpha \in (0, 1)$, see [31]). Throughout this paper, $\Delta(z, r)$ is the disk given by

$$\Delta(z, r) := \{w \in \mathbb{D} : |z - w| < r(1 - |z|)\}, z \in \mathbb{D}, 0 < r < 1.$$

Let

$$\tilde{\omega}(z) := \frac{1}{(1 - |z|^2)^2} \int_{\Delta(z, 1/2)} \omega(\zeta) dA(\zeta), z \in \mathbb{D}.$$

In Section 2.2, we prove the following theorem which is a variant of results given in [2, 3, 10].

Theorem 2.1. *Suppose that $\omega \in \mathcal{C}_{p_0}$ for some $p_0 > 1$. The following assertions are equivalent.*

- (1) $\omega \in \mathcal{HL}$.
- (2) $\tilde{\omega} \in \mathcal{HL}$.
- (3) $\omega_{0,t} \lesssim \tilde{\omega}$ for some $t > 0$.

Note that if ω satisfies (2.1), then $\omega \in \mathcal{C}_p$ for all $p > 1$. Moreover, if $\frac{\omega}{(1-|z|^2)^\eta} \in B_{p_0}(\eta)$ for some $p_0 > 1$ and $\eta > -1$, then $\omega \in \mathcal{C}_{p_0}$ and $\omega_{0,t} \lesssim \tilde{\omega}$ for all $t \geq p_0(\eta + 2) - 2$, see [2, Lemma 2.1].

As a direct consequence of the previous theorem, if $\omega \in \mathcal{C}_{p_0}$ for some $p_0 > 1$, then the condition

$$\int_{\mathbb{D}} \omega(z) \log \frac{1}{1-|z|^2} dA(z) < \infty$$

is necessary to have $\omega \in \mathcal{HL}$ (Corollary 2.1).

Motivated by Theorem 2.1, we consider the following class of weights.

Definition 2.1. *Let $p > 1$ and let $t \geq 0$. We say that a weight ω belongs to the class $\mathcal{C}_{p,t}$ if $\omega \in \mathcal{C}_p$ and $\omega_{0,t} \lesssim \tilde{\omega}$.*

In Section 2.3, we prove that if $\omega \in \mathcal{C}_{p_0,t}$ for some $p_0 > 1$ and $t \geq 0$, then the dilation $f_r(z) = f(rz)$ of each function f in A_ω^p satisfies

$$\|f_r\|_{A_\omega^p} \leq C(\omega) \|f\|_{A_\omega^p}, \quad f \in A_\omega^p,$$

for all $r \in [0, 1)$. Moreover, we prove that polynomials are dense in A_ω^p for all $p > 0$.

Concerning radial weights, various generalizations of the fact that $\omega_\alpha \in \mathcal{HL}$, $\alpha > -1$, are obtained for different classes of weights, see [5, 38, 47] and references therein. Recently, Peláez and Rättyä [44] gave a complete description of radial weights belonging to \mathcal{HL} . Using their result (Theorem 2.5 below), we will prove the following theorem. In the sequel, $\tilde{\omega}$ is the function defined by

$$\tilde{\omega}(r) := \frac{1}{1-r} \int_r^1 \omega(s) ds, \quad r \in [0, 1).$$

Theorem 2.2. *Let ω be a radial weight. The following assertions are equivalent.*

- (1) $\omega \in \mathcal{HL}$.
- (2) $\tilde{\omega}$ is a weight and $\omega_{s,t}(r) \lesssim \tilde{\omega}(r)$, $r \in [0, 1)$, for some $s \in (-1, 0)$ and some $t > 0$.
- (3) $\tilde{\omega} \in \mathcal{W}(0, t)$ for some $t > 0$.

Note that, contrary to $\tilde{\omega}$, $\tilde{\omega}$ can be non-integrable. Note also that in Theorem 2.2, the condition $\omega_{s,t} \lesssim \tilde{\omega}$ for some $s \in (-1, 0)$ and $t > 0$ cannot be replaced by $\omega_{0,t} \lesssim \tilde{\omega}$ for some $t > 0$, even

for weights that satisfy (2.1). Indeed, let ω be the weight given by

$$\omega(r) = \frac{1}{(1-r) \left(\ln \frac{e}{1-r}\right)^\gamma}, \quad \gamma > 1.$$

A direct calculation shows that, for all $t > 0$, we have

$$\omega_{0,t}(r) \asymp \ln \left(\frac{e}{1-r} \right) \omega(r), \quad r \in [0, 1).$$

By Theorem 2.1, $\omega \notin \mathcal{HL}$. Now, for all $\gamma > 1$, we have

$$\tilde{\omega}(r) \asymp \frac{1}{(1-r) \left(\ln \frac{e}{1-r}\right)^{\gamma-1}}.$$

Thus $\tilde{\omega}$ is non-integrable for $1 < \gamma \leq 2$. While for $\gamma > 2$, $\tilde{\omega}$ is integrable and $\omega_{0,t} \asymp \tilde{\omega}$ for all $t > 0$.

In the last section, we consider Toeplitz and integration operators on A_ω^2 for ω in $\mathcal{C}_{p_0,t}$ for some $p_0 > 1$ and $t \geq 0$. We give a complete description of the boundedness, compactness, and membership of such operators to Schatten classes, extending results obtained in [2, 16].

Recall that Toeplitz operator T_μ on A_ω^2 associated with a positive Borel measure μ on \mathbb{D} is the transformation defined by

$$T_\mu f(z) := \int_{\mathbb{D}} f(\zeta) K^\omega(z, \zeta) \omega(\zeta) d\mu(\zeta), \quad z \in \mathbb{D}.$$

The study of T_μ has received an important attention from several authors. The boundedness, compactness, and membership to p -Schatten ideals of the operator T_μ on A_ω^2 have been studied for different weights. For example, Luecking [?] considered standard Bergman spaces A_α^2 . Oleinik-Perel'man weights were studied by Oleinik and Perel'man [35], and later by Arroussi and Pau [8]. Constantin [16] characterized the boundedness, compactness, and membership of T_μ to Schatten class $\mathcal{S}_p(A_\omega^2)$ for $p \geq 1$, where $\frac{\omega}{(1-|z|^2)^\eta} \in B_{p_0}(\eta)$ for some $p_0 > 1$ and $\eta > -1$. In [21], El-Fallah, Mahzouli, Marrhich, and Naqos studied the operator T_μ on A_ω^2 for a large class of weights ω by imposing conditions on the reproducing kernel K^ω . Peláez and Rättyä [39] studied Toeplitz operators on Bergman spaces associated with radial weights, see also [41] and references therein.

Recall that, for $p > 0$, the p -Schatten ideal of a separable Hilbert space \mathcal{H} , denoted by $\mathcal{S}_p(\mathcal{H})$, consists of compact operators T on \mathcal{H} for which the sequence of the singular numbers $(s_n(T))_n$; that is, the sequence of the square root of the eigenvalues of the positive operator T^*T , belongs to ℓ^p .

We will use results from [21] and [20] to obtain a complete description of the boundedness, compactness, and membership of T_μ to $\mathcal{S}_p(A_\omega^2)$ for all $p > 0$ and for any weight $\omega \in \mathcal{C}_{p_0,t}$ for some $p_0 > 1$ and $t \geq 0$. Throughout this paper, we decompose \mathbb{D} using the sets $\Delta(z, r)$, $0 < r < 1$. The Lebesgue measure of a Borel set Δ of \mathbb{D} will be denoted by $|\Delta|$. Let $z_0 \in \mathbb{D}$. For $z \in \Delta(z_0, r)$, we have

- $|\Delta(z, r)| \asymp (1 - |z_0|^2)^2$.
- $|1 - \bar{z}w| \asymp 1 - |z|^2 \asymp 1 - |w|^2$, $w \in \Delta(z_0, r)$.
- $|1 - \bar{z}w| \asymp |1 - \bar{z}\zeta|$, $w, \zeta \in \Delta(z_0, r)$.

In the terminology of [21], the sets $\Delta(z, r)$ give a (ρ, δ) -lattice of \mathbb{D} for $\rho(z) = (1 - |z|^2)/2$ and for some choice of δ . Let $\Delta(z_n, \delta)$ be the corresponding (ρ, δ) -lattice of \mathbb{D} and let $(\Delta_n)_n$ be an enumeration of $\Delta(z_n, \delta)$. Let $b > 1$ such that $b\Delta_n = \Delta(z_n, b\delta)$ is a covering of \mathbb{D} of finite multiplicity, see [21] for more details. We prove the following theorem which improves Constantin's result. In the sequel, for a positive measure μ on \mathbb{D} , we denote by μ_ω the measure

$$d\mu_\omega(z) := \omega(z)d\mu(z), \quad z \in \mathbb{D}.$$

Theorem 2.3. *Let $p > 0$ and let ω be a weight in $\mathcal{C}_{p_0,t}$ for some $p_0 > 1$ and $t \geq 0$. Let μ be a positive Borel measure on \mathbb{D} such that T_μ is compact on A_ω^2 . Then T_μ belongs to $\mathcal{S}_p(A_\omega^2)$ if and only if*

$$\sum_{n=0}^{\infty} \left(\frac{\mu_\omega(\Delta_n)}{A_\omega(\Delta_n)} \right)^p < \infty.$$

Let $g \in \text{Hol}(\mathbb{D})$, the generalized Volterra integration operator V_g associated with g is defined by

$$V_g f(z) := \int_0^z f(\zeta)g'(\zeta)d\zeta, \quad f \in \text{Hol}(\mathbb{D}).$$

The boundedness, compactness, and membership to Schatten classes of V_g have been the object of several papers on various spaces of analytic functions, see for instance [1, 4, 5, 51]. Aleman and Siskakis [5] characterized the boundedness and compactness of V_g on the standard Bergman space A_α^2 . They proved that V_g is bounded (resp., compact) on A_α^2 if and only if g belongs to the Bloch space \mathcal{B} (resp., little Bloch space \mathcal{B}_0). Recall that

$$\mathcal{B} := \{g \in \text{Hol}(\mathbb{D}), \sup_{z \in \mathbb{D}} (1 - |z|^2)|g'(z)| < \infty\},$$

and

$$\mathcal{B}_0 := \{g \in \text{Hol}(\mathbb{D}), \limsup_{|z| \rightarrow 1} (1 - |z|^2)|g'(z)| = 0\}.$$

They also characterized the membership of V_g to $\mathcal{S}_p(A_\alpha^2)$. More precisely, they showed that $V_g \in \mathcal{S}_p(A_\alpha^2)$, for $p > 1$, if and only if g belong to the Besov space

$$\mathcal{B}_p := \{g \in \text{Hol}(\mathbb{D}), \int_{\mathbb{D}} |g'(z)|^p (1 - |z|^2)^{p-2} dA(z) < \infty\},$$

and $V_g \in \mathcal{S}_1(A_\alpha^2)$ if and only if g is constant. If $\frac{\omega}{(1-|z|^2)^\eta} \in B_p(\eta)$, Constantin [16] proved that, for $p \geq 2$, the operator V_g belongs to $\mathcal{S}_p(A_\omega^2)$ if and only if g belongs to \mathcal{B}_p . We have the following

Theorem 2.4. *Suppose that $\omega \in \mathcal{C}_{p_0, t}$ for some $p_0 > 1$ and $t \geq 0$. Let $p > 0$ and let g be an analytic function on \mathbb{D} such that V_g is a compact operator on A_ω^2 . Then the following assertions hold.*

- (1) *If $p > 1$, then V_g belongs to $\mathcal{S}_p(A_\omega^2)$ if and only if g belongs to \mathcal{B}_p .*
- (2) *If $p \leq 1$, then V_g belongs to $\mathcal{S}_p(A_\omega^2)$ if and only if $g' \equiv 0$ on \mathbb{D} .*

Our proofs of Theorem 2.3 and Theorem 2.4 avoid the complex interpolation method used in [16]. Our approach is based on the paper [20].

2.2 Hardy–Littlewood theorem

This section contains the proofs of Theorem 2.1 and Theorem 2.2. We start by giving some properties of the class $\mathcal{W}_{s, t}$.

Lemma 2.1. *Let $s > -1$, $t \geq 0$, and let ω be a weight on \mathbb{D} . Then the following assertions hold.*

- (1) *For all $s' \geq s$ and $t' \geq t$ we have $\omega_{s', t'} \lesssim \omega_{s, t}$. In particular, $\mathcal{W}(s, t) \subset \mathcal{W}(s', t')$ for all $s' \geq s$ and $t' \geq t$.*
- (2) *If $t > 0$, then for all $s' \in (-1, s)$ and $t' \in (0, t)$ such that $\omega_{s', t'}$ is a weight we have $\omega_{s', t'} \in \mathcal{W}(s, t)$.*

To prove Lemma 2.1, we need the following useful inequality [25, Lemma 2.5].

Lemma 2.2. *Let $\alpha > -1$, $\beta, \gamma > 0$ such that $\beta + \gamma - \alpha > 2$ and $\beta < 2 + \alpha < \gamma$. Then*

$$\int_{\mathbb{D}} \frac{(1 - |z|^2)^\alpha}{|1 - \bar{z}\zeta|^\beta |1 - \bar{z}w|^\gamma} dA(z) \lesssim \frac{1}{|1 - \bar{w}\zeta|^\beta (1 - |w|^2)^{\gamma - \alpha - 2}}, \quad w, \zeta \in \mathbb{D}.$$

Proof of Lemma 2.1. For $s' \geq s$ and $t' \geq t$ we have

$$\begin{aligned} & \int_{\mathbb{D}} \frac{\omega(w)(1 - |w|^2)^{s'}(1 - |z|^2)^{t'}}{|1 - \bar{w}z|^{2+s'+t'}} dA(w) \\ &= \int_{\mathbb{D}} \frac{\omega(w)(1 - |w|^2)^s(1 - |z|^2)^t}{|1 - \bar{w}z|^{2+s+t}} \left(\frac{1 - |z|^2}{|1 - \bar{w}z|} \right)^{t'-t} \left(\frac{1 - |w|^2}{|1 - \bar{w}z|} \right)^{s'-s} dA(w) \\ &\lesssim \int_{\mathbb{D}} \frac{\omega(w)(1 - |w|^2)^s(1 - |z|^2)^t}{|1 - \bar{w}z|^{2+s+t}} dA(w). \end{aligned}$$

Therefore $\omega_{s',t'} \lesssim \omega_{s,t}$.

To prove the assertion (2), let $s' \in (-1, s)$ and $t' \in (0, t)$. By Fubini's theorem we have

$$\begin{aligned} & (\omega_{s',t'})_{s,t}(z) \\ &= \int_{\mathbb{D}} \frac{(1 - |w|^2)^s(1 - |z|^2)^t}{|1 - \bar{w}z|^{2+s+t}} \int_{\mathbb{D}} \frac{\omega(\zeta)(1 - |\zeta|^2)^{s'}(1 - |w|^2)^{t'}}{|1 - \bar{w}\zeta|^{2+s'+t'}} dA(\zeta) dA(w) \\ &= \int_{\mathbb{D}} \omega(\zeta)(1 - |\zeta|^2)^{s'}(1 - |z|^2)^t \int_{\mathbb{D}} \frac{(1 - |w|^2)^{s+t'}}{|1 - \bar{w}z|^{2+s+t}|1 - \bar{w}\zeta|^{2+s'+t'}} dA(w) dA(\zeta). \end{aligned}$$

Since $t > t'$ and $s > s'$, by Lemma 2.2, we get

$$\int_{\mathbb{D}} \frac{(1 - |w|^2)^{s+t'}}{|1 - \bar{w}z|^{2+s+t}|1 - \bar{w}\zeta|^{2+s'+t'}} dA(w) \lesssim \frac{1}{(1 - |z|^2)^{t-t'} |1 - \bar{z}\zeta|^{2+s'+t'}}.$$

We obtain

$$(\omega_{s',t'})_{s,t}(z) \lesssim \int_{\mathbb{D}} \frac{\omega(\zeta)(1 - |\zeta|^2)^{s'}(1 - |z|^2)^t}{(1 - |z|^2)^{t-t'} |1 - \bar{z}\zeta|^{2+s'+t'}} dA(\zeta) = \omega_{s',t'}(z). \quad \square$$

For r in $(0, 1)$, let α_r be any real number in $(0, r/(1+r))$ such that

$$\chi_{\Delta(w, \alpha_r)}(z) \leq \chi_{\Delta(z, r)}(w), \quad z, w \in \mathbb{D}.$$

This inequality will be frequently used from here on.

Lemma 2.3. *Let ω be a weight. Let $s > -1$, $t \geq 0$, and $p > 0$. Then $\|f\|_{A_\omega^p} \lesssim \|f\|_{A_{\omega_{s,t}}^p}$ for all $f \in A_{\omega_{s,t}}^p$.*

Proof. Let r in $(0, 1)$. We have

$$\begin{aligned}
\|f\|_{A_{\omega,s,t}^p}^p &= \int_{\mathbb{D}} |f(z)|^p \int_{\mathbb{D}} \frac{\omega(w)(1-|w|^2)^s(1-|z|^2)^t}{|1-\bar{w}z|^{2+s+t}} dA(w)dA(z) \\
&\geq \int_{\mathbb{D}} |f(z)|^p \int_{\Delta(z,r)} \frac{\omega(w)}{(1-|w|^2)^2} dA(w)dA(z) \\
&\geq \int_{\mathbb{D}} \frac{\omega(w)}{(1-|w|^2)^2} \int_{\Delta(w,\alpha_r)} |f(z)|^p dA(z)dA(w) \\
&\gtrsim \|f\|_{A_{\omega}^p}^p.
\end{aligned}$$

The last inequality follows from the subharmonicity of $|f|^p$. \square

Proposition 2.1. *Let ω be a weight such that $\omega \in \mathcal{W}(s, t)$ for some $s \in (-1, 0]$ and $t \geq 0$. Then $\omega \in \mathcal{HL}$.*

Proof. If $\omega \in \mathcal{W}(s, t)$ for some $s \in (-1, 0]$ and some $t \geq 0$ then, by the first assertion of Lemma 2.1, $\omega \in \mathcal{W}(0, t)$. Hence $\frac{\omega}{(1-|z|^2)^t} \in B_{p_0}(t)$ for all $p_0 > 1$ (see the proof of (c) \Rightarrow (b) in [3, Corollary 4.4]). Therefore, by Theorem 3.1 in [2], $\omega \in \mathcal{HL}$. \square

Note that the sufficient condition, $\omega \in \mathcal{W}(s, t)$ for some $s \in (-1, 0]$ and some $t \geq 0$, to have $\omega \in \mathcal{HL}$ is optimal in the sense that there is a weight ω such that for all $s > 0$ and for some $t > 0$ we have $\omega_{s,t} \lesssim \omega$ and $\omega \notin \mathcal{HL}$. Indeed, for $\sigma \geq 2$, Bao, Wulan, and Zhu [10] proved that there exists a positive measure μ such that

$$\omega_{\sigma}(z) := \int_{\mathbb{D}} \left(\frac{(1-|z|^2)(1-|w|^2)}{|1-\bar{z}w|^2} \right)^{\sigma} d\mu(w)$$

is a weight and $\omega_{\sigma} \notin \mathcal{HL}$. Now, for $\sigma = 2$, let $s > 0$ and $t > 2$. By Fubini's theorem and Lemma 2.2, we have

$$\begin{aligned}
(\omega_2)_{s,t}(\zeta) &= (1-|\zeta|^2)^t \int_{\mathbb{D}} (1-|w|^2)^2 \int_{\mathbb{D}} \frac{(1-|z|^2)^{s+2}}{|1-\bar{\zeta}z|^{2+s+t}|1-\bar{z}w|^4} dA(z)d\mu(w) \\
&\lesssim \omega_2(\zeta).
\end{aligned}$$

Throughout, for a weight ω on \mathbb{D} and for $r \in (0, 1)$, let

$$\tilde{\omega}_r(z) := \frac{1}{(1-|z|^2)^2} \int_{\Delta(z,r)} \omega(\zeta) dA(\zeta), \quad z \in \mathbb{D}.$$

Recall that $\tilde{\omega} = \tilde{\omega}_{1/2}$. Note that if $\omega \in \mathcal{C}_p$ for some $p > 1$ then, for all β and r in $(0, 1)$, we have

$$\tilde{\omega}_\beta \asymp \tilde{\omega} \text{ and } \tilde{\omega}_\beta(\zeta) \asymp \tilde{\omega}(z) \text{ if } \zeta \in \Delta(z, r), \quad (2.2)$$

see Lemma 2.2 in [16]. In order to prove Theorem 2.1, we first establish the following lemma.

Lemma 2.4. *Let $p > 0$, $r \in (0, 1)$ and $\omega \in \mathcal{C}_{p_0}$ for some $p_0 > 1$. Then $\tilde{\omega}_r$ is a weight and $A_\omega^p = A_{\tilde{\omega}_r}^p$. Moreover, we have*

$$\|f\|_{A_\omega^p} \asymp \|f\|_{A_{\tilde{\omega}_r}^p}, \quad f \in \text{Hol}(\mathbb{D}).$$

Proof. Let r be in $(0, 1)$. By (2.2) and Fubini's theorem, we have

$$\begin{aligned} \int_{\mathbb{D}} \tilde{\omega}_r(z) dA(z) &\asymp \int_{\mathbb{D}} \tilde{\omega}_{\alpha_r}(z) dA(z) \\ &= \int_{\mathbb{D}} \frac{1}{(1-|z|^2)} \int_{\mathbb{D}} \chi_{\Delta(z, \alpha_r)}(\zeta) \omega(\zeta) dA(\zeta) dA(z) \\ &\leq \int_{\mathbb{D}} \omega(\zeta) \int_{\mathbb{D}} \frac{1}{(1-|z|^2)} \chi_{\Delta(\zeta, r)}(z) dA(z) dA(\zeta) \\ &\asymp \int_{\mathbb{D}} \omega(\zeta) dA(\zeta). \end{aligned}$$

It follows that $\tilde{\omega}_r$ is a weight. If $f \in A_{\tilde{\omega}_r}^p$ then, as in the proof of Lemma 2.3, we have

$$\|f\|_{A_{\tilde{\omega}_r}^p}^p \gtrsim \|f\|_{A_\omega^p}^p.$$

On the other hand, for $f \in A_\omega^p$, by the subharmonicity of $|f|^p$ and Holder's inequality, we have [2]

$$|f(z)|^p \lesssim \frac{1}{\int_{\Delta(z, r)} \omega(w) dA(w)} \int_{\Delta(z, \alpha_r)} |f(\zeta)|^p \omega(\zeta) dA(\zeta), \quad z \in \mathbb{D}. \quad (2.3)$$

Equivalently, we have

$$|f(z)|^p \tilde{\omega}_r(z) \lesssim \frac{1}{(1-|z|^2)^2} \int_{\Delta(z, \alpha_r)} |f(\zeta)|^p \omega(\zeta) dA(\zeta), \quad z \in \mathbb{D}.$$

Therefore

$$\begin{aligned} \|f\|_{A_{\tilde{\omega}_r}^p}^p &\lesssim \int_{\mathbb{D}} \frac{1}{(1-|z|^2)^2} \int_{\mathbb{D}} \chi_{\Delta(z, \alpha_r)}(\zeta) |f(\zeta)|^p \omega(\zeta) dA(\zeta) dA(z) \\ &\leq \int_{\mathbb{D}} \frac{1}{(1-|z|^2)^2} \int_{\mathbb{D}} \chi_{\Delta(\zeta, r)}(z) |f(\zeta)|^p \omega(\zeta) dA(\zeta) dA(z) \\ &= \int_{\mathbb{D}} |f(\zeta)|^p \omega(\zeta) \int_{\Delta(\zeta, r)} \frac{1}{(1-|z|^2)^2} dA(z) dA(\zeta) \\ &\asymp \|f\|_{A_\omega^p}^p. \end{aligned}$$

The desired result is obtained. □

Proof of Theorem 2.1. Since $\omega \in \mathcal{C}_{p_0}$, the weight $\omega_{[np]} := (1 - |z|^2)^{np}\omega$ belongs to \mathcal{C}_{p_0} . Moreover

$$\widetilde{\omega_{[np]}} \asymp (1 - |z|^2)^{np}\tilde{\omega}.$$

By applying Lemma 2.4 to ω and $\omega_{[np]}$, respectively, we deduce that $\omega \in \mathcal{HL}$ if and only if $\tilde{\omega} \in \mathcal{HL}$. Since $\tilde{\omega}$ satisfies (2.2), by Corollary 4.4 in [3], we have $\tilde{\omega} \in \mathcal{HL}$ if and only if $(\tilde{\omega})_{0,t} \lesssim \tilde{\omega}$ for some $t > 0$. Now, we have

$$\begin{aligned} \omega_{0,t}(z) &= \int_{\mathbb{D}} \frac{\omega(w)(1 - |z|^2)^t}{|1 - \bar{w}z|^{2+t}} dA(w) \\ &\asymp \int_{\mathbb{D}} \omega(w)(1 - |z|^2)^t \int_{\mathbb{D}} \frac{\chi_{\Delta(w, \alpha_r)}(\zeta)}{(1 - |\zeta|^2)^2 |1 - \bar{\zeta}z|^{2+t}} dA(\zeta) dA(w) \\ &\lesssim \int_{\mathbb{D}} \frac{\tilde{\omega}_r(\zeta)(1 - |z|^2)^t}{|1 - \bar{\zeta}z|^{2+t}} dA(\zeta) \\ &\asymp \int_{\mathbb{D}} \frac{\tilde{\omega}(\zeta)(1 - |z|^2)^t}{|1 - \bar{\zeta}z|^{2+t}} dA(\zeta) \\ &= (\tilde{\omega})_{0,t}(z). \end{aligned}$$

In a similar manner we also have $\omega_{0,t} \gtrsim (\tilde{\omega})_{0,t}$, and hence $(\tilde{\omega})_{0,t} \asymp \omega_{0,t}$, which completes the proof. \square

Corollary 2.1. *Suppose that $\omega \in \mathcal{C}_{p_0}$ for some $p_0 > 1$. If $\omega \in \mathcal{HL}$, then*

$$\int_{\mathbb{D}} \omega(z) \log \frac{1}{1 - |z|^2} dA(z) < \infty.$$

Proof. By Theorem 2.1, if $\omega \in \mathcal{HL}$, then there exists $t > 0$ such that $\omega_{0,t}$ is a weight on \mathbb{D} . Using Fubini's theorem, we have

$$\begin{aligned} \infty > \int_{\mathbb{D}} \omega_{0,t}(\zeta) dA(\zeta) &= \int_{\mathbb{D}} \int_{\mathbb{D}} \frac{(1 - |\zeta|^2)^t}{|1 - \bar{\zeta}z|^{2+t}} dA(\zeta) \omega(z) dA(z) \\ &\asymp \int_{\mathbb{D}} \omega(z) \log \frac{1}{1 - |z|^2} dA(z). \end{aligned} \quad \square$$

As an example to illustrate the previous corollary, let ω be the function given by

$$\omega(z) = \sum_{n=0}^{\infty} \frac{1}{|1 - \bar{z}a_n|^4 (n+1)^3 \log^2(n+1)}, \quad a_n = n/(n+1).$$

ω is a weight satisfying (2.1) and then $\omega \in \mathcal{C}_p$ for all $p > 1$. Since

$$\begin{aligned}
& \int_{\mathbb{D}} \omega(z) \log \frac{1}{1-|z|^2} dA(z) \\
&= \sum_{n=0}^{\infty} \frac{1}{(n+1)^3 \log^2(n+1)} \int_{\mathbb{D}} \frac{1}{|1-\bar{z}a_n|^4} \log \frac{1}{1-|z|^2} dA(z) \\
&\geq \sum_{n=0}^{\infty} \frac{1}{(n+1)^3 \log^2(n+1)} \int_{\Delta(a_n, 1/2)} \frac{1}{|1-\bar{z}a_n|^4} \log \frac{1}{1-|z|^2} dA(z) \\
&\asymp \sum_{n=0}^{\infty} \frac{1}{(n+1) \log^2(n+1)} \log \frac{1}{1-|a_n|^2} \\
&\gtrsim \sum_{n=0}^{\infty} \frac{1}{(n+1) \log(n+1)} \\
&= \infty,
\end{aligned}$$

it follows that $\omega \notin \mathcal{HL}$.

Now we are going to prove Theorem 2.2. For a radial weight ω on \mathbb{D} , let $\hat{\omega}(r) := \int_r^1 \omega(s) ds$, $r \in [0, 1)$. We say that ω belongs to \hat{D} if $\hat{\omega}(r) \lesssim \hat{\omega}\left(\frac{1+r}{2}\right)$ for all $r \in [0, 1)$, and ω belongs to \check{D} if there exist $K = K(\omega) > 1$ and $C = C(\omega) > 1$ such that $\hat{\omega}(r) \geq C\hat{\omega}\left(1 - \frac{1-r}{K}\right)$ for all $r \in [0, 1)$. Peláez and Rättyä [44] proved the following result.

Theorem 2.5. *Let ω be a radial weight, $0 < p < \infty$, and $n \in \mathbb{N}^*$. Then*

$$\|f\|_{A_{\omega}^p}^p \asymp \sum_{k=0}^{n-1} |f^{(k)}(0)| + \int_{\mathbb{D}} |f^{(n)}(z)|^p (1-|z|^2)^{np} \omega(z) dA(z), \quad f \in \text{Hol}(\mathbb{D}),$$

if and only if $\omega \in \mathcal{D} := \hat{D} \cap \check{D}$.

Note that $\omega \in \hat{D}$ if and only if [43] there is a constant $\alpha > 0$ such that

$$\hat{\omega}(r) \lesssim \left(\frac{1-r}{1-t}\right)^{\alpha} \hat{\omega}(t) \tag{2.4}$$

for all $0 \leq r \leq t < 1$. By similar arguments given in [43], $\omega \in \check{D}$ if and only if there is a constant $\beta > 0$ such that

$$\hat{\omega}(t) \lesssim \left(\frac{1-t}{1-r}\right)^{\beta} \hat{\omega}(r) \tag{2.5}$$

for all $0 \leq r \leq t < 1$ (see also [44]).

Recall that

$$\tilde{\omega}(r) = \frac{1}{1-r} \hat{\omega}(r), \quad r \in [0, 1).$$

Proof of Theorem 2.2. Suppose that $\omega \in \mathcal{HL}$. By (2.5) we have

$$\int_0^1 \tilde{\omega}(t) dt \lesssim \hat{\omega}(0) \int_0^1 (1-t)^{\beta-1} dt \lesssim \hat{\omega}(0) = \int_0^1 \omega(s) ds.$$

It follows that $\tilde{\omega}$ is a weight. Now fix ρ in $[0, 1)$. By Theorem 2.5, the weight ω is in \mathcal{D} . Note that the constant β in (2.5) can be assumed in $(0, 1)$. Let $s \in (-\beta, 0)$ and $t > \alpha$, where α is the constant in (2.4). We have

$$\begin{aligned} \omega_{s,t}(\rho) &\asymp \int_0^{2\pi} \int_0^1 \frac{\omega(r)(1-r)^s(1-\rho)^t}{|1-r\rho e^{i\theta}|^{2+s+t}} r dr d\theta \\ &\asymp \int_0^1 \frac{\omega(r)(1-r)^s(1-\rho)^t}{(1-r\rho)^{1+s+t}} dr \\ &= \int_0^\rho \frac{\omega(r)(1-r)^s(1-\rho)^t}{(1-r\rho)^{1+s+t}} dr + \int_\rho^1 \frac{\omega(r)(1-r)^s(1-\rho)^t}{(1-r\rho)^{1+s+t}} dr \\ &\leq \int_0^\rho \frac{\omega(r)(1-\rho)^t}{(1-r)^{1+t}} dr + \int_\rho^1 \frac{\omega(r)(1-r)^s}{(1-\rho)^{1+s}} dr. \end{aligned}$$

Let $u > 1$ and $N \in \mathbb{N}^*$ such that $u^N(1-\rho) = 1$. Let

$$\rho_j := 1 - u^j(1-\rho), \quad j = 0, 1, \dots, N.$$

We have

$$\begin{aligned} \int_0^\rho \frac{\omega(r)(1-\rho)^t}{(1-r)^{1+t}} dr &= \sum_{j=0}^{N-1} \int_{\rho_{j+1}}^{\rho_j} \frac{\omega(r)(1-\rho)^t}{(1-r)^{1+t}} dr \\ &\leq (1-\rho)^t \sum_{j=0}^{N-1} \frac{1}{(1-\rho_j)^{1+t}} \hat{\omega}(\rho_{j+1}). \end{aligned}$$

By (2.4), we obtain

$$\begin{aligned} \int_0^\rho \frac{\omega(r)(1-\rho)^t}{(1-r)^{1+t}} dr &\lesssim \frac{1}{1-\rho} \sum_{j=0}^{N-1} \frac{1}{u^{j(1+t)}} \left(\frac{1-\rho_{j+1}}{1-\rho} \right)^\alpha \hat{\omega}(\rho) \\ &\lesssim \tilde{\omega}(\rho) \sum_{j=0}^{\infty} \frac{1}{u^{j(t-\alpha)}}. \end{aligned}$$

Hence

$$\int_0^\rho \frac{\omega(r)(1-\rho)^t}{(1-r)^{1+t}} dr \lesssim \check{\omega}(\rho). \quad (2.6)$$

On the other hand, for $k > 1$, let

$$\tilde{\rho}_j := 1 - (1-\rho)/k^j, \quad j \in \mathbb{N}.$$

We have

$$\begin{aligned} \int_\rho^1 \frac{\omega(r)(1-r)^s}{(1-\rho)^{1+s}} dr &= \sum_{j=0}^{\infty} \int_{\tilde{\rho}_j}^{\tilde{\rho}_{j+1}} \frac{\omega(r)(1-r)^s}{(1-\rho)^{1+s}} dr \\ &\lesssim \sum_{j=0}^{\infty} \frac{(1-\tilde{\rho}_j)^s}{(1-\rho)^{1+s}} \int_{\tilde{\rho}_j}^1 \omega(r) dr. \end{aligned}$$

By (2.5), we obtain

$$\int_\rho^1 \frac{\omega(r)(1-r)^s}{(1-\rho)^{1+s}} dr \lesssim \check{\omega}(\rho) \sum_{j=0}^{\infty} \frac{1}{k^{j(\beta+s)}} \leq \check{\omega}(\rho).$$

Therefore $\omega_{s,t}(\rho) \lesssim \check{\omega}(\rho)$. Hence (1) \Rightarrow (2).

Assume next that $\check{\omega}$ is a weight and $\omega_{s,t} \lesssim \check{\omega}$ for some $s \in (-1, 0)$ and $t \geq 0$. It follows that $\omega_{s,t} \asymp \check{\omega}$ since we always have $\omega_{s,t} \gtrsim \check{\omega}$. Indeed

$$\begin{aligned} \omega_{s,t}(\rho) &\asymp \int_0^1 \frac{\omega(r)(1-r)^s(1-\rho)^t}{(1-r\rho)^{1+s+t}} dr \\ &\geq \int_0^1 \frac{\omega(r)(1-\rho)^t}{(1-r\rho)^{1+t}} dr \\ &\geq \int_\rho^1 \frac{\omega(r)(1-\rho)^t}{(1-r\rho)^{1+t}} dr \\ &\gtrsim \check{\omega}(\rho). \end{aligned}$$

Now, $\omega_{s,t}$ is a weight because $\check{\omega}$ is a weight, and for any $t' > t$ we have

$$(\check{\omega})_{0,t'} \asymp (\omega_{s,t})_{0,t'} \lesssim \omega_{s,t} \asymp \check{\omega},$$

by the second assertion of Lemma 2.1. We deduce that $\check{\omega} \in \mathcal{W}(0, t')$ for all $t' > t$. Thus (3) is satisfied.

Finally, assume that (3) holds, that is, $\check{\omega} \in \mathcal{W}(0, t)$ for some $t > 0$. By Proposition 2.1, $\check{\omega}$ belongs to \mathcal{HL} . Therefore, by [40, Theorem 9], ω belongs to \mathcal{HL} . \square

We end this section by the following

Corollary 2.2. *Let ω be a radial weight such that $\check{\omega}(r) \lesssim \omega(r)$ for all $r \in [0, 1)$. The following assertions are equivalent.*

- (1) $\omega \in \mathcal{HL}$ (or equivalently $\omega \in \mathcal{D}$).
- (2) $\omega \in \hat{D}$.
- (3) $\omega \in \mathcal{W}(s, t)$ for some $s \in (-1, 0)$ and $t \geq 0$.
- (4) $\omega/(1 - |z|^2)^\eta \in B_p(\eta)$ for some $\eta \geq 0$ and for all $p > 1$.
- (5) $\omega/(1 - |z|^2)^\eta \in B_p(\eta)$ for some $\eta \geq 0$ and some $p > 1$.

Proof. The implications (1) \Rightarrow (2) and (4) \Rightarrow (5) are trivial. The implication (3) \Rightarrow (4) is contained in the proof of Proposition 2.1. If (5) holds, then $\omega \in \mathcal{HL}$ by Theorem 3.1 in [2]. If $\omega \in \mathcal{HL}$ then, by Theorem 2.2, $\omega \in \mathcal{W}(s, t)$ for some $s \in (-1, 0)$ and $t \geq 0$, and hence (1) \Rightarrow (3). Suppose now that $\omega \in \hat{D}$. Let α be the constant in (2.4) and let $t > \alpha$. We have

$$\int_\rho^1 \frac{\omega(r)(1-\rho)^t}{(1-r\rho)^{1+t}} dr \lesssim \frac{1}{1-\rho} \int_\rho^1 \omega(r) dr = \check{\omega}(\rho), \quad \rho \in (0, 1).$$

This together with (2.6), we get

$$\int_0^1 \frac{\omega(r)(1-\rho)^t}{(1-r\rho)^{1+t}} dr \lesssim \check{\omega}(\rho), \quad \rho \in (0, 1).$$

Equivalently, we have

$$\omega_{0,t}(z) \lesssim \check{\omega}(z), \quad z \in \mathbb{D}.$$

Therefore

$$\|f\|_{A_{\omega_{0,t}}^p} \lesssim \|f\|_{A_\omega^p} \lesssim \|f\|_{A_\omega^p}, \quad f \in A_\omega^p.$$

On the other hand, by Lemma 2.3, we have

$$\|f\|_{A_{\omega_{0,t}}^p} \gtrsim \|f\|_{A_\omega^p}, \quad f \in A_{\omega_{0,t}}^p.$$

We deduce that $A_\omega^p = A_{\omega_{0,t}}^p$ and therefore, by Theorem 9 in [40], $\omega \in \mathcal{D}$. The proof is achieved. \square

2.3 Density of polynomials and properties of reproducing kernels

2.3.1 Density of polynomials in A_ω^p .

It is well-known that if ω is a radial weight on \mathbb{D} then polynomials are dense in A_ω^p , see [29, p. 343] for example. For the non-radial case, it is also known that the fact that polynomials are dense in A_ω^p is not always true, see [18, p. 138]. The density of polynomials in Bergman spaces associated with Bekollé-Bonami weights can be deduced from atomic decomposition for the functions in such spaces in terms of kernel functions of the standard weighted Bergman spaces [2, 15, 31]. Combining this fact, Theorem 2.1, and Lemma 2.4, we obtain the density of the polynomials in A_ω^p for all $p > 0$ if $\omega \in \mathcal{C}_{p_0, t}$ for some $p_0 > 1$ and $t \geq 0$. Indeed, in this case we have $\tilde{\omega} \in \mathcal{HL}$ and $\tilde{\omega}$ satisfies (2.1). Therefore, $\frac{\tilde{\omega}}{(1-|z|^2)^\eta}$ belongs to $B_q(\eta)$ for some $q > 1$ and $\eta > -1$. We deduce that the polynomials are dense in $A_{\tilde{\omega}}^p$. Hence, by Lemma 2.4, the polynomials are dense in A_ω^p . In fact, we prove in the following theorem that the weights considered in this paper have the important property that the family of dilates $f_r(z) := f(rz)$, $r \in [0, 1)$, of each function f in A_ω^p is uniformly bounded, and polynomials are dense in A_ω^p .

Theorem 2.6. *Let $p > 0$ and suppose that $\omega \in \mathcal{C}_{p_0, t}$ for some $p_0 > 1$ and $t \geq 0$. Let $f \in A_\omega^p$. Then*

$$\|f_r\|_{A_\omega^p} \leq C(\omega) \|f\|_{A_\omega^p}, \quad r \in [0, 1),$$

and the polynomials are dense in A_ω^p .

Proof. Since $\omega_{0, t} \lesssim \tilde{\omega}$ for some $t \geq 0$ and since we always have $\omega_{0, t} \gtrsim \tilde{\omega}$, we get $\omega_{0, t} \asymp \tilde{\omega}$. Therefore, by Lemma 2.4, we have $\|f\|_{A_\omega^p} \asymp \|f\|_{A_{\omega_{0, t}}^p}$, for all $f \in \text{Hol}(\mathbb{D})$. On the other hand,

for any r in $(0, 1)$, we have

$$\begin{aligned}
(\omega_{0,t})_{0,t}(z) &\asymp (\tilde{\omega}_{\alpha_r})_{0,t}(z) \\
&= \int_{\mathbb{D}} \frac{(1 - |z|^2)^t}{|1 - \bar{\zeta}z|^{2+t}(1 - |\zeta|^2)^2} \int_{\mathbb{D}} \chi_{\Delta(\zeta, \alpha_r)}(w) \omega(w) dA(w) dA(\zeta) \\
&\leq \int_{\mathbb{D}} \frac{(1 - |z|^2)^t}{|1 - \bar{\zeta}z|^{2+t}(1 - |\zeta|^2)^2} \int_{\mathbb{D}} \chi_{\Delta(w, r)}(\zeta) \omega(w) dA(w) dA(\zeta) \\
&= \int_{\mathbb{D}} \omega(w) (1 - |z|^2)^t \int_{\Delta(w, r)} \frac{1}{|1 - \bar{\zeta}z|^{2+t}(1 - |\zeta|^2)^2} dA(\zeta) dA(w) \\
&\asymp \int_{\mathbb{D}} \frac{\omega(w) (1 - |z|^2)^t}{|1 - \bar{w}z|^{2+t}} dA(w) \\
&= \omega_{0,t}(z).
\end{aligned}$$

We deduce that $\omega_{0,t}$ belongs to $\mathcal{W}(0, t)$. Now, fix f in $A_{\omega_{0,t}}^p$. For all $\zeta \in \mathbb{D}$ we have

$$\begin{aligned}
|f(\zeta)|^p &\lesssim \int_{\Delta(\zeta, 1/2)} \frac{|f(z)|^p}{|1 - \bar{z}\zeta|^2} dA(z) \\
&\asymp \int_{\Delta(\zeta, 1/2)} |f(z)|^p \frac{(1 - |z|^2)^t}{|1 - \bar{z}\zeta|^{2+t}} dA(z).
\end{aligned}$$

Therefore

$$|f(\zeta)|^p \lesssim \int_{\mathbb{D}} |f(z)|^p \frac{(1 - |z|^2)^t}{|1 - \bar{z}\zeta|^{2+t}} dA(z), \quad \zeta \in \mathbb{D}. \quad (2.7)$$

Let $r_0 \in (0, 1)$ close enough to 1 so that $|1 - \bar{z}\zeta| \lesssim |1 - r\bar{z}\zeta|$ for all $r \geq r_0$ and for all $z, \zeta \in \mathbb{D}$. It is not hard to see that $\|f_r\|_{A_{\omega_r}^p} \lesssim \|f\|_{A_{\omega_r}^p}$ for all $f \in A_{\omega_r}^p$ and for all $r < r_0$. For $r \geq r_0$, the Inequality (2.7) gives

$$\begin{aligned}
|f(r\zeta)|^p &\lesssim \int_{\mathbb{D}} |f(z)|^p \frac{(1 - |z|^2)^t}{|1 - r\bar{z}\zeta|^{2+t}} dA(z) \\
&\lesssim \int_{\mathbb{D}} |f(z)|^p \frac{(1 - |z|^2)^t}{|1 - \bar{z}\zeta|^{2+t}} dA(z), \quad \zeta \in \mathbb{D}.
\end{aligned}$$

Let h be the function given by

$$h(\zeta) =: \int_{\mathbb{D}} |f(z)|^p \frac{(1 - |z|^2)^t}{|1 - \bar{z}\zeta|^{2+t}} dA(z), \quad \zeta \in \mathbb{D}.$$

By Fubini's theorem, we have

$$\begin{aligned}
& \int_{\mathbb{D}} h(\zeta) \omega_{0,t}(\zeta) dA(\zeta) \\
&= \int_{\mathbb{D}} \int_{\mathbb{D}} |f(z)|^p \frac{(1-|z|^2)^t}{|1-\bar{z}\zeta|^{2+t}} dA(z) \int_{\mathbb{D}} \frac{\omega(w)(1-|\zeta|^2)^t}{|1-\bar{\zeta}w|^{2+t}} dA(w) dA(\zeta) \\
&= \int_{\mathbb{D}} |f(z)|^p \left(\int_{\mathbb{D}} \int_{\mathbb{D}} \frac{\omega(w)(1-|\zeta|^2)^t}{|1-\bar{\zeta}w|^{2+t}} \frac{(1-|z|^2)^t}{|1-\bar{z}\zeta|^{2+t}} dA(w) dA(\zeta) \right) dA(z) \\
&= \int_{\mathbb{D}} |f(z)|^p (\omega_{0,t})_{0,t}(z) dA(z) \\
&\lesssim \int_{\mathbb{D}} |f(z)|^p \omega_{0,t}(z) dA(z) < \infty.
\end{aligned}$$

We obtain

$$\|f_r\|_{A_{\omega_{0,t}}^p} \lesssim \|f\|_{A_{\omega_{0,t}}^p}, \quad f \in A_{\omega_{0,t}}^p, \quad r \geq r_0.$$

Equivalently, we have $\|f_r\|_{A_{\omega}^p} \lesssim \|f\|_{A_{\omega}^p}$ for all $f \in A_{\omega}^p$ and for all $r \geq r_0$. Also, by Lebesgue's convergence theorem we have $\lim_{r \rightarrow 1^-} \|f_r - f\|_{A_{\omega}^p} = 0$, and hence polynomials are dense in A_{ω}^p . The proof is complete. \square

2.3.2 Reproducing kernel of A_{ω}^2

Notice that, by (2.3), if $\omega \in \mathcal{C}_p$ for some $p > 1$ then for all $f \in A_{\omega}^2$ we have

$$|f(z)|^2 \lesssim \frac{1}{\int_{\Delta(z,1/2)} \omega(w) dA(w)} \|f\|_{A_{\omega}^2}^2, \quad z \in \mathbb{D}.$$

Therefore, A_{ω}^2 is a reproducing kernel Hilbert space with kernel K^{ω} and we have

$$\|K_z^{\omega}\|_{A_{\omega}^2}^2 \lesssim \frac{1}{\int_{\Delta(z,1/2)} \omega(w) dA(w)}, \quad \forall z \in \mathbb{D}.$$

Also, by Lemma 2.4, A_{ω}^2 has a reproducing kernel $K^{\tilde{\omega}}$ with

$$\|K_z^{\tilde{\omega}}\|_{A_{\omega}^2}^2 \lesssim \frac{1}{\int_{\Delta(z,1/2)} \omega(w) dA(w)} = \frac{1}{(1-|z|^2)^2 \tilde{\omega}(z)}, \quad \forall z \in \mathbb{D}. \quad (2.8)$$

We have the following proposition.

Proposition 2.2. *Suppose that $\omega \in \mathcal{C}_{p_0, t}$ for some $p_0 > 1$ and $t \geq 0$. The reproducing kernel $K^{\tilde{\omega}}$ of $A_{\tilde{\omega}}^2$ satisfies*

$$\lim_{z \rightarrow \partial\mathbb{D}} \|K_z^{\tilde{\omega}}\|_{A_{\tilde{\omega}}^2} = \infty. \quad (2.9)$$

Moreover, for every $\zeta \in \mathbb{D}$, we have

$$|K_z^{\tilde{\omega}}(\zeta)| = o\left(\|K_z^{\tilde{\omega}}\|_{A_{\tilde{\omega}}^2}\right), \quad z \rightarrow \partial\mathbb{D}. \quad (2.10)$$

Proof. Since, by Theorem 2.1, $\tilde{\omega}$ belongs to \mathcal{HL} and since

$$(1 - |z|^2)^2 \tilde{\omega}(z) \lesssim \int_{\mathbb{D}} \omega(\zeta) dA(\zeta) \lesssim 1, \quad z \in \mathbb{D},$$

we have

$$\|f\|_{A_{\tilde{\omega}}^2}^2 \lesssim |f(0)|^2 + \int_{\mathbb{D}} |f'(z)|^2 dA(z).$$

It follows that the classical Dirichlet space

$$\mathcal{D}_0 := \{f \in \text{Hol}(\mathbb{D}) : \|f\|_{\mathcal{D}_0}^2 := |f(0)|^2 + \int_{\mathbb{D}} |f'(z)|^2 dA(z) < \infty\}$$

is contained in $A_{\tilde{\omega}}^2$ with $\|f\|_{A_{\tilde{\omega}}^2} \lesssim \|f\|_{\mathcal{D}_0}$, for all $f \in \mathcal{D}_0$. Let R be the reproducing kernel of \mathcal{D}_0 .

It is well-known that

$$R_z(w) = 1 + \log \frac{1}{1 - \bar{z}w}, \quad z, w \in \mathbb{D}.$$

By the properties of reproducing kernels, for all z in \mathbb{D} we have

$$\begin{aligned} \|R_z\|_{\mathcal{D}_0}^2 &= |R_z(z)| = |\langle R_z, K_z^{\tilde{\omega}} \rangle_{A_{\tilde{\omega}}^2}| \leq \|R_z\|_{A_{\tilde{\omega}}^2} \|K_z^{\tilde{\omega}}\|_{A_{\tilde{\omega}}^2} \\ &\lesssim \|R_z\|_{\mathcal{D}_0} \|K_z^{\tilde{\omega}}\|_{A_{\tilde{\omega}}^2}. \end{aligned}$$

Therefore $\lim_{z \rightarrow \partial\mathbb{D}} \|K_z^{\tilde{\omega}}\|_{A_{\tilde{\omega}}^2} = \infty$.

To prove (2.10), it suffices to show that the normalized reproducing kernel $k_z^{\tilde{\omega}} := \frac{K_z^{\tilde{\omega}}}{\|K_z^{\tilde{\omega}}\|_{A_{\tilde{\omega}}^2}}$ of $A_{\tilde{\omega}}^2$ converges to zero weakly as $z \rightarrow \partial\mathbb{D}$. Let p be an analytic polynomial. We have

$$|\langle k_z^{\tilde{\omega}}, p \rangle_{A_{\tilde{\omega}}^2}| = \frac{|p(z)|}{\|K_z^{\tilde{\omega}}\|_{A_{\tilde{\omega}}^2}} \leq \frac{\|p\|_{\infty}}{\|K_z^{\tilde{\omega}}\|_{A_{\tilde{\omega}}^2}}.$$

It follows that $\langle k_z^{\tilde{\omega}}, p \rangle_{A_{\tilde{\omega}}^2} \rightarrow 0$, as $z \rightarrow \partial\mathbb{D}$. By Theorem 2.6 and Lemma 2.4, polynomials are dense in $A_{\tilde{\omega}}^2$. Hence, we obtain the desired result. \square

The next proposition gives estimates of the reproducing kernel of $A_{\tilde{\omega}}^2$ near diagonal.

Proposition 2.3. *Suppose that $\omega \in \mathcal{C}_{p_0, t}$ for some $p_0 > 1$ and $t \geq 0$. The reproducing kernel $K^{\tilde{\omega}}$ of $A_{\tilde{\omega}}^2$ satisfies the following estimates.*

(1) For all $z \in \mathbb{D}$, we have $\|K_z^{\tilde{\omega}}\|_{A_{\tilde{\omega}}^2}^2 \asymp \frac{1}{(1-|z|^2)^2 \tilde{\omega}(z)}$.

(2) There exists $\beta \in (0, 1)$ such that for all $z \in \mathbb{D}$ we have

$$|K_z^{\tilde{\omega}}(w)| \asymp \|K_z^{\tilde{\omega}}\|_{A_{\tilde{\omega}}^2} \|K_w^{\tilde{\omega}}\|_{A_{\tilde{\omega}}^2}, \quad w \in \Delta(z, \beta).$$

Proof. For $a \in \mathbb{D}$, let g_a be the function defined by

$$g_a(z) := 1/(1 - \bar{a}z)^{t/2+1}, \quad z \in \mathbb{D}.$$

By the first assertion of Lemma 2.1, we have

$$\|g_a\|_{A_{\tilde{\omega}}^2}^2 = \int_{\mathbb{D}} \frac{\omega(z)}{|1 - \bar{a}z|^{t+2}} dA(z) = \frac{\omega_{0,t}(a)}{(1 - |a|^2)^t} \lesssim \frac{\tilde{\omega}(a)}{(1 - |a|^2)^t}.$$

Hence, by Lemma 2.4, g_a belongs to $A_{\tilde{\omega}}^2$ with

$$\|g_a\|_{A_{\tilde{\omega}}^2}^2 \lesssim \frac{\tilde{\omega}(a)}{(1 - |a|^2)^t}, \quad \forall a \in \mathbb{D}.$$

Now, By duality we have

$$\|K_z^{\tilde{\omega}}\|_{A_{\tilde{\omega}}^2}^2 = \sup\{|f(z)|^2, \|f\|_{A_{\tilde{\omega}}^2} = 1\} \geq \frac{|g_z(z)|^2}{\|g_z\|_{A_{\tilde{\omega}}^2}^2} \gtrsim \frac{1}{(1 - |z|^2)^2 \tilde{\omega}(z)}.$$

Taking into account (2.8), the first assertion is obtained.

Since $\tilde{\omega}$ satisfies (2.1), it follows from the first assertion that

$$\|K_w^{\tilde{\omega}}\|_{A_{\tilde{\omega}}^2} \asymp \|K_z^{\tilde{\omega}}\|_{A_{\tilde{\omega}}^2}, \quad w \in \Delta(z, 1/2).$$

The proof of the second assertion of the theorem is then similar to the proof of Lemma 3.3 in [22]. \square

Note that (2.9) can be recovered from the first assertion of the previous proposition since we have

$$(1 - |z|^2)^2 \tilde{\omega}(z) = \int_{\mathbb{D}} \chi_{\Delta(z, 1/2)}(\zeta) \omega(\zeta) dA(\zeta) \rightarrow 0, \quad |z| \rightarrow 1^-,$$

by Lebesgue's convergence theorem. We also note that from the previous proposition, Lemma 2.4, and the properties of reproducing kernels, if $\omega \in \mathcal{C}_{p_0, t}$ for some $p_0 > 1$ and $t \geq 0$, then

$$\|K_z^\omega\|_{A_\omega^2}^2 \asymp \|K_z^{\tilde{\omega}}\|_{A_{\tilde{\omega}}^2}^2 \asymp \frac{1}{(1-|z|^2)^2 \tilde{\omega}(z)}, \quad z \in \mathbb{D}.$$

Therefore, for all $r \in (0, 1)$, we have

$$\|K_z^\omega\|_{A_\omega^2}^2 \asymp \frac{1}{\int_{\Delta(z, r)} \omega(\zeta) dA(\zeta)}, \quad z \in \mathbb{D}.$$

This estimate was obtained by Constantin in [16] for ω such that $\frac{\omega}{(1-|z|^2)^\eta} \in B_{p_0}(\eta)$ for some $p_0 > 1$ and $\eta > -1$. Also, notice that since $\|K_w^\omega\|_{A_\omega^2} \asymp \|K_z^\omega\|_{A_\omega^2}$ if $w \in \Delta(z, 1/2)$, there exists $\beta \in (0, 1)$ such that for all $z \in \mathbb{D}$ we have

$$|K_z^\omega(w)| \asymp \|K_z^\omega\|_{A_\omega^2} \|K_w^\omega\|_{A_\omega^2}, \quad w \in \Delta(z, \beta).$$

The proof is similar again to the proof of Lemma 3.3 in [22].

2.4 Toeplitz and Volterra operators on Bergman spaces

In all this section, we suppose that $\omega \in \mathcal{C}_{p_0, t}$ for some $p_0 > 1$ and $t \geq 0$.

2.4.1 Toeplitz operators on A_ω^2 .

The following result describes the boundedness and compactness of Toeplitz operators T_μ on A_ω^2 . As mentioned in the introduction, the case of Bekollé-Bonami weights is considered in [16]. Our proofs are based on Lemma 2.4 and [21]. Notice that the Toeplitz operator T_μ on A_ω^2 associated with a positive Borel measure μ satisfies

$$\langle T_\mu f, f \rangle_{A_\omega^2} = \int_{\mathbb{D}} |f(\zeta)|^2 \omega(\zeta) d\mu(\zeta), \quad f \in A_\omega^2. \quad (2.11)$$

Recall also that μ_ω is the measure $d\mu_\omega(z) := \omega(z) d\mu(z)$, $z \in \mathbb{D}$.

Theorem 2.7. *Let μ be a positive Borel measure on \mathbb{D} . Then*

- (1) T_μ is bounded on A_ω^2 if and only if $\mu_\omega(\Delta_n) = O(A_\omega(\Delta_n))$.
- (2) T_μ is compact on A_ω^2 if and only if $\mu_\omega(\Delta_n) = o(A_\omega(\Delta_n))$, $n \rightarrow \infty$.

Proof. Let ν be the measure defined on \mathbb{D} by $d\nu = \frac{\omega}{\tilde{\omega}}d\mu$. For all $f \in A_\omega^2(= A_{\tilde{\omega}}^2)$ we have

$$\langle T_\mu f, f \rangle_{A_\omega^2} = \int_{\mathbb{D}} |f(z)|^2 \omega(z) d\mu(z) = \int_{\mathbb{D}} |f(z)|^2 \tilde{\omega}(z) d\nu(z) = \langle T_\nu f, f \rangle_{A_{\tilde{\omega}}^2}.$$

It follows that T_μ is bounded on A_ω^2 if and only if T_ν is bounded on $A_{\tilde{\omega}}^2$. Hence, by Proposition 2.2 and Proposition 2.3, T_μ is bounded on A_ω^2 if and only if (see [21])

$$\frac{1}{|\Delta_n|} \int_{\Delta_n} \frac{\omega(z)}{\tilde{\omega}(z)} d\mu(z) = O(1).$$

and, by (2.2), this is equivalent to

$$\int_{\Delta_n} \omega(z) d\mu(z) \lesssim \int_{\Delta_n} \omega(z) dA(z).$$

Similarly, we prove the second assertion of the theorem. \square

An alternative way to prove Theorem 2.7 is by using Theorem 2.1, Lemma 2.4, and [16, Theorems 3.1, 3.3, 4.1, and 4.2] concerning the boundedness and compactness of Toeplitz operators on the weighted Bergman spaces associated with Bekollé-Bonami weights.

Using Proposition 2.2 and Proposition 2.3, we will now prove Theorem 2.3 which describes the membership of T_μ to $\mathcal{S}_p(A_\omega^2)$, $p > 0$.

Proof of Theorem 2.3. Recall that, for all $f \in A_\omega^2(= A_{\tilde{\omega}}^2)$, we have $\langle T_\mu f, f \rangle_\omega = \langle T_\nu f, f \rangle_{\tilde{\omega}}$, where T_ν is the Toeplitz operator acting on $A_{\tilde{\omega}}^2$ which is associated with the measure $d\nu = \frac{\omega}{\tilde{\omega}}d\mu$. By Lemma 2.4, the operator $X : A_\omega^2 \rightarrow A_{\tilde{\omega}}^2$, $Xf = f$, is bounded and invertible. It follows that X^* , the adjoint of X defined by

$$\langle X^* f, g \rangle_{A_\omega^2} = \langle f, Xg \rangle_{A_{\tilde{\omega}}^2},$$

is bounded and invertible on A_ω^2 , moreover

$$\langle X^* T_\nu X f, f \rangle_{A_\omega^2} = \langle T_\nu X f, X f \rangle_{A_{\tilde{\omega}}^2} = \langle T_\nu f, f \rangle_{A_{\tilde{\omega}}^2} = \langle T_\mu f, f \rangle_{A_\omega^2}, \quad f \in A_\omega^2.$$

In particular, $X^* T_\nu X$ is positive and so $X^* T_\nu X = T_\mu$. Therefore, by Weyl's monotonicity principle for compact positive operators, we have

$$\lambda_n(T_\mu) = \lambda_n(X^* T_\nu X) \asymp \lambda_n(T_\nu), \quad n = 0, 1, 2, \dots$$

Hence, as a direct consequence of Proposition 2.2, Proposition 2.3, and [20, Theorem 4.5], $T_\mu \in \mathcal{S}_p(A_\omega^2)$ if and only if

$$\sum_{n=0}^{\infty} \left(\frac{\nu(\Delta_n)}{|\Delta_n|} \right)^p < \infty.$$

Equivalently, $T_\mu \in \mathcal{S}_p(A_\omega^2)$ if and only if

$$\sum_{n=0}^{\infty} \left(\frac{\mu_\omega(\Delta_n)}{A_\omega(\Delta_n)} \right)^p < \infty. \quad \square$$

2.4.2 Volterra operators on A_ω^2 .

In order to study Volterra operators on A_ω^2 , we give in the following lemma a connection between these operators and Toeplitz operators on A_ω^2 .

Lemma 2.5. *For all f in A_ω^2 we have $\langle V_g^* V_g f, f \rangle_{A_\omega^2} \asymp \langle T_{\mu_g} f, f \rangle_{A_\omega^2}$, where*

$$d\mu_g := (1 - |z|^2)^2 |g'(z)|^2 dA(z).$$

Proof. Indeed, by Theorem 2.1, we have

$$\begin{aligned} \langle V_g^* V_g f, f \rangle_{A_\omega^2} &= \|V_g f\|_{A_\omega^2}^2 \asymp \int_{\mathbb{D}} |(V_g f)'(z)|^2 (1 - |z|^2)^2 \omega(z) dA(z) \\ &= \int_{\mathbb{D}} |f(z)|^2 |g'(z)|^2 (1 - |z|^2)^2 \omega(z) dA(z) \\ &= \langle T_{\mu_g} f, f \rangle_{A_\omega^2}. \end{aligned} \quad \square$$

The following result describes the boundedness and compactness of V_g on A_ω^2 .

Theorem 2.8. *Let g be an analytic function on \mathbb{D} . Then, V_g is bounded (resp. compact) on A_ω^2 if and only if $g \in \mathcal{B}$ (resp. $g \in \mathcal{B}_0$).*

Proof. If $g \in \mathcal{B}$ then, by (2.11), T_{μ_g} is bounded on A_ω^2 , and hence V_g is bounded on A_ω^2 by the previous lemma.

Suppose now that V_g is bounded on A_ω^2 . Then, by the previous lemma, T_{μ_g} is bounded on A_ω^2 . By Lemma 2.4, we deduce that T_{μ_g} is bounded on A_ω^2 . Therefore, by Theorem 5.1 in [21], we have

$$\frac{1}{|\Delta_n|} \int_{\Delta_n} (1 - |z|^2)^2 |g'(z)|^2 dA(z) = O(1).$$

The result follows by the subharmonicity of $|g'(z)|^2$. Similarly, we prove that V_g is compact on A_ω^2 if and only if $g \in \mathcal{B}_0$. \square

We need the following lemma for the proof of Theorem 2.4.

Lemma 2.6. *Let $d\mu = (1 - |z|^2)^\alpha |h|^2 dA$ with $h \in \text{Hol}(\mathbb{D})$ and $\alpha \geq 0$. If T_μ is compact on A_ω^2 , then T_μ belongs to $\mathcal{S}_p(A_\omega^2)$, $p > 0$, if and only if*

$$\sum_{n=0}^{\infty} \left(\frac{\mu(\Delta_n)}{A(\Delta_n)} \right)^p < \infty.$$

Proof. By applying Lemma 2.4 for the weight $(1 - |z|^2)^\alpha \omega$, for all $f \in A_\omega^2$, we have

$$\begin{aligned} \langle T_\mu f, f \rangle_{A_\omega^2} &= \int_{\mathbb{D}} |f(z)|^2 (1 - |z|^2)^\alpha \omega(z) |h(z)|^2 dA(z) \\ &\asymp \int_{\mathbb{D}} |f(z)|^2 (1 - |z|^2)^\alpha \tilde{\omega}(z) |h(z)|^2 dA(z) \\ &= \langle T_\mu f, f \rangle_{A_\omega^2} \end{aligned}$$

Taking into account Proposition 2.2 and Proposition 2.3, the desired result is obtained from Theorem 4.5 in [20]. \square

Proof of Theorem 2.4. Note first that, since V_g is compact on A_ω^2 , T_{μ_g} is compact on A_ω^2 by Lemma 2.5. To obtain the first assertion, it is enough to prove that for all $p > 1/2$ we have

$$\sum_{n=0}^{\infty} (s_n(V_g))^{2p} \asymp \int_{\mathbb{D}} |g'(z)|^{2p} (1 - |z|^2)^{2p-2} dA(z).$$

From Lemma 2.5 and Weyl's monotonicity principle for compact positive operators, we have

$$s_n^2(V_g) \asymp \lambda_n(T_{\mu_g}), \quad n = 1, 2, 3, \dots \quad (2.12)$$

where $\lambda_n(T_{\mu_g})$ is the n^{th} eigenvalue of T_{μ_g} . Hence, it suffices to show that

$$\sum_{n=0}^{\infty} (\lambda_n(T_{\mu_g}))^p \asymp \int_{\mathbb{D}} |g'(z)|^{2p} (1 - |z|^2)^{2p-2} dA(z).$$

Using Lemma 2.6, we have

$$\sum_{n=0}^{\infty} (\lambda_n(T_{\mu_g}))^q \asymp \sum_{n=0}^{\infty} \left(\frac{1}{A(\Delta_n)} \int_{\Delta_n} (1 - |z|^2)^2 |g'(z)|^2 dA(z) \right)^q, \quad q > 0.$$

Since

$$(1 - |z|^2) |g'(z)| \lesssim \frac{1}{|\Delta_n|} \int_{b\Delta_n} (1 - |\zeta|^2) |g'(\zeta)| dA(\zeta), \quad \text{for } z \in \Delta_n,$$

we get

$$\frac{1}{A(\Delta_n)} \int_{\Delta_n} (1 - |z|^2)^2 |g'(z)|^2 dA(z) \lesssim \left(\frac{1}{|\Delta_n|} \int_{b\Delta_n} (1 - |\zeta|^2) |g'(\zeta)| dA(\zeta) \right)^2.$$

Therefore, since $2p > 1$, we obtain

$$\begin{aligned} \sum_{n=0}^{\infty} (\lambda_n(T_{\mu_g}))^p &\lesssim \sum_{n=0}^{\infty} \frac{1}{|\Delta_n|} \int_{b\Delta_n} (1 - |\zeta|^2)^{2p} |g'(\zeta)|^{2p} dA(\zeta) \\ &\lesssim \int_{\mathbb{D}} |g'(\zeta)|^{2p} (1 - |\zeta|^2)^{2p-2} dA(\zeta). \end{aligned}$$

On the other hand, we have

$$\begin{aligned} &\int_{\mathbb{D}} |g'(z)|^{2p} (1 - |z|^2)^{2p-2} dA(z) \\ &\asymp \sum_{n=0}^{\infty} \int_{\Delta_n} |g'(z)|^{2p} (1 - |z|^2)^{2p-2} dA(z) \\ &\asymp \sum_{n=0}^{\infty} \frac{1}{|\Delta_n|} \int_{\Delta_n} |g'(z)|^{2p} (1 - |z|^2)^{2p} dA(z) \\ &\lesssim \sum_{n=0}^{\infty} \frac{1}{|\Delta_n|} \int_{\Delta_n} \left(\frac{1}{|\Delta_n|} \int_{b\Delta_n} |g'(\zeta)|^2 (1 - |\zeta|^2)^2 dA(\zeta) \right)^p dA(z) \\ &\asymp \sum_{n=0}^{\infty} \left(\frac{1}{|\Delta_n|} \int_{b\Delta_n} |g'(\zeta)|^2 (1 - |\zeta|^2)^2 dA(\zeta) \right)^p \\ &\asymp \sum_{n=0}^{\infty} \left(\int_{b\Delta_n} |g'(\zeta)|^2 dA(\zeta) \right)^p. \end{aligned}$$

Since $(b\Delta_n)_n = (\Delta(z_n, b\delta))_n$ is a covering of \mathbb{D} of finite multiplicity, there is $N \in \mathbb{N}^*$ such that $b\Delta_n \subset \cup_{k=1}^N \Delta_{n_k}$ for all n . Hence

$$\int_{b\Delta_n} |g'(\zeta)|^2 dA(\zeta) \lesssim \int_{\Delta_{n'_k}} |g'(\zeta)|^2 dA(\zeta),$$

where n'_k is such that

$$\int_{\Delta_{n'_k}} |g'(\zeta)|^2 dA(\zeta) = \max_{1 \leq k \leq N} \int_{\Delta_{n_k}} |g'(\zeta)|^2 dA(\zeta).$$

We deduce that

$$\begin{aligned}
\int_{\mathbb{D}} |g'(z)|^{2p} (1 - |z|^2)^{2p-2} dA(z) &\lesssim \sum_{n=0}^{\infty} \left(\int_{\Delta_{n'_k}} |g'(\zeta)|^2 dA(\zeta) \right)^p \\
&\leq \sum_{n=0}^{\infty} \left(\int_{\Delta_n} |g'(\zeta)|^2 dA(\zeta) \right)^p \\
&\asymp \sum_{n=0}^{\infty} \left(\frac{1}{|\Delta_n|} \int_{\Delta_n} |g'(\zeta)|^2 (1 - |\zeta|^2)^2 dA(\zeta) \right)^p \\
&\asymp \sum_{n=0}^{\infty} (\lambda_n(T_{\mu_g}))^p,
\end{aligned}$$

where the last equivalence is obtained by Lemma 2.6. This completes the proof of the first assertion of the theorem.

If V_g belongs to $\mathcal{S}_1(A_\omega^2)$, then T_{μ_g} belongs to $\mathcal{S}_{1/2}(A_\omega^2)$ by (2.12) and hence, by Lemma 2.6, we obtain

$$\begin{aligned}
\infty > \sum_{n=0}^{\infty} \left(\frac{\mu_g(\Delta_n)}{A(\Delta_n)} \right)^{1/2} &= \sum_{n=0}^{\infty} \left(\frac{1}{A(\Delta_n)} \int_{\Delta_n} |g'(\zeta)|^2 (1 - |\zeta|^2)^2 dA(\zeta) \right)^{1/2} \\
&\gtrsim \sum_{n=0}^{\infty} \frac{1}{A(\Delta_n)} \int_{\Delta_n} |g'(\zeta)| (1 - |\zeta|^2) dA(\zeta) \\
&\asymp \sum_{n=0}^{\infty} \int_{\Delta_n} \frac{|g'(\zeta)|}{1 - |\zeta|^2} dA(\zeta) \\
&\asymp \int_{\mathbb{D}} \frac{|g'(\zeta)|}{1 - |\zeta|^2} dA(\zeta).
\end{aligned}$$

This implies that $g' \equiv 0$ on \mathbb{D} . The proof is achieved. □

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Résumé :

Dans cette thèse, on étudie le comportement des valeurs singulières de l'opérateur de Hankel sur des espaces de Bergman à poids. On considère des opérateurs de Hankel compact à symbole anti-analytique et on donne une estimation de la trace pour toute fonction convexe h . Ceci nous permet de donner une estimation du comportement asymptotique des valeurs singulières. Pour les poids radiaux, on montre à quel moment la décroissance critique est achevée. Ensuite, on donne une condition nécessaire et suffisante pour que les valeurs singulières soient un grand O de la décroissance critique. Quand cette condition est vérifiée, on donne les asymptotiques des valeurs singulières. On étudie aussi des espaces de Bergman à poids qui vérifient le théorème de Hardy-Littlewood. On étend des résultats existants à une large classe de poids. On montre que les polynômes sont denses dans les espaces de Bergman à poids pour des poids dans cette classe. Ensuite on démontre plusieurs propriétés sur le comportement des noyaux reproduisant des espaces de Bergman à poids, ce qui nous permet d'étudier les opérateurs de Toeplitz et d'intégration sur ces espaces

Mots-clefs (5): Espaces de Bergman, opérateurs de Hankel, opérateurs de Toeplitz, opérateurs Volterra, valeurs singulières

Abstract :

In this thesis, we study the behavior of the singular values of Hankel operators on weighted Bergman spaces. We consider compact Hankel operators with anti-analytic symbols, and give a trace estimates for any convex function h . This allows us to give asymptotic estimates of the singular values. For the radial weights, we show when the critical decay is achieved. Then we give a necessary and sufficient condition for the singular values to be big O of this critical decay. When this condition is satisfied, we give asymptotics of the singular values. We also study weighted Bergman spaces that satisfies Hardy-Littlewood theorem. We extend some existing results to a large class of weights. We prove that polynomials are dense in weighted Bergman spaces for weights belonging to this class. Then we prove several proprieties on the behavior of the kernel of the weighted Bergman spaces which allows us to study Toeplitz and Integration operators on these spaces.

Key Words (5): Bergman spaces, Hankel operator, Toeplitz operator, Volterra operator, singular values