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Superharmonically weighted Dirichlet Spaces

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SUPERHARMONICALLY WEIGHTED DIRICHLET SPACES

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Dedicated to my Parents and my Siblings

Lalla Khiti Sadiki Amari
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ABSTRACT

In this thesis, we deal with superharmonically weighted Dirichlet spaces.

In the first part of this dissertation, we determine all supreharmonically weighted Dirichlet spaces which contain no infinite Blaschke products. Next, we give a sufficient condition which ensuring the membership of Blaschke products. However, we propose two methods of regularization of Blaschke products. In fact, the regularization condition is a sufficient condition for zero sets. Our condition improves Shapiro-Shields condition for a large class of superharmonic weight, including standard weights. We end this part by studying the uniqueness sets for the harmonic weights. Actually, we extend the Khavin-Maz'ya uniqueness theorem for all harmonically weighted Dirichlet spaces.

In the second part, we characterize the invariant subspaces of multiplication operator by z on superharmonically weighted Dirichlet spaces. We prove that the extended Brown-Shields conjecture is true, in the countable case. Finally, we give a complete description of closed ideals of the algebra obtained by the intersection of superharmonically weighted Dirichlet spaces and the analytic Lipschitz spaces.

Keywords: Supreharmonically weighted Dirichlet spaces; Blaschke products; Zero sets; Uniqueness sets; Reproducing kernels; Capacity; Invariant subspaces; Cyclicity; Closed ideals.

RÉSUMÉ

Dans cette thèse, nous considérons les espaces de Dirichlet à poids super-harmonique.

Dans la première partie de cette dissertation, nous déterminons tous les espaces de Dirichlet à poids super-harmonique qui ne contiennent aucun produit de Blaschke infini.

Ensuite, nous donnons une condition suffisante de l'appartenance des produits de Blaschke Cependant, nous proposons deux méthodes de régularisations des produits de Blaschke. En fait, cette condition de régularisation est une condition suffisante pour les ensembles zéro. Notre condition améliore celle de Shapiro-Shields pour une grande classe de poids super-harmonique, y compris les poids standards. En terminant cette partie par les ensembles d'unicité pour les poids harmonique. Précisément, nous étendons le théorème d'unicité de Khavin-Maz'ya pour tous les espaces de Dirichlet à poids harmonique.

Dans la deuxième partie, nous caractérisons les sous-espaces invariants de l'opérateur de multiplication par z sur les espaces de Dirichlet à poids super-harmonique. Nous répondons positivement à la conjecture étendue de Brown-Shields dans le cas dénombrable. Finalement, nous donnons une description complète des idéaux fermés de l'algèbre obtenue par l'intersection des espaces de Dirichlet à poids super-harmonique et l'algèbre des fonctions Lipschitzienne.

Mots-clés : Les espaces de Dirichlet à poids super-harmonique; Produits de Blaschke; Ensembles zéro; Ensembles unicité, Noyaux reproduisant; Capacité; Sous-espaces invariant; Cyclicité; Idéaux fermés.

C'est avec la logique que nous prouvons
et avec l'intuition que nous trouvons.

— Henri Poincaré

RÉSUMÉ DE LA THÈSE

Etant donné une fonction $\omega : \mathbb{D} \to (0, +\infty]$ intégrable pour la mesure de Lebesgue planaire dA, une telle fonction est appelée un poids sur $\mathbb{D}$. L'espace de Dirichlet, associé à ω, $\mathcal{D}_\omega$ est l'ensemble de toutes les fonctions holomorphes sur $\mathbb{D}$ telles que le module de leurs dérivées soit de carré intégrable par rapport à la mesure ωdA sur $\mathbb{D}$.
Soit ω un poids super-harmonique sur $\mathbb{D}$. Le théorème de représentation de Jensen-Riesz assure l'existence d'une mesure de Borel positive μ sur $\mathbb{D}$ et une mesure de Borel positive finie ν sur $\mathbb{T}$ telles que :

$$\omega(z) = U_\mu(z) + P_\nu(z) \qquad (z \in \mathbb{D}),$$

où U_μ[1] est le potentiel de Green de μ et P_ν est la transformée de Poisson de ν.

Dans toute la suite, la mesure μ vérifie la condition (1.2).

Remarquons que l'espace de Dirichlet à poids super-harmonique $\mathcal{D}_\omega$ est un sous espace de l'espace de Hardy $\mathrm{H}^2(\omega = (1 - |z|^2))$. Il est bien connu que tous les produits de Blaschke sont des fonctions bornées sur $\mathbb{D}$. Par conséquent, ils sont des éléments de H^2. Cependant, la situation est très différente pour l'espace de Dirichlet classique $\mathcal{D}(\omega = 1)$. En fait, L. Carleson [17] a montré que $\mathcal{D}$ ne contient aucun produit de Blaschke infini.

Le premier chapitre est une présentation des résultats obtenus tout au long de cette dissertation. Nous donnons aussi un aperçu historique sur les différents sujets étudiés dans ce mémoire.

Dans le deuxième chapitre, nous déterminons tous les espaces de Dirichlet à poids super-harmonique qui ne contiennent aucun produit de Blaschke infini. Proprement dit, nous prouvons que $\mathcal{D}$ est le plus grand espace parmi tous les espaces de Dirichlet à poids super-harmonique qui contiennent que les produits de Blaschke finis. Plus précisément, si h est la dérivée de la mesure ν par rapport à m, alors les assertions suivantes sont équivalentes :

m désigne la mesure de Lebesgue normalisée sur $\mathbb{T}$.

i. $\mathcal{D}_\omega$ ne contient aucun produit de Blaschke infini.

ii. Il existe $c > 0$, tel que $h \geq c$ $q.p.$ sur $\mathbb{T}$.

iii. $\liminf_{|z| \to 1^-} \omega(z) > 0$.

1 Notons que U_μ n'est pas identiquement infini si et seulement si (1.2) est satisfaite.

iv. $\mathcal{D}_\omega \subset \mathcal{D}$.

Par ailleurs, les espaces de Dirichlet à poids standard $\mathcal{D}_{\omega_\alpha}$, $\omega_\alpha(z) := (1-|z|^2)^\alpha$ pour $\alpha \in (0,1)$[2], contiennent des produits de Blaschke infinis ([16]). Nous donnons une condition suffisante de l'appartenance des produits de Blaschke à $\mathcal{D}_\omega$. En effet, $B \in \mathcal{D}_\omega$ si

Pour la définition de la fonction Ψ_ω voir (1.4).

$$\sum_n \Psi_\omega(z_n) < \infty,$$

où B le produit de Blaschke associé à la suite $\mathcal{Z} = (z_n)_n$. Si en plus la suite $\mathcal{Z}$ est uniformément séparée, cette condition est aussi nécessaire.

En utilisant ce résultat, il existe un produit de Blaschke B qui n'appartient à aucun espace $\mathcal{D}_\omega$, néanmoins $B(1-z)^2 \in \mathcal{D}_\omega$. Dans ce cas, $(1-z)^2$ est une fonction régularisante du produit de Blaschke B dans $\mathcal{D}_\omega$. L'espace $\mathcal{D}_\omega$ admet la propriété de division [3], par conséquent, une suite de Blaschke $\mathcal{Z}$ est un ensemble zéro pour $\mathcal{D}_\omega$ si et seulement si, il existe une fonction extérieure $f \in \mathcal{D}_\omega$ telle que $Bf \in \mathcal{D}_\omega$. En se basant sur ce principe de régularisation, nous allons proposer deux méthodes de construction des fonctions extérieures dans $\mathcal{D}_\omega$. La première est la méthode classique de Carleson, voir Sous-section 2.4.1. Pour la deuxième construction, nous allons exploiter la théorie du potentiel induite par $\mathcal{D}_\omega$. Évidemment, la fonction obtenue sera une fonction régularisante du produit de Blaschke en question dans $\mathcal{D}_\omega$ sous une condition sur la suite. En tenant compte des résultats donnés dans Corollaire 4.3.1 et Théorème 4.3.2. Il est bien clair que notre condition améliore celle de Shapiro-Shields (1.6) pour une grande classe de poids super-harmoniques y compris les poids ω_α.

Une suite $\mathcal{Z} = (z_n)_n \subset \mathbb{D}$ est un ensemble zéro pour $\mathcal{D}_\omega$, s'il existe une fonction dans $\mathcal{D}_\omega$ qui s'annule sur $\mathcal{Z}$ et nulle part ailleurs dans $\mathbb{D}$.

Le chapitre 3 sera consacré aux ensembles d'unicité sur $\mathbb{T}$ pour les espaces de Dirichlet à poids harmonique ($\omega = P_\nu$). Notre objectif principal, dans ce chapitre, est d'étendre le théorème d'unicité de Khavin-Maz'ya pour tous les espaces de Dirichlet à poids harmonique. Nous donnons une condition suffisante pour les ensembles d'unicité. Nous prouvons qu'un ensemble de Borel $E \subset \mathbb{T}$ est un ensemble d'unicité pour $\mathcal{D}_{P_\nu}$ si il existe une famille d'arcs $(I_n)_n$ deux-à-deux disjoints et une constante $\gamma \in (0,1)$ telles que

Un ensemble $E \subset \mathbb{T}$ est appelé un ensemble d'unicité pour $\mathcal{D}_\omega$ s'il possède la propriété suivante: $f = 0$ si $f \in \mathcal{D}_\omega$ s'annule sur E.

$$\sum_n |I_n| \log \frac{|I_n|}{c_{P_\nu}(E \cap \gamma I_n)} = -\infty.$$

Comme application de ce résultat, nous donnons une mesure atomique positive ν et un ensemble dénombrable fermé E qui est un ensemble d'unicité pour $\mathcal{D}_{P_\nu}$, pour plus de détails voir Section 3.3.
Il est bien connu qu'un ensemble de Borel $E \subset \mathbb{T}$ est un ensemble d'unicité pour $\mathcal{D}_{P_{\delta_\zeta}}$ si et seulement si la mesure de Lebesgue de E est

2 Dans ce qui suit, α désigne un paramètre appartenant à l'intervalle ouvert $(0,1)$.
3 Précisément, en utilisant (2.11), si f est une fonction de H^2 telle que $\theta f \in \mathcal{D}_\omega$, où θ est une fonction intérieure, alors $f \in \mathcal{D}_\omega$ et $\|f\|_{\mathcal{D}_\omega} \lesssim \|\theta f\|_{\mathcal{D}_\omega}$.

positive $(m(E) > 0)$. Ce résultat peut être étendu pour certains poids harmoniques. Soit ν une mesure atomique finie positive sur $\mathbb{T}$ qui satisfait (1.10). Nous prouvons qu'un ensemble de Borel $E \subset \mathbb{T}$ est un ensemble d'unicité pour $\mathcal{D}_{P_\nu}$ si et seulement si $m(E) > 0$.

Le quatrième chapitre traite la structure des sous-espaces invariants de l'opérateur de multiplication par z. Nous obtenons une caractérisation complète des sous-espaces invariants de l'espace $\mathcal{D}_\omega$. Formellement, si $\mathcal{M}$ est un sous-espace invariant de $\mathcal{D}_\omega$ alors il existe une fonction intérieure $\mathcal{I}_\mathcal{M}$ et une fonction extérieure $f \in \mathcal{D}_\omega$ telles que

Un sous-espace fermé $\mathcal{M}$ dans $\mathcal{D}_\omega$ est appelé sous-espace invariant, si la fonction zf appartient à $\mathcal{M}$ dès que $f \in \mathcal{M}$.

$$\mathcal{M} = \mathcal{I}_\mathcal{M}\mathrm{H}^2 \cap [f]_{\mathcal{D}_\omega}. \tag{0.1}$$

$[f]_{\mathcal{D}_\omega}$ désigne le plus petit sous-espace invariant de $\mathcal{D}_\omega$ contenant $f \in \mathcal{D}_\omega$.

Ce résultat est nouveau même pour les espaces de Dirichlet à poids standard $\mathcal{D}_{\omega_\alpha}$. Afin d'obtenir une caractérisation plus explicite, nous devons décrire les sous-espaces invariants générés par les fonctions extérieures. La conjecture étendue de Browen-Shields prétend que, si f est une fonction extérieure dans $\mathcal{D}_\omega$ alors il existe un ensemble $E \subset \mathbb{T}$ tel que

$$[f]_{\mathcal{D}_\omega} = \mathcal{M}_\mathrm{E} := \left\{g \in \mathcal{D}_\omega \;:\; g_{|_\mathrm{E}} = 0 \;\; c_\omega - q.e.\right\}.$$

Nous répondons positivement à cette conjecture dans le cas dénombrable. En effet, si $\mathcal{M}$ un sous-espace invariant de $\mathcal{D}_\omega$ tel que $\underline{\mathcal{Z}}(\mathcal{M}) \cap \mathbb{T}$[4] est dénombrable, $\mathrm{E} := \{\zeta \in \underline{\mathcal{Z}}(\mathcal{M}) \cap \mathbb{T} : c_\omega(\zeta) > 0\}$, alors

$$\mathcal{M} = \mathcal{I}_\mathcal{M}\mathrm{H}^2 \cap \mathcal{M}_\mathrm{E}.$$

Une question naturelle et remarquable se pose à partir de (0.1) : Déterminer toutes les fonctions $f \in \mathcal{D}_\omega$ satisfaisant $[f]_{\mathcal{D}_\omega} = \mathcal{D}_\omega$. Une telle fonction qui vérifie cette propriété est appelée une fonction cyclique dans $\mathcal{D}_\omega$. La conjecture de Brown-Shields affirme que : Une fonction f est cyclique dans $\mathcal{D}_\omega$ si et seulement si f est extérieure et $c_\omega(Z(f)) = 0$. Nous prouvons que cette conjecture est vraie pour $\mathcal{D}_\omega$ dans le cas dénombrable. Plus précisément, soit f une fonction extérieure dans $\mathcal{D}_\omega$ telle que $\underline{\mathcal{Z}}(f) \cap \mathbb{T}$ est dénombrable, f est cyclique dans $\mathcal{D}_\omega$ si et seulement si $c_\omega(Z(f)) = 0$. Nous términons ce chapitre par une condition suffisante pour la cyclicité dans $\mathcal{D}_\omega$. A savoir, soit f une fonction extérieure dans $\mathcal{D}_\omega$, $E := Z(f)$, si

$Z(f)$ est l'ensemble des zéros des limites radiales de f dans $\mathbb{T}$.

$$\int_0 c_\omega(E_t)\frac{\log(1/t)}{t}dt < \infty,$$

alors f est cyclique dans $\mathcal{D}_\omega$. Cette condition est nouvelle même pour les espaces de Dirichlet à poids standard $\mathcal{D}_{\omega_\alpha}$.

Le dernier chapitre s'intéresse principalement à la structure des idéaux fermés de l'algèbre $\mathcal{D}_\omega \cap \Lambda_\alpha$[5]. Par dualité, la description des

4 Pour la définition de $\underline{\mathcal{Z}}(\mathcal{M})$ et $\underline{\mathcal{Z}}(f)$ voir Chapitre 4.

5 Λ_α est l'algèbre des fonctions analytiques sur le disque unité $\mathbb{D}$ qui sont Lipschitzienne d'ordre $\alpha \in (0,1)$ sur $\overline{\mathbb{D}}$.

idéaux fermés de $\mathcal{D}_\omega \cap \Lambda_\alpha$ peut être réduite à un théorème spécial d'approximation. Nous donnons une preuve élémentaire du théorème d'approximation établit par A.L. Matheson [53] (resp. B. Bouya [13]) pour Λ_α (resp. $\mathcal{D} \cap \Lambda_\alpha$). Notre approche nous permet d'étendre ce résultat à l'algèbre $\mathcal{D}_\omega \cap \Lambda_\alpha$. Par conséquent, nous obtenons une description complète des idéaux fermés de $\mathcal{D}_\omega \cap \Lambda_\alpha$. Proprement dit, soit $\mathcal{I} \neq (0)$ un idéal fermé de $\mathcal{D}_\omega \cap \Lambda_\alpha$, alors il existe un ensemble fermé $E_\mathcal{I} \subset \mathbb{T}$ et une fonction intérieure $\theta_\mathcal{I}$ tels que

$$\mathcal{I} = \{f \in \mathcal{D}_\omega \cap \Lambda_\alpha : \; f_{|E_\mathcal{I}} = 0, \text{ et } f \in \theta_\mathcal{I} H^2\}.$$

CONTENTS

1 INTRODUCTION

In this thesis, we consider the superharmonically weighted Dirichlet spaces $\mathcal{D}_\omega$, which is the set of all analytic functions in the unite disk whose derivatives are square area integrable with superharmonic weight ω, as the main subject. This dissertation aims to develop an understanding of certain topics in $\mathcal{D}_\omega$. Especially, we are mainly interested to examine some research questions in these spaces. Namely

- Blaschke products and Zero Sets.
- Uniqueness sets.
- Invariant Subspaces and Cyclicity.
- Closed ideals.

A. Aleman established in [4] a representation theorem for a subclass of Bounded operators on a Hilbert space. More formally, let T be a bounded operator on a separated Hilbert space $\mathcal{H}$ such that

(i) $\sum\limits_{k=0}^{n} (-1)^k \binom{n}{k} \|T^k x\|^2_{\mathcal{H}} \leq 0$ for all $x \in \mathcal{H}$ and for all $n \geq 2$,

(ii) $\bigcap\limits_n T^n \mathcal{H} = \{0\}$,

(iii) $\dim(\mathcal{H} \ominus T(\mathcal{H})) = 1$.

There exists a positive superharmonic function ω on $\mathbb{D}$ such that $(T, \mathcal{H})$ is unitarily equivalent to $(M_z, \mathcal{D}_\omega)$, where M_z is the operator of multiplication by z. This class is called operators of Dirichlet type. This remarkable representation theorem was first obtained by S. Richter[1].

Here $\mathbb{D}$ (resp. $\mathbb{T}$) denotes the open unit disk (resp. the unit circle).

However, let ω be a nonnegative superharmonic weight on $\mathbb{D}$. By Jensen-Riesz representation theorem (see for instance Theorem 1.24′ of [50]), there exists a positive Borel measure μ on $\mathbb{D}$ and a finite positive Borel measure ν on $\mathbb{T}$ such that

A weight ω is a function $\omega : \mathbb{D} \to (0, +\infty]$ which is integrable on $\mathbb{D}$ with respect to the Lebesgue measure.

$$\omega(z) = U_\mu(z) + P_\nu(z), \quad (z \in \mathbb{D}), \tag{1.1}$$

where U_μ is the Green potential of μ and P_ν is the Poisson transform of ν (see Chapter 2). It is well known that U_μ is not identically infinite if and only if

$$\int_{\mathbb{D}} (1 - |z|) d\mu(z) < \infty. \tag{1.2}$$

1 Richter proved in [62] that every cyclic analytic 2–isometry can be represented as a multiplication by z on a harmonically weighted Dirichlet spaces (i.e $\omega = P_\nu$).

Note that the superharmonically weighted Dirichlet space $\mathcal{D}_\omega$ is a subspace of Hardy space H^2. It is known that every Blaschke product is bounded, so it belongs to H^2, see [23]. Nevertheless, for the classical Dirichlet space $\mathcal{D}$ $(\omega = 1)$ the situation is quite different. L. Carleson [18] proved that $\mathcal{D}$ contains only finite Blaschke products.
Our first contribution in this thesis is to determine all superharmonically weighted Dirichlet spaces which contain no infinite Blaschke product. We prove that $\mathcal{D}$ is the largest space among all superharmonically weighted Dirichlet spaces which contains no infinite Blaschke product. Namely, we have the following result.

Hereafter we denote by m the normalized Lebesgue measure on $\mathbb{T}$.

Theorem 1. *Let ω be a superharmonic weight and let P_ν be the harmonic part of ω. Let h be the derivative of ν with respect to m. The following are equivalent:*

(1) *$\mathcal{D}_\omega$ contains no infinite Blaschke product.*

(2) *There exists $c > 0$, such that $h \geq c$ a.e. on $\mathbb{T}$.*

(3) $\liminf_{|z|\to 1^-} \omega(z) > 0.$

(4) $\mathcal{D}_\omega \subset \mathcal{D}.$

However, the standard weighted Dirichel spaces $\mathcal{D}_{\omega_\alpha}$ contain infinite Blaschke products, where $\omega_\alpha(z) := (1-|z|^2)^\alpha$ and $0 < \alpha < 1$. In fact, L. Carleson showed in his thesis [16] that the condition

$$\sum_n \omega_\alpha(z_n) < \infty \tag{1.3}$$

implies that the Blaschke product B, with zeros $\mathcal{Z} := (z_n)$, belongs to $\mathcal{D}_{\omega_\alpha}$. Moreover, if in addition $\mathcal{Z}$ is uniformly separated[2], K.M. Dyakonov [25] proved that (1.3) is also necessary. In this direction, we will extend such results for all superharmonic weights. Befor that, let us introducte the function Ψ_ω given by

The definition of Blaschke product will be given by the formula (2.1).

$$\Psi_\omega(z) = (1-|z|^2)\int_{\mathbb{D}} \frac{1-|w|^2}{|1-\bar{z}w|^2} d\mu(w) + P_\nu(z) \qquad (z \in \mathbb{D}). \tag{1.4}$$

For classical weights one can get an explicit formula of the function Ψ_ω. Especially, we have $\Psi_{\omega_\alpha}(z) \asymp \omega_\alpha(z)$ (for other examples of radial weights, see Section 2.4). In the following theorem we give a condition which will ensuring that Blashcke product be an element of $\mathcal{D}_\omega$. Further, this sufficient condition will be necessary if our sequence is uniformly separated.

2 The definition of uniformly separated sequences will be given by (2.13). Recall that L. Carleson [19] showed that $\mathcal{Z}$ is an interpolating sequence for H^∞ if and only if $\mathcal{Z}$ is uniformly separated (for more details on this, see [38, 69]).

Theorem 2. *Let $\omega = U_\mu + P_\nu$ be a superharmonic weight, where μ is a positive Borel measure on $\mathbb{D}$ satisfying (1.2) and ν be a finite positive Borel measure on $\mathbb{T}$. Let B be the Blaschke product associated with $\mathcal{Z} = (z_n)_n \subset \mathbb{D}$. Then $B \in \mathcal{D}_\omega$, if*

$$\sum_n \Psi_\omega(z_n) < \infty. \tag{1.5}$$

If in addition, $\mathcal{Z}$ is uniformly separated then (1.5) is also necessary.

Using Theorem 2, one can get a Balschke product that does not belong to $\mathcal{D}_\omega$ and it represents the inner part of a function in $\mathcal{D}_\omega$. Indeed, let $\mathcal{Z} = (r_n)_n \subset (0,1)$ be a Blaschke sequence and consider $f(z) := (1-z)^2 B(z)$ where B is the Blaschke product associated with $\mathcal{Z}$. Note that $f' \in \mathrm{H}^\infty$, this implies that $f \in \mathcal{D}_\omega$. Moreover, if $\mathcal{Z}$ is uniformly separated and (1.5) does not satisfy, then the associated Blaschke product B does not belong to $\mathcal{D}_\omega$. In particular, the sequence $\mathcal{Z}$ is a zero set for $\mathcal{D}_\omega$.

A sequence $\mathcal{Z} = (z_n)_n \subset \mathbb{D}$ is said a Blachke sequence if $\sum(1-|z_n|) < 0$.

A sequence $\mathcal{Z} = (z_n)_n \subset \mathbb{D}$ is a zero set for $\mathcal{D}_\omega$ if there is a function in $\mathcal{D}_\omega$ that vanishes on $\mathcal{Z}$ and nowhere else in $\mathbb{D}$.

One of the most important problems for superharmonically weighted Dirichlet spaces is the description of zero sets for $\mathcal{D}_\omega$. This problem is still open even for the classical Dirichlet space. From the above discussion all Blaschke product belongs to H^2. By Jensen's formula (see for instance [23, 38]), the zeros set of a function in Hardy space H^2 satisfies Blaschke condition. Thus, the zero sets of the Hardy space H^2 are exactly the Blaschke sequences. For the classical Dirichlet space $\mathcal{D}$, L. Carleson [17] exploited the potential theory induced by $\mathcal{D}$ implicitly to construct an outer function f in $\mathcal{D}$ such that $Bf \in \mathcal{D}$, if $\mathcal{Z}$ satisfies

$$\sum_n \frac{1}{\log^{1-\varepsilon}(1/(1-|z_n|))} < \infty,$$

for some $\varepsilon \in (0,1)$, in particular $\mathcal{Z}$ is a zero set for $\mathcal{D}$. In [71], H.S. Shapiro and A.L. Shields improve this condition by showing that one may take $\varepsilon = 0$. The idea of Shapiro-Shields is mainly based on some properties of the reproducing kernels of a large class of Hilbert spaces. However, D. Marshall and C. Sundberg [51] observed that Shapiro-Shields proof works for all Hilbert spaces of analytic functions on $\mathbb{D}$ that enjoy Pick property (see [2, 69]). In this direction S. Shimorin [72] proved that every superharmonically weighted Dirichlet space $\mathcal{D}_\omega$ possesses Pick property. So, if $\mathcal{Z} = (z_n)_n$ satisfies Shapiro-Shields condition, namely

$$\sum_n \frac{1}{K^\omega(z_n, z_n)} < \infty, \tag{1.6}$$

By Theorem 2 of [72], we have $K^\omega(z,z) \leq 2/(1-|z|^2)$. Thus, the condition (1.6) implies the Blaschke condition.

where K^ω is the reproducing kernel of $\mathcal{D}_\omega$, then $\mathcal{Z}$ is a zero set for $\mathcal{D}_\omega$. Keep in mind the above discussion, in the case of the standard weight Dirichlet spaces $\mathcal{D}_{\omega_\alpha}$ both conditions (1.6) and (1.3) coincide. This property is satisfied for a large class of radial weights, see Corollary

4.3.1. Take into account Theorem 4.7, Condition (1.5) implies Shapiro-Shields condition (1.6).

In accordance with the Riesz-Nevanlinna factorization every nonzero function $F \in \mathcal{D}_\omega$ admits an essentially unique representation in the form $F := SBf$ where S is a singular inner function associated with a singular measure, B is the Blaschke product associated to some Blaschke sequence and f is the outer factor of F. Furthemore, the space $\mathcal{D}_\omega$ possesses a division property . As a consequence, a Blaschke sequence $\mathcal{Z}$ is a zero set for $\mathcal{D}_\omega$ if and only if there exists an outer function $f \in \mathcal{D}_\omega$ such that $Bf \in \mathcal{D}_\omega$. Based on this remarkable fact, we give two ways to construct functions $f \in \mathcal{D}_\omega$ such that $fB \in \mathcal{D}_\omega$. The first one is the classical Carleson's construction of smooth outer functions (see Theorem 2.4.1). While, the second construction is based on potential theory induced by $\mathcal{D}_\omega$. We will construct outer functions in $\mathcal{D}_\omega$ with large real part on a given sequence of closed subsets of $\mathbb{T}$. Namely, if c_ω[3] denotes the capacity associated with $\mathcal{D}_\omega$, we have the following theorem.

The equivalent (2.11) says that $\mathcal{D}_\omega$ satisfies the division property. That is, given $f \in H^2$, if $\theta f \in \mathcal{D}_\omega$, where θ is an inner function, then $f \in \mathcal{D}_\omega$ and $\|f\|_{\mathcal{D}_\omega} \lesssim \|\theta f\|_{\mathcal{D}_\omega}$.

Theorem 3. *Let ω be a superharmonic weight. Let $(F_n)_n$ be a family of closed subsets of $\mathbb{T}$ and set $E_n = \bigcup_{k\geq n} F_k$. There exists a function $f \in \mathcal{D}_\omega$ such that* $\mathrm{Re} f \geq 0$ *on $\mathbb{D}$ and*

$$\mathrm{Re} f \geq \log \frac{1}{c_\omega(E_n)} \quad c_\omega - q.e. \text{ on } F_n \text{ for all } n \geq 1.$$

To obtain the main result on zero sets for general superharmonic weights ω (radial or nonradial). We use the approach of Regularization of Blaschke product by an outer function in $\mathcal{D}_\omega$. To take advantage of this approach, we exploit Theorem 3 and an upper estimate of the norm (see Section 2.2). In order to highlight the condition (1.5) and for the sake of simplicity, we state our result for weights that satisfy some additional condition.

The condition $\omega(z) = O((1-|z|^2)^\alpha)$ for $\alpha \in [0,1)$ says that $\mathcal{D}_\omega$ contains the classical Dirichlet space.

Theorem 4. *Let ω be a superharmonic weight on $\mathbb{D}$. Suppose that there exists $\alpha \in [0,1)$ such that $\omega(z) = O((1-|z|^2)^\alpha)$. Let $\mathcal{Z} = (z_n)_n$ be a sequence of $\mathbb{D}$ such that $\sum_n (1-|z_n|^2)^{1-\varepsilon} < \infty$, for some $\varepsilon \in (0,1)$. Let $\gamma = \frac{\varepsilon}{1-\alpha}$ and let $E_n = \bigcup_{k\geq n} I(z_k,\gamma)$[4]. If there exists $A > 0$ such that*

$$\sum_n \Psi_\omega(z_n) c_\omega^A(E_n) < \infty, \tag{1.7}$$

then $\mathcal{Z}$ is a zero set for $\mathcal{D}_\omega$.

The latter result is also new even for the standard Dirichlet spaces $\mathcal{D}_{\omega_\alpha}$. That is, since $\Psi_{\omega_\alpha}(z) \asymp \omega_\alpha(z) = 1/K^{\omega_\alpha}(z,z)$, it is clear that

Recall that the reproducing kernel of $\mathcal{D}_{\omega_\alpha}$ is $K^{\omega_\alpha}(z,w) = 1/(1-\overline{w}z)^\alpha$.

3 For more details about c_ω capacity see Subsection 2.4.2.
4 Let $\gamma > 0$ and let $z \in \mathbb{D} \setminus \{0\}$. The closed arc of $\mathbb{T}$ centered at $z/|z|$ and of length ($1-|z|)^\gamma$ will be denoted by $I(z,\gamma)$. We also write $I(0,\gamma) = \mathbb{T}$.

the condition (1.7) improves (1.3) and (1.6). Note also that the condition (1.7) takes into account $Arg(z_n)$ unlike both conditions. A more general results will be stated in Sections 2.4 and 2.5.

The dual problem of describing the zero sets is the problem of describing the uniqueness sets for the superharmonically weighted Dirichlet spaces $\mathcal{D}_\omega$. As mentioned above, the space $\mathcal{D}_\omega$ has the division property. So, let E be a subset of $\mathbb{T}$, E is a uniqueness set for $\mathcal{D}_\omega$ if and only if E is not a zero set for $\mathcal{D}_\omega$.

A subsets E of $\mathbb{T}$ is called a uniqueness set for $\mathcal{D}_\omega$ if it has the property that, $f = 0$ if $f \in \mathcal{D}_\omega$ vanishes at all points of E.

Firstly, L. Carleson [18] proved that, if E is a closed set of $\mathbb{T}$ which satisfies, for some $\alpha > 0$, $c_{\omega_\alpha}(E \cap I) \geq const.m(I)$ for all arcs I with centers belonging to E. Then E is a uniqueness set for $\mathcal{D}$ if and only if

$$\int_{\mathbb{T}} \log dist(\zeta, E) dm(\zeta) = -\infty. \tag{1.8}$$

If (1.8) is not satisfied then E is called a Carleson set.
This result was extended by S.P. Preobrazhenskii [59] to a large class of Dirichlet type spaces included $\mathcal{D}_{\omega_\alpha}$ (see also [56]).
Secondly, V. Khavin and V. Maz'ya [48] showed that a Borel set $E \subset \mathbb{T}$ is a uniqueness set for $\mathcal{D}$ if, there exists a family of pairwise disjoint open arcs $(I_n)_n$ such that

The capacity c_{P_m} (resp. The capacity c_{ω_α}) is comparable to the logarithmic capacity (resp. to Riesz's capacity c_α)

$$\sum_n |I_n| \log \frac{|I_n|}{c_{P_m}(E \cap I_n)} = -\infty. \tag{1.9}$$

They also proved that there exists a uniqueness set of c_{ω_α}–capacity zero (for any $\alpha > 0$) for the classical Dirichlet space (see also [44]). Furthermore, using Jensen's inequality, one can see easly that (1.9) implises (1.8), for more details see [47]. In [44], K. Kellay gave the generalization of these results for the standard weighted Dirichlet spaces $\mathcal{D}_{\omega_\alpha}$. In fact, both conditions (1.9) and (1.8) can be deducted from Theorem 3.1. of [44]. The main purpose, regarding this topic, is to extend Khavin-Maz'ya uniqueness theorem to all harmonically Dirichlet spaces $\mathcal{D}_{P_\nu}$.

Theorem 5. *Let E be a Borel subset of $\mathbb{T}$ of Lebesgue measure zero. Suppose that there exists a family of pairwise disjoint open arcs $(I_n)_n$ and $\gamma \in (0,1)$ such that*

$$\sum_n |I_n| \log \frac{|I_n|}{c_{P_\nu}(E \cap \gamma I_n)} = -\infty,$$

then E is a uniqueness set for $\mathcal{D}_{P_\nu}$.

In general, the characterization of the exeptional sets (we are maily intersting by uniqueness sets) will be presented with respect to "special set functions": Lebesgue measure for Hardy space, Logarithmic capacity for the classical Dirichlet space and Riesz's capacity for the standard weighted Dirichlet spaces. We will take advantage of these special functions to classify uniqueness sets. In particular, a Borel set $E \subset \mathbb{T}$ is a uniqueness set for Hardy space H^2 if and only if $m(E) > 0$.

As a consequence, the inclusion $\mathcal{D} \subset \mathrm{H}^2$ says that positive Lebesgue measure is sufficient to be uniqueness set for the classical Dirichlet space $\mathcal{D}$. On the other hand, L. Carleson [18] established a necessary condition in terms of the capacity. Namely, let E be a closed subset of $\mathbb{T}$, if E is an uniqueness set for $\mathcal{D}$ then $c_{P_m}(E) > 0$. Note also that the countable closed sets can be considered as the smallness exeptional sets with respect to the previous three special functions. Nevertheless, it is well known that each countable set has a zero logarithmic capacity. It follows that every countable subset of $\mathbb{T}$ is not a uniqueness set for $\mathcal{D}(= \mathcal{D}_{P_m})$. However, using Theorem 5 to obtain the first example of countable closed subset of $\mathbb{T}$ that is a uniqueness set for some harmonically weighted Dirichlet spaces.

Theorem 6. *There exists a discrete probability measure ν on the unit circle and a countable closed set $E \subset \mathbb{T}$, such that $\nu(E) = 0$ and E is a uniqueness set for $\mathcal{D}_{P_\nu}$.*

Here δ_ζ denotes the Dirac measure at $\zeta \in \mathbb{T}$.

Moreover, it is well known that a Borel set E is a uniqueness set for $\mathcal{D}_{P_{\delta_\zeta}}$ if and only if E has a positive Lebesgue measure. This result can be extended to some other discrete measures. Let ν be a positive finite atomic measure on $\mathbb{T}$ given by $\nu = \sum\limits_n c_n \delta_{\zeta_n}$. In [39], D. Guillot proved that every Blaschke sequence is a zero set for $\mathcal{D}_{P_\nu}$ if and only if

$$\int_{\mathbb{T}} \log V_2(\nu)(\zeta) dm(\zeta) < \infty, \tag{1.10}$$

where $V_2(\nu)$ is the newtonian potential[5] of the measure ν. Especially, he gave a function which regularizes all Blaschke products in $\mathcal{D}_{P_\nu}$ if (1.10) is satisfied. Taking advantage of this result, we give a necessary and sufficient condition for a Borel set of $\mathbb{T}$ to be a uniqueness set for $\mathcal{D}_{P_\nu}$.

Theorem 7. *Let ν be a positive finite atomic measure on $\mathbb{T}$ that satisfies (1.10). Then a Borel set $E \subset \mathbb{T}$ is a uniqueness set for $\mathcal{D}_{P_\nu}$ if and only if $m(E) > 0$.*

Now, we are mainly interested by the structure of the shift-invariant subspaces in superharmonically weighted Dirichlet spaces. A closed subspace $\mathcal{M}$ in $\mathcal{D}_\omega$ is called invariant subspace (shift-invariant subspace), if the function zf belongs to $\mathcal{M}$ whenever $f \in \mathcal{M}$.
In [64], S. Richter and C. Sundberg gave the structure of shift-invariant subspaces of all harmonically weighted Dirichlet spaces $\mathcal{D}_{P_\nu}$. Their argument is based on two important facts. The first says that, every invariant subspace of $\mathcal{D}_{P_\nu}$ has the form $\phi\mathcal{D}_{P_{\nu_\phi}}$ where $d\nu_\phi = |\phi|^2 d\nu$ and ϕ is a multiplier of $\mathcal{D}_{P_\nu}$. This result has been extended by A. Aleman [4] to all superharmonically weighted Dirichlet spaces. The second fact is an estimate of local Dirichlet integral (with respect to $\mathbb{T}$) for

5 For the definition of the newtonien potential see Section 3.3

cut-off functions. Theorems 4.1.1 and 5.3.1 claim that the second result remains true (with respect to $\mathbb{D}$). Enjoying these facts and following the strategy of Richter and Sundberg we obtain a complete characterization of shift-invariant subspaces of all superharmonically weighted Dirichlet spaces. Namely, we have the following theorem.

Theorem 8. *Let ω be a positive superharmonic weight on $\mathbb{D}$. Let $\mathcal{M}$ be a shift-invariant subspace of $\mathcal{D}_\omega$ and consider $\mathcal{I}_\mathcal{M}$ the greatest common inner divisor of $\mathcal{M}$. Then, there is an outer function $f \in \mathcal{D}_\omega \cap \mathrm{H}^\infty$ such that*

$$\mathcal{M} = \mathcal{I}_\mathcal{M}\mathrm{H}^2 \cap [f]_{\mathcal{D}_\omega}. \tag{1.11}$$

Furthermore, f can be chosen so that f and $\mathcal{I}_\mathcal{M} f$ are multipliers of $\mathcal{D}_\omega$.

Given $f \in \mathcal{D}_\omega$, the smallest invariant subspace of $\mathcal{D}_\omega$ containing f will be denoted by $[f]_{\mathcal{D}_\omega}$.

This characterization is new even for the standard weighted Dirichlet spaces $\mathcal{D}_\alpha$, $\alpha \in (0,1)$. To obtain a more explicit characterization, we need to describe the invariant subspaces generated by outer functions. The extended Brown-Shields conjecture: If f is an outer function in $\mathcal{D}_\omega$ then there exists a subset E of $\mathbb{T}$ such that

$$[f]_{\mathcal{D}_\omega} = \mathcal{M}_\mathrm{E} := \{g \in \mathcal{D}_\omega \ : \ g_{|_\mathrm{E}} = 0 \ \ c_\omega - q.e\}.$$

In [28], O. El-Fallah, Y. Elmadani and K. Kellay responded positively to that conjecture for $\mathcal{D}_{P_\nu}$ when $\operatorname{supp}\nu$ is countable. By following their argument, one can extend that result for all superharmonically weighted Dirichlet spaces. Namely, we have the following result.

This conjecture has been presented as Conjecture 1.1 for classical Dirichlet space [66]. The Brown-Shields conjecture will be stated below.

Theorem 9. *Let $\omega = U_\mu + P_\nu$ be a positive superharmonic weight on $\mathbb{D}$. Let $\mathcal{M}$ be a shift-invariant subspace of $\mathcal{D}_\omega$ such that $\underline{\mathcal{Z}}(\mathcal{M}) \cap \mathbb{T}$ is countable. Then*

$$\mathcal{M} = \mathcal{I}_\mathcal{M}\mathrm{H}^2 \cap \mathcal{M}_\mathrm{E},$$

where $\mathrm{E} = \{\zeta \in \underline{\mathcal{Z}}(\mathcal{M}) \cap \mathbb{T} : c_\omega(\zeta) > 0\}$.

A natural and interesting question arises from (1.11): Determine all functions $f \in \mathcal{D}_\omega$ that satisfy $[f]_{\mathcal{D}_\omega} = \mathcal{D}_\omega$. A function that checks this property is called a cyclic function for $\mathcal{D}_\omega$. The problem of vectors cyclic for $\mathcal{D}_\omega$ is still an open question.
A necessary conditions for cyclicity follow directly from weak-type inequality for capacity c_ω. More precisely, let f be a *cyclic* vector in $\mathcal{D}_\omega$ then f is outer and $c_\omega(\mathcal{Z}(f)) = 0$, where $\mathcal{Z}(f)$ is the zeros set of radial limits of the function f on $\mathbb{T}$. This remark was stated firstly in [15] for the classical Dirichlet space $\mathcal{D}$. In 1984, Brown and Shields [15] conjectured that an outer function $f \in \mathcal{D}$ is cyclic if and only if $c_{P_m}(\mathcal{Z}(f)) = 0$. The Brown-Shields conjecture asserts that an outer function $f \in \mathcal{D}_\omega$ is cyclic if and only if $c_\omega(\mathcal{Z}(f)) = 0$.
It was shown by D. Sarason [68] (see also [39]) that the Brown-Shields conjecture is true for the space $\mathcal{D}_{P_{\delta_\zeta}}$. Recently, O. El-Fallah, Y. Elmadani and K. Kellay [28] proved that this conjecture is true for harmonically weighted Dirichlet space $\mathcal{D}_{P_\nu}$ if $\operatorname{supp}\nu$ is countable. This result remains true for all superharmonically weighted Dirichlet spaces. Formally, we have the following theorem.

Theorem 10. *Let $\omega = U_\mu + P_\nu$ be a positive superharmonic weight on $\mathbb{D}$ and set $\overline{\mu} = \mu + \nu$. Let $f \in \mathcal{D}_\omega$ be an outer function such that $\operatorname{supp}(\overline{\mu}) \cap \mathcal{Z}(f)$ is countable. Then f is cyclic in $\mathcal{D}_\omega$ if and only if $c_\omega(\mathcal{Z}(f)) = 0$.*

Now, let us discuss vectors cyclic under some additional conditions that will involve "capacity zero". H. Hedenmalm and A. Shields [42] proved that, if $\mathcal{Z}(f)$ is countable then f is cyclic in the classical Dirichlet space. While, O. El-Fallah, K. Kellay and T. Ransford [29] replaced "countable" by a condition closer to "capacity zero". They showed precisely that an outer function $f \in \mathcal{D} \cap A(\mathbb{D})$ is cyclic in $\mathcal{D}$, if

$$\int_0 c(\mathrm{E}_t) \frac{\log\log(1/t)}{t\log(1/t)} dt < \infty,$$

where $\mathrm{E} = \mathcal{Z}(f)$ and $\mathrm{E}_t = \{\zeta \in \mathbb{T} : d(\zeta, \mathrm{E}) < t\}$.
To obtain an extension of this result for all superharmonically weighted Dirichlet spaces, we establish a converse of strong-type inequality for c_ω−capacity (see Theorem 4.4.4). Namely, we have the following theorem.

This result was stated in [34] for harmonically weighted Dirichlet spaces.

Theorem 11. *Let ω be a positive superharmonic weight on $\mathbb{D}$. Let f be an outer function in $\mathcal{D}_\omega \cap A(\mathbb{D})$ and set $\mathrm{E} = \mathcal{Z}(f)$. Then f is cyclic in $\mathcal{D}_\omega$, if*

$$\int_0 c_\omega(\mathrm{E}_t) \frac{\log(1/t)}{t} dt < \infty.$$

This result is new even for the standard Dirichlet spaces $\mathcal{D}_{\omega_\alpha}$.

The structure of closed ideals in Banach algebras of analytic functions in the unit disk has been interested by several authors. The first complete description of closed ideals in Banach analytic function algebras was obtained for the disk algebra $A(\mathbb{D})$. The characterization of closed ideals in $A(\mathbb{D})$ was given independently by A. Beurling and W. Rudin [43, 67]. More precisely, they showed that each non-trivial ideal of $A(\mathbb{D})$ will be of the form

Beurling's result on the closed ideals for $A(\mathbb{D})$ has never been published. The ideas are in Beurling [9].

$$\mathcal{I} = \left\{ f \in A(\mathbb{D}) : \; f_{|E_\mathcal{I}} = 0, \text{ and } f \in \theta_\mathcal{I} A(\mathbb{D}) \right\},$$

where $E_\mathcal{I}$ is the intersection of zeros of functions in $\mathcal{I}$ on the unit circle and $\theta_\mathcal{I}$ is the greatest common divisor of the inner parts of the non-zero functions in $\mathcal{I}$. So, we say that the closed ideals, of a given Banach algebra, are standard if every closed ideal is the intersection of that algebra with some closed ideal. Several authors have obtained a complete description of closed ideals in some Banach algebras [12, 13, 49, 52, 53, 70, 73, 74, 76]. They proved that the closed ideals in these spaces are standard. However, J. Esterle [35] showed that there exists a closed ideal in the Winner algebra does not have the above standard form.

For more details about the greatest divisor of the the non-zero inner functions in a given closed ideal see [43, 67].

The last part of this thesis is concerned with the structure of closed ideals of the Banach algebra $\mathcal{D}_\omega \cap \Lambda_\alpha$. Note that A.L. Matheson [53]

Λ_α is the algebra of analytic functions on the unit disk $\mathbb{D}$ that are Lipschitz of ordre $\alpha \in (0,1)$ on $\overline{\mathbb{D}}$.

and F.A. Shamoyan [70] have proved independently that every closed ideal of the Banach algebra Λ_α will be the above standard form. Namely, let $\mathcal{I}$ be a non-trivial closed ideal of Λ_α then

$$\mathcal{I} = \left\{f \in \Lambda_\alpha : \ f_{|E_{\mathcal{I}}} = 0, \text{ and } f \in \theta_{\mathcal{I}} \mathrm{H}^\infty\right\}.$$

Later on, B. Bouya [13] showed an analoguous result for the algebra $\mathcal{D} \cap \Lambda_\alpha$.

The general method of describing the closed ideals in Banach algebras of analytic functions smooth up to the boundary makes use the duality. This reduces the description of the closed ideals to a special approximation theorem. We give an elementary proof of the approximation theorem established by A.L. Matheson in [53] for spaces of Lipschitz analytic functions Λ_α. Our approach allows us to extend this approximation theorem to $\mathcal{D}_\omega \cap \Lambda_\alpha$. As a consequence, we obtain a complete description of closed ideals of the Banach algebra $\mathcal{D}_\omega \cap \Lambda_\alpha$. Let $\mathcal{I} \neq \{0\}$ be a closed ideal of $\mathcal{D}_\omega \cap \Lambda_\alpha$. The inner factor $\theta_{\mathcal{I}}$ of $\mathcal{I}$ is the greatest common divisor of the inner factors of all nonzero elements of $\mathcal{I}$. The zero set $E_{\mathcal{I}}$ of $\mathcal{I}$ is defined by $E_{\mathcal{I}} := \{\zeta \in \mathbb{T} : \ g(\zeta) = 0, \ g \in \mathcal{I}\}$. Formally, we have the following result.

Theorem 12. *Let $\alpha \in (0,1)$ and let ω be a nonnegative superharmonic weight on $\mathbb{D}$. Let $\mathcal{I} \neq (0)$ be a closed ideal of $\mathcal{D}_\omega \cap \Lambda_\alpha$, then*

$$\mathcal{I} = \{f \in \mathcal{D}_\omega \cap \Lambda_\alpha : \ f_{|E_{\mathcal{I}}} = 0, \text{ and } f \in \theta_{\mathcal{I}} H^2\}.$$

Note that the latter result shows that the boundary zero sets (boundary uniqueness sets) of $\mathcal{D}_\omega \cap \Lambda_\alpha$ play a crucial role in the structure of closed ideals in $\mathcal{D}_\omega \cap \Lambda_\alpha$. L. Carleson [18] has shown that the zero sets of Λ_α are the closed sets which not satisfy condition (1.8). In particular, the closed set E_I presented in Theorem 12 is a Carleson set.

Part I

BLASCHKE PRODUCTS, ZERO SETS AND UNIQUENESS SETS

2 BLASCHKE PRODUCTS AND ZERO SETS IN WEIGHTED DIRICHLET SPACES.

In this chapter, we deal with superharmonically weighted Dirichlet spaces $\mathcal{D}_\omega$. First, we prove that the classical Dirichlet space is the largest , among all these spaces, which contains no infinite Blaschke product. Next, we give new sufficient conditions on a Blaschke sequence to be a zero set for $\mathcal{D}_\omega$. Our conditions improves Shapiro-Shields condition for all $\mathcal{D}_\alpha$, when $\alpha \in (0,1)$.

2.1 INTRODUCTION

Let $\mathbb{D}$ be the unit disc of the complex plane $\mathbb{C}$ and let $\mathbb{T} := \partial\mathbb{D}$ be the unit circle. Let dA (resp. dm) be the normalized Lebesgue measure on $\mathbb{D}$ (resp. $\mathbb{T}$). The Hardy space H^2 is the space of analytic functions f on $\mathbb{D}$ such that

$$\|f\|^2_{H^2} := |f(0)|^2 + \int_{\mathbb{D}} |f'(z)|^2 \log(1/|z|) dA(z) < \infty.$$

A weight ω is a function $\omega : \mathbb{D} \to (0, +\infty]$ which is integrable on $\mathbb{D}$ with respect to dA. The weighted Dirichlet space $\mathcal{D}_\omega$ associated with ω is defined by

$$\mathcal{D}_\omega := \left\{ f \in H^2 : \mathcal{D}_\omega(f) := \int_{\mathbb{D}} |f'(z)|^2 \omega(z) dA(z) < \infty \right\}.$$

The space $\mathcal{D}_\omega$ will be endowed by the hilbertian norm $\|f\|^2_\omega := \|f\|^2_{H^2} + \mathcal{D}_\omega(f)$. Let $0 \leq \alpha < 1$ and denote by $\mathcal{D}_\alpha$ the standard Dirichlet space which corresponds to the weight $\omega_\alpha(z) = (1+\alpha)(1-|z|^2)^\alpha$. The classical Dirichlet space is $\mathcal{D}_0$ and will be denoted by $\mathcal{D}$.

In this chapter, we are mainly interested in superhamonically weighted Dirichlet spaces. Several results on these spaces can be found in [4, 8, 28, 32, 33, 72]. In general the description of zero sets remains an open problem, even for the standard Dirichlet spaces $\mathcal{D}_\alpha$ for $\alpha \in [0,1)$. Recall that a sequence $\mathcal{Z} = (z_n)_{n \geq 1} \subset \mathbb{D}$ is a zero set for $\mathcal{D}_\omega$ if there is a function in $\mathcal{D}_\omega$ that vanishes on $\mathcal{Z}$ and nowhere else in $\mathbb{D}$.

We say that $\mathcal{Z} = (z_n)_{n \geq 1} \subset \mathbb{D}$ is a Blaschke sequence if, $\sum_{n \geq 1}(1 - |z_n|^2) < \infty$. The associated Blaschke product B is given by

$$B(z) = \prod_{n \geq 1} \frac{|z_n|}{z_n} \frac{z_n - z}{1 - \overline{z_n} z}, \quad (z \in \mathbb{D}), \tag{2.1}$$

with the convention $|z_n|/z_n = -1$, if $z_n = 0$. It is known [23] that zero sets for H^2 are the Blaschke sequences and that every Blaschke product is bounded, so it belongs in H^2. For the classical Dirichlet space, L. Carleson [17] proved that if $\mathcal{Z}$ satisfies

$$\sum_{n\geq 1} \frac{1}{\log^{1-\varepsilon} 1/(1-|z_n|)} < \infty \tag{2.2}$$

for some $\varepsilon \in (0,1)$, then $\mathcal{Z}$ is a zero set for $\mathcal{D}$. In [71], H.S. Shapiro and A.L. Shields improve this result by proving that if

$$\sum_{n\geq 1} \frac{1}{\log 1/(1-|z_n|)} < \infty,$$

then $\mathcal{Z}$ is a zero set for $\mathcal{D}$. This last result is sharp in the following sense, if $(r_n)_n \subset (0,1)$ such that $\sum 1/|\log(1-r_n)| = \infty$, then there exists a sequence $(\theta_n)_{n\geq 1}$ such that $(r_n e^{i\theta_n})_{n\geq 1}$ is not a zero set for $\mathcal{D}$ (see [55]). Similar results are obtained for all standard Dirichlet spaces $\mathcal{D}_\alpha$ (see for instance [56]).

D. Marshall and C. Sundberg [51] observed that the argument used by Shapiro-Shields works for all Hilbert spaces of analytic functions on $\mathbb{D}$ that enjoy Pick property (see also [2, 69]). Note that S. Shimorin [72] proved that every superharmonically weighted Dirichlet space $\mathcal{D}_\omega$ possesses Pick property. So, if $\mathcal{Z} = (z_n)_{n\geq 1}$ satisfies the Shapiro-Shields condition, namely

$$\sum_{n\geq 1} \frac{1}{K^\omega(z_n, z_n)} < \infty, \tag{2.3}$$

where K^ω is the reproducing kernel of $\mathcal{D}_\omega$, then $\mathcal{Z}$ is a zero set for $\mathcal{D}_\omega$. For $\alpha \in (0,1)$, it is known that the reproducing kernel of $\mathcal{D}_\alpha$ is given by $K^\alpha(z,w) = \frac{1}{(1-z\overline{w})^\alpha}$ and the Shapiro-Shields condition for $\mathcal{D}_\alpha$ becomes

$$\sum_{n\geq 1} (1-|z_n|^2)^\alpha < \infty. \tag{2.4}$$

In fact, condition (2.4) implies that $B \in \mathcal{D}_\alpha$ ([16]). Moreover, if $\mathcal{Z}$ is uniformly separated then condition (2.4) is also necessary ([25, 77]). It is worth mentioning that this result remains true if $\mathcal{Z}$ is only separated ([5, 58]).

For the classical Dirichlet space the situation is quite different. L. Carleson proved in [18] (see also [32]) the following formula

$$\mathcal{D}(Bf) = \mathcal{D}(f) + \sum_{n\geq 1} \int_{\mathbb{T}} \frac{1-|z_n|^2}{|1-\overline{z_n}\zeta|^2} |f(\zeta)|^2 dm(\zeta) \quad (f \in H^2). \tag{2.5}$$

As a consequence, the only Blaschke products in $\mathcal{D}$ are finite Blaschke products.

Our first goal in this chapter is to determine all superharmonically weighted Dirichlet spaces which contain no infinite Blaschke product. We prove that $\mathcal{D}$ is the largest space among all superharmonically weighted Dirichlet spaces which contains no infinite Blaschke product. To state our result let us denote by P_ν the Poisson transform of the positive measure ν on $\mathbb{T}$.

Theorem 2.1.1. *Let ω be a superharmonic weight and let P_ν be the harmonic part of ω. Let h be the derivative of ν with respect to m. The following are equivalent.*

i) *$\mathcal{D}_\omega$ contains no infinite Blaschke product.*

ii) *There exists $c > 0$, such that $h \geq c$ a.e. on $\mathbb{T}$.*

iii) $\liminf_{|z| \to 1^-} \omega(z) > 0.$

iv) $\mathcal{D}_\omega \subset \mathcal{D}.$

The second goal in this chapter is to give some sufficient conditions which ensure that a sequence $\mathcal{Z}$ is a zero set for $\mathcal{D}_\omega$. Taking advantage of the inclusion $\mathcal{D}_\omega \subset H^2$, each zero set for $\mathcal{D}_\omega$ satisfies the Blaschke condition. The converse is in general not true (see Section 2.6). For the standard Dirichlet spaces there are several papers that deal with this problem (see for instance [17, 45, 56, 71]). In Section 2.4, we give two ways to construct functions $f \in \mathcal{D}_\omega$ such that $fB \in \mathcal{D}_\omega$. The first one is the classical Carleson's construction of smooth outer functions (see Theorem 2.4.1). While the second construction is based on potential theory induced by $\mathcal{D}_\omega$. We will construct outer functions in $\mathcal{D}_\omega$ with large real part on a given sequence of closed subsets of $\mathbb{T}$. Namely, if c_ω denotes the capacity associated with $\mathcal{D}_\omega$ (see Subsection 2.4.2), we have the following result which is in the spirit of Theorem 5.1 of [29] (see also [34, 66]).

Theorem 2.1.2. *Let ω be a superharmonic weight. Let $(F_n)_n$ be a family of closed subsets of $\mathbb{T}$ and let $E_n = \cup_{k \geq n} F_k$. There exists a function $f \in \mathcal{D}_\omega$ such that* $\mathrm{Re} f \geq 0$ *on $\mathbb{D}$ and*

$$\mathrm{Re} f \geq \log \frac{1}{c_\omega(E_n)} \; c_\omega - q.e. \text{ on } F_n \text{ for all } n \geq 1.$$

Our main result on zero sets deals with general superharmonic weights ω (radial or nonradial). To state our theorem let us introduce some notations. Let Ψ_ω be the function given by

$$\Psi_\omega(z) = (1 - |z|^2) \int_{\mathbb{D}} \frac{1 - |w|^2}{|1 - \bar{z}w|^2} d\mu(w) + P_\nu(z), \quad (z \in \mathbb{D}),$$

where $\mu = -\Delta\omega$ and P_ν is the harmonic part of ω. Let $\gamma > 0$ and $z \in \mathbb{D} \setminus \{0\}$. The closed arc of $\mathbb{T}$ centered at $z/|z|$ and of length $(1 - |z|)^\gamma$ will be denoted by $I(z, \gamma)$. We also write $I(0, \gamma) = \mathbb{T}$.

Theorem 2.1.3. *Let ω be a superharmonic weight on $\mathbb{D}$. Suppose that there exists $\alpha \in [0, 1)$ such that $\omega(z) = O((1 - |z|^2)^\alpha)$. Let $\mathcal{Z} = (z_n)_{n \geq 1}$ be a sequence of $\mathbb{D}$ such that $\sum_{n \geq 1} (1 - |z_n|^2)^{1-\varepsilon} < \infty$, for some $\varepsilon \in (0, 1)$. Let $\gamma = \frac{\varepsilon}{1-\alpha}$ and let $E_n = \cup_{k \geq n} I(z_k, \gamma)$. If there exists $A > 0$ such that*

$$\sum_{n \geq 1} \Psi_\omega(z_n) c_\omega^A(E_n) < \infty,$$

then $\mathcal{Z}$ is a zero set for $\mathcal{D}_\omega$.

Note that this result allows to give new examples of zero sets for $\mathcal{D}_\omega$, even for the standard spaces $\mathcal{D}_\alpha$. For $\alpha \in (0,1)$, one can see easily that $\Psi_{\omega_\alpha} \asymp \omega_\alpha$. By Theorem 2.1.3, if

$$\sum_{n\geq 1}(1-|z_n|)^\alpha c_\alpha^A(E_n) < \infty,$$

then $\mathcal{Z}$ is a zero set for $\mathcal{D}_\alpha$. So, our condition improves Shapiro-Shields condition and takes into account Arg z_n. A more general result for $\mathcal{D}_\alpha$ is obtained in Theorem 2.4.6.

The chapter is organized as follows. In Section 2, we give an upper estimate of $\|Bf\|_\omega$, where $f \in H^2$ and B is a general Blaschke product. Section 3 is devoted to Blaschke products in $\mathcal{D}_\omega$. We also give in this section the proof of Theorem 2.1.1. In Section 4, we state and prove our main theorems on zero sets for $\mathcal{D}_\omega$. In Section 5, we consider general weighted Dirichlet spaces and give extensions of some previous results. The last section contains some remarks and two problems.

2.2 NORM ESTIMATES

In this section we give an upper estimate of $\|Bf\|_\omega$ which will be useful in the proofs of our main results.

Let ω be a superharmonic weight on $\mathbb{D}$. By Jensen-Riesz representation theorem (see for instance [6, 60]), there exist a positive Borel measure μ on $\mathbb{D}$ and a finite positive Borel measure ν on $\mathbb{T}$ such that

$$\omega(z) = U_\mu(z) + P_\nu(z) \quad (z \in \mathbb{D}), \tag{2.6}$$

where U_μ is the Green potential of μ defined by

$$U_\mu(z) = \int_{\mathbb{D}} \log\left|\frac{1-\overline{w}z}{w-z}\right| d\mu(w)$$

and P_ν is the Poisson transform of ν defined by

$$P_\nu(z) = \int_{\mathbb{T}} \frac{1-|z|^2}{|1-\overline{\zeta}z|^2} d\nu(\zeta).$$

Recall that $\mu = -\Delta\omega$ where Δ is the distributional laplacian operator. It is known [6] that U_μ is not identically infinite if and only if

$$\int_{\mathbb{D}} (1-|z|)d\mu(z) < \infty. \tag{2.7}$$

So, in the sequel we suppose that (2.7) is satisfied. Note that U_μ (resp. P_ν) is called the pure superharmonic (resp. the harmonic) part of ω. We say that ω is purely superharmonic if $\omega = U_\mu$.

Let f be a function in H^2. S. Richter and C. Sundberg extended Carleson's formula (2.5) to all harmonically weighted Dirichlet spaces [32, 63]. In particular, we have

$$\mathcal{D}_{P_\nu}(Bf) = \mathcal{D}_{P_\nu}(f) + \sum_{n\geq 1}(1-|z_n|^2)\int_{\mathbb{T}} \frac{|f(\zeta)|^2}{|1-\overline{z_n}\zeta|^2} d\nu(\zeta), \tag{2.8}$$

where B is the Blaschke product associated with $(z_n)_{n\geq 1}$. On the other hand, using Green's formula [38], it is clear that

$$\int_{\mathbb{D}} |f'(w)|^2 \log\left|\frac{1-z\overline{w}}{z-w}\right| dA(w) = P(|f|^2)(z) - |f(z)|^2 \quad (z \in \mathbb{D}).$$

So, if $\omega = U_\mu$ is a purely superharmonic weight then we have the following formula

$$\begin{aligned} \mathcal{D}_\omega(f) &= \int_{\mathbb{D}}\int_{\mathbb{D}} |f'(w)|^2 \log\left|\frac{1-z\overline{w}}{z-w}\right| dA(w)d\mu(z) \\ &= \int_{\mathbb{D}} \left(P(|f|^2)(z) - |f(z)|^2\right) d\mu(z). \end{aligned} \tag{2.9}$$

In particular, $f \in \mathcal{D}_\omega$ if and only if (2.9) is finite. The latter result was obtained by A. Aleman [3, 4]. See also [8].

Let $\mathcal{Z} = (z_n)_{n\geq 1}$ be a Blaschke sequence of $\mathbb{D}$ and let μ be a positive Borel measure on $\mathbb{D}$. Let $\mathcal{Z}_\mu$ be the function defined by

$$\mathcal{Z}_\mu(\zeta) = \sum_{n\geq 1}(1-|z_n|^2)\int_{\mathbb{D}} \frac{(1-|z|^2)^2}{|1-\bar{z}_n z|^2|1-\bar{z}\zeta|^2} d\mu(z) \quad (\zeta \in \mathbb{T}).$$

The following estimate will be useful in the sequel.

Theorem 2.2.1. *Let $\omega = U_\mu$ be a purely superharmonic weight. Let $f \in \mathcal{D}_\omega$ and let B be the Blaschke product associated with the sequence $\mathcal{Z} = (z_n)_{n\geq 1}$ of $\mathbb{D}$. Then*

$$\|Bf\|_\omega^2 \lesssim \|f\|_\omega^2 + \int_{\mathbb{T}} |f(\zeta)|^2 \mathcal{Z}_\mu(\zeta) dm(\zeta). \tag{2.10}$$

Proof. By equation (2.9), we have

$$\|Bf\|_\omega^2 \asymp \|f\|_\omega^2 + \int_{\mathbb{D}} |f(z)|^2 (1-|B(z)|^2) d\mu(z). \tag{2.11}$$

Let B_k denotes the finite Blaschke product associated with $(z_n)_{1\leq n\leq k}$ and let $B_0 = 1$. We have

$$1-|B_k(z)|^2 = \sum_{n=1}^{k} |B_{n-1}(z)|^2 - |B_n(z)|^2 = \sum_{n=1}^{k} \frac{|B_{n-1}(z)|^2(1-|z_n|^2)(1-|z|^2)}{|1-\bar{z}_n z|^2}.$$

Then

$$1-|B(z)|^2 = \sum_{n=1}^{\infty} \frac{|B_{n-1}(z)|^2(1-|z_n|^2)(1-|z|^2)}{|1-\bar{z}_n z|^2} \leq \sum_{n=1}^{\infty} \frac{(1-|z_n|^2)(1-|z|^2)}{|1-\bar{z}_n z|^2}$$

Combining this inequality with equation (2.11), we obtain

$$\begin{aligned} \|Bf\|_\omega^2 &\lesssim \|f\|_\omega^2 + \sum_{n=1}^{\infty}(1-|z_n|^2)\int_{\mathbb{D}} \frac{|f(z)|^2(1-|z|^2)}{|1-\bar{z}_n z|^2} d\mu(z) \\ &\leq \|f\|_\omega^2 + \sum_{n=1}^{\infty}(1-|z_n|^2)\int_{\mathbb{T}} |f(\zeta)|^2 \int_{\mathbb{D}} \frac{(1-|z|^2)^2}{|\zeta - z|^2|1-\bar{z}_n z|^2} d\mu(z)dm(\zeta) \\ &= \|f\|_\omega^2 + \int_{\mathbb{T}} |f(\zeta)|^2 \mathcal{Z}_\mu(\zeta) dm(\zeta). \end{aligned}$$

The proof is complete. □

Note that if $\omega = U_\mu + P_\nu$ is a general superharmonic weight we have

$$\begin{aligned} \|Bf\|_\omega^2 \quad &\lesssim \quad \|f\|_\omega^2 + \int_{\mathbb{T}} |f(\zeta)|^2 \mathcal{Z}_\mu(\zeta) dm(\zeta) \\ &\quad + \sum_{n\geq 1} (1-|z_n|^2) \int_{\mathbb{T}} \frac{|f(\zeta)|^2}{|1-\overline{z_n}\zeta|^2} d\nu(\zeta), \end{aligned} \tag{2.12}$$

by (2.8) and (2.10).

2.3 BLASCHKE PRODUCTS

Our aim in this section is to prove Theorem 2.1.1. First, we give a sufficient condition which ensures that an infinite Blaschke product belongs to $\mathcal{D}_\omega$.

Let μ be a positive Borel measure on $\mathbb{D}$ satisfying (2.7). The function Ψ_μ is given by

$$\Psi_\mu(z) = (1-|z|^2) \int_{\mathbb{D}} \frac{1-|w|^2}{|1-\overline{w}z|^2} d\mu(w) \quad (z \in \mathbb{D}).$$

Let $\mathcal{Z} = (z_n)_{n\geq 1} \subset \mathbb{D}$ and recall that $\mathcal{Z}$ is said to be uniformly separated if there exists $\delta > 0$ such that

$$\prod_{n:\ n\neq k} \left| \frac{z_n - z_k}{1-\overline{z}_n z_k} \right| \geq \delta, \quad (k \geq 1). \tag{2.13}$$

Theorem 2.3.1. *Let $\omega = U_\mu + P_\nu$ be a superharmonic weight, where μ is a positive Borel measure on $\mathbb{D}$ satisfying (2.7) and ν be a finite positive Borel measure on $\mathbb{T}$. Let B be the Blaschke product associated with $\mathcal{Z} = (z_n)_{n\geq 1} \subset \mathbb{D}$. Then $B \in \mathcal{D}_\omega$, if*

$$\sum_{n\geq 1} \Psi_\mu(z_n) + P_\nu(z_n) < \infty. \tag{2.14}$$

If in addition, $\mathcal{Z}$ is uniformly separated then (2.14) is also necessary.

Proof. Applying (2.12), with $f = 1$, we get

$$\begin{aligned} \|B\|_\omega^2 \quad &\leq \quad 1 + \int_{\mathbb{T}} \mathcal{Z}_\mu(\zeta) dm(\zeta) + \sum_{n\geq 1} \int_{\mathbb{T}} \frac{1-|z_n|^2}{|1-\overline{z}_n\zeta|^2} d\nu(\zeta) \\ &= \quad 1 + \sum_{n\geq 1} \Psi_\mu(z_n) + P_\nu(z_n) < \infty. \end{aligned}$$

This proves that $B \in \mathcal{D}_\omega$. If $\mathcal{Z}$ is uniformly separated, by [25], we have

$$1 - |B(z)|^2 \asymp \sum_{n\geq 1} \frac{(1-|z_n|^2)(1-|z|^2)}{|1-\overline{z}_n z|^2}, \quad (z \in \mathbb{D}).$$

The second assertion comes from (2.8) and (2.9). □

The following lemma will be useful in the proof of the main result.

Lemma 2.3.2. *Let $\omega = U_\mu + P_\nu$ be a superharmonic weight, where μ is a positive Borel measure on $\mathbb{D}$ satisfying (2.7) and ν is a finite positive Borel measure on $\mathbb{T}$. The following are equivalent.*

i) $\mathcal{D}_\omega$ *contains an infinite Blaschke product.*

ii) $\liminf_{|z|\to 1^-} \big(\Psi_\mu(z) + P_\nu(z)\big) = 0.$

Proof. Suppose that there exists an infinite Blaschke sequence $\mathcal{Z} = (z_n)_{n\geq 1} \subset \mathbb{D}$ such that the associated Blaschke product B belongs to $\mathcal{D}_\omega$. Let $(a_n)_{n\geq 1}$ be a uniformly separated infinite subsequence of $\mathcal{Z}$ and let B_1 be the associated Blaschke product. Clearly, from (2.8) and (2.9), $B_1 \in \mathcal{D}_\omega$. By Theorem 2.3.1, we have $\sum_{n\geq 1} \Psi_\mu(a_n) + P_\nu(a_n) < \infty$. In particular, $\liminf_{|z|\to 1^-} \big(\Psi_\mu(z) + P_\nu(z)\big) = 0$.

Conversely, suppose that $\liminf_{|z|\to 1^-} \big(\Psi_\mu(z) + P_\nu(z)\big) = 0$. Then there exists $(z_n)_{n\geq 1}$ such that $\Psi_\mu(z_n) + P_\nu(z_n) < 1/2^n$. By Theorem 2.3.1, the Blaschke product associated with $(z_n)_{n\geq 1}$ belongs to $\mathcal{D}_\omega$. □

Proof of Theorem 2.1.1. The following implications $ii) \implies iii) \implies iv)$ are obvious. $iv) \implies i)$ comes from the fact that $\mathcal{D}$ contains no infinite Blaschke product.

It remains to prove that $i) \implies ii)$. To this end, suppose that $ii)$ is not satisfied and suppose that $\omega = U_\mu + P_\nu$. For $\varepsilon > 0$, we have $m(\{\zeta \in \mathbb{T} : \ h(\zeta) < \varepsilon\}) := a > 0$. Let $(r_n)_{n\geq 1} \in (0,1)$ be a sequence which converges to 1. By Fatou's lemma, we have $\lim_n P_\nu(r_n\zeta) = h(\zeta)$ *a.e.*. Then for n, large enough, we have

$$m(\{\zeta \in \mathbb{T} : \ P_\nu(r_n\zeta) < \varepsilon\}) \geq a/2.$$

Let $u_n(\zeta) = \Psi_\mu(r_n\zeta)$. We have

$$\|u_n\|_1 := \int_{\mathbb{T}} u_n(\zeta)dm(\zeta) \asymp \int_{\mathbb{D}} \frac{(1-r_n)}{1-r_n|z|}(1-|z|)d\mu(z).$$

Since the measure $(1-|z|)d\mu(z)$ is finite, $\lim_n \|u_n\|_1 = 0$. In particular we have, for large n,

$$m(\{\zeta \in \mathbb{T}\ u_n(\zeta) \geq \varepsilon\}) < a/2.$$

Consequently,

$$m(\{\zeta \in \mathbb{T} : \ P_\nu(r_n\zeta) < \varepsilon\} \cap \{\zeta \in \mathbb{T} : \ u_n(\zeta) < \varepsilon\}) > 0.$$

This implies that $\liminf_{|z|\to 1^-} \Psi_\mu(z) + P_\nu(z) = 0$. By applying Lemma 2.3.2 we obtain the result. □

2.4 ZERO SETS FOR SUPERHAMONICALLY WEIGHTED DIRICHLET SPACES

Since $\mathcal{D}_\omega$ possesses a division property it is clear that a Blaschke sequence $\mathcal{Z}$ is a zero set for $\mathcal{D}_\omega$ if and only if there exists an outer function $f \in \mathcal{D}_\omega$ such that $Bf \in \mathcal{D}_\omega$. In this section we give two ways to construct such functions.

2.4.1 *Regularization using smooth functions*

We begin this subsection by recalling some standard facts. Let ρ be an increasing function such that $\rho(0) = 0$. We say that a closed subset E of $\mathbb{T}$ is a $\rho-$Carleson set if

$$\int_{\mathbb{T}} \log \frac{1}{\rho(d(\zeta, E))} dm(\zeta) < \infty.$$

It is known (see for instance [11, 18, 76]) that if $\rho(t) = O(t^\varepsilon)$, for some $\varepsilon > 0$, then there exists an analytic function f on $\mathbb{D}$ with bounded derivatives and such that $|f(z)| = O(\rho(d(z, E))$.
The aim of the following theorem is to exhibit conditions ensuring the membership of fB to $\mathcal{D}_\omega$. For simplicity, we state our result only in the radial case.

Theorem 2.4.1. *Let ρ be an increasing function such that $\rho(t) = O(t^\varepsilon)$ for some $\varepsilon > 0$. Let E be a $\rho-$Carleson set. Let $\omega = U_\mu$ be a purely superharmonic radial weight on $\mathbb{D}$. Let $\mathcal{Z} = (z_n)_{n \geq 1}$ be a Blaschke sequence such that*

$$\sum_{n \geq 1} \Psi_\mu(z_n) \rho(16 d(z_n, E)) < \infty. \tag{2.15}$$

Then $\mathcal{Z}$ is a zero set for $\mathcal{D}_\omega$.

We will need the following lemma, which is due to K. Kellay and J. Mashreghi [45].

Lemma 2.4.2. *Let ρ, E satisfying the conditions of Theorem 2.4.1. Then, for $z \in \mathbb{D}$, we have*

$$\int_{\mathbb{T}} \frac{1 - |z|^2}{|1 - \bar{z}\zeta|^2} \rho(d(\zeta, E)) dm(\zeta) \lesssim \rho(2d(z, E)) + (1 - |z|^2) \int_{2d(z,E)}^{2} \frac{\rho(t)}{t^2} dt.$$

Proof of Theorem 2.4.1. Since $\rho-$Carleson sets and $\rho^{2/\varepsilon}-$Carleson sets are the same, we suppose that $\rho(t) = O(t^2)$. By the above discussion, there exists a function $f \in \mathcal{D}_\omega$ such that $|f(z)|^2 \leq \rho(d(z, E))$. We have

$$\begin{aligned}
& \int_{\mathbb{T}} |f(\zeta)|^2 \mathcal{Z}_\mu(\zeta) dm(\zeta) \\
\leq \quad & \int_{\mathbb{T}} \rho(d(\zeta, E)) \mathcal{Z}_\mu(\zeta) dm(\zeta) \\
= \quad & \sum_{n \geq 1} (1 - |z_n|^2) \int_{\mathbb{D}} \frac{(1 - |z|^2)^2}{|1 - \bar{z}_n z|^2} \int_{\mathbb{T}} \frac{\rho(d(\zeta, E))}{|1 - \bar{z}\zeta|^2} dm(\zeta) d\mu(z).
\end{aligned}$$

By Lemma 2.4.2 we have

$$\int_{\mathbb{T}} \frac{1-|z|^2}{|1-\bar{z}\zeta|^2}\rho(d(\zeta,E))dm(\zeta) \lesssim \rho(2d(z,E)) + (1-|z|^2).$$

Then

$$\begin{aligned} &\int_{\mathbb{D}} \frac{(1-|z|^2)^2}{|1-\bar{z}_n z|^2}\int_{\mathbb{T}} \frac{\rho(d(\zeta,E))}{|1-\bar{z}\zeta|^2}dm(\zeta)d\mu(z) \\ \lesssim\ & \int_{\mathbb{D}} \frac{(1-|z|^2)}{|1-\bar{z}_n z|^2}\rho(2d(z,E))d\mu(z) + \int_{\mathbb{D}} \frac{(1-|z|^2)^2}{|1-\bar{z}_n z|^2}d\mu(z). \end{aligned}$$

Since ω is radial, $d\mu(r\zeta) = d\lambda(r)dm(\zeta)$, where λ is a positive Borel measure on $[0,1]$. Recall that $\omega \not\equiv +\infty$ is equivalent to

$$\int_{\mathbb{D}} (1-|z|)d\mu(z) = \int_0^1 (1-r)d\lambda(r) < \infty.$$

Then we have

$$\int_{\mathbb{D}} \frac{(1-|z|^2)^2}{|1-\overline{z_n}z|^2}d\mu(z) = \int_0^1\int_{\mathbb{T}} \frac{(1-r^2)^2}{|1-r\overline{z_n}\zeta|^2}dm(\zeta)d\lambda(r) \lesssim \int_0^1 (1-r)d\lambda(r) < \infty.$$

Note that $d(z,E) \le \max(2d(z/|z|,E), 2(1-|z|))$. Using Lemma 2.4.2 again, we get

$$\begin{aligned} &\int_{\mathbb{D}} \frac{(1-|z|^2)}{|1-\overline{z_n}z|^2}\rho(2d(z,E))d\mu(z) \\ \le\ & \int_0^1\int_{\mathbb{T}} \frac{(1-r^2)}{|1-\overline{z_n}r\zeta|^2}\rho(4d(\zeta,E))dm(\zeta)d\lambda(r) \\ &+\int_0^1\int_{\mathbb{T}} \frac{(1-r^2)}{|1-\overline{z_n}r\zeta|^2}\rho(4(1-r))dm(\zeta)d\lambda(r) \\ \lesssim\ & \int_0^1 \frac{1-r}{1-r|z_n|}\rho(8d(rz_n,E))d\lambda(r) + \int_0^1 (1-r)d\lambda(r). \end{aligned}$$

Combining these inequalities, we obtain

$$\begin{aligned} \int_{\mathbb{T}} |f(\zeta)|^2 \mathcal{Z}_\mu(\zeta)dm(\zeta) \quad \lesssim \quad & \sum_{n\ge1}(1-|z_n|^2)\int_0^1 \frac{1-r}{1-r|z_n|}\rho(8d(rz_n,E))d\lambda(r) \\ & + \sum_{n\ge1}(1-|z_n|^2). \end{aligned}$$

Since ρ is increasing,

$$\rho(8d(rz_n,E)) \le \rho(16d(z_n,E)) + \rho(8(1-r)) \lesssim \rho(16d(z_n,E)) + (1-r)^2.$$

We get

$$\begin{aligned} &\int_{\mathbb{T}} |f(\zeta)|^2 \mathcal{Z}_\mu(\zeta)dm(\zeta) \\ \lesssim\ & \sum_{n\ge1}(1-|z_n|^2)\left(\int_0^1 \frac{1-r}{1-r|z_n|}d\lambda(r)\right)\rho(16d(z_n,E)) + \sum_{n\ge1}(1-|z_n|^2) \\ \asymp\ & \sum_{n\ge1}\Psi_\mu(z_n)\rho(16d(z_n,E)) + \sum_{n\ge1}(1-|z_n|^2). \end{aligned}$$

By Theorem 2.2.1 and (2.15), $Bf \in \mathcal{D}_\omega$. The proof is complete. □

Let $\Lambda = \{(\alpha,\beta) : \ \alpha \in (0,1), \ \beta \in (-\infty,+\infty)\} \cup \{0\} \times (0,+\infty) \cup \{1\} \times (-\infty,0)$ and consider

$$\omega_{\alpha,\beta}(z) = \frac{(1-|z|^2)^\alpha}{\log^\beta(1/(1-|z|))}, \quad (\alpha,\beta) \in \Lambda. \tag{2.16}$$

Corollary 1. *Let ρ be an increasing function such that $\rho(t) = O(t^\varepsilon)$ for some $\varepsilon > 0$ and let E be a $\rho-$Carleson set. Set $(\alpha,\beta) \in \Lambda$ and suppose that a Blaschke sequence $\mathcal{Z} = (z_n)_{n\geq 1}$ of $\mathbb{D}$ satisfies*

$$\sum_{n\geq 1} \omega_{\alpha,\beta}(z_n)\rho(16d(z_n,E)) < \infty.$$

Then $\mathcal{Z}$ is a zero set for $\mathcal{D}_{\omega_{\alpha,\beta}}$.

Proof. By a straightforward computation we get, for all $(\alpha,\beta) \in \Lambda$, that $\Psi_{\omega_{\alpha,\beta}} \asymp \omega_{\alpha,\beta}$. The desired result comes from Theorem 2.4.1. $\square$

2.4.2 *Regularization using potential theory*

Let $\omega = U_\mu + P_\nu$ be a superharmonic weight on $\mathbb{D}$. Now we introduce the notion of capacity associated with $\mathcal{D}_\omega$. The space $\mathcal{D}^h_\omega$ is given by

$$\mathcal{D}^h_\omega := \{f \in L^2(\mu) : \ \|f\|^2_\omega := \|f\|^2_{L^2(m)} + \mathcal{D}_\omega(f) < \infty\},$$

where

$$\begin{aligned}\mathcal{D}_\omega(f) \ &= \ \int_{\mathbb{T}}\int_{\mathbb{T}} |f(\zeta)-f(\eta)|^2 A_\mu(\zeta,\eta) dm(\zeta)dm(\eta) \\ &\quad + \int_{\mathbb{T}}\int_{\mathbb{T}} \frac{|f(\zeta)-f(\eta)|^2}{|\zeta-\eta|2} d\nu(\zeta)dm(\eta)\end{aligned}$$

and

$$A_\mu(\zeta,\eta) = \int_{\mathbb{D}} \frac{(1-|z|^2)}{|1-\overline{\zeta}z|^2}\frac{(1-|z|^2)}{|1-\overline{z}\eta|^2} d\mu(z).$$

Note that, the real part of $\mathcal{D}^h_\omega$ is a Dirichlet space in the sense of Beurling-Deny (see [10, 37]). So, the capacity c_ω associated with $\mathcal{D}^h_\omega$ is defined by

$$c_\omega(E) = \inf\{\|f\|^2_\omega : \ f \in \mathcal{D}^h_\omega \text{ and } f \geq 1 \ a.e \text{ on a neighborhood of } E \ \}.$$

By the definition, c_ω is inner, that is

$$c_\omega(E) = \inf\{c_\omega(U) : U \text{ is open and } E \subset U\}.$$

We also have the following properties

- $c_\omega(\emptyset) = 0$ and $c_\omega(E_1) \leq c_\omega(E_2)$, whenever $E_1 \subset E_2 \subset \mathbb{T}$.
- Let $(A_n)_n$ be an increasing sequence of subsets of $\mathbb{T}$. Then $c_\omega(\cup_n A_n) = \lim\limits_{n\to\infty} c_\omega(A_n)$.

- Let (K_n) be a decreasing sequence of compacts subset of $\mathbb{T}$. Then $c_\omega(\cap_n K_n) = \lim_{n\to\infty} c_\omega(K_n)$.

According to Choquet's theorem, we have

$$c_\omega(E) = \sup\{c_\omega(K) : K \text{ is compact and } K \subset E\},$$

where E is a Borelian subset of $\mathbb{T}$.
A property is said to be satisfied $c_\omega-$ quasi everywhere, if there exists a subset $N \subset \mathbb{T}$ with $c_\omega(N) = 0$ and such that the property is satisfied on $\mathbb{T} \setminus N$. We say that a function f defined on $\mathbb{T}$ is quasi continuous if for every $\varepsilon > 0$ there exists an open subset U with $c_\omega(U) < \varepsilon$ and such that f is continuous on $\mathbb{T} \setminus U$. It is known, from the general potential theory associated with Dirichlet forms, that for each function $f \in \mathcal{D}^h_\omega$, there exists a quasi continuous function $\tilde{f} \in \mathcal{D}^h_\omega$ such that $\tilde{f} = f$ a.e.. Note also that if f and g are two quasi continuous functions of $\mathcal{D}^h_\omega$ such that $f = g$ a.e. on an open subset U then $f = g\ c_\omega - q.e.$ on U. Using these facts one can see that, for every subset E of $\mathbb{T}$, we have

$$c_\omega(E) = \inf\{\|f\|^2_\omega :\ f \in \mathcal{D}^h_\omega \text{ and } \tilde{f} \geq 1\ c_\omega - q.e.\text{on } E\ \}.$$

Now, we introduce the notion of potential which will play an important role in our construction. We say that a positive Borel measure λ on $\mathbb{T}$ is of finite energy (with respect to $\mathcal{D}^h_\omega$) if there exists a constant $C > 0$ such that

$$\int |f(\zeta)| d\lambda(\zeta) \leq C\|f\|_\omega, \quad (f \in \mathcal{D}^h_\omega \cap \mathcal{C}(\mathbb{T})),$$

where $\mathcal{C}(\mathbb{T})$ is the space of continuous functions on $\mathbb{T}$. In particular the linear form $f \to \int f(\zeta) d\lambda(\zeta)$ can be extended to the whole space $\mathcal{D}^h_\omega$. Consequently, there exists a unique (since $\mathcal{D}^h_\omega \cap \mathcal{C}(\mathbb{T})$ is dense in $\mathcal{D}^h_\omega$) function $g_\lambda \in \mathcal{D}^h_\omega$ such that

$$\langle g_\lambda, f\rangle = \int f(\zeta) d\lambda(\zeta) \quad (f \in \mathcal{D}^h_\omega \cap \mathcal{C}(\mathbb{T})).$$

Let E be a closed subset of $\mathbb{T}$. The set of positive Borel measures supported by E and with finite energy with respect to $\mathcal{D}^h_\omega$, will be denoted by $S_\omega(E)$. It is known that if $\lambda \in S_\omega(\mathbb{T})$, then λ charges no sets of $c_\omega-$ capacity zero. In particular if $f \in \mathcal{D}^h_\omega$, then $\tilde{f} \in L^1(\lambda)$ and

$$\langle g_\lambda, f\rangle = \int \tilde{f}(\zeta) d\lambda(\zeta) \quad (f \in \mathcal{D}^h_\omega). \tag{2.17}$$

The function g_λ is called the potential of λ with respect $\mathcal{D}^h_\omega$. We have the following fundamental theorem.

Theorem 2.4.3. *Let E be a closed subset of $\mathbb{T}$ such that $c_\omega(E) > 0$. There exists a unique measure $\lambda_E \in S_\omega(E)$ such that*

$$c_\omega(E) = \|g_{\lambda_E}\|^2_\omega = \lambda_E(E).$$

Moreover, we have $0 \leq g_{\lambda_E} \leq 1$ and $\tilde{g}_{\lambda_E} = 1\ c_\omega - q.e.$ on E.

A direct and important consequence of this theorem is the following expression of the capacity of closed sets in terms of measures with finite energy

$$c_\omega(E) = \sup\{\lambda(E) :\ \lambda \in S_\omega(E) \text{ and } \tilde{g}_\lambda \leq 1\ c_\omega - q.e.\},$$

where E is a closed subset of $\mathbb{T}$. For more details we refer to [37].

Let K^ω be the reproducing kernel of $\mathcal{D}_\omega$. The Poisson transform of a function $f \in \mathcal{D}^h_\omega$ will be also denoted by f. Let $f^+ = \langle f, K^\omega\rangle$ the analytic part of f. Since $\overline{f^-} := \overline{f - f^+} \in \mathcal{D}_\omega$, we have

$$f(z) = \langle f, 2\mathrm{Re}K^\omega_z - 1\rangle \quad (z \in \mathbb{D}).$$

Let $\lambda \in S_\omega(\mathbb{T})$ and let $f_\lambda =: g^+_\lambda$. We have

$$g_\lambda(z) = \int_{\mathbb{T}} (2\mathrm{Re}K^\omega_z(\zeta) - 1) d\lambda(\zeta) \text{ and } f_\lambda(z) = \int_{\mathbb{T}} K^\omega_z(\zeta) d\lambda(\zeta) \quad (z \in \mathbb{D}).$$

From the above discussion we have the following result.

Lemma 2.4.4. *Let ω be a superharmonic weight and let E be a closed subset of $\mathbb{T}$ such that $c_\omega(E) > 0$. Then $\mathrm{Re}f_{\lambda_E} \geq 0$, $\mathrm{Re}f_{\lambda_E} = 1\ c_\omega - q.e.$ on E and $\|f_{\lambda_E}\|^2_\omega \asymp c_\omega(E)$.*

The following lemma will be used in the proof of Theorem 2.1.2.

Lemma 2.4.5. *Let ω be a superharmonic weight. Let $(E_n)_n$ be a (finite or infinite) decreasing sequence of closed subsets of $\mathbb{T}$ such that $c_\omega(E_n) > 0$ for each n. There exists a function $f \in \mathcal{D}_\omega$ such that $\mathrm{Re}f \geq 0$ on $\mathbb{D}$, $\|f\|_\omega \leq 1$ and*

$$\mathrm{Re}f \geq \frac{1}{4}\log\frac{1}{c_\omega(E_n)} \quad c_\omega - q.e. \text{ on } E_n \text{ for all } n.$$

Proof. Let $\Lambda_p = \{n :\ 2^p \leq \log(1/c_\omega(E_n)) < 2^{p+1}\}$. Write $\{p :\ \Lambda_p \neq \emptyset\} = \{p_j,\ j \geq 1\}$ such that $p_j < p_{j+1}$ and put $n_j = \min \Lambda_{p_j}$. Then we have

$$2^{p_j} \leq \log\frac{1}{c_\omega(E_n)} < 2^{p_j+1}, \quad n_j \leq n < n_{j+1}.$$

Let $f = \sum_{j\geq 1} 2^{p_j - 1} f_j$, where $f_j = f_{\lambda_{E_{n_j}}}$. We have

$$\|f\|_\omega \leq \sum_{j\geq 1} 2^{p_j-1}\|f_j\|_\omega \leq \sum_{j\geq 1} 2^{p_j-1} c_\omega^{1/2}(E_j) \leq \sum_{j\geq 1}\frac{2^{p_j-1}}{2^{2^{p_j-1}}} \leq 1.$$

Let $n \geq 1$ and let j be such that $n_j \leq n < n_{j+1}$. Since $E_n \subset E_{n_j}$ we have $\mathrm{Re}f_j = 1$, $c_\omega-$ q.e. on E_n. We get

$$\mathrm{Re}f \geq 2^{p_j-1} \geq \frac{1}{4}\log\frac{1}{c_\omega(E_n)}, \quad c_\omega\text{–q.e. on } E_n.$$

□

Proof of Theorem 2.1.2. Obviously, one can suppose that $c_\omega(F_n) > 0$, for all $n \geq 1$. Let $E_j^n = \cup_{j\leq k\leq n} F_k$. By Lemma 2.4.5, there exists a sequences $(f_n)_n \subset \mathcal{D}_\omega$ such that $\mathrm{Re} f_n \geq 0$ *a.e.* on $\mathbb{T}$,

$$\sup_n \|f_n\|_\omega < \infty \quad \text{and} \quad \mathrm{Re} f_n \geq \log \frac{1}{c_\omega(E_j^n)}\ c_\omega - q.e. \text{ on } F_j\ (1 \leq j \leq n).$$

Since $E_j^n \subset E_j$, $\mathrm{Re}\ f_n \geq \log \frac{1}{c_\omega(E_j)}\ c_\omega - q.e.$ on F_j $(1 \leq j \leq n)$. One can extract from $(f_n)_n$ a subsequence which converges weakly to a quasi-continuous function $f \in \mathcal{D}_\omega$. Since weak convergence implies uniform convergence on compact subsets of $\mathbb{D}$, we have $\mathrm{Re} f \geq 0$ on $\mathbb{D}$. By a standard argument, it is easy to see that $\mathrm{Re} f \geq \log \frac{1}{c_\omega(E_j)}\ c_\omega - q.e.$ on F_j for $j \geq 1$. Hence the proof is complete. □

Before giving the proof of Theorem 2.1.3 remark that the condition $\omega(z) = U_\mu(z) + P_\nu(z) = O((1-|z|)^\alpha)$ implies that

$$\nu(I) \lesssim (1-|z|)P_\nu(z) = O(m(I)^{1+\alpha}),$$

where I is an arbitrary arc of $\mathbb{T}$ and $z \in \mathbb{D} \setminus \{0\}$ such that $z/|z|$ is the center of I and $1-|z|$ is the length of I. This implies that $\nu = 0$ if $\alpha > 0$ and $d\nu = \phi dm$ for some bounded function ϕ, if $\alpha = 0$.

Proof of Theorem 2.1.3. By Theorem 2.1.2, there exists a function $f \in \mathcal{D}_\omega$ such that $\mathrm{Re} f \geq 0$ *a.e.* on $\mathbb{T}$ and $\mathrm{Re} f \geq \log(1/c_\omega(E_n))\ c_\omega - q.e.$ on $I(z_n, \gamma)$. Clearly, the bounded analytic function $h =: e^{-Af} \in \mathcal{D}_\omega$ and $|h| \leq c_\omega(E_n)^A\ c_\omega - q.e.$ on $I(z_n, \gamma)$. Our goal is to prove that $Bh \in \mathcal{D}_\omega$. By Theorem 2.2.1 and equality (2.9) it suffices to prove that

$$\sum_{n\geq 1} \int_{\mathbb{T}} \int_{\mathbb{D}} |h(\zeta)|^2 \frac{(1-|z_n|^2)((1-|z|^2)^2}{|1-\overline{z_n}z|^2|1-\overline{z}\zeta|^2} d\mu(z) dm(\zeta) < \infty, \tag{2.18}$$

and

$$\sum_{n\geq 1} \int_{\mathbb{T}} |h(\zeta)|^2 \frac{1-|z_n|^2}{|1-\overline{z}_n\zeta|^2} d\nu(\zeta) < \infty. \tag{2.19}$$

Since $|h| \leq c_\omega(E_n)^A\ c_\omega - q.e.$ on $I(z_n, \gamma)$, we have

$$\int_{I(z_n,\gamma)} \int_{\mathbb{D}} |h(\zeta)|^2 \frac{(1-|z_n|^2)((1-|z|^2)^2}{|1-\overline{z_n}z|^2|1-\overline{z}\zeta|^2} d\mu(z) dm(\zeta) \leq \Psi_\mu(z_n) c_\omega^{2A}(E_n)$$

On the other hand, we have

$$\begin{aligned} U_\mu(w) &= \int_{\mathbb{D}} \log \frac{1}{|\varphi_z(w)|} d\mu(z) \\ &\geq \int_{\mathbb{D}} (1-|\varphi_z(w)|) d\mu(z) \\ &\asymp \int_{\mathbb{D}} \frac{(1-|z|^2)(1-|w|^2)}{|1-\overline{z}w|^2} d\mu(z). \end{aligned}$$

Let D_z be the disc centered at z and of radius $\frac{1-|z|}{4}$. We get

$$\begin{aligned}
&\int_{\mathbb{D}} \frac{(1-|z_n|^2)U_\mu(w)}{|1-\overline{z}_n w|^2|1-\overline{w}\zeta|^2} dA(w)\\
\gtrsim\ & \int_{\mathbb{D}}\int_{\mathbb{D}} \frac{(1-|z_n|^2)}{|1-\overline{z}_n w|^2|1-\overline{w}\zeta|^2}\frac{(1-|z|^2)(1-|w|^2)}{|1-\overline{z}w|^2} dA(w)d\mu(z)\\
\geq\ & \int_{\mathbb{D}}\int_{D_z} \frac{(1-|z_n|^2)}{|1-\overline{z}_n z|^2|1-\overline{z}\zeta|^2}\frac{(1-|z|^2)(1-|w|^2)}{|1-\overline{z}w|^2} dA(w)d\mu(z)\\
\asymp\ & \int_{\mathbb{D}} \frac{(1-|z_n|^2)(1-|z|^2)^2}{|1-\overline{z}_n z|^2|1-\overline{z}\zeta|^2} d\mu(z).
\end{aligned}$$

Since h is bounded and $U_\mu(z) \leq \omega(z) = O((1-|z|)^\alpha)$, we deduce that

$$\begin{aligned}
\int_{\mathbb{D}} |h(\zeta)|^2 \frac{(1-|z_n|^2)(1-|z|^2)^2}{|1-\overline{z}_n z|^2|1-\overline{z}\zeta|^2} d\mu(z) &\lesssim \int_{\mathbb{D}} \frac{(1-|z_n|^2)((1-|z|^2)^2}{|1-\overline{z}_n z|^2|1-\overline{z}\zeta|^2} d\mu(z)\\
&\lesssim \int_{\mathbb{D}} \frac{(1-|z_n|^2)U_\mu(z)}{|1-\overline{z}_n z|^2|1-\overline{z}\zeta|^2} dA(z)\\
&\lesssim \frac{(1-|z_n|^2)}{|1-\overline{z}_n\zeta|^{2-\alpha}}.
\end{aligned}$$

Let $J_n := \mathbb{T} \setminus I(z_n,\gamma)$. We have

$$\begin{aligned}
\int_{J_n}\int_{\mathbb{D}} |h(\zeta)|^2 \frac{(1-|z_n|^2)((1-|z|^2)^2}{|1-\overline{z}_n z|^2|1-\overline{z}\zeta|^2} d\mu(z)dm(\zeta) &\lesssim \int_{J_n} \frac{(1-|z_n|^2)}{|1-\overline{z}_n\zeta|^{2-\alpha}} dm(\zeta)\\
&\asymp \frac{1-|z_n|^2}{dist^{1-\alpha}(z_n,J_n)}\\
&\asymp (1-|z_n|^2)^{1-\gamma(1-\alpha)}\\
&= (1-|z_n|^2)^{1-\varepsilon},
\end{aligned}$$

which completes the proof of (2.18).

To prove (2.19) we suppose that $\alpha = 0$, otherwise $\nu = 0$. Let $d\nu = \phi dm$, where ϕ is a bounded function. We have

$$\begin{aligned}
&\sum_{n\geq 1}\int_{\mathbb{T}} |h(\zeta)|^2 \frac{1-|z_n|^2}{|1-\overline{z}_n\zeta|^2} d\nu(\zeta)\\
\lesssim\ & \sum_{n\geq 1}\left(\int_{I(z_n,\gamma)} |h(\zeta)|^2 \frac{1-|z_n|^2}{|1-\overline{z}_n\zeta|^2} d\nu(\zeta) + \int_{J_n} \frac{1-|z_n|^2}{|1-\overline{z}_n\zeta|^2} dm(\zeta)\right)\\
\lesssim\ & \sum_{n\geq 1} P_\nu(z_n) c^A_\omega(E_n) + \sum_{n\geq 1}(1-|z_n|)^{1-\varepsilon},
\end{aligned}$$

and (2.19) is proved. □

Applying the latter result to $\omega_{\alpha,\beta}$, which is given by (2.16), we get

Corollary 2. *Let $(\alpha,\beta) \in \Lambda$. Let $\mathcal{Z} = (z_n)_{n\geq 1}$ be a sequence of $\mathbb{D}$ such that $\sum_{n\geq 1}(1-|z_n|^2)^{1-\varepsilon} < \infty$, for some $\varepsilon \in (0,1)$. Let $\gamma = \frac{\varepsilon}{1-\alpha}$ and let $E_n = \cup_{k\geq n} I(z_k,\gamma)$. If there exists $A > 0$ such that*

$$\sum_{n\geq 1} \omega_{\alpha,\beta}(z_n) c^A_{\omega_{\alpha,\beta}}(E_n) < \infty,$$

then $\mathcal{Z}$ is a zero set for $\mathcal{D}_{\omega_{\alpha,\beta}}$.

Theorem 2.4.6. *Let $\alpha, \varepsilon \in (0,1)$ and let $\gamma = \frac{\varepsilon}{1-\alpha}$. Let $\mathcal{Z} = (z_n)_{n\geq 1}$ be a sequence of $\mathbb{D}$ and let $E_n = \cup_{k\geq n} I(z_k, \gamma)$. Suppose that $\sum_{n\geq 1}(1-|z_n|)^{1-\varepsilon} < \infty$. If*

$$\sum_{n\geq 1}(1-|z_n|)^{\alpha} \exp\left(-\log^{1/\beta}(1/c_\alpha(E_n))\right) < \infty,$$

for some $\beta \in (\alpha, 1)$, then $\mathcal{Z}$ is a zero set for $\mathcal{D}_\alpha$.

Proof. Recall that the reproducing kernel of $\mathcal{D}_\alpha$ is given by $K^\alpha(z,w) = \frac{1}{(1-z\overline{w})^\alpha}, z, w \in \mathbb{D}$. Let $f \in \mathcal{D}_\alpha$ be the function constructed in Theorem 2.1.2. We have $\mathrm{Re} f \geq 0$ *a.e.* on $\mathbb{T}$ and $\mathrm{Re} f \geq \log(1/c_\alpha(\cup_{j\geq n} I_j))$ $c_\alpha - q.e.$ on I_n. Since $|\mathrm{Arg}(K^\alpha(z,w))| < \frac{\pi}{2}\alpha$, it is easy to see that $\mathrm{Re} f^{1/\beta} \geq 0$. This implies that $h =: e^{-f^{1/\beta}} \in \mathcal{D}_\alpha$. We get the result by following the proof of Theorem 2.1.3. □

Note that for the classical Dirichlet space $\mathcal{D}$ the associated capacity c_0 is comparable with the logarithmic capacity and the reproducing kernel is given by

$$K(z,w) = \frac{1}{z\overline{w}} \log \frac{1}{1-z\overline{w}} \qquad (z, w \in \mathbb{D}).$$

Clearly, we have $|\mathrm{Im} K(z,w)| \leq \frac{\pi}{2}$. Our method implies the following result.

Theorem 2.4.7. *Let $\mathcal{Z} = (z_n)_{n\geq 1}$ be a sequence of $\mathbb{D} \setminus \{0\}$. Let $\varepsilon \in (0,1)$ and suppose that $\sum_{n\geq 1}(1-|z_n|)^{1-\varepsilon} < \infty$. Let $E_n = \cup_{j\geq n} I_j(z_n, \varepsilon)$. If*

$$\sum_{n\geq 1} \exp\left(-\frac{A}{c_0(E_n)}\right) < \infty,$$

for some $A > 0$, then $\mathcal{Z}$ is a zero set for $\mathcal{D}$.

Proof. It suffices to replace h in the proof of Theorem 2.1.3 by $\exp(-Ae^f)$. □

Note that Theorem 2.4.7 improves Carleson's condition (2.2). Indeed, suppose that there exists $\varepsilon \in (0,1)$ such that

$$M := \sum_{n\geq 1} (\log(1/(1-|z_n|)))^{\varepsilon-1} < \infty.$$

This implises that $c_0(E_n) \leq M (\log(1/(1-|z_n|)))^{-\varepsilon}$. Therefore, for $N+1 > 1/\varepsilon$, we obtain

$$\sum_{n\geq 1} \exp(-1/c_0(E_n)) \quad \leq \quad M^N \sum_{n\geq 1} (\log(1/(1-|z_n|)))^{\varepsilon-1} < \infty.$$

2.5 GENERAL WEIGHTS

In this section we discuss briefly the case of general positive weights. Let ω be an integrable positive weight on $\mathbb{D}$. Let B be the Blaschke product associated with the sequence $\mathcal{Z} = (z_n)_{n\geq 1}$. Using the fact that $(1-|z|^2)|B'(z)| \leq 1$, we get

$$(1-|z|^2)|B'(z)|^2 \leq \sum_{n\geq 1} \frac{1-|z_n|^2}{|1-\overline{z_n}\zeta|^2}.$$

Now one can show easily the following proposition.

Proposition 2.5.1. *Let $f \in H^2$, and let B be the Blaschke product associated with the Blaschke sequence $\mathcal{Z} = (z_n)_{n\geq 1}$ of $\mathbb{D}$. Then*

$$\|Bf\|^2_\omega \lesssim \|f\|^2_\omega + \int_{\mathbb{T}} |f(\zeta)|^2 \tilde{\mathcal{Z}}_\omega(\zeta) dm(\zeta). \tag{2.20}$$

where

$$\tilde{\mathcal{Z}}_\omega(\zeta) = \sum_{n\geq 1} (1-|z_n|^2) \int_{\mathbb{D}} \frac{\omega(z)}{|1-\overline{z_n}z|^2|1-\overline{z}\zeta|^2} dA(z) \quad (\zeta \in \mathbb{T}).$$

Note that for ω_α we have

$$\tilde{\mathcal{Z}}_{\omega_\alpha}(\zeta) \asymp \sum_{n\geq 1} \frac{1-|z_n|^2}{|1-\overline{z_n}\zeta|^{2-\alpha}} \quad (\zeta \in \mathbb{T}).$$

A comparison between Proposition 2.5.1 and Theorem 2.2.1 is needed. Let $\omega = U_\mu$ be a purely superharmonic weight. To get $\mathcal{Z}_\mu \asymp \tilde{\mathcal{Z}}_\omega$ it suffices to have $(1-|z|^2)^2|\Delta\omega(z)| \asymp \omega(z)$. Note that this condition is satisfied for a large class of regular radial concave weights. For example, it is satisfied by the weights $\omega_{\alpha,\beta}$, when $\alpha \in (0,1)$ and $\beta \in (-\infty, +\infty)$. However, if $\alpha = 0$ and $\beta > 0$ (which means that $\mathcal{D}_{\omega_{\alpha,\beta}}$ is close to the classical Dirichlet space) the situation is quite different. Indeed, we have

$$(1-|z|^2)^2|\Delta\omega_{0,\beta}(z)| \asymp \frac{\omega_{0,\beta}(z)}{\log(\frac{1}{1-|z|})} = \omega_{0,\beta+1}(z), \quad (|z| \to 1^-).$$

and Theorem 2.2.1 is more precise than Proposition 2.5.1. The same phenomenon happens when $\mathcal{D}_\omega$ is close to H^2. It is the case for $\mathcal{D}_{\omega_{\alpha,\beta}}$ with $\alpha = 1$ and $\beta < 0$.

By Proposition 2.5.1, it is clear that $B \in \mathcal{D}_\omega$ if $\sum_{n\geq 1} \Phi_\omega(z_n) < \infty$, where Φ_ω is given by

$$\Phi_\omega(w) = (1-|w|^2) \int_{\mathbb{D}} \frac{\omega(z)}{|1-\overline{z}w|^2(1-|z|^2)} dA(z).$$

An analogue of Theorem 2.4.1 can be stated as follow.

Theorem 2.5.2. *Let ω be a radial weight and let E, ρ satisfying conditions of Theorem 2.4.1. Let $\mathcal{Z} = (z_n)_{n\geq 1}$ be a Blaschke sequence of $\mathbb{D}$. If*

$$\sum_{n\geq 1} \Phi_\omega(z_n)\rho(16d(z_n, E)) < \infty,$$

then $\mathcal{Z}$ is a zero set of $\mathcal{D}_\omega$.

2.6 FINAL REMARKS

Let ω be a superharmonic weight. As mentioned in Introduction 2.1, a Blaschke sequence $\mathcal{Z} = (z_n)_{n\geq 1}$ is a zero set for $\mathcal{D}_\omega$ if $\sum_{n\geq 1} \frac{1}{K^\omega(z_n,z_n)} < \infty$. To take advantage from this result we need to estimate $K^\omega(z,z)$.

Problem 1: Give an estimate of $K^\omega(z,z)$, where $\omega = U_\mu + P_\nu$ is a general superharmonic weight.

The estimate of $K^\omega(z,z)$, when ω is a "regular" radial weight is not difficult to obtain. Indeed, for example if $\omega = \omega_{\alpha,\beta}$, with $(\alpha,\beta) \in \Lambda$, then

$$K^\omega(z,z) \asymp 1/\tilde{\omega}_{\alpha,\beta}(z) \quad (z \in \mathbb{D}),$$

where

$$\tilde{\omega}_{\alpha,\beta} = \begin{cases} \omega_{\alpha,\beta} & \text{if} \quad \alpha \in (0,1], \\ \\ \omega_{\alpha,\beta+1} & \text{if} \quad \alpha = 0. \end{cases}$$

Note also that for the harmonic case $\omega = P_\nu$ it was proved in [28] that

$$K^\omega(z,z) \asymp 1 + \int_0^{|z|} \frac{dr}{(1-r)P_\nu(rz/|z|) + (1-r)^2} \quad (z \in \mathbb{D} \setminus \{0\}).$$

We remark that an answer to this problem will allow to get estimates of the capacity of some borelian subsets of $\mathbb{T}$ (see [28] for the harmonic case).

In [39], D. Guillot considered harmonic weights $\omega = P_\nu$, where $\nu = \sum_{n\geq 1} c_n \delta_{\zeta_n}$ is a positive finite atomic measure on $\mathbb{T}$. He proved that every Blaschke sequence is a zero set for $\mathcal{D}_\omega$ if and only if

$$\int_{\mathbb{T}} \log V_2(\nu)(\zeta) dm(\zeta) < \infty,$$

where $V_2(\nu)(\zeta) = \int_{\mathbb{T}} \frac{d\nu(\lambda)}{|\zeta-\lambda|^2}$ is the newtonian potential. A natural and interesting problem raised from this result is the following.

Problem 2: Determine all superharmonicaly weighted Dirichlet spaces for which every Blaschke sequence is a zero set.

3 UNIQUENESS SETS FOR HARMONICALLY DIRICHLET SPACES

Let ν be a positive finite Borel measure on the unit circle. The associated Dirichlet space $\mathcal{D}_{P_\nu}$ consists of holomorphic functions on the unit disc whose derivatives are square integrable when weighted against the Poisson integral of ν. We give a sufficient condition on a Borel subset E of the unit circle which ensures that E is a uniqueness set for $\mathcal{D}_{P_\nu}$. As application, we give an example of positive measure ν and a countable closed set E such that $\nu(E) = 0$ and E is uniqueness set for $\mathcal{D}_{P_\nu}$.

3.1 HAVIN-MAZYA UNIQUENESS THEOREM

Let ν be a positive finite Borel measure on the unit circle $\mathbb{T}$, the *harmonically weighted Dirichlet space* $\mathcal{D}^h_{P_\nu}$ is the set of all functions $f \in L^2(\mathbb{T})$ for which

$$\mathcal{D}_\nu(f) = \int_{\mathbb{T}} \mathcal{D}_\xi(f) d\nu(\xi) < \infty,$$

where $\mathcal{D}_\xi(f)$ is the local Dirichlet integral of f at $\xi \in \mathbb{T}$ given by

$$\mathcal{D}_\xi(f) := \int_{\mathbb{T}} \Big| \frac{f(\zeta) - f(\xi)}{\zeta - \xi} \Big|^2 \frac{|d\zeta|}{2\pi}.$$

The space $\mathcal{D}^h_{P_\nu}$ is endowed with the norm

$$\|f\|^2_\nu := \|f\|^2_{L^2(\mathbb{T})} + \mathcal{D}_\nu(f).$$

The analytic weighted Dirichlet space $\mathcal{D}(\mu)$ is defined by

$$\mathcal{D}_{P_\nu} = \{f \in \mathcal{D}^h_{P_\nu} : \widehat{f}(n) = 0,\ n < 0\}.$$

Let $\mathbb{D}$ be the open unit disc in the complex plane. Let H^2 denote the Hardy space of analytic functions on $\mathbb{D}$. As usual, we use the identification $H^2 = \{f \in L^2(\mathbb{T}) : \widehat{f}(n) = 0,\ n < 0\}$. Since $\mathcal{D}_{P_\nu} \subset H^2$, every function $f \in \mathcal{D}_{P_\nu}$ has non-tangential limits almost everywhere on $\mathbb{T}$. We denote by $f(\zeta)$ the non-tangential limit of f at $\zeta \in \mathbb{T}$ if it exists. Note that by Douglas Formula the Dirichlet-type space $\mathcal{D}_{P_\nu}$ is the set of analytic functions $f \in H^2$, such that

$$\mathcal{D}_\nu(f) := \int_{\mathbb{D}} |f'(z)|^2 P_\nu(z) dA(z) < \infty,$$

where $dA(z) = dxdy/\pi$ stands for the normalized area measure in $\mathbb{D}$ and P_ν is the Poisson integral of ν

$$P_\nu(z) := \int_{\mathbb{T}} \frac{1-|z|^2}{|\zeta - z|^2} d\nu(\zeta), \qquad z \in \mathbb{D}.$$

For a proof of this fact see [32, Theorem 7.2.5] and for numerous results on the Dirichlet-type space and operators acting thereon see [32, 61, 62, 64, 65].

The capacity associated with $\mathcal{D}_{P_\nu}$ is denoted by c_μ and is given by

$$c_\nu(E) := \inf\left\{ \|f\|_\mu^2 : f \in \mathcal{D}^h_{P_\nu} \text{ , } |f| \geq 1 \text{ a.e. on a neighborhood of } E \right\}.$$

Since the L^2 norm is dominated by the Dirichlet norm $\|.\|_\nu$, it is obvious that capacity 0 implies Lebesgue measure 0. We say that a property holds c_ν-quasi-everywhere (c_ν-q.e.) if it holds everywhere outside a set of zero c_ν capacity. Note that c_ν-q.e. implies a.e. We have

$$c_\nu(E) = \inf\left\{ \|f\|_\nu^2 : f \in \mathcal{D}^h_{P_\nu} \text{ and } |f| \geq 1 \; c_\nu\text{-q.e on } E \right\}.$$

Furthermore every function $f \in \mathcal{D}_{P_\nu}$ has non-tangential limits c_ν-q.e. on $\mathbb{T}$ [21, Theorem 2.1.9]. For more details see [40]

Recall that if $I \subset \mathbb{T}$ be the arc of lenght $|I| = 1 - \rho$ with midpoint $\zeta \in \mathbb{T}$, then

$$c_\nu(I) \asymp 1 + \int_0^\rho \frac{dr}{(1-r)P_\nu(r\zeta) + (1-r)^2}, \tag{3.1}$$

where the implied constants are absolute. For the proof see [28, Theorem 2].

Let E be a subset of $\mathbb{T}$, the set E is said to be a uniqueness set for $\mathcal{D}_{P_\nu}$ if, for each $f \in \mathcal{D}_{P_\nu}$ such that its non-tangential limit $f = 0$ on E, we have $f = 0$.

Note that if $d\nu = dm$ the normalized arc measure on $\mathbb{T}$, then the space $\mathcal{D}_{P_\nu}$ coincides with the classical Dirichlet space $\mathcal{D}$ given by

$$\mathcal{D} = \{f \in H^2 : \int_{\mathbb{D}} |f'(z)|^2 dA(z) < \infty\}.$$

In this case, c_m is comparable to the logarithmic capacity. Namely $c_m(I) \asymp |\log|I||^{-1}$ for every arc $I \subset \mathbb{T}$, [54, Theorem 14].

Khavin and Maz'ya proved in [48] that a Borel subset of $\mathbb{T}$, E, is a uniqueness set for $\mathcal{D}$ if there exists a family of pairwise disjoint open arcs (I_n) such that

$$\sum_n |I_n| \log \frac{|I_n|}{c_m(E \cap I_n)} = -\infty.$$

For other uniqueness results for $\mathcal{D}$ see also [18, 20, 32, 39, 44, 46].

Our aim in this chapter is to extend Khavin and Maz'ya uniqueness theorem to a general Dirichlet spaces $\mathcal{D}_{P_\nu}$.

Let $\gamma \in (0,1)$ and let $I = (e^{ia}, e^{ib})$. The arc γI is given by

$$\gamma I = (e^{i(a+(1-\gamma)\frac{b-a}{2})}, e^{i(b-(1-\gamma)\frac{b-a}{2})}).$$

The main result of this chapter is the following theorem.

Theorem 3.1.1. *Let E be a Borel subset of $\mathbb{T}$ of Lebesgue measure zero. Suppose that there exists a family of pairwise disjoint open arcs $(I_n)_n$ and $\gamma \in (0,1)$ such that*

$$\sum_n |I_n| \log \frac{|I_n|}{c_\nu(E \cap \gamma I_n)} = -\infty,$$

then E is a uniqueness set for $\mathcal{D}_{P_\nu}$.

As application, we give in Section 3, an example of positive measure μ and a countable closed set E such $\nu(E) = 0$ and E is uniqueness set for $\mathcal{D}_{P_\nu}$.
The key of the proof of the main theorem is an upper bound for $\frac{1}{I}\int_I |f|$ in terms of capacity of $E \cap I$, for any open arc I in $\mathbb{T}$.

The next section is devoted to the proof of Theorem 3.1.1. In Section 3 we discuss the case of discrete measures.

3.2 POINCARÉ TYPE INEQUALITY

First, let us introduce some notations which will be useful in the sequel. Let J and L be arcs of $\mathbb{T}$ and let f be a borelian function defined on $\mathbb{T}$. We set

$$\mathcal{D}_{J,L,\nu}(f) := \int_{\zeta \in J} \int_{\xi \in L} \frac{|f(\zeta) - f(\xi)|^2}{|\zeta - \xi|^2} \frac{|d\zeta|}{2\pi} d\nu(\xi),$$

and

$$\langle f \rangle_J := \frac{1}{|J|} \int_J |f(\xi)||d\xi|.$$

The following lemma is the key in the proof of Theorem 3.1.1.

Lemma 3.2.1. *Let $0 < \gamma < 1$ and let $f \in \mathcal{D}_{P_\nu}$ such that $f|_E = 0$ for some borelian subset E of $\mathbb{T}$. Then, for any open arc $I \subset \mathbb{T}$*

$$\langle f \rangle_{\gamma I}^2 \leq \kappa \frac{\mathcal{D}_{I,\mathbb{T},\nu}(f) + \int_I |f|^2}{c_\nu(E \cap \gamma I)},.$$

where κ is a constant depending only on γ.

Proof. Without loss of generality, we can suppose that $I = (e^{-i\theta}, e^{i\theta})$ with $2\theta < \pi$. In this case $\gamma I = (e^{-i\gamma\theta}, e^{i\gamma\theta})$.

Let ϕ be a positive function on $\mathbb{T}$, $0 \leq \phi \leq 1$, such that $\operatorname{supp}\phi = \frac{1+\gamma}{2}I$, $\phi = 1$ on γI and

$$|\phi(\zeta) - \phi(\xi)| \leq \frac{c_1}{|I|}|\zeta - \xi|, \qquad \zeta, \xi \in \mathbb{T}.$$

where c_1 depends only on γ. Set $L = \frac{1+\gamma}{2}I$ and consider

$$F(\zeta) := \phi(\zeta)\Big|1 - \frac{|f(\zeta)|}{\langle f\rangle_L}\Big|, \qquad \zeta \in \mathbb{T}.$$

Hence $F \geq 0$ and $F = 1$ c_ν-q.e on $E \cap \gamma I$. Therefore,

$$c_\nu(E \cap \gamma I) \leq \|F\|_\nu^2.$$

We claim that

$$\|F\|_\nu^2 \leq \kappa \frac{\mathcal{D}_{I,\mathbb{T},\nu}(f) + \int_L |f|^2}{\langle f\rangle_L^2}. \tag{3.2}$$

where κ depends only on γ.

Indeed, we have

$$\begin{aligned}
&\|F\|_\nu^2 = \int_{\mathbb{T}} |F(\zeta)|^2 \frac{|d\zeta|}{2\pi} + \int_{\mathbb{T}}\int_{\mathbb{T}} \frac{|F(\zeta) - F(\xi)|^2}{|\zeta - \xi|^2}\frac{|d\zeta|}{2\pi}d\nu(\xi)\\
&\leq \frac{1}{\langle f\rangle_L^2}\int_L |\langle f\rangle_L - |f(\zeta)||^2 \frac{|d\zeta|}{2\pi}\\
&+ \int_I\int_I \frac{|F(\zeta) - F(\xi)|^2}{|\zeta - \xi|^2}\frac{|d\zeta|}{2\pi}d\nu(\xi)\\
&+ \frac{1}{\langle f\rangle_L^2}\int\limits_{\zeta\in\mathbb{T}\setminus I}\int\limits_{\xi\in L} \frac{|\langle f\rangle_L - |f(\xi)||^2}{|\zeta - \xi|^2}\frac{|d\zeta|}{2\pi}d\nu(\xi)\\
&+ \frac{1}{\langle f\rangle_L^2}\int\limits_{\zeta\in L}\int\limits_{\xi\in\mathbb{T}\setminus I} \frac{|\langle f\rangle_L - |f(\zeta)||^2}{|\zeta - \xi|^2}\frac{|d\zeta|}{2\pi}d\nu(\xi)\\
&= \frac{A}{2\pi\langle f\rangle_L^2} + \frac{B}{2\pi} + \frac{C}{2\pi\langle f\rangle_L^2} + \frac{D}{2\pi\langle f\rangle_L^2}.
\end{aligned} \tag{3.3}$$

By Cauchy-Schwarz inequality we have

$$|\langle f\rangle_L - |f(\xi)||^2 \leq \frac{1}{|L|}\int_L |f(\eta) - f(\xi)|^2|d\eta|. \tag{3.4}$$

Hence

$$\begin{aligned} A &:= \int_L |\langle f\rangle_L - |f(\zeta)||^2 |d\zeta| \\ &\leq \frac{1}{|L|}\int_L\int_L |f(\eta)-f(\xi)|^2|d\eta||d\xi| \\ &\leq \frac{2}{|L|}\int_L\int_L |f(\eta)|^2|d\eta||d\xi| + \frac{2}{|L|}\int_L\int_L |f(\xi)|^2|d\eta||d\xi| \\ &= 4\int_L |f(\zeta)|^2|d\zeta|. \end{aligned} \tag{3.5}$$

Next we estimate B. For $(\zeta,\xi)\in\mathbb{T}\times\mathbb{T}$, we have

$$\begin{aligned} |F(\zeta)-F(\xi)| &= \left|\phi(\zeta)\left(\left|1-\frac{|f(\zeta)|}{\langle f\rangle_L}\right| - \left|1-\frac{|f(\xi)|}{\langle f\rangle_L}\right|\right) + (\phi(\zeta)-\phi(\xi))\left|1-\frac{|f(\xi)|}{\langle f\rangle_L}\right|\right| \\ &\leq \frac{1}{\langle f\rangle_L}|f(\zeta)-f(\xi)| + \frac{c_1}{\langle f\rangle_L}\frac{|\zeta-\xi|}{|I|}|\langle f\rangle_L - |f(\xi)||. \end{aligned} \tag{3.6}$$

By (3.6) and (3.4) we get

$$\begin{aligned} B &:= \int_I\int_I \frac{|F(\zeta)-F(\xi)|^2}{|\zeta-\xi|^2}|d\zeta|d\nu(\xi) \\ &\leq \frac{2}{\langle f\rangle_L^2}\int_I\int_I \frac{|f(\zeta)-f(\xi)|^2}{|\zeta-\xi|^2}|d\zeta|d\nu(\xi) \\ &\quad + \frac{2\times c_1^2}{\langle f\rangle_L^2|I|^2|L|}\int_I\int_I\int_L |f(\eta)-f(\xi)|^2|d\eta||d\zeta|d\nu(\xi) \\ &\leq \frac{2}{\langle f\rangle_L^2}\int_I\int_I \frac{|f(\zeta)-f(\xi)|^2}{|\zeta-\xi|^2}|d\zeta|d\nu(\xi) \\ &\quad + \frac{2\times c_1^2}{\langle f\rangle_L^2|I||L|}\int_I\int_L |f(\eta)-f(\xi)|^2|d\eta|d\nu(\xi) \\ &\leq \frac{2}{\langle f\rangle_L^2}\int_I\int_I \frac{|f(\zeta)-f(\xi)|^2}{|\zeta-\xi|^2}|d\zeta|d\nu(\xi) \\ &\quad + \frac{c_2}{\langle f\rangle_L^2}\int_I\int_I \frac{|f(\eta)-f(\xi)|^2}{|\eta-\xi|^2}|d\eta|d\nu(\xi) \\ &\leq c_3\frac{\mathcal{D}_{I,I,\nu}(f)}{\langle f\rangle_L^2}. \end{aligned} \tag{3.7}$$

where c_2 and c_3 depend only on γ.
Using again (3.4) , we see that

$$\begin{aligned}
C &:= \int_{\zeta\in\mathbb{T}\setminus I}\int_{\xi\in L}\frac{|\langle f\rangle_L-|f(\xi)||^2}{|\zeta-\xi|^2}|d\zeta|d\nu(\xi)\\
&\leq \int_{\zeta\in\mathbb{T}\setminus I}\frac{|d\zeta|}{d(\zeta,L)^2}\int_{\xi\in L}|\langle f\rangle_L-|f(\xi)||^2 d\nu(\xi)\\
&\leq \frac{c_4}{|I|}\int_{\xi\in L}|\langle f\rangle_L-|f(\xi)||^2 d\nu(\xi)\\
&\leq \frac{c_5}{|I|^2}\int_I\int_I|f(\eta)-f(\xi)|^2|d\eta|d\nu(\xi)\\
&\leq c_5\int_I\int_I\frac{|f(\eta)-f(\xi)|^2}{|\eta-\xi|^2}|d\eta|d\nu(\xi)\\
&= 2\pi c_5\mathcal{D}_{I,I,\nu}(f), \qquad (3.8)
\end{aligned}$$

where c_4 and c_5 depend only on γ.
Obviously, $(\xi,\zeta,\eta)\in\mathbb{T}\setminus I\times L\times L$,we have

$$|\zeta-\xi|\geqslant(1-\gamma)|I| \quad \text{and} \quad |\eta-\xi|\leq\frac{\gamma}{1-\gamma}|\zeta-\xi|.$$

Therefore, by (3.4) we oblain

$$\begin{aligned}
D &:= \int_{\xi\in\mathbb{T}\setminus I}\int_{\zeta\in L}\frac{|\langle f\rangle_L-|f(\zeta)||^2}{|\zeta-\xi|^2}|d\zeta|d\nu(\xi)\\
&\leq \int_{\xi\in\mathbb{T}\setminus I}\int_{\zeta\in L}\frac{1}{|L|}\int_{\eta\in L}\frac{|f(\eta)-f(\xi)+f(\xi)-f(\zeta)|^2}{|\zeta-\xi|^2}|d\zeta||d\eta|d\nu(\xi)\\
&\leq 2\int_{\xi\in\mathbb{T}\setminus I}\int_{\eta\in L}\int_{\zeta\in L}\frac{|f(\eta)-f(\xi)|^2}{|\zeta-\xi|^2}|d\zeta||d\eta|d\nu(\xi)\\
&\quad +2\int_{\xi\in\mathbb{T}\setminus I}\int_{\zeta\in L}\frac{|f(\xi)-f(\zeta)|^2}{|\zeta-\xi|^2}|d\zeta|d\nu(\xi)\\
&\leq \frac{c_6}{|I|}\int_{\xi\in\mathbb{T}\setminus I}\int_{\eta\in L}\int_{\zeta\in L}\frac{|f(\eta)-f(\xi)|^2}{|\eta-\xi|^2}|d\zeta||d\eta|d\nu(\xi)\\
&\quad +2\int_{\xi\in\mathbb{T}\setminus I}\int_{\zeta\in L}\frac{|f(\xi)-f(\zeta)|^2}{|\zeta-\xi|^2}|d\zeta|d\nu(\xi)\\
&= c_7\int_{\xi\in\mathbb{T}\setminus I}\int_{\zeta\in L}\frac{|f(\xi)-f(\zeta)|^2}{|\zeta-\xi|^2}|d\zeta|d\nu(\xi)\\
&\leq 2\pi c_7\mathcal{D}_{I,\mathbb{T}\setminus I,\nu}(f), \qquad (3.9)
\end{aligned}$$

where c_6 and c_7 depend only on γ.

Combining (3.3), (3.5), (3.7), (3.8) and (3.9) we get (3.2) and the proof is complete. □

Proof of Theorem 3.1.1 . Let $f \in \mathcal{D}_{P_\nu}$ such that $f|_E = 0$ and set $\ell = \sum_n |I_n|$. By Lemma 3.2.1 and Jensen's inequality it follows that

$$\begin{aligned}
\int_{\bigcup \gamma I_n} \log|f(\xi)||d\xi| &= \sum_n \gamma|I_n| \frac{1}{\gamma|I_n|}\int_{\gamma I_n} \log|f(\xi)||d\xi| \\
&\leq \frac{\gamma}{2}\sum_n |I_n| \log\langle f\rangle^2_{\gamma I_n} \\
&\leq \frac{\gamma}{2}\sum_n |I_n| \log\Big(\kappa \frac{\mathcal{D}_{I_n,\mathbb{T},\mu}(f) + \int_{I_n}|f|^2}{C_\mu(E\cap\gamma I_n)}\Big) \\
&= \frac{\gamma}{2}\ell\log\kappa + \frac{\gamma}{2}\mathcal{I},
\end{aligned}$$

where

$$\mathcal{I} = \sum_n |I_n| \log\frac{|I_n|}{c_\nu(E\cap\gamma I_n)} + \sum_n |I_n|\log\Big(\frac{\mathcal{D}_{I_n,\mathbb{T},\nu}(f) + \int_{I_n}|f|^2}{|I_n|}\Big)$$

Using again Jensen inequality, we get

$$\begin{aligned}
\ell & \sum_n \frac{|I_n|}{\ell}\log\Big(\frac{\mathcal{D}_{I_n,\mathbb{T},\nu}(f) + \int_{I_n}|f|^2}{|I_n|}\Big) \\
\leq & \ \ell\log\Big[\frac{1}{\ell}\Big(\sum_n \mathcal{D}_{I_n,\mathbb{T},\nu}(f) + \int_{I_n}|f|^2\Big)\Big] \\
\leq & \ \ell\log\Big[\frac{1}{\ell}(\mathcal{D}_\nu(f) + \|f\|_2^2)\Big] \\
= & \ \ell\log\frac{\|f\|_\nu^2}{\ell}.
\end{aligned}$$

Therefore $\mathcal{I} = -\infty$. So by Fatou's Theorem we obtain $f = 0$ and the proof is complete. □

3.3 COUNTABLE SUM OF DIRAC MEASURES

First observe that for the Dirac measure δ_ζ it is known, (see [32]), that $\mathcal{D}(\delta_\zeta) = \mathbb{C} + (z-\zeta)H^2$. Then a Borel set E is a uniqueness set for $\mathcal{D}(\delta_\zeta)$ if and only if the E has a positive Lebesgue measure. This result can be extended to some other discrete measures. For a positive Borel measure ν we will denote by $V_2(\nu)$ the Newtonian potential given by

$$V_2(\nu)(\zeta) = \int_0^{2\pi} \frac{d\nu(e^{it})}{|e^{it}-\zeta|^2}.$$

D. Guillot showed in [39, Theorem 2.1] that if there exists $f \in \mathcal{D}_\mu$ such that $f = 0$ ν a.e on $\mathbb{T}$ then

$$\int_{\mathbb{T}} \log V_2(\nu)(\zeta)|d\zeta| < \infty.$$

He also proved that the converse is true for all discrete measures. The following result is an immediate consequence of [39].

Proposition 1. *Consider a positive sequence* $(a_n)_{n\geq 1}$ *such that* $\sum_{n\geq 1} a_n = 1$ *and let* $\mu = \sum_n a_n \delta_{\zeta_n}$ *where* $\zeta_n \in \mathbb{T}$*. If*

$$\int_{\mathbb{T}} \log V_2(\nu)(\zeta)|d\zeta| < \infty,$$

then a Borel set $E \subset \mathbb{T}$ *is a uniqueness set for* $\mathcal{D}_{P_\nu}$ *if and only if* $|E| > 0$*.*

Proof. Since $\mathcal{D}_{P_\nu} \subset H^2$, it is obvious that the condition $|E| > 0$ is sufficient. Conversely, Let $E \subset \mathbb{T}$ be a Borel set such that $|E| = 0$, there exists a function $\varphi \in H^\infty \setminus \{0\}$ such that $\varphi_{|E} = 0$. By [40], there exists $f \in \mathcal{D}_{P_\nu} \cap H^\infty \setminus \{0\}$ such that $f(\zeta_n) = 0,\ (n \geq 1)$. Now it suffices to prove that $\varphi f \in \mathcal{D}_{P_\nu}$. Indeed,

$$\begin{aligned}
\mathcal{D}_\nu(\varphi f) &= \int_{\mathbb{T}}\int_{\mathbb{T}} \left|\frac{\varphi f(\zeta) - \varphi f(\zeta')}{\zeta - \zeta'}\right|^2 |d\zeta| d\nu(\zeta') \\
&= \int_{\mathbb{T}}\int_{\mathbb{T}} \left|\frac{\varphi f(\zeta)}{\zeta - \zeta'}\right|^2 |d\zeta| d\nu(\zeta') \\
&\leq \|\varphi\|_\infty^2 \int_{\mathbb{T}}\int_{\mathbb{T}} \left|\frac{f(\zeta)}{\zeta - \zeta'}\right|^2 |d\zeta| d\nu(\zeta') \\
&= \|\varphi\|_\infty^2 \int_{\mathbb{T}}\int_{\mathbb{T}} \left|\frac{f(\zeta) - f(\zeta')}{\zeta - \zeta'}\right|^2 |d\zeta| d\nu(\zeta') \\
&= \|\varphi\|_\infty^2 \mathcal{D}_\nu(f).
\end{aligned}$$

Which completes the proof. □

Note that for a positive Borel measures $\nu = \sum_n a_n \delta_{\zeta_n}$ where $\zeta_n \in \mathbb{T}$ such that

$$\int_{\mathbb{T}} \log V_2(\nu)(\zeta)|d\zeta| = \infty, \tag{3.10}$$

we have no complete characterization for uniqueness sets for $\mathcal{D}_{P_\nu}$. In this case, by [39] the countable set $\{\zeta \in \mathbb{T} : \nu(\zeta) \neq 0\}$ is a uniqueness set for $\mathcal{D}_{P_\nu}$. The situation is more complicated than the previous case. In fact, Using theorem 3.1.1, we will construct more smaller uniqueness set for $\mathcal{D}_{P_\nu}$. Before stating this result, we give an example of discrete measures μ satisfying (3.10). A closed set $E \subset \mathbb{T}$ is said to be a Carleson set if

$$\int_{\mathbb{T}} \log \operatorname{dist}(\zeta, E)|d\zeta| > -\infty.$$

This condition is equivalent to $|E| = 0$ and

$$\sum_n |\ell_n| \log |\ell_n| > -\infty,$$

where ℓ_n are the complementary intervals of E. For more informations on Carleson sets, see e.g. [32, §4.4].

Proposition 2. *Let E be a countable closed set of the unit circle such that E is non Carleson set. Then there exists a discrete measure μ on the unit circle such that $E = \{\zeta \in \mathbb{T} : \nu(\zeta) \neq 0\}$ and (3.10) is satisfied.*

Proof. Let $\mathbb{T} \setminus E = \bigcup (a_n, b_n)$ and let $\nu = \sum\limits_{n \geq 1} c_n(\delta_{a_n} + \delta_{b_n})$, with $c_n = b_n - a_n$. We have

$$\begin{aligned} \int_{\mathbb{T}} \log V_2(\nu)(\zeta)|d\zeta| &\geq \sum_{n\geq 1} \int_{a_n}^{b_n} \log \frac{c_n}{\operatorname{dist}^2(e^{it}, \{a_n, b_n\})} dt \\ &\geq \sum_{n\geq 1} \int_{a_n}^{b_n} \log \frac{c_n}{(b_n - a_n)^2} dt \\ &= \sum_{n\geq 1} (b_n - a_n) \log \frac{1}{b_n - a_n} = +\infty. \end{aligned}$$

□

Proposition 3. *There exists a discrete probability measure μ on the unit circle and a countable closed set $E \subset \mathbb{T}$, such that $\nu(E) = 0$ and E is a set of uniqueness for $\mathcal{D}_{P_\nu}$.*

To prove this result we need the following lemma.

Lemma 3.3.1. *Let $\zeta \in \mathbb{T}$, $a \in (0, 1/2)$. Let ν be a probability measure on $\mathbb{T}$ such that $\nu \geq \sum_{k\geq 2} ak^{-2}\delta_{\zeta e^{ia^k}}$, then $c_\nu(\{\zeta\}) \gtrsim \sqrt{a}$.*

Proof. We consider the measure λ on $\mathbb{T}$ defined by

$$\lambda = \sum_{k\geq 2} \frac{a}{k^2} \delta_{\zeta e^{ia^k}}.$$

By (3.1), see also [28, Theorem 2], we have

$$\begin{aligned} \frac{1}{c_\nu(\zeta)} &\asymp 1 + \int_0^1 \frac{dr}{(1-r)P_\nu(r\zeta) + (1-r)^2} \\ &\lesssim 1 + \int_0^1 \frac{dr}{(1-r)P_\lambda(r\zeta) + (1-r)^2} \end{aligned}$$

For $a \leq 1 - r$, we have

$$\begin{aligned} (1-r)P_\lambda(r\zeta) &\asymp \sum_{k\geq 2} \frac{a(1-r)^2}{k^2((1-r)^2 + a^{2k})} \\ &\asymp \sum_{k\geq 2} \frac{a}{k^2} \\ &\asymp a. \end{aligned}$$

For $1-r<a$, let k_r be a real number such that $1-r=a^{k_r}$. Since $x \longmapsto x^2a^{2x}$; $x \geqslant 1$, is a decreasing function, we have

$$\begin{aligned}(1-r)P_\lambda(r\zeta) &\asymp \sum_{k\geq 2}\frac{a(1-r)^2}{k^2((1-r)^2+a^{2k})}\\ &= \sum_{k\geq k_r}\frac{a}{k^2}+\sum_{k<k_r}\frac{a(1-r)^2}{k^2a^{2k}}\\ &\asymp \frac{a}{k_r}\\ &= \frac{a\log(1/a)}{\log(1/1-r)}.\end{aligned}$$

Then we obtain

$$\begin{aligned}\frac{1}{c_\nu(\zeta)} &\leq 1+\int_0^{1-a}\frac{dr}{a+(1-r)^2}+\int_{1-a}^{1}\frac{dr}{\frac{a\log(1/a)}{\log(1/1-r)}+(1-r)^2}\\ &\leq 1+\frac{1}{\sqrt{a}}\int_0^{\frac{1}{\sqrt{a}}}\frac{du}{1+u^2}+\int_{1-a}^{1}\frac{\log(1/1-r)dr}{a\log(1/a)}\\ &\asymp \frac{1}{\sqrt{a}}\end{aligned}$$

and the lemma is proved. □

Proof of Proposition 3. Let E be the set define by

$$E := \left\{\zeta_n = e^{i\theta_n} : n \geqslant 3\right\} \cup \{1\},$$

with $\theta_n := 1/\log(n)$. Let $a_n = \varepsilon(\theta_n - \theta_{n+1})$ for some small $\varepsilon > 0$. Consider the sequence $\zeta_{n,k} = e^{i\theta_{n,k}}$ where

$$\theta_{n,k} := \theta_n + a_n^k \qquad k \geqslant 2$$

The measure ν is given by

$$\nu := \sum_{n\geqslant 3}\sum_{k\geqslant 2}\frac{a_n}{k^2}\delta_{\zeta_{n,k}}.$$

To prove that E is a uniqueness set for $\mathcal{D}_{P_\nu}$, we use Theorem 3.1.1. Let $I_n = (\theta_n - a_n, \theta_n + a_n)$. By Lemma 3.3.1, we have $c_\nu(\zeta_n) \geq \sqrt{a_n}$. Then

$$\begin{aligned}\sum_n |I_n|\log\left(\frac{c_\nu(I_n\cap E)}{|I_n|}\right) &= \sum_n |I_n|\log\left(\frac{c_\nu(\zeta_n)}{|I_n|}\right)\\ &\geq \sum_n a_n\log\left(\frac{1}{a_n}\right) = +\infty\end{aligned}$$

and the proof is complete. □

Part II

INVARIANT SUBSPACES, CYCLICITY AND CLOSED IDEALS

4

INVARIANT SUBSPACES AND CYCLICITY IN WEIGHTED DIRICHLET SPACES

In this chapter we deal with structure of shift-invariant subspaces and the vectors cyclic in superharmonically weighted Dirichlet spaces $\mathcal{D}(\mu)$ where μ is a finite positive Borel measure on the closed unit disc $\overline{\mathbb{D}}$. In the light of Richter-Sundberg's characterization [64], we obtain the complete structure of shift-invariant subspaces of all $\mathcal{D}(\mu)$. Next we prove that the Browen-Shields conjecture is true for $\mathcal{D}(\mu)$ if $\operatorname{supp}(\mu) \cap \mathbb{T}$ is countable.

4.1 INTRODUCTION

Let μ be a finite positive Borel measure on the closed unit disc $\overline{\mathbb{D}}$. The restriction of the measure μ on the unit circle $\mathbb{T}$ (resp. on the open unit disc $\mathbb{D}$) is $\mu_{\mathbb{T}}$ (resp. $\mu_{\mathbb{D}}$). The positive superharmonic function U_μ on $\mathbb{D}$ associated with μ is given by

$$\mathrm{U}_\mu(z) := \int_{\mathbb{T}} \frac{1-|z|^2}{|\zeta - z|^2} d\mu_{\mathbb{T}}(\zeta) + \int_{\mathbb{D}} \log\left|\frac{z-w}{1-z\overline{w}}\right|^2 \frac{d\mu_{\mathbb{D}}(w)}{1-|w|^2} \quad (z \in \mathbb{D}).$$

Note that $U_{\mu_{\mathbb{D}}}$ is the Green potential of $\mu_{\mathbb{D}}$ and $U_{\mu_{\mathbb{T}}}$ is the Poisson transform of $\mu_{\mathbb{T}}$.

For a function f in the classical Hardy space H^2 on $\mathbb{D}$, the *weighted Dirichlet integral* is defined by

$$\mathcal{D}_\mu(f) := \int_{\mathbb{D}} |f'(z)|^2 U_\mu(z) dA(z) \tag{4.1}$$

where $dA(z)$ denotes the normalized Lebesgue area measure on $\mathbb{D}$. The radials limits of the function $f \in \mathrm{H}^2$ will be also denoted by f. We define the *local Dirichlet integral* of f at $\xi \in \overline{\mathbb{D}}$ by

$$\mathcal{D}_\xi(f) := \int_{\mathbb{T}} \left|\frac{f(\zeta)-f(\xi)}{\zeta-\xi}\right|^2 dm(\zeta),$$

where m is the normalized Lebesgue measure on $\mathbb{T}$. If $f(\xi)$ does not exists then we set $\mathcal{D}_\xi(f) = \infty$.

Recall that Richter and Sundberg [63] have established the following formula

$$\mathcal{D}_{\mu_{\mathbb{T}}}(f) = \int_{\mathbb{T}} \mathcal{D}_\xi(f) d\mu_{\mathbb{T}}(\xi). \tag{4.2}$$

For $\mu = m$ this result is due to J.Douglas [22]. However, from Green's formula (see (4.15)), we have

$$\mathcal{D}_{\mu_{\mathbb{D}}}(f) = \int_{\mathbb{D}} \mathcal{D}_\xi(f) d\mu_{\mathbb{D}}(\xi).$$

This formula was obtained by Aleman [4]. Put together these formulas, we have

$$\mathcal{D}_\mu(f) = \int_{\overline{\mathbb{D}}} \mathcal{D}_\zeta(f) d\mu(\zeta). \tag{4.3}$$

The superharmonically weighted Dirichlet space $\mathcal{D}(\mu)$ is defined as the space of all functions $f \in \mathrm{H}^2$ satisfying

$$\|f\|_\mu^2 := \|f\|_{\mathrm{H}^2}^2 + \mathcal{D}_\mu(f) < \infty.$$

The space $\mathcal{D}(\mu)$ endowed with the norm $\|.\|_\mu$ is a Hilbert space.
The harmonically weighted Dirichlet spaces $\mathcal{D}(\mu_{\mathbb{T}})$ were introduced by Richter [62]. Several authors have been interested in this spaces see for instance [21, 27, 28, 34, 39, 61, 63, 64, 66, 72].

Let f and g be two outer functions in H^2. Let $f \wedge g$ and $f \vee g$ be the outer functions associated with $|f| \wedge |g|(e^{it}) := \min(|f(e^{it})|, |g(e^{it})|)$ and $|f| \vee |g|(e^{it}) := \max(|f(e^{it})|, |g(e^{it})|)$ respectively, see Subsection 4.2.1.

The estimate of local Dirichlet integral for cut-off functions play a central role in the theory of shift-invariant subspaces of $\mathcal{D}(\mu)$. However, to obtain a complete description of closed ideals of some Banach algebra, H.Bahajji-El Idrissi and O.El-Fallah [7] proved the following estimate.

$$\mathcal{D}_\zeta(f \wedge f^2) \leq 4\, \mathcal{D}_\zeta(f) \qquad (\zeta \in \mathbb{D}). \tag{4.4}$$

This result was obtained by Richter and Sundberg [64] for $\zeta \in \mathbb{T}$. Furthermore, they established two other important estimates. Our first goal in this chapter is to extend these results for $\zeta \in \mathbb{D}$. The two cases ($\zeta \in \mathbb{T}$ and $\zeta \in \mathbb{D}$) will be stated in the following theorem.

Theorem 4.1.1. *Let f, g be an outer functions in H^2 and $\zeta \in \overline{\mathbb{D}}$. Then*

(i) $\mathcal{D}_\zeta(f \vee g) \leq \mathcal{D}_\zeta(f) + \mathcal{D}_\zeta(g)$,

(ii) $\mathcal{D}_\zeta(f \wedge g) \leq \mathcal{D}_\zeta(f) + \mathcal{D}_\zeta(g)$,

Remark that this result improve Proposition 3.3 of [4] for outer functions.

To give the structure of shift-invariant subspaces of $\mathcal{D}(\mu)$, we will exploit Theorem 4.1.1 and (4.4). This technique was firstly used by Richter-Sundberg [64]. They gave the structure of each invariant-subspaces of all harmonically Dirichlet spaces. Namely, both characterizations will be considered, every invariant subspace $\mathcal{M}$ of $\mathcal{D}(\mu)$ has the form

$$\mathcal{M} = \mathcal{I}_{\mathcal{M}} \mathrm{H}^2 \cap [f]_{\mathcal{D}(\mu)}, \tag{4.5}$$

where f is an outer function in $\mathcal{D}(\mu)$ and $\mathcal{I}_{\mathcal{M}}$ is the greatest common inner divisor of $\mathcal{M}$. This result is new even for the space $\mathcal{D}_\alpha$. Furthemore, we get a complete description of $\mathcal{M}$ when $\mathrm{supp}(\mu) \cap \mathbb{T}$ is countable. More precesily, we will show, if c_μ is the capacity associated with $\mathcal{D}(\mu)$ see Section 4.4, that

$$[f]_{\mathcal{D}(\mu)} = \mathcal{M}_{\mathrm{E}} := \{g \in \mathcal{D}(\mu) : g_{|_{\mathrm{E}}} = 0 \ \ c_\mu - q.e\},$$

where $\mathrm{E} = \{\zeta \in \operatorname{supp}(\mu) \cap \underline{\mathcal{Z}}(\mathcal{M}) \cap \mathbb{T} : c_\mu(\zeta) > 0\}$.
The latter result was obtained by O.El-Fallah, Y.Elmadani and K.Kellay [28] for harmonically weigthed Dirichlet space. Namely we have the following result.

Theorem 4.1.2. *Let μ be a positive finite measure on $\overline{\mathbb{D}}$ such that* $\operatorname{supp}(\mu) \cap \mathbb{T}$ *is countable. Let $\mathcal{M}$ be a shift-invariant subspace of $\mathcal{D}(\mu)$. Then*

$$\mathcal{M} = \mathcal{I}_{\mathcal{M}}\mathrm{H}^2 \cap \mathcal{M}_{\mathrm{E}}.$$

This result is a positive answer of extended Browen- Shields conjecture [66, Conjecture 1.1].

A natural and interesting question arises from (4.5). Determine all functions $f \in \mathcal{D}(\mu)$ that satisfy $[f]_{\mathcal{D}(\mu)} = \mathcal{D}(\mu)$. A function that checks this property is called a cyclic function for $\mathcal{D}(\mu)$. The problem of vectors cyclic for $\mathcal{D}(\mu)$ is still an open question. The Brown-Shields conjecture: a function in $\mathcal{D}(\mu)$ is cyclic if and only if it is an outer function whose zero set has zero $c_\mu-$capacity.
Before stating our main result with respect that conjecture, let us fixing some notations. Let f be a function in $\mathcal{D}(\mu)$, we denote by

$$\underline{\mathcal{Z}}(f) = \{w \in \overline{\mathbb{D}} : \liminf_{z \in \mathbb{D},\, z \to w} |f(z)| = 0\},$$

and

$$\mathcal{Z}(f) := \{\zeta \in \mathbb{T} : \lim_{r \to 1^-} f(r\zeta) = 0\}.$$

The following theorem claims that the Brown-Shields conjecture is true for $\mathcal{D}(\mu)$ if $\operatorname{supp}(\mu) \cap \mathbb{T}$ is countable.

Theorem 4.1.3. *Let μ be a positive finite measure on $\overline{\mathbb{D}}$. Let $f \in \mathcal{D}(\mu)$ be an outer function such that* $\operatorname{supp}(\mu) \cap \underline{\mathcal{Z}}(f)$ *is countable. Then f is cyclic in $\mathcal{D}(\mu)$ if and only if $c_\mu(\mathcal{Z}(f)) = 0$.*

In particular, Theorem 1 of [28] will be obtained if μ is supported by $\mathbb{T}$.
In particular, this theorem is an extension of Theorem 1 of [28]. To get the previous theorems 4.1.2 and 4.1.3, we will exploit the argument stated in [28].
However, Theorem 4.6.2 gives an affirmative replies to that conjecture for $\mathcal{D}(\mu_{\mathbb{D}})$ when $\mu_{\mathbb{D}}$ is a Carleson measure for $\mathcal{D}(\mu_{\mathbb{D}})$. For more details see Section 4.6.
Next, we establish a sufficient condition for cyclicity in $\mathcal{D}(\mu)$ in terms of capacity c_μ. That is the aim of Theorem 4.6.5. This result is also hold for $\mathcal{D}_\alpha$ $(0 < \alpha < 1)$.

The estimates of reproducing kernels are known to be a very useful tool for studying some problems arising in the functions theory of Hilbert spaces of holomorphic functions. Namely, zero sets, uniqueness sets and interpolation theory, capacity, ... whitin superharmonically weighted Dirichlet spaces.
In order to state some estimates let us setting some notations. The reproducing kernel of $\mathcal{D}(\mu)$ will be denoted by K^μ. The function Ψ_μ is given by

$$\Psi_\mu(z) := \int_{\mathbb{D}} \frac{1-|z|^2}{|1-z\overline{w}|^2} d\mu(w) \qquad (z \in \mathbb{D}). \tag{4.6}$$

Recall that, O.El-Fallah, Y.Elmadani and K.Kellay [27] have obtained a more precise estimate of the reproducing kernel on the diagonal for the harmonically Dirichlet space. Namely, we have

$$K^{\mu_{\mathbb{T}}}(z,z) \asymp 1 + \int_0^{|z|} \frac{dr}{(1-x)^2 + (1-r)\Psi_{\mu_{\mathbb{T}}}(rz/|z|)}, \qquad (z \in \mathbb{D}). \tag{4.7}$$

Using the mean inequality one can extend the upper inequality for all superharmonically Dirichlet spaces. To get a lower inequality, we will estimate the norm of a test function. This requires an other formula of the weighted Dirichlet integral, see Lemma 4.2.1. The following theorem summarizes that discussion. Before that let us introduce the maximal Cauchy transform $\tilde{\mathcal{C}}_\mu$ of μ. The function $\tilde{\mathcal{C}}_\mu$ is defined by

$$\tilde{\mathcal{C}}_\mu(z) := \int_{\mathbb{D}} \frac{d\mu(w)}{|1-\overline{w}z|} \quad (z \in \mathbb{D}).$$

Theorem 13. *Let μ be a nonnegative finite Borel measure on $\overline{\mathbb{D}}$. We have for $z \in \mathbb{D}$*

$$h_\mu(z) \lesssim K^\mu(z,z) \lesssim 1 + \int_0^{|z|} \frac{dr}{(1-r)^2 + (1-r)\Psi_\mu(rz/|z|)}, \tag{4.8}$$

where $h_\mu(z) := \frac{\log^2 \frac{e}{1-|z|}}{1+\tilde{\mathcal{C}}_\mu(z)} + \frac{1}{(1-|z|^2)+\Psi_\mu(z)}$.

In particullar, by (4.7), the upper estimate is optimal in the harmonic case ($\mu = \mu_{\mathbb{T}}$). To take advantage of the latter result, we considere a purely superharmonic radial weight. In this case it would be more clear that the lower estimate is also optimal. So, we considere λ a finite positive Borel measure on $[0,1)$, one can see that $\Psi_\lambda(z) := \int_0^1 \frac{1-|z|^2}{1-r^2|z|^2} d\lambda(r)$ $(z \in \mathbb{D})$. Hence, we obtain an upper estimate for the integral in the right hand of (4.8). Namely we have

$$K^{\lambda \otimes m}(z,z) \lesssim \frac{\log \frac{1}{1-|z|}}{(1-|z|) + \Psi_\lambda(z)} \qquad (z \in \mathbb{D}). \tag{4.9}$$

Under some additional assumptions we get the equivalence instead of the inequality (4.9). Especially, this inequality becomes an equivalence if the measure λ satisfies one of the two following conditions:

($\mathcal{R}_1$) consider that the measure λ satisfy

$$\tilde{\mathcal{C}}_{\lambda\otimes m}(z)/\Psi_\lambda(z) = O\left(\log(e/(1-|z|))\right), \quad |z| \to 1^-.$$

Moreover in that situation we have

$$K^{\lambda\otimes m}(z,z) \asymp \log^2(e/(1-|z|))/(1+\tilde{\mathcal{C}}_{\lambda\otimes m}(z)). \tag{4.10}$$

($\mathcal{R}_2$) suppose that Ψ_λ is a decreasing function on $(0,1)$, $\Psi_\lambda(1^-) = 0$ and $t \mapsto \Psi_\lambda(1-t)/t$ is decreasing, such that

- $\inf_{t\in(0,1)} \Psi_\lambda(1-t)/t^\varepsilon > 0, \quad$ for all $\quad \varepsilon > 0$
- $\inf_{\delta\in(0,1)} \frac{\Psi_\lambda(1-\delta)}{\log(e/\delta)} \int_\delta^1 \frac{dx}{x\Psi_\lambda(1-x)} > 0.$

Both conditions are different. In particular we consider the measure $\lambda_{0,\beta}$, $\beta < 0$, given by

$$d\lambda_{0,\beta}(r) := (1-r)^{-1}\log^{\beta-1}(1/(1-r))dr, \qquad (r \in [0,1)).$$

The measure $\lambda_{0,\beta}$ satisfies ($\mathcal{R}_2$) but it does not fulfill the condition ($\mathcal{R}_1$) for $\beta \leq -1$. The two estimates, lower-upper, in (4.8) are not equivalents for that measure ($\beta \leq -1$).

However the following equivalence

$$K^{\lambda\otimes m}(z,z) \asymp \frac{1}{(1-|z|^2)+\Psi_\lambda(z)}, \qquad (z \in \mathbb{D}). \tag{4.11}$$

will be obtained if the measure λ satisfies

($\mathcal{R}_3$) there exists $\varepsilon > 0$ such that $x \mapsto \Psi_\lambda(x)/(1-x)^\varepsilon$ is a decreasing function.

Let $(\alpha,\beta) \in (0,1)\times\mathbb{R}$. We consider the measure $\lambda_{\alpha,\beta}$ given by

$$d\lambda_{\alpha,\beta}(r) := (1-r)^{\alpha-1}\log^{\beta}(1/(1-r))dr, \qquad (r \in [0,1)).$$

The measure $\lambda_{\alpha,\beta}$ checks the condition ($\mathcal{R}_3$). For more details see Section 4.3.

4.2 DIRICHLET INTEGRAL

4.2.1 *Cut-off functions*

In this subsection, we are interested in some estimates of local Dirichlet integral of Cut-Off functions. Our main result here extends Lemma 2.2. of [64]. In fact, Richter Sundberg have studied some punctual inequalities to obtain that result. Keeping in mind Theorem 4.1.1, one cannot get this result using the same technique as [64]. However, we are going to establish some integral inequalities to obtain the desired estimates.

Let f and g be two outer functions in H^2. The outer functions $f \wedge g$ and $f \vee g$, associated with $|f| \wedge |g|(e^{it}) := \min(|f(e^{it})|, |g(e^{it})|)$ and $|f| \vee |g|(e^{it}) := \max(|f(e^{it})|, |g(e^{it})|)$ respectively, are defined by

$$f \wedge g(z) = \exp\left(\int_{\mathbb{T}} \frac{e^{it}+z}{e^{it}-z} \log(|f| \wedge |g|(e^{it}))dm(e^{it})\right), \quad (z \in \mathbb{D}),$$

and

$$f \vee g(z) = \exp\left(\int_{\mathbb{T}} \frac{e^{it}+z}{e^{it}-z} \log(|f| \vee |g|(e^{it}))dm(e^{it})\right), \quad (z \in \mathbb{D}).$$

Let f be an outer function and let $\xi \in \mathbb{D}$. Using the fact that $z \mapsto |f(\xi)|^2 - \mathrm{Re}(\overline{f(\xi)}f(z))$ is an harmonic function on $\mathbb{D}$, one can see that

$$\mathcal{D}_\xi(f) = \int_{\mathbb{T}} \frac{|f(\zeta)|^2 - |f(\xi)|^2}{|\zeta - \xi|^2} dm(\zeta).$$

This means that, for $d\sigma_\xi(\zeta) := P_\xi(\zeta)dm(\zeta)$, we have

$$(1-|\xi|^2)\mathcal{D}_\xi(f) = \int_{\mathbb{T}} |f(\zeta)|^2 d\sigma_\xi(\zeta) - \exp\int_{\mathbb{T}} \log|f(\zeta)|^2 d\sigma_\xi(\zeta), \quad (4.12)$$

The latter formula was stated in [4].

Proof of Theorem 4.1.1. Let $d\sigma_\xi = P_\xi dm$. We denote by $A := \{|g| \leq |f|\}$ and $A^c := \{|g| \geq |f|\}$. Set $\alpha = \sigma_\lambda(A)$ and $\beta = 1-\alpha$.
If $\alpha\beta = 0$ then (i) and (ii) are obvious. So one can assume that $\alpha, \beta > 0$ in what follows.
First let us setting some notations. Write

$$A_f := \int_A |f|^2 d\sigma_\lambda \qquad \text{and} \qquad A_f^c := \int_{A^c} |f|^2 d\sigma_\lambda$$
$$x_1(f) := \exp\int_A \log|f|^2 d\sigma_\lambda \qquad \text{and} \quad x_2(f) := \exp\int_{A^c} \log|f|^2 d\sigma_\lambda$$

Write $D := \{0 \leq y_1 \leq x_1 \text{ and } 0 \leq x_2 \leq y_2\}$. We consider the function h defined on D by

$$h(x_1, x_2, y_1, y_2) := x_1x_2 + y_1y_2 - x_2y_1,$$

Note that $\partial_{x_2}h = x_1 - y_1 \geq 0$ on D. Then, for all $(x_1, y_2, y_1, y_2) \in D$, we have

$$h(x_1, x_2, y_1, y_2) \quad \leq \quad h(x_1, y_2, y_1, y_2) = x_1y_2$$

Obviously $(x_1(f), x_2(f), x_1(g), x_2(g)) \in D$. By Jensen's inequality we have

$$x_1(f) \leq (A_f/\alpha)^\alpha \qquad \text{and} \qquad x_2(g) \leq (A_g^c/\beta)^\beta.$$

Therefore, using Young's inequality, we obtain

$$\begin{aligned}
&\exp \int_{\mathbb{T}} \log |f|^2 d\sigma_\xi + \exp \int_{\mathbb{T}} \log |g|^2 d\sigma_\xi - \exp \int_{\mathbb{T}} \log |f \wedge g|^2 d\sigma_\xi \\
&= h(x_1(f), x_2(f), x_1(g), x_2(g)) \\
&\leq (A_f/\alpha)^\alpha (A_g^c/\beta)^\beta \\
&\leq A_f + A_g^c \\
&= \int_{\mathbb{T}} |f|^2 d\sigma_\xi + \int_{\mathbb{T}} |g|^2 d\sigma_\xi - \int_{\mathbb{T}} |f \wedge g|^2 d\sigma_\xi.
\end{aligned}$$

Combining the last inequality and (4.12), we get (i). While the inequality (ii) will be obtained by the same argument as above. This completes the proof. □

Remark 1. *Note that Theorem 4.1.1 and (4.4) can be extended to a probability space (Ω, σ). Namely, let $f, g \in L^2(\sigma)$ such that $\sigma(\{|fg| = 0\}) = 0$. Then*

i) $\mathcal{E}(f \wedge g) \leq \mathcal{E}(f) + \mathcal{E}(g)$,

ii) $\mathcal{E}(f \vee g) \leq \mathcal{E}(f) + \mathcal{E}(g)$,

iii) $\mathcal{E}(f \wedge f^2) \leq 4\mathcal{E}(f)$.

where $\mathcal{E}(f) := \int_\Omega |f|^2 d\sigma - \exp \int_\Omega \log |f|^2 d\sigma$.

Now let us list some results about local Dirichlet integral that will be useful in Section 4.5. These results were previously stated in [64] for $\xi \in \mathbb{T}$, and they will be considered in what follows. In particular let $f \in \mathrm{H}^2$ and let $\mathcal{I}$ be an inner function, we have for $\xi \in \overline{\mathbb{D}}$

$$\begin{aligned}
\mathcal{D}_\xi(\mathcal{I}f) &= \int_{\mathbb{T}} \frac{|f(\zeta)|^2 - |\mathcal{I}(\xi)|^2 |f(\xi)|^2}{|\zeta - \xi|^2} dm(\zeta) \\
&= \int_{\mathbb{T}} \frac{|f(\zeta)|^2 - |f(\xi)|^2}{|\zeta - \xi|^2} dm(\zeta) \\
&\quad + |f(\xi)|^2 \int_{\mathbb{T}} \frac{|\mathcal{I}(\zeta)|^2 - |\mathcal{I}(\xi)|^2}{|\zeta - \xi|^2} dm(\zeta) \\
&= \mathcal{D}_\xi(f) + |f(\xi)|^2 \mathcal{D}_\xi(\mathcal{I}).
\end{aligned} \tag{4.13}$$

Thus, let $f \in \mathcal{D}(\mu)$ and write $f := \mathcal{I}\mathcal{O}$ the inner-outer factorization. From (4.13) and Theorem 4.1.1 we have

$$\begin{aligned}
\mathcal{D}_\xi(\mathcal{I}(\mathcal{O} \wedge 1)) &= \mathcal{D}_\xi(\mathcal{O} \wedge 1) + |(\mathcal{O} \wedge 1)(\lambda)|^2 \mathcal{D}_\xi(\mathcal{I}) \\
&\leq \mathcal{D}_\xi(\mathcal{O}) + |\mathcal{O}(\xi)|^2 \mathcal{D}_\xi(\mathcal{I}) \\
&= \mathcal{D}_\xi(f).
\end{aligned}$$

Moreover, let $f \in \mathrm{H}^2$ and $\varphi \in \mathrm{H}^\infty$, it is easy to see that, $\xi \in \mathbb{D}$

$$\mathcal{D}_\xi(\varphi f) \leq 2\|\varphi\|_{\mathrm{H}^\infty}^2 \mathcal{D}_\xi(f) + 2\mathcal{D}_\xi(\varphi)|f(\xi)|^2$$

and

$$\mathcal{D}_\xi(\varphi)|f(\xi)|^2 \leq 2\|\varphi\|^2_{\mathrm{H}^\infty}\mathcal{D}_\xi(f) + 2\mathcal{D}_\xi(\varphi f)$$

Combining Theorem 4.1.1, (4.4), (4.3) and the above dissucion. We have the following corollary

Corollary 3. *Let μ be a positive finite measure on $\overline{\mathbb{D}}$. Let f and g be an outer functions in H^2. Then*

i) $\mathcal{D}_\mu(f \vee g) \leq \mathcal{D}_\mu(f) + \mathcal{D}_\mu(g)$,

ii) $\mathcal{D}_\mu(f \wedge g) \leq \mathcal{D}_\mu(f) + \mathcal{D}_\mu(g)$,

iii) $\mathcal{D}_\mu(f \wedge f^2) \leq 4\mathcal{D}_\mu(f)$,

iv) let $\mathcal{I}$ be an inner function such that $\mathcal{I}f \in \mathcal{D}(\mu)$, we have

$$\mathcal{D}_\mu(\mathcal{I}(f \wedge 1)) \leq \mathcal{D}_\mu(\mathcal{I}f).$$

4.2.2 *Equivalent Norms*

Our starting point in this subsection is an equivalent norm. Befor that let us setting some notation. Let μ be a nonnegative finite Borel measure on $\overline{\mathbb{D}}$. The following result was stated in [8]. For the sake of completeness we include the proof here.

Lemma 4.2.1. *Let μ be a nonnegative finite Borel measure on $\overline{\mathbb{D}}$. Let $f \in \mathrm{H}^2$. Then*

$$\mathcal{D}_\mu(f) \asymp \int_{\mathbb{D}} |f'(z)|^2 \Psi_\mu(z) dA(z).$$

This equivalent norm will be very useful to obtain (4.8).

Proof. The classical identity of Littlewood-Paley (see for instance [38, P.229]) says that

$$\int_{\mathbb{D}} |f'(z)|^2 \log\frac{1}{|z|} dA(z) \asymp \int_{\mathbb{D}} |f'(z)|^2 (1-|z|^2) dA(z).$$

Thus, using Fubini's theorem, we have

$$\begin{aligned}
\int_{\mathbb{D}} |f'(z)|^2 U_{\mu_{\mathbb{D}}}(z) dA(z) &= \int_{\mathbb{D}}\int_{\mathbb{D}} |f'(z)|^2 \log\frac{1}{|\varphi_w(z)|} dA(z) \frac{d\mu_{\mathbb{D}}(w)}{1-|w|^2} \\
&= \int_{\mathbb{D}}\int_{\mathbb{D}} |(f\circ\varphi_w)'(z)|^2 \log\frac{1}{|z|} dA(z) \frac{d\mu_{\mathbb{D}}(w)}{1-|w|^2} \\
&\asymp \int_{\mathbb{D}}\int_{\mathbb{D}} |(f\circ\varphi_w)'(z)|^2 (1-|z|^2) dA(z) \frac{d\mu_{\mathbb{D}}(w)}{1-|w|^2} \\
&= \int_{\mathbb{D}}\int_{\mathbb{D}} |f'(z)|^2 (1-|\varphi_w(z)|^2) dA(z) \frac{d\mu_{\mathbb{D}}(w)}{1-|w|^2} \\
&= \int_{\mathbb{D}} |f'(z)|^2 \int_{\mathbb{D}} \frac{(1-|\varphi_w(z)|^2)}{1-|w|^2} d\mu_{\mathbb{D}}(w) dA(z),
\end{aligned}$$

where $\varphi_z(w) := \frac{z-w}{1-\bar{z}w}$ for $z, w \in \mathbb{D}$. The disered equivalent follows directly from (4.1) and (4.6). □

As consequence, if $\mu = \lambda \otimes m$ where λ is a given finite positive Borel measure on $[0,1)$, then, for $f \in \mathcal{D}(\lambda \otimes m)$, we have

$$\|f\|^2_{\lambda\otimes m} \asymp \sum_{n\geq 0} \psi_\lambda(n)|\hat{f}(n)|^2 =: \|f\|^2_{\mathcal{D}(\lambda\otimes m)}, \tag{4.14}$$

where $\psi_\lambda(0) = 1$ and $\psi_\lambda(n) := 2n^2 \int_0^1 r^{2n-1}\Psi_\lambda(r)dr$ for $n \geq 1$.

An interesting and useful formula can be obtained from Green's formula (see for instance [38]). Namely we have

$$\int_{\mathbb{D}} |f'(w)|^2 \log\left|\frac{1-\overline{w}\xi}{\xi - w}\right| dA(w) = P(|f|^2)(\xi) - |f(\xi)|^2 \quad (\xi \in \mathbb{D}). \tag{4.15}$$

Put together the last equality, (4.12) and (4.2). Therefore, we get

$$\mathcal{D}_\mu(f) = \int_{\mathbb{D}} \mathcal{D}_\xi(f)d\mu(\xi).$$

4.3 KERNEL ESTIMATES

The main purpose of this section is the estimate of the reproducing kernel on the diagonal. The general result is a lower estimate and an upper estimate. We use the mean inequality to obtain the upper inequality. However, a lower inequality will be obtained by estimating the norm of a test function. That is the conclusion of Thoerem 13. In particular, we get an easier and more precise results in some situations. That is the aim of Corollary 4.3.1 and Theorem 4.3.2. Indeed, if λ be a finite positive Borel measure on $[0,1)$ which satisfy $(\mathcal{R}_1)$, from (4.9) and (4.8) we have

$$K^{\lambda\otimes m}(z,z) \asymp \frac{\log\frac{1}{1-|z|}}{(1-|z|)+\Psi_\lambda(z)}, \qquad (z \in \mathbb{D}).$$

Thus, we get $1 + \tilde{\mathcal{C}}_\mu(z) \asymp ((1-|z|^2) + \Psi_\mu(z))\log(1/(1-|z|))$. This implises (i) of Corollary 4.3.1. However, the second function of the left hand sides of (4.8) is also important. In fact, from the condition $(\mathcal{R}_3)$ one can see that

$$\int_0^{|z|} \frac{dr}{(1-r)^2 + (1-r)\Psi_\lambda(r)} \lesssim \frac{1}{(1-|z|^2)+\Psi_\lambda(z)}, \qquad (z \in \mathbb{D}).$$

As result, we get the desired estimate. That is the second objective of the following corollary.

Corollary 4.3.1. *Let λ be a finite positive Borel measure on $[0,1)$.*

i) If the condition $(\mathcal{R}_1)$ *is satisfied then*

$$K^{\lambda\otimes m}(z,z) \asymp \frac{\log^2 \frac{e}{1-|z|}}{1+\tilde{\mathcal{C}}_{\lambda\otimes m}(z)}, \qquad (z \in \mathbb{D}).$$

ii) If the condition $(\mathcal{R}_3)$ *is satisfied then*

$$K^{\lambda\otimes m}(z,z) \asymp \frac{1}{(1-|z|^2)+\Psi_\lambda(z)}, \qquad (z \in \mathbb{D}).$$

Furthemore, in both cases the reproducing kernel in the diagonal is also equivalent to the integral in the right hand sides of (4.8)*. This means that the two inequalities of* (4.8) *are optimal.*

Now, let us illustrate this discussion by some standard examples.

- Let $(\alpha,\beta) \in (0,1)\times\mathbb{R}\cup\{1\}\times\mathbb{R}^+$. We consider the measure $\mu_{\alpha,\beta}$ given by

$$d\mu_{\alpha,\beta}(z) := \begin{cases} (1-|z|^2)^{\alpha-1}\log^\beta \frac{1}{1-|z|^2}dA(z) & \text{if} \quad \alpha\in(0,1) \\ \\ \log^{\beta-1}\frac{1}{1-|z|^2}dA(z) & \text{if} \quad \alpha=1. \end{cases}$$

Note that in some particular cases our space coincide with some classical spaces. In fact, if $\alpha\in(0,1)$ we have $\mathcal{D}_{\mu_{\alpha,0}} = \mathcal{D}_\alpha$ the standard Dirichlet space, also $\mathcal{D}_{\mu_{1,0}} = \mathrm{H}^2$ the Hardy space.
It is not difficult to see that

$$U_{\mu_{\alpha,\beta}}(z) \asymp (1-|z|^2)^\alpha \log^\beta \frac{1}{1-|z|^2}, \qquad (z\in\mathbb{D}).$$

By an elementary calculation we get

$$\Psi_{\mu_{\alpha,\beta}}(x)/(1-x)^\varepsilon \asymp (1-x)^{\alpha-\varepsilon}\log^\beta\frac{1}{1-x}, \quad \text{for } x\in(0,1),$$

Thus, it follows from (i) of Corollary 4.3.1 that

$$K^{\mu_{\alpha,\beta}}(z,z) \asymp \frac{1}{(1-|z|^2)+\Psi_{\mu_{\alpha,\beta}}(z)}$$

Nevertheless, one can see that $\tilde{\mathcal{C}}_{\mu_{\alpha,\beta}}(a)\asymp 1$, $a\in(0,1)$. And so

$$\frac{\tilde{\mathcal{C}}_{\mu_{\alpha,\beta}}(a)}{\Psi_{\mu_{\alpha,\beta}}(a)\log\frac{1}{1-a}} \asymp \frac{1}{(1-a)^\alpha\log^{1+\beta}\frac{1}{1-a}} \to \infty, \quad a\to 1^-.$$

This means that the measure $\mu_{\alpha,\beta}$ does not satisfy (i) of Corollary 4.3.1.

- We define the measure $\mu_{0,\beta}$, for $\beta \in (-\infty, 0)$, by

$$d\mu_{0,\beta}(z) := \frac{1}{(1-|z|^2)} \log^{\beta-1} \frac{1}{1-|z|^2} dA(z).$$

By an ordinary calculate one can see that

$$\tilde{C}_{\mu_{0,\beta}}(z) \asymp \log^{\beta+1} \frac{1}{1-|z|^2} \quad \text{if } \beta \in (-1,0).$$

And

$$\Psi_{\mu_{0,\beta}}(z) \asymp \log^{\beta} \frac{1}{1-|z|^2} \asymp U_{\mu_{0,\beta}}(z)$$

Remark that $\Psi_{\mu_{0,\beta}}(a)/(1-a)^{\varepsilon} \to +\infty$ as $a \to 1^-$. Moreover, for $\beta \in (-1,0)$, we have $\tilde{C}_{\mu_{0,\beta}}(a) \asymp \Psi_{\mu_{0,\beta}}(a) \log \frac{1}{1-a}$, $a \in (0,1)$. Therefore, by Corollary 4.3.1, we obtain

$$K^{\mu_{0,\beta}}(z,z) \asymp \frac{\log^2 \frac{e}{1-|z|}}{1+\tilde{C}_{\mu_{0,\beta}}(z)} \asymp \frac{\log \frac{e}{1-|z|}}{(1-|z|)+\Psi_{\mu_{0,\beta}(z)}}.$$

However, there exists a measure which not satisfy (i) and (ii). For instance we consider the measure $\mu_{0,\beta}$ for $\beta \leq -1$. In fact, we have

$$\tilde{C}_{\mu_{0,\beta}}(z) \asymp \begin{cases} \log\log \frac{e}{1-|z|^2} & \text{if } \quad \beta = -1 \\ \\ 1 & \text{if } \quad \beta < -1. \end{cases}$$

This implises that

$$\lim_{a\to 1^-} \frac{\Psi_{\mu_{0,\beta}}(a)}{(1-a)^{\varepsilon}} = \lim_{a\to 1^-} \frac{\tilde{C}_{\mu_{0,\beta}}(a)}{\Psi_{\mu_{0,\beta}}(a) \log \frac{1}{1-a}} = +\infty, \quad \varepsilon > 0.$$

That means one cannot give an equivalent of $K^{\mu_{0,\beta}}(z,z)$, $z \in \mathbb{D}$ using Corollary 4.3.1. Also, the failures of (4.8) can be explained by the latter example ($\beta \leq -1$). Indeed, for $a \in (1/2, 1)$, we have $\int_0^a \frac{dr}{(1-r)^2+(1-r)\Psi_{\mu_{0,\beta}}(r)} \asymp \log^{1-\beta} \frac{e}{1-a}$ and

$$h_{\mu_{0,\beta}} \asymp \begin{cases} \log^2 \frac{e}{1-a^2} / \log\log \frac{e}{1-a^2} & \text{if } \quad \beta = -1 \\ \\ \log^2 \frac{e}{1-a^2} & \text{if } \quad \beta < -1. \end{cases}$$

where $h_{\mu_{0,\beta}} := \frac{\log^2 \frac{e}{1-a}}{1+\tilde{C}_{\mu_{0,\beta}}(a)} + \frac{1}{(1-a^2)+\Psi_{\mu_{0,\beta}}(a)}$. This implies that both estimations, upper-lower, in (4.8) are not equivalent.

That means one cannot get an equivalent of reproducing kernel in the diagonal for the space $\mathcal{D}(\mu_{0,\beta})$, $\beta \leq -1$, by (4.8). This equivalent is still desired.

Now we shall establish a result for measures that satisfy $(\mathcal{R}_2)$. In particular, this condition is checked by the measure $\mu_{0,\beta}$, $\beta \leq -1$. And the conclusion of the following theorem gives an answer of the obove question.

Theorem 4.3.2. *Let λ be a finite positive Borel measure on $[0,1)$. Assume that $(\mathcal{R}_2)$ is satisfied. Then we have*

$$K^{\lambda\otimes m}(z,z) \asymp \frac{\log \frac{e}{1-|z|}}{(1-|z|)+\Psi_\lambda(z)}, \qquad (z \in \mathbb{D}).$$

The proof of Theorem 4.3.2 is based on both lemmas below. In lemma 4.3.3 we will give an estimate of the sequence $(\|z^n\|_\mu)_n$. While, within Lemma 2 we get that some series are asymptotically equivalent to more practical functions .

Proof. Now we will use the norm formula given by (4.14). Obviously the sequence $(z^n/\|z^n\|_{\mathcal{D}(\lambda\otimes m)})_n$ is an orthonormal basis for $\mathcal{D}(\mu)$ with respect the norm $\|.\|_{\mathcal{D}(\lambda\otimes m)}$. From lemmas 4.3.3 and 4.3.4 , $1/2 < |z| < 1$, we have

$$\begin{aligned} K^\mu(z,z) = 1 + \sum_{n\geq 1} \frac{|z|^{2n}}{\|z^n\|_{\mathcal{D}(\lambda\otimes m)}} &= 1 + \sum_{n\geq 1} \frac{|z|^{2n}}{\psi_\lambda(n)} \\ &\asymp 1 + \sum_{n\geq 1} \frac{|z|^{2n}}{n\Psi_\lambda(n/(n+1))} \\ &\asymp \frac{\log(1/(1-|z|^2))}{\Psi_\lambda(z)}. \end{aligned}$$

The equivalent is obvious for $|z| \leq 1/2$. □

Firstly, we get an estimate of the sequence $(\psi_\lambda(n))_{n\in\mathbb{N}}$ using only the growth of the function Ψ_λ and the fact that $t \mapsto \frac{\Psi_\lambda(1-t)}{t}$ is a decreasing function.

Lemma 4.3.3. *Let λ be a finite positive Borel measure on $[0,1)$. Suppose that $(\mathcal{R}_2)$ is satisfied. We have*

$$\int_0^1 t^n \Psi_\lambda(t)dt \asymp \frac{\Psi_\lambda(n/(n+1))}{n+1}, \qquad \textit{for all } n \in \mathbb{N}.$$

Remark that, if $t \mapsto \Psi_\lambda(1-t)/t$ is a decreasing function on $(1-a,1)$ for some $0 < a < 1$. We will use the function $\tilde{\Psi}_\lambda$ defined by $\tilde{\Psi}_\lambda(t) := \Psi_\lambda(t)$ for $t \in (1-a,1)$ and $\tilde{\Psi}_\lambda(t) := \Psi_\lambda(a)$ else.

Proof. Using the growth of the function Ψ_λ, we then obtain

$$\int_{\frac{n}{n+1}}^{1} t^n \Psi_\lambda(t)dt \leq \Psi_\lambda(n/(n+1)) \int_{\frac{n}{n+1}}^{1} (1-t)^n dt \leq \frac{\Psi_\lambda(n/(n+1))}{n+1}$$

Since $t \mapsto \Psi_\lambda(t)/(1-t)$ is a decreasing function, so we have

$$\begin{aligned} \int_0^{\frac{n}{n+1}} t^n \Psi_\lambda(t)dt &\leq (n+1)\Psi_\lambda(n/(n+1)) \int_0^{\frac{n}{n+1}} t^n(1-t)dt \\ &\asymp \frac{\Psi_\lambda(n/(n+1))}{n+1}. \end{aligned}$$

On the other hand, using again the monotony of Ψ_λ, we get

$$\int_0^{\frac{n}{n+1}} t^n \Psi_\lambda(t)dt \geq \Psi_\lambda(n/(n+1)) \int_{\frac{n-1}{n+1}}^{\frac{n}{n+1}} t^n dt \asymp \frac{\Psi_\lambda(n/(n+1))}{n+1}$$

Consequently, the desired equivalent is obtained. And the proof is complete. □

The following lemma present an asymptotic equivalent of some series. Keep in mind Lemma 4.3.3 and (4.14). As consequence, we obtain an estimate of the reproducing kernel on the diagonal for that class.

Lemma 4.3.4. *Let λ be a finite positive Borel measure on $[0,1)$. Assume that $(\mathcal{R}_2)$ is satisfied. We have*

$$\sum_{n\geq 0} \frac{t^n}{(n+1)\Psi_\lambda(\frac{n}{n+1})} \asymp \frac{\log(e/(1-t))}{\Psi_\lambda(t)}$$

Proof. The result is obvious for $t \in (0,1/2)$. Let $t \in (1/2,1)$, since $y \mapsto \Psi_\lambda(1-y)/y$ is a decreasing function on $(0,1)$ we have

$$\int_{\frac{1}{1-t}}^{+\infty} \frac{t^x}{x\Psi_\lambda(1-\frac{1}{x})}dx \leq \frac{1-t}{\Psi_\lambda(t)} \int_{\frac{1}{1-t}}^{+\infty} t^x dx = \frac{1-t}{\Psi_\lambda(t)} \frac{t^{\frac{1}{1-t}}}{\log\frac{1}{t}} \asymp \frac{1}{\Psi_\lambda(t)}.$$

On the other hand, using the growth of the function Ψ_λ, we have

$$\int_1^{\frac{1}{1-t}} \frac{t^x}{x\Psi_\lambda(1-\frac{1}{x})}dx \leq \frac{1}{\Psi_\lambda(t)} \int_1^{\frac{1}{1-t}} \frac{1}{x}dx \asymp \frac{\log(e/(1-t))}{\Psi_\lambda(t)}$$

Since λ satisfies $(\mathcal{R}_2)$, we have

$$\int_1^{\frac{1}{1-t}} \frac{t^x}{x\Psi_\lambda(1-\frac{1}{x})}dx \gtrsim \frac{\log(e/(1-t))}{\Psi_\lambda(t)}$$

Consequently, it follows from the monotony of the function $x \mapsto \frac{t^x}{x\Psi_\lambda(1-\frac{1}{x})}$ that

$$\sum_{n\geq 0} \frac{t^n}{(n+1)\Psi_\lambda(\frac{n}{n+1})} \asymp \int_1^{\infty} \frac{t^x}{x\Psi_\lambda(1-\frac{1}{x})}dx \asymp \frac{\log(e/(1-t))}{\Psi_\lambda(t)}.$$

This completes the proof. □

Proof of Theorem 13. Obviously the both inequalities are satisfied for $|z| \leq 1/2$.

Let $a \in (1/2, 1)$ and let $\lambda \in \mathbb{D}$. Let D_a be the disk of radius $\frac{1-a}{8}$ centered at $a - \frac{1-a}{8}$. We denote by

$$\Delta_a := \left\{ x + iy : 0 \leq x \leq a \text{ and } \frac{x-1}{2} \leq y \leq \frac{1-x}{2} \right\}.$$

Note that

$$(1 - |x+iy|) + \Psi_\mu(x+iy) \asymp (1-x) + \Psi_\mu(x), \qquad \text{for all } \; x + iy \in \Delta_a.$$

Using the subharmonicity of $|K_\lambda(a)|$ and Lemma 4.2.1, we have

$$\begin{aligned}
|K_\lambda(a)| &\leq \frac{1}{|D_a|} \int_{D_a} |K_\lambda(x+iy)| dxdy \\
&\leq \frac{1}{|D_a|} \int_{a-\frac{1-a}{4}}^{a} \int_{-\frac{1-x}{2}}^{\frac{1-x}{2}} |K_\lambda(x+iy)| dxdy \\
&\lesssim \frac{1}{1-a} \int_{a-\frac{1-a}{4}}^{a} \int_{-\frac{1}{2}}^{\frac{1}{2}} |K_\lambda(x + i(1-x)y)| dxdy \\
&\lesssim \frac{1}{1-a} \int_{a-\frac{1-a}{4}}^{a} \int_{-\frac{1}{2}}^{\frac{1}{2}} \int_0^x |\partial_t K_\lambda(t + i(1-t)y)| dtdxdy + \|K_\lambda\|_{\mathrm{H}^2} \\
&\lesssim \int_0^a \int_{-\frac{1-u}{2}}^{\frac{1-u}{2}} |\partial_{u+iv} K_\lambda(u+iv)| dv \frac{du}{1-u} + \|K_\lambda\|_{\mathrm{H}^2} \\
&\lesssim \|K_\lambda\|_\omega \left(\int_0^a \int_{-\frac{1-u}{2}}^{\frac{1-u}{2}} \frac{dvdu}{(1-u)^2((1-u^2-v^2) + \Psi_\mu(u+iv))} \right)^{\frac{1}{2}} \\
&\quad + \|K_\lambda\|_{\mathrm{H}^2} \\
&\asymp \|K_\lambda\|_\omega \left(\int_0^a \frac{du}{(1-u)^2 + (1-u)\Psi_\mu(u)} \right)^{\frac{1}{2}} + \|K_\lambda\|_{\mathrm{H}^2}
\end{aligned}$$

For $\lambda = a$, we obtain

$$K(a,a) \lesssim 1 + \int_0^a \frac{du}{(1-u)^2 + (1-u)\Psi_\mu(u)}.$$

Now, we consider the function h_a defined by $h_a := \frac{1}{2}\left(\frac{\varphi_a}{\|\varphi_a\|_\mu} + \frac{\psi_a}{\|\psi_a\|_\mu} \right)$, where

$$\varphi_a(z) := \log \frac{e}{1-az}, \quad \text{ and } \quad \psi := \frac{1}{1-az}; \quad (z \in \mathbb{D}).$$

By Cauchy's integral formula one can see that

$$\int_{\mathbb{D}} \frac{1}{|1-az|^2} \frac{1-|z|^2}{|1-\overline{\zeta}z|^2} dA(z) \lesssim \frac{1}{|1-\overline{\zeta}z|}, \qquad (\zeta \in \overline{\mathbb{D}}).$$

Consequently, using Lemma 4.2.1, we have

$$\begin{aligned}
\|\varphi_a\|_\mu^2 &\asymp |\varphi_a(0)|^2 + \int_{\mathbb{D}} |\varphi_a'(z)|^2 \Psi_\mu(z) dA(z) \\
&= 1 + a^2 \int_{\mathbb{D}} \int_{\mathbb{D}} \frac{1-|z|^2}{|1-aw|^2 |1-\bar{z}w|^2} dA(z) d\mu(w) \\
&\quad + \int_{\mathbb{T}} \int_{\mathbb{D}} \frac{1}{|1-az|^2} \frac{1-|z|^2}{|\zeta - z|^2} dA(z) d\mu(\zeta) \\
&\lesssim 1 + \int_{\mathbb{D}} \frac{1}{|1-aw|} d\mu(w) + \int_{\mathbb{T}} \frac{1}{|\zeta - a|} d\mu(\zeta) \\
&= 1 + \tilde{\mathcal{C}}_\mu(a).
\end{aligned}$$

On the other hand, it follows from Lemma 5.2 of [36] that

$$\int_{\mathbb{D}} \frac{1}{|1-az|^4} \frac{1-|z|^2}{|1-\bar{\zeta}z|^2} dA(z) \lesssim \frac{1}{1-a^2} \frac{1}{|1-\bar{\zeta}a|^2} \qquad (\zeta \in \overline{\mathbb{D}}).$$

Therefore, we get

$$\begin{aligned}
\|\psi_a\|_\mu^2 &\asymp \|\psi_a\|_{\mathrm{H}^2}^2 + \int_{\mathbb{D}} |\psi_a'(z)|^2 \Psi_\mu(z) dA(z) \\
&\asymp \frac{1}{1-a^2} + \int_{\mathbb{D}} \int_{\mathbb{D}} \frac{1-|z|^2}{|1-az|^4 |1-\bar{z}w|^2} dA(z) d\mu(w) \\
&\quad + \int_{\mathbb{T}} \int_{\mathbb{D}} \frac{1}{|1-az|^4} \frac{1-|z|^2}{|\zeta - z|^2} dA(z) d\mu(\zeta) \\
&\lesssim \frac{1}{1-a^2} + \frac{1}{1-a^2} \int_{\mathbb{D}} \frac{1}{|1-aw|^2} d\mu(w) + \frac{1}{1-a^2} \int_{\mathbb{T}} \frac{1}{|\zeta - a|^2} d\mu(\zeta) \\
&= \frac{1}{1-a^2} + \frac{\Psi_\mu(a)}{(1-a^2)^2} = ((1-a^2) + \Psi_\mu(a)) \psi_a^2(a).
\end{aligned}$$

This means that

$$\frac{\psi_a^2(a)}{\|\psi_a\|_\mu^2} \gtrsim \frac{1}{(1-a^2) + \Psi_\mu(a)}.$$

One can see by duality that $\|K_a^\mu\|_\mu^2 = \sup\left\{ |f(a)|^2 : \|f\|_\mu^2 \leq 1 \right\}$. Therefore we have

$$\begin{aligned}
\|K_a\|_\mu^2 \geq |h_a(a)|^2 &\asymp \frac{\varphi_a^2(a)}{\|\varphi_a\|_\mu^2} + \frac{\psi_a^2(a)}{\|\psi_a\|_\mu^2} \\
&\gtrsim \frac{\log^2 \frac{e}{1-a}}{1 + \tilde{\mathcal{C}}_\mu(a)} + \frac{1}{(1-a^2) + \Psi_\mu(a)}.
\end{aligned}$$

And the proof is complete. □

4.4 CAPACITY

The main purpose of this section is to extend the converse strong-type inequality for capacity with respect superharmonically weighted

Dirichlet spaces. That result was established in [29] for classical Dirichlet space (see also [34]).

Let us introduce a notion of capacity. Let μ be a finite positive Borel measure on $\overline{\mathbb{D}}$. The *weighted Dirichlet integral* for a given $f \in H^2$ can be presented by the following formula

$$\mathcal{D}_\mu(f) = \int_{\mathbb{T}}\int_{\mathbb{T}} |f(\zeta)-f(\eta)|^2 A_{\mu_{\mathbb{D}}}(\zeta,\eta)dm(\zeta)dm(\eta) + \int_{\mathbb{T}} \mathcal{D}_\xi(f)d\mu_{\mathbb{T}}(\xi)$$

where $A_{\mu_{\mathbb{D}}}(\zeta,\eta) = \int_{\mathbb{D}} P_z(\zeta)P_z(\eta)d\mu_{\mathbb{D}}(z)$. That result was stated in [8]. The space $\mathcal{D}^h(\mu)$ given by

$$\mathcal{D}^h(\mu) := \{f \in L^2(\mathbb{T}) : \ \|f\|_\mu^2 := \|f\|^2_{L^2(\mathbb{T})} + \mathcal{D}_\mu(f) < \infty\},$$

That space is a Dirichlet space in the sense of Beurling-Deny (see [10, 37]). In particular the capacity c_μ associated with $\mathcal{D}^h(\mu)$ of a subset $E \subset \mathbb{T}$ is defined by

$$c_\mu(E) = \inf\{\|f\|_\mu^2 : \ f \in \mathcal{D}^h(\mu) \text{ and } f \geq 1 \ m-a.e. \text{ on a neighborhood of E }\}.$$

A property is said to be satisfied c_μ-quasi everywhere ($c_\mu-$q.e.), if it is verified everywhere outside a set of $c_\mu-$capacity zero.

To obtain a strong type inequality for capacity c_μ, we will use trancature method . Let φ be a nondecreasing function in $\mathcal{C}_0^\infty(\mathbb{R})$ such that

$$\varphi(t) = \begin{cases} 0 & \text{if} \quad t \leq 1/2, \\ \\ 1 & \text{if} \quad t \geq 1. \end{cases}$$

Consider the smooth truncation $(F_j)_{j\in\mathbb{Z}}$, where

$$F_j(f) = 2^j\varphi(2^{-j}|f|) \quad (j \in \mathbb{Z}).$$

Note that, D.R. Adams used this smooth truncation in [1] to shows a strong-type estimate for Sobolev space on $\mathbb{R}$. Using this technique, one can extend this result for some spaces, see for instance [21, 41, 75, 78]. In particular, it can be obtained for superharmonicaly weighted Dirichlet spaces following by the same argument. For the sake of completeness we include the proof here. It is easy to see that $\|F_j(f)\|^2_{L^2(m)} \leq 2^{2j}m(|f| > 2^{j-1})$. Therefore we have

$$\begin{aligned}\sum_{-\infty}^{+\infty} \|F_j(f)\|^2_{L^2(m)} &\leq \sum_{-\infty}^{+\infty}\int_{2^{j-2}}^{2^{j-1}} 2^{j+2}m(|f| > 2^{j-1})dt \\ &\lesssim \int_0^\infty m(|f| > t)2tdt = \|f\|^2_{L^2(m)}.\end{aligned}$$

Note that, if $2^{j-1} \leq |f| < 2^j$ then, $F_k(f) = 2^k$ for $k < j$ and $F_k(f) = 0$ for $k > j$. Thus, by the mean value theorem, we obtain

$$\sum_{-\infty}^{+\infty} |F_k(f(\zeta)) - F_k(f(\eta))|^2 \lesssim |f(\zeta)-f(\eta)|^2 \qquad (\zeta,\eta \in \mathbb{T}).$$

It is follows that

$$\sum_{-\infty}^{+\infty} \|F_k(f)\|_\mu^2 \lesssim \|f\|_\mu^2. \tag{4.16}$$

As a consequence of the above discussion, we get a strong-type inequality for c_μ-capacity. That is the conclusion of the following theorem.

Theorem 4.4.1. *Let $f \in \mathcal{D}^h(\mu)$, then*

$$\int_0^\infty c_\mu(|f| > t)tdt \lesssim \|f\|_\mu^2.$$

The general version with Dirichlet forms of this result was proved in [37].

Proof. Remark that $2^{-k}F_k(f) \geq 1$ on $\{|f| > 2^k\}$. This means that $2^{-k}F_k(f)$ is a test function for $\{|f| > 2^k\}$. In particular $c_\mu(|f| > 2^k) \leq 2^{-2k}\|F_k(f)\|_\mu^2$. Thus, by (4.16), we have

$$\begin{aligned} \int_0^\infty c_\mu(|f| > t)tdt &= \sum_{-\infty}^{+\infty} \int_{2^k}^{2^{k+1}} c_\mu(|f| > t)tdt \\ &\lesssim \sum_{-\infty}^{+\infty} 2^{2k} c_\mu(|f| > 2^k) \\ &\lesssim \sum_{-\infty}^{+\infty} \|F_k(f)\|_\mu^2 \lesssim \|f\|_\mu^2. \end{aligned}$$

And the proof is complete. □

To state the main result in this section, we need to develop some important properties of the notion of potential. A function f defined on $\mathbb{T}$ is called quasi continuous if for every $\varepsilon > 0$ there exists an open subset U with $c_\mu(U) < \varepsilon$ and such that f is continuous on $\mathbb{T} \setminus U$. It is known that for each function $f \in \mathcal{D}^h(\mu)$, there exists a quasi continuous function $\tilde{f} \in \mathcal{D}^h(\mu)$ such that $\tilde{f} = f$ a.e. on $\mathbb{T}$, see for instance Theorem 2.1.3 of [37].

A positive Borel measure λ on $\mathbb{T}$ is of finite energy (with respect to $\mathcal{D}^h(\mu)$) if the linear from $f \mapsto \int f d\lambda$ is contiuous on $\mathcal{D}^h(\mu)$. The set of positive Borel measures supported by, a given closed subset of $\mathbb{T}$, E and with finite energy (with respect to $\mathcal{D}_\mu^h$) will be denoted by $S_\mu(E)$. Let $\lambda \in S_\mu(\mathbb{T})$. The measure λ charges no sets of $c_\mu-$capacity zero. As a consequence, there exists an unique function $g_\lambda \in \mathcal{D}^h(\mu)$ such that

$$\langle g_\lambda, f \rangle = \int \tilde{f}(\zeta) d\lambda(\zeta), \quad (f \in \mathcal{D}_\mu^h).$$

The function g_λ is called the potential of λ with respect $\mathcal{D}^h(\mu)$. Consequently we have the fundamental theorem of potential theory.

Theorem 4.4.2. *Let E be a closed subset of $\mathbb{T}$ such that $c_\mu(E) > 0$. There exists a unique measure $\lambda_E \in S_\mu(E)$ such that*

$$c_\mu(E) = \|g_{\lambda_E}\|_\mu^2 = \lambda_E(E).$$

Moreover, we have $0 \leq g_{\lambda_E} \leq 1$ and $\tilde{g}_{\lambda_E} = 1$ $c_\mu - q.e.$ on E.

This result was stated in [7] (see also [34, 66]). For more details on the general potential theory associated with Dirichlet forms we refer to [37].

Let K^μ be the reproducing kernel of $\mathcal{D}_\mu$. The Poisson transform of a function $f \in \mathcal{D}_\mu^h$ will be also denoted by f. Let $\lambda \in S_\mu(\mathbb{T})$ we have

$$g_\lambda(z) = \int_{\mathbb{T}} (2\mathrm{Re}K_z^\mu(\zeta) - 1) d\lambda(\zeta),$$

and

$$f_\lambda(z) := \langle g_\lambda, K_z^\mu \rangle = \int_{\mathbb{T}} K_z^\mu(\zeta) d\lambda(\zeta), \quad (z \in \mathbb{D}). \tag{4.17}$$

The following lemma is an immediate result of what has been said above.

Lemma 4.4.3. *Let E be a closed subset of $\mathbb{T}$ such that $c_\mu(E) > 0$. Then* $\mathrm{Re} f_{\lambda_E} \geq 0$, $\mathrm{Re} f_{\lambda_E} = 1$ *$c_\mu - q.e.$ on E and $\|f_{\lambda_E}\|_\mu^2 \asymp c_\mu(E)$.*

Now we are ready to establish the converse of strong-type inequality for c_μ-capacity. Using the same argument proposed in [29]. We obtain the following theorem.

Theorem 4.4.4. *Let E be a closed subset of $\mathbb{T}$ and let $\eta : (0, \pi] \to (0, +\infty)$ be a decreasing, continuous function such that $\eta(0^+) = \infty$. The following are equivalent*

1. *There exists a functions $f \in \mathcal{D}(\mu)$ such that*

$$\liminf_{z \to \zeta} |f(z)| \geq \eta(d(\zeta, \mathrm{E})) \qquad (\zeta \in \mathbb{T}). \tag{4.18}$$

2. *There exists a functions $f \in \mathcal{D}(\mu)$ such that*

$$\liminf_{z \to \zeta} \mathrm{Re} f(z) \geq \eta(d(\zeta, \mathrm{E})) \qquad (\zeta \in \mathbb{T}). \tag{4.19}$$

3. *The function η and the set E satisfy*

$$\int_0^\pi c_\mu(\mathrm{E}_t) |d\eta^2(t)| < \infty. \tag{4.20}$$

This result can be seen as a generalization of Theorem 5.1 of [29] (see also [34]).

Proof. (2) $\Longrightarrow$ (1) is obvious.
Now assume that there is a function $f \in \mathcal{D}(\mu)$ which satisfies (4.18). By Theorem 4.4.1 we have

$$\int_0^{\pi} c_\mu(\mathrm{E}_t)|d\eta^2(t)| \le \int_0^{\pi} c_\mu(\{|f| \ge \eta(t)\})|d\eta^2(t)| \lesssim \|f\|_\mu^2.$$

This means that (1) $\Longrightarrow$ (3).
It remains to prove that (3) $\Longrightarrow$ (2). Let n_0 be the smallest natural integer that checks $n_0 \ge \eta(\pi)$. Set $\delta_n := \eta^{-1}(n)$ for $n \ge n_0$. By the growth of the function η and (4.20) we have

$$\sum_{n=n_0+1}^{+\infty} n c_\mu(\mathrm{E}_{\delta_n}) \lesssim \int_0^{\pi} c_\mu(\mathrm{E}_t)|d\eta^2(t)| < \infty. \tag{4.21}$$

According to Theorem 4.4.2, there exists a unique measure λ_n supported on E_{δ_n} such that $c_\mu(E_{\delta_n}) = \|g_{\lambda_n}\|_\mu^2 = \lambda_n(E_{\delta_n})$ for $n \ge n_0$. Let f be the function defined by

$$f(z) := n_0 + \sum_{n \ge n_0} f_n(z), \qquad (z \in \mathbb{D}),$$

where $f_n := \langle g_{\lambda_n}, K^\mu \rangle$. Using Theorem 2 of [72] to see that $|f_n(z)| \le c_\mu(E_{\delta_n})/d(z, E_{\delta_n})$, $z \in \mathbb{D}$. The series converges locally uniformly in $\mathbb{D}$. Furthemore, for $n \ge m \ge n_0$, we have

$$\left\langle c_\mu^{-1}(E_{\delta_n}) f_n - c_\mu^{-1}(E_{\delta_m}) f_m, c_\mu^{-1}(E_{\delta_m}) f_m \right\rangle = 0.$$

It follows from (4.21) and Lemma 5.4. of [29] that $f \in \mathcal{D}_\omega$. Obviously that function satisfies (4.19). This finishes the proof. □

4.5 INVARIANT SUBSPACES

In this section we are mainly interested by the structure of the shift-invariant subspaces in superharmonically weighted Dirichlet spaces. A closed subspace $\mathcal{M}$ in $\mathcal{D}(\mu)$ is called invariant subspace (M_z-invariant subspace or shift-invariant subspace) if $M_z(\mathcal{M}) \subset \mathcal{M}$ where M_z is the operator of multiplication by z. That is a bounded operator on $\mathcal{D}(\mu)$. The lattice of all closed M_z-invariant subspaces in $\mathcal{D}(\mu)$ will be denoted by $\mathrm{Lat}(\mathrm{M}_z, \mathcal{D}(\mu))$.

The main result here is a complet characterization of $\mathrm{Lat}(\mathrm{M}_z, \mathcal{D}(\mu))$. This result is new even for standard weighted Dirichlet spaces $\mathcal{D}_\alpha = \mathcal{D}(\mu_\alpha)$, $0 < \alpha < 1$.

Given $f \in \mathcal{D}(\mu)$, we denote by $[f]_{\mathcal{D}(\mu)}$ the closed linear span in $\mathcal{D}(\mu)$ of all polynomial multiples of f, that is the smallest invariant subspace which contains f. The two following results will be very useful in several arguments.

Proposition 4.5.1. *([4]) Let $f, g \in \mathcal{D}(\mu)$ such that $|g| \le |f|$ on $\mathbb{D}$. Then $[g]_{\mathcal{D}(\mu)} \subset [f]_{\mathcal{D}(\mu)}$.*

By the same proof given in [30, 32], we obtain the following lemma.

Lemma 4.5.2. *Let μ be a nonnegative finite Borel measure on $\overline{\mathbb{D}}$. Let $\mathcal{M}$ be a closed subspace of $\mathcal{D}(\mu)$. Suppose that there exists a sequence (f_n) in $\mathcal{M}$ such that $\sup_n \mathcal{D}_\mu(f_n) < \infty$ and $f_n \to f$ pointwise in $\mathbb{D}$. Then $f \in \mathcal{M}$.*

Let $\mathcal{M}$ be an invariant subspace of $\mathcal{D}(\mu)$. As an immediate consequense of the latter result, let $\mathcal{I}$ be an inner function, we have $\mathcal{I}\mathcal{M} \cap \mathcal{D}(\mu) \subset \mathcal{M} \cap \mathcal{I}\mathrm{H}^2$. However, A. Aleman [4] poved that there exists a multiplier of $\mathcal{D}(\mu)$ such that $\mathcal{M} = [\phi]_{\mathcal{D}(\mu)} = \phi\mathcal{D}(\mu_\phi)$, where $d\mu_\phi = |\phi|^2 d\mu$. Taking advantage of this representation and (4.13) to get the converse inclusion. Thus we have

$$\mathcal{I}\mathcal{M} \cap \mathcal{D}(\mu) = \mathcal{M} \cap \mathcal{I}\mathrm{H}^2. \tag{4.22}$$

That result was obtained by Richter and Sundberg [64] for harmonically weighted Dirichlet spaces. Write $\phi = \mathcal{I}_\mathcal{M} f$ the inner-outer factorization of the function ϕ. In particular, by (4.13) and (4.3) the function f is a multiplier of $\mathcal{D}(\mu)$. Note that the inner function $\mathcal{I}_\mathcal{M}$ is the greatest common inner divisor of $\mathcal{M}$. It follows from (4.22) that

$$[\phi]_{\mathcal{D}(\mu)} \subset \mathcal{I}_\mathcal{M} [f]_{\mathcal{D}(\mu)} \cap \mathcal{D}(\mu).$$

Next, we consider a function h in $[f]_{\mathcal{D}(\mu)}$ such that $\mathcal{I}_\mathcal{M} h \in \mathcal{D}(\mu)$. It is known that every function in $\mathcal{D}(\mu)$ can be represented as a quotien of two functions in $\mathcal{D}(\mu) \cap \mathrm{H}^\infty$, moreover the denominator is inversible in $\mathcal{D}(\mu)$. So, one can assume that the function h is an outer function in $\mathcal{D}(\mu) \cap \mathrm{H}^\infty$. Taking advantage of Theorem 4.1.1, there exists a bounded sequence in $[\phi]_{\mathcal{D}(\mu)}$. Moreover that sequence converges pointwise in $\mathbb{D}$ to $\mathcal{I}_\mathcal{M} h$. Since the weak convergence in $\mathcal{D}(\mu)$ implises the pointwise convergence in $\mathbb{D}$, then $\mathcal{I}_\mathcal{M} h \in [\phi]_{\mathcal{D}(\mu)}$. For a generalt resul see Lemma 4.5.2. And the converse inclusion is obtained. Therefore we have

$$\mathcal{M} = \mathcal{I}_\mathcal{M} [f]_{\mathcal{D}(\mu)} \cap \mathcal{D}(\mu) = [f]_{\mathcal{D}(\mu)} \cap \mathcal{I}_\mathcal{M}\mathrm{H}^2. \tag{4.23}$$

As a consequence of the above discussion, we obtain a complet characterization of $\mathrm{Lat}(\mathrm{M}_z, \mathcal{D}(\mu))$. In particular, this result was obtained by Richter and Sundberg [64] for harmonic weights. Both results will be considered in the following theorem.

Theorem 4.5.3. *Let μ be a positive finite measure on $\overline{\mathbb{D}}$ and let $\mathcal{M} \in \mathrm{Lat}(\mathrm{M}_z, \mathcal{D}(\mu))$. Let $\mathcal{I}_\mathcal{M}$ be the greatest common inner divisor of $\mathcal{M}$. Then, there is an outer $f \in \mathcal{D}(\mu) \cap \mathrm{H}^\infty$ such that*

$$\mathcal{M} = \mathcal{I}_\mathcal{M}\mathrm{H}^2 \cap [f]_{\mathcal{D}(\mu)}.$$

Furthermore f can be chosen so that f and $\mathcal{I}_\mathcal{M} f$ are multipliers of $\mathcal{D}(\mu)$.

Now, we are developing a few general results about invariant subspaces of $\mathcal{D}(\mu)$ generated by outer functions. Let f be an outer function.

Let $\mathcal{I}$ be an inner function such that both functions $\mathcal{I}f$ and $\mathcal{I}f^2$ belong to $\mathcal{D}(\mu)$. We consider $\varphi_n := \mathcal{I}(f \wedge nf^2)$. This sequence, by (4.13) and (4.3), is bounded in $[\mathcal{I}f^2]_{\mathcal{D}(\mu)}$. Clearly, it converges pointwise in $\mathbb{D}$ to $\mathcal{I}f$. Therefore we have

$$[\mathcal{I}f]_{\mathcal{D}(\mu)} \subset [\mathcal{I}f^2]_{\mathcal{D}(\mu)}.$$

It is known that the invaraint subspaces generated by f and $f \wedge 1$ is the same, see [4]. So, one can assume that f is belongs to the algebra $\mathcal{D}(\mu) \cap \mathrm{H}^\infty$. From Proposition 4.5.1 and the previous inclusion one can see that

$$[f]_{\mathcal{D}(\mu)} = [f^\alpha]_{\mathcal{D}(\mu)}, \qquad (\alpha > 0).$$

Let g be an outer function in $\mathcal{D}(\mu)$. Recall that every invariant subspaces can be represented as a Dirichlet space associated with a multiplier. Take into consideration Theorem 1, we get $[f \vee g]_{\mathcal{D}(\mu)} \subset [f, g]_{\mathcal{D}(\mu)}$. The converse follows directly from Proposition 4.5.1. Namely we have

$$[f \vee g]_{\mathcal{D}(\mu)} = [f, g]_{\mathcal{D}(\mu)}.$$

Now, we assume in addition that $fg \in \mathcal{D}(\mu)$. And so

$$[fg]_{\mathcal{D}(\mu)} = [(fg) \wedge 1]_{\mathcal{D}(\mu)} \subset [(f(g \wedge 1)) \vee f]_{\mathcal{D}(\mu)} = [f(g \wedge 1), f]_{\mathcal{D}(\mu)} = [g]_{\mathcal{D}(\mu)}$$

By symmetry we obtain $[fg]_{\mathcal{D}(\mu)} \subset [f]_{\mathcal{D}(\mu)} \cap [g]_{\mathcal{D}(\mu}$.
Note that, every invariant subspaces is generated by a multiplier. In particular, $[f]_{\mathcal{D}(\mu)} \cap [g]_{\mathcal{D}(\mu)} = [\varphi]_{\mathcal{D}(\mu)}$, where φ is an outer function. By the same argument as above, one can see that $\varphi^2 \in [fg]_{\mathcal{D}(\mu)}$. Taking in account the fact that φ and φ^2 generate the same invariant subspace. And the second inclusion is obtained. Thus we have

$$[f]_{\mathcal{D}(\mu)} \cap [g]_{\mathcal{D}(\mu)} = [fg]_{\mathcal{D}(\mu)}$$

Otherwise, if ever the function fg does not belong to $\mathcal{D}(\mu)$. Note that f and $f \wedge 1$ generate the same invariant subspace. Furthemore $(f \wedge 1)(g \wedge 1) \in \mathcal{D}(\mu) \cap \mathrm{H}^\infty$. Appliying the last equality and Proposition 4.5.1, we obtain

$$[f]_{\mathcal{D}(\mu)} \cap [g]_{\mathcal{D}(\mu)} = [f \wedge g]_{\mathcal{D}(\mu)}.$$

We are gathering these results in the following theorem.

Theorem 4.5.4. *Let $f, g \in \mathcal{D}(\mu)$ be outer functions. Let $\mathcal{I}$ be an inner function in H^2.Then*

(1) $[\mathcal{I}f]_{\mathcal{D}(\mu)} \subset [\mathcal{I}f^2]_{\mathcal{D}(\mu)}$, *if* $\mathcal{I}f, \mathcal{I}f^2 \in \mathcal{D}(\mu)$,

(3) $[f^\alpha]_{\mathcal{D}(\mu)} = [f]_{\mathcal{D}(\mu)}$, *if* $f^\alpha \in \mathcal{D}(\mu)$ *for* $\alpha > 0$,

(2) $[f, g]_{\mathcal{D}(\mu)} = [f \vee g]_{\mathcal{D}(\mu)}$,

(5) $[f]_{\mathcal{D}(\mu)} \cap [g]_{\mathcal{D}(\mu)} = [fg]_{\mathcal{D}(\mu)}$, *if* $fg \in \mathcal{D}(\mu)$,

(4) $[f]_{\mathcal{D}(\mu)} \cap [g]_{\mathcal{D}(\mu)} = [f \wedge g]_{\mathcal{D}(\mu)}$.

The latter theorem and the following lemma will be useful in the Proof of Theorem 4.1.2. Using the same argument given in the proof of Lemma 3.2 of [28], we obtain the following.

Lemma 4.5.5. *Let $\zeta \in \mathbb{T}$. The following properties are equivalent:*

(1) *ζ is a bounded point evaluation of $\mathcal{D}(\mu)$.*

(2) *The polynomial $z - \zeta$ is not cyclic for $\mathcal{D}(\mu)$.*

(3) $c_\mu(\zeta) > 0$.

More speaking, we will give a complete description of the invariant subspace $[f]_{\mathcal{D}(\mu)}$ if, $\mathrm{supp}(\mu) \cap \mathbb{T}$ is countable. That is the conclusion of Theorem 4.1.2. For that, let us introduce some notations. Let $g \in \mathcal{D}(\mu)$ and $\mathcal{M} \in \mathrm{Lat}(\mathrm{M}_z, \mathcal{D}(\mu))$, we denote by

$$\underline{\mathcal{Z}}(g) = \{w \in \overline{\mathbb{D}} : \liminf_{z\in\mathbb{D},\, z\to w} |g(z)| = 0\} \quad \text{and} \quad \underline{\mathcal{Z}}(\mathcal{M}) := \bigcap_{g\in\mathcal{M}} \underline{\mathcal{Z}}(g).$$

Note that, by weak-type inequality for capacity c_μ, the invariant subspace

$$\mathcal{M}_{\mathrm{E}} := \{g \in \mathcal{D}(\mu) : g_{|\mathrm{E}} = 0 \ \ c_\mu - q.e\}$$

is closed in $\mathcal{D}(\mu)$, where $\mathrm{E} := \{\xi \in \mathrm{supp}(\mu) \cap \underline{\mathcal{Z}}(\mathcal{M}) \cap \mathbb{T} : c_\mu(\xi) > 0\}$.

Proof of Theorem 4.1.2. Let $\mathcal{M} \in \mathrm{Lat}(\mathrm{M}_z, \mathcal{D}(\mu))$. By Theorem 4.5.3 there exists an inner function $\mathcal{I}_{\mathcal{M}}$ and an outer function $f \in \mathcal{D}(\mu) \cap \mathrm{H}^\infty$ such that

$$\mathcal{M} = \mathcal{I}_{\mathcal{M}} H^2 \cap [f]_{\mathcal{D}(\mu)}.$$

Then we have, $\mathrm{E} = \{\xi \in \mathrm{supp}(\mu) \cap \underline{\mathcal{Z}}(\mathcal{M}) \cap \mathbb{T} : c_\mu(\xi) > 0\} = \{\xi \in \mathrm{supp}(\mu) \cap \mathcal{Z}(f) : c_\mu(\xi) > 0\}$.
Let $\psi \in [f]_{\mathcal{D}(\mu)}$. From Lemma 4.5.5, the evaluation at ξ is continuous on $\mathcal{D}(\mu)$, for all $\xi \in \mathrm{E}$. Thus $\psi(\xi)$ exists and $\psi(\xi) = 0$. This means that $[f]_{\mathcal{D}(\mu)} \subset \mathcal{M}_{\mathrm{E}}$.
For the reverse inclusion. By Theorem 2 of [4] there exists a multiplier $\phi \in \mathrm{H}^\infty \cap \mathcal{M}_{\mathrm{E}}$ which is an outer function such that $\mathcal{M}_{\mathrm{E}} = [\phi]_{\mathcal{D}(\mu)}$. Let $\mathcal{J}$ be the closed ideal of the Banach algebra $\mathrm{H}^\infty \cap \mathcal{D}(\mu)$ given by

$$\mathcal{J} = \left\{\psi \in \mathrm{H}^\infty \cap \mathcal{D}(\mu) : \psi\phi \in [f]_{\mathcal{D}(\mu)}\right\}$$

Note that $[f]_{\mathcal{D}(\mu)} \cap \mathrm{H}^\infty \subset \mathcal{J}$. Suppose that $\underline{\mathcal{Z}}(\mathcal{J}) = \emptyset$, it follows from Theorem 4.3 of [28] that $\mathcal{D}(\mu) \cap \mathrm{H}^\infty = \mathcal{J}$. Otherwise, if $\underline{\mathcal{Z}}(\mathcal{J}) \neq \emptyset$, using the same argument as in [28, Theorem 1], then we get $\mathcal{D}(\mu) \cap \mathrm{H}^\infty = \mathcal{J}$. Therefore we have $\mathcal{M}_{\mathrm{E}} \subset [f]_{\mathcal{D}(\mu)}$. □

We closing this section by a seconde complete description of $[f]_{\mathcal{D}(\mu_{\mathbb{D}})}$, where f is an outer function in $\mathcal{D}(\mu_{\mathbb{D}})$. Namely, if $\mu_{\mathbb{D}}$ is a Carleson measure for $\mathcal{D}(\mu_{\mathbb{D}})$ then $[f]_{\mathcal{D}(\mu_{\mathbb{D}})} = \mathcal{D}(\mu_{\mathbb{D}})$, for more details see Section 4.6. From Theorem 4.5.3, we have the following result.

Theorem 4.5.6. *Let $\mu_{\mathbb{D}}$ be a positive finite Borel measure on $\mathbb{D}$. Suppose that $\mu_{\mathbb{D}}$ is a Carleson measure for $\mathcal{D}(\mu_{\mathbb{D}})$. Let $\mathcal{M} \in \mathrm{Lat}(\mathrm{M}_z, \mathcal{D}(\mu_{\mathbb{D}}))$. Then*

$$\mathcal{M} = \mathcal{I}_{\mathcal{M}}\mathrm{H}^2 \cap \mathcal{D}(\mu_{\mathbb{D}}),$$

where $\mathcal{I}_{\mathcal{M}}$ is the greatest common inner divisor of $\mathcal{M}$.

4.6 CYCLIC VECTORS

In this section, we are going to investigate cyclic vectors in $\mathcal{D}(\mu)$. Recall that a vector $f \in \mathcal{D}(\mu)$ is called *cyclic* (*cyclic* in $\mathcal{D}(\mu)$) with respect to the multiplication operator M_z, if $[f]_{\mathcal{D}(\mu)} = \mathcal{D}(\mu)$. According to Theorem 4.5.3, every non-zero invariant subspace $\mathcal{M}$ of $\mathcal{D}(\mu)$ is of the form $\mathcal{M} = \mathcal{I}_{\mathcal{M}}\mathrm{H}^2 \cap [f]_{\mathcal{D}(\mu)}$ where f is an outer function in $\mathcal{D}(\mu)$ and $\mathcal{I}_{\mathcal{M}}$ is the greatest common inner divisor of $\mathcal{M}$. Thus, understanding the cyclic vectors in $\mathcal{D}(\mu)$ would be a significant step towards the complete description of the invariant subspaces of $\mathcal{D}(\mu)$.

A necessary conditions for cyclicity follows directly from weak-type inequality for capacity c_μ. Namely, let f be a *cyclic* vector in $\mathcal{D}(\mu)$ then f is outer and $c_\mu(\mathcal{Z}(f)) = 0$, where

$$\mathcal{Z}(f) := \left\{ \zeta \in \mathbb{T} : \lim_{r \to 1^-} f(r\zeta) = 0 \right\}.$$

This remark was stated firstly for classical Dirichlet space $\mathcal{D}$ in [15]. In 1984, Brown and Shields [15] conjectured that an outer function $f \in \mathcal{D}$ is cyclic if and only if $c_m(\mathcal{Z}(f)) = 0$. The Brown-Shields conjecture asserts that an outer function $f \in \mathcal{D}(\mu)$ is cyclic if and only if $c_\mu(\mathcal{Z}(f)) = 0$.
It was shown by D. Sarason [68] (see also [39]) that, if $\mu = \delta_\xi$ the Dirac measure at $\xi \in \mathbb{T}$, the Brown-Shields conjecture is true for the space $\mathcal{D}(\delta_\xi)$. Recently, El-Fallah, Elmadani and Kellay [28] proved that this conjecture is true for harmonically weighted Dirichlet space $\mathcal{D}(\mu_{\mathbb{T}})$ if $\operatorname{supp} \mu_{\mathbb{T}}$ is countable.

Our main result in this section is a positive answer to the Brown-Shields conjecture for $\mathcal{D}(\mu)$ when $\operatorname{supp} \mu \cap \mathbb{T}$ is countable. That is the purpose of the following theorem.

Theorem 4.6.1. *Let μ be a positive finite measure on $\overline{\mathbb{D}}$. Let $f \in \mathcal{D}(\mu)$ be an outer function such that $\operatorname{supp}(\mu) \cap \underline{\mathcal{Z}}(f)$ is countable. Then f is cyclic in $\mathcal{D}(\mu)$ if and only if $c_\mu(\mathcal{Z}(f)) = 0$.*

In particular, Theorem 1 of [28] will be obtained if μ is supported by $\mathbb{T}$.

Now, we consider $\mu_{\mathbb{D}}$ a finite positive Borel measure on $\mathbb{D}$. The balayage function S_μ of the measure μ is defined by

$$S_{\mu_{\mathbb{D}}}(\zeta) = \int_{\mathbb{D}} \frac{1-|z|^2}{|1-\overline{\zeta}z|^2} d\mu_{\mathbb{D}}(z), \ \zeta \in \mathbb{T}.$$

It follows from Fubini's theorem that $S_{\mu_{\mathbb{D}}} \in \mathrm{L}^1(\mathbb{T})$. The weighted Hardy space associated with $S_{\mu_{\mathbb{D}}}$ is given by

$$\mathrm{H}^2{}_{\mu_{\mathbb{D}}} := \left\{ f \in \mathrm{H}^2 \ : \|f\|^2_{\mathrm{H}^2{}_{\mu_{\mathbb{D}}}} := \int_{\mathbb{T}} |f(\zeta)|^2 S_{\mu_{\mathbb{D}}}(\zeta) dm(\zeta) < \infty \right\}.$$

It is well known that the polynomials are dense in the space $\mathrm{H}^2{}_{\mu_{\mathbb{D}}}$. Let $F_{\mu_{\mathbb{D}}}$ be the outer function associated with $\sqrt{S_{\mu_{\mathbb{D}}}}$. Namely

$$F_{\mu_{\mathbb{D}}}(z) = \exp\left(\int_{\mathbb{T}} \frac{\zeta+z}{\zeta-z} \log \sqrt{S_{\mu_{\mathbb{D}}}(\zeta)} dm(\zeta) \right), \ (z \in \mathbb{D}).$$

Let $f \in \mathrm{H}^2{}_\mu$. Clearly we have $\|F_{\mu_{\mathbb{D}}} f\|_{\mathrm{H}^2} = \|f\|_{\mathrm{H}^2{}_{\mu_{\mathbb{D}}}}$. This means that $f \in \mathrm{H}^2{}_{\mu_{\mathbb{D}}}$ if and only if $f \in F_{\mu_{\mathbb{D}}} \mathrm{H}^2$. Therefore f is cyclic in $\mathrm{H}^2{}_{\mu_{\mathbb{D}}}$ if and only if f is outer.
Moreover, it was proved in [8, Theorem 5.3] that $\mathrm{H}^2{}_{\mu_{\mathbb{D}}} = \mathcal{D}(\mu_{\mathbb{D}})$ if and only if $\mu_{\mathbb{D}}$ is a Carleson measure for $\mathcal{D}(\mu_{\mathbb{D}})$.
As a consequence, we have the following theorem.

Theorem 4.6.2. *Let $\mu_{\mathbb{D}}$ be a positive finite Borel measure on $\mathbb{D}$. Suppose that $\mu_{\mathbb{D}}$ is a Carleson measure for $\mathcal{D}(\mu_{\mathbb{D}})$. Then $f \in \mathcal{D}(\mu_{\mathbb{D}})$ is cyclic in $\mathcal{D}(\mu_{\mathbb{D}})$ if and only if f is an outer function.*
Furthemore, the generalized Brown-Shields conjecture is true.

To prove Theorem 1 we need some lemmas. Using the same argument as in [28] to obtain the following lemmas.

Lemma 4.6.3. *Let $f \in \mathcal{D}(\mu) \cap \mathrm{H}^\infty$ and $g \in \mathrm{H}^\infty$ be an outer functions such that $||f||_\infty \leq 1$ and $||g||_\infty \leq 1$. Let h be the outer function given by*

$$|h(\zeta)| = \begin{cases} |f(\zeta)| & a.e. on \quad V, \\ \\ |g(\zeta)| & a.e. on \quad \mathbb{T} \setminus V, \end{cases}$$

where V is a closed neighborhood of $\operatorname{supp} \mu \cap \mathbb{T}$. Then $h \in \mathcal{D}(\mu)$ and

$$\mathcal{D}_\mu(h) \leq \mathcal{D}_\mu(f) + \frac{\mu(\mathbb{D})}{\delta^2} \left(||g||_2^2 + 2|| \log(1/|g|)||_{L^1(\mathbb{T})} \right),$$

where $\delta = d(\mathbb{T} \setminus V, \operatorname{supp} \mu \cap \mathbb{T})$.

Lemma 4.6.4. *Let $f \in \mathcal{D}(\mu) \cap \mathrm{H}^\infty$ be an outer function.*

(1) *If* $\underline{\mathcal{Z}}(f) \cap \operatorname{supp}\mu = \emptyset$ *then* f *is cyclic for* $\mathcal{D}(\mu)$.

(2) $\underline{\mathcal{Z}}([f]_{\mathcal{D}(\mu)}) \subset \underline{\mathcal{Z}}(f) \cap \operatorname{supp}(\mu)$.

(3) $\underline{\mathcal{Z}}([f]_{\mathcal{D}(\mu)}) = \underline{\mathcal{Z}}([f]_{\mathcal{D}(\mu)} \cap \mathrm{H}^\infty)$.

Proof of Theorem 4.6.1. Let f be an outer function in $\mathcal{D}(\mu)$. Without loss of generality we can suppose that $f \in \mathcal{D}(\mu) \cap \mathrm{H}^\infty$. Now suppose that $c_\mu(\mathcal{Z}(f)) = 0$. By Lemma (4.6.4), we have

$$\underline{\mathcal{Z}}([f]_{\mathcal{D}(\mu)}) \subset \underline{\mathcal{Z}}(f) \cap \operatorname{supp}\mu.$$

$\underline{\mathcal{Z}}([f]_{\mathcal{D}(\mu)}) \neq \emptyset$, hence by (4.6.4) $\underline{\mathcal{Z}}([f]_{\mathcal{D}(\mu)} \cap \mathrm{H}^\infty) = \emptyset$ by [28, Theorem 4.3] we have $[f]_{\mathcal{D}(\mu)} \cap \mathrm{H}^\infty) = \mathcal{D}(\mu) \cap \mathrm{H}^\infty$. Then f is cyclic.

Suppose that $\underline{\mathcal{Z}}([f]_{\mathcal{D}(\mu)}) \neq \emptyset$. Since $\underline{\mathcal{Z}}(f) \cap \operatorname{supp}\mu$ is countable set, there exists $V = (e^{ia}, e^{ib})$ a neighborhood of an isolated point $\zeta_0 \in \underline{\mathcal{Z}}([f]_{\mathcal{D}(\mu)})$ such that $V \cap \underline{\mathcal{Z}}([f]_{\mathcal{D}(\mu)}) = \{\zeta_0\}$. Consider the outer functions

$$|h(\zeta)| = \begin{cases} |(\zeta - e^{ia})(e^{ib} - \zeta)||f(\zeta)|, & \zeta \in V \\ |(\zeta - e^{ia})(e^{ib} - \zeta)|, & \zeta \notin V \end{cases}$$

and

$$|g(\zeta)| = \begin{cases} |(\zeta - e^{ia})(e^{ib} - \zeta)|, & \zeta \in V \\ |(\zeta - e^{ia})(e^{ib} - \zeta)||f(\zeta)|, & \zeta \notin V \end{cases}$$

By Lemma 4.6.3, $h, g \in \mathcal{D}(\mu) \cap \mathrm{H}^\infty$ and $hg \in [f]_{\mathcal{D}(\mu)}$. let us consider the following closed ideal of $\mathrm{H}^\infty \cap \mathcal{D}(\mu)$

$$\mathcal{J} = \left\{\psi \in \mathrm{H}^\infty \cap \mathcal{D}(\mu) : \psi g \in [f]_{\mathcal{D}(\mu)}\right\}$$

Since $[f]_{\mathcal{D}(\mu)} \cap \mathrm{H}^\infty \subset \mathcal{J}$ and $h \in \mathcal{J}$,

$$\underline{\mathcal{Z}}(\mathcal{J}) \subset \underline{\mathcal{Z}}([f]_{\mathcal{D}(\mu)}) \cap \underline{\mathcal{Z}}(h) \subset \{\zeta_0\}.$$

Hence, $z - \zeta_0 \in \mathcal{J}$ and then $(z - \zeta_0)g \in [f]_{\mathcal{D}(\mu)}$. And we have $[(z - \zeta_0)g]_{\mathcal{D}(\mu)} = [(z - \zeta_0)]_{\mathcal{D}(\mu)} \cap [g]_{\mathcal{D}(\mu)}$. We get

$$[(z - \zeta_0)]_{\mathcal{D}(\mu)} \cap [g]_{\mathcal{D}(\mu)} \subset [f]_{\mathcal{D}(\mu)}.$$

If $c_\mu(\{\zeta_0\}) = 0$, then $z - \zeta_0$ is cyclic and hence $g \in [f]_{\mathcal{D}(\mu)}$.

If $c_\mu(\{\zeta_0\}) > 0$, using the same argument of [28, Theorem 1], we get $g \in [f]_{\mathcal{D}(\mu)}$.

We are led to a contradiction since $\zeta_0 \in \underline{\mathcal{Z}}[f]_{\mathcal{D}(\mu)}$ and $g(\zeta_0) \neq 0$. Hence $\underline{\mathcal{Z}}([f]_{\mathcal{D}(\mu)}) = \emptyset$, then f is cyclic in $\mathcal{D}(\mu)$. □

Let E be a closed subset of $\mathbb{T}$ such that. In 1952, Carleson [18] proved that, there exists a non-zero function $f \in \mathcal{D}$ (not necessarily in $A(\mathbb{D})$) such that $\mathrm{E} \subset \mathcal{Z}(f)$, if $c_m(\mathrm{E}) = 0$. Later on, in 1985 Brown and Cohn [14] constructed an explicit example of cyclic function in $\mathcal{D} \cap A(\mathbb{D})$ vanishing on E. Their construction is based on a modification of an argument used in [18]. Taking advantage of this argument, we will extend this result to superharmonically weighted Dirichlet spaces. Indeed, assume that $c_\mu(E) = 0$. So, one can find a decreasing sequence $(E_n)_n$ of closed neighborhoods of E such that

$$\sum_n c_\mu^{1/2}(E_n) < \infty.$$

From Theorem 4.4.2, there exists a unique measure λ_n supported on E_n which satisfies $c_\mu(E_n) = \|g_{\lambda_n}\|_\mu^2 = \lambda_n(E_n)$. And consider the function $h(z) := \sum_n \langle g_{\lambda_n}, K_z^\mu \rangle, z \in \mathbb{D}$.
Note that the potential g_{λ_n} admits the integral representation (4.17). Moreover $|K^\mu(z,w)| \leq 2/|1-\overline{w}z|$, $(z, w \in \mathbb{D})$, see [72]. Therefore, the function h belongs to $\mathcal{D}(\mu)$.
Theorem 4.4.2 says that $\liminf_{z\to\zeta} \operatorname{Re} \langle g_{\lambda_n}, K_z^\mu \rangle \geq 1$, $\zeta \in E$. Let $(r_n) \subset (0,1)$ be an increasing sequence to 1 such that $\operatorname{Re} h_n(z) \geq 1/2$, where $h_n(z) := \langle g_{\lambda_n}, K_{r_n z}^\mu \rangle$. In particular $h_n \in A(\mathbb{D})$. Let f be the function defined by

$$f := \exp(-\sum h_n).$$

That function belongs to $\mathcal{D}(\mu) \cap A(\mathbb{D})$ and vanishes on E. The sequence $f_n := \exp(-\sum_{k>n} f_n)$ converges pointwise to 1 on $\mathbb{D}$ and $f_n \in [f]_{\mathcal{D}(\mu)}$. This means that $1 \in [f]_{\mathcal{D}(\mu)}$.

As a result of the above discussion, we have the following theorem.

Theorem 14. *Let μ be a positive finite measure on $\overline{\mathbb{D}}$. Let E be a closed subset of $\mathbb{T}$. If $c_\mu(E) = 0$, then there exists a cyclic function $f \in \mathcal{D}(\mu) \cap A(\mathbb{D})$ that vanishes on E.*

This result was obtained by Elmadani and Labghail [34] for harmonically weighted Dirichlet space (see also [66]).

In 1990, a weaker version of the Brown-Shields conjecture was established by Hedenmalm and Shields [42] if, $\mathcal{Z}(f)$ is countable (see also [65]). In 2006, El-Fallah, Kellay and Ransford [29] replaced "countable" by a condition closer to "capacity zero". Namely, they proved that an outer function $f \in \mathcal{D} \cap A(\mathbb{D})$ is cyclic in $\mathcal{D}$ if

$$\int_0 c(E_t) \frac{\log \log(1/t)}{t \log(1/t)} dt < \infty,$$

where $E_t = \{\zeta \in \mathbb{T} : d(\zeta, E) < t\}$.
An extension of this result was given in [34] for harmonically weighted Dirichlet spaces. Our last main result in this section is stated as follows.

Theorem 4.6.5. *Let μ be a positive finite measure on $\overline{\mathbb{D}}$. Let f be an outer function in $\mathcal{D}(\mu) \cap A(\mathbb{D})$ and $E := \{\zeta \in \mathbb{T} : f(\zeta) = 0\}$, if*

$$\int_0 c_\mu(E_t) \frac{\log(1/t)}{t} dt < \infty, \tag{4.24}$$

then f is cyclic in $\mathcal{D}(\mu)$

To prove this result, we need the following spectral synthesis theorem which was proved by several authors (see for instance [29, 31, 32, 34]).

Theorem 4.6.6. *Let μ be a positive finite measure on $\overline{\mathbb{D}}$. Let f be an outer function in $\mathcal{D}(\mu) \cap A(\mathbb{D})$ and set $\mathrm{E} = \mathcal{Z}(f)$. Suppose that there exists a function $g \in \mathcal{D}(\mu)$ such that*

$$|g(z)| \lesssim d(z, \mathrm{E})^4 \quad (z \in \mathbb{D}), \tag{4.25}$$

then $g \in [f]_{\mathcal{D}(\mu)}$.

Proof of Theorem 4.6.5. The two conditions (4.24) and (4.20) coincide for $\eta(t) := \log \frac{1}{t}$. Thus, by Theorem 4.4.4, there exists a function $h \in \mathcal{D}(\mu)$ such that

$$|g_t(z)| \lesssim d(z, E)^t \qquad (z \in \mathbb{D}),$$

where $g_t := \exp(-th)$ $(t > 0)$. The function g_t is a bounded function and belongs to $\mathcal{D}(\mu)$. From theorems 4.6.6 and 4.5.4, we have $g_t \in [f]_{\mathcal{D}(\mu)}$. Clearly $g_t \to 1$ pointwise on $\mathbb{D}$.
Therefore f is cyclic for $\mathcal{D}(\mu)$. And the proof is complete. $\square$

5 APPROXIMATION AND CLOSED IDEALS IN SOME ANALYTIC SPACES

In this chapter, we give an elementary proof of the approximation theorem established by A. Matheson in [53] for analytic Lipschitz spaces λ_α. Our approach allows us to extend this approximation theorem to $\mathcal{D}_\omega \cap \lambda_\alpha$, where $\mathcal{D}_\omega$ denotes superharmonically weighted Dirichlet spaces (including standard Dirichlet spaces). As a consequence, we give a complete description of closed ideals of the Banach algebra $\mathcal{D}_\omega \cap \lambda_\alpha$. This extends previous results obtained A. Matheson [53] and B. Bouya [13].

5.1 INTRODUCTION

Let $\mathbb{D}$ be the unit disc of the complex plane $\mathbb{C}$ and let $\mathbb{T} := \partial\mathbb{D}$ be the unit circle. Let dA (resp. dm) be the normalized Lebesgue measure on $\mathbb{D}$ (resp. $\mathbb{T}$). The Hardy space H^2 is the space of analytic functions f on $\mathbb{D}$ such that

$$\|f\|_{H^2}^2 := |f(0)|^2 + \int_{\mathbb{D}} |f'(z)|^2 \log(1/|z|) dA(z) < \infty.$$

Let $A(\mathbb{D})$ be the disc algebra which consists of continuous functions on $\overline{\mathbb{D}}$ that are analytic on $\mathbb{D}$. For $\alpha \in (0,1)$, the separable analytic Lipschitz algebra λ_α is given by

$$\lambda_\alpha := \{f \in A(\mathbb{D}) : \ |f(z) - f(w)| = o(|z-w|^\alpha) \text{ as } |z-w| \text{ tends to } 0\}.$$

Equipped with the norm,

$$\|f\|_\alpha = \|f\|_\infty + \sup_{z,w\in\mathbb{D}, z\neq w} \frac{|f(z)-f(w)|}{|z-w|^\alpha},$$

where $\|f\|_\infty = \sup_{z\in\mathbb{D}} |f(z)|$, the algebra λ_α is a Banach algebra.

Let ω be a nonnegative superharmonic weight on $\mathbb{D}$. The weighted Dirichlet space $\mathcal{D}_\omega$ associated with ω is defined by

$$\mathcal{D}_\omega := \left\{ f \in H^2 : \mathcal{D}_\omega(f) := \int_{\mathbb{D}} |f'(z)|^2 \omega(z) dA(z) < \infty \right\}.$$

The standard Dirichlet spaces $\mathcal{D}_\gamma$ with $\gamma \in [0,1]$, correspond to $\omega_\gamma(z) = (\gamma+1)(1-|z|^2)^\gamma$. The classical Dirichlet space $\mathcal{D}_0$ is denoted by $\mathcal{D}$. Note also that $\mathcal{D}_1 = H^2$. Various facts about Hardy and Dirichlet spaces used in this paper can be found in [32, 38].

We will also consider the Banach algebra $\mathcal{D}_\omega \cap \lambda_\alpha$ equipped with the norm

$$\|f\|_{\mathcal{D}_\omega \cap \lambda_\alpha} = \mathcal{D}_\omega^{1/2}(f) + \|f\|_\alpha, \quad f \in \mathcal{D}_\omega \cap \lambda_\alpha.$$

The maximal ideals of $\mathcal{D}_\omega \cap \lambda_\alpha$ can be identified with $\overline{\mathbb{D}}$ by $z \to \chi_z$: $f \in \mathcal{D}_\omega \cap \lambda_\alpha \to f(z)$.
Our goal is to prove the following result which extends earlier work of A. Matheson [53] and B. Bouya [13] (see also [70]).

Theorem 5.1.1. *Let $\alpha \in (0,1)$ and let ω be a nonnegative superharmonic weight on $\mathbb{D}$. Let $\mathcal{I} \neq \{0\}$ be a closed ideal of $\mathcal{D}_\omega \cap \lambda_\alpha$, then there exists an inner function θ and a closed subset E of $\mathbb{T}$ such that*

$$\mathcal{I} = \{f \in \mathcal{D}_\omega \cap \lambda_\alpha : \ f_{|E} = 0, \text{ and } f \in \theta H^2\}.$$

It is somewhat standard, see Section 5.5, that Theorem 5.1.1 can be derived from the following approximation theorem which is the main result in this chapter.

Theorem 5.1.2. *Let $\alpha \in (0,1)$ and let ω be a nonnegative superharmonic weight on $\mathbb{D}$. Let $f \in \mathcal{D}_\omega \cap \lambda_\alpha$ be a function that vanishes on a closed subset E of $\mathbb{T}$. Then, given $M > 0$, there exists $(f_n)_n \subset \mathcal{D}_\omega \cap \lambda_\alpha$ such that*

1. *f_n vanishes on E for all $n \geq 1$,*
2. *$|f_n(z)| = O(\operatorname{dist}^M(z,E))$ for all $n \geq 1$,*
3. *$\lim_{n\to 0} \|f_n f - f\|_{\mathcal{D}_\omega \cap \lambda_\alpha} = 0$.*

To obtain an analogue of Theorem 5.1.2 for λ_α (resp. $\mathcal{D} \cap \lambda_\alpha$), A. Matheson (resp. B. Bouya) used the Korenblum method which is based on subtle estimates of outer functions in various regions of the unit disc. Our approach gives an alternative simpler proof.

5.2 PRELIMINARIES

The norm of λ_α can be expressed in several manners. Here we will use a result of K. Dyakonov [24] (see also [57] for an elementary proof). Namely, let

$$\Lambda_\alpha := \{f \in A(\mathbb{D}) : \ |f(z) - f(w)| = O(|z-w|^\alpha)\}.$$

Given $f \in A(\mathbb{D})$, then $f \in \Lambda_\alpha$ if and only if $||f(\zeta_1)| - |f(\zeta_2)|| = O(|\zeta_1 - \zeta_2|^\alpha)$, $\zeta_1, \zeta_2 \in \mathbb{T}$ and $P(|f|)(z) - |f(z)| = O\left((1-|z|)^\alpha\right)$, $z \in \mathbb{D}$, where $P(|f|)(z) := \displaystyle\int_{\mathbb{T}} \frac{1-|z|^2}{|e^{it}-z|^2}|f(e^{it})|dm(e^{it})$ is the Poisson integral of the restriction of $|f|$ to the unit circle $\mathbb{T}$. For more general

results see [26].
By arguing in the same way as in [57], one can see that $f \in \lambda_\alpha$ if and only if the following two conditions are satisfied:

- $||f(\zeta_1)| - |f(\zeta_2)|| = o(|\zeta_1 - \zeta_2|^\alpha), \quad |\zeta_1 - \zeta_2| \to 0, \quad \zeta_1, \zeta_2 \in \mathbb{T}.$

- $\limsup_{|z| \to 1^-} \dfrac{P(|f|)(z) - |f(z)|}{(1-|z|)^\alpha} = 0.$

In this case we have

$$\|f\|_\alpha \asymp \|f\|_\infty + \sup_{\zeta_1, \zeta_2 \in \mathbb{T}, \zeta_1 \neq \zeta_2} \frac{||f(\zeta_1)| - |f(\zeta_2)||}{|\zeta_1 - \zeta_2|^\alpha} + \sup_{z \in \mathbb{D}} \frac{P(|f|)(z) - |f(z)|}{(1-|z|)^\alpha}.$$

The following lemma is standard and can be obtained easily. We omit the proof here.

Lemma 5.2.1. *Let $\alpha \in (0,1)$. Let $(\phi_n)_n \subset \mathcal{C}(\mathbb{T})$ be such that*

$$|\phi_n(\zeta_1) - \phi_n(\zeta_2)| = o(|\zeta_1 - \zeta_2|^\alpha), \ |\zeta_1 - \zeta_2| \to 0, \quad \zeta_1, \zeta_2 \in \mathbb{T},$$

uniformly in n. Then $(P(\phi_n))_n \subset \mathcal{C}(\overline{\mathbb{D}})$ and

$$|P(\phi_n)(z) - P(\phi_n)(w)| = o(|z-w|^\alpha), \ |z-w| \to 0, \quad z, w \in \mathbb{D}$$

uniformly in n.

The following lemma will be useful in the proof of Theorem 5.1.2.

Lemma 5.2.2. *Let $\alpha \in (0,1)$. Let $(f_n)_n \subset \lambda_\alpha$. Suppose that*

1. *$(f_n)_n$ converges to f uniformly on compact subsets of $\mathbb{D}$,*

2. $||f_n(\zeta_1)| - |f_n(\zeta_2)|| = o(|\zeta_1 - \zeta_2|^\alpha), \quad |\zeta_1 - \zeta_2| \to 0, \quad \zeta_1, \zeta_2 \in \mathbb{T},$ *uniformly in n,*

3. $P(|f_n|)(z) - |f_n(z)| = o((1-|z|)^\alpha), \ |z| \to 1^-$, *uniformly in n.*

Then $f \in \lambda_\alpha$ and $(f_n)_n$ converges to f in λ_α.

Proof. Let $z \in \mathbb{D}$. By an adequate application of the Schwarz lemma (see Lemma 2 of [57]), we have

$$(1-|z|)|f_n'(z)| \le 2 \sup\{|f_n(w)| - |f_n(z)| : \quad |w-z| < 1-|z|\}.$$

Note that

$$\begin{aligned} |f_n(w)| - |f_n(z)| &\le P(|f_n|)(w) - |f_n(z)| \\ &= P(|f_n|)(w) - P(|f_n|)(z) + P(|f_n|)(z) - |f_n(z)|. \end{aligned}$$

We have, by Lemma 5.2.1, $P(|f_n|)(w) - P(|f_n|)(z) = o(|z-w|^\alpha)$, $|z-w| \to 0$, uniformly in n. Then, by assumption (3), we have

$$(1-|z|)|f_n'(z)| = o((1-|z|)^\alpha),$$

uniformly in n. The desired result comes from Lemma 1 of [53]. $\square$

Let ω be a nonnegative superharmonic weight on $\mathbb{D}$. By the Jensen-Riesz representation theorem (see for instance [6, 60]), there exist a positive Borel measure μ on $\mathbb{D}$ and a finite positive Borel measure ν on $\mathbb{T}$ such that

$$\omega(z) = U_\mu(z) + P_\nu(z), \quad z \in \mathbb{D}, \tag{5.1}$$

where U_μ is the Green potential of μ, given by

$$U_\mu(z) = \int_{\mathbb{D}} \log\left|\frac{1-\overline{w}z}{w-z}\right| d\mu(w)$$

and P_ν is the Poisson transform of ν

$$P_\nu(z) = \int_{\mathbb{T}} \frac{1-|z|^2}{|1-\overline{\zeta}z|^2} d\nu(\zeta).$$

Let f be a function in H^2. By Green's formula, we have

$$\int_{\mathbb{D}} |f'(w)|^2 \log\left|\frac{1-z\overline{w}}{z-w}\right| dA(w) = P(|f|^2)(z) - |f(z)|^2, \quad z \in \mathbb{D}.$$

So,

$$\begin{aligned} \mathcal{D}_{U_\mu}(f) &= \int_{\mathbb{D}}\int_{\mathbb{D}} |f'(w)|^2 \log\left|\frac{1-z\overline{w}}{z-w}\right| dA(w)d\mu(z) \\ &= \int_{\mathbb{D}} \left(P(|f|^2)(z) - |f(z)|^2\right) d\mu(z). \end{aligned}$$

The latter formula is due to A. Aleman. For more information on superharmonically weighted Dirichlet spaces we refer to [4].
In the harmonic case ($\omega = P_\nu$), S. Richter and C. Sundberg obtained a more precise formula which extends the famous Carleson formula stated for the classical Dirichlet space $\mathcal{D}$. First, they proved that if θ is an inner function then

$$\mathcal{D}_{P_\nu}(\theta f) = \int_{\mathbb{T}} \mathcal{D}_\zeta(f) d\nu(\zeta) + \int_{\mathbb{T}} |f(\zeta)|^2 \mathcal{D}_\zeta(\theta) dm(\zeta), \tag{5.2}$$

where $\mathcal{D}_\zeta = \mathcal{D}_{P_{\delta_\zeta}}$ and δ_ζ is the Dirac measure at ζ. They also proved that if f is an outer function then

$$\mathcal{D}_\zeta(f) = \int_{\mathbb{T}} \frac{|f(\zeta_1)|^2 - |f(\zeta)|^2 - 2|f(\zeta)|^2 \log(|f(\zeta_1)|/|f(\zeta)|)}{|\zeta_1-\zeta|^2} dm(\zeta_1). \tag{5.3}$$

For Richter and Sundberg's complete formula, we refer to [32, 63].

From the above discussion one can see easily that $\mathcal{D}_\omega \cap \lambda_\alpha$ satisfies the F-property. That is, given $f \in H^2$, if $\theta f \in \mathcal{D}_\omega \cap \lambda_\alpha$, where θ is an inner function, then

$$f \in \mathcal{D}_\omega \cap \lambda_\alpha \quad \text{and} \quad \|f\|_{\mathcal{D}_\omega \cap \lambda_\alpha} \leq C\|\theta f\|_{\mathcal{D}_\omega \cap \lambda_\alpha}.$$

5.3 CUT-OFF FUNCTIONS

Let f, g be two outer functions. Let $f \wedge g$ be the outer function associated with $|f| \wedge |g|(e^{it}) := \min(|f(e^{it})|, |g(e^{it})|)$. Namely,

$$f \wedge g(z) = \exp\left(\int_{\mathbb{T}} \frac{e^{it}+z}{e^{it}-z} \log(|f| \wedge |g|(e^{it}))dm(e^{it})\right), \quad z \in \mathbb{D}.$$

Our starting point is the Richter-Sundberg result on cut-off functions [64]. Indeed, they proved, using (5.3), that

$$\mathcal{D}_{P_\nu}(f \wedge f^2) \leq 4\mathcal{D}_{P_\nu}(f).$$

So, for an integer $k \geq 1$, we have

$$\begin{aligned} \mathcal{D}_{P_\nu}(f \wedge f^{2^k}) &= \mathcal{D}_{P_\nu}\left((f \wedge f^{2^{k-1}}) \wedge (f \wedge f^{2^{k-1}})^2\right) \\ &\leq 4\mathcal{D}_{P_\nu}(f \wedge f^{2^{k-1}}). \end{aligned}$$

Then we get

$$\mathcal{D}_{P_\nu}(f \wedge f^{2^k}) \leq 4^k \mathcal{D}_{P_\nu}(f). \tag{5.4}$$

Our goal in this section is to prove analogous inequalities for λ_α and for $\mathcal{D}_{U_\mu}$. The following is the key result in the proof of Theorem 5.1.2.

Theorem 5.3.1. *Let $\alpha \in (0,1)$, $p \geq 1$. Let $f \in H^2$ be an outer function. We have*

$$P(|f \wedge f^p|)(z) - |f \wedge f^p(z)| \leq p^2\left(P(|f|)(z) - |f(z)|\right), \quad z \in \mathbb{D}.$$

The proof of Theorem 5.3.1 is based on the following elementary lemma.

Lemma 5.3.2. *Let $x \geq 1, y \in [0,1]$. Let $\alpha, \beta \geq 0$ be such that $\alpha + \beta = 1$. We have*

$$x^\alpha(p^2y^\beta - y^{p\beta}) \leq \beta(p^2y - y^p) + \alpha(p^2-1)x.$$

Proof. Let φ be the function given by

$$\varphi(x,y) := x^\alpha(p^2y^\beta - y^{p\beta}) - \beta(p^2y - y^p) - \alpha(p^2-1)x.$$

Note that $\varphi(x,0) \leq 0$. For $y \in (0,1]$, we have

$$\begin{aligned} \partial_y\varphi(x,y) &= \tfrac{1}{y}(\beta x^\alpha(p^2y^\beta - py^{p\beta}) - \beta(p^2y - py^p)) \\ &\geq \beta p(\psi(y^\beta) - \psi(y)), \end{aligned}$$

where $\psi(t) = pt - t^p$. Since ψ is increasing on $[0,1]$, we obtain $\partial_y\varphi(x,y) \geq 0$ and $\varphi(x,y) \leq \varphi(x,1) = (p^2-1)(x^\alpha - \alpha x - \beta)$. Obviously, $x \mapsto \varphi(x,1)$ is decreasing on $[1,+\infty[$. Then $\varphi(x,y) \leq \varphi(1,1) = 0$. □

Proof of Theorem 5.3.1. Let $d\sigma = P_z dm$. Let

$$\alpha = \sigma(\{|f| \geq 1\}) \quad \text{and} \quad \beta = \sigma(\{|f| < 1\}).$$

If $\alpha = 0$ or $\beta = 0$, then the result is obvious. We suppose in the sequel that $\alpha, \beta > 0$. Write

$$x = \exp\left(\int_{\{|f|\geq 1\}} \log|f| \frac{d\sigma}{\alpha}\right),$$

and

$$y = \exp\left(\int_{\{|f|< 1\}} \log|f| \frac{d\sigma}{\beta}\right).$$

With these notations, we have

$$|f(z)| = x^\alpha y^\beta \quad \text{and} \quad |f \wedge f^p(z)| = x^\alpha y^{p\beta}.$$

From Lemma 5.3.2, we obtain

$$p^2|f(z)| - |f \wedge f^p(z)| \leq \beta(p^2 y^\beta - y^{p\beta}) + \alpha(p^2-1)x.$$

Let $A_f = \displaystyle\int_{\{|f|\geq 1\}} |f| d\sigma$ and let $A_f^c = \displaystyle\int_{\{|f|<1\}} |f| d\sigma$. By Jensen's inequality, we have

$$\alpha x = \alpha \exp\left(\int_{\{|f|\geq 1\}} \log|f| \frac{d\sigma}{\alpha}\right) \leq A_f.$$

Since $h(t) = p^2 e^t - e^{pt}$ is convex on $(-\infty, 0)$, we have

$$\begin{aligned}
\beta(p^2 y - y^p) &= \beta h\left(\int_{\{|f|\leq 1\}} \log|f| \frac{d\sigma}{\beta}\right) \\
&\leq \int_{\{|f|\leq 1\}} h(\log|f|) d\sigma \\
&= p^2 A_f^c - A_{f^p}^c.
\end{aligned}$$

Combining these inequalities, we obtain

$$p^2|f(z)| - |f \wedge f^p(z)| \leq p^2 A_f^c - A_{f^p}^c + (p^2-1)A_f.$$

This is equivalent to

$$A_f + A_{f^p}^c - |f \wedge f^p(z)| \leq p^2\left((A_f + A_f^c) - |f(z)|\right),$$

which is the desired inequality. □

The following lemma is due to A. Aleman (see Lemma 3.2 of [4]).

Lemma 5.3.3. *Let $f \in H^2$ be an outer function. We have*

$$P(|1 \wedge f|)(z) - |1 \wedge f(z)| \leq P(|f|)(z) - |f(z)|, \quad z \in \mathbb{D}. \tag{5.5}$$

We also need the following lemma in the proof of Theorem 5.1.2, that can be deduced easily from Lemma 5.3.3, Corollary 2.3 of [64] and Theorem 2 of [24].

Lemma 5.3.4. *Let $f \in H^2$ be an outer function and let θ be an inner function such that $\theta f \in \mathcal{D}_\omega \cap \lambda_\alpha$. Then $\theta(1 \wedge f) \in \mathcal{D}_\omega \cap \lambda_\alpha$ and*

$$\|\theta(1 \wedge f)\|_{\mathcal{D}_\omega \cap \lambda_\alpha} \lesssim \|\theta f\|_{\mathcal{D}_\omega \cap \lambda_\alpha},$$

where the constant involved depends only on α and ω.

5.4 APPROXIMATION THEOREM

Let $f = \theta g \in \mathcal{D}_\omega \cap \lambda_\alpha \setminus \{0\}$, where θ, g are respectively the inner and outer factors of f. Suppose that $f_{|E} = 0$. Since λ_α possesses the F-property, $g \in \lambda_\alpha$. We have $g_{|E} = 0$ and then $|g(z)| = O(\text{dist}(z,E)^\alpha)$.

Let $n \geq 1$, $p = 2^k$ and put $f_n = 1 \wedge n^{p-1}g^{p-1}$. By Lemma 5.3.4, $f_n \in \mathcal{D}_\omega \cap \lambda_\alpha$ and clearly

$$|f_n(z)| = O(\text{dist}(z,E)^M), \quad p\alpha \geq M.$$

We have $ff_n = \theta(g \wedge n^{p-1}g^p)$. It is clear that $f_n \in A(\mathbb{D})$ and that $(ff_n)_n$ converges uniformly to f on any compact subset of $\mathbb{D}$.
Now, we prove that $(ff_n)_n$ converges to f in λ_α. Firstly, we claim that

$$||ff_n(\zeta_1)| - |ff_n(\zeta_2)|| \leq p\,||f(\zeta_1)| - |f(\zeta_2)||, \qquad (\zeta_1, \zeta_2 \in \mathbb{T}).$$

Since $|ff_n| = |g| \wedge n^{p-1}|g|^p$ on $\mathbb{T}$. We get

$$|ff_n(\zeta)| = \begin{cases} |g(\zeta)| & \text{if} \quad |g(\zeta)| \geq 1/n \\ n^{p-1}|g(\zeta)|^p & \text{if} \quad |g(\zeta)| \leq 1/n. \end{cases}$$

Let $\zeta_1, \zeta_2 \in \mathbb{T}$ we have to discuss tree situation.

- If $|g(\zeta_1)| \geq 1/n$ and $|g(\zeta_2)| \geq 1/n$, we have

  $$||ff_n(\zeta_1)| - |ff_n(\zeta_2)|| = ||g(\zeta_1)| - |g(\zeta_2)|| = ||f(\zeta_1)| - |f(\zeta_2)||$$

- If $|g(\zeta_1)| \leq 1/n$ and $|g(\zeta_2)| \leq 1/n$, we have

  $$\begin{aligned} ||ff_n(\zeta_1)| - |ff_n(\zeta_2)|| &= n^{p-1}\,||g(\zeta_1)|^p - |g(\zeta_2)|^p| \\ &\leq p\,||f(\zeta_1)| - |f(\zeta_2)|| \end{aligned}$$

- If $|g(\zeta_2)| \leq 1/n \leq |g(\zeta_1)|$, we have

  $$||ff_n(\zeta_1)| - |ff_n(\zeta_2)|| = |g(\zeta_1)| - n^{p-1}|g(\zeta_2)|^p \leq p\,||f(\zeta_1)| - |f(\zeta_2)||$$

  In order to show the last inequality, we consider the function φ defined by

  $$\varphi(x,y) := (p-1)x + y^p x^{p-1} - py; \qquad 0 < y \leq 1/n \leq x.$$

  Note that $\partial_x \varphi = (p-1)(1 + y^p x^{p-2}) \geq 0$ for all $0 < y \leq 1/n \leq x$. So $\varphi(x,y) \geq \varphi(1/n, y) = (p-1)/n + y^p(1/n)^{p-1}$. Moreover, the function $y \mapsto \varphi(1/n, y)$ is non-increasing on $(0, 1/n]$. Therefore we have

  $$\varphi(x,y) \geq \varphi(1/n, 1/n) = 0, \quad 0 < y \leq 1/n \leq x.$$

  In particular $\varphi(|g(\zeta_1)|, |g(\zeta_2)|) \geq 0$. This gives the desired inequality. By symmetry we get the result for $|g(\zeta_1)| \leq 1/n \leq |g(\zeta_2)|$. The claimed is established.

Secondly, we have

$$\begin{aligned} P(|ff_n|)(z) - |ff_n(z)| &= \big(P(|g \wedge n^{p-1}g^p|)(z) - |g \wedge n^{p-1}g^p(z)|\big) + \\ &\quad + |g \wedge n^{p-1}g^p(z)|(1 - |\theta(z)|) \\ &\leq p^2(P(|g|)(z) - |g(z)|) + |g(z)|(1 - |\theta(z)|) \\ &\leq p^2\left(P(|f|)(z) - |f(z)|\right). \end{aligned}$$

Then by Lemma 5.2.2, $\lim_n \|ff_n - f\|_{\lambda_\alpha} = 0$.
Next, using Theorem 5.3.1 and (5.4), we get

$$\sup_n \mathcal{D}_\omega(ff_n) \leq 4^k \mathcal{D}_\omega(f).$$

Then there exists a subsequence of $(ff_n)_n$ converges weakly in $\mathcal{D}_\omega$. Keep in mind that $ff_n \to f$ pointwise in $\mathbb{D}$, this subsequence noted by (fh_n) converges weakly to f in $\mathcal{D}_\omega$. Let F_j be the convex hull of $\{ff_n : n \geq j\}$. Note that $F_1 = \cup_{j\geq 1} F_j$ which is a convex set, so f is an element of the weak closure of $\overline{F_1}^*$. Since $\mathcal{D}_\omega$ is a Hilbert space, the weak closing and strong closing coincide for the convex subsets of $\mathcal{D}_\omega$. Then there exists a sequence $fg_j \in F_1$ such that $\lim_j \mathcal{D}_\omega(fg_j - f) = 0$. The fact that $(ff_n)_n$ converges to f in λ_α implises that $\lim_j \|fg_j - f\|_{\lambda_\alpha} = 0$. Note that $g_j := \sum_{finite} \beta_{j_n} f_{j_n}$ such that $\sum_{finite} \beta_{j_n} = 1$. Obviously g_j vanishes on E. Therefore we have

$$|g_j(z)| \leq \sum_{finite} \beta_{j_n} |f_{j_n}(z)| \leq C\,\mathrm{dist}^M(z,E) \sum_{finite} \beta_{j_n} = C\,\mathrm{dist}^M(z,E).$$

This completes the proof.

5.5 CLOSED IDIALS

Let $\mathcal{I} \neq \{0\}$ be a closed ideal of $\mathcal{D}_\omega \cap \lambda_\alpha$. The inner factor $\theta_{\mathcal{I}}$ of $\mathcal{I}$ is the greatest common divisor of the inner factors of all nonzero elements of $\mathcal{I}$. We say that $\mathcal{I}$ is without inner factor if $\theta_{\mathcal{I}} = 1$.

The proof of Theorem 5.1.1, is based on Theorem 5.1.2 and on the following spectral synthesis theorem which was proved for several algebras (see for instance [13, 31, 42, 49, 53]). We omit the proof here.

Theorem 5.5.1. *Let $\mathcal{I} \neq \{0\}$ be a closed ideal of $\mathcal{D}_\omega \cap \lambda_\alpha$ without inner factor. Let $f \in \mathcal{D}_\omega \cap \lambda_\alpha$ be such that*

$$|f(z)| = O(\mathrm{dist}(z,E)^4),\ z \in \mathbb{D},$$

where $E_{\mathcal{I}} := \{\zeta \in \mathbb{T} :\ g(\zeta) = 0,\ g \in \mathcal{I}\}$ is the zero set of $\mathcal{I}$. Then $f \in \mathcal{I}$.

Proof of Theorem 5.1.1. Let $\mathcal{I} \neq \{0\}$ be a closed ideal of $\mathcal{D}_\omega \cap \lambda_\alpha$. Let $\theta = \theta_{\mathcal{I}}$ and let E be the zero set of $\mathcal{I}$ on $\mathbb{T}$. Obviously

$$\mathcal{I} \subset \mathcal{M}(\theta, E) := \{g \in \mathcal{D}_\omega \cap \lambda_\alpha : \ g_{|E} = 0 \text{ and } g \in \theta H^2\}.$$

To prove the reverse inclusion, let $f \in \mathcal{M}(\theta, E)$ and consider

$$\mathcal{I}(f) := \{g \in \mathcal{D}_\omega \cap \lambda_\alpha : \ gf \in \mathcal{I}\}.$$

Clearly, $\mathcal{I}(f)$ is a closed ideal of $\mathcal{D}_\omega \cap \lambda_\alpha$ without inner factor and the zero set of $\mathcal{I}(f)$ is a subset of E. By Theorem 5.1.2, there exists $f_n \in \mathcal{D}_\omega \cap \lambda_\alpha$ such that $|f_n(z)| = O(\text{dist}(z, E)^4)$ and $f_n f$ converges to f in $\mathcal{D}_\omega \cap \lambda_\alpha$. Then by Theorem 5.5.1, $f_n \in \mathcal{I}(f)$. In particular $f_n f \in \mathcal{I}$. Since $\mathcal{I}$ is closed, $f \in \mathcal{I}$. The proof is complete.

$\square$

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Abstract:

In this thesis, we deal with superharmonically weighted Dirichlet spaces.

In the first part of this dissertation, we determine all supreharmonically weighted Dirichlet spaces which contain no infinite Blaschke products. Next, we give a sufficient condition which ensuring the membership of Blaschke products. However, we propose two methods of regularization of Blaschke products. In fact, the regularization condition is a sufficient condition for zero sets. Our condition improves Shapiro-Shields condition for a large class of superharmonic weight, including standard weights. We end this part by studying the uniqueness sets for the harmonic weights. Actually, we extend the Khavin-Maz'ya uniqueness theorem for all harmonically weighted Dirichlet spaces.

In the second part, we characterize the invariant subspaces of multiplication operator by z on suparharmonically weighted Dirichlet spaces. We prove that the extended Brown-Shields conjecture is true, in the countable case. Finally, we give a complete description of closed ideals of the algebra obtained by the intersection of suparharmonically weighted Dirichlet spaces and the analytic Lipschitz spaces.

Keywords: Supreharmonically weighted Dirichlet spaces; Blaschke products; Zero sets; Uniqueness sets; Reproducing kernels; Capacity; Invariant subspaces; Cyclicity; Closed ideals.

Résumé:

Dans cette thèse, nous considérons les espaces de Dirichlet à poids super-harmonique.

Dans la première partie de cette dissertation, nous déterminons tous les espaces de Dirichlet à poids super-harmonique qui ne contiennent aucun produit de Blaschke infini. Ensuite, nous donnons une condition suffisante de l'appartenance des produits de Blaschke. Cependant, nous proposons deux méthodes de régularisations des produits de Blaschke. En fait, cette condition de régularisation est une condition suffisante pour les ensembles zéro. Notre condition améliore celle de Shapiro-Shields pour une grande classe de poids super-harmonique, y compris les poids standards. En terminant cette partie par les ensembles d'unicité pour les poids harmonique. Précisément, nous étendons le théorème d'unicité de Khavin-Maz'ya pour tous les espaces de Dirichlet à poids harmonique.

Dans la deuxième partie, nous caractérisons les sous-espaces invariants de l'opérateur de multiplication par z sur les espaces de Dirichlet à poids super-harmonique. Nous répondons positivement à la conjecture étendue de Brown-Shields dans le cas dénombrable. Finalement, nous donnons une description complète des idéaux fermés de l'algèbre obtenue par l'intersection des espaces de Dirichlet à poids super-harmonique et l'algèbre des fonctions Lipschitzienne.

***Mots-clés*:** Les espaces de Dirichlet à poids super-harmonique; Produits de Blaschke; Ensembles zéro; Ensembles unicité, Noyaux reproduisant; Capacité; Sous-espaces invariant; Cyclicité; Idéaux fermés.

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