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Spectral approximation preservers problems

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Dedication

I dedicate my dissertation work to my family and many friends. A special feeling of gratitude to my loving parents, Mr Ahmed **NKHAYLIA** -God rest his soul- and Mrs Fatna **AOUAD**, my dream of completing this work would not have blossomed without their wholehearted support, sacrifice, and prayers. They deserve special mention for providing me with all the opportunities to explore my potentials and pursue my dreams.

With great pleasure, I would like to dedicate this work and to give special thanks to my wife Assia **LOULIDI** for the mental strength, support, prayers and encouragement. Her support and motivation helped me to become strong and capable to complete this thesis.

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Contents

List of Symbols

$\mathbb{R}$	The set of real numbers
$\sup(S)$	The supremum of $S \subset \mathbb{R}$
$\inf(S)$	The infimum of $S \subset \mathbb{R}$
$\max(S)$	The maximum element of $S \subset \mathbb{R}$
$\min(S)$	The minimum element of $S \subset \mathbb{R}$
$\mathbb{C}$	The set of complex numbers
$\mathrm{Im}(z)$	The imaginary part of $z \in \mathbb{C}$
$\mathrm{Re}(z)$	The real part of $z \in \mathbb{C}$
$\overline{z}$	The complex conjugate of $z \in \mathbb{C}$
$\mathbb{T}$	The unit circle of the complex $\mathbb{C}$
$D(a, r)$	The closed disc centered at a and of radius r
∂S	The boundary of $S \subset \mathbb{C}$
$\mathbb{C}^n$	The n-dimensional complex coordinate space
$\mathcal{M}_n(\mathbb{C})$	The algebra of $n \times n$ complex matrices
A^t	The transpose of $A \in \mathcal{M}_n(\mathbb{C})$
$\mathrm{diag}(A)$	The diagonal of $A \in \mathcal{M}_n(\mathbb{C})$
$\det(A)$	The determinant of $A \in \mathcal{M}_n(\mathbb{C})$
$A_1 \oplus A_2$	The direct sum of two square matrices A_1 and A_2
X	The complex Banach space
X^*	The dual X
$\mathcal{B}(X)$	The Banach algebra of all bounded linear operators on X
$\langle x, f \rangle$	The pairing between $x \in X$ and $f \in X^*$
$x \otimes f$	The operator of rank at most one
$\mathcal{F}_1(X)$	The set of all operators of rank at most one in $\mathcal{B}(X)$

$\mathcal{H}$	The complex Hilbert space
$\mathcal{B}(\mathcal{H})$	The algebra of all bounded linear operators on $\mathcal{H}$
$\langle x, y\rangle$	Scalar product of the vectors x, y
$x \otimes y$	The operator of rank at most one
$\mathcal{F}_1(\mathcal{H})$	The set of all operators of rank at most one in $\mathcal{B}(\mathcal{H})$
A^*	The adjoint of A
$\dim(\cdot)$	Algebraic dimension
$\ker(A)$	The kernel of A
$\operatorname{ran}(A)$	The range of A
$\operatorname{rank}(A)$	The dimension of $\operatorname{ran}(A)$
$R_A(\lambda)$	The resolvent of A at $\lambda \in \mathbb{C}$
$\rho(A)$	The resolvent set of A
$\sigma(A)$	The spectrum of A
$r(A)$	The spectral radius of A
$\Lambda_\varepsilon(A)$	The ε-pseudo spectrum of A
$\rho_\varepsilon(A)$	The ε-pseudo spectral radius of A
$\sigma_\varepsilon(A)$	The ε-condition spectrum of A
$r_\varepsilon(A)$	The ε-condition spectral radius of A.

Publications

During this thesis we produced three articles published in journals indexed by SCOPUS.

1. Z. Abdelali and H. Nkhaylia, Maps preserving the pseudo spectrum of skew triple product of operators, Linear Multilinear Algebra. 67, no. 11, (2018), 2297 - 2306.
2. Z. Abdelali and H. Nkhaylia, Multiplicatively pseudo spectrum-preserving maps, Linear and Multilinear Algebra and Function Spaces. Contemporary Mathematics. 750, (2020), 43.
3. Z. Abdelali and H. Nkhaylia, Condition spectrum of rank one operators and preservers of the condition spectrum of skew product of operators, Complex Analysis and Operator Theory. 14, no.7, (2020), 1 - 29.

Abstract

In this work in Operator theory, we are interested in spectral approximation nonlinear preserver problems. Consider $\mathcal{H}$ a complex Hilbert space with dimension of $\mathcal{H}$ is greater than 2 and $\mathcal{B}(\mathcal{H})$ be the algebra of all bounded linear operators on $\mathcal{H}$, and $\mathcal{A}$, $\mathcal{B}$ be two subsets of $\mathcal{B}(\mathcal{H})$ both containing all operators of rank at most one. Let $A \in \mathcal{B}(\mathcal{H})$ and $\varepsilon \in (0, +\infty)$, $\varepsilon' \in (0,1)$ be two given fixed real numbers, denote by $\Lambda_\varepsilon(A)$, $\rho_\varepsilon(A)$, $\sigma_{\varepsilon'}(A)$ and $r_{\varepsilon'}(A)$ the ε-pseudo spectrum of A, the ε-pseudo spectral radius of A, the ε'-condition spectrum of A and the ε'-condition spectral radius of A respectively. In this thesis we first characterize the surjective maps ϕ from $\mathcal{A}$ to $\mathcal{B}$ satisfying $\delta(\phi(A) \bullet \phi(B)) = \delta(A \bullet B)$ for all $A, B \in \mathcal{A}$, where δ stands for $\Lambda_\varepsilon(A)$ or $\rho_\varepsilon(A)$ and $T \bullet S = TS^*T$, or δ stands for $\sigma_{\varepsilon'}(\cdot)$ or $r_{\varepsilon'}(\cdot)$ and $T \bullet S = TS^*$. Second, we determined all pairs of surjective maps $\phi, \psi : \mathcal{A} \longrightarrow \mathcal{B}$ satisfying $\phi(A)\psi(B) = 0$ if and only if $AB = 0$ for all $A, B \in \mathcal{A}$, where $\mathcal{A}$ and $\mathcal{B}$ are two subsets of the algebra of all bounded linear operators on a complex Banach space X, containing all operators of rank at most one. Moreover, we provided a version of this result in the Hilbert space context. This version will be used to describe all pairs of surjective maps ϕ and ψ from $\mathcal{A}$ to $\mathcal{B}$ such that $\Lambda_\varepsilon(\phi(A)\psi(B)) = \Lambda_\varepsilon(AB)$ for all A and B in $\mathcal{A}$.

Keywords : Pseudo spectrum; Pseudo spectral radius; Condition spectrum; Condition spectral radius; Nonlinear preserver.

Résumé

Dans ce travail en théorie des opérateurs, on s'intéresse aux problèmes de préservation d'approximation spectrale. On considère $\mathcal{H}$ un espace de Hilbert complexe de dimension supérieure ou égale à 3 et $\mathcal{B}(\mathcal{H})$ l'algèbre des opérateurs bornés sur h, et soient $\mathcal{A}$ et $\mathcal{B}$ deux sous-ensembles de $\mathcal{B}(\mathcal{H})$ contenant tous les opérateurs de rang au plus un. Soient $T \in \mathcal{B}(\mathcal{H})$ et $\varepsilon \in (0, +\infty)$, $\varepsilon' \in (0, 1)$ deux réels fixes donnés, notons $\Lambda_\varepsilon(A)$, $\rho_\varepsilon(A)$, $\sigma_{\varepsilon'}(A)$ et $r_{\varepsilon'}(A)$ le ε-pseudo spectre de T, le rayon ε-pseudo spectral de T, le spectre de ε'-condition de T et le rayon spectral de ε'-condition de T respectivement. Dans cette thèse, nous caractérisons en premier lieu les applications surjectives ϕ de $\mathcal{A}$ vers $\mathcal{B}$ qui satisfont $\delta(\phi(A) \bullet \phi(B)) = \delta(A \bullet B)$ pour tout $A, B \in \mathcal{A}$, où δ représente $\Lambda_\varepsilon(A)$ ou $\rho_\varepsilon(A)$ et $T \bullet S = TS^*T$, ou δ représente $\sigma_{\varepsilon'}(A)$ ou $r_{\varepsilon'}(A)$ et $T \bullet S = TS^*T$. Second, nous déterminons les applications surjectives ϕ et ψ définies sur deux sous-ensembles $\mathcal{A}$ et $\mathcal{B}$, de l'algèbre des opérateurs linéaires bornés sur un espace de Banach complexe X contenant tous les opérateurs de rang égal à un pour lesquels l'équivalence suivante est vérifiée ȷ$\phi(T)\psi(S) = 0$ si est seulement si $TS = 0$ pour tout $T, S \in \mathcal{A}$". De plus, dans le cas X est un espace de Hilbert, nous caractérisons les applications surjectives ϕ et ψ de $\mathcal{A}$ vers $\mathcal{B}$ qui satisfont $\Lambda_\varepsilon(\phi(T)\psi(S)) = \Lambda_\varepsilon(TS)$ pour tout $T, S \in \mathcal{A}$.

Mots Clés : Pseudo spectre; Rayon pseudo spectral; Spectre de condition; Rayon spectral de condition; Conservation non linéaire.

Résumé etendu

Dans ce travail en théorie des opérateurs, on s'intéresse aux problèmes de préservation d'approximation spectrale. On considère $\mathcal{H}$ un espace de Hilbert complexe de dimension supérieure ou égale à 3 et $\mathcal{B}(\mathcal{H})$ l'algèbre des opérateurs bornés sur $\mathcal{H}$, et soient $\mathcal{A}$ et $\mathcal{B}$ deux sous ensembles de $\mathcal{B}(\mathcal{H})$ contenant tous les opérateurs de rang au plus un, et $\mathcal{A}_I$, $\mathcal{B}_I$ deux sous ensembles de $\mathcal{B}(\mathcal{H})$ contenant tous les opérateurs de la forme $R + \lambda I$, où R est un opérateur de rang au plus un et $\lambda \in \mathbb{C}$. Soient $A \in \mathcal{B}(\mathcal{H})$ et $\varepsilon \in (0, +\infty)$, $\varepsilon' \in (0, 1)$ deux réels fixes donnés, notons $\Lambda_\varepsilon(A)$, $\rho_\varepsilon(A)$, $\sigma_{\varepsilon'}(A)$ et $r_{\varepsilon'}(A)$ le ε-pseudo spectre de A, le rayon ε-pseudo spectral de A, le spectre de ε'-condition de A et le rayon spectral de ε'-condition de A respectivement.

Dans cette thèse, nous caractérisons en premier lieu les applications surjectives ϕ de $\mathcal{A}$ vers $\mathcal{B}$ qui conservent le rayon ε-pseudo spectral du produit semi-triple de deux opérateurs, c'est à dire

$$\rho_\varepsilon(\phi(A)\phi(B)^*\phi(A)) = \rho_\varepsilon(AB^*A)$$

pour tout $A, B \in \mathcal{A}$. Aussi nous caractérisons les applications surjectives ϕ de $\mathcal{A}_I$ vers $\mathcal{B}_I$ qui conservent le ε-pseudo spectre du produit semi-triple de deux opérateurs, c'est à dire

$$\Lambda_\varepsilon(\phi(A)\phi(B)^*\phi(A)) = \Lambda_\varepsilon(AB^*A)$$

pour tout $A, B \in \mathcal{A}_I$. Ensuite nous donnons les expressions de spectre de ε'-condition et de rayon spectral de ε'-condition des opérateurs de rang égal à un. Après, nous donnons des descriptions complètes de certaines classes spéciales de matrices ou d'opérateurs en termes de spectre de ε'-condition ou de rayon spectral de ε'-condition, par exemple soit R un opérateur de rang égal à un, alors

$$\text{R est nilpotent} \iff \sigma_{\varepsilon'}(R) = D\left(0, \frac{\sqrt{\varepsilon'}\|R\|}{1-\varepsilon'}\right) \iff r_{\varepsilon'}(R) = \frac{\sqrt{\varepsilon'}\|R\|}{1-\varepsilon'},$$

et aussi R est un projecteur orthogonal multiplier par un scalaire si est seulement si

$$\sigma_{\varepsilon'}(R) = \bigcup_{\alpha,\beta\in\sigma(R)} D\left(\frac{\alpha-\beta\varepsilon'^2}{1-\varepsilon'^2}\ ,\ \frac{\varepsilon'|\alpha-\beta|}{1-\varepsilon'^2}\right)$$

si est seulement si

$$r_{\varepsilon'}(R) = \frac{\|R\|}{1-\varepsilon'},$$

avec $D\,(a,r)=\{z\in\mathbb{C}:|z-a|\leq r\}$ et $\sigma(R)$ le spectre de l'opérateur R.

Aussi nous caractérisons les applications surjectives ϕ de $\mathcal{A}$ vers $\mathcal{B}$ qui préservent le rayon spectral de ε'-condition du produit semi-double de deux opérateurs i.e.,

$$r_{\varepsilon'}(\phi(A)\phi(B)^*)=r_{\varepsilon'}(AB^*)$$

pour tout $A,B\in\mathcal{A}$. De plus nous décrivons les applications surjectives ϕ de $\mathcal{A}$ vers $\mathcal{B}$ qui préservent le spectre de ε'-condition du produit semi-double de deux opérateurs i.e.,

$$\sigma_{\varepsilon'}(\phi(A)\phi(B)^*)=\sigma_{\varepsilon'}(AB^*)$$

pour tout $A,B\in\mathcal{A}$. Enfin, nous déterminons les applications surjectives ϕ et ψ définies sur deux sous ensembles $\mathcal{A}$ et $\mathcal{B}$, de l'algèbre des opérateurs linéaires bornés sur un espace de Banach complexe X, et contenant tous les opérateurs de rang égal à un pour lesquels l'équivalence suivante est vérifiée

$$"\phi(A)\psi(B)=0 \text{ si est seulement si } AB=0 \text{ pour tout } A,B\in\mathcal{A}".$$

La caractérisation est formulée en séparant deux cas, dimension infinie et dimension finie supérieure ou égale à 3. Aussi, nous présentons une version de ce résultat dans le contexte d'un espace de Hilbert. De plus, dans le cas X est un espace de Hilbert, nous caractérisons les applications surjectives ϕ,ψ définies sur $\mathcal{A}$ à valeurs dans $\mathcal{B}$ telles que le ε-pseudo spectre de $\phi(A)\psi(B)$ coïncide avec celle de AB pour tout $A,B\in\mathcal{A}$. où $\mathcal{A}$, $\mathcal{B}$ deux ensembles de $\mathcal{B}(\mathcal{H})$ contient tout les opérateurs de rang un.

Introduction

Linear preserver problems (abbreviated as LPPs) are an important parts of functional analysis. It has numerous applications in many parts of mathematics and physics including matrix theory, function theory, complex analysis, differential and integral equations, control theory and quantum physics.

Let $\mathcal{M}$ be an algebra of bounded linear operators on a complex Banach space. The first group of LPPs is concerned with the study of those linear transformations on $\mathcal{M}$ which preserve certain functions.

Problem I. Let F be a scalar-valued, vector-valued or set-valued given function on $\mathcal{M}$. Characterize those linear transformations ϕ on $\mathcal{M}$ which satisfy

$$F(\phi(A)) = F(A) \qquad (A \in \mathcal{M}). \tag{1}$$

To give an example for such a problem, we recall a well-known result of Frobenius from 1897 describing the general form of all determinant preserving linear maps on matrix algebras which is commonly considered as the first result on LPPs. Namely, Frobenius proved in [22] that if $\mathcal{M} = \mathcal{M}_n(\mathbb{C})$ (the algebra of all $n \times n$ complex matrices) and $F(A) = \det(A)$, then every linear transformation ϕ which satisfies (1) is either of the form

$$\phi(A) = MAN \qquad (A \in \mathcal{M})$$

or

$$\phi(A) = MA^tN \qquad (A \in \mathcal{M})$$

for some nonsingular matrices $M, N \in \mathcal{M}_n(\mathbb{C})$ with $\det(MN) = 1$ (A^t stands for the transpose of A).

Now we turn to the second group of LPPs which concerns linear transformations that preserve certain subsets.

Problem II. Let $\mathcal{S}$ be a given subset of $\mathcal{M}$. Characterize those linear transformations ϕ on $\mathcal{M}$, which satisfy

$$A \in \mathcal{S} \Longrightarrow \phi(A) \in \mathcal{S} \qquad (A \in \mathcal{M}) \tag{2}$$

or

$$A \in \mathcal{S} \Longleftrightarrow \phi(A) \in \mathcal{S} \qquad (A \in \mathcal{M}). \tag{3}$$

In the first case we say that ϕ preserves the elements of $\mathcal{S}$ in one direction (or, more simply, that ϕ preserves the elements of $\mathcal{S}$), while in the second case we say that ϕ preserves the elements of $\mathcal{S}$ in both directions. A problem which should be certainly mentioned here is the problem of preserving commutativity. Although there are many results on the problem, this is still an active research topic. (For a recent remarkable achievement of the recent years see [49].) The basic result concerning commutativity preserving linear transformations on operator algebras is due to Omladi. In the paper [56] he described the structure of all bijective linear transformations on the whole operator algebra of a Banach space which preserve the commutativity in both directions. In the Hilbert space setting his result reads as follows. Let $\mathcal{H}$ be a Hilbert space of dimension at least 3 and $\mathcal{B}(\mathcal{H})$ be the algebra of all bounded linear operators on $\mathcal{H}$, let $\phi : \mathcal{H} \to \mathcal{H}$ be a bijective linear transformation which preserves commutativity in both directions. Then there exist a nonzero scalar λ, an invertible operator $U \in \mathcal{B}(\mathcal{H})$ and a linear functional f on $\mathcal{B}(\mathcal{H})$ such that ϕ is either of the form

$$\phi(A) = \lambda U A U^{-1} + f(A) I \qquad (A \in \mathcal{B}(\mathcal{H}))$$

or of the form

$$\phi(A) = \lambda U A^t U^{-1} + f(A) I \qquad (A \in \mathcal{B}(\mathcal{H})),$$

where A^t is the transpose of A.

The second example for this type of LPPs, we mention the famous Kaplansky's problem on invertibility preservers although it concerns general Banach algebras not merely operator algebras. Kaplansky's Problem reads as follows:
Let $\mathcal{A}, \mathcal{B}$ be semi-simple Banach algebras and let $\phi : \mathcal{A} \longrightarrow \mathcal{B}$ be a surjective linear transformation with the properties that $\phi(1) = 1$ (i.e., ϕ is unital) and $\phi(A) \in \mathcal{B}$ is invertible whenever so is $A \in \mathcal{A}$ (i.e., ϕ preserves invertibility in one direction). Is it true that ϕ is necessarily a Jordan homomorphism?

The third group of LPPs concerns linear transformations preserving certain relations.

Problem III. Let $\sim$ be a relation on $\mathcal{M}$. Characterize those linear transformations ϕ on $\mathcal{M}$ which satisfy

$$A \sim B \Longrightarrow \phi(A) \sim \phi(B) \qquad (A, B \in \mathcal{M}) \tag{4}$$

or

$$A \sim B \Longleftrightarrow \phi(A) \sim \phi(B) \qquad (A, B \in \mathcal{M}). \tag{5}$$

To give an example for such a problem, we mention the problem of zero product preserving mappings. These are the linear transformations $\phi : \mathcal{M} \longrightarrow \mathcal{M}$ which satisfy

$$AB = 0 \Longleftrightarrow \phi(A)\phi(B) = 0 \qquad (A, B \in \mathcal{M}). \tag{6}$$

Finally, the fourth group of LPPs concerns linear transformations which commute with certain maps on $\mathcal{M}$.

Problem VI. Given a map $F : \mathcal{M} \longrightarrow \mathcal{M}$. Characterize those linear transformations ϕ on $\mathcal{M}$ which satisfy

$$F(\phi(A)) = \phi(F(A)) \qquad (A \in \mathcal{M}). \tag{7}$$

Although in this case we sometimes say that ϕ preserves F, this is not to be confused with Problem I. To mention an interesting problem of this kind, we refer to the paper [14], where all the linear transformations on a full matrix algebra which preserve the kth power (here $F(A) = A^k, k$ being a fixed integer not less than 2) were described. The importance of that problem lies in the fact that when $k = 2$, the corresponding preservers are exactly the Jordan homomorphisms of $\mathcal{M}_n(\mathbb{C})$.

Consider $\mathcal{H}$ a complex Hilbert space with dimension of $\mathcal{H}$ is greater than 2 and $\mathcal{B}(\mathcal{H})$ be the algebra of all bounded linear operators on $\mathcal{H}$, and $\mathcal{A}$, $\mathcal{B}$ be two subsets of $\mathcal{B}(\mathcal{H})$ both containing all operators of rank at most one, and $\mathcal{A}_I$, $\mathcal{B}_I$ be two subsets of $\mathcal{B}(\mathcal{H})$ both containing all operators of the form $R + \lambda I$, where R is an operator in $\mathcal{B}(\mathcal{H})$ of rank at most one and $\lambda \in \mathbb{C}$. Let $A \in \mathcal{B}(\mathcal{H})$ and $\varepsilon \in (0, +\infty)$, $\varepsilon' \in (0, 1)$ be two given fixed real numbers, denote by $\Lambda_\varepsilon(A)$, $\rho_\varepsilon(A)$, $\sigma_{\varepsilon'}(A)$ and $r_{\varepsilon'}(A)$ the ε-pseudo spectrum of A, the ε-pseudo spectral radius of A, the ε'-condition spectrum of A and the ε'-condition spectral radius of A respectively.

This thesis is divided according to the following plan:

In chapter 1, we start by exhibiting some spectral sets of an operator and their properties. Then, we define the ε-pseudo spectrum, the ε-pseudo spectral radius, the ε'-condition spectrum and the ε'-condition spectral radius. Finally, we present complete descriptions of some special classes of matrices or operators in terms of the ε-pseudo spectrum, the ε-pseudo spectral radius, the ε'-condition spectrum or the ε'-condition spectral radius.

In chapter 2, we present fundamental theorems of preserver problems of the ε-pseudo spectrum, the ε-pseudo spectral radius, the ε'-condition spectrum and the ε'-condition spectral radius.

In chapter 3, we first characterize maps ϕ from $\mathcal{A}$ to $\mathcal{B}$ preserving the ε-pseudo spectral radius of the skew triple product of operators in the sense that

$$\rho_\varepsilon(\phi(A)\phi(B)^*\phi(A)) = \rho_\varepsilon(AB^*A) \quad \text{for all} \quad A, B \in \mathcal{A}.$$

Also, we describe the applications ϕ from $\mathcal{A}_I$ to $\mathcal{B}_I$ which preserve the ε-pseudo spectrum of the skew triple product of operators in the sense that

$$\Lambda_\varepsilon(\phi(A)\phi(B)^*\phi(A)) = \Lambda_\varepsilon(AB^*A) \quad \text{for all} \quad A, B \in \mathcal{A}_I.$$

In chapter 4, we provide the expression of the ε'-condition spectrum of any operator of rank one, and give an explicit formula for its ε'-condition spectral radius. Thus we will give complete descriptions of some special classes of matrices or operators in terms of the ε'-condition spectrum or the ε'-condition spectral radius. It is then illustrated that the results can be applied to characterize surjective mappings ϕ from $\mathcal{A}$ to $\mathcal{B}$ preserving the ε'-condition spectrum of operators skew double product in the sense that

$$r_{\varepsilon'}(\phi(A)\phi(B)^*) = r_{\varepsilon'}(AB^*) \quad \text{for all} \quad A, B \in \mathcal{A}.$$

Furthermore, we characterize surjective maps ϕ from $\mathcal{A}$ to $\mathcal{B}$ preserving the ε'-condition spectrum of the skew double product of operators in the sense that

$$\sigma_{\varepsilon'}(\phi(A)\phi(B)^*) = \sigma_{\varepsilon'}(AB^*) \quad \text{for all} \quad A, B \in \mathcal{A}.$$

In chapter 5, let $\mathcal{A}$ and $\mathcal{B}$ be two subsets of the algebra of all bounded linear operators on a complex Banach space X, containing all operators of rank at most one, we determine all pairs of surjective maps ϕ and ψ from $\mathcal{A}$ to $\mathcal{B}$ satisfying

$$\phi(A)\psi(B) = 0 \quad \text{if and only if} \quad AB = 0 \quad \text{for all} \quad A, B \in \mathcal{A}.$$

Moreover, we provide a version of this result in the Hilbert space context. This version will be used to describe all pairs of surjective maps ϕ and ψ defined on $\mathcal{A}$ with values in $\mathcal{B}$ such that the ε-pseudo spectrum of $\phi(A)\psi(B)$ coincides with that of AB for all A and B in $\mathcal{A}$. This chapter contains several applications of this contribution, thus finding certain results established by other authors.

Finally, we shall recapitulate the main findings of the thesis, and propose issues inviting future research.

Chapter 1

Preliminaries

This chapter is devoted to reminders on the spectral approximation theory. First, we start by exhibiting some spectral sets of an operator and their properties. Then, we define the pseudo spectrum, the pseudo spectral radius, the condition spectrum and the condition spectral radius. Finally, we present complete descriptions of some special classes of matrices or operators in terms of the pseudo spectrum, the pseudo spectral radius, the condition spectrum or the condition spectral radius.

1.1 Notations and Reminders

Throughout this thesis, $\mathcal{B}(X)$ will be the Banach algebra of all bounded linear operators on a complex Banach space X with $\dim X \geq 3$, and denote its identity operator by I. Recall that the natural pairing between X and its dual space X^* is defined by

$$\langle\ ,\ \rangle : X \times X^* \to \mathbb{C}, (x, f) \mapsto f(x),$$

and that the adjoint of any operator $A \in \mathcal{B}(X)$ is defined by

$$A^* : X^* \to X^*;\ f \mapsto f \circ A,$$

so that

$$\langle Ax, f\rangle = \langle x, A^* f\rangle \text{ for all } (x, f) \in X \times X^*.$$

Note that if $A : X \to X$ is a bounded conjugate linear operator on X (i.e., A is additive and

$$A(\lambda x) = \overline{\lambda} A(x) \text{ for all } \lambda \in \mathbb{C} \text{ and } x \in X),$$

then its adjoint A^* is the bounded conjugate linear operator on X^* defined by

$$A^*(f) = \overline{f} \circ A \text{ for all } f \in X^*.$$

where $\overline{f}(x) = \overline{f(x)}$ for all $x \in X$. Thus

$$\langle Ax, f\rangle = \overline{\langle x, A^* f\rangle} \text{ for all } (x, f) \in X \times X^*.$$

Recall that a semi-linear map $A : X \to X$ is an additive map for which there is a fixed ring automorphism τ of $\mathbb{C}$ such that

$$A(\lambda x) = \tau(\lambda)A(x)$$

for all $\lambda \in \mathbb{C}$ and $x \in X$. We define an operator of rank one $A \in \mathcal{B}(X)$ as an operator with dimensional range is one, i.e, $\dim A(X) = 1$.

A operator $A \in \mathcal{B}(X)$ of rank one if and only if A is of the form $x \otimes f$ where $x \in X$ and $f \in X^*$. It is defined by

$$(x \otimes f)(z) := \langle z, f\rangle x \text{ for all } z \in X.$$

We will denote by $\mathcal{F}_1(X)$ the set of all operators of rank at most one in $\mathcal{B}(X)$. This means that

$$\mathcal{F}_1(X) := \{x \otimes f \;:\; x \in X \text{ and } f \in X^*\}.$$

For all $x \in X$ and $f \in X^*$, we denote

$$R_f := \{y \otimes f : y \in X\} \text{ and } L_x := \{x \otimes g : g \in X^*\}.$$

L_x and R_f are maximal among additive subgroups of $\mathcal{F}_1(X)$. Note that $L_x = L_{\alpha x}$ and $R_f = R_{\alpha f}$ for every nonzero $\alpha \in \mathbb{C}$. Note also that if x and y are linearly independent (resp, f and g are linearly independent), then

$$L_x \cap L_y = \{0\} \quad (\text{resp, } R_f \cap R_g = \{0\}).$$

Lastly, note that

$$L_x \cap R_f = \{\alpha x \otimes f : \alpha \in \mathbb{C}\}.$$

For a linear subspace M of X (resp, X^*), we denote

$$M^{\perp} := \{f \in X^* : \langle x, f\rangle = 0, x \in M\} \quad (\text{resp, } M^{\perp} := \{x \in X : \langle x, f\rangle = 0, f \in M\}).$$

For any $A \in \mathcal{B}(X)$, we denote by $\ker(A)$, $\mathrm{ran}(A)$ and $\mathrm{rank}(A)$ respectively the kernel, the range of A and the dimension of $\mathrm{ran}(A)$. For a subset $\mathcal{A}$ of $\mathcal{B}(X)$, and $A \in \mathcal{A}$ we denote

$$\{A\}^R := \{B \in \mathcal{A} \setminus \{0\} : BA = 0\} \text{ and } \{A\}^L := \{B \in \mathcal{A} \setminus \{0\} : AB = 0\}.$$

The set $\{A\}^R$(resp, $\{A\}^L$) is said to be maximal, if for any operator $N \in \mathcal{A}$,

$$\{A\}^R \subseteq \{N\}^R \Rightarrow \{A\}^R = \{N\}^R \quad (\text{resp, } \{A\}^L \subseteq \{N\}^L \Rightarrow \{A\}^L = \{N\}^L).$$

If $X = \mathcal{H}$ is a complex Hilbert space, then $(X, \langle\ ,\ \rangle)$ will be identified with the Hilbert space $(\mathcal{H}, \langle\ ,\ \rangle)$ where $\langle\ ,\ \rangle$ is an inner product on $\mathcal{H}$, and $\mathcal{B}(X)$ with $\mathcal{B}(\mathcal{H})$, the algebra of all bounded linear operators on $\mathcal{H}$. The algebra $\mathcal{B}(\mathcal{H})$, equipped with the norm $\|.\|$ induced by the usual vector norm $\|x\| = \langle x, x\rangle^{1/2}$ on $\mathcal{H}$, i.e.,

$$\|A\| = \max\left\{\|Ax\| : x \in \mathcal{H}, 0 < \|x\| \leq 1\right\}.$$

We will denote by $\mathcal{F}_1(\mathcal{H})$ the set of all operators of rank at most one in $\mathcal{B}(\mathcal{H})$. This means that

$$\mathcal{F}_1(\mathcal{H}) := \{x \otimes y \ : \ x \in \mathcal{H} \text{ and } y \in \mathcal{H}\},$$

If X has dimension $n < \infty$, then X will be identified with the Hilbert space $\mathbb{C}^n$ and $\mathcal{B}(X)$ with $\mathcal{M}_n(\mathbb{C})$, the algebra of $n \times n$ complex matrices. According to the above identifications we have

$$\langle x, y\rangle = x_1\overline{y_1} + \cdots x_n\overline{y_n}$$

for all $x = (x_1, ..., x_n)$ and $y = (y_1, ..., y_n)$ in $\mathbb{C}^n$. For a ring automorphism τ of $\mathbb{C}$, we denote

$$x_\tau = (\tau(x_1), \tau(x_2), ..., \tau(x_n)), \ \ x_{\tau^*} = (\overline{\tau(\overline{x_1})}, \overline{\tau(\overline{x_2})}, ..., \overline{\tau(\overline{x_n})}) \quad \text{and} \quad M^\tau = (\tau(t_{ij}))$$

for all $x = (x_1, ..., x_n) \in \mathbb{C}^n$ and $M = (t_{ij}) \in \mathcal{M}_n(\mathbb{C})$.

For any $a \in \mathbb{C}$ and $r > 0$ we denote by $D(a, r)$ the closed disc centered at a and of radius r, and $\mathbb{T}$ stands for the unit circle of the complex $\mathbb{C}$.

1.2 Spectral sets of an operator

In this section we collect basic definitions and properties of the spectral sets of an operator $A \in \mathcal{B}(\mathcal{H})$.

1.2.1 The resolvent and the spectrum of an operator

Definition 1.2.1. *Let $A \in \mathcal{B}(\mathcal{H})$. The resolvent set $\rho(A)$ of A is the set of all complex scalars λ such that the operator $\lambda I - A$ is invertible.*

Proposition 1.2.2. *Let $A \in \mathcal{B}(\mathcal{H})$. $\rho(A)$ is open and the resolvent $R_A(\lambda) = (\lambda I - A)^{-1}$ of A is analytic on $\rho(A)$.*

Proposition 1.2.3. *Let $A \in \mathcal{B}(\mathcal{H})$ with $\|A\| < 1$. Then $I - A$ is invertible and $(I - A)^{-1} = \sum_{k=0}^{k=+\infty} A^k$, where the series is norm convergent. We have $\|(I - A)^{-1}\| \leq \frac{1}{1-\|A\|}$.*

Definition 1.2.4. *Let $A \in \mathcal{B}(\mathcal{H})$. The spectrum of A, denoted $\sigma(A)$ is the set of all complex scalars λ such that the operator $\lambda I - A$ is not invertible. We have $\sigma(A) = \mathbb{C} \setminus \rho(A)$. The spectral radius of A is given by*

$$r(A) := \sup\left\{|\lambda| : \lambda \in \sigma(A)\right\}.$$

Remark 1.2.5. *If* $A \in \mathcal{M}_n(\mathbb{C})$, *then* $\sigma(A)$ *is the set of eigenvalues of* A.

Theorem 1.2.6. *Let* $A \in \mathcal{B}(\mathcal{H})$. *The spectrum* $\sigma(A)$ *is a non empty compact subset of* $\mathbb{C}$.

Theorem 1.2.7. *Let* $A \in \mathcal{B}(\mathcal{H})$. *Then*

$$r(A) = \lim_{n\to+\infty} \|A^n\|^{\frac{1}{n}} = \inf_{n\geq 1} \|A^n\|^{\frac{1}{n}}.$$

For all A *we have that* $r(A) \leq \|A\|$. *If* A *is normal, then* $r(A) = \|A\|$.

Proposition 1.2.8. *The spectrum of a rank one operator* $x \otimes y$ *is*

$$\sigma(x \otimes y) = \{0, \langle x, y\rangle\} \text{ and } r(x \otimes y) = |\langle x, y\rangle|.$$

Proof. Let $x \otimes y$ be a rank one operator and let $u \in \mathcal{H}$ be a nonzero vector and $\lambda \in \mathbb{C}$ such that $(x \otimes y)u = \lambda u$. Then $\langle u, y\rangle x = \lambda u$, thus $\langle u, y\rangle\langle x, y\rangle = \lambda\langle u, y\rangle$. For every $u \in \mathcal{H}$ such that $\langle u, y\rangle = 0$, then $\lambda = 0$. Otherwise, $\lambda = \langle x, y\rangle$. This shows that $\sigma(x \otimes y) = \{0, \langle x, y\rangle\}$, then $r(x \otimes y) = |\langle x, y\rangle|$.

1.2.2 The pseudo spectrum and the pseudo spectral radius

In this section, we define the pseudo spectrum and the pseudo spectral radius. Also, we present complete descriptions of some special classes of matrices or operators in terms of the pseudo spectrum or the pseudo spectral radius.

Definition 1.2.9. *Let* $A \in \mathcal{B}(\mathcal{H})$ *and* $\varepsilon > 0$ *be arbitrary. The* ε*-pseudo spectrum of* A *is defined by*

$$\Lambda_\varepsilon(A) := \left\{\lambda \in \mathbb{C} : \ \|(\lambda I - A)^{-1}\| \geq \frac{1}{\varepsilon}\right\},$$

with the convention that $\|(\lambda I - A)^{-1}\| = \infty$ *if* $\lambda \in \sigma(A)$. *The* ε*-pseudo spectral radius of* A *is defined by*

$$\rho_\varepsilon(A) := \sup\left\{|z| : z \in \Lambda_\varepsilon(A)\right\}.$$

Proposition 1.2.10. *Let* $A \in \mathcal{B}(\mathcal{H})$ *and* $\varepsilon > 0$ *be arbitrary. Then* $\Lambda_\varepsilon(A)$ *is a compact of* $\mathbb{C}$ *and contains* $\sigma(A)$.

Proposition 1.2.11. *Let* $A \in \mathcal{B}(\mathcal{H})$ *and* $\varepsilon > 0$ *be arbitrary. The three following statements are equivalent.*

1) $\lambda \in \Lambda_\varepsilon(A)$.

2) There exists $B \in \mathcal{B}(\mathcal{H})$ *with* $\|B\| \leq \varepsilon$ *such that* $\lambda \in \sigma(A + B)$.

3) $\lambda \in \sigma(A)$ or there exists $v \in \mathcal{H}$ with $\|v\| = 1$ such that $\|(\lambda I - A)v\| \leq \varepsilon$.

Proof. Let us first show that 1) implies 3). Assume $\lambda \in \Lambda_\varepsilon(A)$ and $\lambda \notin \sigma(A)$. Then we can find $u \in \mathcal{H}$ such that $\|(\lambda I - A)^{-1}u\| \geq \varepsilon^{-1}\|u\|$. Let $v = (\lambda I - A)^{-1}u$. Then $\|(\lambda I - A)v\| \leq \varepsilon\|v\|$, and 3) follows by normalizing v. Next we show that 3) implies 2). If $\lambda \in \sigma(A)$, we can take $B = 0$. Thus assume $\lambda \notin \sigma(A)$. Let $v \in \mathcal{H}$ with $\|v\| = 1$ and $\|(\lambda I - A)v\| \leq \varepsilon$. Define a rank one operator B by

$$Bu = -\langle v, u\rangle(\lambda I - A)v.$$

Then $\|B\| \leq \varepsilon$, and $(A - \lambda I + B)v = 0$, so λ is an eigenvalue of $A + B$. Finally let us show that 2) implies 1). Here we use proof by contradiction. Assume that 2) holds and furthermore that $\lambda \notin \sigma(A)$ and $\|(\lambda I - A)^{-1}\| < \varepsilon$. We have $A + B - \lambda I = (I + B(\lambda I - A)^{-1})(A - \lambda I)$. Now our assumptions imply that $\|B(\lambda I - A)^{-1}\| < \varepsilon\varepsilon^{-1} = 1$, thus $I + B(\lambda I - A)^{-1}$ is invertible, see Proposition 1.2.3. Since $(A - \lambda I)$ is invertible, too, it follows that $A + B - \lambda I$ is invertible, contradicting $\lambda \in \sigma(A + B)$.

Remark 1.2.12. *Let $A \in \mathcal{B}(\mathcal{H})$ and $\varepsilon > 0$ be arbitrary.*

$$\Lambda_\varepsilon(A) = \bigcup_{E \in \mathcal{B}(\mathcal{H}), \|E\| \leq \varepsilon} \sigma(A + E).$$

Definition 1.2.13. *Let $A \in \mathcal{M}_n(\mathbb{C})$. The singular values of A are the square roots of non-negative eigenvalues of the self-adjoint operator A^*A.*

Definition 1.2.14. *Let $A \in \mathcal{B}(\mathcal{H})$. The s-numbers of A are the square roots of non-negative eigenvalues of the self-adjoint operator A^*A.*

Proposition 1.2.15 ([44]). *Let $A \in \mathcal{B}(\mathcal{H})$ and $\varepsilon > 0$ be arbitrary. Then $\Lambda_\varepsilon(A) = \{\lambda \in \mathbb{C} : \ s_{\min}(\lambda I - A) \leq \varepsilon\}$. where $s_{\min}$ denotes the minimal singular value in the matrix case and the smallest s-number for an operator.*

Let $A_1 \oplus A_2$ denotes the direct sum of two square matrices A_1 and A_2, whose dimensions need not be equal; in other words it is the block diagonal matrix

$$A_1 \oplus A_2 = \begin{pmatrix} A_1 & 0 \\ 0 & A_2 \end{pmatrix}$$

The following proposition collects some basic properties of the ε-pseudo spectrum.

Proposition 1.2.16 ([31, 35, 44]). *Let $A \in \mathcal{B}(\mathcal{H})$ and $\varepsilon > 0$ be arbitrary. The following statements hold.*

1) If $A \in \mathcal{M}_n(\mathbb{C})$ and $A = A_1 \oplus A_2$, then $\Lambda_\varepsilon(A) = \Lambda_\varepsilon(A_1) \cup \Lambda_\varepsilon(A_2)$.

2) $\sigma(A) + D(0, \varepsilon) \subseteq \Lambda_\varepsilon(A)$. The set equality holds if A is normal.

3) For any $c \in \mathbb{C}$, we have $\Lambda_\varepsilon(A + cI) = c + \Lambda_\varepsilon(A)$.

4) For any nonzero $c \in \mathbb{C}$, we have $\Lambda_\varepsilon(cA) = c\Lambda_{\frac{\varepsilon}{|c|}}(A)$.

5) For every unitary operator $U \in \mathcal{B}(\mathcal{H})$, we have $\Lambda_\varepsilon(UAU^*) = \Lambda_\varepsilon(A)$.

6) For every conjugate unitary operator U, we have $\Lambda_\varepsilon(UAU^*) = \Lambda_\varepsilon(A^*) = \overline{\Lambda_\varepsilon(A)}$.

Proposition 1.2.17 ([35])**.** *Suppose $\varepsilon > 0$ and*

$$A := \begin{pmatrix} a & b \\ 0 & c \end{pmatrix}$$

Then,

$$\Lambda_\varepsilon(A) = \Big\{ z \in \mathbb{C} : \sqrt{(|z-a| + |z-c|)^2 + |b|^2} - \sqrt{(|z-a| - |z-c|)^2 + |b|^2} \le 2\varepsilon \Big\}.$$

Remark 1.2.18. *$b = 0$ if and only if $\Lambda_\varepsilon(A) = D(a,\varepsilon) \cup D(c,\varepsilon)$. If $a = c = 0$, then*

$$\Lambda_\varepsilon(A) = \Big\{ z \in \mathbb{C} : |z| \le \varepsilon(\varepsilon + |b|) \Big\}.$$

By Proposition 1.2.16(2), if A is normal, then $\Lambda_\varepsilon(A) = \sigma(A) + D(0,\varepsilon)$. By Proposition 1.2.17, for $\varepsilon > 0$ and $A \in \mathcal{M}_2(\mathbb{C})$, if $\Lambda_\varepsilon(A)$ is the union of two disks, which may be identical, with radius ε, then A is normal. We do not need to know $\sigma(A)$ in advance to conclude that A is normal. However, the situation for higher dimensions is more delicate. In fact, contrary to the belief of some authors (see [44, Theorem 2.4], the converse of Proposition 1.2.16(2) is not true as shown in the following example.

Exemple 1.2.19 ([35])**.** *Let $\varepsilon = 2$, $w = e^{i2\pi/3}$, and $A = A_1 \oplus A_2$ with $A_1 = \mathrm{diag}(1, w, w^2)$ and*

$$A_2 := \begin{pmatrix} 0 & b \\ 0 & 0 \end{pmatrix}$$

where $b > 0$. Then A is not normal. If $b > 0$ satisfies

$$\Lambda_\varepsilon(A_2) = \Big\{ z \in \mathbb{C} : |z| \le \sqrt{2(2+|b|)} \Big\} \subseteq D\left(0, (\sqrt{13}+1)/2\right).$$

Then

$$\Lambda_\varepsilon(A) = \Lambda_\varepsilon(A_1) \cup \Lambda_\varepsilon(A_2) = \Lambda_\varepsilon(A_1) = \sigma(A) + D(0,\varepsilon).$$

The following proposition, proved in [31, Theorem 2.2] characterizes self-adjoint operators through their pseudo spectral properties.

Proposition 1.2.20. *Let $A \in \mathcal{B}(\mathcal{H})$, $\varepsilon > 0$ be arbitrary and $t \in \mathbb{R}$. Then $e^{it}A$ is self-adjoint if and only if $\Lambda_\varepsilon(A) \subseteq \Big\{ z \in \mathbb{C} : |\operatorname{Im}(e^{it}z)| \le \varepsilon \Big\}$.*

The following proposition gives a characterization of some special classes of matrices in terms of the ε-pseudo spectrum.

Proposition 1.2.21 ([31, 35]). *Let $\varepsilon > 0$ and $A \in \mathcal{B}(\mathcal{H})$. The following statements hold.*

1) *For any $a \in \mathbb{C}$, we have $\Lambda_\varepsilon(A) = D(a, \varepsilon)$ if and only if $A = aI$.*

2) *If $a \in \mathbb{C}$ is a nonzero scalar, then $\Lambda_\varepsilon(A) = D(0, \varepsilon) \cup D(a, \varepsilon)$ if and only if there exists a nontrivial orthogonal projection $P \in \mathcal{B}(\mathcal{H})$ such that $A = aP$.*

The second result describes the ε-pseudo spectrum and the ε-pseudo spectral radius of any operator of rank one.

Proposition 1.2.22 ([35]). *Let $\varepsilon > 0$ and x, y be two vectors in $\mathcal{H}$. Then*

1)

$$\Lambda_\varepsilon(x \otimes y) = \Big\{ z \in \mathbb{C} \ : \ \sqrt{(|z| + |z - \langle x, y\rangle|)^2 + \|x\|^2\|y\|^2 - |\langle x, y\rangle|^2} - \sqrt{(|z| - |z - \langle x, y\rangle|)^2 + \|x\|^2\|y\|^2 - |\langle x, y\rangle|^2} \le 2\varepsilon \Big\}.$$

2)

$$\rho_\varepsilon(x \otimes y) = \frac{1}{2}\left(\sqrt{|\langle x, y\rangle|^2 + 4\varepsilon^2 + 4\varepsilon\|x\|\|y\|} + |\langle x, y\rangle| \right).$$

In the following we present some topological properties of the ε-pseudo spectrum.

Proposition 1.2.23 ([42], Theorem 3.1). *Let $\varepsilon > 0$ and $A \in \mathcal{B}(\mathcal{H})$. Then $\Lambda_\varepsilon(A)$ has no isolated points.*

The next theorem proves a special property of the ε-pseudo spectrum which is not true for the usual spectrum.

Theorem 1.2.24 ([42], Theorem 3.6). *Let $\varepsilon > 0$ and $A \in \mathcal{B}(\mathcal{H})$. Then $\Lambda_\varepsilon(A)$ has a finite number of components and every component of $\Lambda_\varepsilon(A)$ contains an element from $\sigma(A)$.*

Corollary 1.2.25 ([42]). *Let $\varepsilon > 0$ and $M \in \mathcal{M}_n(\mathbb{C})$. Then, the following statements hold.*

1) *$\Lambda_\varepsilon(A)$ has at most n connected components, each containing one or more eigenvalues of A.*

2) *If $\Lambda_\varepsilon(A)$ has n components, then M is diagonalizable.*

1.2.3 The condition spectrum and the condition spectral radius

In this section, we define the condition spectrum and the condition spectral radius. Also, we present, the complete descriptions the condition spectrum and the condition spectral radius of some special classes of operators. For example, the condition spectrum of self-adjoint operators, orthogonal projections and rank one nilpotent operators.

Definition 1.2.26. *Let ε be a fixed positive real number such that $0<\varepsilon<1$. The ε-condition spectrum of an operator $A \in \mathcal{B}(\mathcal{H})$ is*

$$\sigma_\varepsilon(A) := \sigma(A) \cup \left\{ \lambda \in \mathbb{C} \setminus \sigma(A) : \ \|(\lambda I - A)^{-1}\| \|\lambda I - A\| \geq \frac{1}{\varepsilon} \right\}.$$

The ε-condition spectral radius of A is given by

$$r_\varepsilon(A) := \sup \left\{ |z| : z \in \sigma_\varepsilon(A) \right\}.$$

The following proposition collects some useful known properties of the ε-condition spectrum.

Proposition 1.2.27. *Let $0<\varepsilon<1$ and $A \in \mathcal{B}(\mathcal{H})$, the following statements hold.*

(1) $\sigma_\varepsilon(A) = \sigma(A)$ if and only if A is a scalar multiple of the identity.

(2) For every nonzero scalars $\alpha, \beta \in \mathbb{C}$, we have $\sigma_\varepsilon(\alpha I + \beta A) = \{\alpha\} + \beta \cdot \sigma_\varepsilon(A)$.

(3) For every unitary operator $U \in \mathcal{B}(\mathcal{H})$, we have $\sigma_\varepsilon(UAU^*) = \sigma_\varepsilon(A)$.

(4) For every conjugate unitary operator U, we have $\sigma_\varepsilon(UAU^*) = \sigma_\varepsilon(A^*) = \overline{\sigma_\varepsilon(A)}$.

The following result describes self-adjoint operators in terms of the ε-condition spectrum.

Proposition 1.2.28 ([47], Proposition 2.2). *Let $0<\varepsilon<1$ and $A \in \mathcal{B}(\mathcal{H})$ be a self-adjoint operator. Then*

$$\sigma_\varepsilon(A) = \bigcup_{\alpha,\beta \in \sigma(A)} D\left(\frac{\alpha - \beta\varepsilon^2}{1-\varepsilon^2} \, , \, \frac{\varepsilon|\alpha-\beta|}{1-\varepsilon^2} \right).$$

Corollary 1.2.29 ([47]). *Let $0<\varepsilon<1$ and $A \in \mathcal{B}(\mathcal{H})$ be a self-adjoint operator. Then*

$$r_\varepsilon(A) = \max \left\{ \frac{|i(A) - \varepsilon s(A)|}{1-\varepsilon}, \frac{|s(A) - \varepsilon i(A)|}{1-\varepsilon} \right\}.$$

where $i(A)$ and $s(A)$ denote the infimum and the supremum of $\sigma(A)$, respectively.

Corollary 1.2.30 ([47]). *Let* $0<\varepsilon<1$ *and* $P\in\mathcal{B}(\mathcal{H})$ *be a nontrivial self-adjoint projection. Then*

$$\sigma_{\varepsilon}(P)=D\left(\frac{1}{1-\varepsilon^2}\ ,\ \frac{\varepsilon}{1-\varepsilon^2}\right)\bigcup D\left(\frac{-\varepsilon^2}{1-\varepsilon^2}\ ,\ \frac{\varepsilon}{1-\varepsilon^2}\right)$$

and

$$r_{\varepsilon}(P)=\frac{1}{1-\varepsilon}.$$

The following proposition determines the ε-condition spectrum and the ε-condition spectral radius of rank one nilpotent operators.

Proposition 1.2.31 ([47]). *Let* $0<\varepsilon<1$ *and* $R\in\mathcal{B}(\mathcal{H})$ *be a rank one nilpotent operator. Then*

$$\sigma_{\varepsilon}(R)=D\left(0,\frac{\sqrt{\varepsilon}\|R\|}{1-\varepsilon}\right)$$

and

$$r_{\varepsilon}(R)=\frac{\sqrt{\varepsilon}\|R\|}{1-\varepsilon}.$$

In the following we present some topological properties of the ε-condition spectrum.

Proposition 1.2.32 ([43], Theorem 3.1). *Let* $0<\varepsilon<1$. *Then* $\sigma_{\varepsilon}(A)$ *has no isolated points.*

Corollary 1.2.33 ([43]). *Let* $0<\varepsilon<1$ *and* $A\in\mathcal{B}(\mathcal{H})$. *Then, if* $\sigma_{\varepsilon}(A)=D(\lambda_0,\varepsilon)$ *for some* $\lambda_0\in\mathbb{C}$, *then* $A=\lambda_0 I$.

The next theorem proves a special property of the ε-condition spectrum which is not true for the usual spectrum.

Theorem 1.2.34 ([43], Theorem 3.6). *Let* $0<\varepsilon<1$ *and* $A\in\mathcal{B}(\mathcal{H})$. *Then* $\sigma_{\varepsilon}(A)$ *has a finite number of components and every component of* $\sigma_{\varepsilon}(A)$ *contains an element from* $\sigma(A)$.

Corollary 1.2.35 ([43]). *Let* $0<\varepsilon<1$ *and* $A\in\mathcal{M}_n(\mathbb{C})$. *If* $\sigma_{\varepsilon}(A)$ *has* n *components, then* M *is diagonalizable.*

Chapter 2

Elementary results on pseudo spectrum and the condition spectrum preserving problems

Recently, there has been interest in studying maps ϕ on matrices or operators satisfying $F(\phi(A) \bullet \phi(B)) = F(A \bullet B)$. Here, $F(\cdot)$ is a spectral function or a spectral set such as the spectrum, the local spectrum, the numerical radius, the pseudo spectral radius, the pseudo spectrum, the condition spectral radius and the condition spectrum. On the other hand, $A \bullet B$ stands for different kinds of products such as the usual product AB, the triple product ABA, the Jordan product $AB + BA$, the skew product A^*B, the skew triple product AB^*A, the skew Jordan product $AB^* + B^*A$, the Lie product $AB - BA$ and the skew Lie products $AB - BA^*$; see for instance [4, 6, 30, 45, 46, 35, 31, 32, 33] and the references therein.

2.1 Pseudo spectrum preservers on matrices or operators

The pseudo spectrum was mainly developed by L. N. Trefethen in [44] with the aim of widening the classical spectrum which is not a sufficiently efficient tool for the study and the analysis of non-normal operators, even in finite dimension. Thus, the pseudo spectrum is used in various applications, and in the literature a pseudo spectral theory is developed alongside the spectral theory. For example the pseudo spectrum has been proposed as a fundamental mathematical tool in non-Hermitian Quantum Mechanics [40].

In [35, Theorem 3.3] Cui et al. characterized maps on $\mathcal{M}_n(\mathbb{C})$ that preserve the ε-pseudo spectrum of the usual product of matrices. They proved that a map ϕ

from $\mathcal{M}_n(\mathbb{C})$ into itself satisfies

$$\Lambda_\varepsilon\left(\phi(A)\phi(B)\right) = \Lambda_\varepsilon(AB) \quad (\forall A, B \in \mathcal{M}_n(\mathbb{C})) \tag{2.1}$$

if and only if there exist a scalar $c = \pm 1$ and a unitary matrix $U \in \mathcal{M}_n(\mathbb{C})$ such that $\phi(A) = cUAU^*$ for all $A \in \mathcal{M}_n(\mathbb{C})$. This result was extended to the infinite dimensional case by Cui et al. [31, Theorem 4.1], as follows

Theorem 2.1.1. *Let $\mathcal{H}$ be an infinite-dimensional complex Hilbert space and $\varepsilon > 0$. Then a surjective map ϕ from $\mathcal{B}(\mathcal{H})$ into itself satisfies*

$$\Lambda_\varepsilon(\phi(A)\phi(B)) = \Lambda_\varepsilon(AB) \;\; \textit{for all} \;\; A, B \in \mathcal{B}(\mathcal{H})$$

if and only if there exist a unitary operator $U \in \mathcal{B}(\mathcal{H})$ and $\mu \in \{1, -1\}$ such that

$$\phi(A) = \mu UAU^* \textit{ for all } A \in \mathcal{B}(\mathcal{H}).$$

In [31] the authors described surjective mappings on $\mathcal{B}(\mathcal{H})$ that preserve the ε-pseudo spectrum of the Jordan product of operators $AB + BA$, as follows

Theorem 2.1.2. *Let $\mathcal{H}$ be a complex Hilbert spaces with the dimension at least three and $\varepsilon > 0$. Then a surjective map ϕ from $\mathcal{B}(\mathcal{H})$ into itself satisfies*

$$\Lambda_\varepsilon(\phi(A)\phi(B) + \phi(B)\phi(A)) = \Lambda_\varepsilon(AB + BA) \;\; \textit{for all} \;\; A, B \in \mathcal{B}(\mathcal{H})$$

if and only if there exist $\mu \in \{1, -1\}$ and a unitary operator U on $\mathcal{H}$ such that

$$\phi(A) = \mu UAU^* \textit{ for all } A \in \mathcal{B}(\mathcal{H}) \quad \textit{or} \quad \phi(A) = \mu UA^tU^* \textit{ for all } A \in \mathcal{B}(\mathcal{H}),$$

where A^t is the transpose of A with respect to an arbitrary but fixed orthogonal basis of $\mathcal{H}$. Furthermore, when $\mathcal{H}$ is finite dimensional, the surjectivity assumption on ϕ can be removed.

In [33] the authors described surjective mappings on $\mathcal{B}(\mathcal{H})$ that preserve the ε-pseudo spectrum or the ε-pseudo spectral radius of skew products of operators, as follows

Theorem 2.1.3. *Let $\mathcal{H}$ be a complex Hilbert spaces with the dimension at least three, $\mathcal{A}$ and $\mathcal{B}$ be two subsets of $\mathcal{B}(\mathcal{H})$ which containing all operators of rank at most one and $\varepsilon > 0$. Then a surjective map $\phi : \mathcal{A} \to \mathcal{B}$ satisfies*

$$\rho_\varepsilon(\phi(A)^*\phi(B)) = \rho_\varepsilon(A^*B) \;\; \textit{for all} \;\; A, B \in \mathcal{A}$$

if and only if there exist two unitary operators U and V on $\mathcal{B}(\mathcal{H})$ and a functional $h : \mathcal{A} \to \mathbb{T}$ such that

$$\phi(A) = h(A)UAV \;\; \textit{for all} \;\; A \in \mathcal{A} \quad \textit{or} \quad \phi(A) = h(A)UA^*V \;\; \textit{for all} \;\; A \in \mathcal{A}.$$

Theorem 2.1.4. *Let $\mathcal{H}$ be a complex Hilbert spaces with the dimension at least three, $\mathcal{A}$ and $\mathcal{B}$ be two subsets of $\mathcal{B}(\mathcal{H})$ which containing all operators of rank at most one and $\varepsilon > 0$. Then a surjective map ϕ from $\mathcal{B}(\mathcal{H})$ into itself satisfies that*

$$\Lambda_\varepsilon(\phi(A)^*\phi(B)) = \Lambda_\varepsilon(A^*B) \quad \textit{for all} \quad A, B \in \mathcal{B}(\mathcal{H})$$

if and only if there exist a complex unit α and two unitary operators $U, V \in \mathcal{B}(\mathcal{H})$ such that

$$\phi(A) = \alpha UAV \quad \textit{for all} \quad A \in \mathcal{B}(\mathcal{H}).$$

In [32] the authors described surjective mappings on $\mathcal{M}_n(\mathbb{C})$ that preserve the ε-pseudo spectrum or the ε-pseudo spectral radius of and Lie products of matrices (see Theorems 2.1.5 and 2.1.6). Their approach is based on the properties of the ε-pseudo spectrum, the ε-pseudo spectral radius and the main results [59, Theorems 2.1 and 2.2] concerning the maps ϕ from $\mathcal{M}_n(\mathbb{C})$ into itself satisfying

$$\phi(A)\phi(B) - \phi(B)\phi(A) = 0 \Longleftrightarrow AB - AB = 0$$

for all $A, B \in \mathcal{M}_n(\mathbb{C})$.

Theorem 2.1.5. *Suppose $n \geq 3$ and $\varepsilon > 0$. Then a surjective map ϕ from $\mathcal{M}_n(\mathbb{C})$ into itself satisfies*

$$\rho_\varepsilon(\phi(A)\phi(B) - \phi(B)\phi(A)) = \rho_\varepsilon(AB - BA) \quad \textit{for all} \quad A, B \in \mathcal{M}_n(\mathbb{C})$$

if and only if there is a unitary $U \in \mathcal{M}_n(\mathbb{C})$ such that

$$\phi(A) = \mu_A U\psi(A)U^* + \nu_A I_n \quad \textit{for all} \quad A \in \mathcal{M}_n(\mathbb{C}),$$

where $\mu_A, \nu_A \in \mathbb{C}$ with $|\mu_A| = 1$, depending on A, and ψ is one of the following maps: $A \to A, A \to \overline{A}, A \to A^t$ or $A \to A^*$, and $\overline{A}$ is the matrix obtained from A by entrywise complex conjugation.

Theorem 2.1.6. *Suppose $n \geq 3$ and $\varepsilon > 0$. Then a surjective map ϕ from $\mathcal{M}_n(\mathbb{C})$ into itself satisfies*

$$\Lambda_\varepsilon(AB - BA) = \Lambda_\varepsilon(\phi(A)\phi(B) - \phi(B)\phi(A)) \quad \textit{for all} \quad A, B \in \mathcal{M}_n(\mathbb{C})$$

if and only if there exist $\mu \in \{1, -1\}$, a unitary matrix $U \in \mathcal{M}_n(\mathbb{C})$, a set Γ of normal matrices with at most two distinct eigenvalues such that

$$\begin{cases} \mu U\psi(A)U^* + \nu_A I_n \ \textit{if} \ A \in \mathcal{M}_n(\mathbb{C}) \setminus \Gamma \\ -\mu U\psi(A)U^* + \nu_A I_n \ \textit{if} \ A \in \Gamma \end{cases}$$

where $\nu_A \in \mathbb{C}$ depends on A, and ψ is one of the following maps: $A \to A$, or $A \to i\overline{A}$.

The next result is the characterization of maps preserving the skew Lie product of operators, established by Jianlian Cui et al in [33].

Theorem 2.1.7. *Let $\mathcal{H}$ and $\mathcal{K}$ be two complex separable Hilbert spaces with the dimension at least three and $\varepsilon > 0$. Then a bijective map $\phi : \mathcal{B}(\mathcal{H}) \to \mathcal{B}(K)$ satisfies*

$$\Lambda_\varepsilon(AB - BA^*) = \Lambda_\varepsilon(\phi(A)\phi(B) - \phi(B)\phi(A)^*) \quad \textit{for all} \quad A, B \in \mathcal{B}(\mathcal{H})$$

if and only if there exist a unitary operator $U \in \mathcal{B}(\mathcal{H}, \mathcal{K})$ and $\mu \in \{-1, 1\}$ such that

$$\phi(A) = \mu UAU^* \quad \textit{for all} \quad A \in \mathcal{B}(\mathcal{H}).$$

In [45] and [46] the authors described surjective mappings on $\mathcal{B}(\mathcal{H})$ that preserve the ε-pseudo spectrum or the ε-pseudo spectral radius of product of operators or of Jordan triple product of operators (see Theorems 2.1.8, 2.1.9, 2.1.10, 2.1.11 and 2.1.12). Their approach is based on the properties of the ε-pseudo spectrum and the ε-pseudo spectral radius and the main results [19, Lemma 2.2] and [23, Theorems 2.1 and 2.2] concerning the maps ϕ from $\mathcal{B}(\mathcal{H})$ into itself satisfying

$$\phi(A) \bullet \phi(B) = 0 \Longleftrightarrow A \bullet B = 0 \tag{2.2}$$

for all $A, B \in \mathcal{B}(\mathcal{H})$. Here, $A \bullet B$ stands for different kinds of products such as the usual product AB and the Jordan triple product ABA.

Theorem 2.1.8. *Let $n \geq 3$ and $\varepsilon > 0$. A map ϕ from $\mathcal{M}_n(\mathbb{C})$ into itself satisfies*

$$\rho_\varepsilon(\phi(A)\phi(B)\phi(A)) = \rho_\varepsilon(ABA) \quad \textit{for all} \quad A, B \in \mathcal{M}_n(\mathbb{C})$$

if and only if there exist a functional $\xi : \mathcal{M}_n(\mathbb{C}) \to \mathbb{T}$ and a unitary matrix $U \in \mathcal{M}_n(\mathbb{C})$ such that either

$$\phi(A) = \xi(T)UAU^* \ \textit{for all} \ A \in \mathcal{M}_n(\mathbb{C}) \quad \textit{or} \quad \phi(A) = \xi(A)U\overline{A}U^* \ \textit{for all} \ A \in \mathcal{M}_n(\mathbb{C}).$$

Theorem 2.1.9. *Let $\mathcal{H}$ be an infinite-dimensional complex Hilbert space and $\varepsilon > 0$. A surjective map ϕ from $\mathcal{B}(\mathcal{H})$ into itself satisfies*

$$\rho_\varepsilon(\phi(A)\phi(B)\phi(A)) = \rho_\varepsilon(ABA) \ \textit{for all} \ A, B \in \mathcal{B}(\mathcal{H})$$

if and only if there is a functional $\xi : \mathcal{B}(\mathcal{H}) \to \mathbb{T}$ and a unitary or conjugate unitary operator U such that

$$\phi(A) = \xi(A)UAU^* \quad \textit{for all} \quad A \in \mathcal{B}(\mathcal{H}).$$

Theorem 2.1.10. *Let $\mathcal{H}$ be an infinite-dimensional complex Hilbert space and $\varepsilon > 0$. A surjective map ϕ from $\mathcal{B}(\mathcal{H})$ into itself satisfies*

$$\rho_\varepsilon(\phi(A)\phi(B)) = \rho_\varepsilon(AB) \quad \textit{for all} \quad A, B \in \mathcal{B}(\mathcal{H})$$

if and only if there is a functional $\xi : \mathcal{B}(\mathcal{H}) \to \mathbb{T}$ and a unitary or conjugate unitary operator U such that

$$\phi(A) = \xi(A) U A U^*$$

for all $A \in \mathcal{B}(\mathcal{H})$.

The following result refines the above theorem in the finite-dimensional case, the surjectivity assumption on ϕ can be removed.

Theorem 2.1.11. *Let $n \geq 3$ and $\varepsilon > 0$. A map ϕ from $\mathcal{M}_n(\mathbb{C})$ into itself satisfies*

$$\rho_\varepsilon(\phi(A)\phi(B)) = \rho_\varepsilon(AB) \quad \textit{for all} \quad A, B \in \mathcal{M}_n(\mathbb{C})$$

if and only if there exist a functional $\xi : \mathcal{M}_n(\mathbb{C}) \to \mathbb{T}$ and a unitary matrix $U \in \mathcal{M}_n(\mathbb{C})$ such that either

$$\phi(A) = \xi(A) U A U^* \textit{ for all } A \in \mathcal{M}_n(\mathbb{C}) \quad \textit{or} \quad \phi(A) = \xi(A) U \overline{A} U^* \textit{ for all } A \in \mathcal{M}_n(\mathbb{C}).$$

Theorem 2.1.12. *Let $\mathcal{H}$ be a complex Hilbert space with $\dim \mathcal{H} \geq 3$. A surjective map ϕ from $\mathcal{B}(\mathcal{H})$ into itself satisfies*

$$\Lambda_\varepsilon(\phi(A)\phi(B)\phi(A)) = \Lambda_\varepsilon(ABA) \quad \textit{for all} \quad A, B \in \mathcal{B}(\mathcal{H})$$

if and only if there are a third root of unity μ and a unitary or conjugate unitary operator U such that

$$\phi(A) = \mu U A U^* \textit{ for all } A \in \mathcal{B}(\mathcal{H}) \quad \textit{or} \quad \phi(A) = \mu U A^t U^* \textit{ for all } A \in \mathcal{B}(\mathcal{H}),$$

where A^t is the transpose of A with respect to an arbitrary but fixed orthogonal basis of $\mathcal{H}$. Furthermore, when $\mathcal{H}$ is finite dimensional, the surjectivity assumption on ϕ can be removed.

2.2 Condition spectrum preservers on matrices or operators

Let $\mathcal{A}$ and $\mathcal{B}$ be two subsets of $\mathcal{B}(\mathcal{H})$ which containing all operators of rank at most one. We mention the following problems: Which surjective maps $\phi : \mathcal{A} \longrightarrow \mathcal{B}$ satisfying

$$\delta\left(\phi(A) \bullet \phi(B)\right) = \delta(A \bullet B) \quad (\forall A, B \in \mathcal{A})? \tag{2.3}$$

Here, δ stands for $\sigma_\varepsilon(\cdot)$ or $r_\varepsilon(\cdot)$, and $A \bullet B$ stands for different kinds of products such as the usual product AB, the triple product ABA, the Jordan product $AB + BA$, the Lie product $AB - BA$, the skew Lie product $AB - BA^*$, the skew product A^*B, the skew triple product AB^*A and the skew-Jordan product $AB^* + B^*A$; see for instance [5, 26, 47] and the references therein.

In [47] the authors described surjective mappings on $\mathcal{B}(\mathcal{H})$ that preserve the ε-condition spectral radius of product of operators or Jordan triple product of operators (see Theorems 2.2.1, 2.2.2 and 2.2.3). Their approach is based on the properties of the ε-condition spectrum, the ε-condition spectral radius and the main results [19, Lemma 2.2] and [23, Theorems 2.1 and 2.2] concerning the maps ϕ from $\mathcal{B}(\mathcal{H})$ into itself satisfying (2.2).

Theorem 2.2.1. *Let $\mathcal{H}$ be an infinite-dimensional complex Hilbert space and $0 < \varepsilon < 1$. A surjective map ϕ from $\mathcal{B}(\mathcal{H})$ into itself satisfies*

$$r_\varepsilon(\phi(A)\phi(B)) = r_\varepsilon(AB) \quad \textit{for all} \quad A, B \in \mathcal{B}(\mathcal{H})$$

if and only if there exist a functional $h : \mathcal{B}(\mathcal{H}) \to \mathbb{T}$ and a unitary or conjugate unitary operator U such that

$$\phi(A) = h(A)UAU^* \,\textit{for all}\,\; A \in \mathcal{B}(\mathcal{H}).$$

The next result refinesthe above theorem in the finite dimensional case. the surjectivity assumption on ϕ can be removed.

Theorem 2.2.2. *Let $n \geq 3$, and $0 < \varepsilon < 1$. A surjective map ϕ from $\mathcal{M}_n(\mathbb{C})$ into itself satisfies*

$$r_\varepsilon(\phi(A)\phi(B)) = r_\varepsilon(AB) \quad \textit{for all} \quad A, B \in \mathcal{M}_n(\mathbb{C})$$

if and only if there exist a functional $h : \mathcal{M}_n(\mathbb{C}) \to \mathbb{T}$ and a unitary or conjugate unitary operator U such that

$$\phi(A) = h(A)UAU^* \,\textit{for all}\,\; A \in \mathcal{M}_n(\mathbb{C}) \quad \textit{or} \quad \phi(A) = h(A)U\overline{A}U^* \,\textit{for all}\,\; A \in \mathcal{M}_n(\mathbb{C}).$$

Theorem 2.2.3. *Let $\mathcal{H}$ be a complex Hilbert space with $\dim \mathcal{H} \geq 3$ and $0 < \varepsilon < 1$. A surjective map from $\mathcal{B}(\mathcal{H})$ into itself satisfies*

$$r_\varepsilon(\phi(A)\phi(B)\phi(A)) = r_\varepsilon(ABA) \quad \textit{for all} \quad A, B \in \mathcal{B}(\mathcal{H})$$

if and only if there is a functional $h : \mathcal{B}(\mathcal{H}) \to \mathbb{T}$ and a unitary or conjugate unitary operator U such that

$$\phi(A) = h(A)UAU^* \,\textit{for all}\, A \in \mathcal{B}(\mathcal{H}) \quad \textit{or} \quad \phi(A) = h(A)UA^*U^* \,\textit{for all}\, A \in \mathcal{B}(\mathcal{H}).$$

Furthermore, when $\mathcal{H}$ is finite dimensional, the surjectivity assumption on ϕ can be removed.

In [26] H. Benbouziane et al. described surjective mappings $\phi : \mathcal{A} \longrightarrow \mathcal{B}$ that preserve the ε-condition spectrum, the ε-condition spectral radius of the skew triple product AB^*A (see Theorems 2.2.4 and 2.2.5), where $\mathcal{A}$ and $\mathcal{B}$ be two subsets of

$\mathcal{B}(\mathcal{H})$ which containing all operators of rank at most one. Their approach is based on the properties of the ε-condition spectrum, the ε-condition spectral radius and the main result [34, Corollary 3.5] concerning the maps $\phi : \mathcal{A} \longrightarrow \mathcal{B}$ preserve zero-skew triple product of operators in the sense that

$$\phi(A)\phi(B)^*\phi(A) = 0 \Longleftrightarrow AB^*B = 0 \quad \text{for all} \quad A, B \in \mathcal{A},$$

where $\mathcal{A}$ and $\mathcal{B}$ be two subsets of $\mathcal{B}(\mathcal{H})$ which containing all operators of rank at most one.

Theorem 2.2.4. *Let $\mathcal{H}$ be a complex Hilbert spaces with the dimension at least three, $\mathcal{A}$ and $\mathcal{B}$ be two subsets of $\mathcal{B}(\mathcal{H})$ which containing all operators of rank at most one and $0 < \varepsilon < 1$. A surjective map $\phi : \mathcal{A} \longrightarrow \mathcal{B}$ satisfies*

$$r_\varepsilon\left(\phi(B)\phi(A)^*\phi(B)\right) = r_\varepsilon(BA^*B) \quad (\forall A, B \in \mathcal{A})$$

if and only if there exist a unitary or conjugate unitary operator U on $\mathcal{H}$ and a functional $h : \mathcal{A} \longrightarrow \mathbb{T}$ such that either $\phi(A) = h(A)UAU^*$ for all $A \in \mathcal{A}$ or $\phi(A) = h(A)UA^*U^*$ for all $A \in \mathcal{A}$.

Theorem 2.2.5. *Let $\mathcal{H}$ be a complex Hilbert spaces with the dimension at least three and $0 < \varepsilon < 1$. A surjective map ϕ from $\mathcal{B}(\mathcal{H})$ into itself satisfies*

$$\sigma_\varepsilon\left(\phi(B)\phi(A)^*\phi(B)\right) = \sigma_\varepsilon(BA^*B) \quad (\forall A, B \in \mathcal{B}(\mathcal{H}))$$

if and only if either there exists a unitary operator U on $\mathcal{H}$ such that $\phi(A) = UAU^*$ for all $A \in \mathcal{B}(\mathcal{H})$, or there is a conjugate unitary operator U on $\mathcal{H}$ such that $\phi(A) = UA^*U^*$ for all $A \in \mathcal{B}(\mathcal{H})$.

Chapter 3

Maps preserving the pseudo spectrum of skew triple product of operators

The pseudo spectrum preserver problem was initiated by J. Alaminos, J. Extremera and A.R. Villena in [29], and then it was continued by several authors; see for instance [4, 6, 45, 46, 32, 33, 31, 35, 30] and the references therein. In [31], the authors characterized surjective mappings ϕ on $\mathcal{B}(\mathcal{H})$ satisfying

$$\Lambda_{\varepsilon}\left(\phi(A) \bullet \phi(B)\right) = \Lambda_{\varepsilon}(A \bullet B) \quad (\forall A, B \in \mathcal{B}(\mathcal{H})) \tag{3.1}$$

for the product $A \bullet B$ such as the usual product AB, and the Jordan product $AB + BA$. They obtained their result for $\mathcal{H}$ finite dimensional, the surjectivity assumption on ϕ can be removed. In [32], the authors described surjective mappings on $\mathcal{M}_n(\mathbb{C})$ preserving the ε-pseudo spectrum (ε-pseudo spectral radius) of Lie products of matrices. In [33, Theorem 5.1 and Corollary 5.4], the authors determined the structure of all surjective maps $\phi : \mathcal{A} \longrightarrow \mathcal{B}$ preserving the ε-pseudo spectrum (ε-pseudo spectral radius) of the skew product of operators, where $\mathcal{A}$ and $\mathcal{B}$ are two subsets of $\mathcal{B}(\mathcal{H})$ containing all operators of rank at most one. The structure of surjections ϕ on $\mathcal{B}(\mathcal{H})$ preserving the ε-pseudo spectrum (ε-pseudo spectral radius) of product or Jordan triple products of operators is determined in [45, 46].

In this chapter, we settle two important problems in this field. Let $\mathcal{A}$ and $\mathcal{B}$ be two subsets of $\mathcal{B}(\mathcal{H})$ containing all operators of rank at most one. We characterize surjective maps $\phi : \mathcal{A} \longrightarrow \mathcal{B}$ preserving the ε-pseudo spectral radius of skew triple product of operators. It is then illustrated that the results can be applied to characterize mappings $\phi : \mathcal{A} \longrightarrow \mathcal{B}$ preserving the ε-pseudo spectrum of skew triple product of operators, where $\mathcal{A}$ and $\mathcal{B}$ are two subsets of $\mathcal{B}(\mathcal{H})$ which contain all operators of the form $R + \lambda I$, where R is an operator in $\mathcal{B}(\mathcal{H})$ of rank at most one and $\lambda \in \mathbb{C}$.

3.1 Statement of the main results

In this section, we gather the statement of our main results. However, to prove each theorem some further tools are needed which are developed in subsequent sections. Each case is discussed below.

Let $\mathcal{A}$ and $\mathcal{B}$ be two subsets of $\mathcal{B}(\mathcal{H})$ containing all operators of rank at most one. We first characterize all surjective maps $\phi : \mathcal{A} \longrightarrow \mathcal{B}$ preserving the ε-pseudo spectral radius of the skew triple product of operators. Precisely, we establish the following result.

Theorem 3.1.1. *Let $\mathcal{A}$ and $\mathcal{B}$ be two subsets of $\mathcal{B}(\mathcal{H})$ containing all operators of rank at most one. A surjective map $\phi : \mathcal{A} \longrightarrow \mathcal{B}$ satisfies*

$$\rho_\varepsilon\left(\phi(B)\phi(A)^*\phi(B)\right) = \rho_\varepsilon(BA^*B) \quad (\forall A, B \in \mathcal{A}) \tag{3.2}$$

if and only if there exist a unitary or conjugate unitary operator U on $\mathcal{H}$ and a functional $h : \mathcal{A} \longrightarrow \mathbb{T}$ such that either $\phi(A) = h(A)UAU^*$ for all $A \in \mathcal{A}$ or $\phi(A) = h(A)UA^*U^*$ for all $A \in \mathcal{A}$.

Next, we assume that $\mathcal{A}$ and $\mathcal{B}$ are two subsets of $\mathcal{B}(\mathcal{H})$ containing all operators of rank at most one plus scalar operators, and describe the form of all surjective maps $\phi : \mathcal{A} \longrightarrow \mathcal{B}$ preserving the ε-pseudo spectrum of the skew triple product. principally, we obtain the following theorem.

Theorem 3.1.2. *Let $\mathcal{A}$ and $\mathcal{B}$ be two subsets of $\mathcal{B}(\mathcal{H})$ which contain all operators of the form $R+\lambda I$, where R is an operator in $\mathcal{B}(\mathcal{H})$ of rank at most one and $\lambda \in \mathbb{C}$. A surjective map $\phi : \mathcal{A} \longrightarrow \mathcal{B}$ satisfies*

$$\Lambda_\varepsilon\left(\phi(B)\phi(A)^*\phi(B)\right) = \Lambda_\varepsilon(BA^*B) \quad (\forall A, B \in \mathcal{A}) \tag{3.3}$$

if and only if either there exists a unitary operator U on $\mathcal{H}$ such that $\phi(A) = UAU^*$ for all $A \in \mathcal{A}$, or there is a conjugate unitary operator U on $\mathcal{H}$ such that $\phi(A) = UA^*U^*$ for all $A \in \mathcal{A}$.

3.2 Preliminaries and auxiliary results

In this section, we collect some useful lemmas needed for the proof of our main results. We also establish some results which are interesting in their own right.

The first result is a simple consequence of the Proposition 1.2.16-(2) and Proposition 1.2.21-(1), and is needed in the sequel.

Lemma 3.2.1. *An operator $A \in \mathcal{B}(\mathcal{H})$ is zero if and only if $\rho_\varepsilon(A) = \varepsilon$.*

Indeed. If $\rho_\varepsilon(A) = \varepsilon$, then $\sigma(A) + D(0, \varepsilon) \subseteq \Lambda_\varepsilon(A) \subseteq D(0, \varepsilon)$. Hence, $\sigma(A) = 0$, $\Lambda_\varepsilon(A) = D(0, \varepsilon)$ and $A = 0$.

The second lemma, established in [34, Corollary 3.5], determines the structure of mappings that preserve zero-skew triple product of operators.

Lemma 3.2.2 ([34]). *Let $\mathcal{H}$ be a real or complex Hilbert space with $\dim \mathcal{H} \geq 3$, and $\mathcal{A}$ and $\mathcal{B}$ be two subsets of $\mathcal{B}(\mathcal{H})$ containing all operators in $\mathcal{B}(\mathcal{H})$ of rank at most one. Suppose that $\phi : \mathcal{A} \longrightarrow \mathcal{B}$ is a surjective map. Then ϕ satisfies*

$$\phi(A)\phi(B)^*\phi(A) = 0 \Longleftrightarrow AB^*A = 0, \ (A, B \in \mathcal{A})$$

if and only if there exist unitary (or conjugate unitary) operators U, V on $\mathcal{H}$ and a functional $h : \mathcal{A} \longrightarrow \mathbb{T}$ such that $\phi(A) = h(A)UAV$ for all $A \in \mathcal{A}$ or $\phi(A) = h(A)UA^*V$ for all $A \in \mathcal{A}$.

We close this section with the following well known lemma. We include here its proof for the sake of completeness.

Lemma 3.2.3. *An operator $T \in \mathcal{B}(\mathcal{H})$ is a unimodular scalar multiple of the identity if and only if $|\langle Tx, y\rangle| = |\langle x, y\rangle|$ for all $x, y \in \mathcal{H}$.*

Indeed. Suppose that $|\langle Tx, y\rangle| = |\langle x, y\rangle|$ for all $x, y \in \mathcal{H}$, and assume by the way of contradiction that there is $x \in \mathcal{H}$ such that Tx and x are linearly independent. Then there exists a nonzero vector z in $\mathcal{H}$ such that $\langle Tx, z\rangle = 0$ and $\langle x, z\rangle = 1$. This contradiction shows that $T = \mu I$ for some scalar $\mu \in \mathbb{C}$. Such a scalar μ must be of modulus one since $|\mu| = |\langle Tx, x\rangle| = |\langle x, x\rangle| = 1$ for any unit vector $x \in \mathcal{H}$.

Since the reverse implication is trivial, the proof is complete.

3.3 Proof of Theorem 3.1.1.

We only need to prove the "*only if*" part. So, assume that $\phi : \mathcal{A} \longrightarrow \mathcal{B}$ is a surjective map such that

$$\rho_\varepsilon \left(\phi(A)\phi(B)^*\phi(A)\right) = \rho_\varepsilon(AB^*A)$$

for all $A, \ B \in \mathcal{A}$. This and Lemma 3.2.1 imply that

$$\phi(A)\phi(B)^*\phi(A) = 0 \Longleftrightarrow AB^*A = 0, \ (A, B \in \mathcal{A}).$$

By Lemma 3.2.2, there exist a functional $h' : A \mapsto \mathbb{C} \setminus \{0\}$ and two operators U and V on $\mathcal{H}$ such that either both U and V are unitary and $\phi(A) = h'(A)UAV$ for all $A \in \mathcal{A}$, or both U and V are conjugate unitary and $\phi(A) = h'(A)UA^*V$ for all $A \in \mathcal{A}$. So, we may and shall assume that ϕ takes the first form since otherwise we replace ϕ by $A \mapsto \phi(A)^*$. Since $\phi(0) = 0$, we also assume that $h'(0) = 1$.

We need to show that $V = \mu U^*$, for some scalar $\mu \in \mathbb{T}$ and $h'(\mathcal{A}) \subseteq \mathbb{T}$. For the reason of readability of the proof, we will use the following notations. Given

$x, y, z \in \mathcal{H}$, we set

$$\begin{cases} \lambda_{x,y,z}(h') = |h'(x \otimes y)|^2 |h'(x \otimes z)| |\langle z, y\rangle| \|x\|^2, \\ \mu_{x,y,z} = |\langle z, y\rangle| \|x\|^2, \\ \delta_{x,y,z} = \sqrt{\mu_{x,y,z}^2 |\langle x, y\rangle|^2 + 4\varepsilon^2 + 4\varepsilon\mu_{x,y,z}\|x\|\|y\|}. \end{cases}$$

We will complete the proof of the theorem by checking three assertions.

Assertion 1. For every $x, y, z \in \mathcal{H}$, we have

$$\begin{aligned} &\left(\mu_{x,y,z}|\langle x, y\rangle| - \lambda_{x,y,z}(h')|\langle VUx, y\rangle|\right)\left(\mu_{x,y,z}|\langle x, y\rangle| + \delta_{x,y,z}\right) \\ &\qquad = 2\varepsilon\|x\|\|y\|\left(\lambda_{x,y,z}(h') - \mu_{x,y,z}\right). \end{aligned} \tag{3.4}$$

Indeed. Let $x, y, z \in \mathcal{H}$ be arbitrary vectors. By equation (3.2), we have

$$\rho_\varepsilon(\phi(x \otimes y)\phi(x \otimes z)^*\phi(x \otimes y)) = \rho_\varepsilon((x \otimes y)(x \otimes z)^*(x \otimes y)).$$

Since $\phi(A) = h'(A)UAV$ for all $A \in \mathcal{A}$, we get

$$\rho_\varepsilon(|h'(x \otimes y)|^2 |h'(x \otimes z)| |\langle z, y\rangle| \|x\|^2 U(x \otimes y)V) = \rho_\varepsilon(|\langle z, y\rangle| \|x\|^2 x \otimes y). \tag{3.5}$$

By Proposition 1.2.22, we obtain that

$$\begin{aligned} & \rho_\varepsilon(|h'(x \otimes y)|^2 ||h'(x \otimes z)| |\langle z, y\rangle| \|x\|^2 Ux \otimes V^*y) \\ = \ & \rho_\varepsilon(\lambda_{x,y,z}(h')Ux \otimes V^*y) \qquad (3.6) \\ = \ & \frac{1}{2}\sqrt{\lambda_{x,y,z}(h')^2 |\langle Ux, V^*y\rangle|^2 + 4\varepsilon^2 + 4\varepsilon\lambda_{x,y,z}(h')\|Ux\|\|V^*y\|} \\ & + \frac{1}{2}\lambda_{x,y,z}(h')|\langle Ux, V^*y\rangle| \\ = \ & \frac{1}{2}\sqrt{\lambda_{x,y,z}(h')^2 |\langle VUx, y\rangle|^2 + 4\varepsilon^2 + 4\varepsilon\lambda_{x,y,z}(h')\|x\|\|y\|} \\ & + \frac{1}{2}\lambda_{x,y,z}(h')|\langle VUx, y\rangle|. \end{aligned}$$

On the other hand, from Proposition 1.2.22, we have

$$\begin{aligned} \rho_\varepsilon(|\langle z, y\rangle| \|x\|^2 x \otimes y) &= \rho_\varepsilon(\mu_{x,y,z} x \otimes y) \\ &= \frac{1}{2}\sqrt{\mu_{x,y,z}^2 |\langle x, y\rangle|^2 + 4\varepsilon^2 + 4\varepsilon\mu_{x,y,z}\|x\|\|y\|} \\ &\quad + \frac{1}{2}\mu_{x,y,z}|\langle x, y\rangle|. \end{aligned} \tag{3.7}$$

By (3.5), (3.6), (3.7), we conclude that

$$\begin{aligned} &\frac{1}{2}\left(\sqrt{\lambda_{x,y,z}(h')^2 |\langle VUx, y\rangle|^2 + 4\varepsilon^2 + 4\varepsilon\lambda_{x,y,z}(h')\|x\|\|y\|} + \lambda_{x,y,z}(h')|\langle VUx, y\rangle|\right) \\ &\qquad = \frac{1}{2}\left(\sqrt{\mu_{x,y,z}^2 |\langle x, y\rangle|^2 + 4\varepsilon^2 + 4\varepsilon\mu_{x,y,z}\|x\|\|y\|} + \mu_{x,y,z}|\langle x, y\rangle|\right) \\ &\qquad = \frac{1}{2}\left(\delta_{x,y,z} + \mu_{x,y,z}|\langle x, y\rangle|\right). \end{aligned}$$

Thus

$$\begin{aligned}\lambda_{x,y,z}(h')^2|\langle VUx,y\rangle|^2 &+ 4\varepsilon^2+4\varepsilon\lambda_{x,y,z}(h')\|x\|\|y\| \\ &= \left(\delta_{x,y,z}+\mu_{x,y,z}|\langle x,y\rangle|-\lambda_{x,y,z}(h')|\langle VUx,y\rangle|\right)^2 \\ &= \delta_{x,y,z}^2+(\mu_{x,y,z}|\langle x,y\rangle|)^2+(\lambda_{x,y,z}(h')|\langle VUx,y\rangle|)^2 \\ &+ 2\delta_{x,y,z}\mu_{x,y,z}|\langle x,y\rangle| \\ &- 2(\delta_{x,y,z}+\mu_{x,y,z}|\langle x,y\rangle|)\lambda_{x,y,z}(h')|\langle VUx,y\rangle|.\end{aligned}$$

Hence

$$\begin{aligned}2\varepsilon\lambda_{x,y,z}(h')\|x\|\|y\| &= \mu_{x,y,z}^2|\langle x,y\rangle|^2+2\varepsilon\mu_{x,y,z}\|x\|\|y\| \\ &+ \delta_{x,y,z}\mu_{x,y,z}|\langle x,y\rangle| \\ &- (\delta_{x,y,z}+\mu_{x,y,z}|\langle x,y\rangle|)\lambda_{x,y,z}(h')|\langle VUx,y\rangle|.\end{aligned}$$

Clearly from this (3.4) follows.

Assertion 2. For every vectors $x,u\in\mathcal{H}$, we have $|h'(x\otimes u)|=1$ and $|\langle VUx,u\rangle|=|\langle x,u\rangle|$.

Indeed. Let $x,u\in\mathcal{H}$ be two nonzero vectors, and let us discuss two cases.

Case 1. If $u\notin\operatorname{span}\{x,VUx\}$, then there exists a nonzero vector $y\in\mathcal{H}$ such that $\langle VUx,y\rangle=\langle x,y\rangle=0$ and $\langle u,y\rangle\neq0$. Since $\langle VUx,y\rangle=\langle x,y\rangle=0$, the equation (3.4) applied to x, y and $z=y$ implies that

$$(\lambda_{x,y,y}(h')|\langle VUx,y\rangle|-\mu_{x,y,y}|\langle x,y\rangle|)=0.$$

Again by equation (3.4), we get

$$2\varepsilon\|x\|\|y\|\left(\lambda_{x,y,y}(h')-\mu_{x,y,y}\right)=0.$$

Thus

$$|h'(x\otimes y)|^2|h'(x\otimes y)|\|y\|^2\|x\|^2=\|y\|^2\|x\|^2,$$

and $|h'(x\otimes y)|=1$.

Next, let us show that $|h'(x\otimes u)|=1$. Since $\langle x,y\rangle=0$ and $\langle VUx,y\rangle=0$, the equation (3.4) implies that

$$2\varepsilon\|x\|\|y\|\left(\lambda_{x,y,u}(h')-\mu_{x,y,u}\right)=0.$$

Thus

$$|h'(x\otimes y)|^2|h'(x\otimes u)||\langle u,y\rangle|\|x\|^2=|\langle u,y\rangle|\|x\|^2.$$

Since $\langle u, y\rangle \neq 0$ and $|h'(x \otimes y)| = 1$, we deduce that $|h'(x \otimes u)| = 1$.
Now, let us show that $|\langle VUx, u\rangle| = |\langle x, u\rangle|$. We know that $|h'(x \otimes u)| = 1$, which implies

$$\lambda_{x,u,u}(h') = |h'(x \otimes u)|^2 |h'(x \otimes u)| \|u\|^2 \|x\|^2 = \|u\|^2 \|x\|^2 = \mu_{x,u,u}. \tag{3.8}$$

By the equations (3.4) and (3.8), we get

$$\lambda_{x,u,u}(h') \left(|\langle VUx, u\rangle| - |\langle x, u\rangle|\right) \left(\mu_{x,u,u} |\langle x, u\rangle| + \delta_{x,u,u}\right) = 0.$$

Since

$$\lambda_{x,u,u}(h') \left(\mu_{x,u,u} |\langle x, u\rangle| + \delta_{x,u,u}\right) \neq 0,$$

we deduce that $|\langle VUx, u\rangle| = |\langle x, u\rangle|$. This completes the proof in this case.

Case 2. If $u \in \operatorname{span}\{x, VUx\}$, choose a vector $u' \notin \operatorname{span}\{x, VUx\}$ and note that we have $u + tu' \notin \operatorname{span}\{x, VUx\}$ for all nonzero scalars t. By the previous case, we get $|\langle VUx, u + tu'\rangle| = |\langle x, u + tu'\rangle|$. Hence

$$|\langle VUx, u\rangle| = \lim_{t \to 0} |\langle VUx, u + tu'\rangle| = \lim_{t \to 0} |\langle x, u + tu'\rangle| = |\langle x, u\rangle|.$$

This and the formula (3.4) imply that

$$\begin{aligned} |\langle x, u\rangle| \left(\mu_{x,u,u} - \lambda_{x,u,u}(h')\right) &\left(\mu_{x,u,u} |\langle x, u\rangle| + \delta_{x,u,u}\right) \\ &= 2\varepsilon \|x\| \|u\| \left(\lambda_{x,u,u}(h') - \mu_{x,u,u}\right). \end{aligned}$$

Therefore,

$$\left(\mu_{x,u,u} - \lambda_{x,u,u}(h')\right) \left(|\langle x, u\rangle| (\mu_{x,u,u} |\langle x, u\rangle| + \delta_{x,u,u}) + 2\varepsilon \|x\| \|u\|\right) = 0.$$

Since

$$\left(|\langle x, u\rangle| (\mu_{x,u,u} |\langle x, u\rangle| + \delta_{x,u,u}) + 2\varepsilon \|x\| \|u\|\right) > 0,$$

we have $\lambda_{x,u,u}(h') = \mu_{x,u,u}$, and thus $|h'(x \otimes u)| = 1$.

Assertion 3. There exists a functional $h : \mathcal{A} \longrightarrow \mathbb{T}$ such that $\phi(A) = h(A) UAU^*$ for all $A \in \mathcal{A}$.

Indeed. The previous assertion ensures that $|\langle VUx, y\rangle| = |\langle x, y\rangle|$ for all $x, y \in \mathcal{H}$, then by Lemma 3.2.3, we obtain that $VU = \mu I$ for some $\mu \in \mathbb{T}$. Therefore $\phi(A) = h'(A) UAV = h(A) UAU^*$ for all $A \in \mathcal{A}$, where $h = \sigma(\mu) h'$ and $\sigma(\mu) = \mu$ (resp. $\overline{\mu}$) if U is unitary (resp. conjugate unitary). Now let us show that $|h(A)| = 1$

for all $A \in \mathcal{A}$. Let $A \in \mathcal{A}$ be an arbitrary nonzero operator and choose $x, y \in \mathcal{H}$ such that $\langle x, Ay\rangle \neq 0$. By formula (3.2) we have

$$\begin{aligned}\rho_\varepsilon\left(\overline{h(A)}\langle x, Ay\rangle x\otimes y\right) &= \rho_\varepsilon\left(h(x\otimes y)^2\overline{h(A)}\langle x, Ay\rangle x\otimes y\right)\\ &= \rho_\varepsilon\left(h(x\otimes y)^2\overline{h(A)}Ux\otimes yA^*x\otimes yU^*\right)\\ &= \rho_\varepsilon\left(\phi(x\otimes y)\phi(A)^*\phi(x\otimes y)\right)\\ &= \rho_\varepsilon\left((x\otimes y)A^*(x\otimes y)\right)\\ &= \rho_\varepsilon\left(\langle x, Ay\rangle x\otimes y\right).\end{aligned}$$

By similar argument as in the proof of Assertion 1, we get

$$\begin{aligned}&\left(\mu'_{x,Ay}|\langle x, y\rangle| - \lambda'_{x,Ay}(h)|\langle x, y\rangle|\right)\left(\mu'_{x,Ay}|\langle x, y\rangle| + \delta'_{x,y,Ay}\right)\\ &\qquad = 2\varepsilon\|x\|\|y\|\left(\lambda'_{x,Ay}(h) - \mu'_{x,Ay}\right),\end{aligned} \tag{3.9}$$

where

$$\begin{cases}\lambda'_{x,Ay}(h) = |h(A)||\langle x, Ay\rangle|\\ \mu'_{x,Ay} = |\langle x, Ay\rangle|\\ \delta'_{x,y,Ay} = \sqrt{\mu'^2_{x,Ay}|\langle x, y\rangle|^2 + 4\varepsilon^2 + 4\varepsilon\mu'_{x,Ay}\|x\|\|y\|}.\end{cases}$$

The formula (3.9) becomes

$$\begin{aligned}&|\langle x, y\rangle|\left(\mu'_{x,Ay} - \lambda'_{x,Ay}(h)\right)\left(\mu'_{x,Ay}|\langle x, y\rangle| + \delta'_{x,y,Ay}\right)\\ &\qquad = 2\varepsilon\|x\|\|y\|\left(\lambda'_{x,Ay}(h) - \mu'_{x,Ay}\right).\end{aligned}$$

Therefore,

$$\left(\mu'_{x,Ay} - \lambda'_{x,Ay}(h)\right)\left(|\langle x, y\rangle|(\mu'_{x,Ay}|\langle x, y\rangle| + \delta'_{x,y,Ay} + 2\varepsilon\|x\|\|y\|\right) = 0.$$

Since

$$\left(|\langle x, y\rangle|(\mu'_{x,Ay}|\langle x, y\rangle| + \delta'_{x,y,Ay} + 2\varepsilon\|x\|\|y\|\right) > 0,$$

we obtain $\lambda'_{x,Ay}(h) = \mu'_{x,Ay}$, and thus $|h(A)| = 1$. This completes the proof of Theorem 3.1.1.

3.4 proof of Theorem 3.1.2

We only need to prove the necessity. Let $\phi : \mathcal{A} \longrightarrow \mathcal{B}$ be a surjective map such that

$$\Lambda_\varepsilon\left(\phi(A)\phi(B)^*\phi(A)\right) = \Lambda_\varepsilon(AB^*A)$$

for all $A, B \in \mathcal{A}$. Clearly,

$$\rho_\varepsilon\left(\phi(A)\phi(B)^*\phi(A)\right) = \rho_\varepsilon(AB^*A)$$

for all $A, B \in \mathcal{A}$. Then Theorem 3.1.1 ensures that there exist a unitary or conjugate unitary operator U on $\mathcal{H}$ and a functional $h : \mathcal{A} \longrightarrow \mathbb{T}$ such that $\phi(A) = h(A)UAU^*$ for all $A \in \mathcal{A}$ or $\phi(A) = h(A)UA^*U^*$ for all $A \in \mathcal{A}$.
We only need to show that if U is unitary (resp. conjugate unitary) operator, then ϕ has the form $A \mapsto UAU^*$ (resp., $A \mapsto UA^*U^*$). We will complete the proof of the theorem by checking two assertions.

Assertion 1. If U is unitary, then $\phi(A) = h(A)UAU^*$ for all $A \in \mathcal{A}$. If, however, U is conjugate unitary, then $\phi(A) = h(A)UA^*U^*$ for all $A \in \mathcal{A}$.

Indeed. Suppose that U is a unitary operator and let us show that ϕ takes the desired form. We can assume without loss of generality that $h(0) = 1$. Assume by the way of contradiction that $\phi(A) = h(A)UA^*U^*$ for all $A \in \mathcal{A}$. By formula (3.3), we have

$$\Lambda_\varepsilon\left(\overline{h(A)}h(I)^2UAU^*\right) = \Lambda_\varepsilon\left(\phi(I)\phi(A)^*\phi(I)\right) = \Lambda_\varepsilon(IA^*I) = \sigma_\varepsilon(A^*)$$

for all $A \in \mathcal{A}$. This and Proposition 1.2.16-(5) entail that

$$\Lambda_\varepsilon(A^*) = \Lambda_\varepsilon\left(\overline{h(A)}h(I)^2A\right), \quad (A \in \mathcal{A}). \tag{3.10}$$

Take an arbitrary nontrivial orthogonal projection $P = x \otimes x \in \mathcal{A}$ and let

$$A = (2i-1)P + (1-i)I.$$

Then (3.10) and Proposition 1.2.21 imply that

$$D(\overline{h(A)}h(I)^2(1-i), \varepsilon) \cup D(\overline{h(A)}h(I)^2 i, \varepsilon) = D(1+i, \varepsilon) \cup D(-i, \varepsilon).$$

It then follows that either

$$\overline{h(A)}h(I)^2(1-i) = 1+i \text{ and } \overline{h(A)}h(I)^2 i = -i,$$

or

$$\overline{h(A)}h(I)^2(1-i) = -i \text{ and } \overline{h(A)}h(I)^2 i = 1+i.$$

Both cases arise a contradiction, and therefore $\phi(A) = h(A)UAU^*$ for all $A \in \mathcal{A}$.
Lastly, if U is a conjugate unitary operator, then by a similar argument as above, one can conclude that $\phi(A) = h(A)UA^*U^*$ for all $A \in \mathcal{A}$.

Assertion 2. We have $h(A) = 1$ for all $A \in \mathcal{A}$.

Indeed. Assume that $\phi(A) = h(A)UAU^*$ for all $A \in \mathcal{A}$, and let A be a nonzero element of $\mathcal{A}$. Then there is a unit vector $x \in \mathcal{H}$ such that $\langle x, Ax\rangle \neq 0$. By applying the statement (2) of Proposition 1.2.21, we have

$$\begin{aligned}
D(0,\varepsilon)\cup D(1,\varepsilon) &= \Lambda_\varepsilon(x\otimes x)\\
&= \Lambda_\varepsilon\left(x\otimes x(x\otimes x)^* x\otimes x\right)\\
&= \Lambda_\varepsilon\left(\phi(x\otimes x)(\phi(x\otimes x))^*\phi(x\otimes x)\right)\\
&= \Lambda_\varepsilon\left(U\Big(h(x\otimes x)\overline{h(x\otimes x)}h(x\otimes x)x\otimes x(x\otimes x)^* x\otimes x\Big)U^*\right)\\
&= \Lambda_\varepsilon\left(h(x\otimes x)x\otimes x\right)\\
&= D(0,\varepsilon)\cup D(h(x\otimes x),\varepsilon).
\end{aligned}$$

Then $h(x\otimes x)=1$, and

$$\begin{aligned}
D(0,\varepsilon)\cup D(\langle x,Ax\rangle,\varepsilon) &= \Lambda_\varepsilon(\langle x,Ax\rangle x\otimes x)\\
&= \Lambda_\varepsilon(x\otimes xA^* x\otimes x)\\
&= \Lambda_\varepsilon\left(\phi(x\otimes x)\phi(A)^*\phi(x\otimes x)\right)\\
&= \Lambda_\varepsilon\left(U\Big(h(x\otimes x)\overline{h(A)}h(x\otimes x)x\otimes xA^* x\otimes x\Big)U^*\right)\\
&= \Lambda_\varepsilon\left(\overline{h(A)}\langle x,Ax\rangle x\otimes x\right)\\
&= D(0,\varepsilon)\cup D(\overline{h(A)}\langle x,Ax\rangle,\varepsilon).
\end{aligned}$$

This entails that $h(A)=1$; as desired.

Similarly one can prove the assertion if U is conjugate unitary. Thus the proof of Theorem 3.1.2 is completed.

Chapter 4

Condition spectrum of rank one operators and preservers of the condition spectrum of skew product of operators

Throughout this chapter, fix an arbitrary orthogonal basis $(e_i)_{i\in\Gamma}$ of $\mathcal{H}$, for $x \in \mathcal{H}$ write $x = \sum_{i\in\Gamma} \xi_i e_i$, and define the conjugate operator $J : \mathcal{H} \longrightarrow \mathcal{H}$ by $J(x) = \sum_{i\in\Gamma} \overline{\xi_i} e_i$. Let $A \in \mathcal{B}(\mathcal{H})$, we denote by $\overline{A}$ the bounded linear operator JAJ. Notice that $\langle \overline{A} e_i, e_j\rangle = \overline{\langle A e_i, e_j\rangle}$ for all $i, j \in \Gamma$.

Recently, there has been interest in studying maps ϕ on matrices or operators satisfying $F(\phi(A) \bullet \phi(B)) = F(A \bullet B)$. Here, $F(\cdot)$ is a spectral function or a spectral set such as the spectrum, the local spectrum, the numerical radius, the ε-pseudo spectral radius, the ε-pseudo spectrum, the ε-condition spectral radius, and the ε-condition spectrum. On the other hand, $A \bullet B$ stands for different kinds of products such as the usual product AB, the triple product ABA, the Jordan product $AB + BA$, the Lie product $AB - BA$, the skew Lie product $AB - BA^*$, the skew product A^*B, the skew triple product AB^*A and the skew-Jordan product $AB^* + B^*A$; see for instance [3, 4, 6, 5, 60, 61, 1, 46, 47, 10, 9, 8, 11, 33, 31, 35, 34, 12] and the references therein.

In this chapter, we compute the ε-condition spectrum of any operator of rank one, and give an explicit formula for its ε-condition spectral radius. Inparticular, we give complete descriptions of some special classes of operators in terms of the ε-condition spectrum or the ε-condition spectral radius. For example, we prove that an operator of rank one R is nilpotent if and only if

$$r_\varepsilon(R) = \frac{\sqrt{\varepsilon}\|R\|}{1-\varepsilon}$$

if and only if

$$\sigma_\varepsilon(R) = D\left(0, \frac{\sqrt{\varepsilon}\|R\|}{1-\varepsilon}\right).$$

Furthermore, we show that $R = \lambda x \otimes x$ for some $\lambda \in \mathbb{C}$ and $x \in \mathcal{H}$ if and only if

$$r_\varepsilon(R) = \frac{\|R\|}{1-\varepsilon}$$

if and only if

$$\sigma_\varepsilon(R) = \bigcup_{\alpha,\beta\in\sigma(R)} D\left(\frac{\alpha-\beta\varepsilon^2}{1-\varepsilon^2}\ ,\ \frac{\varepsilon|\alpha-\beta|}{1-\varepsilon^2}\right).$$

These result will be used to settle two important problems in the condition spectrum (or the condition spectral radius) preservers. First, let $\mathcal{A}$ and $\mathcal{B}$ be two subsets of $\mathcal{B}(\mathcal{H})$ containing all operators of rank at most one, we characterize surjective maps $\phi : \mathcal{A} \longrightarrow \mathcal{B}$ "preserving condition spectral radius" of skew product of operators. Second, we completely describe the form of surjective mappings $\phi : \mathcal{A} \longrightarrow \mathcal{B}$ preserving the condition spectrum of skew product of operators.

4.1 Statement of the main results

First, let us fix some notations. For every positive real number ε and every complex scalar z, we set

$$\widehat{\varepsilon} := \varepsilon + \frac{1}{\varepsilon} \text{ and } \lfloor z \rfloor := \begin{cases} \dfrac{z}{|z|} & \text{if} \quad z \neq 0 \\ \\ 1 & \text{if} \quad z = 0. \end{cases}$$

For every subset S of $\mathbb{C}$ we denote by ∂S the boundary of S.

4.1.1 Condition spectrum of operators of rank one

Our first main result describes the ε-condition spectrum and the ε-condition spectral radius of any operator of rank one.

Theorem 4.1.1. *Let $x \otimes y \in \mathcal{F}_1(\mathcal{H})$. Then the following statements hold.*

1. *The ε-condition spectrum of $x \otimes y$ is given by*

$$\begin{aligned} \sigma_\varepsilon(x\otimes y) &= \left\{z \in \mathbb{C} :\ |z|^2 + |z - \langle x,y\rangle|^2 + \|x\|^2\|y\|^2 - |\langle x,y\rangle|^2 \geq \widehat{\varepsilon}|z(z-\langle x,y\rangle)|\right\} \\ &= \left\{\tfrac{\langle x,y\rangle}{2} + te^{i\theta}\lfloor\langle x,y\rangle\rfloor\ :\ t\in\mathbb{R}\ ,\ -\frac{\pi}{2} \leq \theta \leq \frac{\pi}{2},\ \textit{and } \varrho_{x,y}(\theta,t) \leq 0\right\}, \end{aligned}$$

where

$$\begin{aligned}\varrho_{x,y}(\theta,t) = (\widehat{\varepsilon}^2-4)t^4 \quad &- \quad \left(4\|x\|^2\|y\|^2 + \frac{|\langle x,y\rangle|^2\left(\widehat{\varepsilon}^2\cos(2\theta)-4\right)}{2}\right)t^2 \\ &+ \quad \left(\frac{\widehat{\varepsilon}^2|\langle x,y\rangle|^4}{16} - (\|x\|^2\|y\|^2 - \frac{|\langle x,y\rangle|^2}{2})^2\right)\end{aligned}$$

for all $(\theta,t)\in\mathbb{R}^2$.

2. *The* ε*-condition spectral radius of* $x\otimes y$ *is given by*

$$r_\varepsilon(x\otimes y) = \frac{|\langle x,y\rangle|}{2} + \sqrt{\left(\frac{|\langle x,y\rangle|}{2}\right)^2 + \frac{\|x\|^2\|y\|^2}{\widehat{\varepsilon}-2}}.$$

As a consequence of Theorem 4.1.1, we obtain the following characterization of some specific rank one operators. This characterization will be proved in Section 4.2 and will be used in the proof of our preserves results.

Corollary 4.1.2. *Let* $R\in\mathcal{F}_1(\mathcal{H})$*. Then*

$$\frac{\sqrt{\varepsilon}\|R\|}{1-\varepsilon} \le r_\varepsilon(R) \le \frac{\|R\|}{1-\varepsilon}. \tag{4.1}$$

Moreover, the three following statements are equivalent

1. R *is a nilpotent operator.*

2. $$r_\varepsilon(R) = \frac{\sqrt{\varepsilon}\|R\|}{1-\varepsilon}.$$

3. $$\sigma_\varepsilon(R) = D\left(0,\frac{\sqrt{\varepsilon}\|R\|}{1-\varepsilon}\right).$$

Finally, the following statements are equivalent

(4) $R=\lambda x\otimes x$ *for some* $\lambda\in\mathbb{C}$ *and* $x\in\mathcal{H}$*.*

(5)

$$r_\varepsilon(R) = \frac{\|R\|}{1-\varepsilon}.$$

(6)

$$\sigma_\varepsilon(R) = \bigcup_{\alpha,\beta\in\sigma(R)} D\left(\frac{\alpha-\beta\varepsilon^2}{1-\varepsilon^2}\,,\,\frac{\varepsilon|\alpha-\beta|}{1-\varepsilon^2}\right).$$

Note that the implication "(1) $\implies$ (3)" in the previous result is known; see [47, Theorem 2.5]. We finish this part of Section 4.1.1 by a geometrical description of ε-condition spectrum of rank one operators. Denote

$$\begin{aligned}\Delta_{x,y,\theta} \;:=\; & \left(4\|x\|^2\|y\|^2 + \frac{|\langle x,y\rangle|^2\left(\widehat{\varepsilon}^2\cos(2\theta)-4\right)}{2}\right)^2 \\ & -4(\widehat{\varepsilon}^2-4)\left(\frac{\widehat{\varepsilon}^2|\langle x,y\rangle|^4}{16} - \left(\frac{|\langle x,y\rangle|^2}{2} + \|x\|^2\|y\|^2\right)^2\right)\end{aligned} \tag{4.2}$$

for all $(x,y) \in \mathcal{H}^2$, and $\theta \in \mathbb{R}$. Observe that $\Delta_{x,y,0} \geq 0$ and let

$$\theta_{x,y} = \max\left\{\theta \in [0,\frac{\pi}{2}] : \Delta_{x,y,\theta} \geq 0\right\}. \tag{4.3}$$

Let $(x,y) \in \mathcal{H}^2$, we also denote

$$\rho^+_{x,y,\theta} := \sqrt{\frac{\left(4\|x\|^2\|y\|^2 - 2|\langle x,y\rangle|^2 + \dfrac{|\langle x,y\rangle|^2\widehat{\varepsilon}^2\cos(2\theta)}{2}\right) + \sqrt{\Delta_{x,y,\theta}}}{2(\widehat{\varepsilon}^2-4)}},$$

and

$$\rho^-_{x,y,\theta} := \sqrt{\frac{\left(4\|x\|^2\|y\|^2 - 2|\langle x,y\rangle|^2 + \dfrac{|\langle x,y\rangle|^2\widehat{\varepsilon}^2\cos(2\theta)}{2}\right) - \sqrt{\Delta_{x,y,\theta}}}{2(\widehat{\varepsilon}^2-4)}}$$

if $|\langle x,y\rangle| > \dfrac{2}{\sqrt{\widehat{\varepsilon}-2}}\|x\|\|y\|$. Otherwise we set $\rho^-_{x,y,\theta} = 0$. Note that, since the spectrum of a rank operator contains at most two elements, [43, Theorem 8] tells us that $\sigma_\varepsilon(x \otimes y)$ possesses at most two connected components. The following result provides a more explicit formula for the ε-condition spectrum of operators of rank at most one.

Proposition 4.1.3. *Let $x \otimes y$ be any rank one operator. Then the set $\sigma_\varepsilon(x \otimes y)$ is symmetrical about the axis $\mathbb{R}\lfloor\langle x,y\rangle\rfloor$, the axis $\dfrac{\langle x,y\rangle}{2} + \mathbb{R}\sqrt{-1}\lfloor\langle x,y\rangle\rfloor$ and the point $\dfrac{\langle x,y\rangle}{2}$. Moreover, we have*

$$\sigma_\varepsilon(x \otimes y) = \left\{\frac{\langle x,y\rangle}{2} + te^{i\theta}\lfloor\langle x,y\rangle\rfloor \;:\; |\theta| \leq \theta_{x,y} \;,\; \rho^-_{x,y,\theta} \leq |t| \leq \rho^+_{x,y,\theta}\right\},$$

and one of the following properties holds :

1. $\sigma_\varepsilon(x \otimes y)$ *is connected, if and only if* $\langle x, y\rangle| \leq \dfrac{2}{\sqrt{\widehat{\varepsilon}-2}}\|x\|\|y\|$ *if and only if*

$$\sigma_\varepsilon(x \otimes y) = \left\{ \frac{\langle x, y\rangle}{2} + te^{i\theta}\lfloor\langle x, y\rangle\rfloor \ : \ |\theta| \leq \frac{\pi}{2} \ , \ |t| \leq \rho^{+}_{x,y,\theta} \right\}.$$

2. $\dfrac{\langle x, y\rangle}{2} \notin \sigma_\varepsilon(x \otimes y)$ *if and only if* $\langle x, y\rangle| > \dfrac{2}{\sqrt{\widehat{\varepsilon}-2}}\|x\|\|y\|$ *if and only if*

$$\left\{ \frac{\langle x, y\rangle}{2} + te^{i\theta}\lfloor\langle x, y\rangle\rfloor \ : \ |\theta| \leq \theta_{x,y} \ , \ \rho^{-}_{x,y,\theta} \leq t \leq \rho^{+}_{x,y,\theta} \right\}$$

and

$$\left\{ \frac{\langle x, y\rangle}{2} + te^{i\theta}\lfloor\langle x, y\rangle\rfloor \ : \ |\theta| \leq \theta_{x,y} \ , \ -\rho^{+}_{x,y,\theta} \leq t \leq -\rho^{-}_{x,y,\theta} \right\}$$

are the two connected components of $\sigma_\varepsilon(x \otimes y)$*.*

4.1.2 Condition spectrum preservers

In this section, we present two results on maps preserving the condition spectrum and the condition spectral radius of the skew product of operators. Let $\mathcal{A}$ and $\mathcal{B}$ be two subsets of $\mathcal{B}(\mathcal{H})$ containing $\mathcal{F}_1(\mathcal{H})$. We characterize all surjective maps $\phi : \mathcal{A} \longrightarrow \mathcal{B}$ preserving the ε-condition spectral radius of the skew product. Precisely, we state:

Theorem 4.1.4. *Let* $\mathcal{A}$ *and* $\mathcal{B}$ *be two subsets of* $\mathcal{B}(\mathcal{H})$ *containing* $\mathcal{F}_1(\mathcal{H})$*. A surjective map* $\phi : \mathcal{A} \longrightarrow \mathcal{B}$ *satisfies*

$$r_\varepsilon\left(\phi(A)^*\phi(B)\right) = r_\varepsilon(A^*B) \ \textit{for all} \ A, B \in \mathcal{A} \tag{4.4}$$

if and only if there exist unitary operators U *and* R *on* $\mathcal{H}$ *and a functional* $h : \mathcal{A} \longrightarrow \mathbb{T}$*, such that either* $\phi(A) = h(A)UAR$ *for all* $A \in \mathcal{A}$ *or* $\phi(A) = h(A)U\overline{A}R$ *for all* $A \in \mathcal{A}$*.*

Next, we assume that $\mathcal{A}_I$ and $\mathcal{B}_I$ are two subsets of $\mathcal{B}(\mathcal{H})$ which contain all operators of the form $R + \lambda I$, where R is an operator in $\mathcal{F}_1(\mathcal{H})$ and $\lambda \in \mathbb{C}$. The following theorem characterizes all surjective maps $\phi : \mathcal{A}_I \longrightarrow \mathcal{B}_I$ preserving the ε-condition spectrum of the skew product.

Theorem 4.1.5. *Let* $\mathcal{A}_I$ *and* $\mathcal{B}_I$ *be two subsets of* $\mathcal{B}(\mathcal{H})$ *which containing all operators of the form* $R + \lambda I$*, where* R *is an operator in* $\mathcal{F}_1(\mathcal{H})$ *and* $\lambda \in \mathbb{C}$*. A surjective map* $\phi : \mathcal{A}_I \longrightarrow \mathcal{B}_I$ *satisfies*

$$\sigma_\varepsilon\left(\phi(A)^*\phi(B)\right) = \sigma_\varepsilon(A^*B) \ \textit{for all} \ A, B \in \mathcal{A}_I \tag{4.5}$$

if and only if there exist unitary operators U, W *on* $\mathcal{H}$ *and* $\alpha \in \mathbb{T}$ *such that* $\phi(A) = \alpha UAW$ *for all* $A \in \mathcal{A}_I$*.*

4.2 Proof of Theorem 4.1.1.

First, we will compute the ε-condition spectrum and the ε-condition spectral radius for the 3×3-matrix

$$A := \begin{pmatrix} a & b & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \tag{4.6}$$

for a given scalars $a, b \in \mathbb{C}$. This involves the following three assertions.

Assertion 1. Let $a, b \in \mathbb{C}$ and $A \in \mathcal{M}_3(\mathbb{C})$ the matrix given by (4.6). For any $z \in \mathbb{C}$, the singular values of the matrix $zI - A$, are given by

$$\begin{cases} s_1(zI-A) &= \dfrac{1}{2}\left(\sqrt{(|z|+|z-a|)^2+|b|^2}+\sqrt{(|z|-|z-a|)^2+|b|^2}\right), \\ s_2(zI-A) &= |z|, \\ s_3(zI-A) &= \dfrac{1}{2}\left(\sqrt{(|z|+|z-a|)^2+|b|^2}-\sqrt{(|z|-|z-a|)^2+|b|^2}\right), \end{cases} \tag{4.7}$$

and we have $s_1(zI - A) \geq s_2(zI - A) \geq s_3(zI - A)$.

Indeed. For $z \in \mathbb{C}$, we denote $\tilde{A}_z = \begin{pmatrix} z-a & -b \\ 0 & z \end{pmatrix}$, and observe that

$$(zI - A)^*(zI - A) = \tilde{A}_z^* \tilde{A}_z \oplus (|z|^2).$$

Let $S^+(\tilde{A}_z)$ and $S^-(\tilde{A}_z)$ be the singular values of the $\tilde{A}_z$ satisfying $S^+(\tilde{A}_z) \geq S^-(\tilde{A}_z)$. Observe that

$$\begin{cases} (S^+(\tilde{A}_z))^2 + (S^-(\tilde{A}_z))^2 &= \operatorname{tr}(\tilde{A}_z{}^* \tilde{A}_z) = |z-a|^2 + |b|^2 + |z|^2, \\ S^+(\tilde{A}_z) S^-(\tilde{A}_z) &= |\det(\tilde{A}_z)| = |(z-a)z|. \end{cases}$$

Then, we immediately obtain

$$\begin{cases} S^+(\tilde{A}_z) &= \dfrac{1}{2}\left(\sqrt{(|z|+|z-a|)^2+|b|^2}+\sqrt{(|z|-|z-a|)^2+|b|^2}\right), \\ S^-(\tilde{A}_z) &= \dfrac{1}{2}\left(\sqrt{(|z|+|z-a|)^2+|b|^2}-\sqrt{(|z|-|z-a|)^2+|b|^2}\right). \end{cases}$$

Clearly $S^+(\tilde{A}_z)$, $S^-(\tilde{A}_z)$ and $|z|$ are the singular values of $zI - A$. Thus, we only need to prove that

$$S^+(\tilde{A}_z) \geq |z| \geq S^-(\tilde{A}_z). \tag{4.8}$$

Indeed, if $a = 0$, $b = 0$ or $z = 0$, then (4.8) is obvious. Now, assume that $a \neq 0$, $b \neq 0$ and $z \neq 0$, then we have

$$\begin{aligned} S^-(\tilde{A}_z) - |z| &= \frac{1}{2}\left(\sqrt{(|z|+|z-a|)^2+|b|^2} - \sqrt{(|z|-|z-a|)^2+|b|^2} - 2|z|\right) \\ &= \frac{1}{2}\frac{\left(\sqrt{(|z|+|z-a|)^2+|b|^2} - \sqrt{(|z|-|z-a|)^2+|b|^2}\right)^2 - 4|z|^2}{\sqrt{(|z|+|z-a|)^2+|b|^2} - \sqrt{(|z|-|z-a|)^2+|b|^2} + 2|z|} \\ &= \frac{|z-a|^2+|b|^2-|z|^2-\sqrt{\left((|z|+|z-a|)^2+|b|^2\right)\left((|z|-|z-a|)^2+|b|^2\right)}}{\sqrt{(|z|+|z-a|)^2+|b|^2} - \sqrt{(|z|-|z-a|)^2+|b|^2} + 2|z|}. \end{aligned}$$

First, it is clear that $\sqrt{(|z|+|z-a|)^2+|b|^2} - \sqrt{(|z|-|z-a|)^2+|b|^2} + 2|z| > 0$. Now, observe that

$$\begin{aligned} |z-a|^2 &+ |b|^2-|z|^2+\sqrt{\left((|z|+|z-a|)^2+|b|^2\right)\left((|z|-|z-a|)^2+|b|^2\right)} \\ &\geq |z-a|^2+|b|^2-|z|^2+|(|z|+|z-a|)(|z|-|z-a|)| \\ &\geq |z-a|^2+|b|^2-|z|^2+|z|^2-|z-a|^2 = |b|^2 \\ &> 0. \end{aligned}$$

Hence

$$\begin{aligned} |z-a|^2 &+ |b|^2-|z|^2-\sqrt{\left((|z|+|z-a|)^2+|b|^2\right)\left((|z|-|z-a|)^2+|b|^2\right)} \\ &= \frac{-4|b|^2|z|^2}{|z-a|^2+|b|^2-|z|^2+\sqrt{\left((|z|+|z-a|)^2+|b|^2\right)\left((|z|-|z-a|)^2+|b|^2\right)}} \\ &\leq 0. \end{aligned}$$

This shows that $S^-(\tilde{A}_z) \leq |z|$. The inequality $S^+(\tilde{A}_z) \geq |z|$ follows from the fact that

$$\begin{aligned} S^+(\tilde{A}_z) - |z| &= \frac{1}{2}\left(\sqrt{(|z|+|z-a|)^2+|b|^2} + \sqrt{(|z|-|z-a|)^2+|b|^2} - 2|z|\right) \\ &\geq \frac{1}{2}\left((|z|+|z-a|)+(|z|-|z-a|)-2|z|\right) \\ &= 0. \end{aligned}$$

This proves (4.8) and completes the proof of Assertion 1.

Assertion 2. Let $a, b \in \mathbb{C}$ and $A \in \mathcal{M}_3(\mathbb{C})$ be the matrix defined by (4.6). Then

$$\sigma_\varepsilon(A) = \left\{z \in \mathbb{C} : |z|^2 + |z-a|^2 + |b|^2 \geq \widehat{\varepsilon}|z(z-a)|\right\}.$$

Indeed. It is well known that

$$\|(zI-A)^{-1}\| = \frac{1}{s_3(zI-A)} \quad \text{and} \quad \|(zI-A)\| = s_1(zI-A) \tag{4.9}$$

for all $z \in \mathbb{C} \setminus \sigma(A)$. Then

$$\begin{cases} \|(zI-A)^{-1}\| &= \dfrac{2}{\sqrt{(|z|+|z-a|)^2+|b|^2}-\sqrt{(|z|-|z-a|)^2+|b|^2}} \\ \\ \|(zI-A)\| &= \dfrac{1}{2}\left(\sqrt{(|z|+|z-a|)^2+|b|^2}+\sqrt{(|z|-|z-a|)^2+|b|^2}\right) \\ &= \dfrac{2|z(z-a)|}{\sqrt{(|z|+|z-a|)^2+|b|^2}-\sqrt{(|z|-|z-a|)^2+|b|^2}} \end{cases}$$

for all $z \in \mathbb{C} \setminus \sigma(A)$. Therefore,

$$\begin{aligned} z \in \sigma_\varepsilon(A) &\iff \|(zI-A)^{-1}\|\|(zI-A)\| \geq \varepsilon^{-1} \\ &\iff \frac{1}{4}\left(\sqrt{(|z|+|z-a|)^2+|b|^2}-\sqrt{(|z|-|z-a|)^2+|b|^2}\right)^2 \ \leq \ \varepsilon|z(z-a)| \\ &\iff \sqrt{(|z|+|z-a|)^2+|b|^2}-\sqrt{(|z|-|z-a|)^2+|b|^2} \ \leq \ 2\sqrt{\varepsilon|z(z-a)|} \\ &\iff \sqrt{\frac{|z|^2+|z-a|^2+2|z(z-a)|+|b|^2}{|z(z-a)|}} \\ &\qquad - \sqrt{\frac{|z|^2+|z-a|^2-2|z(z-a)|+|b|^2}{|z(z-a)|}} \ \leq \ 2\sqrt{\varepsilon} \\ &\iff \sqrt{\frac{|z|}{|z-a|}+\frac{|z-a|}{|z|}+2+\frac{|b|^2}{|z(z-a)|}} \\ &\qquad - \sqrt{\frac{|z|}{|z-a|}+\frac{|z-a|}{|z|}-2+\frac{|b|^2}{|z(z-a)|}} \ \leq \ 2\sqrt{\varepsilon} \end{aligned}$$

for all $z \in \mathbb{C} \setminus \{0, a\}$. Thus

$$\begin{aligned} z \in \sigma_\varepsilon(A) &\iff \sqrt{h_{a,b}(z)+2}-\sqrt{h_{a,b}(z)-2} \leq 2\sqrt{\varepsilon} \\ &\iff \left(\sqrt{h_{a,b}(z)+2}-\sqrt{h_{a,b}(z)-2}\right)^2 \leq 4\varepsilon \\ &\iff h_{a,b}(z)-\sqrt{h_{a,b}^2(z)-4} \leq 2\varepsilon \end{aligned}$$

for all $z \in \mathbb{C} \setminus \{0, a\}$, where

$$h_{a,b}(z) = \frac{|z|}{|z-a|}+\frac{|z-a|}{|z|}+\frac{|b|^2}{|z(z-a)|} \qquad (z \in \mathbb{C} \setminus \{0, a\}).$$

Now, from the fact that $t \mapsto t-\sqrt{t^2-4}$ is a decreasing function on $[2, +\infty)$, and using the fact that $\widehat{\varepsilon}$ is the unique real solution of the equation $t-\sqrt{t^2-4} = 2\varepsilon$, we obtain the equivalence

$$h_{a,b}(z)-\sqrt{h_{a,b}^2(z)-4} \leq 2\varepsilon \iff h_{a,b}(z) \geq \widehat{\varepsilon}.$$

Thus,

$$z \in \sigma_\varepsilon(A) \iff h_{a,b}(z) \geq \widehat{\varepsilon} \tag{4.10}$$

for all $z \in \mathbb{C} \setminus \sigma(A)$. This implies that

$$\sigma_\varepsilon(A) \setminus \sigma(A) = \left\{ z \in \mathbb{C} \setminus \sigma(A) : |z|^2 + |z-a|^2 + |b|^2 \geq \widehat{\varepsilon}|z(z-a)| \right\}.$$

Finally, since $\sigma(A) = \{0, a\} \subset \left\{ z \in \mathbb{C} : |z|^2 + |z-a|^2 + |b|^2 \geq \widehat{\varepsilon}|z(z-a)| \right\}$, we get

$$\sigma_\varepsilon(A) = \left\{ z \in \mathbb{C} : |z|^2 + |z-a|^2 + |b|^2 \geq \widehat{\varepsilon}|z(z-a)| \right\}.$$

Assertion 3. Let $a, b \in \mathbb{C}$ and $A \in \mathcal{M}_3(\mathbb{C})$ be the matrix given by (4.6). Then, the following statements hold.

1. $\sigma_\varepsilon(A)$ is equal to the set of all complex numbers $\frac{a}{2} + te^{i\theta}\lfloor a \rfloor$ such that $t \in \mathbb{R}$, $-\frac{\pi}{2} \leq \theta \leq \frac{\pi}{2}$ and
$$\begin{aligned} (\widehat{\varepsilon}^2 - 4)t^4 \quad &- \quad \left(2|a|^2 + 4|b|^2 + \frac{|a|^2\widehat{\varepsilon}^2 \cos(2\theta)}{2} \right) t^2 \\ &+ \quad \tfrac{\widehat{\varepsilon}^2|a|^4}{16} - (\tfrac{|a|^2}{2} + |b|^2)^2 \qquad\qquad \leq 0. \end{aligned} \tag{4.11}$$
2. The ε-condition spectral radius of A is given by
$$r_\varepsilon(A) = \frac{|a|}{2} + \sqrt{\left(\frac{|a|}{2}\right)^2 + \frac{|a|^2 + |b|^2}{\widehat{\varepsilon} - 2}}.$$

Indeed. If $a \neq 0$, we may replace A by $A' = \frac{1}{a}A$. Observe that $\sigma_\varepsilon(A) = a \cdot \sigma_\varepsilon(A')$, then the property (4.11) of Assertion 3 is equivalent to

$$\left(\widehat{\varepsilon}^2 - 4\right) t^4 - \left(2 + 4|b'|^2 + \frac{\widehat{\varepsilon}^2 \cos(2\theta)}{2} \right) t^2 + \frac{\widehat{\varepsilon}^2}{16} - (\frac{1}{2} + |b'|^2)^2 \leq 0, \tag{4.12}$$

where $b' = b/a$. We claim that (4.1) holds, if and only if $\frac{1}{2} + te^{i\theta} \in \sigma_\varepsilon(A')$ for all $t \in \mathbb{R}$ and $-\frac{\pi}{2} \leq \theta \leq \frac{\pi}{2}$. Indeed, let $t \in \mathbb{R}$ and $-\frac{\pi}{2} \leq \theta \leq \frac{\pi}{2}$, we have

$$\begin{aligned} \tfrac{1}{2} + te^{i\theta} \in \sigma_\varepsilon(A') \quad &\iff \quad h_{1,b'}(te^{i\theta} + \tfrac{1}{2}) \geq \widehat{\varepsilon} \\ &\iff \quad |te^{i\theta} + \tfrac{1}{2}|^2 + |te^{i\theta} - \tfrac{1}{2}|^2 + |b'|^2 \geq \widehat{\varepsilon}|t^2e^{2i\theta} - \tfrac{1}{4}| \\ &\iff \quad t^2 + \cos(\theta)t + \tfrac{1}{4} + t^2 - \cos(\theta)t + \tfrac{1}{4} + |b'|^2 \geq \widehat{\varepsilon}\sqrt{t^4 - \tfrac{\cos(2\theta)t^2}{2} + \tfrac{1}{16}} \\ &\iff \quad 2t^2 + \tfrac{1}{2} + |b'|^2 \geq \widehat{\varepsilon}\sqrt{t^4 - \tfrac{\cos(2\theta)t^2}{2} + \tfrac{1}{16}} \\ &\iff \quad \left(\widehat{\varepsilon}^2 - 4\right) t^4 - \left(2 + 4|b'|^2 + \tfrac{\widehat{\varepsilon}^2 \cos(2\theta)}{2} \right) t^2 + \tfrac{\widehat{\varepsilon}^2}{16} - (\tfrac{1}{2} + |b'|^2)^2 \leq 0. \end{aligned}$$

This proves Assertion 3-(1) in the case when $a \neq 0$. Now, assume that $a = 0$, by Assertion 2, we have

$$\begin{aligned}
\sigma_\varepsilon(A) &= \left\{z \in \mathbb{C} \ : \ 2|z|^2 + |b|^2 \geq \widehat{\varepsilon}|z|^2\right\} \\
&= \left\{z \in \mathbb{C} \ : \ |z| \leq \tfrac{|b|}{\sqrt{\widehat{\varepsilon}-2}}\right\} = D\left(0, \tfrac{|b|}{\sqrt{\widehat{\varepsilon}-2}}\right) \\
&= \left\{te^{i\theta} \ : \ t \in \mathbb{R}, -\tfrac{\pi}{2} \leq \theta \leq \tfrac{\pi}{2} \ , \ |t| \leq \tfrac{|b|}{\sqrt{\widehat{\varepsilon}-2}}\right\} \\
&= \left\{te^{i\theta} \ : \ t \in \mathbb{R}, -\tfrac{\pi}{2} \leq \theta \leq \tfrac{\pi}{2} \ , \ (\widehat{\varepsilon} - 2)t^2 - |b|^2 \leq 0\right\} \\
&= \left\{te^{i\theta} \ : \ t \in \mathbb{R}, -\tfrac{\pi}{2} \leq \theta \leq \tfrac{\pi}{2} \ , \ \left((\widehat{\varepsilon} - 2)t^2 - |b|^2\right)\left((\widehat{\varepsilon} + 2)t^2 + |b|^2\right) \leq 0\right\} \\
&= \left\{\tfrac{a}{2} + te^{i\theta}\lfloor a \rfloor \ : \ t \in \mathbb{R}, -\tfrac{\pi}{2} \leq \theta \leq \tfrac{\pi}{2} \ , \ \left(\widehat{\varepsilon}^2 - 4\right)t^4 - 4|b|^2t^2 - |b|^4 \leq 0\right\}.
\end{aligned}$$

Thus, the proof of Assertion 3-(1) is complete.

Now, we have to prove the second statement of Assertion 3. Observe that if $a = 0$, then we obviously have $r_\varepsilon(A) = \frac{|b|}{\sqrt{\widehat{\varepsilon}-2}}$. Then assume that $a \neq 0$, by the compactness and the description of $\sigma_\varepsilon(A)$, we conclude that there exists $z_0 = \frac{a}{2} + t_0e^{i\theta_0}\lfloor a \rfloor \in \partial\sigma_\varepsilon(A)$ such that

$$r_\varepsilon(A) = |z_0| = \left|\frac{a}{2} + t_0e^{i\theta_0}\lfloor a \rfloor\right| = \left|\frac{|a|}{2} + t_0e^{i\theta_0}\right| = \frac{\sqrt{|a|^2 + 4|a|t_0\cos(\theta_0) + 4t_0^2}}{2}. \tag{4.13}$$

Now, it suffices to compute θ_0 and t_0. From the description of $\sigma_\varepsilon(A)$ given in Assertion 3-(1) and since the expression given in (4.11) is continuous with respect to (t, θ), we conclude that for every $z = \frac{a}{2} + te^{i\theta}\lfloor a \rfloor \in \partial\sigma_\varepsilon(A)$ we have

$$(\widehat{\varepsilon}^2 - 4)t^4 - \left(2|a|^2 + 4|b|^2 + \frac{|a|^2\widehat{\varepsilon}^2\cos(2\theta)}{2}\right)t^2 + \frac{\widehat{\varepsilon}^2|a|^4}{16} - (\frac{|a|^2}{2} + |b|^2)^2 = 0. \tag{4.14}$$

This implies that

$$\begin{aligned}
t^2 &= \frac{1}{2(\widehat{\varepsilon^2}-4)}\left(2|a|^2+4|b|^2+\frac{\widehat{\varepsilon^2}|a|^2\cos(2\theta)}{2}\right) \\
&\quad +\frac{1}{2(\widehat{\varepsilon^2}-4)}\sqrt{\left(2|a|^2+4|b|^2+\frac{\widehat{\varepsilon^2}|a|^2\cos(2\theta)}{2}\right)^2-4\left(\widehat{\varepsilon^2}-4\right)\left(\frac{\widehat{\varepsilon^2}|a|^4}{16}-(\frac{|a|^2}{2}+|b|^2)^2\right)}. \\
&\leq \frac{1}{2(\widehat{\varepsilon^2}-4)}\left(2|a|^2+4|b|^2+\frac{\widehat{\varepsilon^2}|a|^2}{2}\right) \\
&\quad +\frac{1}{2(\widehat{\varepsilon^2}-4)}\sqrt{\left(2|a|^2+4|b|^2+\frac{\widehat{\varepsilon^2}|a|^2}{2}\right)^2-4\left(\widehat{\varepsilon^2}-4\right)\left(\frac{\widehat{\varepsilon^2}|a|^4}{16}-(\frac{|a|^2}{2}+|b|^2)^2\right)} \\
&= \frac{1}{2(\widehat{\varepsilon^2}-4)}\left(2|a|^2+4|b|^2+\frac{\widehat{\varepsilon^2}|a|^2}{2}\right)+\frac{1}{2(\widehat{\varepsilon^2}-4)}\sqrt{4\widehat{\varepsilon^2}(|a|^4+8|a|^2|b|^2+|b|^4)}
\end{aligned}$$

for all $z=\frac{a}{2}+te^{i\theta}\lfloor a\rfloor\in\partial\sigma_\varepsilon(A)$. This together with (4.13) and (4.14) imply that $\theta_0=0$ and

$$\begin{aligned}
t_0^2 &= \frac{1}{2(\widehat{\varepsilon^2}-4)}\left(2|a|^2+4|b|^2+\frac{\widehat{\varepsilon^2}|a|^2}{2}\right)+\frac{1}{2(\widehat{\varepsilon^2}-4)}\sqrt{4\widehat{\varepsilon^2}(|a|^4+8|a|^2|b|^2+|b|^4)} \\
&= \frac{1}{2(\widehat{\varepsilon^2}-4)}\left(2|a|^2+4|b|^2+\frac{\widehat{\varepsilon^2}|a|^2}{2}\right)+\frac{1}{2(\widehat{\varepsilon^2}-4)}\sqrt{(4\widehat{\varepsilon^2}(|a|^2+|b|^2)^2)} \\
&= \frac{1}{2(\widehat{\varepsilon^2}-4)}\left(2|a|^2+4|b|^2+\frac{\widehat{\varepsilon^2}|a|^2}{2}\right)+\frac{\widehat{\varepsilon}(|a|^2+|b|^2)}{2(\widehat{\varepsilon^2}-4)} \\
&= \frac{(\widehat{\varepsilon}-2)^2|a|^2+4(\widehat{\varepsilon}-2)(|a|^2+|b|^2)}{4(\widehat{\varepsilon}-2)^2}.
\end{aligned}$$

Thus,

$$\begin{aligned}
r_\varepsilon(A) &= \frac{1}{2}\sqrt{|a|^2+4|a|t_0\cos(\theta_0)+4t_0^2} \\
&= \frac{1}{2}\sqrt{|a|^2+4|a|t_0+4t_0^2} \\
&= \frac{|a|}{2}+t_0 \\
&= \frac{|a|}{2}+\sqrt{\frac{(\widehat{\varepsilon}-2)^2|a|^2+4(\widehat{\varepsilon}-2)(|a|^2+|b|^2)}{4(\widehat{\varepsilon}-2)^2}}.
\end{aligned}$$

Therefore,

$$r_\varepsilon(A)=\frac{|a|}{2}+\sqrt{\left(\frac{|a|}{2}\right)^2+\frac{|a|^2+|b|^2}{\widehat{\varepsilon}-2}}.$$

Now, we are in position to prove Theorem 4.1.1. Let x and y be two nonzero vectors in $\mathcal{H}$. Choose a unit vector $w \in \mathcal{H}$ such that $\langle w, x\rangle = \langle w, y\rangle = 0$, and let $\mathcal{H}_1 = [u, v, w]$ be the linear space spanned by $u := \frac{x}{\|x\|}$, $v = \frac{y - \langle y,u\rangle u}{\|y-\langle y,u\rangle u\|}$ and w. Using the orthogonal decomposition $\mathcal{H} = \mathcal{H}_1 \oplus \mathcal{H}_2$, where $\mathcal{H}_2 := \mathcal{H}_1^{\perp}$ is the orthogonal complement of $\mathcal{H}_1$ in $\mathcal{H}$, write $x \otimes y = \tilde{A} \oplus O_{\mathcal{H}_2}$. We shall identify $\tilde{A}$ with its matrix with respect to the basis (u, v, w), and which is given by

$$\tilde{A} = \begin{pmatrix} \langle x, y\rangle & \sqrt{\|x\|^2\|y\|^2 - |\langle x, y\rangle|^2} & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

Then for every $z \in \mathbb{C}$, the operator $zI - x \otimes y$ has the following representation

$$(zI_{\mathcal{H}_i} - \tilde{A}) \oplus (zI_{\mathcal{H}_2}),$$

according to the decomposition $\mathcal{H} = \mathcal{H}_1 \oplus \mathcal{H}_2$, where $I_{\mathcal{H}_i}$ is the identity operator on $\mathcal{H}_i$, $i = 1, 2$. Recall that Assertion 1, ensures that

$$s_3^2(zI_{\mathcal{H}_1} - \tilde{A}) \le s_2^2(zI_{\mathcal{H}_1} - \tilde{A}) = |z|^2 \le s_1^2(zI_{\mathcal{H}_1} - \tilde{A})$$

for all $z \in \mathbb{C}$. Clearly, we have

$$(zI_{\mathcal{H}} - A)^*(zI_{\mathcal{H}} - A) = (zI_{\mathcal{H}_1} - \tilde{A})^*(zI_{\mathcal{H}_1} - \tilde{A}) \oplus (|z|^2 I_{\mathcal{H}_2}).$$

Therefore, $\sigma(\tilde{A}) = \{0, \langle x, y\rangle\} = \sigma(A)$ and for every $z \in \mathbb{C} \setminus \{0, \langle x, y\rangle\}$, we have

$$\begin{aligned} \|zI_{\mathcal{H}} - A\|^2 &= \|(zI_{\mathcal{H}} - A)^*(zI_{\mathcal{H}} - A)\| = r\left((zI_{\mathcal{H}} - A)^*(zI_{\mathcal{H}} - A)\right) \\ &= r\left((zI_{\mathcal{H}_1} - \tilde{A})^*(zI_{\mathcal{H}_1} - \tilde{A}) \oplus (|z|^2 I_{\mathcal{H}_2})\right) \\ &= \max\left\{r\left((zI_{\mathcal{H}_1} - \tilde{A})^*(zI_{\mathcal{H}_1} - \tilde{A})\right), r(|z|^2 I_{\mathcal{H}_2})\right\} \\ &= \max\left\{s_1^2(zI_{\mathcal{H}_1} - \tilde{A}), |z|^2\right\} = s_1^2(zI_{\mathcal{H}_1} - \tilde{A}) \\ &= \|zI_{\mathcal{H}_1} - \tilde{A}\|^2, \end{aligned}$$

and

$$\begin{aligned} \|(zI_{\mathcal{H}} - A)^{-1}\|^2 &= \|((zI_{\mathcal{H}} - A)^{-1})^*(zI_{\mathcal{H}} - A)^{-1}\| = r\left(((zI_{\mathcal{H}} - A)^{-1})^*(zI_{\mathcal{H}} - A)^{-1}\right) \\ &= r\left(((zI_{\mathcal{H}_1} - \tilde{A})^{-1})^*(zI_{\mathcal{H}1} - \tilde{A})^{-1} \oplus \frac{1}{|z|^2} I_{\mathcal{H}_2}\right) \\ &= \max\left\{r\left(((zI_{\mathcal{H}_1} - \tilde{A})^{-1})^*(zI_{\mathcal{H}1} - \tilde{A})^{-1}\right), r(\frac{1}{|z|^2} I_{\mathcal{H}_2})\right\} \\ &= \max\left\{\frac{1}{\min\{\lambda : \lambda \in \sigma\left((zI_{\mathcal{H}_1} - \tilde{A})^*(zI_{\mathcal{H}_1} - \tilde{A})\}\right)}, \frac{1}{|z|^2}\right\} \\ &= \max\left\{\frac{1}{s_3^2(zI_{\mathcal{H}_1} - \tilde{A})}, \frac{1}{|z|^2}\right\} = \frac{1}{s_3^2(zI_{\mathcal{H}_1} - \tilde{A})} \\ &= \|(zI_{\mathcal{H}_1} - \tilde{A})^{-1}\|^2. \end{aligned}$$

Hence, from the definition of the ε-condition spectrum, we conclude that

$$\sigma_\varepsilon(A) = (\sigma_\varepsilon(A) \setminus \sigma(A)) \cup \sigma(A) = (\sigma_\varepsilon(\tilde{A}) \setminus \sigma(\tilde{A})) \cup \sigma(A) = \sigma_\varepsilon(\tilde{A}),$$

and that $r_\varepsilon(A) = r_\varepsilon(\tilde{A})$. Therefore, from Assertions 2 and 3, for

$$a = \langle x, y\rangle \text{ and } b = \sqrt{\|x|^2\|y\|^2 - |\langle x, y\rangle|^2},$$

we get the desired forms of $\sigma_\varepsilon(x \otimes y)$ and $r_\varepsilon(x \otimes y)$. Thus, the proof of Theorem 4.1.1 is complete.

4.3 Proof of Corollary 4.1.2.

Let $R = x \otimes y \in \mathcal{F}_1(\mathcal{H})$. We have

$$\begin{aligned}
\frac{\sqrt{\varepsilon}\|R\|}{1-\varepsilon} &= \frac{1}{2(\widehat{\varepsilon}-2)}\sqrt{4(\widehat{\varepsilon}-2)\|x\|^2\|y\|^2} \\
&\leq \frac{1}{2(\widehat{\varepsilon}-2)}\left((\widehat{\varepsilon}-2)|\langle x, y\rangle| + \sqrt{(\widehat{\varepsilon}-2)^2|\langle x, y\rangle|^2 + 4(\widehat{\varepsilon}-2)\|x\|^2\|y\|^2}\right) \\
&\leq \frac{1}{2(\widehat{\varepsilon}-2)}\left((\widehat{\varepsilon}-2)\|x\|\|y\| + \sqrt{(\widehat{\varepsilon}-2)^2\|x\|\|y\|^2 + 4(\widehat{\varepsilon}-2)\|x\|^2\|y\|^2}\right) \\
&\leq \frac{1}{2(\widehat{\varepsilon}-2)}\left((\widehat{\varepsilon}-2) + \sqrt{(\widehat{\varepsilon}-2)^2 + 4(\widehat{\varepsilon}-2)}\right)\|x\|\|y\| \\
&\leq \frac{1}{1-\varepsilon}\|x\|\|y\| = \frac{\|R\|}{1-\varepsilon}.
\end{aligned}$$

So, from this and Theorem 4.1.1-(2), we obtain (4.1).

Now, let $R = x \otimes y \in \mathcal{F}_1(\mathcal{H})$, again by Theorem 4.1.1 we get,

$$R^2 = 0 \Longrightarrow \sigma_\varepsilon(R) = D\left(0, \frac{\sqrt{\varepsilon}\|R\|}{1-\varepsilon}\right) \quad \Longrightarrow \quad r_\varepsilon(R) = \frac{\sqrt{\varepsilon}\|R\|}{1-\varepsilon}.$$

It suffices to prove that $r_\varepsilon(R) = \frac{\sqrt{\varepsilon}\|R\|}{1-\varepsilon} \Longrightarrow R^2 = 0$. Assume that $r_\varepsilon(R) = \frac{\sqrt{\varepsilon}\|R\|}{1-\varepsilon} = \frac{\|x\|\|y\|}{\sqrt{\widehat{\varepsilon}-2}}$, by Theorem 4.1.1, we get

$$\begin{aligned}
r_\varepsilon(R) &= \frac{1}{2(\widehat{\varepsilon}-2)}\left((\widehat{\varepsilon}-2)|\langle x, y\rangle| + \sqrt{(\widehat{\varepsilon}-2)^2|\langle x, y\rangle|^2 + 4(\widehat{\varepsilon}-2)\|x\|^2\|y\|^2}\right) \\
&= \frac{|\langle x, y\rangle|}{2} + \sqrt{|\frac{\langle x, y\rangle}{2}|^2 + \frac{\|x\|^2\|y\|^2}{(\widehat{\varepsilon}-2)}}.
\end{aligned}$$

Thus

$$\frac{\|x\|\|y\|}{\sqrt{\widehat{\varepsilon}-2}} = \frac{|\langle x, y\rangle|}{2} + \sqrt{|\frac{\langle x, y\rangle}{2}|^2 + \frac{\|x\|^2\|y\|^2}{(\widehat{\varepsilon}-2)}}.$$

This implies that $\langle x, y\rangle = 0$. Hence $R^2 = 0$.

Finally, let R be a rank one operator. First assume that $R = \lambda x \otimes x$, for some $\lambda \in \mathbb{C}$ and $x \in \mathcal{H}$. Let $z \in \mathbb{C} \setminus \sigma(x \otimes x)$, and note that Theorem 4.1.1 implies that

$$\begin{aligned}
z \in \sigma_\varepsilon(x \otimes x) \quad & \Longleftrightarrow \quad |z|^2 + |z - \|x\|^2|^2 \geq \widehat{\varepsilon}|z(z - \|x\|^2)| \\
& \Longleftrightarrow \quad \frac{|z|}{|z - \|x\|^2|} + \frac{|z - \|x\|^2|}{|z|} \geq \varepsilon + \frac{1}{\varepsilon} \\
& \Longleftrightarrow \quad \frac{|z|}{|z - \|x\|^2|} \leq \varepsilon \ \mathit{or} \ \frac{|z - \|x\|^2|}{|z|} \leq \varepsilon \\
& \Longleftrightarrow \quad |z|^2 \leq \varepsilon^2 |z - \|x\|^2|^2 \ \mathit{or} \ |z - \|x\|^2|^2 \leq \varepsilon^2 |z|^2.
\end{aligned}$$

Therefore, for every $z = u + iv \in \mathbb{C} \setminus \sigma(x \otimes x)$, we have

$$\begin{aligned}
z \in \sigma_\varepsilon(x \otimes x) \quad & \Longleftrightarrow \quad \begin{cases} u^2 + v^2 \leq \varepsilon^2 (u - \|x\|^2)^2 + \varepsilon^2 v^2 \\ \hfill \mathit{or} \\ (u - \|x\|^2)^2 + v^2 \leq \varepsilon^2 u^2 + \varepsilon^2 v^2 \end{cases} \\
& \Longleftrightarrow \quad \begin{cases} \left(u + \frac{\varepsilon^2 \|x\|^2}{1-\varepsilon^2}\right)^2 + v^2 \leq \left(\frac{\varepsilon \|x\|^2}{1-\varepsilon^2}\right)^2 \\ \hfill \mathit{or} \\ \left(u - \frac{\|x\|^2}{1-\varepsilon^2}\right)^2 + v^2 \leq \left(\frac{\varepsilon \|x\|^2}{1-\varepsilon^2}\right)^2 \end{cases} \\
& \Longleftrightarrow \quad z \in D\left(\frac{-\varepsilon^2 \|x\|^2}{1-\varepsilon^2}, \frac{\varepsilon \|x\|^2}{1-\varepsilon^2}\right) \bigcup D\left(\frac{\|x\|^2}{1-\varepsilon^2}, \frac{\varepsilon \|x\|^2}{1-\varepsilon^2}\right).
\end{aligned}$$

Since,

$$\sigma(x \otimes x) \subset D\left(\frac{-\varepsilon^2 \|x\|^2}{1-\varepsilon^2}, \frac{\varepsilon \|x\|^2}{1-\varepsilon^2}\right) \bigcup D\left(\frac{\|x\|^2}{1-\varepsilon^2}, \frac{\varepsilon \|x\|^2}{1-\varepsilon^2}\right),$$

we have

$$\sigma_\varepsilon(x \otimes x) \ = \ D\left(\frac{-\varepsilon^2 \|x\|^2}{1-\varepsilon^2}, \frac{\varepsilon \|x\|^2}{1-\varepsilon^2}\right) \bigcup D\left(\frac{\|x\|^2}{1-\varepsilon^2}, \frac{\varepsilon \|x\|^2}{1-\varepsilon^2}\right).$$

Thus,

$$\begin{aligned}
\sigma_\varepsilon(R) \ &= \ \sigma_\varepsilon(\lambda x \otimes x) = \lambda \sigma_\varepsilon(x \otimes x) \\
&= \ \lambda \left(D\left(\frac{-\varepsilon^2 \|x\|^2}{1-\varepsilon^2}, \frac{\varepsilon \|x\|^2}{1-\varepsilon^2}\right) \bigcup D\left(\frac{\|x\|^2}{1-\varepsilon^2}, \frac{\varepsilon \|x\|^2}{1-\varepsilon^2}\right) \right) \\
&= \ D\left(\frac{-\lambda \varepsilon^2 \|x\|^2}{1-\varepsilon^2}, \frac{\varepsilon |\lambda| \|x\|^2}{1-\varepsilon^2}\right) \bigcup D\left(\frac{\lambda \|x\|^2}{1-\varepsilon^2}, \frac{\varepsilon |\lambda| \|x\|^2}{1-\varepsilon^2}\right) \\
&= \ \bigcup_{\alpha, \beta \in \sigma(R)} D\left(\frac{\alpha - \beta \varepsilon^2}{1-\varepsilon^2}, \frac{\varepsilon |\alpha - \beta|}{1-\varepsilon^2}\right).
\end{aligned}$$

For the second implication, let $R = x \otimes y \in \mathcal{F}_1(\mathcal{H})$ such that

$$\sigma_\varepsilon(R) = \bigcup_{\alpha,\beta\in\sigma(R)} D\left(\frac{\alpha-\beta\varepsilon^2}{1-\varepsilon^2}\ ,\ \frac{\varepsilon|\alpha-\beta|}{1-\varepsilon^2}\right).$$

It is clear that $r_\varepsilon(R) = \frac{|\langle x,y\rangle|}{1-\varepsilon^2} + \frac{\varepsilon|\langle x,y\rangle|}{1-\varepsilon^2} = \frac{|\langle x,y\rangle|}{1-\varepsilon}$. We claim that $R = \lambda x \otimes x$ for some $\lambda \in \mathbb{C}$. Indeed, by Theorem 4.1.1, we have

$$r_\varepsilon(R) = \frac{|\langle x,y\rangle|}{2} + \sqrt{\left|\frac{\langle x,y\rangle}{2}\right|^2 + \frac{\|x\|^2\|y\|^2}{\widehat{\varepsilon}-2}}.$$

Thus

$$\frac{|\langle x,y\rangle|}{1-\varepsilon} = \frac{|\langle x,y\rangle|}{2} + \sqrt{\left|\frac{\langle x,y\rangle}{2}\right|^2 + \frac{\|x\|^2\|y\|^2}{\widehat{\varepsilon}-2}}.$$

By a simple calculation we obtain that $|\langle x,y\rangle| = \|x\|\|y\|$. Hence, $R = \lambda x \otimes x$, for some $\lambda \in \mathbb{C}$ and $x \in \mathcal{H}$. The proof of the corollary is complete.

4.4 Proof of Proposition 4.1.3.

Let $x \otimes y \in \mathcal{F}_1(\mathcal{H})$, $t \in \mathbb{R}$ and $-\frac{\pi}{2} \le \theta \le \frac{\pi}{2}$. By Theorem 4.1.1, we have

$$\begin{aligned}\frac{\langle x,y\rangle}{2} + te^{i\theta}\lfloor\langle x,y\rangle\rfloor\ \in \sigma_\varepsilon(x\otimes y) \quad &\Longleftrightarrow \quad \varrho_{x,y}(\theta,t) \le 0\\ &\Longleftrightarrow \quad \varrho_{x,y}(-\theta,t) \le 0\\ &\Longleftrightarrow \quad \frac{\langle x,y\rangle}{2} + te^{-i\theta}\lfloor\langle x,y\rangle\rfloor\ \in \sigma_\varepsilon(x\otimes y).\end{aligned}$$

This implies that $\sigma_\varepsilon(x\otimes y)$ is symmetrical about the axis $\frac{\langle x,y\rangle}{2} + \mathbb{R}\lfloor\langle x,y\rangle\rfloor$. Similarly, we have

$$\begin{aligned}\frac{\langle x,y\rangle}{2} + te^{i\theta}\lfloor\langle x,y\rangle\rfloor\ \in \sigma_\varepsilon(x\otimes y) \quad &\Longleftrightarrow \quad \varrho_{x,y}(\theta,t) \le 0\\ &\Longleftrightarrow \quad \varrho_{x,y}(\theta,-t) \le 0\\ &\Longleftrightarrow \quad \frac{\langle x,y\rangle}{2} - te^{i\theta}\lfloor\langle x,y\rangle\rfloor\ \in \sigma_\varepsilon(x\otimes y).\end{aligned}$$

Therefore, $\sigma_\varepsilon(x\otimes y)$ is symmetrical about the point $\frac{\langle x,y\rangle}{2}$. Finally, from the previous facts we conclude that $\sigma_\varepsilon(x\otimes y)$ is symmetrical about the axis $\frac{\langle x,y\rangle}{2} + \mathbb{R}\sqrt{-1}\lfloor\langle x,y\rangle\rfloor$.

The rest of the proof follows from Theorem 4.1.1-(1) together with an easy computation of the solutions of the equation $\varrho(\theta,t) = 0$ (for any fixed $\theta \in [0,\pi/2]$) and the following observations:

- For every fixed $t \in \mathbb{R}$, one can easily see that the function $\theta \mapsto \varrho(\theta, t)$ is not decreasing on $[0, \pi/2]$.
- For every fixed $\theta \in [0, \pi/2]$, the set $\{t \in [0, +\infty) \ : \ \varrho(\theta, t) \leq 0\}$ is either empty or a closed bonded interval. This follows from the fact that the polynomial $t \mapsto \varrho(\theta, t)$ possesses at most two zeros on $[0, +\infty)$.
- If $|\langle x, y\rangle| > 2\|x\|\|y\|$, then $\frac{\partial}{\partial t}\varrho(\frac{\pi}{2}, t) \geq 0$ for all $t \in [0, \infty)$. Thus, the mapping $t \mapsto \varrho(\frac{\pi}{2}, t)$ is not decreasing on $[0, \infty)$. Therefore $\varrho(\frac{\pi}{2}, t) > 0$ for all $t \in [0, \infty)$. Hence,
$$\begin{aligned} \frac{\langle x, y\rangle}{2} \notin \sigma_\varepsilon(x \otimes y) \quad &\Longleftrightarrow \quad \varrho(\tfrac{\pi}{2}, 0) > 0 \\ &\Longleftrightarrow \quad \sqrt{\hat{\varepsilon} - 2} \cdot |\langle x, y\rangle| > 2\|x\|\|y\| \\ &\Longleftrightarrow \quad \sigma_\varepsilon(x \otimes y) \cap \left(\frac{\langle x, y\rangle}{2} + \mathbb{R}\sqrt{-1}\lfloor\langle x, y\rangle\rfloor\right) = \emptyset. \end{aligned}$$

4.5 Proof of Theorem 4.1.4.

First, note that from Proposition 1.2.27 we easily conclude that the ε-condition spectral radius satisfies the following properties:

Proposition 4.5.1. *For every $A \in \mathcal{B}(\mathcal{H})$ the following statements hold.*

1. *For every complex unit μ, and every unitary or conjugate unitary operator U on $\mathcal{H}$, we have*
$$r_\varepsilon(\mu U A U^*) = r_\varepsilon(A) = r_\varepsilon\left(\overline{A}\right).$$
2. $r_\varepsilon(A) = 0 \iff A = 0$.
3. $r_\varepsilon(tA) = t \cdot r_\varepsilon(A)$ *for all* $t \in [0, +\infty)$.

From the above proposition we immediately deduce the "*if*" part of Theorem 4.1.4. Therefore, we only need to prove the "*only if*" part of Theorem 4.1.4. So, assume that $\phi : \mathcal{A} \longrightarrow \mathcal{B}$ is a surjective mapping satisfying (4.4). Then, by Proposition 4.5.1 and [33, Theorem 2.1, Theorem 2.2] we conclude that either

$$\phi(x \otimes y) = Ux \otimes k_x(y) \quad \text{for all } x \otimes y \in \mathcal{F}_1(\mathcal{H}), \tag{4.15}$$

or

$$\phi(x \otimes y) = UJx \otimes k_x(y) \quad \text{for all } x \otimes y \in \mathcal{F}_1(\mathcal{H}), \tag{4.16}$$

where U is a unitary operator in $\mathcal{B}(\mathcal{H})$ and $k : \mathcal{H} \times \mathcal{H} \longrightarrow \mathcal{H}, \ (x, y) \mapsto k_x(y)$ is a map satisfying $\|k_x(y)\| = \|y\|$ for all $x, y \in \mathcal{H}$.

Now, let us point out that from the fact that

$$J(x \otimes y)J = Jx \otimes Jy, \ \left(\overline{A}\right)^* \overline{B} = \overline{A^* B} \text{ and } r_\varepsilon\left(\overline{A}\right) = r_\varepsilon(A)$$

for all $A, B \in \mathcal{A}$ and $x, y \in \mathcal{H}$, we obtain that:

$$\phi \text{ satisfies (4.16) if and only if the mapping } A \mapsto \phi\left(\overline{A}\right) \text{ satisfies (4.15).}$$

For this reason we may and we must restrict our-self to the case when

$$\phi(x \otimes y) = Ux \otimes k_x(y) \text{ for all } x \otimes y \in \mathcal{F}_1(\mathcal{H}).$$

The rest of the proof will be provided by checking four assertions.

Assertion 1. For every vectors $x, y \in \mathcal{H}$ such that $\langle x, y\rangle \neq 0$, we have

$$|\langle k_x(u), k_y(v)\rangle| = |\langle u, v\rangle|, \tag{4.17}$$

for all $u, v \in \mathcal{H}$, and hence

$$\langle u, v\rangle = 0 \Longleftrightarrow \langle k_x(u), k_y(v)\rangle = 0, \tag{4.18}$$

for all $u, v \in \mathcal{H}$.

Indeed. Let x and y be two vectors in $\mathcal{H}$ such that $\langle x, y\rangle \neq 0$ and let $u, v \in \mathcal{H}$. From formula (4.4), we have

$$\begin{aligned} |\langle x, y\rangle| r_\varepsilon\left(k_x(u) \otimes k_y(v)\right) &= r_\varepsilon\left(\left(k_x(u) \otimes xU^*\right)\left(Uy \otimes k_y(v)\right)\right) \\ &= r_\varepsilon\left(\phi(x \otimes u)^* \phi(y \otimes v)\right) \\ &= r_\varepsilon\left((x \otimes u)^*(y \otimes v)\right) \\ &= |\langle x, y\rangle| r_\varepsilon(u \otimes v), \end{aligned}$$

and hence

$$r_\varepsilon\left(k_x(u) \otimes k_y(v)\right) = r_\varepsilon(u \otimes v). \tag{4.19}$$

From Theorem 4.1.1, we know that

$$\begin{cases} r_\varepsilon(u \otimes v) = \dfrac{|\langle u, v\rangle|}{2} + \sqrt{\left(\dfrac{|\langle u, v\rangle|}{2}\right)^2 + \dfrac{\|u\|^2\|v\|^2}{\widehat{\varepsilon} - 2}} \\ r_\varepsilon\left(k_x(u) \otimes k_y(v)\right) = \dfrac{|\langle k_x(u), k_y(v)\rangle|}{2} + \sqrt{\left(\dfrac{|\langle k_x(u), k_y(v)\rangle|}{2}\right)^2 + \dfrac{\|k_x(u)\|^2\|k_y(v)\|^2}{\widehat{\varepsilon} - 2}}. \end{cases}$$

From this together with equation (4.19), we infer that

$$\begin{aligned} \frac{|\langle u, v\rangle|}{2} + \sqrt{\left(\frac{|\langle u, v\rangle|}{2}\right)^2 + \frac{\|u\|^2\|v\|^2}{\widehat{\varepsilon} - 2}} &= \frac{|\langle k_x(u), k_y(v)\rangle|}{2} \\ &\quad + \sqrt{\left(\frac{|\langle k_x(u), k_y(v)\rangle|}{2}\right)^2 + \frac{\|k_x(u)\|^2\|k_y(v)\|^2}{\widehat{\varepsilon} - 2}}. \end{aligned}$$

Since $\|k_x(u)\| = \|u\|$ and $\|k_y(v)\| = \|v\|$, we conclude that

$$\begin{aligned} \frac{1}{2}\left(|\langle u, v\rangle| - |\langle k_x(u), k_y(v)\rangle|\right) &= \beta_{k_x(u),k_y(v)} - \beta_{u,v} \\ &= \frac{|\langle k_x(u), k_y(v)\rangle|^2 - |\langle u, v\rangle|^2}{4(\beta_{k_x(u),k_y(v)} + \beta_{u,v})}, \end{aligned}$$

where

$$\beta_{w,z} = \sqrt{\left(\frac{|\langle w, z\rangle|}{2}\right)^2 + \frac{\|w\|^2\|z\|^2}{\widehat{\varepsilon} - 2}}$$

for all $w, z \in \mathcal{H}$. Therefore,

$$\left(|\langle k_x(u), k_y(v)\rangle| - |\langle u, v\rangle|\right)\left(1 + \frac{|\langle k_x(u), k_y(v)\rangle| + |\langle u, v\rangle|}{2(\beta_{k_x(u),k_y(v)} + \beta_{u,v})}\right) = 0. \tag{4.20}$$

Since

$$1 + \frac{|\langle k_x(u), k_y(v)\rangle| + |\langle u, v\rangle|}{2(\beta_{k_x(u),k_y(v)} + \beta_{u,v})} > 0,$$

we conclude that

$$|\langle k_x(u), k_y(v)\rangle| = |\langle u, v\rangle|,$$

and hence

$$\langle u, v\rangle = 0 \Longleftrightarrow \langle k_x(u), k_y(v)\rangle = 0.$$

Thus, the proof of Assertion 1 is complete.
In the following two assertions we will use essentially some notions and results given in [55, §2.2]. For every nonzero vector $x \in \mathcal{H}$, the ray generated by x will be denoted by $\underline{x}$. The set of all rays in $\mathcal{H}$ will be denoted by $\underline{\mathcal{H}}$, this means that $\underline{\mathcal{H}} := \{\underline{x} : x \in \mathcal{H} \setminus \{0\}\}$. We say that the rays $\underline{x}, \underline{y} \in \underline{\mathcal{H}}$ are orthogonal to each other, in notation $\langle \underline{x}, \underline{y}\rangle = 0$, if $\langle x, y\rangle = 0$. Observe that $\langle x, y\rangle = 0$ if and only if $\langle u, v\rangle = 0$ for all $u \in \underline{x}$ and $v \in \underline{y}$. In the following assertion we discuss some properties of the map

$$\begin{array}{rcccl} \underline{k_x} & : & \underline{\mathcal{H}} & \to & \underline{\mathcal{H}} \\ & & \underline{u} & \mapsto & \underline{k_x}\left(\underline{u}\right) := \underline{k_x(u)} \end{array} \tag{4.21}$$

for each $x \in \mathcal{H} \setminus \{0\}$.

Assertion 2. For each $x \in \mathcal{H} \setminus \{0\}$, we have $\underline{k_x}(\cdot)$ is bijective map and satisfies

$$\langle \underline{u}, \underline{v}\rangle = 0 \Longleftrightarrow \langle \underline{k_x}\left(\underline{u}\right), \underline{k_x}\left(\underline{v}\right)\rangle = 0 \tag{4.22}$$

for all $\underline{u}, \underline{v} \in \underline{\mathcal{H}}$.

Indeed. Let x be a nonzero vector in $\mathcal{H}$, we show that k_x is surjective. It suffices to show that the folowing equality holds.

$$\phi\left(\{x \otimes y : \ y \in \mathcal{H} \setminus \{0\}\}\right) = \{Ux \otimes y : \ y \in \mathcal{H} \setminus \{0\}\}. \tag{4.23}$$

Since $\phi\left(\{x \otimes y : \ y \in \mathcal{H} \setminus \{0\}\}\right) \subseteq \{Ux \otimes y : \ y \in \mathcal{H} \setminus \{0\}\}$ is obvious, we only have to prove the other inclusion. Let $z \in \mathcal{H} \setminus \{0\}$, since ϕ preserves rank one operators in both directions, then there exists $u \otimes v \in \mathcal{F}_1(\mathcal{H})$ such that $Ux \otimes z = \phi(u \otimes v) = Uu \otimes k_u(v)$. Thus Uu and Ux are linearly dependent, and $u = \alpha x$ for some nonzero scalar α. Thus, $\phi(x \otimes \overline{\alpha} v) = Ux \otimes z$, and the equality (4.23) is proved. The surjectivity of k_x follows immediately.

Now, we show that $\underline{k_x}(\cdot)$ is injective and well defined. Let $\underline{y}, \underline{y'} \in \underline{\mathcal{H}}$, by (4.18) and the surjectivity of $k_x(\cdot)$, we have

$$\begin{aligned}
\underline{y} = \underline{y'} \iff & \ \langle y, v\rangle = 0 \quad \text{if and only if} \quad \langle y', v\rangle = 0 \quad \text{for all } v \in \mathcal{H} \\
\iff & \ \langle k_x(y), k_x(v)\rangle = 0 \quad \text{if and only if} \quad \langle k_x(y'), k_x(v)\rangle = 0 \quad \text{for all } v \in \mathcal{H} \\
\iff & \ \langle k_x(y), w\rangle = 0 \quad \text{if and only if} \quad \langle k_x(y'), w\rangle = 0 \quad \text{for all } w \in \mathcal{H} \\
\iff & \ \underline{k_x(y)} = \underline{k_x(y')} \\
\iff & \ \underline{k_x}\left(\underline{y}\right) = \underline{k_x}\left(\underline{y'}\right).
\end{aligned}$$

Consequently $\underline{k_x}(\cdot)$ is injective and well defined. Furthermore, since $k_x(\cdot)$ is surjective, it follows that $\underline{k_x}(\cdot)$ must be surjective too. Therefore, $\underline{k_x}(\cdot)$ is bijective. Now, from (4.18) and Assertion 1, we have

$$\begin{aligned}
\langle \underline{u}, \underline{v}\rangle = 0 \iff & \ \langle u, v\rangle = 0 \\
\iff & \ \langle k_x(u), k_x(v)\rangle = 0 \\
\iff & \ \langle \underline{k_x(u)}, \underline{k_x(v)}\rangle = 0 \\
\iff & \ \langle \underline{k_x}\left(\underline{u}\right), \underline{k_x}\left(\underline{v}\right)\rangle = 0
\end{aligned}$$

for all $u, v \in \mathcal{H}$. This proves (4.22) and completes the proof of Assertion 2.

The following assertion is a consequence of [55, Corollary 2.2.2].

Assertion 3. There exist a unitary or conjugate unitary operator R and a functional $d : \mathcal{H} \setminus \{0\} \times \mathcal{H} \setminus \{0\} \to \mathbb{C} \setminus \{0\}$, $(x, y) \mapsto d_x(y)$ such that

$$k_x(y) = d_x(y) R^* y \ \text{ for all } \ (x, y) \in \mathcal{H} \setminus \{0\} \times \mathcal{H}.$$

Indeed. Let x be a nonzero vector in $\mathcal{H}$. From Assertion 2 we know that $\underline{k_x}$ is a bijective transformation on $\underline{\mathcal{H}}$ satisfying (4.22). Then in light of [55, Corollary 2.2.2], we have the following two possibilities:

1. $\dim(\mathcal{H}) = \infty$, and, there exists an invertible bounded linear or conjugate-linear operator V_x on $\mathcal{H}$ such that
$$\underline{k_x}\,(\underline{u}) = \underline{V_x(u)} = \underline{(V_x^{-1})^*(u)} \quad \text{for all} \quad u \in \mathcal{H}; \tag{4.24}$$
2. $\dim(\mathcal{H}) = n$ for a given integer $n \geq 3$, and, there exist a nonsingular matrix $V_x \in M_n(\mathbb{C})$ and a ring automorphism τ_x of $\mathbb{C}$ such that
$$\underline{k_x}\,(\underline{u}) = \underline{V_x(u_{\tau_x})} = \underline{(V_x^{-1})^*(u_{\tau_x})} \quad \text{for all} \quad u \in \mathcal{H}. \tag{4.25}$$

From (4.24) and (4.25), there exists a map $d'_x(\cdot) : \mathcal{H} \longrightarrow \mathbb{C} \setminus \{0\}$ such that

$$\text{if } \dim(\mathcal{H}) = \infty, \text{ then, } k_x(u) = d'_x(u)V_x(u) \quad \text{for all } u \in \mathcal{H}, \tag{4.26}$$

and

$$\text{if } \dim(\mathcal{H}) = \mathbb{C}^n, \text{ then, } k_x(u) = d'_x(u)V_x(u_{\tau_x}) \quad \text{for all } u \in \mathbb{C}^n. \tag{4.27}$$

Now, in the case $\dim(\mathcal{H}) = \mathbb{C}^n$, we show that τ_x is either the identity or the complex conjugation. By (4.18), (4.25) and (4.27) we get that

$$\langle u, v\rangle = 0 \iff \langle u_{\tau_x}, v_{\tau_x}\rangle = 0 \tag{4.28}$$

for all nonzero vectors $u, v \in \mathcal{H}$. Choose $u = (1, \frac{1}{\mu}, 0, ..., 0)$ and $v = (1, -\overline{\mu}, 0, ..., 0)$ with $\mu \neq 0$. Therefore, by equation (4.28), we get that $\tau_x(\mu) = \overline{\tau_x(\overline{\mu})}$, $\mu \in \mathbb{C} \setminus \{0\}$. Thus, τ_x is map the real numbers into real numbers, and consequently, the restriction of τ_x to $\mathbb{R}$ is the identity map. Clearly that, $\tau_x(i) = \pm i$, and thus τ_x is either the identity or the complex conjugation. Therefore, in both cases that is $\dim(\mathcal{H}) = \infty$ or $\dim(\mathcal{H}) < \infty$, we have

$$k_x(u) = d'_x(u)W'_x(u) \quad \text{for all } u \in \mathcal{H}, \tag{4.29}$$

where W'_x is an invertible bounded linear or conjugate-linear operator on $\mathcal{H}$.

Now, we claim that there exist a nonzero scalar β_x that depends on x and a unitary or conjugate unitary operator W independent of x such that $W'_x = \beta_x W$. Indeed, assume by the way of contradiction that $(W'_x)^* W'_y$ is not a scalar for some $x, y \in \mathcal{H}$ and $\langle x, y\rangle \neq 0$. Then there exists a nonzero vector $v \in \mathcal{H}$ such that v and $(W'_x)^* W'_y v$ are linearly independent. In this case, one can find another nonzero vector $u \in \mathcal{H}$ such that $\langle u, v\rangle = 0$ while $\langle u, (W'_x)^* W'_y v\rangle \neq 0$. Therefore, by (4.18), we have

$$\begin{aligned} \langle u, v\rangle = 0 \quad &\Longrightarrow \quad \langle k_x(u), k_y(v)\rangle = 0 \\ &\Longrightarrow \quad \langle d'_x(u)W'_x u, d'_y(v)W'_y v\rangle = 0 \\ &\Longrightarrow \quad \langle u, (W'_x)^* W'_y v\rangle = 0, \end{aligned}$$

this is a contradiction. Therefore $(W'_x)^*W'_y$ is a scalar and hence W'_x and W'_y are linearly dependent for all $x, y \in \mathcal{H}$ with $\langle x, y\rangle \neq 0$. In other case, if $x, y \in \mathcal{H} \setminus \{0\}$ such that $\langle x, y\rangle = 0$, then there exists $z \in \mathcal{H}$ such that $\langle x, z\rangle \neq 0$ and $\langle y, z\rangle \neq 0$. Then W'_x and W'_z are linearly dependent and W'_y and W'_z are also linearly dependent. Therefore, W'_x and W'_y are linearly dependent for all $x, y \in \mathcal{H} \setminus \{0\}$. Then, we can conclude that, there exist a unitary or conjugate unitary operator W independent of x and a functional $\beta : \mathcal{H} \to \mathbb{C} \setminus \{0\}; x \mapsto \beta_x$ such that $W'_x = \beta_x W \quad \text{for all} \quad x \in \mathcal{H}$; as claimed.

Finally, denote $d_x(y) := \beta_x d'_x(y)$ for all $x, y \in \mathcal{H}$, and $R := W^*$. We conclude that,

$$k_x(y) = d_x(y)R^*(y) \quad \text{for all } y \in \mathcal{H}.$$

Thus the proof of Assertion 3 is complete.

Assertion 4. There exists a functional $h : \mathcal{A} \mapsto \mathbb{T}$ such that $\phi(A) = h(A)UAR$ for all $A \in \mathcal{A}$, and R is a unitary operator.

Indeed. We divide this proof into four steps.

Step 1. There exists a functional $h_1 : \mathcal{F}_1(\mathcal{H}) \mapsto \mathbb{T}$ such that $\phi(x \otimes y) = h(x \otimes y)Ux \otimes R^*y$ for all $x \otimes y \in \mathcal{F}_1(\mathcal{H})$.

Let $x, y \in \mathcal{H} \setminus \{0\}$. From formula (4.4), we have

$$\begin{aligned} \phi(x \otimes y) = Ux \otimes k_x(y) &= Ux \otimes d_x(y)R^*(y) \\ &= \overline{d_x(y)}Ux \otimes R^*y. \end{aligned}$$

We define the map $h_1 : \mathcal{F}_1(\mathcal{H}) \mapsto \mathbb{C} \setminus \{0\}$ by $h_1(x \otimes y) = \overline{d_x(y)}$ if $x \otimes y \in \mathcal{F}_1(\mathcal{H}) \setminus \{0\}$ and $h_1(0) = 1$. Hence, $\phi(x \otimes y) = h_1(x \otimes y)Ux \otimes R^*y$ for all $x \otimes y \in \mathcal{F}_1(\mathcal{H})$. Now, let $x \otimes y \in \mathcal{F}_1(\mathcal{H}) \setminus \{0\}$, by equation (4.17), we get

$$\begin{aligned} \|y\|^2 = |\langle k_x(y), k_x(y)\rangle| &= |\langle d_x(y)R^*(y), d_x(y)R^*(y)\rangle| \\ &= |\langle h_1(x \otimes y)R^*(y), h_1(x \otimes y)R^*(y)\rangle| \\ &= |h_1(x \otimes y)|^2\|R^*(y)\|^2 = |h_1(x \otimes y)|^2\|y\|^2. \end{aligned}$$

Therefore, $|h_1(x \otimes y)| = 1$ for all $x \otimes y \in \mathcal{F}_1(\mathcal{H})$.

Step 2. For every $x, y \in \mathcal{H}$ and $A \in \mathcal{A}$, we have

$$r_\varepsilon(A^*x \otimes y)\left(|\langle A^*x, y\rangle| - |\langle \psi(A)x, y\rangle|\right) = \frac{\|y\|^2}{\widehat{\varepsilon} - 2}\left(\|\psi(A)x\|^2 - \|A^*x\|^2\right) \tag{4.30}$$

where $\psi(A) := R\phi(A)^*U$. Let $A \in \mathcal{A}$ and $x, y \in \mathcal{H}$. From formula (4.4), we have

$$\begin{aligned}
r_\varepsilon\left(\psi(A)x \otimes y\right) &= r_\varepsilon\left(R\phi(A)^*Ux \otimes y\right) \\
&= \frac{|\langle R\phi(A)^*Ux, y\rangle|}{2} + \sqrt{\left(\frac{|\langle R\phi(A)^*Ux, y\rangle|}{2}\right)^2 + \frac{\|R\phi(A)^*Ux\|^2\|y\|^2}{\widehat{\varepsilon}-2}} \\
&= \frac{|\langle \phi(A)^*Ux, R^*y\rangle|}{2} + \sqrt{\left(\frac{|\langle \phi(A)^*Ux, R^*y\rangle|}{2}\right)^2 + \frac{\|\phi(A)^*Ux\|^2\|R^*y\|^2}{\widehat{\varepsilon}-2}} \\
&= r_\varepsilon\left(\phi(A)^*Ux \otimes R^*y\right) \\
&= r_\varepsilon\left(\phi(A)^*\phi(x \otimes y)\right) \\
&= r_\varepsilon(A^*x \otimes y).
\end{aligned}$$

Thus

$$r_\varepsilon\left(\psi(A)x \otimes y\right) = r_\varepsilon(A^*x \otimes y). \tag{4.31}$$

Using the expression of condition spectral radius given in Theorem 4.1.1, we obtain that

$$\begin{aligned}
\frac{|\langle \psi(A)x, y\rangle|}{2} + \sqrt{\left(\frac{|\langle \psi(A)x, y\rangle|}{2}\right)^2 + \frac{\|\psi(A)x\|^2\|y\|^2}{\widehat{\varepsilon}-2}} &= \frac{|\langle A^*x, y\rangle|}{2} \\
&+ \sqrt{\left(\frac{|\langle A^*x, y\rangle|}{2}\right)^2 + \frac{\|A^*x\|^2\|y\|^2}{\widehat{\varepsilon}-2}}.
\end{aligned}$$

Thus

$$\begin{aligned}
\left(\frac{|\langle \psi(A)x, y\rangle|}{2}\right)^2 + \frac{\|\psi(A)x\|^2\|y\|^2}{\widehat{\varepsilon}-2} &= \left(\beta_{A^*x,y} + \frac{|\langle A^*x, y\rangle|}{2} - \frac{|\langle \psi(A)x, y\rangle|}{2}\right)^2 \\
&= \beta^2_{A^*x,y} + \left(\frac{|\langle A^*x, y\rangle|}{2}\right)^2 + \left(\frac{|\langle \psi(A)x, y\rangle|}{2}\right)^2 \\
&\quad +\beta_{A^*x,y}|\langle A^*x, y\rangle| - \beta_{A^*x,y}|\langle \psi(A)x, y\rangle| \\
&\quad -\frac{|\langle A^*x, y\rangle||\langle \psi(A)x, y\rangle|}{2},
\end{aligned}$$

where

$$\beta_{A^*x,y} = \sqrt{\left(\frac{|\langle A^*x, y\rangle|}{2}\right)^2 + \frac{\|A^*x\|^2\|y\|^2}{\widehat{\varepsilon}-2}}.$$

Hence

$$\begin{aligned}
\frac{\|\psi(A)x\|^2\|y\|^2}{\widehat{\varepsilon}-2} &= 2\left(\frac{|\langle A^*x, y\rangle|}{2}\right)^2 + \frac{\|A^*x\|^2\|y\|^2}{\widehat{\varepsilon}-2} \\
&\quad +\beta_{A^*x,y}|\langle A^*x, y\rangle| - \beta_{A^*x,y}|\langle \psi(A)x, y\rangle| \\
&\quad -\frac{|\langle A^*x, y\rangle||\langle \psi(A)x, y\rangle|}{2}.
\end{aligned}$$

By a simple calculation we obtain that

$$r_{\varepsilon}(A^*x \otimes y)\left(|\langle A^*x, y\rangle| - |\langle \psi(A)x, y\rangle|\right) = \frac{\|y\|^2}{\widehat{\varepsilon} - 2}\left(\|\psi(A)x\|^2 - \|A^*x\|^2\right).$$

Step 3. For every $x \in \mathcal{H} \setminus \{0\}$ and $A \in \mathcal{A} \setminus \{0\}$, there exist a nonzero scalar α_x such that $\psi(A)x = \alpha_x A^*x$.

Let $A \in \mathcal{A} \setminus \{0\}$, and assume that there exists $x \in \mathcal{H} \setminus \{0\}$ such that $\psi(A)x$ and A^*x are linearly independent. Choose a nonzero vector $y \in \mathcal{H}$ such that $\langle A^*x, y\rangle = 1$ and $\langle \psi(A)x, y\rangle = 0$. By equation (4.30), and since $r_{\varepsilon}(A^*x \otimes y) > 0$, we have $\|\psi(A)x\| > \|A^*x\|$. On the other hand, let $z \in \mathcal{H}$ be a nonzero vector such that $\langle A^*x, z\rangle = 0$ and $\langle \psi(A)x, z\rangle = 1$, by equation (4.30), we obtain $\|A^*x\| > \|\psi(A)x\|$. Which gives a contradiction, therefore, $\psi(A)x$ and A^*x are linearly dependent for all $x \in \mathcal{H}$, thus there exist a nonzero scalar α_x such that $\psi(A)x = \alpha_x A^*x$ for all $x \in \mathcal{H} \setminus \{0\}$.

Step 4. The statement of Assertion 4 holds.

By equation (4.31), we have $\ker(\psi(A)) = \ker(A^*)$ for all $A \in \mathcal{A} \setminus \{0\}$. Let $A \in \mathcal{A} \setminus \{0\}$, then according to the space decomposition, we have

$$\mathcal{H} = \mathcal{H}_1 \oplus \ker(A^*),$$

where $\mathcal{H}_1$ is an algebraic complement of $\ker(A^*)$. Now, we show that $\psi(A)$ and A^* are linearly dependent. Let $x \in \mathcal{H}_1 \setminus \{0\}$, and note there exists a unique nonzero scalar λ_x such that $\psi(A)x = \lambda_x A^*x$. Now, we claim that the functional $x \mapsto \lambda_x$ is constant on $\mathcal{H}_1 \setminus \{0\}$. Indeed, let $x, y \in \mathcal{H}_1 \setminus \{0\}$, and observe first that if x and y are linearly dependent, then obviously $\lambda_x = \lambda_y$. Thus we may assume that x and y are linearly independent. Then,

$$\psi(A)(x+y) = \psi(A)x + \psi(A)y = \lambda_x A^*x + \lambda_y A^*y.$$

On the other hand,

$$\psi(A)(x+y) = \lambda_{x+y} A^*(x+y) = \lambda_{x+y} A^*x + \lambda_{x+y} A^*y.$$

Therefore,

$$(\lambda_{x+y} - \lambda_x)\, A^*x = (\lambda_y - \lambda_{x+y})\, A^*y.$$

Thus

$$A^*\left((\lambda_{x+y} - \lambda_x)x - (\lambda_y - \lambda_{x+y})y\right) = 0.$$

So, $(\lambda_{x+y} - \lambda_x)x - (\lambda_y - \lambda_{x+y})y = 0$ and thus $\lambda_x = \lambda_{x+y} = \lambda_y$ since x and y are linearly independent vectors in $\mathcal{H}_1$. Thus, there exists a nonzero scalar λ such that $\psi(A)x = \lambda A^*x$ for all $x \in \mathcal{H}_1$, and hence $\psi(A)x = \lambda A^*x$ for all $x \in \mathcal{H}$. Consequently $\psi(A)$ and A^* are linearly dependent, and there exists a functional $h : \mathcal{A} \longrightarrow \mathbb{C} \setminus \{0\}$ such that

$$\phi(A) = h(A)UAR$$

for all $A \in \mathcal{A}$. Clearly, we have $h(A) = h_1(A)$ if A is of rank one. Equation (4.31) implies that $|h(A)| = 1$ for all $A \in \mathcal{A}$. Since U is linear, it follows that R must be linear too.

This completes the proof of Theorem 4.1.4.

4.6 Proof of Theorem 4.1.5.

Keeping in mind the statement of Proposition 1.2.27, we only need to prove the "*only if*" part of Theorem 4.1.5. So, let $\phi : \mathcal{A}_I \longrightarrow \mathcal{B}_I$ be a surjective map such that

$$\sigma_\varepsilon\left(\phi(A)^*\phi(B)\right) = \sigma_\varepsilon(A^*B),$$

for all $A, B \in \mathcal{A}_I$. Clearly,

$$r_\varepsilon\left(\phi(A)^*\phi(B)\right) = r_\varepsilon(A^*B),$$

for all $A, B \in \mathcal{A}_I$. Then, Theorem 4.1.4 ensures that there exist two unitary operators U and W on $\mathcal{H}$ and a functional $h : \mathcal{A}_I \mapsto \mathbb{T}$ such that

$$\phi(A) = h(A)UAW \quad (\forall A \in \mathcal{A}_I), \tag{4.32}$$

or

$$\phi(A) = h(A)U\overline{A}W \quad (\forall A \in \mathcal{A}_I). \tag{4.33}$$

Keep in mind that from Corollary 4.1.2 we have $\sigma_\varepsilon(x \otimes x) = D_\epsilon$ for all unit vectors $x \in \mathcal{H}$, where

$$D_\epsilon := D\left(\frac{-\varepsilon^2}{1-\varepsilon^2}, \frac{\varepsilon}{1-\varepsilon^2}\right) \bigcup D\left(\frac{1}{1-\varepsilon^2}\ ,\ \frac{\varepsilon}{1-\varepsilon^2}\right). \tag{4.34}$$

We will use the set D_ϵ in the reminder of the proof and we need to observe that for every complex number β, we have

$$\beta D_\epsilon = D_\epsilon \iff \beta = 1.$$

First, we claim that the form 4.33 does not occur. Indeed, assume by way of contradiction that $\phi(A) = h(A)U\overline{A}W$ for all $A \in \mathcal{A}_I$. Then, by formula (4.5) and Proposition 1.2.27-(3), we have

$$\begin{array}{rcl} \sigma_\varepsilon(A) & = & \sigma_\varepsilon(I^*A) \\ & = & \sigma_\varepsilon\left(\phi(I)^*\phi(A)\right) \\ & = & \sigma_\varepsilon\left(\overline{h(I)}h(A)W^*\overline{A}W\right) \\ & = & \sigma_\varepsilon\left(\overline{h(I)}h(A)\overline{A}\right) \\ & = & \sigma_\varepsilon\left(\overline{h(I)}h(A)\overline{A}\right) \end{array} \tag{4.35}$$

for all $A \in \mathcal{A}_I$. Take an arbitrary unit vector $x \in \mathcal{H}$, and let

$$A = ix \otimes x - \frac{\varepsilon^2(1-i)}{1-\varepsilon^2} I.$$

Hence $\overline{A} = -ix \otimes x - \frac{\varepsilon^2(1+i)}{1-\varepsilon^2} I$. Then, Equation (4.35), Proposition 4.1.2-(2) and Corollary 1.2.27-(2) imply that

$$\overline{h(I)}h(A) \cdot \left(-iD_\varepsilon + \left\{ -\frac{\varepsilon^2(1+i)}{1-\varepsilon^2} \right\} \right) = iD_\varepsilon + \left\{ -\frac{\varepsilon^2(1-i)}{1-\varepsilon^2} \right\}$$

Since D_ε is the union of two disc, we conclude that either

$$\begin{cases} \overline{h(I)}h(A) \left(\frac{\varepsilon^2 i}{1-\varepsilon^2} - \frac{\varepsilon^2(1+i)}{1-\varepsilon^2} \right) = \frac{-\varepsilon^2 i}{1-\varepsilon^2} - \frac{\varepsilon^2(1-i)}{1-\varepsilon^2} \\ \\ \overline{h(I)}h(A) \left(\frac{-i}{1-\varepsilon^2} - \frac{\varepsilon^2(1+i)}{1-\varepsilon^2} \right) = \frac{i}{1-\varepsilon^2} - \frac{\varepsilon^2(1-i)}{1-\varepsilon^2}, \end{cases}$$

or

$$\begin{cases} \overline{h(I)}h(A) \left(\frac{\varepsilon^2 i}{1-\varepsilon^2} - \frac{\varepsilon^2(1+i)}{1-\varepsilon^2} \right) = \frac{i}{1-\varepsilon^2} - \frac{\varepsilon^2(1-i)}{1-\varepsilon^2} \\ \\ \overline{h(I)}h(A) \left(\frac{-i}{1-\varepsilon^2} - \frac{\varepsilon^2(1+i)}{1-\varepsilon^2} \right) = \frac{-\varepsilon^2 i}{1-\varepsilon^2} - \frac{\varepsilon^2(1-i)}{1-\varepsilon^2}. \end{cases}$$

Both cases arise a contradiction, and therefore, $\phi(A) = h(A)UAW$ for all $A \in \mathcal{A}_I$; as desired.

Now, we know that ϕ takes the form (4.32). Then, it remains to show that there exists $\alpha \in \mathbb{T}$ such that $h(A) = \alpha$ for all nonzero $A \in \mathcal{A}_I$. Indeed, by Equation (4.35) we obtain that

$$\sigma_\varepsilon(A) = \sigma_\varepsilon \left(\overline{h(I)}h(A)A \right) = \overline{h(I)}h(A)\sigma_\varepsilon(A) \tag{4.36}$$

for all $A \in \mathcal{A}_I$. Thus, for every nonzero vector $x \in \mathcal{H}$, we have

$$\|x\|^2 D_\varepsilon = \sigma_\varepsilon(x \otimes x) = h(x \otimes x)\overline{h(I)}\sigma_\varepsilon(x \otimes x) = h(x \otimes x)\overline{h(I)}\|x\|^2 D_\epsilon.$$

Then $h(x \otimes x)\overline{h(I)} = 1$, and $h(x \otimes x) = h(I)$. Now, we claim that $h(x \otimes y) = h(I)$ for all vectors $x, y \in \mathcal{H}$. Indeed, let x and y be two nonzero vectors in $\mathcal{H}$. If $\langle y, x \rangle \neq 0$, then we have

$$\begin{aligned} \langle y, x \rangle \|y\|^2 D_\epsilon &= \sigma_\varepsilon(\langle y, x \rangle y \otimes y) \\ &= \sigma_\varepsilon\left((x \otimes y)^* (y \otimes y) \right) \\ &= \sigma_\varepsilon\left(\phi(x \otimes y)^* \phi(y \otimes y) \right) \\ &= \sigma_\varepsilon\left(h(y \otimes y)\overline{h(x \otimes y)}\langle y, x \rangle y \otimes y \right) \\ &= h(y \otimes y)\overline{h(x \otimes y)}\langle y, x \rangle \|y\|^2 D_\epsilon. \end{aligned}$$

Thus $h(y \otimes x) = h(y \otimes y) = h(I)$. If $\langle y, x\rangle = 0$, then we choose a nonzero vector $z \in \mathcal{H}$ such that $\langle z, x\rangle \neq 0$ and $\langle z, y\rangle \neq 0$. Then we have

$$\begin{aligned}
\langle z, x\rangle \|y\|^2 D_\epsilon &= \sigma_\varepsilon(\langle z, x\rangle y \otimes y) \\
&= \sigma_\varepsilon\left((x \otimes y)^*(z \otimes y)\right) \\
&= \sigma_\varepsilon\left(\phi(x \otimes y)^*\phi(z \otimes y)\right) \\
&= \sigma_\varepsilon\left(h(z \otimes y)\overline{h(x \otimes y)}\langle z, x\rangle y \otimes y\right) \\
&= h(z \otimes y)\overline{h(x \otimes y)}\langle z, x\rangle \|y\|^2 D_\epsilon.
\end{aligned}$$

Hence $h(x \otimes y) = h(z \otimes y) = h(I)$; and the claim is proved.
Finally, we show that $h(A) = h(I)$ for all nonzero $A \in \mathcal{A}_I$. Let $A \in \mathcal{A}_I \setminus \{0\}$, and choose a nonzero vector $x \in \mathcal{H}$ such that $A^*x \neq 0$. Then we have

$$\begin{aligned}
\|A^*x\|^2 D_\epsilon &= \sigma_\varepsilon(A^*x \otimes A^*x) \\
&= \sigma_\varepsilon((x \otimes A^*x)^* A) \\
&= \sigma_\varepsilon(\phi(x \otimes A^*x)^*\phi(A)) \\
&= \sigma_\varepsilon(h(A)\overline{h(x \otimes A^*x)}A^*x \otimes A^*x) \\
&= h(A)\overline{h(x \otimes A^*x)}\|A^*x\|^2 D_\epsilon.
\end{aligned}$$

Then $h(A) = h(x \otimes A^*x) = h(I)$. This finishes the proof of Theorem 4.1.5.

Chapter 5

Multiplicatively preserving maps

Nonlinear preserver problems demand the characterization of maps on subsets of algebras that preserve various spectral quantities or subsets or relations but without assuming any algebraic condition like linearity or multiplicativity. Over the last few decades, these problems have been studied by numerous authors and the first nonlinear preserver problem was considered by Kowalski and Słodkowski who proved in [39] that a complex-valued function f on a Banach algebra $\mathcal{A}$ is linear and multiplicative provided that $f(0) = 0$ and $f(x) - f(y)$ lies in the spectrum of $x - y$ for all x and y in $\mathcal{A}$. This generalizes the well-known theorem of Gleason–Kahane–Żelazko in the theory of Banach algebra [24, 37]. Since then, a number of techniques have been developed to treat nonlinear preserver problems and many results have been obtained mainly in matrix theory and in operator theory; see for instance [7, 57, 2, 36, 48, 16, 15, 18, 17, 31, 20, 21, 25, 28, 12, 27, 13, 52, 54, 55]. In [57], Bhatia, Šemrl and Sourour described the form of all surjective maps on $\mathcal{M}_n(\mathbb{C})$ preserving the spectral radius of the difference of matrices, and thus, in particular, they provided an extension of Marcus and Moyls' result [51] in the absence of the linearity. In [54], Molnár studied maps preserving the spectrum of operator or matrix products and showed, in particular, that a surjective map φ on $\mathcal{B}(\mathcal{H})$, preserves the spectrum of operator products if and only if φ is an automorphism or an automorphism multiplied by -1. His results have been extended in several directions, and, in particular, there has been interest in studying maps ϕ on matrices or operators satisfying $F(\phi(A) \bullet \phi(B)) = F(A \bullet B)$. Here, $F(\cdot)$ is a spectral function or a spectral set such as the spectrum, the numerical range, the ε-pseudo spectral radius, the ε-pseudo spectrum, the ε-condition spectral radius, and the ε-condition spectrum and the local spectrum and $A \bullet B$ stands for different kinds of products such as the usual product AB, the triple product ABA, the Jordan product $AB + BA$, the skew product A^*B, the skew triple product AB^*A, and the skew-Jordan product

$AB^* + B^*A$; see for instance [3, 4, 60, 61, 1, 10, 9, 8, 11, 47, 31, 34, 12] and the references therein.

Let $\mathcal{A}$ and $\mathcal{B}$ be two algebras and let $\phi : \mathcal{A} \longrightarrow \mathcal{B}$ be a map. If, for $A, B \in \mathcal{A}, \phi(A)\phi(B) = 0 \Rightarrow AB = 0$ ($A, B \in \mathcal{A}, \phi(A)\phi(B) = 0 \Longleftrightarrow AB = 0$), we call that ϕ preserves zero products (preserves zero products in both directions). There are many references discussing the classification of linear or additive maps preserving zero products on various function algebras and operator algebras (e.g., for some most recent papers, see [4,10,15]). The importance of the study of zero-product preservers relies upon the fact that the zero-product is a quite basic relation between the elements in operator algebras, and the zero-product is also useful to other topics of both mathematics and physics (e.g., [16,17]). In [53], Molnár characterized the general bijective transformations on $\{P \in \mathcal{F}_1(X) \ : \ P^2 = P\}$ preserving zero product in both directions. This result was used as a main tool for generalizing the Uhlhorn's version of Wigner's theorem, which is a basic result in quantum mechanics. Wigner's Theorem states that every quantum mechanical invariance transformation can be represented by a unitary or an antiunitary operator on a complex Hilbert space. Note that the result of Molnár was given for infinite dimensional case. Here we will cover also the finite dimensional case and we are influenced by the approach, due to Šemrl [58] , using the fundamental theorem of projective geometry (see [58, Theorems 1.1 and 1.2]) we will also use some ideas of [19, Lemma 2.2].

5.1 Multiplicatively zero double product-preserving maps

Let $\mathcal{A}$ and $\mathcal{B}$ be two subsets of $\mathcal{B}(X)$ containing all operators in $\mathcal{B}(X)$ of rank at most one. Our first result characterizes all surjective maps $\phi, \psi : \mathcal{A} \longrightarrow \mathcal{B}$ satisfying

$$\phi(A)\psi(B) = 0 \Longleftrightarrow AB = 0 \quad (\forall A, B \in \mathcal{A}). \tag{5.1}$$

This result was motivated by a theorem concerning the characterization of bijectives transformations on $\{P \in \mathcal{F}_1(X) \ : \ P^2 = P\}$ preserving zero product in both directions due to Molnár [53]. Our result generalizes the main results of [19] and [34], which their authors described the general form of maps preserving zero products and zero skew-product of operators. Note that our results are differs from the above-mentioned works in terms of the techniques used, as it agrees in part with its predecessors and differs in another part, moreover in our siting the map ϕ can be teaks its values randomly outside $\mathcal{F}_1(\mathcal{H})$ (see Example 5.1.3).

5.1.1 Statement of the main results

Theorem 5.1.1. *Let $\mathcal{A}$ and $\mathcal{B}$ be two subsets of $\mathcal{B}(X)$ containing all operators in $\mathcal{B}(X)$ of rank at most one. Suppose that $\phi, \psi : \mathcal{A} \longrightarrow \mathcal{B}$ are two surjective maps satisfying*

$$\phi(A)\psi(B) = 0 \Longleftrightarrow AB = 0 \quad (\forall A, B \in \mathcal{A}).$$

Then, $\phi(\mathcal{F}_1(X)) = \mathcal{F}_1(X)$ and

$$\phi(A)\psi(B) = 0 \Longleftrightarrow AB = 0 \quad (\forall A, B \in \mathcal{F}_1(X)) \tag{5.2}$$

if and only if the following statements hold.

1. *If $\dim(X) = \infty$, there exist two maps $k : X \times X^* \longrightarrow X$, $h : X \times X^* \longrightarrow X^*$ and a bounded invertible linear or conjugate linear operator U on X such that*

   $$\phi(x\otimes f) = k_f(x)\otimes(U^{-1})^* f \text{ and } \psi(x\otimes f) = Ux\otimes h_x(f) \text{ for all } (x, f) \in X\times X^*,$$

   where $k_f(x) := k(x, f)$ and $h_x(f) := h(x, f)$ for all $(x, f) \in X \times X^*$.

2. *If $X = \mathbb{C}^n$ with $n \geq 3$, there exist two maps $k, h : \mathbb{C}^n \times \mathbb{C}^n \longrightarrow \mathbb{C}^n$, a nonsingular matrix $U \in \mathcal{M}_n(\mathbb{C})$ and a ring automorphism τ of $\mathbb{C}$ such that*

   $$\phi(x \otimes y) = k_y(x) \otimes y_{\tau^*}U^{-1} \text{ and } \psi(x \otimes y) = Ux_\tau \otimes h_x(y) \text{ for all } x, y \in \mathbb{C}^n,$$

   where $k_y(x) := k(x, y)$ and $h_x(y) := h(x, y)$ for all $x, y \in \mathbb{C}^n$.

Our next main theorem is given in Hilbert space context. We provide, without proof, the following restatement of Theorem 5.1.1 in the Hilbert space context. This version will be needed for the proof of Theorem 5.2.1.

Theorem 5.1.2. *Let $\mathcal{A}$ and $\mathcal{B}$ be two subsets of $\mathcal{B}(\mathcal{H})$ containing all operators in $\mathcal{B}(\mathcal{H})$ of rank at most one. Suppose that $\phi, \psi : \mathcal{A} \longrightarrow \mathcal{B}$ are two surjective maps satisfying*

$$\phi(A)\psi(B) = 0 \Longleftrightarrow AB = 0 \quad (\forall A, B \in \mathcal{A}).$$

Then, $\phi(\mathcal{F}_1(\mathcal{H})) = \mathcal{F}_1(\mathcal{H})$ and

$$\phi(A)\psi(B) = 0 \Longleftrightarrow AB = 0 \quad (\forall A, B \in \mathcal{F}_1(\mathcal{H}))$$

if and only if the following statements hold.

1. *If $\dim(\mathcal{H}) = \infty$, there exist two maps $h, k : \mathcal{H} \times \mathcal{H} \to \mathcal{H}$ and a bounded invertible linear or conjugate linear operator U on $\mathcal{H}$ such that*

   $$\phi(x\otimes y) = k_y(x)\otimes(U^{-1})^* y \text{ and } \psi(x\otimes y) = Ux\otimes h_x(y) \text{ for all } x, y \in \mathcal{H}, \tag{5.3}$$

   where $k_y(x) := k(x, f)$ and $h_x(y) := h(x, y)$ for all $x, y \in \mathcal{H}$.

2. *If $\mathcal{H} = \mathbb{C}^n$ with $n \geq 3$, there exist two maps $k, h : \mathbb{C}^n \times \mathbb{C}^n \longrightarrow \mathbb{C}^n$, a nonsingular matrix $U \in \mathcal{M}_n(\mathbb{C})$ and a ring automorphism τ of $\mathbb{C}$, such that*

$$\phi(x \otimes y) = k_y(x) \otimes y_{\tau^*} U^{-1} \text{ and } \psi(x \otimes y) = U x_\tau \otimes h_x(y) \text{ for all } x, y \in \mathbb{C}^n, \quad (5.4)$$

where $k_y(x) := k(x, y)$ and $h_x(y) := h(x, y)$ for all $x, y \in \mathbb{C}^n$.

The following example shows that the map ϕ can takes its values randomly outside $\mathcal{F}_1(\mathcal{H})$:

Exemple 5.1.3. *Let f be any surjective map on the set of invertible operators $A \in \mathcal{B}(\mathcal{H})$. We define ϕ, ψ on $\mathcal{B}(\mathcal{H})$ by $\psi(A) = A$ for all $A \in \mathcal{B}(\mathcal{H})$ and*

$$\phi(A) = \begin{cases} A & \text{if } A \text{ is not invertible} \\ f(A) & \text{if } A \text{ is invertible .} \end{cases}$$

It is easy to check that $\phi, \psi : \mathcal{A} \longrightarrow \mathcal{A}$ are surjective and satisfying

$$\phi(A)\psi(B) = 0 \Longleftrightarrow AB = 0 \quad (\forall A, B \in \mathcal{B}(\mathcal{H})).$$

In this section, we provide some applications of our main results. We start with the following theorem which is a direct consequence of Theorem 5.1.1.

Theorem 5.1.4. *Let $\mathcal{A}$ and $\mathcal{B}$ be two subsets of $\mathcal{B}(X)$ containing all operators in $\mathcal{B}(X)$ of rank at most one. Suppose that $\phi, \psi : \mathcal{A} \longrightarrow \mathcal{B}$ are two surjective maps satisfying*

$$AB = 0 \iff \phi(A)\psi(B) = 0 \iff \psi(A)\phi(B) = 0 \quad (\forall A, B \in \mathcal{F}_1(X)) \qquad (5.5)$$

if and only if the following statements hold.

1. *If $\dim(X) = \infty$, there exist two functionals $\alpha, \beta : X \times X^* \longrightarrow \mathbb{C} \setminus \{0\}$ and two bounded invertible linear or conjugate linear operators U and V on X such that*

$$\phi(x \otimes f) = \alpha(x, f) V x \otimes (U^{-1})^* f \text{ and } \psi(x \otimes f) = \beta(x, f) U x \otimes (V^{-1})^* f$$

for all $(x, f) \in X \times X^*$.

2. *If $X = \mathbb{C}^n$ with $n \geq 3$, there exist two functionals $k, h : \mathbb{C}^n \times \mathbb{C}^n \longrightarrow \mathbb{C} \setminus \{0\}$, two nonsingular matrices $U, V \in \mathcal{M}_n(\mathbb{C})$ and two ring automorphisms τ_1 and τ_2 of $\mathbb{C}$ such that*

$$\phi(x \otimes y) = \alpha(x, y) V x_{\tau_1} \otimes y_{\tau_2^*} U^{-1} \quad \text{and} \quad \psi(x \otimes y) = \beta(x, y) U x_{\tau_2} \otimes y_{\tau_1^*} V^{-1}$$

for all $x, y \in \mathbb{C}^n$..

Now, we will apply Theorem 5.1.4 to characterize maps preserving zero products and zero skew products. Let $\mathcal{A}$ and $\mathcal{B}$ be two subsets of $\mathcal{B}(X)$ which containing all operators of rank at most one, and note that surjective maps $\phi : \mathcal{A} \longrightarrow \mathcal{B}$ satisfying

$$\phi(A)\phi(B) = 0 \Longleftrightarrow AB = 0 \quad (A, B \in \mathcal{A}). \tag{5.6}$$

were described in [19, Lemma 2.2]. Such a characterization is now a consequence of Theorem 5.1.4.

Corollary 5.1.5. *Suppose that* $\phi : \mathcal{A} \longrightarrow \mathcal{B}$ *is a surjective map satisfying* (5.6), *then there exists a functional* $l : X \otimes X^* \longrightarrow \mathbb{C} \setminus \{0\}$ *and*

1. *In the case* $\dim(X) = \infty$, *there exists a bounded invertible linear or conjugate linear operator* U *on* X *such that*

   $$\phi(x \otimes f) = l(x, f)U(x \otimes f)U^{-1}$$

   for all rank one operator $x \otimes f$.

2. *In the case* $X = \mathbb{C}^n$, *there exist a nonsingular matrix* $U \in \mathcal{M}_n(\mathbb{C})$ *and a ring automorphism* τ *of* $\mathbb{C}$ *such that*

   $$\phi(x \otimes y) = l(x, y)Ux_\tau \otimes y_{\tau^*}U^{-1}$$

   for all rank one matrix $x \otimes y$.

Next, let $\mathcal{A}$ and $\mathcal{B}$ be two subsets of $\mathcal{B}(\mathcal{H})$ containing all operators of rank at most one. Let $K \in \mathcal{B}(\mathcal{H})$ be an invertible self-adjoint operator and denote $A^\dagger = K^{-1}A^*K$ for all $A \in \mathcal{B}(\mathcal{H})$. Surjective maps $\phi : \mathcal{A} \longrightarrow \mathcal{B}$ satisfying

$$\phi(A)\phi(B)^\dagger = 0 \Longleftrightarrow AB^\dagger = 0 \quad (A, B \in \mathcal{A}), \tag{5.7}$$

and

$$\phi(A)^\dagger\phi(B) = 0 \Longleftrightarrow A^\dagger B = 0 \quad (A, B \in \mathcal{A}), \tag{5.8}$$

were described in [34, Theorem 2.1]. Such a characterization is now a consequence of Theorem 5.1.4. Here, we state this result, in a slightly different way, since the original version in [34, Theorem 2.1] requires a minor correction; see the discussion after the proof of the following result.

Corollary 5.1.6. *Suppose that* $\phi : \mathcal{A} \longrightarrow \mathcal{B}$ *is a surjective map satisfying* (5.7) *and* (5.8), *then there exist a functional* $h : \mathcal{H} \times \mathcal{H} \longrightarrow \mathbb{C} \setminus \{0\}$, *two nonzero real scalars* c *and* d, *and linear or conjugate linear bounded invertible operators* U *and* V *on* $\mathcal{H}$ *such that* $U^*KU = cK$, $V^*KV = dK$ *and*

$$\phi(x \otimes y) = h(x, y)Ux \otimes V^*y \quad (x,\ y \in \mathcal{H}).$$

Note that in the statement of the previous result, U and V need not be simultaneously linear or simultaneously conjugate linear. For example, let $\mathcal{H} = \mathbb{C}^n$ and $K = I$. If

$$\phi(x \otimes y) = x \otimes \overline{y}$$

for all $x, y \in \mathbb{C}^n$, then ϕ satisfies (5.7) and (5.8). While there is no simultaneously linear or simultaneously conjugate linear operators U and V in $\mathcal{B}(\mathcal{H})$ such that ϕ has the form $x \otimes y \mapsto U(x \otimes y)V$. In fact the equality

$$h_1(x \otimes f)Ux \otimes V^*f = h_1(x \otimes f)U(x \otimes f)V$$

claimed in [34, page 2245, line 3, in the proof of Theorem 2.1] is not correct when exactly one of operators U and V is conjugate linear. For this reason, we also give Corollary 5.1.7.

Corollary 5.1.7. *Suppose that $\phi : \mathcal{A} \longrightarrow \mathcal{B}$ is a surjective map satisfying*

$$\phi(A)\phi(B)^* = 0 \iff AB^* = 0 \tag{5.9}$$

for all $A, B \in \mathcal{A}$, and

$$\phi(A)^*\phi(B) = 0 \iff A^*B = 0 \tag{5.10}$$

for all $A, B \in \mathcal{A}$. Then there exist unitary or conjugate unitary operators U and V and a functional $h : \mathcal{H}^2 \longrightarrow \mathbb{C} \setminus \{0\}$ such that

$$\phi(x \otimes y) = h(x, y)Ux \otimes V^*y$$

for all $x, y \in \mathcal{H}$.

5.1.2 Preliminaries

In this section, we establish two auxiliary, but important, result : first, the fundamental theorem of projective geometry as in [58, Proof of Theorems 1.1 and 1.2]; second, a characterization of rank one operators in term of the zero double product. These results needed for the proof of Theorem 5.1.1.

The following lemma, which may be of independent interest, gives a characterization of rank one operators in term of the zero double product.

Lemma 5.1.8. *Let $\mathcal{A}$ be a subset of $\mathcal{B}(X)$ containing $\mathcal{F}_1(X)$, and let $A \in \mathcal{A}$ be a nonzero operator, the following statements are equivalent.*

1. *A is an operator of rank one.*
2. *$\{A\}^R$ is maximal.*
3. *$\{A\}^L$ is maximal.*

Indeed. First, for (1) $\Longleftrightarrow$ (2). Let $A \in \mathcal{A} \setminus \{0\}$ such that the set $\{A\}^R$ is maximal. Suppose on the contrary that $\operatorname{rank}(A) \geq 2$, then there are two vectors x_1 and x_2 in X such that Ax_1 and Ax_2 are linearly independent. We choose $f_1, f_2 \in X^*$ such that $\langle Ax_i, f_j \rangle = \delta_{ij}$ (Kroneckor symbol), $i, j = 1, 2$. Let $P = Ax_1 \otimes f_1 \in \mathcal{A}$, then $\{A\}^R \subseteq \{P\}^R$. Since the set $\{A\}^R$ is maximal, we have $\{P\}^R = \{A\}^R$. Now, let $B = Ax_2 \otimes f_2 \in \mathcal{A}$, we have

$$BA = Ax_2 \otimes f_2 A = Ax_2 \otimes A^* f_2 \neq 0,$$

and

$$BP = (Ax_2 \otimes f_2)(Ax_1 \otimes f_1) = \langle Ax_1, f_2 \rangle Ax_2 \otimes f_1 = 0.$$

Therefore, $B \in \{P\}^R$ and $B \notin \{A\}^R$, and thus $\{P\}^R \neq \{A\}^R$, which is a contradiction. Hence, $\operatorname{rank}(A) = 1$.

Conversely, assume that $\operatorname{rank}(A) = 1$, and set $A = x \otimes f$ for some $x \in X$ and $f \in X^*$. If $N \in \mathcal{A}$ satisfies $\{A\}^R \subseteq \{N\}^R$, then for any operator of rank one $u \otimes h$ with $\langle x, h \rangle = 0$, we have

$$u \otimes hA = (u \otimes h)(x \otimes f) = \langle x, h \rangle u \otimes f = 0.$$

Therefore, $u \otimes h \in \{A\}^R$, and thus $u \otimes h \in \{N\}^R$. Hence $\ker(N^*) \supseteq [h]$ whenever $\langle x, h \rangle = 0$, which implies that $\operatorname{rank}(N^*) \leq 1$. Now, we show that $\operatorname{rank}(N) \leq 1$. Suppose on the contrary that $\operatorname{rank}(N) \geq 2$, then there are two vectors x_1 and x_2 in X such that Nx_1 and Nx_2 are linearly independent. We choose $f_1, f_2 \in X^*$ satisfying $\langle Nx_i, f_j \rangle = \delta_{ij}$, $i, j = 1, 2$. If $g_i = f_i \circ N$, then $\langle x_i, g_j \rangle = \delta_{ij}$, $i, j = 1, 2$. Hence $\operatorname{rank}(N^*) \geq 2$, which is a contradiction. Thus, $\operatorname{rank}(N) \leq 1$. So, let $N = y \otimes g$ for some $(y, g) \in X \times X^*$. We show that x and y are linearly dependent. Suppose on the contrary that x and y are linearly independent and choose $l \in X^*$ such that $\langle x, l \rangle = 0$ and $\langle y, l \rangle = 1$. Then $x \otimes l \in \{A\}^R$ and $x \otimes l \notin \{N\}^R$. This contradicts the fact $\{A\}^R \subseteq \{N\}^R$ and shows that x and y are linearly dependent. Therefore, $\{N\}^R = \{A\}^R$, and the set $\{A\}^R$ is maximal.

Now, we only need to establish the implication (1) $\Rightarrow$ (3). Let $A \in \mathcal{A} \setminus \{0\}$ such that the set $\{A\}^L$ is maximal. Suppose on the contrary that $\operatorname{rank}(A) \geq 2$, then there are two vectors x_1 and x_2 in X such that Ax_1 and Ax_2 are linearly independent. We choose $f \in X^*$ satisfying $\langle Ax_2, f \rangle = 0$ and $\langle Ax_1, f \rangle = 1$. Let $P := x_1 \otimes A^* f \in \mathcal{A}$, and note that $\{A\}^L \subseteq \{P\}^L$. Since the set $\{A\}^L$ is maximal, then $\{P\}^L = \{A\}^L$. Now, if $B = x_2 \otimes A^* f \in \mathcal{A}$, then

$$AB = Ax_2 \otimes A^* f \neq 0,$$

and

$$PB = (x_1 \otimes A^* f)(x_2 \otimes A^* f) = \langle Ax_2, f \rangle x_1 \otimes A^* f = 0.$$

Therefore, $B \in \{P\}^L$ and $B \notin \{A\}^L$. Thus $\{P\}^L \neq \{A\}^L$, which is a contradiction. Hence, $\operatorname{rank}(A) = 1$ and completes the proof.

For the proof of Theorem 5.1.1, we need the following lemma (see [38, Lemma 2, Corollary, page 2] and [50, Lemma B]) that tells us that a bijective semi-linear transformation S between infinite-dimensional complex normed linear spaces X and Y is automatically bicontinuous and linear or conjugate linear provided that S and S^{-1} carry closed hyperplanes to closed hyperplanes.

Lemma 5.1.9. *If $S : X \longrightarrow X$ is a bijective semi-linear transformation between infinite-dimensional complex normed linear space X such that S and S^{-1} carry closed hyperplanes to closed hyperplanes, then S is bicontinuous and linear or conjugate linear.*

5.1.3 Proof of Theorem 5.1.1.

We only need to establish the "*only if*" part whose proof is long and delicate. Hence, we break it into several assertions. Assume that ϕ and ψ satisfying (5.1).

Assertion 1. Let $\phi, \psi : \mathcal{A} \longrightarrow \mathcal{B}$ be two surjective maps satisfying (5.1). Then ϕ and ψ preserves the operators of rank at most one in both directions.

Indeed. By the surjectivity of ϕ, ψ and (5.1), we have $\phi(0) = 0$ and $\psi(0) = 0$. Now, it suffices to prove that ϕ and ψ preserve operators of rank one in both directions. Let $A \in \mathcal{A} \setminus \{0\}$, and let us first show that $\operatorname{rank}(A) = 1$ if and only if $\operatorname{rank}(\psi(A)) = 1$. For any $N \in \mathcal{A} \setminus \{0\}$, we note that (5.1) and the surjectivity of ϕ imply that

$$\{A\}^R \subseteq \{N\}^R \iff \{\psi(A)\}^R \subseteq \{\psi(N)\}^R,$$

and

$$\{A\}^R = \{N\}^R \iff \{\psi(A)\}^R = \{\psi(N)\}^R.$$

This together with Lemma 5.1.8 and the surjectivity of ϕ entail that

$$\begin{aligned} \operatorname{rank}(A) = 1 \quad &\iff \quad \text{the set } \{A\}^R \text{ is maximal} \\ &\iff \forall N \in \mathcal{A} \setminus \{0\}, \quad \text{we have } \{A\}^R \subseteq \{N\}^R \Rightarrow \{A\}^R = \{N\}^R \\ &\iff \forall N \in \mathcal{B} \setminus \{0\}, \quad \text{we have } \{\psi(A)\}^R \subseteq \{N\}^R \Rightarrow \{\psi(A)\}^R = \{N\}^R \\ &\iff \quad \text{the set } \{\psi(A)\}^R \text{ is maximal} \\ &\iff \operatorname{rank}(\psi(A)) = 1. \end{aligned}$$

Similarly, we show that $\operatorname{rank}(A) = 1$ if and only if $\operatorname{rank}(\phi(A)) = 1$. By (5.1), Lemma

5.1.8, and the surjectivity of ψ, we have

$$\begin{aligned}
\operatorname{rank}(A)=1 \quad &\Longleftrightarrow \quad \text{the set } \{A\}^L \text{ is maximal}\\
&\Longleftrightarrow \quad \forall N\in\mathcal{A}\setminus\{0\},\ \ \text{we have } \{A\}^L\subseteq\{N\}^L \Rightarrow \{A\}^L=\{N\}^L\\
&\Longleftrightarrow \quad \forall N\in\mathcal{B}\setminus\{0\},\ \ \text{we have } \{\phi(A)\}^L\subseteq\{N\}^L \Rightarrow \{\phi(A)\}^L=\{N\}^L\\
&\Longleftrightarrow \quad \text{the set } \{\phi(A)\}^L \text{ is maximal}\\
&\Longleftrightarrow \quad \operatorname{rank}(\phi(A))=1.
\end{aligned}$$

The proof of Assertion 1 is now complete.

In the following assertion we will use essentially some notions and results given in [55, §2.2]. For every nonzero vector $x\in X$, the ray generated by x will be denoted by $\underline{x}$, this is the set $\{\lambda x \ :\ \lambda\in\mathbb{C}\setminus\{0\}\}$. The set of all rays in X will be denoted by $\underline{X}$, this means that $\underline{X}:=\{\underline{x}: x\in X\setminus\{0\}\}$. We say that the rays $\underline{x},\underline{y}\in\underline{X}$ are orthogonal to each other, in notation $\langle\underline{x},\underline{y}\rangle=0$, if we have $\langle x,y\rangle=0$. Observe that $\langle x,y\rangle=0$ if and only if $\langle u,v\rangle=0$ for all $u\in\underline{x}$ and $v\in\underline{y}$. We well also need the notation $\underline{x}+\underline{y}:=\{\alpha x+\beta y:\alpha,\beta\in\mathbb{C}\}\setminus\{0\}$ for all nonzero vectors $x,y\in X$.

Assertion 2. Let $\phi,\psi:\mathcal{A}\longrightarrow\mathcal{B}$ be two surjective maps satisfying (5.1). Then there exist a bijective semilinear map $U: X\longrightarrow X$, a bijective semilinear map $V:X^*\longrightarrow X^*$ and two maps $k:X\times X^*\longrightarrow X$ and $h:X\times X^*\longrightarrow X^*$ such that

$$\phi(x\otimes f)=k_f(x)\otimes(Vf) \ \text{ and } \ \psi(x\otimes f)=(Ux)\otimes h_x(f) \ \text{ for all } \ (x,f)\in X\times X^*, \tag{5.11}$$

where $k_f(x):=k(x,f)$ and $h_x(f):=h(x,f)$ for all $(x,f)\in X\times X^*$. Furthermore, we have

$$\langle Ux,Vf\rangle=0 \iff \langle x,f\rangle=0 \tag{5.12}$$

for all $(x,f)\in X\times X^*$.

Indeed. Keep in mind that ϕ and ψ are surjective maps preserving operators of rank at most one in both directions, and let $R=x\otimes f$ and $Q=y\otimes g$ be two operators of rank one. Let $\psi(R)=x'\otimes f'$ and $\psi(Q)=y'\otimes g'$, and note that (5.1) entails that

$$\begin{aligned}
x=\lambda y \text{ for some } \lambda \quad &\Longleftrightarrow \quad \forall P\in\mathcal{F}_1(X),\ \ PR=0\Leftrightarrow PQ=0\\
&\Longleftrightarrow \quad \forall P\in\mathcal{F}_1(X),\ \ \phi(P)\psi(R)=0\Leftrightarrow \phi(P)\psi(Q)=0\\
&\Longleftrightarrow \quad \forall P\in\mathcal{F}_1(X),\ \ \phi(P)(x'\otimes f')=0\Leftrightarrow \phi(P)(y'\otimes g')=0\\
&\Longleftrightarrow \quad \forall z\otimes h\in\mathcal{F}_1(X),\ \ (z\otimes h)(x'\otimes f')=0\Leftrightarrow (z\otimes h)(y'\otimes g')=0\\
&\Longleftrightarrow \quad \forall h\in X^*,\ \ \langle x',h\rangle=0\Leftrightarrow\langle y',h\rangle=0\\
&\Longleftrightarrow \quad x'=\lambda' y' \text{ for some } \lambda' .
\end{aligned}$$

This implies that for every nonzero $x \in X$ there exists a nonzero $z \in X$ such that $\psi(L_x) = L_z$.

Now, we define a map $\varphi : \underline{X} \longrightarrow \underline{X}$ by

$$\varphi\left(\underline{x}\right) = \underline{y} \text{ if and only if } \psi(L_x) = L_y.$$

Sinse ϕ and ψ preserve operators of rank at most one in both directions, φ is a bijective map. We show that for every nonzero vectors $x, u, v \in X$, we have

$$\underline{x} \subseteq \underline{u} + \underline{v} \Longleftrightarrow \varphi(\underline{x}) \subseteq \varphi(\underline{u}) + \varphi(\underline{v}).$$

Let $(x, u, v) \in X^3$ and let $\varphi(\underline{x}) = \underline{x'}$, $\varphi(\underline{v}) = \underline{v'}$, $\varphi(\underline{u}) = \underline{u'}$. We have

$$\begin{aligned}
\underline{x} \subseteq \underline{u} + \underline{v} \quad &\Longleftrightarrow \quad \forall P \in \mathcal{F}_1(X),\ \ (PL_u = \{0\} \text{ and } PL_v = \{0\}) \Rightarrow PL_x = \{0\} \\
&\Longleftrightarrow \quad \forall P \in \mathcal{F}_1(X),\ \ (\phi(P)\psi(L_u) = \{0\} \text{ and } \phi(P)\psi(L_v) = \{0\}) \\
&\qquad\ \ \Rightarrow \phi(P)\psi(L_x) = \{0\} \\
&\Longleftrightarrow \quad \forall Q \in \mathcal{F}_1(X),\ \ (QL'_u = \{0\} \text{ and } QL'_v = \{0\}) \Rightarrow QL'_x = \{0\} \\
&\Longleftrightarrow \quad \varphi(\underline{x}) \subseteq \varphi(\underline{u}) + \varphi(\underline{v}).
\end{aligned}$$

Thus, using the fundamental theorem of projective geometry as in [58, Proof of Theorems 1.1 and 1.2], we conclude that the map φ is induced by a semilinear bijective map $U : X \longrightarrow X$, this means that $\varphi\left(\underline{x}\right) = \underline{Ux}$ for all $x \in X$. Therefore, for every operator of rank at most one $x \otimes f \in \mathcal{F}_1(X)$ there exists a map $h : X \times X^* \longrightarrow X^*$ such that

$$\psi(x \otimes f) = Ux \otimes h_x(f) \text{ for all } (x, f) \in X \times X^*,$$

where $h_x(f) := h(x, f)$ for all $(x, f) \in X \times X^*$.

Applying a similar process to $\Psi : \underline{X^*} \longrightarrow \underline{X^*}$ defined by

$$\Psi\left(\underline{f}\right) = \underline{g} \text{ if and only if } \phi(R_f) = R_g,$$

we prove that there is a semilinear bijective map $V : X^* \longrightarrow X^*$ such that $\Psi\left(\underline{f}\right) = \underline{Vg}$. Hence, there exists a map $k : X \times X^* \longrightarrow X$, $(x, f) \mapsto k_f(x)$ such that

$$\phi(x \otimes f) = k_f(x) \otimes Vf$$

for all $(x, f) \in X \times X^*$. To complete the proof of the assertion, observe that by (5.1) we have

$$\begin{aligned}
\langle Uy, Vf \rangle = 0 \quad &\Longleftrightarrow \quad (k_f(x) \otimes Vf)(Uy \otimes h_y(f)) = 0 \\
&\Longleftrightarrow \quad \phi(x \otimes f)\psi(y \otimes f) = 0 \\
&\Longleftrightarrow \quad (x \otimes f)(y \otimes f) = 0 \\
&\Longleftrightarrow \quad \langle y, f \rangle = 0
\end{aligned}$$

for all $y \in X$ and $f \in X^*$. This finish the proof of assertion 2.

Assertion 3. Let $\phi, \psi : \mathcal{A} \longrightarrow \mathcal{B}$ be two surjective maps satisfying (5.1). Then then there exist two maps $k : X \times X^* \longrightarrow X$, $h : X \times X^* \longrightarrow X^*$ and a bounded invertible linear or conjugate linear operator U on X such that

$$\phi(x \otimes f) = k_f(x) \otimes (U^{-1})^* f \text{ and } \psi(x \otimes f) = Ux \otimes h_x(f) \text{ for all } (x, f) \in X \times X^*,$$

where $h_x(f) := h(x, f)$ for all $(x, f) \in X \times X^*$.

Indeed. Keep in mind that ϕ and ψ satisfying the conditions (5.11) of Assertion 3, and let us show that U is continuous and linear or conjugate linear. We will prove that U and U^{-1} carry every closed hyperplane of X to a closed hyperplane. Pick a nonzero $x \in X$, by (5.12) we know that

$$\langle Uy, Vf \rangle = 0 \iff \langle y, f \rangle = 0$$

for all nonzero $y \in X$ and nonzero $f \in X^*$. Since U and V are bijective we deduce that

$$\langle U^{-1}z, V^{-1}f \rangle = 0 \iff \langle z, f \rangle = 0 \tag{5.13}$$

for all nonzero $z \in X$ and nonzero $f \in X^*$. Now, let $H \subseteq X$ be a closed hyperplane of X. Then, there is a nonzero $f_0 \in X^*$ such that $H = \{x \in X : \langle x, f_0 \rangle = 0\}$. By (5.12) and (5.13), we have

$$U(H) = \{x \in X : \langle x, Vf_0 \rangle = 0\},$$

and

$$U^{-1}(H) = \left\{x \in X : \langle x, V^{-1}f_0 \rangle = 0\right\}.$$

From Lemma 5.1.9, we conclude that U is continuous and linear or conjugate linear. Therefore, from the definition of the adjoint U^* and (5.12) we have

$$\langle y, U^*Vf \rangle = 0 \iff \langle Uy, Vf \rangle = 0 \iff \langle y, f \rangle = 0 \tag{5.14}$$

for all $y \in X$ and $f \in X^*$. Let us show that $(U^{-1})^*$ and V are locally linearly dependent. Assume by way of contradiction that they are locally linearly independent, and let f be a vector in X^* such that U^*Vf and f are linearly independent. Then there exists a nonzero vector $z \in X$ such that $\langle z, U^*Vf \rangle = 0$ and $\langle z, f \rangle = 1$, this contradicts (5.14). Thus U^*Vf and f are linearly dependent for all $f \in X^*$. Thus, there exists a functional $\mu : X^* \mapsto \mathbb{C} \setminus \{0\}$, $f \mapsto \mu_f$ such that $Vf = \mu_f (U^{-1})^* f$ for all $f \in X^*$. Hence $\phi(x \otimes f) = \mu_f k_f(x) \otimes (U^{-1})^* f$ and $\psi(x \otimes f) = Ux \otimes h_x(f)$ for all $(x, f) \in X \times X^*$. Thus, Assertion 3 is proved.

Assertion 4. Let $\phi, \psi : \mathcal{A} \longrightarrow \mathcal{B}$ be two surjective maps satisfying (5.1) and suppose that $X = \mathbb{C}^n$ is a finite-dimensional space with $n \geq 3$, then there exist two maps $k, h : \mathbb{C}^n \times \mathbb{C}^n \longrightarrow \mathbb{C}^n$, a nonsingular matrix $U \in \mathcal{M}_n(\mathbb{C})$ and a ring automorphism τ of $\mathbb{C}$ such that

$$\phi(x \otimes y) = k_y(x) \otimes y_{\tau^*} U^{-1} \text{ and } \psi(x \otimes y) = Ux_\tau \otimes h_x(y) \text{ for all } x, y \in \mathbb{C}^n,$$

where $k_y(x) := k(x, y)$ and $h_x(y) := h(x, y)$ for all $x, y \in \mathbb{C}^n$.

Indeed. Note that Assertion 3, tells us that

$$\phi(x \otimes y) = k_y(x) \otimes Vy \text{ and } \psi(x \otimes y) = Ux \otimes h_x(y),$$

where U and V are two semilinear mapping on $\mathbb{C}$. Thus, there exist two ring automorphisms τ_1 and τ_2 of $\mathbb{C}$ and two matrices B and C such that

$$Ux = Cx_{\tau_1} \text{ and } Vx = Bx_{\tau_2}$$

for all $x \in \mathbb{C}$. First, we show that $\tau_1 = \tau_2^*$. Let $\mu \in \mathbb{C}$ and $x, y \in X$ be two unit vectors such that $\langle x, y \rangle = 0$. Clearly, $\langle \mu x - y, x + \overline{\mu} y \rangle = 0$ and by (5.12), we have

$$\begin{aligned} 0 &= \langle U(\mu x - y), V(x + \overline{\mu} y) \rangle \\ &= \tau_1(\mu)\langle Ux, Vx \rangle - \overline{\tau_2(\overline{\mu})}\langle Uy, Vy \rangle + \tau_1(\mu)\overline{\tau_2(\overline{\mu})}\langle Ux, Vy \rangle - \langle Uy, Vx \rangle \\ &= \tau_1(\mu)\langle Ux, Vx \rangle - \overline{\tau_2(\overline{\mu})}\langle Uy, Vy \rangle. \end{aligned}$$

Therefore,

$$\tau_1(\mu)\langle Cx_{\tau_1}, Bx_{\tau_2} \rangle - \overline{\tau_2(\overline{\mu})}\langle Cy_{\tau_1}, By_{\tau_2} \rangle = 0. \tag{5.15}$$

for all $\mu \in \mathbb{C}$. Choosing $\mu = 1$ in the above equality, we obtain that

$$\langle Cx_{\tau_1}, Bx_{\tau_2} \rangle = \langle Cy_{\tau_1}, By_{\tau_2} \rangle. \tag{5.16}$$

From (5.15) and (5.16), we conclude that

$$\left(\tau_1(\mu) - \overline{\tau_2(\overline{\mu})}\right) \langle Cx_{\tau_1}, Bx_{\tau_2} \rangle = 0,$$

for all $\mu \in \mathbb{C}$ and all unit vectors $x \in X$. Again by (5.12) and the fact that $\langle x, x \rangle \neq 0$, we have $\langle Cx_{\tau_1}, Bx_{\tau_2} \rangle = \langle Ux, Vx \rangle \neq 0$. Then

$$\tau_1(\mu) = \overline{\tau_2(\overline{\mu})} \text{ for all } \mu \in \mathbb{C}.$$

In the rest of the proof of Assertion 6, we set $\tau := \tau_1 = \tau_2^*$. Now, let us show that B^t and C^{-1} are linearly dependent. By (5.1), we have

$$\begin{aligned} \langle y_\tau, x_{\tau^*} \rangle = 0 \quad &\Longleftrightarrow \quad \langle y, x \rangle = 0 \\ &\Longleftrightarrow \quad \langle Uy, Vx \rangle = 0 \\ &\Longleftrightarrow \quad \langle Cy_{\tau_1}, Bx_{\tau_2} \rangle = 0 \\ &\Longleftrightarrow \quad \langle B^*Cy_\tau, x_{\tau^*} \rangle = 0 \\ &\Longleftrightarrow \quad \langle (B^tC)y_\tau, x_{\tau^*} \rangle = 0 \end{aligned}$$

for all $x, y \in \mathbb{C}^n$. Hence

$$\langle y, x\rangle = 0 \iff \langle (B^tC)y, x\rangle = 0 \text{ for all } x, y \in \mathbb{C}^n.$$

Then $(B^tC)y$ and y are linearly independent for all $y \in \mathbb{C}^n$, thus $B^t = \lambda C^{-1}$ for some nonzero scalar $\lambda \in \mathbb{C}$. Therefore,

$$\phi(x \otimes y) = k_y(x) \otimes Vy = k_y(x) \otimes By_{\tau^*} = k_y(x) \otimes y_{\tau^*}B^t = (\lambda k_y(x)) \otimes y_{\tau^*}C^{-1}$$

for all $x, y \in \mathbb{C}^n$, and

$$\psi(x \otimes y) = Ux \otimes h_x(y) = Cx_\tau \otimes h_x(y)$$

for all $x, y \in \mathbb{C}^n$. By writing U instead of C and $k_y(x)$ instead of $\lambda k_y(x)$ we obtain the desired forms of ϕ and ψ.

5.1.4 Proof of Theorem 5.1.4.

Assume first that $\dim(X) = \infty$. Applying Theorem 5.1.1 and (5.5) we conclude that there exist four X-valued maps k, k', h and h' on $X \times X^*$ and two bounded invertible linear or conjugate linear operators U and V on X such that

$$\begin{array}{lllll} \phi(x \otimes f) & = & k_f(x) \otimes (U^{-1})^*f & = & Vx \otimes h'_x(f) \\ \psi(x \otimes f) & = & Ux \otimes h_x(f) & = & k'_f(x) \otimes (V^{-1})^*f \end{array} \tag{5.17}$$

for all $(x, f) \in X \times X^*$, where $k_f(x) := k(x, f)$ and $h_x(f) := h(x, f)$ for all $(x, f) \in X \times X^*$. Therefore, there exists two maps μ and ν from $X \times X^*$ to $\mathbb{C} \setminus \{0\}$ such that

$$k'_f(x) = \nu(x, f)Ux \text{ and } h'_f(x) = \nu(x, f)(V^{-1})^*fx$$

for all $(x, f) \in X \times X^*$. This allows us to conclude that there exist two maps α and β from $X \times X^*$ to $\mathbb{C} \setminus \{0\}$ and two bounded invertible linear or conjugate linear operators U and V on X such that

$$\phi(x \otimes f) = \alpha(x, f)Vx \otimes (U^{-1})^*f \text{ and } \psi(x \otimes f) = \beta(x, f)Ux \otimes (V^{-1})^*f$$

for all $(x, f) \in X \times X^*$.

Now, if $X = \mathbb{C}^n$, for a given integer $n \geq 3$, then by Theorem 5.1.1 there exist four $\mathbb{C}^n$-valued maps $k, , k', h, h'$ on $\mathbb{C}^n \times \mathbb{C}^n$, two nonsingular matrices $U, V \in \mathcal{M}_n(\mathbb{C})$, and two ring automorphisms τ_1 and τ_2 of $\mathbb{C}$ such that

$$\begin{array}{lllll} \phi(x \otimes y) & = & k_y(x) \otimes y_{\tau_2^*}U^{-1} & = & Vx_{\tau_1} \otimes h'_x(y) \\ \psi(x \otimes y) & = & Ux_{\tau_2} \otimes h_x(y) & = & k'_y(x) \otimes y_{\tau_1^*}V^{-1} \end{array} \tag{5.18}$$

for all $x, y \in \mathbb{C}^n$, where $k_y(x) := k(x,y)$ and $h_x(y) := h(x,y)$ for all $x, y \in \mathbb{C}^n$. By a similar argument as above, there exist two functionals α and β on $\mathbb{C}^n \times \mathbb{C}^n$, two nonsingular matrices $U, V \in \mathcal{M}_n(\mathbb{C})$, and two ring automorphisms τ_1 and τ_2 of $\mathbb{C}$ such that

$$\phi(x \otimes y) = \alpha(x,y) V x_{\tau_1} \otimes y_{\tau_2^*} U^{-1} \text{ and } \psi(x \otimes y) = \beta(x,y) U x_{\tau_2} \otimes y_{\tau_1^*} V^{-1}$$

for all $x, y \in \mathbb{C}^n$. This completes the proof of Theorem 5.1.4.

5.1.5 Proof of Corollary 5.1.5.

Applying Theorem 5.1.4 in the case when $\psi = \phi$, we conclude that: If $\dim(X) = \infty$, there exist two functionals $\alpha, \beta : X \times X^* \longrightarrow \mathbb{C}$ and two bounded invertible linear or conjugate linear operators U and V on X such that

$$\alpha(x,f) V x \otimes (U^{-1})^* f = \phi(x \otimes f) = \psi(x \otimes f) = \beta(x,f) U x \otimes (V^{-1})^* f$$

for all $(x,f) \in X \times X^*$. Therefore, $\beta(x,f)Ux$ and $\alpha(x,f)Vx$ are linearly dependent for all $(x,f) \in X \times X^*$. Hence, U and V are linearly dependent, and thus, there exist a functional $l : X \times X^* \longrightarrow \mathbb{C} \setminus \{0\}$ and a bounded invertible linear or conjugate linear operator U on X such that

$$\phi(x \otimes f) = l(x,f) U x \otimes (U^{-1})^* f = l(x,f) U (x \otimes f) U^{-1}$$

for every rank one operator $x \otimes f \in \mathcal{A}$.

If $X = \mathbb{C}^n$ with $n \geq 3$, there exist two functionals $k, h : \mathbb{C}^n \times \mathbb{C}^n \longrightarrow \mathbb{C} \setminus \{0\}$, two nonsingular matrices $U, V \in \mathcal{M}_n(\mathbb{C})$ and two ring automorphisms τ_1 and τ_2 of $\mathbb{C}$ such that

$$\alpha(x,y) V x_{\tau_1} \otimes y_{\tau_2^*} U^{-1} = \phi(x \otimes y) = \psi(x \otimes y) = \beta(x,y) U x_{\tau_2} \otimes y_{\tau_1^*} V^{-1}$$

for all $x, y \in \mathbb{C}^n$. Therefore, $\beta_{x,y} U x_{\tau_2}$ and $\alpha_{x,y} V x_{\tau_1}$ are linearly dependent for all $x, y \in \mathbb{C}^n$. Hence, Ux and Vx are linearly dependent for all $x = (x_1, ..., x_n) \in \{0,1\}^n$, thus $U = \lambda V$ for some scalar λ and $\tau_1 = \tau_2$. Therefore, there exists a functional $l : \mathcal{A} \longrightarrow \mathbb{C} \setminus \{0\}$ and there exist a nonsingular matrix $U \in \mathcal{M}_n(\mathbb{C})$ and a ring automorphism τ of $\mathbb{C}$ such that $\phi(A) = l(A) U (x_\tau \otimes y_{\tau^*}) U^{-1}$ for every all rank one matrices $x \otimes y$.

5.1.6 Proof of Corollary 5.1.6.

Set

$$\psi(A) := K^{-1} \left(\phi(K^{-1} A^* K) \right)^* K$$

for all $A \in \mathcal{A}$, and note that ϕ and ψ satisfy (5.5). Therefore, Theorem 5.1.4 applies, and thus two cases will be discussed.

If $\dim(\mathcal{H}) = \infty$, there exist two functionals α and β from $\mathcal{H}^2$ to $\mathbb{C} \setminus \{0\}$ and two bounded invertible linear or conjugate linear operators U and V on $\mathcal{H}$ such that

$$\phi(x \otimes y) = \alpha(x,y)Vx \otimes (U^{-1})^* y \text{ and } \psi(x \otimes y) = \beta(x,y)Ux \otimes (V^{-1})^* y$$

for all $(x, y) \in \mathcal{H}^2$. Then, for every $x \in \mathcal{H}$, we have

$$\begin{array}{rcl} \beta_{x,x}Ux \otimes (V^{-1})^* x &=& \psi(x \otimes x) \\ &=& K^{-1}\left(\phi(K^{-1}x \otimes Kx)\right)^* J \\ &=& K^{-1}\left(\alpha_{x,x}VK^{-1}x \otimes (U^{-1})^* Kx\right)^* K \\ &=& \overline{\alpha_{x,x}}K^{-1}(U^{-1})^* Kx \otimes KVK^{-1}x. \end{array}$$

Thus, there exists two functionals α' and β' on $\mathcal{H}$ such that

$$K^{-1}(U^{-1})^* Kx = \beta'_x Ux$$

and

$$KVK^{-1}x = (\alpha'_x)(V^{-1})^* x.$$

Therefore, there exist a nonzero scalars c and d such that $U^*KU = cK$ and $V^*KV = dK$. Thus c and d are in fact reals. Hence, there exists a functional $h : \mathcal{H} \times \mathcal{H} \longrightarrow \mathbb{C} \setminus \{0\}$ and two bounded invertible linear or conjugate linear operators U and V on $\mathcal{H}$ such that $\phi(x \otimes y) = h(x, y)Vx \otimes U^* y$, for all rank one operators $x \otimes y \in \mathcal{B}(\mathcal{H})$.

If $\mathcal{H} = \mathbb{C}^n$ with $n \geq 3$, there exist two functionals α and β on $\mathbb{C}^n \times \mathbb{C}^n$, two nonsingular matrices $U, V \in \mathcal{M}_n(\mathbb{C})$, and two ring automorphisms τ_1 and τ_2 of $\mathbb{C}$ such that

$$\phi(x \otimes y) = \alpha(x,y)Vx_{\tau_1} \otimes y_{\tau_2^*}U^{-1} \text{ and } \psi(x \otimes y) = \beta(x,y)Ux_{\tau_2} \otimes y_{\tau_1^*}V^{-1}$$

for all $x, y \in \mathbb{C}^n$. Thus, there exists two functionals $\alpha', \beta' : \mathcal{H} \longrightarrow \mathbb{C} \setminus \{0\}$ such that

$$K^{-1}(U^{-1})^* K^{-1}x_{\tau_2^*} = \beta'_x Ux_{\tau_2}$$

and

$$KVKx_{\tau_1} = \alpha'_x(V^{-1})^* x_{\tau_1^*}.$$

Thus $\tau_i^* = \tau_i$ and hence τ_i is either the identity or the complex conjugation map on $\mathbb{C}$, $i = 1, 2$. By similar arguments as above, we obtain the desired form of ϕ.

5.2 Multiplicatively pseudo spectrum-preserving maps

In [35], the authors proved that every rank one operator $x \otimes y$ in $\mathcal{B}(\mathcal{H})$ be the algebra of all bounded linear operators on a complex Hilbert space $\mathcal{H}$, $\rho_\varepsilon(x \otimes y)$ coincides with

$$\frac{1}{2}\left(\sqrt{|\langle x, y\rangle|^2 + 4\varepsilon^2 + 4\varepsilon\|x\|\|y\|} + |\langle x, y\rangle|\right).$$

This result was used as a main tool for studying maps ϕ on matrices or operators satisfying $\rho_\varepsilon(\phi(A) \bullet \phi(B)) = \rho_\varepsilon(A \bullet B)$. Here, $A \bullet B$ stands for different kinds of products such as the usual product AB, the triple product ABA, the skew product A^*B, the skew triple product AB^*A; see for instance [4, 45, 46, 33] and the references therein.

In [33] J. Cui, C.K. Li and N. S. Sze determined the structure of all surjective maps $\phi : \mathcal{B}(\mathcal{H}) \longrightarrow \mathcal{B}(\mathcal{H})$ preserving the ε-pseudo spectrum of the skew product (AB^*). This paper belongs to this subject, and investigates the form of all surjective maps $\phi, \psi : \mathcal{A} \longrightarrow \mathcal{B}$ such that, the ε-pseudo spectrum of the product $\phi(A)\psi(B)$ coincides with that of the product AB for all $A, B \in \mathcal{A}$, where $\mathcal{A}$ and $\mathcal{B}$ are two subsets of $\mathcal{B}(\mathcal{H})$ both containing all operators of rank at most one (see Theorem 5.2.1). The obtained result generalizes the main results of [33], [31] and [35] where maps preserving products (AB) and skew-product (AB^*) of matrices or operators are described. Our approach is based on the properties of the ε-pseudo spectrum and our Theorem 5.1.2 in Section 5.1.

5.2.1 Statement of the main results

Theorem 5.2.1. *Two surjective maps $\phi, \psi : \mathcal{A} \longrightarrow \mathcal{B}$ satisfying*

$$\Lambda_\varepsilon(\phi(A)\psi(B)) = \Lambda_\varepsilon(AB) \quad (\forall A, B \in \mathcal{A}) \tag{5.19}$$

if and only if there exist a bounded invertible linear operator U on $\mathcal{H}$ and a unitary operator V on $\mathcal{H}$, such that $\phi(A) = \mu VAU^{-1}$ and $\psi(A) = \nu UAV^*$ for all $A \in \mathcal{A}$, where $\mu, \nu \in \mathbb{C}$, such that $\mu\nu = 1$.

In the following two corollaries, we will apply Theorem 5.2.1 to characterize maps preserving ε-pseudo spectrum of the product AB, and maps preserving the ε-pseudo spectrum of the skew product AB^*. Surjective maps ϕ from $\mathcal{B}(\mathcal{H})$ into itself satisfying

$$\Lambda_\varepsilon(\phi(A)\phi(B)) = \Lambda_\varepsilon(AB), \quad (A, B \in \mathcal{A}) \tag{5.20}$$

were described in [31, Theorem 4.1] and [35, Theorem 3.3]. Such characterizations are now a consequence of Theorem 5.2.1.

Corollary 5.2.2. *A surjective map* $\phi : \mathcal{A} \longrightarrow \mathcal{B}$ *satisfies* (5.20) *if and only if there exist a nonzero scalar* λ *with* $\lambda^2 = 1$ *and a unitary operator* V *on* $\mathcal{H}$ *such that* $\phi(A) = \lambda VAV^*$, *for all* $A \in \mathcal{A}$.

Surjective maps ϕ from $\mathcal{B}(\mathcal{H})$ into itself satisfying

$$\Lambda_\varepsilon(\phi(A)^*\phi(B)) = \Lambda_\varepsilon(A^*B) \quad (\forall A, B \in \mathcal{B}(\mathcal{H})),$$

were described in [33, Corollary 5.4]. Such a characterizations is now a consequence of Theorem 5.2.1.

Corollary 5.2.3. *A surjective map* $\phi : \mathcal{A} \longrightarrow \mathcal{B}$ *satisfies*

$$\Lambda_\varepsilon(\phi(A)^*\phi(B)) = \Lambda_\varepsilon(A^*B), \quad (A, B \in \mathcal{A}) \tag{5.21}$$

if and only if there exist a complex unit scalar λ *and unitary operators* $U, V \in \mathcal{A}$ *such that* $\phi(A) = \lambda VAU$ *for all* $A \in \mathcal{A}$.

5.2.2 Preliminaries

In this section, we collect some useful lemmas needed for the proof of our main results. We also establish some results which are interesting in their own right.

The following lemma is a new and natural observation which consists of characterizing nilpotent rank one operators, see Corollary 5.2.5.

Lemma 5.2.4. *Let* $\mathcal{H}$ *be a complex Hilbert space, and let* x *and* y *be two vectors in* $\mathcal{H}$. *If* $\omega \in \mathbb{C}$, *then*

$$\omega + \frac{\langle x, y\rangle}{2} \in \Lambda_\varepsilon(x \otimes y) \iff -\omega + \frac{\langle x, y\rangle}{2} \in \Lambda_\varepsilon(x \otimes y).$$

Indeed. Let x and y be two vectors in $\mathcal{H}$, and consider the function defined by

$$\begin{aligned} h_{x,y}(z) \quad := \quad & \sqrt{(|z| + |z - \langle x, y\rangle|)^2 + \|x\|^2\|y\|^2 - |\langle x, y\rangle|^2} \\ & -\sqrt{(|z| - |z - \langle x, y\rangle|)^2 + \|x\|^2\|y\|^2 - |\langle x, y\rangle|^2} \end{aligned}$$

for all $z \in \mathbb{C}$. Observe that for every complex number ω, we have

$$h_{x,y}\left(\omega + \frac{\langle x, y\rangle}{2}\right) = h_{x,y}\left(-\omega + \frac{\langle x, y\rangle}{2}\right)$$

and by the characterization of the ε-pseudo spectrum given in Proposition 1.2.22, we conclude the lemma.

Corollary 5.2.5. *Let* $\mathcal{H}$ *be a complex Hilbert space, and let* x *and* y *be two vectors in* $\mathcal{H}$. *Then the following statements are equivalent.*

1) $x \otimes y$ *is a nilpotent operator.*

2) $\Lambda_\varepsilon(x \otimes y)$ *is an closed disc of* $\mathbb{C}$ *centered at zero.*

3)

$$\Lambda_\varepsilon(x \otimes y) = D\left(0, \sqrt{\varepsilon^2 + \varepsilon\|x\|\|y\|}\right).$$

Indeed. Since (3) ⇒ (2) is obvious, we only have to prove that (2) ⇒ (1) ⇒ (3). First, for (1) ⇒ (3), let $R := x \otimes y \in \mathcal{F}_1(\mathcal{H})$, and note that Proposition 1.2.22 entails that

$$\langle x, y\rangle = 0 \Longrightarrow \Lambda_\varepsilon(x \otimes y) = D\left(0, \sqrt{\varepsilon^2 + \varepsilon\|x\|\|y\|}\right).$$

Now, for (2) ⇒ (1), let x and y be two vectors in $\mathcal{H}$ such that $\langle x, y\rangle \neq 0$. By Lemma 5.2.4, we note that $\dfrac{\langle x, y\rangle}{2}$ is a symmetry point of $\Lambda_\varepsilon(x \otimes y)$, since $\Lambda_\varepsilon(x \otimes y)$ is an open disc of $\mathbb{C}$ centered at zero. Hence, $\langle x, y\rangle = 0$ and this contradiction shows that (2) ⇒ (1).

5.2.3 Proof of Theorem 5.2.1.

We only need to prove the "*only if*" part. So, assume that $\phi, \psi : \mathcal{A} \longrightarrow \mathcal{B}$ are surjective maps, such that

$$\Lambda_\varepsilon(\phi(A)\psi(B)) = \Lambda_\varepsilon(AB)$$

for all $A, B \in \mathcal{A}$. This and the statement (1) of Proposition 1.2.21 imply that

$$\phi(A)\psi(B) = 0 \Longleftrightarrow AB = 0$$

for all $A, B \in \mathcal{A}$. Thus we can assume that ϕ and ψ have the forms given in Theorem 5.1.2. We will proceed by checking five assertions.

Assertion 1: If x is a unit vector in $\mathcal{H}$, we have

$$\langle u, v\rangle = 0 \Longleftrightarrow \langle k_x(u), h_x(v)\rangle = 0 \tag{5.22}$$

for all $u, v \in \mathcal{H}$.

Indeed. Let x, u and v be three nonzero vectors in $\mathcal{H}$ such that $\|x\| = 1$. By (5.19), (5.3) and (5.4), we get

$$\begin{aligned}\Lambda_\varepsilon(u \otimes v) &= \Lambda_\varepsilon\left((u \otimes x)(x \otimes v)\right)\\ &= \Lambda_\varepsilon\left(\phi(u \otimes x)\psi(x \otimes v)\right)\\ &= \Lambda_\varepsilon\left(k_x(u) \otimes h_x(v)\right).\end{aligned} \tag{5.23}$$

Assume that $\langle u, v\rangle = 0$, and note that Corollary 5.2.5 entails that $\Lambda_\varepsilon(u\otimes v) = D\left(0, \sqrt{\varepsilon^2+\varepsilon\|u\|\|v\|}\right)$. This together with equation (5.23) imply that

$$\Lambda_\varepsilon\left(k_x(u)\otimes h_x(v)\right) = D\left(0, \sqrt{\varepsilon^2+\varepsilon\|u\|\|v\|}\right).$$

Thus, by Corollary 5.2.5, we have $\langle k_x(u), h_x(v)\rangle = 0$. Hence,

$$\langle u, v\rangle = 0 \Longrightarrow \langle k_x(u), h_x(v)\rangle = 0.$$

By a similar way as in the above discussion, we get

$$\langle u, v\rangle = 0 \Longleftarrow \langle k_x(u), h_x(v)\rangle = 0,$$

and hence

$$\langle u, v\rangle = 0 \Longleftrightarrow \langle k_x(u), h_x(v)\rangle = 0.$$

Thus, the proof of Assertion 1 is complete.

In the following assertion we discuss some properties of the maps

$$\begin{array}{cccc} \underline{h_x}(\cdot) & : & \underline{\mathcal{H}} & \to & \underline{\mathcal{H}} \\ & & \underline{y} & \mapsto & \underline{h_x(y)} \end{array} \quad \text{and} \quad \begin{array}{cccc} \underline{k_x}(\cdot) & : & \underline{\mathcal{H}} & \to & \underline{\mathcal{H}} \\ & & \underline{y} & \mapsto & \underline{k_x(y)} \end{array}$$

for each $x \in \mathcal{H}\setminus\{0\}$.

Assertion 2: For each $x \in \mathcal{H}\setminus\{0\}$, we have that $\underline{h_x}(\cdot)$ and $\underline{k_x}(\cdot)$ are a bijective maps satisfying

$$\langle \underline{u}, \underline{v}\rangle = 0 \iff \left\langle \underline{h_x}\left(\underline{u}\right), \underline{k_x}\left(\underline{v}\right)\right\rangle = 0 \tag{5.24}$$

for all $\underline{u}, \underline{v} \in \underline{\mathcal{H}}$.

Indeed. For a nonzero vector x in $\mathcal{H}$, we show that k_x is surjective. It suffices to show that the following equality

$$\psi\left(\{x\otimes y: \ y\in\mathcal{H}\}\right) = \{Ux_\tau\otimes y: \ y\in\mathcal{H}\} \tag{5.25}$$

holds, where

$$x_\tau = \begin{cases} x_{\tau_1} & \text{if} \quad \mathcal{H} = \mathbb{C}^n \\ x & \text{if} \quad \dim(\mathcal{H}) = \infty. \end{cases}$$

Since the inclusion $\psi\left(\{x\otimes y: \ y\in\mathcal{H}\}\right) \subseteq \{Ux_\tau\otimes y: \ y\in\mathcal{H}\}$ is obvious, we only have to prove the reverse inclusion. For $z\in\mathcal{H}$, since ψ preserves operators of rank at most one in both directions, then there exists $u\otimes v\in\mathcal{F}_1(\mathcal{H})$ such that

$$Ux_\tau\otimes z = \phi(u\otimes v) = Uu_\tau\otimes k_u(v).$$

Then Ux_τ and Uu_τ are linearly dependent, and so $u = \alpha x$ for some nonzero scalar α. Thus, $\phi(x \otimes \overline{\alpha} v) = Ux_\tau \otimes z$, and the equality (5.25) is proved. The surjectivity of k_x immediately follows. By a similar argument as above, one can conclude that $h_x(\cdot)$ is surjective as well.

Now, we show that $\underline{h_x}(\cdot)$ and $\underline{k_x}(\cdot)$ are injective and well defined. Let $\underline{y}, \underline{y'} \in \underline{\mathcal{H}}$, by Assertion 1 and the surjectivity of $k_x(\cdot)$, we have

$$\begin{aligned} \underline{y} = \underline{y'} \quad &\Longleftrightarrow \quad \langle v, y\rangle = 0 \quad \text{if and only if} \quad \langle v, y'\rangle = 0 \quad \text{for all } v \in \mathcal{H} \\ &\Longleftrightarrow \quad \langle k_x(v), h_x(y)\rangle = 0 \quad \text{if and only if} \quad \langle k_x(v), h_x(y')\rangle = 0 \quad \text{for all } v \in \mathcal{H} \\ &\Longleftrightarrow \quad \langle w, h_x(y)\rangle = 0 \quad \text{if and only if} \quad \langle w, h_x(y'))\rangle = 0 \quad \text{for all } w \in \mathcal{H} \\ &\Longleftrightarrow \quad \underline{h_x(y)} = \underline{h_x(y')} \\ &\Longleftrightarrow \quad \underline{h_x}\left(\underline{y}\right) = \underline{h_x}\left(\underline{y'}\right). \end{aligned}$$

Consequently, $\underline{h_x}(\cdot)$ is injective and well defined. By a similar argument, one shows that $\underline{k_x}(\cdot)$ is injective and well defined. Furthermore, since $k_x(\cdot)$ is surjective, it follows that $\underline{k_x}(\cdot)$ must be surjective too. Therefore, $\underline{k_x}(\cdot)$ is bijective. Now, from Assertion 1, we have

$$\begin{aligned} \langle \underline{u}, \underline{v}\rangle = 0 \quad &\Longleftrightarrow \quad \langle u, v\rangle = 0 \\ &\Longleftrightarrow \quad \langle h_x(u), k_x(v)\rangle = 0 \\ &\Longleftrightarrow \quad \langle \underline{h_x(u)}, \underline{k_x(v)}\rangle = 0 \\ &\Longleftrightarrow \quad \langle \underline{h_x}\left(\underline{u}\right), \underline{k_x}\left(\underline{v}\right)\rangle = 0 \end{aligned}$$

for all $u, v \in \mathcal{H}$.

The following assertion is a consequence of [55, Corollary 2.2.2].

Assertion 3. There exist a unitary or conjugate unitary operator W and a functional $d : \mathcal{H} \setminus \{0\} \times \mathcal{H} \setminus \{0\} \to \mathbb{C} \setminus \{0\}, \;\; (x, y) \mapsto d_x(y)$ such that

$$h_x(y) = k_x(y) = d_x(y) W y \quad \text{for all} \quad (x, y) \in \mathcal{H} \setminus \{0\} \times \mathcal{H}.$$

Indeed. We first show that $h_u(x)$ and $k_u(x)$ are linearly dependent for all $x, y, u \in \mathcal{H}$ such that $\|u\| = 1$. Let x, y and u be three nonzero vectors in $\mathcal{H}$ such that $\|u\| = 1$. By equations (5.19), (5.3) and (5.4), we get

$$\begin{aligned} \sigma_\varepsilon(x \otimes x) \quad &= \quad \sigma_\varepsilon((x \otimes u)(u \otimes v)) \\ &= \quad \sigma_\varepsilon(\phi(x \otimes u)\psi(u \otimes x)) \\ &= \quad \sigma_\varepsilon\left(k_u(x) \otimes h_u(x)\right). \end{aligned}$$

By Proposition 1.2.21, we conclude that $k_u(x) \otimes h_u(x)$ is an orthogonal projection. Hence there exists a functional $\lambda_u : \mathcal{H} \longrightarrow \mathbb{R} \setminus \{0\}$ such that $k_u(x) = \lambda_u(x) h_u(x)$.

Now, let x be a nonzero vector in $\mathcal{H}$. From Assertion 2, we know that $\underline{h_x}$ and $\underline{k_x}$ are two bijective transformations on $\underline{\mathcal{H}}$ satisfying (5.24). Then in light of [55, Corollary 2.2.2], we have the following two possibilities:

1. $\dim(\mathcal{H}) = \infty$, and there exists an invertible bounded linear or conjugate-linear operator V_x on $\mathcal{H}$ such that
$$\underline{k_x}\,(\underline{u}) = \underline{h_x}\,(\underline{u}) = \underline{V_x(u)} \text{ for all } u \in \mathcal{H}; \tag{5.26}$$
2. $\mathcal{H} = \mathbb{C}^n$, for a given integer $n \geq 3$, and there exist a nonsingular matrix $V_x \in \mathcal{M}_n(\mathbb{C})$ and a ring automorphism τ_x of $\mathbb{C}$ such that
$$\underline{k_x}\,(\underline{u}) = \underline{h_x}\,(\underline{u}) = \underline{V_x(u_{\tau_x})} \text{ for all } u \in \mathcal{H}. \tag{5.27}$$

Thus, in the case $\mathcal{H} = \mathbb{C}^n$, by (4.24) and (4.25), there exist two maps $d'_x(\cdot), l'_x(\cdot) : \mathcal{H} \longrightarrow \mathbb{C} \setminus \{0\}$, a nonsingular matrix $V_x \in \mathcal{M}_n(\mathbb{C})$ and a ring automorphism τ_x of $\mathbb{C}$ such that
$$h_x(u) = d'_x(u)V_x(u_{\tau_x}) \text{ and } k_x(u) = l'_x(u)V_x(u_{\tau_x}) \tag{5.28}$$
for all $u \in \mathcal{H}$. Now, let us show that τ_x is either the identity or the complex conjugation. Observe that by (5.22) and (4.25) we get
$$\langle u, v\rangle = 0 \iff \langle u_{\tau_x}, v_{\tau_x}\rangle = 0 \tag{5.29}$$
for all nonzero vectors $u, v \in \mathcal{H}$. Choose $u = (1, \frac{1}{\mu}, 0, ..., 0)$ and $v = (1, -\overline{\mu}, 0, ..., 0)$ with $\mu \in \mathbb{C} \setminus \{0\}$. Therefore, by equation (5.29), we get that $\tau_x(\mu) = \overline{\tau_x(\overline{\mu})}$, $\mu \in \mathbb{C} \setminus \{0\}$. Thus, τ_x maps the real numbers into real numbers, and consequently, the restriction of τ_x to $\mathbb{R}$ is the identity map. Clearly, $\tau_x(i) = \pm i$, and thus τ_x is either the identity or the complex conjugation.

Therefore, in the both cases $\dim(\mathcal{H}) = \infty$ and $\dim(\mathcal{H}) < \infty$, there exist two maps $d'_x(\cdot), l'_x(\cdot) : \mathcal{H} \longrightarrow \mathbb{C} \setminus \{0\}$, such that
$$h_x(u) = d'_x(u)W'_x(u) \text{ and } k_x(u) = l'_x(u)W'_x(u) \text{ for all } u \in \mathcal{H}, \tag{5.30}$$
where W'_x is an invertible bounded linear or conjugate-linear operator on $\mathcal{H}$.

Now, we will prove that there exist a nonzero scalar β_x depending only on x, and a unitary or conjugate unitary operator W' independent of x such that $W'_x = \beta_x W$. Indeed, assume by the way of contradiction that $(W'_x)^*W'_y$ is not a scalar operator for some non orthogonal vectors $x, y \in \mathcal{H}$. Then there exists a nonzero vector $v \in \mathcal{H}$ such that v and $(W'_x)^*W'_y v$ are linearly independent. In this case, one can find another nonzero vector $u \in \mathcal{H}$ such that $\langle u, (W'_x)^*W'_y v\rangle \neq 0$ and $\langle u, v\rangle = 0$. Therefore, by Assertion 1, we have
$$\begin{aligned}\langle u, (W'_x)^*W'_y v\rangle &= \langle d'_x(u)W'_x u, l'_y(v)W'_y v\rangle \\ &= \langle h_x(u), k_y(v)\rangle \\ &= 0\end{aligned}$$

which is a contradiction. Then $(W'_x)^*W'_y$ is a scalar operator and hence W'_x and W'_y are linearly dependent for all $x, y \in \mathcal{H}$ with $\langle x, y\rangle \neq 0$. In the other case, if $x, y \in \mathcal{H} \setminus \{0\}$ such that $\langle x, y\rangle = 0$, then there exists $z \in \mathcal{H}$ such that $\langle x, z\rangle \neq 0$ and $\langle y, z\rangle \neq 0$. Then W'_x and W'_z are linearly dependent and W'_y and W'_z are also linearly dependent. Therefore, W'_x and W'_y are linearly dependent for all $x, y \in \mathcal{H} \setminus \{0\}$. Then, we conclude that there exist a unitary or conjugate unitary operator W on $\mathcal{H}$ and a functional $\beta : \mathcal{H} \to \mathbb{C} \setminus \{0\}; x \mapsto \beta_x$ such that

$$W'_x = \beta_x W \text{ for all } x \in \mathcal{H}.$$

Denote $d_x(u) = \beta_x d'_x(u)$ and $l_x(u) = \beta_x l'_x(u)$ for all $x, u \in \mathcal{H}$, we conclude that

$$h_x(u) = d_x(u)W(u) \text{ and } k_x(u) = l'_x(u)W(u)$$

for all $x, u \in \mathcal{H}$. This completes the proof of Assertion 3.

In the rest of this section U denotes the invertible linear or conjugate linear operator on $\mathcal{H}$ given in Theorem 5.1.2, and W is the unitary or conjugate unitary operator on $\mathcal{H}$ given in the previous assertion.

Assertion 4. There are two nonzero scalars $r, s \in \mathbb{C}$ with $rs = 1$ such that

$$\phi(A) = \mu W A U^{-1} \text{ and } \psi(A) = \nu U A W^*$$

for all operators of rank one $A \in \mathcal{F}_1(\mathcal{H})$.

Indeed. Note that by Theorem 5.1.2 and Assertion 3, we know that there exist two functionals $h_1, h_2 : \mathcal{F}_1(\mathcal{H}) \longrightarrow \mathbb{C}$ such that

$$\phi(x \otimes u) = h_1(x \otimes u)Wx \otimes (U^{-1})^* u_{\tau_3} \text{ and } \psi(x \otimes u) = h_2(x \otimes u)Ux_\tau \otimes uW^* \quad (5.31)$$

for all $x \otimes u \in \mathcal{F}_1(\mathcal{H})$, where

$$x_{\tau_3} = \begin{cases} x_{\tau_2} & \text{if} \quad \mathcal{H} = \mathbb{C}^n \\ x & \text{if} \quad \dim(\mathcal{H}) = \infty. \end{cases}$$

for all $x \in \mathcal{H}$. We also denote

$$\eta(z) := \begin{cases} z & \text{if} \quad \dim(\mathcal{H}) = \infty \quad \text{and } U \text{ is linear} \\ \overline{z} & \text{if} \quad \dim(\mathcal{H}) = \infty \quad \text{and } U \text{ is conjugate linear} \\ \tau_1(z) & \text{if} \quad \dim(\mathcal{H}) < \infty. \end{cases} \quad (5.32)$$

for all $z \in \mathbb{C}$. To complete the proof of the assertion, we will proceed in two steps.

Step 1. We have to show that $h_1, h_2 : \mathcal{F}_1(\mathcal{H}) \setminus \{0\} \longrightarrow \mathbb{C} \setminus \{0\}$ are in fact constant. Let $x \otimes u$, $y \otimes v \in \mathcal{F}_1(\mathcal{H}) \setminus \{0\}$ with $\langle y, u\rangle \neq 0$. By equations (5.19) and (5.31), we have

$$\begin{aligned}\rho_\varepsilon(h_1(x \otimes u)h_2(y \otimes v)\eta(\langle y, u\rangle)x \otimes v) &= \rho_\varepsilon(\phi(x \otimes u)\psi(y \otimes v))\\ &= \rho_\varepsilon((x \otimes u)(y \otimes v))\\ &= \rho_\varepsilon(\langle y, u\rangle x \otimes v),\end{aligned}$$

Thus, by Proposition 1.2.22, we have

$$\rho_\varepsilon(\langle y, u\rangle x \otimes v) = \frac{1}{2}\left(\sqrt{|\langle y, u\rangle|^2|\langle x, v\rangle|^2 + 4\varepsilon^2 + 4\varepsilon|\langle y, u\rangle|\|x\|\|v\|} + |\langle y, u\rangle||\langle x, v\rangle|\right)$$

$$\begin{aligned}\rho_\varepsilon(\lambda_{x,y,u,v}(h_1, h_2)x \otimes v) &= \frac{1}{2}\sqrt{|\lambda_{x,y,u,v}(h_1, h_2)|^2|\langle x, v\rangle|^2 + 4\varepsilon^2 + 4\varepsilon|\lambda_{x,y,u,v}(h_1, h_2)|\|x\|\|v\|}\\ &\quad + \frac{1}{2}|\lambda_{x,y,u,v}(h_1, h_2)||\langle x, v\rangle|,\end{aligned}$$

where

$$\lambda_{x,y,u,v}(h_1, h_2) = h_1(x \otimes u)h_2(y \otimes v)\eta(\langle y, u\rangle).$$

Thus, we conclude that

$$\left(\sqrt{|\langle y, u\rangle|^2|\langle x, v\rangle|^2 + 4\varepsilon^2 + 4\varepsilon|\langle y, u\rangle|\|x\|\|v\|} + |\langle y, u\rangle||\langle x, v\rangle|\right)$$

$$=$$

$$\sqrt{|\lambda_{x,y,u,v}(h_1, h_2)|^2|\langle x, v\rangle|^2 + 4\varepsilon^2 + 4\varepsilon|\lambda_{x,y,u,v}(h_1, h_2)|\|x\|\|v\|} + |\lambda_{x,y,u,v}(h_1, h_2)||\langle x, v\rangle|.$$

Therefore

$$|\langle y, u\rangle|^2|\langle x, v\rangle|^2 + 4\varepsilon^2 + 4\varepsilon|\langle y, u\rangle|\|x\|\|v\|$$

$$=$$

$$(\gamma_{x,y,u,v}(h_1, h_2) + |\lambda_{x,y,u,v}(h_1, h_2)||\langle x, v\rangle| - |\langle y, u\rangle||\langle x, v\rangle|)^2$$

$$=$$

$$\begin{aligned}&\gamma^2_{x,y,u,v}(h_1, h_2) + |\lambda_{x,y,u,v}(h_1, h_2)|^2\langle x, v\rangle|^2 + |\langle y, u\rangle|^2|\langle x, v\rangle|^2 - 2\gamma_{x,y,u,v}(h_1, h_2)|\langle y, u\rangle||\langle x, v\rangle|\\ &\quad + 2\gamma_{x,y,u,v}(h_1, h_2)|\lambda_{x,y,u,v}(h_1, h_2)\langle x, v\rangle| - 2|\langle y, u\rangle||\lambda_{x,y,u,v}(h_1, h_2)\langle x, v\rangle|^2,\end{aligned}$$

where

$$\gamma_{x,y,u,v}(h_1, h_2) = \sqrt{|\lambda_{x,y,u,v}(h_1, h_2)|^2|\langle x, v\rangle|^2 + 4\varepsilon^2 + 4\varepsilon|\lambda_{x,y,u,v}(h_1, h_2)|\|x\|\|v\|}.$$

Hence

$$\begin{aligned}4\varepsilon\|x\|\|v\|\left(|\langle y,u\rangle|-|\lambda_{x,y,u,v}(h_1,h_2)|\right) &= 2|\lambda_{x,y,u,v}(h_1,h_2)|^2|\langle x,v\rangle|^2\\ &\quad -2\gamma_{x,y,u,v}(h_1,h_2)|\langle y,u\rangle||\langle x,v\rangle|\\ &\quad +2\gamma_{x,y,u,v}(h_1,h_2)|\lambda_{x,y,u,v}(h_1,h_2)\langle x,v\rangle|\\ &\quad -2|\langle y,u\rangle||\lambda_{x,y,u,v}(h_1,h_2)\langle x,v\rangle|^2.\end{aligned}$$

A straightforward computation shows that

$$|\langle x,v\rangle|\left(\lambda_{x,y,u,v}(h_1,h_2)|\langle x,v\rangle|+\gamma_{x,y,u,v}(h_1,h_2)\right)\left(|\lambda_{x,y,u,v}(h_1,h_2)|-|\langle y,u\rangle|\right)$$

$$=$$

$$2\varepsilon\|x\|\|v\|\left(|\langle y,u\rangle|-|\lambda_{x,y,u,v}(h_1,h_2)|\right).$$

Hence,

$$\left(\frac{|\langle x,v\rangle|}{2\varepsilon\|x\|\|v\|}\left(\lambda_{x,y,u,v}(h_1,h_2)|\langle x,v\rangle|+\gamma_{x,y,u,v}(h_1,h_2)\right)+1\right)\left(|\lambda_{x,y,u,v}(h_1,h_2)|-|\langle y,u\rangle|\right)=0.$$

Since

$$\frac{|\langle x,v\rangle|}{2\varepsilon\|x\|\|v\|}\left(\lambda_{x,y,u,v}(h_1,h_2)|\langle x,v\rangle|+\gamma_{x,y,u,v}(h_1,h_2)\right)+1>0,$$

we obtain

$$|\lambda_{x,y,u,v}(h_1,h_2)|=|\langle y,u\rangle|.$$

Thus $|h_1(x\otimes u)h_2(y\otimes v)|=\frac{|\langle y,u\rangle|}{|\eta(\langle y,u\rangle)|}=1$ if $\dim(\mathcal{H})=\infty$. Now, assume that $\mathcal{H}=\mathbb{C}^n$ for some integer $n\geq 3$, clearly we have

$$|h_1(x\otimes u)h_2(y\otimes v)|=\frac{|\langle y,u\rangle|}{|\eta(\langle y,u\rangle)|}=1 \tag{5.33}$$

for all $x,y,u,v\in\mathbb{C}^n\setminus\{0\}$ with $|\langle y,u\rangle|$ is a positive integer. Next, assume that $x,y,u,v\in\mathbb{C}^n\setminus\{0\}$ and $|\langle y,u\rangle|\neq 1$, we have two cases to discuss:

Case 1. If y and u are linearly independent, then

$$\{z\in\mathcal{H}\ :\ \langle z,u\rangle=1\}\text{ and }\{z\in\mathcal{H}\ :\ \langle z,y\rangle=1\}$$

are two non-parallel affine hyperplanes of $\mathbb{C}^n$. So

$$\{z\in\mathcal{H}\ :\ \langle z,u\rangle=1\}\cap\{z\in\mathcal{H}\ :\ \langle z,y\rangle=1\}$$

is a nonempty affine hyperplane, hence there exists $z\in\mathcal{H}$ such that $\langle z,z\rangle$ is a positive integer and

$$\langle z,u\rangle=1\text{ and }\langle z,y\rangle=1.$$

Therefore, we have

$$|h_1(x\otimes u)h_2(z\otimes v)| = 1 \text{ and } |h_1(x\otimes z)h_2(y\otimes v)| = 1.$$

Then

$$|h_1(x\otimes u)h_2(z\otimes v)||h_1(x\otimes z)h_2(y\otimes v)| = 1 \text{ and } |h_1(x\otimes z)h_2(z\otimes v)| = 1,$$

hence $|h_1(x\otimes u)h_2(y\otimes v)| = 1$.

Case 2. If y and u are linearly dependent. Choose $z\in\mathcal{H}$ such $\langle z,z\rangle$ is a positive integer, $\langle z,y\rangle\langle z,u\rangle\neq 0$ and u,y,z are linearly independent. By the previous case we conclude that

$$|h_1(x\otimes u)h_2(z\otimes v)||h_1(x\otimes z)h_2(y\otimes v)| = 1 \text{ and } |h_1(x\otimes z)h_2(z\otimes v)| = 1,$$

hence $|h_1(x\otimes u)h_2(y\otimes v)| = 1$.

We conclude that if $\mathcal{H}$ is a finite or infinite dimensional Hilbert space, we have

$$|h_1(x\otimes u)h_2(y\otimes v)| = 1$$

for all $x,y,u,v\in\mathcal{H}\setminus\{0\}$ with $\langle y,u\rangle\neq 0$. Finally, if $x,y,u,v\in\mathcal{H}\setminus\{0\}$ with $\langle y,u\rangle = 0$, one can choose $z\in\mathbb{C}^n$ such that $\langle z,u\rangle\langle z,y\rangle\neq 0$, then

$$|h_1(x\otimes u)h_2(z\otimes v)| = 1\ ,\ |h_1(x\otimes z)h_2(y\otimes v)| = 1 \text{ and } |h_1(x\otimes z)h_2(z\otimes v)| = 1.$$

This implies that

$$|h_1(x\otimes u)h_2(y\otimes v)| = 1$$

for all $x,y,u,v\in\mathcal{H}\setminus\{0\}$. Consequently, there exist two nonzero scalars μ and ν such that $|\mu\nu| = 1$ and

$$h_1(x\otimes u) = \mu \text{ and } h_2(x\otimes u) = \nu \tag{5.34}$$

for all nonzero $x\otimes u\in\mathcal{F}_1(\mathcal{H})$.

Step 2. The operators U and W are linear and $\mu\nu = 1$. Moreover, if $\dim(\mathcal{H}) < \infty$ then τ_1 is the identity of $\mathbb{C}$.

The equalities (5.34) together with (5.19) and (5.31) tells us that

$$\Lambda_\varepsilon(\mu\nu\eta(\langle y,u\rangle)x\otimes v) = \Lambda_\varepsilon(\langle y,u\rangle x\otimes v) \tag{5.35}$$

for all nonzero $x,v\in\mathcal{H}$. Therefore, if $x\in\mathcal{H}$ is a unit vector, then by Proposition

1.2.21, we have

$$\begin{aligned} D(0,\varepsilon)\cup D(\alpha,\varepsilon) &= \Lambda_\varepsilon(\alpha x\otimes x)\\ &= \Lambda_\varepsilon((\alpha x\otimes x)(x\otimes x))\\ &= \Lambda_\varepsilon(\phi(\alpha x\otimes x)\psi(x\otimes x))\\ &= \Lambda_\varepsilon\left((\mu W\alpha x\otimes (U^{-1})^* x_{\tau_3})(\nu U x_\tau\otimes xW^*)\right)\\ &= \Lambda_\varepsilon\left((W\mu^W\alpha x\otimes (U^{-1})^* x_{\tau_3})(\nu U x_\tau\otimes xW^*)\right)\\ &= \Lambda_\varepsilon\left(\mu^W\nu\eta(\langle x,x\rangle)\alpha x\otimes x\right)^W\\ &= \Lambda_\varepsilon\left(\mu^W\nu\alpha x\otimes x\right)^W\\ &= \left(D(0,\varepsilon)\cup D(\mu^W\nu\alpha,\varepsilon)\right)^W\\ &= D(0,\varepsilon)\cup D(\mu(\nu\alpha)^W,\varepsilon) \end{aligned}$$

for all $\alpha\in\mathbb{C}\setminus\{0\}$, where $\lambda^W=\lambda$ (resp. $\overline{\lambda}$) if W is unitary (resp. conjugate unitary) for all scalars λ. This implies that $\mu(\nu\alpha)^W=\alpha$ for all $\alpha\in\mathbb{C}\setminus\{0\}$. For $\alpha=1$ we obtain $\mu\nu^W=1$, hence $\alpha^W=\alpha$ for all $\alpha\in\mathbb{C}\setminus\{0\}$. Thus W is an unitary operator and $\mu\nu=1$.

Next, if $x\in\mathcal{H}$ is a unit vector, then by Proposition 1.2.21, we have

$$\begin{aligned} D(0,\varepsilon)\cup D(\alpha,\varepsilon) &= \Lambda_\varepsilon(\alpha x\otimes x)\\ &= \Lambda_\varepsilon((x\otimes\overline{\alpha}x)(x\otimes x))\\ &= \Lambda_\varepsilon(\phi(x\otimes\overline{\alpha}x)\psi(x\otimes x))\\ &= \Lambda_\varepsilon\left((\mu W x\otimes (U^{-1})^*(\overline{\alpha}x)_{\tau_3})(\nu U x_\tau\otimes xW^*)\right)\\ &= \Lambda_\varepsilon\left(\eta(\alpha)x\otimes x\right)\\ &= D(0,\varepsilon)\cup D(\eta(\alpha),\varepsilon) \end{aligned}$$

for all $\alpha\in\mathbb{C}\setminus\{0\}$. Hence $\eta(\alpha)=\alpha$ for all $\alpha\in\mathbb{C}$ and so U is a bounded linear operator. Thus, we obtain the desired forms of ϕ and ψ and the proof of Assertion 4 is complete.

Keeping in mind the notations of the previous assertion, the following assertion shows that ϕ and ψ have the desired forms on $\mathcal{A}$.

Assertion 5. We have $\phi(A)=\mu WAU^{-1}$ and $\psi(A)=\nu UAW^*$ for all $A\in\mathcal{A}$.

Indeed. Let A be an arbitrary nonzero operator in $\mathcal{A}$. For any unit vector $x\in\mathcal{H}$, we have

$$\begin{aligned} \Lambda_\varepsilon\left(\nu\phi(A)Ux\otimes WAx\right) &= \Lambda_\varepsilon\left(\phi(A)\psi(x\otimes Ax)\right)\\ &= \Lambda_\varepsilon\left(A(x\otimes Ax)\right)\\ &= D(0,\varepsilon)\cup D(\|Ax\|^2,\varepsilon). \end{aligned}$$

If $Ax \neq 0$, then from Proposition 1.2.21 we obtain that $\nu\phi(A)Ux \otimes WAx = \|Ax\|^2 w \otimes w$ for a unit vector $w \in \mathcal{H}$. Therefore, $w = \frac{1}{\|WAx\|}WAx$ and we have

$$\begin{aligned} \nu\phi(A)Ux \otimes WAx &= \|Ax\|^2 \left(\frac{WAx}{\|WAx\|}\right) \otimes \left(\frac{WAx}{\|WAx\|}\right) \\ &= \|WAx\|^2 \left(\frac{WAx}{\|WAx\|}\right) \otimes \left(\frac{WAx}{\|WAx\|}\right) \\ &= WAx \otimes WAx. \end{aligned}$$

Hence

$$\nu\phi(A)Ux = WAx$$

for all unit vectors $x \in \mathcal{H}$ with $Ax \neq 0$. Thus

$$\phi(A)y = \frac{1}{\nu}WAU^{-1}y \tag{5.36}$$

for all vectors $y \in \mathcal{H}$ such that $AU^{-1}y \neq 0$. Now if $AU^{-1}y = 0$ for some y one can choose any vector y_0 such that $AU^{-1}y_0 \neq 0$, then (5.36) holds for y_0 and $y - y_0$. Consequently,

$$\phi(A) = \frac{1}{\nu}WAU^{-1} = \mu WAU^{-1}$$

for all $A \in \mathcal{A}$.

Similarly, we have

$$\begin{aligned} \Lambda_\varepsilon\left(\mu WA^*x \otimes xU^{-1}\psi(A)\right) &= \Lambda_\varepsilon\left(\phi(A^*x \otimes x)\psi(A)\right) \\ &= \Lambda_\varepsilon\left(A^*x \otimes A^*x\right)) \\ &= D(0,\varepsilon) \cup D(\|A^*x\|^2, \varepsilon) \end{aligned}$$

for all unit vectors $x \in \mathcal{H}$ and $A \in \mathcal{A}$. Thus,

$$\mu WA^*x \otimes xU^{-1}\psi(A) = \|WAx\|^2 \left(\frac{WA^*x}{\|WA^*x\|}\right) \otimes \left(\frac{WA^*x}{\|WA^*x\|}\right)$$

Hence

$$\overline{\mu}\psi(A)^*(U^*)^{-1}x = WA^*x$$

for all unit vectors $x \in \mathcal{H}$ and $A \in \mathcal{A}$. Thus

$$\overline{\mu}\psi(A)^*(U^*)^{-1} = WA^*$$

for all $A \in \mathcal{A}$. Therefore

$$\psi(A) = \frac{1}{\mu}UAW^* = \nu UAW^*$$

for all $A \in \mathcal{A}$.

This completes the proof of Theorem 5.2.1.

5.2.4 Proof of Corollary 5.2.2.

Applying Theorem 5.2.1 for ϕ and $\psi = \phi$, we conclude that there exist a bounded invertible linear operator U on $\mathcal{H}$ and a unitary operator V on $\mathcal{H}$, such that

$$\mu VAU^{-1} = \phi(A) = \psi(A) = \nu UAV^*$$

for all $A \in \mathcal{A}$, where $\mu, \nu \in \mathbb{C}$, such that $\mu\nu = 1$. Therefore, for every $x \otimes u \in \mathcal{F}_1(\mathcal{H})$, we have $\mu Vx \otimes (U^{-1})^* u = \nu Ux \otimes Vu$, then there exists $\beta \in \mathbb{C}$ such that $Ux = \beta Vx$ for all $x \in \mathcal{H}$. Thus, $U = \beta V$, so, $\phi(A) = \lambda VAV^*$, with $\lambda = \nu\beta$. Now, we show that $\lambda^2 = 1$. By equation (5.20) and Proposition 1.2.21, for a unit vector $x \in \mathcal{H}$, we have

$$\begin{aligned}
D(0,\varepsilon) \cup D(\lambda^2,\varepsilon) &= \Lambda_\varepsilon(\lambda^2 x \otimes x) \\
&= \Lambda_\varepsilon(\lambda^2 (x \otimes x)(x \otimes x)) \\
&= \Lambda_\varepsilon(\lambda Vx \otimes xV^* \lambda Vx \otimes xV^*) \\
&= \Lambda_\varepsilon(\phi(x \otimes x)\phi(x \otimes x)) \\
&= \Lambda_\varepsilon((x \otimes x)(x \otimes x)) \\
&= \Lambda_\varepsilon(x \otimes x) \\
&= D(0,\varepsilon) \cup D(1,\varepsilon),
\end{aligned}$$

and then $\lambda^2 = 1$.

5.2.5 Proof of Corollary 5.2.3

Applying Theorem 5.2.1 for $\psi(A) := (\phi(A^*))^*$ $(A \in \mathcal{A})$, clearly, ϕ and ψ satisfy (5.19). Then there exist a bounded invertible linear operator U on $\mathcal{H}$ and a unitary operator V on $\mathcal{H}$, such that $\phi(A) = \mu VAU^{-1}$ and $\psi(A) = \nu UAV^*$ for all $A \in \mathcal{A}$, where $\mu, \nu \in \mathbb{C}$, such that $\mu\nu = 1$. Thus

$$\overline{\mu}(U^{-1})^* AV^* = (\phi(A^*))^* = \psi(A) = \nu UAV^* \tag{5.37}$$

for every $A \in \mathcal{A}$. For $x \in \mathcal{H}$, we choose $A = x \otimes xV$, then, equation (5.37) implies that $\overline{\mu}(U^{-1})^* x \otimes x = \nu Ux \otimes x$, hence $\left(\overline{\mu}(U^{-1})^* - \nu U\right) x = 0$, for all $x \in \mathcal{H}$. Thus, U is a λ multiple of a unitary operator W, where λ is a complex unit scalar. Therefore, $\phi(A) = \lambda WAV^*$ for all $A \in \mathcal{A}$.

Conclusion

The aim of our work is to study maps ϕ on matrices or operators satisfying

$$F(\phi(A) \bullet \phi(B)) = F(A \bullet B).$$

Here, $F(\cdot)$ is a spectral function or a spectral set such as the pseudo spectral radius, the pseudo spectrum, the condition spectral radius and the condition spectrum. On the other hand, $A \bullet B$ stands for different kinds of products such as the usual product AB, the skew product A^*B and the skew triple product AB^*A; see [4, 5, 6] and the references therein.

In detail, consider $\mathcal{H}$ a complex Hilbert space with dimension of $\mathcal{H}$ is greater than 2 and $\mathcal{B}(\mathcal{H})$ be the algebra of all bounded linear operators on $\mathcal{H}$, and $\mathcal{A}$, $\mathcal{B}$ be two subsets of $\mathcal{B}(\mathcal{H})$ both containing all operators of rank at most one, and $\mathcal{A}_I$, $\mathcal{B}_I$ be two subsets of $\mathcal{B}(\mathcal{H})$ both containing all operators of the form $R + \lambda I$, where R is an operator in $\mathcal{B}(\mathcal{H})$ of rank at most one and $\lambda \in \mathbb{C}$. Let $A \in \mathcal{B}(\mathcal{H})$ and $\varepsilon \in (0, +\infty)$, $\varepsilon' \in (0,1)$ be two given fixed real numbers, denote by $\Lambda_\varepsilon(A)$, $\rho_\varepsilon(A)$, $\sigma_{\varepsilon'}(A)$ and $r_{\varepsilon'}(A)$ the ε-pseudo spectrum of A, the ε-pseudo spectral radius of A, the ε'-condition spectrum of A and the ε'-condition spectral radius of A respectively.

In chapter 1, we started by exhibing some spectral sets of an operator and their properties. Then, we gave the definitions of the ε-pseudo spectrum, the ε-pseudo spectral radius, the ε'-condition spectrum and the ε'-condition spectral radius. Finally, we gave complete descriptions of some special classes of matrices or operators in terms of the ε-pseudo spectrum, the ε-pseudo spectral radius, the ε'-condition spectrum or the ε'-condition spectral radius. For example, an operator is equal to $e^{it}A$ for a self-adjoint operator A and $t \in \mathbb{R}$ if and only if its pseudo spectrum lies in the set $\{z \in \mathbb{C} : |\operatorname{Im}(e^{it}z)| < \varepsilon\}$. In particular, an operator $A = \alpha I$ for some scalar α if and only if $\Lambda_\varepsilon(A) = D(\alpha, \varepsilon)$, an operator A is a nontrivial projection if and only if $\Lambda_\varepsilon(A) = D(0, \varepsilon) \cup D(1, \varepsilon)$. Furthermore, if an operator A satisfies $A^2 = 0$, then $\Lambda_\varepsilon(A) = D\left(0, \sqrt{\varepsilon^2 + \varepsilon\|A\|}\right)$.

Then, for $A \in \mathcal{B}(\mathcal{H})$ is a self-adjoint operator, then

$$\sigma_{\varepsilon'}(A) = \bigcup_{\alpha,\beta \in \sigma(A)} D\left(\frac{\alpha - \beta\varepsilon'^2}{1 - \varepsilon'^2}\ ,\ \frac{\varepsilon'|\alpha - \beta|}{1 - \varepsilon'^2}\right)$$

and

$$r_{\varepsilon'}(A) = \max\left\{\frac{|i(A) - \varepsilon' s(A)|}{1-\varepsilon'}, \frac{|s(A) - \varepsilon' i(A)|}{1-\varepsilon'}\right\}$$

where $i(A)$ and $s(A)$ denote the infimum and the supremum of $\sigma(A)$, respectively. In particular, if A is a nontrivial self-adjoint projection, then

$$\sigma_{\varepsilon'}(A) = D\left(\frac{1}{1-\varepsilon'^2}\;,\;\frac{\varepsilon'}{1-\varepsilon'^2}\right)\bigcup D\left(\frac{-\varepsilon'^2}{1-\varepsilon^2}\;,\;\frac{\varepsilon'}{1-\varepsilon'^2}\right)$$

and

$$r_{\varepsilon'}(P) = \frac{1}{1-\varepsilon'}.$$

Let $R \in \mathcal{B}(\mathcal{H})$ be a rank one nilpotent operator, then

$$\sigma_{\varepsilon'}(R) = D\left(0, \frac{\sqrt{\varepsilon'}\|R\|}{1-\varepsilon'}\right),$$

and

$$r_{\varepsilon'}(R) = \frac{\sqrt{\varepsilon'}\|R\|}{1-\varepsilon'}.$$

In chapter 2, we presented the fundamental theorems of preserver problems of the ε-pseudo spectral radius, the ε-pseudo spectrum, the ε'-condition spectral radius, or the ε'-condition spectrum; see [4, 5, 6, 45, 46, 47, 35, 31, 32, 33, 42, 43, 41] and the references therein.

In chapter 3, we characterized the surjective maps ϕ from $\mathcal{A}$ to $\mathcal{B}$ preserving the ε-pseudo spectral radius of the skew triple product of operators in the sense that

$$\rho_\varepsilon(\phi(A)\phi(B)^*\phi(A)) = \rho_\varepsilon(AB^*A) \text{ for all } A, B \in \mathcal{A}.$$

Also, we described the surjective applications ϕ from $\mathcal{A}$ to $\mathcal{B}$ which preserve the ε-pseudo spectrum of the skew triple product of operators in the sense that

$$\Lambda_\varepsilon(\phi(A)\phi(B)^*\phi(A)) = \Lambda_\varepsilon(AB^*A) \text{ for all } A, B \in \mathcal{A}_I.$$

In chapter 4, we provided the expression of the ε'-condition spectrum of any operator of rank one, and give an explicit formula for its ε'-condition spectral radius. Then, we gave complete descriptions of some special classes of matrices or operators in terms of the ε'-condition spectrum or the ε'-condition spectral radius. It is then illustrated that the results can be applied to characterize surjective mappings $\phi : \mathcal{A} \longrightarrow \mathcal{B}$ satisfying

$$\delta(\phi(A) \bullet \phi(B)) = \delta(A \bullet B) \text{ for all } A, B \in \mathcal{A},$$

where δ stands for $\sigma_{\varepsilon'}(\cdot)$ or $r_{\varepsilon'}(\cdot)$; see [5] and the reference therein.

In chapter 5, we determined all pairs of surjective maps $\phi, \psi : \mathcal{A} \longrightarrow \mathcal{B}$ satisfying

$$\phi(A)\psi(B) = 0 \quad \text{if and only if} \quad AB = 0 \quad \text{for all} \quad A, B \in \mathcal{A},$$

where $\mathcal{A}$ and $\mathcal{B}$ are two subsets of the algebra of all bounded linear operators on a complex Banach space X, containing all operators of rank at most one.

Moreover, we provided a version of this result in the Hilbert space context. This version will be used to describe all pairs of surjective maps ϕ and ψ defined on $\mathcal{A}$ with values in $\mathcal{B}$ such that the ε-pseudo spectrum of $\phi(A)\psi(B)$ coincides with that of AB for all A and B in $\mathcal{A}$; see [4, 6, 35, 19, 31, 32, 33, 34] and the references therein.

Finally, we are confident that our results will serve as a basis for future studies of nonlinear preserver problems and should be extended by researchers in this field. For this reason, we are currently in the process of investigating different ideas based on the results of this thesis, and to further our research we intend to generalize the results developed in this work for the algebra of all bounded linear operators on a complex Hilbert space to the case of every algebra of all bounded linear operators on a complex Banach space.

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Abstract

In this work in Operator theory, we are interested in spectral approximation nonlinear preserver problems. Consider H a complex Hilbert space with dimension of H is greater than 2 and B(H) be the algebra of all bounded linear operators on H and A, B be two subsets of B(H) both containing all operators of rank at most one. Let T∈ B(H) and ε ∈ (0,+∞), ε' ∈ (0,1) be two given fixed real numbers, denote by Λ_ε(T), ρ_ε(T), $\sigma_{\varepsilon'}$(T) and $r_{\varepsilon'}$(T) the ε-pseudo spectrum of T, the ε-pseudo spectral radius of T, the ε'-condition spectrum of T and the ε'-condition spectral radius of T respectively. In this thesis we first characterize the surjective maps φ from A to B satisfying δ(φ(T)•φ(S))= δ(T•S) for all T,S ∈ A, where δ stands for Λ_ε(T) or ρ_ε(T) and T•S=TS*T, or δ stands for $\sigma_{\varepsilon'}(\cdot)$ or $r_{\varepsilon'}(\cdot)$ and T•S=TS*. Second, we determined all pairs of surjective maps φ and ψ from A to B satisfying ”φ(T)ψ(S) = 0 si est seulement si TS = 0 for all T,S ∈ A”, where A and B are two subsets of the algebra of all bounded linear operators on a complex Banach space X, containing all operators of rank at most one. Moreover, we provided a version of this result in the Hilbert space context. This version will be used to describe all pairs of surjective maps φ and ψ from A to B such that Λ_ε (φ(T)ψ(S)) = Λ_ε (TS) for all T,S ∈ A.

Keywords : Pseudo spectrum; Pseudo spectral radius; Condition spectrum; Condition spectral radius; Nonlinear preserver.

Résumé

Dans ce travail en théorie des opérateurs, on s’intéresse aux problèmes de préservation d’approximation spectrale. On considère H un espace de Hilbert complexe de dimension supérieure ou égale à 3 et B(H) l’algèbre des opérateurs bornés sur H, et soient A et B deux sous-ensembles de B(H) contenant tous les opérateurs de rang au plus un. Soient T ∈ B(H) et ε ∈ (0,+∞), ε' ∈ (0,1) deux réels fixes donnés, notons Λ_ε(T), ρ_ε(T), $\sigma_{\varepsilon'}$(T), et $r_{\varepsilon'}$(T) le ε-pseudo spectre de T, le rayon ε-pseudo spectral de T, le spectre de ε’-condition de T et le rayon spectral de ε’-condition de T respectivement. Dans cette thèse, nous caractérisons en premier lieu les applications surjectives φ de A vers B qui satisfont δ(φ(T)•φ(S))= δ(T•S) pour tout T,S ∈ A, où δ représente Λ_ε(T) ou ρ_ε(T) et T•S=TS*T, ou δ représente $\sigma_{\varepsilon'}$(T) ou $r_{\varepsilon'}$(T) et T•S=TS*. Second, nous déterminons les applications surjectives φ et ψ définies sur deux sous-ensembles A et B, de l’algèbre des opérateurs linéaires bornés sur un espace de Banach complexe X contenant tous les opérateurs de rang égal à un pour lesquels l’équivalence suivante est vérifiée ”φ(T)ψ(S) = 0 si est seulement si TS = 0 pour tout T,S ∈ A”. De plus, dans le cas X est un espace de Hilbert, nous caractérisons les applications surjectives φ, ψ de A vers B qui satisfont Λ_ε(φ(T)ψ(S)) = Λ_ε(TS) pour tout T,S ∈ A.

Mots Clés : Pseudo spectre; Rayon pseudo spectral; Spectre de condition; Rayon spectral de condition; Conservation non linéaire.

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