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Concrete spectral analysis of twisted Laplacians on some classical and mixed automorphic functions on $\mathbb{C}^n$ and applications.

JURY

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To the memory of my dears whom I've lost during the last couple of years
– My uncles Mohammed and Abderrahim. – My cousins Nassim and Mohammed.
– My grand-father Ahmed and my aunt Fatima of "Thasyath"-Temsamane.
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A. El Fardi

IN MEMORIAM PROFESSOR AHMED INTISSAR

This page is dedicated to the memory of my late second supervisor, Professor Ahmed Intissar, who was one of the very distinguished mathematicians in Morocco. My first contact with Professor Intissar was in my third year of Bachelor (2012) when I attended his lectures on complex analysis. In the same year, he was a member of the committee that examined my Bachelor thesis. Short time foreword, in my first year of Master, I had attended his lectures on differential geometry. I have always been fascinated by his way of explaining mathematics and by his philosophy pro concrete mathematics up to now. After I finished my Master, he accepted to supervise me together with my first supervisor professor Allal Ghanmi who was once under his supervision back when he prepared his Ph.D. in 2005 ([30]). Until few months before he left us, professor Intissar made sure, despite his poor health condition, to show up in nearly every session of the weekly seminar of our research team. Him passing away in July 26, 2017 was grieved by all who have known him personally or from academia.

Born on January 1, 1951 in Ouled-Said-Settat (Morocco), father of three children, Prof. Ahmed Intissar had a bachelor from Faculty of sciences in Rabat, Mohammed V University. He post graduated from University of Nice Sophia Antipolis (France). In 1984, he got his Ph.D. from Massachusetts Institute of Technology (MIT)-(Boston).

Professor Ahmed Intissar had first interested in Harmonic Analysis and Scattering Theory, then a while after he came back to Morocco, he shifted his interest toward automorphic functions and some related topics. He supervised a number of Ph.D. students who become later brilliant professors in different national universities. He had a sincerity that attracted people. His approach to any new phenomenon was notably creative and deep. His impact on scattering theory and Harmonic Analysis will be remembered with admiration. In Marrakech's congress (2014), he received with Professor Takeshi Kawazoe a tribute from all the Moroccan scientific community for their mathematical contributions.

Some selected publications of professor Intissar can be found in the memoriam written by his brother Abdelkader Intissar, professor at Corsica University, Pascal-Paoli (France), *In memory of Professor Ahmed INTISSAR A selection of the publications of his work*[1] and from which I borrowed some lines in this page.

[1] https://www.researchgate.net/publication/319643987_In_memory_of_Professor_Ahmed_INTISSAR_A_selection_of_the_publications_of_his_work

ABSTRACT

We provide in this thesis some generalizations of classical automorphic functions on $\mathbb{C}^n$ with weight ν. We parametrize the automorphic factor in two ways, the first one is by adding a second weight μ into the automorphic factor. These are called mixed automorphic functions of first kind. The second parametrization is given by multiplying the standard factor with its copy transported using an equivariant pair (ρ, τ). These functions are called mixed automorphic functions of second kind. On the spaces of mixed automorphic functions of each kind, obeying a suitable growth condition, we realize the appropriate invariant Laplacians. We show that these spaces are isospectral where the spectrum reduces to discrete eigenvalues. By doing so, we provide a spectral decomposition and construct an orthogonal Hilbert basis for these spaces of mixed automorphic functions.

In the last part of this thesis, we investigate some questions concerning the construction and applications of automorphic functions. More precisely, we study the kernel of Poincaré theta series operator. Last but not least, we study the reproducing kernel function K^{ν}_{Γ} of the space of holomorphic automorphic functions, which happens to be the Poincaré periodization of Fock reproducing kernel function. Thereby, we derive interesting arithmetical identities that appear in chemistry and physics. This provides a novel application of automorphic functions in mathematics as well as in other scientific fields.

Keywords: Automorphic functions; Mixed automorphic functions; Spectral theory; Magnetic Laplacians; Poincaré Theta operator; Arithmetical identities; Lattice Sums.

RÉSUMÉ

Nous proposons dans cette thèse des généralisations des fonctions automorphes classiques sur $\mathbb{C}^n$ à poids ν. Nous avons paramétré le facteur d'automorphie de deux façons, la première était par ajouter un deuxième poids μ dans le facteur d'automorphie. Nous appelons les fonctions résultantes les fonctions automorphes "mixed" de première espèce. La deuxième paramétrisation est faite par multiplier le facteur d'automorphie usuel par sa copie transportée selon un pair équivariant (ρ, τ). Nous appelons ces fonctions les fonctions automorphes "mixed" de seconde espèce. Sur les espaces des fonctions automorphes "mixed" de chaque espèce, obéissant à une condition de croissance, nous réalisons les Laplaciens invariants convenables. Nous montrons par la suite que ces espaces sont isospectraux où le spectre se réduit à des valeurs propres discrètes. Par conséquent, nous fournissons une décomposition spectrale et construisons une base d'Hilbert orthogonale pour ces espaces.

Dans la dernière partie de cette thèse, nous étudions quelques questions concernant la construction et les applications des fonctions automorphes. Plus précisément, nous étudions le noyau de l'opérateur thêta de Poincaré. Enfin, nous traitons la fonction noyau reproduisant K^{ν}_{Γ} de l'espace des fonctions automorphes holomorphes, qui est exactement la périodisation à la Poincaré de la fonction noyau reproduisant de l'espace de Fock. Par conséquent, nous trouvons des identités arithmétiques intéressantes qui apparaissent en chimie et en physique. Cela nous fournit une nouvelle application des fonctions automorphes en mathématiques ainsi qu'en d'autres domaines scientifiques.

Mots Clés : Fonctions automorphes ; Fonctions automorphes "mixed" ; Théorie spectrale ; Laplaciens magnétiques ; Opérateur thêta de Poincaré ; Identités arithmétiques ; Sommation sur les réseaux.

RÉSUMÉ DE LA THÈSE

L'objet principal de ce travail et d'étudier concrètement différents aspects de la théorie spectrale des fonctions automorphes standards et certaines de ces généralisations dites à bi-poids de premier et deuxième espèce. Dans ce manuscrit, on définit les fonctions automorphes standards de poids $\nu > 0$ sur $\mathbb{C}^n$ comme étant des fonctions fonctions propres d'un Laplacien satisfaisant à l'équation fonctionnelle

$$f(z+\gamma) = \chi(\gamma) e^{i\nu \Im \langle z,\gamma \rangle} f(z), \tag{0.1}$$

pour tout $\gamma \in \Gamma$, pour la donnée d'un sous–groupe discret Γ de $(\mathbb{C}^n, +)$ et une application $\chi : \Gamma \to \mathsf{U}(1)$ vérifiant

$$\chi(\gamma\gamma') = \chi(\gamma)\chi(\gamma')\, \mathrm{e}^{-\nu i \Im \left\langle \gamma' \cdot 0, \gamma^{-1} \right\rangle}$$

pour tout $\gamma, \gamma' \in \Gamma$. La définition proposée s'étend naturellement au cadre des orbifolds, i.e. $\Gamma = \mathsf{U}(n) \ltimes \mathbb{C}^n$, agissant sur $\mathbb{C}^n$ par les translations et les rotations, en considérant l'équation fonctionnelle

$$f(z \cdot \gamma) = \chi(\gamma) e^{-i\nu \Im \left\langle z, \gamma^{-1} \cdot 0 \right\rangle} f(z), \quad \forall \gamma \in \Gamma.$$

Les propriétés basiques des fonctions automorphes standards sont décrites au Chapitre 2, après avoir donner au Chapitre 1 un aperçu générale sur les sous groupes discrets Γ de $\mathsf{U}(n) \ltimes \mathbb{C}^n$, et plus généralement, le groupe des isométries euclidiennes. Au Chapitre 3, on introduit la classe des fonctions automorphes à bi-poids (ν, μ) définies comme étant les fonctions satisfaisant

$$F(\gamma \cdot z) = e^{-i\nu \Im m \left\langle z, \gamma^{-1} \cdot z \right\rangle - i\mu \Re e \left\langle z, \gamma^{-1} \cdot z \right\rangle} F(z), \tag{0.2}$$

où μ est nouveau paramètre réel, généralisant ainsi fonctions automorphes standards qui correspondent à $\mu = 0$. On les nomme les fonctions automorphes "mixed" de première espèce. Nous étudions ensuite la théorie spectrale de l'opérateur différentiel

$$\Delta_{\nu,\mu} := \sum_{j=1}^{n} \left\{ 4 \frac{\partial^2}{\partial z_j \partial \overline{z_j}} + 2(\nu + i\mu) z_j \frac{\partial}{\partial z_j} - 2(\nu - i\mu) \overline{z_j} \frac{\partial}{\partial \overline{z_j}} - (\nu^2 + \mu^2)|z_j|^2 + 2i\mu \right\} =: \Delta^{\Gamma}_{\nu,\mu},$$

laissant invariant l'espace des fonctions considérées. Notre Laplacien $\Delta^{\Gamma}_{\nu,\mu} = \Delta_{\nu,\mu}$ se laisse réaliser comme étant la transformée de Fourier partielle du sous-Laplacian associé à un groupe de type Heisenberg $\mathbb{C} \times_\omega \mathbb{C}^n$ obtenu comme une extension centrale du groupe de Heisenberg réel $H_{2n+1} = (\mathbb{R} \times \mathbb{C}^n, \cdot_{\Im m \omega})$. Il se réalise aussi comme un Laplacien de Bochner sur les sections lisses d'un fibré en droites hermitien avec connexion au dessus de $M = \mathbb{C}^n$, ou encore, comme un opérateur de Schrödinger avec un champ magnétique constant sur la variété riemannienne complète orientée $\mathbb{C}^n$ et associée à une phase spéciale de Berry.

On étudie dans un premier temps le spectre de $\Delta_{\nu,\mu}$ lorsqu'il agit sur l'espace de Frechet $\mathcal{C}^\infty(\mathbb{C}^n)$ des fonctions $\mathcal{C}^\infty$ à valeurs complexes sur $\mathbb{C}^n$, ainsi que lorsqu'il agit sur l'espace de Hilbert libre des fonctions à valeurs complexes de carrés intégrables sur $\mathbb{C}^n$. Ensuite, on détermine le spectre de $\Delta_{\nu,\mu}$ lorsqu'il agit sur l'espace $L^{2;\nu}_{\Gamma,\chi}(\mathbb{C}^n)$ des fonctions mesurables au sens de Lebesgue sur $\mathbb{C}^n$ satisfaisant l'équation (0.2) pour presque tout $z \in \mathbb{C}^n$ et tout $\gamma \in \Gamma$, telles que

$$\|f\|^2_\Gamma = \int_{\Lambda(\Gamma)} |f|^2 d\lambda(z) < +\infty, \tag{0.3}$$

où $\Lambda(\Gamma)$ est un domaine fondamental du réseau maximal Γ qui représente l'espace orbital $\mathbb{C}^n/\Gamma$. On montre en particulier que le spectre est purement discret et stable. Plus exactement, on a

$$\sigma(\Delta^\Gamma_{\nu,\mu|L^{2;\nu}_{\Gamma,\chi}(\mathbb{C}^n)}) = \sigma(\Delta_{\nu,\mu}|_{L^{2,\nu}(\mathbb{C}^n)}) = \sigma(\Delta^\Gamma_{\nu,0}|_{L^{2,\nu}(\mathbb{C}^n)}) = \{-2\nu(2\ell+n); \ell = 0,1,2\cdots\}.$$

On montre aussi que les espaces propres de $\Delta^\Gamma_{\nu,\mu}$ agissant sur $L^{2;\nu}_{\Gamma,\chi}(\mathbb{C}^n)$ sont de dimension finie, dont la dimension est calculée explicitement à la Selberg.

Dans le Chapitre 4, on introduit des fonctions automorphes "mixed" de seconde espèce sur le plan complexe, associées à un certain pair équivariant (ρ,τ) et vérifiant l'équation fonctionnelle

$$F(\gamma\cdot z) = \chi(\gamma) e^{-i(\nu \Im m\langle z,\gamma^{-1}\cdot z\rangle + \mu\Im m\langle \tau(z),\rho(\gamma)^{-1}\cdot z\rangle)} F(z).$$

On donne une caractérisation du pair équivariant (ρ,τ) mettant en évidence la relation entre ρ et τ. On montre également un théorème de passage entre le problème des valeurs propres d'un Laplacien de Bochner, construit de sorte qu'il laisse l'espace fonctionnel considéré invariant, et le problème des valeurs propres du Laplacien magnétique $\Delta_{B,0}$ sur l'espace des fonctions automorphes standards. Cela est prouvé en montrant que le Laplacien de Bochner invariant construit est un opérateur magnétique de Schrödinger lié à un champ magnétique constant, ce qui nous a permis de trouver la transformée isomorphe (transformée jauge) qui le reliant au Laplacien magnétique standard. La transformée est de la forme

$$\left[\mathcal{W}^{\mu,\nu}_\tau f\right](z) := e^{\varphi^{\nu,\mu}_\tau(z)} f(z), \tag{0.4}$$

pour certaine fonction réel $\varphi^{\nu,\mu}_\tau(z)$, solution d'une équation aux dérivées partielles qui dépend de τ. L'analyse spectrale de cette classe est réalisée en montrant que la même transformation de jauge relie l'espace des fonctions automorphes "mixed" de seconde espèce à l'espace des fonctions automorphes simples associé à un poids $\nu = B$ et à un pseudo-caractère χ spécifique. Le cas de dimensions supérieurs a été également discuté.

Au Chapitre 5, on considère l'opérateur thêta de Poincaré P_Γ définit par

$$\mathcal{P}_\Gamma(f)(z) := \sum_{\gamma\in\Gamma} \overline{\chi(\gamma)} e^{-2\nu\langle z+\frac{\gamma}{2},\gamma\rangle} f(z+\gamma),$$

où Γ étant un réseau de $(\mathbb{C},+)$ de rang 1 ou 2. Cet opérateur représente un outil fondamental pour la construction des fonctions automorphes, dans le cas où la série converge absolument et uniformément sur tout compact. On démontre que cet opérateur est bien défini et borné sur l'espace

$$\mathcal{F}^{1,\nu}(\mathbb{C}) := \left\{ f \in \mathcal{H}ol(\mathbb{C}),\ \int_{\mathbb{C}} |f(z)|\, e^{-\nu|z|^2} d\lambda(z) < +\infty \right\}.$$

On démontre également que l'image $\mathcal{P}_\Gamma(\mathcal{F}^{1,\nu}(\mathbb{C}))$ est dense dans l'espace de Banach

$$\mathcal{F}^{1,\nu}_{\Gamma,\chi}(\mathbb{C}) := \left\{ f \in \mathcal{O}^{2\nu}_{\Gamma,\chi},\ \|f\|_{1,\Gamma} := \int_{\Lambda(\Gamma)} |f(z)|\, e^{-\nu|z|^2} d\lambda(z) < +\infty \right\},$$

où $\mathcal{O}^{2\nu}_{\Gamma,\chi}$ est l'espace des fonctions holomorphes automorphes standards vérifiant une équation fonctionnelle équivalente à (0.1). On montre aussi que l'opérateur de Poincaré $\mathcal{P}_\Gamma : \mathcal{F}^{1,\nu}(\mathbb{C}) \longrightarrow \mathcal{F}^{1,\nu}_{\Gamma,\chi}(\mathbb{C})$ est surjectif. De plus, la description concrète de son noyau est donnée en terme de certaines fonctions spéciales dépendantes du rang de Γ. On donne aussi des conditions explicites suffisantes et nécessaires sur les coefficients de Fourier pour qu'une fonction f appartienne au noyau de P_Γ. Ces conditions dépendent du rang de Γ et sont sous la forme d' identités spéciales faisant appel aux polynômes d'Hermite à variable réel. À cette fin, nous avons pu déterminer le dual de l'espace de Banach $\mathcal{F}^{1,\nu}_{\Gamma,\chi}(\mathbb{C})$ qui s'avère être l'espace $\mathcal{F}^{\infty,\nu}_{\Gamma,\chi}(\mathbb{C})$ des fonctions $f \in \mathcal{O}^{2\nu}_{\Gamma,\chi}$ telles que

$$\|f\|_{\infty,\Gamma} := \sup_{z\in\mathbb{C}} \left\{ |f(z)|\, e^{-\nu|z|^2} \right\} < +\infty,$$

Le dual est pris par rapport au produit de Petersson

$$\langle f, g\rangle_\Gamma := \int_{\Lambda(\Gamma)} f(z)\overline{g(z)}e^{-2\nu|z|^2} d\lambda(z)$$

pour $f \in \mathcal{F}^{1,\nu}_{\Gamma,\chi}(\mathbb{C})$ et $g \in \mathcal{F}^{\infty,\nu}_{\Gamma,\chi}(\mathbb{C})$.

Dans le dernier chapitre (Chapitre 6), on s'intéresse principalement à la fonction noyau reproduisant

$$K^\nu_{\Gamma,\chi}(z, w) := \left(\frac{\nu}{\pi}\right) \sum_{\gamma\in\Gamma} \chi(\gamma) e^{-\frac{\nu}{2}|\gamma|^2 + \nu\left(z\overline{\gamma} - \overline{w}\gamma + z\overline{w}\right)}$$

de l'espace des fonctions holomorphes automorphes standards, donnée par la série absolument et uniformément convergente sur tout compact. Ici Γ est un réseau maximal et χ un pseudo-caractère de Γ. On étudie certaines propriétés analytiques et arithmétiques de $K^\nu_{\Gamma,\chi}$. Plus précisément, on examine la distribution et la continué de ses zéros. On montre en particulier que les zéros de $K^\nu_{\Gamma,\chi}$ sont symétriques, non-isolés et de distribution uniforme. En effet, $\mathcal{Z}(K^\nu_{\Gamma,\chi})$ est obtenu par des Γ-translations des zéros de $K^\nu_{\Gamma,\chi}(z, w)$ à l'intérieur d'un produit cartésien de deux domaines fondamentaux. L'étude systématique de $\mathcal{Z}(K^\nu_{\Gamma,\chi})$ conduit à une caractérisation des zéros, contenus dans un domaine fondamental, communs entre toutes les fonctions holomorphes appartenant à l'espace de Hilbert "thêta Bargmann-Fock" $\mathcal{O}^\nu_{\Gamma,\chi}(\mathbb{C})$ (l'ensemble analytique de $\mathcal{O}^\nu_{\Gamma,\chi}(\mathbb{C})$). On démontre aussi que cet ensemble analytique est fini de cardinal inférieur ou égal à $\nu Area(\Gamma)/\pi$. Par la suite, en utilisant principalement des propriétés des polynômes d'Hermite complexes, une représentation en série de $K^\nu_{\Gamma,\chi}$ est donnée en fonction des coefficients surnommés les coefficients de Hermite-Gauss complexes. En effet, on montre que $K^\nu_{\Gamma,\chi}$ est développable en série sous la forme

$$K^\nu_{\Gamma,\chi}(z, w) = \left(\frac{\nu}{\pi}\right) \sum_{m,n=0}^{\infty} (-1)^m a_{m,n}(\Gamma|\nu, \chi) \frac{z^n \overline{w}^m}{m!n!},$$

où les coefficients $a_{m,n}(\Gamma|\nu, \chi)$ sont donnés par

$$a_{m,n}(\Gamma|\nu, \chi) := \sum_{\gamma\in\Gamma} \chi(\gamma) e^{-\frac{\nu}{2}|\gamma|^2} H^\nu_{m,n}(\gamma; \overline{\gamma}).$$

De plus, on obtient des sommations de réseau (lattice sums) remarquables en évaluant les coefficients d'Hermite-Gauss complexes $a_{m,n}(\Gamma|\nu, \chi)$. On termine le chapitre par quelques remarques sur l'utilité de ces sommations dans la dérivation des identités qui relient les fonctions thêta de Jacobi. Au reste, on a énoncé une conjecture, basée sur l'approche numérique, affirme que l'équation $a_{0,0}(\Gamma|\nu, \chi) = 0$ caractérise le réseau de von Neumann.

CONTENTS

GENERAL INTRODUCTION

It is a deeper subject than I appreciated and, I begin to suspect, deeper than anyone yet appreciates. To see it whole is certainly a daunting, for the moment even impossible, task.

(Robert Langlands, Writing about the theory of automorphic forms)

Thesis overview

Automorphic functions (or Fuchsian functions in the Henri Poincaré's terminology, named after the mathematician Lazarus Fuchs) generalize in some sort the concept of periodic functions, including both trigonometric and elliptic. According to the Poincaré's original definition, an analytic function is automorphic if it is invariant under the action of a discrete infinite group of linear fractional transformations. More generally, if Γ is a discrete group that acts properly discontinuously on a locally compact complex manifold M (including domains $\mathscr{D}$ of $\mathbb{C}^n$), then an automorphic form f on M is an holomorphic function on M satisfying the functional equation

$$f(\gamma \cdot z) = J(\gamma, z) f(z), \tag{0.5}$$

where $\cdot$ denotes the action of Γ on M and $J(\gamma, z)$ is a mapping that verifies the co-cycle identity

$$J(\gamma\gamma', z) = J(\gamma, \gamma' \cdot z) J(\gamma', z),$$

and which we usually call an *automorphic factor*. Even more generally, a smooth function f, which is not necessarily analytic, is said to be an automorphic function if it verifies Equation (0.5). This analytical definition has, as well, a geometrical relevance. Indeed, when $\Gamma \backslash M$ is a manifold, then such functions represent cross sections of line bundles over it.

The origins of the theory of automorphic functions can be traced back to the works of Riemann, Klein and Poincaré. The extension to the non-analytic setting is then considered by Hans Maass [55] as well as by Alte Selberg in his seminal paper [72]. Since then, the concept of automorphic forms/functions has known many changes and the theory has undergone tremendous developments. Today it is one of the central research areas in mathematics with links to many different fields within and outside mathematics including representation theory, number theory, PDEs, algebraic geometry and differential geometry, as well as cryptography and quantum physics. The interested reader can refer to the survey paper by A. B. Venkov [80] for interesting references and applications.

The eigenvalue problem on spaces of automorphic functions is motivated by doing harmonic analysis on the quotient space $\Gamma \backslash M$ as one of its major aims is to decompose every automorphic function satisfying suitable growth conditions into eigenvalues components of a self-adjoint and invariant differential operator Δ. This modern way to look at the spectral properties of automorphic functions is thanks to the huge influence of the works of Maass and then Selberg.

Automorphic functions can be defined in a number of frameworks depending on the data (M, Γ, J). In the original hyperbolic context, M is replaced by the unit disc or the upper half-plane of $\mathbb{C}$, Γ in this case is a Fuchsian group acting by the linear fractional transformations and $J(\gamma, z) = \gamma'(z)^k$, for a fixed positive integer k. **However, in our context,** M is the whole space $\mathbb{C}^n$ and Γ is a lattice of $\mathbb{C}^n$, or more generally a discrete subgroup of the group of direct isometries of $\mathbb{C}^n$. When Γ is a lattice of $\mathbb{C}^n$, $\mathbb{C}^n/\Gamma$ is a toroidal manifold on which many studies have been pursued [34, 76, 84]. Whereas, when Γ is a discrete subgroup of the group of direct isometries, to the best of our knowledge, automorphic functions in such situation was never considered before in mathematical literature. It is only recently that they have been considered implicitly in Abe et al. [2] in the context of studying field theory on magnetized orbifolds; we recognize them without explicit mention of them by name. The spectral theory in this later situation is the subject of an ongoing project, and for which we include some background on the relevant discrete subgroups in Chapter 1 and some remarks on the eigenvalue problem in the last section of Chapter 2.

The choice of the automorphic factor J will be subject to modification depending on the chapters. Generally speaking, in the mathematical literature, the automorphic factor J is restrictively chosen in a specific form to describe, up to an equivalence relation, classes of line bundles over the quotient space $\Gamma\backslash\mathbb{C}^n$. We refer to this kind of choice as classical automorphic functions on $\mathbb{C}^n$. However, we are not restricting ourselves by this way of choosing J. In fact, automorphic functions appear in many applied areas, associated with different forms of J, and their utility should not be confined into their geometrical pertinence. This interdisciplinary relevance of automorphic functions is one among other motivations that led us to introduce and study exhaustively two classes of automorphic functions associated with bi-weighted automorphic factors. We are calling them mixed automorphic functions of first and of second kind. Moreover, we provide, for each one of these classes, the appropriate invariant Laplacians acting on them, and then precise their spectrum in contexts that nuanced below.

The functions that we are calling here **mixed automorphic functions of first kind** were discovered in [22] after investigating a natural extension of the real Heisenberg group where the symplectic form in the definition of the group's law was replaced by the associated Hermitian form. The resulting group turned out to be a central extension of the real Heisenberg group and its projective representation, in the spirit of Cartier [16], resulted in automorphic functions of the form

$$f(\gamma \cdot z) = J_{\mu,\nu}(\gamma, z) f(z); \qquad (\gamma, z) \in \Gamma \times \mathbb{C}^n.$$

On the other hand, **mixed automorphic functions of second kind** were first introduced by Stiller [77] and extensively investigated by Lee [54]. They arise naturally as holomorphic forms on elliptic varieties [39, 54]. They are defined, Associated with an equivariant pair (ρ, τ), to satisfy the functional equation of the form

$$f(\gamma \cdot z) = J(\gamma, z) J(\rho(\gamma), \tau(z)) f(z); \qquad (\gamma, z) \in \Gamma \times \mathbb{C}^n.$$

In the context of the complex plane $\mathbb{C}$, we are only aware of Ghanmi [28] and El Gourari and Ghanmi [26] where the authors show a lifting result between mixed automorphic functions of second kind and classical automorphic functions. In Chapter 4, we slightly improve the result in [26] by showing that the lifting also stands when the action of two dimensional motion group is considered instead of the action of a lattice of $\mathbb{C}$. We will also show that such lifting fails in general in higher dimensions without imposing sharp restrictions on the equivariant pair (ρ, τ).

In one dimension (i.e., $n = 1$), it will be shown that both mixed automorphic functions of first kind and of second kind and their associated eigenvalue problems can be lifted to the classical automorphic functions and its associated eigenvalue problem. Therefore, in the last two chapters, we are going back to the space of classical automorphic functions. Moreover, in these chapters, **we turn our attention away from the spectral properties** of likewise spaces to **investigate other questions concerning the construction of automorphic functions and their application** in finding some nice arithmetical identities.

The method that we are investigating to construct automorphic functions is the well-known one due to Henri Poincaré, which we often refer to as *"periodization à la Poincaré"*. The method consists of considering

$$\mathcal{P}_\Gamma(f) := \sum_{\gamma \in \Gamma} f(\gamma \cdot z) J(\gamma, z)^{-1},$$

often called ***Poincaré theta series***, provided that the series converges absolutely and uniformly on compact sets. Accordingly, we will investigate important questions that arose naturally. The first one concerns the classes of functions on which $\mathcal{P}_\Gamma$ is well-defined. The second is the characterization of the set of functions f for which $\mathcal{P}_\Gamma f$ vanishes identically as well as those for which $\mathcal{P}_\Gamma f$ does not vanish (vanishing Poincaré problem raised by Poincaré himself in 1882). As far as we know, related works treating these questions are done only in the hyperbolic context. For instance, we highlight the following fundamental works: Bers [9] has introduced a class of holomorphic integrable functions on the unit disk $\mathbb{D}$, with respect to the hyperbolic metric, on which $\mathcal{P}_\Gamma$ is well-defined. Concerning the kernel of $\mathcal{P}_\Gamma$, viewed as an operator on the Bers spaces, Metzger [58] has given a set of functions that closely spans $\ker(\mathcal{P}_\Gamma)$. Later, Kra [49] obtained a characterization of the kernel using the Eichler cohomology for finitely generated Fuchsian group. The result is then improved by Masumoto [57] who gave a concrete characterization of $\ker(\mathcal{P}_\Gamma)$ on Hardy spaces for finitely generated Fuchsian group of the first kind.

Square integrable holomorphic automorphic functions form a reproducing kernel Hilbert space where the kernel function is denoted K^ν_Γ and was constructed in Ghanmi and Intissar [34] as the Poincaré periodization of Fock Space's reproducing kernel. Being so, **in the last chapter** we focus our attention on the properties of the kernel function K^ν_Γ itself. For instance, we give an interesting power series representation of $K^\nu_{\Gamma,\chi}$ by means of the so-called complex Hermite-Gauss coefficients and study the zeros of K^ν_Γ, which will lead to establish remarkable identities that generalize some identities investigated by Perelomov [64] in the framework of coherent states and rediscovered later by Boon and Zak [12]. Beforehand, we will remark that one Perelomov identity in the case of von Neumann lattice [61] follows effortlessly using our knowledge about the considered space of square integrable holomorphic automorphic functions. **The most remarkable element in this chapter, and one of the most in this thesis overall**, is that it provides a novel application of automorphic functions, adding to the long list of their applications, not only in mathematical fields but also in other scientific fields. Indeed, such identities are lattice sums that arise in chemistry and physics.

Organization and description of the main results

The thesis is composed of six chapters. Although the logical sequencing used in ordering them, they are manly self-contained and almost readable in any order. When a result from another chapter is used, clear and appropriate cross-referencing is provided. Besides Chapter 1, which is manly independent the thesis can be seen as divided into two parts as follows. **Part I: Spectral theory on automorphic functions**, composed of chapters 2, 3 and 4, and **Part II: Analytic theory to study the construction and applications of automorphic functions**, composed of Chapters 5 and 6. In the following we provide in summery the material treated in each chapter.

Chapter 1 introduces some backgrounds on the relevant groups Γ in our context, i.e., which acts properly discontinuously on $\mathbb{C}^n \equiv \mathbb{R}^{2n}$. Specifically, such groups are discrete subgroups of Euclidean isometries $\mathrm{Isom}(\mathbb{C}^n)$. The most relevant subgroups of $\mathrm{Isom}(\mathbb{C}^n)$ for us are the lattices of $(\mathbb{C}^n, +)$, which will be revised in the first section. Then, we discuss the discrete subgroups of $\mathrm{Isom}(\mathbb{R}^n)$ and $\mathrm{Isom}^+(\mathbb{R}^n)$, especially the co-compact ones (i.e., $\Gamma\backslash\mathbb{C}^n$ is compact), and which are also known as crystallographic groups. The classification of such groups led us to state some fundamental theorems in theory of n-dimensional crystallographic groups. Namely, Bieberbach's first, second and third theorems as well as Zassenhaus' theorem. The explicit description of discrete subgroups of planar Euclidean motion group $\mathrm{Isom}^+(\mathbb{C}) \equiv \mathsf{U}(1) \ltimes \mathbb{C}$ is then given. Moreover, it is explained when the orbit space $\Gamma\backslash\mathbb{C}^n$ can be supplied with manifold structure versus when it can only supplied with a weaker structure called orbifold.

In **Chapter 2,** we revise the basic properties of two equivalent spaces of classical automorphic functions on toroidal manifolds (i.e., Γ is a lattice of $\mathbb{C}^n$), associated with a pseudo-character $\chi : \Gamma \to \mathsf{U}(1)$ and a parameter $\nu > 0$, and which verify

$$f(z+\gamma) = \chi(\gamma) e^{i\nu \Im\langle z,\gamma\rangle} f(z)$$

or

$$f(z+\gamma)=\chi(\gamma)e^{\frac{\nu}{2}|\gamma|^2+\nu\langle z,\gamma\rangle}f(z), \tag{0.6}$$

Furthermore, the corresponding eigenvalue problems for *magnetic Laplacians* $\Delta_{\nu,0}$ and Δ_ν are considered. Then, we construct an explicit orthogonal basis for each of the corresponding eigenspaces (Landau levels). Such bases involve theta functions in their expression. For instance, we show that for some real numbers α_{1_j} and α_{τ_j} describing the map χ, the functions φ_J; $J=(j_1,\cdots,j_n)\in\{0,1,\cdots,N-1\}^n$, defined by

$$\varphi_J(z):=\varphi_{j_1}^{\alpha_{1_1},\alpha_{\tau_1}}(z_1)\cdots\varphi_{j_n}^{\alpha_{1_n},\alpha_{\tau_n}}(z_n),$$

where

$$\varphi_j^{\alpha_1,\alpha_\tau}(z):=e^{\frac{\nu}{2}z^2}e^{2i\pi(\alpha_1+j)z}\vartheta(Nz+(\alpha_1\tau-\alpha_\tau)+j\tau|N\tau),$$

form an orthogonal basis of the first level functional space $\mathcal{O}^\nu_{\Gamma,\chi}(\mathbb{C}^n)=Ker(\Delta_\nu)$, where $N:=\dim\mathcal{O}^\nu_{\Gamma,\chi}(\mathbb{C})$. Additionally, we give a description of their zeros, which will be useful in the last chapter. Furthermore, an explicit basis is given for the higher *Landau levels* $E^\nu_\ell(\Delta_\nu)=Ker(\Delta_\nu-\nu\ell)$ as well. For instance, for $n=1$, it is proved that a basis is given in terms of the Jacobi theta function and the rescaled complex Hermite polynomials by

$$(-1)^\ell e^{\frac{\nu}{2}z^2+2i\pi(\alpha_1+j)z}\sum_{k=0}^{\ell}\binom{\ell}{k}i^{\ell-k}H^{\nu/2}_{\ell-k}\left(2\Im mz+\frac{2\pi}{\nu}(\alpha_1+j)\right)\times\partial_z^k\vartheta(Nz+(\alpha_1\tau-\alpha_\tau)+j\tau|N\tau).$$

Lastly, we introduce automorphic functions on orbifolds (i.e., Γ is a subgroup of $\mathsf{U}(n)\ltimes\mathbb{C}^n$ with eventual torsion) and give some remarks on a method that can lead to an explicit spectral study on them for $n=1$.

Chapter 3 incorporates the results obtained in El Fardi, Ghanmi, and Intissar [22, 21]. In this chapter, we considers the **space of mixed automorphic functions of first kind** (a.k.a. of bi-weight (ν,μ), where ν and μ are real numbers such that $\mu>0$) defined to be the space of smooth functions $F\in\mathcal{C}^\infty(\mathbb{C}^n)$ satisfying the functional equation

$$F(\gamma\cdot z)=e^{-i\nu\Im m\langle z,\gamma^{-1}\cdot z\rangle-i\mu\Re e\langle z,\gamma^{-1}\cdot z\rangle}F(z)$$

for given discrete subgroup Γ of $\mathsf{U}(n)\ltimes\mathbb{C}^n$. Then we investigate the spectral theory of a second-order differential operator of Laplacian type,

$$\Delta_{\nu,\mu}=\sum_{j=1}^n\left\{4\frac{\partial^2}{\partial z_j\partial\overline{z_j}}+2(\nu+i\mu)z_j\frac{\partial}{\partial z_j}-2(\nu-i\mu)\overline{z_j}\frac{\partial}{\partial\overline{z_j}}-(\mu^2+\nu^2)|z_j|^2+2i\mu\right\},$$

on the introduced space of mixed automorphic functions of second kind.
The operator $\Delta_{\nu,\mu}$ is constructed so that it leaves the space invariant. Simultaneously, it is closely connected to the sub-Laplacian of a group of Heisenberg type $\mathbb{C}\times_\omega\mathbb{C}^n$ using partial Fourier transform with (ν,μ) as dual arguments. Such group is realized as a central extension of the standard Heisenberg group $H_{2n+1}=(\mathbb{R}\times\mathbb{C}^n,\cdot_{\Im m\omega})$. It turns out that $\Delta_{\nu,\mu}$ is interesting in itself. Indeed, it represents a Bochner Laplacian on the smooth sections of a Hermitian line bundle with connection over the manifold $M=\mathbb{C}^n$. It can also be realized as a Schrödinger operator with a uniform magnetic field on the complete oriented Riemannian manifold $\mathbb{C}^n\equiv\mathbb{R}^{2n}$ and associated to a special Berry phase.

For Γ being a lattice of maximal rank, we describe concretely the spectral analysis of $\Delta_{\nu,\mu}$ when acting on both the Frechet space $\mathcal{C}^\infty(\mathbb{C}^n)$ of $\mathcal{C}^\infty$-complex-valued functions on n-complex space $\mathbb{C}^n$ endowed with the compact-open topology and the free Hilbert space of square integrable complex-valued functions on $\mathbb{C}^n$. Moreover, we show a stability theorem for the spectrum in the sense that $\Delta_{\nu,\mu}$ acting on mixed automorphic functions of first kind or on the free Hilbert space as well as $\Delta_{\nu,0}$ acting on classical automorphic functions or on the free Hilbert space are isospectral. Precisely, the spectrum is purely discrete and given by $-2\nu(2\ell+n)$; $\ell = 0,1,2\cdots$. The corresponding L^2-automorphic eigenspaces are of finite dimension whose dimension is computed explicitly *à la* Selberg. In fact, we determinate the traces of integral operators associated with the automorphic kernel functions obtained by averaging reproducing kernels of the free L^2-eigenspaces. General properties of a class of (Γ,χ)-automorphic kernel functions are also investigated. We also construct generators of each eigenspace by means of theta functions.

Chapter 4 introduces **mixed automorphic functions of second kind** on the complex plane, associated with a certain equivariant pair (ρ,τ) and given subgroup Γ of planar Euclidean motion group $\mathrm{U}(1)\ltimes\mathbb{C}$ and map $\chi:\Gamma\to\mathrm{U}(1)$, and which satisfy the functional equation

$$F(\gamma\cdot z)=\chi(\gamma)e^{-i(\nu\Im m\langle z,\gamma^{-1}\cdot z\rangle+\mu\Im m\langle\tau(z),\rho(\gamma)^{-1}\cdot z\rangle)}F(z).$$

Right after the introduction of the space, we give a characterization for the equivariant pairs (ρ,τ) that highlights the relation between ρ and τ. We also show a lifting theorem between the eigenvalue problem of an appropriate invariant Bochner Laplacian, constructed in such a way that it leaves the considered functional space invariant, and the eigenvalue problem of the standard magnetic Laplacian Δ_ν on the space of classical automorphic functions. The proof is handled by showing that the constructed invariant Bochner Laplacian is a magnetic Schrödinger operator associated with a constant magnetic field, which led to find the isomorphic transform (gauge transformation) relating it to the standard magnetic Laplacian with constant magnetic field. The spectral analysis of this class is accomplished by showing that the same isomorphic transform maps $\mathcal{M}_\tau^{\nu,\mu}(\mathbb{C})$ to the space of classical automorphic functions associate with specific data $(B_\tau^{\nu,\mu},\chi_\tau)$. Near the end of the chapter, we discuss the difficulties of generalizing the same results to higher dimensions. The results of this chapter are outlined in El Fardi and Ghanmi [20].

Chapter 5 deals with the study of the **Poincaré theta series operator** $\mathcal{P}_\Gamma(f)$, which is a fundamental tool to construct automorphic functions, provided that the series converges absolutely and uniformly on compact sets. Formally, in the context of the whole complex plane $\mathbb{C}$ and Γ being a lattice of $(\mathbb{C},+)$ of rank 1 or 2, the theta Poincaré series operator is given as follows

$$\mathcal{P}_\Gamma(f)(z):=\sum_{\gamma\in\Gamma}\overline{\chi(\gamma)}e^{-2\nu\langle z+\frac{\gamma}{2},\gamma\rangle}f(z+\gamma).$$

In the first main result of this chapter, we show that this operator is well-defined on

$$\mathcal{F}^{1,\nu}(\mathbb{C}):=\left\{f\in\mathcal{H}ol(\mathbb{C}),\ \int_{\mathbb{C}}|f(z)|\,e^{-\nu|z|^2}d\lambda(z)<+\infty\right\}.$$

Then, we prove a density theorem of the image $\mathcal{P}_\Gamma(\mathcal{F}^{1,\nu}(\mathbb{C}))$ in the Banach space

$$\mathcal{F}^{1,\nu}_{\Gamma,\chi}(\mathbb{C}):=\left\{f\in\mathcal{O}^{2\nu}_{\Gamma,\chi},\ \|f\|_{1,\Gamma}:=\int_{\Lambda(\Gamma)}|f(z)|\,e^{-\nu|z|^2}d\lambda(z)<+\infty\right\},$$

where $\mathcal{O}^{2\nu}_{\Gamma,\chi}$ is the space of classical automorphic holomorphic functions satisfying an equation similar to (0.6). The Poincaré theta operator $\mathcal{P}_\Gamma:\mathcal{F}^{1,\nu}(\mathbb{C})\longrightarrow\mathcal{F}^{1,\nu}_{\Gamma,\chi}(\mathbb{C})$ is shown to be bounded and onto. Moreover, the concrete description of its kernel is given in terms of some special functions that depends on the rank of the lattice Γ. Furthermore, we provide explicit sufficient and necessary conditions on the Fourier coefficients of a function f to belong to the kernel of $\mathcal{P}_\Gamma$. These conditions depend on the rank of Γ and are in the form of special identities that involves the univariate real Hermite polynomials. To this end, we were able to determinate the dual of the Banach space $\mathcal{F}^{1,\nu}_{\Gamma,\chi}(\mathbb{C})$, which is found to be $\mathcal{F}^{\infty,\nu}_{\Gamma,\chi}(\mathbb{C})$, the space of all $f\in\mathcal{O}^{2\nu}_{\Gamma,\chi}$ such that

$$\|f\|_{\infty,\Gamma}:=\sup_{z\in\mathbb{C}}\left\{|f(z)|\,e^{-\nu|z|^2}\right\}<+\infty,$$

under the Petersson inner product

$$\langle f, g \rangle_\Gamma := \int_{\Lambda(\Gamma)} f(z)\overline{g(z)} e^{-2\nu|z|^2} d\lambda(z)$$

for $f \in \mathcal{F}^{1,\nu}_{\Gamma,\chi}(\mathbb{C})$ and $g \in \mathcal{F}^{\infty,\nu}_{\Gamma,\chi}(\mathbb{C})$. The material of this chapter is extracted from our preprint El Fardi et al. [24].

The last chapter (**Chapter 6**) includes our results published in El Fardi et al. [23]. In this chapter, we are concerned by the automorphic reproducing kernel function given by the absolutely and uniformly convergent series on compact subsets of $\mathbb{C} \times \mathbb{C}$,

$$K^\nu_{\Gamma,\chi}(z, w) := \left(\frac{\nu}{\pi}\right) \sum_{\gamma \in \Gamma} \chi(\gamma) e^{-\frac{\nu}{2}|\gamma|^2 + \nu\left(z\overline{\gamma} - \overline{w}\gamma + z\overline{w}\right)}$$

for given co-compact lattice Γ and pseudo-character χ. More precisely, we investigate its analytical and arithmetical properties. We start by examining the distribution and the discreteness of its zeros and show that the set $\mathcal{Z}(K^\nu_{\Gamma,\chi})$ of zeros of $K^\nu_{\Gamma,\chi}$ is symmetric, not isolated, and its distribution is uniform. Indeed, $\mathcal{Z}(K^\nu_{\Gamma,\chi})$ consists of the Γ-translation of the zeros of $K^\nu_{\Gamma,\chi}(z, w)$ contained in a cartesian product of fundamental cells. The systematic study of the zero set of $K^\nu_{\Gamma,\chi}$ inside a cartesian product of fundamental cells leads to a characterization of the common zeros (analytic set) contained in a fundamental cell of all holomorphic functions belonging to the theta Bargmann-Fock Hilbert space $\mathcal{O}^\nu_{\Gamma,\chi}(\mathbb{C})$. It is shown that this analytic set is finite and its cardinal is less or equal to $\nu Area(\Gamma)/\pi$. Subsequently, using mainly properties of the complex Hermite polynomials, we obtain interesting representation series of $K^\nu_{\Gamma,\chi}$ by means of the so-called complex Hermite-Gauss coefficients. Indeed, we show that $K^\nu_{\Gamma,\chi}$ can be expanded as follows

$$K^\nu_{\Gamma,\chi}(z, w) = \left(\frac{\nu}{\pi}\right) \sum_{m,n=0}^{\infty} (-1)^m a_{m,n}(\Gamma|\nu, \chi) \frac{z^n \overline{w}^m}{m!n!},$$

where the coefficients $a_{m,n}(\Gamma|\nu, \chi)$ are given by

$$a_{m,n}(\Gamma|\nu, \chi) := \sum_{\gamma \in \Gamma} \chi(\gamma) e^{-\frac{\nu}{2}|\gamma|^2} H^\nu_{m,n}(\gamma; \overline{\gamma}).$$

Moreover, we obtain some remarkable lattice summations by evaluating the Hermite-Gauss coefficients $a_{m,n}(\Gamma|\nu, \chi)$. We ended the chapter by providing some remarks about the usefulness of these lattice sums in deriving identities relating Jacobi theta functions as well as stating a conjecture, based on numerical approach, that $a_{0,0}(\Gamma|\nu, \chi) = 0$ characterize von Neumann lattice.

Perspectives

The research done while preparing this thesis opened doors toward multiple research questions. Some of these questions are ongoing projects and some of them are projects that we hope to tackle soon. We summarize these projects in the following points

- Study the concrete spectral properties of classical automorphic functions on orbifolds in one and in high dimensions. Where the one dimension case seems accessible by studying exhaustively the averaging of the classical automorphic basis on toroidal manifolds along the rotational group, the high dimensions case seems extremely hard without using other techniques.
- Establish the concrete description of the spectral properties of mixed automorphic functions of first kind on quasi-tori (i.e., Γ is not maximal).
- Study the concrete spectral properties of mixed automorphic functions of second kind in high dimensions, which will require implementing new theory to study the involved Bochner Laplacian with a non-constant magnetic field.

- continue the work done in the last chapter by investigating more identities using the introduced classes of functions (mixed automorphic functions) as well as trying to solve the stated conjecture about the characterization of von Neumann lattice.
- Investigate applications for the spaces introduced in El Fardi et al. [25], which are not related to the core subject of this thesis.

CHAPTER 1

DISCRETE SUBGROUPS OF EUCLIDEAN ISOMETRIES

We introduce some backgrounds on the relevant groups Γ that acts properly discontinuously on $\mathbb{C}^n \cong \mathbb{R}^{2n}$. Namely, such groups are discrete subgroups of the Euclidean isometries group $\mathrm{Isom}(\mathbb{C}^n)$. The most relevant subgroups of $\mathrm{Isom}(\mathbb{C}^n)$ to the theory of automorphic functions on $\mathbb{C}^n$ are the lattices of $(\mathbb{C}^n, +)$, which will be revised in the first section. The discrete subgroups of $\mathrm{Isom}(\mathbb{R}^n)$ and $\mathrm{Isom}^+(\mathbb{R}^n)$ are also discussed especially the co-compact ones (i.e., $\Gamma\backslash\mathbb{C}^n$ is compact), which are also known as crystallographic groups. Fundamental theorems from the theory of n-dimensional crystallographic groups are given. Namely, Bieberbach first, second and third theorems as well as Zassenhaus theorem. The explicit description of discrete subgroups of the planar Euclidean motion group $\mathrm{Isom}^+(\mathbb{C}) \equiv \mathrm{U}(1) \ltimes \mathbb{C}$ is given. Moreover, it is explained when the orbit space $\Gamma\backslash\mathbb{C}^n$ can be supplied with manifold structure in contrast to when it can only be supplied with a weaker structure called of an orbifold.

To define automorphic functions on $\mathbb{C}^n$, one needs to fix a discrete group Γ that acts *properly discontinuously*[1] on $\mathbb{C}^n$. Thus, a smooth function is automorphic if the equation

$$f(\gamma \cdot z) = J(\gamma, z) f(z) \tag{1.1}$$

holds for all $(\gamma, z) \in \Gamma \times \mathbb{C}^n$, where $\cdot$ denotes the action of $\gamma \in \Gamma$ on the elements of $\mathbb{C}^n$ and $J(\gamma, z)$ is a mapping on $\Gamma \times \mathbb{C}^n$ verifying the co-cycle equality

$$J(\gamma\gamma', z) = J(\gamma, \gamma' \cdot z) J(\gamma', z).$$

One can give many arguments why Γ needs to act properly discontinuously on $\mathbb{C}^n$. The main one is to ensure the necessary tools to investigate non-zero measurable automorphic functions. Indeed, such investigation is tightly related to doing analysis on the orbit space $\mathbb{C}^n/\Gamma$. Having a satisfactory structure on $\mathbb{C}^n/\Gamma$ impose upon Γ to act properly discontinuously. If in addition, Γ is acting *freely*[2] the orbit space can be supplied with a manifold structure. However, when the action is not free, which is the case of some groups that will be considered in this chapter, the orbit space cannot be a manifold but still has, in our case, a smooth structure called orbifold. Recall the following definitions

Orbit space: To define the *orbit space* (quotient space) of $\mathbb{C}^n$ modulo Γ. We start by defining the orbit of an element $z \in \mathbb{C}^n$ to be the set $[z] := \{\gamma \cdot z \mid \gamma \in \Gamma\}$. The orbit space $\Gamma\backslash\mathbb{C}^n$ (or $\mathbb{C}^n/\Gamma$ even though the action is denoted left) is the space of all orbits $[z]$ such that $z \in \mathbb{C}^n$.

[1]By a properly discontinuously action of given group G on a topological space X, we mean that for all $x \in X$ and for all non-empty compact subset $K \subset X$ the set $\{g \in G : g \cdot x \in K\}$ is finite.

[2]We say that the action of a group G on a topological space X is free if for all $x \in X, g \cdot x = x$ implies $g = e$ (where e is the neutral element of G).

Fundamental domain: The orbit space $\Gamma\backslash\mathbb{C}^n$ can be represented through a *fundamental domain* $\Lambda(\Gamma)$ (or simply Λ when the context is clear). That is, a connected region of $\mathbb{C}^n$ satisfying the following rules:

1. $\{\gamma \cdot \Lambda\}_{\gamma\in\Gamma}$ is pairwise disjoint,
2. $\{\gamma \cdot \Lambda\}_{\gamma\in\Gamma}$ cover $\mathbb{C}^n$.

Geometrically, the orbit space $\Gamma\backslash\mathbb{C}^n$ can be modeled as the closure of Λ with certain identifications have been made among the boundary points.

In our context, Γ will be seen as a subgroup of a bigger group G that acts on $\mathbb{C}^n$, where G is $(\mathbb{C}^n, +)$ acting by translations or, more generally, the group of Euclidean isometries. In such groups G, a subgroup is acting properly discontinuously or being discrete is exactly the same thing.

In the first section, we review briefly the notion of discrete subgroups of $(\mathbb{C}^n, +)$, which are also known as *lattices of* $(\mathbb{C}^n, +)$. The second section serves as an introduction to the group of isometries of $\mathbb{C}^n \cong \mathbb{R}^{2n}$. Therein, the real space $\mathbb{R}^n$ is considered for all n odd or even. In the third section, we give some elements of the theory of discrete subgroups of the group of Euclidean isometries in real dimension n. Crystallographic groups (i.e., co-compact ones) will get more attention where Bieberbach and Zassenhaus theorems will be given. Also, we will digress upon the structure of the orbit space. In particular, it will be explained that in general, if Γ is a discrete subgroup of the group of Euclidean isometries, $\mathbb{C}^n/\Gamma$ does not have the structure of a manifold but a generalized structure called orbifold. In the last section, we will give an explicit description of the subgroups of the group of direct Euclidean isometries on the complex plane ($\mathsf{U}(1) \ltimes \mathbb{C}^n$).

1.1 Lattices of $(\mathbb{C}^n, +)$

Let $r \in \mathbb{N}$ be a positive integer such that $0 < r \leq 2n$. Then, a lattice L of $(\mathbb{C}^n, +)$ of rank r is any subgroup of $(\mathbb{C}^n, +)$ of the form

$$L_r = \mathbb{Z}\omega_1 + \mathbb{Z}\omega_2 + \cdots + \mathbb{Z}\omega_r \tag{1.2}$$

where $\{\omega_i\}_{i=1,\cdots,r}$ is an $\mathbb{R}$-independent family of elements in $\mathbb{C}^n$. To the coherence of notations, we say that $\{0\}$ is a *lattice* of rank 0. Moreover, when $k = 2n$, L_{2n} is called a lattice of *full-rank* (or maximal rank). Of course, this definition can be adapted for real spaces of any dimension. Notice that lattices of $\mathbb{C}^n$ of any rank are discrete with respect to the usual topology on $\mathbb{C}^n$. Conversely, we can show the following result (see e.g [46])

Proposition 1.1.1. *Any discrete subgroup of $\mathbb{C}^n$ is a lattice*

A lattice L_r acts freely and properly discontinuously on $\mathbb{C}^n$ by translations ($\gamma \cdot z = z + \gamma$). Therefore, the orbit space $\mathbb{C}^n/L_r$ has a manifold structure and refered to as *toroidal manifold*. In the case of maximal rank, $\mathbb{C}^n/L$ is the complex *torus* . Whereas when the rank is less than $2n$, $\mathbb{C}^n/L$ is called *quasi-torus* .

Fundamental domains representing $\mathbb{C}^n/L$ are of finite volume (associated with the Lebesgue measure on $\mathbb{C}^n$) if and only if L is a full-rank lattice. In this case L is said to be co-compact.

The case $n = 1$:

A full rank lattice of $\mathbb{C}$ can be represented (up to an equivalence relation) by a variable $\tau \in \mathbb{C}$ belonging to the upper half-plane ($\Im\tau > 0$) such that $L =: L_\tau := \mathbb{Z} + \tau\mathbb{Z}$ (we use also the notation $\mathbb{Z}[\tau]$). Such representation can be generalized to high dimensions through the Riemann matrices (see e.g. [75]). Moreover, according to τ, lattices of $\mathbb{C}$ can be classified by their geometrical shape. This will be presented near the end of this chapter.

1.2 Group of Euclidean isometries

By definition, an isometry of $\mathbb{R}^n$ is a distance-preserving bijection, and set of isometries of $\mathbb{R}^n$ is denoted by $\mathrm{Isom}(\mathbb{R}^n)$. When $\mathbb{R}^n$ is endowed with the standard euclidean distance, $\mathrm{Isom}(\mathbb{R}^n)$ to is called the Euclidean group. Indeed, it is easy to show that the composition of two isometries is again an isometry and that the inverse of an isometry is also an isometry. Therefore, $\mathrm{Isom}(\mathbb{R}^n)$ is a group under the composition law.

By the cosine rule, the size of the angles of a triangle are determined by the lengths of its sides, so isometries preserve the size of (geometrical) angles. Accordingly, an isometry is said to be direct or indirect according as it preserves or reverses the sense of the angles. Furthermore, we can show that these two cases are the only possible ones.

Direct isometries are also known as motions, rigid motions, proper motions or also displacements and the class of all direct isometries is noted $\mathrm{Isom}^+(\mathbb{R}^n)$. In the other side, indirect isometries are known as improper motions or anti-displacements.

An important theorem in this setting asserts that every isometry of $\mathbb{R}^n$ is a composition of at most $n+1$ reflections. If, in addition, the isometry fixes at least one point, then it is a composition of at most n reflections. Recall that a reflection is the isometry of $\mathbb{R}^n$ that fixes all the points in a chosen hyperplane and interchanges the position of points along each line perpendicular to that hyperplane at equal distance from it.

In the plane (i.e. n=2), we can use the last result to describe four classes of isometries of the plane. It can be shown that every nonzero isometry is a translation, rotation, reflexion or glide-reflexion (a glide reflection is the composition of a reflection and a nonzero translation in a direction parallel to the line of a reflection).

In coordinate terms, any isometry α of $\mathbb{R}^n$ can be expressed as $\alpha(x) = A(x) + b$ for some $A \in O_n(\mathbb{R})$ and $b \in \mathbb{R}^n$. To simplify notation, we write $\alpha = [A, b]$, hence the composition in $\mathrm{Isom}(\mathbb{R}^n)$ translates into the following formula

$$[A,b][A',b'] = A(A'(x)+b') + b = AA'(x) + Ab' + b = [AA', Ab' + b]. \tag{1.3}$$

This shows in particular that $\mathrm{Isom}(\mathbb{R}^n)$ can be realized as the semi-direct product $O_n(\mathbb{R}) \ltimes \mathbb{R}^2$. In a similar fashion, isometries that preserve orientation can be seen as the elements of $\mathsf{SO}_n(\mathbb{R}) \ltimes \mathbb{R}^n$. It is always convenient to see $\mathrm{Isom}(\mathbb{R}^n)$ and $\mathrm{Isom}^+(\mathbb{R}^n)$ as subgroups of $\mathrm{Aff}(\mathbb{R}^n) := \mathsf{GL}_n(\mathbb{R}) \ltimes \mathbb{R}^n$, the group of affine maps of $\mathbb{R}^n$. $\mathsf{GL}_n(\mathbb{R}) \ltimes \mathbb{R}^n$, $\mathrm{Isom}(\mathbb{R}^n)$ and $\mathrm{Isom}^+(\mathbb{R}^n)$ have representations using matrices. Being indeed, one may represent any element of $\mathrm{Aff}(\mathbb{R}^n)$ (*resp.* $\mathrm{Isom}(\mathbb{R}^n)$ and $\mathrm{Isom}^+(\mathbb{R}^n)$) as an $(n+1) \times (n+1)$ homogeneous transformation matrix of the form:

$$\begin{pmatrix} A & b \\ 0^T & 1 \end{pmatrix} \begin{pmatrix} x \\ 1 \end{pmatrix} = \begin{pmatrix} Ax + b \\ 1 \end{pmatrix}, \tag{1.4}$$

where A is some element in $\mathsf{GL}_n(\mathbb{R})$ (*resp.* $\mathsf{O}_n(\mathbb{R})$ or $\mathsf{SO}_n(\mathbb{R})$), and $b \in \mathbb{R}^n$. Notice also that any element $g := [A, b]$ in $\mathrm{Aff}(\mathbb{R}^n)$ can be rewritten as the product of its linear part and its translation part:

$$[A,b] = \mathrm{Lin}(g)\,\mathrm{Trans}(g) := [A,0][1,b]$$

In the 2-dimensions case, we can identify $\mathsf{SO}_2(\mathbb{R})$ with the unit circle in the complex plane $\mathbb{T}$, which is isomorphic to the unitary group $\mathsf{U}(1)$. Therefore, orientation preserving isometries can be seen as $\mathsf{U}(1) \ltimes \mathbb{C}$.

1.3 Discrete subgroups of $\mathrm{Isom}(\mathbb{R}^n)$ and $\mathrm{Isom}^+(\mathbb{R}^n)$

Before studying automorphic functions on $\mathbb{C}^n = \mathbb{R}^{2n}$ associated with motion actions, one needs to concretely identify the subgroups of $\mathrm{Isom}(\mathbb{R}^n)$ that acts properly discontinuously on $\mathbb{R}^n$. By Theorem 1.3.1 below, such subgroups are exactly the discrete subgroups of $\mathrm{Isom}(\mathbb{R}^n)$ with respect to the usual euclidean topology. Moreover, we can distinguish between two kinds of discrete subgroups of $\mathrm{Isom}(\mathbb{R}^n)$, co-compact and non-co-compact. We say that Γ is co-compact if the orbit space $\mathbb{R}^n/\Gamma$ is compact. Otherwise we say that Γ is non-co-compact.

The objective of this section is to state some important notions and results concerning discrete subgroups of $\mathrm{Isom}(\mathbb{R}^n)$. We are interested particularly in co-compact subgroups which are also known as Crystallographic groups.

1.3.1 n-dimensional Crystallographic groups

Definition 1.1 (Crystallographic groups)**.** *A crystallographic group* Γ *is any co-compact discrete subgroup of* $\mathrm{Isom}(\mathbb{R}^n)$.

The following theorem gives a characterization of subgroups of $\mathrm{Isom}(\mathbb{R}^n)$ that acts properly discontinuously on $\mathbb{R}^n$.

Theorem 1.3.1 ([81])**.** *Let* Γ *be a subgroup of the Euclidean group* $\mathrm{Isom}(\mathbb{R}^n)$. *Then*

i) Γ *acts properly discontinuously on the Euclidean space* $\mathbb{R}^n$ *if and only if* Γ *is discrete in* $\mathrm{Isom}(\mathbb{R}^n)$.

ii) *If* Γ *is closed, then it acts freely on* $\mathbb{R}^n$ *if and only if it is torsion-free*[3].

iii) Γ *acts properly discontinuously and with compact quotient on* $\mathbb{R}^n$ *if and only if* Γ *is a discrete uniform subgroup (i.e. its translational part is a full rank lattice) of* $\mathrm{Isom}(\mathbb{R}^n)$.

The cornerstone of the theory of n-dimensional crystallographic groups is based on a series of three theorems of L. Bieberbach (1911 [10] and 1912 [11]), which answer part of the 18th Hilbert's famous problems (see Milnor [59]). For modern versions of the proof see for example Wolf [81, Chapter 3] and Buser [14].

Theorem 1.3.2 (Bieberbach first theorem)**.** *Let* Γ *be a n-dimensional crystallographic group. Then the subgroup* $\mathrm{Trans}(\Gamma)$ *of all elements in* Γ *that are translations is a normal subgroup of finite index in* Γ *and moreover a lattice of* $\mathbb{R}^n$.

Theorem 1.3.3 (Bieberbach second theorem)**.** *Two crystallographic groups are isomorphic if and only if they are affinely equivalent (i.e. they are conjugate by an element of the affine group).*

Theorem 1.3.4 (Bieberbach third theorem)**.** *The number of crystallographic groups in dimension n is finite up to isomorphisms.*

Bieberbach first theorem is a generalization to n-dimensional real space of Schoenflies theorem 1891[69] for the three dimension case. It was proved by Bieberbach [10] in 1911 and implies that Γ is an extension of a maximal lattice of $\mathbb{R}^n$ by some finite group $\Theta \cong \mathrm{Lin}(\Gamma)$, i.e., Γ fits in the exact sequence :

$$0 \longrightarrow \mathrm{Trans}(\Gamma) \cong \mathbb{Z}^n \longrightarrow \Gamma \longrightarrow \Theta \cong \mathrm{Lin}(\Gamma) \longrightarrow 1.$$

Bieberbach second theorem asserts that two crystallographic groups Γ and Γ' are isomorphic, if an only if, they are affinity *conjugate* to each other. That is, if there exists $h \in \mathrm{Aff}(\mathbb{R}^n)$ such that

$$\Gamma' = h \circ \Gamma \circ h^{-1}.$$

In 1948, Zassenhaus gave a constructive proof for the third Bieberbach Theorem. In doing so, he also proved a converse to the first Bieberbach Theorem:

Theorem 1.3.5 (Zassenhaus [82])**.** *A group* Γ *which is an extension of* $\mathbb{Z}^n$ *by a finite* $\Theta \subset \mathsf{GL}_n(\mathbb{Z})$ *can be embedded in* $\mathrm{Isom}(\mathbb{R}^n)$ *as crystallographic group.*

$\mathsf{U}(n) \ltimes \mathbb{C}^n$ the group of motions: In the reminder of this thesis, the spaces considered are complex vector spaces. Accordingly, when dealing with orientation preserving isometries in higher dimensions, we will only consider the part given by the subgroup $\mathsf{U}(n) \ltimes \mathbb{C}^n$ of $\mathrm{Isom}^+(\mathbb{R}^{2n}) = SO_{2n}(\mathbb{R}) \ltimes \mathbb{R}^{2n}$, which is proper once $n > 1$. This is not a restriction if we want to preserve the complex structure, as the set $\mathsf{U}(n) \ltimes \mathbb{C}^n$ contains all the isometries of $\mathbb{C}^n$ with respect to the hermitian product $\langle z, w \rangle = z\overline{w}$ (i.e. bijections t such that $\langle t(z), t(w) \rangle = \langle z, w \rangle$). From now on, when we talk about direct isometries (or motions) we mean exclusively the elements of $\mathsf{U}(n) \ltimes \mathbb{C}^n$.

[3] A group G is torsion-free if the neutral element e is the only element such that $g^n = e$ for some positive integer n.

Bieberbach groups vs Crystallographic groups By *Bieberbach group* we mean a torsion-free crystallographic group $\Gamma \subset \mathrm{Isom}(\mathbb{R}^n)$. For such groups, Theorem 1.3.1 implies that it acts freely and properly discontinuously. Therefore, $\Gamma\backslash\mathbb{R}^n$ can be endowed with a manifold (*flat manifold*) stricture. Whereas, when Γ has torsions, $\Gamma\backslash\mathbb{R}^n$ can no longer be manifold but an orbifold (see subsection below). For general reference on the subject see Charlap [17] and the references therein.

1.3.2 Orbifolds

The word *orbifold* has been used by Thurston [78, Chapter 13] in 1978 as a replacement for V-manifold, concept invented by Ishiro Satake [67] in 1956. The concept was introduced to describe the smooth structure of spaces that look like manifolds, except on a few subsets, where they look like quotients of linear domains by a finite group of linear transformations. The following definition follows the one given in [71, 78]

Definition 1.2. *An n-orbifold is a metrizable topological space $\mathcal{O}$ endowed with a collection $\{(U_i, \tilde{U}_i, \phi_i, \Gamma_i); i\}$, called an atlas, satisfying the conditions*

i) The family $\{U_i\}_i$ is a covering of $\mathcal{O}$, i.e. $\mathcal{O} = \bigcup_i U_i$

ii) EachU_i is an open subset of $\mathcal{O}$, $\tilde{U}_i$ is an open subset of $\mathbb{R}^n$, and Γ_i is a finite group acting on $\tilde{U}_i$ such that $\phi_i : \tilde{U}_i/\Gamma_i \mapsto U_i$ is a homeomorphism.

iii) Whenever $U_i \subset U_j$, here is an injective group homomorphism $f_{ij} : \Gamma_i \hookrightarrow \Gamma_j$ and an embedding $\tilde{\phi}_{ij} : \tilde{U}_i \hookrightarrow \tilde{U}_i$ which is equivariant with respect to f_{ij}, i.e. for each $\gamma \in \Gamma_i$, $\phi_{ij}(\gamma \cdot x) = f_{ij}\phi_{ij}(x)$ such that the diagram below commutes.

$$\begin{tikzcd}
\tilde{U}_i \arrow[r, "\tilde{\phi}_{ij}"] \arrow[d] & \tilde{U}_i \arrow[d] \\
\tilde{U}_i/\Gamma_i \arrow[r, "\phi_{ij}"] & \tilde{U}_j/\Gamma_i \arrow[d, "f_{ij}"] \\
 & \tilde{U}_j/\Gamma_j \\
U_i \arrow[uu] \arrow[r, hook, "\subset"] & U_j \arrow[u]
\end{tikzcd}$$

Figure 1.1: Diagram relating the data of an orbifold

Here the ϕ_{ij} are induced by the monomorphisms (injective homomorphism).

Remark 1.3.6. *Many mathematicians tried to improve Satake's definition because it does not lead to a satisfactory notion of smooth maps, which is pointed by Satake himself in [68]. One of the solutions to this problem is presented in [41].*

1.4 Planar orientation preserving discrete subgroups

Bieberbach and Zassenhaus theorems gave us an unambiguous algebraic description of n-dimensional crystallographic groups, but exhibiting explicit representative of each crystallographic group (up to affine equivalence) is not a simple task in general. Rising the dimension, the number of discrete subgroups grow very fast. Practically, the explicit description becomes impossible by hand once we reach dimension 4. Even with the help of computers, giving names to differentiate between subgroups can impose a severe issue in manipulating them [65]. In this section, we provide the explicit description of discrete subgroups Γ of the planar orientation preserving group, i.e., discrete subgroups of $\mathrm{U}(1) \ltimes \mathbb{C}$.

To this end, notice first that Bieberbach's first theorem asserts that, up to affine conjugations, the subgroups Γ are of the form $\Gamma = U \ltimes L$, where U is some subgroup of finite order of $\mathsf{U}(1)$ and L a lattice of $\mathbb{C}$. Moreover, one can easily check that necessarily and sufficiently $UL \subset L$. Indeed, for necessity, for arbitrary $a \in U$ and $b \in L$ we have

$$[a,0][1,b] = [a,ab] \ \in \Gamma,$$

so that $ab \in L$. We point out that in fact we have equality as the inclusion in the other sense is trivial. Sufficiency is clear. Therefore, using Proposition 1.1.1 we can assert the following

Proposition 1.4.1. *Let Γ of the form $U \times L \subset U(1) \times \mathbb{C}$ be a discrete subgroup of $\mathsf{U}(1) \ltimes \mathbb{C}$. Then, L is in one of the following forms:*
(i) $L = \{0\}$,
(ii) $L = \mathbb{Z}\omega_1$ for some $\omega_1 \in \mathbb{C}^*$, or
(iii) $L = \mathbb{Z}\omega_1 + \mathbb{Z}\omega_2$ for some $\mathbb{R}$-linearly independent vectors $\omega_1, \omega_2 \in \mathbb{C}^*$, and U is the group of qth roots of the unity for some positive integer q that depends on L, $U = \mathsf{U}_q := \{z \in \mathbb{C} | z^q = 1\}$.

Subsequently, by fixing L in one of the three forms given in Proposition 1.4.1, we can determine the corresponding q for which $\Gamma = \mathsf{U}_q \times L$ is a subgroup of G, i.e, the integers q for which $\mathsf{U}_q L \subset L$:

1. **The case $L = \{0\}$:** It is easy to see that $\mathsf{U}_q \{0\} \subset \{0\}$ for every $q = 0, 1, 2, \cdots$, and then, $\mathsf{U}_q \times \{0\}$ is a subgroup of G for every $q = 0, 1, 2, \cdots$.

2. **The case $L = \mathbb{Z}\omega_1$:** For $q = 1$ or 2, we have $\mathsf{U}_q \mathbb{Z}\omega_1 \subset \mathbb{Z}\omega_1$. This is trivially true because $\mathsf{U}_1 = \{1\}$ and $\mathsf{U}_2 = \{-1, 1\}$. If we suppose the existence of some $q \geq 3$ such $\mathsf{U}_q \{0\} \subset \mathbb{Z}\omega_1$, it follows that $e^{\frac{2\pi}{q}} \omega_1 = n\omega_1$ and therefore $e^{\frac{2\pi}{q}} = n$ for some $n \in \mathbb{Z}$, which is impossible. Thus, the only values that can be taken by q are 1 and 2. Hence,

$$\Gamma = \{1\} \ltimes \mathbb{Z}\omega_1 \quad \text{or} \quad \Gamma = \{1, -1\} \ltimes \mathbb{Z}\omega_1 \ .$$

3. **The case $L = \mathbb{Z}\omega_1 + \mathbb{Z}\omega_2$:** We can show that the condition $\mathsf{U}_q(\mathbb{Z}\omega_1 + \mathbb{Z}\omega_1) \subset \mathbb{Z}\omega_1 + \mathbb{Z}\omega_1$ gives a necessary condition on q that restrict it to take no values other than $\{1, 2, 3, 4 \text{ and } 6\}$. This result is known in the field of chemistry by "The crystallographic restriction theorem" (see Coxeter [18, Chapter 4]). For the converse we need to investigate if $\mathsf{U}_q(\mathbb{Z}\omega_1 + \mathbb{Z}\omega_1) \subset \mathbb{Z}\omega_1 + \mathbb{Z}\omega_1$ holds for q taking the values $1, 2, 3, 4, 6$. Similarly, as in the preceding case, for the values 1 and 2 the inclusion holds without any additional condition. On the contrary for the remaining values (i.e $q = 3, 4$ and 6) we have to add some condition on the shape of the lattice $\mathbb{Z}\omega_1 + \mathbb{Z}\omega_1$ to ensure that the inclusion holds. Indeed, for $q = 3$ or 6 we can show that the lattice has to be hexagonal, and for $q = 4$ the lattice has to be square (*see* fig. 1.4, page 14).

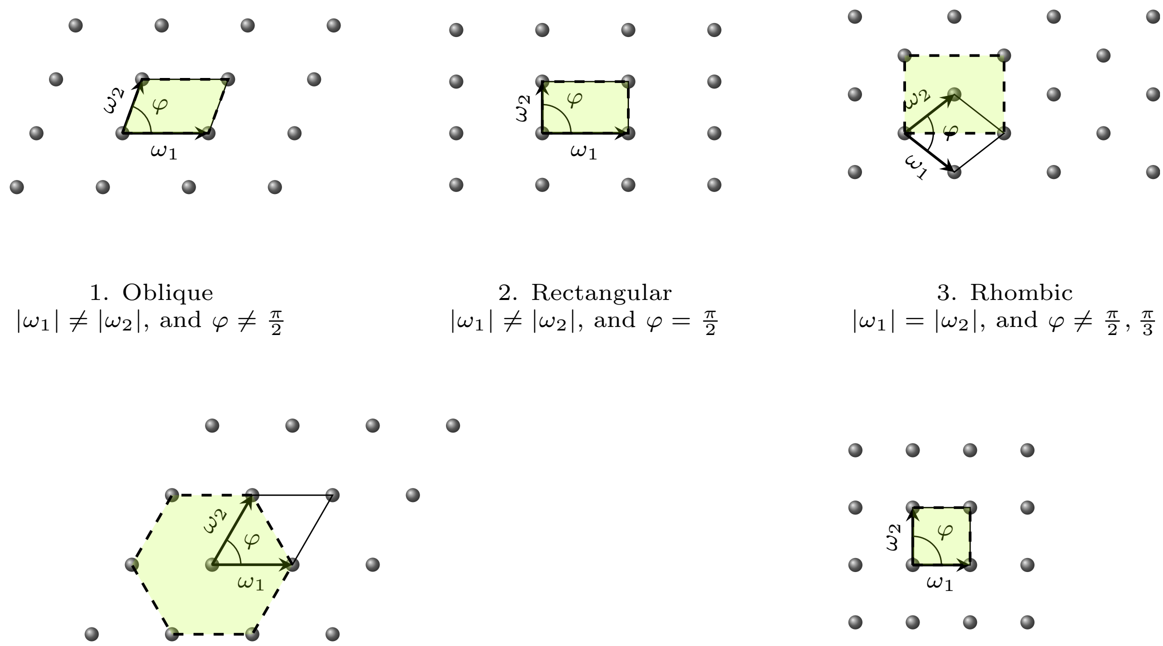
In nutshell, we can state the following theorem

Theorem 1.4.2 (Classification of discrete groups of direct isometries)**.** *Let Γ be a discrete subgroup of the group of direct isometries of $\mathbb{C} \cong \mathbb{R}^2$. Then Γ is affinely conjugate to the group (that has the form $U \ltimes L$) of elements which acts in one form of the following*

- *Rank 0 translational counterpart L (a.k.a. Rosette groups)*
  - $z \mapsto \gamma \cdot z = \mathrm{e}^{2i\pi k/q} z, \qquad k = 0, 1, 2, \cdots, q-1.$
- *Rank 1 translational counterpart L (a.k.a. frieze groups)*
  - $z \mapsto \gamma \cdot z = z + m, \qquad m \in \mathbb{Z}.$
  - $z \mapsto \gamma \cdot z = \pm z + m, \qquad m \in \mathbb{Z}.$
- *Rank 2 translational counterpart L (a.k.a. wallpaper groups or crystallographic groups)*
  - $z \mapsto \gamma \cdot z = z + m + n\tau, \qquad m, n \in \mathbb{Z}.$
  - $z \mapsto \gamma \cdot z = \pm z + m + n\tau, \qquad m, n \in \mathbb{Z}.$

- $z \mapsto \gamma \cdot z = i^k z + m + ni, \qquad m, n \in \mathbb{Z}, k = 0, 1, 2, 3.$
- $z \mapsto \gamma \cdot z = \omega^{2k} z + m + n\omega, \qquad m, n \in \mathbb{Z}, k = 0, 1, 2.$
- $z \mapsto \gamma \cdot z = \omega^{k} z + m + n\omega, \qquad m, n \in \mathbb{Z}, k = 0, 1, 2, 3, 4, 5.$

Here q is any fixed positive integer, $\tau \in \mathbb{C}$ is a fixed number in the complex upper plane ($\Im m(\tau) > 0$) and $\omega = \mathrm{e}^{i\pi/3}$.



1. Oblique $|\omega_1| \neq |\omega_2|$, and $\varphi \neq \frac{\pi}{2}$

2. Rectangular $|\omega_1| \neq |\omega_2|$, and $\varphi = \frac{\pi}{2}$

3. Rhombic $|\omega_1| = |\omega_2|$, and $\varphi \neq \frac{\pi}{2}, \frac{\pi}{3}$

4. Hexagonal $|\omega_1| = |\omega_2|$, and $\varphi = \frac{\pi}{3}$

5. Square $|\omega_1| = |\omega_2|$, and $\varphi = \frac{\pi}{2}$

Figure 1.2: Type of lattices as classified by Bravais. A fundamental paralelograme is surounded by continious side

Fundamental domains In each of the exhibited cases above for Γ, the quotient space $\Gamma\backslash\mathbb{C}$ can be associated with a fundamental domain. There are infinity many choices, but it is always convenient to consider a fundamental domain with simple shape. When Γ is a Lattice (torsion-free), a Dirichtet-Voronoi cell [5] provides an example of such simple shaped fundamental domains. When, in addition, Γ is full rank, a fundamental parallelogram is also a fundamental domain (which is different from the associated Dirichtet-Voronoi cell [1, S3.4]). On the other hand, when Γ has torsion ($L = \mathsf{U}_q, q \geq 2$), a fundamental domain is $1/q$ of the Dirichtet-Voronoi cell, such that the union of its q images by the action of U_q doesn't overlap an gives the entire cell. The same thing can be done in the full rank case for a fundamental parallelogram instead of a Dirichtet-Voronoi cell. Accordingly, in the following we illustrate some possible choices of fundamental domains for the action of Γ on $\mathbb{C}$ in each case of the cases given by Theorem 1.4.2, except the full rank lattice case with $q = 1$, which is already illustrated in Figure 1.4.

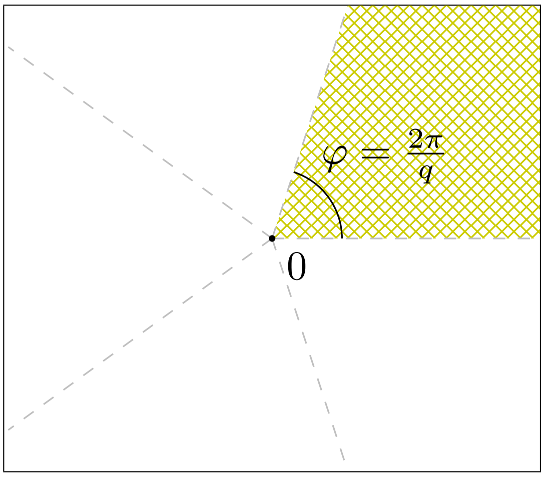


Figure 1.3: —- Fundamental domain in rank 0 case, $q > 0$

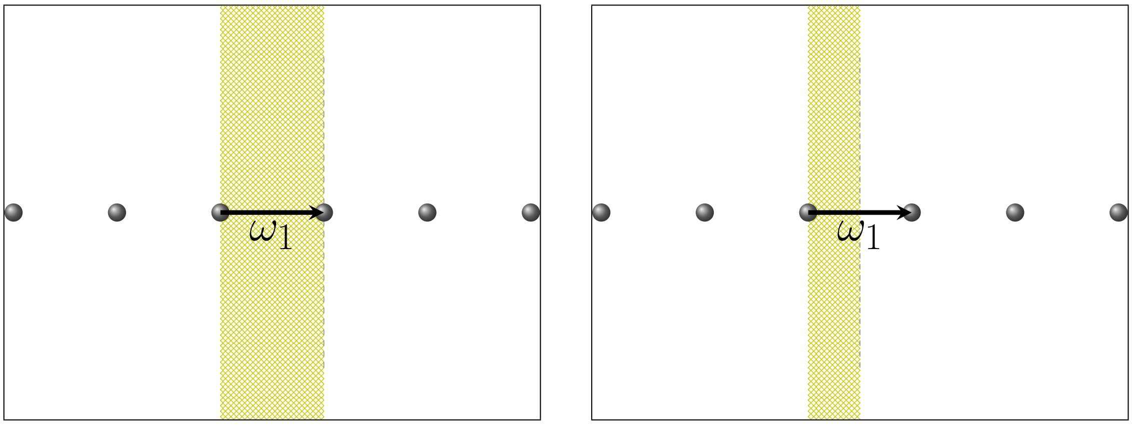


Figure 1.4: —– Fundamental domain in rank 1 case; Left: $q = 1$. Right: $q = 2$

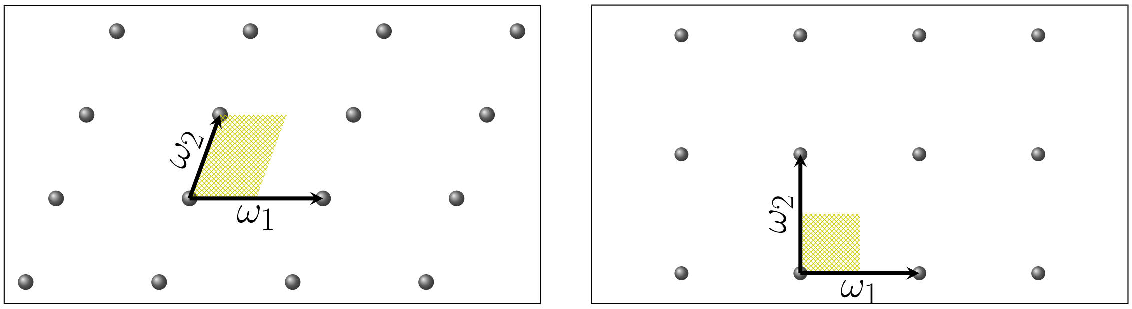


Figure 1.5: —– Fundamental domain in rank 2 case; Left: $q = 2$. Right: $q = 4$

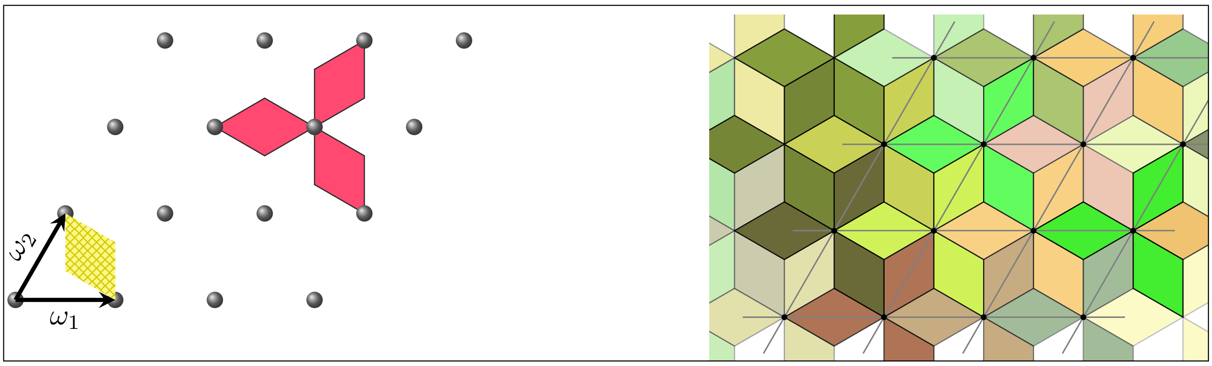


Figure 1.6: —- Fundamental domain in rank 2 case, $q = 3$; Left: A fundamental domain in yellow then its shape after U_3 action in pink. Right: shows pseudorandom coloring with respect to the action of L on the pinkish shape

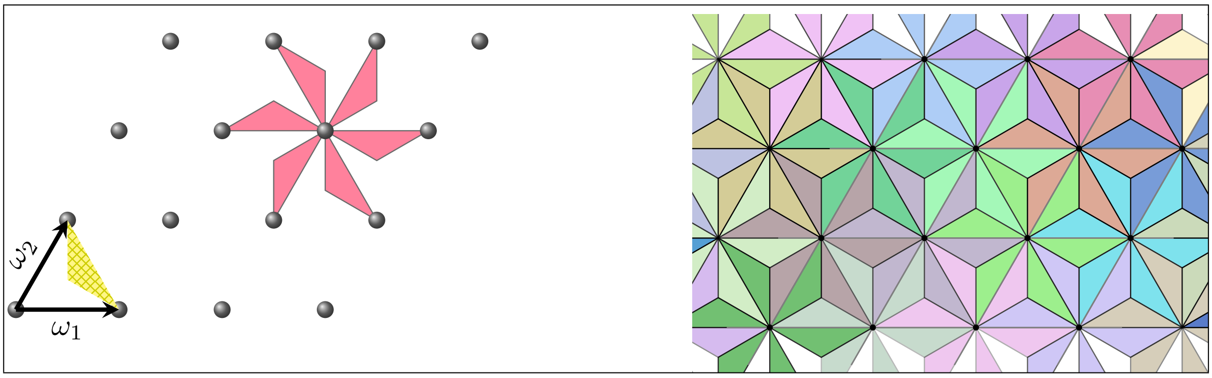


Figure 1.7: —- Fundamental domain in rank 2 case, $q = 6$; Left: A fundamental domain in yellow then its shape after U_6 action in pink. Right: shows pseudorandom coloring with respect to the action of L on the pinkish shape

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CHAPTER 2

AUTOMORPHIC FUNCTIONS ON TOROIDAL MANIFOLDS AND ORBIFOLDS

Basic properties are given for two equivalent spaces of standard automorphic functions on toroidal manifolds (i.e. Γ is a lattice of $\mathbb{C}^n$), associated with a pseudo-character χ and a parameter $\nu > 0$. Furthermore, the corresponding eigenvalue problems are considered. Explicit orthogonal bases are constructed, and which involves the theta functions in their expression. Moreover, a description of the zeros of their elements is given. Last but not least, automorphic functions on orbifolds (i.e. Γ is a subgroup of $\mathsf{U}(n) \ltimes \mathbb{C}^n$ with possible existence of torsion) are considered and some remarks on their spectral properties, when $n = 1$, are given.

In the context of toroidal manifolds, i.e., $\mathbb{C}^n/\Gamma$ is a torus ($\Gamma = L$ is a full-rank lattice) or quasi-torus (Γ not necessary of full-rank), standard automorphic functions are defined with respect to the Appell-Humbert factor, such that they display the functional equation

$$g(z+\gamma) = \chi(\gamma) e^{\frac{\nu}{2}|\gamma|^2 + \nu\langle z,\gamma\rangle} g(z); \qquad \forall \gamma \in \Gamma, \tag{2.1}$$

where $\langle z, w\rangle := \sum_{i=1}^n z_i \overline{w}_i$ is the usual hermitian product on $\mathbb{C}^n$, $\nu > 0$ is a fixed parameter, Γ is a lattice of $\mathbb{R}^{2n} = \mathbb{C}^n$ and χ a mapping from Γ to the unit circle $\{\lambda \in \mathbb{C};\ |\lambda| = 1\} = \mathsf{U}(1)$ satisfying the condition given in Proposition 2.1.1. For full-rank lattices, the automorphic factor involved in (2.1) describes of all holomorphic line bundles over a torus. Throughout this chapter Γ is full-rank unless stated otherwise.

Another interesting class of automorphic functions in this context is the class of functions f displaying the functional equation

$$f(z+\gamma) = \chi(\gamma) e^{i\nu \Im m\langle z,\gamma\rangle} f(z); \qquad \forall \gamma \in \Gamma. \tag{2.2}$$

In this chapter we recall some results on automorphic functions obeying the functional equation (2.1) or (2.2). We also give explicit bases of these spaces. A generalization of automorphic functions on orbifolds is also introduced, where the action of a lattice is replaced by an action of a discrete subgroup of Euclidean motions.

2.1 Spaces of standard automorphic functions

Let us begin this section by introducing the following notations and definitions.

Definition 2.1. *To given data (ν, Γ, χ), we define the spaces*

$\boxed{1}$ $\mathcal{A}ut^{\nu,0}_{\Gamma,\chi}(\mathbb{C}^n)$ *of Lebesgue measurable functions $f : \mathbb{C}^n \to \mathbb{C}$ that satisfy the functional equation* (2.2) *for almost every $z \in \mathbb{C}^n$.*

$\boxed{2}$ $\mathcal{A}ut^{\nu}_{\Gamma,\chi}(\mathbb{C}^n)$ *of Lebesgue measurable functions $g : \mathbb{C}^n \to \mathbb{C}$, satisfying the functional equation* (2.1) *for almost every $z \in \mathbb{C}^n$.*

We refer to this spaces as (ν, Γ, χ)-measurable automorphic functional spaces. Or simply, when the context is clear, measurable automorphic functional spaces.

Definition 2.2. *To given data (ν, Γ, χ), we define the spaces*

$\boxed{1}$ $\mathcal{F}^{\nu,0}_{\Gamma,\chi}(\mathbb{C}^n)$ *of $\mathcal{C}^\infty$ functions $f : \mathbb{C}^n \to \mathbb{C}$ that satisfy the functional equation* (2.2) *for all $z \in \mathbb{C}^n$.*

$\boxed{2}$ $\mathcal{F}^{\nu}_{\Gamma,\chi}(\mathbb{C}^n)$ *(resp. $\mathcal{O}^{\nu}_{\Gamma,\chi}(\mathbb{C}^n)$) of $\mathcal{C}^\infty$* (resp. *holomorphic) functions $g : \mathbb{C}^n \to \mathbb{C}$, satisfying the functional equation* (2.1) *for all $z \in \mathbb{C}^n$.*

We refer to this spaces as (ν, Γ, χ)-$\mathcal{C}^\infty$-automorphic functional spaces, or simply, when the context is clear, $\mathcal{C}^\infty$-automorphic functional spaces. $\mathcal{O}^{\nu}_{\Gamma,\chi}(\mathbb{C}^n)$ is called the space of holomorphic automorphic functions.

Note that the 0 in the notations is motivated by the fact that these spaces will be seen as particular cases of more general spaces that will be introduced in the next chapter associated with bi-wight (ν, μ).

There is traditional going back and forth between the spaces defined using equation (2.1) and the ones defined using (2.2). The first equation has the advantage of involving a holomorphic automorphic factor. Whereas, the other equation involves a unitary automorphic factor. We will give explicitly the transformation that ensures the transition after stating the following proposition, which gives sufficient and necessary condition on the triplet (ν, Γ, χ) in order to $\mathcal{F}^{\nu,0}_{\Gamma,\chi}(\mathbb{C}^n)$ and $\mathcal{F}^{\nu}_{\Gamma,\chi}(\mathbb{C}^n)$ be nonzero vector spaces. Namely, we have

Proposition 2.1.1. *The complex vector space $\mathcal{F}^{\nu}_{\Gamma,\chi}(\mathbb{C}^n)$ and $\mathcal{F}^{\nu,0}_{\Gamma,\chi}(\mathbb{C}^n)$ are nonzero spaces if and only if the triplet (ν, Γ, χ) satisfies the following condition*

$$\chi(\gamma_1 + \gamma_2) = \chi(\gamma_1)\chi(\gamma_2)e^{i\nu \Im m\langle \gamma_1, \gamma_2 \rangle} \tag{RDQ}$$

for every $\gamma_1, \gamma_2 \in \Gamma$. In this case $\mathcal{F}^{\nu}_{\Gamma,\chi}(\mathbb{C}^n)$ and $\mathcal{F}^{\nu,0}_{\Gamma,\chi}(\mathbb{C}^n)$ are infinite dimensional complex vector spaces.

Under the (RDQ) condition, the map χ satisfies the following properties:

$$\chi(0) = 1 \quad \text{and} \qquad \chi(-\gamma) = \overline{\chi(\gamma)}. \tag{2.3}$$

Also, by interchanging the roles of γ_1 and γ_2 in (RDQ) and using the fact that the symplectic form $\Im m \langle \cdot, \cdot \rangle$ is antisymmetric, we have necessarily

$$\nu \Im m \langle \gamma_1, \gamma_2 \rangle \in \pi\mathbb{Z} \tag{2.4}$$

for every $\gamma_1, \gamma_2 \in \Gamma$. Thus, by taking into consideration the orientation of the lattice and according to the above remark, it is highly useful to introduce N such that

$$N := \frac{\nu}{\pi} vol(\Lambda(\Gamma)) \in \mathbb{N},$$

where $vol(\Lambda(\Gamma))$ is the Lebesgue volume measure in $\mathbb{C}^n = \mathbb{R}^{2n}$ of a fundamental domain $\Lambda(\Gamma)$ of Γ.

Before giving the proof of the proposition, we begin by introducing the transformation that links $\mathcal{A}ut^{\nu,0}_{\Gamma,\chi}(\mathbb{C}^n)$ with $\mathcal{A}ut^{\nu}_{\Gamma,\chi}(\mathbb{C}^n)$ (*resp.* $\mathcal{F}^{\nu,0}_{\Gamma,\chi}(\mathbb{C}^n)$ with $\mathcal{F}^{\nu}_{\Gamma,\chi}(\mathbb{C}^n)$). Indeed, these spaces are isomorphic to each other, respectively, through the bijection

$$\begin{gathered} f \in \mathcal{A}ut^{\nu,0}_{\Gamma,\chi} \mapsto \mathfrak{G}f \in \mathcal{A}ut^{\nu}_{\Gamma,\chi}, \quad f \in \mathcal{F}^{\nu,0}_{\Gamma,\chi}(\mathbb{C}^n) \mapsto \mathfrak{G}f \in \mathcal{F}^{\nu}_{\Gamma,\chi}(\mathbb{C}^n); \\ \mathfrak{G}f(z) = e^{\frac{\nu}{2}|z|^2} f(z). \end{gathered} \tag{2.5}$$

Therefore, showing the non-triviality of $\mathcal{F}^{\nu,0}_{\Gamma,\chi}(\mathbb{C}^n)$ is equivalent to showing the the non-triviality of the equivalent space $\mathcal{F}^{\nu}_{\Gamma,\chi}(\mathbb{C}^n)$.

Proof of Proposition 2.1.1. The proof of "only if" follows by assuming that $\mathcal{F}^{\nu}_{\Gamma,\chi}$ is nontrivial space and next by computing $f(z+\gamma_1+\gamma_2)$ in two manners, for a given nonzero function $f \in \mathcal{F}^{\nu}_{\Gamma,\chi}$. Indeed, we have

$$f(z+\gamma_1+\gamma_2) = \chi(\gamma_1+\gamma_2)e^{i\nu\Im m\langle z,\gamma_1+\gamma_2\rangle}f(z), \tag{2.6}$$

and also

$$\begin{aligned} f(z+\gamma_1+\gamma_2) &= f([z+\gamma_1]+\gamma_2) = \chi(\gamma_2)e^{i\nu\Im m\langle z+\gamma_1,\gamma_2\rangle}f(z+\gamma_1) \\ &= \chi(\gamma_1)\chi(\gamma_2)e^{i\nu\Im m\langle \gamma_1,\gamma_2\rangle}e^{i\nu\Im m\langle z,\gamma_1+\gamma_2\rangle}f(z). \end{aligned} \tag{2.7}$$

Next, by equating the right hand sides of (2.6) and (2.7) and using the fact that f is not identically zero, we conclude that

$$\chi(\gamma_1+\gamma_2) = \chi(\gamma_1)\chi(\gamma_2)e^{i\nu\Im m\langle \gamma_1,\gamma_2\rangle} \tag{RDQ}$$

for all $\gamma_1, \gamma_2 \in \Gamma$.

For the converse, one can use some geometrical argument base on the realization of automorphic functions as sections of holomorphic line bundles over $\mathbb{C}^n/\Gamma$, or can use this direct proof based on the usage of Poicaré theta operator, which will be the central object in Chapter 5. In fact by classical analysis, one can pick any arbitrary non zero $\mathcal{C}^\infty$ function ψ on $\mathbb{C}^n$ such that Supp$\psi \subset \Lambda(\Gamma)$, where $\Lambda(\Gamma)$ is a fundamental domain of Γ in $\mathbb{C}^n$. Then, we can use the following lemma to show that the space $\mathcal{F}^{\nu}_{\Gamma,\chi}$ is a nonzero space and is of infinite dimension.

Lemma 2.1.2. *Suppose that the condition (RDQ) holds. Let ψ be a compactly supported $\mathcal{C}^\infty$ function such that Supp$\psi \subset \Lambda(\Gamma)$, and denote by $\mathcal{P}^{\nu}_{\Gamma,\chi}\psi$ the (Γ,χ)-periodization (à la Poincaré) of ψ given by*

$$[\mathcal{P}^{\nu}_{\Gamma,\chi}\psi](z) = \sum_{\gamma\in\Gamma} \overline{\chi(\gamma)}e^{-i\nu\Im m\langle z,\gamma\rangle}\psi(z+\gamma)$$

or equivalently (in view of (2.3)) by

$$[\mathcal{P}^{\nu}_{\Gamma,\chi}\psi](z) = \sum_{\gamma\in\Gamma} \chi(\gamma)e^{i\nu\Im m\langle z,\gamma\rangle}\psi(z-\gamma).$$

Then, we have

i) The function $\mathcal{P}^{\nu}_{\Gamma,\chi}\psi$ is a nonzero $\mathcal{C}^\infty$ function on $\mathbb{C}^n$.

ii) The function $\mathcal{P}^{\nu}_{\Gamma,\chi}\psi$ belongs to the space $\mathcal{F}^{\nu,0}_{\Gamma,\chi}(\mathbb{C}^n)$.

□

For more details we refer the reader to [34], where the proofs were reworked with more details in the Master thesis [40].

2.2 Properties of the spaces of standard automorphic functions

We endow $\mathcal{F}^{\nu,0}_{\Gamma,\chi}(\mathbb{C}^n)$, $\mathcal{F}^{\nu}_{\Gamma,\chi}(\mathbb{C}^n)$ and $\mathcal{O}^{\nu}_{\Gamma,\chi}(\mathbb{C}^n)$ by norms which turn them into pre-Hilbert spaces. Their completion, when Γ is full-rank, are given by L^2-Hilbert spaces of automorphic functions. We give in this section a concise description of these classes of automorphic functions and their properties without proof. For the proofs and more details, one can refer to [76]. We begin by defining appropriate norms on automorphic functional spaces

Lemma 2.2.1. *For any lattice Γ (maximal or not), let $f_1, f_2 \in \mathcal{F}^{\nu,0}_{\Gamma,\chi}(\mathbb{C}^n)$ and $g_1, g_2 \in \mathcal{F}^{\nu}_{\Gamma,\chi}(\mathbb{C}^n)$. Then, $f_1(z)\overline{f_2(z)}$, and $g_1(z)\overline{g_2(z)}\, e^{-\nu\langle z,z\rangle}$ are Γ-periodic.*

This lemma shows that

$$\langle f_1, f_2\rangle_\Gamma = \int_{\Lambda(\Gamma)} f_1(z)\overline{f_2(z)}d\lambda(z)$$

(*resp.*

$$\langle\langle g_1, g_2\rangle\rangle_\Gamma = \int_{\Lambda(\Gamma)} f_1(z)\overline{f_2(z)}\, e^{-\nu\langle z,z\rangle}\, d\lambda(z))$$

are well defined scalar products on $\mathcal{F}^{\nu,0}_{\Gamma,\chi}(\mathbb{C}^n)$ (*resp.* on $\mathcal{F}^{\nu}_{\Gamma,\chi}(\mathbb{C}^n)$), and independent of the choice of the fundamental domain $\Lambda(\Gamma)$. The square root of their evaluations at the diagonal are norms, which turn the respective space into a pre-Hilbert space. Being a sub-space, $\mathcal{O}^{\nu}_{\Gamma,\chi}(\mathbb{C}^n)$ inherit the scalar product of $\mathcal{F}^{\nu}_{\Gamma,\chi}(\mathbb{C}^n)$. Moreover, when Γ is maximal, it can be shown that $\mathcal{O}^{\nu}_{\Gamma,\chi}(\mathbb{C}^n)$ is finite dimensional, hence complete. Notice that $d\lambda(z)$ is the usual Lebesgue measure on $\mathbb{C}^n$.

We associate to $(\mathcal{F}^{\nu,0}_{\Gamma,\chi}(\mathbb{C}^n), \langle,\rangle_\Gamma)$ and $(\mathcal{F}^{\nu}_{\Gamma,\chi}(\mathbb{C}^n), \langle\langle,\rangle\rangle_\Gamma)$, respectively, the Hilbert spaces $(L^{2;\nu,0}_{\Gamma,\chi}(\mathbb{C}^n), \langle,\rangle_\Gamma)$ and $(L^{2;\nu}_{\Gamma,\chi}(\mathbb{C}^n), \langle\langle,\rangle\rangle_\Gamma)$, where $L^{2;\nu,0}_{\Gamma,\chi}(\mathbb{C}^n)$ and $L^{2;\nu}_{\Gamma,\chi}(\mathbb{C}^n)$ are defined as follows

Definition 2.3. *To given data (ν,Γ,χ), we define the Hilbert spaces*

$\boxed{1}$ $$L^{2;\nu,0}_{\Gamma,\chi}(\mathbb{C}^n) := \left\{ f \in \mathcal{A}ut^{\nu,0}_{\Gamma,\chi}(\mathbb{C}^n) \text{ such that } \int_{\Lambda(\Gamma)} |f|^2\, d\lambda(z) < +\infty \right\}$$

$\boxed{2}$ $$L^{2;\nu}_{\Gamma,\chi}(\mathbb{C}^n) := \left\{ g \in \mathcal{A}ut^{\nu}_{\Gamma,\chi}(\mathbb{C}^n) \text{ such that } \int_{\Lambda(\Gamma)} |g|^2\, e^{-\nu\langle z,z\rangle}\, d\lambda(z) < +\infty \right\}$$

When Γ is of maximal rank, $L^{2;\nu,0}_{\Gamma,\chi}(\mathbb{C}^n)$ and $L^{2;\nu}_{\Gamma,\chi}(\mathbb{C}^n)$ are the completions of $\mathcal{F}^{\nu,0}_{\Gamma,\chi}(\mathbb{C}^n)$ and $\mathcal{F}^{\nu}_{\Gamma,\chi}(\mathbb{C}^n)$ respectively. The result seems true also for non-maximal lattices but is not included in the cited reference.

2.3 Eigenvalue problem on the automorphic spaces $\mathcal{F}^{\nu,0}_{\Gamma,\chi}(\mathbb{C}^n)$ and $\mathcal{F}^{\nu}_{\Gamma,\chi}(\mathbb{C}^n)$

In this section we introduce the eigenvalue problem on the spaces $\mathcal{F}^{\nu,0}_{\Gamma,\chi}(\mathbb{C}^n)$ (*resp.* $L^{2;\nu,0}_{\Gamma,\chi}$) and $\mathcal{F}^{\nu}_{\Gamma,\chi}(\mathbb{C}^n)$ (*resp.* $L^{2;\nu}_{\Gamma,\chi}$). To this end, we shall introduce appropriate Laplacians associated with each automorphic factor defining these spaces. Let us begin by explaining what we mean by “appropriate”. Associated with a fixed element $\gamma \in \Gamma$ we define the following operators

$$\mathcal{T}^{\nu,0}_\gamma f(z) := \overline{\chi(\gamma)}\, e^{-\nu\Im m\langle z,\gamma\rangle}\, f(z+\gamma) \qquad \text{and} \qquad \mathcal{T}^{\nu}_\gamma g(z) := \overline{\chi(\gamma)}\, e^{-\frac{\nu}{2}|\gamma|^2-\nu\langle z,\gamma\rangle}\, g(z+\gamma), \tag{2.8}$$

which represent respectively the automorphic equations (2.2) and (2.1). In our wording, an appropriate or invariant Laplacian with respect to an automorphic equation, refer to some geometrical Laplacian such that it commutes with the operator of type $\mathcal{T}_\gamma$ (in concordance with (2.8)) representing that automorphic equation for all $\gamma \in \Gamma$.

Accordingly, if we consider

$$\boxed{\widetilde{\Delta}_{\nu,0} := -\left\{ \sum_{j=1}^n \frac{\partial^2}{\partial z_j\partial\bar{z}_j} + \frac{\nu}{2}(E-\overline{E}) - \frac{\nu^2}{4}|z|^2 \right\}} \tag{2.9}$$

and

$$\boxed{\Delta_\nu := -\left\{ \sum_{j=1}^n \frac{\partial^2}{\partial z_j\partial\bar{z}_j} - \nu\overline{E} \right\},} \tag{2.10}$$

where E is the Euler operator $E := z\frac{\partial}{\partial z}$ and $\overline{E}$ is its formal complex conjugate $\overline{E} := \bar{z}\frac{\partial}{\partial\bar{z}}$, we can show the invariance properties given in the proposition below. Notice that the tilde over $\Delta_{\nu,0}$ in (2.9) is present for an irrelevant rescaling factor in this context ($\widetilde{\Delta}_{\nu,0} := -1/4\,\Delta_{\nu,0}$), though important to unify the notations with a much general differential operator $\Delta_{\nu,\mu}$ introduced in the next chapter.

Proposition 2.3.1. *For all $g \in \Gamma$*

$$\mathcal{T}_\gamma^{\nu,0}\widetilde{\Delta}_{\nu,0} = \widetilde{\Delta}_{\nu,0}\mathcal{T}_\gamma^{\nu,0} \qquad \textit{and} \qquad \mathcal{T}_\gamma^{\nu}\Delta_\nu = \Delta_\nu\mathcal{T}_\gamma^{\nu} \tag{2.11}$$

These commutation property of $\widetilde{\Delta}_{\nu,0}$ and Δ_ν allows the definition of the eigenvalue problems

$$(P)_{\mathcal{F}^{\nu,0}_{\Gamma,\chi}} \begin{cases} \widetilde{\Delta}_{\nu,0} f = \lambda f \\ f \in \mathcal{F}^{\nu,0}_{\Gamma,\chi}(\mathbb{C}^n) \end{cases} \quad \text{and} \quad (P)_{L^{2;\nu,0}_{\Gamma,\chi}} \begin{cases} \widetilde{\Delta}_{\nu,0} f = \lambda f \\ f \in L^{2;\nu,0}_{\Gamma,\chi}(\mathbb{C}^n), \end{cases} \tag{2.12}$$

as well as

$$(P)_{\mathcal{F}^{\nu}_{\Gamma,\chi}} \begin{cases} \Delta_{\nu} f = \lambda f \\ f \in \mathcal{F}^{\nu}_{\Gamma,\chi}(\mathbb{C}^n) \end{cases} \quad \text{and} \quad (P)_{L^{2;\nu}_{\Gamma,\chi}} \begin{cases} \Delta_{\nu} f = \lambda f \\ f \in L^{2;\nu}_{\Gamma,\chi}(\mathbb{C}^n) \end{cases} . \tag{2.13}$$

These two set of problems are related by the same transform $\mathfrak{G}$ given in (2.5), so the solution of one of them can be recovered from the solution of the other. The treatment of $(P)_{\mathcal{F}^{\nu,0}_{\Gamma,\chi}}$ and $(P)_{L^{2;\nu,0}_{\Gamma,\chi}}$ when Γ is a full-rank lattice of $\mathbb{C}^n$ is given in the next chapter as special cases of the eigenvalue problems considered there on mixed automorphic functions of first kind. More broadly, the study of eigenvalue problems of the above type depends hugely on the dimension of the space ($n = 1$ or $n > 1$) and the rank of the lattice Γ. For instance when Γ is maximal and $n \geq 1$, the spectrum can be found using Selberg technique [34], which will be adapted to investigate the spectrum in the next chapter. Alternatively, it can be found by studying the coefficients in a Fourier-like expansion of automorphic functions [44, 63], which allows not only to find the spectrum but also to build an orthogonal basis on the spaces of L^2 automorphic functions. In the next section, we give the method to build such bases in one dimension as well as in high dimensions for some particular full rank lattices of $\mathbb{C}^n$.

2.4 Orthogonal basis of the spaces of automorphic functions on $\mathbb{C}$

When manipulating automorphic functions, one practice keeps appearing in many contexts is to try to transform, somehow, automorphic functions to periodic functions, then use the much richer arsenal we process on periodic functions, then again try to carry back the information found to the original automorphic functions. A good illustration of this practice is demonstrated in this section where we are giving a step by step of the construction of a basis of the spaces $\mathcal{F}^{\nu,0}_{\Gamma,\chi}(\mathbb{C})$ and $\mathcal{F}^{\nu}_{\Gamma,\chi}(\mathbb{C})$. It illustrates also how one factor of automorphy can present advantages in working with in comparison with the other although being in some sort geometrically equivalent.

2.4.1 Basis for $\mathcal{F}^{\nu,0}_{\Gamma,\chi}(\mathbb{C})$

In this subsection we construct explicit orthogonal basis of $\mathcal{F}^{\nu,0}_{\Gamma,\chi}(\mathbb{C})$ by constructing orthogonal basis of each eigenspace of the invariant operator

$$\widetilde{\Delta}^{\Gamma,\chi}_{\nu,0} := -\left\{ \sum_{j=1}^{n} \frac{\partial^2}{\partial z_j \partial \bar{z}_j} + \frac{\nu}{2}(E - \overline{E}) - \frac{\nu^2}{4}|z|^2 \right\}, \tag{2.14}$$

when acting on the space $\mathcal{F}^{\nu,0}_{\Gamma,\chi}(\mathbb{C})$. Notice that $\widetilde{\Delta}^{\Gamma,\chi}_{\nu,0}$ (confounded with its Friedrichs extension) is self-adjoint and positive. Therefor, the family constructed as the union of the basis of each eigenspace is a basis of the hole space $\mathcal{F}^{\nu,0}_{\Gamma,\chi}(\mathbb{C})$. To achieve this, we are adopting a constructive approach based on the Fourier expansion of periodic functions.

For simplification and without loss of generality, being co-compact discrete subgroup of $\mathbb{C}^n$, Γ is chosen to have the form $\mathbb{Z} + \tau\mathbb{Z}$, where τ is some complex number in the upper complex half plane. The automorphy equation

$$f(z + \gamma) = \chi(\gamma)e^{i\nu\Im m\langle z,\gamma\rangle} f(z)$$

gives rise to two sub-equations

$$f(z+1) = \chi(1) e^{i\frac{\pi N}{\Im m\tau}\Im m z} f(z) \tag{2.15a}$$

$$f(z+\tau) = \chi(\tau) e^{i\frac{\pi N}{\Im m\tau}\Im m(\bar{\tau} z)} f(z). \tag{2.15b}$$

Now, since $\chi(1)$ and $\chi(\tau)$ are of modulus 1, they can be respectively represented as

$$\chi(1) = e^{2i\pi\alpha} \quad \text{and} \quad \chi(\tau) = e^{2i\pi\beta} \tag{2.16}$$

for some reals α and β. Next, we transform f to a periodic function in the direction of the real axis. Indeed, we claim the following (whose the proof holds by direct computations)

Lemma 2.4.1. *For every $f \in \mathcal{F}^{\nu,0}_{\Gamma,\chi}(\mathbb{C})$, the function*

$$F(z) = \mathrm{e}^{-i\frac{\pi N}{2\Im m\tau}\Im m(z^2) - 2i\pi z\alpha} f(z), \tag{2.17}$$

verifies the following equations

$$F(z+1) = F(z) \tag{2.18a}$$

$$F(z+\tau) = e^{-2i\pi N\Re(z) - i\pi N\Re(\tau) + 2i\pi(\beta - \tau\alpha)} F(z). \tag{2.18b}$$

Remark 2.4.2. *It is worth pointing out that the function substitution in* (2.17) *is not easy to remark, a systematic way to find such function substitution is provided in the next subsection.*

Being periodic in the direction of x axis, F can be formally expended into Fourier series as follows

$$F(z) = \sum_{n\in\mathbb{Z}} a_n(y)\, \mathrm{e}^{2i\pi n x}. \tag{2.19}$$

Subsequently

$$f(z) = \mathrm{e}^{i\frac{\pi N}{2\Im m\tau}\Im m(z^2) + 2i\pi z\alpha} \sum_{n\in\mathbb{Z}} a_n(y)\, \mathrm{e}^{2i\pi n x}, \tag{2.20}$$

where z is written in the basis $\{1, \tau\}$, $z = x + y\tau$; $x, y \in \mathbb{R}$. Now, using (2.18b) we can prove the following lemma

Lemma 2.4.3. *The coefficients $a_n(y)$ verifies the following equation*

$$a_n(y+1) = \mathrm{e}^{-i\pi N(2y+1)\Re(\tau) + 2i\pi(\beta - \tau\alpha)}\, a_{n+N}(y). \tag{2.21}$$

Proof. Inserting Fourier expansion (2.19) of F in Equation (2.18b), we can query out the following computations

$$\begin{aligned}
\sum_{n\in\mathbb{Z}} a_n(y+1)\, \mathrm{e}^{2i\pi n x} &= e^{-2i\pi N\Re(z) - i\pi N\Re(\tau) + 2i\pi(\beta-\tau\alpha)} \sum_{n\in\mathbb{Z}} a_n(y)\, \mathrm{e}^{2i\pi n x} \\
&= \sum_{n\in\mathbb{Z}} \mathrm{e}^{-i\pi N(2y+1)\Re(\tau) + 2i\pi(\beta-\tau\alpha)}\, a_n(y)\, \mathrm{e}^{2i\pi(n-N)x} \\
&= \sum_{n\in\mathbb{Z}} \mathrm{e}^{-i\pi N(2y+1)\Re(\tau) + 2i\pi(\beta-\tau\alpha)}\, a_{n+N}(y)\, \mathrm{e}^{2i\pi n x}.
\end{aligned}$$

As the expansion is unique for every fixed y, we find the equation (2.21) for the Fourier coefficients $a_n(y)$. □

Lemma 2.14 implies that the Fourier expansion (2.19) is fully determined once N consecutive a_n are fixed.

To give the explicit form of the coefficients $a_n(y)$, we need to utilize the fact that f is an eigenfunction of the operator $\widetilde{\Delta}^{\Gamma,\chi}_{\nu,0}$. We implement the process of the *factorization method* (see Subsection 3.5.1). We begin by providing a basis for the first level (i.e the eigenspace associated with the smallest eigenvalue). Then, by the creation operator, we generate basis for the levels above. To this end, recall that the operator $\widetilde{\Delta}^{\Gamma,\chi}_{\nu,0}$ has a decomposition in the following form

$$\widetilde{\Delta}^{\Gamma,\chi}_{\nu,0} = a^- a^+ - \frac{\nu}{2} = a^+ a^- + \frac{\nu}{2}, \tag{2.22}$$

where

$$a^+ = -\frac{\partial}{\partial z} + \frac{\nu}{2}\bar{z} \qquad \text{and} \qquad a^- = \frac{\partial}{\partial \bar{z}} + \frac{\nu}{2} z, \tag{2.23}$$

which are called respectively creation and annihilation operators. Thus, if the function f is in the first level, then f vanishes when the annihilation operator is applied to it:

$$\left(\frac{\partial}{\partial \bar{z}} + \frac{\nu}{2} z\right) f = 0. \tag{2.24}$$

Replacing f using (2.17) and (2.19) yields

$$\left(\frac{\partial}{\partial \bar{z}} + \frac{\nu}{2} z\right) \mathrm{e}^{i\frac{\pi N}{2\Im m\tau}\Im m(z^2)+2i\pi z\alpha} \sum_{n\in\mathbb{Z}} a_n(y)\, \mathrm{e}^{2i\pi n x} = 0, \tag{2.25}$$

and equivalently

$$\begin{aligned}\sum_{n\in\mathbb{Z}} \Bigg\{ -\frac{\nu}{2}\bar{z}\, \mathrm{e}^{i\frac{\nu}{2}\Im m(z^2)+2i\pi z\alpha}\, a_n(y)\, \mathrm{e}^{2i\pi n x} + e^{i\frac{\nu}{2}\Im m(z^2)+2i\pi z\alpha} \frac{\partial}{\partial \bar{z}}\left(a_n(y)\, \mathrm{e}^{2i\pi n x}\right) \\ + \frac{\nu}{2} z\, \mathrm{e}^{i\frac{\pi N}{2\Im m\tau}\Im m(z^2)+2i\pi z\alpha}\, a_n(y)\, \mathrm{e}^{2i\pi n x} \Bigg\} = 0.\end{aligned} \tag{2.26}$$

Using the change of coordinates with the help of the multi-variable chain rule we can easily find the partial derivatives with respect to z and $\bar{z}$ in terms of ones with respect to x and y. Indeed,

$$\begin{cases} z = x + y\tau \\ \bar{z} = x + y\bar{\tau} \end{cases} \quad \longrightarrow \quad \begin{cases} x = \frac{1}{\bar{\tau}-\tau}(\bar{\tau} z - \tau \bar{z}) = \dfrac{\Im m(\tau\bar{z})}{\Im m(\tau)} \\ y = \frac{1}{\bar{\tau}-\tau}(\bar{z} - z) = \frac{\Im z}{\Im m(\tau)} \end{cases}$$

yields

$$\frac{\partial}{\partial z} = \frac{\bar{\tau}}{\bar{\tau}-\tau}\frac{\partial}{\partial x} - \frac{1}{\bar{\tau}-\tau}\frac{\partial}{\partial y} \quad \text{and} \quad \frac{\partial}{\partial \bar{z}} = \frac{-\tau}{\bar{\tau}-\tau}\frac{\partial}{\partial x} + \frac{1}{\bar{\tau}-\tau}\frac{\partial}{\partial y}.$$

Therefore, (2.26) reduces further to

$$\begin{aligned}\sum_{n\in\mathbb{Z}} \Bigg\{ \frac{\nu}{2}(z-\bar{z})\, \mathrm{e}^{i\frac{\nu}{2}\Im m(z^2)+2i\pi z\alpha}\, a_n(y) \\ + e^{i\frac{\nu}{2}\Im m(z^2)+2i\pi z\alpha} \left(\frac{-2\pi i n\tau}{\bar{\tau}-\tau} a_n(y) + \frac{1}{\bar{\tau}-\tau} a_n'(y)\right) \Bigg\} \mathrm{e}^{2i\pi n x} = 0.\end{aligned}$$

Hence, after the usage of the uniqueness of Fourier expansion and simplifications we find that for any $n \in \mathbb{Z}$, we have

$$a_n'(y) = 2i\Im m\tau(\nu i \Im m(z) + 2\pi i n) a_n(y) = (-2\pi N \Im m(z) + 2i\pi\tau) a_n(y).$$

Moreover, we can prove the following lemma

Lemma 2.4.4. *If $f \in \mathcal{F}^{\nu,0}_{\Gamma,\chi}(\mathbb{C})$ admitting the expansion* (2.20) *is an eigenfunction associated with the smallest eigenvalue of $\widetilde{\Delta}^{\Gamma,\chi}_{\nu,0}$. Then, there exist some constants k_n such that*

$$a_n(y) = k_n\, \mathrm{e}^{-\pi N y^2 \Im m\tau + 2i\pi n y\tau}. \tag{2.27}$$

The constants k_n are given explicitly by

$$k_n = l_n \, \mathrm{e}^{i\pi \frac{n^2\tau}{N} - 2i\pi(\beta-\tau\alpha)\frac{n}{N}} .$$

Furthermore,

$$f(z) = \mathrm{e}^{i\nu z \Im m(z) + 2i\pi z\alpha} \sum_{n\in\mathbb{Z}} l_n \, \mathrm{e}^{i\pi \frac{n^2\tau}{N} - 2i\pi(\beta-\tau\alpha)\frac{n}{N}} \, \mathrm{e}^{2i\pi nz}, \tag{2.28}$$

where l_n are real constants such that $l_{n+N} = l_n$ for every $n \in \mathbb{Z}$.

Proof. The equation (2.27) is clear from the development done above, while (2.28) is a result that can be obtained as follows

$$\begin{aligned} f(z) &= \mathrm{e}^{i\frac{\pi N}{2\Im m\tau}\Im m(z^2) + 2i\pi z\alpha} \sum_{n\in\mathbb{Z}} k_n \, \mathrm{e}^{-\pi N y^2 \Im m\tau + 2i\pi n y\tau} \, \mathrm{e}^{2i\pi nx} \\ &= \mathrm{e}^{i\pi N y^2(\Re(\tau) + i\Im m(\tau)) + i\pi Nxy} \, \mathrm{e}^{2i\pi z\alpha} \sum_{n\in\mathbb{Z}} k_n \, \mathrm{e}^{2i\pi nz} \\ &= \mathrm{e}^{i\pi Ny(y\tau+x)} \, \mathrm{e}^{2i\pi z\alpha} \sum_{n\in\mathbb{Z}} k_n \, \mathrm{e}^{2i\pi nz} \\ &= \mathrm{e}^{i\nu z\Im m(z)} \, \mathrm{e}^{2i\pi z\alpha} \sum_{n\in\mathbb{Z}} k_n \, \mathrm{e}^{2i\pi nz} . \end{aligned}$$

By combining the results in Lemmas 2.4.3 and 2.4.4, we find that

$$\begin{aligned} k_n &= \mathrm{e}^{\pi N(2y+1)\Im m\tau - 2i\pi n\tau} \, \mathrm{e}^{-i\pi N(2y+1)\Re\tau + 2i\pi(\beta-\tau\alpha) + 2i\pi Ny\tau} \, k_{n+N} \\ &= \mathrm{e}^{-i\pi N(2y+1)(\Re\tau + i\Im m\tau) + 2i\pi Ny\tau} \, \mathrm{e}^{-2i\pi n\tau + 2i\pi(\beta-\tau\alpha)} \, k_{n+N} \\ &= \mathrm{e}^{-i\pi N\tau - 2i\pi n\tau + 2i\pi(\beta-\tau\alpha)} \, k_{n+N}. \end{aligned}$$

The equation

$$k_{n+N} = \mathrm{e}^{i\pi N\tau + 2i\pi n\tau - 2i\pi(\beta-\tau\alpha)} \, k_n$$

has solutions of the form

$$k_n = l_n \, \mathrm{e}^{i\pi \frac{n^2\tau}{N} - 2i\pi(\beta-\tau\alpha)\frac{n}{N}},$$

where l_n are real constants such that $l_{n+N} = l_n$; $n \in \mathbb{Z}$. This ends the proof of the lemma. □

Accordingly, we can provide an orthogonal basis in terms of Riemann theta function with characteristics a and b defined by

$$\vartheta_{a,b}(z|\tau) = \sum_{n\in\mathbb{Z}} \mathrm{e}^{i\pi\tau(n+a)^2 + 2i\pi(n+a)(z+b)} . \tag{2.29}$$

Theorem 2.4.5. *An orthogonal basis for the first level is given by the N eigenfunctions:*

$$\widetilde{\varphi}_j^{\alpha,\beta}(z|\tau;N) := \mathrm{e}^{i\nu z\Im m(z) + 2i\pi z\alpha} \, \vartheta_{\frac{j}{N},\alpha\tau-\beta}(Nz|N\tau), \tag{2.30}$$

with

$$\left\| \widetilde{\varphi}_j^{\alpha,\beta}(\cdot|\tau;N) \right\|^2_{L^{2;\nu,0}_{\Gamma,\chi}(\mathbb{C})} = \mathrm{e}^{2\pi\Im m(\tau)\frac{\alpha^2}{N}} \sqrt{\frac{\Im m(\tau)}{2N}}. \tag{2.31}$$

Proof. Starting from (2.28), there exist $l_j; j \in \{1, 2, \cdots, N\}$, such that the Fourier expansion of the function f becomes

$$\begin{aligned}
f(z) &= e^{i\nu z \Im m(z)+2i\pi z\alpha} \sum_{n\in\mathbb{Z}} k_n \, e^{2i\pi nz} \\
&= \sum_{j=0}^{N-1} l_j \quad e^{i\nu z\Im m(z)+2i\pi z\alpha} \sum_{\substack{n\in\mathbb{Z} \\ n=j \mod N}} \quad e^{i\pi \frac{n^2\tau}{N} - 2i\pi(\beta-\tau\alpha)\frac{n}{N}} \, e^{2i\pi nz} \\
&= \sum_{j=0}^{N-1} l_j \quad e^{i\nu z\Im m(z)+2i\pi z\alpha} \sum_{n\in\mathbb{Z}} e^{i\pi\tau\frac{(nN+j)^2}{N} - 2i\pi(\beta-\tau\alpha)\frac{(nN+j)}{N}} \, e^{2i\pi(nN+j)z} \\
&= \sum_{j=0}^{N-1} l_j \quad e^{i\nu z\Im m(z)+2i\pi z\alpha} \sum_{n\in\mathbb{Z}} e^{i\pi N\tau(n+\frac{j}{N})^2 + 2i\pi(n+\frac{j}{N})(Nz-\beta+\tau\alpha)} .
\end{aligned} \tag{2.32}$$

From the above construction, it is clear that $\left\{\widetilde{\varphi}_j^{\alpha,\beta}\right\}_{j=0,\cdots,N-1}$ spans the first level of Δ_ν acting on the Hilbert space of automorphic functions. It remains to show that $\left\{\widetilde{\varphi}_j^{\alpha,\beta}\right\}_{j=1,\cdots,N}$ is an orthogonal family then compute the norm of its elements. Indeed, we have

$$\begin{aligned}
\left\langle \widetilde{\varphi}_j^{\alpha,\beta}, \widetilde{\varphi}_k^{\alpha,\beta} \right\rangle_\Gamma &= \int_{\Lambda(\Gamma)} \widetilde{\varphi}_j^{\alpha,\beta} \overline{\widetilde{\varphi}_k^{\alpha,\beta}} dm \\
&= \Im m(\tau) \int_0^1 \int_0^1 \widetilde{\varphi}_j^{\alpha,\beta} \overline{\widetilde{\varphi}_k^{\alpha,\beta}} dxdy \\
&= \Im m(\tau) \int_0^1 \int_0^1 e^{i\frac{\pi N}{\Im m\tau}\Im m(z)(z-\bar{z})+2i\pi\alpha(z-\bar{z})} \sum_{n,m\in\mathbb{Z}} e^{i\pi N(\tau(n+j/N)^2 - \bar{\tau}(m+k/N)^2} \\
&\qquad e^{2i\pi N(z(n+j/N)-\bar{z}(m+k/N))} \, e^{2i\pi((n+j/N)(\alpha\tau-\beta)-(m+k/N)(\alpha\bar{\tau}-\beta)} .
\end{aligned} \tag{2.33}$$

Then, by Fubini's theorem together with the normal convergence of the series on compacts, we invert the summation and the integral. We then find that the only term that depends on x can be evaluated as follows

$$\int_0^1 e^{2i\pi Nx((n+j/N)-(m+k/N))} = \int_0^1 e^{2i\pi x(Nn-Nm+j-k)} \, dx,$$

which is zero whenever $Nn - Nm + j - k = 0$ and 1 otherwise. Using that $0 \le j, k \le N-1$, we get that

$$\int_0^1 e^{2i\pi x(Nn-Nm+j-k)} \, dx = \delta_{j,k}\delta_{m,n}.$$

Hence the orthogonality of the family $\left\{\widetilde{\varphi}_j^{\alpha,\beta}\right\}_{j=1,\cdots,N}$ follows.

Let us now compute the norm of an element $\widetilde{\varphi}_j^{\alpha,\beta}$. Notice first that the only remaining variable of integration in (2.33) is y, and the summation over the double indices n and m reduces further to a simple summation over n, to wit

$$\begin{aligned}
\left\langle \widetilde{\varphi}_j^{\alpha,\beta}, \widetilde{\varphi}_j^{\alpha,\beta} \right\rangle_\Gamma &= \Im m(\tau) \sum_{n,m\in\mathbb{Z}} \int_0^1 e^{-2\frac{\pi N}{\Im m\tau}(y\Im m(\tau))^2 - 4\pi N\frac{\alpha}{N}y\Im m(\tau)} \, e^{-2\pi N\Im m(\tau)\tau(n+j/N)^2} \, e^{-4\pi N(n+j/N)y\Im m(\tau)} \\
&\qquad e^{-4\pi(n+j/N)\alpha\Im m(\tau)} \, dy \\
&= \Im m(\tau) \sum_{n,m\in\mathbb{Z}} \int_0^1 e^{-2\pi N\Im m(\tau)(y+n+j/N+\alpha/N)^2} \, e^{2\pi N\Im m(\tau)(\alpha/N)^2} \, e^{2\pi N\Im m(\tau)(\frac{\alpha}{N})^2} \, dy.
\end{aligned}$$

By the viable substitution of $y + n$ by y we recognize the Gaussian integral

$$\left\langle \widetilde{\varphi}_j^{\alpha,\beta}, \widetilde{\varphi}_j^{\alpha,\beta} \right\rangle_\Gamma = \Im m(\tau) \, e^{2\pi\Im m(\tau)\frac{\alpha^2}{N}} \int_{\mathbb{R}} e^{-2\pi N\Im m(\tau)(y+\frac{j+\alpha}{N})^2} \, dy = e^{2\pi\Im m(\tau)\frac{\alpha^2}{N}} \sqrt{\frac{\Im m(\tau)}{2N}}.$$

□

Remark 2.4.6. *To get the basis for both the high levels, we only need to iterate each $\widetilde{\varphi}_j^{\alpha,\beta}$ by means of the creation operators. In addition, to generalize to higher dimensions we need to perform special products between copies of the basis in one dimensional case. These ideas will be developed next for the equivalent space $\mathcal{F}^{\nu}_{\Gamma,\chi}(\mathbb{C}^n)$.*

2.4.2 Basis for $\mathcal{F}^{\nu}_{\Gamma,\chi}(\mathbb{C})$

In this subsection we follow the same path as in the above one and build an explicit orthogonal basis for $\mathcal{F}^{\nu}_{\Gamma,\chi}(\mathbb{C})$ by constructing orthogonal basis of each eigenspace of the invariant operator

$$\Delta_{\nu}^{\Gamma,\chi} := -\sum_{j=1}^{n}\left\{\frac{\partial^2}{\partial z_j \partial \bar{z}_j} - \nu\overline{E}\right\}, \tag{2.34}$$

when acting on the space $\mathcal{F}^{\nu}_{\Gamma,\chi}(\mathbb{C})$. We always confound the differential operator with its Friedrichs extension, which exists because $\Delta_{\nu}^{\Gamma,\chi}$ is symmetric and non-negative. The computations are quite similar, with the advantage that the expansion of holomorphic simply periodic (i.e., periodic in one direction) functions in Fourier series is given by constant coefficients versus coefficients that depend on y in the above case.

Recall that the first level is exactly the space of holomorphic automorphic functions $\mathcal{O}^{\nu}_{\Gamma,\chi}(\mathbb{C})$ consisting of entire functions satisfying the functional equation

$$f(z+\gamma) = \chi(\gamma)e^{\nu(z+\frac{\gamma}{2})\bar{\gamma}}f(z) \tag{2.35}$$

for all $z \in \mathbb{C}$ and $\gamma \in \Gamma_\tau = \mathbb{Z} + \tau\mathbb{Z}$.
Let us begin by providing a systematic way to ansatz a function change that leads to periodicity in one direction as we did in the above subsection.

Lemma 2.4.7 (Function substitution). *Let $f \in \mathcal{O}^{\nu}_{\Gamma,\chi}$. Then, there exists a polynomial P such that $h_f(z) = e^{P(z)}f(z)$ is $\mathbb{Z}$-periodic. Moreover, P is given by*

$$P(z) = -\frac{\nu}{2}z^2 - 2i\pi\alpha z.$$

Proof. For given $f \in \mathcal{O}^{\nu}_{\Gamma,\chi}$, $h_f(z) := e^{P_f(z)}f(z)$ and $h_f(z+1) = h_f(z)$ is equivalent to

$$\begin{aligned} h(z+1) &= e^{P(z+1)}f(z+1) = e^{a(z+1)^2+b(z+1)}f(z+1) \\ &= \chi(1)e^{a(az+1)+b}e^{\nu(z+\frac{1}{2})}e^{az^2+bz}f(z) \\ &= \chi(1)e^{(2a+\nu)z+a+b+\frac{\nu}{2}}h_f(z) \\ &= e^{(2a+\nu)z+a+b+\frac{\nu}{2}+2i\pi\alpha}h(z), \end{aligned}$$

where the real α, as in the previous subsection, is such that $\chi(1) = e^{2i\pi\alpha}$. Pick also $\beta \in \mathbb{R}$ such that $\chi(\tau) = e^{2i\pi\beta}$. Then, $h_f(z) = h_f(z+1)$ is equivalent to the linear system

$$\begin{cases} 2a+\nu = 0 \\ a+b+\frac{\nu}{2}+2i\pi\alpha = 0 \end{cases} \Leftrightarrow \begin{cases} 2a+\nu = 0 \\ a+b+\frac{\nu}{2}+2i\pi\alpha = 0 \end{cases} \Leftrightarrow \begin{cases} a = -\frac{\nu}{2} \\ b = -2i\pi\alpha \end{cases}.$$

The lemma follows because P is independent of f. $\square$

Using the above function substitution we find that

$$\begin{aligned} h(z+\tau) &= \chi(\tau)e^{-2i\pi\alpha\tau}e^{-\nu(z+\frac{\tau}{2})(\tau-\bar{\tau})}h(z) \\ &= e^{2i\pi(\beta-\alpha\tau)}e^{-iN\pi(2z+\tau)}h(z). \end{aligned} \tag{2.36}$$

Let us now borrow from Bargmann work (1967) [7, page 79], the following assertion

Lemma 2.4.8. *Let L be a lattice in $\mathbb{R}^n \subset \mathbb{R}^n + i\mathbb{R}^n = \mathbb{C}^n$ and h a holomorphic function on $\mathbb{C}^n$ such that $h_f(z+l) = h(z)$, for every $z \in \mathbb{C}^n$ and $l \in \Gamma$. Then, there exists unique sequence of complex numbers $(a_{l^*})_{l^* \in L^*}$*

$$h(z) = \sum_{l^* \in L^*} a_{l^*} e^{i\langle z, l^* \rangle_{\mathbb{C}^n}},$$

with L^* is the dual lattice of L, $l^* \in L^*$ if and only if $\langle l^*, l \rangle_{\mathbb{C}^n}$ is an integer for all $l \in L$

In our context $L = \mathbb{Z}$ so $L^* = 2\pi\mathbb{Z}$. Thus by the above lemma we expand h as

$$h(z) = \sum_{k \in \mathbb{Z}} a_k e^{izk^*}; \quad k^* = 2\pi k,$$

where the coefficients a_k are given by

$$a_k = \frac{1}{S(\Lambda(\Gamma))} \int_{\Lambda(\Gamma)} h(z) e^{-2i\pi k\bar{z}} d\lambda(z).$$

Therefore,

$$f(z) = e^{\frac{\nu}{2}z^2 + 2i\pi z\alpha} \sum_{k \in \mathbb{Z}} a_k e^{2i\pi zk}. \tag{2.37}$$

Thus (2.36) reads

$$h(z+\tau) = e^{2i\pi(\beta - \alpha\tau)} e^{-iN\pi(2z+\tau)} h(z).$$

In terms of power series

$$\begin{aligned}
\sum_{k \in \mathbb{Z}} a_k e^{2i\pi zk} e^{2i\pi\tau k} &= e^{2i\pi(\beta-\alpha\tau)} e^{-2i\pi zN} e^{-2i\pi\tau\frac{N}{2}} \sum_{k \in \mathbb{Z}} a_k e^{2i\pi zk}, \\
&= \sum_{k \in \mathbb{Z}} a_k e^{2i\pi(\beta-\alpha\tau)} e^{-2i\pi\tau\frac{N}{2}} e^{2i\pi z(k-N)}, \\
&= \sum_{k' \in \mathbb{Z}} \left(a_{(k'+N)} e^{2i\pi(\beta-\alpha\tau)} e^{-2i\pi\tau\frac{N}{2}} \right) e^{2i\pi zk'}.
\end{aligned}$$

By identification, due to the uniqueness of the expansion, we get

$$a_{(k'+N)} = e^{2i\pi\tau(k'+\frac{N}{2})} e^{-2i\pi(\beta-\alpha\tau)} a_{k'}.$$

Hence, f is completely determined by $a_0, a_1, \ldots, a_{N-1}$. The other coefficients are obtained by the recurrence property

$$a_{j+kN} = e^{2i\pi\tau(kj+k^2\frac{N}{2})} e^{-2i\pi k(\beta-\alpha\tau)} a_j,$$

which can be shown by induction. Return back to (2.37)

$$\begin{aligned}
f(z) &= e^{\frac{\nu}{2}z^2 + 2i\pi\alpha z} \sum_{k \in \mathbb{Z}} a_k e^{2i\pi zk} \\
&= e^{\frac{\nu}{2}z^2 + 2i\pi\alpha z} \sum_{j=0}^{N-1} \sum_{k \in \mathbb{Z}} a_{j+kN} e^{2i\pi z(j+kN)} \\
&= e^{\frac{\nu}{2}z^2 + 2i\pi\alpha z} \sum_{j=0}^{N-1} a_j \sum_{k \in \mathbb{Z}} e^{-2i\pi k(\beta-\alpha\tau)} e^{2i\pi\tau(kj+k^2\frac{N}{2})} e^{2i\pi z(j+kN)} \\
&= \sum_{j=0}^{N-1} a_j \varphi_j^{\alpha,\beta}(z|\tau; N),
\end{aligned}$$

where $\varphi_j^{\alpha,\beta}(z|\tau; N)$ stands for

$$\varphi_j^{\alpha,\beta}(z|\tau; N) = e^{\frac{\nu}{2}z^2 + 2i\pi\alpha z} \sum_{k \in \mathbb{Z}} e^{-2i\pi k(\beta-\alpha\tau)} e^{2i\pi\tau(kj+k^2\frac{N}{2})} e^{2i\pi z(j+kN)}.$$

The above function can be reexpressed in terms of $\vartheta_{a,b}$ in (2.29) as follows

$$\begin{aligned}\varphi_j^{\alpha,\beta}(z|\tau;N) &= e^{\frac{\nu}{2}z^2}e^{2i\pi(\alpha+j)z}\vartheta_{0,0}(Nz+(j+\alpha)\tau-\beta|N\tau)\\ &= e^{\frac{\nu}{2}z^2}e^{2i\pi(\alpha+j)z}\vartheta_{0,j\tau}(Nz-\beta+\alpha\tau|N\tau)\\ &= e^{\frac{\nu}{2}z^2}e^{2i\pi(\alpha+j)z}\vartheta_{0,(j+\alpha)\tau-\beta}(Nz|N\tau).\end{aligned}$$

Accordingly, we have proved the following main theorem

Theorem 2.4.9. *The functions*

$$\varphi_j^{\alpha,\beta}(z|\tau;N) := e^{\frac{\nu}{2}z^2}e^{2i\pi(\alpha+j)z}\vartheta_{0,0}(Nz+(\alpha\tau-\beta)+j\tau|N\tau), \tag{2.38}$$

where $j=0,1,\cdots,N-1$, *form a basis of the functional space* $\mathcal{O}^{\nu}_{\Gamma,\chi}(\mathbb{C})$.

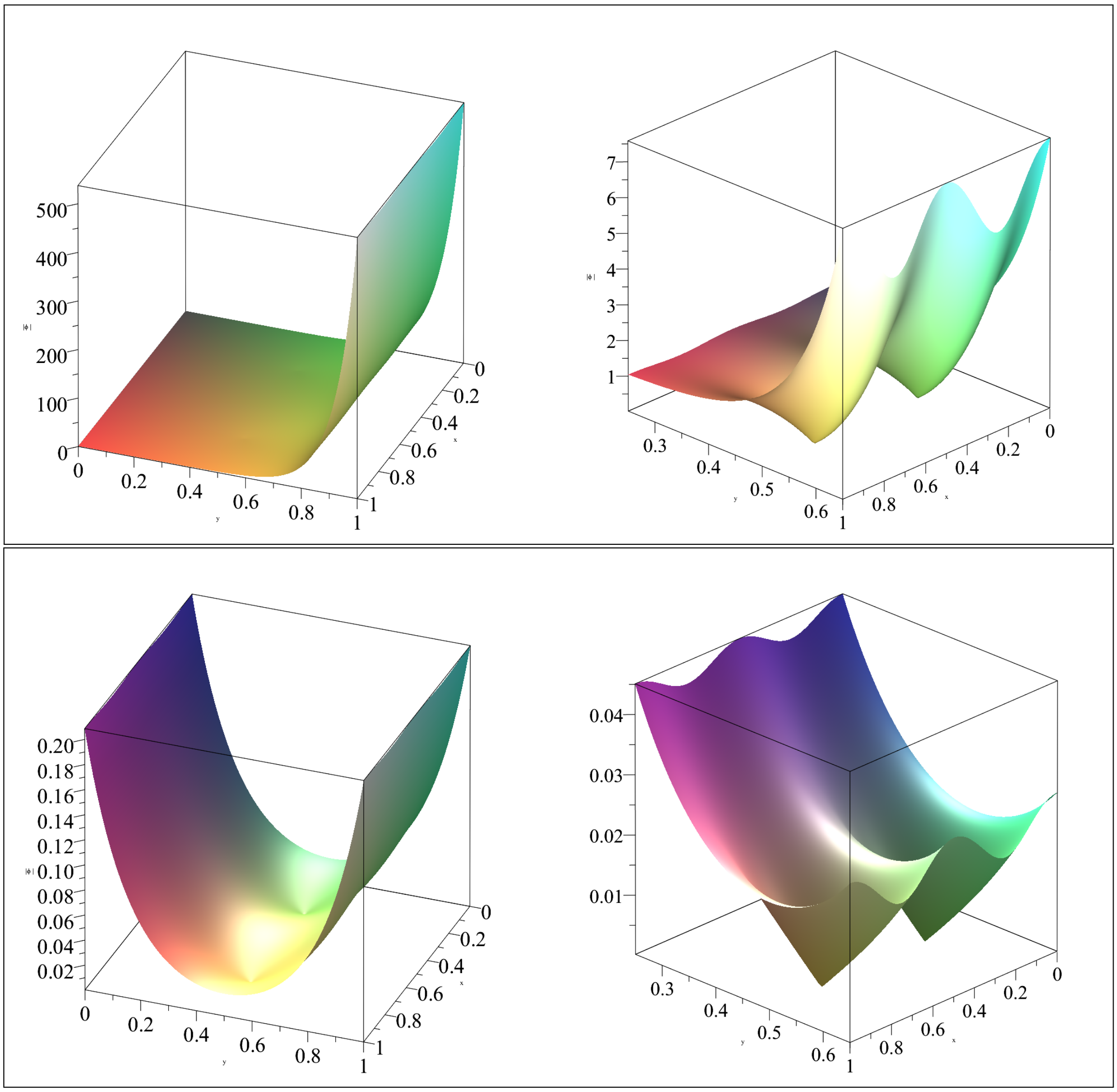

Figure 2.1: —Up: Graph of $|\widetilde{\varphi}_0^{\alpha,\beta}(z|I;2)|$; Up-left: in fundamental domain. Up-right: Close to the strip where zeros are located. —Down: Graph of $|\widetilde{\varphi}_1^{\alpha,\beta}(z|I;2)|$; Down-left: In a fundamental domain. Down-right: Close to the strip where zeros are located.

2.5 Explicit bases for high levels and high dimensions

The aim of this section is the construction of a basis for $\mathcal{O}^{\nu}_{\Gamma,\chi}(\mathbb{C}^n)$ and $\mathcal{E}^2_{\ell}(\Delta^{\Gamma,\chi}_{\nu},\mathbb{C}^n)$, which will generalize the basis constructed in the last section for the first level and one dimension, where $\mathcal{E}^2_{\ell}(\Delta^{\Gamma,\chi}_{\nu},\mathbb{C}^n)$ denotes the eigenspace associated with eigenvalue $\lambda_\ell = \nu\ell$ of

$$\Delta^{\Gamma,\chi}_{\nu} := -\sum_{j=1}^{n}\left\{\frac{\partial^2}{\partial z_j\partial \bar{z}_j} - \nu\overline{E}\right\}, \tag{2.39}$$

acting on $L^{2;\nu}_{\Gamma,\chi}(\mathbb{C}^n)$. Notice that $\Delta^{\Gamma,\chi}_{\nu}$ and Δ_ν are formally the same, though the presence of Γ,χ in the notation of the operator is to highlight its action on automorphic functions in contrast to when it acts on the free Hilbert space $L^2(\mathbb{C}^n, e^{-\nu|z|^2}d\lambda(z))$. It is known that the spectrum of Δ_ν acting on $\mathcal{F}^{\nu}_{\Gamma,\chi}(\mathbb{C}^n)$, on $L^{2;\nu}_{\Gamma,\chi}(\mathbb{C}^n)$, or on the free space $L^2(\mathbb{C}^n, e^{-\nu|z|^2}d\lambda(z))$ are the same and given by

$$\sigma(\Delta_\nu) = \sigma(\Delta^{\Gamma,\chi}_{\nu}) = \nu\ell, \qquad \ell = 0,1,2,\cdots. \tag{2.40}$$

Subsequently, we have $\mathcal{E}^2_{\ell}(\Delta^{\Gamma,\chi}_{\nu},\mathbb{C}^n) = Ker(\Delta^{\Gamma,\chi}_{\nu} - \nu\ell)$, and then $\mathcal{O}^{\nu}_{\Gamma,\chi}(\mathbb{C}^n) = \mathcal{E}^{\Gamma,\chi}_{0}(\Delta_\nu)$.

This construction method is based on the fact that $\Delta^{\Gamma,\chi}_{\nu}$ can be factorized in terms of the first order operators $A_{-,j} = \partial_{\bar{z}_j}$ and their formal adjoint $A_{+,j} = (-\partial_{z_j} + \nu\bar{z}_j)$ with respect to the Gaussian measure. Namely, we have

$$\Delta_\nu = \sum_{j=1}^{n} A_{+,j}A_{-,j}. \tag{2.41}$$

The first result in this context concerns $\mathcal{O}^{\nu}_{\Gamma,\chi}(\mathbb{C}^n)$. Recall that in one dimension, the basis of $\mathcal{O}^{\nu}_{\Gamma,\chi}(\mathbb{C})$ was given in therms of theta function (2.29) by

$$\varphi^{\alpha,\beta}_{k}(z|\tau;N) := e^{\frac{\nu}{2}z^2+2i\pi(\alpha+k)z}\vartheta_{0,(\alpha+k)\tau-\beta}(Nz|N\tau) \qquad z\in\mathbb{C}. \tag{2.42}$$

Subsequently, we can show

Proposition 2.5.1. *There exist some reals $\alpha_1,\beta_1,\cdots\alpha_n,\beta_n$ such that the functions φ_K; $K=(k_1,\cdots,k_n)\in\{0,1,\cdots,N-1\}^n$, defined by*

$$\varphi_K(z) := \varphi^{\alpha_1,\beta_1}_{k_1}(z_1)\cdots\varphi^{\alpha_n,\beta_n}_{k_n}(z_n) \tag{2.43}$$

form an orthogonal basis of the functional space $\mathcal{O}^{\nu}_{\Gamma^n_\tau,\chi}(\mathbb{C}^n)$, where Γ^n_τ is the Cartesian product of n copies of Γ_τ

Proof. The result readily follows by taking the product of n copies of $\varphi^{\alpha,\beta}_{m}$, namely the functions $\varphi_J(z)$ in (2.43) form an orthogonal basis of $\mathcal{O}^{\nu}_{\Gamma,\chi}(\mathbb{C}^n)$. □

The next result concerns the eigenspace $\mathcal{E}^2_{\ell}(\Delta^{\Gamma,\chi}_{\nu},\mathbb{C}^n) = Ker(\Delta^{\Gamma,\chi}_{\nu} - \nu\ell)$ for $\ell>0$. To this end, we denote by H^{σ}_{k}; $\sigma>0$, the rescaled (complex) Hermite polynomial defined by

$$H^{\sigma}_{k}(z) = (-1)^k e^{\sigma z^2}\frac{d^k}{dz^k}e^{-\sigma z^2}, \quad z\in\mathbb{C}.$$

Theorem 2.5.2. *Let φ_K be as in* (2.43)*. Then, the functions $A_{+,J}\varphi_K := A_{+,j_1}A_{+,j_2}\cdots A_{+,j_l}\varphi_K$ for varying $1\le j_1\le\cdots\le j_l\le n$ and $K=(k_1,\cdots,k_n)\in\{0,1,\cdots,N-1\}^n$ is an orthogonal basis of $\mathcal{E}^2_{\ell}(\Delta^{\Gamma,\chi}_{\nu},\mathbb{C}^n)$ for any $\ell=0,1,2,\cdots$, and $n=1,2,\cdots$. For the particular case of $n=1$, they reduces further to*

$$\varphi^{\alpha,\beta}_{\ell,m}(z) := (-1)^\ell e^{\frac{\nu}{2}z^2+2i\pi(\alpha+m)z}\sum_{k=0}^{\ell}\binom{\ell}{k}i^{\ell-k}H^{\nu/2}_{\ell-k}\left(2\Im mz + \frac{2\pi}{\nu}(\alpha+m)\right)\partial^k_z\vartheta_{0,(\alpha\tau-\beta)+m\tau}(Nz|N\tau) \tag{2.44}$$

with varying $m=0,1,\cdots,N-1$, and constitute an orthogonal basis of $\mathcal{E}^2_{\ell}(\Delta^{\Gamma,\chi}_{\nu},\mathbb{C})$ for every ℓ.

The proof of Theorem 2.5.2 is contained in Lemmas 2.5.3, 2.5.4 and 2.5.6 below.

Lemma 2.5.3. *Fix $n = 1$ and let ℓ be a nonnegative integer and set $A_+^\ell = (-\partial_z + \nu\bar{z})^\ell$ and $A_-^\ell = \partial_{\bar{z}}^\ell$.*

i) The operators A_+^ℓ and A_-^ℓ leave invariant the space $\mathcal{C}^\infty_{\Gamma,\chi}(\mathbb{C})$ of $\mathcal{C}^\infty$ functions satisfying the functional equation in (2.1)*.*

ii) We have $\Delta_\nu A_+^\ell = A_+^\ell(\Delta_\nu + \nu\ell)$. Moreover, the A_+ is injective from $\mathcal{E}^2_\ell(\Delta_\nu^{\Gamma,\chi},\mathbb{C})$ into $\mathcal{E}^2_{\ell+1}(\Delta_\nu^{\Gamma,\chi},\mathbb{C})$.

Proof. The proof is straightforward. □

Subsequently, if f is a given eigenfunction of $\Delta_\nu^{\Gamma,\chi}$ in $L^{2;\nu}_{\Gamma,\chi}(\mathbb{C}^n)$ with λ as the corresponding eigenvalue, then $A_+^\ell f$ is again an eigenfunction of $\Delta_\nu^{\Gamma,\chi}$ but with $\lambda + \nu\ell$ as the associated eigenvalue. This is to say that for $n = 1$ the basis of the eigenspace $\mathcal{E}^2_\ell(\Delta_\nu^{\Gamma,\chi},\mathbb{C})$ can be constructed from the ground states (the elements of $\mathcal{O}^{2,\nu}_{\Gamma,\chi}(\mathbb{C}) := \mathcal{E}^2_0(\Delta_\nu^{\Gamma,\chi})$) by applying A_+^ℓ, since in this case all the eigenspaces $\mathcal{E}^2_\ell(\Delta_\nu^{\Gamma,\chi},\mathbb{C})$ have the same dimension. More precisely, we assert

Lemma 2.5.4. *For $n = 1$, the functions $A_+^\ell\varphi_m^{\alpha,\beta}$; $m = 0, 1, \cdots \leq N-1$, form a basis of $\mathcal{E}^2_\ell(\Delta_\nu^{\Gamma,\chi},\mathbb{C})$ for every ℓ. Moreover, they are given by* (2.44)*, i.e., $A_+^\ell\varphi_m^{\alpha,\beta}(z) =: \varphi_{\ell,m}^{\alpha,\beta}(z)$.*

Proof. We need only to prove the $\mathbb{R}$-independency of $A_+^\ell\varphi_m^{\alpha,\beta}$; $m = 0, 1, \cdots \leq N-1$, which immediately follows since this family is orthogonal, indeed we have

$$\left\langle A_+^\ell\varphi_m^{\alpha,\beta}, A_+^\ell\varphi_{m'}^{\alpha,\beta}\right\rangle_{L^{2;\nu}_{\Gamma_\tau,\chi}(\mathbb{C})} = \nu^\ell\ell! \left\langle \varphi_m^{\alpha,\beta}, \varphi_{m'}^{\alpha,\beta}\right\rangle_{L^{2;\nu}_{\Gamma_\tau,\chi}(\mathbb{C})} = \nu^\ell\ell! \left\|\varphi_m^{\alpha,\beta}\right\|^2_{L^{2;\nu}_{\Gamma_\tau,\chi}(\mathbb{C})} \delta_{m,m'}.$$

The explicit expression of $A_+^\ell\varphi_m^{\alpha,\beta}$ can be handled by rewriting the operator A_+^ℓ as

$$A_+^\ell f = (-1)^\ell e^{\nu|z|^2}\partial_z^\ell(e^{-\nu|z|^2}f),$$

and therefore

$$A_+^\ell\varphi_m^{\alpha,\beta}(z) = e^{\frac{\nu}{2}z^2+2i\pi(\alpha+m)z}\sum_{k=0}^{\ell}(-1)^k\binom{\ell}{k}G^{\nu,\frac{\nu}{2}}_{\ell-k}(z,\bar{z}|2i\pi(\alpha+m))\partial_z^k\vartheta_{(0,(\alpha\tau-\beta)+m\tau)}(Nz|N\tau).$$

where the polynomials

$$G_k^{\nu,\sigma}(z,\bar{z}|\xi) = (-1)^k e^{\nu|z|^2-\sigma z^2-\xi z}\partial_z^k e^{-\nu|z|^2+\sigma z^2+\xi z}; \quad \nu > 0, \sigma \neq 0, \xi \in \mathbb{C},$$

are the ones introduced and studied in [8] and showed to can reexpressed as

$$G_k^{\nu,\sigma}(z,\bar{z}|\xi) = (-i)^k H_k^\sigma\left(\frac{2\sigma z - \nu\bar{z} + \xi}{2i\sigma}\right).$$

□

Remark 2.5.5. *The functions involved in the right hand-side of* (2.44) *can be seen as a special kind of generalization of the Jacobi theta functions to the non-holomorphic setting for satisfying the functional equation* (2.1) *and that for $\ell = 0$ they reduce further to Jacobi theta function $\vartheta(Nz + (\alpha\tau - \beta) + m\tau|N\tau)$.*

Lemma 2.5.6. *For $n \geq 2$, the set of functions $A_{+,j_1}A_{+,j_2}\cdots A_{+,j_l}\varphi_K$ for varying $1 \leq j_1 \leq \cdots \leq j_l \leq n$ and $K = (k_1,\cdots,k_n) \in \{0,1,\cdots,N-1\}^n$ is an orthogonal basis of $\mathcal{E}^2_\ell(\Delta_\nu^{\Gamma,\chi},\mathbb{C}^n)$.*

Proof. Notice first that Lemma 2.5.3 remains valid for high dimension and the assertions holds true if we replace there A_+ by $A_{+,j}$ and A_- by $A_{-,j}$. Thus, for every fixed $j = 1, \cdots, n$, the map $A_{+,j} : \mathcal{E}^2_\ell(\Delta^{\Gamma,\chi}_\nu, \mathbb{C}^n) \longrightarrow \mathcal{E}^2_{\ell+1}(\Delta^{\Gamma,\chi}_\nu, \mathbb{C}^n)$ is injective and therefore the function

$$A_{+,J_\ell}\varphi := A_{+,j_1}A_{+,j_2}\cdots A_{+,j_l}\varphi$$

belongs to $\mathcal{E}^2_\ell(\Delta^{\Gamma,\chi}_\nu, \mathbb{C}^n)$ for every given $\varphi \in \mathcal{E}^2_0(\Delta^{\Gamma,\chi}_\nu, \mathbb{C}^n)$. Moreover, by means of Lemma 2.5.4 and the definition of φ_K (see (2.43)) and applying Fubini's theorem, it is clear that the family of functions $A_{+,J_\ell}\varphi_K$ for varying $J_\ell = (j_1, \cdots, j_l) \in \{1, \cdots, n\}^\ell$ and $K = (k_1, \cdots, k_n) \in \{0, 1, \cdots, N-1\}^n$ is orthogonal in $\mathcal{E}^2_\ell(\Delta^{\Gamma,\chi}_\nu, \mathbb{C}^n)$. Then the result of Lemma 2.5.6 follows since

$$\begin{aligned}\#\{A_{+,J_\ell}\varphi_K; J_\ell, K\} &= \left[\left(\frac{\mu}{\pi}\right)^n vol(\Lambda(\Gamma))\right] \times \frac{(n-1+\ell)!}{(n-1)!\ell!} \\ &= \dim \mathcal{E}^2_\ell(\Delta^{\Gamma,\chi}_\nu, \mathbb{C}^n),\end{aligned}$$

where # denotes the cardinality of a set. The last dimension formula can be seen as direct consequence of Theorem 3.6.2. □

2.6 Zeros of automorphic functions on $\mathbb{C}$

If we look thoroughly at the set of zeros of the functions $\widetilde{\varphi}^{\alpha,\beta}_j(z|\tau; N)$ and $\varphi^{\alpha,\beta}_j(z|\tau; N)$ given respectively in (2.38) and (2.30) (see also Figure 2.1), we can see that they have the same number N of zeros. And they occur on the same line inside a fundamental domain, which is parallel to x axis and pass through $\frac{-2\beta+1}{2N}$ mod Γ. More precisely we can show that

Proposition 2.6.1. *The set of zeros of $\varphi^{\alpha,\beta}_j$ and $\widetilde{\varphi}^{\alpha,\beta}_j$ are given by*

$$\mathcal{Z}(\varphi^{\alpha,\beta}_j) = \mathcal{Z}(\widetilde{\varphi}^{\alpha,\beta}_j) = \left\{\frac{N + 2(\alpha + j)}{2N}\tau + \frac{-2\beta + 1}{2N} + \frac{\ell}{N} \mod \Gamma; \ell = 0 \cdots N-1\right\}, \tag{2.45}$$

for $j = 0, \cdots, N-1$.

Proof. The result can be driven easily using Theorems 2.4.5 and 2.4.9 combined with the fact that $\vartheta_{a,b}(z|\tau)$ has zeros at $(a + \frac{1}{2})\tau + (b + \frac{1}{2}) \mod \Gamma$ (see e.g. Mumford [60, p. 12]). □

More generally, we can show that every holomorphic automorphic function $f \in \mathcal{O}^\nu_{\Gamma,\chi}$ has exactly N zeros inside a fundamental domain. Namely, we can establish the following result which will be used in Chapter 6.

Proposition 2.6.2. *We have*

$$\mathcal{Z}(f) = \mathcal{Z}(f|_{\Lambda(\Gamma)}) + \Gamma.$$

Moreover, the number of zeros of f contained in any fundamental cell $\Lambda(\Gamma)$ is constant. More precisely, we have

$$\#(\mathcal{Z}(f|_{\Lambda(\Gamma)})) = \dim \mathcal{O}^\nu_{\Gamma,\chi}(\mathbb{C}) = N.$$

Proof. The first assertion is clear by means of the equation of automorphy. We need only to determine the number of zeros of the holomorphic function $f \in \mathcal{O}^\nu_{\Gamma,\chi}(\mathbb{C})$ inside $\Lambda(\Gamma)$. By a small displacement of $\Lambda(\Gamma)$, say $u + \Lambda(\Gamma)$ for certain $u \in \mathbb{C}$, we are free to assume that $f \neq 0$ along the boundary $\partial\Lambda(\Gamma)$ of a fundamental region $\Lambda(\Gamma)$. Without loss of generality, we can assume that $\partial\Lambda(\Gamma)$ is a piecewise smooth path that runs around each zero of f in $\Lambda(\Gamma)$ exactly one time and that the summits are the origin O, ω_1,

$\omega_1 + \omega_2$ and ω_2. Let denote by $\mathcal{Z}(f|_{\Lambda(\Gamma)})$ the set zeros of f inside such $\Lambda(\Gamma)$. Hence, by applying the principle argument to f, we get

$$\begin{aligned}\#(\mathcal{Z}(f|_{\Lambda(\Gamma)})) &= \frac{1}{2i\pi}\oint_{\partial\Lambda(\Gamma)} \frac{f'(z)}{f(z)}dz = \frac{1}{2i\pi}\left(\int_O^{\omega_1} + \int_{\omega_1}^{\omega_1+\omega_2} + \int_{\omega_1+\omega_2}^{\omega_2} + \int_{\omega_2}^{0}\right)\\ &= \frac{1}{2i\pi}\int_0^1 \left[\omega_1\left(\frac{f'(t\omega_1)}{f(t\omega_1)} - \frac{f'(t\omega_1+\omega_2)}{f(t\omega_1+\omega_2)}\right) + \omega_2\left(\frac{f'(\omega_1+t\omega_2)}{f(\omega_1+t\omega_2)} - \frac{f'(t\omega_2)}{f(t\omega_2)}\right)\right] dt.\end{aligned}$$

Now, according to the well established fact

$$\frac{f'(z+\gamma)}{f(z+\gamma)} - \frac{f'(z)}{f(z)} = \nu\overline{\gamma}$$

provided that $f(z+\gamma') \neq 0; \gamma' \in \Gamma$, and valid for every f satisfying the functional equation (6.1), we can prove the following

$$\#(\mathcal{Z}(f|_{\Lambda(\Gamma)})) = \frac{1}{2i\pi}\int_0^1 \nu\left(\omega_1\overline{\omega_2} - \omega_2\overline{\omega_1}\right) dt = \left(\frac{\nu}{\pi}\right)\Im m(\omega_1\overline{\omega_2}).$$

This completes the proof. $\square$

2.7 Automorphic functions on orbifolds

In this last section we discuss how one can extend the definition of automorphic functions to orbifolds. Let G be the Euclidean motion group given by the semi-direct product $\mathsf{U}(n) \ltimes \mathbb{C}^n$. G can be realized as

$$G = \left\{ g = \begin{pmatrix} A & b \\ 0^T & 1 \end{pmatrix}; \qquad a \in \mathsf{U}(n),\ b \in \mathbb{C}^n \right\},$$

and acts transitively on $\mathbb{C}^n$ by the holomorphic mappings $z \longmapsto g \cdot z := az + b$. $\mathbb{C}^n$ can hence be realized as Hermitian symmetric space $\mathbb{C} = G/\mathsf{U}(n)$. The elements of G are denoted $[A, b]$ where $A \in \mathsf{U}(n)$ is the linear part and $b \in \mathbb{C}^n$ is the translational part. For more details refer to the first chapter.
Let Γ be a discrete subgroup of G. Then, we define automorphic functions in the context of orbifolds to be functions f that verify the generalized equation of automorphy

$$f(\gamma \cdot z) = \chi(\gamma)e^{-i\nu\Im m\langle z, \gamma^{-1}.0\rangle} f(z); \qquad z \in \mathbb{C}^n, \gamma \in \Gamma. \tag{2.46}$$

If we take $\Gamma = 1 \ltimes L$ (where L is a lattice of $\mathbb{C}^n$) and identify $[1, \gamma] \in \Gamma$ with the element $\gamma \in L$, (2.46) reduces to the equation (2.2) given before in the context of toroidal manifolds:

$$f(z+\gamma) = \chi(\gamma)e^{i\nu\Im m\langle z,\gamma\rangle} f(z); \qquad z \in \mathbb{C}^n, \gamma \in L. \tag{2.2}$$

For the sake of explicit formulas and light notations, form now on, let Γ be of the canonical form $U \ltimes L$ where U is a subgroup of $\mathsf{U}(n)$ and L is a lattice of $\mathbb{C}^n$ (Recall from Chapter 1 that this canonical form represents, according to Bieberbach theorems, all discrete subgroups of G). The spaces of automorphic functions on toroidal manifolds are extended to the orbifold case as follows

Definition 2.4. *To given data (ν, Γ, χ), we define the spaces*

$\boxed{1}$ $\mathcal{A}ut^{\nu,0}_{\Gamma,\chi}(\mathbb{C}^n)$ *of Lebesgue measurable functions $f : \mathbb{C}^n \to \mathbb{C}$ that satisfy the functional equation* (2.46) *for almost all $z \in \mathbb{C}^n$.*

$\boxed{2}$ $\mathcal{F}^{\nu,0}_{\Gamma,\chi}(\mathbb{C}^n)$ *of C^∞ functions $f : \mathbb{C}^n \to \mathbb{C}$ that satisfy the functional equation* (2.46) *for all $z \in \mathbb{C}^n$.*

$\boxed{3}$ $L^{2;\nu,0}_{\Gamma,\chi}(\mathbb{C}^n) := \left\{ f \in \mathcal{A}ut^{\nu,0}_{\Gamma,\chi}(\mathbb{C}^n) \text{ such that } \int_{\Lambda(\Gamma)} |f|^2 d\lambda(z) < +\infty \right\}$ *endowed with the inner product*

$$\langle f_1, f_2\rangle_\Gamma = \int_{\Lambda(\Gamma)} f_1(z)\overline{f_2(z)} d\lambda(z).$$

The set $\Lambda(\Gamma)$ is the fundamental domain representing the orbital space $\mathbb{C}/\Gamma$ and is connected to $\Lambda(L)$ by

$$\Lambda(L) = \bigcup_{u \in U \ltimes \{0\}} u \cdot \Lambda(\Gamma). \tag{2.47}$$

This leads to establish the inclusion

$$L^{2;\nu,0}_{\Gamma,\chi}(\mathbb{C}^n) \subset L^{2;\nu,0}_{L,\chi}(\mathbb{C}^n). \tag{2.48}$$

The same operator

$$\widetilde{\Delta}_{\nu,0} := -\left\{ \sum_{j=1}^{n} \frac{\partial^2}{\partial z_j \partial \bar{z}_j} + \frac{\nu}{2}(E - \overline{E}) - \frac{\nu^2}{4}|z|^2 \right\}, \tag{2.49}$$

is invariant with respect to the generalized equation of automorphy (2.46). That is

Proposition 2.7.1. *For all $\gamma \in \Gamma$*

$$\mathcal{T}^{\nu,0}_\gamma \widetilde{\Delta}_{\nu,0} = \widetilde{\Delta}_{\nu,0} \mathcal{T}^{\nu,0}_\gamma, \tag{2.50}$$

where $\mathcal{T}^{\nu,0}_\gamma$ (in (2.8)*) is extended to element $\gamma \in \mathsf{U}(n) \ltimes \mathbb{C}^n$ as follows*

$$\mathcal{T}^{\nu,0}_\gamma f(z) := \overline{\chi(\gamma)}\, \mathrm{e}^{-\nu \Im m \langle z, \gamma^{-1}\cdot 0 \rangle} f(z+\gamma). \tag{2.51}$$

Therefore, the eigenvalue problem is well defined on the spaces of automorphic functions on orbifolds. Accordingly, one can ask about the concrete spectral theory of $\widetilde{\Delta}^{\Gamma,\chi}_{\nu,0}$ acting on these spaces. In view of the inclusion given by the Equation (2.48), the spectrum of $\widetilde{\Delta}_{\nu,0}$ acting on $L^{2;\nu,0}_{\Gamma,\chi}(\mathbb{C}^n)$ is a subset of the spectrum of $\widetilde{\Delta}_{\nu,0}$ acting on $L^{2;\nu,0}_{L,\chi}(\mathbb{C}^n)$, which is given by

$$\nu\ell + \nu n; \qquad \ell = 0, 1, 2, \cdots. \tag{2.52}$$

Now, is this subset proper? In one dimension (n=1) the problem can be approached by taking the solutions of the eigenvalue problem of $\widetilde{\Delta}_{\nu,0}$ acting on $L^{2;\nu,0}_{L,\chi}(\mathbb{C}^n)$ and periodizing theme along the linear part of Γ given by U. This can be done by applying the operator

$$\begin{aligned} \Theta_U[f] &= \sum_{u \in U} \lambda_u f(uz) \\ &= \sum_{m=0}^{q} \lambda^m f(e^{\frac{2i\pi m}{q}} z), \end{aligned}$$

where $\lambda_u := \lambda$ are a factors such that $\lambda^q = 1$. The second line in the equation above is derived from the fact that $U = \mathsf{U}_q$ for some $q \in \mathbb{N}$ (for more details see Section 1.4). This periodization results in solutions for the eigenvalue problem of $\widetilde{\Delta}_{\nu,0}$ acting on $L^{2;\nu,0}_{\Gamma,\chi}(\mathbb{C}^n)$. The problem is now reduced to check for which eigenvalue in (2.52), the periodization of all associated eigenfunctions will result in a vanishing function. In this planar case (i.e., n=1), this approach is affordable thanks to the restriction on q when $L \neq \{0\}$; i.e. L is of rank one or full-rank (see the discussion given in Section 1.4) which allows to exhaustively study all the possible cases. For $L = \{0\}$, although there is no restriction on q we can show the isospectrality of $L^{2;\nu,0}_{\Gamma,\chi}(\mathbb{C}^n)$ and $L^{2;\nu,0}_{L,\chi}(\mathbb{C}^n)$ thanks to the radial solution of the eigenvalue problem on the free space given (in much general situation) in the next chapter by the Proposition 3.5.2

$$\varphi^\nu_\lambda(z) = e^{-\frac{\nu}{2}|z|^2} {}_1F_1(-\lambda; n; \nu|z|^2). \tag{2.53}$$

Nevertheless, for n in general the problem seems difficult to handle as in the one dimensional case due to the huge number, and the lack of a practical characterization, of discrete subgroups of $\mathsf{U}(n) \ltimes \mathbb{C}^n$.

CHAPTER 3

MIXED AUTOMORPHIC FUNCTIONS OF FIRST KIND

We consider mixed automorphic functions of first kind, which is a new space of automorphic functions associated with bi-wighted automorphic factor with two parameters (ν,μ). The appropriate invariant Laplacian is given by the twisted Laplacian $\Delta_{\nu,\mu}$ on the n-complex space associated with the sub-Laplacian of the Heisenberg group $\mathbb{C}\times_{\omega}\mathbb{C}^n$. The later is realized as a central extension of the real Heisenberg group H_{2n+1}. The main results concern the spectral theory of $\Delta_{\nu,\mu}$ when acting on the space of L^2 mixed automorphic functions associated with given full-rank lattice Γ of $\mathbb{C}^n$. We describe its spectrum proving a stability theorem. Using the Selberg's approach, we give the explicit dimension formula for the corresponding L^2-eigenspaces. We also provide a concrete basis of such L^2-eigenspaces.

3.1 Definition of Mixed automorphic functions of first kind

As in Section 2.7, let

$$G := \mathsf{U}(n)\ltimes\mathbb{C}^n = \left\{ g = \begin{pmatrix} A & b \\ 0 & 1 \end{pmatrix} =: [A,b];\;\; A\in\mathsf{U}(n), b\in\mathbb{C}^n \right\}, \tag{3.1}$$

which acts transitively on $\mathbb{C}^n$ via the mappings $g.z = Az+b$ for $g=[A,b]\in G$. Let Γ a discrete subgroup of G. Let μ and ν two reals such that $\nu>0$. And let $\chi:\Gamma\to\mathsf{U}(1)$.

Associated with data $(\Gamma;\chi,\nu,\mu)$, we perform $\mathcal{F}^{\nu,\mu}_{\Gamma,\chi}(\mathbb{C}^n)$ the vector space of smooth complex-valued functions F on $\mathbb{C}^n$ satisfying the functional equation :

$$F(\gamma\cdot z) = \chi(\gamma) j^{\nu,\mu}(\gamma,z)F(z), \tag{3.2}$$

for every $\gamma\in\Gamma$ and $z\in\mathbb{C}$, Where the factor $j^{\nu,\mu}$ is given by the mapping

$$j^{\nu,\mu}(g,z) = \exp\left(-i\nu\Im m(\left\langle z, g^{-1}.0\right\rangle) + i\mu\Re e(\left\langle z, g^{-1}.0\right\rangle)\right). \tag{3.3}$$

on $G\times\mathbb{C}^n$

Definition 3.1. *The space of $\mathcal{F}^{\nu,\mu}_{\Gamma,\chi}(\mathbb{C}^n)$ is called the space of mixed automorphic functions of first kind with bi-weight (ν,μ).*

For simplification, throughout this chapter we take $\chi\equiv 1$. Subsequently, we drop χ from the notation. The particular space $\mathcal{F}^{0,0}_{\Gamma}(\mathbb{C}^n)$, corresponding to $(\nu,\mu)=(0,0)$, is nothing else but the space of Γ-periodic functions which is known to be a non-zero vector space. However, for arbitrary given $(\nu,\mu)\neq(0,0)$ and Γ,additional assumption on the triplet $(\Gamma;\nu,\mu)$ must be imposed in order to ensure the non-triviality of the space $\mathcal{F}^{\nu,\mu}_{\Gamma}(\mathbb{C}^n)$. In fact, similar to the standard automorphic functions we treated in the previous chapter, such space is not trivial if and only if $j^{\nu,\mu}$ satisfies the co-cycle equation

$$j^{\nu,\mu}(\gamma\gamma',z) = j^{\nu,\mu}(\gamma,\gamma' z)j^{\nu,\mu}(\gamma',z) \tag{3.4}$$

for every $\gamma,\gamma' \in \Gamma$. It can be shown that it is the case if and only if

$$\mu\Re e\left\langle \gamma^{-1}\cdot 0, \gamma'\cdot 0\right\rangle - \nu\Im m\left\langle \gamma^{-1}\cdot 0, \gamma'\cdot 0\right\rangle \in 2\pi\mathbb{Z} \tag{3.5}$$

for every $\gamma,\gamma' \in \Gamma$. Throughout this chapter Γ will a full rank lattice of $\mathbb{C}^n$, hence more concrete characterization is stated after the following definition

Definition 3.2. *The triplet $(\Gamma;\nu,\mu)$, of real numbers ν,μ and discrete subgroup Γ in $(\mathbb{R}^{2n},+) = (\mathbb{C}^n,+)$, is said to be quantized if it obeys the Riemann-Dirac quantization (RDQ) condition for mixed automorphic functions of first kind given by the equation* (3.5) *which equivalently reads*

$$\mu\Re e\left\langle \gamma, \gamma'\right\rangle \in 2\pi\mathbb{Z} \quad \text{and} \quad \nu\Im m\left\langle \gamma, \gamma'\right\rangle \in 2\pi\mathbb{Z}, \tag{3.6}$$

for all $\gamma,\gamma' \in \Gamma$.

So far, the acronym (RDQ) was used twice: in the context of standard automorphic functions and now in the context of mixed automorphic functions. In this chapter it is used to mean the later.

Lemma 3.1.1. *Assume that $(\Gamma;\nu,\mu)$ is quantized. Then, the space $\mathcal{F}^{\nu,\mu}_{\Gamma}(\mathbb{C}^n)$ is an infinite dimensional vector space over $\mathbb{C}$.*

Proof. Let $\varphi(z)$ be a compactly supported $\mathcal{C}^\infty$-complex-valued function on $\mathbb{C}^n$ with support contained in the interior of a fundamental domain $\Lambda(\Gamma)$ of Γ. Making a sort of $(\Gamma;\nu,\mu)$-periodization *à la* Poincaré

$$[\mathcal{P}^{\nu,\mu}_{\Gamma}\varphi](z) = \sum_{\gamma\in\Gamma} e^{-i[\mu\Re e(\langle z,\gamma\rangle)-\nu\Im m(\langle z,\gamma\rangle)]}\varphi(z+\gamma). \tag{3.7}$$

It is clear that $\mathcal{P}^{\nu,\mu}_{\Gamma}\varphi|_{\Lambda(\Gamma)}(z) = \varphi(z)$. Moreover, we can check easily that $\mathcal{P}^{\nu,\mu}_{\Gamma}\varphi \in \mathcal{F}^{\nu,\mu}_{\Gamma}(\mathbb{C}^n)$ thanks to the (RDQ)-assumption. Hence, the vector space $\mathcal{F}^{\nu,\mu}_{\Gamma}(\mathbb{C}^n)$ contains all the elements $\mathcal{P}^{\nu,\mu}_{\Gamma}\varphi$ for $\varphi \in \mathcal{C}^\infty_c(\Lambda(\Gamma))$. It follows that $\mathcal{F}^{\nu,\mu}_{\Gamma}(\mathbb{C}^n)$ is not-trivial and it is of infinite dimension. □

From now on, we assume that the triplet $(\Gamma;\nu,\mu)$ is quantized in the sense of Definition 3.2. The quantization condition we required may seem very restrictive. However, we provide below examples of such triplets where the discrete subgroup Γ is full-rank so that $\mathbb{C}^n/\Gamma$ is a compact manifold. Recall that, Γ is full-rank means that there exists some $\mathbb{R}$-linearly independent vectors $u_1, u_2, \cdots, u_{2n}$ in $\mathbb{C}^n$ such that

$$\Gamma = \mathbb{Z}u_1 + \mathbb{Z}u_2 + \cdots + \mathbb{Z}u_{2n}. \tag{3.8}$$

Example 3.1.2. *Let $n = 1$ and b, c be integer numbers such that $c > 0$ and $b^2 - 4c < 0$. Let τ be the solution of the algebraic equation $\tau^2 - b\tau + c = 0$ so that $Im\tau > 0$. Then the lattices $\Gamma_\tau = \mathbb{Z} + \tau\mathbb{Z}$ in $\mathbb{C} = \mathbb{R}^2$ are of special kind and include the Gauss numbers $\Gamma_i = \mathbb{Z} + i\mathbb{Z}$ and $\Gamma_j = \mathbb{Z} + j\mathbb{Z}$. Making varying b and c, we can get other lattices Γ_τ in $\mathbb{C}$ such as those associated to $\tau = \frac{-1+i\sqrt{7}}{2}, \frac{-1+i\sqrt{11}}{2}, \frac{-1+i\sqrt{15}}{2}, \frac{-1+i\sqrt{19}}{2}, \cdots$ and corresponding to $b = -1$ and $c = 2, 3, 4, 5, \cdots$, respectively. The choice of the parameters (ν,μ) such as $\nu = 2\pi k_0$ and $\mu = \pi l_0/Im\tau$ for some $k_0, l_0 \in \mathbb{Z}$, leads to a quantized triplet $(\Gamma_\tau;\nu,\mu)$.*

Example 3.1.3. *Starting from Γ_τ as in Example 3.1.2, we can build a lattice in $\mathbb{C}^n = \mathbb{R}^{2n}$ of maximal rank $2n$, say $\widetilde{\Gamma^n_\tau} = \Gamma_\tau \times \Gamma_\tau \times \cdots \times \Gamma_\tau$ (n-times) such that $\mathbb{C}^n/\widetilde{\Gamma^n_\tau}$ is a torus of $\mathbb{R}^{2n} = \mathbb{C}^n$. Moreover, the triplet $(\widetilde{\Gamma^n_\tau};\nu,\mu)$ is subject to the (RDQ) assumption for suitable conditions on (ν,μ) like the ones provided in the previous example, i.e., $\nu = 2\pi k_0$ and $\mu = \pi l_0/Im\tau$ for some $k_0, l_0 \in \mathbb{Z}$.*

The definition of automorphic functions of first kind will be motivated later on by the introduction of some twisted Laplacian associated with $\mathbb{C} \times_\omega \mathbb{C}^n$, which is a central extension of the real Heisenberg group. This group will be defined in the next section. Then, the introduced functions (mixed automorphic functions of first kind) will be released geometrically in two ways. The first one as sections of a line bundle over $\mathbb{C}^n/\Gamma$, and the second one as equivariant functions of a principal bundle associated with this central extension.

3.2 Central extension of the Heisenberg group

We realize $N_\omega := \mathbb{C} \times_\omega \mathbb{C}^n$ as a central extension of the *Heisenberg group* $H_{2n+1} := \mathbb{R} \times_{Im\omega} \mathbb{C}^n$, where $\omega(z,w)$ denotes the standard Hermitian form on $\mathbb{C}^n$. To this end, we follow the exposition given in [56]. Being indeed, if $(K, \bullet)$ and $(G, \odot)$ are two abelian groups and $\psi : K \times K \to G$ a given mapping. On $G \times K$ we define the $\cdot_\psi$-law by

$$(z_0; z) \cdot_\psi (w_0; w) = (z_0 \odot w_0 \odot \psi(z,w); z \bullet w).$$

We say that $G \times_\psi K$ is a central extension of $(K, \bullet)$ by $(G, \odot)$ associated with ψ if the short sequence

$$0 \to K \to G \times_\psi K \to G \to 0$$

is exact, and such that K is in $Z(G)$, the center of the group E. This holds if one of the following two equivalent assertions is satisfied, to wit

i) ψ preserves the neutral element $\psi(0_K, 0_K) = 0_G$ and verifies the cocycle relation

$$\psi(x,y) \odot \psi(x \bullet y, z) = \psi(x, y \bullet z) \odot \psi(y,z)$$

for every $x, y, z \in K$.

ii) $G \times_\psi K := (G \times K, \cdot_\psi)$ is a group.

Now, let $\mathbb{R}^2 = \mathbb{R}_s \times \mathbb{R}_t$ be the real (s,t)-plane identified with the complex plane $\mathbb{C} = \{z_0 = s + it;\ s,t \in \mathbb{R}\}$ and $\mathbb{C}^n$ denotes the complex n-space endowed with its standard Hermitian form

$$\omega(z,w) := \langle z, w \rangle = \sum_{j=1}^{n} z_j \bar{w}_j$$

for $z = (z_1, z_2, \cdots, z_n)$ and $w = (w_1, w_2, \cdots, w_n)$ in $\mathbb{C}^n$. Thus, we define $N_\omega = \mathbb{C} \times_\omega \mathbb{C}^n$ to be the set $\mathbb{C} \times \mathbb{C}^n$ endowed with the ω-law given by

$$(z_0; z) \cdot_\omega (w_0; w) = (z_0 + w_0 + \langle z, w \rangle; z + w). \tag{3.9}$$

Under (3.9), $N_\omega := \mathbb{C} \times_\omega \mathbb{C}^n$ is a non-commutative nilpotent group of step two with center $Z(N_\omega) = \mathbb{C} \times_\omega \{0\} = (\mathbb{R} \times \mathbb{R}) \times_\omega \{0\}$. The identity element is $(0;0)$ and the symmetric of an element $(z_0; z)$ is $(-z_0 - \langle z, z \rangle; -z)$. Notice for instance that the ω-law given by (3.9) can be rewritten in the coordinates $z_0 = (s,t)$, $w_0 = (s', t')$ and $z \in \mathbb{C}^n$ and $z, w \in \mathbb{C}^n$ as follows

$$((s,t); z) \cdot_\omega ((s', t'); w) = ((s + s' + Re\langle z, w \rangle, t + t' + Im\langle z, w \rangle); z + w). \tag{3.10}$$

Hence, endowing the set $\mathbb{R}_t \times \mathbb{C}^n$ with the $\cdot_{Im\omega}$-law given by

$$(t; z) \cdot_{Im\omega} (t'; w) = (t + t' + Im\langle z, w \rangle; z + w), \tag{3.11}$$

makes $\mathbb{R} \times_{Im\omega} \mathbb{C}^n$ a group, which is nothing else than the classical real Heisenberg group of dimension $2n+1$. One can notice easily that $(\mathbb{C} \times \mathbb{C}^n, \cdot_\omega)$, in addition of being the central extension of $\mathbb{C}^n$ by $\mathbb{C}$ associated with the map $\psi = \omega$, can also be viewed, due to (3.10), as the central extension of $(\mathbb{R}_t \times \mathbb{C}^n, \cdot_{\Im m\omega})$ by $\mathbb{R}_s$ associated with $\psi = \Re\omega$. This can be stated otherwise using directly the definition; if we denote by q the projection mapping from $(\mathbb{R}_s \times \mathbb{R}_t) \times_\omega \mathbb{C}^n$ onto $\mathbb{R}_t \times_{Im\omega} \mathbb{C}^n$ given by $q(s,t;z) = (t;z)$, one check that the mapping q is a homomorphism from the group $N_\omega = \mathbb{C} \times_\omega \mathbb{C}^n$ onto the Heisenberg group $H_{2n+1} = \mathbb{R} \times_{Im\omega} \mathbb{C}^n$ and that the kernel of q is given by

$$\ker q = \{(s,0;0),\ s \in \mathbb{R}\} = (\mathbb{R}_s \times \{0\}) \times_\omega \{0\}.$$

Since $\ker q$ is contained in the center $Z(N_\omega)$ of N_ω, we may say that the group $N_\omega = \mathbb{C} \times_\omega \mathbb{C}^n$ is a central extension of the Heisenberg group $H_{2n+1} = \mathbb{R} \times_{Im\omega} \mathbb{C}^n$ by $(\mathbb{R}_s, +)$; i.e we have $\mathbb{C} \times_\omega \mathbb{C}^n / \ker q = \mathbb{C} \times_\omega \mathbb{C}^n / \mathbb{R}_s = H_{2n+1}$. Accordingly, harmonic analysis on our group $N_\omega = \mathbb{C} \times_\omega \mathbb{C}^n$ will have many links to that on the classical Heisenberg group.

3.3 Geometrical realization of mixed automorphic functions of first kind

In this section, we give geometrical realizations of the space of mixed automorphic functions of first kind as a space of sections of some rank-one bundles over $\mathbb{C}^n/\Gamma$ under the assumption that the (RDQ) "quantization rule" in (3.6) holds. In fact, we will provide two kinds of such realizations, the first one as sections of a *line bundle* and the second one as equivariant functions of a principal bundle. Indeed, under the (RDQ)-assumption (3.6), the automorphic factor $j^{\nu,\mu}(\gamma;z)$ in (3.3) satisfies the chain rule

$$j^{\nu,\mu}(gg',z) = j^{\nu,\mu}(g,g'z)j^{\nu,\mu}(g',z). \tag{3.12}$$

Therefore, the Γ-action on $\mathbb{C}^n$ can be extended to $\mathbb{C}^n \times \mathbb{C}$ by considering the mappings

$$\widetilde{\gamma}(z,v) := (\gamma + z; j^{\nu,\mu}(\gamma,z)v) \tag{3.13}$$

for varying $\gamma \in \Gamma$. The mapping $\widetilde{\gamma}$ is linear on the fiber $\mathbb{C}_v$ and hence the space $\mathbb{C}^n \times \mathbb{C}$ becomes a homogeneous line bundle over $\mathbb{C}^n$ whose fiber is $p^{-1}(z) = \{z\} \times \mathbb{C} \simeq \mathbb{C}$, where p is the projection of $\mathbb{C}^n \times \mathbb{C}$ onto $\mathbb{C}^n$. If π denotes the canonical quotient projection, then the above claims can be pictured by the following commutative diagrams

$$\begin{tikzcd}
{} & {} \arrow[d, phantom, "\circlearrowleft"] & \mathbb{C}^n \times \mathbb{C} \\
\mathbb{C}^n \times \mathbb{C} \arrow[urr, "\widetilde{\gamma+\gamma'}"] \arrow[rr, "\widetilde{\gamma}"] \arrow[d, "p"'] \arrow[drr, phantom, "\circlearrowleft"] & & \mathbb{C}^n \times \mathbb{C} \arrow[u, "\widetilde{\gamma}'"'] \arrow[d, "p"] \\
\mathbb{C}^n \arrow[rr, "\gamma"] \arrow[dr, "\pi"'] & {} \arrow[d, phantom, "\circlearrowleft"] & \mathbb{C}^n \arrow[dl, "\pi"] \\
& \mathbb{C}^n/\Gamma &
\end{tikzcd}$$

Accordingly, there is a line bundle over $\mathbb{C}^n/\Gamma$ (where the fiber is $\mathbb{C}$) so that its cross sections are indeed our space $\mathcal{F}^{\nu,\mu}_{\Gamma}(\mathbb{C}^n)$ of Γ-periodic functions of quantified bi-weight (ν,μ). Therefore, the space $\mathcal{F}^{\nu,\mu}_{\Gamma}(\mathbb{C}^n)$ is of infinite dimension.

We conclude this section by providing another concrete geometrical realization of $\mathcal{F}^{\nu,\mu}_{\Gamma}(\mathbb{C}^n)$ as equivariant functions of a rank-one principal bundle over $\mathbb{C}^n/\Gamma$. To this end, we invoke the group $N_\omega := \mathbb{C} \times_\omega \mathbb{C}^n$ constructed in [22] and realized as a central extension of the Heisenberg group $H_{2n+1} := \mathbb{R} \times_{\Im m\omega} \mathbb{C}^n$, where the mapping ω denotes the standard Hermitian form on $\mathbb{C}^n$ given by $\omega(z,w) = \langle z,w \rangle$. Assume that we are given a lattice Γ_0 in $(\mathbb{C},+) = (\mathbb{R}^2,+)$ and a lattice Γ in $(\mathbb{C}^n,+) = (\mathbb{R}^{2n},+)$ such that ω sends $\Gamma \times \Gamma$ to Γ_0, i.e., for every γ,γ', we have

$$\omega(\gamma,\gamma') = \langle \gamma,\gamma' \rangle \in \Gamma_0. \tag{3.14}$$

Then under the condition (3.14), it is easy to check that the following facts hold.

Lemma 3.3.1. *$\Gamma_0 \times_\omega \Gamma$ is a discrete subgroup of $N_\omega = \mathbb{C} \times_\omega \mathbb{C}^n$ and it acts freely by left translations on $N_\omega = \mathbb{C} \times_\omega \mathbb{C}^n$. The resulting quotient space $\mathbb{P}^{n+1}_\omega := \Gamma_0 \times_\omega \Gamma \backslash \mathbb{C} \times_\omega \mathbb{C}^n$ is a smooth manifold.*

Obviously, the functions $f \in \mathcal{C}^\infty(\mathbb{P}^{n+1}_\omega)$ are those on $\mathbb{C} \times \mathbb{C}^n$ that are $\Gamma_0 \times_\omega \Gamma$-left-invariants, i.e.,

$$f((\gamma_0;\gamma) \cdot_\omega (z_0;z)) = f(\gamma_0 + z_0 + \omega(\gamma,z); \gamma + z) = f(z_0;z). \tag{3.15}$$

for all $(\gamma_0;\gamma) \in \Gamma_0 \times \Gamma$ and $(z_0;z) \in \mathbb{C} \times \mathbb{C}^n$.

Lemma 3.3.2. *The group $(\mathbb{C},+)$ is isomorphic to the center $Z(N_\omega) = \mathbb{C} \times_\omega \{0\}$ of N_ω.*

Subsequently, the group $(\mathbb{C},+)$ acts on $N_\omega = \mathbb{C}\times_\omega \mathbb{C}^n$ via the rule (left translations) as follows

$$\lambda.(z_0;z) = (\lambda + z_0; z). \tag{3.16}$$

The action in (3.16) can be restricted to $\lambda \in \Gamma_0$ and $(z_0,z) = (\gamma_0,\gamma) \in \Gamma_0\times\Gamma$, so that we can see that the torus $\mathbb{C}^n/\Gamma$ acts freely and effectively on the quotient space $(\mathbb{C}/\Gamma_0)\backslash\mathbb{P}^{n+1}_\omega$.

Definition 3.3. *A function $f(z_0;z) \in \mathcal{C}^\infty(\mathbb{C}\times\mathbb{C}^n)$ is said to be equivriant with respect to given pair of real numbers (ν,μ) if it satisfies*

$$f(\lambda+z_0;z) = e^{i[\mu\Re e(\lambda)+\nu\Im m(\lambda)]}f(z_0;z)$$

for every $\lambda\in\mathbb{C}$ and $(z_0;z)\in\mathbb{C}\times\mathbb{C}^n$.

Accordingly, equivariant functions $f(z_0,z)$ are necessary of the form

$$f(z_0;z) = e^{i[\mu\Re e(z_0)+\nu\Im m(z_0)]}F(z), \tag{3.17}$$

where $F(z)$ is a $\mathcal{C}^\infty$ function on $\mathbb{C}^n$.

Proposition 3.3.3. *Let Γ,Γ_0 be lattices in $(\mathbb{C},+)$ and $(\mathbb{C}^n,+)$ respectively, and assume that the condition* (3.14) *holds. Then, we have*

i) Let $f(z_0;z)$ be in $\mathcal{C}^\infty(\mathbb{P}^{n+1}_\omega)$ such that it is equivariant. Then the function $F(z)$ occurring in (3.17) *must satisfies the following "pseudo-periodic condition" with respect to the Γ-lattice of $(\mathbb{C},+)$. Namely, for every $\gamma\in\Gamma$ and $z\in\mathbb{C}^n$, we have*

$$F(z+\gamma) = e^{i\nu\Im m\langle z,\gamma\rangle - i\mu\Re e\langle z,\gamma\rangle}F(z). \tag{3.18}$$

ii) Let ν,μ be in Γ_0^* the dual lattice of Γ_0 in $\mathbb{R}^2=\mathbb{C}$ defined by

$$\Gamma_0^* = \left\{\gamma_0^*\in\mathbb{R}^2;\ \Re e(\gamma_0^*.\overline{\gamma_0}) \in 2\pi\mathbb{Z}\ \ \textit{for all}\ \ \gamma_0\in\Gamma_0\right\}.$$

Then, a function $F(z)$ on $\mathbb{C}^n$ satisfying (3.18) *with respect to Γ gives rise, through the formula* (3.16)*, to a function $f(z_0;z)$ on $\mathbb{P}^{n+1}_\omega$ which is equivariant.*

To conclude, we picture the involved algebraic and geometrical objects by the following commutative diagrams keeping in mind the previous notations.

$$\begin{tikzcd}
{} \arrow[r] & \Gamma_0\times_\omega\{0\} \arrow[r, "inj"] \arrow[d, "i"'] \arrow[dr, phantom, "\circlearrowleft"] & \Gamma_0\times_\omega\Gamma \arrow[r, "p"] \arrow[d, "i"'] \arrow[dr, phantom, "\circlearrowleft"] & \Gamma \arrow[r] \arrow[d, "i"'] & {} \\
{} \arrow[r] & \mathbb{C}\times_\omega\{0\} \arrow[r, "inj"] \arrow[d, "q"'] \arrow[dr, phantom, "\circlearrowleft"] & \mathbb{C}\times_\omega\mathbb{C}^n \arrow[r, "p"] \arrow[d, "q"'] \arrow[dr, phantom, "\circlearrowleft"] & \mathbb{C}^n \arrow[r] \arrow[d, "q"'] & {} \\
{} \arrow[r] & \mathbb{C}/\Gamma_0 = T^2 \arrow[r, "inj"] \arrow[d] & \mathbb{P}^{n+1}_\omega \arrow[r, "\pi"] \arrow[d] & \mathbb{C}^n/\Gamma \arrow[r] \arrow[d] & {} \\
& {} & {} & {} &
\end{tikzcd}$$

Above inj is the natural injection morphism from the subgroup $\Gamma_0\times_\omega\{0\}$ into $\Gamma_0\times_\omega\Gamma$ and similarly for $\mathbb{C}\times_\omega\{0\}$ into the group $N_\omega = \mathbb{C}\times_\omega\mathbb{C}^n$. The mapping p is the projection on the second factor and it is a morphism of groups. The mapping q is the natural quotient projection from $\mathbb{C}\times_\omega\{0\}$ onto the quotient group $\Gamma_0\times_\omega\{0\}\backslash\mathbb{C}\times_\omega\{0\}$ which is isomorphic as a group to the torus $T^2=\mathbb{C}/\Gamma_0$. To precise the projection mapping π in above, we should notice here that T^2 acts on $\mathbb{P}^{n+1}_\omega$ so that the set of its orbits $\mathcal{O}(T^2,\mathbb{P}^{n+1}_\omega)$ is diffeomorphic to the torus $\mathbb{C}^n/\Gamma$. More precisely, we have $\mathcal{O}(T^2,\mathbb{P}^{n+1}_\omega) = \mathbb{P}^{n+1}_\omega/T^2$ and π is then the canonical projection of the action T^2 on $\mathbb{P}^{n+1}_\omega$ for which $\Gamma_0\times_\omega\{0\}\backslash\mathbb{C}\times_\omega\{0\} = T^2$ are the fibers of the principal bundle $\mathbb{P}^{n+1}_\omega$ over $\mathbb{C}^n/\Gamma$.

3.4 Construction of the invariant Laplacian $\Delta_{\nu,\mu}$

3.4.1 Explicit formula for the sub-Laplacian $\mathcal{L}_\omega$ on $N_\omega = \mathbb{C} \times_\omega \mathbb{C}^n$

The group $N_\omega = \mathbb{C} \times_\omega \mathbb{C}^n$ with the ω-law given in (3.9) is a real *Lie group* of dimension $2n+2$, and its tangent space at its neutral element $e = (0;0) \in \mathbb{C} \times \mathbb{C}^n$ is given by $T_{(0;0)}N_\omega = (\mathbb{C}, +) \times (\mathbb{C}^n, +)$ viewed as a real vector space of dimension $2n+2$. In fact, N_ω is naturally equipped with the standard differentiable structure on euclidean spaces generated by the coordinates system $\{(\mathbb{C} \times \mathbb{C}^n, x)\}$, where x is the coordinates map

$$x : \mathbb{C} \times \mathbb{C}^n \longrightarrow \mathbb{R}^{2n+2}; \quad (z_0, z) \longmapsto (s, t, x_1, y_1, x_2, y_2, \cdots, x_n, y_n).$$

The group action and the group symmetric maps are smooth under this differentiable structure. Let denote by $\mathfrak{n}_\omega$ its associated *Lie algebra* composed of all *left-invariant vector fields* on N_ω and endowed with the standard bracket on vector fields. It is a well known fact that $\mathfrak{n}_\omega \cong T_{(0;0)}N_\omega$. For the sack of giving the explicit formula for the sub-Laplacian $\mathcal{L}_\omega$ on $N_\omega = \mathbb{C} \times_\omega \mathbb{C}^n$, we need to build a basis of $\mathfrak{n}_\omega$ which will be constructed as first order differential operators on functions of N_ω. Define the left action by a fixed element $(z_0; z) \in N_\omega$ by

$$\ell_{(z_0;z)} : N_\omega \longrightarrow N_\omega; \quad (w_0; w) \longmapsto (z_0; z) \cdot_\omega (w_0; w).$$

This map is a diffeomorphism with respect to the Lie group structure. Hence, it is possible to extend its push-forward to act on vector fields. Furthermore, its action on a vector field X is given explicitly by

$$\ell_{g_*} X_p f = X_p(f \circ \ell_g),$$

for test data p and f such that $p \in N_\omega$ and f is a smooth function of N_ω. By definition, a vector field X is said to be left-invariant if the equality $\ell_{g_*} X = X$ holds.

In order to construct a left-invariant vector field basis, we take a basis of the tangent vectors at the identity and generate from each vector of the tangent basis, a left-invariant vector field by pushing it forward using $\ell_{(z_0,z)}$. Recall that a basis of the tangent vector space $T_{(0;0)}N_\omega$ acting on smooth functions f is given by

$$\left(\frac{\partial}{\partial x^i}\right)_{(0;0)} f := \partial_i (f \circ x^{-1})\Big|_{x(0,0)}; \quad i = 1, 2, \cdots, 2n+2, \tag{3.19}$$

where ∂_i is the ordinary partial derivative with respect to the i-th variable. We can now carry out the following computation in order to find generators for $\mathfrak{n}_\omega$:

$$\ell_{(z_0;z)_*}\left(\frac{\partial}{\partial x^i}\right)_{(0;0)} f = \left(\frac{\partial}{\partial x^i}\right)_{(0;0)} (f \circ \ell_{(z_0;z)}) = \partial_i((f \circ \ell_{(z_0,z)}) \circ x^{-1})\Big|_{x(0,0)}.$$

We plug in $x^{-1} \circ x$ in the middle of the last equation and we use the multivariable chain rule to get

$$\begin{aligned}
\ell_{(z_0;z)_*}\left(\frac{\partial}{\partial x^i}\right)_{(0;0)} f &= \partial_i((f \circ x^{-1}) \circ (x \circ \ell_{(z_0;z)} \circ x^{-1}))\Big|_{x(0,0)} \\
&= \sum_{m=1}^{2n+2} \partial_m (f \circ x^{-1})\Big|_{x \circ \ell_{(z_0;z)} \circ x^{-1} \circ x(0,0)} \times \partial_i (x^m \circ \ell_{(z_0;z)} \circ x^{-1})\Big|_{x(0,0)} \\
&= \sum_{m=1}^{2n+2} \mathbf{J}_{m,i} \times \left(\frac{\partial}{\partial x^m}\right)_{(z_0;z)},
\end{aligned}$$

where $\mathbf{J}_{m,i} := \partial_i (x^m \circ \ell_{(z_0;z)} \circ x^{-1})\Big|_{x(0,0)}$ and $x^m \circ \ell_{(z_0;z)} \circ x^{-1}$ is the m-th coordinate map of $x \circ \ell_{(z_0;z)} \circ x^{-1}$. Explicitly, we have

$$\begin{aligned}
&x \circ \ell_{(z_0;z)} \circ x^{-1}(s', t', x'_1, y'_1, \cdots, x'_n, y'_n) = \\
&\left(s + s' + \sum_{j=1}^{n}(x_j x'_j + y_j y'_j), t + t' + \sum_{j=1}^{n}(y_j x'_j - x_j y'_j), x_1 + x'_1, y_1 + y'_1, \cdots, y_n + y'_n\right).
\end{aligned}$$

Therefore, it follows that $\mathbf{J}_{m,i}$ can be viewed as the components of the following Jacobian matrix

$$\mathbf{J} := \mathbf{J}|_{(0,\cdots,0)} = \begin{pmatrix} 1 & 0 & x_1 & y_1 & \cdots & x_n & y_n \\ 0 & 1 & y_1 & -x_1 & \cdots & y_n & -x_n \\ \vdots & 0 & \ddots & & & & 0 \\ \vdots & & & \ddots & & & \\ \vdots & & & & \ddots & & \\ \vdots & & & & & \ddots & \\ 0 & 0 & \cdots & & & \cdots & 1 \end{pmatrix}.$$

Reading vertically, column by column, we find the following basis

$$\begin{cases} S = \left(\frac{\partial}{\partial s}\right) \\ T = \left(\frac{\partial}{\partial t}\right) \\ X_j = x_j\left(\frac{\partial}{\partial s}\right) + y_j\left(\frac{\partial}{\partial t}\right) + \left(\frac{\partial}{\partial x_j}\right) \\ Y_j = y_j\left(\frac{\partial}{\partial s}\right) - x_j\left(\frac{\partial}{\partial t}\right) + \left(\frac{\partial}{\partial y_j}\right). \end{cases} \tag{3.20}$$

Note that we are using the coordinates $z_0 = s + it$ and $z_j = x_j + iy_j$ with

$$\frac{\partial}{\partial x^1} = \frac{\partial}{\partial s}, \qquad \frac{\partial}{\partial x^2} = \frac{\partial}{\partial t}, \qquad \frac{\partial}{\partial x^{2j+1}} = \frac{\partial}{\partial x_j}, \qquad \frac{\partial}{\partial x^{2j+2}} = \frac{\partial}{\partial y_j}; \tag{3.21}$$

for $j = 1, \cdots, n$.
In the following proposition, we summarize the above discussion on $N_\omega = \mathbb{C} \times_\omega \mathbb{C}^n$ and its associated Lie algebra $\mathfrak{n}_\omega$ with some additional remarks.

Proposition 3.4.1. *The real vector fields $S = \frac{\partial}{\partial s}$, $T = \frac{\partial}{\partial t}$ together with X_j, Y_j; $j = 1, \cdots, n$ given by*

$$X_j = x_j\left(\frac{\partial}{\partial s}\right) + y_j\left(\frac{\partial}{\partial t}\right) + \left(\frac{\partial}{\partial x_j}\right) \quad \text{and} \quad Y_j = y_j\left(\frac{\partial}{\partial s}\right) - x_j\left(\frac{\partial}{\partial t}\right) + \left(\frac{\partial}{\partial y_j}\right)$$

form a basis for $\mathfrak{n}_\omega$. Moreover, they satisfy the following commutation relations of Heisenberg type

$$\begin{cases} [S, X_j] = [S, Y_j] = 0 \\ [T, X_j] = [T, Y_j] = 0 \\ [S, T] = 0 \\ [X_j, X_k] = [Y_j, Y_k] = 0 \\ [X_j, Y_k] = -2\delta_{jk}T; \end{cases}$$

for all $j, k = 1, \cdots, n$.

Remark 3.4.2. *As expected, we see, in view of the above proposition, that the Lie algebra $\mathfrak{n}_\omega$ of $N_\omega = \mathbb{C} \times_\omega \mathbb{C}^n$ with $\omega(z, w) = \langle z, w\rangle$ is also a central extension of the classical Heisenberg algebra $H_{2n+1} = \mathbb{R} \times_{Im\omega} \mathbb{C}^n$ generated by the vector fields*

$$\left\{ T = \frac{\partial}{\partial t}, \tilde{X}_j = -y_j\frac{\partial}{\partial t} + \frac{\partial}{\partial x_j}, \tilde{Y}_j = x_j\frac{\partial}{\partial t} + \frac{\partial}{\partial y_j} \right\}; \quad j = 1, \cdots, n,$$

with the nontrivial commutation relation $[\tilde{X}_j, \tilde{Y}_k] = -2T$, where (x_j, y_j); $j = 1, \cdots, n$,, are coordinates of $\mathbb{C}^n$.

Remark 3.4.3. *To build such left-invariant vector fields, one can also look for a one parameter group of N_ω, i.e., a group homomorphism $\gamma : (\mathbb{R}, +) \longrightarrow N_\omega$ (curves $\gamma(\varepsilon) \in N_\omega$; $\varepsilon \in \mathbb{R}$) satisfying*

$$\dot{\gamma}(0) = \frac{d\gamma}{d\varepsilon}(\varepsilon)|_{\varepsilon=0} = (v_0; v) \in T_{(0;0)}N_\omega = \mathbb{C} \times \mathbb{C}^n.$$

According to the above discussion, we can introduce the following definition of sub-Laplacian on $N_\omega = \mathbb{C} \times_\omega \mathbb{C}^n$.

Definition 3.4. *Let X_j, Y_j; $j = 1, \cdots, n$, be the vector fields given in Proposition 3.4.1. Then, the operator*

$$\mathcal{L}_\omega = \sum_{j=1}^{n} X_j^2 + Y_j^2$$

is called here the sub-Laplacian of $N_\omega = \mathbb{C} \times_\omega \mathbb{C}^n$.

The following proposition gives the explicit differential expression of $\mathcal{L}_\omega$ in terms of the Laplace-Beltarmi $\Delta_{\mathbb{R}^{2n}}$ of $\mathbb{C}^n = \mathbb{R}^{2n}$,

$$\Delta_{\mathbb{R}^{2n}} := \sum_{j=1}^{n} \frac{\partial^2}{\partial x_j^2} + \frac{\partial^2}{\partial y_j^2}$$

and the first order differential operators $E_{x,y}$ and $F_{x,y}$ defined by

$$E_{x,y} := \sum_{j=1}^{n} x_j \frac{\partial}{\partial x_j} + y_j \frac{\partial}{\partial y_j}$$

and

$$F_{x,y} := \sum_{j=1}^{n} x_j \frac{\partial}{\partial y_j} - y_j \frac{\partial}{\partial x_j}.$$

Namely, we have

Proposition 3.4.4. *The sub-Laplacian $\mathcal{L}_\omega$ prescribed in Definition 3.4 is given explicitly in the coordinates t, s, x_j, y_j; $j = 1, \cdots, n$, of $N_\omega = \mathbb{C} \times_\omega \mathbb{C}^n$ as follows*

$$\mathcal{L}_\omega = \Delta_{\mathbb{R}^{2n}} + 2(E_{x,y} + n)\frac{\partial}{\partial s} - 2F_{x,y}\frac{\partial}{\partial t} + (|x|^2 + |y|^2)\left(\frac{\partial^2}{\partial s^2} + \frac{\partial^2}{\partial t^2}\right), \tag{3.22}$$

where $|x|^2 = \sum\limits_{j=1}^{n} x_j^2$ and $|y|^2 = \sum\limits_{j=1}^{n} y_j^2$.

Proof. The explicit expression of $\mathcal{L}_\omega$ given in Proposition 3.4.4 can be handled by straightforward computations. □

Remark 3.4.5. *If we consider the coordinates $(s, t) \in \mathbb{R}^2 = \mathbb{C}$ and $z = (z_1, \cdots, z_n) \in \mathbb{C}^n$ with $z_j = x_j + iy_j$, then the sub-Laplacian $\mathcal{L}_\omega$ in (3.22) can be rewritten as*

$$\mathcal{L}_\omega = 4\sum_{j=1}^{n} \frac{\partial^2}{\partial z_j \partial \bar{z}_j} + 2(E + \overline{E} + n)\frac{\partial}{\partial s} - 2i(E - \overline{E})\frac{\partial}{\partial t} + |z|^2\left(\frac{\partial^2}{\partial s^2} + \frac{\partial^2}{\partial t^2}\right), \tag{3.23}$$

where $E = \sum\limits_{j=1}^{n} z_j \frac{\partial}{\partial z_j}$ is the complex Euler operator and $\overline{E} = \sum\limits_{j=1}^{n} \bar{z}_j \frac{\partial}{\partial \bar{z}_j}$ is its complex conjugate.

Remark 3.4.6. *The action of $\mathcal{L}_\omega$ on functions $F(t; z)$ on $N_\omega = \mathbb{C} \times_\omega \mathbb{C}^n$ that are independent of the argument s, reduces to that of the sub-Laplacian*

$$\widetilde{\mathcal{L}}_{Im\omega} = 4\sum_{j=1}^{n} \frac{\partial^2}{\partial z_j \partial \bar{z}_j} - 2i(E - \overline{E})\frac{\partial}{\partial t} + |z|^2 \frac{\partial^2}{\partial t^2} \tag{3.24}$$

of the classical Heisenberg group $\mathbb{R} \times_{Im\omega} \mathbb{C}^n = H_{2n+1}$.

We conclude this subsection by mentioning that both operators $\mathcal{L}_\omega$ and $\widetilde{\mathcal{L}}_{Im\omega}$ are not elliptic. But they share many aspects and nice properties of their spectral theory with elliptic operators. We will precise this by giving the concrete description of the spectral eigenfunction problem of the associated elliptic differential operator

$$\boxed{\Delta_{\nu,\mu} = 4\sum_{j=1}^{n} \frac{\partial^2}{\partial z_j \partial \bar{z}_j} + 2(\nu + i\mu)E - 2(\nu - i\mu)\overline{E} - \left(\mu^2 + \nu^2\right)|z|^2 + 2i\mu n.} \tag{3.25}$$

Formally, $\Delta_{\nu,\mu}$ is related to $\mathcal{L}_\omega$ using partial Fourier transform in (s,t) with $(i\nu, i\mu)$ as dual arguments.

In the next section, we will prove that the operator $\Delta_{\nu,\mu}$ can also be regarded as a Schrödinger operator in the presence of a uniform magnetic field $\overrightarrow{B}_\mu = id\theta_{\nu,\mu}$ on $\mathbb{C}^n = \mathbb{R}^{2n}$ associated to a specific differential 1-form $\theta_{\nu,\mu}$.

3.4.2 Realization of $\Delta_{\nu,\mu}$ as a magnetic Schrödinger operator and invariance property

A *Magnetic Schrödinger operator* on a complete oriented Riemannian manifold (M,g) acting on scalar functions has the general form

$$H_\theta = (d + \text{ext }\theta)^*(d + \text{ext }\theta), \tag{3.26}$$

where θ is a given $\mathcal{C}^1$ real differential 1-form on M (potential vector). Here d stands for the usual exterior derivative acting on the space of differential p-forms $\Omega^p(M)$, $\text{ext}\,\theta$ is the operator of exterior left multiplication by θ, i.e., $(\text{ext}\,\theta)\omega = \theta \wedge \omega$ and $(d + \text{ext}\theta)^*$ is the formal adjoint of $d + \text{ext}\theta$ with respect to the Hermitian product on $\Omega^p = \Omega^p(M)$

$$\langle \alpha, \beta \rangle_{\Omega^p} = \int_M \alpha \wedge \star\beta$$

induced by the metric g, where $\star$ denotes the *Hodge star operator* associated with the volume form. From general theory of Schrödinger operators on non-compact manifold M (see for example [74]), it is known that the operator H_θ, viewed as an unbounded operator in $L^2(M, dm)$, is essentially self-adjoint for any smooth measure dm.

In our framework M is the complex n-space $\mathbb{C}^n$ equipped with its Kähler metric

$$g = ds^2 = -\frac{i}{2}\sum_{j=1}^{n} dz_j \otimes d\bar{z}_j = \sum_{j=1}^{n} dx_j \otimes dy_j$$

and the corresponding volume form is $\mathbf{V}\text{ol} = dx_1 dy_1 \cdots dx_n dy_n$. associated with the parameters ν and μ, we consider the potential vector

$$\theta_{\nu,\mu}(z) := -\frac{\nu - i\mu}{2}\sum_{j=1}^{n} \bar{z}_j dz_j + \frac{\nu + i\mu}{2}\sum_{j=1}^{n} z_j d\bar{z}_j. \tag{3.27}$$

Thus, we prove the following result concerning the twisted Laplacian defined by (3.25).

Proposition 3.4.7. *For every complex-valued $\mathcal{C}^\infty$ function f on $\mathbb{C}^n$, we have*

$$\Delta_{\nu,\mu} f = -H_{\theta_{\nu,\mu}} f = -(d + \text{ext}\,\theta_{\nu,\mu})^*(d + \text{ext}\,\theta_{\nu,\mu})f. \tag{3.28}$$

Sketched proof. We start by writing $H_{\theta_{\nu,\mu}} := (d + \text{ext}\,\theta_{\nu,\mu})^*(d + \text{ext}\,\theta_{\nu,\mu})$ as

$$H_{\theta_{\nu,\mu}} = d^* df + d^* \,\text{ext}\,\theta_{\nu,\mu} f + (\text{ext}\,\theta_{\nu,\mu})^* df + (\text{ext}\,\theta_{\nu,\mu})^* \,\text{ext}\,\theta_{\nu,\mu}.$$

Next, using the well-known facts $d^* = -\star d \star$ and $(\operatorname{ext}\theta)^* = \star \operatorname{ext}\theta\star$, we establish the following

$$\begin{aligned}
&d^* d = -4\sum_{j=1}^{n} \frac{\partial}{\partial z_j}\frac{\partial}{\partial \overline{z_j}},\\
&d^* \operatorname{ext}\theta_{\nu,\mu} f + (\operatorname{ext}\theta_{\nu,\mu})^* d = \sum_{j=1}^{n}\left(-2(\nu+i\mu) z_j \frac{\partial}{\partial z_j} + 2(\nu - i\mu)\overline{z_j}\frac{\partial}{\partial \overline{z_j}} - 2i\mu\right),\\
&(\operatorname{ext}\theta_{\nu,\mu})^* \operatorname{ext}\theta_{\nu,\mu} = (\nu^2+\mu^2)\,|z|^2 .
\end{aligned}$$

□

One of the advantages of the formula for $\Delta_{\nu,\mu}$ as given by (3.28) with the differential 1-form $\theta_{\nu,\mu}$ in (3.27) is that we can derive easily some invariance properties of the Laplacian $\Delta_{\nu,\mu}$ with respect to the group G of the motions of the complex Hermitian space $(\mathbb{C}^n, ds^2)$; $ds^2 = \sum\limits_{j=1}^{n} dz_j \otimes d\bar{z}_j$. The pull-back $g^*\theta_{\nu,\mu}$ of the differential 1-form $\theta_{\nu,\mu}$ by the action mapping $z \longmapsto g.z$ is related to $\theta_{\nu,\mu}$ by the following identity for every $g \in G = \mathsf{U}(n) \ltimes \mathbb{C}^n$.

Proposition 3.4.8. *Let $\theta_{\nu,\mu}$ be as in* (3.27). *Then, for every $g \in G = \mathsf{U}(n) \ltimes \mathbb{C}^n$ we have*

$$g^*\theta_{\nu,\mu} = \theta_{\nu,\mu} + \frac{d\overline{j^{\nu,\mu}(g,z)}}{\overline{j^{\nu,\mu}(g,z)}}, \tag{3.29}$$

where

$$j^{\nu,\mu}(g,z) = \exp(i\phi_{\nu,\mu}(g,z)). \tag{3.30}$$

The phase function $\phi_{\nu,\mu}(g,z)$ is given by

$$\phi_{\nu,\mu}(g,z) = \mu\Re e\left(\left\langle z, g^{-1}.0\right\rangle\right) - \nu\Im m\left(\left\langle z, g^{-1}.0\right\rangle\right). \tag{3.31}$$

Proof. The identity (3.29) holds by component-wise straightforward computations. Indeed, direct computation yields

$$\begin{aligned}
g^*\theta_{\nu,\mu}(z) &= \theta_{\nu,\mu}(z) + \frac{i\mu}{2} d\left[\left\langle z, g^{-1}.0\right\rangle + \overline{\langle z, g^{-1}.0\rangle}\right] - \frac{\nu}{2} d\left[\left\langle z, g^{-1}.0\right\rangle - \overline{\langle z, g^{-1}.0\rangle}\right]\\
&= \theta_{\nu,\mu}(z) + id(\phi_{\nu,\mu}(g,z)),
\end{aligned}$$

where g^{-1} is the inverse mapping of $z \longmapsto g.z$ and $g^{-1}.0 = -A^{-1}b = -A^*b$ for $g = [A,b] \in \mathsf{U}(n) \ltimes \mathbb{C}^n$. We conclude since $dj^{\nu,\mu}(g,z) = id(\phi_{\nu,\mu}(g,z))j^{\nu,\mu}(g,z)$. □

Notice that the relation (3.29) reads also as $g^*\theta_{\nu,\mu} = \theta_{\nu,\mu} + d\log(j^{\nu,\mu}(\gamma,z))$ and shows that the differential 1-form $\theta_{\nu,\mu}$ is not G-invariant. Though, $g^*\theta_{\nu,\mu}$ and $\theta_{\nu,\mu}$ still in the same class of the de Rham cohomology group. Also, it gives insight how to make, in view of the expression (3.28), the Laplacian $\Delta_{\nu,\mu}$ invariant with respect to a G-action on functions built with the help of the following automorphic factor $j^{\nu,\mu}(g,z)$ defined through (3.30) and satisfying the chain rule

$$j^{\nu,\mu}(gg',z) = j^{\nu,\mu}(g,g'z)j^{\nu,\mu}(g',z) \tag{3.32}$$

for every $g \in G = \mathsf{U}(n) \ltimes \mathbb{C}^n$ and $z \in \mathbb{C}^n$. associated with $j^{\nu,\mu}$, we define $T_g^{\nu,\mu}$ to be the operator acting on differential p-forms ω of $\mathbb{C}^n$ through the formula

$$T_g^{\nu,\mu}\omega = \overline{j^{\nu,\mu}(g,z)}g^*\omega.$$

On $\mathcal{C}^\infty$ complex-valued functions f on $\mathbb{C}^n$, it reduces further to

$$[T_g^{\nu,\mu}f](z) = \overline{j^{\nu,\mu}(g,z)}g^*f(z) = \overline{j^{\nu,\mu}(g,z)}f(g.z). \tag{3.33}$$

Thus, the following invariance property for $\Delta_{\nu,\mu}$ holds.

Proposition 3.4.9. *For every $g \in \mathsf{U}(n) \ltimes \mathbb{C}^n$, we have*

$$\Delta_{\nu,\mu} T_g^{\nu,\mu} = T_g^{\nu,\mu} \Delta_{\nu,\mu}. \tag{3.34}$$

Proof. Using the well-known facts $g^*d = dg^*$ and $g^*(\alpha \wedge \beta) = g^*\alpha \wedge g^*\beta$, we get

$$T_g^{\nu,\mu}\big((d + \operatorname{ext}\theta_{\nu,\mu})f\big) = \overline{j^{\nu,\mu}(\gamma,z)}\Big(d[g^*f] + [g^*\theta_{\nu,\mu}] \wedge [g^*f]\Big).$$

Now, by means of the identity (3.29), it follows

$$\begin{aligned} T_g^{\nu,\mu}\big((d + \operatorname{ext}\theta_{\nu,\mu})f\big) &= \overline{j^{\nu,\mu}(\gamma,z)}d[g^*f] + \overline{j^{\nu,\mu}(\gamma,z)}\theta_{\nu,\mu}[g^*f] + d(\overline{j^{\nu,\mu}(\gamma,z)})[g^*f] \\ &= d(\overline{j^{\nu,\mu}(\gamma,z)}[g^*f]) + \theta_{\nu,\mu}\overline{j^{\nu,\mu}(\gamma,z)}[g^*f] \\ &= (d + \operatorname{ext}\theta_{\nu,\mu})\big(T_g^{\nu,\mu}f\big). \end{aligned}$$

Moreover, $T_g^{\nu,\mu}$ commutes also with $(d + \operatorname{ext}\theta_{\nu,\mu})^*$ for $T_g^{\nu,\mu}$ being a unitary transformation. Therefore, by means of the expression of $\Delta_{\nu,\mu} = -(d + \operatorname{ext}\theta_{\nu,\mu})^*(d + \operatorname{ext}\theta_{\nu,\mu})$ as a magnetic Schrödinger operator $H_{\theta_{\nu,\mu}}$, we deduce easily that $\Delta_{\nu,\mu}$ and $T_g^{\nu,\mu}$ commute. This ends the proof. $\square$

Remark 3.4.10. *For $g \in \{I_n\} \ltimes \mathbb{C}^n = (\mathbb{C}^n, +)$, where I_n denotes the identity matrix in $\mathbb{C}^{n\times n}$, the unitary operators $T_g^{\nu,\mu}$ given in (3.33) define projective representation of G on the space of C^∞ functions on $\mathbb{C}^n$. In fact, they are the so-called magnetic translation operators that arise in the study of Schrödinger operators in the presence of uniform magnetic field.*

Remark 3.4.11. *The potential vector $\theta_{\nu,\mu}$ given through* (3.27) *can be seen as a specific magnetic vector potential that corresponds to an isotropic magnetic field of constant strength 2ν, since $d(i\theta_{\nu,\mu})(z) = 2\nu\mathbf{Vol}(z)$. It is issued from the holomorphic gauge θ_ν^h and the symmetric gauge θ_ν^s defined respectively by*

$$\theta_\nu^h = -\nu\sum_{j=1}^n \overline{z_j}dz_j \quad \text{and} \quad \theta_\nu^s = -\frac{\nu}{2}\sum_{j=1}^n(\overline{z_j}dz_j - z_jd\overline{z_j}).$$

More exactly, we have the gauge transformations

$$\theta_{\nu,\mu} = \theta_\nu^h + \frac{i\mu+\nu}{2}d\sum_{j=1}^n |z_j|^2 \quad \text{and} \quad \theta_{\nu,\mu} = \theta_\nu^s + \frac{i\mu}{2}d\sum_{j=1} |z_j|^2.$$

Remark 3.4.12. *The connection form $\theta_{\nu,\mu}$ in* (3.27) *reads*

$$\theta_{\nu,\mu}(z) = i\sum_{j=1}^n(\nu y_j + \mu x_j)dx_j - (\nu x_j - \mu y_j)dy_j \tag{3.35}$$

in the real coordinates $(x_1, y_1, x_2, y_2, \cdots, x_n; y_n)$. Its divergence is then showed to be given by $div\theta_{\nu,\mu} = 2in\mu$. Accordingly, $div\theta_{\nu,\mu}$ corresponds to the constant term involved in the expression of the magnetic quantum Hamiltonian $\Delta_{\nu,\mu}$ given through (3.25)*. Therefore, $\theta_{\nu,\mu}$ is not a radiation (coulomb) gauge, unless $\mu = 0$, and therefore $\theta_{\nu,\mu}$ can be seen as a perturbation of the Coulomb gauge. The perturbation operator in $\Delta_{\nu,\mu}$ is given by*

$$D_\mu := 2i\mu(E + \overline{E} + n) - \mu^2|z|^2,$$

so that $\Delta_{\nu,\mu} = H_\nu + D_\mu$.

3.5 Spectral properties of $\Delta_{\nu,\mu}$ acting on $\mathcal{C}^\infty(\mathbb{C}^n)$ and $L^2(\mathbb{C}^n, dm)$

We denote by $\mathcal{C}^\infty(\mathbb{C}^n)$ the Frechet space of complex-valued functions on $\mathbb{C}^n$ endowed with the compact-open topology, while $L^2(\mathbb{C}^n, dm)$ denotes the usual Hilbert space of square integrable complex-valued functions $F(z)$ on $\mathbb{C}^n$ with respect to the usual Lebesgue measure dm. In the sequel, we will give a concrete description of the eigenspaces of $\Delta_{\nu,\mu}$ in both $\mathcal{C}^\infty(\mathbb{C}^n)$ and $L^2(\mathbb{C}^n, dm)$. To this end, let λ be any complex number in $\mathbb{C}$ and $E_\lambda(\Delta_{\nu,\mu})$ be the eigenspace of $\Delta_{\nu,\mu}$ corresponding to the eigenvalue $-2\nu(2\lambda+n)$ in $\mathcal{C}^\infty(\mathbb{C}^n)$; that is

$$E_\lambda(\Delta_{\nu,\mu}) = \left\{F \in \mathcal{C}^\infty(\mathbb{C}^n);\ \Delta_{\nu,\mu}F = -2\nu(2\lambda+n)F\right\}. \tag{3.36}$$

Also, by $A^2_\lambda(\Delta_{\nu,\mu})$ we denote the subspace of $L^2(\mathbb{C}^n, dm)$ whose elements $F(z)$ satisfy $\Delta_{\nu,\mu}F = -2\nu(2\lambda + n)F$. Namely, by elliptic regularity of $\Delta_{\nu,\mu}$, we have

$$A^2_\lambda(\Delta_{\nu,\mu}) = L^2(\mathbb{C}^n, dm) \cap E_\lambda(\Delta_{\nu,\mu}). \tag{3.37}$$

The first result related to $E_\lambda(\Delta_{\nu,\mu})$ and $A^2_\lambda(\Delta_{\nu,\mu})$ is the following.

Proposition 3.5.1. *The eigenspaces $E_\lambda(\Delta_{\nu,\mu})$ and $A^2_\lambda(\Delta_{\nu,\mu})$ are invariants under the $T^{\nu,\mu}$-action given by* (3.33), *in the sense that for every $g \in G = \mathsf{U}(n) \ltimes \mathbb{C}^n$ we have*

$$T^{\nu,\mu}_g\left(E_\lambda(\Delta_{\nu,\mu})\right) \subset E_\lambda(\Delta_{\nu,\mu}) \quad \text{and} \quad T^{\nu,\mu}_g\left(A^2_\lambda(\Delta_{\nu,\mu})\right) \subset A^2_\lambda(\Delta_{\nu,\mu}).$$

Proof. This can be handled easily making use of the invariance property (**??**) of $\Delta_{\nu,\mu}$ by the unitary transformations $T^{\nu,\mu}_g$. □

Proposition 3.5.2. *The set of spherical eigenfuctions of $\Delta_{\nu,\mu}$ with $-2\nu(2\lambda+n)$ as eigenvalue is a one dimensional vector subspace of $E_\lambda(\Delta_{\nu,\mu})$ generated by*

$$\varphi^{\nu,\mu}_\lambda(z) = e^{-\frac{\nu-i\mu}{2}|z|^2}{}_1F_1(-\lambda; n; \nu|z|^2) \tag{3.38}$$

where ${}_1F_1(a;c;x)$ is denoting here the usual confluent hypergeometric function ,

$${}_1F_1(a;c;x) = 1 + \frac{a}{c}\frac{x}{1!} + \frac{a(a+1)}{c(c+1)}\frac{x^2}{2!} + \cdots; \quad x \in \mathbb{C}.$$

Remark 3.5.3. *By a "spherical" (or radial here) eigenfuction of $\Delta_{\nu,\mu}$, we mean $\mathsf{U}(n)$-invariant function f satisfying $f(h.z) = f(z)$ for all $h \in \mathsf{U}(n)$ and $z \in \mathbb{C}^n$.*

Sketched proof. To prove the statement, we write $\Delta_{\nu,\mu}$ in polar coordinates $z = r\theta$ with $r \geq 0$ and $\theta \in S^{2n-1}$ as

$$\Delta_{\nu,\mu} = \frac{\partial^2}{\partial r^2} + \left(\frac{2n-1}{r} + 2i\mu\right)\frac{\partial}{\partial r} - (\mu^2+\nu^2)r^2 + 2i\mu n + L^\theta_{\nu,\mu},$$

where $L^\theta_{\nu,\mu}$ stands for the tangential component of $\Delta_{\nu,\mu}$. The eigenvalue problem $\Delta_{\nu,\mu}f = -2\nu(2\lambda+n)f$ for radial functions $f(z) = \psi(x)$, with $x = r^2$, reduces to the differential equation

$$\left\{x\frac{\partial^2}{\partial x^2} + (n + i\mu x)\frac{\partial}{\partial x} - [\frac{\mu^2+\nu^2}{4}x + \frac{i\mu-\nu}{2}n - \nu\lambda]\right\}\psi = 0.$$

Next, making use of the appropriate change of function $\psi(x) = e^{\frac{i\mu-\nu}{2}x}y(x)$, we see that the previous equation leads to the confluent hypergeometric differential equation [62, page 193]

$$xy'' + (n-\nu x)y' + \nu\lambda y = 0$$

whose regular solution at $x = 0$ is the confluent hypergeometric function ${}_1F_1(-\lambda; n; \nu x)$. □

Remark 3.5.4. *The additional of the gradient of the the gauge function $\varphi(z) = \frac{i\mu}{2}|z|^2$ in the gauge transformation $\theta_{\nu,\mu} = \theta^s_\nu + \frac{i\mu}{2}d|z|^2$, is equivalent to multiplying the eigenstates of the Landau Hamiltonian H_ν; $\mu = 0$, by the phase factor $e^{-\frac{i\nu}{2}|z|^2}$. In fact, this follows by considering the similarity transformation generated by the unitary operator $Sf = e^{-\frac{i\mu}{2}|z|^2}f$ and next showing that*

$$\Delta_{\nu,\mu} = e^{-\frac{i\mu}{2}|z|^2}\Delta_{\nu,0}e^{+\frac{i\mu}{2}|z|^2}. \tag{3.39}$$

This is clear from the proof of Proposition 3.5.2. In other words the operators $\Delta_{\nu,\mu}$ and $\Delta_{\nu,0}$ are unitary equivalent in $L^2(\mathbb{C}^n, dm)$.

Remark 3.5.5. *According to Remark 3.5.4, any physical interpretation or application must take into account the parameter μ that is related to the special gauge transformation we have made.*

Remark 3.5.6. *The key observation contained in the identity* (3.39) *will deserve as outline of the proof of Proposition 3.5.2 as well as the proofs of the assertions below, taking into account the well-established results for $\Delta_{\nu,0}$ (see [6, 4, 32, 79, 48] and the references therein).*

Accordingly, we claim the following

Proposition 3.5.7. *Let $(\nu,\mu) \in \mathbb{R}^2$ with $\nu > 0$ and $\lambda \in \mathbb{C}$. Then, the eigenspace $A^2_\lambda(\Delta_{\nu,\mu})$ as defined (3.37) is non-zero (Hilbert) space if and only if $\lambda = l$ with $l = 0,1,2,\cdots$, is a nominative integer. Moreover, the spaces $A^2_l(\Delta_{\nu,\mu})$, $l = 0,1,2,\cdots$, are pairwise orthogonal in $L^2(\mathbb{C}^n, dm)$ and we have the following orthogonal decomposition in Hilbertian subspaces*

$$L^2(\mathbb{C}^n, dm) = \bigoplus_{l=0}^{\infty} A^2_l(\Delta_{\nu,\mu}).$$

Remark 3.5.8. *A direct proof of Proposition 3.5.7 can be handled using Proposition 3.5.2 and the asymptotic behavior of the confluent hypergeometric function given by [62, page 332]*

$${}_1F_1(a;c;x) = \Gamma(c)\left\{\frac{(-x)^{-a}}{\Gamma(c-a)} + \frac{e^x x^{a-c}}{\Gamma(a)}\right\}\left(1 + O(\frac{1}{x})\right) \tag{3.40}$$

as $x \to +\infty$. This asymptotic behaviour can also be used to show that the radial function $\varphi^{\nu,\mu}_\lambda$ given by (3.38) *is bounded if and only if $\lambda = l$; $l = 0,1,2,\cdots$.*

Remark 3.5.9. *The unitary equivalence of $\Delta_{\nu,\mu}$ and $\Delta_{\nu,0}$ in $L^2(\mathbb{C}^n, dm)$ shows that the spectrum of $\Delta_{\nu,\mu}$ is purely punctual and isolated and coincides with the one of $\Delta_{\nu,0}$. Thus, the energy levels (eigenvalues) are independent of the μ-parameter and are quantized as $-2\nu(2l+n)$ for varying $l \in \mathbb{Z}^+$. This is also contained in Proposition 3.5.7. Moreover, each energy level is infinitely degenerate, since it corresponds to infinite linearly independent states of the quantum system. The states $\psi_{p,q|l}$, given by* (3.41) *below, with index l constitute the l-th Landau level.*

The following result prescribes the elements of the L^2-eigenspaces of $\Delta_{\nu,\mu}$ in terms of the confluent hypergeometric functions ${}_1F_1$ and the harmonic polynomials $h^{pq}(z,\bar{z})$ on $\mathbb{C}^n$ that are homogeneous of degree p in z and degree q in $\bar{z}$ (see [27]).

Proposition 3.5.10. *An orthogonal basis of the infinite dimensional eigenspace $A^2_l(\Delta_{\nu,\mu})$ in (3.37) is given by*

$$\psi^\nu_{p,q|l}(z) = e^{-\frac{\nu+i\mu}{2}|z|^2}{}_1F_1(q-l, n+p+q; \nu|z|^2)h^{pq}(z,\bar{z}), \tag{3.41}$$

for arbitrary nonnegative integers p,q with $q \le l$. Moreover, a function F belongs to $A^2_l(\Delta_{\nu,\mu})$ if and only if it can be expanded as follows

$$F(z) \;=\; \sum_{p=0}^{+\infty}\sum_{q=0}^{l} a_{p,q}\psi^\nu_{p,q|l}(z),$$

where the constants $a_{p,q}$ satisfy the growth condition

$$\|F\|^2_{L^2(\mathbb{C}^n;dm)} = \sum_{p=0}^{+\infty}\sum_{q=0}^{l} |a_{p,q}|^2 \left(\frac{1}{\nu}\right)^{n+p+q} \frac{(l-q)!\Gamma^2(n+p+q)}{2\Gamma(n+p+l)} \|h^{pq}\|^2_{L^2(S^{2n-1})} < +\infty.$$

Proof. The result follows immediately from the similarity transformation (3.39) combined with ii) in Proposition 6 in [32]. □

We conclude this subsection with the following assertion.

Proposition 3.5.11. *Let $(\nu,\mu)\in\mathbb{R}^2$ with $\nu>0$. For fixed $l=0,1,2,\cdots$, let P_l be the orthogonal eigenprojector operator from $L^2(\mathbb{C}^n,dm)$ onto the eigenspace $A^2_l(\Delta_{\nu,\mu})$ with $-2\nu(2l+n)$ as eigenvalue. Then the Schwartz kernel $P_l^{\nu,\mu}(z,w)$ of the operator P_l is given by the following explicit formula*

$$P_l^{\nu,\mu}(z,w)=\left(\frac{\nu}{\pi}\right)^n\frac{(n-1+l)!}{(n-1)!l!}j^{\nu,\mu}(z,w)e^{-\frac{\nu}{2}|z-w|^2}{}_1F_1(-l;n;\nu|z-w|^2), \tag{3.42}$$

where the factor $j^{\nu,\mu}(z,w)$; $z,w\in\mathbb{C}^n$ is given by

$$j^{\nu,\mu}(z,w)=e^{-\frac{i\mu}{2}(|z|^2-|w|^2)+\frac{\nu}{2}(\langle z,w\rangle-\overline{\langle z,w\rangle})}. \tag{3.43}$$

Sketched proof. The proof for $\mu=0$ is contained in [6, 4, 32]. For arbitrary μ, the proof can be handled in a similar way, or by making use of the key observation that in $L^2(\mathbb{C}^n,dm)$, the operators $\Delta_{\nu,\mu}$ and $\Delta_{\nu,0}$ are unitary equivalents and we have

$$\Delta_{\nu,\mu}=e^{-\frac{i\mu}{2}|z|^2}\Delta_{\nu,0}e^{+\frac{i\mu}{2}|z|^2}.$$

□

3.5.1 Factorization of $\Delta_{\nu,\mu}$ and associated Hermite polynomials

In this section, we study the spectral theory of $\Delta_{\nu,\mu}$ on $L^2(\mathbb{C}^n,d\lambda)$ using the *factorization method*. This method finds its origin in the works of Dirac [19] and Schrödinger [70], then was developed by Infeld and Hull [43] in order to solve eigenvalue problems appearing in quantum theory. Notice for instance that the operator $\Delta_{\nu,0}=L_\mu$ is refereed in physic-mathematical literature as the Landau operator on $\mathbb{R}^{2n}=\mathbb{C}^n$ (or Schrödinger operator on $\mathbb{R}^{2n}$ in the presence of a uniform magnetic field $\overrightarrow{B}_\nu=\sqrt{-1}d\theta_{\nu,\mu}$) and for which many of their spectral properties that we are considering go back to Landau's work in 1930 on the Hamiltonian in $\mathbb{R}^2=\mathbb{C}$ given by

$$L_\nu=-4\frac{\partial^2}{\partial z\partial\bar z}-2\nu\left(z\frac{\partial}{\partial z}-\bar z\frac{\partial}{\partial\bar z}\right)+\nu^2|z|^2=4\left(-\frac{\partial}{\partial z}+\frac{\nu}{2}\bar z\right)\left(\frac{\partial}{\partial\bar z}+\frac{\nu}{2}z\right)-2\nu I$$

More generally the Laplacian $\Delta_{\nu,\mu}$ can be rewritten as

$$\Delta_{\nu,\mu}=-4\sum_{j=1}^n\left(-\frac{\partial}{\partial z_j}+\frac{\nu-i\mu}{2}\bar z_j\right)\left(\frac{\partial}{\partial\bar z_j}+\frac{\nu+i\mu}{2}z_j\right)-2\nu nI.$$

Hence, in view of the above remarks, the spectral properties of $\Delta_{\nu,\mu}$ on $L^2(\mathbb{C}^n,dm)$ or on $\mathcal{C}^\infty(\mathbb{C}^n)$ can be derived from Landau's work [52]. To this end, It will be helpful to define

$$\tilde\Delta_{\nu,\mu}:=-\frac{1}{4}\Delta_{\nu,\mu}.$$

We also need to define, for $j=1,2,\cdots,n$, the following first order differential operators

$$a_j^+=-\frac{\partial}{\partial z}+\frac{\nu-imu}{2}\bar z_j=-\,\mathrm{e}^{\frac{\nu-i\mu}{2}|z_j|^2}\frac{\partial}{\partial z_j}\,\mathrm{e}^{-\frac{\nu-i\mu}{2}|z_j|^2}, \tag{3.44}$$

and

$$a_j^-=\frac{\partial}{\partial\bar z}+\frac{\nu+i\mu}{2}z_j=\mathrm{e}^{-\frac{\nu+i\mu}{2}|z_j|^2}\frac{\partial}{\partial\bar z_j}\,\mathrm{e}^{\frac{\nu+i\mu}{2}|z_j|^2}. \tag{3.45}$$

These operators satisfy the commutation relationships $\left[a_j^-, a_k^+\right] = n\nu\delta_{j,k}$, where $\delta_{j,k}$ is the Krönecker symbol. They are linked to the Laplacian $\tilde{\Delta}_{\nu,\mu}$ through

$$\sum_{j=1}^{n} a_j^+ a_j^- = \tilde{\Delta}_{\nu,\mu} - \frac{n}{2}\nu \quad \text{and} \quad \sum_{j=1}^{n} a_j^- a_j^+ = \tilde{\Delta}_{\nu,\mu} + \frac{n}{2}\nu. \tag{3.46}$$

Moreover, we have the following creation and annihilation equalities

$$\tilde{\Delta}_{\nu,\mu} a_j^+ = a_j^+(\tilde{\Delta}_{\nu,\mu} + \nu) \quad \text{and} \quad \tilde{\Delta}_{\nu,\mu} a_j^- = a_j^-(\tilde{\Delta}_{\nu,\mu} - \nu),$$

which allow the determination of the eigenvalues and eigenvectors of $\tilde{\Delta}_{\nu,\mu}$. Indeed, if ψ is an eigenvector of $\tilde{\Delta}_{\nu,\mu}$ associated with the eigenvalue λ we have the following

$$\tilde{\Delta}_{\nu,\mu}(a_j^+\psi) = a_j^+(\tilde{\Delta}_{\nu,\mu} + \nu)\psi = (\lambda + \nu)a_j^+\psi, \tag{3.47}$$

$$\tilde{\Delta}_{\nu,\mu}(a_j^-\psi) = a_j^-(\tilde{\Delta}_{\nu,\mu} - \nu)\psi = (\lambda - \nu)a_j^-\psi. \tag{3.48}$$

Thus, we need only to know those associated with the lowest eigenvalue. In fact, since $\tilde{\Delta}_{\nu,\mu}$ is positive semi-definite, all the eigenvalues are real and nonnegative. Moreover, from symmetry and ellipticity of $\tilde{\Delta}_{\nu,\mu}$ we know that $\tilde{\Delta}_{\nu,\mu}$ has an infinite sequence of nonnegative eigenvalues (see for example [51]):

$$0 \le \lambda_0 < \lambda_1 < \cdots \uparrow \infty.$$

Therefore, if ψ_0 is an eigensolution associated with λ_0, we have necessary $a_j^-\psi_0 = 0$ for every $j = 1, 2, \cdots, n$, thanks to (3.48). This implies, by using the second expression in (3.45), that

$$\psi_0(z) = e^{-\frac{\nu+i\mu}{2}|z|^2} f(z),$$

where f is any arbitrary holomorphic function. Consequently,

$$\mathbf{A}_0 = Ker(a_j^-) = \overline{\text{span}}\left\{z^m\, e^{-\frac{\nu+i\mu}{2}|z|^2};\ m \in (\mathbb{Z}^+)^n\right\}.$$

Here we have used the multivariate notation z^m for given multi-index $m := (m_1, m_2, \cdots, m_n)$ to mean $z^m := z_1^{m_1} z_2^{m_2} \cdots z_n^{m_n}$. Making use of the creation operators leads to the following family of multi-indexed functions $h_{r,s}^{\nu,\mu}$; $r = (r_1, \cdots, r_n); s = (s_1, \cdots, s_n)$,

$$\begin{aligned} h_{r,s}^{\nu,\mu} &= (a_1^+)^{r_1} \cdots (a_n^+)^{r_n} (z^s\, e^{-\frac{\nu+i\mu}{2}|z|^2}) \\ &= (-1)^{|r|} e^{\frac{\nu-i\mu}{2}|z|^2} D_z^r e^{-\frac{\nu-i\mu}{2}|z|^2} (z^s\, e^{-\frac{\nu+i\mu}{2}|z|^2}) \\ &= (-1)^{|r|+|s|} \nu^{-|s|} e^{\frac{\nu-i\mu}{2}|z|^2} D_z^r D_{\bar{z}}^s e^{-\nu|z|^2}, \end{aligned} \tag{3.49}$$

where $|m|$ and $m!$ stand for $|m| := m_1 + \cdots + m_2$ and $m! := m_1! \cdots m_n!$ respectively, and D_z^m and $D_{\bar{z}}^m$ are defined by

$$D_z^m := \frac{\partial^{|m|}}{\partial z_1^{m_1} \cdots \partial z_n^{m_n}} \quad \text{and} \quad D_{\bar{z}}^m := \frac{\partial^{|m|}}{\partial \bar{z}_1^{m_1} \cdots \partial \bar{z}_n^{m_n}}.$$

According to the above discussion, $h_{r,s}^{\nu,\mu}$ are eigensolutions associated with the eigenvalue $\frac{n}{2}\nu + |r|\,\nu$ and $\lambda_l := \nu(\frac{n}{2} + l)$; $l = 0, 1, 2, \cdots$, are, indeed, eigenvalues of $\tilde{\Delta}_{\nu,\mu}$. The following proposition shows that $\left\{\lambda_l := \nu(\frac{n}{2} + l);\ l = 0, 1, 2, \cdots\right\}$ are the only eigenvalues of $\tilde{\Delta}_{\nu,\mu}$.

Proposition 3.5.12. *$h_{r,s}^{\nu,\mu}$ form a complete orthogonal system in the Hilbert space $L^2(\mathbb{C}^n, d\lambda)$. Moreover, we have the following decomposition $L^2(\mathbb{C}^n, d\lambda) = \bigoplus_{l=1}^{\infty} \mathbf{A}_l$, where*

$$\mathbf{A}_l := \overline{\text{span}}\left\{h_{r,s}^{\nu,\mu};\ r, s \in (\mathbb{Z}^+)^n,\ |r| = l\right\}.$$

Proof. The identity (3.49) shows that $h^{\nu,\mu}_{r,s}$, up to $e^{i\nu|z|^2/2}$, are essentially the high-dimensional analogue of the univariate *complex Hermite functions*

$$h^{\sigma}_{m,n}(\xi,\bar{\xi}) := (-1)^{m+n}\, e^{\sigma|\xi|^2}\, \frac{\partial^{m+n}}{\partial\xi^m\partial\bar{\xi}^n}\, e^{-\sigma|\xi|^2};\ \xi\in\mathbb{C},\sigma>0,$$

considered in [45, 29]. The main idea of the proof is then to separate the variable z in the expression (3.49) into its components z_j to get

$$\begin{aligned} h^{\nu,\mu}_{r,s}(z,\bar{z}) &= \nu^{-|s|}\, e^{\frac{-\nu-i\mu}{2}|z|^2}\prod_{j=1}^{n}(-1)^{r_j+s_j}\, e^{\nu|z_j|^2}\,\frac{\partial^{r_j+s_j}}{\partial z_j^{r_j}\partial\bar{z}_j^{s_j}}\, e^{-\nu|z_j|^2}\\ &= \nu^{-|s|}\, e^{\frac{-\nu-i\mu}{2}|z|^2}\prod_{j=1}^{n} h_{r_j,s_j}(\sqrt{\nu}z_j,\sqrt{\nu}\bar{z}_j). \end{aligned}$$

□

This implies that the eigenvalues of $\Delta_{\nu,\mu} = -4\tilde{\Delta}_{\nu,\mu}$ are

$$\{-2\nu(n+2l);\ l=0,1,2,\cdots\},$$

which coincide with the results in Section 3.5. Notice as well that $A^2_l(\Delta_{\nu,\mu})$ and $\mathbf{A}_l$ refer to the same set.

Remark 3.5.13. *The Hermite functions $h^{\nu,\mu}_{r,s}$ are given explicitly by*

$$h^{\nu,\mu}_{r,s} = \nu^{-|s|}\, e^{\frac{-\nu-i\mu}{2}|z|^2}\sum_{|k|=0}^{\min(r,s)}\frac{(\sqrt{\nu}^{|r|+|s|-2|k|})(-1)^{|r-s|}r!s!}{k!(r-k)!(s-k)!}z^{s-k}\bar{z}^{r-k}, \tag{3.50}$$

where $r=(r_1,\cdots,r_n)$, $s=(s1,\cdots,s_n)$ and $\min(r,s) := (\min(r_1,s_1),\cdots,\min(r_n,s_n))$.

We switch now to the study of the spectrum of $\Delta_{\nu,\mu}$ on mixed automorphic functions.

3.6 Spectrum, dimension formula and basis

By analogy to the classical automorphic functions defined in the previous chapter we can perform the following construction. Using the definition of the space $\mathcal{F}^{\nu,\mu}_{\Gamma}(\mathbb{C}^n)$ and the $T^{\nu,\mu}_g$-invariance property of the Laplacian $\Delta_{\nu,\mu}$ in (3.34), we can consider $\Delta_{\nu,\mu}$ as an operator acting on the space $\mathcal{F}^{\nu,\mu}_{\Gamma}(\mathbb{C}^n)$, carrying $\mathcal{F}^{\nu,\mu}_{\Gamma}(\mathbb{C}^n)$ to $\mathcal{F}^{\nu,\mu}_{\Gamma}(\mathbb{C}^n)$. Thus, for given $F_1,F_2\in\mathcal{F}^{\nu,\mu}_{\Gamma}(\mathbb{C}^n)$, the function $\psi(z)=F_1(z)\overline{F_2(z)}$ is Γ-periodic, in the sense that $\psi(z+\gamma)=\psi(z)$ for every $\gamma\in\Gamma$. Therefore, the quantity

$$\langle F_1,F_2\rangle := \int_{\Lambda(\Gamma)} F_1(z)\overline{F_2(z)}dm(z), \tag{3.51}$$

where $\Lambda(\Gamma)$ is a fundamental domain of Γ in $\mathbb{C}^n$, is well-defined and is independent of the choice of the fundamental domain. Moreover, it defines a Hermitian scalar product on $\mathcal{F}^{\nu,\mu}_{\Gamma}(\mathbb{C}^n)$. We define $L^2_{\nu,\mu}(\mathbb{C}^n/\Gamma)$ to be the Hilbert space obtained as completion of the pre-hilbertian space $\mathcal{F}^{\nu,\mu}_{\Gamma}(\mathbb{C}^n)$ with respect to the scalar product (3.51). The operator $\Delta_{\nu,\mu}$, acting with densely domain in $L^2_{\nu,\mu}(\mathbb{C}^n/\Gamma)$, is essentially self-adjoint. Thus, we denote by $\Delta^{\Gamma}_{\nu,\mu}$ its unique self-adjoint realization in the Hilbert space $L^2_{\nu,\mu}(\mathbb{C}^n/\Gamma)$.

Within the above notation, we can state and prove the main results on the spectral properties of $\Delta^{\Gamma}_{\nu,\mu}$ for lattices in $\mathbb{R}^{2n}=\mathbb{C}^n$ and reals ν,μ such that $(\Gamma;\nu,\mu)$ is quantized. The first one determinates the spectrum of $\Delta^{\Gamma}_{\nu,\mu}$ on $L^2_{\nu,\mu}(\mathbb{C}^n/\Gamma)$.

Theorem 3.6.1 (Spectrum stability). *Let Γ be a lattice of $\mathbb{C}^n = \mathbb{R}^{2n}$ such that the triplet $(\Gamma;\nu,\mu)$ satisfies the (RDQ) condition. Then, the spectrum of $\Delta^{\Gamma}_{\nu,\mu}$ acting on $L^2_{\nu,\mu}(\mathbb{C}^n/\Gamma)$ coincides with the spectrum of $\Delta_{\nu,\mu}$ when acting on the free Hilbert space $L^2(\mathbb{C}^n;dm)$. More precisely, we have*

$$\sigma(\Delta^{\Gamma}_{\nu,\mu}) = \{-2\nu(2\ell+n);\ \ \ell = 0,1,2,\cdots\} = \sigma(\Delta_{\nu,\mu}). \tag{3.52}$$

However, each eigenvalue $-2\nu(2\ell+n)$ occurs here with finite degeneracy in contrary to those of $\Delta_{\nu,\mu}$ on $L^2(\mathbb{C}^n;dm)$.

Proof. For any given $g \in G = \mathbb{C}^n \ltimes \mathsf{U}(n)$ and $F(z) \in \mathcal{C}^\infty(\mathbb{C}^n)$, we consider the function defined on $\mathbb{C}^n$ by the "averaging" formula over $\mathsf{U}(n)$,

$$\mathcal{A}^{\nu,\mu}_g F(z) = \int_{\mathsf{U}(n)} T^{\nu,\mu}_{gh} F(z) dh = \int_{\mathsf{U}(n)} j^{\nu,\mu}(gh,z) F(gh.z) dh, \tag{3.53}$$

where $j^{\nu,\mu}(\cdot,z)$ is the automorphic factor given by (3.30) and dh stands for the normalized (left) Haar measure on the unitary group $\mathsf{U}(n)$. By the compactness of the Lie group $\mathsf{U}(n)$, it is easy to see that the output function $\mathcal{A}^{\nu,\mu}_g F$ shares the same smoothness as that of the input function F. Notice, for instance, that $\mathcal{A}^{\nu,\mu}_g F$ is a $\mathcal{C}^\infty$-complex-valued function on $\mathbb{C}^n$ if F is $\mathcal{C}^\infty$ and $\mathcal{A}^{\nu,\mu}_g F$ is a bounded function on $\mathbb{C}^n$ if F is bounded. We also have $\mathcal{A}^{\nu,\mu}_g F(0) = F(g.0)$ and $\mathcal{A}^{\nu,\mu}_g F(k.z) = \mathcal{A}^{\nu,\mu}_g F(z)$ for all $k \in \mathsf{U}(n)$ and $z \in \mathbb{C}^n$. This is to say that $\mathcal{A}^{\nu,\mu}_g F(z)$ is a radial function on $\mathbb{C}^n$. Moreover, the averaging operator $\mathcal{A}^{\nu,\mu}_g$ leaves invariant the eigenspace

$$E_\lambda(\Delta_{\nu,\mu}) = \{f \in \mathcal{C}^\infty(\mathbb{C}^n);\ \Delta_{\nu,\mu} f = -2\nu(2\lambda+n)f\}.$$

This relies on the co-cycle condition satisfied by the automorphic factor $j^{\nu,\mu}(g,z)$ as well as on the invariance property satisfied by $\Delta_{\nu,\mu}$ with respect to the $T^{\nu,\mu}_g$-action defined in (3.33). Accordingly, the explicit expression of $\mathcal{A}^{\nu,\mu}_g F(z)$ is given in terms of the confluent hypergeometric function (*see* Proposition 3.5.2)

$$\mathcal{A}^{\nu,\mu}_g F(z) = F(g.0) e^{(\frac{i\mu-\nu}{2})|z|^2} {}_1F_1(-\lambda;n;\nu|z|^2) = F(g.0)\varphi^{\nu,\mu}_\lambda(z). \tag{3.54}$$

The spherical eigenfuction $\varphi^{\nu,\mu}_\lambda(z)$ is then bounded on $\mathbb{C}^n$ if and only if λ is a nonnegative integer. This immediately follows from the well-known asymptotic behavior of the confluent hypergeometric function ${}_1F_1(\alpha;\gamma;x)$, near $x = +\infty$, given by [62, page 332]

$${}_1F_1(a;c;x) = \Gamma(c)\left\{\frac{(-x)^{-a}}{\Gamma(c-a)} + \frac{e^x x^{a-c}}{\Gamma(a)}\right\}\left(1 + O\left(\frac{1}{x}\right)\right),$$

where $\Gamma(z)$ is the Gamma Euler function on $\mathbb{C}$. Subsequently, the eigenspace

$$\mathcal{E}^b_\lambda(\Delta_{\nu,\mu}) = \big\{F \in E_\lambda(\Delta_{\nu,\mu});\ F \text{ is bounded on } \mathbb{C}^n\big\}$$

formed by bounded eigenfunctions with $-2\mu(2\lambda+n)$ as eigenvalue, is a nonzero space if and only if λ is a nonnegative integer, $\lambda = \ell = 0,1,2,\cdots$. To conclude for Theorem 3.6.1, we need only to observe that for given lattice Γ in $\mathbb{C}^n$ the L^2-eigenspace

$$\mathcal{F}^2_\lambda(\Delta^\Gamma_{\nu,\mu}) = \Big\{F \in L^2_{\nu,\mu}(\mathbb{C}^n/\Gamma);\ \Delta^\Gamma_{\nu,\mu} F = -2\nu(2\lambda+n)F\Big\}$$

is indeed contained in $\mathcal{E}^b_\lambda(\Delta_{\nu,\mu})$. □

Under the (RDQ) condition, the operator $\Delta^\Gamma_{\nu,\mu}$ admits a compact resolvent in $L^2_{\nu,\mu}(\mathbb{C}^n/\Gamma)$. Therefore, the eigeuspaces $\mathcal{F}^2_\ell(\Delta^\Gamma_{\nu,\mu})$ with $\ell = 0,1,2,\cdots$, are of finite dimension, for Γ being a co-compact lattice of $\mathbb{C}^n$ (i.e., $\mathbb{C}^n/\Gamma$ is a compact torus). Therefore, the index Atiyah-Singer theorem might be used to give the dimension of the space $\mathcal{F}^2_0(\Delta^\Gamma_{\nu,\mu})$ associated the lowest eigenvalue of the operator $\Delta^\Gamma_{\nu,\mu}$ viewed as an elliptic differential operator on the compact manifold $\mathbb{C}^n/\Gamma$, acting on sections of the line bundle $\mathcal{L}_{\nu,\mu}$ over $\mathbb{C}^n/\Gamma$ (see the previous section). The general case of $\mathcal{F}^2_\ell(\Delta^\Gamma_{\nu,\mu})$ with $\ell = 1,2,\cdots$, is not easy to handel in an abstract way. However, using the *Selberg approach* we will be able to give the explicit dimension formula for the L^2-eigenspaces $\mathcal{F}^2_\ell(\Delta^\Gamma_{\nu,\mu})$. More precisely, we assert

Theorem 3.6.2 (Dimension formula). *Let $(\Gamma;\nu,\mu)$ be a quantized triplet and $\ell = 0,1,2,\cdots$. The dimension formula of $\mathcal{F}^2_\ell(\Delta^\Gamma_{\nu,\mu})$ is given by*

$$\dim \mathcal{F}^2_\ell(\Delta^\Gamma_{\nu,\mu}) = \left(\frac{\nu}{\pi}\right)^n \frac{(n-1+\ell)!}{(n-1)!\ell!} vol(\Lambda(\Gamma)), \tag{3.55}$$

where $vol(\Lambda(\Gamma))$ is the Lebesgue volume measure in $\mathbb{C}^n = \mathbb{R}^{2n}$ of a fundamental domain $\Lambda(\Gamma)$ of Γ.

Proof. For $\ell = 0,1,2,\cdots$, let $P_\ell(\Delta^\Gamma_{\nu,\mu})$ be the orthogonal projection of $L^2_{\nu,\mu}(\mathbb{C}^n/\Gamma)$ onto the L^2-eigenspace $\mathcal{F}^2_\ell(\Delta^\Gamma_{\nu,\mu})$. Then, under the (RDQ)-condition, the Schwartz kernel $P^{\nu,\mu;\Gamma}_\ell(z,w)$ of the orthogonal projector operator $P_\ell(\Delta^\Gamma_{\nu,\mu})$ from $L^2_{\nu,\mu}(\mathbb{C}^n/\Gamma)$ to $\mathcal{F}^2_\ell(\Delta^\Gamma_{\nu,\mu})$ can be proved to be given explicitly by the formula

$$\begin{aligned} P^{\nu,\mu;\Gamma}_\ell(z,w) = \left(\frac{\nu}{\pi}\right)^n \frac{(n-1+\ell)!}{(n-1)!\ell!} e^{-\frac{i\mu}{2}(|z|^2-|w|^2)+\frac{\nu}{2}(\langle z,w\rangle - \overline{\langle z,w\rangle})} \\ \sum_{\gamma\in\Gamma} e^{-\frac{i\mu}{2}|\gamma|^2+\frac{\nu}{2}(\langle z+w,\gamma\rangle-\overline{\langle z+w,\gamma\rangle})} e^{-\frac{\nu}{2}|z-w-\gamma|^2} {}_1F_1(-\ell;n;\nu|z-w-\gamma|^2). \end{aligned}$$

The kernel function $P^{\nu,\mu;\Gamma}_\ell(z,w)$ is in fact the $T^{\nu,\mu}$-periodization à la Poincaré of the Schwartz kernel $P^{\nu,\mu}_\ell(z,w)$ of the orthogonal projection on the free Hilbert space $L^2_{\nu,\mu}(\mathbb{C}^n;dm)$ on the L^2-eigenspace $\mathcal{F}^2_\ell(\Delta_{\nu,\mu})$ given explicitly in [22]. Therefore, the dimension of $\mathcal{F}^2_\ell(\Delta^\Gamma_{\nu,\mu})$ is given by

$$\begin{aligned} \dim \mathcal{F}^2_\ell(\Delta^\Gamma_{\nu,\mu}) &= Trace(P_\ell(\Delta^\Gamma_{\nu,\mu})) \\ &= \int_{\Lambda(\Gamma)} P^{\nu,\mu;\Gamma}_\ell(z,z)dm(z) \\ &= \left(\frac{\nu}{\pi}\right)^n \frac{(n-1+\ell)!}{(n-1)!\ell!} vol(\Lambda(\Gamma)). \end{aligned}$$

The last equality follows making use of the identity

$$\int_{\Lambda(\Gamma)} e^{\nu(\langle z,\gamma\rangle - \overline{\langle z,\gamma\rangle})} dm(z) = \delta_{0,\gamma} vol(\Lambda(\Gamma)).$$

□

Remark 3.6.3. *The dimension of $\mathcal{F}^2_\ell(\Delta^\Gamma_{\nu,\mu})$ is clearly independent of the parameter μ. Moreover, for $n = 1$, the eigenspaces $\mathcal{F}^2_\ell(\Delta^\Gamma_{\nu,\mu})$ are all of the same dimension $\left(\frac{\nu}{\pi}\right) vol(\Lambda(\Gamma))$.*

3.6.1 On the basis of $\mathcal{F}^2_\ell(\Delta^\Gamma_{\nu,\mu}, \mathbb{C}^n)$

The key observation in obtaining our basis for $\mathcal{F}^2_\ell(\Delta^\Gamma_{\nu,\mu}, \mathbb{C}^n)$ is that the multiplication operator

$$M^{\nu,\mu} : f \longmapsto e^{\frac{i\mu+\nu}{2}|z|^2} f$$

defines an isometric bijection from $L^2_{\nu,\mu}(\mathbb{C}^n/\Gamma)$, the L^2–space of bi–weighted automorphic functions, onto the space $L^{2,\mu}_{\Gamma,\chi_\mu}(\mathbb{C}^n)$ of simple (Γ,χ)-automorphic functions satisfying (2.1) and associated to the character $\chi_\mu(\gamma) := e^{\frac{i\mu}{2}|\gamma|^2}; \gamma \in \Gamma$.

According to the discussion in the beginning of this section, the basis for the eigenspaces of $\Delta^\Gamma_{\nu,\mu}$ are inherited from those of the eigenspaces of the Laplacian $\Delta^{\Gamma,\chi_\mu}_\mu$ acting on $L^2_{\nu,\mu}(\mathbb{C}^n/\Gamma)$. We formulate the result of the above discussion as follows.

Corollary 3.6.4. *Keep notations as above. The functions*

$$e^{-\frac{i\mu+\nu}{2}|z|^2} A_{+,J}\varphi_K = e^{-\frac{i\mu+\nu}{2}|z|^2} A_{+,j_1}A_{+,j_2}\cdots A_{+,j_l}\varphi_K$$

for varying $1 \leq j_1 \leq \cdots \leq j_l \leq n$ and $K = (k_1,\cdots,k_n) \in \{0,1,\cdots,N-1\}^n$, form an orthogonal basis of $\mathcal{F}^2_\ell(\Delta^\Gamma_{\nu,\mu},\mathbb{C}^n)$. Here φ_K is the function in (2.43). *In particular, for $n = 1$, the functions $e^{-\frac{i\mu+\nu}{2}|z|^2}\varphi^{\alpha,\beta}_{\ell,m}(z)$, $m = 0,1,\cdots,N-1$, constitute an orthogonal basis of $\mathcal{F}^2_\ell(\Delta^\Gamma_{\nu,\mu},\mathbb{C})$, where $\varphi^{\alpha,\beta}_{\ell,m}$ are the functions given by* (2.44).

CHAPTER 4

MIXED AUTOMORPHIC FUNCTIONS OF SECOND KIND

We consider the class of mixed automorphic functions of second kind on the complex plane $\mathcal{M}^{\nu,\mu}_{\rho,\tau}(\mathbb{C})$. It is associated with a certain equivariant pair (ρ,τ) and given discrete subgroup $\Gamma \subset \mathrm{U}(1) \ltimes \mathbb{C}$. Precisely, automorphic functions of second kind satisfy a special two-parameter functional equation involving a product of two automorphic factors, where the second one is a perturbation of the first one by a given equivariant pair (ρ,τ). We prove a lifting theorem between the eigenvalue problem of an appropriate invariant Bochner Laplacian and the eigenvalue problem of the standard magnetic Laplacian on the space of classical automorphic functions. This is handled by proving a decomposition of the magnetic Schrödinger operator in terms of deformed Hodge-de Rham differential. The spectral analysis of this class is accomplished by characterizing $\mathcal{M}^{\nu,\mu}_{\rho,\tau}(\mathbb{C})$ as the image by an appropriate isomorphic transform of the space of classical automorphic functions involving specific weight $B^{\nu,\mu}_{\tau}$ and specific pseudo-character χ_τ. In the last part, the high dimension case is also discussed.

The space of mixed automorphic functions of second kind that we are dealing with in this chapter can be seen as another way to generalize the classical automorphic functions. Namely, instead of changing the bilinear form inside the automorphic factor according to an additional parameter μ, we keep the classical automorphic factor as it is and multiply it by its copy transported according to an equivariant transformation pair (ρ,τ). Such notion has been introduced by Stiller [77]. It arises naturally as holomorphic forms on elliptic varieties [39]. In the context of hyperbolic plane, it was extensively studied by M.H. Lee (see [54] and the references therein). To the best of our knowledge, such notion have been introduced in the context of complex plane, subject to the action of a full rank lattice, for the first time in Ghanmi [28] followed by El Gourari and Ghanmi [26] where the authors constructed an explicit transform that lifts mixed automorphic functions and its eigenvalue problem to classical automorphic function. This chapter is based on these works where we provide a slight improvement by showing that the lifting result holds also when we allow rotation actions of a discrete subgroup of $\mathrm{U}(1)$ in addition to the translations given by a lattice of $(\mathbb{C},+)$

4.1 Definition of the space of mixed automorphic functions of second kind

Let the group G as in the previous chapters (see e.g. Section 2.7)

$$G = \left\{ g := [a,b] \equiv \begin{pmatrix} a & b \\ 0 & 1 \end{pmatrix}; \quad a \in \mathrm{U}(1),\, b \in \mathbb{C} \right\},$$

and Γ be a discrete subgroup of G. Recall that G acts transitively on the complex plane $\mathbb{C}$ by the holomorphic mappings $g \cdot z = az + b$, where $g = [a,b] \in G$ and $z \in \mathbb{C}$. Then, we can give the following definition

Definition 4.1. *By G-equivariant pair (ρ,τ), we mean a G-endomorphism ρ and a compatible mapping $\tau:\mathbb{C}\to\mathbb{C}$ such that*

$$\tau(g\cdot z)=\rho(g)\cdot\tau(z);\qquad \forall g\in G,\quad \forall z\in\mathbb{C}. \tag{4.1}$$

Let now (ρ,τ) be an equivariant pair as above and χ a map such that $\chi:\Gamma\to U(1)$. Let ν and μ two reals such that $\nu,\mu>0$. Associated with data $(\Gamma;\chi,\nu,\mu,\rho,\tau)$, we perform $\mathcal{M}^{\nu,\mu}_{\rho,\tau}(\mathbb{C})$ the vector space of smooth complex-valued functions F on $\mathbb{C}$ satisfying the functional equation :

$$F(\gamma\cdot z)=J^{\nu,\mu}_{\rho,\tau}(\gamma,z)F(z) \tag{4.2}$$

for every $\gamma\in\Gamma$ and $z\in\mathbb{C}$, where

$$J^{\nu,\mu}_{\rho,\tau}(\gamma,z):=\chi(\gamma)j^{\nu}(\gamma,z)j^{\mu}(\rho(\gamma),\tau(z)),$$

and

$$j^{\alpha}(g,z):=\mathrm{e}^{-\alpha i\Im m\left\langle z,\gamma^{-1}\cdot 0\right\rangle}.$$

Definition 4.2. *The space of $\mathcal{M}^{\nu,\mu}_{\rho,\tau}(\mathbb{C})$ is called the space of planar mixed automorphic functions of second kind, or also of bi weight (ν,μ) with respect to the equivalent pair (ρ,τ) and the discrete subgroup Γ.*

As in the classical and the first kind cases, the data$(\Gamma;\chi,\nu,\mu,\rho,\tau)$ cannot be chosen freely or else the space will be trivial.

Remark 4.1.1. *The equivariant condition* (4.1) *ensure that $j^{\alpha}(\rho(\cdot),\tau(\cdot))$ is an automorphic factor when $j^{\alpha}(\cdot,\cdot)$ is one too. Indeed, for any $\gamma,\gamma'\in\Gamma$ we have*

$$\begin{aligned}j^{\alpha}(\rho(\gamma\gamma'),\tau(z))&=j^{\alpha}(\rho(\gamma)\rho(\gamma'),\tau(z))\\&=j^{\alpha}(\rho(\gamma),\rho(\gamma')\cdot\tau(z))j^{\alpha}(\rho(\gamma'),\tau(z))\\&=j^{\alpha}(\rho(\gamma),\tau(\gamma'\cdot z))j^{\alpha}(\rho(\gamma'),\tau(z)),\end{aligned}$$

where in the second line we used the co-cycle equation verified by j^{α} from being an automorphic factor, and the third line follows using the equivariant condition (4.1).

Using the last remark and following the same argument as we did in the classical case, we can show that

Proposition 4.1.2. *The functional space $\mathcal{M}^{\nu,\mu}_{\rho,\tau}(\mathbb{C})$ is nontrivial if and only if*

$$\chi(\gamma\gamma')=\chi(\gamma)\chi(\gamma)e^{-2i\phi^{\nu,\mu}_{\rho}(\gamma,\gamma')} \tag{4.3}$$

holds for every $\gamma,\gamma'\in\Gamma$, where $\phi^{\nu,\mu}_{\rho}(g,g')$ is the real-valued function defined on $G\times G$ by

$$\phi^{\nu,\mu}_{\rho}(g,g'):=\Im\Big(\nu\left\langle g^{-1}\cdot 0,g'\cdot 0\right\rangle+\mu\left\langle\rho(g^{-1})\cdot 0,\rho(g')\cdot 0\right\rangle\Big). \tag{4.4}$$

Equivariant pairs are important for our definition of mixed automorphic functions of second kind. Hence, in the next section we will focus on characterizing them in our specific unidimensional complex context.

4.2 On G-equivariant pairs

In this section we digress into a partial characterization of equivariant pairs (ρ,τ), where ρ is a G-endomorphism and $\tau:\mathbb{C}\to\mathbb{C}$ a compatible mapping such that (4.1) holds (see Definition 4.1). At first, the objective was to exploit Equation (4.1) to exhibit the possible maps τ if ρ is fixed, or vice versa the possible ρ if τ is fixed. Such characterization seem nearly impossible without imposing some regularity on both maps. In the following we give our attempt to provide such bridge between the choice of ρ and the choice of τ.

Let (ρ,τ) be a *G-equivariant pair*. The equivarience equation (4.1) implies in particular that $\rho(G_x) \subset G_{\tau(x)}$ for every fixed $x \in \mathbb{C}$, where $G_x := \{g \in G \mid g\cdot x = x\}$. As a special case, we have

$$\rho(G_0) \subset G_{\tau(0)}. \tag{4.5}$$

Next, using the fact that G acts transitively on $\mathbb{C}$, one can exhibit for every $z \in \mathbb{C}$ an element $g_z \in G$ such that $z = g_z.0$. Hence, one can check that $\tau(z) = \rho(g_z).\tau(0)$, and therefore we have

$$\tau(z) = \rho([1,z]).\tau(0). \tag{4.6}$$

Conversely, it can be checked that (4.6) defines all mappings from $\mathbb{C}$ onto $\mathbb{C}$ that satisfy the equivariant condition (4.1) for a given G-endomorphism $\rho : \mathbb{C} \longrightarrow \mathbb{C}$. Precisely, we state the following

Proposition 4.2.1. *Let ρ be a G-endomorphism and define $\Xi_\rho := \left\{\beta \in \mathbb{C} \mid G_\beta \supset \rho(G_0)\right\}$. Then a given mapping $\tau : \mathbb{C} \longrightarrow \mathbb{C}$ satisfies the equivariant condition* (4.1) *if and only if*

$$\tau(z) = \tau_\beta(z) := \rho([1,z])\cdot\beta \tag{4.7}$$

for certain fixed $\beta \in \Xi_\rho$ ($\tau_\beta(0) = \beta$).

Proof. The "only if" is clear, hence one has only to prove "if". To this end, fix $\beta \in \Xi_\rho$ and note that τ_β, as defined by (4.6), satisfies

$$\tau_\beta(z) = \rho(g_z)\cdot\beta \tag{4.8}$$

for every $g_z \in G$, such that $g_z \cdot 0 = z$. Indeed, since $g_z = [a,z] = [1,z][a,0]$ for some arbitrary $a \in \mathrm{U}(1)$, it follows

$$\rho(g_z)\cdot\beta = \rho([1,z][a,0])\cdot\beta = \rho([1,z])\cdot(\rho([a,0])\cdot\beta)\,.$$

Thus, using the fact that $\beta \in \Xi_\rho$, which reads here as $\rho([a,0])\cdot\beta = \beta$ for every $a \in \mathrm{U}(1)$, one then gets

$$\rho(g_z)\cdot\beta = \rho([1,z])\cdot\beta := \tau_\beta(z).$$

Now, since for every $g \in G$ we have $(gg_z)\cdot 0 = g.z$, we obtain

$$\tau(g\cdot z) \overset{(4.8)}{=} \rho(gg_z)\cdot\beta = \rho(g)\cdot(\rho(g_z)\cdot\beta) = \rho(g)\cdot\tau_\beta(z),$$

which means that the equivariant condition (4.1) holds. This completes the proof. □

Remark 4.2.2. *Note that if $\rho(G_0)$ is trivial; i.e. $\rho(G_0) = \{1\}$, then $\Xi_\rho = \mathbb{C}$, and therefore, for the particular case $\rho = \mathrm{Id}_G$ (Id_G being the identity map of G), the mapping τ such that the pair (Id_G,τ) satisfies the equivariant condition* (4.1) *are the translations $\tau_\beta(z) = z+\beta$ with $\beta \in \mathbb{C}$. Note also that such translations satisfy* (4.1) *in the case when the G-endomorphism ρ verifies $\rho([1,z]) = (a_z,z)$.*

Example 4.2.3. *Let $\mathfrak{h} = [\alpha_\mathfrak{h},\beta_\mathfrak{h}]$ be a fixed element in G. Consider the G-endomorphism $\rho_\mathfrak{h}$ given by*

$$\rho_\mathfrak{h}(g) := \mathfrak{h}g\mathfrak{h}^{-1} = [a,(1-a)\beta_\mathfrak{h} + b\alpha_\mathfrak{h}]; \qquad g = [a,b] \in G, \tag{4.9}$$

and the map $\tau_\mathfrak{h} : \mathbb{C} \longrightarrow \mathbb{C}$ defined by

$$\tau_\mathfrak{h}(z) := \mathfrak{h}\cdot z = \alpha_\mathfrak{h} z + \beta_\mathfrak{h}. \tag{4.10}$$

The pair $(\rho_\mathfrak{h},\tau_\mathfrak{h})$ defines an "alteration of $(\mathrm{Id}_G,\mathrm{Id}_\mathbb{C})$" in the sense of W.F. Hammond [36] and satisfies the equivariant condition (4.1)*. This is in correlation with what precede, since for such $\rho_\mathfrak{h}$, we have $\rho_\mathfrak{h}([a,b])\cdot\beta_\mathfrak{h} = \beta_\mathfrak{h}$, i.e. $\beta_\mathfrak{h} \in \Xi_\rho$. Therefore, one can use* (4.7) *to reproduce again $\tau_\mathfrak{h}$ that is given through* (4.10)*.*

Example 4.2.4. *Consider the conjugate endomorphism under G,*

$$\rho([a,b]) = \overline{[a,b]} := [\bar{a},\bar{b}].$$

In this case, one can see easily that $\Xi_\rho = \{0\}$. According to Proposition 4.2.1, $([a,b] \mapsto \overline{[a,b]},\tau)$ is an equivariant pair if and only if :

$$\tau(z) := [1,\bar{z}]\cdot 0 = \bar{z}. \tag{4.11}$$

And then, $([a,b] \mapsto \overline{[a,b]}, z \mapsto \bar{z})$ is another example of equivariant pair.

4.2.1 When ρ separates variables

Here, we deal with the mapping $\rho : G \longrightarrow G$ of the form $\rho = \tilde{\rho} \ltimes \tilde{\tau}$ meaning $\rho([a,b]) := [\tilde{\rho}(a), \tilde{\tau}(b)]$ with $\tilde{\rho} : \mathsf{U}(1) \longrightarrow \mathsf{U}(1)$ and $\tilde{\tau} : \mathbb{C} \longrightarrow \mathbb{C}$. Then, it is clear that ρ is a G-endomorphism if and only if $\tilde{\rho}$ is a $\mathsf{U}(1)$-endomorphism and $\tilde{\tau}$ satisfies

$$\tilde{\tau}(ab + c) = \tilde{\rho}(a)\tilde{\tau}(b) + \tilde{\tau}(c) \tag{4.12}$$

for every $a \in \mathsf{U}(1)$ and $b, c \in \mathbb{C}$. Hence, one can deduce that $\tilde{\tau}(0) = 0$. Therefore, $\tilde{\tau}$ becomes additive and satisfies

$$\tilde{\tau}(ab) = \tilde{\rho}(a)\tilde{\tau}(b). \tag{4.13}$$

Thus, according to Proposition 4.2.1, the mapping $\tau : \mathbb{C} \longrightarrow \mathbb{C}$ satisfying (4.1) relatively to $\rho = \tilde{\rho} \ltimes \tilde{\tau}$ are those given by

$$\tau_\beta(z) = \tilde{\tau}(z) + \beta \tag{4.14}$$

for varying $\beta \in \Xi_\rho$. Moreover, we can show that for all $\rho = \tilde{\rho} \ltimes \tilde{\tau}$ we have $0 \in \Xi_\rho$. Then, from (4.14) with $\beta = 0$, we get that the compatible map is $\tau_0(z) = \tilde{\tau}(z)$. On the other side, it is easy to see that $\beta \neq 0$ belongs to Ξ_ρ if and only if $\tilde{\rho} \equiv 1$. Indeed,

$$\rho([a,0]) \cdot \beta := [\tilde{\rho}(a), \tilde{\tau}(0)] \cdot \beta = [\tilde{\rho}(a), 0] \cdot \beta = \tilde{\rho}(a)\beta.$$

In this case, $\rho(G_0)$ is trivial and $\Xi_\rho = \mathbb{C}$. Furthermore, (4.12) reads simply as

$$\tilde{\tau}(ab + c) = \tilde{\tau}(b) + \tilde{\tau}(c), \tag{4.15}$$

and clearly yields

$$\tau_\beta(az + b) = \tau_\beta(z) + \tau_\beta(b) + \beta \tag{4.16}$$

for all $a \in \mathsf{U}(1)$ and $z, b \in \mathbb{C}$. In conclusion, we assert the following

Proposition 4.2.5. *Keep notations as above and assume that the pair $(\tilde{\rho} \ltimes \tilde{\tau}, \tau)$ satisfies the equivariant condition* (4.1). *Then, we have two possible cases :*

- *$\tilde{\rho}$ is a $\mathsf{U}(1)$-endomorphism, $\tilde{\tau}$ is additive such that $\tilde{\tau}(ab + c) = \tilde{\tau}(b) + \tilde{\tau}(c)$, $a \in \mathsf{U}(1)$, $b, c \in \mathbb{C}$ and $\tau(z) = \tilde{\tau}(z)$.*
- *$\tilde{\rho} \equiv 1, \tilde{\tau}$ is additive such that $\tilde{\tau}(ab + c) = \tilde{\tau}(b) + \tilde{\tau}(c)$, $a \in \mathsf{U}(1)$, $b, c \in \mathbb{C}$ and $\tau(z) = \tilde{\tau}(z) + \beta$ for certain $\beta \in \mathbb{C}$.*

Remark 4.2.6. *In the context of Example 4.2.3, we have*

$$\tau_{\mathfrak{h}}(az + b) = \tilde{\rho}(a)(\tau_{\mathfrak{h}}(z) - \tau_{\mathfrak{h}}(0)) + \tau_{\mathfrak{h}}(b), \tag{4.17}$$

which is clearly similar to (4.16).

Remark 4.2.7. *The equivariant pair given in Example 4.2.4 by $([a,b] \mapsto \overline{[a,b]}, z \mapsto \bar{z})$ fits the first point of Proposition 4.2.5. In fact we have, for all $a \in \mathsf{U}(1)$ and $z \in \mathbb{C}$*

$$\tilde{\rho}(a) = \bar{a}, \text{ and } \tau(z) = \tilde{\tau}(z) = \bar{z}.$$

4.3 Magnetic Schrödinger operator and the reduction of $\mathcal{M}^{\nu,\mu}_{\rho,\tau}(\mathbb{C})$

In the remaining of this chapter we will be interested in the spectral analysis the space of mixed automorphic functions of second kind. Recall from Subsection 3.4.2 that a *Magnetic Schrödinger operator* on a complete oriented Riemannian manifold (M, g) is defined by

$$H_\theta = (d + \text{ext } \theta)^*(d + \text{ext } \theta). \tag{4.18}$$

θ is a given $\mathcal{C}^1$ real differential 1-form on M (potential vector), d stands for the usual exterior derivative acting on the space of differential p-forms $\Omega^p(M)$, $\mathrm{ext}\,\theta$ is the operator of exterior left multiplication by θ, i.e., $(\mathrm{ext}\,\theta)\omega = \theta \wedge \omega$, and $(d + i\mathrm{ext}\theta)^*$ is the formal adjoint of $d + i\mathrm{ext}\theta$ with respect to the Hermitian product

$$\langle \alpha, \beta \rangle_{\Omega^p} = \int_M \alpha \wedge \star\beta$$

induced by the metric g on $\Omega^p = \Omega^p(M)$, where $\star$ denotes the Hodge star operator associated with the volume form.

In this chapter, M is the complex plane $\mathbb{C}$ equipped with its Kähler metric

$$g = ds^2 = -\frac{i}{2} dz \otimes d\bar{z} = dx \otimes dy$$

and the corresponding volume form is $\mathbf{V}\mathrm{ol} = dxdy$.

Associated with the parameters $\mu, \nu, \alpha > 0$ and the equivariant pair (ρ, τ), we consider the following two families of potential vectors

$$\theta_\alpha(z) := -\frac{\alpha}{2}(\bar{z}dz - zd\bar{z}), \tag{4.19}$$

and

$$\theta_\tau^{\nu,\mu}(z) := -\Big\{\frac{\nu}{2}(\bar{z}dz - zd\bar{z}) + \frac{\mu}{2}(\bar{\tau}d\tau - \tau d\bar{\tau})\Big\} = -\Big(\frac{\overline{S_\tau^{\nu,\mu}(z)}}{2}dz - \frac{S_\tau^{\nu,\mu}(z)}{2}d\bar{z}\Big), \tag{4.20}$$

where $S_\tau^{\nu,\mu}(z) := \nu z + \mu\Big(\tau\frac{\partial\bar{\tau}}{\partial\bar{z}} - \bar{\tau}\frac{\partial\tau}{\partial\bar{z}}\Big)$. The first vector potential is the standard Landau gauge associated with a constant magnetic field of intensity α. The second is constructed in order to make the underling operator commutes with the projective representations $\mathcal{T}_g^{\nu,\mu}$ associated with the automorphic factor in this chapter and given below in (4.24). These vector potentials lead respectively to the following Schrödinger operators, given in terms of ordinary partial derivatives,

$$-\Delta_{\alpha,0} = H_{\theta_\alpha(z)} = -4\frac{\partial^2}{\partial z\partial\bar{z}} - 2\alpha\Big(z\frac{\partial}{\partial z} - \bar{z}\frac{\partial}{\partial\bar{z}}\Big) + \alpha^2|z|^2, \tag{4.21}$$

$$\boxed{-\Delta_\tau^{\nu,\mu} := H_{\theta_\tau^{\nu,\mu}(z)} = -4\frac{\partial^2}{\partial z\partial\bar{z}} - 2\Big(S_\tau^{\nu,\mu}\frac{\partial}{\partial z} - \overline{S_\tau^{\nu,\mu}}\frac{\partial}{\partial\bar{z}}\Big) + |S_\tau^{\nu,\mu}|^2 - \mu(\tau\Delta\bar{\tau} - \bar{\tau}\Delta\tau).} \tag{4.22}$$

From the expressions above, it is easy to notice that $\Delta_{\alpha,0}$ can be seen as a special case of $\Delta_\tau^{\nu,\mu}$. The particularity of these two operators manifests in the following result

Proposition 4.3.1. *The operators $\Delta_{\alpha,0}$ and $\Delta_\tau^{\nu,\mu}$ satisfy the invariance properties :*

$$\mathcal{T}_g^\alpha \Delta_{\alpha,0} = \Delta_{\alpha,0}\mathcal{T}_g^\alpha \qquad \mathcal{T}_g^{\nu,\mu}\Delta_\tau^{\nu,\mu} = \Delta_\tau^{\nu,\mu}\mathcal{T}_g^{\nu,\mu},$$

for all $g \in G$, where

$$[\mathcal{T}_g^\alpha f](z) := \overline{j^\alpha(g,z)} f(g\cdot z), \tag{4.23}$$

$$[\mathcal{T}_g^{\nu,\mu} f](z) := \overline{J_{\rho,\tau}^{\nu,\mu}(g,z)} f(g\cdot z), \tag{4.24}$$

These invariant properties imply in particular that we can consider the eigenvalue problem of $\Delta_{\alpha,0}$ and $\Delta_\tau^{\nu,\mu}$ respectively on the space $\mathcal{F}_{\Gamma,\chi}^{\alpha,0}(\mathbb{C})$ of standard automorphic functions and on the space $\mathcal{M}_{\rho,\tau}^{\nu,\mu}(\mathbb{C})$ of mixed automorphic functions of second kind.

Proof of Proposition 4.3.1. We are concerned here with the invariance of $\Delta_\tau^{\nu,\mu}$ as the other case was treated and also follows as a particular case. This proof is analogical to the one given in Proposition 3.4.9 (see also Theorem 3.4 in [26]). Though, it involves some more computational difficulties. For instance, we find ourselves in need for the following handy formulas driven from the equivariant condition $\tau(\gamma\cdot z) = \rho(\gamma)\cdot\tau(z)$. Indeed, by writing the G-endomorphism $\rho : G \to G = \mathrm{U}(1) \ltimes \mathbb{C}$ as $\rho(\gamma) = [\phi(\gamma), \psi(\gamma)]$, and differentiating the equivariant condition, we get the identities

$$\frac{\partial \tau}{z}(\gamma \cdot z) = \bar{a}\phi(\gamma)\frac{\partial \tau}{z}(z) \qquad (4.25) \qquad\qquad \frac{\partial \tau}{\bar{z}}(\gamma \cdot z) = a\phi(\gamma)\frac{\partial \tau}{\bar{z}}(z) \qquad (4.27)$$

$$\frac{\partial \bar{\tau}}{z}(\gamma \cdot z) = \bar{a}\overline{\phi(\gamma)}\frac{\partial \bar{\tau}}{z}(z) \qquad (4.26) \qquad\qquad \frac{\partial \bar{\tau}}{\bar{z}}(\gamma \cdot z) = a\overline{\phi(\gamma)}\frac{\partial \bar{\tau}}{z}(\bar{z}). \qquad (4.28)$$

Let $\theta_\tau^{\nu,\mu}$ as in (4.20). For $g \in G = \mathrm{U}(1) \ltimes \mathbb{C}$ let us denote again by g the mapping $z \mapsto g.z$. Then g^* denotes the pull-back mapping on the space of differential form of $\mathbb{C}$. $\mathcal{T}_g^{\nu,\mu}$ can be naturally extended to the space of differential forms as follows

$$\mathcal{T}_g^{\nu,\mu}\omega = \overline{J_{\rho,\tau}^{\nu,\mu}(g,z)}g^*\omega. \tag{4.29}$$

Now, by component-wise straightforward computations, making use of the equations (4.25)–(4.28), we show that

$$g^*\theta_{\nu,\mu} = \theta_{\nu,\mu} + \frac{d\overline{J^{\nu,\mu}(\gamma,z)}}{\overline{J^{\nu,\mu}(\gamma,z)}}. \tag{4.30}$$

Then we can perform the following algebraic computations using the well known facts $g^*d = dg^*$ as well as $g^*(\alpha \wedge \beta) = g^*\alpha \wedge g^*\beta$, together with the established fact (4.30),

$$\begin{aligned}
\mathcal{T}_g^{\nu,\mu}((d + \mathrm{ext}\,\theta_{\nu,\mu})f) &= \overline{J^{\nu,\mu}(g,z)}\Big[g^*\Big((d + \mathrm{ext}\,\theta_{\nu,\mu})f\Big)\Big] \\
&= \overline{J^{\nu,\mu}(g,z)}\Big(d[g^*f] + [g^*\theta_{\nu,\mu}] \wedge [g^*f]\Big) \\
&\overset{(4.30)}{=} \overline{J^{\nu,\mu}(g,z)}d[g^*f] + \overline{J^{\nu,\mu}(g,z)}\theta_{\nu,\mu}[g^*f] + d(\overline{J^{\nu,\mu}(g,z)})[g^*f] \\
&= d(\overline{J^{\nu,\mu}(g,z)}[g^*f]) + \theta_{\nu,\mu}\overline{J^{\nu,\mu}(g,z)}[g^*f] \\
&= (d + \mathrm{ext}\,\theta_{\nu,\mu})(\mathcal{T}_g^{\nu,\mu}f).
\end{aligned}$$

This completes the proof, since $\mathcal{T}_g^{\nu,\mu}$ commutes also with $(d + \mathrm{ext}\,\theta_{\nu,\mu})^*$ for $\mathcal{T}_g^{\nu,\mu}$ being a unitary transformation. ☐

One of the principal result in this chapter is stated in the following theorem.

Theorem 4.3.2. *The eigenvalue problem of $\Delta_\tau^{\nu,\mu}$ on the space of mixed automorphic functions of second kind $\mathcal{M}_{\rho,\tau}^{\nu,\mu}(\mathbb{C})$ is equivalent to the eigenvalue problem of $\Delta_{\alpha,0}$ on the space of classical automorphic functions $\mathcal{F}_{\Gamma,\chi}^{\alpha,0}(\mathbb{C})$ for certain fixed α that depends on the parameters ν, μ and the equivariant pairs (ρ, τ) through a bijective map relating the two.*

The proof of Theorem 4.3.2 is contained in the following results (Theorem 4.3.3 and Theorem 4.3.5). Let first denote by $B_\tau^{\mu,\nu}$ the quantity

$$\boxed{B_\tau^{\mu,\nu} := \nu + \mu\left(\left|\frac{\partial \tau}{\partial z}(z)\right|^2 - \left|\frac{\partial \tau}{\partial \bar{z}}(z)\right|^2\right),}$$

which will be proven to be real and constant in the next result.

Theorem 4.3.3. *Let $\alpha = B_\tau^{\mu,\nu}$. Then, α is a real constant and the operators $\Delta_{\alpha,0}$ and $\Delta_\tau^{\nu,\mu}$ are intertwining. Namely, there exists a smooth real mapping $\varphi_\tau^{\nu,\mu}$ such that*

$$\Delta_\tau^{\nu,\mu} = e^{-\varphi_\tau^{\nu,\mu}(z)}\Delta_{B_\tau^{\mu,\nu},0}\, e^{\varphi_\tau^{\nu,\mu}(z)}. \tag{4.31}$$

Proof. The proof of $B_\tau^{\mu,\nu}$ being constant is similar to the one given in [28] (see Proposition 4.1 and Theorem 3.1 there). Indeed, by using (4.27) and (4.27), keeping in mind that $\phi(g)(\frac{\partial g \cdot z}{\partial z})$ belongs to $\mathrm{U}(1)$, it follows that $B_\tau^{\mu,\nu}$ is invariant under the action of G , and therefore is constant on $\mathbb{C}$.

In order to show that two magnetic Schrödinger operators are intertwining, it is enough to show that the exterior derivatives of their potential vectors are equal. Indeed, it can be checked that $d\theta_\tau^{\nu,\mu}(z) =$

$d\theta_{B_\tau^{\mu,\nu}}(z)$. Therefore, using the fact that every closed differential form on $\mathbb{C}$ is exact, there exists a smooth mapping $\varphi_\tau^{\nu,\mu}$ such that

$$\theta_\tau^{\nu,\mu}(z) = \theta_{B_\tau^{\mu,\nu}} + d\varphi_\tau^{\nu,\mu}. \tag{4.32}$$

The map $\varphi_\tau^{\nu,\mu}$ can be freely chosen up to a translation by constants. By means of the last equation we can write

$$(d + \mathrm{ext}\ \theta_\tau^{\nu,\mu}) = e^{-\varphi_\tau^{\nu,\mu}}(d + \mathrm{ext}\ \theta_{B_\tau^{\mu,\nu}})e^{\varphi_\tau^{\nu,\mu}}. \tag{4.33}$$

Accordingly, its adjoint $(d + \mathrm{ext}\,\theta_\tau^{\nu,\mu})^*$ holds similar equality. Subsequently,

$$\Delta_\tau^{\nu,\mu} = e^{-\varphi_\tau^{\nu,\mu}(z)}\Delta_{B_\tau^{\mu,\nu},0}\, e^{\varphi_\tau^{\nu,\mu}(z)}. \tag{4.34}$$

Now, it remains to show that $\varphi_\tau^{\nu,\mu}$ can be assumed to be real. We have from the definition of $\varphi_\tau^{\nu,\mu}$ that $d\varphi_\tau^{\nu,\mu} = \theta_\tau^{\nu,\mu}(z) - \theta_{B_\tau^{\mu,\nu}}$. Hence,

$$\frac{\partial\varphi_\tau^{\nu,\mu}}{z}(z) = \frac{1}{2}\left[B_\tau^{\mu,\nu}\bar{z} - S_\tau^{\nu,\mu}(z)\right], \tag{4.35}$$

and

$$\frac{\partial\varphi_\tau^{\nu,\mu}}{\bar{z}}(z) = \frac{1}{2}\left[\overline{S_\tau^{\nu,\mu}(z)} - B_\tau^{\mu,\nu}z\right]. \tag{4.36}$$

Therefor, since

$$\overline{\frac{\partial\varphi_\tau^{\nu,\mu}(z)}{z}} = -\frac{\partial\varphi_\tau^{\nu,\mu}(z)}{\bar{z}},$$

one can check that $\varphi_\tau^{\nu,\mu}$ has a constant imaginary part

$$\frac{\partial\varphi_\tau^{\nu,\mu} - \overline{\varphi_\tau^{\nu,\mu}}}{\partial z}(z) = 0, \qquad \frac{\partial\varphi_\tau^{\nu,\mu} - \overline{\varphi_\tau^{\nu,\mu}}}{\partial\bar{z}}(z) = 0, \tag{4.37}$$

and then can be translated such that it is purely real. □

Remark 4.3.4. *The intertwining property between $\Delta_{\alpha,0}$ and $\Delta_\tau^{\nu,\mu}$ can be argued using physical intuition. Indeed, both $\Delta_{\alpha,0}$ and $\Delta_\tau^{\nu,\mu}$ are observables for the quantum behavior of a charged particle under the influence of a constant magnetic field $B_\tau^{\mu,\nu}$ (see for example [3]), which means that the two operators are related by a gouge transformation. Furthermore, the two operators would rather be unitary equivalent if we had considered a different convention for the Schrödinger operator. Namely, using the connection $(d \pm i\theta)$ instead of $(d \pm \theta)$, which is common in physic literature (*see *e.g. [3]).*

Actually, the function $\varphi_\tau^{\nu,\mu}$ turns out to be the key to establish, as well, a bijection between classical automorphic functions and mixed automorphic functions. Indeed, let us define $\mathcal{W}_\tau^{\mu,\nu}$ to be the transformation given by

$$\boxed{\left[\mathcal{W}_\tau^{\mu,\nu}f\right](z) := e^{\varphi_\tau^{\nu,\mu}(z)}f(z).} \tag{4.38}$$

Then, we assert the following main result

Theorem 4.3.5. *$\mathcal{W}_\tau^{\mu,\nu}$ makes bijection between the space of mixed automorphic functions $\mathcal{M}_{\rho,\tau}^{\nu,\mu}(\mathbb{C})$ and the space of classical (Γ,χ_τ)-automorphic functions $\mathcal{F}_{\Gamma,\chi_\tau}^{B_\tau^{\mu,\nu},0}(\mathbb{C})$*

$$\mathcal{F}_{\Gamma,\chi_\tau}^{B_\tau^{\mu,\nu}}(\mathbb{C}) := \left\{F \in \mathcal{C}^\infty(\mathbb{C}), \quad F(\gamma\cdot z) = \chi_\tau(\gamma)j^{B_\tau^{\mu,\nu}}(\gamma,z)F(z)\right\},$$

where $B_\tau^{\mu,\nu} = \nu + \mu(|\frac{\partial\tau}{\partial z}|^2 - |\frac{\partial\tau}{\partial\bar{z}}|^2) \in \mathbb{R}$ and χ_τ is the pseudo-character defined on Γ by

$$\chi_\tau(\gamma) = \chi(\gamma)\exp\left(\varphi_\tau^{\nu,\mu}(\gamma\cdot 0) - 2i\mu\Im\left\langle\tau(0),\rho(\gamma)^{-1}\cdot 0\right\rangle\right). \tag{4.39}$$

For the proof, we begin with the following

Lemma 4.3.6. *The function $\widehat{\chi_\tau}$ defined on $\mathbb{C} \times \Gamma$ by*

$$\widehat{\chi_\tau}(z;\gamma) := e^{(\varphi_\tau^{\nu,\mu}(\gamma\cdot z) - \varphi_\tau^{\nu,\mu}(z))}\chi(\gamma) j^{\nu - B_\tau^{\mu,\nu}}(\gamma, z) j^{\mu}(\rho(\gamma), \tau(z)), \tag{4.40}$$

is independent of the variable z. Here $j^\alpha = e^{-i\alpha\Im\left\langle z, g^{-1}\cdot 0\right\rangle}$.

Proof. Differentiation of $\widehat{\chi_\tau}(z;\gamma)$ with respect to the z-variable gives

$$\begin{aligned} \frac{1}{\chi\widehat{\chi_\tau}}\frac{\partial\widehat{\chi_\tau}}{\partial z} &= \underbrace{\left(\frac{\partial\varphi_\tau^{\nu,\mu}}{\partial z}(\gamma\cdot z) - \frac{\partial\varphi_\tau^{\nu,\mu}}{\partial z}(z)\right)}_{A} \\ &\quad - \underbrace{\left([\nu - B_\tau^{\mu,\nu}]\frac{\partial i\Im\left\langle z, \gamma^{-1}\cdot 0\right\rangle}{\partial z} + \mu\frac{\partial i\Im\left\langle \tau(z), \rho(\gamma)^{-1}\cdot 0\right\rangle}{\partial z}\right)}_{B}. \end{aligned} \tag{4.41}$$

Component B: It is easy to check that

$$B = \frac{1}{2}\left[(B_\tau^{\mu,\nu} - \nu)a\bar{b} - \mu\Big(\phi(\gamma)\overline{\psi(\gamma)}\frac{\partial\tau}{\partial z} - \overline{\phi(\gamma)}\psi(\gamma)\frac{\partial\bar{\tau}}{\partial z}\Big)\right]. \tag{4.42}$$

Here and elsewhere, we are using the notation $\gamma := [a, b]$ and $\rho(\gamma) = [\phi(\gamma), \psi(\gamma)]$.

Component A: Let us now evaluate to the component A. By means of (4.35), we have

$$\begin{aligned} \frac{\partial\varphi_\tau^{\nu,\mu}(\gamma\cdot z)}{\partial z} &= \frac{a}{2}\left[(B_\tau^{\mu,\nu} - \nu)\overline{(\gamma\cdot z)} - \mu\Big(\overline{\tau(\gamma\cdot z)}\frac{\partial\tau}{\partial z}(\gamma\cdot z) - \tau(\gamma\cdot z)\frac{\partial\bar{\tau}}{\partial z}(\gamma\cdot z)\Big)\right] \\ &= \frac{1}{2}\left[(B_\tau^{\mu,\nu} - \nu)(\bar{z} + a\bar{b}) - \mu\left(\bar{\tau}\frac{\partial\tau}{\partial z} + \phi(\gamma)\overline{\psi(\gamma)}\frac{\partial\tau}{\partial z} - (\tau(z)\frac{\partial\bar{\tau}}{\partial z} + \overline{\phi(\gamma)}\psi(\gamma)\frac{\partial\bar{\tau}}{\partial z})\right)\right] \\ &= \frac{1}{2}\left[(B_\tau^{\mu,\nu} - \nu)\bar{z} - \mu\Big(\bar{\tau}\frac{\partial\tau}{\partial z} - \tau\frac{\partial\bar{\tau}}{\partial z}\Big)\right] + \frac{1}{2}\left[(B_\tau^{\mu,\nu} - \nu)a\bar{b} - \mu\Big(\phi(\gamma)\overline{\psi(\gamma)}\frac{\partial\tau}{\partial z} - \overline{\phi(\gamma)}\psi(\gamma)\frac{\partial\bar{\tau}}{\partial z}\Big)\right] \\ &= \frac{\partial\varphi_\tau^{\nu,\mu}}{z}(z) + \frac{1}{2}\left[(B_\tau^{\mu,\nu} - \nu)a\bar{b} - \mu\Big(\phi(\gamma)\overline{\psi(\gamma)}\frac{\partial\tau}{\partial z} - \overline{\phi(\gamma)}\psi(\gamma)\frac{\partial\bar{\tau}}{\partial z}\Big)\right] \\ &= \frac{\partial\varphi_\tau^{\nu,\mu}}{z}(z) + B. \end{aligned}$$

Notice that, for shortening reasons, we omit sometimes the variable z. The second line follows, after some computations, mainly by making use of the equations (4.25)–(4.28). Hence, $A = B$ and therefore

$$\frac{\partial\widehat{\chi_\tau}}{\partial z} = 0.$$

Moreover, using the facts that $\overline{j^\alpha(\gamma, z)} = j^{-\alpha}(\gamma, z)$ and $\frac{\partial\varphi_\tau^{\nu,\mu}(\gamma\cdot z)}{\partial\bar{z}} = \overline{\frac{\partial\overline{\varphi_\tau^{\nu,\mu}(\gamma\cdot z)}}{\partial z}} = \overline{\frac{\partial\varphi_\tau^{\nu,\mu}(\gamma\cdot z)}{\partial z}}$, it follows that

$$\frac{\partial\widehat{\chi_\tau}}{\partial\bar{z}} = 0. \tag{4.43}$$

This ends the proof. □

Proof of Theorem 4.3.5. We shall prove that $\mathcal{W}_\tau^{\nu,\mu}F$ belongs to

$$\mathcal{F}_{\Gamma,\chi_\tau}^{B_\tau^{\nu,\mu}} := \Big\{F;\, \mathcal{C}^\infty,\ F(\gamma\cdot z) = \chi_\tau(\gamma) j^{B_\tau^{\nu,\mu}}(\gamma, z)F(z)\Big\}$$

whenever $F \in \mathcal{M}_\tau^{\nu,\mu}(\mathbb{C})$, where $\chi_\tau(\gamma)$ is the one defined through (4.39). Indeed, for fixed $\gamma \in H$, we have

$$\begin{aligned}
[\mathcal{W}_\tau^{\nu,\mu}F](\gamma\cdot z) &:= e^{i\varphi_\tau^{\nu,\mu}(\gamma\cdot z)}F(\gamma\cdot z)\\
&= e^{i\varphi_\tau^{\nu,\mu}(\gamma\cdot z)}\chi(\gamma)j^{\nu}(\gamma,z)j^{\mu}(\rho(\gamma),\tau(z))F(z)\\
&= e^{i(\varphi_\tau^{\nu,\mu}(\gamma\cdot z)-\varphi_\tau^{\nu,\mu}(z))}\chi(\gamma)j^{\nu}(\gamma,z)j^{\mu}(\rho(\gamma),\tau(z))[\mathcal{W}_\tau^{\nu,\mu}F](z)\\
&= \widehat{\chi_\tau}(z;\gamma)j^{B_\tau^{\mu,\nu}}(\gamma,z)[\mathcal{W}_\tau^{\nu,\mu}F](z),
\end{aligned}$$

where $\widehat{\chi_\tau}(z;\gamma)$ is defined by (4.40). Hence, as $\widehat{\chi_\tau}(z;\gamma)$ is invariable with respect to z by Lemma 4.3.6, we can see that $\widehat{\chi_\tau}(z;\gamma) = \widehat{\chi_\tau}(0;\gamma) =: \chi_\tau(\gamma)$ and therefore

$$[\mathcal{W}_\tau^{\nu,\mu}F](\gamma\cdot z) = \chi_\tau(\gamma)j^{B_\tau^{\mu,\nu}}(\gamma,z)[\mathcal{W}_\tau^{\nu,\mu}F](z).$$

The proof is completed □

Remark 4.3.7. *The data $(\Gamma; B_\tau^{\mu,\nu}, \chi_\tau(\gamma))$ indeed satisfies the (R.D.Q) condition whenever $\mathcal{M}_{\rho,\tau}^{\nu,\mu}$ is nontrivial, which is true when the condition in Proposition 4.1.2 holds.*

4.4 Spectral theory of $\Delta_\tau^{\nu,\mu}$ on $L^2(\mathbb{C}, d\lambda)$ and $\mathcal{M}_{\rho,\tau}^{\nu,\mu}(\mathbb{C})$

In this section, we provide a some consequences of the results in the previous chapter. Notice that we are restriction ourself for the case of full rank lattice. The general case will follow automatically once its resolved in the case of classical automorphic function, which is not yet the case and is a subject of our ongoing investigation.

The concrete spectral analysis of $\Delta_\tau^{\nu,\mu}$, acting on the free Hilbert space $L^2(\mathbb{C}; d\lambda)$, can be deduced by means of $\mathcal{W}_\tau^{\nu,\mu}$ as follows

Theorem 4.4.1. *Let $A_{\tau;k}^{\nu,\mu}(\mathbb{C}) = \left\{F \in L^2(\mathbb{C}; d\lambda); \quad \Delta_\tau^{\nu,\mu}F = E_kF\right\}$ be the L^2-eigenspace of $\Delta_\tau^{\nu,\mu}$ associated with the eigenvalue $E_k = -2B_\tau^{\mu,\nu}(2k+1)$.*

1. *We have the following orthogonal decomposition*

$$L^2(\mathbb{C}; d\lambda) = \bigoplus_{k=0}^{\infty} A_{\tau;k}^{\nu,\mu}(\mathbb{C}).$$

2. *Let $\psi_\tau^{\nu,\mu}(z,w) := \varphi_\tau^{\nu,\mu}(z) - \varphi_\tau^{\nu,\mu}(w)$. Then the eigenprojector kernel of $A_{\tau;k}^{\nu,\mu}$ satisfies the invariance property*

$$K_{\tau;k}^{\nu,\mu}(z,w) = e^{(\psi_\tau^{\nu,\mu}(g\cdot z, g\cdot w)\psi_\tau^{\nu,\mu}(z,w))}e^{iB_\tau^{\mu,\nu}\Im(z-w,g^{-1}.0)}K_{\tau;k}^{\nu,\mu}(g.z,g.w),$$

and is given explicitly by

$$K_{\tau;k}^{\nu,\mu}(z,w) = \frac{B_\tau^{\mu,\nu}}{\pi}e^{\psi_\tau^{\nu,\mu}(z,w)}e^{iB_\tau^{\mu,\nu}\Im\langle z,w\rangle}e^{-\frac{B_\tau^{\mu,\nu}}{2}|z-w|^2}L_k(2B_\tau^{\mu,\nu}|z-w|^2),$$

where $L_k(x) = L_k^0(x)$ denotes the usual Laguerre polynomial.

The spectral properties of $\Delta_\tau^{\nu,\mu}$ acting on $\mathcal{M}_\tau^{\nu,\mu}(\mathbb{C})$ follows using Theorem 4.3.3 together with Theorem 4.3.5 (invariance and intertwining). Indeed, we have

$$\mathcal{W}_\tau^{\nu,\mu}(\mathcal{E}_{\tau;k}^{\nu,\mu}) = \mathcal{E}_k^{B_\tau^{\mu,\nu}},$$

where for every fixed positive integer $k = 0, 1, 2, \cdots$, the space $\mathcal{E}_{\tau;k}^{\nu,\mu}$ (resp. $\mathcal{E}_k^{B_\tau^{\mu,\nu}}$) is the space of all eigenfunctions of $\Delta_\tau^{\nu,\mu}$ (resp. $\Delta_{B_\tau^{\mu,\nu},0}$) in $\mathcal{M}_\tau^{\nu,\mu}(\mathbb{C})$ (resp. $\mathcal{F}_{\Gamma,\chi_\tau}^{B_\tau^{\mu,\nu}}(\mathbb{C})$) corresponding to the eigenvalue $E_k = -2B_\tau^{\mu,\nu}(2k+1)$, i.e.,

$$\mathcal{E}_{\tau;k}^{\nu,\mu} := \left\{F \in \mathcal{M}_\tau^{\nu,\mu}(\mathbb{C}); \quad \Delta_\tau^{\nu,\mu}F = -2B_\tau^{\mu,\nu}(2k+1)F\right\}, \tag{4.44}$$

and

$$\mathcal{E}_k^{B_\tau^{\mu,\nu}} := \left\{ F \in \mathcal{F}_{\Gamma,\chi_\tau}^{B_\tau^{\mu,\nu}}(\mathbb{C}); \quad \Delta_{B_\tau^{\mu,\nu},0} F = -2B_\tau^{\mu,\nu}(2k+1)F \right\}.$$

Proposition 4.4.2. *The $\mathcal{M}_\tau^{\nu,\mu}(\mathbb{C})$-eigenspaces $\mathcal{E}_{\tau;k}^{\nu,\mu}$ in (4.44) are of finite and the same dimension which is given explicitly by*

$$\dim \mathcal{E}_{\tau;k}^{\nu,\mu} = (B_\tau^{\mu,\nu}/\pi) Area(\mathbb{C}/\Gamma).$$

4.5 Mixed automorphic functions of second kind in high dimensions

In the previous section, a lifting result was given between the space of mixed automorphic functions of second kind and the space of classical automorphic functions in one dimension. Natural investigation arises, which is to attempt a generalization in high dimensions. It turns out that such generalization is not as immediate as one could have been thinking. In the following we explain why this is the case. In parallel, we give the first steps toward a lifting result in high dimensions.

4.5.1 Adaptation of notions to high dimensions

As in the previous chapters, let G be the group $\mathrm{U}(n) \ltimes \mathbb{C}^n$ and Γ be a discrete subgroup of G. An element $\gamma := [A, b]$ of Γ can be expanded component-wise as follows:

* $A := (a_{i,j})_{i,j}$; $i, j = 1, 2, \cdots, n$, such that

$$\sum_{\ell=1} a_{\ell,i}\overline{a_{\ell,j}} = \delta_{i,j} \ \ \forall i, j = 1, 2, \cdots, n. \tag{4.45}$$

* $b := (b_i)_i$; $i = 1, 2, \cdots, n$.

The action of $\gamma \in G$ on $z := (z_1, \cdots, z_n) \in \mathbb{C}^n$ yields :

$$(\gamma \cdot z)_i = \sum_{j=1}^{n} a_{i,j} z_j + b_i. \tag{4.46}$$

By a G-equivariant pair (ρ, τ) in this context, we mean a G-endomorphism ρ and a compatible $\mathcal{C}^1$ mapping $\tau : \mathbb{C}^n \to \mathbb{C}^n$ such that

$$\tau(g \cdot z) = \rho(g) \cdot \tau(z); \qquad g \in G, \quad z \in \mathbb{C}. \tag{4.47}$$

The G-endomorphism ρ can be expressed using its linear and translational counterparts. That is, $\rho : \gamma \mapsto [\alpha(\gamma), \beta(\gamma)]$ such that $\alpha := (\alpha_{i,j})_{i,j}$; $i, j = 1, 2, \cdots, n$, is a matrix where the components are functions in γ with

$$\sum_{\ell=1} \alpha_{i,\ell}\overline{\alpha_{\ell,j}} = \delta_{i,j}; \qquad i, j = 1, 2, \cdots, n,$$

and $\beta := (\beta_i)_i$; $i = 1, 2, \cdots, n$ is a column vector in $\mathbb{C}^n$ with components are functions in γ as well. Notice that the variable γ is usually dropped to lighten the notations.The map τ is $\mathbb{C}^n$ valued, Hence, can be rewritten as follows:

$$\tau(z) = (\tau_1(z), \tau_2(z), \cdots, \tau_n(z)),$$

where its components are $\mathbb{C}$-valued functions on $\mathbb{C}^n$.
Now, using the component expansion of ρ and τ, we can rewrite the equivalence condition (4.47):

$$\tau_\ell = \sum_{j=1}^{n} \alpha_{\ell,j} \tau_j + \beta_\ell. \tag{4.48}$$

It will be of importance in the squeal to express the information within the equivariant condition (4.47) on the partial derivatives of the components of τ. Indeed, by differentiating left-hand side of (4.48) and with the usage of chain rule, for functions of several complex variables (see for example [47]), we get:

$$\frac{\partial \tau_\ell(\gamma \cdot z)}{\partial z_k} = \sum_{j=1}^{n} a_{j,k} \frac{\partial \tau_\ell}{\partial z_j}(\gamma \cdot z), \qquad \text{and} \qquad \frac{\partial \tau_\ell(\gamma \cdot z)}{\partial \overline{z_k}} = \sum_{j=1}^{n} \overline{a_{j,k}} \frac{\partial \tau_\ell}{\partial \overline{z_j}}(\gamma \cdot z),$$

for all $k, \ell = 1, 2, \cdots, n$.
Differentiating right-hand sides of the above equations and identifying the both sides yields

$$\sum_{j=1}^{n} a_{j,k} \frac{\partial \tau_\ell}{\partial z_j}(\gamma \cdot z) = \sum_{j=1}^{n} \alpha_{\ell,j} \frac{\partial \tau_j}{\partial z_k} \tag{4.49}$$

$$\sum_{j=1}^{n} \overline{a_{j,k}} \frac{\partial \tau_\ell}{\partial \overline{z_j}}(\gamma \cdot z) = \sum_{j=1}^{n} \alpha_{\ell,j} \frac{\partial \tau_j}{\partial \overline{z_k}} \tag{4.50}$$

4.5.2 Construction of the invariant Laplacian & the failure of reduction

Following the same scheme as in the 1-dimensional case, we construct an appropriate magnetic Schrödinger operator

$$H_\theta = (d + \text{ext } \theta)^*(d + \text{ext } \theta), \tag{4.51}$$

which is invariant with respect to the automorphic factor. As expected, taking n-copies of the potential vector form $\theta^{\nu,\mu}_{\tau_\ell}$ leads to such invariant operator.

Theorem 4.5.1. *The magnetic Schrödinger operator $H_{\theta^{\nu,\mu}_\tau}$ commutes with the automorphic factor $J^{\nu,\mu}_{\rho,\tau}$ of mixed automorphic functions of second kind, where*

$$\theta^{\nu,\mu}_\tau := -\frac{1}{2} \sum_{\ell=1}^{n} \Big\{ \nu(\bar{z}_\ell dz_\ell - z_\ell d\bar{z}_\ell) + \mu(\overline{\tau_\ell} d\tau_\ell - \tau_\ell d\overline{\tau_\ell}) \Big\}. \tag{4.52}$$

For convenience, we adopt sometimes in the squeal the notation $z_{n+i} := \overline{z_i}; i = 1, \cdots, n$.

Proof. The computations are quite straightforward. They are similar to the ones for the factor containingν which was treated in Chapter 3. $\square$

Now, let us move to the unexpected part. The magnetic field of the operator $H_{\theta^{\nu,\mu}_\tau}$ is not constant in general. This implies in particular that

Theorem 4.5.2. *The intertwining property between $H_{\theta^{\nu,\mu}_\tau}$ and $\Delta_{\nu,0}$ does not hold in general in high dimensions.*

Proof. Let us focus on the factor where τ is present:

$$\begin{aligned}
d(\overline{\tau_\ell} d\tau_\ell - \tau_\ell d\overline{\tau_\ell}) &= 2 \sum_{\ell=1}^{n} d\overline{\tau_\ell} \wedge d\tau_\ell \\
&= 2 \sum_{\ell=1}^{n} \left(\sum_{i=1}^{n} \partial_{z_i} \overline{\tau_\ell} dz_i + \partial_{\overline{z_i}} \overline{\tau_\ell} d\overline{z_i} \right) \wedge \left(\sum_{j=1}^{n} \partial_{z_j} \tau_\ell dz_j + \partial_{\overline{z_j}} \tau_\ell d\overline{z_j} \right) \\
&= 2 \sum_{\ell=1}^{n} \Big(\sum_{i<j} (\partial_{z_i} \overline{\tau_\ell} \partial_{z_j} \tau_\ell - \partial_{z_j} \overline{\tau_\ell} \partial_{z_i} \tau_\ell) dz_i \wedge dz_j \\
&\quad + (\partial_{z_i} \overline{\tau_\ell} \partial_{\overline{z_j}} \tau_\ell - \partial_{\overline{z_j}} \overline{\tau_\ell} \partial_{z_i} \tau_\ell) dz_i \wedge d\overline{z_j} \\
&\quad (\partial_{\overline{z_i}} \overline{\tau_\ell} \partial_{z_j} \tau_\ell - \partial_{z_j} \overline{\tau_\ell} \partial_{\overline{z_i}} \tau_\ell) d\overline{z_i} \wedge dz_j \\
&\quad + (\partial_{\overline{z_i}} \overline{\tau_\ell} \partial_{\overline{z_j}} \tau_\ell - \partial_{\overline{z_j}} \overline{\tau_\ell} \partial_{\overline{z_i}} \tau_\ell) d\overline{z_i} \wedge d\overline{z_j} \\
&\quad + \sum_{k=1}^{n} (|\partial_{\overline{z_k}} \tau_\ell|^2 - |\partial_{z_k} \tau_\ell|^2) dz_k \wedge d\overline{z_k} \Big).
\end{aligned}$$

$\square$

The operator $H_{\theta_\tau^{\nu,\mu}}$ would be associated with a constant magnetic field if $d\theta_\tau^{\nu,\mu}$ is a Kähler 2-form. Accordingly, the terms in factors other than the last one would vanish if and only if for all ℓ fix the determinant

$$\begin{vmatrix} \partial_{z_i}\overline{\tau_\ell} & \partial_{z_i}\tau_\ell \\ \partial_{z_j}\overline{\tau_\ell} & \partial_{z_j}\tau_\ell \end{vmatrix} = 0$$

for all $i, j = 1, \cdots, n$. This implies that the columns are linearly dependent. It follows that (equivalent), there exists $f : \mathbb{C}^n \to \mathbb{C}$ such that :

$$\frac{\partial\overline{\tau_\ell}}{\partial z_i} = f(z)\frac{\partial\tau_\ell}{\partial z_i}. \tag{4.53}$$

The last statement can be checked with fixing $i = 1$ and running j through $1, \cdots, n$. This shows that the condition (4.53) must be verified to ensure the gauge transformation (the intertwining property) between the operator $H_{\theta_\tau^{\nu,\mu}}$ and $\Delta_{\nu,0}$.

This marks the end of the proof and of this chapter.

CHAPTER 5

POINCARÉ THETA SERIES OPERATOR ASSOCIATED WITH A LATTICE OF $\mathbb{C}$

We deal with the study of the Poincaré theta series operator

$$\mathcal{P}_\Gamma(f)(z) := \sum_{\gamma\in\Gamma} \overline{\chi(\gamma)} e^{-2\nu\langle z+\frac{\gamma}{2},\gamma\rangle} f(z+\gamma)$$

which is a fundamental tool of constructing automorphic functions, provided that the series converges absolutely and uniformly on compact sets. Mainely, we answer the questions that arose naturally: what are the classes of functions on which $\mathcal{P}_\Gamma$ is well-defined as well as the vanishing Poincaré problem raised by Poincaré himself (1882). More exactly, we show that this operator is well-defined on the L^1-Bargmann-Fock space $\mathcal{F}^{1,\nu}(\mathbb{C})$. We also provide a density theorem of the image $\mathcal{P}_\Gamma(\mathcal{F}^{1,\nu}(\mathbb{C}))$ in the Banach space $\mathcal{F}^{1,\nu}_{\Gamma,\chi}(\mathbb{C})$ of L^1-holomorphic (Γ,χ) automorphic functions satisfying , and we determinate the dual with respect to a Petersson inner product. Moreover, we prove that the Poincaré theta operator $\mathcal{P}_\Gamma : \mathcal{F}^{1,\nu}(\mathbb{C}) \longrightarrow \mathcal{F}^{1,\nu}_{\Gamma,\chi}(\mathbb{C})$ is onto. We give in addition, explicit, sufficient and necessary condition on the Fourier coefficients of a function f to belong to the kernel of the Poincaré theta operator. This condition is a special identity that depends on the rank of Γ and the involves the univariate real Hermite polynomials.

In the previous chapters we showed that, in one dimension, both mixed automotphic functions of first kind and of second kind and their associated eigenvalue problems can be lifted to the standard automorphic functions and its associated eigenvalue problem. In the remaining two chapters we are going back to the space of classical automorphic function $\mathcal{F}^{\nu}_{\Gamma,\chi}(\mathbb{C})$ where L is a lattice of $(\mathbb{C},+)$ and investigate some pertinent questions in this framework. Namely, in this chapter we investigate the construction of automorphic functions using a method that we often refer to by *"periodization à la Poincaré"* , as it is due to Poincaré. Recall that, roughly, automorphic functions are functions such that $f(\gamma\cdot z) = J(\gamma,z)f(z)$; $z\in D$, $\gamma\in\Gamma$. Then, the method consists of considering

$$\mathcal{P}_\Gamma(f)(z) := \sum_{\gamma\in\Gamma} f(\gamma\cdot z)J(\gamma,z)^{-1},$$

often called Poincaré theta series, provided that the series converges absolutely and uniformly on compact sets. It takes an appropriate smooth function f that guarantees the convergence of the series and generate, by definition, an automorphic function. Accordingly, important questions arose naturally. The first one concerns the classes of functions on which $\mathcal{P}_\Gamma$ is well-defined. The second is the characterization of the set of functions f for which $\mathcal{P}_\Gamma f$ vanishes identically as well as those for which $\mathcal{P}_\Gamma f$ does not vanish (vanishing Poincaré problem raised by Poincaré himself in 1882). As far as we know, related works treating these questions are done in the context of the unit complex hyperbolic disc $\mathbb{D}$ or the upper half-plane $\mathcal{H}$ with Γ being a Fuchsian group acting on by linear fractional transformations and $J(\gamma,z) = \gamma'(z)^k$, for a fixed positive integer k. For example, Bers has introduced a class of holomorphic integrable functions on $\mathbb{D}$, with respect to the hyperbolic metric, on which $\mathcal{P}_\Gamma$ is well-defined [9]. Concerning the kernel of $\mathcal{P}_\Gamma$, viewed as an operator on the Bers spaces, Metzger has given a set of functions that closely spans $\ker(\mathcal{P}_\Gamma)$ [58]. Later, Kra obtained a characterization of the kernel using the Eichler cohomology for finitely generated Fuchsian group [49]. The result is then improved by Masumoto who gave a concrete characterization of $\ker(\mathcal{P}_\Gamma)$ on Hardy spaces for finitely generated Fuchsian group of the first kind [57].

5.1 Context precision and definitions

In our context, we are concerned by the situation where D is the whole complex plane $\mathbb{C}$ and $\Gamma = L$ is a discrete subgroup of $(\mathbb{C}, +)$ of rank 1 or 2.

Hereafter, we perform the space $\mathcal{O}^{2\nu}_{\Gamma,\chi}$ of all holomorphic functions on $\mathbb{C}$ satisfying the functional equation

$$f(z+\gamma) = \chi(\gamma) e^{2\nu\langle z+\frac{\gamma}{2},\gamma\rangle} f(z), \tag{5.1}$$

where χ is a given map on Γ with values in the unit circle such that

$$\chi(\gamma_1+\gamma_2) = \chi(\gamma_1)\chi(\gamma_2) e^{2i\nu\Im\langle\gamma_1,\gamma_2\rangle}. \tag{5.2}$$

The 2ν is present here instead of ν to simplify later formulas and notations. Consequently, the function $z \longmapsto |f(z)|\, e^{-\nu|z|^2}$ is Γ-periodic on $\mathbb{C}$, for every $f \in \mathcal{O}^{2\nu}_{\Gamma,\chi}$, and therefore the quantity

$$\|f\|_{1,\Gamma} := \int_{\Lambda(\Gamma)} |f(z)|\, e^{-\nu|z|^2} d\lambda(z)$$

makes sense and is independent of the choice of the fundamental domain $\Lambda(\Gamma)$. Moreover, it defines a norm on $\mathcal{O}^{2\nu}_{\Gamma,\chi}$.

Associated with the automorphic factor in (5.1), the Poincaré series operator reads

$$\mathcal{P}_\Gamma(f)(z) := \sum_{\gamma\in\Gamma} \overline{\chi(\gamma)} e^{-2\nu\langle z+\frac{\gamma}{2},\gamma\rangle} f(z+\gamma) \tag{5.3}$$

for given entire function f. The absolute and uniform convergence of $\mathcal{P}_\Gamma(f)$ on every compact subset of $\mathbb{C}$ is then needed to ensure that $\mathcal{P}_\Gamma(f)$ belongs to $\mathcal{O}^{2\nu}_{\Gamma,\chi}$. We will see that this is fulfilled if we consider $\mathcal{P}_\Gamma$ on

$$\mathcal{F}^{1,\nu}(\mathbb{C}) := \left\{ f \in \mathcal{H}ol(\mathbb{C}),\ \int_{\mathbb{C}} |f(z)|\, e^{-\nu|z|^2} d\lambda(z) < +\infty \right\},$$

which can be seen as a specific case of the Banach space

$$\mathcal{F}^{1,\nu}_{\Gamma,\chi}(\mathbb{C}) := \left\{ f \in \mathcal{O}^{2\nu}_{\Gamma,\chi},\ \|f\|_{1,\Gamma} := \int_{\Lambda(\Gamma)} |f(z)|\, e^{-\nu|z|^2} d\lambda(z) < +\infty \right\}$$

by taking $\Gamma = \{0\}$ and $\chi = 1$.

In the next sections, we aim to present our original results concerning the concrete description of $\mathcal{P}_\Gamma$ when acting on $\mathcal{F}^{1,\nu}(\mathbb{C})$. Mainly, we prove a density theorem for $\mathcal{P}_\Gamma$. Moreover, the $\mathcal{P}_\Gamma$ is shown to be onto (Theorem 5.4.1). The kernel is also characterized in terms of some special functions derived from the basis of the space of L^2 automorphic functions $\mathcal{F}^{\nu}_{\Gamma,\chi}(\mathbb{C})$, and which depends of the rank of the discrete subgroup Γ (Theorem 5.5.1). Theorem 5.5.2 describes the explicit condition to be satisfied by the Fourier coefficients of a given function to belong in such kernel. To prove the results, we had to determine the dual of $\mathcal{F}^{1,\nu}_{\Gamma,\chi}(\mathbb{C})$ (Theorem 5.3.1).

5.2 Boundedness of $\mathcal{P}_\Gamma$ form $\mathcal{F}^{1,\nu}(\mathbb{C})$ to $\mathcal{F}^{1,\nu}_{\Gamma,\chi}(\mathbb{C})$

In this section we state and prove two main theorems. The first theorem deals with the continuity of $\mathcal{P}_\Gamma$ viewed as an operator form $\mathcal{F}^{1,\nu}(\mathbb{C})$ to $\mathcal{F}^{1,\nu}(\mathbb{C})$. Namely, we show that $\mathcal{P}_\Gamma$ is well-defined on $\mathcal{F}^{1,\nu}(\mathbb{C})$ with range contained in $\mathcal{F}^{1,\nu}_{\Gamma,\chi}(\mathbb{C})$ and that its operator norm is less or equal to one. And the second theorem shows that any automorphic function in $\mathcal{F}^{1,\nu}_{\Gamma,\chi}(\mathbb{C})$ is, in fact, a periodization à la Poincaré.

We begin by stating and proving the firs result

Theorem 5.2.1. *The operator $\mathcal{P}_\Gamma$ is well-defined and continuous from $\mathcal{F}^{1,\nu}(\mathbb{C})$ to $\mathcal{F}^{1,\nu}_{\Gamma,\chi}(\mathbb{C})$.*

Proof. Let $f \in \mathcal{F}^{1,\nu}(\mathbb{C})$. Then

$$\begin{aligned}
\|\mathcal{P}_\Gamma(f)\|_{1,\Gamma} &:= \int_{\Lambda(\Gamma)} \left| \sum_{\gamma\in\Gamma} \overline{\chi(\gamma)} e^{-2\nu\langle z+\frac{\gamma}{2},\gamma\rangle} f(z+\gamma) \right| e^{-\nu|z|^2} d\lambda(z) \\
&\leq \int_{\Lambda(\Gamma)} \sum_{\gamma\in\Gamma} \left| e^{-2\nu\langle z+\frac{\gamma}{2},\gamma\rangle} f(z+\gamma) \right| e^{-\nu|z|^2} d\lambda(z) \\
&\leq \sum_{\gamma\in\Gamma} \int_{\Lambda(\Gamma)} \left| e^{-2\nu\langle z+\frac{\gamma}{2},\gamma\rangle} f(z+\gamma) \right| e^{-\nu|z|^2} d\lambda(z)
\end{aligned}$$

The last inequality follows by means of the monotone convergence theorem. Making the change of variable $w = z+\gamma$ taking into account that $\Lambda(\Gamma)$ is a fundamental domain, we obtain

$$\begin{aligned}
\|\mathcal{P}_\Gamma(f)\|_{1,\Gamma} &\leq \sum_{\gamma\in\Gamma} \int_{\Lambda(\Gamma)+\gamma} |f(w)|\, e^{-\nu|w|^2} d\lambda(w) \\
&\leq \int_{\mathbb{C}} |f(w)|\, e^{-\nu|w|^2} d\lambda(w) = \|f\|_1 . \qquad (5.4)
\end{aligned}$$

Now, since the L^1 convergence implies the normal convergence of holomorphic functions, the inequality (5.4) implies that the series (5.3) converges uniformly on every compact subset of the fundamental domain $\Lambda(\Gamma)$. But, for every compact subset D of $\mathbb{C}$, we can find a compact subset D_0 of $\Lambda(\Gamma)$ and finite sequence $\gamma_0, \cdots, \gamma_n$ such that $D \subset \bigcup_{j=0}^{n} (D_0+\gamma_j)$. Hence $\mathcal{P}_\Gamma(f)$ is a holomorphic function on the whole $\mathbb{C}$. $\square$

The converse of the above theorem is described in Theorem 5.4.1, which shows that all $f \in \mathcal{F}^{1,\nu}_{\Gamma,\chi}(\mathbb{C})$ are periodization à la Poincaré below. But before we can prove that we need to identify the duals of $\mathcal{F}^{1,\nu}_{\Gamma,\chi}(\mathbb{C})$ with respect to a specific Petersson inner product.

5.3 Dual space of $\mathcal{F}^{1,\nu}_{\Gamma,\chi}(\mathbb{C})$

In order to give the surjectivity of $\mathcal{P}_\Gamma$ and also the explicit description of the kernel of $\mathcal{P}_\Gamma$ when acting on $\mathcal{F}^{1,\nu}(\mathbb{C})$, we need to determine the dual space of the Banach space $\mathcal{F}^{1,\nu}_{\Gamma,\chi}(\mathbb{C})$. Namely, we prove that

Theorem 5.3.1. *The dual space of the Banach space $\mathcal{F}^{1,\nu}_{\Gamma,\chi}(\mathbb{C})$ is the space*

$$\mathcal{F}^{\infty,\nu}_{\Gamma,\chi}(\mathbb{C}) := \left\{ f \in \mathcal{O}^{2\nu}_{\Gamma,\chi},\ \|f\|_{\infty,\Gamma} := \sup_{z\in\mathbb{C}} \left\{ |f(z)|\, e^{-\nu|z|^2} \right\} < +\infty \right\}$$

under the scalar product

$$\langle f, g\rangle_\Gamma := \int_{\Lambda(\Gamma)} f(z)\overline{g(z)} e^{-2\nu|z|^2} d\lambda(z), \quad f \in \mathcal{F}^{1,\nu}_{\Gamma,\chi}(\mathbb{C}), g \in \mathcal{F}^{\infty,\nu}_{\Gamma,\chi}(\mathbb{C}). \qquad (5.5)$$

The proof uses the following formula which is of major usefulness for both the proof of Theorem 5.3.1 and Theorem 5.5.1 in the next section.

Lemma 5.3.2. *Let g be a holomorphic function satisfying the automorphy condition* (5.1). *Then*

$$\int_{\Lambda(\Gamma)} \mathcal{P}_\Gamma(f)(z)\overline{g(z)}e^{-2\nu|z|^2}d\lambda(z) = \int_{\mathbb{C}} f(z)\overline{g(z)}e^{-2\nu|z|^2}d\lambda(z), \tag{5.6}$$

for all $f \in \mathcal{F}^{1,\nu}(\mathbb{C})$.

Proof of Lemma 5.3.2.

Let $f \in \mathcal{F}^{1,\nu}(\mathbb{C})$ and g be a function such that $g(z+\gamma) = \chi(\gamma)e^{2\nu\left\langle z+\frac{\gamma}{2},\gamma\right\rangle}g(z)$. Then, we have

$$\begin{aligned}\int_{\Lambda(\Gamma)} \mathcal{P}_\Gamma(f)(z)\overline{g(z)}\,\mathrm{e}^{-2\nu|z|^2}\,d\lambda(z) &= \int_{\Lambda(\Gamma)} \left(\sum_{\gamma\in\Gamma}\overline{\chi(\gamma)}e^{-2\nu\left\langle z+\frac{\gamma}{2},\gamma\right\rangle}f(z+\gamma)\right)\overline{g(z)}\,\mathrm{e}^{-2\nu|z|^2}\,d\lambda(z)\\ &= \sum_{\gamma\in\Gamma}\int_{\Lambda(\Gamma)} \overline{\chi(\gamma)}e^{-2\nu\left\langle z+\frac{\gamma}{2},\gamma\right\rangle}f(z+\gamma)\overline{g(z)}e^{-2\nu|z|^2}d\lambda(w).\end{aligned}$$

Using the change of variable $w = z+\gamma$ as well as the facts $\chi(-\gamma) = \overline{\chi(\gamma)}$, it follows

$$\begin{aligned}\int_{\Lambda(\Gamma)} \mathcal{P}_\Gamma(f)(z)\overline{g(z)}\,\mathrm{e}^{-2\nu|z|^2}\,d\lambda(z) &= \sum_{\gamma\in\Gamma}\int_{\Lambda(\Gamma)+\gamma} f(z)\overline{g(z)}\,\mathrm{e}^{-2\nu|z|^2}\,d\lambda(z)\\ &= \int_{\mathbb{C}} f(z)\overline{g(z)}\,\mathrm{e}^{-2\nu|z|^2}\,d\lambda(z).\end{aligned}$$

This completes the proof of the Lemma. □

Proof of Theorem 5.3.1. Recall first that for any continuous linear functional ℓ on $\mathcal{F}^{1,\nu}_{\Gamma,\chi}(\mathbb{C})$, the bounded function g in (5.9) satisfies

$$\ell(f) = \int_{\Lambda(\Gamma)} f(z)\overline{g(z)}e^{-\nu|z|^2}d\lambda(z).$$

Accordingly, we define ψ_g to be

$$\psi_g(z) := \left(\frac{2\nu}{\pi}\right)\int_{\mathbb{C}} K(z,w)g(w)e^{-\nu|w|^2}d\lambda(w),$$

where $K(z,w) = e^{2\nu z\overline{w}}$ is the Fock reproducing kernel function. Then, ψ_g is holomorphic on $\mathbb{C}$ and verifies

$$\begin{aligned}|\psi_g(z)| &\leqslant \left(\frac{2\nu}{\pi}\right)\int_{\mathbb{C}} |K(z,w)g(w)|\,e^{-\nu|w|^2}d\lambda(w)\\ &\leqslant \left(\frac{2\nu}{\pi}\right)||g||_\infty\int_{\mathbb{C}} |K(z,w)|\,e^{-\nu|w|^2}d\lambda(w)\\ &= 2||g||_\infty e^{\nu|z|^2}.\end{aligned}$$

Now, by using the fact that the function g satisfies (5.9) as well as the explicit form of $K(z,w)$, we can show that

$$\begin{aligned}\psi_g(z+\gamma) &= \left(\frac{2\nu}{\pi}\right)\int_{\mathbb{C}} K(z+\gamma,w)g(w)e^{-\nu|w|^2}d\lambda(w)\\ &= \left(\frac{2\nu}{\pi}\right)\int_{\mathbb{C}} e^{2\nu(z+\gamma)\overline{w}}g(w-\gamma)\chi(\gamma)e^{-i\nu\Im((w-\gamma)\overline{\gamma})}e^{-\nu|w|^2}d\lambda(w)\\ &= \left(\frac{2\nu}{\pi}\right)\chi(\gamma)\int_{\mathbb{C}} e^{2\nu(z+\gamma)\overline{w+\gamma}}g(w)e^{-i\nu\Im((w)\overline{\gamma})-\nu\Re e(w\overline{\gamma})}e^{-\nu|\gamma|^2}e^{-\nu|w|^2}d\lambda(w)\\ &= \left(\frac{2\nu}{\pi}\right)\chi(\gamma)e^{2\nu z\overline{\gamma}+\nu|\gamma|^2}\int_{\mathbb{C}} g(w)e^{2\nu z\overline{w}}e^{-\nu|w|^2}d\lambda(w)\\ &= \chi(\gamma)e^{2\nu\left\langle z+\frac{\gamma}{2},\gamma\right\rangle}\psi_g(z).\end{aligned}$$

Thus, ψ_g belongs to $\mathcal{F}^{\infty,\nu}_{\Gamma,\chi}(\mathbb{C})$. Lemma 5.4.3 implies that

$$\ell(\mathcal{P}_\Gamma(f)) = \int_{\Lambda(\Gamma)} f(z)\overline{g(z)}e^{-\nu|z|^2}d\lambda(z).$$

Using Fubini theorem, we obtain

$$\begin{aligned}
\langle f, \psi_g\rangle_{\{0\}} &= \left(\frac{2\nu}{\pi}\right)\int_{\mathbb{C}} f(z)\overline{\int_{\mathbb{C}} K(z,w)g(w)e^{-\nu|w|^2}d\lambda(w)}e^{-2\nu|z|^2}d\lambda(z)\\
&= \left(\frac{2\nu}{\pi}\right)\int_{\mathbb{C}} \overline{g(w)}\int_{\mathbb{C}} f(z)K(w,z)e^{-2\nu|z|^2}d\lambda(z)e^{-\nu|w|^2}d\lambda(w).
\end{aligned}$$

Now, by the fact (see [83, Theorem 2.10, p. 39])

$$\mathcal{F}^{1,\nu}(\mathbb{C}) \subset \mathcal{F}^{2,2\nu}(\mathbb{C}) := \left\{ f \text{ holomorphic}, \int_{\mathbb{C}} |f(z)|^2 e^{-2\nu|z|^2}d\lambda(z) < \infty \right\}$$

and using the reproducing property of the reproducing kernel $K(z,w)$ of $\mathcal{F}^{2,2\nu}(\mathbb{C})$, we obtain

$$\ell(\mathcal{P}_\Gamma(f)) = \int_{\mathbb{C}} f(w)\overline{g(w)}e^{-\nu|w|^2}d\lambda(w) = \langle f, \psi_g\rangle_{\{0\}} \tag{5.7}$$

for all $f \in \mathcal{F}^{1,\nu}(\mathbb{C})$ and, by Lemma 5.3.2 we conclude that

$$\ell(\varphi) = \langle \varphi, \psi_g\rangle_\Gamma$$

whenever $\varphi \in \mathcal{P}_\Gamma(\mathcal{F}^{1,\nu}(\mathbb{C}))$. By density (Theorem 5.4.1), we get

$$\ell(f) = \langle f, \psi_g\rangle_\Gamma;\ f \in \mathcal{F}^{1,\nu}_{\Gamma,\chi}(\mathbb{C}).$$

For the uniqueness, let $g \in \mathcal{F}^{\infty,\nu}_{\Gamma,\chi}(\mathbb{C})$, such that $\langle f, g\rangle_\Gamma = 0$ for all $f \in \mathcal{F}^{1,\nu}_{\Gamma,\chi}(\mathbb{C})$. Then by Lemma 5.3.2 implies that $\langle \mathcal{P}_\Gamma(\phi), g\rangle_\Gamma = \langle \phi, g\rangle_{\{0\}} = 0$ for all $\phi \in \mathcal{F}^{1,\nu}(\mathbb{C})$ and by ([83, Corollary 2.25]) we have that $g = 0$. The proof of Theorem 5.3.1 is completed. □

5.4 Surjectivity of $\mathcal{P}_\Gamma$ from $\mathcal{F}^{1,\nu}(\mathbb{C})$ to $\mathcal{F}^{1,\nu}_{\Gamma,\chi}(\mathbb{C})$

Now we are able to show the following surjectivity theorem

Theorem 5.4.1. *The image $\mathcal{P}_\Gamma(\mathcal{F}^{1,\nu}(\mathbb{C}))$ is dense in $\mathcal{F}^{1,\nu}_{\Gamma,\chi}(\mathbb{C})$. Furthermore, the Poincaré theta operator $\mathcal{P}_\Gamma$: $\mathcal{F}^{1,\nu}(\mathbb{C}) \longrightarrow \mathcal{F}^{1,\nu}_{\Gamma,\chi}(\mathbb{C})$ is onto.*

For the proof we make use of the following lemma that gives, under some conditions, the existence, the unicity and an estimate of $\bar\partial$-equation on a wighted L^2 space.

Lemma 5.4.2 (Hedenmalm [38]). *Let $\phi : \mathbb{C} \to \mathbb{R}$ be a C^2-smooth map with $\Delta\phi > 0$ everywhere and $f \in L^2(\mathbb{C}, e^{2\phi})$ such that $\int_{\mathbb{C}} f(z)g(z)d\lambda(z) = 0$ for all $g \in \mathcal{F}^{2,\phi}(\mathbb{C})$. Then there exists a solution to the $\bar\partial$-equation $\bar\partial u = f$ with*

$$\int_{\mathbb{C}} |u|^2 e^{2\phi}\Delta\phi d\lambda(z) \le \frac{1}{2}\int_{\mathbb{C}} |f|^2 e^{2\phi}d\lambda(z).$$

Furthermore, if in addition ϕ is C^4-smooth and meets the curvature-type condition

$$\frac{1}{\Delta\phi}\Delta\log\Delta\phi \ge -2 \quad on \ \ \mathbb{C},$$

then the solution u is unique.

Notice that for any continuous linear functional ℓ on $\mathcal{F}^{1,\nu}_{\Gamma,\chi}(\mathbb{C})$, there exists a bounded Borel measurable function g on $\Lambda(\Gamma)$ such that ℓ can be represented as

$$\ell(F) = \int_{\Lambda(\Gamma)} F(z)\overline{g(z)}e^{-\nu|z|^2}d\lambda(z). \tag{5.8}$$

This follows by Hahn-Banach and Riesz theorem. Next, since the fundamental domain $\Lambda(\Gamma)$ is arbitrary, we can extend g to the whole $\mathbb{C}$ by setting

$$g(z+\gamma) = \chi(\gamma)e^{2i\nu\Im\langle z,\gamma\rangle}g(z). \tag{5.9}$$

This extension is well-defined since $\chi(\gamma)e^{2i\nu\Im\langle z,\gamma\rangle}$ is an automorphic factor. Associated with such g, we consider

$$\varphi_g(z) = \overline{g(z)}e^{-\nu|z|^2},$$

which clearly satisfies the functional equation

$$\varphi_g(z) = \chi(\gamma)\, \mathrm{e}^{2\nu\left\langle z+\frac{\gamma}{2},\gamma\right\rangle}\, \varphi_g(z+\gamma). \tag{5.10}$$

Moreover, we can prove the following

Lemma 5.4.3. *For every $f \in \mathcal{F}^{1,\nu}(\mathbb{C})$, we have*

$$\ell(\mathcal{P}_\Gamma(f)) = \int_{\mathbb{C}} f(z)\varphi_g(z)d\lambda(z).$$

Proof. Starting from

$$\begin{aligned}
\ell(\mathcal{P}_\Gamma(f)) &= \int_{\Lambda(\Gamma)} \mathcal{P}_\Gamma(f)(z)\overline{g(z)}e^{-\nu|z|^2}d\lambda(z)\\
&= \int_{\Lambda(\Gamma)} \left(\sum_{\gamma\in\Gamma}\overline{\chi(\gamma)}e^{-2\nu\left\langle z+\frac{\gamma}{2},\gamma\right\rangle}f(z+\gamma)\right)\overline{g(z)}e^{-\nu|z|^2}d\lambda(z)\\
&= \sum_{\gamma\in\Gamma}\overline{\chi(\gamma)}\int_{\Lambda(\Gamma)} e^{-2\nu\left\langle z+\frac{\gamma}{2},\gamma\right\rangle}f(z+\gamma)\overline{g(z)}e^{-\nu|z|^2}d\lambda(z),
\end{aligned}$$

and making the change of variable $w = z+\gamma$, we obtain

$$\ell(\mathcal{P}_\Gamma(f)) = \sum_{\gamma\in\Gamma}\int_{\Lambda(\Gamma)+\gamma} f(w)\overline{g(w)}e^{-\nu|w|^2}d\lambda(w) = \int_{\mathbb{C}} f(w)\varphi_g(w)d\lambda(w).$$

□

In the light of these preparations, we are now able to prove Theorem 5.4.1.

Proof of Theorem 5.4.1. Suppose that $\ell(\mathcal{P}_\Gamma(f)) = 0$ for all $f \in \mathcal{F}^{1,\nu}(\mathbb{C})$ and let show that $\ell(F) = 0$ for all $F \in \mathcal{F}^{1,\nu}_{\Gamma,\chi}(\mathbb{C})$. By Lemma 5.4.3, the assumption $\ell(\mathcal{P}_\Gamma(f)) = 0$ reads

$$\int_{\mathbb{C}} f(z)\varphi_g(z)d\lambda(z) = 0.$$

Therefore,

$$\int_{\mathbb{C}} f(z)\varphi_g(z)d\lambda(z) = 0$$

for all $f \in \mathcal{F}^{2,\nu}$, since $\mathcal{F}^{2,\nu} := \mathcal{H}ol(\mathbb{C}) \cap L^2(\mathbb{C}; e^{-\nu|z|^2}d\lambda(z)) \subset \mathcal{F}^{1,\nu}(\mathbb{C})$. Now, making appeal to Lemma 5.4.2 with $\phi(z) := \frac{\nu}{2}|z|^2$, we conclude that there exists h such that $\bar\partial h = \varphi_g$. Then, we claim that h satisfies the functional equation

$$h(z+\gamma) = \overline{\chi(\gamma)}\,\mathrm{e}^{-2\nu\left\langle z+\frac{\gamma}{2},\gamma\right\rangle}\, h(z).$$

This follows by uniqueness of h, since $\chi(\gamma)\,\mathrm{e}^{2\nu\langle z+\frac{\gamma}{2},\gamma\rangle}\, h(z+\gamma)$ is also a solution of $\bar{\partial} = \varphi_g$, thanks to (5.10). Accordingly, we have

$$\ell(F) = \int_{\Lambda(\Gamma)} F(z)\varphi_g(z)d\lambda(z) = \int_{\Lambda(\Gamma)} F(z)\frac{\partial h}{\partial \bar{z}}(z)d\lambda(z) = \int_{\Lambda(\Gamma)} \frac{\partial F h}{\partial \bar{z}}(z)d\lambda(z).$$

By Stokes formula we find

$$\ell(F) = \int_{\partial(\Lambda(\Gamma))} F(z)h(z)d\lambda(z).$$

In addition, the function $F\cdot h$ is invariant by Γ. Indeed

$$F(z+\gamma)h(z+\gamma) = \chi(\gamma)\,\mathrm{e}^{2\nu\langle z+\frac{\gamma}{2},\gamma\rangle}\, F(z)\overline{\chi(\gamma)}\,\mathrm{e}^{-2\nu\langle z+\frac{\gamma}{2},\gamma\rangle}\, h(z) = F(z)h(z).$$

Using the symmetry of the boundary $\partial(\Lambda(\Gamma))$, we deduce that $\ell(F) = 0$ for all $F \in \mathcal{F}^{1,\nu}_{\Gamma,\chi}(\mathbb{C})$, and therefore $\mathcal{P}_\Gamma(\mathcal{F}^{1,\nu}(\mathbb{C}))$ is dense in $\mathcal{F}^{1,\nu}_{\Gamma,\chi}(\mathbb{C})$.

To show that $\mathcal{P}_\Gamma : \mathcal{F}^{1,\nu}(\mathbb{C}) \longrightarrow \mathcal{F}^{1,\nu}_{\Gamma,\chi}(\mathbb{C})$ is onto, let $F \in \mathcal{F}^{1,\nu}_{\Gamma,\chi}(\mathbb{C})$ and denote by $\mathbb{1}_{\Lambda(\Gamma)}$ the characteristic function of a fundamental domain $\Lambda(\Gamma)$. Then, the function

$$\psi(z) := \left(\frac{2\nu}{\pi}\right)\int_{\mathbb{C}} K(z,w)\mathbb{1}_{\Lambda(\Gamma)}(w)F(w)e^{-2\nu|w|^2}d\lambda(w).$$

is holomorphic on $\mathbb{C}$ and verifies

$$\int_{\mathbb{C}} |\psi(z)|\, e^{-\nu|z|^2}d\lambda(z) \leqslant \|F\|_{1,\Gamma}.$$

This implies that $\psi \in \mathcal{F}^{1,\nu}(\mathbb{C})$. Then, $\mathcal{P}_\Gamma(\psi)$ is well-defined and lies in $\mathcal{F}^{1,\nu}_{\Gamma,\chi}(\mathbb{C})$. We shall show that $\mathcal{P}_\Gamma(\psi) = F$. Let $g \in \mathcal{F}^{\infty,\nu}_{\Gamma,\chi}(\mathbb{C})$, then by using the Lemma 5.3.2 we get

$$\begin{aligned}
\langle \mathcal{P}_\Gamma(\psi), g\rangle_\Gamma = \langle \psi, g\rangle_{\{0\}} &= \left(\frac{2\nu}{\pi}\right)\int_{\mathbb{C}} \overline{g(z)}\int_{\Lambda(\Gamma)} K(z,w)F(w)e^{-2\nu|w|^2}d\lambda(w)e^{-2\nu|z|^2}d\lambda(z)\\
&= \left(\frac{2\nu}{\pi}\right)\int_{\Lambda(\Gamma)} F(w)\int_{\mathbb{C}} \overline{g(z)}K(z,w)e^{-2\nu|z|^2}d\lambda(z)e^{-2\nu|w|^2}d\lambda(w)\\
&= \langle F, g\rangle_\Gamma.
\end{aligned}$$

The last equality follows from the fact that $K(z,w)$ reproduces holomorphic functions satisfying the growth condition $|g(z)| \le Ce^{-\nu|z|^2}$ for some positive constant C (see [34, Lemma 3.9]). Hence $\mathcal{P}_\Gamma(\psi) = F$ by the duality Theorem 5.3.1. $\square$

5.5 The description of the kernel of $\mathcal{P}_\Gamma$

In this section, the description of the kernel of $\mathcal{P}_\Gamma$ is given in terms of the orthogonal of some concrete family of functions. We distinguish two cases of the discrete subgroup Γ, which can be represented by $\Gamma = \mathbb{Z}$ for the non co-compact case and by $\Gamma = \mathbb{Z} + \tau\mathbb{Z}$; $\Im(\tau) > 0$, for the co-compact case (see Chapter 1, Section 1.1 for more details). Notice that in the first case $\Gamma = \mathbb{Z}$, the map χ reduces to a character of $\mathbb{Z}$. Therefore, it is uniquely determined by its value at 1, $\chi(1) = e^{2i\pi\alpha}$ for fixed real α. Then, the following functions [33]

$$e^{\alpha,\nu}_n(z) := e^{\nu z^2 + 2i\pi(\alpha+n)z}; \qquad n \in \mathbb{Z}, \tag{5.11}$$

satisfy the functional equation (5.1). In the other case, i.e. $\Gamma = \mathbb{Z} + \tau\mathbb{Z}$, the assumption (5.2) implies in particular that $\nu\Im(\tau) \in \pi\mathbb{Z}^+$. Moreover, the quantity $m := 2\nu\Im(\tau)/\pi$ turns to be the dimension of $\mathcal{F}^{\infty,\nu}_{\Gamma,\chi}(\mathbb{C})$. Furthermore, a basis for those spaces are given by the following functions [63, 44]

$$\theta^{\nu,\alpha,\beta}_r(z) := e^{\nu z^2}\sum_{\substack{k\in\mathbb{Z}\\ k=r \mod m}} e^{i\frac{\pi(k+\alpha)^2\tau}{m} + 2i\pi(\alpha+k)\left(z-\frac{\beta}{m}\right)}; \qquad r = 1,2,\cdots,m. \tag{5.12}$$

Theorem 5.5.1. *For $\Gamma = \mathbb{Z}$, the kernel of $\mathcal{P}_\Gamma$ is given by $\ker(\mathcal{P}_\Gamma) = \{e_n^{\alpha,\nu}; n \in \mathbb{Z}\}^{\perp_{\langle\cdot,\cdot\rangle_\Gamma}}$ where α is a fixed real number such that $\chi(1) = e^{2i\pi\alpha}$ and*

$$e_n^{\alpha,\nu}(z) := e^{\nu z^2 + 2i\pi(\alpha+n)z}; \qquad n \in \mathbb{Z}, \tag{5.13}$$

while when $\Gamma = \mathbb{Z}[\tau] = \mathbb{Z} + \tau\mathbb{Z}$; $\Im(\tau) > 0$, we have $\ker(\mathcal{P}_\Gamma) = \left\{\theta_r^{\nu,\alpha,\beta}; r = 1, \cdots, m\right\}^{\perp_{\langle\cdot,\cdot\rangle_\Gamma}}$, where

$$\theta_r^{\nu,\alpha,\beta}(z) := e^{\nu z^2} \sum_{\substack{k\in\mathbb{Z} \\ k=r \mod m}} e^{i\frac{\pi(k+\alpha)^2\tau}{m} + 2i\pi(\alpha+k)(z-\frac{\beta}{m})}; \qquad r = 1, 2, \cdots, m. \tag{5.14}$$

Notice that the orthogonal complement of the considered subspaces is taken with respect to (5.5).

Proof. For the first case $\Gamma = \mathbb{Z}$, the family $\{e_n^{\alpha,\nu}\}_{n\in\mathbb{Z}}$ belongs to $\mathcal{F}^{\infty,\nu}_{\Gamma,\chi}(\mathbb{C})$. Then, in view of Lemma 5.3.2 we see that $\mathcal{P}_\Gamma(f) \equiv 0$ if and only if f is orthogonal to $\{e_n^{\alpha,\nu}\}_{n\in\mathbb{Z}}$. Indeed, if $P_\Gamma(f) = 0$ then f is orthogonal to the family $\{e_n^{\alpha,\nu}\}_n$. The converse follows since $\{e_n^{\alpha,\nu}\}_n$ is dense in $F^{\infty,\nu}_{\Gamma,\chi}$. The proof of the second assertion for $\Gamma = \mathbb{Z}[\tau]$ is similar. $\square$

The last main result is a variant of Theorem 5.5.1 and gives an explicit sufficient and necessary condition on the Fourier coefficients of a function f to belong to the kernel of the Poincaré theta operator.

Theorem 5.5.2. *The function f belongs to $\ker \mathcal{P}_\Gamma$ if and only if its Taylor coefficients satisfy the condition*

$$\sum_{k\geq 0} a_k \left(\frac{i}{2\sqrt{\nu}}\right)^k H_k\left(\frac{\pi}{\sqrt{\nu}}(\alpha+n)\right) = 0$$

for each $n \in \mathbb{Z}$, if $\Gamma = \mathbb{Z}$ is rank one discrete lattice. Whereas, coefficients satisfy the condition

$$\sum_{k\geq 0} a_k \left(\frac{i}{2\sqrt{\nu}}\right)^k \sum_{\substack{n\in\mathbb{Z} \\ n=r \mod m}} e^{-i\frac{\pi(n+\alpha)^2\bar{\tau}}{m} - 2i\pi(\alpha+n)\frac{\beta}{m}} H_k\left(\frac{\pi}{\sqrt{\nu}}(\alpha+n)\right) = 0$$

for $r = 0, 1, \cdots m-1$, if $\Gamma = \mathbb{Z}[\tau]$; $\Im(\tau) > 0$. Here H_k are the usual Hermite polynomials.

Proof of Theorem 5.5.2. Let f be a function in $\mathcal{F}^{1,\nu}(\mathbb{C})$ with the Taylor expansion

$$f(z) = \sum_{k=0}^{+\infty} a_k z^k.$$

Then, Theorem 5.5.2 follows by combining the result of Theorem 5.5.1 with the following integral formula

$$\int_{\mathbb{C}} z^k \overline{e^{\nu z^2 + 2i\pi(\alpha+n)z}} e^{-2\nu|z|^2} d\lambda(z) = \left(\frac{\pi}{2\nu 2}\right)\left(\frac{-i}{2\sqrt{\nu}}\right)^k H_k\left(\frac{\pi}{\sqrt{\nu}}(\alpha+n)\right).$$

The integral in the left hand-side is in fact the inverse Bargmann transform of the monomials (see [83, p. 222-223]). $\square$

This marks the end of this chapter.

CHAPTER 6

ANALYTIC AND ARITHMETIC PROPERTIES OF THE REPRODUCING KERNEL $K^{\nu}_{\Gamma,\chi}$

We discuss some analytical and arithmetical properties of the reproducing kernel function $K^{\nu}_{\Gamma,\chi}$. More precisely, we show that the set $\mathcal{Z}(K^{\nu}_{\Gamma,\chi})$ of zeros of $K^{\nu}_{\Gamma,\chi}$ is symmetric, not isolated and its distribution is uniform, in the sense that $\mathcal{Z}(K^{\nu}_{\Gamma,\chi})$ consists of the Γ-translation of the zeros of $K^{\nu}_{\Gamma,\chi}$ contained in a cartesian product of fundamental cells. The systematic study of these zeros leads to a characterization of the common zeros (analytic set) contained in a fundamental cell of all holomorphic functions belonging to the theta Bargmann-Fock Hilbert space $\mathcal{O}^{\nu}_{\Gamma,\chi}(\mathbb{C})$ considered in Chapter 2. It is shown that this analytic set is finite and its cardinal is less or equal to $\nu S(\Gamma)/\pi$. We also provide an interesting power series representation of $K^{\nu}_{\Gamma,\chi}$ by means of the so-called complex Hermite-Gauss coefficients. Moreover, we establish the connection to the Poincaré series associated with the monomials. Determination of zeros of $K^{\nu}_{\Gamma,\chi}$ gives rise to interesting identities. More remarkable identities known as lattice sums are proved by evaluating the complex Hermite-Gauss coefficients, generalizing certain arithmetic identities obtained by Perelomov [64] in the framework of coherent states for the specific case of von Neumann lattice [61] and rediscovered later by Boon and Zak [12]. The proof we provide seems to be new, simpler and more direct. We conclude the chapter by proving that the complex Hermite-Gauss coefficients $a^{p,q}_{m,n}(\Gamma|\nu,\chi)$ are non-trivial lattice's functions in the sense of J-P. Serre [73].

So far, the focus in the previous chapters, except the last one, was the study of spectral properties of classical and mixed automorphic functions. This chapter, like the last one, pursue a different goal. The primary material of this Chapter still the functional space $\mathcal{O}^{\nu}_{\Gamma,\chi}(\mathbb{C})$ of holomrphic automorphic functions, which we considered dealt with in Chapter 2. Recall that for given $\mathbb{C}$-lattice of full rank $\Gamma = L$, the space $\mathcal{O}^{\nu}_{\Gamma,\chi}(\mathbb{C})$ is the L^2-functional space of holomorphic (Γ,χ)-theta functions f on $\mathbb{C}$ of magnitude $\nu > 0$, provided that the functional equation

$$f(z+\gamma) = \chi(\gamma)e^{\nu\left(z+\frac{\gamma}{2}\right)\overline{\gamma}}f(z) \tag{6.1}$$

holds for all $z \in \mathbb{C}$ and all $\gamma \in \Gamma$. The pseudo-character nature of χ, i.e.,

$$\chi(\gamma+\gamma') = \chi(\gamma)\chi(\gamma')e^{\frac{\nu}{2}(\gamma\overline{\gamma'}-\overline{\gamma}\gamma')}, \tag{6.2}$$

is required to ensure that $\mathcal{O}^{\nu}_{\Gamma,\chi}(\mathbb{C})$ is a nonzero vector space. This implies in particular that the parameter ν is quantified such that the imaginary part of $\nu S(\Gamma)$ belongs to $\pi\mathbb{N}$ where $S(\Gamma) = \Im(\omega_1\overline{\omega_2})$ is the cell area of Γ (see the discussion that follows Proposition 2.1.1 for more details). In this case, $\mathcal{O}^{\nu}_{\Gamma,\chi}(\mathbb{C})$ is moreover a reproducing kernel Hilbert space with respect to the hermitian product

$$\langle f,g\rangle_{\nu,\Gamma} = \int_{\Lambda(\Gamma)} f(z)\overline{g(z)}e^{-\nu|z|^2}dm(z), \tag{6.3}$$

which independent of the choice of the fundamental region $\Lambda(\Gamma)$ representing $\mathbb{C}/\Gamma$. As pointed out in [34], the reproducing kernel function is given by the series

$$K^{\nu}_{\Gamma,\chi}(z,w) := \left(\frac{\nu}{\pi}\right)\sum_{\gamma\in\Gamma}\chi(\gamma)e^{-\frac{\nu}{2}|\gamma|^2+\nu\left(z\overline{\gamma}-\overline{w}\gamma+z\overline{w}\right)}, \tag{6.4}$$

which converges absolutely and uniformly on compact subsets of $\mathbb{C}\times\mathbb{C}$. Recall also that the dimension of $\mathcal{O}^{\nu}_{\Gamma,\chi}(\mathbb{C})$ is known to be finite and given by

$$\dim \mathcal{O}^{\nu}_{\Gamma,\chi}(\mathbb{C}) = \left(\frac{\nu}{\pi}\right)S(\Gamma). \tag{6.5}$$

6.1 Automorphic functions and evaluation of some lattice sums

In this section, we digress into our motivations for the investigation provided in this chapter, which is primarily to find applications for the theory of automorphic functions on euclidean spaces, presented here or elsewhere, in annexe fields. One of the applications that was previously investigated for example in [35] is to use automorphic functions to derive arithmetical identities involving infinite summation over multiple indices. We refer to such summations by *lattice sums*, which are of major importance in many areas, especially in chemistry. For general reference on this subject, we refer to e.g. [13].

More specifically, our investigation was motivated at first by the fact that for the von Neumann lattice [61], the concrete description of $\mathcal{O}^{\nu}_{\Gamma,\chi}(\mathbb{C})$ allows the recovery of some arithmetic identities obtained by Perelomov [64] (see also [12]). In fact, for the specific case of $\nu=\pi/S(\Gamma)$ and χ being the "Weierstrass pseudo-character", $\chi(\gamma)=\chi_W(\gamma)=+1$ if $\gamma/2\in\Gamma$ and $\chi(\gamma)=\chi_W(\gamma)=-1$ otherwise, the space $\mathcal{O}^{\nu}_{\Gamma,\chi_W}(\mathbb{C})$ is one-dimensional and is generated by the modified *Weierstrass σ-function* [31]

$$\widetilde{\sigma}_{\mu}(z;\Gamma) := e^{-\frac{1}{2}\mu z^2}\sigma(z;\Gamma), \tag{6.6}$$

where $\sigma(z;\Gamma)$ denotes the classical Weierstrass σ-function

$$\sigma(z;\Gamma) = z\prod_{\gamma\in\Gamma}{}^{'}\left(1-\frac{z}{\gamma}\right)e^{z/\gamma+\frac{1}{2}(z/\gamma)^2}. \tag{6.7}$$

The prime in the product to exclude the term with $\gamma=0$. The parameter $\mu=\mu(\Gamma)$ is an invariant of the lattice $\Gamma=\mathbb{Z}\omega_1+\mathbb{Z}\omega_2$ and given in terms of the *Weierstrass zeta-function* $\zeta(z;\Gamma)=\sigma^{'}(z;\Gamma)/\sigma(z;\Gamma)$ by

$$\mu=\mu(\Gamma)=\frac{2i}{S}\left(\zeta(\omega_1^*;\Gamma)\overline{\omega_2^*}-\zeta(\omega_2^*;\Gamma)\overline{\omega_1^*}\right) \tag{6.8}$$

with $\omega_\ell^*=2\omega_\ell$; $\ell=1,2$.
In this von Neumann lattice situation, the expression of $K^{\nu}_{\Gamma,\chi_W}(z,w)$ reduces further to

$$K^{\nu}_{\Gamma,\chi_W}(z,w) = C\,e^{\frac{1}{2}(\mu z^2+\overline{\mu}\,\overline{w}^2)}\sigma(z;\Gamma)\overline{\sigma(w;\Gamma)},$$

up to nonzero constant C. Therefore, one can conclude easily that the zeros of K^{ν}_{Γ,χ_W} are localized in $(\Gamma\times\mathbb{C})\cup(\mathbb{C}\times\Gamma)$. As immediate consequence, $K^{\nu}_{\Gamma,\chi_W}(0,0)=0$, or equivalently

$$\sum_{\gamma\in\Gamma}\chi_W(\gamma)e^{-\frac{\nu}{2}|\gamma|^2}=0. \tag{6.9}$$

Therefore, we recover the arithmetic identity obtained by Perelomov in [64, Eq. (47) with $k=0$] (see also (15) and (15a) in [12]).

For arbitrary lattice, the lattice sum in (6.9) as well as more arithmetic identities can be derived by considering and evaluating the so-called complex Hermite-Gauss coefficients

$$a_{m,n}(\Gamma|\nu,\chi) := \sum_{\gamma\in\Gamma} \chi(\gamma) e^{-\frac{\nu}{2}|\gamma|^2} H^{\nu}_{m,n}(\gamma;\overline{\gamma}), \tag{6.10}$$

where

$$H^{\nu}_{m,n}(z,\overline{z}) := (-1)^{m+n} e^{\nu|z|^2} \frac{\partial^{m+n}}{\partial\overline{z}^m \partial z^n} \left(e^{-\nu|z|^2}\right). \tag{6.11}$$

denote the rescaled complex Hermite polynomials (see [29] and the references therein). This is possible thanks to the key observation that $K^{\nu}_{\Gamma,\chi}(z,w)$ can be expanded as in (6.28) below.

6.2 Properties of the zero set of $K^{\nu}_{\Gamma,\chi}$

We begin by discussing some properties of $K^{\nu}_{\Gamma,\chi}$ that are common between reproducing kernels. The following result is easy to obtain.

Proposition 6.2.1. *The z-function*

$$\sum_{\gamma\in\Gamma} \chi(\gamma) e^{-\frac{\nu}{2}|\gamma|^2 + \nu\left(z\overline{\gamma} - \overline{z}\gamma\right)}$$

is nonnegative real-valued function on $\mathbb{C}$.

Proof. This follows from the fact that

$$K^{\nu}_{\Gamma,\chi}(z,z) = \left(\frac{\nu}{\pi}\right) \sum_{\gamma\in\Gamma} \chi(\gamma) e^{-\frac{\nu}{2}|\gamma|^2 + \nu\left(z\overline{\gamma} - \overline{z}\gamma + |z|^2\right)}, \tag{6.12}$$

and that the kernel function $K^{\nu}_{\Gamma,\chi}$ is nonnegative real-valued function on the diagonal of $\mathbb{C}\times\mathbb{C}$ for satisfying

$$K^{\nu}_{\Gamma,\chi}(z,z) = \int_{\Lambda(\Gamma)} |K^{\nu}_{\Gamma,\chi}(z,w)|^2 e^{-\nu|w|^2} dm(w) \geq 0 \tag{6.13}$$

which is a particular case of

$$K^{\nu}_{\Gamma,\chi}(z,\xi) = \int_{\Lambda(\Gamma)} K^{\nu}_{\Gamma,\chi}(z,w) K^{\nu}_{\Gamma,\chi}(w,\xi) e^{-\nu|w|^2} dm(w)$$

for every fixed $\xi\in\mathbb{C}$. This follows by making use of the reproducing property ([34])

$$f(z) = \int_{\Lambda(\Gamma)} K^{\nu}_{\Gamma,\chi}(z,w) f(w) e^{-\nu|w|^2} dm(w) \tag{6.14}$$

for every $f\in\mathcal{O}^{\nu}_{\Gamma,\chi}(\mathbb{C})$. □

Now we state one of the main theorems of this chapter. It concerns the *analytic set* (the common zeros) of the theta Bargmann-Fock Hilbert space

$$\mathcal{Z}(\mathcal{O}^{\nu}_{\Gamma,\chi}(\mathbb{C})|\Lambda(\Gamma)) := \bigcap_{f\in\mathcal{O}^{\nu}_{\Gamma,\chi}(\mathbb{C})} \mathcal{Z}(f|_{\Lambda(\Gamma)}).$$

We also denote by $\mathcal{Z}(\varphi_w|_{\Lambda(\Gamma)})$ the set of zeros of the function $\varphi_w(z) := K^{\nu}_{\Gamma,\chi}(z,w)$ contained in an arbitrary fundamental cell $\Lambda(\Gamma)$.

Theorem 6.2.2. *Let $\nu > 0$ and χ any pseudo-character as in (6.2) on a given lattice Γ. Then, the set $\mathcal{Z}(K^{\nu}_{\Gamma,\chi})$ of zeros of $K^{\nu}_{\Gamma,\chi}$ is symmetric and is given by*

$$\mathcal{Z}(K^{\nu}_{\Gamma,\chi}) = \bigcup_{w\in\Lambda(\Gamma)} \Big((\mathcal{Z}(\varphi_w|_{\Lambda(\Gamma)}) + \Gamma) \times \{w\} \Big) \cup \bigcup_{w\in\Lambda(\Gamma)} \Big(\{w\} \times (\mathcal{Z}(\varphi_w|_{\Lambda(\Gamma)}) + \Gamma) \Big).$$

Moreover, the analytic set $\mathcal{Z}(\mathcal{O}^{\nu}_{\Gamma,\chi}(\mathbb{C})|\Lambda(\Gamma))$ is finite with cardinal less or equal to the dimension of $\mathcal{O}^{\nu}_{\Gamma,\chi}(\mathbb{C})$ and reduces further to

$$\mathcal{Z}(\mathcal{O}^{\nu}_{\Gamma,\chi}(\mathbb{C})|\Lambda(\Gamma)) = \Xi := \{w \in \Lambda(\Gamma);\ \varphi_w \equiv 0 \text{ on } \Lambda(\Gamma)\}.$$

The proof of Theorem 6.2.2 is contained in Lemmas 6.2.3 and 6.2.4, Propositions 6.2.5 and 6.2.7, and Remarks 6.2.6 and 6.2.9 below.

Lemma 6.2.3. *The zeros set $\mathcal{Z}(K^{\nu}_{\Gamma,\chi})$ of $K^{\nu}_{\Gamma,\chi}$ is symmetric.*

Proof. The symmetry of $\mathcal{Z}(K^{\nu}_{\Gamma,\chi})$ follows easily since $K^{\nu}_{\Gamma,\chi}(z,w) = \overline{K^{\nu}_{\Gamma,\chi}(w,z)}$. □

Lemma 6.2.4. *Let $(z_0,w_0) \in \mathbb{C}\times\mathbb{C}$. Then, $K^{\nu}_{\Gamma,\chi}(z_0,w_0) = 0$ if and only if $(\{z_0\}+\Gamma)\times(\{w_0\}+\Gamma))$ is contained in the zeros set $\mathcal{Z}(K^{\nu}_{\Gamma,\chi})$ of $K^{\nu}_{\Gamma,\chi}$. Thus, we have*

$$\mathcal{Z}(K^{\nu}_{\Gamma,\chi}) = \mathcal{Z}(K^{\nu}_{\Gamma,\chi}|_{\Lambda(\Gamma)\times\Lambda(\Gamma)}) + \Gamma\times\Gamma. \tag{6.15}$$

Proof. Making use of the Γ-bi-invariance property ([34]),

$$K^{\nu}_{\Gamma,\chi}(z+\gamma, w+\gamma') = \chi(\gamma) e^{\frac{\nu}{2}|\gamma|^2+\nu z\overline{\gamma}} K^{\nu}_{\Gamma,\chi}(z,w)\overline{\chi(\gamma')} e^{\frac{\nu}{2}|\gamma'|^2+\nu\overline{w}\gamma'} \tag{6.16}$$

that holds for every $z,w\in\mathbb{C}$ and $\gamma,\gamma'\in\Gamma$, it follows readily that the elements of the set $(\{z_0\}+\Gamma)\times(\{w_0\}+\Gamma)$ are also zeros of $K^{\nu}_{\Gamma,\chi}$ whenever $K^{\nu}_{\Gamma,\chi}(z_0,w_0)=0$. Then, we can conclude since $(z,w)\in\mathcal{Z}(K^{\nu}_{\Gamma,\chi})$ is equivalent to the existence of unique $(z_0,w_0)\in\Lambda(\Gamma)\times\Lambda(\Gamma)$ and unique $(\gamma_0,\gamma_0')\in\Gamma\times\Gamma$ such that $(z,w) = (z_0,w_0)+(\gamma_0,\gamma_0')$ and $(z_0,w_0)\in\mathcal{Z}(K^{\nu}_{\Gamma,\chi})$. □

The following result describes the set $\mathcal{Z}(\varphi_w)$ of all zeros of φ_w in terms of $\mathcal{Z}(\varphi_w|_{\Lambda(\Gamma)})$. It is almost a restatement of Proposition 2.6.2 for the particular case where f is the kernel function.

Proposition 6.2.5. *We have*

$$\mathcal{Z}(\varphi_w) = \mathcal{Z}(\varphi_w|_{\Lambda(\Gamma)}) + \Gamma.$$

Moreover, the number of zeros of φ_w contained in any fundamental cell $\Lambda(\Gamma)$ is constant and independent of w. More exactly, we have

$$\#(\mathcal{Z}(\varphi_w|_{\Lambda(\Gamma)})) = \dim \mathcal{O}^{\nu}_{\Gamma,\chi}(\mathbb{C}).$$

Remark 6.2.6. *Notice that Proposition 6.2.5 shows that $\mathcal{Z}(\varphi_w)\times\{w\}$ is contained in $\mathcal{Z}(K^{\nu}_{\Gamma,\chi})$ for every arbitrary fixed $w\in\mathbb{C}$. This can be used to show that $\mathcal{Z}(K^{\nu}_{\Gamma,\chi})$ is in particular not discrete.*

Proposition 6.2.7. *We have*

$$\mathcal{Z}(K^{\nu}_{\Gamma,\chi}) = \mathcal{Z}(K^{\nu}_{\Gamma,\chi}|_{\Lambda(\Gamma)\times\Lambda(\Gamma)}) + \Gamma\times\Gamma$$

and the set

$$\Xi = \{w\in\Lambda(\Gamma) \text{ such that } \varphi_w \equiv 0 \text{ on } \Lambda(\Gamma)\} \tag{6.17}$$

is finite with

$$\#(\Xi) \leqslant \dim \mathcal{O}^{\nu}_{\Gamma,\chi}(\mathbb{C}) = \left(\frac{\nu}{\pi}\right) S(\Gamma). \tag{6.18}$$

Moreover, we have

$$\Xi = \bigcap_{f\in\mathcal{O}^{\nu}_{\Gamma,\chi}(\mathbb{C})} \mathcal{Z}(f) = \bigcap_{w\in\mathbb{C}} \mathcal{Z}(\varphi_w) = \bigcap_{w\in\Lambda(\Gamma)} \mathcal{Z}(\varphi_w). \tag{6.19}$$

Proof. The distribution of zeros of $K^{\nu}_{\Gamma,\chi}$ is uniform in the sense that they are completely determined by translating those localized in $\Lambda(\Gamma) \times \Lambda(\Gamma)$. This is contained in 6.2.4. Accordingly, we will concentrate on $\mathcal{Z}(K^{\nu}_{\Gamma,\chi}|_{\Lambda(\Gamma)\times\Lambda(\Gamma)})$. Recall that from Proposition 6.2.5, the number of zeros in $\Lambda(\Gamma)$ of every nonzero f belonging to $\mathcal{O}^{\nu}_{\Gamma,\chi}(\mathbb{C})$ is constant and is given by

$$\#(\mathcal{Z}(f|_{\Lambda(\Gamma)})) = \dim \mathcal{O}^{\nu}_{\Gamma,\chi}(\mathbb{C}) = \left(\frac{\nu}{\pi}\right) S(\Gamma). \tag{6.20}$$

Thus, from the definition of Ξ given through (6.17), we have $\varphi_w(z) = 0$ for all z in $\Lambda(\Gamma)$ and $w \in \Xi$. Therefore,

$$f(w) = \int_{\Lambda(\Gamma)} K^{\nu}_{\Gamma,\chi}(w,z) f(z) e^{-\nu|z|^2} dm(z) = \int_{\Lambda(\Gamma)} \overline{\varphi_w(z)} f(z) e^{-\nu|z|^2} dm(z) = 0.$$

Consequently

$$\Xi \subset \bigcap_{f\in\mathcal{O}^{\nu}_{\Gamma,\chi}(\mathbb{C})} \mathcal{Z}(f) \subset \bigcap_{w\in\mathbb{C}} \mathcal{Z}(\varphi_w) \subset \bigcap_{w\in\Lambda(\Gamma),\varphi_w\neq 0} \mathcal{Z}(\varphi_w) \subset \mathcal{Z}(f) \tag{6.21}$$

for all $w \in \Lambda(\Gamma)$ and for all $f \in \mathcal{O}^{\nu}_{\Gamma,\chi}(\mathbb{C})$. Notice that the set $\{w \in \Lambda(\Gamma), \varphi_w \neq 0\}$ is notrivial since $K^{\nu}_{\Gamma,\chi}$ is not identically zero on $\mathbb{C} \times \mathbb{C}$. This entails in particular that Ξ is finite with

$$0 \leqslant \#(\Xi) \leqslant \dim \mathcal{O}^{\nu}_{\Gamma,\chi}(\mathbb{C}) = \left(\frac{\nu}{\pi}\right) S(\Gamma). \tag{6.22}$$

To prove the inverse inclusion, let $z_0 \in \bigcap_{w\in\Lambda(\Gamma)} \mathcal{Z}(\varphi_w)$, then $\varphi_w(z_0) = 0$ for all $w \in \Lambda(\Gamma)$. Therefore, by hermitian symmetry of the kernel function we see that φ_{z_0} is identically zero on $\Lambda(\Gamma)$. This implies that $\varphi_{z_0} \equiv 0$ on $\Lambda(\Gamma)$, since φ_{z_0} admits an infinite set of zeros in $\Lambda(\Gamma)$. This shows that $z_0 \in \Xi$. In conclusion, we get (6.19). This completes the proof. □

Corollary 6.2.8. *The analytic sets of* $\left(\mathcal{P}^{\nu}_{\Gamma,\chi}(e_m)|_{\Lambda(\Gamma)}\right)_m$ *and* $\left(\varphi_w|_{\Lambda(\Gamma)}\right)_{w\in\Lambda(\Gamma)}$ *coincide and are given by*

$$\bigcap_{m\in\mathbb{N}} \mathcal{Z}(\mathcal{P}^{\nu}_{\Gamma,\chi}(e_m)|_{\Lambda(\Gamma)}) = \bigcap_{w\in\Lambda(\Gamma)} \mathcal{Z}(\varphi_w|_{\Lambda(\Gamma)}) = \{w \in \Lambda(\Gamma);\ \varphi_w \equiv 0\} = \Xi. \tag{6.23}$$

Proof. We have already shown that $\bigcap_{w\in\mathbb{C}} \mathcal{Z}(\varphi_w|_{\Lambda(\Gamma)}) \subset \mathcal{Z}(f)$, for all f in $\mathcal{O}^{\nu}_{\Gamma,\chi}(\mathbb{C})$ (see (6.21)). This holds also true for the specific functions $f = \mathcal{P}^{\nu}_{\Gamma,\chi}(e_m) \in \mathcal{O}^{\nu}_{\Gamma,\chi}(\mathbb{C})$. The converse

$$\bigcap_{m\in\mathbb{N}} \mathcal{Z}(\mathcal{P}^{\nu}_{\Gamma,\chi}(e_m)|_{\Lambda(\Gamma)}) \subset \bigcap_{w\in\mathbb{C}} \mathcal{Z}(\varphi_w|_{\Lambda(\Gamma)}) = \Xi. \tag{6.24}$$

follows making use (6.30) in Theorem 6.3.1 and (6.19) in Proposition 6.2.7. Indeed, if

$$z_0 \in \bigcap_{m\in\mathbb{N}} \mathcal{Z}(\mathcal{P}^{\nu}_{\Gamma,\chi}(e_m)|_{\Lambda(\Gamma)}),$$

then we have

$$\varphi_w(z_0) = K^{\nu}_{\Gamma,\chi}(z_0, w) = \sum_{m\in\mathbb{N}} \mathcal{P}^{\nu}_{\Gamma,\chi}(e_m)(z_0) \frac{\overline{w}^m}{m!} = 0 \tag{6.25}$$

for every $w \in \mathbb{C}$. □

Remark 6.2.9. *The fact that Ξ is a finite set implies that*

$$\bigcup_{w\in\Lambda(\Gamma)} \left(\mathcal{Z}(\varphi_w|_{\Lambda(\Gamma)}) \times \{w\}\right) \tag{6.26}$$

is a uncountable set of zeros of $K^{\nu}_{\Gamma,\chi}$. The zeros set $\mathcal{Z}(K^{\nu}_{\Gamma,\chi}|_{\Lambda(\Gamma)\times\Lambda(\Gamma)})$ admits then condensation points by means of the compactness of $\overline{\Lambda(\Gamma) \times \Lambda(\Gamma)}$. Moreover, it is not-discrete if Ξ is not-trivial. Indeed, we have $K^{\nu}_{\Gamma,\chi}(w_0, w) = \varphi_w(w_0) = 0$ for every $w_0 \in \Xi$ and $w \in \mathbb{C}$.

Remark 6.2.10. *The zero set $\mathcal{Z}(K^{\nu}_{\Gamma,\chi}|_{\Lambda(\Gamma)\times\Lambda(\Gamma)})$ is uncountable and the distribution of such zeros is uniform in the sense that*

$$\mathcal{Z}(K^{\nu}_{\Gamma,\chi}) = \mathcal{Z}(K^{\nu}_{\Gamma,\chi}|_{\Lambda(\Gamma)\times\Lambda(\Gamma)}) + \Gamma \times \Gamma.$$

Such set of zeros is not-discrete if Ξ is not-trivial. Moreover, the analytic set zeros corresponding to the functions $\varphi_w|_{\Lambda(\Gamma)}$; $w \in \Lambda(\Gamma)$, coincides with Ξ.

6.3 Arithmical expansion of the kernel $K^{\nu}_{\Gamma,\chi}$

Recall that the (Γ,χ)-Poincaré theta operator is defined by

$$\mathcal{P}^{\nu}_{\Gamma,\chi}(\psi)(z) := \sum_{\gamma\in\Gamma} \chi(\gamma) e^{-\frac{\nu}{2}|\gamma|^2+\nu z\overline{\gamma}} \psi(z-\gamma), \tag{6.27}$$

provided that the series converges.

In this section we state and prove the second main theorem, which concerns the expansion in power series of $K^{\nu}_{\Gamma,\chi}$ and the connection to $\mathcal{P}^{\nu}_{\Gamma,\chi}(e_m)$, the periodization à la Poincaré of the monomials $e_m(z) := z^m$. More precisely, we assert

Theorem 6.3.1. *The (Γ,χ)-automorphic reproducing kernel function can be expanded in power series as follows*

$$K^{\nu}_{\Gamma,\chi}(z,w) = \left(\frac{\nu}{\pi}\right) \sum_{m,n=0}^{\infty} (-1)^m a_{m,n}(\Gamma|\nu,\chi) \frac{z^n \overline{w}^m}{m!n!}, \tag{6.28}$$

where the coefficients $a_{m,n}(\Gamma|\nu,\chi)$ are given by

$$a_{m,n}(\Gamma|\nu,\chi) := \sum_{\gamma\in\Gamma} \chi(\gamma) e^{-\frac{\nu}{2}|\gamma|^2} H^{\nu}_{m,n}(\gamma;\overline{\gamma}). \tag{6.29}$$

Moreover, its expression in terms of the automorphic functions obtained by periodization à la Poincaré of the monomials is given by

$$K^{\nu}_{\Gamma,\chi}(z,w) = \left(\frac{\nu}{\pi}\right) \sum_{m=0}^{\infty} \left[\mathcal{P}^{\nu}_{\Gamma,\chi}(e_m)\right](z) \frac{(\nu\overline{w})^m}{m!}. \tag{6.30}$$

Proof. The proof of (6.28) lies essentially in the fact that in (6.4) we recognize the exponential generating function ([29])

$$e^{\nu(a\gamma+b\overline{\gamma}-ab)} = \sum_{m,n=0}^{\infty} \frac{a^m b^n}{m!n!} H^{\nu}_{m,n}(\gamma;\overline{\gamma}) \tag{6.31}$$

with $a = -\overline{w}$ and $b = z$. To prove (6.30), we recall that $\left[\mathcal{P}^{\nu}_{\Gamma,\chi}(e_m)\right](z)$ is given through

$$\mathcal{P}^{\nu}_{\Gamma,\chi}(e_m)(z) := \sum_{\gamma\in\Gamma} \chi(\gamma)(z-\gamma)^m e^{-\frac{\nu}{2}|\gamma|^2+\nu z\overline{\gamma}}. \tag{6.32}$$

Thus, making use of the generating function ([29])

$$\nu^n(\overline{\xi}-z)^n e^{\nu\xi z} = \sum_{m=0}^{\infty} \frac{z^m}{m!} H^{\nu}_{m,n}(\xi;\overline{\xi}) \tag{6.33}$$

with $\xi = \overline{\gamma}$, one obtains

$$\left[\mathcal{P}^{\nu}_{\Gamma,\chi}(e_m)\right](z) = \frac{(-1)^m}{\nu^m} \sum_{n=0}^{\infty} a_{m,n}(\Gamma|\nu,\chi) \frac{z^n}{n!}. \tag{6.34}$$

Therefore, the relationship (6.30) follows easily from (6.28) and (6.34). The proof of Theorem 6.3.1 is completed. □

We end this section by giving the following definition

Definition 6.1. *The involved $a_{m,n}(\Gamma|\nu,\chi)$ in the above theorem are called the generalized complex Hermite-Gauss coefficients .*

6.4 Derived arithmetical identities and lattice sums

In Theorem 6.3.1, the generalized complex Hermite-Gauss coefficients $a_{m,n}(\Gamma|\nu,\chi)$ possess interesting arithmetic properties. These kind of identities are the main focus in this section. In fact, if we consider the quantities

$$a^{p,q}_{m,n}(\Gamma|\nu,\chi) := \sum_{\gamma\in\Gamma} \chi(\gamma)\gamma^p\overline{\gamma}^q e^{-\frac{\nu}{2}|\gamma|^2} H^{\nu}_{m,n}(\gamma;\overline{\gamma}) \tag{6.35}$$

for given integers m,n,p,q, the we prove the following

Theorem 6.4.1. *Assume that χ is real-valued. Then, the generalized complex Hermite-Gauss coefficients vanish*

$$\sum_{\gamma\in\Gamma} \chi(\gamma)\gamma^p\overline{\gamma}^q e^{-\frac{\nu}{2}|\gamma|^2} H^{\nu}_{m,n}(\gamma,\overline{\gamma}) = 0 \tag{6.36}$$

for every integers m,n,p,q such that $m+n+p+q$ is odd.

Before giving the proof of Theorem 6.4.1, we begin by discussing some of its immediate consequences. In fact, under the assumption that the pseudo-character is real valued, we get

$$\sum_{\gamma\in\Gamma} \chi(\gamma) e^{-\frac{\nu}{2}|\gamma|^2} H^{\nu}_{2m+1,2n}(\gamma;\overline{\gamma}) = 0 \tag{6.37}$$

as well as

$$\sum_{\gamma\in\Gamma} \chi(\gamma) e^{-\frac{\nu}{2}|\gamma|^2} H^{\nu}_{2m,2n+1}(\gamma;\overline{\gamma}) = 0. \tag{6.38}$$

This follows by setting $p = q = 0$ in (6.36). Changing γ by $-\gamma$ and next taking the complex conjugate, taking into account the fact $\overline{H^{\nu}_{m,n}(z;\overline{z})} = H^{\nu}_{n,m}(z;\overline{z})$ in (6.37) gives rise to the lattice sum (6.38). We can as well stat the following remark

Remark 6.4.2. *Under the assumption that χ is a real-valued pseudo-character, we have $a_{2m+1,2n}(\Gamma|\nu,\chi) = 0 = a_{2m,2n+1}(\Gamma|\nu,\chi)$. In particular, we have the arithmetic identities*

$$\sum_{\gamma\in\Gamma} \chi(\gamma)\gamma^{2k+1} e^{-\frac{\nu}{2}|\gamma|^2} = 0 \tag{6.39}$$

and

$$\sum_{\gamma\in\Gamma} \chi(\gamma)\overline{\gamma}^{2k+1} e^{-\frac{\nu}{2}|\gamma|^2} = 0 \tag{6.40}$$

that follow by taking $p = q = 0$ in (6.36). *These identities generalize those obtained in [64, 12]. Accordingly, the expansion series* (6.28) *of the reproducing kernel function $K^{\nu}_{\Gamma,\chi}$ reduces further to the following*

$$\begin{aligned} K^{\nu}_{\Gamma,\chi}(z,w) = \left(\frac{\nu}{\pi}\right) \Bigg(& \sum_{m,n=0}^{\infty} a^{\nu}_{\Gamma,\chi}(2m,2n) \frac{z^{2n}\overline{w}^{2m}}{(2m)!(2n)!} \\ & - \sum_{m,n=0}^{\infty} a^{\nu}_{\Gamma,\chi}(2m+1,2n+1) \frac{z^{2n+1}\overline{w}^{2m+1}}{(2m+1)!(2n+1)!} \Bigg). \end{aligned} \tag{6.41}$$

Proof of Theorem 6.4.1. Under the assumption that χ is real-valued, we have $\chi(-\gamma) = \overline{\chi(\gamma)} = \chi(\gamma)$. Therefore, the symmetry

$$a^{p,q}_{m,n}(\Gamma|\nu,\chi) = (-1)^{m+n+p+q} a^{p,q}_{m,n}(\Gamma|\nu,\chi) \tag{6.42}$$

follows easily from the definition of $a^{p,q}_{m,n}(\Gamma|\nu,\chi)$ combined with the well-established fact

$$H^{\nu}_{m,n}(-\xi,-\overline{\xi}) = (-1)^{m+n} H^{\nu}_{m,n}(\xi,\overline{\xi}),$$

being indeed

$$\begin{aligned} a^{p,q}_{m,n}(\Gamma|\nu,\chi) &= \sum_{\gamma\in\Gamma} \chi(-\gamma)(-\gamma)^q(-\overline{\gamma})^p e^{-\frac{\nu}{2}|-\gamma|^2} H^{\nu}_{m,n}(-\gamma,-\overline{\gamma}) \\ &= (-1)^{m+n+p+q} a^{p,q}_{m,n}(\Gamma|\nu,\chi). \end{aligned}$$

This entails in particular that $a^{p,q}_{m,n}(\Gamma|\nu,\chi) = 0$ whenever $m+n+p+q$ is odd. □

In the last theorem of this chapter, the lattice sums $a^{p,q}_{m,n}(\Gamma|\nu,\chi)$ will be seen as functions in the lattice $\Gamma \in \mathcal{L}$, where $\mathcal{L}$ denotes the set of all lattices in $\mathbb{C}$. Accordingly, the next theorem shows that the complex Hermite-Gauss coefficients $a_{m,n}(\Gamma|\nu,\chi)$ are lattice's functions.

Theorem 6.4.3. *There exist specific ν and χ such that $a^{p,q}_{m,n}(\Gamma|\nu,\chi)$ are lattice's functions in sense that for every nonzero complex number λ, we have*

$$a^{p,q}_{m,n}(\lambda\Gamma|\nu_{\lambda\Gamma},\chi_{\lambda\Gamma}) = \frac{\lambda^p\overline{\lambda}^q}{\overline{\lambda}^m\lambda^n} a^{p,q}_{m,n}(\Gamma|\nu,\chi). \tag{6.43}$$

Proof. To prove (6.43), let $\nu = \nu_{\Gamma}$ and $\chi = \chi_{\Gamma}$ be such that

$$\nu_{\lambda\Gamma} = |\lambda|^{-2}\nu_{\Gamma} \quad \text{and} \quad \chi_{\lambda\Gamma}(\lambda\gamma) = \chi_{\Gamma}(\gamma) \tag{6.44}$$

for every complex $\lambda \in \mathbb{C}\setminus\{0\}$. Therefore

$$e^{-\frac{\nu_{\lambda\Gamma}}{2}|\lambda\gamma|^2} = e^{-\frac{\nu_{\Gamma}}{2}|\gamma|^2}. \tag{6.45}$$

Moreover, using the explicit expression of the complex Hermite polynomials ([29]), to wit

$$H^{\nu}_{m,n}(\xi,\overline{\xi}) = m!n!\nu^{m+n}\sum_{k=0}^{min(m,n)} \frac{(-1)^k}{\nu^k k!}\frac{\xi^{m-k}}{(m-k)!}\frac{\overline{\xi}^{n-k}}{(n-k)!},$$

we get

$$H^{\nu_{\lambda\Gamma}}_{m,n}(\lambda\gamma;\overline{\lambda}\overline{\gamma}) = \frac{1}{\lambda^n\overline{\lambda}^m} H^{\nu_{\Gamma}}_{m,n}(\gamma;\overline{\gamma}). \tag{6.46}$$

Hence, by inserting (6.44), (6.45) and (6.46) in the definition of $a^{p,q}_{m,n}(\lambda\Gamma|\nu_{\lambda\Gamma},\chi_{\lambda\Gamma})$, we obtain

$$a^{p,q}_{m,n}(\lambda\Gamma|\nu_{\lambda\Gamma},\chi_{\lambda\Gamma}) = \lambda^{p-n}\overline{\lambda}^{q-m} a^{p,q}_{m,n}(\Gamma|\nu,\chi).$$

□

Remark 6.4.4. *The key tool in establishing* (6.43) *is the hypothesis* (6.44)*. The existence of such lattice's functions $\nu = \nu_{\Gamma}$ and $\chi = \chi_{\Gamma}$ is assured by considering for example*

$$\nu = \nu_{\Gamma} = \frac{2i\pi(\overline{\omega_1}^k\omega_2^k - \omega_1^k\overline{\omega_2}^k)}{\overline{\omega_1}^{k+1}\omega_2^{k+1} - \omega_1^{k+1}\overline{\omega_2}^{k+1}}$$

and the Weierstrass pseudo-character

$$\chi_{\Gamma}(\gamma) = \begin{cases} +1 & \text{if} \quad \gamma/2 \in \Gamma \\ -1 & \text{if} \quad \gamma/2 \notin \Gamma \end{cases}.$$

Such ν and χ are not unique.

Remark 6.4.5. *The generalized complex Hermite-Gauss coefficients $a^{p,q}_{m,n}(\Gamma|\nu,\chi)$ are nontrivial examples of the so-called lattice's functions according the Serre terminology [73, Chap. VII].*

We conclude this section by proving further interesting properties satisfied by the generalized complex Hermite-Gauss coefficients $a^{p,q}_{m,n}(\Gamma|\nu,\chi)$.

Lemma 6.4.6. *For any pseudo-character χ and every integers m,n,p,q, we have*

$$\overline{a^{p,q}_{m,n}(\Gamma|\nu,\chi)} = (-1)^{m+n+p+q} a^{p,q}_{m,n}(\Gamma|\nu,\chi). \tag{6.47}$$

Moreover, the generalized complex Hermite-Gauss coefficients $a^{p,q}_{m,n}(\Gamma|\nu,\chi)$ satisfy the following recurrence formulas

$$\nu a^{p,q+1}_{m,n}(\Gamma|\nu,\chi) = a^{p,q}_{m+1,n}(\Gamma|\nu,\chi) + \nu n a^{p,q}_{m,n-1}(\Gamma|\nu,\chi). \tag{6.48}$$

and

$$\nu a^{p+1,q}_{m,n}(\Gamma|\nu,\chi) = a^{p,q}_{m,n+1}(\Gamma|\nu,\chi) + \nu m a^{p,q}_{m-1,n}(\Gamma|\nu,\chi). \tag{6.49}$$

Proof. The proof of the symmetry relationship (6.47) is similar to the one provided for (6.42). We need to use the fact $\overline{\chi(\gamma)} = \chi(-\gamma)$ combined with $H^{\nu}_{m,n}(-\xi,-\overline{\xi}) = (-1)^{m+n}\overline{H^{\nu}_{n,m}(\xi,\overline{\xi})}$. The recurrence formula (6.48) is an immediate consequence of the three-term recurrence formula

$$H^{\nu}_{m+1,n}(\xi,\overline{\xi}) = \nu\overline{\xi} H^{\nu}_{m,n}(\xi,\overline{\xi}) - \nu n H^{\nu}_{m,n-1}(\xi,\overline{\xi})$$

satisfied by the complex Hermite polynomials ([29]). While (6.49) follows easily from (6.48) by taking the complex conjugate, taking into account (6.47). □

6.5 Question and remarks

Notice finally that, for the data of any (Γ,χ,ν) (not necessary satisfying the pseudo-character property (6.2)), we can define the quantities $a_{m,n}(\Gamma|\nu,\chi)$ and in particular

$$a_{0,0}(\Gamma|\nu,\chi) := \sum_{\gamma\in\Gamma} \chi(\gamma) e^{-\frac{\nu}{2}|\gamma|^2}.$$

We conjecture that

$$\sum_{\gamma\in\Gamma} \chi(\gamma) e^{-\frac{\nu}{2}|\gamma|^2} = 0 \Longleftrightarrow \nu = \frac{\pi}{S(\Gamma)}$$

for any given real-valued χ. This is a characterization of von Neumann lattices. In fact, from numerical observation, we assert that $\sum\limits_{\gamma\in\Gamma} \chi(\gamma) e^{-\frac{\nu}{2}|\gamma|^2}$ is positive when $\nu > \pi/S(\Gamma)$, vanishes when $\nu = \pi/S(\Gamma)$ and negative otherwise. When (6.2) is satisfied, the space $\mathcal{O}^{\nu}_{\Gamma,\chi}(\mathbb{C})$ is nontrivial and $a_{0,0}(\Gamma|\nu,\chi)$ coincides with $K^{\nu}_{\Gamma,\chi}(0,0)$. Moreover, we have $\nu = k\pi/S(\Gamma)$ for $k = 1,2,\cdots$.

By numerical approach, it seems that the lattice sum

$$\sum_{\gamma\in\Gamma} \chi(\gamma) e^{-\frac{\nu}{2}|\gamma|^2} = \sum_{m,n=-\infty}^{+\infty} e^{\frac{-\nu}{2}\left(m^2+n^2\right)} =: f(\nu)$$

viewed as function in $\nu \in \mathbb{R}^{*+}$ and associated with $\Gamma = \mathbb{Z} + i\mathbb{Z}$ and $\chi \equiv 1$, is increasing with upper bound equals to 1 and with no lower bound.

values of ν	Evaluation
0.001	-911.395647437
0.01	-100
0.1	$-9.9999939719299899993142273906296 44$
1	0
1.25	0.3608381973529082048783912510840871280853
1.5	0.5852075679820198541304883766877041518512
1.75	0.7276806878628701459775102868169982969575
2	0.8196872998200458995950539646962870101812
3	0.9637440634268266347151459429784258329968
4	0.9925162797522077475130738581932461342102
8	0.9999860505819289371809872750653816298146
8×10	0.9999999999999999999999999999999999999999

The above values are calculated using the following Mathematica code

Code 1: Numerical evaluation of the lattice sum

```
NSum[(−1)^(i+j+i∗j)∗Exp[(−\[Nu]∗Pi)/2∗(i^2+j^2)],{i,−Infinity,Infinity},{j,−Infinity, Infinity },NSumTerms−>30,
    Method−>"AlternatingSigns",WorkingPrecision−>40]
```

From the summation formulas

$$\sum_{\gamma\in\Gamma} \chi(\gamma) e^{-\frac{\nu}{2}|\gamma|^2} = 0, \tag{6.50}$$

when $\nu = \pi/S(\Gamma)$, and

$$\sum_{\gamma\in\Gamma} \chi(\gamma)\gamma^p \overline{\gamma}^q e^{-\frac{\nu}{2}|\gamma|^2} H^{\nu}_{m,n}(\gamma,\overline{\gamma}) = 0$$

for real-valued χ and integers m,n,p,q such that $m+n+p+q$ is odd (see (6.36) in Theorem 6.4.1), one can drive nice identities involving special functions such as *Jacobi theta functions* (see for example [53]). The first summation taken in the case $\Gamma = \mathbb{Z} + i\mathbb{Z}$ gives rise to the following identity

$$\vartheta_2(0,e^{-2\nu})^2 - \vartheta_3(0,e^{-2\nu})^2 - 2\vartheta_2(0,e^{-2\nu})\vartheta_3(0,e^{-2\nu}) = 0. \tag{6.51}$$

Indeed, by splitting such summation over: i) both m and n are odd, ii) both m and n are even and lastly, iii) m and n are of different parity, and next using the following equalities

$$\begin{aligned}
\text{i)} \sum_{m=-\infty}^{\infty}\sum_{n=-\infty}^{\infty} e^{\frac{-\nu}{2}\left((2m+1)^2+(2n+1)^2\right)} &= \nu^{-\frac{1}{2}}\left(\frac{\pi}{2}\right)^{\frac{1}{2}} \vartheta_2(0,e^{-2\nu})\vartheta_3\left(\frac{\pi}{2}, e^{-\frac{\pi^2}{2\nu}}\right) \\
&= \vartheta_2(0,e^{-2\nu})^2. \\
\text{ii)} \sum_{m=-\infty}^{\infty}\sum_{n=-\infty}^{\infty} e^{\frac{-\nu}{2}\left((2m)^2+(2n)^2\right)} &= \vartheta_3(0,e^{-2\nu})^2. \\
\text{iii)} \sum_{m=-\infty}^{\infty}\sum_{n=-\infty}^{\infty} e^{-\nu\left((2m)^2+(2n+1)^2\right)} &= \vartheta_2(0,e^{-2\nu})\vartheta_3(0,e^{-2\nu}).
\end{aligned}$$

The splitting was suggested by the distribution of the signs of the components inside the double summation showed in the following graph. The proof of the identity (6.51) requires detailed knowledge of the relationships between Jacobi theta functions. The proof of the more complicated identities of the type (2.11) by direct methods is evidently nontrivial. This is, word by word, the same remark asserted by Perelomov in [64].

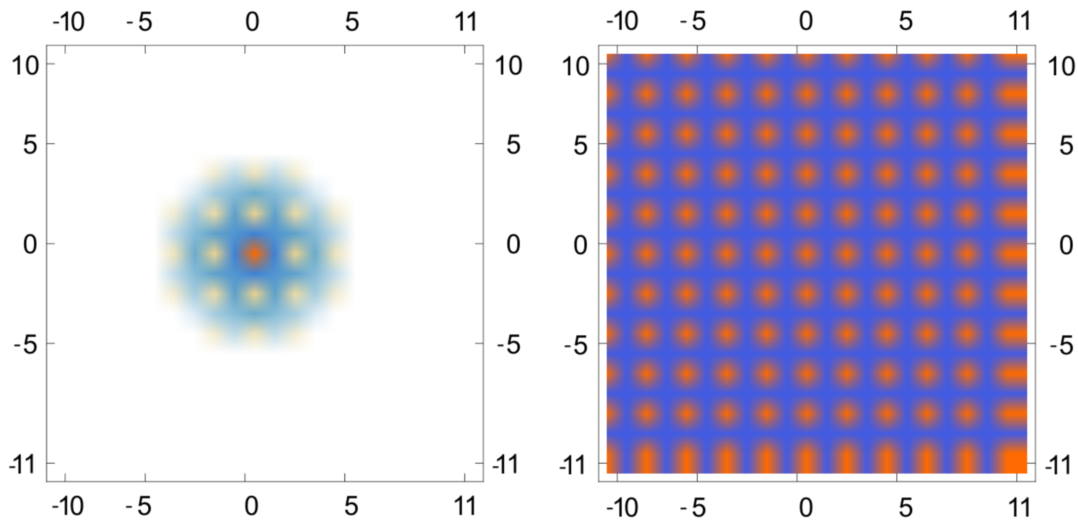

Figure 6.1: – This figures the distribution of the sign of components in (6.50)

Appendices

SOME CODE USED IN THIS THESIS

In this annex chapter we give the verbatim code of some numerical computations and figures used in this thesis. If you are interested in reusing the codes listed in this chapter you may need to retype them, because copying directly from the pdf, by experience, in changing some Unicode characters to some nonstandard ones, which generates compilation errors.

Verbatim TikZ code to draw lattices and their fundamental domains

Here I give the LaTeX source code for drawing a lattice using TikZ. The code is extracted from Figure 1.2 showing type of lattices. Note that, a good practice in implementing figures is to draw your figure in a separate tex file with the document class `standalone`, and, after compilation, include the pdf output in your main document using simply `\includegraphics[ ]{filenameWithoutBlankSpace.pdf}` after loading `graphicx` package.

Code 2: TikZ code to draw a lattice with shape emphasis

```
\documentclass{standalone}
\usepackage{tikz}
\usetikzlibrary{calc,quotes,angles}

\begin{document}
\begin{minipage}{14cm} % This is just for making many tikzpicture in the same line. The 14cm is generic, you can use
    instead e.g.  \textwidth
\hfil % hfil is an horizontal dynamic spacing to separate between elements evenly
\begin{tikzpicture}[scale=0.3,font=\scriptsize] % O B L I Q U E
% Set coordinates of the parallelogram generated by omega_1 and omega_2
\coordinate (A) at (0,0);
\coordinate (B) at (0:3); %Omega_1
\coordinate (D) at (70:2); %Omega_2
\coordinate (C) at ($(B) +(D)$);
%Lattice creation using foreach loop
\foreach \a in {-1,-0,...,2.1}
\foreach \b in {-1,-0,...,2.1}
\shade [ball color = gray] ($\a*(B) + \b*(D)$) circle[radius=7pt];
% Draw the generators of the lattice omega_1 and omega_2 with their nods
\draw[thick,>=stealth,->] (0,0) -- (B) node[midway,below,sloped] {$\omega_1$};
\draw[thick,>=stealth,->] (0,0) -- (D) node[midway,above,sloped] {$\omega_2$};
% Draw fundamental parallelogram
\draw (A) -- (B) -- (C) -- (D) -- cycle;
% Drow the angle phi
\draw pic["$\varphi$",draw,angle eccentricity=1.6,angle radius=3mm] {angle=B--A--D};
% Show the type of lattice with the green like color
\draw[dashed,thick, fill=green!30!yellow, fill opacity=0.2] (A) -- (B) -- (C) -- (D) -- cycle;
% The caption
\node [below=1cm, align= flush center] at ($(1,-1.5)$) {1. Oblique\\$|\omega_1| \neq |\omega_2|$, and $\varphi
    \neq \frac{\pi}{2}$};
\end{tikzpicture}
\hfil
\end{minipage}
\end{document}
```

Draw the graph of the automorphic function basis

In Chapter 2 we gave a basis for holomorphic automorphic functions. Here we explain how to draw the graphs given in Figure 2.1, which were made using the `plot` package from Maple. The drawing of the graphs inside Maple itself is simple but I have found some difficulties in exporting them. I was hoping to export the graphs in a vectorized format (e.g. PostScript or pdf), but after many attempt the output was unsatisfactory, the Maple engine to create such formats is not working properly. My final refuge was to export in png with increased resolution. The main trick is: to first visualize your plotting on Maple, keeping the `plotsetup` as default. When satisfied by the result using the toolbox provided by Maple user interface, retain all the specifics given by the toolbox such as the orientation angle. Then, change the `plotsetup` to the "export mode" (see the code below). Lastly, re-execute the plot with all the specifics you have retained. The image is saved in the project directory as a png file.

Code 3: Maple code for drawing and exporting a complex function graph

```
>   with(plots):
>   tau   := [0+1*I]: # Lattice variable
     S  :=Im(tau(1)):  # Area of a fundamental domain
     alpha := 0: # Assiciated to the psudo-character at 1
     beta  := 0: # Assiciated to the psudo-character at tau
     j  := 1: # The index for elements in the basis
     N  := 2: # Dimension of the first landau level
# The "export mode": This line readjusts plot setup to write the plot execution into an external png file .
# Note that the plot execution will not be printed from now on
>   plotsetup(png, plotoutput = "BasisN1J1.png", plotoptions = ‘portrait,  noborder, width=960, height=970,
    leftmargin=0.5in, bottommargin=0in‘):
# This is the plot of the function with some specifics I retained wile plotting in the default plotsetup
>   plot3d(abs(G(x, y)), x=0..1, y= 0..1, grid=[200, 200], style=surface, lightmodel=light2, orientation=[21, 61, 3],
     shading=xyz, axes=boxed, labels=["x", "y", "|&varphi; |"], thickness=
1, axesfont=[TIMES, 30]);
# Notice that the axesfont size will seem large in the default setup but they are ok in the higher resolution export
    setup defined above
# Now to return to the default plot setup
>   plotsetup(default)
```

Routine using Maple to generate left or right invariant vector fields

The following code illustrate how we can use `DifferentialGeometry` package of Maple to compute left-invariant and right-invariant vector fields. We give below the code for the unidimensional real Heisenberg group, but can extended easily to higher dimensions and to the complex Heisenberg groups. Finally, I thank Torre 237 from MaplePrimes for helping me with the code.

Code 4: Maple code to compute invariant vector fields for Real Heiseberg group

```
>    with(DifferentialGeometry): with(LieAlgebras): with(GroupActions):
# We define a procedure giving the multiplication rule:
> MultRule := (x0, x1, x2, y0, y1, y2) -> [x0 + y0 -(1/1)*x1*y2 + (1/1)*x2*y1, x1 + y1,x2 + y2];
MultRule := (x0, x1, x2, y0, y1, y2) → [-x1 y2 + x2 y1 + x0 + y0, x1 + y1, x2 + y2]
>    L := MultRule(a, b, c, x0, x1, x2);
L := [-bx2 + cx1 + a + x0, b + x1, c + x2]
# First create a coordinate chart for the group.
>    DGsetup([x0, x1, x2], G);
"frame name: G"
# Create the left multiplication map by the group element [a, b, c].
>    L := MultRule(a, b, c, x0, x1, x2);
L := [-bx2 + cx1 + a + x0, b + x1, c + x2]
>    muL := Transformation(G, G, [x0 =L[1], x1 = L[2], x2 = L[3]]);
muL := [x0 = -bx2 + cx1 + a + x0, x1 = b + x1, x2 = c + x2]
# The infinitesimal generators of the left multiplication are the right-invariant vector fields .
# And the infinitesimal generators of the right multiplication are the left -invariant vector fields .
```

```
# Then Let us begin by defining the right multiplication rule by the group element [a, b, c].
>     muR := Transformation(G, G, [x0 = R[1], x1 = R[2], x2 = R[3]]);
muR := [x0 = bx2 - cx1 + a + x0, x1 = b + x1, x2 = c + x2]
>     R := MultRule(x0, x1, x2, a, b, c);
R := [bx2 - cx1 + a + x0, b + x1, c + x2]
# Now the left−invariant vector fields and the right−invariant vector fields are given as follow
>     LeftInvVectorFields := InfinitesimalTransformation(muR,[a, b, c] ,[a = 0,b = 0,c = 0]);
LeftInvVectorFields := (∂x0, x2 ∂x0 + ∂x1, -x1 ∂x0 + ∂x2)
>     RightInvVectorFields := InfinitesimalTransformation(muL,[a, b, c] ,[a = 0, b = 0,c = 0]);
RightInvVectorFields := (∂x0, -x2 ∂x0 + ∂x1, x1 ∂x0 + ∂x2)
```

LIST OF CODES

INDEX

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Abstract:

We provide in this thesis some generalizations of classical automorphic functions on $\mathbb{C}^n$ with weight ν. We parametrize the automorphic factor in two ways, the first one is by adding a second weight μ into the automorphic factor. These are called mixed automorphic functions of first kind. The second parametrization is given by multiplying the standard factor with its copy transported using an equivariant pair (ρ, τ). These functions are called mixed automorphic functions of second kind. On the spaces of mixed automorphic functions of each kind, obeying a suitable growth condition, we realize the appropriate invariant Laplacians. We show that these spaces are isospectral where the spectrum reduces to discrete eigenvalues. By doing so, we provide a spectral decomposition and construct an orthogonal Hilbert basis for these spaces of mixed automorphic functions.

In the last part of this thesis, we investigate some questions concerning the construction and applications of automorphic functions. More precisely, we study the kernel of Poincaré theta series operator. Last but not least, we study the reproducing kernel function of the space of holomorphic automorphic functions, which happens to be the Poincaré periodization of Fock reproducing kernel function. Thereby, we derive interesting arithmetical identities that appear in chemistry and physics. This provides a novel application of automorphic functions in mathematics as well as in other scientific fields.

Keywords: *Automorphic functions; Mixed automorphic functions; Spectral theory; Magnetic Laplacians; Poincaré Theta operator; Arithmetical identities; Lattice Sums.*

Résumé :

Nous proposons dans cette thèse des généralisations des fonctions automorphes classiques sur $\mathbb{C}^n$ à poids ν. Nous avons paramétré le facteur d'automorphie de deux façons, la première était par ajouter un deuxième poids μ dans le facteur d'automorphie. Nous appelons les fonctions résultantes les fonctions automorphes « mixed » de première espèce. La deuxième paramétrisation est faite par multiplier le facteur d'automorphie usuel par sa copie transportée selon un pair équivariant (ρ, τ). Nous appelons ces fonctions les fonctions automorphes mixed de seconde espèce. Sur les espaces des fonctions automorphes mixed de chaque espèce, obéissant à une condition de croissance, nous réalisons les Laplaciens invariants convenables. Nous montrons par la suite que ces espaces sont isospectraux où le spectre se réduit à des valeurs propres discrètes. Par conséquent, nous fournissons une décomposition spectrale et construisons une base d'Hilbert orthogonale pour ces espaces.

Dans la dernière partie de cette thèse, nous étudions quelques questions concernant la construction et les applications des fonctions automorphes. Plus précisément, nous étudions le noyau de l'opérateur thêta de Poincaré. Enfin, nous traitons la fonction noyau reproduisant de l'espace des fonctions automorphes holomorphes, qui est exactement la périodisation à la Poincaré de la fonction noyau reproduisant de l'espace de Fock. Par conséquent, nous trouvons des identités arithmétiques intéressantes qui apparaissent en chimie et en physique. Cela nous fournit une nouvelle application des fonctions automorphes en mathématiques ainsi qu'en d'autres domaines scientifiques.

Mots-clés : *Fonctions automorphes; Fonctions automorphes mixed; Théorie spectrale; Laplaciens magnétiques; Opérateur thêta de Poincaré; Identités arithmétiques; Sommation sur les réseaux.*

Academic year: 2017 – 2018.

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