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Sujet:

EXTENSION DE CERTAINES PROPRIÉTÉS ALGÉBRIQUES À LA THÉORIE DES ANNEAUX GRADUÉS.

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EXTENSION OF SOME ALGEBRAIC PROPERTIES TO GRADED RING THEORY

Dédicace

C'EST AVEC UN ÉNORME PLAISIR,

UNE RECONNAISSANCE ÉTERNELLE

UNE PROFONDE GRATITUDE,

ET UN SINCÈRE AMOUR,

JE DÉDIE CE TRAVAIL ...

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(3) A. Assarrar, N. Mahdou, Ü. Tekir and S. Koç, *Commutative Graded-S-coherent rings*, Czechoslov. Math. J., 73 (2) (2023), 553-564.

(4) A. Assarrar and N. Mahdou, *On graded n-irreducible ideals of commutative graded rings*, Commun. Korean Math. Soc., 38 (4) (2023), 1001-1017.

(5) A. Assarrar and N. Mahdou, *The Generalized principal ideal theorem subject to Graded-Nonnil-Noetherian commutative rings*, Submitted for publication.

Other papers/Autres articles:

(1) A. Assarrar and N. Mahdou, *On a weak version of Graded-coherent rings*, Submitted for publication.

(2) A. Assarrar, H. Kim and N. Mahdou, *On Graded strongly α-n-irreducible ideals*, Submitted for publication.

(3) A. Assarrar, N. Mahdou Ü. Tekir and S. Koç, *On some weak versions of graded-Noetherian modules and rings*, Submitted for publication.

Summary (In French)

En algèbre commutative, notamment dans la théorie des anneaux gradués, l'étude des propriétés gradués est considérée primordiale pour le développement de plusieurs structures en algèbre ainsi qu'en géométrie.

Dans cette thèse, nous généralisons l'étude de certains anneaux et idéaux particuliers au contexte de la théorie des anneaux gradués (les anneaux n-cohérents, les anneaux S-cohérents, les idéaux n-irréductibles et les anneaux non-nils-Noethériens). De plus, nous étudions le transfert possible de certaines de ces propriétés à quelques constructions spécifiques d'anneaux gradués. Cette thèse se compose de cinq chapitres couvrant cinq articles, dont quatre sont publiés [10, 11, 12, 13].

Dans les trois premiers chapitres, nous nous intéressons à l'étude de la version graduée de certaines propriétés liées à la notion d'anneau cohérent. Le premier chapitre présente les résultats de transfert de certaines propriétés de type cohérent qui existent déjà dans la littérature (gradué-Noethérien, gradué-cohérent) et d'autres qu'on vient d'introduire comme les anneaux gradués-v-cohérents, gradués-quasi-cohérents et gradués de conducteurs finis) à certaines extensions triviales d'anneaux gradués spécifiques. Notre objectif est de fournir de nouvelles classes d'anneaux gradués commutatifs satisfaisant ces propriétés. Le deuxième chapitre se porte sur la généralisation de la notion n-cohérence au contexte des anneaux et modules gradués commutatifs. Dans le troisième chapitre, nous généralisons l'étude des anneaux et modules S-cohérents au contexte de la théorie des anneaux gradués. Le chapitre 4 traite les versions graduées des notions d'idéaux n-irréductibles et fortement n-irréductibles. Enfin, le dernier chapitre de cette thèse est consacré à l'étude de la version graduée des anneaux nonnil-Noethériens. Nos résultats génèrent de nouveaux exemples et contre exemples originaux d'anneaux et d'idéaux gradués satisfaisant ces propriétés.

La thèse s'achève par quelques perspectives que nous souhaitons aborder dans nos travaux futurs.

Mots clés: gradué-Noethérien, gradué-Artinien, gradué-cohérent, gradué-v-cohérent, gradué-quasi-cohérent, gradué-de conducteur-fini, gradué-n-cohérent, gradué-S-cohérent, gradué-n-irréductible, gradué-nonnil-Noethérien, extension triviale gradué.

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Summary

In commutative algebra, especially in the theory of graded rings, the study of graded properties is considered primordial to the development of several structures in algebra as well as in geometry.

In this thesis, we generalize the study of some particular rings and ideals to the context of graded ring theory (namely, n-coherent rings, S-coherent rings, n-irreducible ideals and nonnil-Noetherian rings). Moreover, we investigate the possible transfer of some of these properties to some specific graded ring constructions. This thesis consists of five chapters covering five papers, of which four are published [10, 11, 12, 13].

In the first three chapters, we are interested in the study of the graded version of certain properties related to the notion of coherent ring. The first chapter present the results of transfer of some coherent-like properties that already exist in the leterature (graded-Noetherian, graded-coherent) and others that we introduced (graded-v-coherent, graded-quasi-coherent and graded-finite conductor rings) to some specific trivial graded ring extensions. Our aim is to provide new classes of commutative graded rings satisfying these properties. The second chapter focuses on the generalization of n-coherent notion to the context of commutative graded rings and modules. In the third chapter, we generalize the study of S-coherent rings and modules to the context of graded ring theory. Chapter 4, deals with the graded versions of the notions of n-irreducible and stongly n-irreducible ideals. Finally, the last chapter of this thesis is devoted to the study of the graded version of nonnil-Noetherian rings. Our results generate new original examples and counter-examples of graded rings and graded ideals satisfying these properties.

This thesis ends with some perspectives that we wish to discuss in our future works.

Keys words: graded-Noetherian, graded-Artinian, graded-coherent, graded-v-coherent, graded-quasi-coherent, graded-finite-conductor, graded-n-coherent, graded-S-coherent, graded-n-irreducible, graded-nonnil-Noetherian, graded trivial extension.

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Introduction (In French)

La notion de graduation, du moins sous la forme d'exemples particuliers, est une notion assez ancienne. Elle est entrée explicitement en mathématique au 18ème siècle avec Euler. Euler appelle un polynôme (d'une ou de plusieurs variables) homogène si tous ses monômes ont un même degré total, qui est dit le degré de ce polynôme. Ensuite, ils ont considéré (surtout en géométrie et topologie algébrique, par exemple Samuel et Zariski [54]), la notion générale de l'anneau $\mathbb{Z}$-gradué (ou des anneaux avec une graduation un peu plus générale $\left(\mathbb{Z}^2, \mathbb{Z}^3, \cdots, \mathbb{Z}^n, \cdots\right)$). Les anneaux gradués, dont les grades sont contenus dans un groupe abélien, ont été considérés par Chevalley [23]. Mais la première notion relativement (mais pas assez) générale d'anneau gradué a été introduite par Bourbaki [21].

Une première lecture à la théorie des modules gradués sur un anneau gradué pourrait donner l'impression qu'elle n'est rien d'autre qu'une théorie de module ordinaire avec toutes ses déclarations décores avec l'adjectif "gradué". Une fois que la graduation est considérée comme triviale, la théorie graduée se réduit à la théorie des modules ordinaires. Donc, de ce point de vue, la théorie des modules gradués peut être considérée comme une extension de la théorie des modules. Toutefois, cette vision simpliste pourrait masquer le fait que les modules gradués sont équipés d'un déplacement, grâce à la possibilité de partitionner les structures puis de réarranger les partitions. Cela donc, ajoute un avantage supplémentaire à la structure et une complexité à la théorie.

D'autre part, et d'un point de vue géométrique, la graduation est un outil algébrique très puissant qui a permet aux mathématiciens de construire des structures algébriques très précieux qui ont facilité l'étude et la compréhension de plusieurs objets géométriques soit en géométrie algébrique, citons par exemple "les variétés projectifs" basées dans leur constuction sur les polynômes homogènes, et même en géométrie différentielle, en particulier en géométrie Reimannienne à savoir les formes extérieures et les formes différentielles qui sont des algèbres graduées qui élicitent leur graduation de celle définie sur le produit tensoriel des modules gradués.

L'un des problèmes les plus importants de la théorie des anneaux gradués est d'étudier certaines propriétés qui sont déjà connues dans le cas non gradué et comme il est annoncé dans le titre, l'objectif principal de cette thèse est d'étendre et étudier certaines propriétés algébriques dans le contexte de la théorie des anneaux gradués, qui fait également l'objet de divers travaux récents réalisés par plusieurs auteurs.

Introduction

The notion of graduation, at least in the form of particular examples, is a rather old one. It entered mathematics explicitly in the 18th century with Euler. Euler defined a polynomial (of one or more variables) to be homogeneous if all its monomials have the same total degree, which is called the degree of that polynomial. Then they considered (especially in geometry and algebraic topology, for example Samuel and Zariski [54]) the general notion of the $\mathbb{Z}$-graduated ring (or rings with the slightly more general graduation $\left(\mathbb{Z}^2, \mathbb{Z}^3, \cdots, \mathbb{Z}^n, \cdots\right)$. Graded rings, whose degrees are contained in an abelian group, were considered by Chevalley [23]. But the first relatively (but not sufficiently) general notion of graded ring was introduced by Bourbaki [21].

A first view to the theory of graded modules over a graded ring might give the impression that it is nothing but ordinary module theory with all its statements decorated with the adjective graded. Once the grading is considered to be trivial, the graded theory reduces to the usual module theory. From this perspective, the theory of graded modules can be considered as an extension of module theory. However, this simplistic overview might conceal the point that graded modules come equipped with a shift, thanks to the possibility of partitioning the structures and then rearranging the partitions. This adds an extra layer of structure and complexity to the theory.

On the other hand, and from a geometrical point of view, graduation is a very powerful algebraic tool that has allowed mathematicians to build very valuable algebraic structures that have facilitated the study and understanding of several geometric objects, either in algebraic geometry, such as "projective varieties", based on their construction exactly on homogeneous polynomials, and even in differential geometry, especilly in Riemannian geometry such as the exterior forms and the differential forms which are a graded algebras which elicite their graduation by that defined on the tensor product of a graded modules.

One of the most important problems of the theory of graded rings is to study some properties that are already known in the ungraded case and vice versa and as it is announced in the title, the main purpose of this thesis is to extend and study some algebraic

properties in the context of graded ring theory, which is also the subject of various recent works made by several authors.

Preliminaries (in French)

Tous les anneaux considérés ci-dessous sont supposés commutatifs et unitaires, tous les modules sont unitaires, G désignera un monoïde commutatif noté additivement, avec un élément identité noté 0*, et tous les anneaux et modules gradués, sauf indication contraire, sont gradués par G. Dans ce chapitre, nous rappelons quelques éléments de base incluant les définitions de base, les propriétés et les résultats généraux de la théorie des anneaux gradués. (voir par exemple [21], [32] et [45])*

• **Anneaux gradués**

Soit G un monoïde commutatif noté additivement ayant un élément identité noté 0. Par anneau gradué A, on entend un anneau gradué par G, c'est-à-dire une somme directe des sous-groupes A_α de A tels que $A_\alpha A_\beta \subseteq A_{\alpha+\beta}$ pour chaque α, $\beta \in G$. L'ensemble $h(A) = \cup_{\alpha\in G} A_\alpha$ est l'ensemble des éléments homogènes de A. Un élément non nul $x \in A$ est dit homogène s'il appartient à l'un des A_α, homogène de degré α si $x \in A_\alpha$. Chaque $z \in A$ peut être écrit de manière unique comme une somme $z = z_{\alpha_i} + \cdots + z_{\alpha_n}$ d'éléments homogènes $z_{\alpha_i} \in A_{\alpha_i}$ où $\alpha_1, \cdots, \alpha_n$ sont distincts ; z_{α_i} est appelé la composante homogène de degré α_i de z. Chaque anneau A est gradué par une graduation triviale en prenant $A_0 = A$ et $A_\alpha = \{0\}$ pour tout $\alpha \neq 0 \in G$. Le monoïde G est dit simplifiable si tout élément α de G est simplifiable, au sens où $\alpha + \lambda = \alpha + \mu$, pour tous $\lambda, \mu \in G$ implique que $\lambda = \mu$. Si G est simplifiable, alors A_0 est un sous-anneau de A (intuitivement $1 \in A_0$) et chaque A_α est un A_0-module.

• **Modules gradués**

Par un A-module gradué M, on entend un A-module gradué par G, c'est-à-dire une somme directe de sous-groupes M_α de M tel que $A_\alpha M_\beta \subseteq M_{\alpha+\beta}$ pour chaque $\alpha, \beta \in G$. L'ensemble $h(M) = \cup_{\alpha\in G} M_\alpha$ est l'ensemble des éléments homogènes de M. Un A-module gradué M est un A-module gradué-libre (gr-libre) s'il existe une base $(e_i)_{i\in I}$ de M constituée d'éléments homogènes. On note que tout A-module gradué-libre est un A-module libre; mais la réciproque est fausse, voir [45, p. 21]. Si G est simplifiable, les M_α sont des A_0-modules. Évidemment, A est un A-module gradué.

• Homomorphismes gradués

Soient A et A' deux anneaux gradués. Un homomorphisme d'anneaux $f : A \to A'$ est dit gradué si $f(A_\alpha) \subseteq A'_\alpha$ pour tout $\alpha \in G$. Un isomorphisme d'anneaux gradués est un homomorphisme d'anneaux gradués bijectif. Soit M et M' deux A-modules gradués et soit $v : M \to M'$ un homomorphisme de A-modules et $\beta \in G$; v est appelé gradué de degré β si $v(M_\alpha) \subseteq M'_{\alpha+\beta}$ pour tous les $\alpha \in G$. Un homomorphisme de A-modules $v : M \to M'$ est dit gradué s'il existe $\beta \in G$ tel que v soit gradué de degré β. Un isomorphisme de A-modules gradués est un homomorphisme bijectif gradué de degré 0. Si $v \neq 0$ et G est simplifiable, le degré de v est alors déterminé de manière unique. Une suite exacte de A-modules gradués est une suite exacte, où les A-modules et les homomorphismes de A-modules en question sont gradués.

• Sous-modules gradués et idéaux gradués

Un sous-module N de M est dit gradué (ou homogène*) si $N = \oplus_{\alpha \in G}(N \cap M_\alpha)$. On note que les assertions suivantes sont équivalentes pour un sous-module N de M : (1) N est gradué ; (2) les composantes homogènes de tout élément de N appartiennent à N ; (3) N est engendré par des éléments homogènes. Un sous-module gradué de A est appelé un idéal gradué de A.

• Structure du quotient gradué

Si N est un sous-module gradué d'un A-module M gradué, alors M/N est un A-module gradué, où $(M/N)_\alpha := (M_\alpha + N)/N$. Si I est un idéal gradué d'un anneau gradué A, alors A/I est un anneau gradué, où $(A/I)_\alpha := (A_\alpha + I)/I$.

• L'image et le noyau des homomorphismes gradués

Soit A un anneau gradué et $u : M \to M'$ un homomorphisme gradué de A-modules de degré α. $Im(u)$ est un sous-module gradué de N. Si α est un élément simplifiable de G, alors $\ker(u)$ est un sous-module gradué de M. Si $\alpha = 0$, alors la bijection $M/\ker(u) \to \mathrm{Im}(u)$ canoniquement associée à u est un isomorphisme de A-modules gradués. Soient A, B deux anneaux gradués et $u : A \to B$ un homomorphisme d'anneaux gradués. Alors $\mathrm{Im}(u)$ est un sous-anneau gradué de B, $\ker(u)$ un idéal gradué de A et la bijection $A/\ker(u) \to \mathrm{Im}(u)$ canoniquement associée à u est un isomorphisme d'anneaux gradués.

• Opérations sur les sous-modules gradués

Soient A un anneau gradué et M un A-module gradué. Chaque somme et chaque intersection de sous-modules gradués de M est un sous-module gradué. Soit A un anneau gradué, M un A-module gradué et N un sous-module de M. Alors il existe un plus grand sous-module gradué N' de M contenu dans N et un plus petit sous-module gradué N'' de M contenant N.

*Cette appellation est adoptée dans les trois premiers chapitres.

• L'annulateur de sous-modules gradués

Soient A un anneau gradué et M un A-module gradué. Si x est un élément homogène de M de degré simplifiable α, alors l'annulateur $(0:_A x)$ de x est un idéal gradué de A. Si G est simplifiable, alors l'annulateur $(0:_A N)$ d'un sous-module gradué N de M est un idéal gradué de A.

• Produit direct d'anneaux gradués

Soient A_1 et A_2 deux anneaux gradués. Alors $A = A_1 \times A_2$ est un anneau gradué dont les éléments homogènes sont l'ensemble $h(A) = \cup_{\alpha\in G} A_\alpha$, où $A_\alpha = (A_1)_\alpha \times (A_2)_\alpha$ pour tous les $\alpha \in G$. On note qu'un idéal de $A_1 \times A_2$ est de la forme $I_1 \times I_2$ pour certains idéaux I_1 de A_1 et I_2 de A_2. On voit aussi facilement que $I_1 \times I_2$ est un idéal gradué de $A_1 \times A_2$ si et seulement si I_1, I_2 sont des idéaux gradués de A_1 et A_2, respectivement.

• Localisation des anneaux gradués

Soit A un anneau gradué et soit M un A-module gradué. Si le monoïde gradué G est un groupe et si S est un ensemble multiplicative d'éléments homogènes de A, alors $S^{-1}A$ est un anneau gradué et $S^{-1}M$ est un $S^{-1}A$-module gradué, où $\left(S^{-1}A\right)_\alpha = \left\{\frac{a}{s} \mid a \in A_\beta, s \in A_\delta \cap S \text{ et } \beta - \delta = \alpha\right\}$ et $\left(S^{-1}M\right)_\alpha = \left\{\frac{m}{s} \mid m \in M_\beta, s \in A_\delta \cap S \text{ et } \beta - \delta = \alpha\right\}$.

• Produit tensoriel des modules gradués

Soit A un anneau gradué et soit M et M' deux A-modules gradués. Définissons $(M \otimes_A M')_\alpha$ comme le groupe additif de $M \otimes_A M'$ généré par le produit $m_\mu \otimes m'_\nu$, où $m_\mu \in M_\mu$, $m'_\nu \in M'_\nu$ et $\mu + \nu = \alpha$. Alors $((M \otimes_A M')_\alpha)_{\alpha\in G}$ est une graduation de $M \otimes_A M'$ et $M \otimes_A M'$ est un A-module gradué.

• Limite inductive des anneaux gradués

Un système direct $\left(A_\lambda, \phi_{\mu\lambda}\right)$ d'anneaux gradués est un système direct d'anneaux tel que chaque A_λ est un anneau gradué et chaque $\phi_{\mu\lambda}$ est un homomorphisme d'anneaux gradués. Si $\left(A_\lambda^\alpha\right)_{\alpha\in G}$ est la graduation de A_λ et si on a $A = \varinjlim A_\lambda$, alors $A^\alpha = \varinjlim A_\lambda^\alpha$, et $(A^\alpha)_{\alpha\in G}$ est une graduation de A et A est un anneau gradué. Si $\phi_\lambda : A_\lambda \longrightarrow A$ est l'application canonique, ϕ_λ est un homomorphisme d'anneaux gradués.

• Idéaux gradués-premiers, gradués-maximaux et gradués-minimaux

Soit I un idéal gradué, on dit que I est un idéal gradué-premier si $xy \in I$ pour un certain $x, y \in h(A)$, alors $x \in I$ ou $y \in I$, de manière équivalente, si A/I est un domaine-gradué, c'est-à-dire si tout élément homogène non nul de A/I est régulier. Notons que, lorsque G est un monoïde sans torsion (c'est-à-dire si G est simplifiable et que le groupe de fractions de G ; $< G >= \{a - b | a, b \in G\}$ est un groupe abélien sans torsion), alors I est premier si et seulement si I est un idéal gradué-premier. L'ensemble de tous les idéaux gradués-premiers de A sera noté gr-Spec(A). Un idéal gradué I est dit gradué-maximal si $I \neq A$ et s'il est maximal parmi les idéaux gradués, de façon équivalente, si A/I est un corps-gradué, c'est-à-dire si tout élément non nul homogène de A/I est inversible. Un anneau

gradué est dit gradué-local s'il a un unique idéal gradué-maximal. Évidemment, un corps est un corps-gradué, alors qu'un corps-gradué n'est pas nécessairement un corps, voir [45, page 46]. Un idéal gradué I est dit gradué-minimal s'il est minimal parmi les idéaux gradués. L'ensemble de tous les idéaux minimaux gradués de A sera noté gr-Min(A).

• **Localisation par idéaux gradués-premiers**

Si A est un anneau gradué, P est un idéal gradué-premier de A, et $S = h(A) \setminus P$ est l'ensemble multiplicatif saturé composé des éléments homogènes de $A \setminus P$, alors $S^{-1}A$ est un anneau gradué-local avec un unique idéal gradué-maximal $S^{-1}P$, $S^{-1}A$ est appelé la localisation graduée de A et sera noté $A_{[P]}$.

• **Anneaux gradués-Noéthériens et gradués-Artiniens**

Un A-module gradué M est appelé gradué-Noéthérien s'il satisfait la condition de chaine ascendante (a.c.c.) sur les sous-modules gradués; de manière équivalente, si chaque sous-module gradué de A est de type fini. Un anneau gradué A est appelé gradué-Noethérien s'il est gradué-Noethérien en tant que A-module gradué. Un A-module gradué M est appelé gradué-Artinien s'il satisfait la condition de chaine descendante (d.c.c.) sur les sous-modules gradués. Un anneau gradué A est appelé gradué-Artinien s'il est gradué-Artinien en tant que A-module gradué. Il est clair qu'un anneau Noethérien gradué est gradué-Noethérien et qu'un anneau Artinien gradué est gradué-Artinien mais l'inverse n'est pas vrai en général. (voir [32, Example 1.1.22])

• **La dimension de Krull graduée d'un anneau gradué**

La hauteur graduée d'un idéal gradué-premier P, notée $gr\text{-}ht(P)$, est définie comme la longueur de la plus longue chaine d'idéaux gradués-premiers contenus dans P. La dimension de Krull graduée d'un anneau gradué A est notée $gr\text{-}dim(A)$ et définie comme suit :

$$\text{gr-dim(A)} = \max\{\text{gr-ht(P)} \mid P \in \text{gr-Spec(A)}\}.$$

• **Le radical gradué**

Soit I un idéal gradué propre de A. Alors le radical gradué de I est noté $Gr(I)$ et il est défini comme suit :

$$Gr(I) = \left\{ a = \sum_{g\in G} a_g \ \in A : \forall g \in G,\ \exists n_g \in \mathbb{N}^* \ tel\ que\ a_g^{n_g} \in I \right\}.$$

Notez que $Gr(I)$ est un idéal gradué de A. Si G est un groupe, $Gr(I)$ est l'intersection de tous les idéaux gradués-premiers de A contenant I, et nous avons $\text{gr-Nil(A)} = \bigcap_{P\in gr-Spec(A)} P = Gr(0)$, voir [52, Proposition 2.5]. Nous renvoyons le lecteur à [52, Proposition 2.4] pour les propriétés de base du radical gradué. Selon [51, Definition 1.5], un idéal gradué propre I est dit gradué-primaire si $a, b \in h(A)$ avec $ab \in I$ alors $a \in I$ ou $b \in Gr(I)$.

Preliminaries

All rings considered bellow are assumed to be commutative and unitary, all modules are unitary, G will denote a commutative monoid written additively, with an identity element denoted by 0 *and all the graded rings and modules, unless otherwise stated, are graded by G. In this chapter, we recall some backgroung material including the basic definitions, properties and general results of graded ring theory, see for instance [21], [45] and [32].*

• Graded rings

Let G be a commutative monoid written additively with an identity element denoted by 0. By a graded ring A, we mean a ring graded by G, that is, a direct sum of subgroups A_α of A such that $A_\alpha A_\beta \subseteq A_{\alpha+\beta}$ for every α, $\beta \in G$. The set $h(A) = \cup_{\alpha\in G} A_\alpha$ is the set of homogeneous elements of A. A nonzero element $x \in A$ is called homogeneous if it belongs to one of the A_α, homogeneous of degree α if $x \in A_\alpha$. Every $z \in A$ may be written uniquely as a sum $z = z_{\alpha_i} + \cdots + z_{\alpha_n}$ of homogeneous elements $z_{\alpha_i} \in A_{\alpha_i}$ where $\alpha_1, \cdots, \alpha_n$ are distinct; z_{α_i} is called the homogeneous component of degree α_i of z. Every ring A is graded by a trivial graduation by taking $A_0 = A$ and $A_\alpha = \{0\}$ for all $\alpha \neq 0 \in G$. The monoid G is said to be cancellative if every element α of G is cancellable, in the sense that $\alpha + \lambda = \alpha + \mu$, $\lambda \in G$, $\mu \in G$ implies that $\lambda = \mu$. If G is cancellative, then A_0 is a subring of A (intuitionally $1 \in A_0$) and every A_α is an A_0-module.

• Graded modules

By a graded A-module M, we mean an A-module graded by G, that is, a direct sum of subgroups M_α of M such that $A_\alpha M_\beta \subseteq M_{\alpha+\beta}$ for every $\alpha, \beta \in G$. The set $h(M) = \cup_{\alpha\in G} M_\alpha$ is the set of homogeneous elements of M. A graded A-module M is called a graded-free A-module (gr-free) if there exists a basis $(e_i)_{i\in I}$ of M consisting of homogeneous elements. Note that, any graded-free A-module is a free A-module; the converse is false, see[45, p. 21]. When G is cancellative, the M_α are A_0-modules. Obviously, A is a graded A-module.

• **Graded homomorphisms**

Let A and A' be two graded rings. A ring homomorphism $f : A \to A'$ is called graded if $f(A_\alpha) \subseteq A'_\alpha$ for all $\alpha \in G$. A graded ring isomorphism is a bijective graded ring homomorphism. Let M and M' be two graded A-modules and let $v : M \to M'$ be an A-module homomorphism and $\beta \in G$; v is called graded of degree β if $v(M_\alpha) \subseteq M'_{\alpha+\beta}$ for all $\alpha \in G$. An A-module homomorphism $v : M \to M'$ is called graded if there exists $\beta \in G$ such that v is graded of degree β. A graded A-module isomorphism is a bijective graded A-module homomorphism of degree 0. If $v \neq 0$ and G is cancellative, the degree of v is, then determined uniquely. An exact sequence of graded A-modules is an exact sequence, where the A-modules and the A-module homomorphisms in question are graded.

• **Graded submodules and graded ideals**

A submodule N of M is called graded (or homogeneous[†]) if $N = \oplus_{\alpha\in G}(N \cap M_\alpha)$. It is well known that the following are equivalent for a submodule N of M: (1) N is graded; (2) the homogeneous components of every element of N belong to N; (3) N is generated by homogeneous elements. A graded submodule of A is called a graded ideal of A.

• **Graded quotient structure**

If N is a graded submodule of a graded A-module M, then M/N is a graded A-module, where $(M/N)_\alpha := (M_\alpha + N)/N$. If I is a graded ideal of a graded ring A, then A/I is a graded ring, where $(A/I)_\alpha := (A_\alpha + I)/I$.

• **The image and Kernel of graded homomorphisms**

Let A be a graded ring and $u : M \to M'$ a graded A-module homomorphism of degree α. $Im(u)$ is a graded submodule of N. If α is a cancellable degree in G, then $\ker(u)$ is a graded submodule of M. If $\alpha = 0$, then the bijection $M/\ker(u) \to \mathrm{Im}(u)$ canonically associated with u is a graded A-module isomorphism. Let A, B be two graded rings and $u : A \to B$ a graded ring homomorphism. Then $\mathrm{Im}(u)$ is a graded subring of B, $\ker(u)$ a graded ideal of A and the bijection $A/\ker(u) \to \mathrm{Im}(u)$ canonically associated with u is an isomorphism of graded rings.

• **Operations on graded submodules**

Let A be a graded ring and M a graded A-module. Every sum and every intersection of graded submodules of M is a graded submodule. Let A be a graded ring, M a graded R-module and N a submodule of M. Then there exist a largest graded submodule N' of M contained in N and a smallest graded submodule N'' of M containing N.

• **The annihilator of graded submodules**

Let A be a graded ring and M a graded A-module. If x is a homogeneous element of M of cancellable degree α, then the annihilator $(0 :_A x)$ of x is a graded ideal of A. If G is

[†]This naming is adopted in the first three chapters.

cancellative, then the annihilator $(0 :_A N)$ of a graded submodule N of M is a graded ideal of A.

• Direct product of graded rings

Let A_1 and A_2 be two graded rings. Then $A = A_1 \times A_2$ is a graded ring with homogeneous elements $h(A) = \cup_{\alpha\in G} A_\alpha$, where $A_\alpha = (A_1)_\alpha \times (A_2)_\alpha$ for all $\alpha \in G$. It is well known that an ideal of $A_1 \times A_2$ is of the form $I_1 \times I_2$ for some ideals I_1 of A_1 and I_2 of A_2. Also it is easily seen that $I_1 \times I_2$ is a graded ideal of $A_1 \times A_2$ if and only if I_1, I_2 are graded ideals of A_1 and A_2, respectively.

• Localization of graded rings

Let A be a graded ring and let M be a graded A-module. If the grading monoid G is a group and if S is a multiplicatively closed set of homogeneous elements of A, then $S^{-1}A$ is a graded ring and $S^{-1}M$ is a graded $S^{-1}A$-module, where $\left(S^{-1}A\right)_\alpha = \left\{\frac{a}{s} \mid a \in A_\beta, s \in A_\delta \cap S \text{ and } \beta - \delta = \alpha\right\}$ and $\left(S^{-1}M\right)_\alpha = \left\{\frac{m}{s} \mid m \in M_\beta, s \in A_\delta \cap S \text{ and } \beta - \delta = \alpha\right\}$.

• Tensor product of graded modules

Let A be a graded ring and let M and M' be two graded A-modules. Define $(M \otimes_A M')_\alpha$ as the additive group of $M \otimes_A M'$ generated by the product $m_\mu \otimes m'_\nu$, where $m_\mu \in M_\mu$, $m'_\nu \in M'_\nu$ and $\mu + \nu = \alpha$. Then $\left((M \otimes_A M')_\alpha\right)_{\alpha\in G}$ is a graduation of $M \otimes_A M'$ and $M \otimes_A M'$ is a graded A-module.

• Inductive limit of graded rings

A direct system $\left(A_\lambda, \phi_{\mu\lambda}\right)$ of graded rings is a direct system of rings such that each A_λ is a graded ring and each $\phi_{\mu\lambda}$ is a homomorphism of graded rings. If $\left(A_\lambda^\alpha\right)_{\alpha\in G}$ is the graduation of A_λ and if we have that $A = \varinjlim A_\lambda$, then $A^\alpha = \varinjlim A_\lambda^\alpha$, and $(A^\alpha)_{\alpha\in G}$ is a graduation of A and A is a graded ring. If $\phi_\lambda : A_\lambda \longrightarrow A$ is the canonical mapping, ϕ_λ is a homomorphism of graded rings.

• Graded-prime, Graded-maximal and graded-minimal ideals

Let I be a graded ideal, I is said to be a graded-prime ideal if whenever $xy \in I$ for some $x, y \in h(A)$, then $x \in I$ or $y \in I$, equivalently, if A/I is a graded-domain, that is, if every nonzero homogeneous element of A/I is regular. Note that, when G is a torsionless monoid (that is, if G is cancellative and the group of fractions of G; $< G >= \{a - b | a, b \in G\}$ is a torsionfree abelian group), then I is graded-prime if and only if I is a prime ideal. The set of all graded-prime ideals of A will be denoted by gr-Spec(A). A graded ideal I is said to be graded-maximal if $I \neq A$ and if it is maximal among graded ideals, equivalently, if A/I is a graded-field, that is, if every nonzero homogeneous element of A/I is invertible. A graded ring is said to be graded-local if it has a unique graded-maximal ideal. Obviously, a graded field is a gr-field, while gr-field need not be a field, see [45, page 46]. A graded ideal I is said to be graded-minimal if it is minimal among graded ideals. The set of all graded-minimal ideals of A will be denoted by gr-Min(A).

• **Localization using graded-prime ideals**

If A is a graded ring, P is a graded-prime ideal in A, and $S = h(A) \setminus P$ is the saturated multiplicative set consisting of the homogeneous elements of $A \setminus P$, then $S^{-1}A$ is a graded-local ring with unique graded-maximal ideal $S^{-1}P$, $S^{-1}A$ is said to be the graded-localization of A and will be denoted by $A_{[P]}$.

• **Graded-Noetherian and graded-Artinian rings**

A graded A-module M is called a graded-Noetherian if it satisfies the ascending chain condition (a.c.c.) on graded submodules; equivalently, if each graded submodule of A is finitely generated. A graded ring A is called graded-Noetherian if it is graded-Noetherian as a graded A-module. A graded A-module M is called graded-Artinian if it satisfies the descinding chain condition (d.c.c.) on graded submodules. A graded ring A is called graded-Artinian if it is graded-Artinian as a graded A-module. It is clear that a Noetherian graded ring is graded-Noetherian and an Artinian graded ring is graded-Artinian but the converse is not true in general. (see [32, Example 1.1.22])

• **The graded Krull's dimension of a graded ring**

The graded height of a graded-prime ideal P denoted by $gr\text{-}ht(P)$, is defined as the length of the longest chain of graded-prime ideals contained in P. The graded Krull dimension of a graded ring A is denoted by $gr\text{-}dim(A)$ and defined as follows:

$$\text{gr-dim(A)} = \max\{\text{gr-ht(P)} \,/ P \in \text{gr-Spec(A)}\}.$$

• **The graded radical**

Let I be a proper graded ideal of A. Then the graded radical of I is denoted by $Gr(I)$ and it is defined as follows:

$$Gr(I) = \left\{ a = \sum_{g \in G} a_g \ \in A : \forall g \in G,\ \exists n_g \in \mathbb{N}^* \ such\ that\ a_g^{n_g} \in I \right\}.$$

Note that $Gr(I)$ is a graded ideal of A, it is the intersection of all the graded-prime ideals of A containing I, and we have $\text{gr-Nil(A)} = \bigcap_{P \in gr-Spec(A)} P = Gr(0)$, see [52, Proposition 2.5]. We refer the reader to [52, Proposition 2.4] for the basic properties of the graded radical. According to [51, Definition 1.5], a proper graded ideal I is said to be graded-primary if whenever $a, b \in h(A)$ with $ab \in I$ then $a \in I$ or $b \in Gr(I)$.

CHAPTER 1

ON GRADED COHERENT-LIKE PROPERTIES IN TRIVIAL RING EXTENSIONS

Abstract.

Let $A = \oplus_{\alpha\in G} A_\alpha$ be a commutative ring with unity graded by an arbitrary grading commutative monoid G, E be a graded A-module and $R = A \propto E$ the graded trivial extension. In this chapter*, in the second section, we improve some results on graded trivial extension and we give some new ones and the theme throughout is how the homogeneous properties of R are related to those of A and E. Then, in another section, we introduce the notions of graded-v-coherent, graded-quasi coherent and graded-finite conductor rings, then we study their transfer in the graded trivial extension.

1.1 Introduction

Let A be a ring and let E be an A-module. We can make Cartesian product $R := A \propto E$ into a ring with respect to componentwise addition and multiplication defined by $(a_1, e_1)(a_2, e_2) := (a_1a_2, a_1e_2 + a_2e_1)$. This is a commutative ring with identity $(1,0)$, called the idealization of E or the trivial extension of A by E, denoted by $A \propto E$. Note that A naturally embeds into $A \propto E$ via $a \to (a,0)$, if N is a submodule of E, then $0 \propto N$ is an ideal of $A \propto E$, $0 \propto E$ is a nilpotent ideal of $A \propto E$ of index 2, and that $(A \propto E)/(0 \propto E) \simeq A$. Idealization of a module was introduced by Nagata [43]. We usually use trivial extension to reduce results concerning submodules to the ideal case, generalizing results from ring

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to modules and constructing examples of commutative rings with zero divisors. An excellent introduction to the subject is [34] and an excellent summary of work done on trivial extension can be found in [7]. See, for instance [27], [29], [36], [39], [40], [41], [48], [50] and [53].

The construction of trivial extension has many generalizations, see e.g., [6], [58] and [59]. Among these generalizations, we have the graded trivial extension. The notion of graded trivial extension, for rings graded by $\mathbb{N}$, was introduced by Nagata [43, Exercice 1, page 24], or also as a theorem by Anderson [7, Theorem 4.5]. And recently in [58] for rings graded by an abelien group. Let $A = \oplus_{\alpha\in G}A_\alpha$ be a graded ring and $E = \oplus_{\alpha\in G}E_\alpha$ a graded A-module. Then $R = A \propto E = \oplus_{\alpha\in G}R_\alpha$ is a graded ring where $R_\alpha := A_\alpha \oplus E_\alpha$.

The purpose of this chapter is to generalize some results in [58]. More precisely, in section 1.2, we study and improve the graded version of some basic results on trivial extension. Among other things, we show that if I is an ideal of A, N is a submodule of E, then $I \propto N$ is a homogeneous ideal of $A \propto E$ if and only if I is a homogeneous, N is a homogeneous submodule of E and $IE \subseteq N$. We also show that the homogeneous-maximal (respectively prime, radical) ideals of R have the form $I \propto E$ where I is a homogeneous-maximal (respectively prime, radical) ideal of A.

Assume that the grading monoid G is a cancellative torsionfree monoid. Let A be a graded ring. Recall that A is said to be a graded-Noetherian ring (gr-Noetherian, for short) if it satisfies the ascending chain condition (a.c.c.) on homogeneous ideals; equivalently, if each homogeneous-prime ideal of R is finitely generated [57, Lemma 2.3]. Obviously, a Noetherian ring is a gr-Noetherian ring, while gr-Noetherian rings need not be Noetherian. It is known that the monoid ring $R[X;G]$ over a ring R is a Noetherian ring (resp. gr-Noetherian ring) if and only if R is a Noetherian ring and G (resp. each ideal of G (that is, a nonempty subset I of the additive monoid G, such that $g + I \subseteq I$ for each $g \in G$)) is finitely generated [28, Theorem 7.7, p. 75] (resp. [57, Theorem 2.4]). Hence, if $\mathbb{Q}$ is the additive group of rational numbers and D is a Noetherian ring, the group ring; $A = D[X;\mathbb{Q}]$ is a gr-Noetherian ring but not a Noetherian ring. Recently, the notion of coherence has known a graded genelization which is due to Bakkari, Mahdou and Riffi, see [15]. Let A be a graded ring. A is called graded-coherent (gr-coherent) if every finitely generated homogeneous ideal is finitely presented. Obviously, every coherent graded ring is a graded-coherent ring while the converse is false in general, see [15, example 3.2].

We also show that, in the case where the monoid is a cancellative torsionfree monoid, $A \propto E$ is gr-Noetherian, if and only if A is gr-Noetherian, and E is finitely generated. We also show that, in the case where the grading monoid is a group, if (A,M) is a local graded ring and E a graded A-module with $ME = 0$. $R := A \propto E$ is graded-coherent, if and only if A is graded-coherent, and M is finitely generated, and E is an (A/M)-vector space of finite rank.

The concept of coherence has many graded generalizations, see e.g. [15] and [16]. Among these generalizations, we have the graded-v-coherence. In Section 1.3, we introduce and study the notion of graded-v-coherent rings. Among other things, we show that,

in the case where the grading monoid G is a group, if A is a graded integral domain which is not a field, A_H its homogeneous quotient field graded by G and $R := A \propto A_H$ the graded trivial extension ring of A by A_H then R is a graded-v-coherent ring if and only if A is graded-v-coherent.

1.2 On the graded trivial extension ring

We begin with the following result which is an extension of a result given in [58, proposition 3.1] to the context of rings graded by a commutative grading monoid G. Note that in [58, proposition 3.1], the notation $A_\alpha(+)E_\alpha$ stands for the set of cartesian product $A_\alpha \oplus E_\alpha$ and does not indicate the idealization of E_α by A_α since A_α may not be a ring and E_α may not be an A_α-module (Note that A_0 is a ring and E_α is an A_0-module for every $\alpha \in G$).

Proposition 1.2.1 *Let $A = \oplus_{\alpha\in G}A_\alpha$ be a graded ring and $E = \oplus_{\alpha\in G}E_\alpha$ a graded A-module. Then $R := A \propto E$ the trivial extension of A by E is a graded ring $R = \oplus_{\alpha\in G}R_\alpha$ where $R_\alpha = (A \propto E)_\alpha := A_\alpha \oplus E_\alpha$.*

Proof. Additively $R := A \propto E = A \oplus E = \oplus_{\alpha\in G}A_\alpha \oplus \oplus_{\alpha\in G}E_\alpha = \oplus_{\alpha\in G}(A_\alpha \oplus E_\alpha) = \oplus_{\alpha\in G}(A \propto E)_\alpha$. Observe that $(A \propto E)_\alpha(A \propto E)_\beta = (A_\alpha \oplus E_\alpha)\left(A_\beta \oplus E_\beta\right) = A_\alpha A_\beta \oplus \left(A_\alpha E_\beta + A_\beta E_\alpha\right) \subseteq A_{\alpha+\beta} \oplus E_{\alpha+\beta} = (A \propto E)_{\alpha+\beta}$.

In [7], the homogeneous ideals of $A \propto E$ are the ideals of the form $I \propto N$ where I is an ideal of R, N is a submodule of E, and $IE \subseteq N$ respecting to its natural $\mathbb{N}$-grading with $(A \propto E)_0 = A \oplus 0$, $(A \propto E)_1 = 0 \oplus E$ and $(A \propto E)_n = 0$ for $n \geq 2$ which can also be viewed as a $\mathbb{Z}_2$-grading since $(0 \oplus E)^2 = 0$. The next result characterizes certain homogeneous ideals in the graded trivial extension ring graded by an arbirary grading commutative monoid G.

Theorem 1.2.2 *Let A be a graded ring, I an ideal of A, E a graded A-module and N a submodule of E. Then $I \propto N$ is a homogeneous ideal of $A \propto E$ if and only if I is a homogeneous ideal of A, N is a homogeneous submodule of E and $IE \subseteq N$. When $I \propto N$ is a homogeneous ideal, E/N is a graded A/I-module and $(A \propto E)/(I \propto N) \cong (A/I) \propto (E/N)$ is a graded ring isomorphism. In particular, $(A \propto E)/(0 \propto N) \cong A \propto (E/N)$ is a graded ring isomorphism and therefore $(A \propto E)/(0 \propto E) \cong A$ is a graded ring isomorphism. So the homogeneous ideals of $A \propto E$ containing $0 \propto E$ are of the form $J \propto E$ for some homogeneous ideal J of A.*

Proof. Suppose that $I \propto N$ is a homogeneous ideal of $A \propto E$. Since $I \propto N$ is an ideal, we have $IE \subseteq N$. Let $a = \sum_{\alpha\in G}a_\alpha \in I$ for some $a \in A$. Now, we need to show that $a_\alpha \in I$ for every $\alpha \in G$. Since $(a,0) = \sum_{\alpha\in G}(a_\alpha,0) \in I \propto N$ and $I \propto N$ is homogeneous, we

conclude that $(a_\alpha,0) \in I \propto N$ which implies that $a_\alpha \in I$. Thus, I is a homogeneous ideal of A. Likewise, N is a homogeneous submodule of E. Conversely, assume that I is a homogeneous ideal of A, N is a homogeneous submodule of E and $IE \subseteq N$. Then note that $I \propto N$ is an ideal of $A \propto E$. Let $(a,m) = \sum_{\alpha\in G}(a_\alpha,m_\alpha) \in I \propto N$ for some $a \in A$ and $m \in E$. Then we have $a = \sum_{\alpha\in G} a_\alpha \in I$ and $m = \sum_{\alpha\in G} m_\alpha \in N$. Since I and N are homogeneous, we have $a_\alpha \in I$ and $m_\alpha \in N$ which gives $(a_\alpha,m_\alpha) \in I \propto N$. Therefore, $I \propto N$ is a homogeneous ideal of $A \propto E$. Let $I \propto N$ be a homogeneous ideal. Then I is a homogeneous ideal of A and N is a homogeneous submodule of E with $IE \subseteq N$. This implies that E/N is a graded A/I-module. Now, consider the map $f : A \propto E \to (A/I) \propto (E/N)$ given by $f((a,e)) = (a+I, e+N)$. Then f is a graded surjective homomorphism and $Ker(f) = I \propto N$. This implies that $(A \propto E)/(I \propto N) \cong (A/I) \propto (E/N)$ is a graded ring isomorphism. The last statement follows from the correspondence theorem.

Note that Proposition 3.3 in [58] characterizes the homogeneous-maximal and the homogeneous-prime ideals of $A \propto E$ under the assumption that G is a group. We next improve this Proposition by taking G a grading commutative monoid (not necessarily a group), and characterize the homogeneous-maximal, the homogeneous-prime and the homogeneous-radical ideals of the graded trivial extension ring.

Theorem 1.2.3 *Let A be a graded ring, E a graded A-module and $R := A \propto E$ the graded trivial extension of A by E.*

1. *The homogeneous-maximal ideals of R have the form $M \propto E$ where M is a homogeneous-maximal ideal of A.*
2. *The homogeneous-prime ideals of R have the form $P \propto E$ where P is a homogeneous-prime ideal of A.*
3. *The homogeneous-radical ideals of R have the form $I \propto E$ where I is a homogeneous-radical ideal of A.*

Proof. (2): Let I be a homogeneous-prime ideal of R. Then $(0 \propto E)^2 = 0 \subseteq I$ and hence $0 \propto E \subseteq I$. So by Theorem 1.2.2, $I = J \propto E$ for some homogeneous ideal J of A. Now, we will show that J is a homogeneous-prime ideal. Let $ab \in J$ for some $a,b \in h(A)$. Then $(a,0)(b,0) = (ab,0) \in I$ which gives $(a,0) \in I$ or $(b,0) \in I$ since I is homogeneous-prime. Then $a \in J$ or $b \in J$ so that J is a homogeneous-prime ideal of A. Suppose that $I = J \propto E$ for some homogeneous-prime ideal J of A and $(a_1,e_1)(a_2,e_2) \in I$ for some $(a_1,e_1),(a_2,e_2) \in h(R)$. Then we have $a_1a_2 \in J$. Since J is a homogeneous-prime ideal of A, we conclude that $a_1 \in J$ or $a_2 \in J$ which implies that $(a_1,e_1) \in I$ or $(a_2,e_2) \in I$. Thus I is a homogeneous-prime ideal of R.
$(1),(3)$: It is similar to (2).

The next result determines the homogeneous units and the homogeneous idempotents for the graded trivial extension.

Proposition 1.2.4 *Assume that the grading monoid G is a group. Let A be a graded ring and E a graded A-module. Then the homogeneous units of $A \propto E$ are $U(A \propto E) \cap h(A \propto E) = (U(A) \propto E) \cap h(A \propto E)$ and the homogeneous idempotents of $A \propto E$ are $Id(A \propto E) \cap h(A \propto E) = Id(A_0) \propto 0$.*

Proof. Suppose that $(a,e) \in U(A \propto E) \cap h(A \propto E)$. So there exists an (s,n) with $(a,e)(s,n) = (1,0)$. Hence $as = 1$, so $a \in U(A) \cap h(A)$ since the inverse of a homgeneous element is also a homogeneous element, see [45, proposition 1.1.1(2)]. Conversely, suppose that $a \in A$ is a homogeneous unit, say $as = 1$. Then $(a,0)(s,0) = (1,0)$ so $(a,0)$ is a homogeneous unit. For any $m \in E$ with the same degree of a, $(0,m)$ is nilpotent and hence $(a,m) = (a,0) + (0,m)$ is a homogeneous unit.
Certainly if $e \in A$ is a homogeneous idempotent, then $e \in A_0$ and $(e,0) \in A_0 \propto 0$ is a homogeneous idempotent. Conversely, suppose that (a,e) is a homogeneous idempotent element of $A \propto E$. Then $(a,e) = (a,e)^2 = \left(a^2, 2ae\right)$. So $a = a^2 \in A_0$ is idempotent. Also, $e = 2ae$, so $ae = 2a^2e = 2ae$ and hence $ae = 0$ so $e = 2ae = 0$.

The following theorem determines when $A \propto E$ is gr-Noetherian.

Theorem 1.2.5 *Assume that the grading monoid G is a cancellative torsionfree monoid. Let A be a graded ring and E a graded A-module. Then $A \propto E$ is gr-Noetherian, if and only if A is gr-Noetherian, and E is finitely generated.*

Proof. Suppose that $A \propto E$ is gr-Noetherian. Then A being graded homomorphic image of $A \propto E$ is gr-Noetherian. Now $0 \propto E$ is a finitely generated ideal of $A \propto E$ since $A \propto E$ is gr-Noetherian and $0 \propto E$ is a homogeneous ideal. Observe that $(0,e_1), \ldots, (0,e_n)$ generate $0 \propto E$ as a homogeneous ideal if and only if $e_1, \ldots, e_n$ generate E as a graded A-module. Hence E is a finitely generated A-module.
Conversely, suppose that A is gr-Noetherian and E is finitely generated, let $P \propto E$ be a homogeneous-prime ideal of $A \propto E$. Since A is gr-Noetherian, P is finitely generated. Then since E is a finitely generated graded A-module, $P \propto E$ is a finitely generated homogeneous ideal of $A \propto E$. Since every homogeneous-prime ideal of $A \propto E$ is finitely generated, then by [57, Lemma 2.3], $A \propto E$ is gr-Noetherian.

The next theorem characterises the graded-coherence in the graded trivial extension ring in the case where G is a group. Recall that a graded ring R is said to be graded-coherent if every finitely genereted homogeneous ideal is finitely presented.

Theorem 1.2.6 *Assume that the grading monoid G is a group. Let (A,M) be a local graded ring and E a graded A-module with $ME = 0$. Let $R := A \propto E$ be the graded trivial extension ring of A by E. Then R is graded-coherent if and only if A is graded-coherent, M is finitely generated, and E is an (A/M)-vector space of finite rank.*

Before poving this theorem, we first establish the following lemma which is the graded version of lemma 2.7 in [35].

Lemma 1.2.7 *Assume that the grading monoid G is a group. Let (A,M) be a local graded ring and E a graded A-module with $ME = 0$. Let $R := A \propto E$ be the graded trivial extension ring of A by E. $(0 : c)$ is a finitely generated homogeneous ideal of R for each homogeneous element $c \in R$ if and only if $(0 : a)$ is a finitely generated homogeneous ideal of A for each homogeneous element $a \in A$, M is finitely generated, and E is an (A/M)-vector space of finite rank.*

Proof. Assume that $(0 : c)$ is a finitely generated homogeneous ideal of R for each homogeneous element $c \in R$. Let $a \in h(A)$ and set $c := (a,0) \in R$. Then $(0 : c) = (0 : a) \propto E'$, where $E' = \{e \in E \mid ae = 0\}$. Therefore, $(0 : a)$ is a finitely generated homogeneous ideal of A. Let e be a nonzero homogeneous element of E and set $c := (0,e) \in R$. Similar arguments show that M is a finitely generated ideal of A since $(0 : c) = M \propto E$. Further, assume that $M \propto E = \sum_{i=1}^{n} R(x_i, e_i)$, where $x_i \in M$ and $e_i \in E$ for each $i = 1, \ldots, n$. Then $E \subseteq \sum_{i=1}^{n} (A/M)e_i$, and hence E is an (A/M)-vector space of finite rank.

Conversely, let $c := (a,e)$ be a nonzero homogeneous element of R. If a is an invertible homogeneous element of A, then by Proposition 1.2.4 c is an invertible homogeneous element of R. Then, without loss of generality, we may assume that $a \in M$. Hence, $(0 : c) = \{(b,f) \in R \mid (b,f)(a,e) = 0\} = \{(b,f) \in R \mid (ab, be + af) = 0\} = \{(b,f) \in R \mid (ab,be) = 0\} = \{(b,f) \in M \propto E \mid ab = 0\} =$ (since by hypothesis we have $ME = 0$ and if b is invertible element of A, then (b,f) is invertible in R, and so $c = 0$, a contradiction). It is clear that if $a = 0$ then $(0 : c) = M \propto E$, and if $a \neq 0$ then $(0 : c) = (0 : a) \propto E$. In both cases, $(0 : c)$ is a finitely generated ideal of R since M and $(0 : a)$ are finitely generated ideals of A and E is an (A/M)-vector space of finite rank.

Proof of Theorem 1.2.6: Assume that R is a graded-coherent ring. By Lemma 1.2.7 and Theorem 3.3(5) in [15], it remains to show that A is graded-coherent. Let $I = \sum_{i=1}^{n} Aa_i$, be a homogeneous ideal of A where $a_i \in M$, since I is a homogeneous ideal we may assume that the a_i are homogeneous elements of A and set $J := \sum_{i=1}^{n} R(a_i, 0)$ which is also a homogeneous ideal. Recall from [15, proposition 3.11] that any finite product of graded-coherent rings is a graded-coherent ring. Consider the exact sequence of graded R-modules:

$$0 \to \mathrm{Ker}(u) \to R^n = A^n \propto E^n \xrightarrow{u} J \to 0$$

where $u\left((b_i,e_i)_{1\leq i\leq n}\right) = \sum_{i=1}^{n} (b_i,e_i)(a_i,0) = (\sum_{i=1}^{n} a_i b_i, 0)$ since $a_i \in M$ for each $i = 1, \ldots, n$. On the other hand, consider the exact sequence of graded A-modules:

$$0 \to \mathrm{Ker}(v) \to A^n \xrightarrow{v} I \to 0$$

where $v\left((b_i)_{1\leq i\leq n}\right) = \sum_{i=1}^{n} a_i b_i$. Then, $\mathrm{Ker}(u) = \mathrm{Ker}(v) \propto E^n$. But J is finitely presented since J is a homogeneous ideal and R is graded-coherent, so according to the exactness

of the first sequence cited in the proof $\mathrm{Ker}(u)$ is a finitely generated R-module since R^n is finitely generated, hence $\mathrm{Ker}(v)$ is a finitely generated A-module. Therefore, I is a finitely presented ideal of A, so A is graded-coherent. Conversely, let J be a finitely generated homogeneous ideal of R and let $S := \{(a_i, e_i)\}_{1\leq i\leq n}$ be a minimal generating set of homogeneous elements of J, where $a_i \in M$ and $e_i \in E$. Consider the exact sequence of graded R-modules:

$$0 \to \mathrm{Ker}(u) \to R^n \xrightarrow{u} J \to 0$$

where $u\left((b_i,f_i)_{1\leq i\leq n}\right) = \sum_{i=1}^n (a_i,e_i)(b_i,f_i) = (\sum_{i=1}^n a_ib_i, \sum_{i=1}^n b_ie_i)$ since $a_i \in M$ for each $i = 1,\ldots,n$. Further, the minimality of S yields $\mathrm{Ker}(u) = \left\{(b_i,f_i)_{1\leq i\leq n} \in R^n \mid \sum_{i=1}^n a_ib_i = 0\right\}$. Let $I := \sum_{i=1}^n Aa_i$ which is a homogeneous ideal of A and consider the surjective A-module homomorphism v defined above. Since the grading monoid is a group then by [15, proposition 2.3], $\mathrm{Ker}(v)$ is a finitely generated A-module since A is graded-coherent. Consequently, $\mathrm{Ker}(u) = \mathrm{Ker}(v) \propto E^n$ is a finitely generated R-module. Therefore, J is finitely presented ideal and hence R is graded-coherent.

Proposition 1.2.8 *Let A_1 and A_2 be two graded rings, and let E_i be a graded A_i-module, $i = 1,2$. Then $(A_1 \times A_2) \propto (E_1 \times E_2) \cong (A_1 \propto E_1) \times (A_2 \propto E_2)$ is a graded ring isomorphism.*

Proof. It is easily checked that the map $((a_1,a_2),(e_1,e_2)) \to ((a_1,e_1),\ (a_2,\ e_2))$ is a graded ring isomorphism.

Recall from [59] that the notion of trivial extension can be generalized to what is called a semi-trivial extension. Let A be a ring, E an A-module, and $\varphi : E \otimes_A E \to A$ an A-module homomorphism satisfying $\varphi(e\otimes e') = \varphi(e'\otimes e)$ and $\varphi(e\otimes e')e'' = e\varphi(e'\otimes e'')$. Then $A \propto_\varphi E = A\oplus E$ with coordinate-wise addition and multiplication defined by $(a,e)(a',e') = (aa' + \varphi(e\otimes e'), ae' + a'e)$ is a commutative ring, called a semi-trivial extension of A by E. For $\varphi = 0$, we have trivial extension. See [59] for more details. The next result introduce the notion of graded semi-trivial extension and show that it is a graded ring.

Theorem 1.2.9 *Let A be a graded ring, E a graded A-module, and $\varphi : E \otimes_A E \to A$ a graded A-module homomorphism of degree 0 satisfying $\varphi(e\otimes e') = \varphi(e'\otimes e)$ and $\varphi(e\otimes e')e'' = e\varphi(e'\otimes e'')$. Then $R := A \propto_\varphi E$ the semi-trivial extension of A by E is a graded ring $R = \oplus_{\alpha\in G} R_\alpha$ where $R_\alpha = (A \propto_\varphi E)_\alpha := A_\alpha \oplus E_\alpha$.*

Proof. Additively $R := A \propto_\varphi E = A \oplus E = \oplus_{\alpha\in G} A_\alpha \oplus \oplus_{\alpha\in G} E_\alpha = \oplus_{\alpha\in G}(A_\alpha \oplus E_\alpha) = \oplus_{\alpha\in G}(A \propto_\varphi E)_\alpha$. Observe that $(A \propto_\varphi E)_\alpha (A \propto_\varphi E)_\beta = (A_\alpha \oplus E_\alpha)\left(A_\beta \oplus E_\beta\right) = (A_\alpha A_\beta + \varphi(E_\alpha \otimes_A E_\beta)) \oplus \left(A_\alpha E_\beta + A_\beta E_\alpha\right) \subseteq A_{\alpha+\beta} \oplus E_{\alpha+\beta} = (A \propto_\varphi E)_{\alpha+\beta}$ since φ is a graded A-module homomorphism of degree 0.

1.3 Graded-v-coherent rings

Let A be a graded ring, and let $Q(A)$ denote the total ring of quotients of A and H the saturated multiplicative set of nonzero regular homogeneous elements of A. Then, by extending some definitions in [2] to the case where rings are with zerodivisors, A_H, called the homogeneous total ring of quotients of A, is a ring graded by $\langle G\rangle$, where $A_H = \oplus_{\alpha\in\langle G\rangle}(A_H)_\alpha$ with $(A_H)_\alpha = \{\frac{a}{b} \mid a \in A_\beta, b \text{ a regular element of } A_\gamma \text{ and } \beta-\gamma=\alpha\}$. If A is a graded integral domain (An integral domain graded by G), then A_H is called the homogeneous quotient field of A. Clearly, every nonzero homogeneous element of A_H is invertible and $(A_H)_0$ is a field. A fractional ideal I of A is called a homogeneous fractional ideal of A if $I \subseteq A_H$ and $I = \oplus_{\alpha\in\langle G\rangle}(I \cap (A_H)_\alpha)$; equivalently, if $I = \frac{1}{s}J$ for some regular element $s \in H$ and some homogeneous integral ideal J of A, see [2, Proposition 2.4]. It is easy to see that a homogeneous principal fractional ideal of A has the form xA for some homogeneous element x of A_H. Given two nonzero fractional ideals I, J. The set $\{x \in T \mid xJ \subseteq I\}$, where T is an overring of A, is a fractional ideal and denoted by $(I :_T J)$. If $T = Q(A)$, the subscript $Q(A)$ will usually be omitted. The fractional ideal $(A : I)$ is denoted by I^{-1} and $(A : (A : I)) = \left(I^{-1}\right)^{-1}$ by I_v. Also, I is called invertible if $II^{-1} = A$, divisorial ideal or v-ideal if $I = I_v$ and v-finite if $I_v = J_v$ (or, equivalently, if $I^{-1} = J^{-1}$) for some finitely generated fractional ideal J of A. Recall from [35] that a ring A is called v-coherent if $(0 : a)$ and $\cap_{1\le i\le n} Aa_i$ are v-finite ideals of A for any finite set of elements a and $a_1, \cdots, a_n$ of A.

We begin by the following result which extend Proposition 2.5 in [2] to the case where rings are with zerodivisors.

Proposition 1.3.1 *Let A be a graded ring and H the saturated multiplivative set of regular homogeneous elements of A. If I and J are two regular homogeneous fractional ideals, then $(I : J)$ is also a homogeneous fractional ideal and $(I : J) = (I :_{A_H} J)$. In particular, if I is a regular homogeneous fractional ideal, then so are I^{-1} and I_v.*

Proof. Let $x \in (I : J)$, so $x = a/b$ where $a \in A$ and b is a regular element of A. Let j be a regular homogeneous element of J, so $j = c/d$ where $c, d \in H$. Then $(a/b)(c/d) \in I \subseteq A_H$. Then we have $x = a/b \in (d/c)A_H \subseteq A_H$, hence $(I : J) = (I :_{A_H} J)$. Then $(I :_{A_H} J)$ is a homogeneous fractional ideal since I and J are homogeneous A-submodules of A_H. In particular if I is a regular homogeneous fractional ideal, then so is $I_v = (A :_{A_H} (A :_{A_H} I))$.

Definition 1.3.2 *Let A be a graded ring and I be a fractional ideal. I is called homogeneous-v-finite ideal if $I_v = J_v$ (which is equivalent to $I^{-1} = J^{-1}$) for some finitely generated homogeneous fractional ideal J of A.*

Definition 1.3.3 *Let A be a graded ring. A is called graded-v-coherent ring if $(0 : a)$ and $\cap_{1\le i\le n} Aa_i$ are homogeneous-v-finite ideals for any finite set of homogeneous elements a and $a_1, \cdots, a_n$ of A.*

Obviously, every v-coherent graded ring is a graded-v-coherent ring which gives us the first and easiest example of graded-v-coherent rings.

Example 1.3.4 *Let (A,M) be any local graded ring with $M^2 = 0$. Then A is a graded-v-coherent ring.*

Proof. By [35, example 2.4], A is a v-coherent ring.

Proposition 1.3.5 *Let A be a graded ring, and let's consider the following statements:*

1. *I^{-1} is homogeneous-v-finite for any finitely generated homogeneous fractional ideal I of A.*
2. *$I_v \cap J_v$ is homogeneous-v-finite for any two finitely generated homogeneous fractional ideals I and J of A.*
3. *$\bigcap_{1\leq i\leq n} Aa_i$ is homogeneous-v-finite for any finite set of homogeneous elements $a_1,\ldots,a_n$ of A.*

Then $(1) \Rightarrow (2)$. Moreover, if A is a graded integral domain, Then the three assertions are equivalent, which each of them characterizes the graded-v-coherence in graded integral domains.

Proof. Assume that (1) is true and let I and J be any two finitely generated homogeneous fractionals ideals of A. Then there exist two finitely generated homogeneous fractional ideals I_1 and J_1 such that $I_v = I_1^{-1}$ and $J_v = J_1^{-1}$. So, $I^{-1} = (I_1)_v$ and $J^{-1} = (J_1)_v$, then $I_v \cap J_v = I_1^{-1} \cap J_1^{-1} = (I_1 + J_1)^{-1}$ which is homogeneous-v-finite by hypothesis since $I_1 + J_1$ is a finitely generated homogeneous fractional ideal of A. Now, let A be a graded integral domain and assume that (2) is true, and let $I = (x_1,\cdots,x_n)A$ with x_i a nonzero homogeneous element in the homogeneous quotient field of A. Then $I^{-1} = (A : I) = (A : x_1A) \cap \cdots \cap (A : x_nA)$ is homogeneous-v-finite, as desired. $(1) \Longleftrightarrow (3)$ is clear since $(\sum_{i=1}^{n} Aa_i)^{-1} = \bigcap_{1\leq i\leq n} Aa_i^{-1}$ for each homogeneous element $a_i \in A$ and any integer $n \geq 1$.

The next result characterizes the graded-v-coherence in the graded trivial extension of a graded integral domain by its homogeneous quotient field.

Theorem 1.3.6 *Assume that the grading monoid G is a group. Let A be a graded integral domain which is not a field, A_H its homogeneous quotient field graded by G and let $R := A \propto A_H$ be the graded trivial extension of A by A_H, then R is a graded-v-coherent ring if and only if A is graded-v-coherent.*

Proof. Observe that $(a,e) \in R$ is regular if and only if $a \neq 0$, and that $(0 : c) = 0 \propto A_H$ for any $c := (0,e) \neq 0 \in R$. Moreover, [34, Theorem 25.1(4)] yields $\bigcap_{1\leq i\leq n} R(a_i,e_i) = \bigcap_{1\leq i\leq n} (Ra_i \propto A_H) = \left(\bigcap_{1\leq i\leq n} Ra_i\right) \propto A_H$, for every finite set of homogeneous elements

$(a_i,e_i)_{1\leq i\leq n}$ of R (with $a_i\neq 0$ for each i). Now, let I be a nonzero homogeneous ideal of A and E a homogeneous A-submodule of A_H with $IA_H\subseteq E$ and let $J:=I\propto E$. By [34, Theorem 25.10], we have $J^{-1}=(I\propto E)^{-1}=I^{-1}\propto A_H=(I\propto IA_H)^{-1}$. Finally, the conclusion is straightforward since $I\propto IA_H$ is finitely generated homogeneous ideal if I is.

Definition 1.3.7 *Let A be a graded ring. Then A is said to be graded-quasi-coherent (resp., graded-finite conductor) if $(0:a)$ and $\bigcap_{i=1}^{n}Aa_i$ (resp., $Ab\cap Ac$) are finitely generated for every finite set of homogeneous elements $a,a_1,a_2,\ldots,a_n$ (resp., b,c) of A.*

Proposition 1.3.8 *Let (A,M) be a graded-local ring with $M^2=0$. The following statements are equivalent.*

1. *A is a graded-coherent ring.*
2. *A is a graded-quasi-coherent ring.*
3. *A is a graded-finite conductor ring.*
4. *$(0:a)$ is finitely generated for every homogeneous element a of A.*
5. *M is finitely generated.*

Proof. $(1)\Rightarrow(2)$: Follows from [5, Theorem 3.3].

$(2)\Rightarrow(3)\Rightarrow(4)$: It is obvious.

$(4)\Rightarrow(5)$: Let $0\neq a\in h(A)$. Since $M^2=0$, clearly $M\subseteq(0:a)\subseteq M$. Then by assumption, $(0:a)=M$ is finitely generated.

$(5)\Rightarrow(1)$: Let M be finitely generated. Take a finitely generated homogeneous ideal I of A. Then we can choose a minimal generating set $S=\{x_1,x_2,\ldots,x_n\}$ of homogeneous elements of I. Consider the exact sequence $Ker(u)\to A^n\xrightarrow{u}I\to 0$ where $u:A^n\to I$ given by $u((a_1,a_2,\ldots,a_n))=\sum_{i=1}^{n}a_ix_i$. Since $M^2=0$, it is clear that $\prod_{i=1}^{n}M\subseteq Ker(u)$. Also, note that $Ker(u)$ is a homogeneous A-submodule of A^n. Now, let $(a_1,a_2,\ldots,a_n)\in Ker(u)$ be a homogeneous element. Then $\sum_{i=1}^{n}a_ix_i=0$. If $a_t\notin M$ for some $t\in\{1,2,\ldots,n\}$, then we have a_t is unit. This implies that $x_t\in\sum_{\substack{i=1\\ i\neq t}}^{n}Ax_i$ which contradicts with the minimality of S. Thus we have $(a_1,a_2,\ldots,a_n)\in\prod_{i=1}^{n}M$ which yields that $Ker(u)=\prod_{i=1}^{n}M$ because every element of $Ker(u)$ is a sum of its homogeneous elements. Since M is finitely generated, so is $Ker(u)=\prod_{i=1}^{n}M$. Then I is finitely presented and so A is a graded-coherent ring.

Theorem 1.3.9 *Assume that the grading monoid G is a group. Let (A,M) be a local graded ring and E a graded A-module with $ME = 0$. Let $R = A \propto E$ be the graded trivial extension. The following statements are equivalent.*

1. *R is a graded-quasi-coherent (respectively, graded-finite conductor) ring.*
2. *A is a graded-quasi-coherent (respectively, graded-finite conductor) ring, M is finitely generated and E is an (A/M)-vector space of finite rank.*

Proof. $(1) \Rightarrow (2)$: Suppose that R is a graded-quasi-coherent ring. Then $(0:c)$ is a finitely generated ideal for every homogeneous element $c \in R$. Then by Lemma 1.2.7, $(0:a)$ is finitely generated for each homogeneous element $a \in A$, M is finitely generated and E is an (A/M)-vector space of finite rank. Let $a_1, a_2, \ldots, a_n$ be homogeneous elements of A. Now, we must show that $\bigcap_{i=1}^{n} Aa_i$ is finitely generated. Since R is graded-quasi-coherent ring, it is clear that $\bigcap_{i=1}^{n} R(a_i,0) = \left(\bigcap_{i=1}^{n} Aa_i\right) \propto 0$ is finitely generated. Which implies that $\bigcap_{i=1}^{n} Aa_i$ is finitely generated, that is, A is a graded-quasi-coherent ring.

$(2) \Rightarrow (1)$: Suppose that A is a graded-quasi-coherent ring, M is finitely generated and E is an (A/M)-vector space of finite rank. Then by Lemma 1.2.7, $(0:c)$ is finitely generated for every homogeneous element $c \in R$. Now, let $(a_i,e_i) \in R$ be homogeneous elements for each $i = 1,2,\ldots,n$. We need to show that $J = \bigcap_{i=1}^{n} R(a_i,e_i)$ is finitely generated. Since (A,M) is local graded, we may assume that $a_i \in M$ for every i. If $J = R(a_t,m_t)$ for some $t = 1,2,\ldots,n$, then there is nothing to prove. So assume that $J \subsetneq R(a_i,e_i)$ for every i. Let $(x,m) \in J$. Since J is graded, for every $\alpha \in G$, we have the α component $(x,m)_\alpha = (x_\alpha, m_\alpha) \in J \subsetneq R(a_i,e_i)$. Then there exists $(y,n) \in R$ such that $(x_\alpha,m_\alpha) = (y,n)(a_i,e_i)$ which gives $(x_\alpha,m_\alpha) = (y_{h_1},n_{h_1})(a_i,e_i) + (y_{h_2},n_{h_2})(a_i,e_i) + \cdots + (y_{h_k},n_{h_k})(a_i,e_i)$. Since R is a graded ring and G is a group, we may assume that $(x_\alpha,m_\alpha) = (y_{h_t},n_{h_t})(a_i,e_i)$ for some $1 \leq t \leq k$. If $y_{h_t} \notin M$, we have (y_{h_t},n_{h_t}) is a unit element of R which gives $(a_i,e_i) = (y_{h_t},n_{h_t})^{-1}(x_\alpha,m_\alpha) \in J$. Then we get $R(a_i,e_i) = J$ which is a contradiction. Thus we have $y_{h_t} \in M$, which implies that $(x_\alpha,m_\alpha) = (y_{h_t},n_{h_t})(a_i,e_i) = (y_{h_i}a_i,0) \in Aa_i \propto 0$. Then we conclude that $(x,m) = (y,0)(a_i,0)$ where $y = y_{h_1} + y_{h_2} + \cdots + y_{h_k} \in M$. Then we have $(x,m) = (y,0)(a_i,0) \in (M \propto 0)(Aa_i \propto 0) = Ma_i \propto 0$ for each $i = 1,2,\ldots,n$. This gives $J \subseteq \bigcap_{i=1}^{n} (Ma_i \propto 0) = \left(\bigcap_{i=1}^{n} Ma_i\right) \propto 0$. Also, note that $\left(\bigcap_{i=1}^{n} Ma_i\right) \propto 0$ is a homogeneous ideal of $A \propto E$. Take a homogeneous element $(y_\alpha,0) \in \left(\bigcap_{i=1}^{n} Ma_i\right) \propto 0$. This gives $y_\alpha \in Ma_i$ for each $i = 1,2,\ldots,n$. Then $y_\alpha = m_{h_i}a_i$ for some homogeneous element $m_{h_i} \in M$ since A is graded. Then we conclude that $(y_\alpha,0) = (m_{h_i},0)(a_i,e_i) \in R(a_i,e_i)$ for each $i = 1,2,\ldots,n$. This implies that $(y_\alpha,0) \in J$. Since every element of $\left(\bigcap_{i=1}^{n} Ma_i\right) \propto 0$ is a sum of

its homogeneous element, we conclude that $\left(\bigcap_{i=1}^{n} Ma_i\right) \propto 0 \subseteq J$, that is, $J = \left(\bigcap_{i=1}^{n} Ma_i\right) \propto 0$. Since M is finitely generated, Ma_i is finitely generated homogeneous ideal. As A is a graded-quasi-coherent ring, $\bigcap_{i=1}^{n} Ma_i$ is a finitely generated ideal of A. Which implies that J is a finitely generated ideal of R. Thus, R is a graded-quasi-coherent ring.

If we take $n = 2$, then one can similarly prove that R is a graded-finite conductor ring if and only if A is a graded-finite conductor ring, M is finitely generated and E is an (A/M)-vector space of finite rank.

CHAPTER 2

COMMUTATIVE GRADED-n-COHERENT AND GRADED VALUATION RINGS

Abstract.

Let $R = \oplus_{\alpha \in G} R_\alpha$ be a commutative ring with unity graded by an arbitrary grading commutative monoid G. For each positive integer, the notions of a graded-n-coherent module and a graded-n-coherent ring are introduced. In this chapter*, many results are generalized from n-coherent rings to graded-n-coherent rings. In the last section, we provide necessary and sufficient conditions for the graded trivial extension ring to be a graded-valuation ring.

2.1 Introduction

If n is a nonnegative integer, we say that an R-module M is n-presented if there is an exact sequence $F_n \to F_{n-1} \to \ldots \to F_0 \to M \to 0$ of R-modules in which each F_i, is finitely generated and free. (Our usage follows [24]; in [60], such M is said to "have a finite n-presentation"). In particular, "0-presented" means finitely generated and "1-presented" means finitely presented. Following [20], we let $\lambda(M) = \lambda_R(M) = \sup\{n \mid M \text{ is an } n\text{-presented } R\text{-module}\}$, so that $0 \leq \lambda(M) \leq \infty$; the properties of the function λ are recalled in Lemma 2.2.3. Classically, the "n-presented" concept allows both ideal-theoretic and module-theoretic approaches to coherent rings. Indeed (cf. [20], p. 63, Exercise 12), a ring R is said to be coherent if each finitely generated ideal is finitely presented; equivalently if each finitely presented R-module is 2-presented.

Let n be a positive integer. Recall from [26] that R is n-coherent ring if each $(n-1)$-presented ideal of R is n-presented. Thus, the 1-coherent rings are just the coherent rings.

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Note that each Bezout (for instance, valuation) domain R is n-coherent for each $n \geq 1$; indeed, each $(n-1)$-presented ideal of R is principal and hence infinitely-presented (in the obvious sense). Moreover, each Noetherian ring is n-coherent for any $n \geq 1$. An excellent summary of work done on n-coherence can be found in [26]. And for background on coherence, we refer the reader to [29]. The concept of coherence has many graded generalizations, see e.g.[15], [16]. Among these generalizations, we have the graded-n-coherence. Accordingly, like it was done in [26], we use the λ-function to introduce both ideal and module theoretic approaches to "graded-n-coherence" for any positive integer n.

Section 2.2 begins, more generally, by defining graded-n-coherent modules for each integer $n \geq 1$. As one might expect, the graded-1-coherent modules are just the "graded-coherent modules" in [15]. Among other things, we show that, if R is a graded ring and $0 \to P \xrightarrow{u} N \xrightarrow{v} M \to 0$ an exact sequence of graded R-modules. Then if $\lambda(P) \geq n-1, N$ is a graded-n-coherent module and v has a cancellable degree then M is a graded-n-coherent module and if $\lambda(M) \geq n$ and N is a graded-n-coherent module, then P is a graded-n-coherent module. We also show that if $m \geq n$ is a positive integer and $M_0 \xrightarrow{u_1} M_1 \xrightarrow{u_2} M_2 \to \cdots \xrightarrow{u_m} M_m$ an exact sequence of graded-n-coherent R-modules such that the degree of every u_i is cancellable. Then $\mathrm{Im}(u_i), \mathrm{Ker}(u_i)$ and $\mathrm{Coker}(u_i)$ are graded-n-coherent R-modules for each $i = 1, 2, \ldots, m$. We also show that if $n \geq 1$, the canonical graded ring homomorphism $R \to R/I$ satisfy $\lambda_R(R/I) \geq n$, and M is a graded R-module. Then M is graded-n-coherent as a graded R/I-module if and only if M is graded-n-coherent as a graded R-module.

In Section 2.3, we introduce and study the notion of graded-n-coherent rings. Among other things, we show that, if R is a graded-n-coherent ring and I an $(n-1)$-presented homogeneous ideal of R. Then R/I is a graded-n-coherent ring. We also show that, if $(R_i)_{i=1,2,\ldots,m}$ is a family of graded rings. Then $\prod_{i=1}^{m} R_i$ is a graded-n-coherent ring if and only if R_i is a graded-n-coherent ring, for each $i = 1, \ldots, m$.

In section 2.4, we introduce and characterise the notion of graded-valuation rings and then, as a main results of this section, we characterise the graded-valuation property in the graded trivial extension ring, more precisely, we show that, in the case where the grading monoid is a torsionfree abelian group, if A is graded ring and E an nonzero graded A-module and $R := A \propto E$ the graded trivial extension ring of A by E. If E is a torsionfree graded A-module, then R is a gr-valuation ring if and only if A is a gr-valuation domain and E is isomorphic to A_H, the homogeneous quotient field of fractions of A. We also show that if A is a graded ring and E a nonzero graded A-module. Then $R := A \propto E$ the graded trivial extension ring is a gr-valuation ring if ond only if A is a gr-valuation domain, E is a gr-divisible and gr-uniserial A-module.

2.2 Graded-n-coherent modules

Definition 2.2.1 *Let R be a graded ring and let n be a positive integer, we say that a graded R-module M is a graded-n-coherent module if M is n-presented and each $(n-1)$-presented homogeneous submodule of M is n-presented.*

It follows from [15] that the graded-1-coherent modules are just the "graded-coherent modules".

Remark 2.2.2 *Let R be a graded ring and let n be a positive integer. Then following assertions hold:*

1. *Every $(n-1)$-presented homogeneous submodule of a graded-n-coherent R-module is a graded-n-coherent R-module.*

2. *Any n-coherent graded R-module is a graded-n-coherent R-module.*

For reference purposes, it will be helpful to recall the following elementry result [20, p.61, Exercise 6] which summarize some behavior of λ.

Lemma 2.2.3 *Let R be a ring and let $0 \to P \to N \to M \to 0$ be an exact sequence of R-modules. Then:*

1. $\lambda(N) \geq \inf\{\lambda(P), \lambda(M)\}$

2. $\lambda(M) \geq \inf\{\lambda(N), \lambda(P)+1\}$

3. $\lambda(P) \geq inf\,\{\lambda(N), \lambda(M)-1\}$

4. *If $N = P \oplus M$ then $\lambda(N) = inf\{\lambda(M), \lambda(P)\}$.*

Theorem 2.2.4 *Let R be a graded ring and let $0 \to P \xrightarrow{u} N \xrightarrow{v} M \to 0$ be an exact sequence of graded R-modules.*

(1) If $\lambda(P) \geq n-1$, N is a graded-n-coherent module and v has a cancellable degree then M is a graded-n-coherent module.

(2) If $\lambda(M) \geq n$ and N is a graded-n-coherent module, then P is a graded-n-coherent module.

Proof.

(1) P is $(n-1)$-presented and N is n-presented; therefore, M is n-presented by Lemma 2.2.3. Let M_1 be an $(n-1)$-presented homogeneous submodule of M. Since v has a cancellable degree the submodule $v^{-1}(M_1)$ of N is homogeneous. Then

the exact sequence : $0 \to P \xrightarrow{u} v^{-1}(M_1) \xrightarrow{v} M_1 \to 0$ shows that $\lambda\left(v^{-1}(M_1)\right) \geq \inf\{\lambda(P), \lambda(M_1)\} \geq n-1$ (Lemma 2.2.3 (1)); therefore, $\lambda\left(v^{-1}(M_1)\right) \geq n$ since $v^{-1}(M_1) \subseteq N$ and N is graded-n-coherent. We conclude, by Lemma 2.2.3(2), that $\lambda(M_1) \geq \inf\left\{\lambda\left(v^{-1}(M_1)\right), \lambda(P)+1\right\} \geq n$.

(2) M and N are both n-presented; therefore, P is $(n-1)$-presented by Lemma 2.2.3(3). Every $(n-1)$-presented homogeneous submodule of a graded-n-coherent module is a graded-n-coherent module by Remark 2.2.2(1); hence, P is a graded-n-coherent R-module.

Theorem 2.2.5 *Let $m \geq n$ be positive integer and let $M_0 \xrightarrow{u_1} M_1 \xrightarrow{u_2} M_2 \to \cdots \xrightarrow{u_m} M_m$ be an exact sequence of graded-n-coherent R-modules such that the degree of every u_i is cancellable. Then $\mathrm{Im}(u_i), \mathrm{Ker}(u_i)$ and $\mathrm{Coker}(u_i)$ are graded-n-coherent R-modules for each $i = 1, 2, \ldots, m$.*

Proof. It suffices to prove the assertion for $m = n$. Let $M_0 \xrightarrow{u_1} M_1 \xrightarrow{u_2} M_2 \to \ldots \xrightarrow{u_n} M_n$ be an exact sequence of graded-n-coherent R-modules. We then have the exact sequences of graded R-modules:
$0 \to Ker(u_1) \to M_0 \to \mathrm{Im}(u_1) \to 0$
$0 \to \mathrm{Im}(u_i) = \mathrm{Ker}(u_{i+1}) \to M_i \to \mathrm{Im}(u_{i+1}) \to 0$, for each $i = 1, \ldots, n-1$, and
$0 \to \mathrm{Im}(u_n) \to M_n \to \mathrm{Coker}(u_n) \to 0$
Since the degree of u_1 is cancellable, $Im(u_1)$ is a finitely generated homogeneous submodule of M_1 since M_0 is finitely generated (for M_0 is graded-n-coherent) therefore, $\mathrm{Im}(u_2)$ is 1-presented; and by induction, we conclude that $\mathrm{Im}(u_n)$ is $(n-1)$-presented. Thus $\mathrm{Im}(u_n)$ is a graded-n-coherent module by Remark 2.2.2(1) since $\mathrm{Im}(u_n)$ is a homogeneous submodule of the graded-n-coherent module M_n. Therefore $Im(u_i)$ and $\mathrm{Ker}(u_i)$ are graded-n-coherent modules by applying Theorem 2.2.4 to the above exact sequences of graded R-modules. Finally, Theorem 2.2.4 and the exact sequences of graded R-modules of degree 0, $0 \to Im(u_i) \to M_i \to \mathrm{Coker}(u_i) \to 0$ show that $\mathrm{Coker}(u_i)$ is graded-n-coherent module for each $i = 1, \ldots, m$.

Theorem 2.2.6 *Let M be a graded R-module and I be a homogeneous ideal of R such that $IM = 0$. Let $n \geq 1$ and let the canonical graded ring homomorphism $R \to R/I$ satisfy $\lambda_R(R/I) \geq n$. Then M is graded-n-coherent as a graded R/I-module if and only if M is graded-n-coherent as a graded R-module.*

Before establishing this theorem, we first prove the following three Lemmas.

Lemma 2.2.7 *Let $R \to S$ be a graded ring homomorphism such that $\lambda_R(S) \geq n$ and let M be an n-presented graded S-module. Then M is an n-presented graded R-module*

Proof. We proceed by induction on n. Case $n = 0$: If M is a finitely generated graded S-module and S a finitely generated graded R-module, it is clear that M is a finitely generated

graded R-module. Assume that the result is true for n. Let M be an $(n+1)$-presented graded S-module and let $\lambda_R(S) \geq n+1$. We have to show that $\lambda_R(M) \geq n+1$. Let $F_{n+1} \overset{u_{n+1}}{\to} F_n \overset{u_n}{\to} \ldots \to F_1 \overset{u_1}{\to} F_0 \overset{u_0}{\to} M \to 0$ be a finite $(n+1)$-presentation of M as a graded S-module. The exact sequence of S-modules $0 \to Ker(u_0) \to F_0 \to M \to 0$ shows that $\lambda_S(\mathrm{Ker}(u_0)) \geq n$; so by induction we have $\lambda_R(\mathrm{Ker}(u_0)) \geq n$ since $\lambda_R(S) \geq n+1 \geq n$. Moreover $\lambda_R(F_0) \geq n+1$ since $\lambda_R(S) \geq n+1$ and F_0 is a finitely generated free graded S-module. Hence $\lambda_R(M) \geq \inf\{\lambda_R(F_0), \lambda_R(\mathrm{Ker}(u_0))+1\} \geq n+1$ by Lemma 2.2.3(2) as desired.

Lemma 2.2.8 *Let $R \to S$ be a graded ring homomorphism such that $\lambda_R(S) \geq n-1$ and let M be a graded S-module. If M is n-presented as a graded R-module, then it is n-presented as a graded S-module.*

Proof. We proceed by induction on n. Case $n = 0$: If M is a finitely generated graded R-module, then M is also a finitely generated graded S-module. We conclude the proof by induction on n. Let M be a graded S-module such that $\lambda_R(M) \geq n+1$ and $\lambda_R(S) \geq n$. We have to show that $\lambda_S(M) \geq n+1$. By induction, we have $\lambda_S(M) \geq n$. The exact sequence of S-modules $0 \to K \to F_0 \to M \to 0$ (in which F_0 is a finitely generated free S-module), considered as an exact seguence of R-modules, shows that $\lambda_R(K) \geq \inf\{\lambda_R(F_0); \lambda_R(M)-1\} \geq n$ (Lemma 2.2.3(3)). Moreover, we have $\lambda_R(S) \geq n \geq n-1$; then by induction we have $\lambda_S(K) \geq n$; hence, $\lambda_S(M) \geq n+1$ by Lemma 2.2.3(2) as desired.

Lemma 2.2.9 *Let $R \to S$ be a graded ring homomorphism such that $\lambda_R(S) \geq n-1$ and let M be a graded S-module. If M is graded-n-coherent as a graded R-module, then it is graded-n-coherent as a graded S-module.*

Proof. Let $R \to S$ be a graded ring homomorphism such that $\lambda_R(S) \geq n-1$ and let M be a graded S-module such that M is graded-n-coherent as a graded R-module. Lemma 2.2.8 shows that $\lambda_S(M) \geq n$ since $\lambda_R(M) \geq n$ and $\lambda_R(S) \geq n-1$. Let N be a homogeneous submodule of the graded S-module M such that $\lambda_S(N) \geq n-1$. Then by Lemma 2.2.7 we have $\lambda_R(N) \geq n-1$. Thus, $\lambda_R(N) \geq n$ since M is a graded-n-coherent R-module; therefore, $\lambda_S(N) \geq n$ by Lemma 2.2.8 as desired.

Proof of Theorem 2.2.6. Let $R \to R/I$ be the canonical graded ring homomorphism such that $\lambda_R(R/I) \geq n$ and let M be a graded R-module such that $IM = 0$. If M is graded-n-coherent as a graded R-module, then it is graded-n-coherent as a graded R/I-module by Lemma 2.2.9 since $\lambda_R(R/I) \geq n \geq n-1$. Conversely, let M be a graded-n-coherent R/I-module. By Lemma 2.2.7 we have $\lambda_R(M) \geq n$ since $\lambda_R(R/I) \geq n$. Let N be a homogeneous submodule of the graded R-module M such that $\lambda_R(N) \geq n-1$. By Lemma 2.2.8 we have $\lambda_{R/I}(N) \geq n-1$ since $\lambda_R(R/I) \geq n$. Thus $\lambda_{R/I}(N) \geq n$ since M is a graded-n-coherent R/I-module and N is a homogeneous submodule of M as a graded R/I-module. Hence, $\lambda_R(N) \geq n$ by Lemma 2.2.7 ($\lambda_R(R/I) \geq n$) and this completes the proof of Theorem 2.2.6.

Remark 2.2.10 *Let the canonical graded ring homomorphism $R \to R/I$ satisfy $\lambda_R(R/I) \geq n-1$, and let M be a graded R-module such that $IM = 0$, where I is a homogeneous ideal of R. If M is graded-n-coherent as a graded R-module, then it is graded-n-coherent as an R/I-module by Lemma 2.2.9.*

2.3 Graded-n-coherent rings

Definition 2.3.1 *A graded ring R is called graded-n-coherent if it is graded-n-coherent as a graded R-module, that is, if each $(n-1)$-presented homogeneous ideal of R is n-presented.*

Remark 2.3.2 *Obviously, every n-coherent graded ring is a graded-n-coherent ring. The converse is not true in general, example 3.2 in [15] gives an example of graded-1-coherent ring which is not 1-coherent.*

The next result shows that we have already many examples of graded-n-coherent rings.

Example 2.3.3 *1. Every graded-valuation domain is a graded-n-coherent ring for each $n \geq 1$, see [3].*

2. Every graded-Noetherian ring is a graded-n-coherent for each $n \geq 1$, see [22].

Proposition 2.3.4 *Let R be a graded-n-coherent ring and let I be an $(n-1)$-presented homogeneous ideal of R. Then R/I is a graded-n-coherent ring.*

Proof. Since R is a graded-n-coherent R-module, it follows from Theorem 2.2.4(1) that R/I is a graded-n-coherent R-module; therefore, by Theorem 2.2.6, R/I is a graded-n-coherent ring.

Remark 2.3.5 *The case $n = 1$ recovers the known fact that if I is a finitely generated homogeneous ideal of a graded-1-coherent ring R, then R/I is a graded-1-coherent ring ([15], Theorem 3.7(1)).*

Theorem 2.3.6 *Let $(R_i)_{i=1,2,\ldots,m}$ be a family of graded rings. Then $\prod_{i=1}^{m} R_i$ is a graded-n-coherent ring if and only if R_i is a graded-n-coherent ring, for each $i = 1, \ldots, m$.*

To establish this Theorem, we need to prove the following Lemma.

Lemma 2.3.7 *Let R_1 and R_2 be two graded rings. Then R_i is an infinitely presented homogeneous ideal of $R_1 \times R_2$, for $i = 1, 2$*

Proof. The graded rings R_1 and R_2, more precisely $R_1 \times 0$ and $0 \times R_2$, are two finitely generated homogeneous ideals of $R_1 \times R_2$ since $0 \to R_1 \to R_1 \times R_2 \to R_2 \to 0$ and $0 \to R_2 \to R_1 \times R_2 \to R_1 \to 0$ are two exact sequences of graded rings. We finish the proof of this Lemma by induction on the degrees of presentation of the R_i using the above two exact sequences of graded rings.

Proof of Theorem 2.3.6. We proceed by induction on m, it suffices to prove the assertion for $m = 2$. Let R_1 and R_2 be two graded rings such that $R_1 \times R_2$ is a graded-n-coherent ring. Since $R_1 \cong (R_1 \times R_2)/R_2$, $R_2 \cong (R_1 \times R_2)/R_1$ are two graded ring isomorphism, and the R_i are infinitely presented homogeneous ideals of $R_1 \times R_2$ by Lemma 2.3.7, then Proposition 2.3.4 shows that R_i $(i = 1,2)$ are graded-n-coherent rings. Conversely, let R_1 and R_2 be two graded-n-coherent rings and let $I = I_1 \times I_2$ be an $(n-1)$-presented homogeneous ideal of $R_1 \times R_2$, where I_i is a homogeneous ideal of R_i; then for each $i = 1,2$: $\lambda_{R_1 \times R_2}(I_i) \geq \inf\{\lambda_{R_1 \times R_2}(I_1), \lambda_{R_1 \times R_2}(I_2)\} = \lambda_{R_1 \times R_2}(I) \geq n-1$ (Lemma 2.2.3(4)). By Lemma 2.2.8, we have $\lambda_{R_i}(I_i) \geq n-1$ ($\lambda_{R_1 \times R_2}(R_i) = \infty$ (Lemma 2.3.7)). Thus, $\lambda_{R_i}(I_i) \geq n$ since R_i is a graded-n-coherent ring and by Lemma 2.2.7, we have $\lambda_{R_1 \times R_2}(I_i) \geq n$ since $\lambda_{R_1 \times R_2}(R_i) = \infty$ (Lemma 2.3.7). Hence : $\lambda_{R_1 \times R_2}(I) = \lambda_{R_1 \times R_2}(I_1 \times I_2) = \inf\{\lambda_{R_1 \times R_2}(I_1), \lambda_{R_1 \times R_2}(I_2)\} \geq n$ and this completes the proof of Theorem 2.3.6.

2.4 Graded-valuation property in graded trivial extension

Assume that the grading monoid G is torsionless, that is a commutative, cancellative monoid and the quotient group of G is a torionfree abelian group. Let A be a graded ring, and let H denote the saturated multiplicative set of regular homogeneous elements of A. Then, by extending some definitions to the case where rings are with zero divisors, A_H, called the homogeneous total ring of quotients of A, is a ring graded by $\langle G \rangle$, where $A_H = \oplus_{\alpha \in \langle G \rangle} (A_H)_\alpha$ with $(A_H)_\alpha = \{\frac{a}{b} \mid a \in A_\beta, b$ a regular element of A_γ and $\beta - \gamma = \alpha\}$. If A is a graded integral domain (An integral domain graded by G), then A_H is called the homogeneous quotient field of A. Clearly, every nonzero homogeneous element of A_H is invertible and $(A_H)_0$ is a field. We say that A is a graded-valuation ring (gr-valuation ring for short) if either $x \in A$ or $x^{-1} \in A$ for every nonzero homogeneous element $x \in A_H$. Recall that if A is a graded ring and E is a graded A-module, then $A \propto E$ is a graded ring where $A \propto E = \oplus_{\alpha \in G}(A \propto E)_\alpha = \oplus_{\alpha \in G}(A_\alpha \oplus E_\alpha)$. This section gives a result of the transfer of gr-valuation property to graded trivial extension ring. We begin with the following result extending Theorem 1.2 in [3] to the case where rings are with zero divisors and which characterizes gr-valuation rings.

Theorem 2.4.1 *Let $A = \oplus_{\alpha\in G} A_\alpha$ be a graded ring. The following statements are equivalent:*

1. *A is a gr-valuation ring.*
2. *Either $a \mid b$ or $b \mid a$ for every nonzero homogeneous elements $a, b \in A$, one at least of which is regular.*
3. *Every pair of homogeneous (fractional) ideals of A, one at least of which is regular, are totally ordered under inclusion.*
4. *Every pair of principal homogeneous ideals of A, one at least of which is regular, are totally ordered under inclusion.*

Proof. (1) implies (2) by definition.
(2) implies (3): Let I, J be two homogeneous ideals of A, one at least of which is regular, suppose that $I \nsubseteq J$ and $J \nsubseteq I$. Let x be an homogeneous element of $I \setminus J$ and y be an homogeneous element of $J \setminus I$. Since $x = (\frac{x}{y})y \notin J$ we have $\frac{x}{y} \notin A$ and since $y = (\frac{y}{x})x \notin I$, we have $\frac{y}{x} \notin A$; therefore A is not a gr-valuation ring, a contradiction.
(3) implies (4) is trivial.
(4) implies (1): Let $x = \frac{a}{b} \in A_H$, where $a, b \in h(A)$. If $x \notin A$, then $Aa \nsubseteq Ab$, therefore $Ab \subseteq Aa$, hence $x^{-1} = \frac{b}{a} \in A$. Hence A is a gr-valuation ring.

Definition 2.4.2 *Let A be a graded ring, a graded A-module E is said to be graded-uniserial (gr-uniserial for short) if the set of its homogeneous submodules is totally ordered by inclusion.*

Next, we give a characterization for the graded trivial extension to be a gr-valuation ring. Note that here the assumption "G is a torsionfree abelian group" is necessary since the grading monoid of A and E must be the same and the fact that is "torsionfree" is used by Lemma 2.4.4.

Theorem 2.4.3 *Assume that the grading monoid is a torsionfree abelian group. Let A be a graded ring and E a nonzero graded A-module. Let $R := A \propto E$ be the graded trivial extension ring of A by E. Assume that E is a torsionfree graded A-module. Then R is a gr-valuation ring if and only if A is a gr-valuation domain and E is isomorphic to A_H, the homogeneous quotient field of fractions of A.*

Before proving Theorem 2.4.3 we establish the following lemma.

Lemma 2.4.4 *Assume that the grading monoid is a torsionfree abelian group. Let A be a graded ring, E a nonzero graded A-module, and $R := A \propto E$ be the graded trivial extension ring of A by E. If R is a gr-valuation ring, then A is a gr-valuation domain and E is a gr-uniserial A-module.*

Proof. Assume that R is a gr-valuation ring. First we wish to show that A is a gr-valuation ring and E is a gr-uniserial A-module. Let $a,b \in h(A)$, one at least of which is regular, if $(a,0)$ divides $(b,0)$ (resp., $(b,0)$ divides $(a,0)$), then a divides b (resp., b divides a). Hence A is a gr-valuation ring. On the other hand, let $x,y \in h(E)$. If $(0,x)$ divides $(0,y)$ (resp., $(0,y)$ divides $(0,x)$) then there exists $(c,z) \in R$ such that $(0,y) = (c,z)(0,x)$ (resp., $(0,x) = (c,z)(0,y)$) and so $y \in Ax$(resp., $x \in Ay$). Therefore, E is a gr-uniserial A-module.

We prove that A is an integral domain. Deny. Let $a,b \in h(A)$ such that $ab = 0, a \neq 0$ and $b \neq 0$. For each $x \in h(E)$, $(b,0)$ divides $(0,x)$ (since R is a gr-valuation ring and $(0,x)$ does not divide $(b,0)$ (since $b \neq 0$)), and so there exists $y \in E$ such that $by = x$, thus $ax = 0$ and so $a \in (0:E)$. Also, for each $x \in h(E)$, $(a,0)$ divides $(0,x)$ and so $x \in aE = 0$, a contradiction since $E \neq 0$. Therefore since the grading monoid is a torsionfree abelian group. Thus A is an integral domain.

Proof of Theorem 2.4.3. Assume that A is a gr-valuation domain and let $R := A \propto A_H$, where A_H is the homogeneous quotient field of A. Our aim is to show that R is a gr-valuation ring. Let (a,x), $(b,y) \in h(R) - \{(0,0)\}$. Two cases are then possible.

Case 1. $a = b = 0$. There exists then $c \in A$ such that $x = cy$ (resp., $y = cx$) since A_H is the homogeneous quotient field of fractions of A and A is a gr-valuation domain. Hence, $(0,x) = (c,0)(0,y)$ (resp., $(0,y) = (c,0)(0,x)$) as desired.

Case 2. $a \neq 0$ or $b \neq 0$. We may assume that $a \neq 0$ and $b \in Aa$. Let $c \in A$ such that $ac = b$, and let $z \in A_H$ such that $az + cx = y$. Hence, $(a,x)(c,z) = (b,y)$ as desired.

Conversely, assume that E is a torsionfree graded A-module, and $R = A \propto E$ is a gr-valuation ring. By Lemma 2.4.4, A is a gr-valuation domain. It remains to show that $E \simeq A_H$. Let $u \in h(E)$ such that $(0:u) = 0$, and let $f : A_H \otimes Au \to A_H \otimes E$ be the homomorphism of A-module induced by the inclusion map $Au \hookrightarrow E$. Since the homogeneous quotient field of A is a flat A-module, hence f is injective. Let $(\lambda,x) \in h(A_H \times E)$, by Lemma 2.4.4 we get that $x \in Au$ or $u \in Ax$. If $x = au$ for some $a \in A$, then $f(\lambda \otimes au) = \lambda \otimes x$. If $u \in Ax$, then there exists $a \in A$ such that $u = ax$. Thus

$$f\left(\frac{\lambda}{a} \otimes u\right) = \frac{\lambda}{a} \otimes u = \frac{\lambda}{a} \otimes ax = \lambda \otimes x.$$

Since f is an homomorphism of A-module, then for every element $(\lambda,x) \in A_H \times E$, there exists an element $y \in A_H \otimes Au$, such that $f(y) = (\lambda,x)$. Consequently, f is an isomorphism of A-module.

Now, consider the homomorphism of A-module $g : E \to A_H \otimes E$ defined by $g(x) = 1 \otimes x$. If $g(x) = 1 \otimes x = 0$, for some homogeneous element $x \in E$ then there exists $0 \neq a \in A$ such that $ax = 0$. By Lemma 4.5 $x \in Au$ or $u \in Ax$. But $u \notin Ax$ since $ax = 0$, $a \neq 0$ and $(0:u) = 0$. Hence, $x = bu$ for some $b \in A$. Then $abu = 0$, hence $ab = 0$ since $(0:u) = 0$ and so $b = 0$ (since A is a gr-valuation domain and $a \neq 0$); thus $x = 0$. Now if $g(x) = 1 \otimes x = 0$, for some $x = \sum_{\alpha \in G} x_\alpha \in E$. It follows that $1 \otimes x_\alpha = 0$ for all $\alpha \in G$ and then, by the above sentence, we have $x_\alpha = 0$ for all $\alpha \in G$, then $x = 0$. Therefore

g is injective. Let $(\lambda,x) \in h(A_H \times E)$. If $\lambda \in A$, then $\lambda \otimes x = 1 \otimes \lambda x = g(\lambda x)$. Now if $\lambda^{-1} \in A$, then there exists $y \in E$ such that $\lambda^{-1} y = x$, since $\left(\lambda^{-1}, 0\right)$ divides $(0,x)$. Hence

$$\lambda \otimes x = \lambda \otimes \left(\lambda^{-1} y\right) = 1 \otimes y = g(y)$$

Since g is an homomorphism of A-module, then for every element $(\lambda,x) \in A_H \times E$, there exists an element $y \in E$, such that $g(y) = (\lambda,x)$. Consequently, g is an isomorphism of A-module. We deduce that

$$E \simeq_g A_H \otimes_A E \simeq_f A_H \otimes_A Au \simeq A_H \otimes_A A$$

Finally, since for all multiplicatively closed subset S of A, the $S^{-1}A$-modules $S^{-1}E$ and $S^{-1}A \otimes_A E$ are isomorphic; more precisely, the map $\varphi : S^{-1}E \to S^{-1}A \otimes_A E$, where $\varphi\left(\frac{x}{s}\right) = \frac{1}{s} \otimes x$ is isomorphism. we have $A_H \otimes_A A \simeq A_H$. Hence $E \simeq A_H$.

Theorem 4.4 enriches the literature with new examples of gr-valuation rings.

Example 2.4.5 *Let K be a field graded by an arbitrary torsionfree group. Let K_H its homogeneous quotient field of fractions. The trivial ring extension of K by K_H, $K \propto K_H$ is a gr-valuation ring.*

The next theorem characterizes the gr-valuation property in the graded trivial extension ring in a general case. Recall from [44, page 179] that a graded A-module is said to be gr-divisible if $ax = b$ with $a \in h(A)$, $b \in h(E)$ has a solution in E.

Theorem 2.4.6 *Assume that the grading monoid is a torsionfree abelian group. Let A be a graded ring and E a nonzero graded A-module. Then $R := A \propto E$ is a gr-valuation ring if and only if A is a gr-valuation domain, E a gr-divisible and gr-uniserial A-module.*

Proof. Assume that R is a gr-valuation ring, then by Lemma 2.4.4, A is a gr-valuation domain and E is gr-uniserial A-module. It remains to show that E is gr-divisible, let $x \in h(E)$ and $a \in h(A) \setminus \{0\}$, $(0,x)$ and $(a,0)$ are two homogeneous elements, since R is a gr-valuation ring, two cases are then possible:

Case 1: $(0,x)$ divides $(a,0)$, then there exist (b,f) suth that $(a,0) = (0,x)(b,f)$ and so $a = 0$, a contradiction.

Case 2: $(a,0)$ divides $(0,x)$, then there exist (c,z) suth that $(0,x) = (a,0)(c,z)$ implies that $x = az$, then E is gr-divisible.

Conversely, let (a,c) and (b,d) be two elements in $h(R)$. Our aim is to show that R is a gr-valuation ring. Two cases are then possible:

Case 1: $a = b = 0$. Since E is gr-divisible, then there exists $x \in h(E)$ such that $dx = c$ (resp., $d = cx$). Hence, $(0,c) = (x,0)(0,d)$ (resp., $(0,d) = (x,0)(0,c)$) as desired.

Case 2: $a \neq 0$ or $b \neq 0$. Since A is a gr-valuation domain, we may assume that $a \neq 0$ and a divides b. Then there exists $x \in h(A)$ such that $ax = b$, then since E is

gr-divisible and since $d - xc$ is a homogeneous element (we can check this easily since $deg(x) = deg(b) - deg(a) = deg(d) - deg(c)$), there exists $y \in h(E)$ such that $ay = d - xc$. Hence $(a,c)(x,y) = (b,d)$ as desired.

CHAPTER 3

COMMUTATIVE GRADED-S-COHERENT RINGS

Abstract.

Recently, motivated by Anderson and Dumitrescu's S-finiteness. Bennis and El Hajoui introduced the notion of S-coherent rings which is the S-version of coherent rings. Let $R = \oplus_{\alpha \in G} R_\alpha$ be a commutative ring with unity graded by an arbitrary commutative monoid G, and S a multiplicatively closed subset of nonzero homogeneous elements of R. We define R to be a graded-S-coherent ring if every finitely generated homogeneous ideal of R is S-finitely presented. The purpose of this chapter* is to give the graded version of several results proved in [19]. We show the non triviality of our generalization by giving an example of a graded-S-coherent ring which is not S-coherent and an example of a graded-S-coherent ring which is not graded-coherent. As a special result of our study, we give the graded version of the Chase's characterization of S-coherent rings.

3.1 Introduction

Let R be a ring, M an R-module and S a multiplicatively closed subset of R, M is called S-finite if there exists a finitely generated submodule N of M such that $sM \subseteq N$ for some $s \in S$, this notion was introduced by Anderson and Dumitrescu, see [5]. According to [19], E is called an S-finitely presented R-module if there exist an exact sequence $0 \to F_1 \to F_0 \to E \to 0$ of R-modules such that F_0 is a finitely generated free R-module and F_1 is S-finite. Any finitely presented R-module is an S-finitely presented R-module; while the converse is false in general, for more results and details, the reader can refer to [19,

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Section 2]. A finitely generated R-module M is said to be an S-coherent R-module if every finitely generated R-submodule of M is an S-finitely presented R-module; and a ring R is said to be an S-coherent ring if R is S-coherent as an R-module.

The concept of coherent rings is one of the most significant notions in homological algebra. Because of its importance, there have been many generalizations to the notion of coherent rings. Some of them are to the context of graded rings, see [10], [15] and [16].

According to [15], a graded ring R is said to be a graded-coherent ring if every finitely generated homogeneous ideal of R is a finitely presented ideal of R. Clearly, every coherent graded ring is a graded-coherent ring, but the converse is false see [15, Example 3.2]. In this chapter, we are interested in the graded version of S-coherent modules and rings which are called, respectively, graded-S-coherent modules and graded-S-coherent rings (see Definition 3.2.1). In Section 3.2, we introduce and study the notion of graded-S-coherent rings and the more general notion of graded-S-coherent modules (over an arbitrary graded ring). Our main aims in this section are to characterize graded-S-coherent modules and graded-S-coherent rings and to establish some of their basic properties. For rings, we have S-coherent $\Rightarrow$ graded-S-coherent and graded-coherent $\Rightarrow$ graded-S-coherent with neither implication being reversible.

3.2 Main results

This section initiates the study of graded-S-coherent modules and rings. We begin, following the classical case, by giving the definition of graded-S-coherent modules.

Definition 3.2.1 *A graded R-module M is said to be graded-S-coherent, if it is finitely generated and every finitely generated homogeneous submodule of M is S-finitely presented. And a graded ring R is said to be graded-S-coherent, if it is graded-S-coherent as a graded R-module; that is, if every finitely generated homogeneous ideal of R is S-finitely presented.*

We next collect some immediate classes of graded-S-coherent modules and rings.

Remark 3.2.2 *Let R be a graded ring. Then the following statements hold:*

(1) *Every finitely generated homogeneous R-submodule of a graded-S-coherent R-module is graded-S-coherent.*

(2) *Recall from [15] that a graded R-module M is said to be graded-coherent if it is finitely generated and every finitely generated homogeneous submodule of M is finitely presented. Clearly, any graded-coherent R-module is a graded-S-coherent R-module since every finitely presented module is S-finitely presented. Hence, every graded-Noetherian ring, see [22], graded-valuation domain, see [3], and graded Prüfer domain, see [4], is a graded-S-coherent rings.*

(3) *Obviously, every S-coherent graded R-module is a graded-S-coherent R-module and every S-coherent graded ring is a graded-S-coherent ring. But the converse is not true in general, as shown by the following example.*

Example 3.2.3 *If A is a countable direct product of $\mathbb{Q}[[t,u]]$'s, consider the polynomial ring graded by $\mathbb{N}$ via $(A[X])_n = AX^n$ for every $n \in \mathbb{N}$, and let $S = \{1\}$, then $A[X]$ is graded-S-coherent but not S-coherent.*

Proof. By [15, Example 3.2] the polynomial ring $A[X]$ is graded-coherent and then it is graded-S-coherent, but it is not S-coherent. In fact, by [56, Proposition 18] $A[X]$ is not coherent since there exists an ideal I which is not finitely generated and it is the intersection of two finitely generated ideals. Now, since $S = \{1\}$, I is not even S-finite, and then $A[X]$ is not S-coherent by [19, Theorem 3.8(3)].

The following result studies the behavior of graded-S-coherence of graded modules in short exact sequences. It is the graded version of [19, Proposition 3.2].

Proposition 3.2.4 *Let $0 \longrightarrow P \xrightarrow{\alpha} N \xrightarrow{\beta} M \longrightarrow 0$ be an exact sequence of graded R-modules. Then the following statements hold:*

1. If P is finitely generated, N is graded-S-coherent and β has a cancellable degree, then M is graded-S-coherent.

2. If M is graded-coherent, P is graded-S-coherent and α has a cancellable degree, then N is graded-S-coherent.

3. If N is graded-S-coherent and P is finitely generated, then P is graded-S-coherent.

Proof. 1. Since N is finitely generated and $\beta : N \longrightarrow M$ is surjective, we have that M is finitely generated. Let M_1 be a finitely generated homogeneous submodule of M. Since β has a cancellable degree, the submodule $\beta^{-1}(M_1)$ of N is homogeneous, and we have the following exact sequence of graded R-modules:

$$0 \to P \to \beta^{-1}(M_1) \to M_1 \to 0$$

Now, $\beta^{-1}(M_1)$ is finitely generated since M_1 and P are so. Therefore, since N is graded-S-coherent, $\beta^{-1}(M_1)$ is S-finitely presented and so using [19, Theorem 2.4(4)], M_1 is S-finitely presented, as desired.

2. Since M and P are finitely generated modules, we have that N is finitely generated. Now, let N_1 be a finitely generated homogeneous submodule of N. Since α has a cancellable degree, consider the exact sequence of graded R-modules $0 \longrightarrow \operatorname{Ker}\left(\beta_{|N_1}\right) \xrightarrow{\alpha} N_1 \xrightarrow{\beta} \beta(N_1) \longrightarrow 0$. Then, since $\beta(N_1)$ is a finitely generated homogeneous submodule of the graded-coherent module M, $\beta(N_1)$ is finitely presented. Hence, $\operatorname{Ker}\left(\beta_{|N_1}\right)$ is finitely generated, and since P is graded-S-coherent, $\operatorname{Ker}\left(\beta_{|N_1}\right)$ is S-finitely presented since it is homogeneous. Therefore, by [19, Theorem 2.4(2)], N_1 is S-finitely presented.

3. This is an immediate consequence of Remark 3.2.2(1).

Remark 3.2.5 *1. Let G be an abelian group, R be a graded ring and S a multiplicatively closed subset of R_0. Then, recall from [38], that R is said to be graded-S-Noetherian if every homogeneous ideal of R is S-finite. Note that every graded-S-Noetherian ring is graded-S-coherent. Indeed, this follows by applying [19, Proposition 2.3] and from the fact that when R is graded-S-Noetherian, every finitely generated graded R-module is graded-S-Noetherian.*

We next give our definition which generalizes the definition given in [38].

Definition 3.2.6 *Let R be a graded ring and S a multiplicatively closed subset of homogeneous elements of R (not necessarily of R_0). Then, R is said to be graded-S-Noetherian if every homogeneous ideal of R is S-finite.*

The next result presents the graded version of Proposition $2(f)$ in [5]. For an ideal I of R, $IR_S \cap R$ means the S-saturation of I.

Proposition 3.2.7 *Let G be a group, R be a graded ring and $S \subseteq R$ a multiplicatively closed subset of homogeneous elements of R. Then, R is graded-S-Noetherian if and only if R_S is graded-Noetherian and for every finitely generated homogeneous ideal I of R, $IR_S \cap R = (I : s)$ for some $s \in S$.*

Proof. Suppose that R is graded-S-Noetherian and choose a homogeneous ideal J of R_S. Then there exists an ideal I of R such that $I \cap S = \emptyset$ and $J = IR_S$. Then note that $IR_S \cap R$ is a homogeneous ideal of R which is graded-S-Noetherian, then there exists $s \in S$ such that $s(IR_S \cap R) \subseteq K \subseteq IR_S \cap R$ for some finitely generated ideal K of R. This implies that

$$[s(IR_S \cap R)]R_S = IR_S \subseteq KR_S \subseteq [IR_S \cap R]R_S = IR_S.$$

Thus, $J = KR_S$ is finitely generated, that is, R_S is a graded-Noetherian ring. Now, suppose that I is a finitely generated homogeneous ideal of R. Then it is clear that $(I : s') \subseteq IR_S \cap R$ for every $s' \in S$. Put $J = IR_S \cap R$. Then J is S-finite, there exist $x_1, x_2, \ldots, x_n \in R$ and $t \in S$ such that $tJ \subseteq Rx_1 + \cdots + Rx_n \subseteq J$. Since $x_i \in J = IR_S \cap R$, we get $t_i x_i \in I$ for some $t_i \in S$, where $i = 1, 2, \ldots, n$. Fix $s = tt_1t_2 \cdots t_n$. Then note that for every $x \in J$, we have $sx \in I$ which implies that $J \subseteq (I : s)$, as needed.

For the reverse implication, assume that R_S is graded-Noetherian and let J be a homogeneous ideal of R. Then $JR_S \cap R = LR_S \cap R$ for some finitely generated homogeneous ideal $L \subseteq J$. If $s \in S$ satisfies $LR_S \cap R = (L : s)$, then $sJ \subseteq L$, which completes our proof.

The following proposition investigates a change of rings result.

Proposition 3.2.8 *Let $\phi : R \longrightarrow L$ be a graded ring homomorphism, M be a graded L-module and S a multiplicatively subset of homogeneous elements of R such that $0 \notin \phi(S)$.*

Then, if the graded R-module L is finitely generated and M is graded-S-coherent as an R-module, then M is graded-$\phi(S)$-coherent as an L-module.

Proof. Let N be a finitely generated homogeneous L-submodule of M. Then so is N as an R-submodule of M. Hence N is S-finitely presented over R since M is graded-S-coherent over R, then, by [19, Proposition 2.6], N is $\phi(S)$-finitely presented over L since the R-module L is finitely generated.

Proposition 3.2.9 *Let R be a graded ring, I be an S-finite homogeneous ideal of R where S is a multiplicatively closed subset of homogeneous elements of R and M be a graded R/I-module. Assume that $I \cap S = \emptyset$ so that $V := \{s+I \in R/I; s \in S\}$ is a multiplicatively closed subset of homogeneous elements of R/I. Then the following statements hold:*

1. M is graded-V-coherent as an R/I-module if and only if it is graded-S-coherent as an R-module.

2. Let I be a finitely generated ideal of R, if R is graded-S-coherent ring, then R/I is graded-V-coherent ring. The converse holds if R/I is a graded-coherent ring and I is graded-S-coherent as an R-module.

Proof. 1. ($\Rightarrow$) Let M be a graded-V-coherent R/I-module and let N be a finitely generated submodule of M, since M is graded-V-coherent as an R/I-module then N is V-finitely presented R/I-module. Now, since I is S-finite, by [19, Proposition 2.7], N is S-finitely presented R-module and so M is graded-S-coherent R-module.

($\Leftarrow$) Similar to the proof of the direct implication.

2. The direct implication follows from (1) by using Proposition 3.2.4(1). The reverse implication is a direct application of Proposition 3.2.4(2).

The following theorem clarifies the situation for the product of graded-S-coherent modules.

Theorem 3.2.10 *Let M_i be a graded R_i-module and S_i a multiplicatively closed set of homogeneous elements of R_i for $i = 1,2,\ldots,n$. Suppose that $R = R_1 \times R_2 \times \cdots \times R_n$, $M = M_1 \times M_2 \times \cdots \times M_n$ and $S = S_1 \times S_2 \times \cdots \times S_n$. The following statements are equivalent.*

1. M_i is a graded-S_i-coherent R_i-module for each $i = 1,2,\ldots,n$.

2. M is a graded-S-coherent R-module.

Proof. By mathematical induction on n, it suffices to prove the case $n = 2$.

$(1) \Rightarrow (2)$: Suppose that N is a finitely generated homogeneous submodule of M. Then we can write $N = N_1 \times N_2$ for some finitely generated homogeneous submodule N_i of M_i for each $i = 1,2$. Since M_i is a graded-S_i-coherent R_i-module, for each $i = 1,2$, there exists an exact sequence of R_i-modules $0 \to K_i \xrightarrow{\alpha_i} F_i \xrightarrow{\beta_i} N_i \to 0$ where K_i is S_i-finite and F_i is finitely generated free R_i-module. Put $K = K_1 \times K_2$, $F = F_1 \times F_2$. Consider the R-homomorphisms $\alpha_1 \times \alpha_2 : K \to F$ defined by $\alpha(k_1,k_2) = (\alpha_1(k_1), \alpha_2(k_2))$ and also

$\beta_1 \times \beta_2 : F \to N$ defined by $\beta(f_1, f_2) = (\beta_1(f_1), \beta_2(f_2))$. Then $0 \to K \overset{\alpha_1 \times \alpha_2}{\to} F \overset{\beta_1 \times \beta_2}{\to} N \to 0$ is an exact sequence of R-modules, where K is S-finite and F is finitely generated free. Thus, N is S-finitely presented, namely, M is a graded-S-coherent R-module.

$(2) \Rightarrow (1)$: Suppose that M is a graded-S-coherent R-module. Consider the graded ring epimorphism $\pi_1 : R \to R_1$. Note that R_1 is a finitely generated R-module and $0 \notin \pi_1(S) = S_1$. Take a finitely generated homogeneous submodule N_1 of M. Then $N = N_1 \times (0)$ $(\cong N_1)$ is a finitely generated homogeneous submodule of M. Since M is a graded-S-coherent R-module, N $(\cong N_1)$ is an S-finitely presented R-module. Then by [19, Proposition 2.6], N_1 $(\cong N)$ is an S_1-finitely presented R_1-module. Therefore, M_1 is a graded-S_1-coherent R_1-module. Likewise, M_2 is a graded-S_2-coherent R_2-module.

As a consequence of the previous theorem, we have the following result.

Corollary 3.2.11 *Let $R = \prod_{i=1}^{n} R_i$ be a direct product of graded rings $R_i (n \in \mathbb{N})$ and $S = \prod_{i=1}^{n} S_i$ be a cartesian product of multiplicatively closed sets S_i of homogeneous elements of R_i. Then, R is graded-S-coherent if and only if R_i is graded-S_i-coherent for every $i \in \{1, \ldots, n\}$.*

Before giving the following two remarks, we recall from [34] the idealization construction. For a ring R and an R-module E (both not necessarily nontrivially graded). The trivial ring extension of A by E is the ring $A := R \propto E$ whose underlying group is $A \oplus E$ with mulitplication defined by $(r,e)(r',e') = (rr', re' + r'e)$.

Remark 3.2.12 (1). *As a particular case of Remark 3.2.2(2), any graded-coherent ring is a graded-S-coherent ring, see [15]. The converse is not true in general. Consider the graded ring $A = \mathbb{Z} \propto (\mathbb{Z}_2)^{(\mathbb{N})}$ with its natural $\mathbb{Z}_2$-grading; $A_0 = \mathbb{Z} \propto 0$ and $A_1 = 0 \propto (\mathbb{Z}_2)^{(\mathbb{N})}$ and consider the multiplicative set of homogeneous elements $S = \{2^n : n \in \mathbb{N}\} \propto 0$. Since $(2,0)$ is a homogeneous element and $(0 : (2,0)) = 0 \propto (\mathbb{Z}_2)^{(\mathbb{N})}$ is not a finitely generated ideal, then, according to [15, Theorem 3.3] A is not graded-coherent. Then, taking any homogeneous ideal I of A, we have that $(2,0)I$ is finitely generated. Hence, A is graded-S-Noetherian and so graded-S-coherent by Remark 3.2.5.*

(2). *Recall from [19, Remark 3.4(3)] that if M is an S-finitely presented R-module, then M_S is a finitely presented R_S-module. Thus, if R is a graded-S-coherent ring, R_S is a graded-coherent ring.*

Now, we give the graded version of the S-counterpart of the classical Chase's result [25, Theorem 2.2]. For reference purposes, it will be helpful to recall the following elementry lemma.

Lemma 3.2.13 ([29], Lemma 2.3.1) *Let R be a ring, let $I = (u_1, u_2, \ldots, u_r)$ be a finitely generated ideal of $R (r \in \mathbb{N})$ and let $a \in R$. Set $J = I + Ra$. Let F be a free module on generators $x_1, x_2, \ldots, x_{r+1}$ and let $0 \longrightarrow K \longrightarrow F \xrightarrow{f} J \longrightarrow 0$, be an exact sequence*

with $f(x_i) = u_i (1 \leq i \leq r)$ and $f(x_{r+1}) = a$. Then there exists an exact sequence $0 \longrightarrow K \cap F' \longrightarrow K \xrightarrow{g} (I : a) \longrightarrow 0$, where $F' = \bigoplus_{i=1}^{r} Rx_i$.

Theorem 3.2.14 *Let R be a graded ring. The following assertions are equivalent:*

1. *R is graded-S-coherent.*

2. *$(I : a)$ is an S-finite ideal of R, for every finitely generated homogeneous ideal I of R and for every homogeneous element $a \in R$.*

3. *$(0 : a)$ is an S-finite ideal of R for every homogeneous element $a \in R$ and the intersection of two finitely generated homogeneous ideals of R is an S-finite ideal of R.*

Proof. $(1) \Longrightarrow (2)$: Let I be a finitely generated homogeneous ideal of R and let a be a homogeneous element of R. Then $J = I + Ra$ is a finitely generated homogeneous ideal of the graded-S-coherent ring R, and so $J = I + Ra$ is S-finitely presented. Then, there exist an exact sequence of R-modules

$$0 \longrightarrow L \longrightarrow K \longrightarrow J \longrightarrow 0,$$

where L is S-finite. Thus, by Lemma 3.2.13, there exists a surjective homomorphism $L \longrightarrow (I : a)$ which makes $(I : a)$ an S-finite ideal.

$(2) \Longrightarrow (1)$: Let $I = \sum_{i=1}^{n} Rr_i$ be a finitely generated homogeneous ideal of R. We use induction on n to prove that I is S-finitely presented. For $n = 1$, we consider the exact sequence of R-modules: $0 \longrightarrow (0 : r_1) \longrightarrow R \longrightarrow I \longrightarrow 0$. By hypothesis (2), $(0 : r_1)$ is S-finite and so I is S-finitely presented, as desired.

For $n > 1$, let $J = \sum_{i=1}^{n-1} Rr_i$. Consider the exact sequences of R-modules: $0 \longrightarrow \ker f \hookrightarrow R^n \xrightarrow{f} I \longrightarrow 0$ and $0 \longrightarrow \ker g \hookrightarrow R^{n-1} \xrightarrow{g} J \longrightarrow 0$ with $f(e_j) = g(e_j) = r_j, 1 \leq j \leq n-1$ and $f(e_n) = r_n$, where $(e_j)_{j=1}^{n}$ is the canonical basis of R^n. Then, by Lemma 3.2.13, there exists an R-module homomorphism $\alpha : \ker f \longrightarrow (J : r_n)$ such that the sequence of R-modules $0 \longrightarrow \ker g \hookrightarrow \ker f \xrightarrow{\alpha} (J : r_n) \longrightarrow 0$ is exact. By hypothesis (2), $(J : r_n)$ is S-finite, and by induction hypothesis, J is S-finitely presented so that $\ker g$ is S-finite. Therefore, $\ker f$ is S-finite so that I is S-finitely presented, as desired.

$(1) \Longrightarrow (3)$: Consider the exact sequence of R-modules: $0 \longrightarrow (0 : a) \longrightarrow R \longrightarrow Ra \longrightarrow 0$, where a is a homogeneous element of R. Since Ra is a finitely generated homogeneous ideal of the graded-S-coherent ring R, Ra is S-finitely presented. Therefore $(0 : a)$ is S-finite, by [19, Proposition 2.3] as desired. Then, let I and J be two finitely generated homogeneous ideals of R. Then so is $I + J$ and hence $I + J$ is S-finitely presented since R is graded-S-coherent. Therefore $I \cap J$ is S-finite, by [19, Corollary 2.5].

$(3) \Longrightarrow (1)$: Let $I = \sum_{i=1}^{n} Rr_i$ be a finitely generated homogeneous ideal of R. We use induction on n to show that I is S-finitely presented.

For $n = 1$, we consider the exact sequence of R-modules: $0 \longrightarrow (0 : r_1) \longrightarrow R \longrightarrow I \longrightarrow 0$. By hypothesis (3), $(0 : r_1)$ is S-finite and so I is S-finitely presented, as desired. Let $n > 1$. By induction hypothesis, $\sum_{i=1}^{n-1} Rr_i$ and Rr_n are S-finitely presented and, by hypothesis (3), the intersection $\left(\sum_{i=1}^{n-1} Rr_i\right) \cap Rr_n$ is S-finite. Therefore, $I = \sum_{i=1}^{n-1} Rr_i + Rr_n$ is S-finitely presented, by [19, Corollary 2.5], as desired.

Recall from [45] that a homogeneous ideal M of a graded ring R is said to be a homogeneous-maximal ideal if it is maximal among proper homogeneous ideals; equivalently, if every nonzero homogeneous element of R/M is invertible and a graded ring is said to be graded-local if it has a unique homogeneous-maximal ideal.

Proposition 3.2.15 *Assume that the grading monoid G is cancellative and let (R,M) be a graded-local ring such that $M^2 = 0$ and S a multiplicatively closed set of homogeneous elements of R. The following statements are equivalent.*

1. *R is a graded-S-coherent ring.*

2. *$(0 : x)$ is an S-finite ideal for every homogeneous element $x \in R$.*

3. *M is S-finite.*

Proof. $(1) \Rightarrow (2)$: Follows from Theorem 3.2.14.

$(2) \Rightarrow (3)$: Assume that M is not zero. Choose $0 \neq x \in M \cap h(R)$. Since $xM = 0$, we have $M \subseteq (0 : x) \subseteq M$, that is $M = (0 : x)$. Then by assumption, M is an S-finite ideal.

$(3) \Rightarrow (1)$: Suppose that M is S-finite and I is a finitely generated homogeneous ideal of R. Assume that $\{a_1, a_2, \ldots, a_n\}$ is a minimal generator set of homogeneous elements of I. Consider the exact sequence $0 \to Ker\beta \hookrightarrow R^n \xrightarrow{\beta} I \to 0$ where $\beta : R^n \to I$ is defined by $\beta(x_1, x_2, \ldots, x_n) = x_1a_1 + x_2a_2 + \cdots + x_na_n$. Since $M^2 = 0$, we can easily get $\prod_{i=1}^{n} M \subseteq Ker\beta$. Let $(x_1, x_2, \ldots, x_n) \in Ker\beta$ be a homogeneous element. Then $x_1a_1 + x_2a_2 + \cdots + x_na_n = 0$. Assume that for some i, $x_i \notin M$. Then we have x_i is a unit, so we have $a_i = -x_i^{-1}(x_1a_1 + x_2a_2 + \cdots + x_{i-1}a_{i-1} + x_{i+1}a_{i+1} + \cdots + x_na_n)$. This gives $I = (a_1, a_2, \ldots, a_{i-1}, a_{i+1}, \ldots, a_n)$ which is a contradiction. So we have $Ker\beta \subseteq \prod_{i=1}^{n} M$, that is, $\prod_{i=1}^{n} M = Ker\beta$ (since G is a cancellative monoid and so β has a cancellable degree, thus Kerβ is a homogeneous submodule and so every element of Kerβ is a sum of its homogeneous elements). Since M is S-finite, so is $\prod_{i=1}^{n} M = Ker\beta$. Thus, I is S-finitely presented, that is, R is a graded-S-coherent ring.

We end this section with the following result which studies the transfer of graded-S-coherence under localizations.

Proposition 3.2.16 *Assume that the grading monoid G is a group and let R be a graded ring. If R is a graded-S-coherent ring, then R_V is a graded-S_V-coherent ring for every multiplicative set V of homogeneous elements of R.*

Proof. R_V is a graded ring since G is a group and V is a multiplivative set of homogeneous elements of R. Now, let J be a finitely generated homogeneous ideal of R_V. Then, there is a finitely generated homogeneous ideal I of R such that $J = I_V$. Since R is graded-S-coherent, I is S-finitely presented. Then, by [19, Lemma 3.10], the homogeneous ideal $J = I \otimes_R R_V$ of R_V is S_V-finitely presented, as desired.

CHAPTER 4

ON GRADED n-IRREDUCIBLE IDEALS OF COMMUTATIVE GRADED RINGS

Abstract.

Let R be a graded ring with nonzero identity and n a positive integer. Our principal aim in this chapter* is to introduce and study the notions of graded n-irreducible and strongly graded n-irreducible ideals which are generalizations of n-irreducible and strongly n-irreducible ideals to the context of graded rings, respectively. A proper graded ideal I of R is called graded n-irreducible (respectively, strongly graded n-irreducible) if for each graded ideals $I_1, \ldots, I_{n+1}$ of $R, I = I_1 \cap \cdots \cap I_{n+1}$ (respectively, $I_1 \cap \cdots \cap I_{n+1} \subseteq I$) implies that there are n of the I_i 's whose intersection is I (respectively, whose intersection is in I). In order to give a study to this notions, we give the graded version of several other results, some of them are well known. Finally, as a special results, we give an example of a graded n-irreducible ideal which is not an n-irreducible ideal and an example of a graded ideal which is graded n-irreducible, but not graded $(n-1)$-irreducible.

4.1 Introduction

Recently, there have been various generalizations of graded prime ideals in several papers. Among the many recent generalizations of the notion of graded prime ideals in the literature, we find the following, defined first by M. Refai and K. F. Al Zoubi [51, Definition 2.13]. A proper graded ideal I of a commutative graded ring R is graded irreducible if $I = J \cap K$ for some graded ideals I and K of R implies that either $I = J$ or $I = K$. A proper ideal I of R is said to be strongly graded irreducible if for each graded ideals J, K

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of $R, J \cap K \subseteq I$ implies that $J \subseteq I$ or $K \subseteq I$. A graded ideal I is said to be graded maximal if $I \neq R$ and if it is maximal among graded ideals, equivalently, if R/I is a graded field, that is, if every nonzero homogeneous element of R/I is invertible. Our aim in this chapter is to study graded n-irreducible and strongly graded n-irreducible ideals which are generalizations of n-irreducible and strongly n-irreducible ideals to the context of graded rings, respectively. Let n be a positive integer. According to [61], a proper ideal I of a ring R is said to be n-irreducible (respectively, strongly n-irreducible) ideal if whenever $I_1 \cap \cdots \cap I_{n+1} = I$ (respectively, $I_1 \cap \cdots \cap I_{n+1} \subseteq I$) for some ideals $I_1, \ldots, I_{n+1}$ of R, then there are n of the I_i's whose intersection is I (respectively, whose intersection is contained in I). The notion of graded 2-irreducible ideals have been introduced in [55, Definition 2.20], an ideal I is called graded 2-irreducible if whenever $I = J \cap K \cap L$ for some graded ideals J, K and L of R, then $J \cap L \subset I$ or $K \cap L \subset I$ or $J \cap K \subset I$.

Our work is motivated by several definitions and concepts which we will recall some of them. In [55], the authors introduced a generalization of graded prime ideals called graded 2-absorbing ideals, and this idea is generalized also by the authors in a paper [30] to a concept called graded n-absorbing ideals. According to [30, Definition 2.1], a proper graded ideal I of R is called a graded n-absorbing ideal if whenever $x_1 \ldots x_{n+1} \in I$ for $x_1, \ldots, x_{n+1} \in h(R)$, then there are n of the x_i's whose product is in I. Thus a graded 1-absorbing ideal is just a graded prime ideal. We refer the readers to [30] for the module version of graded n-absorbing ideals. Recall that for any graded ideal I of R, the graded radical of I is defined as follows:

$$Gr(I) = \left\{ a = \sum_{g \in G} a_g \in R : \forall g \in G,\ \exists n_g > 0 \text{ such that } a_g^{n_g} \in I \right\}.$$

Note that $Gr(I)$ is a graded ideal of R. We refer the reader to [52, Proposition 2.4] for the basic properties of the graded radical. According to [51, Definition 1.5], a proper graded ideal I is said to be graded-primary if whenever $a, b \in h(R)$ with $ab \in I$ then $a \in I$ or $b \in Gr(I)$. The concept of graded 2-absorbing primary ideals, as a generalization of graded-primary ideals was introduced and investigated in [55].

Let n be a positive integer. A proper graded ideal I of a graded ring R is called graded n-irreducible if whenever $I = I_1 \cap \cdots \cap I_{n+1}$ for some graded ideals $I_1, \ldots, I_{n+1}$ of R, then there are n of the I_i's whose intersection is I. Obviously, any graded irreducible ideal is a graded n-irreducible ideal. A proper graded ideal I of a graded ring R is called strongly graded n-irreducible if for each graded ideals $I_1, \ldots, I_{n+1}$ of R, $I_1 \cap \cdots \cap I_{n+1} \subseteq I$ implies that there are n of the I_i's whose intersection is in I. Clearly, any strongly graded irreducible ideal is a strongly graded n-irreducible ideal.

This chapter is organized as follows, in the second Section, we give the main properties of graded n-irreducible and strongly graded n-irreducible ideals. Among others results in this section, we will prove that a strongly graded n-irreducible ideal is a graded n-irreducible ideal, that the intersection of n strongly graded irreducible ideals is a strongly

graded n-irreducible ideal. Nevertheless, the product of $(n+1)$ graded comaximal ideals need not be a graded n-irreducible ideal. Besides the above results, the stability of strongly graded n-irreducible (and hence graded n-irreducible) ideals with respect to various graded ring-theoretic constructions such as the graded localization and factor of graded rings, we also characterize the strongly graded n-irreducible ideals in the direct product of a finite number of graded rings. In particular, as a main result of this section, the relationship between strongly graded n-irreducible ideals and graded n-absorbing ideals is considered, see Theorem 4.2.15, Example 4.2.14.

We devote Section 4.3 to the study of graded n-irreducible and the strongly graded n-irreducible ideals in several classes of commutative graded rings. Among several other results, we prove that a nonzero graded ideal I of a graded principal ideal domain R (gr-PID for short) is graded n-irreducible if and only if I is strongly graded n-irreducible if and only if I is graded n-absorbing primary if and only if $I = R\left(p_1^{l_1} \dots p_m^{l_m}\right)$, for some distinct homogeneous prime elements $p_1, \dots, p_m$ of R and some natural numbers $l_1, \dots, l_m$ such that $m \leq n$. As a consequence, an example for which a graded ring R has a graded ideal which is graded n-irreducible, but not graded $(n-1)$-irreducible is given. Also, we prove that a proper graded ideal I of a graded von Neumann regular ring R is graded n-irreducible if and only if I is graded n-absorbing. We close this section by proving that if a proper graded ideal I of a graded-Noetherian ring is graded n-irreducible, then either I is graded irreducible or I is the intersection of exactly n graded irreducible ideals of R.

Finally, in order to give the graded version of the above results, we give the graded version of several other results, some of them are well known, see Theorems 4.2.10, 4.2.11, 4.2.12, 4.2.13, 4.3.1 and 4.3.3.

4.2 Graded n-irreducible and strongly graded n-irreducible ideals

Definition 4.2.1 *Let n be a positive integer. A proper graded ideal I of a graded ring R is said to be graded n-irreducible (respectively, strongly graded n-irreducible) ideal if whenever $I_1 \cap \dots \cap I_{n+1} = I$ (respectively, $I_1 \cap \dots \cap I_{n+1} \subseteq I$) for some graded ideals $I_1, \dots, I_{n+1}$ of R, then there are n of the I_i's whose intersection is I (respectively, whose intersection is contained in I).*

By definition, one can easily see that every n-irreducible graded ideal of R is also a graded n-irreducible ideal. However, the converse is not always true as shown by the following example.

Example 4.2.2 *Consider the ideal $I = (2)$ of the Gaussian integer graded ring $\mathbb{Z}[i]$ with $G = \mathbb{Z}_2$. Then $I = (2)$ is a graded prime ideal of $\mathbb{Z}[i]$ since 2 is a homogeneous element*

of R and then by Proposition 4.2.3 given next, I is graded irreducible. But, I is not irreducible.

We next give some basic properties of graded n-irreducible ideals and strongly graded n-irreducible ideals.

Proposition 4.2.3 *Let I be a graded ideal of a graded ring R, and let m and n be positive integers. Then:*

1. *If I is strongly graded n-irreducible, then I is graded n-irreducible.*
2. *$I = \{0\}$ is graded n-irreducible if and only if I is strongly graded n-irreducible.*
3. *If I is a graded prime ideal, then I is strongly graded n-irreducible.*
4. *There is a minimal strongly graded n-irreducible ideal of R over I.*
5. *If I is graded n-irreducible, then I is graded m-irreducible for all integer $m \geq n$.*
6. *I is a graded n-irreducible ideal if and only if whenever $I = I_1 \cap \cdots \cap I_m$ for some graded ideals $I_1, \ldots, I_m$ of R with $m > n$, then there are n of the I_i's whose intersection is I.*
7. *If I is strongly graded n-irreducible and J is a graded ideal of R such that $J \subseteq I$, then I/J is a strongly graded n-irreducible ideal of R/J.*

Proof. (1) Suppose that I is strongly graded n-irreducible and let $I_1, \ldots, I_{n+1}$ be $(n+1)$ graded ideals of R such that $I_1 \cap \cdots \cap I_{n+1} = I$. Then $I_1 \cap \cdots \cap I_{n+1} \subseteq I$, and therefore there are n of the I_i's whose intersection is in I. Otherwise, I is in any intersection of n graded ideals of I_i's, and it then follows that I is the intersection of n graded ideals of the I_i's.

(2) The sufficient condition follows directly from (1). Conversely, suppose that I is graded n-irreducible. Let $I_1, \ldots, I_{n+1}$ be $(n+1)$ graded ideals of R such that $I_1 \cap \cdots \cap I_{n+1} \subseteq I = \{0\}$, then $I_1 \cap \cdots \cap I_{n+1} = I$. It follows by hypothesis that there are n of the I_i's whose intersection is I, therefore this intersection is in I.

(3) Suppose that I is graded prime and let $I_1, \ldots, I_{n+1}$ be a graded ideals of R such that $I_1 \cap \cdots \cap I_{n+1} \subseteq I$. Then $(I_1 \cap \cdots \cap I_n) \cap I_{n+1} \subseteq I$. Therefore, either $(I_1 \cap \cdots \cap I_n) \subseteq I$ or $I_{n+1} \subseteq I$. If $(I_1 \cap \ldots \cap I_n) \subseteq I$, we are done. If $I_{n+1} \subseteq I$, then any intersection of I_{n+1} with $(n-1)$ graded ideals of the other I_i's is in I. This implies that I is strongly graded n-irreducible, as asserted.

(4) Let $Z = \{J \mid J$ is a strongly graded n-irreducible ideal of R containing $I\}$. Since every graded maximal ideal is strongly graded n-irreducible by (3), then $Z \neq \emptyset$. Let $\{J_i\}_{i \in L}$ be a chain in Z, then since any intersection of graded ideals is a graded ideal, $J = \bigcap_{i \in L} J_i$ is a strongly graded n-irreducible ideal containing I. By Zorn's lemma Z has a minimal element.

The proofs of (5) and (6) are clear.

(7) Let $I_1,\dots,I_{n+1}$ be a graded ideals of R containing J such that $(I_1/J)\cap\cdots\cap(I_{n+1}/J) \subseteq I/J$. Hence, $I_1\cap\cdots\cap I_{n+1}\subseteq I$. Therefore, there are n of the I_i's whose intersection is in I. Without loss of generality, assume that $I_1\cap I_2\cap\cdots\cap I_n\subseteq I$, then $(I_1/J)\cap\cdots\cap(I_n/J)\subseteq I/J$. Consequently, I/J is strongly graded n-irreducible.

Remark 4.2.4 *Let I be a proper graded ideal of a graded ring R. By Proposition 4.2.3(5), we have that a graded n-irreducible ideal is also graded m-irreducible ideal for all integers $m \geq n$. If I is a graded n-irreducible ideal of R for some positive integer n, then just like the ungraded case we can define $gr\text{-}\omega_R(I) = \min\{n \mid I$ is a graded n-irreducible ideal of $R\}$; else, set $gr\text{-}\omega_R(I) = \infty$. It is appropriate to define $gr\text{-}\omega_R(R) = 0$. Thus for any graded ideal I of R, $gr\text{-}\omega_R(I) \in \mathbb{N}\cup\{\infty\}$ with $gr\text{-}\omega_R(I) = 1$ if and only if I is a graded irreducible ideal of R and $gr\text{-}\omega_R(I) = 0$ if and only if $I = R$.*

Proposition 4.2.5 *If I_j is a graded n_j-irreducible ideal of a graded ring R for each $1 \leq j \leq m$, then $I_1\cap\cdots\cap I_m$ is a graded n-irreducible ideal of R for $n = n_1+\cdots+n_m$. In particular, if $I_1,\dots,I_n$ are n strongly graded irreducible ideals of a graded ring R. Then $\bigcap_{i=1}^{n} I_i$ is a strongly graded n-irreducible ideal of R.*

Proof. Obvious.

Proposition 4.2.6 *Let R be a graded ring, and let $P_1,\dots,P_{n+1}$ be pairwise comaximal graded prime ideals of R. Then $P_1\dots P_{n+1}$ is not a graded n-irreducible ideal of R.*

Proof. Note that, since the P_i's are pairwise comaximal, then $P_1\dots P_{n+1} = P_1\cap\cdots\cap P_{n+1}$ and the assertion follows by way of contradiction.

Corollary 4.2.7 *If R is a graded ring such that every proper graded ideal of R is graded n-irreducible, then R has at most n graded maximal ideals.*

Proof. Suppose that I and J are distinct graded maximal ideals in R, then $I+J$ is also a graded ideal, and $I\subset I+J\subseteq R$. Since I is graded maximal, the first inclusion is strict and $I+J=R$. Therefore I and J are comaximal and it remains to use Proposition 4.2.6.

In order to prove Theorem 4.2.15, we next give the graded version of several Theorems some of them are well known. But we begin first by introducing the following definitions.

Definition 4.2.8 *Let n be a positive integer. A proper graded ideal I of a graded ring R is said to be graded n-absorbing primary if whenever $x_1x_2\cdots x_{n+1}\in I$ for x_1, x_2, $\cdots$, $x_{n+1}\in h(R)$, then either $x_1x_2\cdots x_n\in I$ or the product of x_{n+1} with $(n-1)$ of the x_i's is in $Gr(I)$.*

Definition 4.2.9 *A graded ideal I of R is said to be graded maximal with respect to the exclusion of S if there is no graded ideal of R which contains I such that $I\cap S=\emptyset$.*

The next result present the graded version of [37, Theorem 1].

Theorem 4.2.10 *Let S be a multiplicatively closed set of homogeneous elements of R and let I be a graded ideal of R, which is graded maximal with respect to the exclusion of S. Then I is graded prime.*

Proof. Given $ab \in I$ for some two homogeneous elements $a, b \in R$. We must claim that a or b lies in I. Suppose the contrary. Then the ideal (I,a) generated by I and a is a stictly larger graded ideal then I and therefore intersects S. Since S is a multiplicative set of homogeneous elements of R, thus there exists a homogeneous element $s_1 \in S$ of the form $s_1 = i_1 + xa$ where i_1 and xa are homogeneous elements of the same degree. Likewise, $s_2 \in S$ of the form $s_2 = i_2 + yb$ where i_2 and yb are homogeneous elements of the same degree. But then $s_1 s_2 = (i_1 + xa)(i_2 + yb) \in I$, which contadicts the fact that I is graded maximal with respect to the exclusion of S. Hence, I is graded prime.

The next result present the graded version of [34, Theorem 2.1].

Theorem 4.2.11 *Let $I \subseteq P$ be two graded ideals of a graded ring R, where P is a graded prime ideal. Then the following statements are equivalents:*

1. *P is a minimal graded prime ideal of I.*
2. *$h(R) \setminus P$ is a multiplicative closed set of homogeneous elements that is maximal with respect to missing I.*
3. *For each homogeneous element $x \in P$, there is a homogeneous element $y \notin P$ and a nonnegative integer i such that $yx^i \in I$.*

Proof. $(1) \Rightarrow (2)$ Develop $h(R) \setminus P$ to a multiplivatively closed set of homogeneous elements S that is maximal with respect to missing I. If Q is a graded ideal containing I that is graded maximal with respect to the exclusion of S, then by Theorem 4.2.10, Q is graded prime. Note that $Q \cap (h(R) \setminus P) = \emptyset$. Now, since, by hypothesis, P is a minimal graded prime ideal of I, we have that $Q = P$. Hence, $h(R) \setminus P = S$.

$(2) \Rightarrow (3)$ Take a nonzero homogeneous element $x \in P$ and let $S = \{yx^i / y \in h(R) \setminus P, i = 0, 1, 2, \cdots\}$. So that, S is a multiplicatively closed set of homogeneous elements that properly contains $h(R) \setminus P$. So there is some $y \in h(R) \setminus P$ and a nonnegative i such that $yx^i \in I$.

$(3) \Rightarrow (1)$ Suppose that $I \subset Q \subseteq P$, where Q is a graded prime ideal. If there exists some $x = \sum_{g \in G} x_g \in P \setminus Q$, since P is a graded ideal, for all $g \in G$ $x_g \in P \setminus Q$, then there is a $y \notin h(P)$ and a positive integer i such that $yx_g^i \in I \subset Q$, since Q is graded prime, $y \in Q$ or $x_g^i \in Q$, a contradiction. Therefore $Q = P$.

The next result present the graded version of [8, Theorem 2.5].

Theorem 4.2.12 *Let I be a graded n-absorbing ideal of a graded ring R. Then there are at most n graded prime ideals of R minimal over I.*

Proof. We may suppose that $n \geq 2$ since a graded 1-absorbing ideal is a graded prime ideal. Suppose that $P_1, \cdots, P_{n+1}$ are distinct graded prime ideals of R minimal over I. Thus, for each $1 \leq i \leq n$ there is an $x_i \in P_i \setminus \left(\cup_{k\neq i} P_k\right) \cup P_{n+1}$. Since all the P_i's are graded, we may suppose that the x_i's are homogeneous. By Theorem 4.2.11, for each $1 \leq i \leq n$, there is a homogeneous element $y_i \in R \setminus P_i$ such that $y_i x_i^{n_i} \in I$ for some integer $n_i \geq 1$. We may assume that all the y_i's have the same degree such that $y_i x_i^{n_i} \in I$ for each $1 \leq i \leq n$. Since $I \subseteq P_{n+1}$ is a graded n-absorbing ideal of R and $x_i \notin P_{n+1}$ for each $1 \leq i \leq n$, we have that $y_i x_i^{n-1} \in I$ for each $1 \leq i \leq n$, and hence $(y_1 + \cdots + y_n) x_1^{n-1} \cdots x_n^{n-1} \in I$. Since $x_i \in P_i \setminus \left(\cup_{k\neq i} P_k\right)$ and $y_i x_i^{n-1} \in I \subseteq P_1 \cap \cdots \cap P_n$ for each $1 \leq i \leq n$, we have $y_i \in \left(\cap_{k\neq i} P_k\right) \setminus P_i$ for each $1 \leq i \leq n$, and thus $(y_1 + \cdots + y_n) \notin P_i$ for each $1 \leq i \leq n$. Hence $(y_1 + \cdots + y_n) \prod_{k\neq i} x_k^{n-1} \notin P_i$ for each $1 \leq i \leq n$; so $(y_1 + \cdots + y_n) \prod_{k\neq i} x_k^{n-1} \notin I$ for each $1 \leq i \leq n$, and thus $x_1^{n-1} \cdots x_n^{n-1} \in I \subseteq P_{n+1}$. Now, since I is a graded n-absorbing ideal of R, we have that $x_i \in P_{n+1}$ for some $1 \leq i \leq n$, which is a contradiction. Therefore there are at most n graded prime ideals of R minimal over I, as desired.

The next result present the graded version of [42, Corollary 2.7].

Theorem 4.2.13 *Let I be a graded n-absorbing primary ideal of R. Then $Gr(I) = P_1 \cap P_1 \cap \cdots \cap P_i$, where $1 \leq i \leq n$ and P_i's are the only distinct graded prime ideals of R that are minimal over I.*

Proof. It follows directly from Theorem 4.2.12, since if I is a graded n-absorbing primary ideal of R, it is clear that $Gr(I)$ is a graded n-absorbing ideal of R.

Note that the concepts of graded n-irreducible ideals and of graded n-absorbing ideals defined first by M. Hamoda and A. Eid Ashour, see [30, Definition 2.1] are different in general as shown by the following example.

Example 4.2.14 *Let $R = \mathbb{Z}[i]$ be the Gaussian integers ring with $G = \mathbb{Z}_2$ and consider the ideal $I = (12)$ of $\mathbb{Z}[i]$. I is a graded ideal since 12 is a homogeneous element of R. Now, since $2.2.3 \in I$, but $2.2 \notin I$ and $2.3 \notin I$, then I is not a graded 2-absorbing ideal of R. On the other hand, Theorem 4.3.5 given heirenafter in Section 4.3 ensures that I is a graded 2-irreducible ideal of R. However, in the following theorem, we show that these concepts are comparable in some cases.*

Theorem 4.2.15 *Let I be a graded radical ideal of a ring R, i.e. $Gr(I) = I$. The following assertions are equivalent:*

1. *I is strongly graded n-irreducible;*

2. *I is graded n-absorbing;*

3. *I is graded n-absorbing primary;*

4. *I is an intersection of exactly n graded prime ideals of R.*

Proof. $(1) \Rightarrow (2)$ Assume that I is strongly graded n-irreducible. Let $I_1, \ldots, I_{n+1}$ be a graded ideals of R such that $I_1 \ldots I_{n+1} \subseteq I$. Hence, $\bigcap_{i=1}^{n+1} I_i \subseteq Gr(\bigcap_{i=1}^{n+1} I_i) \subseteq Gr(I) = I$. Since I is strongly graded n-irreducible, then there are n of the I_i's whose intersection is in I. So, there are n of the I_i's whose product is in I. Consequently I is graded n-absorbing.

$(2) \Rightarrow (3)$ Is trivial.

$(3) \Rightarrow (4)$ Follows from Theorem 4.2.13.

$(4) \Rightarrow (1)$ As a particular case of Proposition 4.2.3(3), we have that any graded prime ideal is strongly graded irreducible, then the result follows from Proposition 4.2.5.

Example 4.2.16 *Consider the ideal $I = (p_1 p_2 \cdots p_n)$ of the Gaussian integer graded ring $\mathbb{Z}[i]$ with $G = \mathbb{Z}_2$, where the p_i's are some distinct homogeneous prime elements of $\mathbb{Z}[i]$ and n is a positive integer. Then $I = (p_1) \cap (p_2) \cap \cdots \cap (p_n)$ is exactly the intersection of n graded prime ideals of $\mathbb{Z}[i]$. Moreover, $Gr(I) = I$, then in light of Theorem 4.2.15, I is a graded n-irreducible (strongly graded n-irreducible).*

Example 4.2.17 *In the light of the previous Example, if we take $p_1 = 2$, I present an example of a graded n-irreducible ideal which is not n-irreducible.*

Recall from the start of this section that the intersection of any two graded ideals is a graded ideal. We next clarify the situation for the stability of graded n-irreducible and strongly graded n-irreducible ideals in various graded ring-theoretic constructions.

Theorem 4.2.18 *Let $f : R \to S$ be a surjective graded ring homomorphism, and let I be a proper graded ideal of R containing* $\ker(f)$. *Then:*

1. *If I is a strongly graded n-irreducible ideal of R, then $f(I)$ is a strongly graded n-irreducible ideal of S.*

2. *I is a graded n-irreducible ideal of R if and only if $f(I)$ is a graded n-irreducible ideal of S. In particular, this holds if f is a graded ring isomorphism.*

Proof. Since f is a surjective graded ring homomorphism, then $f(J \cap R) = J$ for each graded ideal J of S and $f(K \cap L) = f(K) \cap f(L), f(K) \cap R = K$ for every graded ideals K, L of R which contain $\ker(f)$.

(1) Assume that I is a strongly graded n-irreducible ideal of R. If $f(I) = S$, then $I = f(I) \cap R = R$, which is a contradiction. Let $J_1, \ldots, J_{n+1}$ be a graded ideals of S such that $J_1 \cap \cdots \cap J_{n+1} \subseteq f(I)$. Hence, $(J_1 \cap R) \cap \cdots \cap (J_{n+1} \cap R) \subseteq f(I) \cap R = I$, and so there

are n of the $(J_i \cap R)$'s whose intersection is in I. Without loss of generality, suppose that $(J_1 \cap R) \cap \cdots \cap (J_n \cap R) \subseteq I$, therefore $J_1 \cap \cdots \cap J_n \subseteq f(I)$. Then $f(I)$ is strongly graded n-irreducible.

(2) The direct implication is similar to part (1). Conversely, assume that $f(I)$ is a graded n-irreducible ideal of S. Let $I_1, \ldots, I_{n+1}$ be a graded ideals of R such that $I = I_1 \cap \cdots \cap I_{n+1}$, then $f(I) = f(I_1) \cap \cdots \cap f(I_{n+1})$. Therefore, since the image of every graded ideal is a graded ideal, there are n of the $f(I_i)$'s whose intersection is $f(I)$ since $f(I)$ is a graded n-irreducible ideal of S. Without loss of generality, assume that $f(I) = f(I_1) \cap \cdots \cap f(I_n)$, then $I = f(I) \cap R = I_1 \cap \cdots \cap I_n$. So that, I is graded n-irreducible, as asserted.

Define a *graded ring extension* to be a graded ring homomorphism $R \to S$, which makes S a graded R-module. The next result present a direct Corollary of the previous Theorem.

Corollary 4.2.19 (1) *Let $R \subseteq S$ be a graded ring extension and J a graded n-irreducible ideal of S. Then $J \cap R$ is a graded n-irreducible ideal of R.*

(2) *Let $I \subseteq J$ be graded ideals of R. Then J is a graded n-irreducible ideal if and only if J/I is a graded n-irreducible ideal of R/I.*

Let R be a ring and let $\{x_1, x_2, \ldots\}$ be (commuting) algebraically independent indeterminates over R. For $m = (m_1, \cdots, m_n) \in \mathbb{N}^n$, let $x^m = x_1^{m_1} \cdots x_n^{m_n}$. Then the polynomial ring $P = R[x_1, \cdots, x_n]$ is graded by $\mathbb{N}$ via $P_s = \{\sum_{m \in \mathbb{N}^n} r_m x^m | r_m \in R \text{ and } \sum_{i=1}^n m_i = s\}$. Note that respecting this graduation every ideal of R is a graded ideal of the polynomial ring P since $P_0 = R$. The following result examines the extensions of n-irreducible ideals of R in the polynomial ring $R[X]$.

Corollary 4.2.20 *Let I be an ideal of a ring R. Then:*

1. *(I, X) is a graded n-irreducible ideal of $R[X]$ if and only if I is an n-irreducible ideal of R.*
2. *$I[X]$ is a graded n-irreducible ideal of $R[X]$ if and only if I is an n-irreducible ideal of R.*

Proof. (1) Follows directly from Corollary 4.2.19(2) since $(I, X)/(X) \cong I$ (graded ring isomorphism since X is a homogeneous element) in $R[X]/(X) \cong R$ (graded ring isomorphism).

(2) By Corollary 4.2.19(1), if $I[X]$ is a graded n-irreducible ideal of $R[X]$, then I is a graded n-irreducible ideal of R.

Conversely, suppose that I is a n-irreducible ideal of R and let $I_1, \ldots, I_{n+1}$ be graded ideals of $R[X]$ such that $I[X] = I_1 \cap \cdots \cap I_{n+1}$. For all $1 \leq i \leq n+1$, $(I_i \cap R)[X] \subseteq I_i$. So, by taking $I_i' = I_i \cap R$, we have that $I[X] = I_1'[X] \cap \cdots \cap I_{n+1}'[X] = \left(I_1' \cap \cdots \cap I_{n+1}'\right)[X]$, therefore

$I = I[X] \cap R = \left(I'_1 \cap \cdots \cap I'_{n+1}\right) = \left(I'_1 \cap \cdots \cap I'_{n+1}\right)[X] \cap R$. Since I is n-irreducible, then there are n of the I'_i's whose intersection is I. Without loss of generality, assume that $I = I'_1 \cap \cdots \cap I'_n$. Hence, $I[X] = I'_1[X] \cap \cdots \cap I'_n[X]$ and therefore $I[X] = I_1 \cap \cdots \cap I_n$. As a result, $I[X]$ is a graded n-irreducible ideal of $R[X]$.

Let G be a group and S be a multiplicatively closed subset of homogeneous elements of a graded ring R. In the next result, consider the natural graded ring homomorphism $f : R \to S^{-1}R$ defined by $f(x) = x/1$. For each graded ideal I of the graded ring $S^{-1}R$, we consider $I^c = \{x \in R \mid x/1 \in I\} = I \cap R$, which is a graded ideal of R and $C^{gr} = \{I^c \mid I$ is a graded ideal of $S^{-1}R\}$.

Theorem 4.2.21 *let G be a group, R be a graded ring and S be a multiplicatively closed set of homogeneous elements of R. Then there is a one-to-one correspondence between the strongly graded n-irreducible ideals of $S^{-1}R$ and strongly graded n-irreducible ideals of R contained in C^{gr} which do not meet S.*

Proof. Let I be a strongly graded n-irreducible ideal of $S^{-1}R$. It is easy to see that, $I^c \neq R$, $I^c \in C^{gr}$ and $I^c \cap S = \emptyset$. Let $I_1, \ldots, I_{n+1}$ be $(n+1)$ graded ideals of R such that $I_1 \cap \cdots \cap I_{n+1} \subseteq I^c$. Then $\left(S^{-1}I_1\right) \cap \cdots \cap \left(S^{-1}I_{n+1}\right) = S^{-1}\left(I_1 \cap \cdots \cap I_{n+1}\right) \subseteq S^{-1}\left(I^c\right) = I$. Therefore there are n of the $S^{-1}I_i$'s whose intersection is in I since I is strongly graded n-irreducible. Hence, there are n of the I_i's whose intersection is in I^c. As a result, I^c is a strongly graded n-irreducible ideal of R.

Conversely, let I be a strongly graded n-irreducible ideal of R such that $I \cap S = \emptyset$ and $I \in C^{gr}$. Since $I \cap S = \emptyset$, $S^{-1}I \neq S^{-1}R$. Let $I_1 \cap I_2 \cap \cdots \cap I_{n+1} \subseteq S^{-1}I$ where $I_1, I_2, \ldots, I_{n+1}$ are graded ideals of $S^{-1}R$. Therefore $\left(I_1^c\right) \cap \left(I_2^c\right) \cap \cdots \cap \left(I_{n+1}^c\right) = \left(I_1 \cap I_2 \cap \cdots \cap I_{n+1}\right)^c \subseteq \left(S^{-1}I\right)^c$. Now, since $I \in C^{gr}$, then $\left(S^{-1}I\right)^c = I$ and so $\left(I_1^c\right) \cap \left(I_2^c\right) \cap \cdots \cap \left(I_{n+1}^c\right) \subseteq I$. As a result, there are n of the I_i^c's whose intersection is in I. Without loss of generality, suppose that $\left(I_1^c\right) \cap \left(I_2^c\right) \cap \cdots \cap \left(I_n^c\right) \subseteq I$. Therefore, $S^{-1}\left(I_1^c\right) \cap S^{-1}\left(I_2^c\right) \cap \cdots \cap S^{-1}\left(I_n^c\right) \subseteq S^{-1}I$. Hence, $S^{-1}I$ is a strongly graded n-irreducible ideal of $S^{-1}R$.

Corollary 4.2.22 *Let G be a group, R be a graded ring and S be a multiplicatively closed set of homogeneous elements of R. If I is a graded-primary and strongly graded n-irreducible ideal of R which does not meet S, then $S^{-1}I$ is a strongly graded n-irreducible ideal of $S^{-1}R$.*

Proof. Since I is a graded-primary ideal of R, and $I \cap S = \emptyset$, by [51, Proposition 1.15(iii)], we have that $(I : s) = I$ for each $s \in S$, so then $\left(S^{-1}I\right)^c = I$ and hence $I \in C^{gr}$. Now, it remains to use Theorem 4.2.21, to get that $S^{-1}I$ is a strongly graded n-irreducible ideal of $S^{-1}R$.

We close this section by a result concerning the strongly graded n-irreducible ideals in the product of a finite number of graded rings. Recall that a graded ideal of $R_1 \times R_2$ has the form $I_1 \times I_2$ for some graded ideals I_i of R_i.

Proposition 4.2.23 *Let I_1 be a strongly graded n-irreducible ideal of a graded ring R_1 and I_2 be a strongly graded m-irreducible ideal of a graded ring R_2. Then $J = I_1 \times I_2$ is a strongly graded $(n+m)$-irreducible ideal of the graded ring $R = R_1 \times R_2$.*

Proof. If $J = I_1 \times R_2$ for some strongly graded n-irreducible ideal I_1 of R_1 or $R_1 \times I_2$ for some strongly graded m-irreducible ideal I_2 of R_2, therefore it is easy to see that J is a strongly graded n-irreducible or strongly graded m-irreducible ideal of $R_1 \times R_2$. Therefore, assume that $J = I_1 \times I_2$ for some strongly graded n-irreducible ideal I_1 of R_1 and some strongly graded m-irreducible ideal I_2 of R_2. Then $I_1' = I_1 \times R_2$ is a strongly graded n-irreducible ideal of $R_1 \times R_2$ and $I_2' = R_1 \times I_2$ is a strongly graded m-irreducible ideal of $R_1 \times R_2$. Hence, $I_1' \cap I_2' = I_1 \times I_2 = J$ is a strongly graded $(n+m)$-irreducible ideal of $R_1 \times R_2$ by Proposition 4.2.5.

Corollary 4.2.24 *Let I_k be a strongly graded n_k-irreducible ideal of a graded ring R_k for each integer $1 \leq k \leq m$. Let $R = R_1 \times \cdots \times R_m$. Then $I_1 \times \cdots \times I_m$ is a strongly graded n-irreducible ideal of R with $n = n_1 + \cdots + n_m$.*

4.3 Extension to some specific graded Rings

We devote this section to the study of the transfer of graded n-irreducible and strongly graded n-irreducible ideals in several special classes of graded commutative rings. We begin by introducing the notion of *graded arithmetical* ring. A graded ring R is said to be a *graded arithmetical* ring, if for each graded ideals I, J and K of R, $(I+J) \cap K = (I \cap K) + (J \cap K)$ and just like the ungraded case, this condition is equivalent to the condition that for each graded ideals I, J and K of R, $(I \cap J) + K = (I+K) \cap (J+K)$.

Let G be a group. Recall from [47] that a graded integral domain is said to be a graded principal ideal domain (gr-PID, for short) if every graded ideal of R is principal and a graded integral domain is said to be a graded Dedekind domain if every graded ideal is projective. Note that every graded principal ideal domain (gr-PID) is a graded Dedekind domain, see [47, Corollaries 1.2(1.2.1)]. Also, every graded Dedekind domain is a graded arithmetical domain.

In order to prove the main result of this section "Theorem 4.3.5", we first give the following result which present the graded version of [42, Theorem 2.8] (for $\phi = \emptyset$).

Theorem 4.3.1 *Let G be a group and R be a graded ring. Suppose that for every $1 \leq i \leq k, I_i$ is a graded n_i-absorbing primary ideal of R such that $Gr(I_i) = P_i$ is a graded n_i-absorbing ideal of R, respectively. Set $n := n_1 + n_2 + \cdots + n_k$. The following conditions hold:*

1. *$I_1 \cap I_2 \cap \cdots \cap I_k$ is a graded n-absorbing primary ideal of R.*

2. $I_1I_2\cdots I_k$ is a graded n-absorbing primary ideal of R.

Proof. (1) Let $L=I_1\cap I_2\cap\cdots\cap I_k$. Then using [52, Proposition 2.4(4)], we have $Gr(L)=P_1\cap P_2\cap\cdots\cap P_k$. Assume that $a_1a_2\cdots a_{n+1}\in L$ for some $a_1,a_2,\ldots,a_{n+1}\in h(R)$ and $a_1\cdots\widehat{a_i}\cdots a_{n+1}\notin Gr(L)$ for every $1\leq i\leq n$. Since G is a group, using [30, Theorem 2.7], $Gr(L)=P_1\cap P_2\cap\cdots\cap P_k$ is graded n-absorbing, then $a_1a_2\cdots a_n\in P_1\cap P_2\cap\cdots\cap P_k$. We have to show that $a_1a_2\cdots a_n\in L$. For every $1\leq i\leq k, P_i$ is graded n_i-absorbing and $a_1a_2\cdots a_n\in P_i$, then there exist elements $1\leq\alpha_1^i,\alpha_2^i,\ldots,\alpha_{n_i}^i\leq n$ such that $a_{\alpha_1^i}a_{\alpha_2^i}\cdots a_{\alpha_{n_i}^i}\in P_i$. If $\alpha_e^c=\alpha_f^d$ for two couples (c,e) and (d,f), then

$$a_{\alpha_1^1}a_{\alpha_2^1}\cdots a_{\alpha_{n_1}^1}\cdots a_{\alpha_1^c}a_{\alpha_2^c}\cdots a_{\alpha_e^c}\cdots a_{\alpha_{n_c}^c}\cdots$$
$$a_{\alpha_1^d}a_{\alpha_2^d}\cdots\widehat{a_{\alpha_f^d}}\cdots a_{\alpha_{n_d}^d}\cdots a_{\alpha_1^k}a_{\alpha_2^k}\cdots a_{\alpha_{n_k}^k}\in Gr(L).$$

Hence, $a_1\cdots\widehat{a_{\alpha_f^d}}\cdots a_na_{n+1}\in Gr(L)$, a contradiction. So that the α_j^i's are distinct. Therefore

$$\left\{a_{\alpha_1^1},a_{\alpha_2^1},\ldots,a_{\alpha_{n_1}^1},a_{\alpha_1^2},a_{\alpha_2^2},\ldots,a_{\alpha_{n_2}^2},\ldots,a_{\alpha_1^k},a_{\alpha_2^k},\ldots,a_{\alpha_{n_k}^k}\right\}=\{a_1,a_2,\ldots,a_n\}.$$

If $a_{\alpha_1^i}a_{\alpha_2^i}\cdots a_{\alpha_{n_i}^i}\in I_i$ for every $1\leq i\leq k$, then

$$a_1a_2\cdots a_n=a_{\alpha_1^1}a_{\alpha_2^1}\cdots a_{\alpha_{n_1}^1}a_{\alpha_1^2}a_{\alpha_2^2}\cdots a_{\alpha_{n_2}^2}\cdots a_{\alpha_1^k}a_{\alpha_2^k}\cdots a_{\alpha_{n_k}^k}\in L,$$

as desired. Hence, we may assume that $a_{\alpha_1^1}a_{\alpha_2^1}\cdots a_{\alpha_{n_1}^1}\notin I_1$. Since I_1 is graded n_1-absorbing primary and

$$a_{\alpha_1^1}a_{\alpha_2^1}\cdots a_{\alpha_{n_1}^1}a_{\alpha_1^2}a_{\alpha_2^2}\cdots a_{\alpha_{n_2}^2}\cdots a_{\alpha_1^k}a_{\alpha_2^k}\cdots a_{\alpha_{n_k}^k}a_{n+1}=a_1\cdots a_{n+1}\in I_1$$

then we have $a_{\alpha_1^1}\cdots\widehat{a_{\alpha_t^1}}\cdots a_{\alpha_{n_1}^1}a_{\alpha_1^2}a_{\alpha_2^2}\cdots a_{\alpha_{n_2}^2}\cdots a_{\alpha_1^k}a_{\alpha_2^k}\cdots a_{\alpha_{n_k}^k}a_{n+1}\in P_1$ for some $1\leq t\leq n_1$. On the other hand, since $a_{\alpha_1^i}a_{\alpha_2^i}\cdots a_{\alpha_{n_i}^i}\in I_i\subseteq Gr(I_i)=P_i$ for every $2\leq i\leq k$, then

$$a_{\alpha_1^1}\cdots\widehat{a_{\alpha_t^1}}\cdots a_{\alpha_{n_1}^1}a_{\alpha_1^2}a_{\alpha_2^2}\cdots a_{\alpha_{n_2}^2}\cdots a_{\alpha_1^k}a_{\alpha_2^k}\cdots a_{\alpha_{n_k}^k}a_{n+1}\in P_2\cap\cdots\cap P_k.$$

As a result, $a_{\alpha_1^1}\cdots\widehat{a_{\alpha_t^1}}\cdots a_{\alpha_{n_1}^1}a_{\alpha_1^2}a_{\alpha_2^2}\cdots a_{\alpha_{n_2}^2}\cdots a_{\alpha_1^k}a_{\alpha_2^k}\cdots a_{\alpha_{n_k}^k}a_{n+1}\in Gr(L)$, which is a contradiction. Likewise, we deduce that $a_{\alpha_1^i}a_{\alpha_2^i}\cdots a_{\alpha_{n_i}^i}\in I_i$ for every $2\leq i\leq k$. Then $a_1a_2\cdots a_n\in L$.

The proof of (2) is omitted since it is similar to that of part (1).

The following result is a direct Corollary of the previous Theorem. We refer the reader to [51] for more informations about the notion of graded-primary ideals.

Corollary 4.3.2 *Let G be a group and R be a graded ring with $1 \neq 0$ and let $P_1, P_2, \ldots, P_n$ be graded prime ideals of R. Suppose that for every $1 \leq i \leq n$, $P_i^{t_i}$ is a graded P_i-primary ideal of R where t_i is a positive integer. Then $P_1^{t_1} \cap P_2^{t_2} \cap \cdots \cap P_n^{t_n}$ and $P_1^{t_1} P_2^{t_2} \cdots P_n^{t_n}$ are graded n-absorbing primary ideals of R. In particular, $P_1 \cap P_2 \cap \cdots \cap P_n$ and $P_1 P_2 \cdots P_n$ are graded n-absorbing primary ideals of R.*

Recall from [44] that a graded ring R is said to be graded-Noetherian if it satisfies the ascending chain condition on graded ideals of R. The following result present the graded version of [42, Theorem 2.15].

Theorem 4.3.3 *Let G be a group and R be a graded-Noetherian integral domain that is not a graded field. The following conditions are equivalent:*

1. *R is a graded Dedekind domain;*
2. *A nonzero proper graded ideal I of R is a graded n-absorbing primary ideal of R if and only if $I = M_1^{t_1} M_2^{t_2} \cdots M_i^{t_i}$ for some $1 \leq i \leq n$ and some distinct graded maximal ideals $M_1, M_2, \ldots, M_i$ of R and some positive integers $t_1, t_2, \ldots, t_i$;*
3. *If I is a nonzero graded n-absorbing primary ideal of R, then $I = M_1^{t_1} M_2^{t_2} \cdots M_i^{t_i}$ for some $1 \leq i \leq n$ and some distinct graded maximal ideals $M_1, M_2, \ldots, M_i$ of R and some positive integers $t_1, t_2, \ldots, t_i$;*
4. *A nonzero proper graded ideal I of R is a graded n-absorbing primary ideal of R if and only if $I = P_1^{t_1} P_2^{t_2} \cdots P_i^{t_i}$ for some $1 \leq i \leq n$ and some distinct graded prime ideals $P_1, P_2, \ldots, P_i$ of R and some positive integers $t_1, t_2, \ldots, t_i$;*
5. *If I is a nonzero graded n-absorbing primary ideal of R, then $I = P_1^{t_1} P_2^{t_2} \cdots P_i^{t_i}$ for some $1 \leq i \leq n$ and some distinct graded prime ideals $P_1, P_2, \ldots, P_i$ of R and some positive integers $t_1, t_2, \ldots, t_i$.*

Proof. The proof is orgnized as follows, the proof of $(1) \Rightarrow (4)$ is similar to that of $(1) \Rightarrow (2)$ and it is omitted. $(2) \Rightarrow (3)$, $(3) \Rightarrow (5)$ and $(4) \Rightarrow (5)$ are straightforward. It remains to claim $(1) \Rightarrow (2)$ and $(5) \Rightarrow (1)$.

$(1) \Rightarrow (2)$ Suppose that R is a graded Dedekind domain that is not a graded field. Then, by [47, Lemma 1.1(3)], every nonzero graded prime ideal of R is graded maximal. Let I be a nonzero graded n-absorbing primary ideal of R. Since R is a graded Dedekind domain, then by [47, Lemma 1.1(4)], there are distinct graded maximal ideals $M_1, M_2, \ldots, M_i$ of R $(i \geq 1)$ such that $I = M_1^{t_1} M_2^{t_2} \cdots M_i^{t_i}$ in which t_j's are positive integers. Hence, $Gr(I) = M_1 \cap M_2 \cap \cdots \cap M_i$. Since I is graded n-absorbing primary and every graded prime ideal of R is graded maximal, then by Theorem 4.2.13, $Gr(I)$ is the intersection of at most n graded maximal ideals of R. Therefore, $i \leq n$. For the reverse implication, assume that $I = M_1^{t_1} M_2^{t_2} \cdots M_i^{t_i}$ for some $1 \leq i \leq n$ and some distinct graded

maximal ideals $M_1, M_2, \ldots, M_i$ of R and some positive integers $t_1, t_2, \ldots, t_i$. Then I is graded n-absorbing primary, by Corollary 4.3.2.

$(5) \Rightarrow (1)$ Let M be an arbitrary graded maximal ideal of R and I be a graded ideal of R such that $M^2 \subset I \subset M$. Hence $Gr(I) = M$ and so I is graded M-primary. Then I is graded n-absorbing primary, and thus by assertion (5) we have that $I = P_1^{t_1} P_2^{t_2} \cdots P_i^{t_i}$ for some $1 \leq i \leq n$ and some distinct graded prime ideals $P_1, P_2, \ldots, P_i$ of R and some positive integers $t_1, t_2, \ldots, t_i$. Then $Gr(I) = P_1 \cap P_2 \cap \cdots \cap P_i = M$ which prove that I is a power of M, a contradiction. Hence, there are no graded ideals properly between M^2 and M. Consequently R is a graded Dedekind domain, by [47, Lemma 1.1(6)].

The following result is a direct Corollary of the previous Theorem.

Corollary 4.3.4 *Let G be a group and R be a gr-PID and I be a nonzero proper graded ideal of R. Then I is a graded n-absorbing primary ideal of R if and only if $I = R(p_1^{t_1} p_2^{t_2} \cdots p_i^{t_i})$, where p_j's are prime homogeneous elements of R, $1 \leq i \leq n$ and t_j's are some integers.*

Now, we are able to give the main result of this Section 4.3.

Theorem 4.3.5 *Let G be a group and R be a gr-PID and I be a nonzero proper graded ideal of R. The following conditions are equivalent:*

1. *I is graded n-irreducible;*
2. *I is graded n-absorbing primary;*
3. *$I = R\left(p_1^{l_1} \ldots p_m^{l_m}\right)$, for some distinct homogeneous prime elements $p_1, \ldots, p_m$ of R and some natural numbers $l_1, \ldots, l_m$ such that $m \leq n$.*

Proof. $(1) \Rightarrow (3)$ Suppose that $I = Ra$ where a is a nonzero homogeneous element of R and let $a = p_1^{l_1} \ldots p_m^{l_m}$ be a homogeneous prime decomposition for a. We have to claim that $m \leq n$. By contradiction, assume that $m > n$. Since R is a gr-PID and $p_1, \ldots, p_m$ are homogeneous prime elements of R, $I = Rp_1^{l_1} \cap \cdots \cap Rp_m^{l_m}$. Now, since I is graded n-irreducible, there are n of the $Rp_i^{l_i}$'s whose intersection is I. Without loss of generality, suppose that $I = Rp_1^{l_1} \cap \cdots \cap Rp_n^{l_n}$. It follows that $Rp_1^{l_1} \cap \cdots \cap Rp_n^{l_n} \subseteq Rp_{n+1}^{l_{n+1}}$, which is a contradiction.

$(3) \Rightarrow (1)$ Suppose that $I = R\left(p_1^{l_1} \ldots p_n^{l_n}\right)$ for some distinct homogeneous prime elements $p_1, \ldots, p_n$ of R and some natural numbers $l_1, \ldots, l_n$. Let $I = Ra_1 \cap \cdots \cap Ra_{n+1}$ for some homogeneous elements $a_1, \ldots, a_{n+1}$ of R. Hence, $a_1, a_2, \ldots, a_{n+1}$ divide $p_1^{l_1} \cdots p_n^{l_n}$. So that $a_i = p_1^{l_{i,1}} \cdots p_n^{l_{i,n}}$ where $l_{i,1}, \ldots, l_{i,n}$ are some nonnegative integers for each $1 \leq i \leq n+1$. Let $\alpha_i = \max\left\{l_{1,i}, \ldots l_{n+1,i}\right\}$ for each $1 \leq i \leq n$. Hence, $I = Ra_1 \cap \cdots \cap Ra_{n+1} = R[a_1, \ldots, a_{n+1}] = R\left(p_1^{\alpha_1} \cdots p_n^{\alpha_n}\right)$. Without loss of generality, suppose that $\alpha_1 =$

$l_{1,1},\ldots,\alpha_n = l_{n,n}$. Hence, $I = R\left(p_1^{l_{1,1}}\cdots p_n^{l_{n,n}}\right) = Ra_1\cap\cdots\cap Ra_n$. As a result, I is graded n-irreducible.

(2) $\Leftrightarrow$ (3) Follows directly from Corollary 4.3.4.

Proposition 4.3.6 *Let I be a proper graded ideal of a graded arithmetical ring R. The following conditions are equivalent:*

1. *I is a graded n-irreducible ideal of R;*
2. *I is a strongly graded n-irreducible ideal of R.*

Proof. (1) $\Rightarrow$ (2) Assume that $I_1,\ldots,I_{n+1}$ are graded ideals such that $\bigcap_{i=1}^{n+1} I_i \subseteq I$. Therefore, $I = I+\left(\bigcap_{i=1}^{n+1} I_i\right) = (I+I_1)\cap(I+I_2)\cap\cdots\cap(I+I_{n+1})$, since R is a graded arithmetical ring. By hypothesis, I is graded n-irreducible, then there are n of the $(I+I_i)$'s whose intersection is I. Therefore, there are n of the $(I+I_i)$'s whose intersection is in I which implies that I is a strongly graded n-irreducible ideal of R.

(2) $\Rightarrow$ (1) By Proposition 4.2.3(1).

The following Corollary is an immediate consequence of Theorem 4.3.5 and Proposition 4.3.6.

Corollary 4.3.7 *Let G be a group and R be a gr-PID and I be a nonzero proper graded ideal of R. The following conditions are equivalent:*

1. *I is strongly graded n-irreducible;*
2. *I is graded n-irreducible;*
3. *I is graded n-absorbing primary;*
4. *$I = R\left(p_1^{l_1}\ldots p_m^{l_m}\right)$, for some distinct homogeneous prime elements $p_1,\ldots,p_m$ of R and some natural numbers $l_1,\ldots,l_m$ such that $m\leq n$.*

Example 4.3.8 *Let $n\geq 1$ be a positive integer. Then there is a graded n-irreducible, but not graded $(n-1)$-irreducible ideal of a graded ring R. Let $R=\mathbb{Z}[i]$ be the Gaussian integers ring with $G=\mathbb{Z}_2$ and pick $I =< 30 >=< (2.3.5) >$. By Corollary 4.3.7, I is a strongly graded 3-irreducible (graded 3-irreducible) ideal of R. On the other hand, since $Gr(I) =< 2 >\cap< 3 >\cap< 5 >= I$, hence I is a graded radical ideal and so, by Theorem 4.2.15, I is not a graded strongly 2-irreducible (not a graded 2-irreducible) ideal of R. More generally, let $p_1,\ldots,p_n\in\mathbb{Z}[i]$ be distinct positive homogeneous prime elements of $\mathbb{Z}[i]$. Then $I=(p_1\ldots p_n)\mathbb{Z}[i]$ is a graded n-irreducible but not graded $(n-1)$-irreducible ideal of $\mathbb{Z}[i]$.*

Let G be a group. Recall that a graded ring R is said to be a graded von Neumann regular ring (gr-von Neumann regular for short) if for each $a \in h(R)$, there exists $b \in h(R)$ such that $a = a^2b$. In this case, the graded principal ideal (a) of R is generated by a homogeneous idempotent element $e \in R$. It is known by [31, Proposition 2] that a ring R is a gr-von Neumann regular ring if and only if its each graded ideal I of R is idempotent, that is, $I = I^2$ if and only if each graded ideal I of R is graded radical, that is, $I = Gr(I)$ if and only if any finitely generated graded ideal of R is a projective module.

Proposition 4.3.9 *Let G be a group, R be a gr-von Neumann regular ring and I be a nonzero graded ideal of R. Then I is graded n-irreducible if and only if I is graded n-absorbing.*

Proof. It is clear that a commutative graded ring R is a graded von Neumann regular ring if and only if $I_1 \cdots I_{n+1} = I_1 \cap \cdots \cap I_{n+1}$ for any graded ideals $I_1, \ldots, I_{n+1}$ of R. Hence the notions of graded n-irreducible and graded n-absorbing ideals coincide.

Theorem 4.3.10 *Let R be a graded Noetherian ring. If I is a graded n-irreducible ideal of R, then either I is graded irreducible or I is the intersection of exactly n graded irreducible ideals.*

Proof. Assume that I is graded n-irreducible. Since R is graded-Noetherian, then I can be expressed as an intersection of finitely many graded irreducible ideals of R by [51, Proposition 2.14], say $I = I_1 \cap I_2 \cap \cdots \cap I_k$. It remains to prove that $k \leq n$. Assume that $k \geq n+1$, then since I is graded n-irreducible, there are n of the I_i 's whose intersection is I. Without loss of generality, suppose that $I_1 \cap I_2 \cap \cdots \cap I_n = I$, hence $I_1 \cap I_2 \cap \cdots \cap I_n \subseteq I_{n+1}$, which is a contradiction.

CHAPTER 5

ON GRADED-NONNIL-NOETHERIAN COMMUTATIVE RINGS

Abstract.

Our present work is motivated by several papers which have introduced and studied the notion of Nonnil-Noetherian rings. Let $A = \oplus_{g\in G}A_g$ be a commutative ring with unity graded by an arbitrary commutative monoid G. We define A to be graded-Nonnil-Noetherian ring if every nonnil graded ideal of A, that is, a graded ideal which is not contained in $\mathrm{Nil(A)}$, is finitely generated. The purpose of this chapter* is to give the graded version of several characterizations, properties and basic results proved in [14], [33] and [18]. In order to give the graded version of some important characterizations and results, a complete section will be devoted to the study of the graded version of the well known principal ideal theorem without any assumption on G. We also show the non-triviality of our generalization by giving examples of graded-Nonnil-Noetherian rings which are not Nonnil-Noetherian and of graded-Nonnil-Noetherian rings which are not graded-Noetherian.

5.1 Introduction

The notion of Noetherian rings is one of the most important topics in commutative algebra. Due to its importance, many authors have generalize this concept to several others concepts and contexts. Nonnil-Noetherian rings and graded-Noetherian rings are typical generalizations of Noetherian rings. In this chapter, our main purpose is to introduce and study the notion of graded-Nonnil-Noetherian rings. In [14], Badawi introduced and studied the notion of Nonnil-Noetherian rings under the strong hypothesis that the nilradical

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of the ring is a divided prime ideal. In [33], Hizem and Benhissi have extended some properties of Nonnil-Noetherian rings but without any assumption on the nilradical. An ideal I of A is said to be nonnil if $I \nsubseteq Nil(A)$ and a ring A is said to be Nonnil-Noetherian if each nonnil ideal of A is finitely generated.

5.2 The graded version of the Generalized principal ideal theorem

The generalized principal ideal theorem, sometimes attributed to Krull, is one of the important results in commutative algebra. In [49], the authors gave a proof of that theorem, with the assumption that G is a torsion-free monoid. In this section, our main purpose is to give a new graded version of this well known theorem without any assumption on G with a new proof. In order to give our main result, we give the following lemmas extending many properties and basic results of Artinian rings to the context of graded rings.

Lemma 5.2.1 *Let A be a graded-Noetherian ring, every nonzero graded ideal of A contains a finite number of graded-minimal graded-prime ideals. In particular, A has a finite number of graded-minimal graded-prime ideals.*

Proof. Let A be a graded-Noetherian ring and M the set of the nonzero graded ideals of A having infinitely many graded-minimal graded-prime ideals. Suppose that M is not empty. We will show that $(\mathrm{M}, \subseteq)$ has a maximal element. Let $I_0 \in \mathrm{M}$. If I_0 is not maximal, there exists $I_1 \in \mathrm{M}$ such that $I_0 \subset I_1$. If I_1 is not maximal, there exists $I_2 \in \mathrm{M}$ such that $I_0 \subset I_1 \subset I_2$. By hypothesis, the ring A is graded-Noetherian, so the process must stop. Let I be a maximal element of $(\mathrm{M}, \subseteq)$. It is not graded-prime, since else, it will be the only graded-minimal graded-prime over itself. Hence, there exist $a, b \in h(A)\backslash I$ such that $ab \in I$. Since $I \subset I + aA$ and $I \subset I + bA$, and the ideals $I + aA$ and $I + bA$ are graded, then each of the graded ideals $I + aA$ and $I + bA$ has a finite number of graded-minimal graded-prime ideals. Now, we have $(I + aA)(I + bA) = I^2 + aI + bI + abA \subseteq I$, and so each graded-minimal graded-prime ideal over I contains either $I + aA$ or $I + bA$ and then it is a graded-minimal graded-prime over one of them. Hence one of them has infinitely many graded-minimal graded-prime ideals, a contradiction. Therefore, $\mathrm{M} = \emptyset$ and each nonzero graded ideal of A has a finite number of graded-minimal graded-prime ideals.

Lemma 5.2.2 *The graded-nilradical* $\text{gr-Nil}(\mathrm{A}) = Gr(0)$ *of a graded-Artinian ring is a nilpotent graded ideal.*

Proof. Let A be a graded-Artinian ring and $I = \text{gr-Nil}(\mathrm{A})$. The sequence of graded ideals $I \supseteq I^2 \supseteq \ldots$ is stationary. Then, there exists an integer $i \geq 1$ such that $I^i = I^{i+1} = \ldots = J$. Suppose that $J \neq (0)$. The set $F = \{J' \text{ graded ideal of } A \text{ such that } J'J \neq (0)\}$ is nonempty. It contains A and also J. The set F has a graded-minimal element $\hat{J}$ for the inclusion since

in the contrary case, A will have a strictly decreasing sequence of graded ideals. Since $\hat{J}J \neq (0)$, there exists $a \in h(\hat{J})$ such that $aJ \neq (0)$. Hence $(a) \subseteq \hat{J}$ and $(a)J = aJ \neq (0)$, so $(a) \in F$. Now, since $\hat{J}$ is graded-minimal, we have to show that $(a) = \hat{J}$. Otherwise, $aJ \subseteq (a) = \hat{J}$ and $(aJ)J = aJ^2 = aJ \neq (0)$ since $J^2 = J$. Then $aJ \in F$. Since $\hat{J}$ is graded-minimal, we have to show that $aJ = \hat{J}$. Hence $(a) = aJ$. So there exists $b \in h(J)$ such that $a = ab$. Since $J \subseteq I = \text{gr-Nil}(A)$, there exists $n \in \mathbb{N}^*$ such that $b^n = 0$. Then $a = ab = ab^2 = \ldots = ab^n = 0$, which is absurd since $aJ \neq (0)$. Therefore $J = (0)$ and I is a nilpotent graded ideal.

Example 5.2.3 *Let G be a group, (A,M) a graded-Artinian graded-local ring. By Theorem 5.2.7 given hereinafter, we have gr-dim$A = 0$, and then M is the only graded-prime ideal of A. By Lemma 5.2.2, there exists $n \in \mathbb{N}^*$ such that $M^n = (0)$. An homogeneous element of A is either a unity or nilpotent. This is, for example, the case of $A = K[X,X^{-1}]$ the laurent polynomial ring graded by $G = \mathbb{Z}$, where K is a field. It is clear that A is graded-Artinian graded-local ring since it is a graded-field.*

Before giving our next lemma, it will be very helpful to recall the following two results [9, Theorem 3.13] and [9, Corollary 3.15] for $S = \{1\}$.

Theorem 5.2.4 *[9, Theorem 3.13] Let G be an abelian group, M a graded-Artinian A-module and N its graded submodule. Then M is a graded-Artinian module if and only if N and M/N are graded-Artinian A-modules.*

Corollary 5.2.5 *[9, Corollary 3.15] Let G be an abelian group, A a graded-Artinian ring and M a finitely generated graded A-module. Then M is a graded-Artinian A-module.*

Lemma 5.2.6 *Let G be an abelian group and A a graded ring with some graded-maximal ideals not necessarily different $M_1,\ldots,M_n$ such that the product $M_1M_2\ldots M_n = (0)$. Then the graded ring A is graded-Artinian if and only if it is graded-Noetherian.*

Proof. For $1 \leq i \leq n$, take $I_i = M_1M_2\ldots M_i$. Then $(0) = I_n \subseteq I_{n-1} \subseteq \ldots \subseteq I_2 \subseteq I_1 \subseteq I_0 = A$. We will proceed by applying Theorem 5.2.4 several times: The graded ring A is graded-Artinian (resp. graded-Noetherian) if and only if the graded A-module A is graded-Artinian (resp. graded-Noetherian) if and only if the graded A-modules I_1 and I_0/I_1 are graded-Artinian (resp. graded-Noetherian) if and only if the graded A-modules $I_2, I_1/I_2$, and I_0/I_1 are graded-Artinian (resp. graded-Noetherian) ... if and only if the graded A-modules $I_{n-1}/I_n, I_{n-2}/I_{n-1}, \ldots, I_1/I_2, I_0/I_1$ are graded-Artinian (resp. graded-Noetherian). Hence, it remains to show that the graded A-modules $I_{i-1}/I_i, 1 \leq i \leq n$, are graded-Artinian if and only if they are graded-Noetherian.

Note that I_{i-1}/I_i is a graded $K_i = A/M_i$-module for the multiplication: $\bar{a} \cdot \hat{x} = \widehat{ax}$, for each $a \in A$ and $x \in I_{i-1}$. In fact, M_i is graded-maximal and then K_i is a graded-field. Moreover, the multiplication is well defined since if $\bar{a} = \bar{b}$ in A/M_i and $\hat{x} = \hat{y}$ in I_{i-1}/I_i, and

then $a-b \in M_i$ and $x-y \in I_i$, so $ax-by=(a-b)x+b(x-y) \in M_iI_{i-1}+I_i=I_i+I_i=I_i$. Hence $\widehat{ax}=\widehat{by}$ in I_{i-1}/I_i. Note also that a subset of I_{i-1}/I_i is a graded A-submodule if and only if it is a graded K_i-submodule. Indeed, the stability of this subset of I_{i-1}/I_i by multiplication by the elements of A is equivalent to its stability by multiplication by the elements of $K_i=A/M_i$.

As a consequence, the graded A-module I_{i-1}/I_i is graded-Noetherian if and only if the graded $K_i=A/M_i$-module I_{i-1}/I_i is graded-Noetherian. Now, since M_i is a graded-maximal ideal of A, K_i is a graded-field and then it is both graded-Artinian and graded-Noetherian. Hence, the graded $K_i=A/M_i$-module I_{i-1}/I_i is graded-Noetherian if and only if the graded $K_i=A/M_i$-module I_{i-1}/I_i is finitely generated, if and only if, using Corollary 5.2.5, the graded $K_i=A/M_i$-module I_{i-1}/I_i is graded-Artinian if and only if the graded A-module I_{i-1}/I_i is graded-Artinian.

The following Theorem characterizes the graded-Artinian property in terms of the graded-Noetherian property and the graded Krull's dimension.

Theorem 5.2.7 *Le G be an abelian group. A graded ring A is graded-Artinian if and only if it is graded-Noetherian and gr-dim$(A)=0$.*

Proof. For the direct implication, it is well known that for a graded-Artinian ring A, every graded-prime ideal is graded-maximal, and A has a finite number of graded-maximal ideals, see [9, Corollary 3.41], so gr-Spec(A) = gr-Max$(A)=\{M_1,\ldots,M_k\}$ is finite. By Lemma 5.2.2, the graded ideal $I=$ gr-Nil$(A)=M_1\cap\ldots\cap M_k$ is nilpotent. There exists $n\in\mathbb{N}^*$ such that $I^n=(0)$. Since $M_1\ldots M_k\subseteq M_1\cap\ldots\cap M_k=I$, then $(M_1\ldots M_k)^n=(0)$. By the Lemma 5.2.6, the ring A is graded-Noetherian.

For the reverse implication, let A be a graded-zero-dimensional graded-Noetherian ring. By Lemma 5.2.1, we have gr-Spec(A) = gr-Max(A) = gr-Min$(A)=\{M_1,\ldots,M_k\}$ which is finite. Then gr-Nil$(A)=M_1\cap\ldots\cap M_k$ and the product $I=M_1\ldots M_k\subseteq$ gr-Nil(A). Since A is graded-Noetherian, the graded ideal $I=(x_1,\ldots,x_s)$ is finitely generated, where the x_i are homogeneous elements. There exists $m\in\mathbb{N}^*$ such that $x_1^m=\ldots=x_s^m=0$. Let $n=ms$, and then $I^n=(0)$. In fact, the generators of I^n are the monomials of the form $x_1^{\alpha_1}\ldots x_s^{\alpha_s}$ with $\alpha_1+\ldots+\alpha_s=n$. If $\alpha_1<m,\ldots,\alpha_s<m$, then $n=\alpha_1+\ldots+\alpha_s<sm=n$, a contradiction. So, there exists an index i such that $\alpha_i\geq m$, and then $x_1^{\alpha_1}\ldots x_s^{\alpha_s}=0$. As a result, we have that $I^n=M_1^n\ldots M_k^n=(0)$. By Lemma 5.2.6, A is graded-Artinian.

In the following example, we give a way to construct graded-Artinian rings.

Example 5.2.8 *Let G be an abelian group, B a graded ring, M a finitely generated graded-maximal ideal of B, $k\in\mathbb{N}$, and I a graded ideal of B containing M^k. The graded ring $A=B/I$ is graded-Artinian. Indeed, a graded-prime ideal of A is of the form P/I with P a graded-prime ideal of B satisfying $M^k\subseteq I\subseteq P$. Then $M\subseteq P$, so $M=P$. The only*

graded-prime ideal of A is M/I. It is finitely generated. By Theorem 5.2.7, the graded ring A is graded-Artinian.

We are now in a position to prove the main result of this section which present the graded version of the Principal Ideal Theorem. We begin first by giving the next elementary lemma. Note that, we need the assumption that G is an abelian group (not necessarily torsionfree), since Theorem 5.2.7, the graded version of Nakayama's Lemma and the following elementary lemma use this assumption.

Lemma 5.2.9 *Let G be an abelian group, A a graded-Noetherian ring, and let P be a graded-prime ideal of A. For every $n \in \mathbb{N}$, set $P^{(n)} := P^n A_{[P]} \cap A$, it is a graded ideal called the n-th symbolic power of P. Then $P^{(n)}A_{[P]} = (P)^n A_{[P]}$.*

Proof. It is important to notice that $P^{(n)} := P^n A_{[P]} \cap A$ is a graded ideal, since $A_{[P]}$ is a graded ring, P^n is a graded ideal and the intersection of graded ideals is a graded ideal. The rest of the proof is just like the ungraded case.

Theorem 5.2.10 *Let G be an abelian group, A a graded-Notherian ring, and let I be a proper graded principal ideal of A. Then each graded-minimal graded-prime ideal over I has graded krull dimension at most one.*

Proof. Let P be a graded-minimal graded-prime ideal over I. Replacing A by $A_{[P]}$ after graded-localizing A at P, we can assume that A is a graded-local ring with graded-maximal ideal P. Suppose, by way of contradiction, that $gr\text{-}ht(P) \geq 2$. Let Q_0 and Q be two graded-prime ideals in A such that $Q_0 \subsetneq Q \subsetneq P$. Remark that if we replace A by A/Q_0, then we can assume that A is a graded-Noetherian graded-domain that is graded-local with graded-maximal ideal P satisfying that P is a graded-minimal graded-prime over I and $(0) \subsetneq Q \subsetneq P$.
Now, let $r \in h(A)$ such that $I = Ar$ and, for each $n \in \mathbb{N}$, set $Q^{(n)} = Q^n A_{[Q]} \cap A$. Since Q is graded-prime, it is clear that $Q^{(n)}$ is a graded Q-primary ideal for every $n \in \mathbb{N}$. The quotient graded ring A/Ar has only one graded-prime ideal, namely, P/Ar. Therefore it is a graded-zero-dimensional graded-Noetherian graded-local ring, hence, by Theorem 5.2.7, it is a graded-Artinian ring. As a consequence, the chain of graded ideals $\left(\left(Q^{(n)} + Ar\right)/Ar\right)_{n\in\mathbb{N}}$ of A/Ar stabilizes, and so there is an $N \in \mathbb{N}$ such that $Q^{(n)} + Ar = Q^{(n+1)} + Ar$ for every $n \geq N$.
Fix $n \geq N$, and then take $q_n \in h(Q^{(n)})$. Since $Q^{(n)} \subseteq Q^{(n+1)} + Ar$, we can write $q_n = q_{n+1} + ar$ for some $q_{n+1} \in h(Q^{(n+1)})$ and $a \in h(A)$. Note that $ar = q_n - q_{n+1} \in Q^{(n)}$. In addition, $r \notin Q$ because P is a graded-minimal graded-prime over Ar in A. This, using the fact that $Q^{(n)}$ is graded Q-primary, implies that $a \in Q^{(n)}$, since $r, a \in h(A)$. As a result, $Q^{(n)} \subseteq Q^{(n+1)} + Q^{(n)}r$, which implies that $Q^{(n)} = Q^{(n+1)} + Q^{(n)}r$. Therefore the graded A-module $M = Q^{(n)}/Q^{(n+1)}$ satisfies that $M = rM$. So it follows from the graded version of

Nakayama's Lemma, see [45, Corollary 2.9.2] that $M = \{0\}$, hence $Q^{(n)}/Q^{(n+1)} = \{0\}$. Thus, for each $n \geq N$ the equality $Q^{(n)} = Q^{(N)}$ holds, and so $Q^n A_{[Q]} = Q^N A_{[Q]}$ by virtue of Lemma 5.2.9. In particular, $Q^n A_{[Q]} = Q^{n+1} A_{[Q]}$, using the the graded version of Nakayama's Lemma again, we have that $Q^n A_{[Q]} = 0$. Now, take a nonzero homogeneous element $q \in Q$. As A is a graded-domain, q^n is a nonzero homogeneous element of $Q^n A_{[Q]} = 0$, which generates a contradiction and complete the proof.

We next give the graded version of the height Theorem.

Theorem 5.2.11 *Let G be an abelian group, A a graded-Notherian ring, and let I be a proper graded ideal generated by n homogeneous elements of A. Then each graded-minimal graded-prime ideal over I has graded krull dimension at most n.*

Proof. The proof is a consequence of the graded version of the principal ideal theorem by induction on the number of homogeneous elements using the same arguments as in the previous one.

The next result examines the converse of the previous Theorem 5.2.11. It is the converse of the graded version of the Generalized Principal Ideal Theorem.

Theorem 5.2.12 *Let G be an abelian group, A a graded-Noetherian ring and P be a graded-prime ideal of graded-height n in A, $n \in \mathbb{N}$. There exist n homogeneous elements $a_1, \ldots, a_n \in A$ such that P is graded-minimal over the graded ideal $(a_1, \ldots, a_n)$.*

Proof. We proceed by induction on n. If $n = 0$, P is graded-minimal, we take an empty set of homogeneous elements of A. Suppose that $n \geq 1$ and the property is true for any graded-prime ideal P with $gr\text{-}ht(P) \leq n-1$. By Lemma 5.2.1, since A is graded-Noetherian, it has a finite number of graded-minimal graded-prime ideals $Q_1, \ldots, Q_k$. Since $gr\text{-}ht(P) = n \geq 1$ and $gr\text{-}ht(Q_i) = 0$, then $P \nsubseteq Q_i$, $1 \leq i \leq k$. By the graded-prime avoidance lemma, we have $P \nsubseteq Q_1 \cup \ldots \cup Q_k$. Let $a_1 \in h(P) \backslash Q_1 \cup \ldots \cup Q_k$. In the graded-Noetherian quotient ring $A/(a_1)$, $gr\text{-}ht(P/(a_1)) \leq n-1$. In fact, else, there exist a chain of length n of graded-prime ideals of $A/(a_1)$ of the form $P_0/(a_1) \subset P_1/(a_1) \subset \ldots \subset P_n/(a_1) = P/(a_1)$ with $P_0 \subset P_1 \subset \ldots \subset P_n = P$ a chain of length n of graded-prime ideals of A containing (a_1). Since a_1 does not belong to any graded-minimal graded-prime ideal of A, P_0 is not graded-minimal. There exists $i \in \{1, \ldots, k\}$ such that $Q_i \subset P_0 \subset P_1 \subset \ldots \subset P_n = P$. Hence we obtain a chain of length $n+1$ of graded-prime ideals of A arriving at P, which contradicts $gr\text{-}ht(P) = n$. By the induction hypothesis, there exist $a_2, \ldots, a_n \in P$ such that $P/(a_1)$ is graded-minimal over the ideal $(\overline{a_2}, \ldots, \overline{a_n})$ in the graded ring $A/(a_1)$. It remains to show that P is graded-minimal over $(a_1, a_2, \ldots, a_n)$ in A. Let Q be a graded-prime ideal of A such that $(a_1, a_2, \ldots, a_n) \subseteq Q \subseteq P$. In the graded ring $A/(a_1), (\overline{a_2}, \ldots, \overline{a_n}) \subseteq Q/(a_1) \subseteq P/(a_1)$. Then $Q/(a_1) = P/(a_1)$, and thus $P = Q$.

Next, we improve the graded version of the Cohen's type theorem for graded-Noetherian rings. Note that this result in proved in [22] under the hypothesis that G is a cancellative torsion-free monoid, see [22, Lemma 2.3].

Lemma 5.2.13 *Let G be a cancellative monoid. A graded ring A is graded-Noetherian if and only if its graded-prime ideals are finitely generated.*

Proof. Suppose that A is not graded-Noetherian. The set M of the graded ideals that are not finitely generated is not empty. We order M with the inclusion. Let $(J_\lambda)_\lambda$ be a totally ordered family of elements of M. The ideal $J = \bigcup J_\lambda$ is graded. If $J = (a_1, \ldots, a_n)$, there exists λ such that $J_\lambda = (a_1, \ldots, a_n)$, which is absurd. Then M is inductive and so it admits a maximal element P. We will show that P is a graded-prime ideal of A, which contradicts the hypothesis. Suppose that there exist $a, b \in h(A)\backslash P$ such that $ab \in P$. Since $P \subset P + aA$, and $P + aA$ is graded, then it is finitely generated. Put $P + aA = (\alpha_1 + ax_1, \ldots, \alpha_n + ax_n)$ with $\alpha_i \in P$ and $x_i \in A$. Since G is a cancellative monoid, consider the graded ideal $I = [P : a] = \{y \in A; ya \in P\}$. Then $P \subset I$ because of b, then I is finitely generated. Put $I = (\beta_1, \ldots, \beta_s)$. We will show that $P = (\alpha_1, \ldots, \alpha_n, a\beta_1, \ldots, a\beta_s)$. This is a contradiction with $P \in$ M. Since $\beta_i \in I$, $a\beta_i \in P$, and we always have an inclusion. Conversely, let $x \in P \subset P + aA$. There exist $u_1, \ldots, u_n \in A$ such that $x = u_1(\alpha_1 + ax_1) + \ldots + u_n(\alpha_n + ax_n) = u_1\alpha_1 + \ldots + u_n\alpha_n + (u_1x_1 + \ldots + u_nx_n)a$. It suffices to show that $(u_1x_1 + \ldots + u_nx_n)a \in (a\beta_1, \ldots, a\beta_s)$. But $(u_1x_1 + \ldots + u_nx_n)a = x - (u_1\alpha_1 + \ldots + u_n\alpha_n) \in P$, and then $u_1x_1 + \ldots + u_nx_n \in I$, by definition of I. Since $I = (\beta_1, \ldots, \beta_s)$, then $(u_1x_1 + \ldots + u_nx_n)a \in (a\beta_1, \ldots, a\beta_s)$.

We end this section by examining the transfer of the graded-Artinian property in trivial graded ring extentions called also the graded idealization. Let A be a graded ring and M a graded A-module. The graded trivial extension of a graded R-module M is denoted by $R \propto M$ with componentwise addition and multiplication defined by $(a,m)(b,m') := (ab, am' + bm)$ for all $a, b \in R$ and $m, m' \in M$. Then $R \propto M$ is a graded ring by $(R \propto M)_g = R_g \oplus M_g$ for all $g \in G$, see [10]. We begin first by improving [10, Theorem 3] which studies the transfer of the graded-Noetherian property in the graded idealization. The next result present a new version of the last theorem without the torsionfree assumption on G.

Theorem 5.2.14 *Let G be a cancellative monoid, A be a graded ring and M a graded A-module. Then the graded idealization ring $A \propto M$ is graded-Noetherian, if and only if A is graded-Noetherian, and M is finitely generated.*

Proof. Suppose that $A \propto M$ is graded-Noetherian. Then A being graded homomorphic image of $A \propto M$ is graded-Noetherian. Now $0 \propto M$ is a finitely generated graded ideal of $A \propto M$ since $A \propto M$ is graded-Noetherian. Observe that $(0, m_1), \ldots, (0, m_n)$ generate $0 \propto M$ as a graded ideal if and only if $m_1, \ldots, m_n$ generate M as a graded A-module. Hence M is a finitely generated A-module. Conversely, suppose that A is graded-Noetherian and

M is finitely generated. It is known that, gr-Spec$(A \propto M) = \{P \propto M; P \in \text{gr-Spec}(A)\}$, see [10], so let $P \propto M$ be a graded-prime ideal of $A \propto M$. Since A is graded-Noetherian, P is finitely generated. Then, since M is a finitely generated graded A-module, $P \propto M$ is a finitely generated graded ideal of $A \propto M$. Since every graded-prime ideal of $A \propto M$ is finitely generated, then by Lemma 5.2.13, $A \propto M$ is graded-Noetherian.

Next, we examine the transfer of the graded-Artinian property in the graded idealization.

Theorem 5.2.15 *Let G be an abelian group, A a graded ring and M be a graded A-module. Then $A \propto M$ is graded-Artinian if and only if the graded ring A is graded-Artinian and the graded module M is finitely generated.*

Proof. For the direct implication, by [9, Proposition 3.11], A is graded-Artinian since it is graded homomorphic to the graded quotient ring $A \propto M/(0) \propto M$. By Theorem 5.2.7, the graded ring $A \propto M$ is graded-Noetherian. By Lemma 5.2.13, the graded module M is finitely generated.

For the reverse implication, by Theorem 5.2.7, the ring A is graded-Noetherian and graded-zero-dimensional. By Lemma 5.2.13, the graded ring $A \propto M$ is graded-Noetherian. By [58, Proposition 3.3], gr-Spec$(A \propto M) = \{P \propto M; P \in \text{gr-Spec}(A)\}$. Then gr-dim$(A \propto M) = \text{gr-dim}A = 0$. By Theorem 5.2.7, $A \propto M$ is graded-Artinian.

5.3 On graded-nonnil-Noetherian Rings

This section initiates the study of graded-Nonnil-Noetherian rings, we start first by recalling the following definitions and notations. Let A be a graded ring. Then gr-Nil$(A) = Gr(0)$ is the set of all $x = \sum_{g \in G} x_g \in R$ such that x_g is a nilpotent element of A for each $g \in G$. gr-Nil(A) is the intersection of all graded-prime ideals of A, see [52, Proposition 2.5]. gr-Spec(A) is its graded-prime spectrum, and gr-Min(A) is the set of its graded-minimal ideals. An ideal I of A is said to be graded-nonnil if $I \nsubseteq \text{gr-Nil}(A)$. We say that A is graded-Nonnil-Noetherian (gr-Nonnil-Noetherian, for short) if each graded-nonnil graded ideal of A is finitely generated. In this section, our main purpose is to give some characterizations of graded-Nonnil-Noetherian property of A in terms of the graded-Noetherianity of $A/\text{gr-Nil}(A)$ which is a graded ring since gr-Nil(A) is a graded ideal. Then, we give a sequence of corollaries extending many other properties of Nonnil-Noetherian rings to the context of graded rings. It is important to notice that if I is a graded ideal, then I is a graded-nonnil ideal if and only if I is a nonnil graded ideal of A. Indeed, if $x = \sum_{g \in G} x_g \in I \setminus \text{gr-Nil(A)}$, then $\exists g \in G$, such that, $x_g \in I \setminus Nil(A)$, since I is a graded ideal, which implies that I is nonnil. Finally, we say that A is graded-reduced if gr-Nil$(\text{A}) = 0$.

We next collect some immediate classes of graded-Nonnil-Noetherian rings.

Remark 5.3.1 *Let R be a graded ring. Then the following statements hold:*

1. *Every graded-Noetherian ring A is a graded-Nonnil-Noetherian ring. But the converse is not true in general, as shown by Example 5.3.2 and Example 5.3.3. It is clear that the two notions coincide for a reduced ring, i.e.,* $\mathrm{Nil(A)} = (0)$*, or even for a graded-reduced ring, i.e.,* $\text{gr-Nil(A)} = (0)$.

2. *Obviously, every Nonnil-Noetherian graded ring is a graded-Nonnil-Noetherian ring. But the converse is not true in general, as shown by Example 5.3.4. Note that the two notions coincide for a trivially graded ring.*

We next give some examples of graded-Nonnil-Noetherian ring that are not graded-Noetherian. Let A be a ring and let $\{Y_1, Y_2, \ldots\}$ be (commuting) algebraically (respectively, analytically) independent indeterminates over A. For $s = (s_1, \cdots, s_n) \in \mathbb{N}^n$, let $Y^s = Y_1^{s_1} \cdots Y_n^{s_n}$. Then the polynomial ring $P = A[Y_1, \cdots, Y_n]$ is graded by $\mathbb{N}$ via:

$$P_m = \left\{ \sum_{s \in \mathbb{N}^n} r_s Y^s | r_s \in R \text{ and } \sum_{i=1}^{n} s_i = m \right\}.$$

Example 5.3.2 *Let* $(Y_i)_{i \in \mathbb{N}}$ *be a sequence of indeterminates over* $\mathbb{Q}$ *and* $n \geq 2$ *a natural integer. Consider the graded ideal* $I = (Y_i^n; i \in \mathbb{N})$ *of the graded ring* $A = \mathbb{Q}[Y_i; i \in \mathbb{N}]$*, graded by* $\mathbb{N}$ *as the inductive limit of the inductive system of the* $\mathbb{N}$*-graded rings* $\mathbb{Q}[Y_1, \cdots, Y_n]$*,* $n \in \mathbb{N}$ *and let* $B = A/I$*. The only graded-prime ideal of B is* $P = (\bar{Y}_i; i \in \mathbb{N})$*. It is not finitely generated, and then B is not graded-Noetherian. By Proposition 5.3.6 given hereinafter, since* $\text{gr-Nil(B)} = P$*, then B is graded-Nonnil-Noetherian.*

We next give an another $\mathbb{Z}_2$-graded ring counter-example.

Example 5.3.3 *Consider the graded ring* $A = \mathbb{Z} \propto \mathbb{Q}$ *with its natural* $\mathbb{Z}_2$*-grading;* $A_0 = \mathbb{Z} \propto 0$ *and* $A_1 = 0 \propto \mathbb{Q}$*. By [14, Theorem* 3.4*], A is a Nonnil-Noetherian and so, by remark 5.3.1(2), A is a graded-Nonnil-Noetherian ring. Now* $Nil(A) = \text{gr-Nil(A)} = 0 \propto \mathbb{Q}$ *is a graded ideal which is not finitely generated, A is not graded-Noetherian.*

We next give an example of a graded-Nonnil-Noetherian ring which is not Nonnil-Noetherian.

Example 5.3.4 *Assume that the grading monoid is a cancellative torsion-free monoid. Let R be a graded ring. It is known that the monoid ring* $A[X;G]$ *over a ring A is graded-Noetherian ring if and only if A is a Noetherian ring and each ideal of G (that is, a nonempty subset I of the additive monoid G, such that* $g + I \subseteq I$ *for each* $g \in G$*) is finitely generated, see [22, Theorem* 2.4 *and Corollary* 2.5*]. Hence, if* $\mathbb{Q}$ *is the additive group of rational numbers and B is a Noetherian ring, the group ring;* $A = B[X;\mathbb{Q}]$ *is a graded-Noetherian ring but not a Noetherian ring. Therefore, since A is reduced, it is a graded-Nonnil-Noetherian ring which is not Nonnil-Noetherien.*

The next proposition characterizes graded-Nonnil-Noetherian rings by the ascending chain condition on nonnil graded ideals.

Proposition 5.3.5 *The following assertions are equivalent for a graded ring A:*

1. *The graded ring A is graded-Nonnil-Noetherian.*
2. *The graded ring A satisfies the acc for the nonnil graded ideals.*
3. *Each nonempty set of nonnil graded ideals of A has a graded-maximal element for the inclusion.*

Proof. $(1) \Longrightarrow (2)$ Let (I_n) be an increasing sequence of nonnil graded ideals of A. The ideal $I = \bigcup I_n$ is graded and nonnil. Take $I = (x_1, \ldots, x_s)$. There exists $m \in \mathbb{N}$ such that $x_1, \ldots, x_s \in I_m$. For each $n \geq m$, $I_n = I_m$.
$(2) \Longrightarrow (3)$ Let F be a nonempty set of nonnil graded ideals of A. Suppose that F does not have a graded-maximal element. Let $I_1 \in F$. Since I_1 is not maximal in $(F, \subseteq)$, there exists $I_2 \in F$ such that $I_1 \subset I_2$. By induction, we construct a strictly increasing sequence of nonnil graded ideals of A, which is absurd.
$(3) \Longrightarrow (1)$ Let I be a nonnil graded ideal of A and F the set of nonnil finitely generated graded ideals contained in I. Since $I \nsubseteq Nil(A)$, there exists $a \in I \backslash Nil(A)$, since I is graded, we can assume that a is a homogeneous element and then $(a) \in F$. By hypothesis, $(F, \subseteq)$ has a graded-maximal element $J \subseteq I$. Suppose that $I \neq J$ and consider an element $x \in I \backslash J$, since I is graded, we may suppose that x is homogeneous. Then $J + xA \in F$ and $J \subset J + xA$, a contradiction. Hence $I = J$ is a finitely generated graded ideal.

The next proposition present the analogue of the well known Cohen's Theorem for graded-Nonnil-Noetherian rings.

Proposition 5.3.6 *Let G be a cancellative monoid, a graded ring A is graded-Nonnil-Noetherian if and only if its nonnil graded-prime ideals are finitely generated.*

Proof. Suppose that A is not graded-Nonnil-Noetherian. The set F of the nonnil graded ideals that are not finitely generated is not empty. We order F with the inclusion. Let $(J_\lambda)_\lambda$ be a totally ordered family of elements of F. The ideal $J = \bigcup J_\lambda$ is graded and nonnil. If $J = (a_1, \ldots, a_n)$, there exists λ such that $J_\lambda = (a_1, \ldots, a_n)$, which is absurd. Then F is inductive and so it admits a maximal element P. We will show that P is a graded-prime ideal of A, which contradicts the hypothesis. Suppose that there exist $a, b \in h(A) \backslash P$ such that $ab \in P$. Since $P \subset P + aA$, then $P + aA$ is nonnil, and since it is graded, then it is finitely generated. Put $P + aA = (\alpha_1 + ax_1, \ldots, \alpha_n + ax_n)$ with $\alpha_i \in P$ and $x_i \in A$. Since G is a cancellative monoid, Consider the graded ideal $I = [P : a] = \{y \in A; ya \in P\}$. Then $P \subset I$ because of b, then I is graded and nonnil and so finitely generated. Put $I = (\beta_1, \ldots, \beta_s)$. We will show that $P = (\alpha_1, \ldots, \alpha_n, a\beta_1, \ldots, a\beta_s)$. This is a contradiction with $P \in$ F. Since

$\beta_i \in I$, $a\beta_i \in P$, and we always have an inclusion. Conversely, let $x \in P \subset P + aA$. There exist $u_1, \ldots, u_n \in A$ such that $x = u_1(\alpha_1 + ax_1) + \ldots + u_n(\alpha_n + ax_n) = u_1\alpha_1 + \ldots + u_n\alpha_n + (u_1x_1 + \ldots + u_nx_n)a$. It suffices to show that $(u_1x_1 + \ldots + u_nx_n)a \in (a\beta_1, \ldots, a\beta_s)$. But $(u_1x_1 + \ldots + u_nx_n)a = x - (u_1\alpha_1 + \ldots + u_n\alpha_n) \in P$, and then $u_1x_1 + \ldots + u_nx_n \in I$, by definition of I. Since $I = (\beta_1, \ldots, \beta_s)$, then $(u_1x_1 + \ldots + u_nx_n)a \in (a\beta_1, \ldots, a\beta_s)$.

Proposition 5.3.7 *The graded homomorphic image of a graded-Nonnil-Noetherian ring is graded-Nonnil-Noetherian.*

Proof. Let $f : A \longrightarrow B$ be a graded homomorphism onto of rings with A is graded-Nonnil-Noetherian. Let J be a nonnil graded ideal of B. Then $J = f\left(f^{-1}(J)\right)$ with $f^{-1}(J) \nsubseteq \mathrm{Nil}(A)$. Since f is a graded homomorphism, $f^{-1}(J)$ is a graded ideal and so by hypothesis it is finitely generated. Then J is also finitely generated.

Example 5.3.8 *The quotient of a graded-Nonnil-Noetherian ring by a graded ideal is a graded-Nonnil-Noetherian, i.e., if A is a graded-Nonnil-Noetherian ring and I is a graded ideal of A, then A/I is graded-Nonnil-Noetherian.*

The following Theorem is the main result of this section, it is the graded version of the main result due to Benhissi, see [18].

Theorem 5.3.9 *Let G be an abelian group, A a graded ring, and* gr-Nil(A) *its graded Nilradical.*

(*i*) *If* gr-Nil(A) $\notin$ gr-Spec(A)*, the following assertions are equivalent:*

(1) *The graded ring A is graded-Nonnil-Noetherian.*

(2) *The graded ring A is graded-Noetherian.*

(3) *The quotient graded ring* $A/$ gr-Nil(A) *is graded-Noetherian, and all the graded-minimal graded-prime ideals of A are finitely generated.*

(*ii*) *If* gr-Nil(A) $\in$ gr-Spec(A)*, then A is graded-Nonnil-Noetherian if and only if* $A/$ gr-Nil(A) *is graded-Noetherian, and all the graded-prime ideals whose graded Krull's dimention equals* 1 *of A are finitely generated.*

Proof. $(i)(1) \Rightarrow (2)$ For all $P \in$ gr-Spec(A), gr-Nil(A) $\subset P$, so P is finitely generated. Then A is graded-Noetherian by Lemma 5.2.13.
$(2) \Rightarrow (3)$ Obvious.
$(3) \Rightarrow (1)$ Let $P \in$ gr-Spec(A) and $Q \in$ gr-Min(A) with $Q \subseteq P$. Since gr-Nil(A) $\subset Q$, then $(A/\text{gr-Nil(A)})/(Q/\text{gr-Nil(A)})$ is graded isomorphic to A/Q. Since $A/$ gr-Nil(A) is graded-Noetherian, so is A/Q. Then its graded ideal P/Q is finitely generated. Since Q is finitely generated, so is P. By Lemma 5.2.13, A is graded-Noetherian.

(ii) $(\Rightarrow)$ A nonzero graded-prime ideal of $A/\,\text{gr-Nil}(A)$ is of the form $P/\,\text{gr-Nil}(A)$ with $P \in \text{gr-Spec}(A)$ and $\text{gr-Nil}(A) \subset P$. By hypothesis, P is finitely generated, so is $P/\,\text{gr-Nil}(A)$. By Lemma 5.2.13, $A/\,\text{gr-Nil}(A)$ is graded-Noetherian. Since $\text{gr-Nil}(A) \in \text{gr-Spec}(A)$, then $\text{gr-Min}(A) = \{\text{gr-Nil}(A)\}$. If P is a graded-prime ideal with graded Krull's dimension equals 1 of A, then $\text{gr-Nil}(A) \subset P$ so P is finitely generated.

$(\Leftarrow)$ Let P be a nonnil gr-prime ideal of A. Then $\text{gr-Nil}(A) \subset P$ and $\bar{P} = P/\,\text{gr-Nil}(A)$ is a nonzero graded-prime ideal of the graded-domain $\bar{A} = A/\,\text{gr-Nil}(A)$. Let $\overline{0} \neq \bar{x} \in \bar{P}$, where $\bar{x}$ is a homogeneous element of $\bar{P}$ and $\bar{Q} \subseteq \bar{P}$ a graded-prime ideal of $\bar{A}$ that is graded-minimal over $(\bar{x})$. Since $\bar{A}$ is graded-Noetherian, by the graded version of the principal ideal theorem, see Theorem 5.2.10, the graded Krull's dimension of $\bar{Q}$ is 1. On the other hand, $\bar{Q} = Q/\,\text{gr-Nil}(A)$ with $\text{gr-Nil}(A) \subset Q \subseteq P$, and Q is graded-prime ideal of A with graded Krull's dimension 1. By the hypothesis, Q is finitely generated. By the graded isomorphism $(A/\,\text{gr-Nil}(A))/(Q/\,\text{gr-Nil}(A)) \simeq A/Q$, the graded-domain A/Q is graded-Noetherian. Its graded ideal P/Q is then finitely generated. Then P is finitely generated, and A is graded-Nonnil-Noetherian, by Proposition 5.3.6.

Corollary 5.3.10 *Let G be an abelian group and A a graded ring.*

1. *If* $\text{gr-Nil}(A) \notin \text{gr-Spec}(A)$*, then A is graded-Noetherian if and only if A is graded-Nonnil-Noetherian.*

2. *If* $\text{gr-Nil}(A) \in \text{gr-Spec}(A)$*, then A is graded-Noetherian if and only if A is graded-Nonnil-Noetherian and* $\text{gr-Nil}(A)$ *is finitely generated.*

Proof. (1) Obvious.

(2) The dirct implication is clear. Conversely, let $P \in \text{gr-Spec}(A)$. If $P = \text{gr-Nil}(A)$, it is finitely generated by hypothesis. If $P \neq \text{gr-Nil}(A)$, then $\text{gr-Nil}(A) \subset P$ so P is finitely generated, using Proposition 5.3.6, since A is graded-Nonnil-Noetherian.

Lemma 5.3.11 *Let G be an abelian group and $P \subset Q$ the two graded-prime ideals of a graded-Noetherian ring A. If there exists a graded-prime ideal properly between them, then there are infinitely many.*

Proof. By passage to the quotient graded-domain A/P, we may assume that $P = (0)$. Suppose that $P_1, \ldots, P_n$ are the only graded-prime ideals properly between (0) and Q. By the graded-prime avoidance Lemma, we have $Q \nsubseteq P_1 \cup \ldots \cup P_n$. Let $x \in h(Q) \backslash P_1 \cup \ldots \cup P_n$. Then Q is graded-minimal over the graded principal ideal (x). By the graded version of the principal ideal Theorem, see Theorem 5.2.7, the graded Krull's dimension of Q is at most 1. But we have the chains $(0) \subset P_i \subset Q, 1 \leq i \leq n$, of graded-prime ideals of A, which is absurd.

We next give the analogue of the preceding result for the graded-Nonnil-Noetherian rings.

Corollary 5.3.12 *Let G be an abelian group and A a graded-Nonnil-Noetherian ring and $P \subset Q$ the two graded-prime ideals of A. If there exists a graded-prime ideal of A strictly between P and Q, then there exists an infinity.*

Proof. By Theorem 5.3.9, $P/\text{gr-Nil(A)} \subset Q/\text{gr-Nil(A)}$ are two graded-prime ideals of the graded-Noetherian ring $A/\text{gr-Nil(A)}$. If R is a graded-prime ideal of A such that $P \subset R \subset Q$, then $R/\text{gr-Nil}(A)$ is a graded-prime ideal of $A/\text{gr-Nil(A)}$ such that $P/\text{gr-Nil(A)} \subset R/\text{gr-Nil(A)} \subset Q/\text{gr-Nil(A)}$. By Lemma 5.3.11, there exists an infinity of graded-prime ideals of $A/\text{gr-Nil(A)}$ between the ideals $P/\text{gr-Nil(A)}$ and $Q/\text{gr-Nil(A)}$. So there exist an infinity of graded-prime ideals between P and Q.

A graded ring A is said to be graded-chained if its lattice of graded ideals is totally ordered by inclusion. Note that a graded-chained ring is graded-local and its graded nilradical, i.e. gr-Nil(A) is a graded-prime ideal. In fact it is the only graded-minimal graded-prime ideal of A.

Corollary 5.3.13 *Let A be a graded-chained ring with graded-maximal ideal M.*

1. *If* $\text{gr-Nil(A)} = M$*, then A is graded-Nonnil-Noetherian.*
2. *If* $\text{gr-Nil(A)} \neq M$*, then A is graded-Nonnil-Noetherian if and only if* $\text{gr-Spec(A)} = \{\text{Nil(A)}, M\}$ *with M a finitely generated graded ideal.*

Proof.

1. If $\text{gr-Spec(A)} = \{\text{gr-Nil(A)}\}$, then by Proposition 5.3.6, A is gr-Nonnil-Noetherian.
2. ($\Rightarrow$) Since $\text{gr-Nil(A)} \neq M$, by Theorem 5.3.9 (ii), $A/\text{gr-Nil(A)})$ is a gr-Noetherian graded-domain with nonzero graded Krull's dimension. Hence gr-ht(M) is 1 in A and $\text{gr-Spec(A)} = \{\text{gr-Nil(A)}, M\}$. Since A is graded-Nonnil-Noetherian, M is finitely generated.

   ($\Leftarrow$) The only nonnil graded-prime ideal of A is M which is finitely generated. Then A is graded-Nonnil-Noetherian.

A graded-prime ideal P of a ring A is called graded-divided if $P \subseteq xA$ for every $x \in h(A)\backslash P$, equivalently, if P is comparable with every principal graded ideal of A. The next result present the graded version of the main result of Badawi's paper about Nonnil-Noetherian rings, see [14].

Corollary 5.3.14 *Let A be a graded ring. Suppose that* gr-Nil(A) *is a graded-divided and graded-prime ideal. Then A is graded-Nonnil-Noetherian if and only if* $A/\text{gr-Nil(A)}$ *is graded-Noetherian.*

Proof. By Theorem 5.3.9 (ii), it remains to show that if $A/\text{gr-Nil(A)}$ is graded-Noetherian, then each gr-prime ideal P of A with gr-ht(P) $= 1$ is finitely generated. Since $A/\text{gr-Nil(A)}$ is graded-Noetherian, its graded ideal $P/\text{gr-Nil(A)}$ is finitely generated. Take $P/\text{gr-Nil(A)} = (\bar{x}_1, \ldots, \bar{x}_n)$ with $x_1, \ldots, x_n \in h(P)$. If all the $x_i \in \text{gr-Nil(A)}$, then $P/\text{gr-Nil(A)} = (\bar{0})$ so $P = \text{gr-Nil(A)}$ which is impossible since gr-ht(P) $= 1$. So there is $x_{i_0} \notin \text{gr-Nil(A)}$. Since gr-Nil(A) is graded-divided, then gr-Nil(A) $\subset (x_{i_0})$. Hence $P = \text{gr-Nil(A)} + (x_1, \ldots, x_n) \subseteq (x_{i_0}) + (x_1, \ldots, x_n) = (x_1, \ldots, x_n) \subseteq P$. Therefore $P = (x_1, \ldots, x_n)$ which is finitely generated.

Proposition 5.3.15 *Let A be a graded ring. Then A is graded-Nonnil-Noetherian if and only if A satisfies the ascending chain condition for finitely generated nonnil graded ideals.*

Proof. Conversely, Suppose that there is a nonnil graded ideal I of A which is not finitely generated. Let $a_1 \in h(I) \setminus \text{gr-Nil(A)}$. Then $(a_1) \subset I$. Let $a_2 \in h(I) \setminus (a_1)$. Then $(a_1) \subset (a_1, a_2) \subset I$, ... Since $a_1 \notin \text{gr-Nil(A)}$, we have a strictly ascending chain of nonnil finitely generated graded ideals.

Lemma 5.3.16 *Let G be an abelian group, A a graded-Noetherian ring and B a graded overring of A, i.e., $A \subseteq B \subseteq S^{-1}A$, where S is the set of the regular homogeneous elements of A. Then B is graded-Noetherian. Moreover, if* gr-dim(A) $= n < \infty$*, then* gr-dim(B) $\leq n$.

Proof. (1) Let $P \in \text{gr-Spec(B)}$. Then $(P \cap A)B \subseteq P$. Conversely, let $p \in P \subset B \subseteq S^{-1}A$, $p = \frac{a}{s}$ with $a \in A$ and $s \in S$. Then $a = sp \in P \cap A$. So $p = \frac{a}{s} = \frac{1}{s}a \in (P \cap A)B$. Then $P = (P \cap A)B$. Since $P \cap A$ is finitely generated in A, then $P = (P \cap A)B$ is finitely generated in B by the same elements. So B is graded-Noetherian.

(2) Let $P \in \text{gr-Spec(B)}$. Then $\text{gr-ht}_{\text{A}}(\text{P} \cap \text{A}) \leq \text{gr-dim(A)} = n$. By the graded version of the converse of the generalized principal ideal, see Theorem 5.2.12, $P \cap A$ is graded-minimal over an ideal $(p_1, \ldots, p_n)A$ with $p_1, \ldots, p_n \in h(P \cap A)$. We will prove that P is graded-minimal over the ideal $(p_1, \ldots, p_n)B$. Let $Q \in \text{gr-Spec(B)}$ be such that $(p_1, \ldots, p_n)B \subseteq Q \subseteq P$. In particular, $p_1, \ldots, p_n \in Q \cap A$. Then $(p_1, \ldots, p_n)A \subseteq Q \cap A \subseteq P \cap A$. Since $P \cap A$ is graded-minimal over $(p_1, \ldots, p_n)A$, we have to show that $Q \cap A = P \cap A$. By (1), $Q = (Q \cap A)B = (P \cap A)B = P$. By Theorem 5.2.11, $\text{gr-ht}_{\text{B}}(\text{P}) \leq n$. Hence for each $P \in \text{gr-Spec(B)}$, $\text{gr-ht}_{\text{B}}(\text{P}) \leq n$. Therefore gr-dim(B) $\leq n$.

Proposition 5.3.17 *Let A be a graded-Nonnil-Noetherian ring and B a graded overring of A, i.e., $A \subseteq B \subseteq S^{-1}A$ where S is the set of regular homogeneous elements of A. If* gr-dim(A) $= n$*, then* gr-dim(B) $\leq n$.

Proof. Since gr-Nil(A) $= A \cap$ gr-Nil(B), then $\bar{A} = A/\text{gr-Nil(A)} \subseteq B/\text{gr-Nil(B)} = \bar{B}$. The elements of the set consisting only on homogeneous elements of A $\bar{S} = \{\bar{s}; s \in S\}$ are regular homogeneous elements in $\bar{A}$. In fact, let $s \in S$ and $a \in A$ such that $\bar{s}\bar{a} = \bar{0}$. Then sa $\in$ gr-Nil(A). There is $n \in \mathbb{N}^*$ such that $(sa)^n = 0$. Since s^n is regular then $a^n = 0$ so $\bar{a} = \bar{0}$. We have $\bar{B} \subseteq \bar{S}^{-1}\bar{A}$. Thus $\bar{B}$ is a graded overring of $\bar{A}$ with $\bar{A}$ a graded-Noetherian ring. It is clear that gr-dim($\bar{\text{A}}$) = gr-dim(A) $= n$ and gr-dim(B) = gr-dim($\bar{\text{B}}$). By Lemma 5.3.16, gr-dim($\bar{\text{B}}$) $\leq n$. Hence gr-dim(B) $\leq n$.

Proposition 5.3.18 *Let G be an abelian group and S a multiplicative set of homogeneous elements of a graded-Nonnil-Noetherian ring A. Then, $S^{-1}A$ is graded-Nonnil-Noetherian.*

Proof. A nonnil graded ideal of $S^{-1}A$ is of the form $J = S^{-1}I$ where I is a nonnil graded ideal of A. Since I is finitely generated, then so is J.

The following proposition shows that the notions of graded-Noetherian and graded-Nonnil-noetherian coincide for a graded decomposable ring.

Proposition 5.3.19 *Let A and B be two graded rings, then the following assertions are equivalent:*

1. *The graded product ring $A \times B$ is graded-Nonnil-Noetherian.*
2. *The graded product ring $A \times B$ is graded-Noetherian.*
3. *The graded rings A and B are graded-Noetherian.*

Proof. Recall that a graded ideal of the product graded ring $A \times B$ is of the form $I = I_1 \times I_2$, where I_1 is a graded ideal of A and I_2 is a graded ideal of B. It is clear that I is finitely generated if and only if I_1 and I_2 are finitely generated. Then we have the equivalence $(2) \Longleftrightarrow (3)$. The implication $(2) \Longrightarrow (1)$ is obvious. For the implication $(1) \Longrightarrow (3)$, by symmetry, it suffices to show that A is graded-Noetherian. Let I be a graded ideal of A. The graded ideal $I \times B$ is nonnil, and then it is finitely generated in $A \times B$. Hence I is finitely generated in A.

Proposition 5.3.20 *Let G be an abelian group, A a graded-Nonnil-Noetherian ring and $P \in$ gr-Spec(A). Then we have that:*

1. *If P is graded-minimal over a finitely generated graded ideal $I = (a_1, \ldots, a_n)$, where the a_i are homogeneous elements of A, then gr-ht$(P) \leq n$.*
2. *If* gr-ht(P) $= n \in \mathbb{N}^*$, *then there exist n homogeneous elements $a_1, \ldots, a_n \in A$ such that P is graded-minimal over the graded ideal $(a_1, \ldots, a_n)$.*

Proof.

1. Let $P_0 \subset P_1 \subset \ldots \subset P_s = P$ be a chain of graded-prime ideals of A. In the graded ring $A/\text{gr-Nil(A)}$, the graded-prime ideal $P/gr\text{-}Nil(A)$ is graded-minimal over the graded ideal $(\bar{a}_1, \ldots, \bar{a}_n)$, and we have the chain of graded-prime ideals $P_0/\text{gr-Nil(A)} \subset P_1/\text{gr-Nil(A)} \subset \ldots \subset P_s/\text{gr-Nil(A)} = P/\text{gr-Nil(A)}$. By Theorem 5.3.9, the graded ring $A/\text{gr-Nil(A)}$ is graded-Noetherian.

   By Theorem 5.2.11, $gr\text{-}ht(P/\text{gr-Nil(A)}) \leq n$. Then $s \leq n$, hence gr-ht(P) $\leq n$.

2. In the graded-Noetherian ring $A/\text{gr-Nil(A)}$, gr-ht(P/gr-Nil(A)) $= n \geq 1$. By Theorem 5.2.12, there exist $a_1, \ldots, a_n \in h(P)$ such that $P/\text{gr-Nil(A)}$ is graded-minimal over the graded ideal $(\bar{a}_1, \ldots, \bar{a}_n)$. Let Q be a graded-prime ideal of A such that $(a_1, \ldots, a_n) \subseteq Q \subseteq P$. Then $(\bar{a}_1, \ldots, \bar{a}_n) \subseteq Q/\text{gr-Nil(A)} \subseteq P/\text{gr-Nil(A)}$. Hence $Q/\text{gr-Nil(A)} = P/\text{gr-Nil(A)}$ and $Q = P$. Then P is graded-minimal over the graded ideal $(a_1, \ldots, a_n)$.

Proposition 5.3.21 *Let A a be graded-Nonnil-Noetherian ring, then each nonnil graded ideal contains a finite product of graded-prime ideals.*

Proof. Let E be the set of the nonnil graded ideals of A that do not contain finite products of graded-prime ideals. Suppose that E $\neq \emptyset$. By Proposition 5.3.5, E admits a graded-maximal element I. In particular, I is not graded-prime. Let $a, b \in h(A)\backslash I$ be such that $ab \in I$. Since $I \subset I + aA$ and $I \subset I + bA$, and $I + aA$, $I + aA$ are two graded ideals, then $I + aA$ and $I + bA$ contain products of graded-prime ideals. But $(I + aA)(I + bA) = I^2 + aI + bI + abA \subseteq I$. Then I contains a product of graded-prime ideals, a contradiction.

A graded ideal I of a graded ring A is said to be graded-irreducible if it does not admit a decomposition of the form $I = I_1 \cap I_2$ with I_1 and I_2 two graded ideals of A satisfying $I \subset I_1$ and $I \subset I_2$.

Proposition 5.3.22 *Let A be a graded-Nonnil-Noetherian ring. Then, each nonnil graded ideal of A is a finite intersection of graded-irreducible ideals.*

Proof. Let E be the set of the nonnil graded ideals of A that are not finite intersections of graded-irreducible ideals. Suppose that E $\neq \emptyset$. By Proposition 5.3.5, E admits a graded-maximal element I. In particular, I is not graded-irreducible. There exist two graded ideals I_1 and I_2 of A with $I \subset I_1$ and $I \subset I_2$ such that $I = I_1 \cap I_2$. The graded ideals I_1 and I_2 are nonnil and then are finite intersections of graded-irreducible ideals. So is the graded ideal I, which is absurd.

Proposition 5.3.23 *Let G be a cancellative monoid, A a graded-Nonnil-Noetherian ring. Then, each graded irreducible nonnil graded ideal is graded-primary.*

Proof. Suppose that there is a graded-irreducible nonnil ideal I that is not graded-primary. There exist $b,c \in h(A)$ such that $bc \in I$ but $c \notin I$ and $b^n \notin Gr(I)$ for each $n \in \mathbb{N}^*$. Since G be a cancellative monoid, for each $k \in \mathbb{N}^*$, $I_k = \left[I : b^k\right] = \left\{y \in A; yb^k \in I\right\}$ is a graded ideal of A strictly containing I because of c. Then I_k is nonnil. The sequence (I_k) is increasing. By Proposition 5.3.5, there exists $m \in \mathbb{N}^*$ such that $I_m = I_n$ for each integer $n \geq m$. It is easy to observe that $I \subseteq I_m \cap (I + b^m A)$. Conversely, let $x \in I_m \cap (I + b^m A)$. There exist $a \in I$ and $r \in A$ such that $x = a + b^m r$. Since $x \in I_m$, then $xb^m \in I$ so $ab^m + rb^{2m} \in I$, and then $b^{2m} r \in I$. So $r \in I_{2m} = I_m$ and $x = a + b^m r \in I$. We conclude that $I = I_m \cap (I + b^m A)$, now, since $b \in h(A)$ and so $(I + b^m A)$ is a graded ideal, the last fact contradicts the graded-irreducibility of I.

Proposition 5.3.24 *Let A be a graded-Nonnil-Noetherian ring. Then, each nonnil graded ideal I contains a power of its graded radical $Gr(I)$.*

Proof. Since $I \subseteq Gr(I)$, then $Gr(I)$ is finitely generated. Let $Gr(I) = (x_1, \ldots, x_s)$. Since $Gr(I)$ is a graded ideal, we may suppose that the $x_i \in h(Gr(I))$, then there exists $n \in \mathbb{N}^*$ such that $x_1^n, \ldots, x_s^n \in I$. For $m = ns$, the graded ideal $(Gr(I))^m$ is generated by the products $x_1^{r_1} \cdots x_s^{r_s}$ with $r_1 + \cdots + r_s = m$. Hence, there exist an index i such that $r_i \geq n$, and so $x_1^{r_1} \ldots x_s^{r_s} \in I$. Therefore $(Gr(I))^m \subseteq I$.

Perspectives

To end this thesis, we present the following perspectives:

1. Examine the relation between some graded-Noetherian like properties of monoid rings and the grading monoid G.

2. Study the graded version of some new notions related to the Noetherian and Coherent concepts.

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