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Sur quelques résultats d'analyse fonctionnelle dans les espaces de Musielak et applications aux EDP

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Abstract

In the last two decades, there has been an increasingly interest in studying nonlinear partial differential equations with non-standard growth conditions. This interest is justified by their applications in many domains: finance, image restoration, non-Newtonian fluids (characterized by their rapidly change of physical state from the liquid state to the solid state under the influence of different stimuli, such as electric or magnetic fields). Such fluids have many applications in several branches of engineering, including seismic protection, the automotive industry (clutches, shock absorbers, ...), military applications, etc.

It is well known that the classical framework of Lebesgue spaces is not sufficient to capture the models of such non-Newtonian fluids. Indeed in 1996, K. Rajagopal and M. Ružička proposed an empirical model for a class of non-Newtonian fluids, called electrorheological fluids, which can change their viscosity under the effect of an electric field and this via a natural energy with variable exponent depending on the strength of the electric field and therefore varies on its domain. This naturally requires studying these fluids in variable exponents Lebesgue space. This last framework does not seem to be realistic to better understand the behavior of non-Newtonian fluids in general. Indeed, the abrupt change in viscosity in the Cauchy tensor confers to the diffusion operator a non-standard growth described by a Musielak function. Thus the functional framework adapted for the study of the models describing the non-Newtonian fluids is that of Musielak spaces.

Although the mathematical analysis of the Musielak and Musielak-Sobolev spaces has been developed since the 1970s, there are still many questions and problems to study in order to solve PDEs that capture models relating to non-Newtonian fluid mechanics.

In this thesis, we develop some mathematical analysis aspects of Musielak and Musielak-Sobolev spaces. Namely, we establish some basic tools of functional analysis that focus first on the density of regular functions in the spaces of Musielak, then on the duality, the separability as well as the reflexivity of these spaces. These results, which are necessary for the existence theory of some PDEs, are then applied to nonlinear variational elliptic equations with coefficients that grow rapidly/slowly and can be described by Musielak functions.

Résumé

Au cours des deux dernières décennies, l'étude des équations aux dérivées partielles (EDP's) à croissance non standard a suscité un vif intérêt dans diverses directions de la recherche. Cet intérêt est justifié par leurs applications dans de nombreux domaines en : finance, restauration d'image, fluides non-newtoniens (caractérisées par leur changement brutale d'état physique de l'état liquide a l'état solide sous l'influence de différents stimuli externes, comme les champs électriques ou magnétiques). De tels fluides ont de nombreuses applications dans plusieurs branches de l'ingénierie, y compris la protection antisismique, l'industrie automobile (embrayages, amortisseurs, ...), applications militaires, ... etc.

Il est bien connu que le cadre classique des espaces de Lebesgue n'est pas suffisant pour capturer les modèles de tels fluides non-newtonien. En effet en 1996, K. Rajagopal et M. Ružička ont proposé un modèle empirique pour une classe de fluides non-Newtonien, appelés fluides électrorhéologiques, qui peuvent changer leurs viscosité sous l'effet du champ électrique et ce via une énergie naturelle à exposant variable en fonction de la force du champ électrique et varie donc sur son domaine. Ce qui naturellement impose d'étudier ces fluides dans des espaces de Lebesgue à exposants variables. Ce dernier cadre ne semble pas être réaliste pour mieux comprendre le comportement des fluides non-Newtonien en général. En effet, le changement brusque de la viscosité dans le tenseur de Cauchy confère à l'opérateur de diffusion une croissance non standard décrite par une fonction de Musielak. Ainsi le cadre fonctionnel adapté pour l'étude des modèles décrivant les fluides non-Newtonien est celui des espaces de Musielak.

Bien que l'analyse mathématique des espaces de Musielak et de Musielak-Sobolev a été développée depuis les années 1970, il reste néanmoins de nombreuses questions et problèmes à étudier a fin de pouvoir résoudre des EDP's qui modélise des modèles relative à la mécanique des fluides non-Newtonien.

Dans cette thèse, nous développons quelques aspects de l'analyse mathématique des espaces de Musielak et de Musielak-Sobolev. Spécialement, nous établissons quelques outils de base d'analyse fonctionnelle qui portent en premier lieu sur la densité des fonctions régulières dans les espaces de Musielak, ensuite sur la dualité, la séparabilité ainsi que la reflexivité de ces espaces. Ces résultats, nécessaires pour la théorie d'existence de certaines EDP's, sont ensuite appliqués à des équations non linéaires elliptiques variationnelles ayant des coefficients qui croissent rapidement/lentement et peuvent être décrits par des fonctions de Musielak.

ملخص

خلال العقدين الأخيرين، عرفت دراسة المعادلات التفاضلية الجزئية ذات معاملات النمو غير الاعتيادية قدراً كبيراً من الاهتمام في العديد من مجالات البحث. هذا الاهتمام له ما يبرره من خلال تطبيقاتها في: الاقتصاد، معالجة الصور، والسوائل غير النيوتونية (التي تتميز بتغير حالتها المادية بشكل مفاجئ من الحالة السائلة إلى الحالة الصلبة تحت تأثير المحفزات الخارجية المختلفة مثل الحقل الكهربائي أو المغناطيسي). هذه السوائل لها العديد من التطبيقات في العديد من فروع الهندسة، بما في ذلك الحماية من الزلازل، صناعة السيارات، التطبيقات العسكرية، الخ.

من المعروف أن الإطار الكلاسيكي لفضاء لوبك (Lebesgue) غير كاف لدراسة نماذج المعادلات التي تصف سلوك هذه السوائل غير النيوتونية. في الواقع، في عام 1996 ، اقترح رجكبال (Rajagopal) و رجشكا (Ruzicka) نموذجا تجريبيا لفئة من السوائل غير النيوتونية، تسمى السوائل الكهروريولوجية، والتي يمكن أن تغير لزوجتها تحت تأثير الحقل الكهربائي، وهذا عن طريق تعبير طبيعي للطاقة مع أس متغير بتغير قوة المجال الكهربائي. هذا يتطلب بطبيعة الحال دراسة هذه السوائل في فضاءات لوبك مع أس متغير. بشكل عام لا يبدو هذا الإطار الأخير واقعيًا لفهم سلوك السوائل غير النيوتونية بشكل أفضل. في الواقع، التغيير المفاجئ في لزوجة موتر كوشي (Cauchy tensor) يمنح لمشغل الانتشار (diffusion operator) نموًا غير اعتيادي يمكن وصفه بواسطة دالة موشيلك (Musielak) . وبالتالي فإن الإطار الوظيفي الذي تم تكييفه لدراسة النماذج التي تصف السوائل غير النيوتونية هو إطار فضاءات موشيلك. على الرغم من أن التحليل الرياضي لفضاءات موشيلك و موشيلك صوبليف (Musielak–Sobolev) قد تم تطويره منذ سبعينيات القرن الماضي، فلا يزال هناك العديد من الأسئلة والمشاكل التي يجب دراستها من أجل حل المعادلات التفاضلية الجزئية التي تصف النماذج المتعلقة بميكانيكا الموائع غير النيوتونية.

في هذه الاطروحة، ندرس بعض جوانب التحليل الرياضي لفضاءات موشيلك. على وجه التحديد، نأسس بعض الأدوات الأساسية للتحليل الوظيفي التي تركز أولاً على كثافة الدوال المنتظمة في فضاءات موشيلك، ثم على الازدواجية، والانفصال وكذلك انعكاسية هذه الفضاءات، ثم نقوم بتطبيق هذه النتائج التي تعد ضرورية لنظرية وجود بعض المعادلات التفاضلية الجزئية، على المعادلات الإهليلجية المتغيرة غير الخطية ذات معاملات نمو سريعة أو بطيئة ويمكن وصفها بواسطة دوال موشيلك .

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Youssef Ahmida,
USMBA,
July 2019

This thesis is dedicated to my parents
for their love, endless support
and encouragement.

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Notation

General notation

$\mathbb{R}^N$	Euclidean space of dimension, N,
$x := (x_1, x_2, \cdots, x_N)$	generic point in $\mathbb{R}^N$,
$\lvert \cdot \rvert$	Euclidean norm on $\mathbb{R}^N$,
Ω	open subset of $\mathbb{R}^N$, $N \geq 1$,
$\partial\Omega$	boundary of Ω, (1.1)
$\lvert\Omega\rvert$	Lebesgue measure of the set Ω,
$\overline{\Omega}$	closure of the set Ω,
χ_Ω	characteristic function of Ω,
$\operatorname{dist}(x, \Omega)$	distance of x to Ω, (1.3)
$\operatorname{diam}(\Omega)$	diameter of the set Ω, (1.4)
$B(x, r)$, $B_r(x)$	open ball with center x and radius r, (1.5)
$\overline{B(x, r)}$	closed ball, (1.6)
$\operatorname{supp} f$	support of the function f, (1.2)
a.e.	almost everywhere
X^*	topological dual space of the vector space X,
$\sigma(X, X^*)$	weak topology on X, (Section 1.2.4)
$\sigma(X^*, X)$	weak-$*$ topology on X^*, (Section 1.2.4)
$⨍_\Omega f(x)\,dx$	integral mean of f on Ω, (1.17)
p'	$:= \dfrac{p}{p-1}$ conjugate exponent of p,
$p(\cdot)$	variable exponent function,
$p'(\cdot)$	$:= \dfrac{p(\cdot)}{p(\cdot)-1}$ conjugate exponent of $p(\cdot)$,
$A : B$	inner product of two tensors $A, B \in \mathbb{R}^{N\times N}$, (6.4)

Sets of smooth functions

$\mathcal{C}(\Omega)$	set of continuous functions on Ω, (Section 1.2)
$\mathcal{C}_0(\Omega)$	functions in $\mathcal{C}(\Omega)$ compactly supported in Ω, (Section 1.2)
$\mathcal{C}^m(\Omega)$, $0 < m \leq \infty$	functions with continuous partial derivatives of all orders less than or equal to m on Ω, (1.11)
$\mathcal{C}_0^m(\Omega)$, $0 < m \leq \infty$	functions in $\mathcal{C}^m(\Omega)$ compactly supported in Ω, (1.12)
$\mathcal{C}_0^\infty(\overline{\Omega})$	the restriction to Ω of functions belonging to $\mathcal{C}_0^\infty(\mathbb{R}^N)$, (Section 1.2)

$\mathcal{C}^{0,\alpha}(\Omega)$, $0 < \alpha \leq 1$	α-Hölder continuous functions, (Section 1.13)
$\mathcal{C}^{0,\alpha}_{loc}(\Omega)$, $0 < \alpha \leq 1$	locally α-Hölder continuous functions, (Section 1.13)
$\mathcal{B}_c(\Omega)$	bounded functions compactly supported in Ω, (1.14)
$\mathcal{S}$	simple functions, (1.15)
$\mathcal{S}_c$	simple functions with compact support, (1.16)

Operators

$f * h$	convolution product, (1.19)
$\tau_h f(x)$	shift of the function f (1.18)
tr $(\cdot)$	trace operator
$\alpha := (\alpha_1, \alpha_2, \cdots, \alpha_N)$	an N-tuple of nonnegative integers α_i,
$\lvert\alpha\rvert$	length of α, (1.7)
$\partial_i := \partial/\partial x_i$	partial derivative operators, (Section 1.2)
D^α	multi-index differentiation operator, (1.8)
∇f	gradient of the function f, (1.9)
div	divergence with respect to the spatial variables (1.10)

Functional spaces

$L^p(\Omega)$, $1 \leq p \leq \infty$	Lebesgue spaces, (1.21)
$L^1_{loc}(\Omega)$	space of locally integrable functions on Ω, (Definition 1.3.1)
$W^m L^p(\Omega)$	Sobolev space, (1.24)
$W^m_0 L^p(\Omega)$	Sobolev space (Definition 1.3.3)
$W^{-m,p'}(\Omega)$	topological dual space of $W^{m,p}_0(\Omega)$, (Section 1.26)
$L^{p(\cdot)}(\Omega)$	variable Lebesgue space, (Definition 1.3.11)
$W^m L^{p(\cdot)}(\Omega)$	variable Sobolev space, (Definition 1.3.12)
$W^m_0 L^{p(\cdot)}(\Omega)$	variable Sobolev space, (Definition 1.3.13)
$W^{-m,p'(\cdot)}(\Omega)$	topological dual space of $W^{m,p(\cdot)}_0(\Omega)$, (1.3.3)
$L_\varphi(\Omega)$, $E_\varphi(\Omega)$	Orlicz spaces, (Definition 1.3.5)
$W^m L_\varphi(\Omega)$, $W^m E_\varphi(\Omega)$	Orlicz-Sobolev spaces, (Definition 1.3.8)
$W^m_0 E_\varphi(\Omega)$, $W^m_0 L_\varphi(\Omega)$	Orlicz-Sobolev spaces, (Definitions 1.3.9 and 1.3.10)
$L_M(\Omega)$, $E_M(\Omega)$	Musielak spaces, (Definition 1.4.3)
$W^m L_M(\Omega)$, $W^m E_M(\Omega)$	Musielak-Sobolev spaces, (Definition 1.4.8)
$W^m_0 L_M(\Omega)$, $W^m_0 E_M(\Omega)$	Musielak-Sobolev spaces, (Definition 3.4.1)
$W^{-m} L_{M^*}(\Omega)$	topological dual space of $W^m_0 E_M(\Omega)$, (Theorem 3.4.2)
$W^{-m} E_{M^*}(\Omega)$	distribution space, (Definition 3.4.2)
$K^m_0 L_M(\Omega)$	Musielak-Sobolev space, (Definition 1.4.3)
$L_M(\Omega, \mathbb{R}^N)$, $E_M(\Omega, \mathbb{R}^N)$	vector valued Musielak spaces, (Section 6.2)
$W^m L_M(\Omega, \mathbb{R}^N)$, $W^m E_M(\Omega, \mathbb{R}^N)$	vector valued Musielak-Sobolev spaces, (Section 6.2)
$W^m_0 L_M(\Omega, \mathbb{R}^N)$, $W^m_0 E_M(\Omega, \mathbb{R}^N)$	vector valued Musielak-Sobolev spaces, (Section 6.2)
$\Pi L_M := \Pi^d_{i=1} L_M(\Omega)$, $\Pi E_M := \Pi^d_{i=1} E_M(\Omega)$	the d-tuple Cartesian product of $L_M(\Omega)$ and $E_M(\Omega)$ respectively,

Norms, modular and convergence modes

$\|\cdot\|_p$, $\|\cdot\|_{L^p(\Omega)}$	norm on $L^p(\Omega)$, (1.22)
$\|\cdot\|_{m,p}$, $\|\cdot\|_{W^{m,p}(\Omega)}$	norm on $W^mL^p(\Omega)$, (1.25)
$\|\cdot\|_\varphi$, $\|\cdot\|_{L_\varphi(\Omega)}$	norm on $L_\varphi(\Omega)$, (1.28)
$\|\cdot\|_{m,\varphi}$, $\|\cdot\|_{W^mL_\varphi(\Omega)}$	norm on $W^mL_\varphi(\Omega)$, (1.30)
$\|\cdot\|_{p(\cdot)}$, $\|\cdot\|_{L^{p(\cdot)}(\Omega)}$	norm on $L^{p(\cdot)}(\Omega)$, (1.33)
$\|\cdot\|_{m,p(\cdot)}$, $\|\cdot\|_{W^{m,p(\cdot)}(\Omega)}$	norm on $W^mL^{p(\cdot)}(\Omega)$, (1.34)
$\|\cdot\|_M$, $\|\cdot\|_{L_M(\Omega)}$	Luxemburg norm on $L_M(\Omega)$, (1.38)
$\|\cdot\|^o_M$, $\|\cdot\|^o_{L_M(\Omega)}$	Orlicz norm on $L_M(\Omega)$, (1.39)
$\|\cdot\|_{m,M}$, $\|\cdot\|_{W^mL_M(\Omega)}$	norm on $W^mL_M(\Omega)$, (1.43)
$\rho_M(\cdot)$	modular generated by the Φ-function M, (1.37)
$\to$, $\rightharpoonup$, $\overset{*}{\rightharpoonup}$	strong, weak and weak-* convergences, (Definitions 1.4.6 , 1.2.4 and 1.2.5)
$\xrightarrow{M}$, $\xrightarrow{mod}$	modular convergences, (Definitions 1.4.5 and 1.4.9)

General introduction

In this thesis we conducted investigations on some mathematical analysis aspects of Musielak spaces in order to apply them in the theory of existence for some nonlinear elliptic equations associated with operators in divergence form. Such operators are defined upon stress tensors having growths that can be expressed through inhomogeneous Musielak functions $M : \Omega \times \mathbb{R}^+ \to \mathbb{R}^+$.

In the last two decades, partial differential equations with non-standard growth described in terms of inhomogeneous Musielak functions have attracted much interest and attention in various directions of research. The typical examples of EDPs with non-standard growth include models of electrorheological fluids (ERFs) [1, 88, 91], image restoration processing [20], non-Newtonian fluid dynamics [50], Poisson equation [28], elasticity equations [45, 65, 104], and thermistor model [105]. Problems in various types involving Musielak spaces are widely considered from the analytical point of view as well, inter alia the highly modern calculus of variations deals with minimization of the variational integrals [9, 24, 25, 53, 109].

Despite the number of application involving classical Lebesgue spaces in the resolution of PDEs describing a vast range of phenomena like diffusion, heat, elasticity, electrostatics, etc. It is well known that the natural energy

$$\int_\Omega |\nabla u|^p dx, \quad p = const. \tag{1}$$

is insufficient to capture the fluids rapidly transferring their viscosity under several external stimuli. In 1949 Winslow [98] published the first observations of the behavior of the particular non-Newtonian fluids called electrorheological fluids. These later fluids have the ability of increase their viscosity under an electric field and return to their initial state after the interruption of the electric field, namely under the influence of a voltage the particles of these fluids connect to each other to form a parallel structure, with a consistent direction with the direction of the applied field (Fig. 1). Therefore, the change on the viscosity makes these fluids useful in the industry, especially for industrial products that can vary their motion. Various applications can be found in the market of ERFs namely in the automotive industry (shock absorbers, clutches,...etc), robotics. In [88, 91] the authors proposed a model for the (ERF), via the natural energy

$$\int_\Omega |\nabla u|^{p(\cdot)} dx \tag{2}$$

where the exponent $p(\cdot)$ depends on the strength of the electric field and thus varies on its domain. The natural energy expression (2) is related to the generalized Lebesgue spaces where a deep study concerning these functional spaces can be found in [27, 28, 67].

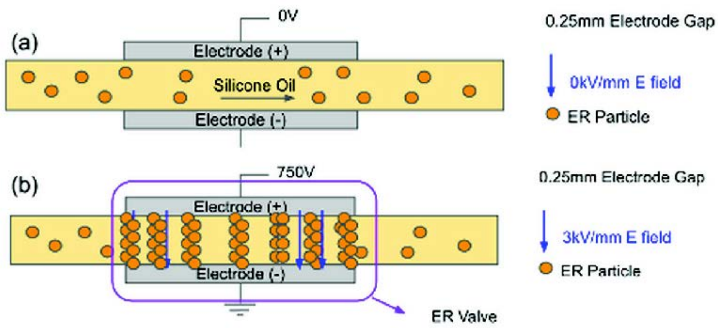

FIGURE 1: Behavior of electrorheological fluid between two electrode plates [86]

(a) The configuration has no potential across the electrodes. (b) The configuration has 750 V of voltage potential across a $0.25 - mm$ gap, which is an electric field (E-field) of $3kV/mm$.

Other examples of materials exhibit a significant behavior in their rheological properties are the shear thickening fluids (STFs) having the ability to become solid under a high rate shear stress (Fig. 2). The commercial applications of these fluids in the industry make their studies and development very attractive in the recent years. The impregnation of these "smart" fluids on multilayer textile materials, used in protection wear (sporting protective, body armor) to reduce their thickness and weight, is an example of application among other industrial applications, like the absorption of shock waves from strong winds or seismic. The STFs can be also integrated in medical equipment, in vehicles (dampers, clutches, shock absorbers), etc. For more applications on STFs in the industry see for instance [43] and references therein.

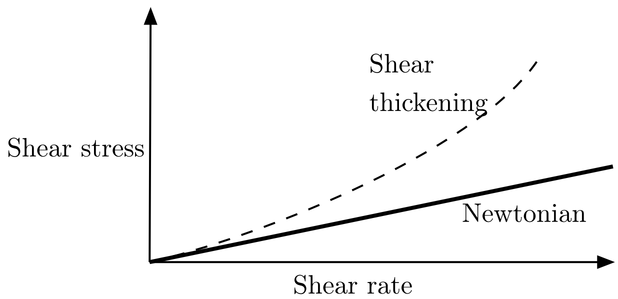

FIGURE 2: Shear thickening and Newtonian fluids behavior

The paramount importance of the non-Newtonian fluid in the new technology and industry, motivate our consideration and extension of the contributions and studies obtained in classical Lebesgue spaces, to evoke differential equations describing by means of more general functions, including for instance stress tensor of the forms

$$S(x,\xi) = \xi \exp(|\xi|^2), \qquad S(x,\xi) = w(x)\frac{\exp(|\xi|)+1}{|\xi|}\xi, \tag{3}$$

describing faster growth compared to polynomial growth and so allow to capture the fast change in the rheological properties of STFs.

Consequently the functional framework for studying models describing non-Newtonian fluids is that of the Musielak space L_M built upon non-homogeneous Φ functions $M : \Omega \times [0,\infty) \to [0,\infty]$ called Musielak function, written $M \in \Phi$, satisfying

- $M(\cdot, s)$ is a measurable function for every $s \geq 0$, $M(x,0) = 0$, $M(x,s) > 0$ for $s > 0$ and $M(x,s) \to \infty$ as $s \to \infty$, $M(x,\cdot)$ is a non-decreasing and convex function for all $x \in \Omega$,
- $\lim_{s\to 0^+} \dfrac{M(x,s)}{s} = 0$ and $\lim_{s\to+\infty} \dfrac{M(x,s)}{s} = +\infty$.

The Musielak spaces $L_M(\Omega)$ (see Definition 1.4.3) and Musielak-Sobolev spaces $W^m L_M(\Omega)$ (see Definition 1.4.8), built upon the Φ-function M, we consider in this thesis allows to capture a several classes of standard modular functional spaces, namely, the classical Lebesgue space governed by the Φ-function

$$M(x,s) = s^p, \ 1 < p < \infty. \tag{4}$$

The generalized (variable exponent) Lebesgue space governed by the Φ-function

$$M(x,s) = s^{p(x)}, \ 1 < p^- \leq p(\cdot) \leq p^+ < \infty. \tag{5}$$

The class of Orlicz spaces generated by Φ-functions not depending on the first variable as

$$M(x,s) = s^p, \qquad M(x,s) = s^p \log(e+s), \qquad M(x,s) = e^{s^2} - 1, \cdots. \tag{6}$$

The space governed by the double phase Φ-function

$$M(x,s) = s^p + a(x)s^q, \ a(x) \geq 0, 1 \leq p < q. \tag{7}$$

The framework of Musielak spaces has been the subject of many investigations, a considerable literature grew upon an account of functional analysis aspects on this functional structure as one can find in the contributions initiated by Musielak [82], the series of papers written by Hudzik [58–61], papers by Kamińska [62–64], and by Skaff [93, 94] basic background of the Musielak and the Musielak-Sobolev spaces was developed. For recent development on this framework space, we refer to [4, 5, 7, 21, 52, 55, 79, 101, 103], while in the context of the existence theory

for problems arising from fluid mechanics can be found, for instance, in [44, 49, 50, 99].
Since the 1970s, the mathematical theory of Musielak and Musielak-Sobolev spaces has been developed. Nevertheless, there are still major questions and problems to be investigated. The Musielak spaces do not inherit all the good properties of the classical Lebesgue and Orlicz spaces. Besides reflexivity and separability, which are not - in general - the properties we deal with, other mathematical problems can also appear. In fact, the acquisition results and techniques obtained for the particular frameworks spaces defined by functions M independent of the variable x called Orlicz spaces may not be transferred, in general, to this more general functional setting. The recent developments for variable exponent Lebesgue and Sobolev spaces governed by the function $M(\cdot, s) = |s|^{p(\cdot)}$, shows unfortunately that the continuity of the shift operator, the density of smooth functions and the poincaré inequalities are properties, among others, that can not hold in these spaces, unless the exponent $p(\cdot)$ is fairly regular. The highly challenging in studying the structure of Musielak spaces is to provide relevant regularities conditions including, among others, those obtained in Orlicz, variable exponent and double phase spaces.

Musielak spaces : difficulties and mathematical challenge.

As Musielak spaces bring together, classical Lebesgue spaces, variable exponent Lebesgue spaces and Orlicz spaces, we provide here several standard properties of these particular classical spaces, which unfortunately in general can not be extended to the Musielak setting unless more regularity assumptions on the Φ-function are assumed. As a starting point, we have :

- **Reflexivity and separability properties.** When we are limited to the classical Lebesgue, variable exponent Lebesgue spaces and double phase spaces governed respectively by the Φ-functions (4), (5) and (7), the noticeable point is that these spaces are separable and reflexive since these Φ-functions satisfy the Δ_2-condition, that is there is a constant $k > 0$ such that

$$M(x, 2s) \leq kM(x, s) + h(x), \tag{8}$$

for all $s \geq 0$ and almost every $x \in \Omega$, where h is a non-negative, integrable function in Ω. Unfortunately for an arbitrary Φ-function M, the Musielak space $L_M(\Omega)$ may be neither reflexive nor separable e.g $M(x, s) = \exp(s^2) - 1$ and $M(x, s) = w(x)(\exp(s) - 1 + s)$. Therefore, another type of problem arises, during the study of PDEs, related to the non-reflexivity and non-separability since we lose some classical tools like the Eberlein-Šmulian theorem (see Theorem 1.2.1). In addition, the weak topology is not, in general, well defined in $L_M(\Omega)$ unless the complementary (in the sense of Young) of the Φ-function M satisfy the Δ_2-condition, for instance $M(x, s) = (1 + s)\log(1 + s) - s$ and its complementary $M^*(x, s) = e^s - s - 1$ that does not satisfy the Δ_2-condition.

In the framework of Sobolev space the weak $\sigma(\Pi L^p, \Pi L^{p'})$, $1 < p < \infty$, $p' = \dfrac{p}{p-1}$, closure is equal to the norm closure in $W^{1,p}(\Omega)$ of the convex set of smooth functions compactly supported in Ω denoted by $W_0^{1,p}(\Omega)$. In contrast to Sobolev spaces, the weak topology is not, in general, well defined in the Orlicz setting. To this end and in order to cover a wide class of Orlicz functions in the definition of Sobolev-Orlicz spaces, the use of the weak-* topology $\sigma(\Pi L_\varphi, \Pi E_{\varphi^*})$ to define the Orlicz spaces $W_0^m L_\varphi(\Omega)$, seems to be more convenient and very interesting in the theory of existence of PDEs in non-reflexive functional spaces.

We shall stress that since we are interested in studying the Dirichlet boundary problem for non-homogeneous PDEs with rapidly growing stress tensors S (see (3)) that can be expressed by Φ-functions which do not necessarily satisfy the Δ_2-condition (8) such as $M(x,s) = \exp(s^2) - 1$ and $M(x,s) = w(x)(\exp(s) - 1 + s)$, we will consider as in the Orlicz-Sobolev framework (see [37, 42]) the space $W_0^m L_M(\Omega)$ defined by means of the weak-* topology in $L_M(\Omega)$.

• **Convolution method and continuity of the shift operator.** It is well-known that the convolution operation is an important tool in the theory of classical Lebesgue spaces. Particularly in building regular functions serving in approximation process. We point out that the operation of convolution is based on the continuity of the shift operator. Unfortunately, the shift operator is not acting in general on Musielak spaces. Indeed, when we put ourselves in the particular case (5), the authors [67] gave the following example : $N = 1$, $\Omega = (-1, 1)$. For $1 \leq r < s < +\infty$ they define the variable exponent

$$p(x) = \begin{cases} r & \text{if } x \in [0,1), \\ s & \text{if } x \in (-1,0) \end{cases}$$

and consider

$$f(x) = \begin{cases} x^{-\frac{1}{s}} & \text{if } x \in [0,1), \\ 0 & \text{if } x \in (-1,0). \end{cases}$$

They show that $\tau_h f \notin L^{p(\cdot)}(\Omega)$ although that $f \in L^{p(\cdot)}(\Omega)$. Therefore, we can not approximate in general identities for any given function in Musielak space.

• **Density and Lavrentiev phenomenon.** In general, smooth functions are not dense in Musielak-Sobolev spaces. The problem of density is related to the so-called Lavrentiev phenomenon firstly observed by [73]. We meet it when the infimum of the variational problem over the smooth functions is strictly greater than infimum taken over the set of all functions satisfying the same boundary conditions. The situation can be illustrated as follows, cf. [107, 109]. Let $H(u) = \displaystyle\int_\Omega h(x, \nabla u)dx$, with integrand $h(x,\xi)$ measurable in $x \in \Omega$ and convex in $\xi \in \mathbb{R}^N$ and satisfies

$$-l_0(x) + k_1|\xi|^{p_1} \leq h(x,\xi) \leq l_0(x) + k_1|\xi|^{p_2}$$

where $l_0 \in L^1(\Omega)$, $k_1 > 0$, $1 < p_1 \le p_2$. Then the variational problem $H_1 = \inf_{u \in W_0^{1,p_1}} H(u)$ has a solution (see [107, p. 465]). Nevertheless, the functional H is not necessarily continuous on $W_0^{1,p_1}(\Omega)$ and it can happen the following inequality

$$\inf_{u \in W_0^{1,p_1}} H(u) < \inf_{u \in C_0^\infty} H(u).$$

The notion of the Lavrentiev phenomenon became naturally generalized to describe the situation where functions from certain spaces cannot be approximated by regular ones (e.g. smooth).

It is known that in the case of the generalized Lebesgue spaces, the Lagrangian function $h(x,\xi) = |\xi|^{p(x)}$ can exhibit the Lavrentiev phenomenon if $p(\cdot)$ is not regular enough, for instance the following example of exponent was given by Zhikov (see [109, Example 3.2])

$$p(x) = \begin{cases} p_2 & \text{for } x_1 x_2 > 0, \\ p_1 & \text{for } x_1 x_2 < 0, \end{cases} \quad 1 \le p_1 < 2 < p_2$$

with $\Omega = \{x \in \mathbb{R}^2,\ |x| < 1\}$. The canonical, but not optimal, assumption ensuring density of smooth functions in norm topology in generalized Sobolev spaces $W^{m,p(\cdot)}(\Omega)$ with

$$1 < p^- = ess \inf_{x \in \Omega} p(x) \le p(x) \le p^+ = ess \sup_{x \in \Omega} p(x) < \infty, \tag{9}$$

is the log-Hölder continuity of the exponent $p(\cdot)$, that is

$$|p(x) - p(y)| \le -C_0 / \log(|x-y|), \quad \text{for every } x,\ y \in \Omega \text{ with } |x-y| \le \frac{1}{2}, \tag{10}$$

for some constant $C_0 > 0$ (see [27, 28, 92, 106, 109]).
The double-phase spaces governed by the Φ-function (7) can also inherit the Lavrentiev phenomenon [109], unless the function a in (7) satisfies the Lipschitz condition i.e

$$|a(x) - a(y)| \le C|x-y|, \ x, y \in \overline{\Omega} \subset \mathbb{R}^N$$

for some constant $C > 0$ and

$$q \le (1 + 1/N)p.$$

See also [23, 25, 32], where the authors provide a sharp result. Therefore, we can not expect the density of smooth functions for any given Musielak-Sobolev space and so more regularity on the Musielak functions are needed.

- **Poincaré-type inequalities.** Another tool that of importance in the study of PDEs and in the framework of Musielak-Sobolev spaces is the Poincaré inequalities. Unfortunately the Poincaré inequalities does not hold in general in the framework of Musielak-Sobolev spaces.

Poincaré norm inequality. In the generalized Sobolev space $W_0^{1,p(\cdot)}(\Omega)$, defined as the norm closure of $C_0^\infty(\Omega)$ functions in $W^{1,p(\cdot)}(\Omega)$, the Poincaré norm inequality was first proved in the paper [67], written about variable exponent Sobolev spaces, under some conditions including the continuity of the variable exponent $p(\cdot)$. Then by using an approach based on the boundedness of the maximal operator on $L^{p(\cdot)}(\Omega)$, the authors in [27, Theorem 6.21] proved the Poincaré norm inequality

$$\|u\|_{L^{p(\cdot)}(\Omega)} \leq c(N, p(\cdot), \Omega)\|\nabla u\|_{L^{p(\cdot)}(\Omega)},$$

for every $u \in W_0^{1,p(\cdot)}(\Omega)$ and for exponents $p(\cdot)$ satisfying (9) and the log-Hölder regularity (10).

In [28, Theorem 8.2.4] the authors defined $W_0^{1,p(\cdot)}(\Omega)$ as the closure of Sobolev functions with compact support in Ω with respect to the norm in $W^{1,p(\cdot)}(\Omega)$. They proved the Poincaré norm inequality for a regular bounded domain and for exponent $p(\cdot)$ satisfying the following two conditions

$$\Big|1/p(x) - 1/p(y)\Big| \leq C_1/\log(e + 1/|x-y|) \tag{11}$$

and for some $p_\infty \in \mathbb{R}$

$$\Big|1/p(x) - p_\infty\Big| \leq C_2/\log(e + |x|) \tag{12}$$

for every x, $y \in \Omega$, where $C_1 > 0$ and $C_2 > 0$ are constants.
Let us note in passing that the above two definitions of the Sobolev space $W_0^{1,p(\cdot)}(\Omega)$ coincide if $p(\cdot)$ is a measurable bounded exponent and if (11) and (12) are fulfilled (see [28, Corollary 11.2.4]). Ciarlet and Dinca [22] proved the Poincaré norm inequality using an approach which does not rely on the density arguments.

Poincaré integral inequality. In the usual Sobolev space denoted $W_0^{1,p}(\Omega)$, $1 \leq p < \infty$, defined as the norm closure of $C_0^\infty(\Omega)$ in $W^{1,p}(\Omega)$, the classical Poincaré integral inequality asserts that

$$\int_\Omega |u(x)|^p dx \leq C(\Omega, p) \int_\Omega |\nabla u(x)|^p dx, \tag{13}$$

for every $u \in W_0^{1,p}(\Omega)$ where $C(\Omega, p)$ is a constant depending on Ω and p.
In general, in the variable exponent Sobolev spaces $W_0^{1,p(\cdot)}(\Omega)$ (defined as the norm closure of $C_0^\infty(\Omega)$ functions in $W^{1,p(\cdot)}(\Omega)$), the Poincaré integral inequality (13) with variable exponent $p(\cdot)$ instead of constant exponent p fails to hold as it was shown in [34], where the authors proved that

$$\lambda = \inf_{0 \neq u \in W_0^{1,p(\cdot)}(\Omega)} \frac{\int_\Omega |\nabla u|^{p(x)} dx}{\int_\Omega |u|^{p(x)} dx}$$

is equal to zero when $\Omega = (-2, 2) \subset \mathbb{R}$ and

$$p(x) = \begin{cases} 3 & \text{if } 0 \leq |x| \leq 1, \\ 4 - |x| & \text{if } 1 \leq |x| \leq 2. \end{cases}$$

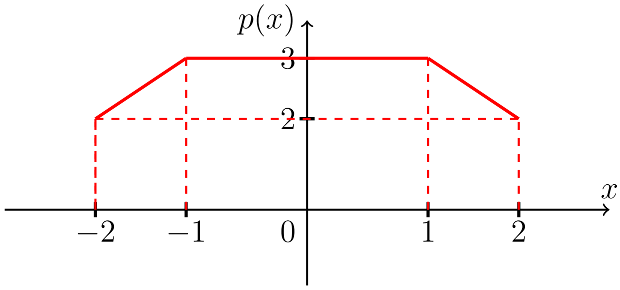

FIGURE 3: Fan-Zhao-Zhang counter-example

Indeed, if the variable exponent $p(\cdot)$ is a continuous function having a minimum or a maximum then an integral version of the Poincaré inequality can not be obtained (see [100]). However, under a suitable monotony property on the variable exponent $p(\cdot)$ Maeda [77] proved the Poincaré integral inequality for $\mathcal{C}_0^1(\Omega)$-functions.
Therefore, in the framework of Musielak spaces, the Poincaré-type inequalities can not in general hold unless more regularity on the Musielak functions is assumed.

Overview of some further approximation results in Musielak-Spaces

An earlier density result of smooth functions in the Musielak-Sobolev spaces $W^m L_M(\mathbb{R}^N)$ with respect to the strong (norm) topology was proved first by Hudzik [60, Theorem 1] assuming the restrictive Δ_2-condition (8) on the Φ-function M and the extra hypothesis

$$\int M(x, |u_\varepsilon|)dx \leq c \int M(x, |u|)dx,$$

where u_ε is the mollification (see (1.20)). Recently in a bounded domain Ω, the same result (without the extra hypothesis) was proved in [52, Theorem 6.5] using the boundedness of the maximal operator.
Density result of smooth functions in Musielak spaces with respect to the modular topology was claimed for the first time in [10] in $\Omega = \mathrm{R}^N$ and then for a bounded star-shaped Lipschitz domain Ω in [11] and under slightly changed assumptions in [47, 48]. The authors assumed that the Φ-function M satisfies, among others, the log-Hölder continuity condition, that is to say there exists a constant $A > 0$ such that for all $s \geq 1$,

$$\frac{M(x,s)}{M(y,s)} \leq s^{-\frac{A}{\log|x-y|}}, \ \forall x, y \in \Omega \text{ with } |x-y| \leq \frac{1}{2}. \tag{14}$$

Nonetheless, the proof involved an essential gap. The Jensen inequality was used for the infimum of convex functions, which obviously is not necessarily convex. This problem is solved recently in [8]. Note that imposing assumption (14) makes sense only for big values of s, since due to x/y symmetry forces the fraction on the left-hand side above has to be estimated from above by the quantity bigger or equal to 1 see also [47].

In the isotropic case (when $M = M(x, |\xi|)$) in [47], for smooth approximation in

$$\{u \in W_0^{1,1}(\Omega) : \ Du \in L_M(\Omega, \mathbb{R}^N)\},$$

it suffices to impose on an Φ-function M continuity condition of log-Hölder-type with respect to x, namely for each $s \geq 0$ and x, y such that $|x - y| < 1/2$ we have

$$\frac{M(x,s)}{M(y,s)} \leq \max\left\{ s^{-\frac{a_1}{\log|x-y|}}, b_1^{-\frac{a_1}{\log|x-y|}} \right\} \qquad \text{with some } a_1 > 0,\ b_1 \geq 1. \tag{15}$$

In the variable exponent case the above condition relates to standard log-Hölder continuity of the exponent. Note that the results in [10, 11, 47, 48, 95] do not cover the functions $e^{sp(x)} - 1$ and $s^{p(x)}/p(x)$ and the two phase function $s^p + a(x)s^q$ unless p and a are constant functions, which are not excluded in our setting.

Overview on the existence theory of elliptic PDEs

In Dauklay notes 1961, Višik [97] gave a compactness method solving some nonlinear elliptic problems. Separately, Minty [80, 81] developed another point of view by observing that suitable assumptions of monotony can avoid the results of compactness. Next, Browder observed that Minty's ideas could be applied to the elliptic equations considered by Višik and developed this idea in a series of papers (see for instance [16, 18]). A statement containing all Višik's results was obtained by Leray-Lions [75] (see also [76]). The idea of Minty-Browder's monotonicity for defined operators from a reflexive Banach space into its dual has proved to be the cornerstone for the development of the theory of mappings of monotone type [14, 19, 57].

Research efforts have focused primarily on the case where the modular function has polynomial growth (i.e. $M(x, s) = |s|^p$ for $1 < p < +\infty$), which naturally requires studying these problems in the Lebesgue space L^p. In particular, the problem

$$\begin{cases} -\text{div}\ a(x, \nabla u) &= f \quad \text{in } \Omega, \\ \qquad\qquad u &= 0 \quad \text{on } \partial\Omega, \end{cases} \tag{16}$$

where $a : \Omega \times \mathbb{R}^N \to \mathbb{R}^N$ is such that

$$\begin{aligned} |a(x,\xi)| &\leq g(x) + c_2|\xi|^{p-1}, \quad g \in L^{\frac{p}{p-1}}(\Omega), \\ a(x,\xi)\cdot\xi &\geq d_1|\xi|^p - h(x), \quad h \in L^1(\Omega), \end{aligned}$$

$1 < p < +\infty$, was studied in [76]. Furthermore, for any given $f \in W^{-1,p'}(\Omega)$, $p' := \dfrac{p}{p-1}$, there exists a unique variational solution u in $W_0^{1,p}(\Omega)$ satisfying

$$\int_\Omega a(x, \nabla u)\nabla v = \langle f, v \rangle, \quad \forall v \in W_0^{1,p}(\Omega).$$

Several materials with inhomogeneities can not be modelled by means of classical Lebesgue spaces. The authors [88, 91] proposed a model for the electrorheological fluids, via the natural energy $\int_\Omega |\nabla u|^{p(\cdot)} dx$ where $p(\cdot)$ is a variable exponent depending on the strength of the electric field and thus varies on its domain. Therefore, we are naturally lead to study these fluids in variable exponent Lebesgue spaces $L^{p(x)}(\Omega)$, $1 < p^- \leq p(x) \leq p^+ < +\infty$ for a.e. in Ω. The noticeable point here is that variable exponent Lebesgue spaces are still reflexive and separable and hence elliptic equations of type (16) can be solved by the same theory (see for instance [33, 67, 90]). But unfortunately, the variable exponent Lebesgue spaces do not inherit all the good properties of the classical Lebesgue spaces used to solve PDEs, like the density of smooth functions and the Poincaré inequalities in $W_0^{1,p(\cdot)}(\Omega)$, unless the exponent $p(\cdot)$ is fairly regular (see [26, 28, 77, 92, 109]).

Later developments for studying nonlinear elliptic boundary value problems was to substitute reflexive Lebesgue spaces L^p by classical Orlicz spaces L_M described by Φ-functions $M(x,s) = M(s)$ -not depending on x- which do not necessarily satisfy the so-called Δ_2-condition (a necessary condition for an Orlicz space to be reflexive [69]). Thus, the studied problems can be handled by non reflexive functional spaces. The theory of monotone operators was extended by Gossez [38] to Banach spaces that are not reflexive in general, which is at the origin of more difficulties. The corresponding mappings of monotone type are not everywhere defined nor bounded and are not generally coercive. The difficulty caused by the lack of reflexivity was avoided by the idea of the complementary systems in Orlicz-Sobolev spaces. Contributions in this direction were initiated by Donaldson [29] and continued by Gossez [36, 37, 39, 41]. Further contributions can be found in [42, 72] and also in [96] where a degree theory is constructed whereas in [83] the authors include further extensions on monotone-like mappings in non reflexive Orlicz-Sobolev spaces.

However, the study of variational nonlinear elliptic equations in Musielak spaces is a hard task. To the best of our knowledge, there are only very few articles in the literature written on the existence and uniqueness of the variational solutions of the problem (16). In addition to the reflexivity that does not take place is added the difficulty of obtaining approximation properties for smooth functions for an appropriate topology in Musielak spaces. Furthermore, Poincaré-type inequalities can not hold in general in Musielak spaces [28, 67, 77]. The approximation in modular topology is a necessary tool to perform complementary systems. This was noticed by Gossez [40] in the setting of classical Orlicz spaces. In Musielak spaces, the first contribution is initiated in [11] but the proof involved a gap at the level of the use of the Jensen inequality. This problem is now fixed in [7]. The study of variational nonlinear elliptic equations in non-reflexive Musielak-Sobolev spaces originated in the work of Benkirane-Valley [12, 13] where the approximation with respect to the modular topology [11] is used. In the anisotropic case, the problem (16) with a datum f is in divergence form is investigated in [46]. The authors prove the existence of a weak solution $u \in W_0^{1,1}(\Omega)$ and $\nabla u \in L_M(\Omega)$ without any further information on the precise space where the solution u itself is expected to be.

Structure of the thesis and contributions

The thesis begins with the preliminary **Chapter 1**. This Chapter contains three sections, in the first one we list some notation and Lebesgue integral results we will use throughout. In the second section we provide an overview on some example of topological vector spaces called modular space consisting of measurable functions $u : \Omega \to \mathbb{R}$ such that

$$\int_\Omega A(x, |u(x)|/\lambda)dx < \infty$$

for some $\lambda > 0$, with $A : \Omega \times [0, \infty) \to [0, \infty)$. Namely, we recall some topological properties of classical Lebesgue, generalized Lebesgue and Orlicz spaces. The third section contains an overview of the more general modular spaces called Musielak and Musielak-Sobolev spaces, in particular, we recall some properties of these functional frameworks, where most of them can be founded in the monograph of Musielak [82] see also [51].

The main results of this thesis are presented in two parts. In the first one, we establish basic approximation results in Musielak and Musielak-Sobolev spaces with respect to the modular and norm convergences, which then allow us to obtain some topological properties that constitute basic tools we use in the theory of existence of PDEs, whareas in the second part we apply the results we have obtained to solve the Dirichlet problem associated with a variational nonlinear elliptic equations. Precisely, our contributions are developed as follows.

Part I

This first part based on [4, 5, 7, 103] by the author et al. contains four chapters.

In **Chapter 2** we introduce and discuss our first regularity assumption, on the Φ-function M that generates the corresponding Musielak spaces $L_M(\Omega)$ and $E_M(\Omega)$, called local integrability condition, that is for any constant number $c > 0$ and for any compact set $K \subset \Omega$ we have

$$\int_K M(x, c)dx < \infty. \tag{17}$$

Obviously, the regularity condition (17) is skipped in the Orlicz setting and is fulfilled if (9) holds when the Φ-function M is equal to $s^{p(\cdot)}$ (see Section 2.2 for more details).
The main purpose of this chapter is to prove the density of several regular subsets in Musielak spaces, that we will use later to provide topological properties of this functional framework. The approach we use to this is based on showing that the continuity of the shift operator remains valid in Musielak spaces if we restraint to bounded functions compactly supported in Ω. The condition (17) we introduce ensure that bounded functions compactly supported in Ω and so smooth functions compactly supported in Ω are contained in Musielak spaces. Thus in terms of the mean continuity of the shift operator we conclude Chapter 2 by establishing some basic approximation results in Musielak spaces with respect to the modular and norm convergences.
In our contribution, the density of smooth functions in Musielak we get is obtained only under the local integrability condition and it unifies the approximation results known in Lebesgue spaces, with a constant or variable exponent (see [2, 67]), and Orlicz spaces (see for instance [2]).

Our aim in **Chapter** 3 is to extend some topological properties of variable exponent Lebesgue and Orlicz spaces to the Musielak spaces. Based on the approximation results obtained in Chapter 2, we prove in Chapter 3 some basic function space properties that constitute the essential tools needed in the theory of existence for partial differential equations involving nonstandard growths described in terms of Musielak functions (see Part II). Such results require to take into account the results earlier studied deeply in the monograph [70] and these presented in the pioneering work by Kovácik and Rákosník [67] concerning the completeness, density, reflexivity and separability of variable exponent Lebesgue and Sobolev spaces.
In particular, we prove the separability of the Musielak space $E_M(\Omega)$ under the condition (17) by a totally different method from that given by Musielak [82, Theorem 7.10], since the author used the density of simple functions assuming that the Φ-function M satisfies the local integrability condition on a measure space (Ω, Σ, μ) where μ is a positive complete measure, that is to say,

$$\int_D M(x,s)d\mu < \infty \text{ for every } s > 0 \text{ and } D \in \Sigma \text{ with } \mu(D) < \infty.$$

Observe that the later condition is stronger than (17) when we limit ourselves to the Borel tribe. The weaker condition (17) and approximation results will be then used to prove a duality and reflexivity results in Musielak spaces at the end of the Chapter 3.

In **Chapter** 4 we introduce relevant structural conditions implying approximation properties in Musielak-Sobolev spaces. Our mains results unify and improve the known results in the Orlicz-Sobolev spaces, as well as the variable exponent

Sobolev spaces. In the seminal paper [40] Gossez proves that weak derivatives in the Orlicz-Sobolev spaces are strong derivatives with respect to the modular topology. In this chapter, we extend the results obtained in [40] for the Orlicz-Sobolev framework, into the Musielak-Sobolev setting, in fact since in general smooth functions are not dense in Musielak-Sobolev spaces we introduce the φ-regularity condition on the Φ-function M, that is to say, there exists a function $\varphi : [0, 1/2] \times \mathbb{R}^+ \to \mathbb{R}^+$ such that $\varphi(\cdot, s)$ and $\varphi(x, \cdot)$ are non-decreasing functions and for all $x, y \in \overline{\Omega}$ with $|x - y| \leq \dfrac{1}{2}$ and for any constant $c > 0$

$$M(x, s) \leq \varphi(|x - y|, s)M(y, s) \qquad \text{with} \qquad \limsup_{\varepsilon \to 0^+} \varphi(\varepsilon, c\varepsilon^{-N}) < \infty. \tag{18}$$

We prove that (18) is a sufficient condition to avoid the Lavrentiev phenomenon and so ensures the density of smooth functions in Musielak-Sobolev spaces. We relax the condition typical in the context of density in the variable exponent spaces, namely the log-Hölder continuity of the exponent (10).

Although it is common to make a big effort to relax growth conditions as much as possible, our method turns out to lead the sharp result when the modular function has at least power-type growth. Since the approximation follows from convolution arguments, we get better regularity in L^p, $p > 1$ than in L^1. In the fully general case we cannot improve (18), because we know only $L_M \subset L^1_{loc}$. Nonetheless, when the modular function has at least power-type growth, we relax (18) and include whole the good double-phase range where the Lavrentiev phenomenon is absent (according to [32, Theorem 3] or [25, Theorem 4.1]). Namely, if

$$M(x, s) \geq c|s|^p \qquad \text{with} \quad p > 1 \text{ and } \quad c > 0,$$

we obtain a smooth approximation of $u \in W_0^{m,p}(\Omega) \cap W^m L_M(\Omega)$ (also of $u \in W_0^{1,p}(\Omega)$, such that $\nabla u \in L_M(\Omega)$) provided that we assume, that there exists a function $\varphi : [0, 1/2] \times \mathbb{R}^+ \to \mathbb{R}^+$ such that $\varphi(\cdot, s)$ and $\varphi(x, \cdot)$ are nondecreasing functions and for all $x, y \in \overline{\Omega}$ with $|x - y| \leq \dfrac{1}{2}$ and for any constant $c > 0$

$$M(x, s) \leq \varphi(|x - y|, s)M(y, s) \quad \text{with} \quad \limsup_{\varepsilon \to 0^+} \varphi(\varepsilon, c\varepsilon^{-\frac{N}{p}}) < \infty.$$

We confirm the precision of the method by showing the lack of the Lavrentiev phenomenon in the double-phase case. Indeed, we get the modular approximation of $W_0^{1,p}(\Omega)$ functions by smooth functions in the double-phase space governed by the modular function $H(x, s) = s^p + a(x)s^q$ with $a \in C^{0,\alpha}(\Omega)$ excluding the Lavrentiev phenomenon within the sharp range $q/p \leq 1 + \alpha/N$. See [25, Theorem 4.1] for the sharpness of the result.

Chapter 5 is devoted to an interesting missing feature which is the Poincaré-type inequalities (norm and integral forms) in Musielak-Sobolev spaces. However, proving the Poincaré integral inequality for functions in $C_0^\infty(\Omega)$ and then extending it by a density argument (as is often done for a constant exponent) is not an easy task since the passage to the limits is not allowed because of the lack in general of density of smooth functions in $W_0^m L_M(\Omega)$.

Our contribution to overcame this problem consists in using the regularity condition (18) on the Φ-function M and the segment property on the domain Ω (see Definition 1.2.7). Those conditions enable us to get the modular density of C_0^∞-functions in $W_0^m L_M(\Omega)$.
Precisely in this chapter we give sufficient conditions on the Φ-function M for the following Poincaré-type inequality

$$\int_\Omega \sum_{|\alpha|<m} M(x, |D^\alpha u(x)|)dx \leq \int_\Omega \sum_{|\alpha|=m} M(x, c|D^\alpha u(x)|)dx \tag{19}$$

for every $u \in W_0^m L_M(\Omega)$ where $c > 0$ is a constant. We also get the same inequality in the subspace $W_0^m E_M(\Omega)$ under minimal assumptions. Such integral inequality yields obviously the Poincaré norm inequality.
We note that among the conditions we assume to obtain (19) we have the following monotony type condition on the first argument of the Φ-function M, that is to say on a segment $[a, b]$ of the real line $\mathbb{R}$, M satisfies

$$\begin{cases} (\mathcal{M}_{on})_0 : \begin{cases} \text{There exist } t_0 \in \mathbb{R}^+ \text{ and } 1 \leq i \leq N \text{ such that the partial} \\ \text{function } x_i \in [a,b] \mapsto M(x,t) \text{ changes constantly its monotony} \\ \text{on both sides of } t_0 \text{ (that is for } t \geq t_0 \text{ and } t < t_0), \end{cases} \\ \text{Or} \\ (\mathcal{M}_{on})_\infty : \begin{cases} \text{There exists } 1 \leq i \leq N \text{ such that for all } t \geq 0, \text{ the partial} \\ \text{function } x_i \in [a,b] \mapsto M(x,t) \text{ is monotone on } [a,b]. \end{cases} \end{cases} \tag{20}$$

where x_i stands for the i^{th} component of $x \in \Omega$. Obviously, the regularity condition (20) is skipped in the Orlicz setting and in the case where $M(x, s) = s^{p(x)}$, the assumption $(\mathcal{M}_{on})_0$ prevents the variable exponent $p(\cdot)$ to get a local extremum while $(\mathcal{M}_{on})_\infty$ is not satisfied unless $p(\cdot)$ is a constant function, therefore, the result we obtain covers the Poincaré integral inequality obtained by Maeda [77] for $\mathcal{C}_0^1$-functions in the case where the variable exponent $p(\cdot)$ is assumed to satisfy a monotony condition (see Section 5.2 for more details).

Part II

This part is based on [6] and contains Chapter 6 as an application of the theoretical results obtained in Part I. In this later Chapter we present the study of the Dirichlet problem associated with a class of nonlinear elliptic equations. Precisely, we are interested in the existence and uniqueness of variational solutions $u : \Omega \to \mathbb{R}^N$ for the following nonlinear elliptic problem

$$\begin{cases} -\mathrm{div}\ a(x, \nabla u) &= f \quad \text{in } \Omega, \\ u &= 0 \quad \text{on } \partial\Omega, \end{cases} \tag{21}$$

considered in a bounded open subset Ω of $\mathbb{R}^N$, $N \geq 1$, where the leading part a is a stress tensor whose growth is expressed through inhomogeneous Musielak function $M : \Omega \times \mathbb{R}^+ \to \mathbb{R}^+$.

We highlight we do not impose any extra growth restriction (neither Δ_2 nor ∇_2), on the Musielak function M as well as on its complementary Musielak function in the sense of Young. Thus, the stress tensor a has the ability to increase rapidly or slowly. We note that partial differential equations with more nonlinearities can be handled by more general functional spaces. In fact, the growth of the vector field a imposes the study of the problem (21) in the natural framework of Musielak spaces. It is worth pointing out that in the absence of restrictive conditions (c.f. Δ_2-condition) on the modular function M, the underlying functional space is not necessarily reflexive and separable. Problem (21) includes many nonlinear equations that arise in various problems in mathematical physics and raises many mathematical difficulties. The situations that we can, for instance, consider for the field a and the related Musielak function M include

$$\begin{array}{lll} a(x,\xi) = p(x)|\xi|^{p(x)-2}\xi & \text{and} & M(x,s) = s^{p(x)}, \\ a(x,\xi) = \xi \exp(|\xi|^2) & \text{and} & M(x,s) = \exp(s^2) - 1, \\ a(x,\xi) = w(x)\dfrac{\exp(|\xi|)+1}{|\xi|}\xi & \text{and} & M(x,s) = w(x)(\exp(s) - 1 + s). \end{array}$$

Within the approach we use, the Galerkin method is coupled with an abstract existence result by F.E. Browder 1966.

Publications of the author

Published

- Y. Ahmida, I. Chlebicka, P. Gwiazda, and A. Youssfi. Gossez's approximation theorems in Musielak-Orlicz-Sobolev spaces. J. Funct. Anal., 275(9): 2538-2571, 2018. https://doi.org/10.1016/j.jfa.2018.05.015

- Y. Ahmida. and A. Youssfi. Poincaré-type inequalities in Musielak spaces. Ann. Acad. Sci. Fenn. Math., (44): 1041-1054, 2019. https://doi.org/10.5186/aasfm.2019.4468

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Chapter 1

Preliminaries and basic facts

1.1 Introduction

In this first chapter, we introduce in Section 1.2 some notation, mathematical facts concerning functional analysis, integration theory, weak and weak-* topologies, that will be needed throughout this thesis, in order to develop the properties of Musielak and Musielak-Sobolev spaces. Section 1.3 is devoted to an overview of particular classes of modular function spaces, namely Lebesgue, generalized Lebesgue and Orlicz spaces. In fact, the section starts by recalling some definitions and properties of classical Lebesgue and Sobolev spaces with a constant exponent p (i.e $L^p(\Omega)$ and $W^mL^p(\Omega)$), contributions on this topic include for instance [2, 15, 70]. These later functional frameworks arise in the modelling of most materials with sufficient accuracy. For certain materials with inhomogeneities, for instance electrorheological fluids, this is not adequate, but rather the exponent should be able to vary. This leads to study those materials in Lebesgue and Sobolev spaces with variable exponent. A deep study of the variable exponent Lebesgue and Sobolev spaces (i.e. $L^{p(\cdot)}(\Omega)$ and $W^mL^{p(\cdot)}(\Omega)$) was conducted see [27, 28, 67] and the references therein. Historically, variable exponent Lebesgue spaces was appeared in the literature for the first time in 1931 in a paper written by W. Orlicz [87], after this paper Orlicz abandoned the study of these spaces for the account of the theory of the function spaces $L_\varphi(\Omega)$, built upon an $\mathcal{N}$-function φ (see Definition 1.3.4 below), that now bear his name for more in this topic see for instance [2, 40, 68–70, 89]. Section 1.4 is devoted to elementary properties of the more general class of modular function space, called Musielak space. Musielak spaces have been the subject of investigation in various directions of research. For instance [58–64, 82, 93, 94] and for more recent research and developments we refer for to [4, 5, 7, 21, 52, 78, 79, 101, 103], while in the context of the existence theory for problems arising from fluid mechanics can be found, for instance, in [44, 49, 50, 99].

1.2 Notation and functional analysis

1.2.1 Notation

In this subsection we summarize some notation, frequently used. By Ω we denote an open subset in the N-dimensional Euclidean real space $\mathbb{R}^N$, $N \geq 1$, and by $\partial\Omega$ the boundary of the set Ω i.e

$$\partial\Omega := \overline{\Omega} \cap \overline{(\mathbb{R}^N - \Omega)}. \tag{1.1}$$

For a measurable function $u : \Omega \to \mathbb{R}$, we denote by supp u the support of the function u i.e

$$\text{supp } u := \overline{\{x \in \Omega : u(x) \neq 0\}} \tag{1.2}$$

where here and below $\overline{D}$ denote the closure of any subset D of $\mathbb{R}^N$.
For $x \in \mathbb{R}^N$, by

$$\text{dist}(x, \Omega) := \inf\{|x - y|,\ y \in \Omega\} \tag{1.3}$$

we mean the distance of x to Ω. Diameter of the set Ω will be denoted by

$$\text{diam}(\Omega) := \sup\{|x - y|,\ x, y \in \Omega\}. \tag{1.4}$$

By

$$B(x, r) := \{y :\ |x - y| < r\} \tag{1.5}$$

we denote the open ball with center x and radius r and the closed ball by

$$\overline{B(x, r)} := \{y :\ |x - y| \leq r\}. \tag{1.6}$$

Given any N-tuple, $\alpha = (\alpha_1, \alpha_2, \cdots, \alpha_N)$ of nonnegative integers α_i, with a degree

$$|\alpha| = \sum_{i=1}^{N} \alpha_i. \tag{1.7}$$

We design, by $\partial_i := \partial/\partial x_i$ the partial derivative operators and by

$$D^\alpha = \partial_1^{\alpha_1} \cdots \partial_N^{\alpha_N} = \frac{\partial^{|\alpha|}}{\partial_1^{\alpha_1} \cdots \partial_N^{\alpha_N}} \tag{1.8}$$

with $D^{(0,\cdots,0)}u = u$ and u a reel function, the gradient of u will be denoted by

$$\nabla u := (\partial_1 u, \cdots, \partial_N u). \tag{1.9}$$

and

$$\text{div } u := \sum_{i=1}^{N} \partial_i u. \tag{1.10}$$

Further we denote by $\mathcal{C}(\Omega)$ the set of continuous functions on Ω and by $\mathcal{C}_0(\Omega)$ the set of $\mathcal{C}(\Omega)$-function compactly supported on Ω. For $0 < m \leq \infty$, let

$$\mathcal{C}^m(\Omega) := \Big\{u : u : \Omega \to \mathbb{R},\ D^\alpha u \in \mathcal{C}(\Omega), |\alpha| \leq m\Big\}, \tag{1.11}$$

$$\mathcal{C}_0^m(\Omega) := \mathcal{C}^m(\Omega) \cap \Big\{u :\ \text{supp } u \text{ compact },\quad \text{supp } u \subset \Omega\Big\}. \tag{1.12}$$

The set $C_0^\infty(\overline{\Omega})$ denote the restriction to Ω of functions belonging to $\mathcal{C}_0^\infty(\mathbb{R}^N)$. For $0 < \alpha \leq 1$ we say that u belongs to $\mathcal{C}^{0,\alpha}(\Omega)$ if there exists a constant $C > 0$ such that

$$|u(x) - u(y)| \leq C|x - y|^\alpha,\ x, y \in \Omega \tag{1.13}$$

and by $\mathcal{C}^{0,\alpha}_{loc}(\Omega)$ if the last inequality holds for x, y in any compact subset of Ω.
We will use $\mathcal{C}(\Omega, \mathbb{R}^N)$, to denote the class of vector value functions $u : \Omega \to \mathbb{R}^N$, such that each of its coordinate belongs to $\mathcal{C}(\Omega)$. Similar notation will be used for the above other sets when the function is of vector-valued.
By $\mathcal{B}_c(\Omega)$ we denote

$$\mathcal{B}_c(\Omega) := \Big\{u : u : \Omega \to \mathbb{R},\ \text{measurable bounded compactly supported in } \Omega\Big\} \tag{1.14}$$

Let $u : \Omega \to \mathbb{R}$ and let B_i, $i = 1, 2, \cdots, p$, be a Lebesgue measurable subsets of Ω. We denote by $\mathcal{S}(\Omega) := \mathcal{S}$ the family of finite linear combinations of characteristic functions of the sets B_i, expressed as follows

$$\mathcal{S} := \Big\{u(x) = \sum_{i=1}^{p} \alpha_i \chi_{B_i}(x),\quad \text{with } \alpha_1, \alpha_2, \cdots, \alpha_p \in \mathbb{R} \text{ and } |B_i| < +\infty\Big\} \tag{1.15}$$

and by $\mathcal{S}_c$ the set

$$\mathcal{S}_c := \mathcal{S} \cap \Big\{\ \cup_{i=1}^p B_i \subset K,\ \text{for some compact } K \text{ of } \Omega\Big\}. \tag{1.16}$$

For a measurable function f, we denote

$$⨍_\Omega f(x)\, dx = \frac{1}{|\Omega|} \int_\Omega f(x)\, dx \tag{1.17}$$

where here and below $|\Omega|$ stands for the Lebesgue measure of the set Ω.
For $h \in \mathbb{R}^N$, let $\tau_h u$ stands for the translation operator defined by

$$\tau_h u(x) = \begin{cases} u(x + h) & \text{if } x \in \Omega \text{ and } x + h \in \Omega, \\ 0 & \text{otherwise in } \mathbb{R}^N. \end{cases} \tag{1.18}$$

We denoted by $c, C, , C_1, C_2, \cdots$ various positive constants that may change from line to line and if they depend on certain variables $n, \Omega, \alpha, \cdots$ we write $C := C(n, \Omega, \alpha, \cdots), \cdots$.

1.2.2 Elementary integration results

In this subsection we recall some theoretical integration results. Missing proofs can be found for instance in [2, 15, 70]. In the sequel $L^p(\Omega)$ and $L^p_{loc}(\Omega)$, $1 \leq p < \infty$, denote the classical Lebesgue spaces, (see Definition 1.3.1 below).

Lemma 1.2.1. *(Dominated convergence theorem)[15, Theorem 4.2]. Let $\{u_n\}_{n\geq 1}$ be a sequence of measurable functions on a measurable set $\Omega \subset \mathbb{R}^N$ with*

- *$u_n \to u$ almost everywhere on Ω,*
- *there exists a function $v \in L^1(\Omega)$ such that $|u_n| \leq v$ a.e for all $n \in \mathbb{N}$ and all most every $x \in \Omega$.*

Then $u \in L^1(\Omega)$ and $\lim_{n\to\infty} \int_\Omega u_n dx = \int_\Omega u dx$.

Lemma 1.2.2. *(Fatou lemma)[15, Lemma 4.1]. Let $\{u_n\}_{n\geq 1}$ be a sequence of measurable non-negative functions almost everywhere on $\Omega \subset \mathbb{R}^N$. Then*

$$\int_\Omega \liminf_{n\to\infty} u_n dx \leq \liminf_{n\to\infty} \int_\Omega u_n dx.$$

Definition 1.2.1. *(Equi-integrability)[31, Définition 1.3.2]. Let $1 \leq p < \infty$. We say that a sequence $\{u_n\} \subset L^p(\Omega)$ is p-equi-integrable if:*

- *$\forall \varepsilon > 0$, there exist $D \subset \Omega$ of finite Lebesgue measure such that for all $n \geq 1$,*
$$\int_{\Omega\setminus D} |u_n|^p dx < \varepsilon,$$
- *$\forall \varepsilon > 0$, $\exists \eta$ such that, $\forall n \geq 1$, $\forall\, \Omega' \subset \Omega$ such that $|\Omega'| \leq \eta$ we have*
$$\int_{\Omega'} |u_n|^p dx < \varepsilon.$$

If $p = 1$ then we say that te sequence $\{u_n\}$ is equi-integrable.

Remark 1.2.1. *Note that any function $u \in L^p(\Omega)$, $1 \leq p < \infty$, is p-equi-integrable (see for instance [31, p.14-15]). Furthermore, if Ω has a finite measure then the second claim of Definition 1.2.1 is fulfilled.*

Lemma 1.2.3. *(Vitali lemma)[110, Theorem 6.28]. Let $1 \leq p < \infty$ and let the sequence $\{u_n\} \subset L^p(\Omega)$ converge almost everywhere on Ω to u. Then u belongs to $L^p(\Omega)$ and*

$$u_n \to u \text{ in } L^p(\Omega) \text{ if and only if } \{u_n\}_{n\geq 1} \text{ is p-equi-integrable.}$$

Lemma 1.2.4. *(Luzin theorem)[70, Theorem 2.1.9]. Let f be a function defined almost everywhere on Ω. Then f is measurable on Ω if and only if for every $\varepsilon > 0$ there exists an open set $D \subset \Omega$, $|D| \leq \varepsilon$, such that the restriction of f to $\Omega \setminus D$ is continuous on $\Omega \setminus D$.*

Lemma 1.2.5. *(Jensen inequality)[70, Theorem 3.2.9]. Let F be a convex function on $\mathbb{R}$ and let $v = v(x)$ be defined and positive almost everywhere on Ω. Then*

$$F\left(\frac{\int_\Omega u(x)v(x)dx}{\int_\Omega v(x)dx}\right) \leq \frac{\int_\Omega F(u(x))v(x)dx}{\int_\Omega v(x)dx}$$

for every nonnegative function u for which the integrals in the last inequality makes sense.

1.2.3 Convolution and mollification

Definition 1.2.2. *(Convolution)[111, Section 1.6].Let f and h be two functions in $L^1_{loc}(\mathbb{R}^N)$, we denote by $f * h$ the convolution product of f with h defined by*

$$f * h(x) = \int_\Omega f(x-y)h(y)dy. \tag{1.19}$$

Let J be a nonnegative, real-valued function such that

$$J \in \mathcal{C}_0^\infty(\mathbb{R}^N), \qquad J(x) = 0 \text{ if } \|x\| \geq 1, \qquad \int_{\mathbb{R}^N} J(x)dx = 1.$$

For example, J may be the function

$$J(x) = ke^{-\frac{1}{1-\|x\|^2}} \text{ if } \|x\| < 1 \text{ and } 0 \text{ if } \|x\| \geq 1,$$

where $k > 0$ is such that $\displaystyle\int_{\mathbb{R}^N} J(x)dx = 1$. For $\varepsilon > 0$ the function

$$J_\varepsilon(x) = \varepsilon^{-N} J(x\varepsilon^{-1})$$

belongs to $\mathcal{C}_0^\infty(\mathbb{R}^N)$ and supp $J_\varepsilon \subset \overline{B(0,\varepsilon)}$, with $B(0,\varepsilon)$ stands for the ball with center 0, and radius ε.

Definition 1.2.3. *(Mollifier, Mollification)[111, Section 1.6]. Let $\varepsilon > 0$, the function J_ε is called mollifier (or regularizer) and the convolution*

$$u_\varepsilon(x) := J_\varepsilon * u(x) = \int_{\mathbb{R}^N} J_\varepsilon(x-y)u(y)dy = \int_{B(0,1)} u(x-\varepsilon y)J(y)dy \tag{1.20}$$

defined for functions u for which the right side of (1.20) has meaning, is called the mollification (or regularization) of u.

Lemma 1.2.6. *(Mollification properties)[2, Theorem 2.29]. Let u be a function which is defined on $\mathbb{R}^N$ and vanishes identically outside Ω.*

- *If $u \in L^1_{loc}(\mathbb{R}^N)$, then $J * u \in \mathcal{C}^\infty(\mathbb{R}^N)$.*
- *If $u \in L^1_{loc}(\Omega)$ and supp u is a compact subset of Ω, then $J * u \in \mathcal{C}_0^\infty(\Omega)$ provided*
$$\varepsilon < dist(supp\ u, \partial\Omega).$$

1.2.4 Weak and weak-* topologies

Let X be a Banach space and X^* its dual, i.e, the space of all continuous linear functionals on X. For $u \in X^*$ and $x \in X$, by $\langle u, x\rangle$ we mean the scalar product for the duality X^*, X.

Weak topology $\sigma(X, X^*)$

The weak topology $\sigma(X, X^*)$ is the coarsest topology on X such that all element of X^* remains a continuous function.

Definition 1.2.4 (Weak convergence). *A sequence $\{u_n\}_n$ in X is said to converge to u in the weak topology $\sigma(X, X^*)$ ($u_n \rightharpoonup u$), if*

$$\langle u_n, f\rangle \to \langle u, f\rangle,\ \forall f \in X^*.$$

The next theorem is an important result in the theory of PDEs for reflexive spaces.

Theorem 1.2.1. *(Eberlein-Šmulian)[15, Theorem 3.18]. Let X be a Banach reflexive space and let $\{u_n\}_n$ be a bounded sequence in X. Then there exists a subsequence $\{u_{n_k}\}$ that converges in the weak topology $\sigma(X, X^*)$.*

Weak topology $\sigma(X^*, X)$

The weak-* topology $\sigma(X^*, X)$ is the coarsest topology on X^* such that all element of X corresponds to a continuous map on X^*.

Definition 1.2.5 (Weak-* convergence). *A sequence $\{f_n\}_n$ in X^* is said to converge to f in the weak-* topology $\sigma(X^*, X)$ ($f_n \overset{*}{\rightharpoonup} f$), if*

$$\langle f_n, u\rangle \to \langle f, u\rangle,\ \forall u \in X.$$

An interesting result in studding PDEs with non reflexive space are the following Theorem and Corollary 1.2.1.

Theorem 1.2.2. *(Banach-Alaoglu-Bourbaki)[15, Theorem 3.16]. The closed unit ball*

$$B_{X^*} := \{u \in X^*,\ \|u\| \leq 1\}$$

is compact in the weak-* topology $\sigma(X^*, X)$.

Corollary 1.2.1. *[15, Corollary 3.30] Let X a Banach separable space and let $\{u_n\}_n$ be a bounded sequence in X^*. Then there exists a subsequence $\{u_{n_k}\}$ that converges in the weak-* topology $\sigma(X^*, X)$.*

1.2.5 Regularity properties on domains

In this subsection we give some smoothness conditions on domains see [2] for more details. Let Ω be an open subset of $\mathbb{R}^N$ and let $\varepsilon > 0$. Set

$$\Omega_\varepsilon = \{x \in \Omega :\ \text{dist } (x, \partial\Omega) < \varepsilon\}.$$

Definition 1.2.6. *(Lipschitz domain)[2, Section 4.9]. A domain Ω is called a Lipschitz domain (or domain with Lipschitz boundary), if there exist positive number ε, a locally finite open cover $\{\theta_i\}$ of $\partial\Omega$, and, for each i a real-valued function f_i of $N-1$ variables, such that the following conditions hold:*

- *For some finite β, every collection of $\beta+1$ of the sets θ_i has empty intersection.*
- *For every pair of points $x, y \in \Omega_\varepsilon$ such that $|x-y| < \varepsilon$, there exists i such that*
$$x, y \in V_i := \{x \in \theta_i :\ dist\ (x, \partial\theta_i) > \varepsilon\}.$$
- *Each function f_i satisfies a Lipschitz condition with constant β: that*
- *For some Cartesian coordinate system $(\xi_{i,1}, \cdots, \xi_{i,n}) \in \theta_i$, $\Omega \cap \theta_i$ is represented by the inequality*

$$\xi_{i,n} < f_i(\xi_{j,1}, \cdots, \xi_{i,n-1}).$$

Definition 1.2.7. *(Segment property)[3, Definition 2.1]. A domain Ω is said to satisfy the segment property, if there exist a finite open covering $\{\theta\}_{i=1}^k$ of $\overline{\Omega}$ and a corresponding non-zero vectors $y_i \in \mathbb{R}^N$ such that $(\overline{\Omega} \cap \theta_i) + ty_i \subset \Omega$ for all $t \in (0,1)$ and $i = 1, \ldots, k$.*

The segment property condition holds, for example, if Ω is a Lipschitz domain (cf. [2]). By convention, the empty set satisfies the segment property.

1.3 An overview on particular classes of modular spaces

In this section we recall some particular examples of modular spaces, studied for the first time in [84] by Nakano. For more details on modular functional spaces see monograph [82] by Musielak.

1.3.1 Lebesgue spaces

In this subsection we give a brief review of some elementary properties of Lebesgue and Sobolev spaces, for the proofs and details see for instance [2, 3, 15, 70, 111].

Definition 1.3.1. *(Lebesgue spaces)[15, Section 4.2]. Let $1 \leq p < \infty$. We shall denote by $L^p(\Omega)$ the Lebesgue space defined as the set of all measurable functions $u : \Omega \to \mathbb{R}$ such that*

$$\int_\Omega |u(x)|^p dx < +\infty. \tag{1.21}$$

The norm on $L^p(\Omega)$ is given by

$$\|u\|_{L^p(\Omega)} := \|u\|_p := \Big(\int_\Omega |u(x)|^p dx\Big)^{\frac{1}{p}}. \tag{1.22}$$

Furthermore, if $p = \infty$, we set

$$L^\infty(\Omega) = \Big\{u : \Omega \to \mathbb{R} \text{ measurable } : \exists C \geq 0, |u(x)| \leq C \text{ a.e on } \Omega\Big\}. \tag{1.23}$$

The space $L^p_{loc}(\Omega)$ define the set of functions u in $L^p(K)$ for every compact subset K of Ω.

Theorem 1.3.1. *[15, Theorems 4.8, 4.10, 3.14] Endowed with the norm (1.22) (resp.(1.23)), $L^p(\Omega)$ is a Banach space for $1 \leq p < \infty$ (resp. $p = \infty$), separable if $1 \leq p < \infty$ and is reflexive if $1 < p < \infty$.*

Lemma 1.3.1. *[15, Theorem 4.11] For $1 \leq p < \infty$ the dual space of $L^p(\Omega)$ is the space $L^{p'}(\Omega)$, with $\dfrac{1}{p'} = 1 - \dfrac{1}{p}$.*

Definition 1.3.2 (Sobolev space)**.** *Let $1 \leq p \leq \infty$ and m a non-negative integer. We shall denote by $W^{m,p}(\Omega)$ the Sobolev space, defined by*

$$W^{m,p}(\Omega) := \{D^\alpha u \in L^p(\Omega),\ |\alpha| \leq m\}. \tag{1.24}$$

The set $W^{m,p}(\Omega)$ will be equipped with the norm

$$\|u\|_{W^{m,p}(\Omega)} = \|u\|_{m,p} := \sum_{|\alpha| \leq m} \|D^\alpha u\|_{L^p(\Omega)}. \tag{1.25}$$

Definition 1.3.3. *Let $1 \leq p < \infty$ and let m a non-negative integer. The norm closure of $\mathcal{C}_0^\infty(\Omega)$ in $W^{m,p}(\Omega)$ is denoted by $W_0^{m,p}(\Omega)$.*

For $1 \leq p < \infty$ and a non-negative integer m, the space $W^{-m,p'}(\Omega)$ denote the dual space of $W_0^{m,p}(\Omega)$, consists of all distributions T of the form

$$T = \sum_{|\alpha| \leq m} (-1)^\alpha D^\alpha v_\alpha, \tag{1.26}$$

where $v_\alpha \in L^{p'}(\Omega)$.

Theorem 1.3.2. *[15, Proposition 9.1] Endowed with the norm (1.22), the spaces $W^{m,p}(\Omega)$, $1 \leq p < \infty$, is a Banach separable space. If $1 < p < \infty$, then $W^{m,p}(\Omega)$ is reflexive.*

1.3.2 Orlicz spaces

In this subsection, we recall some properties of Orlicz spaces, for more details on these functional framework see [2, 37, 69, 70, 89]. The Orlicz spaces are generated by more general convex functions defined as follows.

Definition 1.3.4. *($\mathcal{N}$-functions)[2, p. 262]. A real function φ defined on $[0, \infty)$ is called $\mathcal{N}$-function, written $\varphi \in \mathcal{N}$, if*

- *$\varphi(\cdot)$ is a non-decreasing and convex function, $\varphi(0) = 0$ and $\varphi(s) > 0$ for $s > 0$,*
- *$\varphi(s) \to \infty$ as $s \to \infty$, $\lim_{s \to 0^+} \frac{\varphi(s)}{s} = 0$ and $\lim_{s \to +\infty} \frac{\varphi(s)}{s} = +\infty$.*

Example 1.3.1. *As examples of $\mathcal{N}$-functions, we give*

- $\varphi_1(s) = |s|^p$, $1 < p < \infty$,
- $\varphi_2(s) = |s|^p \log(e + |s|)$, $1 < p < \infty$,
- $\varphi_3(s) = \frac{1}{p}[(1 + |s|^2)^{\frac{p}{2}} - 1]$, $1 < p < \infty$,
- $\varphi_4(s) = e^{|s|^p} - 1$, $1 < p < \infty$.

Definition 1.3.5. *(Orlicz space)[2, p. 268]. Let $\varphi \in \mathcal{N}$, we shall denote by $L_\varphi(\Omega)$ the Orlicz space, defined as the set of all measurable functions $u : \Omega \to \mathbb{R}$ such that*

$$\int_\Omega \varphi(|u(x)|/\lambda)dx < +\infty, \quad \textit{for some } \lambda > 0. \tag{1.27}$$

The spaces $L_\varphi(\Omega)$, equipped with the Luxemburg norm

$$\|u\|_{L_\varphi(\Omega)} = \|u\|_\varphi = \inf\left\{\lambda > 0 : \int_\Omega \varphi(|u(x)|/\lambda)dx \leq 1\right\}, \tag{1.28}$$

is a Banach space (see [2, Theorem 8.10]).

Definition 1.3.6. *[2, p. 270] Let $\varphi \in \mathcal{N}$, the space $E_\varphi(\Omega)$ denotes the norm closure in $L_\varphi(\Omega)$ of bounded function compactly supported in $\overline{\Omega}$.*

Definition 1.3.7. *(Complementary $\mathcal{N}$-Functions)[2, p.264]. Let $\varphi \in \mathcal{N}$ and let $\varphi^* : [0, \infty) \to [0, \infty]$ be defined by*

$$\varphi^*(s) = \sup_{t \geq 0}\{st - \varphi(t)\} \text{ for all } s \geq 0 \text{ and all } x \in \Omega. \tag{1.29}$$

The function φ^* is called the complementary function to φ in the sense of Young.

Lemma 1.3.2. *[2, Theorem 8.19] Let (φ, φ^*) be a pair of complementary $\mathcal{N}$-functions. The dual space of $E_\varphi(\Omega)$ is isometrically isomorphic to $L_{\varphi^*}(\Omega)$.*

Definition 1.3.8. *(Orlicz-Sobolev spaces)[2, p. 281]. Let $\varphi \in \mathcal{N}$ and m a non-negative integer. We shall denote by $W^m L_\varphi(\Omega)$ and $W^m E_\varphi(\Omega)$ the Orlicz-Sobolev spaces, defined by*

$$W^m L_\varphi(\Omega) := \{D^\alpha u \in L_\varphi(\Omega),\ |\alpha| \leq m\},$$

$$W^m E_\varphi(\Omega) := \{D^\alpha u \in E_\varphi(\Omega),\ |\alpha| \leq m\}.$$

The spaces $W^m L_\varphi(\Omega)$ and $W^m L_\varphi(\Omega)$ equipped with the norm

$$\|u\|_{W^m L_\varphi(\Omega)} = \|u\|_{m,\varphi} := \sum_{|\alpha| \leq m} \|D^\alpha u\|_\varphi. \tag{1.30}$$

are Banach spaces. The spaces $W^m L_\varphi(\Omega)$ and $W^m E_\varphi(\Omega)$ will be identified with subspaces of $(N+1)$ copies of the product $\Pi L_\varphi(\Omega)$ and $\Pi E_\varphi(\Omega)$ respectively.

Lemma 1.3.3. *[2, Theorem 8.21 and Remark 8.22]. Let $\varphi \in \mathcal{N}$,*

- *$E_\varphi(\Omega)$ is separable.*
- *$L_\varphi(\Omega)$ is not separable unless if $L_\varphi(\Omega) = E_\varphi(\Omega)$.*
- *$L_\varphi(\Omega)$ is not reflexive unless if $L_\varphi(\Omega) = E_\varphi(\Omega)$ and $L_{\varphi^*}(\Omega) = E_{\varphi^*}(\Omega)$.*

Definition 1.3.9. *[37]. Let $\varphi \in \mathcal{N}$. The norm closure of $\mathcal{C}_0^\infty(\Omega)$ in $W^m L_\varphi(\Omega)$ is denoted by $W_0^m E_\varphi(\Omega)$.*

Let (φ, φ^*) be a pair of complementary $\mathcal{N}$-functions and let m be a non-negative integer, the space $W^{-m}L_{\varphi^*}(\Omega)$ denote the dual space of $W_0^m E_\varphi(\Omega)$, consists of all distributions T of the form

$$T = \sum_{|\alpha| \le m} (-1)^\alpha D^\alpha v_\alpha,$$

where $v_\alpha \in L_{\varphi^*}(\Omega)$.

Since in general, Orlicz space $L_\varphi(\Omega)$ is not reflexive, it can be useful in analysis of PDEs (e.g [37, 42]) to provide instead $W_0^m E_\varphi(\Omega)$ the space $W_0^m L_\varphi(\Omega)$ defined as follows.

Definition 1.3.10. *Let $\varphi \in \mathcal{N}$. The space $W_0^m L_\varphi(\Omega)$ denote the closure of $\mathcal{C}_0^\infty(\Omega)$ with respect to the weak-* topology $\sigma(\Pi L_\varphi, \Pi E_{\varphi^*})$ in $W^m L_\varphi(\Omega)$.*

Note that, in general, $W_0^m L_\varphi(\Omega) \neq W_0^m E_\varphi(\Omega)$.

1.3.3 Generalized Lebesgue spaces

The third class of examples of modular spaces, given in this subsection, is the generalized Lebesgue and generalized Sobolev spaces. See [27, 28, 67] for more details on these spaces. Throughout this subsection we will denote by $p : \Omega \to [0, \infty]$ a measurable function, satisfying

$$1 < p^- = ess \inf_{x \in \Omega} p(x) \le p(x) \le\le p^+ = ess \sup_{x \in \Omega} p(x) < \infty, \tag{1.31}$$

Definition 1.3.11. *[67, Section 2.]. We shall denote by $L^{p(\cdot)}(\Omega)$ the set of all Lebesgue measurable functions $u : \Omega \to \mathbb{R}$ such that*

$$\int_\Omega |u(x)/\lambda|^{p(\cdot)} dx < +\infty, \quad \textit{for some } \lambda > 0. \tag{1.32}$$

The set $L^{p(\cdot)}(\Omega)$ will be called generalized (or variable exponent) Lebesgue space. The norm on $L^{p(\cdot)}(\Omega)$ is given by

$$\|u\|_{L^{p(\cdot)}(\Omega)} = \|u\|_{p(\cdot)} := \inf \left\{ \lambda > 0 : \int_\Omega \Big(|u(x)|/\lambda \Big)^{p(\cdot)} dx \le 1 \right\}. \tag{1.33}$$

Theorem 1.3.3. *[28, Theorem 3.2.7., Lemma 3.4.4. and Theorem 3.4.7.]. Endowed with the norm (1.33), $L^{p(\cdot)}(\Omega)$ is a Banach, separable and reflexive space.*

Definition 1.3.12 (Variable exponent Sobolev space)**.** *Let m a non-negative integer. We shall denote by $W^{m,p(\cdot)}(\Omega)$ the generalized (or variable exponent) Sobolev space, defined by*

$$W^{m,p(\cdot)}(\Omega) := \{ D^\alpha u \in L^{p(\cdot)}(\Omega), \ |\alpha| \le m \}.$$

The space $W^{m,p(\cdot)}(\Omega)$ will be equipped with the norm

$$\|u\|_{W^{m,p(\cdot)}(\Omega)} = \|u\|_{m,p(\cdot)} = \sum_{|\alpha|\leq m} \|D^\alpha u\|_{L^{p(\cdot)}(\Omega)}. \tag{1.34}$$

Definition 1.3.13. *The norm closure of $\mathcal{C}_0^\infty(\Omega)$ in $W^{m,p(\cdot)}(\Omega)$ is denoted by $W_0^{m,p(\cdot)}(\Omega)$.*

Theorem 1.3.4. *[27, 28, 67].*
Endowed with the norm (1.34), the spaces $W^{m,p(\cdot)}(\Omega)$ is a Banach, separable and reflexive space.

For $1 \leq p(\cdot) < \infty$ and a non-negative integer m, the space $W^{-m,p'(\cdot)}(\Omega)$ denote the dual space of $W_0^{m,p(\cdot)}(\Omega)$, consists of all distributions T of the form (1.26) with $v_\alpha \in L^{p'(\cdot)}(\Omega)$.

1.4 Musielak spaces

In the last two decades, a wide literature was grown up for the class of Musielak spaces. In this section we list some facts about these later spaces and their structural properties. By this functional structure we will include, among others, the particular functional spaces given in Section 1.3. For more details we refer to the classical monograph [82] by Musielak. As a starting point we therefore define and give some properties that concern the Musielak functions.

1.4.1 Musielak functions (Φ-functions)

Definition 1.4.1 (ϕ-functions, Φ-functions)**.** *A real function $M : \Omega \times [0,\infty) \to [0,\infty]$ is called a ϕ-function, written $M \in \phi$, if*

- *$M(x,\cdot)$ is a non-decreasing and convex function for all $x \in \Omega$,*
- *$M(x,0) = 0$, $M(x,s) > 0$ for $s > 0$ and $M(x,s) \to \infty$ as $s \to \infty$*
- *$M(\cdot,s)$ is a measurable function for every $s \geq 0$.*

A ϕ-function is called Φ-function or Musielak function, denoted by $M \in \Phi$, if furthermore it satisfies

$$\lim_{s\to 0^+} \frac{M(x,s)}{s} = 0 \ \textit{and} \ \lim_{s\to +\infty} \frac{M(x,s)}{s} = +\infty.$$

Observe that equivalently a Φ-function M can be represented as

$$M(x,t) = \int_0^t a(x,s)ds, \ \text{for } t \geq 0,$$

where $a(x,\cdot)$ is a right continuous and increasing function, $a(x,s) > 0$ for $s > 0$, $a(x,0) = 0$, $a(x,s) \to \infty$ as $s \to \infty$ for every $x \in \Omega$.

Example 1.4.1. *As examples of ϕ-functions, we give*

- $M_1(x,s) = |s|^{p(x)},\ 1 < p(\cdot) < \infty,$
- $M_2(x,s) = |s|^{p(x)} \log(e + |s|),\ 1 < p(\cdot) < \infty,$
- $M_3(x,s) = \dfrac{1}{p(x)}[(1+|s|^2)^{\frac{p(x)}{2}} - 1],\ 1 < p(\cdot) < \infty,$
- $M_4(x,s) = |s|^p + a(x)|s|^q,\ 1 < p < q, 0 \le a(\cdot) \in L^1_{loc}(\Omega),$
- $M_5(x,s) = e^{|s|^{p(x)}} - 1,\ 1 < p(\cdot) < \infty,$
- $M_6(x,s) = \infty\chi_{(1,\infty)}(s).$

Definition 1.4.2. *(Complementary Φ-functions)[82, Definition 13.1]. Let $M \in \Phi$ and let $M^* : \Omega \times [0,\infty) \to [0,\infty]$ be defined by*

$$M^*(x,s) = \sup_{t\ge 0}\{st - M(x,t)\} \text{ for all } s \ge 0 \text{ and all } x \in \Omega. \tag{1.35}$$

The function M^* is called the complementary Φ-function to M in the sense of Young.

The complementary of a Φ-function M can be also expressed as

$$M^*(x,t) = \int_0^t a^*(x,s)ds, \text{ for } t \ge 0,$$

where $a^*(x,s) = \sup\{v, a(x,v) \le s\}$.

Example 1.4.2. *As examples of pair of complementary Φ-functions we give.*

- *If $M_1(x,s) = \dfrac{1}{p}|s|^p$, $1 < p < \infty$, then the complementary Φ-function is defined by*
$$M_1^*(x,s) = \frac{1}{p'}|s|^{p'}$$
where, $p' = \dfrac{p}{p-1}$.
- *If $M_2(x,s) = e^{|s|} - |s| - 1$, then the complementary Φ-function is defined by*
$$M_2^*(x,s) = (1+|s|)\log(1+|s|) - |s|.$$

- *If* $M_3(x,s) = \dfrac{1}{p(x)}|s|^{p(x)},\ 1 < p(x) < \infty$, *then the complementary* Φ*-function is defined by*

$$M_3^*(x,s) = \frac{1}{p'(x)}|s|^{p'(x)}$$

where, $p'(x) = \dfrac{p(x)}{p(x)-1}$.

Proposition 1.4.1. *(Young inequality)[82, Theorem 13.6]. Let* (M, M^*) *be a pair of complementary* Φ*-functions. Then for all* $u, v \geq 0$ *and for all* $x \in \Omega$ *we have*

$$uv \leq M(x,u) + M^*(x,v), \quad \forall u, v \geq 0, \forall x \in \Omega, \tag{1.36}$$

Equality holds in (1.36) when $v = a(x,u)$ *or* $u = a^*(x,v)$.

1.4.2 Musielak spaces: definitions and properties

In this subsection we define the Musielak spaces and provide some of their elementary properties.

Definition 1.4.3. *(Musielak spaces)[82, Definition 7.2]. Let* $M \in \phi$, *we shall denote by* $L_M(\Omega)$ *the set of all measurable functions* $u : \Omega \to \mathbb{R}$ *such that*

$$\rho_M(u(x)/\lambda) := \int_\Omega M(x, |u(x)|/\lambda)dx < +\infty, \quad \textit{for some } \lambda > 0. \tag{1.37}$$

and by $E_M(\Omega)$ *the set of all measurable functions* $u : \Omega \to \mathbb{R}$ *satisfying (1.37) for all* $\lambda > 0$. *The sets* $L_M(\Omega)$ *and* $E_M(\Omega)$ *will be called Musielak spaces.*

Example 1.4.3. *As examples of Musielak spaces we give.*

- *The classical Lebesgue space* $L^p(\Omega)$ *(see Definition 1.3.1).*

- *The variable Lebesgue space* $L^{p(x)}(\Omega)$ *(see Definition 1.3.11).*

- *The weight Lebesgue space defined by*

$$L^p(\Omega, w) := \Big\{u : \Omega \to \mathbb{R},\ \textit{measurable}\ : uw^{\frac{1}{p}} \in L^p(\Omega)\Big\}$$

  where $1 \leq p < \infty$ *and* $w = w(x)$ *is measurable and positive function a.e. in* Ω.

- *The space generated by the* ϕ*-function* $B(x,s) = e^{|s|} - s - 1$ *defined by*

$$L_B(\Omega) = \Big\{u : \Omega \to \mathbb{R},\ \textit{measurable}\ : \int_\Omega B\Big(x, \frac{|u(x)|}{\lambda}\Big)dx < \infty,\ \textit{for some } \lambda > 0\Big\}$$

Remark 1.4.1. *We note here that in the first three examples above the space $L_M(\Omega)$ and $E_M(\Omega)$ are the same. While in the last example generated by the ϕ-function $B(x,s) = e^{|s|} - s - 1$, the spaces $L_M(\Omega)$ and $E_M(\Omega)$ are not equal (see Lemma 3.2.3 below).*

Definition 1.4.4. *(Orlicz and Luxemburg norms)[82, pages 85 and 100]. Let $M \in \phi$ we define the Luxemburg norm by*

$$\|u\|_{L_M(\Omega)} = \|u\|_M := \inf\left\{\lambda > 0 : \int_\Omega M(x, |u(x)|/\lambda)dx \le 1\right\}, \tag{1.38}$$

and the Orlicz norm by

$$\|u\|^o_{M,\Omega} = \sup_{\rho_{M^*(v)} \le 1} \int_\Omega |u(x)v(x)|dx. \tag{1.39}$$

By definition of the Luxemburg norm it's easy to see the following equivalence

$$\|u\|_M \le 1 \Leftrightarrow \int_\Omega M(x, |u(x)|)dx \le 1. \tag{1.40}$$

Definition 1.4.5 (Modular convergence)**.** *A sequence $\{u_n\}_n$ is said to converge modularly to u in $L_M(\Omega)$ ($u_n \xrightarrow[n\to\infty]{M} u$), if there exists $\lambda > 0$ such that*

$$\rho_M((u_n - u)/\lambda) := \int_\Omega M\left(x, |u_n - u|/\lambda\right) dx \to 0 \text{ as } n \to \infty,$$

Definition 1.4.6 (Norm convergence)**.** *A sequence $\{u_n\}$ is said to converge to u in norm ($u_n \to u$) in $L_M(\Omega)$, if $\|u_n - u\|_M \to 0$ as $n \to \infty$.*

Theorem 1.4.1. *[82, Theorem 7.7]. Equipped with the Luxemburg norm (1.38), $L_M(\Omega)$ is a Banach space.*

Proposition 1.4.2. *The space $E_M(\Omega)$ is a closed subset of $L_M(\Omega)$.*

Proof. Let $\{u_n\} \subset E_M(\Omega)$ such that $u_n \to u \in L_M(\Omega)$. For any $\lambda > 0$, we have $\int_\Omega M(x, 2\lambda|u_n - u|)dx \to 0$. This implies $M(x, 2\lambda|u_n - u|) \to 0$ in $L^1(\Omega)$. So that, there is $h \in L^1(\Omega)$ such that $M(x, 2\lambda|u_n - u|) \le h$. Hence,

$$\lambda|u_n(x) - u(x)| \le \frac{1}{2}M^{-1}(x, h(x)),$$

which yields

$$\lambda|u(x)| \le \lambda|u_n(x)| + \frac{1}{2}M^{-1}(x, h(x)).$$

By the convexity of M, we get

$$M(x,\lambda|u(x)|)\leq\frac{1}{2}M(x,2\lambda|u_n(x)|)+\frac{1}{2}h(x).$$

Thus

$$\int_\Omega M(x,\lambda|u(x)|)dx\leq\frac{1}{2}\int_\Omega M(x,2\lambda|u_n(x)|)dx+\frac{1}{2}\int_\Omega h(x)dx<\infty.$$

So that $u\in E_M(\Omega)$. □

By Young inequality (1.36) it's easy to check the following Hölder inequality (see [82, Theorem 13.13])

Proposition 1.4.3. *(Hölder inequality)[82, Theorem 13.13]. Let (M,M^*) be a pair of complementary Φ-functions. for all $u\in L_M(\Omega)$ and $v\in L_{M^*}(\Omega)$ we have that*

$$\int_\Omega |u(x)v(x)|dx\leq\|u\|^o_{M,\Omega}\|v\|_{M^*}\leq 2\|u\|_M\|v\|_{M^*} \tag{1.41}$$

Definition 1.4.7. *(Inverse Φ-functions) Let $M\in\Phi$, we denote by M^{-1} the inverse of the M with respect to its second argument, that is*

$$M^{-1}(x,M(x,s))=M(x,M^{-1}(x,s))=s. \tag{1.42}$$

1.4.3 Introduction to Musielak-Sobolev spaces

In this subsection we introduce the Musielak-Sobolev space.

Definition 1.4.8. *[82, Definition 10.1] Let $M\in\phi$. For a positive integer m, we define the Musielak-Sobolev spaces $W^mL_M(\Omega)$ and $W^mE_M(\Omega)$ as follows*

$$W^mL_M(\Omega)=\{u\in L_M(\Omega):D^\alpha u\in L_M(\Omega),|\alpha|\leq m\},$$

$$W^mE_M(\Omega)=\{u\in E_M(\Omega):D^\alpha u\in E_M(\Omega),|\alpha|\leq m\},$$

where $\alpha=(\alpha_1,\alpha_2,\cdots,\alpha_N)$, $|\alpha|=|\alpha_1|+|\alpha_2|+\cdots+|\alpha_N|$ and D^α denote the distributional derivatives.

The spaces $W^mL_M(\Omega)$ and $W^mE_M(\Omega)$ are endowed with the Luxemburg norm

$$\|u\|_{W^mL_M(\Omega)}=\|u\|_{m,M}:=\inf\left\{\lambda>0:\sum_{|\alpha|\leq m}\int_\Omega M\left(x,\frac{|D^\alpha u|}{\lambda}\right)dx\leq 1\right\}. \tag{1.43}$$

Example 1.4.4. *As examples of Musielak-Sobolev spaces we give*

- *The classical Lebesgue-Sobolev space $W^mL^p(\Omega)$, $1\leq p<\infty$, (see Definition 1.3.2).*

- *The variable Lebesgue-Sobolev space* $W^m L^{p(x)}(\Omega)$ *satisfying (1.31) (see Definition 1.3.12).*
- *The weight Lebesgue space defined by*

$$W^m L^p(\Omega, w) = \Big\{u : \Omega \to \mathbb{R},\ \textit{measurable}\ : \\ uw^{\frac{1}{p}} \in L^p(\Omega)\ \textit{and}\ D^\alpha u w^{\frac{1}{p}} \in L^p(\Omega), |\alpha| \le m\Big\},$$

where $1 \le p < \infty$ *and* $w = w(x)$ *is measurable and positive function a.e. in* Ω.

- *The space generated by the* ϕ*-function* $B(x, s) = e^{|s|} - s - 1$ *defined by*

$$W^m L_B(\Omega) = \Big\{u : \Omega \to \mathbb{R}, \\ \textit{measurable}\ : u \in L_B(\Omega)\ \textit{and}\ D^\alpha u \in L_B(\Omega), |\alpha| \le m\Big\}.$$

Remark 1.4.2. *As in Remark 1.4.1 in the first three examples above we have* $W^m L_M(\Omega) = W^m E_M(\Omega)$. *While in the last example generated by the* ϕ*-function* $B(x, s) = e^{|s|} - s - 1$ *it is not the case (see Lemma 3.2.3 below).*

Definition 1.4.9 (Modular convergence)**.** *A sequence* $\{u_n\}_n$ *is said to converge modularly to* u *in* $W^m L_M(\Omega)$ ($u_n \xrightarrow[n\to\infty]{mod} u$), *if there exists* $\lambda > 0$ *such that*

$$\int_\Omega M\Big(x, \frac{|D^\alpha u_n(x) - D^\alpha u(x)|}{\lambda}\Big)dx \xrightarrow[n\to\infty]{} 0.$$

Definition 1.4.10 (Norm convergence)**.** *A sequence* $\{u_n\}$ *is said to converge to* u *in norm* ($u_n \to u$) *in* $W^m L_M(\Omega)$, *if for all* $|\alpha| \le m$ *we have*

$$\|D^\alpha u_n - D^\alpha u\|_M \to 0 \ \textit{as}\ n \to \infty.$$

Denote by $d = d(m, N)$ the number of multi-indices $\alpha = (\alpha_1, \alpha_2, \cdots, \alpha_N)$ satisfying $|\alpha| \le m$, that is $d = \sum_{|\alpha|\le m} 1$. Let $M \in \Phi$, define $\Pi L_M(\Omega) = \prod_{i=1}^{d} L_M(\Omega)$ as the d-tuple Cartesian product of $L_M(\Omega)$. The mapping

$$P : W^m L_M(\Omega) \to \Pi L_M(\Omega)$$

defined for all $u \in W^m L_M(\Omega)$ by

$$P(u) = (D^\alpha u)_{|\alpha|\le m} \tag{1.44}$$

establishes an isometric isomorphism from $W^m L_M(\Omega)$ onto the subspace $X = P(W^m L_M(\Omega))$ of $\Pi L_M(\Omega)$ for $m > 0$. Thus, we can identify the space $W^m E_M(\Omega)$ with the subspace $P(W^m E_M(\Omega))$ of the product $\Pi E_M(\Omega)$ for $m > 0$.

Remark 1.4.3. *In the particular case of Orlicz-Sobolev spaces other interesting spaces defined by mean of the closure of $\mathcal{C}_0^\infty$-functions are given (see Definitions 1.3.9 and 1.3.10). In general for Φ-functions M depending on x, $\mathcal{C}_0^\infty$-functions do not belong to the space $W^m L_M(\Omega)$. However, in Definition 3.4.1 below, we will give a sufficient condition for which Definitions 1.3.9 and 1.3.10 holds in Musielak spaces.*

Part I

Some functional analysis results

Introduction to Part I

In this Part, we investigate some mathematical properties of Musielak and Musielak-Sobolev spaces. The Part contains four chapters constitute a groundwork to solve PDEs with general growth conditions. These chapters are based on papers [4, 5, 7, 103] by the author et al.

The results we give in this Part are a generalization of several studies obtained in the particular cases of Orlicz space, variable exponent Sobolev space and the space generated by double phase function. More than half of this Part is concerned by the problem of the density of smooth functions in Musielak and Musielak-Sobolev spaces. In fact, the Part starts with Chapter 2 about the continuity of the shift operator, which will be a substantial tool for proving the approximation results in the Musielak spaces given at the end of the chapter under a sufficient regularity condition on the Φ-function of local integrability type (see ($\mathcal{L}^1_{loc}$) below)).

As first application of Chapter 2 we provide Chapter 3 on the problem of duality, separability and reflexivity in booth Musielak and Musielak-Sobolev spaces. The first two topological properties will be obtained only under the local integrability condition. Whereas the reflexivity of the Musielak space will be obtained if, in addition to the local integrability condition, we assume that the pair of complementary Φ-functions (M, M^*) satisfies the condition Δ_2 that we discuss at the beginning of the chapter. This result of reflectivity we get joins that obtained in the case of Orlicz space (see [89]).

In Chapter 4 we address the problem of density in Musielak-Sobolev spaces. In general, smooth functions are not dense in these functional frameworks, in this chapter we provide new systematic regularity assumptions (see ($\mathcal{L}_{hol}$) or (4.8) and ($\mathcal{L}^p_{hol}$) below) in connection with the log-Hölder continuity condition of variable exponent or the sharp range condition for powers in double phase spaces. In this later, chapter we prove modular and norm densities of smooth function compactly supported whether in $\mathbb{R}^N$ or in a sufficient smooth open set Ω in Musielak-Sobolev spaces, in addition, we give a Meyers-Serrin type approximation whereby the weak derivatives are identified to the strong ones with respect to the modular topology. The identification with null trace functions space will also be given.

The remaining chapter of this part is Chapter 5 concerned with Poincaré type inequalities in Musielak-Sobolev spaces. These types of inequalities are generally not verified in these functional frameworks, in this chapter we provide a new systematic regularity assumption (see ($\mathcal{M}_{on}$) below) whereby we prove the Poincaré integral inequality in several Musielak-Sobolev spaces as well as the Poincaré norm inequality. The condition $(\mathcal{M}_{on})$ we introduce is in connection with Maeda monotonicity condition for variable exponent $p(\cdot)$ (see [77]) when $M(\cdot, s) = s^{p(\cdot)}$.

Chapter 2

The shift operator and approximation results in Musielak spaces

2.1 Introduction

In this chapter, we are interested in establishing some basic approximation results in Musielak spaces with respect to the modular and norm convergence. In fact, we give sufficient conditions for the continuity in norm of the shift operator in these functional frameworks and an application to the convergence in norm of approximate identities, whereby we prove the density of smooth functions in L_M, in both modular and norm topologies. The main results of this chapter are contained in Theorem 2.3.1, Theorem 2.3.2 and Theorem 2.4.1.

Unlike the classical Orlicz spaces, the spatial dependence of the Φ-function M does not allow, in general, to smooth functions compactly sported in Ω to belong to Musielak space $L_M(\Omega)$. The chapter starts by introducing and discuss the sufficient condition of locally integrability type (see ($\mathcal{L}^1_{loc}$) below) which allows to get the above inclusion.

In general, if $u \in E_M(\Omega)$ (Definition 1.4.3) we can't expect that $\tau_h u$ belongs to $E_M(\Omega)$ as was proved first by Kamińska [62, Theorem 2.1] (see also [67, Example 2.9 and Theorem 2.10]). In the first main result Theorem 2.3.1, we prove that the translation operator acts on the set of bounded functions compactly supported in Ω. In the case where $M(x,t) = |t|^{p(x)}$, a similar result was proved in [26, p.261] by using the continuous imbedding between variable exponent Lebesgue spaces. Unfortunately, this result is not true in general in variable Lebesgue spaces, as showed in [62, Example 2^0] (see also [28, Proposition 3.6.1]) unless the exponent is constant.

As a consequence of Theorem 2.3.1, we prove a norm convergence of the approximate identities in Theorem 2.3.2. The remaining main result of this chapter Theorem 2.4.1 is a unified generalization of the approximation results known in Lebesgue spaces $L^p(\Omega)$, $1 < p < \infty$ and Orlicz spaces. The approach, now classical, is based upon reducing the study to continuous functions compactly supported in Ω and then using imbedding theorems and a sequence of mollifiers to conclude, see for instance [2, Corollary 2.30 and Theorem 8.21]. This classical approach is based on the fact that the translation operator $u(\cdot + h)$ is continuous in norm when h tends to zero. This fails to hold in Musielak spaces as showed in Remark 2.3.1. Consequently, we can not approximate in general the identities for a given function.

In the framework of variable exponent Lebesgue spaces $L^{p(\cdot)}(\Omega)$, Kovácik and Rákosník [67, Theorem 2.11] proved first the density of the set $\mathcal{C}_0^\infty(\Omega)$ of infinitely differentiable functions compactly supported in Ω, provided only that the variable exponent $p(\cdot) \in L^\infty(\Omega)$. Their idea consists in showing successively that the set of essentially bounded functions $L^\infty(\Omega) \cap L^{p(\cdot)}(\Omega)$ is dense in $L^{p(\cdot)}(\Omega)$ and by means of Luzin's theorem the subset of continuous functions $\mathcal{C}(\Omega) \cap L^{p(\cdot)}(\Omega)$ is dense in $L^{p(\cdot)}(\Omega)$, which finally lead to the density of the set $\mathcal{C}_0^\infty(\Omega)$ in $L^{p(\cdot)}(\Omega)$.

In Musielak spaces, the situation is more complicated. At first, we note that although the assumption $(\mathcal{L}^1_{loc})$ is satisfied, the inclusion $L^\infty(\Omega) \subset L_{M(\cdot,\cdot)}(\Omega)$ does not hold true in general even if Ω is an open of R^N with finite Lebesgue measure. Secondly, the use of the similar idea to that of Cruz-Uribe and Fiorenza [26] will requires additional assumptions. Thirdly, in contrast to what is mentioned above, the translation operator is not acting, in general, between Musielak spaces (see [67, example 2.9 and Theorem 2.10] and [28, Proposition 3.6.1]). For these reasons and for the best of our knowledge, it is not possible to obtain approximation results using classical ideas. The approach we use consists in starting by proving the density of smooth functions $\mathcal{C}_0^\infty(\Omega)$ in $\mathcal{B}_c(\Omega)$ with respect to the norm in $L_M(\Omega)$, (see Theorem 2.3.2), and then the density of bounded functions compactly supported in norm in $E_M(\Omega)$ and in modular in $L_M(\Omega)$, (see Lemma 2.4.1), which allow us to get the density of smooth functions $\mathcal{C}_0^\infty(\Omega)$ in $E_M(\Omega)$ and $L_M(\Omega)$ with respect to the norm and modular convergences respectively. The idea we use in this paper is essentially based upon using the fact that for a function $u \in \mathcal{B}_c(\Omega)$, the translation operator $u(\cdot + h)$ is continuous with respect to the norm $\|\cdot\|_{L_M(\Omega)}$ as h tends to 0 (see Theorem 2.3.1). This constitutes the main reason why we introduce the space $\mathcal{B}_c(\Omega)$.

Outline of the chapter. The chapter is organized as follow, in Section 2.2 we introduce the local integrability condition $(\mathcal{L}^1_{loc})$. Section 2.3 provide the continuity in norm of the translation result in Musielak space. Finally, Section 2.4 is devoted to some density results in Musielak spaces.

2.2 The regularity of local integrability $(\mathcal{L}^1_{loc})$

Unlike the classical Orlicz spaces, the spatial dependence of the Φ-function M does not allow, in general, to bounded functions to belong to Musielak spaces. An extra additional hypothesis is needed. In the sequel, we make use of the following local integrability.

Definition 2.2.1. *We say that $M \in \Phi$ is locally integrable, if for any constant number $c > 0$ and for every compact set $K \subset \Omega$ we have*

$$(\mathcal{L}^1_{loc}) \qquad \int_K M(x,c)dx < \infty.$$

Example 2.2.1. *We have the following examples of Φ-functions satisfying the condition ($\mathcal{L}^1_{loc}$), and thus are admissible in our results below.*

1. *All $\mathcal{N}$-functions (Definition 1.3.4), namely for*
   - $M(x,s) = |s|^p,\ 1 < p < +\infty,$
   - $M(x,s) = (1+|s|)\log(1+|s|) - |s|,$
   - $M(x,s) = e^{|s|} - |s| - 1,$

   satisfy obviously condition.

2. *If $p(\cdot)$ satisfies (1.31), it is obvious to check that the following Φ-functions*
   - $M(x,s) = |s|^{p(x)}$
   - $M(x,s) = |s|^{p(x)}\log(e+|s|),$
   - $M(x,s) = \dfrac{1}{p(x)}[(1+|s|^2)^{\frac{p(x)}{2}} - 1],$
   - $M(x,s) = e^{|s|^{p(x)}} - 1,$

   satisfy the assumption ($\mathcal{L}^1_{loc}$).

3. *If $a \in L^1_{loc}(\Omega)$ then double phase Φ-function*
   - $M(x,s) = |s|^p + a(x)|s|^q,$

   with $1 < p < q < \infty$, $0 \le a(\cdot)$, satisfies ($\mathcal{L}^1_{loc}$).

Comments and remarks on the assumption ($\mathcal{L}^1_{loc}$)

- Inequality ($\mathcal{L}^1_{loc}$) was introduced in [82, Definition 7.5] for a measure space (Ω, Σ, μ) where μ is a positive complete measure, that is to say

$$\int_D M(x,s)d\mu < \infty, \quad \text{for every } s > 0 \text{ and } D \in \Sigma \text{ with } \mu(D) < \infty, \tag{2.1}$$

  (see [82, Definition 7.5]), which is stronger than the condition ($\mathcal{L}^1_{loc}$) when we limit ourselves to the Borel tribe.

- Observe that ($\mathcal{L}^1_{loc}$) is not always satisfied as shown by the following example. Set $\Omega = (-1/2, 1/2)$ and set

$$M(x,s) = \begin{cases} s^{1/x} & x \in (0,1/2), \\ s^2 & x \in (-1/2, 0). \end{cases}$$

  Note that M is a Φ-function. Consider the compact set $K = [0,1/4]$, which is contained in Ω. Then for $c > 1$

$$\int_K M(x,c)dx = \int_0^{1/4} c^{1/x}dx = +\infty.$$

- The condition $(\mathcal{L}^1_{loc})$ ensures that the set of bounded functions compactly supported in Ω belong to $E_M(\Omega)$ and so is the class of smooth functions compactly supported in Ω.
- The functions essentially bounded do not belong to $E_M(\Omega)$ even if $(\mathcal{L}^1_{loc})$ is satisfied.
- If $x \mapsto M(x,s)$ is a continuous function on $\overline{\Omega}$ then, by (1.35), is the complementary Φ-function M^* to M. Thus $(\mathcal{L}^1_{loc})$ holds for the pair of complimentary Φ-functions (M, M^*).

Remark 2.2.1. *It is worth pointing out that if a Φ-function A satisfies*

$$\lim_{s\to\infty} \text{ess inf}_{x\in\Omega} \frac{A(x,s)}{s} = +\infty, \tag{2.2}$$

then its complementary Φ-function A^* satisfies $(\mathcal{L}^1_{loc})$ not only for compact subsets $K \subset \Omega$ but for all measurable subsets of Ω having finite Lebesgue measure. Indeed, assume that (2.2) is fulfilled. Then for an arbitrary $c > 0$, there exists a constant $s_c > 0$ (not depending on x) such that for all $s > s_c$ and for every $x \in \Omega$

$$\frac{A(x,s)}{s} > c + 1.$$

Noticing that $\sup\limits_{s>s_c}(sc - A(x,s)) \le 0$ and from the definition of A^*, we obtain

$$A^*(x,c) \le \sup_{0\le s\le s_c} (sc - A(x,s)) \le cs_c, \text{ for every } x \in \Omega.$$

This proves that A^* satisfies $(\mathcal{L}^1_{loc})$ for any subsets of finite Lebesgue measure.

A useful implication of the condition $(\mathcal{L}^1_{loc})$ is the following equivalence norms result. The equivalence between Orlicz and Luxemburg norms (Definition 1.4.4) is well-known in the Orlicz spaces setting [70, Theorem 3.8.5], while in the Musielak framework this result was proved by Musielak [82, Theorem 13.11] using a local integrability condition upon measurable sets with finite measure.

Lemma 2.2.1. *Let $M \in \Phi$ satisfy $(\mathcal{L}^1_{loc})$. Then, for all $u \in L_M(\Omega)$*

$$\|u\|_M \le \|u\|^o_M \le 2\|u\|_M \tag{2.3}$$

Proof. The inequality in the right-hand side is an easy consequence of the Young inequality. We only need to prove the left-hand side inequality. To this end, it is sufficient to prove that

$$\int_\Omega M(x, |u(x)|/\|u\|^o_M)dx \le 1.$$

This can be done by using ($\mathcal{L}^1_{loc}$) and following exactly the lines of [70, Lemma 3.7.2]. □

2.3 Continuity of the shift operator

We state in this section a continuity result of the shift operator in Musielak space.

2.3.1 M-mean continuity

In this subsection, for $h \in \mathbb{R}^N$, we denote by $\tau_h u$ the shift operator defined by (1.18). Note that if the function u has a compact support, $\tau_h u$ is well-defined provided that $h < dist(supp\, u, \partial\Omega)$. In the following Theorem, we prove that the translation operator acts on the set of bounded functions compactly supported in Ω.

Theorem 2.3.1. *Let $M \in \Phi$ satisfy ($\mathcal{L}^1_{loc}$). Then, any $u \in \mathcal{B}_c(\Omega)$ is M-mean continuous, that is to say for every $\varepsilon > 0$ there exists a $\eta = \eta(\varepsilon) > 0$ such that for $h \in \mathbb{R}^N$ with $|h| < \eta$ we have*

$$\|\tau_h u - u\|_M < \varepsilon.$$

Proof. For $u \in \mathcal{B}_c(\Omega)$, let $supp u = U \subset B_R \cap \Omega$, where by B_R we denote a ball with radius $R > 0$. Let $h \in \mathbb{R}^N$ with $|h| < \min(1, dist(U, \partial\Omega))$. We have $supp\, \tau_h u \subset B := B_{R+1} \cap \Omega$. Let us define $\overline{B} = \overline{B}_{R+1} \cap \Omega$ where $\overline{B}_{R+1}$ stands for the closed ball with radius $R + 1$. Thanks to ($\mathcal{L}^1_{loc}$), for any constant number $C > 0$ and any compact subset $K \subset \Omega$ one has $M(x, C) \in L^1(K)$. Therefore for arbitrary $\varepsilon > 0$, there is $\nu > 0$ such that for every measurable subset $\Omega' \subset K$

$$\int_{\Omega'} M(x, C)dx < \frac{\varepsilon}{2}, \text{ whenever } |\Omega'| < \nu. \tag{2.4}$$

For this ν, there exists $\rho \in (0, 1)$ such that $|H_\rho| < \dfrac{\nu}{4}$ where

$$H_\rho = \{x \in B : dist(x, \partial B) \le \rho\}.$$

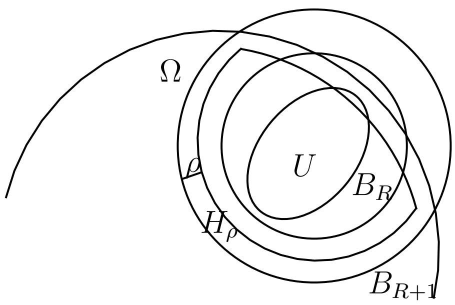

FIGURE 2.1: Domain geometry

Define $U_\rho = B \setminus H_\rho$. Since u is measurable on U_ρ, Luzin's theorem ensures that for $\nu > 0$ there exists a closed set $F_{1,\nu} \subset U_\rho$ such that the restriction of u to $F_{1,\nu}$ is continuous and $|U_\rho \setminus F_{1,\nu}| < \dfrac{\nu}{4}$. We then have, $|B \setminus F_{1,\nu}| < \dfrac{\nu}{2}$. The function u is uniformly continuous on the compact set $F_{1,\nu}$. It follows that for $\varepsilon > 0$, there exists an $\eta \in (0, \rho)$ such that for all x, $x + h \in F_{1,\nu}$ one has

$$|h| < \eta \Rightarrow |u(x+h) - u(x)| < \frac{\varepsilon}{2\Big(\displaystyle\int_U M(x,1)dx + 1\Big)}. \tag{2.5}$$

Define the two sets

$$F_{2,\nu} = \{x \in U, x + h \in F_{1,\nu}\} \text{ and } F_\nu = F_{1,\nu} \cap F_{2,\nu}.$$

The set F_ν is a closed subset of Ω. In addition, we have $|B \setminus F_\nu| < \nu$. Indeed, since the Lebesgue measure is invariant by translation we get $|B \setminus F_{1,\nu}| = |B \setminus F_{2,\nu}|$. Therefore,

$$|B \setminus F_\nu| = |(B \setminus F_{1,\nu}) \cup (B \setminus F_{2,\nu})| \leq |B \setminus F_{1,\nu}| + |B \setminus F_{2,\nu}| < \nu. \tag{2.6}$$

If $x \notin B$ then for $|h| < \eta$ we have $x + h \notin B_R \cap \Omega$. If not, we will get $x \in B$ which contradicts the fact that $x \notin B$. Hence, we obtain

$$\begin{aligned}\int_\Omega M(x, |\tau_h u(x) - u(x)|)dx &= \int_B M(x, |\tau_h u(x) - u(x)|)dx \\ &= \int_{B\cap F_\nu} M(x, |\tau_h u(x) - u(x)|)dx \\ &+ \int_{B\setminus F_\nu} M(x, |\tau_h u(x) - u(x)|)dx.\end{aligned} \tag{2.7}$$

By (2.5) the first term in the right-hand side can be estimated as

$$\int_{B\cap F_\nu} M(x, |\tau_h u(x) - u(x)|)dx \leq \int_{B\cap F_\nu} M\left(x, \frac{\varepsilon}{2\Big(\displaystyle\int_U M(x,1)dx + 1\Big)}\right)dx < \frac{\varepsilon}{2}.$$

As regards the second term in the right-hand side of (2.7), we use the fact that $u \in \mathcal{B}_c(\Omega)$ is bounded by a constant number $c > 0$ and then (2.4) (since $B \setminus F_\nu \subset K := H_\rho \cup \overline{U_\rho} \cup U$) to obtain

$$\int_{B\setminus F\nu} M(x, |\tau_h u(x) - u(x)|)dx \leq \int_{B\setminus F\nu} M(x, 2c)dx \leq \frac{\varepsilon}{2}. \tag{2.8}$$

Putting together (2.7) and (2.8), we get

$$\forall \varepsilon > 0, \exists \eta > 0 : |h| < \eta \Rightarrow \int_\Omega M(x, |\tau_h u(x) - u(x)|)dx < \varepsilon.$$

Let $\delta > 0$ be arbitrary but fixed. As $u/\delta \in \mathcal{B}_c(\Omega)$, we get

$$\exists \eta > 0 : |h| < \eta \Rightarrow \int_\Omega M\Big(x, \frac{|\tau_h u(x) - u(x)|}{\delta}\Big) dx \leq 1,$$

which gives

$$\|\tau_h u - u\|_M \leq \delta \text{ whenever } |h| < \eta.$$

□

Remark 2.3.1. *Note that the boundedness of the function u in Theorem 2.3.1 is necessary, else the result is false. Indeed, when we put ourselves in the particular case $M(x,t) = t^{p(x)}$, the authors [67, example 2.9 and Theorem 2.10] gave the following example : $N = 1$, $\Omega = (-1,1)$. For $1 \leq r < s < +\infty$ they define the variable exponent*

$$p(x) = \begin{cases} r & \text{if } x \in [0,1), \\ s & \text{if } x \in (-1,0) \end{cases}$$

and consider

$$f(x) = \begin{cases} x^{-\frac{1}{s}} & \text{if } x \in [0,1), \\ 0 & \text{if } x \in (-1,0). \end{cases}$$

They show that $\tau_h f \notin L^{p(\cdot)}(\Omega)$ although that $f \in L^{p(\cdot)}(\Omega)$. Observe here, in this example, that the function f is compactly supported but not bounded on Ω.

2.3.2 Application to approximate identities

For $\varepsilon > 0$ and a real function u by u_ε we denote the regularization sequence of u defined by (1.20). As a consequence of Theorem 2.3.1, we give the following density result.

Theorem 2.3.2. *Let $M \in \Phi$ satisfy ($\mathcal{L}^1_{loc}$) and let $u \in \mathcal{B}_c(\Omega)$. For any $\varepsilon > 0$ small enough, we have $u_\varepsilon \in \mathcal{C}^\infty_0(\Omega)$. Furthermore,*

$$\|u_\varepsilon - u\|_M \to 0 \text{ as } \varepsilon \to 0^+.$$

Proof. For $\varepsilon > 0$, the function u_ε defined in (1.20) belongs to $\mathcal{C}^\infty_0(\Omega)$ whenever $\varepsilon < dist(supp\, u, \partial\Omega)$ (see for example [2, Theorem 2.29]). Let M^* stands for the complementary Φ-function of M and let $v \in L_{M^*}(\Omega)$. By Fubini's theorem and Hölder's inequality (1.41) we can write

$$\begin{aligned} \int_\Omega \big|\big(u_\varepsilon(x) - u(x)\big)v(x)\big| dx &\leq \int_{\mathbb{R}^N} \Big(\int_\Omega |u(x-\varepsilon y) - u(x)||v(x)| dx\Big) J(y) dy \\ &\leq 2\|v\|_{M^*} \int_{|y|\leq 1} \|\tau_{-\varepsilon y} u - u\|_M J(y) dy. \end{aligned}$$

Hence, by the definition of the Orlicz norm and the inequality (2.3) we obtain

$$\|u_\varepsilon - u\|_M \leq 2 \int_{|y|\leq 1} \|\tau_{-\varepsilon y} u - u\|_M J(y)dy.$$

We can now use Theorem 2.3.1 Given $\mu > 0$, there exists $\eta > 0$ such that for $\varepsilon < \eta$ we get

$$\|\tau_{-\varepsilon y} u(x) - u(x)\|_M \leq \mu$$

for every y with $|y| \leq 1$. Then we conclude that

$$\|u_\varepsilon - u\|_M \leq 2\mu \int_{|y|\leq 1} J(y)dy = 2\mu,$$

which gives the result. □

2.4 Density results

2.4.1 Auxiliary density results

We give below some density results in $L_M(\Omega)$ and $E_M(\Omega)$. In the following lemma we prove that under the condition $(\mathcal{L}^1_{loc})$ bounded functions compactly supported in Ω are dense in $E_M(\Omega)$ with respect to the strong topology and in $L_M(\Omega)$ with respect to the modular topology.

Lemma 2.4.1. *Assume that $M \in \phi$ satisfies $(\mathcal{L}^1_{loc})$. Then*

1. *$\mathcal{B}_c(\Omega)$ is dense in $E_M(\Omega)$ with respect to the strong topology in $L_M(\Omega)$.*
2. *$\mathcal{B}_c(\Omega)$ is dense in $L_M(\Omega)$ with respect to the modular topology in $L_M(\Omega)$.*

Proof. 1. If $u \in E_M(\Omega)$, then for all $\lambda > 0$ one has $M(x, |u|/\lambda) \in L^1(\Omega)$. Denote by T_j, $j > 0$, the truncation function at levels $\pm j$ defined on $\mathbb{R}$ by $T_j(s) = \max\{-j, \min\{j, s\}\}$. We define the sequence $\{u_j\}$ by

$$u_j = T_j(u)\chi_{K_j}, \tag{2.9}$$

where χ_{K_j} stands for the characteristic function of the set

$$K_j = \left\{x \in \Omega : |x| \leq j, dist(x, \Omega^c) \geq \frac{1}{j}\right\}.$$

Hence, the function $u_j \in \mathcal{B}_c(\Omega)$ and converges almost everywhere to u in Ω. Thus $M(x, |u_j(x) - u(x)|/\lambda) \to 0$ a.e. in Ω and

$$M(x, |u_j(x) - u(x)|/2\lambda) \leq M(x, |u(x)|/\lambda) \in L^1(\Omega). \tag{2.10}$$

So that by the Lebesgue dominated convergence theorem, we obtain

$$\int_\Omega M(x|u_j(x) - u(x)|/2\lambda)dx \leq 1 \text{ for } j \text{ large enough},$$

which yields $\lim_{j\to+\infty} \|u_j - u\|_M \leq \lambda$. Being $\lambda > 0$ arbitrary, we get $\lim_{j\to+\infty} \|u_j - u\|_M = 0$.

2. Now if $u \in L_M(\Omega)$ then for some $\lambda > 0$ one has $M(x, |u|/\lambda) \in L^1(\Omega)$.

   Let $\{u_j\}$ the sequence defined in (2.9). The inequality (2.10) holds for some $\lambda > 0$ and since $u_j \in \mathcal{B}_c(\Omega)$ and converges a.e. to u in Ω, we get $M(x, |u_j(x) - u(x)|/2\lambda) \to 0$ a.e. in Ω. Thus, Lebesgue's dominated convergence theorem yields
$$\int_\Omega M(x, |u_j(x) - u(x)|/2\lambda) \to 0, \text{ as } j \to \infty.$$

□

Remark 2.4.1. *In view of Lemma 3.2.3, observe that if the ϕ-function $M \in \Delta_2$ then bounded functions compactly supported in Ω are dense in $L_M(\Omega)$ for the norm topology.*

Let $\mathcal{S}$ and $\mathcal{S}_c$ stand for the sets defined respectively by (1.15) and (1.16).

Lemma 2.4.2. *Assume that $M \in \phi$ satisfies ($\mathcal{L}^1_{loc}$). Then the set $\mathcal{S}_c$ is dense in $E_M(\Omega)$ with respect to the strong topology in $E_M(\Omega)$.*

Proof. Let $u \in \mathcal{B}_c(\Omega)$. Since u is a measurable function, by classical result [2] we know that there exists a sequence $\{u_n\} \subset \mathcal{S}$ converging pointwise to u in Ω and satisfying $|u_n(x)| \leq |u(x)|$, for all $n \in \mathrm{N}$ and $x \in \Omega$. Since $u \in \mathcal{B}_c(\Omega)$, we can assume that $u_n \in \mathcal{S}_c$. Hence,

$$M(x, |u_n(x) - u(x)|/\lambda) \leq M(x, 2|u(x)|/\lambda) \in L^1(\Omega).$$

As for all $\lambda > 0$
$$M(x, |u_n(x) - u(x)|/\lambda) \to 0 \text{ a.e. in } \Omega,$$
by the Lebesgue dominated convergence theorem we obtain

$$\int_\Omega M(x, |u_n(x) - u(x)|/\lambda)dx \leq 1, \text{ for } n \text{ large enough},$$

which yields $\|u_n - u\|_M \leq \lambda$, for n large enough. Being $\lambda > 0$ arbitrary, we get

$$\|u_n - u\|_M \to 0 \text{ as } n \to \infty.$$

□

2.4.2 Density of smooth functions

The following approximation theorem is the main result of this subsection.

Theorem 2.4.1. *Let $M \in \Phi$ satisfy ($\mathcal{L}^1_{loc}$), then*

1. *$\mathcal{C}_0^\infty(\Omega)$ is dense in $E_M(\Omega)$ with respect to the strong topology in $E_M(\Omega)$.*

2. *$\mathcal{C}_0^\infty(\Omega)$ is dense in $L_M(\Omega)$ with respect to the modular topology in $L_M(\Omega)$.*

Proof. 1. Combining Theorem 2.3.2 and Lemma 2.4.1, we obtain the density of $\mathcal{C}_0^\infty(\Omega)$ in $E_M(\Omega)$ with respect to the strong topology.

2. Let $u \in L_M(\Omega)$. According to Lemma 2.4.1, there exist $w \in \mathcal{B}_c(\Omega)$ and $\lambda > 0$ such that for all $\eta \geq 0$

$$\int_\Omega M(x, |u(x) - w(x)|/\lambda)dx \leq \eta.$$

Then by Theorem 2.3.2, there exists a function $v \in \mathcal{C}_0^\infty(\Omega)$ that converges strongly to w in $L_M(\Omega)$. But we know that the norm topology is strong than the modular one, more precisely we have

$$\int_\Omega M(x, |w(x) - v(x)|)dx \leq \eta.$$

Let us make the choice $\lambda_1 = \max\{1, \lambda\}$ and use the convexity of the Φ-function M, we can write

$$\begin{aligned}\int_\Omega M(x, |u(x) - v(x)|/2\lambda_1)dx \quad &\leq \frac{1}{2}\int_\Omega M(x, |u(x) - w(x)|/\lambda)dx \\ &+\frac{1}{2}\int_\Omega M(x, |w(x) - v(x)|)dx.\end{aligned}$$

This, yields the result.

□

Remark 2.4.2. *Let $M \in \Phi$. If $M \in \Delta_2$, then in view of Theorem 2.4.1 and Lemma 3.2.2, the set of $\mathcal{C}_0^\infty(\Omega)$ function is dense in $L_M(\Omega)$ with respect to the norm $\|\cdot\|_M$.*

2.5 Conclusion

In this chapter we proved the continuity in norm of the shift operator in the Musielak L_M spaces. An application to the convergence in norm of approximate identities is given, whereby we proved density results of the smooth functions in Musielak spaces, in both modular and norm topologies. These densities results are summarized in the following Figure.

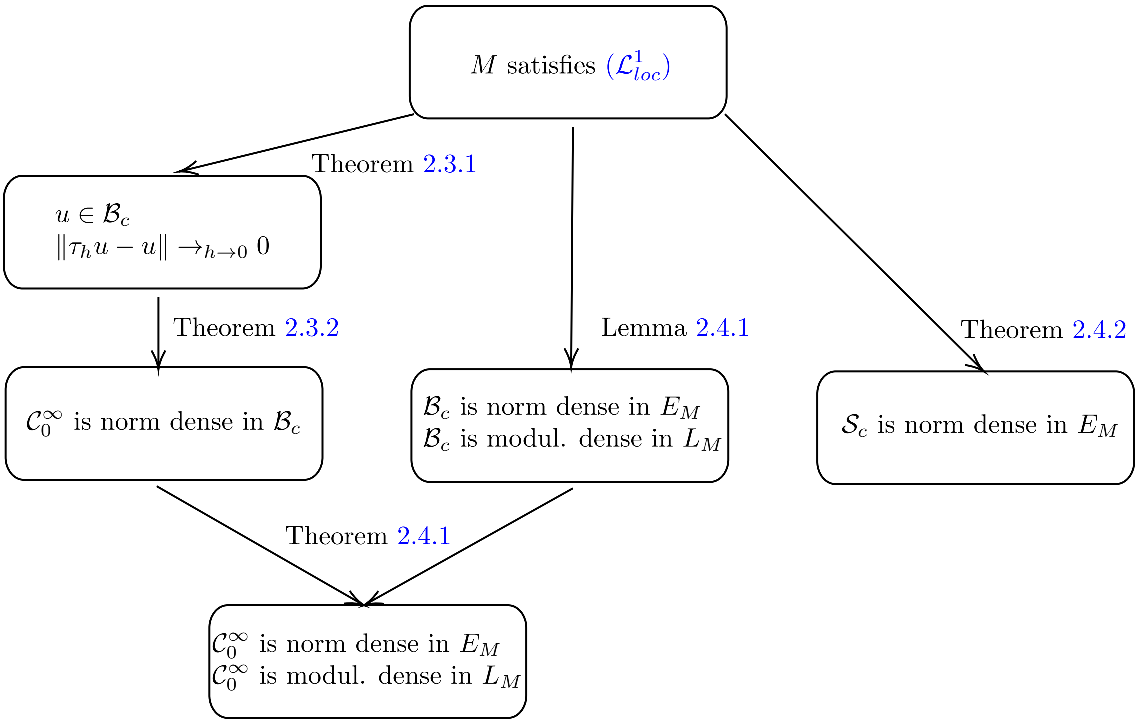

FIGURE 2.2: Some density results in Musielak spaces
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Chapter 3

Topological properties and applications of the approximation results

3.1 Introduction

The aim of this chapter is to extend some topological properties of classical Lebesgue and Orlicz spaces to the Musielak spaces. Using the foregoing approximation results obtained in Chapter 2 we prove here some basic function space properties that constitute essential tools needed in the existence theory for partial differential equations involving nonstandard growths described in terms of Musielak functions (see Part II). Such results requires to take into account the results earlier studied deeply in the monographs [69, 82] and those of the particular framework of variable Lebesgue and Sobolev spaces concerning completeness, density, reflexivity and separability obtained by Kovácik and Rákosník [67].

The chapter deals with Section 3.2 where we consider the Δ_2-condition and discuss the relation between some types of convergences in Musielak spaces. Sections 3.3 and 3.4 are respectively devoted to the proof of some topological properties of Musielak and Musielak-Sobolev spaces, namely duality, separability and reflexivity properties.

In the Orlicz spaces, the density of simple functions, denoted $\mathcal{S}$, in $E_M(\Omega)$ is an important step in the proof of the duality result (see for instance [2, Theorem 8.19]). In the Musielak spaces such approximation result needs to assume a local integrability condition with respect to all measurable finite Lebesgue measure subset of Ω (2.1) (see [82, Theorem 7.6]) which allows to get the inclusion $\mathcal{S} \subset E_M(\Omega)$. In Theorems 3.3.1 and 3.4.2 we prove respectively the duality results in Musielak and Musielak-Sobolev spaces under the weaker condition ($\mathcal{L}^1_{loc}$) and the proof of Theorem 3.3.1 act otherwise, compared to the classic case of Lebesgue spaces, by using the density of the set S_c of simple functions compactly supported in Ω (see Lemma 2.4.2) instead of simple functions only.

In the framework of classical Lebesgue and Orlicz spaces, the separability of the closure of bounded functions compactly supported in $\overline{\Omega}$ is well-known, see for instance [2, Theorem 2.21 and Theorem 8.21], while for bounded variable exponent spaces one can see [67, corollary 2.12]. Here, using the density results obtained in Theorem 2.4.1 we prove in Theorem 3.3.2 and Theorem 3.4.3 respectively the

separability of $E_M(\Omega)$ and $W^mE_M(\Omega)$. The proof of Theorem 3.3.2 is totally different from that given by Musielak [82, Theorem 7.10], since the author used the density of simple functions assuming that the Φ-function M satisfies the local integrability condition for all measurable finite Lebesgue measure subset of Ω (2.1), which is stronger than the condition $(\mathcal{L}^1_{loc})$ when we limit ourselves to the Borel tribe.
Furthermore, in Theorems 3.3.3 and 3.4.4 we give sufficient conditions under which the reflexivity of Musielak and Musielak-Sobolev spaces holds.

3.2 The Δ_2-condition

A special class of Φ-functions if the following.

Definition 3.2.1 (Δ_2-condition). *We say that M satisfies the Δ_2-condition, written $M \in \Delta_2$, if there is a constant $k > 0$ such that*

$$M(x, 2s) \le kM(x, s) + h(x) \tag{3.1}$$

for all $s \ge 0$ and almost every $x \in \Omega$, where h is a nonnegative, integrable function in Ω.

Remark 3.2.1. *The hypothesis (3.1) seems to be natural in the sense that if the Φ-function M is independent of the variable x, i.e. $M\colon [0, \infty) \to [0, \infty]$ is an N-function (see [2]), it is equivalent to the formulation of the Δ_2-condition considered in the Orlicz spaces setting. That is there exist two constants $c > 0$ and $t_0 > 0$ such that*

$$M(2t) \le cM(t) \tag{3.2}$$

for every $t \ge t_0$ or $t \ge 0$ according to whether Ω has finite Lebesgue measure or not successively. Indeed, assume that $|\Omega| < +\infty$, where from now on $|E|$ stands for the Lebesgue measure of a subset E of $\mathbb{R}^N$. Suppose that (3.1) holds but (3.2) is not true. Thus, there exists a sequence $t_n \ge t_0$ such that $t_n \to +\infty$ satisfying $M(2t_n) > 2^nM(t_n)$. Then, by (3.1) we have

$$\int_\Omega h(x)dx \ge (2^n - k)M(t_n)|\Omega|.$$

The passage to the limit as n tends to infinity yields a contradiction with the fact that $h \in L^1(\Omega)$. Conversely, assume that (3.2) is fulfilled. The function f defined by $f(t) = M(2t) - cM(t)$ is continuous and so we get $0 \le \max\limits_{0\le t\le t_0} f(t) = C < +\infty$. Therefore, we obtain

$$M(2t) \le cM(t) + C, \text{ for all } t \ge 0.$$

Hence, we can choose $h(x) = C$. Suppose now that $|\Omega| = +\infty$. Assume that (3.1) holds but (3.2) is not true. So that for $c = k$, there is $s \ge 0$ satisfying

$M(2s) > kM(s)$. From (3.1) we can write $h(x) \geq M(2t) - kM(t)$ for all $t \geq 0$. Substituting t by s and then integrating over Ω one has

$$\int_\Omega h(x)dx \geq (M(2s) - kM(s))|\Omega| = +\infty,$$

which contradicts the fact that $h \in L^1(\Omega)$. Conversely, if (3.2) is true then (3.1) holds for $h = 0$.

Definition 3.2.2. *A Φ-function M is said to satisfy the weak Δ_2-condition, written $M \in \Delta_{2,w}$, if for every $u_n \in L_M(\Omega)$*

$$\int_\Omega M(x, u_n)dx \to 0 \text{ implies } \int_\Omega M(x, 2u_n)dx \to 0 \text{ as } n \to \infty.$$

Lemma 3.2.1. *Let $M \in \Phi$. Then the norm convergence and the modular convergence are equivalent if and only if $M \in \Delta_{2,w}$.*

Proof. Suppose that the norm convergence and the modular convergence are equivalent and let $u_n \in L_M(\Omega)$ be such that $\int_\Omega M(x, u_n)dx \to 0$. Then, $\|u_n\|_M \to 0$ as $n \to \infty$. By the definition of the norm, we can get

$$\int_\Omega M(x, 2u_n)dx \leq 2\|u_n\|_M.$$

Thus, $\int_\Omega M(x, 2u_n)dx \to 0$ as $n \to \infty$.

Conversely, let $u_n \in L_M(\Omega)$ and suppose that $\int_\Omega M\Big(x, \frac{|u_n|}{\lambda}\Big)dx \to 0$ for some $\lambda > 0$. We shall prove that $\int_\Omega M(x, \mu|u_n|)dx \to 0$ as $n \to \infty$ for every $\mu > 0$. Let $m \in \mathrm{N}$ be such that $\lambda\mu \leq 2^m$. Then, we can write

$$\int_\Omega M(x, \mu|u_n|)dx \leq \frac{\lambda\mu}{2^m} \int_\Omega M\Big(x, 2^m\frac{|u_n|}{\lambda}\Big)dx \to 0 \text{ as } n \to \infty.$$

□

Lemma 3.2.2. *Let $M \in \Phi$. If $M \in \Delta_2$, then the norm convergence and the modular convergence are equivalent.*

This lemma was proved in [82, Theorem 8.14] using the local integrability (see [82, Definition 7.5]). This local integrability is not necessary, we give here a proof based on Vitali's theorem.

Proof. We only need to prove that the modular convergence implies the norm convergence, the converse is an easy task. Let $\{u_n\}$ be a sequence of functions belonging to $L_M(\Omega)$ such that $\displaystyle\int_\Omega M(x, u_n(x)/\lambda) \to 0$ as $n \to \infty$, for some $\lambda > 0$. Thus, $M(x, u_n(x)/\lambda) \to 0$ strongly in $L^1(\Omega)$. Hence, for a subsequence still indexed by n, we can assume that $u_n \to u$ a.e. in Ω. Let p be a fixed integer, by the Δ_2-condition we can write

$$M(x, 2^p u_n(x)/\lambda) \leq k^p M(x, u_n(x)/\lambda) + (k^{p-1} + \cdots + k + 1)h(x).$$

Therefore, by Vitali's theorem we get $\displaystyle\lim_{n\to\infty} \int_\Omega M(x, 2^p u_n) = 0$. For an arbitrary $\lambda > 0$, there exists m such that $\lambda \leq 2^m$. Then we can write

$$\int_\Omega M(x, \lambda u_n)dx \leq \frac{\lambda}{2^m} \int_\Omega M(x, 2^m u_n)dx \to 0 \text{ as } n \to \infty,$$

which gives $\|u_n\|_M \to 0$ as $n \to \infty$. □

Lemma 3.2.3. *Let $M \in \phi$, then the following assertions are equivalent*

i) $E_M(\Omega) = L_M(\Omega)$.

ii) $M \in \Delta_2$.

Proof. i)⇒ ii) For any $u \in L_M(\Omega)$ we have $2u \in L_M(\Omega)$. Then it follows

$$L_M(\Omega) \subset L_{\widetilde{M}}(\Omega), \quad \text{where} \quad \widetilde{M}(x, u) = M(x, 2u).$$

Therefore, by [82, Theorem 8.5 (b)] there exist a constant $k > 0$ and a nonnegative function $h \in L^1(\Omega)$ such that

$$\widetilde{M}(x, u) = M(x, 2u) \leq kM(x, u) + h(x).$$

ii)⇒ i) For $u \in L_M(\Omega)$ there exists $\lambda > 0$ such that $\displaystyle\int_\Omega M(x, |u(x)|/\lambda)dx < \infty$. We shall prove that for every $\mu > 0$ we have $\displaystyle\int_\Omega M(x, |u(x)|/\mu)dx < \infty$. Indeed, there exists an integer m such that $\dfrac{\lambda}{\mu} \leq 2^m$. Thus, we can write

$$\begin{aligned}
&\int_\Omega M(x, |u(x)|/\mu)dx \\
&\leq \frac{\lambda}{2^m \mu} \int_\Omega M\Big(x, \frac{2^m |u(x)|}{\lambda}\Big)dx \\
&\leq \frac{\lambda}{2^m \mu} \left(k^m \int_\Omega M\Big(x, \frac{|u(x)|}{\lambda}\Big)dx + (k^{m-1} + \cdots + k + 1) \int_\Omega h(x)dx \right) < \infty.
\end{aligned}$$

□

Remark 3.2.2. *Unlike the Orlicz framework, the equality $E_M(\Omega) = L_M(\Omega)$ **does not imply** that the Φ-function M satisfies $M(x, 2s) \leq kM(x, s)$ for all $s \geq 0$ and almost every $x \in \Omega$ (see [52, Example 4.2]).*

3.3 Musielak spaces

3.3.1 Duality results

In this subsection we state our main duality result Theorem 3.3.1. But before we come to Theorem 3.3.1, we give the following auxiliary results.

Lemma 3.3.1. *Assume that $M \in \Phi$ satisfies $(\mathcal{L}^1_{loc})$. For every nonempty subset $E \subset K$ where K is a compact subset of Ω, there exist two constant numbers c_1, $c_2 \geq 0$ such that*

$$\|\chi_E\|_M \leq \frac{1}{M^{-1}(c_1, \frac{c_2}{|E|})}. \tag{3.3}$$

Proof. Let $x_0 \in \Omega$ be fixed. By the assumption $(\mathcal{L}^1_{loc})$, the measurable function $x \mapsto M\Big(x, M^{-1}(x_0, \dfrac{1}{2|E|})\chi_E\Big)$ belongs to $L^1(\Omega)$. Hence, there is an $\eta > 0$ such that for any measurable subset Ω' of Ω, one has

$$|\Omega'| < \eta \Rightarrow \int_{\Omega'} M\Big(x, M^{-1}(x_0, \frac{1}{2|E|})\chi_E\Big)dx < \frac{1}{2}.$$

As $M(\cdot, s)$ is measurable on E, Luzin's theorem implies that for $\eta > 0$ there exists a closed set $F_\eta \subset E$ such that the restriction of $M(\cdot, s)$ to F_η is continuous and $|E \setminus F_\eta| < \eta$. Let k be the point where the supremum of $M(\cdot, s)$ is reached in the set F_η.

$$\begin{aligned}\int_E M\Big(x, M^{-1}(k, \frac{1}{2|E|})\Big)dx &= \int_{F_\eta} M\Big(x, M^{-1}(k, \frac{1}{2|E|})\Big)dx \\ &\quad + \int_{E\setminus F_\eta} M\Big(x, M^{-1}(k, \frac{1}{2|E|})\Big)dx.\end{aligned}$$

For the first term in the right-hand side of the last equality, we use (1.42) obtaining

$$\int_{F_\eta} M\Big(x, M^{-1}(k, \frac{1}{2|E|})\Big)dx \leq \int_{F_\eta} M\Big(k, M^{-1}(k, \frac{1}{2|E|})\Big)dx \leq \frac{1}{2},$$

while for the second one, since $|E \setminus F_\eta| < \eta$ we have

$$\int_{E\setminus F_\eta} M\Big(x, M^{-1}(k, \frac{1}{2|E|})\Big)dx \leq \frac{1}{2}.$$

Thus, we get

$$\int_\Omega M\Big(x, M^{-1}(k, \frac{1}{2|E|})\chi_E\Big)dx \le 1.$$

□

Remark 3.3.1. *Formula (3.3) remains valid either for every nonempty bounded subset E of Ω or for every nonempty subset E of Ω if Ω is a bounded open.*

Lemma 3.3.2. *Let M and M^* be two complementary Φ-functions. Let $v \in L_{M^*}(\Omega)$ be a fixed function. Define*

$$L_v(u) = \int_\Omega u(x)v(x)dx, \ \forall u \in L_M(\Omega). \tag{3.4}$$

Then L_v defines a linear continuous functional on $L_M(\Omega)$. Furthermore,

$$\|v\|_{M^*} \le \|L_v\| \le 2\|v\|_{M^*},$$

where $\|L_v\| = \sup\{|L_v(u)|, \|u\|_M \le 1\}$.

Proof. We omit the proof, since it's similar to the one given in [2, Lemma 8.17] in the framework of Orlicz spaces. □

Remark 3.3.2. *The above result holds also when L_v is restricted to $E_M(\Omega)$. In general, continuous linear functionals defined on $L_M(\Omega)$ can be expressed in a different form to that defined in (3.4) (see [70, Theorem 3.13.5]). Hence, we can not have the Riesz representation theorem as in classical Lebesgue spaces. In the following lemma we give an "almost complete" analogue of Riesz representation theorem. A similar result in the Orlicz framework can be found in [70, Theorem 3.13.6].*

Lemma 3.3.3. *Assume that $M \in \Phi$ satisfies ($\mathcal{L}^1_{loc}$) and let $L \in [E_M(\Omega)]'$. Then there exists a unique function $v \in L_{M^*}(\Omega)$ such that*

$$L(u) = \int_\Omega u(x)v(x)dx, \quad \forall u \in E_M(\Omega). \tag{3.5}$$

Proof. We begin first by assuming that u belongs to $\mathcal{S}_c$ i.e. u is of the form $\sum_{i=1}^{p} \alpha_i \chi_{B_i}(x)$ where B_i are measurable sets of finite Lebesgue measure such that $\cup_{i=1}^{p} B_i \subset K$ for some compact subset $K \subset \Omega$ and $\alpha_i \in \mathbb{R}$, for $i = 1, 2, \cdots, p$. Let μ be the complex measure defined for a measurable set of finite Lebesgue measure $A \subset K \subset \Omega$ for some compact K as follows

$$\mu(A) = L(\chi_A).$$

By (3.3) there exist two constants c_1, $c_2 \geq 0$ such that

$$|\mu(A)| \leq \|L\| \|\chi_A\|_M \leq \frac{\|L\|}{M^{-1}(c_1, c_2/|A|)} \to 0 \text{ as } |A| \to 0.$$

Thus, the measure μ is absolutely continuous with respect to the Lebesgue measure and it follows by Radon-Nikodym's theorem that there exists a nonnegative measurable function $v \in L^1(\Omega)$, unique up to sets of Lebesgue measure zero, such that

$$\mu(A) = \int_A v(x)dx.$$

Hence, we can write

$$\begin{aligned} L(u) &= \sum_{i=1}^{p} \alpha_i L(\chi_{B_i}) = \sum_{i=1}^{p} \alpha_i \mu(B_i) = \sum_{i=1}^{p} \alpha_i \int_{B_i} v(x)dx \\ &= \sum_{i=1}^{p} \alpha_i \int_\Omega v(x)\chi_{B_i} dx = \int_\Omega u(x)v(x)dx. \end{aligned} \tag{3.6}$$

Now for arbitrary $u \in E_M(\Omega)$, by Lemma 2.4.2 we can found a sequence of functions $u_j \in \mathcal{S}_c$ such that $u_j \to u$ a.e. in Ω and strongly in $E_M(\Omega)$. Therefore, by Fatou's lemma we obtain

$$\begin{aligned} \left|\int_\Omega u(x)v(x)dx\right| &\leq \liminf_{j\to+\infty} \int_\Omega |u_j(x)v(x)|dx = \liminf_{j\to+\infty} L(|u_j|\text{sgn}v) \\ &\leq \|L\| \liminf_{j\to+\infty} \|u_j\|_M \leq \|L\| \|u\|_M. \end{aligned}$$

This implies that $v \in L_{M^*}(\Omega)$. Let $L_v(u) = \displaystyle\int_\Omega u(x)v(x)dx$, the linear functional defined by (3.4). By (3.5), L_v and L coincide on the set $\mathcal{S}_c$ and by Lemma 2.4.2, they coincide everywhere in $E_M(\Omega)$. $\square$

Theorem 3.3.1. *Let $M \in \Phi$ satisfy ($\mathcal{L}^1_{loc}$) and let M^* stands for it's complementary function. Then, the dual space $(E_M(\Omega))'$ of $E_M(\Omega)$ is isomorphic to $L_{M^*}(\Omega)$, denote $(E_M(\Omega))' \simeq L_{M^*}(\Omega)$.*

Proof. It is immediate that from Lemma 3.3.3, we get the isomorphism

$$L_{M^*}(\Omega) \simeq [E_M(\Omega)]'.$$

$\square$

3.3.2 Separability and reflexivity results

We begin this subsection by considering the following result.

Theorem 3.3.2. *Assume that $M \in \Phi$ satisfies ($\mathcal{L}^1_{loc}$). Then, the space $E_M(\Omega)$ is separable.*

Proof. Let $u \in E_M(\Omega)$. By virtue of Theorem 2.4.1, we can assume that $u \in \mathcal{C}_0(\Omega)$ (the set of continuous functions with compact support in Ω). Hence, it's sufficient to show that there exists a countable set dense in $\mathcal{C}_0(\Omega)$ with respect to the strong topology in $E_M(\Omega)$. Let Ω_n be the sequence of compact subsets of $\mathbb{R}^N$ defined by

$$\Omega_n = \{x \in \Omega : |x| \leq n \text{ and } dist(x, \partial\Omega) \geq \frac{1}{n}\}.$$

Recall that $\Omega = \cup_{i=1}^{+\infty}\Omega_n$. Let $\mathcal{P}$ be the set of all polynomials on $\mathbb{R}^N$ with rational coefficients and

$$\mathcal{P}_n = \{v\chi_{\Omega_n} : v \in \mathcal{P}\},$$

where χ_{Ω_n} is the characteristic function of Ω_n. If $dist(supp\ u, \partial\Omega) > \dfrac{1}{n}$ then u belongs to $\mathcal{C}(\Omega_n)$ and by using the density of $\mathcal{P}_n$ in $\mathcal{C}(\Omega_n)$ [2, corollary 1.32], there exists a sequence $u_j \in \mathcal{P}_n$ that converges uniformly to u, that is to say

$$\forall\varepsilon > 0, \exists j_0 \in \mathrm{N}, \forall j \geq j_0, \sup_{x\in\Omega_n} |u_j(x) - u(x)| \leq \frac{\varepsilon}{\displaystyle\int_{\Omega_n} M(x, 1)dx + 1}.$$

So that for every $\varepsilon > 0$, there exists $j_0 \in \mathrm{N}$ such that for any $j \geq j_0$, one has

$$\int_{\Omega_n} M\Big(x, \frac{|u(x) - u_j(x)|}{\varepsilon}\Big)dx \leq 1.$$

Therefore, $\mathcal{P}_n$ is dense in $\mathcal{C}(\Omega_n)$ for the strong topology in $L_M(\Omega)$. Consequently, the countable set $\cup_{n=1}^{\infty}\mathcal{P}_n$ is dense in $E_M(\Omega)$ and $E_M(\Omega)$ is separable. □

Remark 3.3.3. *In view of Theorem 3.3.2 and Lemma 3.2.3, if the Φ-function $M \in \Delta_2$ and satisfies ($\mathcal{L}^1_{loc}$) then $L_M(\Omega)$ is a separable space.*

We conclude this section by the following reflexivity theorem.

Theorem 3.3.3. *Let M, $M^* \in \Phi$ be a pair of complementary Φ-functions satisfying both ($\mathcal{L}^1_{loc}$) and the Δ_2-condition. Then, the Musielak space $L_M(\Omega)$ is reflexive.*

Proof. Since M and M^* satisfy both the Δ_2-condition, by Lemma 3.2.3 and Theorem 3.3.1 we get

$$L_{M^*}(\Omega) \simeq [L_M(\Omega)]' \text{ and } L_M(\Omega) \simeq [L_{M^*}(\Omega)]'$$

and then the conclusion follows. □

3.4 Musielak-Sobolev spaces

Definition 3.4.1. *Let $M \in \Phi$ satisfies $(\mathcal{L}^1_{loc})$.*

- *We denote by $W_0^m E_M(\Omega)$ the norm closure of $\mathcal{C}_0^\infty(\Omega)$ in $W^m L_M(\Omega)$.*
- *If furthermore, M^* satisfies $(\mathcal{L}^1_{loc})$ then we define the space $W_0^m L_M(\Omega)$ as the closure of $\mathcal{C}_0^\infty(\Omega)$ with respect to the weak-* topology $\sigma(\Pi L_M, \Pi E_{M^*})$.*

Remark 3.4.1. *1. Note here that, in Definition 3.4.1, since M satisfies $(\mathcal{L}^1_{loc})$, $\mathcal{C}_0^\infty(\Omega)$-functions are contained in $W^m L_M(\Omega)$ and so the space $W_0^m E_M(\Omega)$ is well defined. Moreover if M^* satisfies $(\mathcal{L}^1_{loc})$ then weak-* topology $\sigma(\Pi L_M, \Pi E_{M^*})$ is well defined (see Theorem 3.3.1) and so the space $W_0^m L_M(\Omega)$.*

2. If the pair of complementary Φ-function (M, M^*) satisfy both the Δ_2-condition and $(\mathcal{L}^1_{loc})$, we have

$$E_{M^*} \overset{M^* \in \Delta_2}{=\!=\!=} L_{M^*} \overset{M \text{ satisfies } (\mathcal{M}1)}{=\!=\!=\!=\!=\!=} (E_M)^* \overset{M \in \Delta_2}{=\!=\!=} (L_M)^*$$

and so the topologies

$$\sigma(L_M, E_{M^*}) = \sigma(L_M, L_{M^*}) = \sigma(L_M, (E_M)^*) = \sigma(L_M, (L_M)^*)$$

agree then. Then Mazur's lemma implies that the closure of $\mathcal{C}_0^\infty(\Omega)$ with respect to the mentioned topologies is equal to norm one in $W^m L_M(\Omega)$.

We summarize Remark 3.4.1 in the following figure.

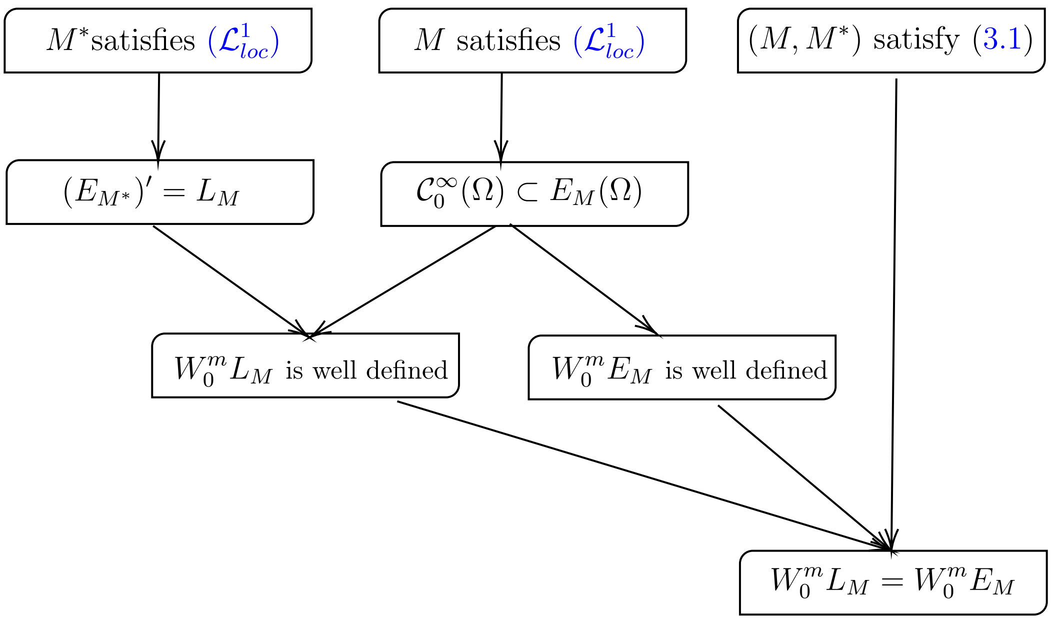

FIGURE 3.1: Relationship between $W_0^m L_M$ and $W_0^m E_M$

3.4.1 A completeness result

The main result of this subsection is Theorem 3.4.1.

Lemma 3.4.1. *Let Ω be a subset of $\mathbb{R}^N$ and $M^* \in \Phi$ satisfies $(\mathcal{L}^1_{loc})$. Then we get the following continuous embedding*

$$L_M(\Omega) \hookrightarrow L^1_{loc}(\Omega).$$

Proof. Let K be an arbitrary compact subset of Ω. For any function $u \in L_M(\Omega)$, we can write using Hölder's inequality

$$\int_K |u(x)|dx \leq 2\|u\|_M \|\chi_K\|_{M^*}.$$

Since M^* satisfies $(\mathcal{L}^1_{loc})$, it is easy to see that

$$\|\chi_K\|_{M^*} \leq \int_K M^*(x,1)dx + 1 < \infty,$$

which yields $L_M(\Omega) \subset L^1_{loc}(\Omega)$. By the same we can get $L_M(\Omega) \subset L^1(\Omega)$. ◻

Lemma 3.4.2. *Let Ω be an open subset of $\mathbb{R}^N$ with finite Lebesgue measure and let (M, M^*) be a pair of complementary Φ-functions. Then we get the following continuous embedding*

$$L_M(\Omega) \hookrightarrow L^1(\Omega)$$

if one of the following conditions holds

1. *M satisfies (2.2).*
2. *M satisfies*

$$\operatorname{ess\,inf}_{x\in\Omega} M(x,1) \geq c > 0. \tag{3.7}$$

Proof. 1. Assume first that $u \neq 0$. By the uniform super linearity assumption (2.2) there exists a constant number $t_0 \geq 0$ such that for all $t \geq t_0$

$$t \leq M(x,t), \quad \text{for every } x \in \Omega.$$

Hence, for all $t \geq 0$ one has $t \leq M(x,t) + t_0$. Substituting t by $\dfrac{|u|}{\|u\|_M}$ and integrating over Ω we get

$$\|u\|_{L^1(\Omega)} \leq (1 + t_0|\Omega|)\|u\|_\Omega.$$

The above inequality trivially holds if $u = 0$.

2. Let $u \in L_M(\Omega)$ and set $\Omega' = \{x \in \Omega, \dfrac{|u(x)|}{\|u\|_M} \geq 1\}$, then by (3.7) we have

$$s \leq \frac{1}{c} M(x,s) \text{ for all } s \geq 1,$$

hence

$$\begin{aligned}\int_\Omega \frac{|u(x)|}{\|u\|_M} dx &= \int_{\Omega'} \frac{|u(x)|}{\|u\|_M} dx + \int_{\Omega \setminus \Omega'} \frac{|u(x)|}{\|u\|_M} dx \\ &\leq \frac{1}{c} \int_{\Omega'} M\big(x, \frac{|u(x)|}{\|u\|_M}\big) dx + |\Omega| \\ &\leq \frac{1}{c} + |\Omega|.\end{aligned}$$

thus,

$$\|u\|_{L^1(\Omega)} \leq \Big(\frac{1}{c} + |\Omega|\Big) \|u\|_\Omega.$$

□

Theorem 3.4.1. *Let (M, M^*) be a pair of complementary Φ-functions. Assume that M^* satisfies $(\mathcal{L}^1_{loc})$, then the spaces $W^m L_M(\Omega)$ and $W^m E_M(\Omega)$ endowed with the Luxemburg norm are Banach spaces.*

Proof. Let $\{u_n\}$ be a Cauchy sequence in $W^m L_M(\Omega)$, thus each $D^\alpha u_n |\alpha| \leq m$ is a Cauchy sequence in $L_M(\Omega)$. Hence by Theorem 1.4.1, $L_M(\Omega)$ is a Banach and so there exist a functions u and u_α such that $u_n \to u$ and $D^\alpha u_n \to u_\alpha$ in norm in $L_M(\Omega)$ as $n \to \infty$. Furthermore, in view of Lemma 3.4.1, $L_M(\Omega) \hookrightarrow L^1_{loc}(\Omega)$. Consequently, the functional

$$T_{u_n}(\psi) = \langle T_{u_n}, \psi \rangle = \int_\Omega u_n(x)\psi(x) dx, \forall \psi \in \mathcal{C}_0^\infty(\Omega),$$

defines a regular distribution and so by Hölder inequality we obtain

$$|T_{u_n}(\psi) - T_u(\psi)| \leq \int_\Omega |u_n(x) - u(x)||\psi(x)| dx \leq \|u_n(x) - u(x)\|_M |\psi(x)|_{M^*,\Omega}.$$

Therefore, $T_{u_n}(\psi) \to T_u(\psi)$ for all $\psi \in \mathcal{C}_0^\infty(\Omega)$ as $n \to \infty$ and by the same $T_{D^\alpha u_n}(\psi) \to T_{u_\alpha}(\psi)$ for all $\psi \in \mathcal{C}_0^\infty(\Omega)$, hence

$$T_{u_\alpha}(\psi) = \lim_{n\to\infty} T_{D^\alpha u_n}(\psi) = \lim_{n\to\infty} (-1)^{|\alpha|} T_{u_n}(D^\alpha \psi) = (-1)^{|\alpha|} T_u(D^\alpha \psi)$$

for all $\psi \in \mathcal{C}_0^\infty(\Omega)$. Thus $D^\alpha u = u_\alpha$, $|\alpha| \leq m$, in the distributional sens on Ω. Hence $W^m L_M(\Omega)$ is a Banach space and by a similar argument $W^m E_M(\Omega)$ is also a Banach space. □

Remark 3.4.2. *Note that, in general, if $L_M(\Omega) \subset L^1_{loc}(\Omega)$ (resp. $E_M(\Omega) \subset L^1_{loc}(\Omega)$), then $W^1L_M(\Omega)$ (resp. $W^1E_M(\Omega)$) endowed with the Luxemburg norm is a Banach space.*

3.4.2 Duality results

The main result of this subsection is Theorem 3.4.2, where we give an identification of the dual of $W_0^mE_M(\Omega)$ with a subspace of $(\mathcal{C}_0^\infty(\Omega))'$. As a starting point we give the following Lemmas

Lemma 3.4.3. *Let $(M, M^*) \in \Phi$ be a pair of complementary Φ-function. Assume that satisfies ($\mathcal{L}^1_{loc}$). Then, for every $L \in (\Pi E_M(\Omega))'$ there exists a unique $v = (v_\alpha)_{|\alpha|\leq m} \in \Pi L_{M^*}(\Omega)$ such that*

$$L(u) = \sum_{|\alpha|\leq m} \langle u_\alpha, v_\alpha \rangle = \sum_{|\alpha|\leq m} \int_\Omega u_\alpha(x)v_\alpha(x)dx, \ \forall u = (u_\alpha)_{|\alpha|\leq m} \in \Pi E_M(\Omega),$$

and the following Hölder inequality holds

$$|\sum_{|\alpha|\leq m} \langle u_\alpha, v_\alpha \rangle| \leq 2\|u\|_{\Pi L_{M^*}}\|v\|_{\Pi L_M}.$$

Proof. The proof flows easily from Lemma 3.3.3. □

The next lemma is an extension to Musielak spaces of the one given in [2, Theorem 3.9].

Lemma 3.4.4. *Let $(M, M^*) \in \Phi$ be a pair of complementary Φ-function. Assume that M satisfies ($\mathcal{L}^1_{loc}$). Then for every $L \in (W^mE_M(\Omega))'$ there exists $v = (v_\alpha)_{|\alpha|\leq m} \in \Pi L_{M^*}(\Omega)$ such that*

$$L(u) = \sum_{0\leq|\alpha|\leq m} \langle D^\alpha u, v_\alpha \rangle, \forall u \in W^mE_M(\Omega). \tag{3.8}$$

Moreover

$$\|L\|_{[W^mE_M(\Omega)]'} \leq 2\|v\|_{\Pi L_{M^*}}\}.$$

Proof. Define $P : W^mE_M(\Omega) \to \Pi E_M(\Omega)$ with $P(u) = (D^\alpha u)_{|\alpha|\leq m}$ for every $u \in W^mE_M(\Omega)$. Let $X = P(W^mE_M(\Omega))$ and consider the functional $\widetilde{L}$ defined on X as follows

$$\widetilde{L}(P(u)) = L(u), \ \forall u \in W^mE_M(\Omega).$$

Let P the isometric isomorphism of $W^mE_M(\Omega)$ onto the closed subspace X of $\Pi E_M(\Omega)$. We easily check that $\widetilde{L} \in X'$ and $\|\widetilde{L}\|_{X'} = \|L\|_{\left(W^mE_M(\Omega)\right)'}$. Thus, by the Hahn-Banach extension Theorem there exists a norm preserving extension $\widehat{L}$

of $\widetilde{L}$ to all of $\Pi E_M(\Omega)$. Then, in view of Lemma 3.4.3 there exists a unique, up to sets of null Lebesgue measure, $v = (v_\alpha)_{|\alpha|\leq m} \in \Pi L_{M^*}(\Omega)$ such that for every $u = (u_\alpha)_{|\alpha|\leq m} \in \Pi E_M(\Omega)$

$$\widehat{L}(u) = \sum_{0\leq|\alpha|\leq m} \langle u_\alpha, v_\alpha\rangle.$$

Thus, for $u \in W^m E_M(\Omega)$ we get

$$L(u) = \widetilde{L}(P(u)) = \widehat{L}(P(u)) = \sum_{0\leq|\alpha|\leq m} \langle D^\alpha u, v_\alpha\rangle.$$

Furthermore,

$$\|L\|_{\left(W^m E_M(\Omega)\right)'} = \|\widetilde{L}\|_{X'} = \|\widehat{L}\|_{\left(\Pi E_M(\Omega)\right)'} \leq \|v\|_{\Pi L_{M^*}}.$$

Now since any element $v \in \Pi L_{M^*}(\Omega)$ for which (3.8) holds for every $u \in W^m E_M(\Omega)$ corresponds to an extension L of $\widetilde{L}$ and so will have norm $\|v\|_{\Pi L_{M^*}}$ superior than $\|L\|_{\left(W^m E_M(\Omega)\right)'}$. □

The next Lemma is generalization of classical Lebesgue space see [2, page 63].

Lemma 3.4.5. *Let $(M, M^*) \in \Phi$ be a pair of complementary Φ-function. Assume that M satisfies $(\mathcal{L}^1_{loc})$. Then, every $L \in (W^m E_M(\Omega))'$ (resp. $L \in (W^m_0 E_M(\Omega))'$) is an extension to $W^m E_M(\Omega)$ of distributions $T \in (\mathcal{C}^\infty_0(\Omega))'$ having the form in (3.10), that is*

$$T = \sum_{|\alpha|\leq m} (-1)^{|\alpha|} D^\alpha v_\alpha, \tag{3.9}$$

where $v = (v_\alpha)_{|\alpha|\leq m} \in \Pi L_{M^*}$.

Proof. By using Lemma 3.4.1 we have $L_{M^*}(\Omega) \subset L^1_{loc}(\Omega)$. Hence, for $f \in L_{M^*}(\Omega)$ the function

$$T_f(\varphi) = \int_\Omega f(x)\varphi(x)dx, \qquad \forall\varphi \in \mathcal{C}^\infty_0(\Omega),$$

defines a regular distribution which we will confuse with f. Let $L \in (W^m E_M(\Omega))'$. By Lemma 3.4.4 there exists a unique $v = (v_\alpha)_{|\alpha|\leq m} \in \Pi L_{M^*}$ such that

$$L(u) = \sum_{|\alpha|\leq m} \langle u_\alpha, v_\alpha\rangle = \sum_{|\alpha|\leq m} \int_\Omega u_\alpha(x)v_\alpha(x)dx, \ \forall u = (u_\alpha)_{|\alpha|\leq m} \in \Pi E_M(\Omega).$$

Hence, for every $u \in W^m E_M(\Omega)$, we can write

$$L(u) = \sum_{|\alpha|\leq m} \int_\Omega D^\alpha u(x)v_\alpha(x)dx = \sum_{|\alpha|\leq m} (-1)^{|\alpha|} \int_\Omega D^\alpha v_\alpha(x)u(x)dx.$$

In particular, for $u \in \mathcal{C}_0^\infty(\Omega)$ we get $L(u) = T(u)$. Now, any $L \in (W_0^m E_M(\Omega))'$ can be extended to $W^m E_M(\Omega)$ by the Hahn-Banach extension theorem and the result follows. $\square$

Theorem 3.4.2. *Suppose that $M \in \Phi$ satisfies $(\mathcal{L}^1_{loc})$ and let M^*, it's complementary Φ-function. Then, the dual space of $W_0^m E_M(\Omega)$, denoted $(W_0^m E_M(\Omega))'$, is isometrically isomorphism to the space $W^{-m}L_{M^*}(\Omega)$ consisting of all distributions $T \in (\mathcal{C}_0^\infty(\Omega))'$ which can be written*

$$T = \sum_{|\alpha| \leq m} (-1)^{|\alpha|} D^\alpha f_\alpha, \tag{3.10}$$

for some $f = (f_\alpha)_{|\alpha|\leq m} \in \Pi L_{M^*}$. Moreover $\|T\|_{W^{-1}L_{M^*}(\Omega)} = \inf\big\{\|f\|_{\Pi L_{M^*(\Omega)}} : f$ satisfies (3.10)$\big\}$.

Proof. Let $L \in (W_0^m E_M(\Omega))'$. By Lemma 3.4.5, the restriction of L in $\mathcal{C}_0^\infty(\Omega)$ is a distribution of the form

$$T = \sum_{|\alpha| \leq m} (-1)^{|\alpha|} D^\alpha v_\alpha \tag{3.11}$$

where $v = (v_\alpha)_{|\alpha|\leq m} \in \Pi L_{M^*}$. Let now T any distribution of the form (3.11), then T has a unique extension $\widetilde{T}$ to $W_0^m E_M(\Omega)$. Indeed, for $u \in W_0^m E_M(\Omega)$ there exists $\varphi_n \in \mathcal{C}_0^\infty(\Omega)$ such that $\varphi_n \to u$ in norm in $W_0^m E_M(\Omega)$. Then, by Hölder's inequality we can write

$$\begin{aligned} |T(\varphi_k) - T(\varphi_n)| &\leq \sum_{|\alpha|\leq m} \int_\Omega |D^\alpha \varphi_k - D^\alpha \varphi_n||v_\alpha| dx \\ &\leq 2 \sum_{|\alpha|\leq m} \|D^\alpha \varphi_k - D^\alpha \varphi_n\|_M \|v_\alpha\|_{M^*} \\ &\leq 2\|\varphi_k - \varphi_n\|_{m,M} \|v\|_{\Pi L_{M^*}}. \end{aligned}$$

This implies that $\{T(\varphi_n)\}$ is a cauchy sequence in $\mathbb{C}$ which converges to the unique limit denoted by $\widetilde{T}(u)$. Let $\{\phi_n\}$ be another sequence in $\mathcal{C}_0^\infty(\Omega)$ with $\|\phi_n - u\|_{W^m E_M(\Omega)} \to 0$ as $n \to \infty$. Then $T(\varphi_n) - T(\phi_n) \to 0$ as $n \to \infty$. It's easy to see that the functional $\widetilde{T}$ is linear and in addition it is continuous. Actually, if $u = \lim_{n\to\infty} \varphi_n$ we get

$$|\widetilde{T}(u)| = \lim_{n\to\infty} |T(\varphi_n)| \leq 2 \lim_{n\to\infty} \|\varphi_n\|_{W_0^m E_M(\Omega)} \|v\|_{\Pi L_{M^*}} \leq 2\|u\|_{W_0^m E_M(\Omega)} \|v\|_{\Pi L_{M^*}}.$$

Hence $\widetilde{T}(u) = L(u)$. $\square$

Similarly to the case Orlicz space (see [42]) we define te flowing distributions space.

Definition 3.4.2. *Let $(M, M^*) \in \Phi$ be a pair of complementary Φ-functions. We denote by $W^{-1}E_{M^*}(\Omega)$ the following space of distributions*

$$W^{-1}E_{M^*}(\Omega) = \Big\{ f \in (\mathcal{C}_0^\infty(\Omega))', \ f = -div\ F, \quad with\ F \in (E_{M^*}(\Omega))^N \Big\}.$$

3.4.3 Separability and reflexivity properties

Theorem 3.4.3. *Assume that $M \in \Phi$ satisfies ($\mathcal{L}^1_{loc}$). Then, $W^m E_M(\Omega)$ is a separable space.*

Proof. By Theorem 3.3.2, the space $E_M(\Omega)$ is separable and so is $\Pi E_M(\Omega)$. Then by means of the operator P defined in (1.44) which is an isometric isomorphism of $W^m E_M(\Omega)$ onto $X = P(W^m E_M(\Omega)) \subset \Pi E_M(\Omega)$, X is a subspace of $\Pi E_M(\Omega)$. Thus X is separable and this is the case also for $W^m E_M(\Omega) = P^{-1}(X)$. □

Remark 3.4.3. *In view of Lemma 3.2.3, if the Φ-function $M \in \Delta_2$ and satisfies ($\mathcal{L}^1_{loc}$) then $W^m L_M(\Omega)$ is a separable space.*

We conclude this section by the following reflexivity theorem of the space $W^m L_M(\Omega)$.

Theorem 3.4.4. *Let $(M, M^*) \in \Phi$ be a pair of complementary Φ-functions satisfying both ($\mathcal{L}^1_{loc}$) and the Δ_2-condition. Then, the Musielak-Sobolev space $W^m L_M(\Omega)$ is reflexive.*

Proof. By Theorem 3.3.3, the space $L_M(\Omega)$ is reflexive and so is $\Pi L_M(\Omega)$. Then by means of the operator P defined in (1.44) which is an isometric isomorphism of $W^m E_M(\Omega)$ onto $X = P(W^m E_M(\Omega)) \subset \Pi E_M(\Omega)$ and since $W^m L_M(\Omega)$ is complete (see Theorem 3.4.1), X is a closed subspace of $\Pi E_M(\Omega)$. Thus X is reflexive. Consequently $W^m L_M(\Omega) = P^{-1}(X)$ is reflexive. □

3.5 Conclusion

We summarize in the following table the sufficient condition to get the main topological results of Musielak as well as Musielak Sobolev spaces obtained in this chapter.

Spaces	M satisfies $(\mathcal{L}^1_{loc})$	M^* satisfies $(\mathcal{L}^1_{loc})$	M satisfies Δ_2	M^* satisfies Δ_2	top. properties	
E_M, $W_0^m E_M$, $W^m E_M$	✓				separable	Theorems 3.3.2 and 3.4.3
L_M, $W^m L_M$	✓		✓			Theorems 3.3.2 and 3.4.3 and Lemma 3.2.3
$W_0^m L_M$	✓	✓	✓			$W_0^m L_M \subset W^m L_M$
E_M	✓				dual: $= L_{M^*}$	Theorem 3.3.1
L_M	✓		✓			Theorem 3.3.1 and Remark 3.3.3
E_M, L_M, $W^m L_M$, $W^m E_M$, $W_0^m L_M$, $W_0^m E_M$	✓	✓	✓	✓	reflexive	Theorem 3.3.3 and Theorem 3.4.4
$W_0^m E_M$	✓				dual: $= W^{-m} L_{M^*}$	Theorem 3.4.2
$W_0^m L_M$	✓	✓	✓	✓		Theorem 3.4.2, Lemma 3.2.3 and Mazur's lemma
$W^m L_M$, $W^m E_M$		✓			Banach	Theorem 3.4.1
$W_0^m L_M$,$W_0^m E_M$	✓	✓				Theorem 3.4.1 and Definition 3.4.1
NB: • If M satisfies $(\mathcal{L}_{hol})$ (resp. (4.8)), then the Φ-function M satisfies $(\mathcal{L}^1_{loc})$ (see Figure 4.1).						

TABLE 3.1: Some topological properties of Musielak and Musielak-Sobolev spaces.

Chapter 4

Approximation results in Musielak spaces in absence of Lavrentiev phenomenon

4.1 Introduction

In this chapter we provide new systematic conditions that guarantee the density of smooth functions in Musielak-Sobolev spaces upon a wide class of Φ-functions $M(x,s)$. We give density results provided that $M(x,s)$ is convex in s and φ-regular (i.e. $M(x,s) \leq \varphi(|x-y|,s)M(y,s)$ under certain type of regularity condition on φ). See conditions (($\mathcal{L}_{hol}$)) or ((4.8) and ($\mathcal{L}^p_{hol}$)) for details. Obviously, φ–regularity is skipped in the Orlicz setting (when $M = M(s)$).
In general, smooth functions are not dense in norm in this type of spaces. In the seminal paper [40] Gossez proves that weak derivatives in the Orlicz-Sobolev spaces are strong derivatives with respect to the modular topology. We extend the idea to the Musielak-Sobolev setting, where the modular function depends also on the spacial variable, i.e. $M = M(x,s)$.

Outline of the chapter. The chapter is organized as follows. In Section 4.2 we provide and discuss the new sufficient condition ($\mathcal{L}_{hol}$), in relation with Zhikov regularity and the log-Hölder continuity in variable exponent Sobolev spaces, and the conditions (4.8) and ($\mathcal{L}^p_{hol}$) in relation with Lavrentiev phenomenon, whereby we prove the density of smooth functions in Musielak-Sobolev spaces.
In Section 4.3 we give several mollification and convolution properties that constitute basic tools we need to prove the main results.
Section 4.4 is devoted to the density of smooth functions compactly supported in $\mathbb{R}^N$, $N \geq 1$, in Musielak-Sobolev spaces, the main result of this Section is Theorem 4.4.1 containing the density of $\mathcal{C}_0^\infty(\mathbb{R}^N)$ in norm in $W^mE_M(\mathbb{R}^N)$ and in $W^mL_M(\mathbb{R}^N)$ with respect to the modular topology, c.f. Definition 1.4.9. In the Orlicz spaces setting, the first density in Theorem 4.4.1 was proved by Donaldson and Trudinger in [30, Theorem 2.1], while in the case of the variable exponent Sobolev spaces, the corresponding result was proved in [92, Theorem 3] for bounded exponent satisfying the log-Hölder condition (10).

Section 4.5 address the approximation results in Ω. The main results of this Section are given by Theorem 4.5.1, Theorem 4.5.2 and Corollary 4.5.1. In the Orlicz-Sobolev framework, the second result of Theorem 4.5.1 was proved by Gossez in [40] assuming that Ω satisfies additionally the cone property. Such a property in [40] guarantees that any element of $W^m L_M(\Omega)$ with compact support in $\overline{\Omega}$ belongs to $W^{m-1}E_M(\Omega)$. The embedding and approximate results obtained in [30] allowed Gossez to prove only the convergence of smooth functions with compact support only for $|\alpha| = m$. Here, we extend the result to the more general setting of Musielak-Sobolev spaces and we enhance it by removing the cone property Theorem 2.4.1. Our approach is based on the mean continuity of the translation operator on the set of bounded functions compactly supported in Ω. While Theorem 4.5.2 and Corollary 4.5.1 are respectively the extension of [40, Theorem 4] and [37, Theorem 1.3] by Gossez in the case of Orlicz spaces.
In Section 4.6 we give a Meyer-Serrin type approximation results. The main results of this section are Theorem 4.6.1 and Theorem 4.6.2. In the first one we deal with the density with respect to the modular topology. Theorem 4.6.1 is an extension and refinement of [40, Theorem 1] obtained in the Orlicz framework. The second result concerns the density of smooth functions in the space $W^m E_M(\Omega)$ with respect to the strong topology in $W^m E_M(\Omega)$. In this generalization we unify and retrieve at a time the density of smooth functions in the Orlicz setting given in [30, Theorem 2.2] and in variable exponent Sobolev spaces given for instance in [26, Theorem 2.5].
Theorem 4.6.2 will follow as an immediate byproduct of Theorem 4.6.1 and its assumptions – as it happens frequently when dealing with Musielak spaces – are not necessarily optimal. The comparison with the assumptions of statements in the same direction is, in general, a delicate matter. For instance, just in the case of the (unweighted) variable exponent Sobolev spaces, in [54] various sets of partially overlapping assumptions are considered and this so special situation deserves some comments (we will not go further and we will not deal with weighted variable exponent Sobolev spaces [108]): from one hand, we noticed that condition $(\mathcal{M})$ is essentially a replacement of the standard log-Hölder continuity condition (see the second claim of Example 4.2.1 below), which is known to be not optimal for the boundedness of the maximal operator, which in turn implies the density of smooth functions in variable exponent Sobolev spaces (see e.g. [27, Theorem 6.14]). On the other hand, one can also realize that the fine technique based on averaging operator estimates (which gives optimality for the boundedness of the maximal operator), used in [51, Theorem 6.4.7] through the assumption called *decay condition (A2),* imposes a restriction in the case of unbounded Ω, against our fairly general assumption (2.2) of uniform superlinearity. This is not a surprise for two reasons. First, in Theorem 4.6.1 we deal with the *modular* topology in $W^m L_M(\Omega)$ and in Theorem 4.6.2 we deal with $W^m E_M(\Omega)$ and not with $W^m L_M(\Omega)$.

We end the chapter by Section 4.7 containing the Theorem 4.7.1 where we prove the following equality

$$W_0^m L_M(\Omega) = W_0^{m,1}(\Omega) \cap W^m L_M(\Omega)$$

and that

$$W_0^1 L_M(\Omega) = \{u \in W^1 L_M(\Omega) : \ \mathrm{tr}(u) = 0 \text{ on } \partial\Omega\}$$

for Ω sufficiently regular.

4.2 Regular Musielak functions

4.2.1 Regularity assumption in connection with Zhikov condition $(\mathcal{L}_{hol})$

In the sequel, we consider the following fundamental assumption.

$(\mathcal{L}_{hol})$ There exists a function $\varphi : [0, 1/2] \times \mathbb{R}^+ \to \mathbb{R}^+$ such that $\varphi(\cdot, s)$ and $\varphi(x, \cdot)$ are nondecreasing functions and for all $x, y \in \overline{\Omega}$ with $|x-y| \leq \dfrac{1}{2}$ and for any constant $c > 0$

$$M(x,s) \leq \varphi(|x-y|, s) M(y,s) \qquad \text{with} \qquad \limsup_{\varepsilon \to 0^+} \varphi(\varepsilon, c\varepsilon^{-N}) < \infty.$$

The assumption $(\mathcal{L}_{hol})$ that we introduce here is more general than (14). The said regularity is widely connected to the question of density of smooth functions in the Musielak-Sobolev spaces. Observe in particular that $\varphi(\tau, s) \geq 1$ for all $(\tau, s) \in [0, 1/2] \times \mathbb{R}^+$. In general the Φ-function M **is not continuous** with respect to its first argument. Actually, only if for all $s \geq 0$ we have $\limsup_{\varepsilon \to 0^+} \varphi(\varepsilon, s) = 1$, then $x \mapsto M(x,s)$ is a continuous function on $\overline{\Omega}$.

Example 4.2.1. *We have the following examples of pairs M and φ satisfying both $(\mathcal{L}_{hol})$, and thus are admissible in our results on density of smooth functions.*

1. *If the Φ-function $M(x,s) = M(s)$ is independent of x, then it satisfies obviously the $(\mathcal{L}_{hol})$ condition by choosing $\varphi(\tau, s) = 1$.*

2. *If $M(x,s) = s^{p(x)}$, $p(x) \geq 1$, then the condition $(\mathcal{L}_{hol})$ with*

   $$\varphi(|x-y|, s) = \max\left\{1, s^{\frac{A}{\log \frac{1}{|x-y|}}}\right\}, \quad s > 0, \quad \textit{for some } A > 0$$

   is equivalent to

   $(\mathcal{LH}_0)$

   $$|p(x) - p(y)| \leq \frac{A}{\log \frac{1}{|x-y|}}, \ \textit{for all } x, y \in \overline{\Omega} \ \textit{with } |x-y| \leq \frac{1}{2}.$$

Indeed, $\Rightarrow$) *Assume that* M *satisfies* $(\mathcal{L}_{hol})$*, that is*

$$M(x,s) \le \varphi(|x-y|,s)M(y,s)$$

is satisfied for all $x, y \in \overline{\Omega}$ *with* $|x-y| \le 1/2$*. For* $s > 0$ *we can write*

$$\frac{M(x,s)}{M(y,s)} \le \varphi(|x-y|,s),\ \ s > 0$$

and recalling that $M(x,s) = s^{p(x)}$ *we have that the exponent* $p(\cdot)$ *satisfies*

$$s^{p(x)-p(y)} \le \varphi(|x-y|,s).$$

As x *and* y *play symmetrical roles, we get*

$$s^{|p(x)-p(y)|} \le \varphi(|x-y|,s),\ \ s > 0. \tag{4.1}$$

Now let us fix $\varepsilon \in (0,1/2)$ *and consider* $x,\ y \in \overline{\Omega}$ *with* $|x-y| = \varepsilon$*. Set* $s = \varepsilon^{-N}$ *and plug these choices into (4.1). The right hand side is* $\varphi(\varepsilon, \varepsilon^{-N}) = e^{AN}$ *which remains bounded as* $\varepsilon \to 0$*. This implies that*

$$(\varepsilon^{-N})^{|p(x)-p(y)|} \le e^{AN}.$$

Then, taking logarithms we obtain

$$|p(x)-p(y)| \le \frac{A}{-\log(\varepsilon)}$$

and recalling that $|x-y| = \varepsilon$*,*

$$|p(x)-p(y)| \le \frac{A}{-\log(|x-y|)}.$$

$\Leftarrow$) *Conversely, assume that* $M(x,s) = s^{p(x)}$ *satisfies* $(\mathcal{LH}_0)$*, that is*

$$|p(x)-p(y)| \le \frac{C}{\log\frac{1}{|x-y|}}, \quad |x-y| \le \frac{1}{2}$$

for some constant $C > 0$*. As* x *and* y *play symmetrical roles, we can assume that* $p(x) - p(y) \ge 0$ *obtaining*

$$0 \le p(x) - p(y) \le \frac{C}{\log\frac{1}{|x-y|}}.$$

Then for $s \ge 1$*, we get*

$$s^{p(x)-p(y)} \le s^{\frac{C}{\log\frac{1}{|x-y|}}}$$

and for $s < 1$, we get

$$s^{p(x)-p(y)} \leq 1.$$

In both cases, we get

$$\frac{M(x,s)}{M(y,s)} := s^{p(x)-p(y)} \leq \varphi(|x-y|,s) := \max\left\{1, s^{\frac{C}{\log\frac{1}{|x-y|}}}\right\}.$$

3. *If* $M(x,s) = \dfrac{1}{p(x)} s^{p(x)}$ *with* $p^- := ess \inf\limits_{x\in\Omega} p(x) \geq 1$ *and* $p^+ := ess \sup\limits_{x\in\Omega} p(x) < \infty$ *then the condition* ($\mathcal{L}_{hol}$) *with*

$$\varphi(|x-y|,s) = \frac{p^+}{p^-} \max\left\{1, s^{\frac{A}{\log\frac{1}{|x-y|}}}\right\}, \quad s > 0, \quad \textit{for some } A > 0$$

is equivalent to

$$|p(x)-p(y)| \leq \frac{C}{\log\frac{1}{|x-y|}}, \ \forall x,y \in \overline{\Omega} \textit{ with } |x-y| \leq \frac{1}{2}, \textit{ for some } C > 0. \tag{4.2}$$

Indeed, $\Rightarrow$) *Assume that M satisfies* ($\mathcal{L}_{hol}$), *that is*

$$M(x,s) \leq \varphi(|x-y|,s) M(y,s)$$

is satisfied for all $x,y \in \overline{\Omega}$ with $|x-y| \leq 1/2$. For $s > 0$ we can write

$$\frac{M(x,s)}{M(y,s)} \leq \varphi(|x-y|,s),$$

which yields

$$s^{p(x)-p(y)} \leq \frac{p^+}{p^-}\varphi(|x-y|,s).$$

As x and y play symmetrical roles, we obtain

$$s^{|p(x)-p(y)|} \leq \frac{p^+}{p^-}\varphi(|x-y|,s), \ s > 0. \tag{4.3}$$

Now let us fix $\varepsilon \in (0,1/2)$ and consider $x,\ y \in \overline{\Omega}$ with $|x-y| = \varepsilon$. Set $s = \varepsilon^{-N}$ and plug these choices into (4.3). The right hand side is $\dfrac{p^+}{p^-}\varphi(\varepsilon,\varepsilon^{-N}) = \left(\dfrac{p^+}{p^-}\right)^2 e^{AN}$ *which remains bounded as $\varepsilon \to 0$. This implies that*

$$(\varepsilon^{-N})^{|p(x)-p(y)|} \leq \left(\frac{p^+}{p^-}\right)^2 e^{AN}.$$

Then, taking logarithms we obtain

$$|p(x) - p(y)| \leq \frac{C}{-\log(\varepsilon)}$$

with $C = \dfrac{\log\left(\left(\frac{p^+}{p^-}\right)^2 e^{AN}\right)}{N}$. *Then since* $|x-y| = \varepsilon$ *we obtain*

$$|p(x) - p(y)| \leq \frac{C}{-\log(|x-y|)}.$$

$\Leftarrow$) *Assume that* $M(x,s) = \dfrac{1}{p(x)} s^{p(x)}$ *satisfies (4.2), that is*

$$|p(x) - p(y)| \leq \frac{C}{\log \frac{1}{|x-y|}}, \quad |x-y| \leq \frac{1}{2}$$

for some constant $C > 0$. *As* x *and* y *play symmetrical roles, we can assume that* $p(x) - p(y) \geq 0$ *obtaining*

$$0 \leq p(x) - p(y) \leq \frac{C}{\log \frac{1}{|x-y|}}.$$

Then for $s \geq 1$, *we get*

$$\frac{p(y)}{p(x)} s^{p(x)-p(y)} \leq \frac{p^+}{p^-} s^{\frac{C}{\log \frac{1}{|x-y|}}}$$

and for $s < 1$, *we get*

$$s^{p(x)-p(y)} \leq 1$$

thus

$$\frac{p(y)}{p(x)} s^{p(x)-p(y)} \leq \frac{p^+}{p^-}.$$

In both cases, we get

$$\frac{M(x,s)}{M(y,s)} := \frac{p(y)}{p(x)} s^{p(x)-p(y)} \leq \varphi(|x-y|, s) := \frac{p^+}{p^-} \max\left\{1, s^{\frac{C}{\log \frac{1}{|x-y|}}}\right\}.$$

4. *Consider* $M(x,s) = s^p + a(x)s^q$, *where* $1 \leq p < q$ *and nonnegative* $a \in C^{0,\alpha}_{loc}(\Omega)$ *with* $\alpha \in (0,1]$ *(i.e.* $|a(x) - a(y)| \leq C_a|x-y|^\alpha$ *locally). Hence we obtain*

$$\frac{M(x,s)}{M(y,s)} = \frac{s^p + a(x)s^q}{s^p + a(y)s^q} = \frac{s^p + a(y)s^q + (a(x) - a(y))s^q}{s^p + a(y)s^q} = 1 + \frac{a(x) - a(y)}{s^{p-q} + a(y)}$$
$$\leq 1 + C_a \frac{|x-y|^\alpha}{s^{p-q} + a(y)} \leq 1 + C_a|x-y|^\alpha s^{q-p}.$$

Thus we have M satisfies the ($\mathcal{L}_{hol}$) condition with

$$\varphi(\tau, s) = C_a \tau^{\alpha} |s|^{q-p} + 1.$$

Furthermore, the assumption $\limsup_{\varepsilon \to 0^+} \varphi(\varepsilon, c\varepsilon^{-N}) < \infty$ forces $q \leq p+\alpha/N$. Below we show how to extend the range.

5. *Let $M(x, s) = s^{p(x)}$ with $p^- := \operatorname{ess\,inf}_{x\in\Omega} p(x) > 1$. If M satisfies the condition ($\mathcal{L}_{hol}$) for every x, $y \in \overline{\Omega}$ with $|x - y| < e^{-1}$, then the variable exponent $p(\cdot)$ satisfies the following* log*-Hölder condition*

($\mathcal{LH}_1$)
$$|p(x) - p(y)| \leq w(|x - y|), \quad \text{for } |x - y| < e^{-1},$$

where the modulus of continuity w is given by

$$w(\tau) = \frac{k \log\log \frac{1}{\tau}}{\log \frac{1}{\tau}}, \quad \text{with } 0 < k \leq p^-/N. \tag{4.4}$$

Indeed, assume that M satisfies ($\mathcal{L}_{hol}$) with $|x-y| < e^{-1}$. So that the following inequality

$$M(x, s) \leq \varphi(|x - y|, s)M(y, s)$$

is satisfied for all $x, y \in \overline{\Omega}$ with $|x - y| < e^{-1}$. For $s > 0$ we can write

$$\frac{M(x, s)}{M(y, s)} \leq \varphi(|x - y|, s).$$

Therefore,

$$s^{p(x)-p(y)} \leq \varphi(|x - y|, s). \tag{4.5}$$

Now fix $\varepsilon \in (0, e^{-1})$ and consider $x, y \in \overline{\Omega}$ with $|x - y| = \varepsilon$. As we have $\lim_{\varepsilon\to 0^+} \sup \varphi(\varepsilon, \frac{1}{\varepsilon^N}) < +\infty$, there is a constant $C > 1$ satisfying $\varphi(\varepsilon, \frac{c}{\varepsilon^N}) \leq C$. Let $\beta \geq 1$ and set $s = e^{-\frac{\beta N \log C \log \frac{1}{\varepsilon}}{p^- \log\log \frac{1}{\varepsilon}}}$. Puting these choices into (4.5) we obtain

$$\left(e^{-\frac{\beta N \log C \log \frac{1}{\varepsilon}}{p^- \log\log \frac{1}{\varepsilon}}}\right)^{p(x)-p(y)} \leq \varphi\left(\varepsilon, e^{-\frac{\beta N \log C \log \frac{1}{\varepsilon}}{p^- \log\log \frac{1}{\varepsilon}}}\right). \tag{4.6}$$

On the other hand, it is obvious that

$$-\frac{\beta N \log C \log \frac{1}{\varepsilon}}{p^- \log\log \frac{1}{\varepsilon}} \leq 0 < N \log \frac{1}{\varepsilon} = \log \frac{1}{\varepsilon^N}$$

(the above inequality holds since $\log \frac{1}{\varepsilon} > 0$ and $\log\log \frac{1}{\varepsilon} > 0$). Then taking the exponential we get

$$e^{-\frac{\beta N \log C \log \frac{1}{\varepsilon}}{p^- \log\log \frac{1}{\varepsilon}}} \le \frac{1}{\varepsilon^N}$$

and since $\varphi(\tau,\cdot)$ *is non-decreasing, we obtain*

$$\varphi\Big(\varepsilon, e^{-\frac{\beta N \log C \log \frac{1}{\varepsilon}}{p^- \log\log \frac{1}{\varepsilon}}}\Big) \le \varphi(\varepsilon,\varepsilon^{-N}) \le C.$$

Therefore,

$$\Big(e^{-\frac{\beta N \log C \log \frac{1}{\varepsilon}}{p^- \log\log \frac{1}{\varepsilon}}}\Big)^{(p(x)-p(y))} \le C.$$

Passing to logarithms in both sides of the above inequality we obtain

$$(p(y)-p(x))\frac{\beta N \log C \log \frac{1}{\varepsilon}}{p^- \log\log \frac{1}{\varepsilon}} \le \log C$$

implying

$$(p(y)-p(x)) \le \frac{p^- \log\log \frac{1}{\varepsilon}}{\beta N \log \frac{1}{\varepsilon}}.$$

As $|x-y|=\varepsilon$, *we get*

$$(p(y)-p(x)) \le \frac{k\log\log \frac{1}{|x-y|}}{\log \frac{1}{|x-y|}},$$

with $k = \dfrac{p^-}{\beta N}$. *By the* x/y *symmetry we obtain*

$$|p(x)-p(y)| \le \frac{k\log\log \frac{1}{|x-y|}}{\log \frac{1}{|x-y|}}.$$

6. *Let* $M(x,s) = \sum_{i=1}^{k} k_i(x)M_i(s) + M_0(x,s)$, *where for every* $i = 1,\cdots,k$ *and function* $k_i : \overline{\Omega} \to (0,+\infty)$ *there exists a nondecreasing function* $\varphi_i : [0,1/2] \to \mathbb{R}^+$ *satisfying* $k_i(x) \le \varphi_i(|x-y|)k_i(y)$ *with* $\limsup_{\varepsilon\to 0^+}\varphi_i(\varepsilon) < \infty$, *whereas* $M_0(x,s)$ *satisfies* $(\mathcal{L}_{hol})$ *with* φ_0. *Then, we can take*

$$\varphi(\tau,s) = \sum_{i=1}^{k}\varphi_i(\tau) + \varphi_0(\tau,s).$$

Indeed

$$
\begin{aligned}
\frac{M(x,s)}{M(y,s)} &= \frac{\sum_{i=1}^{k} k_i(x)M_i(s) + M_0(x,s)}{\sum_{j=0}^{k} k_j(y)M_j(s) + M_0(y,s)} \\
&\le \frac{\sum_{i=1}^{k} \varphi_i(|x-y|)k_i(y)M_i(s)}{\sum_{j=0}^{k} k_j(y)M_j(s)} + \frac{M_0(x,s)}{M_0(y,s)} \\
&\le \sum_{i=1}^{k} \varphi_i(|x-y|)\frac{k_i(y)M_i(s)}{\sum_{j=1}^{k} k_j(y)M_j(s)} + \varphi_0(|x-y|,|s|) \\
&\le \sum_{j=1}^{k} \varphi_j(|x-y|)\frac{\sum_{i=1}^{k} k_i(y)M_i(s)}{\sum_{j=1}^{k} k_j(y)M_j(s)} + \varphi_0(|x-y|,|s|) \\
&= \sum_{j=1}^{k} \varphi_j(|x-y|) + \varphi_0(|x-y|,|s|) = \varphi(|x-y|,|s|).
\end{aligned}
$$

Remark 4.2.1. *Note that in the point five of Example 4.2.1, the variable exponent $p(\cdot)$ satisfies $(\mathcal{LH}_1)$ with w is given by (4.4). Consequently, by [106, Theorem 3.1] the result of Meyers-Serrin (see Theorem 4.6.1 below) we prove include the density result*

$$
\overline{\mathcal{C}^\infty(\Omega) \cap W^{1,p(\cdot)}(\Omega)}^{\|\cdot\|_{W^{1,p(\cdot)}(\Omega)}} = W^{1,p(\cdot)}(\Omega), \tag{4.7}
$$

see also [27, Proposition 6.42] page 264, for more developments.

4.2.2 The sharp assumption in connection with Lavrentiev phenomenon $(\mathcal{L}^p_{hol})$

Although it is common to make a big effort to relax growth conditions as much as possible, our method turn out to lead to the sharp result when the modular function has at least power-type growth. Since the approximation follows from convolution arguments, we get better regularity in L^p, $p > 1$ than in L^1. In the fully general case we cannot improve $(\mathcal{L}_{hol})$, because we know only

$$
L_M \subset L^1_{loc}.
$$

Nonetheless, when the modular function has at least power-type growth, we relax $(\mathcal{L}_{hol})$ and include whole the good double-phase range where the Lavrentiev phenomenon is absent (according to [32, Theorem 3] or [25, Theorem 4.1]).

Namely, if

$$
M(x,s) \ge c|s|^p \qquad \text{with } \; p > 1 \text{ and } \; c > 0, \tag{4.8}
$$

we obtain smooth approximation of $u \in W_0^{m,p}(\Omega) \cap W^m L_M(\Omega)$ (also of $u \in W_0^{1,p}(\Omega)$, such that $Du \in L_M(\Omega)$) provided that we assume

($\mathcal{L}^p_{hol}$) There exists a function $\varphi : [0,1/2] \times \mathbb{R}^+ \to \mathbb{R}^+$ such that $\varphi(\cdot, s)$ and $\varphi(x,\cdot)$ are nondecreasing functions and for all $x, y \in \overline{\Omega}$ with $|x-y| \leq \frac{1}{2}$ and for any constant $c > 0$

$$M(x,s) \leq \varphi(|x-y|, s) M(y,s) \quad \text{with} \quad \limsup_{\varepsilon \to 0^+} \varphi(\varepsilon, c\varepsilon^{-\frac{N}{p}}) < \infty.$$

The celebrated case of the Musielak space, when the modular function has at least power-type growth is the double-phase space. Following e.g. [9, 23–25], we shall consider

$$H(x,s) = |s|^p + a(x)|s|^q \qquad \text{with} \qquad a \in C^{0,\alpha}(\Omega),$$

and a Carathéodory function $F : \Omega \times \mathbb{R} \times \mathbb{R}^N \to \mathbb{R}$ such that for any $x \in \Omega$, $v \in \mathbb{R}$, and $z \in \mathbb{R}^N$ it holds that

$$\nu H(x,z) \leq F(x,v,z) \leq LH(x,z) \qquad \text{with certain} \quad 0 < \nu \leq L.$$

Let us denote the associated space $W^{1,H}(\Omega) = \{u \in W_0^{1,1}(\Omega) : H(\cdot, Du) \in L^1(\Omega)\}$ and class $W^{1,F}(\Omega) = \{u \in W_0^{1,1}(\Omega) : F(\cdot, u, Du) \in L^1(\Omega)\}$. Then we have the following sharp result.

Remark 4.2.2. *Suppose $\Omega \subset \mathbb{R}^N$ has a segment property, $N \geq 1$, $p, q > 1$, $\alpha \in (0,1)$, and nonnegative $a \in C^{0,\alpha}(\Omega)$, where*

$$\frac{q}{p} \leq 1 + \frac{\alpha}{N}. \tag{4.9}$$

Then for any $u \in W_0^{1,p}(\Omega) \cap W^{1,F}(\Omega)$ compactly supported in Ω there exist a sequence $\{u_k\}_k \subset C_0^\infty(\Omega)$ converging to u: $u_k \xrightarrow[k\to\infty]{} u$ in $W^{1,p}(\Omega)$ and $Du_k \xrightarrow[k\to\infty]{M} Du$ modularly in $W^{1,H}(\Omega)$, which entails $F(\cdot, u_k, Du_k) \xrightarrow[k\to\infty]{} F(\cdot, u, Du)$ in $L^1(\Omega)$.

Indeed, when we take into account the double phase modular function $M(x,s) = H(x,s)$, then of course (4.8) and ($\mathcal{L}^1_{loc}$) are satisfied. Further we need to ensure ($\mathcal{L}^p_{hol}$). For $0 \leq a \in C^{0,\alpha}(\Omega)$, a good choice is

$$\varphi(\tau, s) = 1 + c_a \tau^\alpha |s|^{q-p}$$

(see Example 4.2.1) and therefore ($\mathcal{L}^p_{hol}$) is satisfied whenever the parameters satisfy (4.9). See [25, Theorem 4.1] for sharpness.
Note, that the form of φ is computed in the point 4 of Example 4.2.1. Then ($\mathcal{L}^p_{hol}$) holds with $\varphi(\tau, |s|) = C_a \tau^\alpha |s|^{q-p} + 1$ if and only if $\alpha + N(p-q)/p \leq 0$, that is $q/p \leq 1 + \alpha/N$.

Remark 4.2.3. *It is worth pointing out that condition* ($\mathcal{L}_{hol}$) *(and* ($\mathcal{L}^p_{hol}$)*) implies the local integrability* ($\mathcal{L}^1_{loc}$)*. Indeed let* K *be a compact subset of* Ω*. For a fixed* $0 < \varepsilon < \dfrac{1}{2}$*, we can write* $K \subset \cup_{x\in K} B(x,\varepsilon)$*, where* $B(x,\varepsilon)$ *stands for the ball of radius* ε *centered in* x*. Since* K *is compact there exist* $x_1, x_2, \cdots, x_m \in K$ *such that* $K \subset \cup_{i=1}^m B(x_i,\varepsilon)$*. By virtue of* ($\mathcal{L}_{hol}$) *we can write*

$$\begin{aligned}
\int_K M(x,c)dx \leq \sum_{i=1}^m \int_{B(x_i,\varepsilon)} M(x,c)dx &\leq \sum_{i=1}^m \int_{B(x_i,\varepsilon)} \varphi(|x-x_i|,c)M(x_i,c)dx \\
&\leq \sum_{i=1}^m \int_{B(x_i,\varepsilon)} \varphi(\varepsilon,c)M(x_i,c)dx \\
&\leq \sum_{i=1}^m \varphi(\varepsilon,c)M(x_i,c)|B(x_i,\varepsilon)| \\
&\leq \varphi(\frac{1}{2},c)\sum_{i=1}^m M(x_i,c)|B(x_i,\varepsilon)| \\
&< +\infty.
\end{aligned}$$

Let (M, M^*) be a pair of complementary Φ-functions. In the following Figure we illustrate the main dependencies between conditions ($\mathcal{L}_{hol}$) (resp.($\mathcal{L}^p_{hol}$)), ($\mathcal{L}^1_{loc}$) and (2.2).

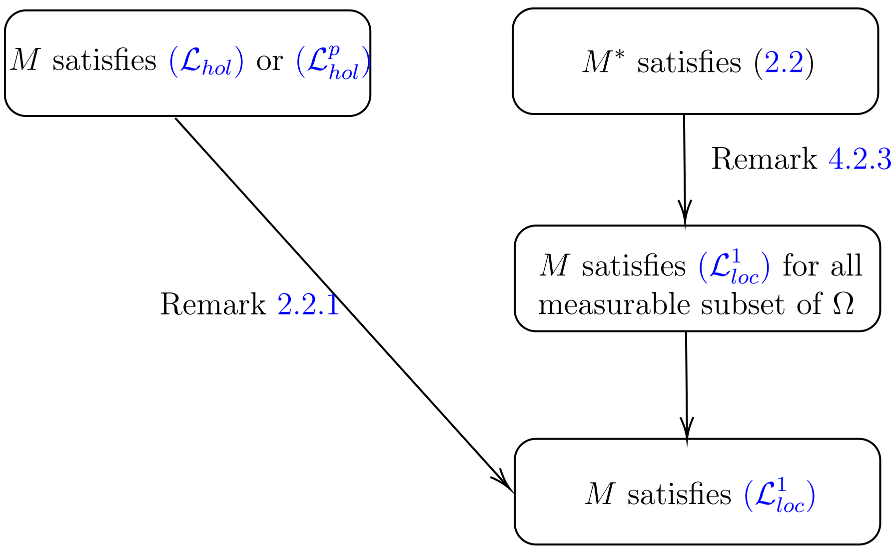

FIGURE 4.1: Relationship between some regularity assumptions on the Φ-functions

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4.3 Mollification and convolution properties in Musielak spaces

For $\varepsilon > 0$, in this section by $u_\varepsilon := J_\varepsilon * u$ we denote the sequence defined by (1.20).

Lemma 4.3.1. *Let Ω be an open subset of $\mathbb{R}^N$ and $M \in \Phi$. Then for every $u \in L_M(\Omega)$ (resp. $u \in L_M(\Omega) \cap L^p(\Omega)$), there exists a constant $c > 0$ depending on $\|u\|_{L^1(\Omega)}$ (resp. $\|u\|_{L^p(\Omega)}$), but not depending on ε such that*

$$|u_\varepsilon(x)| \leq c\varepsilon^{-N} \qquad (\textit{resp. } |u_\varepsilon(x)| \leq c\varepsilon^{-\frac{N}{p}}).$$

Proof. We can write

$$\begin{aligned}
|u_\varepsilon(x)| = \Big| \int_{B(0,1)} u(x-\varepsilon y)J(y)dy \Big| &\leq \max_{|y|\leq 1} |J(y)| \int_{B(0,1)} |u(x-\varepsilon y)|dy \\
&\leq \varepsilon^{-N} \max_{|y|\leq 1} |J(y)| \int_{\Omega\cap B(0,\varepsilon)} |u(x-y)|dy \\
&\leq \varepsilon^{-N} \max_{|y|\leq 1} |J(y)| \int_{\Omega\cap B(x,\varepsilon)} |u(y)|dy \\
&\leq \frac{c}{\varepsilon^N}.
\end{aligned}$$

where $c = \max\limits_{|y|\leq 1} |J(y)| \|u\|_{L^1(\Omega)}$.
As for the assertion for $u \in L^p(\Omega)$ we have

$$\begin{aligned}
&|u_\varepsilon(x)| = \Big| \int_{B(0,1)} u(x-\varepsilon y)J(y)dy \Big| \\
&\leq \left(\int_{B(0,1)} |u(x-\varepsilon y)|^p dy \right)^{\frac{1}{p}} \left(\int_{B(0,1)} |J(y)|^{p'} dy \right)^{\frac{1}{p'}} \\
&\leq c \left(\varepsilon^{-N} \int_{\Omega\cap B(0,\varepsilon)} |u(x-\varepsilon y)|^p dy \right)^{\frac{1}{p}} \\
&\leq c\varepsilon^{-N/p} \left(\int_{\Omega\cap B(x,\varepsilon)} |u(x-y)|^p dy \right)^{\frac{1}{p}} \\
&\leq c\varepsilon^{-N/p}.
\end{aligned}$$

where $c = \left(\int_{B(0,1)} |J(y)|^{p'} dy \right)^{\frac{1}{p'}} \|u\|_{L^p(\Omega)}$. □

Lemma 4.3.2. *Let Ω be an open subset of $\mathbb{R}^N$ with the segment property and let $M \in \Phi$. Consider r_i and ε_i related to θ_i', satisfying (4.28) and (4.30), respectively.*

Assume further that $w \in L_M(\Omega)$ (resp. $w \in L_M(\Omega) \cap L^p(\Omega)$) and $J_{\varepsilon_i} * (w)_{r_i}(x)$ is given by (1.20) and (4.31) below. Then there exists a constant $c > 0$ depending on $\|w\|_{L^1(\Omega)}$ (resp. $\|w\|_{L^p(\Omega)}$) such that

$$|J_{\varepsilon_i} * (w)_{r_i}(x)| \le c\varepsilon_i^{-N} \qquad (\text{resp. } |J_{\varepsilon_i} * (w)_{r_i}(x)| \le c\varepsilon_i^{-\frac{N}{p}}) \tag{4.10}$$

Proof. In the spirit of the proof of Lemma 4.3.1 we notice that

$$\begin{aligned}
|J_{\varepsilon_i} * (w)_{r_i}(x)| &= \Big|\int_{B(0,1)} w(x + r_i z_i - \varepsilon_i y) J(y) dy\Big| \\
&\le \max_{|y|\le 1} |J(y)| \int_{B(0,1)} |w(x + r_i z_i - \varepsilon_i y)| dy \\
&\le \varepsilon_i^{-N} \max_{|y|\le 1} |J(y)| \int_\Omega \chi_{B(0,\varepsilon_i)} |w(x + r_i z_i - y)| dy \\
&\le \varepsilon_i^{-N} \max_{|y|\le 1} |J(y)| \int_\Omega \chi_{B(x+r_i z_i,\varepsilon_i)} |w(y)| dy \\
&\le c\varepsilon_i^{-N}.
\end{aligned}$$

where $c = \max_{|y|\le 1} |J(y)| \|w\|_{L^1(\Omega)}$. We note here that for r_i and ε_i small enough, the segment property ensures that the ball $B(x + r_i z_i, \varepsilon_i)$ is contained in Ω. Applying the Hölder inequality as in the proof of Lemma 4.3.1 we get the second assertion. □

For any function $f : \mathbb{R} \to \mathbb{R}$ the second conjugate function f^{**} (cf. (1.35)), is convex and $f^{**}(x) \le f(x)$. In fact, f^{**} is a convex envelope of f, namely it is the biggest convex function smaller or equal to f.

We give below an observation on the regularity of M, when we define

$$\widetilde{M}(y, s) := \lim_{\delta\to 0^+} \fint_{B(y,\delta)} M(z, s)\, dz \qquad \text{and for } \varepsilon > 0 \quad \widetilde{M}_{x,\varepsilon}(s) := \inf_{y\in B(x,\varepsilon)} \widetilde{M}(y, s) \tag{4.11}$$

and recall that $(\widetilde{M}_{x,\varepsilon})^{**}$ stands for the second conjugate.

Lemma 4.3.3. *Let Ω be an open subset of $\mathbb{R}^N$ and an Φ-function M satisfy ($\mathcal{L}_{hol}$) (resp. (4.8) and ($\mathcal{L}^p_{hol}$)). Let $\varepsilon \in \big(0, 1/2\big]$ be arbitrary. Then, for all x, $y \in \Omega$ such that $y \in B(x, \varepsilon/2)$ we have*

$$\frac{M(y, s)}{(\widetilde{M}_{x,\varepsilon})^{**}(s)} \le 4(\varphi(\varepsilon, s))^2. \tag{4.12}$$

Proof. Note that for a.e. $y \in \Omega$ and $s \in \mathbb{R}_+ \cup \{0\}$, we have

$$\widetilde{M}(y,s) = M(y,s). \tag{4.13}$$

Then from $(\mathcal{L}_{hol})$, for all $x, y \in \Omega$ such that $|x-y| \leq \frac{1}{2}$ one has

$$\widetilde{M}(x,s) \leq \varphi(|x-y|,s)\widetilde{M}(y,s) \tag{4.14}$$

Moreover, $\widetilde{M}$ is locally Lipschitz with respect to s. Indeed, for $s \in (0,R)$, $R > 0$, we can write

$$\sup_{y \in B(x,\frac{\varepsilon}{2}),\, s<R} \left| \frac{\partial M}{\partial s}(y,s) \right| \leq \sup_{y \in B(x,\frac{\varepsilon}{2})} \frac{M(y,R) - M(y,0)}{R-0} \leq \sup_{y \in B(x,\frac{\varepsilon}{2})} \frac{M(y,R)}{R}.$$

By virtue of $(\mathcal{L}_{hol})$, we have

$$\sup_{y \in B(x,\varepsilon/2)} M(y,R) \leq \varphi(\varepsilon,R) M(x,R).$$

So that we obtain

$$\sup_{y \in B(x,\varepsilon/2),\, s<R} \left| \frac{\partial M}{\partial s}(y,s) \right| \leq \frac{\varphi(\frac{1}{2},R)M(x,R)}{R}.$$

Thus, $\widetilde{M}$ and $\widetilde{M}_{x,\varepsilon}$ are continuous in s. Fix arbitrary $y \in \overline{B(x,\varepsilon/2)}$. Then we estimate

$$A := \frac{M(y,s)}{(\widetilde{M}_{x,\varepsilon})^{**}(s)} \leq 4(\varphi(\varepsilon,s))^2. \tag{4.15}$$

Let us start with writing

$$A = \frac{M(y,s)}{\widetilde{M}_{x,\varepsilon}(s)} \cdot \frac{\widetilde{M}_{x,\varepsilon}(s)}{(\widetilde{M}_{x,\varepsilon})^{**}(s)} = A_1 \cdot A_2$$

and noting that for any fixed $s \neq 0$, we consider $\{y_s^n\}_n \subset \overline{B\Big(x,\varepsilon/2\Big)}$, such that for every $n > n(s)$ we have

$$\widetilde{M}_{x,\varepsilon}(s) \geq \widetilde{M}(y_s^n,s) - \frac{1}{n}.$$

If necessary taking bigger n, we can further estimate

$$\widetilde{M}_{x,\varepsilon}(s) \geq \frac{1}{2}\widetilde{M}(y_s^n,s). \tag{4.16}$$

Therefore, for a.e. $y \in \overline{B\Big(x, \varepsilon/2\Big)}$ we have

$$A_1 = \frac{M(y,s)}{\widetilde{M}_{x,\varepsilon}(s)} = \frac{\widetilde{M}(y,s)}{\widetilde{M}_{x,\varepsilon}(s)} \le 2\frac{\widetilde{M}(y,s)}{\widetilde{M}(y_s^n,s)} \le 2\varphi(|y-y_s^n|,s) \le 2\varphi(\varepsilon,s), \tag{4.17}$$

due to (4.13), (4.14), (4.16) and monotonicity of φ. As for A_2, let us remark that if $\widetilde{M}_{x,\varepsilon}$ is convex in s, then $\widetilde{M}_{x,\varepsilon} = (\widetilde{M}_{x,\varepsilon})^{**}$ and $A_2 = 1$. Otherwise there exist $s_1 < s_2$, such that for every $s \in (s_1, s_2)$ we have $\widetilde{M}_{x,\varepsilon}(s) > (\widetilde{M}_{x,\varepsilon})^{**}(s)$ and $\widetilde{M}_{x,\varepsilon}(s_i) = (\widetilde{M}_{x,\varepsilon})^{**}(s_i)$, $i = 1, 2$. Then for every $t \in [0,1]$ we have

$$(\widetilde{M}_{x,\varepsilon})^{**}(ts_1 + (1-t)s_2) = t\widetilde{M}_{x,\varepsilon}(s_1) + (1-t)\widetilde{M}_{x,\varepsilon}(s_2).$$

Let us consider $\{y_{s_1}^n\}_n$, $\{y_{s_2}^n\}_n$ defined similarly to $\{y_s^n\}_n$ and estimate

$$(\widetilde{M}_{x,\varepsilon})^{**}(ts_1 + (1-t)s_2) \ge t\widetilde{M}(y_{s_1}^n, s_1) + (1-t)\widetilde{M}(y_{s_2}^n, s_2) - \frac{1}{n}.$$

We can assume without loss of generality that

$$\widetilde{M}(y_{s_1}^n, s_1) < \widetilde{M}(y_{s_2}^n, s_1)$$

because otherwise we arrive at $\widetilde{M} \le (\widetilde{M})^{**}$ that is $A_2 = 1$. Hence,

$$A_2 = \frac{\widetilde{M}_{x,\varepsilon}(ts_1 + (1-t)s_2)}{(\widetilde{M}_{x,\varepsilon})^{**}(ts_1 + (1-t)s_2)} \le \frac{\widetilde{M}(y_{s_2}^n, ts_1 + (1-t)s_2)}{t\widetilde{M}(y_{s_1}^n, s_1) + (1-t)\widetilde{M}_{x,\varepsilon}(y_{s_2}^n, s_2) - \frac{1}{n}}$$

$$\le \frac{t\widetilde{M}(y_{s_2}^n, s_1) + (1-t)\widetilde{M}(y_{s_2}^n, s_2)}{t\widetilde{M}(y_{s_1}^n, s_1) + (1-t)\widetilde{M}(y_{s_2}^n, s_2) - \frac{1}{n}} = h(t).$$

Since for $t \in (0,1)$ we notice that

$$\begin{aligned} h'(t) &= \frac{(\widetilde{M}(y_{s_2}^n, s_1) - \widetilde{M}(y_{s_1}^n, s_1))\widetilde{M}(y_{s_2}^n, s_2)}{(t(\widetilde{M}(y_{s_1}^n, s_1) - \widetilde{M}(y_{s_2}^n, s_2)) + \widetilde{M}(y_{s_2}^n, s_2))^2} \\ &+ \frac{(\widetilde{M}(y_{s_2}^n, s_2) - \widetilde{M}(y_{s_2}^n, s_1))}{n(t(\widetilde{M}(y_{s_1}^n, s_1) - \widetilde{M}(y_{s_2}^n, s_2)) + \widetilde{M}(y_{s_2}^n, s\xi_2))^2} > 0, \end{aligned}$$

the maximum of h is attained at $t = 1$, which implies

$$A_2 \le \frac{\widetilde{M}(y_{s_2}^n, s_1)}{\widetilde{M}(y_{s_1}^n, s_1) - \frac{1}{n}}.$$

We can restrict ourselves to n sufficiently big to have

$$A_2 \leq 2\frac{\widetilde{M}(y^n_{s_2}, s_1)}{\widetilde{M}(y^n_{s_1}, s_1)} \leq 2\varphi(|y^n_{s_2} - y^n_{s_1}|, s_1) \leq 2\varphi(\varepsilon, s_1) \leq 2\varphi(\varepsilon, s). \tag{4.18}$$

Note that we applied here (4.14). Combining (4.17) with (4.18) gives (4.15). □

Lemma 4.3.4. *Let Ω be an open subset of $\mathbb{R}^N$ and an Φ-function M satisfy $(\mathcal{L}_{hol})$ (resp. (4.8) and $(\mathcal{L}^p_{hol})$). Let $\varepsilon \in \big(0, 1/2\big]$ be arbitrary. Then, for any function $u \in L_M(\Omega)$ (resp. $u \in L_M(\Omega) \cap L^p(\Omega)$), the function $u_\varepsilon \in L_M(\Omega)$. Moreover, the following inequality holds true*

$$\int_\Omega M(x, u_\varepsilon(x)/\lambda)dx \leq 4(\varphi(\varepsilon, \varepsilon^{-N}c/\lambda))^3 \int_\Omega M(x, u(x)/\lambda)dx, \tag{4.19}$$

$$\left(\text{resp.} \quad \int_\Omega M(x, u_\varepsilon(x)/\lambda)dx \leq 4(\varphi(\varepsilon, \varepsilon^{-\frac{N}{p}}c/\lambda))^3 \int_\Omega M(x, u(x)/\lambda)dx\right), \tag{4.20}$$

for some $\lambda > 0$. The constant c is the one that appears in Lemma 4.3.1.

Proof. For $u \in L_M(\Omega)$ there exists $\lambda > 0$ such that $\int_\Omega M(y, |u(y)|/\lambda)dy < +\infty$. Let $x \in \Omega$ be fixed and $\widetilde{M}, \widetilde{M}_{x,\varepsilon}(s)$ be given by (4.11). Using $(\mathcal{L}_{hol})$, Lemma 4.3.1, (4.12) and Jensen's inequality we can write for all $z \in B(x, \varepsilon/2)$

$$\begin{aligned}
M(x, |u_\varepsilon(x)|/\lambda) &= \frac{M(x, |u_\varepsilon(x)|/\lambda)}{M(z, |u_\varepsilon(x)|/\lambda)} \frac{M(z, |u_\varepsilon(x)|/\lambda)}{(\widetilde{M}_{x,\varepsilon})^{**}(|u_\varepsilon(x)|/\lambda)} (\widetilde{M}_{x,\varepsilon})^{**}(|u_\varepsilon(x)|/\lambda) \\
&\leq 4(\varphi(\varepsilon, u_\varepsilon/\lambda))^3 (\widetilde{M}_{x,\varepsilon})^{**}\Big(\frac{1}{\lambda}\int_\Omega J_\varepsilon(x-y)|u(y)|dy\Big) \\
&\leq 4(\varphi(\varepsilon, c\varepsilon^{-N}/\lambda))^3 \int_{|x-y|\leq\varepsilon} J_\varepsilon(x-y)(\widetilde{M}_{x,\varepsilon})^{**}(|u(y)|/\lambda)dy \\
&\leq 4(\varphi(\varepsilon, c\varepsilon^{-N}/\lambda))^3 \int_{|x-y|\leq\varepsilon} J_\varepsilon(x-y)\widetilde{M}_{x,\varepsilon}(|u(y)|/\lambda)dy \\
&\leq 4(\varphi(\varepsilon, c\varepsilon^{-N}/\lambda))^3 \int_{|x-y|\leq\varepsilon} J_\varepsilon(x-y)\widetilde{M}(y, |u(y)|/\lambda)dy.
\end{aligned}$$

Now we integrate both sides of the last inequality with respect to x and use Fubini's theorem, we obtain

$$\begin{aligned}
\int_\Omega M(x, |u_\varepsilon(x)|/\lambda)dx &\leq 4(\varphi(\varepsilon, c\varepsilon^{-N}/\lambda))^3 \int_\Omega \Big(\int_{|x-y|\leq\varepsilon} J_\varepsilon(x-y)\widetilde{M}(y, |u(y)|/\lambda)dy\Big)dx \\
&\leq 4(\varphi(\varepsilon, c\varepsilon^{-N}/\lambda))^3 \int_\Omega \Big(\int_{\mathbb{R}^N} J_\varepsilon(x-y)dx\Big)\widetilde{M}(y, |u(y)|/\lambda)dy \\
&\leq 4(\varphi(\varepsilon, c\varepsilon^{-N}/\lambda))^3 \int_\Omega M(y, |u(y)|/\lambda)dy,
\end{aligned}$$

This proves (4.19).

Applying the second claim of Lemma 4.3.1 instead of the first one and following the same lines we get the second assertion. □

Lemma 4.3.5. *Let Ω be an open subset of $\mathbb{R}^N$ with the segment property and an Φ-function M satisfies $(\mathcal{L}_{hol})$ (resp. (4.8) and $(\mathcal{L}^p_{hol})$). Consider r_i and ε_i related to θ'_i, satisfying (4.28) and (4.30), respectively. Assume further that $w \in L_M(\Omega)$ (resp. $w \in L_M(\Omega) \cap L^p(\Omega)$) and $J_{\varepsilon_i} * (w)_{r_i}(x)$ is given by (1.20) and (4.31) below. Moreover, the following inequality holds true*

$$\int_\Omega M(x, J_{\varepsilon_i} * (w)_{r_i}(x)/\lambda)dx \le 4(\varphi(\varepsilon_i, \varepsilon_i^{-N}c/\lambda))^3 \int_\Omega M(x, w(x)/\lambda)dx,$$

$$\left(\text{resp.} \quad \int_\Omega M(x, J_{\varepsilon_i} * (w)_{r_i}(x)/\lambda)dx \le 4(\varphi(\varepsilon_i, \varepsilon_i^{-\frac{N}{p}}c/\lambda))^3 \int_\Omega M(x, w(x)/\lambda)dx \right),$$

for some $\lambda > 0$, where c is the constant from Lemma 4.3.2.

Proof. Since $w \in L_M(\Omega)$. There exists $\lambda > 0$ such that

$$\int_\Omega M(y, |w(y)|/\lambda)dy < +\infty.$$

Let $x \in \Omega$ be fixed. Let

$$\widetilde{M}(y, s) = \lim_{\delta \to 0^+} \fint_{B(y,\delta)} M(z, s)\, dz \text{ and } \widetilde{M}_{x,\varepsilon_i}(s) = \inf_{y \in B(x, \varepsilon_i/2)} \widetilde{M}(y, s).$$

Using $(\mathcal{L}_{hol})$, (4.10), (4.12) and Jensen's inequality we can write for all $z \in B(x, \varepsilon_i/2)$

$$\begin{aligned}
&M(x, |J_{\varepsilon_i} * (w)_{r_i}(x)|/\lambda) \\
&= \frac{M(x, |J_{\varepsilon_i} * (w)_{r_i}(x)|/\lambda)}{M(z, |J_{\varepsilon_i} * (w)_{r_i}(x)|/\lambda)} \frac{M(z, |J_{\varepsilon_i} * (w)_{r_i}(x)|/\lambda)}{(\widetilde{M}_{x,\varepsilon_i})^{**}(|J_{\varepsilon_i} * (w)_{r_i}(x)|/\lambda)} (\widetilde{M}_{x,\varepsilon_i})^{**}(|J_{\varepsilon_i} * (w)_{r_i}(x)|/\lambda) \\
&\le 4(\varphi(\varepsilon_i, c\varepsilon_i^{-N}/\lambda))^3 (\widetilde{M}_{x,\varepsilon_i})^{**} \Big(\frac{1}{\lambda} \int_\Omega J_{\varepsilon_i}(x + r_i z_i - y)|w(y)|dy \Big) \\
&\le 4(\varphi(\varepsilon_i, c\varepsilon_i^{-N}/\lambda))^3 \int_{|x + r_i z_i - y| \le \varepsilon_i} J_{\varepsilon_i}(x - y)(\widetilde{M}_{x,\varepsilon_i})^{**}(|w(y)|/\lambda)dy \\
&\le 4(\varphi(\varepsilon_i, c\varepsilon_i^{-N}/\lambda))^3 \int_{|x + r_i z_i - y| \le \varepsilon_i} J_{\varepsilon_i}(x - y)\widetilde{M}_{x,\varepsilon_i}(|w(y)|/\lambda)dy \\
&\le 4(\varphi(\varepsilon_i, c\varepsilon_i^{-N}/\lambda))^3 \int_{|x + r_i z_i - y| \le \varepsilon_i} J_{\varepsilon_i}(x - y)\widetilde{M}\Big(y, |w(y)|/\lambda\Big)dy.
\end{aligned}$$

Integrating both sides of the last inequality with respect to x and using Fubini's theorem, we obtain

$$\begin{aligned}
&\int_\Omega M(x, |J_{\varepsilon_i} * (w)_{r_i}(x)|/\lambda)dx \\
&\leq 4(\varphi(\varepsilon_i, c\varepsilon_i^{-N}/\lambda))^3 \int_\Omega \Big(\int_{|x+r_iz_i-y|\leq\varepsilon_i} J_{\varepsilon_i}(x-y)\widetilde{M}(y, |w(y)|/\lambda)dy\Big)dx \\
&\leq 4(\varphi(\varepsilon_i, c\varepsilon_i^{-N}/\lambda))^3 \int_\Omega \Big(\int_{|x+r_iz_i-y|\leq\varepsilon_i} J_{\varepsilon_i}(x-y)dx\Big)\widetilde{M}\big(y, |w(y)|/\lambda\big)dy \\
&\leq 4(\varphi(\varepsilon_i, c\varepsilon_i^{-N}/\lambda))^3 \int_\Omega M(y, |w(y)|/\lambda)dy.
\end{aligned}$$

Applying the second claim of Lemma 4.3.4 instead of the first one and following the same lines we get the second assertion. □

In the following lemma we prove the convergence of the mollification. Such result in the case of Orlicz spaces can be found in [2, Lemma 3.16] .

Lemma 4.3.6. *Assume that an Φ-function M satisfies ($\mathcal{L}_{hol}$) (resp. (4.8) and ($\mathcal{L}^p_{hol}$)). If $u \in W^mL_M(\Omega)$ has a compact support in Ω and u_ε stands for the sequence defined in (1.20), then $u_\varepsilon \xrightarrow[\varepsilon\to 0^+]{mod} u$ in $W^mL_M(\Omega)$.*

If additionally $u \in W^mE_M(\Omega)$, then $u_\varepsilon \to u$ in norm in $W^mE_M(\Omega)$.

Proof. Let $u \in W^mL_M(\Omega)$ with supp$u = \Omega'$ and let $\varepsilon < \mathrm{dist}(\Omega', \partial\Omega)$. Then, $D^\alpha J_\varepsilon * u(x) = J_\varepsilon * D^\alpha u(x)$ in the distributional sense in Ω. We fix arbitrary $\eta > 0$. Since $D^\alpha u \in L_M(\Omega)$ for all $|\alpha| \leq m$, we can assume by Lemma 2.4.1 that there exists a sequence (u_n^α), where for each n the function u_n^α is bounded and compactly supported in Ω, and there exists $\lambda > 0$ such that

$$\int_\Omega M(x, |D^\alpha u(x) - u_n^\alpha(x)|/\lambda)dx \leq \eta. \tag{4.21}$$

Observe that the sequence (u_n^α) does not have to be in the derivative form. By the convexity of the Φ-function M with respect to its second argument, the Jensen inequality enables us to write

$$\begin{aligned}
K &= \int_\Omega M(x, |D^\alpha J_\varepsilon * u(x) - D^\alpha u(x)|/3\lambda)dx \\
&\leq \frac{1}{3}\int_\Omega M(x, |J_\varepsilon * D^\alpha u(x) - J_\varepsilon * u_n^\alpha(x)|/\lambda)dx \\
&+ \frac{1}{3}\int_\Omega M(x, |J_\varepsilon * u_n^\alpha(x) - u_n^\alpha(x)|/\lambda)dx \\
&+ \frac{1}{3}\int_\Omega M(x, |u_n^\alpha(x) - D^\alpha u(x)|/\lambda)dx = K_1 + K_2 + K_3.
\end{aligned} \tag{4.22}$$

The term K_3 is already estimated by (4.21). To conclude the case of K_1, we apply Lemma 4.3.4 and (4.21). We get

$$K_1 \le 4(\varphi(2\varepsilon, c/\varepsilon^N))^3 K_3 \le 4(\varphi(2\varepsilon, c/\varepsilon^N))^3 \eta. \tag{4.23}$$

As for the term K_2, by Jensen's inequality and Fubini's theorem we can write

$$\begin{aligned} K_2 &\le \int_\Omega M\Big(x, |J_\varepsilon * u_n^\alpha(x) - u_n^\alpha(x)|/\lambda\Big) \\ &= \int_\Omega M\Big(x, \int_{B(0,1)} \frac{J(y)}{\lambda} \big|u_n^\alpha(x-\varepsilon y) - u_n^\alpha(x)\big| dy\Big) dx \\ &\le \int_{B(0,1)} J(y) \int_\Omega M\Big(x, \frac{|u_n^\alpha(x-\varepsilon y) - u_n^\alpha(x)|}{\lambda}\Big) dx dy. \end{aligned}$$

Then Theorem 2.3.1 applied to the function u_n^α yields that there exists $\varepsilon_\eta > 0$ such that for every $|y| < 1$ and $\varepsilon \le \varepsilon_\eta$, we have

$$\|u_n^\alpha(\cdot - \varepsilon y) - u_n^\alpha(\cdot)\|_M \le \lambda\eta.$$

Thus, for $\varepsilon < \varepsilon_\eta$ we get

$$\int_\Omega M\Big(x, \frac{|u_n^\alpha(x-\varepsilon y) - u_n^\alpha(x)|}{\lambda}\Big) dx \le \eta,$$

which implies

$$K_2 \le \eta \int_{B(0,1)} J(y) dy = \eta. \tag{4.24}$$

Putting all (4.21), (4.23) and (4.24) together in (4.22), we obtain

$$I \le \eta \left[4(\varphi(2\varepsilon, c/\varepsilon^N))^3 + 1\right].$$

Since η is arbitrary, we get

$$K = \int_\Omega M(x, |D^\alpha J_\varepsilon * u(x) - D^\alpha u(x)|/3\lambda) dx \to 0 \text{ as } \varepsilon \to 0.$$

Therefore,

$$\sum_{|\alpha| \le m} \int_\Omega M(x, |D^\alpha J_\varepsilon * u(x) - D^\alpha u(x)|/3\lambda) dx \to 0 \text{ as } \varepsilon \to 0.$$

Now, if $u \in W^m L_M(\Omega) \cap W^{m,p}(\Omega)$ we obtain the result in a similar way, taking into account the second claim of Lemma 4.3.4. $\square$

4.4 Approximation results when $\Omega = \mathbb{R}^N$

We give below the important observation, that norm convergence in E_M result from the modular one in L_M. An Orlicz version can be found in [68, page 4].

Lemma 4.4.1. *Let $M \in \Phi$-function satisfy ($\mathcal{L}^1_{loc}$) and let $u_k \xrightarrow[k\to\infty]{M} u$ in $L_M(\Omega)$ with every $\lambda > 0$ (cf. Definition 1.4.5), then $u_k \xrightarrow[k\to\infty]{} u$ in the norm topology in $L_M(\Omega)$.*

Proof. For all $\lambda > 0$ we have $\displaystyle\int_\Omega M\Big(x, \lambda|u_k(x) - u(x)|\Big)dx \to 0$ as $k \to \infty$. Thus, $M\Big(x, \lambda|u_n - u|\Big)$ tends to 0 strongly in $L^1(\Omega)$ and so for a subsequence, still indexed by k, we can assume that $u_k \to u$ a.e. in Ω. For an arbitrary $v \in L_{M^*}(\Omega)$, there exists $\lambda_v > 0$ such that $M^*\Big(x, \dfrac{|v|}{\lambda_v}\Big) \in L^1(\Omega)$. By Young's inequality and the convexity of M^* we can write

$$|(u_k(x) - u(x))v(x)| \leq M\Big(x, 2\lambda_v|u_k(x) - u(x)|\Big) + \frac{1}{2}M^*\Big(x, \frac{|v(x)|}{\lambda_v}\Big).$$

Applying Vitali's theorem we obtain $\displaystyle\int_\Omega |(u_k(x) - u(x))v(x)|dx \to 0$ for all $v \in L_{M^*}(\Omega)$ and so

$$\|u_k - u\|^o_M = \sup_{\|v\|_{M^*}\leq 1} \int_\Omega |(u_k(x) - u(x))v(x)|dx \to 0 \text{ as } k \to +\infty,$$

which yields the result. $\square$

We introduce the approximate sequence u_R. Define $\chi \in \mathcal{C}^\infty_0(\mathbb{R}^N)$ by $\chi(x) = 1$ if $\|x\| \leq 1$, $\chi(x) = 0$ if $\|x\| \geq 2$, $0 \leq \chi \leq 1$ and $|D^\alpha\chi(x)| \leq c$ for all $x \in \mathbb{R}^N$ and $|\alpha| \leq m$. Given a function u, we denote by u_r the function

$$u_R = \chi_R u, \quad \text{where} \quad \chi_R(x) = \chi(x/R),\ R > 0. \tag{4.25}$$

Lemma 4.4.2. *Let $M \in \Phi$. If $u \in W^mL_M(\mathbb{R}^N)$ (resp. $u \in W^mE_M(\mathbb{R}^N)$) and u_R is given by (4.25), then $u_R \xrightarrow[r\to\infty]{mod} u$ in $W^mL_M(\mathbb{R}^N)$ (resp. $u_R \to u$ in norm in $W^mE_M(\mathbb{R}^N)$) as $R \to \infty$.*

Proof. For $u \in W^mL_M(\mathbb{R}^N)$, there exists $\lambda > 0$ such that $\displaystyle\int_{\mathbb{R}^N} M(x, |D^\alpha u|/\lambda)dx < \infty$, $\forall|\alpha| \leq m$. Observe first that $u_R \in W^mL_M(\mathbb{R}^N)$. On one hand, when $R \to \infty$ one has

$$M(x, |D^\alpha u - \chi_R(x)D^\alpha u|/2\lambda) \to 0 \quad \text{a.e. in } \mathbb{R}^N,$$

meanwhile on the other hand

$$M(x, |D^\alpha u - \chi_R(x) D^\alpha u|/2\lambda) \leq M(x, |D^\alpha u|/\lambda) \in L^1(\mathbb{R}^N).$$

So that by the Lebesgue Dominated Convergence Theorem, we obtain

$$\lim_{R\to\infty} \int_{\mathbb{R}^N} M(x, |D^\alpha u - \chi_R(x) D^\alpha u|/2\lambda) = 0. \tag{4.26}$$

Now, for $t = \sum\limits_{\beta\neq 0, \beta\leq\alpha} \binom{\alpha}{\beta}$ and by using the Leibnitz formula for $|\alpha| \leq m$ we get

$$\begin{aligned}
I &= \int_{\mathbb{R}^N} M(x, |D^\alpha u - D^\alpha u_R|/4c\lambda t) dx \\
&\leq \frac{1}{2ct} \int_{\mathbb{R}^N} M\Big(x, |D^\alpha u - \chi_R(x) D^\alpha u|/2\lambda\Big) dx \\
&+ \frac{1}{2} \int_{\mathbb{R}^N} M\Big(x, \frac{1}{2ct\lambda} \sum_{\beta\neq 0, \beta\leq\alpha} \binom{\alpha}{\beta} \frac{1}{R^{|\beta|}} \big|(D^\beta \chi)(x/R) D^{\alpha-\beta} u\big|\Big) dx \\
&\leq \frac{1}{2ct} \int_{\mathbb{R}^N} M\Big(x, |D^\alpha u - \chi_R(x) D^\alpha u|/2\lambda\Big) dx \\
&+ \frac{1}{2t} \sum_{\beta\neq 0, \beta\leq\alpha} \binom{\alpha}{\beta} \int_{\mathbb{R}^N} M\Big(x, \frac{1}{2c\lambda R^{|\beta|}} \big|(D^\beta \chi)(x/R) D^{\alpha-\beta} u\big|\Big) dx.
\end{aligned}$$

By using (4.26) the first term in the right hand side of the last inequality tends to zero as $R \to \infty$, while for the second term in the right-hand side we have

$$M\Big(x, \frac{1}{2c\lambda R^{|\beta|}} \big|(D^\beta \chi)(x/R) D^{\alpha-\beta} u\big|\Big) \to 0 \quad \text{as} \quad R \to \infty \quad \text{a.e. in } \mathbb{R}^N$$

and for $R > 1$

$$\int_{\mathbb{R}^N} M\Big(x, \frac{1}{2c\lambda R^{|\beta|}} \big|(D^\beta \chi)(x/R) D^{\alpha-\beta} u\big|\Big) dx \leq \int_{\mathbb{R}^N} M\Big(x, \frac{|D^{\alpha-\beta} u|}{\lambda}\Big) dx < \infty.$$

Hence by using the Lebesgue Dominated Convergence Theorem, we obtain

$$\int_{\mathbb{R}^N} M\Big(x, \frac{1}{2c\lambda R^{|\beta|}} \big|(D^\beta \chi)(x/R) D^{\alpha-\beta} u\big|\Big) dx \to 0 \quad \text{as} \quad R \to \infty.$$

This yields the result for $u \in W^m L_M(\mathbb{R}^N)$. Analogically we get the norm convergence in $W^m E_M(\mathbb{R}^N)$ (cf. Lemma 4.4.1). $\square$

Lemma 4.4.3. *Let M be an Φ-function and $u_n, u \in L_M(\Omega)$. If $u_n \xrightarrow[n\to\infty]{M} u$ modularly, then $u_n \to u$ in $\sigma(L_M, L_{M^*})$.*

Nonetheless, for $M \in \Delta_2$, the weak and modular closures are equal.

Proof. There exists $\lambda > 0$ such that $\int_\Omega M\Big(x, \dfrac{|u_n(x)-u(x)|}{\lambda}\Big)dx \to 0$ as $n \to \infty$. Thus, $M\Big(x, \dfrac{|u_n-u|}{\lambda}\Big)$ tends to 0 strongly in $L^1(\Omega)$ and so for a subsequence, still indexed by n, $u_n \to u$ a.e. in Ω. For an arbitrary $v \in L_{M^*}(\Omega)$, there exists $\lambda_v > 0$ such that $M^*\Big(x, \dfrac{|v|}{\lambda_v}\Big) \in L^1(\Omega)$. Young's inequality allows us to write

$$\frac{1}{\lambda\lambda_v}|(u_n-u)v| \le M\Big(x, \frac{|u_n(x)-u(x)|}{\lambda}\Big) + M^*\Big(x, \frac{|v(x)|}{\lambda_v}\Big).$$

Thus, applying Vitali's theorem we obtain $\int_\Omega (u_n-u)v\,dx \to 0$. □

Lemma 4.4.4. *If $f : \mathbb{R}_+ \to \mathbb{R}_+$ is an increasing convex function, then*

$$f\left(\sum_{i=1}^k \frac{a_i}{2^i}\right) \le \sum_{i=1}^k \frac{2^{k-i}}{2^k-1} f(a_i).$$

Proof.

$$\begin{aligned} f\left(\sum_{i=1}^k \frac{a_i}{2^i}\right) = f\left(\sum_{i=1}^k \frac{2^{k-i}}{2^k-1}\frac{2^k-1}{2^{k-i}}\frac{a_i}{2^i}\right) &\le \sum_{i=1}^k \frac{2^{k-i}}{2^k-1} f\left(\frac{2^k-1}{2^{k-i}}\frac{a_i}{2^i}\right) \\ &\le \sum_{i=1}^k \frac{2^{k-i}}{2^k-1} f(a_i). \end{aligned}$$

□

The main result of this section is stated as follows.

Theorem 4.4.1. *Let (M, M^*) be a pair of complementary Φ-functions. Assume that M satisfies ($\mathcal{L}_{hol}$) (resp. (4.8) and ($\mathcal{L}^p_{hol}$)) and M^* satisfies ($\mathcal{L}^1_{loc}$). Then*

1) *$\mathcal{C}_0^\infty(\mathbb{R}^N)$ is dense in $W^m E_M(\mathbb{R}^N)$ with respect to the norm topology in $W^m L_M(\mathbb{R}^N)$.*

2) *For every $u \in W^m L_M(\mathbb{R}^N)$, there exist a sequence of functions $u_k \in \mathcal{C}_0^\infty(\mathbb{R}^N)$ such that $u_k \xrightarrow[k\to\infty]{mod} u$ in $W^m L_M(\mathbb{R}^N)$.*

Proof. Let $u \in W^m L_M(\mathbb{R}^N)$. We shall find $\lambda > 0$ and $v \in \mathcal{C}_0^\infty(\mathbb{R}^N)$ such that for every $\eta \geq 0$

$$\int_{\mathbb{R}^N} M(x, |D^\alpha u(x) - D^\alpha v(x)|/\lambda)dx \leq \eta \qquad \forall |\alpha| \leq m.$$

According to Lemma 4.4.2, we can assume that u is compactly supported in $\mathbb{R}^N$. So that by Lemma 4.3.6, one can approximate u by a function v in $\mathcal{C}_0^\infty(\mathbb{R}^N)$ and this yields the result. If u belongs to $W^m E_M(\mathbb{R}^N)$, we prove the result following the similar way noting that $\lambda > 0$ can be chosen arbitrary. □

Remark 4.4.1. *If in addition $M \in \Delta_2$, then $\mathcal{C}_0^\infty(\mathbb{R}^N)$ is dense in norm topology in $W^m L_M(\mathbb{R}^N)$, because in this case $W^m L_M(\mathbb{R}^N) = W^m E_M(\mathbb{R}^N)$.*

4.5 Approximation in $\Omega \subset \mathbb{R}^N$

We give the density result on the sets satisfying the segment property. We have the following theorem relating to [40, Theorem 3].

Theorem 4.5.1. *Assume that Ω satisfies the segment property and an Φ-function M satisfies ($\mathcal{L}_{hol}$) (resp. (4.8) and ($\mathcal{L}^p_{hol}$)) and let M^* satisfy ($\mathcal{L}^1_{loc}$). Then*

1) *$\mathcal{C}_0^\infty(\overline{\Omega})$ is dense in $W^m E_M(\Omega)$ with respect to the norm topology in $W^m L_M(\Omega)$.*
2) *For every $u \in W^m L_M(\Omega)$, there exist a sequence of functions $u_k \in \mathcal{C}_0^\infty(\overline{\Omega})$ such that $u_k \xrightarrow[k\to\infty]{mod} u$ in $W^m L_M(\Omega)$.*

Proof. Let $u \in W^m L_M(\Omega)$ and fix arbitrary $\eta \geq 0$. We shall find $v \in \mathcal{C}_0^\infty(\overline{\Omega})$ and $\lambda > 0$ such that

$$\int_\Omega M(x, |D^\alpha u - D^\alpha v|/\lambda)dx \leq \eta, \qquad \forall |\alpha| \leq m. \tag{4.27}$$

We will construct v using a finite sequence of functions $v_i \in \mathcal{C}_0^\infty(\overline{\Omega})$, $i = 0, 1, \cdots, k$, satisfying

$$\int_\Omega M(x, |D^\alpha u - D^\alpha v|/\lambda)dx \leq \int_\Omega M\left(x, \sum_{i=0}^k |D^\alpha u_i - D^\alpha v_i|/\lambda_i^\alpha\right) dx \leq \eta, \ \forall |\alpha| \leq m$$

with some $\lambda_i^\alpha > 0$. The sequence $\{v_i\}_i$ is related to the covering we introduce below.
Without loss of generality we can assume that $\operatorname{supp} u \subset K$ and K is compact (see Lemma 4.4.2). We will distinguish the two cases: either easy part $K \subset\subset \Omega$ or hard part $K \cap \partial\Omega \neq \emptyset$. Since there exist a finite open covering of $\overline{\Omega}$ given by the segment property, and $K \cap \partial\Omega$ is compact, there exists also a finite collection

$\{\widehat{\theta}_i\}_{i=1}^k$ covering $K \cap \partial\Omega$. Let $E = K \setminus \cup_{i=1}^k \widehat{\theta}_i$. Since E is a compact subset of Ω, there exists an open set $\widehat{\theta}_0$ with a compact closure in Ω such that $E \subset \overline{\widehat{\theta}_0} \subset \Omega$. Hence $\{\widehat{\theta}_i\}_{i=0}^k$ is an open covering of K. Moreover, as in the proof of [3, Theorem 1.9], we can construct another open covering $\{\theta_i'\}_{i=0}^k$ of K with θ_i' has a compact closure in $\widehat{\theta}_i$ for $i = 0, 1, \dots, k$.

Let $\{\psi_i\}_{i=1}^k$ be a partition of unity associated to $\{\theta_i'\}_{i=0}^k$, with $\sum_{i=0}^k \psi_i = 1$ on K and let $u_i = u\psi_i$. Then $u = \sum_{i=0}^k u_i$, $\operatorname{supp} u_i \subset \theta_i'$ and u_i belongs to $W^m L_M(\Omega)$, for $i = 0, 1, \dots, k$.

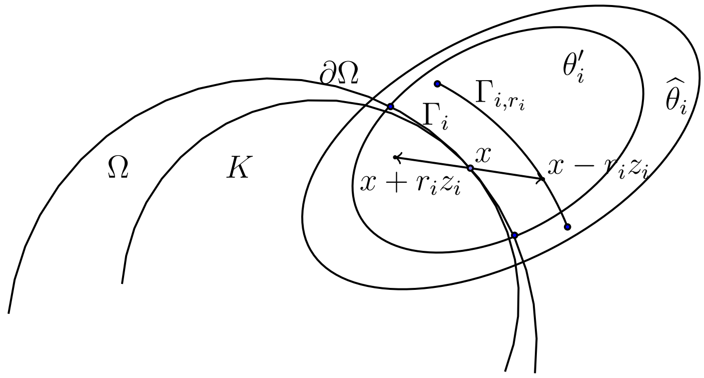

Figure 4.2: Domain geometry

.

For a fixed i such that $1 \leq i \leq k$, we extend u_i to $\mathbb{R}^N$ by zero outside θ_i'. Let z_i be a nonzero vector associated to $\widehat{\theta}_i$ by the segment property and let $r_i \in (0, 1)$ be such that

$$0 < r_i < \min\{1/(|z_i| + 1), \operatorname{dist}(\theta_i', \partial\widehat{\theta}_i)|z_i|^{-1}\}. \tag{4.28}$$

Denote $\Gamma_i = \overline{\theta_i'} \cap \partial\Omega$ and $\Gamma_{i,r_i} = \Gamma_i - r_i z_i$, then we have $\Gamma_{i,r_i} \subset \theta_i$. Indeed, for $x \in \Gamma_{i,r_i}$, we get by (4.28)

$$\operatorname{dist}(x, \theta_i') \leq \operatorname{dist}(x, \Gamma_i) \leq |r_i z_i| < \operatorname{dist}(\theta_i', \partial\widehat{\theta}_i).$$

Furthermore, $\Gamma_{i,r_i} \cap \overline{\Omega} = \emptyset$. Indeed, if $\Gamma_{i,r_i} \cap \overline{\Omega} \neq 0$ then there exists $x \in \Gamma_{i,r_i} \cap \overline{\Omega} \subset \widehat{\theta}_i \cap \overline{\Omega}$ and $x + r_i z_i \in \Gamma_i$ contradicting the segment property.

If $i = 0$, we consider $\operatorname{supp} u_0 \subset \theta_0' \subset \Omega$. Choosing $\varepsilon_0 > 0$ small enough such that $\varepsilon_0 < dist(\theta_0', \partial\Omega)$, the regularized function $v_0 = J_{\varepsilon_0} * u_0$ belongs to $\mathcal{C}_0^\infty(\Omega)$. We fix $\lambda_0, \eta_0 > 0$, which due to Lemma 4.3.6 exist such that for all sufficiently small $\varepsilon_0 > 0$ we have

$$\int_\Omega M\Big(x, \frac{|D^\alpha u_0(x) - D^\alpha J_{\varepsilon_0} * u_0(x)|}{\lambda_0}\Big) dx \leq \frac{\eta_0}{2}, \qquad \forall |\alpha| \leq m. \tag{4.29}$$

Now, according to the decomposition without loss of generality we can assume that u has its support in a compact set $K \subset \overline{\Omega}$ with $K \cap \partial\Omega \neq \emptyset$. By the partition of unity, we arrive to the case $K \subset \widehat{\theta}_i$ for some i. The construction of approximate sequence for a function whose support touches the boundary will be based on the idea of pushing support of u (restricted to a set of a smaller covering) a bit to outside of Ω to cover its uniform neighbourhood (but still remaining in a set from a bigger covering). Note that for this we exploit the fact that we can consider local systems of coordinates associated with the segment property. We would not be able to do our construction e.g. in the presence of external cusps. Afterwards we mollify and prove convergence.

Fix $1 \leq i \leq k$ and define

$$(u_i)_{r_i}(x) = u_i(x + r_i z_i).$$

Since $\mathrm{dist}(\Gamma_{i,r_i}, \overline{\Omega}) > 0$, for

$$\varepsilon_i < \mathrm{dist}(\Gamma_{i,r_i}, \widehat{\theta}_i \cap \overline{\Omega}) \tag{4.30}$$

and extending u_i to $\mathbb{R}^N$ to be identically zero outside $\widehat{\theta}_i$, we define the sequence

$$v_i^{\varepsilon_i,r_i}(x) = J_{\varepsilon_i} * (u_i)_{r_i}(x) = \int_{B(0,1)} J(y) u_i(x + r_i z_i - \varepsilon_i y) dy, \tag{4.31}$$

see (1.20). Our aim is now to estimate

$$J^i = \int_\Omega M\left(x, \frac{|D^\alpha u_i(x) - J_{\varepsilon_i} * (D^\alpha u_i)_{r_i}|}{4\lambda_{i,n}^\alpha}\right) dx.$$

Observe that $\mathrm{supp}\, v_i^{\varepsilon_i,r_i} \subset \overline{\Omega}$ and so the function $v_i^{\varepsilon_i,r_i}$ belongs to $\mathcal{C}_0^\infty(\overline{\Omega})$. Thus, applying Lemma 4.3.2 to the function $(u_i)_{r_i}$, defined on $\overline{\Omega}$ by the segment property, we get

$$|v_i^{\varepsilon_i,r_i}| \leq \frac{c_i}{\varepsilon_i^N} \text{ (resp. } |v_i^{\varepsilon_i,r_i}| \leq \frac{c_i}{\varepsilon_i^{N/p}})$$

where $c_i > 0$ depends on the norm $\|u_i\|_{L^1(\mathrm{supp}\, u)}$. Note that $D^\alpha u_i \in L_M(\Omega)$ for every $|\alpha| \leq m$ and so there exist λ_i^α such that

$$\int_\Omega M(x, |D^\alpha u_i(x)|/\lambda_i^\alpha) dx < +\infty.$$

By Lemma 2.4.1 there exist a sequence of functions $\{u_{i,n}^\alpha\}$, where $u_{i,n}^\alpha$ is bounded and has a compact support in Ω, and $\lambda_i^{\alpha,\eta_0} > 0$ and $n_i^{\alpha,\eta_0} > 0$, such that for every $n > n_i^{\alpha,\eta_0}$

$$I_1 = \int_\Omega M\Big(x, \frac{|D^\alpha u_i(x) - u_{i,n}^\alpha(x)|}{\lambda_i^\alpha}\Big) dx \leq \eta_0. \tag{4.32}$$

The Jensen inequality yields

$$
\begin{aligned}
J^i \leq & \frac{1}{4}\int_\Omega M\left(x, \frac{|D^\alpha u_i(x) - u_{i,n}^\alpha(x)|}{\lambda_i^\alpha}\right) dx \\
& + \frac{1}{4}\int_\Omega M\left(x, \frac{|u_{i,n}^\alpha(x) - (u_{i,n}^\alpha(x))_{r_i}|}{\lambda_i^\alpha}\right) dx \\
& + \frac{1}{4}\int_\Omega M\left(x, \frac{|(u_{i,n}^\alpha(x))_{r_i} - J_{\varepsilon_i} * (u_{i,n}^\alpha(x))_{r_i}|}{\lambda_i^\alpha}\right) dx \\
& + \frac{1}{4}\int_\Omega M\left(x, \frac{|J_{\varepsilon_i} * (u_{i,n}^\alpha(x))_{r_i} - J_{\varepsilon_i} * (D^\alpha u_i(x))_{r_i}|}{\lambda_i^\alpha}\right) dx \\
& = \frac{1}{4}(I_1 + I_2 + I_3 + I_4).
\end{aligned}
\tag{4.33}
$$

We shall show that $J^i \leq C_i^\alpha \eta_0$ for some constant $C_i^\alpha > 0$ whenever ε_i and r_i are sufficiently small.

The case of I_1 is done by (4.32). We deal with I_2 using Theorem 2.3.1, which ensures that there exist $r_{i,n}^{\alpha,\eta_0} > 0$, such that for all $r_i < r_{i,n}^{\alpha,\eta_0}$ we can estimate $I_2 \leq \eta_0$. In order to find a bound on I_3 we notice that the Jensen inequality and the Fubini theorem yield

$$
\begin{aligned}
I_3 &= \int_\Omega M\Big(x, \frac{|(u_{i,n}^\alpha(x))_{r_i} - J_{\varepsilon_i} * (u_{i,n}^\alpha(x))_{r_i}|}{\lambda_i^\alpha}\Big) dx \\
&= \int_\Omega M\Bigg(x, \Bigg| \int_{B(0,1)} \frac{J(y)}{\lambda_i^\alpha}\Big(u_{i,n}^\alpha(x + r_i z_i) - u_n^\alpha(x + r_i z_i - \varepsilon_i y)\Big) dy \Bigg|\Bigg) dx \\
&\leq \int_{B(0,1)} J(y) \int_\Omega M\Big(x, \frac{\Big|u_{i,n}^\alpha(x + r_i z_i) - u_n^\alpha(x + r_i z_i - \varepsilon_i y)\Big|}{\lambda_i^\alpha}\Big) dx dy,
\end{aligned}
\tag{4.34}
$$

where $r_i < r_{i,n}^{\alpha,\eta_0}$ and its relation with z_i is given by (4.28). Using Theorem 2.3.1, for every $|y| < 1$ and $\eta_0 > 0$ there exists $\varepsilon_{i,n}^{\alpha,\eta_0}$ such that for $\varepsilon_i \leq \varepsilon_{i,n}^{\alpha,\eta_0}$ we get

$$
\int_\Omega M\Big(x, \frac{u_{i,n}^\alpha(x + r_i z_i) - u_{i,n}^\alpha(x + r_i z_i - \varepsilon_i y)}{\lambda_i^\alpha}\Big) dx \leq \eta_0,
$$

thus via (4.34) we can estimate $I_3 \leq \eta_0 \displaystyle\int_{B(0,1)} J(y) dy = \eta_0$ (whenever $r_i < r_{i,n}^{\alpha,\eta_0}$ and $\varepsilon_i < \varepsilon_{i,n}^{\alpha,\eta_0}$). In the case of I_4 we apply Lemma 4.3.5 to the expression

$(u^\alpha_{i,n}(x))_{r_i} - (D^\alpha u_i(x))_{r_i}$ and (4.32) to claim that there exist $\overline{\varepsilon}^{\alpha,\eta_0}_{i,n}$, such that for all $\varepsilon_i < \overline{\varepsilon}^{\alpha,\eta_0}_{i,n}$ (and $r_i < r^{\alpha,\eta_0}_{i,n}$) we have

$$\begin{aligned} I_4 &= \int_\Omega M\left(x, \frac{\left|J_{\varepsilon_i} * (u^\alpha_{i,n}(x))_{r_i} - J_{\varepsilon_i} * (D^\alpha u_i(x))_{r_i}\right|}{\lambda^\alpha_i}\right)dx \\ &\leq 4\Big(\varphi\Big(\varepsilon_i, \frac{c_i}{\lambda^\alpha_i \varepsilon_i^N}\Big)\Big)^3 \int_\Omega M\Big(x, \frac{\left|u^\alpha_{i,n}(x) - D^\alpha u_i(x)\right|}{\lambda^\alpha_i}\Big)dx \\ &\leq 4\Big(\varphi\Big(\varepsilon_i, \frac{c_i}{\lambda^\alpha_i \varepsilon_i^N}\Big)\Big)^3 \eta_0. \end{aligned} \tag{4.35}$$

To sum up, we get in (4.33) for $\varepsilon_i \leq \min\{\varepsilon^{\alpha,\eta_0}_{i,n}, \overline{\varepsilon}^{\alpha,\eta_0}_{i,n}\}$ and $r_i < r^{\alpha,\eta_0}_{i,n}$

$$\begin{aligned} \int_\Omega M\Big(x, \frac{|D^\alpha u_i(x) - D^\alpha v_i(x)|}{4\lambda^\alpha_i}\Big)dx &\leq \frac{1}{4}(I_1 + I_2 + I_3 + I_4) \\ &\leq \eta_0\left(\frac{3}{4} + \Big(\varphi\Big(\varepsilon_i, \frac{c_i}{\lambda^\alpha_i \varepsilon_i^N}\Big)\Big)^3\right), \end{aligned}$$

for every α such that $|\alpha| \leq m$. Therefore, for an arbitrary $\bar{\eta} > 0$ and sufficiently small $\varepsilon_i^{\bar{\eta}}$ and $r_i^{\bar{\eta}}$ we have

$$J = \int_\Omega M\Big(x, \frac{|D^\alpha u_i(x) - D^\alpha J_{\varepsilon_i^{\bar{\eta}}} * (u_i)_{r_i^{\bar{\eta}}}(x)|}{4\lambda^\alpha_i}\Big)dx \leq \frac{\bar{\eta}}{2^i}.$$

Let $v = \sum_{i=1}^k J_{\varepsilon_i^{\bar{\eta}}} * (u_i)_{r_i^{\bar{\eta}}} + J_{\varepsilon_0} * u_0.$ and note that v belongs to $\mathcal{C}_0^\infty(\overline{\Omega})$. Take $\lambda = \max\{\lambda_0, 4\lambda^\alpha_i\}$. By (4.29), Lemma 4.4.4, and the last inequality for every α such that $|\alpha| \leq m$ we obtain

$$\begin{aligned} \int_\Omega M\Big(x, \frac{|D^\alpha u(x) - D^\alpha v(x)|}{2^{k+1}\lambda}\Big)dx &\leq \frac{1}{2}\int_\Omega M\Big(x, \sum_{i=1}^k \frac{|D^\alpha u_i(x) - D^\alpha J_{\varepsilon_i^{\bar{\eta}}} * (u_i)_{r_i^{\bar{\eta}}}(x)|}{2^i \lambda}\Big)dx \\ &+ \frac{1}{2^{k+1}}\int_\Omega M\Big(x, \frac{|D^\alpha u_0(x) - J_{\varepsilon_0} * D^\alpha u_0(x)|}{\lambda}\Big)dx \\ &\leq \sum_{i=1}^k \frac{2^{k-i}}{2^k - 1}\int_\Omega M\Big(x, \frac{|D^\alpha u_i(x) - D^\alpha J_{\varepsilon_i^{\bar{\eta}}} * (u_i)_{r_i^{\bar{\eta}}(x)}|}{\lambda}\Big)dx \\ &+ \frac{1}{2^{k+1}}\int_\Omega M\Big(x, \frac{|D^\alpha u_0(x) - J_{\varepsilon_0} * D^\alpha u_0(x)|}{\lambda}\Big)dx \\ &\leq \sum_{i=1}^k \frac{2^{k-i}}{2^k - 1}\frac{1}{2^i}\bar{\eta} + \frac{1}{2^{k+2}}\bar{\eta}. \end{aligned}$$

Choosing $\bar{\eta} = \frac{\eta}{2}\left(\sum_{i=1}^{k} \frac{2^{k-i}}{2^k - 1}\right)^{-1}$ we get (4.27) and hence the second assertion is proven. Indeed, the method of construction gives us a unique v independent of α.

To get the first assertion, it suffices to note that if additionally to the above reasoning u belongs to $W^m E_M(\Omega)$, we obtain (4.27) with arbitrary $\lambda > 0$ (cf. Lemma 4.4.1). □

Remark 4.5.1. *The assumption M^* satisfies ($\mathcal{L}^1_{loc}$) in Theorems 4.4.1 and 4.5.1 is used to get the continuous embedding of L_M into L^1_{loc}. Note here that the assumption M^* satisfies ($\mathcal{L}^1_{loc}$) in this two last theorems can be replaced by the condition (3.7) (see Lemma 3.4.2).*

We give below the extension of [40, Theorem 4] by Gossez.

Theorem 4.5.2. *Assume that Ω satisfies the segment property and an Φ-function M satisfies ($\mathcal{L}_{hol}$) (resp. (4.8) and ($\mathcal{L}^p_{hol}$)) and let M^* satisfy ($\mathcal{L}^1_{loc}$). Then for every $u \in W^m_0 L_M(\Omega)$, there exist a sequence of functions $u_k \in \mathcal{C}^\infty_0(\Omega)$ such that $u_k \xrightarrow[k\to\infty]{mod} u$ in $W^m L_M(\Omega)$.*

Proof. Let $u \in W^m_0 L_M(\Omega)$. Our goal is to find $v \in \mathcal{C}^\infty_0(\Omega)$ such that for some $\lambda > 0$ and all $\eta \geq 0$ we get

$$\int_\Omega M\Big(x, \frac{|D^\alpha u(x) - D^\alpha v(x)|}{\lambda}\Big)dx \leq \eta \qquad \forall |\alpha| \leq m.$$

The proof follows exactly the same lines as the one of the second claim of Theorem 4.5.1, except that we change r_i by $-r_i$ in (4.31) so that

$$v_i = J_{\varepsilon_i} * (u_i)_{-r_i} = \int_{B(0,1)} J(y) u_i(x - r_i z_i - \varepsilon_i y)dy$$

and choose

$$\varepsilon_i < dist\Big((\theta'_i \cap \overline{\Omega}) + r_i z_i, \mathbb{R}^N \setminus \Omega\Big).$$

Therefore, $\operatorname{supp} v_i \subset \Omega$ and the function v_i belongs to $\mathcal{C}^\infty_0(\Omega)$. □

The above theorem has the following consequences being an extension to Musielak-Sobolev spaces of the results proved by Gossez in [37, Theorem 1.3] in the case of Orlicz spaces.

Corollary 4.5.1. *Assume that Ω satisfies the segment property and an Φ-function M satisfies ($\mathcal{L}_{hol}$) (resp. (4.8) and ($\mathcal{L}^p_{hol}$)) and let M^* satisfy ($\mathcal{L}^1_{loc}$). Then*

1) $\mathcal{C}^\infty_0(\overline{\Omega})$ is $\sigma(\Pi L_M, \Pi L_{M^*})$-dense in $W^m L_M(\Omega)$.

2) $\mathcal{C}^\infty_0(\Omega)$ is $\sigma(\Pi L_M, \Pi L_{M^*})$-dense in $W^m_0 L_M(\Omega)$.

Proof. It is a direct consequence of Theorem 4.5.1, Theorem 4.5.2, and Lemma 4.4.3. □

4.6 Non standard Meyers-Serrin results

The main results of this section are Theorem 4.6.1 and Theorem 4.6.2.

Theorem 4.6.1. *Let Ω be an open subset of $\mathbb{R}^N$ and let $M \in \Phi$ satisfy (2.2) and ($\mathcal{L}_{hol}$). Then $\mathcal{C}^\infty(\Omega) \cap W^m L_M(\Omega)$ is dense in $W^m L_M(\Omega)$ with respect to the modular topology in $W^m L_M(\Omega)$.*

Proof. Since the closure of $\mathcal{C}^\infty(\Omega) \cap W^m L_M(\Omega)$ with respect to the modular topology is included in $W^m L_M(\Omega)$, we only need to prove the reverse inclusion. Let us fix $u \in W^m L_M(\Omega)$. We can assume that u has a compact support in $\overline{\Omega}$. Indeed, define $\chi \in \mathcal{C}_0^\infty(\mathbb{R}^N)$ by

- $\chi(x) = 1$ if $\|x\| \le 1$,
- $\chi(x) = 0$ if $\|x\| \ge 2$,
- $|D^\alpha \chi(x)| \le c$ for all $x \in \mathbb{R}^N$ and $|\alpha| \le m$.

For $r > 0$ let $\chi_r(x) = \chi(x/r)$. Then $\chi_r(x) = 1$ if $\|x\| \le r$ and $|D^\alpha \chi_r(x)| \le \dfrac{c}{r^\alpha} \le c$ if $r \ge 1$. Therefore, by the Leibnitz derivative formula the function $u_r := \chi_r u$ belongs to $W^m L_M(\Omega)$. Furthermore, there exists a constant number $\mu > 0$ such that for any $\eta > 0$ and for r large enough we get

$$\sum_{|\alpha| \le m} \int_\Omega M\Big(x, \frac{\big|D^\alpha u(x) - D^\alpha u_r(x)\big|}{\mu}\Big) dx \le \eta. \tag{4.36}$$

This last inequality can be proved following the same lines of [7, Lemma 3], for the sake of completeness, we give here the full argument (which follows, in fact, an idea going back to [40]). Indeed since $u \in W^m L_M(\Omega)$, there exists a constant number $\delta > 0$ such that $\displaystyle\int_\Omega M(x, |D^\alpha u(x)|/\delta) dx < \infty$, $\forall |\alpha| \le m$. On one hand, when $r \to \infty$ one has

$$M\Big(x, \frac{|D^\alpha u(x) - \chi_r(x) D^\alpha u(x)|}{2\delta}\Big) \to 0 \quad \text{a.e. in } \Omega,$$

meanwhile on the other hand

$$M\Big(x, \frac{|D^\alpha u(x) - \chi_r(x) D^\alpha u(x)|}{2\delta}\Big) \le M\Big(x, \frac{|D^\alpha u(x)|}{\delta}\Big) \in L^1(\Omega).$$

So that by the Lebesgue Dominated Convergence Theorem, we obtain

$$\lim_{r\to\infty} \int_\Omega M\Big(x, \frac{|D^\alpha u(x) - \chi_r(x) D^\alpha u(x)|}{2\delta}\Big) dx = 0. \tag{4.37}$$

Now, for $t = \sum_{\beta\neq 0,\beta\leq\alpha} \binom{\alpha}{\beta}$ and by using the Leibnitz formula for $|\alpha| \leq m$ we get

$$\begin{aligned}
&\int_\Omega M\Big(x, \frac{|D^\alpha u(x) - D^\alpha u_r(x)|}{4c\delta t}\Big)dx\\
&\leq \frac{1}{2ct}\int_\Omega M\Big(x, \frac{|D^\alpha u(x) - \chi_r(x)D^\alpha u(x)|}{2\delta}\Big)dx\\
&+\frac{1}{2}\int_\Omega M\Big(x, \frac{1}{2ct\delta}\sum_{\beta\neq 0,\beta\leq\alpha}\tbinom{\alpha}{\beta}\frac{1}{r^{|\beta|}}\left|(D^\beta\chi)(x/r)D^{\alpha-\beta}u(x)\right|\Big)dx\\
&\leq \frac{1}{2ct}\int_\Omega M\Big(x, \frac{|D^\alpha u(x) - \chi_r(x)D^\alpha u(x)|}{2\delta}\Big)dx\\
&+\frac{1}{2t}\sum_{\beta\neq 0,\beta\leq\alpha}\tbinom{\alpha}{\beta}\int_\Omega M\Big(x, \frac{1}{2c\delta r^{|\beta|}}\left|(D^\beta\chi)(x/r)D^{\alpha-\beta}u(x)\right|\Big)dx.
\end{aligned}$$

By using (4.37) the first term in the right hand side of the last inequality tends to zero as $r \to \infty$, while for the second term in the right-hand side we have

$$M\Big(x, \frac{1}{2c\delta r^{|\beta|}}\left|(D^\beta\chi)(x/r)D^{\alpha-\beta}u(x)\right|\Big) \to 0 \quad \text{as} \quad r\to\infty \quad \text{a.e. in } \Omega$$

and for $r > 1$

$$\int_\Omega M\Big(x, \frac{1}{2c\delta r^{|\beta|}}\left|(D^\beta\chi)(x/r)D^{\alpha-\beta}u(x)\right|\Big)dx \leq \int_\Omega M\Big(x, \frac{|D^{\alpha-\beta}u(x)|}{\delta}\Big)dx < \infty.$$

Hence by using the Lebesgue dominated convergence theorem, we obtain

$$\int_\Omega M\Big(x, \frac{1}{2c\delta r^{|\beta|}}\left|(D^\beta\chi)(x/r)D^{\alpha-\beta}u(x)\right|\Big)dx \to 0 \quad \text{as} \quad r\to\infty.$$

Thus for $\mu = 4c\delta t$ we get (4.36). Therefore without loss of generality, we can assume that u has a compact support K in $\overline{\Omega}$. We shall show that there is a constant number $\nu > 0$ such that for any $\eta > 0$ we can find $v \in \mathcal{C}^\infty(\Omega)\cap W^mL_M(\Omega)$ satisfying

$$\sum_{|\alpha|\leq m}\int_\Omega M\Big(x, \frac{\left|D^\alpha u(x) - D^\alpha v(x)\right|}{\nu}\Big)dx \leq \eta. \tag{4.38}$$

In the case $\Omega = \mathbb{R}^N$ we can make the proof, without loss of generality, thinking that Ω is a big ball (hence in this case Ω can be thought bounded and, in particular, its boundary $\partial\Omega$ is not empty) containing K, and finding $v \in \mathcal{C}_0^\infty(\Omega)\cap W^mL_M(\Omega)$, because outside such ball there is no contribution in the integral in (4.38).

For $i = 1, 2, \cdots$, define the following sequence of subsets of Ω

$$\Omega_i = \Big\{ x \in \Omega, |x| < i, \ \mathrm{dist}(x, \partial\Omega) > 1/i \Big\}.$$

Observe that for $i = 1, 2, \cdots$, $\Omega_i \subset \Omega_{i+1}$ and $\overset{\infty}{\underset{i=1}{\cup}} \Omega_i = \Omega$. Set by convention $\Omega_{-1} = \Omega_0 = \emptyset$ and let Θ be the collection of covering open subsets of Ω defined by

$$\Theta = \Big\{ \theta_i, \ \theta_i = \Omega_{i+1} \setminus \overline{\Omega}_{i-1}, i = 1, 2, \cdots \Big\}$$

Since $K \cap \partial\Omega$ is compact there exists an open set V_0 such that $K \cap \partial\Omega \subset V_0$, hence

$$K = (\Omega \cap K) \cup (K \cap \partial\Omega) \subset \cup_{i=1}^{\infty} \theta_i \cup V_0.$$

Furthermore since K is compact we can extract a finite open covering of K such that $K \subset \cup_{i=1}^{k} \theta_i \cup V_0$. Therefore, there exists a partition of unity subordinate to this cover (see for instance [102, p. 61]). That is, there exists a collection of functions $\{\psi_i\}$ such that $\mathrm{supp}(\psi_i) \subset \theta_i$ for $i = 1, \cdots, k$ and $\mathrm{supp}(\psi_0) \subset V_0$, $0 \leq \psi_i \leq 1$ and for every $x \in K$ we have $\displaystyle\sum_{i=0}^{k} \psi_i(x) = 1$. For $i = 0, 1, \cdots, k$ defining $u_i := \psi_i u$ we have $u_i \in W^m L_M(\Omega)$, supp $u_0 \subset V_0 \cap K$ and supp $u_i \subset \theta_i \cap K$ for $i = 1, \cdots, k$. Let J stands for the Friedrichs mollifier kernel defined on $\mathbb{R}^N$ by

$$J(x) = \kappa e^{-\frac{1}{1-\|x\|^2}} \text{ if } \|x\| < 1 \text{ and } 0 \text{ if } \|x\| \geq 1,$$

where $\kappa > 0$ is such that $\displaystyle\int_{\mathbb{R}^N} J(x)dx = 1$. For $\varepsilon > 0$, we define $J_\varepsilon(x) = \varepsilon^{-N} J(x/\varepsilon)$ and

$$u_\varepsilon(x) = J_\varepsilon * u(x) = \int_{\mathbb{R}^N} J_\varepsilon(x-y)u(y)dy = \int_{B(0,1)} u(x - \varepsilon y)J(y)dy.$$

For $i = 1, \cdots, k$ we define

$$J_\varepsilon * u_i(x) = \int_{\theta_i \cap K} u_i(y) J_\varepsilon(x-y)dy.$$

It's clear that $J_\varepsilon * u_i$ is non-zero only if $y \in \theta_i \cap K$ and $|x - y| < \varepsilon$. Choosing

$$0 < \varepsilon < \frac{1}{(i+1)(i+2)}$$

we get $\dfrac{1}{i+2} < \mathrm{dist}(x, \partial\Omega) \leq \dfrac{1}{i-2}$, the second inequality holding if $i \geq 3$. Therefore, $\mathrm{supp}(J_\varepsilon * u_i) \subset U_i := \Omega_{i+2} \setminus \overline{\Omega}_{i-2}$. Note that $\overline{U}_i$ is a compact contained in Ω. Since $D^\alpha u_i$ belongs to $L_M(\Omega)$, for all $|\alpha| \leq m$, by Lemma 2.4.1 for an arbitrary $\eta_0 > 0$, there exist a sequence $\{u^\alpha_{i,n}\}$, where $u^\alpha_{i,n}$ is bounded and has a compact support in $\theta_i \cap K \subset \Omega$, a constant number $\lambda^\alpha_i > 0$ and an integer $n^\alpha_i > 0$ such that for all $n \geq n^\alpha_i$ we have

$$\int_\Omega M\Big(x, \frac{|D^\alpha u_i(x) - u^\alpha_{i,n}(x)|}{\lambda^\alpha_i}\Big)dx \leq \frac{\eta_0}{2^i}.$$

Setting $n_0 =: \max\limits_{|\alpha|\leq m, 1\leq i\leq k} n^\alpha_i$ and $\lambda = \max\limits_{|\alpha|\leq m, 1\leq i\leq k} \lambda^\alpha_i$, we also have

$$\int_\Omega M\Big(x, \frac{|D^\alpha u_i(x) - u^\alpha_{i,n_0}(x)|}{\lambda}\Big)dx \leq \frac{\eta_0}{2^i}. \tag{4.39}$$

For a fixed $j = 1, \cdots,$ we define

$$\begin{aligned} I^j &= \int_{\Omega_j} M\left(x, \Big|\sum_{i=1}^{k} \frac{D^\alpha u_i(x) - D^\alpha J_\varepsilon * u_i(x)}{12\lambda}\Big|\right)dx \\ &= \int_{\Omega_j} M\left(x, \Big|\sum_{i=1}^{j+1} \frac{D^\alpha u_i(x) - D^\alpha J_\varepsilon * u_i(x)}{12\lambda}\Big|\right)dx. \end{aligned}$$

By construction, for any $x \in \Omega_j \cap K$ there exist at most 4 terms in the above sum which are non-zero at x. Using the convexity property we can write

$$\begin{aligned} I^j &\leq \frac{1}{3}\int_{\Omega_j} M\left(x, \sum_{i=1}^{j+1} \frac{\Big|D^\alpha u_i(x) - u^\alpha_{i,n_0}(x)\Big|}{4\lambda}\right)dx \\ &+ \frac{1}{3}\int_{\Omega_j} M\left(x, \sum_{i=1}^{j+1} \frac{\Big|u^\alpha_{i,n_0}(x) - J_\varepsilon * u^\alpha_{i,n_0}(x)\Big|}{4\lambda}\right)dx \\ &+ \frac{1}{3}\int_{\Omega_j} M\left(x, \sum_{i=1}^{j+1} \frac{\Big|J_\varepsilon * u^\alpha_{i,n_0}(x) - J_\varepsilon * D^\alpha u_i(x)\Big|}{4\lambda}\right)dx. \end{aligned}$$

Therefore,

$$\begin{aligned} I^j &\leq \frac{1}{12}\sum_{i=1}^{j+1}\int_{\Omega_j} M\left(x, \frac{\Big|D^\alpha u_i(x) - u^\alpha_{i,n_0}(x)\Big|}{\lambda}\right)dx \\ &+ \frac{1}{12}\sum_{i=1}^{j+1}\int_{\Omega_j} M\left(x, \frac{\Big|u^\alpha_{i,n_0}(x) - J_\varepsilon * u^\alpha_{i,n_0}(x)\Big|}{\lambda}\right)dx \\ &+ \frac{1}{12}\sum_{i=1}^{j+1}\int_{\Omega_j} M\left(x, \frac{\Big|J_\varepsilon * u^\alpha_{i,n_0}(x) - J_\varepsilon * D^\alpha u_i(x)\Big|}{\lambda}\right)dx \\ &= \frac{1}{12}(I^j_1 + I^j_2 + I^j_3). \end{aligned}$$

We shall estimate I_1^j, I_2^j and I_3^j. Starting with the term I_1^j, we get from (4.39)

$$I_1^j \leq \eta_0.$$

For the term I_2^j, by Jensen's inequality and Fubini's theorem we can write

$$\begin{aligned}
&\int_{\Omega_j} M\left(x, \frac{\left|u_{i,n_0}^\alpha(x) - J_\varepsilon * u_{i,n_0}^\alpha(x)\right|}{\lambda}\right) dx \\
&\leq \int_\Omega M\left(x, \int_{B(0,1)} \frac{J(y)}{\lambda} \left|u_{i,n_0}^\alpha(x) - u_{i,n_0}^\alpha(x-\varepsilon y)\right| dy\right) dx \\
&\leq \int_{B(0,1)} J(y) \left[\int_\Omega M\left(x, \frac{\left|u_{i,n_0}^\alpha(x) - u_{i,n_0}^\alpha(x-\varepsilon y)\right|}{\lambda}\right) dx\right] dy.
\end{aligned}$$

Using Theorem 2.3.1 for every $|y| < 1$ and for $\eta_0 > 0$ there exist $\varepsilon_{0,n_0}^{\alpha,\eta_0}$ such that for $\varepsilon \leq \varepsilon_{0,n_0}^{\alpha,\eta_0}$ we get

$$\int_\Omega M\left(x, \frac{\left|u_{i,n_0}^\alpha(x) - u_{i,n_0}^\alpha(x-\varepsilon y)\right|}{\lambda}\right) dx \leq \frac{\eta_0}{2^i}.$$

Therefore,

$$I_2^j \leq \eta_0.$$

We turn now to estimate I_3^j. To do so we follow the lines of the proof of [7, Lemma 9] with the function $u_{i,n_0}^\alpha - D^\alpha u_i$, using the growth of the function φ with respect to its second argument for small enough $\varepsilon \leq \dfrac{1}{2}$ obtaining

$$\begin{aligned}
&\int_{\Omega_j} M\left(x, \frac{|J_\varepsilon * u_{i,n_0}^\alpha(x) - J_\varepsilon * D^\alpha u_i(x)|}{\lambda}\right) dx \\
&\leq 4\left(\varphi\left(\varepsilon, \frac{c_{i,n_0}^j}{\lambda \varepsilon^N}\right)\right)^3 \int_{\Omega_j} M\left(x, \frac{|u_{i,n_0}^\alpha(x) - D^\alpha u_i(x)|}{\lambda}\right) dx
\end{aligned}$$

where

$$c_{i,n_0}^j := \max_{|y|\leq 1} |J(y)| \int_{\Omega_j} |u_{i,n_0}^\alpha(x) - D^\alpha u_i(x)| dx.$$

In view of (4.39) and since the support of u_{i,n_0}^α is contained in $\theta_i \cap K \subset K \cap \Omega$ by Lemma 3.4.1 there exists $t_0 \geq 0$ such that

$$
\begin{aligned}
&\int_{\Omega_j} |u^\alpha_{i,n_0}(x) - D^\alpha u_i(x)| dx \\
&\le \int_{\Omega\cap K} |u^\alpha_{i,n_0}(x) - D^\alpha u_i(x)| dx \\
&\le (1 + t_0|K|)\|u^\alpha_{i,n_0} - D^\alpha u_i\|_{L_M(K\cap\Omega)} \\
&\le \lambda(1 + t_0|K|)\Big(\int_\Omega M\Big(x, \frac{|u^\alpha_{i,n_0}(x) - D^\alpha u_i(x)|}{\lambda}\Big) + 1\Big) \\
&\le c := \lambda(1 + t_0|K|)(\eta_0 + 1).
\end{aligned}
$$

Hence, there is a constant number $c > 0$ satisfying $c^j_{i,n_0} \le c$ for every $j \ge 1$ and every $i \ge 1$. Therefore, by (4.39) one has

$$I^j_3 \le 4\eta_0\Big(\varphi\Big(\varepsilon, \frac{c}{\lambda\varepsilon^N}\Big)\Big)^3.$$

By the assumption ($\mathcal{L}_{hol}$) the sequence $\Big\{\varphi\Big(\varepsilon, \frac{c}{\lambda\varepsilon^N}\Big)\Big\}_\varepsilon$ is bounded. Hence, there is a constant $C > 0$ such that

$$I^j_3 \le 4C\eta_0.$$

Thus, combining those estimates we get for all $|\alpha| \le m$ and for small $\varepsilon \le \min\Big\{\frac{1}{2}, \frac{1}{(k+1)(k+2)}, \varepsilon^{\alpha,\eta_0}_{0,n_0}\Big\}$

$$I^j \le c_0\eta_0,$$

where $c_0 = \dfrac{2C+1}{6}$. Consequently,

$$
\begin{aligned}
&\int_{\Omega_j} M\left(x, \sum_{i=1}^{k} \frac{\big|D^\alpha u_i(x) - J_\varepsilon * D^\alpha u_i(x)\big|}{12\lambda}\right) dx \\
&= \int_{\Omega_j} M\left(x, \sum_{i=1}^{j+1} \frac{\big|D^\alpha u_i(x) - D^\alpha J_\varepsilon * u_i(x)\big|}{12\lambda}\right) dx \\
&\le c_0\eta_0.
\end{aligned}
$$

Then letting $j \to \infty$ we get by using monotone convergence theorem

$$\int_\Omega M\left(x, \sum_{i=1}^{k} \frac{\big|D^\alpha u_i(x) - J_\varepsilon * D^\alpha u_i(x)\big|}{12\lambda}\right) dx \le c_0\eta_0.$$

Let now $i = 0$. For $\varepsilon_0 > 0$ small enough we define the function

$$J_{\varepsilon_0} * u_0(x) = \int_{V_0 \cap K} u_0(y) J_{\varepsilon_0}(x-y)dy$$

which is non-zero only if $y \in V_0 \cap K$ and $|x-y| < \varepsilon_0$. Since $D^\alpha u_0$ belongs to $L_M(\Omega)$, $|\alpha| \le m$, by Lemma 2.4.1 there exists a constant number $\lambda_0 > 0$ and a sequence $\{u_n^\alpha\}$, where u_n^α is bounded and has a compact support in $V_0 \cap K \cap \overline{\Omega}_n \subset \Omega$ such that

$$\int_\Omega M\Big(x, \frac{|D^\alpha u_0(x) - u_n^\alpha(x)|}{\lambda_0}\Big)dx \le \frac{\eta_0}{3}.$$

Furthermore,

$$\begin{aligned}
&\int_\Omega M\Big(x, \frac{|D^\alpha u_0(x) - J_{\varepsilon_0} * D^\alpha u_0(x)|}{3\lambda_0}\Big)dx \\
&\le \frac{1}{3}\int_\Omega M\Big(x, \frac{|D^\alpha u_0(x) - u_n^\alpha(x)|}{\lambda_0}\Big)dx \\
&+\frac{1}{3}\int_\Omega M\Big(x, \frac{|u_n^\alpha(x) - J_{\varepsilon_0} * u_n^\alpha(x)|}{\lambda_0}\Big)dx \\
&+\frac{1}{3}\int_\Omega M\Big(x, \frac{|J_{\varepsilon_0} * u_n^\alpha(x) - J_{\varepsilon_0} * D^\alpha u_0(x)|}{\lambda_0}\Big)dx \\
&= \frac{1}{3}(J_1 + J_2 + J_3).
\end{aligned}$$

As above we have to estimate J_1, J_2 and J_3. This can be done, by slight modifications, as it was done for the terms I_1^j, I_2^j and I_3^j. Hence for $\varepsilon_0 \le \frac{1}{2}$ we can write

$$\int_\Omega M\Big(x, \frac{|D^\alpha u_0(x) - J_{\varepsilon_0} * D^\alpha u_0(x)|}{3\lambda_0}\Big)dx \le \eta_0.$$

Define the function v by

$$v(x) = \sum_{i=1}^{k} J_\varepsilon * u_i(x) + J_{\varepsilon_0} * u_0(x),$$

Note here that if K does not meet $\partial\Omega$ then the term $J_{\varepsilon_0} * u_0(x)$ in the equality does not appear. To sum up, let $\nu = \max\{24\lambda, 6\lambda_0\}$, then by convexity of the Φ-function M we get

$$\begin{aligned}
\sum_{|\alpha|\le m}\int_\Omega M\left(x, \frac{\left|D^\alpha u(x) - v(x)\right|}{\nu}\right)dx \quad &\le \frac{1}{2}\sum_{|\alpha|\le m}\left\{\int_\Omega M\left(x, \sum_{i=1}^{k}\frac{\left|D^\alpha u_i(x) - J_\varepsilon * D^\alpha u_i(x)\right|}{12\lambda}\right)dx\right. \\
&\left.+\frac{1}{2}\int_\Omega M\Big(x, \frac{|D^\alpha u_0(x) - J_{\varepsilon_0} * D^\alpha u_0(x)|}{3\lambda_0}\Big)dx\right\} \\
&\le \frac{\eta_0(c_0+1)\sum_{|\alpha|\le m} 1}{2}.
\end{aligned}$$

which ends the proof of (4.38).

In order to end the proof, we need to gather (4.36) and (4.38). This can be done applying the same kind of substitution of the triangle inequality for norms, that we already applied several times (for instance, in the last chain of inequalities). □

Theorem 4.6.2. *Let Ω be an open subset of $\mathbb{R}^N$ and let $M \in \Phi$ satisfy (2.2) and $(\mathcal{L}_{hol})$. Then $\mathcal{C}^\infty(\Omega) \cap W^m E_M(\Omega)$ is dense in $W^m E_M(\Omega)$ with respect to the norm topology in $W^m E_M(\Omega)$.*

Proof. Let u belongs to $W^m E_M(\Omega)$. Arguing as in the proof of Theorem 4.6.1 we obtain (4.38) with arbitrary $\lambda > 0$ (cf. Lemma 4.4.1). □

These two spaces $W^m E_M(\Omega)$ and $W^m L_M(\Omega)$ coincide in a large class of $\Phi-$functions by virtue of the following proposition, where the Δ_2 assumption is only sufficient (see [52, Example 4.2]).

Proposition 4.6.1. *If M satisfies the Δ_2-condition (see Definition 3.2.1), then $W^m L_M(\Omega) = W^m E_M(\Omega)$ (see [103]).*

Second, still in [54] it has been observed that the density of continuous functions is a lot more common occurrence in Sobolev spaces than the boundedness of the maximal function.

For results of density of smooth functions in variable exponent Sobolev spaces coming from properties of the Riesz potential we mention the paper [66], where other comments concerning the link with the boundedness of the maximal operator appear.

4.7 Approximation results in null trace functions space

For $u \in W^1 L_M(\Omega)$ the trace $tr(u) = u|_{\partial\Omega}$ is well defined. Indeed, if Ω is of finite Lebesgue measure and M satisfies (2.2) one has $W^1 L_M(\Omega) \hookrightarrow W^{1,1}(\Omega)$ (see Lemma 3.4.2) and by the Gagliardo trace theorem (see [35]) we have the embedding $W^{1,1}(\Omega) \hookrightarrow L^1(\partial\Omega)$. Hence, we conclude that for all $u \in W^1 L_M(\Omega)$ there holds $u|_{\partial\Omega} \in L^1(\partial\Omega)$. We give in the following theorem density results in zero trace Musielak-Sobolev spaces.

Theorem 4.7.1. *Let Ω be a bounded open subset in $\mathbb{R}^N$ having the segment property and let (M, M^*) be a pair of complementary Φ-functions. Assume that M satisfies $(\mathcal{L}_{hol})$ and (2.2). Then, we get*

$$W_0^m L_M(\Omega) = W_0^{m,1}(\Omega) \cap W^m L_M(\Omega).$$

If furthermore Ω has a Lipschitz boundary $\partial\Omega$, then we obtain

$$W_0^1 L_M(\Omega) = \{u \in W^1 L_M(\Omega) : \ tr(u) = 0 \ on \ \partial\Omega\}. \tag{4.40}$$

Proof. We begin first by showing that $W_0^m L_M(\Omega) \subset W_0^{m,1}(\Omega) \cap W^m L_M(\Omega)$. Since $W_0^m L_M(\Omega)$ is a subset of $W^m L_M(\Omega)$, it's sufficient to check that for all function u belonging to $W_0^m L_M(\Omega)$ we have $u \in W_0^{m,1}(\Omega)$. Let $u \in W_0^m L_M(\Omega)$. By Theorem 4.5.2 there exist $\lambda > 0$ and a sequence $u_k \in \mathcal{C}_0^\infty(\Omega)$ such that

$$\sum_{|\alpha| \leq m} \int_\Omega M(x, |D^\alpha u_k(x) - D^\alpha u(x)|/\lambda) dx \to 0 \text{ as } k \to \infty.$$

Therefore, for all $|\alpha| \leq m$

$$\int_\Omega M(x, |D^\alpha u_k(x) - D^\alpha u(x)|/\lambda) dx \to 0 \text{ as } k \to \infty.$$

Thus, for a subsequence still denoted by u_k, we can assume

$$D^\alpha u_k \to D^\alpha u \text{ a.e. in } \Omega.$$

Applying Vitali's theorem we obtain

$$\int_\Omega |D^\alpha u_k(x) - D^\alpha u(x)| dx \to 0 \text{ as } k \to \infty,$$

which implies that $u \in W_0^{m,1}(\Omega)$.

Conversely, we should prove that $W_0^{m,1}(\Omega) \cap W^m L_M(\Omega) \subset W_0^m L_M(\Omega)$. We will show that for $u \in W_0^{m,1}(\Omega) \cap W^m L_M(\Omega)$ there exist a sequence $v \in \mathcal{C}_0^\infty(\Omega)$ such that v converges in the modular sense to u in $W^m L_M(\Omega)$ and then we conclude by using Theorem 4.5.2. Let us denote by $\widetilde{u}$ the extension of u by zero outside Ω. Since u belongs to $W_0^{m,1}(\Omega)$ it yields, $\widetilde{u} \in W^{m,1}(\mathbb{R}^N)$ and $\widetilde{D^\alpha u} = D^\alpha \widetilde{u}$ in the distributional sense and a.e. in $\mathbb{R}^N$ (see [2, Lemma 3.27]) and so $\widetilde{u} \in W^m L_M(\mathbb{R}^N)$. Then by Lemma 4.4.2 we can assume that u has compact support $K \subset \overline{\Omega}$. We will distinguish the two cases: either $K \subset \Omega$ or $K \cap \partial\Omega \neq \emptyset$. If $K \subset \Omega$ then we get the desired inclusion by Lemma 4.3.6. If $K \cap \partial\Omega \neq \emptyset$, then, as in the proof of Theorem 4.5.1, there exist a finite collection $\{\widehat{\theta}_i\}_{i=1}^k$ covering the compact set $K \cap \partial\Omega$ and an open covering $\{\theta_i'\}_{i=0}^k$ of K with θ_i' has a compact closure in $\widehat{\theta}_i$ for $i = 0, 1, \cdots, k$. Then u can be splitted into finitely-many pieces u_i, such that $u = \sum_{i=1}^k u_i$ with $\operatorname{supp} u_i \subset \theta_i'$, $i = 0, 1, \cdots, k$.

For $i = 0$, we consider $\operatorname{supp} u_0 \subset \theta_0' \subset \Omega$, then as for the first case by Lemma 4.3.6 there exist $\varepsilon_0 > 0$ small enough ($\varepsilon_0 < dist(\theta_0', \partial\Omega)$), such the regularized function $v_0 = J_{\varepsilon_0} * u_0$ belongs to $\mathcal{C}_0^\infty(\Omega)$ and converge in modular since to u in $W^m L_M(\Omega)$. For $1 \leq i \leq k$ fix. Let z_i be a non-zero vector associated to $\widehat{\theta}_i$ by the segment property and let $r_i \in (0, 1)$ be such that

$$0 < r_i < \min\{1/(|z_i| + 1), dist(\theta_i', \partial\widehat{\theta}_i)|z_i|^{-1}\}.$$

Define

$$(u_i)_{-r_i}(x) = u_i(x - r_i z_i).$$

and choose

$$\varepsilon_i < dist\Big((\theta_i' \cap \overline{\Omega}) + r_i z_i, \mathbb{R}^N \setminus \Omega\Big).$$

We define then the sequences

$$v_i^{\varepsilon_i, r_i}(x) = J_{\varepsilon_i} * (u_i)_{-r_i} = \int_{B(0,1)} J(y) u_i(x - r_i z_i - \varepsilon_i y) dy$$

and

$$v(x) = \sum_{i=1}^{k} v_i^{\varepsilon_i, r_i}(x) + J_{\varepsilon_0} * u_0(x),$$

therefore, $v \in \mathcal{C}_0^\infty(\Omega)$. Then, arguing similarly as in the proof of Theorem 4.5.1, we prove that v converges to u in $W^m L_M(\Omega)$ with respect to the modular convergence. This implies that u belongs to $W_0^m L_M(\Omega)$.
To check (4.40) observe that $\{u \in W^1 L_M(\Omega) : \operatorname{tr}(u) = 0 \text{ on } \partial\Omega\} \subset \{u \in W^{1,1}(\Omega) : \operatorname{tr}(u) = 0 \text{ on } \partial\Omega\} = W_0^{1,1}(\Omega)$. So that for any $v \in \{u \in W^1 L_M(\Omega) : \operatorname{tr}(u) = 0 \text{ on } \partial\Omega\}$ one has $v \in W_0^{1,1}(\Omega) \cap W^1 L_M(\Omega) = W_0^1 L_M(\Omega)$. $\square$

4.8 Conclusion

In this chapter we performed the density of smooth functions in the modular topology in the Musielak-Sobolev spaces essentially extending the results of Gossez [40] obtained in the Orlicz-Sobolev setting. We impose systematic regularity assumptions on M which allows to study the problem of density unifying and improving the known results in the Orlicz-Sobolev spaces, as well as the variable exponent Sobolev spaces.
We summarize in the following table our main density results in Musielak-Sobolev spaces obtained in this chapter.

	M satisfies ($\mathcal{L}_{hol}$)	M satisfies ($\mathcal{L}^1_{loc}$)	M^* satisfies ($\mathcal{L}^1_{loc}$)	M satisfies (2.2)	Domain geometry	Density results	
$\mathcal{C}_0^\infty(\mathbb{R}^\mathbb{N})$	✓	✓				norm dense in $W^m E_M(\mathbb{R}^\mathbb{N})$	Theorem 4.4.1
	✓	✓			Segment property	modular dense in $W^m L_M(\mathbb{R}^N)$	
$\mathcal{C}_0^\infty(\overline{\Omega})$	✓	✓				norm dense in $W^m E_M(\Omega)$	Theorem 4.5.1
	✓	✓				modular dense in $W^m L_M(\Omega)$	
$\mathcal{C}_0^\infty(\Omega)$	✓	✓	✓			modular dense in $W_0^m L_M(\Omega)$	Theorem 4.5.2
$\mathcal{C}^\infty(\Omega) \cap W^m E_M(\Omega)$	✓	✓		✓		norm dense in $W^m E_M(\Omega)$	Theorem 4.6.1
$\mathcal{C}^\infty(\Omega) \cap W^m L_M(\Omega)$	✓	✓		✓		modular dense in $W^m E_M(\Omega)$	Theorem 4.6.2
$W_0^{m,1}(\Omega) \cap W^m L_M(\Omega)$	✓	✓	✓	✓	Segment property	equal to $W_0^m L_M(\Omega)$	Theorem 4.7.1
$u \in W^1 L_M(\Omega)$ $\mathrm{tr}(u) = 0$ on $\partial\Omega$	✓	✓	✓	✓	Lipschitz boundary	equal to $W_0^m L_M(\Omega)$	
NB: if M satisfies ($\mathcal{L}_{hol}$) (resp. (4.8)) or M^* satisfies (2.2), then the Φ-function M satisfies ($\mathcal{L}^1_{loc}$) (see Figure 4.1).							

TABLE 4.1: Summary of density results in Musielak-Sobolev spaces.

Chapter 5

Poincaré-type inequalities in Musielak spaces

5.1 Introduction

In this chapter we investigate the Poincaré-type inequalities in Musielak-Sobolev spaces.

Let Ω be a bounded open subset of $\mathbf{R}^N$, $N \geq 1$. The classical Poincaré integral inequality asserts that

$$\int_\Omega |u(x)|^p dx \leq C(\Omega, p) \int_\Omega |\nabla u(x)|^p dx \tag{5.1}$$

for every $u \in W_0^{1,p}(\Omega)$ where $C(\Omega, p)$ is a constant depending on Ω and p. In fact, this inequality remains valid if Ω is only bounded in one direction. Recalling here that when Ω is regular (see for instance [85, Theorem 4.14]) we have

$$W_0^{1,p}(\Omega) = \left\{u \in W^{1,p}(\Omega) : tr(u) = 0 \text{ on } \partial\Omega\right\} \tag{5.2}$$

and hence $W_0^{1,p}(\Omega) = W_0^{1,1}(\Omega) \cap W^{1,p}(\Omega)$.
Gossez [37, Lemma 5.7] proved the existence of two constants $c_m > 0$ and $c_{m,\Omega} > 0$ such the following Orlicz version of Poincaré integral inequality

$$\int_\Omega \sum_{|\alpha|<m} \varphi(|D^\alpha u(x)|)dx \leq c_m \int_\Omega \sum_{|\alpha|=m} \varphi(c_{m,\Omega}|D^\alpha u(x)|)dx, \tag{5.3}$$

holds for every $u \in W_0^m L_\varphi(\Omega)$. Since no extra condition is assumed on φ, inequality (5.3) proved in $W_0^m L_\varphi(\Omega)$ covers not only (5.1) but it remains valid for a wide class of Orlicz functions.
Unfortunately, in the framework of variable exponent spaces the situation is more complicated and more regularities on the exponent are needed (see General introduction).

In order to generalize the Poincaré-type inequalities in Musielak-Sobolev spaces and cover the different results obtained in the above modular spaces, our main results of this chapter are given in 4 Theorems (see Theorems 5.3.1, 5.4.1, 5.4.2 and 4.7.1).

The chapter is organized as follow, in Section 5.2 we introduce and discuss the new regularity assumption $(\mathcal{M}_{on})$ bellow, to get the Poincaré-type inequalities. Section 5.3 is devoted to the Poincaré-type inequalities in absence of Lavrentiev phenomenon, the main result of this section is Theorems 5.3.1. This later theorem is a generalization of [37, Lemma 5.7] in the framework of Orlicz spaces where only the definition of the space $W_0^m L_\varphi(\Omega)$ (Definition 1.3.10) i.e by means of the weak-$*$ topology, is used to get the Poincaré integral inequality without assuming the segment property on the bounded open Ω. As in the classical way, the Poincaré integral inequality was first proved for smooth functions and then (5.4) follows by a density argument based on mollifications.
In general, the shift operator is not acting between Musielak spaces (see General introduction). So we face a major difficulty in using mollification and then we can not use the same approach as in [37].
Our contribution to overcame this problem consists in using the regularity conditions $(\mathcal{L}^1_{loc})$ and $(\mathcal{L}_{hol})$ on the Φ-function M and the segment property on the domain Ω (see Definition 1.2.7). Those conditions enable us to get the modular density of C_0^∞-functions in $W_0^m L_M(\Omega)$ (see Theorem 4.5.2) which we use in the proof of Theorem 5.3.1 instead the weak-* density as it was done in [37].
In Section 5.4 we give further results on Poincaré-type inequalities, the main results of this section are Theorem 5.4.1, Theorem 5.4.2 and Theorem 5.4.3. In Theorem 5.4.1 we prove the Poincaré-type inequalities in the space $W_0^m E_M(\Omega)$. We note here that by the definition of $W_0^m E_M(\Omega)$, we do not need to assume in the above Theorem 5.4.1 the segment property on Ω and the condition $(\mathcal{L}_{hol})$ on the Φ-function M. Therefore, the result we obtain covers the Poincaré integral inequality obtained by Maeda [77] for $\mathcal{C}_0^1$-functions in the case where the variable exponent $p(\cdot)$ is assumed to satisfy a monotony condition (see Example 5.2.1). While in Theorem 5.4.2 we prove the Poincaré-type inequalities in the space $K_0^m L_M(\Omega)$ (see Definition 5.4.1), for the particular case $M(\cdot, s) = |s|^{p(\cdot)}$ see [28] where a deep study was devoted to the space defined as the norm closure in $W^{1,p(\cdot)}(\Omega)$ of the set of $W^{1,p(\cdot)}(\Omega)$-functions with compact support in Ω. In the end of Section 5.4 we give Theorem 5.4.3 where we address the validity of Poincaré-types inequalities in the zero trace Musielak-Sobolev spaces.

5.2 Regular Musielak functions of monotone type $(\mathcal{M}_{on})$

In this section we introduce a monotony type regularity assumption on the Φ-function with respect to its first argument.

Definition 5.2.1. *A Φ-function M is said to satisfy the $\mathcal{M}_{on}$-condition on a segment $[a,b]$ of the real line $\mathbb{R}$, if*
Either

$$(\mathcal{M}_{on})\quad \begin{cases} (\mathcal{M}_{on})_0 : \begin{cases} \text{There exist } t_0 \in \mathbb{R}^+ \text{ and } 1 \leq i \leq N \text{ such that the partial} \\ \text{function } x_i \in [a,b] \mapsto M(x,t) \text{ changes constantly its} \\ \text{monotony on both sides of } t_0 \text{ (that is for } t \geq t_0 \text{ and } t < t_0), \end{cases} \\ \\ \text{Or} \\ \\ (\mathcal{M}_{on})_\infty : \begin{cases} \text{There exists } 1 \leq i \leq N \text{ such that for all } t \geq 0, \text{ the partial} \\ \text{function } x_i \in [a,b] \mapsto M(x,t) \text{ is monotone on } [a,b]. \end{cases} \end{cases}$$

Here, x_i stands for the i^{th} component of $x \in \Omega$.

Some comments and examples

A Φ-function $M : \Omega \times \mathbb{R}^+ \to \mathbb{R}^+$ satisfies the $(\mathcal{M}_{on})_0$-condition on $[a,b] \subset \Omega$ with respect to x_1 if the following hold:

- $O = \big\{x' = (x_2, \cdots, x_N) \in \mathbb{R}^{N-1}; (x_1, x') \in \Omega$ for any $x_1 \in [a,b]\big\}$ is non-empty.
  (Consequently for $t \in \mathbb{R}^+$, the partial mapping $x_1 \in [a,b] \to M\big((x_1,x'),t\big) \in \mathbb{R}^+$ makes sense for any $x' \in O$).

- There exists $t_0 > 0$ such that for any $x' \in O$, if $t \geq t_0$ the map $x_1 \in [a,b] \to M\big((x_1,x'),t\big) \in \mathbb{R}^+$ has certain type of monotony, for example it is increasing (decreasing), while for $t < t_0$ the map $x_1 \in [a,b] \to M\big((x_1,x'),t\big)$ decreasing (increasing).

Example 5.2.1. *As examples of Φ-functions satisfying the condition $(\mathcal{M}_{on})$ and thus are admissible in our case we give.*

1. *In the case where $M(x,s) = s^{p(x)}$, the assumption $(\mathcal{M}_{on})_0$ prevents the variable exponent $p(\cdot)$ to get a local extremum while $(\mathcal{M}_{on})_\infty$ is not satisfied unless $p(\cdot)$ is a constant function. In fact assume that $p : \Omega \to \mathbb{R}^+$ satisfies $1 < p(\cdot) < \infty$ and the following:*

   - *$O = \{x' = (x_2, \cdots, x_N) \in \mathbb{R}^{N-1}; (x_1, x') \in \Omega$ for any $x_1 \in [a,b]\}$ is non-empty.*

   - *For any $x' \in O$ the mapping $x_1 \in [a,b] \to p(x_1,x')$ is non-decreasing.*
     Take $M(x,s) = s^{p(x)}$, $x \in \Omega$, $s \geq 0$ and let $s_0 = 1$. Then, for $s \geq 1$ $x_1 \in [a,b] \to M(x,s) = s^{p(x_1,x')}$ is non-decreasing for any $x \in \Omega$ while for $s < 1$, $x_1 \in [a,b] \to M(x,s) = s^{p(x_1,x')}$ is non-increasing for any $x \in \Omega$. Consequently, $M(x,s) = s^{p(x)}$ satisfies the $(\mathcal{M}_{on})_0$-condition on $[a,b]$ with $i = 1$.

2. *Let us consider the double phase function* $M(x,s) = s^p + a(x)s^q$.

   - *If there is* $1 \le i \le N$ *such that the function* $x_i \mapsto a(x)$ *is monotone then* M *satisfies obviously* $(\mathcal{M}_{on})_\infty$ *and so* $(\mathcal{M}_{on})$.
   - *If* $x_i \mapsto a(x)$ *is not a constant function then the double phase function* M *can not satisfy* $(\mathcal{M}_{on})_0$.

3. *If* $1 < p(\cdot) < +\infty$ *and there exists* $1 \le i \le N$ *such that the function* $x_i \mapsto p(x)$ *is monotone on a compact subset of the real line* $\mathbb{R}$, *then the following* Φ*-functions*

   - $M_1(x,s) = s^{p(x)}$,
   - $M_2(x,s) = s^{p(x)} \log(e+s)$,
   - $M_3(x,s) = e^{s^{p(x)}} - 1$,

   satisfy $(\mathcal{M}_{on})$.

5.3 Poincaré-type inequalities in absence of Lavrentiev phenomenon

In this subsection we give our first main result of this chapter, concerns Poincaré-type inequalities in the Musielak spaces $W_0^m L_M(\Omega)$.

Theorem 5.3.1. *Let* Ω *be a bounded open subset in* $\mathbb{R}^N$ *having the segment property. Let* (M, M^*) *be a pair of complementary* Φ*-functions. Assume that* M *satisfies* $(\mathcal{L}_{hol})$ *and* $(\mathcal{M}_{on})$ *and* M^* *satisfies* $(\mathcal{L}^1_{loc})$. *Then there exists a constant* $c_{m,\Omega}$ *depending only on* m *and* Ω *such that for every* $u \in W_0^m L_M(\Omega)$

$$\int_\Omega \sum_{|\alpha|<m} M(x, |D^\alpha u|)dx \le \int_\Omega \sum_{|\alpha|=m} M(x, c_{m,\Omega}|D^\alpha u|)dx. \tag{5.4}$$

Moreover, for every $u \in W_0^m L_M(\Omega)$

$$\sum_{|\alpha|<m} \|D^\alpha u\|_M \le C(m,\Omega) \sum_{|\alpha|=m} \|D^\alpha u\|_M, \tag{5.5}$$

where $C(m,\Omega)$ *is a constant depending only on* m *and* Ω.

Proof. Let d be the diameter of Ω. As $(\mathcal{M}_{on})$ is concerned and without loss of generality, we can assume that $i = 1$. Being Ω bounded, using a translation if necessary, we may assume that it is contained in the strip $\Omega \subset \{(x_1, x') \in [0,d] \times \mathbb{R}^{N-1}\}$. Let $\partial_1 := \dfrac{\partial}{\partial x_1}$ stands for the partial derivative operator with respect to x_1 and let us first assume that $u \in \mathcal{C}_0^\infty(\Omega)$.

Part 1 : We assume that there exists $t_0 \in \mathbb{R}^+$ such that the function $x_1 \in [0,d] \mapsto M\Big((x_1,x'),t\Big)$ changes the variation on both sides of t_0.

Case 1. Assume that $x_1 \in [0,d] \mapsto M\Big((x_1,x'),t\Big)$ is non-decreasing for $t \le t_0$ and non-increasing for $t_0 < t$.
Defining the two sets

$$E_1 = \Big\{\xi \in [0,d] : |\partial_1 u(\xi,x')| \le \frac{1}{2d} t_0\Big\} \text{ and } E_2 = \Big\{\xi \in [0,d] : |\partial_1 u(\xi,x')| > \frac{1}{2d} t_0\Big\},$$

we can write

$$\begin{aligned} u(x_1,x') &= u(x_1,x')\chi_{E_1}(x_1) + u(x_1,x')\chi_{E_2}(x_1) \\ &= -\int_{x_1}^{d} \partial_1\Big(u(\xi,x')\chi_{E_1}(\xi)\Big)d\xi + \int_0^{x_1} \partial_1\Big(u(\xi,x')\chi_{E_2}(\xi)\Big)d\xi \\ &= -\int_{[x_1,d]\cap E_1} \partial_1 u(\xi,x')d\xi + \int_{[0,x_1]\cap E_2} \partial_1 u(\xi,x')d\xi. \end{aligned}$$

Thus,

$$|u(x_1,x')| \le \int_{[x_1,d]\cap E_1} \Big|\partial_1 u(\xi,x')\Big|d\xi + \int_{[0,x_1]\cap E_2} \Big|\partial_1 u(\xi,x')\Big|d\xi.$$

Then, the convexity of the Φ-function M and Jensen's inequality enable us to write

$$\begin{aligned} M\Big(x,|u(x_1,x')|\Big) &\le \frac{1}{2d}\int_0^d M\Big((x_1,x'),2d\Big|\partial_1 u(\xi,x')\Big|\chi_{[x_1,d]\cap E_1}(\xi)\Big)d\xi \\ &+ \frac{1}{2d}\int_0^d M\Big((x_1,x'),2d\Big|\partial_1 u(\xi,x')\Big|\chi_{[0,x_1]\cap E_2}(\xi)\Big)d\xi \\ &\le \frac{1}{2d}\int_{[x_1,d]\cap E_1} M\Big((\xi,x'),2d\Big|\partial_1 u(\xi,x')\Big|\Big)d\xi \\ &+ \frac{1}{2d}\int_{[0,x_1]\cap E_2} M\Big((\xi,x'),2d\Big|\partial_1 u(\xi,x')\Big|\Big)d\xi \\ &\le \frac{1}{2d}\int_0^d M\Big((\xi,x'),2d\Big|\partial_1 u(\xi,x')\Big|\Big)d\xi. \end{aligned}$$

Integrating successively with respect to x' and x_1, we obtain

$$\int_\Omega M\Big(x,|u(x)|\Big)dx \le \frac{1}{2}\int_\Omega M\Big(x,2d\Big|\partial_1 u(x)\Big|\Big)dx. \tag{5.6}$$

Case 2. Assume that $x_1 \in [0,d] \mapsto M\Big((x_1,x'),t\Big)$ is non-increasing on $t \leq t_0$ and non-decreasing on $t_0 < t$.
We can write

$$\begin{aligned}
u(x_1,x') &= u(x_1,x')\chi_{E_1}(x_1) + u(x_1,x')\chi_{E_2}(x_1) \\
&= \int_0^{x_1} \partial_1\Big(u(\xi,x')\chi_{E_1}(\xi)\Big)d\xi - \int_{x_1}^{d} \partial_1\Big(u(\xi,x')\chi_{E_2}(\xi)\Big)d\xi \\
&= \int_{[0,x_1]\cap E_1} \partial_1 u(\xi,x')d\xi - \int_{[x_1,d]\cap E_2} \partial_1 u(\xi,x')d\xi,
\end{aligned}$$

which implies

$$|u(x_1,x')| \leq \int_{[0,x_1]\cap E_1} \Big|\partial_1 u(\xi,x')\Big|d\xi + \int_{[x_1,d]\cap E_2} \Big|\partial_1 u(\xi,x')\Big|d\xi.$$

Once again the convexity of the Φ-function M and Jensen's inequality enable us to write

$$\begin{aligned}
M\Big(x,|u(x_1,x')|\Big) &\leq \frac{1}{2d}\int_0^d M\Big((x_1,x'),2d\Big|\partial_1 u(\xi,x')\Big|\chi_{[0,x_1]\cap E_1}(\xi)\Big)d\xi \\
&+ \frac{1}{2d}\int_0^d M\Big((x_1,x'),2d\Big|\partial_1 u(\xi,x')\Big|\chi_{[x_1,d]\cap E_2}(\xi)\Big)d\xi \\
&\leq \frac{1}{2d}\int_{[0,x_1]\cap E_1} M\Big((\xi,x'),2d\Big|\partial_1 u(\xi,x')\Big|\Big)d\xi \\
&+ \frac{1}{2d}\int_{[x_1,d]\cap E_2} M\Big((\xi,x'),2d\Big|\partial_1 u(\xi,x')\Big|\Big)d\xi \\
&\leq \frac{1}{2d}\int_0^d M\Big((\xi,x'),2d\Big|\partial_1 u(\xi,x')\Big|\Big)d\xi.
\end{aligned}$$

Integrating successively with respect to x' and x_1, we obtain

$$\int_\Omega M\Big(x,|u(x)|\Big)dx \leq \frac{1}{2}\int_\Omega M\Big(x,2d\Big|\partial_1 u(x)\Big|\Big)dx.$$

Part 2 : Assume now that for all $t \geq 0$, the function $x_1 \in [0,d] \mapsto M\Big((x_1,x'),t\Big)$ is monotone.
Case 1. Assume first that $x_1 \in [0,d] \mapsto M\Big((x_1,x'),t\Big)$ is non-increasing.

By using Jensen's inequality we get

$$\begin{aligned}
M\Big(x,|u(x_1,x')|\Big) &\leq M\Big((x_1,x'),\int_0^{x_1}\Big|\partial_1 u(\xi,x')\Big|d\xi\Big)\\
&\leq M\Big((x_1,x'),\int_0^{d}\Big|\partial_1 u(\xi,x')\Big|\chi_{[0,x_1]}(\xi)d\xi\Big)\\
&\leq \frac{1}{d}\int_0^{d} M\Big((x_1,x'),d\Big|\partial_1 u(\xi,x')\Big|\chi_{[0,x_1]}(\xi)\Big)d\xi\\
&\leq \frac{1}{d}\int_0^{x_1} M\Big((\xi,x'),d\Big|\partial_1 u(\xi,x')\Big|\Big)d\xi\\
&\leq \frac{1}{d}\int_0^{d} M\Big((\xi,x'),d\Big|\partial_1 u(\xi,x')\Big|\Big)d\xi.
\end{aligned}$$

Integrating successively with respect to x' and x_1, we obtain

$$\int_\Omega M\Big(x,|u(x)|\Big)dx \leq \int_\Omega M\Big(x,d\Big|\partial_1 u(x)\Big|\Big)dx.$$

Case 2. Assume that $x_1 \in [0,d] \mapsto M\Big((x_1,x'),t\Big)$ is non-decreasing. By virtue of Jensen's inequality we can write

$$\begin{aligned}
M\Big(x,|u(x_1,x')|\Big) &\leq M\Big((x_1,x'),\int_{x_1}^{d}\Big|\partial_1 u(\xi,x')\Big|d\xi\Big)\\
&\leq M\Big((x_1,x'),\int_0^{d}\Big|\partial_1 u(\xi,x')\Big|\chi_{[x_1,d]}(\xi)d\xi\Big)\\
&\leq \frac{1}{d}\int_0^{d} M\Big((x_1,x'),d\Big|\partial_1 u(\xi,x')\Big|\chi_{[x_1,d]}(\xi)\Big)d\xi\\
&\leq \frac{1}{d}\int_{x_1}^{d} M\Big((\xi,x'),d\Big|\partial_1 u(\xi,x')\Big|\Big)d\xi\\
&\leq \frac{1}{d}\int_0^{d} M\Big((\xi,x'),d|\partial_1 u(\xi,x')|\Big)d\xi.
\end{aligned}$$

Integrating successively with respect to x' and x_1, we obtain

$$\int_\Omega M\Big(x,|u(x)|\Big)dx \leq \int_\Omega M\Big(x,d\Big|\partial_1 u(x)\Big|\Big)dx. \tag{5.7}$$

To sum up, from (5.6)-(5.7), we obtain

$$\int_\Omega M\Big(x,|u(x)|\Big)dx \leq \int_\Omega M\Big(x,2d\Big|\partial_1 u(x)\Big|\Big)dx, \tag{5.8}$$

for all $u \in \mathcal{C}_0^\infty(\Omega)$.
Let now $u \in W_0^1 L_M(\Omega)$ be arbitrary. By Theorem 4.5.2 there exist $\lambda > 0$ and a sequence of functions $u_k \in \mathcal{C}_0^\infty(\Omega)$ such that

$$\int_\Omega M\Big(x,\frac{|u_k(x)-u(x)|}{\lambda}\Big)dx + \int_\Omega M\Big(x,\frac{|\nabla u_k(x)-\nabla u(x)|}{\lambda}\Big)dx \to 0$$

as $k \to +\infty$. Hence, up to a subsequence still again indexed by k, we can assume that $u_k \to u$ a.e. in Ω. Then, using (5.8) we can write

$$\begin{aligned}
\int_\Omega M\Big(x,\frac{|u(x)|}{4\lambda d}\Big)dx &\leq \liminf_{k\to+\infty}\int_\Omega M\Big(x,\frac{|u_k(x)|}{4\lambda d}\Big)dx \\
&\leq \liminf_{k\to+\infty}\int_\Omega M\Big(x,\frac{1}{2\lambda}\Big|\partial_1 u_k(x)\Big|\Big)dx \\
&\leq \frac{1}{2}\liminf_{k\to+\infty}\int_\Omega M\Big(x,\frac{1}{\lambda}\Big|\partial_1 u_k(x)-\partial_1 u(x)\Big|\Big)dx \\
&+ \frac{1}{2}\int_\Omega M\Big(x,\frac{1}{\lambda}\Big|\partial_1 u(x)\Big|\Big)dx \\
&\leq \frac{1}{2}\int_\Omega M\Big(x,\frac{1}{\lambda}\Big|\partial_1 u(x)\Big|\Big)dx.
\end{aligned}$$

Thus, (5.4) is proved. Let us now prove the inequality (5.5). For $u \in W_0^m L_M(\Omega)$, it can be checked easily from (5.4) that

$$\sum_{|\alpha|<m}\int_\Omega M\Big(x,\frac{|D^\alpha u(x)|}{C(m,\Omega)\sum_{|\beta|=m}\|D^\beta u\|_M}\Big)dx \leq 1,$$

where $C(m,\Omega) = c_{m,\Omega}\Big(1+\sum_{|\beta|=m}1\Big)$ depending only on m and Ω. The proof of Theorem 5.3.1 is then achieved. □

Remark 5.3.1. *A direct consequence of the inequality (5.5), is that the maps*

$$u \in W_0^m L_M(\Omega) \to \sum_{|\alpha|\leq m}\|D^\alpha u\|_M$$

and

$$u \in W_0^m L_M(\Omega) \to \sum_{|\alpha|=m}\|D^\alpha u\|_M$$

define equivalent norms on $W_0^m L_M(\Omega)$.

5.4 Further results on Poincaré-type inequalities

The following theorem concerns the Poincaré integral inequality in the Musielak-Sobolev space $W_0^m E_M(\Omega)$ defined as the norm closure of $\mathcal{C}_0^\infty$-functions in $W^m E_M(\Omega)$.

Theorem 5.4.1. *Let Ω be a bounded open subset in $\mathbb{R}^N$ and let $M \in \Phi$ satisfies ($\mathcal{L}^1_{loc}$) and ($\mathcal{M}_{on}$). Then inequality (5.4) holds true for every $u \in W_0^m E_M(\Omega)$ and then so is (5.5).*

Proof. Let $u \in W_0^m E_M(\Omega)$. By the definition of $W_0^m E_M(\Omega)$, there exists a sequence $\{u_k\}_k$ of $\mathcal{C}_0^\infty(\Omega)$ functions such that $D^\alpha u_k \to D^\alpha u$ for all $|\alpha| \leq m$ with respect to the norm topology in $L_M(\Omega)$. As the norm convergence implies the modular one, one has $D^\alpha u_k \to D^\alpha u$ for all $|\alpha| \leq m$ with respect to the modular topology. Therefore, we get the result by following exactly the same lines of the proof of Theorem 5.3.1. □

Definition 5.4.1. *Let Ω be a bounded open subset of $\mathbb{R}^N$ and $M \in \Phi$ we define the space $K_0^m L_M(\Omega)$ as the norm closure of the set of $W^m L_M(\Omega)$ functions with compact support in Ω.*

In the particular case $M(x,t) = t^{p(x)}$, $K_0^m L_M(\Omega)$ is nothing but the space $W_0^{m,p(x)}(\Omega)$ defined in [28].

Theorem 5.4.2. *Let Ω be an open subset in $\mathbb{R}^N$ and let $M \in \Phi$ satisfies ($\mathcal{L}_{hol}$). Then $K_0^m L_M(\Omega)$ coincides with $W_0^m E_M(\Omega)$. Furthermore, if Ω is bounded and M satisfies ($\mathcal{M}_{on}$) then (5.4) and (5.5) are fulfilled.*

Proof. The embedding $W_0^m E_M(\Omega) \subset K_0^m L_M(\Omega)$ is obviously satisfied. It only remains to show that $K_0^m L_M(\Omega) \subset W_0^m E_M(\Omega)$ holds true. Let $u \in K_0^m L_M(\Omega)$ and let $\eta > 0$ be arbitrary. We will show that there is a sequence $v \in \mathcal{C}_0^\infty(\Omega)$ such

$$\sum_{|\alpha|\leq m} \|D^\alpha u - D^\alpha v\|_M \leq \eta. \tag{5.9}$$

By the definition of $K_0^m L_M(\Omega)$ there exist a sequence $\{u_k\}$ in $W^m L_M(\Omega)$ of compactly supported functions in Ω and $k_\alpha > 0$ such that for all $k > k_\alpha$ and $|\alpha| \leq m$ we have

$$\|D^\alpha u - D^\alpha u_k\|_M \leq \frac{\eta}{2K}.$$

where K is the total number of multi-indices with $|\alpha| \leq m$. Now by using Lemma 4.3.6 there exist a sequence $\{u_k^n\}$ in $\mathcal{C}_0^\infty(\Omega)$ and $n_\eta^{k,\alpha} > 0$ such that for all $n > n_\eta^{k,\alpha}$ and $|\alpha| \leq m$ we have

$$\|D^\alpha u_k - D^\alpha u_k^n\|_M \leq \frac{\eta}{2K}.$$

By the triangle inequality we get for all $k > k_0^\alpha$ and $n > n_\eta^{k,\alpha}$

$$\begin{aligned}\|D^\alpha u - D^\alpha u_k^n\|_M &\leq \|D^\alpha u - D^\alpha u_k)\|_M + \|D^\alpha u_k - D^\alpha u_k^n\|_M \\ &\leq \frac{\eta}{K},\end{aligned}$$

Hence follows (5.9) and then the following identification

$$K_0^m L_M(\Omega) = W_0^m E_M(\Omega).$$

Therefore, Theorem 5.4.2 follows immediately from Theorem 5.4.1. □

Now, the remaining question is how to provide a satisfactory generalization of the Poincaré inequality for constant exponent, because the equality

$$W_0^{1,p}(\Omega) = \big\{u \in W^{1,p}(\Omega) : tr(u) = 0 \text{ on } \partial\Omega\big\}$$

is substituted by the inclusion

$$W_0^1 L_M(\Omega) = \overline{\mathcal{C}_0^\infty(\Omega)}^{\sigma(\Pi L_M, \Pi E_{M^*})} \subset \big\{u \in W^1 L_M(\Omega) : tr(u) = 0 \text{ on } \partial\Omega\big\}$$

which may be strict in general unless additional conditions are imposed on the Φ-function M. We give the answer in the following theorem.

Theorem 5.4.3. *Let Ω be a bounded open subset in $\mathbb{R}^N$ having a Lipschitz boundary $\partial\Omega$ and let (M, M^*) be a pair of complementary Φ-functions. Assume that M satisfies $(\mathcal{L}_{hol})$, $(\mathcal{M}_{on})$ and (2.2). Then, (5.4) and (5.5) hold true in the space*

$$W_0^1 L_M(\Omega) = \{u \in W^1 L_M(\Omega) : \; tr(u) = 0 \text{ on } \partial\Omega\}.$$

Proof. The proof follows immediately from Theorem 4.7.1 and Theorem 5.3.1. □

5.5 Conclusion

In this chapter, we introduced conditions on the Musielak functions under which we proved the Poincaré-type inequalities in the functional Musielak-Sobolev structure. We highlight that the results we get extend the ones already well known in Sobolev, Orlicz and variable exponent Sobolev spaces.
We summarize in the following table the main results obtained in this chapter.

	M satisfies $(\mathcal{M}_{on})$	M satisfies $(\mathcal{L}_{hol})$	M satisfies $(\mathcal{L}^1_{loc})$	M^* satisfies $(\mathcal{L}^1_{loc})$	M satisfies (2.2)	Domain geometry	Poincaré-type ineq. (5.4) and (5.5)	
$W^m_0 L_M(\Omega)$	✓	✓	✓	✓		bounded + segment property	✓	Theorem 5.3.1
$W^m_0 E_M(\Omega)$	✓		✓			bounded	✓	Theorem 5.4.1
$W^{m,1}_0(\Omega) \cap W^m L_M(\Omega)$	✓	✓	✓	✓		bounded + segment property	✓	Theorems 5.3.1 and 4.7.1
$K^m_0 L_M(\Omega)$	✓	✓	✓			bounded	✓	Theorem 5.4.2
$u \in W^1 L_M(\Omega)$ $\mathrm{tr}(u) = 0$	✓	✓	✓	✓	✓	bounded + Lipschitz boundary	✓	Theorem 5.4.3

NB:
- If M satisfies $(\mathcal{L}_{hol})$ (resp. (4.8)), then the Φ-function M satisfies $(\mathcal{L}^1_{loc})$ (see Figure 4.1).
- If M satisfies (2.2) then M^* satisfies $(\mathcal{L}^1_{loc})$ (see Figure 4.1)
- Poincaré's inequalities remains valid if Ω is only bounded in one direction.

TABLE 5.1: Summary of Poincaré-type inequalities in Musielak-Sobolev spaces.

Part II

Application to partial differential equations

Introduction to Part II

Partial differential equations are ubiquitous in several domains and present mathematical tools to describe a wide variety of natural phenomena like electromagnetism, elasticity, diffusion, Hydrodynamics, etc.

This Part contains Chapter 6 which is dedicated to the application of the mathematical results we have obtained in the first part, to solve a class of elliptic equations with rapidly/slowly increasing coefficients arising in some non-Newtonian fluids.

In particular we prove the existence and uniqueness of variational solutions $u : \Omega \to \mathbb{R}^N$ for the following nonlinear elliptic problem

$$\begin{cases} -\text{div } a(x, \nabla u) & = f \quad \text{in } \Omega, \\ \qquad\qquad u & = 0 \quad \text{on } \partial\Omega, \end{cases}$$

where Ω is a bounded open subset of $\mathbb{R}^N$, $N \geq 1$, and a is a stress tensor whose growth may be described by non homogeneous Musielak functions M.

We stress that any extra growth restriction (neither Δ_2 nor ∇_2), on the pair of complementary Φ-functions (M, M^*), is imposed. That is the Φ-functions we consider allow to capture a several class of energy expressions, namely

$$\int_\Omega |\nabla u|^{p(x)} dx, \qquad \int_\Omega |\nabla u|^{p(x)} \log(e + |\nabla u|) dx,$$

$$\int_\Omega |\nabla u|^p + a(x)|\nabla u|^q dx, \qquad \int_\Omega b(x, w)(|\nabla u|^p + a(x)|\nabla u|^q) dx,$$

$$\int_\Omega (e^{p(x)|\nabla u|} - 1) dx, \qquad \int_\Omega (1 + |\nabla u|^{p(x)-2})|\nabla u|^2 dx.$$

However, we assume that the pair of complementary Φ-functions (M, M^*) generate the energy space $W_0^1 L_M$ satisfy some structural assumptions that allow a definition of the variational framework and obtain Poincaré-type inequalities. Within the approach we use, the Galerkin method is coupled with an abstract existence result by F.E. Browder 1966.

Chapter 6

Variational nonlinear elliptic equations in non reflexive Musielak spaces

6.1 Introduction

In this Chapter we present the study of the Dirichlet problem associated with a class of nonlinear elliptic equations. Precisely, we are interested in the existence and uniqueness of variational solutions $u : \Omega \to \mathbb{R}^N$ for the following nonlinear elliptic problem

$$\begin{cases} -\text{div } a(x, \nabla u) & = f \quad \text{in } \Omega, \\ \qquad\qquad u & = 0 \quad \text{on } \partial\Omega, \end{cases} \tag{6.1}$$

considered in a bounded open subset Ω of $\mathbb{R}^N$, $N \geq 1$, where the leading part a is a stress tensor whose growth is expressed through inhomogeneous Musielak function $M : \Omega \times \mathbb{R}^+ \to \mathbb{R}^+$. Precisely, we suppose that $a(x, \xi) : \Omega \times \mathbb{R}^{N\times N} \to \mathbb{R}^{N\times N}$ is a Carathéodory function (that is measurable in x for fixed ξ and continuous in ξ for fixed x in Ω) for which there exist some constants d_i, $c_i > 0$, $i \in \{1, 2\}$, and two functions $l \in E_{M^*}(\Omega)$ and $h \in L^1(\Omega)$ such that for a.e. $x \in \Omega$ and all $\xi, \xi' \in \mathbb{R}^{N\times N}$, with $\xi \neq \xi'$,

($\mathcal{A}_1$) Growth and coercivity.

$$|a(x, \xi)| \leq l(x) + c_1 {M^*}^{-1}(x, M(x, c_2|\xi|)) \tag{6.2}$$

and

$$a(x, \xi) : \xi \geq d_1 M(x, d_2|\xi|) - h(x). \tag{6.3}$$

($\mathcal{A}_2$) The monotony condition.

$$(a(x, \xi) - a(x, \xi')) : (\xi - \xi') > 0.$$

The approach used

It is our purpose in this chapter to solve the problem (6.1) in the variational framework, that is to find a vector-valued function $u \in W_0^1 L_M(\Omega, \mathbb{R}^N)$ that satisfies

$a(x, \nabla u) \in L_{M^*}(\Omega, \mathbb{R}^{N\times N})$ and

$$\langle \mathbf{A}u, v\rangle = \langle f, v\rangle, \qquad \text{for all } v \in W_0^1 L_M(\Omega, \mathbb{R}^N),$$

where $\mathbf{A}u := -\text{div}\ a(x, \nabla u)$ and $\langle \cdot, \cdot\rangle$ stands for the duality pairing. The construction of the variational framework is obtained by means of the sufficient regularity assumptions $(\mathcal{L}^1_{loc})$, $(\mathcal{L}_{hol})$ and $(\mathcal{M}_{on})$) allowing to get some basic properties of density and duality but also some Poincaré-type inequalities. The approach we use consists of coupling Galerkin's method in the separable space $W_0^1 E_M(\Omega, \mathbb{R}^N)$ (Lemma 6.2.3) with an abstract result in [17]. In order to extend into the whole space $W_0^1 L_M(\Omega, \mathbb{R}^N)$ we need to get the density of $W_0^1 E_M(\Omega, \mathbb{R}^N)$ with respect to the topology $\sigma(\Pi L_M, \Pi L_{M^*})$ in $W_0^1 L_M(\Omega, \mathbb{R}^N)$, which is the crucial step in our approach. This density result, that needs more regularity on Musielak function M, is provided by Lemma 6.2.4 (see also [7]).

The chapter is organized as follows. In section 6.2 we give some basic facts on vector-valued Musielak spaces. The main result is stated in section 6.4. In section 6.5 we provide the proof of the main result.

6.2 Vector-valued Musielak spaces

In this section we provide some results about vector-valued Musielak spaces and auxiliary results. For vector or tensor valued functions, we will use the Euclidean norm in $\mathbb{R}^N$ or $\mathbb{R}^{N\times N}$.

For $M \in \Phi$, the vector valued Musielak space, denoted $L_M(\Omega, \mathbb{R}^N)$ (resp. $E_M(\Omega, \mathbb{R}^N)$), is the set of all measurable functions $u := (u_1, u_2, \cdots, u_N) : \Omega \to \mathbb{R}^N$, such that each component u_i of u belongs to $L_M(\Omega)$ (resp. $E_M(\Omega)$). The space $L_M(\Omega, \mathbb{R}^N)$ (resp. $E_M(\Omega, \mathbb{R}^N)$) is endowed with the norm

$$\|u\|_M := \||u|\|_M := \sum_{i=1}^N \|u_i\|_M.$$

The vector-valued Musielak-Sobolev space denoted $W^1 L_M(\Omega, \mathbb{R}^N)$ is defined as the set of functions $u \in L_M(\Omega, \mathbb{R}^N)$ with each of its weak partial derivatives $\{\partial_j u_i\}$, $i, j \in \{1, 2, \cdots, N\}$, belonging to $L_M(\Omega)$. We can endow $W^1 L_M(\Omega, \mathbb{R}^N)$ with the norm

$$\begin{aligned}\|u\|_{W^1 L_M(\Omega,\mathbb{R}^N)} &= \inf\Big\{\lambda > 0 : \sum_{i=1}^N \int_\Omega M(x, |u_i(x)|/\lambda)dx \\ &\quad + \sum_{j=1}^N \sum_{i=1}^N \int_\Omega M(x, |\partial_j u_i(x)|/\lambda)dx \le 1\Big\}.\end{aligned}$$

Observe here that by a simple calculation for a vector valued function $u \in L_M(\Omega, \mathbb{R}^N)$ and $\lambda > 0$, the modular expressions

$$\int_\Omega M(x,|u|/\lambda)dx+\int_\Omega M(x,|\nabla u|/\lambda)dx$$

and

$$\sum_{i=1}^N\int_\Omega M(x,|u_i|/\lambda)dx+\sum_{j=1}^N\sum_{i=1}^N\int_\Omega M(x,|\partial_j u_i|/\lambda)dx$$

are equivalent and so are the norms $\|u\|_{W^1L_M(\Omega,\mathbb{R}^N)}$ and $\|u\|_M+\|\nabla u\|_M := \||u|\|_M+\||\nabla u|\|_M$.

In what follows, we identify the space $W^1L_M(\Omega,\mathbb{R}^N)$ to a subspace of the product ΠL_M of $N(1+N)$ copies of $L_M(\Omega)$ i.e. $\Pi L_M := (L_M(\Omega))^N\times(L_M(\Omega))^{N\times N}$. By $W^{-1}E_{M^*}(\Omega,\mathbb{R}^N)$, we denote the following space of distributions

$$W^{-1}E_{M^*}(\Omega,\mathbb{R}^N)=\Big\{f\in(\mathcal{C}_0^\infty(\Omega,\mathbb{R}^N))',\ f=-\mathrm{div}\ F,\ \text{ with } F\in E_{M^*}(\Omega,\mathbb{R}^{N\times N})\Big\}.$$

We will use the notation $A:B$ for the inner product of two tensors $A,B\in\mathbb{R}^{N\times N}$ given by

$$A:B=\sum_{1\le i,j\le N}A_{i,j}B_{i,j}. \tag{6.4}$$

Lemma 6.2.1. *Let (M,M^*) be a pair of complementary Φ-functions. Assume that M^* satisfies ($\mathcal{L}^1_{loc}$), then the spaces $W^1L_M(\Omega,\mathbb{R}^N)$ and $W^1E_M(\Omega,\mathbb{R}^N)$ endowed with the Luxemburg norm are both Banach spaces.*

Proof. Let K be an arbitrary compact subset of Ω. For any function $u\in L_M(\Omega,\mathbb{R}^N)$, we can write using Hölder's inequality

$$\int_K|u(x)|dx\le 2\|u\|_M\|\chi_K\|_{M^*}.$$

Since M^* satisfies ($\mathcal{L}^1_{loc}$), it is easy to see that

$$\|\chi_K\|_{M^*}\le\int_K M(x,1)dx+1<\infty,$$

which yields $L_M(\Omega,\mathbb{R}^N)\subset L^1_{loc}(\Omega,\mathbb{R}^N)$ (resp. $E_M(\Omega,\mathbb{R}^N)\subset L^1_{loc}(\Omega,\mathbb{R}^N)$). Therefore, by following exactly the same lines of Theorem 3.4.1, the proof follows without any difficulty and thus it is omitted. □

Lemma 6.2.2. *Let $M\in\Phi$ satisfies ($\mathcal{L}^1_{loc}$). Then the space $E_M(\Omega,\mathbb{R}^N)$ is separable.*

Proof. The proof follows from Theorem 3.3.2. The only change we can make is to replace the real function $u:\Omega\to\mathbb{R}$ by the vector valued function $u:=(u_1,u_2,\cdots,u_N)^T:\Omega\to\mathbb{R}^N$. □

Lemma 6.2.3. *Let $M\in\Phi$ satisfies ($\mathcal{L}^1_{loc}$). Then $W^1E_M(\Omega,\mathbb{R}^N)$ is a separable space.*

Proof. Let $u \in W^1E_M(\Omega, \mathbb{R}^N)$ we define the mapping $P : W^1E_M(\Omega, \mathbb{R}^N) \to \Pi E_M$ by

$$Pu = (u, \nabla u).$$

Clearly the mapping P is an isometric isomorphism of $W^1E_M(\Omega, \mathbb{R}^N)$ onto

$$X = PW^1E_M(\Omega, \mathbb{R}^N) \subset \Pi E_M.$$

Furthermore, since the space $E_M(\Omega, \mathbb{R}^N)$ is separable (see Lemma 6.2.2) we have ΠE_M is also separable and so the subspace X is a separable. Therefore $W^1E_M(\Omega, \mathbb{R}^N) = P^{-1}X$ is a separable space.

□

Lemma 6.2.4. *Let Ω be an open subset of $\mathbb{R}^N$ satisfying the segment property and let (M, M^*) be a pair of complementary Φ-functions. Suppose that M satisfies ($\mathcal{L}_{hol}$) and M^* satisfies ($\mathcal{L}^1_{loc}$). Then*

1. *$\mathcal{C}_0^\infty(\Omega, \mathbb{R}^N)$ is dense in $W_0^1L_M(\Omega, \mathbb{R}^N)$ with respect to the modular convergence.*
2. *$\mathcal{C}_0^\infty(\Omega, \mathbb{R}^N)$ is $\sigma(\Pi L_M, \Pi L_{M^*})$ dense in $W_0^1L_M(\Omega, \mathbb{R}^N)$.*

Proof. The proof follows from Theorem 4.5.2 and Corollary 4.5.1. The only change we can make is to replace the real function $u : \Omega \to \mathbb{R}$ by the vector valued function $u := (u_1, u_2, \cdots, u_N)^T : \Omega \to \mathbb{R}^N$. In particular changing the modular expressions

$$\rho_M(u/\lambda) := \int_\Omega M\Big(x, \frac{|u|}{\lambda}\Big) \quad \text{and} \quad \rho_M(\partial_j u/\lambda) := \int_\Omega M\Big(x, \frac{|\partial_j u|}{\lambda}\Big),$$

by

$$\sum_{i=1}^N \rho_M(u_i/\lambda) := \sum_{i=1}^N \int_\Omega M\Big(x, \frac{|u_i|}{\lambda}\Big) \quad \text{and} \quad \rho_M(\partial_j u/\lambda) := \sum_{i=1}^N \int_\Omega M\Big(x, \frac{|\partial_j u_i|}{\lambda}\Big)$$

respectively. Where ∂_j is the partial derivative and $j = 1, \cdots, N$. □

Lemma 6.2.5 (Poincaré-type inequalities)**.** *Let Ω be a bounded open subset of $\mathbb{R}^N$ satisfying the segment property and let (M, M^*) be a pair of complementary Φ-functions. Assume that M satisfy ($\mathcal{L}^1_{loc}$) and ($\mathcal{M}_{on}$). Then there exists a constant c_Ω depends only on Ω such that for every $u \in W_0^1E_M(\Omega, \mathbb{R}^N)$*

$$\int_\Omega M(x, |u|)dx \leq \int_\Omega M(x, c_\Omega |\nabla u|)dx \tag{6.5}$$

and for every $u \in W_0^1E_M(\Omega, \mathbb{R}^N)$

$$\|u\|_M \leq c(\Omega)\|\nabla u\|_M, \tag{6.6}$$

where $c(\Omega)$ is a constant depending only on Ω.

Moreover, if M satisfies ($\mathcal{L}_{hol}$), then (6.5) and (6.6) holds true for all $u \in W_0^1L_M(\Omega,\mathbb{R}^N)$ and the two norms $\|u\|_M$ and $\|\nabla u\|_M$ are equivalent on $W_0^1L_M(\Omega,\mathbb{R}^N)$.

Proof. The proof follows immediately from Theorem 5.3.1 and Theorem 5.4.1 using similar changes as in the proof of Lemma 6.2.4. □

In the following Theorem we give an identification of the dual of $W_0^1E_M(\Omega,\mathbb{R}^N)$ with a subspace of $(\mathcal{C}_0^\infty(\Omega,\mathbb{R}^N))'$. The proof is rather similar to that of Subsection 3.4.2. For the convenience of the reader we give it here.

Theorem 6.2.1. *Let (M, M^*) be a pair of complementary Φ-functions. Assume that M satisfy ($\mathcal{L}^1_{loc}$). Then, the dual space of $W_0^1E_M(\Omega,\mathbb{R}^N)$ denoted $(W_0^1E_M(\Omega,\mathbb{R}^N))'$, is isometrically isomorphic to the space $W^{-1}L_{M^*}(\Omega,\mathbb{R}^N)$ consisting of all distributions $T \in (\mathcal{C}_0^\infty(\Omega,\mathbb{R}^N))'$ which can be written*

$$T = f_0 - div\ F, \tag{6.7}$$

for some $f = (f_0, F) = (f_0, f_1, \cdots, f_{N(N+1)-1}) \in \Pi L_{M^*}$. Moreover

$$\|T\|_{W^{-1}L_{M^*}(\Omega,\mathbb{R}^N)} = \inf\big\{\|f\|_{\Pi L_{M^*}} : f \text{ satisfies } (6.7)\big\}.$$

In addition, if Ω is bounded and M satisfies ($\mathcal{M}_{on}$) we can take $f_0 = 0$.

The proof is quite similar to one given in the classical Sobolev spaces (see for instance [2]). For the convenience of the reader we give it here. We begin by the following two lemmas.

Lemma 6.2.6. *Let (M, M^*) be a pair of complementary Φ-functions. Assume that M satisfies ($\mathcal{L}^1_{loc}$). Then, $(\Pi E_M)'$ is isometrically isometric to ΠL_{M^*}. That is, for every $L \in (\Pi E_M)'$ there exists a unique vector of $(N + N^2)$ components $v = \Big((v_i)_{i=1,\cdots,N}, ((v_{i,j})_{j=1,\cdots,N})_{i=1,\cdots,N}\Big) \in \Pi L_{M^*}$ such that for every $u = \Big((u_i)_{i=1,\cdots,N}, ((u_{i,j})_{j=1,\cdots,N})_{i=1,\cdots,N}\Big) \in \Pi E_M$*

$$L(u) = \sum_{i=1}^N \langle u_i, v_i\rangle + \sum_{i,j=1}^N \langle u_{i,j}, v_{i,j}\rangle := \sum_{i=1}^N \int_\Omega u_i(x)v_i(x)dx + \sum_{i,j=1}^N \int_\Omega u_{i,j}(x)v_{i,j}(x)dx,$$

and the following Hölder inequality holds

$$|\sum_{i=1}^N \langle u_i, v_i\rangle + \sum_{i,j=1}^N \langle u_{i,j}, v_{i,j}\rangle| \le 2\|u\|_{\Pi L_M}\|v\|_{\Pi L_{M^*}}.$$

Proof. The proof follows easily from Lemma 3.3.3. □

The next Lemma is an extension to Musielak spaces of the one given in [2, Theorem 3.9].

Lemma 6.2.7. *Let (M, M^*) be a pair of complementary Φ-functions. Assume that M satisfies $(\mathcal{L}^1_{loc})$. Then for every $L \in (W^1E_M(\Omega, \mathbb{R}^N))'$ there exists $v = \Big((v_i)_{i=1,\cdots,N}, ((v_{i,j})_{j=1,\cdots,N})_{i=1,\cdots,N}\Big) \in \Pi L_{M^*}$ such that*

$$L(u) = \sum_{i=1}^{N} \langle u_i, v_i\rangle + \sum_{i,j=1}^{N} \langle \partial_{x_j} u_{i,j}, v_{i,j}\rangle, \qquad \forall u \in W^1E_M(\Omega, \mathbb{R}^N). \tag{6.8}$$

Moreover

$$\|L\|_{\left(W^1E_M(\Omega,\mathbb{R}^N)\right)'} \leq 2\|v\|_{\Pi L_{M^*}},$$

Proof. Define $P : W^1E_M(\Omega, \mathbb{R}^N) \to \Pi E_M$ with

$$P(u) = (u, \nabla u) := \Big((u_i)_{i=1,\cdots,N}, ((\partial_{x_j} u_i)_{j=1,\cdots,N})_{i=1,\cdots,N}\Big).$$

for every $u \in W^1E_M(\Omega, \mathbb{R}^N)$. Let $X = P(W^1E_M(\Omega, \mathbb{R}^N))$ and consider the functional $\widetilde{L}$ defined on X as follows

$$\widetilde{L}(P(u)) = L(u), \ \forall u \in W^1E_M(\Omega, \mathbb{R}^N).$$

The functional P is an isometric isomorphism of $W^1E_M(\Omega, \mathbb{R}^N)$ onto the subspace X of ΠE_M. We easily check that $\widetilde{L} \in X'$ and $\|\widetilde{L}\|_{X'} = \|L\|_{\left(W^1E_M(\Omega,\mathbb{R}^N)\right)'}$. Thus, by the Hahn-Banach extension Theorem there exists a norm preserving extension $\widehat{L}$ of $\widetilde{L}$ to all of ΠE_M. Then, in view of Lemma 6.2.6 there exists a unique, up to sets of null Lebesgue measure, $v = \Big((v_i)_{i=1,\cdots,N}, ((v_{i,j})_{j=1,\cdots,N})_{i=1,\cdots,N}\Big) \in \Pi L_{M^*}$ such that for any element $u = \Big((u_i)_{i=1,\cdots,N}, ((u_{i,j})_{j=1,\cdots,N})_{i=1,\cdots,N}\Big) \in \Pi E_M$

$$\widehat{L}(u) = \sum_{i=1}^{N} \langle u_i, v_i\rangle + \sum_{i,j=1}^{N} \langle u_{i,j}, v_{i,j}\rangle.$$

Thus, for $u \in W^1E_M(\Omega, \mathbb{R}^N)$ we get

$$L(u) = \widetilde{L}(P(u)) = \widehat{L}(P(u)) = \sum_{i=1}^{N} \langle u_i, v_i\rangle + \sum_{i,j=1}^{N} \langle \partial_{x_j} u_i, v_{i,j}\rangle.$$

Furthermore,

$$\|L\|_{\left(W^1E_M(\Omega,\mathbb{R}^N)\right)'} = \|\widetilde{L}\|_{X'} = \|\widehat{L}\|_{(\Pi E_M)'} \leq 2\|v\|_{\Pi L_{M^*}}.$$

Now since any element $v \in \Pi L_{M^*}$ for which (6.8) holds for every $u \in W^1E_M(\Omega,\mathbb{R}^N)$ corresponds to an extension $\widehat{L}$ of $\widetilde{L}$ and so will have norm $\|v\|_{\Pi L_{M^*}}$ superior than $\|L\|_{\left(W^1E_M(\Omega,\mathbb{R}^N)\right)'}$. □

We are now able to prove Theorem 6.2.1. Let $L \in (W_0^1E_M(\Omega,\mathbb{R}^N))'$ given by (6.8) for some $v = \Big((v_i)_{i=1,\cdots,N}, ((v_{i,j})_{j=1,\cdots,N})_{i=1,\cdots,N}\Big) \in \Pi L_{M^*}$ and let T be defined on $\mathcal{C}_0^\infty(\Omega,\mathbb{R}^N)$ by

$$T(\varphi) = \sum_{i=1}^{N}\int_\Omega \varphi_i(x)v_i(x)dx + \sum_{i,j=1}^{N}\int_\Omega \partial_{x_j}\varphi_i(x)v_{i,j}(x)dx, \tag{6.9}$$

thus for any $\varphi := (\varphi_i)_{i=1,\cdots,N} \in \mathcal{C}_0^\infty(\Omega,\mathbb{R}^N) \subset W_0^1E_M(\Omega,\mathbb{R}^N)$ we have $T(\varphi) = L(\varphi)$. Hence, the restriction of L to $\mathcal{C}_0^\infty(\Omega,\mathbb{R}^N)$, is the distribution T.

Let now T any distribution of the form (6.9), then T has a unique extension $\widetilde{T}$ to $W_0^1E_M(\Omega,\mathbb{R}^N)$. Indeed, for $u = (u_i)_{i=1,\cdots,N} \in W_0^1E_M(\Omega,\mathbb{R}^N)$ there exists $\varphi_n = (\varphi_i^n)_{i=1,\cdots,N} \in \mathcal{C}_0^\infty(\Omega,\mathbb{R}^N)$, $i = 1,\cdots,N$, such that $\varphi_n \to u$ in norm in $W_0^1E_M(\Omega,\mathbb{R}^N)$. Then, by Hölder's inequality we can write

$$\begin{aligned}
|T(\varphi_k) - T(\varphi_n)| &\leq \sum_{i=1}^{N}\int_\Omega |\varphi_i^k - \varphi_i^n||v_i|dx + \sum_{i,j=1}^{N}\int_\Omega |\partial_{x_j}\varphi_i^k - \partial_{x_j}\varphi_i^n||v_{i,j}|dx \\
&\leq 2\sum_{i=1}^{N}\|\varphi_i^k - \varphi_i^n\|_M\|v_i\|_{M^*} \\
&+ 2\sum_{i,j=1}^{N}\|\partial_{x_j}\varphi_i^k - \partial_{x_j}\varphi_i^n\|_M\|v_{i,j}\|_{M^*} \\
&\leq 2\|\varphi_k - \varphi_n\|_{W_0^1E_M(\Omega,\mathbb{R}^N))}\|v\|_{\Pi L_{M^*}}.
\end{aligned}$$

This implies that $\{T(\varphi_n)\}$ is a Cauchy sequence in $\mathbb{C}$ which converges to the unique limit denoted by $\widetilde{T}(u)$. Let $\{\phi_n\}$ be another sequence in $\mathcal{C}_0^\infty(\Omega,\mathbb{R}^N)$ with $\|\phi_n - u\|_{W^1E_M(\Omega,\mathbb{R}^N)} \to 0$ as $n \to \infty$. Then $T(\varphi_n) - T(\phi_n) \to 0$ as $n \to \infty$. It's easy to see that the functional $\widetilde{T}$ is linear and in addition it is continuous since we have

$$\begin{aligned}
|\widetilde{T}(u)| = \lim_{n\to\infty}|T(\varphi_n)| &\leq 2\lim_{n\to\infty}\|\varphi_n\|_{W_0^1E_M(\Omega,\mathbb{R}^N)}\|v\|_{\Pi L_{M^*}} \\
&= 2\|u\|_{W_0^1E_M(\Omega,\mathbb{R}^N)}\|v\|_{\Pi L_{M^*}}.
\end{aligned}$$

Therefore, $\widetilde{T} \in (W_0^1 E_M(\Omega, \mathbb{R}^N))'$. Furthermore, $\widetilde{T}(u) = L(u)$.

Now if Ω is bounded and M satisfies ($\mathcal{M}_{on}$), then thanks to the Poincaré inequality (6.6) we can equip the space $W_0^1 E_M(\Omega, \mathbb{R}^N)$ by the equivalent norm $\|\nabla u\|_M$ and so the space $W_0^1 E_M(\Omega, \mathbb{R}^N)$ can be identified to a subspace of ΠE_M of only N^2 copies. Hence, repeating the same arguments as above with ΠE_M and ΠL_M of N^2 copies we get the results. $\square$

6.3 Auxiliary results

Definition 6.3.1 (Equi-integrability). *A sequence of measurable functions $u_n : \Omega \to \mathbb{R}$ is called equi-integrable if*

- $\forall \varepsilon > 0$, $\exists \eta > 0$ *such that* $\forall\ \Omega' \subset \Omega$ *such that* $|\Omega'| \leq \eta$ *we have*

$$\int_{\Omega'} |u_n| dx < \varepsilon, \forall n \geq 1.$$

- $\forall \varepsilon > 0$, *there exist* $D \subset \Omega$ *of finite Lebesgue measure such that for all* $n \geq 1$

$$\int_{\Omega \setminus D} |u_n| dx < \varepsilon.$$

Remark 6.3.1. *Let Ω be an open subset of $\mathbb{R}^N$ of finite Lebesgue measure, (M, M^*) be a pair of complementary Φ-functions such that M satisfies (2.2) and let $\{u_n\}_n$ be a uniformly bounded sequence in $L_M(\Omega)$. Then the second claim of Definition 6.3.1 is fulfilled. Indeed, since M satisfies (2.2) by Remark 2.2.1, there exist $c > 0$ such that $\displaystyle\int_\Omega M^*(x, 1) dx \leq c|\Omega|$. Then by Hölder inequality we have*

$$\int_\Omega |u_n| dx \leq 2\|u_n\|_M \|\chi_\Omega\|_{M^*} \leq 2\|u_n\|_M(1 + c|\Omega|).$$

Therefore, the sequence $\{u_n\}$ is uniformly bounded in $L^1(\Omega)$ and so since Ω is of finite Lebesgue measure, it's obvious that the second claim of Definition 6.3.1 is fulfilled.

Now we give an analogous result to the de La Vallée-Poussin theorem with functions of two variables.

Lemma 6.3.1. *Let Ω be an open subset of $\mathbb{R}^N$ with finite Lebesgue measure and let $M \in \Phi$ satisfies (2.2). Assume that there exists a constant $c > 0$ such that $\displaystyle\int_\Omega M(x, |u_n|) dx \leq c$ for all $n \in \mathbb{N}$. Then the sequence $\{u_n\}$ is equi-integrable.*

Proof. By virtue of Remark 6.3.1, it sufficient to check the first claim of Definition 6.3.1. Indeed, let $\varepsilon > 0$ be arbitrary. By (2.2) there exists a constant $C > 0$ (not depending on x) such that

$$\frac{M(x,s)}{s} \geq \frac{2c}{\varepsilon}, \text{ for every } s \geq C.$$

Using the fact that $\int_\Omega M(x,|u_n|)dx \leq c$, we can have

$$\begin{aligned}\int_{\{|u_n|\geq C\}} |u_n|dx &\leq \frac{\varepsilon}{2c}\int_{\{|u_n|\geq C\}} M\big(x,|u_n(x)|\big)dx \\ &\leq \frac{\varepsilon}{2}.\end{aligned}$$

Let $\eta = \dfrac{\varepsilon}{2C}$ and let Ω' be a subset of Ω such that $|\Omega'| \leq \eta$. Then

$$\begin{aligned}\int_{\Omega'} |u_n|dx &\leq \int_{\{\Omega'\cap\{|u_n|\geq C\}\}} |u_n|dx + \int_{\Omega'\cap\{|u_n|<C\}} |u_n|dx \\ &\leq \varepsilon.\end{aligned}$$

This proves that the sequence $\{u_n\}$ is equi-integrable. □

Lemma 6.3.2. *Let Ω be an open subset of $\mathbb{R}^N$ with finite Lebesgue measure and let (M, M^*) be a pair of complementary Φ-functions. Assume that M satisfies (2.2). Let $\{u_n\}_n$ be a uniformly bounded sequence in $L_M(\Omega)$, such that $u_n \to u$ almost everywhere in Ω (or at least in measure in Ω). Then u belongs to $L_M(\Omega)$ and $u_n \rightharpoonup u$ for the weak-* topology $\sigma(L_M(\Omega), E_{M^*}(\Omega))$.*

Proof. The proof is rather similar to that of [69, Theorem 14.6] in the framework of Orlicz spaces. For the convenience of the reader we give it again here. On one hand, by Lemma 6.3.1, the sequence $\{u_n\}_n$ is equi-integrable and hence by Vitali's theorem $\{u_n\}_n$ converges to u in $L^1(\Omega)$ and in measure as well. On the other hand, as the sequence $\{u_n\}_n$ bounded in $L_M(\Omega)$, there exists $\lambda > 0$ such that

$$\int_\Omega M\Big(x, \frac{|u_n(x)|}{\lambda}\Big)dx \leq 1.$$

Thus by Fatou's lemma $u \in L_M(\Omega)$. Moreover, since the sequence $\{u_n\}$ is uniformly bounded in $L_M(\Omega)$, there exists a subsequence $\{u_{n_k}\}$ such that u_{n_k} converges to u_0 in $L_M(\Omega)$ with respect to the weak-* topology $\sigma(L_M(\Omega), E_{M^*}(\Omega))$. Our aim is to prove that $u(x) = u_0(x)$ for almost every $x \in \Omega$. To do so, we will estimate the integral $\int_\Omega |u(x) - u_0(x)|dx$. Let $\varepsilon > 0$ be arbitrary. In view of Lemma 6.3.1, there exists $\eta > 0$ such that for every subset $\Omega' \subset \Omega$ with $|\Omega'| \leq \eta$ on has

$$\int_{\Omega'} |u_0(x)|dx \leq \frac{\varepsilon}{6}, \text{ and } \int_{\Omega'} |u_{n_k}(x)|dx \leq \frac{\varepsilon}{6}, \ \forall k = 1, 2, \cdots. \tag{6.10}$$

Since u and u_0 belong both to $L_M(\Omega)$, a constant $c > 0$ can be found such that

$$\int_\Omega M\Big(x, \frac{|u(x) - u_0(x)|}{c}\Big)dx \leq 1.$$

By (2.2) there exists a constant $C_0 > 0$ (not depending on x) such that

$$\frac{M(x,s)}{s} \geq \frac{6c}{\varepsilon}, \text{ for every } s \geq C_0.$$

Then defining $B = \{x \in \Omega, |u(x) - u_0(x)| > cC_0\}$, we obtain

$$\int_B |u(x) - u_0(x)|dx \leq \frac{\varepsilon}{6}\int_\Omega M\Big(x, \frac{|u(x) - u_0(x)|}{c}\Big)dx \leq \frac{\varepsilon}{6}. \tag{6.11}$$

On the other hand, we can write

$$\int_\Omega |u(x) - u_0(x)|dx = \int_B |u(x) - u_0(x)|(x)dx + \int_{\Omega\setminus B} |u(x) - u_0(x)|(x)dx := I_1 + I_2.$$

Note that (6.11) implies

$$I_1 \leq \frac{\varepsilon}{6}.$$

In order to estimate I_2, we have to split the integral according to suitable subsets of Ω. Let us denote by $s_0(x)$ the function $\operatorname{sgn}(u(x) - u_0(x))$ and let us choose $\varepsilon > 0$ to be such that $\eta \leq \frac{\varepsilon}{6cC_0}$. Then we can write

$$\begin{aligned} I_2 &= \int_{\Omega\setminus B} |u(x) - u_0(x)|dx = \int_{\Omega\setminus B} (u(x) - u_0(x))s_0(x)dx \\ &= \int_{\Omega\setminus B} (u(x) - u_{n_k}(x))s_0(x)dx + \int_{\Omega\setminus B} (u_{n_k}(x) - u_0(x))s_0(x)dx \\ &\leq \int_\Omega |u(x) - u_{n_k}(x)|\chi_{\Omega\setminus B}dx + \Big|\int_\Omega (u_{n_k}(x) - u_0(x))s_0(x)\chi_{\Omega\setminus B}dx\Big| \\ &:= J_1(n_k) + J_2(n_k), \end{aligned}$$

where $\chi_{\Omega\setminus B}$ denotes the characteristic function of the subset $\Omega \setminus B$. In view of Remark 2.2.1, the function $\chi_{\Omega\setminus B}$ belongs to $E_{M^*}(\Omega)$. As the sequence $\{u_{n_k}\}$ convergences to u_0 for the weak-* topology $\sigma(L_M(\Omega), E_{M^*}(\Omega))$, there exists $k_1 > 0$ such that for every $k > k_1$

$$J_2(n_k) = \Big|\int_\Omega (u_{n_k}(x) - u_0(x))s_0(x)\chi_{\Omega\setminus B}(x)dx\Big| \leq \frac{\varepsilon}{6}.$$

Let us consider the following subset

$$\Omega_k = \Big\{x \in \Omega : |u_{n_k}(x) - u(x)| \geq \frac{\varepsilon}{6|\Omega|}\Big\}.$$

We can estimate $J_1(n_k)$ as follows

$$\begin{aligned}
J_1(n_k) \ &\leq \int_{\Omega\setminus\Omega_k} |u(x) - u_{n_k}(x)|\chi_{\Omega\setminus B}dx + \int_{\Omega_k} |u(x) - u_{n_k}(x)|\chi_{\Omega\setminus B}dx \\
&\leq \int_{\Omega\setminus\Omega_k} |u(x) - u_{n_k}(x)|dx + \int_{\Omega_k} |u(x) - u_0(x)|\chi_{\Omega\setminus B}dx \\
&+ \int_{\Omega_k} |u_0(x) - u_{n_k}(x)|\chi_{\Omega\setminus B}dx \\
&\leq \int_{\Omega\setminus\Omega_k} |u(x) - u_{n_k}(x)|dx + \int_{\Omega_k} |u(x) - u_0(x)|\chi_{\Omega\setminus B}dx \\
&+ \int_{\Omega_k} |u_0(x)|dx + \int_{\Omega_k} |u_{n_k}(x)|dx \\
&:= J_1^1(n_k) + J_1^2(n_k) + J_1^3(n_k) + J_1^4(n_k).
\end{aligned}$$

The first integral can be estimated as

$$J_1^1(n_k) \leq \frac{\varepsilon}{6|\Omega|}|\Omega \setminus \Omega_k| \leq \frac{\varepsilon}{6},$$

It follows from the convergence in measure of the subsequence $\{u_{n_k}\}$ to the function u that there exists $k_0 > 0$ not depending on ε such that for all $k > k_0$, we have $|\Omega_k| \leq \eta$. Hence, we get

$$J_1^2(n_k) \leq cC_0|\Omega_k| \leq cC_0\eta \leq \frac{\varepsilon}{6}.$$

By virtue of (6.10), the remaining terms can be estimated as

$$J_1^3(n_k) + J_1^4(n_k) \leq \frac{\varepsilon}{3}.$$

Hence, for $k \geq \max\{k_0, k_1\}$, we obtain

$$I_2 \leq \frac{5\varepsilon}{6}.$$

In conclusion, for all $\varepsilon > 0$ we have

$$\int_{\Omega} |u(x) - u_0(x)| dx \leq \varepsilon.$$

So as ε is arbitrary, we have $u = u_0$ almost every where in Ω. This achieves the proof. □

Finally, we give below the abstract existence result in finite dimensional spaces that we use for the accomplishment of our work.

Lemma 6.3.3. *(see [17, Lemmas 2 and 3], [56, Lemma 1]) Let V be a finite dimensional real Banach space, T a continuous mapping of V to V^*, and $\psi : V \to (-\infty, +\infty]$ a convex, lower semi-continuous function with $\psi(0) = 0$. Assume $R > 0$ is such that*

$$(Tw, w) + \psi(w) > 0, \ \forall w \in V \ \textit{with} \ \|w\|_V = R.$$

Then there exists $u \in V$, $\|u\|_V < R$, solving the inequality

$$(Tu, w - u) + \psi(w) - \psi(u) \geq 0, \ \forall w \in V.$$

6.4 Main result

Let Ω be a bounded open subset of $\mathbb{R}^N$ satisfying the segment property. Let (M, M^*) be a pair of complementary Φ-functions. Assume that M satisfies $(\mathcal{L}_{hol})$ and $(\mathcal{M}_{on})$ and M^* satisfies $(\mathcal{L}^1_{loc})$ and (2.2).

Define $\mathbf{A}u = -\mathrm{div}\, a(x, \nabla u)$. The mapping $\mathbf{A} : \mathcal{D}(\mathbf{A}) \subset W^1_0 L_M(\Omega, \mathbb{R}^N) \to W^{-1} L_{M^*}(\Omega, \mathbb{R}^N)$ is defined from its domain

$$\mathcal{D}(\mathbf{A}) = \Big\{ u \in W^1_0 L_M(\Omega, \mathbb{R}^N), \ a(\cdot, \nabla u) \in L_{M^*}(\Omega, \mathbb{R}^{N\times N}) \Big\}$$

into the dual space $W^{-1} L_{M^*}(\Omega, \mathbb{R}^N)$ of $W^1_0 E_M(\Omega, \mathbb{R}^N)$ (see Theorem 6.2.1). For every $u \in \mathcal{D}(\mathbf{A})$, the action of $\mathbf{A}u$ on an element $v \in W^1_0 L_M(\Omega, \mathbb{R}^N)$ is given by

$$\langle \mathbf{A}u, v \rangle = \int_{\Omega} a(x, \nabla u) : \nabla v dx.$$

Our main result states the following.

Theorem 6.4.1. *Let Ω be a bounded open subset of $\mathbb{R}^N$, $N \geq 1$, satisfying the segment property. Let (M, M^*) be a pair of complementary Φ-functions. Assume that M satisfies $(\mathcal{L}_{hol})$ and $(\mathcal{M}_{on})$ and M^* satisfies $(\mathcal{L}^1_{loc})$ and (2.2). Assume further that the stress tensor a associated with problem (6.1) satisfies $(\mathcal{A}_1)$ and $(\mathcal{A}_2)$. Then for any given $f \in W^{-1} E_{M^*}(\Omega, \mathbb{R}^N)$ the problem (6.1) admits a unique variational solution. Namely, there exists a function $u \in W^1_0 L_M(\Omega, \mathbb{R}^N)$ that satisfies*

$$\langle \boldsymbol{A}u, v \rangle = \langle f, v \rangle, \quad \textit{for all } v \in W^1_0 L_M(\Omega, \mathbb{R}^N). \tag{6.12}$$

Some comments and remarks

1. Let (M, M^*) be the pair of complementary Φ-functions of Theorem 6.4.1. Since M^* satisfies (2.2), we point out that in view of Remark 2.2.1 the Φ-function M satisfies $(\mathcal{L}^1_{loc})$ and so the space $W^1_0E_M(\Omega, \mathbb{R}^N)$ is well defined. In addition, since M^* satisfies $(\mathcal{L}^1_{loc})$ the weak-* topology $\sigma(\Pi L_M, \Pi E_{M^*})$ is well defined (see 3.3.1) and so is the Musielak space $W^1_0L_M(\Omega, \mathbb{R}^N)$.

2. As far as the problem (6.1) is concerned, we use $W^1_0L_M(\Omega, \mathbb{R}^N)$ as an energy space. Unfortunately, in the above variational formulation of the solution we will deal with integral quantities containing sequences in $W^1_0E_M(\Omega)$ as integrands and so there appears the problem of passing to the limit obtaining the variational formulation (6.12) in the whole space $W^1_0L_M(\Omega, \mathbb{R}^N)$. This problem can be seen as an easy task if $M^* \in \Delta_2$, since in this case we have $L_{M^*} = E_{M^*}$ and then both of the two topologies $\sigma(\Pi L_M, \Pi E_{M^*})$ and $\sigma(\Pi L_M, \Pi L_{M^*})$ coincide.
   Therefore, we find that despite the fact that M^* does not satisfy the Δ_2-condition (3.1), the problem of crossing the passage to the limit can be exceeded by using the topology $\sigma(\Pi L_M, \Pi L_{M^*})$. To be more precise, we need to get the density of $W^1_0E_M(\Omega)$ with respect to the topology $\sigma(\Pi L_M, \Pi L_{M^*})$ in the space $W^1_0L_M(\Omega)$. Such a density can hold true provided that M satisfies the structural assumption $(\mathcal{L}_{hol})$ and its complementary Musielak function M^* satisfies $(\mathcal{L}^1_{loc})$. It is worth recalling here that without assuming the Δ_2-condition neither on M nor on M^*, then $\mathcal{C}^\infty_0(\Omega, \mathbb{R}^N)$ is dense in $W^1_0L_M(\Omega, \mathbb{R}^N)$ with respect to the topology $\sigma(\Pi L_M, \Pi L_{M^*})$ (see Lemma 6.2.4 below).

3. We shall stress that since we do not assume the restrictive Δ_2-condition neither on M nor on M^*, we are then lead to deal with non reflexive Banach spaces and therefore we cannot, in general, apply the Eberlein-Šmulian theorem in the space $W^1_0L_M(\Omega, \mathbb{R}^N)$. This is the main reason why we introduce the energy space $W^1_0L_M(\Omega, \mathbb{R}^N)$ as the completion of the set of smooth functions with respect of the weak-* topology $\sigma(\Pi L_M, \Pi E_{M^*})$. This overcomes the crucial difficulty of extracting convergent subsequences in the compactness method by applying the Banach-Alaoglu-Bourbaki theorem.

4. The particular case $M(\cdot, s) = s^{p(\cdot)}$ leads to the variable exponent Sobolev space $W^{1,p(\cdot)}_0(\Omega)$. The norm Poincaré inequality in the space $W^{1,p(\cdot)}_0(\Omega)$ is proved first in [67] under some conditions including the continuity on the variable exponent $p(\cdot)$. The norm Poincaré inequality in the space $W^{1,p(\cdot)}_0(\Omega)$ is also obtained in [28] where the authors use an approach based on the boundedness of the maximal operator on $L^{p(\cdot)}$ imposing the so-called log-Hölder continuity on the exponent $p(\cdot)$; while in [77] the author deduce the norm Poincaré inequality from the integral inequality assuming that the exponent is monotone with respect to one of its component on Ω.

5. In Chapter 5 we used, among others, the structural assumption $(\mathcal{M}_{on})$ as a sufficient condition to prove the integral Poincaré inequality (see 6.5 below) and so the norm Poincaré inequality (see 6.6 below) in $W_0^1L_M(\Omega,\mathbb{R}^N)$. As in the standard way of Sobolev spaces with constant/variable exponent, another approach can be used to prove the norm Poincaré inequality. In the spirit of the space variable exponent Sobolev spaces, we think that the remaining challenge is to obtain the norm Poincaré inequality avoiding the hypothesis $(\mathcal{M}_{on})$.

Example 6.4.1. *As examples of equations and corresponding Musielak functions to which we can apply the result of Theorem 6.4.1, we can give*

1) $-div\,(|\nabla u|^{p-2}\nabla u) = f$, $M(x,s) := \frac{1}{p}|s|^p$, $1 < p = const < +\infty$ *and* $f \in W^{-1}E_{M^*}(\Omega) = W^{-1,p'}(\Omega)$, $p' = \dfrac{p}{p-1}$. *Note that M satisfies trivially $(\mathcal{L}_{hol})$ and $(\mathcal{M}_{on})$.*

2) $-div\,(|\nabla u|^{p(x)-2}\nabla u) = f$, $M(x,s) = |s|^{p(x)}$, *with* $1 < p^- \leq p(x) \leq p^+ < \infty$, *where* $p^- := ess\inf_{x\in\Omega} p(x)$ *and* $p^+ := ess\sup_{x\in\Omega} p(x)$, *and* $f \in W^{-1}E_{M^*}(\Omega) = W^{-1,p'(x)}(\Omega)$, $p'(x) := \dfrac{p(x)}{p(x)-1}$. *Note that by $(\mathcal{L}_{hol})$ the exponent $p(\cdot)$ satisfies the* log*-Hölder condition (see Example 4.2.1), while $(\mathcal{M}_{on})$ implies that $p(\cdot)$ has a monotone partial function which provides the Poincaré inequality (see 5.2.1, Theorem 5.3.1 and [77]).*

3) $-div\left(|\nabla u|^{p-2}\nabla u + w(x)|\nabla u|^{q-2}\nabla u\right) = f$, $M(x,s) = \frac{1}{p}|s|^p + \frac{1}{q}w(x)|s|^q$, *with* $1 < p < q < \infty$, $f \in W^{-1}E_{M^*}(\Omega)$, $w \in C^{0,\alpha}_{loc}(\Omega)$, $\alpha \in (0,1]$. *We point out that $(\mathcal{L}_{hol})$ forces $q \leq p + \alpha/N$ (see Example 4.2.1) while $(\mathcal{M}_{on})$ implies that w is monotone with respect to one of its component (see Example 5.2.1).*

4) $-div\left(\dfrac{e^{|\nabla u|}-1}{|\nabla u|}\nabla u\right) = f$, $M(x,s) = e^{|s|} - |s| - 1$, *and* $f \in W^{-1}E_{M^*}(\Omega)$. *Observe that M satisfies trivially $(\mathcal{L}_{hol})$ and $(\mathcal{M}_{on})$.*

6.5 Proof of Theorem 6.4.1

In this section, we prove the main result. We divide the proof in tree steps.

Proof. **Step1: Galerkin approximation**. In this part we prove the existence of approximate solution in finite dimensional spaces using the Galerkin method. The space $W_0^1E_M(\Omega,\mathbb{R}^N)$ being separable (see Theorem 6.2.3) there exists a non-decreasing sequence of finite dimensional subspaces $\{V_k\}$ of $W_0^1E_M(\Omega,\mathbb{R}^N)$ (see for instance [74, Lemma 4.1]), equipped with the norm $\|\cdot\|_{V_k} = \|\cdot\|_{W_0^1L_M(\Omega,\mathbb{R}^N)}$, such that $\cup_{k=1}^{\infty} V_k$, is norm dense in $W_0^1 E_M(\Omega,\mathbb{R}^N)$. Denote by V_k^* the dual space of V_k. For each k let j_k (resp. j_k^*) be the identity mapping (resp. the dual projection) from V_k into $W_0^1 E_M(\Omega,\mathbb{R}^N)$ (resp. from $W^{-1}L_{M^*}(\Omega,\mathbb{R}^N)$ into V_k^*). Let

$$\mathbf{A}_k : V_k \to V_k^*$$

be the sequence of Galerkin approximation mappings of $\mathbf{A}$ defined by $\mathbf{A}_k u = j_k^* \mathbf{A} j_k u$. We note that for all u and v in V_k we have $\langle \mathbf{A}_k u, v\rangle = \langle \mathbf{A} u, v\rangle$. Our aim is to prove that for $f \in W^{-1}E_{M^*}(\Omega,\mathbb{R}^N)$ there exist $u_k \in V_k$ solving the inequality

$$\langle \mathbf{A}_k u_k - f, u_k - v\rangle = \langle \mathbf{A} u_k - f, u_k - v\rangle \leq 0 \tag{6.13}$$

for all $v \in V_k$. To do this, we use the standard finite dimensional argument (see Lemma 6.3.3). At first, we shall show that the following two assertions are fulfilled

1. $\mathbf{A}$ is continuous mapping of V_k into V_k^*.

2. For R large enough, we have $\langle \mathbf{A}v - f, v\rangle > 0$, for all $v \in V_k$ with $\|v\|_{V_k} = R$.

Let us check the first assertion (1). Let n_k denotes the dimension of the finite space V_k and let $B = \{w_1, \cdots, w_{n_k}\}$ be a fixed basis in V_k. Without loss of generality for an arbitrary $\xi \in V_k$ we can write

$$\xi = \sum_{i=1}^{n_k} \beta_i w_i, \text{ with } \sum_{i=1}^{n_k} |\beta_i| = 1.$$

Then by (6.2) we get

$$\begin{aligned}
\|a(x,\nabla\xi)\|_{M^*} &\leq \|l\|_{M^*} + c_1\Big(1 + \int_\Omega M\big(x, c_2\big|\sum_{i=1}^{n_k}\beta_i\nabla w_i\big|\big)dx\Big) \\
&\leq \|l\|_{M^*} + c_1\Big(1 + \sum_{i=1}^{n_k}|\beta_i| \int_\Omega M(x, c_2|\nabla w_i|)dx\Big) \\
&\leq \|l\|_{M^*} + c_1\Big(1 + \int_\Omega M(x, c_2 \max_{1\leq i\leq n_k} |\nabla w_i|)dx\Big).
\end{aligned} \tag{6.14}$$

This shows that $a(x,\nabla\xi)$ is bounded in $L_{M^*}(\Omega,\mathbb{R}^{N\times N})$ when ξ runs over V_k. Let now $\{\xi_n\}_n$ be a sequence of vectors in V_k and let $\xi \in V_k$ such that $\xi_n \to \xi$ in V_k. In particular, $\|\nabla\xi_n - \nabla\xi\|_M$ tends to zero as $n \to \infty$. Thus, for n large enough we have

$$\int_\Omega M(x, |\nabla\xi_n - \nabla\xi|)dx \leq \|\nabla\xi_n - \nabla\xi\|_M.$$

So that $M(\cdot, |\nabla\xi_n - \nabla\xi|) \to 0$ in norm in $L^1(\Omega)$ as n tends to $+\infty$ and for a subsequence, indexed again by n, we have $\nabla\xi_n \to \nabla\xi$ a.e. in Ω. Therefore, $a(x,\nabla\xi_n) \to a(x,\nabla\xi)$ a.e. in Ω and as the sequence $\{a(x,\nabla\xi_n)\}_n$ is uniformly bounded in $L_{M^*}(\Omega,\mathbb{R}^{N\times N})$ we deduce by Lemma 6.3.2 that

$$a(x,\nabla\xi_n)\rightharpoonup a(x,\nabla\xi)\in L_{M^*}(\Omega,\mathbb{R}^{N\times N}),$$

for the weak-* topology $\sigma(L_{M^*}(\Omega,\mathbb{R}^{N\times N}),E_M(\Omega,\mathbb{R}^{N\times N}))$. This shows that the mapping $\mathbf{A}$ is continuous from V_k to its dual V_k^* and the assumption (1) is fulfilled.

Let now check that (2) is filled. We know that any $f\in W^{-1}E_{M^*}(\Omega,\mathbb{R}^N)$ can be written

$$f=-\text{div } F$$

where $|F|\in E_{M^*}(\Omega,\mathbb{R}^{N\times N})$. If (2) were false, there would exists an element $v\in V_k$ with $\|v\|_{V_k}=R$ such that

$$\langle \mathbf{A}v-f,v\rangle\leq 0.$$

Then according to (6.3) and Young's inequality it yields

$$\begin{aligned} d_1\int_\Omega M(x,d_2|\nabla v|)dx &\leq \int_\Omega F:\nabla v dx+\int_\Omega h(x)dx\\ &\leq \int_\Omega M(x,\lambda|\nabla v|)dx+\int_\Omega M^*\Big(x,\frac{|F|}{\lambda}\Big)dx+\int_\Omega h(x)dx\end{aligned}$$

with $\lambda>0$ choosing small as needed. Then For $\lambda<\min\Big\{d_1d_2,d_2\Big\}$, by the convexity of M we get

$$d_1\int_\Omega M(x,d_2|\nabla v|)dx \leq \frac{\lambda}{d_2}\int_\Omega M(x,d_2|\nabla v|)dx+\int_\Omega M^*\Big(x,\frac{|F|}{\lambda}\Big)dx+\|h\|_{L^1(\Omega)}.$$

Hence, we get

$$\|v\|_{W_0^1L_M(\Omega,\mathbb{R}^N)}:=\|\nabla v\|_M\leq C:=\frac{1}{d_1d_2-\lambda}\Big(\int_\Omega M^*\Big(x,\frac{|F|}{\lambda}\Big)dx+\|h\|_{L^1(\Omega)}\Big)+\frac{1}{d_2},$$

a contradiction with $R>0$ large enough proving the assertion (2).
Finally, applying Lemma 6.3.3 with $R=C+1$, we deduce the existence of a sequence $\{u_k\}_k\subset V_k$ such that

$$\|u_k\|_{W_0^1L_M(\Omega,\mathbb{R}^N)}:=\|\nabla u_k\|_M\leq C+1 \tag{6.15}$$

and satisfying the inequality

$$\langle \mathbf{A}u_k-f,u_k-v\rangle\leq 0, \tag{6.16}$$

for every $v\in V_k$.

Step2: Passage to the limit. At first we shall prove that the inequality (6.16) remains valid for every $v\in W_0^1L_M(\Omega,\mathbb{R}^N)$. To do this, let j be a fixed integer.

Since the sequence of finite subspace V_k is non-decreasing, for all $k \geq j$ we have $v_j \in V_k$. Hence, v_j is an admissible test function in (6.16). Taking it so it yields

$$\langle \mathbf{A}u_k, u_k \rangle \leq \langle \mathbf{A}u_k, v_j \rangle + \langle f, u_k - v_j \rangle. \tag{6.17}$$

Now, by virtue of (6.14) the sequence $\{a(x, \nabla u_k)\}_k$ is uniformly bounded in $L_{M^*}(\Omega, \mathbb{R}^{N\times N})$. Thus, there exists $\vartheta \in L_{M^*}(\Omega, \mathbb{R}^{N\times N})$ such that, up to a subsequence if necessary, we can assume

$$a(x, \nabla u_k) \rightharpoonup \vartheta \in L_{M^*}(\Omega, \mathbb{R}^{N\times N}),$$

for $\sigma(L_{M^*}(\Omega, \mathbb{R}^{N\times N}), E_M(\Omega, \mathbb{R}^{N\times N}))$. Thus, we have

$$\int_\Omega a(x, \nabla u_k) : \nabla v_j \to \int_\Omega \vartheta : \nabla v_j dx.$$

On the other hand, by (6.15) there exist a measurable function $u \in W_0^1 L_M(\Omega, \mathbb{R}^N)$ and a subsequence, still indexed by n, such that $u_k \rightharpoonup u$ in $W_0^1 L_M(\Omega, \mathbb{R}^N)$ for $\sigma(\Pi L_M, \Pi E_{M^*})$. Thus, we can pass to the limit as k tends to ∞ in (6.17) obtaining

$$\lim_{k\to\infty} \sup \langle \mathbf{A}u_k, u_k \rangle \leq \int_\Omega \vartheta : \nabla v_j dx + \langle f, u - v_j \rangle, \forall j \in \mathbb{N}.$$

We note that since the right hand side of the last inequality is linear with respect to v_j, the inequality

$$\lim_{k\to\infty} \sup \langle \mathbf{A}u_k, u_k \rangle \leq \int_\Omega \vartheta : \nabla v dx + \langle f, u - v \rangle,$$

also remains true for all $v \in \cup_{k=1}^\infty V_k$ and by the norm density of $\cup_{k=1}^\infty V_k$ in $W_0^1 E_M(\Omega, \mathbb{R}^N)$ it holds for all $v \in W_0^1 E_M(\Omega, \mathbb{R}^N)$. Indeed, if $v \in W_0^1 E_M(\Omega, \mathbb{R}^N)$ there exists a sequence $\{w_j\}_j$ in $\cup_{k=1}^\infty V_k$ that converges to v in norm in $W_0^1 E_M(\Omega, \mathbb{R}^N)$. Then by Hölder's inequality one has

$$\left| \int_\Omega \vartheta : (\nabla v - \nabla w_j) dx - \langle f, v - w_j \rangle \right| \leq \|\vartheta\|_{M^*} \|v - \nabla w_j\|_M + \|f\|_{M^*} \|v - w_j\|_M$$

where the right-hand side of this inequality tends to zero as $j \to \infty$. Hence

$$\lim_{k\to\infty} \sup \langle \mathbf{A}u_k, u_k \rangle \leq \int_\Omega \vartheta : \nabla v dx + \langle f, u - v \rangle,$$

for all $v \in W_0^1 E_M(\Omega, \mathbb{R}^N)$. Furthermore, the last inequality holds true for all $v \in W_0^1 L_M(\Omega, \mathbb{R}^N)$. Indeed, since the space $W_0^1 E_M(\Omega, \mathbb{R}^N)$ is $\sigma(\Pi L_M, \Pi L_{M^*})$ dense in $W_0^1 L_M(\Omega, \mathbb{R}^N)$ (see Lemma 6.2.4), for a given $v \in W_0^1 L_M(\Omega, \mathbb{R}^N)$ there exists a sequence $\{y_j\}_j \subset W_0^1 E_M(\Omega, \mathbb{R}^N)$ converging to v with respect to the

$\sigma(\Pi L_M, \Pi L_{M^*})$ topology. Then it is easy to see that

$$\int_\Omega \vartheta : \nabla y_j dx - \langle f, y_j\rangle \to \int_\Omega \vartheta : \nabla v dx - \langle f, v\rangle \ \text{ as } j \to \infty.$$

Therefore,

$$\lim_{k\to\infty} \sup\langle \mathbf{A}u_k, u_k\rangle \leq \int_\Omega \vartheta : \nabla v dx + \langle f, u - v\rangle,$$

for all $v \in W_0^1 L_M(\Omega, \mathbb{R}^N)$. In particular for $v = u$ we obtain

$$\lim_{k\to\infty} \sup\langle \mathbf{A}u_k, u_k\rangle \leq \int_\Omega \vartheta : \nabla u dx. \tag{6.18}$$

Now we shall prove that $\vartheta = a(x, \nabla u)$. To do this, we shall prove that

$$\nabla u_k \to \nabla u \text{ a.e. in } \Omega.$$

This will follows if we show that for a.e. $x \in \Omega$

$$J_k(x) = (a(x, \nabla u_k(x)) - a(x, \nabla u(x))) : (\nabla u_k(x) - \nabla u(x))$$

tends to zero as $k \to \infty$. It will follows from [71, Lemma 6] that the sequence $\{\nabla u_k(x)\}$ remains bounded in $\mathbb{R}^{N\times N}$ and $\nabla u_k \to \nabla u$ for a.e. $x \in \Omega$. Indeed, since $\int_\Omega J_k(x)dx \geq 0$, it is sufficient to check that

$$\limsup_k \int_\Omega J_k(x)dx \to 0.$$

Defining $\Omega_j = \Big\{x \in \Omega : |\nabla u(x)| \leq j\Big\}$, for $j \in \mathbb{N}$, we can write

$$\begin{aligned}
\int_{\Omega_j} \big(a(x, \nabla u_k) - a(x, \nabla u)\big) : \big(\nabla u_k - \nabla u\big) dx &= \int_{\Omega_j} a(x, \nabla u_k) : \big(\nabla u_k - \nabla u\big) dx \\
&- \int_{\Omega_j} a(x, \nabla u) : \big(\nabla u_k - \nabla u\big) dx \\
&= I_1(k, j) - I_2(k, j).
\end{aligned}$$

We start by estimating $I_1(k, j)$.

$$\begin{aligned}
I_1(k, j) &= \int_\Omega a(x, \nabla u_k) : \nabla u_k dx - \int_{\Omega\setminus\Omega_j} a(x, \nabla u_k) : \nabla u_k dx \\
&- \int_{\Omega_j} a(x, \nabla u_k) : \nabla u dx \\
&= E_1(k) + E_2(k, j) + E_3(k, j).
\end{aligned}$$

Thanks to (6.18) we have

$$\lim_{k\to\infty} \sup E_1(k) \leq \int_\Omega \vartheta : \nabla u dx. \tag{6.19}$$

For the term $E_2(k,j)$, by (6.3) we get

$$E_2(k,j) \leq \int_{\Omega\setminus\Omega_j} h(x)dx. \tag{6.20}$$

Furthermore, since

$$a(x,\nabla u_k) \rightharpoonup \vartheta \in L_{M^*}(\Omega,\mathbb{R}^{N\times N}) \text{ for } \sigma(L_{M^*}(\Omega,\mathbb{R}^{N\times N}), E_M(\Omega,\mathbb{R}^{N\times N}))$$

and $\chi_j\nabla u$ belongs to $E_M(\Omega,\mathbb{R}^{N\times N})$ (Remark 2.2.1 with $A=M^*$), we obtain

$$\lim_{k\to\infty} E_3(j,k) = -\int_\Omega \vartheta : \nabla u\chi_j dx. \tag{6.21}$$

Finally, combining (6.19), (6.20) and (6.21) we obtain

$$\lim_{k\to\infty} \sup I_1(k,j) \leq \varepsilon_j \text{ with } \varepsilon_j \to 0 \text{ as } j\to\infty.$$

For the term $I_2(k,j)$, since

$$\nabla u_k \to \nabla u \text{ for } \sigma(L_M(\Omega,\mathbb{R}^{N\times N}), E_{M^*}(\Omega,\mathbb{R}^{N\times N}))$$

and since by (6.2) and Remark 2.2.1 we have $\chi_j a(x,\nabla u)$ belongs to $E_{M^*}(\Omega,\mathbb{R}^{N\times N})$ we get

$$\lim_{k\to\infty} I_2(k,j) = 0,\ \forall j\geq 1. \tag{6.22}$$

Hence, we conclude that

$$\limsup_k \int_{\Omega_j} J_k(x) \leq \varepsilon_j \text{ with } \varepsilon_j\to 0 \text{ as } j\to\infty.$$

As a consequence of [71, Lemma 6] we obtain $\nabla u_k \to \nabla u$ a.e. in Ω and so by Lemma 6.3.2 we have the identification

$$a(x,\nabla u) = \vartheta \in L_{M^*}(\Omega,\mathbb{R}^{N\times N}),\ \text{a.e. in } \Omega.$$

Now we shall prove that $\langle \mathbf{A}(u_k), u_k\rangle \to \langle \mathbf{A}(u), u\rangle$, as k tends to infinity. Indeed, by (6.18) and the above equality we immediately have

$$\lim_{k\to\infty} \sup\langle \mathbf{A}(u_k), u_k\rangle \leq \int_\Omega a(x,\nabla u) : \nabla u dx.$$

It remains to show the reverse inequality, that is

$$\liminf_{k\to\infty} \int_\Omega a(x,\nabla u_k) : \nabla u_k dx \geq \int_\Omega a(x,\nabla u) : \nabla u dx. \tag{6.23}$$

Let j be fixed and Ω_j as defined above. Using $(\mathcal{A}_2)$ we can write

$$\begin{aligned}
\int_\Omega a(x,\nabla u_k) : \nabla u_k dx &= \int_{\Omega_j} a(x,\nabla u_k) : \nabla u_k dx + \int_{\Omega\setminus\Omega_j} a(x,\nabla u_k) : \nabla u_k dx \\
&\geq \int_{\Omega_j} a(x,\nabla u) : \Big(\nabla u_k - \nabla u\Big) dx + \int_{\Omega_j} a(x,\nabla u_k) : \nabla u dx \\
&+ \int_{\Omega\setminus\Omega_j} a(x,\nabla u_k) : \nabla u_k dx \\
&= I_2(k,j) + \int_{\Omega_j} a(x,\nabla u_k) : \nabla u dx - E_2(k,j).
\end{aligned}$$

Since $\chi_j \nabla u$ belongs to $E_M(\Omega,\mathbb{R}^{N\times N})$ (by Remark 2.2.1), we get

$$\lim_{k\to\infty} \int_{\Omega_j} a(x,\nabla u_k) : \nabla u dx = \int_{\Omega_j} a(x,\nabla u) : \nabla u dx.$$

Using (6.20) and (6.22) and the fact that $h \in L^1(\Omega)$ we have

$$\liminf_{k\to\infty} \int_\Omega a(x,\nabla u_k) : \nabla u_k dx \quad \geq \quad \int_{\Omega_j} a(x,\nabla u) : \nabla u dx - \varepsilon_j, \text{ with } \varepsilon_j \to 0,$$

as $j \to \infty$. Then, letting $j \to \infty$ in the last inequality we get (6.23), that is

$$\langle \mathbf{A} u_k, u_k \rangle \to \int_\Omega a(x,\nabla u) : \nabla u dx.$$

Finally, we are able to pass to the limit as k tends to $+\infty$ in (6.13) and obtain

$$\langle \mathbf{A} u, u - v \rangle \leq \langle f, u - v \rangle,$$

for all $v \in W_0^1 L_M(\Omega,\mathbb{R}^N)$. Therefore, the equation

$$\langle \mathbf{A} u, v \rangle = \langle f, v \rangle, \forall v \in W_0^1 L_M(\Omega,\mathbb{R}^N),$$

has at leas one solution in $u \in W_0^1 L_M(\Omega,\mathbb{R}^N)$ with $a(x,\nabla u) \in L_{M^*}(\Omega,\mathbb{R}^{N\times N})$.

Step3: Uniqueness. Assume that there exists another solution v of problem (6.1). Then multiplying the equation corresponding to the solution u by the function $(u - v)$ and that corresponding to the solution v by the function $(v - u)$ and then summing up, we obtain

$$\int_{\Omega} \big(a(x, \nabla u) - a(x, \nabla v)\big) : (\nabla u - \nabla v) dx = 0.$$

Thanks to $(\mathcal{A}_2)$, we get $\nabla u = \nabla v$ a.e in Ω and so by the Poincaré inequality (Lemma 6.2.5) we conclude that $u = v$ a.e in Ω. $\square$

Bibliography

[1] E. Acerbi and G. Mingione. “Regularity results for stationary electrorheological fluids”. In: *Arch. Ration. Mech. Anal.* 164.3 (2002), pp. 213–259.

[2] R. A. Adams and John J. F. Fournier. *Sobolev spaces*. Second. Vol. 140. Pure and Applied Mathematics (Amsterdam). Elsevier/Academic Press, Amsterdam, 2003, pp. xiv+305.

[3] S. Agmon. *Lectures on elliptic boundary value problems*. Prepared for publication by B. Frank Jones, Jr. with the assistance of George W. Batten, Jr. Van Nostrand Mathematical Studies, No. 2. D. Van Nostrand Co., Inc., Princeton, N.J.-Toronto-London, 1965, pp. v+291.

[4] Y. Ahmida, A. Fiorenza, and A. Youssfi. “$H = W$ Musielak spaces framework”. In: *submitted* (2019).

[5] Y. Ahmida and A. Youssfi. “Poincaré-type inequalities in Musielak spaces”. In: *Ann. Acad. Sci. Fenn. Math.* 44 (2019), pp. 1041–1054.

[6] Y. Ahmida and A. Youssfi. “Variational nonlinear elliptic equations in non reflexive Musielak spaces”. In: *submitted* (2019).

[7] Y. Ahmida, I. Chlebicka, P. Gwiazda, and A. Youssfi. “Gossez’s approximation theorems in Musielak–Orlicz–Sobolev spaces”. In: *J. Funct. Anal.* 275.9 (2018), pp. 2538–2571.

[8] M. Ait Khellou and A. Benkirane. “Correction to: Elliptic inequalities with L^1 data in Musielak-Orlicz spaces [MR3634344]”. In: *Monatsh. Math.* 187.1 (2018), pp. 181–187.

[9] P. Baroni, M. Colombo, and G. Mingione. “Regularity for general functionals with double phase”. In: *Calc. Var. Partial Differential Equations* 57.2 (2018), Art. 62, 48.

[10] A. Benkirane, J. Douieb, and M. Ould Mohamedhen Val. “An approximation theorem in Musielak-Orlicz-Sobolev spaces”. In: *Comment. Math.* 51.1 (2011), pp. 109–120.

[11] A. Benkirane and M. Ould Mohamedhen Val. “Some approximation properties in Musielak-Orlicz-Sobolev spaces”. In: *Thai J. Math.* 10.2 (2012), pp. 371–381.

[12] A. Benkirane and M. Sidi El Vally. “An existence result for nonlinear elliptic equations in Musielak-Orlicz-Sobolev spaces”. In: *Bull. Belg. Math. Soc. Simon Stevin* 20.1 (2013), pp. 57–75.

[13] A. Benkirane and M. Sidi El Vally. “Variational inequalities in Musielak-Orlicz-Sobolev spaces”. In: *Bull. Belg. Math. Soc. Simon Stevin* 21.5 (2014), pp. 787–811.

[14] H. Brezis. “Équations et inéquations non linéaires dans les espaces vectoriels en dualité”. In: *Ann. Inst. Fourier (Grenoble)* 18.fasc. 1 (1968), pp. 115–175.

[15] H. Brezis. *Functional analysis, Sobolev spaces and partial differential equations*. Universitext. Springer, New York, 2011, pp. xiv+599.

[16] F. E. Browder. “Nonlinear elliptic boundary value problems”. In: *Bull. Amer. Math. Soc.* 69 (1963), pp. 862–874.

[17] F. E. Browder. “On the unification of the calculus of variations and the theory of monotone nonlinear operators in Banach spaces”. In: *Proc. Nat. Acad. Sci. U.S.A.* 56 (1966), pp. 419–425.

[18] F. E. Browder. “Variational boundary value problems for quasi-linear elliptic equations of arbitrary order”. In: *Proc. Nat. Acad. Sci. U.S.A.* 50 (1963), pp. 31–37.

[19] F. E. Browder and P. Hess. “Nonlinear mappings of monotone type in Banach spaces”. In: *J. Functional Analysis* 11 (1972), pp. 251–294.

[20] Y. Chen, S. Levine, and M. Rao. “Variable exponent, linear growth functionals in image restoration”. In: *SIAM J. Appl. Math.* 66.4 (2006), 1383–1406 (electronic).

[21] I. Chlebicka. “A pocket guide to nonlinear differential equations in Musielak-Orlicz spaces”. In: *Nonlinear Anal.* 175 (2018), pp. 1–27.

[22] P. G. Ciarlet and G. Dinca. “A Poincaré inequality in a Sobolev space with a variable exponent”. In: *Chin. Ann. Math. Ser. B* 32.3 (2011), pp. 333–342.

[23] M. Colombo and G. Mingione. “Bounded minimisers of double phase variational integrals”. In: *Arch. Ration. Mech. Anal.* 218.1 (2015), pp. 219–273.

[24] M. Colombo and G. Mingione. “Calderón-Zygmund estimates and non-uniformly elliptic operators”. In: *J. Funct. Anal.* 270.4 (2016), pp. 1416–1478.

[25] M. Colombo and G. Mingione. “Regularity for double phase variational problems”. In: *Arch. Ration. Mech. Anal.* 215.2 (2015), pp. 443–496.

[26] D. Cruz-Uribe and A. Fiorenza. “Approximate identities in variable L^p spaces”. In: *Mathematische Nachrichten* 280.3 (2007), pp. 256–270.

[27] D. Cruz-Uribe and A. Fiorenza. *Variable Lebesgue spaces.* Applied and Numerical Harmonic Analysis. Foundations and harmonic analysis. Birkhäuser/Springer, Heidelberg, 2013, pp. x+312.

[28] L. Diening, P. Harjulehto, P. Hästö, and M. Ružička. *Lebesgue and Sobolev spaces with variable exponents.* Vol. 2017. Lecture Notes in Mathematics. Springer, Heidelberg, 2011, pp. x+509.

[29] T. Donaldson. "Nonlinear elliptic boundary value problems in Orlicz-Sobolev spaces". In: *J. Differential Equations* 10 (1971), pp. 507–528.

[30] T. K. Donaldson and N. S. Trudinger. "Orlicz-Sobolev spaces and imbedding theorems". In: *J. Funct. Anal.* 8 (1971), pp. 52–75.

[31] J. Droniou. "Intégration et Espaces de Sobolev a Valeurs Véctorielles," in: *Lecture Notes, Université de Provence, Marseille,* (2001).

[32] L. Esposito, F. Leonetti, and G. Mingione. "Sharp regularity for functionals with (p, q) growth". In: *J. Differential Equations* 204.1 (2004), pp. 5–55.

[33] X. Fan and Q. Zhang. "Existence of solutions for $p(x)$-Laplacian Dirichlet problem". In: *Nonlinear Anal.* 52.8 (2003), pp. 1843–1852.

[34] X. Fan and D. Zhao. "On the spaces $L^{p(x)}(\Omega)$ and $W^{m,p(x)}(\Omega)$". In: *J. Math. Anal. Appl.* 263.2 (2001), pp. 424–446.

[35] E. Gagliardo. "Caratterizzazioni delle tracce sulla frontiera relative ad alcune classi di funzioni in n variabili". In: *Rend. Sem. Mat. Univ. Padova.* 27 (1957), pp. 284–305.

[36] J.-P. Gossez. "A remark on strongly nonlinear elliptic boundary value problems". In: *Bol. Soc. Brasil. Mat.* 8.1 (1977), pp. 53–63.

[37] J.-P. Gossez. "Nonlinear elliptic boundary value problems for equations with rapidly (or slowly) increasing coefficients". In: *Trans. Amer. Math. Soc.* 190 (1974), pp. 163–205.

[38] J.-P. Gossez. "Opérateurs monotones non linéaires dans les espaces de Banach non réflexifs". In: *J. Math. Anal. Appl.* 34 (1971), pp. 371–395.

[39] J.-P. Gossez. "Orlicz-Sobolev spaces and nonlinear elliptic boundary value problems". In: (1979), pp. 59–94.

[40] J.-P. Gossez. "Some approximation properties in Orlicz-Sobolev spaces". In: *Studia Math.* 74.1 (1982), pp. 17–24.

[41] J.-P. Gossez. "Surjectivity results for pseudo-monotone mappings in complementary systems". In: *J. Math. Anal. Appl.* 53.3 (1976), pp. 484–494.

[42] J.-P. Gossez and V. Mustonen. "Variational inequalities in Orlicz-Sobolev spaces". In: *Nonlinear Anal.* 11.3 (1987), pp. 379–392.

[43] S. Gürgen, M.C. Kuşhan, and W. Li. "Shear thickening fluids in protective applications: A review". In: *Progress in Polymer Science* 75 (2017). cited By 28, pp. 48–72.

[44] P. Gwiazda and A. Świerczewska Gwiazda. "On steady non-Newtonian fluids with growth conditions in generalized Orlicz spaces". In: *Topol. Methods Nonlinear Anal.* 32.1 (2008), pp. 103–113.

[45] P. Gwiazda, F. Z. Klawe, and A. Świerczewska-Gwiazda. "Thermo-visco-elasticity for Norton-Hoff-type models". In: *Nonlinear Anal. Real World Appl.* 26 (2015), pp. 199–228.

[46] P. Gwiazda, P. Minakowski, and A. Wróblewska-Kamińska. "Elliptic problems in generalized Orlicz-Musielak spaces". In: *Cent. Eur. J. Math.* 10.6 (2012), pp. 2019–2032.

[47] P. Gwiazda, I. Skrzypczak, and A. Zatorska-Goldstein. "Existence of renormalized solutions to elliptic equation in Musielak-Orlicz space". In: *J. Differential Equations* 264.1 (2018), pp. 341–377.

[48] P. Gwiazda, I. Skrzypczak, and A. Zatorska-Goldstein. "Well-posedness of parabolic equations in the non-reflexive and anisotropic Musielak-Orlicz spaces in the class of renormalized solutions". In: *J. Differential Equations* 265.11 (2018), pp. 5716–5766.

[49] P. Gwiazda and A. Świerczewska-Gwiazda. "On non-Newtonian fluids with a property of rapid thickening under different stimulus". In: *Math. Models Methods Appl. Sci.* 18.7 (2008), pp. 1073–1092.

[50] P. Gwiazda, A. Świerczewska-Gwiazda, and A. Wróblewska. "Monotonicity methods in generalized Orlicz spaces for a class of non-Newtonian fluids". In: *Math. Methods Appl. Sci.* 33.2 (2010), pp. 125–137.

[51] P. Harjulehto and P. Hästö. *Orlicz spaces and generalized Orlicz spaces*. Vol. 2236. Lecture Notes in Mathematics. Springer, Heidelberg, 2019, pp. X, 169.

[52] P. Harjulehto, P. Hästö, and R. Klén. "Generalized Orlicz spaces and related PDE". In: *Nonlinear Anal.* 143 (2016), pp. 155–173.

[53] P. Harjulehto, P. Hästö, and O. Toivanen. "Hölder regularity of quasiminimizers under generalized growth conditions". In: *Calc. Var. Partial Differential Equations* 56.2 (2017), 56:22.

[54] P. Hästö. "On the density of continuous functions in variable exponent Sobolev space". In: *Rev. Mat. Iberoam.* 23.1 (2007), pp. 213–234.

[55] P. A. Hästö. "The maximal operator on generalized Orlicz spaces". In: *J. Funct. Anal.* 269.12 (2015), pp. 4038–4048.

[56] P. Hess. "On a class of strongly nonlinear elliptic variational inequalities". In: *Math. Ann.* 211 (1974), pp. 289–297.

[57] P. Hess. "On nonlinear mappings of monotone type with respect to two Banach spaces". In: *J. Math. Pures Appl. (9)* 52 (1973), pp. 13–26.

[58] H. Hudzik. "A generalization of Sobolev spaces. I". In: *Funct. Approximatio Comment. Math.* 2 (1976), pp. 67–73.

[59] H. Hudzik. "A generalization of Sobolev spaces. II". In: *Funct. Approximatio Comment. Math.* 3 (1976), pp. 77–85.

[60] H. Hudzik. "Density of $C_0^\infty(\mathbf{R}^n)$ in generalized Orlicz-Sobolev space $W_M^k(\mathbf{R}^n)$". In: *Funct. Approx. Comment. Math.* 7 (1979), pp. 15–21.

[61] H. Hudzik. "On problem of density of $C_0^\infty(\Omega)$ in generalized Orlicz-Sobolev space $W_M^k(\Omega)$ for every open set $\Omega \subset \mathbf{R}^n$". In: *Comment. Math. Prace Mat.* 20.1 (1977/78), pp. 65–78.

[62] A. Kamińska. "On some compactness criterion for Orlicz subspace $E_\Phi(\Omega)$". In: *Comment. Math. Prace Mat.* 22.2 (1980/81), pp. 245–255.

[63] A. Kamińska. "On some convexity properties of Musielak-Orlicz spaces". In: *Proceedings of the 12th winter school on abstract analysis (Srní, 1984)*. Suppl. 5. 1984, pp. 63–72.

[64] A. Kamińska. "Some convexity properties of Musielak-Orlicz spaces of Bochner type". In: *Proceedings of the 13th winter school on abstract analysis (Srní, 1985)*. 10. 1985, 63–73 (1986).

[65] F. Z. Klawe. "Thermo-visco-elasticity for models with growth conditions in Orlicz spaces". In: *Topol. Methods Nonlinear Anal.* 47.2 (2016), pp. 457–497.

[66] T. Kostopoulos and N. Yannakakis. "Density of smooth functions in variable exponent Sobolev spaces". In: *Nonlinear Anal.* 127 (2015), pp. 196–205.

[67] O. Kováčik and J. Rákosník. "On spaces $L^{p(x)}$ and $W^{k,p(x)}$". In: *Czechoslovak Math. J.* 41(116).4 (1991), pp. 592–618.

[68] W. M. Kozlowski. *Modular function spaces*. Vol. 122. Monographs and Textbooks in Pure and Applied Mathematics. Marcel Dekker, Inc., New York, 1988, pp. x+252.

[69] M. A. Krasnoselśkiĭ and Ja. B. Rutickiĭ. *Convex functions and Orlicz spaces*. Translated from the first Russian edition by Leo F. Boron. P. Noordhoff Ltd., Groningen, 1961, pp. xi+249.

[70] A. Kufner, O. John, and S. Fučík. *Function spaces*. Monographs and Textbooks on Mechanics of Solids and Fluids; Mechanics: Analysis. Noordhoff International Publishing, Leyden; Academia, Prague, 1977, pp. xv+454.

[71] R. Landes. "On Galerkin's method in the existence theory of quasilinear elliptic equations". In: *J. Funct. Anal.* 39.2 (1980), pp. 123–148.

[72] R. Landes and V. Mustonen. "Pseudomonotone mappings in Sobolev-Orlicz spaces and nonlinear boundary value problems on unbounded domains". In: *J. Math. Anal. Appl.* 88.1 (1982), pp. 25–36.

[73] M. Lavrentiev. “Sur quelques problèmes du calcul des variations”. In: *Ann. Mat. Pura Appl.* 41 (1927), pp. 107–124.

[74] H. Le Dret. *Nonlinear elliptic partial differential equations*. Universitext. An introduction, Translated from the 2013 French edition. Springer, Cham, 2018, pp. x+253.

[75] J. Leray and J.-L. Lions. “Quelques résultats de Višik sur les problèmes elliptiques non linéaires par les méthodes de Minty-Browder”. In: *Bulletin de la Société Mathématique de France* 93 (1965), pp. 97–107.

[76] J.-L. Lions. *Quelques méthodes de résolution des problèmes aux limites non linéaires.* Dunod; Gauthier-Villars, Paris, 1969, pp. xx+554.

[77] F.-Y. Maeda. “Poincaré type inequalities for variable exponents”. In: *JIPAM. J. Inequal. Pure Appl. Math.* 9.3 (2008), Article 68, 5.

[78] F.-Y. Maeda, Y. Mizuta, T. Ohno, and T. Shimomura. “Approximate identities and Young type inequalities in Musielak-Orlicz spaces”. In: *Czechoslovak Math. J.* 63(138).4 (2013), pp. 933–948.

[79] F.-Y. Maeda, Y. Mizuta, T. Ohno, and T. Shimomura. “Boundedness of maximal operators and Sobolev’s inequality on Musielak-Orlicz-Morrey spaces”. In: *Bull. Sci. Math.* 137.1 (2013), pp. 76–96.

[80] G. J. Minty. “Monotone (nonlinear) operators in Hilbert space”. In: *Duke Math. J.* 29 (1962), pp. 341–346.

[81] G. J. Minty. “on a “monotonicity” method for the solution of non-linear equations in Banach spaces”. In: *Proc. Nat. Acad. Sci. U.S.A.* 50 (1963), pp. 1038–1041.

[82] J. Musielak. *Orlicz spaces and modular spaces*. Vol. 1034. Lecture Notes in Mathematics. Springer-Verlag, Berlin, 1983, pp. iii+222.

[83] V. Mustonen and M. Tienari. “On monotone-like mappings in Orlicz-Sobolev spaces”. In: *Math. Bohem.* 124.2-3 (1999), pp. 255–271.

[84] H. Nakano. *Modulared Semi-Ordered Linear Spaces*. Maruzen Co., Ltd., Tokyo, 1950, pp. i+288.

[85] J. Nečas. *Les méthodes directes en théorie des équations elliptiques*. Masson et Cie, Éditeurs, Paris; Academia, Éditeurs, Prague, 1967, p. 351.

[86] Q.-A. Nguyen, S.J. Jorgensen, J. Ho, and L. Sentis. “Characterization and testing of an electrorheological fluid valve for control of ERF actuators”. In: *Actuators* 4.3 (2015). cited By 6, pp. 135–155.

[87] W. Orlicz. “Über Konjugierte Exponentenfolgen”. In: Studia Math., 1931, pp. 200–211.

[88] K. Rajagopal and M. Ružička. “On the modeling of electrorheological materials”. In: *Mech. Res. Commun.* 23 (1996), pp. 401–407.

[89] M. M. Rao and Z. D. Ren. *Theory of Orlicz spaces*. Vol. 146. Monographs and Textbooks in Pure and Applied Mathematics. Marcel Dekker, Inc., New York, 1991, pp. xii+449.

[90] V. D. Rădulescu and D. D. Repovš. *Partial differential equations with variable exponents*. Monographs and Research Notes in Mathematics. Variational methods and qualitative analysis. CRC Press, Boca Raton, FL, 2015, pp. xxi+301.

[91] M. Ružička. *Electrorheological fluids: modeling and mathematical theory*. Vol. 1748. Lecture Notes in Mathematics. Springer-Verlag, Berlin, 2000, pp. xvi+176.

[92] S. Samko. "Denseness of $C_0^\infty(\mathbf{R}^N)$ in the generalized Sobolev spaces $W^{M,P(X)}(\mathbf{R}^N)$". In: *Direct and inverse problems of mathematical physics (Newark, DE, 1997)*. Vol. 5. Int. Soc. Anal. Appl. Comput. Kluwer Acad. Publ., Dordrecht, 2000, pp. 333–342.

[93] M. S. Skaff. "Vector valued Orlicz spaces generalized N-functions. I". In: *Pacific J. Math.* 28 (1969), pp. 193–206.

[94] M. S. Skaff. "Vector valued Orlicz spaces generalized N-functions. II". In: *Pacific J. Math.* 28 (1969), pp. 413–430.

[95] A. Świerczewska-Gwiazda. "Nonlinear parabolic problems in Musielak-Orlicz spaces". In: *Nonlinear Anal.* 98 (2014), pp. 48–65.

[96] M. Tienari. "A degree theory for a class of mappings of monotone type in Orlicz-Sobolev spaces". In: *Ann. Acad. Sci. Fenn. Ser. A I Math. Dissertationes* 1.97 (1994), p. 68.

[97] M.I. Višik. "Boundary value problems for quasilinear strongly elliptic systems of divergent form." In: *Sov. Math., Dokl.* 2 (1961), pp. 643–647.

[98] W.M. Winslow. "Induced fibration of suspensions". In: *Journal of Applied Physics* 20.12 (1949). cited By 1089, pp. 1137–1140.

[99] A. Wróblewska. "Steady flow of non-Newtonian fluids—monotonicity methods in generalized Orlicz spaces". In: *Nonlinear Anal.* 72.11 (2010), pp. 4136–4147.

[100] Q. Zhang X. Fan and D. Zhao. "Eigenvalues of $p(x)$-Laplacian Dirichlet problem". In: *J. Math. Anal. Appl.* 302.2 (2005), pp. 306 –317.

[101] S. Yang, D. Yang, and W. Yuan. "New characterizations of Musielak-Orlicz-Sobolev spaces via sharp ball averaging functions". In: *Front. Math. China* 14.1 (2019), pp. 177–201.

[102] K. Yosida. *Functional analysis*. Classics in Mathematics. Reprint of the sixth (1980) edition. Springer-Verlag, Berlin, 1995, pp. xii+501.

[103] A. Youssfi and Y. Ahmida. "Some approximation results in Musielak-Orlicz spaces". In: *To appear in Czechoslovak Math. J.* (2019).

[104] V. V. Zhikov. "Meyer-type estimates for solving the nonlinear Stokes system". In: *Differ. Uravn.* 33.1 (1997), pp. 107–114, 143.

[105] V. V. Zhikov. "On some variational problems". In: *Russian J. Math. Phys.* 5.1 (1997), 105–116 (1998).

[106] V. V. Zhikov. "On the density of smooth functions in Sobolev-Orlicz spaces". In: *Zap. Nauchn. Sem. S.-Peterburg. Otdel. Mat. Inst. Steklov. (POMI)* 310.Kraev. Zadachi Mat. Fiz. i Smezh. Vopr. Teor. Funkts. 35 [34] (2004), pp. 67–81, 226.

[107] V. V. Zhikov. "On variational problems and nonlinear elliptic equations with nonstandard growth conditions". In: *J. Math. Sci. (N.Y.)* 173.5 (2011). Problems in mathematical analysis. No. 54, pp. 463–570.

[108] V. V. Zhikov and M. D. Surnachëv. "On the density of smooth functions in weighted Sobolev spaces with a variable exponent". In: *Algebra i Analiz* 27.3 (2015), pp. 95–124.

[109] V.V. Zhikov. "On Lavrentiev's phenomenon". In: *Russian J. Math. Phys.* 3.2 (1995), pp. 249–269.

[110] W. P. Ziemer. *Modern real analysis*. Second. Vol. 278. Graduate Texts in Mathematics. With contributions by Monica Torres. Springer, Cham, 2017, pp. xi+382.

[111] W. P. Ziemer. *Weakly differentiable functions*. Vol. 120. Graduate Texts in Mathematics. Sobolev spaces and functions of bounded variation. Springer-Verlag, New York, 1989, pp. xvi+308.

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