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Stabilization of infinite-dimensional dynamical systems governed by semilinear differential equations with distributed delay

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ملخص الرسالة

تعالج هذه الأطروحة قضية استقرار النظم الشبه الخطية اللانهائية الأبعاد ذات التأخر الموزع. ينصب التركيز بالضبط على تطوير استراتيجيات التحكم وما يسمى بشروط الملاحظة لتحقيق مستويات مختلفة من الاستقرار، بما في ذلك الاستقرار الضعيف والقوي والمتسارع.

في الفصل الأول، تم تجميع مختلف المفاهيم والنتائج المحتاجة في هذه الأطروحة وتهم بالأساس بعض التعاريف، المبرهنات والخاصيات المتعلقة بنصف الزمرة الخاصة بالتطبيقات الخطية ومختلف النتائج المتصلة بالمعادلات التفاضلية من وجود الحل ووحدانيته وكذا بعض مفاهيم وخصائص التحكم والاستقرار الخاصة بالمعادلات التفاضلية.

في الفصل الثاني، تم اقترح نوع من التحكم المحدود لدراسة الاستقرار الضعيف، القوي و المتسارع للنظم ثنائية الخطية مع تأخر منفصل في فضاء هيلبرت. اولا، تم البرهنة على وجود ووحدانية الحل الكلي للنظمة المدروسة في هذا العمل. ثانيًا، تم اقتراح بعض الشروط الكافية لدراسة استقرار النظمة ثنائية الخطية. بعد ذالك، في حالة الاستقرار القوي، تم اعطاء تقدير اضمحلال صريح للحالة المستقرة. في النهاية، تم تقديم أمثلة مع عمليات المحاكاة العددية لإظهار قابلية تطبيق نتائجنا النظرية.

في الفصل الثالث، تم تقديم سلسلة من انواع التحكم التي اثبتت فعاليتها للحصول على استقرار ضعيف و المتسارع بالنسبة للنظم شبه الخطية ذات التأخرات المتعددة و المنفصلة. بالإضافة إلى ذلك، تم تقديم حالة شرط ≪ شرط الملاحظة ≫ أبسط، سهل التحقق في التطبيقات العملية.

يركز الفصل الرابع على دراسة استقرار النظم شبه الخطية ذات التأخر الموزع. باستخدام شرط ≪ شرط الملاحظة ≫ مبني أساسا على نصف زمرة حل للجزء الخطي من النظمة المعتبرة في هذا العمل. ثم الحصول على نتائج جيدة تخص الاستقرار الضعيف والقوي والمتسارع للنظمة المعتبرة باستخدام التحكم الرباعي والمعياري. زيادة على ذالك تم ايضا مناقشة استقرار النظم ثنائية الخطية باستخدام طريقة تقسيم فضاء الحالة.

في الفصل الخامس، تم اقتراح مجموعة جديدة من عناصر التحكم لتحقيق الاستقرار الخاص بفئة من النظم شبه الخطية غير المحدودة الأبعاد مع تأخر موزع. تتيح هذه المجموعة من عناصر التحكم الحصول على استقرار قوي وضعيف لهذه النظم الشبه الخطية. علاوة على ذلك، يتضح أنه

من بين هذه المجموعة من عناصر التحكم هناك عنصر وحيد من هذه العائلة يمكن وصفه بأنه الحل الوحيد لمشكلة التصغير، مما يجعلها قابلة للتطبيق على الأنظمة الفيزيائية والبيولوجية.

يركز الفصل الأخير على الاستقرار القوي والضعيف لفئة من النظم ثنائية الخطية مع تأخر موزع في فضاء باناخ باستخدام التحكم ، من خلال تقديم شرط ملاحظة معبرًا عنها بدلالة نصف زمرة حل الجزء الخطي للمعادلة المعتبرة. ثم إثبات الاستقرار القوي للمعادلة الخطية في كليتها باستعمال شرط ملاحظة جديد في فضاء باناخ. ثم ايضا البرهنة على ضعف الاستقرار باستخدام شرط أخر مختلف عن شرط الملاحظة المقترح في الاستقرار القوي. في المجال التطبيقي، ثم القيام بعمليات المحاكاة العددية للتحقق من صحة النتائج المحصل عليها نظريا.

Abstract

This thesis explores the feedback stabilization of infinite-dimensional semi-linear systems with distributed delays. The focus is on developing control strategies and observability conditions to achieve various levels of stabilization, including weak, strong, and exponential stabilization.

In the first chapter, we present the main results needed through this thesis. It is mainly about some definitions, theorems and properties concerning the C_0-semigroup theory, and it's application in delay differential equations. Moreover, we give some results and definitions about control and stabilization of semilinear system with delay.

In the second chapter, we consider the decomposition of the Hilbert phase space and we propose a bounded feedback control to study the weak, strong and exponential stabilization of bilinear systems with discrete delay in a Hilbert space. Firstly, we discuss the well possedness of mild solutions of the considered systems. Secondly, some sufficient conditions are given to guarantee the feedback stabilization for the bilinear systems. The stabilization results are given in terms of observation estimates. Furthermore, an explicit decay estimate of the stabilized state is obtained in the strong stabilization case.

In the third chapter, a sequence of feedback control is introduced to prove the exponentially and weakly stabilization of the semi-linear systems with discrete multi-delays. Additionally, a simple observability condition is presented, which is easy to verify in practical applications.

The fourth chapter concentrates on the feedback stabilization analysis of the semilinear system with distributed delay. General results are provided, utilizing an observability condition based on the semigroup solution of the linear part of the considered system. Results on the weak, strong, and exponential stabilization of the system are obtained using quadratic and normalized controls. The feedback stabilization of bilinear systems is also discussed, employing the decomposition method of the state space.

In the fifth chapter, a novel family of feedback controls is proposed to stabilize a class of infinite-dimensional semilinear systems with distributed delay. This family of controls enables strong and weak stabilization of the closed-loop semilinear system. Furthermore, it is shown that one specific control within this family can be characterized as the unique solution of an appropriate minimization problem, making it applicable to physical and biological systems.

The final chapter focuses on the strong and weak stabilization of a class of bilinear systems with distributed delay in the Banach space using feedback control. By imposing an observability condition expressed in terms of the semigroup solution of the linear part of the system considered, strong stabilization of the system is demonstrated, along with ex-

plicit decay estimate. Moreover, the weak stabilization is also proven using the conditions other than the observability condition used in strong stabilization.
We mention that at the end of each chapter, some examples of application are provided. Moreover, numerical simulations are given to show the applicability of our theoretical results.

Introduction

0.1 General introduction

Many practical dynamical systems exhibit delays, but they are often neglected for simplicity. Sensor or actuator reaction times, data transmission times, molecular transfer periods, and measurement times are all examples of factors that cause delays. However, the presence of delays can induce poor performance and instabilities in control systems, so it is important to take them into account.

Delay dynamical systems form a class of infinite-dimensional systems widely used for modeling and analyzing transport and propagation phenomena (of matter, energy, or information) see, e.g. [7, 32, 34]. This class of equations was initiated in control theory due to their simple forms and their applicability to many different real-life problems such as biology, economics, engineering, chemistry. For further details on this subject, the reader can refer to, for example, [62, 76]. Furthermore, even if the process itself does not contain delays, the closed-loop system can generate significant delays (with sensors, actuators, and computation time) . A delayed type system is modeled by a system in which the derivative of the "state" at the present time is a function of the values of the "past state".

Systems with distributed delays are an important class of mathematical models that arise in a variety of real-world applications. These systems are characterized by time delays that are not fixed, but rather distributed over a range of values. Such delays can occur in a wide range of physical systems, including transportation networks, chemical processes, heat conduction, and many others. The distributed nature of these delays can have a significant impact on the behavior of the system, often leading to complex dynamics that are difficult to analyze using traditional techniques. As a result, the modeling and analysis of systems with distributed delays is an active area of research with many important applications. This field draws on a range of mathematical tools, including partial differential equations and integral equations, as well as numerical techniques such as

finite element methods and spectral methods. By developing accurate models of systems with distributed delays and understanding their behavior, researchers can develop effective control strategies that can improve the performance and efficiency of these systems in a wide range of applications.

0.2 Some examples of dynamic systems with distrubated delay

0.2.0.1 Integro-differential equation of Volterra

In 1934, V. Volterra published the first work that used a logistic equation with a distributed delay. In addition, numerous experiments with laboratory populations of many organisms with short generation times were conducted in 1930. As the populations died out, attempts to apply logistic models to these trials were often unsuccessful, one cause being the overpopulation of waste and dead organisms in the closed environment. V. Volterra, in [89], examined the cumulative influence of the mortality rate of a species using an integral or distributed lag term, as a function of the population at any time since the beginning of the experiment. An integro-differential equation is used to model the situation

$$\dot{w}(t) = \alpha w(t) \left(1 - \frac{1}{\beta} \int_{-\infty}^{t} k(t-\theta) w(t)\, d\theta\right),$$

where $k(t)$ denotes the lag kernel, is a weighting factor that determines how much weight to give to the population size at previous times when computing the current effect on resource availability. We refer to [12, 16, 80] for more general equations concerning delay integrodifferential Volterra reaction-diffusion equations.

0.2.0.2 Predator-prey model

In 1928, the Italian mathematician Vito Volterra presented one of the first models for combining predator-prey interactions that took into account the time lag. Volterra introduced the following model, which assumed that the development of the predator in interaction with the prey is not immediate.

$$\begin{cases} \dot{w_1}(t) = (\beta - \alpha w_2(t)) w_1(t), \\ \dot{w_2}(t) = \left(-\beta_1 + \alpha_1 w_1(t) + \displaystyle\int_{-r}^{0} k(\theta) w_1(t+\theta)\, d\theta\right) w_2(t), \end{cases}$$

such that w and v represent the number of predators and prey respectively. α, α_1, β and β_1 are positive constants. The positive function k denotes the method that expresses the

relationship between profit and the abundance of predators and prey in the time period $[t-r, t]$.

0.2.0.3 Dynamic model of pluripotent stem cells

Hematopoiesis, or the process of blood production, is one of the most important biological processes in the human body. It occurs in the bone marrow, where pluripotent stem cells generate mature cells. These cells pass into the circulation after ejecting their nuclei and become blood cells. Since the end of the 1970s, researchers have been studying the dynamics of this cell population, notably in [57, 78]. On the other hand, in [1], the authors propose the following model:

$$\dot{w}(t) = -(\alpha + g(w(t))w(t) + 2\int_{\varpi}^{\omega} e^{-\gamma r}\beta(r)g(w(t-s))w(t-s)\,ds,$$

which describes the dynamics of a population of pluripotent stem cells involved in the bone marrow blood production process. $w(t)$ is the density of pluripotent stem cells (cells/kg) at time $t > 0$. The first term in the above equation, on the right side, represents the loss of cells due to cell death and differentiation. The second term expresses the proliferating cells that divide into two daughter cells during mitosis and the duration of the cell cycle is represented by the time r. The proliferating cells are actually resting cells pushed into the proliferating phase one generation earlier, resulting in a time delay of $g(w(t-r))w(t-r)$. The second factor is required for each proliferating cell to divide into two daughter cells. $e^{-\gamma r}$ denotes apoptosis-induced population attenuation during the proliferative phase. β is a density function that describes the division density of proliferating cells, including $\int_{\varpi}\beta(s)\,ds = 1$, g is the reentry rate into the proliferating compartment. Furthermore, the authors established delay-independent conditions and proved that the distributed delay can destabilize the whole system. In particular, they demonstrated that Hopf bifurcations can appear.

0.2.0.4 Model of distributed population dynamics

Population dynamics is concerned with the evolution, over time, of populations of living beings (e.g. cells) often with their own characteristics. Among the current research axes in this field, we quote the one launched by the group of Adimy and his collaborators, in the framework of mathematical modelling in medicine and biology, which is interested in the dynamics of the prion, i.e. how the good prion proteins interact with the bad ones, leading to serious diseases, as well as in a more general reflection concerning the notion of adaptation in a population.

$$\begin{cases} \dfrac{\partial w(x,t)}{\partial t} = \delta \dfrac{\partial^2 w(x,t)}{\partial x^2} + cw(x,t)\left(1 - \displaystyle\int_{-\infty}^{0} h(\theta) w(x,t+s)ds\right) \text{ for } 0 \leq x \leq \pi \text{ and } t \geq 0 \\ \dfrac{\partial w(0,t)}{\partial t} = \dfrac{\partial w(\pi,t)}{\partial t} = 0 \text{ for } t > 0 \\ w(x,s) = \phi(s)(x) \text{ for } 0 \leq x \leq \pi \text{ and } -\infty < s \leq 0 \end{cases}$$

describes the dynamics of a distributed population of a single species, where $w(x,t)$ represents the size of the total population at position x and time t, the two constants δ and c are positive measuring respectively the diffusion rate and the intrinsic growth and $h :]-\infty, 0] \to \mathbb{R}^+$ is an integrable function verifying

$$\int_{-\infty}^{0} h(s)ds = 1$$

The model has been studied by several authors for example, see [12, 16, 80].

0.2.0.5 Competition model of Lotka-Volterra

We place ourselves in the line of the mathematical ecology work initiated in the 1920s by the work of Lotka and Volterra, work in which the representation of interactions between species by systems of differential equations was introduced. The predator-prey models developed by these two scientists separately are the founding models of modern ecology. The purpose of developing Volterra's model was to explain a widespread phenomenon in real predator-prey systems, namely phase-shift oscillations in prey and predator abundances. The following model was proposed in 1928 to describe the interaction between a prey population and a predator population. It is described by the following system of ordinary differential equations:

$$\begin{cases} \dfrac{dw_1(t)}{dt} = (a - bw_2(t))\, w_1(t) \text{ for } t \geq 0 \\ \dfrac{dw_2(t)}{dt} = \left(-c + dw_1(t) + \displaystyle\int_{-r}^{0} k(s) w_1(t+s)ds\right) w_2(t) \text{ for } t \geq 0 \\ w_1(\theta) = \phi_1(\theta) \text{ for } -r \leq \theta \leq 0 \text{ and } w_2(0) = \phi_2(0) \end{cases}$$

where w_1 and w_2 denote prey and predator populations, respectively, and the following parameters denote:

a is the natural growth rate of prey in the absence of predation, b is the mortality rate of prey due to predation, c is the natural mortality rate of predators in the absence of prey, d is the natural growth rate of predators in the presence of prey. k is a kernel function that describes how the predator's gain at each time t depends on the prey population

abundance w_1 in a past time interval $[t-r,t], r>0$. The conversion of prey eaten by predators occurs continuously over a time r, starting from the death of the prey, which is described by the last term of the second equation that shows a continuous finite delay distributed.

In [15], Brelot then introduced delay into the Volterra model to model a hereditary biological problem:

$$\begin{cases} \dfrac{dw_1(t)}{dt} = w_1(t)\left(a - bw_2(t) - \displaystyle\int_0^{+\infty} k(s)w_2(t-s)ds\right) \text{ for } t \geq 0 \\ \dfrac{dw_2(t)}{dt} = w_2(t)\left(-c + dw_1(t) - \displaystyle\int_0^{+\infty} k(s)w_1(t-s)ds\right) \text{ for } t \geq 0 \\ w_1(\theta) = \phi_1(\theta) \text{ and } w_2(\theta) = \phi_2(\theta) \text{ for } \theta \leq 0 \end{cases}$$

Spatial circulation and migration of species from one location to another can complicate ecological interactions and influence their evolutionary behavior. For example, if food is provided homogeneously across space, spatial diffusion may result from the tendency of species to migrate from regions of higher population density to regions of lower population density.

0.3 Differential equations with finite delay

Differential equations with finite delays are a powerful mathematical tool used to model and analyze dynamical systems in various scientific and engineering disciplines [11, 68]. These equations incorporate the influence of past values of a function on its current behavior, making them suitable for describing systems with memory effects or time delays. This introductory overview will explore the fundamental aspects of differential equations with finite delays, their mathematical formulation, solution techniques, and notable applications [33].

The study of differential equations with finite delays has a rich history and has attracted considerable attention from researchers due to their widespread applicability. Numerous real-world phenomena exhibit delayed responses, such as chemical reactions with time-dependent rate constants [44], biological systems with time delays [18, 91], and control systems [92, 97]. Understanding the dynamics of such systems is critical for accurate modeling, analysis and prediction.

Solving differential equations with finite delays is a challenging task due to the nonlocal nature of the problem. Traditional methods for solving ordinary differential equations may not be directly applicable, necessitating specialized techniques. Analytical approaches such as the method of steps, Laplace transforms, and series expansions have been developed to handle specific classes of delay equations see, e.g [37, 52, 102]. Additionally,

numerical approximation methods including finite difference, finite element, and spectral methods are commonly employed for obtaining numerical solutions. The study of differential equations with finite delays has found wide application across numerous scientific and engineering domains. In ecology, these equations are used to model population dynamics with time delays in reproductive processes or migration patterns. In neuroscience, delay equations are employed to describe the propagation of signals in neural networks and the synchronization of neuronal activity. Control theory utilizes delay equations to analyze the stability and performance of systems with time delays in feedback loops. Other areas of application include epidemiology, physics, economics, and engineering. Several comprehensive references provide in-depth coverage of differential equations with finite delays. One highly regarded book is "Functional Differential Equations: Advances and Applications" by Constantin Corduneanu [25], which presents theoretical foundations, solution techniques, and applications of delay differential equations. Another valuable resource is "Time Delay Systems: Methods, Applications, and New Trends" edited by Rifat Sipahi [82], which offers a collection of articles covering different aspects of delay equations, including numerical methods and stability analysis.

Using the delay in the description of a dynamic allows us to obtain a model even closer to reality. Consequently, distributed delay equations intervene in several fields of mathematical applications, and in particular it is used to model the phenomenon of time lag in thermodynamics [35, 98], in ecology as well as in epidemiology such as predator-prey systems [36], in mechanics, in electronics, where many phenomena are non-local, in the time variable but sometimes also in the space variable (age, size, etc).

0.4 Bilinear system and semilinear system

Bilinear systems and semilinear systems are important classes of mathematical models that have been extensively studied in the field of control theory. These systems play a crucial role in understanding and analyzing the behavior of various physical, engineering and biological systems, for more details on this subject see references [5, 17, 63, 64, 93, 94]. The study of bilinear systems dates back to the mid-20th century. Bilinear systems are characterized by equations that involve the product of two variables or the product of a variable and its derivative. These systems exhibit a combination of linear and nonlinear behaviors, making them particularly intriguing and challenging to analyze. Notable contributions to the theory of bilinear systems include the works of R. E. Kalman and R. S. Bucy in the 1960s [48], who developed powerful techniques for control and filtering of bilinear systems. Semilinear systems, on the other hand, have a long history in the field of partial differential equations (PDEs). Semilinear PDEs involve linear terms and nonlinear terms that depend on the solution itself or its derivatives. The study of

semilinear systems began in the early 20th century and has since evolved into a rich and diverse field of research. Prominent mathematicians such as J. Leray, L. Nirenberg, and J. Serrin [54, 55, 67, 85] made significant contributions to the understanding and analysis of semilinear PDEs. Both bilinear systems and semilinear systems have found widespread applications in various disciplines. Bilinear systems are commonly used to model systems with interactions between different variables, such as chemical reactions, population dynamics with limited resources, and economic systems with nonlinear interactions. Semilinear systems, on the other hand, provide a mathematical framework for describing phenomena such as wave propagation, heat transfer, and population dynamics [93]. The analysis and control of bilinear systems and semilinear systems have been the subject of extensive research. Researchers have developed a wide range of techniques and approaches to study the behavior, stability, and control of these systems. These include analytical methods, numerical simulations, and advanced control strategies. The works of P. Khargonekar and J. Lygeros [28, 29, 53, 58, 79, 103] have made significant contributions to the theory and control of bilinear systems.

0.5 Short historical note for the research area

The stabilization of dynamical systems is one of the most important aspect of control theory which has been studied by several authors, we refer to [81], [83] and [87] and references therein for more details on this subject. The fundamental theory of stabilization established by Lyapunov is intensively developed for finite-dimensional dynamical systems. In the case without delay, the semilinear system can be written in the following form:

$$\dot{w}(t) = \mathcal{A}w(t) + v(t)Fw(t) \text{ and } w(0) = w_0 \in \mathcal{X}, \tag{0.1}$$

where F is a linear bounded operator from $\mathcal{X}$ to $\mathcal{X}$. The feedback stabilization of the class of equations (0.1) has been studied by many researchers, see for example [4, 6, 10, 24, 26, 69, 73, 74, 75, 81, 83, 100] and references therein. In [6], the authors considered the case where the state space $\mathcal{X}$ is of finite dimension, they supposed that $S(t)$ is a semigroup of contraction and F is a sequentially continuous operator from $\mathcal{X}$ to $\mathcal{X}$ such that

$$\langle FS(t)\xi, S(t)\xi\rangle \;=\; 0 \quad \text{for all } t \geq 0 \;\Rightarrow\; \xi = 0,$$

then the feedback

$$v(t) \;=\; -\langle w(t)\;,\; Fw(t)\rangle, \tag{0.2}$$

weakly stabilizes the system (0.1). In [10], the authors assumed that the operator $\mathcal{A}$ has a compact resolvent and generates a semigroup of contraction and F is a linear positive

and self-adjoint operator satisfying

$$\int_0^T \|F^{\frac{1}{2}} S(t)\xi\|_{\mathcal{X}}^2 dt \ \geq \ \delta\|\xi\|_{\mathcal{X}}^2 \ \text{ for all } \xi \ \in \ \mathcal{X},$$

for some constants $\delta, T > 0$. Then, the control (0.2) strongly stabilizes the system (0.1) with the following decay estimate

$$\|w(t)\|_{\mathcal{X}} \ = \ O\left(\frac{1}{\sqrt{t}}\right) \ \text{as} \quad \text{t} \ \longrightarrow +\infty. \tag{0.3}$$

Moreover, the authors in [71] showed that if $S(t)$ is a semigroup of contraction and if the operator F is locally Lipschitz from $\mathcal{X}$ to $\mathcal{X}$ and satisfies the following observability condition

$$\int_0^T |\langle FS(t)\xi, S(t)\xi\rangle| dt \geq \delta\|\xi\|_{\mathcal{X}}^2 \quad \text{for all } \xi \ \in \ \mathcal{X},$$

for some constants $\delta, T > 0$. Then, the control (0.2) strongly stabilizes the system (0.1) and the estimate (0.3) holds. In the case where the state space $\mathcal{X}$ is of infinite dimension, the authors in [4] supposed that the operator $\mathcal{A}$ generates a semigroup $S(t)$ of contraction on the Banach space $\mathcal{X}$ and F is a linear bounded operator on $\mathcal{X}$. They showed that if there exist $\delta, T > 0$ such that

$$\int_0^T |\langle FS(t)\xi, S(t)\xi\rangle| dt \geq \delta\|\xi\|_{\mathcal{B}}^2 \ \text{ for all } \xi \ \in D(\mathcal{A}),$$

where $\mathcal{B}$ is a Banach space such that $D(\mathcal{A}) \subset \mathcal{B} \subset \mathcal{X}$. Then, for all initial data $w(0) \in D(\mathcal{A})$, the feedback (0.2) guarantees the strong stabilization of bilinear system (0.1). Moreover, they showed that the solution satisfies the following decay estimate

$$\|w(t)\|_{\mathcal{X}}^2 \ = \ O\left(t^{-\frac{\theta}{2-\theta}}\right) \ \text{as} \quad \text{t} \ \longrightarrow \ +\infty$$

with $\theta \in (0, 1)$ and verifies some interpolation property. Furthermore, in [75], the author discussed the uniform stabilization of bilinear systems using multiplicative control inputs. In the general case where the space $\mathcal{X}$ is Banach space, there are a few works discussing the stabilization question for a bilinear system without delay. Recently in [9], the author studied the strong stabilization of (0.1) in the case where $\mathcal{X}$ is a Banach space. They showed that if the semigroup $S(t)$ is of contraction and if the operator F is linear bounded from $\mathcal{X}$ to $\mathcal{X}$ and the duality map $\mathcal{J}$ is Lipschitz-continuous such that

$$\int_0^T |\langle FS(\sigma)w, \mathcal{J}(S(\sigma)w)\rangle| d\sigma \geq \eta\|w\|_{\mathcal{X}}^2 \quad \text{for all } w \ \in \ \mathcal{X},$$

for some $\eta, T > 0$. Thus, the following feedback control

$$v(t) \ = \ -\langle \, Fw(t), \mathcal{J}(w(t))\rangle,$$

strongly stabilizes the system (0.1) and the estimate (0.3) holds. In the case of discrete delay, the system (0.8) is written in the following form

$$\dot{w}(t) = \mathcal{A}w(t) + v(t)Fw(t-r) \text{ and } w_0 = \phi \in \mathcal{C}. \tag{0.4}$$

There are many works dealing with the feedback stabilization of this class of bilinear systems. In [30] and [39], the authors used the decomposition method to investigate the strong stabilization of bilinear system (0.4) with discrete delay. They showed, when r less than some positive constant and when the space $\mathcal{X}$ is decomposed into two invariant subspaces $\mathcal{X}^u$ and $\mathcal{X}^s$ with respect to $S(t)$ and F such that the dimension of $\mathcal{X}^u$ is finite and $S(t)$ induces on $\mathcal{X}^s$ an exponentially stable C_0-semigroup $S^s(t)$, that the bilinear system (0.4) with discrete delay is strongly stabilizable under the following quadratic feedback control $v(t) = -\rho\langle w^u(t), F^u w^u(t-r)\rangle$ with $\rho \geq 0$. More generally, in [19], the authors investigated the feedback stabilization of bilinear system (0.4) with discrete delay. They proved the strong stabilization of this system for any $r \geq 0$ and for any arbitrary dimension of $\mathcal{X}^u$ and $\mathcal{X}^s$.

During the last decades the optimal control theory has gained a great importance among engineers, mathematicians and other researchers in the world see e.g [62]. Many articles have been developed and published, see for instance [3, 5, 19, 20, 21, 59, 70, 101] for the stabilizationanalysis and [46, 47, 60, 70, 96, 104] for the problem of optimal feedback control for a class of bilinear systems without delay. In the case of the system without delay ($r = 0$), the authors in [95], it was established that if the semigroup $(S(t))_{t\geq 0}$ is of contraction and F is a sequentially continuous operator such that

$$\langle PFS(t)\zeta, S(t)\zeta\rangle\langle FS(t)\zeta, S(t)\zeta\rangle = 0 \quad \text{for all } t \geq 0 \Rightarrow \zeta = 0,$$

and

$$Re\langle P\mathcal{A}S(t)\zeta, \zeta\rangle = 0, \quad \text{for all } \zeta \in D(\mathcal{A}),$$

where P is a compact operator mapping from $\mathcal{X}$ to $\mathcal{X}$. Then, the feedback control $v(t) = -\langle PFw(t), w(t)\rangle$ weakly stabilizes system (0.1) and minimizes the cost function defined by

$$J(v) = \int_0^{+\infty} |\langle PFw(t), w(t)\rangle|^2 dt + \int_0^{+\infty} |v(t)|^2 dt$$

where $v \in V_{ad} = \{v \text{ such that } w(t) \rightharpoonup 0 \text{ as } t \to +\infty \text{ and } J(v) < \infty\}$. In [70], the authors proved that if $(S(t))_{t\geq 0}$ is a semigroup of isometries such that

$$\int_0^1 |\langle FS(s)\zeta, S(s)\zeta\rangle| ds \geq \delta \|\zeta\|_{\mathcal{X}}^2 \text{ for all } \zeta \in \mathcal{X},$$

for some $\delta > 0$, then, the feedback control

$$v(t) = -\langle Fw(t), w(t)\rangle, \tag{0.5}$$

strongly stabilizes system (0.8). Moreover, they obtained the following decay estimate

$$\|w(t)\|_{\mathcal{X}} = O\left(t^{\frac{-1}{2}}\right), \text{ as } \quad t \longrightarrow +\infty.$$

Furthermore, they should that the feedback control (0.5) is a solution of the following minimization cost function

$$J(v) = \int_0^{+\infty} |\langle Fw(t), w(t)\rangle|^2 dt \; + \; \int_0^{+\infty} |v(t)|^2 dt$$

where $v \in V_{ad} \; = \; \{v : \; J(v) \; < \; \infty\}$.
For systems with multi-delays, many researchers have studied these systems using serval methods and approaches such as the Lyapunov-Kravskii method, for more details on this subject, we refer to [45, 51, 61] and references therein. In general, there are several difficult aspects in the applicability of the Lyapunov-Kravskii approach to system stabilization analysis, one of them is the lack of efficient algorithms to correctly construct the corresponding Lyapunov-Kravskii functionals. In the case of finite or infinite dimension without delay, these types of the systems studied extensively throughout the past two decades, we refer the reader to [4, 6, 10, 38, 41, 42, 43, 56, 65, 66, 71, 72] for more details.

0.6 Presentation of the thesis

The main objective of the second chapter is to study the weak, strong and exponential stabilization of bilinear systems with discrete delay in a Hilbert space of the following system:

$$\begin{cases} \dot{w}(t) = \mathcal{A}w(t) + v(t)Fw(t-r), \, t > 0, \\ \quad w_0 = \phi \in \mathcal{C} = C([-r, 0]; \mathcal{X}), \end{cases} \tag{0.6}$$

where $\mathcal{A}$ is the infinitesimal generator of a linear C_0-semigroup $S(t)$ on a real Hilbert space $(\mathcal{X},\langle .,. \rangle \;)$ and for all $w \in \mathcal{X} \; \|w\| = \sqrt{\langle w, w\rangle}$. The operator $F : \mathcal{X} \to \mathcal{X}$ is linear and bounded, while the scalar valued function $t \; \to \; v(t)$ represents the control, the positive real number r denotes the delay and $\mathcal{C} = C([-r, 0]; \mathcal{X})$ is the Banach space of continuous $\mathcal{X}$ -valued functions on $[-r, 0]$ with $\|\phi\|_{\mathcal{C}} = \sup_{-r \leq s \leq 0} \|\phi(s)\|_{\mathcal{X}}$. For $t \in [0, +\infty)$, the history function w_t is an element of $\mathcal{C}$ such that $w_t(s) = w(t + s) \; for \; all \; s \; \in \; [-r, 0]$. We consider the decomposition of the phase space and we propose a bounded feedback control to study the weak, strong and exponential stabilization. Firstly, we discuss the well possedness of mild solutions of the considered systems. Secondly, some sufficient conditions are given to guarantee the feedback stabilization for the bilinear systems. The stabilization results are given in term of observation estimates. Furthermore, an explicit decay estimate of the stabilized state is obtained in the strong stabilization case. At

the end, examples with numerical simulations are given to show the applicability of our theoretical results.

The third chapter deals with the problem of weak and exponential stabilization by a sequence of feedback control for the following class of multi-delays semi-linear systems

$$\begin{cases} \dot{w}(t) = \mathcal{A}w(t) + \sum_{i=0}^{m} v_i(t)F_i w(t-r_i) \ \text{ for } t > 0, \\ w_0 = \phi \in \mathcal{C} := C([-r, 0]; \mathcal{X}). \end{cases} \tag{0.7}$$

where $\mathcal{A}$ is the infinitesimal generator of a linear C_0 -semigroup $S(t)$ on a real Hilbert space $\mathcal{X}$, endowed with inner product $\langle ., . \rangle$ and corresponding norm $\|.\|$. The operator $F_i : \mathcal{X} \to \mathcal{X}$ is locally Lipschitz such that $F_i(0) = 0$ for all $i \in \{0, ..., m\}$, the positive real numbers r_i are delays with $0 = r_0 < r_1 < r_2 < ... < r_m = r$ and $v_i : \mathbb{R}^+ \to \mathbb{R}$ for $i \in \{0, ..., m\}$ are sequences of feedback control. First, we analyze the existence, uniqueness of the global mild solution of the system (0.7). In addition, we establish an estimate of the decay which will be used to study the exponential and weak stabilization of our system (0.7). Moreover, applications with numerical simulation to wave equations and heat equation are presented.

In the fourth chapter, we interest to the stabilization of the following class of infinite dimensional semilinear systems with distributed delay:

$$\begin{cases} \dot{w}(t) = \mathcal{A}w(t) + v(t)F(w_t) \quad \text{for } t > 0, \\ w_0 = \phi \in \mathcal{C} := C([-r, 0]; \mathcal{X}). \end{cases} \tag{0.8}$$

where $\mathcal{A}$ is the infinitesimal generator of a C_0-semigroup $(S(t))_{t\geq 0}$ on a Hilbet space $\mathcal{X}$. F is a locally Lipschitz function from $\mathcal{C}$ to $\mathcal{X}$ such that $F(0) = 0$. The control v, of type feedback, is a defined function from $\mathbb{R}^+$ to $\mathbb{R}$. First, we consider an observation condition in terms of the semigroup solution of the linear part of the considered system and parameterized by a positive constant. Some sufficient conditions are given with respect to this parameter and the bounded feedback control to guarantee the feedback stabilization of the semilinear system with distributed delay. Moreover, when this parameter is greater than or equal to 2, an explicit polynomial decay rate of the stabilization state is estimated in the strong stabilization case. In the bilinear case with distributed delay and by using the decomposition method of the state space, we investigate the feedback stabilization of the considered system using some suitable conditions like observability assumption. In the case of strong stabilization, we obtain the same explicit decay estimate of the stabilized state. Furthermore, when the parameter is equal to 2, we show that the normalized feedback control exponentially stabilizes the semilinear system. The obtained results are illustrated by three examples and numerical simulations for wave

equation with distributed delay.

The objective of the fifth chapter is to characterize the feedback control which guaranties the strong stabilization of system (0.8) and minimizing the functional cost

$$\min_{v \in V_{ad}} J(v), \tag{0.9}$$

where

$$J(v) = \int_0^{+\infty} |\langle F(w_s), w(s)\rangle|^2 ds \; + \; \int_0^{+\infty} |v(s)|^2 ds,$$

and V_{ad} is the set of all $v \in L^2(0,\infty)$ such that the corresponding solution $w(.)$ to system (0.8) exists on the interval $[0,+\infty)$ and satisfies $\|w(t)\|_{\mathcal{X}} \leq K$, for all $t \geq 0$, where $K \geq 0$ and $J(v) < +\infty$.

As a normal continuation of the above chapters, The sixth chapter focuses on a general class of bilinear systems. Especially, we discuss the feedback stabilization of a class of infinite-dimensional bilinear systems with distributed delay in the Banach space $\mathcal{X}$. We accomplish this through the use of a decomposition method for the Banach phase space $\mathcal{X}$, dividing it into two invariant subspaces, denoted $\mathcal{X}^u$ and $\mathcal{X}^s$. Our approach involves two main steps. First, we analyze the global mild solution of the system (0.8), studying the existence and uniqueness of this solution. Second, we investigate the strong stabilization of our system within the Banach space, by considering an observation condition that expressed by $(U(t))_{t\geq 0}$ semigroup solution of system (0.8) with $F = 0$, and by a positive constant $\delta \geq 2$. Third, we discuss the weak stabilization of system (0.8) in Banach space by using author weaker assumptions than the observability condition. Finally, an example with distributed delay is given and some numerical simulations are also included to illustrate the applicability of the obtained theoretical results.

Chapter 1

Preliminaries

In this chapter we recall some classical notions and results that will be used in the following chapters. Let us first start with a few reminders about the theory of semigroups

1.1 C_0 semigroups

1.1.1 Definitions and Proposals

Definition 1.1.1. *[77] Let $\mathcal{X}$ be a Banach space. We call C_0 a semigroup (or strongly continuous semigroup) a family $(S(t))_{t\in\mathbb{R}^+}$ of bounded linear operators from $\mathcal{X}$ to $\mathcal{X}$, verifying:*

- $S(t+s) = S(t)S(s), \quad$ *for all* $t, s \in \mathbb{R}^+$,
- $S(0) = I$,
- $\lim\limits_{t\to 0^+} \|S(t)w - w\|_{\mathcal{X}} = 0, \quad$ *for all* $w \in \mathcal{X}$.

Definition 1.1.2. *[77] Let $(S(t))_{t\in\mathbb{R}^+}$ be a semigroup C_0. The infinitesimal generator of $(S(t))_{t\in\mathbb{R}^+}$ is an unbounded linear operator defined by:*

$$\mathcal{A}w = \lim_{t\to 0^+} \frac{S(t)w - w}{t}$$

The domain of $\mathcal{A}$, denoted $\mathcal{D}(\mathcal{A})$, is given by

$$\mathcal{D}(\mathcal{A}) = \left\{ w \in \mathcal{X} : \lim_{t\to 0^+} \frac{S(t)w - w}{t} \text{ exists} \right\}$$

Proposition 1. *[77] Every C_0 strongly continuous semigroup has a unique infinitesimal generator $(\mathcal{A}, \mathcal{D}(\mathcal{A}))$.*

Definition 1.1.3. *[77] A C_0-semigroup $(S(t))_{t\in\mathbb{R}^+}$ is said to be :*

- *of contraction if $\|S(t)\|_{\mathcal{L}(\mathcal{X})} \leq 1$ for all $t \geq 0$,*
- *compact if $S(t)$ is a compact operator for all $t > 0$.*

The purpose of the following Proposition is to give the relations between a C_0 semigroup and its infinitesimal generator.

Proposition 2. *[77] Let $(S(t))_{t\in\mathbb{R}^+}$ be a C_0 semigroup and $\mathcal{A}$ its infinitesimal generator, then:*

- *for all $w \in \mathcal{D}(\mathcal{A})$ we have $S(t)w \in \mathcal{D}(\mathcal{A})$, $\quad \mathcal{A}S(t)w = S(t)\mathcal{A}w$ and*

$$\frac{d}{dt}S(t)w = \mathcal{A}S(t)w = S(t)\mathcal{A}w, \quad \text{for tout } t \geq 0$$

- *for all $t \geq 0$ and $w \in \mathcal{X}$ we have $\displaystyle\int_0^t S(s)xds \in \mathcal{D}(\mathcal{A})$ and $S(t)w - w = \mathcal{A}\int_0^t S(s)wds$*
- $\displaystyle(\lambda I - \mathcal{A})^{-1} = \int_0^\infty e^{-\lambda t}S(t)dt, \quad \text{for all } \lambda \in \rho(\mathcal{A})$
- $\displaystyle S(t)w = \lim_{n\to+\infty}\left(I - \frac{t\mathcal{A}}{n}\right)^{-n} w, \quad \text{for all } w \in \mathcal{X}, \text{ for all } t \geq 0$

The first property, shows that if $w \in \mathcal{D}(\mathcal{A})$, then $t \mapsto S(t)w$ is a solution of the following evolution equation:

$$\begin{cases} \frac{dw}{dt} = \mathcal{A}w \\ w(0) = w_0 \end{cases} \tag{1.1}$$

when $w \in \mathcal{D}(\mathcal{A}), t \mapsto S(t)w$ is called the strong solution of (1.1). Note however that $S(t)w$ exists even if $w \notin \mathcal{D}(\mathcal{A})$. In this case $t \mapsto S(t)w$ is called a mild solution of (1.1). This solution is not necessarily strongly differentiable, in the sense that $\displaystyle\lim_{h\to0^+}\frac{w(t+h)-w(t)}{h}$ does not necessarily exist. But it "verifies the equation (1.1) in a weaker sense".

1.1.2 Hille-Yosida Theorem

The following theorem gives a necessary and sufficient condition for an operator $\mathcal{A}$ to generate a semigroup of contraction.

Theorem 1.1.4. *[77] (Hille-Yosida). An unbounded linear operator $\mathcal{A}$ generates a semi-group of contraction $(S(t))_{t\in\mathbb{R}^+}$ if and only if:*

- *$\mathcal{A}$ is closed and $\overline{\mathcal{D}(\mathcal{A})} = \mathcal{X}$.*
- *The resolvent set of $\mathcal{A}$, given by $\rho(\mathcal{A}) = \{\lambda \in \mathbb{R} : (\lambda I - \mathcal{A})^{-1} \in \mathcal{L}(X)\}$, contains $]0, +\infty[$ and for all $\lambda > 0$ we have*

$$\left\|(\lambda I - \mathcal{A})^{-1}\right\| \leq \frac{1}{\lambda}$$

The following theorem generalizes the Hille-Yosida theorem, and is applicable to any C_0-semigroup.

Theorem 1.1.5. *[77] An unbounded linear operator $\mathcal{A}$ generates a C_0 semigroup $(S(t))_{t\in\mathbb{R}^+}$ if and only if:*

- *$\mathcal{A}$ is closed and $\overline{\mathcal{D}(\mathcal{A})} = \mathcal{X}$*
- *There exists $M \geq 1$ and $\omega \in \mathbb{R}$, such that the resolvent set $\rho(\mathcal{A})$ of $\mathcal{A}$ contains $]\omega, +\infty[$, and for all $\lambda > \omega$ we have*

$$\left\|(\lambda I - \mathcal{A})^{-n}\right\| \leq \frac{M}{(\lambda - \omega)^n}, \quad n = 1, 2 \dots$$

In this case, $(S(t))_{t\in\mathbb{R}^+}$ verifies $\|S(t)\| \leq Me^{\omega t}$, for all $t \geq 0$.

We end this section with some examples of C_0 semigroups.

1.1.3 Examples of C_0 semigroup

Example 1. *[31] Let $\mathcal{A}$ be a bounded operator, then $\mathcal{A}$ generates the C_0 semigroup given by*

$$S(t) = \sum_{n=0}^{\infty} \frac{(t\mathcal{A})^n}{n!} = e^{t\mathcal{A}}$$

In this case we have $\mathcal{D}(\mathcal{A}) = \mathcal{X}$.

Example 2. *[31] Consider on $\mathbb{R}$ the following transport equation*

$$\begin{cases} \dfrac{\partial w}{\partial t}(x,t) = \dfrac{\partial w}{\partial x}(x,t) \\ w(x,0) = w_0(x) \end{cases}$$

where $w_0 \in L^p(\mathbb{R})$, *with* $1 \leq p < \infty$. *By positing* $\mathcal{X} = L^p(\mathbb{R})$ *and* $\mathcal{A} = \frac{\partial}{\partial x}$ with $\mathcal{D}(\mathcal{A}) = W^{1,p}(\mathbb{R})$, $\mathcal{A}$ *generates the translation semigroup given by*

$$[S(t)w](x) = w(x+t), \quad \text{for all } w \in L^p(\mathbb{R})$$

and the solution of the transport equation is written $w(x,t) = [S(t)w_0](x)$.

Example 3. *[31] Consider the following transport equation on* $]0, L[$.

$$\begin{cases} \dfrac{\partial w}{\partial t}(x,t) = \dfrac{\partial w}{\partial x}(x,t) \\ w(0,t) = w(L,t) \\ w(x,0) = w_0(x) \end{cases}$$

where $w_0 \in L^2(0,L)$. *On the pose* $\mathcal{X} = L^2(0,L)$ *and* $\mathcal{A} = \dfrac{\partial}{\partial x}$ *with* $\mathcal{D}(\mathcal{A}) = \{w \in H^1(0,L) \ : \ w(0) = w(L)\}$. *for all* $w \in L^2(0,L)$, *we define* $z_w : \mathbb{R} \to \mathbb{R}$ *by*

$$z_w(x) = w\left(x - L\ E\left(\frac{x}{L}\right)\right) \qquad a.e. \text{ on } \mathbb{R}$$

where $E\left(\frac{x}{L}\right)$ *denotes the integer part of* $\frac{x}{L}$. *Then* $\mathcal{A}$ *generates the* C_0 *semigroup given by*

$$[S(t)w](x) = z_w(x+t), \quad \text{for all } t \geq 0, \text{ for all } w \in L^2(0,L)$$

and the solution of the transport equation is written $w(x,t) = [S(t)w_0](x)$.

Example 4. *[31] Consider the following heat equation on* $[0, \pi[$.

$$\begin{cases} \dfrac{\partial w}{\partial t}(x,t) = \dfrac{\partial^2 w}{\partial x^2}(x,t) \\ w(0,t) = w(\pi,t) = 0 \\ w(x,0) = w_0(x) \end{cases}$$

We suppose that $w_0 \in L^2(0,\pi)$. *By positing* $\mathcal{X} = L^2(0,\pi)$ *and* $\mathcal{A} = \frac{\partial^2}{\partial x^2}$ with $\mathcal{D}(\mathcal{A}) = H^2(0,\pi) \cap H_0^1(0,\pi)$, $\mathcal{A}$ *generates the* C_0 *semigroup given by*

$$S(t)w = \sum_{n=1}^{\infty} e^{-n^2 t} \langle w, \phi_n \rangle_{L^2(0,\pi)} \phi_n, \quad \text{for all } w \in L^2(0,\pi)$$

where $\phi_n(x) = \sin(nx), (\phi_n)_{n \in \mathbb{N}^*}$ *is the basis of eigenfunctions du Laplacian* $\Delta = \dfrac{\partial^2}{\partial x^2}$, *with Dirichlet boundary conditions.*
The solution of the heat equation is therefore $w(x,t) = [S(t)w_0](x)$.

Example 5. *[31] We consider on $\mathbb{R}^n$ the following heat equation*

$$\begin{cases} \dfrac{\partial w}{\partial t}(x,t) = \Delta w(x,t) \\ w(x,0) = w_0(x) \end{cases}$$

where $w_0 \in L^2(\mathbb{R}^n)$. By posing $\mathcal{X} = L^2(\mathbb{R}^n)$ and $\mathcal{A} = \Delta$ with $\mathcal{D}(\mathcal{A}) = H^2(\mathbb{R}^n)$. $\mathcal{A}$ generates the C_0 semigroup given by

$$S(0) = I, \quad [S(t)w](x) = \frac{1}{(4\pi t)^{\frac{n}{2}}} \int_{\mathbb{R}^n} e^{\frac{-|x-s|^2}{4t}} w(s)ds, \quad \text{for all } t > 0, \quad \text{for all } w \in L^2(\mathbb{R}^n)$$

The solution of the heat equation is thus written $w(x,t) = [S(t)w_0](x)$.

1.2 Existence results of the solution of systems with delay

Let $\mathcal{X}$ be a Banach space and let r be a positive constant. Let $\mathcal{C} = C([-r,0];\mathcal{X})$ be the Banach space of continuous functions with value in $\mathcal{X}$ on $[-r,0]$ with norm $\|.\|$. For any real number such that $a \leq b$ and any continuous function $w : [a-r, b] \to \mathcal{X}$, w_t denotes the element of $\mathcal{C}$ given by $w_t(\theta) = w(t+\theta)$ for any $\theta \in [-r, 0]$.
Now we recall some classical existence results for certain classes of non-autonomous semi-linear systems.

1.2.1 Case of non-autonomous systems

Consider the following system:

$$\begin{cases} \dfrac{dw(t)}{dt} = Aw(t) + F(t, w_t), \quad t > a \\ w_a = \phi \in \mathcal{C}. \end{cases} \tag{1.2}$$

We start with an existence result of the mild solution of the system (1.2).

Theorem 1.2.1. *[86] Let $F : [a,b] \times \mathcal{C} \to X$ a continuous function satisfying the following Lipschitz condition:*

$$\|F(t,w) - F(t,\tilde{w})\|_X \leq L\|w - \tilde{w}\|_{\mathcal{C}}, \text{ for all } t \in [a,b], \text{ and } w, \tilde{w} \in \mathcal{C}$$

where L is a positive constant. Suppose that $(S(t))_{t\geq 0}$ is a C_0-semigroup of bounded linear operators on X satisfying $\|S(t)\| \leq e^{\omega t}$, for any $t \geq 0$, with ω a constant. Then for any given $\phi \in \mathcal{C}$, there exists a unique function solution of the system (1.2) given by :

$$\begin{cases} w(t) = S(t-a)\phi + \int_a^t S(t-s)F\left(s, x_s\right) ds \quad t > a \\ w_a = \phi \in \mathcal{C}. \end{cases}$$

1.2.2 Case of autonomous systems

Consider the following system

$$\begin{cases} \frac{dw(t)}{dt} = Aw(t) + F\left(w_t\right), \quad t > a \\ y_a = \varphi \in \mathcal{C}. \end{cases} \tag{1.3}$$

Theorem 1.2.2. *[90] Let's assume that the two assumptions*

- *A generates a C_0-semigroup of bounded linear operators.*
- *$F : \mathcal{C} \to X$ is locally Lipschitzian continuous.*

are verified. Then we have :

- *for all $\phi \in \mathcal{C}$, there exists a maximal existence interval $[-r, T_\varphi)$ and a unique mild solution $w(t, \varphi)$ of the system (1.3) defined on $[-r, T_\varphi)$ with $T_\varphi = +\infty$ or $\lim_{t \to T_\varphi^-} \|w(t, \varphi)\| = +\infty$.*
- *Moreover, $w(t, \varphi)$ is a continuous function with respect to φ, in the following sense: if $\varphi \in \mathcal{C}$ and $t \in [0, T_\varphi)$, then there exist two positive constants L and ϵ such that for all $\psi \in \mathcal{C}$ with $\|\varphi - \psi\|_{\mathcal{C}} < \epsilon$ and $T_\psi \leq T_\varphi$ we have:*

$$\|w(s, \varphi) - w(s, \psi)\| \leq L\|\varphi - \psi\|_{\mathcal{C}}, \forall s \in [-r, T_\psi]$$

1.3 Some useful results for semilinear systems

In this section, we give some useful definitions and properties which are needed in the whole of this thesis.

Definition 1.3.1. *Let $T > 0$. A function $w \in C([-r, T]; \mathcal{X})$ is said to be a mild solution of the system (0.8) if it satisfies the following variation of constants formula*

$$\begin{aligned} w(t) &= S(t)w(0) + \int_0^t S(t-s)v(s)F(w_s)ds \quad \text{for } t \in [0, T] \\ w_0 &= \phi \in C([-r, 0]; \mathcal{X}), \end{aligned}$$

where $S(t)$ denotes the semigroup generated by the linear operator $\mathcal{A}$.

Definition 1.3.2. *The system (0.8) is strongly (resp. weakly) stabilizable if there exists a feedback control $v(t) = g(w_t)$, g is a functional defined from $\mathcal{C}$ to $\mathbb{R}$ such that the corresponding mild solution $w(t)$ of the system (0.8) satisfies the following properties:*
(a) For each initial state w_0 of the system (0.8), there exists a unique mild solution of the system (0.8) defined on $[-r, +\infty)$.
(b) $\{0\}$ is a stable equilibrium of the system (0.8).
(c) $w(t) \to 0$ strongly (resp. $w(t) \to 0$ weakly) as $t \to +\infty$ for all $\phi \in \mathcal{C}$.

Definition 1.3.3. *Suppose that the two conditions (a) and (b) of the previous definition are satisfied, then, the system (0.8) is said to be exponentially stabilizable, if there exist $\sigma, N > 0$ such that*

$$\|w(t)\|_{\mathcal{X}} \leq Ne^{-\sigma t}\|\phi\|_{\mathcal{C}} \quad \text{for all } t \geq 0.$$

Remark 1.3.4. *It is clear that exponential stability $\Rightarrow$ strong stability $\Rightarrow$ weak stability.*

Lemma 1.3.5. *[3] Let (w_k) be a sequence of positive real numbers satisfying*

$$w_{k+1} \leq w_k - Rw_{k+1}^{2+m} \quad \text{for all } k \geq 0,$$

where $R > 0$ and $m > -1$ are constants. Then, there exists $M \geq 0$ (depending on m and R) such that

$$w_k \leq \frac{M}{(k+1)^{\frac{1}{(1+m)}}} \quad \text{for all } k \geq 0.$$

For all $t \geq 0$, we define the linear operator $U(t)$ on $\mathcal{C}$ by

$$U(t)\phi = x_t(., \phi),$$

where $x(., \phi)$ is the solution of the following equation:

$$\begin{cases} \dot{w}(t) = \mathcal{A}w(t) \text{ for } t > 0 \\ w_0 = \phi \end{cases}$$

Proposition 1.3.6. *[90] $(U(t))_{t\geq 0}$ is a linear strongly continuous semigroup on $\mathcal{C}$, moreover $(U(t))_{t\geq 0}$ satisfies for $t \geq 0$ and $\theta \in [-r, 0]$ the translation property:*

$$(U(t)\xi)(\theta) = \begin{cases} [U(t+\theta)\xi](0) & \text{if } t+\theta \geq 0 \\ \xi(t+\theta) & \text{if } t+\theta \leq 0 \end{cases}$$

Now, we define the duality mapping, denoted by $\mathcal{J}$, as follows :

$$\mathcal{J}(w) = \{w^* \in \mathcal{X}^* : \langle w, w^* \rangle = \|w^*\|^2 = \|w\|^2 \}, \tag{1.4}$$

where $\mathcal{X}^*$ denotes the set of all bounded linear forms on $\mathcal{X}$. Any component of $\mathcal{J}(w)$ is indicated by f_w for $w \in \mathcal{X}$. Moreover, we have the following definition.

Definition 1.3.7. *[49] We say that $\mathcal{J}$ is Lipschitz-continuous, if $\mathcal{J}$ is single valued and the mapping $w \to f_w$ is Lipschitz-continuous.*

Lemma 1.3.8. *[49] Let $t \in \mathbb{R} \to w(t) \in \mathcal{X}$ be a function which satisfies:*
(1) the function $t \to \|w(t)\|_{\mathcal{X}}$ is a.e. differentiable on $\mathbb{R}$,
(2) the weak derivative w' of w exists a.e. on $\mathbb{R}$.
Then, for almost every $t \in \mathbb{R}$, we have

$$\|w(t)\|_{\mathcal{X}} \frac{d}{dt}\|w(t)\|_{\mathcal{X}} = \langle w'(t), w^* \rangle, \quad \text{for all } w^* \in \mathcal{J}(w(t)).$$

Chapter 2

Feedback stabilization of infinite dimensional bilinear systems with discrete delay

2.1 Introduction

In this chapter, we suppose that the state space $\mathcal{X}$ can be decomposed as the direct sum of two closed subspaces $\mathcal{X}^u$ and $\mathcal{X}^s$ of $\mathcal{X}$, which are orthogonal to each one and invariants under $S(t)$,

$$\mathcal{X} = \mathcal{X}^u \ + \ \mathcal{X}^s,$$

and denote by $S^u(t)$ and $S^s(t)$ the restrictions of $S(t)$ to $\mathcal{X}^u$ and $\mathcal{X}^s$ respectively (see e.g. [50, 87] for more details about this decomposition). In the sequel, we suppose that

$$F\mathcal{X}^u \subseteq \mathcal{X}^u \text{ and } F\mathcal{X}^s \subseteq \mathcal{X}^s. \tag{2.1}$$

Then, the system (0.6) can be decomposed into the form

$$\begin{cases} \dot{w}^u(t) = \mathcal{A}^u w^u(t) + v(t)F^u w^u(t-r), \quad \text{for all } t > 0, \\ \qquad w_0^u = \phi^u \in \mathcal{C}^u = C([-r,0];\mathcal{X}^u), \end{cases} \tag{2.2}$$

with $F^u = P^u F P^u$ and P^u is the orthogonal projection on $\mathcal{X}^u$,
and

$$\begin{cases} \dot{x}^s(t) = \mathcal{A}^s w^s(t) + v(t)F^s w^s(t-r), \text{ for all } t > 0, \\ w_0^s = \phi^s \in \mathcal{C}^s = C\left([-r,0];\mathcal{X}^s\right), \end{cases}$$

with $F^s = P^s F P^s$ and P^s is the orthogonal projection on $\mathcal{X}^s$.

We consider the following new bounded feedback control defined, for $p \in \{0,1,2\}$, as follows

$$v(t) = \begin{cases} -\dfrac{\langle w^u(t), F^u w^u(t-r)\rangle}{\max\{\|w^u(t)\|_{\mathcal{X}}^p, \|w^u(t-r)\|_{\mathcal{X}}^p\}} & if \ \max\{\|w^u(t)\|_{\mathcal{X}}, \|w^u(t-r)\|_{\mathcal{X}}\} \neq 0, \\ 0 & else. \end{cases}$$

At the beginning, we prove the existence and uniqueness of the global mild solution of the system (0.6) on $[-r, +\infty)$. After that, we investigate the stabilization problem and we show that the system (0.6) is stabilizable for any $r \geq 0$ and for any arbitrary dimension of $\mathcal{X}^u$ and $\mathcal{X}^s$. In fact, we prove that the feedback control $v(t)$ stabilizes weakly and strongly the systems for $p \in \{0,1\}$ and stabilize exponentially the system for $p = 2$. Furthermore, in the strong stabilization case, we establish an explicit polynomial decay estimate of the stabilized state. We should mention here that the obtained stabilization results are given in terms of the observability conditions. Moreover, we develop another result on the exponential stabilization of system (0.6), in the case where just $\mathcal{X}^u$ is invariant with respect to F and without the exponential stability condition on $S^s(t)$. To illustrate our theoretical results, we give at the end some applications and numerical simulations.

2.2 Strong stabilization

In this section, we study the strong stabilization of the system (0.6) with decay estimate.

Theorem 2.2.1. *Let $\mathcal{A}$ be the generator of a C_0-semigroup $S(t)$. Suppose that condition (2.1) holds and*
1. $S^u(t)$ is a contraction semigroup,
2. $S^s(t)$ is exponentially stable,
3. there exist positive constants T and δ such that

$$\int_0^T |\langle F^u S^u(s)\xi, S^u(s+r)\xi\rangle| \, ds \geq \delta \|\xi\|_{\mathcal{X}}^2, \ \text{for all } \xi \in \mathcal{X}^u. \tag{2.3}$$

Then for all $p \in \{0,1,2\}$, the feedback

$$v(t) = -\frac{\langle w^u(t), F^u w^u(t-r)\rangle}{\max\{\|w^u(t)\|_{\mathcal{X}}^p, \|w^u(t-r)\|_{\mathcal{X}}^p\}} \mathbf{1}_E, \tag{2.4}$$

where $E = \{t \geq 0 : \max\{\|w^u(t)\|_{\mathcal{X}}, \|w^u(t-r)\|_{\mathcal{X}}\} \neq 0\}$, strongly stabilizes the system (0.6), and we have the following decay estimate

$$\|w(t)\|_{\mathcal{X}} = O\left(\frac{1}{\sqrt{t}}\right), \ as \quad t \longrightarrow +\infty.$$

Proof: With the feedback (2.4), system (2.2) becomes

$$\begin{cases} \dot{w^u}(t) = \mathcal{A}^u w^u(t) + \mathcal{F}\left(w_t^u\right), \text{ for all } t > 0, \\ w_0^u = \phi \in \mathcal{C}^u, \end{cases} \tag{2.5}$$

where

$$\mathcal{F}(\phi) \;=\; \begin{cases} -\dfrac{\langle \phi(0), F^u\phi(-r)\rangle}{\max\{\|\phi(0)\|_{\mathcal{X}}^p, \|\phi(-r)\|_{\mathcal{X}}^p\}} F^u\phi(-r), & if \quad \max\{\|\phi(0)\|_{\mathcal{X}}, \|\phi(-r)\|_{\mathcal{X}}\} \neq 0, \\ 0, & else. \end{cases}$$

First, we will show that F is locally lipschitz, i.e. for any $K > 0$ there exists a constant Γ_K such that

$$\|F(\psi) - F(\varphi)\|_{\mathcal{X}} \leq \Gamma_K \|\psi - \varphi\|_{\mathcal{C}}, \tag{2.6}$$

where $\psi, \varphi \in \mathcal{C}^u$ are such that $\|\psi\|_{\mathcal{C}} \leq K$, $\|\varphi\|_{\mathcal{C}} \leq K$. We study the three cases for the integer p.
Case 1, let $p = 0$. Then $\mathcal{F}(\phi) = -\langle \phi(0), F^u\phi(-r)\rangle F^u\phi(-r)$ and

$$\begin{aligned} \|\mathcal{F}(\psi) - \mathcal{F}(\varphi)\|_{\mathcal{X}} &= \|\langle \psi(0), F^u\psi(-r)\langle F^u\psi(-r) - \langle \varphi(0), F^u\varphi(-r)\rangle F^u\varphi(-r)\|_{\mathcal{X}} \\ &\leq \|\langle \psi(0), F^u\psi(-r)\rangle (F^u\psi(-r) - F^u\varphi(-r))\|_{\mathcal{X}} \\ &\quad + \|\langle \psi(0) - \varphi(0), F^u\psi(-r)\rangle F^u\varphi(-r)\|_{\mathcal{X}} \\ &\quad + \|\langle \varphi(0), F^u\varphi(-r) - F^u\psi(-r)\rangle F^u\varphi(-r)\|_{\mathcal{X}} \\ &\leq 3K^2\|F^u\|^2\|\varphi - \psi\|_{\mathcal{C}^u}. \end{aligned}$$

In the sequel, we set $\delta_\phi = \max\{\|\phi(0)\|_{\mathcal{X}}, \|\phi(-r)\|_{\mathcal{X}}\}$ for all $\phi \in \mathcal{C}$.
Case 2, let $p = 1$. The inequality (2.6) is verified in the case $\delta_\psi = 0$ or $\delta_\varphi = 0$ as in this case we have: $F(\psi) = 0$ or $F(\varphi) = 0$.

If $0 < \delta_\psi \leq \delta_\varphi$, then we have :

$$\begin{aligned}
\|\mathcal{F}(\psi) - \mathcal{F}(\varphi)\|_{\mathcal{X}} &= \left\| \frac{\langle \psi(0), F^u\psi(-r)\rangle}{\delta_\psi} F^u\psi(-r) - \frac{\langle \varphi(0), F^u\varphi(-r)\rangle}{\delta_\varphi} F^u\varphi(-r) \right\|_{\mathcal{X}} \\
&= \frac{\|\delta_\varphi\langle \psi(0), F^u\psi(-r)\rangle F^u\psi(-r) - \delta_\psi\langle \varphi(0), F^u\varphi(-r)\rangle F^u\varphi(-r)\|_{\mathcal{X}}}{\delta_\psi\delta_\varphi} \\
&\leq \frac{\|\delta_\varphi\langle \psi(0), F^u\psi(-r)\rangle F^u\psi(-r) - \delta_\psi\langle \psi(0), F^u\psi(-r)\rangle F^u\psi(-r)\|_{\mathcal{X}}}{\delta_\psi\delta_\varphi} \\
&\quad + \frac{\|\delta_\psi\langle \psi(0), F^u\psi(-r)\rangle F^u\psi(-r) - \delta_\psi\langle \varphi(0), F^u\varphi(-r)\rangle F^u\varphi(-r)\|_{\mathcal{X}}}{\delta_\psi\delta_\varphi} \\
&\leq \|F^u\|^2 \frac{\|\psi(0)\|_{\mathcal{X}}\|\psi(-r)\|_{\mathcal{X}}^2}{\delta_\psi\delta_\varphi}|\delta_\varphi - \delta_\psi| \\
&\quad + \frac{\|\langle \psi(0), F^u\psi(-r)\rangle F^u\psi(-r) - \langle \varphi(0), F^u\varphi(-r)\rangle F^u\varphi(-r)\|_{\mathcal{X}}}{\delta_\varphi} \\
&\leq \|F^u\|^2 K\|\varphi - \psi\|_{\mathcal{C}} + \frac{\|\langle \psi(0), F^u\psi(-r)\rangle (F^u\psi(-r) - F^u\varphi(-r))\|_{\mathcal{X}}}{\delta_\varphi} \\
&\quad + \frac{\|\langle \psi(0) - \varphi(0), F^u\psi(-r)\rangle F^u\varphi(-r)\|_{\mathcal{X}}}{\delta_\varphi} \\
&\quad + \frac{\|\langle \varphi(0), F^u\varphi(-r) - F^u\psi(-r)\rangle F^u\varphi(-r)\|_{\mathcal{X}}}{\delta_\varphi} \\
&\leq \ 4K\|F^u\|^2\|\varphi - \psi\|_{\mathcal{C}^u}.
\end{aligned}$$

Case 3, let $p = 2$. It follows that :

$$\begin{aligned}
\|\mathcal{F}(\psi) - \mathcal{F}(\varphi)\|_{\mathcal{X}} &= \left\| \frac{\langle \psi(0), F^u\psi(-r)\rangle}{\delta_\psi^2} F^u\psi(-r) - \frac{\langle \varphi(0), F^u\varphi(-r)\rangle}{\delta_\varphi^2} F^u\varphi(-r) \right\|_{\mathcal{X}} \\
&= \frac{\|\delta_\varphi^2\langle \psi(0), F^u\psi(-r)\rangle F^u\psi(-r) - \delta_\psi^2\langle \varphi(0), F^u\varphi(-r)\rangle F^u\varphi(-r)\|_{\mathcal{X}}}{\delta_\psi^2\delta_\varphi^2} \\
&\leq \frac{\|\delta_\varphi^2\langle \psi(0), F^u\psi(-r)\rangle F^u\psi(-r) - \delta_\psi^2\langle \psi(0), F^u\psi(-r)\rangle F^u\psi(-r)\|_{\mathcal{X}}}{\delta_\psi^2\delta_\varphi^2} \\
&\quad + \frac{\|\delta_\psi^2\langle \psi(0), F^u\psi(-r)\rangle F^u\psi(-r) - \delta_\psi^2\langle \varphi(0), F^u\varphi(-r)\rangle F^u\varphi(-r)\|_{\mathcal{X}}}{\delta_\psi^2\delta_\varphi^2} \\
&\leq \|F^u\|^2 \frac{\|\psi(0)\|_{\mathcal{X}}\|\psi(-r)\|_{\mathcal{X}}^2}{\delta_\psi^2\delta_\varphi^2}|\delta_\varphi^2 - \delta_\psi^2| \\
&\quad + \frac{\|\langle \psi(0), F^u\psi(-r)\rangle F^u\psi(-r) - \langle \varphi(0), F^u\varphi(-r)\rangle F^u\varphi(-r)\|_{\mathcal{X}}}{\delta_\varphi^2}
\end{aligned}$$

$$
\begin{aligned}
\|\mathcal{F}(\psi)-\mathcal{F}(\varphi)\|_{\mathcal{X}} &\leq \|F^u\|^2\frac{\delta_\varphi\delta_\psi^2}{\delta_\psi^2\delta_\varphi^2}(\delta_\varphi+\delta_\psi)\,|\delta_\varphi-\delta_\psi| \\
&\quad+\frac{\|\langle\psi(0),F^u\psi(-r)\rangle(F^u\psi(-r)-F^u\varphi(-r))\|_{\mathcal{X}}}{\delta_\varphi^2} \\
&\quad+\frac{\|\langle\psi(0)-\varphi(0),F^u\psi(-r)\rangle F^u\varphi(-r)\|_{\mathcal{X}}}{\delta_\varphi^2} \\
&\quad+\frac{\|\langle\varphi(0),F^u\varphi(-r)-F^u\psi(-r)\rangle F^u\varphi(-r)\|_{\mathcal{X}}}{\delta_\varphi^2} \\
&\leq\ 2\|F^u\|^2\frac{\delta_\varphi}{\delta_\varphi^2}\delta_\varphi\|\varphi-\psi\|_{\mathcal{C}^u}+3\|F^u\|^2\|\varphi-\psi\|_{\mathcal{C}^u}. \\
&\leq\ 5\|F^u\|^2\|\varphi-\psi\|_{\mathcal{C}^u}.
\end{aligned}
$$

Then, the system (2.2) controlled by (2.4), possesses a unique mild solution $w^u \in C([-r,T_{max}),\mathcal{X}^u)$ given by the variation of constants formula

$$
w^u(t)=\begin{cases} S^u(t)w^u(0)+\int_0^t S^u(t-\sigma)\mathcal{F}\left(w_\sigma^u\right)d\sigma, & t\in[0,t_{\max}), \\ \phi(t), & t\in[-r,0], \end{cases}
$$

where $S^u(t)$ is the semigroup generated by the operator $\mathcal{A}^u$.
Let $0<T<T_{max}$. Since $g(t)=\mathcal{F}\left(w_t^u\right)$ is continuous in $[0,T)$, there exists a sequence $(g_n)_n\subset\ C^1\left([0,T];\mathcal{X}^u\right)$ such that

$$
g_n\longrightarrow g \qquad \left(C\left([0,T];\mathcal{X}^u\right),\|.\|_\infty\right),
$$

Moreover, the operator $\mathcal{A}^u$ generates a semigroup of contraction, so the domain of $\mathcal{A}^u$ is dense in $\mathcal{X}^u$ and consequently there exists a sequence $(\xi_n)\subset D\left(\mathcal{A}^u\right)$ such that $(\xi_n)_n$ converges to $w^u(0)$ in $\mathcal{X}^u$, and define the function

$$
w_n^u(t)=S^u(t)\xi_n+\int_0^t S^u(t-s)g_n(s)ds,\ \ for\ all\ t\ \in\ [0,T]. \tag{2.7}
$$

Then, $x_n(t)\in D\left(\mathcal{A}\right)$, $w_n^u\in C^1\left([0,T],\mathcal{X}^u\right)$ and we have

$$
\dot{w^u}_n(t)=\mathcal{A}^u w_n^u(t)+g_n(t),\ t\in(0,T),\quad w_n^u(0)=\xi_n.
$$

Using the dissipativity of $\mathcal{A}^u$, we obtain

$$
\frac{d}{dt}\left\|w_n^u(t)\right\|_{\mathcal{X}}^2\leq 2\left\langle g_n(t),w_n^u(t)\right\rangle,\ \text{for all } t\in(0,T).
$$

Integrating this inequality over the interval $[\tau,t]$, we get

$$
\|w_n^u(t)\|_{\mathcal{X}}^2-\|w_n^u(\tau)\|_{\mathcal{X}}^2\leq 2\int_\tau^t\langle g_n(s),w_n^u(s)\rangle\,ds,\ \text{for all } 0\leq\tau\leq t\leq T. \tag{2.8}
$$

Using equation (2.7) and the variation of constants formula, we deduce that

$$\|w_n^u(t) - w^u(t)\|_{\mathcal{X}} \quad \leq \|\xi_n - w^u(0)\|_{\mathcal{X}} + T \sup_{s\in[0,T]} \|g_n(s) - g(s)\|_{\mathcal{X}}\,.$$

Moreover,

$$\|w_n^u(t) - w^u(t)\|_{\mathcal{X}} \to 0, \text{ as } n \to +\infty.$$

Then, by passing to the limit in the estimate (2.8), we get via the dominated convergence theorem

$$\|w^u(\tau)\|_{\mathcal{X}}^2 - \|w^u(t)\|_{\mathcal{X}}^2 \geq 2\int_\tau^t \frac{\langle w^u(s), F^u w^u(s-r)\rangle^2}{\max\{\|w^u(s)\|_{\mathcal{X}}^p, \|w^u(s-r)\|_{\mathcal{X}}^p\}} \mathbf{1}_E ds, \quad 0 \leq \tau \leq t \leq T. \tag{2.9}$$

for $p \in \{0, 1, 2\}$. In particular, we have

$$\|w^u(t)\|_{\mathcal{X}} \leq \|\phi\|_{\mathcal{C}}, \quad -r \leq t \leq T_{max}. \tag{2.10}$$

Finally, $w^u(t)$ is a global solution (i.e $T_{max} = +\infty$), which satisfies the following variation of constants formula, for all $t \geq 0$

$$w^u(t) - S^u(t)w^u(0) = -\int_0^t S^u(t-s) \frac{\langle F^u w^u(s-r), w^u(s)\rangle}{\max\{\|w^u(s)\|_{\mathcal{X}}^p, \|w^u(s-r)\|_{\mathcal{X}}^p\}} \mathbf{1}_E F^u w^u(s-r) ds.$$

It yields by Holder's inequality that for all $t \in [0, r+T]$, we have

$$\|w^u(t) - S^u(t)w^u(0)\|_{\mathcal{X}} \leq \sqrt{r+T}\|F\|\|\phi\|_{\mathcal{C}}^{1-\frac{p}{2}} \left(\int_0^t \frac{|\langle Lw^u(s-r), w^u(s)\rangle|^2}{\max\{\|w^u(s)\|_{\mathcal{X}}^p, \|w^u(s-r)\|_{\mathcal{X}}^p\}} \mathbf{1}_E ds\right)^{\frac{1}{2}}. \tag{2.11}$$

In view of

$$\begin{aligned}\langle S^u(s+r)w^u(0), F^u S^u(s)w^u(0)\rangle &= \langle w^u(s+r), F^u S^u(s)w^u(0) - F^u w^u(s)\rangle \\ &\quad - \langle w^u(s+r) - S^u(s+r)w^u(0), F^u S^u(s)w^u(0)\rangle \\ &\quad + \langle w^u(s+r), F^u w^u(s)\rangle.\end{aligned}$$

We obtain

$$\begin{aligned}|\langle S^u(s+r)w^u(0), F^u S^u(s)w^u(0)\rangle| &\leq \|F\|\|w^u(0)\|_{\mathcal{X}}\|w^u(s) - S^u(s)w^u(0)\|_{\mathcal{X}} \\ &\quad + \|F\|\|w^u(0)\|_{\mathcal{X}}\|w^u(s+r) - S^u(s+r)w^u(0)\|_{\mathcal{X}} \\ &\quad + |\langle w^u(s+r), F^u w^u(s)\rangle|.\end{aligned}$$

Employing inequality (2.11), we obtain

$$\begin{aligned}&|\langle S^u(s+r)w^u(0), F^u S^u(s)w^u(0)\rangle| \\ &\quad \leq 2\sqrt{T+r}\|F\|^2\|\phi\|_{\mathcal{C}}^{1-\frac{p}{2}}\|w^u(0)\|_{\mathcal{X}} \left(\int_0^{r+T} \frac{|\langle F^u w^u(s-r), w^u(s)\rangle|^2}{\max\{\|w^u(s)\|_{\mathcal{X}}^p, \|w^u(s-r)\|_{\mathcal{X}}^p\}} \mathbf{1}_E ds\right)^{\frac{1}{2}} \\ &\quad\quad + |\langle w^u(s+r), F^u w^u(s)\rangle|.\end{aligned} \tag{2.12}$$

Replacing $w^u(0)$ by $w^u(t)$ in (2.12), we obtain

$$
\begin{aligned}
&|\langle S^u(s+r)w^u(t), F^u S^u(s)w^u(t)\rangle| \\
&\quad\leq\ 2\sqrt{T+r}\|F\|^2\|\phi\|_{\mathcal{C}}^{1-\frac{p}{2}}\|w^u(t)\|_{\mathcal{X}}\left(\int_t^{t+r+T}\frac{|\langle F^u w^u(s-r), w^u(s)\rangle|^2}{\max\{\|w^u(s)\|_{\mathcal{X}}^p, \|w^u(s-r)\|_{\mathcal{X}}^p\}}\mathbf{1}_E ds\right)^{\frac{1}{2}} \\
&\qquad + |\langle w^u(s+t+r), F^u w^u(s+t)\rangle|.
\end{aligned}
$$

Integrating the last inequality over the interval $[0,T]$ and using the Holder's inequality we obtain

$$
\begin{aligned}
&\int_0^T |\langle S^u(s+r)w^u(t), F^u S^u(s)w^u(t)\rangle| ds \\
&\qquad\leq\ 2T\sqrt{T+r}\|F\|^2\|\phi\|_{\mathcal{C}}^{1-\frac{p}{2}}\|w^u(t)\|_{\mathcal{X}}\left(\int_t^{t+r+T}\frac{|\langle F^u w^u(s-r), w^u(s)\rangle|^2}{\max\{\|w^u(s)\|_{\mathcal{X}}^p, \|w^u(s-r)\|_{\mathcal{X}}^p\}}\mathbf{1}_E ds\right)^{\frac{1}{2}} \\
&\qquad + \int_0^T |\langle w^u(s+t+r), F^u w^u(s+t)\rangle| ds.
\end{aligned}
\tag{2.13}
$$

From (2.9) it follows that

$$
\max\{\|w^u(s)\|_{\mathcal{X}}, \|w^u(s-r)\|_{\mathcal{X}}\} \leq \|w^u(t)\|_{\mathcal{X}}, \quad \text{for all } t+r \ \leq\ s\ \leq\ t+r+T,
$$

and by the variable change, we get that

$$
\begin{aligned}
\int_0^T |\langle w^u(s+t+r), F^u w^u(s+t)\rangle| ds &= \int_{t+r}^{t+r+T} |\langle w^u(s), F^u w^u(s-r)\rangle|\mathbf{1}_E ds \\
&\leq \sqrt{T}\|w^u(t)\|_{\mathcal{X}}^{\frac{p}{2}}\left(\int_{t+r}^{t+r+T}\frac{|\langle w^u(s), F^u w^u(s-r)\rangle|^2}{\max\{\|w^u(s)\|_{\mathcal{X}}^p, \|w^u(s-r)\|_{\mathcal{X}}^p\}}\mathbf{1}_E ds\right)^{\frac{1}{2}}.
\end{aligned}
\tag{2.14}
$$

So, we conclude that

$$
\int_0^T |\langle S^u(s+r)w^u(t), F^u S^u(s)w^u(t)\rangle| ds \leq M\left(\int_t^{t+r+T}\frac{|\langle w^u(s), F^u w^u(s-r)\rangle|^2}{\max\{\|w^u(s)\|_{\mathcal{X}}^p, \|w^u(s-r)\|_{\mathcal{X}}^p\}}\mathbf{1}_E ds\right)^{\frac{1}{2}},
\tag{2.15}
$$

with $M = 2\sqrt{r+T}\|F\|^2 T\|\phi\|_{\mathcal{C}}^{2-\frac{p}{2}} + \sqrt{T}\|\phi\|_{\mathcal{C}}^{\frac{p}{2}}$.
Now, using the observability condition (2.3), we derive

$$
\begin{aligned}
\delta\|w^u(t)\|_{\mathcal{X}}^2 &\leqslant \int_0^T \langle S^u(s+r)w^u(t), F^u S^u(s)w^u(t)\rangle\, ds \\
&\leq M\left(\int_t^{t+r+T}\frac{|\langle w^u(s), F^u w^u(s-r)\rangle|^2}{\max\{\|w^u(s)\|_{\mathcal{X}}^p, \|w^u(s-r)\|_{\mathcal{X}}^p\}}\mathbf{1}_E ds\right)^{\frac{1}{2}}.
\end{aligned}
$$

If $\|\phi\|_{\mathcal{C}} = 0$ then we use the inequality (2.10) to find $\|w^u(t)\|_{\mathcal{X}} = 0$, otherwise, we have

$$\left(\frac{\delta}{M}\right)^2 \|w^u(t)\|_{\mathcal{X}}^4 \leq \int_t^{t+T+r} \frac{|\langle w^u(s), F^u w^u(s-r)\rangle|^2}{\max\{\|w^u(s)\|_{\mathcal{X}}^p, \|w^u(s-r)\|_{\mathcal{X}}^p\}} \mathbf{1}_E ds,$$

and

$$-2\int_t^{t+T+r} \frac{|\langle w^u(s), F^u w^u(s-r)\rangle|^2}{\max\{\|w^u(s)\|_{\mathcal{X}}^p, \|w^u(s-r)\|_{\mathcal{X}}^p\}} \mathbf{1}_E ds \leq -2(\frac{\delta}{M})^2 \|w^u(t)\|_{\mathcal{X}}^4,$$

and inequality (2.9), we obtain that

$$\|w^u(t+T+r)\|^2 - \|w^u(t)\|^2 \leq -2(\frac{\delta}{M})^2 \|w^u(t)\|^4.$$

Let $(\nu_k)_k$ be the sequence defined by $\nu_k = \|w^u(k(T+r))\|_{\mathcal{X}}^2$. Observing that $\nu_{k+1} \leq \nu_k$, we derive, from the last inequality that

$$\nu_{k+1} \leq \nu_k - \frac{1}{M_1}\nu_{k+1}^2,$$

with $M_1 = \dfrac{M^2}{2\delta^2}$. From Lemma 1.3.5, we obtain

$$\nu_k \leq \frac{M_2}{k+1}, \text{ for all } k \geq 0.$$

Let us now set $k = E(\dfrac{t}{T+r})$, where $E(.)$ designs the integer part. Then using again the fact that $\|w^u(t)\|$ decreases, we obtain that

$$\|w^u(t)\|_{\mathcal{X}}^2 \leq \|w^u(k(T+r))\|_{\mathcal{X}}^2 \leq \frac{M_2}{k+1} \leq M_2 \frac{(T+r)}{t},$$

we conclude that

$$\|w^u(t)\|_{\mathcal{X}} = O\left(\frac{1}{\sqrt{t}}\right), \quad as \;\; t \longrightarrow +\infty.$$

For the component $w^s(t)$, we shall show that $w^s(t)$ is defined for all $t > 0$ and exponentially converges to 0, as $t \to +\infty$. Using the same method as in the proof of the existence and uniqueness of mild solution of system (2.5), we can prove that

$$\mathcal{F}(\phi) = \begin{cases} -\dfrac{\langle \phi^u(0), F^u\phi^u(-r)\rangle}{\max\{\|\phi^u(0)\|_{\mathcal{X}}^p, \|\phi^u(-r)\|_{\mathcal{X}}^p\}} F\phi(-r), & if \quad \max\{\|\phi^u(0)\|_{\mathcal{X}}, \|\phi^u(-r)\|_{\mathcal{X}}\} \neq 0, \\ 0, & else, \end{cases}$$

$\mathcal{F}$ is locally lipschitz from $\mathcal{C}$ into $\mathcal{X}$. The system (0.6), excited by the control (2.4), admits a unique mild solution defined for all t in a maximal interval $[0, t_{max})$ given by

$$w(t) = S(t)w(0) + \int_0^t S(t-s)v(s)Fw(s-r)ds.$$

Thus

$$w^s(t) = S^s(t)w^s(0) + \int_0^t S^s(t-s)v(s)F^s w^s(s-r)ds, \quad for\ all\ t \in [0, t_{max}).$$

We use the second condition of our theorem, so there exist $\sigma, R > 0$ such that

$$\|S^s(t)\| \leq Re^{-\sigma t},\ for\ all\ t \geq 0.$$

Then

$$\|w^s(t-r)\|_{\mathcal{X}} \leq Re^{-\sigma(t-r)} \|\phi^s\|_{\mathcal{C}} + e^{-\sigma t} \int_0^{t-r} Re^{\sigma r} e^{\sigma s} |v(s)| \, \|F^s w^s(s-r)\|_{\mathcal{X}} \, ds,$$

and

$$\|w^s(t-r)\|_{\mathcal{X}} \, e^{\sigma t} \leq Re^{\sigma r} \|\phi^s\|_{\mathcal{C}} + e^{\sigma r} \int_0^{t-r} Re^{\sigma s} |v(s)| \, \|F^s w^s(s-r)\|_{\mathcal{X}} \, ds.$$

Using Gronwall's inequality, we have

$$\|w^s(t-r)\|_{\mathcal{X}} e^{\sigma t} \leq Re^{\sigma r} \|\phi^s\|_{\mathcal{C}} \, e^{\left(\int_0^{t-r} Re^{\sigma r} |v(s)| \|F\| ds \right)}.$$

Since $\displaystyle\int_0^{+\infty} |v(t)|^2 dt \leq \frac{\|\phi^u\|_{\mathcal{C}}^2}{2}$, it follows from Holder's inequality that

$$\|w^s(t-r)\|_{\mathcal{X}} e^{\sigma t} \leq Re^{\sigma r} \|\phi^s\| \, e^{\left(Re^{\sigma r} \|F\| M \sqrt{t-r} \right)},\ for\ all\ t\ \geq\ r,$$

with $M = \dfrac{\|\phi^u\|}{\sqrt{2}}$.

$$\|w^s(t-r)\|_{\mathcal{X}} \leq Re^{\sigma r} \|\phi^s\|_{\mathcal{C}} \, e^{\left(Re^{\sigma r} \|F\| M \sqrt{t-r} - \sigma t \right)},\ for\ all\ t\ \geq\ r.$$

It follows that

$$\|w^s(t)\|_{\mathcal{X}} \leq R \|\phi^s\|_{\mathcal{C}} \, e^{\left(MRe^{\sigma r} \|F\| \sqrt{t} - \sigma t \right)}, \quad for\ all\ t\ \geq\ 0,$$

and

$$\frac{\sigma t}{2} - MRe^{\sigma r} \|F\| \sqrt{t} + \frac{(MRe^{\sigma r} \|F\|)^2}{2\sigma} = \frac{\sigma}{2} (\sqrt{t} - \frac{MRe^{\sigma r} \|F\|}{\sigma})^2 \geq 0.$$

Then

$$e^{MRe^{\sigma r} \|F\| \sqrt{t} - \dfrac{\sigma t}{2}} \ \leq\ e^{\dfrac{(MRe^{\sigma r} \|F\|)^2}{2\sigma}},$$

and hence

$$\|w^s(t)\|_{\mathcal{X}} \leq R \|\phi^s\|_{\mathcal{C}} \, e^{\dfrac{(MRe^{\sigma r} \|F\|)^2}{2\sigma}} \, e^{-\frac{\sigma}{2} t},\ for\ all\ t \geq 0,$$

it follows that $w^s(t)$ is bounded on $[0, t_{max})$, so that $t_{max} = +\infty$.

Remark 2.2.2.
From condition (2.3), if δ and T satisfy the inequality $\delta > T\|F^u\|$, then $\mathcal{X}^u = \{0\}$ and $\mathcal{X}^s = \mathcal{X}$.

Corollary 2.2.3. *Assume that assumptions (1) and (2) in Theorem 2.2.1 hold. If we suppose in addition that,*
1. the space $\mathcal{X}^u$ is finite dimensional.
2. for all $\xi \in \mathcal{X}^u$,

$$\langle F^u S^u(s)\xi, S^u(s+r)\xi\rangle = 0, \quad for\ all\ s \geq 0 \Rightarrow \xi = 0. \tag{2.16}$$

Then, the feedback (2.4) strongly stabilizes the system (0.6) and we have the following decay estimate:

$$\|w(t)\|_{\mathcal{X}} = O\left(\frac{1}{\sqrt{t}}\right),\ as\ t \longrightarrow +\infty.$$

Proof: It is clear that the inequality (2.3) is verified for $\xi = 0$. Now, let $\xi \in \mathcal{X}^u\backslash\{0\}$ and for any $T > 0$, assume that

$$\inf_{\|\xi\|_{\mathcal{X}}=1} \int_0^T |\langle F^u S^u(s)\xi, S^u(s+r)\xi\rangle|\, ds = 0.$$

Since $\displaystyle\int_0^T |\langle F^u S^u(s)\xi, S^u(s+r)\xi\rangle|\, ds$ depends continuously on ξ and $\dim \mathcal{X}^u < \infty$, then there exists $\xi \in \mathcal{X}^u$ such that

$$\|\xi\|_{\mathcal{X}} = 1 \text{ and } \int_0^T |\langle F^u S^u(s)\xi, S^u(s+r)\xi\rangle|\, ds = 0.$$

We then deduce that

$$|\langle F^u S^u(s)\xi, S^u(s+r)\xi\rangle| = 0 \ \ for\ all\ s \in [0,T].$$

Or T is arbitrary, then (2.16), implies that $\xi = 0$, which gives the contradiction. Then, there exists a constant $\delta > 0$, such that

$$\inf_{\|\xi\|_{\mathcal{X}}=1} \int_0^T |\langle F^u S^u(s)\xi, S^u(s+r)\xi\rangle|\, ds = \delta.$$

Now, let $\xi \in \mathcal{X}^u\backslash\{0\}$ and take $\widetilde{\xi} = \dfrac{\xi}{\|\xi\|_{\mathcal{X}}}$. Then, we have

$$\int_0^T |\langle F^u S^u(s)\xi, S^u(s+r)\xi\rangle|\, ds \geq \delta\|\xi\|_{\mathcal{X}}^2.$$

Then condition (2.3) holds and consequently the feedback (2.4) strongly stabilizes the system (0.6) and we have the following decay estimate

$$\|w(t)\|_{\mathcal{X}} = O\left(\frac{1}{\sqrt{t}}\right),\ \text{as}\ t \longrightarrow +\infty.$$

2.3 Weak stabilization

In the following, we use a weaker assumption than the observability inequality condition (2.3) to ensures the weak stabilization of the system (0.6).

Theorem 2.3.1. *Suppose that condition (2.1) holds. If*
1. $S^u(t)$ is a contraction semigroup,
2. $S^s(t)$ is exponentially stable,
4. for any sequence $(\xi_n)_n \subset \mathcal{X}^u$ such that $\xi_n \rightharpoonup \xi$ in $\mathcal{X}^u$, we have

$$\langle S^u(t+r)\xi_n, F^u S^u(t)\xi_n\rangle \to 0 \text{ as } n \to +\infty, \text{ for all } t \geq 0 \Rightarrow \xi = 0. \tag{2.17}$$

Then, the system (0.6) is weakly stabilizable by the control (2.4).

Proof: From inequality (2.10), it follows that $\|w^u(t)\|$ is bounded in the reflexive space $\mathcal{X}^u$. Then, there exists a sequence $(t_n)_n$ with $t_n \to +\infty$ such that

$$w^u(t_n) \rightharpoonup \varphi \quad \text{for some } \varphi \in \mathcal{X}^u.$$

Moreover, for any $T > 0$, we have from (2.15), that for $p \in \{0, 1, 2\}$

$$\int_0^T |\langle S^u(s+r)w^u(t_n), F^u S^u(s)w^u(t_n)\rangle|\, ds \leq M \left(\int_{t_n}^{t_n+T+r} \frac{|\langle w^u(s), F^u w^u(s-r)\rangle|^2}{\max\{\|w^u(s)\|^p_{\mathcal{X}}, \|w^u(s-r)\|^p_{\mathcal{X}}\}} \mathbf{1}_E ds \right)^{\frac{1}{2}},$$

with $M = 2\sqrt{r+T}\|F\|^2 T \|\phi\|_{\mathcal{C}}^{2-\frac{p}{2}} + \sqrt{T}\|\phi\|_{\mathcal{C}}^{\frac{p}{2}}$. Using the inequality (2.9), we obtain that

$$M \left(\int_t^{t+T+r} \frac{|\langle w^u(s), F^u w^u(s-r)\rangle|^2}{\max\{\|w^u(s)\|^p_{\mathcal{X}}, \|w^u(s-r)\|^p_{\mathcal{X}}\}} \mathbf{1}_E ds \right)^{\frac{1}{2}} \leq \frac{M}{2} \left(\|w^u(t)\|^2_{\mathcal{X}} - \|w^u(t+T+r)\|^2_{\mathcal{X}} \right),$$

we have

$$\int_0^T |\langle S^u(s+r)w^u(t_n), F^u S^u(s)w^u(t_n)\rangle|\, ds \to 0 \quad \text{as } n \to +\infty.$$

So according to the dominated convergence theorem, we deduce that

$$\int_0^T \lim_{n\to+\infty} |\langle S^u(s+r)w^u(t_n), F^u S^u(s)w^u(t_n)\rangle| ds = 0.$$

Then

$$\lim_{n\to+\infty} |\langle S^u(s+r)w^u(t_n), F^u S^u(s)w^u(t_n)\rangle| = 0 \text{ for all } s \in [0, T].$$

Or T is arbitrary, then assumption (2.17) implies that $\varphi = 0$.
Using the fact that $S^s(t)$ is exponentially stable as in the proof of the Theorem 2.2.1 we can see that there exist $N, \sigma > 0$ such that

$$\|w^s(t)\|_{\mathcal{X}} \leq N e^{-\sigma t}\|\phi^s\|_{\mathcal{C}}, \text{ for all } t \geq 0,$$

Then, $w(t) = w^u(t) + w^s(t)$ tends to 0 weakly as $t \to +\infty$.

2.4 Exponential stabilization

In cases p =0,1 and 2, we proved in the previous sections, the strong and weak stabilization of the system (0.6) with the feedback control (2.3). However, the feedback low can not guarantee the exponential stabilization of the system (0.6) for $p = 0$ or $p = 1$. The next result treat the case $p = 2$.

Theorem 2.4.1. *Let $\mathcal{A}$ be a generator of a C_0-semigroup $S(t)$ and suppose that condition (2.1) holds. We further assume that:*
1. $S^u(t)$ is a contraction semigroup,
2. $S^s(t)$ is exponentially stable,
3. the observability condition (2.3) holds.
Then, the feedback

$$v(t) = -\frac{\langle w^u(t), F^u w^u(t-r)\rangle}{\max\{\|w^u(t)\|_{\mathcal{X}}^2, \|w^u(t-r)\|_{\mathcal{X}}^2\}}\mathbf{1}_E, \tag{2.18}$$

where $E = \{t \geq 0 : \max\{\|w^u(t)\|_{\mathcal{X}}, \|w^u(t-r)\|_{\mathcal{X}}\} \neq 0\}$, exponentially stabilizes the system (0.6).

Proof : Using the observability condition (2.3) for $\xi = w^u(j(r+T))$ where w^u is the restriction of the solution w on w^u, we obtain

$$\delta\|w^u(j(r+T))\|_{\mathcal{X}}^2 \leq \int_0^T |\langle S^u(s+r)w^u(j(r+T)), F^u S^u(s)w^u(j(r+T))\rangle| ds.$$

Inequalities (2.13) and (2.14) for $p = 2$ imply that

$$\int_0^T |\langle S^u(s+r)w^u(j(r+T)), F^u S^u(s)w^u(j(r+T))\rangle| ds$$
$$\leq \widetilde{M}\|w^u(j(r+T))\|_{\mathcal{X}} \left(\int_{j(r+T)}^{(j+1)(r+T)} \frac{|\langle F^u w^u(s-r), w^u(s)\rangle|^2}{\max\{\|w^u(s)\|_{\mathcal{X}}^2, \|w^u(s-r)\|_{\mathcal{X}}^2\}}\mathbf{1}_E ds\right)^{\frac{1}{2}},$$

with $\widetilde{M} = 2\sqrt{r+T}\|F\|^2 T + \sqrt{T}$. So we have

$$\frac{\delta^2}{\widetilde{M}^2}\|w^u(j(r+T))\|_{\mathcal{X}}^2 \leq \int_{j(r+T)}^{(j+1)(r+T)} \frac{|\langle F^u w^u(s-r), w^u(s)\rangle|^2}{\max\{\|w^u(s)\|_{\mathcal{X}}^2, \|w^u(s-r)\|_{\mathcal{X}}^2\}}\mathbf{1}_E ds.$$

Using inequality (2.9), we obtain

$$\|w^u((j+1)(r+T))\|_{\mathcal{X}}^2 - \|w^u(j(r+T))\|_{\mathcal{X}}^2 \leq -\frac{2\delta^2}{\widetilde{M}^2}\|w^u(j(r+T))\|_{\mathcal{X}}^2,$$

and then

$$\|w^u((j+1)(r+T))\|_{\mathcal{X}}^2 + \frac{2\delta^2}{\widetilde{M}^2}\|w^u((j)(r+T))\|_{\mathcal{X}}^2 \leq \|w^u(j(r+T))\|_{\mathcal{X}}^2.$$

Letting $\nu_j = \|w^u(j(r+T))\|^2_{\mathcal{X}}$ and observing that $\nu_{j+1} \leq \nu_j$ we derive that

$$\nu_{j+1} + \frac{2\delta^2}{\widetilde{M^2}}\nu_{j+1} \leq \nu_j.$$

That is to say for all $j \in \mathbb{N}$

$$\nu_{j+1} \leq \frac{1}{1+\frac{2\delta^2}{\widetilde{C^2}}}\nu_j,$$

and

$$\nu_j \leq \frac{1}{(1+\frac{2\delta^2}{\widetilde{C^2}})^j}\nu_0,$$

with $1+\frac{2\delta^2}{\widetilde{C^2}} > 1$. Let us now set $j = E(\frac{t}{r+T})$ (where $E(.)$ designs the integer part) and using again the fact that $\|w^u(t)\|$ decreases, we have

$$\|w^u(t)\|^2_{\mathcal{X}} \leq \|w^u(j(r+T))\|^2_{\mathcal{X}} \leq \frac{1}{(1+\frac{2\delta^2}{\widetilde{C^2}})^{\frac{t}{r+T}-1}}\|w^u(0)\|^2_{\mathcal{X}}.$$

We conclude the following exponential decay

$$\|w^u(t)\| \leq \sqrt{1+\frac{2\delta^2}{\widetilde{C^2}}}\, e^{\frac{-\ln\left(1+\frac{2\delta^2}{\widetilde{C^2}}\right)t}{2\,(r+T)}}\|\phi^u\|_{\mathcal{C}}, \quad \textit{for all } t \geq 0.$$

Then

$$\|w^u(t)\| \leq R\, e^{-\omega t}\|\phi^u\|_{\mathcal{C}} \ \textit{ for all } t \geq 0,$$

where

$$R = \sqrt{1+\frac{2\delta^2}{\widetilde{C^2}}} \quad \textit{and} \quad \omega = \frac{\ln(1+\frac{2\delta^2}{\widetilde{C^2}})}{2(r+T)}.$$

Moreover, since $S^s(t)$ is exponentially stable, we can see as in the proof of Theorem 2.2.1 that there exist $N, \sigma > 0$ such that

$$\|w^s(t)\|_{\mathcal{X}} \leq N\, e^{-\sigma t}\|\phi^s\|_{\mathcal{C}}, \ \textit{for all } t \geq 0.$$

In the next, without using the exponential stability condition of $S^s(t)$ and without using the invariance condition of $\mathcal{X}^s$ with respect to F, we have the following result on the exponential stabilization of the system (0.6).

Theorem 2.4.2. *Let $\mathcal{A}$ be a generator of a C_0-semigroup $S(t)$. Suppose that the condition $F\mathcal{X}^u \subseteq \mathcal{X}^u$ holds and that*

1. $S^u(t)$ is a contraction semigroup,
2. there exist positive constant T and δ such that

$$\int_0^T |\langle F^u S(s)\xi, S(s+r)\xi\rangle|\, ds \geq \delta\|\xi\|_{\mathcal{X}}^2, \text{ for all } \xi \in \mathcal{X}. \tag{2.19}$$

Then the feedback

$$v(t) = -\frac{\langle w^u(t), F^u w^u(t-r)\rangle}{\max\{\|w^u(t)\|_{\mathcal{X}}^2, \|w^u(t-r)\|_{\mathcal{X}}^2\}}\mathbf{1}_E, \tag{2.20}$$

where $E = \{t \geq 0 : \max\{\|w^u(t)\|_{\mathcal{X}}, \|w^u(t-r)\|_{\mathcal{X}}\} \neq 0\}$, exponentially stabilizes the system (0.6) (resp. $\|w(t)\|_{\mathcal{X}} = 0$, for all $t \geq 0$), if $\delta \leq T\|F^u\|$ (resp. $\delta > T\|F^u\|$).

Proof: Let $\xi = w(j(r+T))$. Since $\|w^u(j(r+T))\|_{\mathcal{X}} \leq \|w(j(r+T))\|_{\mathcal{X}}$ and by using the observability condition (2.19), it follows that :

$$\delta\|w^u(j(r+T))\|_{\mathcal{X}}^2 \leq \int_0^T |\langle S^u(s+r)w^u(j(r+T)), F^u S^u(s)w^u(j(r+T))\rangle| ds.$$

Inequalities (2.13) and (2.14) for $p = 2$ imply that

$$\begin{aligned}\int_0^T &|\langle S^u(s+r)w^u(j(r+T)), F^u S^u(s)w^u(j(r+T))\rangle| ds \\ &\leq \widetilde{M}\|w^u(j(r+T))\|_{\mathcal{X}} \left(\int_{j(r+T)}^{(j+1)(r+T)} \frac{|\langle F^u w^u(s-r), w^u(s)\rangle|^2}{\max\{\|w^u(s)\|_{\mathcal{X}}^2, \|w^u(s-r)\|_{\mathcal{X}}^2\}}\mathbf{1}_E ds\right)^{\frac{1}{2}},\end{aligned}$$

with $\widetilde{M} = 2\sqrt{r+T}\|F\|^2 T + \sqrt{T}$. So that

$$\frac{\delta^2}{\widetilde{M}^2}\|w^u(j(r+T))\|_{\mathcal{X}}^2 \leq \|w^u(j(r+T))\|_{\mathcal{X}} \left(\int_{j(r+T)}^{(j+1)(r+T)} \frac{|\langle F^u w^u(s-r), w^u(s)\rangle|^2}{\max\{\|w^u(s)\|_{\mathcal{X}}^2, \|w^u(s-r)\|_{\mathcal{X}}^2\}}\mathbf{1}_E ds\right)^{\frac{1}{2}},$$

Using the same procedure for w^u as in Theorem 2.4.1, we obtain that

$$\|w^u(t)\|_{\mathcal{X}} \leq R\, e^{-\omega t}\|\phi^u\|_{\mathcal{C}} \quad \text{for all } t \geq 0,$$

where

$$R = \sqrt{1 + \frac{2\delta^2}{\widetilde{C}^2}} \quad \text{and} \quad \omega = \frac{\ln(1 + \frac{2\delta^2}{\widetilde{C}^2})}{2(r+T)}.$$

Using the observability condition (2.19) with $\xi = w(t)$ we have

$$\delta\|w(t)\|_{\mathcal{X}}^2 \leq T\|F^u\|\|w^u(t)\|_{\mathcal{X}}^2.$$

If $\delta > T\|F^u\|$, we obtain that

$$\left(1 - \frac{T\|F^u\|}{\delta}\right)\|w(t)\|_{\mathcal{X}}^2 \leq 0.$$

Moreover, in this case we have that $\mathcal{X} = \{0\}$.
If $\delta = T\|F^u\|$, then $\mathcal{X}^u = \mathcal{X}$ and $\mathcal{X}^s = \{0\}$ and consequently

$$\|w(t)\|_{\mathcal{X}} = \|w^u(t)\|_{\mathcal{X}} \leq R\,e^{-\omega t}\|\phi\|_{\mathcal{C}} \quad for\ all\ t \geq 0,$$

If $\delta < T\|F^u\|$, we obtain that

$$\|w(t)\|_{\mathcal{X}} \leq \left(\frac{T\|F^u\|}{\delta}\right)^{\frac{1}{2}} R\,e^{-\omega t}\|\phi\|_{\mathcal{C}} \quad for\ all\ t \geq 0.$$

Remark 2.4.3. *From the proof of Theorem 2.4.2, it's clear that the condition $\delta = T\|F^u\|$ implies that $\mathcal{X}^u = \mathcal{X}$ and $\mathcal{X}^s = \{0\}$. But, in general the reverse is not true, for more details, see example 3.*

2.5 Applications and Numerical simulations

Understanding requires us to put theory into practice and to allow our practices to enrich our theoretical and see its applicability, so we will study the following three examples.

Example 1: Let us consider the following mono-dimensional bilinear heat equation :

$$\begin{cases} \dfrac{\partial w(x,t)}{\partial t} = \dfrac{\partial^2 w(x,t)}{\partial x^2} + v(t)w(x,t-r), & for\ x \in (0,1)\ and\ t \geq 0, \\ \dfrac{\partial w(0,t)}{\partial x} = \dfrac{\partial w(1,t)}{\partial x} = 0, & for\ t \in [-r,+\infty), \\ w(x,t) = \cos(t), & for\ t \in [-r,0]\ and\ x \in (0,1). \end{cases} \tag{2.21}$$

Consider the space $\mathcal{X} = L^2(0,1)$, and the operator $\mathcal{A} : \mathcal{D}(\mathcal{A}) \subset \mathcal{X} \to \mathcal{X}$ defined by

$$\mathcal{A}w = \frac{\partial^2 w}{\partial x^2},\ with\ \mathcal{D}(\mathcal{A}) = \left\{w \in H^2(0,1); \frac{\partial w(0,t)}{\partial x} = \frac{\partial w(1,t)}{\partial x} = 0\right\},$$

and

$$F = Id_{\mathcal{X}}.$$

The spectrum of $\mathcal{A}$ is given by the simple eigenvalues $\lambda_n = -\pi^2 n^2, n \geq 0$ with its corresponding eigenfunctions defined as follows

$$\phi_n(x) = \begin{cases} \sqrt{2}\cos(n\pi x), & \text{if } n \geq 1, \\ 1, & \text{else .} \end{cases}$$

From [31] the semigroup $S(t)$ generated by operator $\mathcal{A}$ is given by

$$S(t)w = \sum_{n=0}^{+\infty} e^{\lambda_n t} \langle w, \phi_n \rangle \phi_n.$$

The state space $\mathcal{X}$ can be decomposed according to $\mathcal{X} = \mathcal{X}^u \oplus \mathcal{X}^s$, where $\mathcal{X}^u = \text{span}\{\phi_0\}$. Here, we have that $S^u(t) = Id_{\mathcal{X}^u}$ and $F^u = Id_{\mathcal{X}^u}$. Then

$$\langle S^u(t+r)w^u, F^u S^u(t)w^u \rangle \;=\; \|w^u\|^2_{\mathcal{X}}.$$

Thus, (2.16) holds. Then, by applying Corollary 2.2.3, we conclude that, for $p \in \{0, 1, 2\}$, the feedback control

$$v(t) = \begin{cases} -\dfrac{\langle w^u(t), F^u w^u(t-r) \rangle}{\max\{\|w^u(t)\|^p_{\mathcal{X}}, \|w^u(t-r)\|^p_{\mathcal{X}}\}}, & if \;\; \max\{\|w^u(t)\|_{\mathcal{X}}, \|w^u(t-r)\|_{\mathcal{X}}\} \neq 0, \\ 0, & else, \end{cases} \tag{2.22}$$

strongly stabilizes the system (2.21) and we have the following decay estimate

$$\|w(t)\|_{\mathcal{X}} = O\left(\frac{1}{\sqrt{t}}\right), \;\text{ as } \quad \text{t} \;\longrightarrow +\infty.$$

Let us consider the system (2.21) and the command (2.22) with $p = 0$. We choose parameters as : $t_{max} = 5$, $r = \frac{\pi}{2}$. Two axes are divided into sections as $n_x = 5$, $n_t = 200$. Then, in the figures below, we get the following results:
Figures 2.1 and 2.3 show the evolution of the state and the norm of the free state ($v(t) = 0$), figures 2.2 and 2.4 show the evolution and norm of the stabilized state with the feedback control (2.22) and the last figure represents the evolution of control (2.22).

Example 2: Let $\Omega = (0, 1)$ and consider the following coupled PDE with discrete delay:

$$\begin{cases} \dfrac{\partial^2 w(x,t)}{\partial^2 t} = \dfrac{\partial^2 w(x,t)}{\partial x^2} + v(t)\dfrac{\partial w(x,t-3)}{\partial t}, & for\ x \in (0,1)\ and\ t \geq 0, \\ \dfrac{\partial y(x,t)}{\partial t} = \dfrac{\partial^2 y(x,t)}{\partial x^2} + v(t)(y(x,t-3) \;+\; w(x,t-3)), & for\ x \in (0,1)\ and\ t \geq 0, \\ w(0,t) = w(1,t) = 0, & for\ t \in [-3, +\infty), \\ \dfrac{\partial y(0,t)}{\partial x} = \dfrac{\partial y(1,t)}{\partial x} = 0, & for\ t \in [-3, +\infty), \\ w(x,t) = \sin(\dfrac{\pi}{2}t)\sin(\pi x), & for\ t \in [-3, 0]\ and\ x \in (0,1), \\ y(x,t) = \cos(t)\sin(\pi x), & for\ t \in [-3, 0]\ and\ x \in (0,1), \end{cases} \tag{2.23}$$

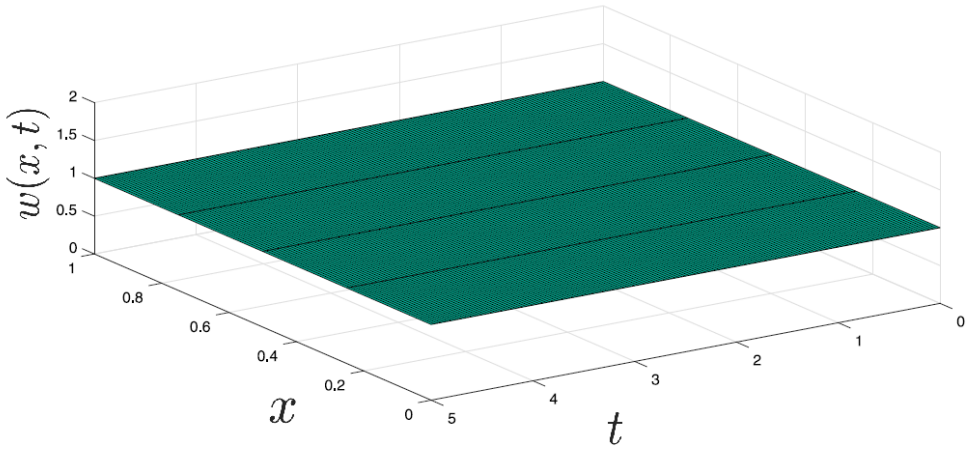

Figure 2.1: The evolution of the free state (v(t)=0).

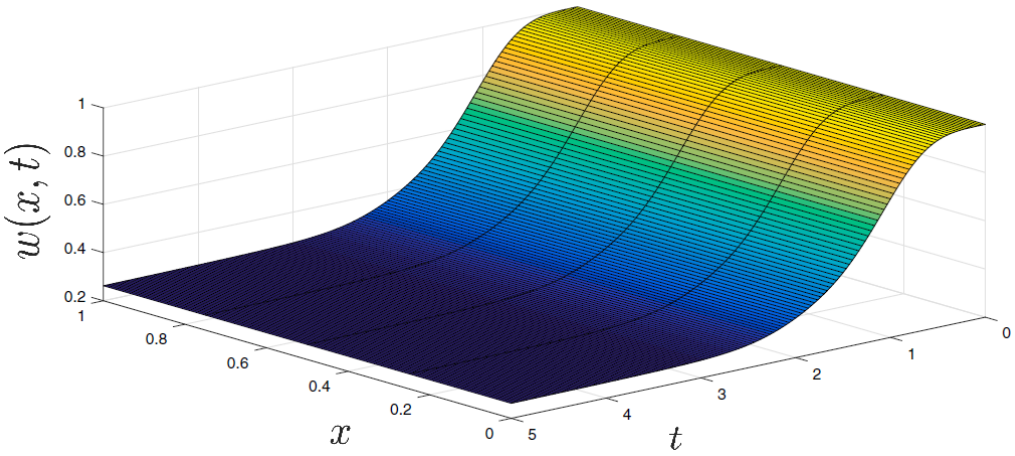

Figure 2.2: The evolution of the $w(x,t)$ by control (2.22).

The space $\mathcal{X} = (L^2(0,1))^3$ can be decomposed as $\mathcal{X} = \mathcal{X}^u + \mathcal{X}^s$, where

$$\mathcal{X}^u = (L^2(0,1))^2 \times \{0\},\ \mathcal{X}^s = \{0\} \times L^2(0,1),$$

with the inner product

$$\langle (x_1, x_2, x_2), (y_1, y_2, y_3) \rangle = \langle x_1, y_1 \rangle_{L^2(0,1)} + \langle x_2, y_2 \rangle_{L^2(0,1)} + \langle x_3, y_3 \rangle_{L^2(0,1)},$$

we define the operator $\mathcal{A} : \mathcal{D}(\mathcal{A}) \subset \mathcal{X} \to \mathcal{X}$, as follows

$$\mathcal{A} = \begin{pmatrix} 0 & I & 0 \\ \frac{\partial^2}{\partial^2 x} & 0 & 0 \\ 0 & 0 & \frac{\partial^2}{\partial^2 x} \end{pmatrix}$$

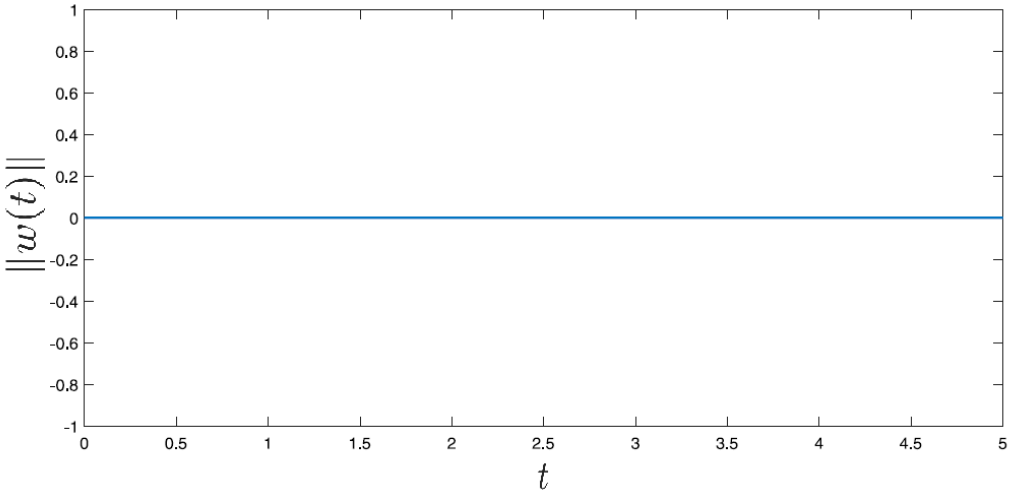

Figure 2.3: The evolution of the state norm $\|w(t)\|$ (v(t)=0).

with

$$\mathcal{D}(\mathcal{A}) = \left(H^2(0,1) \cap H_0^1(0,1)\right) \times H_0^1(0,1) \times \left\{ y \in H^2(0,1) \ : \ \frac{\partial y(0,t)}{\partial x} = \frac{\partial y(1,t)}{\partial x} = 0 \right\},$$

and the operator L defined from $\mathcal{X}$ to $\mathcal{X}$ by:

$$F = \begin{pmatrix} 0 & 0 & 0 \\ 0 & I & 0 \\ I & 0 & I \end{pmatrix}.$$

Then, by this transformation, system (2.25) takes the abstract from (0.6)

$$z = \begin{pmatrix} x \\ \frac{\partial x}{\partial t} \\ y \end{pmatrix}.$$

The set of function $(\phi_n)_{n\geq 1}$ expressed by $\phi_n(x) = \sqrt{2}\sin(n\pi x)$, for all $n \geq 1$ is a complete orthonormal eigenfunctions system of $\frac{\partial^2}{\partial x^2}$ associated to the eigenvalues $\lambda_n = -(n\pi)^2, (n \in \mathbb{N}^*)$.

Let $w = (w_1, w_2, w_3) \in \mathcal{X}$ and $w^u = (w_1, w_2, 0)$, $w^s = (0, 0, w_3)$, with $w_1 = \sum_{n=1}^{\infty} \alpha_n \phi_n$ and $w_2 = \sum_{n=1}^{\infty} (-\lambda_n)^{\frac{1}{2}} \mu_n \phi_n$, where $(\alpha_n, \mu_n) \in \mathbb{R}^2$ for all $n \geq 1$. It is well known that (see e.g. [31]) the operator $\mathcal{A}^u$ is an infinitesimal generator of a semigroup of isometries, which is given by :

$$S^u(t)w^u = \sum_{n=1}^{+\infty} \begin{pmatrix} \alpha_n \cos(\sqrt{-\lambda_n}t) + \mu_n \sin(\sqrt{-\lambda_n}t) \\ -\alpha_n\sqrt{-\lambda_n}\sin(\sqrt{-\lambda_n}t) + \mu_n\sqrt{-\lambda_n}\cos(\sqrt{-\lambda_n}t) \\ 0 \end{pmatrix} \phi_n,$$

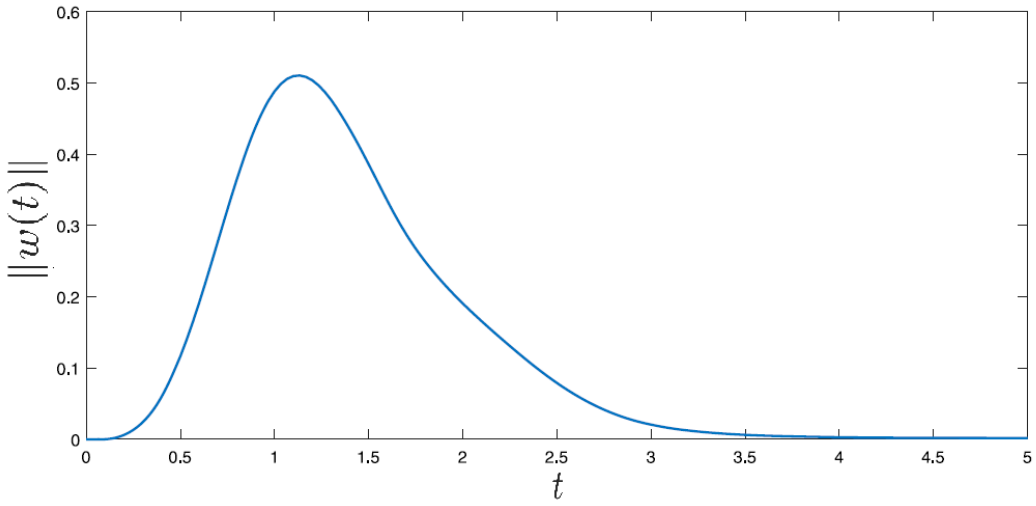

Figure 2.4: The evolution of the state norm $\|w(t)\|$ by control (2.22).

and

$$S^u(t+3)w^u = \sum_{n=1}^{+\infty} \begin{pmatrix} \alpha_n \cos(\sqrt{-\lambda_n}(t+3)) + \mu_n \sin(\sqrt{-\lambda_n}(t+3)) \\ -\alpha_n\sqrt{-\lambda_n} \sin(\sqrt{-\lambda_n}(t+3)) + \mu_n\sqrt{-\lambda_n} \cos(\sqrt{-\lambda_n}(t+3)) \\ 0 \end{pmatrix} \phi_n.$$

Then

$$\begin{aligned} \langle F^u S^u(t)w^u, S^u(t+3)w^u\rangle = \sum_{n=1}^{+\infty} & \lambda_n(\alpha_n^2 \sin(\sqrt{-\lambda_n}t) \sin(\sqrt{-\lambda_n}(t+3)) \\ & - \alpha_n\mu_n \sin(\sqrt{-\lambda_n}t) \cos(\sqrt{-\lambda_n}(t+3)) \\ & - \alpha_n\mu_n \sin(\sqrt{-\lambda_n}(t+3)) \cos(\sqrt{-\lambda_n}t) \\ & + \mu_n^2 \cos(\sqrt{-\lambda_n}t) \cos(\sqrt{-\lambda_n}(t+3))). \end{aligned}$$

Integrating the last equality from 0 to 3, we have

$$\int_0^3 |\langle F^u S^u(t)w^u, S^u(t+3)w^u\rangle| dt = \frac{3}{2} \sum_{n=1}^{+\infty} -\lambda_n(\alpha_n^2 + \mu_n^2) \geq \frac{3}{2}\|w^u\|_{\mathcal{X}}^2.$$

Thus, condition (2.3) holds and by applying Theorem 2.4.1, we conclude that the feedback control

$$v(t) = \begin{cases} -\dfrac{\langle w^u(t), F^u w^u(t-3)\rangle}{\max\{\|w^u(t)\|_{\mathcal{X}}^2, \|w^u(t-3)\|_{\mathcal{X}}^2\}}, & if \max\{\|w^u(t)\|_{\mathcal{X}}, \|w^u(t-3)\|_{\mathcal{X}}\} \neq 0, \\ 0, & else, \end{cases} \tag{2.24}$$

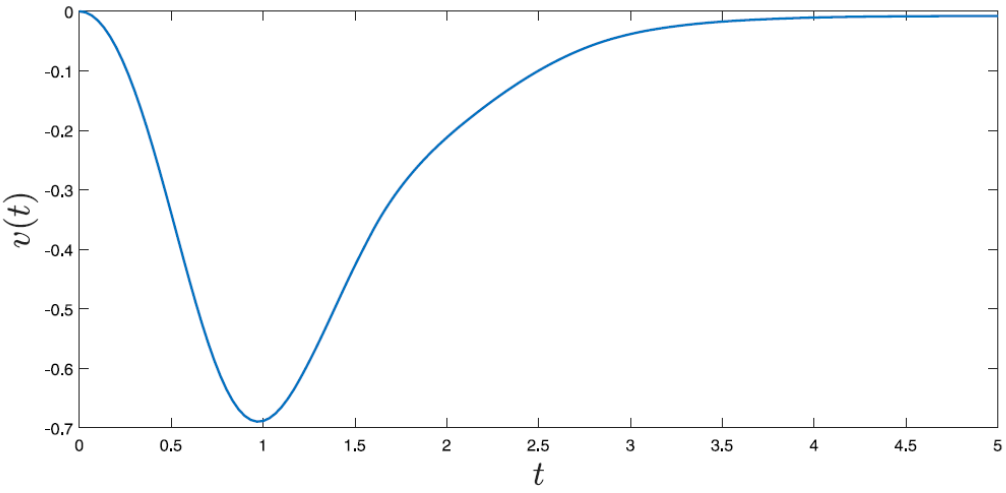

Figure 2.5: The evolution of the control (2.22).

exponentially stabilizes the system (2.23).
To understand this example we will pass to apply it in the following simulation, we choose $p = 2$, and the parameters as : $t_{max} = 40$, $r = 3$. Two axes are divided into sections as $n_x = 4$, $n_t = 2000$.

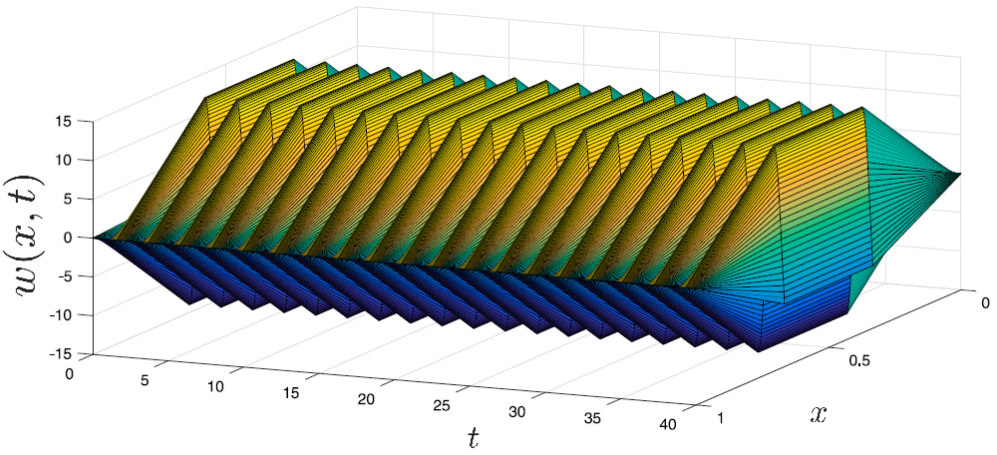

Figure 2.6: The evolution of $w(x,t)$ by control (v(t)=0).

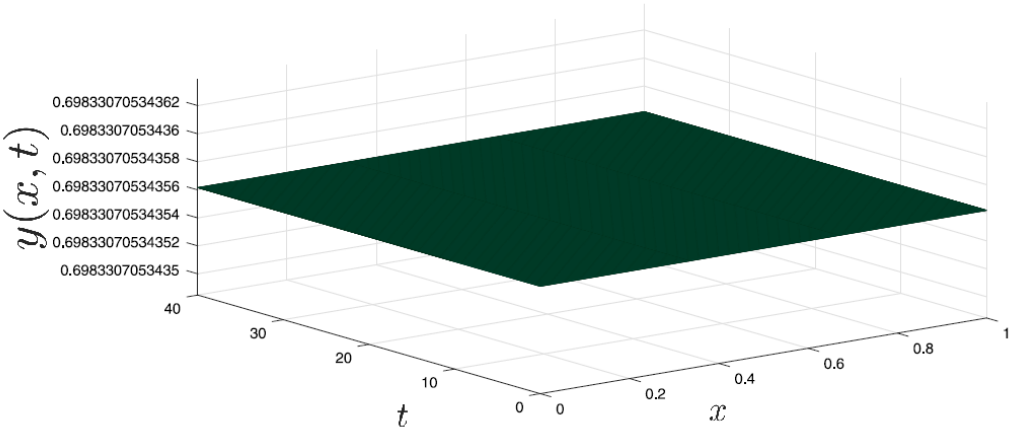

Figure 2.7: The evolution of $y(x,t)$ by control (v(t)=0).

Example 3: Let $\Omega = (0,1)$ and consider the system

$$\begin{cases} \dfrac{\partial^2 w(x,t)}{\partial t^2} = \dfrac{\partial^2 w(x,t)}{\partial x^2} + v(t)\dfrac{\partial w(x,t-3)}{\partial t}, & for\ x \in (0,1)\ and\ t \geq 0, \\ w(0,t) = w(1,t) = 0, & for\ t \in [-3,+\infty), \\ w(x,t) = \sin(\dfrac{\pi}{2}t)\sin(\pi x), & for\ t \in [-3,0]\ and\ x \in (0,1). \end{cases} \tag{2.25}$$

The space $\mathcal{X} = \mathcal{H}_0^1(0,1) \times L^2(0,1)$ can be decomposed as $\mathcal{X} = \mathcal{X}^u + \mathcal{X}^s$, where $\mathcal{X}^u = \mathcal{H}_0^1(0,1) \times L^2(0,1)$, $\mathcal{X}^s = \{0\}$ with the inner product

$$\langle (x_1,x_2), (y_1,y_2) \rangle = \langle x_1, y_1 \rangle_{H_0^1(0,1)} + \langle x_2, y_2 \rangle_{L^2(0,1)},$$

we define the operator $\mathcal{A} : \mathcal{D}(\mathcal{A}) \subset \mathcal{X} \to \mathcal{X}$ as follows

$$\mathcal{A} = \begin{pmatrix} 0 & I \\ \frac{\partial^2}{\partial^2 x} & 0 \end{pmatrix} \ with\ \mathcal{D}(\mathcal{A}) = \left(H^2(0,1) \cap H_0^1(0,1)\right) \times H_0^1(0,1),$$

and $F = \begin{pmatrix} 0 & 0 \\ 0 & I \end{pmatrix}$. Then, the system (2.25) is equivalent to (0.6) with

$$y = \begin{pmatrix} w \\ \frac{\partial w}{\partial t} \end{pmatrix}.$$

The set of function $(\phi_n)_{n\geq 1}$ expressed by $\phi_n(x) = \sqrt{2}\sin(n\pi x)$, for all $n \geq 1$ is a complete orthogonal eigenfunctions system of $\frac{\partial^2}{\partial x^2}$ associated to the eigenvalues $\lambda_n = -(n\pi)^2, (n \in \mathbb{N}^*)$.

Let $w = (w_1, w_2) \in \mathcal{X}$ and $w^u = (w_1, w_2)$, $w^s = (0,0)$, with $w_1 = \sum_{n=1}^{\infty} \alpha_n \phi_n$ and

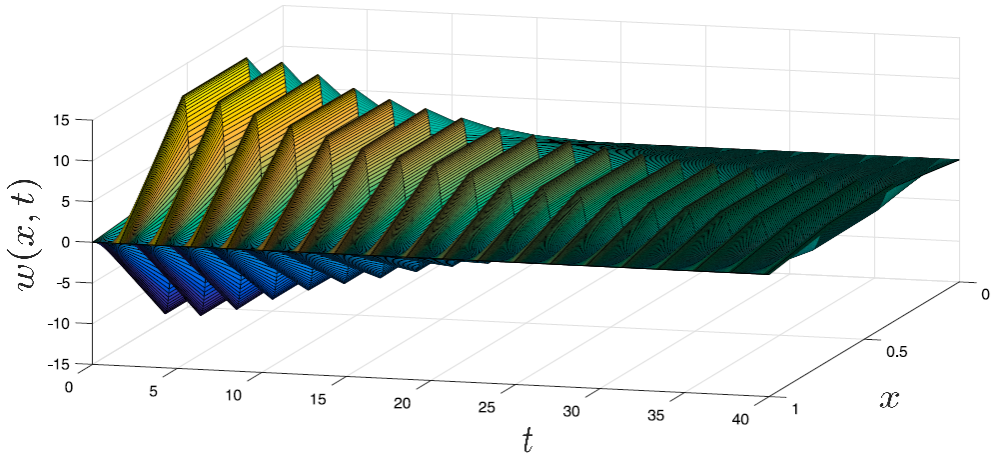

Figure 2.8: The evolution of $w(x,t)$ by control (2.24).

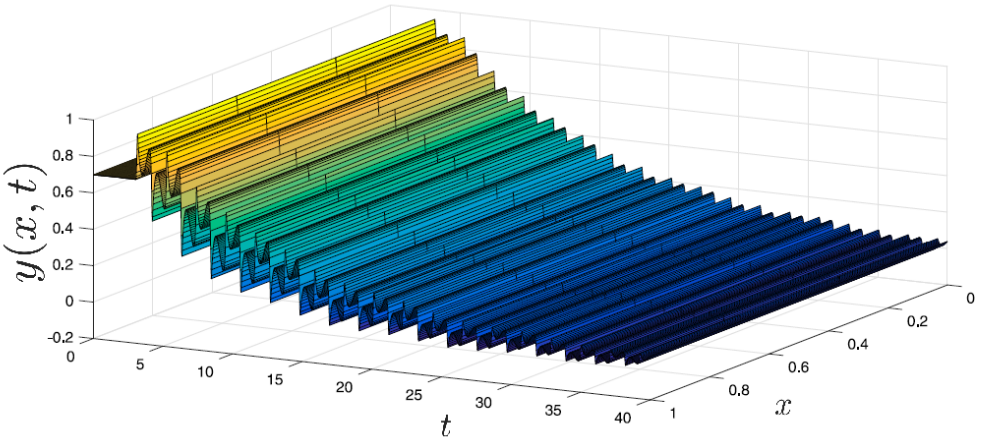

Figure 2.9: The evolution of $y(x,t)$ by control (2.24).

$w_2 = \sum_{n=1}^{\infty}(-\lambda_n)^{\frac{1}{2}}\mu_n\phi_n$, where $(\alpha_n, \mu_n) \in \mathbb{R}^2 \quad$ for all $n \geq 1$. It is well known that (see e.g. [31]) the operator $\mathcal{A}$ is an infinitesimal generator of a semigroup of isometries, which is given by :

$$S(t)w = \sum_{n=1}^{+\infty} \begin{pmatrix} \alpha_n \cos(\sqrt{-\lambda_n}t) + \mu_n \sin(\sqrt{-\lambda_n}t) \\ -\alpha_n\sqrt{-\lambda_n}\sin(\sqrt{-\lambda_n}t) + \mu_n\sqrt{-\lambda_n}\cos(\sqrt{-\lambda_n}t) \end{pmatrix} \phi_n,$$

and

$$S(t+3)w = \sum_{n=1}^{+\infty} \begin{pmatrix} \alpha_n \cos(\sqrt{-\lambda_n}(t+3)) + \mu_n \sin(\sqrt{-\lambda_n}(t+3)) \\ -\alpha_n\sqrt{-\lambda_n}\sin(\sqrt{-\lambda_n}(t+3)) + \mu_n\sqrt{-\lambda_n}\cos(\sqrt{-\lambda_n}(t+3)) \end{pmatrix} \phi_n.$$

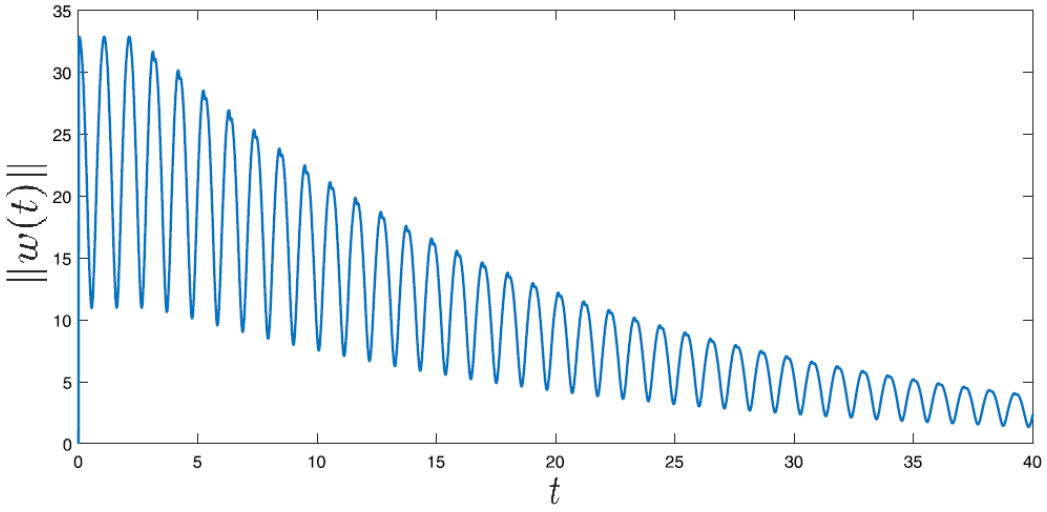

Figure 2.10: The evolution of the stabilized state norm $\|w(t)\|$ by control (2.24).

Then

$$\begin{aligned}\langle F^u S(t)w, S(t+3)w\rangle = &\sum_{n=1}^{+\infty} \lambda_n(\alpha_n^2 \sin(\sqrt{-\lambda_n}t)\sin(\sqrt{-\lambda_n}(t+3))\\ &- \alpha_n\mu_n \sin(\sqrt{-\lambda_n}t)\cos(\sqrt{-\lambda_n}(t+3))\\ &- \alpha_n\mu_n \sin(\sqrt{-\lambda_n}(t+3))\cos(\sqrt{-\lambda_n}t)\\ &+ \mu_n^2 \cos(\sqrt{-\lambda_n}t)\cos(\sqrt{-\lambda_n}(t+3))).\end{aligned}$$

Integrating the last equality from 0 to 3, we have

$$\int_0^3 |\langle F^u S(t)w, S(t+3)w\rangle| dt = \frac{3}{2}\sum_{n=1}^{+\infty} -\lambda_n(\alpha_n^2+\mu_n^2) \geq \frac{3}{2}\|w\|_{\mathcal{X}}^2.$$

Thus, condition (2.3) holds and by applying Theorem 2.2.1 (resp. Theorem 2.4.2), we conclude that the feedback control

$$v(t) = \begin{cases} -\dfrac{\langle w^u(t), F^u w^u(t-3)\rangle}{\max\{\|w^u(t)\|_{\mathcal{X}}^p, \|w^u(t-3)\|_{\mathcal{X}}^p\}}, & if \max\{\|w^u(t)\|_{\mathcal{X}}, \|w^u(t-3)\|_{\mathcal{X}}\} \neq 0,\\ 0, & else,\end{cases}$$

strongly stabilizes the system (2.25) for all $p \in \{0,1,2\}$, with the following decay estimate

$$\|w(t)\|_{\mathcal{X}} = O\left(\frac{1}{\sqrt{t}}\right) \text{ as } \quad t \longrightarrow +\infty,$$

and exponentially stabilizes the system (2.25) for $p = 2$.

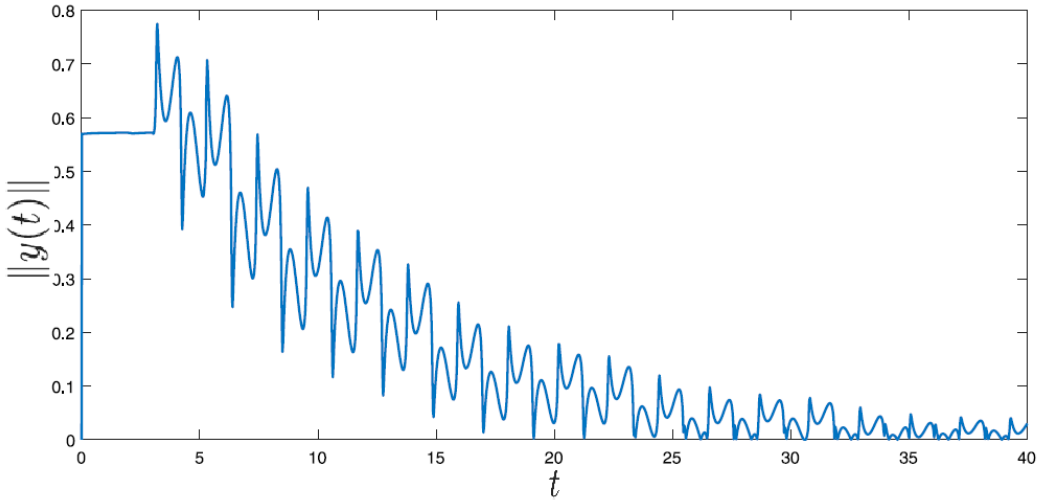

Figure 2.11: The evolution of the stabilized state norm $\|y(t)\|$ by control (2.24).

Conclusion. In this chapter, we investigated the feedback stabilization of bilinear systems with discrete delay. We provided stabilizing feedback laws which are either continuous or bounded with respect to initial states. The conventional degrees of stability (i.e. weak, strong or exponential) are studied. We showed that the feedback control (2.4) stabilizes strongly, weakly the bilinear systems (0.6) with a polynomial estimate of the stabilized state for $p = 0, 1$ and 2. Furthermore, for $p = 2$, we proved that the feedback control (2.4) stabilizes exponentially the bilinear systems (0.6). The stabilization assumptions rely on the observation conditions in the sense of linear systems theory.

Chapter 3

Weak and exponential stabilization for a semi-linear systems with discrete multi-delays

3.1 Introduction

The present chapter would be a continuation of works [39, 40]. In this chapter, we are concerned with the weak and exponential stabilization of the system (0.7). First, we analyze the existence, uniqueness of the global mild solution of the system (0.7). In addition, we establish an estimate of the decay which will be used to study the exponential and weak stabilization of our system (0.7). We consider the following observability condition

$$\sum_{i=0}^{i=m} \int_r^T |\langle F_i S(t)\xi, S(t+r_i)\xi\rangle| dt \geq \delta \|\xi\|_{\mathcal{X}}^2 \text{ for all } \xi \in \mathcal{X}\ , \tag{3.1}$$

for some $T > r$ and $\delta > 0$, to prove that the system (0.7) is exponential stabilizable using the new following sequence feedback control, for $i \in \{0, ..., m\}$

$$v_i(t) = \begin{cases} -\dfrac{\langle w(t), F_i w(t-r_i)\rangle}{\max\limits_{0\leq j\leq m} \|w(t-r_j)\|_{\mathcal{X}}^2}, & \text{if } w(t) \neq 0, \\ 0, & \text{else,} \end{cases} \tag{3.2}$$

On the other hand, we discuss the weak stabilization of the system (0.7). We give some weaker conditions than (3.1) to guarantee the weak stabilization of the system (0.7). Finally, we apply our theoretical results to the wave equation and heat equation. Moreover, some numerical simulations are given at the end of this chapter.

3.2 Existence, uniqueness of the global mild solution

We start this section by the following theorem on the existence, uniqueness of the global mild solution of the system (0.7).

Theorem 3.2.1. *Let $\mathcal{A}$ generates a semigroup of contraction $S(t)$. Suppose that for $i \in \{0, ..., m\}$, F_i is a locally lipschitz function from $\mathcal{X}$ into $\mathcal{X}$, i.e: for each $k > 0$ there exists a constant Γ_k^i such that*

$$\|F_i x_1 - F_i x_2\|_{\mathcal{X}} \leq \Gamma_k^i \|x_1 - x_2\|_{\mathcal{X}} \quad \text{for all } (x_1, x_2) \in \mathcal{X}^2 \text{ with } \|x_1\|_{\mathcal{X}} \leq k, \|x_2\|_{\mathcal{X}} \leq k.$$

Then, system (0.7) controlled by (3.2) admits a unique global mild solution $w(t)$ defined on $\mathbb{R}^+$. Moreover, for all $t \geq \tau \geq 0$

$$\|w(\tau)\|_{\mathcal{X}}^2 - \|w(t)\|_{\mathcal{X}}^2 \geq 2 \sum_{i=0}^{m} \int_{\tau}^{t} \frac{\langle w(s), F_i w(s - r_i)\rangle^2}{\max\limits_{0 \leq j \leq m} \|w(s - r_j)\|_{\mathcal{X}}^2} \mathbf{1}_E ds. \tag{3.3}$$

with $E = \{ t \geq 0 : \ w(t) \neq 0 \}$.

Proof. Using the sequence of feedback control (3.2), the system (0.7) takes the following abstract form

$$\begin{cases} \dot{w}(t) = \mathcal{A}w(t) + F(w_t) \ \text{ for } t > 0, \\ w_0 = \phi \in \mathcal{C}, \end{cases}$$

where F is a function defined from $\mathcal{C}$ to $\mathcal{X}$ given by

$$F(\psi) = \begin{cases} -\sum\limits_{i=0}^{m} \dfrac{\langle \psi(0), F_i \psi(-r_i)\rangle}{\max\limits_{0 \leq j \leq m} \|\psi(-r_j)\|_{\mathcal{X}}^2} F_i \psi(-r_i), & \text{if } \psi(0) \neq 0 \\ 0, & \text{else.} \end{cases}$$

Let δ_ψ be such that $\delta_\psi = \max\limits_{0 \leq j \leq m} \|\psi(-r_j)\|_{\mathcal{X}}$. First, F is a locally Lipschitz function on $C([-r, 0], \mathcal{X}) = \mathcal{C}$. In fact, for any $k \geq 0$ and ψ, φ in $\mathcal{C}$ such that $\|\psi\|_{\mathcal{X}} \leq k$, $\|\varphi\|_{\mathcal{X}} \leq k$. If $\delta_\psi = 0$ or $\delta_\varphi = 0$, suppose that $\delta_\varphi = 0$, then $\varphi(-r_i) = 0$ for all $i \in \{0, ..., m\}$ and $F(\varphi) = 0$.
Then, we have

$$\begin{aligned} \|F(\psi) - F(\varphi)\|_{\mathcal{X}} &\leq \left\| \sum_{i=0}^{m} \frac{\langle \psi(0), F_i \psi(-r_i)\rangle}{\delta_\psi^2} F_i \psi(-r_i) \right\|_{\mathcal{X}} \\ &\leq \sum_{i=0}^{m} \left(\Gamma_k^i\right)^2 \frac{\|\psi(0)\|_{\mathcal{X}} \|\psi(-r_i)\|_{\mathcal{X}}}{\delta_\psi^2} \|\psi(-r_i)\|_{\mathcal{X}} \\ &\leq \sum_{i=0}^{m} \left(\Gamma_k^i\right)^2 \frac{\delta_\psi \delta_\psi}{\delta_\psi^2} \|(\psi - \varphi)(-r_i)\|_{\mathcal{X}} \quad (\textit{because } \varphi(-r_i) = 0 \text{ for all } i \in \{0, ..., m\}) \\ &\leq \sum_{i=0}^{m} \left(\Gamma_k^i\right)^2 \|\psi - \varphi\|_{\mathcal{C}}. \end{aligned}$$

If not, we can assume that $\delta_\psi \leq \delta_\varphi$. Now, we will show that $\delta_\varphi - \delta_\psi \leq \|\varphi - \psi\|_{\mathcal{C}}$. We have two cases to discuss.
Case 1 : if $\delta_\psi = \|\psi(-r_{j_0})\|_{\mathcal{X}}$ and $\delta_\varphi = \|\varphi(-r_{j_0})\|_{\mathcal{X}}$ for $j_0 \in \{0, ..., m\}$.

$$\begin{aligned}\delta_\varphi - \delta_\psi &= \|\varphi(-r_{j_0})\|_{\mathcal{X}} - \|\psi(-r_{j_0})\|_{\mathcal{X}} \\ &\leq \|\varphi(-r_{j_0}) - \psi(-r_{j_0})\|_{\mathcal{X}} \\ &\leq \|\varphi - \psi\|_{\mathcal{C}}.\end{aligned}$$

Case 2 : if there exists $i \neq j$ with $i, j \in \{0, ..., m\}$ such that $\delta_\psi = \|\psi(-r_j)\|_{\mathcal{X}}$ and $\delta_\varphi = \|\varphi(-r_i)\|_{\mathcal{X}}$. Then, we have $\|\psi(-r_i)\|_{\mathcal{X}} \leq \delta_\psi = \|\psi(-r_j)\|_{\mathcal{X}}$ and then $-\|\psi(-r_j)\|_{\mathcal{X}} \leq -\|\psi(-r_i)\|_{\mathcal{X}}$. Hence

$$\begin{aligned}\|\varphi(-r_i)\|_{\mathcal{X}} - \|\psi(-r_j)\|_{\mathcal{X}} &\leq \|\varphi(-r_i)\|_{\mathcal{X}} - \|\psi(-r_i)\|_{\mathcal{X}} \\ &\leq \|\varphi(-r_i) - \psi(-r_i)\|_{\mathcal{X}} \\ &\leq \|\varphi - \psi\|_{\mathcal{C}}.\end{aligned}$$

Then

$$\begin{aligned}\|F(\psi) - F(\varphi)\|_{\mathcal{X}} &= \left\| \sum_{i=0}^{m} \frac{\langle \psi(0), F_i\psi(-r_i)\rangle}{\delta_\psi^2} F_i\psi(-r_i) - \sum_{i=0}^{m} \frac{\langle \varphi(0), F_i\varphi(-r_i)\rangle}{\delta_\varphi^2} F_i\varphi(-r_i) \right\|_{\mathcal{X}} \\ &= \frac{\left\| \delta_\varphi^2 \sum_{i=0}^{m} \langle \psi(0), F_i\psi(-r_i)\rangle F_i\psi(-r_i) - \delta_\psi^2 \sum_{i=0}^{m} \langle \varphi(0), F_i\varphi(-r_i)\rangle F_i\varphi(-r_i) \right\|_{\mathcal{X}}}{\delta_\psi^2 \delta_\varphi^2} \\ &\leq \sum_{i=0}^{m} \frac{\left\| \delta_\varphi^2 \langle \psi(0), F_i\psi(-r_i)\rangle F_i\psi(-r_i) - \delta_\psi^2 \langle \psi(0), F_i\psi(-r_i)\rangle F_i\psi(-r_i) \right\|_{\mathcal{X}}}{\delta_\psi^2 \delta_\varphi^2} \\ &\quad + \sum_{i=0}^{m} \frac{\left\| \delta_\psi^2 \langle \psi(0), F_i\psi(-r_i)\rangle F_i\psi(-r_i) - \delta_\psi^2 \langle \varphi(0), F_i\varphi(-r_i)\rangle F_i\varphi(-r_i) \right\|_{\mathcal{X}}}{\delta_\psi^2 \delta_\varphi^2} \\ &\leq \frac{\|\psi(0)\|_{\mathcal{X}} \sum_{i=0}^{m} \left(\|F_i\psi(-r_i)\|_{\mathcal{X}}\right)^2}{\delta_\psi^2 \delta_\varphi^2} |\delta_\varphi^2 - \delta_\psi^2| \\ &\quad + \sum_{i=0}^{m} \frac{\|\langle \psi(0), F_i\psi(-r_i)\rangle F_i\psi(-r_i) - \langle \varphi(0), F_i\varphi(-r_i)\rangle F_i\varphi(-r_i)\|_{\mathcal{X}}}{\delta_\varphi^2}.\end{aligned}$$

Since $\delta_\psi \leq \delta_\varphi$, we have

$$\begin{aligned}
\|F(\psi)-F(\varphi)\|_{\mathcal{X}} &\leq \sum_{i=0}^{m}\left(\Gamma_k^i\right)^2\frac{\delta_\varphi\delta_\psi^2}{\delta_\psi^2\delta_\varphi^2}(\delta_\varphi+\delta_\psi)\,|\delta_\varphi-\delta_\psi| \\
&+\sum_{i=0}^{m}\frac{\|\langle\psi(0),F_i\psi(-r_i)\rangle(F_i\psi(-r_i)-F_i\varphi(-r_i))\|_{\mathcal{X}}}{\delta_\varphi^2} \\
&+\sum_{i=0}^{m}\frac{\|\langle\psi(0)-\varphi(0),F_i\psi(-r_i)\rangle F_i\varphi(-r_i)\|_{\mathcal{X}}}{\delta_\varphi^2} \\
&+\sum_{i=0}^{m}\frac{\|\langle\varphi(0),F_i\varphi(-r_i)-F_i\psi(-r_i)\rangle F_i\varphi(-r_i)\|_{\mathcal{X}}}{\delta_\varphi^2} \\
&\leq\ 2\sum_{i=0}^{m}\left(\Gamma_k^i\right)^2\frac{\delta_\varphi}{\delta_\varphi^2}\delta_\varphi\|\varphi-\psi\|_{\mathcal{C}}\ +\ 3\sum_{i=0}^{m}\left(\Gamma_k^i\right)^2\|\varphi-\psi\|_{\mathcal{C}} \\
&\leq\ 5\sum_{i=0}^{m}\left(\Gamma_k^i\right)^2\|\varphi-\psi\|_{\mathcal{C}}.
\end{aligned}$$

Then, the system (0.7) possesses a unique mild solution $w \in C([-r,T_{max}),\mathcal{X})$ which is given by the following variation of constants formula (see [90], p 51)

$$w(t)=\begin{cases} S(t)w(0)+\displaystyle\int_0^t S(t-\sigma)F\left(w_\sigma\right)d\sigma, & t\in[0,T_{\max}) \\ \phi(t), & t\in[-r,0] \end{cases}$$

where $S(t)$ is the semigroup generated by the operator $\mathcal{A}$.
Let $T \in (0,T_{max})$ and consider the function $h:[0,T]\to\mathcal{X}$ defined by

$$h(t)=F\left(w_t\right).$$

It's clear that $h \in C([0,T],\mathcal{X})$, so there exist a sequence $(h_n)_n \subset C^1([0,T];\mathcal{X})$ such that

$$h_n \longrightarrow h \quad \text{uniformly in } C\left([0,T];\mathcal{X}\right). \tag{3.4}$$

The domain of $\mathcal{A}$ dense in $\mathcal{X}$ since $\mathcal{A}$ generates a semigroup of contraction, then there exists a sequence $(\mu_n)\subset D(\mathcal{A})$ such that

$$(\mu_n)_n\ \to\ w(0) \text{ in } \mathcal{X}, \tag{3.5}$$

and consider the function w_n defined by

$$w_n(t)=S(t)\mu_n+\int_0^t S(t-s)h_n(s)ds\ \text{ for all } t\ \in[0,T]. \tag{3.6}$$

Then, $w_n(t)\in D(\mathcal{A})$ and w_n is continuously differentiable in $[0,T]$ (see [77], p.187) and w_n is the unique solution of the system

$$\begin{cases} \dot{w}_n(t)=\mathcal{A}w_n(t)+h_n(t)\ for\ all\ t\ \in\ (0,T), \\ w_n(0)=\mu_n. \end{cases}$$

Since $\mathcal{A}$ is dissipative, *i.e* : $\langle A\psi, \psi\rangle \leq 0$ for all $\psi \in \mathcal{D}(\mathcal{A})$, we obtain that

$$\frac{d}{dt}\|w_n(t)\|_{\mathcal{X}}^2 \leq 2\langle h_n(t), w_n(t)\rangle \quad \text{for all } t \in (0,T). \tag{3.7}$$

Integrating the inequality (3.7) from τ to t, we derive

$$\|w_n(t)\|_{\mathcal{X}}^2 - \|w_n(\tau)\|_{\mathcal{X}}^2 \leq 2\int_\tau^t \langle h_n(s), w_n(s)\rangle\, ds \quad \textit{for all } 0 \leq \tau \leq t \leq T. \tag{3.8}$$

Second, we have to show that $w_n \to w$ uniformly in $C([0,T];\mathcal{X})$. Since $S(t)$ is a semigroup of contraction, using equation (3.6) and the variation of constants formula, we get that

$$\|w_n(t) - w(t)\|_{\mathcal{X}} \leq \|\mu_n - w(0)\|_{\mathcal{X}} + T\sup_{s\in[0,T]}\|h_n(s) - h(s)\|_{\mathcal{X}}.$$

Using (3.4), (3.5) and the last inequality, we get that

$$\|w_n(t) - w(t)\|_{\mathcal{X}} \to 0 \text{ as } n \to +\infty.$$

By passing to the limit in the inequality (3.8), we get via the dominated convergence theorem that

$$\|w(\tau)\|_{\mathcal{X}}^2 - \|w(t)\|_{\mathcal{X}}^2 \geq 2\sum_{i=0}^{m}\int_\tau^t \frac{\langle w(s), F_i w(s-r_i)\rangle^2}{\max\limits_{0\leq j\leq m}\|w(s-r_j)\|_{\mathcal{X}}^2}\mathbf{1}_E ds \quad \text{for } 0 \leq \tau \leq t \leq T,$$

where $E = \{\, t \geq 0 : w(t) \neq 0\,\}$. In particular, we get

$$\|w(t)\|_{\mathcal{X}} \leq \|w(0)\|_{\mathcal{X}}, \quad 0 \leq t \leq T_{max}. \tag{3.9}$$

Using the fact that $t \to w(t)$ is continuous in $[-r, 0]$, then, there exists $K > 0$, such that

$$\|w(t)\|_{\mathcal{X}} \leq K, \quad -r \leq t \leq 0. \tag{3.10}$$

From (3.9) and (3.10), we get

$$\|w(t)\|_{\mathcal{X}} \leq K, \quad -r \leq t \leq T_{max}. \tag{3.11}$$

Therefore, by the blow-up phenomena, $w(t)$ is global solution of system (0.7) (see [90]) ■

3.3 Exponential stabilization

In this section, we study the exponential stability of the system (0.7). We have the following result.

Theorem 3.3.1. *Let $\mathcal{A}$ generates a semigroup of contraction* S(t) *on $\mathcal{X}$. Suppose that for $i \in \{0, ..., m\}$, F_i is a locally Lipschitz function from $\mathcal{X}$ into $\mathcal{X}$ such that*

$$\sum_{i=0}^{i=m} \int_r^T |\langle F_i S(t)\xi, S(t+r_i)\xi\rangle| dt \geq \delta \|\xi\|_{\mathcal{X}}^2 \text{ for all } \xi \in \mathcal{X}\ , \tag{3.12}$$

for some $T > r$ and $\delta > 0$. Then, the sequence of feedback control (3.2) exponentially stabilizes the system (0.7).

Proof. Using Theorem 3.2.1, the system (0.7) possesses a unique global mild solution $w(t)$ defined on $[-r, +\infty[$. Let w be the function defined, for $t \geq 0$ and $\sigma \in [0, r+T]$, by

$$z(\sigma, t) = w(\sigma + t) - S(\sigma)w(t).$$

Then, for all $t \geq 0$ and $\sigma \in [0, r+T]$, we have that

$$z(\sigma, t) = -\int_t^{\sigma+t} S(\sigma + t - s) \sum_{i=0}^{m} \frac{\langle F_i w(s-r_i), w(s)\rangle}{\max\limits_{0\leq j\leq m} \|w(s-r_j)\|_{\mathcal{X}}^2} \mathbf{1}_E F_i w(s-r_i) ds,$$

the semigroup $S(t)$ is of contraction, then by Schwartz's inequality, we get for all $\sigma \in [0, r+T]$, that

$$\begin{aligned}
\|z(\sigma,t)\| &\leq \int_t^{t+r+T} \sum_{i=0}^{m} \frac{|\langle F_i w(s-r_i), w(s)\rangle|}{\max\limits_{0\leq j\leq m} \|w(s-r_j)\|_{\mathcal{X}}} \mathbf{1}_E \Gamma_K^i ds \\
&\leq \int_t^{t+r+T} \left(\sum_{i=0}^{m} \left(\Gamma_K^i\right)^2\right)^{\frac{1}{2}} \left(\sum_{i=0}^{m} \frac{|\langle F_i w(s-r_i), w(s)\rangle|^2}{\max\limits_{0\leq j\leq m} \|w(s-r_j)\|_{\mathcal{X}}^2} \mathbf{1}_E\right)^{\frac{1}{2}} ds \\
&\leq \left(\sum_{i=0}^{m} \left(\Gamma_K^i\right)^2\right)^{\frac{1}{2}} \int_t^{t+r+T} \left(\sum_{i=0}^{m} \frac{|\langle F_i w(s-r_i), w(s)\rangle|^2}{\max\limits_{0\leq j\leq m} \|w(s-r_j)\|_{\mathcal{X}}^2} \mathbf{1}_E\right)^{\frac{1}{2}} ds,
\end{aligned}$$

using for a second time the Schwartz's inequality, we obtain that

$$\|z(\sigma,t)\| \leq \sqrt{T+r} \left(\sum_{i=0}^{m} \left(\Gamma_K^i\right)^2\right)^{\frac{1}{2}} \left(\sum_{i=0}^{m} \int_t^{t+r+T} \frac{|\langle F_i w(s-r_i), w(s)\rangle|^2}{\max\limits_{0\leq j\leq m} \|w(s-r_j)\|_{\mathcal{X}}^2} \mathbf{1}_E ds\right)^{\frac{1}{2}}. \tag{3.13}$$

For $i \in \{0, ..., m\}$ and $s \in [0, T]$, we have that

$$\begin{aligned}
\langle S(s+r_i)w(t), F_i S(s)w(t)\rangle &= \langle w(s+r_i+t), F_i S(s)w(t) - F_i w(s+t)\rangle \\
&\quad - \langle w(s+r_i+t) - S(s+r_i)w(t), F_i S(s)w(t)\rangle \\
&\quad + \langle w(s+r_i+t), F_i w(s+t)\rangle.
\end{aligned}$$

By using the fact that F_i is a locally Lipschitz function for all $i \in \{0,...,m\}$ and the function $t \to \|w(t)\|$ is decreasing on $\mathbb{R}^+$, we deduce that

$$|\langle S(s+r_i)w(t), F_iS(s)w(t)\rangle| \leq \Gamma_K^i\|w(t)\|_{\mathcal{X}}\|z(s,t)\|_{\mathcal{X}} + \Gamma_K^i\|w(t)\|_{\mathcal{X}}\|z(s+r_i,t)\|_{\mathcal{X}} + |\langle w(s+r_i+t), F_iw(s+t)\rangle|,$$

and hence

$$\sum_{i=0}^{m}|\langle S(s+r_i)w(t), F_iS(s)w(t)\rangle| \leq \left(\sum_{i=0}^{m}\Gamma_K^i\right)\|w(t)\|_{\mathcal{X}}\|z(s,t)\|_{\mathcal{X}} + \left(\sum_{i=0}^{m}\Gamma_K^i\|w(t)\|_{\mathcal{X}}\|z(s+r_i,t)\|_{\mathcal{X}}\right) + \sum_{i=0}^{m}|\langle w(s+r_i+t), F_iw(s+t)\rangle|.$$

Then, from inequality (3.13), we get

$$\begin{aligned}&\sum_{i=0}^{m}|\langle S(s+r_i)w(t), F_iS(s)w(t)\rangle| \\ &\leq 2\sqrt{T+r}\left(\sum_{i=0}^{m}\Gamma_K^i\right)\left(\sum_{i=0}^{m}\left(\Gamma_K^i\right)^2\right)^{\frac{1}{2}}\|w(t)\|_{\mathcal{X}}\left(\sum_{i=0}^{m}\int_t^{t+r+T}\frac{|\langle F_iw(s-r_i), w(s)\rangle|^2}{\max\limits_{0\leq j\leq m}\|w(s-r_j)\|_{\mathcal{X}}^2}\mathbf{1}_E ds\right)^{\frac{1}{2}} \\ &\quad + \sum_{i=0}^{m}|\langle w(s+r_i+t), F_iw(s+t)\rangle|.\end{aligned}$$

Integrating the last inequality over the interval $[r,T]$ we get

$$\begin{aligned}&\sum_{i=0}^{m}\int_r^T|\langle S(s+r_i)w(t), F_iS(s)w(t)\rangle|ds \\ &\leq 2(T-r)\sqrt{T+r}\left(\sum_{i=0}^{m}\Gamma_K^i\right)\left(\sum_{i=0}^{m}\left(\Gamma_K^i\right)^2\right)^{\frac{1}{2}}\|w(t)\|_{\mathcal{X}}\left(\sum_{i=0}^{m}\int_t^{t+r+T}\frac{|\langle F_iw(s-r_i), w(s)\rangle|^2}{\max\limits_{0\leq j\leq m}\|w(s-r_j)\|_{\mathcal{X}}^2}\mathbf{1}_E ds\right)^{\frac{1}{2}} \\ &\quad + \sum_{i=0}^{m}\int_r^T|\langle w(s+t+r_i), F_iw(s+t)\rangle|ds.\end{aligned} \tag{3.14}$$

From the inequality (3.3), we can see that

$$\max_{0\leq j\leq m}\|w(s-r_j)\|_{\mathcal{X}} \leq \|w(t)\|_{\mathcal{X}} \text{ for all } s \in [t+r, t+r+T], \tag{3.15}$$

and by using the variable change $\sigma = s+t+r_i$, we get

$$\begin{aligned}\sum_{i=0}^{m}\int_r^T|\langle w(s+t+r_i), F_iw(s+t)\rangle|ds &= \sum_{i=0}^{m}\int_{t+r_i+r}^{t+r_i+T}|\langle w(\sigma), F_iw(\sigma-r_i)\rangle|d\sigma \\ &\leq \sum_{i=0}^{m}\int_{t+r}^{t+r+T}|\langle w(\sigma), F_iw(\sigma-r_i)\rangle|d\sigma.\end{aligned}$$

Using the Schwartz's inequality, we obtain that

$$\sum_{i=0}^{m}\int_r^T |\langle w(s+t+r_i), F_i w(s+t)\rangle| ds$$
$$\leq \int_{t+r}^{t+r+T}\left(\sum_{i=0}^{m}\frac{|\langle w(\sigma), F_i w(\sigma-r_i)\rangle|^2}{\max\limits_{0\leq j\leq m}\|w(\sigma-r_j)\|_{\mathcal{X}}^2}\mathbf{1}_E\right)^{\frac{1}{2}}\left(\sum_{i=0}^{m}\max_{0\leq j\leq m}\|w(\sigma-r_j)\|_{\mathcal{X}}^2\right)^{\frac{1}{2}} d\sigma. \tag{3.16}$$

From the inequality (3.15) and by using for a second time the Schwartz's inequality, we get

$$\sum_{i=0}^{m}\int_r^T |\langle w(s+t+r_i), F_i w(s+t)\rangle| ds$$
$$\leq \sqrt{m+1}\|w(t)\|_{\mathcal{X}}\int_{t+r}^{t+r+T}\left(\sum_{i=0}^{m}\frac{|\langle w(\sigma), F_i w(\sigma-r_i)\rangle|^2}{\max\limits_{0\leq j\leq m}\|w(\sigma-r_j)\|_{\mathcal{X}}^2}\mathbf{1}_E\right)^{\frac{1}{2}} d\sigma$$
$$\leq \sqrt{m+1}\sqrt{T}\|w(t)\|_{\mathcal{X}}\left(\sum_{i=0}^{m}\int_t^{t+r+T}\frac{|\langle w(\sigma), F_i w(\sigma-r_i)\rangle|^2}{\max\limits_{0\leq j\leq m}\|w(\sigma-r_j)\|_{\mathcal{X}}^2}\mathbf{1}_E d\sigma\right)^{\frac{1}{2}}. \tag{3.17}$$

Combining inequalities (3.14) and (3.17), we obtain the following inequality

$$\sum_{i=0}^{m}\int_r^T |\langle S(s+r_i)w(t), F_i S(s)w(t)\rangle| ds \leq \widetilde{C}\|w(t)\|_{\mathcal{X}}\left(\sum_{i=0}^{m}\int_t^{t+r+T}\frac{|\langle F_i w(s-r_i), w(s)\rangle|^2}{\max\limits_{0\leq j\leq m}\|w(s-r_j)\|_{\mathcal{X}}^2}\mathbf{1}_E ds\right)^{\frac{1}{2}}, \tag{3.18}$$

with $\widetilde{C} = 2(T-r)\sqrt{T+r}\left(\sum\limits_{i=0}^{m}\Gamma_K^i\right)\left(\sum\limits_{i=0}^{m}\left(\Gamma_K^i\right)^2\right)^{\frac{1}{2}} + \sqrt{m+1}\sqrt{T}$. Using the observability condition (3.12) with $\xi = w(j(r+T))$, we get

$$\delta\|w(j(r+T))\|_{\mathcal{X}}^2 \leq \sum_{i=0}^{m}\int_r^T |\langle S(s+r_i)w(j(r+T)), F_i S(s)w(j(r+T))\rangle| ds$$
$$\leq \widetilde{C}\|w(j(r+T))\|_{\mathcal{X}}\left(\sum_{i=0}^{m}\int_{j(r+T)}^{j(r+T)+r+T}\frac{|\langle F_i w(s-r_i), w(s)\rangle|^2}{\max\limits_{0\leq j\leq m}\|w(s-r_j)\|_{\mathcal{X}}^2}\mathbf{1}_E ds\right)^{\frac{1}{2}}.$$

Thus

$$\frac{\delta^2}{\widetilde{C}^2}\|w(j(r+T))\|_{\mathcal{X}}^2 \leq \sum_{i=0}^{m}\int_{j(r+T)}^{(j+1)(r+T)}\frac{|\langle F_i w(s-r_i), w(s)\rangle|^2}{\max\limits_{0\leq j\leq m}\|w(s-r_j)\|_{\mathcal{X}}^2}\mathbf{1}_E ds.$$

Using inequality (3.3), we obtain that

$$\|w((j+1)(r+T))\|_{\mathcal{X}}^2 - \|w(j(r+T))\|_{\mathcal{X}}^2 \leq -\frac{2\delta^2}{\widetilde{C}^2}\|w(j(r+T))\|_{\mathcal{X}}^2.$$

Then

$$\|w((j+1)(r+T))\|_{\mathcal{X}}^2 + \frac{2\delta^2}{\widetilde{C}^2}\|w(j(r+T))\|_{\mathcal{X}}^2 \leq \|w(j(r+T))\|_{\mathcal{X}}^2.$$

Observing that $\|w((j+1)(r+T)))\|_{\mathcal{X}}^2 \leq \|w(j(r+T)))\|_{\mathcal{X}}^2$, we derive that

$$\|w((j+1)(r+T))\|_{\mathcal{X}}^2 + \frac{2\delta^2}{\widetilde{C}^2}\|w((j+1)(r+T))\|_{\mathcal{X}}^2 \leq \|w(j(r+T))\|_{\mathcal{X}}^2,$$

which means that for all $j \in \mathbb{N}$, we have that

$$\|w((j+1)(r+T))\|_{\mathcal{X}}^2 \leq \frac{1}{1+\frac{2\delta^2}{\widetilde{C}^2}}\|w(j(r+T))\|_{\mathcal{X}}^2,$$

and then

$$\|w(j(r+T))\|_{\mathcal{X}}^2 \leq \frac{1}{\left(1+\frac{2\delta^2}{\widetilde{C}^2}\right)^j}\|w(0)\|_{\mathcal{X}}^2.$$

Let us now set $j = E(\frac{t}{r+T})$ where $E(.)$ designs the integer part. Thus

$$\frac{1}{\left(1+\frac{2\delta^2}{\widetilde{C}^2}\right)^j} \leq \frac{1}{\left(1+\frac{2\delta^2}{\widetilde{C}^2}\right)^{\frac{t}{r+T}-1}}.$$

Since $t \to \|w(t)\|_{\mathcal{X}}$ is a decreasing function, then

$$\|w(t)\|_{\mathcal{X}} \leq \|w(j(r+T))\|_{\mathcal{X}} \leq \sqrt{1+\frac{2\delta^2}{\widetilde{C}^2}}\left(1+\frac{2\delta^2}{\widetilde{C}^2}\right)^{-\frac{t}{2(r+T)}}\|w(0)\|_{\mathcal{X}},$$

and hence, we obtain the following exponential decay estimates

$$\|w(t)\|_{\mathcal{X}} \leq \sqrt{1+\frac{2\delta^2}{\widetilde{C}^2}}\exp\left(\frac{-\ln(1+\frac{2\delta^2}{\widetilde{C}^2})t}{2(r+T)}\right)\|\phi\|_{\mathcal{C}} \quad \text{for all } t \geq 0 \ \blacksquare$$

Remark 3.3.2. *If there exist $i \in \{0,...,m\}$, $T_1 > r$ and $\delta_1 > 0$ such that*

$$\int_r^{T_1} |\langle F_i S(t)\xi, S(t+r_i)\xi\rangle| dt \geq \delta_1 \|\xi\|_{\mathcal{X}}^2 \text{ for all } \xi \in \mathcal{X}\ .$$

then, the observability condition (3.12) holds.

3.4 Weak stabilization

This section is reserved to discuss the weak stabilization of the system (0.7). We give a weaker observability condition than the one given by (3.12), which ensures the weak stabilization of the system (0.7).

Theorem 3.4.1. *Let $\mathcal{A}$ generate a semigroup of contraction $S(t)$. Suppose that for all $i \in \{0,...,m\}$, F_i is a locally Lipschitz function from $\mathcal{X}$ into $\mathcal{X}$. If there exist $j \in \{0,...,m\}$ and $T > r$ such that for all sequence $(\xi_n)_n \subset \mathcal{X}$ with $\xi_n \rightharpoonup \xi$ in $\mathcal{X}$, we have*

$$\langle S(t+r_j)\xi_n, F_jS(t)\xi_n\rangle \to 0 \text{ as } n \to +\infty \text{ (for all } t \in [r,T]) \quad \Rightarrow \xi = 0. \tag{3.19}$$

Then, the system (0.7) is weakly stabilizable by the sequence of feedback control (3.2).

Proof. From inequality (3.11), we have that $\|w(t)\|$ is bounded in the reflexive space $\mathcal{X}$, then there exists $t_n \to +\infty$ as $n \to +\infty$ and $\psi \in \mathcal{X}$ such that:

$$w(t_n) \rightharpoonup \psi \quad \text{as } n \to +\infty.$$

From inequality (3.18), we have that for $j \in \{0,...,m-1\}$

$$\begin{aligned}\int_r^T |\langle S(s+r_j)w(t), F_jS(s)w(t)\rangle| ds &\leq \sum_{i=0}^m \int_r^T |\langle S(s+r_i)w(t), F_iS(s)w(t)\rangle| ds \\ &\leq \widetilde{C}K\left(\sum_{i=0}^m \int_t^{t+r+T} \frac{|\langle F_iw(s-r_i), w(s)\rangle|^2}{\max\limits_{0\leq j\leq m}\|w(s-r_j)\|_{\mathcal{X}}^2}\mathbf{1}_E ds\right)^{\frac{1}{2}},\end{aligned}$$

such that $\widetilde{C} = 2(T-r)\sqrt{T+r}\left(\sum_{i=0}^m \Gamma_K^i\right)\left(\sum_{i=0}^m \left(\Gamma_K^i\right)^2\right)^{\frac{1}{2}} + \sqrt{m+1}\sqrt{T}$ with K is the constant given in (3.11). Then

$$\int_r^T |\langle S(s+r_j)w(t_n), F_jS(s)w(t_n)\rangle| ds \leq \widetilde{C}K\left(\sum_{i=0}^m \int_{t_n}^{t_n+r+T} \frac{|\langle F_iw(s-r_i), w(s)\rangle|^2}{\max\limits_{0\leq j\leq m}\|w(s-r_j)\|_{\mathcal{X}}^2}\mathbf{1}_E ds\right)^{\frac{1}{2}},$$

Using inequality (3.3), we have that

$$\sum_{i=0}^m \int_{t_n}^{t_n+r+T} \frac{|\langle F_iw(s-r_i), w(s)\rangle|^2}{\max\limits_{0\leq j\leq m}\|w(s-r_j)\|_{\mathcal{X}}^2}\mathbf{1}_E ds \leq \frac{1}{2}\left(\|w(t_n)\|_{\mathcal{X}}^2 - \|w(t_n+r+T)\|_{\mathcal{X}}^2\right).$$

and the function $t \to (\|w(t)\|_{\mathcal{X}})$ is decreasing. Then, $(\|w(t_n)\|_{\mathcal{X}})_n$ converge and we have that

$$\int_r^T |\langle S(s+r_j)w(t_n), F_jS(s)w(t_n)\rangle| ds \to 0 \quad \text{as } n \to +\infty.$$

The sequence $(|\langle S(s+r_j)w(t_n), F_jS(s)w(t_n)\rangle|)_{n\in\mathbb{N}}$ is bounded in $\mathbb{R}$, then, there exists a subsequence of $(|\langle S(s+r_j)w(t_n), F_jS(s)w(t_n)\rangle|)_{n\in\mathbb{N}}$, denoted also by $(|\langle S(s+r_j)w(t_n), F_jS(s)w(t_n)\rangle|)_{n\in\mathbb{N}}$ which is convergent. According to the dominated convergence theorem, we get that

$$\int_r^T \lim_{n\to+\infty} |\langle S(s+r_j)w(t_n), F_jS(s)w(t_n)\rangle| ds = 0.$$

Then

$$\lim_{n\to+\infty} |\langle S(s+r_j)w(t_n), F_jS(s)w(t_n)\rangle| = 0 \quad \text{for all } s \in [r,T].$$

and hence condition (3.19) implies that $\psi = 0$. Then, $w(t)$ tends to 0 weakly as $t \to +\infty$. ■

3.5 Application and Numerical Simulation

To show the applicability of our theoretical results, we consider the following two examples.
Example 1: The heat equation.
Consider, now, the following bilinear heat equation in one dimension with Neumann boundary conditions:

$$\begin{cases} \dfrac{\partial w(x,t)}{\partial t} = \dfrac{\partial^2 w(x,t)}{\partial x^2} + v_0(t)F_0w(x,t) + v_1(t)F_1w(x,t-r) \text{ for } x \in (0,1),\ t \geq 0,\ r \geq 0 \\ \dfrac{\partial w(0,t)}{\partial t} = \dfrac{\partial w(1,t)}{\partial t} = 0 \qquad \text{for } t \in [-r,+\infty), \\ w(x,t) = \phi(x,t), \qquad \text{for } x \in (0,1),\ t \in [-r,0], \end{cases} \tag{3.20}$$

Let $\mathcal{X} = L^2(0,1)$ be the state space and define the operator $\mathcal{A}$ from $\mathcal{X}$ to $\mathcal{X}$ by :

$$\mathcal{A}w = \frac{\partial^2 w(\xi,t)}{\partial \xi^2} \text{ with } D(\mathcal{A}) = \{\, w \in \mathcal{X}^2(0,1) \,:\, \frac{\partial w(0,t)}{\partial t} = \frac{\partial w(1,t)}{\partial t} = 0 \,\}.$$

The eigenfunctions of $\mathcal{A}$ are defined as follows :

$$\varphi_m(x) = \begin{cases} \sqrt{2}\cos((m-1)\pi x) & \text{if } m \geq 2 \\ 1 & \text{if } m = 1 \end{cases}$$

Moreover, $(\varphi_m)_{m\geq 1}$ is a complete orthonormal eigenfunctions system of $\mathcal{A}$ associated to the eigenvalues $\lambda_m = -\pi^2(m-1)^2$ for $m \geq 1$. Moreover, the operator F_0 and F_1 defined by

$$F_0w = \sum_{m=0}^{+\infty} \frac{1}{m^2}\langle w, \varphi_m\rangle\varphi_m \text{ and } F_1 = I.$$

Then

$$\langle F_0 S(t)w, S(t)w\rangle \;=\; \sum_{m=0}^{+\infty} \frac{e^{2\lambda_m t}}{m^2} |\langle w, \varphi_m\rangle|^2.$$

Then, condition (3.19) of Theorem 3.4.1 is verified, which implies that the system (3.20) is weakly stabilizable using the following sequence of feedback control

$$v_i(t) = \begin{cases} -\dfrac{\langle w(t), F_i w(t-r_i)\rangle}{\max\limits_{0\le j\le 1} \|w(t-r_j)\|^2}, & \text{if } w(t) \neq 0, \\ 0, & \text{else,} \end{cases}$$

for $i \in \{0, 1\}$.

Example 2 Let $\Omega = (0, 1)$ and consider the following system

$$\begin{cases} \dfrac{\partial^2 w(x,t)}{\partial t^2} = \dfrac{\partial^2 w(x,t)}{\partial x^2} + v_1(t)h_1(x)\dfrac{\partial w(x,t-1)}{\partial t} + v_2(t)h_2(x)\dfrac{\partial w(x,t-2)}{\partial t} + \\ \qquad\qquad v_3(t)h_3(x)\dfrac{\partial w(x,t-3)}{\partial t} & \text{for } x \in \Omega,\ t \ge 0, \\ w(0,t) = w(1,t) = 0 & \text{for } t \in [-3, +\infty), \\ w(x,t) = \phi_0(x,t),\ \dfrac{\partial w(x,t)}{\partial t} = \phi_1(x,t) & \text{for } x \in \Omega,\ t \in [-3, 0] \end{cases} \tag{3.21}$$

where $h_i \in L^\infty(\Omega)$ for all $i \in \{1, 2, 3\}$ such that $h_3(\xi) \ge c$, for some positive constant c. Consider the space $\mathcal{X} = H_0^1(\Omega) \times L^2(\Omega)$, with the inner produt

$$\langle (w_1, w_2), (y_1, y_2)\rangle = \langle w_1, y_1\rangle_{H_0^1(0,1)} + \langle w_2, y_2\rangle_{L^2(0,1)},$$

and define the operator $\mathcal{A} : D(\mathcal{A}) \subset \mathcal{X} \to \mathcal{X}$ by

$$\mathcal{A} = \begin{pmatrix} 0 & I \\ \Delta & 0 \end{pmatrix} \; with\; D(\mathcal{A}) = \left(H^2(0,1) \cap H_0^1(0,1)\right) \times H_0^1(0,1),$$

and

$$F_i = \begin{pmatrix} 0 & 0 \\ 0 & h_i(.) \end{pmatrix} \text{ for } i \in \{1, 2, 3\}.$$

By considering those operations, system (3.21) takes the abstract for (0.7) with

$$y = \begin{pmatrix} w \\ \frac{\partial w}{\partial t} \end{pmatrix}.$$

The eigenfunctions of the operator $\Delta = \dfrac{\partial^2}{\partial x^2}$ with $D(\Delta) = \{\, x \in H^2(0,1) \;:\; w(0,t) = w(1,t) = 0 \,\}$ are defined as follows :

$$\psi_m(\xi) = \sqrt{2}\sin(m\pi x), \text{ for all } m \ge 1.$$

Moreover, $(\psi_m)_{m\geq 1}$ is a complete orthonormal eigenfunctions system of Δ associated to the eigenvalues $\lambda_m = -(m\pi)^2, (m \in \mathbb{N}^*)$.

Let $y = (y_1, y_2) \in \mathcal{X}$, then $y_1 = \sum_{m=1}^{\infty} \alpha_m \psi_m$ and $y_2 = \sum_{m=1}^{\infty} (-\lambda_m)^{\frac{1}{2}} \gamma_m \psi_m$, where α_m and γ_m are real numbers for all $m \geq 1$.

From [31], the skew adjoint operator $\mathcal{A}$ generates a semigroupe of isometries $S(t)$ and using the separation of variables, we obtain

$$S(t)y = \sum_{m=1}^{+\infty} \begin{pmatrix} \alpha_m \cos(\sqrt{-\lambda_m}t) + \gamma_m \sin(\sqrt{-\lambda_m}t) \\ -\alpha_m\sqrt{-\lambda_m} \sin(\sqrt{-\lambda_m}t) + \gamma_m\sqrt{-\lambda_m} \cos(\sqrt{-\lambda_m}t) \end{pmatrix} \psi_m,$$

It follows that

$$S(t+3)y = \sum_{m=1}^{+\infty} \begin{pmatrix} \alpha_m \cos(\sqrt{-\lambda_m}(t+3)) + \gamma_m \sin(\sqrt{-\lambda_m}(t+3)) \\ -\alpha_m\sqrt{-\lambda_m} \sin(\sqrt{-\lambda_m}(t+3)) + \gamma_m\sqrt{-\lambda_m} \cos(\sqrt{-\lambda_m}(t+3)) \end{pmatrix} \psi_m.$$

Moreover, F_i is locally Lipschitz function for $i \in \{1, 2, 3\}$. In fact, for all $w = (w_1, w_2) \in \mathcal{X}$ and $y = (y_1, y_2) \in \mathcal{X}$

$$\begin{aligned} \|F_i w - F_i y\| &= \|h_i(.)w_2 - h_i(.)y_2\| \\ &\leq \|h_i\|_{L^\infty(\Omega)} \|w - y\|. \end{aligned}$$

Let r_i be such that $r_i = i$ for $i \in \{0, 1, 2, 3\}$. Then, for $t \geq 0$

$$\begin{aligned} \langle F_3 S(t)y, S(t+r_3)y\rangle \geq c \sum_{m=1}^{+\infty} & \lambda_m(\alpha_m^2 \sin(\sqrt{-\lambda_m}t) \sin(\sqrt{-\lambda_m}(t+3)) \\ & - \alpha_m\gamma_m \sin(\sqrt{-\lambda_m}t) \cos(\sqrt{-\lambda_m}(t+3)) \\ & - \alpha_m\gamma_m \sin(\sqrt{-\lambda_m}(t+3)) \cos(\sqrt{-\lambda_m}t) \\ & + \gamma_m^2 \cos(\sqrt{-\lambda_m}t) \cos(\sqrt{-\lambda_m}(t+3))). \end{aligned} \tag{3.22}$$

Integrating the inequality (3.22) from 3 to 6, we have

$$\int_3^6 |\langle F_3 S(t)y, S(t+r_3)y\rangle| dt \geq \frac{3c}{2} \sum_{m=1}^{+\infty} -\lambda_m(\alpha_m^2 + \gamma_m^2) \geq \frac{3c}{2} \|y\|_{\mathcal{X}}^2.$$

Thus, (3.12) holds, by applying Theorem 3.3.1, we conclude that the sequence of feedback control

$$v_i(t) = \begin{cases} -\dfrac{\langle y(t), F_i y(t-r_i)\rangle}{\max\limits_{0\leq j\leq 3} \|y(t-r_j)\|_{\mathcal{X}}^2}, & \text{if } y(t) \neq 0, \\ 0, & \text{else,} \end{cases} \tag{3.23}$$

for $i \in \{1, 2, 3\}$ exponentially stabilizes the system (3.21).

Now, we give numerical simulations of the system (3.21) with $h_1(x) = e^{-x}$, $h_2(x) = \sin(x)$, $h_3(x) = 1$ and $\phi_0(x,t) = \sin(\frac{\pi}{2}t)\sin(\pi x)$. Then, we get the results shown in figures 3.1, 3.2, 3.3, 3.4, 3.5, 3.6 and 3.7.

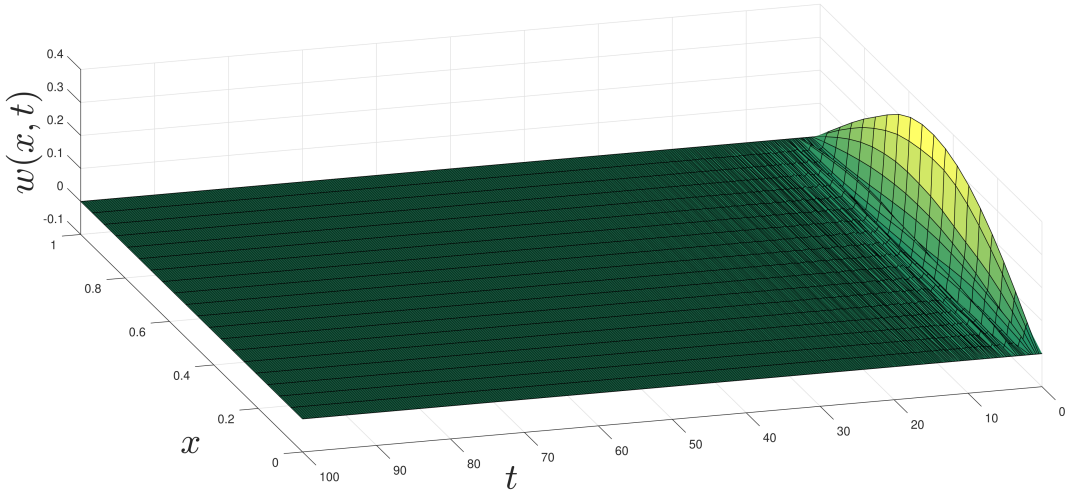

Figure 3.1: The evolution of $w(x,t)$ by the sequence of feedback control (3.23).

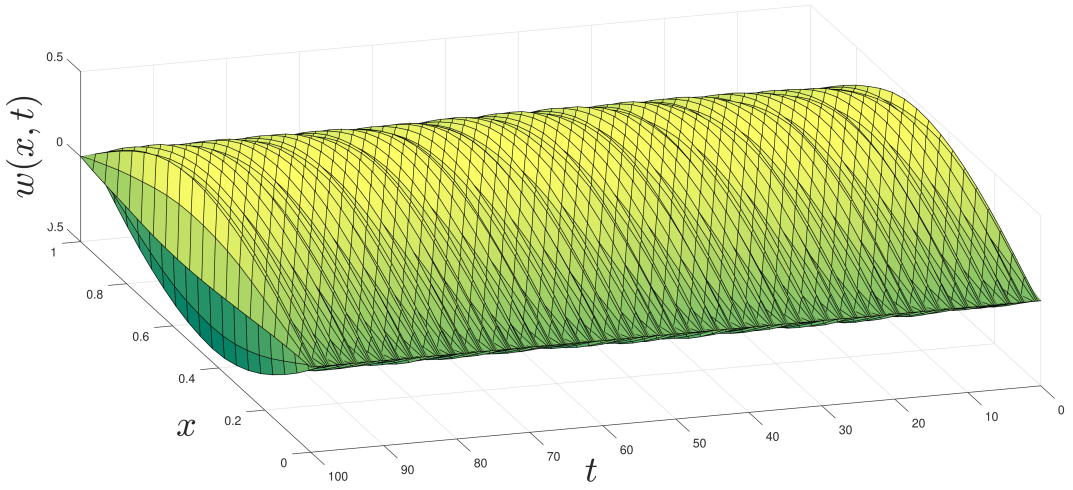

Figure 3.2: The evolution of $w(x,t)$ ($v_i(t) = 0$ for $i \in \{1, 2, 3\}$).

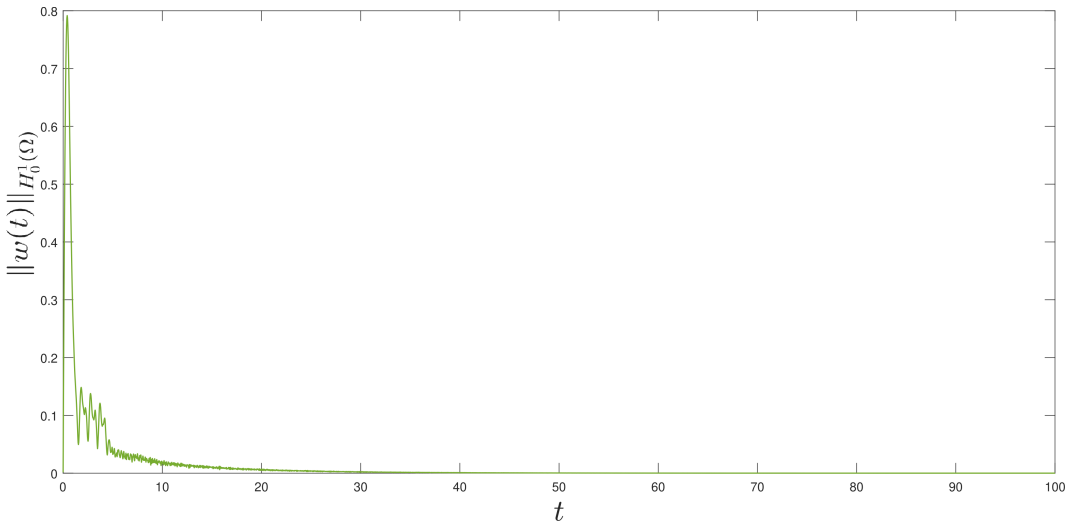

Figure 3.3: The evolution of the stabilized state norm $\|w(t)\|_{H_0^1(\Omega)}$.

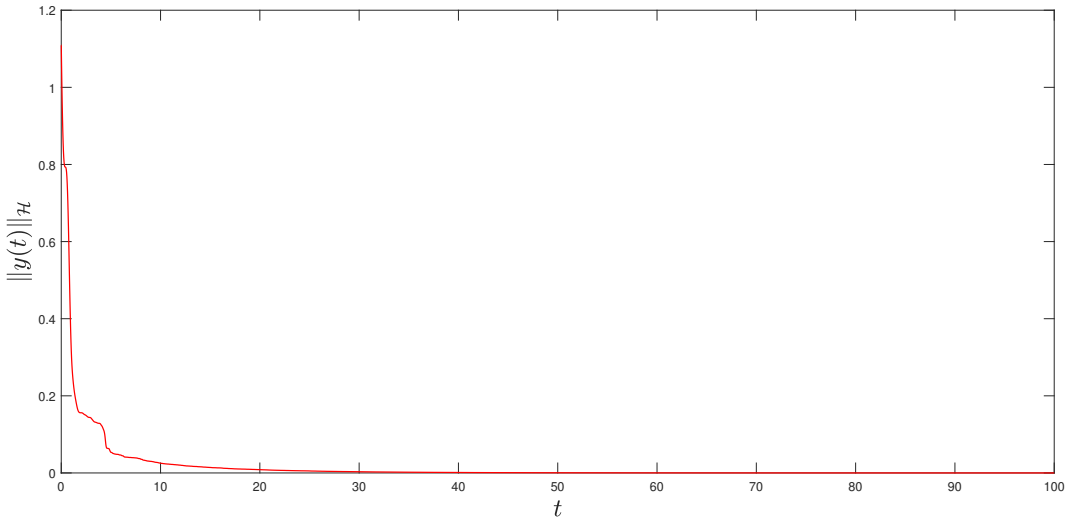

Figure 3.4: The evolution of the stabilized state norm $\|y(t)\|_{\mathcal{X}}$.

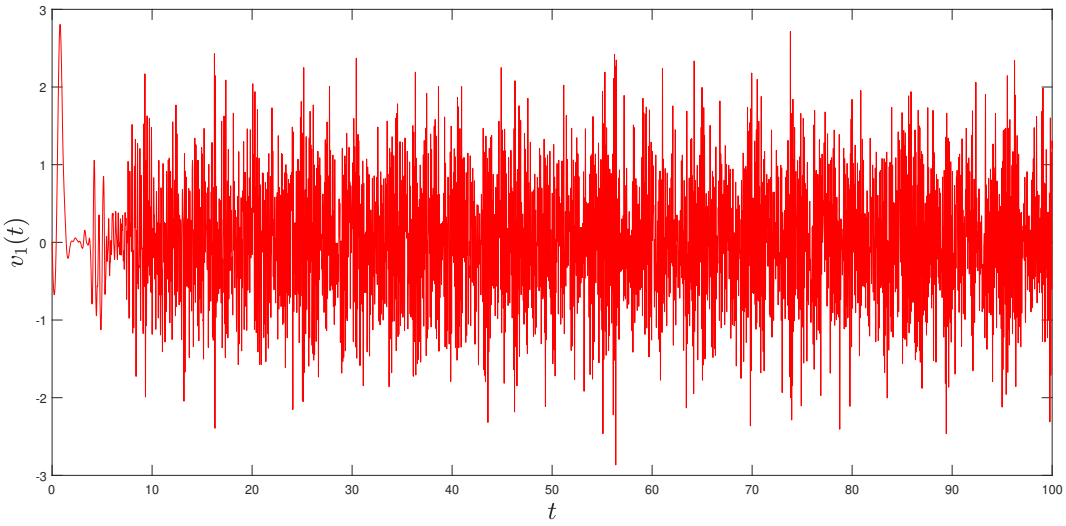

Figure 3.5: The evolution of the control $v_1(t)$.

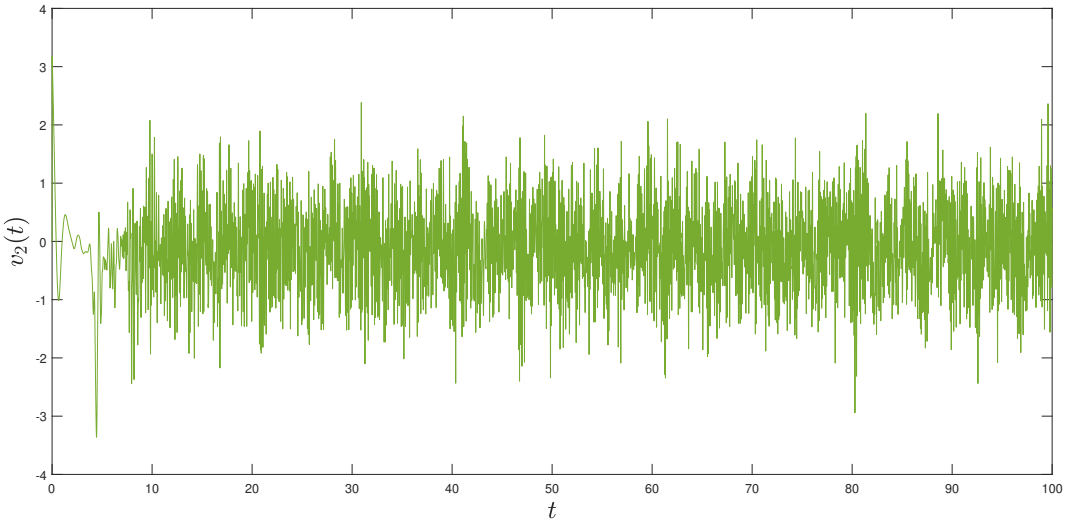

Figure 3.6: The evolution of the control $v_2(t)$.

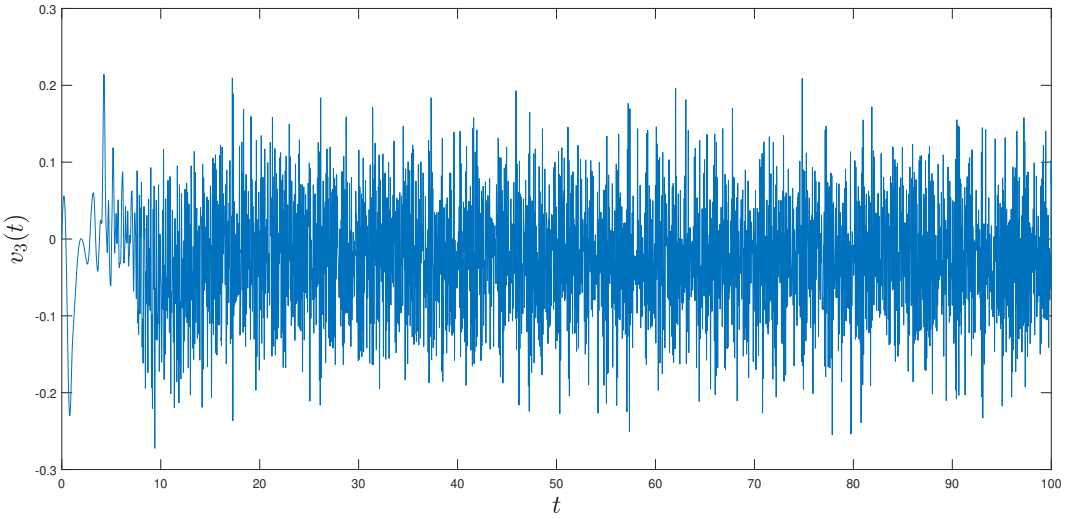

Figure 3.7: The evolution of the control $v_3(t)$.

Conclusion. This chapter is concerned with the feedback stabilization of infinite-dimensional semi-linear systems with discrete multi-delays. We proved that the sequence of feedback control (3.2) stabilizes exponentially and weakly the semi-linear systems (0.7). Moreover, in the exponential stabilization case, we give another observability condition that implies the one presented in Theorem 3.3.1. This observability condition is more simple and very easy to verify in practice.

Chapter 4

Feedback stabilization of semilinear system with distributed delay

4.1 Introduction

The present chapter is concerned with the feedback stabilization of the semilinear system (0.8) with distributed delay. We consider an observation condition in terms of the semigroup solution of system (0.8) with $F = 0$ and parameterized by a positive constant γ. More precisely, we consider the following observation condition for some constants T, $\delta > 0$ and $\gamma \geq 2$ such that

$$\int_0^T |\langle F(U(s)\xi), (U(s)\xi)(0)\rangle| ds \geq \delta \|\xi(0)\|_{\mathcal{X}}^{\gamma} \quad \text{for all } \xi \in \mathcal{C}.$$

where $(U(t))_{t>0}$ is the semigroup solution of system (0.8) with $F = 0$. We show that under this observation condition with $\gamma \geq 2$, the quadratic feedback control strongly stabilizes the system (0.8). An explicit polynomial decay rate of the stabilization state is estimated as follows:

$$\|w(t)\|_{\mathcal{X}} = O\left(t^{-\frac{1}{2(\gamma-1)}}\right) \text{ as } t \longrightarrow +\infty. \tag{4.1}$$

Moreover, under some suitable conditions, we prove the weak stabilization of system (0.8) under the quadratic feedback control. Furthermore, when $\gamma = 2$, we show that the normalized feedback control exponentially stabilizes the system (0.8). In the special case where the state space $\mathcal{X}$ is of finite dimension, we give another condition to guarantee the exponentially stabilization of system (0.8). In the bilinear case, we consider the decomposition method of the state space $\mathcal{X}$ to discuss the strong and weak stabilization. We show that for $\gamma \geq 2$ and under the observability condition

on the space $\mathcal{C}^u$, the quadratic feedback control $v^u(t) = -\langle w^u(t), F^u(w_t^u)\rangle$ strongly stabilizes the system (0.8) with the same estimation (4.1). Moreover, if $\mathcal{X}^u$ is of finite dimension, we prove with another condition, weaker than observability condition, that system (0.8) is strongly stabilizable such that the estimate (4.1) is satisfied with $\gamma = 2$. Moreover, without using the exponential stability condition of the restriction of the semigroup $S(t)$ on $\mathcal{X}^s$, we prove the strong stabilization of system (0.8) by using the following quadratic feedback control $v^u(t) = -\langle w^u(t), F^u(w_t^u)\rangle$ and the estimation (4.1) is satisfied with $\gamma = 2$. Finally, three examples are given to illustrate our theoretical results. Especially, numerical simulations of wave equation with distributed delay are given.

Through this chapter, we use the following new feedback control, for all $p \in (-\infty, 2]$

$$v_p(t) = -\frac{\langle w(t), F(w_t)\rangle}{\|w_t\|_{\mathcal{C}}^p}\mathbf{1}_E \tag{4.2}$$

where $\mathbf{1}_E$ is the characteristic function of the set $E = \{t \geq 0 : \ w_t \ \neq 0\}$.

4.2 Existence and uniqueness of the global mild solution

This section is concerned with the global existence and uniqueness of the mild solution of the system (0.8).

Theorem 4.2.1. *Assume that:*
1. $\mathcal{A}$ generates a $\mathcal{C}_0-$semigroup of contraction on $\mathcal{X}$,
2. F is a locally Lipschitz from $\mathcal{C}$ to $\mathcal{X}$, such that $F(0) = 0$,
hold. Then, system (0.8) admits a unique global mild solution $w(t)$ on $[-r, +\infty[$ corresponds to

$$v_p(t) = -\frac{\langle w(t), F(w_t)\rangle}{\|w_t\|_{\mathcal{C}}^p}\mathbf{1}_E \text{ for all } p \in (-\infty, 2].$$

Moreover, we have the following inequality

$$\|w(\tau)\|_{\mathcal{X}}^2 - \|w(t)\|_{\mathcal{X}}^2 \geq 2\int_\tau^t \frac{|\langle w(s), F(w_s)\rangle|^2}{\|w_s\|_{\mathcal{C}}^p}\mathbf{1}_E ds \quad \text{for all } t \geq \tau \geq 0. \tag{4.3}$$

Proof. Taking into account the feedback control (4.2), the system (0.8) becomes

$$\begin{cases} \dot{w}(t) = \mathcal{A}w(t) + \widehat{F}_p(w_t) \ \text{ for all } t > 0, \\ w_0 = \phi \in \mathcal{C}, \end{cases}$$

where $\widehat{F}_p$ is the function defined from $\mathcal{C}$ to $\mathcal{X}$ by

$$\widehat{F}_p(\psi) = \begin{cases} -\dfrac{\langle \psi(0), F(\psi)\rangle}{\|\psi\|_{\mathcal{C}}^p} F(\psi) & \psi \neq 0 \\ 0 & \psi = 0 \end{cases}$$

The function $\widehat{F}_p$ is locally Lipschitz on $\mathcal{C}$ for all $p \in (-\infty, 2]$. In fact, for any $\eta \geq 0$ and ψ, φ in $\mathcal{C}$ such that $\|\psi\|_{\mathcal{C}} \leq \eta$, $\|\varphi\|_{\mathcal{C}} \leq \eta$.
If $\psi = 0$ or $\varphi = 0$, we suppose that $\varphi = 0$, then $\widehat{F}_p(\varphi) = 0$ and we have that

$$\left\| \widehat{F}_p(\psi) - \widehat{F}_p(\varphi) \right\|_{\mathcal{X}} = \left\| \frac{\langle \psi(0), F(\psi)\rangle}{\|\psi\|_{\mathcal{C}}^p} F(\psi) \right\|_{\mathcal{X}}$$

Since F is a locally Lipschitz function, then there exists a constant Γ_η such that

$$\|F(\varphi_1) - F(\varphi_2)\|_{\mathcal{X}} \leq \Gamma_\eta \|\varphi_1 - \varphi_2\|_{\mathcal{C}}$$

for all $(\varphi_1, \varphi_2) \in \mathcal{C}^2$ such that $\|\varphi_1\|_{\mathcal{C}} \leq \eta$ and $\|\varphi_2\|_{\mathcal{C}} \leq \eta$. Then

$$\begin{aligned} \left\| \widehat{F}_p(\psi) - \widehat{F}_p(\varphi) \right\|_{\mathcal{X}} &\leq \Gamma_\eta^2 \|\psi\|_{\mathcal{C}}^{2-p} \|\psi\|_{\mathcal{C}} \\ &\leq \Gamma_\eta^2 \eta^{2-p} \|\psi - \varphi\|_{\mathcal{C}} \end{aligned}$$

If not, we suppose that $0 < \|\psi\|_{\mathcal{C}} \leq \|\varphi\|_{\mathcal{C}}$, then

$$\begin{aligned} \|\widehat{F}_p(\psi) - \widehat{F}_p(\varphi)\|_{\mathcal{X}} &= \left\| \frac{\langle \psi(0), F(\psi)\rangle}{\|\psi\|_{\mathcal{C}}^p} F(\psi) - \frac{\langle \varphi(0), F(\varphi)\rangle}{\|\varphi\|_{\mathcal{C}}^p} F(\varphi) \right\|_{\mathcal{X}} \\ &= \frac{\| \|\varphi\|_{\mathcal{C}}^p \langle \psi(0), F(\psi)\rangle F(\psi) - \|\psi\|_{\mathcal{C}}^p \langle \varphi(0), F(\varphi)\rangle F(\varphi)\|_{\mathcal{X}}}{\|\psi\|_{\mathcal{C}}^p \|\varphi\|_{\mathcal{C}}^p} \end{aligned}$$

Adding and subtracting the term $\|\psi\|_{\mathcal{C}}^p \langle \psi(0), F(\psi)\rangle F(\psi)$ in $\|\varphi\|_{\mathcal{C}}^p \langle \psi(0), F(\psi)\rangle F(\psi) - \|\psi\|_{\mathcal{C}}^p \langle \varphi(0), F(\varphi)\rangle F(\varphi)$, we get that

$$\begin{aligned} \|\widehat{F}_p(\psi) - \widehat{F}_p(\varphi)\|_{\mathcal{X}} \leq & \frac{\| \|\varphi\|_{\mathcal{C}}^p \langle \psi(0), F(\psi)\rangle F(\psi) - \|\psi\|_{\mathcal{C}}^p \langle \psi(0), F(\psi)\rangle F(\psi)\|_{\mathcal{X}}}{\|\psi\|_{\mathcal{C}}^p \|\varphi\|_{\mathcal{C}}^p} \\ & + \frac{\| \|\psi\|_{\mathcal{C}}^p \langle \psi(0), F(\psi)\rangle F(\psi) - \|\psi\|_{\mathcal{C}}^p \langle \varphi(0), F(\varphi)\rangle F(\varphi)\|_{\mathcal{X}}}{\|\psi\|_{\mathcal{C}}^p \|\varphi\|_{\mathcal{C}}^p} \\ \leq & \frac{| \|\varphi\|_{\mathcal{C}}^p - \|\psi\|_{\mathcal{C}}^p | \, \|\langle \psi(0), F(\psi)\rangle F(\psi)\|}{\|\psi\|_{\mathcal{C}}^p \|\varphi\|_{\mathcal{C}}^p} \\ & + \frac{\|\psi\|_{\mathcal{C}}^p \, \|\langle \psi(0), F(\psi)\rangle F(\psi) - \langle \varphi(0), F(\varphi)\rangle F(\varphi)\|_{\mathcal{X}}}{\|\psi\|_{\mathcal{C}}^p \|\varphi\|_{\mathcal{C}}^p} \end{aligned}$$

Since F is a locally Lipschitz function, then

$$\begin{aligned}\|\widehat{F}_p(\psi)-\widehat{F}_p(\varphi)\|_{\mathcal{X}} \leq\ & \Gamma_\eta^2\frac{\|\psi\|_{\mathcal{C}}^3}{\|\psi\|_{\mathcal{C}}^p\|\varphi\|_{\mathcal{C}}^p}|\|\varphi\|_{\mathcal{C}}^p-\|\psi\|_{\mathcal{C}}^p| \\ &+\frac{\|\langle\psi(0),F(\psi)\rangle F(\psi)-\langle\varphi(0),F(\varphi)\rangle F(\varphi)\|_{\mathcal{X}}}{\|\varphi\|_{\mathcal{C}}^p}.\end{aligned}$$

Since

$$\begin{aligned}\langle\psi(0),F(\psi)\rangle F(\psi)-\langle\varphi(0),F(\varphi)\rangle F(\varphi) =&\langle\psi(0),F(\psi)\rangle(F(\psi)-F(\varphi))+\langle\psi(0)-\varphi(0),F(\psi)\rangle F(\varphi) \\ &+\langle\varphi(0),F(\varphi)-F(\psi)\rangle F(\varphi),\end{aligned}$$

it follows that

$$\begin{aligned}\|\widehat{F}_p(\psi)-\widehat{F}_p(\varphi)\|_{\mathcal{X}} \ \leq\ & \Gamma_\eta^2\frac{\|\psi\|_{\mathcal{C}}^3}{\|\psi\|_{\mathcal{C}}^p\|\varphi\|_{\mathcal{C}}^p}|\|\varphi\|_{\mathcal{C}}^p-\|\psi\|_{\mathcal{C}}^p| \\ &+\frac{\|\langle\psi(0),F(\psi)\rangle(F(\psi)-F(\varphi))\|_{\mathcal{X}}}{\|\varphi\|_{\mathcal{C}}^p} \\ &+\frac{\|\langle\psi(0)-\varphi(0),F(\psi)\rangle F(\varphi)\|_{\mathcal{X}}}{\|\varphi\|_{\mathcal{C}}^p} \\ &+\frac{\|\langle\varphi(0),F(\varphi)-F(\psi)\rangle F(\varphi)\|_{\mathcal{X}}}{\|\varphi\|_{\mathcal{C}}^p}.\end{aligned}$$

Moreover, by using the formula

$$a^p-b^p=(a-b)\sum_{k=0}^{p-1}a^{p-1-k}b^k,$$

we get that

$$\begin{aligned}\|\widehat{F}_p(\psi)-\widehat{F}_p(\varphi)\|_{\mathcal{X}} \ \leq\ & \Gamma_\eta^2\|\psi\|_{\mathcal{C}}^{2-p}\frac{\|\psi\|_{\mathcal{C}}}{\|\varphi\|_{\mathcal{C}}^p}\left(\sum_{k=0}^{p-1}\|\varphi\|_{\mathcal{C}}^{p-1-k}\|\psi\|_{\mathcal{C}}^k\right)|\|\varphi\|_{\mathcal{C}}-\|\psi\|_{\mathcal{C}}| \\ &+\frac{\|\langle\psi(0),F(\psi)\rangle(F(\psi)-F(\varphi))\|_{\mathcal{X}}}{\|\varphi\|_{\mathcal{C}}^p} \\ &+\frac{\|\langle\psi(0)-\varphi(0),F(\psi)\rangle F(\varphi)\|_{\mathcal{X}}}{\|\varphi\|_{\mathcal{C}}^p} \\ &+\frac{\|\langle\varphi(0),F(\varphi)-F(\psi)\rangle F(\varphi)\|_{\mathcal{X}}}{\|\varphi\|_{\mathcal{C}}^p}.\end{aligned}$$

Using the fact that $\|\psi\|_{\mathcal{C}}\leq\|\varphi\|_{\mathcal{C}}\leq\eta$, we get that

$$\begin{aligned}\left\|\widehat{F}_p(\psi)-\widehat{F}_p(\varphi)\right\|_{\mathcal{X}} &\leq p\Gamma_\eta^2\|\psi\|_{\mathcal{C}}^{2-p}\|\varphi-\psi\|_{\mathcal{C}} \\ &\quad+\ 3\Gamma_\eta^2\eta^{2-p}\|\varphi-\psi\|_{\mathcal{C}} \\ &\leq(p+3)\Gamma_\eta^2\eta^{2-p}\|\varphi-\psi\|_{\mathcal{C}}.\end{aligned}$$

From [90], there exists a maximal interval of existence $[-r, T_{max})$ and a unique mild solution $w \in C([-r, T_{max}), \mathcal{X})$ of system (0.8). Consider the following function

$$\begin{aligned} f^p : [0, T_{max}) &\longrightarrow \mathcal{X} \\ t &\longmapsto \widehat{F}_p(w_t) \end{aligned}$$

Let $0 < T < T_{max}$. Since $f^p \in C([0, T_{max}); \mathcal{X})$, there exists a sequence $(f_n^p)_n \subset C^1([0, T]; \mathcal{X})$ such that

$$f_n^p \longrightarrow f^p \text{ as } n \to +\infty \text{ uniformly in } C([0, T]; \mathcal{X}).$$

Let $(\xi_n) \subset D(\mathcal{A})$ such that $(\xi_n)_n$ converges to $w(0)$ in $\mathcal{X}$ and define the following function

$$w_n(t) = T(t)\xi_n + \int_0^t T(t-s) f_n^p(s) ds, \quad t \in [0, T]. \tag{4.4}$$

Then, from [21, p. 187], $w_n(t) \in D(A)$, $w_n \in C^1([0, T], \mathcal{X})$ and it follows that

$$\dot{w}_n(t) = \mathcal{A} w_n(t) + f_n^p(t) \; t \in (0, T), \text{ with } \quad w_n(0) = \xi_n.$$

By using the dissipativity of operator $\mathcal{A}$, we obtain

$$\frac{d}{dt} \|w_n(t)\|_{\mathcal{X}}^2 \leq 2 \langle f_n^p(t), w_n(t) \rangle \text{ for all } t \in [0, T].$$

Integrating this last inequality, we derive

$$\|w_n(t)\|_{\mathcal{X}}^2 - \|w_n(\tau)\|_{\mathcal{X}}^2 \leq 2 \int_\tau^t \langle f_n^p(s), w_n(s) \rangle \, ds, \tag{4.5}$$

for all $0 \leq \tau \leq t \leq T$. Furthermore, we have to show that $w_n \to w$ as $n \to +\infty$ uniformly in $C([0, T]; \mathcal{X})$. Since $S(t)$ is a semigroup of contraction, we get using equation (4.4) and the variation of constants formula

$$\|w_n(t) - w(t)\|_{\mathcal{X}} \leq \|\xi_n - w(0)\|_{\mathcal{X}} + T \sup_{s \in [0,T]} \|f_n^p(s) - f^p(s)\|_{\mathcal{X}}.$$

Then

$$\|w_n(t) - w(t)\|_{\mathcal{X}} \to 0 \; as \; n \to +\infty.$$

Hence, letting n to $+\infty$ in (4.5), we get by the dominated convergence theorem that

$$\|w(\tau)\|_{\mathcal{X}}^2 - \|w(t)\|_{\mathcal{X}}^2 \geq 2 \int_\tau^t \frac{|\langle w(s), F(w_s) \rangle|^2}{\|w_s\|_{\mathcal{C}}^p} \mathbf{1}_E ds$$
$$\text{for all } 0 \leq \tau \leq t \leq T.$$

Then, the function $t \to \|w(t)\|_{\mathcal{X}}$ is a decreasing function on $[0, +\infty[$ and using the fact that $t \to w(t)$ is continuous in $[-r, 0]$, then, there exists $K > 0$, such that

$$\|w(t)\|_{\mathcal{X}} \leq K \quad \text{for } -r \leq t < T_{max}. \tag{4.6}$$

Finally, by the blow-up phenomena we have $T_{max} = +\infty$, which implies that $w(t)$ is a global mild solution of system (0.8) on $[-r, +\infty[$ ■

Remark 4.2.2. *Using the estimation (4.3), we have that the function $t \to \|w(t)\|_{\mathcal{X}}$ is a decreasing function on $[0, +\infty[$. Moreover, there exist $t^* \geq 0$ such that $w(t^*) = 0$ if and only if $w(t) = 0$ for all $t \geq t^*$.*

4.3 Strong and weak Stabilization with quadratic control

In this section, we discuss the strong stabilization of the system (0.8) by using Theorem 4.2.1. Then, we have the following theorem.

Theorem 4.3.1. *Assume that:*
1. $\mathcal{A}$ generates a $\mathcal{C}_0-$semigroup of contraction on $\mathcal{X}$,
2. F is a locally Lipschitz from $\mathcal{C}$ to $\mathcal{X}$, such that $F(0) = 0$,
3. there exist T, $\delta > 0$ and $\gamma \geq 2$ such that

$$\int_0^T |\langle F(U(s)\xi), (U(s)\xi)(0)\rangle| ds \geq \delta \|\xi(0)\|_{\mathcal{X}}^{\gamma} \quad \text{for all } \xi \in \mathcal{C}, \tag{4.7}$$

hold. Then, the feedback control

$$v_0(t) = -\langle w(t), F(w_t)\rangle$$

strongly stabilizes the system (0.8). Moreover, we have the following decay estimate

$$\|w(t)\|_{\mathcal{X}} = O\left(t^{-\frac{1}{2(\gamma-1)}}\right) \text{ as } t \longrightarrow +\infty.$$

Proof. The system (0.8) controlled by $v_0(t) = -\langle w(t), F(w_t)\rangle$ has a unique global mild solution on $[-r, +\infty[$, this solution is given by the following variation of constants formula

$$w(s) = S(s)w(0) - \int_0^s S(s-\sigma)\langle w(\sigma), F(w_\sigma)\rangle F(w_\sigma) d\sigma,$$

for all $s \geq 0$.

Since the semigroup $S(s)$ is of contraction, we get, by Holder's inequality, that for $s \in [0,T]$.

$$\|w(s) - S(s)w(0)\|_{\mathcal{X}} \leq K\Gamma_K\sqrt{T}\left(\int_0^s |\langle w(\sigma), F(w_\sigma)\rangle|^2 d\sigma\right)^{\frac{1}{2}},$$

where K is the constant given in inequality (4.6) and Γ_K is the Lipschitz constant of F. Then, for all $t \geq 0$

$$\|w(s+t) - S(s)w(t)\|_{\mathcal{X}} \leq K\Gamma_K\sqrt{T}\left(\int_t^{t+s} |\langle w(\sigma), F(w_\sigma)\rangle|^2 d\sigma\right)^{\frac{1}{2}}. \tag{4.8}$$

Moreover, for all $s \in [0,T]$, we have

$$\begin{aligned}
\langle (U(s)w_t)(0), F(U(s)w_t)\rangle &= \langle S(s)w(t), F(U(s)w_t)\rangle \\
&= \langle w(s+t), F(U(s)w_t) - F(w_{s+t})\rangle \\
&\quad - \langle w(s+t) - S(s)w(t), F(U(s)w_t)\rangle \\
&\quad + \langle w(s+t), F(w_{s+t})\rangle.
\end{aligned}$$

Then, for $s \in [0,T]$, we have

$$\begin{aligned}
|\langle (U(s)w_t)(0), F(U(s)w_t)\rangle| &\leq K\Gamma_K \|w_{s+t} - U(s)w_t\|_{\mathcal{C}} \\
&\quad + K\Gamma_K \|w(s+t) - S(s)w(t)\|_{\mathcal{X}} \\
&\quad + |\langle w(s+t), F(w_{s+t})\rangle| \\
&\leq 2K\Gamma_K \|w_{s+t} - U(s)w_t\|_{\mathcal{C}} \\
&\quad + |\langle w(s+t), F(w_{s+t})\rangle|.
\end{aligned} \tag{4.9}$$

Let $s \in [0,T]$ and $\theta \in [-r,0]$. From inequality (4.8), it follows that if $s+\theta \geq 0$, we have that

$$\begin{aligned}
\|(w_{s+t} - U(s)w_t)(\theta)\|_{\mathcal{X}} &= \|w(s+t+\theta) - (U(s)w_t)(\theta)\|_{\mathcal{X}} \\
&= \|w(s+t+\theta) - S(s+\theta)w(t)\|_{\mathcal{X}} \\
&\leq K\Gamma_K\sqrt{T}\left(\int_t^{t+s+\theta} |\langle w(\sigma), F(w_\sigma)\rangle|^2 d\sigma\right)^{\frac{1}{2}} \\
&\leq K\Gamma_K\sqrt{T}\left(\int_t^{t+T} |\langle w(\sigma), F(w_\sigma)\rangle|^2 d\sigma\right)^{\frac{1}{2}}
\end{aligned}$$

and if $s+\theta \leq 0$, then

$$\begin{aligned}
\|(w_{s+t} - U(s)w_t)(\theta)\|_{\mathcal{X}} &= \|w(s+t+\theta) - (U(s)w_t)(\theta)\|_{\mathcal{X}} \\
&= \|w(s+t+\theta) - w(s+t+\theta)\|_{\mathcal{X}} \\
&= 0.
\end{aligned}$$

Hence

$$\|w_{s+t} - U(s)w_t\|_{\mathcal{C}} \leq K\Gamma_K\sqrt{T}\left(\int_t^{t+T} |\langle w(\sigma), F(w_\sigma)\rangle|^2 d\sigma\right)^{\frac{1}{2}}. \tag{4.10}$$

From inequalities (4.9) and (4.10) , it follows that

$$\begin{aligned}|\langle (U(s)w_t)(0), F(U(s)w_t)\rangle| \leq 2\left(K\Gamma_K\right)^2\sqrt{T}\left(\int_t^{t+T} |\langle w(\sigma), F(w_\sigma)\rangle|^2 d\sigma\right)^{\frac{1}{2}} \\ + |\langle w(s+t), F(w_{s+t})\rangle|.\end{aligned}$$

Integrating the last inequality over $[0, T]$ with respect to s, then

$$\begin{aligned}\int_0^T |\langle (U(s)w_t)(0), F(U(s)w_t)\rangle| ds \leq 2T\left(K\Gamma_K\right)^2\sqrt{T}\left(\int_t^{t+T} |\langle w(\sigma), F(w_\sigma)\rangle|^2 d\sigma\right)^{\frac{1}{2}} \\ + \int_0^T |\langle w(s+t), F(w_{s+t})\rangle| ds.\end{aligned}$$

Using the variable change and the Holder's inequality, we get that

$$\begin{aligned}\int_0^T |\langle w(s+t), F(w_{s+t})\rangle| ds &= \int_t^{t+T} |\langle w(\sigma), F(w_\sigma)\rangle| ds \\ &\leq \sqrt{T}\left(\int_t^{t+T} |\langle w(\sigma), F(w_\sigma)\rangle|^2 d\sigma\right)^{\frac{1}{2}},\end{aligned}$$

and hence

$$\int_0^T |\langle (U(s)w_t)(0), F(U(s)w_t)\rangle| ds \leq R\left(\int_t^{t+T} |\langle w(\sigma), F(w_\sigma)\rangle|^2 d\sigma\right)^{\frac{1}{2}}, \tag{4.11}$$

where $R = \left(2\left(K\Gamma_K\right)^2 T + 1\right)\sqrt{T}$. Using the observability condition (4.7) with $\xi = w_t$, we derive

$$\begin{aligned}\delta\|w(t)\|_{\mathcal{X}}^{\gamma} &\leq \int_0^T |\langle (U(s)w_t)(0), F(U(s)w_t)\rangle| ds \\ &\leq R\left(\int_t^{t+T} |\langle w(\sigma), F(w_\sigma)\rangle|^2 d\sigma\right)^{\frac{1}{2}}.\end{aligned}$$

Thus

$$\left(\frac{\delta}{R}\right)^2 \|w(t)\|_{\mathcal{X}}^{2\gamma} \leq \int_t^{t+T} |\langle w(\sigma), F(w_\sigma)\rangle|^2 d\sigma. \tag{4.12}$$

Using inequality (4.3) with $p = 0$, we obtain that

$$\|w(t)\|_{\mathcal{X}}^2 - \|w(t+T)\|_{\mathcal{X}}^2 \geq 2\int_t^{t+T} |\langle w(\sigma), F(w_\sigma)\rangle|^2 d\sigma,$$

and then

$$\|w(t+T)\|_{\mathcal{X}}^2 - \|w(t)\|_{\mathcal{X}}^2 \leq -2\int_t^{t+T} |\langle w(\sigma), F(w_\sigma)\rangle|^2 d\sigma.$$

Consequently, inequality (4.12) implies that

$$\|w(t+T)\|_{\mathcal{X}}^2 - \|w(t)\|_{\mathcal{X}}^2 \leq -2\left(\frac{\delta}{R}\right)^2 \|w(t)\|_{\mathcal{X}}^{2\gamma}.$$

Letting $u_k = \|w(kT)\|_{\mathcal{X}}^2$. Then

$$u_{k+1} - u_k \leq -2\left(\frac{\delta}{R}\right)^2 u_k^\gamma.$$

Since the function $t \to \|w(t)\|_{\mathcal{X}}$ is decreasing on $[0, +\infty[$, it follows that $u_{k+1} \leq u_k$ and hence

$$u_{k+1} \leq u_k - 2\left(\frac{\delta}{R}\right)^2 u_{k+1}^\gamma.$$

By applying Lemma 1.3.5 with $m = \gamma - 2 > -1$, we obtain that

$$u_k \leq \frac{M}{(k+1)^{\frac{1}{\gamma-1}}} \text{ for all } k \geq 0.$$

Finally, let $k = E\left(\frac{t}{T}\right)$, where $E(.)$ designs the integer part and using again the fact that $t \to \|w(t)\|_{\mathcal{X}}$ is a decreasing function on $\mathbb{R}^+$, we deduce the following estimate

$$\|w(t)\|_{\mathcal{X}} = O\left(t^{-\frac{1}{2(\gamma-1)}}\right) \text{ as } t \longrightarrow +\infty \blacksquare$$

Remark 4.3.2. *1- For $0 \leq \gamma < 2$, the observability condition (4.7) is not well defined. In fact, suppose that there exist T, $\delta > 0$ and $0 \leq \gamma < 2$ such that*

$$\int_0^T |\langle F(U(s)\xi), (U(s)\xi)(0)\rangle| ds \geq \delta \|\xi(0)\|_{\mathcal{X}}^\gamma \quad \text{for all } \xi \in \mathcal{C}.$$

Let $y \in \mathcal{X}\backslash\{0\}$ such that $x(t) = e^{-(t+r)}\frac{y}{\|y\|_{\mathcal{X}}}$ for $t \in [-r, +\infty[$. Since F is a locally Lipschitz function, then for $t \geq 0$, there exists a constant Γ_1 such that

$$\|F(x_t)\|_{\mathcal{X}} \leq \Gamma_1 \|x_t\|_{\mathcal{C}}.$$

The semigroup $S(t)$ is of contraction, then for $\xi = x_t$ with $t \geq 0$, we get that

$$\begin{aligned}\delta\|x_t(0)\|_{\mathcal{X}}^{\gamma} &\leq \int_0^T \langle F(U(s)x_t), (U(s)x_t)(0)\rangle|ds \\ &\leq \Gamma_1 \int_0^T \|U(s)x_t\|_{\mathcal{C}} ds \|x(t)\|_{\mathcal{X}} \\ &\leq \Gamma_1 \int_0^T e^{-(t+s)} ds\ e^{-(t+r)} \\ &\leq \Gamma_1 e^{-t}(1-e^{-T})e^{-(t+r)}\end{aligned}$$

and consequently

$$\delta e^{(2-\gamma)t} e^{(1-\gamma)r} \leq \Gamma_1(1-e^{-T})$$

which gives the contradiction since t is arbitrary in $\mathbb{R}^+$.
2-Note that by taking $F(\varphi) = L^*L\varphi(0)$ in the estimate (4.7) where L is an observation operator, one recovers the classical concept of exact observability with $\gamma = 2$ in the sense of linear system theory (see [27]).

In finite-dimensional space, we obtain the strong stabilization of system (0.8) using another condition weaker than condition (4.7).

Corollary 4.3.3. *Assume that:*
1. the state space $\mathcal{X}$ is of finite dimensional,
2. $\mathcal{A}$ generates a $\mathcal{C}_0-$semigroup of contraction on $\mathcal{X}$,
3. F is a linear bounded operator from $\mathcal{C}$ to $\mathcal{X}$,
4. there exist $T_2 > T_1 > r$ and for all $\xi \in \mathcal{C}$ such that

$$\langle F(U(s)\xi), (U(s)\xi)(0)\rangle = 0 \quad \text{for all } s \in [T_1, T_2] \Rightarrow \xi(0) = 0, \tag{4.13}$$

hold. Then, the feedback control

$$v_0(t) = -\langle w(t), F(w_t)\rangle$$

strongly stabilizes the system (0.8) and we have the following decay estimate:

$$\|w(t)\|_{\mathcal{X}} = O\left(t^{-\frac{1}{2}}\right) \text{ as } t \longrightarrow +\infty.$$

Proof . Clearly, the inequality (4.7) is verified for all $\xi \in \mathcal{C}$ such that $\xi(0) = 0$. We know that for all $s \in [T_1, T_2]$

$$\langle F(U(s)\xi), (U(s)\xi)(0)\rangle = \langle F(S(s+.)\xi(0)), S(s)\xi(0)\rangle$$

Now, assume that

$$\inf_{\|x\|_{\mathcal{X}}=1} \int_{T_1}^{T_2} |\langle F(S(s+.)x), S(s)x\rangle|\, ds \;=\; 0.$$

Since $\displaystyle\int_{T_1}^{T_2} |\langle F(S(s+.)x), S(s)x\rangle|\, ds$ depends continuously x and $\dim \mathcal{X} < +\infty$, then, there exists $x_0 \in \mathcal{X}$ such that

$$\|x_0\|_{\mathcal{X}} = 1 \text{ and } \int_{T_1}^{T_2} |\langle F(S(s+.)x_0), S(s)x_0\rangle|\, ds = 0.$$

So there exists $\xi \in \mathcal{C}$ such that $\xi(0) = x_0$ and hence

$$|\langle F(S(s+.)\xi(0)), S(s)\xi(0)\rangle| = 0 \;\text{ for all } s \in [T_1, T_2].$$

Condition (4.13) implies that $\xi(0) = 0$, which gives the contradiction. Then, there exists a constant $\delta > 0$ such that

$$\inf_{\|x\|_{\mathcal{X}}=1} \int_{T_1}^{T_2} |\langle F(S(s+.)x), S(s)x\rangle|\, ds \;=\; \delta.$$

Now, let $x = \dfrac{1}{\|\xi(0)\|_{\mathcal{X}}}\xi(0)$, then $\|x\|_{\mathcal{X}} = 1$ and we have

$$\int_{T_1}^{T_2} |\langle F(U(s)\xi), (U(s)\xi)(0)\rangle|\, ds \geq \delta \|\xi(0)\|_{\mathcal{X}}^2.$$

Since

$$\int_{0}^{T_2} |\langle F(U(s)\xi), (U(s)\xi)(0)\rangle|\, ds \geq \int_{T_1}^{T_2} |\langle F(U(s)\xi), (U(s)\xi)(0)\rangle|\, ds.$$

Then, by applying Theorem 4.3.1, the feedback control $v_0(t) = -\langle w(t), F(w_t)\rangle$ strongly stabilizes the system (0.8), with the following decay estimate

$$\|w(t)\|_{\mathcal{X}} = O\left(t^{-\frac{1}{2}}\right) \quad \text{as } t \longrightarrow +\infty \blacksquare$$

The following theorem gives sufficient conditions to get the weak stabilization of system (0.8).

Theorem 4.3.4. *Assume that:*
1. $\mathcal{A}$ generates a $\mathcal{C}_0-$semigroup of contraction on $\mathcal{X}$,
2. F is a locally Lipschitz from $\mathcal{C}$ to $\mathcal{X}$, such that $F(0) = 0$,

3. there exist $T_2 > T_1 > 0$ *such that for all sequence* $(\xi_n)_n \subset \mathcal{C}$ *with* $\xi_n(0) \rightharpoonup \xi(0)$ *in* $\mathcal{X}$, *we have*

$$\langle (U(t)\xi_n)(0), F(U(t)\xi_n)\rangle \to 0 \text{ as } n \to +\infty \quad \text{for all } t \in [T_1, T_2] \Rightarrow \xi(0) = 0, \tag{4.14}$$

hold. Then, the system (0.8) is weakly stabilizable by the feedback control $v_0(t) = -\langle w(t), F(w_t)\rangle$.

Proof . From inequality (4.3), it follows that the function $t \to \|w(t)\|_{\mathcal{X}}$ is bounded. Since the space $\mathcal{X}$ is reflexive, we deduce that there exists $t_n \to +\infty$ and $x \in \mathcal{X}$ such that

$$w(t_n) \to x \text{ weakly.}$$

Since, for any $T_2 > 0$, we have that

$$\int_{T_1}^{T_2} |\langle (U(s)w_{t_n})(0), F(U(s)w_{t_n})\rangle| \, ds \leq \int_0^{T_2} |\langle (U(s)w_{t_n})(0), F(U(s)w_{t_n})\rangle| \, ds,$$

and from inequality (4.11), it follows that

$$\int_{T_1}^{T_2} |\langle (U(s)w_{t_n})(0), F(U(s)w_{t_n})\rangle| \, ds \leq R \left(\int_{t_n}^{t_n+T_2} |\langle w(s), F(w_s)\rangle|^2 \, ds \right)^{\frac{1}{2}},$$

with $R = \left(2\left(K\Gamma_K\right)^2 T_2 + 1\right)\sqrt{T_2}$. From inequality (4.3), we have that

$$\int_{t_n}^{t_n+T_2} |\langle w(s), F(w_s)\rangle|^2 ds \leq \frac{1}{2}\left(\|w(t_n)\|_{\mathcal{X}}^2 - \|w(t_n+T_2)\|_{\mathcal{X}}^2\right).$$

Since $t \to \|w(t)\|_{\mathcal{X}}$ is a decreasing function on $[0, +\infty[$, it follows that the sequence $(\|w(t_n)\|_{\mathcal{X}})_n$ converges and hence

$$\int_0^{T_2} |\langle (U(s)w_{t_n})(0), F(U(s)w_{t_n})\rangle| \, ds \to 0 \quad \text{as } n \to +\infty.$$

The sequence $(|\langle (U(s)w_{t_n})(0), F(U(s)w_{t_n})\rangle|)_{n\in\mathbb{N}}$ is bounded in $\mathbb{R}$, then, there exists a subsequence of $(|\langle (U(s)w_{t_n})(0), F(U(s)w_{t_n})\rangle|)_{n\in\mathbb{N}}$, denoted also by $(|\langle (U(s)w_{t_n})(0), F(U(s)w_{t_n})\rangle|)_{n\in\mathbb{N}}$ which is convergent. Using the dominated convergence theorem, we get that

$$\int_{T_1}^{T_2} \lim_{n\to+\infty} |\langle (U(s)w_{t_n})(0), F(U(s)w_{t_n})\rangle| ds = 0,$$

and consequently

$$\lim_{n\to+\infty} |\langle (U(s)w_{t_n})(0), F(U(s)w_{t_n})\rangle| = 0 \quad \text{for all } s \in [T_1, T_2].$$

Then condition (4.14) implies that $x = 0$. Finally $w(t)$ tends to 0 weakly as t goes to $+\infty$ ■

4.4 Strong and weak stabilization through the decomposition of the state space $\mathcal{X}$ in the bilinear case with quadratic control

Through this section, we suppose that

> $\mathcal{X}$ can be decomposed as a direct sum of two closed subspaces $\mathcal{X}^u$and $\mathcal{X}^s$of $\mathcal{X}$, which are orthogonal to each one and invariants under the semigroup $S(t)$.

Let $S^u(t)$ and $S^s(t)$ denote the restrictions of $S(t)$ on $\mathcal{X}^u$ and $\mathcal{X}^s$ respectively. The operators $S^u(t)$ and $S^s(t)$ are C_0-semigroups on $\mathcal{X}^u$ and $\mathcal{X}^s$ respectively and let $\mathcal{A}^u$ and $\mathcal{A}^s$ be their generators respectively. Furthermore, we suppose that

$$F(\mathcal{C}^u) \subset \mathcal{X}^u \text{ and } F(\mathcal{C}^s) \subset \mathcal{X}^s,$$

where $\mathcal{C}^u = C([-r,0],\mathcal{X}^u)$ and $\mathcal{C}^s = C([-r,0],\mathcal{X}^s)$. Then, the system (0.8) can be decomposed as follows

$$\begin{cases} \dot{w}^u(t) = \mathcal{A}^u w^u(t) + v(t)F^u(w_t^u) \quad \text{for all } t>0, \\ w_0^u = \phi^u \in \mathcal{C}^u = C\left([-r,0];\mathcal{X}^u\right), \end{cases} \tag{4.15}$$

with $F^u = P^u F$ and P^u is the orthogonal projection on $\mathcal{X}^u$ and

$$\begin{cases} \dot{w}^s(t) = \mathcal{A}^s w^s(t) + v(t)F^s(w_t^s) \text{ for all } t>0, \\ w_0^s = \phi^s \in \mathcal{C}^s = C\left([-r,0];\mathcal{X}^s\right), \end{cases} \tag{4.16}$$

with $F^s = P^s F$ and P^s is the orthogonal projection on $\mathcal{X}^s$.

Theorem 4.4.1. *Assume that:*
1. F is a linear bounded operator from $\mathcal{C}$ to $\mathcal{X}$,
2. $S^u(t)$ is a contraction semigroup on $\mathcal{X}^u$,
3. $S^s(t)$ is exponentially stable on $\mathcal{X}^s$,
4. there exist positive constants T, δ and $\gamma \geq 2$ such that

$$\int_0^T |\langle F^u(U^u(s)\xi^u), (U^u(s)\xi^u)(0)\rangle|\, ds \geq \delta \|\xi^u(0)\|_{\mathcal{X}}^{\gamma}, \quad \textit{for all } \xi^u \in \mathcal{C}^u, \tag{4.17}$$

hold. Then, the feedback control

$$v_0^u(t) = -\langle w^u(t), F^u(w_t^u)\rangle \tag{4.18}$$

strongly stabilizes the system (0.8). Moreover, we have the following decay estimate

$$\|w(t)\|_{\mathcal{X}} = O\left(t^{-\frac{1}{2(\gamma-1)}}\right), \text{ as } t \longrightarrow +\infty.$$

Proof. According to Theorem 4.3.1, since $\gamma \geq 2$, the feedback control (4.18) strongly stabilizes the system (4.15) and we have the following estimate

$$\|w^u(t)\|_{\mathcal{X}} = O\left(t^{-\frac{1}{2(\gamma-1)}}\right) \text{ as } t \longrightarrow +\infty.$$

Furthermore, from Theorem 4.2.1, we get that

$$\|w^u(\tau)\|^2_{\mathcal{X}} - \|w^u(t)\|^2_{\mathcal{X}} \geq 2\int_\tau^t |\langle w^u(s), F(w^u_s)\rangle|^2 ds \quad \text{for all } t \geq \tau \geq 0. \tag{4.19}$$

Using the feedback control (4.18), system (0.8) takes the following form :

$$\begin{cases} \dot{w}(t) = \mathcal{A}w(t) + \widehat{F}(w_t) \quad \text{for all } t > 0, \\ w_0 = \phi \in \mathcal{C} = C([-r, 0]; \mathcal{X}), \end{cases} \tag{4.20}$$

where $\widehat{F}$ is a function defined from $\mathcal{C}$ to $\mathcal{X}$ by

$$\widehat{F}(\psi) \;=\; -\langle \psi^u(0), F^u(\psi^u)\rangle F(\psi).$$

It is clear that F is a locally Lipschitz function, then, the system (0.8) admits a unique mild solution defined for all $t \in [0, t_{max})$ given by the following variation of constants formula

$$w(t) = S(t)w(0) + \int_0^t S(t-\sigma)v^u(\sigma)F(w_\sigma)d\sigma \quad \text{for all } t \in [0, t_{max}).$$

Thus

$$w^s(t) = S^s(t)w^s(0) + \int_0^t S^s(t-\sigma)v^u(\sigma)F^s(w^s_\sigma)d\sigma \text{ for all } t \in [0, t_{max}).$$

Since $S^s(t)$ is exponentially stable, so there exist $\beta > 0$ and $C \geq 1$ such that

$$\|S^s(t)\| \leq Ce^{-\beta t} \quad \text{for all } t \geq 0.$$

Then, for all $t \in [0, t_{max})$ and $\theta \in [-r, 0]$ such that $t + \theta \geq 0$

$$\|w^s(t+\theta)\|_{\mathcal{X}} \leq Ce^{-\beta(t+\theta)} \|\phi^s\|_{\mathcal{C}} + e^{-\beta t} \int_0^{t+\theta} Ce^{-\beta\theta} e^{\beta\sigma} |v^u(\sigma)| \, \|F^s(w^s_\sigma)\|_{\mathcal{X}} \, d\sigma.$$

Thus

$$\|w^s(t+\theta)\|_{\mathcal{X}} e^{\beta t} \leq Ce^{\beta r} \|\phi^s\|_{\mathcal{C}} + e^{\beta r} \int_0^t C|v^u(\sigma)|\|F^s\|\|w^s_\sigma\|_{\mathcal{C}} e^{\beta\sigma} d\sigma, \tag{4.21}$$

and for all $t \in [0, t_{max})$ and $\theta \in [-r, 0]$ such that $t + \theta \leq 0$, we have $\|w^s(t+\theta)\|_{\mathcal{X}} = \|\phi^s(t+\theta)\|_{\mathcal{X}}$. Moreover

$$\begin{aligned} \|w^s(t+\theta)\|_{\mathcal{X}} e^{\beta t} &\leq e^{-\beta\theta} \|\phi^s\|_{\mathcal{C}} \\ &\leq e^{\beta r} \|\phi^s\|_{\mathcal{C}}. \end{aligned} \tag{4.22}$$

From (4.21) and (4.22), we have for all $t \geq 0$

$$\|w^s_t\|_{\mathcal{C}} e^{\beta t} \leq Ce^{\beta r} \|\phi^s\|_{\mathcal{C}} + e^{\beta r} \int_0^t C|v^u(\sigma)|\|F^s\|\|w^s_\sigma\|_{\mathcal{C}} e^{\beta\sigma} d\sigma.$$

Using Gronwall's inequality, we have

$$\|w^s_t\|_{\mathcal{C}} e^{\beta t} \leq C e^{\beta r} \|\phi^s\|_{\mathcal{C}} e^{\left(\int_0^t Ce^{\beta r}|v^u(\sigma)|\|F\|d\sigma\right)}.$$

From inequality (4.19), we have

$$\int_0^{+\infty} |v^u(t)|^2 dt \leq \frac{K^2}{2}$$

and consequently by using Holder's inequality, we get that

$$\|w^s_t\|_{\mathcal{C}} e^{\beta t} \leq C e^{\beta r} \|\phi^s\|_{\mathcal{C}} e^{\left(Ce^{\beta r}\|F\|\frac{K}{\sqrt{2}}\sqrt{t}\right)} \text{ for all } t \geq 0,$$

and then

$$\|w^s_t\|_{\mathcal{C}} \leq C e^{\beta r} \|\phi^s\|_{\mathcal{C}} e^{\left(Ce^{\beta r}\|F\|\frac{K}{\sqrt{2}}\sqrt{t}-\beta t\right)} \text{ for all } t \geq 0.$$

Thus

$$\|w^s(t)\|_{\mathcal{X}} \leq C e^{\beta r} \|\phi^s\|_{\mathcal{C}} e^{\left(Ce^{\beta r}\|F\|\frac{K}{\sqrt{2}}\sqrt{t}-\beta t\right)} \text{ for all } t > 0.$$

Then, there exist two positive constants N and σ such that

$$\|w^s(t)\|_{\mathcal{X}} \leq \|\phi^s\|_{\mathcal{C}} Ne^{-\sigma t} \text{ for all } t > 0. \tag{4.23}$$

It follows that $w^s(t)$ is bounded on $[0, t_{max})$, so that $t_{max} = +\infty$ ■

In the following, we discuss the relation between the stability of the system (0.8) and the stability of it's projection on the subspace $\mathcal{X}^u$ in the case where $\mathcal{X}^u$ is of finite dimensional. This result is a particular case of the previous theorem with another condition that guarantees stability.

Corollary 4.4.2. *Assume that:*
1. *the state space $\mathcal{X}^u$ is of finite dimensional,*
2. *F is a linear bounded operator from $\mathcal{C}$ to $\mathcal{X}$,*
3. *$S^u(t)$ is a contraction semigroup on $\mathcal{X}^u$,*
4. *$S^s(t)$ is exponentially stable on $\mathcal{X}^s$,*
5. *there exists $T_2 > T_1 > r$ and for all $\xi \in \mathcal{C}^u$ such that*

$$\langle F^u(U^u(s)\xi), (U^u(s)\xi)(0)\rangle = 0 \quad \text{for all } s \in [T_1, T_2] \Rightarrow \xi(0) = 0,$$

hold. Then the feedback control

$$v_0^u(t) = -\langle w^u(t), F^u(w_t^u)\rangle$$

strongly stabilizes the system (0.8) and we have the following decay estimate:

$$\|w(t)\|_{\mathcal{X}} = O\left(t^{-\frac{1}{2}}\right) \text{ as } t \longrightarrow +\infty\,.$$

Proof . We apply Corollary 4.3.3 to system (4.15), we obtain that the feedback control (4.18) strongly stabilizes the system (4.15) and we have the following decay estimate:

$$\|w^u(t)\|_{\mathcal{X}} = O\left(t^{-\frac{1}{2}}\right) \text{ as } t \longrightarrow +\infty$$

Moreover, since $S^s(t)$ is exponentially stable on $\mathcal{X}^s$, then (4.23) holds which means that

$$\|w^s(t)\|_{\mathcal{X}} \leq \|\phi^s\|_{\mathcal{C}} Ne^{-\sigma t} \text{ for all } t > 0,$$

this complete the proof ■ In the following theorem, we prove the strong stabilization of the system (0.8) without using the exponential stability condition of $S^s(t)$ on $\mathcal{X}^s$.

Theorem 4.4.3. *Assume that:*
1. *F is a linear bounded operator from $\mathcal{C}$ to $\mathcal{X}$,*

2. $S^u(t)$ is a contraction semigroup on $\mathcal{X}^u$,
3. there exist positive constants T and δ such that

$$\int_0^T |\langle F^u(U^u(s+r)\xi), (U^u(s+r)\xi)(0)\rangle| ds \geq \delta \|\xi(0)\|_{\mathcal{X}}^2 \quad \textit{for all } \xi \in \mathcal{C}, \tag{4.24}$$

hold. Then, the feedback control

$$v_0^u(t) = -\langle w^u(t), F^u(w_t^u)\rangle$$

strongly stabilizes the system (0.8) and we have the following decay estimate

$$\|w(t)\|_{\mathcal{X}} = O\left(t^{-\frac{1}{2}}\right) \quad as \quad t \longrightarrow +\infty.$$

Proof . Since $\mathcal{X}^u$ and $\mathcal{X}^s$ are orthogonal, we have that $\|\xi^u(0)\|_{\mathcal{X}} \leq \|\xi(0)\|_{\mathcal{X}}$. Then, it follows from the observability condition (4.24) that

$$\begin{aligned}
\delta\|\xi^u(0)\|_{\mathcal{X}}^2 &\leq \int_0^T |\langle F^u(U^u(s+r)\xi^u), (U^u(s+r)\xi^u)(0)\rangle| \, ds \\
&\leq \int_r^{T+r} |\langle F^u(U^u(s)\xi^u), (U^u(s)\xi^u)(0)\rangle| \, ds \\
&\leq \int_0^{T+r} |\langle F^u(U^u(s)\xi^u), (U^u(s)\xi^u)(0)\rangle| \, ds,
\end{aligned}$$

and hence

$$\int_0^{\widetilde{T}} |\langle F^u(U^u(s)\xi^u), (U^u(s)\xi^u)(0)\rangle| \, ds \geq \delta\|\xi^u(0)\|_{\mathcal{X}}^2 \quad \text{for all } \xi \in \mathcal{C},$$

with $\widetilde{T} = T + r$. Applying Theorem 4.3.1, we deduce that the feedback control (4.18) strongly stabilizes the system (4.15). Moreover, we have that

$$\|w^u(t)\|_{\mathcal{X}} \leq \frac{M}{\sqrt{t}} \quad \text{for all } t > 0, \tag{4.25}$$

Using the observability condition (4.24), we obtain that

$$\delta\|\xi(0)\|_{\mathcal{X}}^2 \leq T\|F^u\|\|\xi^u(0)\|_{\mathcal{X}}^2. \tag{4.26}$$

Thus

$$(\delta - T\|F^u\|) \, \|\xi(0)\|_{\mathcal{X}}^2 \leq 0.$$

If $\mathcal{X} = \{0\}$, then $w(t) = 0$ for all $t \geq 0$. If not, then $\mathcal{X} \neq \{0\}$ and $\delta \leq T\|F^u\|$. Moreover, using the inequality (4.25) and (4.26), we get that

$$\|w(t)\|_{\mathcal{X}} \leq \left(\frac{T\|F^u\|}{\delta}\right)^{\frac{1}{2}} \frac{M}{\sqrt{t}} \text{ for all } t > 0.$$

Finally, the feedback control (4.18) strongly stabilizes the system (0.8) and we have the following decay estimate

$$\|w(t)\|_{\mathcal{X}} = O\left(t^{-\frac{1}{2}}\right) \quad \text{as} \quad t \longrightarrow +\infty \ \blacksquare$$

In the following theorem, we use the decomposition method of state space to prove the weak stabilization of system (0.8) with the feedback control (4.18).

Theorem 4.4.4. *Assume that:*
1. F is a linear bounded operator from $\mathcal{C}$ to $\mathcal{X}$,
2. $S^u(t)$ is a contraction semigroup on $\mathcal{X}^u$,
3. $S^s(t)$ is exponentially stable on $\mathcal{X}^s$,
4. there exist $T_2 > T_1 > 0$ and for all sequence $(\xi_n) \subset \mathcal{C}^u$ such that $\xi_n(0) \rightharpoonup \xi(0)$ in $\mathcal{X}^u$ we have

$$\langle (U^u(t)\xi_n)(0), F^u(U^u(t)\xi_n)\rangle \to 0, \ \text{as } n \to +\infty \ \text{ for all } t \in [T_1, T_2] \Rightarrow \xi(0) = 0, \quad (4.27)$$

then, the system (0.8) is weakly stabilizable by the feedback control $v_0^u(t) = -\langle w^u(t), F^u(w_t^u)\rangle$.

Proof . Theorem 4.3.4 implies that the feedback control (4.18) weakly stabilizes the system (4.15). Moreover, since $S^s(t)$ is exponentially stable, then (4.23) holds which means that

$$\|w^s(t)\|_{\mathcal{X}} \leq \|\phi^s\|_{\mathcal{C}} Ne^{-\sigma t} \text{ for all } t > 0.$$

Since $w(t) = w^s(t) + w^u(t)$, then, the system (0.8) is weakly stabilizable by the feedback control (4.18) $\blacksquare$

4.5 Exponential Stabilization with normalized control

We can establish the exponential stability of system (0.8) based on Theorem 4.2.1, which leads to the following theorem.

Theorem 4.5.1. *Assume that:*
1. $\mathcal{A}$ generates a $\mathcal{C}_0-$semigroup of contraction on $\mathcal{X}$,
2. F is a locally Lipschitz from $\mathcal{C}$ to $\mathcal{X}$, such that $F(0) = 0$,
3. there exist $T > 0$ and $\delta > 0$ such that

$$\int_0^T |\langle F(U(s+r)\xi), (U(s+r)\xi)(0)\rangle| ds \geq \delta \|\xi(0)\|_{\mathcal{X}}^2 \ \text{ for all } \xi \in \mathcal{C}, \quad (4.28)$$

hold. Then, the feedback control

$$v_2(t) = -\frac{\langle w(t), F(w_t)\rangle}{\|w_t\|_{\mathcal{C}}^2}\mathbf{1}_E$$

exponentially stabilizes the system (0.8).

Proof . According to Theorem 4.2.1, system (0.8) controlled by $v_2(t) = -\frac{\langle w(t),F(w_t)\rangle}{\|w_t\|_{\mathcal{C}}^2}\mathbf{1}_E$ possesses a unique global mild solution defined on $[-r, +\infty[$ given by the following variation of the constant formula

$$w(s) = S(s)w(0) - \int_0^s S(s-\sigma)\frac{\langle w(\sigma), F(w_\sigma)\rangle}{\|w_\sigma\|_{\mathcal{C}}^2}\mathbf{1}_E F(w_\sigma)d\sigma \text{ for all } s \geq 0.$$

The semigroup $S(s)$ is of contraction and by the Holder's inequality, one has

$$\|w(s) - S(s)w(0)\|_{\mathcal{X}} \leq \Gamma_K\sqrt{T+r}\left(\int_0^s \frac{|\langle w(\sigma), F(w_\sigma)\rangle|^2}{\|w_\sigma\|_{\mathcal{C}}^2}\mathbf{1}_E d\sigma\right)^{\frac{1}{2}} \quad \text{for all } s \in [0, T+r].$$

Then, for all $t \geq 0$

$$\begin{aligned}&\|w(s+t) - S(s)w(t)\|_{\mathcal{X}}\\ &\qquad \leq \Gamma_K\sqrt{T+r}\left(\int_t^{t+s} \frac{|\langle w(\sigma), F(w_\sigma)\rangle|^2}{\|w_\sigma\|_{\mathcal{C}}^2}\mathbf{1}_E d\sigma\right)^{\frac{1}{2}} \qquad (4.29)\\ &\qquad\qquad \text{for all } s \in [0, T+r].\end{aligned}$$

Moreover, for all $s \geq 0$, we have that

$$\begin{aligned}\langle (U(s+r)w_t)(0), F(U(s+r)w_t)\rangle &= \langle S(s+r)w(t), F(U(s+r)w_t)\rangle\\ &= \langle w(s+r+t), F(U(s+r)w_t) - F(w_{s+r+t})\rangle\\ &\quad - \langle w(s+r+t) - S(s+r)w(t), F(U(s+r)w_t)\rangle\\ &\quad + \langle w(s+r+t), F(w_{s+r+t})\rangle.\end{aligned}$$

Then, for $s \in [0, T]$, we get that

$$\begin{aligned}|\langle (U(s+r)w_t)(0), F(U(s+r)w_t)\rangle| \leq\ &\|w(t)\|\Gamma_K \|w_{s+r+t} - U(s+r)w_t\|_{\mathcal{C}}\\ &+ \|w(t)\|\Gamma_K \|w(s+r+t) - S(s+r)w(t)\|_{\mathcal{X}}\\ &+ |\langle w(s+r+t), F(w_{s+r+t})\rangle|.\end{aligned}$$

From the last inequality and (4.29), It follow that

$$\begin{aligned}|\langle (U(s+r)w_t)(0), F(U(s+r)w_t)\rangle| \leq\ &2\|w(t)\|\,(\Gamma_K)^2\sqrt{T+r}\left(\int_t^{t+r+T} \frac{|\langle w(\sigma), F(w_\sigma)\rangle|^2}{\|w_\sigma\|_{\mathcal{C}}^2}\mathbf{1}_E d\sigma\right)^{\frac{1}{2}}\\ &+ |\langle w(s+r+t), F(w_{s+r+t})\rangle|.\end{aligned}$$

Integrating the last inequality over $[0,T]$ with respect to s and using Holder's inequality, we obtain that

$$\begin{aligned}\int_0^T |\langle (U(s+r)w_t)(0), F(U(s+r)w_t)\rangle| ds \\ &\leq 2T\|w(t)\|_{\mathcal{X}} \ (\Gamma_K)^2 \sqrt{T+r} \left(\int_t^{t+r+T} \frac{|\langle w(\sigma), F(w_\sigma)\rangle|^2}{\|w_\sigma\|_{\mathcal{C}}^2} \mathbf{1}_E d\sigma \right)^{\frac{1}{2}} \\ &+ \int_0^T |\langle w(s+r+t), F(w_{s+r+t})\rangle| \, ds.\end{aligned}$$

Since $t \to \|w(t)\|$ is a decreasing function on $\mathbb{R}^+$, we can see that

$$\|w_s\|_{\mathcal{C}} \leq \|w(t)\|_{\mathcal{X}} \ \text{ for all } s \in [t+r, t+r+T],$$

and by using the variable change, we get

$$\begin{aligned}\int_0^T |\langle w(s+r+t), F(w_{s+r+t})\rangle| ds &\leq \int_{t+r}^{t+r+T} |\langle w(s), F(w_s)\rangle| ds \\ &\leq \int_{t+r}^{t+r+T} \frac{|\langle w(s), F(w_s)\rangle|}{\|w_s\|_{\mathcal{C}}} \mathbf{1}_E \|w_s\|_{\mathcal{C}} ds \\ &\leq \sqrt{T}\|w(t)\|_{\mathcal{X}} \left(\int_t^{t+r+T} \frac{|\langle w(s), F(w_s)\rangle|^2}{\|w_s\|_{\mathcal{C}}^2} \mathbf{1}_E ds \right)^{\frac{1}{2}}\end{aligned}$$

Then

$$\int_0^T |\langle (U(s+r)w_t)(0), F(U(s+r)w_t)\rangle| ds \leq \widetilde{C}\|w(t)\|_{\mathcal{X}} \left(\int_t^{t+r+T} \frac{|\langle w(s), F(w_s)\rangle|^2}{\|w_s\|_{\mathcal{C}}^2} \mathbf{1}_E ds \right)^{\frac{1}{2}},$$

where $\widetilde{C} = 2T\,(\Gamma_K)^2 \sqrt{T+r} + \sqrt{T}$. Using the observability condition (4.28) with $\xi = w_t$, we get that

$$\begin{aligned}\delta\|w(t)\|_{\mathcal{X}}^2 &\leq \int_0^T |\langle (U(s+r)w_t)(0), F(U(s+r)w_t)\rangle| d\sigma \\ &\leq \widetilde{C}\|w(t)\|_{\mathcal{X}} \left(\int_t^{t+r+T} \frac{|\langle w(s), F(w_s)\rangle|^2}{\|w_s\|_{\mathcal{C}}^2} \mathbf{1}_E ds \right)^{\frac{1}{2}}.\end{aligned}$$

$$\frac{\delta^2}{\widetilde{C}^2}\|w(j(r+T))\|_{\mathcal{X}}^2 \leq \int_{j(r+T)}^{(j+1)(r+T)} \frac{|\langle w(s), F(w_s)\rangle|^2}{\|w_s\|_{\mathcal{C}}^2} \mathbf{1}_E ds. \tag{4.30}$$

Using inequality (4.3) with $p=2$, we obtain that

$$\|w(j(r+T))\|_{\mathcal{X}}^2 - \|w((j+1)(r+T))\|_{\mathcal{X}}^2 \geq 2 \int_{j(r+T)}^{(j+1)(r+T)} \frac{|\langle w(s), F(w_s)\rangle|^2}{\|w_s\|_{\mathcal{C}}^2} \mathbf{1}_E ds,$$

and

$$\|w((j+1)(r+T))\|_{\mathcal{X}}^2 - \|w(j(r+T))\|_{\mathcal{X}}^2 \leq -2\int_{j(r+T)}^{(j+1)(r+T)} \frac{|\langle w(s), F(w_s)\rangle|^2}{\|w_s\|_{\mathcal{C}}^2}\mathbf{1}_E ds.$$

Using inequality (4.30), we get that

$$\|w((j+1)(r+T))\|_{\mathcal{X}}^2 - \|w(j(r+T))\|_{\mathcal{X}}^2 \leq -\frac{2\delta^2}{\widetilde{C}^2}\|w(j(r+T))\|_{\mathcal{X}}^2,$$

and

$$\|w((j+1)(r+T))\|_{\mathcal{X}}^2 + \frac{2\delta^2}{\widetilde{C}^2}\|w(j(r+T))\|_{\mathcal{X}}^2 \leq \|w(j(r+T))\|_{\mathcal{X}}^2.$$

Observing that $\|w((j+1)(r+T)))\|_{\mathcal{X}}^2 \leq \|w(j(r+T)))\|_{\mathcal{X}}^2$. Then, we derive that

$$\begin{aligned}\|w((j+1)(r+T))\|_{\mathcal{X}}^2 + \frac{2\delta^2}{\widetilde{C}^2}\|w((j+1)(r+T))\|_{\mathcal{X}}^2 \\ \leq \|w(j(r+T))\|_{\mathcal{X}}^2,\end{aligned}$$

which means that for all $j \in \mathbb{N}$, we have that

$$\|w((j+1)(r+T))\|_{\mathcal{X}}^2 \leq \frac{1}{1+\frac{2\delta^2}{\widetilde{C}^2}}\|w(j(r+T))\|_{\mathcal{X}}^2$$

and then

$$\|w(j(r+T))\|_{\mathcal{X}}^2 \leq \frac{1}{\left(1+\frac{2\delta^2}{\widetilde{C}^2}\right)^j}\|w(0)\|_{\mathcal{X}}^2.$$

Let us now set $j = E(\frac{t}{r+T})$ where $E(.)$ designs the integer part. Thus

$$\frac{1}{\left(1+\frac{2\delta^2}{\widetilde{C}^2}\right)^j} \leq \frac{1}{\left(1+\frac{2\delta^2}{\widetilde{C}^2}\right)^{\frac{t}{r+T}-1}}.$$

Since $t \to \|w(t)\|_{\mathcal{X}}$ is a decreasing function, then

$$\begin{aligned}\|w(t)\|_{\mathcal{X}} &\leq \|w(j(r+T))\|_{\mathcal{X}} \\ &\leq \sqrt{1+\frac{2\delta^2}{\widetilde{C}^2}}\left(1+\frac{2\delta^2}{\widetilde{C}^2}\right)^{-\frac{t}{2(r+T)}}\|w(0)\|_{\mathcal{X}},\end{aligned}$$

and hence, we obtain the following exponential decay estimates

$$\|w(t)\|_{\mathcal{X}} \leq \sqrt{1+\frac{2\delta^2}{\widetilde{C}^2}}\exp\left(\frac{-\ln(1+\frac{2\delta^2}{\widetilde{C}^2})t}{2(r+T)}\right)\|\phi\|_{\mathcal{C}} \text{ for all } t \geq 0 \ \blacksquare$$

In the following result, we show the exponential stability of the system (0.8) in the case of finite-dimensional space.

Theorem 4.5.2. *Assume that:*
1. the state space $\mathcal{X}^u$ is of finite dimensional,
2. $\mathcal{A}$ generates a $\mathcal{C}_0-$semigroup of contraction on $\mathcal{X}$,
3. F is a locally Lipschitz from $\mathcal{C}$ to $\mathcal{X}$, such that $F(0)=0$,
4. there exists $\mu>0$ such that

$$|\langle F(\varphi),\varphi(0)\rangle| \geq \mu\|\varphi(0)\|_{\mathcal{X}}^2 \quad \text{for all } \varphi \in \mathcal{C}, \tag{4.31}$$

hold. Then, the feedback control

$$v_2(t) = -\frac{\langle w(t), F(w_t)\rangle}{\|w_t\|_{\mathcal{C}}^2}\mathbf{1}_E$$

exponentially stabilizes the system (0.8).

Proof. Using the inequality (4.31) where $\varphi = U(s+r)\xi$ for $s\in[0,T]$ and $\xi\in\mathcal{C}$, then

$$|\langle F(U(s+r)\xi),(U(s+r)\xi)(0)\rangle| \geq \mu\|(U(s+r)\xi)(0)\|_{\mathcal{X}}^2.$$

Since $S(s)$ is a semigroup of contraction, then $\|S(s+r)\xi(0)\|_{\mathcal{X}}$ decreases with respect to $s\in[0,T]$ and we have

$$\begin{aligned}|\langle F(U(s+r)\xi),(U(s+r)\xi)(0)\rangle| \geq& \mu\|S(s+r)\xi(0)\|_{\mathcal{X}}^2\\ \geq& \widetilde{\mu}\|\xi(0)\|_{\mathcal{X}}^2\end{aligned}$$

where $\widetilde{\mu} = \mu \inf\limits_{\|y\|_{\mathcal{X}}=1}\left(\|S(T+r)y\|_{\mathcal{X}}\right)$.
Integrating the last inequality over $[0,T]$ with respect to s, we have that

$$\int_0^T |\langle F(U(s+r)\xi),(U(s+r)\xi)(0)\rangle| ds \geq \delta\|\xi(0)\|_{\mathcal{X}}^2$$

with $\delta = \widetilde{\mu}T$. Then, Theorem 4.5.1 implies that the feedback control $v_2(t) = -\frac{\langle w(t),F(w_t)\rangle}{\|w_t\|_{\mathcal{C}}^2}\mathbf{1}_E$ exponentially stabilizes the system (0.8) ■

Remark 4.5.3. *In the bilinear case without delay, the condition (4.31) means that the operator F is coercive.*

4.6 Application and Numerical Simulation

In this section, we present three examples in order to improve the theoretical results obtained in previous sections.

Example 1: Let us consider the following system with distributed delay defined by

$$\begin{cases} \dfrac{\partial w(x,t)}{\partial t} = -w(x,t) + v(t)\|w(t)\|_{\mathcal{X}} \displaystyle\int_{-2}^{0} k(s)w(x,t+s)ds & \\ & \text{for } x \in (0,1),\ t > 0, \\ w(x,t) = \phi(x,t), & \text{for } x \in (0,1),\ t \in [-2,0], \end{cases} \tag{4.32}$$

where k is a continuous function from $[-2,0]$ to $\mathbb{R}^+$ such that $k(s) > 0$. Let $\mathcal{X}$ be such that $\mathcal{X} = L^2(0,1)$. The operator $\mathcal{A}$ is defined on $\mathcal{X}$ by $\mathcal{A}w = -w$ and the operator F is defined from $C([-2,0],\mathcal{X})$ to $\mathcal{X}$ as follows:

$$F(\xi) = \|\xi(0)\|_{\mathcal{X}} \int_{-2}^{0} k(s)\xi(s)ds.$$

The operator $\mathcal{A}$ generates a semigroup of contraction on $\mathcal{X}$ given by $S(t)w = e^{-t}w$. Clearly, F is a locally Lipschitz function.
Let $c = \min\limits_{s\in[-2,0]} k(s)$. For all $\xi \in \mathcal{C}$ and $t \in [2,3]$

$$\begin{aligned} |\langle F(U(t)\xi), (U(t)\xi)(0)\rangle| &= |\langle \|(U(t)\xi)(0)\|_{\mathcal{X}} \int_{-2}^{0} k(s)(U(t)\xi)(s)ds, (U(t)\xi)(0)\rangle| \\ &= \|S(t)\xi(0)\|_{\mathcal{X}}\ |\langle \int_{-2}^{0} k(s)S(s+t)\xi(0)ds, S(t)\xi(0)\rangle| \end{aligned}$$

Integrating the last inequality over $[2,3]$ with respect to t, we have

$$\begin{aligned} \int_2^3 |\langle F(U(t)\xi), (U(t)\xi)(0)\rangle| dt &= \int_2^3 \|S(t)\xi(0)\|_{\mathcal{X}}\ |\langle \int_{-2}^{0} k(s)S(s+t)\xi(0)ds, S(t)\xi(0)\rangle| dt \\ &\geq \frac{c}{3}(e^{-6} - e^{-9})(e^2 - 1)\|\xi(0)\|_{\mathcal{X}}^3. \end{aligned}$$

Then, condition (4.7) is verified. Consequently, Theorem 4.3.1 assures that the feedback control

$$v_0(t) = -\langle w(t), F(w_t)\rangle$$

for all $t \geq 0$, strongly stabilizes the system (4.32) and we have the estimate :

$$\|w(t)\|_{\mathcal{X}} = O\left(t^{-\frac{1}{4}}\right) \text{ as } t \longrightarrow +\infty.$$

Example 2: Let us consider the following mono-dimensional bilinear system with distributed delay and Neumann boundary conditions:

$$\begin{cases} \dfrac{\partial w(x,t)}{\partial t} = \dfrac{\partial^2 w(x,t)}{\partial x^2} + v(t)\Big(\displaystyle\int_{-3}^{0} k(s)w(x,t+s)ds & \\ \qquad\qquad +\alpha w(x,t)\Big) & \text{for } x \in (0,1),\ t \geq 0, \\ \dfrac{\partial w(0,t)}{\partial x} = \dfrac{\partial w(1,t)}{\partial x} = 0 & \text{for } t \in [-3,+\infty), \\ w(x,t) = \phi(x,t), & \text{for } x \in (0,1),\ t \in [-3,0]. \end{cases} \tag{4.33}$$

where k is a continuous function from $[-3,0]$ to $\mathbb{R}^+$ such that $k(s) > 0$ and $\alpha \geq 0$. Here, we take the state space $\mathcal{X} = L^2(0,1)$ and the operator $\mathcal{A}$ is defined by $\mathcal{A}w = \dfrac{\partial^2 w}{\partial x^2}$, with

$$\mathcal{D}(\mathcal{A}) = \left\{ w \in H^2(0,1);\ \frac{\partial w(0,t)}{\partial x} = \frac{\partial w(1,t)}{\partial x} = 0 \right\}.$$

The operator $\mathcal{A}$ is an infinitesimal generator of a semigroup $S(t)$ of contraction (see [31]). The spectrum of $\mathcal{A}$ is given by simple eigenvalues $\lambda_m = -\pi^2(m-1)^2$, for $m \geq 1$ with its corresponding eigenfunctions $\varphi_1(x) = 1$ and $\varphi_m(x) = \sqrt{2}\cos((m-1)\pi x)$, for $m \geq 2$. The operator F defined from $C([-3,0],\mathcal{X})$ to $\mathcal{X}$ by

$$F(\xi) \ = \ \int_{-3}^{0} k(s)\xi(s)ds + \alpha\xi(0).$$

Clearly, F is a linear bounded operator. The semigroup $S(t)$ defined from $\mathcal{X}$ to $\mathcal{X}$ by

$$S(t)w = \sum_{m=1}^{+\infty} e^{\lambda_m t}\langle w, \varphi_m\rangle\varphi_m.$$

Let $c = \min\limits_{s\in[-3,0]} k(s)$. For all $\xi \in \mathcal{C}$ and $t \geq 3$

$$\begin{aligned}\langle F(U(t)\xi), (U(t)\xi)(0)\rangle \ \geq \ & 3c\,|\langle \xi(0), \varphi_1\rangle|^2 \\ & + c\sum_{m=2}^{+\infty} e^{2\lambda_m t}\left(\frac{1-e^{-3\lambda_m}}{\lambda_m}\right)|\langle \xi(0), \varphi_m\rangle|^2 \\ & + \alpha\|S(t)\xi(0)\|_{\mathcal{X}}^2.\end{aligned}$$

Let $a_m = e^{2\lambda_m t}\left(\frac{1-e^{-3\lambda_m}}{\lambda_m}\right) > 0$ for all $m \geq 2$. Then, condition (4.14) holds and consequently, system (4.33) is weakly stabilizable using the following feedback control

$$v_0(t) = -\int_0^1 w(x,t)\left(\int_{-3}^{0} k(s)w(x,t+s)ds + \alpha w(x,t)\right)dx.$$

Now, the objective is to use the decomposition method to show the weak stabilization of system (4.33). The state space $\mathcal{X}$ can be decomposed as follows to $\mathcal{X} = \mathcal{X}^u \oplus \mathcal{X}^s$, where $\mathcal{X}^u = \text{span}\{\varphi_1\}$ and $\mathcal{X}^s = \text{span}\{\varphi_m;\ m \geq 2\}$. Then, $S^u(t) = Id_{\mathcal{X}^u}$ and we have for $\xi^u \in \mathcal{C}^u$ and $t \geq 3$

$$\langle F^u(U^u(t)\xi^u), (U^u(t)\xi^u)(0)\rangle \ \geq \ 3c\,|\langle \xi^u(0), \varphi_1\rangle|^2 + \alpha\|S^u(t)\xi^u(0)\|_{\mathcal{X}}^2$$

Then, condition (4.27) holds and we have

$$S^s(t)w^s \ = \ \sum_{m=2}^{+\infty} e^{2\lambda_m t}\,\langle w^s, \varphi_m\rangle\,\varphi_m$$

and

$$\|S^s(t)w^s\|_{\mathcal{X}} \leq e^{2\lambda_2 t}\|w^s\|_{\mathcal{X}}.$$

Consequently $S^s(t)$ is exponentially stable. Then the conditions of Theorem 4.4.4 is satisfied, we conclude that system (4.33) is weakly stabilizable using the following feedback control

$$v_0^u(t) = -\int_0^1 w^u(x,t)\left(\int_{-3}^0 k(s)w^u(x,t+s)ds + \alpha w^u(x,t)\right)dx.$$

Example 3: Let $\Omega = (0,1)$ and consider the following system mono-dimensional wave equation with distributed delay:

$$\begin{cases} \dfrac{\partial^2 w(x,t)}{\partial t^2} = \dfrac{\partial^2 w(x,t)}{\partial x^2} + v(t)\left(\displaystyle\int_{-2}^0 \frac{\partial w(x,t+s)}{\partial t}ds + \frac{\partial w(x,t)}{\partial t}\right), \\ \qquad\qquad \text{for } x \in \Omega,\ t \geq 0, \\ w(0,t) = w(1,t) = 0, \qquad \text{for } t \in [-2,+\infty), \\ w(x,t) = \phi_1(x,t),\ \dfrac{\partial w(x,t)}{\partial t} = \phi_2(x,t), \quad \text{for } x \in \Omega, t \in [-2,0]. \end{cases} \tag{4.34}$$

Consider the following space $\mathcal{X} = H_0^1(\Omega) \times L^2(\Omega)$ with the following inner product:

$$\langle (x_1,x_2),(y_1,y_2)\rangle = \langle x_1,y_1\rangle_{H_0^1(0,1)} + \langle x_2,y_2\rangle_{L^2(0,1)}.$$

Let

$$z = \begin{pmatrix} w \\ \dfrac{\partial w}{\partial t} \end{pmatrix}$$

then, the system (4.34) takes the following abstract form:

$$\begin{cases} \dot{z}(t) = \mathcal{A}z(t) + v(t)F(z_t) \quad \text{for all } t > 0, \\ z_0 = \phi \in \mathcal{C} = C([-r,0];\mathcal{X}), \end{cases} \tag{4.35}$$

where the operator $\mathcal{A} : \mathcal{D}(\mathcal{A}) \subset \mathcal{X} \to \mathcal{X}$ defined by

$$\mathcal{A} = \begin{pmatrix} 0 & I \\ \Delta & 0 \end{pmatrix}$$

with

$$\mathcal{D}(\mathcal{A}) = \left(H^2(\Omega) \cap H_0^1(\Omega)\right) \times H_0^1(\Omega),$$

and $\Delta = \dfrac{\partial^2}{\partial x^2}$ is the Laplacian operator with eigenvalues given by $\lambda_m = -(m\pi)^2, (m \in \mathbb{N}^*)$ associated to the eigenfunctions $(\varphi_m)_{m\geq 1}$ expressed by

$$\varphi_m(x) = \sqrt{2}\sin(m\pi x), \text{ for all } m \geq 1.$$

And the operator F defined from $\mathcal{C} = C([-2,0],\mathcal{X})$ to $\mathcal{X}$ by

$$F(\xi) = \begin{pmatrix} 0 \\ \int_{-2}^{0} \xi_2(s)ds + \xi_2(0) \end{pmatrix}.$$

for $\xi = (\xi_1, \xi_2) \in \mathcal{C}$. It is clear that F is a locally Lipschitz function. Let $w = (w_1, w_2) \in \mathcal{X}$ with $w_1 = \sum_{m=1}^{\infty} \alpha_m \varphi_m$, $w_2 = \sum_{m=1}^{\infty} (-\lambda_m)^{\frac{1}{2}} \beta_m \varphi_m$ and $(\alpha_m, \beta_m) \in \mathbb{R}^2$ for all $m \geq 1$. Using the separation of variables of the semigroup of isometries $S(t)$ generated by the skew adjoint operator $\mathcal{A}$ (for more details see [31]), we obtain that

$$S(t)w = \sum_{m=1}^{+\infty} \begin{pmatrix} \alpha_m \cos(\sqrt{-\lambda_m}t) + \beta_m \sin(\sqrt{-\lambda_m}t) \\ -\alpha_m\sqrt{-\lambda_m}\sin(\sqrt{-\lambda_m}t) + \beta_m\sqrt{-\lambda_m}\cos(\sqrt{-\lambda_m}t) \end{pmatrix} \varphi_m$$

Then, for all $t \in [2,3]$ and $\xi \in \mathcal{C}$ we have

$$\begin{aligned} F(U(t)\xi) = &\sum_{m=1}^{+\infty} \begin{pmatrix} 0 \\ \int_{-2}^{0} -\alpha_m\sqrt{-\lambda_m}\sin(\sqrt{-\lambda_m}(t+s))ds \end{pmatrix} \varphi_m \\ &+ \begin{pmatrix} 0 \\ \int_{-2}^{0} \beta_m\sqrt{-\lambda_m}\cos(\sqrt{-\lambda_m}(t+s))ds \end{pmatrix} \varphi_m \\ &+ \begin{pmatrix} 0 \\ -\alpha_m\sqrt{-\lambda_m}\sin(\sqrt{-\lambda_m}t) + \beta_m\sqrt{-\lambda_m}\cos(\sqrt{-\lambda_m}t) \end{pmatrix} \varphi_m, \end{aligned}$$

and

$$\begin{aligned} \langle F(U(t)\xi), (U(t)\xi)(0)\rangle \;=\; &\sum_{m=1}^{+\infty} -\lambda_m \Big(\alpha_m^2 \sin^2(\sqrt{-\lambda_m}t) \;- 2\alpha_m\beta_m \sin(\sqrt{-\lambda_m}t)\cos(\sqrt{-\lambda_m}t) \\ &+\beta_m^2 \cos^2(\sqrt{-\lambda_m}t)\Big). \end{aligned}$$

Integrating the last equation from 2 to 3, we obtain

$$\begin{aligned} \int_2^3 |\langle F(U(t)\xi), (U(t)\xi)(0)\rangle| dt \;&=\; \frac{1}{2}\sum_{m=1}^{+\infty} -\lambda_m \left(\alpha_m^2 + \beta_m^2\right) \\ &= \frac{1}{2}\|\xi(0)\|_{\mathcal{X}}^2. \end{aligned}$$

Since

$$\int_0^3 |\langle F(U(t)\xi), (U(t)\xi)(0)\rangle| dt \;\geq\; \int_2^3 |\langle F(U(t)\xi), (U(t)\xi)(0)\rangle| dt.$$

Then, condition (4.7) holds. Hence, Theorem 4.3.1 implies that the feedback control

$$v_0(t) = -\int_0^1 \frac{\partial w(x,t)}{\partial t}\left(\int_{-2}^0 \frac{\partial w(x,t+s)}{\partial t}ds + \frac{\partial w(x,t)}{\partial t}\right)dx \tag{4.36}$$

for all $t \geq 0$, strongly stabilizes the system (4.34) and we have the estimate :

$$\|z(t)\|_{\mathcal{X}} = O\left(t^{-\frac{1}{2}}\right) \text{ as } t \longrightarrow +\infty.$$

Now, we will show that the system (4.34) is exponentially stabilizable using the renormalized control. Let $\xi \in \mathcal{C}$

$$\begin{aligned}\int_0^1 |\langle F(U(t+3)\xi), (U(t+3)\xi)(0)\rangle|dt &= \int_3^4 |\langle F(U(t)\xi), (U(t)\xi)(0)\rangle|dt \\ &= \sum_{m=1}^{+\infty} -\lambda_m\left(\alpha_m^2 + \beta_m^2\right) \\ &= \frac{1}{2}\|\xi(0)\|_{\mathcal{X}}^2.\end{aligned}$$

Then, condition (4.28) holds. Hence, Theorem 4.5.1 implies that the feedback control

$$v_2(t) = -\frac{\langle z(t), F(z_t)\rangle}{\|z_t\|_{\mathcal{C}}^2}\mathbf{1}_E \tag{4.37}$$

exponentially stabilizes the system (4.34).

To illustrate the obtained theoretical results of example 3, we consider the following initial condition

$$\phi_1(x,t) = \frac{\pi^2}{2}\, t\, e^{-(1-x)}.$$

Figures 4.1 and 4.2 illustrate the evolution of the first component $w(x,t)$ and the second component $\dfrac{\partial w(x,t)}{\partial t}$ of solution of system (4.34) in the case where the feedback control $v(t) = 0$. Moreover, from the fact that the semigroup $S(t)$ is isometries, this means that $\langle \mathcal{A}z, z\rangle = 0$ for all $z \in \mathcal{D}(\mathcal{A})$, we get that $\dfrac{d}{dt}\left\|(w(t), \dfrac{\partial w(t)}{\partial t})\right\|_{\mathcal{X}}^2 = 0$ and consequently $\left\|(w(t), \dfrac{\partial w(t)}{\partial t})\right\|_{\mathcal{X}} = \left\|(w(0), \dfrac{\partial w(0)}{\partial t})\right\|_{\mathcal{X}}$ for all $t > 0$. We mention that this theoretical result is supported by the numerical result obtained in figure 4.3. Moreover, if we consider the feedback control (4.37), figures 4.4 and 4.5 illustrate the evolution of the first and the second components of the solution of system (4.34) using the feedback control (4.37). The last figure shows clearly the speed of convergence to 0 as $t \to +\infty$ of the norm of the solution of system (4.34) controlled by the both feedback controls

(4.36) and (4.37). This means that the feedback control (4.37) exponentially stabilizes the system (4.34).

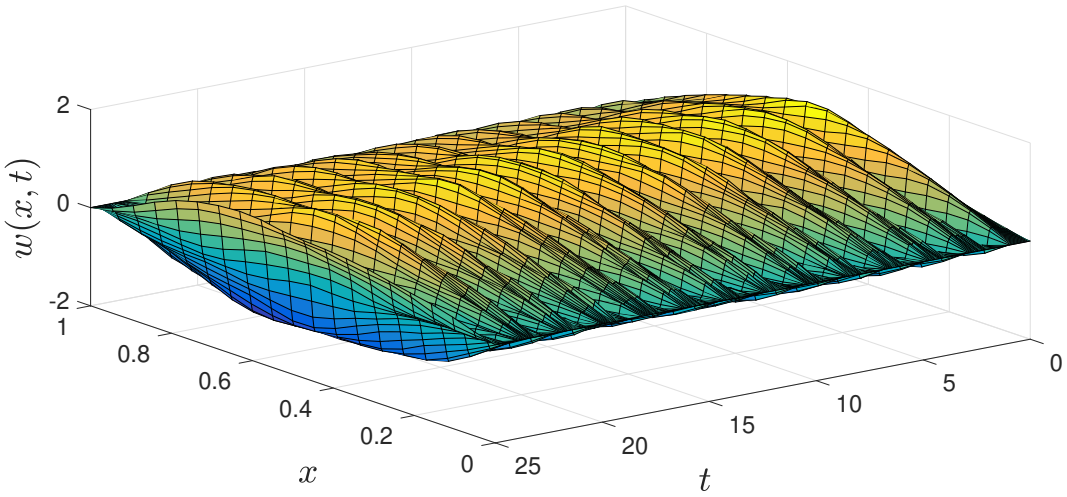

Figure 4.1: The evolution of the first $w(x,t)$ component of the free state.

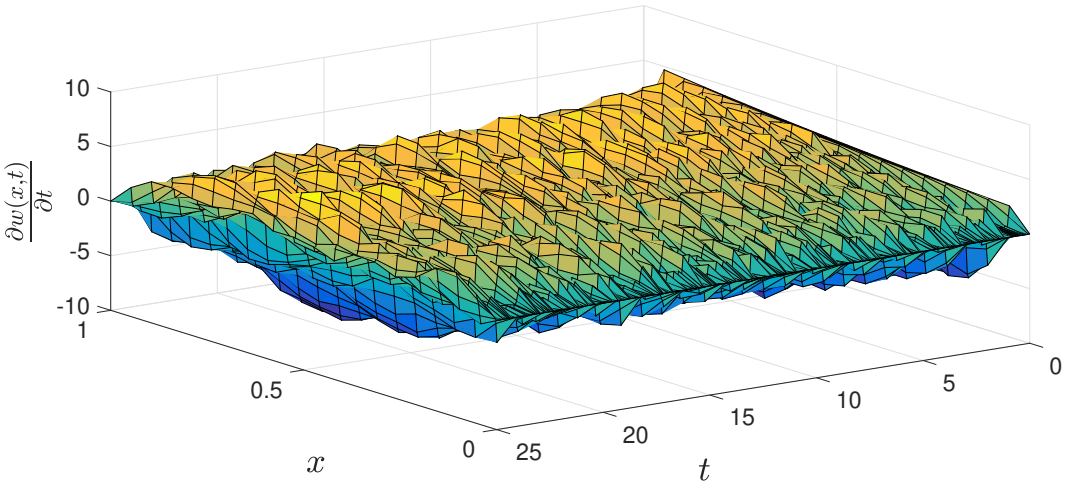

Figure 4.2: The evolution of the second component $\dfrac{\partial w(x,t)}{\partial t}$ of the free state.

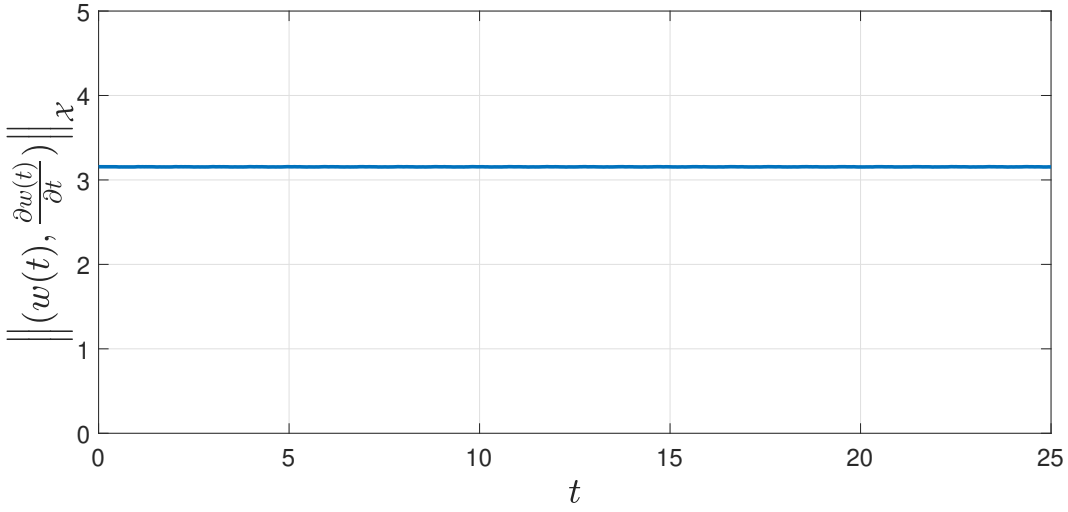

Figure 4.3: The evolution of the stabilized state norm $\left\|(w(t), \frac{\partial w(t)}{\partial t})\right\|_{\mathcal{X}}$ of the free state.

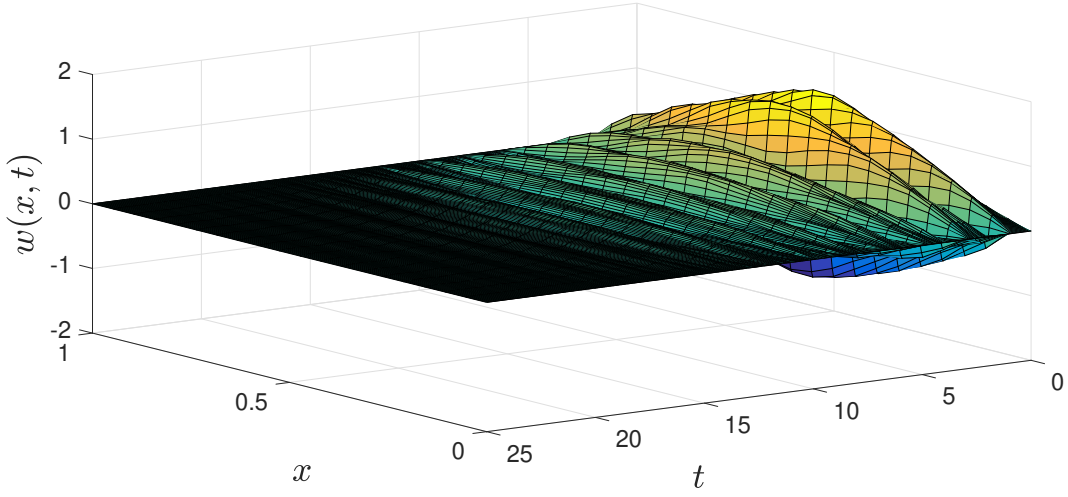

Figure 4.4: The evolution of the first component $w(x,t)$ of the stabilized state.

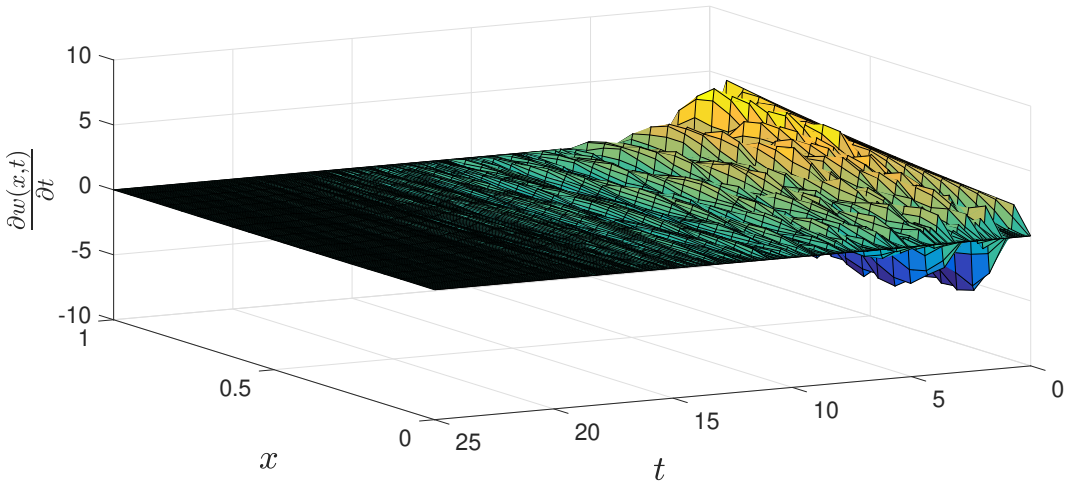

Figure 4.5: The evolution of the second component $\frac{\partial w(x,t)}{\partial t}$ of the stabilized state.

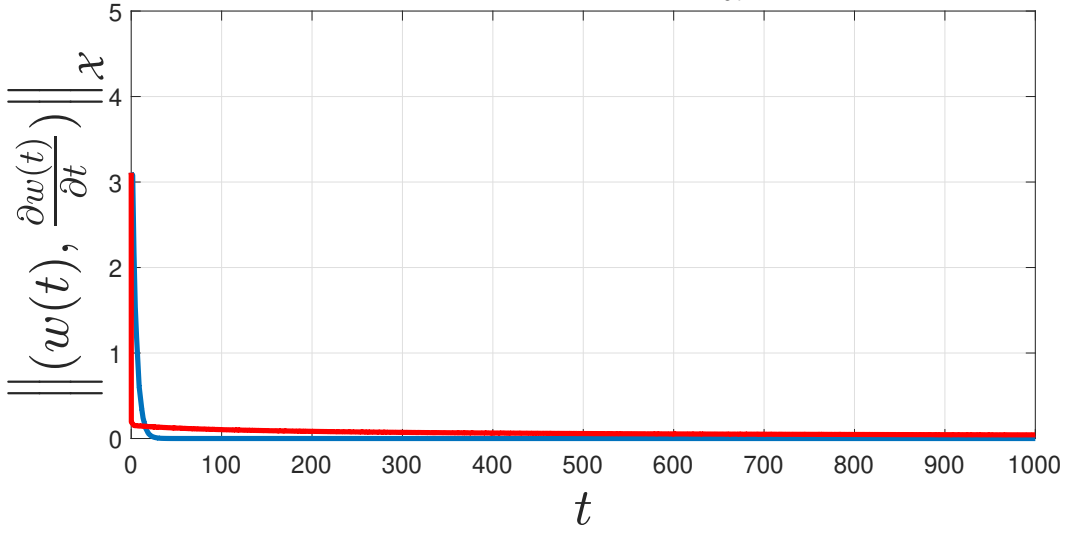

Figure 4.6: The evolution of the stabilized state norm $\left\|(w(t), \frac{\partial w(t)}{\partial t})\right\|_{\mathcal{X}}$ by control (4.37) (colored in blue) and by control (4.36) (colored in red).

Conclusion. This chapter focuses on the feedback stabilization analysis of the semilinear system (0.8) with distributed delay. We presented more general results on the feedback stabilization of system (0.8). We considered an observability condition in terms of the semigroup solution of the system (0.8) with $F = 0$ and parameterized by a positive constant γ. According to the parameter γ, we gave results on the feedback stabilization of system (0.8) (weak, strong and exponential stabilization) using the quadratic and normalized controls. Moreover, in the special case of the bilinear system, we discussed the feedback stabilization using the decomposition method of the state space $\mathcal{X}$.

Chapter 5

Optimal feedback control for a class of infinite dimensional semilinear systems with distributed delay

5.1 Introduction

This chapter deals with the feedback stabilization of the semilinear system (0.8) with distributed delay. More precisely, we consider an observation condition in terms of the semigroup solution of system (0.8) with $F = 0$ and parameterized by a positive constant p. We show that under this observation condition with $p \geq 2$, the feedback control strongly stabilizes system (0.8). An explicit polynomial decay rate of the stabilization state is estimated by

$$\|w(t)\|_{\mathcal{X}} = O\left(t^{-\frac{1}{2(p-1)}}\right) \text{ as } t \longrightarrow +\infty.$$

Moreover, under some suitable conditions, we prove the weak stabilization of system (0.8). Furthermore, we prove that one of the used family of feedback controls is the unique solution of an appropriate minimization problem. In the case where the state space $\mathcal{X}$ is of finite dimension, we give another condition to guarantee that the solution of the given minimization problem exists and unique and strongly stabilizes system (0.8). Moreover, in the case where F is a bounded linear operator and $\mathcal{X}$ is of finite dimensional, we give another condition weaker than the considered observation condition to prove that the unique solution of the given minimization problem strongly stabilizes system (0.8). Finally, as examples, two physical systems with distributed delay and with numerical simulations are given to illustrate the obtained theoretical results.

5.2 Strong stabilization

The following result will be used to get the strong stabilization result. We consider the family of feedback controls defined by :

$$v_\alpha(t) \;=\; -\frac{\langle F(w_t), w(t)\rangle}{1+\alpha|\langle F(w_t), w(t)\rangle|},\ \alpha \;\geq\; 0 \tag{5.1}$$

Here, the control $v_\alpha(t)$ depends on the parameter α, We will show numerically the impact of varying this parameter on the strong stabilization of the system (0.8), see figure 2 in section 6.

Theorem 5.2.1. *Let $\mathcal{A}$ generate a $\mathcal{C}_0-$semigroup of contraction $(S(t))_{t\geq 0}$ on $\mathcal{X}$. Suppose that F is a locally Lipschitz function from $\mathcal{C}$ to $\mathcal{X}$, i.e: for each $K>0$ there exists a constant Γ_K such that*

$$\|F(w_1)-F(w_2)\|_{\mathcal{X}} \;\leq\; \Gamma_K\|w_1-w_2\|_{\mathcal{C}} \quad \textit{for all}\ (w_1,w_2)\;\in\;\mathcal{C}^2$$

with $\|w_1\|_{\mathcal{C}} \leq K, \|w_2\|_{\mathcal{C}} \leq K$. Then, system (0.8) admits a unique global mild solution $w(t)$ defined on $[-r,+\infty[$ corresponding to feedback control $v_\alpha(t)$ for all $\alpha\geq 0$. Moreover, $v_\alpha(t)$ for all $\alpha\geq 0$, allows the following estimate

$$\left(\int_0^T |\langle F(U(s)w_t),(U(s)w_t)(0)\rangle|ds\right)^2 = O\left(\int_t^{t+T}\frac{|\langle F(w_s),w(s)\rangle|^2}{1+\alpha|\langle F(w_s),w(s)\rangle|}ds\right)\ \textit{as}\ t\;\to\;+\infty. \tag{5.2}$$

Proof. We consider the following closed-loop system :

$$w(t)=\mathcal{A}w(t)\;+\;f(w_t),\ w_0=\phi, \tag{5.3}$$

where

$$f(\varphi)\;=\;-\frac{\langle F(\varphi),\varphi(0)\rangle}{1+\alpha|\langle F(\varphi),\varphi(0)\rangle|}F(\varphi)\ \text{for all}\ \varphi\;\in\;\mathcal{C}.$$

Since F is a locally Lipschitz function, we obtain that

$$\|f(\varphi_1)-f(\varphi_2)\|_{\mathcal{X}} \;\leq\; K^2\Gamma_K^2(3+5\alpha K^2\Gamma_K)\|\varphi_1-\varphi_2\|_{\mathcal{C}}$$

for all $(\varphi_1,\varphi_2)\;\in\;\mathcal{C}^2$ such that $\|\varphi_1\|_{\mathcal{C}}\;\leq\;K$ and $\|\varphi_2\|_{\mathcal{C}}\;\leq\;K$. Then, f is locally Lipschitz, and consequently, system (5.3) has a unique mild solution $w(t)$ defined on a maximal interval $[0,T_{max}[$ by

$$w(t)=S(t)w(0)+\int_0^t S(t-s)f(w_s)ds. \tag{5.4}$$

The operator $\mathcal{A}$ generates a semigroup of contraction, then

$$\frac{d}{dt}\|w(t)\|_{\mathcal{X}}^2 \;\leq\; -2\frac{|\langle F(w_t), w(t)\rangle|^2}{1+\alpha|\langle F(w_t), w(t)\rangle|}.$$

Integrating the above inequality, we get

$$\|w(t)\|_{\mathcal{X}}^2 - \|w(\sigma)\|_{\mathcal{X}}^2 \;\leq\; -2\int_{\tau}^{t} \frac{|\langle F(w_s), w(s)\rangle|^2}{1+\alpha|\langle F(w_s), w(s)\rangle|}ds \text{ for all } t \geq \sigma \geq 0, \tag{5.5}$$

and hence the function $t \to \|w(t)\|_{\mathcal{X}}$ is a decreasing function on $[0, T_{max}[$ and using the fact that $t \to w(t)$ is continuous in $[-r, 0]$, then, there exists $R > 0$, such that

$$\|w(t)\|_{\mathcal{X}} \leq R \quad \text{for} \quad -r \;\leq t < T_{max}. \tag{5.6}$$

By the Blow-up phenomena, we obtain that $T_{max} = +\infty$ (see [90]). Since the semigroup $(S(t))_{t\geq 0}$ is of contraction, then it follow from formula (5.4) and the Schwartz's inequality that

$$\|w(\sigma+t) - S(\sigma)w(t)\|_{\mathcal{X}} \leq R\Gamma_R \left(T\int_{t}^{\sigma+t} \frac{|\langle F(w_s), w(s)\rangle|^2}{1+\alpha|\langle F(w_s), w(s)\rangle|}ds\right)^{\frac{1}{2}} \quad \text{for all } \sigma \in [0, T]. \tag{5.7}$$

For all $s \geq 0$ and $t \geq 0$, we have

$$\begin{aligned}\langle F(U(s)w_t), (U(s)w_t)(0)\rangle &= \langle F(U(s)w_t), S(s)w(t)\rangle \\ &= \langle F(U(s)w_t) - F(w_{s+t}), w(s+t)\rangle \\ &\quad - \langle F(U(s)w_t), w(s+t) - S(s)w(t)\rangle \\ &\quad + \langle F(w_{s+t}), w(s+t)\rangle.\end{aligned}$$

Using the fact that F is a locally lipschitz function, then from inequality (5.6), we deduce that

$$\begin{aligned}|\langle F(U(s)w_t), (U(s)w_t)(0)\rangle| \leq\; & R\Gamma_R \|w_{s+t} - U(s)w_t\|_{\mathcal{C}} \\ & + R\Gamma_R \|w(s+t) - S(s)w(t)\|_{\mathcal{X}} \\ & + |\langle F(w_{s+t}), w(s+t)\rangle\end{aligned} \tag{5.8}$$

Moreover, from inequality (5.7), we get

$$\begin{aligned}|\langle F(U(s)w_t), (U(s)w_t)(0)\rangle| \;\leq\; & 2\left(R\Gamma_R\right)^2 \left(T\int_{0}^{t+T} \frac{|\langle F(w_s), w(s)\rangle|^2}{1+\alpha|\langle F(w_s), w(s)\rangle|}ds\right)^{\frac{1}{2}} \\ & + |\langle F(w_{s+t}), w(s+t)\rangle| \quad \text{for all } s \in [0, T].\end{aligned}$$

Integrating the above inequality over $[0,T]$ and using Cauchy-Schwart's, we obtain that

$$\int_0^T |\langle F(U(s)w_t),(U(s)w_t)(0)\rangle| ds \leq \left(2\left(R\Gamma_R\right)^2 T^{\frac{3}{2}} + T^{\frac{1}{2}}\left(1+\alpha R^2\Gamma_R\right)^{\frac{1}{2}}\right)\left(\int_t^{t+T} \frac{|\langle F(w_s),w(s)\rangle|^2}{1+\alpha|\langle F(w_s),w(s)\rangle|} ds\right)^{\frac{1}{2}}. \tag{5.9}$$

The following result gives sufficient conditions to get the strong stabilization of system (0.8) with a decay estimate.

Theorem 5.2.2. *Let $\mathcal{A}$ generate a $\mathcal{C}_0-$semigroup $(S(t))_{t\geq 0}$ of contraction on $\mathcal{X}$ and F is a locally Lipschitz function from $\mathcal{C}$ into $\mathcal{X}$. If there exist $T,\gamma\geq 0$ and $p\geq 2$, such that*

$$\int_0^T |\langle F(U(s)\zeta),(U(s)\zeta)(0)\rangle| ds \geq \gamma\|\zeta(0)\|^p \text{ for all } \zeta \in \mathcal{C}. \tag{5.10}$$

Then, for $\alpha\geq 0$, the control $v_\alpha(t)$ strongly stabilizes system (0.8) with the following decay estimate

$$\|w(t)\|_{\mathcal{X}} = O\left(t^{-\frac{1}{2(p-1)}}\right) \text{ as } t\longrightarrow +\infty.$$

Proof . Using inequality (5.5), we deduce that

$$\|w(kT)\|_{\mathcal{X}}^2 - \|w((k+1)T)\|_{\mathcal{X}}^2 \geq 2\int_{kT}^{T(k+1)} \frac{|\langle F(w_s),w(s)\rangle|^2}{1+\alpha|\langle F(w_s),w(s)\rangle|} ds.$$

Using the inequalities (5.9) and (5.10), we have

$$\|w(kT)\|_{\mathcal{X}}^2 - \|w((k+1)T)\|_{\mathcal{X}}^2 \geq \beta\|w(kT)\|_{\mathcal{X}}^{2p}. \tag{5.11}$$

where

$$\beta = \frac{2\gamma^2}{\left(2\left(R\Gamma_R\right)^2 T^{\frac{3}{2}} + T^{\frac{1}{2}}\left(1+\alpha R^2\Gamma_R\right)^{\frac{1}{2}}\right)^2}$$

Taking $s_k = \|w(kT)\|_{\mathcal{X}}^2$, the inequality (5.11) can be written as the following form

$$s_{k+1} \leq s_k - \beta s_{k+1}^{(p-2)+2}.$$

By applying Lemma 1.3.5, we obtain that

$$s_k \leq \frac{M}{(k+1)^{\frac{1}{p-1}}} \text{ for all } k\geq 0.$$

Let $k = E\left(\frac{t}{T}\right)$, where $E(.)$ designs the integer part. Since $t \to \|w(t)\|_{\mathcal{X}}$ is a decreasing function on $[0, +\infty)$, we deduce the following estimate

$$\|w(t)\|_{\mathcal{X}} = O\left(t^{-\frac{1}{2(p-1)}}\right) \text{ as } t \longrightarrow +\infty \blacksquare$$

5.3 Weak stabilization

The following theorem provides sufficient conditions for weak stabilization of system (0.8) with distributed delay by using the feedback control $v_\alpha(t)$ defined by (5.1).

Theorem 5.3.1. *Let $\mathcal{A}$ generate a $\mathcal{C}_0-$semigroup $(S(t))_{t\geq 0}$ of contraction on $\mathcal{X}$ and F is a locally Lipschitz function from $\mathcal{C}$ to $\mathcal{X}$. If there exist $T_2 > T_1 > 0$ such that for all sequence $(\zeta_n)_n \subset \mathcal{C}$ with $\zeta_n(0) \rightharpoonup \zeta(0)$ in $\mathcal{X}$ we have that*

$$\langle F(U(t)\zeta_n), (U(t)\zeta_n)(0)\rangle \to 0 \text{ as } n \to +\infty \quad \text{for all } t \in [T_1, T_2] \Rightarrow \zeta(0) = 0. \tag{5.12}$$

Then, system (0.8) is weakly stabilizable by the feedback control $v_\alpha(t)$ for all $\alpha \geq 0$.

Proof. Inequality (5.6), implies that the function $t \to \|w(t)\|_{\mathcal{X}}$ is bounded on $[0, +\infty[$. Moreover, since the space $\mathcal{X}$ is reflexive, we conclude that there exists $t_n \to +\infty$ and $\varphi \in \mathcal{X}$ such that

$$w(t_n) \text{ converges weakly to } \varphi.$$

Since, for any $T_2 > T_1 > 0$, one has

$$\int_{T_1}^{T_2} |\langle F(U(s)w_{t_n}), (U(s)w_{t_n})(0)\rangle|\, ds \leq \int_0^{T_2} |\langle F(U(s)w_{t_n}), (U(s)w_{t_n})(0)\rangle|\, ds,$$

then, it follow from inequality (5.9) that

$$\begin{aligned}\int_{T_1}^{T_2} &|\langle F(U(s)w_t), (U(s)w_t)(0)\rangle| ds \\ &\leq \left(2(R\Gamma_R)^2 T^{\frac{3}{2}} + T^{\frac{1}{2}}\left(1 + \alpha R^2 \Gamma_R\right)^{\frac{1}{2}}\right)\left(\int_{t_n}^{t_n+T_2} \frac{|\langle F(w_s), w(s)\rangle|^2}{1+\alpha|\langle F(w_s), w(s)\rangle|} ds\right)^{\frac{1}{2}}.\end{aligned}$$

Moreover, from inequality (5.5), we get that

$$\int_{t_n}^{t_n+T_2} \frac{|\langle F(w_s), w(s)\rangle|^2}{1+\alpha|\langle F(w_s), w(s)\rangle|} ds \leq \frac{1}{2}\left(\|w(t_n)\|_{\mathcal{X}}^2 - \|w(t_n+T_2)\|_{\mathcal{X}}^2\right).$$

Furthermore, since $t \to \|w(t)\|_{\mathcal{X}}$ is a decreasing function on $[0, +\infty[$, it follows that the sequence $(\|w(t_n)\|_{\mathcal{X}})_n$ converges as n goes to $+\infty$ and

$$\int_0^T |\langle F(U(s)w_{t_n}), (U(s)w_{t_n})(0)\rangle|\, ds \;\to\; 0 \quad as\ n \to +\infty.$$

The sequence $(|\langle F(U(s)w_{t_n}), (U(s)w_{t_n})(0)\rangle|)_{n\in\mathbb{N}}$ is bounded in $[0, +\infty)$, then, there exists a subsequence of $(|\langle F(U(s)w_{t_n}), (U(s)w_{t_n})(0)\rangle|)_{n\in\mathbb{N}}$, denoted also by $(|\langle (U(s)w_{t_n})(0), F(U(s)w_{t_n})\rangle|)_{n\in\mathbb{N}}$ which is convergent. Then, by applying the dominated convergence theorem, we obtain that

$$\int_{T_1}^{T_2} \lim_{n\to+\infty} |\langle F(U(s)w_{t_n}), (U(s)w_{t_n})(0)\rangle| ds = 0,$$

and hence

$$\lim_{n\to+\infty} |\langle F(U(s)w_{t_n}), (U(s)w_{t_n})(0)\rangle| = 0 \quad \text{for all } s \in [T_1, T_2].$$

Then condition (5.12) implies that $\varphi = 0$ and finally $w(t)$ goes to 0 weakly as t goes to $+\infty$ ■

5.4 Optimal control

The following result concerns with the existence and uniqueness of solution for the minimization problem (16). This solution strong stabilizes system (0.8).

Theorem 5.4.1. *Let $\mathcal{A}$ generate a $\mathcal{C}_0-$semigroup of isometries on $\mathcal{X}$ and F is a locally Lipschitz function from $\mathcal{C}$ to $\mathcal{X}$. Suppose that there exist $T, \gamma > 0$ and $p \geq 2$, such that*

$$\int_0^T |\langle F(U(t)\zeta), (U(t)\zeta)(0)\rangle|\, dt \geq \gamma \|\zeta(0)\|_{\mathcal{X}}^p \text{ for all } \zeta \in \mathcal{C}, \tag{5.13}$$

Then, the feedback control $v_0(t)$ is the unique solution of minimization problem (0.9) which strongly stabilizes system (0.8).

Proof . Consider the system (0.8), then we have

$$\frac{d}{dt}\|w(t)\|_{\mathcal{X}}^2 \;=\; 2\langle \mathcal{A}w(t), w(t)\rangle \;+\; 2v_0(t)\langle F(w_t), w(t)\rangle.$$

Hence

$$\frac{d}{dt}\|w(t)\|_{\mathcal{X}}^2 \;=\; 2\langle \mathcal{A}w(t), w(t)\rangle \;-\; 2|\langle F(w_t), w(t)\rangle|^2.$$

Since $(S(t))_{t\geq 0}$ is a isometries, we get that

$$\frac{d}{dt}\|w(t)\|_{\mathcal{X}}^2 \;=\; -2|\langle F(w_t), w(t)\rangle|^2. \tag{5.14}$$

Integrating the above inequality, we get that

$$\int_0^t |\langle F(w_s), w(s)\rangle|^2 ds \;\leq\; \frac{1}{2}\|\phi\|_{\mathcal{C}}^2,$$

consequently $J(v_0)$ is finite for all $\phi \in \mathcal{C}$ and hence $v_0 \in V_{ad}$ which means that $V_{ad} \neq \emptyset$. Now, the objective is to prove that the control $v_0(t)$ is a solution of the minimization problem (0.9). Let $v \in V_{ad}$, we have that

$$\frac{d}{dt}\|w(t)\|_{\mathcal{X}}^2 \;=\; |\langle F(w_t), w(t)\rangle \;+\; v(t)|^2 - \;|\langle F(w_t), w(t)\rangle|^2 - |v(t)|^2.$$

Integrating above inequality, we get that

$$\int_0^t \left(|\langle F(w_s), w(s)\rangle|^2 \;+\; |v(s)|^2\right) ds \;+\; \|w(t)\|_{\mathcal{X}}^2 \;-\; \|w(0)\|_{\mathcal{X}}^2 = \int_0^t |\langle F(w_s), w(s)\rangle \;+\; v(s)|^2 \, ds. \tag{5.15}$$

Using the variation of constants formula, we get that

$$w(s) \;=\; S(s)w(0) \;+\; \int_0^s v(\sigma)S(s-\sigma)F(w_\sigma)d\sigma, \quad \text{for all } s \geq 0. \tag{5.16}$$

Since $\|w(t)\|_{\mathcal{X}} \leq K$, for all $t \geq 0$ and the function $t \to w(t)$ is continuous in $[-r, 0]$, then, there exists $M > 0$ such that

$$\|w(t)\|_{\mathcal{X}} \leq M \;\text{ for all } t \;\in\; [-r, +\infty).$$

The function F is locally Lipschitz, then by the Schwartz's inequality, we get for all $s \in\; [0, T]$

$$\|w(s+t) \;-\; S(s)w(t)\|_{\mathcal{X}} \;\leq\; M\Gamma_M\sqrt{T}\sqrt{\lambda(t)}, \tag{5.17}$$

where $\lambda(t) \;=\; \int_t^{t+T} |v(s)|^2 ds$. Moreover

$$\begin{aligned}\langle F(U(s)w_t, (U(s)w_t)(0))\rangle \;&= \langle F(U(s)w_t), S(s)w(t)\rangle \\ &= \langle F(U(s)w_t) - F(w_{s+t}), w(s+t)\rangle \\ &\quad - \;\langle F(U(s)w_t), w(s+t) - S(s)w(t)\rangle \\ &\quad + \;\langle F(w_{s+t}), w(s+t)\rangle.\end{aligned}$$

Using inequality (5.17), we deduce that

$$|\langle F(U(s)w_t, (U(s)w_t)(0))\rangle| \;\leq\; 2(M\Gamma_M)^2\sqrt{T}\sqrt{\lambda(t)} + |\langle F(w_{s+t}), w(s+t)\rangle|, \quad \text{for all } s \;\in [0, T].$$

Integrating this inequality over $[0,T]$, yields

$$\int_0^T |\langle F(U(s)w_t,(U(s)w_t)(0))\rangle|ds \leq 2(M\Gamma_M)^2T\sqrt{T}\sqrt{\lambda(t)} + \int_0^T |\langle F(w_{s+t}),w(s+t)\rangle|ds. \tag{5.18}$$

It follows from (5.13) and (5.18) that

$$\gamma\|w(t)\|_{\mathcal{X}}^p \leq 2(M\Gamma_M)^2T\sqrt{T}\sqrt{\lambda(t)} + \sqrt{T}\sqrt{\widetilde{\lambda}(t)},$$

with $\widetilde{\lambda}(t) = \displaystyle\int_t^{t+T} |\langle F(w_s),w(s)\rangle|^2ds$. Since λ and $\widetilde{\lambda}$ converge to zero as $t \to +\infty$, then $\|w(t)\|_{\mathcal{X}} \to 0$ as $t \to +\infty$. So the control v_0 strongly stabilizes system (0.8). Letting t to $+\infty$ in inequality (5.15), we obtain that

$$J(v) = \|w(0)\|_{\mathcal{X}}^2 + \int_0^{+\infty} |\langle F(w_s),w(s)\rangle + v(s)|^2ds, \tag{5.19}$$

which implies that $J(v) \geq \|w(0)\|_{\mathcal{X}}^2 = J(v_0)$, so v_0 is an optimal control. Now, let $v_i(t)$, $i = 1,2$, be two solutions of problem (0.9). From (5.19) we deduce that the feedback control $v_i(t) = -\langle F(w_t^i),w^i(t))\rangle$ where $w^i(t)$ verifies

$$\begin{aligned}\dot{w}(t) =& Aw(t) - \langle F(w_t),w(t)\rangle F(w_t)\\ w_0 =& \phi,\end{aligned}$$

thus $w^i(t) = w(t)$ $i = 1,2$ and hence $v_1(t) = v_2(t)$. Then, v_0 is an optimal control of the problem (0.9), strongly stabilizes system (0.8)■

In finite-dimensional space, we obtain the strong stabilization of system (0.8) by solving the minimization problem (0.9) using another condition.

Corollary 5.4.2. *Let $\mathcal{A}$ generate a C_0-semigroup of isometries on $\mathcal{X}$ and F is a locally Lipschitz function from $\mathcal{C}$ to $\mathcal{X}$. If $\mathcal{X}$ of finite dimensional space and suppose that there exist $p \geq 2$ and $\mu > 0$ such that*

$$|\langle F(\xi),\xi(0)\rangle| \geq \mu\|\xi(0)\|_{\mathcal{X}}^p \quad \textit{for all } \xi \in \mathcal{C}, \tag{5.20}$$

Then, the feedback control $v_0(t)$ is the unique solution of minimization problem (0.9) which strongly stabilizes system (0.8).

Proof. Using the inequality (5.20) with $\xi = U(s+r)\zeta$ for $s \in [0,T]$ and $\zeta \in \mathcal{C}$, then

$$|\langle F(U(s+r)\zeta),(U(s+r)\zeta)(0)\rangle| \geq \mu\|(U(s)\zeta)(0)\|_{\mathcal{X}}^p.$$

Since $(S(s))_{s\geq 0}$ is a semigroup of isometries, then, we have that

$$\begin{aligned}|\langle F(U(s+r)\zeta),(U(s+r)\zeta)(0)\rangle| \geq & \mu\|S(s+r)\zeta(0)\|_{\mathcal{X}}^{p}\\ \geq & \mu\|\zeta(0)\|_{\mathcal{X}}^{p}.\end{aligned}$$

Integrating above inequality over $[0,T]$ with respect to s, we get that

$$\int_0^T |\langle F(U(s+r)\zeta),(U(s+r)\zeta)(0)\rangle| ds \geq \mu T\|\zeta(0)\|_{\mathcal{X}}^{p},$$

and

$$\int_0^{T+r} |\langle F(U(s)\zeta),(U(s)\zeta)(0)\rangle| ds \geq \delta\|\zeta(0)\|_{\mathcal{X}}^{p}.$$

Theorem 5.4.1 implies that v_0 is an optimal control of the minimization problem (0.9), strongly stabilizes system (0.8)■

As a special case of Theorem 5.4.1, we consider the case where the space $\mathcal{X}$ is of finite dimensional and F a bounded linear operator. Here, we use a condition weaker than the condition (5.13) used in Theorem 5.4.1.

Corollary 5.4.3. *Let $\mathcal{A}$ generate a C_0-semigroup of isometries on $\mathcal{X}$ and F be a linear bounded operator from $\mathcal{C}$ to $\mathcal{X}$. If $\mathcal{X}$ a finite dimensional space and if there exist $T_2 > T_1 > r$ and for all $\zeta \in \mathcal{C}$ such that*

$$\langle F(U(s)\zeta),(U(s)\zeta)(0)\rangle = 0 \quad \text{for all } s \in [T_1,T_2] \Rightarrow \zeta(0) = 0, \tag{5.21}$$

Then, the feedback control $v_0(t)$ is the unique solution of minimization problem (0.9), strongly stabilizes system (0.8).

Proof. Evidently, for all $\zeta \in \mathcal{C}$ such that $\zeta(0) = 0$, the inequality (5.13) is verified. For all $s \in [T_1,T_2]$, we have

$$\langle F(U(s)\zeta),(U(s)\zeta)(0)\rangle = \langle F(S(s+.)\zeta(0)),S(s)\zeta(0)\rangle\,.$$

Now, suppose that

$$\inf_{\|y\|_{\mathcal{X}}=1} \int_{T_1}^{T_2} |\langle F(S(s+.)y),S(s)y\rangle|\, ds \;=\; 0.$$

Since $\displaystyle\int_{T_1}^{T_2} |\langle F(S(s+.)y),S(s)y\rangle|\, ds$ depends continuously to y and $\dim \mathcal{X} < +\infty$, then, there exists $y_0 \in \mathcal{X}$ such that

$$\|y_0\|_{\mathcal{X}} = 1 \text{ and } \int_{T_1}^{T_2} |\langle F(S(s+.)y_0),S(s)y_0\rangle|\, ds = 0.$$

Then, there exists $\zeta \in \mathcal{C}$ such that $\zeta(0) = y_0$ and hence

$$|\langle F(S(s+.)\zeta(0)), S(s)\zeta(0)\rangle| = 0 \text{ for all } s \in [T_1, T_2].$$

Condition (5.21) implies that $\zeta(0) = 0$, which gives the contradiction. Then, there exists a constant $\delta > 0$ such that

$$\inf_{\|y\|_{\mathcal{X}}=1} \int_{T_1}^{T_2} |\langle F(S(s+.)y), S(s)y\rangle| \, ds = \delta.$$

Let $y = \dfrac{1}{\|\zeta(0)\|_{\mathcal{X}}}\zeta(0)$, then $\|y\|_{\mathcal{X}} = 1$ and we obtain that

$$\int_{T_1}^{T_2} |\langle F(U(s)\zeta), (U(s)\zeta)(0)\rangle| \, ds \geq \delta \|\zeta(0)\|_{\mathcal{X}}^2.$$

Since

$$\int_0^{T_2} |\langle F(U(s)\zeta), (U(s)\zeta)(0)\rangle| \, ds \geq \int_{T_1}^{T_2} |\langle F(U(s)\zeta), (U(s)\zeta)(0)\rangle| \, ds.$$

Then, the conditions of Theorem 5.4.1 is satisfied, so the feedback control $v_0(t)$ strongly stabilizes system (0.8) and also it's the unique solution of problem (0.9)■

5.5 Application and Numerical Simulation

To enhance the value of the previous theoretical part, we proceed to apply the previous results to some physical systems with distributed delay.

Example: Consider the system mono-dimensional wave equation with distributed delay on $\Omega = (0, 1)$ described by following system

$$\begin{cases} \dfrac{\partial^2 w(x,t)}{\partial t^2} = \dfrac{\partial^2 w(x,t)}{\partial x^2} + v(t) \left\| \left(w(t), \dfrac{\partial w(t)}{\partial t} \right) \right\|_{\mathcal{X}} \left(\displaystyle\int_{-2}^{0} \dfrac{\partial w(x,t+s)}{\partial t} ds + \dfrac{\partial w(x,t)}{\partial t} \right), \\ \qquad\qquad \text{for } x \in (0,1), \ t \geq 0, \\ w(0,t) = w(1,t) = 0, \qquad \text{for } t \in [-2, +\infty), \\ w(x,t) = \phi_1(x,t), \qquad \text{for } x \in (0,1), t \in [-2, 0] \\ \dfrac{\partial w(x,t)}{\partial t} = \phi_2(x,t), \qquad \text{for } x \in (0,1), t \in [-2, 0]. \end{cases} \tag{5.22}$$

Consider the space $\mathcal{X} = H_0^1(0,1) \times L^2(0,1)$ with the following inner product

$$\langle (x_1, x_2), (y_1, y_2) \rangle = \langle x_1, y_1 \rangle_{H_0^1(0,1)} + \langle x_2, y_2 \rangle_{L^2(0,1)}.$$

Let $\mathcal{A} : \mathcal{D}(\mathcal{A}) \subset \mathcal{X} \to \mathcal{X}$ the operator defined by

$$\mathcal{A} = \begin{pmatrix} 0 & I \\ \Delta & 0 \end{pmatrix}$$

with

$$\mathcal{D}(\mathcal{A}) = \left(H^2(0,1) \cap H_0^1(0,1)\right) \times H_0^1(0,1),$$

and the Laplacian operator $\Delta = \dfrac{\partial^2}{\partial x^2}$ with $D(\Delta) = \{\, w \in H^2(0,1) : w(0,t) = w(1,t) = 0 \,\}$ and eigenvalues given by $\lambda_m = -(m\pi)^2, (m \in \mathbb{N}^*)$ associated to the eigenfunctions $(\varphi_m)_{m\geq 1}$ expressed by

$$\varphi_m(x) = \sqrt{2}\sin(m\pi x), \text{ for all } m \geq 1.$$

The operator F defined from $\mathcal{C} = C([-2,0], \mathcal{X})$ to $\mathcal{X}$ by

$$F(\zeta) = \begin{pmatrix} 0 \\ \|\zeta(0)\|_{\mathcal{X}} \left(\displaystyle\int_{-2}^{0} \zeta_2(s)ds + \zeta_2(0) \right) \end{pmatrix}.$$

for $\zeta = (\zeta_1, \zeta_1) \in \mathcal{C}$. System (5.22) may be written in the form (0.8) with

$$z = \begin{pmatrix} w \\ \dfrac{\partial w}{\partial t} \end{pmatrix}.$$

Let $w = (w_1, w_2) \in \mathcal{X}$ with $w_1 = \displaystyle\sum_{m=1}^{\infty} \alpha_m \varphi_m$, $w_2 = \displaystyle\sum_{m=1}^{\infty} (-\lambda_m)^{\frac{1}{2}} \beta_m \varphi_m$ and $(\alpha_m, \beta_m) \in \mathbb{R}^2$ for all $m \geq 1$. Using the separation of variables of the semigroup of isometries $(S(t))_{t\geq 0}$ generated by the skew adjoint operator $\mathcal{A}$ (for more details see [31]) and we have that

$$S(t)w = \sum_{m=1}^{+\infty} \begin{pmatrix} \alpha_m \cos(\sqrt{-\lambda_m}t) + \beta_m \sin(\sqrt{-\lambda_m}t) \\ -\alpha_m\sqrt{-\lambda_m}\sin(\sqrt{-\lambda_m}t) + \beta_m\sqrt{-\lambda_m}\cos(\sqrt{-\lambda_m}t) \end{pmatrix} \varphi_m$$

Then, for all $\zeta \in \mathcal{C}$ and $t \in [2,3]$ we have

$$F(U(t)\zeta) = \|U(t)\zeta(0)\|_{\mathcal{X}} \left[\sum_{m=1}^{+\infty} \begin{pmatrix} 0 \\ \beta_m\sqrt{-\lambda_m}\cos(\sqrt{-\lambda_m}t) - \alpha_m\sqrt{-\lambda_m}\sin(\sqrt{-\lambda_m}t) \end{pmatrix} \varphi_m \right.$$

$$\left. + \begin{pmatrix} 0 \\ \displaystyle\int_{-2}^{0} (-\alpha_m\sqrt{-\lambda_m}\sin(\sqrt{-\lambda_m}(t+s)))ds + \int_{-2}^{0} (\beta_m\sqrt{-\lambda_m}\cos(\sqrt{-\lambda_m}(t+s)))ds \end{pmatrix} \varphi_m \right]$$

Since the semigroup $(S(t))_{t\geq 0}$ is of isometries, then, we have

$$\begin{aligned}\langle F(U(t)\zeta),(U(t)\zeta)(0)\rangle &= \|\zeta(0)\|_{\mathcal{X}} \sum_{m=1}^{+\infty} -\lambda_m \Big(\alpha_m^2 \sin^2(\sqrt{-\lambda_m}t) + \beta_m^2 \cos^2(\sqrt{-\lambda_m}t) \\ &\quad -2\alpha_m\beta_m \sin(\sqrt{-\lambda_m}t)\cos(\sqrt{-\lambda_m}t)\Big).\end{aligned}$$

Integrating from 2 to 3, we obtain that

$$\begin{aligned}\int_2^3 |\langle F(U(t)\zeta),(U(t)\zeta)(0)\rangle| dt &= \frac{\|\zeta(0)\|_{\mathcal{X}}}{2} \sum_{m=1}^{+\infty} -\lambda_m \left(\alpha_m^2+\beta_m^2\right) \\ &= \frac{1}{2}\|\zeta(0)\|_{\mathcal{X}}^3\end{aligned}$$

and since

$$\int_0^3 |\langle F(U(t)\zeta),(U(t)\zeta)(0)\rangle| dt \geq \int_2^3 |\langle F(U(t)\zeta),(U(t)\zeta)(0)\rangle| dt.$$

Then, condition (5.13) holds. So, Theorem 5.4.1, implies that the feedback control

$$v_0(t) = -\left\|\left(w(t), \frac{\partial w(t)}{\partial t}\right)\right\|_{\mathcal{X}} \int_0^1 \frac{\partial w(x,t)}{\partial t}\left(\int_{-2}^0 \frac{\partial w(x,t+s)}{\partial t} ds + \frac{\partial w(x,t)}{\partial t}\right) dx$$

for all $t \geq 0$, is the unique solution of following minimization problem

$$\min_{v\in V_{ad}} J(v), \tag{5.23}$$

where

$$J(v) = \int_0^{+\infty} |\langle F(w_s), w(s)\rangle|^2 ds + \int_0^{+\infty} |v(s)|^2 ds,$$

and strongly stabilizes system (5.22) with the following decay estimate

$$\|z(t)\|_{\mathcal{X}} = O\left(t^{-\frac{1}{4}}\right) \text{ as } t \longrightarrow +\infty.$$

Now, if we choose $\alpha = 0, 50, 100, 200$ and $\phi_1(x,t) = \frac{\pi}{2}\exp(x-1))\sin(t\pi-\pi)$, we obtain the evolution of the feedback control $v_0(t)$ which is the unique solution of the minimization problem (5.23) and strongly stabilizes system (5.22) in figure 5.1.

From figure 5.2, we observe that the stabilized state norm $\|z(t)\|_{\mathcal{X}}$ goes to 0 for different values of α. Moreover, we can see that when the value of α is very small, the state norm $\|z(t)\|_{\mathcal{X}}$ tends rapidly to 0.

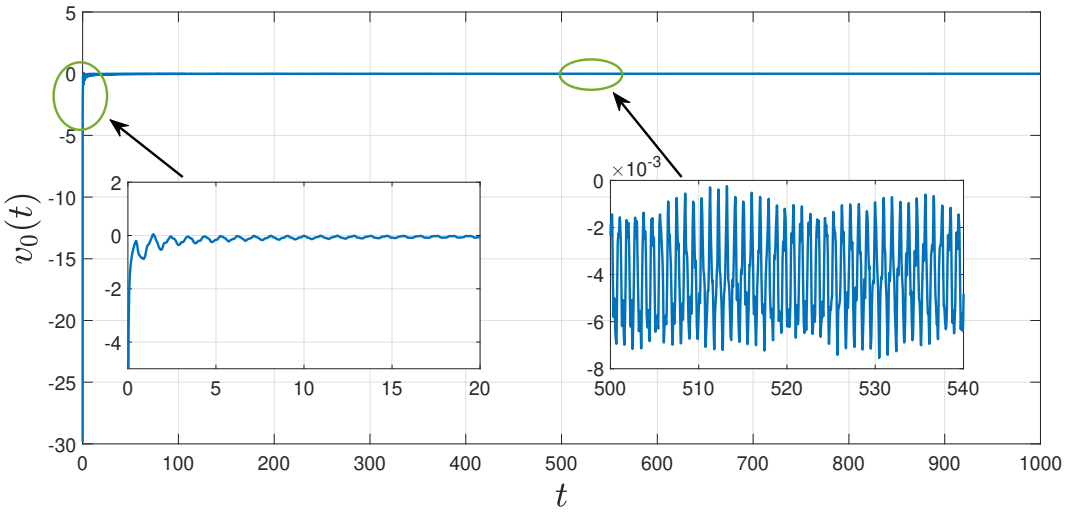

Figure 5.1: The evolution of the feedback control $v_0(t)$.

Moreover in figure 5.2, we give the evolution of the stabilized state norm $\|z(t)\|_{\mathcal{X}}$ corresponding to the feedback control $v_\alpha(t)$ for different values of α.

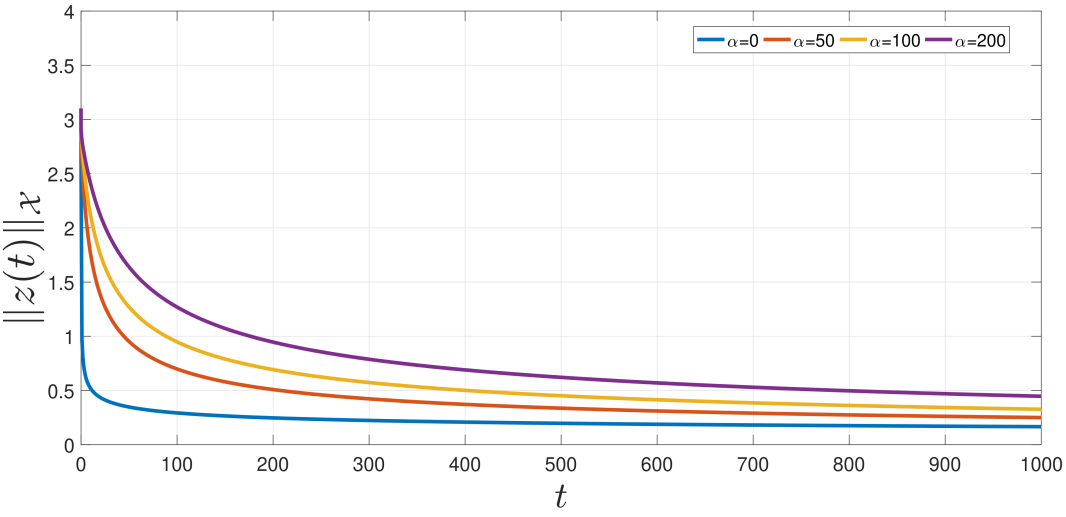

Figure 5.2: The evolution of the stabilized state norm $\|z(t)\|_{\mathcal{X}}$ corresponding to feedback control $v_\alpha(t)$ for $\alpha = 0, 50, 100, 200$.

Conclusion. In this chapter, a new family of feedback controls is proposed to stabilize a class of infinite dimensional semilinear system with distributed delay. The proposed feedback control allows us to get the strong and weak stabilization of the closed-loop semilinear system with distributed delay. Moreover, among this family of feedback controls we showed that one control can be characterized as the unique solution of an appropriate minimization problem and can be applied to physical and biological systems.

Chapter 6

Strong and weak stabilization through the decomposition method of the Banach state space for bilinear systems with distributed delay

6.1 Introduction

In this chapter, we study the strong and weak stabilization of the bilinear system (0.8) with distributed delay in the Banach space $\mathcal{X}$. We accomplish this through the use of a decomposition method for the Banach phase space $\mathcal{X}$, dividing it into two invariant subspaces, denoted $\mathcal{X}^u$ and $\mathcal{X}^s$. Our approach involves two main steps. First, we analyze the global mild solution of the system (0.8), studying the existence and uniqueness of this solution. Second, we investigate the strong stabilization of our system within the Banach space, by considering an observation condition that expressed by $(U(t))_{t\geq 0}$ semigroup solution of system (0.8) with $F = 0$, and by a positive constant $\delta \geq 2$. This observation condition is defined by

$$\int_0^T |\langle F^u(U^u(\sigma)\zeta^u), \mathcal{J}((U^u(\sigma)\zeta^u)(0))\rangle|\, d\sigma \geq \eta \|\zeta^u(0)\|^{\delta}_{\mathcal{X}^u} \quad \text{for every } \zeta^u \in \mathcal{C}^u, \tag{6.1}$$

for some T, $\eta > 0$. $(U^u(t))_{t\geq 0}$ is the restriction of $(U(t))_{t\geq 0}$ on the subspace $\mathcal{X}^u$ and $\mathcal{J}$ is the duality mapping defined by (1.4). We prove that the system (0.8) is strongly stabilizable using the following feedback control

$$v^u(t) = -\frac{\langle F(w^u_t), \mathcal{J}(w^u(t))\rangle}{1 + |\langle F(w^u_t), \mathcal{J}(w^u(t))\rangle|}.$$

Furthermore, it is possible to estimate a polynomial decay rate for the stabilization state explicitly. This is given by the expression:

$$\|w(t)\|_{\mathcal{X}}^2 = O\left(t^{-\frac{1}{\delta-1}}\right) \text{ as } t \longrightarrow +\infty.$$

Third, we discuss the weak stabilization of system (0.8) in Banach space by using author weaker assumptions than the observability condition (6.1). Finally, an example with distributed delay is given and some numerical simulations are also included to illustrate the applicability of the obtained theoretical results.

Through this chapter, we assume that:

- $\mathcal{J}$ is Lipschitz-continuous in the sense of the definition (1.3.7).

- The space $\mathcal{X}$ can be decomposed as a direct sum of two closed subspaces $\mathcal{X}^u$ and $\mathcal{X}^s$ of $\mathcal{X}$ (i.e. $\mathcal{X} = \mathcal{X}^u \oplus \mathcal{X}^s$), which are invariants under the semigroup $S(t)$ and orthogonal to each one.

The operators $S^u(t)$ and $S^s(t)$ are defined as the restrictions of $S(t)$ on $\mathcal{X}^u$ and $\mathcal{X}^s$ respectively. They are C_0-semigroups on $\mathcal{X}^u$ and $\mathcal{X}^s$ respectively. Let $\mathcal{A}^u$ and $\mathcal{A}^s$ denote their generators respectively.

Furthermore, we suppose that

$$F(\mathcal{C}^u) \subset \mathcal{X}^u \text{ and } F(\mathcal{C}^s) \subset \mathcal{X}^s,$$

where $\mathcal{C}^s = C([-r, 0], \mathcal{X}^s)$ and $\mathcal{C}^u = C([-r, 0], \mathcal{X}^u)$. Then, the system (0.8) can be decomposed as follows

$$\begin{cases} \dot{w}^u(t) = \mathcal{A}^u w^u(t) + v(t)F^u(w_t^u) \quad \text{for } t > 0, \\ w_0^u = \phi^u \in \mathcal{C}^u, \end{cases} \tag{6.2}$$

where $F^u = P^u F$ and $P^u : Y \to \mathcal{X}^u$ is the orthogonal projection on $\mathcal{X}^u$ and

$$\begin{cases} \dot{w}^s(t) = \mathcal{A}^s w^s(t) + v(t)F^s(w_t^s) \text{ for } t > 0, \\ w_0^s = \phi^s \in \mathcal{C}^s, \end{cases} \tag{6.3}$$

where $F^s = P^s F$ and $P^s : Y \to \mathcal{X}^s$ is the orthogonal projection on $\mathcal{X}^s$.

6.2 Existence, uniqueness of the global mild solution

In this section, we discuss the existence and uniqueness of the global mild solution of the system (0.8) by using the following feedback control:

$$v^u(t) \;=\; -\frac{\langle F(w_t^u), \mathcal{J}(w^u(t))\rangle}{1+|\langle F(w_t^u), \mathcal{J}(w^u(t))\rangle|}. \tag{6.4}$$

Additionally, we get an estimate that will be used in the next to obtain the strong stabilization result of the system (0.8).

Theorem 6.2.1. *If $(S^u(t))_{t\geq 0}$ is a contraction semigroup on $\mathcal{X}^u$, then, the system (6.2) admits a unique global mild solution $w^u(t)$ defined on $[-r,+\infty[$ corresponding to the feedback control $v^u(t)$ defined by (6.4) and we have*

$$\left(\int_0^T |\langle F(U^u(\sigma)w_t^u), \mathcal{J}((U^u(\sigma)w_t^u)(0))\rangle| d\sigma\right)^2 = O\left(\int_t^{t+T} \frac{|\langle F(w_\sigma^u), \mathcal{J}(w^u(\sigma))\rangle|^2}{1+|\langle F(w_\sigma^u), \mathcal{J}(w^u(\sigma))\rangle|} d\sigma\right) \text{ as } t \to +\infty. \tag{6.5}$$

Proof: We take into consideration the next closed-loop system:

$$w^u(t) = \mathcal{A}w^u(t) \;+\; f(w_t^u) \text{ and } w_0^u = \phi, \tag{6.6}$$

where the function $f : \mathcal{C}^u \to \mathcal{X}$ is defined by

$$f(\zeta) \;=\; -\frac{\langle F(\zeta), \mathcal{J}(\zeta(0))\rangle}{1+|\,\langle F(\zeta), \mathcal{J}(\zeta(0))\rangle\,|} F(\zeta) \text{ for every } \zeta \in \mathcal{C}^u.$$

First, we prove that f is locally Lipschitz: for all $(\zeta_1,\zeta_2) \in (\mathcal{C}^u)^2$ such that $\|\zeta_1\|_{\mathcal{C}^u} \leq K$, $\|\zeta_2\|_{\mathcal{C}^u} \leq K$,

$$\begin{aligned}
&\|f(\zeta_1)-f(\zeta_2)\|_{\mathcal{X}} \leq \left\| \frac{\langle F(\zeta_1), \mathcal{J}(\zeta_1(0))\rangle}{1+|\,\langle F(\zeta_1), \mathcal{J}(\zeta_1(0))\rangle\,|} F(\zeta_1) - \frac{\langle F(\zeta_2), \mathcal{J}(\zeta_2(0))\rangle}{1+|\,\langle F(\zeta_2), \mathcal{J}(\zeta_2(0))\rangle\,|} F(\zeta_2) \right\|_{\mathcal{X}} \\
&\leq \left\| \frac{\langle F(\zeta_1), \mathcal{J}(\zeta_1(0))\rangle F(\zeta_1)\,(1+|\,\langle F(\zeta_2), \mathcal{J}(\zeta_2(0))\rangle\,|) - \langle F(\zeta_2), \mathcal{J}(\zeta_2(0))\rangle F(\zeta_2)\,(1+|\,\langle F(\zeta_1), \mathcal{J}(\zeta_1(0))\rangle\,|)}{(1+|\,\langle F(\zeta_1), \mathcal{J}(\zeta_1(0))\rangle\,|)\,(1+|\,\langle F(\zeta_2), \mathcal{J}(\zeta_2(0))\rangle\,|)} \right\|_{\mathcal{X}} \\
&\leq \|\langle F(\zeta_1), \mathcal{J}(\zeta_1(0))\rangle F(\zeta_1) - \langle F(\zeta_2), \mathcal{J}(\zeta_2(0))\rangle F(\zeta_2)\|_{\mathcal{X}} + \\
&\|\langle F(\zeta_1), \mathcal{J}(\zeta_1(0))\rangle F(\zeta_1)|\,\langle F(\zeta_2), \mathcal{J}(\zeta_2(0))\rangle\,| - \langle F(\zeta_2), \mathcal{J}(\zeta_2(0))\rangle F(\zeta_2)|\,\langle F(\zeta_1), \mathcal{J}(\zeta_1(0))\rangle\,|\|_{\mathcal{X}} \\
&\leq \|\langle F(\zeta_1), \mathcal{J}(\zeta_1(0))\rangle F(\zeta_1) - \langle F(\zeta_2), \mathcal{J}(\zeta_2(0))\rangle F(\zeta_2)\|_{\mathcal{X}} + \\
&\|\langle F(\zeta_2), \mathcal{J}(\zeta_2(0))\rangle\|_{\mathcal{X}}\, \|\langle F(\zeta_1), \mathcal{J}(\zeta_1(0))\rangle F(\zeta_1) - \langle F(\zeta_2), \mathcal{J}(\zeta_2(0))\rangle F(\zeta_2)\|_{\mathcal{X}} + \\
&\|\langle F(\zeta_2), \mathcal{J}(\zeta_2(0))\rangle F(\zeta_2)\|_{\mathcal{X}}\, \|\,|\,\langle F(\zeta_2), \mathcal{J}(\zeta_2(0))\rangle\,| - |\,\langle F(\zeta_1), \mathcal{J}(\zeta_1(0))\rangle\,|\|_{\mathcal{X}}\,.
\end{aligned}$$

Moreover, we have

$$\begin{aligned}\langle F(\zeta_1), \mathcal{J}(\zeta_1(0))\rangle F(\zeta_1) - \langle F(\zeta_2), \mathcal{J}(\zeta_2(0))\rangle F(\zeta_2) &= \langle F(\zeta_1) - F(\zeta_2), \mathcal{J}(\zeta_1(0))\rangle F(\zeta_1) \\ &+ \langle F(\zeta_2), \mathcal{J}(\zeta_1(0)) - \mathcal{J}(\zeta_2(0))\rangle F(\zeta_1) \\ &+ \langle F(\zeta_2), \mathcal{J}(\zeta_2(0))\rangle (F(\zeta_1) - F(\zeta_2)).\end{aligned}$$

Then

$$\|\langle F(\zeta_1), \mathcal{J}(\zeta_1(0))\rangle F(\zeta_1) - \langle F(\zeta_2), \mathcal{J}(\zeta_2(0))\rangle F(\zeta_2)\|_{\mathcal{X}} \leq 3K^2\|F\|^2_{\mathcal{C}^u} L\|\zeta_1 - \zeta_2\|_{\mathcal{C}^u},$$

where L is the Lipschitz constant of $\mathcal{J}$, and we get

$$\begin{aligned}\|f(\zeta_1) - f(\zeta_2)\|_{\mathcal{X}} &\leq 3K^2\|F\|^2_{\mathcal{C}^u} L\left(1 + K^2\|F\|_{\mathcal{C}^u} L\right)\|\zeta_1 - \zeta_2\|_{\mathcal{C}^u} + \\ &\quad K^3\|F\|^2_{\mathcal{C}^u} L\left(\|\langle F(\zeta_2) - F(\zeta_1), \mathcal{J}(\zeta_2(0))\rangle + \langle F(\zeta_1), \mathcal{J}(\zeta_2(0)) - \mathcal{J}(\zeta_1(0))\rangle\|_{\mathcal{X}}\right) \\ &\leq \left[3K^2\|F\|^2_{\mathcal{C}^u} L\left(1 + K^2\|F\|_{\mathcal{C}^u} L\right) + 2K^4\|F\|^3_{\mathcal{C}^u} L^2\right]\|\zeta_1 - \zeta_2\|_{\mathcal{C}^u}.\end{aligned}$$

Then, the system (6.6) has a unique mild solution $w^u(t)$ defined on $[0, T_{max})$ and satisfies

$$w^u(t) = S^u(t)w^u(0) + \int_0^t S^u(t-\sigma)f(w^u_\sigma)d\sigma. \tag{6.7}$$

Since, the semigroup $(S(t))_{t\geq 0}$ is of contraction, then

$$\frac{d}{dt}\|w^u(t)\|^2_{\mathcal{X}} \leq -2\frac{|\langle F(w^u_t), \mathcal{J}(w^u(t))\rangle|^2}{1 + |\langle F(w^u_t), \mathcal{J}(w^u(t))\rangle|},$$

and by integrating the last inequality, we get

$$\|w^u(t_2)\|^2_{\mathcal{X}} - \|w^u(t_1)\|^2_{\mathcal{X}} \leq -2\int_{t_1}^{t_2} \frac{|\langle F(w^u_\sigma), \mathcal{J}(w^u(\sigma))\rangle|^2}{1 + |\langle F(w^u_\sigma), \mathcal{J}(w^u(\sigma))\rangle|}d\sigma \text{ for every } t_2 \geq t_1 \geq 0. \tag{6.8}$$

Consequently, $\sigma \to \|w^u(\sigma)\|_{\mathcal{X}}$ is a decreasing function on $[0, T_{max})$, moreover, since the function $\sigma \to w^u(\sigma)$ is continuous in $[-r, 0]$, we conclude that there exists a constant $R > 0$, such that

$$\|w^u(\sigma)\|_{\mathcal{X}} \leq R \quad \text{for } -r \leq \sigma < T_{max}. \tag{6.9}$$

The Blow-up phenomena implies that $T_{max} = +\infty$ (see [90]). Since the semigroup $(S(t))_{t\geq 0}$ is of contraction, it follows from formula (6.7) and the Schwartz's inequality that

$$\|w^u(s+t) - S^u(s)w^u(t)\|_{\mathcal{X}} \leq R\|F\|_{\mathcal{C}}\left(T\int_t^{s+t} \frac{|\langle F(w^u_\sigma), \mathcal{J}(w^u(\sigma))\rangle|^2}{1 + |\langle F(w^u_\sigma), \mathcal{J}(w^u(\sigma))\rangle|}d\sigma\right)^{\frac{1}{2}} \quad \text{for all } s \in [0, T]. \tag{6.10}$$

for every $\sigma \geq 0$ and $t \geq 0$, we have

$$\begin{aligned}\langle F(U^u(\sigma)w_t^u), \mathcal{J}((U^u(\sigma)w_t^u)(0))\rangle &= \langle F(U(\sigma)w_t^u), \mathcal{J}(S^u(\sigma)w^u(t))\rangle \\ &= \langle F(U^u(\sigma)w_t^u) - F(w_{\sigma+t}^u), \mathcal{J}(w^u(\sigma+t))\rangle \\ &\quad - \langle F(U^u(\sigma)w_t^u), \mathcal{J}(w^u(\sigma+t)) - \mathcal{J}(S^u(\sigma)w^u(t))\rangle \\ &\quad + \langle F(w_{\sigma+t}^u), \mathcal{J}(w^u(\sigma+t))\rangle.\end{aligned}$$

Using inequality (6.9), we deduce that

$$\begin{aligned}|\langle F(U^u(\sigma)w_t^u), \mathcal{J}((U^u(\sigma)w_t^u)(0))\rangle| &\leq \|F\|_{\mathcal{C}} LR \left\|w_{\sigma+t}^u - U^u(\sigma)w_t^u\right\|_{\mathcal{C}} \\ &\quad + \|F\|_{\mathcal{C}} LR \left\|w^u(\sigma+t) - S^u(\sigma)w^u(t)\right\|_{\mathcal{X}} \\ &\quad + \left|\langle F(w_{\sigma+t}^u), \mathcal{J}(w^u(\sigma+t))\rangle\right|.\end{aligned}$$

Moreover, from inequality (6.10), we get

$$\begin{aligned}|\langle F(U^u(\sigma)w_t^u), \mathcal{J}((U^u(\sigma)w_t^u)(0))\rangle| \;\leq\; & 2L\left(R\|F\|_{\mathcal{C}}\right)^2 \left(T\int_t^{s+t} \frac{|\langle F(w_\sigma^u), \mathcal{J}(w^u(\sigma))\rangle|^2}{1+|\langle F(w_\sigma^u), \mathcal{J}(w^u(\sigma))\rangle|} d\sigma\right)^{\frac{1}{2}} \\ & + \left|\langle F(w_{\sigma+t}^u), \mathcal{J}(w^u(\sigma+t))\rangle\right| \text{ for all } \sigma \in [0,T].\end{aligned}$$

Integrating the above inequality over $[0,T]$ and using Cauchy-Schwart's inequality, we obtain that

$$\begin{aligned}&\int_0^T |\langle F(U^u(\sigma)w_t), \mathcal{J}((U^u(\sigma)w_t)(0))\rangle| d\sigma \\ &\quad\leq \left(2L\left(R\|F\|_{\mathcal{C}}\right)^2 T^{\frac{3}{2}} + \left(T + T\|F\|_{\mathcal{C}} LR^2\right)^{\frac{1}{2}}\right)\left(\int_t^{t+T} \frac{|\langle F(w_\sigma^u), \mathcal{J}(w^u(\sigma))\rangle|^2}{1+|\langle F(w_\sigma^u), \mathcal{J}(w^u(\sigma))\rangle|} d\sigma\right)^{\frac{1}{2}}.\end{aligned} \tag{6.11}$$

which gives the desired estimate.

6.3 Strong and weak stabilization

Now, we will present a theorem that establishes the strong stabilization of the system (6.2) as well as the decay estimation.

Theorem 6.3.1. *Suppose that:*
1. $(S^u(t))_{t\geq 0}$ *is a contraction semigroup on* $\mathcal{X}^u$,
2. there exist $\delta \geq 2$ *and* T, $\eta > 0$ *such that*

$$\int_0^T |\langle F^u(U^u(\sigma)\zeta^u), \mathcal{J}((U^u(\sigma)\zeta^u)(0))\rangle| \, d\sigma \geq \eta \|\zeta^u(0)\|_{\mathcal{X}^u}^{\delta} \quad \text{for every } \zeta^u \in \mathcal{C}^u, \tag{6.12}$$

thus, the feedback control $v^u(t)$ defined by (6.4) strongly stabilizes the system (6.2). Furthermore, we obtain

$$\|w^u(t)\|_{\mathcal{X}}^2 = O\left(t^{-\frac{1}{\delta-1}}\right) \text{ as } t \longrightarrow +\infty.$$

Proof. Using inequality (6.11) and the observability condition (6.12), we conclude that

$$\begin{aligned}\eta\|w^u(t)\|_{\mathcal{X}}^{\delta} &\leq \int_0^T |\langle F(U(\sigma)w_t^u)\mathcal{J}((U(\sigma)w_t^u)(0))\rangle| d\sigma \\ &\leq \left(2L\left(R\|F\|_{\mathcal{C}}\right)^2 T^{\frac{3}{2}} + \left(T + T\|F\|_{\mathcal{C}} LR^2\right)^{\frac{1}{2}}\right)\left(\int_t^{t+T} \frac{|\langle F(w_\sigma^u), \mathcal{J}(w^u(\sigma))\rangle|^2}{1+|\langle F(w_\sigma^u), \mathcal{J}(w^u(\sigma))\rangle|} d\sigma\right)^{\frac{1}{2}}.\end{aligned}$$

and thus

$$\left(\frac{\eta}{2L\left(R\|F\|_{\mathcal{C}}\right)^2 T^{\frac{3}{2}} + \left(T + T\|F\|_{\mathcal{C}} LR^2\right)^{\frac{1}{2}}}\right)^2 \|w^u(t)\|_{\mathcal{X}}^{2\delta} \leq \int_t^{t+T} \frac{|\langle F(w_\sigma^u), \mathcal{J}(w^u(\sigma))\rangle|^2}{1+|\langle F(w_\sigma^u), \mathcal{J}(w^u(\sigma))\rangle|} d\sigma. \tag{6.13}$$

Using inequalities (6.8) and (6.13), we obtain that

$$\begin{aligned}\|w^u(t+T)\|_{\mathcal{X}}^2 - \|w^u(t)\|_{\mathcal{X}}^2 &\leq -2\int_t^{t+T} \frac{|\langle F(w_\sigma^u), \mathcal{J}(w^u(\sigma))\rangle|^2}{1+|\langle F(w_\sigma^u), \mathcal{J}(w^u(\sigma))\rangle|} d\sigma \\ &\leq -2\left(\frac{\eta}{2L\left(R\|F\|_{\mathcal{C}}\right)^2 T^{\frac{3}{2}} + \left(T + T\|F\|_{\mathcal{C}} LR^2\right)^{\frac{1}{2}}}\right)^2 \|w^u(t)\|_{\mathcal{X}}^{2\delta} \\ &\leq -2\left(\frac{\eta}{2L\left(R\|F\|_{\mathcal{C}}\right)^2 T^{\frac{3}{2}} + \left(T + T\|F\|_{\mathcal{C}} LR^2\right)^{\frac{1}{2}}}\right)^2 \|w^u(t+T)\|_{\mathcal{X}}^{2\delta}.\end{aligned}$$

Then, we have

$$\|w^u((j+1)T)\|_{\mathcal{X}}^2 \leq \|w^u(jT)\|_{\mathcal{X}}^2 - 2\left(\frac{\eta}{2L\left(R\|F\|\right)^2 T^{\frac{3}{2}} + \left(T + T\|F\|_{\mathcal{C}} LR^2\right)^{\frac{1}{2}}}\right)^2 \|w^u((j+1)T)\|_{\mathcal{X}}^{2\delta}.$$

Using Lemma 1.3.5 with $p = \delta - 2, >, -1$, we get

$$\|w^u(jT)\|_{\mathcal{X}}^2 \leq \frac{M}{(j+1)^{\frac{1}{\delta-1}}} \text{ for every } j \geq 0.$$

Finally, since the function $t \to \|w^u(t)\|_{\mathcal{X}}$ is decreasing on $\mathbb{R}^+$ and by letting $j = \left[\frac{t}{T}\right]$, where $[\,.\,]$ denotes the integer part, we conclude that:

$$\|w^u(t)\|_{\mathcal{X}}^2 = O\left(t^{-\frac{1}{\delta-1}}\right) \text{ as } t \longrightarrow +\infty \blacksquare$$

Next, using the exponentially stable condition on the semigroup $S^s(t)$ on $\mathcal{X}^s$, we get the following theorem.

Theorem 6.3.2. *Suppose that $(S^s(t))_{t\geq 0}$ is exponentially stable on $\mathcal{X}^s$, thus, there exist $\nu, N \geq 0$ such that*

$$\|w^s(t)\|_{\mathcal{X}} \leq \|\phi^s\|_{\mathcal{C}} N \exp(-\nu t) \quad \textit{for all } \ t > 0. \tag{6.14}$$

Proof. The system (0.8) can be rewritten by using the feedback control (6.4) as the following form:

$$\begin{cases} \dot{w}(t) = \mathcal{A}w(t) + \widehat{F}(w_t) \quad \text{for } t > 0, \\ w_0 = \phi \in \mathcal{C} = C([-r, 0]; \mathcal{X}^u), \end{cases} \tag{6.15}$$

where the function $\widehat{F} : \mathcal{C} \longrightarrow \mathcal{X}$ is defined by:

$$\widehat{F}(\psi) \ = \ -\frac{\langle F^u(\psi^u), \mathcal{J}(\psi^u(0))\rangle}{1 + |\,\langle F^u(\psi^u), \mathcal{J}(\psi^u(0))\rangle\,|} F(\psi).$$

the operator $\widehat{F}$ is locally Lipschitz. In fact, for all $(\zeta_1, \zeta_2) \in \mathcal{C}^2$ such that $\|\zeta_1\|_{\mathcal{C}} \leq K$, $\|\zeta_2\|_{\mathcal{C}} \leq K$, we have

$$\begin{aligned}
&\|\widehat{F}(\zeta_1) - \widehat{F}(\zeta_2)\|_{\mathcal{X}} \ \leq \left\| \frac{\langle F(\zeta_1^u), \mathcal{J}(\zeta_1^u(0))\rangle}{1 + |\,\langle F(\zeta_1^u), \mathcal{J}(\zeta_1^u(0))\rangle\,|} F(\zeta_1) - \frac{\langle F(\zeta_2^u), \mathcal{J}(\zeta_2^u(0))\rangle}{1 + |\,\langle F(\zeta_2^u), \mathcal{J}(\zeta_2^u(0))\rangle\,|} F(\zeta_2) \right\|_{\mathcal{X}} \\
&\leq \left\| \frac{\langle F(\zeta_1^u), \mathcal{J}(\zeta_1^u(0))\rangle\, F(\zeta_1)\,(1 + |\,\langle F(\zeta_2^u), \mathcal{J}(\zeta_2^u(0))\rangle\,|) - \langle F(\zeta_2^u), \mathcal{J}(\zeta_2^u(0))\rangle\, F(\zeta_2)\,(1 + |\,\langle F(\zeta_1^u), \mathcal{J}(\zeta_1^u(0))\rangle\,|)}{(1 + |\,\langle F(\zeta_1^u), \mathcal{J}(\zeta_1^u(0))\rangle\,|)\,(1 + |\,\langle F(\zeta_2^u), \mathcal{J}(\zeta_2^u(0))\rangle\,|)} \right\|_{\mathcal{X}} \\
&\leq \|\langle F(\zeta_1^u), \mathcal{J}(\zeta_1^u(0))\rangle\, F(\zeta_1) - \langle F(\zeta_2^u), \mathcal{J}(\zeta_2^u(0))\rangle\, F(\zeta_2)\|_{\mathcal{X}} + \\
&\|\langle F(\zeta_1^u), \mathcal{J}(\zeta_1^u(0))\rangle\, F(\zeta_1)|\,\langle F(\zeta_2^u), \mathcal{J}(\zeta_2^u(0))\rangle\,| - \langle F(\zeta_2^u), \mathcal{J}(\zeta_2^u(0))\rangle\, F(\zeta_2)|\,\langle F(\zeta_1^u), \mathcal{J}(\zeta_1^u(0))\rangle\,|\|_{\mathcal{X}} \\
&\leq \|\langle F(\zeta_1^u), \mathcal{J}(\zeta_1^u(0))\rangle\, F(\zeta_1) - \langle F(\zeta_2^u), \mathcal{J}(\zeta_2^u(0))\rangle\, F(\zeta_2)\|_{\mathcal{X}} + \\
&\|\langle F(\zeta_2^u), \mathcal{J}(\zeta_2^u(0))\rangle\|_{\mathcal{X}}\, \|\langle F(\zeta_1^u), \mathcal{J}(\zeta_1^u(0))\rangle\, F(\zeta_1) - \langle F(\zeta_2^u), \mathcal{J}(\zeta_2^u(0))\rangle\, F(\zeta_2)\|_{\mathcal{X}} + \\
&\|\langle F(\zeta_2^u), \mathcal{J}(\zeta_2^u(0))\rangle\, F(\zeta_2)\|_{\mathcal{X}}\, \|\,|\,\langle F(\zeta_2^u), \mathcal{J}(\zeta_2^u(0))\rangle\,| - |\,\langle F(\zeta_1^u), \mathcal{J}(\zeta_1^u(0))\rangle\,|\,\|_{\mathcal{X}}\,.
\end{aligned}$$

Moreover,

$$\begin{aligned}
\langle F(\zeta_1^u), \mathcal{J}(\zeta_1^u(0))\rangle\, F(\zeta_1) - \langle F(\zeta_2^u), \mathcal{J}(\zeta_2^u(0))\rangle\, F(\zeta_2) &= \langle F(\zeta_1^u) - F(\zeta_2^u), \mathcal{J}(\zeta_1^u(0))\rangle\, F(\zeta_1) \\
&+ \langle F(\zeta_2^u), \mathcal{J}(\zeta_1^u(0)) - \mathcal{J}(\zeta_2^u(0))\rangle\, F(\zeta_1) \\
&+ \langle F(\zeta_2^u), \mathcal{J}(\zeta_2^u(0))\rangle\, (F(\zeta_1) - F(\zeta_2)),
\end{aligned}$$

and hence

$$\|\langle F(\zeta_1^u), \mathcal{J}(\zeta_1^u(0))\rangle\, F(\zeta_1) - \langle F(\zeta_2^u), \mathcal{J}(\zeta_2^u(0))\rangle\, F(\zeta_2)\|_{\mathcal{X}} \leq 3K^2 \|F\|_{\mathcal{C}}^2 L \|\zeta_1 - \zeta_2\|_{\mathcal{C}}.$$

where L is the Lipschitz constant of $\mathcal{J}$. So, we get

$$\begin{aligned}\|\widehat{F}(\zeta_1^u) - \widehat{F}(\zeta_2^u)\|_{\mathcal{X}} \;\leq\; & 3K^2\|F\|_{\mathcal{C}}^2 L\left(1 + K^2\|F\|_{\mathcal{C}} L\right)\|\zeta_1 - \zeta_2\|_{\mathcal{C}} + \\ & K^3\|F\|_{\mathcal{C}}^2 L\left(\|\langle F(\zeta_2^u) - F(\zeta_1^u), \mathcal{J}(\zeta_2^u(0))\rangle + \langle F(\zeta_1^u), \mathcal{J}(\zeta_2^u(0)) - \mathcal{J}(\zeta_1^u(0))\rangle\|_{\mathcal{X}}\right) \\ & \leq \left[3K^2\|F\|_{\mathcal{C}}^2 L\left(1 + K^2\|F\|_{\mathcal{C}} L\right) + 2K^4\|F\|_{\mathcal{C}}^3 L^2\right]\|\zeta_1 - \zeta_2\|_{\mathcal{C}}.\end{aligned}$$

Therefore, the system (0.8) has a unique mild solution, which is defined on $[0, t_{max})$ and can be expressed using following variation of constants formula:

$$w(t) = S(t)w(0) + \int_0^t S(t-\sigma)\widehat{G}(w_\sigma)d\sigma \quad \text{for } t \in [0, t_{max}),$$

and thus

$$w^s(t) = S^s(t)w^s(0) + \int_0^t S^s(t-\sigma)v^u(\sigma)F^s(w_\sigma^s)d\sigma \;\; \text{for } t \in [0, t_{max}).$$

Since $(S^s(t))_{t\geq 0}$ is exponentially stable, there exist constants $\beta > 0$ and $C \geq 1$ such that

$$\|S^s(t)\|_{\mathcal{X}^s} \leq C\exp(-\beta t) \quad \text{for all } t \geq 0.$$

Then, for every $\varrho \in [-r, 0]$ and $t \in [0, t_{max})$ such that $t + \varrho \geq 0$

$$\begin{aligned}\|w^s(t+\varrho)\|_{\mathcal{X}} \leq\; & C\exp(-\beta(t+\varrho))\|\phi^s\|_{\mathcal{C}} \\ & + \exp(-\beta t)\int_0^{t+\varrho} C\exp(-\beta\varrho)\exp(\beta\sigma)|v^u(\sigma)|\,\|F^s(w_\sigma^s)\|_{\mathcal{X}}\,d\sigma,\end{aligned}$$

and

$$\|w^s(t+\varrho)\|_{\mathcal{X}}\exp(\beta t) \leq C\exp(\beta r)\|\phi^s\|_{\mathcal{C}} + \exp(\beta r)\int_0^t C|v^u(\sigma)|\|F^s\|\|w_\sigma^s\|_{\mathcal{C}}\exp(\beta\sigma)\,d\sigma.$$

Moreover, for $\varrho \in [-r, 0]$ and $t \in [0, t_{max})$ such that $t + \varrho \leq 0$, we get

$$\|w^s(t+\varrho)\|_{\mathcal{X}}\exp(\beta t) \leq \|\phi^s\|_{\mathcal{C}}\exp(\beta r).$$

Finally, for every $t \geq 0$

$$\|w_t^s\|_{\mathcal{C}}\exp(\beta t) \leq C\exp(\beta r)\|\phi^s\|_{\mathcal{C}} + e^{\beta r}\int_0^t C|v^u(\sigma)|\|F^s\|\|w_\sigma^s\|_{\mathcal{C}}\exp(\beta\sigma)\,d\sigma.$$

Appliying the Gronwall's lemma, we get

$$\|w_t^s\|_{\mathcal{C}}\exp(\beta t) \leq C\exp(\beta r)\|\phi^s\|_{\mathcal{C}}\exp\left(C\exp(\beta r)\|F\|_{\mathcal{C}}\int_0^t |v^u(\sigma)|d\sigma\right).$$

Moreover,

$$\begin{aligned}\int_0^t |v^u(\sigma)|d\sigma &\leq \int_0^t \frac{|\langle F(w^u_\sigma), \mathcal{J}(w^u(\sigma))\rangle|}{1+|\langle F(w^u_\sigma), \mathcal{J}(w^u(\sigma))\rangle|} d\sigma \\ &\leq \int_0^t \frac{|\langle F(w^u_\sigma), \mathcal{J}(w^u(\sigma))\rangle|}{(1+|\langle F(w^u_\sigma), \mathcal{J}(w^u(\sigma))\rangle|)^{\frac{1}{2}}} \frac{1}{(1+|\langle F(w^u_\sigma), \mathcal{J}(w^u(\sigma))\rangle|)^{\frac{1}{2}}} d\sigma \\ &\leq \int_0^t \frac{|\langle F(w^u_\sigma), \mathcal{J}(w^u(\sigma))\rangle|}{(1+|\langle F(w^u_\sigma), \mathcal{J}(w^u(\sigma))\rangle|)^{\frac{1}{2}}} d\sigma.\end{aligned}$$

By applying the Holder's inequality, we get

$$\int_0^t \frac{|\langle F(w^u_\sigma), \mathcal{J}(w^u(\sigma))\rangle|}{(1+|\langle F(w^u_\sigma), \mathcal{J}(w^u(\sigma))\rangle|)^{\frac{1}{2}}} d\sigma \leq \left(\int_0^t \frac{|\langle F(w^u_\sigma), \mathcal{J}(w^u(\sigma))\rangle|^2}{1+|\langle F(w^u_\sigma), \mathcal{J}(w^u(\sigma))\rangle|} d\sigma\right)^{\frac{1}{2}} \sqrt{t},$$

and then

$$\int_0^t \frac{|\langle F(w^u_\sigma), \mathcal{J}(w^u(\sigma))\rangle|}{1+|\langle F(w^u_\sigma), \mathcal{J}(w^u(\sigma))\rangle|} d\sigma \leq \frac{R}{\sqrt{2}}\sqrt{t}.$$

Then, we get for $t \geq 0$

$$\|w^s_t\|_{\mathcal{C}} \exp(\beta t) \leq C \exp(\beta r) \|\phi^s\|_{\mathcal{C}} \exp\left(C\exp(\beta r)\|F\|_{\mathcal{C}} \frac{R}{\sqrt{2}}\sqrt{t}\right),$$

i.e.

$$\|w^s_t\|_{\mathcal{C}} \leq C \exp(\beta r) \|\phi^s\|_{\mathcal{C}} \exp\left(C\exp(\beta r)\|F\|_{\mathcal{C}} \frac{R}{\sqrt{2}}\sqrt{t} - \beta t\right).$$

Thus, for every $t \geq 0$, we get

$$\|w^s(t)\|_{\mathcal{X}} \leq C \exp(\beta r) \|\phi^s\|_{\mathcal{C}} \exp\left(C\exp(\beta r)\|F\| \frac{R}{\sqrt{2}}\sqrt{t} - \beta t\right)$$

which means that there exist $\nu, N \geq 0$ such that

$$\|w^s(t)\|_{\mathcal{X}} \leq \|\phi^s\|_{\mathcal{C}} N \exp(-\nu t) \quad \text{for all} \quad t > 0. \tag{6.16}$$

Finally, $w^s(t)$ is bounded on $[0, t_{max})$, which means that $t_{max} = +\infty$ ■

As a direct result by combining Theorems 6.3.1 and 6.3.2, we get the following result dealing with the strong stabilization of the system (0.8).

Theorem 6.3.3. *Suppose that:*
1. $(S^s(t))_{t\geq 0}$ *is exponentially stable on* $\mathcal{X}^s$,

2. $(S^u(t))_{t\geq 0}$ *is a contraction semigroup on* $\mathcal{X}^u$,
3. there exist T, $\eta > 0$ *and* $\delta \geq 2$ *such that*

$$\int_0^T |\langle F^u(U^u(\sigma)\zeta^u), \mathcal{J}((U^u(\sigma)\zeta^u)(0))\rangle|\, d\sigma \geq \eta \|\zeta^u(0)\|^\delta_{\mathcal{X}^u} \quad \text{for every } \zeta^u \in \mathcal{C}^u, \tag{6.17}$$

then, the feedback control (6.4) strongly stabilizes the system (0.8). Furthermore, we obtain

$$\|w(t)\|^2_{\mathcal{X}} = O\left(t^{-\frac{1}{\delta - 1}}\right) \text{ as } t \longrightarrow +\infty.$$

In the situation where $\mathcal{X}^u = \mathcal{X}$ and $\mathcal{X}^s = \{0\}$, we obtain the following result.

Corollary 6.3.4. *Suppose if* $(S^u(t))_{t\geq 0}$ *is a contraction semigroup on* $\mathcal{X}$ *and there exist* $T > 0$ $\eta > 0$ *and* $\delta \geq 2$ *such that*

$$\int_0^T |\langle F(U(\sigma)\zeta), \mathcal{J}((U(\sigma)\zeta)(0))\rangle|\, d\sigma \geq \eta \|\zeta(0)\|^\delta_{\mathcal{X}} \quad \text{for every } \zeta \in \mathcal{C}, \tag{6.18}$$

then, the feedback control

$$v(t) = -\frac{\langle F(w_t), \mathcal{J}(w(t))\rangle}{1 + |\langle F(w_t), \mathcal{J}(w(t))\rangle|}$$

strongly stabilizes the system (6.2). Furthermore, we obtain

$$\|w(t)\|^2_{\mathcal{X}} = O\left(t^{-\frac{1}{\delta - 1}}\right) \text{ as } t \longrightarrow +\infty.$$

Remark 6.3.5. *1- In the context of Hilbert spaces,* $\mathcal{X}$ *can be identified with its dual space* $\mathcal{X}^*$, *and the duality pairing between the two is determined by the same inner product that exists in* $\mathcal{X}$.

2- Note that the situation where $r = 0$ *and* $\mathcal{X}$ *is a Hilbert space, the traditional concept of exact observability in the sense of linear system theory is recovered (see e.g [27]) where* $F = D^*D$ *in the estimate (6.18) and* D *is an observation operator. Moreover, the estimate (6.18) is more general than the estimates given in [9, 19, 21].*

Lemma 6.3.6. *Let* $\mathcal{X}$ *be a finite-dimensional space. Then, the duality mapping* $\mathcal{J} = Id_{\mathcal{X}}$. *Moreover, the following condition:*
If there exist $T_2 > T_1 > r$ *and for every* $\zeta \in \mathcal{C}$ *such that*

$$\langle F(U(\sigma)\zeta), (U(\sigma)\zeta)(0)\rangle = 0 \quad \text{for all } \sigma \in [T_1, T_2] \Rightarrow \zeta(0) = 0, \tag{6.19}$$

implies that the observation condition (6.18) *is satisfied with* $\mathcal{J} = Id_{\mathcal{X}}$.

Proof . Let $\mathcal{X}$ be a finite-dimensional space where $\dim(\mathcal{X}) = n$, then there exists a basis $\{e_1, e_2, ..., e_n\}$ of $\mathcal{X}$ and a corresponding basis $\{e_1^*, e_2^*, ..., e_n^*\}$ of the dual space $\mathcal{X}^*$ such that any linear functional f on $\mathcal{X}$ can be written as a linear combination of $e_1^*, e_2^*, ..., e_n^*$ with coefficients given by $f(e_i)$ for $i = 1, 2, ..., n$. In addition, the duality pairing between $\mathcal{X}$ and $\mathcal{X}^*$ can be identified with the standard inner product in the Euclidean space $\mathbb{R}^n$, via the identification of $\mathcal{X}$ with $\mathbb{R}^n$ induced by the basis $\{e_1, e_2, ..., e_n\}$ and hence $\mathcal{J} = Id_{\mathcal{X}}$. Now, let us make the following assumption:

$$\inf_{\|w\|_{\mathcal{X}}=1} \int_{T_1}^{T_2} |\langle F(U(\sigma + .)w), S(\sigma)w\rangle| \, d\sigma \;=\; 0.$$

Clearly, for any $\sigma \in [T_1, T_2]$, the following equality holds true:

$$\langle F(U(\sigma)\zeta), (U(\sigma)\zeta)(0)\rangle = \langle F(S(\sigma + .)\zeta(0)), S(\sigma)\zeta(0)\rangle \, .$$

Since $\dim(\mathcal{X}) < \infty$, the set $\{w \in Y : \|w\|_{\mathcal{X}} = 1\}$ is compact on Y. Moreover, the application $w \to \int_{T_1}^{T_2} |\langle F(S(\sigma + .)w), S(\sigma)w\rangle| \, d\sigma$ is continuous on $\mathcal{X}$. Thus, there exists $w_0 \in \mathcal{X}$ such that

$$\|w_0\|_{Y=1} \text{ and } \int_{T_1}^{T_2} |\langle F(S(\sigma + .)w_0), S(\sigma)w_0\rangle| \, d\sigma = 0.$$

Thus, there exists $\zeta \in \mathcal{C}$ such that $\zeta(0) = w_0$ and

$$|\langle F(S(\sigma + .)\zeta(0)), S(\sigma)\zeta(0)\rangle| = 0 \;\text{ for all } \sigma \;\in\; [T_1, T_2].$$

However, based on condition (6.19), it can be inferred that $\zeta(0) = 0$, which contradicts the fact that $\|\zeta(0)\|_{\mathcal{X}} = \|w_0\|_{\mathcal{X}} = 1$. Therefore, there exists a constant $\eta > 0$ such that

$$\inf_{\|w\|_{\mathcal{X}}=1} \int_{T_1}^{T_2} |\langle F(U(\sigma + .)w, S(\sigma)w\rangle| \, d\sigma \;=\; \eta.$$

Let $w = \dfrac{1}{\|\zeta(0)\|_{\mathcal{X}}}\zeta(0)$, then $\|w\|_{\mathcal{X}} = 1$ and we have

$$\int_{T_1}^{T_2} |\langle F(U(\sigma + .)\zeta(0)), S(\sigma)\zeta(0)\rangle| \, d\sigma \geq \eta\|\zeta(0)\|_{\mathcal{X}}^2 \quad \blacksquare$$

The weak stabilization of system (0.8) can be obtained by satisfying the conditions provided in the following theorem.

Theorem 6.3.7. *Suppose that conditions (1) and (2) of Theorem 6.3.3 are satisfied. If there exist two positive constants $T_2 > T_1 > 0$ and for every $(\zeta_n)_n \subset \mathcal{C}^u$ such that $\zeta_n(0) \rightharpoonup \zeta(0)$ in $\mathcal{X}^u$, we get*

$$\langle F(U^u(\sigma)\zeta_n), \mathcal{J}((U^u(\sigma)\zeta_n)(0))\rangle \to 0 \text{ as } n \to +\infty \quad \text{for every } \sigma \in [T_1, T_2] \Rightarrow \zeta(0) = 0, \tag{6.20}$$

then, the control $v^u(t)$ defined by (6.4) weakly stabilizes the system (0.8).

Proof. Let $(t_n)_n$ be a sequence such that $t_n \to +\infty$. Inequality (6.8) implies that $\|w^u(t)\|_{\mathcal{X}^u}$ is bounded. As the space $\mathcal{X}$ is reflexive, thus, we can extract a subsequence of $(t_n)_n$, denoted by $(t_n)_n$ such that $w^u(t_n) \to w \in \mathcal{X}^u$ weakly as $n \to +\infty$. From inequality (6.11), we have for any $T_2 > T_1 > 0$ that

$$\int_{T_1}^{T_2} |\langle F(U^u(\sigma)w_t^u), \mathcal{J}((U^u(\sigma)w_{t_n}^u)(0)\rangle| d\sigma$$
$$\leq \left(2L\left(R\|F\|_{\mathcal{C}}\right)^2 T_2^{\frac{3}{2}} + \left(T + T\|F\|_{\mathcal{C}} LR^2\right)^{\frac{1}{2}}\right) \left(\int_t^{t+T_2} \frac{|\langle F(w_\sigma^u), \mathcal{J}(w^u(\sigma))\rangle|^2}{1 + |\langle F(w_\sigma^u), \mathcal{J}(w^u(\sigma))\rangle|} d\sigma\right)^{\frac{1}{2}}.$$

From inequality (6.8), we have that

$$\int_{t_n}^{t_n+T_2} \frac{|\langle F(w_\sigma^u), \mathcal{J}(w^u(\sigma))\rangle|^2}{1 + |\langle F(w_\sigma^u), \mathcal{J}(w^u(\sigma))\rangle|} d\sigma \leq \frac{1}{2}\left(\|w^u(t_n)\|_{\mathcal{X}^u}^2 - \|w^u(t_n + T_2)\|_{\mathcal{X}^u}^2\right).$$

Then

$$\int_0^{T_2} \left|\langle F(U^u(\sigma))w_{t_n}^u), \mathcal{J}((U^u(\sigma)w_{t_n}^u)(0))\rangle\right| d\sigma \to 0 \quad as\ n \to +\infty.$$

Based on the dominated convergence theorem, it follows that

$$\int_{T_1}^{T_2} \lim_{n\to+\infty} |\langle F(U^u(\sigma)w_t^u), \mathcal{J}((U^u(\sigma)w_{t_n}^u)(0)\rangle| d\sigma = 0,$$

and consequently

$$\lim_{n\to+\infty} |\langle F(U^u(\sigma)w_t^u), \mathcal{J}((U^u(\sigma)w_{t_n}^u)(0)\rangle| = 0 \quad \text{for all } \sigma \in [T_1, T_2].$$

Thus condition (6.20) implies that $w = 0$, i.e $w^u(t) \to 0$ weakly as $t \to +\infty$.
Applying Theorem 6.3.2, we conclude that $w(t) = w^u(t) + w^s(t) \longrightarrow 0$ weakly as $t \to +\infty$
■

Now, we provide sufficient conditions for achieving weak stabilization of system (0.8) in the particular case where $\mathcal{X}^u = \mathcal{X}$ and $\mathcal{X}^s = \{0\}$. This corollary follows directly from Theorem 6.3.7.

Corollary 6.3.8. *Suppose that:*
1. $(S(t))_{t\geq 0}$ *is a contraction semigroup on* $\mathcal{X}$,
2. there exist two positive constants $T_2 > T_1 > 0$ *and for every* $(\zeta_n)_n \subset \mathcal{C}$ *such that* $\zeta_n(0) \rightharpoonup \zeta(0)$ *in* $\mathcal{X}$, *we get*

$$\langle F(U(t)\zeta_n), \mathcal{J}((U(t)\zeta_n)(0))\rangle \to 0 \ as\ n \to +\infty \quad for\ every\ t \in [T_1, T_2] \Rightarrow \zeta(0) = 0, \tag{6.21}$$

then, the control

$$v(t) = -\frac{\langle F(w_t), \mathcal{J}(w(t))\rangle}{1 + |\langle F(w_t^u), \mathcal{J}(w^u(t))\rangle|}$$

weakly stabilizes the system (0.8).

6.4 Apllication and Numerical Simulation

We apply the previous theoretical results to the following example. Moreover, the numerical simulation is presented at the end of this example.

Example 1. Let $\mathcal{X}$ be the space such that $\mathcal{X} = \mathcal{X}^u \oplus \mathcal{X}^s$ with $\mathcal{X}^u = L^p(0,+\infty)\times \{0\}$, $p > 1$ and $\mathcal{X}^s = \{0\} \times L^2(0,1)$ and the following system :

$$\begin{cases} w_t(x,t) = w_{xx}(x,t) - \dfrac{\left(\displaystyle\int_0^{+\infty} |z(x,t)|^p dx\right)^{\frac{2}{p}}}{1+\left(\displaystyle\int_0^{+\infty} |z(x,t)|^p dx\right)^{\frac{2}{p}}} \displaystyle\int_{-2}^{0} w(x,t+s)ds & \text{for } (x,t) \in (0,1)\times(0,+\infty) \\ z_t(x,t) = -z(x,t) - \dfrac{\left(\displaystyle\int_0^{+\infty} |z(x,t)|^p dx\right)^{\frac{2}{p}}}{1+\left(\displaystyle\int_0^{+\infty} |z(x,t)|^p dx\right)^{\frac{2}{p}}} z(x,t) & \text{for } (x,t) \in (0,+\infty)\times(0,+\infty) \\ w(0,t) = w(1,t) = 0 & \text{for } t \in [-2,+\infty) \\ z(0,t) = \lim\limits_{x\to+\infty} z(x,t) = 0 & \text{for } t \in [0,+\infty) \\ w(x,t) = \phi_1(x,t) & \text{for } (x,t) \in (0,1)\times[-2,0] \\ z(x,0) = \phi_2(x,0) & \text{for } x \in (0,+\infty) \end{cases} \tag{6.22}$$

Putting $y(t) = \begin{pmatrix} z(t) \\ w(t) \end{pmatrix}$, then the above system takes the following abstract form :

$$\begin{cases} \dot{y}(t) = \mathcal{A}y(t) + v(t)F(y_t) \ \text{ for } t > 0, \\ y_0 = \phi \in \mathcal{C}, \end{cases} \tag{6.23}$$

the operator $\mathcal{A} : D(\mathcal{A}) \subset \mathcal{X} \to \mathcal{X}$ is defined by

$$\mathcal{A} = \begin{pmatrix} -I & 0 \\ 0 & \frac{\partial^2}{\partial x^2} \end{pmatrix} \text{ and } D(\mathcal{A}) = \left\{ z \in L^p(0,+\infty) : z(0) = \lim_{x\to+\infty} z(x) = 0 \right\} \times (H_0^1(0,1) \cap H^2(0,1)),$$

where

$$\begin{aligned} H^2(0,1) =& \{z \in H^1(0,1) : z' \in H^1(0,1)\}, \\ H_0^1(0,1) =& \{z \in H^1(0,1) : z(0) = z(1) = 0\}, \\ W^{1,p}(0,+\infty) =& \{z \in L^p(0,+\infty) : z' \in L^p(0,+\infty)\}. \end{aligned}$$

The operator $\mathcal{A}_1 z = -z, \quad D(\mathcal{A}_1) = \left\{ z \in L^p(0,+\infty),\ z(0) = \lim\limits_{x\to+\infty} z(x) = 0 \right\}$ generates the semigroup of contraction $(S_1(t))_{t\geq 0}$ defined by

$$S_1(t)z = \exp(-t)z$$

then, the operator $\mathcal{A}^u : D(\mathcal{A}^u) \subset \mathcal{X}^u \to \mathcal{X}^u$ is defined by

$$\mathcal{A}^u = \begin{pmatrix} \mathcal{A}_1 & 0 \\ 0 & 0 \end{pmatrix}, \quad D(\mathcal{A}^u) = \left\{ z \in W^{1,p}(0,+\infty), z(0) = \lim_{x \to +\infty} z(x) = 0 \right\} \times \{0\}$$

generates the semigroup of contraction

$$S^u(t) = \begin{pmatrix} S_1(t) & 0 \\ 0 & 0 \end{pmatrix}.$$

The operator $\mathcal{A}_2 = \dfrac{\partial^2}{\partial x^2}$, $D(\mathcal{A}_2) = H_0^1(0,1) \cap H^2(0,1)$ is the Laplacian operator with eigenvalues given by $\lambda_m = -(m\pi)^2, (m \in \mathbb{N}^*)$ associated to the eigenfunctions $(\varphi_m)_{m \geq 1}$ expressed by $\varphi_m(x) = \sqrt{2}\sin(m\pi x)$, for all $m \geq 1$, generates the semigroup of exponentially stable $(S_2(t))_{t \geq 0}$ defined by

$$S_2(t)w = \sum_{m=1}^{+\infty} \exp(\lambda_m t)\langle w, \varphi_m \rangle \varphi_m$$

then, the operator $\mathcal{A}^s : D(\mathcal{A}^s) \subset \mathcal{X}^s \to \mathcal{X}^s$ is defined by

$$\mathcal{A}^s = \begin{pmatrix} 0 & 0 \\ 0 & \mathcal{A}_2 \end{pmatrix}, \quad D(\mathcal{A}^s) = \{0\} \times H_0^1(0,1) \cap H^2(0,1)$$

generates the semigroup of exponentially stable

$$S^s(t) = \begin{pmatrix} 0 & 0 \\ 0 & S_2(t) \end{pmatrix}.$$

Thus, the operator $\mathcal{A}$ generates the semigroup $S(t) = S^u(t) + S^s(t)$. Furthermore, we define the operator F from $\mathcal{C} = C([-2,0], Y)$ to $\mathcal{X}$ by

$$F(\zeta) = \begin{pmatrix} \zeta_1(0) \\ \displaystyle\int_{-2}^{0} \zeta_2(s)ds \end{pmatrix}.$$

for every $\zeta \in \mathcal{C}$ (i.e $\zeta(\varrho) = (\zeta_1(\varrho), \zeta_2(\varrho)) \in Y$, for $\varrho \in [-2,0]$) and the feedback control

$$v^u(t) = -\frac{\left(\displaystyle\int_0^{+\infty} |z(x,t)|^p dx\right)^{\frac{2}{p}}}{1 + \left(\displaystyle\int_0^{+\infty} |z(x,t)|^p dx\right)^{\frac{2}{p}}}.$$

From [8, p. 82], the duality mapping $\mathcal{J}$ is Lipschitz defined on $L^p(0,+\infty)$ and given by the following formula :

$$\mathcal{J}(z) = \left\{ \frac{|z|^{p-2}}{\|z\|^{p-2}} z \right\} \quad \text{for } z \in L^p(0,+\infty).$$

Thus, for all $z \in L^p(0,+\infty)$ and for all $u \in L^q(0,+\infty)$ with $\frac{1}{p}+\frac{1}{q}=1$, we have

$$\langle u, \mathcal{J}(z)\rangle = \int_0^{+\infty} \frac{|z|^{p-2}(x)}{\|z\|^{p-2}} z(x)u(x)dx.$$

Then, for every $\zeta^u \in \mathcal{C}^u$

$$\begin{aligned}\int_0^T |\langle F^u(U^u(\sigma)\zeta^u), \mathcal{J}^u((U^u(\sigma)\zeta^u)(0))\rangle|\, d\sigma &\geq T\|S^u(T)\zeta^u(0)\|^2_{\mathcal{X}^u} \\ &\geq T\exp(-pT)\|\zeta^u(0)\|^2_{\mathcal{X}^u}.\end{aligned}$$

where $\mathcal{J}^u(z,w) = \begin{pmatrix} \mathcal{J}(z) \\ 0 \end{pmatrix}$. So all assumptions of Theorem 6.3.1 hold. Then, the feedback control $v^u(t)$ strong stabilizes the system (6.22) and we have

$$\|y(t)\|^2_{\mathcal{X}} = O\left(t^{-1}\right) \text{ as } t \longrightarrow +\infty.$$

Remark 6.4.1. *The restriction of the solution y of the system (6.22) on $\mathcal{X}^u$ and $\mathcal{X}^s$ satisfy respectively the following equalities*

$$\|y^u(t)\|_{\mathcal{X}^u} = \left(\int_0^{+\infty} |z(x,t)|^p dx\right)^{\frac{1}{p}}$$

and

$$\|y^s(t)\|_{\mathcal{X}^s} = \left(\int_0^1 |w(x,t)|^2 dx\right)^{\frac{1}{2}}.$$

To provide further insight into the theoretical findings presented in Example 1, we examine a specific initial condition: $\phi_1(x,t) = \frac{\pi\,(1+x)\,t}{2}$ and $\phi_2(x,t) = \sin(\pi x)\mathbf{1}_{[0,4]}$. The first and second components of the system's solution under the feedback control $v^u(t)$ are plotted in Figures 6.1 and 6.2. Furthermore, we demonstrate the convergence of the norm of these components to 0 as t approaches infinity in the third figure, using feedback control $v^u(t)$ where $p = \frac{3}{2}$. In the last figure, we show the evolution of the feedback control $v^u(t)$.

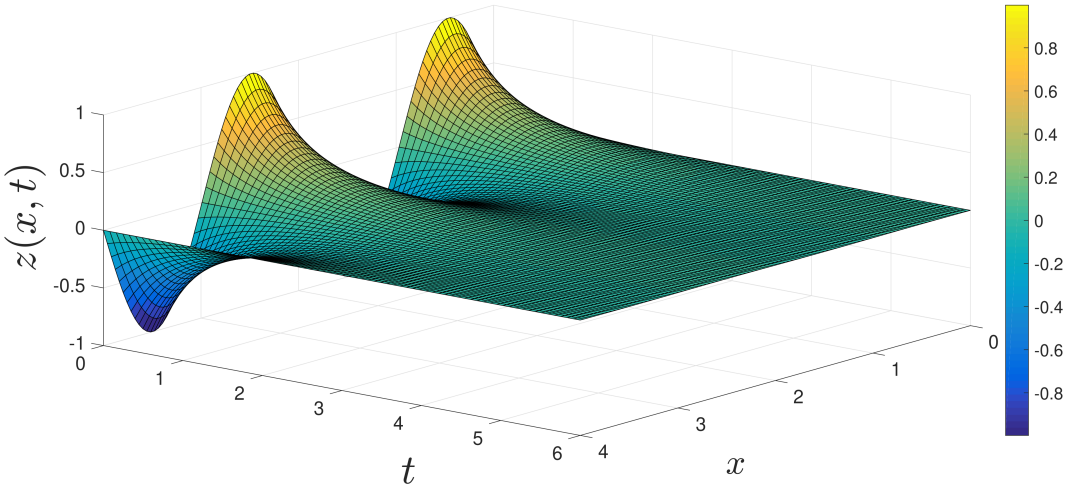

Figure 6.1: The evolution of $z(x,t)$ the first component of the solution $y(x,t)$ of system (6.23) corresponding to the feedback control $v^u(t)$.

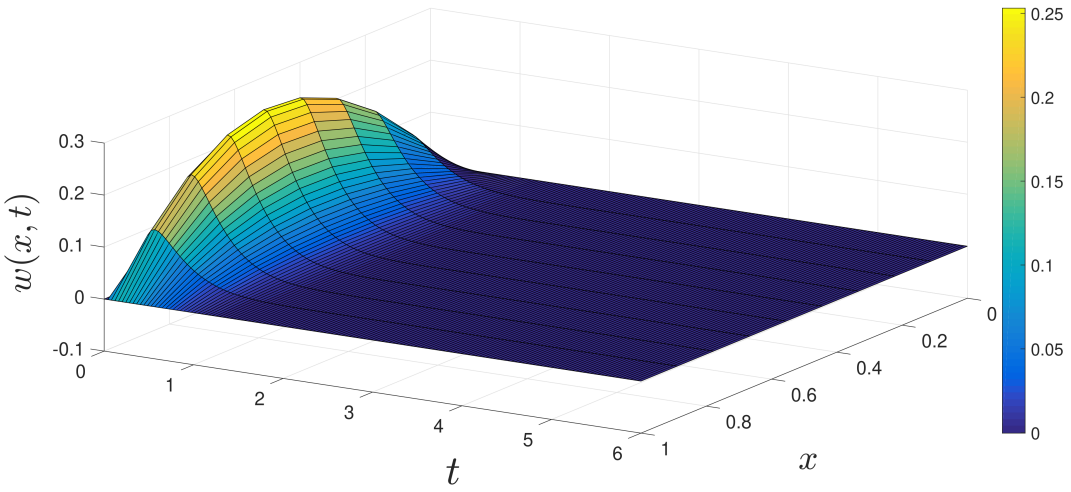

Figure 6.2: The evolution of $w(x,t)$ the second component of the solution $y(x,t)$ of system (6.23) corresponding to the feedback control $v^u(t)$.

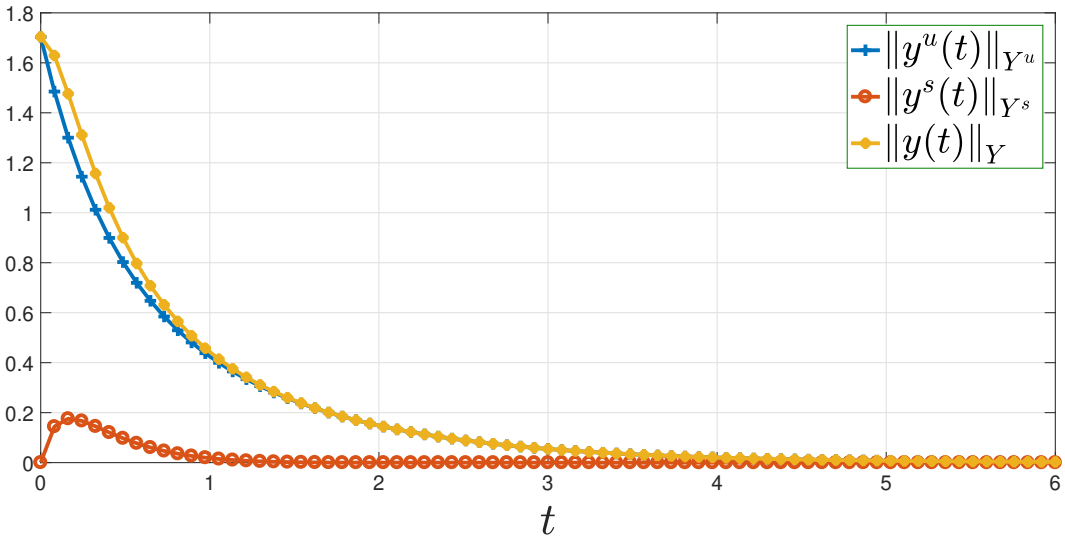

Figure 6.3: The evolution of $\|y^u(t)\|_{\mathcal{X}^u}$, $\|y^s(t)\|_{\mathcal{X}^s}$ and $\|y(t)\|_{\mathcal{X}}$.

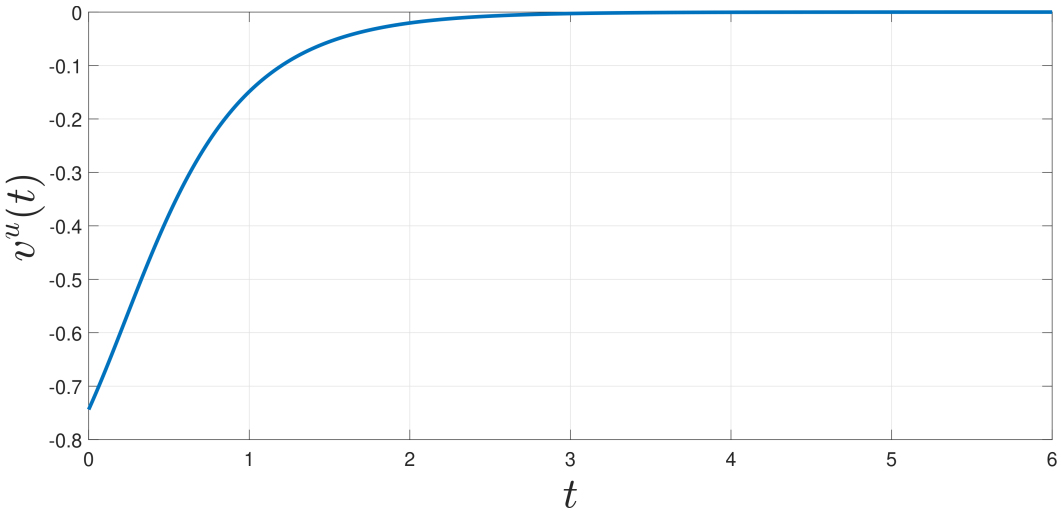

Figure 6.4: The evolution of $v^u(t)$.

Conclusion. In this chapter, we discuss the strong and weak stabilization for a class of bilinear systems with distributed delay in the Banach space by using the feedback control. By imposing an observability condition expressed in terms of the semigroup solution of system (0.8) with $F = 0$ and the parameter $\delta \geq 2$, we demonstrate the strong stabilization of the system (0.8). Moreover, the speed of decay has been also explicitly established. Furthermore, we proved the weak stabilization by utilizing some weaker conditions. An application with distributed delay is considered. Moreover, some numerical simulations are provided to verify the validity of the obtained theoretical results.

Conclusion and perspectives

Thought this thesis, we discussed the feedback stabilization of semilinear systems with distributed delay. Especially, we study the weak, strong and exponential stabilization of these classes of equations. We presented some sufficient conditions to get the desired results using different observation conditions and/ or the decomposition method of the state space. Moreover, we studied the minimization problem and its relation with the stabilization of the considered systems.

The introduced control strategies and observability conditions provide valuable tools for designing and analyzing control systems in various applications. In perspective, future research can further explore the applicability of these strategies in different domains and investigate more advanced control techniques for infinite-dimensional systems. Additionally, experimental validations and practical implementations of the proposed approaches would be beneficial to demonstrate their effectiveness and robustness in real-world scenarios. Moreover, possible future developments concerning the theoretical part can be summarized as follows:

- The stabilization of non-homogeneous semilinear systems with delay.

- The stabilization of the semilinear systems of neutral type.

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