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## *Sampling and interpolation in Bergman type spaces*

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# Sampling and interpolation in Bergman type spaces

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October 01, 2022

# Dedication

To my family,

To my father and my mother,

To all my friends.

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# Abstract

This thesis treats two topics. In the first part, we study sampling and interpolation with unbounded multiplicity in the classical standard weighted Bergman spaces of analytic functions in the unit disk of the complex plane. The main results obtained are a necessary condition and a sufficient condition in both sampling and interpolation cases, with a small gap. It is known that analogue results hold in the classical Fock space of entire functions. We mention that our results may be applied even for bounded multiplicities, as a weak alternative for Seip characterization of sampling and interpolation sequences known by the presence of a Beurling-Landau type densities which are difficult to check in general. The second part is devoted to the study of some properties of zero sets in the classical Fock space. To our knowledge, a complete characterization of zero sets for Fock spaces doesn't exist yet. Also, the gap between necessary and sufficient conditions remains without explanation. One of our results is a response to a question asked by K. Zhu in his book "Analysis on Fock Spaces". As an answer, we prove that two Banach Fock spaces with different exponents doesn't share the same zero sets. Moreover, it is known that in Hardy spaces and in Bergman spaces of the unit disk every sub-sequence of a zero set remains again a zero set, this hereditary property doesn't take place in Fock spaces. In this perspective, we show that a zero set for the Fock space has the property; every sub-sequence is a zero set too if and only if it satisfies the sufficient condition.

**Keywords**: Bergman spaces, Fock spaces, zero sets, uniqueness divisor, sampling divisor, interpolating divisor, $\overline{\partial}$-method, entire functions.

# Résumé

Cette thèse contient deux parties. Dans la première, nous étudions les suites d'échantillonnage et d'interpolation multiples dans les espaces de Bergman à poids standards de fonctions holomorphes sur le disque unité. La multiplicité est supposée non bornée ( peut *ê*tre uniformément bornée ou croissante quand la suite s'approche du bord). Notre objectif est d'obtenir une caractérisation en parallèle à des résultas déjà obtenus dans différents espaces de fonctions analytiques. Dans chaque situation, d'interpolation ou d'échantillonnage, nous obtenons une condition nécessaire et une condition suffisante avec un petit gap difficile à franchir. Des résultats similaires ont été déjà obtenus dans le cas des espaces de Fock dans le plan complexe.
Dans la deuxième partie nous étudions quelques propriétés des ensembles de zéros des espaces de Fock des fonctions entières. Jusqu'à présent, les suites de zéros pour ces espaces de fonctions entières ne sont pas encore caractérisées! Un gap reste entre la condition nécessaire et suffisante. Nous obtenons un résultat qui affirme que deux espaces (Banach) de Fock à exposants différents n'ont pas les mêmes ensembles de zéros. C'est une réponse à une question posée par Kehe Zhu dans son livre " Analysis on Fock Spaces".

**Mots-clés**: espaces de Bergman, espaces de Fock, ensembles de zéro, diviseur d'unicité, diviseur d'échantillonnage, diviseur d'interpolation, $\overline{\partial}$-Méthode, fonctions entières.

# Résumé détaillé

Dans la première partie nous considérons les espaces de Bergman à poinds standard. Pour $\alpha > -1$, *l'espace de Bergman* $\mathcal{A}^2_\alpha(\mathbb{D})$ est donné par

$$\mathcal{A}^2_\alpha(\mathbb{D}) := \left\{ f : \mathbb{D} \to \mathbb{C} \text{ holomorphe}, \ \|f\|_{\alpha,2} = \left( \int_{\mathbb{D}} |f(z)|^2 dA_\alpha(z) \right)^{\frac{1}{2}} < +\infty \right\},$$

où $dA_\alpha(z) = (\alpha+1)(1-|z|^2)^\alpha dA(z)$ où $dA$ est la mesure d'aire normalisée sur le disque unité $\mathbb{D}$. Il est connue que $\mathcal{A}^2_\alpha(\mathbb{D})$ muni de la norme $\|.\|_{\alpha,2}$ est un espace de Hilbert. De plus, $\mathcal{A}^2_\alpha(\mathbb{D})$ à une base orthonormale obtenue par les monômes normalisés, c'est-à-dire,

$$e_j(z) := \frac{z^j}{\|z^j\|_{\alpha,2}} = \sqrt{\frac{\Gamma(j+2+\alpha)}{j!\Gamma(2+\alpha)}} z^j, \qquad j \geq 0,$$

où $\Gamma(s)$ est la fonction Gamma, voir la référence [HKZ00] pour une présentation complète de $\mathcal{A}^2_\alpha(\mathbb{D})$. Aussi, $\mathcal{A}^2_\alpha(\mathbb{D})$ est un espace à noyau reproduisant. Le noyau est explicitement donné par la formule

$$K_w(z) = \frac{1}{(1-\overline{w}z)^{\alpha+2}}, \qquad (z, w \in \mathbb{D}).$$

Une des approches des suites d'interpolation et d'échantillonnage, dans les espaces de Hilbert, est motivée par le dévelopement en série de Fourier. En effet, si $\mathcal{H}$ est un espace de Hilbert séparable, avec produit sclaire noté $\langle ., . \rangle$, il possède une base orthonormale $\{e_j\}_{j\geq 0}$, et on peut identifier $\mathcal{H}$ à $\ell^2$ via $f \longmapsto \{\langle f, e_j \rangle\}_j$. Ainsi, nous obtenons une interpolation des coefficients de Fourier. Un raisonement simillaire peut s'appliquer, sans perdre de généralité, dans un espace de Hilberst de fonction analytiques à noyau reproduisant $k_\lambda$ sur le disque unité $\mathbb{D}$ (ou un domaine $\Omega$ de $\mathbb{C}^d$ ). Si pour une suite $\Lambda = \{\lambda_n\}_n$, la famille des noyaux normalisés $\{k_{\lambda_n}\}_n$ est une base inconditionnelle et le système $(\{\varphi_n\}, \{k_{\lambda_n}\}_n)$ est bio-rthogonal, c'est-à-dire, $\langle \varphi_k, k_{\lambda_n} \rangle = \delta_{n,k}$ ( ici $\delta_{n,k}$ est le symbol de Kroneker), alors $f \longmapsto \{\langle f, k_{\lambda_n} \rangle\}_n$, $\mathcal{H} \to \ell^2(\|\varphi\|^2)$ est une bijection. Ainsi, on obtient un procéde d'interpolation. Par d'autres système orthogonaux $(\{\varphi_n\}_n, \{\phi_{\lambda_n}\}_n)$ nous obtenons d'autres type d'interpolation, par exemple $\phi_n = \frac{\partial^m}{\partial^m \lambda} k_\lambda$, $m \geq 1$, dans l'espace de Hardy $\mathcal{H}^2(\mathbb{D})$, nous pouvons interpoler les dérivées $m$-ième de la fonction $f$, ceci nous amène à l'interpolation de type Hermite. Autrement dit, les valeurs sont imposées à la fonction ont ces dérivées aussi. De plus, pour chaque point $\lambda \in \mathbb{D}$, la differance entre la fonction $f$ et son développement de Taylor $T(z) = \sum_{k=0}^{m-1} \frac{f^{(k)}(\lambda)}{k!}(z-\lambda)^k$ a un zéro de multiplicité $m$, on peut écrire $f - T \in b_\lambda \mathcal{H}^2(\mathbb{D})$. Cette formulation nous conduit à l'interpolation généralisées ou à l'interpolation multiple qui va nous intéresser dans le cadre des espaces de Bergman $\mathcal{A}^2_\alpha(\mathbb{D})$.

Etant donné une suite de points distincts $\Lambda \subset \mathbb{D}$ avec multiplicité $m_\lambda$ associée à chaque $\lambda \in \Lambda$. Nous appelons un *diviseur* la collection de paires $X = \{(\lambda, m_\lambda)\}_{\lambda\in\Lambda}$. L'espace de Bergman $\mathcal{A}^2_\alpha(\mathbb{D})$ est invariant par l'action du groupe de Möbius de disque unité $\mathbb{D}$. Par conséquent, l'operateur linéaire $T_\lambda$ définie sur $\mathcal{A}^2_\alpha(\mathbb{D})$, $T_\lambda f(z) = \left[\varphi'_\lambda(z)\right]^{\frac{\alpha+2}{2}} f(\varphi_\lambda(z))$ est une isométrie involutive et surjective – $\varphi_\lambda$ c'est la transformation de Möbius qui échange $\lambda$ et 0 – On expoitant les propriétés de l'operateur $T_\lambda$ nous définissons les diviseurs d'interpolation et d'échantillonnage pour $\mathcal{A}^2_\alpha(\mathbb{D})$ comme suit:

- Un diviseur $X = \{(\lambda, m_\lambda)\}_{\lambda\in\Lambda}$ est dit **d'échantillonnage** pour $\mathcal{A}^2_\alpha(\mathbb{D})$ s'il existe deux constantes $c, C > 0$, tel que

$$c\|f\|^2_{\alpha,2} \leq \sum_{\lambda\in\Lambda} \sum_{0\leq j<m_\lambda} |\langle f, T_\lambda e_j\rangle|^2 \leq C\|f\|^2_{\alpha,2}, \qquad f \in \mathcal{A}^2_\alpha(\mathbb{D}).$$

  La définition au dessus motive l'introduction de l'espace $\ell^2$ des suites à interpoler (comme donnée a représenter). Pour un diviseur $X = \{(\lambda, m_\lambda)\}_{\lambda\in\Lambda}$ soit

$$\ell^2(X) = \{(v^j_\lambda)_{\lambda\in\Lambda,\ 0\leq j<m_\lambda} : \|v\|^2_2 := \sum_{\lambda\in\Lambda} \sum_{j<m_\lambda} |v^j_\lambda|^2 < \infty\}.$$

- Un diviseur $X = \{(\lambda, m_\lambda)\}_{\lambda\in\Lambda}$ est dit **d'interpolation** pour $\mathcal{A}^2_\alpha(\mathbb{D})$ si pour toute suite $v \in \ell^2(X)$, il existe $f \in \mathcal{A}^2_\alpha(\mathbb{D})$ tel que

$$\langle f, T_\lambda e_j\rangle = v^j_\lambda, \qquad (\lambda \in \Lambda, j < m_\lambda).$$

Dans le cas de multiplicités uniformes $m_\lambda = 1$, nous retrouvons les problèmes d'interpolation et d'échantillonnage classiques, c'est-à-dire $\{k_\lambda\}_{\lambda\in\Lambda}$ est une suite de Reiz ou c'est une frame respectivement, ces deux problèmes ont été résolus dans les années 1990 par K. Seip à l'aide des densités hyperboliques de type Beurling (ce qui est difficile à vérifier en générale).

Dans notre travail, nous considérons l'interpolation et l'échantillonnage multiples comme c'est définis ci-dessus. Le problème devient plus technique dans ce cas, surtout lorsque les multiplicités ne sont pas bornées pour les deux cas (d'interpolation / échantillonnage), nous obtenons des conditions optimales (nécessaires et suffisantes) pour l'interpolation et l'échantillonnage (avec un petit écart), en termes de conditions de séparation (resp. de recouvrement) de certains disque pseudo-hyperbolique liées à la suite de depart (ces résultats restent encore valable également pour le cas multiplicités bornées, et même lorsque $m_\lambda = 1$). Pour quelques mots sur les techniques utilisées dans notre travail. L'ingrédient principale pour l'échantillonnage est un résultat d'unicité basé sur la redistribution uniforme des masses de Dirca $\Delta(\log|f|) = \sum_{\lambda\in\Lambda} m_\lambda \delta_\lambda$, $f \in N_{\lambda,m_\lambda}$ sur des disques pseudohyperboliques centrés sur les zéros de $f$ avec un rayon spécial adapté à la multiplicité, cette redistribution est faite à l'aide de l'identité de Green. Une autre clé pour la condition nécessaire du résultat d'interpolation et aussi pour la condition suffisante du résultat d'échantillonnage est que; si $f$ est une fonction de norme $L^2$, inférieur ou égale à 1 sur un disque pseudohyperbolique et aussi, $f$ est de norme quotient par rapport au sous-espace fermé $N_{\lambda,m_\lambda}$ petite, alors elle sera petite dans un disque avec un rayon légèrement plus petit. Afin d'obtenir ce type de résultats, nous utilisons le principe du maximum et certaines estimations de la fonction betta incomplète. La condition suffisante pour l'interpolation est basée sur la technique $\overline{\partial}$ à l'aide du théorème d'Oshawa (variante de Théorème de Hörmander) qui lui nécessitait une construction d'un poids à singularités bien prescrites afin de s'adapter à des multiplicités arbitrairement grandes.

Pour la deuxème partie. Fixons un paramètre $\alpha > 0$. Pour $0 < p < \infty$ l'espace de Fock $\mathcal{F}^p_\alpha$ est constitué des fonctions entires $f$ qui vérifient la condition

$$\|f\|^p_{p,\alpha} := \frac{\alpha p}{2\pi} \int_{\mathbb{C}} |f(z)|^p e^{-\frac{\alpha p}{2}|z|^2} dA(z) \ < \infty,$$

où $dA$ est dans ce cas désigne la mesure de Lebesgue dans le plan complexe. Un ensemble denombrable $\Lambda \subseteq \mathbb{C}$ est dit un ensemble (ou suite) de zéro pour $\mathcal{F}^p_\alpha$ si il existe une fonction $f \in \mathcal{F}^p_\alpha$ tel que $Z(f) = \Lambda$, où $Z(f) := \{z \in \mathbb{C} \ : \ f(z) = 0\}$. K. Zhu dans son livre [Zhu12]) à posé la question suivante: Est ce que deux espace de Fock $\mathcal{F}^p_\alpha$ et $\mathcal{F}^q_\alpha$, $p \neq q$, ont les même ensembles de zéro? Nous donnons une réponse positive à cette question. Pour ceci on considère un résaux carré perturbé soiesusement sur la droite réelle de plus ayant une densité de Beurling uniform. Noter resultat principale est le suivant: pour $0 < p < q < \infty$ et $0 < \alpha < \infty$, il existe une suite $\Lambda \subset \mathbb{C}$ tel que

1. $\mathcal{D}^+(\Lambda) = \mathcal{D}^-(\Lambda) = \frac{\alpha}{\pi}$ ( ici $\mathcal{D}^+$ et $\mathcal{D}^-$ désigne les desnsités Beurling-Landau),
2. $\Lambda$ est une suite de zéros pour $\mathcal{F}^q_\alpha$,
3. $\Lambda$ n'est pas une suite de zéro pour $\mathcal{F}^p_\alpha$.

L'idée de perturber le réseau carré vient après une longue enquête sur les ensembles de zéros et ceux d'unicité également, la compréhension du comportement des produits de Weierstrass. De plus, le fait que la perturbation donne une suite relativement proche du réseau carré d'origine garantie la condition densité uniforme. Par suite, la preuve est achevée par l'estimation de certains produits infinie.

# Contents

# List of Symbols

Throughout this thesis we use the following notations :

| | |
|---|---|
| $\mathbb{Z}$ | The set of integer numbers; |
| $\mathbb{C}$ | The set of complex numbers; |
| $\mathbb{R}$ | The set of real numbers; |
| $\sup_S$ | The supremum of $S \subset \mathbb{R}$; |
| $\inf_S$ | The infimum of $S \subset \mathbb{R}$; |
| $\max_S$ | The maximum element of $S \subset \mathbb{R}$; |
| $\min_S$ | The minimum element of $S \subset \mathbb{R}$; |
| $\mathrm{Im}(z)$ | The imaginary part of $z \in \mathbb{C}$; |
| $\mathrm{Re}(z)$ | The real part of $z \in \mathbb{C}$; |
| $\overline{z}$ | The complex conjugate of $z \in \mathbb{C}$; |
| $\mathbb{T}$ | The unit circle of the complex $\mathbb{C}$; |
| $\mathbb{D}$ | The open unit disk in the complex plane $\mathbb{C}$; |
| $d(z, \Lambda)$ | The Euclidean distance from $z$ to the set $\Lambda$; |
| $\rho(z, \Lambda)$ | The pseudo-hyperbolic distance from $z$ to the set $\Lambda$; |
| $D_e(a, r)$ | The open Euclidean disk centered at $a$ and of radius $r$, also it can be denoted $\mathbb{D}(a, r)$; |
| $D(\lambda, r_\lambda)$ | The open pseudo-hyperbolic disk centered at $\lambda$ and of hyperbolic radius $r_\lambda$; |
| $\mathbb{C}^n$ | The $n$-dimensional complex coordinate space; |
| $\mathrm{Hol}(\Omega)$ | The collection of holomorphic (analytic) functions on a domain $\Omega$ of $\mathbb{C}$; |
| $X$ | The complex Banach space; |
| $X^*$ | The dual $X$; |
| $B(X)$ | The Banach algebra of all bounded linear operators on $X$; |
| $\mathcal{H}$ | The complex Hilbert space; |

- $A \lesssim B$ means that there is an absolute constant $C$ such that $A \leq CB$.
- $A \asymp B$ if both $A \lesssim B$ and $B \lesssim A$ hold.

# Introduction

This thesis deals with some problems on spaces of analytic functions. First, sampling and interpolation problems in standard weighted Bergman spaces of analytic functions in the unit disk. Second, zero sets problem in Fock spaces of entire functions in the complex plane.

We start by a brief history of sampling and interpolation problems. Even these problems had been extensively studied in spaces of analytic functions at the end of the 20th century, their birth goes back to the beginning of the century. In 1916 R. Nevanlinna [Nev19] and G. Pick [Pic15] solved independently a problem very close to the classical Lagrange interpolation one. The Nevanlinna-Pick problem is the following: given two finite sequences $\{z_k\}_{k=1}^n$ and $\{w_k\}_{k=1}^n$ in $\mathbb{D}$, under which conditions the interpolation problem

$$f(z_i) = w_i, \qquad i = 1, 2, ..., n$$

has a solution $f$ analytic in $\mathbb{D}$ and $\sup_{z\in\mathbb{D}} |f(z)| \leq 1$. They proved that the interpolation problem has a solution if and only if the matrix

$$\left(\frac{1-\overline{w_j}w_k}{1-\overline{z_j}z_k}\right)_{j,k=1,2,\ldots,n} \tag{1}$$

is positive and semi-definite. The solution $f$ can be taken to be a Blaschke product. Hereafter, L. Carleson (1958), motivated by the corona problem, considered a problem closely related to the previous one; given an infinite sequence $\{z_n\}_{n\geq 0}$ in the unit disk $\mathbb{D}$, under which conditions the interpolation problem

$$f(z_n) = a_n, \qquad n \geq 0, \tag{2}$$

has a solution $f$ in the algebra $\mathcal{H}^\infty(\mathbb{D})$ of bounded analytic functions on $\mathbb{D}$, for every bounded sequence $\{a_n\}_{n\geq 0} \in l^\infty(\mathbb{N})$. Carleson's Theorem says that a sequence $Z = \{z_n\}_{n\geq 1}$ is interpolating for $\mathcal{H}^\infty(\mathbb{D})$ if and only if it is uniformly separated, i.e. there exists a constant $\delta > 0$ such that

$$\inf_{i\geq 1} \prod_{i\neq j} \left|\frac{z_i - z_j}{1-\overline{z_i}z_j}\right| > \delta > 0. \tag{3}$$

The sampling problem was initiated by Whittaker-Kotelnikov-Shannon Theorem's [Sha49] in communication theory, it was very prominent around 1950, but probably the problem had been treated mathematically already even earlier. The result says that any $L^2(\mathbb{R}, dx)$ function $f$ whose Fourier transform $\mathcal{F}(f)(\xi) = \frac{1}{(2\pi)^{\frac{1}{2}}} \int f(x)e^{-ix\xi}dx$ is supported in $[-\pi, \pi]$ can be represented as

$$f(x) = \sum_{n=-\infty}^{\infty} f(n)\frac{\sin \pi(x-n)}{\pi(x-n)}, \qquad x \in \mathbb{R}. \tag{4}$$

The convergence is both in $L^2(\mathbb{R}, dx)$ and uniform on $\mathbb{R}$. We have also the energy identity

$$\int_{\mathbb{R}} |f(x)|^2 dx = \sum_{n=-\infty}^{\infty} |f(n)|^2. \tag{5}$$

In language of reproducing kernel (4) tells that the normalized reproducing kernel functions at points $\mathbb{Z}$, $\left\{k_{z_n=n}(z) = \frac{\sin \pi(x-n)}{\pi(x-n)}\right\}_{n\in\mathbb{Z}}$ is a Riesz Basis (actually, is an orthonormal basis) in Paley–Wiener space. The sampling theorem plays fundamental importance in digital signal processing. It provides the theoretical foundation for digital coding of continuous waveforms and decoding signals. For their applications and theoretical considerations in mathematics interpolating and sampling sequences have been studied in a broad variety of settings, we refer to the books [Sei04; HKZ00; DS04a] for an account on these problems. The particular situation of the Bergman spaces was completely solved by K. Seip in [Sei93]. We mention that the $L^2$ *standard weighted Bergman space* $\mathcal{A}^p_\alpha := \mathcal{A}^p_\alpha(\mathbb{D})$, $\alpha > -1$, $0 < p < \infty$, is defined by

$$\mathcal{A}^p_\alpha = \left\{ f : \mathbb{D} \longmapsto \mathbb{C} \text{ analytic}, \; \|f\|^p_{\alpha,p} := (\alpha+1)\int_{\mathbb{D}} |f(z)|^p (1-|z|^2)^\alpha dA(z) < +\infty \right\}.$$

For the uniform case, define $\mathcal{A}^\infty_\alpha = \mathcal{A}^\infty_\alpha(\mathbb{D})$

$$\mathcal{A}^\infty_\alpha = \left\{ f : \mathbb{D} \longmapsto \mathbb{C} \text{ analytic}, \; \|f\|_{\alpha,\infty} := \sup_{z\in\mathbb{D}} |f(z)|(1-|z|^2)^{\frac{\alpha}{2}} < \infty \right\}.$$

Known also as the growth space and it is very connected to $\mathcal{A}^p_\alpha$. Namely, in the study of zero sets, sampling and interpolating sequences. Here we refer to the work of B. Korenblum, K. Seip and H. Hedenmalm (see again [HKZ00], where $\mathcal{A}^\infty_\alpha$ is denoted by $\mathcal{A}^{-\frac{\alpha}{2}}$ in their situation).

For a brief definition of multiple sampling and interpolation in Bergman spaces. Given a set of discrete points $\Lambda \subset \mathbb{D}$ with multiplicity $m_\lambda$ associated to each $\lambda \in \Lambda$, we call *divisor* a set of pairs $X = \{(\lambda, m_\lambda)\}_{\lambda\in\Lambda}$. To each $\lambda, m_\lambda$ associate the closed vanishing subspace of $\mathcal{A}^p_\alpha$

$$N^2_\lambda := N^{2,\alpha}_{\lambda,m_\lambda} = \{f \in \mathcal{A}^2_\alpha : f^{(j)}(\lambda) = 0, \quad \forall j < m_\lambda\}.$$

Then $X = \{(\lambda, m_\lambda)\}_{\lambda\in\Lambda}$ is called a *sampling divisor* for $\mathcal{A}^2_\alpha$ if there exist $c > 0$ and $C > 0$ such that

$$c\|f\|^2_{\alpha,2} \leq \sum_{\lambda\in\Lambda} \|f\|^2_{\mathcal{A}^2_\alpha/N^2_\lambda} \leq C\|f\|^2_{\alpha,2}, \qquad f \in \mathcal{A}^2_\alpha. \tag{6}$$

Here, $\|f\|_{\mathcal{A}^2_\alpha/N^2_\lambda} := \inf_{g\in N^2_\lambda} \|f-g\|_{\alpha,2}$ is the quotient norm. Similarly, $X = \{(\lambda, m_\lambda)\}_{\lambda\in\Lambda}$ is an *interpolating divisor* for $\mathcal{A}^2_\alpha$ if for any sequence $(f_\lambda)_{\lambda\in\Lambda} \subset \mathcal{A}^2_\alpha$ such that

$$\sum_{\lambda\in\Lambda} \|f_\lambda\|^2_{\mathcal{A}^2_\alpha/N^2_\lambda} < \infty,$$

there exists $f \in \mathcal{A}^2_\alpha$ such that

$$f - f_\lambda \in N^2_\lambda, \qquad \lambda \in \Lambda.$$

For motivation and more understanding of the definition, we cite the following formulas, for a given $\lambda \in \mathbb{D}$ with multiplicity $m_\lambda$ we have for every $f \in \mathcal{A}^2_\alpha$

$$\langle f, T_\lambda e_j\rangle = \sum_{k=0}^{j} a_{j,k}(\lambda) f^{(k)}(\lambda), \qquad a_{j,k}(\lambda) \in \mathbb{C},\ j \in \mathbb{N}, \tag{7}$$

$$\|f\|^2_{\mathcal{A}^2_\alpha/N^2_\lambda} = \sum_{0\le k<m_\lambda} |\langle f, T_\lambda e_k\rangle|^2 . \tag{8}$$

Where the angles brackets $\langle .,.\rangle$ stand for the scalar product and $\{e_k\}_{k\geq 0}$ is the orthonormal basis obtained by normalizing monomials. Therefore, (7) and (8) tell that, to apprehend $f(\lambda)$, $f'(\lambda)$, ... , $f^{m_\lambda-1}(\lambda)$ is equivalent to know $\langle f, T_\lambda e_0\rangle$, $\langle f, T_\lambda e_1, \rangle$ ,...,$\langle f, T_\lambda e_{m_\lambda}\rangle$, up to a resolution of a linear triangular system because the coefficients $a_{j,k}(.)$ in the formula (7) are independent of the function $f$.

In relation with sampling and interpolating divisor we need the following preciseness on uniqueness. A divisor $X = \{(\lambda, m_\lambda)\}_{\lambda\in\Lambda}$ is called a *uniqueness divisor* for $\mathcal{A}^2_\alpha$ (or - with the same idea for $\mathcal{A}^\infty_\alpha$- ) if

$$\bigcap_{\lambda\in\Lambda} N^{2,\alpha}_{\lambda,m_\lambda} = \Big\{0\Big\}. \tag{9}$$

Otherwise, $X = \{(\lambda, m_\lambda)\}_{\lambda\in\Lambda}$ is called to be a *zero divisor* for $\mathcal{A}^2_\alpha$ . K. Seip in [Sei93] had characterized completely both sampling and interpolating sequences. That is, divisor $X = \{(\lambda_j, m_{\lambda_j} = 1)\}_{j\geq 1}$ in our setting) for $\mathcal{A}^\infty_{2\alpha}$ ( noted by $\mathcal{A}^{-n}$ in his original paper) in terms of densities. The solution is inspired by B. Korenblum's $\mathcal{A}^{-\infty}$ theory and Beurling's work on band-limited functions, that is functions of exponential type $\leq a$, bounded on the real line. In the same paper, K. Seip modified the techniques so as to apply it to the standard weighted Bergman $\mathcal{A}^2_\alpha$ and obtained the characterization for the corresponding sampling and interpolating sequences $X = \{(\lambda_j, 1)\}_{j\geq 1}$. Later many authors had carried out these modifications in the more general context of $\mathcal{A}^p_\alpha$, $1 < p < \infty$. The upper and lower Beurling-Landau Type density for the unit disk used by K. Seip in his achievement are, respectively,

$$\mathcal{D}^+(\Lambda) = \limsup_{r\to 1^-} \sup_{\varphi\in \mathrm{Aut}(\mathbb{D})} \frac{\sum_{\lambda\in\Lambda:\frac{1}{2}<|\varphi(\lambda)|<r} \log\frac{1}{|\varphi(\lambda)|}}{|\log(1-r)|}, \qquad \Lambda = \{\lambda_j\}_{j\geq 1}. \tag{10}$$

And

$$\mathcal{D}^-(\Lambda) = \liminf_{r\to 1^-} \inf_{\varphi\in \mathrm{Aut}(\mathbb{D})} \frac{\sum_{\lambda\in\Lambda:\frac{1}{2}<|\varphi(\lambda)|<r} \log\frac{1}{|\varphi(\lambda)|}}{|\log(1-r)|}, \qquad \Lambda = \{\lambda_j\}_{j\geq 1}. \tag{11}$$

Where $\mathrm{Aut}(\mathbb{D})$ denotes the Mobius group of transformations and $\Lambda = \{\lambda_j\}_{j\geq 1}$ is a discrete sequence (or we might think to $X = \{(\lambda_j, 1)\}_{j\geq 1}$). K. Seip's result states that $\Lambda = \{\lambda_j\}_{j\geq 1}$ is a sampling sequence for $\mathcal{A}^2_\alpha$ if and only if it is a finite union of hyperbolically separated sequences and its contains a sequence $\Lambda'$ with $\mathcal{D}^-(\Lambda') > \frac{\alpha+1}{2}$. And $\Lambda = \{\lambda_j\}_{j\geq 1}$ is an interpolating sequence if and only if it is hyperbolically separated and $\mathcal{D}^+(\Lambda) < \frac{\alpha+1}{2}$. And the same results hold for $\mathcal{A}^\infty_\alpha$ with a minor modifications ($\mathcal{D}^-(\Lambda') > \frac{\alpha}{2}$ for sampling and $\mathcal{D}^+(\Lambda) < \frac{\alpha}{2}$ interpolation).
Subsequently, the case of multiple interpolation, but not sampling, with uniformly bounded multiplicities had been studied for instance by Krosky and Schuster [KS01] using also extremal functions (we mention related work by A. Hartmann who considered in [Har01] finite unions of Bergman interpolating sequences based on extremal functions, in the case of multiple interpolation being in a sense a limit case of finite unions). In the Fock space, besides considering the case of simple interpolation and sampling problems, Seip – in particular with Brekke – was interested in the situation of higher multiplicities. Again, the density conditions obtained by these authors imply that there are no simultaneous sampling and interpolating sequences, neither in the simple case nor in the multiple case. Brekke and Seip in [BS93] also asked whether there could be simultaneous sampling and interpolating sequences when the uniform boundedness condition on the multiplicities is relaxed. In [Bor+17] it was shown that at least when the multiplicities tend to infinity, this is not possible (see also [EHR21] for the case of bounded multiplicites in the weighted Fock space).

One difficulty occuring in the case of unbounded multiplicities is the lack of a reasonable definition of densities. In [Bor+17], the authors introduce covering and separation conditions related with critical radii suitably related with the multiplicities to circumvent densities. Though those conditions do not characterize multiple interpolation and sampling, they get in a sense closer and closer to a characterization when the multiplicities grow (indeed the difference between necessary and sufficient conditions of the radii remains bounded while the radii tend to infinity). Bergman and Fock spaces share many properties also many open problems, and techniques often translate from one setting to the other.

Our principal aim is to study the situation concerning multiple interpolating and sampling with unbounded multiplicities in the Bergman space $\mathcal{A}^p_\alpha$ and the growth space $\mathcal{A}^\infty_\alpha$. New difficulties and challenges appear in order to adapt the situation from the underlying euclidean metric in the Fock space to pseudohyperbolic metric in the Bergman space. While this might be rather direct for simple interpolation and sampling the situation requires a quite delicate analysis of the criticial radii in the pseudo-hyperbolic metric in particular when the multiplicities are not uniformly bounded. On the technical side, replacing the incomplete $\Gamma$-function by the incomplete $\beta$-function gives rise to other difficulties.

It is mentionable that generalized interpolation problems (but not sampling problems) have been considered long ago in the Hardy space for which a complete answer is given by the so-called generalized Carleson condition (see [Nik78; Vas78]). In this situation, the case of interpolating sequences with unbounded multiplicities is completely understood (see also earlier work by Vinogradov-Rukshin [VR82]).

Therefore, without claiming exaustivity, our work on multiple sampling and interpolation is in the same weave as many well known and powerful results on spaces of analytic functions as cited before, mentioning work on interpolating sequences with uniformly bounded multiplicity in the Korenblum space, see [Mas99] and for weighted spaces of entire functions see [Oun07; Oun08].

To state our main results we need the following overlap condition.

**Definition 0.0.1.** *A divisor $X$ satisfies the finite overlap condition for $\mathcal{A}^2_\alpha$ if*

$$S_X = \sup_{z\in\mathbb{D}} \sum_{\lambda\in\Lambda} \chi_{D\left(\lambda,\sqrt{\frac{m_\lambda}{m_\lambda+\alpha+1}}\right)}(z) < \infty.$$

We should mention that this finite overlap condition is intimately related to the Carleson measure condition.

Now we are in a position to state the geometric condition for sampling divisors.

**Theorem 0.0.2.** *Let $\alpha > -1$.*

*(a) If $X$ is a sampling divisor for $\mathcal{A}^2_\alpha$, then $X$ satisfies the finite overlap condition and there exists $0 < C_X < \alpha + 1$ such that*

$$\bigcup_{\lambda\in\Lambda} D\left(\lambda, \sqrt{\frac{m_\lambda + C_X}{m_\lambda+\alpha+1}}\right) = \mathbb{D}.$$

*(b) Conversely, suppose the divisor $X$ satisfies the finite overlap condition. There is a constant $C > 1$ depending on $S_X$ such that if for some compact $K$ of $\mathbb{D}$ we have*

$$\bigcup_{\lambda\in\Lambda, m_\lambda > C} D\left(\lambda, \sqrt{\frac{m_\lambda - C}{m_\lambda+\alpha+1}}\right) = \mathbb{D}\setminus K,$$

*then $X$ is a sampling divisor for $\mathcal{A}^2_\alpha$.*

This theorem tells us that if disks with slightly smaller radii than the critical one already cover the unit disk, then we have a sampling divisor. And if a divisor is sampling then at least disks with slightly bigger radii cover the unit disk (up to a compact set).

One could be tempted to complain about the constant $C$ appearing in (b) above. Note that the theorem is completely general and applies even in the case of uniformly bounded multiplicites where the result proved in [BS93] requires density conditions. So there is no hope getting a sufficient condition only from the covering without additional conditions for instance on the critical radius.
In the analogous situation for interpolating divisors the covering condition is replaced by a separation condition of disks with slightly bigger or smaller radii than the criticial ones.

**Theorem 0.0.3.** *Let $\alpha > -1$.*

*(a) If $X$ is an interpolating divisor for $\mathcal{A}^2_\alpha$, then there exists $C_X > 0$ such that the hyperbolic disks*

$$\left\{ D\left(\lambda, \sqrt{\frac{m_\lambda - C_X}{m_\lambda + \alpha + 1}}\right)\right\}_{\lambda\in\Lambda, m_\lambda > C_X}$$

*are pairwise disjoint.*

*(b) Conversely, if for some $C_X$ such that $(\alpha+1)(1-e^{-1}) < C_X < \alpha+1$, the hyperbolic disks*

$$\left\{ D\left(\lambda, \sqrt{\frac{m_\lambda + C_X}{m_\lambda + \alpha + 1}}\right)\right\}_{\lambda\in\Lambda}$$

*are pairwise disjoint, then $X$ is an interpolating divisor for $\mathcal{A}^2_\alpha$.*

Notice that the separation condition appearing in the statement (a) implies the finite overlap condition (the overlap is actually void), which is again related to the Carleson measure condition.

Again, we should point out that additional conditions on the constant $C$ are required in (b) since the theorem is completely general covering the case when the multiplicities are uniformly bounded in which case the result [BS93] involves again density conditions. So, separation alone for $C$ arbitrary close to 0 cannot be sufficient for interpolation.

Concerning both Theorems 0.0.2 and 0.0.3, we would also like to emphasize the fact that densities, even if they provide characterizations for simple or uniformly bounded multiplicites, are hard to compute in a general situation (particularly in the pseudohyperbolic metric), while our overlapping and separation conditions are much easier to apprehend.
Here is another observation: in case $X$ is a sampling divisor, the finite overlap condition is necessary. In case $X$ is an interpolating divisor, we get a separation condition, which obviously also implies the finite overlap (there is actually no overlap and now $S_X = 1$). So in both cases, the area of pseudohyperbolic disks centered at $\lambda$ and with radius comparable to $\sqrt{(m_\lambda - C)/(m_\lambda + \alpha + 1)}$ add up to a finite sum,

yielding the following Blaschke type condition which seems new:

$$\sum_{\lambda\in\Lambda}(m_\lambda(1-|\lambda|^2))^2 < +\infty. \tag{12}$$

The result which affirms that in the Fock space (with Gaussian weight) there are no Riesz bases (simultaneously interpolating and sampling) is quite expensive to obtain. First in [Sei93] for simple multiplicity, and later for uniformly bounded multiplicity in [BS93] this requires the density characterizations of interpolating and sampling sequences. More recently the third and fourth authors discussed this problem in [Bor+17] when the multiplicities go to infinity. In this case, there are no characterizations available, but the gap between necessary conditions for multiple sampling and multiple interpolation (given by the corresponding results to Theorems 0.0.2 and 0.0.3), together with some geometric lemma, allowed to conclude. The situation in Bergman spaces is dramatically simpler. The main reason is due to the fact that the multiplier algebra for standard Bergman spaces is not trivial and contains bounded analytic functions in the unit disk. As a consequence, any interpolating divisor is a zero divisor (pick a function vanishing in all the points $\lambda$ up to the order $m_\lambda - 1$ except for one point $\lambda_0$ in which we interpolate the value 1, then multiply the interpolating function by $b_{\lambda_0}^{m_\lambda}$), and can thus not be sampling since sampling divisors are uniqueness.

In view of the above discussions, the central role of zero and uniqueness sets should be clear. In this connection, we will formulate here a necessary condition for zero divisors which does not seem to follow from those known so far. A fairly precise results on zero sets in $\mathcal{A}^2_\alpha$

**Theorem 0.0.4.** *Let $\alpha > 0$, $\varepsilon > 0$, and $X = \{(\lambda, m_\lambda)\}_{\lambda\in\Lambda}$ be a divisor such that*

$$\bigcup_{\lambda\in\Lambda} D\left(\lambda, \sqrt{\frac{m_\lambda}{m_\lambda+\alpha+2+\varepsilon}}\right) = \mathbb{D}\setminus K \tag{13}$$

*for some compact set $K \subset \mathbb{D}$. Then $X$ is a uniqueness divisor for $\mathcal{A}^2_\alpha$.*

We recall that a uniqueness divisor is a non zero divisor.
And define

$$\mathcal{A}^\infty_\alpha = \left\{ f:\mathbb{D}\longmapsto\mathbb{C} \text{ analytic, } : \|f\|^2_{\alpha,\infty} := \sup_{z\in\mathbb{D}}(1-|z|_2)^{\frac{\alpha}{2}}|f(z)| < +\infty \right\}.$$

In this setting, the results are completely analogous – replacing essentially $\alpha+1$ by $\alpha$ in the theorems cited above – Note that it follows immediately from the fact that $\mathcal{A}^2_\alpha \subset \mathcal{A}^\infty_{\alpha+2}$ which allows to connect some results between both situations. However, in general the results in $\mathcal{A}^\infty_\alpha$ do not follow immediately from those for $\mathcal{A}^2_\alpha$ and the proofs have to be rerun. We also point out a curious phenomenon. Indeed the above embedding works for $\alpha > -1$, but there is no $A^2_\beta$ which embeds into $A^\infty_\alpha$ when $\alpha \in (0,1]$. We would also like to comment on the uniqueness result on Hilbert case and uniform case (which has essentially the same statement just replacing $\alpha + 2$ by $\alpha$). In [Sei95b], Seip gave fairly precise sufficient and necessary conditions exhibiting a small gap between these. His conditions are based on the Korenblum density which is difficult to check in general. The condition appearing in (13) yields maybe a more transparent necessary condition for zero divisors (for $X$ to be a zero divisor it is necessary that the covering condition (13) does not hold for any compact $K$ and any $\varepsilon > 0$).

Now, we deal with the second part of this thesis; zero set problem. In the literature the Fock space $\mathcal{F}^2_\pi(\mathbb{C})$ of entire functions (see definition below) appears in many situation we cite, it can be seen (by isomorphism) as the eigenspace associated with the pure point spectrum $\lambda = 2\pi$ in $L^2(\mathbb{R}^2, dxdy)$ of the magnetic Schrödinger operator associated to the vector potential $A = (\pi y, -\pi x, 0)$, given in $x$ and $y$ coordinates,

$$H := -\Big(\frac{\partial^2}{\partial^2 x} + i\pi y\Big)^2 - \Big(\frac{\partial^2}{\partial^2 y} - i\pi x\Big)^2 \tag{14}$$

($H$ is also called Landau Hamiltonians). Fock space $\mathcal{F}^2_\pi(\mathbb{C})$ is also linked to the signal processing. In 1946, Gabor introduced a special collection of functions named after him Gabor system [Gab93]. To define them, for a given sequence $\Lambda = \{\lambda_{m,n}\}_{m,n}$ in the complex plane $\mathbb{C}$ and a window function $g \in L^2(\mathbb{R}, dx)$ a Gabor system $\mathcal{G}(\Lambda, g)$ is defined as

$$\mathcal{G}(\Lambda, g) = \Big\{\varrho_{x,-y} g(t) = e^{ix} g(t-y) \ : \ \lambda_{m,n} = x + iy \in \Lambda\Big\}. \tag{15}$$

Originally introduced for applications in signal processing, and they have been widely used since then. However, one of the most fundamental question concerning Gabor systems which asks for the description of all time-frequency sequences $\Lambda$ such that $\mathcal{G}(\Lambda, g)$ is complete in $L^2(\mathbb{R}, dx)$ (and more difficult to be complete and minimal ) remained widely open. A complete description is known only in the case of a Gaussian window $g(t) = e^{-\pi t^2}$ and $\Lambda$ being a lattice. In this classical case, it was von Neumann [Neu55] who first observed (without proof) that the system $\mathcal{G}(\Lambda, g)$ is complete when $\Lambda$ is the integer lattice. Latter many different proofs were given [BGZ75; Bar+71; SW92] treating also the case of $\Lambda$ being an arbitrary lattice. However, the case of irregular systems ($\Lambda$ is not a lattice) remains a complete mystery to date, even in this classical case. And recently a complete description of irregular Gabor frames was given by Seip and Wallsten [SW92; Sei92] by translating the problem to once of sampling sequences in the Fock space (By Bargmann transform below) and making use of complex analysis, that is what makes the case of a Gaussian window $g(t) = e^{-\pi t^2}$ simpler to handle. As we have mentioned before, the tools that always used to translate the problem is the Bargman transform which is defined by the following formula:

$$\mathcal{B}[f](z) := e^{-i\pi xy}\int_{\mathbb{R}} f(t)\overline{(\varrho_{x,-y}g)(t)}dt = 2^{\frac{1}{4}}\int_{\mathbb{R}} e^{-\pi t^2}e^{2\pi tz}e^{-\frac{\pi}{2}t}dt, \qquad z = x+iy. \tag{16}$$

The transform $\mathcal{B}$ maps $L^2(\mathbb{R}, dx)$ isometrically to The Fock space defined by

$$\mathcal{F}^2_\pi(\mathbb{C}) := \left\{ f : \mathbb{C} \longmapsto \mathbb{C} \text{ entire}, \ \|f\|^2_{\pi,2} := \int_{\mathbb{C}} |f(z)|^2 e^{-\pi|z|^2} dA(z) < +\infty \right\}.$$

Where $dA$ is the planar Lebesgue measure It is well know that $\mathcal{F}^2_\pi(\mathbb{C})$ is a reproducing kernel Hilbert space, with kernel functions at $\lambda \in \mathbb{C}$ given by $K_\lambda(z) = e^{\pi\bar{\lambda}z}$, $z \in \mathbb{C}$, i.e.

$$f(\lambda) = \langle f, K_\lambda\rangle, \qquad f \in \mathcal{F}^2_\pi(\mathbb{C}) \ \lambda \in \mathbb{C}.$$

Furthermore

$$\mathcal{B}\left[\varrho_{Re(\lambda),-Im(\lambda)}g\right](.) = e^{-\frac{\pi}{2}|\lambda|^2}k_{\bar{\lambda}}(.), \qquad (\lambda \in \mathbb{C}), \tag{17}$$

where $k_\lambda$ design the normalized reproducing kernel for Fock space $\mathcal{F}^2_\pi$.
The bridge between Fock space $\mathcal{F}^2_\pi(\mathbb{C})$ and Gabor system $\mathcal{G}(\Lambda, g)$ can now follows by a duality argument and one part is the following :

*The Gabor system $\mathcal{G}(\Lambda, g)$ is complete and minimal in $L^2(\mathbb{R}, dx)$ if and only if $\Lambda$ is a set of uniqueness of zero excess for $\mathcal{F}^2_\pi(\mathbb{C})$ –If we remove a point from $\Lambda$ we get a zero set for $\mathcal{F}^2_\pi(\mathbb{C})$– [ALS09].*

Our study is concerned with Banach Fock spaces, $\mathcal{F}^p_\alpha$, $\alpha > 0$, $0 < p < \infty$, defined by

$$\mathcal{F}^p_\alpha := \mathcal{F}^p_\alpha(\mathbb{C}) = \left\{ f : \mathbb{C} \to \mathbb{C} \text{ entire}, \ \|f\|^p_{p,\alpha} := \frac{\alpha p}{2\pi}\int_{\mathbb{C}} \left| f(z)e^{-\frac{\alpha}{2}|z|^2}\right|^p dA(z)\} < \infty \right\}. \tag{18}$$

For $\alpha > 0$ and $p = \infty$. The Fock space $\mathcal{F}^\infty_\alpha$ is the space of entire functions which satisfy

$$\|f\|_{\infty,\alpha} := \sup_{z\in\mathbb{C}} |f(z)|e^{-\frac{\alpha}{2}|z|^2} < \infty. \tag{19}$$

A countable set $\Lambda \subseteq \mathbb{C}$ is called a zero set for $\mathcal{F}^p_\alpha$ if there exists a function $f \in \mathcal{F}^p_\alpha$ such that $Z(f) = \Lambda$, where $Z(f) := \{z \in \mathbb{C} \ : \ f(z) = 0\}$. A central question arise when we deal with space of analytic functions, is to characterize, zero sets (or zero sequences). In contrast with the Blaschke condition in Hardy spaces. For the case of Fock spaces $\mathcal{F}^p_\alpha$ it is known, from K. Zhu work [Zhu93], that if $\{z_n\}_{n\geq 1}$ is the zeros set of a function $f \in \mathcal{F}^p_\alpha$, $f(0) \neq 0$, then we have the following necessary condition

$$\sum_{n\geq 1} \frac{1}{|z_n|^{2+\epsilon}} < \infty, \qquad \epsilon > 0,$$

and we can't take $\epsilon = 0$ in general.
On the other hand, a sufficient condition is given by

$$\sum_{n\geq 1} \frac{1}{|z_n|^2} < \infty.$$

The upper and lower uniform Beurling-Landau density of a sequence $\Lambda \subset \mathbb{C}$ are defined, respectively, by

$$\mathcal{D}^+(\Lambda) := \limsup_{\rho\to\infty} \sup_{z\in\mathbb{C}} \frac{N_\Lambda(z,\rho)}{\pi\rho^2}$$

and

$$\mathcal{D}^-(\Lambda) := \liminf_{\rho\to\infty} \inf_{z\in\mathbb{C}} \frac{N_\Lambda(z,\rho)}{\pi\rho^2},$$

where the quantity $N_\Lambda(z,\rho)$ design the number of the elements in the intersection of $\Lambda$ and the Euclidean open disk $D(z,\rho)$ of center $z \in \mathbb{C}$ and radius $\rho > 0$. Our main result is the following:

**Theorem 0.0.5.** *Let $p$ and $q$ be positive numbers such that $q < p$. There exists a sequence $\Lambda$ in $\mathbb{C}$ satisfying*

$$\mathcal{D}^+(\Lambda) = \mathcal{D}^-(\Lambda) = \frac{\alpha}{\pi}, \tag{20}$$

*and such that $\Lambda$ is a zero set for $\mathcal{F}^p_\alpha$ but it is not a zero set for $\mathcal{F}^q_\alpha$. ($\mathcal{F}^q_\alpha \subset \mathcal{F}^p_\alpha$),*

In the proof of Theorem 0.0.5, the condition (20) seems to be optimal for a sequence $\Lambda$ with a uniform density, that is $\mathcal{D}^+(\Lambda) = \mathcal{D}^-(\Lambda)$, to be a zero set for $\mathcal{F}^p_\alpha$ but not for $\mathcal{F}^q_\alpha$. It turns out that $\alpha/\pi$ is a critical number within the interpolating and sampling theorems on Fock spaces, given by K. Seip and R. Wallstén [SW92].

Our second result on zero set for Fock spaces is related to some stability. A phenomena which is one of the main difference between Fock spaces and Hardy spaces and even Bergman spaces of the unit disk where zero sets are well stable [DS04b; HKZ00].
This theorem gives a complete description of zero sets for which all their subsequences are also zero sets for $\mathcal{F}^p_\alpha, 0 < p \leq \infty$.

**Theorem 0.0.6.** *Let $Z = \{z_n\}_{n\in\mathbb{N}}$ be a zero set for $\mathcal{F}^p_\alpha$, $0 < p < \infty$. The following statements are equivalents*

1. *Every subset of $Z$ is a zero set for $\mathcal{F}^p_\alpha$,*
2. *$Z$ satisfies*

$$\sum_{n\in\mathbb{N}} \frac{1}{|z_n|^2} < \infty. \tag{21}$$

Finally we want to add a comment; that is, the problem of uniqueness set in the Fock space $\mathcal{F}^p_\alpha$, which is sometimes seems to be closed to the one of zero sets, remains also an open problem; and maybe more delicate. We mention that some partial results was obtained by G. Ascensi, Y. Lyubarskii, and K. Seip [ALS09], and also Y. Omari in [Oma19].

# Chapter 1

# Bergman spaces, classical sampling and interpolation

The chapter contains necessary tools needed for studying multiple sampling and multiple interpolation in Bergman spaces, in the next chapter. To this end, we will review pseudo-hyperbolic geometry, Bergman spaces and Grouth spaces. It will be end by recalling K. Seip and B. Korenblum results on zero sets ($\kappa$-density), even if they are the best results that we find in this direction is not enough for us to deal with multiple sampling and multiple interpolation problems which are very connected to the zero set one. We mention that this chapter is based on the references [HKZ00] and [DS04b].

### 1.0.1 The Poincaré disk model

Along this thesis $\mathbb{C}$ denote complex plane and let us denote the open unit disk

$$\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$$

and the unit circle

$$\mathbb{T} = \{z \in \mathbb{C} : |z| = 1\}.$$

As we are going to work we complex analysis. We shall use the Wirtinger differential operators

$$\frac{\partial}{\partial z} = \frac{1}{2}\Big(\frac{\partial}{\partial x} - i\frac{\partial}{\partial y}\Big), \qquad \frac{\partial}{\partial \bar{z}} = \frac{1}{2}\Big(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\Big).$$

Usually when we study analytic space on unit disk $\mathbb{D}$ we keep in mind that it is intimately related to the Möbius group $Aut(\mathbb{D})$ and the pseudohyperbolic distance which is a concept more natural than the euclidean metric for various problem that connect Bergman space and unit disk. Let $\lambda \in \mathbb{D}$, consider the Möbius map $\varphi_\lambda$ of the disk that interchanges $\lambda$ and 0,

$$\varphi_\lambda(z) = \frac{\lambda - z}{1 - \bar{\lambda} z}, \qquad (z \in \mathbb{D}).$$

And pseudo hyperbolic distance between $\lambda$ and $z$ is defined by

$$\rho(\lambda, z) = |\varphi_\lambda(z)|.$$

The function $\varphi_\lambda$ is a Möbius transformation, a conformal automorphism of the disk with $\varphi_\lambda(\lambda) = 0$. Also not that $\rho(\lambda, 0) = |\lambda|$. It is not hard to verify that $\rho$ is a true metric on $\mathbb{D}$. The two first property are obvious, perhaps the triangle inequality is less obvious, but we will prove it later in a stronger form. Direct calculations show that $\varphi_\lambda$, has three additional properties, which we are going to use frequently along the next chapter:

(a) $\varphi_\lambda$ is one to one and onto with $\varphi_\lambda^{-1} = \varphi_\lambda$ ( an involution).

(b) The real Jacobian determinant of $\varphi_\lambda$ at $z$ is $|\varphi_\lambda'(z)|^2 = \frac{(1-|\lambda|^2)^2}{|1-z\bar{\lambda}|^4}$.

(c) For all $z \in \mathbb{D}$, $1 - |\varphi_\lambda(z)|^2 = \frac{(1-|z|^2)(1-|\lambda|^2)}{|1-z\bar{\lambda}|^2}$.

The *hyperbolic metric* $\beta$, also called *Poincaré metric* or *Bergman metric* is given by

$$\beta(\lambda, z) = \log \frac{1 + \rho(\lambda, z)}{1 - \rho(\lambda, z)} = 2 \tanh^{-1} \rho(\lambda, z). \qquad (\lambda, z \in \mathbb{D})$$

To verify the triangle inequality for the pseudo hyperbolic metric, we can show the sharp inequalities, called *strong triangle inequality* for the *pseudohyperbolic metric.* For any three points $\lambda, z, \xi \in \mathbb{D}$,

$$\frac{|\rho(\lambda, z) - \rho(z, \xi)|}{1 - \rho(\lambda, z)\rho(z, \xi)} \leq \rho(\lambda, z) \leq \frac{\rho(\lambda, z) + \rho(z, \xi)}{1 + \rho(z, \xi)\rho(\lambda, z}). \tag{1.1}$$

For a brief proof, it suffices to assume $\lambda = 0$. The inequality reduces to

$$\frac{||z| - |\xi||}{1 - |z||\xi|} \leq \left| \frac{\xi - z}{1 - \xi z} \right| \leq \frac{||z| + |\xi||}{1 + |z||\xi|}$$

By symmetry, we may assume that $\xi$ is real and positive, and that $|z| = r < \xi$. Then $\varphi_\xi$ maps the circle $|z| = r$ onto a circle symmetric with respect to the real axis, intersecting the real axis at the points $\frac{\xi - r}{1 - r\xi}$ and $\frac{\xi + r}{1 + r\xi}$. In particular,

$$\frac{\xi - r}{1 - r\xi} \leq |\varphi_\xi(z)| \leq \frac{r + \xi}{1 + r\xi}$$

which proves the desired results. A basic consequence of the strong form of the triangle inequality that we will use later frequently, is that the estimates $\rho(\lambda, z) \leq r$ and $\rho(\xi, \lambda) \leq s$ imply

$$\rho(z, \xi) \leq \frac{r + s}{1 + rs}.$$

Because the function $f(x,y) = \frac{x+y}{1+xy}$ attains a maximum value in the rectangle $[0,r] \times [0,s]$ at the point $(r,s)$.
Furthermore, by the property (c) above. We have the identity

$$|\varphi_\lambda'(z)| = \frac{1 - |\varphi_\lambda(z)|^2}{1 - |z|^2}$$

which says that the hyperbolic density $\frac{|dz|}{1-|z|^2}$ is Möbius invariant. The identity can also derived from the transformation formula for the Bergman kernel function of the unit disk

$$\frac{\varphi_\lambda'(z)\overline{\varphi_\lambda'(\xi)}}{(1 - \varphi_\lambda(z)\overline{\varphi_\lambda(\xi)})^2} = \frac{1}{(1 - z\overline{\xi})^2}.$$

Define the *pseudohyperbolic disk*

$$D(\lambda, r) = \{z \in \mathbb{D} : \rho(\lambda, z) < r\} = \varphi_\lambda(D(0,r)),$$

(sometimes we will just call it *hyperbolic disk* to simplify). For instance the the euclidean disk of center $\lambda$ and radius $r > 0$ will be denoted by $D_e(\lambda, r)$. Notice that the hyperbolic disks are true euclidean disks (see [Gar07, p. 3]), in fact

$$D(\lambda, r) = D_e\left(\frac{1 - r^2}{1 - r^2|\lambda|^2}\lambda, \frac{1 - |\lambda|^2}{1 - r^2|\lambda|^2}r\right).$$

Later we will need the following result. Any pair of pseudo hyperbolically concentric circles are at constant pseudo hyperbolic distance from each other. Namely, if we have two circles $C$ and $C'$ defined by $\rho(\lambda, z) = r$ and $\rho(\lambda, z) = r'$, then each point $z \in C$ has pseudo hyperbolic distance

$$\min_{z' \in C'} \rho(z, z') = \rho(r, r') \tag{1.2}$$

from the circle $C'$.For the proof, it suffices to take $\lambda = 0$, by Möbius invariance. Then the problem reduces to showing by elementary calculus that $\rho(r, r'e^{i\theta})$ is smallest for $\theta = 0$. In connection with Möbius transformation we will need the invariant *hyperbolic measure*

$$d\mathcal{V}(z) = \frac{dm(z)}{(1 - |z|^2)^2}, \tag{1.3}$$

where $dm$ is the normalized Lebesgue measure such that $m(\mathbb{D}) = 1$. Hence, the hyperbolic area of measurable subset $\Omega \subset \mathbb{D}$ is given by

$$\mathcal{V}(\Omega) = \int_\Omega \frac{dm(\xi)}{(1 - |\xi|^2)^2}.$$

It is easily seen to be Möbius invariant. As a consequence, a pseudohyperbolic disk has hyperbolic area

$$\mathcal{V}(D(\lambda, r)) = \mathcal{V}(D(0, r)) = \frac{1}{\pi}\int_0^r \int_0^{2\pi} \frac{s}{(1 - s^2)^2} ds d\theta = \frac{r^2}{1 - r^2}.$$

Thus hyperbolic area depends only on the radius $r$ on not on the center $\lambda$. We mention that the invariant *hyperbolic measure* can be seen as the the Laplacian of the *Poincaré metric* (1.3) in distributional sens

$$\phi(z) = \log \frac{1}{1-|z|^2}. \tag{1.4}$$

Notice, the measure $d\mathcal{V}$ is invariant under the Möbius group. The unit disk $\mathbb{D}$ together with the hyperbolic metric is called the Poincaré model of the hyperbolic plane.

### 1.0.2 The Bergman space $\mathcal{A}^p_\alpha$

Let $\alpha > -1$, consider the probability measure on the unit disk

$$dA_\alpha(z) = (\alpha+1)(1-|z|^2)^\alpha dA(z), \qquad z = x+iy,$$

where $dA(z) = \frac{1}{\pi}dxdy$ is the normalized area measure on $\mathbb{D}$. For $0 < p < +\infty$, the *$L^p$ weighted Bergman space* $\mathcal{A}^p_\alpha(\mathbb{D})$ is defined by

$$\mathcal{A}^p_\alpha := \mathcal{A}^p_\alpha(\mathbb{D}) = \left\{ f \ : \ \mathbb{D} \to \mathbb{C} \text{ holomorphic } : \|f\|_{\alpha,p} = \left( \int_{\mathbb{D}} |f(z)|^p A_\alpha(z) \right)^{\frac{1}{p}} < +\infty \right\}.$$

For the uniform case, define

$$\mathcal{A}^\infty_\alpha := \mathcal{A}^\infty_\alpha(\mathbb{D}) := \left\{ f \ \mathbb{D} \to \mathbb{C} \text{ holomorphic } : \|f\|_{\alpha,\infty} = \sup_{z\in\mathbb{D}} |f(z)|(1-|z|^2)^{\frac{\alpha}{2}} < \infty \right\}.$$

These definitions are the usual one that we can be found in [BM+07], [DS04a] or [HKZ00] ( with a minor modification for $\mathcal{A}^\infty_\alpha$ also called sometimes namely in [HKZ00] ). These standard weighted Bergman spaces are usually the starting point where we can perform exact computations before to achieve more complicated weighted Bergman space. Therefore, many problems on these space ($\mathcal{A}^\infty_\alpha$, $\mathcal{A}^p_\alpha$) were well studied like Bergman projection, Inner factorization, Sampling and interpolating sequence, Composition and Toeplitz operator. However, many phenomena still not understood yet for example zeros set and invariant sub-space for the shift operator... .

Besides for $\alpha - 1$ we get the Hardy space $\mathcal{H}^p(\mathbb{D})$ (which is not treated in this thesis) and for $p = \infty$ the classical space $\mathcal{A}^\infty_\alpha$ is also known as the *Growth space* and also denoted by $\mathcal{A}^{-\alpha}$ in the literature (see for example [HKZ00, Ch. 4]).

It is interesting for us that the evaluation functional can be controlled by the $L^p$ norm taken in a disk. The proof can be found in [HKZ00, p. 20] or [JMT95, Lemma 1.3].

**Lemma 1.0.1.** *Let $f$ be a holomorphic function in $\mathbb{D}$, $0 < p < \infty$, $\alpha > 0$ and $0 < s < 1$. Then, for all $\lambda \in \mathbb{D}$ there is a constant $c = c(\alpha, p, s)$ such that*

$$|f(\lambda)|^p(1-|\lambda|^2)^{\alpha+2} \leq c \int_{D(\lambda,s)} |f(z)|^p dA_\alpha(z).$$

*As a consequence*

$$\lim_{|z|\to 1^-} |f(z)|(1-|z|^2)^{\frac{\alpha+2}{p}} = 0, \qquad f \in \mathcal{A}^p_\alpha.$$

By the lemma above one can shows that the Bergman space $\mathcal{A}^p_\alpha$ is a Banach space when $1 \leq p < +\infty$ with a nice polynomial approximation, and a complete metric space, we site this result from [HKZ00], the proof is easy and based on the lemma 1.0.1.

**Proposition 1.0.2** (see [HKZ00] ). *(a) For every $0 < p < +\infty$ and $-1 < \alpha < +\infty$, the weighted Bergman space $\mathcal{A}^p_\alpha$ is closed in $L^p(\mathbb{D}, dA_\alpha)$.*

*(b) For every $f \in \mathcal{A}^p_\alpha$,$0 < p < +\infty$ and $-1 < \alpha < +\infty$, there exists a sequence of polynomials $\{p_n\}_n$ such that $\|p_n - f\| \to 0$ as $n \to \infty$.*

**Remark 1.** *Notice this result gives an inclusion between the previous spaces $\mathcal{A}^p_\alpha \subset \mathcal{A}^\infty_{\frac{2}{p}(\alpha+2)}$ for all $\alpha > 0$ and $0 < p < \infty$.*

**Lemma 1.0.3.** *For $1 \leq p \leq \infty$, the translation operator in $\mathcal{A}^p_\alpha$ defined by $T_\lambda f(z) = \left[\varphi'_\lambda(z)\right]^{\frac{\alpha+2}{p}} f(\varphi_\lambda(z))$, i.e.,*

$$\begin{aligned} T_\lambda : \mathcal{A}^p_\alpha &\longrightarrow \mathcal{A}^p_\alpha \\ f &\longmapsto T_\lambda f(.) := T_{\lambda,\alpha,p} f(.) = \left[\frac{|\lambda|^2 - 1}{(1-\overline{\lambda}.)^2}\right]^{\frac{\alpha+2}{p}} f(\varphi_\lambda(\cdot)). \end{aligned}$$

*Notice that $\varphi_\lambda$ and $T_\lambda$ are involutions, in fact, $T_\lambda$ *is a self-adjoint operator.*

Furthermore, these analytic spaces have a natural group of unitary operators with non trivial characterization. This kind of results were extensively studied (see [For73], [LRW60], [Nag59] or [Rud76] e.g. for the firsts results). The characterization of this isometries can be found in [DS04a, Section 2.8] for unweighted and [Kol82, Theorem 4] for weighted Bergman spaces for $2 < p < \infty$. Since is just a matter of a computation like in previous chapters the proof is omitted.

*The sub-index $p$ (or $\alpha$) will be omitted if it is clear which $p$ (or $\alpha$) is used to simplify notation.

### 1.0.3 The Hilbert case $\mathcal{A}^2_\alpha$

We now focus our attention to the special case $p = 2$, the associated Bergman space $\mathcal{A}^2_\alpha$ is a Hilbert space equipped with the inner product

$$\langle f, g\rangle := \int_{\mathbb{D}} f(z)\overline{g(z)}dA_\alpha(z). \tag{1.5}$$

In mind $dA_\alpha(z) = (1+\alpha)(1-|z|^2)^\alpha \frac{dxdy}{\pi}$, $z = x+iy$. With the help of lemma 1.0.1 we can show that the evaluation functional at the point $z \in \mathbb{D}$

$$\begin{aligned} E_z : \mathcal{A}^2_\alpha &\longrightarrow \mathbb{C} \\ f &\longmapsto E_z(f) = f(z). \end{aligned}$$

is a continuous linear operator. By the Riesz representation theorem, for any $z \in \mathbb{D}$, the evaluation functional is represented by a certain function $K_z \in \mathcal{A}^2_\alpha$, that is,

$$E_z(f) = f(z) = \langle f, K_z\rangle, \qquad f \in \mathcal{A}^2_\alpha.$$

We call the function $K_z$ *Bergman* or *reproducing kernel* (see [Ber70] for more information), these kernel function play an essential role in theory of Bergman. The general theory of reproducing kernels was established in a complete form by N. Aronszajn [Aro50] in 1950. One important result of the theory is that we can compute the the reproducing kernels functions in particular case, exactly when the Hilbert space has an explicit orthonormal bais $\{e_j\}_{j\geq 0}$ with the formula 1.6 below. Thus, For the Bergman space $\mathcal{A}^2_\alpha$ see for instance [HKZ00, p.5] or [Zhu07] the reproducing kernel $K_w$ is given by

$$K_w(z) = \sum_{j\geq 0} \overline{e_j(w)}e_j(z) = \frac{1}{(1-\overline{w}z)^{\alpha+2}}, \tag{1.6}$$

For simplicity, since the weighted $dA_\alpha$ is radial the monomials $\{z_n\}_n$ are orthogonal with respect to the inner product 1.5 we can construct an orthonormal basis by the normalized monomials. With this purpose, recall the definition of the $\Gamma$ and $\beta$ functions. For $\mathrm{Re}(z) > 0$, we define the $\Gamma$*-function*

$$\Gamma(z) = \int_0^\infty t^{z-1}e^{-t}dt.$$

And for $\mathrm{Re}(a) > 0$ and $\mathrm{Re}(b) > 0$, we define the $\beta$*-function*

$$\beta(a, b) = \int_0^1 t^{a-1}(1-t)^{b-1}dt.$$

We also recall that

$$\beta(a, b) = \frac{\Gamma(a)\Gamma(b)}{\Gamma(a+b)}, \quad z\Gamma(z) = \Gamma(z+1).$$

**Lemma 1.0.4.** *The functions*

$$e_j(z) = \left(\frac{\Gamma(j+\alpha+2)}{\Gamma(\alpha+1)j!}\right)^{\frac{1}{2}} z^j, \quad j \geq 0, z \in \mathbb{D}.$$

*form an orthonormal basis on* $\mathcal{A}^2_\alpha$.

*Proof.* We have seen before that the polynomials are dense in $\mathcal{A}^2_\alpha$, and $e_i$ and $e_j$ are orthogonal if $i \neq j$, so they form an orthogonal basis. Also

$$\begin{aligned}\|z^j\|^2_\alpha = (\alpha+1)\int_{\mathbb{D}} |z^j|^2(1-|z|^2)^\alpha dm(z) &= 2(\alpha+1)\int_0^1 t^{2j+1}(1-t^2)^\alpha dt \\ &= (\alpha+1)\beta(j+1,\alpha+1) \\ &= (\alpha+1)\frac{\Gamma(\alpha+1)\Gamma(j+1)}{\Gamma(j+\alpha+2)} \\ &= \frac{\Gamma(\alpha+2)\Gamma(j+1)}{\Gamma(j+\alpha+2)}.\end{aligned}$$

□

Lemma 2.1 allows to compute the reproducing kernel, which is given by *

$$\begin{aligned}K_w(z) = \sum_{j\geq 0} \overline{e_j(w)} e_j(z) &= \sum_{j\geq 0} \frac{(z\overline{w})^j}{(\alpha+1)\beta(j+1,\alpha+1)} \\ &= \sum_{j\geq 0} \frac{\Gamma(\alpha+j+2)}{\Gamma(\alpha+2)\Gamma(j+1)} = \frac{1}{(1-\overline{w}z)^{\alpha+2}}.\end{aligned} \tag{1.7}$$

A particular case of the precious identity gives $K_z(z) = \langle K_z, K_z\rangle = \|K_z\|^2$. And we get the normalized Bergman kernel

$$k_w(z) = \frac{K_w(z)}{\|K_w\|_\alpha} = \left(\frac{(1-|w|^2)^{\frac{1}{2}}}{1-\overline{w}z}\right)^{\alpha+2}.$$

## 1.1 Sampling and interpolation

### Classical sampling in $\mathcal{A}^2_\alpha$

For simplicity we start with the unweighted Bergman space $\mathcal{A}^2_0 := \mathcal{A}^2_0(\mathbb{D})$, which has the normalized reproducing kernel *

$$k_z(w) = \frac{K_z(w)}{\|K_z\|} = \frac{1-|z|^2}{(1-\overline{z}w)^2}. \tag{1.8}$$

*make use of the following power series $\frac{1}{(1-x)^\lambda} = \sum_{=0}^{+\infty} \frac{\Gamma(j+\lambda)}{j!\Gamma(\lambda)} x^j$.

*It should we don't make no confusion for the unweighted Bergman space $\mathcal{A}^2_0$, i.e. $\alpha = 0$ we may note the norm of a function as $\|f\|_{0,p}$ or $\|f\|_p$ or just $\|f\|$.

A sequence of distinct points $\Lambda = \{z_n\}_{n\geq 1}$ is called sampling sequence for $\mathcal{A}_0^2$, if there exist two constants $A$ and $B$ such that

$$A\|f\|_{\mathcal{A}^2}^2 \leq \sum_{n\geq 1} |\langle f, k_{z_n}\rangle|^2 \leq B\|f\|_{\mathcal{A}^2}^2, \qquad f \in \mathcal{A}^2(\mathbb{D}). \tag{1.9}$$

This inequalities are called sampling inequalities.
One motivation of sampling sequence comes form the theory of frames and wavelets, which is inspired bay its large applications in signal theory and approximation theory in analysis and optimization. We refer to the textbook of Ole Chritiansen [Chr+03] where a better presentation of this theory can be found. For a brief idea, it is known that if $\mathcal{H}$ is any separable Hilbert space and $\{e_n\}_n$ an orthonormal basis, Fourier and Parseval's formula state that

$$x = \sum_n \langle x, e_n\rangle e_n, \qquad \text{and} \qquad \|x\|^2 = \sum_n \left|\langle x, e_n\rangle,\right|^2 \qquad x \in \mathcal{H}. \tag{1.10}$$

More generally, a sequence $\{u_n\}_n$ is said to be a frame for $\mathcal{H}$ if there exist constants $A$ and $B$ such that

$$A\|x\|^2 \leq \sum_{n\geq 1} |\langle x, u_n\rangle|^2 \leq B\|x\|^2, \qquad (x \in \mathcal{H}). \tag{1.11}$$

It is known that for every frame $\{u_n\}$ in $\mathcal{H}$, there exist a bi-orthogonal family $\{v_n\}$ in the sens that

$$\langle u_i, v_j\rangle = \delta_{ij} := \begin{cases} 1 & \text{if} \quad i = j \\ 0 & \text{if} \quad i \neq j \end{cases}$$

And such that

$$x = \sum_n \langle x, v_n\rangle u_n = \sum_n \langle x, u_n\rangle v_n, \qquad (x \in \mathcal{H}). \tag{1.12}$$

The family $\{v_n\}_n$ is called the dual frame of the frame $\{u_n\}_n$.

If we come back to the setting of Bergman space we know that the reproducing kernels functions has the property: If $z_n, z_m$ two distinct point in $\mathbb{D}$

$$\langle K_{z_n}, K_{z_m}\rangle = K_{z_n}(z_m). \tag{1.13}$$

Since kernels function in Bergman space do not have zeros at the unit disk (A fact that remains true for most weighted Bergman space $\mathcal{A}_w^2$), then no sequence $\{z_n\}_n$ of distinct point in unit disk $\mathbb{D}$ such that the normalized Bergman kernels $\left\{\frac{K_{z_n}}{\|K_{z_n}\|}\right\}_n$ form an orthonomal basis. However, for certain choice of $\{z_n\}_n$, the normalized kernels functions $\left\{\frac{K_{z_n}}{\|K_{z_n}\|}\right\}_n$ will form a frame for $\mathcal{A}^2$, and this is what we call $\{z_n\}_n$ asampling sequence. The problem of sampling sequence was studied in various Hilbert space of holomorphic function in some domain of $\mathbb{C}^n$, like the Fock spaces

$\mathcal{F}^2_\pi(\mathbb{C}^n)$ where the problem was motivated by the study of Gabor systems generated by the Gaussian function function on the real line. This is the famous work of K. Seip and Walstan one can see [Sei04] where a complete characterization was given, later K. Siep state and solves the same problem in Bergman space. Surprisingly, there is no counter part of sampling sequence of the Hardy space of the unit disk, $0 < p < \infty$

$$\mathcal{H}^p(\mathbb{D}) = \left\{ f \ : \mathbb{D} \to \mathbb{C} \text{ holomorhpic}, \quad \|f\|_p^p := \sup_{0<r<1} \int_0^{2\pi} |f(e^{i\theta})|^p d\theta < \infty \right\},$$

the reason can be driven from the two sampling inequalities (1.9).

## Sampling, interpolation and the frame operator

It might be interesting and clarifying to state the notion of *sampling* and *interpolation* in more functional analytic terms. suppose $mathcal H$ is a reproducing kernel Hilbert space of analytic function on the unit disk $\mathbb{D}$ with kernel denoted $K_z$ and consider a sequence $Z = \{z_n\}_{n\geq 0}$ for which the restriction operator

$$R_{Z,2} : \mathcal{H} \longrightarrow l^2(\mathbb{N}) \tag{1.14}$$

$$f \longmapsto R_Z(f) = \left\{ \frac{K_{z_n}}{\|K_{z_n}\|} \right\}_{n\geq 1}. \tag{1.15}$$

is bounded from a reproducing kernel Hilbert space $\mathcal{H}$ to the trace space $l^2(\mathbb{N})$. Then the sequence $Z$ is interpolating if and only if $R_{Z,2}$ is onto (or right invertible ) and $Z$ is sampling if and only if $R_{Z,2}$ is onto with closed range (or lift invertible). Thus in functional analytic, sampling appears as a dual notion of interpolation. By translating this properties of the restriction operator to the Banach case $\mathcal{A}^p_\alpha$, $0 < p < \infty$. *Sampling* and *Interpolation* sequence find their analogue.
A sequence $\Lambda = \{z_n\}_{n\geq 1}$ of distinct points in $\mathbb{D}$ an called an interpolation sequence for $\mathcal{A}^p_\alpha(\mathbb{D})$ if for ever sequence ( mean data) $\{w_n\}_{n\geq 1}$ of complex number with

$$\sum_{n\geq 1} (1 - |z_n|^2)^{\alpha+2} |w_n|^p < \infty,$$

there exists a function $f \in \mathcal{A}^p_\alpha$ such that

$$f(z_n) = w_n, \qquad n \geq 1.$$

This definition is equivalent to the fact $l^p(\mathbb{N}) \subset R_{Z,p}\big(\mathcal{A}^p_\alpha\big)$, where $R_{Z,p}$ is the weighted restriction operator

$$R_{Z,p} : \mathcal{A}^p_\alpha \longrightarrow l^p(\mathbb{N}) \tag{1.16}$$

$$f \longrightarrow R_{Z,p}(f) = \left\{ (1 - |z_n|^2)^{\frac{\alpha+2}{p}} f(z_n) \right\}_{n\geq 1}. \tag{1.17}$$

And A sequence $\Lambda = \{z_n\}_{n\geq 1}$ of distinct points in $\mathbb{D}$ an called a sampling sequence for $\mathcal{A}^p_\alpha(\mathbb{D})$ if there exists two constant $c, C > 0$ such that

$$c\|f\|^p_{p,\alpha} \leq \sum_{n\geq 1}(1-|z_n|^2)^{\alpha+2}|f(z_n)|^p \leq C\|f\|^p_{p,\alpha} \qquad f \in \mathcal{A}^p_\alpha(\mathbb{D}).$$

Or equivalently the weighted restriction operator (1.16) is bounded and onto with closed range.

## Classical interpolation in Hardy spaces $\mathcal{H}^p(\mathbb{D})$

Even there is no sampling sequence for Hardy space the of the unit disk $\mathcal{H}^p(D)$, Interpolating sequence in modern way find there birth on the Hardy space and exactly the algebra of bounded analytic functions

$$\mathcal{H}^\infty(\mathbb{D}) = \left\{ f \ : \mathbb{D} \to \mathbb{C} \text{ holomorhpic}, \|f\|_\infty := \sup_{z\in\mathbb{D}} |f(z)| < \infty \right\}$$

by L. Carleson. Actually, before Carleson, In 1916 Nevalinna [Nev19] and Pick [Pic15] solve independently a problem a problem very closed to the classical Langrange interpolation polynomials. The Nevalinna-Pick problem was: Given two finite sequence in $\mathbb{D}$, $\{z_k\}_{k=1}^n$ and $\{w_k\}_{k=1}^n$, it ask for condition under which the interpolating problem

$$f(z_i) = w_i, \qquad i = 1, 2, ..., n$$

has an analytic solution $f$ in $\mathbb{D}$ and $\|f\|_\infty \leq 1$? There answer was that, the interpolation problem has a solution if and only if the matrix

$$\left( \frac{1-\overline{w_j}w_k}{1-\overline{z_j}z_k} \right)_{j,k=1,2,...,n} \tag{1.18}$$

is positive and semi definite. And the function $f$ can always be taken to be a Blacshke product. Latter in 1958, L. Carleson motivated by Nevalinna-Pick problem and the corona problem, consider a closely interpolating problem which consist of: Given an infinite sequence $\{z_n\}_{n\geq 0}$ in the unit disk $\mathbb{D}$ Carleson ask under which condition the interpolation problem

$$f(z_n) = a_n, \qquad n \geq 0. \tag{1.19}$$

has a solution $f$ in $\mathcal{H}^\infty(\mathbb{D})$ for every bounded sequence $\{a_n\}_{n\geq 0}$, $(\{a_n\}_{n\geq 1} \in l^\infty(\mathbb{N})$. Carleson characterization states that a sequence $Z = \{z_n\}_{n\geq 0}$ is interpolating for $\mathcal{H}^\infty(\mathbb{D})$ if and only if is uniformly separated, i.e. there exist a constant $\delta > 0$ such that

$$\inf_{i\geq 0} \prod_{i\neq j} \left| \frac{z_i - z_j}{1-\overline{z_i}z_j} \right| > \delta > 0. \tag{1.20}$$

## 1.2 Seip's description for sampling and interpolation sequences in Bergman spaces

We shall recall that the case of interpolation and sampling sequences with multiplicity 1 are characterized by Seip for $p = 2$ in [Sei93] and later by Hedenmalm, Richter and Seip for $0 < p < \infty$ in [HRS96], and Schuster in [Sch00] for sampling sequences with $1 \leq p < \infty$. These are also characterized in a more general setting by Berndtsson and Ortega-Cerdà in [OB95] for Hilbert spaces of analytic functions. For the interpolating sequences with uniformly bounded multiplicity in the Korenblum space see [Mas99]. A well exposed proofs qnd study of the case with multiplicity 1 can be found in [Sei04, Chapter 3] for $L^2$ weighted Bergman spaces and [HKZ00, Chapter 5] for the $L^\infty$ case.

Let us recall these classical characterizations for the case $p = 2$ taken almost verbatim from [Sei04]. Consider the *upper* and *lower Beurling-Landau densities*, for $\Lambda = \{\lambda_n\}_{n\geq 1}$

$$\mathcal{D}^+(\Lambda) = \limsup_{r\to 1} \sup_{z\in\mathbb{D}} \frac{\sum_{\lambda\,:\,\frac{1}{2}<|\varphi_\lambda(z)|<r} (1 - |\varphi_\lambda(z)|)}{\log \frac{1}{1-r}},$$

$$\mathcal{D}^-(\Lambda) = \liminf_{r\to 1} \inf_{z\in\mathbb{D}} \frac{\sum_{\lambda\,:\,\frac{1}{2}<|\varphi_\lambda(z)|<r} (1 - |\varphi_\lambda(z)|)}{\log \frac{1}{1-r}}.$$

These densities can be interpreted as an average of the number of points per unit area. More precisely, let $n_z(r)$ denote the number of points of $\varphi_z(\lambda)$ contained in the disk $D(0, r)$ and set

$$N_z(r) = \int_0^r n_z(t)dt$$

and

$$A(r) = 2\int_0^r \frac{t^2}{1-t^2}dt.$$

Then the densities can be rewritten as

$$\mathcal{D}^+(\Lambda) = \limsup_{r\to 1} \sup_{z\in\mathbb{D}} \frac{N_z(r)}{A(r)},$$

$$\mathcal{D}^-(\Lambda) = \liminf_{r\to 1} \inf_{z\in\mathbb{D}} \frac{N_z(r)}{A(r)}.$$

Recall we say that a sequence $\Lambda \subset \mathbb{D}$ is a *separated sequence* (in the hyperbolic metric) if

$$\inf_{\lambda,\lambda'\in\Lambda} \rho(\lambda, \lambda') = \inf_{\lambda,\lambda'\in\Lambda} \left|\frac{\lambda - \lambda'}{1 - \overline{\lambda}\lambda'}\right| > 0, \quad (\lambda \neq \lambda').$$

Now we can present the characterization for sampling and interpolation with multiplicity 1.

**Theorem 1.2.1** (Seip [Sei93] ). *For $0 < \alpha < \infty$ A separated sequence of points $\Lambda$ in $\mathbb{D}$ is*

*(a) an interpolating sequence for $\mathcal{A}^2_\alpha$ if and only if $\mathcal{D}^+(\Lambda) < \frac{\alpha+1}{2}$,*

*(b) a sampling sequence for $\mathcal{A}^2_\alpha$ if and only if $\mathcal{D}^-(\Lambda) > \frac{\alpha+1}{2}$.*

One can see [HKZ00] for a very exposed proofs.

**Remark 2.** *It is clear by the Theorem 1.2.1 that any sequence (separated) cannot be by simultaneously sampling and interpolating.*

In the general case for $0 < p < \infty$, a characterization of interpolating and sampling sequences can be found in [HRS96, Theorem 3.1 and Theorem 3.2]. This is independent of $p$, more precisely

**Theorem 1.2.2** (Seip [Sei93] or [HKZ00]). *suppose $1 \leq 1 < \infty$, $-1 \leq \alpha < \infty$,*

*(a) A sequence $\Lambda = \{\lambda_j\}_j$ of points in $\mathbb{D}$ is interpolating for $\mathcal{A}^p_\alpha$ if and only if it is separated and $\mathcal{D}^+(\Lambda) < \frac{\alpha+1}{p}$.*

*(b) A sequence $\Lambda = \{\lambda_j\}_j$ of points in $\mathbb{D}$ is sampling for $\mathcal{A}^p_\alpha$ if and only if it can be expressed as a finite union of separated sequences and there exists a separated sub-sequence $\Lambda' \subset \Lambda$ for which $\mathcal{D}^-(\Lambda) > \frac{\alpha+1}{p}$.*

Since these densities "count" the number of points uniformly, this suggest that we can not use them to obtain conditions when the multiplicity is unbounded. We want to obtain another geometric conditions such that we can prove that there is no sampling and interpolating divisor for $\mathcal{A}^2_\alpha$ with unbounded multiplicity.

In many books, I learned that the successful characterization of interpolation and interpolation sequence sequence for the space $\mathcal{A}^\infty_\alpha$, and $\mathcal{A}^p_\alpha$, was strongly influenced by Beurling's results on interpolation and sampling for the Banach space of functions of exponential type $\leq \alpha$, bounded on the real line. In link with Seip's density. Counting points in hyperbolic space is a bit tricky because the space expands faster than in Euclidean geometry: the hyperbolic area of the (hyperbolic) annulus between the disks of radii $R$ and $R+1$ is comparable to that of the whole disk of radius $R$. But Seip's sampling and interpolation have been applied in many problems. Thomson applies Seip's sampling to show that the closure of polynomials in $L^p(\mathbb{D}, d\mu)$ can change quite dramatically with the parameter $p$.

## 1.3 $\mathcal{A}^\infty_\alpha$ zero divisors

In this section we present the Korenblum's conditions in terms of $\kappa$-densities for zero set in $\mathcal{A}^\infty_\alpha$. For preciseness, a sequence $\Lambda = \{\lambda_j\}_j$ in $\mathbb{D}$ is called a *zero set* or *zero divisor* for $\mathcal{A}^\infty_\alpha$ if there exists a nonzero function $f \in \mathcal{A}^\infty_\alpha$ such that $\Lambda = Z(f)$, counting multiplicities, where

$$Z(f) = \{z \in \mathbb{D} : f(z) = 0\}.$$

### 1.3.1 Korenblum's conditions

In the Hardy space $\mathcal{H}^p(\mathbb{D})$, the zero sets are characterized by mean of the Blaschke condition, they are independent of $p$, and every subset of an $\mathcal{H}^p(D)$ zero set remain a zero set for $\mathcal{H}^p(\mathbb{D})$. Also the union of two $\mathcal{H}^p(\mathbb{D})$ zero sets is always an $\mathcal{H}^p(\mathbb{D})$ zero set. In 1974 Charles Horowitz worked in the similar three questions for the Bergman space $\mathcal{A}_0^p$. He showed that the $\mathcal{A}_0^p$ vary with $p$ , and every subset of an $\mathcal{A}_0^p$ zero set is also a zeros set, but zero sets in $\mathcal{A}_0^p$ need not to be stable by union. However as we will see later in chapter (3 or 4), The situation in the Fock Space $\mathcal{F}_\alpha^p(\mathbb{C})$ is dramatically different.
All the results obtained on the moduli of a zeros sequence $\{z_k\}_{k\geq 1}$, comes from Jensen's formula

$$\log|f(0)| + \sum_{k=1}^{n} \log\frac{r}{|z_k|} = \frac{1}{2\pi}\int_0^{2\pi} \log|f(re^{i\theta})|d\theta, \qquad 0<r<1, \tag{1.21}$$

for $f$ analytic in $\mathbb{D}$ with $Z(f)=\{z_k\}_{k\geq 1}$ satisfy

$$0<|z_1|\leq|z_2|\leq ...\leq|z_n|<r\leq|z_{n+1}|\leq ....$$

the zeros are repeated according to multiplicity.
Multiply the Jensen formula by $p>0$, composition by exponential function and apply the arithmetic geometric mean inequality we obtain

$$|f(0)|^p\prod_{k=1}^{n}\frac{r^p}{|z_k|^p} = \exp\left(\frac{1}{2\pi}\int_0^{2\pi}\log|f(re^{i\theta})|^p d\theta\right) \leq \frac{1}{2\pi}\int_0^{2\pi}|f(re^{i\theta})|d\theta,\ 0<r<1,$$

inequality holds for each $r<1$ and for arbitrary counting $n$. indeed, the product $\prod_{k=1}^{m}\frac{r^p}{|z_k|^p}$ increase to a maximum value as $m$ goes from 1 to $n$, since the factors $\frac{r}{|z_k|}$ are greater than 1 when $|z_k|<r$. It actually decreases for $m>n$, as soon as $|z_k|<r$ and so the extra factors are less than 1. Thus if $f\in\mathcal{A}_0^p$, integration yield

$$2|f(0)|^p\prod_{k=1}^{n}\frac{1}{|z_k|^p}\int_0^1 r^{np+1}dr \leq \frac{1}{2\pi}\int_0^1 |f(re^{i\theta})|^p r dr d\theta = \|f\|_{p,0}^p.$$

That is,

$$\left(\frac{2}{np+2}\right)^{\frac{1}{p}}|f(0)|\prod_{k=1}^{n}\frac{1}{|z_k|} \leq \|f\|_{p,0}.$$

Therefore, we arrived to the following theorem obtained by Horowitz in [Hor74]

**Theorem 1.3.1** ([Hor74]). *If $\{z_k\}_{k\geq 1}$ is the ordered zeros of a function $f\in\mathcal{A}_0^p$ with $f(0)\neq 0$, then*

$$\prod_{k=1}^{n}\frac{1}{|z_k|} = O(n^{\frac{1}{p}}), \qquad n\to\infty. \tag{1.22}$$

One can see that the constant $\frac{1}{p}$ in the above theorem is the best possible. However, the condition is far from being sufficient for a sequence $\{z_k\}_{k\geq 1}$ to be a zero set for $\mathcal{A}_0^p$. For instance, if $f \in \mathcal{A}_0^p$ has it zeros in a finite union of a Stolz angles, then they must satisfy Blaschke condition. However, the condition in the theorem 1.22 admits non-Blaschke sequences, even if they are aligned in a single ray, as we will see later, (or by Shapiro shields result) any necessary and sufficient condition must take in account of arguments as well as moduli of the points $z_k$. In term of modulus we can deduce the following corollary from theorem by method of summation by part. However, the details will be omitted since we have already derived the result from Jensen's formula. Let $\{z_k\}_{k\geq 1}$ be the zero set of a function $f$ belonging to $\mathcal{A}_0^p$ for some $p > 0$ with $f(0) \neq 0$. Then for every $\varepsilon > 0$

$$\sum_{k\geq 1}(1-|z_k|)\Big(\log\frac{1}{(1-|z_k|)}\Big)^{-1-\varepsilon} < \infty.$$

In his work Hor, Horowitz provides a special infinite product that converges uniformly on compact sets of $\mathbb{D}$ and so represents an analytic function.

$$f(z) = \prod_{k=1}^{\infty}(1 - bz^{m^k}), \qquad b > 1 \quad m = 2, 3, 4 \tag{1.23}$$

for a careful choice of parameters $b$ and $m$ depending on $p$ and $q$ Horowitz obtain the following theorem for a well exposed proof see [DS04a].

**Theorem 1.3.2** ([Hor74]). *If $0 < p < q < \infty$, there exists an $\mathcal{A}_0^p$ zero set that is not $\mathcal{A}_0^q$ zero set.*

We summarize, in his proof, Horowitz showed that in case $0 < p \leq 2$ if

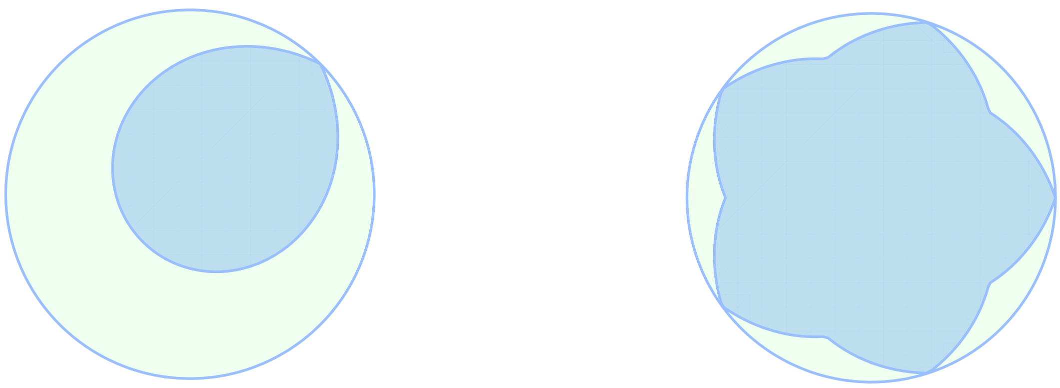
$$p < \frac{2\log(m)}{\log(1+b^2)} \qquad \text{and} \qquad \frac{\log(m)}{\log(b)} < q,$$

then the function $f$ in (1.23) belong to $\mathcal{A}_0^p$ but its zero set $\{z_k\}_{k\geq 1}$ do not form an $\mathcal{A}_0^q(\mathbb{D})$ zero set. For the case $p > 2$, let $p' = \frac{p}{p-1}$ be the conjugate index. The choice of $b > 1$ and $m \geq 1$ such that

$$\frac{\log(1+b^{p'})}{\log(m)} < p' - 1, \qquad \text{or} \qquad p < \frac{\log(m)}{\frac{p-1}{p}\log(1+b^{\frac{p}{p-1}})} < q \qquad \text{and} \qquad \frac{\log(m)}{\log(b)} < a$$

(both inequality can be satisfied for certain choices of $b > 1$ and $m = 2, 3, \ldots$ ). Any such choice produces a function 1.23 $f \in \mathcal{A}_0^p$ whose zeros do not constitute an $\mathcal{A}_0^q$. In of Horowitz construction the following corollaries was obtained.

**Corollary 1.3.3.** *1. There is a function $f \in \mathcal{A}_0^1$ which cannot be factored as a product of two $\mathcal{A}_0^2$ functions, one of them nonvanishing.*



(a) Example of Stolz angle at the point $\zeta = e^{\frac{\pi i}{4}}$.

(b) Stolz star for 5th primitive roots of unity.

Figure 1.1: Here is an example of a star and Stolz'angle.

2. *The condition in the corollary 1.3.1 is the best possible. For every $p$ with $0 < p < \infty$, there exist a function $f \in \mathcal{A}_0^p$ with zero set satisfying*

$$\sum_{k\geq 1}(1-|z_k|)\left(\log\frac{1}{(1-|z_k|)}\right)^{-1-\varepsilon} = \infty.$$

There is no characterization for zero sets for $\mathcal{A}_\alpha^\infty$ known for the author. We present some conditions obtained by Korenblum (see [Kor75]) and Seip (see **[Seip93]** and [Sei95a]) from [HKZ00, Chapter 4].

Fix a point $\zeta \in \partial\mathbb{D}$, we let $\mathfrak{s}_\zeta$ denote the *Stolz angle*[*] with vertex $\zeta$ and aperture $\pi/2$ (see figure 1.1a), this is,

$$\mathfrak{s}_\zeta = \left\{ z \in \mathbb{D} : |1-\bar{\zeta}z| < \frac{\pi}{2}(1-|z|^2) \right\}.$$

Let $F$ be a finite set of points in the boundary $\partial\mathbb{D}$, we define the *Stolz star domain* (see figure 1.1b)

$$\mathfrak{s}_F = \bigcup_{\zeta\in F}\mathfrak{s}_\zeta.$$

For an arc $I \subset \partial\mathbb{D}$, let $|I|_n$ be its *normalized arc length*, i.e.[*]

$$|I|_n = \int_I \frac{|dz|}{2\pi}$$

[*] *Privalov ice cream cone* is also used to call this region.

[*] Given a curve $\gamma$ parametrizied by $\gamma = \gamma(t) : [a,b] \to \mathbb{C}$ we define the *element of arc* $|dz|$ by

$$\int_\gamma f(z)|dz| = \int_0^b f(\gamma(t))|\gamma'(t)|dt.$$

For a closed and proper subset $A$ of $\partial\mathbb{D}$ with complementary arcs $\{I_j\}_j$, we define the *Beurling-Carleson characteristic*[*] of $A$ as

$$\tilde{\kappa}(A) = \sum_{j\geq 1} |I_j|_n \left(1 - \log |I_j|_n\right).$$

Let $\Lambda = \{\lambda\}_{\lambda\in\Lambda}$ be a sequence of points in $\mathbb{D}$ and $B$ an arbitrary set of $\mathbb{D}$. Let us consider

$$\Sigma(\Lambda, B) = \sum_{\lambda:\ \lambda\in B} (1 - |\lambda|^2).$$

We call $\kappa$-*density* of the sequence $\Lambda$ in the Stolz star $\mathfrak{s}_F$ to the quantity

$$D(\Lambda, \mathfrak{s}_F) = \frac{\Sigma(\Lambda, F)}{\tilde{\kappa}(F)}.$$

Finally, let us define the *upper* and *lower asymptotic $\kappa$-densities* of $\Lambda$, respectively, as the quantities

$$\mathcal{D}^+_\kappa(\Lambda) = \limsup_{\tilde{\kappa}(F)\to\infty} D(\Lambda, \mathfrak{s}_F),$$
$$\mathcal{D}^-_\kappa(\Lambda) = \liminf_{\tilde{\kappa}(F)\to\infty} D(\Lambda, \mathfrak{s}_F).$$

In [HKZ00, Chapter 4] we can find some sufficient and necessary conditions for zero sets of $\mathcal{A}^\infty_\alpha$ in terms of $\Sigma(\Lambda, B)$ or $\tilde{\kappa}(F)$. Also, a relation between the $\kappa$-densities and the analogue Beurling-Landau was established. However we omit these points, since it is enough to present the following result for completeness.

**Theorem 1.3.4.** *Let $\Lambda$ be a sequence in $\mathbb{D}$,*

*(a) if $\Lambda$ is a $\mathcal{A}^\infty_\alpha$ zero set, then $\mathcal{D}^+_\kappa(\Lambda) \leq \alpha$.*

*(b) Suppose that $\mathcal{D}^+_\kappa(\Lambda) < \alpha$, then is a $\mathcal{A}^\infty_\alpha$ zero set.*

For $\alpha = 0$ and $p = \infty$, there is the well-known *Blaschke condition*, which is necessary and sufficient, for a set of zeros counting the multiplicity, in case of Hardy spaces

$$\sum_{\lambda\in\Lambda} (1 - |\lambda|) < \infty.$$

In [Kor75], Korenblum made a penetrating study of zero sets in $\mathcal{A}^p_0$ and found further conditions that take account of angular distribution – in a weaker way by introducing Carleson set and the $\kappa-$density– Seip [Sei94], [Sei95b] used more refined methods to obtain sharper results. A fascinating exposition of these developments can be found in book of Korenblum, Hedenmalm, and Zhu [HKZ00]. The lack of a good characterization of zero sets in these Bergman space was for us a motivation to look for an alternative. In the next chapter we will obtain a new Jensen's formula, and a weak necessary condition for zero divisor, that is zero set with multiplicities.

[*] Also known as *entropy* in the literature.

# Chapter 2

# Multiple sampling and interpolation in Bergman spaces

This chapter contains essentially, my work in collaboration with C. Cruz, A. Hartmann and K. Kellay [Aad+22]. As the title indicate it, the subject is about sampling with samples and derivatives until some order of functions in a given Bergman space. Also, interpolating function and their derivatives to some given order. The idea of the work goes back to a question by supervisor professor Omar El-Fallah, when he asked me to look for an alternative description of sampling and interpolation sequences in Bergman spaces because Seip's characterization is difficult to check in general. I haven't made a lot of change compared to the paper version, just added and extended some proofs. In the idea that any reader can read this chapter independently, even to the chapter (1).

## 2.1 Introduction

Interpolating and sampling sequences of the Bergman space had been completely characterized by K. Seip in [Sei93] using density conditions, and in more general Hilbert spaces of analytic functions by Berndtsson and Ortega-Cerdà in [OB95]. Subsequently, the case of multiple interpolation (but not sampling) with uniformly bounded multiplicites had been studied for instance by Krosky and Schuster [KS01] using also extremal functions (we mention related work by the third named author who considered in [Har01] finite unions of Bergman interpolating sequences based on extremal functions, the case of multiple interpolation being in a sense a limite case of finite unions). In view of his density characterizations, Seip's results imply that there are no simultaneous sampling and interpolating sequences in the Bergman spaces.

In the Fock space, besides considering the case of simple interpolation and sampling problems, Seip – in particular with Brekke – was interested in the situation of higher multiplicities. Again, the density conditions obtained by these authors imply that there are no simultaneous sampling and interpolating sequences, neither in the simple case nor in the multiple case. Brekke and Seip in [BS93] also asked whether there could be simultaneous sampling and interpolating sequences when the uniform boundedness condition on the multiplicities is relaxed. In [Bor+17] it was shown that at least when the multiplicities tend to infinity, this is not possible (see also [EHR21] for the case of bounded multiplicites in the weighted Fock space).

One difficulty occuring in the case of unbounded multiplicities is the lack of a reasonable definition of densities. In [Bor+17], the authors introduce covering and separation conditions related with critical radii suitably related with the multiplicities to circumvent densities. Though those conditions do not characterize multiple interpolation and sampling, they get in a sense closer and closer to a characterization when the multiplicities grow (indeed the difference between necessary and sufficient conditions of the radii remains bounded while the radii tend to infinity).

Bergman and Fock spaces share many properties, and techniques often translate from one setting to the other. The aim is to study the situation concerning multiple interpolating and sampling with unbounded multiplicities in the Bergman space. New difficulties and challenges appear in order to adapt the situation from the underlying euclidean metric in the Fock space to pseudohyperbolic metric in the Bergman space. While this might be rather direct for simple interpolation and sampling the situation requires a quite delicate analysis of the criticial radii in the pseudohyperbolic metric in particular when the multiplicities are not uniformly bounded. On the technical side, replacing the incomplete $\Gamma$-function by the incomplete $\beta$-function gives rise to other difficulties.

It is mentionable that generalized interpolation problems (but not sampling problems) have been considered long ago in the Hardy space for which a complete answer is given by the so-called generalized Carleson condition (see [Nik78; Vas78]). In this situation, the case of interpolating sequences with unbounded multiplicities is completely understood (see also earlier work by Vinogradov-Rukshin [VR82]).

Without claiming exaustivity, we finish this first tour on multiple interpolation problems, mentioning work on interpolating sequences with uniformly bounded multiplicity in the Korenblum space, see [Mas99] and for weighted spaces of entire functions see [Oun07; Oun08].

We now introduce the necessary notation. Let $\alpha > -1$, we consider the $L^2$ weighted Bergman space

$$\mathcal{A}^2_\alpha = \left\{ f : \mathbb{D} \to \mathbb{C} \text{ holomorphic, } \|f\|^2_{\alpha,2} = \int_{\mathbb{D}} |f(z)|^2 dA_\alpha(z) < +\infty \right\},$$

where $dA_\alpha(z) = (1+\alpha)(1-|z|^2)^\alpha dxdy/\pi$, $z = x+iy$. The space $\mathcal{A}^2_\alpha$ is a reproducing kernel Hilbert space with the scalar product

$$\langle f, g \rangle := \int_{\mathbb{D}} f(z)\overline{g(z)} dA_\alpha(z).$$

The normalized monomials form an orthonormal basis for $\mathcal{A}^2_\alpha$ and are given by

$$e_n(z) = \sqrt{\frac{\Gamma(n+2+\alpha)}{n!\Gamma(2+\alpha)}} z^n, \qquad n \geq 0, \tag{2.1}$$

where $\Gamma(s)$ stands for the usual Gamma function [HKZ00, p.4]. Thus, the reproducing kernel of $\mathcal{A}^2_\alpha$ is

$$K_w(z) = \sum_{j \geq 0} \overline{e_j(w)} e_j(z) = \frac{1}{(1-\overline{w}z)^{\alpha+2}},$$

and the normalized Bergman kernel is $k_w(z) = K_w(z)/\|K_w\|_{\alpha,2}$. The reproducing kernel gives rise in a standard way to a growth condition in the Bergman space:

$$|f(\lambda)|^2 = |\langle f, K_\lambda \rangle|^2 \leq \left( \frac{1}{1-|\lambda|^2} \right)^{\alpha+2} \|f\|^2_{\alpha,2}, \quad f \in \mathcal{A}^2_\alpha, \lambda \in \mathbb{D}. \tag{2.2}$$

We consider the Möbius transform

$$\varphi_\lambda(z) = \frac{\lambda - z}{1 - \overline{\lambda} z} \tag{2.3}$$

and the isometric translation operators in $\mathcal{A}^2_\alpha$ given by $T_\lambda f(z) = \left[\varphi'_\lambda(z)\right]^{\frac{2+\alpha}{2}} f(\varphi_\lambda(z))$, i.e.,

$$\begin{aligned} T_\lambda : \ \mathcal{A}^2_\alpha &\longrightarrow \mathcal{A}^2_\alpha \\ f &\longmapsto T_\lambda f := \left[ \frac{|\lambda|^2 - 1}{(1-\overline{\lambda}\cdot)^2} \right]^{\frac{2+\alpha}{2}} f(\varphi_\lambda(\cdot)). \end{aligned}$$

Notice that $\varphi_\lambda$ and $T_\lambda$ are involutions, in fact, $T_\lambda$ is a self-adjoint operator.

In the Bergman space, the underlying metric on the unit disk is the pseudo-hyperbolic distance which is defined *via* the already mentioned Möbius transform (2.3):

$$\rho(u, v) = |\varphi_u(v)|, \quad u, v \in \mathbb{D}.$$

We also associate the pseudohyperbolic disk with this distance: $D(\lambda, r) = \{z \in \mathbb{D} : \rho(\lambda, z) < r\}$ for $\lambda \in \mathbb{D}$ and $0 < r < 1$. In order to be interpolating in the Bergman space, a sequence has to be separated in this metric (see [Sei93]). Also, in order to be sampling (at least in the Hilbertian case under consideration here), it can be deduced that any pseudohyperbolic neighborhood with fixed radius can contain at most a uniformly bounded number of points (Carleson measure condition).

In order to better understand multiple interpolation and sampling problems we shall comment a little bit more on the multiplicity one situation. In this case, given a set of points $\Lambda \subset \mathbb{D}$, one is interested in the values of a function in given points $f(\lambda)$, $\lambda \in \Lambda$. When $\Lambda$ is separated in the pseudohyperbolic metric, it can be shown that for $f \in \mathcal{A}^2_\alpha$, we have

$$\sum_{\lambda \in \Lambda} \frac{|f(\lambda)|^2}{\|K_\lambda\|^2} = \sum_{\lambda \in \Lambda} (1-|\lambda|^2)^{2+\alpha} |f(\lambda)|^2 < \infty. \tag{2.4}$$

Conversely, if the sequence $\Lambda$ is sufficiently separated, if can be shown that every sequence of values $(v_\lambda)_{\lambda \in \Lambda}$ the square of which is summable against the weight $(1-|\lambda|^2)^{2+\alpha}$ can be interpolated by a function $f \in \mathcal{A}^2_\alpha$. Observe that

$$\langle f, T_\lambda e_0 \rangle = T_\lambda f(0) = \left[ \frac{|\lambda|^2 - 1}{(1 - \overline{\lambda} \times 0)^2} \right]^{\frac{2+\alpha}{2}} f(\varphi_\lambda(0)) = (|\lambda|^2 - 1)^{\frac{2+\alpha}{2}} f(\lambda),$$

so that (2.4) translates to

$$\sum_{\lambda \in \Lambda} |\langle f, T_\lambda e_0 \rangle|^2 < +\infty.$$

Notice that, intuitively, a sequence of interpolation must be sufficiently sparse, and should be a set of zeros of a holomorphic function (at least up to one point). Analogously, any sampling sequence should have sufficiently big density and is in general not a set of zeros (except in spaces which admit complete interpolating sequences). This naive relation points at the connection between our problems and uniqueness questions which will be studied in the next Section 2.2.

Let us now switch to the multiple case. Instead of studying only the values of a function in the points of the given sequence, it is natural to consider germs of functions in those points, i.e. to consider also derivatives of the function up to a certain order depending on the point (Hermite type interpolation). As long as the multiplicites are uniformly bounded, the definition of the target space can be based on point evaluations and their derivatives with suitable weights. However, when we allow multiplicities to grow to infinity, the weights and constants have to be chosen in a very precise way. In the Hilbertian case, since $(e_j)_{j \geq 0}$ is an orthonormal basis and $T_\lambda$ an isometry, it is natural to consider $\langle f, T_\lambda e_j \rangle$, where $\lambda \in \Lambda$ and $j$ is bounded by the multiplicity. With this in mind, we can now define sampling and interpolation in the general case. Given a set of points $\Lambda \subset \mathbb{D}$ with multiplicity $m_\lambda$, we call *divisor* a set of pairs $X = \{(\lambda, m_\lambda)\}_{\lambda \in \Lambda}$.

**Definition 2.1.1.** *A divisor $X$ is sampling for $\mathcal{A}^2_\alpha$ if there exists a constant $C > 0$, such that for all $f \in \mathcal{A}^2_\alpha$*

$$\frac{1}{C} \sum_{\lambda \in \Lambda} \sum_{j < m_\lambda} |\langle f, T_\lambda e_j \rangle|^2 \leq \|f\|^2_{\alpha,2} \leq C \sum_{\lambda \in \Lambda} \sum_{j < m_\lambda} |\langle f, T_\lambda e_j \rangle|^2 .$$

Note that $P_\lambda f = \sum_{j<m_\lambda} \langle f, T_\lambda e_j \rangle T_\lambda e_j$ is an orthogonal projection and it can be shown that

$$\ker P_\lambda = N^{2,\alpha}_{\lambda,m_\lambda} := \{f \in \mathcal{A}^2_\alpha : f^{(j)}(\lambda) = 0, \quad \forall j < m_\lambda\}, \tag{2.5}$$

(see also equation (4) in [BS93, p.114] where this matter is discussed in the Fock space), so that in particular

$$\sum_{j<m_\lambda} |\langle f, T_\lambda e_j \rangle|^2 = \|f\|^2_{\mathcal{A}^2_\alpha / N^{2,\alpha}_{\lambda,m_\lambda}}.$$

It is useful to recall the weaker notion of uniqueness. A divisor $X$ is called a *uniqueness divisor* (or simply $X$ is uniqueness) if every function vanishing up to the order $m_\lambda - 1$ is $\lambda$ is necessarily the zero function. Clearly, a sampling divisor is uniqueness, but the converse is in general not true.

The above definition gives rise to a natural definition of the following target space needed for interpolation.

$$\ell^2(X) = \left\{ (v^j_\lambda)_{\lambda\in\Lambda,\ 0\le j<m_\lambda} : \|v\|_2^2 := \sum_{\lambda\in\Lambda} \sum_{j<m_\lambda} |v^j_\lambda|^2 < \infty \right\}.$$

**Definition 2.1.2.** *The divisor $X$ is interpolating for $\mathcal{A}^2_\alpha$ if for all sequences $v \in \ell^2(X)$, there exists $f \in \mathcal{A}^2_\alpha$ such that*

$$\langle f, T_\lambda e_j \rangle = v^j_\lambda. \qquad (\lambda \in \Lambda, j < m_\lambda) \tag{2.6}$$

Note again that the above interpolation condition is equivalent to interpolation by germs of $f$ in $\lambda$ up to the order $m_\lambda - 1$ (see equation (4) in [BS93, p.114]). Clearly, if $f$ interpolates $v$ in the above way, then $\sum_{j<m_\lambda} |v^j_\lambda|^2 = \|f\|^2_{\mathcal{A}^2_\alpha / N^{2,\alpha}_{\lambda,m_\lambda}}$. The reinterpretation in terms of quotient norms will be useful later when considering the situation in $\mathcal{A}^\infty_\alpha$. Furthermore, Note that in the case of interpolation divisor $X$, an application of open mapping theorem to the (multiple) restriction operator, i.e.,

$$\begin{aligned} R_X :\ \mathcal{A}^2_\alpha &\longrightarrow \ell^2(X) \\ f &\longmapsto R_X(f) := \{\langle , T_\lambda e_j \rangle\}_{\lambda\in\Lambda,\ 0\le j<m_\lambda}. \end{aligned}$$

shows that the solution $f$ in (2.6) can always be chosen in such a way that $\|f\|_{2,\alpha} \le M\|v\|_2$, with $M$ depending only on $X$. The smallest such constant will be denoted $M_X$ and is called the interpolation constant of $X$ (Truthfully $M_X = \|\tilde{R}_X^{-1}\|$, $\tilde{R}_X :\ \mathcal{A}^2_\alpha / \big( \cap_\Lambda N^{2,\alpha}_{\lambda,m_\lambda} \big) \longrightarrow \ell^2(X)$ ).

In the case of the classical Fock space, multiple interpolation and sampling was related to some critical radius. More precisely, since in the Fock space the underlying metric is euclidean, given a multiplicity $m_\lambda$ in a point $\lambda \in \mathbb{C}$ the "influence zone" of $\lambda$ meaning the knowledge of $f(\lambda)$,...,$f^{m_\lambda-1}(\lambda)$ — or equivalently that of $\langle f, T_\lambda e_0\rangle$,...,$\langle f, T_\lambda e_{m_\lambda}\rangle$ — was the euclidean disk $D_e(\lambda, \sqrt{m_\lambda}) = \{z \in \mathbb{C} : |z-\lambda| < \sqrt{m_\lambda}\}$ (see [Bor+17]). This corresponds more or less to redistributing the multiplicity in a regular way in an euclidean disk. It is a priori not so clear how to define this redistribution in the pseudohyperbolic case in particular when we authorize the multiplicity to tend to infinity. The corresponding critical radius appears in the following overlap condition; it will be clear from later discussions where this radius comes from. Our sampling and interpolating conditions are all expressed with respect to this critical value (slightly increasing or decreasing it). We will need an overlap condition that we introduce now.

**Remark 3.** *There is a relation between the k-th derivative of a $\mathcal{A}^2_\alpha$-function $f$ and the value $\langle f, T_\lambda e_k\rangle$. This can be peered by the development of $f$ around $\lambda$ in terms of $\{T_\lambda e_k\rangle\}_{k\geq 0}$. However, let us compute these terms, take $f \in \mathcal{A}^2_\alpha$ and $\lambda \neq 0$,*

$$\langle f, T_\lambda e_k\rangle = \langle T_\lambda f, e_k\rangle = \frac{\|z^k\|_{2,\alpha}}{k!}\partial^k\left(T_\lambda f\right)(0).$$

*By the Leibniz's rule and the simplification $f^{(k)}(x) = \partial^k f(x)$*

$$\begin{aligned}\partial^k\big(T_\lambda f\big)(z) &= \sum_{j=0}^{k}\binom{k}{j}\varphi_\lambda^{(j+1)}(z)\big(f\circ\varphi_\lambda\big)^{(k-j)}(z)\\ &= \varphi_\lambda'(z)\big(f\circ\varphi_\lambda\big)^{(k)}(z) + \sum_{j=1}^{k}\binom{k}{j}\varphi_\lambda^{(j+1)}(w)\big(f\circ\varphi_\lambda\big)^{k-j}(z).\end{aligned}$$

*The Faà di Bruno's formula yields*

$$\big(f\circ\varphi_\lambda\big)^{(m)}(z) = \sum_{S_m}\frac{m!}{\prod_{i=1}^m b_i!}f^{(b_1+\ldots+b_m)}\circ\varphi_\lambda(z)\prod_{i=1}^{m}\left(\frac{\varphi_\lambda^{(i)}(z)}{i!}\right)^{b_i}$$

*where $S_m$ is the set of all solutions of nonnegative integers $b_1, \cdots, b_m$ of the Diophantine equation $b_1 + 2b_2 + \ldots + mb_m = m$, i.e.*

$$S_m = \{(b_1, ..., b_m) \in \mathbb{N}^m : b_1 + 2b_2 + ... + mb_m = m\}.$$

*Hence, we get*

$$\big(f\circ\varphi_\lambda\big)^{(m)}(0) = \sum_{S_m}\frac{m!}{\prod_{i=1}^m b_i!}f^{(b_1+\ldots+b_m)}(\lambda)\prod_{i=1}^{m}\left(\frac{\varphi_\lambda^{(i)}(0)}{i!}\right)^{b_i}.$$

*Notice that*

$$\varphi_\lambda^{(i)}(0) = i!(|\lambda|^2-1)(-\overline{\lambda})^{i-1}, \qquad i \geq 1$$

*implies*

$$\begin{aligned}\big(f\circ\varphi_\lambda\big)^{(m)}(0) &= \sum_{S_m}\frac{m!}{\prod_{i=1}^m b_i!}f^{(b_1+\ldots+b_m)}(\lambda)\prod_{i=1}^m(|\lambda|^2-1)^{b_i}(-\overline{\lambda})^{b_i(i-1)}\\ &= \sum_{S_m}\frac{m!}{\prod_{i=1}^m b_i!}(|\lambda|^2-1)^{b_1+\ldots+b_m}(-\overline{\lambda})^{m-(b_1+\cdots+b_m)}f^{(b_1+\ldots+b_m)}(\lambda)\\ &= m!(-\overline{\lambda})^m\sum_{S_m}\frac{1}{\prod_{i=1}^m b_i!}\left(\frac{1-|\lambda|^2}{\overline{\lambda}}\right)^{b_1+\cdots+b_m}f^{(b_1+\cdots+b_m)}(\lambda)\\ &= m!(-\overline{\lambda})^m\sum_{b=(b_1,\cdots,b_m)\in S_m}\frac{1}{b!}\left(\frac{1-|\lambda|^2}{\overline{\lambda}}\right)^{|b|}f^{(|b|)}(\lambda)\end{aligned}$$

*where* $|b| = b_1+\cdots+|b_m|$ *and* $b! = b_1!\cdots b_m!$. *Finally we obtain*

$$\begin{aligned}\frac{k!}{\|z^k\|_{2,\alpha}}\langle f,T_\lambda e_k\rangle &= \varphi_\lambda'(z)\big(f\circ\varphi_\lambda\big)^{(k)}(z)+\sum_{j=1}^k\binom{k}{j}\varphi_\lambda^{(j+1)}(w)\big(f\circ\varphi_\lambda\big)^{k-j}(z)\Big|_{z=0}\\ &= (|\lambda|^2-1)\big(f\circ\varphi_\lambda\big)^{(k)}(0)+\sum_{j=1}^k\binom{k}{j}(j+1)!(|\lambda|^2-1)(-\overline{\lambda})^j\big(f\circ\varphi_\lambda\big)^{(k-j)}(0)\\ &= k!(-\overline{\lambda})^{k+1}\sum_{b=(b_1,\ldots,b_k)\in S_k}\frac{1}{b!}\left(\frac{1-|\lambda|^2}{\overline{\lambda}}\right)^{|b|+1}f^{(|b|)}(\lambda)\\ &\quad+\sum_{j=1}^k k!(j+1)(-\overline{\lambda})^{k+1}\sum_{b=(b_1,\ldots,b_{k-j})\in S_{k-j}}\frac{1}{b!}\left(\frac{1-|\lambda|^2}{\overline{\lambda}}\right)^{|b|+1}f^{(|b|)}(\lambda)\\ &= k!(-\overline{\lambda})^{k+1}\sum_{j=0}^k(j+1)\sum_{b=(b_1,\ldots,b_{k-j})\in S_{k-j}}\frac{1}{b!}\left(\frac{1-|\lambda|^2}{\overline{\lambda}}\right)^{|b|+1}f^{(|b|)}(\lambda).\end{aligned}$$

*Therefore, we can observe the relation sought,*

$$\langle f,T_\lambda e_j\rangle = \sum_{k=0}^j c_{j,k}(1-|\lambda|^2)f^{(k)}(\lambda) = \sum_{k=0}^j a_{j,k,\lambda}f^{(k)}(\lambda), \qquad j\in\mathbb{N}. \tag{2.7}$$

**Definition 2.1.3.** *A divisor $X$ satisfies the finite overlap condition for $\mathcal{A}^2_\alpha$ if*

$$S_X = \sup_{z\in\mathbb{D}}\sum_{\lambda\in\Lambda}\chi_{D\left(\lambda,\sqrt{\frac{m_\lambda}{m_\lambda+\alpha+1}}\right)}(z)<\infty.$$

We should mention that this finite overlap condition is intimately related to the Carleson measure condition.

Now we are in a position to state the geometric condition for sampling divisors.

**Theorem 2.1.4.** *Let* $\alpha > -1$.

*(a) If* $X$ *is a sampling divisor for* $\mathcal{A}^2_\alpha$*, then* $X$ *satisfies the finite overlap condition and there exists* $0 < C_X < \alpha + 1$ *such that*

$$\bigcup_{\lambda\in\Lambda} D\left(\lambda, \sqrt{\frac{m_\lambda + C_X}{m_\lambda + \alpha + 1}}\right) = \mathbb{D}.$$

*(b) Conversely, suppose the divisor* $X$ *satisfies the finite overlap condition. There is a constant* $C > 1$ *depending on* $S_X$ *such that if for some compact* $K$ *of* $\mathbb{D}$ *we have*

$$\bigcup_{\lambda\in\Lambda, m_\lambda > C} D\left(\lambda, \sqrt{\frac{m_\lambda - C}{m_\lambda + \alpha + 1}}\right) = \mathbb{D} \setminus K,$$

*then* $X$ *is a sampling divisor for* $\mathcal{A}^2_\alpha$.

This theorem tells us that if disks with slightly smaller radii than the critical one already cover the unit disk, then we have a sampling divisor. And if a divisor is sampling then at least disks with slightly bigger radii cover the unit disk (up to a compact set).

One could be tempted to complain about the constant $C$ appearing in (b) above. Note that the theorem is completely general and applies even in the case of uniformly bounded multiplicites where the result proved in [BS93] requires density conditions. So there is no hope getting a sufficient condition only from the covering without additional conditions for instance on the critical radius.

In the analogous situation for interpolating divisors the covering condition is replaced by a separation condition of disks with slightly bigger or smaller radii than the criticial ones.

**Theorem 2.1.5.** *Let* $\alpha > -1$.

*(a) If* $X$ *is an interpolating divisor for* $\mathcal{A}^2_\alpha$*, then there exists* $C_X > 0$ *such that the hyperbolic disks*

$$\left\{ D\left(\lambda, \sqrt{\frac{m_\lambda - C_X}{m_\lambda + \alpha + 1}}\right)\right\}_{\lambda\in\Lambda, m_\lambda > C_X}$$

*are pairwise disjoint.*

*(b) Conversely, if for some* $C_X$ *such that* $(\alpha+1)(1-e^{-1}) < C_X < \alpha + 1$*, the hyperbolic disks*

$$\left\{ D\left(\lambda, \sqrt{\frac{m_\lambda + C_X}{m_\lambda + \alpha + 1}}\right)\right\}_{\lambda\in\Lambda}$$

*are pairwise disjoint, then* $X$ *is an interpolating divisor for* $\mathcal{A}^2_\alpha$.

Notice that the separation condition appearing in the statement (a) implies the finite overlap condition (the overlap is actually void), which is again related to the Carleson measure condition.

Again, we should point out that additional conditions on the constant $C$ are required in (b) since the theorem is completely general covering the case when the multiplicities are uniformly bounded in which case the result [BS93] involves again density conditions. So, separation alone for $C$ arbitrary close to 0 cannot be sufficient for interpolation.

Concerning both Theorems 2.1.4 and 2.1.5, we would also like to emphasize the fact that densities, even if they provide characterizations for simple or uniformly bounded multiplicites, are hard to compute in a general situation (particularly in the pseudohyperbolic metric), while our overlapping and separation conditions are much easier to apprehend.

Here is another observation: in case $X$ is a sampling divisor, the finite overlap condition is necessary. In case $X$ is an interpolating divisor, we get a separation condition, which obviously also implies the finite overlap (there is actually no overlap and now $S_X = 1$). So in both cases, the area of pseudohyperbolic disks centered at $\lambda$ and with radius comparable to $\sqrt{(m_\lambda - C)/(m_\lambda + \alpha + 1)}$ add up to a finite sum, yielding the following Blaschke type condition which seems new:

$$\sum_{\lambda \in \Lambda} (m_\lambda (1 - |\lambda|^2))^2 < +\infty. \tag{2.8}$$

The result which affirms that in the Fock space (with Gaussian weight) there are no Riesz bases (simultaneously interpolating and sampling) is quite expensive to obtain. First in [Sei93] for simple multiplicity, and later for uniformly bounded multiplicity in [BS93] this requires the density characterizations of interpolating and sampling sequences. More recently the third and fourth authors discussed this problem in [Bor+17] when the multiplicities go to infinity. In this case, there are no characterizations available, but the gap between necessary conditions for multiple sampling and multiple interpolation (given by the corresponding results to Theorems 2.1.4 and 2.1.5), together with some geometric lemma, allowed to conclude. The situation in Bergman spaces is dramatically simpler. The main reason is due to the fact that the multiplier algebra for standard Bergman spaces is not trivial and contains bounded analytic functions in the unit disk. As a consequence, any interpolating divisor is a zero divisor (pick a function vanishing in all the points $\lambda$ up to the order $m_\lambda - 1$ except for one point $\lambda_0$ in which we interpolate the value 1, then multiply the interpolating function by $b_{\lambda_0}^{m_\lambda}$), and can thus not be sampling since sampling divisors are uniqueness.

In view of the above discussions, the central role of zero and uniqueness sets should be clear. In this connection, we will formulate here a necessary condition for zero divisors which does not seem to follow from those known so far. A fairly precise results on zero sets in $\mathcal{A}$

**Theorem 2.1.6.** *Let $\alpha > 0$, $\varepsilon > 0$, and $X = \{(\lambda, m_\lambda)\}_{\lambda\in\Lambda}$ be a divisor such that*

$$\bigcup_{\lambda\in\Lambda} D\left(\lambda, \sqrt{\frac{m_\lambda}{m_\lambda + \alpha + 2 + \varepsilon}}\right) = \mathbb{D} \setminus K \tag{2.9}$$

*for some compact set $K \subset \mathbb{D}$. Then $X$ is uniqueness divisor for $\mathcal{A}^2_\alpha$.*

We recall that a uniqueness divisor is a non zero divisor.

Without going into more details in this introduction, we mention that the same problems can be considered in the uniform norm: let $\alpha > 0$, and define

$$\mathcal{A}^\infty_\alpha = \left\{ f \in \mathrm{Hol}(\mathbb{D}) : \|f\|_{\alpha,\infty} := \sup_{z\in\mathbb{D}} (1 - |z|_2)^{\frac{\alpha}{2}} |f(z)| < +\infty \right\}.$$

In this setting, the results are completely analogous – replacing essentially $\alpha + 1$ by $\alpha$ in the theorems cited above – and will be discussed in Section 4. Note that it follows immediately from (2.2) that $\mathcal{A}^2_\alpha \subset \mathcal{A}^\infty_{\alpha+2}$ which allows to connect some results between both situations. However, in general the results in $\mathcal{A}^\infty_\alpha$ do not follow immediately from those for $\mathcal{A}^2_\alpha$ and the proofs have to be rerun. We also point out a curious phenomenon. Indeed the above embedding works for $\alpha > -1$, but there is no $A^2_\beta$ which embeds into $A^\infty_\alpha$ when $\alpha \in (0, 1]$.

We would also like to comment on the uniqueness result Theorem 2.1.6 and its corresponding result Theorem 2.2.2 below in $\mathcal{A}^\infty_\alpha$ (which has essentially the same statement just replacing $\alpha + 2$ by $\alpha$). In [Sei95b], Seip gave fairly precise sufficient and necessary conditions exhibiting a small gap between these. His conditions are based on the Korenblum density which is difficult to check in general. The condition appearing in (2.9) (or in (2.13) below) yields maybe a more transparent necessary condition for zero divisors (for $X$ to be a zero divisor it is necessary that the covering condition (2.9) does not hold for any compact $K$ and any $\varepsilon > 0$).

The rest of the chapter is organized as follows: in Section 2, we prove the $\mathcal{A}^\infty_\alpha$ uniqueness Theorem. Section 3 is devoted to the proofs of Theorem 2.1.4 and 2.1.5 respectively. In Section 4 we discuss the uniform case.

## 2.2 Zeros and a Jensen type formula

In this section we will prove a uniqueness result for $\mathcal{A}^\infty_\alpha$ (which will also be useful for the case $p = 2$).

Let us introduce the invariant measure on the unit disk $d\mathcal{V}(z) = (1-|z|^2)^{-2} dm(z)$, where $m$ is the normalized Lebesgue measure such that $m(\mathbb{D}) = 1$. It it well-known that $d\mathcal{V}$ is invariant under Möbius transforms. We mention that a direct calculation shows that

$$\int_{D(\zeta,r)} d\mathcal{V}(z) = \int_{D(0,r)} d\mathcal{V}(z) = \frac{r^2}{1-r^2}, \quad 0<r<1.$$

We observe here that morally speaking, the critical radius $r$ has to be chosen more or less in such a way that this mass corresponds to the multiplicity. To be more precise, and as we will see below (see (2.12) below), the critical radius has to be chosen *via* redestributing the mass of the laplacian of the logarithm of an $\mathcal{A}^\infty_\alpha$-function (as was done in the euclidean metric appearing in the setting of the Fock space).

We will discuss the situation here in $\mathcal{A}^\infty_\alpha$ which requires $\alpha>0$ contrarily to $\mathcal{A}^2_\alpha$ where $\alpha>-1$.

Now, in the spirit of our observation above (and in particular (2.12) below), for fixed $\varepsilon>0$, let

$$r_\lambda = r_{\lambda,\alpha,\varepsilon} := \sqrt{\frac{m_\lambda}{m_\lambda+\alpha+\varepsilon}} \tag{2.10}$$

and set $D_\lambda = D(\lambda, r_\lambda)$. (In a sense, the critical radius in $\mathcal{A}^\infty_\alpha$ corresponds to the situation when $\varepsilon=0$.)

**Lemma 2.2.1.** *Let $\alpha>0$, if $X=\{(\lambda_k,m_k)\}_{k\geq 1}$ is a zero divisor for $\mathcal{A}^\infty_\alpha$, then for $\varepsilon>0$,*

$$\int_{D(0,r)} \sum_{k\geq 1} \chi_{D_\lambda}(z)\log\frac{r}{|z|}d\mathcal{V}(z) \leq \frac{\alpha}{2(\alpha+\varepsilon)}\log\frac{1}{1-r^2}+O(1), \quad r\to 1.$$

*Proof.* Let $X=\{(\lambda_k,m_k)\}_{k\geq 1}$ be a zero divisor for $\mathcal{A}^\infty_\alpha$ then there exists a non zero function $f\in\mathcal{A}^\infty_\alpha$ such that

$$f^{(k)}(\lambda)=0, \qquad 0\leq k<m_\lambda,\ \lambda\in\Lambda.$$

We will obtain our condition by redistributing the mass $\Delta(\log|f|)$ on the hyperbolic disks $D_\lambda$.

The function $\log|f|$ is subharmonic and not identically $-\infty$ and we have for all $z\in\mathbb{D}$

$$\log|f(z)| \leq \log\|f\|_{\alpha,\infty} + \frac{\alpha}{2}\log\frac{1}{1-|z|^2} =: s(z).$$

We use the same inductive method as in [Bor+17]. First we construct a new subharmonic function $h$ in $\mathbb{D}$ such that

$$\log|f(z)|\leq h(z)\leq s(z).$$

Later, we will obtain our bound using Green's identity. Let $h_0=\log|f|$ and recall $r_{\lambda_k}=\sqrt{\frac{m_k}{m_k+\alpha+\varepsilon}}$, then

$$h_0(z) = m_1\log\frac{|\varphi_{\lambda_1}(z)|}{r_{\lambda_1}} + \underbrace{\log|f(z)| - m_1\log\frac{|\varphi_{\lambda_1}(z)|}{r_{\lambda_1}}}_{=U_0}.$$

Since $f/\varphi_{\lambda_1}^{m_{\lambda_1}}$ is holomorphic in $\mathbb{D}$, $U_0$ is a subharmonic function on $\mathbb{D}$ and harmonic in a small neighborhood of $\lambda_1$. In fact, in a small neighborhood $V$ of $\lambda_1$ not containing any other zero of $f$ than $\lambda_1$, we have

$$|f(z)| = |\varphi_{\lambda_1}(z)|^{m_1}|g(z)|,$$

where $g$ is a holomorphic function with no zeros in $V$.

Therefore

$$\Delta \log|f(z)| = m_1 \Delta \log|\varphi_{\lambda_1}(z)| + \underbrace{\Delta \log|g|}_{=0}, \quad z \in V.$$

Now we will modify the term $m_1 \log \frac{|\varphi_{\lambda_1}(z)|}{r_{\lambda_1}}$ to obtain a constant function with respect to the invariant laplacian. Let us consider

$$v_1(z) = \begin{cases} \frac{\alpha+\varepsilon}{2} \log \frac{1-r_{\lambda_1}^2}{1-|\varphi_{\lambda_1}(z)|^2}, & z \in D(\lambda_1, r_{\lambda_1}), \\ \\ m_1 \log \frac{|\varphi_{\lambda_1}(z)|}{r_{\lambda_1}}, & \text{otherwise.} \end{cases} \tag{2.11}$$

Then $v_1$ is harmonic outside $D(\lambda_1, r_{\lambda_1})$ and $v_1 \in C^1(\mathbb{D})$. Using $\Delta = 4\partial\overline{\partial}$, and the fact that $\log|1-\overline{\lambda}z|$ is harmonic, we have inside $D(\lambda_1, r_{\lambda_1})$

$$\begin{aligned} \Delta v_1 &= \Delta\left(\frac{\alpha+\varepsilon}{2} \log \frac{1-r_{\lambda_1}^2}{1-|\varphi_{\lambda_1}(z)|^2}\right) = \frac{\alpha+\varepsilon}{2} \Delta\left(\log \frac{|1-\overline{\lambda_1}z|^2}{(1-|z|^2)(1-|\lambda_1|^2)}\right) \\ &= (\alpha+\varepsilon)\Delta(\log|1-\overline{\lambda_1}z|) - \frac{\alpha+\varepsilon}{2}\Delta(\log(1-|z|^2)) \\ &= \frac{2(\alpha+\varepsilon)}{(1-|z|^2)^2}. \end{aligned}$$

(We have used that $\Delta \log(1-|z|^2) = -\dfrac{4}{(1-|z|^2)^2}$.) Observe that with the definition of the invariant laplacian $\widetilde{\Delta} u = (1-|z|^2)^2 \Delta u$, the preceding computation shows that $\widetilde{\Delta} v_1$ is constant. We thus obtain the total mass of the measure $\Delta v_1$ on $D(\lambda_1, r_{\lambda_1})$ which is equal to $2(\alpha+\varepsilon)\mathcal{V}(D(\lambda_1, r_{\lambda_1}))$:

$$\begin{aligned} \int_{\mathbb{D}} \Delta v_1 dm &= \int_{D(\lambda_1, r_{\lambda_1})} 2(\alpha+\varepsilon)\frac{dm}{(1-|z|^2)^2} = 2(\alpha+\varepsilon)\mathcal{V}(D(\lambda_1, r_{\lambda_1})) \\ &= 2(\alpha+\varepsilon)\frac{r_{\lambda_1}^2}{1-r_{\lambda_1}^2}. \end{aligned} \tag{2.12}$$

By the specific definition of $r_{\lambda_1}$ this is equal to $2m_{\lambda_1}$.

On the other hand, since $\Delta \log|z| = 2\pi\delta_0$,

$$\begin{aligned} \int_{\mathbb{D}} \Delta\left(m_1 \log \frac{|\varphi_{\lambda_1}(z)|}{r_{\lambda_1}}\right) dm(z) &= m_1 \int_{\mathbb{D}} \Delta(\log|\lambda_1 - z|) dm(z) \\ &= 2m_1. \end{aligned}$$

In particular, by the very definition of $r_{\lambda_1}$, both total masses coincide so that in terms of Laplacians, we can replace $m_1 \log \frac{|\varphi_{\lambda_1}(.)|}{r_{\lambda_1}}$ by $v_1(.)$ in $D(\lambda_1, r_{\lambda_1})$. This yields the function $h_1 = v_1 + U_0$. Obviously,

$$h_1(z) = \frac{\alpha+\varepsilon}{2} \log \frac{1-r_{\lambda_1}^2}{1-|\varphi_{\lambda_1}(z)|^2} \chi_{D(\lambda_1, r_{\lambda_1})}(z) + m_1 \log \frac{|\varphi_{\lambda_1}(z)|}{c_{\lambda_1}} \chi_{\mathbb{D}\setminus D(\lambda_1, r_{\lambda_1})}(z) + U_0.$$

Let us show that

(a) $h_1(z) \leq s(z)$, $z \in \mathbb{D}$,

(b) $h_0(z) \leq h_1(z)$, $z \in \mathbb{D}$.

We start proving $h_1 \leq s$. This is clear for $z \notin D(\lambda_1, r_{\lambda_1})$, because $h_1 = h_0$ in $\mathbb{D} \setminus D(\lambda_1, r_{\lambda_1})$. For $z \in D(\lambda_1, r_{\lambda_1})$, we consider the function $w_1 = v_1 + U_0 - s$, we have

$$\Delta w_1 = \Delta \left( v_1 + U_0 - \frac{\alpha}{2} \log \frac{1}{1-|z|^2} \right) = \Delta U_0 + \Delta \frac{\varepsilon}{2} \log \frac{1}{1-|z|^2} \geq 0.$$

Hence $w_1$ is subharmonic on $D(\lambda_1, r_{\lambda_1})$, and for $\xi \in \partial D(\lambda_1, r_{\lambda_1})$, since $v_1(\xi) = 0$, we have

$$w_1(\xi) = U_0(\xi) - s(\xi) = h_0(\xi) - s(\xi) = \log |f(\xi)| - s(\xi) \leq 0.$$

So in the boundary $w_1$ is non-positive, so that it is non-positive throughout the disc by the maximum principle.

It remains to see $h_0 \leq h_1$. Again outside the hyperbolic disc $D(\lambda_1, r_{\lambda_1})$ we have $h_0 = h_1$. In the disc $D(\lambda_1, r_{\lambda_1})$ we need to compare the following functions

$$\varphi(z) = \frac{\alpha+\varepsilon}{2} \log \frac{1-r_{\lambda_1}^2}{1-|\varphi_{\lambda_1}(z)|^2},$$

and

$$\psi(z) = m_1 \log \frac{|\varphi_{\lambda_1}(z)|}{r_{\lambda_1}}.$$

More precisely, we have to show that $\psi \leq \varphi$ on $D(\lambda_1, r_1)$. For this, we use the auxiliary functions

$$\varphi : x \longmapsto \frac{\alpha+\varepsilon}{2} \log \frac{1-r_{\lambda_1}^2}{1-x^2},$$

and

$$\psi : x \longmapsto m_1 \log \frac{x}{r_{\lambda_1}}.$$

The function $\psi$ is concave, while $\varphi$ is convex and $\psi(r_{\lambda_1}) = \varphi(r_{\lambda_1}) = 0$. Moreover

$$\varphi'(r_{\lambda_1}) = \psi'(r_{\lambda_1}) = \sqrt{m_1(m_1+\alpha+\varepsilon)}$$

thus the two functions touch smoothly at $x = r_{\lambda_1}$, and $\psi \leq \varphi$ on $(0, r_{\lambda_1}]$. As a consequence $h_0 \leq h_1$ in $D(\lambda_1, r_{\lambda_1})$.

Now we construct $h_2$ in the same way as before. We have $h_1 = v_1 + U_0$, so we can write

$$h_1(z) = m_2 \log \frac{|\varphi_{\lambda_2}(z)|}{r_{\lambda_2}} + \underbrace{v_1 + U_0 - m_2 \log \frac{|\varphi_{\lambda_2}(z)|}{r_{\lambda_2}}}_{U_1}$$

$$= m_2 \log \frac{|\varphi_{\lambda_2}(z)|}{r_{\lambda_2}} + U_1,$$

where, since $f/(\varphi_{\lambda_1}^{m_1}\varphi_{\lambda_2}^{m_2})$ is holomorphic, $U_1$ is a subharmonic function that is harmonic in a small neighborhood of $\lambda_2$. Again we modify the term $m_2 \log \frac{|\varphi_{\lambda_2}(z)|}{r_{\lambda_2}}$ in the hyperbolic disc $D(\lambda_2, r_{\lambda_2})$, and set

$$v_2(z) = \begin{cases} \frac{\alpha+\varepsilon}{2} \log \frac{1-r_{\lambda_2}^2}{1-|\varphi_{\lambda_2}(z)|^2}, & z \in D(\lambda_2, r_{\lambda_2}) \\ \\ m_2 \log \frac{|\varphi_{\lambda_2}(z)|}{r_{\lambda_2}}, & \text{otherwise.} \end{cases}$$

And as in the first step, set $h_2 = v_2 + U_1$.
Iterating this procedure we obtain a sequence of sub-harmonic functions $(h_n)_n$, such that for every $z \in \mathbb{D}$, the sequence $(h_n(z))_n$ is increasing and

$$\log |f(z)| \leq h_n(z) \leq s(z).$$

So the pointwise limit $h$ of the sequence $(h_n)_n$, which is still subharmonic on $\mathbb{D}$, is comprised between $\log |f|$ and $s(z)$.

Observe that $h$ has been obtained from $\log |f|$ by replacing around each zero $\lambda$ of $f$ the function $m \log |\varphi_\lambda|/r_\lambda$ by $\frac{\alpha+\varepsilon}{2} \log(1 - r_\lambda^2)/(1 - |\varphi_\lambda|^2)$ and by a harmonic function far from the zeros so that the laplacian of $\log |f|$ is given by the sum of the laplacians of $v_k$.

Since $h(z) \leq s(z)$, by Green's formula ([HKZ00, Theorem 3.6, p. 59]), for $0 \leq r < 1$

$$\int_{D(0,r)} \Delta h(z) \log \frac{r}{|z|} dm(z) = -2h(0) + \frac{1}{r\pi} \int_{|z|=r} h(z) d|z|$$
$$\leq 2 \left( -h(0) + \log \|f\|_{\alpha,\infty} + \frac{\alpha}{2} \log \frac{1}{1-r^2} \right).$$

On the other hand,

$$\int_{D(0,r)} \Delta h(z) \log \frac{r}{|z|} dm(z) \geq 2(\alpha+\varepsilon) \int_{D(0,r)} \sum_{k\geq 1} \chi_{D_{\lambda_k}}(z) \log \frac{r}{|z|} d\mathcal{V}(z).$$

Hence,

$$\int_{D(0,r)} \sum_{k\geq 1} \chi_{D_{\lambda_k}}(z) \log \frac{r}{|z|} d\mathcal{V}(z) \leq \frac{\alpha}{2(\alpha+\varepsilon)} \log \frac{1}{1-r^2} + O(1), \qquad r \longrightarrow 1,$$

as required. □

We are now in a position to prove the uniqueness result.

**Theorem 2.2.2.** *Let $\alpha > 0$, $\varepsilon > 0$, and $X = \{(\lambda, m_\lambda)\}_{\lambda\in\Lambda}$ be a divisor such that*

$$\bigcup_{\lambda\in\Lambda} D\left(\lambda, \sqrt{\frac{m_\lambda}{\alpha+\varepsilon+m_\lambda}}\right) = \mathbb{D}\setminus K \tag{2.13}$$

*for some compact $K \subset \mathbb{D}$. Then $X$ is a uniqueness divisor for $\mathcal{A}^\infty_\alpha$.*

Before giving the proof of this result, we mention that Theorem 2.1.6 easily follows from this. Indeed, suppose $X$ *is* a zero divisor in $\mathcal{A}^2_\alpha \subset \mathcal{A}^\infty_{\alpha+2}$, then (2.13) does not hold for any compact $K$ where $\alpha$ is replaced by $\alpha+2$ as required in Theorem 2.1.6.

*Proof.* By contradiction, suppose $X$ is a zero divisor for $\mathcal{A}^\infty_\alpha$. Recall $D_\lambda = D\left(\lambda, \sqrt{\frac{m_\lambda}{\alpha+\varepsilon+m_\lambda}}\right)$. By Lemma 2.2.1,

$$c + \frac{\alpha}{2(\alpha+\varepsilon)}\log\frac{1}{1-r^2} \geq \int_{D(0,r)} \sum_{\lambda\in\Lambda}\chi_{D_\lambda}(z)\log\frac{r}{|z|}\frac{dm(z)}{(1-|z|^2)^2}. \tag{2.14}$$

Hence

$$\int_{D(0,r)} \sum_{\lambda\in\Lambda}\chi_{D_\lambda}(z)\log\frac{r}{|z|}d\mathcal{V}(z) = \int_{D(0,r)} \log\frac{r}{|z|}d\mathcal{V}(z)$$
$$+ \int_{D(0,r)\cap K}\left[\sum_{\lambda\in\Lambda}\chi_{D_\lambda}(z) - 1\right]\log\frac{r}{|z|}d\mathcal{V}(z) + \int_{D(0,r)\setminus K}\left[\sum_{\lambda\in\Lambda}\chi_{D_\lambda}(z) - 1\right]\log\frac{r}{|z|}d\mathcal{V}(z).$$

Notice that

(a) when $z \in D(0,r)\setminus K$, then $\sum_{\lambda\in\Lambda}\chi_{D_\lambda}(z) - 1 \geq 0$,

(b) integration inside $K$ only contributes at most as an additive (negative) constant,

so that

$$\int_{D(0,r)} \sum_{\lambda\in\Lambda}\chi_{D_\lambda}(z)\log\frac{r}{|z|}d\mathcal{V}(z) \geq \int_{D(0,r)} \log\frac{r}{|z|}d\mathcal{V}(z) + O(1) = \frac{1}{2}\log\frac{1}{1-r^2} + O(1).$$

But this is in contradiction with (2.14) which concludes the proof. □

We should stop here for a little observation. In the case of the Fock space, it was enough to consider the critical radius in order to get the corresponding uniqueness result. This was related to the observation that when we cover the whole plane by disks we will necessarily encounter "big" overlaps of these disks however far we are from the origin, and that in this case $\log R/|z|$ can be arbitrarily big. The situation changes in the unit disk where $\log R/|z|$ tends to zero, and much more subtle overlap conditions have to be discussed. For the purpose of our discussions here it is sufficient to play on the parameter $\varepsilon$. For that reason we have to impose a covering condition with smaller radii involving $\varepsilon$ (equivalently the critical radius gives the uniqueness result in all weighted Bergman spaces $\mathcal{A}^{\infty}_{\alpha'}$ with weight $\alpha' > \alpha$). It is easy to check that for given $\varepsilon > 0$ there exist $C_2 > C_1 > 0$ (actually $C_2 > \varepsilon > C_1 > 0$), such that for every $m_\lambda > C_2$,

$$\frac{m_\lambda - C_2}{m_\lambda + \alpha} \leq \frac{m_\lambda}{m_\lambda + \alpha + \varepsilon} \leq \frac{m_\lambda - C_1}{m_\lambda + \alpha}. \tag{2.15}$$

According to (2.15), playing on $C$, this is comparable to the same expression with $\varepsilon$ in the denominator, but no constant subtracted in the numerator. This will allow us to treat at the same time the situation with finite overlap (coming from sampling), or the separation condition (coming from interpolation). The value of $C$ will not be important in our estimates.

We finish this section with the short argument leading to (2.8). Denoting by $D_e(u, r) = \{z \in \mathbb{C} : |z - u| < r\}$ a Euclidean disk, we have (see [Gar07, p.4])

$$D(\lambda, r_\lambda) = D_e\left(\frac{1 - r_\lambda^2}{1 - r_\lambda^2|\lambda|^2}\lambda, \frac{1 - |\lambda|^2}{1 - r_\lambda^2|\lambda|^2} r_\lambda\right) = D_e(Z_\lambda, R_\lambda). \tag{2.16}$$

Using the finite overlap, an area argument shows that

$$\sum_{\lambda \in \Lambda} \left(\frac{1 - |\lambda|^2}{1 - r_\lambda^2|\lambda|^2} r_\lambda\right)^2 \lesssim \pi^2, \tag{2.17}$$

where

$$r_\lambda = \sqrt{\frac{m_\lambda - C}{m_\lambda + \alpha + 1}}.$$

The finite overlap implies that the Euclidean radii $R_\lambda$ tend to zero when $\lambda$ approaches the boundary $\partial\mathbb{D}$:

$$R_\lambda = \frac{1 - |\lambda|^2}{1 - r_\lambda^2|\lambda|^2} r_\lambda, \to 0, \quad \text{as} \quad |\lambda| \to 1. \tag{2.18}$$

Since $r_\lambda = \sqrt{\frac{m_\lambda - C}{m_\lambda + \alpha + 1}} \sim 1$, we have from (2.17)

$$\left(\frac{1 - |\lambda|^2}{1 - r_\lambda^2|\lambda|^2}\right)^2 \to 0, \text{ as } |\lambda| \to 1.$$

On the other hand

$$\begin{aligned} \frac{1 - |\lambda|^2}{1 - r_\lambda^2|\lambda|^2} &= (1 - |\lambda|^2)\frac{1}{1 - |\lambda|^2\left(\frac{m_\lambda - C}{m_\lambda + \alpha + 1}\right)} = (1 - |\lambda|^2)\frac{1}{1 - |\lambda|^2\left(1 - \frac{C + \alpha + 1}{m_\lambda + \alpha + 1}\right)} \\ &= \frac{1}{1 + \left(\frac{(C + \alpha + 1)|\lambda|^2}{(m_\lambda + \alpha + 1)(1 - |\lambda|^2)}\right)} \to 0, \text{ as } |\lambda| \to 1. \end{aligned}$$

Necessarily

$$(m_\lambda + \alpha + 1)(1 - |\lambda|^2) \sim m_\lambda(1 - |\lambda|^2) \to 0, \text{ as } |\lambda| \to 1,$$

and thus

$$\frac{1 - |\lambda|^2}{1 - r_\lambda^2|\lambda|^2} \simeq m_\lambda(1 - |\lambda|^2),$$

which in view of (2.17) leads to (2.8).

### 2.2.1 Lack of sampling and interpolating divisors in $\mathcal{A}_\alpha^2$

The result which affirm that there are no Riesz bases (simultaneously interpolating and sampling) in the Fock spaces, with Gaussian weight, was very expensive to obtain. First in [Sei93] for simple multiplicity, and later for uniformly bounded multiplicity in [BS93], and recently by the third and fourth authors in [Bor+17], even when the multiplicities go to infinity. However, the situation in Bergman spaces is different. The negative result on Riesz bases is very easy to obtain. The main reason is due to the fact that the multiplier algebra for standard Bergman space is not trivial and contains bounded analytic functions in unit disk. We present a brief proof below.

**Proposition 2.2.3.** *A divisor $X = \{(\lambda, m_\lambda)\}_{\lambda\in\Lambda}$ cannot be interpolating and sampling simultaneously for $\mathcal{A}_\alpha^2$.*

*Proof.*
Without loss of generality we assume that $\alpha = 0$. The proof goes as in the case of finite multiplicities $m_\lambda = 1$. The idea is, every interpolating sequence is also a zero sequence. We will see that the same phenomena remains valid even with higher multiplicities.
Suppose that $X = \{(\lambda, m_\lambda)\}_{\lambda\in\Lambda}$ is an interpolating divisor for $\mathcal{A}_\alpha^2$. Set

$$N^2_{\lambda,m_\lambda} := \{f \in \mathcal{A}_\alpha^2 : \partial_z^j f(\lambda) = 0, \quad \forall j < m_\lambda\}.$$

Let $\lambda' \in \Lambda$. We can interpolate special functions $\{f_\lambda\}_{\lambda\in\Lambda}$. Hence, there exists a function $f \in \mathcal{A}_\alpha^2$ such that

1. $f - 0 \in N^{2,\alpha}_{\lambda,m_\lambda}$, $\lambda \in \Lambda \setminus \{\lambda'\}$,
2. $f - 1 \in N^{2,\alpha}_{\lambda',m_{\lambda'}}$.

Thus

$$f \in \bigcap_{\lambda\in\Lambda\setminus\{\lambda'\}} N^2_{\lambda,m_\lambda}.$$

Put

$$F(z) = [\varphi_{\lambda'}(z)]^{m_{\lambda'}} f(z).$$

Since, the multiplier algebra of $\mathcal{A}_\alpha^2$ contains bounded analytic function, we get

(a) $F \in \bigcap_{\lambda \in \Lambda} N^{2,\alpha}_{\lambda, m_\lambda}$,

(b) $F \in \mathcal{A}^2_\alpha$.

Hence $X = \{(\lambda, m_\lambda)\}_{\lambda \in \Lambda}$ is a zero divisor for Bergman space $\mathcal{A}^2_\alpha$.
Furthermore, it is not hard to see that any sampling divisor is a uniqueness one. Indeed, assume $X = \{(\lambda, m_\lambda)\}_{\lambda \in \Lambda}$ is a sampling divisor for $\mathcal{A}^2_\alpha$ and let $f \in \mathcal{A}^2_\alpha$ with

$$\partial_z^j f(\lambda) = 0, \ 0 \leq j < m_\lambda, \ \lambda \in \Lambda.$$

In other words $f \in \bigcap_{\lambda \in \Lambda} N^{2,\alpha}_{\lambda, m_\lambda}$. We will figure out that

$$\langle f, T_\lambda e_k \rangle = 0, \ 0 \leq k < m_\lambda, \ \lambda \in \Lambda.$$

And

$$\sum_{\lambda \in \Lambda} \|f\|^2_{\mathcal{A}^2_\alpha / N^2_\lambda} = \sum_{k < m_\lambda} |\langle f, T_\lambda e_k \rangle|^2 = 0.$$

Let us compute these terms, $\langle f, T_\lambda e_k \rangle$, assume $\lambda \neq 0$,

$$\langle f, T_\lambda e_k \rangle = \langle T_\lambda f, e_k \rangle = \frac{1}{k! \|z^k\|_{2,\alpha}} \partial^k \left[ T_\lambda f(.) \right] \Big|_{z=0}.$$

Using The Faà di Bruno's formula, and the same computation leading to (2.7), we get again

$$\langle f, T_\lambda e_k \rangle = \sum_{j=0}^{k} c_{\alpha,j,k} (1 - |\lambda|^2) \partial_z^j f(\lambda)$$

where the coefficients $c_{\alpha,j,k}$ are independent of $f$ and $\lambda$. Therefore, if

$$\partial_z^j f(\lambda) = 0, \ 0 \leq j < m_\lambda, \ \lambda \in \Lambda.$$

Then

$$\sum_{k < m_\lambda} \langle f, T_\lambda e_k \rangle = 0, \ 0 \leq k < m_\lambda, \ \lambda \in \Lambda.$$

And the right sampling inequality

$$\|f\|^2_{\alpha,2} \lesssim \sum_{\lambda \in \Lambda} \sum_{j < m_\lambda} |\langle f, T_\lambda e_j \rangle|^2 .$$

Ensure that $f$ is the null function. Therefore, every sampling divisor is a uniqueness divisor also for $\mathcal{A}^2_\alpha$. □

## 2.3 Sampling and interpolation in $\mathcal{A}^2_\alpha$

### 2.3.1 Local $L^2$-estimates

We will obtain the results *via* a local control of the $\mathcal{A}^2_\alpha$-functions. Let us consider the $L^2$-norm in a domain $\Omega$, denoted by $\|\cdot\|_\Omega$. With this norm, we can obtain a control of the norm of the basis elements in terms of the regularized $\beta$-function.

For Re $a > 0$ and Re $b > 0$, we define the *$\beta$-function*

$$\beta(a,b) = \int_0^1 t^{a-1}(1-t)^{b-1}dt. \tag{2.19}$$

Recall that

$$\beta(a,b) = \frac{\Gamma(a)\Gamma(b)}{\Gamma(a+b)}, \tag{2.20}$$

and so by Stirling's formula (see [ASM65])

$$\beta(n+1,\alpha+1) \sim \frac{\Gamma(\alpha+1)}{n^{1+\alpha}}, \tag{2.21}$$

The following sharp inequality is sometimes useful (see [Gau59])

$$\left(\frac{x}{x+s}\right)^{1-s} \leq \frac{\Gamma(x+s)}{x^s\Gamma(x)} \leq 1, \;\; x>0, \;\; 0<s<1. \tag{2.22}$$

For $x \in (0,1]$, we define the *incomplete $\beta$-function*

$$\beta(x;a,b) = \int_0^x t^{a-1}(1-t)^{b-1}dt, \tag{2.23}$$

and the *regularized incomplete $\beta$-function* (or *regularized $\beta$-function* for short)

$$I(x;a,b) = \frac{\beta(x;a,b)}{\beta(a,b)}. \tag{2.24}$$

With these notations in mind we can compute the norm of the orthonormal basis $\{e_j\}_j$ on smaller disks (we will of course be interested in disks of type $D_\lambda$). For this we need the following result.

**Lemma 2.3.1.** *Let $\{e_j\}_j$ be the orthonormal basis for $\mathcal{A}^2_\alpha$. Then, for all $j \geq 0$ and $0<r<1$*

$$\|e_j\|^2_{D(0,r)} = \|e_j\chi_{D(0,r)}\|^2_{\alpha,2} = I(r^2; j+1, \alpha+1).$$

*Proof.* Recall first from (2.1) that

$$e_n(z) = \sqrt{\frac{\Gamma(n+2+\alpha)}{n!\Gamma(2+\alpha)}}z^n, \qquad n \geq 0.$$

Since $\Gamma(n) = (n-1)!$, and with (2.20) in mind, we can rewrite

$$\beta(n+1, \alpha+1) = \frac{n!\Gamma(1+\alpha)}{\Gamma(n+2+\alpha)} = \frac{n!\Gamma(2+\alpha)}{(1+\alpha)\Gamma(n+2+\alpha)},$$

and

$$e_n(z) = \sqrt{\frac{1}{(1+\alpha)\beta(n+1, \alpha+1)}} z^n, \qquad n \geq 0 \tag{2.25}$$

(which can be found also directly by computing the norm $\|z^n\|^2_{\alpha,2}$).

Now, the lemma follows from the following computation with the obvious substitution $s = t^2$,

$$\begin{aligned} \|z^j\|^2_{D(0,r)} &= (\alpha+1) \int_{D(0,r)} |z^j|^2 (1-|z|^2)^\alpha dm(z) = 2(\alpha+1) \int_0^r t^{2j+1}(1-t^2)^\alpha dt \\ &= (\alpha+1)\beta(r^2; j+1, \alpha+1). \end{aligned}$$

□

For our later discussions we will thus need estimates on the regularized incomplete $\beta$-function.

**Lemma 2.3.2.** *Let $\alpha > -1$ and $r_m = \sqrt{\frac{m}{m+\alpha+1}}$ then*

*(a) For every $c > 0$, there is $\varepsilon = \varepsilon(\alpha, c) > 0$ such that for all $m \geq c+1$ and $n < m$*

$$I\left(\frac{m-c}{m+\alpha+1}; n+1, \alpha+1\right) \geq \varepsilon.$$

*(b) For $t < r_m$, there exists $\varepsilon = \varepsilon(\alpha)$ such that*

$$F_{m,\alpha}(t) = \left[1-t^2\right]^{\alpha+2} \sum_{0 \leq j < m} \frac{t^{2j}}{(\alpha+1)\beta(j+1, \alpha+1)} \geq \varepsilon.$$

*(c) Given $0 < \eta < 1$, there exists $a_\alpha(\eta) > 0$ such that for all $j \geq m \geq a_\alpha(\eta)$*

$$I\left(\frac{m-a(\eta)}{m+\alpha+1}; j+1, \alpha+1\right) \leq \eta I\left(\frac{m}{m+\alpha+1}; j+1, \alpha+1\right).$$

The lemma does not appeal to a specific zero divisor. Later on, $m$ will correspond to $m_\lambda$ and $r_m$ to $r_\lambda$. Compare the critical radius $r_m$ appearing in this lemma with the one given in (2.10) (as mentioned there, the critical value corresponds to the situation when $\varepsilon = 0$): the term $\alpha$ appearing there turns into $\alpha+1$ here.

*Proof.* (a) Since $\beta(x, n+1, \alpha+1)$ is increasing in $x$, for $m > n$

$$I\left(\frac{m-c}{m+\alpha+1}; n+1, \alpha+1\right) \geq I\left(\frac{n-c}{n+\alpha+1}; n+1, \alpha+1\right).$$

We need to treat two cases.
First, suppose $\alpha > 0$, then

$$\begin{aligned}\beta\left(\frac{n-c}{n+\alpha+1}; n+1, \alpha+1\right) &= \int_0^{\frac{n-c}{n+\alpha+1}} t^n(1-t)^\alpha dt \\ &\geq \left(1-\frac{n-c}{n+\alpha+1}\right)^\alpha \int_0^{\frac{n-c}{n+\alpha+1}} t^n dt \\ &\geq \left(\frac{\alpha+1+c}{n+\alpha+1}\right)^\alpha \frac{1}{n+1}\left(1-\frac{\alpha+1+c}{n+\alpha+1}\right)^{n+1},\end{aligned}$$

which is comparable to $\dfrac{1}{(n+1)^{(\alpha+1)}}$ for $n \geq c+1$ (uniformly in $n$ and $m$). Now, By (2.21), this yields inequality (a) at least when $n \geq c+1$.

Second, suppose $\alpha \in (-1, 0]$. Note that for $r \in (0,1), n \geq 0$ we have

$$\begin{aligned}\int_0^r t^n(1-t)^\alpha dt &= \frac{r^{n+1}}{n+1}(1-r)^\alpha + \frac{\alpha}{n+1}\int_0^r t^{n+1}(1-t)^{\alpha-1}dt \\ &\geq \frac{r^{n+1}}{n+1}(1-r)^\alpha + \frac{\alpha}{n+1}\frac{r}{1-r}\int_0^r t^n(1-t)^\alpha dt,\end{aligned}$$

where we have also used the fact that $\alpha r/(1-r)$ is decreasing in $r$ since $\alpha < 0$. Hence

$$\int_0^r t^n(1-t)^\alpha dt \geq \frac{1}{1+\frac{-\alpha}{n+1}\frac{r}{1-r}}\frac{r^{n+1}}{n+1}(1-r)^\alpha. \tag{2.26}$$

Let $r = \frac{n-c}{n+\alpha+1}$ (the interesting case being $n \geq c+1$). In this situation we have

$$1+\frac{-\alpha}{n+1}\times\frac{r}{1-r} = 1+\frac{-\alpha}{\alpha+1+c}\times\frac{n-c}{n+1} \leq 1+\frac{-\alpha}{\alpha+1+c}.$$

Thus, from (2.26)

$$\begin{aligned}\int_0^{\frac{n-c}{n+\alpha+1}} t^n(1-t)^\alpha dt &\geq \frac{1}{[1+\frac{-\alpha}{\alpha+1+c}]}\times\frac{1}{(n+1)}\times\left(\frac{\alpha+1+c}{n+\alpha+1}\right)^\alpha\times\left(\frac{n-c}{n+\alpha+1}\right)^{n+1} \\ &\geq \frac{(\alpha+1+c)^\alpha e^{-(\alpha+1+c)(2+c)}}{[1+\frac{-\alpha}{\alpha+1+c}]}\times\frac{1}{\left(n+1\right)^{\alpha+1}}.\end{aligned}$$

at the second inequality we have used $\frac{t-1}{t} \leq \log t,\ t > 0$.
On the other hand, by (2.22) we get

$$\frac{1}{\beta(n+1,\alpha+1)} = \frac{\Gamma(n+1+\alpha+1)}{\Gamma(\alpha+1)\Gamma(n+1)} \geq \frac{\left(n+1\right)^{\alpha+1}}{\Gamma(\alpha+1)}.$$

Put

$$\varepsilon = \varepsilon(\alpha, c) := \frac{1}{\Gamma(\alpha+1)}\times\frac{(\alpha+1+c)^\alpha e^{-(\alpha+1+c)(2+c)}}{[1+\frac{-\alpha}{\alpha+2+c}]}.$$

Hence it follows from the inequalities above

$$I\left(\frac{n-c}{n+\alpha+2}; n+1, \alpha+1\right) = \frac{\int_0^{\frac{n-c}{n+\alpha+2}} t^n(1-t)^\alpha dt}{\beta(n+1,\alpha+1)} \geq \varepsilon, \tag{2.27}$$

which is the desired result.
Finally, for $n < c+1$, independently whether $\alpha \geq 0$ or $\alpha \in (-1,0)$, the desired estimate follows from the fact that the integration interval $[0, \frac{m-c}{m+\alpha+1}]$ of the incomplete $\beta$-function contains a fixed interval $[0, \frac{1}{c+\alpha+2}]$ and both powers of $t$ and $(1-t)$ appearing in the definition of $\beta(x; n+1, \alpha+1)$ are controlled. Dividing by $\beta(n+1, \alpha+1)$ does not change this control since $n$ is bounded.

(b) Recall that the reproducing kernel of $\mathcal{A}^2_\alpha$, $K(t,t) = (1-t^2)^{-\alpha-2}$ satisfies also

$$K(t,t) = \sum_{j\geq 0} \frac{t^{2j}}{(\alpha+1)\beta(j+1,\alpha+1)}, \qquad 0 \leq t < 1$$

Hence, it suffices to prove that there exists $\epsilon = \epsilon(\alpha)$ and $m_0$ such that for $t < r_m$

$$R_{m,\alpha}(t) = (1-t^2)^{\alpha+2} \sum_{j\geq m} \frac{t^{2j}}{(\alpha+1)\beta(j+1,\alpha+1)} \leq 1-\epsilon, \qquad m \geq m_0,$$

By (2.21),

$$\frac{1}{(\alpha+1)\beta(j+1,\alpha+1)} = \frac{1}{(\alpha+1)\Gamma(\alpha+1)/j^{\alpha+1}(1+o_j(1))} = \frac{j^{1+\alpha}}{\Gamma(2+\alpha)}(1+o_j(1)),$$

where $o_j(1)$ tends to zero when $j$ tends to infinity. So

$$R_{m,\alpha}(t) = \frac{(1-t^2)^{\alpha+2}}{\Gamma(\alpha+2)} \sum_{j\geq m} j^{1+\alpha} t^{2j}(1+o_j(1)) \leq \frac{(1-t^2)^{\alpha+2}}{\Gamma(\alpha+2)}(1+o_m(1)) \sum_{j\geq m} j^{1+\alpha} t^{2j}.$$

We will pass to an integral. For that, note that when $x \in [j-1, j]$ and $t \in [0,1)$, we have $t^{2j} \leq t^{2x}$, and obviously $j^{1+\alpha} = x^{1+\alpha}(1+o_j(1))$. We deduce

$$\begin{aligned} R_{m,\alpha}(t) &\leq (1+o_m(1))\frac{(1-t^2)^{\alpha+2}}{\Gamma(\alpha+2)} \int_{m-1}^{\infty} x^{1+\alpha} t^{2x} dx \\ &= (1+o_m(1))\frac{(1-t^2)^{\alpha+2}}{\Gamma(\alpha+2)(\log 1/t^2)^{2+\alpha}} \int_{(m-1)(\log 1/t^2)}^{\infty} u^{1+\alpha} e^{-u} du. \end{aligned}$$

Note that the function $t \longmapsto (1-t^2)/\ln(1/t^2)$ is bounded by 1 on $(0,1)$. Also, observe that $0 < t < r_m = \sqrt{\frac{m}{m+\alpha+1}}$, and hence

$$\log\frac{1}{t^2} > \log\frac{m+\alpha+1}{m} = \log(1+\frac{\alpha+1}{m+\alpha+1}) = \frac{\alpha+1}{m+\alpha+1}(1+o_m(1)).$$

We deduce

$$R_{m,\alpha}(t) \;\leq\; \frac{1+o_m(1)}{\Gamma(\alpha+2)} \int_{\frac{(\alpha+1)(m-1)}{m+\alpha+1}(1+o_m(1))}^{\infty} u^{1+\alpha} e^{-u} du.$$

Let $\beta = \frac{(\alpha+1)(m-1)}{m+\alpha+1}(1+o_m(1))$, then using the notation $\Gamma(a,b) := \int_b^\infty t^{a-1}e^{-t}dt$ for the incomplete Gamma function, we have

$$R_{m,\alpha}(t) \leq \frac{1+o_m(1)}{\Gamma(\alpha+2)}\Gamma(\alpha+2,\beta).$$

Now observe that for finite $m$ the estimate in (b) is trivially true. We can thus assume that $m$ is big enough so that $\beta \geq \alpha+1-1 = \alpha$. So

$$R_{m,\alpha}(t) \leq \frac{1+o_m(1)}{\Gamma(\alpha+2)}\Gamma(\alpha+2,\alpha).$$

Since $\alpha$ is fixed, we obviously have $q = \Gamma(\alpha+2,\alpha)/\Gamma(\alpha+2) < 1$, and for sufficiently big $m$ we have $1+o_m(1) < (1+q)/(2q)$ implying that $R_{m,\alpha} \leq (1+q)/2 < 1$, and we can pick $\varepsilon = (1-q)/2$.

(c) Clearly, setting

$$\rho = \frac{m-a(\eta)}{m+\alpha+1}, \quad r = \frac{m}{m+\alpha+1},$$

it is enough to show the estimate for the incomplete $\beta$-function:

$$\begin{aligned} \beta\left(\frac{m-a(\eta)}{m+\alpha+1}; j+1, \alpha+1\right) &= \int_0^\rho t^j(1-t)^\alpha dt \\ &\leq \eta \int_0^r t^j(1-t)^\alpha dt \\ &= \eta\beta\left(\frac{m}{m+\alpha+1}; j+1, \alpha+1\right) \end{aligned}$$

First <u>assume $\alpha > 0$</u>.
Note that

$$\beta_2 := \int_0^r t^j(1-t)^\alpha dt \geq (1-r)^\alpha \frac{r^{j+1}}{j+1}, \tag{2.28}$$

and an integration by part, as well as the the fact that $t/(1-t)$ is increasing in $t$, yield

$$\begin{aligned} \int_0^\rho t^j(1-t)^\alpha dt &= \frac{\rho^{j+1}}{j+1}(1-\rho)^\alpha + \frac{\alpha}{j+1}\int_0^\rho t^{j+1}(1-t)^{\alpha-1}dt \\ &\leq \frac{\rho^{j+1}}{j+1}(1-\rho)^\alpha + \frac{\alpha}{j+1}\frac{\rho}{1-\rho}\int_0^\rho t^j(1-t)^\alpha dt. \end{aligned}$$

So, if $\frac{\alpha}{j+1}\frac{\rho}{1-\rho} < 1$, we get

$$\beta_1 := \int_0^\rho t^j(1-t)^\alpha dt \le \frac{1}{1-\frac{\alpha}{j+1}\frac{\rho}{1-\rho}}\frac{\rho^{j+1}}{j+1}(1-\rho)^\alpha. \tag{2.29}$$

Given $0 < \eta < 1$ and $a > 0$, it follows from (2.28) and (2.29)

$$\frac{\beta_1}{\beta_2} \le \frac{1}{1-\frac{\alpha}{j+1}\frac{\rho}{1-\rho}}\left(\frac{1-\rho}{1-r}\right)^\alpha\left(\frac{\rho}{r}\right)^{j+1} \tag{2.30}$$

Simple computations show

$$\frac{\rho}{r} = \frac{m-a(\eta)}{m} = 1-\frac{a(\eta)}{m} < 1,$$

and

$$\frac{1-\rho}{1-r} = \frac{\alpha+1+a(\eta)}{\alpha+1} \quad \text{and} \quad \frac{\rho}{1-\rho} = \frac{m-a(\eta)}{\alpha+1+a(\eta)}.$$

Now for $j \ge m \ge a(\eta)$,

$$\frac{\alpha}{j+1}\frac{\rho}{1-\rho} = \frac{\alpha}{j+1}\frac{m-a(\eta)}{\alpha+1+a(\eta)} \le \frac{\alpha m}{(m+1)(\alpha+1)+(m+1)a(\eta)} \le \frac{\alpha}{\alpha+1+a(\eta)},$$

therefore,

$$\frac{1}{1-\frac{\alpha}{j+1}\frac{m-a(\eta)}{\alpha+1+a(\eta)}} \le \frac{\alpha+1+a(\eta)}{1+a(\eta)}.$$

It remains the term

$$\left(\frac{\rho}{r}\right)^{j+1} = \left(1-\frac{a(\eta)}{m}\right)^{j+1} \le \left(1-\frac{a(\eta)}{m}\right)^m \le e^{-a(\eta)} \tag{2.31}$$

Hence

$$\frac{\beta_1}{\beta_2} \le \frac{\alpha+1+a(\eta)}{1+a(\eta)} \times \left(1+\frac{a(\eta)}{1+\alpha}\right)^\alpha \times e^{-a(\eta)}.$$

Since the exponential decrease dominates the polynomial growth, we have

$$\lim_{a\to+\infty} \frac{\alpha+1+a}{1+a} \times \left(1+\frac{a}{1+\alpha}\right)^{\alpha+1} \times e^{-a} = 0,$$

which proves the claim: for every $\eta > 0$, we can find $a$, such that $\beta_1/\beta_2 < \eta$ independently on $j \ge m \ge a$, and the same is true for $I_1/I_2$.

Second, assume $\alpha \in (-1, 0]$. The exact same arguments as in the case $\alpha > 0$ allow to reverse the inequalities in (2.28) and (2.29) (since $\alpha$ is negative, the expression $\alpha r/(1-r)$ is decreasing and $(1-r)^\alpha$ is increasing in $r$). Hence,

$$\beta_2 := \int_0^r t^j(1-t)^\alpha dt \geq \frac{1}{1+\frac{-\alpha}{j+1}\frac{r}{1-r}} \frac{r^{j+1}}{j+1}(1-r)^\alpha, \tag{2.32}$$

and

$$\beta_1 := \int_0^\rho t^j(1-t)^\alpha dt \leq \ (1-\rho)^\alpha \frac{\rho^{j+1}}{j+1}. \tag{2.33}$$

As in (2.30) for $\alpha > 0$, it follows from (2.32) and (2.33)

$$\frac{\beta_1}{\beta_2} \leq \left(1 - \frac{\alpha}{j+1}\frac{r}{1-r}\right)\left(\frac{1-\rho}{1-r}\right)^\alpha \left(\frac{\rho}{r}\right)^{j+1},$$

where the first factor is now in the numerator instead of the denominator. Hence, with $\rho = \frac{m-a(\eta)}{m+\alpha+1}$ and $r = \frac{m}{m+\alpha+1}$ in mind, we get as before

$$\frac{\beta_1}{\beta_2} \leq \left[1 + \frac{-\alpha}{j+1} \times \frac{m}{\alpha+1}\right] \times \left(1 + \frac{a(\eta)}{\alpha+1}\right)^\alpha \times \left(1 - \frac{a(\eta)}{m}\right)^{j+1}.$$

Since $j+1 > m$ and $\alpha < 0$, we get

$$1 + \frac{-\alpha}{j+1} \times \frac{m}{\alpha+1} \leq 1 + \frac{-\alpha}{\alpha+1} = \frac{1}{1+\alpha}.$$

Again

$$\left(1 - \frac{a(\eta)}{m}\right)^{j+1} \leq e^{-a},$$

and now

$$\left(1 + \frac{a(\eta)}{\alpha+1}\right)^\alpha < 1,$$

so that

$$\frac{\beta_1}{\beta_2} \leq \frac{1}{1+\alpha}e^{-a},$$

which again goes to 0, proving the claim also in this case. □

As mentioned earlier, we will need to switch form the orthonormal basis $\{e_n\}_n$ in $\mathcal{A}^2_\alpha$ to an orthonormal basis on a smaller disk $D(0,r)$. The following lemma recalls this simple fact.

**Lemma 2.3.3.** *Let* $f = \sum_{n\geq 0} a_n e_n \in \mathcal{A}^2_\alpha$. *Then*

$$(\alpha+1)\int_{D(0,r)} |f(z)|^2(1-|z|^2)^\alpha dm(z) = \sum_{n\geq 0} I(r^2, n+1, \alpha+1)|a_n|^2,$$

*for every* $r \in (0,1)$

*Proof.* Set
$$f_n = \frac{e_n}{\sqrt{I(r^2, n+1, \alpha+1)}},$$
then, by Lemma (2.3.1), $(f_n)$ is orthonormal with respect to the measure $dA_\alpha$ on $D(0, r)$. Writing now
$$f(z) = \sum_{n\geq 0} a_n e_n = \sum_{n\geq 0} (a_n \sqrt{I(r^2, n+1, \alpha+1)}) f_n,$$
we get the required equality.

□

**Lemma 2.3.4.** *Let $f \in \mathcal{A}^2_\alpha$, for every $c > 0$ there exist constants $A = A(c) > 0$ such that for every $m \geq c+1$, we have*
$$\sum_{j<m} |\langle f, T_\lambda e_j\rangle|^2 \leq A(\alpha+1) \int_{D\left(\lambda, \sqrt{\frac{m-c}{m+\alpha+1}}\right)} |f(z)|^2 (1-|z|^2)^\alpha dm(z),$$
*for any $\lambda \in \mathbb{D}$.*

*Proof.* Set $r = r_{m,\alpha,c} := \sqrt{\frac{m-c}{m+\alpha+1}}$. Taking $g = T_\lambda f$, the statement can be rewritten as
$$\sum_{0\leq j<m} |\langle g, e_j\rangle|^2 \lesssim (\alpha+1) \int_{D(0,r)} |g(z)|^2 (1-|z|^2)^\alpha dm(z).$$
Since $g \in \mathcal{A}^2_\alpha$,
$$g(z) = \sum_{j\geq 0} a_j e_j = \sum_{j\geq 0} a_j \frac{z^j}{\sqrt{(\alpha+1)\beta(j+1, \alpha+1)}}$$
and
$$|\langle g, e_j\rangle| = |a_j|^2, \; j \geq 0.$$
Using Lemma 2.3.3 applied to $g$, the claim is equivalent to
$$\sum_{0\leq j<m} |a_j|^2 \lesssim \sum_{j\geq 0} |a_j|^2 I\left(\frac{m-c}{m+\alpha+1}; j+1, \alpha+1\right),$$
where we have substituted back the value of $r$. Therefore it is enough to prove
$$I\left(\frac{m-c}{m+\alpha+1}; j+1, \alpha+1\right) \geq \varepsilon(\alpha, c) > 0$$
for $j < m$ and $m \geq c+1$, but this is given by Lemma 2.3.2(a). □

The next lemma relates the finite overlap condition to a kind of Carleson measure type condition.

**Lemma 2.3.5.** *Let $X = \{(\lambda, m_\lambda)\}_{\lambda\in\Lambda}$ be a divisor. Then $X$ satisfies the finite overlap condition if and only if there exists a constant $C > 0$ satisfying*

$$\sum_{\lambda\in\Lambda}\sum_{j<m_\lambda} |\langle f, T_\lambda e_j\rangle|^2 \le C\|f\|_{\alpha,2}^2, \quad f \in \mathcal{A}_\alpha^2.$$

*Proof.* Suppose that the estimate holds. Given $z \in \mathbb{D}$, set $f_z = T_z 1$, observe that $|T_z 1| = |k_z|$, so $\|f_z\|_{\alpha,2} = 1$ and

$$\begin{aligned}|\langle f_z, T_\lambda e_j\rangle|^2 &= |\langle T_z 1, T_\lambda e_j\rangle|^2 = |\langle e_0, T_z T_\lambda e_j\rangle|^2 = |T_z T_\lambda e_j(0)|^2 \\ &= \frac{1}{(\alpha+1)\beta(j+1,\alpha+1)}\left[1-|\varphi_\lambda(z)|^2\right]^{2+\alpha}|\varphi_\lambda(z)|^{2j}, \end{aligned} \tag{2.34}$$

where we have used the form (2.25) of $e_j$. Then, by assumption,

$$\begin{aligned}1 = \|f\|_{\alpha,2}^2 &\gtrsim \sum_{\lambda\in\Lambda}\sum_{j<m_\lambda}\frac{1}{(\alpha+1)\beta(j+1,\alpha+1)}\left[1-|\varphi_\lambda(z)|^2\right]^{2+\alpha}|\varphi_\lambda(z)|^{2j} \\ &\ge \sum_{\lambda\in\Lambda} F_{m_\lambda,\alpha}(|\varphi_\lambda(z)|)\chi_{D\left(\lambda,\sqrt{\frac{m_\lambda}{m_\lambda+\alpha+1}}\right)}(z) \ge \epsilon\sum_{\lambda\in\Lambda}\chi_{D\left(\lambda,\sqrt{\frac{m_\lambda}{m_\lambda+\alpha+1}}\right)}(z),\end{aligned}$$

where the function $F$ is defined in Lemma 2.3.2(b) and the last bound also comes from that result. Since the constants do not depend on $z$, we conclude that

$$1 \gtrsim \sum_{\lambda\in\Lambda}\chi_{D\left(\lambda,\sqrt{\frac{m_\lambda}{m_\lambda+\alpha+1}}\right)}(z).$$

In the opposite direction, if $X$ satisfies the finite overlap condition, we just apply Lemma 2.3.4, which gives

$$\begin{aligned}\sum_{\lambda\in\Lambda}\sum_{j<m_\lambda}|\langle f, T_\lambda e_j\rangle|^2 &\lesssim \sum_{\lambda\in\Lambda}(\alpha+1)\int_{D\left(\lambda,\sqrt{\frac{m_\lambda}{m_\lambda+\alpha+1}}\right)}|f(z)|^2(1-|z|^2)^\alpha dm(z) \\ &= \sum_{\lambda\in\Lambda}(\alpha+1)\int_{\mathbb{D}}\chi_{D\left(\lambda,\sqrt{\frac{m_\lambda}{m_\lambda+\alpha+1}}\right)}(z)|f(z)|^2(1-|z|^2)^\alpha dm(z) \\ &\le S_X\|f\|_{\alpha,2}^2,\end{aligned}$$

where $S_X$ is the overlap constant introduced in Definition (2.1.3).

□

### 2.3.2 Sampling for $\mathcal{A}_\alpha^2$

In order to obtain our geometric conditions we need the following local control of $\mathcal{A}_\alpha^2$-functions.

**Lemma 2.3.6.** *Given $0 < \eta \leq 1$ there exists $a(\eta) > 0$ such that if $f \in \mathcal{A}^2_\alpha$, and $m \geq a(\eta)$, and if*

$$\sum_{j<m} |\langle f, e_j\rangle|^2 \leq \eta/2,$$

$$(\alpha+1)\int_{D\left(0,\sqrt{\frac{m}{m+\alpha+1}}\right)} |f(z)|^2(1-|z|^2)^\alpha dm(z) \leq 1,$$

*then*

$$(\alpha+1)\int_{D\left(0,\sqrt{\frac{m-a(\eta)}{m+\alpha+1}}\right)} |f(z)|^2(1-|z|^2)^\alpha dm(z) \leq \eta.$$

As we will see from the proof, we have $a(\eta) = a_\alpha(\eta/2)$ where $a_\alpha$ appears in Lemma 2.3.2(c).

*Proof.* If we write

$$f(z) = \sum_j a_j e_j(z),$$

by the first assumption and (2.1), using the orthogonality of $\{e_n\}_n$ with respect to the measure $dA_\alpha$ on any disk $D(0,r)$, $0 < r < 1$, and Lemma 2.3.3,

$$\begin{aligned}
&(\alpha+1)\int_{D\left(0,\sqrt{\frac{m-a(\eta)}{m+\alpha+1}}\right)} |f(z)|^2(1-|z|^2)^\alpha dm(z) \\
&= \sum_{j\geq 0} |a_j|^2 I\left(\frac{m-a(\eta)}{m+\alpha+1}; j+1, \alpha+1\right) \\
&\leq \sum_{j<m} |a_j|^2 + \sum_{j\geq m} |a_j|^2 I\left(\frac{m-a(\eta)}{m+\alpha+1}; j+1, \alpha+1\right) \\
&\leq \frac{\eta}{2} + \sum_{j\geq m} |a_j|^2 I\left(\frac{m-a(\eta)}{m+\alpha+1}; j+1, \alpha+1\right).
\end{aligned}$$

Now, by Lemma 2.3.2 (c) for $j \geq m \geq a_\alpha(\eta/2)$, another application of Lemma 2.3.3 and the hypothesis,

$$\begin{aligned}
\sum_{j\geq m} |a_j|^2 I\left(\frac{m-a(\eta/2)}{m+\alpha+1}; j+1, \alpha+1\right) &\leq \frac{\eta}{2}\sum_{j\geq m} |a_j|^2 I\left(\frac{m}{m+\alpha+1}; j+1, \alpha+1\right) \\
&\leq (1+\alpha)\frac{\eta}{2}\int_{D\left(0,\sqrt{\frac{m}{m+\alpha+1}}\right)} |f(z)|^2(1-|z|^2)^\alpha dm(z) \leq \frac{\eta}{2},
\end{aligned}$$

and the result follows. □

We are now in a position to prove Theorem 2.1.4 which we restate here for convenience.

**Theorem.** *(a) If $X$ is a sampling divisor for $\mathcal{A}^2_\alpha$, then $X$ satisfies the finite overlap condition and there exists $0 < C < \alpha + 1$ such that*

$$\bigcup_{\lambda\in\Lambda} D\left(\lambda, \sqrt{\frac{m_\lambda + C}{m_\lambda + \alpha + 1}}\right) = \mathbb{D}.$$

*(b) Conversely, let the divisor $X$ satisfy the finite overlap condition and if there exists $C = C(S_X) > 0$ such that for some compact $K$ of $\mathbb{D}$ we have*

$$\bigcup_{\lambda\in\Lambda, m_\lambda > C} D\left(\lambda, \sqrt{\frac{m_\lambda - C}{m_\lambda + \alpha + 1}}\right) = \mathbb{D} \setminus K,$$

*then $X$ is a sampling divisor for $\mathcal{A}^2_\alpha$.*

*Proof. Necessary Part (a)*

Let $X = \{(\lambda, m_\lambda)\}_{\lambda\in\Lambda}$ be a sampling divisor. By Lemma 2.3.5, it satisfies the finite overlap condition. Suppose that for every $0 < C < \alpha + 1$

$$\bigcup_{\lambda\in\Lambda} D\left(\lambda, \sqrt{\frac{m_\lambda + C}{m_\lambda + \alpha + 1}}\right) \neq \mathbb{D}.$$

Then there exists a sequence $0 < C_j \uparrow (\alpha + 1)$ and $z_j \in \mathbb{D}$ such that

$$z_j \notin \bigcup_{\lambda\in\Lambda} D\left(\lambda, \sqrt{\frac{m_\lambda + C_j}{m_\lambda + \alpha + 1}}\right).$$

Put $r_{\lambda,C_j} = \sqrt{\frac{m_\lambda + C_j}{m_\lambda+\alpha+1}}$ and $r_\lambda = \sqrt{\frac{m_\lambda}{m_\lambda+\alpha+1}}$. Thus

$$z_j \in \mathbb{D} \setminus \left[\bigcup_{\lambda\in\Lambda} D\left(\lambda, r_{\lambda,C_j}\right)\right] \subset \mathbb{D} \setminus \left[\bigcup_{\lambda\in\Lambda} D\left(\lambda, r_\lambda\right)\right].$$

Let

$$\zeta \in \bigcup_{\lambda\in\Lambda} D\left(\lambda, r_\lambda\right)$$

that is $\zeta \in D\big(\lambda_0, r_{\lambda_0}\big)$ for some $\lambda_0 \in \Lambda$ and we have for all $j$

$$\rho(\zeta, \lambda_0) < r_{\lambda_0}, \qquad \text{and} \qquad \rho(z_j, \lambda_0) > r_{\lambda_0,C_j}.$$

By the triangular inequality for the pseudohyperbolic metric (see [Gar07, p.4]), we get

$$\begin{aligned} \rho(z_j, \zeta) &\geq \frac{\rho(z_j, \lambda_0) - \rho(\zeta, \lambda_0)}{1 - \rho(z_j, \lambda_0)\rho(\zeta, \lambda_0)} \geq \frac{r_{\lambda_0,C_j} - r_{\lambda_0}}{1 - r_{\lambda_0,C_j} r_{\lambda_0}} \\ &\geq \frac{(r_{\lambda_0,C_j})^2 - (r_{\lambda_0})^2}{1 - (r_{\lambda_0} r_{\lambda_0,C_j})^2} \geq \frac{C_j m_{\lambda_0} + C_j(\alpha + 1)}{(2\alpha + 2 - C_j)m_{\lambda_0} + (\alpha + 1)^2} \to 1, \qquad j \to \infty, \end{aligned}$$

and this latter convergence is uniform in $m$. Since we have chosen $\zeta$ arbitrary in $\bigcup_{\lambda\in\Lambda} D(\lambda, r_\lambda)$, there exists a sequence $(z_j)_j \subset \mathbb{D}$ such that :

$$\rho_j := \operatorname{dist}\left(z_j, \bigcup_{\lambda\in\Lambda} D\big(\lambda, r_\lambda\big)\right) \to 1.$$

Set $f_j = T_{z_j}1$, observe that $|T_{z_j}1| = |k_{z_j}|$, and applying Lemma 2.3.4 we obtain

$$\begin{aligned}
\sum_{\lambda\in\Lambda}\sum_{k<m_\lambda} |\langle f_j, T_\lambda e_k\rangle|^2 &\lesssim \sum_{\lambda\in\Lambda}(\alpha+1)\int_{D\left(\lambda,\sqrt{\frac{m_\lambda}{m_\lambda+\alpha+1}}\right)} \frac{(1-|z_j|^2)^{2+\alpha}}{|1-\overline{z_j}\zeta|^{2(2+\alpha)}}(1-|\zeta|^2)^\alpha dm(\zeta) \\
&\lesssim (\alpha+1)S_X\int_{\cup_{\lambda\in\Lambda}D\left(\lambda,\sqrt{\frac{m_\lambda}{m_\lambda+\alpha+1}}\right)} \left[1-|\varphi_{z_j}(\zeta)|^2\right]^\alpha |\varphi_{z_j}(\zeta)|^2 dm(\zeta) \\
&\lesssim (\alpha+1)\int_{|w|\geq\rho_j}(1-|w|^2)^\alpha dm(w) \to 0,
\end{aligned}$$

since $\rho_j \to 1$. This contradicts the sampling inequality.

*Sufficiency part (b)*

Suppose the divisor is not sampling. Then there exists a sequence $(f_n)_{n\geq1}$ such that $\|f_n\|_{\alpha,2} = 1$ and

$$\sum_{\lambda\in\Lambda}\sum_{0\leq j<m_\lambda} |\langle f_n, T_\lambda e_j\rangle|^2 \to 0 \quad \text{as } n\to\infty.$$

Passing to a weakly convergent sub-sequence denoted again by $(f_n)_{n\geq1}$ we have two possibilities: either (i) $(f_n)$ converges weakly to $f \neq 0$ or (ii) $(f_n)$ converges weakly to 0.

In the case (i), $X$ is a zero divisor for a function $f \in \mathcal{A}^2_\alpha$. By Theorem 2.1.6, for every $\varepsilon > 0$,

$$\mathbb{D} \setminus \bigcup_{\lambda\in\Lambda} D\left(\lambda, \sqrt{\frac{m_\lambda}{\alpha+2+\varepsilon+m_\lambda}}\right)$$

cannot be compact. Similarly as in (2.15), given $\varepsilon > 0$ for every $C_1 > \varepsilon + 1 > 0$ such that for every $m \geq 1$,

$$\frac{m - C_1}{m+\alpha+1} \leq \frac{m}{m+\alpha+\varepsilon+2}$$

Hence

$$D\left(\lambda, \sqrt{\frac{m_\lambda - C_1}{m_\lambda+\alpha+1}}\right) \subset D\left(\lambda, \sqrt{\frac{m_\lambda}{m_\lambda+\alpha+2+\varepsilon}}\right),$$

for every $\lambda \in \Lambda$ with $m_\lambda > C_1$. And

$$\mathbb{D} \setminus \bigcup_{\lambda\in\Lambda, m_\lambda > C_1} D\left(\lambda, \sqrt{\frac{m_\lambda - C_1}{m_\lambda+\alpha+1}}\right)$$

cannot be compact, leading to a contradiction.

In the second case (ii), we define $\eta = \frac{1}{S_X+1}$ and set the constant $C = a(\eta)$ where $a(\eta)$ is given by $\leq C(S_X)$ ($C$ to be fixed later in agreement with $\eta$ in l in Lemma 2.3.6. Denote

$$\Lambda_1 = \{\lambda \in \Lambda : m_\lambda - C > 0\}.$$

In order to reach a contradiction we will assume that the disks $D\left(\lambda, \sqrt{\frac{m_\lambda - a(\eta)}{m_\lambda+\alpha+1}}\right)$, $\lambda \in \Lambda_1$, cover the unit disk up to a compact set, i.e. there is $R = R(\eta) \in [0,1)$ such that

$$\mathbb{D} \setminus D(0,R) \subset \bigcup_{\lambda\in\Lambda_1} D\left(\lambda, \sqrt{\frac{m_\lambda - a(\eta)}{m_\lambda+\alpha+1}}\right).$$

We get for every $n \geq 1$

$$1 = \int_{\mathbb{D}} |f_n(z)|^2 dA_\alpha(z) \leq \int_{D(0,R)} |f_n(z)|^2 dA_\alpha(z) + \sum_{\lambda\in\Lambda_1} \int_{D\left(\lambda, \sqrt{\frac{m_\lambda - a(\eta)}{m_\lambda+\alpha+1}}\right)} |f_n(z)|^2 dA_\alpha(z).$$

Denote by $\Lambda_2$ the set of $\lambda \in \Lambda_1$ such that

$$\sum_{j<m_\lambda} |\langle f_n, T_\lambda e_j\rangle|^2 \leq \frac{\eta}{2}(\alpha+1) \int_{D\left(\lambda, \sqrt{\frac{m_\lambda}{m_\lambda+\alpha+1}}\right)} |f_n(z)|^2 (1-|z|^2)^\alpha dm(z) =: \frac{\eta'}{2}.$$

By Lemma 2.3.6 applied for $\lambda \in \Lambda_2$, $\eta'$ and $f_n$ with $\|f_n\|_{\alpha,2} = 1$, we obtain

$$\begin{aligned}
1 \leq& (\alpha+1) \int_{D(0,R)} |f_n(z)|^2 (1-|z|^2)^\alpha dm(z) \\
&+ \sum_{\lambda\in\Lambda_2} (\alpha+1) \int_{D\left(\lambda, \sqrt{\frac{m_\lambda - a(\eta)}{m_\lambda+\alpha+1}}\right)} |f_n(z)|^2 (1-|z|^2)^\alpha dm(z) \\
&+ \sum_{\lambda\in\Lambda_1\setminus\Lambda_2} (\alpha+1) \int_{D\left(\lambda, \sqrt{\frac{m_\lambda - a(\eta)}{m_\lambda+\alpha+1}}\right)} |f_n(z)|^2 (1-|z|^2)^\alpha dm(z) \\
&\leq (\alpha+1) \int_{D(0,R)} |f_n(z)|^2 (1-|z|^2)^\alpha dm(z) \\
&+ \eta \sum_{\lambda\in\Lambda_2} (\alpha+1) \int_{D\left(\lambda, \sqrt{\frac{m_\lambda}{m_\lambda+\alpha+1}}\right)} |f_n(z)|^2 (1-|z|^2)^\alpha dm(z) \\
&+ \sum_{\lambda\in\Lambda_1\setminus\Lambda_2} (\alpha+1) \int_{D\left(\lambda, \sqrt{\frac{m_\lambda - a(\eta)}{m_\lambda+\alpha+1}}\right)} |f_n(z)|^2 (1-|z|^2)^\alpha dm(z).
\end{aligned}$$

For the first term on the right hand side of the last inequality, the weak convergence, *via* dominated convergence, yields

$$(\alpha+1) \int_{D(0,R)} |f_n(z)|^2 (1-|z|^2)^\alpha dm(z) = o(1). \quad (n \to \infty).$$

And by the definition of $\Lambda_2$

$$\begin{aligned}
&\sum_{\lambda\in\Lambda_1\setminus\Lambda_2}(\alpha+1)\int_{D\left(\lambda,\sqrt{\frac{m_\lambda-a(\eta)}{m_\lambda+\alpha+1}}\right)}|f_n(z)|^2(1-|z|^2)^\alpha dm(z)\\
&\leq \sum_{\lambda\in\Lambda_1\setminus\Lambda_2}(\alpha+1)\int_{D\left(\lambda,\sqrt{\frac{m_\lambda}{m_\lambda+\alpha+1}}\right)}|f_n(z)|^2(1-|z|^2)^\alpha dm(z)\\
&\leq \frac{2}{\eta}\sum_{\lambda\in\Lambda_1\setminus\Lambda_2}\sum_{j<m_\lambda}|\langle f_n,T_\lambda e_j\rangle|^2=o(1)\quad(n\to\infty)
\end{aligned}$$

Finally,

$$\begin{aligned}
1&\leq o(1)+\eta\sum_{\lambda\in\Lambda}(\alpha+1)\int_{D\left(\lambda,\sqrt{\frac{m_\lambda}{m_\lambda+\alpha+1}}\right)}|f_n(z)|^2(1-|z|^2)^\alpha dm(z)\\
&\leq o(1)+\eta S_X(\alpha+1)\int_{\mathbb{D}}|f_n(z)|^2(1-|z|^2)^\alpha dm(z)=o(1)+\frac{S_X}{S_X+1},\quad n\to\infty.
\end{aligned}$$

We have reached a contradiction.

□

### 2.3.3 Interpolation for $\mathcal{A}^2_\alpha$

As in [Bor+17], we obtain the geometric condition by a $\overline{\partial}$-scheme and a local control of the functions in the space. To do so, we adapt the same technique as in [OB95]. We will need the following version of Hörmander's $L^2$-estimates for $\overline{\partial}$ due to Ohsawa [Ohs94]. Recall the definition of the invariant laplacian

$$\widetilde{\Delta}=(1-|z|^2)^2\Delta$$

and define the invariant convolution of two functions $f$, $g$

$$(f\star g)(z)=\int_{\mathbb{D}}g(\varphi_z(\zeta))f(\zeta)d\mathcal{V}(\zeta).$$

**Theorem 2.3.7.** ***(Ohsawa)[Ohs94]*** *Let $\psi$ be any subharmonic function in the disk such that $\widetilde{\Delta}\psi>\delta>0$. Then there is a solution $u$ to the equation $\overline{\partial}u=g$ such that*

$$\int_{\mathbb{D}}|u(z)|^2\frac{e^{-\psi(z)}}{1-|z|^2}dm(z)\leq C_\delta\int_{\mathbb{D}}|g(z)|^2e^{-\psi(z)}(1-|z|^2)dm(z). \tag{2.35}$$

We need to regularize the weight in such a way that we will not destroy the interpolation after the $\overline{\partial}$-surgery. This will be achieved by Lemma 2.3.8 below. We recall from Lemma 2.3.2 the critical radius in $\mathcal{A}^2_\alpha$, $r_\lambda=\sqrt{\frac{m_\lambda}{m_\lambda+\alpha+1}}$, and its dilation from Theorem 2.1.5 $r'_\lambda=\sqrt{\frac{m_\lambda+C_X}{m_\lambda+\alpha+1}}$, and denote the associated hyperbolic disks $D_\lambda=D(\lambda,r_\lambda)$ and $D'_\lambda=D\left(\lambda,r'_\lambda\right)$ respectively.

We will also need the following auxiliary function (see [OB95, p.119])

$$\xi(\zeta)=\xi_\lambda(\zeta)=\begin{cases}0, & 0\le|\zeta|<r_\lambda,\\ \frac{\log\frac{1}{|\zeta|^2}}{K(m_\lambda,c,\alpha)}, & r_\lambda<|\zeta|<r'_\lambda,\\ 0, & r'_\lambda<|\zeta|<1,\end{cases}$$

where

$$K=K(m_\lambda,C_X,\alpha)=\int_{r_\lambda<|\zeta|<r'_\lambda}\log\frac{1}{|\zeta|^2}d\mathcal{V}(\zeta).$$

(so that the $L^1$-norm of $\xi$ is one, for some more precise estimates on $K$ see below). Consider the weight

$$\begin{aligned}w_{\Lambda,\alpha}(z):&=\sum_{\lambda\in\Lambda}m_\lambda\left[\log|\varphi_\lambda(z)|^2-\frac{1}{K}\int_{r_\lambda<|\zeta|<r'_\lambda}\log|\varphi_{\varphi_\lambda(z)}(\zeta)|^2\log\frac{1}{|\zeta|^2}d\mathcal{V}(\zeta)\right]\chi_{D'_\lambda}(z)\\ &=\sum_{\lambda\in\Lambda}m_\lambda\left[\log|\varphi_\lambda(z)|^2-\int_{\mathbb{D}}\log|\varphi_{\varphi_\lambda(z)}(\zeta)|^2\xi_\lambda(\zeta)d\mathcal{V}(\zeta)\right]\chi_{D'_\lambda}(z)\end{aligned}\tag{2.36}$$

**Lemma 2.3.8.** *Let $X=\{(\lambda,m_\lambda)\}_{\lambda\in\Lambda}$ be a divisor and let $C_X$ be such that $(\alpha+1)(1-e^{-1})<C_X<\alpha+1$ the hyperbolic disks*

$$\left\{D\left(\lambda,\sqrt{\frac{m_\lambda+C_X}{m_\lambda+\alpha+1}}\right)\right\}_{\lambda\in\Lambda}$$

*are pairwise disjoint. Then the weight $w_{\Lambda,\alpha}$ above satisfies*

- *(a)* $w_{\Lambda,\alpha}\le 0$,
- *(b)* $-w_{\Lambda,\alpha}\le A(\alpha)$ *in* $D'_\lambda\setminus D_\lambda$,
- *(c) and* $\widetilde{\Delta}w_{\Lambda,\alpha}\ge -4(\alpha+1-\varepsilon)$ *for some* $\varepsilon$ *depending on* $C_X$.

*Proof.* To see $(a)$, since $\log|\varphi_a|$ is sub-harmonic, we have

$$\log|a|\times\log\frac{1}{r^2}\le\frac{1}{2\pi}\int_0^{2\pi}\log|\varphi_a(re^{i\theta}))|\log\frac{1}{r^2}d\theta.$$

Integrating from $r_\lambda$ to $r'_\lambda$ with respect to the measure $\frac{rdr}{(1-r^2)^2}$ and dividing by $K$ yields the required result.

For $(b)$, observe that the separation condition implies that it is sufficient to consider only one term of the sum. Let us also set $a=\varphi_\lambda(z)$, and notice that $z\in D_{\lambda'}\setminus D_\lambda$ implies in particular $|a|=|\varphi_\lambda(z)|>r_\lambda$. Hence

$$\begin{aligned}-w_{\Lambda,\alpha}(z)&=m_\lambda\left[\frac{1}{K(m_\lambda,C_X,\alpha)}\int_{r_\lambda<|\zeta|<r'_\lambda}\underbrace{\log|\varphi_a(\zeta)|^2\log\frac{1}{|\zeta|^2}}_{<0}d\mathcal{V}(\zeta)-\log|a|^2\right]\\ &\le m_\lambda\log\frac{1}{|a|^2}<m_\lambda\log\frac{1}{r_\lambda^2}=m_\lambda\log\frac{m_\lambda+\alpha+1}{m_\lambda}\\ &\le \alpha+1.\end{aligned}$$

Let us discuss $(c)$. Setting $a = \varphi_\lambda(z)$, we have

$$h(z) := \int_{\mathbb{D}} \log|\varphi_{\varphi_\lambda(z)}(\zeta)|^2 \xi(\zeta) d\mathcal{V}(\zeta) = \int_{\mathbb{D}} \log|\varphi_a(\zeta)|^2 \xi(\zeta) d\mathcal{V}(\zeta) = (\xi \star E)(a),$$

with $E(u) = \log|u|^2$. Observe that $\widetilde{\Delta}(h \circ \varphi_\lambda) = (\widetilde{\Delta} h) \circ \varphi_\lambda$ (see e.g. [OB95, p.120]), $\widetilde{\Delta}(\mu \star \log|\cdot|^2) = 4\mu$ for any measure (notice that in [OB95], the authors define $\Delta = \partial\overline{\partial}$ so that in our setting we have to introduce an additional factor 4), and hence

$$\widetilde{\Delta} h = \widetilde{\Delta}[(\xi \star E) \circ \varphi_\lambda] = 4 \times (\xi \circ \varphi_\lambda).$$

Hence, for $z \in D'_\lambda \setminus D_\lambda$

$$\begin{aligned}\widetilde{\Delta} w_{\Lambda,\alpha}(z) &= m_\lambda \left( \underbrace{\widetilde{\Delta} \log|\varphi_\lambda(z)|^2}_{4\pi(1-|z|^2)\delta_\lambda} - \widetilde{\Delta} h(z) \right) \\ &\geq -4m_\lambda (\xi \circ \varphi_\lambda)(z).\end{aligned}$$

Since $\xi$ is decreasing, we get

$$\widetilde{\Delta} w_{\Lambda,\alpha}(z) \geq -4m_\lambda \xi(r_\lambda) = -4\frac{m_\lambda}{K} \log \frac{m_\lambda + \alpha + 1}{m_\lambda} \geq -4\frac{\alpha+1}{K}.$$

Let us estimate

$$K = 2\int_{r_\lambda}^{r'_\lambda} \frac{-r \ln r^2}{(1-r^2)^2} dr$$

The function $h(r) = \dfrac{r^2}{1-r^2} \log \dfrac{1}{r^2}$ is increasing on $(0,1)$, we get

$$\begin{aligned}K &= \int_{(r_\lambda)^2}^{(r'_\lambda)^2} \log \frac{1}{t} \frac{dt}{(1-t)^2} = \left[ \frac{1}{1-t} \log \frac{1}{t} \right]_{r_\lambda^2}^{r_\lambda'^2} + \int_{r_\lambda^2}^{r_\lambda'^2} \frac{1}{t(1-t)} dt \\ &= h(r'_\lambda) - h(r_\lambda) + \log \frac{1-r_\lambda^2}{1-r_\lambda'^2} \\ &\geq \log \frac{\alpha+1}{\alpha+1-C_X} \end{aligned} \tag{2.37}$$

Since $C_X > (1+\alpha)(1-e^{-1})$, we have $K > 1$ as required. □

We are ready to establish our conditions for interpolating divisors. We recall the statement of the corresponding Theorem 2.1.5 here for the convenience of the reader.

**Theorem.** *Let $\alpha > -1$.*

(a) *If $X$ is an interpolating divisor for $\mathcal{A}^2_\alpha$, then there exists $C_X > 0$ such that the hyperbolic disks*

$$\left\{ D\left(\lambda, \sqrt{\frac{m_\lambda - C_X}{m_\lambda + \alpha + 1}}\right) \right\}_{\lambda\in\Lambda, m_\lambda > C_X}$$

*are pairwise disjoint.*

(b) *Conversely, if for some $C_X$ such that $(\alpha+1)(1-e^{-1}) < C_X < \alpha + 1$, the hyperbolic disks*

$$\left\{ D\left(\lambda, \sqrt{\frac{m_\lambda + C_X}{m_\lambda + \alpha + 1}}\right) \right\}_{\lambda\in\Lambda}$$

*are pairwise disjoint, then $X$ is an interpolating divisor for $\mathcal{A}^2_\alpha$.*

*Proof. Sufficiency part.*

The proof is based on a $\overline{\partial}$-method which consists, as usual, in constructing first a smooth interpolating function, and then to use Hörmander's solution to the $\overline{\partial}$-equation with $L^2$-estimates to make the interpolating function holomorphic without destroying the interpolation.

Given $v = (v^j_\lambda)_{\lambda\in\Lambda,\, \alpha < m_\lambda} \in \ell^2(X)$, take polynomials $p_\lambda$, $\lambda \in \Lambda$, with $\deg p_\lambda \leq m_\lambda - 1$, such that

$$\langle p_\lambda, e_j \rangle = v^j_\lambda, \quad \lambda \in \Lambda,\ j < m_\lambda.$$

We recall that the above interpolation condition means that we interpolate germs in $\lambda$ and that it is thus sufficient to guarantee that the interpolating function and its derivatives take the values $v^j_\lambda$ in $\lambda$, $0 \leq j \leq m_\lambda - 1$. Recall from (2.5) that $N^{2,\alpha}_{\lambda,m_\lambda}$ denotes the set of functions $f$ in $\mathcal{A}^2_\alpha$ vanishing up to the order $m_\lambda - 1$ in $\lambda$. Since $v \in \ell^2(X)$, we have

$$\|p_\lambda\|_{\mathcal{A}^2_\alpha/N^{2,\alpha}_{0,m_\lambda}} = \|p_\lambda\|_{\alpha,2}, \quad \sum_{\lambda\in\Lambda} \|p_\lambda\|^2_{\mathcal{A}^2_\alpha/N^{2,\alpha}_{0,m_\lambda}} = \|v\|^2_2 < \infty.$$

Let us denote $Q_\lambda = T_\lambda p_\lambda$, $r_\lambda = \sqrt{\frac{m_\lambda}{m_\lambda+\alpha+1}}$, $r'_\lambda = \sqrt{\frac{m_\lambda+C_X}{m_\lambda+\alpha+1}}$, $D_\lambda = D(\lambda, r_\lambda)$ and $D'_\lambda = D\left(\lambda, r'_\lambda\right)$. Notice that $\{D'_\lambda\}_{\lambda\in\Lambda}$ are pairwise disjoint by hypothesis. Consider the smooth interpolating function

$$F(z) = \sum_{\lambda\in\Lambda} Q_\lambda(z)\eta\left(|\varphi_\lambda(z)| - r'_\lambda\right), \tag{2.38}$$

where $\eta = \eta_\lambda$ is a smooth cut-off function on $\mathbb{R}$, so that

(a) $\operatorname{supp} \eta \subset (-\infty, 0]$,

(b) $\eta \equiv 1$ on $(-\infty, r_\lambda - r'_\lambda]$,

(c) $|\eta'| \lesssim \frac{1}{r'_\lambda - r_\lambda} \simeq 2\frac{m_\lambda+\alpha+1}{C_X}$.

The separation condition implies that in each $z$, $F(z)$ is given by at most one term.

Notice that $\operatorname{supp} F \subset \bigcup_{\lambda\in\Lambda} D'_\lambda$. Also

$$\operatorname{supp} \overline{\partial} F \subset \bigcup_{\lambda\in\Lambda} \left(D'_\lambda \setminus D_\lambda\right),$$

since $\eta\left(|\varphi_\lambda(z)| - r'_\lambda\right)$ is constant outside $\bigcup_{\lambda\in\Lambda}\left(D'_\lambda \setminus D_\lambda\right)$.

A direct calculation shows that

$$|\overline{\partial}\eta\left(|\varphi_\lambda(z)| - r'_\lambda\right)| \leq \frac{1}{2}\|\eta'\|_\infty \frac{1-|\varphi_\lambda(z)|^2}{1-|z|^2}, \tag{2.39}$$

so that for $z \in D'_\lambda \setminus D_\lambda$, property (c) yields

$$\begin{aligned} |\overline{\partial}F(z)| &\leq |Q_\lambda(z)| \times |\overline{\partial}\eta| \times |\overline{\partial}\big(|\varphi_\lambda(z)|\big)| \\ &\lesssim \frac{|Q_\lambda(z)|}{r'_\lambda - r_\lambda}\frac{1-r_\lambda^2}{1-|z|^2}. \end{aligned} \tag{2.40}$$

Since $T_\lambda$ is an isometry of $\mathcal{A}^2_\alpha$,

$$\|Q_\lambda\|^2_{\alpha,2} = \|p_\lambda\|^2_{\alpha,2} = \|p_\lambda\|^2_{\mathcal{A}^2_\alpha/N^2_{0,m_\lambda}},$$

and therefore $F$ has the growth of $\mathcal{A}^2_\alpha$:

$$\begin{aligned} (\alpha+1)\int_{\mathbb{D}} |F(z)|^2(1-|z|^2)^\alpha dm(z) &\leq \sum_{\lambda\in\Lambda}(\alpha+1)\int_{D'_\lambda} |Q_\lambda(z)|^2(1-|z|^2)^\alpha dm(z) \\ &\leq \sum_{\lambda\in\Lambda} \|Q_\lambda\|^2_{\mathcal{A}^2_\alpha} = \|v\|_2^2. \end{aligned}$$

Next we construct a holomorphic interpolating function using a $\overline{\partial}$-technique. As in the scheme used in [OB95], we are looking for a holomorphic interpolating function of the from $f = F - u$, where $u$ is a solution to the $\overline{\partial}$-problem $\overline{\partial}u = \overline{\partial}F$ with the conditions

$$\int_{\mathbb{D}} |u(z)|^2(1-|z|^2)^\alpha dm(z) < \infty,$$

and

$$\partial^j u(\lambda) = 0,\ \forall j < m_\lambda.$$

This last condition will ensure that

$$\partial^j f(\lambda) = \partial^j F(\lambda), \qquad j < m_\lambda,$$

and we remind that the interpolation condition $\langle f, T_\lambda e_j\rangle = v^j_\lambda$ translates into an interpolation by germs.

We will apply Ohsawa's Theorem 2.3.7 with the subharmonic weight

$$\phi(z) = (\alpha+1)\log\frac{1}{1-|z|^2} + w_{\Lambda,\alpha}(z),$$

where $w_{\Lambda,\alpha}$ is the weight in Lemma 2.3.8. We need to compute $\widetilde{\Delta}\phi$ as in Ohsawa's theorem:

$$\begin{aligned}\widetilde{\Delta}\phi &= (1-|z|^2)^2\Delta\left((\alpha+1)\log\frac{1}{1-|z|^2} + w_{\Lambda,\alpha}\right)\\ &= 4(\alpha+1) + (1-|z|^2)^2\Delta w_{\Lambda,\alpha}\\ &\geq 4(\alpha+1) - 4(\frac{\alpha+1}{K}) > \varepsilon.\end{aligned}$$

The last inequality is due to Lemma 2.3.8 (c).
The properties of the weight $w_{\Lambda,\alpha}$ and Ohsawa's estimate (2.35) yield

$$\begin{aligned}\int_{\mathbb{D}}|u(z)|^2(1-|z|^2)^\alpha dA_\alpha(z) &= (\alpha+1)\int_{\mathbb{D}}|u(z)|^2\frac{e^{-(\alpha+1)\log\frac{1}{1-|z|^2}}}{1-|z|^2}\frac{dm(z)}{\pi}\\ &\leq (\alpha+1)\int_{\mathbb{D}}|u(z)|^2\frac{e^{-\phi(z)}}{1-|z|^2}\frac{dm(z)}{\pi} \quad \text{(Lemma 2.3.8(a))}\\ &\lesssim \int_{\mathbb{D}}|\overline{\partial}F(z)|^2e^{-\phi(z)}(1-|z|^2)dm(z) \quad \text{(Ohsawa)}\\ &\lesssim \sum_{\lambda\in\Lambda}\int_{D'_\lambda\setminus D_\lambda}\frac{|Q_\lambda(z)|^2}{(r'_\lambda-r_\lambda)^2}\left(\frac{1-r_\lambda^2}{1-|z|^2}\right)^2(1-|z|^2)^{\alpha+2}dm(z) \quad \text{(Lemma 2.3.8(b) \& (2.40))}\end{aligned}$$

Now

$$\frac{1-r_\lambda^2}{r'_\lambda-r_\lambda} = \frac{(r'_\lambda+r_\lambda)(1-r_\lambda^2)}{r'^2_\lambda-r_\lambda^2} \leq 2\frac{\alpha+1}{C_X}, \qquad \lambda\in\Lambda, \tag{2.41}$$

so that

$$\int_{\mathbb{D}}|u(z)|^2(1-|z|^2)^\alpha dA_\alpha(z) \lesssim \sum_{\lambda\in\Lambda}\|Q_\lambda\|^2_{\mathcal{A}^2_\alpha/N^2_{0,m_\lambda}} < \infty.$$

Hence, $f = F-u \in \mathcal{A}^2_\alpha$.

Finally, we want to see that $\langle f, T_\lambda e_j\rangle = v^j_\lambda$, $j<m_\lambda$. We have already mentioned that for this we need $u$ to vanish at order $m_\lambda$ in each $\lambda$, so let us examine the order of the singularity near $\lambda$. For each $z\in D_\lambda$

$$w(z) = m_\lambda\log|\varphi_\lambda(z)|^2 + C_\lambda.$$

and therefore

$$\begin{aligned}+\infty > \int_{D_\lambda}|u(z)|^2e^{-w(z)}dm(z) &\gtrsim \int_{D_\lambda}|u(z)|^2e^{-\log|\varphi_\lambda(z)|^{2m_\lambda}}dm(z)\\ &= \int_{D_\lambda}|u(z)|^2\frac{1}{|\varphi_\lambda(z)|^{2m_\lambda}}dm(z).\end{aligned}$$

This forces $u$ to vanish at order $m_\lambda$ on $\lambda \in \Lambda$. Therefore

$$\langle f, T_\lambda e_j \rangle = v_\lambda^j, \qquad j \leq m_\lambda, \lambda \in \Lambda.$$

*Necessary part.*

Let $X = \{(\lambda, m_\lambda)\}_{\lambda\in\Lambda}$ be an interpolating divisor and assume that the discs $\left\{D\left(\lambda, \sqrt{\frac{m_\lambda - C_X}{m_\lambda+\alpha+1}}\right)\right\}_{\lambda,\ m_\lambda > C_X}$ are not separated for any $C_X > 0$. Let $r_\lambda = \sqrt{\frac{m_\lambda - C_X}{m_\lambda+\alpha+1}}$ and $r'_\lambda = \sqrt{\frac{m_\lambda-(C_X-1)}{m_\lambda+\alpha+1}}$. There exists $\lambda_1, \lambda_2 \in \Lambda$, $\lambda_1 \neq \lambda_2$ and $m_{\lambda_1}, m_{\lambda_2} > C_X$ such that

$$D\left(\lambda_1, r_{\lambda_1}\right) \cap D\left(\lambda_2, r_{\lambda_2}\right) \neq \emptyset.$$

And also by the same argument

$$D\left(\lambda_1, \sqrt{\frac{m_{\lambda_1} - (C_X-1)}{m_{\lambda_1}+\alpha+1}}\right) \cap D\left(\lambda_2, \sqrt{\frac{m_{\lambda_2} - (C_X-1)}{m_{\lambda_2}+\alpha+1}}\right) \neq \emptyset.$$

Let $\zeta \in \partial D\left(\lambda_1, r_{\lambda_1}\right)$ and $\zeta' \in \partial D\left(\lambda_1, r'_{\lambda_1}\right)$, we have

$$\begin{aligned}\rho\left(\zeta, \zeta'\right) &\geq \frac{\rho(\zeta', \lambda_1) - \rho(\lambda_1, \zeta)}{1-\rho(\zeta', \lambda_1)\rho(\lambda_1, \zeta)} = \rho(r_\lambda, r'_\lambda) = \frac{r'_{\lambda_1} - r_{\lambda_1}}{1 - r_{\lambda_1} r'_{\lambda_1}} \geq \frac{(r'_{\lambda_1})^2 - (r_{\lambda_1})^2}{1-(r_{\lambda_1} r'_{\lambda_1})^2} \\ &= \frac{m_{\lambda_1} + \alpha + 1}{m_{\lambda_1}(2\alpha + 2C_X + 1) + (\alpha+2)^2 - C_X^2 + C_X} \geq \frac{1}{2\alpha + 2C_X + 1} =: \delta > 0.\end{aligned}$$

Hence, the estimate of the hyperbolic distance between $\partial D\left(\lambda_1, r_{\lambda_1}\right)$ and $\partial D\left(\lambda_1, r'_{\lambda_1}\right)$, is bounded from below by $\delta$. Thus, if $w \in D\left(\lambda_1, r_{\lambda_1}\right) \cap D\left(\lambda_2, r_{\lambda_2}\right) \subset D\left(\lambda_1, r'_{\lambda_1}\right) \cap D\left(\lambda_2, r'_{\lambda_2}\right)$, and

$$\varepsilon = \frac{1}{2}\delta,$$

then we have

$$D(w, \varepsilon) \subset \left(\lambda_1, \sqrt{\frac{m_{\lambda_1} - C_X}{m_{\lambda_1}+\alpha+1}}\right) \cap D\left(\lambda_2, \sqrt{\frac{m_{\lambda_2} - C_X}{m_{\lambda_2}+\alpha+1}}\right).$$

Since $X$ is an interpolating divisor, there exists $f \in \mathcal{A}^2_\alpha$ such that

(a) $f \in N^2_{\lambda_1, m_{\lambda_1}}$,

(b) $f - T_w 1 \in N^2_{\lambda_2, m_{\lambda_2}}$,

(c) $\|f\|_{\alpha,2} \leq M_X$, where $M_X$ is a fixed (interpolating) constant depending only on $X$.

By Lemma 2.3.6 applied to both $f$ and $f - T_w 1$ (for which the sum of the squares of the corresponding Fourier coefficients $|\langle g, e_j\rangle|$ vanish and the norm on the disks are in particular bounded by $M_X$) we have

$$\int_{D\left(\lambda_1, \sqrt{\frac{m_{\lambda_1} - C_X + 1}{m_{\lambda_1} + \alpha + 1}}\right)} |f(z)|^2 dA_\alpha(z) + \int_{D\left(\lambda_2, \sqrt{\frac{m_{\lambda_2} - C_X + 1}{m_{\lambda_2} + \alpha + 1}}\right)} |(f - T_w 1)(z)|^2 dA_\alpha(z) = o(1) \cdot M_X^2, \quad C_X \to \infty,$$

and therefore

$$\int_{D(w,\epsilon)} |f(z)|^2 dA_\alpha(z) + \int_{D(w,\epsilon)} |(f - T_w 1)(z)|^2 dA_\alpha(z) = o(1) \cdot M_X^2, \quad C_X \to \infty.$$

On the other hand,

$$(\alpha + 1) \int_{D(w,\epsilon)} |(T_w 1)(z)|^2 dA_\alpha(z) = (\alpha + 1) \int_{D(0,\epsilon)} (1 - |z|^2)^\alpha dm(z) = \left(1 - \left(1 - \epsilon^2\right)^{\alpha+1}\right) > 0,$$

which gives a contradiction when $C_X > C_X(M_X)$.

□

## 2.4 The $\mathcal{A}_\alpha^\infty$-case

Let $\alpha > 0$, we now consider

$$\mathcal{A}_\alpha^\infty = \left\{ f \in \mathrm{Hol}(\mathbb{D}) : \|f\|_{\alpha,\infty} := \sup_{z \in \mathbb{D}} (1 - |z|^2)^{\frac{\alpha}{2}} |f(z)| < +\infty \right\}.$$

We shall start recalling the reformulation of interpolation and sampling met in the situation $p = 2$ in terms of vanishing subspaces.

For each $\lambda, m_\lambda$ we have already introduced the subspace

$$N_\lambda^2 := N_{\lambda,m}^{2,\alpha} = \{ f \in \mathcal{A}_\alpha^2 : f^{(j)}(\lambda) = 0, \quad \forall j < m \}.$$

Observe that $\sum_{j<m_\lambda} |\langle f, T_\lambda e_j\rangle|^2 = \|f\|^2_{\mathcal{A}_\alpha^2/N_\lambda^2}$. Then it becomes clear that $X$ is a *sampling divisor* for $\mathcal{A}_\alpha^2$ if, for all $f \in \mathcal{A}_\alpha^2$,

$$\|f\|^2_{\alpha,2} \simeq \sum_{\lambda \in \Lambda} \sum_{j<m_\lambda} |\langle f, T_\lambda e_j\rangle|^2 = \sum_{\lambda \in \Lambda} \|f\|^2_{\mathcal{A}_\alpha^2/N_\lambda^2}.$$

Similarly, $X$ is *interpolating* for $\mathcal{A}^2_\alpha$, if for all sequence $(f_\lambda)_{\lambda\in\Lambda} \subset \mathcal{A}^2_\alpha$ such that

$$\sum_{\lambda\in\Lambda} \|f_\lambda\|^2_{\mathcal{A}^2_\alpha/N^2_\lambda} < \infty,$$

there exists $f \in \mathcal{A}^2_\alpha$ so that

$$f - f_\lambda \in N^2_\lambda.$$

In order to consider the corresponding $L^\infty$ sampling and interpolation problems, we associate to each $\lambda \in \mathbb{D}$ the subspace

$$N^\infty_\lambda = N^{\infty,\alpha}_{\lambda,m_\lambda} := \{f \in \mathcal{A}^\infty_\alpha : \partial^j f(\lambda) = 0, \forall j < m_\lambda\}.$$

**Definition 2.4.1.** *A divisor is called sampling for $\mathcal{A}^\infty_\alpha$, if there exists $L > 0$ such that*

$$\|f\|_{\alpha,\infty} \leq L \sup_{\lambda\in\Lambda} \|f\|_{\mathcal{A}^\infty_\alpha/N^\infty_\lambda}.$$

In a similar way we define generalized interpolation.

**Definition 2.4.2.** *The divisor $X$ is called interpolating for $\mathcal{A}^\infty_\alpha$ if for every sequence $(f_\lambda)_{\lambda\in\Lambda}$ with*

$$\sup_{\lambda\in\Lambda} \|f_\lambda\|_{\mathcal{A}^\infty_\alpha/N^\infty_\lambda} < \infty,$$

*there exists a function $f \in \mathcal{A}^\infty_\alpha$ such that*

$$f - f_\lambda \in N^\infty_\lambda, \quad \lambda \in \Lambda.$$

### 2.4.1 Local $L^\infty$-estimates

As in the $L^2$ case we need a local control of the functions of the space $\mathcal{A}^\infty_\alpha$ with small quotient norm. Here is the result corresponding to Lemma 2.3.6 for $\mathcal{A}^\infty_\alpha$ $(\alpha > 0)$.

**Lemma 2.4.3.** *(i) For every $\eta, \varepsilon \in (0,1)$, there exists $C > 0$ such that if $f \in \mathcal{A}^\infty_\alpha$ satisfies $\|f\|_{\alpha,\infty} \leq 1$, $m \geq C$, $\|f\|_{\mathcal{A}^\infty_\alpha/N^{\infty,\alpha}_{0,m}} < \varepsilon$, then*

$$|f(z)| \left(1 - |z|^2\right)^{\frac{\alpha}{2}} \leq \eta + \varepsilon, \quad z \in D\left(0, \sqrt{\frac{m-C}{m+\alpha}}\right).$$

*(ii) For every $C > 0$ there exist $\eta, \varepsilon \in (0,1)$, such that if $f \in \mathcal{A}^\infty_\alpha$ satisfies $\|f\|_{\alpha,\infty} \leq 1$, $m \geq C$, $\|f\|_{\mathcal{A}^\infty_\alpha/N^{\infty,\alpha}_{0,m}} < \varepsilon$, then*

$$|f(z)| \left(1 - |z|^2\right)^{\frac{\alpha}{2}} \leq 1 - \eta, \quad z \in D\left(0, \sqrt{\frac{m-C}{m+\alpha}}\right).$$

The result in (i) is of course of interest when $\eta+\varepsilon<1$, and in particular when $f\in N_{0,m}^{\infty,\alpha}$ in which case we can pick $\varepsilon$ arbitrarily small.

Note that the critical radius $\sqrt{\frac{m}{m+\alpha}}$ is different from the one appearing for $p=2$. We have already met this radius in Theorem 2.2.2.

*Proof.* Claim (i):

Since $\|f\|_{\mathcal{A}_\alpha^\infty/N_{0,m}^{\infty,\alpha}}<\varepsilon$, there exist a function $g\in N_{0,m}^{\infty,\alpha}$ such that

(a) $\|f-g\|_{\mathcal{A}_\alpha^\infty}\le\varepsilon$,

(b) $g(z)=z^m h(z)$, where $h$ is a holomorphic function in the unit disk.

Since $\|f\|_{\alpha,\infty}\le 1$ we have the bound

$$|g(z)|\le|g(z)-f(z)|+|f(z)|\le(1+\varepsilon)\frac{1}{(1-|z|^2)^{\frac{\alpha}{2}}},\qquad z\in\mathbb{D}.$$

and in terms of $h$ and the functions $\vartheta_{m,\alpha}(t)=\log\dfrac{1}{t^m(1-t^2)^{\frac{\alpha}{2}}}$

$$|h(z)|\le(1+\varepsilon)e^{\vartheta_{m,\alpha}(|z|)},\qquad z\in\mathbb{D}.$$

Using the maximum principal we obtain

$$\max_{z\in D\left(0,\sqrt{\frac{m}{m+\alpha}}\right)}|h(z)|=(1+\varepsilon)e^{\left[\vartheta_{m,\alpha}\left(\sqrt{\frac{m}{m+\alpha}}\right)-\vartheta_{m,\alpha}\left(\sqrt{\frac{m-C}{m+\alpha}}\right)+\vartheta_{m,\alpha}\left(\sqrt{\frac{m-C}{m+\alpha}}\right)\right]}$$

Observe that

$$\vartheta_{m,\alpha}\left(\sqrt{\frac{m-C}{m+\alpha}}\right)-\vartheta_{m,\alpha}\left(\sqrt{\frac{m}{m+\alpha}}\right)=\frac{C}{2}+o(1)-\log\left(\frac{\alpha+2C}{\alpha}\right)^{\frac{\alpha}{2}}.\qquad(2.42)$$

Since the term $o(1)$ goes to 0 when $m$ goes to infinity, and $m\ge C$, the above expression can be made arbitrarily big. Let $\delta=\delta(C)$ be the corresponding constant (thus with $\lim_{C\to+\infty}\delta(C)=+\infty$), we get

$$\max_{z\in D\left(0,\sqrt{\frac{m}{m+\alpha}}\right)}|h(z)|\le(1+\varepsilon)e^{-\delta}e^{\vartheta_{m,\alpha}\left(\sqrt{\frac{m-C}{m+\alpha}}\right)}.\qquad(2.43)$$

Since $\delta(C)\to+\infty$ when $C\to+\infty$ there exists a $C$ such that $(1+\varepsilon)e^{-\delta}\le\eta$. Then

$$|h(z)|\le\eta e^{\vartheta_{m,\alpha}\left(\sqrt{\frac{m-C}{m+\alpha}}\right)},\quad z\in\partial D\left(0,\sqrt{\frac{m}{m+\alpha}}\right).$$

Now, by the maximum principal again, restricting the estimate to the smaller disk $D\left(0,\sqrt{\frac{m-C}{m+\alpha}}\right)$, and using the fact that $\vartheta_{m,\alpha}$ is decreasing on $(0,\sqrt{\frac{m}{m+\alpha}})$ we get

$$|h(z)|\le\eta e^{\vartheta_{m,\alpha}\left(\sqrt{\frac{m-C}{m+\alpha}}\right)}\le\eta e^{\vartheta_{m,\alpha}(|z|)},\quad z\in D\left(0,\sqrt{\frac{m-C}{m+\alpha}}\right).\qquad(2.44)$$

Finally, for $z \in D\left(0, \sqrt{\frac{m-C}{m+\alpha}}\right)$

$$\begin{aligned}
|f(z)|\left(1-|z|^2\right)^{\frac{\alpha}{2}} &\leq |f(z)-g(z)|\left(1-|z|^2\right)^{\frac{\alpha}{2}} + |g(z)|\left(1-|z|^2\right)^{\frac{\alpha}{2}} \\
&\leq \varepsilon + |z|^m \left(1-|z|^2\right)^{\frac{\alpha}{2}} |h(z)| \\
&= \varepsilon + e^{-\vartheta_{m,\alpha}(|z|)}|h(z)| \\
&\leq \varepsilon + \eta. \qquad (2.45)
\end{aligned}$$

Claim (ii):

The proof follows exactly the same lines and ideas. First one should observe that given $C > 0$, the difference appearing in (2.42) is uniformly bounded from below by some $\delta > 0$ (this is clear when $m$ is big, say $m \geq m_0$, and for $1 \leq m < m_0$ we just take the smallest of finitely many stricly positive numbers). Then looking at (2.43), we have to convince ourselves that there are $\varepsilon, \eta > 0$ such that $(1+\varepsilon)e^{-\delta} < 1-\eta-\varepsilon$ which is easily seen to be true. Finally, the same estimates as in (2.45) lead to

$$|f(z)|\left(1-|z|^2\right)^{\frac{\alpha}{2}} \leq \varepsilon + (1-\eta-\varepsilon) = 1-\eta.$$

□

### 2.4.2 Sampling for $\mathcal{A}_\alpha^\infty$

Now we are ready to establish our conditions for sampling conditions.

**Theorem 2.4.4.** *Let $\alpha > 0$.*

*(a) If $X$ is a sampling divisor for $\mathcal{A}_\alpha^\infty$, then there exists $0 < C < \alpha$ such that*

$$\bigcup_{\lambda \in \Lambda} D\left(\lambda, \sqrt{\frac{m_\lambda + C}{m_\lambda + \alpha}}\right) = \mathbb{D}.$$

*(b) Conversely, if there exists $C = C(S_X) > 0$ such that for some compact $K$ of $\mathbb{D}$ we have*

$$\bigcup_{\lambda \in \Lambda, m_\lambda > C} D\left(\lambda, \sqrt{\frac{m_\lambda - C}{m_\lambda + \alpha}}\right) = \mathbb{D} \setminus K,$$

*then $X$ is a sampling divisor for $\mathcal{A}_\alpha^\infty$.*

*Proof.*
(a)*Necessary Condition.*
Suppose that for every $0 < C < \alpha$, we have

$$\bigcup_{\lambda \in \Lambda} D\left(\lambda, \sqrt{\frac{m_\lambda + C}{m_\lambda + \alpha}}\right) \neq \mathbb{D}.$$

Thus, there exists an increasing sequence of positive numbers $(C_k)$ tending to $\alpha$ and a sequence $(z_k)$ with $z_k \in \mathbb{D}$ such that :

$$z_k \in \mathbb{D} \setminus \left[ \bigcup_{\lambda\in\Lambda} D\left(\lambda, \sqrt{\frac{m_\lambda + C_k}{m_\lambda+\alpha}}\right)\right] \subset \mathbb{D} \setminus \left[ \bigcup_{\lambda\in\Lambda} D\left(\lambda, \sqrt{\frac{m_\lambda}{m_\lambda+\alpha}}\right)\right].$$

As in the proof of the necessary condition of the sampling theorem in the Hilbertian case, we will show that

$$d_k := \operatorname{dist}\left(z_k, \bigcup_{\lambda\in\Lambda} D\big(\lambda, \sqrt{\frac{m_\lambda}{m_\lambda+\alpha}}\big)\right) \to 1.$$

Put $r_{\lambda,C_k} = \sqrt{\frac{m_\lambda+C_k}{m_\lambda+\alpha}}$ and $r_\lambda = \sqrt{\frac{m_\lambda}{m_\lambda+\alpha}}$. Let

$$\zeta \in \bigcup_{\lambda\in\Lambda} D\left(\lambda, \sqrt{\frac{m_\lambda}{m_\lambda+\alpha}}\right).$$

Then there exists $\lambda_0 \in \Lambda$ such that $\zeta \in D(\lambda_0, r_{\lambda_0})$ and $\rho(z_k,\lambda_0) > r_{\lambda_0,C_k}$, $k \geq 1$. Since $\alpha > 0$, we get as in the Hilbertian situation

$$\begin{aligned}\rho(z_k,\zeta) &\geq \frac{\rho(z_k,\lambda_0) - \rho(\zeta,\lambda_0)}{1-\rho(z_k,\lambda_0)\rho(\zeta,\lambda_0)} \geq \frac{(r_{\lambda_0,C_k})^2 - (r_{\lambda_0})^2}{1-(r_{\lambda_0})^2(r_{\lambda_0,C_k})^2}\\ &= \frac{C_k(m_{\lambda_0}+\alpha)}{m_{\lambda_0}(2\alpha - C_k)+\alpha^2}.\end{aligned}$$

Observe that this last expression is decreasing in $m_{\lambda_0}$ so that passing to the limit $m_{\lambda_0} \to +\infty$, we get

$$\rho(z_k,\zeta) \geq \frac{C_k}{2\alpha - C_k}.$$

Thus, $d_k \geq C_k/(2\alpha - C_k) \to 1$ when $k \to +\infty$.

Now pick $f_k(z) = T_{z_k}(1)$. We will show that we cannot sample uniformly $f_k$ meaning that $\sup_{\lambda\in\Lambda} \|f_k\|_{\mathcal{A}^\infty_\alpha/N^{\alpha,\infty}_{\lambda,m_\lambda}} \to 0$ (while $\|f_k\|_{\mathcal{A}^\infty_\alpha} = 1$). In view of the construction it is enough to show that when $u \geq \sqrt{\frac{m+C_k}{m+\alpha}}$ (In mind $u := u_k = \varphi_\lambda(z_k)$ with this way we translate to $\lambda = 0$), then $\|T_u 1\|_{\mathcal{A}^\infty_\alpha/N^{\alpha,\infty}_{0,m}} \to 0$ uniformly in $m$ when $C_k \to \alpha$. Recall that

$$T_u 1(z) = \left(\frac{1-|u|^2}{(1-\overline{u}z)^2}\right)^{\alpha/2} = (1-|u|^2)^{\alpha/2}\frac{1}{(1-\overline{u}z)^\alpha}.$$

Using the standard Taylor series for power functions we get

$$\frac{1}{(1-\overline{u}z)^\alpha} = \sum_{n\geq 0}\binom{-\alpha}{n}(-\overline{u})^n z^n = \sum_{n=0}^{m-1}\binom{-\alpha}{n}(-\overline{u})^n z^n + z^m h(z) = f_0(z) + z^m h(z).$$

The following etimate is well known

$$\binom{-\alpha}{n} = \frac{(-1)^n}{\Gamma(\alpha)n^{1-\alpha}}(1+o(1)).$$

Hence

$$|f_0(z)| \leq C \sum_{n=0}^{m-1} \frac{1}{n^{1-\alpha}} u^n |z|^n$$

(we remind that $u > 0$). Here $C$ is some irrelevant universal constant. Hence

$$\|T_u 1\|_{\mathcal{A}^\infty_\alpha / N^{\alpha,\infty}_{0,m}} \leq (1-|u|^2)^{\alpha/2} \|f_0\|_{\mathcal{A}^\infty} \leq C(1-|u|^2)^{\alpha/2} \sup_{|z|<1} (1-|z|^2)^{\alpha/2} \sum_{n=0}^{m-1} \frac{1}{n^{1-\alpha}} u^n |z|^n$$

The function $\varphi_n(x) = (1-x^2)^{\alpha/2} x^n$ admits a maximum in $x_n = \sqrt{n/(n+\alpha/2)}$ which, up to a multiplicative constant, behaves like $1/n^{\alpha/2}$. Hence

$$\|T_u 1\|_{\mathcal{A}^\infty_\alpha / N^{\alpha,\infty}_{0,m}} \leq C(1-|u|^2)^{\alpha/2} \sum_{n=0}^{m-1} n^{\alpha/2-1} \leq C(1-|u|^2)^{\alpha/2} m^{\alpha/2}$$

where in the above inequalities $C$ are different universal constants. On the other hand

$$(1-|u|^2)^{\alpha/2} \leq \left(1 - \frac{m+C_k}{m+\alpha}\right)^{\alpha/2} = \left(\frac{\alpha - C_k}{m+\alpha}\right)^{\alpha/2},$$

so that

$$\|T_u 1\|_{\mathcal{A}^\infty_\alpha / N^{\alpha,\infty}_{0,m}} \leq C\left(m\frac{\alpha - C_k}{m+\alpha}\right)^{\alpha/2} \leq C(C_k - \alpha)^{\alpha/2}$$

uniformly in $m$. Since $C_k \to \alpha$ the above expression goes to 0 (uniformly in $m$), and we reach the desired conclusion.

(b)*Sufficient Condition.*
Suppose that there exists a sequence $(f_n)_n$ such that $\|f_n\|_{\alpha,\infty} = 1$, and

$$\sup_{\lambda \in \Lambda} \|f_n\|_{\mathcal{A}^\infty_\alpha / N^\infty_\lambda} \to 0, \qquad n \to \infty.$$

Passing to a sub-sequence converging uniformly on compact subsets denoted again by $(f_n)_n$, we have two possibilities: either (A) the sequence $(f_n)_n$ converges to $f \neq 0$ or (B) the sequence $(f_n)_n$ converges to 0.

(A): In this case $X$ is a zero divisor for $\mathcal{A}^\infty_\alpha$. Then, by the Uniqueness Theorem 2.2.2, $\mathbb{D} \setminus \left[\bigcup_{\lambda\in\Lambda} D\left(\lambda, \sqrt{\frac{m_\lambda}{m_\lambda+\alpha+\varepsilon}}\right)\right]$ cannot be compact for any $\varepsilon > 0$. On the other hand, for every $C > 0$ and for every $0 < \varepsilon < C$ we have

$$\sqrt{\frac{m_\lambda - C}{m_\lambda + \alpha}} < \sqrt{\frac{m_\lambda}{m_\lambda + \alpha + \varepsilon}}. \tag{2.46}$$

This yields,

$$\mathbb{D} \setminus \left[ \bigcup_{\lambda \in \Lambda} D \left( \lambda, \sqrt{\frac{m_\lambda}{m_\lambda + \alpha + \varepsilon}} \right) \right] \subset \mathbb{D} \setminus \left[ \bigcup_{\lambda \in \Lambda} D \left( \lambda, \sqrt{\frac{m_\lambda - C}{m_\lambda + \alpha}} \right) \right].$$

Therefore, for no $C > 0$, $\mathbb{D} \setminus \left[ \bigcup_{\lambda \in \Lambda} D \left( \lambda, \sqrt{\frac{m_\lambda - C}{m_\lambda + \alpha}} \right) \right]$ can be compact, contradicting the hypothesis.

(B): In this case, by contradiction we will assume that for some compact set $K \subset \mathbb{D}$ we have

$$\Omega = \bigcup_{\lambda \in \Lambda} D\Big( \lambda, \sqrt{\frac{m_\lambda - C}{m_\lambda + \alpha}} \Big) = \mathbb{D} \setminus K.$$

Since by assumption $(f_n)_n$ converges to 0 on compact subsets, there exists $n_0 \in \mathbb{N}$ such that

$$|f_n(z)| \left(1 - |z|^2\right)^{\frac{\alpha}{2}} < \frac{1}{2}, \qquad z \in K, n \geq n_0.$$

Next, for the given $C$, Lemma 2.4.3(ii) implies the existence of $\eta, \varepsilon > 0$ ensuring a control on $f$. Since $\sup_{\lambda \in \Lambda} \|f_n\|_{\mathcal{A}_\alpha^\infty / N_\lambda^\infty} \to 0$, there exists $n_1$ such that for $n \geq n_1$, these quotient norms are stricly smaller than $\varepsilon$ (uniformly in $\lambda$) as required by the lemma. Since moreover $\|f_n\|_{\alpha,\infty} = 1$, Lemma 2.4.3 implies that

$$|f_n(z)| \left(1 - |z|^2\right)^{\frac{\alpha}{2}} < 1 - \eta, \quad z \in \bigcup_{\lambda \in \Lambda, m_\lambda > C} D\Big( \lambda, \sqrt{\frac{m_\lambda - C}{m_\lambda + \alpha}} \Big).$$

Hence, $\|f_n\|_{\alpha,\infty} < 1$ for $n > \max(n_0, n_1)$ and we get a contradiction.

□

### 2.4.3 Interpolation for $\mathcal{A}_\alpha^\infty$

We need the following result by Berndtsson [**B**] (see [OB95, Theorem G] ) for the uniform estimates in the $\bar{\partial}$-surgery.

**Theorem 2.4.5.** *Let $\psi$ be a subharmonic function and*

$$\varphi(z) = \min \left\{ (1 - |z|)\Delta\psi(z), \frac{1}{1 - |z|} \right\}.$$

*Let $f$ be a function in $\mathbb{D}$ such that*

$$\sup \frac{|f(z)|}{\varphi(z)} e^{-\psi(z)/2} < \infty.$$

*Let $u \in L^2(\mathbb{D}, e^{-\psi} dm)$ be the canonical solution to $\bar{\partial} u = f$. Then*

$$\sup |u(z)| e^{-\widetilde{\psi}(z)/2} \leq \sup \frac{|f(z)|}{\varphi(z)} e^{-\psi(z)/2},$$

*where $\widetilde{\psi}(z) = \sup_{|z - \zeta| < 1/2(1-|z|)} \psi(\zeta)$.*

The corresponding result for interpolation in $\mathcal{A}^\infty_\alpha$ reads as follows.

**Theorem 2.4.6.** *Let $\alpha > 0$.*

*(a) If $X$ is an interpolating divisor for $\mathcal{A}^\infty_\alpha$, then there exists $C_X > 0$ such that the hyperbolic disks*

$$\left\{ D\left(\lambda, \sqrt{\frac{m_\lambda - C_X}{m_\lambda + \alpha}}\right)\right\}_{\lambda\in\Lambda, m_\lambda > C_X}$$

*are pairwise disjoint.*

*(b) Conversely, if for some $C_X$ such that $\alpha(1-e^{-1}) < C_X < \alpha$, the hyperbolic disks*

$$\left\{ D\left(\lambda, \sqrt{\frac{m_\lambda + C_X}{m_\lambda + \alpha}}\right)\right\}_{\lambda\in\Lambda}$$

*are pairwise disjoint, then $X$ is an interpolating divisor for $\mathcal{A}^\infty_\alpha$.*

*Proof. Necessary part*

Let $X = \{(\lambda, m_\lambda)\}_{\lambda\in\Lambda}$ be an interpolating divisor and assume that the discs

$$\left\{ D\left(\lambda, \sqrt{\frac{m_\lambda - C_X}{m_\lambda + \alpha}}\right)\right\}_{\lambda\in\Lambda, m_\lambda > C_X}$$

are not pairwise disjoint for any $C_X$. Arguing as in the proof of Theorem 2.1.5, we see that there exist $\lambda, \lambda' \in \Lambda$ and $w \in \mathbb{D}$ such that

$$D(w,\epsilon) \subset D\left(\lambda, \sqrt{\frac{m_\lambda - C_X + 1}{m_\lambda + \alpha}}\right) \cap D\left(\lambda', \sqrt{\frac{m_{\lambda'} - C_X + 1}{m_{\lambda'} + \alpha}}\right).$$

Since $X$ is an interpolating divisor, there exists a function $f \in \mathcal{A}^\infty_\alpha$ such that

(a) $f \in N^\infty_{\lambda, m_\lambda}$,

(b) $f - T_w 1 \in N^\infty_{\lambda', m_{\lambda'}}$,

(c) $\|f\|_{\alpha,\infty} \le M_X$.

Let us denote $\|\cdot\|_{\infty,U}$ the norm with a supremum taken in the set $U \subset \mathbb{D}$. By Lemma 2.4.3(i) applied to $f$ and $f - T_w 1$ we have

$$\|f\|_{\infty, D\left(\lambda, \sqrt{\frac{m_\lambda - C_X + 1}{m_\lambda + \alpha}}\right)} + \|f - T_w 1\|_{\infty, D\left(\lambda', \sqrt{\frac{m_{\lambda'} - C_X + 1}{m_{\lambda'} + \alpha}}\right)} < 2\eta,$$

where we can pick $\eta < 1/2$ when $C_X$ is sufficiently big (note that since $f$ and $f - T_w 1$ are zero in the corresponding quotient spaces, we can consider $\varepsilon = 0$). Therefore

$$\|f\|_{\infty, D(w,\varepsilon)} + \|f - T_w 1\|_{\infty, D(w,\varepsilon)} < 2\eta$$

However

$$\sup_{z\in D(w,\varepsilon)} \|T_w 1\|_{\infty,D(w,\varepsilon)} = 1,$$

so $X$ cannot be interpolating.

*Sufficient part*

Here we use the same scheme as in the $L^2$-case: we construct a smooth interpolating function and we modify it to obtain a holomorphic one. However, now we need an $L^\infty$-estimate for the solution to the $\overline{\partial}$-equation which will be provided by Theorem 2.4.5.

Let $(\rho_{\lambda_j})_{j\geq 1}$ be holomorphic data (polynomials) with $\sup_j \|\rho_{\lambda_j}\|_{\alpha,\infty} \leq 1$. Given any $N \geq 1$ we look for functions $f_N \in \mathrm{Hol}(\mathbb{D})$ and $M$ independent of $N$ such that

$$\begin{aligned} f_N - \rho_j \in N^\infty_{\lambda_j}, \quad j = 1,\dots,N; \\ \|f_N\|_{\alpha,\infty} \leq M, \quad \forall N \in \mathbb{N}. \end{aligned}$$

Then, by Montel's theorem, the limit $f = \lim_N f_N$ gives the desired result. For this set, $m_j = m_{\lambda_j}$,

$$D_j = D\left(\lambda_j, r_j\right), \quad r_j = \sqrt{\frac{m_j}{m_j+\alpha}} \quad \text{and} \quad D_j' = D\left(\lambda_j, r_j'\right), \quad r_j' = \sqrt{\frac{m_j + C_X}{m_j+\alpha}}.$$

Define the smooth interpolating function,

$$F_N(z) = \sum_{j=1}^N \rho_j(z)\eta(|\varphi_{\lambda_j}(z)| - r_j'),$$

where $\eta = \eta_\lambda$ is a smooth cut-off function on $\mathbb{R}$, with

(a) $\operatorname{supp} \eta \subset (-\infty, 0]$,

(b) $\eta \equiv 1$ on $(-\infty, r_j - r_j']$,

(c) $|\eta'| \lesssim \frac{1}{r_j'-r_j} \simeq \frac{m_j+\alpha}{C_X}$.

By the separation hypothesis we have

$$\operatorname{supp} F_N \subset \bigcup_{j=1}^N D_j' \subset \bigcup_{j\geq 1} D_j'.$$

Furthermore, $F_N$ has the characteristic growth of $\mathcal{A}^\infty_\alpha$, due to the property (b) and the separation hypothesis again, Namely

$$\begin{aligned} \sup_{z\in\mathbb{D}} \left(1-|z|^2\right)^{\frac{\alpha}{2}} |F_N(z)| &= \max_{1\leq j\leq N} \sup_{z\in\mathbb{D}} \left(1-|z|^2\right)^{\frac{\alpha}{2}} |\rho_j(z)| && (2.47) \\ &\leq \sup_{j\geq 1} \|\rho_j\|_{\mathcal{A}^\infty_\alpha} \leq 1 && (2.48) \end{aligned}$$

uniformly in $N$. On the other hand,

$$\operatorname{supp} \overline{\partial} F_N \subset \bigcup_{j=1}^{N} \left( D'_j \setminus D_j \right).$$

And

$$\overline{\partial} F_N(z) = \sum_{j \leq N} \rho_j(z) \eta'(|\varphi_{\lambda_j}(z)| - r'_j) \overline{\partial} |\varphi_{\lambda_j}(z)| \chi_{D'_j \setminus D_j}(z).$$

Hence, for $z \in D'_j \setminus D_j$, as in (2.39),

$$\begin{aligned} |\overline{\partial} F_N(z)| &= |\rho_j(z)| \left| \eta' \left( |\varphi_{\lambda_j}(z)| - r'_j \right) \right| \left| \overline{\partial} |\varphi_{\lambda_j}(z)| \right| \\ &\leq \frac{1}{2} |\rho_j(z)| \|\eta'\|_\infty \frac{1 - |\varphi_{\lambda_j}(z)|^2}{1 - |z|^2}. \end{aligned}$$

Therefore, by (b) and (c), we get for $z \in D'_j \setminus D_j$

$$\begin{aligned} |\overline{\partial} F_N(z)| \left( 1 - |z|^2 \right)^{\frac{\alpha}{2} + 1} &\lesssim \|\rho_j\|_{\alpha,\infty} \|\eta'\|_\infty (1 - |\varphi_{\lambda_j}(z)|^2) \\ &\lesssim \|\rho_j\|_{\alpha,\infty} \frac{1 - r_j^2}{r'_j - r_j} \\ &\lesssim \frac{2(\alpha + 1)}{C_X} \|\rho_j\|_{\alpha,\infty}, \end{aligned}$$

where we have used a similar estimate as in (2.41).

This leads to

$$\sup_{z \in \mathbb{D}} \frac{|\overline{\partial} F_N(z)|}{\frac{1}{1-|z|^2}} e^{-\frac{\alpha}{2} \log\left(\frac{1}{1-|z|^2}\right)} \lesssim \sup_{j \geq 1} \|\rho_j\|_{\mathcal{A}^\infty_\alpha} < \infty, \tag{2.49}$$

where underlying constants are independant on $N$.

Again the holomorphic interpolating function in $\mathcal{A}^\infty_\alpha$ will be obtained *via* the solution to a $\overline{\partial}$-problem: $f_N = F_N - u_N$, where $\overline{\partial} u_N = \overline{\partial} F_N$ with the conditions

$$\sup_{z \in \mathbb{D}} |u_N(z)| \left( 1 - |z|^2 \right)^{\frac{\alpha}{2}} < \infty,$$

and

$$\partial^k u_N(\lambda_j) = 0, \ \forall k < m_{\lambda_j}.$$

The last condition ensures that $\partial^k f_N(\lambda_j) = \partial^k F_N(\lambda_j)$, for $k < m_{\lambda_j}$, and then

$$\partial^k \Big( F_N - u_N - \rho_j \Big)(\lambda_j) = 0, \quad k < m_{\lambda_j}, 1 \leq j \leq N. \tag{2.50}$$

We will use a similar weight function $w$ as in (2.36) where now

$$r_\lambda = \sqrt{\frac{m_\lambda}{m_\lambda + \alpha}}, \quad \text{and} \quad r'_\lambda = \sqrt{\frac{m_\lambda + C_X}{m_\lambda + \alpha}}.$$

More precisely, set

$$w := w_{\Lambda,\alpha,N}(z) = \sum_{j=1}^{N} m_{\lambda_j}\left[E(.) - E \star \xi_{\lambda_j}(.)\right](\varphi_{\lambda_j}(z))\chi_{D'_{\lambda_j}}(z), \quad z \in \mathbb{D},$$

where $E(z) = \log|z|^2$, and for $\lambda \in \Lambda$

$$\xi_\lambda(\zeta) = \begin{cases} 0 & \text{if } 0 \le |\zeta| < r_\lambda, \\ \dfrac{\log \frac{1}{|\zeta|^2}}{K(m_\lambda, C_X, \alpha)} & \text{if } r_\lambda < |\zeta| < r_\lambda^{'}, \\ 0 & \text{if } |\zeta| > r_\lambda^{'}. \end{cases}$$

Again $K := K(m_\lambda, C_X, \alpha) = \int_{r_\lambda < |\zeta| < r'_\lambda} \log(1/|\zeta|^2)d\mathcal{V}(\zeta)$. As in (2.37) we see that $K \ge \log(\alpha/(\alpha - C_X)) > 1$ when $C_X > \alpha(1 - e^{-1})$. Let

$$\psi(z) := \psi_{\Lambda,\alpha}(z) = \alpha \log \frac{1}{1-|z|^2} + w(z), \quad z \in \mathbb{D}.$$

By the same arguments as in the proof of lemma 2.3.8, we have

(a) $w \le 0$,

(b) $-w \le A(\alpha)$ in $D'_j \setminus D_j$,

(c) $\widetilde{\Delta} w \ge -4(\alpha - \varepsilon)$ for some $\varepsilon$ depending on $C_X$.

Clearly from (a) and the definition of $\psi$ we have $\psi(z) \le \alpha \log \frac{1}{1-|z|^2}$ for $z \in \mathbb{D}$, and with (c) we get that under the condition $\alpha(1 - e^{-1}) < C_X < \alpha$, for every $z \in \mathbb{D}$,

$$(1 - |z|^2)\Delta\psi(z) = \frac{\widetilde{\Delta}\psi(z)}{1-|z|^2} \gtrsim \frac{\varepsilon(C_X)}{1-|z|^2}.$$

Thus,

$$\varphi(z) := \min\{(1 - |z|)\Delta\psi(z), \frac{1}{1-|z|}\} \asymp \frac{1}{1-|z|^2}, \qquad z \in \mathbb{D}.$$

Now applying Theorem 2.4.5, we see that the there exists $u_N \in L^2(\mathbb{D}, e^{-\psi}dm)$, a canonical solution of the $\overline{\partial}$-equation $\overline{\partial}u_N = \overline{\partial}F_N$ satisfying

$$\sup_{z\in\mathbb{D}} |u_N(z)|e^{-\frac{1}{2}\widetilde{\psi(z)}} \le \sup_{z\in\mathbb{D}} \frac{|\overline{\partial}F_N(z)|}{\varphi(z)} e^{-\frac{1}{2}\psi(z)},$$

where $\widetilde{\psi(z)} := \sup_{|z-\zeta|<1/2(1-|z|)} \psi(z)$. By (2.49) and (b)

$$\begin{aligned}\sup_{z\in\mathbb{D}} \frac{|\overline{\partial}F_N(z)|}{\varphi(z)} e^{-\frac{1}{2}\psi(z)} &\lesssim \sup_{z\in\mathbb{D}} \frac{|\overline{\partial}F_N(z)|}{\frac{1}{1-|z|^2}} e^{-\frac{\alpha}{2}\log(\frac{1}{1-|z|^2}) - w(z)} \\ &\lesssim \sup_{j=1,\dots,N} \sup_{z\in D'_j\setminus D_j} \frac{|\overline{\partial}F_N(z)|}{\frac{1}{1-|z|^2}} e^{-\alpha \log \frac{1}{1-|z|^2}} e^{A(\alpha)} \\ &\lesssim \sup_{j\ge1} \|\rho_j\|_{\mathcal{A}^\infty_\alpha} \lesssim 1.\end{aligned}$$

Thus, uniformly in $N$

$$\sup_{z\in\mathbb{D}} |u_N(z)|e^{-\frac{1}{2}\widetilde{\psi(z)}} < \infty.$$

On the other hand, by (a)

$$\begin{aligned}\widetilde{\psi(z)} := \sup_{|z-\zeta|<1/2(1-|z|)} \psi(\zeta) &= \sup_{|z-\zeta|<1/2(1-|z|)} \left(\alpha \log \frac{1}{1-|\zeta|^2} + w(\zeta)\right) \\ &\leq \alpha \log \frac{1}{1-|z|^2} + \log \frac{3}{2}.\end{aligned}$$

We obtain finally

$$\sup_{z\in\mathbb{D}} |u_N(z)|e^{-\frac{\alpha}{2}\log\frac{1}{1-|z|^2}} \leq \frac{2}{3} \sup_{z\in\mathbb{D}} |u_N(z)|e^{-\frac{1}{2}\widetilde{\psi}(z)} \lesssim 1. \tag{2.51}$$

Hence, by (2.47) and (2.50), $f_N = F_N - u_N \in \mathcal{A}_\alpha^\infty$. This completes the proof.

□

# Chapter 3

# Results on zero sets for Fock spaces

The chapter can stand alone. However, the concepts treated are very linked to those in the previous chapter. The two first sections present an essential review of the theory of entire functions and some needed tools to the study. The section is also an an introduction to Fock spaces and some of their properties. I emphasis that some elementary proofs are given for the completeness which are taken from [Zhu12; Boa54; Lev96; Zhu11; Rud+75]. The third section (3.2) is the essential one to this chapter. Indeed, it is my paper work with B. Bouya and Y. Omari in [ABO18]. The last 3.3 is a recent result in the paper [AO22].

## 3.1 Entire functions and Fock spaces

### 3.1.1 Entire functions

First, we collect and highlight several results about entire functions and zero sets in Fock spaces. The purpose is to fix notations and to give quickly some interesting properties of functions belonging to Fock spaces. All these results will be used to give a first idea about the distributions of zero sets in Fock space. Actually, at the present there is no complete characterization of such sets in Fock space. In addition, it is well known to experts that conditions depending on merely the moduli of the zeros, are far from being characterizations. The elusive interaction between arguments plays the key role and is not well understood. The truth is that zero sets for Fock spaces enjoy some mystery that we will discover, and make the subject more fascinating and attractive. Maybe this first section will be elementary lessons for experts. They can skip it and move forward to the section (3.2) where our results are exposed.

Let $\mathbb{C}$ denote the complex plane. We say that a function $f$ is an entire function when it is analytic on the complex plane $\mathbb{C}$. If $f$ is an entire function we define the zero set of $f$ by

$$Z(f) := \{z \in \mathbb{C} : f(z) = 0\}. \tag{3.1}$$

A fundamental result in complex analysis is the following identity theorem.

**Theorem 3.1.1.** *Let $f$ be an entire function. If the zero set of $f$, $Z(f)$ has a limit point in $\mathbb{C}$, then $f \equiv 0$ on $\mathbb{C}$.*

As a consequence of the above identity theorem, it follows that the zero set of an entire function which is not identically zero is a discrete sequence and cannot have any finite limit point also no value occurs infinitely many times in that sequence. Hence, the zero sequence $\{z_n\}_{n\geq 1}$ of an entire function is either (i) a finite sequence or (ii) satisfies the condition that $|z_n| \to \infty$ as $n \to \infty$. In particular, we can always arrange the zeros in the increasing order of their moduli; $|z_1| \leq |z_2| \leq ... \leq |z_n| \leq ....$ As we are concerned by Fock spaces, one of the key difference between this space and the classical Hardy and Bergman spaces is the lack of bounded entire functions as it tells **Liouville's theorem**: a bounded entire function is necessarily constant. In the shame of the study of zeros of analytic function in specific function spaces, an important tool in any such study is the classical **Jensen's formula** which we can drive from the **Green's formula** that we always meet in an elementary course on distribution or about Sobolev spaces.

**Theorem 3.1.2** (Jensen's formula)**.** *Suppose that*

**(a)** *$f$ is analytic in the closed disc $\overline{D}(0, r)$,*

**(b)** *$f$ does not vanish on the circle $|z| = r$,*

**(c)** *$f(0) = 1$, and*

**(d)** *the zeros of $f$ in the disk $D(0, r)$ are $\{z_1, ..., z_N\}$, with multiple zeros repeated according to multiplicity.*

*Then,*

$$\sum_{k=1}^{N} \log \frac{r}{|z_k|} = \frac{1}{2\pi} \int_0^{2\pi} \log |f(re^{i\theta})| d\theta. \tag{3.2}$$

If $f(0)$ is nonzero but not necessarily 1, Jensen's formula take the form

$$\log |f(0)| + \sum_{k=1}^{N} \log \frac{r}{|z_k|} = \frac{1}{2\pi} \int_0^{2\pi} \log |f(re^{i\theta})| d\theta, \tag{3.3}$$

here $\{z_1, ..., z_N\}$ are the zeros of $f$ in $0 < |z| < r$. More generally, if $f$ has a zero of order $k$ at the origin, we just apply the above formula to the function $\frac{f}{z^k}$ and we drive a variant formula. A well known procedure in the theory of entire function is the **Weierstrass factorization**, which allows to factor out the zeros of every entire function $f$ in a canonical way. The Weierstrass factorization is based on a collection of simple entire functions called elementary factors. Namely, we define

$$E_0(z) = 1 - z, \qquad (z \in \mathbb{C})$$

and for every positive integer $n$,

$$E_p(z) = (1-z)\exp(z + \frac{z^2}{2} + ... + \frac{z^p}{p}), \qquad (z \in \mathbb{C}).$$

Here we state the famous Weierstrass's Theorem.

**Theorem 3.1.3** ( [Boa54] **Weierstrass's theorem**). *Let $\{z_n\}_{n\geq 1}$ be a sequence of nonzero complex numbers such that $\{|z_n|\}$ is nondecreasing and tends to $\infty$. Then it is possible to choose a sequence $\{p_n\}_{n\geq 1} \subset \mathbb{N}$ such that*

$$\sum_{n=1}^{\infty}\Big(\frac{r}{|z_n|}\Big)^{p_n+1} < \infty, \qquad r > 0. \tag{3.4}$$

*Furthermore, the infinite product*

$$P(z) = \prod_{n=0}^{\infty} E_{p_n}(\frac{z}{z_n}) \tag{3.5}$$

*converge uniformly on every compact subset of $\mathbb{C}$, the function $P$ is an entire, and the zeros of $P$ are exactly $Z(P) = \{z_n\}_{n\geq 1}$, counting multiplicity.*

Even if a lot of results and their proof are well known in the literature, we will present some proofs that I found very interesting, and helpful later. Further, they don't need a lot of work. For the proof of Weierstrass theorem we need this elementary lemma.

**Lemma 3.1.4.** *For every $|z| < 1$ and any $p \geq 0$ we have*

$$|1 - E_p(z)| \leq |z|^{p+1}.$$

*Proof.* For $p = 0$ the result is evident. For $p \geq 1$, let consider the auxiliary function

$$z \longmapsto \phi_p(z) := \frac{1 - E_p(z)}{z^{p+1}}$$

$\phi_p$ is an entire function, because the singularity at the origin is removable. Indeed, 0 is a zero of $1 - E_p(z)$ of order $p+1$, to see this it is enough to compute the derivative. For every $z \in \mathbb{C}$ we have

$$(1 - E_p(z))' = -E_p'(z) = -z^p \exp(\sum_{n=1}^{k}\frac{z^n}{n}) = -z^p \sum_{n\geq 0} a_n z^n,$$

hence $z = 0$ is a zero of order $p$. Furthermore, the series above converge uniformly on every compact subset of $\mathbb{C}$ with positive Taylor coefficients $a_n$. Thus we can write

$$\phi_p(z) = \frac{1 - E_p(z)}{z^{p+1}} = \frac{1}{z^{p+1}} \int_{[0,z]} E_p'(z)dz = -\sum_{n\geq 0} \frac{a_n}{n+p+1} z^n$$

consequently, for every $|z| < 1$ we have

$$|\phi_p(z)| = \frac{|1 - E_p(z)|}{|z|^{p+1}} \leq \sum_{n\geq 0} \frac{a_n}{n+p+1} = \phi_p(1) = 1.$$

This completes the proof. □

Now we can prove Weierstrass's Theorem.

*Proof.* (*Weierstrass's theorem*) In order to prove that the infinite product (3.5) converge uniformly on every compact set of the complex plane, it is enough to prove that the series

$$\sum_{n\geq 1} \Big(1 - E_{p_n}(\frac{z}{z_n})\Big)$$

converges normally on every compact set of the complex plane. For this, if $K$ is a compact subset in $\mathbb{C}$, we can find some $r > 0$ such that $K \subseteq D(0, r)$, and after a bigger rank $N$ we have $|z_n| > r$, thus $\Big|\frac{z}{z_n}\Big| < 1$, for every $z \in K$, by the lemma 3.1.4 we have

$$\begin{aligned}
\sum_{n=1}^{\infty} \sup_{z\in K} \Big|1 - E_{p_n}(\frac{z}{z_n})\Big| &= \sum_{n=1}^{N} \sup_{z\in K} \Big|1 - E_{p_n}(\frac{z}{z_n})\Big| + \sum_{n=N+1}^{\infty} \sup_{z\in K} \Big|1 - E_{p_n}(\frac{z}{z_n})\Big| \\
&\leq \sum_{n=1}^{N} \sup_{z\in K} \Big|1 - E_{p_n}(\frac{z}{z_n})\Big| + \sum_{n=N+1}^{\infty} \sup_{z\in K} \Big|\frac{z}{z_n}\Big|^{p_n+1} \\
&\leq \sum_{n=1}^{N} \sup_{z\in K} \Big|1 - E_{p_n}(\frac{z}{z_n})\Big| + \sum_{n=N+1}^{\infty} \Big(\frac{r}{|z_n|}\Big)^{p_n+1} \quad < \infty.
\end{aligned}$$

Thus the series $\sum_{n\geq 1} \Big(1 - E_{p_n}(\frac{z}{z_n})\Big)$ converge normally and uniformly on very compact set in $\mathbb{C}$. □

Let's, give some interesting comments on the Weierstrass's Theorem (3.1.3). Note that if $|z_n| \to \infty$, the choice $p_n = n - 1$ will always satisfy condition (3.4). However, there are better choices in many cases. In particular $\{z_n\}_n$ is the zero sequence of an entire function $f$ and if there exists an integer $p$ such that

$$\sum_{n=1}^{\infty} \frac{1}{|z_n|^{p+1}} < \infty, \tag{3.6}$$

then $f$ is called to be of finite rank. Furthermore, if $p$ is the smallest integer such that (3.6) is satisfied, then $f$ is said to be of rank $p$. A function is of infinite rank if it is not of finite rank. If $\{z_n\}$ is the zero sequence of an entire function of rank $p \in \mathbb{N}$, then (3.6) is satisfied with $p_n = p$. The product $P$ associated with this canonical choice of $\{p_n\}_n$ will be called the standard form. Here we state the classical Hadamard's factorization Theorem.

**Theorem 3.1.5** ( [Boa54] **Hadamard's factorization).** *Let $f$ be an entire function of finite rank $p$. If $P$ is the standard product associated with the zero sequence $\{z_n\}_n$ of $f$. Then there exist a nonzero integer $m$ and an entire function $g$ such that*

$$f(z) = z^m P(z) e^{g(z)}, (z \in \mathbb{C}), \tag{3.7}$$

*where*

$$P(z) = \prod_{n \geq 1} E_p(\frac{z}{z_n}).$$

*The integer $m$ is unique, and the entire function $g$ is unique up to an additive constant of the form $2k\pi i$.*

It is evident from Jensen's formula that the more zeros an entire function has, the faster it must increase. In a general Jensen's formula seems to tell the whole truth unless some special restriction is imposed on the growth of the function in various directions or the position of the zeros (we can also think about classical Hardy spaces and Bergman spaces). This tells us that there is a connection between the rate of growth and the distribution of zeros of an entire function. Indeed, we are going to present some tools and functional that measure the rate of growth. Let $f$ be an entire function. For any $r > 0$, define the maximum modulus of $f$ in the circle $|z| = r$

$$M(r) := M_f(r) = \sup\{|f(z)| : |z| = r\}.$$

The quantity

$$\rho(f) := \rho = \limsup_{r \longrightarrow \infty} \frac{\log \log M_f(r)}{\log r} = \inf\{a > 0, \quad |f(z)| = O(e^{|z|^a})\}, \tag{3.8}$$

is called the order of the function $f$. It is clear that $0 \leq \rho \leq \infty$. We say that $f$ is of finite order when $\rho < \infty$. Otherwise, $f$ is said to be of infinite order. If $f$ is of finite order $\rho$, we define

$$\tau = \limsup_{r \longrightarrow \infty} \frac{\log M(r)}{r^\rho} = \inf\{a > 0, \quad |f(z)| \leq e^{a|z|^\rho}, \quad |z| \to \infty\}. \tag{3.9}$$

If $\tau < \infty$, we say that $f$ is of finite type. More specifically, we say that $f$ is of order $\rho$ and type $\tau$. If $\tau = \infty$ we say that $f$ is of maximum type or infinite type.

Let $\{z_n\}_n$ be the zero sequence, except 0, of an entire function $f$. The infimum of all positive numbers $s$ such that

$$\sum_{n=1}^{\infty} \frac{1}{|z_n|^s} < \infty \tag{3.10}$$

will be denoted $\rho_1 = \rho_1(f)$ and is called exponent of convergence. For useful idea the smallest positive **integer** $s$ satisfying the convergence condition (3.10) above will be denoted $m+1$. The following theorem from [Boa54, p. 17 ] (see also [Zhu12; Lev96]) gives some relation among the quantities defined above.

**Theorem 3.1.6.** *For any entire function $f$ that is not identically zero, we have the following relations:*

**(a)** $\rho_1(f) \leq m \leq \rho_f$.

**(b)** *If $\rho$ is not an integer, then $\rho_f = \rho_1(f)$.*

**(c)** $m = [\rho_1(f)]$ *if $\rho_1(f)$ is not an integer.*

**(d)** $\rho_1(f) - 1 \leq m \leq \rho_1(f) \leq \rho_f$ *in all case.*

*Here, $[x]$ means the the integer part.*

The theory of entire functions confirms that there is a lack of similarity (surprising of compatibility) between the zeros of those with non-integral orders, and those of integral order which dominates much of the theory. For example, every entire function with non-integral orders has an infinite set of zeros. The following result named after **Lindelöf's theorem** is one of the tools that we need in our study. Lindelof's Theorem is not standard in the sens it does not appears in most elementary complex analysis texts. For complete proof we refer to [Boa54] or [Lev96].

**Theorem 3.1.7** ( [Boa54; Lev96] **Lindelöf's theorem** )**.** *Suppose $f$ is an entire function of integral order $\rho$ ($\rho \in \mathbb{N}$), $f(0) \neq 0$ and $\{z_n\}$ is the zero sequence of $f$. Then $f$ is of finite type ($\tau_f < \infty$) if and only if the following two condition hold:*

**(a)** *$n(r) = O(r^\rho)$ as $r \to \infty$, where $n(r)$ is the number (counting multiplicity) of the zeros of $f$ in the disk $|z| \leq r$.*

**(b)** *The partial sums*

$$S(r) = \sum_{|z_n| \leq r} \frac{1}{z_n^\rho} \tag{3.11}$$

*are bounded in $r$.*

We should mention that there are several results in complex analysis that are called Lindelöf's Theorem. The one we state here is very interesting in our study. Furthermore, It has been used to provide an interpolation sequence which is not a zero sequence in Fock space. We refer to the work of Tung in [Tun06].

### 3.1.2 The Fock spaces

Let $dA$ denote the Euclidean area measure on the complex plane, that is $dA(z) = dxdy$, $z = x + iy$. We introduce Fock spaces of entire functions. Following the reference [Zhu12].

**Definition 3.1.8.** *For a positive parameter $\alpha$. The Fock space $\mathcal{F}_\alpha^p$, $0 < p \leq \infty$ is defined by*

$$\mathcal{F}_\alpha^p := \mathcal{F}_\alpha^p(\mathbb{C}) = \left\{ f : \mathbb{C} \to \mathbb{C} \ \textit{entire}, \ \|f\|_{p,\alpha}^p := \frac{\alpha p}{2\pi} \int_{\mathbb{C}} \left| f(z) e^{-\frac{\alpha}{2}|z|^2} \right|^p dA(z) \} < \infty \right\}. \tag{3.12}$$

*For $\alpha > 0$ and $p = \infty$. The Fock space $\mathcal{F}_\alpha^\infty$ is the space of entire functions which satisfy*

$$\|f\|_{\infty,\alpha} := \sup_{z\in\mathbb{C}} |f(z)| e^{-\frac{\alpha}{2}|z|^2} < \infty. \tag{3.13}$$

Sometimes it is useful to see Fock space $\mathcal{F}_\alpha^p$, $0 < p \leq \infty$ as

$$\mathcal{F}_\alpha^p := \mathcal{F}_\alpha^p(\mathbb{C}) = \left\{ f : \mathbb{C} \to \mathbb{C} \text{ entire} \right\} \cap L^p(\mathbb{C}, d\lambda_{\frac{p\alpha}{2}}),$$

where $L^p(\mathbb{C}, d\lambda_{\frac{p\alpha}{2}})$ the classical $L^p$ Lebesgue space of complex valued measurable functions associated with the **Gaussian** measure

$$d\lambda_\beta(z) = \frac{\beta}{\pi} e^{-\beta|z|^2} dA(z), \quad \beta > 0. \tag{3.14}$$

**Proposition 3.1.9.** *Let $w \in \mathbb{C}$. The application point evaluation*

$$\begin{array}{rccc} \Lambda_w : & \mathcal{F}_\alpha^p & \longrightarrow & \mathbb{C} \\ & f & \longmapsto & f(w), \end{array}$$

*is a bounded linear functional on $\mathcal{F}_\alpha^p$, $0 < p \leq \infty$, with norm $\|\Lambda_w\| = e^{\frac{\alpha}{2}|w|^2}$.*

*Proof.* First, we work with $0 < p < \infty$. Apply the mean value inequality to the sub-harmonic function $|f|^p$ and integrate on polar coordinates, we get

$$|f(0)|^p \leq \frac{\alpha p}{2\pi} \int_{\mathbb{C}} \left| f(z) e^{-\frac{\alpha}{2}|z|^2} \right|^p dA(z). \tag{3.15}$$

Then by translation argument we replace $f$ bay $h(z) = f(w - z) e^{\alpha z\overline{w} - \frac{\alpha}{2}|w|^2}$ and the formula (3.15) becomes

$$\begin{aligned} |f(w) e^{-\frac{\alpha}{2}|w|^2}|^p &\leq \frac{\alpha p}{2\pi} \int_{\mathbb{C}} |f(w-z) e^{\alpha z\overline{w} - \frac{\alpha}{2}|w|^2} e^{-\frac{\alpha}{2}|z|^2}|^p dA(z) \\ &= \frac{\alpha p}{2\pi} \int_{\mathbb{C}} |f(w-z) e^{-\frac{\alpha}{2}|w-z|^2}|^p dA(z) \\ &= \frac{\alpha p}{2\pi} \int_{\mathbb{C}} |f(z) e^{-\frac{\alpha}{2}|z|^2}|^p dA(z) \quad = \quad \|f\|_{p,\alpha}^p. \end{aligned}$$

Thus,

$$|f(w)| \leq e^{\frac{\alpha}{2}|w|^2} \|f\|_{p,\alpha}.$$

And

$$\|\Lambda_w\| \leq e^{\frac{\alpha}{2}|w|^2}.$$

On the other hand, consider the function $h(z) = e^{\alpha z\overline{w} - \frac{\alpha}{2}|w|^2}$, that belongs to $\mathcal{F}^p_\alpha$. We have

$$\|h\|^p_{p,\alpha} = \frac{\alpha p}{2\pi} \int_{\mathbb{C}} |e^{\alpha z\overline{w} - \frac{\alpha}{2}|w|^2} e^{-\frac{\alpha}{2}|z|^2}|^p dA(z) = \frac{\alpha p}{2\pi} \int_{\mathbb{C}} e^{-\frac{p\alpha}{2}|z-w|^2} dA(z) = 1.$$

$\Lambda_w(h) = h(w) = e^{\frac{\alpha}{2}|w|^2}$. As a consequence

$$\|\Lambda_w\| = e^{\frac{\alpha}{2}|w|^2}.$$

For the case $p = \infty$. For every $f \in \mathcal{F}^\infty_\alpha$ we have

$$|f(w)|e^{-\frac{\alpha}{2}|w|^2} \leq \sup_{u\in\mathbb{C}} |f(u)|e^{-\frac{\alpha}{2}|u|^2} = \|f\|_{\infty,\alpha},$$

thus $\|\Lambda_w\| \leq e^{\frac{\alpha}{2}|w|^2}$. Again the function (or any normalized polynomial)

$$h(z) := e^{\alpha z\overline{w} - \frac{\alpha}{2}|w|^2}, \qquad z \in \mathbb{C},$$

gives the desired result. □

**Proposition 3.1.10.** *If $1 \leq p \leq \infty$, then $\mathcal{F}^p_\alpha$ endowed with the norm $\|.\|_{p,\alpha}$ is a vector Banach space. Furthermore, the particular case $p = 2$, $\mathcal{F}^2_\alpha$ is a Hilbert space with the fallowing inner product*

$$\langle f, g\rangle = \int_{\mathbb{C}} f(z)\overline{g(z)} e^{-\frac{\alpha}{2}|z|^2} dA(z), \qquad f, g \in \mathcal{F}^2_\alpha.$$

*inherited from $L^2(\mathbb{C}, d\lambda_\alpha)$. When $0 < p < 1$, $\mathcal{F}^p_\alpha$ is a complete metric space with the distance $d(f, g) = \|f - g\|^p_{p,\alpha}$.*

*Proof.* For the proof, it is enough to show that $\mathcal{F}^p_\alpha$ is a closed subspace of $L^p_\alpha$. Let $(f_n)_{n\in\mathbb{N}}$ be a sequence in $\mathcal{F}^p_\alpha$ which converge to $f \in L^p_\alpha$. This shows that $(f_n)_{n\in\mathbb{N}}$ is a Cauchy sequence in $L^p_\alpha$. Since for every $z \in \mathbb{C}$ and $n, m \in \mathbb{N}$ we have

$$|f_n(z) - f_m(z)|e^{-\frac{\alpha}{2}|z|^2} \leq \|f_n - f_m\|_{p,\alpha}.$$

$(f_n)_{n\in\mathbb{N}}$ is a Cauchy sequence for the topology of convergence uniform on compact set of $\mathbb{C}$. Consequently $(f_n)_{n\in\mathbb{N}}$ is convergent to an entire function. By the uniqueness of limit, we conclude that $f$ is an entire function. Hence $f \in \mathcal{F}^p_\alpha$. □

**Proposition 3.1.11.** *If $1 \leq p < q \leq \infty$, then $\mathcal{F}^p_\alpha \subset \mathcal{F}^q_\alpha$, and the inclusion is proper and continuous.*

*Proof.* For every function $f \in \mathcal{F}^p_\alpha$, using the point-wise estimate we get

$$\begin{aligned}
\|f\|^q_{q,\alpha} &= \frac{q\alpha}{2\pi}\int_{\mathbb{C}} |f(z)|^q e^{-\frac{q\alpha}{2}|z|^2} dA(z) \\
&= \frac{q\alpha}{2\pi}\int_{\mathbb{C}} |f(z)|^{q-p}|f(z)|^p e^{-\frac{q\alpha}{2}|z|^2} dA(z) \\
&\leq \frac{q\alpha}{2\pi}\|f\|^{q-p}_{p,\alpha}\int_{\mathbb{C}} |f(z)|^p e^{-\frac{p\alpha}{2}|z|^2} dA(z) \\
&= \frac{q}{p}\|f\|^{q-p}_{p,\alpha}\|f\|^p_{p,\alpha} \quad = \quad \frac{q}{p}\|f\|^q_{p,\alpha}.
\end{aligned}$$

This shows the continuous inclusion $\mathcal{F}^p_\alpha$ in $\mathcal{F}^q_\alpha$.
Suppose that the inclusion $\mathcal{F}^p_\alpha \subset \mathcal{F}^q_\alpha$ is not proper, then the identity application

$$I : \mathcal{F}^p_\alpha \longrightarrow \mathcal{F}^q_\alpha$$

is bounded and invertible. The open mapping theorem ensure the existence of $C > 0$ such that

$$\frac{1}{C}\|f\|_{p,\alpha} \leq \|f\|_{q,\alpha} \leq C\|f\|_{p,\alpha}, \qquad \forall f \in \mathcal{F}^p_\alpha. \tag{3.16}$$

And for every $n \in \mathbb{N}$,

$$\begin{aligned}
\|z^n\|^p_{p,\alpha} &= \frac{\alpha p}{2\pi}\int_{\mathbb{C}} |z|^{np} e^{-\frac{\alpha p}{2}|z|^2} dA(z) \\
&= \frac{\alpha p}{2\pi}\int_0^\infty \int_0^{2\pi} r^{np} e^{-\frac{\alpha p}{2}r^2} r dr d\theta \\
&= (\frac{2}{\alpha p})^{np}\int_0^\infty x^{\frac{np}{2}} e^{-x} dx \\
&= (\frac{2}{\alpha p})^{np}\Gamma(\frac{np}{2}+1).
\end{aligned}$$

And a computation with Sterling's formula shows that *

$$\|z^n\|_{p,\alpha} \asymp (\frac{n}{\alpha p})^n n^{\frac{1}{2p}}, \qquad \|z^n\|_{q,\alpha} \asymp (\frac{n}{\alpha q})^n n^{\frac{1}{2q}}, \qquad n \in \mathbb{N}.$$

Thus, with a little analysis we get a contradiction with (3.16). □

The study of zeros sets for Fock spaces have a lot of motivations. Let us recall the definition in order to fix ideas.

**Definition 3.1.12.** *1. A sequence $Z = \{z_n\}_n$ of complex numbers is called* ***a zero sequence*** *(or zero set) for Fock space $\mathcal{F}^p_\alpha$, $0 < p \leq \infty$, if there exist a nonzero function $f \in \mathcal{F}^p_\alpha$, such that $Z$ is the zero sequence of $f$, counting multiplicities.*

*The Sterling's formula: $\Gamma(z) \asymp \sqrt{2\pi} z^{z-\frac{1}{2}} e^{-z}, \qquad |arg(z)| < \pi$.

2. *A sequence $Z = \{z_n\}_n$ of complex numbers is called* ***a uniqueness sequence*** *(or uniqueness set) for Fock space $\mathcal{F}^p_\alpha$, $0 < p \leq \infty$, if every function $f$ in $\mathcal{F}^p_\alpha$ that vanishes on $Z$ must be identically zero.*

The following result establishes the maximum order and type for functions in Fock spaces.

**Theorem 3.1.13.** *Let $\alpha$ and $0 < p \leq \infty$. If $f \in \mathcal{F}^p_\alpha$ then $f$ is of order less than or equal to $2$. Furthermore, if $f$ is of order $2$, then $f$ must be of type less than or equal to $\frac{\alpha}{2}$.*

*Proof.* Let $f \in \mathcal{F}^p_\alpha$, then

$$|f(z)| \leq Ce^{\frac{\alpha}{2}|z|^2}, \qquad (z \in \mathbb{C}).$$

Hence

$$\rho(f) = \limsup_{r\to\infty} \frac{\log\log M_f(r)}{\log r} \leq \limsup_{r\to\infty} \frac{\log(C + \frac{\alpha}{2}r^2)}{\log r} = 2.$$

And if $\rho(f) = 2$ then

$$\tau(f) = \limsup_{r\to\infty} \frac{\log M_f(r)}{r^2} \leq \limsup_{r\to\infty} \frac{\frac{\alpha}{2}r^2 + C}{r^2} = \frac{\alpha}{2}.$$

□

We state the necessary condition for zero sets in the Fock spaces $\mathcal{F}^p_\alpha$ obtained by K. Zhu in [Zhu93]. A well exposed proof is in [Zhu12].

**Theorem 3.1.14** ([Zhu12] Necessary condition)**.** *Let $f \in \mathcal{F}^p_\alpha$, $0 < p \leq \infty$. Suppose $Z = \{z_n\}_n$ is the zero sequence of $f$, then there exists a positive constant $c$ and a rearrangement of $\{z_n\}$ such that*

$$|z_n| \geq C\sqrt{n}, \qquad (n \in \mathbb{N}).$$

*Proof.* Arrange the sequence $\{z_n\}$ such that $|z_1| \leq |z_2| \leq ... \leq |z_n| \leq |z_{n+1}|$. Without loss of generality we can assume $f(0) = 1$, then a direct application of Jensen's formula (3.1.2) gives

$$\log \prod_{k=1}^{n(r)} \frac{r}{|z_n|} = \int_0^{2\pi} \log|f(re^{i\theta})| \frac{d\theta}{2\pi}.$$

Therefore, for any $n \geq 1$ we have

$$\log \prod_{k=1}^{n} \frac{r}{|z_n|} \leq \log \prod_{k=1}^{n(r)} \frac{r}{|z_n|}.$$

Hence

$$\log(\frac{r}{|z_n|})^n \leq \log\prod_{k=1}^{n}\frac{r}{|z_n|} \leq \frac{\alpha}{2}r^2 + C.$$

Consequently

$$\frac{1}{|z_n|} \leq \frac{1}{r}e^{\frac{\alpha r^2}{2n}+\frac{C}{n}}, \qquad r > 0, n \geq 1.$$

Apply the last formula to a nicely chosen sequence $(r_k)$ which converge to $\sqrt{n}$ and such that $f$ has no zero on each $|z| = r_{n,k}$. Therefore, as $k \to \infty$ we obtain

$$\frac{1}{|z_n|} \lesssim \frac{1}{\sqrt{n}}e^{\frac{\alpha}{2}+\frac{C}{n}}, \qquad (\forall n \geq 1). \tag{3.17}$$

It is clear that there is some positive constant $c'$ such that $|z_n| \geq c'\sqrt{n}$, for every $n \in \mathbb{N}$. □

Therefore, from Theorem 3.1.14 we obtain the following condition, that we will refer sometimes as the necessary condition also.

**Corollary 3.1.15.** *If $f \in \mathcal{F}^p_\alpha$ and $Z(f) = \{z_n\}$, such that $f(0) \neq 0$, then*

$$\sum_{n\geq 1}\frac{1}{|z_n|^{2+\epsilon}} < \infty, \quad \textit{for any } \epsilon > 0. \tag{3.18}$$

The function

$$f(z) = \frac{\sin(\delta z^2)}{\delta z^2} = \prod_{n=1}^{\infty}\Big(1 - \frac{\delta^2 z^4}{n^2\pi^2}\Big), \tag{3.19}$$

shows that the estimate in Theorem 3.1.14 is the best possible ( for some choice of $\delta > 0$). More precisely, we can find a constant $c > 0$ such that

$$\frac{1}{c}\sqrt{n} \leq |z_n| \leq c\sqrt{n}, \qquad (n \geq 1).$$

**Theorem 3.1.16** ([Zhu12] Sufficiency condition)**.** *Let $Z = \{z_n\}_{n\geq 1}$ be a sequence of complex numbers ( excluding zero) such that*

$$\sum_{n\geq 1}\frac{1}{|z_n|^2} < \infty. \tag{3.20}$$

*Then $Z = \{z_n\}_{n\geq 1}$ is a zero sequence for Fock space $\mathcal{F}^p_\alpha$, $0 < p \leq \infty$. Moreover*

$$f(z) = \prod_{n\geq 1}E_1\big(\frac{z}{z_n}\big) \in \mathcal{F}^p_\alpha.$$

*Proof.* Just for technical adjustment, we may assume that $\{z_n\}$ has been ordered in such a way that $\{|z_n|\}_n$ is nondecreasing. By the Theorem 3.1.3 introduced at the begging of this chapter the Weierstrass product

$$f(z) = \prod_{n\geq 1} E_1(\frac{z}{z_n})$$

converge and define an entire function $f$ with zero sequence $\{z_n\}$, under the condition (3.20). It's remain to show that the function $f$ belongs to all Fock spaces $\mathcal{F}^p_\alpha$, $0 < p \leq \infty$ and $\alpha > 0$. If $|z| < \frac{1}{2}$,we have

$$\begin{aligned} \log|E_1(z)| &= Re(z + \log(1-z)) \\ &= Re(-\sum_{n\geq 2} \frac{z^n}{n}) \\ &\leq \sum_{n\geq 2} \frac{|z|^n}{n} \\ &\leq |z|^2 \sum_{n\geq 2} \frac{1}{2^n} = |z|^2. \end{aligned}$$

On the other hand,

$$|E_1(z)| \leq (1+|z|)e^{|z|}, \qquad (z \in \mathbb{C}), \tag{3.21}$$

$$\log|E_1(z)| \leq \log(1+z) + |z|, \qquad (z \in \mathbb{C}) \tag{3.22}$$

Then for any $A > 0$, there exists a radius $R > 0$ such that

$$\log|E_1(z)| \leq A|z|^2, \qquad |z| \geq R.$$

On the other side, the function $\frac{\log|E_1(z)|}{|z|^2}$ is continuous on the corona $\{\frac{1}{2} \leq |z| \leq R\}$ except at $z = 1$ where it takes $-\infty$. Hence, there exists a constant $B$ such that

$$\log|E_1(z)| \leq B|z|^2, \qquad \frac{1}{2} \leq |z| \leq R.$$

Consequently

$$\log|E_1(z)| \leq M|z|^2, \qquad z \in \mathbb{C}$$

where $M = \max\{1, A, B\}$. Under the condition (3.20), for $\varepsilon > 0$ there exists $N \in \mathbb{N}$ such that $\sum_{n\geq N+1} \frac{1}{|z_n|^2} \leq \frac{\varepsilon}{M}$, Thus

$$\sum_{n\geq N+1} \log|E_1(\frac{z}{z_n})| \leq \varepsilon|z|^2.$$

From the inequality (3.22) there exists $R > 0$ such that

$$\log|E-1(z)| \leq \frac{\varepsilon}{S}|z|^2, \qquad |z| \geq R$$

where $S = \sum_{n=1}^{N} \frac{1}{|z|^2}$. Hence, for $R' = R|z_N|$, we have

$$\sum_{n=1}^{N} \log |E_1(\frac{z}{z_n})| \leq \varepsilon, \qquad |z| \geq R'.$$

As a result

$$\log |f(z)| \leq 2\varepsilon |z|^2, \qquad |z| \to \infty.$$

Hence it is enough to take $\varepsilon$ small enough in the way that $2\varepsilon < \frac{\alpha}{2}$, and we obtain $f \in \mathcal{F}^p_\alpha$. □

### 3.1.3 Some pathological properties of zero sets in Fock spaces

In this section, we will present some surprising pathological properties of zero sets in Fock spaces in order to motivate an open question that we will gives a positive answer. In the contrast with the situation in Hardy space $\mathcal{H}^p(\mathbb{D})$, $0 < p < \infty$ for the unit disk we know that the union of two zero sequences is again a zero sequence and every sub-sequence of a zero sequence for is again a zero sequence for the same space. However, in the Bergman space $\mathcal{A}^2_0(\mathbb{D})$ the union of two zero sequences is not necessarily a zero sequence and a famous result of Horowetz in 1974 assert that every sub-sequence of a zero sequence is also a zero sequence in the case of Bergman space. This proposition from [Zhu12] gives us the situation in Fock spase case.

**Proposition 3.1.17** ([Zhu12]). *Let $\alpha > 0$ and $0 < p \leq \infty$.*

1. *There exist two zero sequences $Z_1$ and $Z_2$ for $\mathcal{F}^p_\alpha$ whose union is no longer a zero sequence for $f\mathcal{F}^p_\alpha$.*
2. *There exist a zero sequence $Z = \{z_n\}$ for $\mathcal{F}^p_\alpha$ and sub-sequence $\{z_{n_k}\}$ of $Z$ which is not a zero sequence anymore for $\mathcal{F}^p_\alpha$.*

*Proof.* 1. Let $\delta \in (\frac{\alpha\pi}{8}, \frac{\alpha}{2})$, and consider the two sequences

$$Z_1 := \{e^{\frac{ik\pi}{2}} \sqrt{\frac{n\pi}{\delta}}, \quad k = 0, 1, 2, 3, n \geq 1\}, \qquad Z_2 = e^{\frac{i\pi}{4}} Z_1.$$

The two entire functions

$$f(z) = \frac{\sin(\delta z^2)}{\delta z^2}, \qquad g(z) = \frac{\sin(\delta e^{\frac{i\pi}{2}} z^2)}{\delta z^2}$$

have there zero sets as $Z_1$ and $Z_2$ respectively. Moreover,

$$|f(z)| \leq e^{\delta |z|^2}, \qquad , |g(z)| \leq e^{\delta |z|^2}, \qquad (z \in \mathbb{C}). \tag{3.23}$$

Since $\delta < \frac{\alpha}{2}$, then $f$ and $g$ belongs to the Fock space $\mathcal{F}^p_\alpha$, for every $0 < p \leq \infty$. Suppose that $Z_1 \cup Z_2 = \{z_n\}$ becomes a zero sequence for Fock space $\mathcal{F}^p_\alpha$. If we come back to the proof of the Theorem 3.1.14 it follows that there exists a constant $C > 0$ such that

$$\prod_{k=1}^{n} \frac{r}{|z_k|} \leq Ce^{\frac{\alpha}{2}r^2}, \quad n \geq 1, \quad r > 0.$$

We take the last inequality for $8n$ instead of $n$ and square both sides inequality we obtain

$$\prod_{k=1}^{8n} \frac{r^2}{|z_k|^2} \leq Ce^{\alpha r^2}, \quad n \geq 1, \quad r > 0.$$

Now, we integrate both sides of the above inequality form 0 to $+\infty$ with respect to the measure $e^{-\beta r^2} r dr$ (for $\beta > \alpha$), we get

$$\left(\frac{\delta}{\beta\pi}\right)^{8n} \frac{\left(8n\right)!}{\left(n!\right)^8} = \left(\frac{\delta}{\beta\pi}\right)^{8n} (8n)! \left(\prod_{k=1}^{n} \frac{1}{k}\right)^k \leq C.$$

We use the Stirling's formula $n! \sim \sqrt{n} n^n e^{-n}$, to get

$$(\frac{8\delta}{\beta\pi})^{8n} \frac{\sqrt{n}}{n^4} \leq C,$$

for $n$ large enough, for the last expression we must have $\frac{8\delta}{\beta\pi} \leq 1$, and since $\beta > \alpha$ and arbitrarily close to $\delta \leq \frac{\pi\alpha}{8}$, this contradict the fact that $\delta \in (\frac{\pi\alpha}{8}, \frac{\alpha}{2})$.

2. Let consider again the sequence

$$Z := \{e^{\frac{ik\pi}{2}} \sqrt{\frac{2n\pi}{\delta}}, \quad k = 0, 1, 2, 3, n \geq 1\}$$

which is a zero sequence $\mathcal{F}^p_\alpha$, $0 < p \leq \infty$. However, if the sub-sequence

$$Z' = \{\pm \sqrt{\frac{2\pi n}{\delta}}; \quad n \geq 1\}$$

remains a zero sequence for $\mathcal{F}^p_\alpha$, then there exists $f \in \mathcal{F}^p_\alpha$ such $Z(f) = Z'$. Thus

$$m(f) = \inf\{s \in \mathbb{N}; \quad \sum_{n} \frac{1}{|z_n|^{s+1}} < \infty\} = 2 \leq \rho(f) \leq 2$$

this ensures, that $f$ is of order 2 and of type at least $\frac{\alpha}{2}$. However,

$$S(r) = \sum_{|z_n| \leq r} \frac{1}{z_n^2} \simeq \sum_{n \leq r^2} \frac{1}{n} \simeq \log(r)$$

which is not bounded in $r$, this contradict the Lindelof's Theorem 3.1.7.

□

This instability of zeros sequence for Fock spaces $f \in \mathcal{F}^p_\alpha$ leads as to ask a question: *What those zero sets of $\mathcal{F}^p_\alpha$ which enjoy the property: every sub-sequence remains also a zero sequence?*

### 3.1.4 Weierstrass $\sigma-$function

In this section we will give same examples of zero sets in Fock space by the prototypes of several Weierstrass's $\sigma-$functions. In the first step we come back to the construction of this Weierstrass function by periodicity or quasi-periodicity. Further, the same idea can be used for the modified $\sigma-$function which has played a crucial role in the characterization of interpolating and sampling sequence in Fock space, by K. Seip achievement. Our goal is to discus the dependence of zero sequences with the parameters $\alpha > 0$ and $0 < p \leq \infty$ for different Fock spaces $\mathcal{F}^p_\alpha$. Actually, this is our point of departure to give an answer to a question asked by K. Zhu in [Zhu12, p.209]. Namely,

*for $p \neq q$, do $\mathcal{F}^p_\alpha$ and $\mathcal{F}^q_\alpha$ always have different zero sequences?*

We will give a positive answer to this question in the section (3.2).
Again, the most part in this section is taken from [Zhu12, p.209] and [Zhu11] (Including proofs). All examples that we will give are constructed from lattices. The simplest example is the standard integer lattice in $\mathbb{C}$

$$\mathbb{Z}^2 := \{m + in : m \in \mathbb{Z}, n \in \mathbb{Z}\}.$$

In general, let $w_1$ and $w_2$ any nonzero complex numbers (exactly two independent vectors). The set

$$\Lambda = \Lambda(0, w_1, w_2) = \{w_{m,n} = mw_1 + nw_2, m, n \in \mathbb{Z}\} \tag{3.24}$$

is called the lattice generated by 0, $w_1$ and $w_2$. To every such lattice we associate the function $P := P_\Lambda$ (named Weierstrass's $P-$function)

$$P(z) = P_\Lambda(z) := \frac{1}{z^2} + \sum_{m,n}{}' \left( \frac{1}{(z - w_{m,n})^2} - \frac{1}{w_{m,n}^2} \right) \tag{3.25}$$

and the $\zeta$ Weierstrass function

$$\zeta(z) = \zeta_\Lambda(z) := \frac{1}{z} + \sum_{m,n}{}' \left( \frac{1}{z - w_{m,n}} + \frac{1}{w_{m,n}} + \frac{z}{w_{m,n}^2} \right). \tag{3.26}$$

Where the summation (with a prime) is taken over all integers $m$ and $n$ with $(m, n) \neq (0, 0)$. The following proposition lists some interesting properties of the Weierstrass's $P-$function.

**Proposition 3.1.18** ([Zhu12]).

1. *$P$ is an even, meromorphic function in the complex plane $\mathbb{C}$ whose poles are the points in the lattice $\Lambda$. Furthermore, $P$ is doubly periodic with periods $w_1$ and $w_2$.*

$$P(z + w_k) = P(z), \qquad k = 1, 2, \qquad z \in \mathbb{C} \setminus \Lambda.$$

2. *$\zeta$ is an odd, meromorphic function in the complex plane $\mathbb{C}$ whose poles are the points in the lattice $\Lambda$. Furthermore, $\zeta$ is "quasi-periodic", i.e. for $k = 1, 2$ we have*

$$\zeta(z + w_k) = \zeta(z) + \eta_k, \qquad z \in \mathbb{C} \setminus \Lambda,$$

*where $\eta_k = 2\zeta(\frac{w_k}{2})$.*

*Proof.* 1. Let $\varepsilon > 0$ small enough. We set

$$U_\varepsilon := \{z \in \mathbb{C} / \quad d(z, \Lambda) > \varepsilon, \qquad |z| \leq \frac{1}{\varepsilon}\}.$$

For every $z \in U_\varepsilon$ we have

$$\frac{1}{(z - w_{m,n})^2} - \frac{1}{w_{m,n}^2} = O\Big(\frac{1}{|w_{m,n}|^3}\Big),$$

for $|w_{m,n}|$ large enough. Moreover

$$\sum_{m,n}{}' \frac{1}{|w_{m,n}|^3} < \infty.$$

Hence, the series (3.25) converge uniformly and absolutely ( on compact subset) to an analytic function on $U_\varepsilon$, for every $\varepsilon > 0$. Hence on $\mathbb{C} \setminus \Lambda$ and define a holomorphic on $\mathbb{C} \setminus \Lambda$. It is clear that $P$ has a double pole at each point $w_{m,n} \in \Lambda$. Hence, $P$ is meromorphic with double poles at precisely the points of $\Lambda$.
Note that the two sequences $\{w_{m,n}\}$ and $\{-w_{m,n}\}$ are the same.

$$\begin{aligned} P(-z) &= \frac{1}{z^2} + \sum_{m,n}{}' \frac{1}{(-z - w_{m,n})^2} - \frac{1}{w_{m,n}^2} \\ &= \frac{1}{z^2} + \sum_{m,n}{}' \frac{1}{(z - (-w_{m,n}))^2} - \frac{1}{(-w_{m,n})^2} \\ &= \frac{1}{z^2} + \sum_{m,n}{}' \frac{1}{(z - w_{m,n})^2} - \frac{1}{w_{m,n}^2} \quad = \quad P(z). \end{aligned}$$

Hence $P$ is an even function. By the uniform convergence of the series on $\mathbb{C} \setminus \Lambda$, we have for every $z \in \mathbb{C} \setminus \Lambda$,

$$P'(z) = -2 \sum_{m,n} \frac{1}{(z - w_{m,n})^3}.$$

So for $k = 1, 2$ we have

$$P'(z + w_k) = -2 \sum_{m,n} \frac{1}{(z + w_k - w_{m,n})^3} = -2 \sum_{m,n} \frac{1}{(z - w_{m,n})^3} = P'(z).$$

This imply (by integration),

$$P(z+w_k) = P(z) + c_k, \qquad k = 1,2, \qquad z \in \mathbb{C} \setminus \Lambda.$$

Since $P$ is even, then for $z = -\frac{w_k}{2}$ on a

$$P(\frac{w_k}{2}) = P(-\frac{w_k}{2} + w_k) = P(-\frac{w_k}{2}) + c_k = P(\frac{w_k}{2}) + c_k.$$

Hence ,$c_k = 0$ and $P$ is doubly periodic with periods $w_1$ and $w_2$.

2. Let $\varepsilon > 0$. For $z \in U_\varepsilon$, we have

$$\frac{1}{z - w_{m,n}} - \frac{1}{w_{m,n}} + \frac{z}{w_{m,n}^2} = O\Big(\frac{1}{|w_{m,n}|^3}\Big),$$

as $|w_{m,n}| \to \infty$. Therefore, the series in (3.26) is uniformly convergent on compact subset of $\mathbb{C}\setminus\Lambda$ and define a holomorphic function on $\mathbb{C}\setminus\Lambda$. It is clear that $P$ has a simple pole at each point $w_{m,n} \in \Lambda$. Hence, $P$ is meromorphic with simple poles at precisely the points of $\lambda$. Furthermore,

$$\begin{aligned}
\zeta(-z) &= -\frac{1}{z} + {\sum_{m,n}}' \frac{1}{-z - w_{m,n}} + \frac{1}{w_{m,n}} - \frac{z}{w_{m,n}^2} \\
&= -\frac{1}{z} - {\sum_{m,n}}' \frac{1}{z - (-w_{m,n})} + \frac{1}{-w_{m,n}} + \frac{z}{(-w_{m,n})^2} \\
&= -\frac{1}{z} - {\sum_{m,n}}' \frac{1}{z - w_{m,n}} + \frac{1}{w_{m,n}} + \frac{z}{w_{m,n}^2} \quad = \quad -\zeta(z).
\end{aligned}$$

Hence $\zeta$ is odd. By the uniform convergence of the series on $\mathbb{C} \setminus \Lambda$, we have

$$\zeta'(z) = -\frac{1}{z^2} + {\sum_{m,n}}' \frac{-1}{(z - w_{m,n})^2} + \frac{1}{w_{m,n}^2} = -P(z),$$

for every $z \in \mathbb{C} \setminus \Lambda$, coupled with the condition

$$\lim_{z\to 0} \Big(\zeta(z) - \frac{1}{z} = 0\Big).$$

Therefore,

$$\zeta'(z+w_k) = \zeta'(z), \qquad k = 1,2.$$

Which say

$$\zeta(z+w_k) = \zeta(z) + \eta_k, \qquad z \in \mathbb{C} \setminus \Lambda.$$

By the parity of $\zeta$, we en replace $z$ by $-\frac{w_k}{2}$ in the last equation, we get $\eta_k = 2\zeta(\frac{w_k}{2})$.

□

**Lemma 3.1.19.** *The periods $w_k$ and the constants $\eta_k$ are related by the following equation:*

$$\eta_1 w_2 - \eta_2 w_1 = 2i\pi. \tag{3.27}$$

*Proof.* We translate the parallelogram spanned by $w_1$ and $w_2$ and centered at $\frac{w_1+w_2}{2}$ by the vector $-\frac{w_1+w_2}{2}$, we come back to another parallelogram $R = R_\Lambda$ with center 0 and vertex

$$\frac{w_1+w_2}{2}, \qquad \frac{w_1-w_2}{2}, \qquad \frac{w_2-w_1}{2}, \qquad -\frac{w_1+w_2}{2}.$$

$R$ is called the fundamental region of the lattice $\Lambda$. By the construction $\zeta$ is analytic on $R$, up to the boundary, except a simple pole at the center of R (which is the origin) with residue 1. Therefore,

$$\int_{\partial R} \zeta(z)d(z) = 2i\pi.$$

Or

$$\int_{\partial R} \zeta(z)d(z) = \int_{I_1} \zeta(z)d(z) + \int_{I_2} \zeta(z)d(z) + \int_{I_3} \zeta(z)d(z) + \int_{I_4} \zeta(z)d(z),$$

Where $(I_k)_{1\leq k\leq 4}$ are the boundary segments of the parallelogram $R$ with direct orientation started from $\frac{w_1+w_2}{2}$. One remark that

$$I_4 = I_2 + w_2, \qquad I_1 = I_3 + w_1.$$

And using the quasi-periodicity of $\zeta$, we obtain the desired result. □

To every lattice $\Lambda = \Lambda(0, w_1, w_2)$, we associate another function called Weierstrass's $\sigma-$function defined by

$$\sigma(z) := \sigma_\Lambda(z) = z\prod_{m,n}{}' E_2\left(\frac{z}{w_{m,n}}\right) = z\prod_{m,n}{}'(1 - \frac{z}{w_{m,n}})e^{\frac{z}{w_{m,n}}+\frac{z^2}{2w_{m,n}^2}}, \quad z \in \mathbb{C}. \tag{3.28}$$

Where the product is taken over all pair over integers such that $(m,n) \neq (0,0)$. The following proposition gives some fundamental properties of the Weierstrass $\sigma-$function.

**Proposition 3.1.20** ([Zhu12]). *For every lattice $\Lambda = \{w_{m,n}\}$ the associated Weierstrass function $\sigma_\Lambda$ is an entire function whose zero set is exactly the lattice $\Lambda$. Furthermore, $\sigma$ is odd and quasi-periodic in the following sense:*

$$\sigma(z + w_k) = e^{\eta_k(z+\frac{w_k}{2})}(z), \tag{3.29}$$

*where $\eta_k$ are the constants from (3.1.18).*

*Proof.* Just recall the Weierstrass product introduced in the section (3.1.1). Since we have

$$\sum_{m,n}{}' \frac{1}{|w_{m,n}|^{2+1}} < \infty.$$

The infinite product in (3.28) converges uniformly absolutely on any compact subset of the complex plane to an entire function by Weierstrass' theorem (3.5). It is also clear that the zero set of $\sigma$ is exactly the lattice $\Lambda = \{w_{m,n}\}$. Again observe that the lattice $\{w_{m,n}\}$ and $\{-w_{m,n}\}$ are the same

$$\begin{aligned}
\sigma(-z) &= -z\prod_{m,n}{}'(1 - \frac{-z}{w_{m,n}})e^{\frac{-z}{w_{m,n}}+\frac{z^2}{2w_{m,n}^2}} \\
&= -z\prod_{m,n}{}'(1 - \frac{z}{-w_{m,n}})e^{\frac{z}{-w_{m,n}}+\frac{z^2}{2(-w_{m,n})^2}} \\
&= -z\prod_{m,n}{}'(1 - \frac{z}{w_{m,n}})e^{\frac{z}{w_{m,n}}+\frac{z^2}{2w_{m,n}^2}} \quad = \quad -\sigma(z).
\end{aligned}$$

Hence, $\sigma$ is an odd function. On the other hand,

$$\frac{d}{dz}\log(\sigma(z)) = \frac{1}{z} + \sum_{m,n}{}' \frac{\frac{-1}{w_{m,n}}}{1-\frac{z}{w_{m,n}}} + \frac{1}{w_{m,n}} + \frac{z}{w_{m,n}^2} = \zeta(z).$$

Thus

$$\frac{d}{dz}\log(\sigma(z+w_k)) = \frac{d}{dz}\log(\sigma(z)) + \eta_k, \qquad k = 1, 2.$$

This imply

$$\log(\sigma(z+w_k)) = \log(\sigma(z)) + \eta_k z + c_k, \qquad k = 1, 2.$$

Therefore,

$$\sigma(z+w_k) = c_k' e^{\eta_k z}\sigma(z), \qquad k = 1, 2.$$

By the parity of $\sigma$ is even, we can substitute in the equation above $z = -\frac{w_k}{2}$, we obtain

$$c_k' = -e^{\eta_k \frac{w_k}{2}}.$$

Consequently, $\sigma$ is quasi-periodic in the sens

$$\sigma(z+w_k) = -e^{\eta_k(z+\frac{w_k}{2})}\sigma(z), \qquad k = 1, 2.$$

□

Let $\alpha$ be a positive parameter. Consider the special lattice generated by $w_1 = \sqrt{\frac{\pi}{\alpha}}$ and $w_2 = iw_1 = i\sqrt{\frac{\pi}{\alpha}}$

$$\Lambda = \Lambda_\alpha = \sqrt{\frac{\pi}{\alpha}}\mathbb{Z}^2 = \left\{w_{m,n} = \sqrt{\frac{\pi}{\alpha}}(m+in), \qquad m,n \in \mathbb{Z}\right\}. \tag{3.30}$$

In this particular case, we can compute the constant $\eta_k$ and relate the quasi-periodicity of $_\Lambda$ to a certain isometry on Fock spaces.

**Proposition 3.1.21.** *[Zhu12] Let $\sigma_\alpha$ the associated Weierstrass $\sigma-$function to the lattice $\Lambda_\alpha$ in (3.30). We have*

$$\eta_1 = \sqrt{\alpha\pi}, \qquad \eta_2 = -i\sqrt{\alpha\pi}. \tag{3.31}$$

*Furthermore,*

$$e^{\alpha\overline{w_{m,n}}z-\frac{\alpha}{2}|w_{m,n}|^2}\sigma(z-w_{m,n}) = (-1)^{m+n+mn}\sigma(z), \qquad m,n\in\mathbb{Z}, (z\in\mathbb{C}). \tag{3.32}$$

*Proof.* Note that for this particular case, for every $m,n\in\mathbb{Z}$ we have

$$w_{m,n} = \sqrt{\frac{\pi}{\alpha}}(m+in) = i\sqrt{\frac{\pi}{\alpha}}(n-im) = iw_{n,m'},$$

where $m' = -m$. Also we know that $\eta_1 = 2\zeta(\frac{1}{2}\sqrt{\frac{\pi}{\alpha}})$ and $\eta_2 = 2\zeta(\frac{i}{2}\sqrt{\frac{\pi}{\alpha}})$. It follows that for every $z\in\mathbb{C}$

$$\begin{aligned}
\zeta(iz) &= \frac{1}{iz} + \sum_{m,n}{}' \frac{1}{iz-w_{m,n}} + \frac{1}{w_{m,n}} + \frac{iz}{w_{m,n}^2} \\
&= \frac{1}{iz} + \sum_{m,n}{}' \frac{1}{iz-iw_{n,m'}} + \frac{1}{iw_{n,m'}} + \frac{iz}{(iw_{n,m'})^2} \\
&= \frac{1}{iz} + \frac{1}{i}\sum_{n,m'}{}' \frac{1}{z-w_{n,m'}} + \frac{1}{w_{n,m'}} + \frac{z}{w_{n,m'}^2} \quad = \quad \frac{1}{i}\zeta(z).
\end{aligned}$$

Therefore, $\zeta(i\frac{1}{2}\sqrt{\frac{\pi}{\alpha}}) = \frac{1}{i}\zeta(\frac{1}{2}\sqrt{\frac{\pi}{\alpha}})$, this along with 3.31 gives $\eta_2 = \frac{1}{i}\eta_1$. Coming back to the equation

$$\eta_1 w_2 - \eta_2 w_1 = 2i\pi,$$

we obtain $\eta_1 = \sqrt{\alpha\pi}$ and $\eta_2 = -i\sqrt{\alpha\pi}$. To prove related to the translation in (3.32) observe that

$$\sigma(z+w_k) = -e^{\eta_k(z+\frac{w_k}{2})}\sigma(z).$$

And by induction

$$\sigma(z+mw_k) = (-1)^m e^{m\eta_k z+\frac{m^2}{2}w_k\eta_k}\sigma(z), \qquad m\in\mathbb{Z}.$$

Therefore

$$\begin{aligned}
\sigma(z+w_{m,n}) &= \sigma(z+mw_1+nw_2) \\
&= (-1)^n e^{(n\eta_2(z+mw_1)+\frac{n^2}{2}w_2\eta_2)}\ \sigma(z+mw_1) \\
&= (-1)^{m+n} e^{n\eta_2(z+mw_1)+m\eta_1 z+\frac{m^2}{2}\eta_1 w_1+\frac{n^2}{2}w_2\eta_2}\ \sigma(z).
\end{aligned}$$

Since $\eta_2 = -i\eta_1 = -i\sqrt{\alpha\pi}$ and $w_2 = iw_1 = \sqrt{\frac{\pi}{\alpha}}$, then

$$n\eta_2(z+mw_1) + m\eta_1 z + \frac{m^2}{2}\eta_1 w_1 + \frac{n^2}{2}w_2\eta_2 = \alpha\overline{w_{m,n}}z - \frac{\alpha}{2}|w_{m,n}|^2 + mni\pi.$$

Thus,

$$e^{\alpha\overline{w_{m,n}}z-\frac{\alpha}{2}|w_{m,n}|^2}\ \sigma(z-w_{m,n}) = (-1)^{m+n+mn}\sigma(z),$$

for every $w_{m,n}\in\Lambda_\alpha$ and $z\in\mathbb{C}$. $\square$

**Corollary 3.1.22** ([Zhu12]). *Let $\alpha$ be a positive parameter. Consider Weierstrass's function $\sigma_\alpha$ (3.28) associated with the lattice $\Lambda_\alpha$ in (3.30). Then,*

1. *$z \mapsto |\sigma(z)|e^{-\frac{\alpha}{2}|z|^2}$ is a doubly periodic function with periods $w_1 = \sqrt{\frac{\pi}{\alpha}}$ and $w_2 = \sqrt{\frac{\pi}{\alpha}}i$,*

2. *For every $z \in \mathbb{C}$*
$$|\sigma(z)|e^{-\frac{\alpha}{2}|z|^2} \asymp d(z, \Lambda_\alpha), \tag{3.33}$$
*where $d(z, \Lambda_\alpha)$ denotes the Euclidean distance from $z$ to the lattice $\Lambda_\alpha$).*

*Proof.* 1. It follows from the quasi-periodicity of $\sigma_\alpha$ as in (3.32). In general for every $z \in \mathbb{C}$, we have $d(z, \Lambda) = |z - w_{m,n}|$, where $w_{m,n}$ the nearest point of $\Lambda$ to $z$. Since $|\sigma(z)|e^{-\frac{\alpha}{2}|z|^2}$ is periodic, then
$$\frac{|\sigma(z)|e^{-\frac{\alpha}{2}|z|^2}}{d(z, \Lambda)} \asymp \frac{|\sigma(z)|e^{-\frac{\alpha}{2}|z|^2}}{|z - w_{m,n}|},$$
which is a bounded function in $\mathbb{C}$. This leads to the desired result.

□

Now we come to a position, to provide some elementary examples of zeros sets (and uniqueness sets). Let $\alpha$ be a positive parameter. Consider the lattice

$$\Lambda = \Lambda_\alpha = \sqrt{\frac{\pi}{\alpha}}\mathbb{Z}^2 = \left\{\omega_{mn} = \sqrt{\frac{\pi}{\alpha}}(m + in), \qquad m, n \in \mathbb{Z}\right\}. \tag{3.34}$$

The set $\Omega_\alpha$ is called the fundamental region

$$\Omega_\alpha = \left\{z = x + iy, |x| \leq \frac{1}{2}\sqrt{\frac{\pi}{\alpha}}, |y| \leq \frac{1}{2}\sqrt{\frac{\pi}{\alpha}}\right\}. \tag{3.35}$$

We recall the Weierstrass function

$$\sigma_\alpha(z) = \sigma_{\Lambda_\alpha}(z) := z \prod_{w_{m,n}\Lambda_\alpha \setminus (0,0)} \left(1 - \frac{z}{w_{m,n}}\right) \exp\left(\frac{z}{w_{m,n}} + \frac{z^2}{w_{m,n}^2}\right), (z \in \mathbb{C}). \tag{3.36}$$

$$\sigma_\alpha(z) = z{\prod_{m,n}}'(1 - \frac{z}{w_{m,n}}) \exp(\frac{z}{w_{m,n}} + \frac{z^2}{2w_{m,n}^2}), \qquad z \in \mathbb{C}, \tag{3.37}$$

The most examples of zero sets for Fock space $\mathcal{F}^p_\alpha$, that we can construct are the lattices or sequences uniformly closed to some lattices an the function is the Weierstrass's $\sigma-$function.

**Lemma 3.1.23** ([Zhu12]). *Let $\alpha_1 < \alpha < \alpha_2$. We have,*

1. *$\sigma_\alpha \in \mathcal{F}^p_{\alpha_2}$ for any $0 < p \leq \infty$.*

2. $\sigma_\alpha \notin \mathcal{F}^p_{\alpha_1}$ *for any* $0 < p \leq \infty$.

3. $\sigma_\alpha \in \mathcal{F}^\infty_\alpha$.

*Proof.* 1. If $p = \infty$, we make use of corollary (3.1.22)

$$|\sigma(z)|e^{-\frac{\alpha}{2}|z|^2} \asymp d(z, \Lambda_\alpha), \qquad z \in \mathbb{C}.$$

Hence,

$$|\sigma(z)|e^{-\frac{\alpha_2}{2}|z|^2} \asymp d(z, \Lambda_\alpha)e^{-\frac{\alpha_2-\alpha}{2}|z|^2}, \qquad z \in \mathbb{C}.$$

Also,

$$\sup_{z\in\mathbb{C}} |\sigma(z)|e^{-\frac{\alpha_2}{2}|z|^2} < \infty.$$

So $\sigma_\alpha \in \mathcal{F}^\infty_{\alpha_2}$.
If $0 < p < \infty$, for every $z \in \mathbb{C}$ we have

$$|\sigma(z)|^p e^{-p\frac{\alpha}{2}|z|^2} \leq \|\sigma_\alpha\|^p_{\infty,\alpha} e^{-p\frac{\alpha_2-\alpha}{2}|z|^2}.$$

Integration leads to

$$\|\sigma_\alpha\|^p_{p,\alpha} \lesssim \int_{\mathbb{C}} e^{-p\frac{\alpha_2-\alpha}{2}|z|^2} dA(z) < \infty.$$

2. If $p = \infty$. For every $z \in \mathbb{C}$ we have

$$|\sigma_\alpha(z)|e^{-\frac{\alpha_1}{2}|z|^2} \asymp d(z, \Lambda_\alpha)e^{\frac{\alpha-\alpha_1}{2}|z|^2}, \qquad \alpha - \alpha_1 > 0.$$

Hence $\sigma_\alpha \notin \mathcal{F}^\infty_{\alpha_1}$. On the other hand, since $\mathcal{F}^p_{\alpha_1} \subset \mathcal{F}^\infty_{\alpha_1}$ for all $0 < p \leq \infty$, we get $\sigma_\alpha \notin \mathcal{F}^p_{\alpha_1}$ for all $0 < p \leq \infty$.

3. Follows from and the periodicity property of $|\sigma(z)|e^{-\frac{\alpha}{2}|z|^2}$.

□

**Lemma 3.1.24.** *Let* $\alpha > 0, 0 < p < \infty$ *and* $f \in \mathcal{F}^p_\alpha$. *Consider the lattice* $\Lambda_\alpha$ *given in* (3.34). *If* $f(\lambda) = 0$ *for every* $\lambda \in \Lambda_\alpha$, *then* $f$ *is identically zero.*

*Proof.* By Weierstrass-Hadamard's factorization Theorem 3.7 we can write $f(z) = \sigma_\alpha(z)h(z)$, for some entire function $h$. In view of the quasiperiodicity of $\sigma_\alpha$ (3.32) it follows

$$
\begin{aligned}
\int_{\mathbb{C}} |f(z)|^p e^{-\frac{p\alpha}{2}|z|^2} dA(z) &= \sum_{m,n} \int_{\Omega_\alpha + \omega_{mn}} \left| |\sigma_\alpha(z)| e^{-\frac{\alpha}{2}|z|^2} |h(z)| \right|^p dA(z) \\
&= \sum_{m,n} \int_{\Omega_\alpha} \left| |\sigma_\alpha(z-\omega_{mn})| e^{-\frac{\alpha}{2}|z-\omega_{mn}|^2} |h(z-\omega_{m,n})| \right|^p dA(z) \\
&= \sum_{m,n} \int_{\Omega_\alpha} \left| \frac{|\sigma_\alpha(z-\omega_{mn})| e^{-\frac{\alpha}{2}|z-\omega_{m,n}|^2}}{|z-\omega_{mn}|} |z-\omega_{mn}| |h(z-\omega_{m,n})| \right|^p dA(z) \\
&= \sum_{m,n} \int_{\Omega_\alpha} \left| \frac{|\sigma_\alpha(z+\omega_{mn})| e^{-\frac{\alpha}{2}|z+\omega_{m,n}|^2}}{d(z,\Lambda_\alpha)} |z-\omega_{mn}| |h(z-\omega_{m,n})| \right|^p dA(z) \\
&\asymp \sum_{mn} \int_{\Omega_\alpha} |z+\omega_{mn}|^p |h(z+\omega_{m,n})|^p dA(z) \\
&\asymp \int_{\mathbb{C}} |zh(z)|^p dA(z) < \infty.
\end{aligned}
$$

This is impossible unless $h$ is identically zero. Another way, to do this, is to use the the sub-harmonicity of the function $z \mapsto |h(z+w_{m,n})|^p$. □

**Theorem 3.1.25.** *Suppose* $0 < p, q \leq \infty$ *and* $\alpha_1 \neq \alpha_2$*. Then* $\mathcal{F}^p_{\alpha_1}$ *and* $\mathcal{F}^q_{\alpha_2}$ *have different zero sets.*

*Proof.* Let $\alpha_1 < \alpha < \alpha_2$. From Lemma 3.1.23 we have $\sigma_\alpha \in \mathcal{F}^q_{\alpha_2}$, hence $\Lambda_\alpha$ is a zero set for $\in \mathcal{F}^q_{\alpha_2}$. Therefore, if $f \in \mathcal{F}^p_{\alpha_1} \subset \mathcal{F}^2_\alpha$ and $f_{|\Lambda_\alpha} = 0$, the Lemma 3.1.24 asserts $f \equiv 0$. Therefore, $\Lambda_\alpha$ is not a zero set for $\mathcal{F}^p_{\alpha_1}$. □

We state again the Zhu question for zero sets is the following: Do two Fock space $\mathcal{F}^p_\alpha$ and $\mathcal{F}^q_\alpha$ have different zero sets whenever $p \neq q$?(as asked in [Zhu12] ). Until 2017 this question was open. However, for the case $0 < p < \infty$ and $q = \infty$ it is easy to find such pairs that do not have the same zero sets. The easy example is the lattice $Z = \Lambda_\alpha$, which is a zero set $\mathcal{F}^\infty_\alpha$, but it is uniqueness set for $\mathcal{F}^p_\alpha$ for every $0 < p < \infty$. In fact, it is a consequence of (3.1.23) and (3.1.24).

Also, $Z^1 = \Lambda_\alpha \backslash \{0\}$ $Z(\mathcal{F}^p_\alpha)$ for any $p > 2$, because $f(z) = \frac{\sigma_\alpha(z)}{z}$ belongs to $\mathcal{F}^p_\alpha$ if and only if$p > 2$. In fact,

$$
\begin{aligned}
\int_{\mathbb{C}} |f(z)|^p e^{-\frac{p\alpha}{2}|z|^2} dA(z) &= \int_{|z|\leq 1} \left| \frac{\sigma_\alpha(z)}{z} \right|^p e^{-\frac{p\alpha}{2}|z|^2} dA(z) \\
&+ \int_{|z|>1} \left| \frac{\sigma_\alpha(z)}{z} \right|^p e^{-\frac{p\alpha}{2}|z|^2} dA(z) \\
&= O(1) + \int_{|z|>1} \frac{\left| \sigma_\alpha(z) e^{-\frac{\alpha}{2}|z|^2} \right|^p}{|z|^p} dA(z) \\
&\asymp O(1) + \int_1^\infty \frac{1}{r^{p-1}} dr.
\end{aligned}
$$

And the last integral converge if and only if $p > 2$. It remains to shows that $Z$ is a uniqueness set for $\mathcal{F}^2_\alpha$. To do so, assume there exists $f \in \mathcal{F}^2_\alpha$ such that $Z(f) = Z^1 = \Lambda_\alpha \backslash \{0\}$. Writing $f(z) = \frac{\sigma_\alpha(z)}{z} g(z)$ where $g$ is an entire function. Then, as in the proof of Lemma 3.1.24

$$\begin{aligned}
\int_{\mathbb{C}} |f(z)|^2 e^{-\frac{2\alpha}{2}|z|^2} dA(z) &= \sum_{m,n} \int_{\Omega_\alpha + \omega_{mn}} \Big| \frac{|\sigma_\alpha(z)| e^{-\frac{\alpha}{2}|z|^2}}{z} |g(z)| \Big|^2 dA(z) \\
&= \sum_{m,n} \int_{\Omega_\alpha} \Big| \frac{|\sigma_\alpha(z-\omega_{mn})| e^{-\frac{\alpha}{2}|z-\omega_{mn}|^2}}{z-\omega_{mn}} |g(z-\omega_{mn})| \Big|^2 dA(z) \\
&\asymp \sum_{m,n} \int_{\Omega_\alpha} \Big| \frac{|\sigma_\alpha(z-\omega_{mn})| e^{-\frac{\alpha}{2}|z-\omega_{mn}|^2}}{d(z,\Lambda_\alpha)} |g(z-\omega_{mn})| \Big|^2 dA(z) \\
&\asymp \sum_{m,n} \int_{\Omega_\alpha} |g(z-\omega_{mn})| \Big|^2 dA(z) \\
&\asymp \int_{\mathbb{C}} |g(z)|^2 dA(z) < \infty.
\end{aligned}$$

We get that $g$ is identically zero. Hence $Z^1$ is a uniqueness set for $\mathcal{F}^2_\alpha$.

On the other hand, the sequence $Z^2 = \Lambda_\alpha \backslash \{a, b\}$ where $a$ and $b$ are any points from $\Lambda$, is a zero set for $\mathcal{F}^2_\alpha$. In fact, the function

$$f(z) = \frac{\sigma_\alpha(z)}{(z-a)(z-b)}$$

belongs to $\mathcal{F}^2_\alpha$ and $Z(f) = Z^2$.

Examples show that, generally it is very delicate to distinguish between zero sets for $\mathcal{F}^p_\alpha$ and $\mathcal{F}^q_\alpha$. Namely, for any integer $N$ such that $pN > 2$, if $Z$ is a zero set for $\mathcal{F}^q_\alpha$ and if $N$ points $\{z_1, ..., z_N\}$ are removed from $Z$, then $Z^{'} = Z \backslash \{z_1, ..., z_N\}$ is a zero set for $\mathcal{F}^p_\alpha$. To do so, if $f \in \mathcal{F}^q_\alpha$ a non zero function with $Z(f) = Z$, then $Z^{'}$ is a zero set for the function

$$g(z) = \frac{f(z)}{(z-z_1)...(z-z_N)}$$

which belongs to $\mathcal{F}^p_\alpha$ (Easy). Therefore, the zero set for $\mathcal{F}^p_\alpha$ and $\mathcal{F}^q_\alpha$ may be different, but they are not too much different. Hence the characterization of zero sets $\mathcal{F}^p_\alpha$ remains a great challenge!

In the Hardy space of unit disk, i.e.

$$H^2(\mathbb{D}) := \{ f \in \mathrm{Hol}(\mathbb{D}) \ / \ \|f\|_2^2 = \sup_{0<r<1} \int_0^{2\pi} |f(re^{i\theta})|^2 d\theta < \infty \}$$

Blaschke's Theorem says

$$\Lambda = \{a_n\}_{n \geq 1} \in Z(H^2(\mathbb{D})) \Longleftrightarrow \sum_{n \geq 1} 1 - |a_n| < \infty.$$

For every zero set $\Lambda=\{a_n\}_{n\geq 1}$ of the Hardy space $\mathcal{H}^2(\mathbb{D})$, we associate the Blaschke product

$$B(z)=\prod_{n=1}^{\infty}\frac{|a_n|}{a_n}\frac{z-a_n}{1-z\bar{a}_n}.$$

Consider the shift invariant subspace of$\mathcal{H}^2(\mathbb{D})$:

$$BH^2=\{f\in H^2(\mathbb{D})\ \ /\ \ f_{|\Lambda}=0\}.$$

We can shows that always that $BH^2$ is a closed infinite dimensional subspace of $H^2(\mathbb{D})$. We mention that the same result remain valid for Bergman space $\mathcal{A}^2_0(\mathbb{D})$.This is no longer true for Fock spaces $\mathcal{F}^p_\alpha$.

**Definition 3.1.26.** *Let $\alpha>0$, $0<p\leq\infty$. For every zero set $Z$ of $\mathcal{F}^p_\alpha$ we define*

$$I^P_Z:=\{f\in\mathcal{F}^p_\alpha\ \ /\ \ f(z)=0,\ z\in Z\},$$

*which is a subspace of $\mathcal{F}^p_\alpha$.*

**Theorem 3.1.27** ([Zhu12],[Zhu11])**.** *For every $0<p\leq\infty$ and $k\in\{1,2,3,...\}\cup\{+\infty\}$, there exists a zero set $Z$ for $\mathcal{F}^p_\alpha$ such that $dim(I^P_Z)=k$.*

*Proof.* For $k=+\infty$ it is trivial, any finite sequence can be taken. Assume $k<+\infty$. For the case $p=+\infty$ and $k>1$. Let $Z^k=\Lambda_\alpha\backslash\{a_1,...,a_{k-1}\}$ where $a_1,...,a_{k-1}$ are any distinct points (removed) from $\Lambda_\alpha$ and

$$f(z)=\frac{\sigma_\alpha(z)}{(z-a_1)....(z-a_{k-1})}$$

hence, $f\in\mathcal{F}^\infty_\alpha$ and $Z(f)=Z^k$. Therefore, if $P$ is a polynomial of degree less than or equal to $k-1$, then

$$Pf\in\mathcal{F}^\infty_\alpha\qquad\text{and}\qquad (Pf)(z)=0\ ,z\in Z^k.$$

Hence, $dim(I^\infty_{Z^k})\geq k$.

On the other hand, if $F\in\mathcal{F}^\infty_\alpha$ is vanishing on $Z^k$, then we can write

$$F(z)=f(z)g(z)=\frac{\sigma_\alpha(z)g(z)}{(z-a_1)....(z-a_{k-1})},$$

with $g$ is an entire function ( that we will show is a polynomial of of degree less than or equal $k-1$). For any $n\in\mathbb{N}^*$, let $C_n=\partial Q(\sqrt{\frac{\pi}{\alpha}}\frac{2n+1}{2})$ the boundary of the square centered at 0 with and vertical and horizontal side length $l=\sqrt{\frac{\pi}{\alpha}}\frac{2n+1}{2}$. It is evident

$$dist(C_n,\Lambda_\alpha)\geq\frac{1}{2}\sqrt{\frac{\pi}{\alpha}},\qquad n\geq 1.$$

So, there exists a constant $C$ such that

$$|\sigma_\alpha(z)|e^{-\frac{\alpha}{2}|z|^2} > C, \qquad z \in C_n, \qquad n \geq 1. \tag{3.38}$$

Since, for every $z \in \mathbb{C}$ we have

$$|F(z)|e^{-\frac{\alpha}{2}|z|^2} = \frac{|\sigma_\alpha(z)|e^{-\frac{\alpha}{2}|z|^2}|g(z)|}{|z-a_1|...|z-a_{k-1}|}$$

which gives

$$\frac{|g(z)|}{|z-a_1|...|z-a_{k-1}|} = \frac{|F(z)|e^{-\frac{\alpha}{2}|z|^2}}{|\sigma_\alpha(z)|e^{-\frac{\alpha}{2}|z|^2}}.$$

In particular, (using $F \in \mathcal{F}_\alpha^\infty$)

$$|g(z)| \leq C|z-a_1|...|z-a_{k-1}|, \qquad z \in C_n, \qquad n \geq 1.$$

By Cauchy integral estimates , $g$ is a polynomial of of degree less than or equal $k-1$. In conclusion. In case $p=\infty$ and $k>1$ and $Z^k = \Lambda_\alpha \setminus \{a_1,...,a_{k-1}\}$. We have showed that $F \in \mathcal{F}_\alpha^\infty$ vanishes on $Z^k$ if and only if

$$F(z) = \frac{\sigma_\alpha(z)g(z)}{(z-a_1)....(z-a_{k-1})},$$

where $g$ is a polynomial of of degree less than or equal $k-1$. So $dim(I_{Z^k}^\infty) = k$. Furthermore, there exist a function $f \in \mathcal{F}_\alpha^\infty$ such that

$$I_{z^k}^\infty = f\mathbf{P}_{k-1}, \qquad Z(f) = Z^k.$$

Where $\mathbf{P}_{k-1}$ is the vector space of polynomials of of degree less than or equal to $k-1$. For the case $p=\infty$ and $k=1$,we work with $Z=\Lambda_\alpha$ the same argument above can be relaxed to prove that $F \in I_Z^\infty$ if and only if $F(z) = c\sigma_\alpha(z)$, where $c$ is a constant.

For the case $0 < p < +\infty$. Let $N$ the smallest integer such that $Np > 2$ or equivalent to

$$\int_{|z|>1} \left|\frac{\sigma_\alpha(z)e^{-\frac{\alpha}{2}|z|^2}}{z^N}\right|^p < \infty. \tag{3.39}$$

Remove $N+k-1$ points $\{a_1, a_2, ..., a_{N+k-11}\}$ from $\Lambda_\alpha$ and set $Z = \Lambda_\alpha \setminus \{a_1, a_2, ..., a_{N+k-11}\}$. Then $Z$ is the zero sequence of the function

$$f(z) = \frac{\sigma_\alpha(z)}{(z-a_1)...(z-a_{N+k-1})} \in I_Z^p \subset \mathcal{F}_\alpha^p.$$

Therefore, for any polynomial $Q$ of degree less than or equal to $k-1$, we have $Q(z)f(z) \in I_Z^p \subset \mathcal{F}_\alpha^p$. Hence, $dim(I_Z^p) \geq k$.

Conversely,if $F$ is any function which lives in $\mathcal{F}^p_\alpha$ and vanishes on $Z$, then we can write

$$F(z) = \frac{\sigma_\alpha(z)g(z)}{(z-a_1)...(z-a_{N+k-1})},$$

where $g$ is an entire function. Since $\mathcal{F}^p_\alpha \subset \mathcal{F}^\infty_\alpha$, by estimates of Cauchy integral formula 3.38 we obtain that $g$ is a polynomial of of degree less than or equal $N+K-1$. if the degree of $g$ is $j > k-1$, then

$$\frac{g(z)}{(z-a_1)...(z-a_{N+k-1})} \asymp \frac{1}{z^{N+k-j-1}}, \qquad z \longrightarrow \infty.$$

So,

$$\begin{aligned} |F(z)|e^{-\frac{\alpha}{2}|z|^2} &\asymp \frac{|\sigma_\alpha(z)|e^{-\frac{\alpha}{2}|z|^2}}{|z|^{N+k-j-1}}, \\ &\asymp \frac{d(z,Z)}{|z|^{N+k-j-1}}, \qquad z \longrightarrow \infty. \end{aligned}$$

Since $F \in \mathcal{F}^p_\alpha$ we must have

$$\int_{|z|>R} \frac{1}{|z|^{p(N+k-j-1)}} dA(z) < \infty$$

that is true if and only if $p(N+k-j-1) > 2$, which contradict the choice of $N$ in 3.39. Therefore, $dim(I^p_Z) \leq k$. We conclude, $dim(I^p_Z) = k$. □

The following result gives the structure of $I^p_Z$ when is of finite dimension.

**Theorem 3.1.28.** *If $Z$ is a zero set for$\mathcal{F}^p_\alpha$ and $dim(I^p_Z) = k$, a positive integer, then there exist $g \in I^p_Z$ such that $I^p_Z = g\mathbf{P}_{k-1}$, where $\mathbf{P}_{k-1}$ is the vector space of polynomials with degree less than or equal to $k-1$.*

*Proof.* We will come back on the proof later. □

**Lemma 3.1.29.** *Let $Z$ be a zero set for $\mathcal{F}^p_\alpha$ and $dim(I^p_Z) = k < \infty$. Then $Z \cup \{a_1, ..., a_k\}$ is always a uniqueness set for $\mathcal{F}^p_\alpha$.*

*Proof.* Assume $Z' = Z \cup \{a_1, ..., a_k\}$ is not of uniqueness for $\mathcal{F}^p_\alpha$, there exist $f \in \mathcal{F}^p_\alpha$ not identically zero such that $f_{|Z'} = 0$. Hence the function

$$f(z), \qquad \frac{f(z)}{z-a_1}, \qquad \frac{f(z)}{z-a_2}, \qquad ..., \qquad \frac{f(z)}{z-a_k},$$

are linearly independent in $I^p_Z$, Hence $dim(I^p_Z) \geq k+1$, leads to a contradiction. □

**Theorem 3.1.30.** *Let $Z$ be a zero set for $\mathcal{F}^p_\alpha$. If $dim(I^p_Z) > k$, a then $Z \cup \{a_1, ..., a_k\}$ will never be a uniqueness set for $\mathcal{F}^p_\alpha$.*

*Proof.* If $dim(I_Z^p) > k$, take $k+1$ linearly independent functions in $I_Z^p$, says $f_1, .., f_{k+1}$. Let $\{a_1, .., a_k\}$ be a collection complex numbers. Looking for a functions $f \in \mathcal{F}_\alpha^p$ which vanishes on $Z' = Z \cup \{a_1, ..., a_k\}$. Consider the combination

$$f(z) = c_1 f_1(z) + c_2 f_2(z) + ... + c_{k+1} f_{k+1}(z) \tag{3.40}$$

And the system (with unknown $c_j$)

$$c_1 f_1(a_j) + c_2 f_2(a_j) + ... + c_{k+1} f_{k+1}(a_j) = 0, \qquad 1 \geq j \leq k. \tag{3.41}$$

up to some modification when the the zeros have are of higher multiplicities. The system (3.41) is of $k$ equations and $k+1$ unknown $(c_j)$. Hence, it has always a solution $c_j, 1 \geq j \leq k+1$. For a choice of $C_j$ the function $f$ (3.40) vanishes on $Z \cup \{a_1, ..., a_k\}$. Hence $Z \cup \{a_1, ..., a_k\}$ is not a uniqueness set for $\mathcal{F}_\alpha^p$. □

**Corollary 3.1.31.** *Let $Z$ be a zero set for $\mathcal{F}_\alpha^p$ and $k$ a positive integer. The following assertions are equivalent:*

1. *$dim(I_Z^p) \leq k.$,*
2. *The sequence $Z \cup \{a_1, ..., a_k\}$ is a uniqueness set for $\mathcal{F}_\alpha^p$ for any $\{a_1, ..., a_k\}$,*
3. *The sequence $Z \cup \{a_1, ..., a_k\}$ is a uniqueness set for $\mathcal{F}_\alpha^p$ for some $\{a_1, ..., a_k\}$.*

*Proof.* The implication (1) to (2) is due to the lemma **??**, and to the fact . (2) imply (3) is trivial. The Theorem3.1.30 shows that (3) imply (1). □

**Corollary 3.1.32.** *Let $Z$ be a zero set for $\mathcal{F}_\alpha^p$ and $k$ positive integer. The following assertions are equivalent:*

1. *$dim(I_Z^p) = k.$*
2. *For any $\{a_1, ..., a_k\}$ the sequence $Z \cup \{a_1, ..., a_{k-1}\}$ is not a uniqueness set for $\mathcal{F}_\alpha^p$ but $\{a_1, ..., a_k\}$ is a uniqueness set.*
3. *For some $\{a_1, ..., a_k\}$ the sequence $Z \cup \{a_1, ..., a_{k-1}\}$ is not a uniqueness set for $\mathcal{F}_\alpha^p$ but $\{a_1, ..., a_k\}$ is uniqueness set.*

*Proof.* Following with the same ideas as before. □

Moreover, if $dim(I_Z^p) = k$, then adding $k$ points to $Z$ we get a uniqueness set for $\mathcal{F}_\alpha^p$, but adding less than $k$ points to $Z$ remain a zero set. And we have the extends result.

**Corollary 3.1.33.** *If $Z$ is a zero set for $\mathcal{F}_\alpha^p$, $dim(I_Z^p) = k \in \mathbb{N}^*$ and $j < k$, then $Z \cup \{a_1, ..., a_j\}$ is always a zero set for $\mathcal{F}_\alpha^p$.*

*Proof.* By the corollary 3.1.32 $Z' = Z \cup \{a_1, ..., a_j\}$ is not a uniqueness set for $\mathcal{F}^p_\alpha$. Hence, there exists $f \in \mathcal{F}^p_\alpha$ which vanishes on $Z'$. Again by 3.1.32 the execs number of zeros is

$$N = Card(Z(f) \backslash Z') \leq k - j,$$

denote $Z(f) \backslash Z' = \{b_j\}_{1 \leq j \leq N}$. Then

$$g(z) = \frac{f(z)}{(z - b_1)...(z - b_N)} \in \mathcal{F}^p_\alpha \text{ and } Z(g) = Z'.$$

Therefore, $Z'$ is a zero set of $\mathcal{F}^p_\alpha$. □

We summarize, if $Z$ is a zero set in $\mathcal{F}^p_\alpha$ and $dim(I^p_Z) = k$, then adding less than $k$ points to $Z$ remain always a zero set for $\mathcal{F}^p_\alpha$, but adding exactly (or more) $k$ points to $Z$, The resulting set is of uniqueness. Again, this shows the unpredictability of zero sets in $\mathcal{F}^p_\alpha$.

## 3.2 Contribution on zero sets for Fock spaces $\mathcal{F}^p_\alpha$

In this section we will present our main contribution for zero sets in Fock spaces, with complete proofs. As we have seen $\mathcal{F}^p_\alpha$ and $\mathcal{F}^p_\beta$ always possess different zeros set in the case $\alpha \neq \beta$. It remains to investigate the case of distinct positive numbers $p$ and $q$. That is, the spaces $\mathcal{F}^p_\alpha$ and $\mathcal{F}^q_\alpha$ possess different zero sets or not; as K. Zhu asked in [Zhu12, Page. 209]. We answer positively to this question by considering a special translation of the square lattice with uniform density. we mention that the notion of density (see, [SW92; Zhu12]) is the principal tools in the characterization of Seip and Walston of sampling and interpolation sequences in Fock space $\mathcal{F}^p_\alpha$. We recall that a sequence of distinct complex numbers $\Lambda = \{\lambda_j\}_{j \geq 1}$ is called interpolating sequence for $\mathcal{F}^p_\alpha$ $(0 < p < \infty)$ if for every sequence $\{a_j\}_{j \geq 1}$ such that

$$\sum_{j \geq 1} \left| a_j e^{-\frac{\alpha}{2}|\lambda_j|^2} \right|^p < \infty,$$

there is a function $f \in \mathcal{F}^p_\alpha$ solving the interpolation problem

$$f(\lambda_j) = a_j, \qquad j = 1, 2, ....$$

The sequence $\Lambda = \{\lambda_j\}_{j \geq 1}$ is called to be sampling for $\mathcal{F}^p_\alpha$ if there exist two constants $c, C > 0$ such that for every function $f \in \mathcal{F}^p_\alpha$ we have the following double inequalities

$$c\|f\|^p_{p,\alpha} \leq \sum_{j \geq 1} \left| f(\lambda_j) e^{-\frac{\alpha}{2}|\lambda_j|^2} \right|^p \leq C\|f\|^p_{p,\alpha}.$$

Also a sequence $\Lambda = \{\lambda_j\}_{j \geq 1}$ is called separated (or uniformly discrete) if there exists $\delta > 0$ such that

$$\left| \lambda_j - \lambda_k \right| > \delta, \qquad j \neq k.$$

The upper and lower uniform Beurling-Landau density of a sequence $\Lambda \subset \mathbb{C}$ is known respectively as the following

$$\mathcal{D}^+(\Lambda) := \limsup_{\rho \to \infty} \sup_{z \in \mathbb{C}} \frac{N_\Lambda(z, \rho)}{\pi \rho^2}$$

and

$$\mathcal{D}^-(\Lambda) := \liminf_{\rho \to \infty} \inf_{z \in \mathbb{C}} \frac{N_\Lambda(z, \rho)}{\pi \rho^2},$$

where $N_\Lambda(z, \rho)$ design the number of the elements in the intersection of $\Lambda$ and the Euclidean open disk $D(z, \rho)$ of center $z \in \mathbb{C}$ and radius $\rho > 0$ (see [Beu89; Lan67] for more about the notion of density). Seip and Walston characterize Sampling and interpolation in [SW92] by mean of the above densities, as follows.

**Theorem 3.2.1** ([SW92])**.** *Let $\alpha > 0$ and $0 < p < \infty$.*

1. *A sequence $\Lambda = \{\lambda_j\}$ is an interpolating set for $\mathcal{F}^p_\alpha$ if and only if is uniformly discrete and $\mathcal{D}^+(\Lambda) < \frac{\alpha}{\pi}$.*
2. *A sequence $\Lambda = \{\lambda_j\}$ is a sampling set for $\mathcal{F}^p_\alpha$ if and only if it is a union of finitely many separated sequences and its contains a sequence $\Lambda'$ with $\mathcal{D}^-(\Lambda) > \frac{\alpha}{\pi}$.*

A well exposed prove of the above Theorem 3.2.1 can be found in [Zhu12] and we mention that there is other proofs with different methods.
Before we state our main theorem we recall the following notion and a useful lemma from [Zhu12]. For a fixed lattices $\Lambda = \{\lambda_{mn}\}$, let $Z = \{w_{mn}\}$ be a sequence of distinct points in $\mathbb{C}$. If there exists $Q > 0$ (not necessary small ) such that

$$\left|\lambda_{mn} - w_{mn}\right| \leq Q$$

for all $w_{mn} \in Z$ then we say that the sequence $Z$ is uniformly closed to the lattice $\Lambda$.

Our main result is the following theorem.

**Theorem 3.2.2.** *Let $p$ and $q$ be positive numbers such that $p > q$. There exists a sequence $\Lambda$ in $\mathbb{C}$ satisfying*

$$\mathcal{D}^+(\Lambda) = \mathcal{D}^-(\Lambda) = \alpha/\pi, \tag{3.42}$$

*and such that $\Lambda$ is a zero set for $\mathcal{F}^p_\alpha$ but it is not for $\mathcal{F}^q_\alpha$.*

In the proof of Theorem 3.2.2, the condition (3.42) seems to be optimal for a sequence $\Lambda$ with a uniform density, that is $\mathcal{D}^+(\Lambda) = \mathcal{D}^-(\Lambda)$, to be a zero set for $\mathcal{F}^p_\alpha$ but not for $\mathcal{F}^q_\alpha$. It turns out that $\alpha/\pi$ is a critical number within the interpolating and sampling theorems on Fock spaces, given by K. Seip and R. Wallstén [**Sei**; SW92].

### 3.2.1 Proof of Theorem 3.2.2

We start this section with some known preliminaries. We consider the following square lattice

$$\Lambda := \{z_{m,n} := a(m+in) \ : \ m,n \in \mathbb{Z}\},$$

where $a$ is a positive number (It can be denoted $_a$ ).Clearly, $\Lambda$ possesses the imaginary axis as a line of symmetry. By translating the positive real points of $\Lambda$ away from 0 and keeping this symmetry unchanged, we define the following modified lattice

$$\Lambda_R := \{w_{m,n} \ : \ m,n \in \mathbb{Z}\},$$

where $R$ is a positive number and

$$w_{m,n} := \begin{cases} z_{m,n}, & \text{if } \ n \neq 0 \text{ or } m = n = 0, \\ a(m + R\frac{m}{|m|}), & \text{if } \ n = 0 \text{ and } m \neq 0. \end{cases} \tag{3.43}$$

We observe that for an integer $N \in \mathbb{N} \setminus \{0\}$, $\Lambda_N$ is then obtained from $\Lambda$ by just removing the following finite symmetric set $\{\pm am \ : \ m \in \{1,2,...,N\}\}$. The Weierstrass function associated to $\Lambda$ is defined by

$$\sigma_a(z) := \sigma_\Lambda(z) = z \prod_{\substack{(m,n)\in\mathbb{Z}^2 \\ (m,n)\neq(0,0)}} \left(1 - \frac{z}{z_{m,n}}\right) \exp\left\{\frac{z}{z_{m,n}} + \frac{z^2}{2z_{m,n}^2}\right\},$$

see the textbooks [Boa54; Lev96]. The modified Weierstrass $\sigma-$function associated to $\Lambda_R$ is known as follows

$$\sigma_{a,R}(z) := z \prod_{\substack{(m,n)\in\mathbb{Z}^2 \\ (m,n)\neq(0,0)}} \left(1 - \frac{z}{w_{m,n}}\right) \exp\left\{\frac{z}{w_{m,n}} + \frac{z^2}{2z_{m,n}^2}\right\},$$

see for instance [Zhu12, Chap. 4]. By using the symmetry enjoyed by the lattices $\Lambda$ and $\Lambda_R$, we simply compute

$$\begin{aligned} \frac{\sigma_a(z)}{\sigma_{a,R}(z)} &= \prod_{m\neq 0} \frac{1 - z/z_m}{1 - z/w_m} \times \frac{\exp(z/z_m)}{\exp(z/w_m)} \\ &= \prod_{m\geq 1} \frac{1 - (z/z_m)^2}{1 - (z/w_m)^2}, \qquad z \in \mathbb{C} \setminus \Lambda_R. \end{aligned} \tag{3.44}$$

The following lemma below gives the first condition in the Theorem 3.2.2, because $\Lambda_R$ is relatively closed to $\Lambda_a$.

**Lemma 3.2.3** (see [Zhu12]). *If $Z$ is a sequence of complex numbers uniformly closed to the lattice $\Lambda_{a=\sqrt{\frac{\pi}{\alpha}}}$, then*

$$\mathcal{D}^-(Z) = \mathcal{D}^+(Z) = \frac{\alpha}{\pi}.$$

The key ingredient in our proof is the following lemma. The proof is stated in the next section 3.2.2.

**Lemma 3.2.4.** *Let $R$ be a positive number. We have*

$$\left|\prod_{m\geq 1}\frac{1-(z/z_m)^2}{1-(z/w_m)^2}\right| \simeq \frac{|z|^{2R}d(z,\Lambda)}{d(z,\Lambda_R)}, \qquad z\notin \Lambda_R\cup\mathbb{D}(0,1).$$

Therefore, by using (3.44) and Lemma 3.2.4,

$$\left|\frac{\sigma_a(z)}{\sigma_{a,R}(z)}\right| \simeq \frac{|z|^{2R}d(z,\Lambda)}{d(z,\Lambda_R)}, \qquad z\notin \Lambda_R\cup\mathbb{D}(0,1). \tag{3.45}$$

We fix a positive number $\alpha$ and consider the lattice $\Lambda$ generated by $a=\sqrt{\pi/\alpha}$. It is known that

$$|\sigma_a(z)|\exp\left(-\frac{\alpha}{2}|z|^2\right)\simeq d(z,\Lambda), \qquad z\in\mathbb{C}, \tag{3.46}$$

see for instance [Zhu12, Corollary 1.21]. Thus, using (3.45) and (3.46),

$$\left|\sigma_{a,R}(z)\right|\exp\left(-\frac{\alpha}{2}|z|^2\right)\simeq d(z,\Lambda_R)|z|^{-2R}, \qquad |z|\geq 1. \tag{3.47}$$

We now let $p$ and $q$ be positive numbers such that $\infty>p>q>0$. We take a number $R$ satisfying $\frac{1}{p}<R<\frac{1}{q}$. By using (3.47),

$$\left|\sigma_{a,R}(z)\right|^p\exp^{-\frac{p\alpha}{2}|z|^2}|z|\lesssim 1/|z|^{2pR-1}, \qquad |z|\geq 1. \tag{3.48}$$

Since $2pR-1>1$, we then obtain $\sigma_{a,R}\in\mathcal{F}^p_\alpha$, and hence $\Lambda_R$ is a zero set for $\mathcal{F}^p_\alpha$. A standard argue by contradiction shows that $\Lambda_R$ cannot be a zero sequence in $\mathcal{F}^q_\alpha$. For the sake of completeness, we sketch here the proof. We suppose that there exists a function $f\in\mathcal{F}^q_\alpha\setminus\{0\}$ with zero set $\Lambda_R$. By Hadamard's factorization theorem, we have

$$f(z)=z\exp\left(Q(z)\right)\prod_{\substack{(m,n)\in\mathbb{Z}^2\\(m,n)\neq(0,0)}}\left(1-\frac{z}{w_{m,n}}\right)\exp\left\{\frac{z}{w_{m,n}}+\frac{z^2}{2w_{m,n}^2}\right\},$$

where $Q$ is a polynomial of degree at most 2. We also obviously have

$$\sum_{m\geq 1}\frac{R(2m+R)}{m^2(m+R)^2}=M_R<+\infty.$$

Thus,

$$\begin{aligned} f(z) &= \exp\left(Q(z)\right)\sigma_{a,R}(z)\prod_{m\neq 0}\exp\left\{\left(\frac{1}{2w_m^2}-\frac{1}{2z_m^2}\right)z^2\right\} \\ &= \sigma_{a,R}(z)\exp\left(Q(z)\right)\exp\left\{-\frac{M_R}{a^2}z^2\right\} \\ &= \sigma_{a,R}(z)\exp\left(L(z)\right), \end{aligned} \tag{3.49}$$

where $L$ is a polynomial of degree at most 2. Using (3.47) and (3.49)

$$\begin{aligned}
\|f\|_q^q &= \frac{\alpha q}{2}\int_{\mathbb{C}} |f(z)|^q \exp^{-\frac{\alpha q}{2}|z|^2}\, dA(z) \\
&= \frac{\alpha q}{2}\int_{\mathbb{C}} |\sigma_{a,R}^q(z)| \exp\big(-\frac{\alpha q}{2}|z|^2\big)| \exp\big(qL(z)\big)|dA(z) \\
&\gtrsim \int_{\mathbb{C}\setminus\mathbb{D}(0,1)} d^q(z,\Lambda_R)|z|^{-2Rq}| \exp\big(qL(z)\big)|dA(z) \\
&\gtrsim \int_{\mathbb{C}\setminus\mathbb{D}^*(\Lambda_R,a/8)} |z|^{-2Rq}| \exp\big)\big(L(z)\big)|dA(z), \qquad (3.50)
\end{aligned}$$

where

$$\mathbb{D}^*(\Lambda_R, a/8) := \bigcup_{\lambda\in\Lambda_R\setminus\{0\}} \mathbb{D}(\lambda, a/8).$$

For $\lambda \in \Lambda_R \setminus \{0\}$ and a point $w \in \mathbb{D}(\lambda, a/8)$, the sub-harmonicity of the function $z \mapsto \phi(z) := |z|^{-2Rq}| \exp\big(qL(z)\big)|$ in $\mathbb{C}\setminus\{0\}$, gives

$$\phi(w) \lesssim \int_{a/4\le|z-w|\le 3a/8} \phi(z)dA(z) \le \int_{\mathbb{D}(\lambda,a/2)\setminus\mathbb{D}(\lambda,a/8)} \phi(z)dA(z).$$

It follows

$$\int_{\mathbb{D}(\lambda,a/8)} \phi(w)dA(w) \lesssim \int_{\mathbb{D}(\lambda,a/2)\setminus\mathbb{D}(\lambda,a/8)} \phi(z)dA(z).$$

Therefore

$$\int_{\mathbb{D}^*(\Lambda,a/8)} \phi(w)dA(w) \lesssim \int_{\mathbb{C}\setminus\mathbb{D}^*(\Lambda,a/8)} \phi(z)dA(z), \qquad (3.51)$$

since

$$\mathbb{D}(\lambda_1, a/2)\cap\mathbb{D}(\lambda_2, a/2) = \emptyset, \qquad \lambda_1 \neq \lambda_2.$$

By combining (3.50) and (3.51)

$$\int_{\mathbb{C}\setminus\mathbb{D}(0,1)} \phi(z)dA(z) \lesssim \|f\|_q^q. \qquad (3.52)$$

Using again the subharmonicity of $\phi$ and taking account of (3.52), we deduce that $\phi$ is bounded at $\infty$, and hence $z \mapsto \exp\big(qL(z)\big)$ possesses a polynomial growth at $\infty$. Thus, $L$ must be a constant and by consequence

$$\int_1^\infty |z|^{-2Rq} dA(z) \lesssim \|f\|_q^q. \qquad (3.53)$$

Which is impossible, since $2Rq - 1 < 1$. Hence $\Lambda_R$ is not a zero set for $\mathcal{F}_\alpha^q$. This completes the proof of Theorem 3.2.2.

### 3.2.2 Proof of Lemma 3.2.4

Let $\alpha$ be a positive number and consider the lattice generated by $a = \sqrt{\pi/a}$. By using the symmetry with respect the the imaginary axis enjoyed by the lattices $\Lambda$ and $\Lambda_R$, we simply compute [R]

$$\begin{aligned}\sigma_{a,R}(z) &= \sigma_a(z) \prod_{m\neq 0} \frac{1 - z/w_m}{1 - z/z_m} \times \frac{\exp(z/w_m)}{\exp(z/z_m)} \\ &= \sigma_a(z) \prod_{m\geq 1} \frac{1-(z/w_m)^2}{1-(z/z_m)^2}, \qquad z \in \mathbb{C} \setminus a\mathbb{Z}, \end{aligned} \tag{3.54}$$

where $w_m := w_{m,0}$ and $z_m := z_{m,0}$. For proving Lemma 3.2.4, we claim that it is sufficient to show

$$\psi_R(z) := \prod_{m\geq 1} \left| \frac{m+R-z}{m-z} \right| \frac{m}{m+R} \asymp \frac{d(z, \mathbb{Z}_R^+)}{d(z, \mathbb{Z}^+)(1+|z|)^R}, \qquad z \in \mathbb{C} \setminus \mathbb{Z}^+, \tag{3.55}$$

where $\mathbb{Z}^+$ is the set of non-negative integers and

$$\mathbb{Z}_R^+ := \{m + R,\ m \in \mathbb{Z}^+\}/$$

Indeed, assume that 3.55 holds. Then

$$\psi_R(z/a) := \prod_{m\geq 1} \left| \frac{1-(z/a(m+R))}{1-(z/am)} \right| \asymp \frac{d(z, a\mathbb{Z}_R^+)}{d(z, a\mathbb{Z}^+)(1+|z|)^R}, \qquad z \in \mathbb{C} \setminus \mathbb{Z}^+,$$

and

$$\psi_R(-z/a) := \prod_{m\geq 1} \left| \frac{1+(z/a(m+R))}{1+(z/am)} \right| \asymp \frac{d(z, a\mathbb{Z}_R^-)}{d(z, a\mathbb{Z}^-)(1+|z|)^R}, \qquad z \in \mathbb{C} \setminus a\mathbb{Z}^-,$$

where where $\mathbb{Z}^- := -\mathbb{Z}^+$ and $\mathbb{Z}_R^- := -\mathbb{Z}_R^+$. We clearly have

$$\frac{d(z, a\mathbb{Z}_R^+)}{d(z, a\mathbb{Z}^+)} \times \frac{d(z, a\mathbb{Z}_R^-)}{d(z, a\mathbb{Z}^-)} \asymp \frac{d(z, \Lambda_R)}{d(z, \Lambda)} \qquad z \in \mathbb{C} \setminus a\mathbb{Z}$$

Thus

$$\prod_{m\geq 1} \left| \frac{1-(z/a(m+R))^2}{1-(z/am)^2} \right| \asymp \frac{d(z, \Lambda_R)}{d(z, \Lambda)(1+|z|)^{2R}} \qquad z \in \mathbb{C} \setminus a\mathbb{Z}. \tag{3.56}$$

By using 3.54 and 3.56 we deduce

$$\left|\sigma_{a,R}(z)\right| \asymp \frac{\left|\sigma_a(z)\right| d(z, \Lambda_R)}{d(z, \Lambda)(1+|z|)^{2R}}, \quad z \in \mathbb{C} \setminus a\mathbb{Z}. \tag{3.57}$$

On the other hand, it is known that

$$|\sigma_a(z)|e^{-\frac{\alpha}{2}|z|^2} \asymp d(z,\Lambda), \qquad z\in\mathbb{C}\setminus\mathbb{Z}, \tag{3.58}$$

see for instance [Zhu12, Corollary 1.21 ]. Hence

$$|\sigma_{a,R}(z)|e^{-\frac{\alpha}{2}|z|^2} \asymp \frac{d(z,\Lambda_R)}{(1+|z|)^{2R}}, \qquad z\in\mathbb{C}\setminus\mathbb{Z}, \tag{3.59}$$

which proves Lemma 3.2.4. Let us proves 3.55. For this aim it sufficient to consider only the situation when $[R]$, the integer part of $R$, is equals zero. Tndee, we fix a number $R>1$. We can factorize $\psi_R$ as follows

$$\begin{aligned}\psi_R(z) &= \psi_R(z-[R])\prod_{m=1}^{[R]}\frac{m+\beta}{|m-z|}\\ &\asymp \psi_\beta(z-[R])\prod_{m=1}^{[R]}\frac{1}{|m-z|}, \qquad z\in\mathbb{C}\setminus\mathbb{Z}^+,\end{aligned}$$

where $\beta:=R-[R]$. We have also

$$\prod_{m=1}^{[R]}\frac{1}{|m-z|}\asymp\frac{1}{d(z,\{1,2,...,[R]\})(1+|z|)^{[R]-1}} \qquad z\in\mathbb{C}\setminus\{1,2,...,[R]\}.$$

If we show

$$\psi_\beta(z)\asymp\frac{d(z,\mathbb{Z}^+_\beta)}{d(z,\mathbb{Z}^+)(1+|z|)^\beta}, \qquad z\in\mathbb{C}\setminus\mathbb{Z}^+,$$

Then

$$\psi_\beta(z-[R])\asymp\frac{d(z,\mathbb{Z}^+_R)}{d(z,\mathbb{Z}^+_{[R]})(1+|z|)^\beta}, \qquad z\in\mathbb{C}\setminus\mathbb{Z}^+_{[R]}.$$

Therefore

$$\psi_R(z)\asymp\frac{d(z,\mathbb{Z}^+_R)}{d(z,\mathbb{Z}^+)(1+|z|)^R}, \qquad z\in\mathbb{C}\setminus\mathbb{Z}^+,$$

which proves (3.55). So, in the sequel we suppose $[R]=0$. We know set for $z=x+iy\in\mathbb{C}$

$$\mathbb{N}_z:=\{m_z-2,m_z-2,m_z\}\cap\mathbb{Z}^+,$$

where

$$m_z:=\min\{m\in\mathbb{Z}^+ \ : m-x\geq 0\},$$

where $x=: Re(z)$ is the real part of $z$. Since $\mathbb{N}_z$ contains at most three elements, we obviously get

$$\prod_{m\in\mathbb{N}_z}\frac{m}{m+R}\asymp 1, \qquad z\in\mathbb{C}.$$

For $z \in \mathbb{C}$. If $x = Re(z) \leq 1$ we the obtain $m_z = 1$, $d(z, \mathbb{Z}^+) = |1 - z|$ and $d(z, \mathbb{Z}_R^+) = |1 + R - z|$, and if

$$x = Re(z) > 1$$

, then $m_z \geq 2$, $d(z, \mathbb{Z}^+) = \min\{|m_z - z|, |m_z - 1 - z| = d(z, \mathbb{N}_z)$ and $d(z, \mathbb{Z}_R^+) = d(z - R, \mathbb{N}_z)$. Thus

$$\prod_{m \in \mathbb{N}_z} \left| \frac{m + R - z}{m - Z} \right| \frac{m}{m + R} \asymp \frac{d(z, \mathbb{Z}_R^+)}{d(z, \mathbb{Z}^+)}, \qquad z \in \mathbb{C} \setminus \mathbb{Z}^+. \tag{3.60}$$

Taking account of 3.60, for proving 3.55 it remains to show that

$$\prod_{m \in \mathbb{Z}^+ \setminus \mathbb{N}_z} \left| \frac{m + R - z}{m - Z} \right| \frac{m}{m + R} \asymp \frac{1}{(1 + |z|)^R}, \qquad z \in \mathbb{C},$$

for which it it is necessary and sufficient to show that

$$\prod_{m \in \mathbb{Z}^+ \setminus \mathbb{N}_z} \left| \frac{m + R - z}{m - z} \right| \frac{m}{m + R} \asymp \frac{1}{(1 + |z|)^R}, \qquad |z| \to \infty. \tag{3.61}$$

In the sequel we set

$$\varphi_1(z) := \prod_{m > 2|z|} \left| \frac{m + R - z}{m - z} \right| \frac{m}{m + R} \asymp \frac{1}{(1 + |z|)^R}, \quad z \in \mathbb{C},$$

If $m > 2|z|$ then $|m + R - z| \geq |m - z| > \frac{m}{2}$, and by using the following usual inequality

$$\log(1 + u) \leq u, \quad u \geq 0,$$

we compute

$$\begin{aligned}
|\log \varphi_1(z)| &\leq \sum_{m > 2|z|} \left| \log \left| \frac{1 - z/(m + R)}{1 - z/m} \right| \right| \\
&= \sum_{m > 2|z|} \max \left\{ \log \left| \frac{1 - z/(m + R)}{1 - z/m} \right|, \log \left| \frac{1 - z/m}{1 - z/(m + R)} \right| \right\} \\
&\leq \sum_{m > 2|z|} \max \left\{ \left| \frac{Rz}{(m + R)(m - z)} \right|, \left| \frac{Rz}{m(m + R - z)} \right| \right\} \\
&\leq 2R|z| \sum_{m > 2|z|} \frac{1}{m^2} = O(1).
\end{aligned}$$

We then have

$$\varphi_1(z) \asymp 1, \qquad z \in \mathbb{C}. \tag{3.62}$$

With 3.62 in mind, for proving 3.61 it rmains to show

$$\varphi_1(z) := \prod_{m\in\mathbb{M}_z} \left|\frac{m+R-z}{m-Z}\right| \frac{m}{m+R} \asymp \frac{1}{(1+|z|)^R}, \quad |z|\to\infty, \tag{3.63}$$

where

$$\mathbb{M}_z := \{m\in\mathbb{Z}^+\setminus\mathbb{N}_z \ : \ m\le 2|z|\}.$$

We recall the following classical equality

$$\log|1+u| = Re(u) + O(|u|^2), \qquad Re(u)\ge -\frac{1}{2} \text{ and } |u|\le 1. \tag{3.64}$$

By using 3.64,

$$\begin{aligned}\sum_{m\in\mathbb{M}_z}\log\frac{m}{m+R} &= -\sum_{m\in\mathbb{M}_z}\log\big(1+\frac{R}{m}\big)\\ &= -\sum_{m\in\mathbb{M}_z}\frac{R}{m}+O(1)\\ &= -R\log|z|+O(1),\end{aligned}$$

which gives

$$\prod_{m\in\mathbb{M}_z}\frac{m}{m+R}\asymp|z|^{-R}\asymp\frac{1}{(1+|z|)^R}, \qquad |z|\to\infty. \tag{3.65}$$

On the other hand, we have

$$\sum_{m\in\mathbb{M}_z}\left|\frac{R}{m-z}\right|^2 \le \sum_{m\in\mathbb{M}_z}\frac{1}{|m-x|^2}\le\sum_{m\ge1}\frac{1}{m^2}<\infty,$$

and since

$$-\frac{1}{2}\le Re\big(\frac{R}{m-z}\big) \text{ and } \frac{R}{|m-z|}\le 1, \qquad m\in\mathbb{Z}_z,$$

then by using 3.64,

$$\sum_{m\in\mathbb{M}_z}\log\left|1+\frac{R}{m-z}\right|^2 = R\sum_{m\in\mathbb{M}_z}\frac{m-x}{(m-x)^2+y^2}+O(1), \qquad z-x+iy, \tag{3.66}$$

For $|z|>3/2$,

$$\mathbb{M}_z := \{m\in\mathbb{M}_z \ : \ m-x\ge 0\} = \{m_z+1, m_z+2, ..., [2|z|]\}\ne\emptyset,$$

and by a simple computation

$$\sum_{m\in\mathbb{M}_z^+}\frac{m-x}{(m-x)^2+y^2} = \log\frac{([2|z|]x)^2+y^2}{(m_z+1-x)^2+y^2}+O(1) \tag{3.67}$$

$$= \begin{cases} O(1) & \text{if } x\le 0\\ \log\frac{1+|y|}{|z|}+O(1), & \text{if} x>0\end{cases} \tag{3.68}$$

We need to distinguish between two different cases. In the case where $x \leq 3$, we obtain $m_z \leq 3$ and hence $\mathbb{M}_z^+ = \mathbb{M}_z$. In this case, we either have $x \leq 0$ or $|x| \asymp |y|$, for $|z| \geq 3/2$. In both situation we deduce the desired estimate 3.63 by combining estimates 3.65, 3.66 and 3.68. In the case $x > 3$, we obtain $m_z \geq 4$ and by consequence

$$\mathbb{M}_z \setminus \mathbb{M}_z^+ = \{1, 2, ..., m_z - 3\} \neq \emptyset.$$

In this case

$$\begin{aligned} \sum_{m \in \mathbb{M}_z^+} \frac{m - x}{(m-x)^2 + y^2} &= \log \frac{([2|z|]x)^2 + y^2}{(m_z + 1 - x)^2 + y^2} + O(1) \\ &= \log \frac{1 + |y|}{|z|} + O(1). \end{aligned} \tag{3.69}$$

Finally we deduce 3.63 by joining 3.65, 3.66, 3.68 and the last estimate 3.69. The proof of Lemma 3.2.4 is completed.

## 3.3 Results on zero sets and zero sub-sets

In [Hor74], C. Horowitz showed that every subset of an $\mathcal{A}_0^p(\mathbb{D})$ zero set is an $\mathcal{A}_0^p(\mathbb{D})$ zero set too (see chapter 2). The situation for Fock spaces is completely different. Indeed, Consider the sin-cardinal type function

$$s(z) = \frac{\sin\left(\frac{\delta}{2} z^2\right)}{z^2} \in \mathcal{F}_\alpha^p, \qquad 0 < \delta < \alpha.$$

The zero set of the function $s$, denoted by $Z(s)$, is a zero sequence for $\mathcal{F}_\alpha^p$, for every $0 \leq p < \infty$. However, if we remove the subset which belongs to the imaginary axis from $Z(s)$, the remaining part is not a zero set anymore for $\mathcal{F}_\alpha^p$. Such result can be viewed as a consequence of the Lindelöf's theorem, see [Boa54, Theorem 2.10.1]. Therefore, a natural question is: which zero set for $\mathcal{F}_\alpha^p$ remains a zero set too for $\mathcal{F}_\alpha^p$, even if an infinite subset was removed?

Actually, the example above is a variant to the one given by Zhu in [Zhu93]. This phenomena is one of the main deference between Fock spaces and Hardy spaces and even Bergman spaces of the unit disk where zero sets are well stable [DS04b; HKZ00].

In the following theorem, we give a complete description of zero sets for which all their sub-sequences are also zero sets for $\mathcal{F}_\alpha^p$.

**Theorem 3.3.1** ([AO22]). *Let $Z = \{z_n\}_{n \in \mathbb{N}}$ be a zero set for $\mathcal{F}_\alpha^p$, $0 < p < \infty$. The following statements are equivalents*

1. *Every subset of $Z$ is a zero set for $\mathcal{F}^p_\alpha$,*

2. *$Z$ satisfies*

$$\sum_{n\in\mathbb{N}} \frac{1}{|z_n|^2} < \infty. \tag{3.70}$$

### 3.3.1 Proof of Theorem 3.3.1

The proof of Theorem 3.3.1 is essentially based on Lindelöf's theorem 3.1.7 which characterizes entire functions of integral order and of finite type. First, we recall some main tools very useful for our proof, we refer to [Boa54; Lev96] for more details. If $f$ is an entire function and $r$ a positive real number, we denote $M(r,f)$ the maximum modulus of $f$ on the circle $|z| = r$

$$M(r,f) = \max_{|z|=r} |f(z)|.$$

The order of $f$ is given by the quantity

$$\rho_f := \limsup_{r\to\infty} \frac{\log\log M(r,f)}{\log(r)}.$$

Always, we have $0 \le \rho_f \le \infty$. In the case $0 < \rho_f < \infty$, the type of $f$ is defined by

$$\tau_f := \limsup_{r\to\infty} \frac{\log M(r,f)}{r^{\rho_f}}.$$

Let $Z = \{z_n\}_{n\in\mathbb{N}}$ be the zero set of an entire function $f$. Following [Boa54], the convergence exponent of the sequence $Z = \{z_n\}_{n\in\mathbb{N}}$ (excluding 0 if it belongs to $Z$) is defined as the infimum of all positive numbers $s$ such that

$$\sum_{n\in\mathbb{N}} \frac{1}{|z_n|^s} < \infty, \tag{3.71}$$

it will be denoted by $\rho_1$ (for short, the convergence exponent of $f$). A consequence of Jensen's formula gives the following relations among the order $\rho_f$ and the exponent of convergence $\rho_1$ of an entire function $f$ (see, [Boa54] for complete proof):

$$\rho_1 \le \rho_f. \tag{3.72}$$

In the sequel, our constructions are based on dividing the complex plane into sectors with some defined opening aperture. To this end, for a given two angles $\beta \in (-\pi, \pi]$ and $\theta \in (0, \pi]$ define

$$\mathcal{S}(\beta,\theta) := \{z \in \mathbb{C} \ :\ |\arg(z) - \beta| \le \theta\} \cup \{z \in \mathbb{C} \ :\ |\arg(-z) - \beta| \le \theta\}.$$

The following lemma will be of prominent role in the proof of Theorem 3.3.1 later on.

**Lemma 3.3.2.** *Let $1 \leq p < \infty$. If $Z = \{z_n\}_{n\in\mathbb{N}}$ is a zero set for $\mathcal{F}^p$ such that $Z \subset \mathcal{S}(\beta, \theta)$ for some $0 \leq \theta < \frac{\pi}{4}$ and $\beta \in (-\pi, \pi]$, then $Z$ satisfies*

$$\sum_{n\in\mathbb{N}} \frac{1}{|z_n|^2} < \infty. \tag{3.73}$$

*Proof.* Without loss of generality, we may suppose $\beta = 0$. Aiming to come to a contradiction, assume that

$$\sum_{n\in\mathbb{N}} \frac{1}{|z_n|^2} = \infty. \tag{3.74}$$

If $g$ is a function in $\mathcal{F}^p$ with $Z(g) = Z$, then by [Zhu12, Theorem 5.1], for every $\epsilon > 0$ we have

$$\sum_{n\in\mathbb{N}} \frac{1}{|z_n|^{2+\epsilon}} < \infty. \tag{3.75}$$

Thus, $Z = \{z_n\}_{n\in\mathbb{N}}$ is of convergence exponent $\rho_1 = 2$. Combining (3.72), (3.75) and (3.74) we obtain

$$2 = \rho_1 \leq \rho_g \leq 2.$$

Hence, $g$ is of order 2, and of type $\tau_g$ less than or equal to $\frac{\alpha}{2}$. Since $g$ is of integral order, Lindelöf's Theorem applies . Writing $z_n = |z_n|e^{i\theta_n}$, $n \in \mathbb{N}$, a straightforward calculation gives

$$\begin{aligned}
|S(r)|^2 &= \left| \sum_{|z_n|\leq r} \frac{1}{z_n^2} \right|^2 = \left| \sum_{|z_n|\leq r} \frac{e^{-2i\theta_n}}{|z_n|^2} \right|^2 \\
&= \left| \sum_{|z_n|\leq r} \frac{\cos\left(2\theta_n\right) - i\sin\left(2\theta_n\right)}{|z_n|^2} \right|^2 \\
&= \left( \sum_{|z_n|\leq r} \frac{\cos\left(2\theta_n\right)}{|z_n|^2} \right)^2 + \left( \sum_{|z_n|\leq r} \frac{\sin\left(2\theta_n\right)}{|z_n|^2} \right)^2 \\
&\geq \left( \sum_{|z_n|\leq r} \frac{\cos\left(2\theta_n\right)}{|z_n|^2} \right)^2 \\
&\geq \cos^2(2\theta) \left( \sum_{|z_n|\leq r} \frac{1}{|z_n|^2} \right)^2 \longrightarrow \infty, \qquad \text{as } r \to \infty.
\end{aligned}$$

This contradicts Lindelöf's Theorem 3.1.7 As a conclusion

$$\sum_{n=0}^{\infty} \frac{1}{|z_n|^2} < \infty.$$

□

**Remark 4.** *We mention that the constant $\frac{\pi}{4}$, that appears in Lemma 3.3.2, is the best possible. A counterexample is given by the zero set of the sin-cardinal function*

$$s(z) = \frac{\sin(\frac{\pi}{2}z^2)}{z^2} \in \mathcal{F}^p_\pi.$$

Now, we can prove Theorem 3.3.1.

## Proof of Theorem 3.3.1.

Let $Z = \{z_n\}_{n\in\mathbb{N}}$ be a sequence of complex numbers which is a zero sequence for $\mathcal{F}^p$ and satisfies (3.70). If $Z'$ is any sub-sequence of $Z$, then it is a zero sequence too for $\mathcal{F}^p$ by the sufficient condition [Zhu12, Theorem 5.3]. Therefore, $Z$ belongs to the desired class.
Conversely, let $Z$ be a zero sequence for $\mathcal{F}^p$ with the property: every subset of $Z$ is also a zero set for $\mathcal{F}^p$. Dividing $Z$ into eight subset, by writing

$$Z = \bigcup_{k=0}^{7} Z_k,$$

where for each $0 \leq k \leq 7$

$$Z_k := Z \cap \{z \in \mathbb{C} \ : \ -\frac{\pi}{8} \leq \arg(z) - \frac{k\pi}{4} < \frac{\pi}{8}\} \subset \mathcal{S}(\frac{\pi}{8}, \frac{\pi}{8} + \varepsilon),$$

and $\varepsilon$ is an arbitrary small positive number in $(0, \frac{\pi}{8})$. By the assumption, each $Z_k$, $k \in \{0, 1, \cdots, 7\}$ is a zero set for $\mathcal{F}^p$. We conclude by Lemma 3.3.2.

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## *Abstract :*

This thesis treats two topics. In the first part, we study sampling and interpolation, with unbounded multiplicity in the classical standard weighted Bergman spaces of analytic functions in the unit disk of the complex plane. The main results obtained are a necessary condition and a sufficient condition in both sampling and interpolation cases, with a small gap. It is known that analogue results hold in the classical Fock space of entire functions. We mention that our results may be applied even for bounded multiplicities, as a weak alternative for Seip's characterization of sampling and interpolation sequences known by the presence of a Beurling-Landau type densities which are difficult to check in general.

The second part is devoted to the study of some properties of zero sets in the classical Fock space. To our knowledge, a complete characterization of zero sets for Fock spaces doesn't exist yet. Also, the gap between necessary and sufficient conditions remains without explanation. One of our results is a response to a question asked by K. Zhu in his book "Analysis on Fock Spaces". As an answer, we prove that two Banach Fock spaces with different exponents doesn't share the same zero sets. Moreover, it is known that in the Hardy spaces and in Bergman spaces of the unit disk every sub-sequence of a zero set remains again a zero set, this hereditary property doesn't take place in Fock spaces. In this perspective, we show that a zero set for the Fock space has the property; every sub-sequence is a zero set too if and only if it satisfies the sufficient condition.

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**Key Words (7):** Bergman spaces, Fock spaces, Zero sets, Uniqueness divisor, Sampling divisor, Interpolating divisor, Dbar-method, Entire functions.

## *Résumé :*

Cette thèse contient deux parties. Dans la première, nous étudions les suites d'échantillonnage et d'interpolation multiples dans les espaces de Bergman à poids standards de fonctions holomorphes sur le disque unité. La multiplicité est supposée non bornée ( peut être uniformément bornée ou croissante quand la suite s'approche du bord). Notre objectif est d'obtenir une caractérisation, en parallèle à des résultats déjà obtenus dans différents espaces de fonctions analytiques. Dans chaque situation, d'interpolation ou d'échantillonnage, nous obtenons une condition nécessaire et une condition suffisante avec un petit gap difficile à franchir. Des résultats similaires ont été déjà obtenus dans le cas des espaces de Fock dans le plan complexe. Dans la deuxième partie nous étudions quelques propriétés des ensembles de zéros des espaces de Fock des fonctions entières. Jusqu'à présent, les suites de zéros pour ces espaces de fonctions entières ne sont pas encore caractérisées! Un gap reste entre la condition nécessaire et suffisante. Nous obtenons un résultat qui affirme que deux espaces (Banach) de Fock à exposant différent n'ont pas les mêmes ensembles de zéros. C'est une réponse à une question posée par Kehe Zhu dans son livre " Analysis on Fock Spaces".

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**Mots-clefs (7):** Espaces de Bergman, Espaces de Fock, Ensembles de zéro, Diviseur d'unicité, Diviseur d'échantillonnage, Diviseur d'interpolation, Dbar-méthode.