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Sujet:

ÉTUDE DE LA STRUCTURE D'UN ANNEAU QUOTIENT R/P VIA L'ACTION DE CERTAINES APPLICATIONS ADDITIVES SUR L'IDÉAL PREMIER P

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Dédicace

Toutes les lettres ne sauraient trouver les mots qu'il faut...
Tous les mots ne sauraient exprimer la gratitude, l'amour, le respect, la reconnaissance...

Je vous dédie ce modeste travail ...

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A mes chers frères et sœurs,

A tous nos Professeurs qui m'ont enseigné tout au long de ce parcours,

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(1) K. Bouchannafa, A. Mamouni and L. Oukhtite, *Structure of a quotient ring R/P and its relation with generalized derivations of R*, Proyecciones Journal of Mathematics. 41 (2022), no. 3, pp. 623-642.

(2) K. Bouchannafa, M. A. Idrissi and L. Oukhtite, *Relationship between the structure of a quotient ring and the behavior of certain additive mappings*, Commun. Korean Math. Soc. 37 (2022), no. 2, pp. 359-370.

(3) L. Oukhtite and K. Bouchannafa, *Commutativity of rings with constraints on endomorphisms*, Rend. Circ. Mat. Palermo, II. Ser (2022). https://doi.org/10.1007/s12215-022-00754-7.

(4) L. Oukhtite, K. Bouchannafa and M. A. Idrissi, *Relationship between the structure of a factor ring R/P and derivations of R*, Bull. Korean Math. Soc. (In press).

(5) K. Bouchannafa, M. A. Idrissi and L. Oukhtite, *Commutativity criteria for a factor ring R/P arising from P-centralizers*, Kyungpook Math. J. (In press).

(6) K. Bouchannafa, A. Hermas and L. Oukhtite, *Lie ideals and generalized derivations in prime rings and Banach algebras*, Submitted for publication.

Other papers:

(1) K. Bouchannafa, M. A. Idrissi and L. Oukhtite, *Commutativity of some factor rings arising from additive mappings*, Asian-European Journal of Mathematics. (In press).

(2) K. Bouchannafa and L. Oukhtite, *New classes of centralizers and their connection with commutativity of rings*, Submitted for publication.

Summary (In French)

La présente thèse s'inscrit dans le cadre de l'algèbre non commutative et porte précisément sur l'étude de la structure d'un anneau quotient R/P où P est un idéal premier de l'anneau R. En fait, son objectif principal consiste à établir une relation entre la structure algébrique d'un anneau quotient et le comportement de quelques applications additives spécifiques sur l'anneau de base ou localement sur des sous-ensembles appropriés de cet anneau. L'originalité de ce travail réside dans le fait que nous avons caractérisé la structure de l'anneau R/P via des dérivations, des dérivations généralisées et des endomorphismes de l'anneau R sans considérer des applications additives de l'anneau R/P.

L'approche adoptée dans cette thèse est d'une importance extrême puisque la primalité (semi-primalité) de l'anneau de base, sur lequel les applications additives en considérations sont définies, n'est pas exigée contrairement aux travaux publiés dans cette direction, par plusieurs auteurs. Cette nouvelle approche nous a permis d'une part, de généraliser et/ou améliorer des travaux de plusieurs auteurs dans cet axe de recherche, d'autre part, elle nous a permis d'étendre aux anneaux semi-premiers certains résultats déjà prouvés sur les anneaux premiers.

Nous avons contribué à l'enrichissement de la littérature par l'introduction et l'initiation d'une nouvelle classe d'applications additives appelées I-centralisateurs (où I est un idéal d'un anneau R) qui couvrent différents concepts classiques. Nous avons donné également une classification complète des dérivations généralisées vérifiant des identités fonctionnelles localement sur un idéal de Lie. Comme application, nous avons étendu plusieurs résultats aux algèbres de Banach premières.

Finalement, nous avons construit des contre-exemples pour justifier que les restrictions

imposées dans les hypothèses de nos théorèmes ne sont pas superflues.

Mots clés: Algèbre de Banach, Anneau premier, Anneau semi-premier, Commutativité, Idéal premier, Dérivation, Dérivation généralisée, Endomorphisme, I-Centralisateur.

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Summary

This thesis falls within the framework of noncommutative algebra and presents a contribution to the study of the algebraic structure of a factor ring R/P, where P is a prime ideal of a ring R, via the behavior of certain additive maps defined on R. It consists of six chapters covering six articles of which three are published, two accepted and another submitted for publication, which can be evaluated and summarized as follows:

The aim of *the first chapter* is to study the structure of quotient ring R/P. Informally this approach is used to establish a connection between the structure of the quotient ring R/P and the behavior of some generalized derivations of the ring R. Thanks to the above mentioned tool, one might be able to extend various results, proven for prime rings, to semi-prime rings.

The second chapter is devoted to the study of derivations and generalized derivations according to the quotient ring R/P. Many examples and results are given to highlight the relationships between this new approach and other consequences that has already existed in the literature.

The third chapter comes to investigate the notions of factor rings and their relations with derivations. Namely, this technique allowed us to extend some results demonstrated for prime rings to semi-prime rings.

The fourth chapter exhibit a commutativity criteria of prime rings R/P through the action of endomorphisms (resp. epimorphisms) of R. The corresponding conclusions characterize the behavior of endomorphisms acting on the center of R/P.

The fifth chapter is motivated by the previous findings and it namely contributes in this direction by examining new concepts of I-centralizers where I is an ideal of the ring under

consideration. Furthermore, we give a full description for some of these I-centralizers.

In *the sixth chapter*, new differential identities are considered only on Lie ideals (not even square closed). As a consequence, some classifications of the involved generalized derivations have been given, except probably when it comes to some isolated cases ($R \subset M_2(K)$, for some field K). Furthermore, our results have been extended to prime Banach algebras.

Keys words: Banach algebras, Prime rings, Semi-prime rings, Commutativity, Prime ideals, Derivations, Generalized derivations, Endomorphisms, I-Centralizers.

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Introduction (In French)

Tout au long de cette thèse, R représentera un anneau associatif non nécessairement commutatif de centre $Z(R)$.

La théorie des anneaux est un domaine de recherche qui porte sur l'étude de structures algébriques réunissant plusieurs branches du sujet et créant une machine puissante pour l'étude des problèmes d'une importance historique et mathématique considérable. Dans ce sens, beaucoup de chercheurs ont examiné la correspondance entre la structure d'un anneau et les propriétés des dérivations. Rappelons qu'une application additive $d : R \to R$ est une dérivation d'un anneau R s'elle satisfait la formule de Leibnitz (i.e. $d(xy) = d(x)y + xd(y)$ pour tout $x, y \in R$). Un exemple très classique d'une dérivation est la dérivation intérieure $d_a : R \to R$ associée à un élément $a \in R$, définie par $d_a(x) = [a,x]^*$ pour tout $x \in R$.

De nombreux résultats dans la littérature indiquent que la structure globale d'un anneau R est souvent étroitement liée au comportement des applications additives définies sur R. Dans cette direction, plusieurs auteurs ont étudié les relations entre les dérivations et la structure des anneaux, ceci grâce à E. C. Posner [61] qui en 1957 a établi deux fameux résultats sur les dérivations. En effet, il a prouvé que si la composition de deux dérivations d'un anneau premier[†] sans 2-torsion[‡] est une dérivation, alors au moins l'une

[*] $[x,y]$ représente le commutateur $xy - yx$ et $x \circ y$ représente l'anti-commutateur $xy + yx$.
[†] Un anneau R est dit premier si pour tout $a, b \in R$, $aRb = (0)$ implique $a = 0$ ou $b = 0$.
[‡] Un anneau est dit sans n-torsion, où $n \in \mathbb{N}^*$, si $na = 0$, avec $a \in R$ entraîne $a = 0$.

d'entre elles est nulle. Ce résultat, appelé le premier Théorème de Posner, est développé et généralisé de manières différentes (voir, par exemple, B. Hvala [48], M. A. Chebotar [33] and C. Lanski [51]). Rappelons qu'une application $f : R \to R$ est dite centralisante sur un sous-ensemble S de R si $[f(x),x] \in Z(R)$ pour tout $x \in S$. Dans le cas particulier, où $[f(x),x] = 0$ pour tout $x \in S$, on dit que f est commutante sur S. Dans le même article [61], Posner a prouvé que l'existence d'une dérivation centralisante non nulle d sur un anneau premier force l'anneau d'être commutatif. Ce résultat, appelé le second Théorème de Posner, est d'une influence considérable sur de nombreuses questions abordées dans plusieurs articles. Un certain nombre d'auteurs ont prolongé ce résultat en considérant des applications centralisantes sur un sous-ensemble bien approprié de l'anneau. A titre d'exemple, Awtar [15] a étendu ce résultat aux idéaux de Jordan (resp. de Lie[§]) non nuls stables par carré. En 1987, Bell et Martindale III [22] ont considéré la même identité différentielle mais sur un idéal à gauche non nul d'un anneau premier R. Dans [28, Théorème B], Brešar a généralisé ce dernier résultat en étudiant un contexte plus général qui consiste à montrer que pour un idéal à gauche non nul U d'un anneau premier R, s'il existe des dérivations $d \neq 0$ et g de R satisfaisant $d(u)u - ug(u) \in Z(R)$ pour tout $u \in U$, alors R est commutatif. Par la suite, Oukhtite [57] a étendu le second Théorème de Posner au contexte des idéaux de Jordan d'un anneau à involution.

En 1991, Brešar a initié un nouveau axe de recherche en introduisant la notion d'une dérivation généralisée qui couvre la notion des dérivations classiques. Une application additive $F : R \to R$ est appelée une dérivation généralisée s'il existe une dérivation $d : R \to R$, appelée dérivation associée à F, telle que $F(xy) = F(x)y + xd(y)$ pour tout $x, y \in R$. Dans le cas particulier, où $d = 0$, on dit que F est un multiplicateur à gauche. Un exemple classique d'une dérivation généralisée est la dérivation intérieure généralisée $F_{a,b} : R \to R$ définie par $F_{a,b}(x) = ax + xb$ avec a, b deux éléments de R. Ceci a impressionné plusieurs auteurs à étendre divers résultats aux dérivations généralisées. Dans cette direction, Hvala [48] a étendu le premier Théorème de Posner concernant la composée de dérivations au cas des dérivations généralisées.

Théorème 0.0.1 ([48], Théorème 1) Soient R un anneau premier sans 2-torsion, $R_C = RC$ la clôture centrale et $F_1, F_2 : R \to R$ deux dérivations généralisées. La composée F_1F_2 est

[§]Un sous-groupe additif U de R est dit un idéal de Lie (resp. de Jordan) si $[u,r] \in U$ (resp. $u \circ r \in U$) pour tout $u \in U$, $r \in R$.

aussi une dérivation généralisée si, et seulement si, l'une des propriétés suivantes est vérifiée:
1. il existe $\lambda \in C$ tel que soit $F_1(x) = \lambda x$ ou bien $F_2(x) = \lambda x$;
2. il existe $a, b \in Q_r(R_C)$ tels que $F_1(x) = xa$ et $F_2(x) = xb$;
3. il existe $a, b \in Q_r(R_C)$ tels que $F_1(x) = ax$ et $F_2(x) = bx$;
4. il existe $a, b \in Q_r(R_C)$ et $\lambda, \mu \in C$ tels que $F_1(x) = ax + xb$ et $F_2(x) = \lambda x + \mu(ax - xb)$.

Récemment, Oukhtite et Mamouni [59] ont exploré la commutativité d'un anneau $*$-premier[¶] admettant une dérivation généralisée centralisante sur un idéal non nul de Jordan.

Le lien entre les identités algébriques et la commutativité des anneaux revient à Divinsky qui a prouvé (1955) qu'un anneau artinien simple admettant un automorphisme commutant est commutatif. Deux ans après, Posner a montré son second théorème pour les dérivations. Postérieurement, l'un des plus célèbres résultats dans cette voie est le fameux Théorème de Mayne, qui a obtenu une conclusion similaire à celui de Posner pour les automorphismes centralisants différents de l'identité. Des résultats analogues étendus au cas des idéaux de Lie ont été étudié par Lanski, Lee, et au cas des idéaux à gauche par Bell et Martindale. Néanmoins, certains auteurs ont pu exprimé la forme explicite d'un anneau admettant une dérivation vérifiant quelques identités algébriques spécifiques.

Théorème 0.0.2 ([25], Théorème) Soient R un anneau unitaire et d une dérivation sur R vérifiant, pour tout $x \in R$, $d(x) = 0$ ou $d(x)$ est inversible dans R. Alors, il existe un anneau à division D tel que:
1. $R = D$ ou,
2. R est l'anneau des matrices carrées d'ordre 2 à coefficient dans D ou,
3. $R = D[x]/(x^2)$, avec $Car(D) = 2$, $d(D) = (0)$ et $d(x) = 1 + ax$ pour un certain $a \in Z(D)$.

Dans le théorème suivant, Brešar a donné la forme explicite des dérivations d, g et h d'un anneau premier vérifiant la condition $f(x) = ag(x) + h(x)b$ pour tout $x \in R$ où a et b des éléments de R.

Théorème 0.0.3 ([28], Théorème 2.1) Soient R un anneau premier, f, g et h des dérivations de R. On suppose qu'il existe $a, b \in R$ tels que $f(x) = ag(x) + h(x)b$ pour tout $x \in R$. Si

[¶] R est $*$-premier si et seulement si, $aRb = aRb^* = (0) \Longrightarrow a = 0$ ou $b = 0$, pour tout $a, b \in R$.

$a \notin Z(R)$ et $b \notin Z(R)$, alors il existe $\lambda \in C$, tels que $d(x) = \lambda[ab,x]$, $g(x) = \lambda[b,x]$ et $h(x) = \lambda[a,x]$ pour tout $x \in R$.

La recherche relative à la commutativité des anneaux, plus particulièrement les anneaux premiers ou semi-premiers[‖], a été la source essentielle du développement de plusieurs théories mathématiques, à savoir la théorie des identités différentielles et la théorie des équations fonctionnelles sur les algèbres. En 1955 Singer et Wermer [67] ont prouvé que toute dérivation continue sur une algèbre de Banach commutative A est forcément à image dans $rad(A)$. Particulièrement, elle est nulle quand A est semi-simple (i.e., $rad(A) = 0$). Bien sûr, ce résultat n'est pas valable dans le cas d'une algèbre de Banach non-commutative (disant à cause des dérivations intérieures), mais il y a plusieurs manières d'établir une version non-commutative du Théorème de Singer-Wermer. Dans cette brève note, la considération des dérivations sur les algèbres de Banach est d'une importance extrême. En effet, les questions abordées dans cette direction nous permettent d'obtenir plus d'informations sur les dérivations et parfois sur la caractéristique de l'anneau. Le premier résultat dans ce sens a été obtenu par Brešar et Vukman [31], qui ont prouvé qu'une dérivation continue est *rad*-commutante sur une algèbre de Banach A (i.e., $[d(x),x] \in rad(A)$ pour tout $x \in A$) est certainement à image dans le $rad(A)$. Une autre extension non-commutative du Théorème Singer-Wermer a été prouvée par Sinclair [66] en 1969: Si d est une dérivation continue sur une algèbre de Banach A alors les idéaux primitifs de A sont invariants par d. L'un des objectifs de notre sujet de doctorat était l'approfondissement de l'étude des dérivations, des dérivations généralisées, des endomorphismes et des centralisateurs à gauche ainsi leurs connexion avec les anneaux quotients R/P (ceci sera présenté dans les Chapitres 1, 2, 3, 4 et 5). Dans le même contexte, nous avons aussi étudié la forme de quelques dérivations généralisées agissant sur les idéaux de Lie, puis leurs applications aux algèbres de Banach (ceci sera présenté dans le Chapitre 6).

Cette thèse présente notre contribution à l'étude de la structure de certaines classes d'anneaux et d'algèbres via l'action de quelques applications additives sur des sous-ensembles appropriés de l'anneau en considération, à savoir les dérivations, les dérivations généralisées et les endomorphismes.

[‖]Un anneau R est dit semi-premier s'il ne contient aucun idéal non nul nilpotent.

Dans le **premier chapitre** nous avons étudié la structure des classes d'anneaux et leurs relations avec certaines identités différentielles algébriques. Plus spécifiquement, nous avons introduit l'anneau quotient R/P où R est un anneau quelconque et P est un idéal premier de R impliquant des dérivations généralisées définies sur l'anneau R. Cette technique nous a permis de retrouver, d'améliorer et de généraliser différents résultats majeurs prouvés dans la littérature.

Théorème 0.0.4 *Soient R un anneau, I un idéal non nul de R et P un idéal premier de R tels que $P \subsetneq I$ et $Car(R/P) \neq 2$. Si F est une dérivation généralisée de R associée à une dérivation d, vérifiant la condition $\overline{F(x \circ y)} \in Z(R/P)$ pour tout $x, y \in I$, alors $F(R) \subseteq P$ ou R/P est un domaine commutatif.*

En tant qu'application directe du théorème ci-dessus, nous avons prolongé un résultat démontré par Quadri et al. dans [63] aux anneaux semi-premiers.

Corollaire 0.0.5 *Soient R un anneau semi-premier et I un idéal non nul de R. Si R admet une dérivation généralisée F associée à une dérivation d non nulle, qui vérifie l'une des propriétés suivantes:*
(1) $F(x \circ y) = 0$ *pour tout* $x, y \in I$;
(2) $F(x \circ y) = \pm x \circ y$ *pour tout* $x, y \in I$;
alors R contient un idéal non nul central. De plus, si R est commutatif alors $Car(R) = 2$.

Ashraf et al. [12, Théorème 2.8] ont établi que, si un anneau semi-prime sans 2-torsion R admet une dérivation généralisée F associée à une dérivation non nulle d telle que $F[x,y] = [F(x),y] + [d(y),x]$ pour tout $x, y \in I$, où I est un idéal non nul de R, alors R contient un idéal central non nul. Nous avons réussi à généraliser ce résultat en considérant l'anneau quotient R/P et une dérivation généralisée de l'anneau de base R.

Théorème 0.0.6 *Soient R un anneau, I un idéal non nul de R et P un idéal premier de R tels que $P \subsetneq I$. Si R admet une dérivation généralisée F associée à une dérivation d satisfaisant la condition $\overline{F[x,y] - [F(x),y] - [d(y),x]} \in Z(R/P)$ pour tout $x, y \in I$, alors l'une des assertions suivantes est vraie:*
(1) $Car(R/P) = 2$;

(2) *d est à image dans P;*
(3) *R/P est un domaine commutatif.*

Dans [68], Dhara et al. ont montré qu'un anneau premier R est commutatif s'il admet deux dérivations généralisées F et G telles que $F(x)F(y)+G(xy)\pm yx \in Z(R)$ pour tout $x,y \in I$, où I est un idéal bilatère non nul de R. Nous avons adopté quasiment la même identité algébrique mais agissant sur le centre de l'anneau quotient R/P, ensuite nous avons montré que l'anneau quotient R/P est nécessairement un domaine commutatif. Par ailleurs, nous sommes parvenus à prouver que le résultat du Dhara ([68], Théorème 1) peut s'étendre aux anneaux semi-premiers. Plus précisément, nous avons démontré le théorème suivant:

Théorème 0.0.7 *Soient R un anneau, I un idéal non nul de R et P un idéal premier de R tels que $P \subsetneq I$. Si F et G deux dérivations généralisées de R associées respectivement aux dérivations d et g. Alors les assertions suivantes sont équivalentes:*
(1) *$\overline{F(x)F(y)+G(xy)\pm yx} \in Z(R/P)$ pour tout $x,y \in I$;*
(2) *R/P est un domaine commutatif.*

Le théorème suivant nous donne plus de détails sur les images des dérivations d et g.

Théorème 0.0.8 *Soient R un anneau, I un idéal non nul de R et P un idéal premier de R tels que $P \subsetneq I$. Si F et G sont des dérivations généralisées de R associées respectivement aux dérivations d et g, satisfaisant la condition $\overline{F(x)F(y)\pm G(xy)\pm xy} \in Z(R/P)$ pour tout $x,y \in I$, alors d et g sont à image dans P ou R/P est un domaine commutatif.*

Finalement, nous avons réussi à démontrer via des contre-exemples que les restrictions imposées dans nos différents théorèmes ne sont pas superflues. En particulier, nos théorèmes ne peuvent pas s'étendre aux idéaux semi-premiers.

Dans le **deuxième chapitre** nous avons traité des dérivations généralisées impliquant des idéaux premiers. Motivé par le fameux Théorème de Posner, nous avons considéré la condition $\overline{F(x)\circ x} \in Z(R/P)$ pour tout x de R. Dans ce contexte, nous avons non seulement caractérisé la structure de l'anneau R/P mais nous avons aussi prouvé que l'image de la dérivation généralisée F est nécessairement dans P.

Comme application de notre théorème, nous avons prouvé les résultats suivants:

Corollaire 0.0.9 *Soit R un anneau premier sans* 2*-torsion. Si F est une dérivation généralisée non nulle de R associée à la dérivation d, alors les affirmations suivantes sont équivalentes:*
(1) $F(x) \circ x \in Z(R)$ *pour tout* $x \in R$*;*
(2) $F(x) \circ x \pm d(x) \circ x \in Z(R)$ *pour tout* $x \in R$*;*
(3) *R est commutatif.*

Proposition 0.0.10 *Soit R un anneau semi-premier. Si R admet une dérivation généralisée F non nulle associée à une dérivation d, vérifiant l'une des conditions suivantes:*
(1) $F(x) \circ x \in Z(R)$ *pour tout* $x \in R$*;*
(2) $F(x) \circ x \pm d(x) \circ x \in Z(R)$ *pour tout* $x \in R$*;*
alors soit R est commutatif ou (il existe un idéal minimal premier P de R tel que $Car(R/P) = 2$ *ou* $F(R) \subseteq P$*).*

Dans la deuxième section nous avons abordé de nouvelles identités différentielles dans le but de généraliser des travaux concernant les dérivations qui agissent comme des homomorphismes (resp. anti-homomorphismes)[**]. Dans [21], Bell et Kappe ont montré le résultat suivant: Soient R un anneau premier et U un idéal à droite non nul de R. Si d est une dérivation de R qui agit comme un homomorphisme ou un anti-homomorphisme, alors $d = 0$ sur R. Asma Ali et al. [14] ont étendu ce dernier résultat à un idéal de Lie stable par carré. Notre objectif dans le théorème suivant est d'étudier un contexte plus général d'identités différentielles faisant intervenir deux dérivations au lieu d'une seule dérivation. De plus, en omettant l'hypothèse de la semi-primalité (primalité) imposée sur l'anneau R.

Théorème 0.0.11 *Soient R un anneau et P un idéal premier de R. Si R admet deux dérivations d et g, alors les assertions suivantes sont équivalentes:*
(1) $\overline{d(x)d(y) - g(yx)} \in Z(R/P)$ *pour tout* $x, y \in R$*;*
(2) $\overline{d(x)d(y) + g(yx)} \in Z(R/P)$ *pour tout* $x, y \in R$*;*
(3) $(d(R) \subseteq P$ *and* $g(R) \subseteq P)$ *ou* R/P *est un domaine commutatif.*

Comme conséquence de ce théorème, nous avons donné une version améliorée du résultat de Bell et Kappe dans le cas où R est un anneau premier. Nous avons étendu également le résultat de Bell et Kappe aux anneaux semi-premiers.

[**]Soient R un anneau et $\emptyset \neq S \subseteq R$, d agit comme un homomorphisme sur S si $d(xy) = d(x)d(y)$ pour tout $x, y \in S$ et d agit comme un anti-homomorphisme sur S si $d(xy) = d(y)d(x)$ pour tout $x, y \in S$.

Dans le **troisième chapitre**, motivé par les travaux d'Ashraf et Rehman [10], nous avons exhibé des conditions nécessaires et suffisantes pour répondre à la conjecture suivante:

Conjecture. Soient R un anneau semi-premier et I un idéal non nul de R. Si R admet des dérivations d et g telles que $d(x)y + xg(y) \pm xy \in Z(R)$ pour tout $x, y \in I$ ou $d(x)y + xg(y) \pm yx \in Z(R)$ pour tout $x, y \in I$, alors R est commutatif.

Nous avons pu fournir une réponse affirmative à la conjecture ci-dessus dans le théorème suivant.

Théorème 0.0.12 *Soient R un anneau semi-premier et I un idéal non nul de R. Si d et g deux dérivations de R, alors les affirmations suivantes sont équivalentes:*
(1) $d(x)y + xg(y) \pm xy \in Z(R)$ *pour tout* $x, y \in I$;
(2) $d(x)y + xg(y) \pm yx \in Z(R)$ *pour tout* $x, y \in I$;
(3) $d(x)y + xg(y) \pm [x, y] \in Z(R)$ *pour tout* $x, y \in I$;
(4) $d(x)y + xg(y) \pm x \circ y \in Z(R)$ *pour tout* $x, y \in I$;
(5) *R est commutatif.*

Le deuxième résultat principal de ce chapitre offre une correction et une généralisation du résultat de Ali et Shuliang qui ont montré que si d est une dérivation d'un anneau semi-premier sans 2-torsion R satisfaisant $d(x)d(y) \pm xy = 0$ ou $d(x)d(y) \pm yx = 0$ pour tout $x, y \in I$, où I est un idéal non nul de R, alors d est commutante sur I. De plus, si $d(I) \neq (0)$ alors R contient un idéal non nul central. Plus spécifiquement, nous avons démontré le théorème suivant:

Théorème 0.0.13 *Soient R un anneau, I un idéal non nul de R et P un idéal premier de R tels que $P \subsetneq I$. Si d et g sont des dérivations de R, alors les assertions suivantes sont équivalentes:*
(1) $\overline{d(x)g(y) \pm xy} \in Z(R/P)$ *pour tout* $x, y \in I$;
(2) $\overline{d(x)g(y) \pm yx} \in Z(R/P)$ *pour tout* $x, y \in I$;
(3) *R/P est un domaine commutatif.*
De plus, si R/P est sans 2-torsion et les dérivations d et g vérifiant l'une des propriétés $\overline{d(x)g(y) \pm [x, y]} \in Z(R/P)$ ou $\overline{d(x)g(y) \pm x \circ y} \in Z(R/P)$ pour tout $x, y \in I$, alors forcément R/P est un domaine commutatif.

Comme application de notre théorème, nous avons obtenu une version améliorée du résultat de Ali et Shuliang dans le **Corollaire 3.2.5** sans utiliser l'hypothèse $d(I) \neq (0)$.

Nous avons montré aussi le théorème suivant qui caractérise la commutativité de R/P et donne aussi quelques propriétés sur les images des dérivations d et g.

Théorème 0.0.14 *Soient R un anneau, I un idéal non nul de R et P un idéal premier de R tels que $P \subsetneq I$ et R/P est sans 2-torsion. Si d et g sont des dérivations de R, alors les assertions suivantes sont équivalentes:*
(1) $\overline{d(xy) \pm g(yx)} \in Z(R/P)$ *pour tout* $x, y \in I$;
(2) d *et* g *sont à image dans* P;
(3) R/P *est un domaine commutatif.*

Remarquons que le Théorème 0.0.14 permet de trouver le résultat amélioré de Bell ci-dessous, en tenant compte du fait que $P = 0$ lorsque l'anneau R est premier.

Théorème 0.0.15 *([18], Théorème 2.3) Si R est un anneau premier sans 2-torsion ayant une dérivation non nulle d telle que $d([x,y]) \in Z(R)$ pour tout $x, y \in R$, alors R est commutatif.*

Le but fondamental du **quatrième chapitre** est d'établir des critères de commutativité de l'anneau quotient R/P à partir de certains endomorphismes (resp. épimorphismes) satisfaisant quelques propriétés algébriques. Dans la première section, on s'est appuyé sur le résultat de Bell et Daif ([19], Théorème 2), où ils ont examiné la commutativité d'un anneau premier admettant un endomorphisme qui préserve fortement la commutativité (SCP)[††] sur un idéal à droite non nul. D'une part, nous avons démontré les théorèmes suivants:

Théorème 0.0.16 *Soient R un anneau, I un idéal non nul de R et P un idéal premier tels que $P \subsetneq I$. Si R admet une application f et un endomorphisme T qui satisfont $[f(x), T(y)] - [x,y] \in P$ pour tout $x, y \in I$, alors l'une des affirmations suivantes est vérifiée:*
(1) $[f(x), x] \in P$ *pour tout* $x \in I$ *et* $(T-1)(R) \subseteq P$;
(2) R/P *est un domaine commutatif.*

[††] Une application f est dite (SCP) sur un sous-ensemble S d'un anneau R si $[f(x), f(y)] = [x,y]$ pour tout $x, y \in S$.

Théorème 0.0.17 *Soient R un anneau, I un idéal non nul de R et P un idéal premier tels que $P \subsetneq I$. Si R admet une application additive f et un épimorphisme g qui satisfont $f(x) \circ g(y) - x \circ y \in P$ pour tout $x, y \in I$, alors l'une des affirmations suivantes est satisfaite:*
(1) *$[f(x), x] \in P$ pour tout $x \in I$ et $(g-1)(R) \subseteq P$;*
(2) *R/P est un domaine commutatif. De plus, si $Car(R/P) \neq 2$, alors $(f-1)(R) \subseteq P$ et $(g-1)(R) \subseteq P$.*

D'autre part, nous avons présenté dans les **Théorèmes 4.2.9 et 4.2.11** de nouvelles identités différentielles impliquant à la fois le commutateur et l'anti-commutateur.

Dans la deuxième section, nous avons généralisé les travaux de [10] et [41] concernant l'étude de la structure des anneaux en considérant un endomorphisme satisfaisant des identités algébriques spécifiques. En effet, nous avons traité des identités algébriques faisant intervenir deux endomorphismes.

Dans le **cinquième chapitre**, nous avons introduit trois nouveaux concepts concernant les centralisateurs à gauche, les centralisateurs à droite et les centralisateurs comme suit.

Définition 0.0.18 *Soit I un idéal d'un anneau R et $f : R \longrightarrow R$ une application additive.*
1) *f est dite un I-centralisateur à gauche si $f(xy) - f(x)y \in I$ pour tout $x, y \in R$.*
2) *f est dite un I-centralisateur à droite si $f(xy) - xf(y) \in I$ pour tout $x, y \in R$.*
3) *f est dite un I-centralisateur si, et seulement si, f est à la fois I-centralisateur à gauche et à droite.*

En considérant ces notions, nous avons généralisé plusieurs résultats concernant la commutativité des anneaux premiers.

Dans le **sixième et dernier chapitre**, nous avons exposé de nouveaux résultats sur les dérivations généralisées agissant sur un idéal de Lie[‡‡] d'un anneau premier. Nous avons obtenu une classification des dérivations généralisées considérées. Nous avons par la suite donné des applications aux algèbres de Banach premières. Plus exactement, nous avons montré les résultats suivants:

[‡‡]Tout idéal de Lie stable par le carré d'un anneau premier sans 2-torsion contient un idéal non nul. Cela signifie que l'étude des identités sur les idéaux de Lie peut être considérée comme le cas le plus optimal.

Théorème 0.0.19 *Soient R un anneau premier de caractéristique différente de 2, L un idéal de Lie non central de R, F_1, F_2 et F_3 trois dérivations généralisées de R. Si $[F_1(x), y] = F_2(x)y + yF_3(x)$ (resp. $F_1(x) \circ y = F_2(x)y + yF_3(x)$) pour tout $x, y \in L$, alors l'une des assertions suivantes est valable:*
(1) *ils existe $\lambda, \mu \in C$ tels que $F_2(x) = \lambda F_1(x)$ et $F_3(x) = \mu F_1(x)$ pour tout $x \in R$;*
(2) *R s'injecte dans l'anneau des matrices carrées d'ordre 2 à coefficients dans un corps.*

Théorème 0.0.20 *Soient A une algèbre de Banach première, O_1, O_2 deux ouverts non vides de A, F_1, F_2 et F_3 trois dérivations généralisées continues non nulles de A et n un entier positif fixé. Si $[F_1(x), y]^n = F_2(x)y + yF_3(x)$ (resp. $(F_1(x) \circ y)^n = F_2(x)y + yF_3(x)$) pour tout $(x, y) \in O_1 \times O_2$, alors A est commutative.*

Théorème 0.0.21 *Soient A une algèbre de Banach première non-commutative, O_1, O_2 deux ouverts non vides de A, F_1, F_2 et F_3 trois dérivations généralisées continues de R. Si $[F_1(x^r), y^s] = F_2(x^r)y^s + y^s F_3(x^r)$ (resp. $F_1(x^r) \circ y^s = F_2(x^r)y^s + y^s F_3(x^r)$) pour tout $(x, y) \in O_1 \times O_2$, où r et s sont des entiers non nuls dépendant du couple d'éléments (x, y), alors l'une des assertions suivantes est valable:*
(1) *il existe $\lambda, \mu \in C$ tels que $F_2(x) = \lambda F_1(x)$ et $F_3(x) = \mu F_1(x)$ pour tout $x \in A$;*
(2) *R s'injecte dans l'anneau des matrices carrées d'ordre 2 à coefficients dans un corps.*

CHAPTER 1

STRUCTURE OF A QUOTIENT RING R/P AND ITS RELATION WITH GENERALIZED DERIVATIONS OF R

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Abstract.

In this chapter, we investigate the structure of a quotient ring R/P where R is an arbitrary ring and P is a prime ideal of R. More precisely, we will characterize the commutativity of R/P via the behavior of generalized derivations, defined on R, satisfying algebraic identities involving the prime ideal P. Moreover, various well-known results characterizing commutativity of prime (semi-prime) rings have been extended. Furthermore, examples are given to prove that the restrictions imposed on the hypothesis of the various theorems were not superfluous.

1.1 Introduction

Recently several authors have investigated the relationship between the commutativity i.e. the structure of the ring R and some concrete additive mappings (such as derivations, automorphisms and generalized derivations) acting on appropriate subsets of the

rings. Herstein [45] showed that a prime ring R with nonzero derivation d satisfying $[d(x),d(y)]=0$ for all $x,y\in R$, must be a commutative integral domain if its characteristic is not two, and, if the characteristic equals two, then the ring must be commutative or an order in a simple algebra which is 4-dimensional over its center. The classical result of Posner [61] states that the existence of a nonzero centralizing derivation on a prime ring forces the ring to be commutative. Mayne [54] proved the analogous result for centralizing automorphisms. A number of authors have extended these theorems of Posner and Mayne in several ways. For example, see [22, 51, 59, 57].

One may observe that the concept of generalized derivations cover both the concepts of derivations and the left multipliers when $d=0$. Hence it should be interesting to extend some results concerning these notions to generalized derivations. More specifically, Brešar in [27] introduce the notion of generalized derivation as follows: An additive map $F:R\longrightarrow R$ is said to be a generalized derivation if there exists a derivation d of R such that $F(xy)=F(x)y+xd(y)$ for all $x,y\in R$. Generalized derivations have been primarily studied on operator algebras. For $a,b\in R$, the mapping $F:R\to R$ defined as $F(x)=ax+xb$ for all $x\in R$ is an example of generalized derivation of R, which is called as inner generalized derivation of R. It is obvious that every derivation (or left multiplier) is a generalized derivation but the converse is not true in general.

The present chapter is motivated by the previous results and we here continue this line of investigation by considering a more general concept rather than the ring R is prime or semi-prime in the hypothesis of our theorems. More precisely, we will establish a relationship between the structure of quotient rings R/P and the behavior of generalized derivations satisfying algebraic identities involving prime ideals.

1.2 Basic results

We will make frequent use of the following facts whose proofs will be left to the reader.

Fact 1 *Let R be a ring, I a nonzero ideal of R and P a prime ideal of R such that $P\subsetneq I$. If $aIb\subseteq P$, where $a,b\in R$, then $a\in P$ or $b\in P$.*

In [5, Theorem 2.2] it is proved that if P is a prime ideal of a ring R and d a derivation of R such that $\big[[x,d(x)],y\big]\in P$ for all $x,y\in R$, then $d(R)\subseteq P$ or R/P is a commutative

ring. Using similar arguments with some modifications, we get the following lemma which plays a crucial role in developing the proofs of our main results.

Lemma 1.2.1 *Let R be a ring, I a nonzero ideal of R and P be a prime ideal of R such that $P \subsetneq I$. If d is a derivation of R satisfying $\overline{[d(x),x]} \in Z(R/P)$ for all $x \in I$, then either $d(R) \subseteq P$ or R/P is a commutative integral domain.*

We recall some related known results in literature: In [11] Ashraf and Rehman proved that if R is a 2-torsion free prime ring and U a nonzero Lie ideal of R such that $u^2 \in U$, for all $u \in U$ and d a derivation which satisfies $d(u \circ v) = u \circ v$, for all $u, v \in U$, then $U \subseteq Z(R)$. Later, Quadri et al. [63], have extended the mentioned result by considering a generalized derivation F acting on a nonzero ideal I of R and without 2-torsion freeness hypothesis. More precisely, they proved that a prime ring must be commutative if it admits a generalized derivation F, associated with a nonzero derivation d, such that $F(x \circ y) = x \circ y$ for all x, y in a nonzero ideal I.

In the following theorem we will consider the situation of a ring R having a generalized derivation F satisfying the more general condition $\overline{F(x \circ y)} \in Z(R/P)$ for all $x, y \in I$. Further, our goal is to confirm that there is a relationship between the structure of the ring R/P and generalized derivations of R, where P is a prime ideal of R.

Theorem 1.2.2 *Let R be a ring, I a nonzero ideal of R and P a prime ideal of R such that $P \subsetneq I$ and $char(R/P) \neq 2$. If F is a generalized derivation of R associated with a derivation d satisfies the condition $\overline{F(x \circ y)} \in Z(R/P)$ for all $x, y \in I$, then $F(R) \subseteq P$ or R/P is a commutative integral domain.*

Proof. We are given that

$$\overline{F(x \circ y)} \in Z(R/P) \quad \text{for all } x, y \in I. \tag{1.1}$$

If $Z(R/P) = \{\overline{0}\}$, then R/P is non-commutative and the relation (1.1) becomes

$$F(x \circ y) \in P \quad \text{for all } x, y \in I. \tag{1.2}$$

Replacing y by yx in (1.2), we find that

$$F(x \circ y)x + (x \circ y)d(x) \in P \quad \text{for all } x, y \in I$$

in such a way that

$$(x \circ y)d(x) \in P \quad \text{for all } x, y \in I. \tag{1.3}$$

Substituting ry for y in (1.3), one can easily verify that

$$r(x \circ y)d(x) + [x, r]yd(x) \in P \quad \text{for all } x, y, r \in I$$

which implies that

$$[x, r]Id(x) \subseteq P \quad \text{for all } x, r \in I. \tag{1.4}$$

Applying Fact 1, it follows that either $[x, I] \subseteq P$ or $d(x) \in P$ holds for all $x \in I$. Let us set $H = \{x \in I / [x, I] \subseteq P\}$ and $K = \{x \in I / d(x) \in P\}$. Then it can be seen that H and K are two additives subgroups of I whose union is I. Using Brauer's trick we have either $I = H$ or $I = K$. Because of R/P is a non-commutative ring, we get necessarily $d(R) \subseteq P$. In this case replacing y by yr in (1.2), we obtain

$$F(y)[r, x] \in P \quad \text{for all } x, y, r \in I.$$

Hence the last expression proves that $F(y)I[r, x] \subseteq P$, by view of the primeness of P we conclude that $F(I) \subseteq P$ and thus $F(R) \subseteq P$.
Now if $Z(R/P) \neq \{\overline{0}\}$, then there exists $\overline{z} \in Z(R/P)$ such that $\overline{z} \neq \overline{0}$. Substituting yz for y in (1.1), we can obviously get

$$\overline{F\big((x \circ y)z - y[x, z]\big)} \in Z(R/P) \quad \text{for all } x, y \in I$$

which reduces to

$$\overline{(x \circ y)d(z) - y[x, d(z)]} \in Z(R/P) \quad \text{for all } x, y \in I. \tag{1.5}$$

Putting ry instead of y in (1.5), it is obvious to verify that

$$\overline{r\big((x \circ y)d(z) - y[x, d(z)]\big) + [x, r]yd(z)} \in Z(R/P) \quad \text{for all } x, y, r \in I$$

we readily see from the above relation that

$$\big[[x, r]yd(z), r\big] \in P \quad \text{for all } x, y, r \in I \tag{1.6}$$

this equation can be rewritten as

$$[x,r]yd(z)r - r[x,r]yd(z) \in P \quad \text{for all } x,y,r \in I. \tag{1.7}$$

Writing xt instead of x in (1.7) and using it, we arrive at

$$x[t,r]yd(z)r - rx[t,r]yd(z) \in P \quad \text{for all } x,y,r,t \in I. \tag{1.8}$$

On the other hand, if we replace x by t in (1.7) and then left multiplying it by x, we obtain

$$x[t,r]yd(z)r - xr[t,r]yd(z) \in P \quad \text{for all } x,y,r,t \in I. \tag{1.9}$$

Using (1.8) together with (1.9), we can also write

$$[x,r][t,r]yd(z) \in P \quad \text{for all } x,y,r,t \in I$$

particularly, we have

$$[x,r]I[x,r]Id(z) \subseteq P \quad \text{for all } x,r \in I.$$

In light of primeness, we get either $[x,r] \in P$ for all $x,r \in I$ or $d(z) \in P$ and thus, we conclude that R/P is an integral domain or $\overline{d(z)} = \overline{0}$. Now if we take $y = z$ in (1.1), then we can see that

$$\overline{\big(F(x)+d(x)\big)z + F(z)x} \in Z(R/P) \quad \text{for all } x \in I. \tag{1.10}$$

The substitution xy for x in (1.10) gives $2\overline{[xd(y),y]}\overline{z} = \overline{0}$ for all $x,y \in I$. Putting rx instead of x, it follows that

$$2\overline{[r,y]xd(y)}\overline{z} = \overline{0} \quad \text{for all } x,y,r \in I.$$

Using 2-torsion freeness, we find that $[r,y]Id(y) \subseteq P$ for all $y,r \in I$. Therefore, we get $d(R) \subseteq P$ or R/P is commutative. By the first case, our hypothesis leads to

$$\overline{F(x)y + F(y)x} \in Z(R/P) \quad \text{for all } x,y \in I. \tag{1.11}$$

Replacing y by yr in (1.11), one can verify that

$$\overline{\big(F(x)y + F(y)x\big)r - F(y)[x,r]} \in Z(R/P) \quad \text{for all } x,y,r \in I$$

thereby obtaining

$$\big[F(y)[x,r],r\big] \in P \quad \text{for all } x,y,r \in I.$$

Putting $yF(y)$ instead of y, we get

$$F(y)\big[F(y)[x,r],r\big] + [F(y),r]F(y)[x,r] \in P \quad \text{for all } x,y,r \in I$$

which leads to $[F(y),r]F(y)[x,r] \in P$ for all $x,y,r \in I$. Replacing x by tx, we obtain

$$[F(y),r]F(y)t[x,r] \in P \quad \text{for all } x,y,r,t \in I.$$

Substituting $xF(y)$ for x, we deduce that

$$[F(y),r]F(y)tx[F(y),r] \in P \quad \text{for all } x,y,r,t \in I.$$

On the other hand, we have $[F(y),r]F(y)tx[F(y),r]F(y) \in P$. As a special case of the latter expression, we may write

$$[F(y),r]F(y)I[F(y),r]F(y)I[F(y),r]F(y) \subseteq P \quad \text{for all } y,r \in I.$$

According to Fact 1, it follows that $[F(y),r]F(y) \in P$ for all $y,r \in I$. Substituting rt for r in the last relation, we obtain

$$[F(y),r]t[F(y),r] \in P \quad \text{for all } y,r,t \in I.$$

Hence the above equation assures that $[F(y),r] \in P$ for all $y,r \in I$. Now writing yt instead of y, we find that $F(y)[t,r] \in P$ and therefore we conclude that $F(R) \subseteq P$ or R/P is commutative. Consequently, in any cases, it follows that $F(R) \subseteq P$ or R/P is a commutative integral domain. ■

Letting R be a prime ring in the previous theorem, then $P = (0)$ is a prime ideal of R, in this case we obtain the commutativity criteria for category of prime rings.

Corollary 1.2.3 *Let R be a 2-torsion free prime ring and I a nonzero ideal of R. If R admits a nonzero generalized derivation F associated with a derivation d satisfies the*

condition $F(x\circ y)\in Z(R)$ for all $x,y\in I$, then R is a commutative integral domain.

As an application of Corollary 1.2.3 we have the following result.

Corollary 1.2.4 *Let R be a 2-torsion free prime ring and I a nonzero ideal of R. If R admits a generalized derivation F associated with a derivation d such that $F\neq id_R$ and $F(x\circ y)-x\circ y\in Z(R)$ $\big($resp. $F\neq -id_R$ and $F(x\circ y)+x\circ y\in Z(R)\big)$ for all $x,y\in I$, then R is a commutative integral domain.*

Proof. Assume that $F\neq\pm id_R$, then $\mathscr{F}=F-id_R$ (resp. $\tilde{\mathscr{F}}=F+id_R$) is also a nonzero generalized derivation satisfying $\mathscr{F}(x\circ y)\in Z(R)$ $\big($resp. $\tilde{\mathscr{F}}(x\circ y)\in Z(R)\big)$ for all $x,y\in I$. However by virtue of Corollary 1.2.3, we conclude that R is commutative. ■

With no further assumption to the characteristic of the considered ring, the following proposition gives an improved version of some known results obtained in [11] and [63] for semi-prime ring.

Proposition 1.2.5 *Let R be a semi-prime ring and I a nonzero ideal of R. If R admits a generalized derivation F associated with a nonzero derivation d, satisfies one of the following properties:*
1) $F(x\circ y)=0$ *for all* $x,y\in I$
2) $F(x\circ y)\pm x\circ y=0$ *for all* $x,y\in I$
then R contains a nonzero central ideal. Moreover, if R is commutative then $char(R)=2$.

Proof. (1) Suppose that $F(x\circ y)=0$ for all $x,y\in I$. The ring R is semi-prime then there exists a family of prime ideals $\mathscr{P}=\{P_\alpha\,/\,\alpha\in\Lambda\}$ such that $\bigcap\limits_{\alpha\in\Lambda}P_\alpha=(0)$. Using the proof of Theorem 1.2.2, by expression (1.4) we get $[d(x),x]I[d(x),x]\subseteq P_\alpha$ for all $\alpha\in\Lambda$ and for all $x\in I$. Therefore, $[d(x),x]I[d(x),x]=0$ and the semi-primeness of R forces that $[d(x),x]=0$ for all $x\in I$. Accordingly, by [22, Theorem 3], we conclude that R contains a nonzero central ideal. Given that R is commutative and suppose that $char(R)\neq 2$, then main assumption becomes $F(xy)=0$ for all $x,y\in I$. Simple arguments give $d=0$, a contradiction.
(2) Using the same technics as in the preceding proof, we can prove the same conclusion holds for $F(x\circ y)\pm x\circ y=0$ for all $x,y\in I$. ■

The authors in [12, Theorem 2.8] established that, if a 2-torsion free semi-prime ring R admits a generalized derivation F associated with a nonzero derivation d such that $F[x,y] = [F(x),y] + [d(y),x]$ for all $x,y \in I$, where I is a nonzero ideal of R, then R contains a nonzero central ideal. Moreover, Ashraf and Almas Khan [9] considered the same identity, but for Lie ideals in $*$-prime rings. More specifically, they proved that, if R is a 2-torsion free $*$-prime ring, $F : R \to R$ is a generalized derivation with a nonzero derivation d which commutes with $*$ and U is a $*$-Lie ideal of R such that $F[u,v] = [F(u),v] + [d(v),u]$ for all $u,v \in U$, then $U \subseteq Z(R)$.

Motivated by the above results, the aim of the next theorem is to study the more general case when the same relation contained on center of R/P. More precisely we will prove the following result.

Theorem 1.2.6 *Let R be a ring, I a nonzero ideal of R and P a prime ideal of R such that $P \subsetneq I$. If R admits a generalized derivation F associated with a derivation d satisfying the condition $\overline{F[x,y] - [F(x),y] - [d(y),x]} \in Z(R/P)$ for all $x,y \in I$, then one of the following assertions holds:*
1) *$char(R/P) = 2$;*
2) *$d(R) \subseteq P$;*
3) *R/P is a commutative integral domain.*

Proof. Suppose that $char(R/P) \neq 2$. We are given that

$$\overline{F[x,y] - [F(x),y] - [d(y),x]} \in Z(R/P) \quad \text{for all } x,y \in I. \tag{1.12}$$

If $Z(R/P) = \{\overline{0}\}$, then the relation (1.12) reduces to

$$F[x,y] - [F(x),y] - [d(y),x] \in P \quad \text{for all } x,y \in I. \tag{1.13}$$

Replacing y by yx in (1.13), we get

$$[x,y]d(x) - y[F(x),x] - [yd(x),x] \in P \quad \text{for all } x,y \in I.$$

Substituting ry for y in the above relation and using it, we can verify that

$$2[x,r]Id(x) \subseteq P \quad \text{for all } x,r \in I.$$

Whence, using 2-torsion freeness with Fact 1 we get $d(R) \subseteq P$ or R/P is commutative, a contradiction. Therefore, we obviously obtain $d(R) \subseteq P$.
Now if $Z(R/P) \neq \{\overline{0}\}$, then there exists $\overline{z} \in Z(R/P)$ such that $\overline{z} \neq \overline{0}$. Substituting yz for y in our hypothesis, we get

$$\overline{[x,y]d(z) - y[F(x),z] - [yd(z),x]} \in Z(R/P) \quad \text{for all } x,y \in I$$

which proves that

$$\big[[x,r]yd(z),r\big] \in P \quad \text{for all } x,y,r \in I.$$

Since, this relation is exactly (1.6), then arguing as before, we find that either R/P is a commutative integral domain or $\overline{d(z)} = \overline{0}$. On the other hand, if we replace y by $-x$ in (1.12), then it is obvious to see that

$$\overline{[F(x),x] + [d(x),x]} \in Z(R/P) \quad \text{for all } x \in I. \tag{1.14}$$

A linearization of relation (1.14), leads to

$$\overline{[F(z),x]} \in Z(R/P) \quad \text{for all } x \in I. \tag{1.15}$$

Writing $xF(z)$ instead of x in (1.15), we get $\overline{F(z)} \in Z(R/P)$. Now substituting zx for x in (1.14), we obtain

$$\overline{[F(z)x,zx]} + 2\overline{[zd(x),zx]} \in Z(R/P) \quad \text{for all } x \in I$$

and therefore

$$2\overline{[d(x),x]}\,\overline{z}^2 \in Z(R/P) \quad \text{for all } x \in I.$$

Hence from the last relation, we get $\overline{[d(x),x]} \in Z(R/P)$ for all $x \in I$. Invoking Lemma 1.2.1, we conclude that $d(R) \subseteq P$ or R/P is commutative. Consequently, in both cases, we have either $d(R) \subseteq P$ or R/P is a commutative integral domain. ■

As an application of our theorem, the following corollary improves the result of [9] for the case when the underlying identity belongs to the center of a prime ring.

Corollary 1.2.7 *Let R be a 2-torsion free prime ring and I a nonzero ideal of R. If R admits a generalized derivation F associated with a nonzero derivation d satisfying $F[x,y]-[F(x),y]-[d(y),x]\in Z(R)$ for all $x,y\in I$, then R is a commutative integral domain.*

Proposition 1.2.8 *[12, Theorem 2.8] Let R be a 2-torsion free semi-prime ring and I a nonzero ideal of R. If R admits a generalized derivation F associated with a nonzero derivation d satisfying $F[x,y]=[F(x),y]+[d(y),x]$ for all $x,y\in I$, then R contains a nonzero central ideal.*

In [68] Dhara et al. showed that, a prime ring R must be commutative if it admits two generalized derivations F and G associated with derivations d and g respectively and satisfies the properties $F(x)F(y)\pm G(xy)\pm yx\in Z(R)$ for all $x,y\in I$, where I is a nonzero two sided ideal of R.

Motivated by the above results, our aim in the following theorem is to investigate a more general context of differential identities with generalized derivations acting in a center of quotient ring R/P by omitting the primeness assumption imposed on the ring.

Theorem 1.2.9 *Let R be a ring, I a nonzero ideal of R and P a prime ideal of R such that $P\subsetneq I$. If (F,d) and (G,g) two generalized derivations of R associated with derivations d and g satisfying the condition $\overline{F(x)F(y)+G(xy)\pm yx}\in Z(R/P)$ for all $x,y\in I$, then R/P is a commutative integral domain.*

Proof. Assume that

$$\overline{F(x)F(y)+G(xy)-yx}\in Z(R/P) \quad \text{for all } x,y\in I. \tag{1.16}$$

Replacing y by yr in (1.16), we have

$$\big[F(x)yd(r)+xyg(r)+y[x,r],r\big]\in P \quad \text{for all } x,y,r\in I. \tag{1.17}$$

Substituting xr for x in (1.17), we get

$$\big[F(x)ryd(r)+xd(r)yd(r)+xryg(r)+y[x,r]r,r\big]\in P \quad \text{for all } x,y,r\in I. \tag{1.18}$$

Using (1.17) together with (1.18), we find that

$$\big[xd(r)yd(r)+[y[x,r],r],r\big]\in P \quad \text{for all } x,y,r\in I. \tag{1.19}$$

Right multiplying (1.19) by r and combining it with the last relation, it follows that

$$\big[x[d(r)yd(r),r],r\big]\in P \quad \text{for all } x,y,r\in I. \tag{1.20}$$

Writing $d(r)yd(r)x$ instead of x in (1.20), we obtain

$$[d(r)yd(r),r]x[d(r)yd(r),r]\in P \quad \text{for all } x,y,r\in I.$$

Applying Fact 1, we get $[d(r)yd(r),r]\in P$ for all $y,r\in I$, that is

$$d(r)yd(r)r-rd(r)yd(r)\in P \quad \text{for all } y,r\in I. \tag{1.21}$$

Replacing y by $yd(r)t$ in (1.21), we get

$$d(r)yd(r)td(r)r-rd(r)yd(r)td(r)\in P \quad \text{for all } y,r,t\in I. \tag{1.22}$$

Putting t instead of y in (1.21) and left multiplying it by $d(r)y$, we arrive at

$$d(r)yd(r)td(r)r-d(r)yrd(r)td(r)\in P \quad \text{for all } y,r,t\in I. \tag{1.23}$$

Combining (1.22) with (1.23), one can verify that

$$d(r)yrd(r)td(r)-rd(r)yd(r)td(r)\in P \quad \text{for all } y,r,t\in I. \tag{1.24}$$

On the other hand, right multiplying (1.21) by $td(r)$ and then subtracting it from (1.24), it is obvious to see that

$$d(r)y[d(r),r]td(r)\in P \quad \text{for all } y,r,t\in I$$

which forces that

$$[d(r),r]I[d(r),r]I[d(r),r] \subseteq P \quad \text{for all } r \in I.$$

According to Fact 1, it follows that $[d(r),r] \in P$ for all $r \in I$, which proves that $\big[[d(r),r],t\big] \in P$ for all $r,t \in I$. By virtue of Lemma 1.2.1, the last equation implies that either $d(R) \subseteq P$ or R/P is an integral domain. Now if we take $d(R) \subseteq P$, then the expression (1.17) becomes

$$x[yg(r),r] + [x,r]yg(r) + \big[y[x,r],r\big] \in P \quad \text{for all } x,y,r \in I. \tag{1.25}$$

Putting xy instead of y in (1.25), one can see that

$$[x,r]xyg(r) + [x,r]y[x,r] \in P \quad \text{for all } x,y,r \in I. \tag{1.26}$$

Substituting $r+x$ for r in (1.26), we obtain

$$[x,r]xyg(x) \in P \quad \text{for all } x,y,r \in I.$$

Accordingly,

$$[g(x),x]t[g(x),x]y[g(x),x] \in P \quad \text{for all } x,y,t \in I.$$

Whence, using again Fact 1, we conclude that either $g(R) \subseteq P$ or R/P is commutative. Thus, in the first case the expression (1.26) reduces to $[x,r]y[x,r] \in P$ for all $x,y,r \in I$. Since P is prime, the last equation implies that $[R,R] \subseteq P$ and therefore R/P is a commutative integral domain.
By similar manner, the same conclusion holds for $\overline{F(x)F(y)+G(xy)+yx} \in Z(R/P)$ for all $x,y \in I$. This completes the proof of our theorem. ■

As an application of Theorem 1.2.9, the following corollary extended the results of Dhara [68, Theorem 1] for semi-prime ring.

Corollary 1.2.10 *Let R be a semi-prime ring and I a nonzero ideal of R. If (F,d) and (G,g) two generalized derivations of R associated with derivations d and g. Then the following assertions are equivalent:*
1) $F(x)F(y) \pm G(xy) \pm yx \in Z(R)$ *for all* $x,y \in I$;
2) R *is commutative.*

Proof. We need only prove that $(1) \Longrightarrow (2)$. Assume that

$$F(x)F(y)+G(xy)\pm yx \in Z(R) \quad \text{for all } x,y \in I. \tag{1.27}$$

By view of the semi-primeness of the ring R, there exists a family of prime ideals $\mathscr{P} = \{P_\alpha / \alpha \in \Lambda\}$ such that $\bigcap\limits_{\alpha\in\Lambda} P_\alpha = (0)$, thereby obtaining $[F(x)F(y)+G(xy)\pm yx, r] \in P_\alpha$ for all $x,y,r \in I$ and for all $\alpha \in \Lambda$. Hence, it follows that $\overline{F(x)F(y)+G(xy)\pm yx} \in Z(R/P_\alpha)$ for all $\alpha \in \Lambda$. Invoking Theorem 1.2.9, we conclude that R/P_α is a commutative integral domain which, because of $\bigcap\limits_{\alpha\in\Lambda} P_\alpha = (0)$, assures that R is commutative.
We notice that, if (G,g) is a generalized derivation on R, then $(-G,-g)$ is also a generalized derivation on R. Thus by putting $(-G,-g)$ instead of (G,g) in the expression (1.27), we get the required result. ■

If we replace G by $G \pm id_R$ in the Corollary 1.2.10, then one can obviously obtain the following result.

Corollary 1.2.11 *Let R be a semi-prime ring and I a nonzero ideal of R. If (F,d) and (G,g) two generalized derivations of R associated with derivations d and g. Then the following assertions are equivalent:*
1) $F(x)F(y)\pm G(xy)\pm [x,y] \in Z(R)$ *for all* $x,y \in I$;
2) $F(x)F(y)\pm G(xy)\pm x\circ y \in Z(R)$ *for all* $x,y \in I$;
3) R *is commutative.*

In [8] Ashraf et al. established that if R is a prime ring, I is a nonzero ideal of R and F is a generalized derivation of R associated with nonzero derivation d such that $F(xy)\pm xy \in Z(R)$ or $F(x)F(y)\pm xy \in Z(R)$ for all $x,y \in I$, then R is commutative.

Our fundamental aim is to generalize this result in two directions. First of all, we will treat a more general differential identity involving two generalized derivations. More specifically, we will study the more general case by considering the following situations: (i) $\overline{F(x)F(y)\pm G(xy)} \in Z(R/P)$ for all $x,y \in I$, (ii) $\overline{[F(x),y]\pm G(xy)} \in Z(R/P)$ for all $x,y \in I$ and (iii) $\overline{F(x)\circ y \pm G(xy)} \in Z(R/P)$ for all $x,y \in I$. Secondly, we will assume that the above algebraic identities belong to $Z(R/P)$, where P is any prime ideal rather than the zero ideal.

Theorem 1.2.12 *Let R be a ring, I a nonzero ideal of R and P a prime ideal of R such that $P \subsetneq I$. If F and G are generalized derivations of R associated with derivations d and g respectively, satisfying one of the following properties:*
1) $\overline{F(x)F(y) \pm G(xy)} \in Z(R/P)$ *for all* $x, y \in I$;
2) $\overline{[F(x), y] \pm G(xy)} \in Z(R/P)$ *for all* $x, y \in I$;
3) $\overline{F(x) \circ y \pm G(xy)} \in Z(R/P)$ *for all* $x, y \in I$;
then $\big(d(R) \subseteq P$ *and* $g(R) \subseteq P\big)$ *or R/P is a commutative integral domain.*

Proof. (1) We are given that

$$\overline{F(x)F(y) + G(xy)} \in Z(R/P) \quad \text{for all } x, y \in I. \tag{1.28}$$

Replacing y by yr in (1.28), we find that

$$\overline{\big(F(x)F(y) + G(xy)\big)r + F(x)yd(r) + xyg(r)} \in Z(R/P) \quad \text{for all } x, y, r \in I$$

and therefore

$$[F(x)yd(r) + xyg(r), r] \in P \quad \text{for all } x, y, r \in I. \tag{1.29}$$

Substituting xt for x in (1.29) and subtracting it with (1.29), we arrive at

$$[xd(t)yd(r), r] \in P \quad \text{for all } x, y, r, t \in I. \tag{1.30}$$

Putting ux instead of x in (1.30), we obtain

$$[u, r]xd(t)yd(r) + u[xd(t)yd(r), r] \in P \quad \text{for all } x, y, r, t, u \in I$$

in such a way that

$$[u, r]Id(t)yd(r) \subseteq P \quad \text{for all } y, r, t, u \in I \tag{1.31}$$

which because of primeness, gives that either $[I, r] \subseteq P$ or $d(t)yd(r) \in P$ for all $y, r, t \in I$. The sets of r for which theses conditions holds are additive subgroups of I with union equal to I; so that by Brauer's trick, we have R/P is commutative or $d(R) \subseteq P$. In the later case the relation (1.29) yields

$$[xyg(r), r] \in P \quad \text{for all } x, y, r \in I. \tag{1.32}$$

Substituting wx for x in (1.32) where $w \in R$, we get $[w,r]xyg(r) \in P$ for all $x,y,r \in I$. As a special case of the last equation, we may write

$$[w,r]xg(r)yg(r) \in P \quad \text{for all } x,y,r \in I \text{ and } w \in R. \tag{1.33}$$

On the other hand, taking $w = g(r)$ in the above relation and combining it with (1.33), it is obvious to see that

$$[g(r),r]x[g(r),r]y[g(r),r] \in P \quad \text{for all } x,y,r \in I.$$

Since P is prime, the last equation assures that $[g(r),r] \in P$ which leads to $\big[[g(r),r],t\big] \in P$ for all $r,t \in I$. Applying Lemma 1.2.1, it follows that either $g(R) \subseteq P$ or R/P is commutative. Now assume that $\overline{F(x)F(y) - G(xy)} \in Z(R/P)$. Thus by putting $(-G,-g)$ instead of (G,g) in the relation (1.28), we get the required result.

(2) Suppose that

$$\overline{[F(x),y] + G(xy)} \in Z(R/P) \quad \text{for all } x,y \in I. \tag{1.34}$$

Replacing y by yr in (1.34), we get

$$\overline{\big([F(x),y] + G(xy)\big)r + y[F(x),r] + xyg(r)} \in Z(R/P) \quad \text{for all } x,y,r \in I$$

which leads to

$$[y[F(x),r] + xyg(r),r] \in P \quad \text{for all } x,y,r \in I. \tag{1.35}$$

Writing ry instead of y in (1.35) and subtracting it from (1.35), we arrive at

$$\big[[x,r]yg(r),r\big] \in P \quad \text{for all } x,y,r \in I. \tag{1.36}$$

Since the expression (1.36) is similar as relation (1.6), reasoning in the same manner as above, we find that

$$[x,r][t,r]yg(r) \in P \quad \text{for all } x,y,r,t \in I$$

in particular

$$[g(r),r]I[g(r),r]I[g(r),r] \subseteq P \quad \text{for all } r \in I.$$

Invoking Fact 1, we get either R/P is an integral domain or $g(R) \subseteq P$. By the second case

the relation (1.35) reduces to

$$\big[y[F(x),r],r\big] \in P \quad \text{for all } x,y,r \in I. \tag{1.37}$$

Putting $F(x)y$ instead of y in the expression (1.37) and using it, we obtain

$$[F(x),r] \in P \quad \text{for all } x,r \in I. \tag{1.38}$$

Replacing x by xr in the expression (1.38), we find that $[xd(r),r] \in P$ for all $x,r \in I$. The substitution tx for x in the last equation gives

$$[t,r]Id(r) \subseteq P \quad \text{for all } r,t \in I.$$

Finally, we claim that either $d(R) \subseteq P$ or R/P is an integral domain.
Furthermore, if we have $\overline{[F(x),y] - G(xy)} \in Z(R/P)$, then arguing as above, we arrive at $\big(d(R) \subseteq P$ and $g(R) \subseteq P\big)$ or R/P is a commutative integral domain.
(3) Using the same techniques as in the second case with a slight modifications, one can see that the same conclusion holds for $\overline{F(x) \circ y \pm G(xy)} \in Z(R/P)$ for all $x,y \in I$. Whence, the proof of our theorem is complete. ■

The following corollary is an immediate consequence of the above theorem.

Corollary 1.2.13 *Let R be a prime ring and I a nonzero ideal of R. If F and G are generalized derivations of R associated with derivations d and g respectively such that at least one is nonzero, then the following assertions are equivalent:*
1) $F(x)F(y) \pm G(xy) \in Z(R)$ *for all* $x,y \in I$;
2) $[F(x),y] \pm G(xy) \in Z(R)$ *for all* $x,y \in I$;
3) $F(x) \circ y \pm G(xy) \in Z(R)$ *for all* $x,y \in I$;
4) $F(x)F(y) \pm G(xy) \pm xy \in Z(R)$ *for all* $x,y \in I$;
5) $[F(x),y] \pm G(xy) \pm xy \in Z(R)$ *for all* $x,y \in I$;
6) $F(x) \circ y \pm G(xy) \pm xy \in Z(R)$ *for all* $x,y \in I$;
7) *R is a commutative integral domain.*

As an application of Corollary 1.2.13, we have the following result.

Corollary 1.2.14 *Let R be a prime ring and I a nonzero ideal of R. If F and G are generalized derivations of R associated with derivations d and g respectively such that at least one is nonzero, then the following assertions are equivalent:*
1) $[F(x),y] \pm G(xy) \pm yx \in Z(R)$ *for all* $x,y \in I$;
2) $[F(x),y] \pm G(xy) \pm [x,y] \in Z(R)$ *for all* $x,y \in I$;
3) $[F(x),y] \pm G(xy) \pm x \circ y \in Z(R)$ *for all* $x,y \in I$;
4) $F(x) \circ y \pm G(xy) \pm yx \in Z(R)$ *for all* $x,y \in I$;
5) $F(x) \circ y \pm G(xy) \pm [x,y] \in Z(R)$ *for all* $x,y \in I$;
6) $F(x) \circ y \pm G(xy) \pm x \circ y \in Z(R)$ *for all* $x,y \in I$;
7) *R is a commutative integral domain.*

As a consequence of Theorem 1.2.12, the next proposition gives a commutativity criteria for semi-prime ring.

Proposition 1.2.15 *Let R be a semi-prime ring and I a nonzero ideal of R. If R admits two generalized derivations F and G associated with nonzero derivations d and g respectively, satisfying one of the following conditions:*
1) $F(x)F(y) \pm G(xy) \in Z(R)$ *for all* $x,y \in I$;
2) $[F(x),y] \pm G(xy) \in Z(R)$ *for all* $x,y \in I$;
3) $F(x) \circ y \pm G(xy) \in Z(R)$ *for all* $x,y \in I$;
then R contains a nonzero central ideal.

Proof. Assume that $F(x)F(y) \pm G(xy) \in Z(R)$ for all $x,y \in I$. The ring R is semi-prime then there exists a family of prime ideals $\mathscr{P} = \{P_\alpha / \alpha \in \Lambda\}$ such that $\bigcap_{\alpha\in\Lambda} P_\alpha = (0)$. Therefore $[F(x)F(y) \pm G(xy), r] \in P_\alpha$ for all $\alpha \in \Lambda$. Using the proof of Theorem 1.2.12, by equation (1.31) we get

$$[d(r),r]I[d(r),r]I[d(r),r] = 0 \quad \text{for all } \ r \in I.$$

In light of the semi-primeness of R, we easily obtain $[d(r),r] = 0$ for all $r \in I$. According to [22, Theorem 3], we conclude that R contains a nonzero central ideal.
Using the same technics as in the preceding proof, the same conclusion holds for the identities $[F(x),y] \pm G(xy) \in Z(R)$ and $F(x) \circ y \pm G(xy) \in Z(R)$ for all $x,y \in I$. ■

In the following proposition we will extend [68, Corollary 6] for semi-prime ring.

Proposition 1.2.16 *Let R be semi-prime ring and I a nonzero ideal of R. Suppose that R admits two generalized derivations F and G associated with derivations d and g respectively such that at least one is nonzero. If the condition $F(x)F(y) \pm G(xy) \pm xy \in Z(R)$ holds for all $x, y \in I$, then R contains a nonzero central ideal.*

The following example proves that the condition "`R/P is 2-torsion free`" is necessary in Theorem 1.2.2.

Example 1. Let us set $R = M_2(\mathbb{Z}_2)$ and $P = (0)$. It is straightforward to check that R is a prime ring with $char(R) = 2$ and P is a prime ideal of R. Define $F : R \to R$ by $F(X) = X \circ A$, where $A = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$, then F is a nonzero generalized derivation of R associated with the inner derivation $d(X) = [X, A]$ satisfying

$$F(X \circ Y) = \begin{pmatrix} ca' + dc' + c'a + d'c & 0 \\ 0 & ca' + dc' + c'a + d'c \end{pmatrix} \in Z(R)$$

for all $X = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ and $Y = \begin{pmatrix} a' & b' \\ c' & d' \end{pmatrix}$ in R. However, R is non-commutative.

The following example proves that the condition of the "`primeness`" imposed on the ideal is crucial in our Theorems.

Example 2. Consider $R = \left\{ \begin{pmatrix} a & b \\ 0 & c \end{pmatrix} / \, a, b, c \in \mathbb{Z} \right\}$ and $P = (0)$. Let I the ideal of R defined by $I = \left\{ \begin{pmatrix} 0 & a \\ 0 & 0 \end{pmatrix} / \, a \in \mathbb{Z} \right\}$. Define the maps on R as follows $F(x) = 2e_{11}x - xe_{11}$ and $G(x) = e_{12}x + xe_{11}$. Then it is clearly to see that F and G are generalized derivations of R associated with nonzero derivations d and g respectively, where $d(x) = e_{11}x - xe_{11}$ and $g(x) = -e_{11}x + xe_{11}$. Moreover F and G satisfies the conditions of all Theorems, but R is not commutative.

CHAPTER 2

RELATIONSHIP BETWEEN THE STRUCTURE OF A QUOTIENT RING AND THE BEHAVIOR OF CERTAIN ADDITIVE MAPPINGS

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Abstract.

The principal aim of this chapter is to study the connection between the structure of a quotient ring R/P and the behavior of certain additive mappings of R. More precisely, we characterize the commutativity of R/P using derivations (generalized derivations) of R satisfying algebraic identities involving the prime ideal P. Furthermore, we provide examples to show that the various restrictions imposed in the hypothesis of our theorems are not superfluous.

2.1 Introduction

Many results in the literature indicate how the global structure of a ring R is often tightly connected to the behaviour of additive mappings defined on R (for example, see

[28]-[55]). A well known result due to Posner [61] states that if d is a derivation of a prime ring R such that $[d(x),x] \in Z(R)$ for all $x \in R$, then either $d = 0$ or R is commutative. In [51] Lanski generalized the result of Posner by considering a derivation d such that $[d(x),x] \in Z(R)$ for all x in a nonzero Lie ideal U of R. A number of authors have extended the theorem of Posner in several directions (for example, see [22] and [57]).

In [11], Ashraf and Rehman proved that if R is a prime ring with a nonzero ideal I of R and d is a derivation of R such that $d(x) \circ d(y) = x \circ y$ for all $x, y \in I$, then R is commutative. Moreover, if d is nonzero and $d(x) \circ d(y) = 0$ for all $x, y \in I$, then R is commutative. Recently, H. E. Bell and Nadeem-Ur Rehman [23] have studied the situation by replacing the derivation d with a generalized derivation F. More precisely, they proved that if R is a prime ring with 1 and $char(R) \neq 2$ such that $F(x) \circ F(y) = 0$ for all $x, y \in R$, then $F = 0$ where, F is a generalized derivation of R associated with a nonzero derivation d. In this line of investigation, Asma Ali et al. [14] have studied the following situations: If R is a 2-torsion free prime ring, U a nonzero Lie ideal of R and $u^2 \in U$, for all $u \in U$ such that d is a derivation of R which acts as an homomorphism or an anti-homomorphism on U, then either $d = 0$ or $U \subseteq Z(R)$. Recently, Rehman [64] studied the above mentioned results of Asma Ali et al. for prime ring with nonzero generalized derivation F. More specifically, he proved that, if R is a 2-torsion free prime ring, $I \neq 0$ an ideal of R and F a nonzero generalized derivation of R with a nonzero derivation d such that either $F(xy) = F(x)F(y)$ or $F(xy) = F(y)F(x)$ for all $x, y \in I$, then R is commutative. Many related generalizations of these results can be found in the literature (see for instance [21] and [2], where further references can be found).

The purpose of the present chapter is to continue this line of investigation and study the structure of a quotient ring R/P admitting specific algebraic identities defined in the ring R. Moreover, we consider a more general concept rather than the ring R is prime or semi-prime in the hypothesis of our theorems.

2.2 Generalized derivations involving prime ideals

A well known result of Posner [61] states that the existence of a derivation d of a prime ring R such that $[d(x),x] \in Z(R)$ for all $x \in R$, forces that either $d = 0$ or R is commutative.

Inspired by the above result, our aim in the following theorem is to study the case when the generalized derivations satisfies some conditions involving anti-commutators

(instead of commutators). More specifically, we will treat the special identity $F(x) \circ x$ belongs to the center of a quotient ring. Indeed, our results are of more specific interest because we will characterize not only the structure of the ring R/P but we will also prove that the generalized derivation F has its rang in the prime ideal P.

Theorem 2.2.1 *Let R be a ring and P a prime ideal of R such that R/P is 2-torsion free. If R admits a generalized derivation F associated with a derivation d such that $\overline{F(x) \circ x} \in Z(R/P)$ for all $x \in R$, then $F(R) \subseteq P$ or R/P is a commutative integral domain.*

Proof. We are given that

$$\overline{F(x) \circ x} \in Z(R/P) \quad \text{for all } x \in R. \tag{2.1}$$

If $Z(R/P) = \{\overline{0}\}$, then R/P is non-commutative and the relation (2.1) reduces to

$$F(x) \circ x \in P \quad \text{for all } x \in R.$$

Linearizing the above expression, we get

$$F(x) \circ y + F(y) \circ x \in P \quad \text{for all } x, y \in R. \tag{2.2}$$

Substituting yx for y in (2.2), we find that

$$-y[F(x), x] + y[d(x), x] + (y \circ x)d(x) \in P \quad \text{for all } x, y \in R. \tag{2.3}$$

Replacing y by ry in (2.3), we obviously see that

$$[x, r]yd(x) \in P \quad \text{for all } r, x, y \in R$$

which implies that

$$[x, r]Rd(x) \subseteq P \quad \text{for all } r, x \in R. \tag{2.4}$$

According to the primeness of P, we get $[x, R] \subseteq P$ or $d(x) \in P$ for all $x \in R$. The sets of x for which theses conditions hold are additive subgroups of R with union equal to R; using Brauer's trick we conclude that $d(R) \subseteq P$. Hence (2.3) yields $y[x, F(x)] \in P$. Thus, $[x, F(x)] \in P$ for all $x \in R$ (since a prime ideal is proper). In view of the hypothesis we

find that $2F(x)x \in P$. Applying 2-torsion freeness, we get $F(x)x \in P$ for all $x \in R$. A linearization of the preceding relation gives

$$F(x)y + F(y)x \in P \quad \text{for all } x, y \in R. \tag{2.5}$$

Replacing y by yr in (2.5) and combining it with the above expression, we may write

$$F(y)[x,r] \in P \quad \text{for all } r,x,y \in R \tag{2.6}$$

thereby obtaining

$$F(y)R[x,r] \subseteq P \quad \text{for all } r,x,y \in R. \tag{2.7}$$

Once again invoking the primeness of P, we conclude that either $F(R) \subseteq P$ or R/P is an integral domain, contrary to our initial hypothesis; hence $F(R) \subseteq P$.
Now if $Z(R/P) \neq \{\overline{0}\}$, then for $\overline{z} \in Z(R/P) \setminus \{\overline{0}\}$, take $x = z$ in (2.1), and by appropriate expansion, obtain $\overline{F(z)} \in Z(R/P)$.
On the other hand, linearizing equation (2.1), we arrive at

$$2\overline{(F(x)z + F(z)x)} \in Z(R/P) \quad \text{for all } x \in R \tag{2.8}$$

in such a way that

$$\overline{[F(x)z + F(z)x, r]} = \overline{0} \quad \text{for all } r,x \in R. \tag{2.9}$$

Writing xr instead of x in (2.9) and using it, we then get

$$\overline{[xd(r)z, r]} = \overline{0} \quad \text{for all } r,x \in R. \tag{2.10}$$

Substituting tx for x in (2.10), we arrive at

$$\overline{[t,r]xd(r)z} = \overline{0} \quad \text{for all } r,t,x \in R \tag{2.11}$$

which implies that $[t,r]Rd(r) \subseteq P$ for all $r,t \in R$. So that R/P is an integral domain or $d(R) \subseteq P$. In the latter case, once again linearizing (2.1), we thereby obtain

$$\overline{F(x) \circ y + F(y) \circ x} \in Z(R/P) \quad \text{for all } x,y \in R. \tag{2.12}$$

Substituting yx for y in (2.12), we obviously get

$$\overline{[y[F(x),x],x]} = \overline{0} \quad \text{for all } x,y \in R. \tag{2.13}$$

If we write $F(x)y$ instead of y in (2.13), then it follows that $\overline{[F(x),x]} = \overline{0}$. In particular $\overline{[F(x),x]} \in Z(R/P)$ for all $x \in R$. Invoking again the hypothesis, it is obvious to see that $\overline{F(x)x} \in Z(R/P)$. Arguing as above, we are forced to conclude that $\overline{[F(y)[x,r],r]} = \overline{0}$ for all $r,x,y \in R$. Accordingly, putting tx instead of x and using it, we find that

$$\overline{F(y)[x,r][t,r]} = \overline{0} \quad \text{for all } r,t,x,y \in R.$$

It now follows from the above expression that

$$F(y)R[x,r]R[t,r] \subseteq P \quad \text{for all } r,t,x,y \in R.$$

Hence, $F(R) \subseteq P$ or R/P is commutative which completes the proof of our theorem. ■

The following corollary is an immediate consequence of the preceding theorem.

Corollary 2.2.2 *Let R be a 2-torsion free prime ring. If R admits a nonzero generalized derivation F associated with a derivation d, then the following assertions are equivalent:*
(1) $F(x) \circ x \in Z(R)$ *for all* $x \in R$;
(2) $F(x) \circ x + d(x) \circ x \in Z(R)$ *for all* $x \in R$;
(3) $F(x) \circ x - d(x) \circ x \in Z(R)$ *for all* $x \in R$;
(4) R *is a commutative integral domain.*

The next proposition extends Corollary 2.2.2 to semi-prime rings.

Proposition 2.2.3 *Let R be a semi-prime ring. If R admits a nonzero generalized derivation F associated with a derivation d satisfying any one of the following conditions:*
(1) $F(x) \circ x \in Z(R)$ *for all* $x \in R$;
(2) $F(x) \circ x + d(x) \circ x \in Z(R)$ *for all* $x \in R$;
(3) $F(x) \circ x - d(x) \circ x \in Z(R)$ *for all* $x \in R$;
then either R is commutative or $\big($there exists a minimal prime ideal P of R such that $char(R/P) = 2$ or $F(R) \subseteq P\big)$.

Proof. (1) Suppose on the contrary that $char(R/P) \neq 2$ and $F(R) \not\subseteq P$ for any minimal prime ideal P of R such that $F(x) \circ x \in Z(R)$ for all $x \in R$, then $[F(x) \circ x, y] = 0$ for all $x, y \in R$. According to semi-primeness, there exists a family $\mathscr{P}$ of prime ideals P such that $\bigcap_{P \in \mathscr{P}} P = (0)$ and therefore $[F(x) \circ x, y] \in P$ for all $P \in \mathscr{P}$. That is

$$\overline{F(x) \circ x} \in Z(R/P) \text{ for all } x \in R \text{ and for all } P \in \mathscr{P}.$$

Accordingly, Theorem 2.2.1 yields that R/P is commutative. However, for all $x, y \in R$ we get $[x, y] \in P$ (for all $P \in \mathscr{P}$) so that $[x, y] = 0$ proving that R is commutative.
(2) Now if $F(x) \circ x + d(x) \circ x \in Z(R)$ for all $x \in R$ or $F(x) \circ x - d(x) \circ x \in Z(R)$ for all $x \in R$, then using the same techniques as used above with slight modifications we get the required result. ■

In 2011 Ashraf and Almas Khan [9] showed that if a 2-torsion free $*$-prime ring R with a $*$-Lie ideal U such that $F[u, v] = [F(u), v]$ or $F(u \circ v) = F(u) \circ v$ for all $u, v \in U$; where F is a generalized derivation associated with a nonzero derivation d, then $U \subseteq Z(R)$.

Motivated by the above results, our aim in the following theorem is to investigate a more general context of differential identities involving a prime ideal by omitting the primeness (semi-primeness) assumption imposed on the ring R and with no further assumption on the characteristic of R. Indeed, we will prove the following result.

Theorem 2.2.4 *Let R be a ring and P a prime ideal of R. If R admits a generalized derivation F associated with a derivation d satisfying any one of the following conditions:*
(1) $\overline{F[x, y] - [F(x), y]} \in Z(R/P)$ *for all $x, y \in R$;*
(2) $\overline{F(x \circ y) - F(x) \circ y} \in Z(R/P)$ *for all $x, y \in R$;*
then $d(R) \subseteq P$ or R/P is a commutative integral domain.

Proof. (1) By given assumption, we have

$$\overline{F[x, y] - [F(x), y]} \in Z(R/P) \text{ for all } x, y \in R. \tag{2.14}$$

Replacing x by xr in (2.14), we obtain

$$\overline{F\big([x,y]r+x[r,y]\big)-[F(x),y]r-F(x)[r,y]-[xd(r),y]}\in Z(R/P) \quad \text{for all } r,x,y\in R$$

by expanding the above equation, we get

$$\big[[x,y]d(r)+xd[r,y]-[xd(r),y],r\big]\in P \quad \text{for all } r,x,y\in R.$$

This can be rewritten as

$$\big[x[r,d(y)],r\big]\in P \quad \text{for all } r,x,y\in R. \tag{2.15}$$

Substituting $d(y)x$ for x in (2.15), we find that

$$[r,d(y)]x[r,d(y)]\in P \quad \text{for all } r,x,y\in R.$$

The primeness of P, leads to that $[r,d(y)]\in P$ for all $r,y\in R$. As a special case of the last equation, we may write that $\overline{[r,d(r)]}\in Z(R/P)$ for all $r\in R$. By view of Lemma 1.2.1, we conclude that $d(R)\subseteq P$ or R/P is a commutative integral domain.
(2) Now if we consider

$$\overline{F(x\circ y)-F(x)\circ y}\in Z(R/P) \quad \text{for all } x,y\in R$$

then proceeding as in (1) with necessary variations, we arrive at $d(R)\subseteq P$ or R/P is a commutative integral domain. ■

As an application of Theorem 2.2.4, we get the following result.

Corollary 2.2.5 *Let R be a prime ring. If R admits a generalized derivation F associated with a nonzero derivation d, then the following assertions are equivalent:*
(1) $F[x,y]-[F(x),y]\in Z(R)$ *for all* $x,y\in R$*;*
(2) $F(x\circ y)-F(x)\circ y\in Z(R)$ *for all* $x,y\in R$*;*
(3) R *is a commutative integral domain.*

Proposition 2.2.6 *Let R be a semi-prime ring. If R admits a generalized derivation F*

associated with a nonzero derivation d satisfying any one of the following conditions:
(1) $F[x,y]-[F(x),y]\in Z(R)$ *for all* $x,y\in R$*;*
(2) $F(x\circ y)-F(x)\circ y\in Z(R)$ *for all* $x,y\in R$*;*
then either R is commutative or $d(R)\subseteq P$ *for some minimal prime ideal P.*

2.3 Some special derivations

In [21], Bell and Kappe studied derivations acting as an homomorphism and an anti-homomorphism on a nonempty subset of a ring R. They proved that if R is a prime ring, U a nonzero right ideal of R and d a derivation of R which acts as an homomorphism or (an anti-homomorphism) on U, i.e. $d(xy)-d(x)d(y)=0$ for all $x,y\in U$ $\big($resp. $d(xy)-d(y)d(x)=0$ for all $x,y\in U\big)$, then $d=0$ on R. Further Asma Ali et al. [14, Theorem 3.1] extended this results to a Lie ideal.

Motivated by the above results, our next aim is to suggest a more general situation by considering differential identities involving two derivations d and g satisfying $d(x)d(y)\pm g(yx)$ belongs to the center of a quotient ring.

Theorem 2.3.1 *Let R be a ring and P a prime ideal of R. If R admits two derivations d and g satisfying any one of the following conditions:*
(1) $\overline{d(x)d(y)-g(yx)}\in Z(R/P)$ *for all* $x,y\in R$*;*
(2) $\overline{d(x)d(y)+g(yx)}\in Z(R/P)$ *for all* $x,y\in R$*;*
then, $\big(d(R)\subseteq P$ *and* $g(R)\subseteq P\big)$ *or* R/P *is a commutative integral domain.*

Proof. (1) Suppose that

$$\overline{d(x)d(y)-g(yx)}\in Z(R/P)\quad\text{for all } x,y\in R. \tag{2.16}$$

If $Z(R/P)=\{\overline{0}\}$, then the hypothesis reduces to

$$d(x)d(y)-g(yx)\in P\quad\text{for all } x,y\in R. \tag{2.17}$$

Replacing y by yr in (2.17), and using (2.17), we find that

$$d(x)yd(r)+g(y)[x,r]+y[g(x),r]-yg(r)x\in P\quad\text{for all } r,x,y\in R. \tag{2.18}$$

Substituting rx for r in (2.18), we obviously get

$$d(x)yd(r)x+d(x)yrd(x)+g(y)[x,r]x+y[g(x),r]x+yr[g(x),x]-yg(r)x^2-yrg(x)x\in P. \tag{2.19}$$

Using (2.18) together with (2.19), we arrive at

$$d(x)yrd(x)-yrxg(x)\in P \quad \text{for all } r,x,y\in R. \tag{2.20}$$

Left multiplying the above expression by t and subtracting from (2.20), it follows that

$$[d(x),t]yrd(x)\in P \quad \text{for all } r,t,x,y\in R. \tag{2.21}$$

Writing $yd(x)$ instead of y in (2.21), we obtain

$$[d(x),t]yd(x)rd(x)\in P \quad \text{for all } r,t,x,y\in R.$$

In particular,

$$[d(x),t]R[d(x),t]R[d(x),t]\subseteq P \quad \text{for all } t,x\in R. \tag{2.22}$$

The primeness of P assures that, $[d(x),t]\in P$ for all $t,x\in R$. Hence Lemma 1.2.1 proving that $d(R)\subseteq P$. In this case the relation (2.17) reduces to $g(yx)\in P$ for all $x,y\in R$. If we write xr for x, then we get $yxg(r)\in P$ for all $r,x,y\in R$. Letting $y=g(r)$, we have $g(r)Rg(r)\subseteq P$ for all $r\in R$. Hence, it follows that $g(R)\subseteq P$.
Now suppose that $Z(R/P)\neq\{\overline{0}\}$, then there exists $\overline{z}(\neq\overline{0})\in Z(R/P)$; substituting yz for y in (2.16), one can easily verify that

$$\overline{d(x)yd(z)-y[g(z),x]-yxg(z)}\in Z(R/P) \quad \text{for all } x,y\in R. \tag{2.23}$$

Putting ry instead of y in (2.23), we get

$$\overline{\big[[d(x),r]yd(z),r\big]}=\overline{0} \quad \text{for all } r,x,y\in R \tag{2.24}$$

which may be restated as

$$\overline{[d(x),r]y[d(z),r]}+\overline{\big[[d(x),r]y,r\big]d(z)}=\overline{0} \quad \text{for all } r,x,y\in R. \tag{2.25}$$

Replacing y by $yd(z)$ in (2.25), it is obvious to see that

$$\overline{[d(x),r]yd(z)[d(z),r]} = \overline{0} \quad \text{for all } r,x,y \in R.$$

In light of the primeness of P, we get for each $r \in R$ either $[d(x),r] \in P$ or $d(z)[d(z),r] \in P$. Let us set $H = \{r \in R / [d(x),r] \in P \text{ for all } x \in R\}$ and $K = \{r \in R / d(z)[d(z),r] \in P\}$. Then it can be seen that H and K are two additives subgroups of R whose union is R. Using Brauer's trick we have either $R = H$ or $R = K$.
Assume that $R = K$, then $d(z)[d(z),r] \in P$. Accordingly $[d(z),r]R[d(z),r] \subseteq P$, by virtue of the primeness, we can see that $\overline{d(z)} \in Z(R/P)$. In this case, expression (2.24) becomes $\overline{\big[[d(x),r]y,r\big]d(z)} = \overline{0}$ for all $r,x,y \in R$. Therefore, $\overline{d(z)} = \overline{0}$ or $[d(x),r] \in P$. By the first case, the hypothesis leads to that $\overline{g(z)x + zg(x)} \in Z(R/P)$ for all $x \in R$, now if we put xr instead of x, then we find that $\overline{[x,r]xg(r)z} = \overline{0}$. So that $[R,R] \subseteq P$ or $g(R) \subseteq P$. On the other hand, if $g(R) \subseteq P$, then expanding the expression (2.16), we are forced to get R/P is commutative or $d(R) \subseteq P$.

(2) Suppose that $\overline{d(x)d(y) + g(yx)} \in Z(R/P)$ for all $x,y \in R$, since $(-g)$ is also a derivation of R. Then, we have by assertion (1) either $\big(d(R) \subseteq P \text{ and } g(R) \subseteq P\big)$ or R/P is an integral domain. ■

As an application of Theorem 2.3.1, we obtain the following corollary which constitutes an improved version of [14, Theorem 1.2].

Corollary 2.3.2 *Let R be a prime ring. If R admits two derivations d and g such that either d or g is nonzero, then the following assertions are equivalent:*
(1) $d(x)d(y) - g(yx) \in Z(R)$ *for all* $x,y \in R$;
(2) $d(x)d(y) + g(yx) \in Z(R)$ *for all* $x,y \in R$;
(3) *R is a commutative integral domain.*

The following proposition gives a generalization of Bell and Kappe's result.

Proposition 2.3.3 *Let d and g be derivations of a semi-prime ring R. If $d(x)d(y) \pm g(yx) = 0$ for all $x,y \in R$; then either $d = g = 0$ or R contains a nonzero central ideal.*

Proof. Assume that $d(x)d(y) \pm g(yx) = 0$ for all $x,y \in R$. By view of the semi-primeness of the ring R, there exists a family Γ of prime ideals such that $\bigcap_{P \in \Gamma} P = (0)$, thereby

obtaining $d(x)d(y) \pm g(yx) \in P$ for all $P \in \Gamma$. Invoking the proof of Theorem 2.3.1, which in view of (2.22), reduces to

$$[d(x),x]R[d(x),x]R[d(x),x] \subseteq P \quad \text{for all } x \in R \text{ and for all } P \in \Gamma.$$

Therefore, one can see that

$$[d(x),x] \in \bigcap_{P \in \Gamma} P = (0) \quad \text{for all } x \in R.$$

Applying [22, Theorem 3], it follows that $d = 0$ or R contains a nonzero central ideal. In the first case, our hypothesis reduces to $g(yx) = 0$ for all $x, y \in R$. Replacing x by xr, we deduce that $yxg(r) = 0$ so that $g = 0$. ■

In [11] Ashraf and Rehman proved that, if R is a 2-torsion free prime ring, I is a nonzero ideal of R and d is a derivation of R such that $d(x) \circ d(y) = x \circ y$ for all $x, y \in I$; then R is commutative. In fact, this result is false because in the particular case when $d = 0$, we get $R = \{0\}$, which is a contradiction.

The fundamental aim of the next theorem is to establish a generalization of the above result by investigating the behavior of the more general expressions. More precisely, we will treat the following special identities:

(i) $\overline{d(x) \circ d(y) - g(x) \circ y} \in Z(R/P)$ for all $x, y \in R$;
(ii) $\overline{d(x) \circ d(y) + g(x) \circ y} \in Z(R/P)$ for all $x, y \in R$.

Theorem 2.3.4 *Let R be a ring and P a prime ideal of R such that R/P is a 2-torsion free. If R admits two derivations d and g satisfying any one of the following conditions:*
(1) $\overline{d(x) \circ d(y) - g(x) \circ y} \in Z(R/P)$ for all $x, y \in R$;
(2) $\overline{d(x) \circ d(y) + g(x) \circ y} \in Z(R/P)$ for all $x, y \in R$;
then, $\big(d(R) \subseteq P$ and $g(R) \subseteq P\big)$ or R/P is a commutative integral domain.

Proof. We have only to prove assertion (1), while the assertion (2) can be proved similarly.

(1) Assuming that

$$\overline{d(x)\circ d(y)-g(x)\circ y}\in Z(R/P) \quad \text{for all } x,y\in R. \tag{2.26}$$

If $Z(R/P)=\{\overline{0}\}$, then the expression (2.26) becomes

$$d(x)\circ d(y)-g(x)\circ y\in P \quad \text{for all } x,y\in R. \tag{2.27}$$

Replacing y by yr in (2.27) and applying it, we obtain

$$-d(y)[d(x),r]+[d(x),y]d(r)+y\big(d(x)\circ d(r)\big)+y[g(x),r]\in P \quad \text{for all } r,x,y\in R. \tag{2.28}$$

Putting ty instead of y in (2.28), and subtracting it from (2.28), we find that

$$-d(t)y[d(x),r]+[d(x),t]yd(r)\in P \quad \text{for all } r,t,x,y\in R. \tag{2.29}$$

Taking $r=d(x)$ in (2.29), we obviously get

$$[d(x),t]yd^2(x)\in P \quad \text{for all } t,x,y\in R.$$

Using the primeness of P, we deduce that either $[d(x),t]\in P$ or $d^2(x)\in P$ for all $x\in R$. Clearly $R=R_1\cup R_2$ with $R_1=\{x\in R \ / \ [d(x),t]\in P \text{ for all } t\in R\}$ and $R_2=\{x\in R \ / \ d^2(x)\in P\}$. Since a group cannot be union of its subgroups then $R=R_1$ in which, one can see from Lemma 1.2.1 that $d(R)\subseteq P$. Now if $R=R_2$ i.e., $d^2(x)\in P$ for all $x\in R$, then replacing x by xy, one can verify that $2d(x)d(y)\in P$ for all $x,y\in R$. Substituting xr for x in the last expression, we get $2d(x)rd(y)\in P$ for all $r,x,y\in R$. In view of 2-torsion freeness, we may conclude that $d(R)\subseteq P$. On the other hand, the equation (2.27) forces that $g(x)\circ y\in P$ for all $x,y\in R$. Writing xy instead of x in this relation, we get $[y,x]g(y)\in P$ and thus by putting rx instead of x, we obtain $[y,r]Rg(y)\subseteq P$ for all $r,y\in R$. So that, we have necessarily $g(R)\subseteq P$. However, $\big(d(R)\subseteq P \text{ and } g(R)\subseteq P\big)$.
Analogously, if $Z(R/P)\neq\{\overline{0}\}$, then writing yz instead of y in (2.26), where $\overline{z}\in Z(R/P)\setminus\{\overline{0}\}$, we find that

$$\overline{\big(d(x)\circ d(y)-g(x)\circ y\big)z+y(d(x)\circ d(z))+[d(x),y]d(z)}\in Z(R/P)$$

this may be restated as

$$\overline{y(d(x)\circ d(z)) + [d(x), y]d(z)} \in Z(R/P). \tag{2.30}$$

Now replacing y by ry in (2.30), one can verify that

$$\overline{\big[[d(x), r]yd(z), r\big]} = \overline{0} \quad \text{for all } r, x, y \in R. \tag{2.31}$$

Since (2.31) is the same as (2.24), reasoning in the same manner as in the proof of Theorem 2.3.1, we arrive at R/P is commutative or $d(R) \subseteq P$ or $\overline{d(z)} = \overline{0}$. By the last case taking $y = z$ in our hypothesis; and expanding it, one can verify that $\overline{g(x)} \in Z(R/P)$ for all $x \in R$. In light of Lemma 1.2.1, we conclude that $g(R) \subseteq P$ or R/P is an integral domain. On the other hand, if $d(R) \subseteq P$ then by developing equation (2.26), we get $g(R) \subseteq P$ or R/P is commutative. Consequently, it follows that either $\big(d(R) \subseteq P$ and $g(R) \subseteq P\big)$ or R/P is a commutative integral domain. ■

Applying Theorem 2.3.4, we get an improved result of Ashraf and Rehman as follows:

Corollary 2.3.5 *Let R be a 2-torsion free prime ring. If R admits two derivations d and g such that either d or g is nonzero, then the following assertions are equivalent:*
(1) $d(x)\circ d(y) - g(x)\circ y \in Z(R)$ *for all* $x, y \in R$;
(2) $d(x)\circ d(y) + g(x)\circ y \in Z(R)$ *for all* $x, y \in R$;
(3) R *is a commutative integral domain.*

The following example proves that the condition "`R/P is 2-torsion free`" in the Theorems 2.2.1 and 2.3.4 is necessary.

Example.
(1) Let us consider $R = M_2(\mathbb{Z}/2\mathbb{Z})$ and $P = \{0\}$, it is straightforward to check that R/P is a prime ring with $char(R/P) = 2$. Moreover, if we take the generalized derivation defined by $F = id_R$, where id_R denote the identity map defined on R by $id_R(r) = r$ for all $r \in R$. Then, we have $\overline{F(X)\circ X} \in Z(R/P)$ for all $X \in R$. Hence F satisfies the condition of Theorem 2.2.1, but R/P is not commutative.

(2) Let us consider R and P as in the preceding example. Furthermore, we define the

derivations d and g by

$$d\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} 0 & b \\ -c & 0 \end{pmatrix} \text{ and } g\begin{pmatrix} a & b \\ c & d \end{pmatrix} = 0_R$$

then it is obvious to verify that d and g satisfying the condition of Theorem 2.3.4. However, R/P is a non-commutative ring.

CHAPTER 3

RELATIONSHIP BETWEEN THE STRUCTURE OF A FACTOR RING R/P AND DERIVATIONS OF R

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Abstract.

The purpose of this chapter is to study the relationship between the existence of certain differential identities with derivations and the commutativity of a quotient rings. More precisely, we investigate commutativity of quotient rings R/P, where R is an arbitrary ring without assumption the primeness or semi-primeness imposed in the ring and P is any prime ideal of R. Moreover, we provide an example to demonstrate that our results cannot be extended to semi-prime ideals.

3.1 Introduction

The history of commuting and centralizing mappings goes back to 1955 when Divinsky [40] proved that a simple artinian ring is commutative if it has a commuting nontrivial automorphism. Two years later Posner [61] has proved that if a prime ring R admits a nonzero derivation d such that $[d(x),x] \in Z(R)$ for all $x \in R$, then R is commutative

Posner's second theorem. Over the last few decades, several authors have subsequently refined and extended these results in various directions (viz., [22, 57], where further references can be found).

Long ago Herstien [45] established that a 2-torsion free prime ring R must be a commutative integral domain if it admits a nonzero derivation d satisfying $d(x)d(y) = d(y)d(x)$ for all $x, y \in R$. Motivated by this result Bell and Daif [20] showed that if R is a prime ring admitting a nonzero derivation d such that $d(xy) = d(yx)$ for all $x, y \in U$, where U is a nonzero two-sided ideal of R, then R is commutative. Daif [36] extended this result for semi-prime rings. Further, in the year 1992 Daif and Bell [37] established commutativity of semi-prime ring satisfying $d([x,y]) = [x,y]$ for all x, y in a nonzero ideal of R, and d a derivation of R. These results are generalized, in [60], for $*$-prime ring R satisfying any one of the properties: $(i)\ d[x,y] = 0$, $(ii)\ d([x,y]) - [x,y] \in Z(R)$, $(iii)\ d([x,y]) + [x,y] \in Z(R)$, $(iv)\ d(x \circ y) = 0$, $(v)\ d(x \circ y) - x \circ y \in Z(R)$, $(vi)\ d(x \circ y) + x \circ y \in Z(R)$ for all $x, y \in J$, where J is a nonzero Jordan ideal of R. More recently, many authors $\big($viz. for example, [10, 11, 22, 50, 57]$\big)$ have investigated the connection between the commutativity for category of prime (semi-prime) rings and some concrete additive mappings satisfying certain differential identities.

In this chapter we continue this line of investigation by considering a more general concept that consists to establish a relationship between the structure of quotient rings R/P and derivations of R satisfying algebraic identities involving the prime ideal P.
We will make frequent use of the following fact which is very crucial for developing the proofs of our main results.

Fact 2 *Let R be a ring, I a nonzero ideal of R and P a prime ideal of R such that $P \subsetneq I$. If $[I,I] \subseteq P$, then R/P is a commutative ring.*

3.2 Identities involving a pair of derivations

In [10] it is proved that a prime ring R must be commutative if it admits a derivation d satisfying one of the following conditions

$$\big(d(xy) \pm xy \in Z(R) \ \text{ for all } \ x, y \in I\big) \ \text{ or } \ \big(d(xy) \pm yx \in Z(R) \ \text{ for all } \ x, y \in I\big)$$

where I is a nonzero ideal I of R.
The question arises whether a similar result holds if we consider algebraic identities involving two derivations. In this context, it is also interesting to examine if the result of [10] can be extended to semi-prime rings.

The above questions lead to the following conjecture.

Conjecture 1: Let R be a semi-prime ring and I a nonzero ideal of R. If R admits derivations d and g such that $\big(d(x)y+xg(y)\pm xy \in Z(R)$ for all $x,y \in I\big)$ or $\big(d(x)y+xg(y)\pm yx \in Z(R)$ for all $x,y \in I\big)$, then R is commutative.

To solve this conjecture, the basic concept upon which our approach is based is to study the structure of a quotient ring R/P, where P is a prime ideal of R, equipped with derivations d and g satisfying

$$\overline{d(x)y+xg(y)\pm xy} \in Z(R/P) \text{ for all } x,y \in I$$

where I is an ideal of R such that $P \subsetneq I$.

Theorem 3.2.1 *Let R be a ring, I a nonzero ideal of R and P a prime ideal of R such that $P \subsetneq I$. If d and g are derivations of R, then the following assertions are equivalent:*
(1) $\overline{d(x)y+xg(y)\pm xy} \in Z(R/P)$ *for all* $x,y \in I$;
(2) $\overline{d(x)y+xg(y)\pm yx} \in Z(R/P)$ *for all* $x,y \in I$;
(3) $\overline{d(x)y+xg(y)\pm [x,y]} \in Z(R/P)$ *for all* $x,y \in I$;
(4) $\overline{d(x)y+xg(y)\pm x\circ y} \in Z(R/P)$ *for all* $x,y \in I$;
(5) R/P *is a commutative integral domain.*

Proof. We need to prove that $(1) \Rightarrow (5)$, $(2) \Rightarrow (5)$, $(3) \Rightarrow (5)$, and $(4) \Rightarrow (5)$.
$(1) \Rightarrow (5)$ We are given that

$$\overline{d(x)y+xg(y)\pm xy} \in Z(R/P) \quad \text{for all } x,y \in I. \tag{3.1}$$

Substituting yr for y in (3.1), one can see that

$$\overline{\big(d(x)y+xg(y)\pm xy\big)r+xyg(r)} \in Z(R/P) \quad \text{for all } x,y,r \in I$$

thereby obtaining

$$[xyg(r), r] \in P \quad \text{for all } x, y, r \in I. \tag{3.2}$$

Replacing x by tx in (3.2), we get

$$[t, r]Iyg(r) \subseteq P \quad \text{for all } y, r, t \in I.$$

Using Fact 1, we obviously obtain $[t, r] \in P$ or $yg(r) \in P$ for all $r \in I$. Consequently, I is a union of two additive subgroups I_1 and I_2 where

$$I_1 = \{r \in I / [I, r] \subseteq P\} \quad \text{and} \quad I_2 = \{r \in I / Ig(r) \subseteq P\}.$$

Since a group cannot be a union of two of its proper subgroups, then either $I = I_1$ or $I = I_2$. If $I = I_1$, then $[I, I] \subseteq P$ and thus Fact 2 assures that R/P is a commutative integral domain. Now, if $I = I_2$, then $Ig(r) \subseteq P$ for all $r \in I$ and therefore $g(r) \in P$ for all $r \in I$. Hence, relation (3.1) yields $\big[(d(x) \pm x)y, r\big] \in P$ for all $x, y, r \in I$. Replacing x by xt in the last expression, it follows that

$$[xd(t)y, r] \in P \quad \text{for all } x, y, r, t \in I. \tag{3.3}$$

Substituting ux for x in (3.3), we arrive at $[u, r]xd(t)y \in P$ so that

$$[u, r]Id(t)yd(t) \subseteq P \quad \text{for all } y, r, t, u \in I.$$

Invoking Fact 1, the above relation assures that either $d(R) \subseteq P$ or $[I, I] \subseteq P$ in which case R/P is commutative by Fact 2. Now if $d(R) \subseteq P$ and $g(R) \subseteq P$, then our hypothesis becomes $\overline{xy} \in Z(R/P)$ for all $x, y \in I$, which implies that R/P is a commutative integral domain.

(2) $\Rightarrow$ (5) Suppose that

$$\overline{d(x)y + xg(y) + yx} \in Z(R/P) \quad \text{for all } x, y \in I. \tag{3.4}$$

Putting yr instead of y in the expression (3.4), we obtain

$$\overline{(d(x)y + xg(y) + yx)r + xyg(r) + y[r, x]} \in Z(R/P) \quad \text{for all } x, y \in I$$

henceforth,

$$\big[xyg(r)+y[r,x],r\big]\in P \quad \text{for all } x,y,r\in I \tag{3.5}$$

which leads to

$$[x,r]yg(r)+x[yg(r),r]+\big[y[r,x],r\big]\in P \quad \text{for all } x,y,r\in I.$$

Substituting xy for y in the preceding relation and applying (3.5), one can see that

$$[x,r]xyg(r)-[x,r]y[x,r]\in P \quad \text{for all } x,y,r\in I. \tag{3.6}$$

Replacing r by $r+x$ in (3.6), we arrive at

$$[x,r]xyg(x)\in P \quad \text{for all } x,y,r\in I. \tag{3.7}$$

Writing rt instead of r in (3.7), one can easily verify that

$$[x,r]I[x,r]Ig(x)\subseteq P \quad \text{for all } x,r\in I.$$

Consequently, either $[x,r]\in P$ or $g(x)\in P$ for all $x,r\in I$. The sets of r for which theses conditions hold are additive subgroups of I with union equal I; thus by Brauer's trick either $[I,I]\subseteq P$ or $g(R)\subseteq P$. But, in the later case, equation (3.6) reduces to

$$[x,r]y[x,r]\in P \quad \text{for all } x,y,r\in I. \tag{3.8}$$

Invoking the primeness of P, equation (3.8) assures that $[I,I]\subseteq P$. Consequently, in both cases we arrive at $[I,I]\subseteq P$, so again an appeal to Fact 2, gives R/P is a commutative integral domain.
For the remaining implications, that is $(3)\Rightarrow(5)$ and $(4)\Rightarrow(5)$, using the same reasoning as above we get the required result. ■

The following Theorem gives an affirmative answer to Conjecture 1.

Theorem 3.2.2 *Let R be a semi-prime ring and I a nonzero ideal of R. If d and g are derivations of R, then the following assertions are equivalent:*

(1) $d(x)y+xg(y)\pm xy\in Z(R)$ *for all* $x,y\in I$;
(2) $d(x)y+xg(y)\pm yx\in Z(R)$ *for all* $x,y\in I$;
(3) $d(x)y+xg(y)\pm[x,y]\in Z(R)$ *for all* $x,y\in I$;
(4) $d(x)y+xg(y)\pm x\circ y\in Z(R)$ *for all* $x,y\in I$;
(5) R *is commutative.*

Proof. For the nontrivial sense, we will proceed by only proving $(1)\Rightarrow(5)$ and the same technique remains valid for the other implications.
Assume that

$$d(x)y+xg(y)\pm xy\in Z(R) \text{ for all } x,y\in I.$$

By view of the semi-primeness of the ring R, there exists a family $\mathscr{P}$ of prime ideals $\mathscr{P}=\{P_\alpha/\alpha\in\Lambda\}$ such that $\bigcap\limits_{\alpha\in\Lambda}P_\alpha=(0)$, thereby obtaining $[d(x)y+xg(y)\pm xy,r]\in P_\alpha$ for all $\alpha\in\Lambda$. Invoking Theorem 3.2.1 we conclude that R/P_α is a commutative integral domain for all $\alpha\in\Lambda$ which, because of $\bigcap\limits_{\alpha\in\Lambda}P_\alpha=(0)$, assures that R is commutative. ■

Remark 3.2.3 *If we set $g=d$ in Theorem 3.2.2, then we get a generalization of ([10], Theorems 2.1 and 2.3) for semi-prime rings.*

In [4] it is proved that if d is a derivation of a 2-torsion free semi-prime ring R satisfying one of the following statements:

$$\big(d(x)d(y)\pm xy=0 \text{ for all } x,y\in I\big) \text{ or } \big(d(x)d(y)\pm yx=0 \text{ for all } x,y\in I\big)$$

where I is an ideal of R, then d is commuting on I. Furthermore, if $d(I)\neq 0$ then R has a nonzero central ideal.
Using a similar approach as in the preceding theorem, our next goal is to generalize the above result in two directions. First of all we will treat algebraic identities involving two derivations rather that one derivation. Secondly, we will consider the case when the underlying identity in not necessarily null but belongs to the center of the ring R, that is

$$\big(d(x)g(y)\pm xy\in Z(R) \text{ for all } x,y\in I\big) \text{ or } \big(d(x)g(y)\pm yx\in Z(R) \text{ for all } x,y\in I\big).$$

Theorem 3.2.4 *Let R be a ring, I a nonzero ideal of R and P a prime ideal of R such that $P\subsetneq I$. If d and g are derivations of R, then the following assertions are equivalent:*

(1) $\overline{d(x)g(y) \pm xy} \in Z(R/P)$ *for all* $x, y \in I$;
(2) $\overline{d(x)g(y) \pm yx} \in Z(R/P)$ *for all* $x, y \in I$;
(3) R/P *is a commutative integral domain.*
Moreover, if R/P *is* 2*-torsion free and the derivations* d *and* g *satisfying one of the conditions* $\overline{d(x)g(y) \pm [x,y]} \in Z(R/P)$ *or* $\overline{d(x)g(y) \pm x \circ y} \in Z(R/P)$ *for all* $x, y \in I$*, then* R/P *is a commutative integral domain.*

Proof. (1) $\Rightarrow$ (3) Suppose that

$$\overline{d(x)g(y) \pm xy} \in Z(R/P) \quad \text{for all } x, y \in I. \tag{3.9}$$

Substituting yr for y in (3.9), we get

$$\overline{\big(d(x)g(y) \pm xy\big)r + d(x)yg(r)} \in Z(R/P) \quad \text{for all } x, y, r \in I$$

thereby obtaining

$$[d(x)yg(r), r] \in P \quad \text{for all } x, y, r \in I. \tag{3.10}$$

Replacing x by xt in the above relation and subtracting it from (3.10), we get

$$[xd(t)yg(r), r] \in P \quad \text{for all } x, y, r, t \in I. \tag{3.11}$$

Writing ux instead of x in (3.11), we obtain

$$[u, r]Id(t)yg(r) \subseteq P \quad \text{for all } y, r, t, u \in I.$$

Using Fact 1, the above relation implies that $[u, r] \in P$ or $d(t)Ig(r) \subseteq P$ for all $r, t, u \in I$. In the later case, one again invoking Fact 1, we get $g(r) \in P$ or $d(I) \subseteq P$. If $d(I) \subseteq P$ or $g(I) \subseteq P$, then our hypothesis leads to $\overline{xy} \in Z(R/P)$ for all $x, y \in I$ so that R/P is a commutative integral domain. If $[I, I] \subseteq P$, then Fact 2 yields that R/P is commutative.
(2) $\Rightarrow$ (3) We are given that

$$\overline{d(x)g(y) + yx} \in Z(R/P) \quad \text{for all } x, y \in I. \tag{3.12}$$

We claim that $Z(R/P) \neq \{\overline{0}\}$. Otherwise, we have

$$d(x)g(y) + yx \in P \quad \text{for all } x, y \in I. \tag{3.13}$$

Substituting yx for y in (3.13), we arrive at

$$d(x)yg(x) \in P \quad \text{for all } x, y \in I. \tag{3.14}$$

Therefore $d(I) \subseteq P$ or $g(I) \subseteq P$ in which case equation (3.13) assures that $P = I$, a contradiction. Consequently, we conclude that $Z(R/P) \neq \{\overline{0}\}$.
Substituting yz for y in (3.12), where $\overline{z} \in Z(R/P) \setminus \{\overline{0}\}$, we easily get

$$[d(x)yg(z), r] \in P \quad \text{for all } x, y, r \in I. \tag{3.15}$$

Since equation (3.15) is the same as equation (3.10), arguing as before we find that $[I, I] \subseteq P$ or $d(I) \subseteq P$ or $g(z) \in P$. In all that follows, we find that R/P is a commutative integral domain.
Now if $\overline{d(x)g(y) - yx} \in Z(R/P)$ for all $x, y \in I$, then using similar techniques with essential variations, we get the required result.
Assume that

$$\overline{d(x)g(y) \pm [x, y]} \in Z(R/P) \quad \text{for all } x, y \in I. \tag{3.16}$$

Taking $x = y$ in (3.16), we see that

$$\overline{d(x)g(x)} \in Z(R/P) \quad \text{for all } x \in I. \tag{3.17}$$

Replacing y by yx in (3.16), one obtains

$$[d(x)yg(x), x] \in P \quad \text{for all } x, y \in I. \tag{3.18}$$

Substituting $g(x)y$ for y in (3.18) and invoking (3.17), we find that

$$d(x)g(x)[yg(x), x] \in P \quad \text{for all } x, y \in I.$$

Since P is prime, we conclude that

$$\text{either } \big(d(x)g(x) \in P\big) \text{ or } \big([yg(x),x] \in P\big) \text{ for all } x,y \in I. \tag{3.19}$$

If $[yg(x),x] \in P$ for all $y \in I$, then putting ry instead of y, we thereby get $[r,x]yg(x) \in P$ for all $y,r \in I$ so that either $g(x) \in P$ and a fortiori $d(x)g(x) \in P$ or $\bar{x} \in Z(R/P)$. On the other hand, if $\bar{x} \in Z(R/P)$, then our hypothesis becomes $\overline{d(x)g(y)} \in Z(R/P)$ for all $y \in I$. Replacing y by yx in the last relation, we get $\overline{d(x)yg(x)} \in Z(R/P)$ for all $y \in I$. In particular, $\overline{d(x)g(x)yd(x)g(x)} \in Z(R/P)$. The primeness of P implies that $d(x)g(x) \in P$ or $\overline{yd(x)g(x)} \in Z(R/P)$ for all $y \in I$. Hence, R/P is commutative or $d(x)g(x) \in P$.
Accordingly, equation (3.19) reduces to $d(x)g(x) \in P$ for all $x \in I$. Linearizing this expression, we obtain

$$d(x)g(y)+d(y)g(x) \in P \quad \text{for all } \ x,y \in I. \tag{3.20}$$

Writing yr for y in (3.20), it is obvious to see that

$$\big(d(x)g(y)+d(y)g(x)\big)r+d(x)yg(r)+d(y)rg(x)+yd(r)g(x)-d(y)g(x)r \in P \tag{3.21}$$

thereby obtaining

$$d(x)yg(r)+d(y)[r,g(x)]+yd(r)g(x) \in P \quad \text{for all } \ x,y,r \in I. \tag{3.22}$$

Replacing y by $d(x)y$ in (3.22) and combining it with (3.22), one can verify that

$$d^2(x)y[r,g(x)] \in P \quad \text{for all } \ x,y,r \in I.$$

Using Fact 1 together with Brauer's trick (that is a group cannot be a union of two of its proper subgroups), we conclude that $[r,g(x)] \in P$ for all $x,r \in I$ or $d^2(x) \in P$ for all $x \in I$. Substituting xy for x in the last relation, we find that

$$d^2(x)y+2d(x)d(y)+xd^2(y) \in P \quad \text{for all } \ x,y \in I.$$

Applying the fact that $d^2(R) \subseteq P$, we get $2d(x)d(y) \in P$ for all $x,y \in I$. Writing xr for x and using 2-torsion freeness hypothesis, we arrive at $d(x)rd(y) \in P$. By virtue of Fact

1, the last equation implies that $d(R) \subseteq P$ and by expression (3.16), we have necessarily $\overline{[x,y]} \in Z(R/P)$ for all $x,y \in I$. Clearly, we conclude that R/P is commutative.
Now if $[r,g(x)] \in P$ for all $x,r \in I$; putting xr instead of x, one can verify that

$$[r,x]g(r) \in P \text{ for all } x,r \in I.$$

By appropriate expansion, obtain

$$[r,x]Ig(r) \subseteq P \text{ for all } x,r \in I.$$

Therefore, we claim that $g(R) \subseteq P$ or R/P is an integral domain. In light of the first case, the hypothesis forces that R/P is a commutative integral domain.
Now consider the case

$$\overline{d(x)g(y) \pm x \circ y} \in Z(R/P) \quad \text{for all } x,y \in I. \tag{3.23}$$

Analogously, if $Z(R/P) = \{\overline{0}\}$, then using the same techniques as in the proof of assertion (2), it is obvious that R/P is a commutative integral domain. Hence, necessary $Z(R/P) \neq \{\overline{0}\}$.
Returning to the main equation and writing yz for y in (3.23), where $\overline{z} \in Z(R/P) \setminus \{\overline{0}\}$, then we get $d(R) \subseteq P$ or $g(z) \in P$ or R/P is an integral domain. On the other hand, the hypothesis leads to $\pm\overline{2xz} \in Z(R/P)$ for all $x \in I$. By view of 2-torsion freeness, it is easy to see that $\overline{x} \in Z(R/P)$ for all $x \in I$. Whence, R/P is a commutative integral domain. ■

As an application of Theorem 3.2.4, we get the desired generalization of [4].

Corollary 3.2.5 *Let R be a semi-prime ring and I a nonzero ideal of R. If d and g are derivations of R, then the following assertions are equivalent:*
(1) $d(x)g(y) \pm xy \in Z(R)$ *for all* $x,y \in I$;
(2) $d(x)g(y) \pm yx \in Z(R)$ *for all* $x,y \in I$;
(3) *R is commutative.*

Proposition 3.2.6 *Let R be a semi-prime ring and I a nonzero ideal of R. If d and g are derivations of R such that $d(x)g(y) \pm [x,y] \in Z(R)$ or $d(x)g(y) \pm x \circ y \in Z(R)$ for all $x,y \in I$, then R is commutative or there exist a minimal prime ideal P of R such that*

$char(R/P) = 2$.

Proof. Suppose on the contrary that $char(R/P) \neq 2$ for any minimal prime ideal P of R. By hypothesis, we have $d(x)g(y) \pm [x,y] \in Z(R)$ for all $x,y \in I$, then $[d(x)g(y) \pm [x,y], r] = 0$ for all $x,y,r \in I$. According to semi-primeness, there exists a family $\mathscr{P}$ of prime ideals P such that $\bigcap_{P\in\mathscr{P}} P = (0)$ and therefore $[d(x)g(y) \pm [x,y], r] \in P$ for all $P \in \mathscr{P}$. That is

$$\overline{d(x)g(y) \pm [x,y]} \in Z(R/P) \quad \text{for all } x,y \in I \quad \text{and for all } P \in \mathscr{P}.$$

Applying Theorem 3.2.4, it follows that R/P is commutative. which, because of $\bigcap_{P\in\mathscr{P}} P = (0)$, assures that R is commutative.
By similar way, the same conclusion holds for $d(x)g(y) \pm x \circ y \in Z(R)$ for all $x,y \in I$. ■

In [37] it is proved that a prime ring R must be commutative if it admits a derivation d such that either $xy + d(xy) = yx + d(yx)$ or $xy - d(xy) = yx - d(yx)$ for all x,y in a nonzero ideal I of R. Motivated by this result, Ashraf and Rehman [11] showed that if a 2-torsion free prime ring R admits a derivation d satisfying $d(x \circ y) \pm (x \circ y) = 0$ for all x,y in a nonzero ideal I of R, then R is commutative. However this result is incorrect because if R is commutative then the hypothesis $d(x \circ y) \pm (x \circ y) = 0$ for all $x,y \in I$ leads to $R = \{0\}$, a contradiction. In fact, one can prove that if R is a 2-torsion free prime ring then there is no derivation d of R such that $d(x \circ y) \pm (x \circ y) = 0$ for all $x,y \in I$.
Inspired by the result of [37], our next aim is to study the relationship between the commutativity of a semi-prime ring and the existence of derivations d and g such that either

$$\big(d(xy) - g(yx) \pm [x,y] \in Z(R)\big) \quad \text{or} \quad \big(d(xy) - g(yx) \pm x \circ y \in Z(R)\big)$$

for all x,y in a nonzero ideal I of R.

Theorem 3.2.7 *Let R be a ring, I a nonzero ideal of R and P a prime ideal of R such that $P \subsetneq I$ and R/P is a 2-torsion free. If d and g are derivations of R, then the following assertions are equivalent:*
(1) $\overline{d(xy) - g(yx) \pm [x,y]} \in Z(R/P)$ *for all* $x,y \in I$;
(2) $\overline{d(xy) - g(yx) \pm x \circ y} \in Z(R/P)$ *for all* $x,y \in I$;
(3) R/P *is a commutative integral domain.*

Proof. $(1) \Rightarrow (2)$ Assuming that

$$\overline{d(xy) - g(yx) \pm [x,y]} \in Z(R/P) \quad \text{for all } x,y \in I. \tag{3.24}$$

If $Z(R/P) = \{\overline{0}\}$, then the relation (3.24) becomes

$$d(xy) - g(yx) \pm [x,y] \in P \quad \text{for all } x,y \in I. \tag{3.25}$$

Replacing y by yx in (3.25), we get

$$xyd(x) - yxg(x) \in P \quad \text{for all } x,y \in I. \tag{3.26}$$

Substituting ry for y in (3.26) and combining it with (3.26), we arrive at

$$[x,r]yd(x) \in P \quad \text{for all } x,y,r \in I$$

so Fact 1 yields $d(R) \subseteq P$ or R/P is commutative, a contradiction. Now if $d(R) \subseteq P$ then (3.26) reduces to $yg(x)x \in P$ and using the fact that a prime ideal is proper, we conclude that $g(x)x \in P$ for all $x \in I$. A linearization of this expression yields

$$g(x)y + g(y)x \in P \quad \text{for all } x,y \in I. \tag{3.27}$$

Writing yr instead of y in (3.27), we find that

$$g(y)[r,x] + yg(r)x \in P \quad \text{for all } x,y,r \in I. \tag{3.28}$$

If we replace y by ty in (3.28), then one can see that

$$g(t)I[r,x] \subseteq P \quad \text{for all } x,r,t \in I.$$

In light of primeness, it follows that $g(R) \subseteq P$ and thus the hypothesis reduces to $[x,y] \in P$ for all $x,y \in I$. Therefore R/P is a commutative integral domain, a contradiction.
Now if $Z(R/P) \neq \{\overline{0}\}$, then there exists $\overline{z} \in Z(R/P)$ such that $\overline{z} \neq \overline{0}$. Substituting yz for y

in (3.24), we obtain

$$[xyd(z) - yxg(z), r] \in P \quad \text{for all } x, y, r \in I. \tag{3.29}$$

Putting ry instead of y in (3.29) and subtracting it from (3.29), we arrive at

$$\big[[x, r]yd(z), r\big] \in P \quad \text{for all } x, y, r \in I. \tag{3.30}$$

Substituting xt for x in (3.30) and using it, we can see that

$$\big[x[t, r]yd(z), r\big] \in P \quad \text{for all } x, y, r, t \in I. \tag{3.31}$$

Replacing x by tx in (3.31), one obtains

$$[t, r]x[t, r]yd(z) \in P \quad \text{for all } x, y, r, t \in I.$$

Accordingly, R/P is an integral domain or $\overline{d(z)} = \overline{0}$. In the later case, the expression (3.29) forces that $[yxg(z), r] \in P$ for all $x, y, r \in I$. If we replace y by ty, then we get $[t, r]yxg(z) \in P$. In particular, $[t, r]yg(z)xg(z) \in P$. By view of Fact 1, we find that either R/P is commutative or $\overline{g(z)} = \overline{0}$. Now putting $y = z$ in our hypothesis, one can easily verify that $\overline{\big(d(x) - g(x)\big)z} \in Z(R/P)$ for all $x \in I$. Then, we have $\overline{(d - g)(R)} \subseteq Z(R/P)$. So, the last equation assures that

$$\overline{g[x, y] \pm [x, y]} \in Z(R/P) \quad \text{for all } x, y \in I. \tag{3.32}$$

Replacing x by $[x, r]$, we get

$$\overline{\big[g[x, r] \pm [x, r], y\big] + \big[[x, r], g(y)\big]} \in Z(R/P) \quad \text{for all } x, y, r \in I.$$

In view of equation (3.32), then one can see that

$$\overline{\big[[x, g(y)], g(y)\big]} \in Z(R/P) \quad \text{for all } x, y \in I. \tag{3.33}$$

Substituting $xg(y)$ for x in (3.33), one can verify that

$$\overline{\big[[x,g(y)],g(y)\big]g(y)} \in Z(R/P) \quad \text{for all } x,y \in I$$

in such a way that

$$\overline{\big[[x,g(y)],g(y)\big]} = \bar{0} \quad \text{or} \quad \overline{g(y)} \in Z(R/P) \quad \text{for all } x,y \in I.$$

Putting rx instead of x, we find that

$$\overline{[r,g(y)][x,g(y)]} = \bar{0} \quad \text{or} \quad \overline{g(y)} \in Z(R/P) \quad \text{for all } x,y,r \in I.$$

Writing tx instead of x in the above relation, we arrive at

$$[x,g(y)]I[x,g(y)] \subseteq P \quad \text{or} \quad \overline{g(y)} \in Z(R/P) \quad \text{for all } x,y \in I$$

hence, $\overline{g(R)} \subseteq Z(R/P)$. On the other hand, expression (3.32) becomes $\overline{[x,y]} \in Z(R/P)$ for all $x,y \in I$. Thus, we obviously get R/P is a commutative integral domain.
The proof of $(2) \Rightarrow (3)$ is similar and requires only slight modifications. ■

Corollary 3.2.8 *Let R be a 2-torsion free prime ring and I a nonzero ideal of R. If d and g are derivations of R, then the following assertions are equivalent:*
(1) $d(xy) \pm g(yx) \pm [x,y] \in Z(R)$ *for all* $x,y \in I$;
(2) $d(xy) \pm g(yx) \pm x \circ y \in Z(R)$ *for all* $x,y \in I$;
(3) *R is a commutative integral domain.*

The following proposition gives an improved version of ([6], Theorem 3.8) and ([46], Theorem 1).

Proposition 3.2.9 *Let R be a semi-prime ring and I a nonzero ideal of R. If d and g are derivations of R satisfies one of the following conditions:*
(1) $d(xy) \pm g(yx) \pm [x,y] \in Z(R)$ *for all* $x,y \in I$;
(2) $d(xy) \pm g(yx) \pm x \circ y \in Z(R)$ *for all* $x,y \in I$;
then either R is commutative or there exist a minimal prime ideal P of R such that $char(R/P) = 2$.

Proof. We have only to prove the assertion (1), while the assertion (2) can be proved similarly. Suppose on the contrary that $char(R/P) \neq 2$ for any minimal prime ideal P of R such that $d(xy) \pm g(yx) \pm [x,y] \in Z(R)$ for all $x,y \in I$, then we obtain $\big[d(xy) \pm g(yx) \pm [x,y], r\big] = 0$ for all $x,y,r \in I$. According to semi-primeness, there exists a family $\mathscr{P}$ of prime ideals P such that $\bigcap\limits_{P \in \mathscr{P}} P = (0)$ and therefore $\big[d(xy) \pm g(yx) \pm [x,y], r\big] \in P$ for all $P \in \mathscr{P}$. So, we deduce that

$$\overline{d(xy) \pm g(yx) \pm [x,y]} \in Z(R/P) \quad \text{for all } x,y \in I \quad \text{and for all } P \in \mathscr{P}.$$

Accordingly, Theorem 3.2.7 yields that R/P is commutative. However, for all $x,y \in I$ we get $[x,y] \in P$ (for all $P \in \mathscr{P}$) so that $[x,y] = 0$ proving that R is commutative. ■

Reasoning in the same manner as in the proof of Theorem 3.2.7, one can easily prove the following result.

Theorem 3.2.10 *Let R be a ring, I a nonzero ideal of R and P a prime ideal of R such that $P \subsetneq I$ and R/P is a 2-torsion free. If d and g are derivations of R satisfying the condition $\overline{d(xy) \pm g(yx)} \in Z(R/P)$ for all $x,y \in I$, then $\big(d(R) \subseteq P$ and $g(R) \subseteq P\big)$ or R/P is a commutative integral domain.*

Corollary 3.2.11 *Let R be a 2-torsion free prime ring and I a nonzero ideal of R. If d is a nonzero derivation of R, then the following assertions are equivalent:*
(1) $d[x,y] \in Z(R)$ *for all* $x,y \in I$;
(2) $d(x \circ y) \in Z(R)$ *for all* $x,y \in I$;
(3) *R is a commutative integral domain.*

The following example demonstrates that the `primeness` hypothesis imposed on the ideal P in our Theorems is not superfluous.

Example. Let us consider the ring $R = \left\{ \begin{pmatrix} a & b \\ 0 & c \end{pmatrix} \big| \, a,b,c \in \mathbb{Z} \right\}$ and set $I = \left\{ \begin{pmatrix} 0 & a \\ 0 & 0 \end{pmatrix} \big| \, a \in \mathbb{Z} \right\}$. Since R is not prime, the ideal $P = \{0\}$ is not prime. Moreover, the mappings d and g

defined from R into itself by

$$d\begin{pmatrix} a & b \\ 0 & c \end{pmatrix} = \begin{pmatrix} 0 & a+b-c \\ 0 & 0 \end{pmatrix} \text{ and } g\begin{pmatrix} a & b \\ 0 & c \end{pmatrix} = \begin{pmatrix} 0 & a-b-c \\ 0 & 0 \end{pmatrix}$$

are derivations of R which satisfy conditions of Theorems 3.2.1, 3.2.2, 3.2.4, 3.2.7 and 3.2.10; but R is a non-commutative ring.

CHAPTER 4

COMMUTATIVITY OF RINGS WITH CONSTRAINTS ON ENDOMORPHISMS

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Abstract.

Our purpose in this chapter is to connect the commutativity of a quotient ring R/P with endomorphisms of R satisfying certain algebraic identities involving prime ideals. The obtained results cover many known theorems (cf. [19, 39, 57]).

4.1 Introduction

Many authors have established a connection between the structure of a ring R and derivations (resp. endomorphisms) of R satisfying certain specific algebraic identities (see for instance [22, 51, 58, 61]).

In [19], Bell and Daif investigated commutativity in rings possessing a derivation or an endomorphism which is strong commutativity preserving on a nonzero right ideal. Indeed, they proved that if a semi-prime ring R admits an endomorphism T which is SCP on a nonzero two-sided ideal U and not the identity on the ideal $U \cap T^{-1}(U)$, then R contains a nonzero central ideal. They also proved that a prime ring must be commutative if

it admits a non-identity endomorphism T which is SCP on an essential right ideal of R. Later, Deng and Ashraf [39] have studied a ring R equipped with mappings F and G such that $[F(x),G(y)] = [x,y]$ for all x,y in a subset S of R. More precisely, they established commutativity of the ring R when G is assumed to be a derivation or an endomorphism of R. In [65], Samman proved that an epimorphism of a semi-prime ring is strong commutativity preserving if and only if it is centralizing. More elegant versions proving various extensions of the preceding results have been established by several authors (see for example [30, 4]).

Motivated by the previous results, our principal aim in this chapter is to consider mappings satisfying some algebraic identities with endomorphisms (epimorphisms) acting on prime ideals and there connection with commutativity of a quotient ring R/P where R is an arbitrary ring (i.e. primeness (semi-primeness) is not required for the ring R).

4.2 Algebraic identities involving prime ideals

We will use frequently the following lemma which is very crucial for developing the proofs of our main results.

Lemma 4.2.1 *Let R be a ring, I a nonzero ideal of R and P a prime ideal such that $P \subsetneq I$. Let g an endomorphism of R. Then, the following assertions are equivalent:*
(1) $[g(x),x] \in P$ *for all* $x \in I$;
(2) $(g-1)(I) \subseteq P$ *or* $\overline{g(I)} \subseteq Z(R/P)$.

Proof. For the nontrivial implication, assume that $[g(x),x] \in P$ for all $x \in I$. The linearization of this relation gives

$$[g(x),y] + [g(y),x] \in P \text{ for all } x,y \in I. \tag{4.1}$$

Substituting xy for x in (4.1) we obtain

$$x[g(x),y] + g(x)[g(y),x] \in P \text{ for all } x,y \in I. \tag{4.2}$$

Left multiplying (4.1) by x and then subtracting from (4.2), we get

$$(g(x)-x)[g(x),y] \in P \text{ for all } x,y \in I. \tag{4.3}$$

Writing ry for y in (4.3) where $r \in I$, we obtain $(g(x)-x)r[g(x),y] \in P$ so that

$$(g(x)-x)I[g(x),y] \subseteq P \text{ for all } x,y \in I.$$

Using Fact 1, the last relation assures that $g(x)-x \in P$ or $\overline{g(x)} \in Z(R/P)$ for all $x \in I$. Since a group cannot be the union of two proper subgroups, then either $(g-1)(I) \subseteq P$ or $\overline{g(I)} \subseteq Z(R/P)$. ■

In ([19], Theorem 2), Bell and Daif investigated the commutativity of rings admitting an endomorphism that is SCP on a nonzero right ideal. Indeed, they proved that a prime ring R must be commutative if it admits a non-identity endomorphism T satisfying $[T(x),T(y)] = [x,y]$ for all x,y in a right ideal U of R.

Our fundamental aim in the next theorem is to generalize Theorem 2 of [19] in two directions. First of all, we will omit the primeness hypothesis by considering an arbitrary ring. On the other hand, we will investigate a more general algebraic identity $[f(x),T(y)]-[x,y]$ where T is an endomorphism and f is an arbitrary mapping.

Theorem 4.2.2 *Let R be a ring, I a nonzero ideal of R and P a prime ideal such that $P \subsetneq I$. If R admits a mapping f and an endomorphism T such that $[f(x),T(y)]-[x,y] \in P$ for all $x,y \in I$, then one of the following assertions holds:*
(1) *$[f(x),x] \in P$ for all $x \in I$ and $(T-1)(R) \subseteq P$;*
(2) *R/P is an integral domain.*

Proof. We are given that

$$[f(x),T(y)]-[x,y] \in P \text{ for all } x,y \in I. \tag{4.4}$$

Replacing y by yr in (4.4), we get

$$[f(x),T(y)]T(r)-[x,y]r+T(y)[f(x),T(r)]-y[x,r] \in P \text{ for all } x,y \in I. \tag{4.5}$$

Right multiplying (4.4) by $T(r)$ and subtracting it from (4.5), we arrive at

$$[x,y](T(r)-r)+T(y)[f(x),T(r)]-y[x,r] \in P \text{ for all } x,y,r \in I \tag{4.6}$$

thereby obtaining

$$[x,y](T(r)-r)+(T(y)-y)[x,r]\in P \quad \text{for all } x,y,r\in I. \tag{4.7}$$

Substituting xr for x in (4.7), we find that

$$[x,y](rT(r)-r^2)+x[r,y](T(r)-r)+(T(y)-y)[x,r]r\in P \quad \text{for all } x,y,r\in I. \tag{4.8}$$

Using equations (4.7) and (4.8), one can easily see that

$$[x,y][T(r),r]+x[r,y](T(r)-r)\in P \quad \text{for all } x,y,r\in I.$$

Putting tx instead of x in the above expression, we get

$$[t,y]I[T(r),r]\subseteq P \quad \text{for all } y,r,t\in I. \tag{4.9}$$

Applying Fact 1, we have either R/P is commutative or $[T(r),r]\in P$ for all $r\in I$. In the latter case, using Lemma 4.2.1 we get the required result. ■

Remark 4.2.3 *Theorem 4.2.2 is also valid for a nonzero right ideal.*

We will now state an improved version of ([19], Theorem 2) which follows at once from Theorem 4.2.2.

Corollary 4.2.4 *Let R be a prime ring and I a nonzero right ideal of R. If R admits a mapping f and a non-identity endomorphism T such that $[f(x),T(y)]-[x,y]=0$ for all $x,y\in I$, then R is an integral domain.*

Proposition 4.2.5 *Let R be a semi-prime ring and I a nonzero ideal of R. If R admits a mapping f and an endomorphism T such that $[f(x),T(y)]-[x,y]=0$ for all $x,y\in I$, then T is commuting on I.*

Proof. Suppose that $[f(x),T(y)]-[x,y]=0$ for all $x,y\in I$. By view of the semi-primeness of the ring R, there exists a family Γ of prime ideals such that $\bigcap_{P\in\Gamma}P=(0)$, which means that $[f(x),T(y)]-[x,y]\in P$ for all $P\in\Gamma$. Invoking the proof of Theorem 4.2.2, expression (4.9) implies that

$$[t,y]x\big(T(r)-r\big)\in P \quad \text{for all } x,y,r,t\in I. \tag{4.10}$$

Accordingly

$$[r,y]x\big(T(r)r-r^2\big)\in P \quad \text{for all } x,y,r\in I \text{ and for all } P\in\Gamma. \tag{4.11}$$

Replacing x by xr in (4.10) and combining with (4.11), we obviously get

$$[t,y]x[T(r),r]\in P \quad \text{for all } x,y,r,t\in I. \tag{4.12}$$

Substituting $T(r)t$ for t in (4.12), we obtain $[T(r),y]tx[T(r),r]\in P$ so that

$$[T(r),r]tx[T(r),r]\in P \quad \text{for all } x,r,t\in I.$$

Hence it follows that from the preceding equation, $[T(r),r]I[T(r),r]I[T(r),r]\subseteq P$. By view of primeness, we get $[T(r),r]\in P$ for all $P\in\Gamma$ and therefore $[T(r),r]=0$ for all $r\in I$. Accordingly, T is commuting on I. ■

The purpose of the following theorem is to investigate a similar algebraic identity involving anti-commutator.

Theorem 4.2.6 *Let R be a ring, I a nonzero ideal of R and P a prime ideal such that $P\subsetneq I$. If f is an additive mapping of R and g is an epimorphism of R such that $f(x)\circ g(y)-x\circ y\in P$ for all $x,y\in I$, then we have one of the following assertions:*
(1) *$[f(x),x]\in P$ for all $x\in I$ and $(g-1)(R)\subseteq P$;*
(2) *R/P is an integral domain. Moreover, if $char(R/P)\neq 2$, then $(f-1)(R)\subseteq P$ and $(g-1)(R)\subseteq P$.*

Proof. Suppose that

$$f(x)\circ g(y)-x\circ y\in P \quad \text{for all } x,y\in I. \tag{4.13}$$

Substituting ys for y in (4.13), we get

$$\big(f(x)\circ g(y)\big)g(s)-g(y)[f(x),g(s)]-(x\circ y)s+y[x,s]\in P \quad \text{for all } x,y,s\in I. \tag{4.14}$$

Left multiplying (4.13) by $g(r)$ and subtracting it from the above relation, we arrive at

$$(x\circ y)(g(s)-s)-g(y)[f(x),g(s)]+y[x,s]\in P \quad \text{for all } x,y,s\in I. \tag{4.15}$$

Replacing y by xy in (4.15), we find that

$$\big(x-g(x)\big)g(y)[f(x),g(s)]\in P \quad \text{for all } x,y,s\in I.$$

Once again replacing y by yr, where $r\in R$, and using the fact that g is an epimorphism we get

$$\big(x-g(x)\big)g(y)R[f(x),g(s)]\subseteq P \quad \text{for all } x,y,s\in I. \tag{4.16}$$

Since P is prime, equation (4.16) leads to $x-g(x)\in P$ or $[f(x),g(s)]\in P$ for all $s\in I$. Accordingly, by Brauer's trick $x-g(x)\in P$ for all $x\in I$ or $[f(x),g(s)]\in P$ for all $x,s\in I$. In the latter case, equation (4.15) reduces to

$$(x\circ y)(g(s)-s)+y[x,s]\in P \quad \text{for all } x,y,s\in I. \tag{4.17}$$

Writing ty instead of y in (4.17), we easily get

$$[x,t]y(g(s)-s)\in P \quad \text{for all } x,y,s,t\in I. \tag{4.18}$$

Therefore, we obtain R/P is commutative or $g(s)-s\in P$ for all $s\in I$. Now if $(g-1)(I)\subseteq P$, then our hypothesis yields

$$(f(x)-x)\circ y\in P \quad \text{for all } x,y\in I. \tag{4.19}$$

Replacing y by xy in (4.19), we get $[f(x),x]y\in P$ so that

$$[f(x),x]y[f(x),x]\in P \quad \text{for all } x,y\in I.$$

Applying Fact 1, the last relation yields $[f(x),x]\in P$ for all $x\in I$.
Now if R/P is commutative and $char(R/P)\neq 2$, then using the hypothesis one can verify that

$$f(x)g(y)-xy\in P \quad \text{for all } x,y\in I. \tag{4.20}$$

Putting yu instead of y in (4.20), it follows that

$$xI(g(u)-u) \subseteq P \quad \text{for all } u,x \in I.$$

By view of the primeness of P, we conclude that $(g-1)(R) \subseteq P$ and thus Eq. (4.20) gives also $(f-1)(R) \subseteq P$. ■

Corollary 4.2.7 *Let R be a prime ring and I a nonzero ideal of R. If R admits an additive mapping f and an epimorphism g such that $f(x) \circ g(y) - x \circ y = 0$ for all $x,y \in I$, then we have one of the following assertions:*
(1) *f is commuting on I and $g = 1$;*
(2) *R is an integral domain. Moreover, if $char(R) \neq 2$, then $f = g = 1$.*

As an application of Theorem 4.2.6, the following proposition shows that the condition f is an endomorphism in ([47], Theorem 5) is not necessary.

Proposition 4.2.8 *Let R be a semi-prime ring and I a nonzero ideal of R. If R admits an additive mapping f and an epimorphism g such that $f(x) \circ g(y) - x \circ y = 0$ for all $x,y \in I$, then g is commuting on I.*

Proof. Assuming that $f(x) \circ g(y) - x \circ y = 0$ for all $x,y \in I$. In light of semi-primeness, there exists a family Γ of prime ideals such that $\bigcap_{P \in \Gamma} P = (0)$, which proves that $f(x) \circ g(y) - x \circ y \in P$ for all $P \in \Gamma$. Applying the proof of Theorem 4.2.6 and by expanding expression (4.18), we arrive at $[g(s),s] \in P$ for all $P \in \Gamma$. Consequence of which, $[g(s),s] = 0$ for all $s \in I$, proving that g is commuting on I. ■

Theorem 4.2.9 *Let R be a ring, I a nonzero ideal of R and P a prime ideal such that $P \subsetneq I$. If f is an additive mapping of R and g is an epimorphism of R such that $[f(x),g(y)] - x \circ y \in P$ for all $x,y \in I$, then we have one of the following assertions:*
(1) *$[f(x),x] \in P$ for all $x \in I$ and $(g-1)(R) \subseteq P$;*
(2) *R/P is an integral domain with $char(R/P) = 2$.*

Proof. Suppose that

$$[f(x),g(y)] - x \circ y \in P \quad \text{for all } x,y \in I. \tag{4.21}$$

Replacing y by ys in (4.21), we get

$$g(y)[f(x),g(s)]+(x\circ y)(g(s)-s)+y[x,s]\in P \quad \text{for all } x,y,s\in I.$$

Combining the last two equations, we may write

$$(g(x)-x)g(y)[f(x),g(s)]\in P \quad \text{for all } x,y,s\in I. \tag{4.22}$$

Putting yr for y in (4.22), we find that

$$\big(g(x)-x\big)g(y)R[f(x),g(s)]\subseteq P \quad \text{for all } x,y,s\in I.$$

In light of Brauer's trick, for each $x\in I$ we have $g(x)-x\in P$ or $[f(x),g(s)]\in P$ for all $s\in I$. In the latter case equation (4.21) reduces to $x\circ s\in P$ for all $x,s\in I$ so that R/P is commutative. Now, if $(g-1)(R)\subseteq P$, then our hypothesis yields

$$[f(x),y]-x\circ y\in P \quad \text{for all } x,y\in I. \tag{4.23}$$

Replacing y by xy in (4.23), we find that $[f(x),x]y\in P$ and hence

$$[f(x),x]I[f(x),x]\subseteq P \quad \text{for all } x\in I.$$

Invoking Fact 1, we conclude that $[f(x),x]\in P$ for all $x\in I$.
Now, assume that R/P is a 2-torsion free prime ring, as we've already established R/P is an integral domain then our hypothesis leads to $x\circ y\in P$ for all $x,y\in I$ and thus $I\subset P$, which contradicts our assumption. ■

Corollary 4.2.10 *Let R be a prime ring and I a nonzero ideal of R. If f is an additive mapping of R and g is a non-identity epimorphism of R such that $[f(x),g(y)]-x\circ y=0$ for all $x,y\in I$, then R is an integral domain and $char(R)=2$.*

Adopting the same arguments, with suitable modification, we get the following result.

Theorem 4.2.11 *Let R be a ring, I a nonzero ideal of R and P a prime ideal such that $P\subsetneq I$. If f is an additive mapping of R and g is an epimorphism of R such that $f(x)\circ g(y)-[x,y]\in P$ for all $x,y\in I$, then we have one of the following assertions:*

(1) $[f(x),x] \in P$ *for all* $x \in I$ *and* $(g-1)(R) \subseteq P$;
(2) R/P *is an integral domain. Moreover, if* $char(R/P) \neq 2$, *then* $f(R) \subseteq P$.

Corollary 4.2.12 *Let R be a prime ring and I a nonzero ideal of R. If f is an additive mapping of R and g is a non-identity epimorphism of R such that $f(x) \circ g(y) - [x,y] = 0$ for all $x,y \in I$, then R is an integral domain. Moreover, if $char(R) \neq 2$, then $f = 0$.*

Proposition 4.2.13 *Let R be a semi-prime ring and I a nonzero ideal of R. If R admits an additive mapping f and an epimorphism g such that either $\big([f(x),g(y)] - x \circ y = 0$ for all $x,y \in I\big)$ or $\big(f(x) \circ g(y) - [x,y] = 0$ for all $x,y \in I\big)$, then g is commuting on I.*

4.3 Endomorphisms acting on the center $Z(R/P)$

In [58] it is proved that a prime ring R must be commutative if it possesses an endomorphism g satisfying any one of the following situations:

$$\begin{aligned} g(xy) - [x,y] &\in Z(R) \quad \text{for all } x,y \in R; \\ g(xy) + [x,y] &\in Z(R) \quad \text{for all } x,y \in R; \\ g(xy) - x \circ y &\in Z(R) \quad \text{for all } x,y \in R; \\ g(xy) + x \circ y &\in Z(R) \quad \text{for all } x,y \in R. \end{aligned}$$

Motivated by the previous results, our purpose in this section is to examine more general algebraic identities involving two endomorphisms g_1 and g_2. More precisely, we will consider the identities :

$$\overline{g_1(x)g_2(y) \pm xy} \in Z(R/P) \text{ for all } x,y \in R, \ \overline{g_1(x)g_2(y) \pm yx} \in Z(R/P) \text{ for all } x,y \in R$$

$$\overline{g_1(x)g_2(y) \pm [x,y]} \in Z(R/P) \text{ for all } x,y \in R, \ \overline{g_1(x)g_2(y) \pm x \circ y} \in Z(R/P) \text{ for all } x,y \in R.$$

Theorem 4.3.1 *Let R be a ring, P a prime ideal of R and g_1, g_2 two endomorphisms of R.*
(1) *If $\overline{g_1(x)g_2(y) - xy} \in Z(R/P)$ for all $x,y \in R$, then $\big(\big(g_1 - 1\big)(R) \subseteq P$ and $(g_2 - 1)(R) \subseteq P\big)$ or R/P is an integral domain.*
(2) *If $\overline{g_1(x)g_2(y) + xy} \in Z(R/P)$ for all $x,y \in R$, then either $(g_1 + g_2)(R) \subseteq P$ or R/P is an integral domain.*

Proof. (1) Assume that

$$\overline{g_1(x)g_2(y)-xy} \in Z(R/P) \quad \text{for all } x,y \in R. \tag{4.24}$$

If $Z(R/P)=\{\overline{0}\}$, then equation (4.24) becomes

$$g_1(x)g_2(y)-xy \in P \quad \text{for all } x,y \in R. \tag{4.25}$$

Substituting yr for y in (4.25), we get

$$g_1(x)g_2(y)g_2(r)-xyr \in P \quad \text{for all } x,y,r \in R. \tag{4.26}$$

Right multiplying (4.25) by $g_2(r)$ and subtracting it from (4.26), we arrive at

$$xy(g_2(r)-r) \in P \quad \text{for all } x,y,r \in R. \tag{4.27}$$

Setting $x=g_2(r)-r$, equation (4.27) yields

$$(g_2(r)-r)R(g_2(r)-r) \subseteq P \quad \text{for all } r \in R \tag{4.28}$$

which, by virtue of primeness, forces $(g_2-1)(R) \subseteq P$. Since equation (4.25) can be rewritten as $g_1(x)\big(g_2(y)-y\big)+\big(g_1(x)-x\big)y \in P$, then we get

$$(g_1(x)-x)y \in P \quad \text{for all } x,y \in R$$

which assures that $(g_1-1)(R) \subseteq P$.
Now if $Z(R/P) \neq \{\overline{0}\}$, then there exists an non-zero element $\overline{z}_0 \in Z(R/P)$. Letting $x=y=z_0$ in (4.24), we obtain $\overline{g_1(z_0)g_2(z_0)} \in Z(R/P)$. Replacing x by xz_0 and y by z_0y in our hypothesis, we get

$$\overline{g_1(x)g_1(z_0)g_2(z_0)g_2(y)-xyz_0^2} \in Z(R/P) \quad \text{for all } x,y \in R$$

which implies that

$$\overline{g_1(z_0)g_2(z_0)g_1(x)g_2(y)-xyz_0^2} \in Z(R/P) \quad \text{for all } x,y \in R.$$

Commuting the above relation with r where $r \in R$, one can see that

$$g_1(z_0)g_2(z_0)[g_1(x)g_2(y),r] - [xy,r]z_0^2 \in P \quad \text{for all } x,y \in R. \tag{4.29}$$

On the other hand, the hypothesis gives

$$[g_1(x)g_2(y),r] - [xy,r] \in P \quad \text{for all } x,y,r \in R. \tag{4.30}$$

Left multiplying equation (4.30) by $g_1(z_0)g_2(z_0)$ and subtracting it from (4.29), we find that

$$\big(g_1(z_0)g_2(z_0) - z_0^2\big)[xy,r] \in P \quad \text{for all } x,y,r \in R. \tag{4.31}$$

Writing y for yt in (4.31), we obtain $\big(g_1(z_0)g_2(z_0) - z_0^2\big)xy[t,r] \in P$ so that

$$\big(g_1(z_0)g_2(z_0) - z_0^2\big)Ry[t,r] \subseteq P \quad \text{for all } y,r,t \in R.$$

Accordingly, $g_1(z_0)g_2(z_0) - z_0^2 \in P$ or R/P is an integral domain. In the first case, setting $x = xz_0$ and $y = z_0$ in (4.24), thereby obtaining

$$\overline{g_1(x)g_1(z_0)g_2(z_0) - xz_0^2} \in Z(R/P) \quad \text{for all } x \in R.$$

This may be rewritten as

$$\overline{(g_1(x) - x)z_0^2} \in Z(R/P) \quad \text{for all } x \in R.$$

Hence $[g_1(x),x] \in P$ for all $x \in R$ and Lemma 4.2.1 assures that $(g_1 - 1)(R) \subseteq P$ or $\overline{g_1(R)} \subseteq Z(R/P)$.

Assume that $g_1(x) \in Z(R/P)$ for all $x \in R$; then our hypothesis reduces to

$$g_1(x)[g_2(y),r] - [xy,r] \in P \quad \text{for all } x,y,r \in R. \tag{4.32}$$

Substituting rx for x in (4.32), we find that

$$g_1(r)g_1(x)[g_2(y),r] - r[xy,r] \in P \quad \text{for all } x,y,r \in R. \tag{4.33}$$

Left multiplying equation (4.32) by $g_1(r)$ and subtracting it from (4.33), we get

$$(g_1(r)-r)[xy,r]\in P \quad \text{for all } x,y,r\in R.$$

which leads to $(g_1-1)(R)\subseteq P$ or R/P is commutative. Consequently, in both cases we see that either $(g_1-1)(R)\subseteq P$ or R/P is commutative.
If $(g_1-1)(R)\subseteq P$, then equation (4.24) implies that $\big[x(g_2(y)-y),r\big]\in P$ for all $x,y,r\in R$. Accordingly, $[t,r]R(g_2(y)-y)\subseteq P$ which, by view of primeness, assures that $(g_2-1)(R)\subseteq P$ or R/P is a commutative integral domain.
(2) Using similar arguments as in the first case with a suitable slight modification, we get the required result. ■

Corollary 4.3.2 *Let R be a prime ring. If R admits two endomorphisms g_1 and g_2, one of them is non-trivial, satisfying the condition $g_1(x)g_2(y)-xy\in Z(R)$ for all $x,y\in R$, then R is commutative.*
If $g_1(x)g_2(y)+xy\in Z(R)$ for all $x,y\in R$ and $char(R)\neq 2$, then R is commutative.

A slight modification in the proof of Theorem 4.3.1 yields the following result.

Theorem 4.3.3 *Let R be a ring and P a prime ideal of R. If g_1 and g_2 are two endomorphisms of R satisfying the condition $\overline{g_1(x)g_2(y)-yx}\in Z(R/P)$ for all $x,y\in R$, then R/P is an integral domain.*

Corollary 4.3.4 *Let R be a semi-prime ring. If R admits two endomorphisms g_1 and g_2, then the following assertions are equivalent:*
(1) *$g_1(x)g_2(y)-yx\in Z(R)$ for all $x,y\in R$;*
(2) *$g_1(x)g_2(y)+yx\in Z(R)$ for all $x,y\in R$;*
(3) *R is commutative.*

Proof. For the non-trivial implications, we only need to prove $(1)\Longrightarrow(3)$ while the $(2)\Longrightarrow(3)$ can be proved similarly. Assume that

$$g_1(x)g_2(y)-yx\in Z(R) \quad \text{for all } x,y\in R.$$

By view of the semi-primeness of the ring R, there exists a family $-$ of prime ideals Γ such that $\bigcap_{P\in\Gamma}P=(0)$. Thereby obtaining $[g_1(x)g_2(y)-yx,r]\in P$ for all $P\in\Gamma$. Invoking

Theorem 4.3.3 we conclude that R/P is a commutative integral domain for all $P \in \Gamma$ which, because of $\bigcap_{P\in\Gamma} P = (0)$, assures that R is commutative. ■

Theorem 4.3.5 *Let R be a ring and P a prime ideal of R. If g_1 and g_2 are two endomorphisms of R, then the following assertions are equivalent:*
(1) $\overline{g_1(x)g_2(y)-[x,y]} \in Z(R/P)$ *for all* $x,y \in R$;
(2) $\overline{g_1(x)g_2(y)-x\circ y} \in Z(R/P)$ *for all* $x,y \in R$;
(3) R/P *is an integral domain.*

Proof. (1) $\Longrightarrow$ (3) Assume that

$$\overline{g_1(x)g_2(y)-[x,y]} \in Z(R/P) \quad \text{for all } x,y \in R. \tag{4.34}$$

If $Z(R/P) = \{\overline{0}\}$, then equation (4.34) may be restated as

$$g_1(x)g_2(y)-[x,y] \in P \quad \text{for all } x,y \in R. \tag{4.35}$$

Replacing y by yx in (4.35), we get

$$[x,y](g_2(x)-x) \in P \quad \text{for all } x,y \in R. \tag{4.36}$$

thereby obtaining

$$[x,r]R(g_2(x)-x) \subseteq P \quad \text{for all } x,r \in R.$$

Accordingly, either R/P is a commutative integral domain or $(g_2-1)(R) \subseteq P$. In the latter case, equation (4.34) forces $g_1(x)y-[x,y] \in P$ for all $x,y \in R$. Writing yt for y, we find that $y[r,x] \in P$ for all $x,y,r \in R$ which implies that R/P is commutative. Hence, in both cases, R/P is a commutative integral domain, a contradiction. Consequently, $Z(R/P) \neq \{\overline{0}\}$. Let $\overline{z_0} \in Z(R/P)\setminus\{\overline{0}\}$ and taking $x=y=z_0$ in (4.34) we easily get $\overline{g_1(z_0)g_2(z_0)} \in Z(R/P)$. Substituting xz_0 for x and z_0y for y in hypothesis, we find that

$$\overline{g_1(z_0)g_2(z_0)g_1(x)g_2(y)-[x,y]z_0^2} \in Z(R/P) \quad \text{for all } x,y \in R. \tag{4.37}$$

Commuting (4.37) with r where $r \in R$, one obtains

$$g_1(z_0)g_2(z_0)[g_1(x)g_2(y),r]-\big[[x,y],r\big]z_0^2 \in P \quad \text{for all } x,y \in R$$

thereby obtaining

$$\left(g_1(z_0)g_2(z_0)-z_0^2\right)\left[[x,y],r\right]\in P \quad \text{for all } x,y,r\in R.$$

Accordingly, either $\left[[x,y],r\right]\in P$ for all $x,y,r\in R$ in which case R/P is commutative or $g_1(z_0)g_2(z_0)-z_0^2\in P$.
Assume that $g_1(z_0)g_2(z_0)-z_0^2\in P$; setting $x=xz_0$ and $y=z_0$ in (4.34) and using appropriate expansion, we obtain

$$\overline{g_1(x)z_0^2}\in Z(R/P) \quad \text{for all } x\in R.$$

Hence, $\overline{g_1(x)}\in Z(R/P)$ for all $x\in R$. Replacing x by xy in our hypothesis, we find that

$$\left[[x,y](g_1(y)-y),r\right]\in P \quad \text{for all } x,y,r\in R. \tag{4.38}$$

Replacing x by yx in (4.38), we obviously get

$$[y,r][x,y](g_1(y)-y)\in P \quad \text{for all } x,y,r\in R$$

in particular

$$[y,r]R[x,y]R(g_1(y)-y)\subseteq P \quad \text{for all } x,y,r\in R.$$

Since P is prime, the above equation assures that either R/P is an integral domain or $(g_1(y)-y)\in P$ for all $y\in R$. So again an appeal to (4.34) gives $\overline{xg_2(y)-[x,y]}\in Z(R/P)$ for all $x,y\in R$, which leads to $\left[[r,y]x,r\right]\in P$ for all $x,y,r\in R$. Replacing x by xt in the last expression we get $[r,y]R[t,r]\subseteq P$ for all $y,r,t\in R$, proving that R/P is a commutative integral domain.
(2) Suppose that $\overline{g_1(x)g_2(y)-x\circ y}\in Z(R/P)$ for all $x,y\in R$, then the same reasoning proves that R/P is an integral domain and this completes our proof.

Corollary 4.3.6 *Let R be a semi-prime ring. If g_1 and g_2 are two endomorphisms of R, then the following assertions are equivalent:*
(1) $g_1(x)g_2(y)-[x,y]\in Z(R)$ *for all* $x,y\in R$;
(2) $g_1(x)g_2(y)-x\circ y\in Z(R)$ *for all* $x,y\in R$;
(3) R *is commutative.*

Remark 4.3.7 *If we set $g_1 = g_2$ in the last corollary, then we get a generalization of ([58], Theorem 2.4) for semi-prime rings.*

The following example proves that the primeness hypothesis imposed on the ideal P is not superfluous in Theorem 4.3.1. More specifically, our result we cannot extends to a semi-prime ring.

Example. Let us consider $R = \left\{ \begin{pmatrix} a & b \\ 0 & 0 \end{pmatrix} \mid a, b \in \mathbb{Z} \right\}$ and set $P = (0)$. It is clear that P is not a prime ideal of R. Furthermore, if we define endomorphisms g_1 and g_2 by $g_1 = 1$ and $g_2 \begin{pmatrix} a & b \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} a & 0 \\ 0 & 0 \end{pmatrix}$, then the conditions of Theorem 4.3.1 are satisfied, but R is not commutative.

CHAPTER 5

COMMUTATIVITY CRITERIA FOR A FACTOR RING R/P ARISING FROM P-CENTRALIZERS

K. Bouchannafa, M. A. Idrissi and L. Oukhtite, *Commutativity criteria for a factor ring R/P arising from P-centralizers* . Kyungpook Math. J. (In press).

Abstract.

This chapter considers a more general class of centralizers called I-centralizers. More precisely, given a prime ideal P of an arbitrary ring R we establish a connection between certain algebraic identities involving a pair of P-left centralizers and the structure of the factor ring R/P.

5.1 Introduction

Throughout R will be a ring with center $Z(R)$. Let $x, y \in R$. The commutator $xy - yx$ will be denoted by $[x,y]$ and the anti-commutator $xy + yx$ will be represented by $x \circ y$. Recall that an ideal P of R is prime if for all $x, y \in R$, $xRy \subseteq P$ implies $x \in P$ or $y \in P$. An additive mapping $d : R \longrightarrow R$ is called a derivation if $d(xy) = d(x)y + xd(y)$ holds for all $x, y \in R$. An additive mapping $F : R \longrightarrow R$ is called a generalized derivation if there exists a derivation $d : R \longrightarrow R$ such that $F(xy) = F(x)y + xd(y)$ for all $x, y \in R$, and d is called the associated derivation of F. During the past few decades, there has been an ongoing

interest concerning the relationship between the commutativity of a ring and the existence of certain specific types of derivations of R.

An additive mapping $T : R \to R$ is said to be a left centralizer (resp. right centralizer) of R if $T(xy) = T(x)y$ (resp. $T(xy) = xT(y)$) for all $x, y \in R$. An additive mapping T is called a centralizer in case T is a left and a right centralizer of R. In ring theory it is more common to work with module homomorphisms. Ring theorists would write that $T : R_R \to R_R$ is a homomorphism of a ring module R into itself. For a semi-prime ring R all such homomorphisms are of the form $T(x) = qx$ for all $x \in R$, where q is an element of Martindale left ring of quotients Q_r (see Chapter 2 in [17]). If R has the identity element then $T : R \to R$ is a left centralizer if T is of the form $T(x) = ax$ for all $x \in R$ and some fixed element $a \in R$. Very recently, S. Ali and N. A. Dar [3], managed the commutativity of prime rings with involution involving left centralizers. For example, a result proven by them is as below: Let R be a prime ring with involution $*$ such that $char(R) \neq 2$ and $S(R) \cap Z(R) \neq (0)$. If R admits a nonzero left centralizer $T : R \to R$ such that $T([x, x^*]) = 0$ for all $x \in R$, then R is commutative. Moreover, in [1] some related results involving left centralizers have also been discussed.

In the following definition, we have initiated the concept of I-centralizers in rings, where I is an ideal, and extended several known results.

Definition. Let I be an ideal of a ring R and $f : R \longrightarrow R$ an additive mapping.
1) f is called an I-left centralizer if $f(xy) - f(x)y \in I$ for all $x, y \in R$.
2) f is called an I-right centralizer if $f(xy) - xf(y) \in I$ for all $x, y \in R$.
3) f is called an I-centralizer if and only if f is both an I-left centralizer and I-right centralizer.

Example.

1. The zero function Θ_R is a I-centralizer on R.

2. The I_d and $-I_d$ are a I-left centralizer (resp. I-right centralizer) on R, where I_d denotes the identity function.

3. Let us consider $R = \left\{ \begin{pmatrix} x & y & 0 \\ 0 & 0 & 0 \\ 0 & z & 0 \end{pmatrix} \mid x,y,z \in \mathbb{Z} \right\}$ and I is a nonzero ideal of R defined by $I = \left\{ \begin{pmatrix} \alpha & \beta & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \mid \alpha,\beta \in \mathbb{Z} \right\}$. Define the additive mapping $T : R \to R$ by:

$$T \begin{pmatrix} x & y & 0 \\ 0 & 0 & 0 \\ 0 & z & 0 \end{pmatrix} = \begin{pmatrix} z & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

this makes it easy to verify that T is a I-centralizer on R but not a centralizer.

The main goal of this work is to continue on this line of investigation and study the relationship between the structure of quotient rings R/P and the behavior of P-centralizers satisfying specific algebraic identities.

5.2 Identities involving a pair of left P-centralizers

In what follows, $\overline{x}$ for x in R denotes $x+P$ in R/P.
In ([16], Theorem 2.3), Aydin proved that if R be a non-commutative prime ring, F generalized derivation of R associated with a nonzero derivation d and $a \notin Z(R)$ such that $F(x)a = aF(x)$ for all $x \in I$, then $d(x) = \lambda[x,a]$, for all $x \in I$, where I is an ideal of R.

Inspired by above result, we here consider a more general algebraic identity involving two P-left centralizers by omitting the primeness assumption imposed on the ring R.

Theorem 5.2.1 *Let R be a ring, I a nonzero ideal of R and P a prime ideal of R such that $P \subsetneq I$. Suppose that T_1 and T_2 are two P-left centralizers on R, satisfying the condition $\overline{T_1(x)a - aT_2(x)} \in Z(R/P)$ for all $x \in I$, where $a \in R$, then one of the following assertions holds:*
(1) *$T_1(R) \subseteq P$ and $aT_2(R) \subseteq P$;*
(2) *R/P is a commutative integral domain;*
(3) *$[a,R] \subset P$.*

Proof. By assumption, we have

$$[T_1(x)a - aT_2(x), r] \in P \text{ for all } r, x \in I. \tag{5.1}$$

Replacing x by xy in (5.1), we obtain

$$[T_1(x)ya, r] - [aT_2(x), r]y - aT_2(x)[y, r] \in P \text{ for all } r, x, y \in I$$

in such a way that

$$(T_1(x)a - aT_2(x))[y, r] + \big[T_1(x)[y, a], r\big] \in P \text{ for all } r, x, y \in I. \tag{5.2}$$

Substituting yr for y in (5.2), we get

$$\big[T_1(x)y[r, a], r\big] \in P \text{ for all } r, x, y \in I.$$

That is

$$T_1(x)y\big[[r, a], r\big] + [T_1(x), r]y[r, a] + T_1(x)[y, r][r, a] \in P \text{ for all } r, x, y \in I. \tag{5.3}$$

Putting $T_1(x)y$ instead of y in (5.3) and using it, one can see that

$$[T_1(x), r]T_1(x)y[r, a] \in P \text{ for all } r, x, y \in I.$$

According to Fact 1, we obtain for each $r \in I$, either $[T_1(x), r]T_1(x) \in P$ or $[r, a] \in P$. Define $A = \{r \in I \,/\, [T_1(x), r]T_1(x) \in P \text{ for all } x \in I\}$ and $B = \{r \in I \,/\, [r, a] \in P\}$. Clearly, A and B are additive subgroups of I whose union is I. Hence by Brauer's trick, we have either $A = I$ or $B = I$.
In the second case, namely $[I, a] \subseteq P$. Since $RI \subseteq I$, then $[R, a] \subseteq P$.
Now consider $A = I$, in this situation

$$[T_1(x), r]T_1(x) \in P \text{ for all } r, x \in I.$$

Substituting sr for r in the above expression, we arrive at

$$[T_1(x),s]rT_1(x) \in P \text{ for all } r,s,x \in I. \tag{5.4}$$

Right multiplying the above equation by s and combining it with (5.4), it follows that

$$[T_1(x),s]I[T_1(x),s] \subseteq P \text{ for all } s,x \in I.$$

Applying Fact 1, we conclude that $\overline{T_1(x)} \in Z(R/P)$ for all $x \in I$. Writing xt for x in the last expression, where $t \in R$, we arrive at $\bar{t} \in Z(R/P)$ or $\overline{T_1(x)} = \overline{0}$. i.e., R/P is commutative or $T_1(R) \subseteq P$ and our hypothesis reduces to

$$[r,aT_2(x)] \in P \text{ for all } r,x \in I$$

which means that

$$a[r,T_2(x)] + [r,a]T_2(x) \in P \text{ for all } r,x \in I. \tag{5.5}$$

Replacing x by xt in (5.5), on can see that

$$a[r,T_2(x)]t + aT_2(x)[r,t] + [r,a]T_2(x)t \in P \text{ for all } x,t \in I. \tag{5.6}$$

Right multiplying (5.5) by t and subtracting it from (5.6), we get

$$aT_2(x)[r,t] \in P \text{ for all } r,t,x \in I. \tag{5.7}$$

Substituting r by ru in (5.7) and employing it, we obtain

$$aT_2(x)I[u,t] \subseteq P \text{ for all } t,u,x \in I. \tag{5.8}$$

Once again invoking Fact 1, it follows from equation (5.8) that $aT_2(R) \subseteq P$ or $[R,R] \subseteq P$. Finally, we have either ($T_1(R) \subseteq P$ and $aT_2(R) \subseteq P$) or $[a,R] \subseteq P$. ■

As an application of our Theorem, we get the following result.

Corollary 5.2.2 *Let R be a non-commutative prime ring and I a nonzero ideal of R. Suppose that T_1 and T_2 are two left centralizers on R such that $T_1(x)a \pm aT_2(x) \in Z(R)$ for*

all $x \in I$, where $a \notin Z(R)$, then $T_1 = 0$ and $aT_2 = 0$.

In ([7], Theorem 2.1), it is showed that if a prime ring R admits a nonzero left centralizer T, with $T(x) \neq x$ for all x in a nonzero ideal I of R, such that $T([x,y]) = [x,y]$ for all $x,y \in I$, then R must be commutative. The author in [56] with addition of 2-torsion freeness hypothesis, extended the preceding result to a Jordan ideal.

Motivated by the preceding results we investigate a more general context which allows us to generalize the above result in two ways. First of all, we will assume that $T([x,y])$ belong to center of R/P rather than $T([x,y]) = 0$. Secondly we will investigate the behavior of the more general expression $T_1(xy) - T_2(yx) \in Z(R/P)$ involving two P-left centralizers instead of the expression $T(xy) - T(yx) = 0$.

Theorem 5.2.3 *Let R be a ring, I a nonzero ideal of R and P a prime ideal of R such that $P \subsetneq I$. Suppose that T_1 and T_2 are two P-left centralizers on R, then the following assertions are equivalent:*
(1) $\overline{T_1(xy) - T_2(yx)} \in Z(R/P)$ for all $x,y \in I$;
(2) ($T_1(R) \subseteq P$ and $T_2(R) \subseteq P$) or R/P is a commutative integral domain.

Proof. By given assumption, we have

$$\overline{T_1(xy) - T_2(yx)} \in Z(R/P) \text{ for all } x,y \in I. \tag{5.9}$$

Substituting yr for y in (5.9), and by expanding this equation, we get

$$\big[T_2(y)[x,r],r\big] \in P \text{ for all } r,x,y \in I. \tag{5.10}$$

Replacing y by $yT_2(y)$ in (5.10), we find that

$$T_2(y)\big[T_2(y)[x,r],r\big] + [T_2(y),r]T_2(y)[x,r] \in P \text{ for all } r,x,y \in I. \tag{5.11}$$

In light of (5.10), Eq. (5.11) yields

$$[T_2(y),r]T_2(y)[x,r] \in P \text{ for all } r,x,y \in I. \tag{5.12}$$

Writing tx for x in (5.12), one can easily to see that

$$[T_2(y),r]T_2(y)t[x,r] \in P \text{ for all } r,t,x,y \in I.$$

According to Fact 1, we obtain either R/P is an integral domain or $[T_2(y),r]T_2(y) \in P$ for all $r,y \in I$. Arguing as above, the last relation assures that $\overline{T_2(y)} \in Z(R/P)$ for all $y \in I$ and our hypothesis becomes

$$T_1(x)[y,x] + [T_1(x),x]y \in P \text{ for all } x,y \in I. \tag{5.13}$$

Putting yu instead of y in (5.13), we get

$$T_1(x)y[u,x] \in P \text{ for all } u,x,y \in I.$$

By the primeness of P, we conclude that $T_1(R) \subseteq P$ or R/P is an integral domain. Now if $T_1(R) \subseteq P$, then equation (5.9) yields $\overline{T_2(y)x} \in Z(R/P)$ for all $x,y \in I$. Commuting this expression with r, we find that $T_2(y)I[x,r] \subseteq P$. Once again applying Fact 1, it follows that $T_2(R) \subseteq P$ or R/P is a commutative integral domain. ■

As an application of Theorem 5.2.3, the following corollary gives a generalization of some results in [7] and [56].

Corollary 5.2.4 *Let R be a prime ring and I a nonzero ideal of R. Suppose that T_1 and T_2 are nonzero two left centralizers on R, then the following assertions are equivalent:*
(1) *$T_1(xy) \pm T_2(yx) \in Z(R)$ for all $x,y \in I$;*
(2) *R is a commutative integral domain.*

Corollary 5.2.5 *Let R be a prime ring and I a nonzero ideal of R. Suppose that T is a nonzero left centralizer on R, then the following assertions are equivalent:*
(1) *$T([x,y]) \in Z(R)$ for all $x,y \in I$;*
(2) *$T(x \circ y) \in Z(R)$ for all $x,y \in I$;*
(3) *R is a commutative integral domain.*

In ([7], Theorem 3.1 and Theorem 3.3) it is proved that a prime ring R must be a commutative integral domain if it admits a non trivial left centralizer T such that $T(xy) -$

$xy \in Z(R)$ or $T(xy) - yx \in Z(R)$ for all x, y in a nonzero ideal I of R. This result can be obtained as an immediate application of Corollary 5.2.5.

Corollary 5.2.6 *Let R be a prime ring and I a nonzero ideal of R. Suppose that T is a non trivial left centralizer on R, then the following assertions are equivalent:*
(1) $T(xy) \pm xy \in Z(R)$ *for all* $x, y \in I$;
(2) $T(xy) \pm yx \in Z(R)$ *for all* $x, y \in I$;
(3) R *is a commutative integral domain.*

The following theorem exhibits a connection between the commutativity of R/P and range inclusion results of a pair of P-left centralizers.

Theorem 5.2.7 *Let R be a ring, I a nonzero ideal of R and P a prime ideal of R such that $P \subsetneq I$. If T_1 and T_2 are two P-left centralizers on R, then the following assertions are equivalent:*
(1) $\overline{T_1(x)T_2(x)} \in Z(R/P)$ *for all* $x \in I$;
(2) $T_1(R) \subseteq P$ *or* $T_2(R) \subseteq P$ *or* R/P *is a commutative integral domain.*

Proof. For non-trivial implications. Assume that

$$\overline{T_1(x)T_2(x)} \in Z(R/P) \text{ for all } x \in I. \tag{5.14}$$

A Linearization of (5.14) gives

$$\overline{T_1(x)T_2(y) + T_1(y)T_2(x)} \in Z(R/P) \text{ for all } x, y \in I.$$

This means that

$$[T_1(x), r]T_2(y) + T_1(x)[T_2(y), r] + T_1(y)[T_2(x), r] + [T_1(y), r]T_2(x) \in P \text{ for all } r, x, y \in I. \tag{5.15}$$

Substituting $yT_2(x)$ for y in (5.15) and combining it from the above expression, we get

$$\big(T_1(x)T_2(y) + T_1(y)T_2(x)\big)[T_2(x), r] \in P \text{ for all } r, x, y \in I. \tag{5.16}$$

Putting tr instead of r in (5.16), we obtain

$$\big(T_1(x)T_2(y)+T_1(y)T_2(x)\big)t[T_2(x),r]\in P \text{ for all } r,t,x,y\in I.$$

In view of the primeness of P, we find that either $T_1(x)T_2(y)+T_1(y)T_2(x)\in P$ for all $x,y\in I$ or $[T_2(x),r]\in P$ for all $r,x\in I$.
In the latter case, taking $x=xs$, it is obviously to see that

$$T_2(x)[s,r]\in P \text{ for all } r,s,x\in I. \tag{5.17}$$

Writing xu for x and using Fact 1, we arrive at $T_2(R)\subseteq P$ or R/P is commutative.
Now consider the first case, i.e., $T_1(x)T_2(y)+T_1(y)T_2(x)\in P$ for all $x,y\in I$. Replacing y by yw in this equation, it follows that $T_1(y)(T_2(x)w-wT_2(x))\in P$ for all $w,x,y\in I$. Thereby obtaining,

$$T_1(y)z(T_2(x)w-wT_2(x))\in P \text{ for all } w,x,y,z\in I.$$

Therefore, either $T_1(R)\subseteq P$ or $T_2(x)w-wT_2(x)\in P$ for all $w,x\in I$. In the last case, putting $x=xy$, we easily get $T_2(x)[w,y]\in P$ for all $w,x,y\in I$ proving that $T_2(R)\subseteq P$ or R/P is an integral domain. ■

Corollary 5.2.8 *Let R be a prime ring and I a nonzero ideal of R. If T_1 and T_2 are two nonzero left centralizers on R such that $T_1(x)T_2(x)\in Z(R)$ for all $x\in I$, then R is a commutative integral domain.*

Theorem 5.2.9 *Let R be a ring, I a nonzero ideal of R and P a prime ideal of R such that $P\subsetneq I$. If T_1 and T_2 are two P-left centralizers on R, then the following assertions are equivalent:*
(1) $\overline{[T_1(x),T_2(y)]}\in Z(R/P)$ for all $x,y\in I$;
(2) $\overline{T_1(x)\circ T_2(y)}\in Z(R/P)$ for all $x,y\in I$;
(3) $T_1(R)\subseteq P$ or $T_2(R)\subseteq P$ or R/P is commutative integral domain.

Proof. Wee only need to prove (1) $\Longrightarrow$ (3) and (2) $\Longrightarrow$ (3).
(1) $\Longrightarrow$ (3) For all $x,y\in I$, we suppose that

$$\overline{[T_1(x),T_2(y)]}\in Z(R/P). \tag{5.18}$$

This may be rewritten as

$$\big[[T_1(x),T_2(y)],r\big] \in P \text{ for all } r,x,y \in I. \tag{5.19}$$

Analogously, replacing yt for y, where $t \in R$ in (5.19), and by appropriate expansion, get

$$[T_1(x),T_2(y)][t,r]+T_2(y)\big[[T_1(x),t],r\big]+[T_2(y),r][T_1(x),t] \in P \tag{5.20}$$

Letting $t = T_1(x)$ in (5.20), one can see that

$$[T_1(x),T_2(y)][T_1(x),r] \in P \text{ for all } r,x,y \in I.$$

Keeping in mind that $\overline{[T_1(x),T_2(y)]} \in Z(R/P)$, we get

$$[T_1(x),T_2(y)]I[T_1(x),r] \subseteq P \text{ for all } r,x,y \in I. \tag{5.21}$$

In light of the primeness of P, we find that either $[T_1(x),T_2(y)] \in P$ or $[T_1(x),r] \in P$ for all $x \in I$. Consequently, I is a union of two additive subgroups I_1 and I_2, where

$$I_1 = \{x \in I \,/\, [T_1(x),T_2(y)] \in P \text{ for all } y \in I\} \quad \text{and} \quad I_2 = \{x \in I \,/\, [T_1(x),I] \subseteq P\}$$

According to Brauer's trick, we are forced to conclude that either $I = I_1$ or $I = I_2$.
If $I = I_1$, i.e. $[T_1(x),T_2(y)] \in P$ for all $x,y \in I$, then replacing y by ys, one obtains

$$T_2(y)[T_1(x),s] \in P \text{ for all } s,x,y \in I. \tag{5.22}$$

Substituting yu for y in (5.22), we obviously get

$$T_2(y)u[T_1(x),s] \in P \text{ for all } s,u,x,y \in I.$$

So again an appeal to Fact 1, gives either $T_2(R) \subseteq P$ or $[T_1(x),s] \in P$ for all $x,s \in I$.
Now if $I = I_2$, that is $[T_1(x),r] \in P$ for all $x,r \in I$, then putting xw instead of x, we obtain

$$T_1(x)[z,r] \in P \text{ for all } x,y,z \in I. \tag{5.23}$$

Writing xw for x in (5.23), we get

$$T_1(x)w[z,r] \in P \text{ for all } r,w,x,z \in I.$$

Accordingly, it follows that $T_1(R) \subseteq P$ or R/P is a commutative integral domain.
$(2) \Longrightarrow (3)$ Can be proved by using the same steps as we did before. ■

Corollary 5.2.10 *Let R be a prime ring and I a nonzero ideal of R. If T_1 and T_2 are two nonzero left centralizers on R, then the following assertions are equivalent:*
(1) $[T_1(x), T_2(y)] \in Z(R)$ *for all* $x,y \in I$;
(2) $T_1(x) \circ T_2(y) \in Z(R)$ *for all* $x,y \in I$;
(3) *R is commutative integral domain.*

Using similar arguments as above with necessary variation, we can prove the following theorem.

Theorem 5.2.11 *Let R be a ring, I a nonzero ideal of R and P a prime ideal of R such that $P \subsetneq I$. Suppose that T_1 and T_2 are two P-left centralizers on I of R, then the following assertions are equivalent:*
(1) $\overline{T_1(x)T_2(y) - [x,y]} \in Z(R/P)$ *for all* $x,y \in I$;
(2) $\overline{T_1(x)T_2(y) - x \circ y} \in Z(R/P)$ *for all* $x,y \in I$;
(3) *R/P is a commutative integral domain.*

Let R be a prime ring. Letting $P = (0)$ in the previous theorem, we deduce that, if $T_1(x)T_2(y) - [x,y] \in Z(R)$ or $T_1(x)T_2(y) - x \circ y \in Z(R)$ for all $x,y \in I$, then R is commutative. The following corollary shows that the same conclusion remains satisfied for semi-prime rings.

Corollary 5.2.12 *Let R be a semi-prime ring and I a nonzero ideal of R. Suppose that T_1 and T_2 are two left centralizers on R, then the following assertions are equivalent:*
(1) $T_1(x)T_2(y) \pm [x,y] \in Z(R)$ *for all* $x,y \in I$;
(2) $T_1(x)T_2(y) \pm x \circ y \in Z(R)$ *for all* $x,y \in I$;
(3) *R is commutative.*

Proof. We have only to prove $(1) \Longrightarrow (3)$, while the implication $(2) \Longrightarrow (3)$ can be proved similarly. The ring R is semi-prime, then there exists a family $\mathscr{P}$ of prime ideals

such that $\bigcap_{P\in\mathscr{P}} P = (0)$. Then we may suppose existence of a two left centralizers T_1 and T_2 satisfying $T_1(x)T_2(y) \pm [x,y] \in Z(R)$ for all $x,y \in I$. We thereby get $[T_1(x)T_2(y) \pm [x,y], r] = 0 \in \bigcap_{P\in\mathscr{P}} P$ for all $r,x,y \in I$, therefore, Theorem 5.2.11 yields that for all $P \in \mathscr{P}$, R/P is commutative combining this fact with $R \hookrightarrow \prod_{P\in\mathscr{P}} R/P$, it follows that R is commutative. ■

CHAPTER 6

LIE IDEALS AND GENERALIZED DERIVATIONS IN PRIME RINGS AND BANACH ALGEBRAS

K. Bouchannafa, A. Hermas and L. Oukhtite, *Lie ideals and generalized derivations in prime rings and Banach algebras*. Submitted for publication.

Abstract.

Let R be a prime ring and L a non-central Lie ideal of R. The main purpose of this chapter is to describe generalized derivations of R satisfying some algebraic identities on L. Moreover, our results have been extended to Banach algebras.

6.1 Introduction

Numerous results in literature shows how the global structure of a ring R is often tightly linked to the behavior of some special additive mappings defined on R. A popular result in this area is due to Posner [61] who proved that a prime ring equipped with a non-zero centralizing derivation is a commutative integral domain. This remarkable theorem of Posner has been influential and it have played a key role in the development of various notions. This result was subsequently refined and extended by a number of algebraists. More specifically, they studied the commutativity of rings admitting suitably constrained

generalized derivation verifying specific identities.

In [42, Theorem 2.7] it is demonstrated that if R is a prime ring of characteristic different from two, admitting two generalized derivations F_1 and F_2 such that $F_1(x)F_2(x) + F_2(x)F_1(x) = 0$ for all $x \in R$ then $F_1 = 0$ or $F_2 = 0$. An interesting result proved in [48, Theorem 2] by Hvala, it states that if F_1 and F_2 are two generalized derivations on a prime ring R of characteristic different from two, verifying $[F_1(x), F_2(x)] = 0$ for all $x \in R$, then there exists $\lambda \in C$ such that $F_1(x) = \lambda F_2(x)$ for all $x \in R$. Later Demir et al. in [38] managed to get the same classification by only considering the main identity on a non-central Lie ideal of a prime ring R, except possibly when R satisfy the standard identity s_4 of degree 4.

When it comes to extending various results on prime Banach algebras. The authors in [13, Theorem 3.1] showed that if $\mathscr{A}$ is a unital prime Banach algebra, F a non-zero continuous generalized derivation with associated derivation d and G_1, G_2 two nonvoid open subsets of $\mathscr{A}$ satisfying $F((xy)^m) - x^m y^m \in Z(\mathscr{A})$ or $F((xy)^m) - y^m x^m \in Z(\mathscr{A})$ for all $(x,y) \in G_1 \times G_2$ and $m = m(x,y) > 1$, then $\mathscr{A}$ is commutative under the additional assumption that $d(Z(\mathscr{A})) \neq 0$.

Motivated by the previous results, we here continue this line of investigation by describing generalized derivations satisfying some specific identities on a non-central Lie ideal of a prime ring. Moreover, as an application, we study continuous generalized derivations satisfying similar algebraic identities locally on nonvoid open subsets of a prime Banach algebra $\mathscr{A}$. Our topological approach is based on Baire's category theorem and some properties of functional analysis.

6.2 Generalized derivations acting on Lie ideals

We permit the following well known facts:

Fact 3 *[32, Main Theorem] Let R be a prime ring of characteristic different from 2, L a non-central Lie ideal of R and F a generalized derivation of R such that $F(x) \in Z(R)$ for any $x \in L$. Then either $F = 0$ or R embeds in a 2×2 matrix ring over a field.*

Fact 4 *Let R be a non-commutative prime ring of characteristic different from 2 and F a generalized derivation of R such that $F(x) \in Z(R)$ for any $x \in R$. Then $F = 0$.*

Fact 5 *[24, Lemma 2] Let R be a prime ring of characteristic different from 2, L a Lie ideal of R and $C_R(L) = \{a \in R : [a,x] = 0 \ \forall x \in L\}$. If L is not central then $C_R(L) = Z(R)$.*

Lemma 6.2.1 *Let R be a prime ring of characteristic different from 2, L a non-central Lie ideal of R and F, G generalized derivations of R such that $F(x)y + yG(x) = 0$ for all $x, y \in L$. Then either $F = G = 0$ or R embeds in a 2×2 matrix ring over a field.*

Proof. Suppose that R doesn't embed in a 2×2 matrix ring over a field and

$$F(x)y + yG(x) = 0 \text{ for all } x, y \in L. \tag{6.1}$$

By [24, Lemma 1] there exists a non-zero ideal I of R such that $[I,R] \subseteq L$. Replacing y by $[u,r]$ in equation (6.1), with $u \in I, r \in R$

$$F(x)[u,r] + [u,r]G(x) = 0 \text{ for all } u \in I, x \in L, r \in R. \tag{6.2}$$

Substituting u by ur, we get

$$F(x)[u,r]r + [u,r]rG(x) = 0 \text{ for all } u \in I, x \in L, r \in R. \tag{6.3}$$

Right multiplying equation (6.2) by r and subtracting it from equation (6.3), we get $[u,r][G(x),r] = 0$ for all $u \in I, x \in L, r \in R$. Taking ut instead of u, with $t \in I$, we get

$$[u,r]I[G(x),r] = 0 \text{ for all } u \in I, x \in L, r \in R.$$

The primeness of R yields $[G(x),r] = 0$ for all $x \in L$, $r \in R$. That is $G(L) \subseteq Z(R)$, it follows from Fact 3 that $G = 0$ which forces equation (6.1) to yield $F = 0$. ■

Theorem 6.2.2 *Let R be a prime ring of characteristic different from 2, L a non-central Lie ideal of R, F_1, F_2 and F_3 be generalized derivations of R. If*

$$[F_1(x), y] = F_2(x)y + yF_3(x) \quad \text{for all } x, y \in L, \tag{6.4}$$

then one of the following holds:
(1) *there exist $\lambda, \mu \in C$ such that $F_2(x) = \lambda F_1(x)$, $F_3(x) = \mu F_1(x)$ for any $x \in R$;*
(2) *R embeds in a 2×2 matrix ring over a field.*

Proof. Assume that R does not embed in a 2×2 matrix ring over a field.
Firstly we point out that, if $F_1(L)\subseteq Z(R)$, then Fact 3 implies $F_1=0$ and relation (6.4) reduces to

$$F_2(x)y+yF_3(x)=0 \text{ for all } x,y\in L.$$

Thus, by Lemma 6.2.1, $F_2=F_3=0$ follows.
Hence, in the sequel we assume that there exists $u_0\in L$ such that $F_1(u_0)\notin Z(R)$ and denote $a=F_1(u_0)$, $b=F_2(u_0)$, $c=F_3(u_0)$. In this sense, relation (6.4) yields

$$[a,y]=by+yc \text{ for all } y\in L. \tag{6.5}$$

In particular,

$$[a,[l,r]]=b[l,r]+[l,r]c \text{ for all } l\in L, r\in R. \tag{6.6}$$

Substituting r by rl, we get

$$[a,[l,r]]l+[l,r][a,l]=b[l,r]l+[l,r]lc \text{ for all } l\in L, r\in R. \tag{6.7}$$

Right multiplying equation (6.6) by l and subtracting it from equation (6.7), we get
$[l,r][a,l]=[l,r][l,c]$ for all $l\in L, r\in R$. That is

$$[l,r][a+c,l]=0 \text{ for all } l\in L, r\in R. \tag{6.8}$$

It follows that $a+c\in Z(R)$. In particular, we have $[F_1(u_0),F_3(u_0)]=0$.
Furthermore, setting lr instead of r in equation (6.6), we get

$$[a,l][l,r]+l[a,[l,r]]=bl[l,r]+l[l,r]c \text{ for all } l\in L, r\in R. \tag{6.9}$$

Left multiplying equation (6.6) by l and subtracting it from equation (6.9), we get
$[a-b,l][l,r]=0$ for all $l\in L, r\in R$.
It follows that $a-b\in Z(R)$. In particular, we have $[F_1(u_0),F_2(u_0)]=0$.
Generally, in all cases we have $[F_1(u),F_2(u)]=0$ and $[F_1(u),F_3(u)]=0$ for any $u\in L$.
Using ([38], Theorem 1.1), there exist $\lambda,\mu\in C$ such that $F_2(x)=\lambda F_1(x)$, $F_3(x)=\mu F_1(x)$ for any $x\in R$. ■

Remark 6.2.3 *In case* $[F_1(x),y]=F_2(x)y+yF_3(x)$ for all $x,y\in R$, *one might expect R to*

be commutative. Instead this would be false as the following example proves:
For K a field, $R = M_2(K)$ and F a generalized derivation of R. It is obvious that $F_1 = F_2 = F$ and $F_3 = -F$ verify the above relation although the ring is not commutative.

Theorem 6.2.4 *Let R be a prime ring of characteristic different from 2, L a non-central Lie ideal of R, F_1, F_2 and F_3 be generalized derivations of R. If*

$$F_1(x) \circ y = F_2(x)y + yF_3(x) \quad \text{for all } x, y \in L, \tag{6.10}$$

then one of the following holds:
(1) *there exist $\lambda, \mu \in C$ such that $F_2(x) = \lambda F_1(x)$, $F_3(x) = \mu F_1(x)$ for any $x \in R$;*
(2) *R embeds in a 2×2 matrix ring over a field.*

Proof. Assume that R does not embed in a 2×2 matrix ring over a field.
If $F_1(L) \subseteq Z(R)$, then Fact 3 forces $F_1 = 0$ and (6.10) becomes

$$F_2(x)y + yF_3(x) = 0 \text{ for all } x, y \in L.$$

Thus, by Lemma 6.2.1, $F_2 = F_3 = 0$ follows.
Hence, in what comes next, we assume that there exists $u_0 \in L$ such that $F_1(u_0) \notin Z(R)$ and denote $a = F_1(u_0)$, $b = F_2(u_0)$, $c = F_3(u_0)$. As a matter of fact, relation (6.10) implies

$$a \circ y = by + yc \text{ for all } y \in L. \tag{6.11}$$

In particular, one can see that

$$a \circ [l, r] = b[l, r] + [l, r]c \text{ for all } l \in L, r \in R. \tag{6.12}$$

Replacing r by rl, we get

$$(a \circ [l, r])l - [l, r][a, l] = b[l, r]l + [l, r]lc \text{ for all } l \in L, r \in R. \tag{6.13}$$

Right multiplying equation (6.12) by l and subtracting it from equation (6.13), we get $[l, r][a, l] = [l, r][l, c]$ for all $l \in L, r \in R$. That is

$$[l, r][a + c, l] = 0 \text{ for all } l \in L, r \in R. \tag{6.14}$$

It follows that $a+c \in Z(R)$. Particularly, we have $[F_1(u_0), F_3(u_0)] = 0$.
On the other hand, taking lr instead of r in equation (6.12), we get

$$l(a \circ [l,r]) + [a,l][l,r] = bl[l,r] + l[l,r]c \text{ for all } l \in L, r \in R. \tag{6.15}$$

Left multiplying equation (6.12) by l and subtracting it from equation (6.15), we obtain $[a-b,l][l,r] = 0$ for all $l \in L, r \in R$.
It follows that $a-b \in Z(R)$. In particular, we have $[F_1(u_0), F_2(u_0)] = 0$.
Generally, in all cases we get $[F_1(u), F_2(u)] = 0$ and $[F_1(u), F_3(u)] = 0$ for any $u \in L$. Using ([38], Theorem 1.1), there exist $\lambda, \mu \in C$ such that $F_2(x) = \lambda F_1(x)$, $F_3(x) = \mu F_1(x)$ for any $x \in R$. ■

6.3 Applications on prime Banach Algebras

Throughout this section, $\mathscr{A}$ denotes a real or complex Banach algebra. To prove our main results we need the following Lemma.

Lemma 6.3.1 [26] *Let $\mathscr{A}$ be a Banach algebra, if $P(t) = \sum_{k=0}^{n} b_k t^k$ is a polynomial in the real variable t with coefficients in $\mathscr{A}$, and if for an infinite set of real values of t, $P(t) \in M$, where M is a closed linear subspace of $\mathscr{A}$, then every b_k lies in M.*

Theorem 6.3.2 *Let $\mathscr{A}$ be a prime Banach algebra, O_1, O_2 nonvoid open subsets on $\mathscr{A}$, F_1, F_2, F_3 non-zero continuous generalized derivations of $\mathscr{A}$ and n a fixed positive integer. If*

$$[F_1(x), y]^n = F_2(x)y + yF_3(x) \text{ for all } (x,y) \in O_1 \times O_2,$$

then $\mathscr{A}$ is commutative.

Proof. Assume that

$$[F_1(x), y]^n - F_2(x)y - yF_3(x) = 0 \text{ for all } (x,y) \in O_1 \times O_2. \tag{6.16}$$

Let $u \in \mathscr{A}$ and $x \in O_1$, then $x + tu \in O_1$ for a sufficiently small real t.
F_1, F_2, F_3 being continuous, one can obviously see that $F_i(ru) = rF_i(u)$ for all $u \in A$,

$r \in \mathbb{R}$, $i \in \{1,2,3\}$. Replacing x by $x+tu$ in equation (6.16), we get

$$\big([F_1(x),y]+[F_1(u),y]t\big)^n - \big(F_2(x)y+yF_3(x)+(F_2(u)y+yF_3(u))t\big) = 0. \tag{6.17}$$

Let $P_{n,m}(u,x,y)$ denotes the sum of all monic monomials with n occurrences of $[F_1(x),y]$ and m occurrences of $[F_1(u),y]$. It follows from equation (6.17) that

$$Q(t) = \sum_{k=0}^{n} P_{n-k,k}(u,x,y)t^k - \big(F_2(x)y+yF_3(x)+(F_2(u)y+yF_3(u))t\big) = 0.$$

Setting $Q(t) = \sum\limits_{k=0}^{n} q_k(u,x,y)t^k$; with $q_0(u,x,y) = [F_1(x),y]^n - F_2(x)y - yF_3(x)$, $q_1(u,x,y) = P_{n-1,1}(u,x,y) - F_2(u)y - yF_3(u)$ and $q_k(u,x,y) = P_{n-k,k}(u,x,y)$ for all $k \in \{2,...,n\}$. As (0) is a closed linear subspace of $\mathscr{A}$, then Lemma 6.3.1 yields $q_k(u,x,y) = 0$ for all $k \in \{0,...,n\}$. In particular $q_n(u,x,y) = 0$, thus

$$[F_1(u),y]^n = 0 \text{ for all } (u,y) \in \mathscr{A} \times O_2.$$

Similarly, one can show that

$$[F_1(u),v]^n = 0 \text{ for all } u,v \in \mathscr{A}. \tag{6.18}$$

In view of (6.18), equation (6.16) reduces to $F_2(x)y+yF_3(x) = 0$ for all $(x,y) \in O_1 \times O_2$. Using the same techniques as above, we find that

$$F_2(x)y+yF_3(x) = 0 \text{ for all } x,y \in \mathscr{A}. \tag{6.19}$$

Thus

$$[F_2(x),r]y = 0 \text{ and } y[r,F_3(x)] = 0 \text{ for all } r,x,y \in \mathscr{A}.$$

That is $F_2(x) \in Z(R)$ and $F_3(x) \in Z(R)$ for all $x \in \mathscr{A}$. F_2, F_3 being non-zero it follows from Fact 4 that $\mathscr{A}$ is commutative. ■

A similar proof with minor adjustment is sufficient to prove the following theorem.

Theorem 6.3.3 *Let $\mathscr{A}$ be a prime Banach algebra, O_1, O_2 nonvoid open subsets on $\mathscr{A}$,*

F_1, F_2, F_3 non-zero continuous generalized derivations of $\mathscr{A}$ and n a fixed positive integer. If

$$(F_1(x) \circ y)^n = F_2(x)y + yF_3(x) \text{ for all } (x,y) \in O_1 \times O_2,$$

then $\mathscr{A}$ is commutative.

Theorem 6.3.4 *Let $\mathscr{A}$ be a non-commutative prime Banach algebra, O_1, O_2 nonvoid open subsets on $\mathscr{A}$, F_1, F_2 and F_3 continuous generalized derivations of $\mathscr{A}$. If*

$$[F_1(x^r), y^s] = F_2(x^r)y^s + y^s F_3(x^r) \text{ for all } (x,y) \in O_1 \times O_2,$$

where r,s are non-zero integers depending on the pair of elements x and y, then one of the following holds:
(1) *there exist $\lambda, \mu \in C$ such that $F_2(x) = \lambda F_1(x)$, $F_3(x) = \mu F_1(x)$ for any $x \in \mathscr{A}$;*
(2) *$\mathscr{A}$ embeds in a 2×2 matrix ring over a field.*

Proof. Let us fix $x \in O_1$ and set

$$K_{r,s} = \{y \in \mathscr{A} \mid [F_1(x^r), y^s] - F_2(x^r)y^s - y^s F_3(x^r) \neq 0\}.$$

We claim that each $K_{r,s}$ is open in $\mathscr{A}$ or equivalently its complement $K_{r,s}^c$ is closed. For this, we consider a sequence $(y_k)_{k>1} \subset K_{r,s}^c$ converging to y and prove that $y \in K_{r,s}^c$.
As $(y_k)_{k>1} \subset K_{r,s}^c$ then $[F_1(x^r), y_k^s] - F_2(x^r)y_k^s + y_k^s F_3(x^r) = 0$ for all $k > 1$.
Hence

$$\begin{aligned} \lim_{k\to\infty} [F_1(x^r), y_k^s] - F_2(x^r)y_k^s - y_k^s F_3(x^r) &= [F_1(x^r), (\lim_{k\to\infty} y_k)^s] - F_2(x^r)(\lim_{k\to\infty} y_k)^s - (\lim_{k\to\infty} y_k)^s F_3(x^r) \\ &= [F_1(x^r), y^s] - F_2(x^r)y^s - y^s F_3(x^r) \\ &= 0 \end{aligned}$$

Therefore $y \in K_{r,s}^c$, thus $K_{r,s}$ is open. Suppose now that all the $K_{r,s}$ are dense in $\mathscr{A}$ then the intersection of the $K_{r,s}$ is also dense by Baire category theorem, a contradiction with the fact that $O_2 \neq \emptyset$. Hence there exist some positive integers p,q depending on x, such that $K_{p,q}$ is not dense. Accordingly, there exists a nonvoid open subset O_3 in $K_{p,q}^c$. Therefore

$$[F_1(x^p), y^q] - F_2(x^p)y^q - y^q F_3(x^p) = 0 \text{ for all } y \in O_3. \tag{6.20}$$

Let's consider $z \in O_3$ and $v \in \mathscr{A}$, $z+tv \in O_3$ for all sufficiently small real t.
Replacing y by $z+tv$ in (6.20), we obtain

$$[F_1(x^p),(z+tv)^q] - F_2(x^p)(z+tv)^q - (z+tv)^q F_3(x^p) = 0. \tag{6.21}$$

Let $P_{i,j}(x,u)$ denotes the sum of all monic monomials with i occurrences of x and j occurrences of u.
As $(z+tv)^q = P_{q,0}(z,v) + P_{q-1,1}(z,v)t + ... + P_{1,q-1}(z,v)t^{q-1} + P_{0,q}(z,v)t^q$ it follows from equation (6.21) that

$$[F_1(x^p), \sum_{i=0}^{q} P_{q-i,i}(z,v)t^i] - F_2(x^p)(\sum_{i=0}^{q} P_{q-i,i}(z,v)t^i) - (\sum_{i=0}^{q} P_{q-i,i}(z,v)t^i)F_3(x^p) = 0. \tag{6.22}$$

It follows from (6.22) that

$$\sum_{i=0}^{q} \Big([F_1(x^p), P_{q-i,i}(z,v)] - F_2(x^p)P_{q-i,i}(z,v) - P_{q-i,i}(z,v)F_3(x^p) \Big) t^i = 0.$$

Thus $Q(t) = \sum_{i=0}^{q} a_i(v,x,z)t^i = 0$; with $a_i(v,x,z) = [F_1(x^p), P_{q-i,i}(z,v)] - F_2(x^p)P_{q-i,i}(z,v) - P_{q-i,i}(z,v)F_3(x^p)$. Using Lemma 6.3.1, we get $a_i(v,x,z) = 0$ for all $i \in \{0,...,q\}$. In particular $a_q(v,x,z) = 0$, that is $[F_1(x^p), v^q] - F_2(x^p)v^q - v^q F_3(x^p) = 0$. In conclusion, we have proved that for a given $x \in O_1$, there exist some positive integers p and q depending on x, such that

$$[F_1(x^p), v^q] - F_2(x^p)v^q - v^q F_3(x^p) = 0 \text{ for all } v \in \mathscr{A}.$$

Let us fix $v \in \mathscr{A}$. Using a similar approach, we arrive at

$$[F_1(u^p), v^q] = F_2(u^p)v^q + v^q F_3(u^p) \text{ for all } u, v \in \mathscr{A}.$$

Now let's consider H_1 and H_2 the additive subgroups generated by $\{a^p \mid a \in \mathscr{A}\}$ and $\{a^q \mid a \in \mathscr{A}\}$ respectively, it follows that

$$[F_1(x), y] = F_2(x)y + yF_3(x) \text{ for all } (x,y) \in H_1 \times H_2. \tag{6.23}$$

Equation (6.23) along with [34] yield that either $a^p \in Z(\mathscr{A})$ for all $a \in \mathscr{A}$ or H_1 contains

a non-central Lie ideal J_1. If $a^p \in Z(\mathscr{A})$ for all $a \in \mathscr{A}$, then $\mathscr{A}$ is commutative by [62]. A contradiction.

Now suppose that H_1 contains a non-central Lie ideal J_1, similarly H_2 contains also another non-central Lie ideal J_2. Let $I_k = \{x \in \mathscr{A} \mid [x, \mathscr{A}] \subset J_k\}$ with $k = 1, 2$. It follows from ([44], Lemma 1.4) that I_1, I_2 are both subrings and Lie ideals of $\mathscr{A}$. Therefore equation (6.23) becomes

$$[F_1(x), y] = F_2(x)y + yF_3(x) \text{ for all } (x, y) \in [I_1, \mathscr{A}] \times [I_2, \mathscr{A}]. \tag{6.24}$$

As $[I_1, \mathscr{A}]$ and $[I_2, \mathscr{A}]$ are dense submodules of $[\mathscr{A}, \mathscr{A}]$ then by ([52], Theorem 2), $[\mathscr{A}, \mathscr{A}]$ satisfy the same identity as $[I_1, \mathscr{A}]$ and $[I_2, \mathscr{A}]$, thus equation (6.24) becomes

$$[F_1(x), y] - F_2(x)y - yF_3(x) = 0 \text{ for all } x, y \in [\mathscr{A}, \mathscr{A}]. \tag{6.25}$$

Since $[\mathscr{A}, \mathscr{A}]$ is a non-central Lie ideal, applying Theorem 6.2.2 we get the required result. ■

Using the same above arguments, with suitable modification, application of Theorem 6.2.4 yields the following result.

Theorem 6.3.5 *Let $\mathscr{A}$ be a non-commutative prime Banach algebra, O_1, O_2 nonvoid open subsets on $\mathscr{A}$, F_1, F_2 and F_3 continuous generalized derivations of $\mathscr{A}$. If*

$$F_1(x^r) \circ y^s = F_2(x^r)y^s + y^s F_3(x^r) \text{ for all } (x, y) \in O_1 \times O_2,$$

where r,s are non-zero integers depending on the pair of elements x and y, then one of the following holds:
(1) *there exist $\lambda, \mu \in C$ such that $F_2(x) = \lambda F_1(x)$, $F_3(x) = \mu F_1(x)$ for any $x \in \mathscr{A}$;*
(2) *$\mathscr{A}$ embeds in a 2×2 matrix ring over a field.*

The following example shows that the primeness hypothesis in Theorems 6.2.2 and 6.2.4 is not superfluous.

Example. Let's consider the ring $\mathscr{R} = M_2(\mathbb{R}) \times \mathbb{R}$ with operations coordinatewise addition and multiplication. It is obvious that $\mathscr{R}$ is a non prime ring.

Consider the generalized derivation

$$F_\alpha((A,a)) = (\alpha A + A\alpha, 0) \text{ with } \alpha \in [M_2(\mathbb{R}), M_2(\mathbb{R})]$$

with associated derivation d_α defined by $d_\alpha((A,a)) = (A\alpha - \alpha A, 0)$.
Setting $L = [M_2(\mathbb{R}), M_2(\mathbb{R})] \times \mathbb{R}$ along with $F_1 = 0$; $F_2 = F_\alpha$ and $F_3 = -F_\alpha$. A simple computation shows that

$$[F_1((A,a)),(B,b)] = F_1((A,a)) \circ (B,b) = F_2((A,a))(B,b) + (B,b)F_3((A,a)) = 0$$

for all $(A,a),(B,b) \in L$. However, none of the assertions of Theorems 6.2.2 and 6.2.4 is satisfied.

Bibliography

[1] A. Abbasi, C. Abdioglu, S. Ali and M. R. Muzibur, *A characterization of Jordan left $*$-centralizers via skew Lie and Jordan products*. Bull. Iranian Math. Soc. 48 (2022), no. 5, 2765-2778.

[2] E. Albaş, *Generalized derivations on ideals of prime rings*. Miskolc Math. Notes 14 (2013), no. 1, 3-9.

[3] S. Ali and N. A. Dar, *On centralizers of prime rings with involution*. Bull. Iranian Math. Soc. 41 (2015), no. 6, 1465-1475.

[4] S. Ali and S. Huang, *On derivations in semiprime rings*. Algebr. Represent. Theory 15 (2012), no. 6, 1023-1033.

[5] F. A. A. Almahdi, A. Mamouni and M. Tamekkante. *A Generalization of Posner's Theorem on Derivations in Rings*. Indian J. Pure Appl. Math. 51 (2020), no. 1, 187-194.

[6] N. Argaç, *On prime and semiprime rings with derivations*. Algebra Colloq. 13 (2006), no. 3, 371-380.

[7] M. Ashraf and S. Ali, *On left multipliers and the commutativity of prime rings*. Demonstratio Math. 41 (2008), no. 4, 763-771.

[8] M. Ashraf, A. Ali and S. Ali, *Some commutativity theorems for rings with generalized derivations*. Sud-Est. Math. 31 (2007), no. 3, 415-421.

[9] M. Ashraf and A. Khan, *Commutativity of $*$-prime rings with generalized derivations*. Rend. Semin. Mat. Univ. Padova 125 (2011), 71-79.

[10] M. Ashraf and N. Rehman, *On derivation and commutativity in prime rings*. East-West J. Math. 3 (2001), no. 1, 87-91.

[11] M. Ashraf and N. Rehman, *On commutativity of rings with derivations*. Results Math. 42 (2002), no. 1-2, 3-8.

[12] M. Ashraf, N. Rehman, A. Shakir and M. Rahman Mozumder, *On semiprime rings with generalized derivations*. Bol. Soc. Parana. Mat. (3) 28 (2010), no. 2, 25-32.

[13] M. Ashraf and B. A. Wani, *On commutativity of rings and Banach algebras with generalized derivations*. Adv. Pure. Appl. Math. 10 (2019), no. 2, 155-163.

[14] A. Asma, N. Rehman and S. Ali, *On Lie ideals with derivations as homomorphisms and anti-homomorphisms*. Acta Math. Hungar. 101 (2003), no. 1-2, 79-82.

[15] R. Awtar, *Lie and Jordan structures in prime rings with derivations*. Proc. Amer. Math. Soc. 41 (1973), 67-74.

[16] N. Aydin, *A note on generalized derivations of prime rings*. Int. J. Algebra 5 (2011), no. 1-4, 17-23.

[17] K. I. Beidar, W. S. Martindale III and A. V. Mikhalev, *Rings with generalized identities*. Marcel Dekker, Inc. New York. Basel. Hong Kong, (1996).

[18] H. E. Bell, *On the commutativity of prime rings with derivation*. Quaestiones Math. 22 (1999), 329-335.

[19] H. E. Bell and M. N. Daif, *On commutativity and strong commutativity-preserving maps*. Canad. Math. Bull. 37 (1994), no. 4, 443-447.

[20] H. E. Bell and M. N. Daif, *On derivations and commutativity in prime rings*. Acta Math. Hungar. 66 (1995), no. 4, 337-343.

[21] H. E. Bell and L. C. Kappe, *Rings in which derivations satisfy certain algebraic conditions*. Acta Math. Hungar. 53 (1989), no. 3-4, 339-346.

[22] H. E. Bell and W. S. Martindale III, *Centralizing mappings of semiprime rings*. Canad. Math. Taureau. 30 (1987), no. 1, 92-101.

[23] H. E. Bell and N. Rehman, *Generalized derivations with commutativity and anti-commutativity conditions*. Math. J. Okayama Univ. 49 (2007), 139-147.

[24] J. Bergen, I. N. Herstein and J. W. Keer, *Lie ideals and derivations of prime rings*. J. Algebra 71 (1981), no. 1, 259-267.

[25] J. Bergen, I. N. Herstein and C. Lanski, *Derivations with invertible values*. Canadian J. Math. 35 (1983), no. 2, 300-310.

[26] F. F. Bonsall and J. Duncan, *Complete normed algebras*. Springer-Verlag, New York, (1973).

[27] M. Brešar, *On the distance of the composition of two derivations to the generalized derivations*. Glasgow Math. J. 33 (1991), no. 1, 89-93.

[28] M. Brešar, *Centralizing mappings and derivations in prime rings*. J. Algebra 156 (1993), no. 2, 385-394.

[29] M. Brešar, *Introduction to Noncommutative Algebra*. University of Ljubljana and Maribor Slovenia, March (2014).

[30] M. Brešar and C. R. Miers, *Strong commutativity preserving maps of semiprime rings*. Canad. Math. Bull. 37 (1994), no. 4, 457-460.

[31] M. Brešar and J. Vukman, *On left derivations and related mappings*. Proc. Amer. Math. Soc. 110 (1990), 7-16.

[32] J. C. Chang, *Generalized Skew Derivations with Power Central Values on Lie Ideals*. Comm. Algebra 39 (2011), no. 6, 2241-2248.

[33] M. A. Chebotar, *On a composition of derivations of prime rings*. Moscow Univ. Math. Bull. 50 (1995), 22-25.

[34] C. L. Chuang, *The additive subgroup generated by a polynomial*. Israel J. Math. 59 (1987), no. 1, 98-106.

[35] J. C. Chuang and T. K. Lee, *Invariance of minimal prime ideals under derivations*. Proc. Amer. Math. Soc. 113 (1991), no. 3, 613-616.

[36] M. N. Daif, *Commutativity results for semiprime rings with derivations*. Internat. J. Math. Math. Sci. 21 (1998), no. 3, 471-474.

[37] M. N. Daif and H. E. Bell, *Remarks on derivations on semiprime rings*. Internat. J. Math. Math. Sci. 15 (1992), no. 1, 205-206.

[38] Ç. Demir, V. De Filippis and N. Argaç, *Quadratic differential identities with generalized derivations on Lie ideals*. Linear Multilinear Algebra 68 (2020), no. 9, 1835-1847.

[39] Q. Deng and M. Ashraf, *On strong commutativity preserving mappings*. Results Math. 30 (1996), no. 3-4, 259-263.

[40] N. J. Divinsky, *On commuting automorphisms of rings*. Trans. Roy. Soc. Canada Sect. III 49 (1955), 19-22.

[41] H. El Mir, A. Mamouni and L. Oukhtite, *Special mappings with central values on prime rings*. Algebra Colloq. 27 (2020), no. 3, 405-414.

[42] M. Fosner and J. Vukman, *Identities with generalized derivations in prime rings*. Mediterr. J. Math. 9 (2012), no. 4, 847-863.

[43] I. N. Herstein, *Rings with involution*. Chicago Lectures in Mathematics. University of Chicago Press, Chicago, Ill.-London, (1976).

[44] I. N. Herstein, *Topics in ring theory*. The University of Chicago Press, Chicago, Ill.-London (1969).

[45] I. N. Herstein, *A note on derivations*. Canad. Math. Bull. 21 (1978), no. 3, 369-370.

[46] M. Hongan, *A note on semiprime rings with derivation*. Internat. J. Math. & Math. Sci. 20 (1997), no. 2, 413-415.

[47] Sh. Huang, Ö. Gölbaşı and E. Koç, *On centralizing and strong commutativity preserving maps of semiprime rings*. Ukrainian Math. J. 67 (2015), no. 2, 323-331.

[48] B. Hvala, *Generalized derivations in rings*. Comm. Algebra 26 (1998), no. 4, 1147-1166.

[49] M. A. Idrissi and L. Oukhtite, *Structure of a quotient ring R/P with generalized derivations acting on the prime ideal P and some applications*. Indian J. Pure Appl. Math. 53 (2022), 792-800.

[50] M. A. Idrissi, L. Oukhtite and N. Muthana, *Center-like subsets in rings with derivations or endomorphisms*. Comm. Alg. 47 (2019), no. 9, 3794-3799.

[51] C. Lanski, *Differential identities, Lie ideals and Posner's theorems*. Pacific J. Math. 134 (1988), no. 2, 275-297.

[52] T. K. Lee, *Semiprime rings with differential identities*. Bull. Inst. Math. Acad. Sinica 20 (1992), no. 1, 27-38.

[53] T. K. Lee, *Generalized derivations of left faithful rings*. Comm. Algebra 27 (1999), no. 8, 4057-4073.

[54] J. Mayne, *Centralizing automorphisms of prime rings*. Canad. Math. Bull. 19 (1976), no. 1, 113-115.

[55] B. Nejjar, A. Kacha, A. Mamouni and L. Oukhtite, *Commutativity theorems in rings with involution*. Comm. Algebra 45 (2017), no. 2, 698-702.

[56] L. Oukhtite, *Left multipliers and Jordan ideals in rings with involution*. Afr. Diaspora J. Math. 11 (2011), no. 1, 24-28.

[57] L. Oukhtite, *Posner's second theorem for Jordan ideals in rings with involution*. Expo. Math. 29 (2011), no. 4, 415-419.

[58] L. Oukhtite, H. EL Mir and B. Nejjar, *Endomorphisms with central values on prime rings with involution*. Int. Electron. J. Algebra 28 (2020), 127-140.

[59] L. Oukhtite and A. Mamouni, *Generalized derivations centralizing on Jordan ideals of rings with involution*. Turkish J. Math. 38 (2014), no. 2, 225-232.

[60] L. Oukhtite, A. Mamouni and M. Ashraf, *Commutativity theorems for rings with differential identities on Jordan ideals*. Comment. Math. Univ. Carolin. 54 (2013), no. 4, 447-457.

[61] E. C. Posner, *Derivations in prime rings*. Proc. Amer. Math. Soc. 8 (1957), 1093-1100.

[62] M. A. Quadri, M. Khan and M. Ashraf, *Some elementary commutativity theorems for rings*. Math. Student 56 (1988), no. 1-4, 223-226.

[63] M. A. Quadri, M. S. Khan and N. Rehman, *Generalized derivations and commutativity of prime rings*. Indian J. Pure Appl. Math. 34 (2003), no. 9, 1393-1396.

[64] N. Rehman, *On generalized derivations as homomorphisms and anti-homomorphisms*. Glas. Mat. Ser. III 39 (59) (2004), no. 1, 27-30.

[65] M. S. Samman, *On strong commutativity-preserving maps*. Int. J. Math. Math. Sci. (2005), no. 6, 917-923.

[66] A. M. Sinclair, *Continuous derivations on Banach algebras*. Proc. Amer. Math. Soc. 20 (1969), 166-170.

[67] I. M. Singer and J. Wermer, *Derivations on commutative normed algebras*. Math. Ann. 129 (1955), 260-264.

[68] S.K. Tiwari, R.K. Sharma and B. Dhara, *Identities related to generalized derivation on ideal in prime rings*. Beitr. Algebra Geom. 57 (2016), no. 4, 809-821.