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Modeling tensor networks and Clifford structures in physics

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Dedication

I dedicate my dissertation work to my family and my friends. A special feeling of gratitude to my loving parents, **Abdellah** and **Malika LAFFIGH** whose words of encouragement and push for tenacity ring in my ears. My sisters **Najat**, **Khadija** and **Naima**, and my brother **Mohammed** have never left my side and they are so special.

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Abstract

Using the toroidal compactification of string theory on n-dimensional tori $\mathbb{T}^n$, we investigate dyonic objects in arbitrary dimensions. First, we present a class of dyonic black solutions formed by two different D-branes using a correspondence between toroidal cycles and objects possessing both magnetic and electric charges, belonging to $U(1)_e^{2^{n-1}} \times U(1)_m^{2^{n-1}}$ dyonic gauge symmetry. This symmetry could be associated with electrically charged magnetic monopole solutions in stringy model buildings of the standard model (SM) extensions. Then, we consider in some detail such black hole classes obtained from even-dimensional toroidal compactifications, and we find that they are linked to $Cl(n)$ Clifford algebras using the vee product. It is believed that this analysis could be extended to dyonic objects which can be obtained from local Calabi–Yau manifold compactifications.

On the other hand, motivated by particle physics results, we investigate certain dyonic solutions in arbitrary dimensions. Concretely, we study the stringy constructions of such objects from concrete compactifications. Then, we elaborate their tensor network realizations using multistate particle formalism.

Keywords: Superstring theory, toroidal compactification, standard model, Clifford algebras, dyonic black solutions, string theory, tensor network.

Résumé

En utilisant la compactification toroïdale de la théorie des cordes sur des tores à n-dimensions $\mathbb{T}^n$, nous étudions des objets dyoniques dans des dimensions arbitraires. Tout d'abord, nous présentons une classe de solutions noires dyoniques formées de deux D-branes différentes, en exploitant une correspondance entre des cycles toroïdaux et des objets possédant à la fois des charges magnétiques et électriques, avec la symétrie de jauge $U(1)_e^{2^{n-1}} \times U(1)_m^{2^{n-1}}$. Cette symétrie de jauge pourrait être injectée dans le modèle standard étendu des particules. En examinant en détail les objets noirs, nous établissons un lien entre ces objets et les algèbres de Clifford $Cl(n)$. Cette analyse pourrait être étendue aux objets dyoniques qui peuvent être obtenus à partir des compactifications sur variétés locales de Calabi-Yau.

Motivés par les résultats de la physique des particules, nous réalisons certaines représentations graphiques des solutions dyoniques dans des dimensions arbitraires. En utilisant la théorie des cordes, nous exposons certaines constructions branaires à partir des compactifications concrètes sur des espaces projectifs. Finalement, nous élaborons leurs réalisations en termes de réseaux de tenseurs en exploitant un formalisme associé aux particules multi-états.

Mots-clefs: Théorie des supercordes, compactification toroïdale, modèle standard, algèbres de Clifford, solutions noires dyoniques, réseau de tenseurs.

Résumé générale

Généralemment cette thèse porte sur les objets dyonics dérivés des théories de cordes et leurs différentes applications en théorie de l'information et physique des particules. Ces objets peuvent être obtenus par compactification toroidique des théories des cordes en des dimensions supérieurs. Les représentation en termes de réseaux de tenseurs est aussi abordée et est appliquée pour la représentation des états multipartites, très utiles en théorie de l'information quantique. Les réalisations basées sur les algèbres de Clifford ont aussi été abordées dans cette thèse.

Plus précisemment, cette thèse est consacrée à l'étude de l'aspect physico-mathématique des objets dyoniques prevenant de la théorie des cordes. Ce travail, se divise en deux parties.

Dans la première partie, on propose quatre chapitres. Après une introduction présentant le problématique de la thèse, le chapitre 1 est consacré à l'étude de la théorie des graphes. Dans le deuxième chapitre, il expose les algèbres de Clifford en présentant la classification graphique. Les réseaux de tenseurs sont discutés dans le troisième chapitre. Le quatrième chapitre contient une présentation détaillée sur les objets dyoniques noirs en théorie des cordes. Ce chapitre traite également leurs propriétés physiques dans un espace de dimension arbitraire.

Dans la deuxième partie, on présente les résultats obtenus dans ce travail qui sont récemment publiés dans des journaux internationaux spécialisés. Ce travail entre dans le détail des modélisations des objets dyoniques de façon approfondie en exploitant un formalise physico-mathématique sophistiqué utilisé en physique des hautes énergies. En effet, le but principal de cette thèse concerne l'étude de certains aspects physiques et mathématiques des tels objets en relation avec les propriétés des algèbres de Clifford et

les tenseurs.

L'importance de ce travail apparait au niveau des comportements physiques et mathématiques des objets dyoniques promettant d'énormes révolutions futures dans le domaine de la physique théorique et hautes énergies.

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Introduction

Tensors are a mathematical concept that encapsulates and generalizes the idea of multilinear maps between vector spaces, functions of multiple vector parameters that are linear with respect to every parameter. A tensor network is simply a countable collection of tensors connected by contractions. Tensor network methods is the term sometimes given to the entire collection of associated tools, which are regularly employed in modern quantum information science, condensed matter physics, mathematics and computer science.

Tensor networks come with an intuitive graphical language that can be used in formal reasoning and in proofs. This diagrammatic language found applications in physics at least as early as the 1970s by Roger Penrose [1]. These methods have recently seen many advancements and adaptations to different domains of physics, mathematics and computer science. An important milestone was David Deutsch's use of the diagrammatic notation in quantum computing, developing the so called quantum circuit model [2]. Quantum circuits are a special class of tensor networks, in which the arrangement of the tensors and their types are restricted. A related diagrammatic language slightly before that is due to Richard Feynman [3]. The quantum circuit model is widely used to describe quantum algorithms and their experimental implementations, to quantify the resources they use, to classify the entangling properties and computational power of specific gate families, and more.

Recently, several interesting investigations of tensor network algorithms have been developed [4–18]. Some of the best known applications of such method are Matrix Product States (MPS), Tensor Trains (TT) [19], Tree Tensor Networks (TTN), the Multi-scale Entanglement Renormalization Ansatz (MERA), Projected Entangled Pair States (PEPS), and various other renormalization methods [5–8, 12, 15, 20]. The excitement is based on the fact that certain classes of quantum systems can now be simulated more efficiently, studied in greater detail, and this has opened new avenues for a greater understanding of certain physical systems.

Indeed, tensor networks are factorizations of very large tensors into networks of smaller tensors. In the tensor diagram notation, a tensor is drawn as labelled shape such as a points(vertices), with zero or more legs(edges). Therefore, from this point of view, the tensor network can be considered as a graph where all graph theory properties should be adopted.

Graph theory is the study of graphs, which are mathematical structures used to model pairwise relations between objects. A graph in this context is made up of vertices (also called nodes or points) which are connected by edges (also called links or lines). Moreover graphs can be used to model many types of relations and processes in physical systems [21]. Many practical problems can be represented by graphs. Emphasizing their application to real-world systems, the term network is sometimes defined to mean a graph in which attributes are associated with the vertices and edges.

Graph theory is used in physics, precisely, in condensed matter physics, the three-dimensional structure of complicated simulated atomic structures can be studied quantitatively by gathering statistics on graph-theoretic properties related to the topology of the atoms. Also, the Feynman graphs and rules of calculation summarize quantum field theory in a form in close contact with the experimental numbers one wants to understand. In statistical physics, graphs can represent local connections between interacting parts of a system, as well as the dynamics of a physical process on such systems [22].
Similarly, in this thesis we use graph theory in Clifford algebra, by associate them together, this has lead to the birth of a "Clifford graph algebras". This construction can be used to build models of representations of simply-laced compact Lie groups [23].

Clifford algebras have numerous important applications in physics. Physicists usually consider a Clifford algebra as an algebra with a basis generated by the matrices $\gamma_0, \ldots, \gamma_3$, called Dirac matrices which have the property that $\gamma_i\gamma_j + \gamma_j\gamma_i = 2\eta_{ij}$ where η is the matrix of a quadratic form of signature (1,3) (or (3,1) corresponding to the two equivalent choices of metric signature). These are exactly the defining relations for the Clifford algebra $Cl_{1,3}(\mathbf{R})$, whose complexification is $Cl_{1,3}(\mathbf{R})_{\mathbf{C}}$ which, by the classification of Clifford algebras, is isomorphic to the algebra of 4×4 complex matrices $Cl_4(\mathbf{C}) \cong M_4(\mathbf{C})$. However, it is best to retain the notation $Cl_{1,3}(\mathbf{R})_{\mathbf{C}}$ since any transformation that takes the bilinear form to the canonical form is not a Lorentz transformation of the underlying spacetime.

The Clifford algebra of spacetime used in physics thus has more structure than $Cl_4(\mathbf{C})$. It has in addition a set of preferred transformations - Lorentz transformations. Whether complexification is necessary to begin with depends in part on conventions used and in part on how much one wants to incorporate straightforwardly, but complexification is most often necessary in quantum mechanics where the spin representation of the Lie algebra $\mathbf{so}(1,3)$ sitting inside the Clifford algebra conventionally requires a complex Clifford algebra [24].

Moreover, in this thesis we investigate dyonic objects in arbitrary dimensions by using the toroidal compactification of string theory on $n-$dimensional tori T^n. First, we present a class of dyonic black solutions formed by two different D-branes using a correspondence between toroidal cycles and objects possessing both magnetic and electric charges, belonging to $U(1)_e^{2^{n-1}} \times U(1)_m^{2^{n-1}}$ dyonic gauge symmetry. This symmetry could be associated with electrically charged magnetic monopole solutions in stringy model buildings of the standard model (SM) extensions. Then, we consider in some details such black hole classes obtained from even dimensional toroidal compactifications, and we find that they are linked to $Cl(n)$ Clifford algebras using the vee product. It is believed that this analysis could be extended to dyonic objects which can be obtained from local Calabi-Yau manifold compactifications.

On the other hand, since the development of string theory framework and its efficient technique for studying non-perturbative phenomena, a certain number of relevant links with other theories have been comprehended [25, 26]. Specially, in such a framework, the formerly familiar group of solitonic p-brane solutions of type II supergravities in 10 dimensions are known to be characterized in terms of D-branes [26, 27]. Non-perturbative properties of supersymmetric Yang–Mills theory, black holes entropy, dyons, magnetic monopoles and a number of appealing issues in various dimensions have been handled in the context of D-brane objects and their dynamics [28, 29]. All these relevant connections have promoted such a string theory aspect to one of the most important and promising side to be explored. It is known that in higher dimensions many physical quantities are reinterpreted from the usual four-dimensional spacetime viewpoint as various faces of possible one quantity. In particular, it has been shown that the single electric and magnetic charges in 10 dimensions could be viewed from the four-dimensional non-compact spacetime standpoint as dyonic charges, being particles carrying simultaneous existence of the electric and the magnetic charges [30]. A dyon with a zero electric charge is usually referred to as a magnetic monopole [31, 32]. These objects are hypothetical particles predicted in many extended models and grand unified theories [33, 34]. Indeed, besides to magnetic monopole solutions, dyonic solutions appear in extended Yang-Mills-Higgs modeling taking places in theories going beyond the ordinary physics associated with material points. It is therefore interesting to see how the D-brane framework automatically encodes this feature by using stringy constructions and computations.

Although there appears to have been no precedent concrete experimental search for dyonic objects, it has been now become possible given the progress made in the direct search of magnetic monopoles at accelerators which has a long history [35]. Actually, seen that dyons possess an electric charge that can, in principle, be significant, with the recent LHC results. For instance, ATLAS and MoEDAL results have reported limits on monopole productions as well as on stable objects with electric charge [36, 37]. Thus, it turns out that LHC could have a right place for the search of dyons.

The aim of this work is to investigated stringy dyonic objects and tensor network formalism. Inspired and motivated by known results obtained in non-trivial theories including string theory, we provide shortly the origin and certain related concepts. After that, we elaborate a general framework of dyonic solutions in the brane structure showing how are naturally obtained in various dimensions within stringy compactifications on a concrete complex geometry. Then, we study dyonic solution representations using tensor network formalism by revealing some characteristics associated with extra dimensions. Concretely, we investigate dyonic black objects in arbitrary dimensions. Inspired by real Hodge diagrams of the toroidal compactifcations on $\mathbb{T}^n$, we propose a new class of doublets of dyonic black solutions formed by two different D-branes given by $\begin{pmatrix} p \\ 6-n-p \end{pmatrix}$ objects. More precisely, we deal with some details such dyonic black classes obtained from lower-dimensional toroidal compactification. In particular, we show that they are linked to $Cl(n)$ Clifford algebra using the vee product.

This thesis is organized as follows. The first part contains 4 chapters. In chapter 1, we present the essential notions and concepts of graph theory, then we treat some linear algebra. Furthermore in chapter 2, we define the Clifford algebra, and we give some examples, then we show how to associate a Clifford algebra to a graph. In chapter 3, we present some basic definitions and concepts of the tensor network, then we show that the tensor network can be used to represent quantum many-body states. In chapter 4, we discuss dyonic objects, we give the notions of the string theory, then we present the black objects in this theory. Finally in the second part, we show our results and we give the scientific contributions.

Part I

Dyonic Objects, Clifford Structures and Tensor Network

Chapter 1

Graph Theory in Physics

In this chapter, we describe briefly graph theory. In particular we discuss the graphs, the subgraphs, the matrix representation of graphs, the graph operation, and the language of graphs and networks. It is recalled that graph theory is the study of graphs, which are mathematical structures used to model pairwise relations between objects. A graph in this context is made up of vertices, nodes, or points which are connected by edges, arcs, or lines. A graph may be undirected, meaning that there is no distinction between the two vertices associated with each edge, or its edges may be directed from one vertex to another. Graphs are one of the prime objects of study in discrete mathematics.

1.1 Generalities on graph theory

1.1.1 Graphs and subgraphs

A graph is formed by vertices (the points), and edges (the lines) connecting the vertices, as shown in the (Figure 1.1).

A graph is a pair of sets (V, E), where V is the set of vertices and E is the set of edges, formed by pairs of vertices. Often, we label the vertices with letters (for example: a,b,c,... or v_1, v_2,...) or numbers 1,2,... Throughout this work, we will label the elements of V in this way [38]. We have $V = (v_1, \ldots, v_6)$ for the vertices and $E = (v_1, v_2), (v_2, v_3), (v_3, v_4), (v_3, v_6), (v_4, v_5), (v_5, v_1)$ for the edges. We label the edges with letters (for example: $a, b, c, \ldots$ or $e_1, e_2, \ldots$) or numbers $1, 2, \ldots$ for simplicity, like

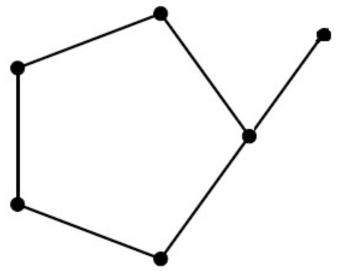


Figure 1.1: The conceptual definition of a graph

shown in the (Figure 1.2).

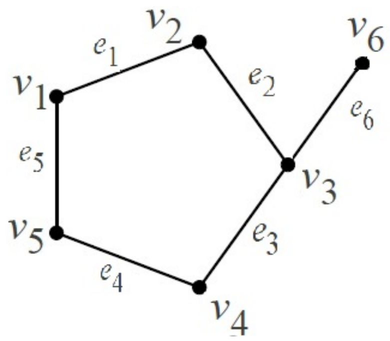


Figure 1.2: The formally definition of a graph

The two edges (u, v) and (v, u) are the same. In other words, the pair is not ordered. We have the following terminologies:

1. The two vertices u and v are end vertices of the edge (u,v).

2. Edges that have the same end vertices are parallel.

3. An edge of the form (v,v) is a loop.

4. A graph is simple if it has no parallel edges or loops.

5. A graph with no edges (i.e. E is empty) is empty.

6. A graph with no vertices (i.e. V and E are empty) is a null graph.

7. A graph with only one vertex is trivial.

8. Edges are adjacent if they share a common end vertex.

9. Two vertices u and v are adjacent if they are connected by an edge, where (u,v) is an edge.

A simple graph that contains every possible edge between all the vertices is called a complete graph. A complete graph with n vertices is denoted as K_n. Some sources claim that the letter K in this notation stands for the German word *komplett* [39], but the German name for a complete graph, **vollständiger Graph**, does not contain the letter K, and other sources state that the notation honors the contributions of *Kazimierz Kuratowski* to graph theory [40]. The first four complete graphs are given as examples in (Figure 1.3).

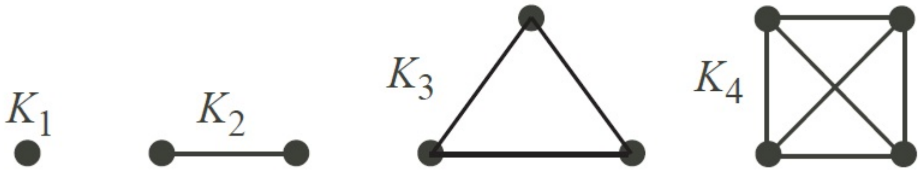


Figure 1.3: A complete graph with 4 vertices

The graph G_1=(V_1,E_1) is a subgraph of G_2=(V_2,E_2) if :

1. $V_1 \subseteq V_2$.
2. every edge of G_1 is also an edge of G_2.

We give some properties concerning the subgraph. In particular, we quote the following:

- the subgraph of $G = (V, E)$ induced by edge set $E_1 \subseteq E$ is

$$G_1 = (V_1, E_1) =< E_1 >$$

where V_1 consists of every end vertex of the edges in E_1.

- the subgraph of $G = (V, E)$ induced by edge set $V_1 \subseteq V$ is

$$G_1 = (V_1, E_1) =< V_1 >$$

where E_1 consists of every end vertex of the vertices in V_1.

- a complete subgraph of G is called a cyclique of G.

Any graph G corresponds to a $n \times m$ matrix called the adjacency matrix of G, that we will see in the section 1.2.

1.1.2 Graph isomorphism

The two drawings shown in (Figure 1.4) both represent the same graph: Their vertex and edge sets are identical, and their edge-endpoint functions are the same. We call this graph G.

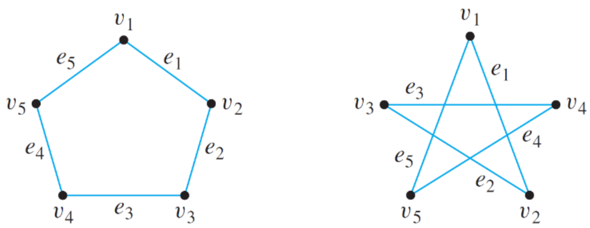


Figure 1.4: Two graphs with the same number of vertices and edges.

Now, consider the graph G' represented in (Figure 1.5)

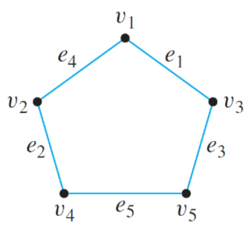


Figure 1.5: The graph G'

It is observed that G' is a different graph from G (for instance, in G the endpoints of e_1 are v_1 and v_2, whereas in G' the endpoints of e_1 are v_1 and v_3).

Two graphs that are the same except for the labeling of their vertices and edges are called isomorphic. *The word isomorphism comes from the Greek, meaning "same form".* Isomorphic graphs are those that have essentially the same form.

Now, we can define the isomorphic graphs, let G and G' be graphs with vertex set $V(G)$ and $V(G')$ and edge sets $E(G)$ ans $E(G')$, respectively. G is isomorphic to G' if, and only if, there exist one-to-one correspondences $g : V(G) \to V(G')$ and $h : E(G) \to E(G')$

that preserve the edge-endpoint functions of G and G' in the sense that for all $v \in V(G)$ and $e \in E(G)$,

$$v \text{ is an endpoint of } e \Leftrightarrow g(v) \text{ is an endpoint of } h(e) \tag{1.1}$$

In other words, G is isomorphic to G' if, and only if, the vertices and edges of G and G' can be matched up by one-to-one, onto functions such that the edges between corresponding vertices correspond to each other.

1.1.3 Tress

One of the important classes of graphs is the trees. The importance of trees is evident from their applications in various areas, especially theoretical computer science and molecular evolution. A tree is a connected graph without any cycles, or a tree is a connected acyclic graph. The edges of a tree are called branches. It follows immediately from the definition that a tree has to be a simple graph (because self-loops and parallel edges both form cycles). A graph having no cycles is said to be acyclic. A forest is an acyclic graph. The (Figure 1.6) displays all trees with fewer than six vertices.

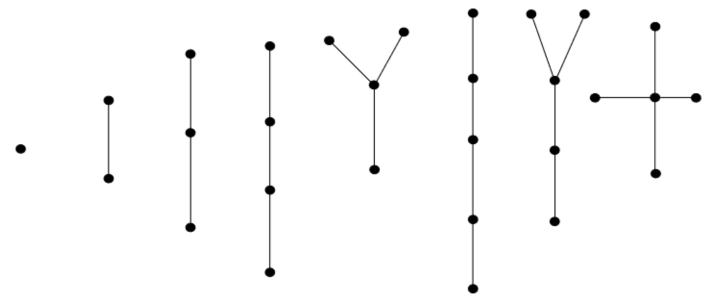


Figure 1.6: Trees with fewer than six vertices.

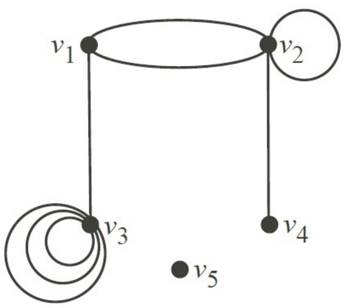
The following result characterises trees. For a tree graph we have the following properties:

- A graph is a tree if and only if there is exactly one path between every pair of its vertices.
- A tree with n vertices has $n - 1$ edges.
- Any connected graph with n vertices and $n - 1$ edges is a tree.

- A graph is a tree if and only if it is minimally connected[1].
- A graph G with n vertices, $n-1$ edges and no cycles is connected.
- Any tree with at least two vertices has at least two pendant vertices.
- A forest of k trees which have a total of n vertices has $n-k$ edges.

1.2 Matrix representation of graph

In this section, we give four different definitions of a matrix representation of graph. The first one is the adjacency matrix of the graph $G = (V, E)$ that is an $n \times n$ matrix $A = (a_{ij})$, where n is the number of vertices in G. It is denoted that $V = \{v_1, ..., v_n\}$ and d_{ij} indicate the number of edges between v_i and v_j. To illustrate these notions we give an example of this case in the (Figure 1.7)

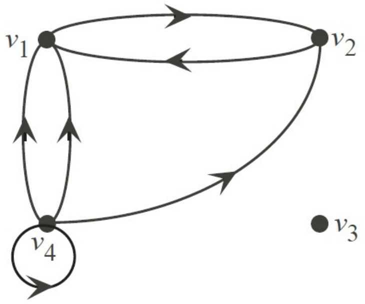


$$\mathbf{A} = \begin{pmatrix} 0 & 2 & 1 & 0 & 0 \\ 2 & 1 & 0 & 1 & 0 \\ 1 & 0 & 3 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

Figure 1.7: The adjacency matrix of a graph having 5 vertices.

The second definition is the adjacency matrix of a directed graph G which is $A = (a_{ij})$, where a_{ij} is the number of arcs that come out of vertex v_i and go into vertex v_j. We give also an example of this case in the (Figure 1.8)

[1]A graph is said to be minimally connected if removal of any one edge from it disconnects the graph. Clearly, a minimally connected graph has no cycles.



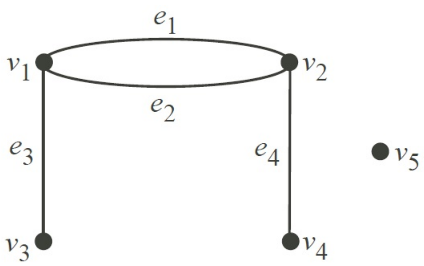
$$\mathbf{A} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 2 & 1 & 0 & 1 \end{pmatrix}$$

Figure 1.8: The adjacency matrix of a directed graph

We define also the all-vertex incidence matrix of a non-empty and loop-less graph $G = (V, E)$ who is an $n \times m$ matrix $A = (a_{ij})$, where n is the number of vertices in G, m is the number of edges in G and

$$a_{ij} = \begin{cases} 1 & \text{if } v_i \text{ is an end vertex of } e_j \\ 0 & \text{otherwise.} \end{cases}$$

The example of this case is given in the (Figure 1.9):



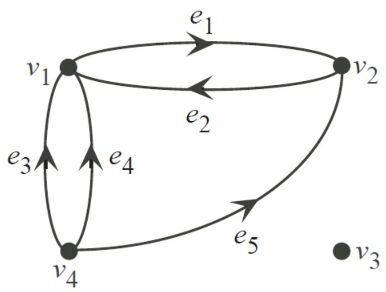
$$\mathbf{A} = \begin{array}{c} \begin{array}{cccc} e_1 & e_2 & e_3 & e_4 \end{array} \\ \begin{pmatrix} 1 & 1 & 1 & 0 \\ 1 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix} \end{array} \begin{array}{c} \\ v_1 \\ v_2 \\ v_3 \\ v_4 \\ v_5 \end{array}$$

Figure 1.9: The all-vertex incidence matrix of a non-empty and loopless graph

The last definition is the all-vertex incidence matrix of a non-empty and loopless directed graph G that is $A = (a_{ij})$, where

$$a_{ij} = \begin{cases} 1 & \text{if } v_i \text{ is the initial vertex of } e_j \\ -1 & \text{if } v_i \text{ is the initial vertex of } e_j \\ 0 & \text{otherwise} \end{cases}$$

The example illustrat this case is given in the (Figure 1.10)



$$
\mathbf{A} = \begin{array}{c}
\begin{array}{ccccc} e_1 & e_2 & e_3 & e_4 & e_5 \end{array} \\
\begin{pmatrix}
1 & -1 & -1 & -1 & 0 \\
-1 & 1 & 0 & 0 & -1 \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 1 & 1
\end{pmatrix}
\end{array}
\begin{array}{c} \\ v_1 \\ v_2 \\ v_3 \\ v_4 \end{array}
$$

Figure 1.10: The all-vertex incidence matrix of a non-empty and loopless directed graph

1.3 More on graphs

1.3.1 Graph Operations

Graph operations produce new graphs from the initial ones. They may be separated into the following major categories. Elementary operations or editing operations create a new graph from the initial one by a simple local change, such as addition or deletion of a vertex or of an edge, merging and splitting of vertices, edge contraction, etc. The graph edit distance between a pair of graphs is the minimum number of elementary operations required to transform one graph into the other. The complement of the simple graph $G = (V, E)$ is the simple graph $\bar{G} = (\bar{V}, \bar{E})$, where the edges in E are exactly the edges not in G (Figure 1.11). Obviously, we have $\bar{\bar{G}} = G$. If the graphs $G = (V, E)$ and $G' = (V', E')$ are simple and $V' \subseteq V$.

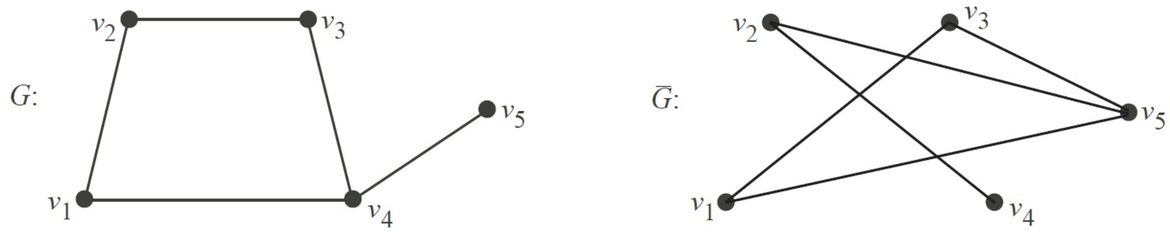


Figure 1.11: A graph G and its complements.

The difference graph is $G - G' = (V, E'')$, where E'' contains those edges from G that are not in G' (simple graph) that is shown in the (Figure 1.12)

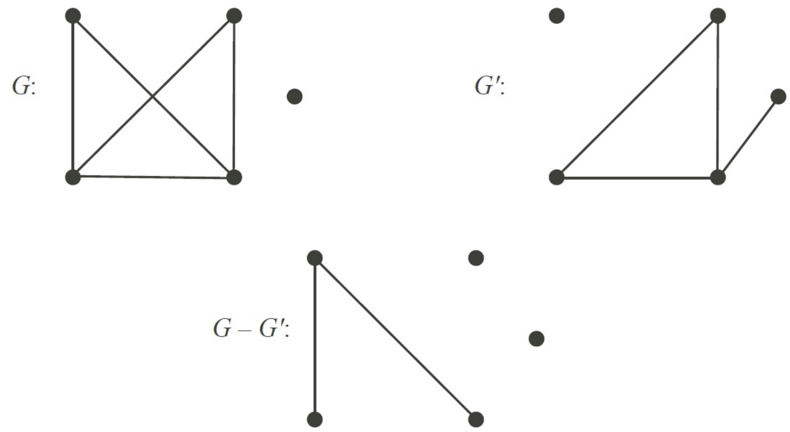


Figure 1.12: Examples of graph operations.

Here, we have a some binary operations between two simple graphs $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$:

- the union is $G_1 \cup G_2 = (V_1 \cup V_2, E_1 \cup E_2)$.
- the intesection is $G_1 \cap G_2 = (V_1 \cap V_2, E_1 \cap E_2)$.
- the ring sum $G_1 \oplus G_2$ is the subgraph of $G_1 \cup G_2$ induced by the edge set $E_1 \oplus E_2$. The set operation $\oplus$ is the symmetric difference.

$$E_1 \oplus E_2 = (E_1 - E_2) \cup (E_2 - E_1).$$

Since the ring sum is a subgraph induced by an edge set, there are no isolated vertices. All three operations are commutative and associative. Binary operations create a new graph from two initial ones as shown in the (Figure 1.13) and (Figure 1.14).

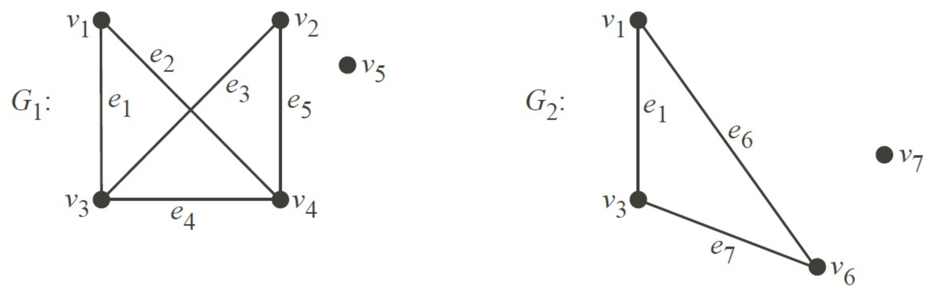


Figure 1.13: Example than two graphs

For the graphs in the (Figure 1.13), the operations are given in the (Figure 1.14).

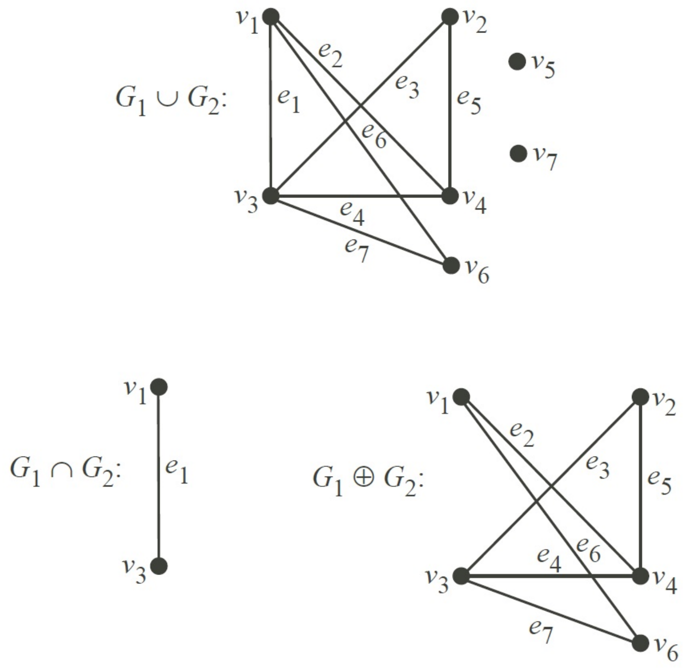


Figure 1.14: Examples of a graph operations

The operations $\cup$, $\cap$ and $\oplus$ can also be defined for more general graphs other than simple graphs. Naturally, we have to "keep track" of the multiplicity of the edges:

- $\cup$: the multiplicity of an edge in $G_1 \cup G_2$ is the larger of its multiplicities in G_1 and G_2.

- $\cap$: the multiplicity of an edge in $G_1 \cap G_2$ is the smaller of its multiplicities in G_1 and G_2.

- $\oplus$: the multiplicity of an edge in $G_1 \oplus G_2$ is $|m_1 - m_2|$, where m_1 is its multiplicity in G_1 and m_2 is its multiplicity in G_2.

$$m_1 - m_2 = \begin{cases} m_1 - m_2 \text{ if } m_1 \geq m_2 \\ 0 \text{ if } m_1 < m_2 \end{cases}$$

where m_1 and m_2 are the multiplicities of edge in G_1 and G_2 (respectively).

1.3.2 Degree of vertex

The degree of the vertex v, written as $d(v)$, is the number of edges with v as an end vertex. By convention, we count a loop twice and parallel edges contribute separately. Considering any graph $G = (E, V)$, the minimun degree of the vertices in a graph G is denoted $\delta(G)$ (=0 if there is an isolated vertex in G). Similarly, we write $\Delta(G)$ as the maximum degree of vertices in G. Then, we have

$$\delta(G) \leq d(v) \leq \Delta(G). \tag{1.2}$$

The graph $G = (V, E)$, where $V = (v_1, ..., v_n)$ and $E = (e_1, ..., e_m)$ satisfy:

$$\sum_{i=1}^{n} d(v_i) = 2m$$

Every graph has an even number of vertices of odd degree.

1.4 Graphs and networks

For the basic concepts of graph theory, the reader is recommended the introductory book by Harary (1967) [41, 42]. Indeed we start by defining formally what a graph is. Consider a finite set $V = \{v_1, v_2, \ldots, v_n\}$ of unspecified elements and let $V \otimes V$ be the set of all ordered pairs $[v_i, v_j]$ of the elements of V. A relation on the set V is any subset $E \subseteq V \otimes V$. The relation E is symmetric if $[v_i, v_j] \in E$ implies $[v_j, v_i] \in E$ and it is reflexive if $\forall v \in V, [v, v] \in E$. The relation E is antireflexive if $[v_i, v_j] \in E$ implies $v_i \neq v_j$. Now, we can define a simple graph as the pair $G = (V, E)$ where V is a finite set of nodes, vertices or points in a plane and E is a symmetric and antireflexive relation on V, whose elements are known as the edges or links of the graph.

In a directed graph, the relation E is non-symmetric. In many physical applications, it is desired that the edges of the graphs support some weights, i.e., real numbers indicating a specific property of the edge. In this case the following more general definition is convenient. A weighted graph is the quadruple $G = (V, E, W, f)$ where V is a finite set of nodes, $E \subseteq V \otimes V = \{e_1, e_2, \ldots, e_m\}$ is a set of edges, $W = \{w_1, w_2, \ldots, w_r\}$ is a set of weights such that $w_i \in \mathbb{R}$ and $f : E \to W$ is a surjective mapping that assigns a weight to each edge. If the weights are positive integer numbers then the resulting graph is a multigraph in which there could be multiple edges between pairs of vertices.

In an undirected graph, we say that wo nodes p and q are adjacent if they are joined by an edge $e = \{p, q\}$. In this case, we say that the nodes p and q are incident with the link e, and the link e is incident with the nodes p and q . The two nodes are referred as the end nodes of the edge. Two edges $e_1 = \{p, q\}$ and $e_2 = \{r, s\}$ are adjacent if they are both incident with at least one node. A simple but important characteristic of a node is its degree, which is defined as the number of edges which are incident with it or similarly the number of nodes adjacent to it. Some slightly different definitions apply for the case of directed graphs. The node p is adjacent to node q if there is a directed link from p to q, $e = \{p, q\}$. We also say that a link from p to q is incident from p and incident to q ; p is incident to e and q is incident from e. Consequently, we have two different kinds of degrees in directed graphs. The in-degree of a node is the number of links incident to it and its out-degree is the number of links incident from it [43, 44].

1.4.1 Graph operators

The incidence and adjacency relations in graphs allow us to define the following graph operators. We start by considering an arbitrary orientation of every edge in a graph. That is, for every edge $\{p, q\}$ we can consider that p is the positive (head) and q the negative (tail) end of the oriented link. We represent the graph through an oriented incidence matrix $\nabla(G)$:

$$\nabla_{ij} = \begin{cases} +1 & \text{node } v_i \text{ is the the head of link } e_j \\ -1 & \text{node } v_i \text{ is the tail of link } e_j \\ 0 & \text{otherwise} \end{cases}$$

We remark that the results obtained below are independent on the orientation of the links but assume that once the links are oriented this orientation is not changed. Now, we consider the vertex L_V and edge L_E spaces as the free linear span of V and E, respectively. That is, the vector spaces of all real-valued functions defined on V and E, respectively. The incidence operator of the graph is then defined as

$$\nabla(G) : L_V \to L_E, \tag{1.3}$$

such that for an arbitrary function $f : V \to R, \nabla f : E \to \mathbb{R}$ is given by

$$(\nabla f)(e) = f(p) - f(q), \tag{1.4}$$

where p is the starting (head) and q the ending (tail) point of the oriented link e. We remain here that a vector field is a function on a given interval with an orientation of the interval. In this case, the interval corresponds to the edge in the graph, which together with its orientation forms a vector field. Then, the continuous analogous of $\nabla(G)$ is the gradient $(\nabla f) = \partial f/\partial x_1, \partial f/\partial x_2, \ldots \partial f/\partial x_n$, which gives the maximum rate of change of the function with direction. That is, we can consider the incidence operator as the gradient operator of the graph.

On the other hand, the adjacency operator is an operator acting on the vertex space L_V and defined as

$$(Af)(p) := \sum_{u,v \in E} f(q), \quad f \in H, \quad i \in V. \tag{1.5}$$

The adjacency operator of an undirected network is a self-adjoint operator, which is bounded on $l^2(V)$. However, the adjacency operator of a directed network might not be self-adjoint. The matrix representation of this operator is the adjacency matrix $\mathbf{A}$, which if the graph does not contain any self-loop is defined as

$$A_{ij} = \begin{cases} 1 & \text{if} \quad i, j \in E, \\ 0 & \text{otherwise.} \end{cases} \tag{1.6}$$

A third operator which is related to the previous two ones and which plays a fundamental role in the applications of graph theory in physics is the Laplacian operator. This operator is defined as

$$\mathbf{L}f = -\nabla.(\nabla f). \tag{1.7}$$

The network version of the Laplacian operator is given by

$$\nabla^2 f = \frac{\partial^2 f}{\partial x_1^2} + \frac{\partial^2 f}{\partial x_2^2} + \ldots + \frac{\partial^2 f}{\partial x_n^2} \tag{1.8}$$

Then, the Laplacian operator acting on $\mathbf{L} : \mathbb{R}^{|V|} \to \mathbb{R}^{|V|}$ reads as

$$(\mathbf{L}f)(u) = \sum_{u \sim v} [f(u) - f(v)]. \tag{1.9}$$

Using a matrix form, it is given by

$$L_{uv} = \sum_{e \in E} \nabla_{eu} \nabla_{ev} = \begin{cases} -1 & \text{if} \quad u, v \in E, \\ k_u & \text{if} \quad u = v, \\ 0 & \text{otherwise.} \end{cases} \tag{1.10}$$

Using the degree matrix which is a diagonal matrix of the degree of the nodes in the graph, the Laplacian and adjacency matrices of a graph are related as follows

$$\mathbf{L{=}K{-}A}. \tag{1.11}$$

1.4.2 General graph concepts

Other important general concepts of graphs theory which are fundamental for the study of graphs and networks in physics are the following.

Walk: A walk in the graph $G = (V, E)$ is a finite sequence of the form:

$$v_{i0},\ e_{j_1},\ v_{i1},\ e_{j_2}, \ldots,\ e_{j_k},\ v_{ik}$$

which consists of alternating vertices and edges of G. The walk starts at a vertex. Vertices v_{it-1} and v_{it} are end vertices of e_{j_t} $(t = 1, ..., k)$, v_{i0} is the initial vertex and v_{ik} is the terminal vertex, k is the length of the walk. A zero length walk is just a single vertex v_{i0}, and a walk is open if $v_{i0} \neq v_{ik}$. Otherwise it is closed. To illustrate this idea, we consider the example graph in the (Figure 1.15),

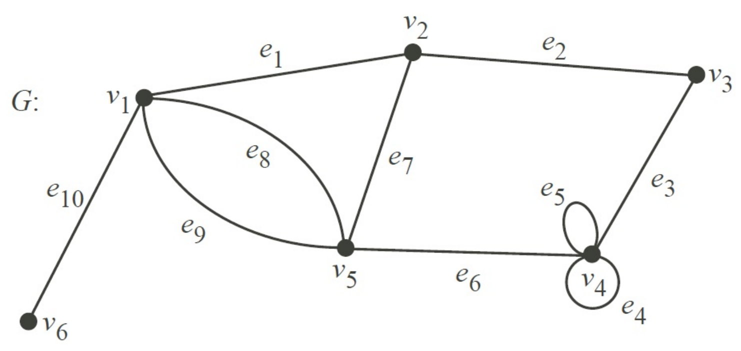


Figure 1.15: A graph G

The vertices $v_2, e_7, v_5, e_8, v_1, e_8, v_5, e_6, v_4, e_5, v_4, e_5$ and v_4 describe an open walk, while $v_4, e_5, v_4, e_3, v_3, e_2, v_2, e_7, v_5, e_6,$ and v_4 concern a closed one.

Trail: A walk is a trail if any edge is traversed at most once. Then, the number of times that the vertex pair (u, v) can appear as consecutive vertices in a trail is at most the number of parallel edges connecting u and v. We consider the graph in the (Figure 1.15), the walk $v_1, e_8, v_5, e_9, v_1, e_1, v_2, e_7, v_5, e_6, v_5, e_5, v_4, e_4, v_4$ is a trail.

Path and Cycle: A trail is a path if any vertex is visited at most once except possibly the initial and terminal vertices when they are the same. A closed path is a cycle. Note that any loop or pair of multiple edges is a cycle. In the (Figure 1.15), the walk $v_2, e_7, v_5, e_6, v_4, e_3, v_3$ is a path, and the walk $v_2, e_7, v_5, e_6, v_4, e_3, v_3, e_2, v_2$ is a cycle.

Connectivity: A graph is connected if only if there is a path between each pair of vertices. The walk starting at u and ending at v is called $u - v$ walk, u and v are connected if there is a $u - v$ walk in the graph. We give these properties of a connectivity graphs. If u and v are connected, and v and w are connected, then u and w are also connected. And if there is a $u - v$ walk and a $v - w$ walk, then there is also $u - w$ walk. The graph is connected if all the vertices are connected to each other.
In general a graph can be connected or not. A graph is connected if there is a path between any pair of nodes in the graph. Otherwise it is disconnected. Every connected subgraph is a connected component of the network. The analogous concept in a directed graph is that of strongly connected graph. A directed graph is strongly connected if there is a directed path between each pair of nodes. The strongly connected components of a directed graph are its maximal strongly connected subgraphs.

Component: The subgraph G_1 (not a null graph) of the graph G is a component of G if:

1. G_1 is connected.

2. G_1 is trivial or G_1 is not trivial and G_1 is the subgraph induced by those edges of G that have one end vertex in G_1.

Rank and nullity of the graph: A graph $G = (E, V)$ with n vertices, m edges and k components has the rank

$$\rho(G) = n - k. \tag{1.12}$$

The nullity of the graph i:

$$\mu(G) = m - n + k. \tag{1.13}$$

We see that $\rho(G) \geq 0$ and $\rho(G) + \mu(G) = m$. In addition $\mu(G) \geq 0$, because $\rho(G) \geq m$.

In an undirected graph the shortest path distance $d(p, q) = d_{pq}$ is the number of edges in the shortest path between the nodes p and q in the graph. If p and q are in different connected components of the graph the distance between them is set to infinite, $d(p, q) := \infty$.
In a directed graph it is typical to consider the directed distance $\overrightarrow{d}(p, q)$ between a pair of nodes p and q as the length of the directed shortest path from p to q. However, in general $\overrightarrow{d}(p, q) \neq \overrightarrow{d}(q, p)$, which violates the symmetry property of metric functions, such that $\overrightarrow{d}(p, q)$ is not a distance but a pseudo-distance or pseudo-metric. The distance between all pairs of nodes in a graph can be arranged in a distance matrix $\mathbf{D}$ which for undirected graphs is a square symmetric matrix. The maximum entry for a given row/column of the distance matrix of an undirected (strongly connected directed) graph is known as the eccentricity $e(p)$ of the node p, $e(p) = \max_{x \in V(G)} \{d(p, x)\}$. The maximum eccentricity among the nodes of a graph is known as the diameter of the graph, which is $diam(G) = \max_{x,y \in V(G)} \{d(x, y)\}$. The average path length $\bar{l}$ of a graph with n node is defined as follows

$$\bar{l} = \frac{1}{n(n-1)} \sum_{x,y} d(x, y). \tag{1.14}$$

An important metric for the study of networks was introduced by Watts and Strogatz (1998) as a way of quantifying how clustered a node is. For a given node i the clustering coefficient is the number of triangles connected to this node $|C_3(i)|$ divided by the number of triples centred on it

$$C_i = \frac{2|C_3(i)|}{k_i(k_i - 1)}, \tag{1.15}$$

where k_i is the degree of the node. The average value of the clustering for all nodes in a network $\overline{C}$ has been extensively used in the analysis of complex networks

$$\overline{C} = \frac{1}{n} \sum_{i=1}^{n} C_i \tag{1.16}$$

A second clustering coefficient has been introduced as a global characterization of network cliquishness (Newman et al., 2001). This index which is also known as network transitivity, is defined as the ratio of three times the number of triangles divided by the number of connected triples (2-paths):

$$C = \frac{3|C_3|}{|P_2|}. \tag{1.17}$$

1.5 Connection with linear algebras

Here, we provide a link with linear algebra. Let $G = (V, E)$ be a graph with n vertices and m edges, say $V = \{v_1, \ldots, v_n\}$ and $E = \{e_1, \ldots, e_m\}$. The *vertex space* $\underline{v}(G)$ of G is the vector space over the $2-$element field $\mathbb{F}_2 = \{0, 1\}$, of all functions $V \to \mathbb{F}_2$. Every element of $\underline{v}(G)$ corresponds naturally to a subset of V, the set of those vertices to which it assigns a 1, and every subset of V is uniquely represented in $\underline{v}(G)$ by its indicator function. We may thus think of $\underline{v}(G)$ as the power set of V made into a vector space: the sum $U + U'$ of two vertex sets $U, U' \subseteq V$ is their symmetric difference, and $U = -U$ for all $U \subseteq V$. The zero in $\underline{v}(G)$, viewed in this way, is the empty (vertex) set $\emptyset$. Since $\{\{v_1\}, \ldots, \{v_n\}\}$ is a basis of $\underline{v}(G)$, its *standard basis*, we have $dim\underline{v}(G) = n$.

In the same way as above, the functions $E \to \mathbb{F}_2$ form the *edge space* $\xi(G)$ of G: its elements are the subsets if E, vector addition amounts to symmetric difference, $\emptyset \subseteq E$ is the zero, and $F = -F$ for all $F \subseteq E$. As befor, $\{\{e_1\}, \ldots, \{e_m\}\}$ is the *standard basis* of $E(G)$, and $dimE(G) = m$.

Since the edges of a graph carry its essential structure, we shall mostly be concerned with the edges space. Given two edge sets $F, F' \in \xi(G)$ and their coefficients $\lambda_1, \ldots, \lambda_m$ and $\lambda'_1, \ldots, \lambda'_m$ with respect to the standard basis, we write

$$< F, F' >:= \lambda_1\lambda'_1 + \ldots + \lambda_m\lambda'_m \in \mathbb{F}_2.$$

Note that $< F, F' >= 0$ may hold even when $F = F' \neq \emptyset$: indeed, $< F, F' >= 0$ if and only if F and F' have an even number of edges in common. Given a subspace $\underline{f}$ of $\xi(G)$, we write

$$\underline{f}^{\perp} := \{D \in \xi(G) \mid < F, D >= 0 \quad \text{for all} \quad F \in \underline{f}\}. \tag{1.18}$$

This is again a subspace of $\xi(G)$ (the space of all vectors solving a certain set of linear equations), and we have

$$dim\underline{f} + dim\underline{f}^{\perp} = m. \tag{1.19}$$

The *cycle space* $\underline{c} = \underline{c}(G)$ is the subspace of $\xi(G)$ spanned by all the cycles in G—more precisely, by their edge sets[2]. The dimension of $\underline{c}(G)$ is the *cyclomatic number* of G. We have the following properties

- The induced cycles in G generate its entire cycle space.
- An edge set $F \subseteq E$ lies in $\underline{c}(G)$ if and only if every vertex of (V, F) has even degree.

If $\{V_1, V_2\}$ is a partition of V, the set $E(V_1, V_2)$ of all the edges of G crossing this partition is called a *cut*. Recall that for $V_1 = \{v\}$ this cut is denoted by $E(v)$.

- Together with $\emptyset$, the cuts in G form a subspace $\underline{c*}$ of $\xi(G)$. This space is generated by cuts of the form $E(\underline{v})$.

Our second assertion, that the cuts $E(\underline{v})$ generate all of $\underline{c}*$, follows from the fact that every edge $xy \in G$ lies exactly two such cuts (in $E(x)$ and $E(y)$); thus every partition $\{V_1, V_2\}$ of V satisfies $E(V_1, V_2) = \sum_{v \in V_1} E(\underline{v})$.

[2]For simplicity, we shall not normally distinguish between cycles and their edge sets in connection with the cycle space

Chapter 2

Graphic Representation of Clifford Structures

Clifford algebras are routinely used to compute particle scattering cross-sections, and form the basic tool for quantum electrodynamics. Lipschitz (1886) was the first to define groups constructed from 'Clifford numbers' and use them to represent rotations in a Euclidean space [45]. Cartan discovered representations of the Lie algebras $so_n(\mathbb{C})$ and $son(\mathbb{R})$, $n > 2$, that do not lift to representations of the orthogonal groups [46]. In physics, Clifford algebras and spinors appear for the first time in Pauli's nonrelativistic theory of the 'magnetic electron'. Dirac (1928), in his work on the relativistic wave equation of the electron, introduced matrices that provide a representation of the Clifford algebra of Minkowski space [47]. Brauer and Weyl (1935) connected the Clifford and Dirac ideas with Cartan's spinorial representations of Lie algebras; they found, in any number of dimensions, the spinorial, projective representations of the orthogonal groups [48].

In this chapter, we introduce the basic concepts of Clifford algebras, its representation, and the notions of Clifford graph algebra.

2.1 Clifford algebras

2.1.1 General concepts and examples

Clifford algebras are defined on vector spaces on which a "metric"[1], i.e. a non-degenerate quadratic form Q has been chosen. Different quadratic forms will generally lead to different algebras. The most important property of the quadratic form is its signature: the numbers of positive and negative eigenvalues.

This is for real spaces. For complex vector spaces, it turns out that the signature is unimportant and that the Clifford algebra depends only on the dimension of the space. For this reason, complex Clifford algebras are of much less importance in physics and will only rarely be discussed here. We will be concerned only with real Clifford algebras. Although complex spaces are the traditional settings for many areas in modern physics, we will see that many of the important Clifford algebras have either an inherent complex structure (even though they are real algebras), or can be "complexified" in a trivial fashion. An example of "complexification" is $M(4, R)$, the algebra of all 4×4 real matrices. We will see below that this is a Clifford algebra and it has a trivial complexification $M(4, C)$ where C is now the field of complex numbers. This algebra is important in the Dirac theory.

All (real) Clifford algebras are defined on the underlying real vector space, R^n, the space of vectors with n real components. The vectors should be regarded as column vectors

$$x = \begin{bmatrix} x^1 \\ x^2 \\ \vdots \\ x^n \end{bmatrix}$$

though sometimes for typographical reasons we will write them in the usual way one writes the coordinates of a point, i.e. as $(x_1, x_2, \ldots, x_n)$, (Of course we can also write $x = [x_1, x_2, \ldots, x_n]^T$ where x^T is the transpose of x.)

Clifford algebras are often described in terms of tensor-product spaces, modulo a

[1]Mathematicians would call this a 'pseudo-metric' as we do not usually assume that the length of a vector is positive. But the term is standard in relativity theory and we abide by this usage.

quadratic form [49]. Mathematically this has the appeal of providing a basis free definition. But from a practical viewpoint (and our approach will always be a practical one), this approach is unnecessarily complicated and obscure, as well as disguising much of the geometry which is inherent in Clifford algebras and which makes their study so rewarding. (It also often makes algebraic manipulations more cumbersome, which is hardly a useful feature). It is possible to define Clifford algebras in terms of basis vectors in R^n always bearing in mind that the choice of basis is immaterial.

Begin with the vector space R^n on which we are given a non-degenerate quadratic form Q. Such a form can always be written as $Q(x) = x^T Qx$ where with a small abuse of notation, we also use Q to denote a symmetric $n \times n$ matrix. Since Q is symmetric, its eigenvalues are real and the fact that Q is non-degenerate means that Q has p positive and q negative eigenvalues with $p + q = n$. This pair (p, q) is called the signature of Q and is the only important property of Q in defining the associated Clifford algebra. This algebra will be written as $Cl(p, q)$ or sometimes as $Cl(Q)$.

Choose an orthonormal basis $e_1, e_2, \ldots, e_n$ in R^n . Here orthonormal means relative to Q so that we require

$$e_i^T Qe_j = 0, \qquad i \neq j \tag{2.1}$$

$$e_i^T Qe_j = \begin{cases} +1, & \text{i=1...p} \\ -1, & \text{i=p+1...n} \end{cases} \tag{2.2}$$

The inner product $xTQy$ will also be written as $\langle x, y\rangle$.

Define multiplication to be an associative operation which satisfies the two conditions

1.
$$e_i^2 = \begin{cases} +1, & \text{i=1...p} \\ -1, & \text{i=p+1...n} \end{cases} \tag{2.3}$$

2.
$$e_i e_j = -e_j e_i, \quad \text{if} \quad i \neq j. \tag{2.4}$$

These two rules[2] allow us to define unambiguously any product $e_{i_1} e_{i_2} \ldots e_{i_r}$ where $A = i_1, i_2, \ldots i_r$ is a subset of $1, 2, \ldots, n$. Further, we can always assume that $i_1 < i_2 < \ldots < i_r$

[2]together with associativity

because by condition (2.4) above we can always reorder the indices. For example, one has

$$e_3e_2e_1 = (e_3e_2)e_1 = -(e_2e_3)e_1 = -e_2(e_3e_1) = +e_2(e_1e_3) = (e_2e_1)e_3 = -e_1e_2e_3.$$

We write e_A for the product $e_{i_1}e_{i_2}\dots e_{i_r}$ (this includes the case where the indices are not necessarily ordered and may even be repeated.)

Example. *In* $Cl(2,0)$ *(so that* $e_1^2 = e_2^2 = -1$*),*

$$e_{121} = e_1e_2e_1 = -e_1e_1e_2 = -(-1)e_2 = e_2.$$

The Clifford algebra $Cl(p,q)$ is the vector space spanned by the products e_A where A is an arbitrary subset of $\{1,2,\dots,n\}$ whose elements can always be written in increasing order.[3]

It follows from this that $Cl(p,q)$ has dimension 2^n.

A point of notation. In special relativity, we work in R^4 and it is conventional in that case to write the coordinates of a vector as (x^0,x^1,x^2,x^3). We write a basis for R^4 as e_0,e_1,e_2,e_3. In the case where the metric is denoted by physicists as $+---$ the Clifford algebra will then be $Cl(1,3)$ so that $e_0^2 = +1, e_1^2 = e_2^2 = e_3^2 = -1$. The other case is where the metric is written as $-+++$ and the Clifford algebra will then be $Cl(3,1)$ so that $e_0^2 = -1, e_1^2 = e_2^2 = e_3^2 = +1$. As we will see later, the two Clifford algebras $Cl(1,3)$ and $Cl(3,1)$ are not isomorphic. This has been used to suggest that there might be some physics behind such aspects.

Now, We give examples associated with $n = 1,2,3$ and 4. Higher dimensional algebras do crop up from time to time in physics but are of lesser importance. The cases $n = 6$ is useful however in shedding some geometric light on the groups $SU(3)$ and $SU(4)$.[4]

n=1

There are two Clifford algebras defined on R^1, $Cl(1,0)$ and $Cl(0,1)$. These both have dimension $2^1 = 2$. Then they are useful in representing points in the plane.

[3] By convention when ϕ is the null set, we define $e_\phi = 1$.

[4] Basically because $SU(4)$ and the two-fold covering group of $SO(6)$ are isomorphic. This is an interesting result we will look at later.

1. $Cl(1,0)$ has the single basis vector e_1 with $e_1^2 = 1$. This is not a division algebra because there are non-zero elements such as $1 \pm e_1$ which do not have inverses.

2. $Cl(0,1)$ again has the single basis vector e_1 this time with $e_1^2 = -1$. Clearly $Cl(0,1)$ is isomorphic to the algebra of complex numbers with $a + be_1 \leftrightarrow a + bi$.

n=2

Here, there are three Clifford algebras; $Cl(2,0)$, $Cl(1,1)$ and $Cl(0,2)$. Their dimensions are all $2^2 = 4$.

1. Consider firstly $Cl(2,0)$. Take as basis e_1, e_2 with $e_1^2 = e_2^2 = 1$. Note that

$$(e_{12})^2 = e_1e_2e_1e_2 = -e_1e_1e_2e_2 = -e_1^2e_2^2 = -1.$$

We can represent $Cl(2,0)$ by the algebra $M(2,R)$ of all 2×2 real matrices by taking

$$1 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad e_1 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \quad e_2 = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \quad e_{12} = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}. \qquad (2.5)$$

This gives an isomorphism between $Cl(2,0)$ and $M(2,R)$.

2. For $Cl(1,1)$ we have the basis e_1, e_2 with $e_1^2 = +1, e_2^2 = -1$. Working as above we now find that $e_{12}^2 = +1$. This time we can represent the basis elements by

$$1 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad e_1 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \quad e_2 = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}, \quad e_{12} = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}. \qquad (2.6)$$

and we have immediately the same result as in 1. Above, namely that $Cl(1,1)$ is isomorphic to $M(2,R)$.

$Cl(2,0)$ and $Cl(1,1)$ are isomorphic algebras. In fact, this is merely an accidental isomorphism and has no real significance. This is an important point to note because it shows the algebraic properties of a Clifford algebra are not enough to describe the algebra fully. In these last two examples, the Clifford algebras are defined by two quite different metrics. One is positive definite, describing Euclidean geometry, while the other is hyperbolic. They could hardly be more different.

3. Before considering $Cl(0,2)$, let us briefly remind ourselves of an important division algebra[5], the quaternions H[6].

   To define the quaternions, we should start with $H = R^4$ with a basis written $1, i, j, k$ and define an associative multiplication to satisfy the properties

$$i^2 = j^2 = k^2 = -1 \tag{2.7}$$

$$ij = k, jk = i, ki = j. \tag{2.8}$$

   This algebra is non-commutative since for example

$$ji = (ki)i = ki^2 = -k = -ij \quad and \quad similarly \quad kj = -jk, ik = -ki.$$

   Returning now to $Cl(0,2)$, choose a basis e_1, e_2 with $e_1^2 = e_2^2 = -1$. It follows (as above) that $e_{12}^2 = -1$. It is now trivial to verify that if we make the correspondence $e_1 \leftrightarrow i, e_2 \leftrightarrow j, e_{12} \leftrightarrow k$, then $Cl(0,2)$ is isomorphic to the quaternions H.

   However, this is not a particularly interesting result since while e_1 and e_2 are vectors in R^2, e_{12} is not. This means that i, j, k in a sense have different standings with k singled out as the different one. There is nothing like this in the quaternions.

n=3

There are two important algebras here, $Cl(3,0)$ and $Cl(0,3)$[7]. Either algebra can be usefully applied to describe geometrical ideas in R^3 such as rotations and reflections.

1. Consider $Cl(3,0)$ with basis e_1, e_2, e_3 satisfying $e_1^2 = e_2^2 = e_3^2 = +1$. The Pauli spin matrices $\sigma_i, i = 1,2,3$ also have this property and we can use the correspondence

$$1 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad e_1 = \sigma_1 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \quad e_2 = \sigma_2 = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}, \quad e_3 = \sigma_3 = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}. \tag{2.9}$$

[5]There are in fact only three associative division algebras over the reals; the reals themselves, the complex numbers and the quaternions.

[6]Discovered independently by William Hamilton and Olinde Rodrigues. Hamilton, an Irish mathematician and amateur bridge carver, is usually credited with their discovery but the French mathematician Rodrigues had described many of their properties some years earlier

[7]The other two algebras are based on hyperbolic geometry and play a lesser role in physics.

It follows that we need to define

$$e_{12} = e_1 e_2 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix} = \begin{bmatrix} i & 0 \\ 0 & -i \end{bmatrix}$$

and similarly

$$e_{23} = \begin{bmatrix} 0 & i \\ i & 0 \end{bmatrix}, \quad e_{31} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$$

and finally

$$e_{123} = \begin{bmatrix} i & 0 \\ 0 & i \end{bmatrix}.$$

It is clear that taking real linear combinations of these eight matrices, we obtain all 2×2 complex matrices. It follows that $Cl(3,0)$ is isomorphic to $M(2,C)$. Because of the connection between the basis elements and the Pauli matrices, $Cl(3,0)$ is often called the Pauli algebra.

Note that $e_{123}^2 = -1$, a result which has led some to prefer this algebra to its rival $Cl(0,3)$ because it gives us another possible representation for i.

Matrices in $M(2,C)$ and therefore too, elements of $Cl(3,0)$ are useful when describing rotations in R^3. To see this, we write a point (x^1, x^2, x^3) in R^3 as a matrix

$$(x^1, x^2, x^3) = x^1 e_1 + x^2 e_2 + x^3 e_3 \leftrightarrow \begin{bmatrix} x^3 & x^1 - x^2 i \\ x^1 + x^2 i & -x^3 \end{bmatrix}.$$

as a trace-zero anti-Hermitian matrix.

This representation has the disadvantage that the x^3 component of a point is singled out for (unfair) special consideration. We will see shortly that $Cl(0,3)$ does not suffer from this problem.

2. For $Cl(0,3)$ the basis vectors e_1, e_2, e_3 satisfy $e_1^2 = e_2^2 = e_3^2 = -1$. This immediately brings to mind the defining properties of the quaternions, namely $i^2 = j^2 = k^2 = -1$. We cannot just identify the basis vectors with i, j, k since the quaternion algebra is four dimensional while $Cl(0,3)$ has dimension $2^3 = 8$. Instead, we can take the representation to be

$$e_1 = \begin{bmatrix} 0 & i \\ i & 0 \end{bmatrix}, \quad e_2 = \begin{bmatrix} 0 & j \\ j & 0 \end{bmatrix}, \quad e_3 = \begin{bmatrix} 0 & k \\ k & 0 \end{bmatrix}$$

where one has

$$e_{23} = \begin{bmatrix} 0 & j \\ j & 0 \end{bmatrix} \begin{bmatrix} 0 & k \\ k & 0 \end{bmatrix} = \begin{bmatrix} i & 0 \\ 0 & i \end{bmatrix}.$$

Similarly, we have

$$e_{31} = \begin{bmatrix} j & 0 \\ 0 & j \end{bmatrix}, \quad e_{12} = \begin{bmatrix} k & 0 \\ 0 & k \end{bmatrix}.$$

Finally, one has

$$e_{123} = e_{12}e_3 = \begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix}.$$

By considering linear combinations of the eight basis vectors for $Cl(0,3)$, we find that the algebra is isomorphic to the algebra of 2×2 quaternions of the form

$$\begin{bmatrix} q_1 & q_2 \\ q_2 & q_1 \end{bmatrix}.$$

It is denoted that $Cl(0,3)$ is simply $H \oplus H$.

In this representation, there is no preferred direction as there is in the case of $Cl(3,0)$.

n=4

This is the dimension we need to consider when discussing the role Clifford algebras might play in special relativity. The two metrics $- + ++$ and $+ - --$ lead to two different algebras $Cl(3,1)$ and $Cl(1,3)$. They are not isomorphic has led to some speculations. There might be some interesting physics in the choice of metric.

We will also briefly look at $Cl(0,4)$ and $Cl(4,0)$ since R^4 plays a role in describing Pauli spinors in a real space. This will be dealt with more thoroughly later.

1. We discuss $Cl(3,1)$ first since it can be represented by 4×4 matrices. This provides a link with the Dirac equation and the Dirac algebra. Because of this, it is perhaps the Clifford algebra preferred by some physicists.

   Here, we need a basis e_0, e_1, e_2, e_3 which satisfies the conditions

   $$e_0^2 = -1, \quad e_1^2 = +1, \quad e_2^2 = +1, \quad e_3^2 = +1.$$

Before continuing, it is worthwhile asking ourselves what type of representation we might be after. Since $Cl(3,1)$ obviously contains $Cl(3,0)$ as a sub-algebra and $Cl(3,0)$ is generated by the Pauli matrices, it would be highly desirable if $Cl(3,1)$ also contained a copy of the Pauli matrices. After all, Pauli matrices generate rotations in R^3 rotations are important examples of Lorentz transformations.

It is easy to show that there is no non-trivial 2×2 matrix e_0 which anticommutes with the Pauli matrices. We would need to increase the dimensions of the matrices. One way to proceed is to start with the 2×2 the Pauli matrices σ_i and then define the generators $e_i (i = 1, 2, 3)$ by

$$e_i = \begin{bmatrix} \sigma_i & 0 \\ 0 & -\sigma_i \end{bmatrix}.$$

The last generator e_0 can then be found using the conditions $e_0 e_i = -e_i e_0$ as well as $e_0^2 = -1$.

The problem with this approach is that we end up with a matrix representation for $Cl(3,1)$ where it is difficult to identify important geometric concepts. For example it is not at all obvious how Lorentz transformations should be described. Even rotations, which are generated by the Pauli matrices, are now generated by the $e_i s (i = 1, 2, 3)$ and each e_i contains two different versions of σ_i.

Finally, the matrix algebra itself is not particularly attractive. The matrices are not all real but clearly (on dimensional grounds) not all complex matrices are included either. Describing it is not straightforward.

There is another approach which yields better results. Start with the Clifford algebra $Cl(2,0)$ which we already know to be isomorphic to the algebra $M(2,R)$ of all 2×2 real matrices. Take the generators to be

$$f_1 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \quad f_2 = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}.$$

Now, we define $e_\mu, \mu = 0, 1, 2, 3$ via

$$e_1 = \begin{bmatrix} f_1 & 0 \\ 0 & -f_1 \end{bmatrix}, \quad e_2 = \begin{bmatrix} f_2 & 0 \\ 0 & -f_2 \end{bmatrix} e_3 = \begin{bmatrix} 0 & I_2 \\ I_2 & 0 \end{bmatrix}, \quad e_0 = \begin{bmatrix} 0 & -I_2 \\ I_2 & 0 \end{bmatrix}.$$

It is now easy to verify that the $e_\mu s$ all anti-commute and that $e_0^2 = -1, e_i^2 = +1, i = 1, 2, 3$. A little more calculation shows that the 16 matrices $e_A, A \subset 1, 2, 3$ in fact span all of $M(4, R)$. Concretely, we have a representation of $Cl(3, 1)$ as the algebra of all 4×4 real matrices.

$Cl(p + 1, q + 1)$ is isomorphic to the algebra of all 2×2 matrices whose elements are the elements of $Cl(p, q)$.

This representation has some advantages over the previous one. For a start, it is easier to describe. Furthermore it bears a direct relationship with the Dirac algebra: the algebra of all 4×4 complex matrices. (This is a good example of a real Clifford algebra having an obvious 'complexification'). The price to pay when we complexify, is that any geometric structure in the real algebra, is generally lost or at least becomes obscured.

Another disadvantage with this representation is that the Pauli matrices have largely disappeared (or at least, have become lost). On the positive side however, is that Lorentz transformations (being 4×4 matrices) can be nicely represented in this algebra.

As we will now see, the Clifford algebra $Cl(1, 3)$ avoids these problems and seems the more appropriate algebra when dealing with ideas in special relativity.

2. The $+ - --$ metric gives the Clifford algebra $Cl(1, 3)$ which in many respects appears superior to $Cl(3, 1)$ for the simple reason that it describes rotations and Lorentz transformations in a more natural and geometric manner. For the present, let us look at how this algebra can be described (as usual) as a matrix algebra.

   $Cl(1, 3)$ contains $Cl(0, 3)$ in an obvious way and the latter algebra will prove highly useful in describing rotations and spin. Concretely, we seek a representation for Cl(1,3) which extends the representation we already have for $Cl(0, 3)$. By this, we mean that we will take the (quaternion) matrix representations for e_1, e_2, e_3 which are already present. As for e_0, we need only choose a matrix which anti-commutes with each $e_i, i = 1, 2, 3$ and which squares to 1. This then leads to the following representation for $Cl(1, 3)$.

$$1 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \tag{2.10}$$

$$e_0 = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \quad e_1 = \begin{bmatrix} 0 & i \\ i & 0 \end{bmatrix}, \quad e_2 = \begin{bmatrix} 0 & j \\ j & 0 \end{bmatrix}, \quad e_3 = \begin{bmatrix} 0 & k \\ k & 0 \end{bmatrix}, \tag{2.11}$$

$$e_{23} = \begin{bmatrix} i & 0 \\ 0 & i \end{bmatrix}, \quad e_{31} = \begin{bmatrix} j & 0 \\ 0 & j \end{bmatrix}, \quad e_{12} = \begin{bmatrix} k & 0 \\ 0 & k \end{bmatrix}, \tag{2.12}$$

$$e_{01} = \begin{bmatrix} 0 & i \\ -i & 0 \end{bmatrix}, \quad e_{02} = \begin{bmatrix} 0 & j \\ -j & 0 \end{bmatrix}, \quad e_{03} = \begin{bmatrix} 0 & k \\ -k & 0 \end{bmatrix}, \tag{2.13}$$

$$e_{023} = \begin{bmatrix} i & 0 \\ 0 & -i \end{bmatrix}, \quad e_{031} = \begin{bmatrix} j & 0 \\ 0 & -j \end{bmatrix}, \quad e_{012} = \begin{bmatrix} k & 0 \\ 0 & -k \end{bmatrix}, \quad e_{123} = \begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix}, \tag{2.14}$$

$$e = e_{0123} = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}. \tag{2.15}$$

It is clear now that these 16 matrices provide a basis for $M(2, H)$ the algebra of all 2×2 matrices with quaternion components.

3. Finally, let us consider the Clifford algebras which arise from an euclidean metric. Start with $Cl(4, 0)$. It would be nice if we could extend the representation of the Pauli algebra $Cl(3, 0)$ which we know to be isomorphic to $M(2, C)$, perhaps to something like $M(n, C)$. It is clear on dimensional grounds that this is impossible. Quaternions, however, seem to give better hope and we can start with something like

$$e_1 = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}, \quad e_2 = \begin{bmatrix} 0 & -j \\ j & 0 \end{bmatrix}, \quad e_3 = \begin{bmatrix} 0 & -k \\ k & 0 \end{bmatrix}. \tag{2.16}$$

To find e_4, the properties that $e_4^2 = +1$ and that e_4 anti-commutes with e_1, e_2, e_3 are enough to show that we can choose

$$e_4 = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}. \tag{2.17}$$

4. Finally, the case of $Cl(0,4)$ can be treated in a similar way. We can start with

$$e_1 = \begin{bmatrix} 0 & i \\ i & 0 \end{bmatrix}, \quad e_2 = \begin{bmatrix} 0 & j \\ j & 0 \end{bmatrix}, \quad e_3 = \begin{bmatrix} 0 & k \\ k & 0 \end{bmatrix}. \tag{2.18}$$

   and find e_4 by a similar process to be (for example)

$$e_4 = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}. \tag{2.19}$$

Three of the cases studied when $n = 4$, give the same representation for the Clifford algebra, namely $M(2, H)$. Nothing should be read into this result. The underlying geometries are quite different and it just goes to show, yet again, that algebra is only half the story. Still, it illustrates why, if you think algebraically alone, it is easy to be led astray.

2.1.2 Clifford algebras over $\mathbb{R}$

In quantum mechanics, we use the complex numbers. Nevertheless, in theories of fermions, it is often important to take into account reality constraints, and hence the properties of the Clifford algebras over $\mathbb{R}$ is quite relevant to physics, particularly when we discuss supersymmetric field theories and string theories in various spacetime dimensions. Moreover, in the physics-geometry interaction the beautiful structure of the Clifford algebras over $\mathbb{R}$ plays an important role. It is thus well worth the extra effort to understand the structures here.

By Sylvester's theorem, when working over $\mathbb{R}$ it suffices to consider

$$Q(e^\mu, e^\nu) = \eta^{\mu\nu} \tag{2.20}$$

where $\eta^{\mu\nu}$ is diagonal with ± 1 entries. Since there are different conventions in the literature, we will denote the Clifford algebra with

$$\begin{aligned} (e^1)^2 = \ldots = (e^{r_+})^2 = +1 \\ (e^{r_+ + 1})^2 = \ldots = (e^{r_+ + s_-})^2 = -1 \end{aligned} \tag{2.21}$$

by $Cl(r_+, s_-)$. The Clifford algebra is thus specified by a pair of nonnegative integers labelling the number of $+1$ and -1 eigenvalues. If the form is definite we simply denote $Cl(r_+)$ or $Cl(s_-)$.[8]

Now, we discuss about the real Clifford algebras in low dimension. In 0 dimensions, we have $Cl(0) = \mathbb{R}$.

We take the $Cl(1_-)$, to see this note that the general element in $Cl(1_-)$ is $a + be^1$. However we have a faithful matrix representation $e^1 \to \sqrt{-1}$. (Note that in fact we have two inequivalent representations, depending on the sign of the squareroot). Thus, one has $a + be^1 \to a + ib$ defines as isomorphism to $\mathbb{C}$, regarded as an algebra over $\mathbb{C}$

$$Cl(1_-) \cong \mathbb{C} \tag{2.22}$$

There is only one real irreducible representation up to equivalence: $V \cong \mathbb{R}^2$ and

$$\rho(a + be^1) = \begin{pmatrix} a & b \\ -b & a \end{pmatrix} \tag{2.23}$$

It is noted that

$$\rho'(a + be^1) = \begin{pmatrix} a & -b \\ b & a \end{pmatrix} \tag{2.24}$$

defines an equivalent representation.

Over $\mathbb{C}$, these matrices can be diagonalized, leading to two inequivalent complex representations of $Cl(1)$. Applying the rule for the complexification of a real vector space with complex structure we have $\mathbb{C} \otimes_{\mathbb{R}} \mathbb{C} \cong \mathbb{C} \oplus \overline{\mathbb{C}}$. The inequivalent representations are

$$\rho_+(a + be^1) = a + bi, \tag{2.25}$$

where one has

$$\rho_-(a + be^1) = a - bi. \tag{2.26}$$

[8] I find it impossible to remember whether the first or second entry should be the number of positive or negative signs. Therefore, I will explicitly put a subscript$\pm$ to indicate the signature. Lawson-Michelsohn take $Cl(r, s) = Cl(r_-, s_+)$.

For $Cl(1_+)$, the multiplication law of elements $a + be^1$ is simply

$$(a + be^1).(a' + b'e^1) = (aa' + bb') + (ab' + a'b)e^1. \tag{2.27}$$

In this dimension, we can introduce central projection operators

$$P_\pm = \frac{1}{2}(1 \pm e^1). \tag{2.28}$$

Then, $P_+Cl(1_+)$ and $P_-Cl(1_+)$ are subalgebras and we can write a direct sum of algebras:

$$Cl(1_+) = P_+Cl(1_+) \oplus P_-Cl(1_+). \tag{2.29}$$

In this case, each of the subalgebras is one-dimensional

$$a + be^1 = (a + b)(\frac{1 + e^1}{2}) + (a - b)(\frac{1 - e^1}{2}). \tag{2.30}$$

In particular, we have

$$Cl(1_+) = \mathbb{R} \oplus \mathbb{R}. \tag{2.31}$$

This algebra is also known as the "double numbers." There are two inequivalent real representations:

$$\rho_+(a + be^1) = a + b \tag{2.32}$$

$$\rho_-(a + be^1) = a - b \tag{2.33}$$

These representations are not faithful. We have a faithful matrix

$$a + be^1 \rightarrow \begin{pmatrix} a & b \\ b & a \end{pmatrix}. \tag{2.34}$$

Of course, the representation(2.34) is in fact reducible. It is equivalent to matrix of the form

$$\begin{pmatrix} a + b & 0 \\ 0 & a - b \end{pmatrix} \tag{2.35}$$

However, one needs both diagonal entries to get a faithful representation. Later, we will talk about $\mathbb{Z}_2$ graded representations. The minimal $\mathbb{Z}_2$ graded representation is 2−dimensional given by (2.34).

Finally, over $\mathbb{C}$, for Cl_1 we could also have taken $(e^1)^2 = +1$

$$\rho_+(a + be^1) = a + b \tag{2.36}$$

and where one has

$$\rho_-(a + be^1) = a - b \tag{2.37}$$

with $a, b \in \mathbb{C}$ reflecting the complexification of the two representations of $Cl(1_+)$. Thus

$$Cl(1) \cong \mathbb{C} \oplus \mathbb{C}. \tag{2.38}$$

For two dimensions, let us introduce the very useful notation

$$\mathbb{K}(n) := \mathrm{Mat}_{n\times n}(\mathbb{K}) \tag{2.39}$$

where one has $\mathbb{K} = \mathbb{R}, \mathbb{C}, \mathbb{H}$. This is an algebra over $\mathbb{R}$ of real dimension $n^2, 2n^2, 4n^2$, respectively.

In two dimensions, we have

$$\begin{aligned} Cl(2_+) &= \mathbb{R}(2) \\ Cl(1_+, 1_-) &= \mathbb{R}(2) \\ Cl(2_-) &= \mathbb{H} \end{aligned} \tag{2.40}$$

To see this, consider first $Cl(2_+)$. Then we give a faithful matrix defied as follows

$$e^1 \to \sigma_1 \quad e^2 \to \sigma_3. \tag{2.41}$$

Then, one has

$$e^1e^2 = -i\sigma_2 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}. \tag{2.42}$$

Now, we can write an arbitrary 2×2 real matrix as a linear combination of $1, \sigma_1, \sigma_3, -i\sigma_2$

$$\rho(a + be^1 + ce^2 + de^1e^2) = \begin{pmatrix} a+c & b-d \\ b+d & a-c \end{pmatrix}. \tag{2.43}$$

Consider the next model $Cl(2-)$. We can use the following mapping associated with the imaginary unit quaternions

$$\begin{aligned} e^1 &\to i \\ e^2 &\to j \\ e^1e^2 &\to k \end{aligned} \tag{2.44}$$

Finally, one has $Cl(1,1)$. Again we can provide a faithful matrix as follows

$$e^1 \to \sigma_1 \quad e^2 \to i\sigma_2. \tag{2.45}$$

Thus, one has

$$\rho(a + be^1 + ce^2 + de^1e^2) = \begin{pmatrix} a+d & b+c \\ b-c & a-d \end{pmatrix}. \tag{2.46}$$

The algebra is that of $\mathbb{R}(2)$. Where one has the following remarks

- When we complexify there is no distinction between the signatures. Any of the above three algebras can be used to show that

$$Cl(2) \cong \mathbb{C}(2). \tag{2.47}$$

- The representation matrices are always denoted as Γ matrices in the physics literature. Thus, for example, what we are saying above is that in $1+1$ dimensions we could choose an irreducible real representation

$$\Gamma^0 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \quad \Gamma^1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}. \tag{2.48}$$

2.1.3 Clifford algebras over $\mathbb{C}$

When we change the ground field from $\mathbb{R}$ to $\mathbb{C}$ the structure of the Clifford algebras simplifies considerably.

Now there is no need to take account of the signature. We can take $e_ie_j + e_je_i = 2\delta_{ij}$. Or we could instead take $e_ie_j + e_je_i = -2\delta_{ij}$. Both conventions lead to equivalent results. Since we are working over the complex numbers we could also send $e_i \to \sqrt{-1}e_i$.

We have already shown that in low dimensions

$$\begin{aligned} \mathbb{C}l(0) &= \mathbb{C} \\ \mathbb{C}l(1) &\cong \mathbb{C} \oplus \mathbb{C} \\ \mathbb{C}l(2) &\cong \mathbb{C}(2) \end{aligned} \tag{2.49}$$

Next, the periodicity is simplified considerably. We now have the following

$$\mathbb{C}l(d+2) \cong \mathbb{C}l(d) \otimes_{\mathbb{C}} \mathbb{C}l(2) \cong \mathbb{C}l(d) \otimes_{\mathbb{C}} \mathbb{C}(2) \tag{2.50}$$

from which we obtain the basic result:

$d = 0\text{mod}2$:

$$\mathbb{C}l(d) = \mathbb{C}(2^{[d/2]}) = \mathbb{C}(2^{d/2}) \tag{2.51}$$

$d = 1\text{mod}2$:

$$\mathbb{C}l(d) = \mathbb{C}(2^{[d/2]}) \oplus \mathbb{C}(2^{[d/2]}) = \mathbb{C}(2^{(d-1)/2}) \oplus \mathbb{C}(2^{(d-1)/2}) \tag{2.52}$$

It is recalled that we define the Clifford volume form to be $\omega := e_1 e_2 \dots e_d$, and show moreover that

$$\omega^2 = (-1)^{\frac{1}{2}d(d-1)} = \begin{cases} +1 & d = 0, 1\text{mod}4 \\ -1 & d = 2, 3\text{mod}4 \end{cases} \tag{2.53}$$

(If we had taken $e_i e_j + e_j e_i = -2\delta_{ij}$ then we would have gotten $(-1)^{\frac{1}{2}d(d+1)}$). When working over $\mathbb{C}$ we can always find a scalar ξ that $\omega_c := \xi\omega$ satisfies $(\omega_c)^2 = 1$. Explicitly, one has

$$\omega_c = \begin{cases} \omega & d = 0, 1\text{mod}4 \\ i\omega & d = 2, 3\text{mod}4 \end{cases} \tag{2.54}$$

Then, we can form projection operators given by

$$P_\pm = \frac{1}{2}(1 \pm w_c) \tag{2.55}$$

When d is odd these projection operators are central "they commute with everything in the algebra". This gives a decomposition of the algebra into a direct sum of subalgebras as in (2.52). That is elements of the form $P_+\phi$ form a subalgebra, not just a vector subspace, because P_+ is central. This subalgebra is isomorphic to $\mathbb{C}l(2^{[d/2]})$ and similarly for the subalgebra of elements of the form $P_-\phi$.

2.2 Representation of the Clifford algebras

Let us now consider representations of the Clifford algebras. This assigns $\phi \in Cl \to \rho(\phi)$, where $\rho(\phi)$ is a linear transformation such that $\rho(\phi_1)\rho(\phi_2) = \rho(\phi_1\phi_2)$. The matrix representations of the generators $\rho(e^\mu) = \Gamma^\mu$ are called $\Gamma-$matrices in the physics literature.

We have seen how to write all the Clifford algebras in terms of matrix algebras over the division algebras. That is, they are all of the form $\mathbb{K}(n)$ or $\mathbb{K}(n) \oplus \mathbb{K}(n)$.

Now we only need a standard result of algebra which says that $\mathbb{K}(n)$ is a simple algebra (it has no nontrivial two-sided ideals) and hence has a unique representation (up to isomorphism) when $\mathbb{K}$ is a division algebra.

If V is an irreducible representation then we have a homomorphism $A \to \mathrm{End}(V)$. Since V is irreducible the image cannot commute with any nontrivial projection operators and hence must be the full algebra $\mathrm{End}(V)$. On the other hand, the kernel would be a two-sided ideal in A. Now, if $A = M_n(D)$, where D is a division algebra then it is simple. To prove this, consider any ideal $I \subset M_n(D)$. If $X = x_{ij}e_{ij}$ is in I with some $x_{kl} \neq 0$ then $e_{kk}X_{e_{ll}} \in I$, but this means $e_{kl} \in I$ but now the ideal generated by e_{kl} is all of $M_n(D)$.

The nature of the representation depends very much on the fields we are working over.

Consider first the complex Clifford algebras, $\mathbb{C}l(d)$. These are of the form $\mathbb{C}(n)$ or $\mathbb{C}(n) \oplus \mathbb{C}(n)$. The unique irrep of $\mathbb{C}(n)$ is just the defining representation space $V = \mathbb{C}^n$. Thus, $\mathbb{C}l(d)$ has a unique irrep for d even, $V = \mathbb{C}^n$ with

$$n = 2^{d/2} \quad d \quad \text{even} \tag{2.56}$$

While for d odd, $\mathbb{C}l(d)$ has two inequivalent irreps $V_\pm$. As vector spaces $V_\pm \cong \mathbb{C}^n$ with

$$n = 2^{(d-1)/2} \quad d \quad \text{odd} \tag{2.57}$$

where $\rho(\phi_1 \oplus \phi_2) = \phi_1$ on V_+ and $\rho(\phi_1 \oplus \phi_2) = \phi_2$ on V_-. Another way to characterize these is to consider ω_c, with $(\omega_c)^2 = 1$. Then, one has

$$\rho_+(\omega_c) = +1 \quad \text{on} \quad V_+ \tag{2.58}$$

$$\rho_-(\omega_c) = -1 \quad \text{on} \quad V_-. \tag{2.59}$$

Now let us consider the Clifford algebras $Cl(r_+, s_-)$ over $\mathbb{R}$. These are all of the form $\mathbb{K}(n)$ or $\mathbb{K}(n) \oplus \mathbb{K}(n)$. Now, one can form an $\mathbb{K}-$linear representation of $\mathbb{K}(n)$ by having the matrices act on $\mathbb{K}^n$. Here $\mathbb{K} = \mathbb{R}, \mathbb{C}, \mathbb{H}$. Note that for the quaternions we must take care that if the matrix multiplication acts from the left on a column vector then the scalar multiplication by $\mathbb{H}$ acts on the right. Again $\mathbb{K}(n)$ is a simple algebra and the unique irrep up to isomorphism is $\mathbb{K}^n$.

Of course, we can consider $\mathbb{K}^n$ to be a real vector space of real dimension $n, 2n, 4n$ for $\mathbb{K} = \mathbb{R}, \mathbb{C}, \mathbb{H}$.

Conversely, a real vector space V is said to have a complex structure if there is a linear transformation $I \in End_{\mathbb{R}}(V)$ which satisfies $I^2 = -1$. With a choice of I, V can be made into a complex vector space of dimension $\frac{1}{2}\dim_{\mathbb{R}}(V)$.

Similarly, a real vector space V is said to have a quaternionic structure if there are linear transformations I, J, K with

$$I^2 = J^2 = k^2 = IJK = -1. \tag{2.60}$$

A choice of I, J, K makes V have a multiplication by quaternions, and defines an isomorphism to $\mathbb{H}^d$ with $d = \frac{1}{4}\dim_{\mathbb{R}}(V)$. For example on $\mathbb{H}^n$ the linear operators I, J, K would be right-multiplication by $\mathbf{i}$, $\mathbf{j}$, $\mathbf{k}$, respectively.

We say that the real representation V of $Cl(r_+, s_-)$ has a complex structure I if $[\rho(\phi), I] = 0$. Similarly, we say that it has a quaternionic structure if

$$[\rho(\phi), I] = [\rho(\phi), J] = [\rho(\phi), K] = 0 \tag{2.61}$$

We can now trivially read off the basic properties of Clifford algebra representations from the above tables. It recalled that $l := [d/2]$.

- If $d_T = s - r \neq 3\text{mod}4$ then $Cl(r_+, s_-)$ is a simple algebra and there is a unique irrep. It is $\mathbb{K}(N)$ acting on $\mathbb{K}^N$ where $N = 2^l$ or $N = 2^{l-1}$.

- If $d_T = s - r = 3\text{mod}4$ then $Cl(r_+, s_-)$ is sum of simple algebras and there are two inequivalent irreps $\rho_\pm$. Note that this is the case where the volume element satisfies $\omega^2 = 1$ and ω is central. We can characterize the representations as $\rho_\pm(\omega) = \pm 1$. Since ω is central these rep's cannot be equivalent.

- For $d_T = 6, 7, 8\text{mod}8$, $\dim_{\mathbb{R}} V = 2^l$. For $d_T = 6, 7, 8\text{mod}8$ we can represent Γ^μ by $2^l \times 2^l$ real matrices. In physics, this is called a Majorana representation of the Clifford algebra.

- Multiplying Γ matrices by a factor of i changes the signature. Therefore for $d_T = 0, 1, 2\text{mod}8$ we can represent Γ^μ by $2^l \times 2^l$ pure imaginary matrices. In physics this is called a Pseudo-Majorana representation.

- For $d_T = 1, 5\text{mod}8$ (i.e. $d_T = 1\text{mod}4$) $\dim_{\mathbb{C}} V = 2^l$ and hence $\dim_{\mathbb{R}} V = 2^{l+1}$, and V carries a complex structure commuting with the Clifford action. However, if we use $2^l \times 2^l$ matrices they must be complex.

- For $d_T = 2, 3, 4 \mathrm{mod} 8$, $\dim_{\mathbb{H}} V = 2^{l-1}$, $\dim_{\mathbb{C}} V = 2^l$, and hence $\dim_{\mathbb{R}} V = 2^{l+1}$, and V carries a quaternionic structure commuting with the Clifford action. Thus we can write a reresentation by $2^{l-1} \times 2^{l-1}$ quaternionic matrices. If we choose to write the quaternions as 2×2 complex matrices A such that $A^* = \sigma_2 A \sigma_2$, then we can represent the Γ^μ by $2^l \times 2^l$ complex matrices such that

$$\Gamma^{\mu,*} = J \Gamma^\mu J^- 1 \tag{2.62}$$

with $J = i\sigma^2 \oplus \ldots \oplus i\sigma^2$. note that $JJ^* = J^2 = -1$ and $J^{tr} = -J$.

Thus the pattern of representations is the following: ν =number of irreps of the algebra, d =real dimension of the irrep,

s	$Cl(s_-)$	$\nu(s)$	$d(s)$	Structure	K	$\dim_{\mathbb{C}}$(Irreps of $\mathbb{C}l(s)$)
1	$\mathbb{C}$	1	2	$\mathbb{C}$	$\mathbb{Z}$	1
2	$\mathbb{H}$	1	4	$\mathbb{H}$	$\mathbb{Z}$	2
3	$\mathbb{H} \oplus \mathbb{H}$	2	4	$\mathbb{H}$	$\mathbb{Z} \oplus \mathbb{Z}$	2
4	$\mathbb{H}(2)$	1	8	$\mathbb{H}$	$\mathbb{Z}$	4
5	$\mathbb{C}(4)$	1	8	$\mathbb{C}$	$\mathbb{Z}$	4
6	$\mathbb{R}(8)$	1	8	$\mathbb{R}$	$\mathbb{Z}$	8
7	$\mathbb{R}(8) \oplus \mathbb{R}(8)$	2	8	$\mathbb{R}$	$\mathbb{Z} \oplus \mathbb{Z}$	8
8	$\mathbb{R}(16)$	1	16	$\mathbb{R}$	$\mathbb{Z}$	16
9	$\mathbb{C}(16)$	1	32	$\mathbb{C}$	$\mathbb{Z}$	16
10	$\mathbb{H}(16)$	1	64	$\mathbb{H}$	$\mathbb{Z}$	32
11	$\mathbb{H}(16) \oplus \mathbb{H}(16)$	2	64	$\mathbb{H}$	$\mathbb{Z} \oplus \mathbb{Z}$	32
12	$\mathbb{H}(32)$	1	128	$\mathbb{H}$	$\mathbb{Z}$	64

$d = s + 1$	$Cl(s_+, 1_-)$	$\nu(s)$	$d(s)$	Structure	K
0+1	$\mathbb{C}$	1	2	$\mathbb{C}$	$\mathbb{Z}$
1+1	$\mathbb{R}(2)$	1	2	$\mathbb{R}$	$\mathbb{Z}$
2+1	$\mathbb{R}(2) \oplus \mathbb{R}(2)$	2	2	$\mathbb{R}$	$\mathbb{Z} \oplus \mathbb{Z}$
3+1	$\mathbb{R}(4)$	1	4	$\mathbb{R}$	$\mathbb{Z}$
4+1	$\mathbb{C}(4)$	1	8	$\mathbb{C}$	$\mathbb{Z}$
5+1	$\mathbb{H}(4)$	1	16	$\mathbb{H}$	$\mathbb{Z}$
6+1	$\mathbb{H}(4) \oplus \mathbb{H}(4)$	2	16	$\mathbb{H}$	$\mathbb{Z} \oplus \mathbb{Z}$
7+1	$\mathbb{H}(8)$	1	32	$\mathbb{H}$	$\mathbb{Z}$
8+1	$\mathbb{C}(16)$	1	32	$\mathbb{C}$	$\mathbb{Z}$
9+1	$\mathbb{R}(32)$	1	32	$\mathbb{R}$	$\mathbb{Z}$
10+1	$\mathbb{R}(32) \oplus \mathbb{R}(32)$	2	32	$\mathbb{R}$	$\mathbb{Z} \oplus \mathbb{Z}$
11+1	$\mathbb{R}(64)$	1	64	$\mathbb{R}$	$\mathbb{Z}$

Note that for $Cl(s_-), d(s + 8k) = 16^k d(s)$.

2.3 Clifford graph algebras

Let G be a graph with n vertices and no multiple edges and no loops. We associate with this graph a unital algebra A_G over $\mathbb{C}$, with n generators $e_1, e_2, \ldots, e_n$ corresponding to the vertices; relations $e_i^2 = -1$ for any i; and relations $e_i e_j = -e_j e_i$, if there is an edge between the i^{th} and j^{th} vertices, and $e_i e_j = e_j e_i$, if there is no edge between the i^{th} and j^{th} vertices.

We call the Clifford algebra associated with a given graph a Clifford graph algebra.

The classical Clifford algebra in Section 1 is a Clifford graph algebra of the complete graph with n vertices. If the graph does not have any edges, then its Clifford graph algebra is commutative.

2.3.1 Clifford adjacency matrix

What the adjacency matrix fails to provide, however, is a method of counting self-avoiding walks and cycles in G. On the other hand, labeling vertices with the nilpotent generators of Cl_n^{nil} and constructing a "nil-Clifford adjacency matrix" overcomes this problem. We define the nil-Clifford adjacency matrix associated with G by

$$A_{ij} = \begin{cases} v_i & \text{if} \quad (v_i, v_j) \in E(G) \\ 0 & \text{otherwise.} \end{cases} \tag{2.63}$$

Observe that $A \in \mathrm{Mat}(Cl_n^{nil}, n)$, the algebra of $n \times n$ matrices with entries in the abelian nilpotent-generated algebra Cl_n^{nil}.

Let A be the nil-Clifford adjacency matrix of any simple graph G on n vertices. For any $m > 1$ and $i \neq j$, summing the coefficients of $(A^m)_{ii}$ yields the number of m-cycles based at v_i occurring in G.

2.3.2 Center of a Clifford graph algebra

As in the case of the classical Clifford algebra, a Clifford graph algebra has dimension 2^n over $\mathbb{C}$, and has a basis of monomials e_α, over all binary strings α of length n, or, correspondingly subsets of the set $\{1, 2, \ldots, n\}$.

We would like to describe the center of a Clifford graph algebra. Often, understanding the center of a ring is a first step towards understanding its structure and the structre of its representations. In case of Clifford graph algebras the dimension of the center uniquely determines the structure of the given Clifford algebra. We will denote the center of algebra A by $Z(A)$.

Suppose that the element $c = \sum a_\alpha e_\alpha$ is in the center, where e_α are monomials. As each monomial either commutes or anticommutes with the basis elements e_i, every monomial e_α lies in the center. Hence, to analyze the center it is enough to analyze central monomials. A monomial e_α is central iff for each vertex i, the number of edges connecting it to the set of vertices α is even.

Let us describe monomials in the center in terms of the adjacency matrix. It is recalled that the adjacency matrix of a graph G is the matrix $(a_{ij}^n)_{i,j=1}$ such that $a_{ij} = 1$ if the vertices i and j are connected and $a_{ij} = 0$ if they are not connected. A central monomial e_α corresponds to a subset of vertices α. If we add up rows of the adjacency matrix corresponding to this subset, the resulting row vector will have only even entries. That is, this vector is the zero vector modulo 2.

The elements of the center form a subalgebra, and the dimension of the center is always a power of 2 [23].

2.3.3 Examples

Let us consider path graphs and star graphs as examples. Not only providing examples can help the readers to understand the construction, but we also have other reasons for providing these particular examples, which we will reveal at the end of this section.

Let us start with a path graph P on n vertices. Vertices i and j are connected by an edge iff $\mid i - j \mid = 1$ (Figure 2.1).

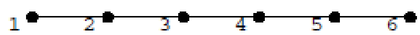


Figure 2.1: Path Graph on 6 Vertices.

If n is even, the center of the corresponding Clifford graph algebra A_P is one-dimensional: $Z(A_P) \simeq \mathbb{C}$. If n is odd, then the center also contains the monomial $e_1 e_3 \ldots e_n$ — the product of odd-numbered generators. In (Figure 2.2), it has been observed that a path graph with 7 vertices. The monomial $e_1 e_3 e_5 e_7$ is in the center of this Clifford graph algebra.

Figure 2.2: Path Graph on 7 Vertices.

Let us compare the Clifford algebras of the complete graph and the path graph with n vertices. If we denote the generators of the Clifford algebra of the complete graph as

e_i and the generators of the Clifford algebra of the path graph as e'_i, we can show that the Clifford algebras of these two graphs are isomorphic by presenting an isomorphism:

$$e'_1 = e_1, \quad \text{and} \quad e'_i = e_{i-1}e_i, \quad \text{for} \quad i \neq 1.$$

The other way is

$$e_i = e'_i e'_{i-1} \dots e'_1.$$

Let us consider the star graph on n vertices (Figure 2.3).

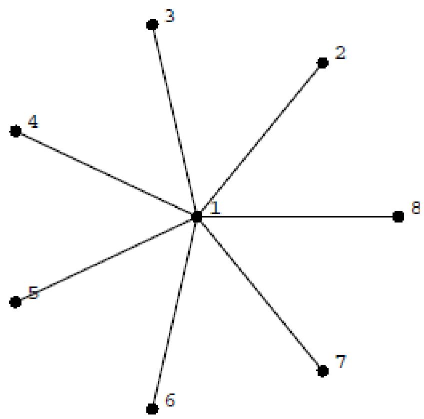


Figure 2.3: Star Graph on 8 Vertices.

In this case, the vertex number 1 connects to every other vertex, and there are no other edges. The center is a linear combination of monomials of even length such that they do not contain e_1. The dimension of the center is 2^{n-2}. Let us show that the Clifford algebra of the star graph is isomorphic to the Clifford algebra of a graph with n vertices and one edge. We denote the generators of the Clifford algebra of the star graph as e_i and the generators of the Clifford algebra of the graph with n vertices and one edge as e'_i, where we number the vertices adjacent to the edge as 1 and 2, so that $e'_1 e'2 = -e'_2 e'_1$. All other pairs of generators commute. We can show that the Clifford algebras of these two graphs are isomorphic by presenting an isomorphism

$$e'_1 = e_1, e'_2 = e_1 e_2 \quad \text{and for} \quad i > 2, e'_i = e_2 e_i,$$

where we have

$$e_1 = e'_1, e_2 = e'_1 e'_2 \quad \text{and for} \quad i > 2, e_i = e'_i e'_1 e'_2.$$

Now it is time to explain this formalism via two examples. The path graph and the star graph with n vertices are quite similar. They have the same number of edges — $n-1$, not to mention that for n equal 2 or 3 the path and the star graphs are isomorphic. On the other hand, the Clifford algebra of a path graph has the smallest possible center and the Clifford algebra of a star graph has largest center for a non-commutative Clifford graph algebra of this size.

2.4 Clifford graph algebra structure

Every central monomial in a Clifford graph algebra corresponds to a central idempotent. Suppose e_α is a central monomial. Then, $e_\alpha^2 = \pm 1$. Let us denote by f_α a multiple of e_α such that $f_\alpha^2 = 1$. If $e_\alpha^2 = 1$, then $f_\alpha = e_\alpha$ and if $e_\alpha^2 = -1$ then $f_\alpha = ie_\alpha$. The element $c = (1 + f_\alpha)/2$ is a central idempotent. Being a central idempotent means that c is in the center and $c^2 = c$, which is easy to check. The central idempotent c gives rise to a decomposition of our Clifford graph algebra A as a direct sum of Ac and $A(1-c)$ [50].

Let us take an index j that belongs to the set α, that is, e_j is one of the generators that the monomial e_α contains. Denote the graph G with the vertex corresponding to j removed by G_{-j}. The algebra $A_{G_{-j}}$ is naturally embedded into A_G. The map $x \to xc$ is an isomorphism between this embedding of $A_{G_{-j}}$ and Ac. Similarly, the map $x \to x(1-c)$ is an isomorphism between the embedding of $A_{G_{-j}}$ and $A(1-c)$. Hence, A_G is a direct sum of two copies of $A_{G_{-j}}$. We can continue the decomposition until we get a graph such that its Clifford algebra has a one-dimensional center.

If the dimension of the center of A_G is 2^k, then A_G is a direct sum of 2^k copies of the Clifford graph algebra of a subgraph G' of G, such that A'_G has a one-dimensional center.

By the Artin-Wedderburn theorem [51] we can show that a Clifford graph algebra is isomorphic to a direct sum of matrix algebras. Therefore, any Clifford graph algebra is a direct sum of 2^k copies of an $m \times m$ matrix algebra, for some k. We can deduce from here that $m = 2^{(n-k)/2}$. In particular, we see that for graphs with odd number of vertices, the corresponding Clifford graph algebra has to have a center of dimension more than 1.

In the examples, we saw that the Clifford graph algebra of a complete graph is isomorphic to a Clifford algebra of a path graph. For n even, both of these algebras are isomorphic to a matrix algebra over the $2^{n/2}$-dimensional space. For odd n, the Clifford algebra of the complete graph is a direct sum of two Clifford graph algebras of the complete graph for $n-1$. The same goes for the path graph.

The Clifford algebra corresponding to a star graph is the direct sum of 2^{n-2} copies of a 2×2 matrix algebra.

2.4.1 Small graphs

Previously, we covered complete graphs, path graphs and star graphs in the examples. Now, we would like to cover all graphs with small number of vertices and see how their Clifford graph algebras are different from each other [23].

Let us denote by $Cliff_k(n)$ the number of graphs with n vertices such that the center of their Clifford graph algebras has dimension 2^k.

For $n = 1$, there is only one graph and its algebra is $\mathbb{C} \oplus \mathbb{C}$. $Cliff_2(1) = 1$.

For $n = 2$, there are two graphs — a complete graph whose Clifford algebra is a 2×2 matrix algebra $Mat(2)$ and the dual graph without edges whose Clifford algebra is $\mathbb{C} \oplus \mathbb{C} \oplus \mathbb{C} \oplus \mathbb{C}$. Thus, $Cliff_1(2) = 1$ and $Cliff_4(2) = 1$.

For $n = 3$, there are four graphs. The Clifford algebra of a graph with no edges is commutative, thus it has an 8-dimensional center and is equal to the direct sum of 8 copies of $\mathbb{C}$. All other graphs (Figure 2.4) have a two-dimensional center, hence they are isomorphic to $Mat(2) \oplus Mat(2)$. This, $Cllif_2(3) = 3$ and $Cllif_8(3) = 1$.

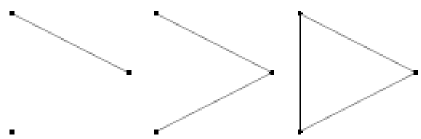


Figure 2.4: Graphs with 3 vertices and non-commutative Clifford algebras.

For $n = 4$, there are 11 graphs. Below we describe all the graphs with 4 vertices and present the dimension of the center of the corresponding Clifford algebra.

1. There is one graph without edges — the dimension is 16.

2. There is one graph with one edge — the dimension is 4.

3. There are two graphs with two edges: the graph with two adjacent edges — the dimension is 4; the graph with 2 non-adjacent edges — the dimension is 1.

4. There are 3 graphs with 3 edges: the complete graph with 3 vertices plus 1 isolated vertex — the dimension is 4; the path graph — the dimension is 1; the star graph — the dimension is 4.

5. There are two graphs with 4 edges: the cycle graph — the dimension is 4; the kite graph which is the dual graph to the graph with two edges adjacent to each other — the dimension is 1.

6. There is one graph with 5 edges — the dimension is 4.

7. The only graph with 6 edges is the complete graph — the dimension is 1.

We can see graphs with 4 vertices and their corresponding dimensions in (Figure 2.5). As we calculated, $Cliff_1(4) = 4$, $Cliff_4(4) = 6$, and $Cliff_{16}(4) = 1$.

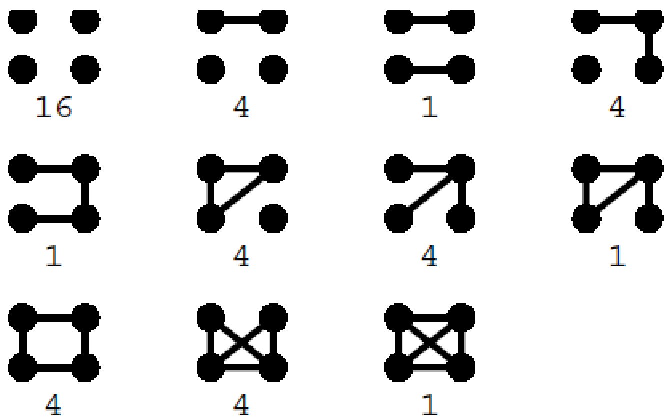


Figure 2.5: Graphs with four vertices.

2.4.2 Clifford Class

Let us say that two graphs with the same number of vertices belong to the same Clifford class if the centers of their Clifford algebras have the same dimension. If two graphs are in the same Clifford class then their Clifford algebras are isomorphic.

Let us consider a special graph with $2k+m$ vertices and k edges which has m isolated vertices and $2k$ vertices of degree one. we denote this graph as $G(k,m)$. This graph is the union of k K_2-graphs and m K_1-graphs. (Here we use the standard definition, K_n, for a complete graph with n vertices.) It is easy to check that the center of the Clifford algebra of this graph has dimension 2^m and the algebra itself is isomorphic to the direct sum of 2^m matrix algebras of size $2^k \times 2^k$.

Suppose G is a graph with at least one edge, then there is a graph G' such that G' is a union of $K2$ and some graph and A_G is isomorphic to A'_G. To demonstrate that, let us number the vertices in graph G in such a way that the vertices numbered 1 and 2 are connected. Suppose e_i are the generators in A_G — the Clifford algebra of the graph G. We build new generators in the algebra A_G. We have $e'_1 = e_1$ and $e'_2 = e_2$ [23]. For every $i > 2$ let us choose a new generator e'_i in the following way:

- If the vertex i is not connected to either vertex 1 or vertex 2, then $e'_i = e_i$.
- If the vertex i is connected to vertex 1 and is not connected to vertex 2, then $e'_i = ie_ie_2$.
- If the vertex i is not connected to vertex 1 and is connected to vertex 2, then $e'_i = ie_ie_1$.
- If the vertex i is connected to both vertices 1 and 2, then $e'_i = e_ie_2e_1$.

The new generators e'_i have the property $e'^2_i = -1$ and $e_ie_j = \pm e_je_i$. Hence, we can build a graph G' for which the generators e'_i generate its Clifford graph algebra.

The algebra A'_G is isomorphic to A_G. The generators e'_i commute with both e'_1 and e'_2 for $i > 2$. This means that vertices numbered 1 and 2 in the graph G' are isolated from the rest of the graph G'. From here we see that G' is the union of K_2 and another graph.

Each Clifford class has exactly one representative of type $G(k,m)$, and a Clifford algebra of a graph is isomorphic to a direct sum of 2^m copies of the matrix algebra over 2^k-dimensional space for some k and m such that $n = 2k + m$.

2.5 More on Clifford algebras and graphs

2.5.1 Odd-determinant graphs

Let us call a graph an odd-determinant graph if its Clifford graph algebra has a one-dimensional center. And The odd-determinant graphs are the graphs whose adjacency matrix has an odd-determinant.

By the discussion above, the parity of the determinant of the adjacency matrix is invariant under basic construction, and by using the basic construction we can replace a given graph G with a canonical graph $G(k, 0)$ such that their Clifford algebras are isomorphic.

Obviously, an odd-determinant graph does not have isolated vertices. Also, all odd-determinant graphs are in the same Clifford class. No graphs in the Clifford class of an odd-determinant graph have isolated vertices. And vice versa, if no graphs in the Clifford class of a graph have isolated vertices then all the graphs in this class are odd-determinant graphs.

From the classification theorem it follows that an odd-determinant graph has a complete graph on even number of vertices in its class.

In the next section we discuss other sets of graphs that are related to odd-determinant graphs.

2.5.2 Mating graphs

By definition, a mating graph, sometimes called a point-determining graph [52], is a graph such that no two vertices have identical sets of neighbors.

An odd-determinant graphs are mating graphs*. If two vertices i and j of the graph G have the same set of neighbors, then the product of the corresponding generators $e_i e_j$ is in the center of the Clifford graph algebra A_G. Hence, odd-determinant graphs can not have two vertices with the same set of neighbors.

The converse of the(*) is not true. There are mating graphs that are not odd-determinant graphs. The smallest examples have three vertices. Graphs with an odd number of vertices can not be odd-determinant. At the same time there are 2 mating graphs with 3 vertices: K_3, and the union of K_2 and an isolated point(Figure 2.6).

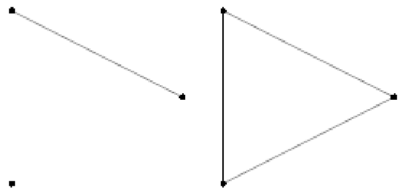


Figure 2.6: Mating graphs with 3 vertices.

It is easy to see that complete graphs are always mating graphs. The set of neighbors of each vertex is uniquely determined by the vertex itself: the neighbors of the vertex are the set of all the other vertices. That means the complete graphs with an odd number of vertices give an infinite set of examples of mating and not odd-determinant graphs. Similarly, the union of a complete graph with an isolated point is a mating graph.

Below we present all mating graphs with 4 vertices. Only one of them is not an odd-determinant graph — the union of the complete graph K_3 and an isolated vertex.

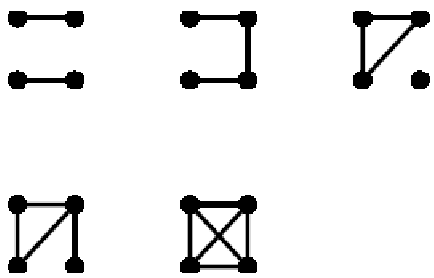


Figure 2.7: Mating graphs with 4 vertices.

We see that if we want to estimate the number of odd-determinant graphs we have a natural bound. The number of such graphs is not more than the number of mating graphs.

There is a natural set of graphs that is somewhat in between odd-determinant graphs and mating graphs. This is the set of graphs with invertible adjacency matrices.

Chapter 3

Tensor Network Theory

In his chapter, we introduce some basic definitions and concepts of tensor network. We show that the tensor network can be used to represent quantum many-body states, where we explain Matrix Product State (MPS) in $1D$ and Projective Entanglement Pair State (PEPS) in $2D$ systems, as well as the generalizations to thermal states and operators. The quantum entanglement properties of the tensor network states including the area law of entanglement entropy will also be discussed. Finally, we present several special tensor network's that can be exactly contracted, and demonstrate the difficulty of contracting tensor network's in general cases.

3.1 Generalities on tensors

Generally speaking, a tensor is defined as a series of numbers labeled by N indexes, with N called the order of the tensor. In this context, a scalar, which is one number and labeled by zero index, is a 0th-order tensor. Many physical quantities are scalars, including energy, free energy, magnetization and so on. Graphically, we use a dot to represent a scalar (Figure 3.1).

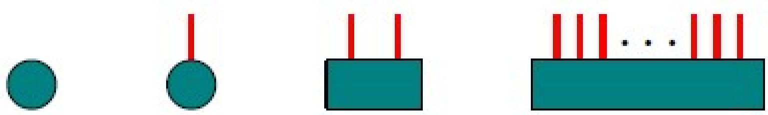


Figure 3.1: The graphic representations of a scalar, vector, matrix, and tensor

A D-component vector consists of D numbers labeled by one index, and thus is a 1st-order tensor. For example, one can write the state vector of a spin-1/2 in a chosen basis (say the eigenstates of the spin operator $\hat{S}^{[z]}$) as

$$|\psi\rangle = C_0|0\rangle + C_1|1\rangle = \sum_{s=0,1} C_s|s\rangle, \tag{3.1}$$

where the coefficients C are two-component vector. Here, we use $|0\rangle$ and $|1\rangle$ to represent spin up and down states. Graphically, we use a dot with one open bond to represent a vector (Figure 3.1).

A matrix is in fact a 2nd-order tensor. Considering two spins as an example, the state vector can be written under a irreducible representation as a four-dimension vector. Instead, under the local basis of each spin, we write it as

$$|\psi\rangle = C_{00}|0\rangle|0\rangle + C_{01}|0\rangle|1\rangle + C_{10}|1\rangle|0\rangle + C_{11}|1\rangle|1\rangle = \sum_{ss'=0}^{1} C_{ss'}|s\rangle|s'\rangle, \tag{3.2}$$

where $C_{ss'}$ is a matrix with two indexes. Here, one can see that the difference between a $(D * D)$ matrix and a D^2-component vector in our context is just the way of labeling the tensor elements. Transferring among vector, matrix and tensor like this will be frequently used later. Graphically, we use a dot with two bonds to represent a matrix and its two indexes (Figure 3.1).

It is then natural to define an N-th order tensor. Considering, N spins, the 2^N coefficients can be written as a N-th order tensor C, [1] satisfying

$$|\psi\rangle = \sum_{s_1 \ldots s'_N=0}^{1} C_{s_1 \ldots s'_N}|s_1\rangle \ldots |s'_N\rangle. \tag{3.3}$$

Similarly, such a tensor can be reshaped into a 2^N-component vector. Graphically, an N-th order tensor is represented by a dot connected with N open bonds (Figure 3.1).

In above, we use states of spin-1/2 as examples, where each index can take two values. For a spin-S state, each index can take $d = 2S + 1$ values, with d called the bond dimension. Besides quantum states, operators can also be written as tensors. A spin-1/2 operator $\hat{S}^{\alpha}$ $(\alpha = x, y, z)$ is a $(2 * 2)$ matrix by fixing the basis, where we have $S^{\alpha}_{s'_1 s'_2 s_1 s_2} =< s'_1 s'_2|\hat{S}^{\alpha}|s_1 s_2 >$. In the same way, an N-spin operator can be written as a $2N$-th order tensor, with N bra and N ket indexes.[2]

[1] If there is no confuse, we use the symbol without all its indexes to refer to a tensor for conciseness, use C to represent $C_{s_1 \ldots s_N}$.

[2] Note that here, we do not distinguish *bra* and *ket* indexes deliberately in a tensor, if not necessary.

We would like to stress some conventions about the "indexes" of a tensor (including matrix) and those of an operator. A tensor is just a group of numbers, where their indexes are defined as the labels labeling the elements. Here, we always put all indexes as the lower symbols, and the upper "indexes" of a tensor (if exist) are just a part of the symbol to distinguish different tensors. For an operator which is defined in a Hilbert space, it is represented by a hatted letter, and there will be no "true" indexes, meaning that both upper and lower "indexes" are just parts of the symbol to distinguish different operators.

3.2 Tensor network and tensor network states

3.2.1 A simple example of two spins and Schmidt decomposition

After introducing tensor (and it diagram representation), now we are going to talk about tensor network, which is defined as the contraction of many tensors. Let us start with the simplest situation, with two spins. For instance, we consider to study the quantum entanglement properties, which can be defined by the Schmidt decomposition [53–55] of the state (Figure 3.2) as

$$|\psi\rangle = \sum_{ss'}^{1} |s\rangle |s\rangle' = \sum_{ss'=0}^{1} \sum_{\alpha=0}^{\chi} U_{s\alpha} \lambda_{\alpha\alpha'} V^{*}_{\alpha s'} |s\rangle |s\rangle' , \tag{3.4}$$

where U and V are unitary matrices, λ is a positive-defined diagonal matrix in descending order,[3] and χ is called the Schmidt rank. λ is called the Schmidt coefficients since in the new basis after the decomposition, the state is written in a summation of χ product states as $|\psi\rangle = \sum_{\alpha} \lambda_{\alpha} |u\rangle_{\alpha} |v\rangle_{\alpha}$, with the new basis $|u\rangle_{\alpha} = \sum_{s} U_{s\alpha} |s\rangle$ and $|v\rangle_{\alpha} = \sum_{s'} V^{*}_{s\alpha} |s\rangle'$.

Graphically, we have a small tensor network, where we use green squares to represent the unitary matrices U and V, and a red diamond to represent the diagonal matrix λ. There are two bonds in the graph shared by two objects, standing for the summations (contractions) of the two indexes in (3.4), α and α'. Unlike s (or s'). The space of the index α (or α') is not from any physical Hilbert space. To distinguish these two kinds, we call the indexes like s the physical indexes and those like α the geometrical or virtual

[3] Sometime, λ is treated directly as a χ-component vector.

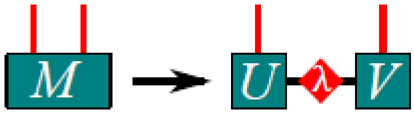


Figure 3.2: The graphic representation of the Schmidt decomposition (singular value decomposition of a matrix). The positive-defined diagonal matrix λ, which gives the entanglement spectrum (Schmidt numbers), is defined on a virtual bond (dumb index) generated by the decomposition.

indexes. Meanwhile, since each physical index is only connected to one tensor, it is also called an open bond.

Some simple observations can be made from the Schmidt decomposition. Generally speaking, the index α (also α' since λ is diagonal) contracted in a tensor network carry the quantum entanglement [56]. In quantum information sciences, entanglement is regarded as a quantum version of correlation [56], which is crucially important to understand the physical implications of tensor network. One usually uses the entanglement entropy to measure the strength of the entanglement, which is defined as $S = -2\sum_{\alpha=1}^{\chi} \lambda_\alpha^2 \ln \lambda_\alpha$. Since the state should be normalized, we have $\sum_{\alpha=1}^{\chi} \lambda_\alpha^1 = 1$. For dim(a)$= 1$, obviously $|\psi\rangle = \lambda_1 |u\rangle_1 |v\rangle_1$ is a product state with zero entanglement $S = 0$ between the two spins. For dim(a)$= \chi$, the entanglement entropy $S \leq \ln \chi$, where S takes its maximum if and only if $\lambda_1 = \ldots = \lambda\chi$. In other words, the dimension of a geometrical index determines the upper bound of the entanglement.

Instead of Schmidt decomposition, it is more convenient to use another language to present later the algorithms: singular value decomposition (SVD), a matrix decomposition in linear algebra. The Schmidt decomposition of a state is the SVD of the coefficient matrix C, where λ is called the singular value spectrum and its dimension χ is called the rank of the matrix. In linear algebra, SVD gives the optimal lower-rank approximations of a matrix, which is more useful to the tensor network algorithms. Specifically speaking, with a given matrix C of rank$-\chi$, the task is to find a rank$-\widehat{\chi}$ matrix $C'(\widehat{\chi} \leq \chi)$ that minimizes the norm

$$\mathcal{D} =\mid M - M' \mid = \sqrt{\sum_{ss'} (M_{ss'} - M'_{ss'})^2}. \tag{3.5}$$

The optimal solution is given by the SVD as

$$M'_{ss'} = \sum_{\alpha=0}^{\chi'-1} U_{s\alpha}\lambda_{\alpha\alpha}V^*_{s'\alpha}. \tag{3.6}$$

In other words, M' is the optimal rank$-\chi'$ approximation of M, and the error is given by

$$\varepsilon = \sqrt{\sum_{\alpha=\chi'}^{\chi-1} \lambda_\alpha^2}, \tag{3.7}$$

which will be called the truncation error in the tensor network algorithms.

3.2.2 Matrix product state

Now we take a N-spin state as an example to explain the matrix product state (MPS), a simple but powerful $1D$ tensor network state. In an MPS, the coefficients are written as a tensor network given by the contraction of N tensors. Schollwöck in his review [57] provides a straightforward way to obtain such a TN is by repetitively using SVD or QR decomposition (Figure 3.3). First, we group the first $N-1$ indexes together as one large index, and write the coefficients as a $2^{N-1}\times 2$ matrix. Then implement SVD or any other decomposition (for example QR decomposition) as the contraction of $C^{[N-1]}$ and $A[N]$

$$C_{s_1\ldots s_{N-1}s_N} = \sum_{\alpha_{N-1}} C^{[N-1]}_{s_1\ldots s_{N-1};\alpha_{N-1}} A^{[N]}_{s_N;\alpha_{N-1}}. \tag{3.8}$$

Note that as a convention in this paper, we always put the physical indexes in front of geometrical indexes and use a comma to separate them. For the tensor $C[N-1]$, one can do the similar thing by grouping the first $N-2$ indexes and decompose again as

$$C_{s_1\ldots s_{N-1}\alpha_{N-1}} = \sum_{\alpha_{N-2}} C^{[N-2]}_{s_1\ldots s_{N-2};\alpha_{N-2}} A^{[N-1]}_{s_N;\alpha_{N-2}\alpha_{N-1}}. \tag{3.9}$$

Then the total coefficients becomes the contraction of three tensors as

$$C_{s_1\ldots s_{N-1}s_N} = \sum_{\alpha_{N-2}\alpha_{N-1}} C^{[N-2]}_{s_1\ldots s_{N-2};\alpha_{N-2}} A^{[N-1]}_{s_{N-1};\alpha_{N-2}\alpha_{N-1}} A^{[N]}_{s_N;\alpha_{N-1}}. \tag{3.10}$$

Repeat decomposing in the above way until each tensor only contains one physical index, we have the MPS representation of the state as

$$C_{s_1\ldots s_{N-1}s_N} = \sum_{\alpha_1\ldots\alpha_{N-1}} A^{[1]}_{s_1\ldots,\alpha_1} A^{[2]}_{s_2\ldots;\alpha_1\alpha_2} \ldots A^{[N-1]}_{s_{N-1},\alpha_{N-2}\alpha_{N-1}} A^{[N]}_{s_N,\alpha_{N-1}}. \tag{3.11}$$

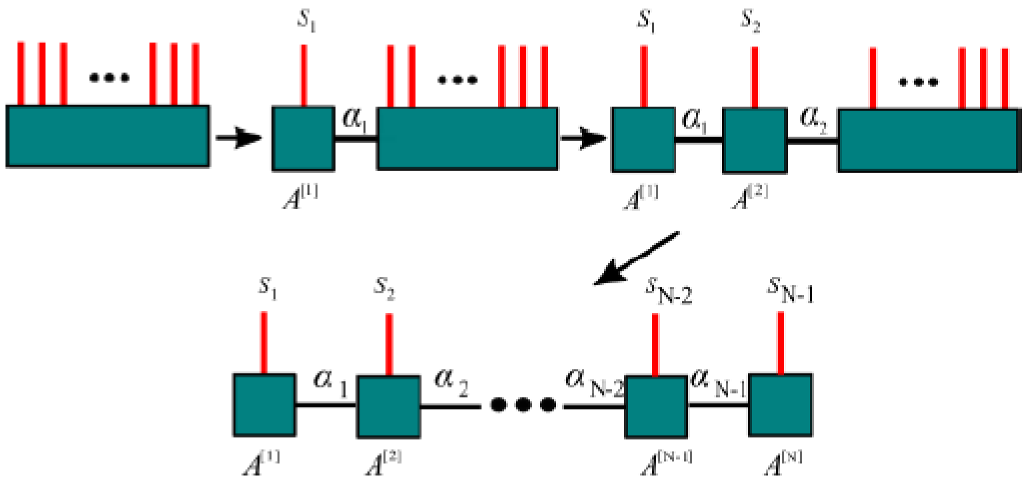


Figure 3.3: An impractical way to obtain an MPS from a many-body wave-function is to repetitively use the SVD.

One can see that an MPS is a tensor network formed by the contraction of N tensors. Graphically, MPS is represented by a $1D$ graph with N open bonds. In fact, an MPS given by (3.11) has open boundary condition, and can be generalized to periodic boundary condition (Figure 3.4) as

$$C_{s_1 \dots s_{N-1} s_N} = \sum_{\alpha_1 \dots \alpha_{N-1}} A^{[1]}_{s_1 \dots, \alpha_N \alpha_1} A^{[2]}_{s_2 \dots ; \alpha_1 \alpha_2} \dots A^{[N-1]}_{s_{N-1}, \alpha_{N-2} \alpha_{N-1}} A^{[N]}_{s_N, \alpha_{N-1} \alpha_N}, \tag{3.12}$$

where all tensors are third-order. Moreover, one can introduce translational invariance to the MPS, $A^{[n]} = A$ for $n = 1, 2, \dots, N$. We use χ, dubbed as virtual bond dimension of the MPS, to represent the dimension of each geometrical index.

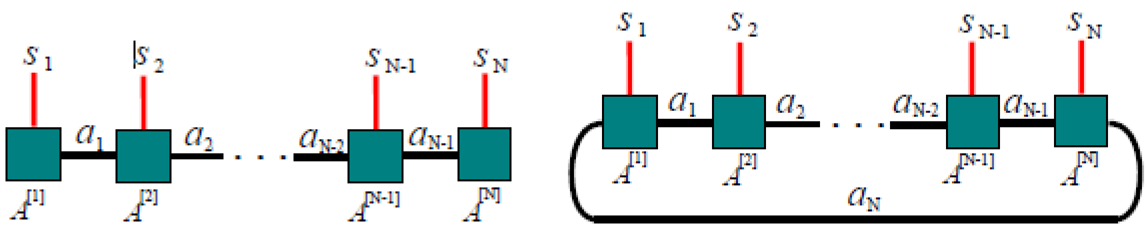


Figure 3.4: The graphic representations of the matrix product states with open (left) and periodic (right) boundary conditions.

MPS is an efficient representation of a many-body quantum state. For a N-spin state, the number of the coefficients is 2^N which increases exponentially with N. For an MPS given by (3.12), it is easy to count that the total number of the elements of all tensors is $Nd\chi^2$ which increases only linearly with N. The above way of obtaining MPS with decompositions is also known as tensor train decomposition (TTD) in MLA, and MPS is also called tensor-train form [58]. The main aim of TTD is investigating the algorithms to obtain the optimal tensor train form of a given tensor, so that the number of parameters can be reduced with well-controlled errors.

In physics, the above procedure shows that any states can be written in an MPS, as long as we do not limit the dimensions of the geometrical indexes. However, it is extremely impractical and inefficient, since in principle, the dimensions of the geometrical indexes $\{a\}$ increase exponentially with N. In the following sections, we will directly applying the mathematic form of the MPS without considering the above procedure.

Now we introduce a simplified notation of MPS that has been widely used in the community of physics. In fact with fixed physical indexes, the contractions of geometrical indexes are just the inner products of matrices (this is how its name comes from). In this sense, we write a quantum state given by (3.11) as

$$|\psi\rangle = tTrA^{[1]}A^{[2]}\dots A^{[N]}\,|s_1s_2\dots s_N\rangle = tTr\prod_{n=1}^{N}A^{[n]}\,|s\rangle_n\,, \tag{3.13}$$

tTr stands for summing over all shared indexes. The advantage of (3.13) is to give a general formula for an MPS of either finite or infinite size, with either periodic or open boundary condition.

3.2.3 Afflect-Kennedy-Lieb-Tasaki state

MPS is not just a mathematic form. It can represent non-trivial physical states. One important example can be found with AKLT model proposed in 1987, a generalization of spin-1 Heisenberg model [59]. For $1D$ systems, Mermin-Wagner theorem forbids any spontaneously breaking of continuous symmetries at finite temperature with sufficiently short-range interactions. For the ground state of AKLT model called AKLT state, it possesses the sparse anti-ferromagnetic order (Figure 3.5), which provides a non-zero excitation gap under the framework of Mermin-Wagner theorem. Moreover, AKTL state provides us a precious exactly-solvable example to understand edge states and (symmetry-protected) topological orders.

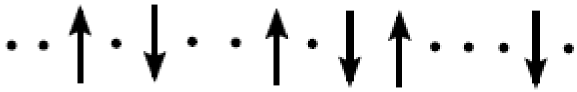


Figure 3.5: One possible configuration of the sparse anti-ferromagnetic ordered state. A dot represents the $S = 0$ state. Without looking at all the $S = 0$ states, the spins are arranged in the anti-ferromagnetic way.

AKLT state can be exactly written in an MPS with $\chi = 2$(see [60] for example). Without losing generality, we assume periodic boundary condition. Let us begin with the AKLT Hamiltonian that can be given by spin-1 operators as

$$\widehat{H} = \sum_n \left[\frac{1}{2}\widehat{S}_n.\widehat{S}_{n+1} + \frac{1}{6}\left(\left(\widehat{S}_n.\widehat{S}_{n+1}\right)^2 + \frac{1}{3}\right)\right] \tag{3.14}$$

By introducing the non-negative-defined projector $\widehat{P}_2\left(\widehat{S}_n + \widehat{S}_{n+1}\right)$ that projects the neighboring spins to the subspace of $S = 2$, (3.14) can be rewritten in the summation of projectors as

$$\widehat{H} = \sum_n \left(\widehat{S}_n + \widehat{S}_{n+1}\right) \tag{3.15}$$

Thus, the AKLT Hamiltonian is non-negative-defined, and its ground state lies in its kernel space, satisfying $\widehat{H}|\Psi_{AKLT} >= 0$ with a zero energy.

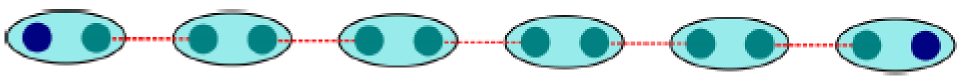


Figure 3.6: An intuitive graphic representation of the AKLT state. The big circles representing $S = 1$ spins, and the small ones are effective $S = \frac{1}{2}$ spins. Each pair of spin-1/2 connecting by a red bond forms a singlet state. The two “free” spin-1/2 on the boundary give the edge state.

Now we construct a wave-function which has a zero energy. As shown in (Figure 3.6), we put on each site a projector that maps two (effective) spins-1/2 to a triplet, i.e. the physical spin-1, where the transformation of the basis obeys

$$|+\rangle = |00\rangle \tag{3.16}$$

$$|\tilde{0}\rangle = \frac{1}{\sqrt{2}}(|01\rangle + |10\rangle), \tag{3.17}$$

$$|-\rangle = |11\rangle . \tag{3.18}$$

The corresponding projector is determined by the Clebsch-Gordan coefficients [61], and is a (3×4) matrix. Here, we rewrite it as a $(3 \times 2 \times 2)$ tensor, whose three components (regarding to the first index) are the ascending, z-component and descending Pauli matrices of spin-1/2,[4]

$$\sigma^+ = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \quad \sigma^z = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \quad \sigma^- = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}. \tag{3.19}$$

In the language of MPS, we have the tensor A satisfying

$$A_{0,\alpha\alpha'} = \sigma^+_{\alpha\alpha'}, \quad A_{1,\alpha\alpha'} = \sigma^z_{\alpha\alpha'}, \quad A_{2,\alpha\alpha'} = \sigma^-_{\alpha\alpha'}. \tag{3.20}$$

Then we put another projector to map two spin-1/2 to a singlet, a spin-0 with

$$|\tilde{0}\rangle = \frac{1}{\sqrt{2}}(|01\rangle - |10\rangle). \tag{3.21}$$

The projector is in fact a (2×2) identity with the choice of (3.19),

$$I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}. \tag{3.22}$$

Now, the MPS of the AKLT state with periodic boundary condition (up to a normalization factor) is obtained by (3.12), with every tensor A given by (3.20). For such an MPS, every projector operator $\widehat{P}_2\left(\widehat{S}_n + \widehat{S}_{n+1}\right)$ in the AKLT Hamiltonian is always acted on a singlet, then we have $\widehat{H}\,|\psi_{AKLT}\rangle = 0$.

3.3 Tree tensor network state (TTNS) and projected entangled pair state (PEPS)

TTNS is a generalization of the MPS that can code more general entanglement states. Unlike an MPS where the tensors are aligned in a $1D$ array, a TTNS is given by a tree

[4]Here, one has some degrees of freedom to choose different projectors, which is only up to a gauge transformation. But once one projector is fixed, the other is also fixed.

graph. (Figure 3.7 (a) and (b)) show two examples of TTNS with the coordination number $z = 3$. The red bonds are the physical indexes and the black bonds are the geometrical indexes connecting two adjacent tensors. The physical ones may locate on each tensor or put on the boundary of the tree. A tree is a graph that has no loops, which leads to many simple mathematical properties that parallel to those of an MPS. For example, the partition function of a TTNS can be efficiently exactly computed. A similar but more power tensor network state called MERA also has such a property (Figure 3.7 (c)). Note an MPS can be treated as a tree with $z = 2$.

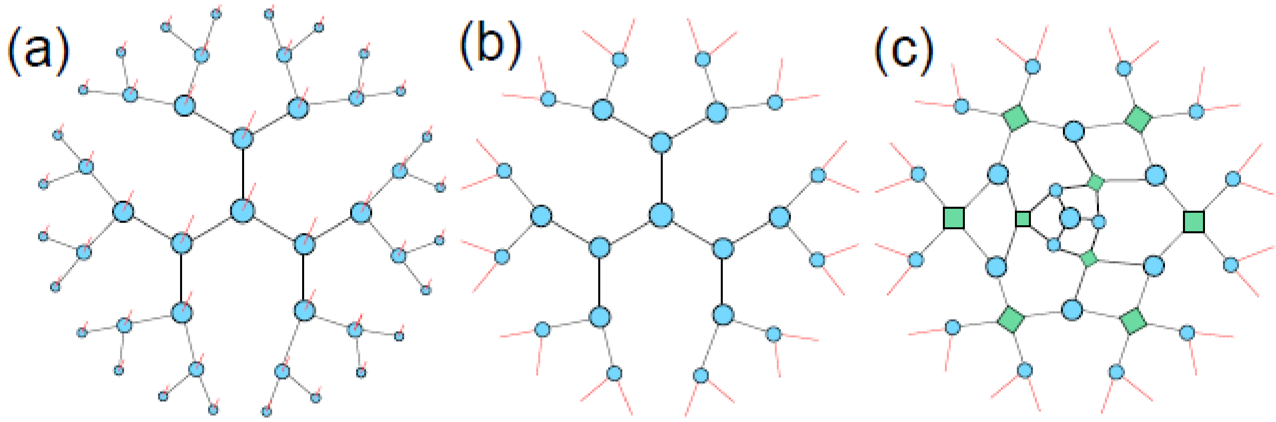


Figure 3.7: The illustration of (a) and (b) two different TTNS's and (c) MERA.

An important generalization to the tensor network's of loopy structures is known as projected entangled pair state (PEPS), proposed by Verstraete and Cirac [62, 63]. The tensors of a PEPS are located in, instead of a $1D$ chain or a tree graph, a d-dimensional lattice, thus graphically forming a d-dimensional tensor network. An intuitive picture of PEPS is given in (Figure 3.8), the tensors can be understood as projectors that map the physical spins into virtual ones. The virtual spins form the maximally entangled state in a way determined by the geometry of the tensor network. Note that such an intuitive picture was firstly proposed with PEPS [62], but it also applies to TTNS.

Similar to MPS, a TTNS or PEPS can be formally written as

$$|\psi\rangle = tTr \prod_n P^{[n]} |S_n\rangle , \tag{3.23}$$

where tTr means to sum over all geometrical indexes. Usually, we do not write the formula of a TTNS or PEPS, but give the graph instead to clearly show the contraction relations.

Such a generalization makes a lot of senses in physics. One key factor regards the area law of entanglement entropy [64–69]. In the following as two straightforward examples, we show that PEPS can indeed represents non-trivial physical states including nearest-neighbor resonating valence bond (RVB) and Z_2 spin liquid states. Note these two types of states on trees can be similarly defined by the corresponding TTNS.

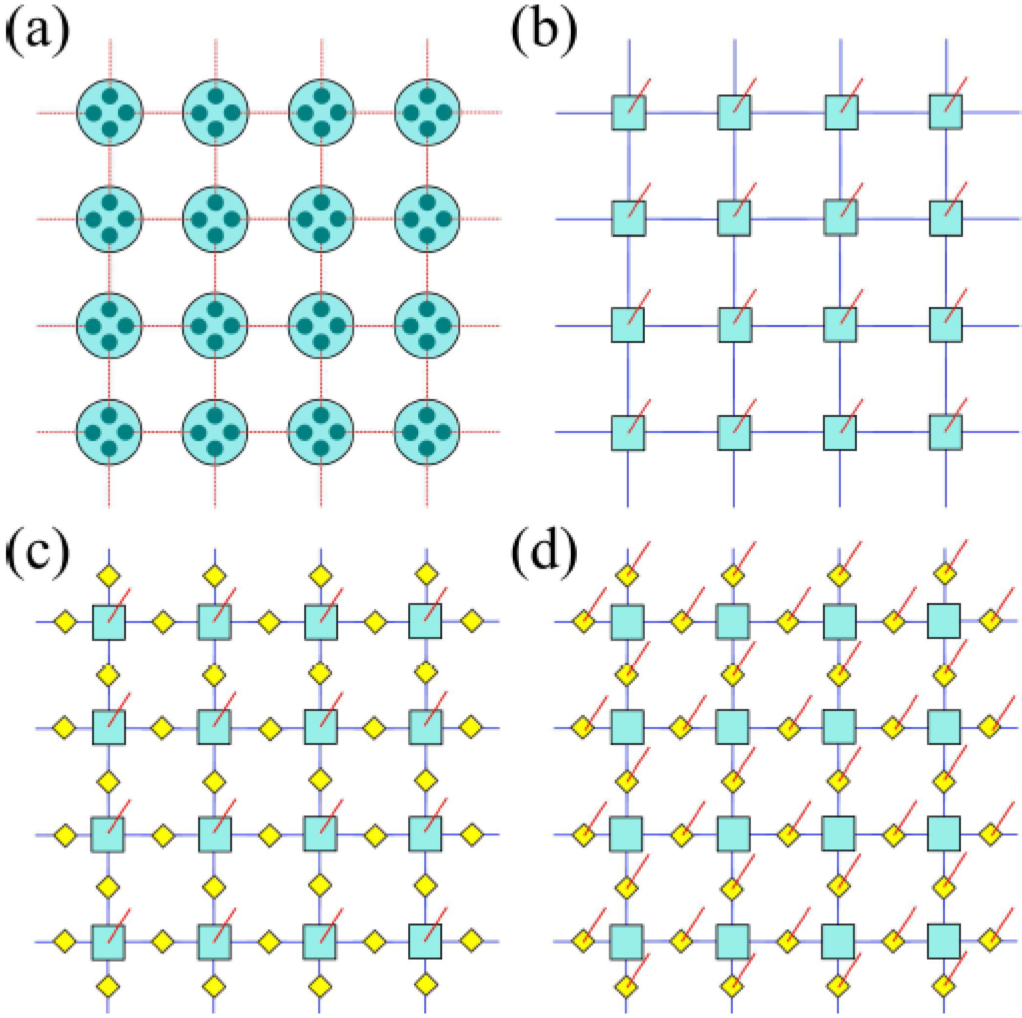


Figure 3.8: (a) An intuitive picture of the projected entangled pair state. The physical spins (big circles) are projected to the virtual ones (small circles), which form the maximally entangled states (red bonds). (b)-(d) Three kinds of frequently used PEPS's.

3.4 PEPS can represent non-trivial many-body states: examples

RVB state was firstly proposed by Anderson to explain the possible disordered ground state of the Heisenberg model on triangular lattice [70, 71]. RVB state is defined as the superposition of macroscopic configurations where all spins are paired to form the singlet states (dimers). The strong fluctuations are expected to restore all symmetries and lead to a spin liquid state without any local orders. The distance between two spins in a dimer can be short range or long range. For nearest-neighbor RVB, the dimers are only be nearest neighbors (Figure 3.9, also see [72]). RVB state is supposed to relate to high-T_c copper-oxide-based superconductor. By doping the singlet pairs, the insulating RVB state can translate to a charged superconductive state [73–75].

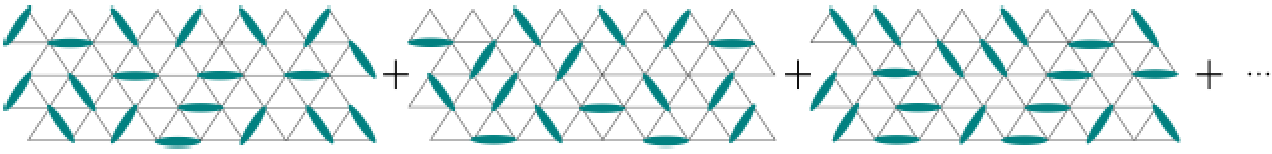


Figure 3.9: The nearest-neighbor RVB state is the superposition of all possible configurations of nearest-neighbor singlets.

For the nearest-neighbor situation, an RVB state (defined on an infinite square lattice, for example) can be exactly written in a PEPS of $\chi = 3$. Without losing generality, we take the translational invariance, the tensor network is formed by infinite copies of several inequivalent tensors. Two different ways have been proposed to construct the nearest-neighbor RVB and PEPS [76, 77]. In addition, Wang et al. proposed a way to construct the PEPS with long-range dimers [78]. In the following, we explain the way proposed by Verstraete et al. to construct the nearest-neighbor one [76]. There are two inequivalent tensors: the tensor defined on each site whose dimensions are $(2 \times 3 \times 3 \times 3 \times 3)$ only has eight non-zero elements,

$$P_{0,0222} = P_{0,2022} = P_{0,2202} = P_{0,2220} = 1 \tag{3.24}$$

$$P_{1,1222} = P_{1,2122} = P_{1,2212} = P_{1,2221} = 1. \tag{3.25}$$

The two-dimensional index of P is a physical index with $s = 0$ representing spin up and $s = 1$ spin down. The extra dimension for each of the other four geometrical indexes is used for carrying the vacuum state. The tensor P is acting as projector that maps the occupied geometrical index (either up or down) to a physical spin. For example, $P_{1,2122}$ means to map a virtual spin up which occupies the second geometrical index to a real spin up. The rest elements are all zero, which means the corresponding projections are forbidden.

Then a projector B is introduced for building spin singlets between two nearest-neighbor sites connected by a shared geometrical bond in the RVB structure. B is a (3×3) matrix with only three non-zero elements

$$B_{01} = 1, \quad B_{10} = 1, \quad B_{22} = 1. \tag{3.26}$$

Matrix B plays as a router, which only lets the singlet state defined as $|10\rangle - |01\rangle$ and vacuum state go through the path.

Then the infinite PEPS (iPEPS) of the nearest-neighbor RVB is given by the contraction of infinite copies of P's on the sites and B's (Fig.3.8) on the bonds as

$$|\psi\rangle = \sum_{\{s,\alpha\}} \prod_{n \in sites} P_{s_n,\alpha_n^1\alpha_n^2\alpha_n^3\alpha_n^4} \prod_{m \in bonds} B_{\alpha_m^1\alpha_m^2} \prod_{j \in sites} |s_j\rangle\,, \tag{3.27}$$

After the contraction of all geometrical indexes, the state is the super-position of all possible configurations consisting of nearest-neighbor dimers. This iPEPS looks different from the one given in (3.23) but they are essentially the same, because one can contract the B's into P's so that the PEPS is only formed by tensors defined on the sites.

Another example is the Z_2 spin liquid state, which is one of simplest string-net states [79–81], firstly proposed by Levin and Wen to characterize gapped topological orders [82]. Similarly with the picture of strings, the Z_2 state is the super-position of all configurations of string loops. Writing such a state with tensor network, the tensor on each vertex is $(2 \times 2 \times 2 \times 2)$ satisfying

$$P_{\alpha_1 \ldots \alpha_N} = \begin{cases} 1, & \alpha_1 \ldots \alpha_N = even \\ 0, & otherwise. \end{cases} \tag{3.28}$$

The tensor P forces the fusion rules of the strings: the number of the strings connecting to a vertex must be even, so that there are no loose ends and all strings have to form loops.

It is also called in some literatures the ice rule [83, 84] or Gauss law [85]. In addition, the square tensor network formed solely by the tensor P gives the famous eight-vertex model, where the number "eight" corresponds to the eight non-zero elements (allowed sting configurations) on a vertex [86].

The tensors B are defined on each bond to project the strings to spins, whose non-zero elements are

$$B_{0,00} = 1, \quad B_{1,11} = 1. \tag{3.29}$$

The tensor B is a projector that maps the spin-up (spin-down) state to the occupied (vacuum) state of a string.

3.5 Tensor network operators

The MPS or PEPS can be readily generalized from representations of states to those of operators called MPO [87–94] or projected entangled pair operator (PEPO)[5] [95–104]. Let us begin with MPO, which is also formed by the contraction of local tensors as (Figure 3.10)

$$\hat{O} = \sum_{\{s,\alpha\}} \prod_n W^{[n]}_{s_n s'_n, \alpha_n \alpha_{n+1}} |s_n\rangle \langle s'_n| . \tag{3.30}$$

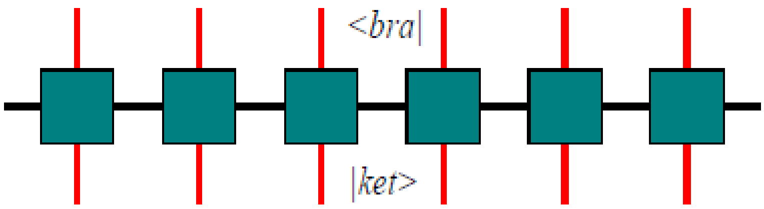


Figure 3.10: The graphic representation of amatrix product operator, where the upward and downward indexes represent the *bra* and *ket* space, respectively.

Different from MPS, each tensor has two physical indexes, of which one is a *bra* and the other is a *ket* index (Figure 3.11). An MPO may represent several non-trivial physical

[5]Generally, a representation of an operator with a tensor network can be called tensor product operator (TPO). MPO and PEPO are two examples.

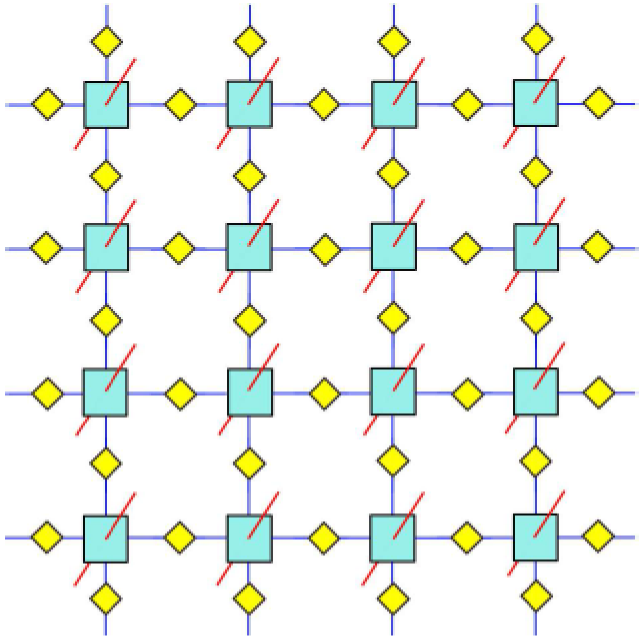


Figure 3.11: The graphic representation of a projected entangled pair operator, where the upward and downward indexes represent the *bra* and *ket* space, respectively

models, for example the Hamiltonian. Crosswhite and Bacon [105] proposed a general way of constructing an MPO from called automata. Now we show how to construct the MPO of an Hamiltonian using the properties of a triangular MPO. Let us start from a general lower-triangular MPO satisfying $W^{[n]}_{::,00} = C^{[n]}$, $W^{[n]}_{::,01} = B^{[n]}$, and $W^{[n]}_{::,11} = B^{[n]}$ with $A^{[n]}$, $B^{[n]}$, and $C^{[n]}$ some $d \times d$ square matrices. We can write $W^{[n]}$ in a more explicit 2×2 block-wise form as

$$W^{[n]} = \begin{pmatrix} C^{[n]} & 0 \\ B^{[n]} & A^{[n]} \end{pmatrix}. \tag{3.31}$$

If one puts such a $W^{[n]}$ in (3.30), it will give the summation of all terms in the form of

$$\begin{aligned} O &= \sum_{n=1}^{N} A^{[1]} \otimes \ldots \otimes A^{[n-1]} \otimes B^{[n]} \otimes C^{[n+1]} \otimes \ldots \otimes C^{[N]} \\ &= \sum_{n=1}^{N} \prod_{\otimes i=1}^{n-1} A^{[i]} \otimes B^{[n]} \otimes \prod_{\otimes j=n+1}^{N} C^{[j]}, \end{aligned} \tag{3.32}$$

with N the total number of tensors and $\prod_{\otimes}$ the tensor product.[6] Such a property can be

[6]For $n = 0, A^{[0]}$ (or $B^{[0]}, C^{[0]}$) does not exist but can be defined as a scalar 1, for simplicity of the formula.

easily generalized to a W formed by $D \times D$ blocks.

Imposing (3.32), we can construct as an example the summation of one-site local terms, $\sum_n X^{[n]}$,[7] with

$$W^{[n]} = \begin{pmatrix} I & 0 \\ X^{[n]} & I \end{pmatrix}. \tag{3.33}$$

with $X^{[n]}$ a $d \times d$ matrix and I the $d \times d$ identity.

If two-body terms are included, such as $\sum_m X^{[m]} + \sum_n Y^{[n]} Z^{[n+1]}$, we have

$$W^{[n]} = \begin{pmatrix} I & 0 & 0 \\ Z^{[n]} & 0 & 0 \\ X^{[n]} & Y^{[n]} & I \end{pmatrix}. \tag{3.34}$$

This can be obviously generalized to L-body terms. With open boundary conditions, the left and right tensors are

$$W^{[1]} = \begin{pmatrix} I & 0 & 0 \end{pmatrix}, \tag{3.35}$$

$$W^{[N]} = \begin{pmatrix} 0 \\ 0 \\ I \end{pmatrix}. \tag{3.36}$$

Now we apply the above technique on a Hamiltonian of, the Ising model in a transverse field

$$\widehat{H} = \sum_n \widehat{S}^z_n \widehat{S}^z_{n+1} + h \sum_n \widehat{S}^x_m. \tag{3.37}$$

Its MPO is given by

$$W^{[n]} = \begin{pmatrix} I & 0 & 0 \\ \widehat{S}^z & 0 & 0 \\ h\widehat{S}^x & \widehat{S}^z & I \end{pmatrix}. \tag{3.38}$$

Such a way of constructing an MPO is very useful. Another example is the Fourier transformation to the number operator of Hubbard model in momentum space $\widehat{n}_k = \widehat{b}^\dagger_k \widehat{b}_k$. The Fourier transformation is written as

$$\widehat{n}_k = \sum_{m,n=1}^{N} e^{i(m-n)k} \widehat{b}^\dagger_m \widehat{b}_n, \tag{3.39}$$

[7]Note that $X^{[n_1]}$ and $X^{[n_2]}$ are not defined in a same space with $n_1 \neq n_2$. Thus, precisely speaking, $\sum$ here is the direct sum. We will not specify this when it causes no confuse.

with $\widehat{b}_n(\widehat{b}_n^\dagger)$ the annihilation (creation) operator on the n-th site. The MPO representation of such a Fourier transformation is given by

$$\widehat{W}_n = \begin{pmatrix} \widehat{I} & 0 & 0 & 0 \\ \widehat{b}^\dagger & e^{ik}\widehat{I} & 0 & 0 \\ \widehat{b} & 0 & e^{-ik}\widehat{I} & 0 \\ \widehat{b}^\dagger\widehat{b} & e^{ik}\widehat{b}^\dagger & e^{-ik}\widehat{b}\widehat{I} & \end{pmatrix}, \tag{3.40}$$

with $\widehat{I}$ the identical operator in the corresponding Hilbert space.

The MPO formulation also allows for a convenient and efficient representation of the Hamiltonians with longer range interactions [106]. The geometrical bond dimensions will in principle increase with the interaction length. Surprisingly, a small dimension is needed to approximate the Hamiltonian with long-range interactions that decay polynomially [98].

Besides, MPO can be used to represent the time evolution operator $\widehat{U}(t) = e^{-\tau\widehat{H}}$ with Trotter-Suzuki decomposition, where τ is a small positive number called Trotter-Suzuki step [107,108]. Such an MPO is very useful in calculating real, imaginary, or even complex time evolutions, which we will present later in detail. An MPO can also give a mixed state.

Similarly, PEPS can also be generalized to projected entangled pair operator (PEPO, Figure 3.11), which on a square lattice for instance can be written as

$$\widehat{O} = \sum_{\{s,\alpha\}} \prod_n W^{[n]}_{s_n s'_n, \alpha_n^1 \alpha_n^2 \alpha_n^3 \alpha_n^4} |s_n\rangle \langle s'_n| \,. \tag{3.41}$$

Each tensor has two physical indexes (bra and ket) and four geometrical indexes. Each geometrical bond is shared by two adjacent tensors and will be contracted.

Chapter 4

Dyonic Objects from String Theory

In this chapter, we study dyonic objects in supergravity models. In particular, we try to give a possible realisation interns of branes. This can be made using the scenario of compactifications.

4.1 From zero to one dimensional objects

4.1.1 Particle physics

Particle physics concerns the study of particles and fundamental interactions. Many studies dealing such a physics have been elaborate. It has been shown that there is a nice interplay between such a physics and Lie algebra structures. It is interesting to note this bridge. This shows that the symmetry is considered as an important tool to study physics. Roughly we recall the following links.

- The strong interaction is linked with $su(3)$ Lie algebra
- The weak interaction corresponds to $su(2)$ Lie algebra
- The electromagnetic interaction is associated with $u(1)$ Lie algebra
- The gravitation theory corresponds to geometric deformations of the univers

Other Lie symmetries have been used in gauge theories. In particular, we have

- Grand Unified theory (GUT) : E_6, $so(10)$, $su(5)$ Lie algebras

- $A_n, D_n, E_{6,7,8}$ gauge theories.

Certain properties of particle physics are associated with the following remarks. A close inspection show that particle physics data are hidden in Lie algebra structures. However, the main problem is in the gravity theory. More details on such problems can be found in many works [109, 110].

4.2 String theory

The motion of particles (zero dimensional objects) could be extended to the motion of higher dimensional objects : strings and branes. Two theories have been dealt with:

- Open string theory
- Closed string theory.

4.2.1 Classical and quantum version of string theory

In string theory, the action of the classical theory involves two contributions

$$S = S_b + S_f, \tag{4.1}$$

where S_b and S_f represent bosonic and fermionic terms, respectively. Evincing the fermionic terms, this generates the following bosonic equation of motion

$$(\partial_\tau^2 - \partial_\sigma^2) X^\mu(\tau, \sigma) = 0. \tag{4.2}$$

In this way, the quantization provides massless states given by the following spectrum

$$(g_{\mu\nu}, B_{\mu\nu}, A_\mu, \phi, ...)$$

Gravitons, photons, and others particles could take places in such a theory. A close examination show that string theory could be considered as a possible unified theory of

- strong interaction,
- weak interaction,

- electromagnetic interaction,
- gravitation interaction.

It has been revealed that two configurations have been investigated to provide non-trivial results. One has the following theories

- Open string theory: (D-branes) $\rightarrow$ Gauge theories
- Closed string theory: $\rightarrow$ Gravity theory.

4.2.2 Bosonic string theory problems

It is interesting to recall that the bosonic string theory suffers from many problems. They are classified as follows.

- It contains tachyons.
- It does not have fermionic states describing matter fields.
- It can not be used to unify particle physics and gravity.
- It does not involve non abelain symmetries.

To overcome such problems, one should add fermions $\psi^{\mu}(\tau, \sigma)$ on the world-sheet.

4.2.3 Supersymmetric string or (superstrings)

It is needed to go beyond the bosonic theory. To do that, we consider superstring theory. The action of this theory can be obtained by adding a fermionic term to the bosonic string action. Indeed, we can write

$$S = \frac{1}{2\pi\alpha'} \int_{\Sigma=WorldSheet} d\sigma d\tau (\partial_{\alpha} X^{\mu} \partial^{\alpha} X_{\mu} \;+\; i\overline{\psi}^{\mu} \rho^{\alpha} \partial_{\alpha} \psi_{\mu}) \tag{4.3}$$

where $\psi^{\mu} = \psi^{\mu}_{A} = \begin{pmatrix} \psi^{\mu}_{+} \\ \psi^{\mu}_{-} \end{pmatrix}$ are $Majorana$ spinors (real two-compent) with respect to world-sheet and vectors with respect to the space-time. It is recalled that ρ^{α} are two-dimensional spin matrices satisfying the following anti-commutator relations

$$\{\rho^{\alpha}, \rho^{\beta}\} = 2\eta^{\alpha\beta}.$$

The above action involves many symmetries. one should note the following ones

1. The formal supersymmetric transformations are illustrated by

$$X^\mu \Longleftrightarrow \psi^\mu.$$

2. The superstring action is invariant under

$$\delta X^\mu = \overline{\chi}\psi^\mu \delta\psi^\mu = -i\rho^\alpha \partial_\alpha X^\mu \epsilon$$

   where χ is a Majorana spinor.

Using similar methods used in the previous chapters, one can find the equations of motion given by

$$\partial_\alpha \partial^\alpha X^\mu = 0, \qquad \rho^\alpha \partial_\alpha \psi^\mu = 0.$$

The boundary conditions of open superstring can be fixed by

$$\delta S_F = \int d\tau [\psi_+^\mu \delta\psi_+^\mu - \psi_-^\mu \delta\psi_-^\mu]_{\sigma=0}^{\sigma=\pi} = 0. \tag{4.4}$$

The associated solutions read as

$$\begin{aligned} \psi_+^\mu &= \pm\psi_-^\mu \\ \delta\psi_+^\mu &= \pm\delta\psi_-^\mu \end{aligned} \tag{4.5}$$

providing two kinds of fermions for open superstrings. The first one is given by

$$\begin{aligned} \psi_+(\tau, 0) &= \psi_-(\tau, 0) \\ \psi_+(\tau, \pi) &= \psi_-(\tau, \pi). \end{aligned} \tag{4.6}$$

This sector is called **Ramond, R** . The second one defined

$$\begin{aligned} \psi_+(\tau, 0) &= -\psi_-(\tau, 0) \\ \psi_+(\tau, \pi) &= -\psi_-(\tau, \pi) \end{aligned} \tag{4.7}$$

called **Neveu-Schwarz (NS)**.

Many sectors could be built in string theory. One has the following blocks.

1. The **R-sector** which is associated with the periodic fermions generates the following models

$$\begin{aligned} \Psi_+^\mu(\tau, \sigma) &= \frac{1}{\sqrt{2}} \sum_{n=-\infty}^{+\infty} d_n^\mu \mathrm{e}^{-in(\tau+\sigma)} \\ \Psi_-^\mu(\tau, \sigma) &= \frac{1}{\sqrt{2}} \sum_{n=-\infty}^{+\infty} d_n^\mu \mathrm{e}^{-in(\tau-\sigma)},\ n \in \mathbb{Z}. \end{aligned} \tag{4.8}$$

2. The **NS-sector** which corresponds to the anti-periodic fermions provides the following modes

$$\begin{aligned}\Psi_{+}^{\mu}(\tau,\sigma) &= \frac{1}{\sqrt{2}}\sum_{r=-\infty}^{+\infty} b_r^{\mu}\mathrm{e}^{-ir(\tau+\sigma)}\\ \Psi_{-}^{\mu}(\tau,\sigma) &= \frac{1}{\sqrt{2}}\sum_{r=-\infty}^{+\infty} b_r^{\mu}\mathrm{e}^{-ir(\tau-\sigma)},\ r\in\mathbb{Z}_{+\frac{1}{2}}.\end{aligned} \tag{4.9}$$

3. For closed superstrings, one should add $\tilde{d_n}^{\mu}$ and $\tilde{b_r}^{\mu}$ modes, needed to build the associated spectrum.

It turns out that for closed strings, one can have four sectors

$$\begin{pmatrix} R \\ NS \end{pmatrix} \otimes \begin{pmatrix} R \\ NS \end{pmatrix} \tag{4.10}$$

controlled by the following modes

R-R	d_n^{μ},$\tilde{d_n^{\mu}}$
R-NS	d_n^{μ}, $\tilde{b_r^{\mu}}$
NS-R	b_r^{μ}, $\tilde{d_n^{\mu}}$
NS-NS	b_r^{μ}, $\tilde{b_r^{\mu}}$

It should be noted that calculations show that superstrings live in 10-dimensional space-time

$$D = 10.$$

It is interesting to indicate that other superstring theory models can be constructed by combining a bosonic string in $d = 26$ and a fermionic string in $d = 10$. This is represented by

$$\overbrace{\begin{pmatrix} Bosonic \\ string \\ in \\ D=26 \end{pmatrix}}^{Left} \otimes \overbrace{\begin{pmatrix} Fermionic \\ string \\ in \\ D=10 \end{pmatrix}}^{Right}$$

where one should use the following classical field content

$$X_L^{\mu}, \mu = 0,\dots,25 \qquad \psi_R^{\mu}, X_R^{\mu}, \mu = 0,\dots,9. \tag{4.11}$$

To get a possible theory, one can should compactify the bosonic extra directions

$$X_L^\mu \;\; \mu = 0, \dots, 9 \qquad X_R^\mu, \mu = 0, \dots, 9 \tag{4.12}$$

$$X_L^I \;\; I = 10, \dots, 25 \qquad \psi_R^\mu, \mu = 0, \dots, 9. \tag{4.13}$$

The combined theory is called hetorotic superstring. After compactifications, the action of such superstrings takes the following form

$$S = \int d\tau d\sigma (\partial_\alpha X \partial^\alpha X_\mu - 2i\psi_-^\mu \partial_+ \psi_- - 2i\lambda_+^A \partial \lambda_+^A). \tag{4.14}$$

In fact, one has two hetorotic superstrings. In particular, we cite

- hetorotic superstring, with the $SO(32)$ Lie symmetry
- hetorotic string, with the $E_8 \times E_8$ Lie symmetry.

4.3 Spectrum of open superstrings from quantization

In this section, we would like to discuss the spectrum of superstrings. For open superstrings, one has two types of quantized relations.

4.3.1 Open superstrings in NS-sector

For the **NS** sector, one has

$$\{b_r^\mu, b_s^\mu\} = \eta^{\mu\nu}\delta_{r+s,0}.$$

The associated right moving relations are

$$\{\tilde{b_r}^\mu, \tilde{b_s}^\mu\} = \eta^{\mu\nu}\delta_{r+s,0},$$

where one has the following configurations.

- b_{-r}^μ and b_{+r}^μ are the creation and annihilation modes, respectively.
- $\tilde{b}_{-r}^\mu$ and $\tilde{b}_{+r}^\mu$ are the creation and annihilation operators, respectively.

It has been shown that the spectrum of such sector is given by

$$\alpha_n^\mu |0 >= 0, \qquad\qquad b_r^\mu |0 >= 0 \qquad n, r > 0.$$

The general state reads as

$$|\psi>=\prod_i(\alpha_{-n_i})^{k_j}\prod_j(b_{-r_j})^{k_j}|0>. \tag{4.15}$$

Precisely, it has been revealed that the mass term is given by

$$M^2=\sum_{n=1}^{\infty}\alpha_{-n}\alpha_n+\sum_{r=\frac{1}{2}}^{+\infty}rb_{-r}b_r-\frac{1}{2}. \tag{4.16}$$

The ground state of the **NS** sector is a tachyon. However, the first excited level generates a massless vector

$$A^\mu\equiv b^\mu_{-\frac{1}{2}}|0> \tag{4.17}$$

Then, this sector behaves a bosonic one.

4.3.2 Open superstrings in R-sector

Concerning the Ramond sector, the commutation relations are given by

$$\begin{aligned}\{d^\mu_m,d^\nu_n\}&=\eta^{\mu\nu}\delta_{m+n,0}\\ \{\tilde{d^\nu_m},\tilde{d^\mu_n}\}&=\eta^{\mu\nu}\delta_{m+n,0}.\end{aligned} \tag{4.18}$$

The associated mass term reads

$$M^2=\sum_{n=1}^{\infty}\alpha_{-n}\alpha_n+\sum_{n=1}^{+\infty}nd_{-n}d_n. \tag{4.19}$$

For $n,m=0$, we have

$$\{d^\mu_0,d^\nu_0\}=\eta^{\mu\nu}.$$

In this way, the ground state of the **R-sector** is massless fermion

$$|0>.$$

Non-massive fermions can be obtained by

$$(d^\mu_0)^k|0>.$$

In $D=10$, the matrices d^μ_0 are 32×32 matrices.

The ground state of the **NS sector** is a tachyon. One should remove it using the G.S.O projection. It acts as follows

$$G.S.O|\psi >= |\psi > .$$

For **NS sector**, one has

$$(-1)^F_{NS} = -(-1)^{\sum_{r=\frac{1}{2}} b_{-r}b_r}.$$

For **R-sector**, the G.S.O act as follows

$$(-1)^F_R = \Gamma^{11}(-1)^{\sum_n d_{-n}d_{-n}}.$$

4.4 Spectrum of superstring models

A close inspection shows that there is a nice classification of superstrings. It is given as follows.

1. Models with $N = 2$ supersymmetry
   (a) Type II A
   (b) Type II B
2. Models with $N = 1$ supersymmetry
   (a) Heterotic, $SO(32)$
   (b) Heterotic, $E_8 \times E_8$
   (c) Type I, $SO(32)$.

4.4.1 Type IIA and Type IIB superstrings

To get the spectrum of such models, it is useful to recall a formal definition of superalgebra. It reads as

$$BB = B, \quad BF = F, \quad FB = F, \quad FF = B. \tag{4.20}$$

This gives the following statements.

- **NS-NS**, being is a bosonic sector
- **R-R**, being is a bosonic sector
- **NS-R**, being a fermionic sector
- **R-NS**, being a fermionic sector.

The states sectors can be built from the modes

$$\alpha, \quad \tilde{\alpha}, \quad b, \quad \tilde{b} \quad d, \quad \tilde{d}. \tag{4.21}$$

For supersymmetric theories, one can deal with only the bosonic sector. For the bosonic sector **NS-NS**, one can have the spectrum

$$8_v \otimes 8_v = 1 \oplus 35 \oplus 28, \tag{4.22}$$

for both theories IIA and IIB. One can interpret the above decomposition as

$$1 \rightarrow \quad \phi \tag{4.23}$$

$$35 \rightarrow \quad g_{\mu\nu} \tag{4.24}$$

$$28 \rightarrow \quad B_{\mu\nu}. \tag{4.25}$$

For the **R-R-sector**, one has two possibilities

$$\begin{aligned} &8_s \otimes 8_c \rightarrow 8 \oplus 56 \rightarrow A^\mu,\, C^{\mu\nu\sigma}, \quad \text{(odd orders)} \\ &8_s \otimes 8_s \rightarrow 1 \oplus 28 \oplus 35, \rightarrow \chi,\, \tilde{B},\, D^{\mu\nu\rho\sigma}, \quad \text{(even orders).} \end{aligned} \tag{4.26}$$

4.4.2 Heterotic superstrings $E_8 \times E_8$ and SO(32)

It is recalled that the heterotic superstring can be constructed from a hybrid of a bosonic string and a fermionic one. In this way, the symmetry group of the bosonic sector can be composed

$$\begin{aligned} SO(1,25) &\longrightarrow SO(1,9) \times SO(16) \\ 26 &\longrightarrow (10,1) \oplus (1,16) \\ P_L^{26} &\longrightarrow P_L^{10} \oplus P_L^{16} \end{aligned} \tag{4.27}$$

where P_L^{10} is identified with P_R. However, 16 of P_L^{16} can be interpreted as the generator of Cartan super Lie algebra of Lie algebra. Calculations show that these Lie algebras can be identified with $so(32)$ and $E_8 \times E_8$. It is recalled that

$$rank(so(32)) \; = \; rank(E_8 \times E_8) \; = \; 16.$$

The mass of states is given by

$$M^2 \; = \; \frac{1}{2} P_A^2 \; + \; N_L \; - \; 1 \; = \; N_R \; - \; a_R$$

where $a_R \;\; = \;\; 0, \; 1/2$ for **R** and **NS sectors**, respectively. For the right moving the massless states of such superstrings contain

$$\underline{V} \; \oplus \; \underline{S}$$

where V is a vector of $SO(1,9)$ and S is a spinor of $SO(10)$ with a fixed chirality. For the left moving massless states, we have two solutions

1. $N_L = 1$ and $P_A^2 = -m^2 = 0$

   The massless states are given by

   $$\alpha_{-1}^A|0> \qquad A = 1, ..., 16$$

   identified with $U(1)^{16}$ gauge symmetry.

2. $N_L = 0$ and $P^2 = 2$

   We have the following sum

   $$\underline{V} \oplus \underline{adjG}$$

   where G is $E_8 \times E_8$ or $SO(32)$ symmetries. The closed superstring spectrum can be obtained by

   $$(\underline{V} \oplus \underline{adjG}) \otimes (\underline{V} \; \oplus \; \underline{S}).$$

   Indeed, the bosonic states are given by

   $$(\underline{V} \oplus \underline{adjG}) \otimes \underline{V}$$

   providing $B_{\mu\nu}, \;\; g_{\mu\nu}, \;\; \Phi$ and A_μ in the adjoint representation of G

   $$A_\mu = A_\mu^a T_a, \qquad a = 1, \ldots, \dim G.$$

The fermionic sector is given by

$$(V \oplus adjG) \otimes \underline{S}. \tag{4.28}$$

4.4.3 Type I superstring

This model (theory) can be obtained from Type IIB by using a symmetry (reflexion) Ω acting on the word-sheet as follows

$$\Omega : \sigma \longrightarrow 2\pi - \sigma, \qquad \Omega^2 = 1, \tag{4.29}$$

This symmetry acts on the fields also follows

$$X_L^\mu \to X_R^\mu \tag{4.30}$$

$$\Omega \alpha_n^\mu \Omega^{-1} \to \tilde{\alpha}_n^\mu. \tag{4.31}$$

For the **NS-NS sector**, this symmetry acts as follows.

$$\Omega \alpha_{-n}^\mu \tilde{\alpha}_{-n}^\nu \Omega^{-1} = \tilde{\alpha}_{-n}^\mu \alpha_{-n}^\nu \tag{4.32}$$

leaving only symmetric fields given by

$$g_{\mu\nu}, \Phi. \tag{4.33}$$

For the **R-R sector**, it leaves only the anti-symmetric fields. Indeed, one has

$$\tilde{B}_{\mu\nu}.$$

Certain calculations show that the gauge symmetry of this theory coincides with the $SO(32)$ symmetry.

4.4.4 Classification of superstrings

The spectrum of superstring contains dilaton ϕ, where $g_s = e^\phi$ is the coupling constant, the graviton $g_{\mu\nu}$ (spin two), the antisymmetric tensor $B_{\mu\nu}$ and the gauge tensors $A_{\mu_1 \ldots \mu_{p+1}}$. One can write

$$(g_{\mu\nu}, \phi, B_{\mu\nu}) \oplus A_{\mu_1 \ldots \mu_{p+1}}. \tag{4.34}$$

The fields $A_{\mu_1 \ldots \mu_{p+1}}$ generalizing the vector potential A_μ to the tensors with $p+1$ (($p+1$)-forms, $p = 1, 2, \ldots$) can couple to p-branes. One can give the following classifications (Table 4.1).

Superstring	Type IIB	Type IIA	Heterotic $E_8 \times E_8$	Heterotic $SO(32)$	Type I
String Type	closed	closed	closed	closed	open (and closed)
Supersymmmetry of space-time	$N = 2$ chiral	$N = 2$ non-chiral	$N = 1$	$N = 1$	$N = 1$
Lie Symmetry	-	-	$E_8 \times E_8$	$SO(32)$	$SO(32)$
D-branes	$-1, 1, 3, 5, 7$	$0, 2, 4, 6$	-	-	$1, 5, 9$

Table 4.1: Classification of strings and branes.

4.5 String theory revolutions

The first revolution concerns the classification of superstring models. Here, we give the main results.

- Models go beyond particle physics
- 5 models are involved
  - Type IIA
  - Type IIB
  - Heterotic superstring theory with the SO(32) = D_{16} gauge symmetry
  - Heterotic superstring theory with the $E_8 \times E_8$ gauge group
  - Type I with SO(32) = D_{16}. gauge symmetry.

We give some problems. Among others, one has

- Dimension of the space-time: $D = 10 > 4$
- Gauge symmetries (Lie algebras) :
  - $E_8 \times E_8$
  - SO(32) = D_{16}.

The second one concerns the superstring problems and solutions. Many problems appear in superstring models

1. Space-time dimension : $D = 10 > 4$
2. Five models
3. Large gauge symmetries : $\mathrm{SO}(32) = D_{16}$, $E_8 \times E_8$
4. Other problems which could be evinced in the present work.

An examination reveals that partial solutions have been reported in many places. One of them is the compactification

$$D = 10 \to D = 4.$$

The string dualities provides connection between string models in lower dimensions. Moreover, new theories are developed. We recall the following ones:

1. M-theory in $D = 11$, (Witten, 1995)
2. F-theory in $D = 12$, (Vafa, 1996).

4.6 Compactification of higher dimensional theories

To solve certain problems, we can exploit the compactification mechanism. Indeed, four dimensional models can be obtained using the following scenario

$$R^{1,D-1} \to R^{1,3} \times X^{D-4}$$

where X^{D-4} is a $(D-4)$ dimensional compact manifold (finite size). One has the following possibilities depending on the theory in question:

1. String theory lives in $D = 10$
$$10 = 4 + 6$$
2. M-theory lives in $D = 11$
$$11 = 4 + 7$$
3. F-theory lives in $D = 12$
$$12 = 4 + 8$$

4.6.1 Toroidal compactification of superstring theory

Here, one considers the simplest compactification. This is given by

$$R^{1,9} \rightarrow R^{1,3} \times X^6$$

where $X^6 = 6$ is a 6-dimensional compact manifold given by

$$X^6 = T^2 \times T^2 \times T^2.$$

This compactification provides models going beyond the standard models of particles.

4.6.2 Connection with particle physics and Calabi-Yau manifolds

It has been shown that Calabi-Yau compactifications could be needed for establishing contact with particle physics. In this way, one has the following scenario

$$R^{1,9} \rightarrow R^{1,3} \times X^6 \tag{4.35}$$

where X^6 is a 3-dimensional Calabi-Yau manifold. The Calabi-Yau geometries are defined by

1. Kahler (complex) manifolds
2. SU(3) holonomy group
3. $\Omega = dz_1 dz_2 dz_3$.

There are many constructions of Calabi-Yau manifolds. Among others, we have

1. Orbifolds : $T^6/G, \quad \subset SU(3)$
2. Hypersurfaces in $CP^4(z_1, z_2, z_3, z_4, z_5)$ projective pace

$$P_5(z_1, z_2, z_3, z4, z_5) = 0 \tag{4.36}$$

3. Hypersurfaces in toric varieties, using toric geometry method.

4.6.3 Compactification of M-theory

It is possible to consider M-theory living in 11 dimensional space-time. The bosonic sector of the corresponding supergravity effective theory at low energies is given by

$$16\pi G_{11}\, S = \int d^{11}x\sqrt{-h}\left(R - \frac{1}{2}|F_4|^2\right) - \frac{1}{6}\int A_3 \wedge F_4 \wedge F_4, \tag{4.37}$$

where h is the space-time metric determinant and R represents the scalar curvature. The field strength $F_4 = dA_3$ originates from the gauge potential 3-form A_3 and G_{11} is the eleven dimensional gravitational constant. A close inspection reveals that M-theory involves two brane solutions, the M2 and the M5-branes. The associated compactifications read as

$$R^{1,10} \to R^{1,3} \times X^7 \tag{4.38}$$

X^7 where are 7-dimensional compact manifolds. Possible models are

1. T^7, ...

2. G_2 manifolds (Standard model)

It is recalled that G_2 manifolds are

1. 7-dimensional real manifolds

2. G_2 holonomy group

3. Special 3-form

$$w = f_{ijk} dw_i dx_j dx_k \tag{4.39}$$

This form is linked to the octonian algebra given by

$$t_i t_j = -\delta_{ij} + f_{ijk} t_k. \tag{4.40}$$

More details on compact manifolds can be found in [111]

4.7 Black objects in string theory

Many works dealing with black holes in string theory have been elaborated. In particular, $N = 2$ four-dimensional (4D) black hole physics has attracted much attention and been investigated using different connections with branes and compactifications. One nice result in this context is the Ooguri, Strominger and Vafa conjecture relating the microstates counting of 4D $N = 2$ BPS black hole in type II superstrings on Calabi-Yau (CY) threefolds to the topological string partition function on the same geometry [112]. This conjecture has been extended in [113, 114] to open topological strings, which capture information on BPS states on D-branes wrapped on Lagrangian submanifolds of CY 3-folds, and to various toric CY threefolds [115].

On the other hand, the 4D $N = 2$ black hole exhibits an interesting phenomenon called the attractor mechanism [116–128] which is the topic of interest here. At the horizon, the moduli scalar fields take fixed values given in terms of the black hole charges. This can be understood in terms of an effective potential which depends on the charges and the moduli [129]. Extremising the potential with respect to the moduli, the minimum gives such fixed values.

The 4D $N = 2$ attractors have been extended to other supergravities in 4D as well as to higher dimensions [130–137]. In five dimensions, for instance, an M-theory realisation of such black hole attractors has been studied in [132]. The corresponding low-energy theory also has $N = 2$ supersymmetry involving real special geometry. It turns out that the 4D $N = 2$ black hole attractors are connected to the 5D attractors [138]. More recently, the black object attractors in 6D and 7D have been given in [133]. In particular, the extremal BPS and non BPS black attractors in $N = 2$ supergravity, embedded in type IIA superstring theory and 11D M-theory on the K3 surface, respectively, have been studied. The attractor mechanism equations and their solutions have been treated using the criticality condition of the attractor potential of such black objects.

Here, we consider type IIA superstring on CY n-folds and discuss the corresponding extremal p-dimensional black brane attractors. In this scenario, the attractor near-horizon geometries are given by products of Ads_{p+2} and $(8 - 2n - p)$-dimensional real spheres S^{8-2n-p}. In the context of the type IIA attractor mechanism on CY n-folds, we conjecture that:

$$
\begin{array}{cccc}
p = 0 & \text{black hole charges fix} & \rightarrow & \text{geometric moduli} \\
p = 3 - n & \text{dyonic black brane charges fix} & \rightarrow & \text{the dilaton} \\
p \neq 0, 3 - n & \text{black brane charges fix} & \rightarrow & \textbf{R-R} \text{ stringy moduli.}
\end{array}
$$

4.7.1 Type IIA superstring on Calabi-Yau spaces and black brane configurations

We start with some general observations about type IIA superstrings on CY manifolds and black brane configurations in the compactified theory. It is noted that a n-dimensional CY manifold is defined by the following conditions: Complex, Kähler, and the existence of a global nonvanishing holomorphic n-form. Equivalently, it is a Kähler manifold with a vanishing first Chern class $c_1 = 0$ ($SU(n)$ Holonolmy group). After compactification, it preserves only $\frac{1}{2^{n-1}}$ of the initial supercharges. Note that each manifold involves a Hodge diagram playing a crucial role in the determination of the string theory spectrum in lower dimensions [139–141]. For later reference, we table the Hodge diagrams for $n = 1, 2, 3$ corresponding to the T^2 torus, the K3 surface and the Calabi-Yau threefolds, respectively:

$n = 1$	$\begin{array}{ccc} & h^{0,0} & \\ h^{1,0} & & h^{0,1} \\ & h^{1,1} & \end{array}$	$\begin{array}{ccc} & 1 & \\ 1 & & 1 \\ & 1 & \end{array}$
$n = 2$	$\begin{array}{ccccc} & & h^{0,0} & & \\ & h^{1,0} & & h^{0,1} & \\ h^{2,0} & & h^{1,1} & & h^{0,2} \\ & h^{2,1} & & h^{1,2} & \\ & & h^{2,2} & & \end{array}$	$\begin{array}{ccccc} & & 1 & & \\ & 0 & & 0 & \\ 1 & & 20 & & 1 \\ & 0 & & 0 & \\ & & 1 & & \end{array}$
$n = 3$	$\begin{array}{ccccccc} & & & h^{0,0} & & & \\ & & h^{1,0} & & h^{0,1} & & \\ & h^{2,0} & & h^{1,1} & & h^{0,2} & \\ h^{3,0} & & h^{2,1} & & h^{1,2} & & h^{0,3} \\ & h^{3,1} & & h^{2,2} & & h^{1,3} & \\ & & h^{3,2} & & h^{2,3} & & \\ & & & h^{3,3} & & & \end{array}$	$\begin{array}{ccccccc} & & & h^{0,0} & & & \\ & & 0 & & 0 & & \\ & 0 & & h^{1,1} & & 0 & \\ 1 & & h^{2,1} & & h^{1,2} & & 1 \\ & 0 & & h^{1,1} & & 0 & \\ & & 0 & & 0 & & \\ & & & 1 & & & \end{array}$

From this table, we can see that each diagram contains two central orthogonal lines, obtained by deleting the zeros. The vertical line represents parameters of the Kähler deformations and the horizontal one encodes the moduli of the complex structure of the Calabi-Yau n-folds. The number of parameters representing the Kähler deformations of the metric is fixed by $h^{1,1}$. The number of the complex structure deformations is given by $h^{n-1,1}$. The mirror version is obtained by interchanging the vertical and horizontal lines. It turns out that the Hodge diagram of a Calabi-Yau manifold carries not only geometric information (Kähler and complex deformations) but also physical information. Indeed, the moduli space of Calabi-Yau type IIA superstring compactification involves three different contributions depending on the entries of the Hodge diagrams. They are given by:

- The geometric deformations of the Calabi-Yau space including the antisymmetric B-field of the **NS-NS** sector. This involves the complex structure deformations, and the complexified Kähler deformations or both.
- The dilaton defining the string coupling constant.
- The scalar moduli coming from the **R-R** gauge fields on non trivial cycles of the Calabi-Yau spaces.

It is noted that in the case of the K3 surface, the last contribution does not appear due the fact that $h^{1,0} = h^{1,2} = h^{0,1} = h^{2,1} = 0$. Then, we have only two parts. Indeed, the 10D perturbative bosonic massless sector reads

$$\textbf{NS-NS} : g_{MN}, \quad B_{MN}, \quad \phi \qquad \textbf{R-R} : A_M, \quad C_{MNK} \tag{4.41}$$

where one has used $M, N, K = 0, \ldots, 9$. The K3 surface compactification gives $N = 2$ supergravity in six dimensions [141]. Its spectrum reads

$$g_{\mu\nu}, \quad B_{\mu\nu}, \quad \phi, \quad A_\mu, C_{\mu\nu\rho}, \quad C_{\mu ij}, X_s, \tag{4.42}$$

where $g_{\mu\nu}$ is the 6D metric, $B_{\mu\nu}$ and $C_{\mu\nu\rho}$ are the 6D antisymmetric gauge fields, A_μ is 6D gravi-photon and $C_{\mu ij}$ are Maxwell gauge fields coming from the compactification of $C_{\mu\nu\rho}$ on the real 2-cycles of the K3 surface. Since $C_{\mu\nu\rho}$ is dual to a vector in six dimensions, the theory has an $U(1)^{24}$ abelian gauge symmetry. In addition to the 6D dilaton ϕ, there are 80 scalar fields $X_s,\ s = 1, \ldots, 80$. The total moduli space of type IIA superstring on the K3 surface is given by

$$\frac{SO(4,20)}{SO(4) \times SO(20)} \times SO(1,1). \tag{4.43}$$

The first factor $\frac{SO(4,20)}{SO(4)\times SO(20)}$ determines the geometric deformations of the K3 surfac in the presence of the antisymmetric B-field of the **NS-NS** sector, while the groupe $SO(1,1)$ corresponds to the dilaton scalar field [133]. This factorization is related to the existence of two black object solutions in six dimensions, required by the electric/magnetic duality represented by the condition [133]

$$p+q=2. \tag{4.44}$$

This condition can be solved by two black object configurations

$$\begin{array}{lll} p=0 & \text{black hole with near-horizon geometry} & Ads_2\times S^4 \\ p=1 & \text{black string with near-horizon geometry} & Ads_3\times S^3. \end{array}$$

It is noted that $p=2$, which is dual to $p=0$, corresponds to a black 2-brane with the $AdS_4\times S^4$ near-horizon geometry. In six dimensions, we have the following properties:

- $\frac{SO(4,20)}{SO(4)\times SO(20)}$ corresponds to the black hole with 24 charges which is exactly the entries appearing in the K3 Hodge diagram. Its brane realization is given by the following Hodge diagram configuration

$$\begin{array}{ccc} & 1 & \\ 1 & 20 & 1 \\ & 1 & \end{array} = \begin{array}{ccc} & D0 & \\ D2 & D2 & D2. \\ & D4 & \end{array} \tag{4.45}$$

  Using the electric-magnetic duality, this factor describes also the moduli space of the black 2-brane, realized by the following Hodge brane configurations

$$\begin{array}{ccc} & 1 & \\ 1 & 20 & 1 \\ & 1 & \end{array} = \begin{array}{ccc} & D2 & \\ D4 & D4 & D4. \\ & D6 & \end{array} \tag{4.46}$$

- $SO(1,1)$ corresponds to the black string solution with brane charges

$$\begin{array}{ccccccc} & 1 & & & & F1 & \\ 1 & 20 & 1 & = & . & . & . \\ & 1 & & & & NS5 & \end{array} \tag{4.47}$$

Having discussed the compactification of type IIA superstrings on Calab-Yau manifolds, in what follows we will present a conjecture concerning the corresponding attractor mechanism for extremal black branes.

4.7.2 Type IIA extremal black branes on Calabi-Yau manifolds

Before going ahead, let us recall some results obtained in this context. Consider type IIA superstring propagating on a Calabi-Yau 3-folds. This leads to a 4D $N = 2$ supergravity coupled to the $U(1)^{h_{1,1}}$ abelian gauge symmetry. In this theory, the scalars belonging to the vector multiplets are given by the complexified Kähler moduli space. The complex structure deformations correspond to 3-cycles, and the **R-R** gauge fields contribute by scalars in the hypermultiplets. However, this sector has no role in the study of the 4D black hole attractors. Indeed, several studies of the 4D type IIA super attractor mechanism reveal that in the near horizon geometry only the complexified Kähler moduli can be fixed by the abelian black hole charges [116–128]. This can be obtained by minimizing the scalar effective potential of the 4D black hole [129]. This potential appears in the action of the $N = 2$ supergravity theory coupled to the Maxwell theory in which the abelian gauge vectors come from the reduction of the antisymmetric 3-form on 2-cycles of Calabi-Yau 3-folds[1]. The other moduli, corresponding either to the complex structure deformations or to the moduli coming from the **R-R** gauge fields on Calabi-Yau cycles, remain free and take arbitrary values in the near horizon limit of black hole. We believe that this is due to the absence of higher-dimensional black objects in 4D[2]. However, the situation in six dimensions is somewhat different. It is obtained by

[1]Note that the parameters of the complex structure has been fixed in the context of type IIB superstrings using mirror symmetry.

[2]This can be seen from the electric-magnetic duality equation.

a compctification on the K3 complex surface. The latter has a mixed geometric moduli space involving both the complexified Kähler and the complex structure deformations. The corresponding attractor mechanism for the black objects has been studied in [133](see also [134–136]). Indeed, using a matrix formulation, the geometric moduli of the K3 surface including the **NS-NS** B-field are fixed by the abelian black hole charges. However, the dilaton has been fixed by the black fundamental string charges (the electric and magnetic charges).

Having discussed the known results, we now present our conjecture for the Calabi-Yau attractor mechanism. The main idea is that the moduli of type IIA superstrings on a Calabi-Yau manifold could be fixed by the charges of extremal black branes appearing in the compactified theory. Indeed, consider type IIA superstrings on a n-dimensional Calabi-Yau manifold. This compactification gives supergravity models with 2^{6-n} supercharges coupled to an abelian gauge symmetry [3]. The near horizon of the corresponding black objects is defined by the product of AdS spaces and spheres:

$$AdS_{p+2} \times S^{8-2n-p}. \tag{4.48}$$

The integers n and p satisfy the following constraints

$$2 \leq 8 - 2n - p. \tag{4.49}$$

The electric/magnetic duality relating the p-dimensional electrical black branes to q-dimensional magnetic ones reads

$$p + q = 6 - 2n. \tag{4.50}$$

This equation can be solved in different ways. Here, we give the possibles configurations. In particular, we have:

- $(p, q) = (0, 6 - 2n)$, describing an electrical charged BH.
- $(p, q) = (3 - n, 3 - n)$, describing dyonic black branes.
- $(p, q) \neq (0, 6 - 2n)$ and $(p, q) \neq (3 - n, 3 - n)$, describing black objects like strings, membranes and higher-dimensional branes.

[3]Notice that the non abelian gauge theories could be obtained from the singular limit of such geometries.

More precisely, a mapping can be established. In particular, we have the following statement:

- The black hole charges fix only the geometric deformations of the Calabi-Yau space including the **NS-NS** B-field. This involves the complex structure deformations, the complexified Kähler deformations or both.

$$p = 0 \qquad \text{black hole charges fix} \rightarrow \quad \text{geometric moduli}$$

- The dilaton could be fixed by the dyonic object.

$$p = 3 - n \qquad \text{dyonic black brane charges fix} \rightarrow \quad \text{the dilaton}$$

- The moduli coming from **R-R** gauge fields on Calabi-Yau cycles should be fixed by the higher-dimensional black object charges, like strings and branes.

$$p \neq 0, 3 - n \qquad \text{black brane charges fix} \rightarrow \quad \textbf{R-R} \text{ stringy moduli.}$$

4.7.3 Motivations of dyonic objects

Before dealing with dyonic objects associated with the electromagnetic duality in higher dimensions, we would like to shed light on some related concepts. Indeed, a similar one goes back to the neutrino discovery, where the proton and the neutron can be regarded as two states of a single particle motivated by the observation that they have approximately equal masses $m_p \simeq m_n = m$ [142], which in turn conducted, according to the mass-energy equivalence, to an energy degeneracy of the underlying interaction. This mass degeneracy led then to the existence of a corresponding symmetry whose the interaction obeys. That is to say, these two particles have an identical behavior under the underlying interaction and that their charge content is their solely difference. In such a case, where these two particles are to be viewed as two linearly independent states of the same particle, i.e., nucleus N, it is unexceptional to depict them in the form of a two component vector like

$$N : \begin{pmatrix} \underline{p} \\ \\ \underline{n} \end{pmatrix}_{m_{\underline{p},\underline{n}}} \tag{4.51}$$

where $\underline{p}$ and $\underline{n}$ refer to the proton and the neutron particles making such a nucleus state. More fundamentally, analogous to the spin-up and spin-down states of a spin-one half particle, the concept of isospin symmetry is introduced and is governed by an $SU(2)_I$ group rotating doublet components, i.e., proton and neutron, into each other in abstract isospin space. In particular, in the modern formulation, the isospin is defined as a vector quantity in which up and down quarks have a value of 1/2, with the 3rd-component being $+1/2$ for up quarks, and $-1/2$ for down quarks. In this picture, denoting the total isospin I and its 3rd component $I_3 = \pm 1/2$, an up-down quarks pair can be assembled in a state of the total isospin 1/2 as

$$\begin{pmatrix} \underline{u} \\ \\ \underline{d} \end{pmatrix}_{I=1/2} \tag{4.52}$$

where, again, $\underline{u}$ and $\underline{d}$ refer to the up and down quarks forming such a doublet state. Albeit such a particle state classification is a good approximate one with a symmetry dealing with nuclear interactions. It remains a useful and large concept. In fact, one could go further and extend this view to the electromagnetic charge where each particle P could be represented according to its charge content by the doublet

$$\mathrm{P} : \begin{pmatrix} Q_e \\ \\ P_m \end{pmatrix}_{Q_{em}} , \tag{4.53}$$

whose components are the particle electric and magnetic charges. At first sight, this picture will seem to be strange or even inconsistent with the symmetry of the electromagnetic interaction as dictated by the corresponding $U(1)_{Q_{em}}$ group. However, this apparent inconsistency could be simply alleviated by assuming a hidden symmetry group associated with the magnetic charge. Within the the standard model framework, a simple and economic way to do so is the consider an extended electroweak symmetry $SU(2)_L \times U(1)_e \times U(1)_m$ involving both electric $U(1)_e$ and magnetic $U(1)_m$ symmetries being broken at a certain high energy resulting in the standard low energy electromagnetic symmetry $U(1)_{Q_{em}}$. In this picture, and at a certain energy scale, the particle charge representation in ref gives rise to the following physical objects, for instance

$$\begin{pmatrix} 0 \\ 0 \end{pmatrix}_0 , \quad \begin{pmatrix} Q_e \\ 0 \end{pmatrix}_{Q_e} , \quad \begin{pmatrix} 0 \\ P_m \end{pmatrix}_{Q_m} , \quad \begin{pmatrix} Q_e \\ P_m \end{pmatrix}_{Q_{em}} . \tag{4.54}$$

These four objects are nothing but all known possible particles, discovered and hypothetical. In particular, the first neutral doublet is nothing but the neutrino, being neutral, the second doublet corresponds to the ordinary charged particles, the third doublet is that of theoretical magnetic monopoles with a single magnetic charge, while the four doublet corresponds to a dyonic particle carrying both electric and magnetic charges. For more clarity we list them in (Table 4.2). We can now see that this particle classifi-

object	Q_e	P_m	nature
neutrinos	no	no	real
ordinary (electrically) charged particles	yes	no	real
magnetic monopoles	no	yes	hypothetical
dyons	yes	yes	hypothetical

Table 4.2: The four possible particles, observed and hypothetical, in terms of their electromagnetic charges.

cation view has given rise to the well motivated hypothetical objects, namely magnetic monocles charged only magnetically, and dyons carrying both the electric and the magnetic charges associated with the electromagnetic duality in higher dimensions. These objects are predicted by most grand unified theories in high energy physics and are now one of the active field of research in many theoretical and experimental works [37]. As we shall see in what follows, such a view can be considered as a part of a larger concept being of great utility to classify physical object families.

Part II

Results and Scientific Contributions

In this part, we present our results and contributions. This involves the following publications:

1. Y. El Maadi et al. "**Stringy Dyonic Solutions and Clifford Structures**", *IJGMMP*, Vol. 16, No. 9 (2019) 1950138,

2. Y. El Maadi et al. **Dyonic Objects and Tensor Network Representation**, *MPLA*, Vol. 36, No. 2 (2021) 2050336,

3. Y. El Maadi et al. **Toric Realisation of Quantum Systems via Clifford Structures**, *JAMP*, Vol. 10, No.2, (2022) 527-537.

Contribution 1

Title: Stringy Dyonic Solutions and Clifford Structure.

Athors: A. Belfakir, A. Belhaj, Y. El Maadi, S. E. Ennadifi, Y. Hassouni ans A. Segui

Journal: International Journal of Geometric Methods in Modern Physics.

In this work, we investigate dyonic objects in arbitrary dimensions, by using the toroidal compactification of string theory on n-dimensional tori $\mathbb{T}^n$. First, we present a class of dyonic black solutions formed by two different D-branes using a correspondence between toroidal cycles and objects possessing both magnetic and electric charges, belonging to $U(1)_e^{2^{n-1}} \times U(1)_m^{2^{n-1}}$ dyonic gauge symmetry. This symmetry could be associated with electrically charged magnetic monopole solutions in stringy model buildings of the standard model (SM) extensions. Then, we consider in some detail such black hole classes obtained from even-dimensional toroidal compactifications, and we find that they are linked to $Cl(n)$ Clifford algebras using the vee product. It is believed that this analysis could be extended to dyonic objects which can be obtained from local Calabi–Yau manifold compactifications.

1 Introduction

Recently, black holes and their extensions have been extensively investigated in connection with higher dimensional supergravity models compactified on Calabi-Yau (CY) manifolds [143–147]. In certain compactifications, the scalar field contributions can be fixed in terms of the black brane potential by optimizing the stringy moduli parameterized by physical and geometrical deformations. In this way, the corresponding entropy functions have been computed using the duality symmetries which act on the invariant black object charges. Considering D-brane physics, several compactfications producing some black brane solutions embedded in type II superstrings have been studied [148–150]. It has been shown that these black objects can be linked to many subjects including quantum information theory by exploiting the qubit mathematical formalism [151–158]. Concretely, a remarkable correspondence between qubit systems and black holes in superstring theory has been established. The relevant findings concern a nice link between the $N = 2$ STU black hole with eight charges and three-qubit system states [155, 159–161]. The analysis has been developed to describe structures going beyond such extremal black hole solutions in terms of higher-dimensional qubit systems. Using string dualities between type IIA and heterotic superstrings, four-qubit systems in the context of type II superstring compactifications have been investigated [162, 163]. It has been revealed that such qubit systems are related to a stringy moduli space $\frac{SO(4,4)}{SO(4)\times SO(4)}$ considered as a reduction of the moduli space of six-dimensional supergravity model obtained from the compactification of the heterotic superstring on $\mathbb{T}^4$ [164]. Moreover, the string/string duality has been exploited to show that there exists an interplay between four-qubit states and a particular ordinary dyonic black holein six dimensions having eight electric and eight magnetic charges.

More recently, electric and magnetic charges have been reconsidred to discuss a dark matter sector using the SL(2,$\mathbb{Z}$) duality [165]. It is recalled that such a sector involves a massive dark photon and dark magnetic monopoles investigated in terms of a kinetic mixing of the dark and usual photons. These activities could be related to black objects in the presence of Dark Energy being the hardest puzzles in modern and new physics. In this regard, any serious attempt corresponding to such an ambiguous dark sector is welcome.

2 Stringy dyonic black objects in arbitrary dimensions

In this section, we reconsider the study of dyonic solutions in type II superstrings compactified on n-dimensional real manifolds X^n. It has been observed that each compact manifold possesses some geometric information which can play a crucial role in the determination of the superstring theory spectrum in $10-n$ dimensions [164]. In particular, they can be exploited to produce all physical data in lower dimensional superstring models. In connections with extremal black objects, it has been suggested that the black p-branes can be built using a system involving $(p+k)$-branes wrapping k-cycles of X^n. It has been remarked that the corresponding near horizon geometries are given by a product of AdS spaces and spheres $\mathrm{Ads}_{p+2} \times S^{8-n-p}$ where n and p are integers constrained by $1 \leq n$ and $2 \leq 8-n-p$. It is noted that these black objects are coupled to $(p+1)$-gauge field C_{p+1}. To measure their electric charges, one should use the field strength dC_{p+1}.This field will play the same role of the field strength $F = dA$ where A is the 1-form gauge potential coupled to the ordinary particle associated with the following action term:

$$S \sim \int F \wedge \star F, \tag{1}$$

where $\star F$ is the dual of F. It is known that the electric charge is measured by the integration

$$Q_e = \int_{\mathbb{S}^{10-n-2}} \star F, \tag{2}$$

where $\mathbb{S}^{10-n-2}$ is the $(10-n-2)$-dimensional real sphere. Similarly, one can compute the magnetic charge using the relation

$$P_m = \int_{S^2} F. \tag{3}$$

These particle charge equations can be generalized to p-branes being p-dimensional objects moving in the string theory space-time.In this case, the electric charge is calculated as follows

$$Q_e = \int_{\mathbb{S}^{10-n-p-2}} \star dC_{p+1}. \tag{4}$$

where $\star dC_{p+1}$ is the dual of the dC_{p+1} gauge field. However, the magnetic charge can be given by the following integration:

$$P_m = \int_{\mathbb{S}^{p+2}} dC_{p+1}. \tag{5}$$

where $\mathbb{S}^{p+2}$ is the $(p+2)$-dimensional real sphere. In higher-dimensional theories including string theory, one may classify the black brane solutions using the extended electric/magnetic duality connecting a p-dimensional electrical black brane to a q-dimensional magnetic one via the following relation:

$$p + q = 6 - n. \tag{6}$$

This constraint can be solved in different manners using appropriate (p, q) couple values including the ones representing the dyonic solutions in arbitrary dimensions. However, a deeper inspection in string compactifications shows that dyonic solutions carrying electric and magnetic charges can be arranged as object like doublets

$$\begin{pmatrix} p \\ q \end{pmatrix}. \tag{7}$$

This doublet configuration is motivated and supported by the SL(2, Z) electric-magnetic duality [165]. It has been remarked that these dyonic solutions are associated with a gauge field symmetry involving two factors corresponding to electric and magnetic sectors

$$G_{dyon} = G_e \times G_m. \tag{8}$$

In the compactification on n-dimensional real manifolds, two different dyonic solutions could appear with electric and magnetic charges. They are classified in what follows.

2.1 Ordinary dyonic solution

This solution is described by the constraint

$$p = q. \tag{9}$$

It turns out that such a solution finds a place only in even-dimensional string theory compactifications. A simple calculation shows that one has

$$p = q = 3 - \frac{n}{2}. \tag{10}$$

This generates dyons represented by the following doublets:

$$\begin{pmatrix} p \\ q \end{pmatrix} = \begin{pmatrix} 3 - \frac{n}{2} \\ 3 - \frac{n}{2} \end{pmatrix}. \tag{11}$$

consisting of the same object which appears in even-dimensional superstring models. This solution can be described only by a single factor related to both electric and magnetic charges. In this way, the above dyonic symmetry can be reduced to

$$G_{dyon} = G_e. \tag{12}$$

2.2 Nonordinary dyonic solution

The second solution concerns the case

$$p \neq q \tag{13}$$

associated with two different D-branes. This involves dyonic objects consisting of an electrically charged black object and its magnetic dual one. In such a case, p and q are constrained by

$$q = 6 - n - p, \qquad p \neq 3 - \frac{n}{2}. \tag{14}$$

This solution generates doublets of the following form:

$$\begin{pmatrix} p \\ q \end{pmatrix} = \begin{pmatrix} p \\ 6 - n - p \end{pmatrix}. \tag{15}$$

carrying electric and magnetic charges associated with two gauge symmetry factors given in (8). This new dyonic solution could be worked out to provide a possible support of monopoles in the context of string theory compactifications.

We wish to add some comments on these dyonic black solutions. The first comment concerns the fact that the nonordinary solution is considered as a single object sharing similar features of the ordinary one described by the same object, appearing in even-dimensional string theory. The associated dyonic physics is invariant under the mapping

$$p \to 6 - n - p. \tag{16}$$

interpreted as an electric-magnetic symmetry in string theory compactifications. The second comment concerns the brane configurations of such dyonic black solutions. In string theory, they should be constructed from the following dyonic brane doublets living in ten-dimensional space-time:

$$\begin{pmatrix} D0 \\ D6 \end{pmatrix}, \quad \begin{pmatrix} NS1 \\ NS5 \end{pmatrix}, \quad \begin{pmatrix} D1 \\ D5 \end{pmatrix}, \quad \begin{pmatrix} D2 \\ D4 \end{pmatrix}, \quad \begin{pmatrix} D3 \\ D3 \end{pmatrix}, \tag{17}$$

Before examining a particular supergravity theory solution, we should recall that the black object charges depend on the choice of the internal space. It turns out that each compactification involves a Hodge diagram carrying not only geometric information but also physical data providing the above mentioned dyonic black solutions.

3 Dyonic black solutions from toroidal compactifications

For simplicity reasons, we consider the toroidal compactification having a nice real Hodge diagram playing a primordial role in the elaboration of dyonic stringy solutions in $10-n$ dimensions. It is recalled that $\mathbb{T}^n$ is a flat compact space which can be constructed using different approaches. One of them is to exploit the trivial circle fibration given by the following identifications $x_i \equiv x_i + 1, \quad i = 1, \ldots, n$. It is remarked that the general real Hodge diagram of $\mathbb{T}^n$ can encode all possible non trivial cycles describing geometric data including the size and the shape parameters. Using a binary number notation, the number $h^{e_1,\ldots,e_n}$ will be associated with a real differential form of degree k (k-differential form) $\prod_{\ell=1}^{n}(\overline{e_\ell} + e_\ell dx_\ell)$, where e_ℓ takes either 0 or 1, and where $\overline{e_\ell}$ is its conjugate. In this way, k equals to $\sum_{i=1}^{n} e_i$. To illustrate such real Hodge diagrams, one may consider lower dimensional cases corresponding to $n=1$, $n=2$ and $n=3$. They are given by, respectively

$n=1$	h^1 h^0	1 1
$n=2$	$h^{1,1}$ $h^{1,0}$ $h^{0,1}$ $h^{0,0}$	1 1 1 1
$n=3$	$h^{1,1,1}$ $h^{1,1,0}$ $h^{1,0,1}$ $h^{0,1,1}$ $h^{1,0,0}$ $h^{0,1,0}$ $h^{0,0,1}$ $h^{0,0,0}$	1 1 1 1 1 1 1 1

3.1 Dyonic black object on $\mathbb{S}^1$

To establish a link with string theory, we consider some dyonic black objects obtained from the compactification on 1-dimensional circle $\mathbb{S}^1$. Indeed, the nine dimensional dyonic black objects involve one electric charge Q_0 and one magnetic charge under P_0 associated with the dyonic gauge symmetry

$$G_{dyon} = U(1)_e \times U(1)_m. \tag{18}$$

The corresponding brane representations can be obtained from the ten dimensional dyonic ones illustrated in the previous section. Using the Hodge diagram representation, the brane configurations are constrained by

$$p + q = 5. \tag{19}$$

In this case, the dyonic solutions can be classified as

$$\begin{pmatrix} D0 \\ D5 \end{pmatrix}, \quad \begin{pmatrix} D1 \\ D4 \end{pmatrix}, \quad \begin{pmatrix} D2 \\ D3 \end{pmatrix}. \tag{20}$$

These objects can be built from the following wrapped D-branes, respectively

$$\begin{array}{ccc} \mathrm{D0} & \mathrm{D1} & \mathrm{D2} \\ D6/S^1_1 & D5/S^1_1 & D4/S^1_1 \end{array} \tag{21}$$

The charges of these solutions are associated with the cycles dual to the following real forms on the circle

$$1, \quad dx. \tag{22}$$

3.2 Dyonic black objects on $\mathbb{T}^2$

Eight-dimensional dyonic black solutions can be obtained from the compactification on $\mathbb{T}^2$ constrained by

$$p + q = 4. \tag{23}$$

In this case, the corresponding type II superstrings involve three dyonic black solutions given by

$$\begin{pmatrix} 0 \\ 4 \end{pmatrix}, \quad \begin{pmatrix} 1 \\ 3 \end{pmatrix}, \quad \begin{pmatrix} 2 \\ 2 \end{pmatrix}. \tag{24}$$

For instance, the $\begin{pmatrix} 0 \\ 4 \end{pmatrix}$ dyonic solution can be represented by the following D-brane charge configurations:

$$\begin{array}{ccc} & \mathrm{D0} & \\ \mathrm{D1}/\mathbb{S}^1_1 & & \mathrm{D5}/\mathbb{S}^1_2 \\ & \mathrm{D6}/\mathbb{T}^2 & \end{array} . \tag{25}$$

Exploiting string theory links including the S-duality, an equivalent D-brane configuration can be obtained by the following mapping:

$$
\begin{aligned}
D1 &\leftrightarrow NS1 \\
D5 &\leftrightarrow NS5
\end{aligned}
\tag{26}
$$

However, the remaining $\begin{pmatrix} p \\ q \end{pmatrix}$ dyonic solutions such that $p+q=4$ can be represented by the following D-brane charge configurations:

$$
\begin{array}{ccc}
 & \mathrm{D}p & \\
\mathrm{D}p+1/\mathbb{S}^1_1 & & \mathrm{D}q+1/\mathbb{S}^1_2 \\
 & \mathrm{D}q+2/\mathbb{T}^2 &
\end{array}
\tag{27}
$$

having two electric charges and two magnetic charges. These charges associated with the cycles belonging to $\mathbb{T}^2 = \mathbb{S}^1_1 \times \mathbb{S}^1_2$ dual to the following real differential forms on $\mathbb{T}^2$ can be organized as follows:

$$
\begin{array}{ccc}
 & 1 & \\
dx_1 & & dx_2 \\
 & dx_1 dx_2 &
\end{array}
\tag{28}
$$

In this representation, the forms $(1,\, dx_1)$ correspond to the $(\mathrm{D}p,\, \mathrm{D}p+1/\mathbb{S}^1_1)$ brane system involving two electric charges associated with $G_e = U(1)^2_e$. However, the dual forms $(dx_2,\, dx_1 dx_2)$ are linked to the $(\mathrm{D}q+1/\mathbb{S}^1_2,\, \mathrm{D}q+2/\mathbb{T}^2)$ brane configuration having two magnetic charges corresponding to $G_m = U(1)^2_m$. For $p=q=2$, it is worth noting that the dyonic gauge symmetry $G_{dyon} = U(1)^2_e \times U(1)^2_m$ reduces to $G_{dyon} = U(1)^2$ for both electric and magnetic charges.

3.3 Dyonic black objects on $\mathbb{T}^3$

In this subsection, we consider three copies of $\mathbb{S}^1$ corresponding to the compactification of superstring theory on $\mathbb{T}^3$. The manifold is determined by three circles $\mathbb{S}^1_1$, $\mathbb{S}^1_2$ and $\mathbb{S}^1_3$ coordinated by x_1, x_2 and x_3, respectively. This compactification produces seven-dimensional dyonic black solutions having four electric charges and four magnetic ones, under $G_{dyon} = U^4(1)_e \times U^4(1)_m$ dyonic gauge symmetry. In this scenario, one has two fundamental solutions given by the following doublets:

$$\begin{pmatrix} 0 \\ 3 \end{pmatrix}, \quad \begin{pmatrix} 1 \\ 2 \end{pmatrix}. \tag{29}$$

Using the T-duality and the real Hodge diagram of $\mathbb{T}^3$, the solution $\begin{pmatrix} 0 \\ 3 \end{pmatrix}$ can be built using the following D-brane configuration

$$\begin{array}{ccc} & \mathrm{D0} & \\ \mathrm{D1}/\mathbb{S}^1_1 & \mathrm{D1}/\mathbb{S}^1_2 & \mathrm{D1}/\mathbb{S}^1_3 \\ \mathrm{D5}/\star\mathbb{S}^1_1 & \mathrm{D5}/\star\mathbb{S}^1_2 & \mathrm{D5}/\star\mathbb{S}^1_3 \\ & \mathrm{D6}/\mathbb{T}^3 & \end{array} \tag{30}$$

where $\star\mathbb{S}^1_i$ are dual to $\mathbb{S}^1_i$ in the $\mathbb{T}^3$. However, the solution $\begin{pmatrix} 1 \\ 2 \end{pmatrix}$ involves only one Hodge D-brane configuration given by

$$\begin{array}{ccc} & \mathrm{D1} & \\ \mathrm{D2}/\mathbb{S}^1_1 & \mathrm{D2}/\mathbb{S}^1_2 & \mathrm{D2}/\mathbb{S}^1_3 \\ \mathrm{D4}/\star\mathbb{S}^1_1 & \mathrm{D4}/\star\mathbb{S}^1_2 & \mathrm{D4}/\star\mathbb{S}^1_3 \\ & D5/\mathbb{T}^3 & \end{array} \tag{31}$$

These four states $\{D1, D2/\mathbb{S}^1_1, D2/\mathbb{S}^1_2, D2/\mathbb{S}^1_3\}$ represent seven-dimensional dyonic solutions carrying electric charges. The remaining four states having magnetic charges are obtained using the Hodge duality which has been used to build the real Hodge diagram illustrated in section 2. An examination reveals that these states can be linked with the following real Hodge diagram of $\mathbb{T}^3$:

$$\begin{array}{ccccc} & 1 & & & \\ & & & & \\ dx_1 & dx_2 & & dx_3 & \\ & & & & \\ \star dx_1 & \star dx_2 & & \star dx_3 & \\ & & & & \\ & dx_1 dx_2 dx_3 & & & \end{array} \tag{32}$$

where $\star dx_i$ are Hodge duals to dx_i in $\mathbb{T}^3$.

3.4 Dyonic black objects on $\mathbb{T}^n$ and dark sectors

The compactification of superstring theory on $\mathbb{T}^n$ generates dyonic black solutions in $10-n$ dimensions. In this way, the corresponding real Hodge diagram can be built in terms of the forms

$$1, \quad dx_1, \quad dx_2, \ldots, \quad dx_1 \ldots dx_n. \tag{33}$$

Concretely, the dynonic black object states can be associated with such a set of differential forms dual to cycles in which D-branes can be wrapped on. The presence of dyonic black objects $\begin{pmatrix} p \\ 6-n-p \end{pmatrix}$, in such a superstring theory compactification, suggest that there are 2^{n-1} states associated with the electric charges. By using the Hodge mapping between forms

$$k - forms \Leftrightarrow (n-k) - forms, \tag{34}$$

we can show that there are also 2^{n-1} states corresponding to the magnetic charges assured by the decomposition

$$2^n = 2^{n-1} \times 2^{n-1}. \tag{35}$$

supported by the dyonic symmetry

$$G_{dyon} = U(1)_e^{2^{n-1}} \times U(1)_m^{2^{n-1}}. \tag{36}$$

This symmetry, associated with the electrically charged magnetic monopole solutions within the context of superstring theory, could be relevant in stringy model buildings of the SM extensions [167, 168]. Precisely, interesting extensions of SM can be constructed from compactified superstring models where the geometry and the topology of the internal space give rise to extra abelian gauge symmetries $U_{X_i}(1)'s$ corresponding to some conserved charges X_i along with new scalar fields S_i. This can be usually achieved in the context of intersecting D-branes wrapping nontrivial cycles embedded in compact manifolds [169–171]. Thus, it could be shown that there are many roads to handle the physics underlying such a dyonic symmetry Gdyon in the stringy inspired extended $SM's$. For the charge identification $X_i \equiv Q_e, P_m$, we can consider the assumed symmetry such as

$$U_{X_i}(1)^{2n} \equiv U(1)_e^{2^{n-1}} \times U(1)_m^{2^{n-1}}, \tag{37}$$

where the new scalar fields S_i can play an important role in the underlying scale as well as the resulting mass spectrum. On the basis of such motivations, one can deal with the extended gauge symmetry

$$G_{SM+dyon} = SU_C(3) \times SU_L(2) \times G_{dyon}, \tag{38}$$

where the first piece $SU_C(3)$ refers to the strong interaction associated with the color charge C. The second one $SU_L(2)$ refers to the weak interaction corresponding to the left isospin charge L. However, the sector $SU_L(2) \times G_{dyon}$ refers now to the extended electroweak dark sector. Roughly, this extended electroweak symmetry is expected to be broken by the vacuum expectation value (vev) of a new scalar S down to the four-dimensional SM one. This, in turn, is broken down to the electromagnetic symmetry $U_{Qem}(1)$ by the vev of the standard Higgs scalar field H in the following scheme:

$$SU_L(2) \times U(1)_e^{2^{n-1}} \times U(1)_m^{2^{n-1}} \xrightarrow{<\mathrm{S}>} SU_2(L) \times U(1)_Y \xrightarrow{<\mathrm{H}>} U_{Qem}(1), \tag{39}$$

where now the new complex scalar S, associated with the dyonic symmetry breaking, corresponds to a 2^{n-1}-tuple with 2^n real components as $S = s_i^1 + is_i^2, \quad i = 1, \ldots, 2^{n-1}$. At this point, seen that the dyonic symmetry breaking scale is expected to be above the SM electroweak scale $\langle S \rangle > \langle H \rangle \sim 10^2 Gev$, this would make such dyons heavy as being proportional somehow to the new scalar vev. Therefore, they are far to be observed at the current experiments unless the corresponding proportionality is highly suppressing. In this case, the best way of probing the existence of such dyons remains their observation in cosmic rays as recently inspected by the actual telescopes [172–175].

Having discussed how a possible link with extended SM is provided by using extra symmetries and scalar fields. It has been shown that these symmetries can be considered as structures of division algebraic ladder operators. In what follows, we elaborate a relation with Clifford algebraic structures reported in [176, 177].

4 Link with differential form structures

A close inspection show that the dyonic black objects obtained from torroidal compactifications can be linked to a nontrivial structures corresponding to the real differential forms on $\mathbb{T}^n$.

To do so, let us first recall such a structure [176–180]. Let V be a vector space over the field of real numbers. In this space, we can define a quadratic form using the map $Q : V \to R$, where Q satisfies

$$Q(\alpha v) = \alpha^2 Q(v) \tag{40}$$

for all α in R and v in V. One can also define the Grassmann-Cartan exterior product $\wedge$ of differential forms as

$$dx^\mu \wedge dx^\nu = -dx^\nu \wedge dx^\mu, \quad \mu \neq \nu, \qquad dx^\mu \wedge dx^\mu = 0. \tag{41}$$

The Grassman product is an associative product $(dx^\mu \wedge dx^\nu) \wedge dx^\lambda = dx^\mu \wedge (dx^\nu \wedge dx^\lambda)$. However, The $\vee$ product of two diffrential forms is defined as

$$dx^\mu \vee dx^\nu = (dx^\mu, dx^\nu) + dx^\mu \wedge dx^\nu \tag{42}$$

Here, (dx^μ, dx^ν) denotes the scalar product between dx^μ and dx^ν and $(dx^\mu, dx^\nu) = g^{\mu\nu}$ where $g^{\mu\nu}$ is the vector space metric. In what follows, we will take $g^{\mu\nu} = \delta^{\mu\nu}$ by considering only the euclidien metrics associated with the toridal compactification using the vee product. The Cliford algebra $Cl(n)$ generated by the differential forms is defined as

$$dx^\mu \vee dx^\nu + dx^\nu \vee dx^\mu = 2\delta^{\mu\nu} Q(dx^\mu) \tag{43}$$

where $Q(dx^\mu) = \{dx^\mu, dx^\mu\} = 1$ and $\mu = 1, \ldots, \frac{n}{2}$ and n is an even number. Using the string theory toroidal compactifaction, we will see that one can construct a set of operators a_ℓ satisfying

$$\{a_\ell, a_{\ell'}\} = \{a_\ell^\dagger, a_{\ell'}^\dagger\} = 0, \quad \{a_\ell, a_{\ell'}^\dagger\} = \delta_{\ell\ell'}. \tag{44}$$

4.1 $Cl(n)$ and differential forms on $\mathbb{T}^n$

To make contact with the above structure, we will be interested in complex geometry associated with n even. For $n=2$, the operator a and $a^\dagger$ are given in terms of one-forms on $\mathbb{T}^2$. Indeed, one has the following identifications:

$$a = \frac{1}{2}(dx^1 + idx^2), \quad a^\dagger = \frac{1}{2}(-dx^1 + idx^2). \tag{45}$$

It is remarked that $\dagger$ maps $i \longrightarrow -i$ and $dx^j \longrightarrow -dx^j$ for $j=(1,2)$. The $Cl(4)$ Clifford algebra could be built using the real differential form set $\{dx^1, dx^2, dx^3, dx^4\}$. From these forms, we construct a set of operators satisfying the above structure. In this case, we have four generators which are a_1, a_2 and thier adjoints $a_1^\dagger$, $a_2^\dagger$. Once again the operation $\dagger$ maps $i \longrightarrow -i$ and $dx^i \longrightarrow -dx^i$. Using the compactification on $\mathbb{T}^4$, the generators can be constructed as follows

$$a_1 = \frac{1}{2}(-dx^1 + idx^2), \quad a_2 = \frac{1}{2}(-dx^3 + idx^4) \tag{46}$$

and

$$a_1^\dagger = \frac{1}{2}(dx^1 + idx^2), \quad a_2^\dagger = \frac{1}{2}(dx^3 + idx^4) \tag{47}$$

Similarly as $Cl(2)$ and $Cl(4)$, the set of ladder operators of $Cl(6)$ is $\{a_1, a_2, a_3, a_1^\dagger, a_2^\dagger, a_3^\dagger\}$ where

$$a_1 = \frac{1}{2}(-dx^5 + idx^4), \quad a_2 = \frac{1}{2}(-dx^3 + idx^1) \quad a_3 = \frac{1}{2}(-dx^6 + idx^2) \tag{48}$$

and

$$a_1^\dagger = \frac{1}{2}(dx^5 + idx^4), \quad a_2^\dagger = \frac{1}{2}(dx^3 + idx^1) \quad a_3^\dagger = \frac{1}{2}(dx^6 + idx^2) \tag{49}$$

satisfying the above Clifford structures.

4.2 $Cl(n)$ and dyonic black objects

The above dyonic black object construction can be made using D-branes moving on non trivial cycles. In the associated compactification, one can distinguish two types of cycles namely electric and magnetic cycles noted by $\mathcal{C}^a$ and $\mathcal{D}^a$, respectively

$$\{[\mathcal{C}^\alpha],\ \alpha = 0, \ldots, 2^{n-1}-1\}, \quad \{[\mathcal{D}^\alpha],\ \alpha = 0, \ldots, 2^{n-1}-1\}. \tag{50}$$

These cycles are dual to ζ and η forms on $\mathbb{T}^n$ defined by

$$\int_{\mathcal{C}^\alpha} \zeta^\beta = \delta^\alpha_\beta, \qquad \int_{\mathcal{D}^\alpha} \eta^\beta = \delta^\alpha_\beta. \tag{51}$$

such that

$$\int_{\mathbb{T}^n} \zeta^\alpha \wedge \eta^\beta = \delta^\beta_\alpha. \tag{52}$$

The electric vectors determine a basis for the 2^{n-1} abelian $(p+1)$-gauge field C_{p+1} obtained by integrating the ten dimensional forms on $\mathcal{C}^\alpha$. Similarly, the magnetic vectors determine a basis for the 2^{n-1} abelian vector fields $(q+1)$-gauge field C_{q+1} obtained by integrating the ten dimensional forms on $\mathcal{D}^\alpha$. Under these abelian gauge fields, associated with $G_{dyon} = U(1)_e^{2^{n-1}} \times U(1)_e^{2^{n-1}}$ dyonic gauge symmetry, the dyonic black objects in generic cohomology class $[X]$, on $\mathbb{T}^n$, are given by

$$[X] = \sum_{\alpha=0}^{2^{n-1}-1} P_\alpha[\mathcal{C}^\alpha] + Q_\alpha[\mathcal{D}^\alpha], \tag{53}$$

which carry P_α electric and Q_α magnetic charges. The dual of this cycle is a general element of $Cl(n)$ with real coefficient associated with physical charges. Motivated by ideals of $Cl(n)$ used in SM of particle physics [176, 177], we associate to each $[X]$ the following black object state:

$$|\psi\rangle = Cl(n)|0\rangle \tag{54}$$

where $|0\rangle$ is considered here as a vacuum state. To be precise, the state $|\psi\rangle$ is mapped to the class $[X]$

$$[X] \leftrightarrow |\psi\rangle. \tag{55}$$

In terms of the ladder operators a_i, the state can take the following form

$$|\psi\rangle = \sum_{e_1,\ldots,e_n=0,1} p_{e_1\ldots e_n} \prod_{\ell=1}^{\frac{n}{2}} a_\ell^{e_\ell} \vee {a^\dagger}_\ell^{e_{\ell+\frac{n}{2}}} |0\rangle \tag{56}$$

where $p_{e_1\ldots e_n}$ are real coefficients describing electric and magnetic charges of dyonic solutions. It should be noted that the ordering problem can be absorbed by negative charges carried by D-brane objects. Moreover, the electric/magnetic charge duality can be assured by

$$p_{e_1\ldots e_n} \leftrightarrow p_{\overline{e_1}\ldots\overline{e_n}} \tag{57}$$

originated from the Hodge duality of $\mathbb{T}^n$. In this way, these charges can be nicely organized as follows

$$\begin{pmatrix} Q_\alpha \\ P_\alpha \end{pmatrix} \equiv \begin{pmatrix} p_{e_1 \ldots e_n} \\ p_{\overline{e_1} \ldots \overline{e_n}} \end{pmatrix} \qquad \alpha = 0, \ldots, 2^{n-1} - 1. \tag{58}$$

In order to illustrate this procedure in detail we consider lower dimensional cases. We believe that the general case can be dealt with without ambiguities. For $n = 2$ associated with $Cl(2)$ structure, the situation is simple. The charges of dyonic black objet can be organized as follows:

$$p_{e_1 e_2} \equiv (Q_\alpha, P_\alpha), \quad \alpha = 0, 1, \quad e_i = 0, 1, \tag{59}$$

A quick examination shows that one can use the following ordering index

$$Q_\alpha = p_{\alpha 0,} \qquad P_\alpha = p_{1\alpha}, \quad \alpha = 0, 1. \tag{60}$$

This involves two electric charges and two magnetic charges corresponding to the following graphic operator representation:

$$\begin{array}{ccc} & 1 & \\ a \vee 1 & & 1 \vee a^\dagger \\ & a \vee a^\dagger & \end{array} \tag{61}$$

In this graphic representation, the operators $(1,\ a \vee 1\ )$ correspond to the (Dp, Dp + $1/\mathbb{S}^1_1$) brane system carrying electric charges, under the $U(1)^2_e$ electric gauge symmetry. However, the dual operators $(1 \vee a^\dagger, a \vee a^\dagger)$ are asssociated with the (D$q + 1/\mathbb{S}^1_2$, Dq + $2/\mathbb{T}^2$) brane configuration having two magnetic charges, under the $U(1)^2_m$ magnetic gauge symmetry..

For $n = 4$, the situation is quite different involving extra indices. It is not obvious to find an elegant index notation. However, we will consider a simple one inspired by the previous case. The charges are indexed as

$$p_{e_1 e_2 e_3 e_4} \equiv (Q_\mu, P_\mu), \quad \mu = 0, \ldots, 7. \tag{62}$$

In this way, they can be arranged as follows:

$$\begin{pmatrix} p_{0000} & p_{1000} & p_{0100} & p_{0010} & p_{0001} & p_{1100} & p_{1010} & p_{1001} \\ p_{1111} & p_{0111} & p_{1011} & p_{1101} & p_{1110} & p_{0011} & p_{0101} & p_{0110} \end{pmatrix} \equiv \begin{pmatrix} Q_0 & Q_1 & Q_2 & Q_3 & Q_4 & Q_5 & Q_6 & Q_7 \\ P_0 & P_1 & P_2 & P_3 & P_4 & P_5 & P_6 & P_7 \end{pmatrix} \tag{63}$$

generating 6-dimensional dyonic black objets obtained from the compactification of string theory on $\mathbb{T}^4$. They are charged under gauge fields associated with the $G_{dyon} = U^8(1)_e \times U^8(1)_m$ dyonic gauge symmetry.

5 Conclusion

In this work, we have investigated dyonic black objects in arbitrary dimensions. Inspired by Hodge diagrams of toroidal compactifications on $\mathbb{T}^n$, we have proposed a new class of doublets of dyonic black solutions formed by two different $D-$branes given by $\begin{pmatrix} p \\ 6-n-p \end{pmatrix}$ objects. In particular, we have pointed out a correspondence between toroidal cycles and the black solutions carrying electric and magnetic charge under $G_{dyon} = U(1)_e^{2^{n-1}} \times U(1)_m^{2^{n-1}}$ dyonic gauge symmetry. It has been suggested that this symmetry could be associated with electrically charged magnetic monopole solutions in string string model buildings of the SM extensions. More precisely, we have considered in some details such dyonic black classes obtained from lower dimensional toroidal compactification, and we have found that they are linked to $Cl(n)$ Clifford algebras using the vee product. It has been observed that the analysis presented here might be extended to a class of local Calabi-Yau manifolds. These manifolds have been extensively studied in the geometric engineering method of quantum field theory, obtained from the compactification of higher dimensional models including string theory, M-theory, and F-theory [181, 182]. Some of them are known by canonical bundles on $n-$dimensional compact spaces given by trivial fibrations of n copies of the projective space $\mathbb{CP}^1$ namely

$$M^n = \underbrace{\mathbb{CP}^1 \times \ldots \mathbb{CP}^n}_{n}$$

The compactification of string theory on such manifolds produces dyonic black objects in $8-2n$-dimensional space-time. Like toroidal compactifications, we are thus expecting that a similar analysis can take place by replacing the circle $\mathbb{S}^1$ by the one-dimensional complex space $\mathbb{CP}^1$

$$\mathbb{S}^1 \rightarrow \mathbb{CP}^1.$$

Concretely, one should also expect to be able to engineer dyonic black objects in terms of even-dimensional D-branes producing models with less supersymmetric charges.

Contribution 2

Title: Dyonic Objects and Tensor Network Representation.

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In this work, we investigate stringy dyonic objects and the corresponding tensor network realizations. Inspired and motivated by known results obtained in non-trivial theories including string theory, we first provide the origin and certain related concepts associated with such objects. After that, we present a general framework of dyonic solutions in the brane structure by showing how they are naturally built from inspired stringy compactifications on certain complex manifolds. Then, we reveal that such dyonic solutions involve tensor network representations.

1 Introduction

Since the development of string theory framework and its efficient technique for studying non-perturbative phenomena, a certain number of relevant links with other theories have been comprehended [25, 26]. Specially, in such a framework, the formerly familiar group of solitonic p-brane solutions of Type II supergravities in 10 dimensions are known to be characterized in terms of D-branes [26, 27]. Non-perturbative properties of supersymmetrric Yang-Mills theory, black holes entropy, dyons, magnetic monopoles and a number of appealing issues in various dimensions have been handled in the context of D-brane objects and their dynamics [28, 29]. All these relevant connections have promoted such a string theory aspect to one of the most important and promising side to be explored. It is known that in higher dimensions many physical quantities are reinterpreted from the usual four-dimensional spacetime viewpoint as various faces of possible one quantity. In particular, it has been shown that the single electric and magnetic charges in ten dimensions could be viewed from the 4-dimensional non-compact spacetime standpoint as dyonic charges, being particles carrying simultaneous existence of the electric and the magnetic charges [30]. A dyon with a zero electric charge is usually referred to as a magnetic monopole [31, 32]. These objects are hypothetical particles predicted in many extended and grand unified theories [33,34]. Indeed, besides to magnetic monopole solutions, dyonic solutions appear in extended Yang-Mills-Higgs models taking places in theories going beyond the ordinary physics associated with material points. It is therefore interesting to see how the D-brane framework automatically encodes this feature by using stringy constructions and computations.

2 Motivations of dyonic objects

Before dealing with dyonic objects associated with the electromagnetic duality in higher dimensions, we would like to shed light on some related concepts. Indeed, a similar one goes back to the neutrino discovery, where the proton and the neutron can be regarded as two states of a single particle motivated by the observation that they have approximately equal masses $m_p \simeq m_n = m$ [142], which in turn conducted, according to the mass-energy equivalence, to an energy degeneracy of the underlying interaction. This mass degeneracy led then to the existence of a corresponding symmetry whose the interaction obeys. That is to say, these two particles have an identical behavior under the underlying interaction and that their charge content is their solely difference. In such a case, where these two particles are to be viewed as two linearly independent states of the same particle, i.e., nucleus N, it is unexceptional to depict them in the form of a two component vector like

$$N : \begin{pmatrix} \underline{p} \\ \\ \underline{n} \end{pmatrix}_{m_{\underline{p},\underline{n}}} \tag{1}$$

where $\underline{p}$ and $\underline{n}$ refer to the proton and the neutron particles making such a nucleus state. More fundamentally, analogous to the spin-up and spin-down states of a spin-one half particle, the concept of isospin symmetry is introduced and is governed by an $SU(2)_I$

group rotating doublet components, i.e., proton and neutron, into each other in abstract isospin space. In particular, in the modern formulation, the isospin is defined as a vector quantity in which up and down quarks have a value of 1/2, with the 3rd-component being +1/2 for up quarks, and −1/2 for down quarks. In this picture, denoting the total isospin I and its 3rd component $I_3 = \pm 1/2$, an up-down quarks pair can be assembled in a state of the total isospin 1/2 as

$$\begin{pmatrix} \underline{u} \\ \\ \underline{d} \end{pmatrix}_{I=1/2} \tag{2}$$

where, again, $\underline{u}$ and $\underline{d}$ refer to the up and down quarks forming such a doublet state. Albeit such a particle state classification is a good approximate one with a symmetry dealing with nuclear interactions. It remains a useful and large concept. In fact, one could go further and extend this view to the electromagnetic charge where each particle P could be represented according to its charge content by the doublet

$$\mathrm{P}: \begin{pmatrix} Q_e \\ \\ P_m \end{pmatrix}_{Q_{em}} , \tag{3}$$

whose components are the particle electric and magnetic charges. At first sight, this picture will seem to be strange or even inconsistent with the symmetry of the electromagnetic interaction as dictated by the corresponding $U(1)_{Q_{em}}$ group. However, this apparent inconsistency could be simply alleviated by assuming a hidden symmetry group associated with the magnetic charge. Within the the standard model framework, a simple and economic way to do so is the consider an extended electroweak symmetry $SU(2)_L \times U(1)_e \times U(1)_m$ involving both electric $U(1)_e$ and magnetic $U(1)_m$ symmetries being broken at a certain high energy resulting in the standard low energy electromagnetic symmetry $U(1)_{Q_{em}}$. In this picture, and at a certain energy scale, the particle charge representation in ref gives rise to the following physical objects, for instance

$$\begin{pmatrix} 0 \\ 0 \end{pmatrix}_0 , \quad \begin{pmatrix} Q_e \\ 0 \end{pmatrix}_{Q_e} , \quad \begin{pmatrix} 0 \\ P_m \end{pmatrix}_{Q_m} , \quad \begin{pmatrix} Q_e \\ \\ P_m \end{pmatrix}_{Q_{em}} . \tag{4}$$

These four objects are nothing but all known possible particles, discovered and hypothetical. In particular, the first neutral doublet is nothing but the neutrino, being neutral, the second doublet corresponds to the ordinary charged particles, the third doublet is that of theoretical magnetic monopoles with a single magnetic charge, while the four doublet corresponds to a dyonic particle carrying both electric and magnetic charges. For more clarity we list them in (Table 3). We can now see that this particle classification view has given rise to the well motivated hypothetical objects, namely magnetic monocles charged only magnetically, and dyons carrying both the electric and the magnetic charges associated with the electromagnetic duality in higher dimensions. These objects are predicted by most grand unified theories in high energy physics and are now one of the active field of research in many theoretical and experimental works [37]. As we shall see in what follows, such a view can be considered as a part of a larger concept being of great utility to classify physical object families.

object	Q_e	P_m	nature
neutrinos	no	no	real
ordinary (electrically) charged particles	yes	no	real
magnetic monopoles	no	yes	hypothetical
dyons	yes	yes	hypothetical

Table 3: The four possible particles, observed and hypothetical, in terms of their electromagnetic charges.

3 Stringy dyonic solutions

In lower dimensional string compactifications, a certain number of dyonic objects can be generated. However, these solutions could be obtained from the geometry of a compact real manifold X^n on which higher dimensional theories are compactified. The geometric information of such compactifications can be deployed to construct such dyonic objects as doublets carrying both electric and magnetic charges [183]. Indeed, they can be generally arranged as

$$\begin{pmatrix} p \\ q \end{pmatrix}. \tag{5}$$

It is worth noting that now p and q denote the electric and the magnetic solitonic objets, with p and q dimensions respectively, forming the dynoic solutions. To see how this could be constructed, consider a gauge field C_{p+1} coupled to a p-brane associated with the field strengtht $F_{p+2} = dC_{p+1}$. The corresponding electric charge Q_e can be computed using the integration over a S^{p+2} ($(p+2)$-cycle)

$$Q_e = \int_{(p+2)-cycle} F_{p+2}. \tag{6}$$

However, the magnetic charge, associated with the magnetic object, is calculated via the duality of the field strengtht F_{p+2} in d

$$P_m = \int_{(d-p-2)-cycle} *dC_{p+1}.$$

To build a dyonic object, one put together the electric and magnetic objects associated with the electromagnetic duality

$$F \longleftrightarrow *F. \tag{7}$$

In string theory, when a magnetic dual of an electric p-brane is a q-brane, where q is related to p by the constraint relation

$$p + q = 6 - n. \tag{8}$$

This includes two different kinds of the dyonic objects. We refer to them as fundamental dyonic solutions ($p = q$) and non-fundamental dyonic solutions ($p \neq q$), respectively. The first one gives rise to the ordinary dyonic solution required by

$$p = q = 3 - \frac{n}{2}, \tag{9}$$

which allows to present any fundamental dyonic state in the form of a doublet of electromagnetic charges with same dimension

$$\begin{pmatrix} p \\ p \end{pmatrix}_{p=3-\frac{n}{2}}. \tag{10}$$

This case is analogue of what we call pure state involving only the charged objects of the same spatial dimension appearing in even dimensional space-time and a compact space respectively. It is recalled that one has three families of dyonic objects such as a dyonic particle in a four dimensional spacetime, a dyonic string in a six dimensional spacetime and a dyonic membrane in an eight dimensional spacetime given respectively by

$$\begin{pmatrix} 0 \\ 0 \end{pmatrix}, \quad \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \quad \begin{pmatrix} 2 \\ 2 \end{pmatrix} \tag{11}$$

The second family, defined by $p \neq q$, are presented by the following constraints

$$q = 6 - n - p, \qquad p \neq 3 - \frac{n}{2}.$$

In this way, these dyonic objects can be formed by two different branes in any compact space. Such dyonic objects are presented by a doublet of the couple (p, q)

$$\begin{pmatrix} p \\ q \end{pmatrix}_{p+q=6-n}. \tag{12}$$

The unusual solutions can be built in terms of the ten dimensional space-time theory either from type II superstrings or heterotic brane doublets

$$\begin{pmatrix} D0 \\ D6 \end{pmatrix}, \quad \begin{pmatrix} D2 \\ D4 \end{pmatrix}, \quad \begin{pmatrix} D1 \\ D5 \end{pmatrix}, \quad \begin{pmatrix} D3 \\ D3 \end{pmatrix}, \quad \begin{pmatrix} NS1 \\ NS5 \end{pmatrix}. \tag{13}$$

These configurations of dyonic states represent the maximally number of dyonic objects in ten dimensions [183]. It turns out that the number of such dyonic objects will be decreased by lowering the dimensional space-time obtained from the compactification X^n.

Now, we are in position to discuss explicit models from concrete compactifications. Working with M-theory/Superstring Inspired Models in d dimensions, we consider the compactification on a n-dimensional compact manifold product of the identical manifolds of dimensions m such that

$$n = km. \tag{14}$$

We refer to such manifolds as

$$X^n = \underbrace{Y^m \times \ldots \times Y^m}_{k-time} \tag{15}$$

where Y^m is a m-dimensional manifold compact space. In connection with such theories in the presence of brane solitonic objects, the d-dimensional space-time geometry can be split as follows

$$AdS_{p+2} \times \mathbb{S}^{d-p-2-n} \times X^n \tag{16}$$

To make contact with dyonic solutions certain conditions should be imposed on Y^m. A close inspection shows that Y^m should satisfy the following conditions

$$\begin{cases} b_0(Y^m) = 1 \\ b_i(Y^m) = 0 \qquad 1 < i < m-1 \\ b_m(Y^m) = 1 \end{cases} \tag{17}$$

where $b_m(r)$ denote the associated Betti numbers. Performing such compactifications, we can build dyonic solutions from the brane doublets in d-dimensional inspired stringy. models. Forgetting for while the connection with string theory models, the compactification can produce 2^{k-1} dyonic double states carrying both electric and magnetic charges. These states are obtained from branes wrapping a-cycles C_a in X-manifolds. These cycles are given in terms of the volume forms ω_i of Y^m-manifolds. Indeed, dyonic stringy solutions, involving the electric and the magnetic charges, allow one to consider two indices associated with binary numbers. Using such a property, the cycles can be denoted by

$$C_a \equiv C_{e_1 \dots e_k}. \tag{18}$$

where e_i is a binary number taking either 0 or 1. The associated volume form are given by

$$(\omega_1)^{e_1} \wedge \dots \wedge (\omega_k)^{e_k} \tag{19}$$

such that

$$\int_{C_{e_1 \dots e_k}} (\omega_1)^{e'_1} \wedge \dots \wedge (\omega_k)^{e'_k} = \delta^{e'_1}_{e_1} \dots \delta^{e'_k}_{e_k}. \tag{20}$$

The dyonic doublet states can be obtained from D-branes wrapping cycles dual to the following doublet volume form

$$\begin{pmatrix} p \\ q \end{pmatrix} \leftrightarrow \begin{pmatrix} \omega_1^{e_1} \wedge \dots \wedge \omega_k^{e_k} \\ \omega_1^{\overline{e_1}} \wedge \dots \wedge \omega_k^{\overline{e_k}} \end{pmatrix} \tag{21}$$

These dyonic states correspond to D-branes with charges

$$\left(Q^{e_1 \dots e_k}, P^{\overline{e_1} \dots \overline{e_k}}\right) \tag{22}$$

which can be obtained by the following integration

$$Q^{e_1 \dots e_k} = \int_{C_{e_1 \dots e_k}} F^2, \qquad P^{\overline{e_1} \dots \overline{e_k}} = \int_{C_{\overline{e_1} \dots \overline{e_k}}} *F^2 \tag{23}$$

To provide a concrete model, we consider the following compact manifold X^{2k} where $n = 2k$

$$X^{2k} = \underbrace{\mathbb{S}^2 \times \mathbb{S}^2 \times \ldots \times \mathbb{S}^2}_{k} \tag{24}$$

where $\mathbb{S}^2$ is 2-dimensional real sphere. Since $\mathbb{S}^2$ is isomorph to $\mathbb{CP}^1$, it is useful to consider the complex geometry. In this way, we take the following factorisation

$$X^{2k} = \underbrace{\mathbb{CP}^1 \times \mathbb{CP}^1 \times \ldots \times \mathbb{CP}^1}_{k} \tag{25}$$

According [181, 184], a nice way to understand such a geometry is to exploit the $N = 2$ sigma model by embedding the involved compact manifold in local Calabi-Yau manifolds given by

$$\mathcal{O}(-2, \ldots, -2) \to \underbrace{\mathbb{CP}^1 \times \mathbb{CP}^1 \times \ldots \times \mathbb{CP}}_{k}. \tag{26}$$

The get the associated cycles, one can use the result of the trivial fibration of two spaces $M \times N$. According to [185], the needed Hodge numbers can be obtained by the following identity

$$h^{(p,q)}(M \times N) = \sum_{\substack{u+r=p \\ v+s=q}} \tag{27}$$

It is recalled that the Hodge diagrams associated with $k = 1, 2, 3$ are listed in (Table 4). It has been revealed that the invariant volume forms belong to the cohomology class $H_+^{j,j}$ formed by $\prod_{\ell=1}^{j} dz_\ell \wedge d\overline{z_\ell}$ where

$$w_\ell = dz_\ell \wedge d\overline{z_\ell} \tag{28}$$

denotes the volume form associated with ℓ-th space. It is easy to calculate that the corresponding Hodge numbers are

$$\dim \; H_+^{j,j} = h_+^{j,j} = \frac{k!}{j!(k-j)!}. \tag{29}$$

It is clear that one has the following relation

$$\dim \; H(X^n) = \sum_{j=0}^{k} h_+^{j,j} = 2^k. \tag{30}$$

$k=1$	$\begin{array}{ccc} & h^{0,0} & \\ h^{1,0} & & h^{0,1} \\ & h^{1,1} & \end{array}$	$\begin{array}{ccc} & 1 & \\ 0 & & 0 \\ & 1 & \end{array}$
$k=2$	$\begin{array}{ccccc} & & h^{0,0} & & \\ & h^{1,0} & & h^{0,1} & \\ h^{2,0} & & h^{1,1} & & h^{0,2} \\ & h^{2,1} & & h^{1,2} & \\ & & h^{2,2} & & \end{array}$	$\begin{array}{ccccc} & & 1 & & \\ & 0 & & 0 & \\ 0 & & 2 & & 0 \\ & 0 & & 0 & \\ & & 1 & & \end{array}$
$k=3$	$\begin{array}{ccccccc} & & & h^{0,0} & & & \\ & & h^{1,0} & & h^{0,1} & & \\ & h^{2,0} & & h^{1,1} & & h^{0,2} & \\ h^{3,0} & & h^{2,1} & & h^{1,2} & & h^{0,3} \\ & h^{3,1} & & h^{2,2} & & h^{1,3} & \\ & & h^{3,2} & & h^{2,3} & & \\ & & & h^{3,3} & & & \end{array}$	$\begin{array}{ccccccc} & & & 1 & & & \\ & & 0 & & 0 & & \\ & 0 & & 3 & & 0 & \\ 0 & & 0 & & 0 & & 0 \\ & 0 & & 3 & & 0 & \\ & & 0 & & 0 & & \\ & & & 1 & & & \end{array}$

Table 4: Hodge diagrams

4 Tensor network representation of dyonic solutions

In this section, we would like to investigate dyonic objects using tensor network formalism [186, 187]. In particular, we exploit such a formalism to unveil certain dyonic object properties. It is recalled that usually tensors are complex having order k and size L [188, 189]. For any tensor, it is interesting to control the associated degrees of freedom as well as its organization. Such data could be used to encode certain physical quantities corresponding to fundamental and non fundamental objects including dyonic ones. For later use, we will consider real tensors with a fixed size being $L = 2$.

Roughly speaking, the tensors, will be used here, is defined as a series of real numbers labeled by k indexes associated with with legs in graph representattions. In such a context, a scalar, which is one number and labeled by zero index, is a 0th-order tensor. Graphically, we represent a scalar by dot as illustrated in (Figure 1).

A two-component vector consists of 2 real numbers labeled by one index, being a

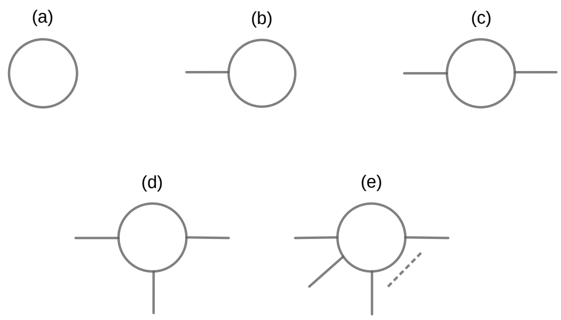

Figure 1: Tensor representation. (a) scalar, (b) vector, and (c-d-e) tensors associated with polyvalent vertices

1st-order tensor. In this way, one can write the state vector of a spin-1/2 as follows

$$|\Psi\rangle = C_1|0\rangle + C_2|1\rangle = \sum_{e=0,1} C_e|e\rangle, \tag{31}$$

where the coefficients C_i are associated with a two-component vector. It is recalled that $|0\rangle$ and $|1\rangle$ are usually used to represent spin up and down states. Graphically, we use a dot with one open bond to represent such a vector (Figure 1 (a)). However, a matrix is in fact a 2nd-order tensor. Considering two spins as an example, the state vector can be written under a irreducible representation as a four-dimensional vector. Using the local basis of each spin associated with binary notation, the state can be written as follows

$$|\Psi\rangle = C_{00}|00\rangle + C_{01}|01\rangle + C_{10}|10\rangle + C_{11}|11\rangle = \sum_{e_1e_2=0}^{1} C_{e_1e_2}|e_1e_2\rangle, \tag{32}$$

where $C_{e_1e_2}$ represent now a matrix with two indexes. It is worth noting that the difference between a (2×2) matrix and a 2^2-component vector is just the way of labeling the tensor elements. Graphically, we use a dot with two bonds to represent a matrix and its two indexes (Figure 1). It is then natural to define an k-th order tensor. Considering k spins, the 2^k coefficients can be written as a k-th order tensor C, satisfying

$$|\Psi\rangle = \sum_{e_1\ldots e_k=0}^{1} C_{e_1\ldots e_k}|e_1\ldots e_k\rangle. \tag{33}$$

Graphically, such a k-th order tensor is represented by a dot connected with k open bonds (polyvalent vertex of order k, Figure 1). Now we can define tensor network (TN), as the contraction of many tensors. A tensor network is a set of tensors where some, or all, of its indices are contracted according to some pattern. Contracting the indices of a tensor network is called, for simplicity, contracting the tensor network. In general, the contraction of a tensor network with some open indices gives as a result another tensor, and in the case of not having any open indices the result is a scalar.

Having given a concise review on tensors, we move to make contact with dyons. Indeed, a close inspection, in the above concrete string models, shows that the dyonic state can be associated with a tensor $T_{e_1\dots e_k}$ defined by the state

$$|\Psi\rangle = \sum_{e_i=0,1} T_{e_1\dots e_k}|e_1\dots e_k\rangle, \tag{34}$$

where the vector $|e_1\dots e_k\rangle$ is given by $|e_1\dots e_k\rangle = |e_1\rangle \otimes \dots \otimes |e_k\rangle$.
To give a tensor network realisation of such dyonic objects, we introduce a new tensor defined by

$$T_{e_1\dots e_k} \to \bar{T}_{e_1\dots e_k} = T_{\bar{e}_1\dots\bar{e}_k} \tag{35}$$

such that $e_i + \bar{e}_i = 1$.

In this way, the above state can be written as

$$|\psi\rangle = \sum_{e_i} T_{e_1\dots e_k}|e_1\dots e_k\rangle + \sum_{\bar{e}_i} T_{\bar{e}_1\dots\bar{e}_k}|\bar{e}_1\dots\bar{e}_k\rangle \tag{36}$$

where the tensor values correspond to the involved charges. Indeed, one proposes

$$\begin{aligned} T_{e_1\dots e_k} &= Q_{e_1\dots e_k} \\ T_{\bar{e}_1\dots\bar{e}_k} &= P_{\bar{e}_1\dots\bar{e}_k} \end{aligned} \tag{37}$$

subject to $e_i + \bar{e}_i = 1$. In this way, the degrees of freedom of T can be split as

$$2^k = 2^{k-1} + 2^{k-1}. \tag{38}$$

Now, we are in position to build a new tensor carrying information on dyonic objects by combing T and $\bar{T}$ tensor defined previously. The suggested notation is given by

$$X^{e_1\dots e_k}_{\bar{e}_1\dots\bar{e}_k} = \begin{pmatrix} T^{e_1\dots e_k} \\ T_{\bar{e}_1\dots\bar{e}_k} \end{pmatrix} \tag{39}$$

In this tensor realization, the dyonic state can be expressed as as

$$|\psi\rangle = \sum_{e_i+\bar{e}_i=1} X^{e_1\dots e_k}_{\bar{e}_1\dots\bar{e}_k}|e_1\bar{e}_1\rangle \otimes \dots \otimes |e_k\bar{e}_k\rangle \tag{40}$$

$$= \sum_{e_i+\bar{e}_i=1} X^{e_1\dots e_k}_{\bar{e}_1\dots\bar{e}_k} \prod_i \otimes_i |e_i\bar{e}_i\rangle. \tag{41}$$

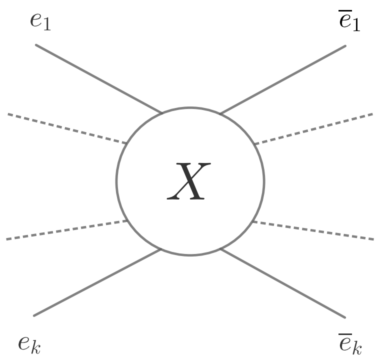

Figure 2: Tensor state representation

Graphically, this state can be illustrated by the tensor given in (Figure 2)

Roughly, we expect that the theory that supports these objects is a color tensor model with a gauge group $U(2)^{\otimes k}$ which acts on each slot by complex rotation. In fact, multiparticle states are given by linear combinations of the full contraction of r copies of the tensor T with r copies of the tensor $\overline{T}$, providing that the states are gauge invariant. To simplify notation, consider a tensor with three indices, namely $k = 3$. The results can be generalized straightforwardly. For one particle, we have

$$\mathcal{O}^{(1)} = T_{ijk}\overline{T}^{ijk}, \tag{42}$$

which is the unique scalar that can be constructed (remember that in color tensor models the contractions must always happen on indices of the same slot). For two indices, there are four choices given by

$$\begin{aligned} \mathcal{O}_1^{(2)} &= T_{i_1j_1k_1}T_{i_2j_2k_2}\overline{T}^{i_1j_1k_1}\overline{T}^{i_2j_2k_2}, \quad \mathcal{O}_2^{(2)} = T_{i_1j_1k_1}T_{i_2j_2k_2}\overline{T}^{i_2j_1k_1}\overline{T}^{i_1j_2k_2} \\ \mathcal{O}_3^{(2)} &= T_{i_1j_1k_1}T_{i_2j_2k_2}\overline{T}^{i_1j_2k_1}\overline{T}^{i_2j_1k_2}, \quad \mathcal{O}_4^{(2)} = T_{i_1j_1k_1}T_{i_2j_2k_2}\overline{T}^{i_1j_1k_2}\overline{T}^{i_2j_2k_1}. \end{aligned} \tag{43}$$

According to [186], r copies we have as many invariants as

$$\mathcal{O}_{\alpha\beta\gamma} = T_{i_1j_1k_1}\dots T_{i_rj_rk_r}\overline{T}^{i_{\alpha(1)}j_{\beta(1)}k_{\gamma(1)}}\dots\overline{T}^{i_{\alpha(r)}j_{\beta(r)}k_{\gamma(r)}}| \quad \alpha,\beta,\gamma \in S_r, \tag{44}$$

subject to the equivalence

$$\mathcal{O}_{\alpha\beta\gamma} \sim \mathcal{O}_{\alpha'\beta'\gamma'} \quad \text{if} \quad \alpha' = \tau\alpha\sigma,\ \beta' = \tau\beta\sigma,\ \gamma' = \tau\gamma\sigma, \tag{45}$$

for some $\sigma, \tau \in S_r$. The equivalence takes into account the freedom to shuffle T and $\overline{T}$ slots in (44). As a comment, note that the same invariants could have been constructed with r copies of (39) and patterns of contraction. We are expressing the same, although we believe that this construction is simpler.

Besides, for $L < r$ (which is going to be the case since $L = 2$), the number of invariants gets reduce in account of redundancies due to the small number of degrees of freedom. Following [187, 190–192], the exact number of invariants has been computed and is

$$\sharp \text{ of invariants} = \sum_{\substack{\mu,\nu,\lambda \vdash r \\ l(\mu),l(\nu),l(\lambda) \leq 2}} g^2_{\mu\nu\lambda}, \tag{46}$$

where $g_{\mu\nu\lambda}$ are the Kronecker coefficients labeled by three Young diagrams or partitions of r, and the fact that $L = 2$ translates into the number of the rows of each Young diagram being equal or less than 2, as indicated in the sum (46).

It is remarked that Kronecker coefficients are hard to compute in general. However the restriction on the partitions to have at most two rows should make a dramatic simplification. This case could be studied somewhere and some neat formulas found for the counting. It would be interesting to find it.

Now, we could interpret (35) as the dyonic condition. It is actually a restriction in the number of degrees of freedom since given T_{ijk} one immediately knows what is $T_{\bar{i}\bar{j}\bar{k}}$, which is the conjugate number. It is noted that thanks to (35), the invariants are real since

$$\begin{aligned}
\overline{\mathcal{O}}_{\alpha\beta\gamma} &= \overline{T_{i_1j_1k_1}\dots T_{i_rj_rk_r}\overline{T}^{i_{\alpha(1)}j_{\beta(1)}k_{\gamma(1)}}\dots\overline{T}^{i_{\alpha(r)}j_{\beta(r)}k_{\gamma(r)}}} \\
&= \overline{T}_{i_1j_1k_1}\dots\overline{T}_{i_rj_rk_r}T^{i_{\alpha(1)}j_{\beta(1)}k_{\gamma(1)}}\dots T^{i_{\alpha(r)}j_{\beta(n)}k_{\gamma(r)}} \\
&= T_{\bar{i}_1\bar{j}_1\bar{k}_1}\dots T_{\bar{i}_r\bar{j}_r\bar{k}_r}\overline{T}^{\bar{i}_{\alpha(1)}\bar{j}_{\beta(1)}\bar{k}_{\gamma(1)}}\dots\overline{T}^{\bar{i}_{\alpha(r)}\bar{j}_{\beta(r)}\bar{k}_{\gamma(n)}} \\
&= T_{i_1j_1k_1}\dots T_{i_rj_rk_r}\overline{T}^{i_{\alpha(1)}j_{\beta(1)}k_{\gamma(1)}}\dots\overline{T}^{i_{\alpha(r)}j_{\beta(r)}k_{\gamma(r)}} = \mathcal{O}_{\alpha\beta\gamma}.
\end{aligned} \tag{47}$$

5 Conclusion

In this work, we have investigated stringy dyonic objects and tensor network formalism. Inspired and motivated by known results obtained in non-trivial theories including string theory, we have provided shortly the origin and certain related concepts. After that, we have elaborated a general framework of dyonic solutions in the brane structure and show how are naturally obtained in various dimensions within stringy compactifications on a concrete complex geometry. Then, we have discussed dyonic solution representations using tensor network formalism by revealing some characteristics associated with extra dimensions.

Assuming the existence of such dimensions, we believe that such a theoretical investigation could shed, somehow, some light on experimental searches for dyons, through either their direct production or indirect detection benchmarks at the current and future accelerators.

Contribution 3

Title: Toric Realisation of Quantum Systems via Clifford Structures.

Athors: M. Bensed and Y. El Maadi

Journal: Journal of Applied Mathematics and Physics.

In this work, we present a geometric approach to describe quantum information systems. Precisely, we elaborate a genuine correspondence between toric geometry and qubit systems using the construction of Clifford graph algebras. This mapping may be explored to investigate qubit information applications using toric geometry considered as a potent tool to understand high energy physics including black holes and string theory. Explicitly, we examine in some details the cases of two, and three qubits, and we find that they are associated with certain Clifford graph structures. Mixed states are also discussed.

1 Introduction

Quantum information combined with toric geometry is considered as a powerful tool to study complex varieties used in modern physics as string theory and related models [56, 193]. As well, toric geometry has been also used to build mirror manifolds providing an outstanding way to understand the extension of T-duality in the presence of D-branes moving near toric Calabi-Yau singularities using combinatorial calculations. Also, it has been remarked that such a combination can be exploited to understand a special class of black hole solutions obtained from type II superstrings on local Calabi–Yau manifolds [194, 195]. Recently, the black hole physics has found a place in quantum information theory using qubit building blocks. Precisely, many connections have been established in the context of STU black holes as proposed in [159, 196].

Studies have been realized to develop new geometric approach to deal with qubit information systems using hypercube graph theory and toric geometry [160, 161]. These investigations have brought new understanding of the fundamental physics associated with qubits and theirs supersymmetric extensions. These latter are connected to various theories including D-branes, toric geometry and supermanifolds. Precisely, a nice interplay between the black holes and qubits have been discussed using higher dimensional supergravity models [152, 154]. Moreover, Clifford algebras play a key role in the link between quantum information and physics of black holes, rigorously dyonic black classes obtained from lower dimensional toroidal compactification are mapped to Clifford algebras using vee product [30]. Moreover, the superpotential depending on four quaternionic fields has been considered as an element of a particular Clifford class [164].

2 Construction of Clifford graph algebras

The strong character of Clifford algebras gives as the possibility of using them as a mapping between quantum computation, and high energy physics [197]. They can play a key role in supersymmetry as supersymmetric description of qubits systems [198]. To make a link between graph theory and quantum information systems, we present an overview on a useful construction of Clifford graph algebras.

Let A is a classical Clifford algebra, then it is a unital algebra over $\mathbb{C}$ with n generators $e_1, e_2, ..., e_n$ which verify the following relations

$$\begin{cases} e_i^2 = -1 & for\ any\ i \\ e_i e_j = -e_j e_i & for\ i \neq j \end{cases} \tag{1}$$

Every monomial in this algebra either commute or anticommutes with generators e_i. The center of this algebra is none other than the subalgebra of elements that commute with all elements. Moreover, a graph G with m vertices and no multiple edges and no loops can be associated with an algebras A_G over $\mathbb{C}$, with m generators corresponding to the vertices as

$$\begin{cases} e_i^2 = 1 & for\ any\ i \\ e_ie_j = -e_je_i & if\ there\ is\ an\ edge\ between\ the\ i^{th}\ and\ j^{th}\ vertices \\ e_ie_j = e_je_i & if\ there\ is\ no\ edge\ between\ the\ i^{th}\ and\ j^{th}\ vertices \end{cases} \tag{2}$$

In the present work, we are interested in such a construction of graphs called Clifford graph algebras which may be used to describe quantum information systems. Precisely, we focus on the dimension of the center of Clifford algebra. More precisely, it has been showed that two graphs with the same number of vertices belong to the same Clifford class if the centers of their Clifford algebras have the same dimension. Their Clifford algebras are isomorphic [23].

To assimilate the purpose of this, let a Clifford graph algebras has dimension 2^n over $\mathbb{C}$, and has a basis of monomials e_α, where every elements $c = \sum a_\alpha e_\alpha$ of the center form a subalgebra and also the dimension of the center is always a power of 2. For instance as we can notice in (Figure 3) if the generators of the Clifford graph algebra of the complete graph are e_i and the generators of the Clifford of path graph are $e_i^{'}$. Then, it has been verified that the Clifford algebras of these two graphs are isomorphic by giving an isomorphism as

$$\begin{cases} e_1^{'} = e_1 \\ e_2^{'} = e_1e_2 \\ e_i^{'} = e_2e_i & for\ i > 2. \end{cases} \tag{3}$$

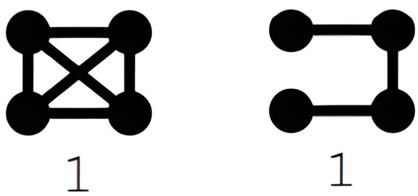

Figure 3: Isomorphism between Clifford graph algebras of complete and path graph.

These graphs with the same dimension of Clifford algebras belong to the same Clifford class and can be used to graphically describe and present the physics of n-qubit systems.

3 Toric geometry and Clifford graph algebra

Toric geometry has been considered as a potent tool in context of study of complex Calabi-Yau manifolds used in the string theory compactification and associated subjects [199].

Firstly, $n/2$-complex dimensional toric manifold, which we denote as $M_{\triangle}^{n/2}$, is obtained by considering the $(\frac{n}{2}+r)$-dimensional complex spaces $C^{\frac{n}{2}+r}$ parameterized by homogeneous coordinates $x = (x_1; x_2, x_3; ::::; x_{\frac{n}{2}} + r)$, and r toric transformations $\hat{T}_a$ acting on the x_i's as follows

$$T_a : x_i \to x_i(\lambda_a^{q_i^a}). \tag{4}$$

Here, we consider the λ_a as r non vanishing complex parameters. Then, the q_i^a are integers called Mori vectors for each a which encode several geometrical information on the manifold and its applications to modern physics. Thus these toric manifolds can be identified with the coset space $C^{\frac{n}{2}+r}/C^{*r}$. In fact, toric graphic realization is generally represented by an integral polytope Δ, namely a toric diagram which is spanned by $(\frac{n}{2}+r)$ vertices v_i of the standard lattice Z^n. The toric data $\{v_i; q_{ia}\}$ must satisfy the following r relations

$$\sum_{i=0}^{\frac{n}{2}+r-1} q_i^a v_i = 0 \quad a = 1, ..., r. \tag{5}$$

These equations encode geometric data of $M_{\triangle}^{n/2}$. The q_i^a integers are interpreted in lower dimensional field theory, namely in the $N = 2$ gauge linear sigma model language, as the $U(1)^r$ gauge charges of $N = 2$ chiral multiples. Furthermore, they have also a geometric perception in terms of the intersections of complex curves C_a and divisors D_i of $M_{\triangle}^{n/2}$ [181, 200]. This exceptional connection has been explored in many fields in modern physics. In particular, it has been used to construct type IIA local geometry.

The basic example in toric geometry is CP^1 which can play an essential role in the building block of higher dimensional toric varieties. It is defined by $r = 1$ and the Mori vector charge takes the values $q_i = (1, 1)$. This geometry has an $U(1)$ toric action CP^1 acting as follows

$$z = \exp^{i\theta} z, \tag{6}$$

where $z = x_1 = x_2$, with two fixed points v_0 and v_1 placed on the real line. These later are the North and south poles respectively which describe such a geometry, considered as the (real) two-sphere $S^2 \sim CP^1$, satisfying the following constraint toric equation

$$v_0 + v_1 = 0. \tag{7}$$

Then, we consider a class of toric varieties that we are interested in by giving a trivial product of one-dimensional projective spaces CP^1's admitting a similar description. We will show later that this class can be used to elaborate a special mapping between quantum information systems and graph theory. We treat the cases of $CP^1 \times CP^1$ and $CP^1 \times CP^1 \times CP^1$ for simplicity reason and in the case of higher dimensional geometries $\otimes_{i=1}^{\frac{n}{2}} CP_i^1$, the toric descriptions can be given by a similar way. In fact, the $\frac{n}{2}$-dimensional toric manifolds present $U(1)^{\frac{n}{2}}$ toric actions. A close inspection shows that there is a similarity between toric graphs of such manifolds and qubit systems using a link with an interesting construction of Clifford graph algebras [23]. To assimilate this mapping, we present G as a graph with n vertices and no multiple edges and no loops. This later is associated with algebra A_G over $\mathbb{C}$, with n generators $e_1, e_2, ..., e_n$ corresponding to the vertices, thus the Clifford algebra associated with a given graph it's called a Clifford graph algebra, precisely we are interested in the number of graphs with n vertices such that the center of their Clifford graph algebras has dimension 2^k being denoted by

$$Cliff_{2^k}(n), \tag{8}$$

where k is an integer which fix the dimension of the center of Clifford algebras.

The connection that we are after is given by considering a special class of toric manifolds associated with $\otimes_{i=1}^{\frac{n}{2}} = CP_i^1$ and with $U(1)^{\frac{n}{2}}$ toric actions exhibiting 2^n fixed points. Then, roughly speaking, we can propose a correspondence connecting two different subjects apparently separate which are Clifford graph and toric geometry

$$2^n \text{ fixed points} \longleftrightarrow n \text{ generators}.$$

Furthermore, in toric geometry language the manifolds are represented by 2^n vertices belonging to the Z^n lattice satisfying n toric equations. Then these graphs share a strong resemblance with Clifford graph algebra formed by n nodes connected with a precise number of edges which are equal or less than $\dfrac{n(n-1)}{2}$. Precisely when we fix the number of vertices $n = k$ and the number of edges equal to $n-1$. Thus, we can propose the following notation

$$Cliff_{2^n}(2^n, 2^n - 1), \tag{9}$$

which can be associated to the number of toric actions as following: number of toric actions $\longleftrightarrow Cliff_{2^n}(2^n, 2^n - 1)$.

We believe that this type of toric manifolds can be used to describe graphically the physics of quantum systems by the help of Clifford graph algebras.

4 Qubits Systems as Clifford Graphs Algebra

Motivated from combinatorial computations in quantum physics, we explore toric geometry to deal with qubit information systems [30,154]. Precisely, we elaborate a toric description in terms of a trivial fibration of one dimensional projective space CP^1's. First, it is recalled that the qubit is a two level system which can be realized, for instance, by a 1/2 spin particle. The one qubit is in a superposition of two states generally written in Dirac notation as

$$|\psi\rangle = a_0|0\rangle + a_1|1\rangle, \tag{10}$$

where a_i are complex numbers satisfying the normalization condition $|a_0|^2 + |a_1|^2 = 1$.

It has been revealed that this constraint can be interpreted geometrically in terms of the Bloch sphere, identified with the quotient group $SU(2)/U(1)$ [199,200]. The analysis can be extended to the case of bipartite systems which can be entangled states playing a fundamental role in the applications of quantum information. Using the usual notation $|i_1 i_2\rangle = |i_1\rangle|i_2\rangle$, the corresponding state in superposition can be expressed as follows

$$\begin{aligned} |\psi\rangle =& \sum_{i_1 i_2} a_{i_1 i_2}|i_1 i_2\rangle \\ =& a_{00}|00\rangle + a_{10}|10\rangle + a_{01}|01\rangle + a_{11}|11\rangle \end{aligned}, \tag{11}$$

where a_{ij} are complex numbers verifying the normalization condition

$$|a_{00}|^2 + |a_{01}|^2 + |a_{10}|^2 + |a_{11}|^2 = 1, \tag{12}$$

describing the CP^3 projective space.

For n-qubits, the general state has the following form

$$|\psi\rangle = \sum_{i_1...i_n=0,1} a_{i_1...i_n}|i_1...i_n\rangle, \tag{13}$$

where a_{ij} satisfy the normalization condition

$$\sum_{i_1...i_n=0,1} | \, a_{i_1.....i_n} \, |^2 = 1. \tag{14}$$

In fact this condition defines the CP^{2^n-1} projective space generalizing the Bloch sphere associated with $n = 1$.

Roughly, the qubit systems can be described by toric manifolds $M_\Delta^{n/2}$ having a strong resemblance with Clifford graph algebras. A close inspection in Clifford graph algebras and toric geometry shows that when we fix the number of qubit $n = k$ and the number of edges is $2^n - 1$, we can propose the following correspondence connecting quantum systems and toric geometry via Clifford graph algebras (Table 5).

Qubits systems	Toric Geometry	Clifford Graph algebra
Basis state	Fixed point	Vertices such as the dim of the center of corresponding Clifford algebra is 2^n
Number of qubits	Number of toric actions	$Cliff_{2^n}(2^n, 2^n - 1) \equiv$ number of graphs with 2^n vertices such that the center of their Clifford graph algebras has dimension 2^n and $2^n - 1$ edges.

Table 5: Correspondence between toric geometry, and quantum systems via Clifford graphs algebras.

To see how this works in practice, we present at first the usual toric geometry notation. Inspired by combinatorial formalism used in quantum information theory, the toric data can be written as follows

$$\sum_{i_1...i_n} q^a_{i_1...i_n} v_{i_1...i_n} = 0, \quad a = 1, ..., r, \tag{15}$$

where the vertex subscripts indicate the corresponding quantum states. To illustrate this notation, we present the model associated with $CP^1 \times CP^1$ toric variety where

$\frac{n}{2} = 2$. This model is related to $n = 4$ classification of Clifford graph algebras. Here, the combinatorial Mori vectors can take the following form

$$\begin{aligned} q^1_{i_1 i_2} &= (q^1_{00}, q^1_{01}, q^1_{10}, q^1_{11}) = (1, 0, 0, 1), \\ q^2_{i_1 i_2} &= (q^1_{00}, q^1_{01}, q^1_{10}, q^1_{11}) = (0, 1, 1, 0). \end{aligned} \tag{16}$$

The corresponding manifold to this toric equations is

$$\sum_{i_1 i_2} q^a_{i_1 i_2} v_{i_1 i_2} = 0, \quad a = 1, 2. \tag{17}$$

The toric data require the following vertices

$$\begin{aligned} v_{00} =&(-1, 0), \quad v_{01} =(0, 1), \\ v_{01} =&(0, 1), \qquad v_{11} =(1, 0), \end{aligned} \tag{18}$$

which can be connected to the Clifford graph algebra denoted as $Cliff_{2^n=4}(2^n = 4, 2^{n-1} = 3)$.

These data can be encoded in Clifford graph algebras describing the case of two qubits illustrated below in (Figure 4). These two graphs have been chosen among eleven graphs in the case of $n = 4$ nodes of Clifford graph algebras, and it has belongs to the same Clifford class so their Clifford algebras are isomorphic [23].

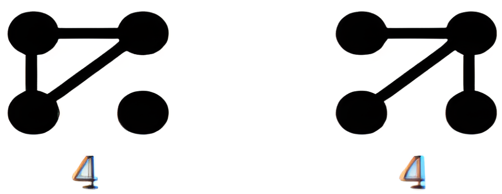

Figure 4: Clifford graph representation of two qubits $n = 2$.

Due to this representation, we can link the basis of n-qubit with the toric geometry. The mapping is given by

$$v_{i_1 \ldots i_n} \to |i_1 \ldots i_n\rangle, \tag{19}$$

and the number of qubits is related as follow in (20) to the number of Clifford graph algebras with 2^n vertices and $2^n - 1$ edges such the dimension of their center is none other than 2^n,

$$n \to Cliff_{2^n}(2^n, 2^n - 1). \tag{20}$$

In the case of 3-qubit $n = 3$, it is remarked that the geometry can be identified with the blow up of $CP^1 \times CP^1 \times CP^1$ toric manifold. In toric geometry language, this manifold is described by the following equations

$$\sum_{i_1 i_2 i_3} q^a_{i_1 i_2 i_3} v_{i_1 i_2 i_3} = 0, \quad a = 1, ..., 5, \tag{21}$$

where 2^3 vertices $v_{i_1 i_2 i_3}$ belong to Z^3.

These combinatorial equations can be solved by the following Mori vectors

$$\begin{aligned}
q^1_{i_1 i_2 i_3} =& (1, 0, 0, 1, 0, 0, 0, 0), \\
q^2_{i_1 i_2 i_3} =& (0, 1, 0, 0, 1, 0, 0, 0), \\
q^3_{i_1 i_2 i_3} =& (0, 0, 1, 0, 0, 1, 0, 0), \\
q^4_{i_1 i_2 i_3} =& (1, -1, 0, 0, 0, 0, 1, 0), \\
q^5_{i_1 i_2 i_3} =& (0, 0, 1, 1, 0, 0, 0, 1).
\end{aligned} \tag{22}$$

The corresponding vertices $v_{i_1 i_2 i_3}$ are given by

$$\begin{aligned}
v_{000} =& (1, 0, 0), \quad v_{100} = (0, 1, 0), \\
v_{010} =& (1, 0, 0), \quad v_{001} = (-1, 0, 0), \\
v_{110} =& (1, 0, 0), \quad v_{101} = (0, 0, -1), \\
v_{110} =& (1, 0, 0), \quad v_{111} = (0, -1, -1),
\end{aligned} \tag{23}$$

which can be connected to the Clifford graphs algebra denoted as $Cliff_{2^n=8}(2^n = 8, 2^n - 1 = 7)$.

The study above is limited to the case of pure states, however the study of mixed states is very important in modern physics and have a strong utility in many applications of quantum information. To deal with it we consider the case of two qubits described in (Figure 4). It is denoted that in the graph of the left side, we have four 0−simplex, three 1−simplex and one 2−simplex. Moreover, the graph in the right side of (Figure 4) is composed by four 0−simplex and three 1−simplex. Then it has been revealed that this composition of simplex form the space of mixed states $\mathcal{M}$, and the space of pure states $\mathcal{P}$ is nothing but the vertices [201, 202].

We can remark that the number of 0−simplex in both graphs of (Figure 4) is related to the number of vertices of Clifford graphs algebra wich is obvious, also the number of 1−simplex correspond to the number of edges in these later graphs.

$$\mathcal{P} \equiv 0\text{−simplex.}$$

$$\mathcal{M} \equiv \text{simplex.}$$

$$E \equiv 1\text{−simplex.}$$

Another link is between the dimension of complex projective space of qubits systems CP^{2^n-1} and the number of 1−simplex in corresponding clifford graphs algebras.

To push further the desired link, we make contact with quantum computational using the representation of a quantum n-gate as unitary $2^n \times 2^n$ matrix which act on n-qubits in $2n$-dimensional complex vector space [203]. The isomorphism exists between a complex Clifford algebra with 2^n generators and the algebra of all $2^n \times 2^n$ matrices

$$Cl(2n, \mathbb{C}) \cong \mathbb{C}(2^n \times 2^n) \tag{24}$$

Precisely, the action of $Cl(2n, \mathbb{C})$ on n-qubit systems is none other than the action of $2^n \times 2^n$ matrix on complex vector in 2^n-dimensional complex space. To assimilate this, we take a basis of 2^n vector space

$$e_L = e_{l_1} \otimes e_{l_2} \otimes ... \otimes e_{l_n}, \quad l_k = 0, 1 \tag{25}$$

where one used in quantum information science

$$|i\rangle = |i_1 i_2 ... i_n\rangle = |i_1\rangle \otimes |i_2\rangle \otimes ... \otimes |i_n\rangle, \tag{26}$$

and where one has

$$|0\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad |1\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix}. \tag{27}$$

4^n-dimensional basis of Clifford algebra is given by

$$g_I = \sigma_{l_1} \otimes \sigma_{l_2} \otimes ... \otimes \sigma_{l_n}, \quad \sigma_{l_k} \in 1, \sigma_x, \sigma_y, \sigma_z. \tag{28}$$

It has been revealed that the action is defined for an element on 2^nvector space by

$$g_I = (\sigma_{l_1}|i_1\rangle) \otimes (\sigma_{l_2}|i_2\rangle) \otimes ... \otimes (\sigma_{l_n}|i_n\rangle), \tag{29}$$

where $\sigma_{l_k}|i_k\rangle$ is the action of Pauli 2×2 matrix on $2D$ complex vector $|0\rangle$ or $|1\rangle$.

Thus, any linear transformation of 2^n-dimensional space can be represented by elements of Clifford algebra with $2n$ generators

$$Cl(2n, \mathbb{C}) \cong \mathbb{C}(2^n \times 2^n) \; : \; \mathbb{C}^{2n} \to \mathbb{C}^{2n}. \tag{30}$$

Here, we can represent the edges as the actions of $Cl(2n, \mathbb{C})$.

This mapping can be explored to study many related applications of quantum information including quantum computing. We expect that this analysis can be pushed further to deal with other toric manifolds having essential utility in modern physics.

5 Conclusion

Employing toric geometry and Clifford graph algebras correspondence, we have discussed qubit information systems. Precisely, we have presented a one to one correspondence between three apparently separate subjects namely toric geometry, Clifford graph algebras and quantum information theory. We think that this work may be explored to deal with qubit system problems using geometry considered as a strong tool to understand modern physics. In particular, we have considered in some details the cases of two and three qubits, and we find that they are associated with $CP^1 \times CP^1$ and $CP^1 \times CP^1 \times CP^1$ toric manifolds, respectively. Developing a geometric procedure, we have revealed that the qubit physics can be converted into a scenario turning toric data of such varieties by the help of a construction of Clifford graph algebras. Pure and mixed states are dealt with.

In this work, we have completed partial results which comes up with many open questions. A obvious one is to examine toric Calabi-Yau supermanifolds. We believe that in this issue, we can deal with superqubit systems. Furthermore, an important question is to investigate the entanglement states in the context of toric geometry and its application including mirror symmetry.

Conclusion

In this thesis, we have investigated dyonic black objects in arbitrary dimensions. Inspired by Hodge diagrams of toroidal compactifications on $\mathbb{T}^n$, we have proposed a new class of doublets of dyonic black solutions formed by two different $D-$branes given by $\begin{pmatrix} p \\ 6-n-p \end{pmatrix}$ objects. In particular, we have pointed out a correspondence between toroidal cycles and the black solutions carrying electric and magnetic charge under $G_{dyon} = U(1)_e^{2^{n-1}} \times U(1)_m^{2^{n-1}}$ dyonic gauge symmetry. It has been suggested that this symmetry could be associated with electrically charged magnetic monopole solutions in string string model buildings of the SM extensions. More precisely, we have considered in some details such dyonic black classes obtained from lower dimensional toroidal compactification, and we have found that they are linked to $Cl(n)$ Clifford algebras using the vee product. It has been observed that the analysis presented here might be extended to a class of local Calabi-Yau manifolds. These manifolds have been extensively studied in the geometric engineering method of quantum field theory, obtained from the compactification of higher dimensional models including string theory, M-theory, and F-theory [181, 182]. Some of them are known by canonical bundles on $n-$dimensional compact spaces given by trivial fibrations of n copies of the projective space $\mathbb{CP}^1$ namely

$$M^n = \underbrace{\mathbb{CP}^1 \times \ldots \mathbb{CP}^n}_{n}$$

The compactification of string theory on such manifolds produces dyonic black objects in $8-2n$-dimensional space-time. Like toroidal compactifications, we are thus expecting that a similar analysis can take place by replacing the circle $\mathbb{S}^1$ by the one-dimensional complex space $\mathbb{CP}^1$

$$\mathbb{S}^1 \to \mathbb{CP}^1.$$

Concretely, one should also expect to be able to engineer dyonic black objects in terms of even-dimensional D-branes producing models with less supersymmetric charges.

On the other hand, we have investigated dyonic objects and tensor network formalism. Inspired and motivated by know result obtained from non-trivial theories including string theory, we provide briefly th origin and certain related concepts. After that, we have elaborated a general framework of dyonic solutions in the brane structure and show how are naturally obtained in various dimensions within stringy compactifications on a concrete complex geometry. Then, we have discussed dyonic solution representations using tensor network formalism by revealing some characteristics associated with extra dimensions. Assuming the existence of such dimensions, we believe that such a theoretical investigation could shed, somehow, some light on experimental searches for dyons, through either their direct production or indirect detection benchmarks at the current and future accelerators.

This thesis opens up for further discussions. We could cute some of them. First, it would be interesting to investigate geometries based on odd-dimensional toroidal compactifications. Second, it should be nice to find a link with fermionic coherent state theory shearing certain similarities with the dyonic black objects dealt with in this thesis. On the other hand, one should also try to establish a relation or a link between graph theory and such dyonic black objects via Clifford algebra structures.

Moreover, a natural question is make contact with MERA being a kind of tensor network states explored in the study of black holes.

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Résumé

En utilisant la compactification toroïdale de la théorie des cordes sur des tores à n-dimensions $\mathbb{T}^n$, nous étudions des objets dyoniques dans des dimensions arbitraires. Tout d'abord, nous présentons une classe de solutions noires dyoniques formées de deux D-branes différentes, en exploitant une correspondance entre des cycles toroïdaux et des objets possédant à la fois des charges magnétiques et électriques, avec la symétrie de jauge $U(1)_e^{2^{n-1}} \times U(1)_m^{2^{n-1}}$. Cette symétrie de jauge pourrait être injectée dans le modèle standard étendu des particules. En examinant en détail ces solutions noirs, nous établissons un lien avec les algèbres de Clifford de type $Cl(n)$. Cette analyse pourrait être étendue aux objets dyoniques qui peuvent être obtenus à partir des compactifications sur variétés locales de Calabi-Yau.

Motivés par les résultats de la physique des particules, nous exposons certaines représentations graphiques des solutions dyoniques dans des dimensions arbitraires. En utilisant la théorie des cordes, nous présentons certaines constructions branaires à partir des compactifications concrètes sur des espaces projectifs. Finalement, nous élaborons leurs réalisations en termes de réseaux de tenseurs en exploitant un formalisme associé aux particules multi-états.

Mots-clefs (5): Théorie des supercordes, compactification toroïdale, modèle standard, algèbres de Clifford, solutions noires dyoniques, réseau de tenseurs.

Abstract

Using the toroidal compactification of string theory on n-dimensional tori $\mathbb{T}^n$, we investigate dyonic objects in arbitrary dimensions. First, we present a class of dyonic black solutions formed by two different D-branes using a correspondence between toroidal cycles and objects possessing both magnetic and electric charges, belonging to $U(1)_e^{2^{n-1}} \times U(1)_m^{2^{n-1}}$ dyonic gauge symmetry. This symmetry could be associated with electrically charged magnetic monopole solutions in stringy model buildings of the standard model (SM) extensions. Then, we consider in some detail such black hole classes obtained from even-dimensional toroidal compactifications, and we find that they are linked to $Cl(n)$ Clifford algebras using the vee product. It is believed that this analysis could be extended to dyonic objects which can be obtained from local Calabi-Yau manifold compactifications.

Motivated by particle physics results, we investigate certain dyonic solutions in arbitrary dimensions. Concretely, we study the stringy constructions of such objects from concrete compactifications. Then, we elaborate their tensor network realizations using multistate particle formalism.

Keywords (5): Superstring theory, toroïdal compactification, standard model, Clifford algebra, black dyonic solutions, tensor network.