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Relative Homological Notions in Graded Module Category and Sheaf Category

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RELATIVE HOMOLOGICAL NOTIONS IN GRADED MODULE CATEGORY AND SHEAF CATEGORY

A PhD thesis in Mathematics by

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Dedication

To my beloved husband, Ahmed Darkaoui Es-sbar

To my beloved children, Sife-ddine, Nouh and Judy

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Abstract

The goal of this thesis goes in three directions. The first two parts focus on relative homological algebra in the category of graded modules, while the third part concerns a specific relative domain of flatness in categories of sheaves.

First, we introduce and study the notions of Gorenstein n -FP-gr-injective and Gorenstein n -gr-flat modules with respect to the notion of special finitely presented graded modules in the category of graded modules, particularly over n -coherent rings. Several fundamental concepts of relative Gorenstein homological algebra are characterized over such rings. For instance, any graded module in R -gr (resp. $\text{gr-}R$) admits Gorenstein n -FP-gr-injective (resp. Gorenstein n -gr-flat) cover and preenvelope.

In the second part, we introduce and explore the notions of n -copresented graded right modules, n -gr-cocoherent right rings and n -FCP-gr-projective right modules as the dual concepts of n -presented graded left modules, n -gr-coherent left rings and n -FP-gr-injective left modules, respectively. We then investigate under what conditions the pair $(\text{gr-}\mathcal{FCP}_n, \text{gr-}\mathcal{FCP}_n^\perp)$ forms a perfect hereditary cotorsion pair, where $\text{gr-}\mathcal{FCP}_n$ denotes the class of n -FCP-gr-projective right modules. Furthermore, we examine several conditions equivalent to each right R -module being n -FCP-gr-projective.

Finally, we develop a new treatment of the geometric flatness in categories of sheaves which shed more light on some of its various important aspects. Namely, in terms of this new approach, some classical schemes are characterized, some new results are established and also known ones are generalized.

⁰**Key Words.** n -gr-coherent ring; Gorenstein n -FP-gr-injective module; Gorenstein n -gr-flat module; n -gr-cocoherent ring, special gr-copresented module, n -FCP-gr-projective module; absolutely pure domain, locally absolutely pure domain, f -indigent, locally f -indigent, flat sheaf, flat quasi-coherent sheaf



Résumé

Les anneaux et les modules gradués sont des sujets classiques en algèbre, et la théorie homologique des anneaux gradués revêt une grande importance en géométrie algébrique, notamment dans le contexte des catégories de faisceaux. Au cours des dernières années, la théorie homologique relative des anneaux gradués a été l'objet d'études par de nombreux chercheurs et est devenue un domaine de recherche particulièrement actif. En outre, nous remarquons qu'au fil des dernières années, de nombreux auteurs ont adopté une nouvelle approche dans l'analyse de la projectivité, de l'injectivité et de la platitude classiques des modules. Ils ont ainsi introduit des domaines relatifs de modules comme un moyen d'évaluer le niveau de platitude d'un tel module, plutôt que de se limiter à déterminer s'il est projectif ou non. L'objectif de cette thèse se répartit en trois directions. Les deux premières parties se concentrent sur l'algèbre homologique relative dans la catégorie des modules gradués, tandis que la troisième partie aborde un domaine spécifique de platitude relatif dans les catégories de faisceaux. Premièrement, nous introduisons et étudions les notions de modules n -FP-gr-injectifs de Gorenstein et de modules n -gr-plats de Gorenstein. Plusieurs concepts fondamentaux de l'algèbre homologique de Gorenstein relative sont caractérisés sur les anneaux n -gr-cohérents. Dans la deuxième partie, nous introduisons et explorons la notions de modules n -FCP-gr-projectifs. Ensuite, nous étudions dans quelles conditions cette classe de modules et sa classe orthogonale droite forment une paire cotorsion parfaite héréditaire. Enfin, nous développons un nouveau traitement de la platitude géométrique dans les catégories de faisceaux qui éclaire certains de ses aspects importants. Notamment, en termes de cette nouvelle approche, certains schémas classiques sont caractérisés et de nouveaux résultats sont établis

⁰**Mots Clés.** Anneau n -gr-cohérent; module n -FP-gr-injectif de Gorenstein; module n -gr-plat de Gorenstein; anneau n -gr-cocohérent; module n -FCP-gr-projectif; domaine absolument pur; domaine localement absolument pur; f -indigent; Localement f -indigent; faisceau plat.



Résumé étendu

Les anneaux et les modules gradués sont des sujets classiques en algèbre, et la théorie homologique des anneaux gradués revêt une grande importance en géométrie algébrique [14], notamment dans le contexte des catégories de faisceaux. Au cours des dernières années, la théorie homologique relative des anneaux gradués a été l'objet d'études par de nombreux chercheurs et est devenue un domaine de recherche particulièrement actif (voir, par exemple, [13, 14, 15, 76, 77]).

En outre, nous remarquons qu'au fil des dernières années, de nombreux auteurs ont adopté une nouvelle approche dans l'analyse de la projectivité, de l'injectivité et de la platitude classiques des modules (voir, par exemple, [17, 6, 27, 8]). Ils ont ainsi introduit des domaines relatifs de modules comme un moyen de mesurer, d'une certaine manière, le degré de platitude d'un tel module, plutôt que de se limiter à déterminer s'il est projectif ou non

L'objectif de cette thèse se répartit en trois directions. Les deux premières parties se concentrent sur l'algèbre homologique relative dans la catégorie des modules gradués, tandis que la troisième partie aborde un domaine spécifique de platitude relatif dans les catégories de faisceaux.

Premièrement, nous introduisons et étudions les notions de modules Gorenstein n -FP-gr-injectifs et de modules Gorenstein n -gr-plats par rapport à la notion de modules gradués spécialement finis présentés dans la catégorie des modules gradués, en particulier sur les anneaux n -cohérents. Plusieurs concepts fondamentaux de l'algèbre homologique de Gorenstein relative sont caractérisés sur de tels anneaux. Par exemple, tout module gradué dans R -gr (resp. gr- R) admet une couverture et une préenveloppe Gorenstein n -FP-gr-injectives (resp. Gorenstein n -gr-plats).

Dans la deuxième partie, nous introduisons et explorons les notions de modules gradués n -coprésentés à droite, d'anneaux n -gr-cocohérents à droite et de modules n -FCP-gr-projectifs à droite en tant que concepts duaux des modules gradués n -présentés à gauche, des anneaux n -gr-cohérents à gauche et des modules n -FP-gr-injectifs à gauche, respectivement. Ensuite, nous étudions dans quelles conditions la paire $(gr\text{-}\mathcal{FCP}_n, gr\text{-}\mathcal{FCP}_n^\perp)$ forme une paire cotorsion parfaite héréditaire, où $gr\text{-}\mathcal{FCP}_n$ désigne la classe des modules n -FCP-gr-projectifs à droite. De plus, nous examinons plusieurs conditions équivalentes pour que chaque R -module à droite soit n -FCP-gr-projectif.

Enfin, nous développons un nouveau traitement de la platitude géométrique dans les catégories de faisceaux qui éclaire certains de ses aspects importants. Notamment, en termes de cette nouvelle approche, certains schémas classiques sont caractérisés, de nouveaux résultats sont établis et des résultats connus sont généralisés.

⁰**Mots Clés.** Anneau n -gr-cohérent; module n -FP-gr-injectif de Gorenstein; module n -gr-plat de Gorenstein; anneau n -gr-cocohérent; module spécial gr-copprésenté; module n -FCP-gr-projectif; domaine absolument pur; domaine localement absolument pur; f -indigent; Localement f -indigent; faisceau plat; faisceau quasi-cohérent plat

Author's papers involved in this thesis

1. M. Amini, D. Bennis and S. Mamdouhi, *n -gr-Coherent rings and Gorenstein graded modules*, Czechoslovak Mathematical **72** (2022), 125-148.
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Introduction

Graded rings play a central role in algebraic geometry and commutative algebra. For example, if R is a commutative ring which is generated by a finite number of elements of degree 1, then by the celebrated work of Serre [69], the category of quasi coherent sheaves on the projective scheme over $\mathbb{P}(R)$, denoted by $\mathcal{Q}\text{coh}(\mathbb{P}(R))$, is equivalent to $\text{Gr-}R$, where $\text{Gr-}R$ is the category of graded modules over R . In particular when $R = K[x_0, x_1, \dots, x_n]$, where K is a field, then $\mathcal{Q}\text{coh}(\mathbb{P}^n(R))$ is equivalent to $\text{Gr-}R[x_0, x_1, \dots, x_n]$ (see [69], [70], [44] and [7] for more precise statements and relations with noncommutative algebraic geometry).

The concept of grading in algebra, in particular graded modules, is crucial in the study of homological aspect of rings. Recently, rings with a group-graded structure have become increasingly important and consequently, the graded analogues of a variety of concepts are widely studied (see [55], [49], [63], [59], [14], [15]). As a result, in the area of homological algebra, graded analogs of various concepts are being developed in recent research.

This thesis is built on three topics in the realm of (Relative) Homological Algebra. Firstly, it delves into the study of a generalization of the Gorenstein graded modules, particularly focusing on these over graded n -coherent rings. Secondly, it investigates the relative homological theory, focusing on the dual notions of n -presented graded left modules, n -gr-coherent left rings, and n -FP-gr-injective left modules. Lastly, it introduces and examines a new perspective on flatness in categories of sheaves.

Generalized Gorenstein injective graded modules:

One of the primary motivations behind the study of homological dimensions stems from a pivotal result dating back to 1956, credited to Auslander-Buchsbaum and Serre. This result establishes that a commutative Noetherian ring R , with residue field k , is regular if and only if every R -module M has finite projective dimension. This result

initiated a trend, highlighting that finiteness of a homological dimension for all modules characterizes rings with special properties. In line with these ideas, in 1969, Auslander and Bridger introduced the notion of G -dimension as a homological dimension for finitely generated modules over commutative Noetherian rings, providing a characterization of local Gorenstein rings, similar to the Auslander-Buchsbaum-Serre theorem. Subsequently, in the mid-1990s, Enochs, Jenda, and Torrecillas extended G -dimension to modules (finitely generated or not) over arbitrary rings, defining Gorenstein projective and Gorenstein flat modules (see [31, 33]). The definition of Gorenstein projective modules is modeled on that of modules with G -dimension zero, while Gorenstein flat modules represent another generalization of modules with G -dimension zero. The dual notion of a Gorenstein projective module – the Gorenstein injective module – was introduced by Enochs and Jenda. Gorenstein homological algebra is based on these classes of modules, which extend and generalize the classes of projective, injective, and flat modules.

We recall that a module E is called absolutely pure (or FP-injective) if $\text{Ext}^1(P, E) = 0$ for all finitely presented R -modules P . The FP-injective dimension of an R -module N is defined to be the least integer $n \geq 0$ such that $\text{Ext}^{n+1}(P, N) = 0$ for all finitely presented R -modules P .

In [39], the author noted that the work of Ding, Mao, and Li (see [54], [56], [25]), provides a natural way to extend notions from Gorenstein homological algebra from noetherian to coherent rings. In the process, the Gorenstein modules are replaced by Ding modules. Like the Gorenstein injectives, the Ding injective modules are also cycles of exact complexes of injective modules, in this case complexes that remain exact when applying a functor $\text{Hom}(E, -)$ for any FP-injective module E . Thus, a Ding injective module is, in particular, Gorenstein injective.

The Ding counterpart of a Gorenstein flat module is a Ding flat module. A module M is said to be Ding flat if there exists an exact sequence of flat modules

$$\mathcal{F} = \cdots \rightarrow F_1 \rightarrow F_0 \rightarrow F_{-1} \rightarrow \cdots$$

such that $E \otimes \mathcal{F}$ is still exact for any FP-injective module E and M is one of the cycles of the complex F . Gillespie showed [39, (Proposition 7.2)] that, over any ring R , a Ding flat module is nothing more than a Gorenstein flat module.

While the Gorenstein modules have very nice properties over Gorenstein rings, the Ding modules have nice properties over the so-called Ding-Chen rings. A Ding-Chen ring is a coherent ring R (i.e., left and right coherent) which has finite FP-injective dimension as both a left and right module over itself. These rings were introduced by Ding and Chen in [24] and called n -FC rings. More precisely, they call a coherent ring R an n -FC ring if the FP-injective dimension of R as both a left and right module over itself is less than or equal to n . Later, Gillespie (in [39]), renamed these rings Ding-Chen rings. So, by Gillespie's definition, a ring R is called a Ding-Chen ring if it is an n -FC ring for some $n \geq 0$.

Since a coherent ring is Noetherian if and only if the FP-injective modules coincide with the injective modules, then the Ding-Chen rings are nothing more than Gorenstein rings whenever R is a Noetherian ring. Gillespie recently proved (in [39]) that, over a Ding-Chen ring, the Ding injective (projective) modules are simply the cycles of exact complexes of injective (projective) modules. For this reason, over a Ding-Chen ring, the class of Ding injective (Ding projective, Ding flat, respectively) modules coincides with that of Gorenstein injective (Gorenstein projective, Gorenstein flat) modules.

Since Ding injective modules are stronger than the Gorenstein injective modules, and in general an FP-injective module is not necessarily Ding injective (see [56, Proposition 2.7]), Gao and Wang introduced and studied in [37] another notion called Gorenstein FP-injective modules. A left R -module N is called Gorenstein FP-injective if there exists an exact sequence $\cdots \rightarrow E_1 \rightarrow E_0 \rightarrow E_0 \rightarrow E_1 \rightarrow \cdots$ of FP-injective left R -modules, with $N = \text{Ker}(E_0 \rightarrow E_1)$, and such that for any finitely presented left R -module U with $\text{pd}_R(U) < \infty$, the functor $\text{Hom}_R(U, -)$ preserves exactness when applied to the sequence. Under this definition, the Gorenstein FP-injective modules are weaker than the usual Gorenstein injective modules, and all FP-injective modules are in the class of Gorenstein FP-injective modules. Due to these modules, several well-known classes of rings were characterized.

The homological theory of graded rings is very important because of its applications in algebraic geometry. Several authors have investigated the graded aspect of some notions in relative homological algebra. For example, Asensio, López Ramos and Torrecillas in [14, 15] introduced the notions of Gorenstein gr-projective, gr-injective

and gr-flat modules. In the recent years, the Gorenstein homological theory for graded rings have become an important area of research (see for instance [16, 38]). The notions of FP-gr-injective modules was introduced in [12], and the homological behavior of FP-gr-injective modules on gr-coherent rings was investigated in [76]. Along the same lines, it is natural to generalize the notion of FP-gr-injective modules and gr-flat modules to n -FP-gr-injective modules and n -gr-flat modules. This was done by Zhao, Gao and Huang in [77] basing on the notion of special finitely presented graded modules which they defined via projective resolutions of n -presented graded modules. Recently, in 2017, Mao via FP-gr-injective modules gave a definition of Ding gr-injective and Ding gr-projective modules over a graded ring modules (see[55]), which are the analogue of the non-graded Ding injective and Ding projective modules. the author showed that these modules have very nice properties over gr-Ding-Chen rings (analogue of the non-graded Ding-Chen ring). For examples, over gr-Ding-Chen rings: the class of Ding gr-injective and Gorenstein gr-flat modules are pre-enveloping, gr-projective modules are nothing more than a Gorenstein gr-flat modules. The fact that Ding gr-injective modules are also stronger than the Gorenstein gr-injective modules, and an FP-gr-injective module is not necessarily Ding gr-injective in general [55, Corollary 3.7], provides a natural way to extend the notions of n -FP-gr-injective and n -gr-flat modules to that of Gorenstein n -FP-gr-injective and Gorenstein n -gr-flat modules, respectively. Under this extension, every n -FP-gr-injective module is Gorenstein n -FP-gr-injective module, and similarly, every n -gr-flat module is Gorenstein n -gr-flat module.

The purpose of this section is to introduce the concepts of Gorenstein n -gr-FP-injective modules and Gorenstein n -gr-flat modules, respectively. Additionally, the section aims to investigate self Gorenstein n -gr-FP-injective coherent rings.

Investigating the relative homological theory of some dual notions in category of graded modules: In 1969, Jans in [51] gave a definition of finitely cogenerated modules as a dual notion of finitely generated modules when he introduced co-Noetherian rings as a dual notion of Noetherian rings. A right R -module M is said to be finitely cogenerated if for every family $\{M_i\}_{i \in I}$ of submodules of M with $\bigcap_{i \in I} M_i = 0$, there is a finite subset $J \subset I$ such that $\bigcap_{i \in J} M_i = 0$. In 1994, Costa in [21] introduced the notion of n -coherent rings for a nonnegative integer n . A left R -module M is

said to be n -presented if it has a finite n -presentation, that is, there exists an exact sequence $F_n \rightarrow F_{n-1} \rightarrow \cdots \rightarrow F_1 \rightarrow F_0 \rightarrow M \rightarrow 0$ with each F_i is finitely generated free R -module, and a ring R is called left n -coherent if every n -presented left R -module is $(n + 1)$ -presented, for more details see [22, 26]. As we know, co-coherent rings as a dual notion of coherent rings have been characterized in various ways, and many nice properties were obtained for such rings in [52, 79, 72]. In 1999, Weimin Xue [73], via finitely cogenerated modules introduced n -copresented modules and n -cocoherent rings as a dual notion of n -presented modules and n -coherent rings, respectively. A right R -module M is said to be n -copresented if there is an exact sequence $0 \rightarrow M \rightarrow E^0 \rightarrow E^1 \rightarrow \cdots \rightarrow E^n$ of right R -modules, where each E^i is a finitely cogenerated injective module. A ring R is called right n -cocoherent if every n -copresented R -module is $(n + 1)$ -copresented. For more details, we refer the reader to [4, 5, 18, 78]. The goal of this topic is to introduce and study n -copresented graded right modules, n -gr-cocoherent right rings and n -FCP-gr-projective right modules as a dual notion of n -presented graded left modules, n -gr-coherent left rings and n -FP-gr-injective left modules, respectively. We then delve into the relative homological theory of these modules.

A new perspective on flatness in categories of sheaves:

Many studies are done every year on projective, injective and flat modules. Many of them involve concepts derived from relative projectivity, injectivity and flatness. Rather than saying whether a module has a certain property or not, each module is assigned a relative domain that, somehow, evaluates the module's proximity to possessing this particular property.

In [27], an alternative perspective on the flatness of a module was introduced. Recall that, for two modules M and N , N is said to be absolutely M -pure if $M \otimes N \rightarrow M \otimes K$ is a monomorphism for every extension K of N . The absolutely pure domain of any module M , $\mathcal{A}p(M)$, is the class of all modules N such that N is absolutely M -pure. As an opposite to flatness, Durğun also introduced in [27] the concept of flat indigent modules as follows: a module is called flat indigent (or an f -indigent module) in case $\mathcal{A}p(M) = \{N \in A - Mod : N \text{ is absolutely pure}\}$. The purpose of [27] was to introduce a new approach on the analysis of the flatness of a module. And thus, it provides

an interesting new side on some other known notions. In this thesis, we investigate these notions in categories of sheaves.

Now, we summarize the contents of this memory.

Chapter I: Preliminaries

This chapter is the preliminaries parts of the memory. We recall some basic terminologies and results which will be used in the rest of the thesis.

Chapter II: n -gr-coherent rings and Gorenstein graded modules

Throughout this chapter, all rings considered are associative with identity element and the modules are unital, and $R\text{-Mod}$ (resp. $\text{Mod-}R$) will denote the category of all left (resp. of all right) R -modules. We denote by $R\text{-gr}$ (resp. $\text{gr-}R$) the category of graded left (resp. of graded right) R -modules, and by $\text{gr-pd}_R(M)$ and $\text{gr-fd}_R(M)$ we will denote the gr-projective and gr-flat dimension of a graded module M in $R\text{-gr}$, respectively. We use $\text{gr-}\mathcal{I}$, $\text{gr-}\mathcal{FI}_n$ and $\mathcal{G}_{\text{gr-}\mathcal{FI}_n}$ to denote the classes of gr-injective, n -FP-gr-injective and Gorenstein n -FP-gr-injective modules in $R\text{-gr}$, respectively, while $\text{gr-}\mathcal{F}$, $\text{gr-}\mathcal{F}_n$, and $\mathcal{G}_{\text{gr-}\mathcal{F}_n}$ to represent the classes of gr-flat, n -gr-flat, and Gorenstein n -gr-flat modules in $\text{gr-}R$, respectively.

In this chapter, we introduce the concepts of Gorenstein n -FP-gr-injective and Gorenstein n -gr-flat modules by utilizing n -FP-gr-injective modules and n -gr-flat modules, respectively, for any $n \geq 1$. Under this framework, Gorenstein n -FP-gr-injective and Gorenstein n -gr-flat modules are characterized as weaker versions of Gorenstein gr-injective and Gorenstein gr-flat modules, respectively. Furthermore, for any $n \geq 1$, all gr-injective and n -FP-gr-injective modules, as well as gr-flat and n -gr-flat modules, are classified as Gorenstein n -FP-gr-injective and Gorenstein n -gr-flat, respectively. However, it is worth noting that in general, Gorenstein n -FP-gr-injective and Gorenstein n -gr-flat R -modules may not necessarily coincide with n -FP-gr-injective and n -gr-flat modules, unless in specific cases, as shown in Proposition 2.1.17.

The chapter is organized as follows:

In Section 2.1, we introduce the concepts of Gorenstein n -FP-gr-injective and Gorenstein n -gr-flat modules for an integer $n \geq 1$, followed by presenting some characterizations of these modules. Notably, we establish several results, including a proof that for an exact sequence $0 \rightarrow A \rightarrow B \rightarrow C \rightarrow 0$ of graded left R -modules, if A and B are Gorenstein n -FP-gr-injective, then C is Gorenstein n -FP-gr-injective if and only if every n -presented module U in $R\text{-gr}$ with $\text{gr-pd}_R(U) < \infty$ is $(n+1)$ -presented. Consequently, we demonstrate that $({}^\perp\mathcal{G}_{gr\text{-}\mathcal{FI}_n}, \mathcal{G}_{gr\text{-}\mathcal{FI}_n})$ forms a hereditary cotorsion pair if and only if every n -presented module U in $R\text{-gr}$ with $\text{gr-pd}_R(U) < \infty$ is $(n+1)$ -presented, and every $M \in ({}^\perp\mathcal{G}_{gr\text{-}\mathcal{FI}_n})^\perp$ has an exact left $(gr\text{-}\mathcal{FI}_n)$ -resolution. Furthermore, for a graded left (or right) R -module M over a left n -gr-coherent ring R , M is Gorenstein n -FP-gr-injective (or Gorenstein n -gr-flat) if and only if its character $R\text{-mod } M^*$ is Gorenstein n -gr-flat (or Gorenstein n -FP-gr-injective). Additionally, we establish that the class of Gorenstein n -FP-gr-injective (or Gorenstein n -gr-flat) modules is closed under direct limit (or direct product). In this section, we provide examples to illustrate that Gorenstein m -FP-gr-injectivity (or Gorenstein m -gr-flatness) does not imply Gorenstein n -FP-gr-injectivity (or Gorenstein n -gr-flatness) for any $m > n$. Moreover, we present examples demonstrating that Gorenstein n -FP-gr-injectivity does not imply gr-injectivity.

In Section 2.2, we show that the class of Gorenstein n -FP-gr-injective and Gorenstein n -gr-flat modules are covering and preenveloping on n -gr-coherent rings. We also establish some equivalent characterizations of n -gr-coherent rings in terms of Gorenstein n -FP-gr-injective and Gorenstein n -gr-flat modules.

Chapter III: Category of n -FCP-gr-projective modules with respect to special copresented graded modules

Throughout this chapter, all rings considered are associative with identity element and the modules are unital, and $R\text{-Mod}$ (resp. $\text{Mod-}R$) will denote the category of all left (resp. of all right) R -modules. We denote by $R\text{-gr}$ (resp. $\text{gr-}R$) the category of

graded left (resp. of graded right) R -modules, and $\text{gr-pd}_R(M)$ will denote the graded projective dimension of M in $R\text{-gr}$. We use gr-FCP_n to denote the class of n -FCP- gr -projective modules in $\text{gr-}R$.

In this chapter, we introduce and examine n -copresented graded right modules, n - gr -cocoherent right rings, and n -FCP- gr -projective right modules, serving as the dual concepts to n -presented graded left modules, n - gr -coherent left rings, and n -FP- gr -injective left modules, respectively. Subsequently, we delve into the relative homological theory of these modules. Additionally, we highlight the significance of special finitely copresented graded modules, which are defined through finitely cogenerated gr -injective resolutions of n -copresented graded modules. These special graded modules play a pivotal role in our exploration.

This chapter is organized in two sections as follows:

In Section 3.1, we first introduce the notions of n -copresented graded right modules, special gr -cogenerated and special gr -copresented modules with respect to any n -copresented graded right module, and also n -cocoherent graded right rings (or, right n - gr -cocoherent rings). Then, via n -copresented graded modules, we present the concept of n -FCP- gr -projective right modules and investigate some characterizations of these modules. In this section, examples are given in order to show that m -FCP- gr -projectivity does not imply n -FCP- gr -projectivity for any $m > n$.

In Section 3.2, we prove that there exist some equivalent characterizations of graded right modules of n -FCP- gr -projective dimension at most k on right n - gr -coherent rings. We obtain some equivalent characterizations of right n - gr -cocoherent rings in terms of n -FCP- gr -projective right modules on the short exact sequences. For example, R is a right n - gr -cocoherent ring if and only if for every exact sequence

$$0 \rightarrow A \rightarrow B \rightarrow C \rightarrow 0$$

of graded right R -modules, where B and C are n -FCP- gr -projective, it follows that A is n -FCP- gr -projective if and only if $(\text{gr-FCP}_n, \text{gr-FCP}_n^\perp)$ is a hereditary cotorsion pair, where gr-FCP_n denotes the class of n -FCP- gr -projective right modules. Moreover, if gr-FI_n is a class of n -FP- gr -injective left modules, then $(\text{gr-FCP}_n)^* \subseteq \text{gr-FI}_n$. Hence, we prove that every graded right R -module is n -FCP- gr -projective if and only if $(\text{gr-FCP}_n, \text{gr-FCP}_n^\perp)$ is a perfect hereditary cotorsion pair and $N \in \text{gr-FCP}_n^\perp$ has

an n -FCP-gr-projective cover with the unique mapping property if and only if R is right n -gr-cocoherent and N is n -FCP-gr-projective if and only if $N(\sigma)$ is gr-injective for any $\sigma \in G$.

Chapter IV: A new approach to flatness in categories of sheaves

Throughout this chapter, all rings are commutative with identity, \mathcal{A} will denote a closed symmetric monoidal Grothendieck category and X will denote a scheme. We denote by $\mathcal{O}_X\text{-Mod}$ and $\mathfrak{Qcoh}(X)$ the category of \mathcal{O}_X -modules and the category of quasi-coherent \mathcal{O}_X -modules, respectively. When U is an open subset of X , we use \mathcal{O}_U to denote the restriction of \mathcal{O}_X to U , and we denote by $\mathcal{O}_U\text{-Mod}$ the category of \mathcal{O}_U -modules. As usual, for a point x in X , we denote by $\mathcal{O}_{X,x}\text{-Mod}$ the category of $\mathcal{O}_{X,x}$ -modules.

The goal of this chapter is to introduce a new approach on the analysis of the flatness in categories of sheaves. As both $\mathcal{O}_X\text{-Mod}$ and $\mathfrak{Qcoh}(X)$ are closed symmetric monoidal Grothendieck categories, we first work in the general setting of a closed symmetric monoidal Grothendieck category. Subsequently, we delve into a detailed study in the categories $\mathcal{O}_X\text{-Mod}$ and $\mathfrak{Qcoh}(X)$. This study provides new concepts that contribute to the study of geometric flatness (known as \otimes -flatness or simply flatness if there is no ambiguity). Indeed, we obtain, for instance, generalizations of several results using new methods which give a different light to the way they are seen now, which in addition, gives new perspectives.

The chapter is organized as follows:

In Section 4.1, we start by recalling essential concepts for this article. Subsequently, we make a preliminary study on absolutely pure domains of objects in closed symmetric monoidal Grothendieck categories, aiming to establish fundamental and common results for $\mathcal{O}_X\text{-Mod}$ and $\mathfrak{Qcoh}(X)$ that we will use in subsequent sections. We generalize the study done in [27] and we give new results. The main result of this section is the elaboration of a new way to treat the flatness of objects in terms of factorizations of morphisms. This contribution allows to easily establish throughout the paper new and interesting results. For instance, Proposition 4.1.9 shows that when an object is

embedded in a flat object, its absolutely pure domain contains a larger class than that of the absolutely pure objects. Additionally, Corollary 4.1.10 shows that if M is a strongly Gorenstein flat object in \mathcal{A} , which has enough flat objects, then $\mathcal{A}p_{\mathcal{A}}(M) = {}^{\perp}(M^+)$. At the end, we establish that every locally finitely presented symmetric monoidal closed Grothendieck category has an f -indigent object, that is, an object whose absolutely pure domain consists only of absolutely pure objects.

In Section 4.2, using the results of Section 4.1, we show in Proposition 4.2.2 that the notion of absolutely pure domains is local in $\mathcal{O}_X\text{-Mod}$. We show that at any two objects \mathcal{M} and \mathcal{N} in $\mathcal{O}_X\text{-Mod}$, \mathcal{N} is absolutely \mathcal{M} -pure if and only if $\mathcal{N}|_U$ is absolutely $\mathcal{M}|_U$ -pure for any open $U \subseteq X$, which is equivalent to \mathcal{N}_x being absolutely \mathcal{M}_x -pure for any $x \in X$. However, these equivalences, in general, do not hold in the category $\mathfrak{Qcoh}(X)$. Nevertheless, when $X = \text{Spec}A$ is an affine scheme and A is coherent, these equivalences do hold. Consequently, it is reasonable to introduce and explore the notion of locally absolute pure domains for quasi-coherent sheaves on any scheme, particularly on locally coherent schemes and provide characterizations. Recall that a scheme X is called locally coherent if, for each affine open subset U of X , $\mathcal{O}_X(U)$ is a coherent ring. In Proposition 4.2.9, we see that if X is a semi-separated quasi-compact and locally coherent scheme, then for any two objects \mathcal{N} and \mathcal{M} in $\mathfrak{Qcoh}(X)$, \mathcal{N} is locally absolutely \mathcal{M} -pure is equivalent to \mathcal{N} being absolutely \mathcal{M} -pure in $\mathfrak{Qcoh}(X)$. At the end of this section, we present a new characterization of locally coherent schemes (see Proposition 4.2.11).

In Section 4.3, we shed light on f -indigent objects in both $\mathcal{O}_X\text{-Mod}$ and $\mathfrak{Qcoh}(X)$. The category $\mathcal{O}_X\text{-Mod}$ on a quasi separated scheme (resp. the category $\mathfrak{Qcoh}(X)$ on a quasi-compact and quasi-separated scheme) is locally finitely presented (see [64, Definition 0.5] and [41, Corollary 6.9.12], respectively). This leads to characterizing f -indigent objects in $\mathcal{O}_X\text{-Mod}$ and, respectively, in $\mathfrak{Qcoh}(X)$ on the mentioned schemes. For an object \mathcal{M} in $\mathcal{O}_X\text{-Mod}$, Proposition 4.3.3 shows that \mathcal{M} is f -indigent in $\mathcal{O}_X\text{-Mod}$ if and only if its restriction to any open set $U \subseteq X$ is f -indigent in $\mathcal{O}_U\text{-Mod}$. This condition is further equivalent to \mathcal{M}_x being f -indigent in $\mathcal{O}_{X,x}\text{-Mod}$ for any $x \in X$. However, these conditions are not equivalent in the category $\mathfrak{Qcoh}(X)$, so we introduce the notion of locally f -indigent objects in $\mathfrak{Qcoh}(X)$ and provide characterizations. We

establish in Corollaries 4.3.9 and 4.3.10 that locally f -indigent objects in $\mathfrak{Qcoh}(X)$ are f -indigent in both $\mathfrak{Qcoh}(X)$ and $\mathcal{O}_X\text{-Mod}$. Propositions 4.3.13 and 4.3.14 give new characterizations of a Noetherian scheme and a boolean affine scheme, respectively.

Finally, in Section 4.3, we delve deeper into the study of absolutely pure domains in $\mathcal{O}_X\text{-Mod}$, focusing specifically on locally coherent schemes. We introduce, in Proposition 4.4.2, a novel approach to characterizing absolutely pure domains of objects in $\mathcal{O}_X\text{-Mod}$. This proposition characterizes the absolutely pure domains of \mathcal{O}_X -modules by examining factorizations of morphisms in $\mathcal{O}_X\text{-Mod}$ through flat objects in $\mathcal{O}_X\text{-Mod}$. Moreover, Proposition 4.4.3 proves that the inverse of Proposition 4.1.9 also holds when X is a locally coherent scheme. In other words, an \mathcal{O}_X -module \mathcal{M} being embedded in a flat object is equivalent to every flat \mathcal{O}_X -module is \mathcal{M} -absolutely pure. Consequently, when X is, moreover, semi-separated, the existence of an f -indigent object in $\mathcal{O}_X\text{-Mod}$ that embeds in a flat object is equivalent to every flat object in $\mathcal{O}_X\text{-Mod}$ being absolutely pure, which, in turn, is equivalent to every injective object in $\mathcal{O}_X\text{-Mod}$ being flat.

PRELIMINARIES

This chapter is the introductory part of the memory. We recall some basic terminology and results (without proofs) which will be used in the next two chapters. Also, in this chapter we extend and prove some well known results from the category of modules to any abelian category. The reader is supposed to be familiar with the language of categories.

1.1 Graded rings and graded modules

This section discusses graded rings and the category of graded modules over a graded ring. This category is characterized as an abelian category, more specifically, a Grothendieck category. Numerous classical invariants developed for the module category can also be extended to this graded setting.

1.1.1 Graded rings

Definition 1.1.1 ([44]). *A ring R is called a G -graded ring if $R = \bigoplus_{\gamma \in G} R_\gamma$, where G is an abelian group, each R_γ is an additive subgroup of R , and $R_\gamma R_\sigma \subseteq R_{\gamma+\sigma}$ for all $\gamma, \sigma \in G$.*

The set $R^h = \bigcup_{\gamma \in G} R_\gamma$ is called the set of homogeneous elements of R . The additive group R_γ is called the γ -component of R , and the nonzero elements of R_γ are called

homogeneous of degree γ . We write $\deg(a) = \gamma$ (or $|r| = \gamma$) if $a \in R_\gamma \setminus \{0\}$. We call the set

$$G_R := \{\gamma \in G \mid R_\gamma \neq 0\}$$

the support of R .

When there is no danger of confusion, we may refer to a G -graded ring simply as a graded ring or a ring graded by a group G .

By definition, $R_0 R_\gamma = R_\gamma$ and $R_\gamma R_0 = R_\gamma$ for any $\gamma \in G$. If these equalities hold for any two arbitrary elements of G , we call the ring a **strongly graded ring**. Namely, a G -graded ring $R = \bigoplus_{\gamma \in G} R_\gamma$ is called a strongly graded ring if $R_\gamma R_\sigma = R_{\gamma+\sigma}$ for all $\gamma, \sigma \in G$.

Examples 1.1.2. • Consider $K[x] = \bigoplus_{n \in \mathbb{Z}} Kx^n$ where $Kx^n = 0$ if $n < 0$. Then $K[x] = \dots 0 \oplus \dots \oplus 0 \oplus K \oplus Kx \oplus \dots$

- Let $R = T[x]$ and G be an abelian group. Set $|x| = \gamma$ for some $\gamma \in G$. For $\sigma \in G$, we see $R_\sigma = \bigoplus_{i \in \mathbb{Z}, i\gamma = \sigma} Tx^i$, where $T \subseteq R_0$. If $G = \mathbb{Z}$ and $|x| = 1$, then for $n \in \mathbb{Z}$, $R_\sigma = \begin{cases} Tx^\sigma & \text{if } n \geq 0; \\ 0 & \text{if } n < 0. \end{cases}$ This is \mathbb{N} -grading.

- Let $R = T[x_1, \dots, x_d]$ and G be any abelian group. Set $|x_i| = \gamma_i$. For $\sigma \in G$, we have $R_\sigma = \bigoplus_{\alpha_1 \gamma_1 + \dots + \alpha_d \gamma_d = \sigma} Tx_1^{\alpha_1} \dots x_d^{\alpha_d}$.

1. If $R = T[x, y]$, $G = \mathbb{Z}$, and $|x| = |y| = 1$, then for $n \in \mathbb{Z}$, $R_n = \bigoplus_{i+j=n, i, j \geq 0} Tx^i y^j$.

2. If $R = T[x, y]$, $G = \mathbb{Z}$, and $|x| = 2$, $|y| = 3$, then $R_n = \bigoplus_{2i+3j=n} Tx^i y^j$.

- Let A be a G -graded and B a H -graded ring. Then $A \otimes_{\mathbb{Z}} B$ has a natural $G \times H$ -graded ring structure as follows. Since A_γ and B_ω , $\gamma \in G$, $\omega \in H$, are \mathbb{Z} -modules, then $A \otimes_{\mathbb{Z}} B$ can be decomposed as a direct sum

$$A \otimes_{\mathbb{Z}} B = \bigoplus_{(\gamma, \omega) \in G \times H} A_\gamma \otimes B_\omega$$

(to be precise, $A_\gamma \otimes B_\omega$ is the image of $A_\gamma \otimes_{\mathbb{Z}} B_\omega$ in $A \otimes_{\mathbb{Z}} B$).

1.1. GRADED RINGS AND GRADED MODULES

Now, if $G = H$ and

$$\begin{aligned} f : G \times G &\longrightarrow G, \\ (\gamma_1, \gamma_2) &\longmapsto \gamma_1 + \gamma_2, \end{aligned}$$

then we get a natural G -graded structure on $A \otimes_{\mathbb{Z}} B$. Namely,

$$A \otimes_{\mathbb{Z}} B = \bigoplus_{\gamma \in G} (A \otimes B)_{\gamma},$$

where

$$(A \otimes B)_{\gamma} = \left\{ \sum_i a_i \otimes b_i \mid a_i \in A^h, b_i \in B^h, \deg(a_i) + \deg(b_i) = \gamma \right\}.$$

1.1.2 Graded modules

Definition 1.1.3 ([44]). Let R be a G -graded ring. A graded left R -module N is defined to be a left R -module N with a direct sum decomposition $N = \bigoplus_{\gamma \in G} N_{\gamma}$, where each N_{γ} is an additive subgroup of N such that $R_{\gamma} N_{\sigma} \subseteq N_{\gamma+\sigma}$ for all $\gamma, \sigma \in G$.

By definition, if $x \in N$, then there exist unique elements $x_{\gamma} \in N_{\gamma}$ for each $\gamma \in G$, all but finitely many of which are zero, such that

$$x = \sum_{\gamma \in G} x_{\gamma}.$$

The element x_{γ} is called the γ -th homogeneous component of x . If $x = x_{\gamma} \neq 0$ for some $\gamma \in G$, then x is called G -homogeneous of degree γ , or just homogeneous if G is understood, and we denote the degree of x by $\deg(x) = \gamma$ or $|x| = \gamma$.

Examples 1.1.4. • If R is a graded ring, then R is a graded R -module over itself.

- Let $\{N_{\lambda}\}$ be a family of graded R -modules, then $\bigoplus_{\lambda} N_{\lambda}$ is a graded R -module. Consequently, $R^n = R \oplus \dots \oplus R$ (n times) is a graded R -module for any $n \geq 1$.
- Let R be a graded ring and S a multiplicatively closed set of homogeneous elements of R . So, R_S is a graded ring, where

$$(R_S)_{\gamma} = \left\{ \frac{r}{s} \in R_S \mid r \text{ and } s \text{ are homogeneous and } \deg r - \deg s = \gamma \right\}.$$

Similarly, if N is a graded R -module, then N_S is graded both as an R -module and as an R_S -module.

For any graded left R -modules M and N , set $\text{Hom}_{R\text{-gr}}(M, N) := \{f : M \rightarrow N \mid f \text{ is } R\text{-linear map and } f(M_\gamma) \subseteq N_\gamma \text{ for any } \gamma \in G\}$, which is the group of all morphisms from M to N in the class $R\text{-gr}$ of all graded left R -modules ($\text{gr-}R$ will denote the class of all graded right R -modules). It is well known that $R\text{-gr}$ is a Grothendieck category.

An R -linear map $f : M \rightarrow N$ is said to be a *graded morphism of degree* τ with $\tau \in G$ if $f(M_\gamma) \subseteq N_{\gamma+\tau}$ for all $\gamma \in G$. Graded morphisms of degree γ build an additive subgroup $\text{HOM}_R(M, N)_\gamma$ of $\text{Hom}_R(M, N)$. Then, $\text{HOM}_R(M, N) = \bigoplus_{\gamma \in G} \text{HOM}_R(M, N)_\gamma$ is a graded abelian group of type G .

We will denote by $\text{Ext}_{R\text{-gr}}^i$ and EXT_R^i the right derived functors of $\text{Hom}_{R\text{-gr}}$ and HOM_R , respectively.

Given a graded left R -module N , the *graded character module* of N is defined as $N^* := \text{HOM}_{\mathbb{Z}}(N, \mathbb{Q}/\mathbb{Z})$, where \mathbb{Q} is the rational numbers field and \mathbb{Z} is the integers ring. It is easy to see that $N^* = \bigoplus_{\gamma \in G} \text{HOM}_{\mathbb{Z}}(N_{-\gamma}, \mathbb{Q}/\mathbb{Z})$.

Let M be a graded right R -module and N a graded left R -module. The abelian group $M \otimes_R N$ may be graded by putting $(M \otimes_R N)_\gamma$ with $\gamma \in G$ to be the additive subgroup generated by elements $x \otimes y$ with $x \in M_\alpha$ and $y \in N_\beta$ such that $\alpha + \beta = \gamma$. The objects of $\mathbb{Z}\text{-gr}$ thus defined will be called the *graded tensor product* of M and N .

If M is a graded left R -module and $\gamma \in G$, then $M(\gamma)$ is the graded left R -module obtained by putting $M(\gamma)_\tau = M_{\tau+\gamma}$ for any $\tau \in G$. The graded module $M(\gamma)$ is called the γ -*suspension* of M . We may regard the γ -suspension as an isomorphism of categories $T_\gamma : R\text{-gr} \rightarrow R\text{-gr}$, given on objects as $T_\gamma(M) = M(\gamma)$ for any $M \in R\text{-gr}$.

Let $R\text{-Mod}$ be the category of left R -modules. The forgetful functor $U : R\text{-gr} \rightarrow R\text{-Mod}$ associates to M the underlying ungraded R -module. This functor has a right adjoint F which associated to $M \in R\text{-Mod}$ the graded R -module $F(M) = \bigoplus_{\gamma \in G} {}^\gamma M$, where each ${}^\gamma M$ is a copy of M written $\{\gamma x : x \in M\}$ with R -module structure defined by $r * {}^\gamma x = {}^{\gamma\tau} (rx)$ for each $r \in R_\gamma$ and $\tau \in G$. If $f : M \rightarrow N$ is R -linear, then $F(f) : F(M) \rightarrow F(N)$ is a graded morphism given by $F(f)(\gamma x) = \gamma f(x)$.

A graded R -module N is called a graded injective (resp. graded flat) module if it is an injective (resp. flat) object in the category $\text{gr-}R$.

For a graded ring R , let \mathcal{F} be a class of graded left R -modules and M be a graded

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left R -module. Following [13, 77], we say that a graded morphism $f : F \rightarrow M$ is an \mathcal{F} -precover of M if $F \in \mathcal{F}$ and $\text{Hom}_{R\text{-gr}}(F', F) \rightarrow \text{Hom}_{R\text{-gr}}(F', M) \rightarrow 0$ is exact for all $F' \in \mathcal{F}$. Moreover, if whenever a graded morphism $g : F \rightarrow F$ such that $fg = f$ is an automorphism of F , then $f : F \rightarrow M$ is called an \mathcal{F} -cover of M . The class \mathcal{F} is called (pre) covering, if each object in $R\text{-gr}$ has an \mathcal{F} -(pre)cover. Dually, the notions of \mathcal{F} -preenvelopes, \mathcal{F} -envelopes and (pre)enveloping are defined.

To any given class \mathcal{Y} of graded right R -modules, denote the classes

$$\mathcal{Y}^\perp = \{X \in \text{gr-}R : \text{Ext}_{\text{gr-}R}^1(Y, X) = 0 \text{ for all } Y \in \mathcal{Y}\}$$

and

$${}^\perp\mathcal{Y} = \{X \in \text{gr-}R : \text{Ext}_{\text{gr-}R}(X, Y) = 0 \text{ for all } Y \in \mathcal{Y}\}.$$

Given two classes of graded right R -modules \mathcal{F} and \mathcal{C} , then we say that the pair $(\mathcal{F}, \mathcal{C})$ is a cotorsion pair in $\text{gr-}R$ if $\mathcal{F}^\perp = \mathcal{C}$ and $\mathcal{F} = {}^\perp\mathcal{C}$. A cotorsion pair $(\mathcal{F}, \mathcal{C})$ is called hereditary if whenever $0 \rightarrow F' \rightarrow F \rightarrow F'' \rightarrow 0$ is exact in $\text{gr-}R$ with $F, F'' \in \mathcal{F}$, then F' is also in \mathcal{F} , or equivalently, if $0 \rightarrow C' \rightarrow C \rightarrow C'' \rightarrow 0$ is an exact sequence in $R\text{-gr}$ with $C, C' \in \mathcal{C}$, then C'' is also in \mathcal{C} .

A duality pair over a graded ring R is a pair $(\mathcal{M}, \mathcal{C})$, where \mathcal{M} is a class of graded left (or right) R -modules and \mathcal{C} is a class of graded right (or left) R -modules, subject to the following conditions:

- (1) For any graded module M , one has $M \in \mathcal{M}$ if and only if $M^* \in \mathcal{C}$.
- (2) \mathcal{C} is closed under direct summands and finite direct sums.

A duality pair $(\mathcal{M}, \mathcal{C})$ is called (co)product-closed, if the class \mathcal{M} is closed under graded direct (co)products, and $(\mathcal{M}, \mathcal{C})$ is called perfect, if it is coproduct-closed, \mathcal{M} is closed under extensions and R belongs to \mathcal{M} .

1.2 Relative FP-gr-injective and gr-flat modules

In [77], the concepts of relative n -FP-gr-injective and n -gr-flat modules were introduced. In this section, we gather some results needed for this thesis that can be found in [77]. We start by giving some general properties of these modules.

Definition 1.2.1 ([77], Defintion 3.1). *Let $n \geq 0$ be an integer. A graded left R -module M is called n -presented if there exists an exact sequence*

$$F_n \rightarrow \cdots \rightarrow F_1 \rightarrow F_0 \rightarrow M \rightarrow 0$$

in R -gr with each F_i is finitely generated projective. A graded ring R is called left n -gr-coherent if each n -presented graded left R -module is $(n + 1)$ -presented.

If M is an n -presented graded left R -module, then there exists an exact sequence

$$F_n \rightarrow \cdots \rightarrow F_1 \rightarrow F_0 \rightarrow M \rightarrow 0$$

in R -gr with each F_i is finitely generated projective. Set $K_n := \text{Im}(F_n \rightarrow F_{n-1})$ and $K_{n-1} := \text{Im}(F_{n-1} \rightarrow F_{n-2})$. Then, we get a short exact sequence

$$(\Delta) : 0 \rightarrow K_n \rightarrow F_{n-1} \rightarrow K_{n-1} \rightarrow 0$$

in R -gr with F_{n-1} is finitely generated projective. It follows that $\text{EXT}_R^n(M, A) \cong \text{EXT}_R^1(K_{n-1}, A)$ for any graded left R -module A .

The modules K_n and K_{n-1} will be called *special finitely gr-generated* and *special finitely gr-presented R -modules*, respectively, and we shall say that the sequence $(\Delta) : 0 \rightarrow K_n \rightarrow F_{n-1} \rightarrow K_{n-1} \rightarrow 0$ in R -gr is a special short exact sequence. Furthermore, a short exact sequence $0 \rightarrow A \rightarrow B \rightarrow C \rightarrow 0$ in R -gr is called *special gr-pure* if the derived sequence

$$0 \rightarrow \text{HOM}_R(K_{n-1}, A) \rightarrow \text{HOM}_R(K_{n-1}, B) \rightarrow \text{HOM}_R(K_{n-1}, C) \rightarrow 0$$

is exact for all special finitely presented graded left R -modules K_{n-1} . In this case, A is referred to as *special gr-pure in B* .

Ungraded n -presented modules have been used by many authors in order to extend some homological notions. For example, in [20], N is called *FP_n -injective* if $\text{Ext}_R^1(M, N) = 0$ for all n -presented left R -modules M . In 2018, Zhao, Gao and Huang in [77] showed that if we similarly use the derived functor EXT^1 to define the *FP_n -gr-injective* and *FP_∞ -gr-injective* modules, then they are just the *FP_n -injective* and *FP_∞ -injective* objects in the class of graded modules, respectively. If M is an n -presented graded left R -module with $n \geq 2$, then $\text{EXT}_R^1(M, N) = \text{Ext}_R^1(M, N)$ for any

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graded R -module N . For this reason, they introduced the concept of n -FP-gr-injective modules as follows:

Definition 1.2.2 ([77], Defintion 3.4). *Let $n \geq 1$ be an integer. A module N in R -gr is called n -FP-gr-injective if $\text{EXT}_R^n(M, N) = 0$ for any n -presented graded left R -module M . A module L in gr - R is called n -gr-flat if $\text{Tor}_n^R(L, M) = 0$ for any n -presented graded left R -module M .*

We denote by $\text{gr-}\mathcal{FI}_n$ (respectively, $\text{gr-}\mathcal{F}_n$) the subcategory of all n -FP-gr-injective (respectively, n -gr-flat) graded left (respectively, right) R -modules.

In general, whenever $1 \leq m \leq n$, every m -FP-gr-injective left R -module is n -FP-gr-injective, and every m -gr-flat right R -module is n -gr-flat. Indeed, there are the following inclusions:

$$\text{gr-}\mathcal{I} \subseteq \text{gr-}\mathcal{FI}_1 \subseteq \text{gr-}\mathcal{FI}_2 \subseteq \cdots \subseteq \text{gr-}\mathcal{FI}_m \subseteq \text{gr-}\mathcal{FI}_{m+1} \subseteq \cdots$$

$$\text{gr-}\mathcal{F} = \text{gr-}\mathcal{F}_1 \subseteq \text{gr-}\mathcal{F}_2 \subseteq \cdots \subseteq \text{gr-}\mathcal{F}_m \subseteq \text{gr-}\mathcal{F}_{m+1} \subseteq \cdots$$

Proposition 1.2.3 ([77], Proposition 3.7). *1. Let $\{N_i\}_{i \in I}$ be a family of graded left R -modules. Then $\prod_{i \in I} N_i$ is n -FP-gr-injective if and only if each N_i is n -FP-gr-injective.*

2. Let $\{L_i\}_{i \in I}$ be a family of graded right R -modules. Then $\bigoplus_{i \in I} L_i$ is n -gr-flat if and only if each L_i is n -gr-flat.

Proposition 1.2.4 ([77], Proposition 3.8). *Let L be a graded right R -module. Then L is n -gr-flat if and only if the graded character R -module L^* is n -FP-gr-injective.*

Proposition 1.2.5 ([77], Proposition 3.9). *The class of n -gr-flat right R -modules is closed under gr-pure submodules and gr-pure quotients.*

Proposition 1.2.6 ([77], Proposition 3.10). *Let $n \geq 1$ be an integer and R be a ring graded by a group G . Then, the following statements are equivalent for a graded left R -module N .*

1. N is n -FP-gr-injective.

2. N is *gr-injective* with respect to all special short exact sequences in $R\text{-gr}$.
3. $N(\sigma)$ is *n-FP-gr-injective* for any $\sigma \in G$.
4. N is *special gr-pure* in any graded left R -module containing it.
5. N is *special gr-pure* in any *gr-injective* left R -module containing it.
6. N is *special gr-pure* in $E_g(N)$, where $E_g(N)$ is the graded injective envelope of N .
7. $N(\sigma)$ is *gr-injective* with respect to all special short exact sequences in $R\text{-gr}$ for any $\sigma \in G$.

We now are in a position to give the main result in this section.

Theorem 1.2.7 ([77], Theorem 3.17). *Let $n \geq 1$ be an integer. Then, the following conditions are equivalent for a graded ring R .*

1. R is *left n-gr-coherent*.
2. Any graded direct product of R is an *n-gr-flat* in $gr\text{-}R$.
3. Any graded direct product of *n-gr-flat* right R -modules is *n-gr-flat* in $gr\text{-}R$.
4. Any direct limit of *n-FP-gr-injective* left R -modules is *n-FP-gr-injective* in $R\text{-gr}$.
5. $\text{Tor}_n^R(\text{HOM}_S(B, C), A) \rightarrow \text{HOM}_S(\text{EXT}_R^n(A, B), C)$ is an isomorphism for any graded ring S with A is *n-presented* in $R\text{-gr}$, $B \in R\text{-gr-}S$, and C is *gr-injective* in $gr\text{-}S$.
6. $\varinjlim_{i \rightarrow m} \text{EXT}_R^n(M, N_i) \rightarrow \text{EXT}_R^n(M, \varinjlim_{i \rightarrow m} N_i)$ is an isomorphism for any *n-presented* graded left R -module M and any direct system $\{N_i\}_{i \in I}$ of graded left R -modules with I directed.
7. $\text{Tor}_n^R(\bigoplus_{i \in I} N_i, M) \rightarrow \bigoplus_{i \in I} \text{Tor}_n^R(N_i, M)$ is an isomorphism for any *n-presented* graded left R -module M and any family of graded right R -modules $\{N_i\}_{i \in I}$.
8. $N \in R\text{-gr}$ is *n-FP-gr-injective* if and only if N^* is *n-gr-flat*.

9. $N \in R\text{-gr}$ is n -FP- gr -injective if and only if N^{**} is n -FP- gr -injective.

10. $N \in \text{gr-}R$ is n - gr -flat if and only if N^{**} is n - gr -flat.

1.3 Gorenstein injective objects

The categorical setting for this Section is that of Grothendieck categories for which our main reference is [71].

Recall that a Grothendieck category is a cocomplete abelian category with a generating set, and with exact direct limits. Throughout this section, \mathcal{G} will denote a Grothendieck category.

To orient the reader, we summarize some standard facts about Grothendieck categories. First, a Grothendieck category is always complete and every object $B \in \mathcal{G}$ has an injective envelope. In particular, \mathcal{G} has enough injectives. A useful fact is that any Grothendieck category is locally small, meaning the lattice of subobjects of any given object A (that is, the class of subobjects of A), denoted by $\mathcal{L}(A)$, is in fact a set (see [71, Proposition IV.6.6]).

To any given class of objects \mathcal{L} of \mathcal{G} we associate its right Ext-orthogonal class,

$$\mathcal{L}^\perp = \{X \in \mathcal{G} \mid \text{Ext}_{\mathcal{G}}^1(L, X) = 0, L \in \mathcal{L}\},$$

and its left Ext-orthogonal class,

$${}^\perp\mathcal{L} = \{X \in \mathcal{G} \mid \text{Ext}_{\mathcal{G}}^1(X, L) = 0, L \in \mathcal{L}\}.$$

In particular, if $\mathcal{L} = \{M\}$ then we simply write ${}^\perp\mathcal{L} = {}^\perp M$ and $\mathcal{L}^\perp = M^\perp$.

Definition 1.3.1 ([31], Definition 2.1). *A complete injective resolution of an object M is an exact sequence of injective objects*

$$\mathbf{E} : \cdots \rightarrow E_1 \rightarrow E_0 \rightarrow E_{-1} \rightarrow \cdots$$

which is $\text{Hom}_{\mathcal{G}}(\mathcal{I}\text{nj}_{\mathcal{G}}, -)$ -exact, with $M = Z_0(\mathbf{E})$, here $Z_0(\mathbf{E})$ is the 0-cycle object of \mathbf{E} , that is, $\text{Ker}(E_0 \rightarrow E_{-1})$ and $\mathcal{I}\text{nj}_{\mathcal{G}}$ is the class of injective objects in \mathcal{G} .

An object G is called Gorenstein injective, if it has a complete injective resolution. We use $\mathcal{GE}_{\mathcal{G}}$ to denote the class of all Gorenstein injective objects.

If \mathbf{E} is a complete injective resolution of an object, then by symmetry, all the kernels of \mathbf{E} are Gorenstein injective objects. It is clear that every injective objects is Gorenstein injective.

Using the definition, we immediately get the following characterization of Gorenstein injective objects which is well known in the case of the category of modules.

Proposition 1.3.2 ([31], Proposition 2.2). *An object G is Gorenstein injective if and only if there exists an exact complex of injectives E_i :*

$$\cdots \rightarrow E_1 \rightarrow E_0 \rightarrow G \rightarrow 0$$

which is $\text{Hom}_{\mathcal{G}}(\text{Inj}_{\mathcal{G}}, -)$ -exact, and $\text{Ext}_{\mathcal{G}}^i(E, G) = 0$ for all $i \geq 1$ and all injective objects E .

In [19], a particular case of Gorenstein injective modules was introduced and these modules were called strongly Gorenstein injectives. The reason behind the introduction of this class of modules was giving a nice characterization for Gorenstein injective modules. Namely, the authors proved that a module is Gorenstein injective if and only if it is a direct summand of a strongly Gorenstein injective module. Here, we extend this fact to Grothendieck categories.

Definition 1.3.3 ([19]). *An object M of \mathcal{G} is said to be strongly Gorenstein injective if it has a complete injective resolution of the form*

$$\cdots \longrightarrow E \xrightarrow{f} E \xrightarrow{f} E \xrightarrow{f} \cdots$$

We use $S\mathcal{GE}_{\mathcal{G}}$ to denote the class of all strongly Gorenstein injective objects of \mathcal{G} .

We establish the following results of strongly Gorenstein injective objects by employing the dual of [19, Theorem 2.7 and Proposition 2.9] for modules.

Proposition 1.3.4. *The following conditions are equivalent for an object M .*

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1. M is strongly Gorenstein injective.
2. There is a short exact sequence $0 \rightarrow M \rightarrow E \rightarrow M \rightarrow 0$ which is $\text{Hom}_{\mathcal{G}}(\text{Inj}_{\mathcal{G}}, -)$ -exact and with E injective.
3. There is a short exact sequence $0 \rightarrow M \rightarrow E \rightarrow M \rightarrow 0$ with E injective and $\text{Ext}_{\mathcal{G}}^1(E', M) = 0$ for every $E' \in \text{Inj}_{\mathcal{G}}$.

Proof. 1. \Leftrightarrow 2. is clear.

For 2. \Leftrightarrow 3. we consider the exact sequence $\cdots \rightarrow \text{Hom}_{\mathcal{G}}(E', E) \rightarrow \text{Hom}_{\mathcal{G}}(E', M) \rightarrow \text{Ext}_{\mathcal{G}}^1(E', M) \rightarrow \text{Ext}_{\mathcal{G}}^1(E', E) = 0 \rightarrow \cdots$ for every $E' \in \text{Inj}_{\mathcal{G}}$. Then, $0 \rightarrow M \rightarrow E \rightarrow M \rightarrow 0$ is $\text{Hom}_{\mathcal{G}}(E', -)$ -exact if and only if $\text{Ext}_{\mathcal{G}}^1(E', M) = 0$ for every injective object E' . ■

Proposition 1.3.5. *An object is Gorenstein injective in \mathcal{G} if and only if it is a direct summand of a strongly Gorenstein injective one.*

Proof. Let G be a Gorenstein injective object. Then, G has a complete injective resolution

$$\mathbf{E} : \cdots \rightarrow E_1 \rightarrow E_0 \rightarrow E_{-1} \rightarrow \cdots$$

Consider the exact sequences

$$0 \rightarrow Z_n(\mathbf{E}) \rightarrow E_n \rightarrow Z_{n-1}(\mathbf{E}) \rightarrow 0$$

and let

$$0 \rightarrow \bigoplus_n Z_n(\mathbf{E}) \rightarrow \bigoplus_n E_n \rightarrow \bigoplus_n Z_{n-1}(\mathbf{E}) \rightarrow 0$$

be their direct sum which is exact since direct sums are exact. Now, let E' be an injective object in \mathcal{G} and consider the following commutative diagram

$$\begin{array}{ccc} \text{Hom}_{\mathcal{G}}(E', \bigoplus_n E_n) & \longrightarrow & \text{Hom}_{\mathcal{G}}(E', \bigoplus_n Z_n(\mathbf{E})) \\ \cong \downarrow & & \downarrow \cong \\ \prod_n \text{Hom}_{\mathcal{G}}(E', E_n) & \longrightarrow & \prod_n \text{Hom}_{\mathcal{G}}(E', Z_n(\mathbf{E})) \end{array}$$

The second row is epic since $0 \rightarrow Z_n(\mathbf{E}) \rightarrow E_n \rightarrow Z_{n-1}(\mathbf{E}) \rightarrow 0$ is $\text{Hom}_{\mathcal{G}}(E', -)$ -exact for every n . Then, the first one is also epic, thus $0 \rightarrow \bigoplus_n Z_n(\mathbf{E}) \rightarrow \bigoplus_n E_n \rightarrow$

$\oplus_n Z_{n-1}(\mathbf{E}) \rightarrow 0$ is also $\text{Hom}_G(E', -)$ -exact. Therefore, $\oplus_n Z_n(\mathbf{E})$ is strongly Gorenstein injective.

The converse holds since the class of Gorenstein injective objects is closed under direct summands (see the proof of [45, Theorem 2.5]) and every strongly Gorenstein injective object is Gorenstein injective. ■

1.4 Gorenstein flat objects in a closed symmetric monoidal Grothendieck category

The categorical setting for this section is that of closed symmetric monoidal Grothendieck categories for which our main reference is [28].

Definition 1.4.1 ([28]). *A monoidal category $(\mathcal{A}, \otimes, 1, \theta, s, r)$ consists of a category \mathcal{A} and*

A1) *a bifunctor $\otimes : \mathcal{A} \times \mathcal{A} \rightarrow \mathcal{A}$,*

A2) *a functorial isomorphism*

$$\theta_{U,V,W} : (U \otimes V) \otimes W \cong U \otimes (V \otimes W)$$

(associativity isomorphism) of functors $\mathcal{A} \times \mathcal{A} \times \mathcal{A} \rightarrow \mathcal{A}$,

A3) *a unit object $1 \in \mathcal{A}$ and functorial isomorphisms*

$$s_V : 1 \otimes V \cong V,$$

$$r_V : V \otimes 1 \cong V$$

for $V \in \mathcal{A}$.

subject to two coherence axioms:

A4) *Pentagon axiom: The following diagram*

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$$\begin{array}{ccc}
 & ((U \otimes V) \otimes W) \otimes Y & \\
 \theta_{U,V,W} \otimes id_Y \swarrow & & \searrow \theta_{(U \otimes V),W,Y} \\
 (U \otimes (V \otimes W)) \otimes Y & & (U \otimes V) \otimes (W \otimes Y) \\
 \theta_{U,(V \otimes W),Y} \downarrow & & \downarrow \theta_{U,V,(W \otimes Y)} \\
 U \otimes (V \otimes W) \otimes Y & \xrightarrow{id_U \otimes \theta_{V,W,Y}} & U \otimes (V \otimes (W \otimes Y))
 \end{array}$$

is commutative for all $U, V, W, Y \in \mathcal{A}$.

A5) Triangle axiom: For any $U, V \in \mathcal{A}$, the diagram

$$\begin{array}{ccc}
 ((U \otimes 1) \otimes V) & \xrightarrow{\theta_{U,1,V}} & U \otimes (1 \otimes V) \\
 \searrow r_U \otimes id_V & & \swarrow id_U \otimes s_V \\
 & U \otimes V &
 \end{array}$$

is commutative.

Definition 1.4.2 ([28]). A symmetry δ for a monoidal category \mathcal{A} is a natural isomorphism $\delta_{UV} : U \otimes V \rightarrow V \otimes U$ in such a way that the following diagrams are commutative:

$$\begin{array}{ccc}
 U \otimes V & \xrightarrow{\delta_{UV}} & V \otimes U \\
 \searrow id_{U \otimes V} & & \downarrow \delta_{VU} \\
 & & U \otimes V
 \end{array}$$

$$\begin{array}{ccccc}
 (U \otimes V) \otimes W & \xrightarrow{\theta_{U,V,W}} & U \otimes (V \otimes W) & \xrightarrow{\delta_{U(V \otimes W)}} & (V \otimes W) \otimes U \\
 \delta_{UV} \otimes id_W \downarrow & & & & \downarrow \theta_{V,W,U} \\
 (V \otimes U) \otimes W & \xrightarrow{\theta_{V,U,W}} & V \otimes (U \otimes W) & \xrightarrow{id_V \otimes \delta_{UW}} & V \otimes (W \otimes U)
 \end{array}$$

$$\begin{array}{ccc}
 1 \otimes V & \xrightarrow{\delta_{1V}} & V \otimes 1 \\
 \searrow s_V & & \swarrow r_V \\
 & V &
 \end{array}$$

A monoidal category \mathcal{A} together with a symmetry is called a symmetric monoidal category.

Interesting results arise when the functor $- \otimes V$ has a right adjoint for any object V in \mathcal{A} .

Definition 1.4.3 ([28]). *A closed monoidal category is a monoidal category \mathcal{A} such that for every object V in \mathcal{A} the functor $- \otimes V : \mathcal{A} \rightarrow \mathcal{A}$ has a right adjoint $\mathcal{H}om_{\mathcal{A}}(V, -) : \mathcal{A} \rightarrow \mathcal{A}$. This means that there exists a bijection between the Hom-sets*

$$\phi : \text{Hom}_{\mathcal{A}}(U \otimes V, W) \xrightarrow{\sim} \text{Hom}_{\mathcal{A}}(U, \mathcal{H}om_{\mathcal{A}}(V, W)) .$$

The functor $\mathcal{H}om_{\mathcal{A}}(-, -) : \mathcal{A}^{op} \times \mathcal{A} \rightarrow \mathcal{A}$ is called the internal Hom functor, and the object $\mathcal{H}om_{\mathcal{A}}(U, W)$ is called the internal Hom of $U, W \in \mathcal{A}$.

Now, we define the flat objects in the category \mathcal{A} .

Definition 1.4.4 ([47]). *An object F in \mathcal{A} is said to be flat if the covariant functor $F \otimes - : \mathcal{A} \rightarrow \mathcal{A}$ preserves monomorphisms.*

Let I be an injective cogenerator for \mathcal{A} and $(-)^+ := \mathcal{H}om_{\mathcal{A}}(-, I) : \mathcal{A} \rightarrow \mathcal{A}$. We use the contravariant functor $(-)^+$ to get the following important result.

Proposition 1.4.5 ([47], theorem 2.3). *The following conditions are equivalent.*

- (i) F is flat in \mathcal{A} .
- (ii) F^+ is an injective object in \mathcal{A} .

Starting now, we assume that \mathcal{A} is a symmetric monoidal closed Grothendieck category with enough flat objects.

Definition 1.4.6 ([33]). *A complete flat resolution of an object M in \mathcal{A} is an exact sequence of flat objects*

$$\mathbf{F} : \cdots \rightarrow F_1 \rightarrow F_0 \rightarrow F_{-1} \rightarrow \cdots$$

which is $(\text{Inj}_{\mathcal{A}} \otimes_{\mathcal{A}} -)$ -exact with, $M = Z_0(\mathbf{F})$.

An object G is called Gorenstein flat, if it has a complete flat resolution. We use $\mathcal{GF}_{\mathcal{A}}$ to denote the class of all Gorenstein flat objects.

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If \mathbf{F} is a complete flat resolution of an object, then by symmetry, all the kernels of \mathbf{F} are Gorenstein flat objects. It is clear that every flat object is Gorenstein flat.

In [19], a particular case of Gorenstein flat modules was introduced and these modules were called strongly Gorenstein flats. The reason behind the introduction of this class of modules was giving a nice characterization for Gorenstein flat modules. Namely, the authors proved that a module is Gorenstein flat if it is a direct summand of a strongly Gorenstein flat module. Here, we extend this fact to closed symmetric monoidal Grothendieck categories.

Definition 1.4.7 ([19], Definition 3.1). *An object M of \mathcal{A} is said to be strongly Gorenstein flat if it has a complete flat resolution of the form*

$$\cdots \longrightarrow F \xrightarrow{f} F \xrightarrow{f} F \xrightarrow{f} \cdots$$

We use $SG\mathcal{F}_{\mathcal{A}}$ to denote the class of all strongly Gorenstein flat objects of \mathcal{A} .

Consequently, the strongly Gorenstein flat objects are simple particular cases of Gorenstein flat objects.

We prove the following characterization of strongly Gorenstein flat objects, which was given in [19, Proposition 3.6] for modules.

Proposition 1.4.8. *The following conditions are equivalent for an object M in \mathcal{A} .*

1. M is strongly Gorenstein flat.
2. There is a short exact sequence $0 \rightarrow M \rightarrow F \rightarrow M \rightarrow 0$ which is $(\text{Inj}_{\mathcal{A}} \otimes_{\mathcal{A}} -)$ -exact and with F flat.

Proof. 1. \Leftrightarrow 2. is clear. ■

Proposition 1.4.9. *[[19], Theorem 3.5] An object is Gorenstein flat in \mathcal{A} if it is a direct summand of a strongly Gorenstein flat one.*

Proof. Let G be a Gorenstein flat object. Then, G has a complete flat resolution

$$\mathbf{F} : \cdots \rightarrow F_1 \rightarrow F_0 \rightarrow F_{-1} \rightarrow \cdots$$

Consider the exact sequences

$$0 \rightarrow Z_n(\mathbf{F}) \rightarrow F_n \rightarrow Z_{n-1}(\mathbf{F}) \rightarrow 0$$

and let

$$0 \rightarrow \bigoplus_n Z_n(\mathbf{F}) \rightarrow \bigoplus_n F_n \rightarrow \bigoplus_n Z_{n-1}(\mathbf{F}) \rightarrow 0$$

be their direct sum which is exact since direct sums are exact and commute with tensor products. Now, let E be an injective object in \mathcal{A} and consider the following commutative diagram

$$\begin{array}{ccc} E \otimes_{\mathcal{A}} (\bigoplus_n Z_n(\mathbf{F})) & \longrightarrow & E \otimes_{\mathcal{A}} (\bigoplus_n F_n) \\ \cong \downarrow & & \downarrow \cong \\ E \otimes_{\mathcal{A}} (\bigoplus_n (Z_n(\mathbf{F}))) & \longrightarrow & E \otimes_{\mathcal{A}} (\bigoplus_n (F_n)) \end{array}$$

The second row is monic since $0 \rightarrow Z_n(\mathbf{E}) \rightarrow F_n \rightarrow Z_{n-1}(\mathbf{E}) \rightarrow 0$ is $(E \otimes_{\mathcal{A}} -)$ -exact for every n . Then, the first one is also monic, thus $0 \rightarrow \bigoplus_n Z_n(\mathbf{F}) \rightarrow \bigoplus_n F_n \rightarrow \bigoplus_n Z_{n-1}(\mathbf{F}) \rightarrow 0$ is also $(E \otimes_{\mathcal{A}} -)$ -exact. Therefore, $\bigoplus_n Z_n(\mathbf{F})$ is strongly Gorenstein flat. ■

We use Proposition 1.4.5 and prove the following important interpretation of the relationship between strongly flat and strongly injective objects in \mathcal{A} .

Proposition 1.4.10. *Let M be an object in \mathcal{A} . If M is strongly Gorenstein flat object in \mathcal{A} , then M^+ is strongly Gorenstein injective object.*

Proof. Let M be a strongly Gorenstein flat object. Then, there exists a short exact sequence of the form $0 \longrightarrow M \xrightarrow{f} F \xrightarrow{g} M \longrightarrow 0$, so we get the following short exact sequence

$$0 \longrightarrow M^+ \xrightarrow{g^+} F^+ \xrightarrow{f^+} M^+ \longrightarrow 0. \text{ Due to the adjunction, the surjectivity of}$$

$$\text{Hom}_{\mathcal{A}}(E \otimes F, I) \xrightarrow{\text{Hom}_{\mathcal{A}}(Id_E \otimes f, I)} \text{Hom}_{\mathcal{A}}(E \otimes M, I)$$

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for every injective object E , implies the surjectivity of

$$\mathrm{Hom}_{\mathcal{A}}(E, F^+) \xrightarrow{\mathrm{Hom}_{\mathcal{A}}(E, f^+)} \mathrm{Hom}_{\mathcal{A}}(E, M^+)$$

Hence, M^+ is strongly Gorenstein injective. ■

1.5 Geometric purity in a closed symmetric monoidal Grothendieck category

In this section, we focus on purity, which arises from a closed symmetric monoidal structure.

In [35], Fox has introduced the notion of pure or \otimes -pure morphisms with respect to a monoidal structure.

Definition 1.5.1 ([35]). *A monomorphism $f : U \rightarrow V$ is called \otimes -pure if $W \otimes f$ is a monomorphism for all $W \in \mathcal{A}$.*

The following results, focusing on geometric purity (\otimes -purity) in \mathcal{A} , have been collected from [47].

Proposition 1.5.2 ([47], Proposition 2.1). *A short exact sequence \mathcal{E} in \mathcal{A} is called \otimes -pure (or simply pure) if for any object M in \mathcal{A} , $M \otimes \mathcal{E}$ remains exact.*

By using the contravariant functor $(-)^+ = \mathcal{H}\mathrm{om}_{\mathcal{A}}(-, I)$, where I is an injective cogenerator of \mathcal{A} , the following result hold.

Proposition 1.5.3 ([47], Proposition 2.1.). *A short exact sequence $\mathcal{E} : 0 \rightarrow A \rightarrow B \rightarrow C \rightarrow 0$ is pure if and only if \mathcal{E}^+ splits*

Definition 1.5.4. *An object E in \mathcal{A} is called pure injective if it is injective with respect to all pure short exact sequences in \mathcal{A} .*

Proposition 1.5.5 ([47], Lemma 2.2). *For any object M in \mathcal{A} , M^+ is pure injective.*

Theorem 1.5.6 ([62], Theorem 5.2.6). *Let \mathcal{A} be a closed symmetric monoidal Grothendieck category. Then, every object can be purely embedded in a pure-injective object, that is, every object has a geometrical pure-injective preenvelope.*

The following result gives an important interpretation of flatness in \mathcal{A} .

Theorem 1.5.7 ([47], theorem 2.3). *The following conditions are equivalent.*

- (i) F is flat in \mathcal{A} .
- (ii) F^+ is an injective object in \mathcal{A} .
- (iii) Any short exact sequence ending in F is pure.

Now, we introduce and study the absolutely pure objects in \mathcal{A} .

Definition 1.5.8. *An object A in \mathcal{A} is called absolutely pure if any monomorphism $A \rightarrow K$ in \mathcal{A} is pure.*

Proposition 1.5.9. *An object A in \mathcal{A} is absolutely pure if and only if it is a pure subobject of an injective object.*

Proof. Assume that A is a pure subobject of an injective object E . Let M be any object of \mathcal{A} , and K be an extension of A . Then, apply the functor $M \otimes -$ to the pushout diagram

$$\begin{array}{ccccc} & & 0 & & \\ & & \downarrow & & \\ 0 & \longrightarrow & A & \longrightarrow & K \\ & & \downarrow & & \downarrow \\ 0 & \longrightarrow & E & \longrightarrow & P \end{array}$$

to get

$$\begin{array}{ccc} M \otimes A & \longrightarrow & M \otimes K \\ \downarrow & & \downarrow \\ M \otimes E & \longrightarrow & M \otimes P \end{array}$$

Thus, $M \otimes A \rightarrow M \otimes K$ is a monomorphism since the morphism $M \otimes A \rightarrow M \otimes E$ is monic by hypothesis, and the morphism $M \otimes E \rightarrow M \otimes P$ is monic since E is injective. The converse is clear. ■

1.6 Geometric purity of sheaves

In this section, we focus on the category of sheaves and the category of quasi-coherent sheaves. To begin, we offer a summary of terminology, with many of these terms can be found in [40].

Let (X, \mathcal{O}_X) be a scheme. We denote by $\mathcal{O}_X\text{-Mod}$ and $\mathcal{Qcoh}(X)$ the categories of \mathcal{O}_X -modules and quasi-coherent \mathcal{O}_X -modules, respectively.

The categories $\mathcal{O}_X\text{-Mod}$ and $\mathcal{Qcoh}(X)$ are known to be Grothendieck. For when $\mathcal{O}_X\text{-Mod}$ and $\mathcal{Qcoh}(X)$ are locally finitely presented, see [64] and [41], respectively.

For a presheaf \mathcal{M} over X , remember that the stalk of \mathcal{M} at $x \in X$ consists of the equivalence classes of the disjoint union of $\mathcal{M}(U)$, where U runs through all open neighborhoods of x ,

$$\mathcal{M}_x = \bigsqcup_{x \in U} \mathcal{M}(U) / \sim$$

such that $r \sim s$ for $r \in \mathcal{M}(U)$ and $s \in \mathcal{M}(V)$ if and only if there exists an open neighborhood $W \subseteq U \cap V$ such that $r|_W = s|_W$. So, an element of \mathcal{M}_x is represented by $t_x := \langle U, t \rangle$ for some open neighborhood U of x such that $t \in \mathcal{M}(U)$. Actually, t_x is the image of the section $t \in \mathcal{M}(U)$ in the stalk \mathcal{M}_x .

A scheme is called quasi-compact if it is quasi-compact as a topological space, meaning that every open cover has a finite subcover. Therefore, $X = \text{Spec}A$, where A is a ring, is a quasi-compact topological space. However, it is not compact because $\text{Spec}A$ is not Hausdorff in general.

Definition 1.6.1. *Let X be a scheme. Then, it is described as follows:*

- **Locally Noetherian:** *If X can be covered by open affine subsets $\text{Spec}(R_i)$ of X , where each R_i is a Noetherian ring.*
- **Noetherian:** *If X is both quasi-compact and locally Noetherian.*

- **Locally coherent:** If X can be covered by open affine subsets $\text{Spec}(R_i)$ of X , where each R_i is a coherent ring.
- **coherent:** If X is both quasi-compact and locally coherent.
- **Quasi-separated:** If the intersection of any two quasi-compact open subsets (or even open affines) is quasi-compact.
- **Semi-separated:** If the intersection of two affine open subsets is again affine.
- **Concentrated:** If it is both quasi-compact and quasi-separated.

Note: All types of schemes given in the definition have a local property, that is, for example, a scheme is locally Noetherian if and only if for every affine open subset $\text{Spec}(R)$ of X , R is a Noetherian ring.

For a morphism of schemes $f : X \rightarrow Y$, if \mathcal{M} is a presheaf on X , then the direct image $f_*(\mathcal{M})$ is a presheaf defined on an open subset U of Y as $f_*(\mathcal{M})(U) := \mathcal{M}(f^{-1}(U))$. If \mathcal{M} is either a sheaf or an \mathcal{O}_X -module, then so is $f_*(\mathcal{M})$. But in general the direct image functor f_* does not preserve quasi-coherence unless X is Noetherian or f is quasi-compact and quasi-separated morphism.

If \mathcal{N} is a presheaf on X , then the presheaf $f^+(\mathcal{N})$ is defined on an open subset $U \subseteq X$ as

$$f^+(\mathcal{N})(U) := \varinjlim_{f(U) \subseteq V} \mathcal{N}(V), \quad \text{where } V \subseteq Y$$

The sheafification functor defined from the category of presheaves to the category of sheaves, preserves the property of being \mathcal{O}_X -module and stalks due to its construction.

For a morphism $f : X \rightarrow Y$ of schemes, if \mathcal{N} is a sheaf on Y , then $f^+(\mathcal{N})$ may not be a sheaf. So we consider its sheafification denoted by $f^{-1}(\mathcal{N})$. It is called the inverse image of \mathcal{N} under f . Then, $(f^{-1}(\mathcal{N}))_x \cong \mathcal{N}_{f(x)}$. If X is an open subset of Y , then $f^{-1}(\mathcal{N}) \cong \mathcal{N}|_X$.

If $\mathcal{N} \in \mathcal{O}_Y\text{-Mod}$, then $f^{-1}(\mathcal{N})$ is an $f^{-1}(\mathcal{O}_Y)$ -module. However, the map $\mathcal{O}_Y \rightarrow$

1.6. GEOMETRIC PURITY OF SHEAVES

$f_*(\mathcal{O}_X)$ corresponds to a map $f^{-1}\mathcal{O}_Y \rightarrow \mathcal{O}_X$ due to the adjoint property. Therefore, to transform it into an \mathcal{O}_X -module, it is tensorized with \mathcal{O}_X ,

$$f^*(\mathcal{N}) := f^{-1}(\mathcal{N}) \otimes_{f^{-1}(\mathcal{O}_Y)} \mathcal{O}_X.$$

Then, we have a functor $f^* : \mathcal{O}_Y\text{-Mod} \rightarrow \mathcal{O}_X\text{-Mod}$ whose stalk at $x \in X$ is $(f^*(\mathcal{N}))_x \cong \mathcal{N}_{f(x)} \otimes_{\mathcal{O}_{Y,f(x)}} \mathcal{O}_{X,x}$. Note that the inverse image functor preserves tensor product.

Recall that if $x \in X$ and M is an $\mathcal{O}_{X,x}$ -module, then the skyscraper sheaf $i_{x,*}(M)$ is an \mathcal{O}_X -module defined as:

$$i_{x,*}(M)(U) = \begin{cases} M & \text{if } x \in U \\ 0 & \text{otherwise} \end{cases} \quad \forall U \subseteq X$$

The stalks of $i_{x,*}(M)$ are $\{0\}$ for all points except at x , where $(i_{x,*}(M))_x = M$. This construction yields a functor $i_{x,*} : \mathcal{O}_{X,x}\text{-Mod} \rightarrow \mathcal{O}_X\text{-Mod}$, which is fully faithful, left exact, preserves injectives and serves as a right adjoint to taking stalks at x .

Let \mathcal{M}, \mathcal{N} be \mathcal{O}_X -modules. The tensor product $\mathcal{M} \otimes_{\mathcal{O}_X} \mathcal{N}$ is defined as the sheafification of the presheaf $U \mapsto \mathcal{M}(U) \otimes_{\mathcal{O}_X(U)} \mathcal{N}(U)$, for each open subset $U \subseteq X$.

There is also an internal hom-functor in $\mathcal{O}_X\text{-Mod}$, $\mathcal{H}om_{\mathcal{O}_X}(-, -)$. The image $\mathcal{H}om_{\mathcal{O}_X}(\mathcal{M}, \mathcal{N})(U)$ on an open subset $U \subseteq X$ is $\mathcal{H}om_{\mathcal{O}_X}(\mathcal{M}|_U, \mathcal{N}|_U)$.

For any \mathcal{O}_X -modules \mathcal{M}, \mathcal{N} , and \mathcal{H} , there are canonical isomorphisms:

$$\begin{aligned} \mathcal{H}om_{\mathcal{O}_X}(\mathcal{M}, \mathcal{H}om_{\mathcal{O}_X}(\mathcal{N}, \mathcal{H})) &\cong \mathcal{H}om_{\mathcal{O}_X}(\mathcal{M} \otimes_{\mathcal{O}_X} \mathcal{N}, \mathcal{H}), \\ \mathcal{H}om_{\mathcal{O}_X}(\mathcal{M}, \mathcal{H}om_{\mathcal{O}_X}(\mathcal{N}, \mathcal{H})) &\cong \mathcal{H}om_{\mathcal{O}_X}(\mathcal{M} \otimes_{\mathcal{O}_X} \mathcal{N}, \mathcal{H}), \\ \mathcal{H}om_{\mathcal{O}_X}(\mathcal{O}_X, \mathcal{M}) &\cong \mathcal{M}(X), \\ \mathcal{H}om_{\mathcal{O}_X}(\mathcal{O}_X, \mathcal{M}) &\cong \mathcal{M}. \end{aligned}$$

That is, the category $\mathcal{O}_X\text{-Mod}$ carries a closed symmetric monoidal structure.

Since $\mathfrak{Q}\text{coh}(X)$ and $\mathcal{O}_X\text{-Mod}$ are Grothendieck categories and the inclusion functor $\mu : \mathfrak{Q}\text{coh}(X) \rightarrow \mathcal{O}_X\text{-Mod}$ preserves limits, colimits, and has a right adjoint, $\mathbf{C} : \mathcal{O}_X\text{-Mod} \rightarrow \mathfrak{Q}\text{coh}(X)$ known as the coherator functor. It contributes a closed symmetric monoidal structure on $\mathfrak{Q}\text{coh}(X)$ because the tensor product of two quasi-coherent sheaves is quasi-coherent. The internal hom on $\mathfrak{Q}\text{coh}(X)$ is defined as $\mathbf{C}(\mathcal{H}\text{om}_{\mathcal{O}_X}(-, -))$. Then, geometrical pure monomorphisms in $\mathcal{O}_X\text{-Mod}$ and $\mathfrak{Q}\text{coh}(X)$ have a special characterization.

Proposition 1.6.2 ([29], Proposition 3.2). *Let $0 \rightarrow \mathcal{M} \rightarrow \mathcal{N}$ be an exact sequence in $\mathcal{O}_X\text{-Mod}$. The following conditions are equivalent:*

1. *The sequence is pure exact.*
2. *For each $x \in X$, the monomorphism $0 \rightarrow \mathcal{M}_x \rightarrow \mathcal{N}_x$ in $\mathcal{O}_{X,x}\text{-Mod}$ is pure.*

The subsequent proposition demonstrates that geometrically pure monomorphisms in $\mathcal{O}_X\text{-Mod}$, with both their domains and codomains belonging to $\mathfrak{Q}\text{coh}(X)$, maintain the purity property over sections.

Proposition 1.6.3 ([29], Propositions 3.3 and 3.4). *Let X be a scheme and $\mathcal{M}, \mathcal{N} \in \mathfrak{Q}\text{coh}(X)$. The following conditions are equivalent:*

1. *$0 \rightarrow \mathcal{M} \rightarrow \mathcal{N}$ is pure exact.*
2. *$0 \rightarrow \mathcal{M}(U) \rightarrow \mathcal{N}(U)$ is pure in $\mathcal{O}_X(U)\text{-Mod}$ for each open affine $U \subseteq X$.*
3. *There exists an open covering of X by affine open sets $\mathcal{U} = \{U_i\}$ such that $0 \rightarrow \mathcal{M}(U_i) \rightarrow \mathcal{N}(U_i)$ is pure in $\mathcal{O}_X(U_i)\text{-Mod}$.*
4. *$0 \rightarrow \mathcal{M}_x \rightarrow \mathcal{N}_x$ is pure in $\mathcal{O}_{X,x}\text{-Mod}$, for each $x \in X$.*

Proposition 1.6.4. [29, Corollary 4.6] *For any $\mathcal{M} \in \mathcal{O}_X\text{-Mod}$ the evaluation map $\nu : \mathcal{M} \rightarrow \mathcal{M}^{++}$ is a pure monomorphism.*

1.7 Absolutely pure sheaves

In this section we will recall some results of (locally) absolutely pure sheaves in both $\mathcal{O}_X\text{-Mod}$ and in $\mathfrak{Qcoh}(X)$ in terms of stalkwise pure exact sequences.

Definition 1.7.1 ([29], Definition 5.1). *Let X be a scheme.*

1. *Let \mathcal{M} be in $\mathcal{O}_X\text{-Mod}$. \mathcal{M} is absolutely pure in $\mathcal{O}_X\text{-Mod}$ if every exact sequence $0 \rightarrow \mathcal{M} \rightarrow \mathcal{N}$ in $\mathcal{O}_X\text{-Mod}$ is pure exact in $\mathcal{O}_X\text{-Mod}$.*
2. *Let \mathcal{M} be a quasi-coherent sheaf on X . \mathcal{M} is called absolutely pure in $\mathfrak{Qcoh}(X)$ if every exact sequence $0 \rightarrow \mathcal{M} \rightarrow \mathcal{N}$ in $\mathfrak{Qcoh}(X)$ is pure exact.*
3. *Let \mathcal{M} be a quasi-coherent sheaf on X . \mathcal{M} is called locally absolutely pure if $\mathcal{M}(U)$ is absolutely pure over $\mathcal{O}_X(U)$ for every affine open $U \subseteq X$.*

Proposition 1.7.2 ([29], Lemma 5.3). *Let \mathcal{M} be an \mathcal{O}_X -module. The following assertions are equivalent:*

1. *\mathcal{M} is absolutely pure in $\mathcal{O}_X\text{-Mod}$.*
2. *$\mathcal{M}|_{U_i}$ is absolutely pure in $\mathcal{O}_{U_i}\text{-Mod}$ for a cover $\{U_i\}$ of X .*

Proposition 1.7.3 ([29], Lemma 5.4). *Let \mathcal{M} be an \mathcal{O}_X -module. If \mathcal{M}_x is absolutely pure for all $x \in X$, then \mathcal{M} is absolutely pure in $\mathcal{O}_X\text{-Mod}$.*

Let $X = \text{Spec}(A)$ be an affine scheme. The next proposition shows that in order to check that a quasi-coherent \mathcal{O}_X -module \widetilde{M} is absolutely pure, it suffices that its restrictions $\widetilde{M}|_{D(f_i)}$, $i = 1, \dots, n$, be absolutely pure, where $\bigcup_{i=1}^n D(f_i) = X$, and $f_1, \dots, f_n \in A$.

Proposition 1.7.4. [29, Proposition 5.5] *Let A be a ring and f_1, f_2, \dots, f_n a finite number of elements of A which generate the unit ideal. Let M be an A -module. If M_{f_i} is absolutely pure over A_{f_i} for every $i = 1, \dots, n$, then M is absolutely pure over A .*

Let $X = \text{Spec}(A)$ be an affine scheme. The following result shows that in order to check that a quasi-coherent \mathcal{O}_X -module \widetilde{M} is absolutely pure, it suffices to check that, for each $\mathfrak{p} \in X$, each stalk $\widetilde{M}_{\mathfrak{p}}$ is an absolutely pure $\mathcal{O}_{X,\mathfrak{p}}$ -module.

Proposition 1.7.5. *[[29], Proposition 5.6] If $\widetilde{M}_{\mathfrak{p}}$ is absolutely pure over $A_{\mathfrak{p}}$ for every prime ideal \mathfrak{p} , then \widetilde{M} is absolutely pure over A .*

Neither Propositions 1.7.4 and 1.7.5 do not assume any condition on the ring A . Their converses are not true in general. However, they are if A is coherent. So, it makes sense to define a notion of locally absolutely pure quasi-coherent sheaves over a locally coherent scheme. So, over a locally coherent scheme, the next proposition states that in order to prove whether a quasi-coherent sheaf is locally absolutely pure, it is sufficient to look at some cover by affine subsets of X . And these show that locally absolute purity is a stalkwise property.

Proposition 1.7.6 ([29], Proposition 5.7). *Let (X, \mathcal{O}_X) be a locally coherent scheme. Then, the following conditions are equivalent for a quasi-coherent sheaf \mathcal{M} :*

1. $\mathcal{M}(U)$ is absolutely pure for every affine U .
2. $\mathcal{M}(U_i)$ is absolutely pure for all $i \in I$ for some cover $\{U_i\}_{i \in I}$ of affine open subsets.
3. \mathcal{M}_x is absolutely pure for all $x \in X$.

Proposition 1.7.7. *[29, Proposition 5.10]*

Let X be a locally coherent scheme. If the class of injective sheaves in $\mathcal{O}_X\text{-Mod}$ is equal to the class of absolutely pure sheaves in $\mathcal{O}_X\text{-Mod}$, then X is a locally Noetherian scheme.

1.8 Flat sheaves

In this section, we will address only some basic results on flat objects in both $\mathcal{O}_X\text{-Mod}$ and $\mathfrak{Qcoh}(X)$.

Recall, from Definition 1.4.4, that an object \mathcal{M} in $\mathcal{O}_X\text{-Mod}$ (resp. in $\mathfrak{Qcoh}(X)$) is flat if the functor $\mathcal{M} \otimes_{\mathcal{O}_X} -$ is exact in $\mathcal{O}_X\text{-Mod}$ (resp. in $\mathfrak{Qcoh}(X)$).

In the following result, we can characterize flatness in $\mathcal{O}_X\text{-Mod}$ by looking at the stalks.

1.8. FLAT SHEAVES

Proposition 1.8.1. *Let (X, \mathcal{O}_X) be a scheme and \mathcal{M} be an object in $\mathcal{O}_X\text{-Mod}$. The following conditions are equivalent.*

- (1) \mathcal{M} is flat in $\mathcal{O}_X\text{-Mod}$.
- (2) For each open subset U of X , $\mathcal{M}|_U$ is flat in $\mathcal{O}_U\text{-Mod}$.
- (3) For any cover $\mathcal{U} = \{U_i\}_{i \in I}$ of X , $\mathcal{M}|_{U_i}$ is flat in $\mathcal{O}_{U_i}\text{-Mod}$.
- (4) There exists an open cover $\mathcal{U} = \{U_i\}_{i \in I}$ of X such that $\mathcal{M}|_{U_i}$ is flat in $\mathcal{O}_{U_i}\text{-Mod}$ for every open subset $U_i \in \mathcal{U}$.
- (5) For each $x \in X$, \mathcal{M}_x is a flat $\mathcal{O}_{X,x}$ -module.

Proof. (1) \Rightarrow (2): Let $\mathcal{E} : 0 \rightarrow \mathcal{A} \rightarrow \mathcal{B} \rightarrow \mathcal{C} \rightarrow 0$ be a short exact sequence in $\mathcal{O}_U\text{-Mod}$, and let $i_U : U \hookrightarrow X$ be the inclusion map. Then, we apply the direct image functor $i_{U,*}$ on \mathcal{E} to get the following short exact sequence in $\mathcal{O}_X\text{-Mod}$:

$$i_{U,*}(\mathcal{E}) : 0 \rightarrow i_{U,*}(\mathcal{A}) \rightarrow i_{U,*}(\mathcal{B}) \rightarrow i_{U,*}(\mathcal{C}) \rightarrow 0.$$

But, \mathcal{M} is flat in $\mathcal{O}_X\text{-Mod}$, then $\mathcal{M} \otimes i_{U,*}(\mathcal{E})$ is short exact in $\mathcal{O}_X\text{-Mod}$. Since the restriction functor $(-)|_U$ is exact, commutes with tensor product and for every $\mathcal{N} \in \mathcal{O}_U\text{-Mod}$, $(i_{U,*}(\mathcal{N})|_U) \cong \mathcal{N}$, then $\mathcal{M}|_U \otimes \mathcal{E}$ is a short exact sequence in $\mathcal{O}_U\text{-Mod}$. Hence, $\mathcal{M}|_U$ is flat in $\mathcal{O}_U\text{-Mod}$.

(2) \Rightarrow (3) \Rightarrow (4): are clear.

(4) \Rightarrow (5): Let $x \in X$, and $\mathcal{U} = \{U_i\}_{i \in I}$ be an open cover of X that verifies (4). Then, there exists an open subset U_i such that $x \in U_i$. We consider the short exact sequence $E : 0 \rightarrow A \rightarrow B \rightarrow C \rightarrow 0$ in $\mathcal{O}_{X,x}\text{-Mod}$. We get that the sequence $i_{x,*}E$ is short exact in $\mathcal{O}_{U_i}\text{-Mod}$ by applying the skyscraper functor $i_{x,*}$ on E . So $\mathcal{M}|_{U_i} \otimes i_{x,*}E$ is short exact by hypothesis. Consequently, $\mathcal{M}_x \otimes (i_{x,*}E)_x$ is also short exact in $\mathcal{O}_{X,x}\text{-Mod}$. But, $(i_{x,*}E)_x \cong E$, which implies that $\mathcal{M}_x \otimes E$ is a short exact sequence. Thus, \mathcal{M}_x is flat in $\mathcal{O}_{X,x}\text{-Mod}$.

(5) \Rightarrow (1): clear. ■

Definition 1.8.2 ([29], Definition 3.10). *A quasi-coherent \mathcal{O}_X -module \mathcal{M} is called tensor flat if every short exact sequence in $\mathcal{D}\text{co}\mathfrak{h}(X)$ ending in \mathcal{M} is pure exact.*

Proposition 1.8.3 ([29], Proposition 3.11). *Let $\mathcal{M} \in \mathfrak{Qcoh}(X)$. If \mathcal{M} is flat, then it is also tensor flat. In case X is semi-separated, the converse also holds.*

n -GR-COHERENT RINGS AND GORENSTEIN GRADED MODULES

Let R be a ring graded by an (abelian) group G and $n \geq 1$ an integer. In this chapter, we introduce and study the notions of Gorenstein n -FP-gr-injective and Gorenstein n -gr-flat modules by using the notion of special finitely presented graded modules. On n -gr-coherent rings, we investigate the relationships between Gorenstein n -FP-gr-injective and Gorenstein n -gr-flat modules. Among other results, we prove that any graded module in R -gr (resp. gr- R) admits Gorenstein n -FP-gr-injective (resp. Gorenstein n -gr-flat) cover and preenvelope.

2.1 Gorenstein n -FP-gr-injective and Gorenstein n -gr-flat modules

In this section, we introduce and study Gorenstein n -FP-gr-injective and Gorenstein n -gr-flat modules which are defined as follows:

Definition 2.1.1. *Let R be a graded ring and $n \geq 1$ an integer. Then, a module M in R -gr is called Gorenstein n -FP-gr-injective if there exists an exact sequence of n -FP-gr-injective modules in R -gr of this form:*

$$\mathbf{A} = \cdots \longrightarrow A_1 \longrightarrow A_0 \longrightarrow A^0 \longrightarrow A^1 \longrightarrow \cdots$$

with $M = \ker(A^0 \rightarrow A^1)$ such that $\text{HOM}_R(K_{n-1}, \mathbf{A})$ is an exact sequence whenever K_{n-1} is special gr -presented with $gr\text{-pd}_R(K_{n-1}) < \infty$. The class of Gorenstein n -FP- gr -injective will be denoted by $\mathcal{G}_{gr\text{-}\mathcal{FI}_n}$.

A module N in $gr\text{-}R$ is called Gorenstein n - gr -flat if there exists an exact sequence of n - gr -flat modules in $gr\text{-}R$ of this form:

$$\mathbf{F} = \cdots \longrightarrow F_1 \longrightarrow F_0 \longrightarrow F^0 \longrightarrow F^1 \longrightarrow \cdots$$

with $N = \ker(F^0 \rightarrow F^1)$ such that $\mathbf{F} \otimes_R K_{n-1}$ is an exact sequence whenever K_{n-1} is special gr -presented in $R\text{-}gr$ with $gr\text{-fd}_R(K_{n-1}) < \infty$.

The class of Gorenstein n -FP- gr -flat will be denoted by $\mathcal{G}_{gr\text{-}\mathcal{F}_n}$.

In the ungraded case, the R -modules A_i and A^i (resp. F_i and F^i) as in the definition above, are called n -FP-injective (resp. n -flat). Also, R -modules M and N are called Gorenstein n -FP-injective and Gorenstein n -flat, respectively, and K_{n-1} is special presented with respect to any n -presented left R -module U .

Remark 2.1.2. *Let R be a graded ring. Then:*

- (1) $gr\text{-}\mathcal{I} \subseteq gr\text{-}\mathcal{FI}_1 \subseteq gr\text{-}\mathcal{FI}_2 \subseteq \cdots \subseteq gr\text{-}\mathcal{FI}_n \subseteq \mathcal{G}_{gr\text{-}\mathcal{FI}_n}$. But, Gorenstein n -FP- gr -injective R -modules need not be gr -injective, see Example 2.1.3(1). Also, $gr\text{-}\mathcal{F} \subseteq gr\text{-}\mathcal{F}_1 \subseteq gr\text{-}\mathcal{F}_2 \subseteq \cdots \subseteq gr\text{-}\mathcal{F}_n \subseteq \mathcal{G}_{gr\text{-}\mathcal{F}_n}$.

In general, every Gorenstein n -FP- gr -injective (resp. Gorenstein n - gr -flat) R -module is not n -FP- gr -injective (resp. n - gr -flat), except in a certain state, see Proposition 2.1.17.

- (2) $\mathcal{G}_{gr\text{-}\mathcal{FI}_1} \subseteq \mathcal{G}_{gr\text{-}\mathcal{FI}_2} \subseteq \cdots \subseteq \mathcal{G}_{gr\text{-}\mathcal{FI}_n}$ and $\mathcal{G}_{gr\text{-}\mathcal{F}_1} \subseteq \mathcal{G}_{gr\text{-}\mathcal{F}_2} \subseteq \cdots \subseteq \mathcal{G}_{gr\text{-}\mathcal{F}_n}$. But for any integers $m > n$, Gorenstein m -FP- gr -injective (resp. Gorenstein m - gr -flat) R -module need not be Gorenstein n -FP- gr -injective (resp. Gorenstein n - gr -flat), see Example 2.1.3(2, 3).

- (3) In Definition 2.1.1, it is clear that $\ker(A_i \rightarrow A_{i-1})$ and $\ker(A^i \rightarrow A^{i+1})$ are Gorenstein n -FP- gr -injective, and $\ker(F_i \rightarrow F_{i-1})$, $\ker(F^i \rightarrow F^{i+1})$ are Gorenstein n - gr -flat for any $i \geq 1$.

2.1. GORENSTEIN N -FP-GR-INJECTIVE AND GORENSTEIN N -GR-FLAT MODULES

It is known that the trivial extension of a commutative ring A by an A -module M , $R = A \ltimes M$, is a \mathbb{Z}_2 -graded ring, see [9, 10].

Example 2.1.3. (1) Let K be a field with characteristic $p \neq 0$ and let $G = \cup_{k \geq 1} G_k$, where G_k is the cyclic group with generator a_k , the order of a_k is p^k and $a_k = a_{k+1}^p$. Let $R = K[G]$. Then, by Remark 2.1.2, $R[H]$ is Gorenstein n -FP-gr-injective for every group H , since by [12, Example iii], $R[H]$ is n -FP-gr-injective but it is not gr-injective.

(2) Let A be a field, E a nonzero A -vector space and $R = A \ltimes E$ be a trivial extension of A by E . If $\dim_A E = 1$, then by Remark 2.1.2, every R -module in R -gr is Gorenstein n -FP-gr-injective, see [1, Corollary 2.2]. If E is an A -vector space with infinite rank, then by [53, Theorem 3.4], every 2-presented in R -gr is projective. So, every module in R -gr is 2-FP-gr-injective and hence, every module in R -gr is Gorenstein 2-FP-gr-injective. If every module in R -gr is Gorenstein 1-FP-gr-injective, then R is gr-regular, contradiction.

(3) Let R be a graded ring with l -FP-gr-dim $R \leq 1$ but not gr-regular, for example, let $R = K[X]$, where K is a field. Then, by Theorem 2.1.15, every module is Gorenstein 2-gr-flat, and there is a module that is not Gorenstein 1-gr-flat, since l -FP-gr-dim(R) ≤ 1 , see Proposition 2.1.17 and [77, Example 3.6].

We start with the result which proves that the behaviour of Gorenstein n -FP-gr-injective (resp. Gorenstein n -gr-flat) modules in short exact sequences is the same as the one of the classical homological notions.

Proposition 2.1.4. *Let R be a graded ring. Then:*

- (1) *For every short exact sequence $0 \rightarrow A \rightarrow B \rightarrow C \rightarrow 0$ in R -gr, B is Gorenstein n -FP-gr-injective if A and C are Gorenstein n -FP-gr-injective.*
- (2) *For every short exact sequence $0 \rightarrow A \rightarrow B \rightarrow C \rightarrow 0$ in gr - R , B is Gorenstein n -gr-flat if A and C are Gorenstein n -gr-flat.*

Proof. (1) By Definition 2.1.1, there is an exact sequence $\cdots \rightarrow A_1 \rightarrow A_0 \rightarrow A^0 \rightarrow A^1 \rightarrow \cdots$ of n -FP-gr-injective modules in R -gr, where $A = \text{Ker}(A^0 \rightarrow A^1)$, $K'_i = \text{Ker}(A_i \rightarrow A_{i-1})$ and $(K^i)' = \text{Ker}(A^i \rightarrow A^{i+1})$. Also, there is an exact sequence $\cdots \rightarrow C_1 \rightarrow C_0 \rightarrow C^0 \rightarrow C^1 \rightarrow \cdots$ of n -FP-gr-injective modules in R -gr, where $C =$

$\text{Ker}(C^0 \rightarrow C^1)$, $K_i'' = \text{Ker}(C_i \rightarrow C_{i-1})$ and $(K^i)'' = \text{Ker}(C^i \rightarrow C^{i+1})$. For any n -presented graded left module P , $\text{EXT}_R^n(P, A_i \oplus C_i) = \text{EXT}_R^n(P, A_i) \oplus \text{EXT}_R^n(P, C_i) = 0$, then $A_i \oplus C_i$ is n -FP-gr-injective for any $i \geq 0$. Similarly, $A^i \oplus C^i$ is n -FP-gr-injective for any $i \geq 0$. Therefore, there is an exact sequence

$$\mathcal{Y} = \cdots \longrightarrow A_1 \oplus C_1 \longrightarrow A_0 \oplus C_0 \longrightarrow A^0 \oplus C^0 \longrightarrow A^1 \oplus C^1 \longrightarrow \cdots$$

of n -FP-gr-injective modules in $R\text{-gr}$, where $B = \text{Ker}(A^0 \oplus C^0 \rightarrow A^1 \oplus C^1)$, $K_i = K_i' \oplus K_i'' = \text{Ker}(A_i \oplus C_i \rightarrow A_{i-1} \oplus C_{i-1})$ and $K^i = (K^i)' \oplus (K^i)'' = \text{Ker}(A^i \oplus C^i \rightarrow A^{i+1} \oplus C^{i+1})$. Let K_{n-1} be a special gr-presented with $\text{gr-pd}_R(K_{n-1}) < \infty$. Then, $\text{EXT}_R^1(K_{n-1}, B) = 0$, and also we have: $\text{EXT}_R^1(K_{n-1}, K_i) = \text{EXT}_R^1(K_{n-1}, K_i' \oplus K_i'') = 0$. Similarly, $\text{EXT}_R^1(K_{n-1}, K^i) = 0$. Consequently, $\text{HOM}_R(K_{n-1}, \mathcal{Y})$ is exact and so, B is Gorenstein n -FP-gr-injective.

(2) By Definition 2.1.1, there is an exact sequence $\cdots \rightarrow A_1 \rightarrow A_0 \rightarrow A^0 \rightarrow A^1 \rightarrow \cdots$ of n -gr-flat modules in $\text{gr-}R$, where $A = \text{Ker}(A^0 \rightarrow A^1)$, $K_i' = \text{Ker}(A_i \rightarrow A_{i-1})$ and $(K^i)' = \text{Ker}(A^i \rightarrow A^{i+1})$. Also, there is an exact sequence $\cdots \rightarrow C_1 \rightarrow C_0 \rightarrow C^0 \rightarrow C^1 \rightarrow \cdots$ of n -gr-flat modules in $\text{gr-}R$, where $C = \text{Ker}(C^0 \rightarrow C^1)$, $K_i'' = \text{Ker}(C_i \rightarrow C_{i-1})$ and $(K^i)'' = \text{Ker}(C^i \rightarrow C^{i+1})$. Similarly to (1), there is an exact sequence

$$\mathcal{Y} = \cdots \longrightarrow A_1 \oplus C_1 \longrightarrow A_0 \oplus C_0 \longrightarrow A^0 \oplus C^0 \longrightarrow A^1 \oplus C^1 \longrightarrow \cdots$$

of n -gr-flat modules in $\text{gr-}R$, where $B = \text{Ker}(A^0 \oplus C^0 \rightarrow A^1 \oplus C^1)$, and if K_{n-1} is special gr-presented with $\text{gr-fd}_R(K_{n-1}) < \infty$, then $\mathcal{Y} \otimes_R K_{n-1}$ is exact and so, B is Gorenstein n -gr-flat. ■

Transfer results of n -FP-injective and Gorenstein n -FP-injective modules with respect to the functor F is given in the following result.

Proposition 2.1.5. *Let R be a ring graded by a group G .*

- (1) *If M is an n -FP-injective left R -module, then $F(M)$ is n -FP-gr-injective.*
- (2) *If M is a Gorenstein n -FP-injective left R -module, then $F(M)$ is Gorenstein n -FP-gr-injective.*

2.1. GORENSTEIN N -FP-GR-INJECTIVE AND GORENSTEIN N -GR-FLAT MODULES

Proof. (1) If $0 \rightarrow K_n \rightarrow F_{n-1} \rightarrow K_{n-1} \rightarrow 0$ is special short exact sequence in R -gr with respect to an n -presented graded left R -module U , then similarly to the proof of [76, Lemma 2.3], $0 = \text{Ext}_{R\text{-gr}}^1(K_{n-1}, F(M)(\sigma)) = \text{Ext}_{R\text{-gr}}^n(U, F(M)(\sigma))$, and hence by [77, Proposition 3.10], $F(M)$ is n -FP-gr-injective.

(2) Let M be a Gorenstein n -FP-injective left R -module. Then, there exists an exact sequence of n -FP-injective left modules :

$$\mathbf{B} = \cdots \rightarrow B_1 \rightarrow B_0 \rightarrow B^0 \rightarrow B^1 \rightarrow \cdots$$

with $M = \ker(B^0 \rightarrow B^1)$ such that $\text{Hom}_R(K'_{n-1}, \mathbf{B})$ is an exact sequence whenever K'_{n-1} is special finitely presented in R -gr with $\text{pd}_R(K'_{n-1}) < \infty$. By (1), $F(B_i)$ and $F(B^i)$ are n -FP-gr-injective for any $i \geq 0$. Since the functor F is exact, we get the following exact sequence

$$\mathbf{F}(\mathbf{B}) = \cdots \rightarrow F(B_1) \rightarrow F(B_0) \rightarrow F(B^0) \rightarrow F(B^1) \rightarrow \cdots$$

of n -FP-gr-injective left R -modules with $F(M) = \ker(F(B^0) \rightarrow F(B^1))$. If K_{n-1} is special gr-presented left module with $\text{gr-pd}_R(K_{n-1}) < \infty$, then $U(K_{n-1})$ is finitely presented with $\text{pd}_R(U(K_{n-1})) < \infty$. By hypothesis, $\text{Hom}_R(U(K_{n-1}), \mathbf{B})$ is exact. So, $\text{Hom}_R(U(K_{n-1}), \mathbf{B}) = \text{Hom}_{R\text{-gr}}(K_{n-1}, \mathbf{F}(\mathbf{B}))$ implies that $\text{Hom}_{R\text{-gr}}(K_{n-1}, \mathbf{F}(\mathbf{B}))$ is exact and consequently, the isomorphism

$$\text{HOM}_R(K_{n-1}, \mathbf{F}(\mathbf{B})) = \bigoplus_{\sigma \in G} \text{HOM}_R(K_{n-1}, \mathbf{F}(\mathbf{B}))_{\sigma} \cong \bigoplus_{\sigma \in G} \text{Hom}_{R\text{-gr}}(K_{n-1}, \mathbf{F}(\mathbf{B})(\sigma))$$

implies that $F(M)$ is Gorenstein n -FP-gr-injective. ■

Now, we give a characterization of a graded ring R on which n -presented modules in R -gr with $\text{gr-pd}_R(U) < \infty$ (resp. $\text{gr-fd}_R(U) < \infty$) are $(n+1)$ -presented. For this, we need the following lemma.

Lemma 2.1.6. *Assume that every n -presented module in R -gr with $\text{gr-fd}_R(U) < \infty$ is $(n+1)$ -presented. Then, for any $t \geq 1$:*

- (1) $\text{EXT}_R^t(K_{n-1}, M) = 0$ for any Gorenstein n -FP-gr-injective module M and any special gr-presented module K_{n-1} in R -gr with $\text{gr-pd}_R(K_{n-1}) < \infty$.

- (2) $\text{Tor}_t^R(M, K_{n-1}) = 0$ for any Gorenstein n -gr-flat module M and any special gr-presented module K_{n-1} in R -gr with $\text{gr-fd}_R(K_{n-1}) < \infty$.

Proof. (1) Assume that K_{n-1} is a special gr-presented module with $\text{gr-pd}_R(K_{n-1}) \leq m$ respect to any n -presented module U in R -gr. If M is a Gorenstein n -FP-gr-injective left R -module, then, there is a left n -FP-gr-injective resolution of M in R -gr. So, we have:

$$0 \longrightarrow N \longrightarrow E_{m-1} \longrightarrow \cdots \longrightarrow E_0 \longrightarrow M \longrightarrow 0,$$

where every E_j is n -FP-gr-injective for every $0 \leq j \leq m-1$. Since $\text{gr-fd}_R(U) < \infty$, U is $(n+1)$ -presented, and so $\text{EXT}_R^{i+1}(K_{n-1}, E_j) = 0$ for any $i \geq 0$. Hence, $\text{EXT}_R^{i+1}(K_{n-1}, M) \cong \text{EXT}_R^{m+i+1}(K_{n-1}, N)$, and since $\text{gr-pd}_R(K_{n-1}) \leq m$, it follows that, $\text{EXT}_R^{i+1}(K_{n-1}, M) = 0$ for any $i \geq 0$.

(2) Assume that K_{n-1} is a special gr-presented module with $\text{gr-fd}_R(K_{n-1}) \leq m$ for any n -presented module U in R -gr. If M is a Gorenstein n -gr-flat right R -module, then there is a right n -gr-flat resolution of M in $\text{gr-}R$ of the form:

$$0 \longrightarrow M \longrightarrow F^0 \longrightarrow \cdots \longrightarrow F^{m-1} \longrightarrow N \longrightarrow 0,$$

where every F^j is n -gr-flat for every $0 \leq j \leq m-1$. Since U is $(n+1)$ -presented, we have $\text{Tor}_{i+1}^R(F^j, K_{n-1}) = 0$ for any $i \geq 0$. If $\text{gr-fd}_R(K_{n-1}) \leq m$, so $\text{Tor}_{i+1}^R(M, K_{n-1}) \cong \text{Tor}_{m+i+1}^R(N, K_{n-1}) = 0$, and hence $\text{Tor}_{i+1}^R(M, K_{n-1}) = 0$ for any $i \geq 0$. ■

Theorem 2.1.7. *Let R be a graded ring. Then, the following statements are equivalent:*

- (1) *Every n -presented module in R -gr with $\text{gr-pd}_R(U) < \infty$ is $(n+1)$ -presented;*
- (2) *For every short exact sequence $0 \rightarrow A \rightarrow B \rightarrow C \rightarrow 0$ in R -gr, C is Gorenstein n -FP-gr-injective if A and B are Gorenstein n -FP-gr-injective.*

Proof. (1) \implies (2) If B is a Gorenstein n -FP-gr-injective module in R -gr, then by Definition 2.1.1 and Remark 2.1.2, there is an exact sequence $0 \rightarrow K \rightarrow B_0 \rightarrow B \rightarrow 0$ in R -gr, where B_0 is n -FP-gr-injective and K is Gorenstein n -FP-gr-injective. Consider,

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the following commutative diagram with exact rows exists:

$$\begin{array}{ccccccc}
 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \\
 0 & \longrightarrow & K & \xlongequal{\quad} & K & & \\
 & & \downarrow & & \downarrow & & \\
 0 & \longrightarrow & D & \longrightarrow & B_0 & \longrightarrow & C \longrightarrow 0 \\
 & & \downarrow & & \downarrow & & \parallel \\
 0 & \longrightarrow & A & \longrightarrow & B & \longrightarrow & C \longrightarrow 0 \\
 & & \downarrow & & \downarrow & & \downarrow \\
 & & 0 & & 0 & & 0
 \end{array}$$

By Proposition 2.1.4(1), D is Gorenstein n -FP-gr-injective, and so we have a commutative diagram in R -gr:

$$\begin{array}{ccccccccccc}
 \cdots & \longrightarrow & D_1 & \longrightarrow & D_0 & \longrightarrow & B_0 & \longrightarrow & E^0 & \longrightarrow & E^1 & \longrightarrow & \cdots \\
 & & \searrow & & \swarrow & & \searrow & & \swarrow & & \searrow & & \\
 & & & & L_0 & & & & D & & & & \\
 & & \swarrow & & \searrow & & \swarrow & & \searrow & & \swarrow & & \\
 0 & & & & 0 & & 0 & & 0 & & 0 & & 0
 \end{array}$$

where D_i and B_0 are n -FP-gr-injective, E^i is gr-injective, $C = \text{Ker}(E^0 \rightarrow E^1)$, $D = \text{Ker}(B_0 \rightarrow C)$, $L_i = \text{Ker}(D_i \rightarrow D_{i-1})$ and $L^i = \text{Ker}(E^i \rightarrow E^{i+1})$. By Remark 2.1.2, E^i and L_i are Gorenstein n -FP-gr-injective and hence by Lemma 2.1.6(1), $\text{EXT}_R^t(K_{n-1}, L_i) = \text{EXT}_R^t(K_{n-1}, D) = 0$ for any special gr-presented K_{n-1} module in R -gr with $\text{gr-pd}_R(K_{n-1}) < \infty$. Therefore, we have the following exact commutative diagram :

$$\begin{array}{ccccccc}
 \text{Hom}(K_{n-1}, D_1) & \longrightarrow & \text{Hom}(K_{n-1}, D_0) & \longrightarrow & \cdots \\
 & \searrow & \swarrow & & \searrow & & \\
 & & \text{Hom}(K_{n-1}, L_0) & & \text{Hom}(K_{n-1}, D) & & \\
 & \swarrow & \searrow & & \swarrow & & \\
 0 & & & & 0 & & 0
 \end{array}$$

Hence C is Gorenstein n -FP-gr-injective.

(2) \implies (1) Let U be an n -presented graded left R -module with $\text{gr-pd}_R(U) < \infty$, and let $0 \rightarrow K_n \rightarrow F_{n-1} \rightarrow K_{n-1} \rightarrow 0$ be a special short exact sequence in $R\text{-gr}$ with respect to U , where K_n is special gr-generated. We show that K_n is special gr-presented. Let M be a Gorenstein n -FP-injective module and $0 \rightarrow M \rightarrow E \rightarrow L \rightarrow 0$ an exact sequence in $R\text{-Mod}$, where E is injective. Then, $0 \rightarrow F(M) \rightarrow F(E) \rightarrow F(L) \rightarrow 0$ is exact, where $F(M)$ and $F(E)$ are Gorenstein n -FP-gr-injective in $R\text{-gr}$ by Proposition 2.1.5. So by (2), we deduce that $F(L)$ is Gorenstein n -FP-gr-injective. We have:

$$0 = \text{Ext}_{R\text{-gr}}^1(F_{n-1}, F(M)) \longrightarrow \text{Ext}_{R\text{-gr}}^1(K_n, F(M)) \longrightarrow \text{Ext}_{R\text{-gr}}^2(K_{n-1}, F(M)) \longrightarrow 0.$$

So, $\text{Ext}_{R\text{-gr}}^1(K_n, F(M)) \cong \text{Ext}_{R\text{-gr}}^2(K_{n-1}, F(M))$. On the other hand,

$$0 = \text{Ext}_{R\text{-gr}}^1(K_{n-1}, F(E)) \longrightarrow \text{Ext}_{R\text{-gr}}^1(K_{n-1}, F(L)) \longrightarrow \text{Ext}_{R\text{-gr}}^2(K_{n-1}, F(M)) \longrightarrow 0.$$

Hence, $\text{Ext}_{R\text{-gr}}^1(K_{n-1}, F(L)) \cong \text{Ext}_{R\text{-gr}}^2(K_{n-1}, F(M))$. Since $F(L)$ is Gorenstein n -FP-gr injective, we get $0 = \text{EXT}_R^1(K_{n-1}, F(L))_\sigma \cong \text{Ext}_{R\text{-gr}}^1(K_{n-1}, F(L)(\sigma))$ for any $\sigma \in G$. This implies that $\text{Ext}_{R\text{-gr}}^1(K_{n-1}, F(L)) = 0$ and hence $\text{Ext}_{R\text{-gr}}^1(K_n, F(M)) = 0$. So, the following commutative diagram exists:

$$\begin{array}{ccccccc} 0 & \longrightarrow & \text{Hom}_{R\text{-gr}}(K_n, F(M)) & \longrightarrow & \text{Hom}_{R\text{-gr}}(K_n, F(E)) & \longrightarrow & \text{Hom}_{R\text{-gr}}(K_n, F(L)) \longrightarrow 0 \\ & & \downarrow \cong & & \downarrow \cong & & \downarrow \cong \\ 0 & \longrightarrow & \text{Hom}_R(K_n, M) & \longrightarrow & \text{Hom}_R(K_n, E) & \longrightarrow & \text{Hom}_R(K_n, L) \end{array}$$

So, $\text{Ext}_{R\text{-gr}}^1(K_n, F(M)) \cong \text{Ext}_R^1(K_n, M) = 0$ for any Gorenstein n -FP-injective left R -module M . Since every FP-injective left module is Gorenstein n -FP-injective, $\text{Ext}_{R\text{-gr}}^1(K_n, F(N)) \cong \text{Ext}_R^1(K_n, N) = 0$ for any FP-injective left module N and so K_n is 1-presented. Therefore, U is $(n + 1)$ -presented in $R\text{-gr}$. \blacksquare

Corollary 2.1.8. *Let every n -presented module in $R\text{-gr}$ with $\text{gr-pd}_R(U) < \infty$ be $(n + 1)$ -presented. Then, a module M in $R\text{-gr}$ is Gorenstein n -FP-gr-injective if and only if every gr-pure submodule and any gr-pure epimorphic image of M are Gorenstein n -FP-gr-injective.*

Proof. (\implies) Let M be Gorenstein n -FP-gr-injective in $R\text{-gr}$. If the exact sequence $0 \rightarrow K \rightarrow M \rightarrow \frac{M}{K} \rightarrow 0$ is gr-pure, then by [12, Proposition 2.2], $\text{EXT}_R^1(K_{n-1}, K) = 0$ for

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every special gr-presented module K_{n-1} in $R\text{-gr}$. So, we have $0 = \text{EXT}_R^1(K_{n-1}, K) \cong \text{EXT}_R^n(U, K)$ for any n -presented module U in $R\text{-gr}$. Thus, K is n -FP-gr-injective, and hence K is Gorenstein n -FP-gr-injective by Remark 2.1.2. Therefore, by Theorem 2.1.7, $\frac{M}{K}$ is Gorenstein n -FP-gr-injective.

(\Leftarrow) Assume that the exact sequence $0 \rightarrow K \rightarrow M \rightarrow L \rightarrow 0$ in $R\text{-gr}$ is gr-pure, where L and K are Gorenstein n -FP-gr-injective. Then, by Proposition 2.1.4(1), M is Gorenstein n -FP-gr-injective. \blacksquare

Corollary 2.1.9. *Let R be a graded ring. Then, the following statements are equivalent:*

- (1) $({}^\perp \mathcal{G}_{gr\text{-}\mathcal{FI}_n}, \mathcal{G}_{gr\text{-}\mathcal{FI}_n})$ is a hereditary cotorsion pair;
- (2) Every n -presented module in $R\text{-gr}$ with $\text{gr-pd}(U) < \infty$ is $(n+1)$ -presented and every $M \in ({}^\perp \mathcal{G}_{gr\text{-}\mathcal{FI}_n})^\perp$ has an exact left $(gr\text{-}\mathcal{FI}_n)$ -resolution.

Proof. (1) \implies (2) Let M be a Gorenstein n -FP-injective module and $0 \rightarrow M \rightarrow E \rightarrow L \rightarrow 0$ an exact sequence in $R\text{-Mod}$, where E is injective. Then, $0 \rightarrow F(M) \rightarrow F(E) \rightarrow F(L) \rightarrow 0$ is exact in $R\text{-gr}$, where $F(M)$ and $F(E)$ are Gorenstein n -FP-gr-injective by Proposition 2.1.5. So by hypothesis, $F(L)$ is Gorenstein n -FP-gr-injective. If U is an n -presented graded left R -module with $\text{gr-pd}_R(U) < \infty$, then similar to the proof (2) \implies (1) of Theorem 2.1.7, it follows that U is $(n+1)$ -presented. Since $({}^\perp \mathcal{G}_{gr\text{-}\mathcal{FI}_n})^\perp = \mathcal{G}_{gr\text{-}\mathcal{FI}_n}$ and every $N \in \mathcal{G}_{gr\text{-}\mathcal{FI}_n}$ has an left exact $(gr\text{-}\mathcal{FI}_n)$ -resolution, then $M \in ({}^\perp \mathcal{G}_{gr\text{-}\mathcal{FI}_n})^\perp$ as well.

(2) \implies (1) Note that we have to show that $({}^\perp \mathcal{G}_{gr\text{-}\mathcal{FI}_n})^\perp = \mathcal{G}_{gr\text{-}\mathcal{FI}_n}$. If $M \in ({}^\perp \mathcal{G}_{gr\text{-}\mathcal{FI}_n})^\perp$, then n -FP-gr-injective resolution $\cdots \rightarrow A_3 \rightarrow A_1 \rightarrow A_0 \rightarrow M \rightarrow 0$ of M in $R\text{-gr}$ exists. Also, we have an exact sequence $0 \rightarrow M \rightarrow E_0 \rightarrow E_1 \rightarrow \cdots$ in $R\text{-gr}$, where any E_i is gr-injective. So, there exists an exact sequence

$$\mathcal{Y} : \cdots \longrightarrow A_3 \longrightarrow A_1 \longrightarrow A_0 \longrightarrow E_0 \longrightarrow E_1 \longrightarrow \cdots$$

of n -FP-gr-injective modules in $R\text{-gr}$ with $M = \text{Ker}(E_0 \rightarrow E_1)$. Let $0 \rightarrow K_n \rightarrow F_{n-1} \rightarrow K_{n-1} \rightarrow 0$ be a special short exact sequence in $R\text{-gr}$ with $\text{gr-pd}_R(K_{n-1}) < \infty$. Then, by hypothesis, K_n is gr-presented with $\text{gr-pd}_R(K_n) < \infty$. So, by [68, Theorem 6.10] and by using the inductive presumption on $\text{gr-pd}_R(K_{n-1})$, we deduce

that $\text{HOM}_R(K_{n-1}, \mathcal{B})$ is exact. Thus, M is Gorenstein n -FP-gr-injective and hence $M \in \mathcal{G}_{gr\text{-}FL_n}$.

Now, if $0 \rightarrow A \rightarrow B \rightarrow C \rightarrow 0$ is a short exact sequence in $R\text{-gr}$, where $A, B \in \mathcal{G}_{gr\text{-}FL_n}$, then, by Theorem 2.1.7, $C \in \mathcal{G}_{gr\text{-}FL_n}$. Hence, the pair $({}^\perp \mathcal{G}_{gr\text{-}FL_n}, \mathcal{G}_{gr\text{-}FL_n})$ is a hereditary cotorsion pair. ■

Proposition 2.1.10. *Assume that every n -presented module in $R\text{-gr}$ with $\text{gr-fd}_R(U) < \infty$ is $(n+1)$ -presented. Then, for every short exact sequence $0 \rightarrow A \rightarrow B \rightarrow C \rightarrow 0$ in $gr\text{-}R$, A is Gorenstein n -gr-flat if B and C are Gorenstein n -gr-flat.*

Proof. If B is Gorenstein n -gr-flat in $gr\text{-}R$, then by Definition 2.1.1 and Remark 2.1.2, there is an exact sequence $0 \rightarrow B \rightarrow F^0 \rightarrow L \rightarrow 0$ in $gr\text{-}R$, where F^0 is n -gr-flat and L is Gorenstein n -gr-flat. We have the following pushout diagram with exact rows:

$$\begin{array}{ccccccc}
 & & 0 & & 0 & & 0 \\
 & & \downarrow & & \downarrow & & \downarrow \\
 0 & \longrightarrow & A & \longrightarrow & B & \longrightarrow & C \longrightarrow 0 \\
 & & \parallel & & \downarrow & & \downarrow \\
 0 & \longrightarrow & A & \longrightarrow & F^0 & \longrightarrow & D \longrightarrow 0 \\
 & & & & \downarrow & & \downarrow \\
 & & & & L & = & L \\
 & & & & \downarrow & & \downarrow \\
 & & & & 0 & & 0
 \end{array}$$

By Proposition 2.1.4(2), D is Gorenstein n -gr-flat, and so we have the following commutative diagram in $gr\text{-}R$:

$$\begin{array}{ccccccccccc}
 \cdots & \longrightarrow & P_1 & \longrightarrow & P_0 & \longrightarrow & F^0 & \longrightarrow & D^0 & \longrightarrow & D^1 & \longrightarrow & \cdots \\
 & & \searrow & & \nearrow & & \searrow & & \nearrow & & \searrow & & \\
 & & & & L_0 & & A & & D & & L^1 & & \\
 & & \nearrow & & \searrow & & \nearrow & & \searrow & & \nearrow & & \\
 & & 0 & & 0 & & 0 & & 0 & & 0 & &
 \end{array}$$

where D^i and F^0 are n -gr-flat modules, P_i is gr-flat, $A = \text{Ker}(F^0 \rightarrow D)$, $D = \text{Ker}(D^0 \rightarrow D^1)$, $L_i = \text{Ker}(P_i \rightarrow P_{i-1})$ and $L^i = \text{Ker}(D^i \rightarrow D^{i+1})$. By Remark 2.1.2,

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P_i and L^i are Gorenstein n -gr-flat and hence by Lemma 2.1.6(2), $\text{Tor}_t^R(L^i, K_{n-1}) = \text{Tor}_t^R(D, K_{n-1}) = 0$ for any special gr-presented module K_{n-1} , with $\text{gr-fd}_R(K_{n-1}) < \infty$ and any $t \geq 0$. So, similar to the proof (1) \implies (2) of Theorem 2.1.7, it follows that $-\otimes_R K_{n-1}$ on the above horizontal sequence in diagram is exact and so A is Gorenstein n -gr-flat. ■

Corollary 2.1.11. *Let every n -presented module in R -gr with $\text{gr-fd}_R(U) < \infty$ be $(n + 1)$ -presented. Then, a module M in $\text{gr-}R$ is Gorenstein n -gr-flat if and only if every gr-pure submodule and any gr-pure epimorphic image of M are Gorenstein n -gr-flat.*

Proof. (\implies) Let M be Gorenstein n -gr-flat in $\text{gr-}R$ and K a gr-pure in M . Then, the exact sequence $0 \rightarrow K \rightarrow M \rightarrow \frac{M}{K} \rightarrow 0$ is gr-pure. So, if K_{n-1} is special gr-presented module in R -gr, then $\text{Tor}_1^R(\frac{M}{K}, K_{n-1}) = 0$ and consequently by [36, Lemma 2.1], $\text{Tor}_1^R(\frac{M}{K}, K_{n-1})^* \cong \text{EXT}_R^1(K_{n-1}, (\frac{M}{K})^*) = 0$. Therefore, the exact sequence $0 \rightarrow (\frac{M}{K})^* \rightarrow M^* \rightarrow K^* \rightarrow 0$ is special gr-pure in R -gr, and using [77, Proposition 3.10], we deduce that $(\frac{M}{K})^*$ is n -FP-gr-injective. By [77, Proposition 3.8], $\frac{M}{K}$ is n -gr-flat, and then Proposition 2.1.10 shows that K is Gorenstein n -gr-flat.

(\impliedby) Let K be a gr-pure in M . Then, the exact sequence $0 \rightarrow K \rightarrow M \rightarrow \frac{M}{K} \rightarrow 0$ is gr-pure. So, it follows, by Proposition 2.1.4(2), that M is Gorenstein n -gr-flat. ■

Also, as for the classical injective (resp. flat) notion, the class $\mathcal{G}_{\text{gr-}\mathcal{FI}_n}$ in R -gr (resp. $\mathcal{G}_{\text{gr-}\mathcal{F}_n}$ in $\text{gr-}R$) is closed under direct products (resp. direct sums).

Proposition 2.1.12. *Let R be a graded ring. Then:*

- (1) *The class $\mathcal{G}_{\text{gr-}\mathcal{FI}_n}$ in R -gr is closed under direct products.*
- (2) *The class $\mathcal{G}_{\text{gr-}\mathcal{F}_n}$ in $\text{gr-}R$ is closed under direct sums.*

Next definition contains some general remarks about resolving classes of graded modules which will be useful in Sections 3 and 4. We use $\text{gr-}\mathcal{I}(R)$ to denote the class of finite injective graded left modules and the symbol $\text{gr-}\mathcal{P}(R)$ denotes the class of finite projective graded right modules (the graded version of [45, 1.1. Resolving classes]).

Definition 2.1.13. *Let R be a graded ring and \mathcal{X} a class of graded modules. Then:*

- (1) We call \mathcal{X} *gr-injectively resolving* if $\text{gr-}\mathcal{I}(R) \subseteq \mathcal{X}$, and for every short exact sequence $0 \rightarrow A \rightarrow B \rightarrow C \rightarrow 0$ with $A \in \mathcal{X}$ the conditions $B \in \mathcal{X}$ and $C \in \mathcal{X}$ are equivalent.
- (2) We call \mathcal{X} *gr-projectively resolving* if $\text{gr-}\mathcal{F}(R) \subseteq \mathcal{X}$, and for every short exact sequence $0 \rightarrow A \rightarrow B \rightarrow C \rightarrow 0$ with $C \in \mathcal{X}$ the conditions $A \in \mathcal{X}$ and $B \in \mathcal{X}$ are equivalent.

By Definition 2.1.13, Propositions 2.1.4, 2.1.10, 2.1.12, Theorem 2.1.7 and the graded version of [45, Proposition 1.4], we have the following easy observations.

Proposition 2.1.14. *Assume that every n -presented module in $R\text{-gr}$ with $\text{gr-fd}_R(U) < \infty$ is $(n + 1)$ -presented. Then:*

- (1) The class $\mathcal{G}_{\text{gr-}\mathcal{FI}_n}$ is *gr-injectively resolving*.
- (2) The class $\mathcal{G}_{\text{gr-}\mathcal{FI}_n}$ is *closed under direct summands*.
- (3) The class $\mathcal{G}_{\text{gr-}\mathcal{F}_n}$ is *gr-projectively resolving*.
- (4) The class $\mathcal{G}_{\text{gr-}\mathcal{F}_n}$ is *closed under direct summands*.

We know that, if R is a left n -gr-coherent ring, then every n -presented module in $R\text{-gr}$ with $\text{gr-fd}_R(U) < \infty$ is $(n + 1)$ -presented. So, in the following theorem according to previous results, we investigate the relationships between Gorenstein n -FP-gr-injective and Gorenstein n -gr-flat modules on n -gr-coherent rings.

Theorem 2.1.15. *Let R be a left n -gr-coherent ring. Then,*

- (1) A module M in $R\text{-gr}$ is *Gorenstein n -FP-gr-injective* if and only if M^* is *Gorenstein n -gr-flat* in $\text{gr-}R$.
- (2) A module M in $\text{gr-}R$ is *Gorenstein n -gr-flat* if and only if M^* is *Gorenstein n -FP-gr-injective* in $R\text{-gr}$.

Proof. (1) (\implies) By Definition 2.1.1, there is an exact sequence $\cdots \rightarrow A_1 \rightarrow A_0 \rightarrow M \rightarrow 0$ in $R\text{-gr}$, where every A_i is n -FP-gr-injective, and by [77, Theorem 3.17],

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every $(A_i)^*$ is n -gr-flat in $\text{gr-}R$. So by [68, Lemma 3.53], there is an exact sequence $0 \rightarrow M^* \rightarrow (A_0)^* \rightarrow (A_1)^* \rightarrow \cdots$ in $\text{gr-}R$. Hence, we have:

$$\mathcal{X} : \cdots \longrightarrow P_1 \longrightarrow P_0 \longrightarrow (A_0)^* \longrightarrow (A_1)^* \longrightarrow \cdots ,$$

where P_i is gr-projective and n -gr-flat by Remark 2.1.2 and also $M^* = \ker((A_0)^* \rightarrow (A_1)^*)$. Let $0 \rightarrow K_n \rightarrow F_{n-1} \rightarrow K_{n-1} \rightarrow 0$ be a special short exact sequence in $R\text{-gr}$ with $\text{gr-fd}_R(K_{n-1}) < \infty$. Then, K_n is gr-presented with $\text{gr-fd}_R(K_n) < \infty$, since R is n -gr-coherent. By [68, Theorem 6.10] and by using the inductive presumption on $\text{gr-fd}_R(K_{n-1})$, we deduce that $\mathcal{X} \otimes_R K_{n-1}$ is exact, and then M^* is Gorenstein n -gr-flat.

(\Leftarrow) Let M^* be Gorenstein n -gr-flat. Then, by (2)(\Rightarrow), M^{**} is Gorenstein n -FP-gr-injective. By [75, Proposition 2.3.5], M is gr-pure in M^{**} , and so by Corollary 2.1.8, M is Gorenstein n -FP-gr-injective.

(2) (\Rightarrow) By Definition 2.1.1, there is an exact sequence $0 \rightarrow M \rightarrow F^0 \rightarrow F^1 \rightarrow \cdots$ of n -gr-flat modules in $\text{gr-}R$. By [77, Proposition 3.8], $(F^i)^*$ is n -FP-gr-injective for any $i \geq 0$. So by [68, Lemma 3.53], there is an exact sequence $\cdots \rightarrow (F^1)^* \rightarrow (F^0)^* \rightarrow M^*$ in $R\text{-gr}$. For a module M^* , there is an exact sequence $0 \rightarrow M^* \rightarrow E_0 \rightarrow E_1 \rightarrow \cdots \rightarrow 0$ in $R\text{-gr}$, where E_i is gr-injective. Consider the following exact sequence:

$$\cdots \longrightarrow (F^1)^* \longrightarrow (F^0)^* \longrightarrow E_0 \longrightarrow E_1 \longrightarrow \cdots$$

with $M^* = \ker(E_0 \rightarrow E_1)$. Hence by analogy with the proof (2) \Rightarrow (1) of Corollary 2.1.9, we obtain that M^* is Gorenstein n -FP-gr-injective.

(\Leftarrow) Let M^* be Gorenstein n -FP-gr-injective in $R\text{-gr}$. Then, by (1)(\Rightarrow), M^{**} is Gorenstein n -gr-flat in $\text{gr-}R$. By [75, Proposition 2.3.5], M is gr-pure in M^{**} , and so by Corollary 2.1.11, M is Gorenstein n -gr-flat. \blacksquare

Next, we are given other results of Gorenstein n -FP-gr-injective and Gorenstein n -gr-flat modules on n -gr-coherent rings.

Proposition 2.1.16. *Let R be a left n -gr-coherent ring. Then,*

- (1) *The class $\mathcal{G}_{\text{gr-}\mathcal{FI}_n}$ in $R\text{-gr}$ is closed under direct limits.*
- (2) *The class $\mathcal{G}_{\text{gr-}\mathcal{F}_n}$ in $\text{gr-}R$ is closed under direct products.*

Proof. (1) Let $U \in R\text{-gr}$ be an n -presented and let $\{A_i\}_{i \in I}$ be a family of n -FP-gr-injective modules in $R\text{-gr}$. Then, by [77, Theorem 3.17], $\varinjlim A_i$ is n -FP-gr-injective. So, if $\{M_i\}_{i \in I}$ is a family of Gorenstein n -FP-gr-injective modules in $R\text{-gr}$, then the following n -FP-gr-injective complex

$$\mathcal{Y}_i = \cdots \longrightarrow (A_i)_1 \longrightarrow (A_i)_0 \longrightarrow (A_i)^0 \longrightarrow (A_i)^1 \longrightarrow \cdots ,$$

where $M_i = \ker((A_i)^0 \rightarrow (A_i)^1)$, induces the following exact sequence of n -FP-gr-injective modules in $R\text{-gr}$:

$$\varinjlim \mathcal{Y}_i = \cdots \longrightarrow \varinjlim (A_i)_1 \longrightarrow \varinjlim (A_i)_0 \longrightarrow \varinjlim (A_i)^0 \longrightarrow \varinjlim (A_i)^1 \longrightarrow \cdots ,$$

where $\varinjlim M_i = \ker(\varinjlim (A_i)^0 \rightarrow \varinjlim (A_i)^1)$. Assume that K_{n-1} is special gr-presented module in $R\text{-gr}$ with $\text{gr-pd}_R(K_{n-1}) < \infty$, then by [77, Proposition 3.13],

$$\text{HOM}_R(K_{n-1}, \varinjlim \mathcal{Y}_i) \cong \varinjlim \text{HOM}_R(K_{n-1}, \mathcal{Y}_i).$$

By hypothesis, $\text{HOM}_R(K_{n-1}, \mathcal{Y}_i)$ is exact, and consequently $\varinjlim M_i$ is Gorenstein n -FP-gr-injective.

(2) Let $U \in R\text{-gr}$ be an n -presented and let $\{F_i\}_{i \in I}$ be a family of n -gr-flat modules in $\text{gr-}R$. Then, by [77, Theorem 3.17], $\prod_{i \in I} F_i$ is n -gr-flat. So, if $\{M_i\}$ is a family of Gorenstein n -gr-flat modules in $\text{gr-}R$, then the following n -gr-flat complex

$$\mathcal{X}_i = \cdots \longrightarrow (F_i)_1 \longrightarrow (F_i)_0 \longrightarrow (F_i)^0 \longrightarrow (F_i)^1 \longrightarrow \cdots ,$$

where $M_i = \ker((F_i)^0 \rightarrow (F_i)^1)$, induces the following exact sequence of n -gr-flat modules in $\text{gr-}R$:

$$\prod_{i \in I} \mathcal{X}_i = \cdots \longrightarrow \prod_{i \in I} (F_i)_1 \longrightarrow \prod_{i \in I} (F_i)_0 \longrightarrow \prod_{i \in I} (F_i)^0 \longrightarrow \prod_{i \in I} (F_i)^1 \longrightarrow \cdots ,$$

where $\prod_{i \in I} M_i = \ker(\prod_{i \in I} (F_i)^0 \rightarrow \prod_{i \in I} (F_i)^1)$. If K_{n-1} is special gr-presented, then

$$\left(\prod_{i \in I} \mathcal{X}_i \otimes_R K_{n-1} \right) \cong \prod_{i \in I} (\mathcal{X}_i \otimes_R K_{n-1}).$$

By hypothesis, $\mathcal{X}_i \otimes_R K_{n-1}$ is exact, and consequently $\prod_{i \in I} M_i$ is Gorenstein n -gr-flat. ■

2.2. COVERS AND PREENVELOPES BY GORENSTEIN GRADED MODULES

In the following proposition, we show that if R is n -gr-coherent, then every Gorenstein n -FP-gr-injective in R -gr is n -FP-gr-injective if $l.n$ -FP-gr-dim(R) $< \infty$, and every Gorenstein n -gr-flat in gr - R is n -gr-flat if $r.n$ -gr-dim(R) $< \infty$.

Proposition 2.1.17. *Let R be a left n -gr-coherent ring.*

- (1) *If $l.n$ -FP-gr-dim(R) $< \infty$, then every Gorenstein n -FP-gr-injective in R -gr is n -FP-gr-injective.*
- (2) *If $r.n$ -gr-dim(R) $< \infty$, then every Gorenstein n -gr-flat in gr - R is n -gr-flat.*

Proof. (1) Let $l.n$ -FP-gr-dim(R) $\leq k$. If M is Gorenstein n -FP-gr-injective in R -gr, then there exists an exact sequence

$$0 \longrightarrow N \longrightarrow A_{k-1} \longrightarrow A_{k-2} \longrightarrow \cdots \longrightarrow A_0 \longrightarrow M \longrightarrow 0$$

in R -gr, where every A_i is n -FP-gr-injective for any $0 \leq i \leq k-1$. Since R is n -gr-coherent for any $t \geq 1$, $\text{EXT}_R^t(K_{n-1}, A_i) = 0$ for all special gr-presented modules K_{n-1} with respect to every n -presented U in R -g. Let $L_i = \ker(A_i \rightarrow A_{i-1})$. Then, we have

$$\text{EXT}_R^{k+1}(K_{n-1}, N) \cong \text{EXT}_R^k(K_{n-1}, L_{k-2}) \cong \cdots \cong \text{EXT}_R^2(K_{n-1}, L_0) \cong \text{EXT}_R^1(K_{n-1}, M).$$

Since n -FP-gr-id $_R(N) \leq k$, then $0 = \text{EXT}_R^{k+1}(K_{n-1}, N) \cong \text{EXT}_R^1(K_{n-1}, M) \cong \text{EXT}_R^1(U, M)$ and consequently M is n -FP-gr-injective.

- (2) The proof is similar to that of (1). ■

2.2 Covers and preenvelopes by Gorenstein graded modules

In this section, by using of duality pairs on n -gr-coherence rings, we show that the classes $\mathcal{G}_{gr-\mathcal{FI}_n}$ (resp. $\mathcal{G}_{gr-\mathcal{F}_n}$) or other signs are covering and preenveloping.

Proposition 2.2.1. *Let R be a left n -gr-coherent ring. Then, the pair $(\mathcal{G}_{gr-\mathcal{FI}_n}, \mathcal{G}_{gr-\mathcal{F}_n})$ is a duality pair.*

Proof. Let M be an R -module in $R\text{-gr}$. Then, by Theorem 2.1.15(1), $M \in \mathcal{G}_{gr\text{-}\mathcal{FI}_n}$ if and only if $M^* \in \mathcal{G}_{gr\text{-}\mathcal{F}_n}$. By Proposition 2.1.12(2), any finite direct sum of Gorenstein n -gr-flat modules is Gorenstein n -gr-flat. Also, by Proposition 2.1.14(4), $\mathcal{G}_{gr\text{-}\mathcal{F}_n}$ is closed under direct summands. So, the pair $(\mathcal{G}_{gr\text{-}\mathcal{FI}_n}, \mathcal{G}_{gr\text{-}\mathcal{F}_n})$ is a duality pair. ■

Proposition 2.2.2. *Let R be a left n -gr-coherent ring. Then, the pair $(\mathcal{G}_{gr\text{-}\mathcal{F}_n}, \mathcal{G}_{gr\text{-}\mathcal{FI}_n})$ is a duality pair.*

Proof. Let M be an R -module in $gr\text{-}R$. Then, by Theorem 2.1.15(2), $M \in \mathcal{G}_{gr\text{-}\mathcal{F}_n}$ if and only if $M^* \in \mathcal{G}_{gr\text{-}\mathcal{FI}_n}$. By Proposition 2.1.12(1), any finite direct sum of Gorenstein n -gr-FP-injective modules is Gorenstein n -FP-gr-injective and by Proposition 2.1.14(2), $\mathcal{G}_{gr\text{-}\mathcal{FI}_n}$ is closed under direct summands. So, the pair $(\mathcal{G}_{gr\text{-}\mathcal{F}_n}, \mathcal{G}_{gr\text{-}\mathcal{FI}_n})$ is a duality pair. ■

Theorem 2.2.3. *Let R be a left n -gr-coherent ring. Then:*

- (1) *The class $\mathcal{G}_{gr\text{-}\mathcal{FI}_n}$ is covering and preenveloping.*
- (2) *The class $\mathcal{G}_{gr\text{-}\mathcal{F}_n}$ is covering and preenveloping.*

Proof. (1) Every direct limit of Gorenstein n -FP-gr-injective modules and every direct product of Gorenstein n -FP-gr-injective modules in $R\text{-gr}$ are Gorenstein n -FP-gr-injective by Propositions 3.1.11(1) and 2.1.12(1), respectively. Also, by Corollary 2.1.8, the class of Gorenstein n -FP-gr-injective modules in $R\text{-gr}$ is closed under gr-pure submodules, gr-pure quotients and gr-pure extensions. So, by Proposition 2.2.1 and [77, Theorem 4.2], we deduce that every R -module in $R\text{-gr}$ has a Gorenstein n -FP-gr-injective cover and a Gorenstein n -FP-gr-injective preenvelope.

(2) Every direct sum of Gorenstein n -gr-flat modules and every direct product of Gorenstein n -gr-flat modules in $gr\text{-}R$ are Gorenstein n -gr-flat by Propositions 2.1.12(2) and 3.1.11(2), respectively. Also, by Corollary 2.1.11, the class of Gorenstein n -gr-flat modules in $gr\text{-}R$ is closed under gr-pure submodules, gr-pure quotients and gr-pure extensions. So, by Proposition 2.2.2 and [77, Theorem 4.2], we deduce that every R -module in $gr\text{-}R$ has a Gorenstein n -gr-flat cover and a Gorenstein n -gr-flat preenvelope. ■

2.2. COVERS AND PREENVELOPES BY GORENSTEIN GRADED MODULES

Now, we give some equivalent characterizations for ${}_R R$ being Gorenstein n -FP-gr-injective in terms of the properties of Gorenstein n -FP-gr-injective and Gorenstein n -gr-flat modules.

Theorem 2.2.4. *Let R be a left n -gr-coherent ring. Then, the following statements are equivalent:*

- (1) ${}_R R$ is Gorenstein n -FP-gr-injective;
- (2) Every graded module in $\text{gr-}R$ has a monic Gorenstein n -gr-flat preenvelope;
- (3) Every gr-injective module in $\text{gr-}R$ is Gorenstein n -gr-flat;
- (4) Every n -FP-gr-injective module in $\text{gr-}R$ is Gorenstein n -gr-flat;
- (5) Every flat module in $R\text{-gr}$ is Gorenstein n -FP-gr-injective;
- (6) Every graded module in $R\text{-gr}$ has an epic Gorenstein n -FP-gr-injective cover.

Moreover, if $l.n\text{-FP-gr-dim}(R) < \infty$, then the above conditions are also equivalent to:

- (7) Every Gorenstein gr-flat module in $R\text{-gr}$ is Gorenstein n -FP-gr-injective;
- (8) Every graded module in $R\text{-gr}$ is Gorenstein n -FP-gr-injective;
- (9) Every Gorenstein gr-injective module in $\text{gr-}R$ is Gorenstein n -gr-flat.

Proof. (8) \implies (7), (7) \implies (5) and (9) \implies (3) are obvious.

(1) \implies (2) By Theorem 2.2.3(2), every module M in $\text{gr-}R$ has a Gorenstein n -gr-flat preenvelope $f : M \rightarrow F$. By Theorem 2.1.15(1), R^* is Gorenstein n -gr-flat, and so $\prod_{i \in I}^{\text{gr-}R} R^*$ is Gorenstein n -gr-flat by Proposition 3.1.11. On the other hand, $({}_R R)^*$ is a cogenerator in $\text{gr-}R$. Therefore, exact sequence of the form $0 \rightarrow M \xrightarrow{g} \prod_{i \in I}^{\text{gr-}R} R^*$ exists, and hence homomorphism $0 \rightarrow F \xrightarrow{h} \prod_{i \in I}^{\text{gr-}R} R^*$ such that $hf = g$ shows that f is monic.

(2) \implies (3) Let E be a gr-injective module in $\text{gr-}R$. Then, by (2), homomorphism $f : E \rightarrow F$ is a monic Gorenstein n -gr-flat preenvelope of E . So, the split exact

sequence $0 \rightarrow E \rightarrow F \rightarrow \frac{F}{E} \rightarrow 0$ exists, and so E is direct summand of F . Hence by Proposition 2.1.14, E is Gorenstein n -gr-flat.

(3) \implies (1) By (3), R^* is Gorenstein n -gr-flat, since R^* is gr-injective. So, R is Gorenstein n -FP-gr-injective by Theorem 2.1.15(1).

(3) \implies (4) Let M be an n -FP-gr-injective module in $\text{gr-}R$. Then, by [77, Proposition 3.10], the exact sequence $0 \rightarrow M \rightarrow E^g(M) \rightarrow \frac{E^g(M)}{M} \rightarrow 0$ is special gr-pure. Since by (3), $E^g(M)$ is Gorenstein n -gr-flat, from Corollary 2.1.11, we deduce that M is Gorenstein n -gr-flat.

(4) \implies (5) Let F be a flat module in R -gr. Then, F^* is gr-injective in $\text{gr-}R$, so F^* is Gorenstein n -gr-flat by (4), and hence F is Gorenstein n -FP-gr-injective by Theorem 2.1.15(1).

(5) \implies (6) By Theorem 2.2.3(1), every module M in R -gr has a Gorenstein n -FP-gr-injective cover $f : A \rightarrow M$. By hypothesis, ${}_R R$ is Gorenstein n -FP-gr-injective. Then, similar to the proof (1) \implies (6) of [77, Theorem 4.8], f is an epimorphism.

(6) \implies (1) By hypothesis, R has an epic Gorenstein n -FP-gr-injective cover $f : D \rightarrow R$, then we have a split exact sequence $0 \rightarrow \text{Ker } f \rightarrow D \rightarrow R \rightarrow 0$ with D is a Gorenstein n -FP-gr-injective module in R -gr. So, by Proposition 2.1.14, R is Gorenstein n -FP-gr-injective in R -gr.

(1) \implies (8) Let M be a graded left R -module. Then, there is an exact sequence $\cdots \rightarrow F_1 \rightarrow F_0 \rightarrow M \rightarrow 0$ in R -gr, where each F_i is gr-flat. If R is a Gorenstein n -FP-gr-injective module in R -gr, then by Proposition 2.1.17(1), R is n -FP-gr-injective. Hence by [77, Proposition 3.16], we deduce that every F_i is n -FP-gr-injective. Also, for module M , there is an exact sequence $0 \rightarrow M \rightarrow E_0 \rightarrow E_1 \rightarrow \cdots \rightarrow 0$ in R -gr, where every E_i is gr-injective. So, we have:

$$\cdots \longrightarrow F_1 \longrightarrow F_0 \longrightarrow E_0 \longrightarrow E_1 \longrightarrow \cdots ,$$

where F_i and E_i are n -FP-gr-injective and $M = \ker(E_0 \rightarrow E_1)$. Thus, similar to proof (2) \implies (1) of Corollary 2.1.9, we get that M is Gorenstein n -FP-gr-injective.

(8) \implies (9) If M is a Gorenstein gr-injective module in $\text{gr-}R$, then M^* is in R -gr. So by hypothesis, M^* is Gorenstein n -FP-gr-injective, and hence by Theorem 2.1.15, it follows that M is Gorenstein n -gr-flat. ■

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Example 2.2.5. *Let R be a commutative, Gorenstein Noetherian, complete, local ring, \mathfrak{m} its maximal ideal. Let $E = E(R/\mathfrak{m})$ be the R -injective hull of the residue field R/\mathfrak{m} of R . By [67, Theorem A], $\lambda\text{-dim}(R \times E) = \dim R$, where $\dim R$ is the Krull dimension of R . We suppose that $\dim R = n$, then $(R \times E)$ is n -gr-coherent. And if we take in [57, Theorem 4.2] $n = 1$ and $B = \{0\}$, we get $\text{Hom}_R(E, E) = R$. Then, by [34, Corollary 4.37], $(R \times E)$ is self gr-injective which implies that $(R \times E)$ is a left n -FP-gr-injective module over itself. Hence, $R \times E$ is n -FC graded ring (n -gr-coherent and n -FP-gr-injective), and then by Remark 2.1.2, $(R \times E)$ is Gorenstein n -FP-gr-injective. For example, the ring $R = K[[X_1, \dots, X_n]]$ of formal power series in n variables over a field K which is commutative, Gorenstein Noetherian, complete, local ring, with $\mathfrak{m} = (X_1, \dots, X_n)$ its maximal ideal. We obtain $\lambda\text{-dim}(R \times E(R/\mathfrak{m})) = n$, that is, $R \times E(R/\mathfrak{m})$ is n -gr-coherent ring. So according to the above $R \times E(R/\mathfrak{m})$ is n -FC graded ring. So, every left $R \times E(R/\mathfrak{m})$ -module is Gorenstein n -FP-gr-injective.*

Proposition 2.2.6. *Let R be a left n -gr-coherent. Then, $(\mathcal{G}_{gr-\mathcal{F}_n}, \mathcal{G}_{gr-\mathcal{F}_n}^\perp)$ is hereditary perfect cotorsion pair.*

Proof. Let $\mathcal{G}_{gr-\mathcal{F}_n}$ be a class of Gorenstein n -gr-flat modules. Then, by Corollary 2.1.11, $\mathcal{G}_{gr-\mathcal{F}_n}$ is closed under gr-pure submodules, gr-pure quotients and gr-pure extensions. On the other hand, $R \in \mathcal{G}_{gr-\mathcal{F}_n}$ by Remark 2.1.2, and $\mathcal{G}_{gr-\mathcal{F}_n}$ is closed under graded direct sums by Proposition 2.1.12. So, it follows that duality pair $(\mathcal{G}_{gr-\mathcal{F}_n}, \mathcal{G}_{gr-\mathcal{F}_n}^\perp)$ is perfect. Consequently by [77, Theorem 4.2], $(\mathcal{G}_{gr-\mathcal{F}_n}, \mathcal{G}_{gr-\mathcal{F}_n}^\perp)$ is perfect cotorsion pair. Consider the short exact sequence $0 \rightarrow A \rightarrow B \rightarrow C \rightarrow 0$ in $\text{gr-}R$, where B and C are Gorenstein n -gr-flat. Then, by Proposition 2.1.10, A is Gorenstein n -gr-flat and hence perfect cotorsion pair $(\mathcal{G}_{gr-\mathcal{F}_n}, \mathcal{G}_{gr-\mathcal{F}_n}^\perp)$ is hereditary. ■

CATEGORY OF n -FCP-GR-PROJECTIVE MODULES WITH RESPECT TO SPECIAL COPRESENTED GRADED MODULES

Let R be a ring graded by an abelian group G and $n \geq 1$ an integer. We introduce the notion of n -FCP-gr-projective R -modules and by using special finitely copresented graded modules. Our investigation reveals the following:

1. We establish several equivalent characterizations of n -FCP-gr-projective modules and graded right modules of n -FCP-gr-projective dimension at most k over n -gr-cocoherent rings.
2. The ring R is right n -gr-cocoherent if and only if the class of n -FCP-gr-projective modules is closed under the kernel of epimorphisms, which is equivalent that the pair $(\text{gr-}\mathcal{FCP}_n, \text{gr-}\mathcal{FCP}_n^\perp)$ is a hereditary cotorsion pair.

Subsequently, we investigate various conditions that are equivalent to each right R -module being n -FCP-gr-projective.

3.1 n -FCP-gr-projective modules

In this section, we first introduce the special gr-copresented and special gr-cogenerated modules via n -copresented graded right modules. Then, by using of these modules, some properties of n -FCP-gr-projective modules are discussed.

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Definition 3.1.1. Let $n \geq 0$ be an integer. Then, a graded right R -module U is called n -copresented if there exists an exact sequence

$$0 \longrightarrow U \longrightarrow E^0 \longrightarrow E^1 \longrightarrow \cdots \longrightarrow E^{n-1} \longrightarrow E^n$$

in $gr\text{-}R$, with each E^i being finitely cogenerated injective. Set $K^{n-1} = \text{Coker}(E^{n-2} \rightarrow E^{n-1})$ and $K^n = \text{Coker}(E^{n-1} \rightarrow E^n)$. Then, we shall say the sequence

$$\Delta' : 0 \longrightarrow K^{n-1} \longrightarrow E^n \longrightarrow K^n \longrightarrow 0$$

in $gr\text{-}R$ is a special short exact sequence. Moreover, we call the objects K^n and K^{n-1} special finitely gr -cogenerated and special finitely gr -copresented (special gr -cogenerated and special gr -copresented for short) right R -modules, respectively. Then, it follows that $\text{EXT}_R^1(M, K^{n-1}) \cong \text{EXT}_R^n(M, U)$ for any graded right R -module M . Also, a short exact sequence $0 \rightarrow A \rightarrow B \rightarrow C \rightarrow 0$ in $gr\text{-}R$ is called special gr -copure, if for every special gr -copresented K^{n-1} , there exists the following exact sequence:

$$0 \longrightarrow \text{HOM}_R(C, K^{n-1}) \longrightarrow \text{HOM}_R(B, K^{n-1}) \longrightarrow \text{HOM}_R(A, K^{n-1}) \longrightarrow 0,$$

where A is said to be special gr -copure in B . A graded ring R is called right n - gr -cocoherent if each n -copresented graded right R -module is $(n+1)$ -copresented.

The following lemma is the graded version of [74, Theorem 3].

Lemma 3.1.2. Let $0 \rightarrow A \rightarrow B \rightarrow C \rightarrow 0$ in $gr\text{-}R$ be a short exact sequence. Then, the following statements hold for any $n \geq 0$.

1. If A and C are n -copresented, then so is B .
2. If C is n -copresented and B is $(n+1)$ -copresented, then A is $(n+1)$ -copresented.
3. If A and B are $(n+1)$ -copresented, then C is n -copresented.

Definition 3.1.3. Let $n \geq 1$ be an integer. A graded right R -module M is called n -FCP- gr -projective if $\text{EXT}_R^n(M, U) = 0$ for any n -copresented graded right R -module U .

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Notice that for ungraded, a right R -module M is called n -FCP-projective if $\text{Ext}_R^n(M, U) = 0$ for any n -copresented right R -module U , and in case $n = 1$, M is called FCP-projective, see [78].

Remark 3.1.4. 1. Every m -copresented graded right R -module is n -copresented for any $m \geq n$.

2. If $n = 1$, then every 1-FCP-gr-projective is FCP-gr-projective.

3. If M is n -FCP-gr-projective and U is $(n+1)$ -copresented graded right R -module, then there exists an exact sequence $0 \rightarrow U \rightarrow E^0 \rightarrow C \rightarrow 0$ in $\text{gr-}R$, where E^0 is finitely cogenerated injective and C is n -copresented by Lemma 3.1.2. So we deduce that the sequence $0 \rightarrow \text{EXT}_R^n(M, C) \rightarrow \text{EXT}_R^{n+1}(M, U) \rightarrow 0$ is exact. But, $\text{EXT}_R^n(M, C) = 0$ since M is n -FCP-gr-projective and C is n -copresented. Hence, $\text{EXT}_R^{n+1}(M, U) = 0$ and consequently M is $(n+1)$ -FCP-gr-projective. Therefore,

$$\text{gr-FCP}_1 \subseteq \text{gr-FCP}_2 \subseteq \dots \subseteq \text{gr-FCP}_n \subseteq \text{gr-FCP}_{n+1} \subseteq \dots$$

In general, m -FCP-gr-projective right R -modules need not be n -FCP-gr-projective whenever $m > n$, see Example 3.1.6(1).

4. Every gr-projective right R -module is n -FCP-gr-projective.

Definition 3.1.5. 1. The n -FCP-gr-projective dimension of a graded right module M is defined by

$$n.\text{FCP-gr-pd}M = \inf\{k : \text{EXT}_R^{k+1}(M, K^{n-1}) = 0 \text{ for every special gr-copresented } K^{n-1}\}.$$

2. The n -FCP-gr-projective global dimension of a graded ring R is defined by

$$r.n.\text{FCP-gr-dim}R = \sup\{n.\text{FCP-gr-pd}M \mid M \text{ is a graded right } R\text{-module}\}.$$

A graded ring R is called a right gr- V -ring if every simple graded right R -module is an injective. A graded ring R is called right gr-hereditary if every submodule of

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a projective graded right R -module is projective. A graded ring R is called right gr-cosemihereditary if every submodule of a projective graded right R -module is FCP-gr-projective, see the graded version of [78, Definition 3.6]. It is clear that gr-hereditary rings are gr-cosemihereditary.

Example 3.1.6. *Let R be a gr-cosemihereditary ring but not gr- V -ring, for example gr-hereditary ring $R = k[X]$ where k is field. Then, by graded version of [78, Theorem 3.9], there exists a graded R -module which is not 1-FCP-gr-projective. Also, by graded version of [78, Theorem 3.7], $r.FCP\text{-}gr\text{-}dim R \leq 1$. So, $FCP\text{-}gr\text{-}dim M \leq 1$ for any graded right R -module M . Hence, by Lemma 3.2.2, there is an exact sequence $0 \rightarrow P_1 \rightarrow P_0 \rightarrow M \rightarrow 0$ in $gr\text{-}R$, where P_0 and P_1 are 1-FCP-gr-projective. If U is a 2-copresented right R -module, then we have $EXT_R^1(P_1, U) = 0$, since every 2-copresented right R -module is 1-copresented. Therefore, from $EXT_R^1(P_1, U) \cong EXT_R^2(M, U)$, we get that every graded right R -module M is 2-FCP-gr-projective.*

We have the following lemma before the next proposition.

Lemma 3.1.7. *Let R be a graded ring, M a graded right R -module, and k a non-negative integer. Then, $EXT_R^k(M, -)_\sigma \cong Ext_{gr\text{-}R}^k(M(\sigma^{-1}), -)$ for any $\sigma \in G$.*

Proof. It is clear that for $k = 0$, $HOM_R(M, -)_\sigma \cong Hom_{gr\text{-}R}(M(\sigma^{-1}), -)$. Let N be a graded right R -module. Then, there exists an exact sequence $0 \rightarrow N \rightarrow E \rightarrow L \rightarrow 0$, where E is gr-injective. Consider the following commutative diagram with exact rows:

$$\begin{array}{ccccccc} 0 & \longrightarrow & EXT_R^{k-1}(M, L)_\sigma & \longrightarrow & EXT_R^k(M, N)_\sigma & \longrightarrow & 0 \\ & & \downarrow \alpha & & \downarrow \beta & & \\ 0 & \longrightarrow & Ext_{gr\text{-}R}^{k-1}(M(\sigma^{-1}), L) & \longrightarrow & Ext_{gr\text{-}R}^k(M(\sigma^{-1}), N) & \longrightarrow & 0 \end{array}$$

By induction hypothesis, α is isomorphism, and then so is β . ■

In the following proposition, we give some equivalent characterizations of n -FCP-gr-projective modules with respect to special gr-copure short exact sequences.

Proposition 3.1.8. *Let R be a graded ring and $n \geq 1$ an integer. Then, the following statements are equivalent for a graded right R -module M .*

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- (1) M is n -FCP- gr -projective.
- (2) M is gr -projective with respect to all special short exact sequences in gr - R .
- (3) The exact sequence $0 \rightarrow A \rightarrow B \rightarrow M \rightarrow 0$ in gr - R is special gr -copure.
- (4) $M(\sigma^{-1})$ is n -FCP- gr -projective for any $\sigma \in G$.
- (5) There exists a special gr -copure short exact sequence $0 \rightarrow K \rightarrow P \rightarrow M \rightarrow 0$ in gr - R , where P is gr -projective.
- (6) There exists a special gr -copure short exact sequence $0 \rightarrow K \rightarrow P \rightarrow M \rightarrow 0$ in gr - R , where P is n -FCP- gr -projective.
- (7) $M(\sigma^{-1})$ is gr -projective with respect to all special short exact sequences in gr - R for any $\sigma \in G$.

Proof. (1) \Rightarrow (2) Let $0 \rightarrow K^{n-1} \rightarrow E^n \rightarrow K^n \rightarrow 0$ be a special short exact sequence with respect to any n -copresented graded right R -module U . Then, $\text{EXT}_R^1(M, K^{n-1}) \cong \text{EXT}_R^n(M, U) = 0$

(2) \Rightarrow (3), (3) \Rightarrow (1), (5) \Rightarrow (6), and (6) \Rightarrow (1) are clear.

(4) \Rightarrow (1) It is clear, since by Lemma 3.1.7, $\text{EXT}_R^n(M, U) \cong \text{EXT}_{gr-R}^n(M(\sigma^{-1}), U)$ for any n -copresented graded right R -module U and any $\sigma \in G$.

(1) \Rightarrow (5) Let M be a graded right R -module. Then, there is an exact sequence $0 \rightarrow K \rightarrow P \rightarrow M \rightarrow 0$ in gr - R with P is gr -projective. By (1), $\text{EXT}_R^1(M, K^{n-1}) \cong \text{EXT}_R^n(M, U) = 0$ for any special gr -copresented K^{n-1} and any n -copresented graded right module U . So, (5) follows.

(2) \Rightarrow (7) It is trivial, since $\text{HOM}_{gr-R}(M(\sigma^{-1}), -) \cong \text{HOM}_R(M, -)_\sigma$.

(2) \Rightarrow (4) Assume that U is an n -copresented graded right R -module and $0 \rightarrow K^{n-1} \rightarrow E^n \rightarrow K^n \rightarrow 0$ is a special short exact sequence in gr - R , where E^n is finitely cogenerated injective. Then, we have the following

$$0 \longrightarrow \text{HOM}_R(M, K^{n-1})_\tau \longrightarrow \text{HOM}_R(M, E^n)_\tau \longrightarrow \text{HOM}_R(M, K^n)_\tau \longrightarrow 0.$$

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Consider the following commutative diagram:

$$\begin{array}{ccccccc}
 0 & \longrightarrow & \text{HOM}_R(M, K^{n-1})_{\tau\sigma} & \longrightarrow & \text{HOM}_R(M, E^n)_{\tau\sigma} & \longrightarrow & \text{HOM}_R(M, K^n)_{\tau\sigma} \rightarrow 0 \\
 & & \downarrow \cong & & \downarrow \cong & & \downarrow \cong \\
 0 & \longrightarrow & \text{Hom}_{\text{gr-}R}(M((\tau\sigma)^{-1}), K^{n-1}) & \longrightarrow & \text{Hom}_{\text{gr-}R}(M((\tau\sigma)^{-1}), E^n) & \longrightarrow & \text{Hom}_{\text{gr-}R}(M((\tau\sigma)^{-1}), K^n) \\
 & & \downarrow \cong & & \downarrow \cong & & \downarrow \cong \\
 0 & \longrightarrow & \text{HOM}_R(M(\sigma^{-1}), K^{n-1})_{\tau} & \longrightarrow & \text{HOM}_R(M(\sigma^{-1}), E^n)_{\tau} & \longrightarrow & \text{HOM}_R(M(\sigma^{-1}), K^n)_{\tau}
 \end{array}$$

with the upper row exact for every $\tau \in G$. So, we deduce that

$$0 \longrightarrow \text{HOM}_R(M(\sigma^{-1}), K^{n-1})_{\tau} \longrightarrow \text{HOM}_R(M(\sigma^{-1}), E^n)_{\tau} \longrightarrow \text{HOM}_R(M(\sigma^{-1}), K^n)_{\tau} \longrightarrow 0$$

is exact, which gives rise to the exactness of

$$0 \longrightarrow \text{HOM}_R(M(\sigma^{-1}), K^{n-1}) \longrightarrow \text{HOM}_R(M(\sigma^{-1}), E^n) \longrightarrow \text{HOM}_R(M(\sigma^{-1}), K^n) \longrightarrow 0$$

Hence, $0 = \text{EXT}_R^1(M(\sigma^{-1}), K^{n-1}) \cong \text{EXT}_R^n(M(\sigma^{-1}), U)$ and consequently, $M(\sigma^{-1})$ is n -FCP-gr-projective. ■

Transfer result of n -FCP-gr-projective modules with respect to the functor F is given in the following result.

Proposition 3.1.9. *Let R be a graded ring. If M is an n -FCP-projective right R -module, then $F(M)$ is n -FCP-gr-projective.*

Proof. Let $0 \rightarrow K^{n-1} \rightarrow E^n \rightarrow K^n \rightarrow 0$ be a special short exact sequence in gr- R and $f : F(M) \rightarrow K^n$ a graded morphism. Since F is a right adjoint functor of the forgetful functor, we have the commutative diagram:

$$\begin{array}{ccccccc}
 & & & & M & & \\
 & & & & \downarrow & \searrow & \\
 0 & \longrightarrow & K^{n-1} & \longrightarrow & E^n & \longrightarrow & K^n \longrightarrow 0
 \end{array}$$

Now, again by the adjoint situation between the forgetful functor and F we have a graded morphism $F(M) \rightarrow E^n$ such that the following diagram is commutative:

$$\begin{array}{ccccccc}
 & & & & F(M) & & \\
 & & & & \downarrow & \searrow & \\
 0 & \longrightarrow & K^{n-1} & \longrightarrow & E^n & \xrightarrow{g} & K^n \longrightarrow 0
 \end{array}$$

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which shows that $F(M)$ is gr-projective with respect to all the special short exact sequences. Let $f : F(M)(\sigma^{-1}) \rightarrow K^n$ be a graded morphism for any $\sigma \in G$. Since, the exact sequence

$$0 \longrightarrow K^{n-1}(\sigma) \longrightarrow E^n(\sigma) \xrightarrow{T_\sigma(g)} K^n(\sigma) \longrightarrow 0$$

exists and $K^{n-1}(\sigma)$ is special gr-copresented. So, there exists a graded morphism $h : F(M) \rightarrow E^n(\sigma)$ such that $T_\sigma(g)h = T_\sigma(f)$, and so $gT_{\sigma^{-1}}(h) = f$ for $T_{\sigma^{-1}}(h) : F(M)(\sigma^{-1}) \rightarrow E^n$. Therefore, for any $\sigma \in G$, $F(M)(\sigma^{-1})$ is gr-projective with respect to all the special short exact sequences and consequently by Proposition 3.1.8, $F(M)$ is n -FCP-gr-projective. ■

Also, as for the classical projective notion, the class $gr\text{-}\mathcal{FCP}_n$ in $gr\text{-}R$ is closed under direct limits.

Proposition 3.1.10. *Let R be a graded ring, K^{n-1} a special gr-copresented and $\{M_i\}_{i \in I}$ a direct system of modules in $gr\text{-}R$ with I directed. Then:*

$$(1) \text{HOM}_R(\varinjlim M_i, K^{n-1}) \cong \varprojlim \text{HOM}_R(M_i, K^{n-1}).$$

$$(2) \text{EXT}_R^1(\varinjlim M_i, K^{n-1}) \cong \varprojlim \text{EXT}_R^1(M_i, K^{n-1}).$$

Proof. (1) For any $\sigma \in G$, we have

$$\text{HOM}_R(\varinjlim M_i, K^{n-1}) = \bigoplus_{\sigma \in G} \text{HOM}_R(\varinjlim M_i, K^{n-1})_\sigma \cong \bigoplus_{\sigma \in G} \text{Hom}_{gr\text{-}R}(\varinjlim M_i(\sigma^{-1}), K^{n-1}).$$

By [68, Proposition 5.26],

$$\bigoplus_{\sigma \in G} \text{Hom}_{gr\text{-}R}(\varinjlim M_i(\sigma^{-1}), K^{n-1}) \cong \varprojlim \bigoplus_{\sigma \in G} \text{Hom}_{gr\text{-}R}(M_i(\sigma^{-1}), K^{n-1}).$$

So,

$$\varprojlim \bigoplus_{\sigma \in G} \text{Hom}_{gr\text{-}R}(M_i(\sigma^{-1}), K^{n-1}) \cong \varprojlim \bigoplus_{\sigma \in G} \text{HOM}_R(M_i, K^{n-1})_\sigma = \varprojlim \text{HOM}_R(M_i, K^{n-1}).$$

(2) Let the exact sequence $0 \rightarrow K^{n-1} \rightarrow E^n \rightarrow K^n \rightarrow 0$ be a special short exact sequence in $gr\text{-}R$. Then, by (1), the following commutative diagram exists:

$$\begin{array}{ccccccc} \text{HOM}_R(\varinjlim M_i, E^n) & \longrightarrow & \text{HOM}_R(\varinjlim M_i, K^n) & \longrightarrow & \text{EXT}_R^1(\varinjlim M_i, K^{n-1}) & \longrightarrow & 0 \\ \downarrow \cong & & \downarrow \cong & & \downarrow & & \\ \varprojlim \text{HOM}_R(M_i, E^n) & \longrightarrow & \varprojlim \text{HOM}_R(M_i, K^n) & \longrightarrow & \varprojlim \text{EXT}_R^1(M_i, K^{n-1}) & \longrightarrow & 0 \end{array}$$

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Therefore, $\text{EXT}_R^1(\varinjlim M_i, K^{n-1}) \cong \varinjlim \text{EXT}_R^1(M_i, K^{n-1})$. ■

Corollary 3.1.11. *Let R be a graded ring. Then, the class $gr\text{-FCP}_n$ in $gr\text{-}R$ is closed under direct limits.*

Proof. Let U be an n -copresented graded right module, and let $\{M_i\}_{i \in I}$ be a family of n -FCP- gr -projective right modules. Then, by Proposition 3.1.10,

$$\text{EXT}_R^n(\varinjlim M_i, U) \cong \text{EXT}_R^1(\varinjlim M_i, K^{n-1}) \cong \varprojlim \text{EXT}_R^1(M_i, K^{n-1}) \cong \varprojlim \text{EXT}_R^n(M_i, U),$$

where K^{n-1} is special gr -copresented. ■

3.2 n - gr -cocoherent rings

In this section, some characterizations of n -FCP- gr -projective right modules on right n - gr -cocoherent rings are given.

We state the following lemmas that are derived from [78, Theorem 2.12].

Lemma 3.2.1. *Let R be a right n - gr -cocoherent ring and M a graded right R -module. Then, the following statements are equivalent:*

- (1) $n\text{-FCP-}gr\text{-pd}(M) \leq k$;
- (2) $\text{EXT}_R^{k+1}(M, K^{n-1}) = 0$ for any special gr -copresented K^{n-1} .

Proof. (2) \implies (1) is trivial by Definition 2.1.17.

(1) \implies (2) Use induction on k . Clear if $n\text{-FCP-}gr\text{-pd}M = k$. Let $n\text{-FCP-}gr\text{-pd}M \leq k - 1$. If $0 \rightarrow K^{n-1} \rightarrow E^n \rightarrow K^n \rightarrow 0$ is a special short exact sequence in $gr\text{-}R$ with respect to any n -copresented graded right R -module U , then from the n - gr -cocoherence of R we deduce that K^n is special gr -copresented. Also, we have $\text{EXT}_R^k(M, K^n) \cong \text{EXT}_R^{k+1}(M, K^{n-1})$. So by induction hypothesis, $\text{EXT}_R^k(M, K^n) = 0$, and consequently $\text{EXT}_R^{k+1}(M, K^{n-1}) = 0$, which completes the proof. ■

Lemma 3.2.2. *Let R be a right n - gr -cocoherent ring, M a graded right R -module and k a non-negative integer. Then, the following statements are equivalent:*

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- (1) n -FCP- gr - $pd(M) \leq k$.
- (2) $\text{EXT}_R^{k+l}(M, K^{n-1}) = 0$ for any special gr -copresented K^{n-1} and all positive integers l .
- (3) $\text{EXT}_R^{k+1}(M, K^{n-1}) = 0$ for any special gr -copresented K^{n-1} .
- (4) There exists an exact sequence

$$0 \rightarrow P_k \xrightarrow{f_k} P_{k-1} \xrightarrow{f_{k-1}} \dots \xrightarrow{f_1} P_0 \xrightarrow{f_0} M \rightarrow 0$$

in gr - R with P_0, P_1, \dots, P_k are n -FCP- gr -projective.

- (5) n -FCP- gr - $pd(M(\sigma^{-1})) \leq k$ for any $\sigma \in G$.

Proof. (1) \implies (2) If n -FCP- gr - $pd(M) \leq k$, then n -FCP- gr - $pd(M) \leq k + l - 1$. So, by Lemma 3.2.1, $\text{EXT}_R^{k+l}(M, K^{n-1}) = 0$.

(4) \implies (1) Since R is right n - gr -cocoherent, by Lemma 4.1, $\text{EXT}_R^j(P_i, K^{n-1}) = 0$ for any special gr -copresented K^{n-1} , all positive integers j , and any $0 \leq i \leq k$. So, by (4), we have:

$$\text{EXT}_R^{k+1}(M, K^{n-1}) \cong \text{EXT}_R^k(\ker(f_0), K^{n-1}) \cong \dots \cong \text{EXT}_R^1(P_k, K^{n-1}).$$

Hence, by Lemma 3.2.1, n -FCP- gr - $pd M \leq k$.

(2) \implies (3) It is obvious.

(3) \implies (4) For every graded right R -module M , there exists an exact sequence

$$0 \longrightarrow P_k \longrightarrow P_{k-1} \longrightarrow \dots \longrightarrow P_1 \longrightarrow P_0 \longrightarrow M \longrightarrow 0$$

in gr - R with P_0, P_1, \dots, P_{k-1} are gr -projective. Therefore, for any positive integer l , $\text{EXT}_R^l(P_i, K^{n-1}) = 0$ for all special gr -copresented modules K^{n-1} and any $0 \leq i \leq k - 1$. Let $K_i = \ker(P_i \rightarrow P_{i-1})$. Then,

$$\text{EXT}_R^{k+1}(M, K^{n-1}) \cong \text{EXT}_R^k(K_0, K^{n-1}) \cong \text{EXT}_R^{k-1}(K_1, K^{n-1}) \cong \text{EXT}_R^1(P_k, K^{n-1}).$$

By (3), $\text{EXT}_R^{k+1}(M, K^{n-1}) = 0$, and so $\text{EXT}_R^1(P_k, K^{n-1}) = 0$, which means that P_k is n -FCP- gr -projective.

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(1) \iff (5) Use induction on k . If $k = 0$, then by Proposition 3.1.8, M is n -FCP-gr-projective if and only if $M(\sigma^{-1})$ is n -FCP-gr-projective for any $\sigma \in G$. Assume that $k > 0$. There is an exact sequence $0 \rightarrow L \xrightarrow{f} P \rightarrow M \rightarrow 0$ in $\text{gr-}R$, where P is gr-projective. For any $\sigma \in G$, it follows that the exact sequence

$$0 \longrightarrow L(\sigma^{-1}) \xrightarrow{T_{\sigma^{-1}}(f)} P(\sigma^{-1}) \longrightarrow M(\sigma^{-1}) \longrightarrow 0,$$

where $P(\sigma^{-1})$ is gr-projective exists. So, by Lemma 3.1.7, for every special gr -copresented K^{n-1} and any $\tau \in G$, we have:

$$\begin{aligned} \text{EXT}_R^{k+1}(M(\sigma^{-1}), K^{n-1})_\tau &\cong \text{EXT}_R^k(L(\sigma^{-1}), K^{n-1})_\tau \\ &\cong \text{Ext}_{\text{gr-}R}^k(L(\tau\sigma)^{-1}, K^{n-1}) \\ &\cong \text{EXT}_R^k(L, K^{n-1})_{\tau\sigma} \\ &\cong \text{EXT}_R^{k+1}(M, K^{n-1})_{\tau\sigma}. \end{aligned}$$

By induction hypothesis, $n\text{-FCP-gr-pd}(L(\sigma^{-1})) \leq k - 1$ if and only if $n\text{-FCP-gr-pd}(L) \leq k - 1$. So, we deduce that $n\text{-FCP-gr-pd}(M(\sigma^{-1})) \leq k$ if and only if $n\text{-FCP-gr-pd}(M) \leq k$. ■

Corollary 3.2.3. *Let R be a right n -gr-coherent ring. Then, the following statements are equivalent:*

(1) *If $\phi : N \rightarrow M$ is an n -FCP-gr-projective preenvelope, then N has an epic n -FCP-gr-projective preenvelope.*

(2) *The cokernel of any n -FCP-gr-projective preenvelope of a graded right R -module is n -FCP-gr-projective.*

Moreover, if every submodule of an n -FCP-gr-projective graded right R -module has an n -FCP-gr-projective preenvelope, the above are equivalent to:

(3) *$r.n\text{-FCP-gr-dim}(R) \leq 1$.*

Proof. (1) \implies (2) Let $\phi : A \rightarrow B$ be an n -FCP-gr-projective preenvelope. Then, $f : \text{Im}(\phi) \rightarrow B$ is an n -FCP-gr-projective preenvelope. By (1), there is an epic n -FCP-gr-projective preenvelope $g : \text{Im}(\phi) \rightarrow C$. Consider the following commutative

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diagram, where D is a pushout of two maps f and g :

$$\begin{array}{ccccccc} 0 & \longrightarrow & \text{Im}(\phi) & \xrightarrow{f} & B & \longrightarrow & \text{Coker}(\phi) \longrightarrow 0 \\ & & \downarrow g & & \downarrow \beta & & \downarrow id \\ & & C & \xrightarrow{\alpha} & D & \longrightarrow & \text{Coker}(\phi) \longrightarrow 0 \end{array}$$

By [68, Exercise 5.10], α is injective and β is surjective. On the other hand, $D = \alpha(C) + \beta(B)$. Since β is surjective, $D = \alpha(C) + D$ and so, $\alpha(C) \subseteq D$. Also, by using of preenvelopes f and g , there is a graded morphism $h : D \rightarrow C$ such that $h\alpha = 1_C$. Hence, $D \cong C \oplus \text{Coker}(\phi)$. Similarly, $D \cong M$. Therefore, from n -FCP-gr-projectivity M , we deduce that $\text{Coker}(\phi)$ is n -FCP-gr-projective.

(2) \implies (1) Let $\phi : N \rightarrow M$ be an n -FCP-gr-projective preenvelope. It is enough to show that $\text{Im}(\phi)$ is n -FCP-gr-projective. Consider the exact sequence $0 \rightarrow \text{Im}(\phi) \rightarrow M \rightarrow \text{Coker}(\phi) \rightarrow 0$. By hypothesis, $\text{Coker}(\phi)$ is n -FCP-gr-projective, and so for any special gr-copresented K^{n-1} , we have:

$$0 = \text{EXT}_R^1(M, K^{n-1}) \longrightarrow \text{EXT}_R^1(\text{Im}(\phi), K^{n-1}) \longrightarrow \text{EXT}_R^2(\text{Coker}(\phi), K^{n-1}).$$

Hence, by Lemma 3.2.2 and (2), $\text{EXT}_R^2(\text{Coker}(\phi), K^{n-1}) = 0$. Therefore,

$$0 = \text{EXT}_R^1(\text{Im}(\phi), K^{n-1}) \cong \text{EXT}_R^n(\text{Im}(\phi), U)$$

for any n -copresented U in $\text{gr-}R$, and then $\text{Im}(\phi)$ is n -FCP-gr-projective.

(2) \implies (3) Let M be a graded right R -module. Then, there exists an exact sequence $0 \rightarrow K \rightarrow P \rightarrow M \rightarrow 0$ in $\text{gr-}R$, where P is a projective graded right R -module. If K is n -FCP-gr-projective, then $r.n.\text{FCP-gr-dim}(R) \leq 1$. So, we show that K is n -FCP-gr-projective. If $\phi : K \rightarrow N$ is an n -FCP-gr-projective preenvelope, then ϕ is injective. So similar to the proof of (2) \implies (1), we get that K is n -FCP-gr-projective.

(3) \implies (1) Let $\phi : N \rightarrow M$ be an n -FCP-gr-projective preenvelope. Then, by Lemma 3.2.2, the exact sequence $0 \rightarrow \text{Im}(\phi) \rightarrow M \rightarrow \text{Coker}(\phi) \rightarrow 0$ implies that $\text{Im}(\phi)$ is n -FCP-gr-projective, and so, $N \rightarrow \text{Im}(\phi)$ is an epic n -FCP-gr-projective preenvelope of N . ■

In the following, we present one of the main results in this paper.

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Theorem 3.2.4. *Let R be a graded ring. Then, the following statements are equivalent:*

- (1) R is right n -gr-cocoherent;
- (2) For every exact sequence $0 \rightarrow A \rightarrow B \rightarrow C \rightarrow 0$ in $\text{gr-}R$, A is n -FCP-gr-projective if B and C are n -FCP-gr-projective.

Proof. (1) \implies (2) Let U be an n -copresented graded right R -module. Then, by Lemma 3.2.2(2), we have: $0 = \text{EXT}_R^n(B, U) \rightarrow \text{EXT}_R^n(A, U) \rightarrow \text{EXT}_R^{n+1}(C, U) = 0$. So, $\text{EXT}_R^n(A, U) = 0$, and hence A is n -FCP-gr-projective.

(2) \implies (1) Let U be an n -copresented graded right R -module and $0 \rightarrow K^{n-1} \rightarrow E^n \rightarrow K^n \rightarrow 0$ a special short exact sequence in $\text{gr-}R$ with respect to U . We show that U is $(n+1)$ -copresented. For this, it is enough to say that K^n is special gr-copresented. Let M be 1-FCP-projective and $0 \rightarrow K \rightarrow P \rightarrow M \rightarrow 0$ an exact sequence in $\text{Mod-}R$ with P is projective. Then, $0 \rightarrow F(K) \rightarrow F(P) \rightarrow F(M) \rightarrow 0$ is exact in $\text{gr-}R$, where by Proposition 3.1.9, $F(P)$ and $F(M)$ are 1-FCP-gr-projective. Hence, Remark 3.1.4 imply that $F(P)$ and $F(M)$ are n -FCP-gr-projective. So, by hypothesis, it follows that $F(K)$ is n -FCP-gr-projective. Since E^n is gr-injective, we have:

$$0 = \text{Ext}_{\text{gr-}R}^1(F(M), E^n) \longrightarrow \text{Ext}_{\text{gr-}R}^1(F(M), K^n) \longrightarrow \text{Ext}_{\text{gr-}R}^2(F(M), K^{n-1}) \longrightarrow 0.$$

So, $\text{Ext}_{\text{gr-}R}^1(F(M), K^n) \cong \text{Ext}_{\text{gr-}R}^2(F(M), K^{n-1})$. Also, we have:

$$0 = \text{Ext}_{\text{gr-}R}^1(F(P), K^{n-1}) \longrightarrow \text{Ext}_{\text{gr-}R}^1(F(K), K^{n-1}) \longrightarrow \text{Ext}_{\text{gr-}R}^2(F(M), K^{n-1}) \longrightarrow 0.$$

Consequently $\text{Ext}_{\text{gr-}R}^1(F(K), K^{n-1}) \cong \text{Ext}_{\text{gr-}R}^2(F(M), K^{n-1})$, and so, by Lemma 3.1.7, n -FCP-gr-projectivity $F(K)$ implies that $\text{Ext}_{\text{gr-}R}^1(F(K)(\sigma^{-1}), K^{n-1}) = 0$ for any $\sigma \in G$. Thus,

$$0 = \text{Ext}_{\text{gr-}R}^1(F(K), K^{n-1}) \cong \text{Ext}_{\text{gr-}R}^2(F(M), K^{n-1}) \cong \text{Ext}_{\text{gr-}R}^1(F(M), K^n).$$

Now, consider the following commutative diagram:

$$\begin{array}{ccccccc} 0 & \longrightarrow & \text{Hom}_{\text{gr-}R}(F(M), K^n) & \longrightarrow & \text{Hom}_{\text{gr-}R}(F(P), K^n) & \longrightarrow & \text{Hom}_{\text{gr-}R}(F(K), K^n) \longrightarrow 0 \\ & & \downarrow \cong & & \downarrow \cong & & \downarrow \cong \\ 0 & \longrightarrow & \text{Hom}_R(M, K^n) & \longrightarrow & \text{Hom}_R(P, K^n) & \longrightarrow & \text{Hom}_R(K, K^n) \end{array}$$

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with the upper row exact. Therefore, $\text{Ext}_{\text{gr-}R}^1(F(M), K^n) \cong \text{Ext}_R^1(M, K^n) = 0$ for any 1-FCP-projective R -module M , and it follows that K^n is 1-copresented. Hence, U is $(n + 1)$ -copresented. ■

Corollary 3.2.5. *Let R be a right n -gr-cocoherent ring. Then, a graded right R -module M is n -FCP-gr-projective if and only if every copure epimorphic images and copure submodules of M is n -FCP-gr-projective.*

Proof. (\implies) Let N be a copure submodule of n -FCP-gr-projective graded right R -module M . Then, the exact sequence $0 \rightarrow N \rightarrow M \rightarrow \frac{M}{N} \rightarrow 0$ is special gr-copure. So, by Proposition 3.1.8, $\frac{M}{N}$ is n -FCP-gr-projective, and hence, by Theorem 3.2.4, N is n -FCP-gr-projective.

(\impliedby) is clear. ■

Theorem 3.2.6. *The pair $(\text{gr-FCP}_n, \text{gr-FCP}_n^\perp)$ is hereditary cotorsion pair if and only if R is right n -gr-cocoherent.*

Proof. (\implies) Let M be a n -FCP-gr-projective right R -module. Then, there is an exact sequence $0 \rightarrow K \rightarrow P \rightarrow M \rightarrow 0$ in $\text{gr-}R$, where P is gr-projective. Thus, by Remark 3.1.4, P is n -FCP-gr-projective, too. Since $(\text{gr-FCP}_n, \text{gr-FCP}_n^\perp)$ is a hereditary cotorsion pair, we deduce that K is n -FCP-gr-projective, and then, by Theorem 3.2.4, it follows that R is n -gr-cocoherent.

(\impliedby) Note that we have to show that ${}^\perp(\text{gr-FCP}_n^\perp) = \text{gr-FCP}_n$. Let K^{n-1} be a special gr-copresented with respect to any n -copresented graded right R -module U and $M \in {}^\perp(\text{gr-FCP}_n^\perp)$. Then, $K^{n-1} \in \text{gr-FCP}_n^\perp$ and $M(\sigma^{-1}) \in {}^\perp(\text{gr-FCP}_n^\perp)$ for all $\sigma \in G$ by analogy with the proof of Proposition 3.1.9. Therefore, by Lemma 3.1.7, $\text{EXT}_R^1(M, K^{n-1})_\sigma \cong \text{Ext}_{\text{gr-}R}^1(M(\sigma^{-1}), K^{n-1}) = 0$, and consequently by Lemma 3.2.2, M is n -FCP-gr-projective. Let $0 \rightarrow F' \rightarrow F \rightarrow F'' \rightarrow 0$ be a exact sequence of modules in $\text{gr-}R$, where F and F'' are n -FCP-gr-projective. Then, by Theorem 3.2.4, F' is n -FCP-gr-projective, since R is n -gr-cocoherent. So, it follows that the pair $(\text{gr-FCP}_n, \text{gr-FCP}_n^\perp)$ is a hereditary cotorsion pair. ■

We denote $(\text{gr-FCP}_n)^* = \{M^* \mid M \in \text{gr-FCP}_n\}$. The following lemma shows the connection between n -FCP-gr-projective and n -FP-gr-injective modules.

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Lemma 3.2.7. *Let R be a graded ring of type G .*

- (1) *If U is n -presented graded left R -module, then U^* is n -copresented graded right R -module.*
- (2) $(gr\text{-}\mathcal{FCP}_n)^* \subseteq gr\text{-}\mathcal{FI}_n$

Proof. (1) Let U be n -presented graded left R -module. Then, there exists an exact sequence

$$F_n \longrightarrow F_{n-1} \longrightarrow \cdots \longrightarrow F_0 \longrightarrow U \longrightarrow 0$$

in $R\text{-gr}$ with each F_i is finitely generated free. So, by graded version of [68, Lemma 3.53], there is an exact sequence

$$0 \longrightarrow U^* \longrightarrow F_0^* \longrightarrow \cdots \longrightarrow F_{n-1}^* \longrightarrow F_n^*$$

in $gr\text{-}R$. It suffices to show that any F_i^* is finitely cogenerated. So, we prove that every F_i^* is injective, too. Consider the short exact sequence $0 \rightarrow A \rightarrow B \rightarrow C \rightarrow 0$ in $gr\text{-}R$. Then, there exists the following commutative diagram with the upper row exact for any $\sigma \in G$:

$$\begin{array}{ccccccc} 0 & \longrightarrow & \text{HOM}_R(C, F_i^*) & \longrightarrow & \text{HOM}_R(B, F_i^*) & \longrightarrow & \text{HOM}_R(A, F_i^*) \\ & & \downarrow \cong & & \downarrow \cong & & \downarrow \cong \\ 0 & \longrightarrow & \bigoplus_{\sigma \in G} \text{Hom}_{gr\text{-}R}(C, F_i^*)_{\sigma} & \longrightarrow & \bigoplus_{\sigma \in G} \text{Hom}_{gr\text{-}R}(B, F_i^*)_{\sigma} & \longrightarrow & \bigoplus_{\sigma \in G} \text{Hom}_{gr\text{-}R}(A, F_i^*)_{\sigma} \\ & & \downarrow \cong & & \downarrow \cong & & \downarrow \cong \\ 0 & \longrightarrow & \bigoplus_{\sigma \in G} \text{Hom}_{R\text{-}gr}(F_i, C^*)_{\sigma} & \longrightarrow & \bigoplus_{\sigma \in G} \text{Hom}_{R\text{-}gr}(F_i, B^*)_{\sigma} & \longrightarrow & \bigoplus_{\sigma \in G} \text{Hom}_{R\text{-}gr}(F_i, A^*)_{\sigma} \\ & & \downarrow \cong & & \downarrow \cong & & \downarrow \cong \\ 0 & \longrightarrow & \text{HOM}_R(F_i, C^*) & \longrightarrow & \text{HOM}_R(F_i, B^*) & \longrightarrow & \text{HOM}_R(F_i, A^*) \rightarrow 0 \end{array}$$

So, $\text{EXT}_R^1(C, F_i^*) = 0$, and hence any F_i^* is injective.

(2) By [77, Definition 3.1], Let $0 \rightarrow K_{n-1} \rightarrow P_{n-1} \rightarrow K_n \rightarrow 0$ be a special short exact sequence with respect to any n -presented graded left R -module U . Then, by (1), K_{n-1}^* be special gr -copresented. So, if M is n -FCP- gr -projective right R -module, then similar to the proof (1), $0 = \text{EXT}_R^1(M, K_{n-1}^*) \cong \text{EXT}_R^1(K_{n-1}, M^*)$. On the other hand, $\text{EXT}_R^n(U, M^*) \cong \text{EXT}_R^1(K^{n-1}, M^*)$ by [77, Definitions 3 and 4]. Therefore, M^* is n -FP- gr -injective right R -module, and then we conclude that $(gr\text{-}\mathcal{FCP}_n)^* \subseteq gr\text{-}\mathcal{FI}_n$. ■

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In the following theorem, by using the previous results, we present some equivalent characterizations to that each module in $\text{gr-}R$ is n -FCP- gr -projective.

Theorem 3.2.8. *Let R be a graded ring of type G . Then, the following statements are equivalent:*

- (1) *Every module in $\text{gr-}R$ is n -FCP- gr -projective;*
- (2) *$\text{gr-id}(U) \leq n - 1$ for any n -copresented graded right R -module U ;*
- (3) *Every special gr -copresented module in $\text{gr-}R$ is gr -injective;*
- (4) *$(\text{gr-}\mathcal{FCP}_n, \text{gr-}\mathcal{FCP}_n^\perp)$ is perfect hereditary cotorsion pair and N has an n -FCP- gr -projective cover with the unique mapping property for any $N \in \text{gr-}\mathcal{FCP}_n^\perp$;*
- (5) *N is gr -injective for any $N \in \text{gr-}\mathcal{FCP}_n^\perp$;*
- (6) *$N(\sigma)$ is gr -injective for any $N \in \text{gr-}\mathcal{FCP}_n^\perp$ and any $\sigma \in G$;*
- (7) *Every graded right R -module has an n -FCP- gr -projective cover with the unique mapping property*
- (8) *R is right n - gr -cocoherent and N is n -FCP- gr -projective for any $N \in \text{gr-}\mathcal{FCP}_n^\perp$;*

Proof. (1) \implies (2), (2) \implies (3) and (3) \implies (1) are clear by Proposition 3.1.8.

par (1) \implies (5) and (5) \implies (3) are obvious.

(5) \iff (6) it follows from $\text{HOM}_R(-, N)_\sigma \cong \text{Hom}_{\text{gr-}R}(-, N(\sigma))$.

(1) \implies (7) First, we show that the class $\text{gr-}\mathcal{FCP}_n$ is covering. If $M \in \text{gr-}\mathcal{FCP}_n$, then by Lemma 3.2.7, $M^* \in \text{gr-}\mathcal{FP}_n$. Contrary, if $M^* \in \text{gr-}\mathcal{FP}_n$, then [77, Proposition 3.8] implies that, M is n - gr -flat. Thus by (1) $M \in \text{gr-}\mathcal{FCP}_n$. On the other hands, the class $\text{gr-}\mathcal{FP}_n$ is closed under direct summands and direct sums by [77, Propositions 3.7 and 3.16]. So, we obtain that $(\text{gr-}\mathcal{FCP}_n, \text{gr-}\mathcal{FI}_n)$ is a duality pair. Also, by (1), it follows that the class $\text{gr-}\mathcal{FCP}_n$ is closed under gr -pure submodules, gr -pure quotients and gr -pure extensions. Therefore, by Corollary 3.1.11 and [77, Theorem 4.2], the class $\text{gr-}\mathcal{FCP}_n$ is covering and hence by hypothesis, (7) follows.

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(7) \implies (1) Let N be a graded module in $\text{gr-}R$. Then, there is a commutative diagram with exact rows:

$$\begin{array}{ccccccc}
 & & & M' & & & \\
 & & \phi \swarrow & \downarrow \alpha\phi & \searrow 0 & & \\
 0 & \longrightarrow & K & \xrightarrow{\alpha} & M & \xrightarrow{\psi} & N \longrightarrow 0 \\
 & & \downarrow & & & & \\
 & & 0 & & & &
 \end{array}$$

where, ψ and ϕ are n -FCP-gr-projective cover with the unique mapping property. Since $\psi\alpha\phi = 0 = \psi$, we have $\alpha\phi = 0$ by (7). Therefore, $K = \text{im}(\phi) \subseteq \ker(\alpha) = 0$, and so $K = 0$. Thus, $N \cong M$, and hence every graded module N is n -FCP-gr-projective.

(1) \implies (8) Let M be a n -FCP-gr-projective right R -module. Then, there is an exact sequence $0 \rightarrow K \rightarrow P \rightarrow M \rightarrow 0$ in $\text{gr-}R$, where P and K are n -FCP-gr-projective by (1). So, by Theorem 3.2.4, it follows that R is n -gr-cocoherent. Also, by hypothesis, N is n -FCP-gr-projective for any $N \in \text{gr-}\mathcal{FCP}_n^\perp$.

(8) \implies (3) Consider the special short exact sequence $\Delta' : 0 \rightarrow K^{n-1} \rightarrow E^{n-1} \rightarrow K^n \rightarrow 0$ in $\text{gr-}R$ with respect to any n -copresented graded right module U , where K^{n-1} is gr-copresented and K_n is n -gr-cogenerated. But K^n is gr-copresented, too, since R is n -gr-cocoherent. Thus, $K^n \in \text{gr-}\mathcal{FCP}_n$ and consequently $0 = \text{EXT}_R^n(K^n, U) \cong \text{EXT}_R^1(K^n, K^{n-1})$, since K^n is n -FCP-gr-projective by (8). Therefore, Δ' is split, and we deduce that K^{n-1} is gr-injective.

(4) \implies (8) By Theorem 3.2.6, R is n -gr-cocoherent. Let $N \in \text{gr-}\mathcal{FCP}_n^\perp$. If $\phi : M \rightarrow N$ is an n -FCP-gr-projective cover with the unique mapping property, then $\ker\phi \in \text{gr-}\mathcal{FCP}_n^\perp$. Thus, similar to the proof of (7) \implies (1), we get that N is n -FCP-gr-projective.

(8) \implies (4) By Theorem 3.2.6, $(\text{gr-}\mathcal{FCP}_n, \text{gr-}\mathcal{FCP}_n^\perp)$ is hereditary cotorsion pair. Also, $R \in \text{gr-}\mathcal{FCP}_n$ and by Corollary 3.1.11 and (8) \implies (3) \implies (1), $\text{gr-}\mathcal{FCP}_n$ is closed under direct sum and extensions. Therefore, we deduce that $(\text{gr-}\mathcal{FCP}_n, \text{gr-}\mathcal{FCP}_n^\perp)$ is a perfect hereditary cotorsion pair. If N is n -FCP-gr-projective for any $N \in \text{gr-}\mathcal{FCP}_n^\perp$, then it is clear that N has an n -FCP-gr-projective cover with the unique mapping property. ■

A NEW APPROACH TO FLATNESS IN CATEGORIES OF SHEAVES

In this chapter, we develop a new treatment of the geometric flatness in categories of sheaves which shed more light on some of its various important aspects. Namely, in terms of this new approach, some classical schemes are characterized, some new results are established and also known ones are generalized.

4.1 Absolutely pure domains in a closed symmetric monoidal Grothendieck category: basic results

The aim of this section is thus to explore the meaning of absolutely pure domains in the closed symmetric monoidal Grothendieck category \mathcal{A} (see Section 1.4), aiming to establish fundamental and common results for $\mathcal{O}_X\text{-Mod}$ and $\Omega\text{coh}(X)$ that we will use in subsequent sections. We generalize the study done in [27] and we give new results.

In what follows, $\text{Flat}(\mathcal{A})$ will denote the class of flat objects in \mathcal{A} and $\text{Abs}(\mathcal{A})$ will denote the class of absolutely pure objects in \mathcal{A} .

Definition 4.1.1. *Given two objects M and N in \mathcal{A} , N is said to be absolutely M -pure if $M \otimes N \rightarrow M \otimes K$ is a monomorphism for every monomorphism $N \rightarrow K$.*

The absolutely pure domain of M is defined as the class

$$\mathcal{A}p_{\mathcal{A}}(M) := \{N \in \mathcal{A} : N \text{ is absolutely } M\text{-pure}\}.$$

The absolutely pure domain can be used to see when an object in \mathcal{A} is flat. This can be seen from the following result which simply follows from the definition.

Proposition 4.1.2. *Let M be an object in \mathcal{A} . Then, the following conditions are equivalent:*

(i) $M \in \text{Flat}(\mathcal{A})$.

(ii) $\mathcal{A}p_{\mathcal{A}}(M) = \mathcal{A}$.

The following result gives a simple characterization of the notion of absolutely pure domains.

Proposition 4.1.3. *The following statements are equivalent for any given objects M and N in \mathcal{A} .*

(i) $N \in \mathcal{A}p_{\mathcal{A}}(M)$.

(ii) $M \otimes N \rightarrow M \otimes E(N)$ is a monomorphism, where $E(N)$ is the injective envelope of N .

(iii) There exists a monomorphism $N \rightarrow A$ such that A is an absolutely pure object, and $M \otimes N \rightarrow M \otimes A$ is a monomorphism.

Proof. (i) \Rightarrow (ii) \Rightarrow (iii) clear.

(iii) \Rightarrow (i) Let K be an extension of N . Then, apply the functor $M \otimes -$ to the pushout diagram

$$\begin{array}{ccccc}
 & & 0 & & \\
 & & \downarrow & & \\
 0 & \longrightarrow & N & \longrightarrow & K \\
 & & \downarrow & & \downarrow \\
 0 & \longrightarrow & A & \longrightarrow & P
 \end{array}$$

to get

$$\begin{array}{ccc}
 M \otimes N & \longrightarrow & M \otimes K \\
 \downarrow & & \downarrow \\
 M \otimes A & \longrightarrow & M \otimes P
 \end{array}$$

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Thus, $M \otimes N \rightarrow M \otimes K$ is a monomorphism since the morphism $M \otimes N \rightarrow M \otimes A$ is monic by hypothesis, and the morphism $M \otimes A \rightarrow M \otimes P$ is monic since A is absolutely pure.

■

As a consequence, it is clear that $\mathcal{A}bs(\mathcal{A}) \subseteq \mathcal{A}p_{\mathcal{A}}(M)$ for any object $M \in \mathcal{A}$. Moreover, the intersection of absolutely pure domains of all objects of \mathcal{A} is the class of all absolutely pure objects.

It is a recognized fact that the tensor product commutes with the direct sum. From this fact, we have the following results.

Proposition 4.1.4. *Let $(M_i)_{i \in I}$ be a family of objects in \mathcal{A} for any index set I . Then, $\mathcal{A}p_{\mathcal{A}}(\bigoplus_{i \in I} M_i) = \bigcap_{i \in I} \mathcal{A}p_{\mathcal{A}}(M_i)$.*

Proof. Let $\mathcal{N} \in \bigcap_{i \in I} \mathcal{A}p_{\mathcal{A}}(\mathcal{M}_i)$, that is, $\mathcal{M}_i \otimes \mathcal{N} \rightarrow \mathcal{M}_i \otimes \mathcal{K}$ is monomorphism for every extension \mathcal{K} of \mathcal{N} and for each $i \in I$. Since direct sum is an exact functor and commutes with tensor products, then $\bigoplus_{i \in I} \mathcal{M}_i \otimes \mathcal{N} \rightarrow \bigoplus_{i \in I} \mathcal{M}_i \otimes \mathcal{K}$ is a monomorphism for every extension \mathcal{K} of \mathcal{N} . So, $\mathcal{N} \in \mathcal{A}p_{\mathcal{A}}(\bigoplus_{i \in I} \mathcal{M}_i)$, thus $\bigcap_{i \in I} \mathcal{A}p_{\mathcal{A}}(\mathcal{M}_i) \subseteq \mathcal{A}p_{\mathcal{A}}(\bigoplus_{i \in I} \mathcal{M}_i)$. The other inclusion will be verified by the universal property of direct sum.

■

Proposition 4.1.5. *If M is an object of \mathcal{A} and M' a pure subobject of M , then $\mathcal{A}p_{\mathcal{A}}(M) = \mathcal{A}p_{\mathcal{A}}(M') \cap \mathcal{A}p_{\mathcal{A}}(M/M')$.*

Proof. Consider the exact sequences

$$0 \rightarrow N \rightarrow E(N) \rightarrow E(N)/N \rightarrow 0$$

and

$$0 \rightarrow M' \rightarrow M \rightarrow M/M' \rightarrow 0.$$

We have the following commutative diagram:

$$\begin{array}{ccccccc}
 0 & \longrightarrow & M' \otimes N & \xrightarrow{\alpha} & M \otimes N & \xrightarrow{\beta} & (M/M') \otimes N \longrightarrow 0 \\
 & & \downarrow \gamma & & \downarrow \delta & & \downarrow \sigma \\
 0 & \longrightarrow & M' \otimes E(N) & \xrightarrow{\theta} & M \otimes E(N) & \xrightarrow{\lambda} & (M/M') \otimes E(N) \longrightarrow 0 \\
 & & \downarrow & & \downarrow & & \downarrow \\
 0 & \longrightarrow & M' \otimes (E(N)/N) & \longrightarrow & M \otimes (E(N)/N) & \longrightarrow & (M/M') \otimes (E(N)/N) \longrightarrow 0 \\
 & & \downarrow & & \downarrow & & \downarrow \\
 & & 0 & & 0 & & 0
 \end{array}$$

First, we assume that N is absolutely M -pure. Then, δ is a monomorphism. For the proof we need to show that γ and σ are monomorphisms. Since M' is a pure subobject of M , the rows are exact. Then, $\theta\gamma = \delta\alpha$ is a monomorphism, and so γ is a monomorphism. Now, the left two columns are exact, and so the right column is exact by the (3x3)-Lemma. Hence σ is a monomorphism. The converse follows by Five-Lemma. ■

Recall that for two objects M and N in an abelian category \mathcal{C} , M is said to be N -subinjective if for every morphism $f : N \rightarrow M$ and every monomorphism $g : N \rightarrow L$, there exists a morphism $h : L \rightarrow M$ such that $hg = f$.

The *subinjectivity domain*, or *domain of subinjectivity*, of an object M of \mathcal{C} is defined as

$$\underline{\mathfrak{In}}_{\mathcal{C}}^{-1}(M) := \{N \in \mathcal{C} : M \text{ is } N\text{-subinjective}\}.$$

The tensor product of objects $M, N \in \mathcal{A}$ satisfies symmetry (see Section 4.1), namely $M \otimes N \cong N \otimes M$, which follows that $\text{Hom}_{\mathcal{A}}(M \otimes N, I) \cong \text{Hom}_{\mathcal{A}}(N, M^+)$ since $\text{Hom}_{\mathcal{A}}(N \otimes M, I) \cong \text{Hom}_{\mathcal{A}}(N, M^+)$.

Proposition 4.1.6. *Let M and N be two objects of \mathcal{A} . Then, $N \in \mathcal{A}p_{\mathcal{A}}(M)$ if and only if $N \in \underline{\mathfrak{In}}_{\mathcal{A}}^{-1}(M^+)$.*

Proof. $N \in \mathcal{A}p_{\mathcal{A}}(M)$ means that the morphism $M \otimes N \rightarrow M \otimes E(N)$ is monic, which is equivalent to the surjectivity of the map $\text{Hom}_{\mathcal{A}}(E(N), M^+) \rightarrow \text{Hom}_{\mathcal{A}}(N, M^+)$. Thus, $N \in \mathcal{A}p_{\mathcal{A}}(M)$ if and only if $N \in \underline{\mathfrak{In}}_{\mathcal{A}}^{-1}(M^+)$. ■

The Proposition 4.1.8 provides new ways to treat the notion of absolutely pure domains. The following lemma will be useful to prove Proposition 4.1.6:

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Lemma 4.1.7. *Let M and N be objects of \mathcal{A} . Then, M is N -subinjective if and only if every morphism $N \rightarrow M$ factors through an injective object.*

Proof. The necessary condition is clear since \mathcal{A} has enough injectives.

Conversely, suppose that every morphism $N \rightarrow M$ factors through an injective object, consider any morphism $f : N \rightarrow M$ and any monomorphism $g : N \rightarrow K$. Then, by the assumption, there exists an injective object E and a commutative diagram:

$$\begin{array}{ccc} M & \xleftarrow{\alpha} & E \\ \uparrow f & \nearrow & \\ 0 \longrightarrow N & \xrightarrow{g} & K \end{array}$$

And by the injectivity of E the diagram

$$\begin{array}{ccc} M & \xleftarrow{\alpha} & E \\ \uparrow f & \nearrow & \uparrow \\ 0 \longrightarrow N & \xrightarrow{g} & K \end{array}$$

can be completed commutatively. ■

Proposition 4.1.8. *Let M and N be two objects of \mathcal{A} . Then, $N \in \mathcal{A}p_{\mathcal{A}}(M)$ if and only if every morphism $N \rightarrow M^+$ factors through an injective object.*

Proof. Lemma 4.1.7 gives that M^+ is N -subinjective if and only if every morphism $N \rightarrow M^+$ factors through an injective object. Therefore, we get the desired result from Proposition 4.1.6. ■

In the following result, we will investigate the absolute pure domain of an object that is embedded in a flat object.

In what follows, $\mathcal{I}nj(\mathcal{A})$ will denote the class of injective objects in \mathcal{A} .

Proposition 4.1.9. *Assume that \mathcal{A} has enough flat objects, and let $0 \rightarrow M \rightarrow F \rightarrow M' \rightarrow 0$ be a short exact sequence with F flat. Then, ${}^{\perp}M'^+ \subseteq \mathcal{A}p_{\mathcal{A}}(M)$. If moreover $\mathcal{I}nj(\mathcal{A}) \subseteq {}^{\perp}M'^+$, then ${}^{\perp}M'^+ = \mathcal{A}p_{\mathcal{A}}(M)$.*

Proof. Let $N \in {}^\perp M^+$. We apply the functor $\text{Hom}_{\mathcal{A}}(N, -)$ to the short exact sequence $0 \rightarrow M'^+ \rightarrow F^+ \rightarrow M^+ \rightarrow 0$, resulting in the following long exact sequence:

$$\longrightarrow \text{Hom}_{\mathcal{A}}(N, F^+) \longrightarrow \text{Hom}_{\mathcal{A}}(N, M^+) \longrightarrow \text{Ext}_{\mathcal{A}}^1(N, M'^+) \longrightarrow \text{Ext}_{\mathcal{A}}^1(N, F^+) \longrightarrow$$

Since $N \in {}^\perp M'^+$, $\text{Ext}_{\mathcal{A}}^1(N, M'^+) = 0$, which implies that $\text{Hom}_{\mathcal{A}}(N, F^+) \rightarrow \text{Hom}_{\mathcal{A}}(N, M^+)$ is epic. In other words, any morphism $N \rightarrow M^+$ can be factored through the injective object F^+ . Thus, by Proposition 4.1.8, $N \in \mathcal{A}p_{\mathcal{A}}(M)$.

Additionally, suppose that $\mathcal{I}nj(\mathcal{A}) \subseteq {}^\perp M'^+$. Let $N \in \mathcal{A}p_{\mathcal{A}}(M)$, and let $N \rightarrow E$ be a monomorphism, with E being injective object. We apply the functors $\text{Hom}_{\mathcal{A}}(E, -)$ and $\text{Hom}_{\mathcal{A}}(N, -)$ to the morphism $F^+ \rightarrow M^+$, resulting in the following commutative diagram:

$$\begin{array}{ccccccc} \text{Hom}_{\mathcal{A}}(E, F^+) & \longrightarrow & \text{Hom}_{\mathcal{A}}(E, M^+) & \longrightarrow & \text{Ext}_{\mathcal{A}}^1(E, M'^+) & \longrightarrow & \text{Ext}_{\mathcal{A}}^1(E, F^+) \\ \downarrow & & \downarrow & & & & \\ \text{Hom}_{\mathcal{A}}(N, F^+) & \longrightarrow & \text{Hom}_{\mathcal{A}}(N, M^+) & \longrightarrow & \text{Ext}_{\mathcal{A}}^1(N, M'^+) & \longrightarrow & \text{Ext}_{\mathcal{A}}^1(N, F^+) \end{array}$$

Since F^+ is injective, $\text{Ext}_{\mathcal{A}}^1(N, F^+) = 0$. To prove that $\text{Ext}_{\mathcal{A}}^1(N, M'^+) = 0$, it is sufficient to show that $\text{Hom}_{\mathcal{A}}(N, F^+) \rightarrow \text{Hom}_{\mathcal{A}}(N, M^+)$ is epic. Since $N \in \mathcal{A}p_{\mathcal{A}}(M)$, the morphism $\text{Hom}_{\mathcal{A}}(E, M^+) \rightarrow \text{Hom}_{\mathcal{A}}(N, M^+)$ is an epimorphism, and $\text{Hom}_{\mathcal{A}}(E, F^+) \rightarrow \text{Hom}_{\mathcal{A}}(E, M^+)$ is epic, (as $\text{Ext}_{\mathcal{A}}^1(E, M'^+) = 0$ by assumption). This implies that $\text{Hom}_{\mathcal{A}}(N, F^+) \rightarrow \text{Hom}_{\mathcal{A}}(N, M^+)$ is epic. ■

An example of an object satisfying the condition of Proposition 4.1.9 which can be found among strongly Gorenstein flat objects.

Corollary 4.1.10. *Assume that \mathcal{A} has enough flat objects. If M is a strongly Gorenstein flat object, then ${}^\perp M^+ = \mathcal{A}p_{\mathcal{A}}(M)$.*

Proof. Let M be a strongly Gorenstein flat object in \mathcal{A} . Then, $\mathcal{I}nj(\mathcal{A}) \subseteq {}^\perp M^+$ is strongly Gorenstein injective by Proposition 1.4.10. Hence, $\mathcal{I}nj(\mathcal{A}) \subseteq {}^\perp M^+$. So, we apply Proposition 4.1.9 to get that ${}^\perp M^+ = \mathcal{A}p_{\mathcal{A}}(M)$. ■

4.1.1 Flat indigent objects

Durğun in [27], has explored the concept of flat indigent modules, as the opposite concept of flat modules. These are modules whose absolutely pure domain consists only of absolutely pure modules. Here, the objective is to explore the meaning and significance of flat indigent objects in a closed symmetric monoidal Grothendieck category.

Definition 4.1.11. *An object M is said to be flat indigent (or an f -indigent object) if $Ap_{\mathcal{A}}(M) = \{N \in \mathcal{A} : N \text{ is absolutely pure}\}$.*

In Proposition 4.1.13, we will show that every locally finitely presented closed symmetric monoidal Grothendieck category has an f -indigent object.

Lemma 4.1.12. *Let \mathcal{A} be a locally finitely presented closed symmetric monoidal Grothendieck category. Then, an object \mathcal{N} is absolutely pure if and only if $\mathcal{M} \otimes \mathcal{N} \rightarrow \mathcal{M} \otimes E(\mathcal{N})$ is a monomorphism for each finitely presented object \mathcal{M} , i.e. \mathcal{N} is absolutely \mathcal{M} -pure for each finitely presented object \mathcal{M} .*

Proof. The direct implication is clear.

Conversely, let \mathcal{G} be an object in \mathcal{A} , $\mathcal{G} = \varinjlim \mathcal{M}_i$, where $\{\mathcal{M}_i\}_{i \in I}$ is a direct system of finitely presented objects in \mathcal{A} . By assumption, we have the following exact sequences

$$\mathcal{E}_i : 0 \rightarrow \mathcal{M}_i \otimes \mathcal{N} \rightarrow \mathcal{M}_i \otimes E(\mathcal{N}) \quad \forall i \in I.$$

Since the functor direct limit is exact and commutes with tensor product, then we get the following exact sequence

$$\varinjlim \mathcal{E}_i : 0 \rightarrow \mathcal{G} \otimes \mathcal{N} \rightarrow \mathcal{G} \otimes E(\mathcal{N}).$$

So, \mathcal{N} is absolutely pure object. ■

Proposition 4.1.13. *If \mathcal{A} is locally finitely presented closed symmetric monoidal Grothendieck category, then \mathcal{A} has an f -indigent object.*

Proof. Let Λ be a complete set of representatives of finitely presented objects in \mathcal{A} . Set $G := \bigoplus_{T_i \in \Lambda} T_i$. Then, G is an f -indigent object by Proposition 4.1.4 and Lemma 4.1.12.

Starting now, we assume that \mathcal{A} is a locally finitely presented closed symmetric monoidal Grothendieck category.

Proposition 4.1.5 yields the following results.

Proposition 4.1.14. *If an object in \mathcal{A} has an f -indigent subobject which is pure, then it is f -indigent itself.*

Proposition 4.1.15. *Let P be a subobject of M . If M/P is flat, then P is f -indigent if and only if M is f -indigent.*

Proof. Since M/P is flat, P is pure in M and $\mathcal{A}p_{\mathcal{A}}(M/P) = \mathcal{A}$. Then, $\mathcal{A}p_{\mathcal{A}}(P) = \mathcal{A}p_{\mathcal{A}}(P) \cap \mathcal{A}p_{\mathcal{A}}(M/P) = \mathcal{A}p_{\mathcal{A}}(M)$ by Proposition 4.1.5. This implies that P is f -indigent if and only if M is f -indigent. ■

Recall that an object M is said to be indigent if $\mathfrak{In}_{\mathcal{A}}^{-1}(M) = \mathcal{I}nj(\mathcal{A})$.

The following result establishes a connection between f -indigent and indigent objects in the context where the class of absolutely pure objects coincides with the one of injectives.

Proposition 4.1.16. *Let M be an object in \mathcal{A} . If the class of absolutely pure and the class of injective objects are equal in \mathcal{A} , then M is f -indigent if and only if M^+ is indigent.*

Proof. The proof is a direct consequence of Proposition 4.1.6. ■

Proposition 4.1.16 shows that if the class of absolutely pure objects coincides with that of injectives, then the category \mathcal{A} has an indigent object that is pure injective. The following result shows that the converse also holds true.

Proposition 4.1.17. *The class of absolutely pure and the class of injective objects are equal in \mathcal{A} if and only if \mathcal{A} has a pure injective indigent object.*

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Proof. Let Λ be a complete set of representatives of finitely presented objects in \mathcal{A} . As indicated in the proof of Proposition 4.1.13, $G := \bigoplus_{T_i \in \Lambda} T_i$ is f -indigent. Now, let $N \in \mathfrak{I}\mathfrak{n}_{\mathcal{A}}^{-1}(G^+)$. By Proposition 4.1.6, $N \in \mathcal{A}p_{\mathcal{A}}(G)$, which implies that N is absolutely pure. So, N is injective by hypothesis. Therefore, G^+ is a pure injective indigent object.

The converse sens comes from Proposition 4.1.6. ■

Corollary 4.1.18. *If \mathcal{A} has a pure injective indigent, then \mathcal{A} has also a pure injective f -indigent object.*

Proof. Let M be a pure injective indigent object. Then, $M^{++} \cong M \oplus M^{++}/M$. So, $\mathfrak{I}\mathfrak{n}_{\mathcal{A}}^{-1}(M^{++}) = \mathfrak{I}\mathfrak{n}_{\mathcal{A}}^{-1}(M) \cap \mathfrak{I}\mathfrak{n}_{\mathcal{A}}^{-1}(M^{++}/M) = \mathcal{I}\mathfrak{n}\mathfrak{j}(\mathcal{A})$. Thus, M^{++} is indigent. Then, M^+ is f -indigent by Proposition 4.1.6. ■

We conclude the discussion of f -indigent objects with a result that constructs f -indigent objects from other known ones.

Proposition 4.1.19. *If M is an f -indigent object of \mathcal{A} , then $M \oplus G$ is f -indigent for any object $G \in \mathcal{A}$.*

Proof. Since $\mathcal{A}p_{\mathcal{A}}(M \oplus G) = \mathcal{A}p_{\mathcal{A}}(M) \cap \mathcal{A}p_{\mathcal{A}}(G)$ and $\mathcal{A}p_{\mathcal{A}}(M)$ consists only of absolutely pure objects, then $\mathcal{A}p_{\mathcal{A}}(M \oplus G) = \mathcal{A}p_{\mathcal{A}}(M)$. Hence, $M \oplus G$ is f -indigent. ■

4.2 Absolutely pure domains in $\mathcal{O}_X\text{-Mod}$ and $\mathfrak{Q}\mathfrak{C}\mathfrak{O}\mathfrak{H}(X)$

In this section, the aim is to study absolutely pure domains in both $\mathcal{O}_X\text{-Mod}$ and $\mathfrak{Q}\mathfrak{C}\mathfrak{O}\mathfrak{H}(X)$. To accomplish this, we will utilize the results related to absolutely pure domains in Section 4.1, taking into consideration that $\mathcal{O}_X\text{-Mod}$ and $\mathfrak{Q}\mathfrak{C}\mathfrak{O}\mathfrak{H}(X)$ are closed symmetric monoidal Grothendieck categories.

Given two \mathcal{O}_X -modules \mathcal{M} and \mathcal{N} , it is natural to ask whether $\mathcal{N} \in \mathcal{A}p_{\mathcal{O}_X\text{-Mod}}(\mathcal{M})$ is equivalent to $\mathcal{N}|_U \in \mathcal{A}p_{\mathcal{O}_U\text{-Mod}}(\mathcal{M}|_U)$ for every open subset U of X . Proposition 4.2.2 shows that it is always true.

The following lemma will be useful to prove Proposition 4.2.2.

Lemma 4.2.1. *Let \mathcal{K} be an object in $\mathcal{O}_X\text{-Mod}$ and \mathcal{J}^x be the injective envelope of \mathcal{K}_x for every $x \in X$. Then, \mathcal{K} is embedded in $\prod_{x \in X} i_{x,*} \mathcal{J}^x$.*

Proof. Let $\varphi^x : \mathcal{K}_x \rightarrow \mathcal{J}^x$ be the injective envelope of \mathcal{K}_x for every $x \in X$. Then, $\prod_{x \in X} \varphi^x : \prod_{x \in X} i_{x,*} \mathcal{K}_x \rightarrow \prod_{x \in X} i_{x,*} \mathcal{J}^x$ is monic. By [64, Lemma 2.19], there exists a monomorphism $\phi : \mathcal{K} \rightarrow \prod_{x \in X} i_{x,*} \mathcal{K}_x$, which implies that the composition $\prod_{x \in X} \varphi^x \circ \phi : \mathcal{K} \rightarrow \prod_{x \in X} i_{x,*} \mathcal{J}^x$ is a monomorphism. ■

Proposition 4.2.2. *Let \mathcal{M} and \mathcal{N} be \mathcal{O}_X -modules. Then, the following conditions are equivalent.*

- (i) $\mathcal{N} \in \mathcal{A}p_{\mathcal{O}_X\text{-Mod}}(\mathcal{M})$.
- (ii) For every open subset U of X , $\mathcal{N}|_U \in \mathcal{A}p_{\mathcal{O}_U\text{-Mod}}(\mathcal{M}|_U)$.
- (iii) For any open cover $\mathcal{U} = \{U_i\}_{i \in I}$ of X , $\mathcal{N}|_{U_i} \in \mathcal{A}p_{\mathcal{O}_{U_i}\text{-Mod}}(\mathcal{M}|_{U_i})$.
- (iv) There exists an open cover $\mathcal{U} = \{U_i\}_{i \in I}$ of X such that $\mathcal{N}|_{U_i} \in \mathcal{A}p_{\mathcal{O}_{U_i}\text{-Mod}}(\mathcal{M}|_{U_i})$ for every $i \in I$.
- (v) For any $x \in X$, $\mathcal{N}_x \in \mathcal{A}p_{\mathcal{O}_{X,x}\text{-Mod}}(\mathcal{M}_x)$.

Proof. (i) \Rightarrow (ii): Let U be an open subset of X , and assume that $\mathcal{N} \in \mathcal{A}p_{\mathcal{O}_X\text{-Mod}}(\mathcal{M})$. Then, $\mathcal{M} \otimes \mathcal{N} \rightarrow \mathcal{M} \otimes E(\mathcal{N})$ is a monomorphism in $\mathcal{O}_X\text{-Mod}$, where $E(\mathcal{N})$ is the injective envelope of \mathcal{N} . Consequently, $(\mathcal{M} \otimes \mathcal{N})|_U \rightarrow (\mathcal{M} \otimes E(\mathcal{N}))|_U$ is a monomorphism in $\mathcal{O}_U\text{-Mod}$. The commutativity of the following diagram:

$$\begin{array}{ccc} (\mathcal{M} \otimes \mathcal{N})|_U & \longrightarrow & (\mathcal{M} \otimes E(\mathcal{N}))|_U \\ \cong \downarrow & & \downarrow \cong \\ \mathcal{M}|_U \otimes \mathcal{N}|_U & \longrightarrow & \mathcal{M}|_U \otimes E(\mathcal{N})|_U \end{array}$$

implies that $\mathcal{M}|_U \otimes \mathcal{N}|_U \rightarrow \mathcal{M}|_U \otimes E(\mathcal{N})|_U$ is monic. Moreover, since $E(\mathcal{N})|_U$ is an injective \mathcal{O}_U -module, then $\mathcal{N}|_U \in \mathcal{A}p_{\mathcal{O}_U\text{-Mod}}(\mathcal{M}|_U)$ by Proposition 4.1.3.

(ii) \Rightarrow (iii) \Rightarrow (iv): These implications are straightforward.

(iv) \Rightarrow (v): Let $x \in X$, and $\mathcal{U} = \{U_i\}_{i \in I}$ be an open cover of X that verifies (iv). Then, there exists an open subset U_i such that $x \in U_i$. For every $y \in U_i$, we

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consider the monomorphism $\mathcal{N}|_{U_i} \rightarrow \prod_{y \in U_i} i_{y,*} J^y$ from Lemma 4.2.1, where each J^y is an injective envelope of \mathcal{N}_y . Consequently, $\mathcal{M}|_{U_i} \otimes \mathcal{N}|_{U_i} \rightarrow \mathcal{M}|_{U_i} \otimes \prod_{y \in U_i} i_{y,*} J^y$ is monic by assumption. Thus, $\mathcal{M}_x \otimes \mathcal{N}_x \rightarrow \mathcal{M}_x \otimes J^x$ is a monomorphism. Since $\prod_{y \in U_i} i_{y,*} J^y$ is injective, we get that $\mathcal{N}_x \in \mathcal{A}p_{\mathcal{O}_{X,x}\text{-Mod}}(\mathcal{M}_x)$ by Proposition 4.1.3.

(v) \Rightarrow (i): Let $\mathcal{N} \rightarrow \mathcal{K}$ be a monomorphism in $\mathcal{O}_X\text{-Mod}$. For each $x \in X$, $\mathcal{N}_x \rightarrow \mathcal{K}_x$ is monic in $\mathcal{O}_{X,x}\text{-Mod}$. Thus, $\mathcal{M}_x \otimes \mathcal{N}_x \rightarrow \mathcal{M}_x \otimes \mathcal{K}_x$ is monic for any $x \in X$ by assumption. Since the stalks commute with tensor product, we deduce that $(\mathcal{M} \otimes \mathcal{N})_x \rightarrow (\mathcal{M} \otimes \mathcal{K})_x$ is monic for each $x \in X$, which implies that $\mathcal{M} \otimes \mathcal{N} \rightarrow \mathcal{M} \otimes \mathcal{K}$ is monic in $\mathcal{O}_X\text{-Mod}$. Therefore, $\mathcal{N} \in \mathcal{A}p_{\mathcal{O}_X\text{-Mod}}(\mathcal{M})$. ■

Given a quasi-coherent \mathcal{O}_X -module \mathcal{M} , it can be shown that $\mathcal{A}p_{\mathcal{O}_X\text{-Mod}}(\mathcal{M}) \cap \mathfrak{Q}\mathfrak{C}\mathfrak{O}\mathfrak{H}(X) \subseteq \mathcal{A}p_{\mathfrak{Q}\mathfrak{C}\mathfrak{O}\mathfrak{H}(X)}(\mathcal{M})$. This inclusion holds because every monomorphism in $\mathfrak{Q}\mathfrak{C}\mathfrak{O}\mathfrak{H}(X)$ is also monic in $\mathcal{O}_X\text{-Mod}$. However, it is essential to note that this inclusion is not an equality. To illustrate, if we assume that $\mathcal{A}p_{\mathfrak{Q}\mathfrak{C}\mathfrak{O}\mathfrak{H}(X)}(\mathcal{M}) \subseteq \mathcal{A}p_{\mathcal{O}_X\text{-Mod}}(\mathcal{M})$, then the statement “if $\mathcal{N} \in \mathcal{A}p_{\mathfrak{Q}\mathfrak{C}\mathfrak{O}\mathfrak{H}(X)}(\mathcal{M})$, then $\mathcal{N} \in \mathcal{A}p_{\mathcal{O}_X\text{-Mod}}(\mathcal{M})$ ” would follow. Applying Proposition 4.2.2, we would conclude that “if $\mathcal{N} \in \mathcal{A}p_{\mathfrak{Q}\mathfrak{C}\mathfrak{O}\mathfrak{H}(X)}(\mathcal{M})$, then $\mathcal{N}_x \in \mathcal{A}p_{\mathcal{O}_{X,x}\text{-Mod}}(\mathcal{M}_x)$ for any $x \in X$ ”. However, this assertion is not universally true (see Remark 4.3.5).

In light of the discussion above, it is natural to consider absolutely pure domains of quasi-coherent \mathcal{O}_X -modules in $\mathfrak{Q}\mathfrak{C}\mathfrak{O}\mathfrak{H}(X)$ and to inquire whether the statements of Proposition 4.2.2 also hold in $\mathfrak{Q}\mathfrak{C}\mathfrak{O}\mathfrak{H}(X)$. For example, Proposition 4.2.5 and Proposition 4.2.6 demonstrate that in the case where $X = \text{Spec}(A)$ and $\tilde{N}, \tilde{M} \in \mathfrak{Q}\mathfrak{C}\mathfrak{O}\mathfrak{H}(X)$, one of the two assertions:

- $\tilde{N}|_{D_A(f_i)} \in \mathcal{A}p_{\mathfrak{Q}\mathfrak{C}\mathfrak{O}\mathfrak{H}(D_A(f_i))}(\tilde{M}|_{D_A(f_i)})$, for $i = 1, \dots, n$, where $f_1, \dots, f_n \in A$ and $X = \cup_{i=1}^n D_A(f_i)$.
- $\tilde{N}_x \in \mathcal{A}p_{\mathcal{O}_{X,x}\text{-Mod}}(\tilde{M}_x)$ for every $x \in X$

implies that $\tilde{N} \in \mathcal{A}p_{\mathfrak{Q}\mathfrak{C}\mathfrak{O}\mathfrak{H}(X)}(\tilde{M})$.

The following two lemmas are useful to prove Proposition 4.2.5 and Proposition 4.2.6.

Lemma 4.2.3. *Let M and N be A -modules, and f_1, \dots, f_n be elements generating the unit ideal. If N_{f_i} is absolutely M_{f_i} -pure over A_{f_i} for every $i = 1, \dots, n$, then N is absolutely M -pure over A .*

Proof. Let $N \rightarrow S$ be a monomorphism, and let $K = \text{Ker}(M \otimes N \rightarrow M \otimes S)$. For every $i \in \{1, \dots, n\}$, $K_{f_i} \cong \text{Ker}(M_{f_i} \otimes N_{f_i} \rightarrow M_{f_i} \otimes S_{f_i}) = 0$ since N_{f_i} is absolutely M_{f_i} -pure over A_{f_i} by assumption. Thus, for every $x \in K$ and any $i \in \{1, \dots, n\}$, there exists some $s_i \geq 0$ such that $f_i^{s_i} x = 0$.

On the other hand, since $1 = h_1 f_1 + \dots + h_n f_n$ for some $h_1, \dots, h_n \in A$ (because $\{f_1, \dots, f_n\}$ generates A), for any $x \in K$, we can write $x = 1 \cdot x = 1^s x = (h_1 f_1 + \dots + h_n f_n)^s x = 0$, where s is an integer satisfying $s > s_1 + \dots + s_n - 1$. Thus, $K = 0$, and then N is absolutely M -pure. ■

Lemma 4.2.4. *Let M and N be A -modules. If N_x is absolutely M_x -pure over A_x for every prime ideal x , then N is absolutely M -pure over A .*

Proof. Let $N \rightarrow K$ be a monomorphism. Then, for any $x \in X$, $M_x \otimes N_x \rightarrow M_x \otimes K_x$ is monic by hypothesis. The tensor product functor commutes with the stalk functor, and it reflects monomorphisms. Therefore, $M \otimes N \rightarrow M \otimes K$ is injective. Consequently, N is absolutely M -pure over A . ■

Proposition 4.2.5. *Let $X = \text{Spec}(A)$, $f_1, \dots, f_n \in A$ such that $X = \cup_{i=1}^n D_A(f_i)$, and $\widetilde{N}, \widetilde{M} \in \mathcal{Q}\text{coh}(X)$. If $\widetilde{N}|_{D_A(f_i)} \in \mathcal{A}p_{\mathcal{Q}\text{coh}(D_A(f_i))}(\widetilde{M}|_{D_A(f_i)})$ for any $i \in \{1, \dots, n\}$, then $\widetilde{N} \in \mathcal{A}p_{\mathcal{Q}\text{coh}(X)}(\widetilde{M})$.*

Proof. Note that the category of quasi-coherent \mathcal{O}_X -modules is equivalent to the category of modules over A . Therefore, by Lemma 4.2.3, the result follows. ■

Proposition 4.2.6. *Let $X = \text{Spec}(A)$, and $\widetilde{N}, \widetilde{M} \in \mathcal{Q}\text{coh}(X)$. If $\widetilde{N}_x \in \mathcal{A}p_{\mathcal{O}_{X,x}\text{-Mod}}(\widetilde{M}_x)$ for every $x \in X$, then $\widetilde{N} \in \mathcal{A}p_{\mathcal{Q}\text{coh}(X)}(\widetilde{M})$.*

Proof. The result follows from Lemma 4.2.3. ■

In general, the converse statements of Propositions 4.2.5 and 4.2.6 are not valid. Nevertheless, when A is coherent, these converse statements do hold, as we will see in Proposition 4.2.9. Consequently, it is reasonable to introduce and explore the notion of

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locally absolute pure domains for quasi-coherent sheaves on any scheme, particularly on locally coherent schemes.

Definition 4.2.7. *Let \mathcal{M} and \mathcal{N} be quasi-coherent \mathcal{O}_X -modules. We say that \mathcal{N} is locally absolutely \mathcal{M} -pure if $\mathcal{N}(U) \in \mathcal{A}p_{\mathcal{O}_X(U)\text{-Mod}}(\mathcal{M}(U))$ for every affine open subset $U \subseteq X$.*

The locally absolutely pure domain of \mathcal{M} is defined as the class

$$\mathcal{L}Ap_{\mathfrak{Qcoh}(X)}(\mathcal{M}) := \{\mathcal{N} \in \mathfrak{Qcoh}(X) : \mathcal{N} \text{ is locally absolutely } \mathcal{M}\text{-pure}\}.$$

The following proposition is the counterpart of Proposition 4.2.2 in the context of a locally coherent scheme. It states that to establish whether a quasi-coherent sheaf \mathcal{N} is locally absolutely \mathcal{M} -pure over a locally coherent scheme, it suffices to examine a cover consisting of affine open subsets of X . Furthermore, this local property holds stalkwise.

Proposition 4.2.8. *Let X be a locally coherent scheme and $\mathcal{M}, \mathcal{N} \in \mathfrak{Qcoh}(X)$. Then, the following conditions are equivalent:*

- (i) $\mathcal{N} \in \mathcal{L}Ap_{\mathfrak{Qcoh}(X)}(\mathcal{M})$.
- (ii) *For any cover $\mathcal{U} = \{U_i\}_{i \in I}$ of affine open subsets of X , $\mathcal{N}(U_i) \in \mathcal{A}p_{\mathcal{O}_X(U_i)\text{-Mod}}(\mathcal{M}(U_i))$ for any $i \in I$.*
- (iii) *There exists a cover $\mathcal{U} = \{U_i\}_{i \in I}$ of affine open subsets of X such that, for any $i \in I$, $\mathcal{N}(U_i) \in \mathcal{A}p_{\mathcal{O}_X(U_i)\text{-Mod}}(\mathcal{M}(U_i))$.*
- (iv) $\mathcal{N}_x \in \mathcal{A}p_{\mathcal{O}_{X,x}\text{-Mod}}(\mathcal{M}_x)$ for all $x \in X$.

Proof. The implications (i) \Rightarrow (ii) and (ii) \Rightarrow (iii) are straightforward.

(iii) \Rightarrow (iv): Let $x \in X$ and choose an affine open subset $U_i \in \mathcal{U}$ such that $x \in U_i$. Consider $\mathcal{N}(U_i) \rightarrow E$ to be an injective envelope in the category of $\mathcal{O}(U_i)$ -modules. By assumption, $\mathcal{M}(U_i) \otimes \mathcal{N}(U_i) \rightarrow \mathcal{M}(U_i) \otimes E$ is monic, which implies that $\mathcal{M}_x \otimes \mathcal{N}_x \rightarrow \mathcal{M}_x \otimes E_x$ is also monic. Since E_x is absolutely pure (as shown in [65, Theorem 3.21] for $\mathcal{O}(U_i)$ being coherent), Proposition 4.1.3 yields $\mathcal{N}_x \in \mathcal{A}p_{\mathcal{O}_{X,x}\text{-Mod}}(\mathcal{M}_x)$.

(iv) \Rightarrow (i): Let U be an affine open subset of X , and consider a monomorphism $\mathcal{N}(U) \rightarrow K$ in the category of $\mathcal{O}(U)$ -modules. Then, for any $x \in U$, $\mathcal{M}_x \otimes \mathcal{N}_x \rightarrow \mathcal{M}_x \otimes K_x$ is a monomorphism in $\mathcal{O}_{X,x}\text{-Mod}$ since $\mathcal{N}_x \rightarrow K_x$ is monic and $\mathcal{N}_x \in \mathcal{A}p_{\mathcal{O}_{X,x}\text{-Mod}}(\mathcal{M}_x)$. Thus, $\mathcal{M}(U) \otimes \mathcal{N}(U) \rightarrow \mathcal{M}(U) \otimes K$ is a monomorphism. Therefore, \mathcal{N} is locally absolutely \mathcal{M} -pure. \blacksquare

At this point, we can explore the relationship between the absolutely pure domain of an object in $\mathfrak{Q}\text{coh}(X)$ and its locally absolutely pure domain over locally coherent scheme.

Proposition 4.2.9. *Let X be a locally coherent scheme. Then, for every quasi-coherent \mathcal{O}_X -module \mathcal{M} , $\mathcal{L}Ap_{\mathfrak{Q}\text{coh}(X)}(\mathcal{M}) \subseteq Ap_{\mathfrak{Q}\text{coh}(X)}(\mathcal{M})$. If, moreover, X is semi-separated and quasi-compact, then $\mathcal{L}Ap_{\mathfrak{Q}\text{coh}(X)}(\mathcal{M}) = Ap_{\mathfrak{Q}\text{coh}(X)}(\mathcal{M})$.*

Proof. For any $\mathcal{N} \in \mathcal{L}Ap_{\mathfrak{Q}\text{coh}(X)}(\mathcal{M})$, Proposition 4.1.3 and Proposition 4.2.8 imply that $\mathcal{N} \in Ap_{\mathcal{O}_X\text{-Mod}}(\mathcal{M})$. Since $Ap_{\mathcal{O}_X\text{-Mod}}(\mathcal{M}) \cap \mathfrak{Q}\text{coh}(X) \subseteq Ap_{\mathfrak{Q}\text{coh}(X)}(\mathcal{M})$, $\mathcal{N} \in Ap_{\mathfrak{Q}\text{coh}(X)}(\mathcal{M})$.

Assuming that X is semi-separated and quasi-compact, and $\mathcal{N} \in Ap_{\mathfrak{Q}\text{coh}(X)}(\mathcal{M})$, we aim to prove that \mathcal{N} is locally absolutely \mathcal{M} -pure. To establish this, it suffices to demonstrate that $\mathcal{N}_x \in Ap_{\mathcal{O}_{X,x}\text{-Mod}}(\mathcal{M}_x)$ for every $x \in X$ (see Proposition 4.2.8). To proceed, consider $\mathcal{N} \rightarrow \mathcal{E}$ as an injective envelope in $\mathfrak{Q}\text{coh}(X)$. Then, $\mathcal{M} \otimes \mathcal{N} \rightarrow \mathcal{M} \otimes \mathcal{E}$ is monic, which implies that $\mathcal{M}_x \otimes \mathcal{N}_x \rightarrow \mathcal{M}_x \otimes \mathcal{E}_x$ for every $x \in X$. Since \mathcal{E}_x is absolutely pure for every $x \in X$, by [3, Proposition 3.3], then $\mathcal{N}_x \in Ap_{\mathcal{O}_{X,x}\text{-Mod}}(\mathcal{M}_x)$ for every $x \in X$ by Proposition 4.1.3. \blacksquare

The following result establishes the connection between the locally absolutely pure domains of a quasi-coherent \mathcal{O}_X -module and its absolutely pure domains in $\mathcal{O}_X\text{-Mod}$.

Proposition 4.2.10. *Let X be a locally coherent scheme. Then, for every quasi-coherent \mathcal{O}_X -module \mathcal{M} , $Ap_{\mathcal{O}_X\text{-Mod}}(\mathcal{M}) \cap \mathfrak{Q}\text{coh}(X) = \mathcal{L}Ap_{\mathfrak{Q}\text{coh}(X)}(\mathcal{M})$.*

Proof. It follows from Proposition 4.2.2 and Proposition 4.2.8. \blacksquare

Now, we explore the relationship between absolutely pure domains in $\mathcal{O}_X\text{-Mod}$ and those in $\mathfrak{Q}\text{coh}(X)$. As discussed earlier in this section, we observed that the containment $Ap_{\mathcal{O}_X\text{-Mod}}(\mathcal{M}) \cap \mathfrak{Q}\text{coh}(X) \subseteq Ap_{\mathfrak{Q}\text{coh}(X)}(\mathcal{M})$ is indeed proper. However, with

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Proposition 4.2.9 and Proposition 4.2.10 in hands, we can establish that if X is a semi-separated quasi-compact scheme, then the statement “If X is locally coherent, then $\mathcal{A}p_{\mathcal{O}_X\text{-Mod}}(\mathcal{M}) \cap \mathfrak{Qcoh}(X) = \mathcal{A}p_{\mathfrak{Qcoh}(X)}(\mathcal{M})$ for any $\mathcal{M} \in \mathfrak{Qcoh}(X)$ ” holds true.

The following result gives a new characterization of coherent scheme.

Proposition 4.2.11. *Let X be a semi-separated quasi-compact scheme. The following are equivalent.*

(i) X is locally coherent.

(ii) For every quasi-coherent \mathcal{O}_X -module \mathcal{M} , $\mathcal{A}p_{\mathfrak{Qcoh}(X)}(\mathcal{M}) \subseteq \mathcal{L}\mathcal{A}p_{\mathfrak{Qcoh}(X)}(\mathcal{M})$.

Proof. (i) \Rightarrow (ii): Proposition 4.2.9.

(ii) \Rightarrow (i): We show that any absolutely pure quasi coherent sheaf is locally absolutely pure. Since X is semi-separated, the direct image functor $i_{U,*}$ preserves quasi-coherence. Let \mathcal{N} be an absolutely pure quasi-coherent \mathcal{O}_X -module, and U an affine open subset of X . For any $\mathcal{O}_X(U)$ -module M , \mathcal{N} is absolutely $i_{U,*}\widetilde{M}$ -pure. Since $i_{U,*}\widetilde{M}(U) = M$, by hypothesis, $\mathcal{N}(U)$ is absolutely M -pure, meaning that $\mathcal{N}(U)$ is absolutely pure in $\mathcal{O}_X(U)\text{-Mod}$. As a result, \mathcal{N} is locally absolutely pure quasi-coherent \mathcal{O}_X -module. ■

4.3 Flat indigent objects in $\mathcal{O}_X\text{-Mod}$ and $\mathfrak{Qcoh}(X)$

We studied f -indigent objects in a general setting in Section 4.1 within a closed symmetric monoidal Grothendieck category. In this section, our focus shifts to a more detailed study of f -indigent objects in both $\mathcal{O}_X\text{-Mod}$ and $\mathfrak{Qcoh}(X)$, employing the results obtained in Section 4.1.1.

Recall that the category $\mathcal{O}_X\text{-Mod}$ on a quasi separated scheme (resp. the category $\mathfrak{Qcoh}(X)$ on a quasi-compact and quasi-separated scheme) is locally finitely presented (see [64, Definition 0.5] and [41, Corollary 6.9.12], respectively).

Using Proposition 4.1.13, we get the following result.

Proposition 4.3.1. *Let X be a quasi-separated scheme. Then, the category $\mathcal{O}_X\text{-Mod}$ has an f -indigent object. Moreover, if X is quasi-compact, then the category $\mathfrak{Qcoh}(X)$ has an f -indigent object.*

Given an object $\mathcal{M} \in \mathcal{O}_X\text{-Mod}$. The natural question that arises is: If \mathcal{M} is f -indigent, can we establish the equivalence that $\mathcal{M}|_U$ is also f -indigent in $\mathcal{O}_U\text{-Mod}$ for every open subset U of X , and does this property extend to the stalk level? Proposition 4.3.3 shows that this fact is always true.

To establish Proposition 4.3.3, it will be useful to give the following lemma.

Lemma 4.3.2. *Let N be an $\mathcal{O}_{X,x}$ -module. If $i_{x,*}N$ is absolutely pure in $\mathcal{O}_X\text{-Mod}$, then N is absolutely pure $\mathcal{O}_{X,x}$ -module.*

Proof. Let $N \rightarrow J$ be an injective envelope in $\mathcal{O}_{X,x}\text{-Mod}$. Since the skyscraper functor $i_{x,*}$ is left exact and preserves injective objects, $i_{x,*}N \rightarrow i_{x,*}J$ is a pure monomorphism by assumption. But the stalk functor preserves pure monomorphism, then $N \rightarrow J$ is pure monomorphism, which implies that N is absolutely pure. ■

Proposition 4.3.3. *Let X be a quasi-separated scheme and \mathcal{M} be an object in $\mathcal{O}_X\text{-Mod}$. The following conditions are equivalent:*

- (i) \mathcal{M} is f -indigent in $\mathcal{O}_X\text{-Mod}$.
- (ii) $\mathcal{M}|_U$ is f -indigent in $\mathcal{O}_U\text{-Mod}$ for every open subset U of X .
- (iii) $\mathcal{M}|_{U_i}$ is f -indigent in $\mathcal{O}_{U_i}\text{-Mod}$ for any cover $\mathcal{U} = \{U_i\}_{i \in I}$ of X .
- (iv) There exists an open cover $\mathcal{U} = \{U_i\}_{i \in I}$ of X such that $\mathcal{M}|_{U_i}$ is f -indigent in $\mathcal{O}_{U_i}\text{-Mod}$ for every open subset $U_i \in \mathcal{U}$.
- (v) \mathcal{M}_x is f -indigent in $\mathcal{O}_{X,x}\text{-Mod}$ for all $x \in X$.

Proof. (ii) \Rightarrow (iii) \Rightarrow (iv) are obvious.

(iv) \Rightarrow (i) Let $\mathcal{N} \in \mathcal{A}p_{\mathcal{O}_X\text{-Mod}}(\mathcal{M})$. By Proposition 4.2.2, $\mathcal{N}|_{U_i} \in \mathcal{A}p_{\mathcal{O}_{U_i}\text{-Mod}}(\mathcal{M}|_{U_i})$ for each open $U_i \in \mathcal{U}$, which implies that $\mathcal{N}|_{U_i}$ is absolutely pure \mathcal{O}_{U_i} -module for each $U_i \in \mathcal{U}$ by assumption. Hence, \mathcal{N} is absolutely pure in $\mathcal{O}_X\text{-Mod}$.

(i) \Rightarrow (v). Let $x \in X$ and $N \in \mathcal{A}p_{\mathcal{O}_{X,x}\text{-Mod}}(\mathcal{M}_x)$. Then, for every $y \in X$, $(i_{x,*}N)_y \in \mathcal{A}p_{\mathcal{O}_{X,y}\text{-Mod}}(\mathcal{M}_y)$ since $(i_{x,*}N)_y \cong N$ if $y = x$ and $(i_{x,*}N)_y \cong 0$ otherwise. By Proposition 4.2.2, $i_{x,*}N \in \mathcal{A}p_{\mathcal{O}_X\text{-Mod}}(\mathcal{M})$. Since \mathcal{M} is f -indigent, then $i_{x,*}N$ is

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absolutely pure in $\mathcal{O}_X\text{-Mod}$. Hence by Lemma 4.3.2, N is absolutely pure in $\mathcal{O}_{X,x}\text{-Mod}$.

(v) \Rightarrow (ii) Let U be an open subset of X and $\mathcal{N} \in \mathcal{A}p_{\mathcal{O}_U\text{-Mod}}(\mathcal{M}|_U)$. Then, by Proposition 4.2.2, $\mathcal{N}_x \in \mathcal{A}p_{\mathcal{O}_{X,x}\text{-Mod}}(\mathcal{M}_x)$ for every $x \in U$. By assumption, \mathcal{N}_x is absolutely pure for every $x \in U$, so \mathcal{N} is absolutely pure object in $\mathcal{O}_U\text{-Mod}$ by [29, Lemma 5.4].

■

Corollary 4.3.4. *Let X be a quasi-separated scheme and \mathcal{N} be an object in $\mathcal{O}_X\text{-Mod}$. The following conditions are equivalent:*

- (i) \mathcal{N} is absolutely pure in $\mathcal{O}_X\text{-Mod}$.
- (ii) $\mathcal{N}|_U$ is absolutely pure in $\mathcal{O}_U\text{-Mod}$ for every open subset U of X .
- (iii) $\mathcal{N}|_{U_i}$ is absolutely pure in $\mathcal{O}_{U_i}\text{-Mod}$ for any cover $\mathcal{U} = \{U_i\}_{i \in I}$ of X .
- (iv) There exists an open cover $\mathcal{U} = \{U_i\}_{i \in I}$ of X such that $\mathcal{N}|_{U_i}$ is absolutely pure in $\mathcal{O}_{U_i}\text{-Mod}$ for every open subset $U_i \in \mathcal{U}$.
- (v) \mathcal{N}_x is absolutely pure in $\mathcal{O}_{X,x}\text{-Mod}$ for all $x \in X$.

Remark 4.3.5. *Now, with Proposition 4.3.3, we can prove that the converse of Proposition 4.2.5 (similarly, Proposition 4.2.6) does not hold in general. To illustrate, let M be an f -indigent A -module. As per Proposition 4.3.3, M_{f_i} is an f -indigent A_{f_i} -module for $i = 1, \dots, n$. If the converse of Proposition 4.2.5 were true, it would imply that, for every absolutely pure module N , N_{f_i} is absolutely pure A_{f_i} -module for each $i = 1, \dots, n$, which is not true in general. However, this is not the case in general.*

Now, we will introduce and study locally f -indigent objects in $\mathfrak{Qcoh}(X)$.

Definition 4.3.6. *Let \mathcal{M} be an object in $\mathfrak{Qcoh}(X)$. \mathcal{M} is said to be locally f -indigent if $\mathcal{M}(U)$ is f -indigent in $\mathcal{O}_X(U)\text{-Mod}$ for every affine open $U \subseteq X$.*

The next proposition states that in order to prove whether a quasi-coherent \mathcal{O}_X -module on a locally coherent scheme is locally f -indigent, it is sufficient to look at some cover by affine open subsets of X .

The following Lemma will assist us in establishing proposition 4.3.8.

Lemma 4.3.7. *Let $X = \text{Spec}(A)$ be an affine scheme and \mathcal{E} be an injective \mathcal{O}_X -module. Then, $\mathcal{E}(X)$ is an injective A -module.*

Proof. It is known that the pair $(\widetilde{(-)}, \Gamma(X, -))$ is a pair of adjoint functors (see [43, Ex. II.5.3]). Since the functor $\widetilde{(-)}$ is exact, then $\Gamma(X, -)$ preserves injective objects by [58, Proposition 25]. hence, $\mathcal{E}(X) = \Gamma(X, \mathcal{E})$ is an injective object. ■

Proposition 4.3.8. *Let X be a locally coherent scheme, and \mathcal{M} be a quasi-coherent \mathcal{O}_X -module. The following assertions are equivalent:*

- (i) $\mathcal{M}(U)$ is f -indigent for every affine open $U \subseteq X$.
- (ii) $\mathcal{M}(U_i)$ is f -indigent for all $i \in I$ for some cover $\mathcal{U} = \{U_i\}_{i \in I}$ of affine open subsets.
- (iii) \mathcal{M}_x is f -indigent for all $x \in X$.

Proof. (i) \Rightarrow (ii) is obvious.

(ii) \Rightarrow (iii) For a given $x \in X$, there exists an affine open subset $U_i \in \mathcal{U}$ such that $x \in U_i$. Let $N \in \mathcal{A}p_{\mathcal{O}_{X,x}\text{-Mod}}(\mathcal{M}_x)$. Then, for every $y \in U_i$, $(i_{x,*}N)_y \in \mathcal{A}p_{\mathcal{O}_{X,y}\text{-Mod}}(\mathcal{M}_y)$ since $(i_{x,*}N)_y \cong N$ if $y = x$ and $(i_{x,*}N)_y \cong 0$ otherwise. So, by Proposition 4.2.2, $i_{x,*}N \in \mathcal{A}p_{\mathcal{O}_{U_i}\text{-Mod}}(\mathcal{M}|_{U_i})$. Let $i_{x,*}N \rightarrow \mathcal{E}$ be an injective envelope in $\mathcal{O}_{U_i}\text{-Mod}$. Then, $\mathcal{M}|_{U_i} \otimes i_{x,*}N \rightarrow \mathcal{M}|_{U_i} \otimes \mathcal{E}$ is monic, which allows that $\mathcal{M}(U_i) \otimes i_{x,*}N(U_i) \rightarrow \mathcal{M}(U_i) \otimes \mathcal{E}(U_i)$ is monic. Since $\mathcal{E}(U_i)$ is injective by Lemma 4.3.7, then by Proposition 4.1.3, $i_{x,*}N(U_i) \in \mathcal{A}p_{\mathcal{O}_X(U_i)\text{-Mod}}(\mathcal{M}(U_i))$. By hypothesis, $i_{x,*}N(U_i)$ is absolutely pure $\mathcal{O}_X(U_i)$ -module, so $N \cong (i_{x,*}N)_x$ is absolutely pure since $\mathcal{O}_X(U_i)$ is coherent.

(iii) \Rightarrow (i) Let U be an affine open of X , $N \in \mathcal{A}p_{\mathcal{O}_X(U)\text{-Mod}}(\mathcal{M}(U))$, and $N \rightarrow E$ be an injective envelope in $\mathcal{O}_X(U)\text{-Mod}$. Then, $\mathcal{M}(U) \otimes N \rightarrow \mathcal{M}(U) \otimes E$ is monic, which implies that for every $x \in U$, $\mathcal{M}_x \otimes N_x \rightarrow \mathcal{M}_x \otimes E_x$ is monic. But the ring $\mathcal{O}_X(U)$ is coherent, then E_x is absolutely pure for every $x \in U$. So, by Proposition 4.1.3, $N_x \in \mathcal{A}p_{\mathcal{O}_{X,x}\text{-Mod}}(\mathcal{M}_x)$ for every $x \in U$. But by assumption, N_x is absolutely pure for every $x \in U$, then N is absolutely pure. ■

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The next Corollary shows that a quasi-coherent \mathcal{O}_X -module that is f -indigent in $\mathfrak{Qcoh}(X)$ on a locally coherent scheme X is also f -indigent in $\mathcal{O}_X\text{-Mod}$.

Corollary 4.3.9. *Let X be a quasi-separated, quasi-compact and locally coherent scheme, and let \mathcal{M} be an object in $\mathfrak{Qcoh}(X)$. If \mathcal{M} is locally f -indigent in $\mathfrak{Qcoh}(X)$, then \mathcal{M} is also f -indigent in $\mathcal{O}_X\text{-Mod}$.*

Proof. It follows from Proposition 4.3.8 and Proposition 4.3.3. ■

Now, we will investigate under what conditions a locally f -indigent quasi-coherent object becomes f -indigent in $\mathfrak{Qcoh}(X)$.

Corollary 4.3.10. *Let X be a semi-separated quasi-compact locally coherent scheme and \mathcal{M} be a quasi-coherent \mathcal{O}_X -modules. Then, \mathcal{M} is f -indigent in $\mathfrak{Qcoh}(X)$ if \mathcal{M} is locally f -indigent.*

Proof. Let \mathcal{N} be an object of $\mathcal{A}p_{\mathfrak{Qcoh}(X)}(\mathcal{M})$. Then, by Proposition 4.2.9, $\mathcal{N}_x \in \mathcal{A}p_{\mathcal{O}_{X,x}\text{-Mod}}(\mathcal{M}_x)$. But, based on the hypothesis and using Proposition 4.3.8, \mathcal{M}_x is f -indigent for every $x \in X$, so \mathcal{N}_x is absolutely pure for every $x \in X$. Thus, \mathcal{N} is absolutely pure. ■

In the following result, we will provide a new characterization of a Noetherian scheme.

Proposition 4.3.11. *Let X be a locally coherent scheme. Then, X is a locally Noetherian scheme if there exists a pure injective indigent object in the category $\mathcal{O}_X\text{-Mod}$.*

Proof. Let \mathcal{M} be a pure injective indigent \mathcal{O}_X -module in $\mathcal{O}_X\text{-Mod}$. The class of absolutely pure \mathcal{O}_X -modules in $\mathcal{O}_X\text{-Mod}$ is contained in the subinjectivity domain of \mathcal{M} , which coincides with the class of injective \mathcal{O}_X -modules since \mathcal{M} is indigent. Therefore, by [29, Proposition 5.10], X is locally Noetherian. ■

Remark 4.3.12. *On a locally Noetherian scheme X , all classes of locally absolutely pure, absolutely pure, locally injective and injective quasi-coherent sheaves are equal, by [42, II, proposition 7.17, theorem 7.18].*

Summing up Proposition 4.1.17 and Remark 4.3.12, we get the following result.

Proposition 4.3.13. *Let X be a quasi-compact locally coherent scheme. Then, the following assertions are equivalent:*

- (i) X is a Noetherian scheme.
- (ii) There exists a pure injective indigent quasi-coherent \mathcal{O}_U -module for every affine open U of X .
- (iii) There exists a pure injective indigent quasi-coherent \mathcal{O}_{U_i} -module for all $i \in I$ for some cover $\{U_i\}_{i \in I}$ of affine open subsets of X .

Proof. (i) \Rightarrow (ii) Let U be an affine open subset of X . Since X is Noetherian scheme, U is also a Noetherian scheme (see [40, Corollary 3.22]). Therefore, the result follows from Proposition 4.1.17 and Remark 4.3.12.

(ii) \Rightarrow (iii) This implication is obvious.

(iii) \Rightarrow (i) Let M be an absolutely pure $\mathcal{O}_X(U_i)$ -module. Since $\mathcal{O}_X(U_i)$ is a coherent ring, M_x is an absolutely pure $\mathcal{O}_{X,x}$ -module for every $x \in U_i$. Let \widetilde{M} be the associated quasi-coherent \mathcal{O}_{U_i} -module of M . Since $\widetilde{M}_x \cong M_x$ for every $x \in U_i$, then by [29, Lemma 5.4.], \widetilde{M} is absolutely pure \mathcal{O}_{U_i} -module. By hypothesis, the category $\Omega\text{coh}(U_i)$ has a pure injective indigent quasi-coherent \mathcal{O}_{U_i} -module. Consequently, the class of absolutely pure quasi-coherent \mathcal{O}_{U_i} -modules is equal to the class of injective quasi-coherent \mathcal{O}_{U_i} -modules. Thus, \widetilde{M} is injective, which allows that M is injective. Hence, X is a locally Noetherian scheme. Since X is quasi-compact, then X is Noetherian. ■

We recall that if a ring A is von Neumann regular, then the spectrum of A , denoted as $\text{Spec}(A)$, is a Boolean space, as shown in [11]. Furthermore, Durğun's work in [27, Proposition 3.3] has established that a ring A is von Neumann regular if and only if every A -module is f-indigent.

In our context, we extend this result to demonstrate that for an affine scheme X to be Boolean such that for every $x \in X$, $\mathcal{O}_{X,x}$ is a field, it is necessary and sufficient that every quasi-coherent \mathcal{O} -module exhibits the property of being f-indigent.

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Proposition 4.3.14. *Let $X = \text{Spec}(A)$ be an affine scheme. Then, the following are equivalent:*

1. *Every quasi-coherent \mathcal{O}_X -module is f -indigent.*
2. *A is von Neumann regular.*
3. *X is a Boolean space and $\mathcal{O}_{X,x}$ is a field for each $x \in X$.*

Proof. (1) \Rightarrow (2): To show that A is von Neumann regular, consider an A -module M and $N \in \mathcal{A}p_{A\text{-Mod}}(M)$. Then, $\widetilde{N} \in \mathcal{A}p_{\Omega\text{coh}(X)}(\widetilde{M})$. By hypothesis, \widetilde{M} is an f -indigent object in $\Omega\text{coh}(X)$, which implies that \widetilde{N} is an absolutely pure object in $\Omega\text{coh}(X)$. Consequently, for any A -module H and any extension K of N , the map $\widetilde{H} \otimes \widetilde{N} \rightarrow \widetilde{H} \otimes \widetilde{K}$ is a monomorphism in $\Omega\text{coh}(X)$. This implies that $H \otimes N \rightarrow H \otimes K$ is a monomorphism in $A\text{-Mod}$. Consequently, we deduce that N is indeed an absolutely pure A -module, which implies that M is f -indigent. Thus, every A -module is f -indigent, as shown in [27, Proposition 3.3], which implies that A is von Neumann regular.

(2) \Rightarrow (3) It is known that for each $x \in X$, $\mathcal{O}_{X,x} \cong A_x$. Since A is von Neumann regular, A_x is a field for each $x \in X$, and so $\text{Spec}(A)$ is a Boolean space (see [11, Remark 2]).

(3) \Rightarrow (2): Refer to [11, Proposition 21]).

(2) \Rightarrow (1): Refer to [27, Proposition 3.3]. ■

4.4 Absolutely pure domains in $\mathcal{O}_X\text{-Mod}$ on locally coherent schemes

This section is devoted to study some homological properties of absolutely pure domains and flat character \mathcal{O}_X -modules over a locally coherent scheme. It is well-known that if A is a commutative coherent ring, then $\text{Hom}_A(I, J)$ is flat for each pair I, J of injective A -modules. Recently, Hosseini and Zarei have proven in [48, Theorem 4.1] that if X is a locally coherent scheme, \mathcal{E} is an absolutely pure \mathcal{O}_X -module and \mathcal{J} is an injective \mathcal{O}_X -module, then $\mathcal{H}om_{\mathcal{O}_X}(\mathcal{E}, \mathcal{J})$ is a flat \mathcal{O}_X -module. Consequently, the following result holds.

Proposition 4.4.1. *Let X be a locally coherent scheme and $\mathcal{M} \in \mathcal{O}_X\text{-Mod}$. If $\varphi : \mathcal{M} \rightarrow \mathcal{A}$ is an injective preenvelope, then $\varphi^* : \mathcal{A}^+ \rightarrow \mathcal{M}^+$ is a flat precover.*

Proof. The proof is similar to the proof of [30, Theorem 3.1]. ■

The following result provides a new characterization of absolutely pure domains in $\mathcal{O}_X\text{-Mod}$ on a locally coherent scheme.

Proposition 4.4.2. *Let X be a locally coherent scheme. Then, for any objects $\mathcal{M}, \mathcal{N} \in \mathcal{O}_X\text{-Mod}$, $\mathcal{N} \in \mathcal{A}p_{\mathcal{O}_X\text{-Mod}}(\mathcal{M})$ if and only if every morphism $\mathcal{M} \rightarrow \mathcal{N}^+$ factors through a flat object in $\mathcal{O}_X\text{-Mod}$.*

Proof. Let $0 \rightarrow \mathcal{N} \rightarrow \mathcal{E}$ be an injective preenvelope of \mathcal{N} in $\mathcal{O}_X\text{-Mod}$. Since X is locally coherent, by Proposition 4.4.1, $\mathcal{E}^+ \rightarrow \mathcal{N}^+ \rightarrow 0$ is a flat precover. We have the following commutative diagram:

$$\begin{array}{ccc} \text{Hom}_{\mathcal{O}_X}(\mathcal{M}, \mathcal{E}^+) & \longrightarrow & \text{Hom}_{\mathcal{O}_X}(\mathcal{M}, \mathcal{N}^+) \\ \downarrow \cong & & \downarrow \cong \\ \text{Hom}_{\mathcal{O}_X}(\mathcal{M} \otimes \mathcal{E}, \mathcal{F}) & \longrightarrow & \text{Hom}_{\mathcal{O}_X}(\mathcal{M} \otimes \mathcal{N}, \mathcal{F}) \end{array}$$

Hence, $\text{Hom}_{\mathcal{O}_X}(\mathcal{M}, \mathcal{E}^+) \rightarrow \text{Hom}_{\mathcal{O}_X}(\mathcal{M}, \mathcal{N}^+)$ is epic if and only if $\mathcal{M} \otimes \mathcal{N} \rightarrow \mathcal{M} \otimes \mathcal{E}$ is monic. Therefore, $\mathcal{N} \in \mathcal{A}p_{\mathcal{O}_X\text{-Mod}}(\mathcal{M})$ if and only if every morphism $\mathcal{M} \rightarrow \mathcal{N}^+$ factors through the flat object \mathcal{E}^+ . ■

In Proposition 4.1.9, we proved that when an \mathcal{O}_X -module \mathcal{M} is embedded in a flat object, it implies that $\text{Flat}(\mathcal{O}_X\text{-Mod}) \subseteq \mathcal{A}p_{\mathcal{O}_X\text{-Mod}}(\mathcal{M})$. The subsequent result establishes the converse over locally coherent scheme.

Proposition 4.4.3. *Let X be a locally coherent scheme and let \mathcal{M} be an object in $\mathcal{O}_X\text{-Mod}$. The following assertions are equivalent:*

- (i) \mathcal{M} embeds in a flat object of $\mathcal{O}_X\text{-Mod}$.
- (ii) Every flat object in $\mathcal{O}_X\text{-Mod}$ is absolutely \mathcal{M} -pure.

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Proof. (i) \Rightarrow (ii): Assume that \mathcal{M} embeds in a flat \mathcal{O}_X -module \mathcal{F} . Let \mathcal{N} be a flat \mathcal{O}_X -module. Consider the following commutative diagram:

$$\begin{array}{ccc} \mathcal{M} \otimes \mathcal{N} & \xrightarrow{\alpha} & \mathcal{M} \otimes E(\mathcal{N}) \\ \downarrow \beta & & \downarrow \gamma \\ \mathcal{F} \otimes \mathcal{N} & \xrightarrow{\sigma} & \mathcal{F} \otimes E(\mathcal{N}) \end{array}$$

Since \mathcal{F} and \mathcal{N} are flat, σ and β are monomorphisms. Thus, $\gamma\alpha$ is a monomorphism, which implies that α is a monomorphism. By Proposition 4.1.3, \mathcal{N} is absolutely \mathcal{M} -pure.

(ii) \Rightarrow (i): Let $\nu_{\mathcal{M}} : \mathcal{M} \rightarrow \mathcal{M}^{++}$ be the evaluation morphism of \mathcal{M} . $\nu_{\mathcal{M}}$ is a pure monomorphism (see [29, Corollary 4.6]). Let $\alpha : \mathcal{F} \rightarrow \mathcal{M}^+$ be an epimorphism, where \mathcal{F} is a flat \mathcal{O}_X -module. Then, $\alpha^+ : \mathcal{M}^{++} \rightarrow \mathcal{F}^+$ is a monomorphism. Since $\mathcal{F} \in \mathcal{A}p_{\mathcal{O}_X\text{-Mod}}(\mathcal{M})$, every morphism $\mathcal{M} \rightarrow \mathcal{F}^+$ factors through a flat \mathcal{O}_X -module. This means that there exists a flat \mathcal{O}_X -module \mathcal{P} such that the following diagram commutes:

$$\begin{array}{ccc} & \mathcal{M} & \\ & \downarrow \nu_{\mathcal{M}} & \\ & \mathcal{M}^{++} & \\ \beta \swarrow & & \downarrow \alpha^+ \\ \mathcal{P} & \xrightarrow{\gamma} & \mathcal{F}^+ \end{array}$$

Since $\alpha^+\nu_{\mathcal{M}}$ is monic, then β is monic, so \mathcal{M} embeds in the flat \mathcal{O}_X -module \mathcal{P} . ■

Corollary 4.4.4. *Let X be a quasi-separated locally coherent scheme. The following assertions are equivalent:*

- (i) *Every injective object in $\mathcal{O}_X\text{-Mod}$ is flat.*
- (ii) *There is an f -indigent object in $\mathcal{O}_X\text{-Mod}$ which embeds in a flat object.*
- (iii) *Every flat object in $\mathcal{O}_X\text{-Mod}$ is absolutely pure.*

Proof. It follows from Proposition 4.4.3 Proposition 4.3.3 and Proposition 4.4.3. ■

As an example satisfying the conditions of Corollary 4.4.4, we give the following.

Example 4.4.5. *Let A be a von Neumann regular ring and $X = \text{Spec}A$ be an affine scheme. Then, every injective \mathcal{O}_X -module is flat.*

Indeed, let \mathcal{N} be an injective \mathcal{O}_X -module. For any $x \in X$, \mathcal{N}_x is a flat A_x -module since A_x is a von Neumann regular ring (see [11, Proposition 5]). Hence, \mathcal{N} is a flat \mathcal{O}_X -module. Consequently, every injective \mathcal{O}_X -module is flat.

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Résumé

Les anneaux et les modules gradués sont des sujets classiques en algèbre, et la théorie homologique des anneaux gradués revêt une grande importance en géométrie algébrique, notamment dans le contexte des catégories de faisceaux. Au cours des dernières années, la théorie homologique relative des anneaux gradués a été l'objet d'études par de nombreux chercheurs et est devenue un domaine de recherche particulièrement actif. En outre, nous remarquons qu'au fil des dernières années, de nombreux auteurs ont adopté une nouvelle approche dans l'analyse de la projectivité, de l'injectivité et de la platitude classiques des modules. Ils ont ainsi introduit des domaines relatifs de modules comme un moyen d'évaluer le niveau de platitude d'un tel module, plutôt que de se limiter à déterminer s'il est projectif ou non. L'objectif de cette thèse se répartit en trois directions. Les deux premières parties se concentrent sur l'algèbre homologique relative dans la catégorie des modules gradués, tandis que la troisième partie aborde un domaine spécifique de platitude relatif dans les catégories de faisceaux. Premièrement, nous introduisons et étudions les notions de modules n -FP- gr -injectifs de Gorenstein et de modules n - gr -plats de Gorenstein. Plusieurs concepts fondamentaux de l'algèbre homologique de Gorenstein relative sont caractérisés sur les anneaux n - gr -cohérents. Dans la deuxième partie, nous introduisons et explorons la notions de modules n -FCP- gr -projectifs. Ensuite, nous étudions dans quelles conditions cette classe de modules et sa classe orthogonale droite forment une paire cotorsion parfaite héréditaire. Enfin, nous développons un nouveau traitement de la platitude géométrique dans les catégories de faisceaux qui éclaire certains de ses aspects importants. Notamment, en termes de cette nouvelle approche, certains schémas classiques sont caractérisés et de nouveaux résultats sont établis.

Mots-clés : Anneau n - gr -cohérent; module n -FP- gr -injectif de Gorenstein; module n - gr -plat de Gorenstein; anneau n - gr -cocoherent; module n -FCP- gr -projectif; domaine absolument pur; domaine localement absolument pur; f -indigent; Localement f -indigent; faisceau plat.

Abstract

Graded rings and graded modules are classical topics in algebra, and the homological theory of graded rings has very important applications in algebraic geometry, particularly in categories of sheaves. In recent years, relative homological theory for graded rings has been studied by many authors and has become a vigorously active area of research. Also, we see that in the last few years, many authors have introduced a new approach in the study of the classical projectivity, injectivity and flatness of modules. This way, they introduced relative domains of modules as a tool to evaluate the flatness level of such a module (so not just to determine whether or not the module is projective). The goal of this thesis goes in three directions. The first two parts focus on relative homological algebra in the category of graded modules, while the third part concerns a specific relative domain of flatness in categories of sheaves. First, we introduce and study the notions of Gorenstein n -FP- gr -injective and Gorenstein n - gr -flat modules. Several fundamental concepts of relative Gorenstein homological algebra are characterized over n -coherent rings. In the second part, we introduce and explore the notions of n -FCP- gr -projective modules. We then investigate under what conditions this class of modules and its right orthogonal form a perfect hereditary cotorsion pair. Finally, we develop a new treatment of the geometric flatness in categories of sheaves which shed more light on some of its various important aspects. Namely, in terms of this new approach, some classical schemes are characterized and some new results are established.

Keywords: n - gr -coherent ring; Gorenstein n -FP- gr -injective module; Gorenstein n - gr -flat module; n - gr -cocoherent ring; n -FCP- gr -projective module; absolutely pure domain; locally absolutely pure domain; f -indigent, locally f -indigent; flat sheaf.