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Fouad TARAZA

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On zero-divisor graphs of commutative rings

JURY

Hassane ESSANNOUNI	PES, Université Mohammed V, FSR Rabat	Président
Ali OUADFEL	PES, Université Mohammed V, FSR Rabat	Examineur
Jilali MIKRAM	PES, Université Mohammed V, FSR Rabat	Examineur
Driss BENNIS	PH, Université Mohammed V, FSR Rabat	Examineur
Driss KARIM	PES, Université Hassan II, FST Mohammedia	Examineur
Khalid ABDELMOUMEN	Professeur au CRMEF de Fès	Invité

Directeur (s) de Thèse :

Jilali MIKRAM et Driss BENNIS

Rapporteurs :

Hassane ESSANNOUNI, Ali OUADFEL et Driss KARIM

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ON ZERO-DIVISOR GRAPHS OF
COMMUTATIVE RINGS

DEDICATION

To my parents and my parents-in-Law

To my wife MAJDA,
whose encouragement and constant support have been
invaluable,

and

my daughter YASMINE

Avant Propos

Les travaux présentés dans cette thèse ont été effectués au laboratoire Mathématiques, Informatique et Applications de la Faculté des Sciences de Rabat. Cette thèse a été réalisée sous la direction du professeur Jilali MIKRAM et codirection du professeur Driss BENNIS.

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RESUME

La notion de graphe des diviseurs de zéro $\Gamma(R)$ d'un anneau commutatif R a été introduite par D.F. Anderson et Livingston dans (The zero-divisor graph of a commutative ring, J. Algebra, **217**, (1999), 434–447), qui a pour but d'étudier la relation entre les propriétés algébriques de l'anneau R et les propriétés graphiques du graphe $\Gamma(R)$.

Motivé par le succès de ce nouveau domaine de recherche, plusieurs auteurs ont introduit d'autres graphes associés à certaines propriétés algébriques de l'anneau. Dans ce cadre nous avons introduit et étudié durant notre étude doctorale une extension du graphe des diviseurs de zéro classique d'un anneau commutatif R , noté $\bar{\Gamma}(R)$ et appelé graphe des diviseurs de zéro prolongé.

Nous avons établi une étude générale de ce nouveau type de graphes. Notamment, nous avons caractérisé quand $\bar{\Gamma}(R)$ et $\Gamma(R)$ coïncident. Nous avons étudié, entre autres, le diamètre et la maille de $\bar{\Gamma}(R)$. En outre, une étude des propriétés des graphes des diviseurs de zéro prolongé des idéalizations est aussi effectuée. Nous avons aussi étudié la relation entre le graphe des diviseurs de zéro prolongé et le graphe annulateur qui est aussi une extension du graphe classique. Enfin, nous avons récemment initié un nouveau projet de recherche qui consiste à étudier les codes binaires associés aux graphes des diviseurs de zéro.

Mots Clés. Graphes des diviseurs de zéro des anneaux commutatifs, Graphes des diviseurs de zéro prolongés des anneaux commutatifs, Idéalizations, Diamètre, Périmètre, Graphes Annihilateurs des anneaux commutatifs, Codes binaires associés aux graphes.

SUMMARY

The notion of zero-divisor graph $\Gamma(R)$ of a commutative ring R has been introduced by D. F. Anderson and Livingston, in (The zero-divisor graph of a commutative ring, *J. Algebra*, **217**, (1999), 434–447), which aims to study the relationship between the ring-theoretic properties of R and the graph-theoretic properties of $\Gamma(R)$.

Motivated by the success of this new area of research several authors have introduced other graphs associated to some ring theoretic properties. In this context we have introduced and studied an extension of the classical zero-divisor graph of a commutative ring R , denoted by $\bar{\Gamma}(R)$ and called extended zero-divisor graph.

We have established a general study of this new type of graphs. Namely, we characterized when $\bar{\Gamma}(R)$ and $\Gamma(R)$ coincide. We studied, among other things, the diameter and girth of $\bar{\Gamma}(R)$. Also, a study of the properties of the extended zero-divisor graph of idealizations is also carried out. We also studied the relation between the extended zero-divisor graph and the annihilator graph which is also an extension of the classical graph. Finally, we have recently initiated a new research project which consists of studying the binary codes associated to zero-divisor graphs.

Key Words. Zero-divisor graphs of commutative rings, extended zero-divisor graphs of commutative rings, Idealizations, diameter, girth, Annihilator graphs of commutative rings, Binary codes from graphs.

Contents

Dedication	ii
Avant Propos	iii
Résumé	iv
Summary	v
Contents	vi
Introduction (Français)	1
Introduction (English)	10
1 Zero-divisor graph of a commutative ring (Preliminaries)	18
1.1 Definitions and some results	18
1.2 Diameter of zero-divisor graphs of rings	24
1.3 Girth of zero-divisor graphs of rings	28
1.4 Annihilator graph of a commutative ring	34
2 On the extended zero-divisor graph of commutative rings	37
2.1 When $\bar{\Gamma}(R)$ and $\Gamma(R)$ coincide?	37
2.2 Diameter of extended graphs of rings	42
2.3 Cycles in extended graphs of rings	46
3 Extended zero-divisor graphs of idealizations	50
3.1 When $\bar{\Gamma}(R \times M)$ and $\Gamma(R \times M)$ coincide?	50
3.2 The Girth of extended zero-divisor graphs of idealizations	53
3.3 Diameter of extended zero-divisor graphs of idealizations	55
4 Comparison of some extensions of zero-divisor graphs	59
4.1 Main results	59
4.2 Annihilator graph of some ring constructions	62

5	Examples of codes from extended zero-divisor graphs with large parameters	66
5.1	Background and notation	66
5.2	Linear binary codes	67
5.3	Codes from incidence matrices of graphs	68
5.4	Examples of codes from extended zero-divisor graphs with large parameters	69
	Appendices	73
	Bibliography	78

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4. K. Abdelmoumen, D. Bennis, J. Mikram and F. Taraza, *Examples of codes from extended zero-divisor graphs with large parameters*. In preparation.

Other papers :

1. D. Bennis, F. Taraza and A. Shaiea, *Directed zero-divisor graphs of idealizations*. In preparation.
2. D. Bennis and F. Taraza, *On extended zero-divisor graphs of small finite commutative rings*. In preparation.

Introduction

Dans cette thèse, tous les anneaux considérés sont commutatifs et unitaires, et les modules sont unitaires.

Notation. Soit R un anneau. Nous utilisons $Z(R)$ pour désigner l'ensemble de tous les diviseurs de zéro de R et $Z(R)^* := Z(R) \setminus \{0\}$. On note par $\text{reg}(R)$ l'ensemble des éléments réguliers de R et par $\text{Ann}(x)$ l'annulateur de l'élément x de R . Pour un idéal I de R , \sqrt{I} désigne le radical de I , en particulier, $\text{Nil}(R) := \sqrt{0}$ est le nilradical de R . Pour un élément nilpotent non nul x de R , n_x désigne l'indice de nilpotence de x . L'anneau $\mathbb{Z}/n\mathbb{Z}$ des résidus modulo un nombre entier strictement positif $n \in \mathbb{N}^*$ sera noté par \mathbb{Z}_n . Nous utilisons le symbole \subset pour signifier l'inclusion et le symbole \subsetneq pour signifier l'inclusion stricte. L'anneau quotient total $S^{-1}R$ de R sera noté par $T(R)$, où S est l'ensemble des éléments réguliers.

Le lecteur sera supposé familier avec les notions de base de la théorie des anneaux commutatifs avec diviseurs de zéro. Pour plus de détails nous renvoyons le lecteur au livre de Huckaba [26]. Nous supposons que le lecteur est aussi familier avec les notions élémentaires de la théorie des graphes. Cependant, nous rappelons quelques notions utilisées dans cette thèse dans une annexe (voir le livre [21] pour plus de détails).

Le graphe des diviseurs de zéro d'un anneau R , noté par $\Gamma(R)$, est un graphe simple associé à R tel que son ensemble de sommets se compose de tous ses diviseurs de zéro non nul et que deux sommets distincts sont adjacents si et seulement si le produit de ces deux sommets est nul. L'idée d'associer des graphes à des structures algébriques remonte à 1988 avec Beck dans [18], où il était principalement intéressé par les colorations. Dans son travail, tous les éléments de l'anneau étaient des sommets du graphe (voir aussi [2]). Ce sont D.F. Anderson et Livingston, dans [11], qui ont introduit le graphe des diviseurs de zéro d'un anneau commutatif et qui ont initié l'étude de l'interaction entre la théorie des graphes et la théorie des anneaux. Depuis lors, les graphes des diviseurs de zéro des anneaux commutatifs ont attiré l'attention de plusieurs chercheurs (voir, par exemple, [4, 6, 7, 10, 11, 13, 15, 16, 28, 29]).

Motivé par le succès de ce nouveau domaine de recherche, plusieurs chercheurs

ont récemment introduit d'autres graphes associés à certaines propriétés algébriques de l'anneau (voir, par exemple, [8, 9, 14, 17, 31, 32]). L'objectif principal de l'étude de ces graphes est d'établir une liaison entre la théorie des graphes et la théorie des anneaux. Dans ce contexte, nous avons introduit un nouveau graphe des diviseurs de zéro, appelé graphe des diviseurs de zéro prolongé qui sert, entre autres, à donner une conditions suffisante pour que l'anneau totale des fractions d'un anneau est 0-dimensionnel (voir Proposition 2.3.8). En effet, l'objectif principal de notre étude doctorale est d'étudier profondément les graphes des diviseurs de zéro prolongés. Dans cette thèse, nous présentons quelques travaux établis sur les graphes des diviseurs de zéro prolongés.

Cette thèse est divisée en cinq chapitres.

* * * * * * * * * * * 1 * * * * * * * * * *

Dans le **Chapitre 1**, nous présentons une vue d'ensemble sur les résultats du graphe des diviseurs de zéro classique d'un anneau. Nous nous concentrerons principalement sur certains résultats classiques dont nous avons besoin dans cette thèse.

* * * * * * * * * * * 2 * * * * * * * * * *

Dans le **Chapitre 2**, nous présentons une extension du graphe des diviseurs de zéro classique d'un anneau commutatif R , noté par $\bar{\Gamma}(R)$, que l'on appelle le *graphe des diviseurs de zéro prolongé* de R , tel que son ensemble de sommets est constitué de tous les diviseurs de zéro non nul de R et que deux sommets distincts x et y sont adjacents par une arête si et seulement s'il existe deux entiers strictement positifs n et m tels que $x^n y^m = 0$ avec $x^n \neq 0$ et $y^m \neq 0$. Ainsi, évidemment, le graphe classique $\Gamma(R)$ est un sous graphe de $\bar{\Gamma}(R)$. Notez aussi que $\bar{\Gamma}(R)$ est un graphe vide si et seulement si R est un anneau intègre.

Dans la section 2, nous nous sommes intéressés à étudier où $\bar{\Gamma}(R)$ et $\Gamma(R)$ coïncident. Le résultat principal de cette section est le théorème 2.1.1. Il donne certaines conditions sur R qui caractérisent où $\bar{\Gamma}(R)$ et $\Gamma(R)$ coïncident :

Théorème. *Soit R un anneau. Les assertions suivantes sont équivalentes:*

1. $\bar{\Gamma}(R) = \Gamma(R)$.
2. R satisfait les deux conditions suivantes:

(i) Si $\text{Nil}(R) \neq \{0\}$, alors l'indice de nilpotence de chaque élément nilpotent non nul est 2,

(ii) Pour chaque $x \in Z(R) \setminus \text{Nil}(R)$, $\text{Ann}(x^2) = \text{Ann}(x)$.

3. R satisfait les deux conditions suivantes:

(i) Si $\text{Nil}(R) \neq \{0\}$, alors l'indice de nilpotence de chaque élément nilpotent non nul est 2,

(ii) Pour chaque $x \in Z(R)$, $\sqrt{\text{Ann}(x)} \setminus \text{Nil}(R) \subset \text{Ann}(x)$.

Dans ce contexte, plusieurs exemples sont donnés (voir Exemples 2.1.3, Proposition 2.1.4 et Exemple 2.1.5). Ensuite, nous étudions quand Γ et $\bar{\Gamma}$ coïncident pour le produit direct fini des anneaux (Proposition 2.1.8) :

Proposition. Soit $(R_i)_{1 \leq i \leq n}$ une famille finie d'anneaux avec $n \in \mathbb{N}^* \setminus \{1\}$. Alors $\bar{\Gamma}(\prod_{i=1}^n R_i) = \Gamma(\prod_{i=1}^n R_i)$ si et seulement si R_i est réduit pour chaque $1 \leq i \leq n$.

La section 3 est consacrée à l'étude du diamètre des graphes des diviseurs de zéro prolongés des anneaux commutatifs. Evidemment, comme une extension du graphe classique $\Gamma(R)$ d'un anneau R , $\bar{\Gamma}(R)$ est également connecté et a un diamètre inférieur ou égal à 3. Dans le Théorème 2.2.2 nous caractérisons le cas où $\bar{\Gamma}(R)$ a un sommet adjacent à tous les autres sommets :

Théorème. Soient R un anneau et $x \in R$. Alors, x est adjacent à tous les autres sommets dans $\bar{\Gamma}(R)$ si et seulement si soit $R \cong \mathbb{Z}_2 \times D$, où D est un anneau intègre, soit $Z(R) = \sqrt{\text{Ann}(x^{n_x-1})}$.

Cela permet de caractériser quand le graphe $\bar{\Gamma}(R)$ est complet (voir Théorème 2.2.3) :

Théorème. Soit R un anneau. Alors $\bar{\Gamma}(R)$ est un graphe complet si et seulement si soit $R \cong \mathbb{Z}_2 \times \mathbb{Z}_2$, soit $Z(R) = \text{Nil}(R)$ et pour chaque $x, y \in Z(R)^*$ $x^{n_x-1}y^{n_y-1} = 0$.

Dans le Théorème 2.2.5, nous étudions le diamètre de $\bar{\Gamma}(R)$ dans le cas où $Z(R) = \text{Nil}(R) \neq \{0\}$:

Théorème. Soit R un anneau avec $Z(R) = \text{Nil}(R) \neq \{0\}$. Alors, $\text{diam}(\bar{\Gamma}(R)) \leq 2$ et exactement l'une des trois assertions soit vérifiée.

1. $|Z(R)^*| = 1$. Dans ce cas, R est isomorphe à \mathbb{Z}_4 ou à $\mathbb{Z}_2[X]/(X^2)$, et $\text{diam}(\bar{\Gamma}(R)) = 0$.
2. $|Z(R)^*| \geq 2$ et $Z(R)^2 = \{0\}$. Dans ce cas, $\bar{\Gamma}(R)$ est un graphe complet.
3. $|Z(R)^*| \geq 2$ et $Z(R)^2 \neq \{0\}$. Si $\bar{Z}(R)^2 = 0$, alors $\bar{\Gamma}(R)$ est un graphe complet. Sinon $\text{diam}(\bar{\Gamma}(R)) = 2$.

Nous étudions également le diamètre du produit direct fini d'anneaux (voir Proposition 2.2.6) :

Proposition. Soit $R = \prod_{i=1}^n R_i$ où $(R_i)_{1 \leq i \leq n}$ est une famille finie d'anneaux avec $n \in \mathbb{N}^* \setminus \{1\}$.

1. Si $n = 2$, nous avons les assertions suivantes:
 - (a) $\text{diam}(\Gamma(R)) = \text{diam}(\bar{\Gamma}(R)) = 1$ si et seulement si $R_1 \cong R_2 \cong \mathbb{Z}_2$.
 - (b) Si R_1 et R_2 sont des anneaux intègres avec $|R_1| \geq 3$ ou $|R_2| \geq 3$, alors $\Gamma(R) = \bar{\Gamma}(R)$ et $\text{diam}(\Gamma(R)) = 2$. Dans ce cas, $\Gamma(R)$ est un graphe biparti complet.
 - (c) Si au moins l'un des anneaux R_1 et R_2 contient un diviseur de zéro non nilpotent, alors $\text{diam}(\Gamma(R)) = \text{diam}(\bar{\Gamma}(R)) = 3$.
 - (d) Si au moins l'un des anneaux R_1 et R_2 n'est pas intègre tel que tous les diviseurs de zéro sont nilpotents dans chaque anneau, alors $\text{diam}(\Gamma(R)) = 3$ et $\text{diam}(\bar{\Gamma}(R)) = 2$.
2. Si $n \geq 3$, alors $\text{diam}(\Gamma(R)) = \text{diam}(\bar{\Gamma}(R)) = 3$.

Dans la section 4, nous étudions la maille du graphe $\bar{\Gamma}(R)$. Aussi, puisque $\Gamma(R)$ est un sous graphe de $\bar{\Gamma}(R)$ et d'après Théorème 1.3.1, on déduit que $\text{gr}(\bar{\Gamma}(R)) \in \{3, 4, \infty\}$. Dans le Théorème 2.3.1, nous montrons que $\bar{\Gamma}(R)$ contient un cycle lorsque $\bar{\Gamma}(R) \neq \Gamma(R)$.

Dans le Théorème 2.3.4, nous étudions la maille du graphe $\bar{\Gamma}(R)$ dans le cas où $Z(R) = \text{Nil}(R) \neq \{0\}$:

Théorème. Soit R un anneau où $Z(R) = \text{Nil}(R) \neq \{0\}$. Alors exactement l'une des trois assertions soit vérifiée.

1. Si $|Z(R)^*| = 1$, alors R est isomorphe à \mathbb{Z}_4 ou $\mathbb{Z}_2[X]/(X^2)$, et $\text{gr}(\bar{\Gamma}(R)) = \infty$.

2. Si $|Z(R)^*| = 2$, alors R est isomorphe à \mathbb{Z}_9 ou $\mathbb{Z}_3[X]/(X^2)$, et $\text{gr}(\bar{\Gamma}(R)) = \infty$.
3. Si $|Z(R)^*| = 3$, alors R est isomorphe à \mathbb{Z}_8 , $\mathbb{Z}_2[X]/(X^3)$, $\mathbb{Z}_4[X]/(2X, X^2 - 2)$, $\mathbb{Z}_2[X, Y]/(X, Y)^2$, $\mathbb{Z}_4[X]/(2, X)^2$, $\mathbb{Z}_4[X]/(X^2 + X + 1)$, ou $\mathbb{F}_4[X]/(X^2)$, et $\text{gr}(\bar{\Gamma}(R)) = 3$.
4. Si $|Z(R)^*| \geq 4$, alors $\text{gr}(\bar{\Gamma}(R)) = 3$.

Dans les Théorèmes 2.3.5 et 2.3.6, nous traitons le cas où $\text{gr}(\bar{\Gamma}(R)) = 4$:

Théorème. Soit R un anneau avec $\text{Nil}(R) \neq \{0\}$ et $\text{gr}(\Gamma(R)) = 4$. Alors $\bar{\Gamma}(R) \neq \Gamma(R)$ implique que $\text{gr}(\bar{\Gamma}(R)) = 4$ et $\bar{\Gamma}(R)$ est un graphe biparti complet.

Théorème. Soit R un anneau avec $\text{Nil}(R) \neq \{0\}$ et $\text{gr}(\Gamma(R)) = \infty$. Alors exactement l'une des assertions suivantes soit vérifiée :

1. $\bar{\Gamma}(R) = \Gamma(R)$ est un singleton ou un graphe étoile. Dans ce cas $\text{gr}(\bar{\Gamma}(R)) = \infty$.
2. $\Gamma(R) = \bar{K}^{1,3}$ (i.e., $R \cong \mathbb{Z}_2 \times \mathbb{Z}_4$ ou $R \cong \mathbb{Z}_2 \times \mathbb{Z}_2[X]/(X^2)$). Dans ce cas $\text{gr}(\bar{\Gamma}(R)) = 4$ et $\bar{\Gamma}(R) \neq \Gamma(R)$.

Dans la Proposition 2.3.8, nous montrons que si $\bar{\Gamma}(R) \neq \Gamma(R)$ et que le graphe $\bar{\Gamma}(R)$ est complété, alors l'anneau quotient total de R est de dimension de Krull zéro.

A la fin du chapitre, nous étudions la maille du graphe du produit direct fini d'anneaux (voir Proposition 2.3.10).

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Dans le **Chapitre 3**, nous sommes intéressés par l'étude des graphes des diviseurs de zéro prolongés des idéalizations. Rappelons que l'idéalisation d'un R -module M (appelée aussi une extension trivial de R par M), notée par $R \times M$, est l'anneau dont le groupe additif est $R \times M$ et la multiplication est définie par : $(r_1, m_1)(r_2, m_2) = (r_1 r_2, r_1 m_2 + r_2 m_1)$ pour tout (r_1, m_1) et (r_2, m_2) dans $R \times M$. Rappelons également que $\text{Nil}(R \times M) = \text{Nil}(R) \times M$ et $Z(R \times M) = (Z(R) \cup Z(M)) \times M$, où $Z(M) := \{r \in R \mid \exists m \in M \setminus \{0\}, rm = 0\}$. A partir de [16, Proposition 1.1], nous avons aussi $Z(R \times M)^* = \{(0, m) \mid m \in M \setminus \{0\}\} \cup \{(a, n) \mid a \in R \setminus \{0\}, n \in M \text{ et il existe un } m \in M \setminus \{0\},$

$am = 0\} \cup \{(a, n) \mid a \in Z(R)^*, n \in M\}$ (voir, par exemple, [3] et [24] pour plus de détails sur cette construction d'anneaux).

Dans la Section 2, nous sommes intéressés par l'étude de la relation entre $\bar{\Gamma}(R \times M)$ et $\Gamma(R \times M)$. Dans le résultat principal de cette section (Théorème 3.1.1), nous caractérisons quand $\bar{\Gamma}(R \times M)$ et $\Gamma(R \times M)$ coïncident :

Théorème. *Les deux graphes $\bar{\Gamma}(R \times M)$ et $\Gamma(R \times M)$ coïncident si et seulement si les conditions suivantes sont vérifiées.*

1. $(2\text{Nil}(R))M = 0$.
2. $\bar{\Gamma}(R) = \Gamma(R)$.
3. $\bigcup_{a \in \Lambda} \text{Ann}(a) \subseteq \text{Ann}(M)$, où $\Lambda = Z(R) \setminus \text{Nil}(R)$.
4. $\text{Ann}_M(a^2) = \text{Ann}_M(a)$ pour tous $a \in R \setminus \text{Nil}(R)$.

Puis, comme corollaires, nous donnons divers exemples d'idéalisations $R \times M$ lorsque $\bar{\Gamma}(R \times M)$ et $\Gamma(R \times M)$ coïncident (voir Corollaires de 3.1.2 à 3.1.6).

Dans la Section 3, nous caractérisons complètement la maille du graphe des diviseurs de zéro prolongé d'une idéalisation. Enfin, la section 4 traite le diamètre du graphe $\bar{\Gamma}(R \times M)$. Nous caractérisons quand le graphe des diviseurs de zéro prolongé d'une idéalisation est complet (Théorème 3.3.2) :

Théorème. *Supposons que $\bar{\Gamma}(R \times M) \neq \Gamma(R \times M)$. Alors $\bar{\Gamma}(R \times M)$ est complet si et seulement si les conditions suivantes sont vérifiées.*

1. $Z(R) = Z(M) = \text{Nil}(R)$.
2. $\bar{Z}(R)^2 = \{0\}$.
3. $r^{n_r-1}M = 0$ pour tous $r \in \text{Nil}(R) \setminus \{0\}$.

Et nous caractérisons quand $\text{diam}(\bar{\Gamma}(R \times M)) \leq 2$ (Théorème 3.3.4) :

Théorème. *$\text{diam}(\bar{\Gamma}(R \times M)) \leq 2$ si et seulement si, pour tous $x, y \in Z(R) \cup Z(M)$, l'une des deux assertions soit vérifiée.*

1. Il existe un $z \in \text{Ann}(M) \setminus \{0\}$ tel que $x, y \in \sqrt{\text{Ann}(z)}$.
2. Il existe un $m \in M \setminus \{0\}$ tel que $x, y \in \sqrt{\text{Ann}_R(m)}$.

* *

Dans le **Chapitre 4**, nous étudions la relation entre le graphe des diviseurs de zéro, le graphe des diviseurs de zéro prolongé et le graphe annulateur. Rappelons que le graphe annulateur d'un anneau R est le graphe noté $AG(R)$ tel que son ensemble de sommets se compose de tous ses diviseurs de zéro non nul de R et que deux sommets distincts x et y sont adjacents si et seulement si $\text{Ann}_R(xy) \neq \text{Ann}_R(x) \cup \text{Ann}_R(y)$.

Dans la section 1, nous montrons que le graphe des diviseurs de zéro prolongé $\bar{\Gamma}(R)$ est un sous graphe du graphe annulateur $AG(R)$, ainsi $\Gamma(R) \subset \bar{\Gamma}(R) \subset AG(R)$ (Théorème 4.1.2):

Théorème. *Soit R un anneau commutatif. Alors $\bar{\Gamma}(R)$ est un sous graphe de $AG(R)$.*

Avant de donner une caractérisation de $AG(R) = \bar{\Gamma}(R)$, nous présentons des cas particuliers des anneaux sur lesquels $AG(R) = \bar{\Gamma}(R)$ (voir Proposition 4.1.3, 4.1.6, 4.1.8 and 4.1.9). Le résultat principal de cette section (Théorème 4.1.11) :

Théorème. *Soit R un anneau commutatif où $\Gamma(R) \neq \bar{\Gamma}(R)$. Alors $AG(R) = \bar{\Gamma}(R)$ si et seulement si soit :*

- $\text{gr}(AG(R)) = 4$; ou
- Si $Z(R) = \text{Nil}(R)$, alors $x^{n_x-1}y^{n_y-1} = 0$ pour chaque $x, y \in \text{Nil}(R)^*$; ou
- Si $Z(R) \neq \text{Nil}(R)$, alors les conditions suivantes sont vérifiées :
 - Pour chaque $x, y \in \text{Nil}(R)^*$, nous avons $x^{n_x-1}y^{n_y-1} = 0$, et
 - S'il existe $z \in Z(R) \setminus \text{Nil}(R)$ et $t \in Z(R)^*$ avec $z^{n_t}t^m \neq 0$ pour tous $n, m \in \mathbb{N}^*$, alors $\text{Ann}_R(zt) = \text{Ann}_R(z)$ ou $\text{Ann}_R(zt) = \text{Ann}_R(t)$.

Dans la section 2, nous étudions le graphe annulateur des anneaux de produits finis (Théorème 4.2.1):

Théorème. *Soit $R = \prod_{i=1}^n R_i$, où $(R_i)_{1 \leq i \leq n}$ est une famille finie d'anneaux avec $n \in \mathbb{N}^* \setminus \{1\}$.*

1. Si $n = 2$, nous avons les assertions suivantes :

- (a) Si $R_1 \cong R_2 \cong \mathbb{Z}_2$, alors $\Gamma(R) = \bar{\Gamma}(R) = AG(R)$, $\text{gr}(\Gamma(R)) = \infty$ and $\text{diam}(\Gamma(R)) = 1$.
- (b) Si R_1 et R_2 sont des anneaux intègres et l'un d'entre eux est isomorphe à \mathbb{Z}_2 , alors $\text{gr}(\Gamma(R)) = \text{gr}(\bar{\Gamma}(R)) = \text{gr}(AG(R)) = \infty$ et $\text{diam}(\Gamma(R)) = \text{diam}(\bar{\Gamma}(R)) = \text{diam}(AG(R)) = 2$. Dans ce cas $\Gamma(R) = \bar{\Gamma}(R) = AG(R)$ est un graphe étoile.
- (c) Si R_1 et R_2 sont des anneaux intègres avec $|R_1| \geq 3$ et $|R_2| \geq 3$, alors $\Gamma(R) = \bar{\Gamma}(R) = AG(R)$, $\text{diam}(\Gamma(R)) = 2$ et $\text{gr}(\Gamma(R)) = 4$. Dans ce cas $\Gamma(R)$ est un graphe biparti complet.
- (d) Si R_1 ou R_2 n'est pas intègre, alors $\text{gr}(\Gamma(R)) = \text{gr}(\bar{\Gamma}(R)) = \text{gr}(AG(R)) = 3$, $\text{diam}(\Gamma(R)) = \text{diam}(\bar{\Gamma}(R)) = 3$ et $\text{diam}(AG(R)) = 2$.
- (e) Si R_1 ou R_2 n'est pas intègre tel que tous les diviseurs de zéro sont nilpotents dans chaque anneau, alors $\text{gr}(\Gamma(R)) = \text{gr}(\bar{\Gamma}(R)) = \text{gr}(AG(R)) = 3$, $\text{diam}(\Gamma(R)) = 3$ et $\text{diam}(\bar{\Gamma}(R)) = \text{diam}(AG(R)) = 2$.
2. Si $n \geq 3$, alors $\text{gr}(\Gamma(R)) = \text{gr}(\bar{\Gamma}(R)) = \text{gr}(AG(R)) = 3$, $\text{diam}(\Gamma(R)) = \text{diam}(\bar{\Gamma}(R)) = 3$ et $\text{diam}(AG(R)) = 2$.

A la fin de cette section, nous étudions où $\bar{\Gamma}(R \times M)$ et $AG(R \times M)$ coïncide (Théorème 4.2.5):

Théorème. Soit R un anneau intègre et M un R -module. Alors $\bar{\Gamma}(R \times M) = AG(R \times M)$ si et seulement si les conditions suivantes sont vérifiées.

1. Pour tous $m \in M$ et $a \in R^*$, $\text{Ann}_R(m) = \text{Ann}_R(am)$ si $a \notin \sqrt{\text{Ann}_R(m)}$.
2. Pour tous $a, b \in R$, $\text{Ann}_M(a) \cup \text{Ann}_M(b) = \text{Ann}_M(ab)$.

*** * 5 * ***

Dans le **Chapitre 5**, nous présentons une première étude sur les codes basés sur des graphes de diviseurs de zéro des anneaux commutatifs. Nous construisons des codes binaires à partir de matrices d'incidence des graphes des diviseurs de zéro prolongé des anneaux de la forme \mathbb{Z}_{p^α} avec des paramètres larges, où p est un nombre premier et α est un nombre entier supérieure à 3.

Après avoir rappelé certaines notions de base et fixé quelques notations dans la section 1, nous présentons, dans la section 2, un aperçu de la théorie du

codage. Nous nous concentrerons principalement sur les résultats dont nous avons besoin dans ce chapitre.

La section 3 présente quelques résultats sur les codes linéaires associés aux matrices d'incidence des graphes.

Dans la section 4 du chapitre 5, nous présentons le premier exemple d'une famille de codes binaires construits à partir de matrices d'incidence de graphe des diviseurs de zéro prolongé de \mathbb{Z}_{p^α} (Proposition 5.4.3):

Proposition. *Soit p un nombre premier et $\alpha > 2$ un entier. $G_{p,\alpha}$ une matrice d'incidence $\bar{\Gamma}(\mathbb{Z}_{p^\alpha})$. Le code binaire $C_2(G_{p,\alpha})$ est de longueur $\frac{(p^{\alpha-1}-1)(p^{\alpha-1}-2)}{2}$, de dimension $p^{\alpha-1}-2$, de distance minimale de Hamming $p^{\alpha-1}-2$, et les mots minimum sont les lignes de $G_{p,\alpha}$ de poids $p^{\alpha-1}-2$.*

Introduction

Throughout this thesis all rings are commutative with identity and all modules are unital.

Setup and Notation. Let R be a ring. We use $Z(R)$ to denote the set of all zero-divisors of R and $Z(R)^* := Z(R) \setminus \{0\}$. We denote by $\text{reg}(R)$ the set of regular elements of R and by $\text{Ann}(x)$ the annihilator of an element x of R . For an ideal I of R , \sqrt{I} means the radical of I , in particular, $\text{Nil}(R) := \sqrt{0}$ is the nilradical of R . For a nonzero nilpotent element x of R , n_x denotes the index of nilpotency of x . The ring $\mathbb{Z}/n\mathbb{Z}$ of the residues modulo a non negative integer $n \in \mathbb{N}^*$ will be noted by \mathbb{Z}_n . We use \subset to mean “is a not necessarily proper subset of” and \subsetneq to mean “is a proper subset of.” Finally, $T(R) = S^{-1}R$, where S is the set of regular elements, is the total quotient ring of R .

The reader will be assumed to be familiar with the basic notions of the theory of commutative rings with zero-divisors. For further details, we refer the reader to the book of Huckaba [26]. We assume that the reader is also familiar with the basic notions of graph theory. However, we recall certain notions used in this thesis in an appendix (see the book [21] for more details).

The zero-divisor graph of a ring R , denoted by $\Gamma(R)$, is a simple graph associated to R such that its vertex set consists of all its nonzero zero-divisors and that two distinct vertices are joined by an edge if and only if the product of these two vertices is zero. The idea of associating graphs with algebraic structures goes back to Beck in [18], where he was mainly interested in colorings. In his work all elements of the ring were vertices of the graph (see also [2]). It was Anderson and Livingston, in [11], who introduced the zero-divisor graph of a commutative ring and initiated the study of the relation between ring-theoretic properties and graph theoretic ones. Since then, the zero-divisor graphs of commutative rings have attracted the attention of several researchers (see, for instance, [4, 6, 7, 10, 11, 13, 15, 16, 28, 29]).

Motivated then by the success of this new area of research several authors have recently introduced other graphs associated to some ring theoretic properties (see, for instance, [8, 9, 14, 17, 31, 32]). The main aim of studying

these graphs is that one may find some results about the algebraic structures and vice versa. In this context, we have introduced a new zero-divisor graphs, called the extended zero-divisors graph, which serves, among other things, to characterize the 0-dimensional ring (see Proposition 2.3.8). Indeed, the main aim in our PhD studies is to study deeply the extended zero-divisor graphs. In this thesis, we present some works established on the extended zero-divisor graph.

This thesis is divided into five chapters.

* * * * * * * * * * * 1 * * * * * * * * * *

In **Chapter 1**, we present an overview on the classical zero-divisor graph of a ring. We will mainly focus on some classical results that we need in this thesis.

* * * * * * * * * * * 2 * * * * * * * * * *

In **Chapter 2**, we present an extension of the classical zero-divisor graph of a commutative ring R , denoted by $\bar{\Gamma}(R)$, which we call *the extended zero-divisor graph* of R , such that its vertex set consists of all its nonzero zero-divisors and that two distinct vertices x and y are joined by an edge if and only if there exist two non negative integers n and m such that $x^n y^m = 0$ with $x^n \neq 0$ and $y^m \neq 0$. Thus, obviously the classical graph $\Gamma(R)$ is a subgraph of $\bar{\Gamma}(R)$. Note also that $\bar{\Gamma}(R)$ is the empty graph if and only if R is an integral domain.

In Section 2, we are interested in studying when $\bar{\Gamma}(R)$ and $\Gamma(R)$ coincide. The main result in this section is Theorem 2.1.1. It gives some conditions on R which characterize when $\bar{\Gamma}(R)$ and $\Gamma(R)$ coincide:

Theorem. *Let R be a ring. The following statements are equivalent:*

1. $\bar{\Gamma}(R) = \Gamma(R)$.
2. R satisfies the two following conditions:
 - (i) If $\text{Nil}(R) \neq \{0\}$, then every nonzero nilpotent element has index 2, and
 - (ii) For every $x \in Z(R) \setminus \text{Nil}(R)$, $\text{Ann}(x^2) = \text{Ann}(x)$.
3. R satisfies the two following conditions:
 - (i) If $\text{Nil}(R) \neq \{0\}$, then every nonzero nilpotent element has index 2, and

(ii) For every $x \in Z(R)$, $\sqrt{\text{Ann}(x)} \setminus \text{Nil}(R) \subset \text{Ann}(x)$.

In this context several examples are given (see Examples 2.1.3, Proposition 2.1.4 and Example 2.1.5). Then, we study this property for finite direct product of rings (Proposition 2.1.8):

Proposition. *Let $(R_i)_{1 \leq i \leq n}$ be a finite family of rings with $n \in \mathbb{N}^* \setminus \{1\}$. Then $\overline{\Gamma}(\prod_{i=1}^n R_i) = \Gamma(\prod_{i=1}^n R_i)$ if and only if R_i is reduced for every $1 \leq i \leq n$.*

Section 3 is devoted to the study of the diameter of extended zero-divisor graphs of commutative rings. Obviously, as an extension of the classical graph $\Gamma(R)$ of ring R , $\overline{\Gamma}(R)$ is also connected and has diameter at most 3. In Theorem 2.2.2 we characterize the case where $\overline{\Gamma}(R)$ has a vertex adjacent to every other vertex:

Theorem. *Let R be a ring and $x \in R$. Then x is adjacent to every other vertex of $\overline{\Gamma}(R)$ if and only if either $R \cong \mathbb{Z}_2 \times D$, where D is an integral domain, or $Z(R) = \sqrt{\text{Ann}(x^{n_x-1})}$.*

This allows to characterize when the graph $\overline{\Gamma}(R)$ is complete (see Theorem 2.2.3):

Theorem. *Let R be a ring. Then $\overline{\Gamma}(R)$ is a complete graph if and only if either $R \cong \mathbb{Z}_2 \times \mathbb{Z}_2$ or $Z(R) = \text{Nil}(R)$ and, for every $x, y \in Z(R)^*$, $x^{n_x-1}y^{n_y-1} = 0$.*

In Theorem 2.2.5 we studies the diameter of $\overline{\Gamma}(R)$ in the case where $Z(R) = \text{Nil}(R) \neq \{0\}$:

Theorem. *Let R be a ring with $Z(R) = \text{Nil}(R) \neq \{0\}$. Then $\text{diam}(\overline{\Gamma}(R)) \leq 2$ and exactly one of the following three cases must occur.*

1. $|Z(R)^*| = 1$. Then R is isomorphic to \mathbb{Z}_4 or $\mathbb{Z}_2[X]/(X^2)$, and $\text{diam}(\overline{\Gamma}(R)) = 0$.
2. $|Z(R)^*| \geq 2$ and $Z(R)^2 = \{0\}$. Then $\overline{\Gamma}(R)$ is a complete graph.
3. $|Z(R)^*| \geq 2$ and $Z(R)^2 \neq \{0\}$. If $\overline{Z}(R)^2 = 0$ then $\overline{\Gamma}(R)$ is a complete graph. If not $\text{diam}(\overline{\Gamma}(R)) = 2$.

We also study the diameter of the graph of finite direct products of rings (see Proposition 2.2.6):

Proposition. Let $R = \prod_{i=1}^n R_i$, where $(R_i)_{1 \leq i \leq n}$ is a finite family of rings with $n \in \mathbb{N}^* \setminus \{1\}$.

1. If $n = 2$, we have the following assertions:

- (a) $\text{diam}(\Gamma(R)) = \text{diam}(\overline{\Gamma}(R)) = 1$ if and only if $R_1 \cong R_2 \cong \mathbb{Z}_2$.
- (b) If R_1 and R_2 are integral domains with $|R_1| \geq 3$ or $|R_2| \geq 3$, then $\Gamma(R) = \overline{\Gamma}(R)$ and $\text{diam}(\Gamma(R)) = 2$. In this case $\Gamma(R)$ is a complete bipartite graph.
- (c) If at least one of R_1 and R_2 contains a non-nilpotent zero-divisor, then $\text{diam}(\Gamma(R)) = \text{diam}(\overline{\Gamma}(R)) = 3$.
- (d) If at least one of R_1 and R_2 is not integral domains such that all zero-divisors are nilpotent in each ring with nonzero zero-divisors, then $\text{diam}(\Gamma(R)) = 3$ and $\text{diam}(\overline{\Gamma}(R)) = 2$.

2. If $n \geq 3$, $\text{diam}(\Gamma(R)) = \text{diam}(\overline{\Gamma}(R)) = 3$.

Finally, in section 4, we study the girth of $\overline{\Gamma}(R)$. Also, since $\Gamma(R)$ is a subgraph of $\overline{\Gamma}(R)$ and by Theorem 1.3.1, we deduce that $\text{gr}(\overline{\Gamma}(R)) \in \{3, 4, \infty\}$. In Theorem 2.3.1, we show that $\overline{\Gamma}(R)$ contains a cycle when $\overline{\Gamma}(R) \neq \Gamma(R)$.

In Theorem 2.3.4, we studies the girth of $\overline{\Gamma}(R)$ in the case where $Z(R) = \text{Nil}(R) \neq \{0\}$:

Theorem. Let R be a ring with $Z(R) = \text{Nil}(R) \neq \{0\}$. Then exactly one of the following three cases must occur.

- 1. If $|Z(R)^*| = 1$, then R is isomorphic to \mathbb{Z}_4 or $\mathbb{Z}_2[X]/(X^2)$, and $\text{gr}(\overline{\Gamma}(R)) = \infty$.
- 2. If $|Z(R)^*| = 2$, then R is isomorphic to \mathbb{Z}_9 or $\mathbb{Z}_3[X]/(X^2)$, and $\text{gr}(\overline{\Gamma}(R)) = \infty$.
- 3. If $|Z(R)^*| = 3$, then R is isomorphic to \mathbb{Z}_8 , $\mathbb{Z}_2[X]/(X^3)$, $\mathbb{Z}_4[X]/(2X, X^2 - 2)$, $\mathbb{Z}_2[X, Y]/(X, Y)^2$, $\mathbb{Z}_4[X]/(2, X)^2$, $\mathbb{Z}_4[X]/(X^2 + X + 1)$, or $\mathbb{F}_4[X]/(X^2)$, and $\text{gr}(\overline{\Gamma}(R)) = 3$.
- 4. If $|Z(R)^*| \geq 4$, then $\text{gr}(\overline{\Gamma}(R)) = 3$.

In Theorems 2.3.5 and 2.3.6, situations where $\text{gr}(\overline{\Gamma}(R)) = 4$ are given:

4. $\text{Ann}_M(a^2) = \text{Ann}_M(a)$ for all $a \in R \setminus \text{Nil}(R)$.

Then, as corollaries, we give various examples of idealizations $R \times M$ such that $\bar{\Gamma}(R \times M)$ and $\Gamma(R \times M)$ coincide (see Corollaries from 3.1.2 to 3.1.6).

In Section 3, we completely characterize the girth of the extended zero-divisor graph of an idealization. Finally, Section 4 deals with the diameter of the graph $\bar{\Gamma}(R \times M)$. We characterize when the extended zero-divisor graph of an idealization is complete (Theorem 3.3.2):

Theorem. *Assume that $\bar{\Gamma}(R \times M) \neq \Gamma(R \times M)$. Then $\bar{\Gamma}(R \times M)$ is complete if and only if the following conditions hold.*

1. $Z(R) = Z(M) = \text{Nil}(R)$.
2. $\bar{Z}(R)^2 = \{0\}$.
3. $r^{n_r-1}M = 0$ for all $r \in \text{Nil}(R) \setminus \{0\}$.

And we characterize when $\text{diam}(\bar{\Gamma}(R \times M)) \leq 2$ (Theorem 3.3.4):

Theorem. *$\text{diam}(\bar{\Gamma}(R \times M)) \leq 2$ if and only if, for all $x, y \in Z(R) \cup Z(M)$, one of the following two assertions holds.*

1. There is a $z \in \text{Ann}(M) \setminus \{0\}$ such that $x, y \in \sqrt{\text{Ann}(z)}$.
2. There is an $m \in M \setminus \{0\}$ such that $x, y \in \sqrt{\text{Ann}_R(m)}$.

***** ✱ 4 ✱ *****

In **Chapter 4**, we investigate the relation between the classical zero-divisor graph, extended zero-divisor graph and annihilator graph of a ring. Recall that the annihilator graph of R is the graph $AG(R)$, such that its vertex set consists of all its nonzero zero-divisors and that two distinct vertices x and y are adjacent if and only if $\text{Ann}_R(xy) \neq \text{Ann}_R(x) \cup \text{Ann}_R(y)$.

In section 1, we show that the extended graph $\bar{\Gamma}(R)$ is in fact a subgraph of the annihilator graph $AG(R)$, and so $\Gamma(R) \subset \bar{\Gamma}(R) \subset AG(R)$ (Theorem 4.1.2):

Theorem. *Let R be a commutative ring. Then $\bar{\Gamma}(R)$ is a subgraph of $AG(R)$.*

Before giving a characterization of when $AG(R) = \bar{\Gamma}(R)$, we give some particular simple case of rings over which $AG(R) = \bar{\Gamma}(R)$ (see Proposition 4.1.3,

4.1.6, 4.1.8 and 4.1.9). So we give one of the main result of this chapter which characterizes the case when $AG(R) = \bar{\Gamma}(R)$ of a commutative ring R (Theorem 4.1.11):

Theorem. *Let R be a commutative ring with $\Gamma(R) \neq \bar{\Gamma}(R)$. Then $AG(R) = \bar{\Gamma}(R)$ if and only if either:*

- $\text{gr}(AG(R)) = 4$; or
- If $Z(R) = \text{Nil}(R)$, then $x^{n_x-1}y^{n_y-1} = 0$ for each $x, y \in \text{Nil}(R)^*$; or
- If $Z(R) \neq \text{Nil}(R)$, then the following conditions hold.
 - For each $x, y \in \text{Nil}(R)^*$ distinct, we have $x^{n_x-1}y^{n_y-1} = 0$, and
 - If there are $z \in Z(R) \setminus \text{Nil}(R)$ and $t \in Z(R)^*$ distinct with $z^n t^m \neq 0$ for all $n, m \in \mathbb{N}^*$, we have $\text{Ann}_R(zt) = \text{Ann}_R(z)$ or $\text{Ann}_R(zt) = \text{Ann}_R(t)$.

In section 2, we study the annihilator graph of the finite direct products of rings (Theorem 4.2.1):

Theorem. *Let $R = \prod_{i=1}^n R_i$, where $(R_i)_{1 \leq i \leq n}$ is a finite family of rings with $n \in \mathbb{N}^* \setminus \{1\}$.*

1. *If $n = 2$, the following assertions hold:*

- (a) *If $R_1 \cong R_2 \cong \mathbb{Z}_2$, then $\Gamma(R) = \bar{\Gamma}(R) = AG(R)$, $\text{gr}(\Gamma(R)) = \infty$ and $\text{diam}(\Gamma(R)) = 1$.*
- (b) *If R_1 and R_2 are integral domains and one of them is isomorphic to \mathbb{Z}_2 , then $\text{gr}(\Gamma(R)) = \text{gr}(\bar{\Gamma}(R)) = \text{gr}(AG(R)) = \infty$ and $\text{diam}(\Gamma(R)) = \text{diam}(\bar{\Gamma}(R)) = \text{diam}(AG(R)) = 2$. In this case $\Gamma(R) = \bar{\Gamma}(R) = AG(R)$ is a star graph.*
- (c) *If R_1 and R_2 are integral domains with $|R_1| \geq 3$ and $|R_2| \geq 3$, then $\Gamma(R) = \bar{\Gamma}(R) = AG(R)$, $\text{diam}(\Gamma(R)) = 2$ and $\text{gr}(\Gamma(R)) = 4$. In this case $\Gamma(R)$ is a complete bipartite graph.*
- (d) *If R_1 or R_2 is not an integral domain, then $\text{gr}(\Gamma(R)) = \text{gr}(\bar{\Gamma}(R)) = \text{gr}(AG(R)) = 3$, $\text{diam}(\Gamma(R)) = \text{diam}(\bar{\Gamma}(R)) = 3$ and $\text{diam}(AG(R)) = 2$.*

- (e) If R_1 or R_2 is not an integral domain such that all zero-divisors are nilpotent in each ring with nonzero zero-divisors, then $\text{gr}(\Gamma(R)) = \text{gr}(\overline{\Gamma}(R)) = \text{gr}(AG(R)) = 3$, $\text{diam}(\Gamma(R)) = 3$ and $\text{diam}(\overline{\Gamma}(R)) = \text{diam}(AG(R)) = 2$.
2. If $n \geq 3$, then $\text{gr}(\Gamma(R)) = \text{gr}(\overline{\Gamma}(R)) = \text{gr}(AG(R)) = 3$, $\text{diam}(\Gamma(R)) = \text{diam}(\overline{\Gamma}(R)) = 3$ and $\text{diam}(AG(R)) = 2$.

We end this section in studying when $\overline{\Gamma}(R \times M)$ and $AG(R \times M)$ coincide (Theorem 4.2.5):

Theorem. *Let R be an integral domain and M an R -module. Then, $\overline{\Gamma}(R \times M) = AG(R \times M)$ if and only if the following conditions hold.*

1. For all $m \in M$ and $a \in R^*$, $\text{Ann}_R(m) = \text{Ann}_R(am)$ if $a \notin \sqrt{\text{Ann}_R(m)}$.
2. For all $a, b \in R$, $\text{Ann}_M(a) \cup \text{Ann}_M(b) = \text{Ann}_M(ab)$.

* * * * * * * * * * * 5 * * * * * * * * * *

In **Chapter 5**, we present the first study on the codes based on zero-divisor graphs of commutative rings. Namely, we construct binary codes from incidence matrices of extended zero-divisor graphs of rings of the form \mathbb{Z}_{p^α} with large parameters, where p is a prime number and α is a nonnegative integer.

After recalling some basic notions and fixing some notations in section 1, we give, in section 2, an overview on the coding theory. We will mainly focus on some results that we need in this chapter.

Section 3 presents some results on linear codes associated to incidence matrices of graphs.

In section 4 of chapter 5, we present the first example of a family of binary codes constructed from incidence matrices of extended zero-divisor graphs of \mathbb{Z}_{p^α} (Proposition 5.4.3):

Proposition. *Let p be a prime integer and $\alpha > 2$ an integer. $G_{p,\alpha}$ an incidence matrix for $\overline{\Gamma}(\mathbb{Z}_{p^\alpha})$. The binary code $C_2(G_{p,\alpha})$ is of length $\frac{(p^{\alpha-1}-1)(p^{\alpha-1}-2)}{2}$, dimension $p^{\alpha-1} - 2$, minimum Hamming distance $p^{\alpha-1} - 2$, and the minimum words are the rows of $G_{p,\alpha}$ of weight $p^{\alpha-1} - 2$.*

Zero-divisor graph of a commutative ring (Preliminaries)

In this chapter, we give an overview on the classical zero-divisor graph of a commutative ring. We will mainly focus on some classical results that we need in this thesis.

1.1 Definitions and some results

Definition 1.1.1 *Let R be a commutative ring with nonzero identity. We define the zero-divisor graph of R , denoted $\Gamma(R)$, to be a simple graph with vertex set being the set of nonzero zero-divisors of R and with $x - y$ an edge if and only if $x \neq y$ and $xy = 0$.*

Note that $\Gamma(R)$ is the empty graph if and only if R is an integral domain.

Let us give some simple examples.

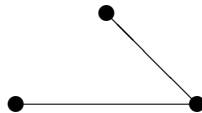
Example 1.1.2



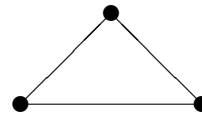
$\Gamma(\mathbb{Z}_4)$ or $\Gamma(\mathbb{Z}_2[X]/(X^2))$



$\Gamma(\mathbb{Z}_9)$, $\Gamma(\mathbb{Z}_2 \times \mathbb{Z}_2)$ or $\Gamma(\mathbb{Z}_3[X]/(X^2))$



$\Gamma(\mathbb{Z}_6)$, $\Gamma(\mathbb{Z}_8)$ or $\Gamma(\mathbb{Z}_2[X]/(X^3))$



$\Gamma(\mathbb{Z}_2[X, Y]/(X^2, XY, Y^2))$ or $\Gamma(\mathbb{F}_4[X]/(X^2))$

Now we give the comforting result from [11] that (nonempty) finite zero-divisor graphs come from finite rings.

Theorem 1.1.3 ([25], Theorem 1 and [11], Theorem 2.2) *Let R be a commutative ring.*

Then $\Gamma(R)$ is finite if and only if either R is finite or R is an integral domain.

In particular, if $1 \leq |\Gamma(R)| < \infty$, then R is finite and not a field. Moreover, $|R| \leq |Z(R)|^2$ if R is not an integral domain.

Proof. It is sufficient to prove the moreover statement. Let $x \in Z(R)^*$. Then the R -module homomorphism $f : R \rightarrow R$ given by $f(r) = rx$ has kernel $\text{Ann}_R(x)$ and image xR . Thus $|R| = |\text{Ann}_R(x)||xR| \leq |Z(R)|^*$. ■

In [34] (see [35]), all graphs on $n = 6, 7, \dots, 14$ vertices which can be realized as the zero-divisor graphs of a commutative rings with 1, and the list of all rings (up to isomorphism) which produce these graphs, were given. Below we list the established ring and graphs for $n = 3$ to 6. The tables for $n = 1, 2, 3$ and 4 can be found in [12]. The results for $n = 5$ can be found in [36].

| Vertices | R | $ R $ | Graph | Type |
|----------|---------------------------------|-------|-----------|---------|
| 3 | \mathbb{Z}_6 | 6 | $K^{1,2}$ | reduced |
| | \mathbb{Z}_8 | 8 | $K^{1,2}$ | local |
| | $\mathbb{Z}_2[X]/(X^3)$ | 8 | $K^{1,2}$ | local |
| | $\mathbb{Z}_4[X]/(2X, X^2 - 2)$ | 8 | $K^{1,2}$ | local |
| | $\mathbb{Z}_2[X, Y]/(X, Y)^2$ | 8 | K^3 | local |
| | $\mathbb{Z}_4[X]/(2, X)^2$ | 8 | K^3 | local |
| | $\mathbb{F}_4[X]/(X^2)$ | 16 | K^3 | local |
| | $\mathbb{Z}_4[X]/(X^2 + X + 1)$ | 16 | K^3 | local |

| Vertices | R | $ R $ | Graph | Type |
|----------|------------------------------------|-------|-----------|---------|
| 4 | $\mathbb{Z}_2 \times \mathbb{F}_4$ | 8 | $K^{1,3}$ | reduced |
| | $\mathbb{Z}_3 \times \mathbb{Z}_3$ | 9 | $K^{2,2}$ | reduced |
| | \mathbb{Z}_{25} | 25 | K^4 | local |
| | $\mathbb{Z}_5[X]/(X^2)$ | 25 | K^4 | local |

| Vertices | R | $ R $ | Graph | Type |
|----------|---|-------|-----------|---------|
| 5 | $\mathbb{Z}_2 \times \mathbb{Z}_5$ | 10 | $K^{1,4}$ | reduced |
| | $\mathbb{Z}_3 \times \mathbb{F}_4$ | 12 | $K^{2,3}$ | reduced |
| | $\mathbb{Z}_2 \times \mathbb{Z}_4$ | 8 | Fig. 1 | mixed |
| | $\mathbb{Z}_2 \times \mathbb{Z}_2[X]/(X^2)$ | 8 | Fig. 1 | mixed |



Fig. 1. The graph for $\mathbb{Z}_2 \times \mathbb{Z}_4$ and $\mathbb{Z}_2 \times \mathbb{Z}_2[X]/(X^2)$.

| Vertices | R | $ R $ | Graph | Type |
|----------|--|-------|-----------|---------|
| 6 | $\mathbb{Z}_3 \times \mathbb{Z}_5$ | 15 | $K^{2,4}$ | reduced |
| | $\mathbb{F}_4 \times \mathbb{F}_4$ | 16 | $K^{3,3}$ | reduced |
| | $\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$ | 8 | Fig. 2 | reduced |
| | \mathbb{Z}_{49} | 49 | K^6 | local |
| | $\mathbb{Z}_7[X]/(X^2)$ | 49 | K^6 | local |

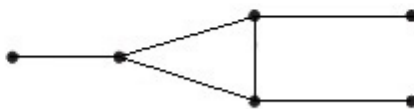
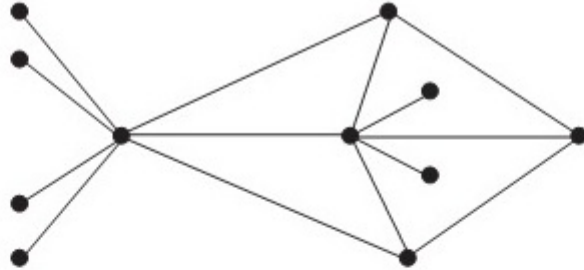


Fig. 2. The graph for $\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$.

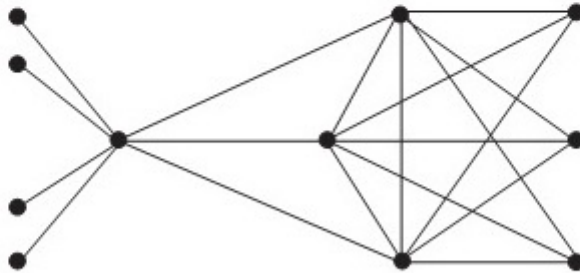
Here are some more graphs.



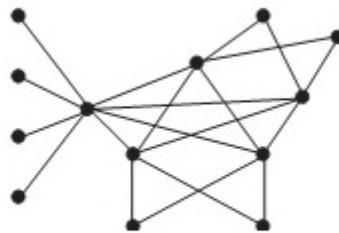
The graph for $\mathbb{Z}_5 \times \mathbb{Z}_4$ and $\mathbb{Z}_5 \times \mathbb{Z}_2[X]/(X^2)$.



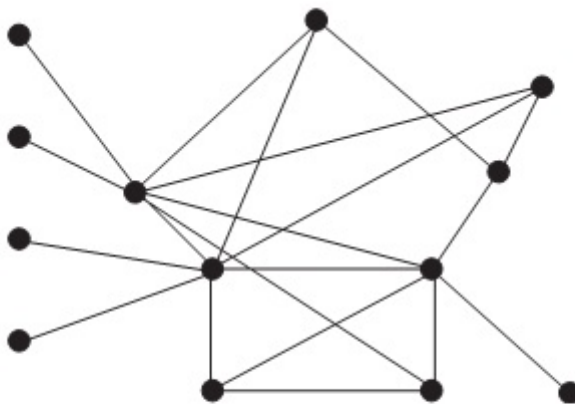
The graph for $\mathbb{Z}_2 \times \mathbb{Z}_8$, $\mathbb{Z}_2 \times \mathbb{Z}_2[X]/(X^3)$,
and $\mathbb{Z}_2 \times \mathbb{Z}_4[X]/(2X, X^2 - 2)$.



The graph for $\mathbb{Z}_2 \times \mathbb{Z}_2[X, Y]/(X, Y)^2$
and $\mathbb{Z}_2 \times \mathbb{Z}_4[X]/(2, X)^2$.



The graph for $\mathbb{Z}_2 \times \mathbb{Z}_3 \times \mathbb{Z}_3$.



. The graph for $\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_4$ and $\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2[X]/(X^2)$.

We next turn to characterize the zero-divisor graphs of commutative rings with von Neumann regular total quotient rings. Recall that a commutative ring R is von Neumann regular if for each $x \in R$, there is a $y \in R$ with $x^2y = x$ (equivalently, R is reduced and $\dim(R) = 0$ [26, Remark, p. 5]). The simplest examples of von Neumann regular rings are direct products of fields.

Zero-divisor graph has been used to characterize various ring properties. Here we give one of them. For that we need to recall some notions.

Let Γ be a (undirected) graph. As in [28], for vertices a and b of Γ , we define $a \leq b$ if a and b are not adjacent and each vertex of Γ adjacent to a is also adjacent to b ; and define $a \sim b$ if $a \leq b$ and $b \leq a$. Thus $a \sim b$ if and only if a and b are adjacent to exactly the same vertices. Clearly \sim is an equivalence relation on Γ . For distinct vertices a and b of Γ , we say that a and b are orthogonal, written $a \perp b$, if a and b are adjacent and there is no vertex c of Γ which is adjacent to both a and b , i.e., the edge $a - b$ is not part of any triangle in Γ . We say that Γ is complemented if for each vertex a of Γ , there is a vertex b of Γ (called a complement of a) such that $a \perp b$, and that Γ is uniquely complemented if Γ is complemented and whenever $a \perp b$ and $a \perp c$, then $b \sim c$. For $a, b \in Z(R)^*$, we have $a \sim b$ in $\Gamma(R)$ if and only if $\text{Ann}(a) - \{a\} = \text{Ann}(b) - \{b\}$.

Theorem 1.1.4 ([10], Theorem 3.5) *The following statements are equivalent for a reduced commutative ring R .*

1. $T(R)$ is von Neumann regular.
2. $\Gamma(R)$ is uniquely complemented.
3. $\Gamma(R)$ is complemented.

Moreover, a nonempty $\Gamma(R)$ is a star graph if and only if $R \cong D \times \mathbb{Z}_2$ for some integral domain D .

To prove this theorem, we need the following lemmas.

Lemma 1.1.5 ([10], Lemma 3.3) *Let R be a commutative ring and $a, b \in Z(R)^*$. Then the following statements are equivalent.*

1. $a \perp b$, $a^2 \neq 0$ and $b^2 \neq 0$.
2. $ab = 0$ and $a + b$ is regular element of R

Lemma 1.1.6 ([10], Lemma 3.4) *Let R be a reduced commutative ring and $a, b, c \in Z(R)^*$. If $a \perp b$ and $a \perp c$, then $b \sim c$. Thus $\Gamma(R)$ is uniquely complemented if and only if $\Gamma(R)$ is complemented.*

Proof of Theorem 1.1.4. (1) \Rightarrow (2) Suppose that $T(R)$ is von Neumann regular. Let $a \in Z(R)^*$. By the comments preceding Theorem 1.1.4, there exists $b \in T(R)$, necessarily nonzero, such that $a \perp b \in \Gamma(T(R))$. Choose $s \in R \setminus Z(R)$ such that $sb \in R$. One can then easily verify that $a \perp b$ in $\Gamma(R)$. Hence $\Gamma(R)$ is complemented, and thus uniquely complemented by Lemma 1.1.6 since R is reduced.

(2) \Rightarrow (3) This is true for any graph.

(3) \Rightarrow (1) Let $x \in R$; we may assume that $x \in Z(R)^*$. Then there is a $y \in Z(R)^*$ such that $x \perp y$. By Lemma 1.1.5, $xy = 0$ and $x + y$ is a regular element of R . Hence $T(R)$ is von Neumann regular by [6, Theorem 2.3]. The moreover statement follows from the remarks after Theorem 1.2.2. ■

For a non reduced rings, we have the following characterizations.

Theorem 1.1.7 ([10], Theorem 3.9) *Let R be a commutative ring with $\text{Nil}(R)$ nonzero. If $\Gamma(R)$ is uniquely complemented, then either $\Gamma(R)$ is a star graph with at most two edges or $\Gamma(R)$ is an infinite star graph with center x , where $\text{Nil}(R) = \{0, x\}$.*

Theorem 1.1.8 ([10], **Theorem 3.14**) *Let R be a commutative ring. Then $\Gamma(R)$ is complemented, but not uniquely complemented, if and only if R is isomorphic to $D \times B$, where D is an integral domain and B is either \mathbb{Z}_4 or $\mathbb{Z}_2[X] = (X^2)$.*

1.2 Diameter of zero-divisor graphs of rings

In this section, we present results that study the diameter of $\Gamma(R)$. However, let us begin with the result in [11] showed that $\Gamma(R)$ is always connected.

Theorem 1.2.1 ([11], **Theorem 2.3**) *Let R be a commutative ring. Then $\Gamma(R)$ is connected with $\text{diam}(\Gamma(R)) \leq 3$.*

Proof. Let $x, y \in Z(R)^*$ be distinct. We show that $d(x, y) \leq 3$. If $xy = 0$, then $d(x, y) = 1$. So suppose that xy is nonzero. There are $z, w \in Z(R)^*$ such that $xz = wy = 0$. If $zw \neq 0$, then $x - zw - y$ is a path of length 2; so $d(x, y) = 2$. If $zw = 0$, then $x - z - w - y$ is a path of length at most 3 (we could have $x = z$ or $w = y$). Thus $d(x, y) \leq 3$, and hence $\Gamma(R)$ is connected and $\text{diam}(\Gamma(R)) \leq 3$. ■

We next determine when $\Gamma(R)$ has a vertex adjacent to every other vertex (i.e., when $\Gamma(R)$ has a spanning tree which is a star graph). Special cases of this are when either $\Gamma(R)$ is a complete graph or a star graph. This is the key concept in characterizing these graphs.

Theorem 1.2.2 ([11], **Theorem 2.5**) *Let R be a commutative ring. Then there is a vertex of $\Gamma(R)$ which is adjacent to every other vertex if and only if either $R \cong \mathbb{Z}_2 \times A$, where A is an integral domain, or $Z(R)$ is an annihilator ideal (and hence is prime).*

Proof. (\Rightarrow) Suppose that $Z(R)$ is not an annihilator ideal and $0 \neq a \in Z(R)$ is adjacent to every other vertex. Now $a \notin \text{Ann}(a) = I$, for otherwise $Z(R) = I$ would be an annihilator ideal. Thus I is maximal among annihilator ideals, and hence is prime [27, Theorem 6]. If $a^2 \neq a$, then $a^3 = a^2a = 0$, and hence $a \in I$, a contradiction. Thus $a^2 = a$; so $R = Ra \oplus R(1 - a)$. Hence we may assume that $R = R_1 \times R_2$ with $(1, 0)$ adjacent to every other vertex. For any $1 \neq c \in R_1$, $(c, 0)$ is a zero-divisor, so $(c, 0) = (c, 0)(1, 0) = (0, 0)$, a contradiction unless $c = 0$. Hence, $R_1 \cong \mathbb{Z}_2$. If R_2 is not an integral domain, then there is a nonzero $b \in Z(R_2)$. Then $(1, b)$ is a zero-divisor of R which is not adjacent to $(1, 0)$, a contradiction. Thus R_2 must be an integral domain. Note that if $Z(R)$ is an annihilator ideal, then it is certainly maximal among annihilator ideals, and hence is prime.

(\Leftarrow) If $R \cong \mathbb{Z}_2 \times A$, where A is an integral domain, then $(\bar{1}, 0)$ is adjacent to every other vertex. If $Z(R) = \text{Ann}(x)$ for some nonzero $x \in R$, then x is adjacent to every other vertex. \blacksquare

We next determine when $\Gamma(R)$ is a complete graph (i.e., any two vertices are adjacent). By definition, $\Gamma(R)$ is complete if and only if $xy = 0$ for all distinct $x, y \in Z(R)$. Except for the case when $R \cong \mathbb{Z}_2 \times \mathbb{Z}_2$, next theorem shows that we must also have $x^2 = 0$ for all $x \in Z(R)$ when $\Gamma(R)$ is complete. So, except for that one case, nilpotent elements are detected by complete graphs.

Theorem 1.2.3 ([11], **Theorem 2.8**) *Let R be a commutative ring. Then $\Gamma(R)$ is complete if and only if either $R \cong \mathbb{Z}_2 \times \mathbb{Z}_2$ or $xy = 0$ for all $x, y \in Z(R)$.*

Proof. (\Leftarrow) By definition.

(\Rightarrow) Suppose that $\Gamma(R)$ is complete, but there is an $x \in Z(R)$ with $x^2 \neq 0$. We show that $x^2 = x$. If not, then $x^3 = x^2x = 0$. Hence $x^2(x + x^2) = 0$ with $x^2 \neq 0$, so $x + x^2 \in Z(R)$. If $x + x^2 = x$, then $x^2 = 0$, a contradiction. Thus $x + x^2 \neq x$, so $x^2 = x^2 + x^3 = x(x + x^2) = 0$ since $\Gamma(R)$ is complete, again a contradiction. Hence $x^2 = x$. As in the proof of Theorem 1.2.2, we have $R \cong \mathbb{Z}_2 \times A$, and necessarily $A \cong \mathbb{Z}_2$. \blacksquare

Next theorem studies the diameter of $\bar{\Gamma}(R)$ when $Z(R) = \text{Nil}(R)$. First, we need the following lemma.

Lemma 1.2.4 ([7], **Lemma 2.1**) *Let R be a ring, and let $x, y \in 2\text{Nil}(R)^*$ be distinct with $xy \neq 0$. Then $(0 : (x, y)) \neq \{0\}$, and moreover, there is a path of length 2 from x to y in $\text{Nil}(R)^* \subseteq \Gamma(R)$. In particular, if $Z(R) = \text{Nil}(R)$, then $\text{diam}(\Gamma(R)) \leq 2$.*

Proof. Since $xy \neq 0$ and $x \in \text{Nil}(R)^*$, let $n(\geq 2)$ be the least positive integer such that $x^n y = 0$. Also, since $x^{n-1}y \neq 0$ and $y \in \text{Nil}(R)^*$, let $m(\geq 2)$ be the least positive integer such that $x^{n-1}y^m = 0$. Then $0 \neq x^{n-1}y^{m-1} \in \text{Nil}(R)$ and $x^{n-1}y^{m-1} \in (0 : (x, y))$. Thus $x - x^{n-1}y^{m-1} - y$ is a path of length 2 from x to y in $\text{Nil}(R)^*$. The ‘‘particular’’ statement is clear. \blacksquare

Theorem 1.2.5 ([7], **Theorem 2.2**) *Let R be a ring with $Z(R) = \text{Nil}(R) \neq \{0\}$. Then exactly one of the following three cases must occur.*

1. $|Z(R)^*| = 1$. In this case, R is isomorphic to \mathbb{Z}_4 or $\mathbb{Z}_2[X]/(X^2)$, and $\text{diam}(\Gamma(R)) = 0$.
2. $|Z(R)^*| \geq 2$ and $Z(R)^2 = \{0\}$. In this case, $\Gamma(R)$ is a complete graph, and $\text{diam}(\Gamma(R)) = 1$.

3. $Z(R)^2 \neq \{0\}$. In this case, $\text{diam}(\Gamma(R)) = 2$.

Proof.

(1) If $|Z(R)^*| = 1$, then $R \cong \mathbb{Z}_4$ or $\mathbb{Z}_2[X]/(X^2)$ [18, Proposition 2.2]. Thus $\text{diam}(\Gamma(R)) = 0$.

(2) If $Z(R)^2 = \{0\}$, then $xy = 0$ for all $x, y \in Z(R)$. Thus $\Gamma(R)$ is a complete graph with $\text{diam}(\Gamma(R)) = 1$ since $|Z(R)^*| \geq 2$.

(3) Suppose that $Z(R)^2 \neq \{0\}$. Then $\Gamma(R)$ is not complete Theorem 1.2.3, and thus $\text{diam}(\Gamma(R)) \geq 2$. Hence $\text{diam}(\Gamma(R)) = 2$ by Lemma 1.2.4. ■

Example 1.2.6 ([16], **Example 3.1**) Consider $\mathbb{Z}_9 \rtimes \mathbb{Z}_9$ under the usual module operation. Clearly $\text{diam}(\Gamma(\mathbb{Z}_9)) = 1$. Observe that if $r \in \text{reg}(\mathbb{Z}_9)$, then $rm = 0$ if and only if $m = 0$. Thus for $r \in \text{reg}(\mathbb{Z}_9)$, we have $(r, m) \notin \text{reg}(\mathbb{Z}_9 \rtimes \mathbb{Z}_9)^*$. Now, $(0, 3)(r, m) = 0$ for any $(r, m) \in \text{reg}(\mathbb{Z}_9 \rtimes \mathbb{Z}_9)^*$ and $(3, 1)(3, 3) \neq (0, 0)$; hence, $\text{diam}(\Gamma(\mathbb{Z}_9 \rtimes \mathbb{Z}_9)) = 2$.

Example 1.2.7 ([16], **Example 3.2**) Let $R = \mathbb{Z}[x]/(x^2)$. Then $\text{diam}(\Gamma(R)) = 1$ since $Z(R)^* = \{ax \mid a \in \mathbb{Z}^*\}$. Let $M = \mathbb{Z}_6$, and define $(a + bx)m = am$. Consider $(2, 1), (3, 1) \in Z(R \rtimes M)^*$. Clearly, $\text{Ann}((2, 1)) = \{(0, 3), (0, 0)\}$, while $\text{Ann}((3, 1)) = \{(0, 2), (0, 0), (0, 4)\}$. Hence, $d((2, 1), (3, 1)) = 3$, so $\text{diam}(\Gamma(R \rtimes M)) = 3$. We can also consider \mathbb{Z}_4 as an R -module under the same operation, in which case $\text{diam}(\Gamma(R \rtimes \mathbb{Z}_4)) = 2$.

We now determine necessary and sufficient conditions on R and M to ensure that $\Gamma(R \rtimes M)$ is complete. Theorem 1.2.3 shows that, if $R \neq \mathbb{Z}_2 \times \mathbb{Z}_2$, then $\Gamma(R)$ is complete if and only if $Z(R)^2 = 0$. Notice in the previous two examples $Z(R)^2 = 0$, yet $Z(R \rtimes M)^2 \neq 0$. So, simply requiring $\Gamma(R)$ to be complete will not guarantee that $\Gamma(R \rtimes M)$ will be complete.

Theorem 1.2.8 ([16], **Theorem 3.3**) Let $\Gamma(R) \neq \emptyset$. Then, $\Gamma(R \rtimes M)$ is complete if and only if $R \rtimes M$ satisfies the following properties :

1. $(Z(R))^2 = 0$;
2. For every $r \in \text{reg}(R)$, $rm \neq 0$ for all $m \in M^*$;
3. If $r \in Z(R)^*$, then $rM = 0$.

Proof. (\Rightarrow) Assume $\Gamma(R \rtimes M)$ is complete. Let $r, s \in Z(R)^*$ and $m \in M^*$. Then $(r, m), (s, 0) \in Z(R \rtimes M)^*$; hence, $(r, m)(s, 0) = (0, 0)$. So, $rs = 0$ for all $r, s \in Z(R)^*$, and $(Z(R))^2 = 0$. If for some $r \in \text{reg}(R)$ we had that $rm = 0$ for some $m \in M^*$, then $(r, 0) \in Z(R \rtimes M)^*$ since $(r, 0)(0, m) = (0, 0)$. So, let $a \in Z(R)^*$. Since $\Gamma(R \rtimes M)$ is complete, we have $(r, 0)(a, 0) = (0, 0)$, a

contradiction. If for some $a \in Z(R)^*$ there exists an $m \in M^*$ so that $am \neq 0$, then $(a, 0), (0, m) \in Z(R \times M)^*$, but $(a, 0)(0, m) = (0, am) \neq (0, 0)$, another contradiction.

(\Leftarrow) Assume that conditions (1), (2), and (3) hold. Therefore $Z(R \times M)^* = \{(0, m) | m \in M^*\} \cup \{(a, n) | a \in Z(R)^*, n \in M\}$. Let $(r, m), (s, n) \in Z(R \times M)^*$. If $r = s = 0$, then clearly $(r, m)(s, n) = (0, 0)$. If $s = 0$ and $r \in Z(R)^*$, then by (3) we have that $rM = 0$. So, $(r, m)(0, n) = (0, 0)$. If $r, s \in Z(R)^*$, then $rs = 0$ by (1) and $rM = sM = 0$ by (3), so $(r, m)(s, n) = (0, 0)$. Thus $\Gamma(R \times M)$ is complete. \blacksquare

We remark that in Example 1.2.6, the ring \mathbb{Z}_9 does not satisfy condition (3), and in Example 1.2.7, the ring $\mathbb{Z}[x]/(x^2)$ does not satisfy condition (2).

Corollary 1.2.9 ([16], Corollary 3.9) $\Gamma(R \times M)$ is complete if and only if $(Z(R \times M))^2 = 0$.

Proof. One direction is trivial while the other follows from the proof of Theorem 1.2.8. \blacksquare

Theorem 1.2.10 ([13], Theorem 3.9) Let R be a commutative ring and M a nonzero R -module. Then $\text{diam}(\Gamma(R \times M)) \leq 2$ if and only if, for all $x, y \in Z(R) \cup Z(M)$, either

1. there is a nonzero $z \in \text{Ann}(M)$ such that $xz = yz = 0$, or
2. there is $m \in M^*$ such that $xm = ym = 0$.

Proof. Suppose that $\text{diam}(\Gamma(R \times M)) \leq 2$. Let $x, y \in Z(R) \cup Z(M)$, and suppose that (2) fails. Then $xm = ym = 0$ for $m \in M$ implies $m = 0$. First suppose that $x \in Z(M)$. Then $xm = 0$ for some $m \in M^*$, and thus $ym \neq 0$. Then $(x, m), (y, m) \in Z(R \times M)^*$ are distinct and not adjacent; so there is a $(z, b) \in Z(R \times M)^*$ adjacent to both. Hence $xz = yz = 0$ implies $x(zn) = y(zn) = 0$ for all $n \in M$. by above comments, $zn = 0$; so $z \in \text{Ann}(M)$. Thus $xb = yb = 0$ implies $b = 0$; so $z \neq 0$. If $x \in Z(R) - Z(M)$, then just repeat the above argument for $(x, 0), (y, m)$ with any $0 \neq m \in M$. Thus (1) holds.

Conversely, suppose that, for all $x, y \in Z(R) \cup Z(M)$, either (1) or (2) holds. Let $(x, a), (y, b) \in Z(R \times M)^*$ be distinct and not adjacent; then $x, y \in Z(R) \cup Z(M)$. If (1) holds, then there is a nonzero $(z, 0)$ adjacent to both; and if (2) holds, then there is a nonzero $(0, m)$ adjacent to both. Thus $\text{diam}(\Gamma(R \times M)) \leq 2$. \blacksquare

1.3 Girth of zero-divisor graphs of rings

In this section, we present some classical result on the girth of $\Gamma(R)$. We begin with the fundamental result of [11].

Theorem 1.3.1 ([11], **Theorem 2.4**) *Let R be a commutative ring. If $\Gamma(R)$ contains a cycle, then $\text{gr}(\Gamma(R)) \leq 4$.*

Proof. Assume by way of contradiction that $n = \text{gr}(\Gamma(R))$ is 5, 6, or 7. Let $x_1 - x_2 - \cdots - x_n - x_1$ be a cycle of minimum length. So, $x_1x_3 \neq 0$. If $x_1x_3 \neq x_i$ for $1 \leq i \leq n$, then $x_2 - x_3 - x_4 - x_1x_3 - x_2$ is a 4-cycle, a contradiction. Thus $x_1x_3 = x_i$ for some $1 \leq i \leq n$. If $x_1x_3 = x_1$, then $x_1 - x_2 - x_3 - x_4 - x_1$ is a 4-cycle. If $x_1x_3 = x_2$, then $x_2 - x_3 - x_4 - x_2$ is a 3-cycle. If $x_1x_3 = x_n$, then $x_1 - x_2 - x_n - x_1$ is a 3-cycle. Hence $x_1x_3 \neq x_1, x_2$, or x_n . However, $x_1 - x_2 - x_1x_3 - x_n - x_1$ is then a 4-cycle, a contradiction. Therefore, there must be a shorter cycle in $\Gamma(R)$, and $\text{gr}(\Gamma(R)) \leq 4$. ■

Another characterization of girth was given in [13] using the fact that R and $T(R)$ have isomorphic zero-divisor graphs. The following theorems explicitly characterize when the girth of a zero-divisor graph is 4 or ∞ , and thus implicitly when the girth is 3.

Theorem 1.3.2 ([10], **Theorem 2.2**) *Let R be a commutative ring with total quotient ring $T(R)$. Then the graphs $\Gamma(T(R))$ and $\Gamma(R)$ are isomorphic.*

Theorem 1.3.3 ([13], **Theorem 2.2**) *The following statements are equivalent for a reduced commutative ring R .*

1. $\text{gr}(\Gamma(R)) = 4$.
2. $T(R) = K_1 \times K_2$, where each K_i is a field with $|K_i| \geq 3$.
3. $\Gamma(R) = K^{m,n}$ with $m, n \geq 2$.

Proof. (1) \Rightarrow (2) Suppose that $\text{gr}(\Gamma(R)) = 4$. Then $\Gamma(R)$ is complemented. Thus $T = T(R)$ is von Neumann regular by Theorem 1.1.4 and not a field. Hence T has a nontrivial idempotent, and thus $T = T_1 \times T_2$. Suppose that there are $0 \neq x, y \in T_1$ with $xy = 0$ (note that $x \neq y$ since R , and hence T , is reduced). Then $(x, 0) - (y, 0) - (0, 1) - (x, 0)$ is a triangle in $\Gamma(T)$, a contradiction since $\text{gr}(\Gamma(T)) = \text{gr}(\Gamma(R)) = 4$ by Theorem 1.3.2. Thus T_1 is an integral domain, in fact, a field. Similarly, T_2 must also be a field. Hence $T = K_1 \times K_2$ for fields K_1 and K_2 has only 2 elements, then $\Gamma(T)$ is a star graph.

In this case, $\text{gr}(\Gamma(T)) = \infty$, a contradiction since $\text{gr}(\Gamma(T)) = \text{gr}(\Gamma(R)) = 4$ by Theorem 1.3.2.

(2) \Rightarrow (3) This follows since the graphs $\Gamma(T)$ and $\Gamma(R)$ are isomorphic by Theorem 1.3.2 and $\Gamma(K_1 \times K_2) = K^{m,n}$, where $m = |K_1| - 1$ and $n = |K_2| - 1$.

(3) \Rightarrow (1) This is clear. ■

Theorem 1.3.4 ([13], Theorem 2.3) *The following statements are equivalent for a commutative ring R with $\text{Nil}(R)$ nonzero.*

1. $\text{gr}(\Gamma(R)) = 4$.
2. $R \cong D \times B$, where D is an integral domain with $|D| \geq 3$ and $B = \mathbb{Z}_4$ or $\mathbb{Z}_2[X]/(X^2)$. (Thus $T(R) \cong T(D) \times B$.)
3. $\Gamma(R) = \overline{K}^{m,3}$ with $m \geq 2$.

Proof. (1) \Rightarrow (2) Suppose that $\text{gr}(\Gamma(R)) = 4$. Then $\Gamma(R)$ is complemented. If $\Gamma(R)$ is uniquely complemented, then $\Gamma(R)$ is a star graph Theorem 1.1.7, and hence $\text{gr}(\Gamma(R)) = \infty$, a contradiction. Thus $R \cong D \times B$, where D is an integral domain and $B = \mathbb{Z}_4$ or $\mathbb{Z}_2[X]/(X^2)$ by Theorem 1.1.8. Hence $\Gamma(R) = \overline{K}^{m,3}$, where $m = |D| - 1$. We must have $|D| \geq 3$ since otherwise $\text{gr}(\Gamma(R)) = \infty$, a contradiction.

The implications (2) \Rightarrow (3) and (3) \Rightarrow (1) are both clear. ■

Theorem 1.3.5 ([13], Theorem 2.4) *The following statements are equivalent for a reduced commutative ring R .*

1. $\Gamma(R)$ is nonempty with $\text{gr}(\Gamma(R)) = \infty$.
2. $T(R) = \mathbb{Z}_2 \times K$, where K is a field.
3. $\Gamma(R) = K^{1,n}$ for some $n \geq 1$.

Proof. (1) \Rightarrow (2) Suppose that $\text{gr}(\Gamma(R)) = \infty$. Then $\Gamma(R) \neq \emptyset$. Then $|\Gamma(R)| \geq 2$ since R is reduced, and thus $\Gamma(R)$ is complemented. As in the proof of (1) \Rightarrow (2) of Theorem 1.3.3, we have $T(R) = K_1 \times K_2$ for fields K_1 and K_2 . If each field has at least three elements, then $\text{gr}(\Gamma(R)) = 4$ by Theorem 1.3.3, a contradiction. Hence we may assume that K_1 has 2 elements; so $K_1 = \mathbb{Z}_2$.

(2) \Rightarrow (3) This follows since the graphs $\Gamma(T)$ and $\Gamma(R)$ are isomorphic by Theorem 1.3.2 and $\Gamma(\mathbb{Z}_2 \times K) = K^{1,n}$, where $n = |K| - 1$.

(3) \Rightarrow (1) This is clear. ■

Theorem 1.3.6 ([13], **Theorem 2.5**) *The following statements are equivalent for a commutative ring R with $\text{Nil}(R)$ nonzero.*

1. $\text{gr}(\Gamma(R)) = \infty$.
2. $R \cong B$ or $R \cong \mathbb{Z}_2 \times B$, where $B = \mathbb{Z}_4$ or $\mathbb{Z}_2[X]/(X^2)$, or $\Gamma(R)$ is a star graph.
3. $\Gamma(R)$ is a singleton, a $\overline{K}^{1,3}$, or a $K^{1,n}$ for some $n \geq 1$.

Proof. (1) \Rightarrow (2) Suppose that $\text{gr}(\Gamma(R)) = \infty$. If $\Gamma(R)$ is a point, then $R = \mathbb{Z}_4$ or $\mathbb{Z}_2[X]/(X^2)$. So assume that $\Gamma(R)$ has at least 2 elements. Then $\Gamma(R)$ is complemented. If $\Gamma(R)$ is uniquely complemented, then $\Gamma(R)$ is a star graph by Theorem 1.1.7. If $\Gamma(R)$ is not uniquely complemented, then $R \cong D \times B$, where D is an integral domain and $B = \mathbb{Z}_4$ or $\mathbb{Z}_2[X]/(X^2)$ by Theorem 1.1.8. If $|D| \geq 3$, then $\text{gr}(\Gamma(R)) = 4$ as in Theorem 1.3.4, a contradiction. Thus $|D| = 2$; so $D = \mathbb{Z}_2$.

The implications (2) \Rightarrow (3) and (3) \Rightarrow (1) are both clear. ■

Theorem 1.3.7 ([7], **Theorem 2.11**) *Let R be a ring with $Z(R) = \text{Nil}(R) \neq \{0\}$. Then exactly one of the following four cases must occur.*

1. $|Z(R)^*| = 1$. In this case, R is isomorphic to \mathbb{Z}_4 or $\mathbb{Z}_2[X]/(X^2)$, and $\text{gr}(\Gamma(R)) = \infty$.
2. $|Z(R)^*| = 2$. In this case, R is isomorphic to \mathbb{Z}_9 or $\mathbb{Z}_3[X]/(X^2)$, and $\text{gr}(\Gamma(R)) = \infty$.
3. $|Z(R)^*| = 3$. If R is isomorphic to \mathbb{Z}_8 , $\mathbb{Z}_2[X]/(X^3)$, or $\mathbb{Z}_4[X]/(2X, X^2 - 2)$, then $\text{gr}(\Gamma(R)) = \infty$. Otherwise, R is isomorphic to $\mathbb{Z}_2[X, Y]/(X, Y)^2$, $\mathbb{Z}_4[X]/(2, X)^2$, $\mathbb{Z}_4[X]/(X^2 + X + 1)$, or $F_4[X]/(X^2)$; and in this case, $\text{gr}(\Gamma(R)) = 3$.
4. $|Z(R)^*| \geq 4$. In this case, $\text{gr}(\Gamma(R)) = 3$.

Proof. By Theorem 1.3.4, $\text{gr}(\Gamma(R)) \neq 4$ when $Z(R) = \text{Nil}(R)$. Thus $\text{gr}(\Gamma(R)) = 3$ or ∞ . The theorem then follows from Theorem 1.3.6, [13, Remark 2.6(a)], and Example 1.1.2. ■

We end this section with some results on the zero-divisor graph of idealizations. In [16], Axtell and Stickles noticed that if $|M| \geq 4$, then $\text{gr}(\Gamma(R \times M)) = 3$. This because $(0, m_1) - (0, m_2) - (0, m_3) - (0, m_1)$ is a cycle of length three, where m_1, m_2 and m_3 are distinct nonzero elements of M . Thus, we only need to consider when the module M has only two or three elements.

Theorem 1.3.8 ([16], **Theorem 2.1**) *Let R be a commutative ring with identity and $M \cong \mathbb{Z}_3$ an R -module. Then*

1. $\text{gr}(\Gamma(R \times \mathbb{Z}_3)) = 3$ if and only if $\text{Ann}(\mathbb{Z}_3) \neq \{0\}$.
2. $\text{gr}(\Gamma(R \times \mathbb{Z}_3)) = \infty$ if and only if $\text{Ann}(\mathbb{Z}_3) = \{0\}$. This occurs precisely when $R \cong \mathbb{Z}_3$.

Proof. (1) Assume there exists a nonzero element $r \in R$ such that $r\mathbb{Z}_3 = 0$. Since $(r, 0) - (0, 1) - (0, 2) - (r, 0)$ is a cycle of length 3, the result is obvious. The other direction is proven by using the converse of the implication proven below.

(2) Assume that $r\mathbb{Z}_3 \neq 0$ for every nonzero element $r \in R$. Then $r\bar{1} \neq 0$ for all $r \in R^*$. Thus $\text{Ann}((0, 1)) = \text{Ann}((0, 2)) = \{(0, 0), (0, 1), (0, 2)\}$. Since $\Gamma(R \times \mathbb{Z}_3)$ is connected, we see that R has no nonzero zero-divisors; hence, R is an integral domain. In light of the remark preceding the theorem, $R \cong \mathbb{Z}_3$. Since $Z(R \times \mathbb{Z}_3)^* = \{(0, 1), (0, 2)\}$, we have $\text{gr}(\Gamma(R \times \mathbb{Z}_3)) = \infty$. The other direction is proven by using the contrapositive of the implication proven in (1). Note that $\mathbb{Z}_3 \times \mathbb{Z}_3 \cong \mathbb{Z}_3[x]/(x^2)$. ■

The above result classifies the girth of $\Gamma(R \times \mathbb{Z}_3)$. We now consider the situation when $M \cong \mathbb{Z}_2$. We will classify when the girth of $\Gamma(R \times \mathbb{Z}_2)$ is 3 and when it is infinite. We begin with the “girth 3” case.

Theorem 1.3.9 ([16], **Theorem 2.2**) *The girth of $\Gamma(R \times \mathbb{Z}_2)$ is three if and only if one of the following assertions hold:*

1. The girth of $\Gamma(R)$ is three.
2. There exists an $r \in R^*$ such that $r^2 = 0$.
3. There exist distinct $a, b \in Z(R)^*$ such that $ab = 0 = a\mathbb{Z}_2 = b\mathbb{Z}_2$.

Proof. (\Leftarrow) If (1) holds, the result is clear. If (2) holds, note that $r\bar{1} = 0$, lest $r \cdot (r\bar{1}) \neq r^2\bar{1}$. Then, $(r, \bar{0}) - (r, \bar{1}) - (0, \bar{1}) - (r, \bar{0})$ is a cycle of length three. If (3) holds, then $(a, \bar{0}) - (b, \bar{0}) - (0, \bar{1}) - (a, \bar{0})$ is a cycle of length 3.

(\Rightarrow) Case 1: The element $(0, \bar{1})$ is part of a minimal length cycle. Then the cycle has the form $(0, \bar{1}) - (a, \bar{i}) - (b, \bar{j}) - (0, \bar{1})$. If $a \neq b$, we have distinct $a, b \in Z(R)^*$, $ab = 0$, and $a\mathbb{Z}_2 = b\mathbb{Z}_2 = 0$; if $a = b$, we have $a \in R^*$ such that $a^2 = 0$.

Case 2: The element $(0, \bar{1})$ is not part of a minimal length cycle. Then, the cycle has the form $(a, \bar{i}) - (b, \bar{j}) - (c, \bar{k}) - (a, \bar{i})$. If a, b , and c are all distinct, then $a - b - c - a$ is a cycle in $\Gamma(R)$, and $\text{gr}(\Gamma(R)) = 3$. If not, then either $a^2 = 0$ or $b^2 = 0$. ■

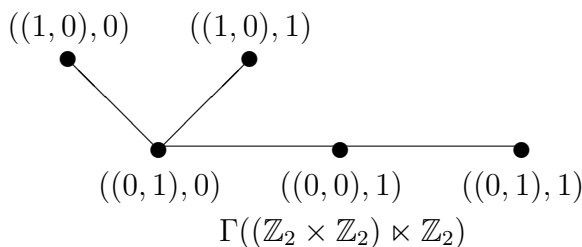
CHAPTER 1. ZERO-DIVISOR GRAPH OF A COMMUTATIVE RING
(PRELIMINARIES)

Now we give the necessary and sufficient conditions for the girth of $R \rtimes \mathbb{Z}_2$ to be infinite. We need some results to use later.

Lemma 1.3.10 ([16], **Lemma 2.3**) *Let $R \cong \mathbb{Z}_2 \times F$, where F is a field. Then, any module operation from R to \mathbb{Z}_2 is a canonical extension of a module operation either from \mathbb{Z}_2 to \mathbb{Z}_2 , or from F to \mathbb{Z}_2 in the case where F is \mathbb{Z}_2 .*

Proof. The annihilator of \mathbb{Z}_2 as an R -module is an ideal of R ; thus $\text{Ann}(\mathbb{Z}_2) = I_1 \times I_2 = I$, where I_1 is an ideal of \mathbb{Z}_2 , and I_2 is an ideal of F . If $I_1 \times I_2 = \{0\}$ then $(1, 0) \cdot 1 = 1 = (0, 1) \cdot 1$, but this would then result in $(1, 1) \cdot 1 = ((1, 0) + (0, 1)) \cdot 1 = (1, 0) \cdot 1 + (0, 1) \cdot 1 = 0$, a contradiction. More easily, $I_1 \times I_2 \neq R$ since the module is unitary. Thus, $I = \{0\} \times F$ or $I = \mathbb{Z}_2 \times \{0\}$. If $I = \{0\} \times F$, then the operation is a canonical extension of the module operation from \mathbb{Z}_2 to \mathbb{Z}_2 . Similarly, if $I = \mathbb{Z}_2 \times \{0\}$, then the operation is a canonical extension of the module operation from F to \mathbb{Z}_2 . However, if $|F| \geq 3$, then there is no module operation from F to \mathbb{Z}_2 since there are nonzero sums of units (which in turn are units), but in the module $u \cdot 1 = 1$. ■

Example 1.3.11 ([16], **Example 2.4**) *Using Lemma 1.3.10, let $R \cong \mathbb{Z}_2 \times \mathbb{Z}_2$ and consider $R \rtimes \mathbb{Z}_2$. Without loss of generality, the module operation is defined by $(0, 0) \cdot 1 = (0, 1) \cdot 1 = 0$ and $(1, 0) \cdot 1 = (1, 1) \cdot 1 = 1$. Note that $R \rtimes \mathbb{Z}_2 \cong \mathbb{Z}_2 \times \mathbb{Z}_2[x]/(x^2)$. Then, $\text{gr}(\Gamma(R \rtimes \mathbb{Z}_2)) = \infty$, as the zero-divisor graph below shows:*



Proposition 1.3.12 ([16], **Proposition 2.5**) *Let $R \cong \mathbb{Z}_2 \times F$, where F is a field and $|F| \geq 3$. Then, $\text{gr}(\Gamma(R \rtimes \mathbb{Z}_2)) = 4$.*

Proof. Since F is a field and $|F| \geq 3$, by Lemma 1.3.10 the module operation from R to \mathbb{Z}_2 is an extension of the module operation from \mathbb{Z}_2 to \mathbb{Z}_2 . We have $((0, 0), 1) - ((0, 1), 0) - ((1, 0), 1) - ((0, a), 0) - ((0, 0), 1)$ is a cycle of length 4 (where $a \in F$ is nonzero and not equal to 1). By Theorem 1.3.9, $\Gamma(R \rtimes \mathbb{Z}_2)$ cannot contain any cycles of length 3, since $\Gamma(R)$ is a star graph centered at $(1, 0)$. Hence $\text{gr}(\Gamma(R \rtimes \mathbb{Z}_2)) = 4$. ■

Lemma 1.3.13 ([16], Lemma 2.6) *If $\text{diam}(\Gamma(R)) = 3$, then the girth of $\Gamma(R \times \mathbb{Z}_2)$ is finite.*

Proof. Let $a - b - c - d$ be a path in $\Gamma(R)$ with a, b, c and d distinct. If $b\mathbb{Z}_2 \neq 0$ and $c\mathbb{Z}_2 \neq 0$, then $b.1 = 1$ and $c.1 = 1$, but $(bc).1 = 0$, a contradiction. Thus, we must have either $b\mathbb{Z}_2 = 0$ or $c\mathbb{Z}_2 = 0$, or both. Assume $b\mathbb{Z}_2 = 0$. If $\mathbb{Z}_2 = 0$, then we get the cycle $(b, 0) - (c, 0) - (b, 1) - (c, 1) - (b, 0)$. If $c\mathbb{Z}_2 \neq 0$, then $d\mathbb{Z}_2 = 0$; hence $(b, 0) - (c, 0) - (d, 0) - (c, 1) - (b, 0)$ is a cycle. ■

Given the idealization $R \times \mathbb{Z}_2$, it is easy to see that $|R/\text{Ann}(\mathbb{Z}_2)| = 2$. Otherwise, let $r_1 + \text{Ann}(\mathbb{Z}_2)$ and $r_2 + \text{Ann}(\mathbb{Z}_2)$ be two cosets distinct from $0 + \text{Ann}(\mathbb{Z}_2)$. Thus $r_1, r_2 \notin \text{Ann}(\mathbb{Z}_2)$ and so $r_1.\bar{1} = r_2.\bar{2} = 1$. Therefore $(r_1.r_2) \in \text{Ann}(\mathbb{Z}_2)$ and so $r_1 + \text{Ann}(\mathbb{Z}_2) = r_2 + \text{Ann}(\mathbb{Z}_2)$. This result will be useful in the proof of the following result.

Theorem 1.3.14 ([16], Theorem 2.7) *The girth of $\Gamma(R \times \mathbb{Z}_2)$ is infinite if and only if $R \cong \mathbb{Z}_2 \times \mathbb{Z}_2$ or R is an integral domain.*

Proof. (\Leftarrow) If $R \cong \mathbb{Z}_2 \times \mathbb{Z}_2$, Example 1.3.11 shows that $\Gamma(R \times \mathbb{Z}_2)$ has no cycles. If R is an integral domain, then $\Gamma(R \times \mathbb{Z}_2)$ is a star graph with center $(0, \bar{1})$.

(\Rightarrow) Lemma 1.3.13 shows $\text{diam}(\Gamma(R)) \leq 2$ or $Z(R)^* = \emptyset$. If $Z(R)^* = \emptyset$, we are done. If $\text{diam}(\Gamma(R)) = 0$, then by [12, Theorem 3.2], we have $R \cong \mathbb{Z}_4$ or $R \cong \mathbb{Z}_2[x]/(x^2)$. In either case there exists a nonzero nilpotent element, and by Theorem 1.3.9, $\text{gr}(\Gamma(R \times \mathbb{Z}_2)) = 3$. If $\text{diam}(\Gamma(R)) = 1$, then $\Gamma(R)$ is complete. Thus, if $R \not\cong \mathbb{Z}_2 \times \mathbb{Z}_2$, then R contains a nilpotent element by Theorem 1.2.3, and by Theorem 1.3.9, $\text{gr}(\Gamma(R \times \mathbb{Z}_2)) = 3$, a contradiction. If $\text{diam}(\Gamma(R)) = 2$ and $\Gamma(R)$ is not a star graph, then $\text{gr}(\Gamma(R)) < \infty$, a contradiction. Thus, by Theorem 1.2.2, the only possibilities for R are $\mathbb{Z}_2 \times D$, where D is an integral domain, or $Z(R)$ is an annihilator ideal. If $Z(R)$ is an annihilator ideal, then R contains a nilpotent element, and we appeal to Theorem 1.3.9. Hence, $R \cong \mathbb{Z}_2 \times D$. If $|D| = 2$, we are done. If D is a finite integral domain, then D is a field and by Proposition 1.3.12, $\text{gr}(\Gamma(R \times \mathbb{Z}_2)) = 4$, a contradiction. The remaining case to investigate is when D is an infinite integral domain. In R , $\text{Ann}(\mathbb{Z}_2)$ is an ideal and hence of one of the following three forms: $\mathbb{Z}_2 \times \{0\}$, $\{0\} \times I$, or $\mathbb{Z}_2 \times I$, where I is nonzero ideal of D . If $\text{Ann}(\mathbb{Z}_2) = \mathbb{Z}_2 \times \{0\}$, then $|R/\text{Ann}(\mathbb{Z}_2)| > 2$ which contradicts the remarks preceding this result. Again using the coset argument, if $\text{Ann}(\mathbb{Z}_2) = \{0\} \times I$ or $\text{Ann}(\mathbb{Z}_2) = \mathbb{Z}_2 \times I$, then there exist distinct, nonzero $a, b \in I$ such that $(0, \bar{a}), (0, \bar{b}) \in \text{Ann}(\mathbb{Z}_2)$. Thus we form a cycle $((1, 0), \bar{0}) - ((0, a), \bar{0}) - ((1, 0), \bar{1}) - ((0, b), \bar{0}) - ((1, 0), \bar{0})$. This contradicts $\text{gr}(\Gamma(R \times \mathbb{Z}_2)) = \infty$. ■

1.4 Annihilator graph of a commutative ring

In this section, we give some results on the annihilator graphs that were introduced by Badawi in [17]. Recall that the annihilator graph of R is the graph denoted by $AG(R)$ such that its vertex set consists of all its nonzero zero-divisors and that two vertices x and y are adjacent if and only if $\text{Ann}_R(xy) \neq \text{Ann}_R(x) \cup \text{Ann}_R(y)$. It is clear that the graph $AG(R)$ is an extension of the zero-divisor graph $\Gamma(R)$.

We begin with a lemma containing several useful properties of $AG(R)$.

Lemma 1.4.1 ([17], Lemma 2.1) *Let R be a commutative ring.*

1. *Let x, y be distinct elements of $Z(R)^*$. Then $x - y$ is not an edge of $AG(R)$ if and only if $\text{Ann}_R(xy) = \text{Ann}_R(x)$ or $\text{Ann}_R(xy) = \text{Ann}_R(y)$.*
2. *If $x - y$ is an edge of $\Gamma(R)$ for some distinct $x, y \in Z(R)^*$, then $x - y$ is an edge of $AG(R)$. In particular, if P is a path in $\Gamma(R)$, then P is a path in $AG(R)$.*
3. *If $x - y$ is not an edge of $AG(R)$ for some distinct $x, y \in Z(R)^*$, then $\text{Ann}_R(x) \subseteq \text{Ann}_R(y)$ or $\text{Ann}_R(y) \subseteq \text{Ann}_R(x)$.*
4. *If $\text{Ann}_R(x) \not\subseteq \text{Ann}_R(y)$ and $\text{Ann}_R(y) \not\subseteq \text{Ann}_R(x)$ for some distinct $x, y \in Z(R)^*$, then $x - y$ is an edge of $AG(R)$.*
5. *If $d_{\Gamma(R)}(x, y) = 3$ for some distinct $x, y \in Z(R)^*$, then $x - y$ is an edge of $AG(R)$.*
6. *If $x - y$ is not an edge of $AG(R)$ for some distinct $x, y \in Z(R)^*$, then there is a $w \in Z(R)^* - \{x, y\}$ such that $x - w - y$ is a path in $\Gamma(R)$, and hence $x - w - y$ is also a path in $AG(R)$.*

In view of (6) in the preceding lemma, we have the following result.

Theorem 1.4.2 ([17], Theorem 2.2) *Let R be a commutative ring with $|Z(R)^*| \geq 2$. Then $AG(R)$ is connected and $\text{diam}(AG(R)) \leq 2$.*

The following is an example of a nonreduced commutative ring R such that $AG(R) \neq \Gamma(R)$ and if $x - y$ is an edge of $AG(R)$ that is not an edge of $\Gamma(R)$ for some distinct $x, y \in Z(R)^*$, then there is no path in $AG(R)$ of length two from x to y .

- Example 1.4.3** ([17], **Example 2.8**)
1. Let $R = \mathbb{Z}_2 \times \mathbb{Z}_4$ and let $a = (0, 1)$, $b = (1, 2)$ and $c = (0, 3)$. Then $a - b$ and $c - b$ are the only two edges of $AG(R)$ that are not edges of $\Gamma(R)$, but there is no path in $AG(R)$ of length two from a to b and there is no path in $AG(R)$ of length two from c to b . Notice that $AG(R) = K^{2,3}$, $\Gamma(R) = \overline{K}^{1,3}$, $\text{gr}(AG(R)) = 4$, $\text{gr}(\Gamma(R)) = \infty$, $\text{diam}(AG(R)) = 2$, and $\text{diam}(\Gamma(R)) = 3$.
 2. Let $R = \mathbb{Z}_2 \times \mathbb{Z}_2[X]/(X^2)$. Let $x = X + (X^2) \in \mathbb{Z}_2[X]/(X^2)$, $a = (0, 1)$, $b = (1, x)$ and $c = (0, 1 + x)$. Then $a - b$ and $c - b$ are the only two edges of $AG(R)$ that are not edges of $\Gamma(R)$, but there is no path in $AG(R)$ of length two from a to b and there is no path in $AG(R)$ of length two from c to b . Again, notice that $AG(R) = K^{2,3}$, $\Gamma(R) = \overline{K}^{1,3}$, $\text{gr}(AG(R)) = 4$, $\text{gr}(\Gamma(R)) = \infty$, $\text{diam}(AG(R)) = 2$, and $\text{diam}(\Gamma(R)) = 3$.

Theorem 1.4.4 ([17], **Theorem 2.9**) *Let R be a commutative ring and suppose that $AG(R) \neq \Gamma(R)$. Then the following statements are equivalent:*

1. $\text{gr}(AG(R)) = 4$;
2. $\text{gr}(AG(R)) \neq 3$;
3. *If $x - y$ is an edge of $AG(R)$ that is not an edge of $\Gamma(R)$ for some distinct $x, y \in Z(R)^*$, then there is no path in $AG(R)$ of length two from x to y .*
4. *There are some distinct $x, y \in Z(R)^*$ such that $x - y$ is an edge of $AG(R)$ that is not an edge of $\Gamma(R)$ and there is no path in $AG(R)$ of length two from x to y ;*
5. *R is ring-isomorphic to either $\mathbb{Z}_2 \times \mathbb{Z}_4$ or $\mathbb{Z}_2 \times \mathbb{Z}_2[X]/(X^2)$.*

Corollary 1.4.5 ([17], **Corollary 2.11**) *Let R be a commutative ring such that $AG(R) \neq \Gamma(R)$. Then $\text{gr}(AG(R)) \in \{3, 4\}$.*

The following result concerns the induced graph of $AG(R)$ with vertices $\text{Nil}(R)^*$.

Theorem 1.4.6 ([17], **Theorem 3.10**) *Let R be a nonreduced commutative ring with $|\text{Nil}(R)^*| \geq 2$ and let $AG_N(R)$ be the (induced) subgraph of $AG(R)$ with vertices $\text{Nil}(R)^*$. Then $AG_N(R)$ is complete.*

Proof. Suppose there are nonzero distinct elements $a, b \in \text{Nil}(R)$ such that $ab \neq 0$. Assume that $\text{Ann}_R(ab) = \text{Ann}_R(a) \cup \text{Ann}_R(b)$. Hence $\text{Ann}_R(ab) = \text{Ann}_R(a)$ or $\text{Ann}_R(ab) = \text{Ann}_R(b)$. Without loss of generality, we may assume that $\text{Ann}_R(ab) = \text{Ann}_R(a)$. Let n be the least positive integer such that $b^n = 0$.

Suppose that $ab^k \neq 0$ for each k , $1 \leq k < n$. Then $b^{n-1} \in \text{Ann}_R(ab) - \text{Ann}_R(a)$, a contradiction. Hence assume that k , $1 \leq k < n$ is the least positive integer such that $ab^k = 0$. Since $ab \neq 0$, $1 < k < n$. Hence $b^{k-1} \in \text{Ann}_R(ab) - \text{Ann}_R(a)$, a contradiction. Thus $a - b$ is an edge of $AG_N(R)$. ■

We end this section with the following result which give the form of the annihilator graph when $\Gamma(R)$ is a star graph.

Theorem 1.4.7 ([17], Theorem 3.17) *Let R be a commutative ring such that $AG(R) \neq \Gamma(R)$. Then the following statements are equivalent:*

1. $\Gamma(R)$ is a star graph;
2. $\Gamma(R) = K^{1,2}$;
3. $AG(R) = K^3$.

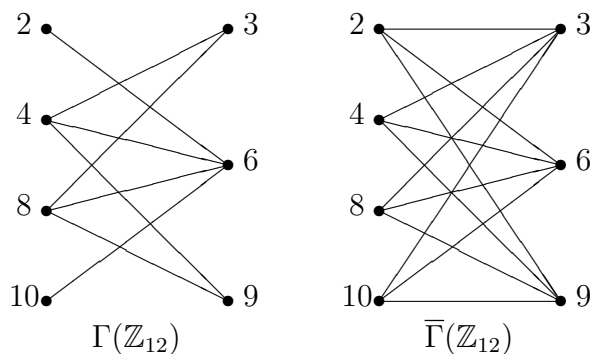
On the extended zero-divisor graph of commutative rings

In this chapter we present a new approach of studying the relation between zero-divisors. In our case two zero-divisors x and y of a commutative ring R are adjacent whenever there exist two non negative integers n and m such that $x^n y^m = 0$ with $x^n \neq 0$ and $y^m \neq 0$. This yield an extension of the classical zero-divisor graph $\Gamma(R)$ of R which will be denoted by $\bar{\Gamma}(R)$.

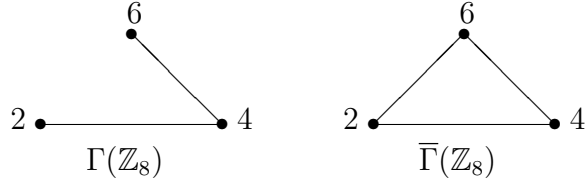
At first, we distinguish when $\bar{\Gamma}(R)$ and $\Gamma(R)$ coincide. We also show that when $\bar{\Gamma}(R)$ and $\Gamma(R)$ are different, then $\bar{\Gamma}(R)$ contains necessarily a cycle and if, moreover, $\bar{\Gamma}(R)$ is complemented then the total quotient ring of R is 0-dimensional. Among other things, diameter and girth of $\bar{\Gamma}(R)$ are also studied.

2.1 When $\bar{\Gamma}(R)$ and $\Gamma(R)$ coincide?

Let us at first give some simple examples which show some differences between the two graphs.



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We begin with the main result in this section which studies when $\bar{\Gamma}(R)$ and $\Gamma(R)$ coincide.

Theorem 2.1.1 *Let R be a ring. The following statements are equivalent:*

1. $\bar{\Gamma}(R) = \Gamma(R)$.
2. R satisfies the two following conditions:
 - (i) If $\text{Nil}(R) \neq \{0\}$, then every nonzero nilpotent element has index 2, and
 - (ii) For every $x \in Z(R) \setminus \text{Nil}(R)$, $\text{Ann}(x^2) = \text{Ann}(x)$.
3. R satisfies the two following conditions:
 - (i) If $\text{Nil}(R) \neq \{0\}$, then every nonzero nilpotent element has index 2, and
 - (ii) For every $x \in Z(R)$, $\sqrt{\text{Ann}(x)} \setminus \text{Nil}(R) \subset \text{Ann}(x)$.

To prove this theorem, we need the following lemma.

Recall that, for a nonzero nilpotent element x of R , we use n_x to denote the index of nilpotency of x .

Lemma 2.1.2 *Let R be a ring and let $x \in R \setminus \{0\}$. Then:*

1. If x is nilpotent, then $\text{Ann}(x) \subsetneq \text{Ann}(x^n)$ for every integer $n \geq 2$.
2. If x is not nilpotent, then we have the equivalence:
 $\text{Ann}(x^2) = \text{Ann}(x)$ if and only if $\text{Ann}(x^n) = \text{Ann}(x)$ for every integer $n \geq 2$.

Proof. 1. Let x be a nonzero nilpotent element of R . If $n_x = 2$, then for every integer $n \geq 2$, $\text{Ann}(x^n) = \text{Ann}(0) = R \supsetneq \text{Ann}(x)$. Now consider $n_x \geq 3$ and suppose by contradiction that there is $n \geq 2$ such that $\text{Ann}(x^n) = \text{Ann}(x)$. Since, for $n \geq n_x$, we have $\text{Ann}(x^n) = \text{Ann}(0) = R$, n must be between 2 and $n_x - 1$. Then, we have $x^{n_x - n} \in \text{Ann}(x^n) = \text{Ann}(x)$. Thus, $x^{n_x - n}x = x^{n_x - n + 1} = 0$ which is absurd since $2 \leq n_x - n + 1 \leq n_x - 1$.

2. Let x be a non-nilpotent element such that $\text{Ann}(x^2) = \text{Ann}(x)$. Let $y \in \text{Ann}(x^n)$ for some integer $n \geq 2$, then $yx^n = 0$ which implies that $yx \in \text{Ann}(x^{n-1})$. By induction $\text{Ann}(x^{n-1}) = \text{Ann}(x)$, hence $y \in \text{Ann}(x^2) = \text{Ann}(x)$, as desired. ■

Proof of Theorem 2.1.1.

(1) \Rightarrow (2). Suppose that there exists a nilpotent element x such that $n_x \geq 3$. By Lemma 2.1.2, $\text{Ann}(x) \subsetneq \text{Ann}(x^n)$ for every integer $n \geq 2$. We may assume that $2 \leq n < n_x$. Consider an element $y \in \text{Ann}(x^n) \setminus \text{Ann}(x)$, then $x^n y = 0$ and $xy \neq 0$ which contradicts the fact that $\bar{\Gamma}(R) = \Gamma(R)$. Now let $x \in Z(R) \setminus \text{Nil}(R)$. Since $\text{Ann}(x) \subset \text{Ann}(x^2)$ it remains to show the other inclusion. Let $y \neq x$ be an element of $\text{Ann}(x^2)$. Then, x and y are adjacent in $\bar{\Gamma}(R)$ which equal to $\Gamma(R)$. Hence $xy = 0$, and therefore $y \in \text{Ann}(x)$.

(2) \Rightarrow (3). Let $y \in \sqrt{\text{Ann}(x)} \setminus \text{Nil}(R)$. Then there exists $n \in \mathbb{N}^*$ such that $y^n x = 0$. Then $x \in \text{Ann}(y^n) = \text{Ann}(y)$ by lemma 2.1.2 and therefore $xy = 0$.

(3) \Rightarrow (1). Let x and y be two adjacent vertices in $\bar{\Gamma}(R)$. Then there exist two positive integers n and m such that $x^n y^m = 0$ with $x^n \neq 0$ and $y^m \neq 0$. Three cases occur:

Case 1: If $x, y \in \text{Nil}(R)$ such that $n_x = n_y = 2$, then $n = m = 1$ (by 2(i)). This means that x and y are adjacent vertices in $\Gamma(R)$.

Case 2: If $x \notin \text{Nil}(R)$ and $y \in \text{Nil}(R)$, then $m = 1$ (by 2(i)). Hence $x \in \sqrt{\text{Ann}(y)} \setminus \text{Nil}(R)$ and by hypothesis $xy = 0$. Thus x and y are adjacent vertices in $\Gamma(R)$.

Case 3: If $x \notin \text{Nil}(R)$ and $y \notin \text{Nil}(R)$, then $x \in \sqrt{\text{Ann}(y^m)} \setminus \text{Nil}(R) \subset \text{Ann}(y^m)$ and so $xy^m = 0$. Thus $y \in \sqrt{\text{Ann}(x)} \setminus \text{Nil}(R) \subset \text{Ann}(x)$, as desired.

■

One can consider, for example, \mathbb{Z}_{24} , $\mathbb{Z}_2 \times \mathbb{Z}_8$ and $\mathbb{Z}_2[X, Y]/(XY^2, X^3)$ to get an example of a ring R that contains a nilpotent element with index at least three and such that there is an element $x \in Z(R) \setminus \text{Nil}(R)$ with $\text{Ann}(x^2) \neq \text{Ann}(x)$.

To show that the conditions (i) and (ii) of both (2) and (3) of Theorem 2.1.1 are independent, we give the following examples. First note that one can show easily that 2(ii) is equivalent to 3(ii).

Example 2.1.3 1. To give an example of a ring R that contains a nilpotent element with index two and contains an element $x \in Z(R) \setminus \text{Nil}(R)$ such that $\text{Ann}(x^2) \neq \text{Ann}(x)$, we can consider the following rings: \mathbb{Z}_{12} , \mathbb{Z}_{18} , \mathbb{Z}_{36} , $\mathbb{Z}_2[X, Y]/(XY^2, X^2)$ and $\mathbb{Z}_2 \times \mathbb{Z}_4$.

2. The following rings can be used as an example of a ring R that contains a nilpotent element with index at least three and such that $\text{Nil}(R) = Z(R)$: \mathbb{Z}_{2^m} (with $m \geq 3$) and $\mathbb{Z}_2[X]/(X^3)$.

However, it seems not easy to get an example of a ring R that contains a nilpotent element with index at least three, and such that $Z(R) \neq \text{Nil}(R)$ with

$\text{Ann}(x^2) = \text{Ann}(x)$ for every element $x \in Z(R) \setminus \text{Nil}(R)$. Then, in order to construct such example, one should establish at first some of its properties. For that, we set the following result.

Proposition 2.1.4 *Let R be a ring which satisfies the following properties: $\text{Nil}(R) \neq \{0\}$, $\text{Nil}(R) \subsetneq Z(R)$ and $\text{Ann}(x^2) = \text{Ann}(x)$ for every $x \in Z(R) \setminus \text{Nil}(R)$.*

Then, for every $y \in \text{Nil}(R)$ and every $x \in Z(R) \setminus \text{Nil}(R)$, $\text{Ann}(x) \subset \text{Ann}(y)$.

Consequently, $\text{Ann}(x)\text{Nil}(R) = \{0\}$ for every $x \in Z(R) \setminus \text{Nil}(R)$.

If furthermore, there exists an element $t \in \text{Nil}(R)$ such that $t^2 \neq 0$, then, for every $x \in Z(R) \setminus \text{Nil}(R)$, $t \notin \text{Ann}(x)$.

Consequently, For every $x \in Z(R) \setminus \text{Nil}(R)$, $\text{Ann}(x) \subset \text{Ann}(t) \subset \text{Nil}(R)$ such that $z^2 = 0$ for every $z \in \text{Ann}(x)$.

Proof. First, consider $y \in \text{Nil}(R)$ and $x \in Z(R) \setminus \text{Nil}(R)$ suppose that $\text{Ann}(x) \not\subset \text{Ann}(y)$. Then, there is an element $a \in R$ such that $ax = 0$ and $ay \neq 0$. Then, $a(x+y) = ay \neq 0$ However, for $n \in \mathbb{N}$ with $y^n = 0$, we have $a(x+y)^n = 0$. This means that $\text{Ann}(x+y) \neq \text{Ann}(x+y)^n$ which is absurd (by hypothesis and Lemma 2.1.2).

This shows that $z^2 = 0$ for every nilpotent element $z \in \text{Ann}(x)$. Thus it remains to prove that $\text{Ann}(t) \subset \text{Nil}(R)$. If not, there is then an element $a \in \text{Ann}(y)$ such that $a \notin \text{Nil}(R)$. Thus, $\text{Ann}(a) \subset \text{Ann}(y)$. But, $y \in \text{Ann}(a)$ implies that $y^2 = 0$, absurd. ■

Now, we are in position to give the desired example. For that we use a new ring construction recently introduced in [1].

Let R be a commutative ring with $1 \neq 0$ and let M_1 and M_2 be R -submodules of a commutative R -algebra L such that $(M_1)^2 := \{xy | x, y \in M_1\} \subset M_2$.

Then, we call 2-trivial extension of R by (M_1, M_2) the ring denoted by $R \times_2 M_1 \times M_2$ whose underling group is $A \times M_1 \times M_2$ with multiplication given by

$$(a, m_1, m_2)(b, n_1, n_2) = (ab, an_1 + bm_1, an_2 + bm_2 + m_1n_1).$$

Note that this construction is an extension of the well-known trivial extension of a ring by a module.

Note that $\text{Nil}(R \times_2 M_1 \times M_2) = \text{Nil}(R) \times_2 M_1 \times M_2$ and $Z(R \times_2 M_1 \times M_2) = \{(r, m_1, m_2) \in R \times M_1 \times M_2 | r \in Z(R) \cup Z(M_1) \cup Z(M_2)\}$. Also note that if M_1 contains an element m such that $m^2 \neq 0$, then $(0, m, 0)$ is a nilpotent element of index 3.

Example 2.1.5 *Let $R = \mathbb{Z} \times \mathbb{Z}_2$. Then, $\mathbb{Z} \cong (\mathbb{Z} \times \mathbb{Z}_2) / (\{0\} \times \mathbb{Z}_2)$ is an R -module with the scalar multiplication defined as follows: $(a, \bar{n})x := ax$ for*

every $(a, n, x) \in \mathbb{Z}^3$.

The ring $S = R \rtimes_2 \mathbb{Z} \rtimes \mathbb{Z}$ satisfies the following properties:

- S contains a nilpotent element with index three.
- $\text{Nil}(R) \subsetneq Z(R)$.
- For every $x \in Z(R) \setminus \text{Nil}(R)$, $\text{Ann}(x^2) = \text{Ann}(x)$.

Proof. To get the result it suffices to show that, for every $x \in Z(S) \setminus \text{Nil}(S)$, $\text{Ann}(x) = \{((0, \bar{0}), 0, 0); (0, \bar{1}), 0, 0\}$. This equality is a simple consequence of the fact that $\text{Nil}(S) = \{((0, \bar{n}), s, t) \mid (n, s, t) \in \mathbb{Z}^3\}$ and $Z(S) \setminus \text{Nil}(S) = \{((2k, \bar{n}), s, t) \mid k \in \mathbb{Z}^* \text{ and } (n, s, t) \in \mathbb{Z}^3\}$. ■

The following particular cases are simple consequences of Theorem 2.1.1.

Corollary 2.1.6 *Let R be a ring. If R contains a nilpotent element of index 3, then $\bar{\Gamma}(R) \neq \Gamma(R)$.*

Corollary 2.1.7 *Let R be a reduced ring. Then $\bar{\Gamma}(R) = \Gamma(R)$.*

Proof. Assume that there is an element $x \in Z(R)^*$ such that $\text{Ann}(x) \neq \text{Ann}(x^2)$. Then there is $z \in Z(R)^*$ such that $zx^2 = 0$ and $zx \neq 0$, hence $zx \in \text{Nil}(R) \setminus \{0\}$, a contradiction since R is reduced. ■

Now, we show when $\bar{\Gamma}(R) = \Gamma(R)$ for the finite direct product of rings.

Proposition 2.1.8 *Let $(R_i)_{1 \leq i \leq n}$ be a finite family of rings with $n \in \mathbb{N}^* \setminus \{1\}$.*

Then $\bar{\Gamma}(\prod_{i=1}^n R_i) = \Gamma(\prod_{i=1}^n R_i)$ if and only if R_i is reduced for every $1 \leq i \leq n$.

Proof. It suffices to prove the case where $n = 2$.

(\Rightarrow) Suppose that R_1 is not reduced. Then there is an element $x_1 \neq 0$ such that $x_1^2 = 0$. We have $(1, 0)(x_1, 1) = (x_1, 0) \neq (0, 0)$ but $(1, 0)(x_1, 1)^2 = (0, 0)$. Then, by Theorem 2.1.1, $\bar{\Gamma}(R_1 \times R_2) \neq \Gamma(R_1 \times R_2)$, a contradiction.

(\Leftarrow) Use Corollary 2.1.7. ■

As a simple consequence of Proposition 2.1.8, we determine when the graph $\bar{\Gamma}(\mathbb{Z}_n)$ coincides with $\Gamma(\mathbb{Z}_n)$.

Corollary 2.1.9 *Let $n = \prod_{i=1}^k P_i^{\alpha_i}$ be the prime factorization of an integer n with $k \in \mathbb{N}^*$. Consider $m := \sup\{\alpha_i \mid 1 \leq i \leq k\}$. Then $\bar{\Gamma}(\mathbb{Z}_n) \neq \Gamma(\mathbb{Z}_n)$ if and*

only if either $m \geq 3$ or ($m = 2$ and $k \geq 2$).

Consequently, $\bar{\Gamma}(\mathbb{Z}_n) = \Gamma(\mathbb{Z}_n)$ if and only if either $n = p^2$ for some prime p or n is square free. In particular, if \mathbb{Z}_n has nonzero nilpotent elements, then $\bar{\Gamma}(\mathbb{Z}_n) = \Gamma(\mathbb{Z}_n)$ if and only if $\Gamma(\mathbb{Z}_n)$ is complete.

2.2 Diameter of extended graphs of rings

In this section, we study the diameter of extended graphs of rings.

Certainly, as an extension of the classical graph of Theorem 1.2.1, $\bar{\Gamma}(R)$ has diameter at most 3.

Theorem 2.2.1 Let R be a ring. Then $\bar{\Gamma}(R)$ is connected with $\text{diam}(\bar{\Gamma}(R)) \leq 3$.

Now we determine some situations where $\text{diam}(\bar{\Gamma}(R)) \leq 2$.

In the following result, as an analogue of Theorem 1.2.2, we characterize when $\bar{\Gamma}(R)$ has a vertex adjacent to every other vertex (i.e., when $\bar{\Gamma}(R)$ has a spanning tree which is a star graph).

Theorem 2.2.2 Let R be a ring. Then, there is a vertex x of $\bar{\Gamma}(R)$ which is adjacent to every other vertex if and only if either $R \cong \mathbb{Z}_2 \times D$, where D is an integral domain, or $Z(R) = \sqrt{\text{Ann}(x^{n_x-1})}$.

Proof. (\Rightarrow) Suppose that x is adjacent to every other vertex of $\bar{\Gamma}(R)$. If x is a nilpotent element, then for every nonzero zero-divisor element $y \neq x$ there is two positive integers α, β such that $y^\alpha x^\beta = 0$ with $y^\alpha \neq 0$ and $x^\beta \neq 0$ (since x and y are adjacent in $\bar{\Gamma}(R)$), thus $\beta < n_x$ and $y^\alpha x^{n_x-1} = 0$ hence $y \in \sqrt{\text{Ann}(x^{n_x-1})}$. Finally, since x is nilpotent, $x \in \sqrt{\text{Ann}(x^{n_x-1})}$, and therefore $Z(R) = \sqrt{\text{Ann}(x^{n_x-1})}$.

If $x \notin \text{Nil}(R)^*$ then $x^2 = x$, if not there are two positive integers α and β such that $(x^2)^\alpha x^\beta = x^{2\alpha+\beta} = 0$, a contradiction since $x \notin \text{Nil}(R)^*$, so $R = Rx \oplus R(1-x)$. Hence we may assume that $R = R_1 \times R_2$ with $(1,0)$ adjacent to every other vertex. For any $1 \neq z \in R_1$, $(z,0)$ is a zero-divisor, so there are $n, m \in \mathbb{N}^*$ such that $(z,0)^n(1,0)^m = (0,0)$ and $(z,0)^n \neq (0,0)$, a contradiction. Hence, $R_1 \cong \mathbb{Z}_2$. If R_2 is not an integral domain, then there is a nonzero $t \in Z(R_2)$. Then $(1,t)$ is a zero-divisor of R which is not adjacent to $(1,0)$, a contradiction. Thus R_2 must be an integral domain.

(\Leftarrow). If $R \cong \mathbb{Z}_2 \times D$ for D an integral domain, then $(\bar{1},0)$ is adjacent to every

other vertex. If $Z(R) = \sqrt{\text{Ann}(x^{n_x-1})}$ for some nonzero $x \in R$, then x is adjacent to every other vertex. ■

We next determine when $\bar{\Gamma}(R)$ is a complete graph (i.e., where the diameter of $\bar{\Gamma}(R)$ is one). In Theorem 1.2.3, it is proved that the graph $\Gamma(R)$ is complete if and only if either $R \cong \mathbb{Z}_2 \times \mathbb{Z}_2$ or $xy = 0$ for every $x, y \in Z(R)^*$ (i.e., $Z(R)^2 = 0$). For our case, we have the following result.

Theorem 2.2.3 *Let R be a ring. Then $\bar{\Gamma}(R)$ is a complete graph if and only if either $R \cong \mathbb{Z}_2 \times \mathbb{Z}_2$ or $Z(R) = \text{Nil}(R)$ and for every $x, y \in Z(R)^*$ $x^{n_x-1}y^{n_y-1} = 0$.*

Proof. (\Leftarrow) By definition.

(\Rightarrow) Suppose that $\bar{\Gamma}(R)$ is complete.

If $Z(R) = \text{Nil}(R)$ then, by definition, $x^{n_x-1}x^{n_x-1} = 0$ for every element $x \in \text{Nil}(R)$. And, since $\bar{\Gamma}(R)$ is complete, for all distinct elements $x, y \in Z(R)^*$ there are two positive integers n and m such that $x^n y^m = 0$ with $x^n \neq 0$ and $y^m \neq 0$. Necessarily $n < n_x$ and $m < m_y$, therefore $x^{n_x-1}y^{n_y-1} = 0$.

Now suppose that $Z(R) \neq \text{Nil}(R)$. Since $\bar{\Gamma}(R)$ is complete and by Theorem 2.2.2, we have $R \cong \mathbb{Z}_2 \times D$, where D is an integral domain. Hence for distinct elements $a, b \in D \setminus \{0\}$, $(0, a)$ and $(0, b)$ are adjacent in $\bar{\Gamma}(R)$, then there are two positive integers n and m such that $(0, a)^n (0, b)^m = (0, 0)$, so $a = 0$ or $b = 0$ thus necessarily $D \cong \mathbb{Z}_2$. ■

To establish an analogy with the classical case we set $\bar{Z}(R) := \{x^{n_x-1} | x \in \text{Nil}(R)^*\}$ and $\bar{Z}(R)^2 := \{x^{n_x-1}y^{n_y-1} | x, y \in \text{Nil}(R)^*\}$.

Corollary 2.2.4 *Let R be a ring such that $\bar{\Gamma}(R) \neq \Gamma(R)$. Then, $\bar{\Gamma}(R)$ is complete if and only if $Z(R) = \text{Nil}(R)$ and $\bar{Z}(R)^2 = \{0\}$.*

In Theorem 1.2.5, the diameter of $\Gamma(R)$ is studied when $Z(R) = \text{Nil}(R) \neq \{0\}$. For our case, we have the following result which is slightly different from Theorem 1.2.5.

Theorem 2.2.5 *Let R be a ring with $Z(R) = \text{Nil}(R) \neq \{0\}$. Then $\text{diam}(\bar{\Gamma}(R)) \leq 2$ and exactly one of the following three cases must occur.*

1. $|Z(R)^*| = 1$. Then R is isomorphic to \mathbb{Z}_4 or $\mathbb{Z}_2[X]/(X^2)$, and $\text{diam}(\bar{\Gamma}(R)) = 0$.
2. $|Z(R)^*| \geq 2$ and $Z(R)^2 = \{0\}$. Then $\bar{\Gamma}(R)$ is a complete graph, and $\text{diam}(\bar{\Gamma}(R)) = 1$.

3. $|Z(R)^*| \geq 2$ and $Z(R)^2 \neq \{0\}$. If $\overline{Z}(R)^2 = 0$ then $\overline{\Gamma}(R)$ is a complete graph, and $\text{diam}(\overline{\Gamma}(R)) = 1$. If not $\text{diam}(\overline{\Gamma}(R)) = 2$.

Proof. 1. If $|Z(R)^*| = 1$ in this case $\overline{\Gamma}(R) = \Gamma(R)$, then $R \cong \mathbb{Z}_4$ or $\mathbb{Z}_2[X]/(X^2)$ by [18, proposition 2.2].

2. If $|Z(R)^*| \geq 2$ and $Z(R)^2 = \{0\}$, then $xy = 0$ for all $x, y \in Z(R)$. Thus $\overline{\Gamma}(R)$ is a complete graph and $\text{diam}(\overline{\Gamma}(R)) = 1$.

3. By Corollary 2.2.4, $\overline{\Gamma}(R)$ is a complete graph, hence $\text{diam}(\overline{\Gamma}(R)) = 1$. If not, there is $x, y \in Z(R)^*$ such that $x^{n_x-1}y^{n_y-1} \neq 0$ hence $xy \notin \{0, x, y\}$, so $x - xy - y$ is a path between x and y of length 2. ■

Now we study the diameter of the graph of finite direct products of rings.

Proposition 2.2.6 Let $R = \prod_{i=1}^n R_i$ where $(R_i)_{1 \leq i \leq n}$ is a finite family of rings with $n \in \mathbb{N}^* \setminus \{1\}$.

1. If $n = 2$, we have the following assertions:

- (a) $\text{diam}(\Gamma(R)) = \text{diam}(\overline{\Gamma}(R)) = 1$ if and only if $R_1 \cong R_2 \cong \mathbb{Z}_2$.
- (b) If R_1 and R_2 are integral domains with $|R_1| \geq 3$ or $|R_2| \geq 3$, then $\Gamma(R) = \overline{\Gamma}(R)$ and $\text{diam}(\Gamma(R)) = 2$. In this case $\Gamma(R)$ is a complete bipartite graph.
- (c) If at least one of R_1 and R_2 contains a non-nilpotent zero-divisor, then $\text{diam}(\Gamma(R)) = \text{diam}(\overline{\Gamma}(R)) = 3$.
- (d) If at least one of R_1 and R_2 is not integral domains such that all zero-divisors are nilpotent in each ring with nonzero zero-divisors, then $\text{diam}(\Gamma(R)) = 3$ and $\text{diam}(\overline{\Gamma}(R)) = 2$.

2. If $n \geq 3$, $\text{diam}(\Gamma(R)) = \text{diam}(\overline{\Gamma}(R)) = 3$.

Proof. 1. The proof of each (a) and (b) is trivial.

We prove the assertion (c). Suppose that R_1 contains a non-nilpotent zero-divisor z . Then, there is an element $z' \in R_1$ such that $zz' = 0$. Then using the following path, in both $\Gamma(R)$ and $\overline{\Gamma}(R)$, $(1, 0) - (0, 1) - (z', 0) - (z, 1)$ and the fact that there is no vertex adjacent to both $(1, 0)$ and $(z, 1)$, we conclude that $\text{diam}(\Gamma(R)) = \text{diam}(\overline{\Gamma}(R)) = 3$.

(d). We prove only the case where, for instance, R_1 is not integral domains such that all zero-divisors are nilpotent and R_2 is integral domains. First, using the same path as above, we have $\text{diam}(\Gamma(R)) = 3$. However, in $\overline{\Gamma}(R)$, $d((1, 0), (z, 1)) = 1$ for every $z \in Z(R_1)^* = \text{Nil}(R_1)$. Now, we have: $Z(R)^* = T_1 \cup T_2 \cup T_3 \cup T_4$ where $T_1 = \{(a, 0) | a \text{ is regular}\}$, $T_2 = \{(b, 0) | b \in Z(R_1)\}$,

$T_3 = \{(0, x) | x \in R_2\}$ and $T_4 = \{(a, x) | a \in Z(R_1), x \in R_2\}$. A simple study of the distance between any two elements shows that $\text{diam}(\overline{\Gamma}(R)) = 2$.

2. Note that $(0, 1, 1, \dots, 1) - (1, 0, 0, \dots, 0) - (0, 0, \dots, 0, 1) - (1, 1, \dots, 1, 0)$ is a shortest path between $(0, 1, 1, \dots, 1)$ and $(1, 1, 1, \dots, 0)$. ■

Proposition 2.2.6 helps to determine the diameter of \mathbb{Z}_n .

Proposition 2.2.7 *For a positive integer $n \in \mathbb{N}^*$, the following assertions hold true:*

1. If $n = 2^2$, then $\text{diam}(\overline{\Gamma}(\mathbb{Z}_n)) = 0$.
2. If either $n = 2^m$ with $m > 2$ or $n = p^m$ with p is an odd prime and $m \geq 2$, then $\text{diam}(\overline{\Gamma}(\mathbb{Z}_n)) = 1$. In this case $\overline{\Gamma}(\mathbb{Z}_n)$ is a complete graph.
3. If $n = p^\alpha q^\beta$ with p and q are distinct primes, then $\text{diam}(\overline{\Gamma}(\mathbb{Z}_n)) = 2$. In this case,
 - If $p = 2$ and $\alpha = \beta = 1$, then $\overline{\Gamma}(\mathbb{Z}_n)$ is a star graph.
 - If either $(p = 2, \alpha = 2$ and $\beta = 1)$ or $n = pq$, then $\overline{\Gamma}(\mathbb{Z}_n)$ is a complete bipartite graph.
4. If $n = \prod_{i=1}^k P_i^{\alpha_i}$ is the prime factorization of n with $p_i \neq p_j$, for $i \neq j$, and $k \geq 3$, then $\text{diam}(\overline{\Gamma}(R)) = 3$.

Proof. 1. Let $n = 2^2$, then $Z(\mathbb{Z}_4)^* = \{\overline{2}\}$ hence $\text{diam}(\overline{\Gamma}(\mathbb{Z}_4)) = 0$.

2. Let $n = 2^m$ with $m > 2$ or $n = p^m$ where p is an odd prime and $m \geq 2$. Then all zero-divisors of \mathbb{Z}_n are multiple of $\overline{2}$ for the first and \overline{p} for the second. It is clear that all zero-divisors are adjacent to each other, so $\text{diam}(\overline{\Gamma}(\mathbb{Z}_n)) = 1$ and $\overline{\Gamma}(R)$ is a complete graph.

3. Let $n = p^\alpha q^\beta$ with p and q are distinct primes. Then, by Proposition 2.2.6, $\text{diam}(\overline{\Gamma}(\mathbb{Z}_n)) = 2$.

If $n = 2q$, then $Z(\mathbb{Z}_n)^* = \{\overline{2h}/0 < h < q\} \cup \{\overline{q}\}$. Hence \overline{q} is adjacent to every other vertex and $d(\overline{2h}, \overline{2h'}) = 2$ with $0 < h < q$ and $0 < h' < q$. Therefore, $\overline{\Gamma}(\mathbb{Z}_{2q})$ is a star graph.

If $n = 4q$, then $Z(\mathbb{Z}_n)^* = \{\overline{2h}/0 < h < 2q\} \cup \{\overline{q}\}$. The two following sets $A := \{\overline{2h}/0 < h < 2q$ and $h \neq q\}$ and $B := \{\overline{kq}/0 < k < 4\}$ form a partition of $Z(\mathbb{Z}_n)^*$ and shows that $\overline{\Gamma}(\mathbb{Z}_n)$ is a complete bipartite graph.

If $n = pq$, then $Z(\mathbb{Z}_n)^* = \{\overline{hp}/0 < h < q\} \cup \{\overline{kq}/0 < k < p\}$. The two following sets $A := \{\overline{hp}/0 < h < q\}$ and $B := \{\overline{kq}/k < p\}$ form a partition of $Z(\mathbb{Z}_n)^*$ and shows that $\overline{\Gamma}(\mathbb{Z}_n)$ is a complete bipartite graph.

4. Follows from Proposition 2.2.6. ■

2.3 Cycles in extended graphs of rings

In this section, we study the girth of $\bar{\Gamma}(R)$.

Also, since $\Gamma(R)$ is a subgraph of $\bar{\Gamma}(R)$ and by Theorem 1.3.1, we have $\text{gr}(\bar{\Gamma}(R)) \in \{3, 4, \infty\}$.

In the classical case there are some examples of rings R such that $\text{gr}(\Gamma(R)) = \infty$. The following result shows that when $\bar{\Gamma}(R) \neq \Gamma(R)$, we have $\text{gr}(\bar{\Gamma}(R)) \in \{3, 4\}$.

Theorem 2.3.1 *Let R be a ring. If $\bar{\Gamma}(R) \neq \Gamma(R)$, then $\bar{\Gamma}(R)$ contains a cycle.*

Proof. Since $\bar{\Gamma}(R) \neq \Gamma(R)$ and by Theorem 2.1.1, there is either a nilpotent element x with $n_x \geq 3$ or an element $x \in Z(R) \setminus \text{Nil}(R)$ such that $\text{Ann}(x) \neq \text{Ann}(x^2)$. For the first case, we have $x - (x + x^{n_x-1}) - x^{n_x-1} - x$ is a cycle of length 3. For the second case, there exists $y \in Z(R)^*$ such that $yx^2 = 0$ and $yx \neq 0$. If $y^2 = 0$ then $y - (x + y) - xy - y$ is a cycle of length 3. If not $x - yx - x^2 - y - x$ is a cycle of length 4. ■

Corollary 2.3.2 *If R contains a nilpotent element of index greater than or equal to three, then $\text{gr}(\bar{\Gamma}(R)) = 3$.*

Corollary 2.3.3 *If there are elements x and z of $Z(R)^*$ such that $x \notin \text{Nil}(R)$, $z^2 = 0$, $zx \neq 0$ and $zx^2 = 0$ then $\text{gr}(\bar{\Gamma}(R)) = 3$.*

In Theorem 1.3.7, the girth of $\Gamma(R)$ is studied when $Z(R) = \text{Nil}(R) \neq \{0\}$. For our case, we have the following slightly different result.

Theorem 2.3.4 *Let R be a ring with $Z(R) = \text{Nil}(R) \neq \{0\}$. Then exactly one of the following three cases must occur.*

1. If $|Z(R)^*| = 1$, then R is isomorphic to \mathbb{Z}_4 or $\mathbb{Z}_2[X]/(X^2)$, and $\text{gr}(\bar{\Gamma}(R)) = \infty$.
2. If $|Z(R)^*| = 2$, then R is isomorphic to \mathbb{Z}_9 or $\mathbb{Z}_3[X]/(X^2)$, and $\text{gr}(\bar{\Gamma}(R)) = \infty$.
3. If $|Z(R)^*| = 3$, then R is isomorphic to \mathbb{Z}_8 , $\mathbb{Z}_2[X]/(X^3)$, $\mathbb{Z}_4[X]/(2X, X^2 - 2)$, $\mathbb{Z}_2[X, Y]/(X, Y)^2$, $\mathbb{Z}_4[X]/(2, X)^2$, $\mathbb{Z}_4[X]/(X^2 + X + 1)$, or $\mathbb{F}_4[X]/(X^2)$, and $\text{gr}(\bar{\Gamma}(R)) = 3$.
4. If $|Z(R)^*| \geq 4$, then $\text{gr}(\bar{\Gamma}(R)) = 3$.

Proof. All assertions follow from Theorem 1.3.7 except the following cases: for $R \cong \mathbb{Z}_8$, $2-4-6-2$ is a cycle of length 3. For $R \cong \mathbb{Z}_2[X]/(X^3)$, $X-X^2-(X^2+X)-X$ is a cycle of length 3. And finally, for $R \cong \mathbb{Z}_4[X]/(2X, X^2-2)$, $2-X-(X-2)-2$ is a cycle of length 3. ■

Theorems 1.3.4 and 1.3.6 allow us to establish situations where $\text{gr}(\bar{\Gamma}(R)) = 4$. Namely we have the two following results.

Theorem 2.3.5 *Let R be a commutative ring with $\text{Nil}(R) \neq \{0\}$ and $\text{gr}(\Gamma(R)) = 4$. Then $\bar{\Gamma}(R) \neq \Gamma(R)$ implies that $\text{gr}(\bar{\Gamma}(R)) = 4$ and $\bar{\Gamma}(R)$ is a complete bipartite graph.*

Proof. From Theorem 1.3.4, $R \cong D \times B$ where D is an integral domain with $|D| \geq 3$ and $B = \mathbb{Z}_4$ or $\mathbb{Z}_2[X]/(X^2)$. Then, $Z(R) = A \cup B$ where $A = \{(a, x) | a \in D \text{ and } x \in \{0; 2\}\}$ and $B = \{(0, n) | n \in \mathbb{Z}_4\}$. One can show that all elements of A are connected to all elements of B in $\bar{\Gamma}(R)$ such that $\bar{\Gamma}(R)$ is a complete bipartite graph, and therefore $\text{gr}(\bar{\Gamma}(R)) = 4$. ■

Theorem 2.3.6 *Let R be a commutative ring with $\text{Nil}(R) \neq \{0\}$ and $\text{gr}(\Gamma(R)) = \infty$. Then exactly one of the following holds:*

1. $\bar{\Gamma}(R) = \Gamma(R)$ is a singleton or a star graph. In this case $\text{gr}(\bar{\Gamma}(R)) = \infty$.
2. $\Gamma(R) = \bar{K}^{1,3}$ (i.e., $R \cong \mathbb{Z}_2 \times \mathbb{Z}_4$ or $R \cong \mathbb{Z}_2 \times \mathbb{Z}_2[X]/(X^2)$). In this case $\text{gr}(\bar{\Gamma}(R)) = 4$ and $\bar{\Gamma}(R) \neq \Gamma(R)$.

Proof. The first assertion flows from Theorem 1.3.6.

When $R \cong \mathbb{Z}_2 \times \mathbb{Z}_4$ or $R \cong \mathbb{Z}_2 \times \mathbb{Z}_2[X]/(X^2)$, we have the following cycle $(1, 0) - (0, 2) - (1, 2) - (0, 1) - (1, 0)$ in $\bar{\Gamma}(R)$ of length 4. ■

In Theorem 1.1.4, the notion of complemented graph is used to characterize when the total quotient ring of a reduced ring R is von Neumann regular. In Proposition 2.3.8 below we attempt to give a similar result. In fact, when $\bar{\Gamma}(R) \neq \Gamma(R)$, we only show that $\bar{\Gamma}(R)$ is complemented is a sufficient condition so that $T(R)$ be zero-dimensional. The converse remains an open problem. For that, we need the following definition and lemma.

Let us say, as in [28], that distinct vertices x and y of $\bar{\Gamma}(R)$ are orthogonal, written $x \perp_{\bar{\Gamma}(R)} y$, if x and y are adjacent and there is no vertex z of $\bar{\Gamma}(R)$ which is adjacent to both x and y , i.e., the edge $x - y$ is not a part of any triangle of $\bar{\Gamma}(R)$. We say that $\bar{\Gamma}(R)$ is complemented if for each vertex x of $\bar{\Gamma}(R)$, there is a vertex y of $\bar{\Gamma}(R)$ (called a complement of x) such that $x \perp_{\bar{\Gamma}(R)} y$.

Lemma 2.3.7 *Let R be a ring. If there are orthogonal elements $x, y \in Z(R)^*$ and there are $n, m \in \mathbb{N}^* \setminus \{1\}$ such that $x^n y^m = 0$ with $x^n \neq 0$ and $y^m \neq 0$, then, $x^n + y^m$ is a regular element of R .*

Proof. Suppose that $z(x^n + y^m) = 0$ for some $z \in R \setminus \{0\}$. Let $t = zx^n = -zy^m$, then $tx^n = ty^m = 0$. If $t = x$ then $x^{n+1} = 0$. With $t = zx^n$, we get $x^n = zx^{2n-1} = 0$, a contradiction since $x^n \neq 0$. Similarly we prove that $t \neq y$. Also, if $t \neq 0$, then t is adjacent to both x and y , a contradiction since $x \perp_{\bar{\Gamma}(R)} y$. Then $t = 0$ and so $zx^n = -zy^m = 0$. Then, it remains to prove that $z \neq x$ and $z \neq y$. Indeed, if $z = x$ then $x^{n+1} = 0$. Then, x is adjacent to x^2 (since $x^{n+1} = x^2 x^{n-1}$ and $x^2 \neq x$). Now with $xy^m = 0$ we have $x^2 y^m = 0$. Then x^2 is adjacent to both x and y , a contradiction. Similarly we prove that $z \neq y$. ■

Proposition 2.3.8 *Let R be a ring with $\bar{\Gamma}(R) \neq \Gamma(R)$. If $\bar{\Gamma}(R)$ is complemented, then $T(R)$ is zero-dimensional.*

Proof. First, note that all nilpotent elements has index 2 (since $\bar{\Gamma}(R)$ is complemented). Then by Theorem 2.1.1, there is an element $x_0 \in Z(R) \setminus \text{Nil}(R)$ such that $\text{Ann}(x_0) \neq \text{Ann}(x_0^2)$. This implies that there is $z_0 \in Z(R)$ such that $z_0 x_0 \neq 0$ and $z_0 x_0^2 = 0$. Also, note that, from Corollary 2.3.3, $z_0 \notin \text{Nil}(R)$. Now, to show that $T(R)$ is zero-dimensional, it is sufficient to show that each non-minimal prime ideal Q of R contains a regular element of R . Let $P \subset Q$ be distinct prime ideals of R . Then there is $x \in Q \setminus P$. Note that $x \notin \text{Nil}(R)$. We have the following possible situations:

Case 1 (x is adjacent to x_0): Then $x_0 \in Q$ and, by Lemma 2.3.7, there exist $\alpha, \beta \in \mathbb{N}^*$ such that $x_0^\alpha + x_0^\beta$ is a regular element of R which belongs to Q .

Case 2 (x is adjacent to z_0): The proof is the same as above.

Case 3 ($x_0 \notin P$): Then $xx_0 \in Q \setminus P$. With xx_0 is adjacent to z_0 and by Lemma 2.3.7, $x^2 x_0^2 + z_0$ is regular and belongs to Q .

Case 4 ($x_0 \in P$): If $z_0 \notin P$, then $xz_0 \in Q \setminus P$ and so xz_0 is adjacent to x_0 . By Lemma 2.3.7, $xz_0 + x_0^2$ is regular and belongs to Q . If $z_0 \in P$, then $x_0^2 + z_0$ is regular and belongs to Q since $x_0^2 z_0 = 0$. ■

Now we study the girth of the graph of finite direct products of rings.

Proposition 2.3.9 *Let $R = \prod_{i=1}^n R_i$ where $(R_i)_{1 \leq i \leq n}$ is a finite family of rings with $n \in \mathbb{N}^* \setminus \{1\}$.*

1. *If $n = 2$, we have the following assertions:*

- (a) $\text{gr}(\Gamma(R)) = \text{gr}(\overline{\Gamma}(R)) = \infty$ if and only if R_1 and R_2 are integral domains and at least one is isomorphic to \mathbb{Z}_2 .
- (b) If R_1 and R_2 are integral domains with $|R_1| \geq 3$ and $|R_2| \geq 3$, then $\Gamma(R) = \overline{\Gamma}(R)$ and $\text{gr}(\Gamma(R)) = 4$.
- (c) If at least one of R_1 and R_2 is not integral domain, then $\text{gr}(\Gamma(R)) = \text{gr}(\overline{\Gamma}(R)) = 3$.

2. If $n \geq 3$, then $\text{gr}(\Gamma(R)) = \text{gr}(\overline{\Gamma}(R)) = 3$.

Proof. 1. The proof of each (a) and (b) is trivial.

We prove the assertion (c). Suppose that R_1 contains a zero-divisor z . Then, there is an element $z' \in R_1$ such that $zz' = 0$. Then $(z, 0) - (z', 1) - (0, 1) - (z, 0)$ is a cycle of length 3. Thus $\text{gr}(\Gamma(R)) = \text{gr}(\overline{\Gamma}(R)) = 3$.

2. $(1, 0, 0, \dots, 0) - (0, 1, 0, \dots, 0) - (0, 0, 1, \dots, 0) - (1, 0, \dots, 0, 0)$ is a cycle of length 3. Thus $\text{gr}(\Gamma(R)) = \text{gr}(\overline{\Gamma}(R)) = 3$. ■

Proposition 4.2.1 can be used to determine the girth of \mathbb{Z}_n .

Proposition 2.3.10 For a positive integer $n \in \mathbb{N}^*$, the following assertions hold true:

1. If $n = 2^2$ or $n = 3^2$ or $n = 2p$ with p is an odd prime, then $\text{gr}(\overline{\Gamma}(\mathbb{Z}_n)) = \infty$.
2. If $n = pq$ or $n = 4p$ with p and q are odd primes, then $\text{gr}(\overline{\Gamma}(\mathbb{Z}_n)) = 4$.
3. We have $\text{gr}(\overline{\Gamma}(\mathbb{Z}_n)) = 3$, if one of the three following assertions holds true:
 - (a) $n = p^m$ with $m > 2$ and p is prime,
 - (b) $n = p^2$ with $p > 3$ is prime,
 - (c) $n = \prod_{i=1}^k P_i^{\alpha_i}$ is the prime factorization of n with $p_i \neq p_j$, for $i \neq j$, and $k \geq 3$.

Proof. 1. If $n = 2^2$ or $n = 3^2$, then $|Z(R)^*| = 1$ or $|Z(R)^*| = 2$ respectively, hence $\text{gr}(\overline{\Gamma}(R)) = \infty$. If $n = 2p$, then $\overline{\Gamma}(R)$ is a star graph (see Proposition 2.2.7), hence $\text{gr}(\overline{\Gamma}(R)) = \infty$.

2. If $n = pq$ or $n = 4p$ with p and q are odd primes. Then, $\overline{\Gamma}(R)$ is a complete bipartite graph (see Proposition 2.2.7). Thus $\text{gr}(\overline{\Gamma}(R)) = 4$.

3. For both assertions (a) and (b), we have that $\overline{\Gamma}(R)$ is complete (see Proposition 2.2.7), hence $\text{gr}(\overline{\Gamma}(R)) = 3$. For the third assertion, we have the following cycle of length 3: $\overline{p_1^{\alpha_1} p_3^{\alpha_3}} - \overline{p_2^{\alpha_2} p_3^{\alpha_3}} - \overline{p_1^{\alpha_1} p_2^{\alpha_2}} - \overline{p_1^{\alpha_1} p_3^{\alpha_3}}$. ■

Extended zero-divisor graphs of idealizations

In this chapter, we present an extended zero-divisor graphs of idealizations $R \times M$ (where M is an R -module). At first, we distinguish when $\overline{\Gamma}(R \times M)$ and the classical zero-divisor graph $\Gamma(R \times M)$ coincide. Various examples in this context are given. Among other things, the diameter and the girth of $\overline{\Gamma}(R \times M)$ are also studied.

3.1 When $\overline{\Gamma}(R \times M)$ and $\Gamma(R \times M)$ coincide?

In this section, we investigate when $\overline{\Gamma}(R \times M)$ and $\Gamma(R \times M)$ coincide. A general result is proved in Theorem 2.1.1 as follows: The two graphs $\overline{\Gamma}(R)$ and $\Gamma(R)$ coincide if and only if every nonzero nilpotent element (if it exists) has index 2, and $\text{Ann}(x^2) = \text{Ann}(x)$ for every $x \in Z(R) \setminus \text{Nil}(R)$.

We start with the main result of this section.

Theorem 3.1.1 *The two graphs $\overline{\Gamma}(R \times M)$ and $\Gamma(R \times M)$ coincide if and only if the following conditions hold.*

1. $(2\text{Nil}(R))M = 0$.
2. $\overline{\Gamma}(R) = \Gamma(R)$.
3. $\bigcup_{a \in \Lambda} \text{Ann}(a) \subseteq \text{Ann}(M)$, where $\Lambda = Z(R) \setminus \text{Nil}(R)$.
4. $\text{Ann}_M(a^2) = \text{Ann}_M(a)$ for all $a \in R \setminus \text{Nil}(R)$.

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Proof. (\Rightarrow) 1. We show that $(2\text{Nil}(R))M = 0$. Clearly we may assume that $\text{Nil}(R) \neq \{0\}$. Suppose that $(2\text{Nil}(R))M \neq 0$. Then there are an $a \in \text{Nil}(R)$ and an $m \in M$ such that $2am \neq 0$. We have $(a, m)^2 = (a^2, 2am) \neq (0, 0)$, a contradiction since $(a, m) \in \text{Nil}(R \times M)$ and $\bar{\Gamma}(R \times M) = \Gamma(R \times M)$.

2. We show that $\bar{\Gamma}(R) = \Gamma(R)$. Let $a \in \text{Nil}(R)$. Then $(a, 0) \in \text{Nil}(R \times M)$. Hence $(a, 0)^2 = (0, 0)$, and therefore $a^2 = 0$. Let $a \in Z(R) \setminus \text{Nil}(R)$. We prove that $\text{Ann}(a^2) \subseteq \text{Ann}(a)$. Let $x \in \text{Ann}(a^2)$. Then $(x, 0) \in \text{Ann}((a, 0)^2) = \text{Ann}((a, 0))$ by hypothesis. Therefore, $(x, 0)(a, 0) = (xa, 0) = (0, 0)$; so $x \in \text{Ann}(a)$. Hence, by Theorem 2.1.1, $\bar{\Gamma}(R) = \Gamma(R)$.

3. We may assume that $Z(R) \neq \text{Nil}(R)$. Suppose that $\bigcup_{a \in \Lambda} \text{Ann}(a) \not\subseteq \text{Ann}(M)$.

Then there is a $b \in \bigcup_{a \in \Lambda} \text{Ann}(a)$ with $b \notin \text{Ann}(M)$. Hence, there is an $a \in$

$Z(R) \setminus \text{Nil}(R)$ such that $ba = 0$, and there is an $n \in M$ such that $bn \neq 0$. Thus $(a, n)(b, 0) \neq (0, 0)$ and $(a, n)^2(b, 0) = (0, 0)$, a contradiction since $\bar{\Gamma}(R \times M) = \Gamma(R \times M)$.

4. Let $a \in R \setminus \text{Nil}(R)$, and consider $0 \neq m \in \text{Ann}_M(a^2)$. Then $a^2m = 0$, and hence $(0, m)(a, 0)^2 = (0, a^2m) = (0, 0)$. Then $(0, m)(a, 0) = (0, 0)$ (since $(a, 0) \notin \text{Nil}(R \times M)$ and $\bar{\Gamma}(R \times M) = \Gamma(R \times M)$). Therefore, $am = 0$; so $m \in \text{Ann}_M(a)$.

(\Leftarrow) We show that $\bar{\Gamma}(R \times M) = \Gamma(R \times M)$. Let $(a, m) \in \text{Nil}(R \times M)$. Then $a \in \text{Nil}(R)$, and hence $a^2 = 0$ since $\bar{\Gamma}(R) = \Gamma(R)$. Thus $(a, m)^2 = (a^2, 2am) = (0, 0)$ since $(2\text{Nil}(R))M = 0$ by hypothesis.

Let $(a, m) \in Z(R \times M) \setminus \text{Nil}(R \times M)$. We only need to show that $\text{Ann}((a, m)^2) \subseteq \text{Ann}(a, m)$. Let $(b, n) \in \text{Ann}((a, m)^2)$. Then $(b, n)(a, m)^2 = (ba^2, 2abm + a^2n) = (0, 0)$. Hence, $b \in \text{Ann}(a^2) = \text{Ann}(a) \subseteq \text{Ann}(M)$ (by hypotheses 2 and 3); so $ba = 0$ and $bm = 0$. Also, $n \in \text{Ann}_M(a^2) = \text{Ann}_M(a)$; so $(b, n) \in \text{Ann}(a, m)$, as desired. ■

Now we give some corollaries which can be used to construct examples of particular rings that satisfy the conditions of Theorem 3.1.1.

We begin with the case where M is a faithful R -module, that is, $aM \neq 0$ for all $a \in R \setminus \{0\}$. Then, using Theorems 2.1.1 and 3.1.1, we get the following result.

Corollary 3.1.2 *Assume M is faithful. Then $\bar{\Gamma}(R \times M) = \Gamma(R \times M)$ if and only if $2\text{Nil}(R) = 0$ and $a^2 = 0$ for all $a \in Z(R)$.*

As a simple consequence of the result above, we get $\bar{\Gamma}(\mathbb{Z}_4 \times (\mathbb{Z}_4)^{(I)}) = \Gamma(\mathbb{Z}_4 \times (\mathbb{Z}_4)^{(I)})$ for an arbitrary set I . Also, for $M = R$, we get the following result.

Corollary 3.1.3 *$\bar{\Gamma}(R \times R) = \Gamma(R \times R)$ if and only if $2\text{Nil}(R) = 0$ and $x^2 = 0$ for every $x \in Z(R)$.*

In particular, if R is reduced, then $\bar{\Gamma}(R \times R) = \Gamma(R \times R)$ if and only if R is an integral domain.

As examples of rings that satisfy the conditions of the above result, one can consider the rings $R = \mathbb{Z}_4$ and $R = \mathbb{Z}_2[X]/(X^2)$.

Notice that if $\bar{\Gamma}(R \times R) = \Gamma(R \times R)$ and $\text{Nil}(R) \neq \{0\}$, then $2 \in Z(R) = \text{Nil}(R)$ and so $4 \cdot 1_R = 0$ (since $\bar{\Gamma}(R) = \Gamma(R)$). Then, if $2 \neq 0$ the vertex 2 is adjacent to all other vertices (since we have $\text{Ann}(2) = \text{Nil}(R) = Z(R)$). Finally, it is also worth noting that if $\bar{\Gamma}(R \times R) = \Gamma(R \times R)$, then the characteristic of R is either 0; 2 or 4.

From Corollary 2.1.7, we have that if R is reduced, then $\bar{\Gamma}(R) = \Gamma(R)$. Using this fact, we get the following result.

Corollary 3.1.4 *If R is reduced, then $\bar{\Gamma}(R \times M) = \Gamma(R \times M)$ if and only if $Z(R) \subseteq \text{Ann}(M)$ and $\text{Ann}_M(a^2) = \text{Ann}_M(a)$ for all $a \in R \setminus \{0\}$.*

In particular,

1. *If R is an integral domain, then $\bar{\Gamma}(R \times M) = \Gamma(R \times M)$ if and only if $\text{Ann}_M(a^2) = \text{Ann}_M(a)$ for all $a \in R \setminus \{0\}$.*
2. *If R is a von Neumann regular ring, then $\bar{\Gamma}(R \times M) = \Gamma(R \times M)$ if and only if R is a field.*
3. *If R is reduced and $M = R/I$ for a radical ideal I , then $\bar{\Gamma}(R \times M) = \Gamma(R \times M)$ if and only if $Z(R) \subseteq I$.*

Proof. Only the assertions 2 and 3 merit a proof.

2. First note that the assertion “ $\text{Ann}_M(a^2) = \text{Ann}_M(a)$ for all $a \in R \setminus \{0\}$ ” holds for every von Neumann regular ring R . Indeed, consider, $m \in M$ such that $a^2m = 0$. Since R is a von Neumann regular ring, there is an $x \in R$ such that $a = xa^2$. Then $am = xa^2m = 0$, as desired. Thus, it remains to prove that if $\bar{\Gamma}(R \times M) = \Gamma(R \times M)$, then R is a field. In fact, if R is not a field, then there is a nontrivial idempotent e . Then we have two zero-divisors e and $1 - e$ such that $e + (1 - e) = 1$. This shows that $Z(R)$ cannot be included in a proper ideal. This completes the proof since M is a nonzero module.

3. Let $a \in R \setminus \{0\}$. We claim that $\text{Ann}_M(a^2) = \text{Ann}_M(a)$. Let $x \in R$ such that $a^2\bar{x} = \bar{0}$. Then $(ax)^2 \in I$. Since I is a radical ideal, $ax \in I$. Therefore, $\bar{x} \in \text{Ann}_M(a)$, as desired. ■

The case where $Z(R)$ is an ideal of R is of interest.

Corollary 3.1.5 *Assume that $Z(R)$ is an ideal of R . Then $\bar{\Gamma}(R \times R/Z(R)) = \Gamma(R \times R/Z(R))$ if and only if $\bar{\Gamma}(R) = \Gamma(R)$.*

We end this section with the case where $M = R/\text{Nil}(R)$.

Corollary 3.1.6 $\bar{\Gamma}(R \times R/\text{Nil}(R)) = \Gamma(R \times R/\text{Nil}(R))$ if and only if the following conditions hold.

1. $\bar{\Gamma}(R) = \Gamma(R)$.
2. $\bigcup_{a \in \Lambda} \text{Ann}(a) \subseteq \text{Nil}(R)$, where $\Lambda = Z(R) \setminus \text{Nil}(R)$.

Proof. It suffices to show that if $\bar{\Gamma}(R) = \Gamma(R)$, then the fourth assertion in Theorem 3.1.1 holds. Indeed, consider a non-nilpotent element a of R . If a is regular, then $\text{Ann}_M(a^2) = \{\bar{0}\} = \text{Ann}_M(a)$, where $M = R/\text{Nil}(R)$. Now suppose that $a \in Z(R) \setminus \text{Nil}(R)$, and let $m \in R$ with $a^2\bar{m} = \bar{0}$. If $\bar{m} = \bar{0}$, then there is nothing to prove. If $\bar{m} \neq \bar{0}$, then a^2m is nilpotent. Thus, a and m are adjacent in $\bar{\Gamma}(R)$ (since they are not nilpotent). And, since $\bar{\Gamma}(R) = \Gamma(R)$, a and m are adjacent in $\Gamma(R)$, as desired. ■

As a ring that satisfies the assertion 2 in Corollary 3.1.6, one can consider the ring $R = \mathbb{R}[X, Y]/(X^2, XY)$, where $\text{Nil}(R) = \bar{X}R$.

3.2 The Girth of extended zero-divisor graphs of idealizations

In this section, we study the girth of extended zero-divisor graphs of $R \times M$. As in the classical case [16], if $|M| \geq 4$, then $\text{gr}(\bar{\Gamma}(R \times M)) = 3$, since $(0, m_1) - (0, m_2) - (0, m_3) - (0, m_1)$ is a cycle of length three, where m_1, m_2 and m_3 are distinct nonzero elements of M . Thus, we only need to consider when the module M has only two or three elements. Also, since $\Gamma(R \times M)$ is a subgraph of $\bar{\Gamma}(R \times M)$, then Theorem 1.3.8 remains valid for our case. Namely, if $M = \mathbb{Z}_3$, then the following two assertions hold.

- $\text{gr}(\bar{\Gamma}(R \times \mathbb{Z}_3)) = 3$ if and only if $\text{Ann}(\mathbb{Z}_3) \neq \{0\}$.
- $\text{gr}(\bar{\Gamma}(R \times \mathbb{Z}_3)) = \infty$ if and only if $\text{Ann}(\mathbb{Z}_3) = \{0\}$.

Thus, only the case where $M \cong \mathbb{Z}_2$ is of interest. We begin with the "extended" counterpart of Theorem 1.3.9.

Theorem 3.2.1 Assume that $M \cong \mathbb{Z}_2$ as an R -module. Then the girth of $\bar{\Gamma}(R \times \mathbb{Z}_2)$ is three if and only if one of the following assertions holds.

1. The girth of $\bar{\Gamma}(R)$ is three.
2. There exists an $r \in R \setminus \{0\}$ such that $r^n = 0$ for some integer $n \geq 2$.
3. There exist distinct $a, b \in Z(R)^*$ and $n, m \in \mathbb{N} \setminus \{0\}$ such that $a^n b^m = 0$ and $a^n \mathbb{Z}_2 = b^m \mathbb{Z}_2 = \{0\}$ with $a^n \neq 0$ and $b^m \neq 0$.

Proof. (\Rightarrow) The proof is similar to that of Theorem 1.3.9. There are two cases to discuss.

Case 1. The element $(0, \bar{1})$ is a part of a minimal length cycle. Then the cycle has the form $(0, \bar{1}) - (a, \bar{i}) - (b, \bar{j}) - (0, \bar{1})$. If $a \neq b$, we have distinct $a, b \in Z(R)^*$ such that there are $n, m \in \mathbb{N}$ and $a^n b^m = 0$ with $a^n \neq 0$, $b^m \neq 0$ and $a^n \mathbb{Z}_2 = b^m \mathbb{Z}_2 = \{0\}$. If $a = b$, we have $a \in R \setminus \{0\}$ such that $a^n = 0$.

Case 2. The element $(0, \bar{1})$ is not a part of a minimal length cycle. Then the cycle has the form $(a, \bar{i}) - (b, \bar{j}) - (c, \bar{k}) - (a, \bar{i})$ (with $a \neq 0$). If a, b and c are all distinct, the path $a - b - c - a$ is a cycle in $\bar{\Gamma}(R)$, and so $\text{gr}(\bar{\Gamma}(R)) = 3$. If not, then either $a^n = 0$ or $b^m = 0$, where $n, m \geq 2$.

(\Leftarrow) If assertion (1) holds, then clearly the girth of $\bar{\Gamma}(R \times \mathbb{Z}_2)$ is three.

Suppose that assertion (2) holds. Then either $r \cdot 1 = \bar{0}$ or $r \cdot 1 = \bar{1}$. If $r \cdot 1 = \bar{1}$, then $r^{n-1}(r \cdot 1) = \bar{1}$, a contradiction (since $r^n = 0$). Thus $r \cdot 1 = \bar{0}$, and so $(r, \bar{0}) - (r, \bar{1}) - (0, \bar{1}) - (r, \bar{0})$ is a cycle of length three in $\bar{\Gamma}(R \times \mathbb{Z}_2)$.

Finally, suppose that assertion (3) holds. Then $(a, \bar{0}) - (b, \bar{0}) - (0, \bar{1}) - (a, \bar{0})$ is a cycle of length three in $\bar{\Gamma}(R \times \mathbb{Z}_2)$. ■

Examples 3.2.2 1. Let $R = \mathbb{Z}_2[X]/(X^3)$. Then $\text{gr}(\bar{\Gamma}(R)) = 3$ and $(\bar{X}, \bar{0}) - (X^2, \bar{0}) - (\bar{X} + X^2, \bar{0}) - (\bar{X}, \bar{0})$ is a cycle of length three in $\bar{\Gamma}(R \times \mathbb{Z}_2)$, but it is not a cycle in $\Gamma(R \times \mathbb{Z}_2)$, although $\text{gr}(\Gamma(R \times \mathbb{Z}_2)) = 3$.

2. Let $R = \mathbb{Z}_{12}$. Then $\bar{6}$ is a nilpotent element and $(\bar{2}, \bar{0}) - (\bar{6}, \bar{0}) - (\bar{6}, \bar{1}) - (\bar{2}, \bar{0})$ is a cycle of length three in $\bar{\Gamma}(R \times \mathbb{Z}_2)$ and also in $\Gamma(R \times \mathbb{Z}_2)$.

3. Let $R = \mathbb{Z}_8$. Then $(\bar{2}, \bar{0}) - (\bar{4}, \bar{0}) - (\bar{6}, \bar{0}) - (\bar{2}, \bar{0})$ is a cycle of length three in $\bar{\Gamma}(R \times \mathbb{Z}_2)$, but it is not a cycle in $\Gamma(R \times \mathbb{Z}_2)$, although $\text{gr}(\Gamma(R \times \mathbb{Z}_2)) = 3$.

Now we give a result that characterizes when the girth of the extended graph $\bar{\Gamma}(R \times \mathbb{Z}_2)$ is infinite. Compare with Theorem 1.3.14.

Theorem 3.2.3 *The girth of $\bar{\Gamma}(R \times \mathbb{Z}_2)$ is infinite if and only if R is an integral domain.*

Proof. (\Leftarrow) If R is an integral domain, then $\bar{\Gamma}(R \times \mathbb{Z}_2) = \Gamma(R \times \mathbb{Z}_2)$ is a star graph with center $(0, \bar{1})$. This shows that the girth of $\bar{\Gamma}(R \times \mathbb{Z}_2)$ is infinite.

(\Rightarrow) If $\text{gr}(\bar{\Gamma}(R \times \mathbb{Z}_2)) = \infty$, then $\text{gr}(\Gamma(R \times \mathbb{Z}_2)) = \infty$ since $\Gamma(R \times \mathbb{Z}_2)$ is a subgraph of $\bar{\Gamma}(R \times \mathbb{Z}_2)$. By Theorem 1.3.14, $R \cong \mathbb{Z}_2 \times \mathbb{Z}_2$ or R is an integral

domain. If $R \cong \mathbb{Z}_2 \times \mathbb{Z}_2$, one can show that $((\bar{1}, \bar{0}), \bar{0}) - ((\bar{0}, \bar{1}), \bar{0}) - ((\bar{0}, \bar{0}), \bar{1}) - ((\bar{0}, \bar{1}), \bar{1}) - ((\bar{1}, \bar{0}), \bar{0})$ is a cycle of length four in $\bar{\Gamma}(R \times \mathbb{Z}_2)$, a contradiction. Therefore, R is an integral domain. ■

Note that Example 1.3.11 shows that, when $R \cong \mathbb{Z}_2 \times \mathbb{Z}_2$, the graph $\Gamma(R \times \mathbb{Z}_2)$ has no cycles. It is also worth noting that the only case in which $\text{gr}(\bar{\Gamma}(R \times M))$ can be four is when $M = \mathbb{Z}_2$ and R does not satisfy any of the conditions of the above results.

We end this section with a result concerning the particular case $M = R$. Recall that the set of nonzero zero-divisors of $R \times R$ is $Z(R \times R)^* = T_1 \cup T_2$, where $T_1 = \{(0, r) | r \in R \setminus \{0\}\}$ and $T_2 = \{(a, r) | a \in Z(R)^* \text{ and } r \in R\}$ (see [16, Proposition 1.1]). Note that if $|R| = 2$ or $|R| = 3$, then $\bar{\Gamma}(R \times R) = \Gamma(R \times R)$, but there are examples of rings R with $|R| \geq 4$ such that $\bar{\Gamma}(R \times R) \neq \Gamma(R \times R)$ (see Section 3.1). Nevertheless, one can show easily that the girth of both $\bar{\Gamma}(R \times R)$ and $\Gamma(R \times R)$ are always the same. Namely, we have the following result.

Corollary 3.2.4 *The following assertions hold.*

1. If $|R| = 2$ or $|R| = 3$, then $\text{gr}(\Gamma(R \times R)) = \text{gr}(\bar{\Gamma}(R \times R)) = \infty$.
2. If $|R| \geq 4$, then $\text{gr}(\Gamma(R \times R)) = \text{gr}(\bar{\Gamma}(R \times R)) = 3$.

3.3 Diameter of extended zero-divisor graphs of idealizations

In this section, we study the diameter of extended zero-divisor graphs of idealizations.

We start with a result that determines when $\bar{\Gamma}(R \times M)$ is a complete graph (i.e., where the diameter of $\bar{\Gamma}(R \times M)$ is at most one).

Recall that, for a nonzero nilpotent element r of R , we use n_r to denote the index of nilpotency of r . The following sets are used in the characterization of a complete extended zero-divisor graph (see Corollary 2.2.4): $\bar{Z}(R) := \{x^{n_x-1} | x \in \text{Nil}(R) \setminus \{0\}\}$ and $\bar{Z}(R)^2 := \{x^{n_x-1}y^{n_y-1} | x, y \in \text{Nil}(R) \setminus \{0\}\}$. Namely, it is proved that when $\bar{\Gamma}(R) \neq \Gamma(R)$, $\bar{\Gamma}(R)$ is complete if and only if $Z(R) = \text{Nil}(R)$ and $\bar{Z}(R)^2 = \{0\}$.

The proof of the desired result uses the following lemma. Note that for an $(r, m) \in R \times M$ and an $n \in \mathbb{N}$, $(r, m)^n = (r^n, nr^{n-1}m)$. Then, one can easily show that $n_{(r,m)}$ is equal to either n_r or $n_r + 1$.

Lemma 3.3.1 $\overline{Z}(R \times M)^2 = \{0\}$ if and only if $\overline{Z}(R)^2 = \{0\}$ and $r^{n_r-1}M = 0$ for all $r \in \text{Nil}(R) \setminus \{0\}$.

Proof. (\Rightarrow) Assume that $\overline{Z}(R \times M)^2 = \{0\}$. Let $r, s \in \text{Nil}(R) \setminus \{0\}$. Then $(0, 0) = (r, 0)^{n_{(r,0)}-1}(s, 0)^{n_{(s,0)}-1}$, which implies that $r^{n_r-1}s^{n_s-1} = 0$. Therefore, $\overline{Z}(R)^2 = \{0\}$. Now, let $r \in \text{Nil}(R) \setminus \{0\}$ and $m \in M \setminus \{0\}$. Then $(0, 0) = (r, 0)^{n_r-1}(0, m) = (r^{n_r-1}, 0)(0, m)$. Hence, $r^{n_r-1}m = 0$; so $r^{n_r-1}M = 0$.

(\Leftarrow) First note that the condition “ $r^{n_r-1}M = 0$ for all $r \in \text{Nil}(R) \setminus \{0\}$ ” implies that $n_{(r,n)} = n_r$ for every $(r, n) \in R \times M$ with $r \neq 0$.

Now consider $(r, n)^{n_{(r,n)}-1}(s, m)^{n_{(s,m)}-1} \in \overline{Z}^2(R \times M)$ with $r, s \in \text{Nil}(R)$ and $n, m \in M$ such that (r, n) and (s, m) are nonzero. We claim that

$$(r, n)^{n_{(r,n)}-1}(s, m)^{n_{(s,m)}-1} = (0, 0).$$

The case where $r = s = 0$ is clear. If $r \neq 0$ and $s = 0$, then

$$(r, n)^{n_{(r,n)}-1}(0, m) = (r^{n_r-1}, (n_r - 1)r^{n_r-2}n)(0, m) = (0, 0).$$

If $r \neq 0$ and $s \neq 0$, then also $(r, n)^{n_{(r,n)}-1}(s, m)^{n_{(s,m)}-1} = (0, 0)$ since $\overline{Z}(R)^2 = \{0\}$ and $r^{n_r-1}M = 0$. ■

Now we give the desired result. Note that it treats only the case where $\overline{\Gamma}(R \times M) \neq \Gamma(R \times M)$, since the other case corresponds to the classical one which was already studied in Theorem 1.2.8.

Theorem 3.3.2 Assume that $\overline{\Gamma}(R \times M) \neq \Gamma(R \times M)$. Then $\overline{\Gamma}(R \times M)$ is complete if and only if the following conditions hold.

1. $Z(R) = Z(M) = \text{Nil}(R)$.
2. $\overline{Z}(R)^2 = \{0\}$.
3. $r^{n_r-1}M = 0$ for all $r \in \text{Nil}(R) \setminus \{0\}$.

Proof. (\Rightarrow) Assume $\overline{\Gamma}(R \times M)$ to be complete and $\overline{\Gamma}(R \times M) \neq \Gamma(R \times M)$. Then the assertions (2) and (3) follow by Lemma 3.3.1, and by Corollary 2.2.4, $Z(R \times M) = \text{Nil}(R \times M)$ and $\overline{Z}(R \times M)^2 = \{0\}$. Then $Z(R) \cup Z(M) = \text{Nil}(R)$ (since $\text{Nil}(R \times M) = \text{Nil}(R) \times M$ and $Z(R \times M) = (Z(R) \cup Z(M)) \times M$). Thus $Z(R) = \text{Nil}(R)$ and $Z(M) \subseteq \text{Nil}(R)$. The converse inclusion $\text{Nil}(R) \subseteq Z(M)$ follows using the assertion 3.

(\Leftarrow) This follows by Lemma 3.3.1 and Corollary 2.2.4. ■

As examples of idealizations that satisfy the conditions of Theorem 3.3.2, one may consider the rings $\mathbb{Z}_8 \times \mathbb{Z}_2$ and $\mathbb{Z}_2[X]/(X^3) \times \mathbb{Z}_2$.

As a simple consequence of Theorem 3.3.2, we get the following result.

Corollary 3.3.3 *If $\bar{\Gamma}(R \times M) \neq \Gamma(R \times M)$ and $\bar{\Gamma}(R \times M)$ is complete, then $\bar{\Gamma}(R)$ is complete.*

In [16], the authors posed a question concerning the characterization of idealizations $R \times M$ of diameter 2. Later this question was resolved (see Theorem 1.2.10). Namely, $\text{diam}(\Gamma(R \times M)) \leq 2$ if and only if, for all $x, y \in Z(R) \cup Z(M)$, either (1) there is a $z \in \text{Ann}(M) \setminus \{0\}$ such that $x, y \in \text{Ann}(z)$, or (2) there is an $m \in M \setminus \{0\}$ such that $x, y \in \text{Ann}_R(m)$. Here, we give the following “extended” counterpart of this result.

Theorem 3.3.4 *$\text{diam}(\bar{\Gamma}(R \times M)) \leq 2$ if and only if, for all $x, y \in Z(R) \cup Z(M)$, one of the following two assertions holds.*

1. *There is a $z \in \text{Ann}(M) \setminus \{0\}$ such that $x, y \in \sqrt{\text{Ann}(z)}$.*
2. *There is an $m \in M \setminus \{0\}$ such that $x, y \in \sqrt{\text{Ann}_R(m)}$.*

Proof. (\Rightarrow) Let $x, y \in Z(R) \cup Z(M)$, and suppose that (2) does not hold. We may assume that $x \neq 0$ and $y \neq 0$. First suppose that $x \in Z(M)$. Then $xm = 0$ for some $m \in M \setminus \{0\}$, and thus $y^\beta m \neq 0$ for every $\beta \in \mathbb{N} \setminus \{0\}$. Then $(x, m), (y, 0) \in Z(R \times M)^*$ are distinct and not adjacent in $\bar{\Gamma}(R \times M)$; so there is a $(z, b) \in Z(R \times M)^*$ adjacent to both (x, m) and $(y, 0)$ in $\bar{\Gamma}(R \times M)$. Hence, there are $\alpha, \beta, \gamma \in \mathbb{N} \setminus \{0\}$ such that

$$(x, m)^\alpha (z, b)^\gamma = 0 = (y, 0)^\beta (z, b)^\gamma,$$

with $(x, m)^\alpha \neq 0$, $(y, 0)^\beta \neq 0$ and $(z, b)^\gamma \neq 0$. This implies that $x, y \in \sqrt{\text{Ann}(z^\gamma)}$. It remains to prove that $z^\gamma \in \text{Ann}(M) \setminus \{0\}$. We have $z^\gamma \neq 0$. Indeed, consider the element $b' \in M$ which satisfies $(z, b)^\gamma = (z^\gamma, b')$, and assume that $z^\gamma = 0$. Then, $b' \neq 0$ and $x^\alpha b' = 0 = y^\beta b'$. This is absurd since (2) does not hold. Now, for all $n \in M$, $x^\alpha (z^\gamma n) = y^\beta (z^\gamma n) = 0$. Since (2) does not hold, $z^\gamma n = 0$. This shows that $z^\gamma \in \text{Ann}(M) \setminus \{0\}$, as desired.

Assume now that $x \in Z(R)$. Then we get the result using the same argument as above for $(x, 0), (y, m) \in Z(R \times M)^*$ with $m \in M \setminus \{0\}$.

(\Leftarrow) The proof is similar to the proof of Theorem 1.2.10. For completeness, we give a proof here. Suppose that for every $x, y \in Z(R) \cup Z(M)$, either (1) or (2) holds. Consider two distinct elements $(x, a), (y, b) \in Z(R \times M)^*$ which are not adjacent in $\bar{\Gamma}(R \times M)$. If assertion (1) holds, then there is a

nonzero $z \in \sqrt{\text{Ann}(M)}$ such that $(z, 0)$ is adjacent to both (x, a) and (y, b) ; and if assertion (2) holds, then there is a nonzero $m \in M$ such that $(0, m)$ is adjacent to both (x, a) and (y, b) . Therefore $\text{diam}(\overline{\Gamma}(R \times M)) \leq 2$. ■

One can show that $R = \mathbb{Z}_{12} \times \mathbb{Z}_3$ satisfies the conditions of Theorem 3.3.4. Then, $\text{diam}(\overline{\Gamma}(R \times M)) = 2$; however, $\text{diam}(\Gamma(R \times M)) = 3$. It is worth noting that, as in [13, Remark 3.10 (a)], if $\text{diam}(\overline{\Gamma}(R \times M)) \leq 2$ then one can show that either $Z(R) \subseteq Z(M)$ or $Z(M) \subseteq Z(R)$.

We end this chapter with a result giving the diameter of the particular case $R \times R$. Note that when R is an integral domain, $\overline{\Gamma}(R \times R) = \Gamma(R \times R)$ and $Z(R \times R) = \{(0, r) | r \in R\}$. So in this case, the result is known. Namely, the following two cases occur:

1. If $|R| = 2$, then $\text{diam}(\Gamma(R \times R)) = 0$. In this case, $\Gamma(R \times R)$ is a complete graph.
2. If $|R| \geq 3$, then $\text{diam}(\Gamma(R \times R)) = 1$. In this case, $\Gamma(R \times R)$ is also a complete graph.

Thus, here, only the case where R is not an integral domain is of interest.

Proposition 3.3.5 *If R is not an integral domain, then $\text{diam}(\overline{\Gamma}(R \times R)) \geq 2$.*

Proof. Let $(0, 1), (z, 0) \in R \times R$, where $z \in Z(R)^*$. Then, $(0, 1)(z, 0) = (0, z) \neq (0, 0)$. Hence, $d((0, 1), (z, 0)) \geq 2$. Therefore, $\text{diam}(\overline{\Gamma}(R \times R)) \geq 2$. ■

We conclude with an example of a ring R such that $\text{diam}(\overline{\Gamma}(R \times R)) = 3$. Let $R = \mathbb{Z}_{30}$. In $\overline{\Gamma}(R \times R)$, $d((\overline{2}, 0), (\overline{5}, 0)) = 3$ (consider the following path $(\overline{2}, 0) - (\overline{15}, 0) - (\overline{6}, 0) - (\overline{5}, 0)$). Then, $\text{diam}(\overline{\Gamma}(R \times R)) = 3$. Note that since R is reduced, $\overline{\Gamma}(R \times R) \neq \Gamma(R \times R)$ (by Corollary 3.1.3).

Comparison of some extensions of zero-divisor graphs

Extended zero-divisor graph and annihilator graph of a ring are two extensions of the classical zero-divisor graph. In this chapter, we investigate the relation between these graphs.

4.1 Main results

In [17], the annihilator graph was introduced as an extension of the classical zero-divisor graph (see Chapter 1). Thus, in the same context, one would naturally introduced an extension $\overline{AG}(R)$ of the annihilator graph $AG(R)$ as follows: vertices x and y are adjacent in $\overline{AG}(R)$ whenever there exist two non-negative integers n and m such that $\text{Ann}_R(x^n y^m) \neq \text{Ann}_R(x^n) \cup \text{Ann}_R(y^m)$. However, because of the following result, we deduce that $\overline{AG}(R)$ and $AG(R)$ coincide.

Proposition 4.1.1 *Let R be a commutative ring, x and y two nonzero zero-divisors. If $\text{Ann}_R(xy) = \text{Ann}_R(x)$ then $\text{Ann}_R(x^n y^m) = \text{Ann}_R(x^n)$ for every $n, m \in \mathbb{N}^*$.*

Proof. Assume that $\text{Ann}_R(xy) = \text{Ann}_R(x)$ with x and $y \in Z(R)^*$. Let $n, m \in \mathbb{N}^*$, show that $\text{Ann}_R(x^n y^m) \subset \text{Ann}_R(x^n)$. Let $r \in \text{Ann}_R(x^n y^m)$, then $rx^n y^m = 0$, hence $rx^{n-1} y^{m-1} \in \text{Ann}_R(xy) = \text{Ann}_R(x)$; therefore $rx^n y^{m-1} = 0$ and hence $rx^n = 0$ since $rx^{n-1} \in \text{Ann}_R(xy) = \text{Ann}_R(x)$ hence the result. ■

As a surprising fact we show that the extended graph $\overline{\Gamma}(R)$ is in fact a partial graph of the annihilator graph $AG(R)$, and so $\Gamma(R) \subset \overline{\Gamma}(R) \subset AG(R)$.

Theorem 4.1.2 *Let R be a commutative ring. Then $\overline{\Gamma}(R)$ is a subgraph of $AG(R)$.*

This work is submitted for publication (in collaboration with M. Bataineh, D. Bennis and J. Mikram).

Proof. Suppose that $x - y$ is an edge of $\bar{\Gamma}(R)$, then there are $n, m \in \mathbb{N}^*$ such that $x^n \neq 0, y^m \neq 0$ and $x^n y^m = 0$. Assume that $x - y$ is not an edge of $AG(R)$, then $\text{Ann}_R(xy) = \text{Ann}_R(x) \cup \text{Ann}_R(y)$. Hence by Lemma 1.4.1 $\text{Ann}_R(xy) = \text{Ann}_R(x)$ or $\text{Ann}_R(xy) = \text{Ann}_R(y)$. Without loss of generality, we may assume that $\text{Ann}_R(xy) = \text{Ann}_R(x)$. Since $x^n y^m = 0$, we have $x^{n-1} y^{m-1} \in \text{Ann}_R(xy) = \text{Ann}_R(x)$. Hence $x^n y^{m-1} = 0$, then $x^{n-1} y^{m-2} \in \text{Ann}_R(xy) = \text{Ann}_R(x)$. Then, $x^n y^{m-2} = 0$, and so $x^n y = 0, x^{n-1} \in \text{Ann}_R(xy) = \text{Ann}_R(x)$. Therefore, $x^n = 0$, a contradiction. ■

In [17] Badawi investigated when $\Gamma(R) = AG(R)$. In our context, we investigate when $\bar{\Gamma}(R) = AG(R)$ under the condition $\Gamma(R) \neq \bar{\Gamma}(R)$.

We start with a necessary condition.

Proposition 4.1.3 *Let R be a commutative ring. If $AG(R) = \bar{\Gamma}(R)$, then for all $x \in Z(R) \setminus \text{Nil}(R)$ and for every $n \in \mathbb{N}^*$, we have $\text{Ann}_R(x^n) = \text{Ann}_R(x^{n+1})$.*

Proof. Suppose there is an element $x \in Z(R) \setminus \text{Nil}(R)$ such that $\text{Ann}_R(x^n) \neq \text{Ann}_R(x^{n+1})$ for some $n \in \mathbb{N}^*$. Then, $x - x^n$ is an edge of $AG(R)$ (since $\text{Ann}(x) \subseteq \text{Ann}(x^n)$). Hence, $x - x^n$ is an edge of $\bar{\Gamma}(R)$ (since $AG(R) = \bar{\Gamma}(R)$). Therefore, $x \in \text{Nil}(R)$, a contradiction. ■

Corollary 4.1.4 *Let R be a commutative ring. If there is an element $x \in Z(R) \setminus \text{Nil}(R)$ such that $\text{Ann}_R(x^n) \neq \text{Ann}_R(x^{n+1})$ for some $n \in \mathbb{N}^*$. Then $AG(R) \neq \bar{\Gamma}(R) \neq \Gamma(R)$.*

Examples 4.1.5 1. *If $R = \mathbb{Z}_{12}$, then $2 - 10$ is an edge of $AG(R)$ but is not an edge of $\bar{\Gamma}(R)$.*

We have $\Gamma(R) \subsetneq \bar{\Gamma}(R) \subsetneq AG(R)$, $\text{gr}(\Gamma(R)) = \text{gr}(\bar{\Gamma}(R)) = 4$, $\text{gr}(AG(R)) = 3$, $\text{diam}(\Gamma(R)) = 3$ and $\text{diam}(\bar{\Gamma}(R)) = \text{diam}(AG(R)) = 2$.

2. *If $R = \mathbb{Z}_{24}$, then $2 - 4$ is an edge of $AG(R)$ but is not an edge of $\bar{\Gamma}(R)$.*

We have $\Gamma(R) \subsetneq \bar{\Gamma}(R) \subsetneq AG(R)$, $\text{gr}(\Gamma(R)) = \text{gr}(\bar{\Gamma}(R)) = \text{gr}(AG(R)) = 3$, $\text{diam}(\Gamma(R)) = 3$ and $\text{diam}(\bar{\Gamma}(R)) = \text{diam}(AG(R)) = 2$.

Before giving a characterization of when $AG(R) = \bar{\Gamma}(R)$, we give some particular simple cases of rings over which $AG(R) = \bar{\Gamma}(R)$.

Proposition 4.1.6 *Let R be a non reduced quasi-local commutative ring with maximal ideal $\text{Nil}(R)$ such that $|\text{Nil}(R)^*| \geq 2$ and $x^{n_x-1} y^{n_y-1} = 0$ for each $x, y \in \text{Nil}(R)^*$. Then $AG(R) = \bar{\Gamma}(R)$ is a complete graph.*

Proof. The result follows since $\bar{\Gamma}(R)$ is a complete graph (by Theorem 2.2.3). ■

Proposition 4.1.7 *Let R be a ring and M an R -module with $Z(R) = Z(M) = \text{Nil}(R)$. Then $AG(R \times M)$ is a complete graph.*

Proof. If $Z(R) = Z(M) = \text{Nil}(R)$, then $Z(R \times M) = (Z(R) \cup Z(M)) \times M = \text{Nil}(R) \times M = \text{Nil}(R \times M)$; therefore $AG(R \times M)$ is a complete graph. ■

Proposition 4.1.8 *Let R be a commutative ring with $\Gamma(R) \neq \bar{\Gamma}(R)$. Then the following statements are equivalent:*

1. $\text{gr}(AG(R)) = 4$.
2. $AG(R) = \bar{\Gamma}(R) = K^{2,3}$ is a complete bipartite graph of R .

Proof. (1) \Rightarrow (2) Since $\text{gr}(AG(R)) = 4$ and $\Gamma(R) \neq \bar{\Gamma}(R)$, R is ring-isomorphic to either $\mathbb{Z}_2 \times \mathbb{Z}_4$ or $\mathbb{Z}_2 \times \mathbb{Z}_2[X]/(X^2)$ (by Theorem 1.4.4). Then $AG(R) = \bar{\Gamma}(R) = K^{2,3}$.

(2) \Rightarrow (1) It is clear. ■

Proposition 4.1.9 *Let R be a commutative ring such that $\Gamma(R) \neq \bar{\Gamma}(R)$. Then $\Gamma(R)$ is a star graph if and only if $AG(R) = \bar{\Gamma}(R) = K^3$.*

Proof. (\Rightarrow) Suppose that $\Gamma(R)$ is a star graph and $\Gamma(R) \neq \bar{\Gamma}(R)$, Then $AG(R) = \bar{\Gamma}(R) = K^3$ by Theorem 1.4.7.

(\Leftarrow) By Theorem 1.4.7, $\Gamma(R) = K^{1,2}$. Therefore, $\Gamma(R)$ is a star graph. ■

Corollary 4.1.10 *Let R be a non reduced commutative ring with $|Z(R)^*| \geq 2$. Then $AG(R)$ is a star graph if and only if $AG(R) = \bar{\Gamma}(R) = \Gamma(R) = K^{1,1}$ or $K^{1,\infty}$.*

Now we give the main result of this chapter which characterizes the case when $AG(R) = \bar{\Gamma}(R)$ of a commutative ring R .

Theorem 4.1.11 *Let R be a commutative ring with $\Gamma(R) \neq \bar{\Gamma}(R)$. Then $AG(R) = \bar{\Gamma}(R)$ if and only if either:*

- $\text{gr}(AG(R)) = 4$; or
- If $Z(R) = \text{Nil}(R)$, then $x^{n_x-1}y^{n_y-1} = 0$ for each $x, y \in \text{Nil}(R)^*$; or
- If $Z(R) \neq \text{Nil}(R)$, then the following conditions hold.
 - For each $x, y \in \text{Nil}(R)^*$ distinct, we have $x^{n_x-1}y^{n_y-1} = 0$, and

- If there are $z \in Z(R) \setminus \text{Nil}(R)$ and $t \in Z(R)^*$ distinct with $z^n t^m \neq 0$ for all $n, m \in \mathbb{N}^*$, we have $\text{Ann}_R(z) = \text{Ann}_R(z)$ or $\text{Ann}_R(z) = \text{Ann}_R(t)$.

Proof. (\Leftarrow) If $\text{gr}(AG(R)) = 4$, then $AG(R) = \bar{\Gamma}(R)$ (by Proposition 4.1.8). If $Z(R) = \text{Nil}(R)$ and $x^{n_x-1}y^{n_y-1} = 0$ for each $x, y \in \text{Nil}(R)^*$, then $\bar{\Gamma}(R)$ is a complete graph (by Theorem 2.2.3). Then, $AG(R) = \bar{\Gamma}(R)$.

Now, assume that $Z(R) \neq \text{Nil}(R)$. Let $x - y$ be an edge of $AG(R)$. Then, if $x, y \in \text{Nil}(R)^*$, we have $x^{n_x-1}y^{n_y-1} = 0$, hence $x - y$ is an edge of $\bar{\Gamma}(R)$ by Theorem 2.2.3.

Now, let $x \in Z(R) \setminus \text{Nil}(R)$ and $y \in Z(R)^*$. Suppose that $x - y$ is not an edge of $\bar{\Gamma}(R)$. Then, $x^n y^m \neq 0$ for all $n, m \in \mathbb{N}^*$, hence $\text{Ann}_R(xy) = \text{Ann}_R(x)$ or $\text{Ann}_R(xy) = \text{Ann}_R(y)$, a contradiction.

(\Rightarrow) Assume that $AG(R) = \bar{\Gamma}(R)$ with $\Gamma(R) \neq \bar{\Gamma}(R)$. Then $\text{gr}(AG(R)) \in \{3, 4\}$ by Corollary 1.4.5. If $\text{gr}(AG(R)) = 4$ there is nothing to prove. If $\text{gr}(AG(R)) = 3$. Then, there are two cases: $\text{diam}(AG(R)) = 1$ or $\text{diam}(AG(R)) = 2$ (by Theorem 1.4.2).

If $\text{diam}(AG(R)) = 1$, $\bar{\Gamma}(R)$ is a complete graph, and hence $Z(R) = \text{Nil}(R)$ and for every $x, y \in \text{Nil}(R)^*$, $x^{n_x-1}y^{n_y-1} = 0$ (by Theorem 2.2.3), as desired.

If $\text{diam}(AG(R)) = 2$, then $Z(R) \neq \text{Nil}(R)$; otherwise $AG(R)$ is complete (by Theorem 1.4.6), a contradiction. Using the (induced) subgraph, $AG_N(R)$, of $AG(R)$ with vertices $\text{Nil}(R)^*$, $AG_N(R) = \bar{\Gamma}_N(R)$ is complete (by Theorem 1.4.6). Hence, for every $x, y \in \text{Nil}(R)^*$, we have $x^{n_x-1}y^{n_y-1} = 0$. Since $\text{diam}(AG(R)) = 2$, there are $z \in Z(R) \setminus \text{Nil}(R)$ and $t \in Z(R)^*$ such that $z - t$ is not an edge of $\bar{\Gamma}(R) = AG(R)$, hence $z^n t^m \neq 0$ for all $n, m \in \mathbb{N}^*$ and $\text{Ann}_R(z) = \text{Ann}_R(z)$ or $\text{Ann}_R(z) = \text{Ann}_R(t)$. We are done. \blacksquare

4.2 Annihilator graph of some ring constructions

We study the annihilator graph of the finite direct products of rings and trivial extension.

The following result studies the annihilator graph of finite direct products of rings.

Theorem 4.2.1 Let $R = \prod_{i=1}^n R_i$, where $(R_i)_{1 \leq i \leq n}$ is a finite family of rings with $n \in \mathbb{N}^* \setminus \{1\}$.

1. If $n = 2$, the following assertions hold:

- (a) If $R_1 \cong R_2 \cong \mathbb{Z}_2$, then $\Gamma(R) = \bar{\Gamma}(R) = AG(R)$, $\text{gr}(\Gamma(R)) = \infty$ and $\text{diam}(\Gamma(R)) = 1$.
- (b) If R_1 and R_2 are integral domains and one of them is isomorphic to \mathbb{Z}_2 , then $\text{gr}(\Gamma(R)) = \text{gr}(\bar{\Gamma}(R)) = \text{gr}(AG(R)) = \infty$ and $\text{diam}(\Gamma(R)) = \text{diam}(\bar{\Gamma}(R)) = \text{diam}(AG(R)) = 2$. In this case $\Gamma(R) = \bar{\Gamma}(R) = AG(R)$ is a star graph.
- (c) If R_1 and R_2 are integral domains with $|R_1| \geq 3$ and $|R_2| \geq 3$, then $\Gamma(R) = \bar{\Gamma}(R) = AG(R)$, $\text{diam}(\Gamma(R)) = 2$ and $\text{gr}(\Gamma(R)) = 4$. In this case $\Gamma(R)$ is a complete bipartite graph.
- (d) If R_1 or R_2 is not an integral domain, then $\text{gr}(\Gamma(R)) = \text{gr}(\bar{\Gamma}(R)) = \text{gr}(AG(R)) = 3$, $\text{diam}(\Gamma(R)) = \text{diam}(\bar{\Gamma}(R)) = 3$ and $\text{diam}(AG(R)) = 2$.
- (e) If R_1 or R_2 is not an integral domain such that all zero-divisors are nilpotent in each ring with nonzero zero-divisors, then $\text{gr}(\Gamma(R)) = \text{gr}(\bar{\Gamma}(R)) = \text{gr}(AG(R)) = 3$, $\text{diam}(\Gamma(R)) = 3$ and $\text{diam}(\bar{\Gamma}(R)) = \text{diam}(AG(R)) = 2$.

2. If $n \geq 3$, then $\text{gr}(\Gamma(R)) = \text{gr}(\bar{\Gamma}(R)) = \text{gr}(AG(R)) = 3$, $\text{diam}(\Gamma(R)) = \text{diam}(\bar{\Gamma}(R)) = 3$ and $\text{diam}(AG(R)) = 2$.

Proof. 1. The proof of each (a) and (b) is trivial.

We prove the assertion (c). Suppose that R_1 and R_2 are integral domains, then $Z(R_1 \times R_2)^* = T_1 \cup T_2$ where $T_1 = \{(0, a); a \in R_2^*\}$ and $T_2 = \{(b, 0); b \in R_1^*\}$. So we have $\text{Ann}(0, a) = T_2$ for all $a \in R_2^*$ and $\text{Ann}(b, 0) = T_1$ for all $b \in R_1^*$. Then, $\Gamma(R)$ is a complete bipartite graph, therefore $\Gamma(R) = \bar{\Gamma}(R) = AG(R)$.

(d). We have $\text{gr}(\Gamma(R)) = \text{gr}(\bar{\Gamma}(R)) = \text{gr}(AG(R)) = 3$ and $\text{diam}(\Gamma(R)) = \text{diam}(\bar{\Gamma}(R)) = 3$ from Propositions 2.3.9 and 2.2.6. Since $\text{diam}(\bar{\Gamma}(R)) = 3$ and by Theorem 1.4.2, $\text{diam}(AG(R)) \leq 2$. It remains to show that $\text{diam}(AG(R)) \neq 1$. We may assume that R_2 is not an integral domain, then 1 and -1 are two regular elements of R_2 . Then $\text{Ann}((0, 1)(0, -1)) = \text{Ann}(0, 1) \cup \text{Ann}(0, -1)$, hence $(0, 1) - (0, -1)$ is not an edge of $AG(R)$, therefore $\text{diam}(AG(R)) = 2$.

(e) By Propositions 2.2.6 and 2.3.9 and the previous demonstration.

2. Since $(0, 0, 0, \dots, 1) - (0, 1, 1, 1, \dots, 1)$ is not an edge of $AG(R)$, then $\text{diam}(AG(R)) = 2$. The other assertions follow from Propositions 2.2.6 and 2.3.9. ■

As a consequence, we determine the girth and the diameter of $AG(\mathbb{Z}_n)$.

Corollary 4.2.2 Let $R = \mathbb{Z}_n$, where $n = \prod_{i=1}^k p_i^{\alpha_i}$ is the prime factorization of n with $k \in \mathbb{N}^*$ then:

1. If $n = 2^2$, then $\text{diam}(\bar{\Gamma}(R)) = \text{gr}(AG(R)) = 0$.
2. If $n = 2^m$ with $m > 2$ or $n = p^m$ with p is an odd prime integer and $m \geq 2$, then $\text{diam}(\bar{\Gamma}(R)) = 1$. In this case $\bar{\Gamma}(R) = AG(R)$ is a complete graph.
3. If $n = p^\alpha q^\beta$ with p and q are distinct primes, then $\text{diam}(\bar{\Gamma}(R)) = 2$.
Namely,
 - If $p = 2$ with $\alpha = \beta = 1$, then $\bar{\Gamma}(R) = AG(R)$ is a star graph.
 - If $p = 2$ with $\alpha = 2$ and $\beta = 1$ or if $n = pq$, then $\bar{\Gamma}(R) = AG(R)$ is a complete bipartite graph.
4. If $n = \prod_{i=1}^k p_i^{\alpha_i}$ with $p_i \neq p_j$ for $i \neq j$ and $k \geq 3$, then $\bar{\Gamma}(R) \neq AG(R)$,
 $\text{diam}(AG(R)) = 2$ and $\text{gr}(AG(R)) = 3$.

We end this chapter with a result that characterizes when $\bar{\Gamma}(R \times M)$ and $AG(R \times M)$ coincide in the case where R is an integral domain.

We need the following lemmas.

Lemma 4.2.3 *Let R be an integral domain and M an R -module. Then $(a, n) - (b, m)$ is not an edge of $AG(R \times M)$ if and only if $\text{Ann}_M(a) \cup \text{Ann}_M(b) = \text{Ann}_M(ab)$.*

Proof. We have $\text{Ann}_R(a, n) = \{(0, k) | k \in \text{Ann}_M(a)\}$, $\text{Ann}_R(b, m) = \{(0, t) | t \in \text{Ann}_M(b)\}$ and $\text{Ann}_R((a, n)(b, m)) = \text{Ann}_M(ab, am+bn) = \{(0, h) | h \in \text{Ann}_M(ab)\}$. Then $(a, n) - (b, m)$ is an edge of $AG(R \times M)$ if and only if $\text{Ann}_M(a) \cup \text{Ann}_M(b) \neq \text{Ann}_M(ab)$. ■

Lemma 4.2.4 *Let R be an integral domain and M an R -module. Then*

1. For $a \in R$, $\text{Ann}_M(a) = \text{Ann}_M(a^2)$ if and only if $(a, n) - (a, m)$ is not an edge of $AG(R \times M)$ for all $n, m \in M^*$.
2. If there is an element $a \in R$ such that $\text{Ann}_M(a) \neq \text{Ann}_M(a^2)$, then $\bar{\Gamma}(R \times M) \neq AG(R \times M)$.

Proof. (1) This is a particular case of the previous lemma where $a = b$.
(2) If there is an element $a \in R$ such that $\text{Ann}_M(a) \neq \text{Ann}_M(a^2)$, then, by (1), $(a, n) - (a, m)$ is an edge of $AG(R \times M)$ but it is not an edge of $\bar{\Gamma}(R \times M)$ since R is an integral domain, therefore $\bar{\Gamma}(R \times M) \neq AG(R \times M)$. ■

Theorem 4.2.5 *Let R be an integral domain and M an R -module. Then $\overline{\Gamma}(R \times M) = AG(R \times M)$ if and only if the following conditions hold.*

1. *For all $m \in M$ and $a \in R^*$, $\text{Ann}_R(m) = \text{Ann}_R(am)$ if $a \notin \sqrt{\text{Ann}_R(m)}$.*
2. *For all $a, b \in R$, $\text{Ann}_M(a) \cup \text{Ann}_M(b) = \text{Ann}_M(ab)$.*

Proof. (\Rightarrow) Assume that $\overline{\Gamma}(R \times M) = AG(R \times M)$, since R is an integral domain, then $Z(R \times M) = Z(M) \times M = T_1 \cup T_2$ such that $T_1 = \{(0, m) | m \in M\}$ and $T_2 = \{(a, n) | a \in R^*, n \in M \text{ and for some } m \in M^*, am = 0\}$.

(1) Let $(0, m) \in T_1$ and $(a, n) \in T_2$ where $(0, m) - (a, n)$ is not an edge of $\overline{\Gamma}(R \times M)$, then for all $\alpha \in \mathbb{N}^*$, $(a, n)^\alpha(0, m) \neq (0, 0)$, hence $a \notin \sqrt{\text{Ann}_R(m)}$. Since $\overline{\Gamma}(R \times M) = AG(R \times M)$, $(0, m) - (a, n)$ is not an edge of $AG(R \times M)$, hence $\text{Ann}((0, m)(a, n)) = \text{Ann}(0, am) = \text{Ann}(0, m) \cup \text{Ann}(a, n)$. We have $\text{Ann}(0, m) = \{(b, k) | b \in \text{Ann}_R(m), k \in M\}$; $\text{Ann}(a, n) = \{(0, k) | k \in \text{Ann}_M(a)\}$ and $\text{Ann}(0, am) = \{(b, k) | b \in \text{Ann}_R(am), k \in M\}$; therefore $\text{Ann}_R(m) = \text{Ann}_R(am)$.

(2) Let $(a, n), (b, m) \in T_2$. Then, $(a, n) - (b, m)$ is not an edge of $\overline{\Gamma}(R \times M)$ since R is an integral domain, hence $(a, n) - (b, m)$ is not an edge of $AG(R \times M)$ since $\overline{\Gamma}(R \times M) = AG(R \times M)$. Therefore, $\text{Ann}_M(a) \cup \text{Ann}_M(b) = \text{Ann}_M(ab)$, by Lemma 4.2.3.

(\Leftarrow) Suppose that $\overline{\Gamma}(R \times M) \neq AG(R \times M)$, then there is an edge $(a, n) - (b, m)$ of $AG(R \times M)$ that is not an edge of $\overline{\Gamma}(R \times M)$. Two cases occur:

Case 1: Without lost of generality, we may assume that $(b, m) \in T_1$ and $(a, n) \in T_2$. Then, for all $\alpha \in \mathbb{N}^*$, $a \notin \sqrt{\text{Ann}_R(m)}$, hence $\text{Ann}_R(m) = \text{Ann}_R(am)$ by hypothesis. Then $(a, n) - (b, m)$ is not an edge of $AG(R \times M)$, a contradiction.

Case 2: $(a, n), (b, m) \in T_2$, then $\text{Ann}_M(a) \cup \text{Ann}_M(b) = \text{Ann}_M(ab)$ hence $(a, n) - (b, m)$ is not an edge of $AG(R \times M)$ by Lemma 4.2.3 a contradiction.

■

Corollary 4.2.6 *Let R be an integral domain. Then $\Gamma(R \times R) = \overline{\Gamma}(R \times R) = AG(R \times R)$ is a complete graph.*

Examples of codes from extended zero-divisor graphs with large parameters

In this chapter, we present the first study on the codes based on zero-divisor graphs of commutative rings. Namely, we construct binary codes from incidence matrices of extended zero-divisor graphs of \mathbb{Z}_{p^α} with large parameters, where p is a prime number and α is a nonnegative integer.

5.1 Background and notation

The adjacency matrix $A = a_{ij}$ of a graph Γ is a $|V| \times |V|$ symmetric matrix defined as follows : $a_{ij} = 1$ if $\{x_i, x_j\} \in E$, and $a_{ij} = 0$ otherwise. An incidence matrix of Γ is an $|V| \times |E|$ matrix $G = (g_{v,e})$ with $g_{v,e} = 1$ if the vertex v is on the edge e and $g_{v,e} = 0$ otherwise.

If $\Gamma = (V, E)$ is a connected graph, then an *edge-cut* of Γ is a subset $S \subseteq E$ such that removing the edges in S renders the new graph $\Gamma - S$ disconnected. The *edge-connectivity* of Γ , denoted by $\lambda(\Gamma)$, is the minimum cardinality of an edge-cut of Γ . A *bridge* of a connected graph is an edge whose removal disconnects the graph. So Γ has a bridge if and only if $\lambda(\Gamma) = 1$. For every graph, we have $\lambda(\Gamma) \leq \delta(\Gamma)$, where $\delta(\Gamma)$ the minimum degrees of vertices of Γ . In particular, if Γ is connected k -regular, $k \geq 2$, then $\lambda(\Gamma) \leq k$. And Γ is maximally edge-connected if $\lambda(\Gamma) = k$. If $\lambda(\Gamma) = \delta(\Gamma)$ and, in addition, the only edge sets of cardinality $\lambda(\Gamma)$ whose removal disconnects Γ are the sets of edges incident with a vertex of degree $\delta(\Gamma)$, then Γ is called *super- λ* .

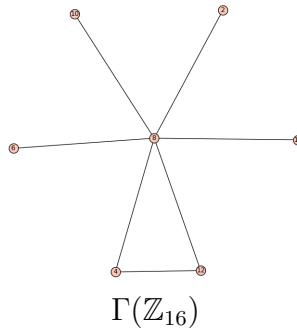
Let us observe an example of the classical zero-divisor graph of the ring \mathbb{Z}_n of the residues modulo a nonnegative integer n and its extended graph.

This work is in preparation (in collaboration with K. Abdelmoumen, D. Bennis and J. Mikram).

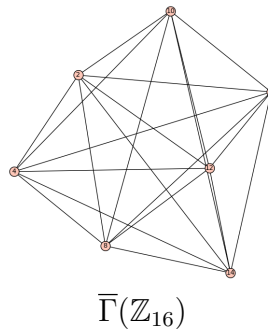
Example 5.1.1 We consider $\mathbf{Z}(\mathbb{Z}_{16}) = \{0, 2, 4, 6, 8, 10, 12, 14\}$.

We use the free mathematics software system Sagemath** to draw graphs.

$\Gamma(\mathbb{Z}_{16})$ is the classical zero-divisor graph of \mathbb{Z}_{16} :



$\bar{\Gamma}(\mathbb{Z}_{16})$ is the extended graph of $\Gamma(\mathbb{Z}_{16})$:



5.2 Linear binary codes

We recall some basic notions of the coding theory (see [30] and [37]).

A linear binary code \mathcal{C} of length n and dimension k (denoted by $[n, k]$), is a k -dimensional linear subspace of the n -dimensional binary vector space \mathbb{F}_2^n . A generator matrix for a linear code \mathcal{C} is a matrix whose rows form a basis for \mathcal{C} . The dual code \mathcal{C}^\perp of an $[n, k]$ code \mathcal{C} is the $[n, n - k]$ code being the orthogonal space of \mathcal{C} with respect to the inner product in \mathbb{F}_2^n . Any generator matrix of \mathcal{C}^\perp is called a parity check matrix of \mathcal{C} . The Hamming distance between two vectors $x = (x_1, \dots, x_n)$, $y = (y_1, \dots, y_n)$ (denoted by $d_H(x, y)$) is equal to the

**www.doc.sagemath.org/html/en/reference/index.html

number of its number of indices i such that $x_i \neq y_i$. The Hamming distance is a metric on \mathbb{F}_2^n . The Hamming weight of a vector is the number of its nonzero coordinates. The smallest distance between pairs of distinct code words is called a minimum distance of a code (denoted by $d_H(\mathcal{C})$ or simply d). A code of minimum distance d can detect $d - 1$ errors and can correct up to $\lfloor \frac{d-1}{2} \rfloor$ errors. An $[n, k, d]$ code is a binary linear code with length n , dimension k and minimum distance d . The quality of an $[n, k, d]$ code is indicated by the Information Rate $\frac{k}{n}$, the relative minimum distance $\frac{d}{n}$ and the complexity of the encoding and decoding procedures. The goal of coding theory is to provide codes with high information rate, high error-correction rate and low complexity of encoding and decoding.

5.3 Codes from incidence matrices of graphs

Binary codes generated by the rows of $|V| \times |E|$ incidence matrices of a connected graph $\Gamma = (V, E)$ were investigated in [22]. It was shown that certain properties of the binary codes can be directly derived from properties of the graphs implying edge-connectivity. In this section we present some of these result.

Let $\Gamma = (V, E)$ be an undirected graph with no loops and no multiple edges. The code of a graph Γ over a finite field \mathbb{F}_q with q elements, where q is a power prime, is the row span of an adjacency matrix A over the field \mathbb{F}_q , denoted by $C_{\mathbb{F}_q}(\Gamma)$ or $C_{\mathbb{F}_q}(A)$. The dimension of this code is the rank of A over \mathbb{F}_q , also written of $\text{rank}_2(A)$ if $q = 2$, in which case we will speak of 2-rank of A or Γ , and write $C_2(\Gamma)$ or $C_2(A)$. If G is an incidence matrix for Γ , $C_{\mathbb{F}_q}(G)$ denotes the row span of G over \mathbb{F}_q , also written $C_2(G)$ in the binary case.

Result 1 ([22], Result 1) *Let $\Gamma = (V, E)$ be a connected graph, G an incidence matrix for Γ , and $C_p(G)$ the row-span of G over Galois field \mathbb{F}_p where p is prime. Then $\dim(C_2(G)) = |V| - 1$. For odd p , $\dim(C_p(G)) = |V|$ if Γ has a closed path of odd length (i.e., if Γ is not bipartite), and $\dim(C_p(G)) = |V| - 1$ if Γ has a no closed path of odd length (i.e., if Γ is bipartite).*

We consider now the binary code $C_2(G)$ of a connected graph Γ .

Theorem 5.3.1 ([22], Theorem 1) *Let $\Gamma = (V, E)$ be a connected graph, G an incidence matrix for Γ . Then*

1. $C_2(G)$ is of length $|E|$, dimension $|V| - 1$ and minimum Hamming distance $\lambda(\Gamma)$.

2. if Γ is super- λ , then $C_2(G)$ is of length $|E|$, dimension $|V| - 1$, minimum Hamming distance $\delta(\Gamma)$, and the minimum words are the rows of G of weight $\delta(\Gamma)$.

Now we present the Result 2 of [22]. It will be used to construct an example of codes from extended zero-divisor graphs.

Result 2 ([22], Result 2) *Let $\Gamma = (V, E)$ be a connected graph and $\lambda(\Gamma) = k$ if one of the following conditions hold:*

1. Γ has diameter at most 2, i.e., any two vertices of Γ are adjacent or have a neighbour in common.
2. $k \geq \frac{(|V(\Gamma)| - 1)}{2}$.
3. Γ has girth gr and $\text{diam}(\Gamma) \leq \text{gr}(\Gamma) - 1$ if $\text{gr}(\Gamma)$ is odd, or $\text{diam}(\Gamma) \leq \text{gr}(\Gamma) - 2$ if $\text{gr}(\Gamma)$ is even.

Further, Γ is super- λ if one of the following conditions are satisfied :

- (i) Γ has diameter at most 2, and in addition Γ has no complete subgraph of order k ;
- (ii) $k \geq \frac{(|V(\Gamma)| + 1)}{2}$;
- (iii) Γ has girth gr and $\text{diam}(\Gamma) \leq \text{gr}(\Gamma) - 1$ if $\text{gr}(\Gamma)$ is odd, or $\text{diam}(\Gamma) \leq \text{gr}(\Gamma) - 2$ if $\text{gr}(\Gamma)$ is even.

These results, together with Theorem 5.3.1, imply the following result.

Corollary 5.3.2 ([22], Corollary 1) *Let $\Gamma = (V, E)$ be a connected k -regular graph on $|V| = n$ vertices, G an $n \times \frac{nk}{2}$ incidence matrix for Γ . If any of the conditions 1 – 3 of Result 2 holds, then the binary code $C_2(G)$ has minimum weight k . If any one of the conditions (i) – (iii) holds, then the only words of weight k are the rows of the incidence matrix.*

5.4 Examples of codes from extended zero-divisor graphs with large parameters

The main result of this section is to construct examples of codes from extended zero-divisor graphs with large parameters. We are interested in codes

from zero-divisor graphs of the rings \mathbb{Z}_n , more precisely the prime ring \mathbb{Z}_{p^α} where p is a prime integer and α is a nonnegative integer.

In the following result, p would be a prime integer and $\alpha \geq 2$ an integer. Then, we recall some results on the zero-divisor graphs of the prime ring \mathbb{Z}_{p^α} which will be useful in the following.

Let us first give the following observations.

Proposition 5.4.1 *Let n be a positive integer.*

\mathbb{Z}_n is a field if and only if n is a prime number. In this case, $\Gamma(\mathbb{Z}_n)$ will be empty.

Let us now observe when we have $\alpha \geq 2$.

We denote by $(p) = \{mp \mid m \in \mathbb{Z}_{p^\alpha}\}$ the ideal of \mathbb{Z}_{p^α} generated by p .

Proposition 5.4.2 $Z(\mathbb{Z}_{p^\alpha}) = \text{Nil}(\mathbb{Z}_{p^\alpha}) = (p)$.

Consequently $|Z(\mathbb{Z}_{p^\alpha})| = p^{\alpha-1} - 1$.

In the following result, we give some example of codes from incidence matrices of classical and extended zero-divisor graphs of \mathbb{Z}_{p^α} . For that, we use Sagemath system.

Sagemath code

```
#Zero divisors of  $\mathbb{Z}_{p^\alpha}$ 
def zero_divisor(p,alpha):
    D=[]
    for k in range(1,p**alpha):
        for j in range(1,p**alpha):
            if k*j%p**alpha==0:
                D.append(j)
    return set(D)

#Parameters of code from incidence matrix of classical
# zero-divisor graph of  $\mathbb{Z}_{p^\alpha}$ .
#First we define the dictionary  $D$  of vertices and edges
# of classical zero-divisor graph of  $\mathbb{Z}_{p^\alpha}$ .
def Parameters_classical(p,alpha):
    D={}
    for k in zero_divisor(p,alpha):
        l=[]
        for j in zero_divisor(p,alpha):
            if k*j%p**alpha==0 and k!=j:
```

```

                                l.append(j)
                                D[k]=1
G=Graph(D)
MS=MatrixSpace(GF(2),len(G.vertices()),len(G.edges()))
M=MS(G.incidence_matrix())
C=LinearCode(M);
[n,k,d]=copy([C.length(),C.dimension(),C.minimum_distance()])
return [n,k,d]

#Parameters of extended zero-divisor graph of  $Z_p^\alpha$ 
#First we define the dictionary D of vertices and edges
# of extended zero-divisor graph of  $Z_p^\alpha$ .
def Parameters_extended(p,alpha):
    D={}
    for k in zero_divisor(p,alpha):
        l=[]
        for j in zero_divisor(p,alpha):
            if k!=j:
                l.append(j)
        D[k]=l
G=Graph(D)
MS=MatrixSpace(GF(2),len(G.vertices()),len(G.edges()))
M=MS(G.incidence_matrix())
C=LinearCode(M);
[n,k,d]=copy([C.length(),C.dimension(),C.minimum_distance()])
return [n,k,d]

```

By observing the table below, in which we have given the parameters of the two types of codes (Type I for classical zero-divisor graphs and type II for extended zero-divisor graphs), view that the zero-divisor graphs are connected and by the Theorem 5.3.1, we denoted an augmentation in the length and the minimum distance for codes of type II, whilst conserving the dimension but by decreasing the information rate.

CHAPTER 5. EXAMPLES OF CODES FROM EXTENDED
ZERO-DIVISOR GRAPHS WITH LARGE PARAMETERS

| p | α | parameters of Type I : $[n, k, d]$ | parameters of Type II : $[n, k, d]$ |
|-----|----------|------------------------------------|-------------------------------------|
| 2 | 3 | [2,2,1] | [3,2,2] |
| 2 | 4 | [7,6,1] | [21,6,6] |
| 2 | 5 | [23,14,1] | [105,14,14] |
| 2 | 6 | [61,30,1] | [465,30,30] |
| 3 | 3 | [13,7,2] | [28,7,7] |
| 3 | 4 | [64,25,2] | [325,25,25] |
| 5 | 3 | [86,23,4] | [276,23,23] |

Note that for all n nonnegative integers, if $\bar{\Gamma}(\mathbb{Z}_n) \neq \Gamma(\mathbb{Z}_n)$ we have the number of edges of $\bar{\Gamma}(\mathbb{Z}_n)$ is strictly higher than that of $\Gamma(\mathbb{Z}_n)$, they have the same vertices and $\lambda(\bar{\Gamma}(\mathbb{Z}_n)) \geq \lambda(\Gamma(\mathbb{Z}_n))$. (Since Γ is a subgraph of $\bar{\Gamma}$).

The following result gives the parameters of the binary code constructed from an incidence matrix of an extended zero-divisor graph of \mathbb{Z}_{p^α} .

Proposition 5.4.3 *Let p be a prime integer and $\alpha > 2$ an integer. Let $G_{p,\alpha}$ an incidence matrix for $\bar{\Gamma}(\mathbb{Z}_{p^\alpha})$. The binary code $C_2(G_{p,\alpha})$ is of length $\frac{(p^{\alpha-1}-1)(p^{\alpha-1}-2)}{2}$, dimension $p^{\alpha-1} - 2$, minimum Hamming distance $p^{\alpha-1} - 2$, and the minimum words are the rows of $G_{p,\alpha}$ of weight $p^{\alpha-1} - 2$.*

Proof. According to Proposition 5.4.3, we have $\bar{\Gamma}(\mathbb{Z}_{p^\alpha})$ is a connected graph. Since p is a prime and $\alpha > 2$ by Proposition 2.2.7, we have $\text{diam}(\bar{\Gamma}(\mathbb{Z}_{p^\alpha})) = 1$. So $\bar{\Gamma}(\mathbb{Z}_{p^\alpha})$ is a complete graph. By Proposition 5.4.2, the number of vertices of $\bar{\Gamma}(\mathbb{Z}_{p^\alpha})$ is $p^{\alpha-1} - 1$, then $\bar{\Gamma}(\mathbb{Z}_{p^\alpha})$ is $(p^{\alpha-1} - 2)$ -regular.

On the other hand, the graph $\bar{\Gamma}(\mathbb{Z}_{p^\alpha})$ verify the condition 1. of Result 2, then $\lambda(\bar{\Gamma}(\mathbb{Z}_{p^\alpha})) = p^{\alpha-1} - 2$. Also $\bar{\Gamma}(\mathbb{Z}_{p^\alpha})$ verify the condition (ii) of the Result 2. Indeed, since p is a prime and $\alpha > 2$, we have $p^{\alpha-1} \geq 2^{\alpha-1} \geq 4$, so $p^{\alpha-1} - 2 \geq \frac{(p^{\alpha-1})+1}{2}$. Thus $\bar{\Gamma}(\mathbb{Z}_{p^\alpha})$ is super- λ regular. By applying Theorem 5.3.1 and Corollary 5.3.2 we conclude the result. ■

Appendices

Appendix 1 : Théorie des graphes

Un graphe non orienté $\Gamma = (V, E)$ est défini par la donnée d'un ensemble $V = \{v_i\}_{i=1}^n$ de sommets et d'un ensemble E d'arêtes, chaque arête étant une paire de sommets. Les sommets v_i et v_j sont adjacents si $v_i - v_j$ est une arête.

Un graphe simple est un graphe sans boucle dont chaque couple de sommets est relié par au plus une arête.

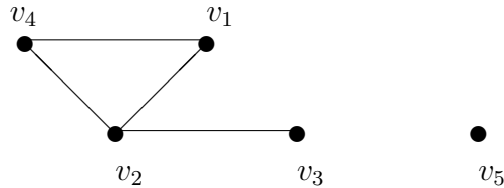


Fig 1 : graphe non orienté

un graphe complet est un graphe simple dont tous les sommets sont adjacents, c'est-à-dire que tout couple de sommets disjoints est relié par une arête. On note un graphe complet non orienté de n sommets par K^n .

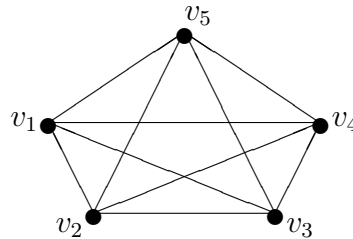


Fig 2 : K^5 graphe simple complet

Un sous graphe Δ d'un graphe Γ est un graphe tel que tous les sommets de Δ sont des sommets de Γ et tous les arêtes de Δ sont des arêtes de Γ . Un graphe vide est un graphe qui ne contient aucune arête. Une clique est un sous graph complet.

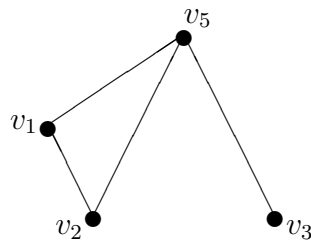


Fig 3 : un sous graphe du
graphe de la fig 2

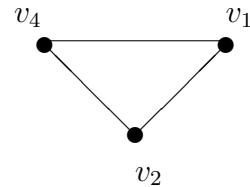


Fig 4 : 3 - clique du
graphe de la fig 1

Un chemin de longueur n du sommet x au sommet y est une séquence de $n + 1$ sommets distincts, commençant par x et se terminant par y , de sorte

que les sommets consécutifs sont adjacents. Si pour chaque paire de sommets distincts $\{x, y\}$ existe un chemin commençant par x et se terminant par y , alors le graphe est dit connecté, sinon, il est déconnecté (voir figure 1).

La distance entre deux sommets x et y est la longueur du chemin le plus court de x à y , et sera notée $d(x, y)$. Si aucun chemin n'existe entre x et y , alors $d(x, y) = \infty$, dans ce cas le graphe est déconnecté.

Le diamètre d'un graphe Γ est la distance maximale entre deux sommets distincts. Ce sera noté $\text{diam}(\Gamma)$.

Un cycle de longueur n est un chemin de la forme $x_1 - x_2 - \dots - x_n - x_1$, où $x_i \neq x_j$ lorsque $i \neq j$. La maille de Γ , noté par $\text{gr}(\Gamma)$, est la longueur du plus court cycle de Γ à condition que Γ contient un cycle, sinon $\text{gr}(\Gamma) = \infty$ (sur la figure 1, $v_1 - v_4 - v_2 - v_1$ est un cycle de longueur 3).

Un graphe est biparti si l'ensemble de ses sommets V peut être partitionné en deux sous-ensembles disjoints A et B tels que chaque arête relie un sommet de A à un sommet de B . Un graphe biparti est complet si tous les sommets de A sont connectés à tous les sommets de B que l'on note par $K^{m,n}$, où $|A| = m$ and $|B| = n$. Lorsque $\Gamma = K^{1,n}$, on dit que Γ est un graphe étoile.

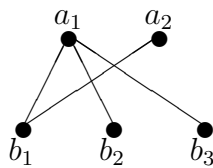


Fig 5 : graphe biparti

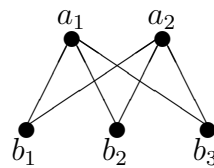


Fig 6 : graphe biparti complet

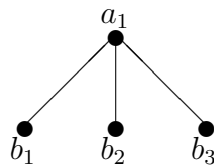


Fig 7 : graphe étoile

Enfin, $\overline{K}^{m,3}$ est le graphe formé en joignant le graphe $\Gamma_1 = K^{m,3}$ ($= A \cup B$ où $|A| = m$ et $|B| = 3$) au graphe étoile $\Gamma_2 = K^{1,m}$ en identifiant le centre de Γ_2 et un point de B .

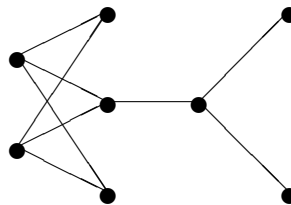


Fig 8 : le graphe $\overline{K}^{2,3}$

Appendix 2 : Graph theory

An undirected graph $\Gamma = (V, E)$ is a set $V = \{v_i\}_{i=1}^n$ of vertices together with a set of edges, where each edge is an unordered pair of vertices. The vertices v_i and v_j are adjacent if $v_i - v_j$ is an edge.

A simple graph, is an undirected graph in which both multiple edges and loops are disallowed. In a simple graph the edges form a set (rather than a multiset) and each edge is an unordered pair of distinct vertices. In a simple graph with n vertices, the degree of every vertex is at most $n - 1$.

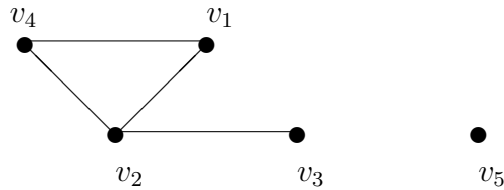


Fig 1 : undirected graph

A graph in which every pair of distinct vertices is adjacent is called a complete graph. We will denote the complete graph on n vertices by K^n .

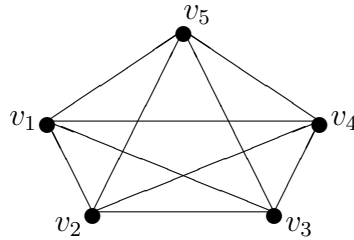
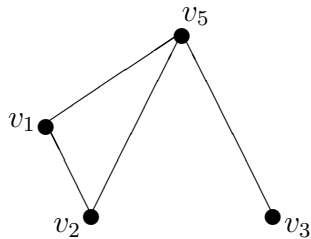
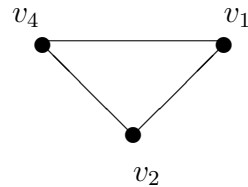


Fig 2 : K^5 complete graph

A subgraph Δ of a graph Γ is a graph such that all the vertices of Δ are vertices of Γ and all the edges of Δ are edges of Γ . The empty graph, consisting of one vertex from Γ and no edges, a clique is a complete subgraph.



**Fig 3 : subgraph of
the graph of fig 2**



**Fig 4 : 3 - clique of the
graph of fig 1**

A path of length n from vertex x to vertex y is a sequence of $n + 1$ distinct vertices, starting with x and ending with y , such that consecutive vertices are

adjacent. If for every pair of distinct vertices $\{x, y\}$ there exists a path starting at x and ending at y , then the graph is referred to as connected; otherwise, it is disconnected (see figure 1).

The distance between two vertices x and y is the length of the shortest path from x to y , and will be denoted $d(x, y)$. The diameter of a graph Γ is the maximum distance between two distinct vertices. This will be denoted $\text{diam}(\Gamma)$. If no path exists between x and y , we say that $d(x, y) = \infty$. In this case the graph is disconnected.

A cycle of length $n \in \mathbb{N}^*$ in Γ is a path of the form $x_1 - x_2 - \dots - x_n - x_1$, where $x_i \neq x_j$ when $i \neq j$. We define the girth of Γ , denoted by $\text{gr}(\Gamma)$, as the length of a shortest cycle in Γ , provided Γ contains a cycle, otherwise, $\text{gr}(\Gamma) = \infty$ (in figure 1, $v_1 - v_4 - v_2 - v_1$ is a cycle of length 3).

One particular type of graph is a bipartite graph, a graph Γ whose vertex set $V(\Gamma)$ can be partitioned into two parts A and B , such that every edge has one end in A and one in B . A complete bipartite graph is a bipartite graph in which every vertex of A is adjacent to every vertex of B . We denote the complete bipartite graph by $K^{m,n}$, where $|A| = m$ and $|B| = n$. When $\Gamma = K^{1,n}$, Γ is called a star graph.

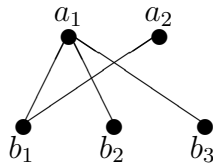


Fig 5 : graphe biparti

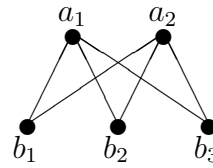


Fig 6 : graphe biparti complet

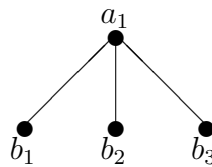


Fig 7 : star graph

Finally, $\overline{K}^{m,3}$ is the graph formed by joining a graph $\Gamma_1 = K^{m,3}$ ($= A \cup B$ with $|A| = m$ and $|B| = 3$) to the star graph $\Gamma_2 = K^{1,m}$ by identifying the center of Γ_2 and a point of B .

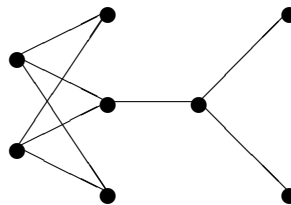


Fig 8 : the graph $\overline{K}^{2,3}$

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Prénom NOM : Fouad TARAZA

Structure de Recherche : Mathématiques, Informatique et Applications

Discipline : Mathématiques

Spécialité : Algèbre et applications

Résumé

La notion de graphe des diviseurs de zéro $\Gamma(R)$ d'un anneau commutatif R a été introduite par D.F. Anderson et Livingston dans (The zero-divisor graph of a commutative ring, J. Algebra, 217, (1999), 434 - 447), qui a pour but d'étudier la relation entre les propriétés algébriques de l'anneau R et les propriétés graphiques du graphe $\Gamma(R)$.

Motivé par le succès de ce nouveau domaine de recherche, plusieurs auteurs ont introduit d'autres graphes associés à certaines propriétés algébriques de l'anneau. Dans ce cadre nous avons introduit et étudié durant notre étude doctorale une extension du graphe des diviseurs de zéro classique d'un anneau commutatif R , noté $\bar{\Gamma}(R)$ et appelé graphe des diviseurs de zéro prolongé.

Nous avons établi une étude générale de ce nouveau type de graphes. Notamment, nous avons caractérisé quand $\Gamma(R)$ et $\bar{\Gamma}(R)$ coïncident. Nous avons étudié, entre autres, le diamètre et la maille de $\bar{\Gamma}(R)$. En outre, une étude des propriétés des graphes des diviseurs de zéro prolongé des idéalizations est aussi effectuée. Nous avons aussi étudié la relation entre le graphe des diviseurs de zéro prolongé et le graphe annihilateur qui est aussi une extension du graphe classique. Enfin, nous avons récemment initié un nouveau projet de recherche qui consiste à étudier les codes binaires associés aux graphes des diviseurs de zéro.

Abstract

The notion of zero-divisor graph $\Gamma(R)$ of a commutative ring R has been introduced by D. F. Anderson and Livingston, in (The zero-divisor graph of a commutative ring, J. Algebra, 217, (1999), 434--447), which aims to study the relationship between the ring-theoretic properties of R and the graph-theoretic properties of $\Gamma(R)$.

Motivated by the success of this new area of research several authors have introduced other graphs associated to some ring theoretic properties. In this context we have introduced and studied an extension of the classical zero-divisor graph of a commutative ring R , denoted by $\bar{\Gamma}(R)$ and called extended zero-divisor graph.

We have established a general study of this new type of graphs. Namely, we characterized when $\bar{\Gamma}(R)$ and $\Gamma(R)$ coincide. We studied, among other things, the diameter and girth of $\bar{\Gamma}(R)$. Also, a study of the properties of the extended zero-divisor graph of idealizations is also carried out. We also studied the relation between the extended zero-divisor graph and the annihilator graph which is also an extension of the classical graph. Finally, we have recently initiated a new research project which consists of studying the binary codes associated to zero-divisor graphs.