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THÈSE de DOCTORAT

Fractional Calculus in Variable Exponent Variational Problems and Epidemiological Models

Présentée par :

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Résumé de la thèse

Cette thèse s'intéresse au calcul fractionnaire en abordant ses bases théoriques dans les problèmes variationnels á exposants variables et ses applications aux prédictions épidémiologiques. Nous analysons les problèmes isotropes fractionnaires, en mettant l'accent sur une diversité de méthodes variationnelles employées pour établir l'existence de solutions faibles pour divers types de problèmes, incluant les équations de type Kirchhoff, les problèmes bi-nonlocaux, les conditions aux limites de Neumann non locales et les problèmes singuliers.

S'appuyant sur cette base, la dissertation introduit un nouveau cadre pour les espaces de Sobolev fractionnaires anisotropiques á exposants variables, afin de mieux appréhender des comportements directionnels. Nous analysons les propriétés élémentaires de ces espaces ainsi que le Laplacien fractionnaire anisotropique, démontrant leur adaptabilité aux problèmes variationnels avec des dépendances directionnelles.

De surcroît, nous considérons l'application du calcul fractionnaire dans la modélisation des dynamiques épidémiologiques et des maladies agricoles. Un modèle d'ordre fractionnaire pour la maladie mosaïque du manioc (CMD) est élaboré en utilisant la dérivée de Caputo pour intégrer les effets de mémoire. Nous étudions également les dynamiques de l'abus de substances chez les femmes en utilisant des modèles fractionnaires avec les dérivées de Caputo et d'Atangana-Baleanu, intégrant des stratégies de contrôle dépendantes du temps. Des simulations numériques appliquées aux deux modèles illustrent la précision et l'adaptabilité des méthodologies fractionnaires dans la prédiction des dynamiques épidémiologiques, tout en apportant des éclairegaes enrichis sur les systèmes agricoles et les problèmes de santé publique.

Mots clés : Calcul fractionnaire, Espaces de Sobolev fractionnaires, Problème singulier, Problème aux limites non local et non linéaire, Espaces anisotropiques á exposants variables, Problémes de type Kirchhoff, Méthodes variationnelles, Modèles épidémiologiques, Analyse numérique.

Sidi Mohamed Ben Abdellah University-Fez
Faculty of Sciences Dhar El Mahraz
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Fractional Calculus in Variable Exponent Variational Problems and Epidemiological Models

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By

Nezha KAMALI

A Ph. D. Thesis

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In Partial Fulfillment of the Requirements for the Degree of

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Dedication

To my family, whose unwavering support and encouragement have guided me through this journey.
Your belief in me has been my greatest source of strength.

To my professors, whose guidance and knowledge have deepened my understanding and inspired my love for learning.
I am truly grateful for your wisdom.

To my friends, who have shared in both the challenges and successes of this path.
Your presence has made this journey meaningful and joyful.

This thesis is dedicated to all of you. Your support has been invaluable.

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Abstract

This thesis provides a thorough analysis of fractional calculus, including a comprehensive investigation of its theoretical foundations in variational problems with variable exponents and its applications in epidemiological predictions. We begin with fractional isotropic problems, focusing on Kirchhoff-type equations involving the fractional $p(x,\cdot)$-Laplacian operator and its generalized integro-differential formulations. Using various variational methods, we analyze the existence and multiplicity of weak solutions for bi-nonlocal problems, extending to nonlinear nonlocal Neumann boundary conditions and singular terms, which underscore the significance of our approaches in the broader context of fractional calculus.

Building on this foundation, the dissertation introduces a novel framework for fractional anisotropic Sobolev spaces with variable exponents, addressing directional behaviors. We analyze the qualitative properties of these spaces, emphasizing the fractional anisotropic Laplacian operator and its application to elliptic-type problems, demonstrating their adaptability to variational problems with directional dependencies.

Furthermore, we investigate the application of fractional calculus in modeling epidemiological and agricultural disease dynamics. A fractional-order model for Cassava Mosaic Disease is developed using the Caputo derivative to incorporate memory effects, analyzing the model's existence, uniqueness, stability, sensitivity, and numerical behavior. We also examine substance abuse dynamics among women using fractional models with Caputo and Atangana-Baleanu derivatives, integrating time-dependent control strategies. Rigorous numerical simulations for both models highlight the precision and adaptability of fractional methodologies in predicting disease dynamics and providing insights into agricultural and public health issues.

Keywords: Fractional calculus, Fractional Sobolev spaces, Singular problem, Nonlocal boundary problem, Anisotropic variable exponents, Kirchhoff-type problems, Variational methods, Epidemiological models, Numerical analysis.

RÉSUMÉ

Cette thèse s'intéresse au calcul fractionnaire en abordant ses bases théoriques dans les problèmes variationnels á exposants variables et ses applications aux prédictions épidémiologiques. Nous analysons les problèmes isotropes fractionnaires, en mettant l'accent sur une diversité de méthodes variationnelles employées pour établir l'existence de solutions faibles pour divers types de problèmes, incluant les équations de type Kirchhoff, les problèmes bi-nonlocaux, les conditions aux limites de Neumann non locales et les problèmes singuliers.
S'appuyant sur cette base, la dissertation introduit un nouveau cadre pour les espaces de Sobolev fractionnaires anisotropiques á exposants variables, afin de mieux appréhender des comportements directionnels. Nous analysons les propriétés élémentaires de ces espaces ainsi que le Laplacien fractionnaire anisotropique, démontrant leur adaptabilité aux problèmes variationnels avec des dépendances directionnelles.
De surcroît, nous considérons l'application du calcul fractionnaire dans la modélisation des dynamiques épidémiologiques et des maladies agricoles. Un modèle d'ordre fractionnaire pour la maladie mosaïque du manioc (CMD) est élaboré en utilisant la dérivée de Caputo pour intégrer les effets de mémoire. Nous étudions également les dynamiques de l'abus de substances chez les femmes en utilisant des modèles fractionnaires avec les dérivées de Caputo et d'Atangana-Baleanu, intégrant des stratégies de contrôle dépendantes du temps. Des simulations numériques appliquées aux deux modèles illustrent la précision et l'adaptabilité des méthodologies fractionnaires dans la prédiction des dynamiques épidémiologiques, tout en apportant des éclairegaes enrichis sur les systèmes agricoles et les problèmes de santé publique.

Mots-clés: Calcul fractionnaire, Espaces de Sobolev fractionnaires, Problème singulier, Problème aux limites non local et non linéaire, Espaces anisotropiques á exposants variables, Problémes de type Kirchhoff, Méthodes variationnelles, Modèles épidémiologiques, Analyse numérique.

الملخص

تتناول هذه الأطروحة الحساب التفاضلي والتكاملي الكسري من خلال دراسة أسسه النظرية في المسائل غير المحلية ذات الأس المتغير وكذا تطبيقاته في النماذج الوبائية. حيث نقوم بتحليل عدد من المسائل الكسرية غير المحلية ذات الأس ذو الاتجاه الوحيد باستخدام طرق مختلفة في إثبات وجود حلول ضعيفة لها، بما في ذلك معادلات كيرشوف، والمسائل غير المحلية الثنائية مع شروط الحدود من نوع نيومان غير المحلية.

استنادًا إلى هذه الأسس، تقدم الأطروحة إطارًا جديدًا لفضاءات سوبوليف الكسرية ذات الأساسات مختلفة الاتجاه، من أجل فهم أعمق للسلوكيات الاتجاهية. كما قمنا بتأسيس بعض خصائصها النوعية مثل الاكتمال، قابلية الفصل، الانعكاسية، و التحدب المنتظم. فضلا عن ذلك قدمنا مشغل لابلاسيان الكسري ذو الأساسات غير متساوية الاتجاه.

بالإضافة إلى ذلك درسنا تطبيق الحساب التفاضلي و التكاملي الكسري في نمذجة الديناميات الوبائية والأمراض الزراعية. حيث قمنا بتطوير نموذج كسري لمرض تبرقش الكاسافا باستخدام مشتقة كابوتوالكسرية لدمج تأثيرات الذاكرة. كما درسنا نموذجا لديناميات تعاطي المواد المخدرة بين النساء باستخدام مشتقات كابوتو وأتانغانا باليانو، متضمنة استراتيجيات تحكم زمنية. توضح المحاكاة الرقمية المطبقة على النموذجين دقة وملاءمة المنهجيات الكسرية في التنبؤ بالديناميات الوبائية، وتوفر رؤى إضافية حول النظم الزراعية وقضايا الصحة العامة.

PUBLICATIONS PRE-PUBLICATIONS

This Ph. D. Thesis is strongly based on the following papers:

1- E. Azroul, **N. Kamali**, and M. Shimi, *On a singular $p(x,\cdot)$-integro-differential elliptic problem* Journal of Pseudo-Differential Operators and Applications, vol. 15(2024), no. 3. https://doi.org/10.1007/s11868-024-00626-x.

2- R. P. Agarwal, E. Azroul, **N. Kamali**, and M. Shimi, *On a class of $p(x,.)$-integrodifferential Kirchhoff-type problem with singular kernel,* Journal of Applied Analysis & Computation, vol.15 (2025), no.1, pp.333–353. doi: 10.11948/20240104.

3- E. Azroul, **N. Kamali**, M. A. Ragusa, and M. Shimi, *On a $p(x,.)$-integrodifferential problem with Neumann boundary conditions* Zeitschrift für Analysis und ihre Anwendungen 2024.

4- E. Azroul, **N. Kamali**, M. A. Ragusa, and M. Shimi, *Variational methods for a fractional $p(x,.)$-bi-nonlocal problem of elliptic type,* Rendiconti del Circolo Matematico di Palermo (In press).

5- E. Azroul, **N. Kamali**, and M. Shimi, *Novel insights into Cassava mosaic disease using Caputo fractional derivative: Modeling and Analysis,* Applicable analysis (In revision state).

6- E. Azroul, **N. Kamali**, M. A. Ragusa, and M. Shimi, *Multiple solutions for anisotropic nonlocal problems with variable exponents* (Preprint).

7- E. Azroul, A. Barbara, **N. Kamali**, and M. Shimi, *Anisotropic fractional Sobolev spaces with variable exponent and application to nonlocal problems* (Preprint).

8- E. Azroul, **N. Kamali**, and M. Shimi, *Empowering women against substance abuse: A fractional mathematical model and optimal control analysis* (Preprint).

9- E. Azroul, **N. Kamali**, and M. Shimi, *Fractional calculus meets public health: analyzing substance abuse dynamics among women with the Atangana-Baleanu derivative* (Preprint).

• Works in final phase of preparation:

10- E. Azroul, **N. Kamali**, and M. Shimi, *Multiple solutions for a variable-order $p(x,\cdot)$-Kirchhoff-type problem with weight* .

11- E. Azroul, **N. Kamali**, and M. Shimi, *On an anisotropic fractional boundary value problem of Kirchhoff type* .

12- E. Azroul, **N. Kamali**, and M. Shimi, *Fractional dynamics of human Melioidosis transmission with asymptomatic cases* .

LIST OF SYMBOLS

Abbreviations	**Meaning**
PDEs	Partial Differential Equations.
FDEs	Fractional Differential Equations.
DFE	Disease-free equilibrium.
EE	Endemic equilibrium.
ER	Electrorheological.
CMD	Cassava Mosaic Disease.
SAW	Substance Abuse among Women.
$(PS)_a$.	Palais-Smale condition at level a.
EVP	Ekeland's Variational Principle.
a.e.	almost everywhere.
$p.v.$	principle value.
$diam$	diamter.

Mathematical dymbols

$\forall$	For all.
$\exists$	There exists.
$\rightarrow$	Strong convergence.
$\rightharpoonup$	Weak convergence.
$\hookrightarrow$	Continuous embedding.
$\hookrightarrow\hookrightarrow$	Compact embedding.

Functions and operators	**Definition**
$\overline{p}(x)$	$p(x,x)$.
$p_{\max}(x)$	$\max\{\overline{p}_1(x), \overline{p}_2(x)\}$.
$p_{\min}(x)$	$\min\{\overline{p}_1(x), \overline{p}_2(x)\}$.
$\vec{p}(x,y)$	$\{p_1(x,y), p_2(x,y), \ldots, p_N(x,y)\}$.
$\overline{p}_i(x)$	$p_i(x,x)$ for all $i = 1,2,\ldots,N$.
$p_M(x)$	$\max\limits_{1\leqslant i\leqslant N} \overline{p}_i(x)$.
$\vec{P}_+$	$\left(p_1^+, \ldots, p_N^+\right)$.
$\vec{P}_-$	$\left(p_1^-, \ldots, p_N^-\right)$.
P_+^+	$\max\{p_1^+, \ldots, p_N^+\}$.
P_+^-	$\min\{p_1^+, \ldots, p_N^+\}$.
P_-^+	$\max\{p_1^-, \ldots, p_N^-\}$.
P_-^-	$\min\{p_1^-, \ldots, p_N^-\}$.
$p_s^*(x)$	$\frac{N\overline{p}(x)}{N - s\overline{p}(x)}$ (Fractional critical exponent).
Δ	Laplacian operator.
$(-\Delta)^s$	Fractional Laplacian operator with $s \in (0,1)$.
$(-\Delta)^s_{p(x,y)}$	Fractional Laplacian operator with variable exponents.
$(-\Delta)^s_{\vec{p}(x,y)}$	Anisotropic fractional Laplacian operator with variable exponents.
${}^{C}D_t^{\alpha}$	The Caputo fractional derivative.
${}^{ABC}D_t^{\alpha}$	The Atangana-Baleanu fractional derivative in Caputo sense.

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PREFACE

"The important thing is not to stop questioning.
Curiosity has its own reason for existing."

Albert Einstein

Mathematics has long held an unwavering allure for me, its elegance and intricacy igniting a deep intellectual curiosity. Through the progression of my academic journey, this fascination has evolved into a profound engagement with increasingly complex and challenging problems. This thesis stands as the culmination of years of rigorous study, exploration, and scholarly inquiry, reflecting both the breadth and depth of my immersion in the field.

My fascination was further nurtured by the pioneering work of our esteemed Professor Abdelmoujib BENKIRANE and his dedicated team, whose contributions, from the early development of the theory of distributions to their profound advancements in nonlinear analysis, partial differential equations, and functional analysis, have significantly shaped the mathematical landscape. Their relentless pursuit of knowledge has inspired me to contribute to this rich area of mathematical theory, where the boundaries between abstraction and application continue to blur in profound and unexpected ways.

This thesis, grounded in the exploration of the fractional $p(x,.)$-Laplacian operator, embodies my pursuit of understanding complex variational problems, in which the intricate interplay between non-linearity and non-locality reveals new avenues for both theoretical insights and practical applications. This investigation, far from being an act of unnecessary complication, emerges from an earnest aspiration to rigorously comprehend the fundamental principles that govern a wide array of systems. By engaging in this meticulous exploration, I aim to unravel the intricate relationships and dynamics inherent within these frameworks. Such diligence not only cultivates a richer

theoretical understanding but also lays the groundwork for transformative applications, even when their implications may not be immediately apparent.

Throughout history, mathematicians have ventured into abstract territories, propelled by an insatiable intellectual curiosity and an unwavering desire to expand the boundaries of knowledge. These endeavors, while seemingly detached from immediate practical concerns, have birthed a multitude of profound theories with far-reaching implications. For instance, the field of fractional calculus has unveiled significant applications in epidemiology, where it offers valuable insights into the dynamics of disease transmission and control. Similarly, discussions surrounding variable exponents have proven instrumental in image processing, enhancing algorithms that optimize image quality and recognition. Furthermore, the intricate interplay of arithmetic and number theory has found critical applications in cryptography and security systems, safeguarding sensitive information in our increasingly digital world. Thus, what may initially appear as abstract exploration often culminates in groundbreaking advancements that shape various domains of science and technology.

In addition to theoretical pursuits, an innate curiosity has fostered exploration into the realm of applied mathematics, with a particular focus on the applications of fractional calculus in predicting transmission dynamics. This exploration has illuminated how the properties of non-locality enhance predictive capabilities, enabling more accurate modeling of complex phenomena across various fields, including epidemiology. This work aims to demonstrate the pivotal role that fractional calculus plays in shaping our understanding of dynamic systems and their behavior over time, thereby highlighting the profound impact of mathematics in addressing critical challenges within public health and agricultural systems.

As a navigator of this thesis, you are cordially invited to immerse yourself in these pages, which not only reflect years of tireless effort but also embody the essence of profound passions, struggles, and commitments. Each word and every idea stands as a testament to a unique journey, eagerly awaiting to be shared with you.

GENERAL INTRODUCTION

0.1 History and motivation

Imagine a mathematical world where the very rules of calculus are etched in shifting sands, where differentiation and integration evolve in ways that challenge our conventional understanding. This is the area of fractional calculus, a fascinating and complex branch of mathematics that redefines our traditional notion of derivatives and integrals. Central to this field is the fractional operators, that capture the essence of anomalous diffusion and long-range interactions, adding depth and nuance to our mathematical toolkit. Just as a chameleon adapts its color to its surroundings, fractional calculus adapts their behavior to the intricacies of the problem at hand, revealing an intriguing interplay between mathematical theory and real-world applications. In this context, we analyze how these advanced mathematical concepts can enhance our analysis of complex variational problems and offer new perspectives on disease transmission.

`How might this groundbreaking idea alter our comprehension of intricate scientific processes, ranging from fractional variational problems with variable exponents to fractional epidemiological models?`

To fully grasp the importance of this concept, one must consider the field of fractional calculus, a branch of mathematics that generalizes the familiar principles of differentiation and integration beyond integer orders. Unlike traditional calculus, where derivatives and integrals are confined to whole numbers, fractional calculus allows for differentiation and integration of any real order. This expanded framework offers a more versatile and refined mathematical language, ideal for modeling processes with memory effects, hereditary characteristics, or anomalous diffusion-complex phenomena that standard calculus cannot effectively capture.

The concept of non-integer order derivatives dates back to September 1695, in a correspondence to the mathematician L'Hôpital, Leibniz introduced his innovative notation $\frac{d^n f}{dt^n}$, where n is an element of the natural numbers to represent the n-th derivative of a function f [119]. In response, L'Hôpital intriguingly questioned the meaning of this notation when $n = \frac{1}{2}$ gave rise to the concept of fractional derivatives and delved into the broader implications of Leibniz's symbolic framework. This question laid the groundwork for analyzing fractional derivatives and inspired further examination of Leibniz's symbolic framework.

- **Euler's interpolation (1738):** In 1738, Euler extended the study to non-integer derivatives using interpolation methods, a significant step in understanding fractional calculus [84].

- **Lagrange's Law of exponents (1772):** The law of exponents proposed by Lagrange in 1772 catalyzed further inquiries into the nature of derivatives beyond integer orders [115].

- **Formalization by Laplace and Lacroix (1819):** In the early 19th century, Laplace and Lacroix systematically formalized the notion of fractional derivatives through the integration of Gamma function Γ, advancing the mathematical framework of fractional calculus [114, 117].

- **Fourier's formula (1822):** Fourier [91] applied the means of integral representation to express the fractional derivative as follows

$$\frac{d^n u(t)}{dt^n} = \frac{1}{2\pi} \int_{-\infty}^{\infty} u(\tau) d\tau \int_{-\infty}^{\infty} \omega^n \cos\left(\omega t - \omega\tau + n\frac{\pi}{2}\right) d\omega.$$

These foundational contributions were pivotal in shaping the development of fractional calculus, though the field remained relatively obscure for some time. It wasn't until Abel's work on the tautochrone (isochrone) problem that fractional operators found formal representation, marking a turning point in their mathematical characterization [2].

- **Joseph Liouville's contributions (1832):** Joseph Liouville revitalized the study of fractional calculus in the early 19th century. In 1832, he published several seminal memoirs that introduced two principal definitions of fractional derivatives. The first extended the integer-order derivatives, although it faced convergence challenges, while the second used the gamma function to provide a more generalized approach [124].

Furthermore, due to their properties, the Riemann-Liouville, Caputo, and Grünwald-Letnikov derivatives are among the most frequently used definitions in fractional calculus. These derivatives are often favored because they closely resemble non-integer derivatives in various contexts, effectively capturing the essence of fractional dynamics.

- Riemann-Liouville fractional order derivative:

$$ {}^{R}_{a}\mathrm{D}^{\alpha}_{t} f(t) = \begin{cases} \frac{1}{\Gamma(m-\alpha)} \frac{\mathrm{d}^m}{\mathrm{d}t^m} \int_a^t (t-\tau)^{m-\alpha-1} f(\tau)\mathrm{d}\tau, & m-1 < \alpha < m, \\ \frac{\mathrm{d}^m}{\mathrm{d}t^m} f(t), & \alpha = m \in \mathbb{N}. \end{cases} $$

- Caputo fractional order derivative:

$$ {}^{C}_{a}\mathrm{D}^{\alpha}_{t} f(t) = \begin{cases} \frac{1}{\Gamma(m-\alpha)} \int_a^t (t-\tau)^{m-\alpha-1} f^m(\tau)\mathrm{d}\tau, & m-1 < \alpha < m, \\ \frac{\mathrm{d}^m}{\mathrm{d}t^m} f(t), & \alpha = m \in \mathbb{N}. \end{cases} $$

- Grünwald-Letnikov fractional order derivative:

$$ {}^{G}_{a}\mathrm{D}^{\alpha}_{t} f(t) = \lim_{h \to 0} h^{-\alpha} \sum_{r=0}^{m} (-1)^r \binom{\alpha}{r} f(t - rh), \quad \alpha \in \mathbb{R}. $$

This historical progression laid the groundwork for the modern development of fractional calculus, setting the stage for its application in diverse areas, including epidemiological modeling. This field is characterized as a specialized branch of mathematical modeling that focuses on understanding the dynamics of infectious diseases within populations. In general, mathematical models are constructed to represent real-world phenomena through intricate equations and quantitative frameworks, enabling researchers to systematically analyze and predict the behavior of complex systems under diverse conditions. Within the realm of epidemiology, these models are meticulously designed to encapsulate the transmission dynamics, progression, and control mechanisms of diseases [51]. For a concise historical overview of mathematical modeling, we direct the reader to Part III of this thesis.

In that context of epidemiology, fractional-order differential equations, have become increasingly valuable due to their ability to incorporate historical data into the modeling of disease dynamics. The key advantage of using these equations is their non-local property, which allows the current state of the epidemic to depend on both present and past conditions, providing a more realistic portrayal of how infections spread over time. Unlike integer-order models, which only consider the immediate state, fractional-order models capture the memory effect inherent in disease transmission. This results in a more comprehensive analysis of disease patterns, enabling researchers to better predict and

control outbreaks by reflecting the persistence and variability of infection rates. The flexibility offered by adjusting the fractional order of differentiation extends the applicability of these models to a broader range of epidemiological scenarios, enhancing their effectiveness in understanding and mitigating the spread of diseases.
Alternatively, it is crucial to recognize that when a theory begins to take shape, its development must be sustained until its foundational properties are thoroughly established. This continuous progression is indispensable to understanding its scope and implications deeply. Furthermore, the pursuit of knowledge compels a consistent extension of more general aspects, leading to an investigation of alternative branches of fractional calculus, particularly when the functions under consideration depend on more than one variable. Around 1938, the Riemann-Louiville integral, given by

$$(I_a^\alpha u)^{RL}(x) = \frac{1}{\Gamma(\alpha)} \int_a^x (x-t)^{\alpha-1} u(t)\, dt, \quad \alpha > 0,$$

was extended to the N-dimensional Euclidean space $\mathbb{R}^N$, with $N \in \mathbb{N}$, by Marcel Riesz [158]. Riesz introduced what is now known as the "Riesz potential", expressed in the following form

$$(I^\alpha u)(x) = \frac{1}{\gamma_N(\alpha)} \int_{\mathbb{R}^N} K_\alpha(x-y) u(y)\, dy,$$

where $\gamma_N(\alpha)$ is a normalizing constant dependent on N and the parameter α. The function K_α, referred to as the "Riesz kernel", is defined as follows

$$K_\alpha(x) := \begin{cases} \frac{1}{|x|^{N-\alpha}}, & \alpha \neq N, N+2, N+4, \cdots, \\ \frac{1}{|x|^{N-\alpha}} \log\left(\frac{1}{|x|}\right), & \alpha = N, N+2, N+4, \cdots. \end{cases}$$

In the framework of fractional calculus, our focus is directed toward the case when $\alpha \notin \{N, N+1, N+2, \ldots\}$. Utilizing Fourier analysis alongside spherical coordinate decomposition, the Riesz potential reveals itself not merely as a function but as a fractional operator that is inherently connected to the positive and negative powers of the Laplacian, denoted as $(-\Delta)^{\frac{\alpha}{2}}$ and $(-\Delta)^{\frac{-\alpha}{2}}$ respectively. In view of the direct and inverse Fourier transform, we write for $0 < \alpha < 2$

$$\begin{cases} {}^R D^\alpha u \equiv (-\Delta)^{\alpha/2} u = \mathscr{F}^{-1} |\xi|^\alpha \mathscr{F} u, \\ I^\alpha u \equiv (-\Delta)^{-\alpha/2} u = \mathscr{F}^{-1} |\xi|^{-\alpha} \mathscr{F} u, \end{cases}$$

with ${}^R D^\alpha$ is the Riesz derivative, ξ is a vector on $\mathbb{R}^N$, and $|\xi|$ is the L^2-norm. Riesz differentiation is interpreted as a convolution with $|x|^{-\alpha-N}$. This convolution, termed

a hyper singular integral, diverges and requires proper definition. For $0 < \alpha < 1$, the convergence can be ensured by introducing the convolution in this way

$$\int_{\mathbb{R}^N} \frac{u(y)-u(x)}{|y-x|^{N+\alpha}} dy = -\int_{\mathbb{R}^N} \frac{u(x)-u(x-y)}{|y|^{N+\alpha}} dy.$$

However, for $1 < \alpha < 2$, the inverse Fourier transform of $|\xi|^\alpha$ fails to exist as a distribution, making the representation invalid. Yet, for negative α, Landkof's analytic [116] continuation yields the fractional Laplacian in real space

$$(-\Delta)^{\alpha/2} u(x) = C(N,\alpha) \text{ p.v.} \int_{\mathbb{R}^N} \frac{u(x)-u(y)}{|x-y|^{N+\alpha}} dy \quad \text{for all } x \in \mathbb{R}^N,$$

where $C(N,\alpha)$ is a normalization constant.

The Riesz potential stands as one of the pivotal concepts that contributed to the birth of the notion of the fractional Laplacian. This integral operator generalizes the classical Laplacian by capturing non-local interactions. It has become prominent in fields such as fractional quantum mechanics [118], fluid dynamics [64], in modeling of cardiac electrical propagation [53] and of anomalous diffusion [54], and in stochastic processes [44], particularly in the context of Lévy processes. Furthermore, the diagram (0.1) showcases

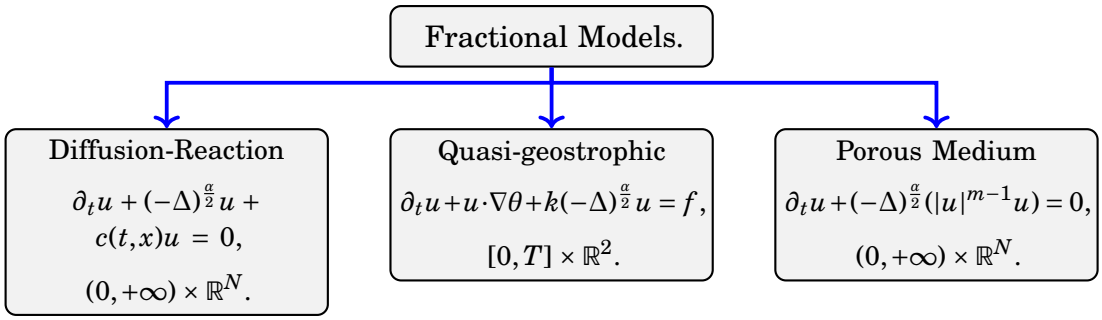


Figure 0.1: Examples of fractional models.

examples of models that implement the fractional Laplacian operator.

The fractional Laplacian serves as a paradigm for the broad class of non-local operators and it is most commonly defined for any $0 < s < 1$ as follows

$$(-\Delta)^s u(x) = C(N,s) \text{p.v.} \int_{\mathbb{R}^N} \frac{u(x)-u(y)}{|x-y|^{N+2s}} dy, \quad \forall x \in \mathbb{R}^N,$$

where $u : \mathbb{R}^N \longrightarrow \mathbb{R}$ is a function that supposed to be regular enough and $C(N,s)$ is a dimensional constant. Furthermore, in [113] the authors bring definitions ranging from the Fourier and distributional approaches to singular integrals, semigroups, and harmonic extensions, proving their equivalence across various mathematical contexts.

To study problems involving this operator, we transition to fractional Sobolev spaces $W^{s,p}$, this spaces are also referred to as Aronszajn, Gagliardo, or Slobodeckij spaces, named after the scholars who introduced them nearly simultaneously [13, 92, 173]. For any $s \in (0,1)$ and $p \in [1,+\infty)$, the fractional Sobolev space is defined for an open subset Ω of $\mathbb{R}^N$ as follows

$$W^{s,p}(\Omega) := \left\{ u \in L^p(\Omega) : \frac{|u(x)-u(y)|}{|x-y|^{\frac{N}{p}+s}} \in L^p(\Omega \times \Omega) \right\},$$

and it is equipped with the Gagliardo norm given by

$$\|u\|_{W^{s,p}(\Omega)} := \left(\int_\Omega |u|^p dx + \int_\Omega \int_\Omega \frac{|u(x)-u(y)|^p}{|x-y|^{N+sp}} dxdy \right)^{\frac{1}{p}}.$$

In [74], Di Nezza et al. offer a thorough study of fractional Sobolev spaces, defining them through the Gagliardo approach and analyzing their core properties. They examine the case $p = 2$, linking it to the fractional Laplacian, and discuss the asymptotic behavior of constants in its definition. They also address the extension of functions from bounded domains, highlighting complexities compared to integer-order spaces, and cover Sobolev inequalities, continuous, and compact embeddings. This comprehensive work opens the door to multiple problems involving the fractional Laplacian; for more details, we refer to the original studies [37, 46, 166].

As previously stated, the fractional Laplacian is distinguished by its inherent non-locality. This non-local characteristic implies that the value of the fractional Laplacian operators at any given point $x \in \Omega$ is influenced not only by the values of the function u within the domain Ω, but also by its behavior throughout the entire space $\mathbb{R}^N$. Such extended dependence underscores the limitations of the function space $W^{s,p}(\Omega)$, which is traditionally employed in local settings. Specifically, $W^{s,p}(\Omega)$ fails to fully capture the boundary conditions required for accurately reflecting the influence of points external to the domain Ω, a critical aspect in the analysis of non-local problems. In this regard, the paper [165] by Raffaella Servadei and Enrico Valdinoci presents significant insights. To address the challenges posed by non-locality, the authors propose a novel framework that generalizes the functional analytic landscape typically reserved for local problems. They consider the set Q defined as follows

$$Q := \mathbb{R}^{2N} \setminus (\Omega^c)^2, \quad \text{where } \Omega^c = \mathbb{R}^N \setminus \Omega.$$

The set Q is introduced for studying problems under boundary conditions on $\mathbb{R}^N \setminus \Omega$ and considered an adequate framework to correctly encode this kind of boundaries in the

variational formulation.
The theory of fractional Sobolev spaces has undergone a significant expansion to include variable exponents, since the domain of variable exponents has consistently proven its vital importance in the modeling of inverse phenomena that are intricately connected to real-world scenarios [10, 12, 55]. This extension led to the introduction of fractional Sobolev spaces with variable exponents $W^{s,p(x,y)}$ [105]. This theoretical evolution has not only refined our understanding of fractional operators but has also paved the way for novel existence and uniqueness results for solutions to nonlinear equations involving fractional and non-local operators. Furthermore, the extension reaches a higher level with the introduction of a generalized version of these spaces, denoted by $W_K^{s,p(x,y)}$ [21], which further broadens the scope of analysis and applicability in various mathematical contexts.
Nevertheless, numerous challenges in the field of nonlinear nonlocal problems continue to raise pertinent questions. For example, how can we effectively implement Neumann boundary conditions within the framework $W_K^{s,p(x,y)}$. Is the integration of a singular term source approached in the same traditional manner within the nonlinear scenario? An intriguing question arises when we consider extending this framework into an anisotropic case. Lastly, how can fractional calculus transcend its limitations in public health systems and agricultural practices, thereby providing innovative solutions to contemporary challenges in these critical areas?

0.2 Objective and outline of the thesis

This dissertation presents several aspects of fractional calculus, encompassing variational problems with variable exponents in both isotropic and anisotropic contexts, along with a substantial analysis of its applications in epidemiological models. We begin with a preliminary chapter 1 that introduces essential mathematical concepts, such as fractional Sobolev spaces with variable exponents, variational methods, and properties of fractional operators. This chapter also establishes a framework for fractional differential equations and their significance in epidemiology, providing a solid theoretical foundation for the subsequent discussions. The thesis is organized into three distinct parts, each addressing a unique facet of fractional calculus and its implications:

★ In Part I, we investigate fractional isotropic problems, where the properties of the system remain invariant across all directions. This part contains two chapters that uti-

lize variational methods to address Kirchhoff-type problems involving the fractional $p(x,\cdot)$-Laplacian operator and its generalized integro-differential version. We also investigate nonlinear nonlocal Neumann boundary conditions and singular terms, treated separately to establish the existence of solutions.

- Chapter 2 addresses a class of Kirchhoff-type problems dealing with a class of examples demonstrating the conditions needed in the application of each variational method, it includes:

  ✓ Employing variational methods, specifically Ekeland's variational principle and the mountain pass theorem, we investigate the existence of weak solutions for a class of bi-nonlocal Kirchhoff-type problems that incorporate the fractional $p(x,\cdot)$-Laplacian operator. We focus on three distinct classes of Kirchhoff functions to derive our findings.

  ✓ Through the use of the mountain pass theorem, the fountain theorem, and the dual fountain theorem, we demonstrate the existence of weak solutions for a variational problem associated with an integro-differential operator. Our investigation specifically addresses a particular class of conditions related to the second member of the problem.

- Chapter 3 is devoted to the investigation of integro-differential problems with variable exponents in two different contexts:

  ✓ The first section addresses a singular $p(x,\cdot)$-integro-differential elliptic problem, where we establish the existence of at least two solutions. This is achieved through the application of Nehari manifolds and minimization techniques, complemented by fiber mapping methods.

  ✓ The second section is devoted to the study of $p(x,\cdot)$-integro-differential problems under nonlinear nonlocal Neumann boundary conditions, incorporating the presence of a bi-nonlocal term. Utilizing Clarke's theorem, we establish the existence of multiple solutions within the functional framework that will be introduced taking into account the boundary conditions used.

★ In Part II, we introduce a novel fractional anisotropic Sobolev space with variable exponents, inspired by a specific case. This part comprises two chapters dedicated to:

- Chapter 4 provides a general introduction to anisotropic problems as well as a historical review and motivation presented in a special case. More precisely:

- ✓ We first state a historical background of this kind of space showing that Anisotropic Sobolev spaces have been key to understanding directional behavior in various fields, evolving to include spaces with variable exponents for greater adaptability.
- ✓ Second, a motivation is presented based on proving the existence of multiple solutions for anisotropic nonlocal problems in a traditional way giving birth lately to the fractional anisotropic fractional Sobolev spaces with variable exponents.

• The new fractional anisotropic Sobolev spaces with variable exponents is introduced in chapter 5 including;

- ✓ The formal definition of the fractional anisotropic Sobolev spaces with variable exponents and the proof of its qualitative properties. Furthermore, we introduce the fractional $\vec{p}(x,\cdot)$-Laplacian operator with its basic properties.
- ✓ An application of the form of fractional anisotropic problems with variable exponent of elliptic type is provided to demonstrate the existence of weak solutions in this framework.

★ Part III takes another direction namely applying fractional calculus in epidemiology with the aim of demonstrating the versatility of fractional calculus through case studies and simulations in epidemiology and agricultural disease modeling. It highlights its potential as a transformative tool for addressing challenges in public health and agricultural sustainability. It contains two chapters that address:

• Chapter 6 offers valuable insights into the application of fractional order models to understand the transmission of Cassava Mosaic Disease, emphasizing the significance of these models in contributing to agricultural systems. In this chapter:

- ✓ We present a novel fractional-order model for the transmission dynamics of CMD, using the Caputo derivative to capture memory effects. Our analysis establishes the existence, uniqueness, positivity, and boundedness of the model's solutions, ensuring biological relevance. Notably, we derive the basic reproduction number R_0 with the next-generation matrix and analyze the local stability of equilibrium points.
- ✓ A sensitivity analysis identifies critical parameters influencing the transmissibility of CMD, guiding targeted intervention strategies. Additionally,

numerical simulations using a generalized predictor-corrector approach illustrate the impact of fractional-order derivatives on disease spread, validating theoretical results and highlighting the potential of fractional calculus in understanding CMD transmission while offering new perspectives and control strategies.

- Chapter 7 is devoted to the use of fractional mathematical models to improve the understanding of the dynamics of substance abuse among women (SAW). The chapter includes an analysis summarized as follows.

  ✓ We present fractional models of substance abuse among women (SAW) using Caputo and Atangana-Baleanu derivatives, demonstrating the existence and uniqueness of solutions. We also analyze the positivity, boundedness, equilibrium points, basic reproduction number, and their asymptotic stability.

  ✓ We conduct a sensitivity analysis of key parameters and perform numerical simulations for each model, comparing the fractional derivatives to demonstrate their greater flexibility and adaptability in the study.

  ✓ We refine our fractional model in the Caputo sense by introducing two time-dependent controls. The first control focuses on public outreach programs to raise awareness of the risks and societal impacts of substance abuse among women. The second control enhances the rehabilitation rate of women through medical and psychological interventions. A numerical simulation demonstrates that applying both controls effectively promotes recovery from addiction.

This thesis is accomplished with a conclusion that summarizes the contributions presented throughout the work, emphasizing the significance of our findings in the context of fractional calculus. This conclusion will also shed light on potential future work that occupies our imagination, exploring pathways for further research and investigation.

CHAPTER 1

PRELIMINARIES

The present chapter reviews the fundamental mathematical concepts and tools that will form the basis for the analysis in the following thesis chapters. Specifically, we will focus on two major topics: fractional Sobolev spaces with variable exponents and fractional calculus, both of which are critical for understanding the variational problems and epidemiological models investigated in this work. For more detailed references and an extended theoretical background, [21, 23, 75, 87, 105, 110, 147, 151] provide further details of these mathematical frameworks.

1.1 Fractional Sobolev spaces with variable exponents

Fractional Sobolev spaces with variable exponents are gaining attention due to their ability to model non-homogeneous media and phenomena with variable characteristics. By allowing the exponent to vary spatially, these spaces extend the classical fractional Sobolev spaces $W^{s,p}$ and provide a flexible framework for analyzing complex problems.

1.1.1 Brief history and motivation

Function spaces with variable exponents were first introduced by Orlicz in 1931 [148], where he investigated function spaces that generalize traditional Lebesgue spaces by allowing the exponent to vary with position. Nakano further developed the theory in the

1950s by introducing modular function spaces that explicitly included variable exponent Lebesgue spaces [142]. Subsequent research, particularly by Russian mathematicians like Tsenov [178] and Sharapudinov [167], focused on the minimization of functionals in variable exponent Lebesgue spaces. The fundamental properties of $L^{p(x)}$ and $W^{1,p(x)}$ spaces with variable exponents were established by Kováčik and Rákosník in 1991 [110], with extensions by Fan and Zhao within the framework of Sobolev spaces [87].
The analysis of PDEs with variable exponents has been significantly inspired by this foundational work in function spaces, particularly in the context of modeling in electrorheological and thermorheological fluids, these are liquids whose viscosity changes significantly, often dramatically, when exposed to an electric field. Linear models like Bingham and Casson assume constant shear rates, while nonlinear models, such as the Power law, incorporate variable exponents, making them more realistic by accounting for spatial variations in electric fields. Variable exponents are important because of their ability to capture the complex behavior of ER fluids under varying electric fields, as demonstrated by the mathematical models of Rajagopal and Ruzicka, which adapt physical laws for these conditions. For further information on these various models, refer to the references [3, 4]. Variable exponents are also pivotal in image restoration processes [59], in magnetostriction [55], and in mathematical modeling of other physical phenomena such as quasi-Newtonian fluids [192].zzzzzzzzzzzzzzzzzzzzzzzzzz

1.1.2 Lebesgue spaces with variable exponents

This subsection is devoted to providing essential results on variable exponent Lebesgue spaces, which will be used in this thesis. For more comprehensive information, the reader is referred to [67, 75, 87, 110, 159] and the references cited therein.
Consider $\Omega \subset \mathbb{R}^N$ as a bounded open domain with Lipschitz boundary, and define the set $C^+(\overline{\Omega})$ as follows

$$C^+(\overline{\Omega}) = \{p \in C(\overline{\Omega}) : p(x) > 1 \text{ for all } x \in \overline{\Omega}\}.$$

For any $p \in C^+(\overline{\Omega})$, we identify p^+ and p^- in this way

$$p^+ = \sup_{x\in\overline{\Omega}} p(x) \quad \text{and} \quad p^- = \inf_{x\in\overline{\Omega}} p(x).$$

The Lebesgue space with variable exponent is identified for any $p \in C^+(\overline{\Omega})$ as

$$L^{p(x)}(\Omega) = \left\{ u : \Omega \to \mathbb{R} \text{ measurable such that: } \int_\Omega |u(x)|^{p(x)} dx < +\infty \right\},$$

and it is equipped with the following Luxemburg norm

$$\|u\|_{L^{p(x)}(\Omega)} = \|u\|_{p(x)} = \inf\left\{\mu > 0; \int_\Omega \left|\frac{u(x)}{\mu}\right|^{p(x)} \mathrm{d}x \leq 1\right\}.$$

Proposition 1.1 ([87]). *The space $\left(L^{p(\cdot)}(\Omega), \|\cdot\|_{p(x)}\right)$ is a separable, reflexive, and Banach space.*

It is widely recognized that the modular plays a critical role in manipulating spaces with variable exponents. Therefore, the modular introduced on $L^{p(x)}(\Omega)$ as follows

$$\begin{aligned} \rho_{p(x)} : L^{p(x)}(\Omega) &\longrightarrow \mathbb{R} \\ u &\longmapsto \int_\Omega |u(x)|^{p(x)} dx. \end{aligned}$$

The modular $\rho_{p(x)}$ and the Luxemburg norm $\|\cdot\|_{L^{p(x)}(\Omega)}$ associated with the generalized Lebesgue space $L^{p(x)}(\Omega)$ are connected by the following results.

Proposition 1.2. *Suppose that $u \in L^{p(x)}(\Omega)$ and $\{u_n\} \subset L^{p(x)}(\Omega)$, it follows that*

i. $\|u\|_{L^{p(x)}(\Omega)} < 1 \ \ (resp. > 1, = 1) \Leftrightarrow \rho_{p(x)}(u) < 1 \ \ (resp. > 1, = 1)$,

ii. $\|u\|_{L^{p(x)}(\Omega)} = \lambda \Leftrightarrow \rho_{p(x)}\left(\frac{u}{\lambda}\right) \equiv 1 \quad$ *for all* $u \in L^{p(x)}(\Omega)\backslash\{0\}$,

iii. $\|u\|_{L^{p(x)}(\Omega)} < 1 \Rightarrow \|u\|^{p^+}_{L^{p(x)}(\Omega)} \leqslant \rho_{p(x)}(u) \leqslant \|u\|^{p^-}_{L^{p(x)}(\Omega)}$,

iv. $\|u\|_{L^{p(x)}(\Omega)} > 1 \Rightarrow \|u\|^{p^-}_{L^{p(x)}(\Omega)} \leqslant \rho_{p(x)}(u) \leqslant \|u\|^{p^+}_{L^{p(x)}(\Omega)}$,

v. $\lim_{n\to+\infty} \|u_n - u\|_{L^{p(x)}(\Omega)} = 0 \Leftrightarrow \lim_{n\to+\infty} \rho_{p(x)}(u_n - u) = 0.$

Let $L^{\hat{p}(x)}(\Omega)$ be the conjugate space of $L^{p(x)}(\Omega)$, where $\frac{1}{p(x)} + \frac{1}{\hat{p}(x)} = 1$, for all $x \in \overline{\Omega}$. Hölder's inequality holds similarly to the classical Lebesgue spaces, as formulated in the following lemma.

Lemma 1.1. *For any $u \in L^{p(x)}(\Omega)$ and $v \in L^{\hat{p}(x)}(\Omega)$, one has the following Hölder type inequality*

$$\left|\int_\Omega uv \ \mathrm{d}x\right| \leqslant \left(\frac{1}{p^-} + \frac{1}{(\hat{p})^-}\right) \|u\|_{L^{p(x)}(\Omega)} \|v\|_{L^{\hat{p}(x)}(\Omega)} \leqslant 2\|u\|_{L^{p(x)}(\Omega)} \|v\|_{L^{\hat{p}(x)}(\Omega)}. \tag{1.1}$$

1.1.3 Weighted variable exponent Lebesgue space

Let w be a measurable function on Ω that verifies

$$(w_1): w(x) > 0 \text{ for a.e } x \in \Omega.$$

$$(w_2): w \in L^1(\Omega) \text{ and } \operatorname*{essinf}_{x\in\Omega} w(x) = w_0 > 0.$$

The weighted variable exponent Lebesgue space is defined as

$$L_w^{p(x)}(\Omega) = \left\{ u : \Omega \to \mathbb{R} \text{ measurable such that: } \int_\Omega w(x)|u(x)|^{p(x)} dx < +\infty \right\},$$

with respect to the norm

$$\|u\|_{p(x),w} = \inf\left\{ \mu > 0 : \int_\Omega w(x) \left| \frac{u(x)}{\mu} \right|^{p(x)} dx \leqslant 1 \right\}.$$

Proposition 1.3 ([17]). *The space* $\left(L_w^{p(\cdot)}(\Omega), \|\cdot\|_{p(x),w}\right)$ *is a separable and a reflexive Banach space.*

The modular is introduced on $L_w^{p(x)}(\Omega)$ as follows

$$\begin{aligned} \rho_{p(x),w} : L_w^{p(x)}(\Omega) &\longrightarrow \mathbb{R} \\ u &\longmapsto \int_\Omega w(x)|u(x)|^{p(x)} dx. \end{aligned}$$

The norm $\|\cdot\|_{p(x),w}$ and the modular $\rho_{p(x),w}$ are related by the following results, analogous to Proposition 1.2.

Proposition 1.4. *Consider* $u \in L_w^{p(x)}(\Omega)$ *and* $\{u_n\} \subset L_w^{p(x)}(\Omega)$. *Hence*

- $\|u\|_{p(x),w} < 1 (\text{ resp. } > 1, = 1) \Leftrightarrow \rho_{p(x),w}(\cdot) < 1 (\text{ resp. } > 1, = 1)$,
- $\|u\|_{p(x),w} < 1 \Rightarrow \|u\|_{p(x),w}^{p^+} \leqslant \rho_{p(x),w}(u) \leqslant \|u\|_{p(x),w}^{p^-}$,
- $\|u\|_{p(x),w} > 1 \Rightarrow \|u\|_{p(x),w}^{p^-} \leqslant \rho_{p(x),w}(u) \leqslant \|u\|_{p(x),w}^{p^+}$,
- $\lim_{n\to+\infty} \|u_n\|_{p(x),w} = 0 \Leftrightarrow \lim_{n\to+\infty} \rho_{p(x),w}(u_n) = 0$,
- $\lim_{n\to+\infty} \|u_n\|_{p(x),w} = \infty \Leftrightarrow \lim_{n\to+\infty} \rho_{p(x),w}(u_n) = \infty$.

For a comprehensive analysis of weighted variable exponent Lebesgue spaces, refer to [17].

1.1.4 Fractional Sobolev spaces with variable exponents

In the intricate world of PDEs, classical tools often fall short of capturing the complexity of natural phenomena, particularly when dealing with nonlocal behavior or spatial heterogeneity. This is where fractional Sobolev spaces with variable exponents step in, first introduced by Kaufmann, Rossi, and Vidal [105], for $0 < s < 1$ in this way

$$W^{s,q(x),p(x,y)}(\Omega) = \left\{ \begin{array}{l} u : \Omega \longrightarrow \mathbb{R} \;\; u \in L^{q(x)}(\Omega) \text{ with} \\ \int_\Omega \int_\Omega \frac{|u(x)-u(y)|^{p(x,y)}}{\mu^{p(x,y)}|x-y|^{N+sp(x,y)}} dxdy < +\infty, \text{ for some } \mu > 0 \end{array} \right\},$$

where $q : \overline{\Omega} \to (1,\infty)$ and $p : \overline{\Omega} \times \overline{\Omega} \to (1,\infty)$ are two continuous functions such that $q(x) > p(x,x)$ for $x \in \overline{\Omega}$. They established results for continuous and compact embeddings based on this condition. Furthermore, the authors in [35] explore key structural aspects of the spaces, such as reflexivity, separability, and density, while proving several properties of the operator associated with fractional Sobolev spaces with variable exponents. These theoretical results are applied to nonlocal problems using a direct variational method, demonstrating the practical relevance of the concepts developed. The paper by Ho and Kim [97] addresses the limitation of the condition considered in [105] and provides a solution to eliminate the restriction by proving the crucial embeddings from these spaces into variable exponent Lebesgue spaces under the condition $q(x) \geqslant p(x,x)$, as an application, they established a priori bounds multiplicity of solutions for certain nonlinear elliptic problems involving the fractional $p(x,\cdot)$-Laplacian operator. Thus far, it can be argued that these definitions do not serve as a straightforward generalization of the classical fractional Sobolev space $W^{s,p}(\Omega)$. This is due to the inclusion of an additional variable q, which must be greater than p. This additional dependency differentiates these spaces from the traditional fractional Sobolev spaces. Afterward, a significant contribution to trace theory is discussed in [73], where the authors considered the definition of fractional Sobolev spaces with variable exponents in the case $q(x) = p(x,x)$ and demonstrated that $W^{s,p(x,y)}(\Omega)$ is continuously embedded in $L^{q(x)}(\partial\Omega)$, given that $q : \partial\Omega \to (1,\infty)$ is a continuous function and satisfies

$$\frac{(n-1)p(x,x)}{n-sp(x,x)} > q(x) \quad \text{in } \partial\Omega \cap \{x \in \overline{\Omega} : n - sp(x,x) > 0\}.$$

Additionally, The definition of the fractional Sobolev spaces with variable exponents provided in [105] extend the classical approach in a rigorous manner, especially when considered alongside these studies and other foundational works, including [20].
For problems with boundary conditions on $\mathbb{R}^N \setminus \Omega$, the space $W^{s,p(x,y)}(\Omega)$ may not provide

an adequate framework. This limitation is discussed in Section 0.1, where the non-local property is integrated into the variational formulation to address such cases effectively. Given that our problems lie within a non-local context, we will review and focus on the definition involving the set Q, and we adopt here the definition that is suitable for our analysis.

Let $p : \overline{Q} \longrightarrow (0, +\infty)$ be a continuous, bounded, and symmetric function, expressly

$$1 < p^- = \min_{(x,y)\in\overline{Q}} p(x,y) \leqslant p(x,y) \leqslant p^+ = \max_{(x,y)\in\overline{Q}} p(x,y) < +\infty, \tag{1.2}$$

and

$$p(x,y) = p(y,x) \text{ for all } (x,y) \in \overline{Q}. \tag{1.3}$$

Set $\overline{p}(x) = p(x,x)$ for all $x \in \overline{\Omega}$, then for $s \in (0,1)$ the fractional Sobolev space with variable exponents is introduced in [23] as follows

$$W = W^{s,p(x,y)}(Q) = \left\{ \begin{array}{l} u : \mathbb{R}^N \longrightarrow \mathbb{R} \text{ measurable such that } u_{|\Omega} \in L^{\overline{p}(x)}(\Omega) \text{ with} \\ \int_Q \frac{|u(x)-u(y)|^{p(x,y)}}{\mu^{p(x,y)}|x-y|^{N+sp(x,y)}} dxdy < +\infty, \text{ for some } \mu > 0. \end{array} \right\}.$$

For all measurable function $u \in W$, the Gagliardo seminorm is defined as follows

$$[u]_{s,p(x,y)} = \inf\left\{\mu > 0 : \int_Q \frac{|u(x)-u(y)|^{p(x,y)}}{\mu^{p(x,y)}|x-y|^{sp(x,y)+N}} \, \mathrm{d}x \, \mathrm{d}y \leqslant 1\right\},$$

therefore the fractional Sobolev space with variable exponent equipped with the norm

$$\|u\|_{W^{s,p(x,y)}(Q)} = \|u\|_{s,p(x,y)} = \|u\|_{L^{\overline{p}(x)}(\Omega)} + [u]_{s,p(x,y)}, \tag{1.4}$$

is a Banach, reflexive, and separable space [23].

Since spaces with variable exponents are governed by the modular, its definition on W is given by

$$\begin{aligned} \rho_{p(x,y)} : W &\longrightarrow \mathbb{R} \\ u &\longmapsto \int_Q \frac{|u(x)-u(y)|^{p(x,y)}}{|x-y|^{N+sp(x,y)}} dxdy + \int_\Omega |u|^{\overline{p}(x)} dx. \end{aligned}$$

In the following, we consider the linear subspace of W given by

$$W_0 := \left\{u \in W : u = 0 \text{ a.e. in } \Omega^c\right\}.$$

The Gagliardo seminorm defines a norm on W_0 equivalent to (1.4), indicating that W_0 is a Banach space, reflexive, and separable. For a straightforward proof, see [23, Lemma

2.3]. Furthermore, W_0 is characterised by the modular mapping defined in this way

$$\rho^0_{p(x,y)} : W_0 \longrightarrow \mathbb{R}$$
$$u \longrightarrow \int_Q \frac{|u(x)-u(y)|^{p(x,y)}}{|x-y|^{N+sp(x,y)}} dxdy.$$

Similar to Proposition 1.2, the modular and the norm on W_0 that will be denoted $\|\cdot\|_{W_0} = [\cdot]_{s,p(x,y)}$ to avoid confusion, are related by the inequalities presented in Proposition 1.5.

Proposition 1.5. *For any $u \in W_0$ and $\{u_n\} \subset W_0$, we have*

(a)– $\|u\|_{W_0} < 1$ *(resp.* $= 1, > 1) \Leftrightarrow \rho^0_{p(x,y)}(u) < 1$ *(resp.* $= 1, > 1)$,

(b)– $1 \leq \|u\|_{W_0} \Rightarrow \|u\|^{p^-}_{W_0} \leq \rho^0_{p(x,y)}(u) \leq \|u\|^{p^+}_{W_0}$,

(c)– $\|u\|_{W_0} \leq 1 \Rightarrow \|u\|^{p^+}_{W_0} \leq \rho^0_{p(x,y)}(u) \leq \|u\|^{p^-}_{W_0}$,

(d)– $\lim_{n\to+\infty} \|u_n\|_{W_0} = 0$ *(resp* $= \infty) \Leftrightarrow \lim_{n\to+\infty} \rho^0_{p(x,y)}(u_n) = 0$ *(resp* $= \infty$).

In the study of fractional Sobolev spaces with variable exponents, embedding theorems play a fundamental role in establishing the functional framework necessary for analyzing variational problems. Specifically, the continuous and compact embedding theorems provide crucial insights into how these spaces relate to more familiar function spaces, such as Lebesgue spaces with variable exponents. In what follows, we will present the relevant continuous and compact embedding results for the fractional Sobolev spaces with variable exponents used throughout this work.

Theorem 1.1 ([23, Theorem 2.2])**.** *Let Ω be a Lipschitz bounded domain in $\mathbb{R}^N$ and $s \in (0,1)$. Consider $p : \overline{Q} \longrightarrow (0,+\infty)$ as a continuous exponent variable that verifies conditions (1.2) and (1.3) with $sp^+ < N$. Moreover, let $r : \overline{\Omega} \longrightarrow (0,+\infty)$ be a continuous variable exponent such that*

$$1 < r^- \leqslant r(x) < p^*_s(x) = \frac{N\overline{p}(x)}{N - s\overline{p}(x)} \text{ for all } x \in \overline{\Omega}.$$

Hence, the space W is continuously embedded in $L^{r(x)}(\Omega)$, which means that there exists a constant $C = C(N,s,p,r,\Omega) > 0$ such that for any $u \in W$,

$$\|u\|_{L^{r(x)}(\Omega)} \leqslant C\|u\|_W.$$

Moreover, this embedding is compact.

Remark 1.1 ([23, Remark 2.2]). *If we replace W with W_0, Theorem 1.1 holds true.*

Let $\mathscr{C}_b^m(\Omega)$ be the subset of $\mathscr{C}^m(\Omega)$ consisting of the functions whose partial derivatives of order m are bounded and uniformly continuous on Ω with $\mathscr{C}_b^0(\Omega) := \mathscr{C}_b(\Omega)$.

Theorem 1.2 ([22, Theorem 2.9]). *Let Ω be a Lipschitz bounded domain in $\mathbb{R}^N$ and $s \in (0,1)$. Consider $p : \overline{\Omega} \times \overline{\Omega} \longrightarrow (1, +\infty)$ as a continuous function satisfies (1.2) and (1.3) with $sp^- > N$. Then $W^{s,p(x,y)}(\Omega)$ is continuously embedded in $L^\infty(\Omega)$, and more precisely,*

$$W^{s,p(x,y)}(\Omega) \hookrightarrow \mathscr{C}_b^{s-N/p^-}(\Omega),$$

where $\mathscr{C}_b^{s-N/p^-}(\Omega)$ denotes the space of Hölder continuous functions of order $s - N/p^-$ on Ω, defined as follows

$$\mathscr{C}_b^{s-N/p^-}(\Omega) = \left\{ \begin{array}{c} \varphi \in C_b(\Omega) : \exists c > 0 \text{ for all } (x,y) \in \Omega^2, \\ |\varphi(x) - \varphi(y)| \leqslant c|x-y|^{s-N/p^-} \end{array} \right\}.$$

In the following lemma, we compare the spaces $W^{s,p(x,y)}(\Omega)$ and W. Moreover, this result is fundamental to proving embedding results in the space W.

Lemma 1.2 ([23, Theorem 2.2]). *Let Ω be a Lipschitz bounded domain in $\mathbb{R}^N$ and let $s \in (0,1)$. Let $p \in C^+(\overline{Q})$ satisfies conditions (1.2) and (1.3). Then, if $u \in W$, we have $u \in W^{s,p(x,y)}(\Omega)$. Moreover,*

$$\|u\|_{W^{s,p(x,y)}(\Omega)} \leqslant \|u\|_W.$$

Remark 1.2. *For any θ_1, θ_2 such that $0 < \theta_1 < \theta_2 < 1$, we have*

$$\mathscr{C}_b^{0,\theta_2}(\Omega) \hookrightarrow \mathscr{C}_b^{0,\theta_1}(\Omega) \hookrightarrow \mathscr{C}_b^0(\Omega) \hookrightarrow \mathscr{C}^0(\overline{\Omega}) = \mathscr{C}(\overline{\Omega}).$$

Where $\mathscr{C}_b^{0,\theta}(\Omega)$ is the space of Hölder continuous functions of order θ on Ω with $\theta = \theta_1$ or $\theta = \theta_2$.

Now, we are ready to state the following embedding result.

Corollary 1.1. *Let Ω be a Lipschitz bounded domain in $\mathbb{R}^N$ and $s \in (0,1)$. Consider $p : \overline{Q} \longrightarrow (1, +\infty)$ as a continuous function satisfies (1.2) and (1.3) with $sp^- > N$. Then W is continuously embedded in $L^\infty(\Omega)$, and more precisely,*

$$W \hookrightarrow \mathscr{C}_b^{s-N/p^-}(\Omega),$$

Proof. From Theorem 1.2 and Lemma 1.2, we can easily deduce the desired result. ■

Corollary 1.2. *Let Ω be a Lipschitz bounded domain in $\mathbb{R}^N$ and $s \in (0,1)$. Consider $p \in C^+(\overline{Q})$ satisfying (1.2) and (1.3) with $sp^- > N$. Then the space W is continuously embedded in $\mathscr{C}(\overline{\Omega})$.*

Corollary 1.3. *Let Ω be a Lipschitz bounded domain in $\mathbb{R}^N$ and $s \in (0,1)$. Consider $p \in C^+(\overline{Q})$ satisfying (1.2) and (1.3) with $sp^- > N$. Then the embedding of W into $\mathscr{C}_b^{0,\theta}(\Omega)$ is compact for any $\theta < s - N/p^-$.*

Remark 1.3. *By Remark 1.2 and Corollary 1.3, if $sp^- > N$, we have that the embedding $W \hookrightarrow \mathscr{C}(\overline{\Omega})$ is compact .*

1.2 Generalized fractional Sobolev spaces with variable exponents

As discussed previously in the literature, the fractional Sobolev spaces with variable exponents $W^{s,p(x,y)}$ provide a powerful framework for addressing nonlocal problems with variable growth conditions. Traditionally, these spaces are associated with operators involving specific singular kernels, such as $|x-y|^{-(N+sp(x,y))}$, which capture the nonlocal interactions between points. However, to tackle more complex and realistic models, it becomes essential to extend this framework by generalizing the kernel to a more flexible and symmetric singular kernel. This extension allows us to investigate a broader class of nonlocal problems involving integro-differential operators.
Consider the kernel $K : \mathbb{R}^N \times \mathbb{R}^N \longrightarrow (0,+\infty)$ belongs to the set of measurable functions that verify

- $(K_1) : K(x,y) = K(y,x)$ for any $(x,y) \in \mathbb{R}^N \times \mathbb{R}^N$.
- (K_2) : There exist two positive constants k_0 and k_1 such that

$$k_1|x-y|^{-N-sp(x,y)} \geqslant K(x,y) \geqslant k_0|x-y|^{-N-sp(x,y)}, \quad \forall (x,y) \in \mathbb{R}^N \times \mathbb{R}^N \text{ with } x \neq y. \tag{1.5}$$

The general fractional Sobolev space with variable exponent introduced in [21] as follows

$$X = W_K^{s,p(x,y)}(Q) = \left\{ \begin{array}{c} u : \mathbb{R}^N \longrightarrow \mathbb{R} \text{ measurable such that } u_{|\Omega} \in L^{\overline{p}(x)}(\Omega) \text{ with} \\ \int_Q \frac{|u(x)-u(y)|^{p(x,y)}}{\mu^{p(x,y)}} K(x,y)dxdy < +\infty, \text{ for some } \mu > 0 \end{array} \right\}.$$

The space X is endowed with the following norm

$$\|u\|_X = \|u\|_{W_K^{s,p(x,y)}(Q)} = \|u\|_{K,p(x,y)} = \|u\|_{\overline{p}(x)} + [u]_{K,p(x,y)},$$

where the generalized Gagliardo semi-norm with variable exponent is defined by

$$[u]_{K,p(x,y)} = \inf\left\{\mu > 0 : \int_Q \frac{|u(x)-u(y)|^{p(x,y)}}{\mu^{p(x,y)}} K(x,y)dxdy \leqslant 1\right\}. \tag{1.6}$$

$(X, \|.\|_X)$ is a separable, reflexive and Banach space [21, Corollary 1].
In the following, we consider the linear subspace of X defined as follows

$$X_0 = W^{s,p(x,y)}_{K,0}(Q) = \left\{u \in W^{s,p(x,y)}_K(Q) \text{ such that } u = 0 \text{ in } \mathbb{R}^N \setminus \Omega\right\}.$$

Consequently, the norm related to this functional framework is the generalized Gagliardo semi-norm with variable exponent defined by (1.6) and denoted $\|\cdot\|_{X_0}$. Hence, $\left(X_0, \|\cdot\|_{X_0}\right)$ is a separable, reflexive and Banach space [21, Lemma 8].
A fundamental tool in the analysis of spaces with variable exponents is the modular, as previously mentioned, which in this context is identified as

$$\begin{aligned} \rho^0_{K,p(x,y)} : X_0 &\longrightarrow \mathbb{R} \\ u &\longmapsto \int_Q |u(x)-u(y)|^{p(x,y)} K(x,y)dxdy. \end{aligned}$$

Similar to the results mentioned above Propositions 1.2 and 1.5, the general fractional Sobolev space with variable exponent X_0 realized also the following connections between $\rho^o_{K,p(.,.)}$ and the generalized Gagliardo semi norm $\|\cdot\|_{X_0}$.

Proposition 1.6. *Let $p : \overline{Q} \longrightarrow (1, +\infty)$ be a continuous variable exponent verifying conditions (1.2) and (1.3). $K : \overline{Q} \longrightarrow (0, +\infty)$ is a measurable and symmetric function satisfying (K_1) and (K_2). Then for any $u \in X_0$ and $(u_n) \subset X_0$, we have*

1. $1 \leqslant \|u\|_{X_0} \Rightarrow \|u\|^{p^-}_{X_0} \leqslant \rho^0_{K,p(\cdot,\cdot)}(u) \leqslant \|u\|^{p^+}_{X_0}$,
2. $\|u\|_{X_0} \leqslant 1 \Rightarrow \|u\|^{p^+}_{X_0} \leqslant \rho^0_{K,p(\cdot,\cdot)}(u) \leqslant \|u\|^{p^-}_{X_0}$,
3. $\lim_{n\to+\infty} \|u_n - u\|_{X_0} = 0 \Longleftrightarrow \lim_{n\to+\infty} \rho^0_{K,p(\cdot,\cdot)}(u_n - u) = 0$.

Following this, proceeding by stating a significant embedding result that forms an essential part of upcoming sections.

Theorem 1.3 ([21, Theorem 3.1]). *Let Ω be a Lipschitz bounded domain in $\mathbb{R}^N$ and $s \in (0,1)$. Let $p : \overline{Q} \longrightarrow (1, +\infty)$ be a continuous variable exponent satisfies conditions (1.2) and (1.3) with $sp^+ < N$. Consider the kernel $K : \overline{Q} \longrightarrow (0, +\infty)$ as a measurable and*

symmetric function with condition (K_2). Furthermore, let $q:\overline{\Omega}\longrightarrow(1,+\infty)$ be a continuous bounded variable exponent such that

$$1<q^{-}\leqslant q(x)<p_s^*(x)=\frac{N\overline{p}(x)}{N-s\overline{p}(x)},\ \textit{for all}\ x\in\overline{\Omega}.$$

Then, the space X is continuously embedded in $L^{q(x)}(\Omega)$. Mathematically expressed by the existence of a positive constant $C=C(N,p,q,s,\Omega)$, such that for any $u\in X$, we have

$$\|u\|_{L^{q(x)}(\Omega)}\leqslant C\|u\|_X\leqslant C\max\{1,\tilde{k}_0\}\|u\|_X,$$

where $\tilde{k_0}=\max\left\{k_0^{-\frac{1}{p^-}},k_0^{-\frac{1}{p^+}}\right\}$. Moreover, this embedding is compact.

Remark 1.4.

(1)- *The continuous and compact embedding results of Theorem 1.3 remain true if we replace X by X_0.*

(2)- *The norms $\|.\|_{X_0}$ and $\|.\|_X$ are equivalent on X_0.*

Theorem 1.4 ([19, Theorem 2.3]). *Let Ω be a Lipschitz bounded domain in $\mathbb{R}^N$ and $s\in(0,1)$. Let $p:\overline{Q}\longrightarrow(1,+\infty)$ be a continuous variable exponent satisfying conditions (1.2) and (1.3) with $sp^+<N$. Assume that $w:\Omega\to(0,+\infty)$ is a measurable function such that $w\in L^{\gamma(x)}(\Omega)$ with $\gamma\in C^+(\overline{\Omega})$ and verify*

$$\frac{N\overline{p}(x)}{\overline{p}(x)-q(x)(N-s\overline{p}(x))}>\gamma(x)>\frac{\overline{p}(x)}{\overline{p}(x)-q(x)}\quad\textit{for all}\ x\in\overline{\Omega}.$$

Then the embedding from W into $L_w^{q(x)}(\Omega)$ is compact. Moreover, there is a constant $\tilde{C}_3>0$ such that

$$\rho_{q(x),w}(u)\leqslant\tilde{C}_3\left(\|u\|_{s,p(x,y)}^{q^-}+\|u\|_{s,p(x,y)}^{q^+}\right).\tag{1.7}$$

Corollary 1.4 ([19, Corollary 2.1]). *Given the same assumptions as in Theorem 1.4, then there exist two positive constants $\tilde{C}_4$, and $\tilde{C}_5$ such that the following inequalities hold for $u\in W$*

$$\rho_{q(x),w}(u)\leqslant\begin{cases}\tilde{C}_4\|u\|_W^{q^+} & \textit{if}\ \|u\|_W>1,\\ \tilde{C}_5\|u\|_W^{q^-} & \textit{if}\ \|u\|_W<1.\end{cases}$$

Corollary 1.5. *Given the same assumptions as in Theorem 1.4, then there exist two positive constants $\tilde{C}_6$, and $\tilde{C}_7$ such that the following inequalities hold for $u\in W_0$*

$$\rho_{q(x),w}(u)\leqslant\begin{cases}\tilde{C}_6\|u\|_{W_0}^{q^+} & \textit{if}\ \|u\|_{W_0}>1,\\ \tilde{C}_7\|u\|_{W_0}^{q^-} & \textit{if}\ \|u\|_{W_0}<1.\end{cases}$$

1.3 Overview of basic properties of the fractional $p(x,\cdot)$-Laplacian and other integro-differential operators

In this section, we recall the fundamental properties of the integro-differential operators used throughout this thesis. These operators, which blend integral and differential calculus, serve as the mathematical foundation for addressing various variational problems.

Primarily, we consider the operator associated with the fractional $p(x,\cdot)$-Laplacian operator $(-\Delta)^s_{p(x,\cdot)}$, which will be denoted L and defined as follows

$$L(u)=\int_Q \frac{|u(x)-u(y)|^{p(x,y)}}{p(x,y)|x-y|^{N+sp(x,y)}}dxdy \quad ,\forall u\in W_0,$$

and

$$\mathscr{L}: W_0 \longrightarrow W_0^*$$

such that

$$\langle \mathscr{L}(u),v\rangle=\int_Q \frac{|u(x)-u(y)|^{p(x,y)-2}(u(x)-u(y))(v(x)-v(y))}{|x-y|^{N+sp(x,y)}}dxdy \quad ,\forall u,v\in W_0\times W_0.$$

Lemma 1.3 ([35, Lemma 4.2]). *Let Ω be a smooth bounded domain in $\mathbb{R}^N$ and $s\in(0,1)$. Let $p:\overline{Q}\longrightarrow(1,+\infty)$ be a continuous variable exponent verifies (1.2) and (1.3), then we have the following assertions*

- *L is well-defined and $L\in\mathscr{C}^1(W_0,\mathbb{R})$. Moreover, the derivative of L is given by*

$$\left\langle L'(u),v\right\rangle=\langle \mathscr{L}(u),v\rangle,$$

- *$\mathscr{L}$ is a bounded and strictly monotone operator.*
- *$\mathscr{L}$ is a mapping of type (S_+), that is,*

$$\begin{cases} u_k \rightharpoonup u \text{ in } W_0,\\ \limsup_{k\to+\infty}\langle \mathscr{L}(u_k)-\mathscr{L}(u),u_k-u\rangle\leqslant 0, \end{cases} \Rightarrow \quad \lim_{k\to+\infty} u_k=u \text{ on } W_0.$$

- *$\mathscr{L}$ is a homeomorphism.*

Next, we present the generalized integro-differential operator with variable exponent which is introduced in [21] as follows

$$\begin{aligned}\mathscr{L}_K^{p(x,.)} : X_0 &\longrightarrow X_0^* \\ u &\longrightarrow \mathscr{L}_K^{p(x,.)}(u) : X \longrightarrow \mathbb{R} \\ &\qquad v \longrightarrow \langle \mathscr{L}_K^{p(x,.)}(u), v\rangle,\end{aligned}$$

where

$$\langle \mathscr{L}_K^{p(x,.)}(u), v\rangle = \int_Q |u(x)-u(y)|^{p(x,y)-2}(u(x)-u(y))(v(x)-v(y))K(x,y)dxdy.$$

Lemma 1.4 ([21, Lemma 11])**.** *Let $p : Q \longrightarrow (1,+\infty)$ be a continuous variable exponent satisfying (1.2) and (1.3). Consider $K : Q \longrightarrow (0,+\infty)$ as a measurable function satisfying (K_1) and (K_2). Then, the following assertions hold*

i. $\mathscr{L}_K^{p(x,\cdot)}$ *is well defined and bounded,*

ii. $\mathscr{L}_K^{p(x,\cdot)}$ *is a strictly monotone operator,*

iii. $\mathscr{L}_K^{p(x,\cdot)}$ *is a mapping of type* (S_+).

1.4 Variational methods for nonlinear nonlocal PDEs

Boundary value problems can be expressed in the following abstract form

$$\begin{cases}\mathscr{A}[u] = 0 & \text{in } \Omega, \\ B[u] = 0 & \text{on } \partial\Omega \text{ or on } \mathbb{R}^N \setminus \Omega.\end{cases} \tag{1.8}$$

where

- $\mathscr{A}[u]$ represents the given partial differential equation for the unknown function u,
- $B[u]$ specifies the boundary conditions.

It is well known that no universal method exists for solving such problems. However, the **Calculus of Variations** introduces a significant class of problems that can be resolved through relatively straightforward techniques in nonlinear nonlocal functional analysis. Specifically, these are **variational problems**, where the operator $\mathscr{A}[u]$ can be

formulated as the first **variation** (or "derivative") of an appropriate energy functional $\mathscr{J}$ defined on a Banach space $\mathbb{H}$. In this context, the operator $\mathscr{A}[u]$ is expressed as

$$\mathscr{A}[u] = \mathscr{J}'(u),$$

where $\mathscr{A} : \mathbb{H} \to \mathbb{H}^*$ and the equation $\mathscr{A}[u] = 0$ is reformulated in its weak form in this way

$$\langle \mathscr{J}'(u), v \rangle = 0, \quad \forall v \in \mathbb{H}.$$

The key advantage of this reformulation is that solving problem (1.8) (at least in a weak sense) is equivalent to finding the critical points of the functional $\mathscr{J}$ on the space $\mathbb{H}$. The minimization method for variational problems seeks to solve the problem by identifying the minimizers of the associated energy functional. In this and the following chapter, we will focus exclusively on variational problems within the fractional Sobolev spaces with variable exponents.

In this section, we will review key concepts essential to the study of nonlinear nonlocal elliptic problems. We will discuss direct variational methods, which help in finding minimizers of energy functionals, and the Mountain Pass Theorem, which ensures the existence of saddle points in critical point theory. Additionally, we will cover Ekeland's Variational Principle, providing conditions for approximate minimizers, and the Fountain and Dual Fountain Theorems, which are instrumental in identifying multiple critical points. Lastly, we will explore genus theory, used for classifying critical points based on symmetry, and topological degree theory, a valuable tool for proving the existence of solutions in nonlinear elliptic frameworks.

Before proceeding with the exposition of the essential tools and methods employed in this work, it is imperative to first introduce and formalize the fundamental definitions that underpin the subsequent analysis.

Definition 1.1. Let $\mathbb{H}$ be a Banach space and $\mathscr{J} \in \mathscr{C}^1(\mathbb{H}, \mathbb{R})$.

- A point $x \in \mathbb{H}$ is called a critical point of $\mathscr{J}$ if $\mathscr{J}'(x) = 0$.
- A real number $c \in \mathbb{R}$ is referred to as a critical value of $\mathscr{J}$ if there exists a critical point $x \in \mathbb{H}$ such that $c = \mathscr{J}(x)$.
- If $c \in \mathbb{R}$ is not a critical value, we call c a regular value of $\mathscr{J}$.

Definition 1.2. Let $\mathscr{J}$ be a functional defined on a Banach space $\mathbb{H}$, where $\mathscr{J} \in \mathscr{C}^1(\mathbb{H}, \mathbb{R})$, and let $a \in \mathbb{R}$. The functional $\mathscr{J}$ is said to satisfy the *Palais-Smale condition* at level a

if every sequence $(u_n)_n \subset X$ satisfying the conditions $\mathcal{J}(u_n) \to a$ and $\mathcal{J}'(u_n) \to 0$ in $\mathbb{H}^*$, possesses a subsequence that converges strongly in $\mathbb{H}$.

In what follows, we shall begin by introducing one of the most well-known and foundational variational methods, namely **Ekeland's Variational Principle** (EVP). This principle not only plays a pivotal role in the broader landscape of variational analysis but also serves as a powerful tool in the study of optimization and critical point theory.

Theorem 1.5 ([82, Ekeland's variational principle]). *Let $(\mathbb{H}, d_{\mathbb{H}})$ be a complete metric space and $\mathcal{J} : \mathbb{H} \to \mathbb{R} \cup \{+\infty\}$ lower semi-continuous function that is bounded below. Then for every $\varepsilon > 0$, there exists $v_\varepsilon \in \mathbb{H}$ such that*

$$\inf_{\mathbb{H}} \mathcal{J} \leq \mathcal{J}(v_\varepsilon) \leq \inf_{\mathbb{H}} \mathcal{J} + \varepsilon$$

and for all $w \neq v_\varepsilon$,

$$\mathcal{J}(v_\varepsilon) < \mathcal{J}(w) + \varepsilon d_X(v_\varepsilon, w).$$

This refinement of Dirichlet's principle enables the systematic construction of minimizing sequences for such functionals. In this framework, each element of the sequence is a minimizer of an auxiliary functional, where the family of functionals is carefully chosen to converge locally uniformly to the original functional. Yet, the rich landscape of nonlinear analysis often presents challenges where the solutions are not global minimizers but rather critical points of a different nature, such as saddle points. In such cases, the mountain pass theorem plays a crucial and complementary role. While Ekeland's principle excels in the construction of minimizing sequences, the mountain pass theorem extends the scope of variational methods by guaranteeing the existence of critical points located along mountain pass-like paths, where there is neither strict minimization nor maximization.

Theorem 1.6 ([186, Mountain pass theorem]). *Let $\mathbb{H}$ be a real Banach space and let $\mathscr{C}^1(\mathbb{H}, \mathbb{R})$ satisfies the Palais-Smale condition. Suppose that $\mathcal{J}(0) = 0$ and*

($\mathbf{R_1}$) : *There exist two positive real numbers r and δ such that $\mathcal{J}(u) \geqslant \delta$ for all $u \in \mathbb{H}$ with $\|u\|_{\mathbb{H}} = r$.*

($\mathbf{R_2}$) : *There exists $\overline{u} \in \mathbb{H}$ with $||\overline{u}||_{\mathbb{H}} > r$, such that $\mathcal{J}(\overline{u}) < 0$.*

Then, $\mathscr{J}$ possesses a critical value $c \geq \delta$. Moreover, c can be characterized as

$$c = \inf_{g\in\Gamma} \max_{z\in[0,1]} \mathscr{J}(g(z)),$$

where

$$\Gamma = \{g \in \mathscr{C}([0,1],\mathbb{H}) : g(0) = 0, g(1) = \overline{u}\}.$$

Now, we will present additional methods that are expressly aimed to prove the existence of multiple solutions in the context of variational problems. We start with the following version of Ricceri's three critical points theorem, which will be used in our subsequent analysis.

Theorem 1.7 ([157, Ricceri's critical point theory])**.** *Let $(\mathbb{H}, \|.\|_{\mathbb{H}})$ be a separable and reflexive real Banach space, $\Phi : \mathbb{H} \to \mathbb{R}$ a continuously Gâteaux differentiable and sequentially weakly lower semicontinuous functional whose Gâteaux derivative admits a continuous inverse on $\mathbb{H}^*$, $\Psi : \mathbb{H} \to \mathbb{R}$ a continuously Gâteaux differentiable functional whose Gâteaux derivative is compact. Assume that*

($\mathbf{D_1}$): $\lim_{\|u\|_{\mathbb{H}}\to+\infty}(\Phi(u) + \lambda\Psi(u)) = +\infty$ *for all* $\lambda > 0$,

($\mathbf{D_2}$): *There are* $r \in \mathbb{R}$ *and* $u_0, u_1 \in \mathbb{H}$ *such that* $\Phi(u_0) < r < \Phi(u_1)$,

($\mathbf{D_3}$): $\inf_{u\in\Phi^{-1}((-\infty,r])} \Psi(u) > \frac{(\Phi(u_1)-r)\Psi(u_0)+(r-\Phi(u_0))\Psi(u_1)}{\Phi(u_1)-\Phi(u_0)}$.

Then there exist an open interval $\Lambda \subset (0,\infty)$ and a positive real number δ such that for each $\lambda \in \Lambda$, and every continuously Gâteaux differentiable functional $\mathscr{J} : \mathbb{H} \to \mathbb{R}$ with compact derivative, there exists $\mu^* > 0$ such that for each $\mu \in [0, \mu^*]$, the equation

$$\Phi'(u) + \lambda\Psi'(u) + \mu\mathscr{J}'(u) = 0$$

has at least three solutions in $\mathbb{H}$ whose norms are less than δ.

Next, For any separable and reflexive Banach space $\mathbb{H}$, we can find $e_j \in \mathbb{H}$ and $e_j^* \in \mathbb{H}^*$ such that

$$\mathbb{H} = \overline{\text{span}\{e_j : j = 1,2,\ldots\}}, \quad \mathbb{H}^* = \overline{\text{span}\{e_j^* : j = 1,2,\ldots\}},$$

and

$$\langle e_i^*, e_j\rangle = \begin{cases} 1, & i = j, \\ 0, & i \neq j. \end{cases}$$

For convenience, we constitute $X_j := \text{span}\{e_j\}, Y_k := \oplus_{j=1}^{k} X_j, Z_k := \overline{\oplus_{j=k}^{\infty} X_j}$.

Lemma 1.5 ([190, Lemma 4.9]). *Let $q \in C^+(\overline{\Omega})$ such that $q(x) < p_s^*(x)$ for all $x \in \overline{\Omega}$. We define*

$$\beta_k = \sup\left\{\|u\|_{L^{q(x)}(\Omega)}; \|u\|_{\mathbb{H}_0} = 1, u \in Z_k\right\}.$$

Then $\beta_k \longrightarrow 0 \quad as \quad k \longrightarrow \infty$.

Definition 1.3. Let $\mathscr{J} \in \mathscr{C}^1(\mathbb{H}, \mathbb{R})$, and $c \in \mathbb{R}$. The function $\mathscr{J}$ satisfies $(PS)_c^*$ (with respect to Y_n), if any sequence $\{u_{n_j}\} \subset E$ such that

$$u_{n_j} \in Y_{n_j}, \quad \mathscr{J}(u_{n_j}) \to c, \quad \mathscr{J}|'_{Y_{n_j}}(u_{n_j}) \to 0 \quad \text{in } X^* \text{ as } n_j \to \infty,$$

admits a convergent subsequence.

Theorem 1.8 ([186, Fountain theorem]). *Assume that $\mathbb{H}$ is a separable Banach space, and $\mathscr{J} \in \mathscr{C}^1(\mathbb{H}, \mathbb{R})$ is an even functional. Moreover, suppose that for each $k = 1, 2, \cdots$ there exist $\delta_k > r_k > 0$ such that*

$(\mathbf{A_1})$: $\inf\limits_{u \in Z_k, \|u\|_{\mathbb{H}} = r_k} \mathscr{J}(u) \to +\infty$ *as* $k \to +\infty$.

$(\mathbf{A_2})$: $\max\limits_{u \in Y_k, \|u\|_{\mathbb{H}} = \delta_k} \mathscr{J}(u) \leqslant 0$.

In addition, if $\mathscr{J}$ verifies the Palais-Smale condition $(PS)_c$ for every $c > 0$. Then $\mathscr{J}$ has an unbounded sequence of critical values.

Theorem 1.9 ([186, Dual fountain theorem]). *Suppose that $\mathscr{J} \in \mathscr{C}^1(\mathbb{H}, \mathbb{R})$ is an even function where $\mathbb{H}$ is a separable Banach space, and for every $k \geqslant k_0 > 0$, there exist $\delta_k > r_k > 0$ such that*

$(\mathbf{B_1})$: $\inf\limits_{u \in Z_k, \|u\|_{\mathbb{H}} = \delta_k} \mathscr{J}(u) \geqslant 0$.

$(\mathbf{B_2})$: $\max\limits_{u \in Y_k, \|u\|_{\mathbb{H}} = r_k} \mathscr{J}u < 0$.

$(\mathbf{B_3})$: $a_k = \inf\limits_{u \in Z_k, \|u\|_{\mathbb{H}} \leqslant \delta_k} \mathscr{J}(u) \longrightarrow 0$ *as* $k \longrightarrow +\infty$.

Furthermore, $\mathscr{J}$ verifies the Palais-Smale condition $(PS)_c^*$ for every $c \in [a_{k_0}, 0)$. Then, $\mathscr{J}$ possesses a sequence of negative critical values converging to 0.

Following our exploration of those theorems, we turn our attention to genus theory, which is associated with topological and geometric considerations.
Denote by $\mathscr{R}$ the class of al closed subset $A \subset \mathbb{H} \setminus \{0\}$, that is symmetric with respect to the origin (i.e if $u \in A \Rightarrow -u \in A$).

Definition 1.4. Let $A \in \mathscr{R}$ and $\mathbb{H} = \mathbb{R}^N$. The genus $\gamma(A)$ of A is defined as follows

$$\gamma(A) = \min\left\{k \geq 1; \text{ there exists an odd continuous mapping } \varphi : A \to \mathbb{R}^k \setminus \{0\}\right\}.$$

If such a k does not exist, we set $\gamma(A) = \infty$.

Remark 1.5. *i. if A is a subset, which consists of finitely many pairs of points, then $\gamma(A) = 1$.*

ii. $\gamma(\varnothing) = 0$ (by definition).

Theorem 1.10. *Let $\mathbb{H} = \mathbb{R}^N$, and let Ω be an open, symmetric and bounded subset of $\mathbb{R}^N$, which $\partial\Omega$ its boundary with $0 \in \Omega$. Then $\gamma(\partial\Omega) = N$.*

Corollary 1.6. *Let us denote by S^{N-1} the $(N-1)$-dimensional sphere in $\mathbb{R}^N$. Then $\gamma(S^{N-1}) = N$.*

Having established the foundational concepts of genus theory, we consider in this section Clarke's theorem which serves as a critical result in understanding the existence of critical points for functionals that exhibit certain topological properties, particularly the study of nonlocal problems under Neumann boundary conditions as will be treated in this thesis.

Theorem 1.11. *[63] Let $\mathscr{J} \in \mathscr{C}^1(\mathbb{H}, \mathbb{R})$ be a functional satisfying the Palais-Smale condition (PS). Furthermore, we suppose that*

(G_1) *$\mathscr{J}$ is even and bounded from below,*

(G_2) *There exist a compact set $K \in \mathscr{R}$ such that $\gamma(K) = k$ and $\sup_{x \in K} \mathscr{J}(x) < \mathscr{J}(0)$.*

Then $\mathscr{J}$ possesses at least k pairs of distinct critical points and their corresponding critical values are less than $\mathscr{J}(0)$.

1.5 Fractional calculus in the context of epidemiological models

This section is devoted to the foundational definitions and fundamental concepts of fractional calculus, specifically focusing on derivatives and integrals of non-integer orders that will be used in the analysis of part III. These mathematical tools are integral to the models and methods employed in this thesis, particularly in the context of biological

and dynamical systems. By exploring the Riemann-Liouville and Caputo derivatives, as well as other relevant fractional operators, this section lays the groundwork for the analytic and numerical approaches used to investigate the intricate behavior of the systems under study. For a more detailed exploration of fractional calculus theory, the reader is directed to consult the works cited in references [147, 151].

1.5.1 Special functions

Given the centrality of fractional calculus in the third part of this thesis, it is necessary to introduce important special functions, including the gamma and Mittag-Leffler functions, which are frequently employed in this area.

Definition 1.5 (Gamma function)**.** The Gamma function, represented by $\Gamma(z)$, extends the concept of the factorial function to non-integer values of z. Specifically, for $n \in \mathbb{N}$, it holds that $\Gamma(n) = (n-1)!$. For $z > 0$, the Gamma function is defined as

$$\Gamma(z) = \int_0^\infty t^{z-1} e^{-t}\, dt. \tag{1.9}$$

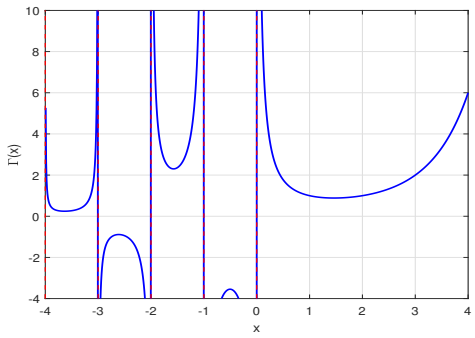


Figure 1.1: The Gamma function $\Gamma(x)$

Definition 1.6 (Mittag-Leffler Function)**.** Let $\alpha, \beta > 0$, the two-parameter version of the Mittag-Leffler function is defined as

$$\mathscr{E}_{\alpha,\beta}(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\alpha k + \beta)}, \quad z \in \mathbb{C}. \tag{1.10}$$

When $\beta = 1$, the function $\mathscr{E}_\alpha(z)$ is referred to as the one-parameter Mittag-Leffler function.

The Mittag-Leffler function generalizes the exponential function.

1.5.2 Useful non-integer derivatives

In this subsection, we present the elementary definitions of fractional-order derivatives used in the analysis of Part III.

Definition 1.7 (The Riemann-Liouville fractional integral)**.** For a given locally integrable real function $f:\mathbb{R}^+ \longrightarrow \mathbb{R}$, the Riemann-Liouville fractional integral of order $\alpha > 0$ is defined as follows

$$I_t^\alpha f(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-\tau)^{\alpha-1} f(\tau) d\tau. \tag{1.11}$$

Definition 1.8 (The Caputo derivative)**.** The Caputo fractional derivative for a function $f \in \mathscr{C}^n$ of order α is given by

$${}^C D_t^\alpha (f(t)) = \frac{1}{\Gamma(n-\alpha)} \int_0^t \frac{f^{(n)}(\tau)}{(t-\tau)^{\alpha+n-1}} d\tau,$$

which is well-defined for absolutely continuous functions with $n-1 < \alpha < n \in \mathbb{N}$.

Definition 1.9 (The ABC derivative)**.** Consider $f \in H^1[0,T]$ and $\alpha \in (0,1)$. The Atangana-Baleanu derivative in the Caputo sense (ABC) is defined as follows [15]

$${}^{ABC} D_t^\alpha f(t) = \frac{ABC(\alpha)}{1-\alpha} \int_0^t \mathscr{E}_\alpha \left(-\alpha \frac{(t-\tau)^\alpha}{1-\alpha} \right) f'(\tau) d\tau, \tag{1.12}$$

where ABC refers to the normalization function that satisfies

$$ABC(\alpha) = 1 - \alpha + \frac{\alpha}{\Gamma(\alpha)} \text{ and } ABC(0) = ABC(1) = 1.$$

Definition 1.10. Let $f \in H^1[0,T]$ and $\alpha \in (0,1)$. The fractional integral in ABC sense is given by

$${}^{ABC} I_t^\alpha f(t) = \frac{1-\alpha}{ABC(\alpha)} f(t) + \frac{\alpha}{ABC(\alpha)\Gamma(\alpha)} \int_0^t f(\tau)(t-\tau)^{\alpha-1} d\tau. \tag{1.13}$$

Definition 1.11. Let f be defined for $t \geq 0$. The Laplace transform of f denoted by $\mathscr{L}(f)$ is an integral transform expressed as follows

$$\mathscr{L}[f(t)](\tau) = \int_0^{+\infty} e^{-\tau t} f(t) dt.$$

Below are the Laplace transforms for functions applied in Part III. For a function f, we have

$$\mathscr{L}\left[{}^{C}D_{t}^{\alpha}f(t)\right] = \tau^{\alpha}\mathscr{L}(\tau) - \sum_{k=0}^{n-1}\tau^{\alpha-k-1}f^{(k)}(0). \tag{1.14}$$

$$\mathscr{L}\left[{}^{ABC}D_{t}^{\alpha}f(t)\right] = \frac{ABC(\alpha)}{\tau^{\alpha-1}(1-\alpha)+\alpha}\left(\tau^{\alpha}\mathscr{L}[f(t)] - \tau^{\alpha-1}f(0)\right). \tag{1.15}$$

$$\mathscr{L}\left(t^{\beta-1}\mathscr{E}_{\alpha,\beta}\left(\pm\lambda t^{\alpha}\right)\right) = \frac{\tau^{\alpha-\beta}}{\tau^{\alpha}\mp\lambda}. \tag{1.16}$$

1.5.3 Theoretical Framework of FDEs

In this subsection, we address the foundational principles essential for the analysis of initial value problems associated with differential equations formulated using fractional derivatives. Let $0 < \alpha < 1$ and $f : \mathbb{R}\times\mathbb{R}^{N} \longrightarrow \mathbb{R}^{N}$ where $N \geqslant 1$. Consider the initial value problem expressed as follows

$$\begin{cases} D_{t_0}^{\alpha}x(t) = f(t,x(t)), \\ x(t_0) = x_0. \end{cases} \tag{1.17}$$

Depending on the context of this thesis, $D_{t_0}^{\alpha}$ may refer to the Caputo derivative or the ABC derivative depending on the problem at hand.
The presence of nonlinearities in many dynamical systems often prevents us from obtaining analytical solutions. To understand the long-term behavior of these systems, it is crucial to first identify basic solutions that serve as a foundation for analyzing the behavior of all other solutions.

Definition 1.12 ([76])**.** A constant x^{*} is an equilibrium point of the system (1.17) if and only if $f(t,x^{*}) = 0$.

An equilibrium solution or point in a dynamical system remains constant over time, often referred to as a fixed, steady-state, or stationary solutions. There are typically two types of equilibrium in epidemiological systems: the disease-free equilibrium (DFE) and the endemic equilibrium. The DFE is characterized by the absence of infection within the population, where all individuals are in the susceptible state and there are no active cases of the disease. The endemic equilibrium represents a state where the disease persists within the population at a constant level.
We set up the foundation for the stability analysis by explaining the different equilibrium points that the system can reach. The focus then shifts to understanding how these points behave when the system experiences small disturbances.

Stability analysis of equilibrium points:

The main objective of stability analysis is to determine whether these equilibrium points are stable or unstable, which indicates whether small perturbations, such as variations in infection rates or changes in control strategies, will lead to a return to equilibrium or cause significant deviations. We now present a fundamental result that plays a crucial role in the stability analysis related to fractional differential equations.

Lemma 1.6 ([130]). *An equilibrium point of system* (1.17) *is locally asymptotically stable if all the eigenvalues* $\lambda_j (j = 1,2,\ldots,n)$ *of the Jacobian matrix* $J = \frac{\partial f}{\partial x}$ *evaluated at the equilibrium satisfy* $\left|\arg(\lambda_j)\right| > \frac{\alpha\pi}{2}$, *and unstable if there exist an eigenvalue* λ_j *such that* $\left|\arg(\lambda_j)\right| < \frac{\alpha\pi}{2}$.

Next, we logically lead to a brief discussion on the basic reproduction number R_0, as it is the key parameter that determines the stability of these equilibrium points.

Basic reproduction number R_0:

The basic reproduction number, denoted as R_0, is a critical threshold parameter in infectious disease transmission. It quantifies the average number of secondary infections produced by a single infectious individual in a completely susceptible population during their infectious period. If $R_0 < 1$, the disease is likely to disappear from the population; conversely, if $R_0 > 1$, the disease may become established. The critical case R_0 represents the situation in which the disease reproduces itself thereby leaving the community with a similar number of infection cases at any time. Therefore, R_0 provides a direct quantitative assessment of the intervention efforts required for disease eradication.
Using the next-generation matrix approach [181], R_0 is determined by decomposing the infected subsystem into two matrices: one representing the rates of new secondary infections $\mathbb{F}$ and the other representing the rates of disease progression, recovery, and mortality $\mathbb{V}$. The basic reproduction number is the spectral radius of the next-generation matrix $(\mathbb{F}.\mathbb{V}^{-1})$.

Key theorems and results

Following our analysis of equilibrium points and the stability of the system, it is crucial to present key lemmas that ensure the existence, uniqueness, boundedness, and positivity of solutions for FDEs under consideration. Although the proofs of these results fall beyond the scope of this preliminary section, stating these lemmas lays the

groundwork for the theoretical framework necessary to analyze the system's dynamics comprehensively.

Lemma 1.7 ([122]). *Suppose that f, the second member of (1.17) satisfies the following conditions*

- *$f(x)$ and $(\partial f/\partial x)(x)$ are continuous on $\mathbb{R}^N$.*
- *$\|f(x)\| \leq \delta_1 + \delta_2\|x\|$ for all $x \in \mathbb{R}^n$, where δ_1 and δ_2 are positive constants.*

Then, the system (1.17) has a unique solution defined on $[0,+\infty)$

Lemma 1.8 (Generalized Mean Value Theorem [146]). *Suppose that $f \in \mathscr{C}([a,b])$ and ${}^{C}D_t^{\alpha} f \in \mathscr{C}((a,b]), 0 < \alpha \leq 1$. Then, one has*

$$f(x) = f(a) + \frac{1}{\Gamma(\alpha)}\left({}^{C}D_t^{\alpha} f\right)(z)(x-a)^{\alpha},$$

with $0 \leq z \leq x$, for all $x \in (a,b]$.

As a consequence of Lemma 1.8, we have the following corollary.

Corollary 1.7. *Suppose that $f \in \mathscr{C}([a,b])$, and ${}^{C}D_t^{\alpha} f(t) \in \mathscr{C}((a,b]), 0 < \alpha \leq 1$. Then,*

(1)- If ${}^{C}D_t^{\alpha} f(t) \geq 0, \forall t \in (a,b]$, then f is increasing.

(2)- If ${}^{C}D_t^{\alpha} f(t) \leq 0, \forall t \in (a,b]$, then f is decreasing.

Lemma 1.9 ([120]). *Let f be a continuous function on $[\tau_0,+\infty)$ with $\tau_0 \geq 0$, we assume that f verifies the following inequality*

$$ {}^{C}D_t^{\alpha} f(t) \leqslant -af(t) + b, \qquad f(\tau_0) = f_0 \in \mathbb{R}, \tag{1.18}$$

where $a,b \in \mathbb{R}$, and $a \neq 0$. Therefore,

$$ f(t) \leqslant \left(f_0 - \frac{b}{a}\right)\mathscr{E}_{\alpha}\left[-a(t-\tau_0)^{\alpha}\right] + \frac{b}{a}. \tag{1.19}$$

Lemma 1.10 ([61]). *Let $0 < \alpha \leq 1$ and $\lambda < 0$. Then, $\mathscr{E}_{\alpha,\alpha}(\lambda t^{\alpha})$ and $\mathscr{E}_{\alpha,\alpha+1}(\lambda t^{\alpha})$ tend monotonically to zero as $t \to \infty$.*

Lemma 1.11. *[1] The solution of the following problem for* $\alpha \in (0,1)$*,*

$$\begin{cases} {}^{ABC}D_t^{\alpha} A(t) = F(t), \quad 0 < t < T, \\ A(0) = A_0, \end{cases} \tag{1.20}$$

is given by

$$A(t) = A_0 + \frac{1-\alpha}{ABC(\alpha)} F(t) + \frac{\alpha}{ABC(\alpha)} \frac{1}{\Gamma(\alpha)} \int_0^t (t-x)^{\alpha-1} F(x) dx. \tag{1.21}$$

1.5.4 Numerical methods for FDEs

For most fractional differential equations, finding an exact analytical solution is quite challenging, so we often rely on numerical methods. While these methods are inspired by those used for regular differential equations, the non-local nature of fractional derivatives makes them different and introduces new challenges that don't appear in the classical case.

This subsection highlights the numerical approaches specifically developed for handling fractional differential equations in the Caputo and ABC sense, shedding light on their underlying algorithms and fundamental principles. We will focus on two numerical methods that are well-adapted and extensively used in our analysis. While we concentrate on these approaches, it is important to note that the literature offers a broader range of numerical techniques for solving FDEs [151, 184].

Numerical methods for Fractional differential equations in Caputo sense

The fractional Adams-Bashforth-Moulton method was selected for this study due to its efficiency and versatility in solving dynamical systems with fractional derivatives. Indeed, the fractional Adams-Bashforth-Moulton method is a reasonable choice, as its stability properties allow for safe use in solving differential equations without significant propagation of rounding errors, while its implementation is not overly time-consuming. This numerical technique was introduced by Diethelm and Freed [77]. The core idea is to apply the Riemann-Liouville operator to the following fractional differential equation

$$\begin{cases} {}^{C}D_t^{\alpha}(y(t)) = f(t, y(t)), \quad 0 < \alpha \leq 1 \\ y(0) = y_0, \end{cases} \tag{1.22}$$

to acquire the following Volterra integral equation

$$y(t) = \sum_{k=0}^{[\alpha]-1} \frac{y_0^k}{k!} t^k + \frac{1}{\Gamma(\alpha)} \int_0^t (t-\tau)^{\alpha-1} f(\tau, y(\tau)) d\tau. \tag{1.23}$$

We assume that y_j is the approximation of $y(t_j)$ for $j = 1,\ldots,n$ within the interval $[0,T]$. To compute y_{n+1} , we substitute the integral in the Volterra equation (1.23) using the trapezoidal quadrature rule. We set $h = \frac{T}{n}$, with $t_j = jh$ for $j = 1,2,3,\ldots,n$. Therefore, the previous equation is discretized to get the corrector formula (the fractional one-step Adams-Moulton method)

$$y_{n+1} = \sum_{k=0}^{\lceil\alpha\rceil-1} \frac{t_{n+1}^k}{k!} y_0^{(k)} + \frac{h^\alpha}{\Gamma(\rho+2)} \sum_{j=0}^{n} a_{j,+n1} f\left(t_j, y_j\right) + \frac{h^\alpha}{\Gamma(\alpha+2)} f\left(t_{n+1}, y_{n+1}^p\right), \tag{1.24}$$

where y^p represents the predictor formula obtained using the Adams-Bashforth method, and it is expressed as follows

$$y_{n+1}^p = \sum_{k=0}^{\lceil\alpha\rceil-1} \frac{y_0^k}{k!} t_{n+1}^k + \frac{1}{\Gamma(\alpha)} \sum_{j=0}^{n} b_{j,n+1} f\left(t_j, y_j\right),$$

with

$$a_{j,n+1} = \begin{cases} k^{\alpha+1} - (n-\alpha)(n+1)^\alpha, & \text{if } j = 0, \\ (n-j+2)^{\alpha+1} + (n-j)^{\alpha+1} - 2(n-j+1)^{\alpha+1}, & \text{if } 0 < j \le n, \\ 1, & \text{if } j = n+1, \end{cases}$$

and

$$b_{j,n+1} = \frac{h^\alpha}{\alpha}\left((n+1-j)^\alpha - (n-j)^\alpha\right).$$

Numerical methods for Fractional differential equations in ABC sense

Now, we present a numerical method specifically designed to approximate the ABC derivative, as introduced by Toufik and Atangana [176]. This method has proven to be particularly effective in handling the non-singular kernel properties of the ABC fractional derivative, making it well-suited for our analysis. Consider the fractional differential equation in ABC sens of the following form

$$\begin{cases} {}^{ABC}D_t^\alpha(y(t)) = f(t,y(t)) \quad 0 < \alpha \le 1, \\ y(0) = y_0. \end{cases} \tag{1.25}$$

Applying the integral in ABC sens to the equation (1.25) to get

$$y(t) - y(0) = \frac{1-\alpha}{ABC(\alpha)} f(t,y(t)) + \frac{\alpha}{ABC(\alpha)\gamma(\alpha)} \int_0^t f(\tau,y(\tau))(t-\tau)^{\alpha-1} d\tau,$$

if we take $t = t_{n+1}$, $n = 0,1,2,\cdots$ we acquire

$$\begin{aligned} y(t_{n+1}) - y(0) &= \frac{(1-\alpha)}{ABC(\alpha)} f(t_n, y(t_n)) + \frac{\alpha}{ABC(\alpha)\times\Gamma(\alpha)} \int_0^{t_{n+1}} f(\tau,y(\tau))(t_{n+1}-\tau)^{\alpha-1}\, \mathrm{d}\tau \\ &= \frac{(1-\alpha)}{ABC(\alpha)} f(t_n, y(t_n)) + \frac{\alpha}{ABC(\alpha)\times\Gamma(\alpha)} \sum_{j=0}^{n} \int_{t_j}^{t_{j+1}} f(\tau,y(\tau))(t_{n+1}-\tau)^{\alpha-1}\, \mathrm{d}\tau. \end{aligned}$$

Within the interval $[t_j, t_{j+1}]$, the function $f(\tau, y(\tau))$ can be approximated through two-step Lagrange polynomial interpolation. By simplifying the integral approximations, we ultimately arrive to the following expression

$$\begin{aligned} y(t) - y(0) = {} & \frac{1-\alpha}{ABC(\alpha)} f(t_n, y_n) \\ & + \frac{\alpha}{ABC(\alpha)} \sum_{j=0}^{n} \left(\frac{h^\alpha f(t_j, y_j)}{\Gamma(\alpha+2)} \left((n+1-j)^\alpha (n-j+2+\alpha) - (n-j)^\alpha (n-j+2+2\alpha) \right) \right. \\ & \left. - \frac{h^\alpha f(t_{j-1}, y_{j-1})}{\Gamma(\alpha+2)} \left((n+1-j)^{\alpha+1} - (n-j)^\alpha (n-j+1+\alpha) \right) \right). \end{aligned}$$

Part I

Applications to fractional isotropic problems

CHAPTER 2

STUDY OF SOME FRACTIONAL $p(x,\cdot)$-KIRCHHOFF-TYPE PROBLEMS

Main objective

This chapter is divided into two main sections, each addressing significant aspects of Kirchhoff-type problems in fractional Sobolev space with variable exponents and its generalized versions. The first section is dedicated to examining the existence of weak solutions for a category of bi-nonlocal Kirchhoff-type problems incorporating the fractional $p(x,\cdot)$-Laplacian operator through variational methods. We address distinct classes of Kirchhoff functions, employing the mountain pass theorem and Ekeland's variational principle to achieve our results. In the second section, attention is directed to a specialized class of $p(x,\cdot)$-integro-differential Kirchhoff-type problems under Dirichlet boundary conditions. Through the application of various variational techniques, we establish the existence of multiple solutions. Our analysis considers different scenarios related to the problem's nonlinearity and growth conditions, providing a comprehensive understanding of solution behavior across diverse cases.

The results presented in this chapter are derived from the study outlined in [5, 29].

2.1 Introduction

Kirchhoff-type problems have become a cornerstone in mathematical physics, offering powerful models for understanding various phenomena. These equations extend classical models by incorporating the influence of the material's geometry and other nonlocal effects, making them particularly valuable for studying complex behaviors. In 1883 [109], Kirchhoff introduced the original one-dimensional Kirchhoff equation to model the oscillations of an elastic string clamped at both ends, it is considered an extension of the classical d'Alembert's wave equation. This extension takes into account the changes in string length during free vibrations. Such vibrations produce steady amplitude when no external forces are applied to the body after the initial displacement. For the one-dimensional scenario, the Kirchhoff-type equation is expressed as follows

$$\rho\frac{\partial^2 u}{\partial t^2}-\left(\frac{\rho_0}{h}+\frac{E}{2L}\int_0^L\left|\frac{\partial u}{\partial x}\right|^2 dx\right)\frac{\partial^2 u}{\partial x^2}=0, \tag{2.1}$$

where $u=u(x,t)$ denotes the lateral displacement at time t and spatial coordinate x, ρ,E,L,h, and ρ_0 are parameters described the mass density, the Young modulus, the length of the string, the area of cross-section, and the initial tension respectively. Alternatively, the stationary form of equation (2.1) is expressed as follows

$$\begin{cases} -\left(a+b\int_\Omega |\nabla u|^2\,\mathrm{d}x\right)\Delta u=f(x,u) & \text{in } \Omega \\ u=0 & \text{on } \partial\Omega. \end{cases} \tag{2.2}$$

This formulation has garnered considerable attention since Lions's seminal work [123] by proposing a functional analysis to attack it. Following this work, other researchers have investigated various Kirchhoff-type problems, starting from the classical case and expanding to more general cases.

- In the classical case, the authors in [8] established the existence of positive solutions for a quasilinear elliptic equation of Kirchhoff-type of the form

$$-M\left(\int_{\mathscr{O}}|\nabla u|^2 dx\right)\Delta u=f(x,u) \text{ in } \Omega, \quad u=0 \quad \text{on } \partial\Omega,$$

  where M represents the Kirchhoff function that should verify certain assumptions. As well as a particular case of the previous problem stated as follows

$$\begin{cases} \left(a+b\displaystyle\int_\Omega |\nabla u|^2 dx\right)\Delta u=f(x,u) & \text{in } \Omega, \\ u=0 & \text{on } \partial\Omega, \end{cases}$$

where $a, b > 0$, has been widely studied [9, 126, 149, 191]. Kirchhoff-type problems of this form have been the focus of extensive research and have attracted considerable interest due to their diverse applications, including:

- Modeling bacterial population growth in biological contexts.
- Analyzing beam deformation under bending stress.
- Studying vibrations in plates or tensioned cables.

- In the case where p is constant, the investigation of Kirchhoff-type equations has been extended to encompass cases involving the p-Laplacian operator, represented in the following form

$$-M\left(\int_\Omega \frac{1}{p}|\nabla u|^p dx\right)\Delta_p u = f(x,u) \text{ in } \Omega \text{ with } u = 0 \text{ on the boundary } \partial\Omega, \tag{2.3}$$

where M and f must satisfy certain conditions. Further details on these conditions can be found in references [66, 81, 103].

- In the other case (when p is variable), $p(x)$-Kirchhoff-type problems were initially studied by Dai and Hao [70], who extended Kirchhoff's problems to include the variable exponent involved by the $p(x)$-Laplacian operator. As well as the paper [87] investigates nonlocal $p(x)$-Laplacian Dirichlet problems in both non-variational and variational forms, using (S_+) mapping theory and variational methods. Furthermore, Chung in [62] used the mountain pass theorem along with Ekeland's variational principle to obtain at least two distinct nontrivial weak solutions for the problem

$$-M\left(\int_\Omega \frac{1}{p(x)}|\nabla u|^{p(x)} dx\right)\Delta_{p(x)} u = \lambda f(x,\Omega) \text{ in } \Omega \text{ with } u = 0 \text{ on } \partial\Omega. \tag{2.4}$$

See [71] for further studies.

On the other hand, the incipience of a generalization including the fractional case was by Fiscella and Valdinocci [90], the authors have proposed a stationary Kirchhoff model in the fractional scenario arising from an interesting physical interpretation. In their correction of the initial (one-dimensional) model, they replaced the local spacial second derivative with the nonlocal operator $(-\Delta)^s$, according to this change, the tension on the string which classically has a "nonlocal" nature from the mean of the kinetic energy $\frac{|\partial_x u|^2}{2}$ on $[0, L]$, has a further nonlocal behavior provided by the H^s-norm of the function

u. Moreover, the authors investigated the existence of multiple solutions for a Kirchhoff-type problem involving a nonlocal integro-differential operator. More precisely, they studied the following problem

$$\text{(2.5)}\quad \begin{cases} -M\left(\int_{\mathbb{R}^{2N}} |u(x)-u(y)|^2 K(x-y)dxdy\right)\mathscr{L}_K u = \lambda f(x,u)+|u|^{2^*-2}u & \text{in} \quad \Omega, \\ u=0 & \text{in} \quad \mathbb{R}^N\setminus\Omega, \end{cases}$$

with $2^* = \frac{2N}{N-2s}$, λ is a positive parameter, the two functions M and f are continuous and satisfying some hypothesis.

Furthermore, other researchers have extended the scalar case to address Kirchhoff-type problems involving the fractional p-Laplacian operator. To enrich the references in this context, we draw attention to these works [25, 26, 45, 88, 133, 152, 187], each addressing a unique problem despite the variational methodologies employed, the specific non-local operators considered, the conditions satisfied by the source term, and the appropriate frameworks considered to handle the problem. This approach not only demonstrates the comparative nature of these problems but also underscores their intrinsic significance.

Top on that, we shall not overlook that this class of problems inherently combines both nonlocal and nonlinear behavior, wherein fractional Kirchhoff-type problems with variable exponents are examined. This area of study has witnessed a significant surge in interest in recent years, as evidenced by various scholarly works and references [7, 24, 183]. Nevertheless, the existing research remains relatively limited, underscoring the potential for further investigation.

In the context of the bi-nonlocal problems, Fan considered this type of problem for the first time in [85]. These are also known as doubly nonlocal problems because non-locality appears in both the integro-differential operator and the coefficient of the source term f. This type of problem describes many systems, for example in physics, the term source could be considered as a function depending on the average of the potential energy. Such problems have been studied under Dirichlet and Neumann boundary conditions [56, 65, 66], but their investigation involving the fractional $p(x,\cdot)$-Laplacian operator remains limited.

More generally, Corrêa and Costa [103] studied the following problem

$$\text{(2.6)}\quad \begin{cases} -M\left(\int_\Omega \frac{1}{p(x)}|\nabla u|^{p(x)}\right)\Delta_{p(x)}u &= \lambda f(x,u)\left[\int_\Omega F(x,u)dx\right]^r \quad \text{in} \quad \Omega, \\ u &= 0 \quad \text{on } \partial\Omega. \end{cases}$$

The authors dealt with the case where $f(x,u)=|u|^{q(x)-2}u$ and used variational methods, namely the mountain pass theorem and Ekeland's variational principle, to prove several

results of the existence of weak solutions. When $\lambda = 1$, the problem (2.6) has been studied in [66], via Krasnoselkii's genus theory, where the authors established the existence and multiplicity of solutions following some hypotheses on M and f.

2.2 On a bi-nonlocal $p(x,\cdot)$-fractional Kirchhoff-type problem

Given the above discussion, the primary aim of the subsequent section is to establish the existence of weak solutions for a bi-nonlocal fractional $p(x,\cdot)$-Kirchhoff-type problems involving integro-differential operator.

2.2.1 Hypothesis and problem statement

We consider the following problem

$$(\mathscr{P}_\lambda)\begin{cases} M\left(\sigma_{p(x,y)}(u)\right)(-\Delta)^s_{p(x,\cdot)}(u) = \lambda f(x,u)\left[\int_\Omega F(x,u)\,dx\right]^\alpha, & \text{in } \Omega,\\ u = 0, & \text{in } \mathbb{R}^N \setminus \Omega, \end{cases}$$

where

- $\sigma_{p(x,y)}(u)$ is defined by the following formula

$$\sigma_{p(x,y)}(u) = \int_Q \frac{1}{p(x,y)} \frac{|u(x)-u(y)|^{p(x,y)}}{|x-y|^{N+sp(x,y)}}\,dx\,dy,$$

- $M : \mathbb{R}^+ \to \mathbb{R}^+$ is the so-called Kirchhoff function that is supposed to be continuous.
- λ and α are two positive real parameters.
- $f : \Omega \times \mathbb{R} \to \mathbb{R}$ is a Carathéodory function, ensuring measurability in x and continuity in t, where $F(x,t) = \int_0^t f(x,\tau)\,d\tau$ is its primitive.

To establish the existence of solutions, we assume that the function f satisfies the following assumptions

- (f_1): There exist constants $0 < l_1 \le l_2$ such that

$$l_1|t|^{q(x)-1} \le f(x,t) \le l_2|t|^{q(x)-1}, \quad \forall (x,t) \in \Omega \times \mathbb{R},$$

where $q \in C^+(\overline{\Omega})$ and satisfies $q(x) < p_s^*(x) = \frac{N\bar{p}(x)}{N - s\bar{p}(x)}$ for all $x \in \overline{\Omega}$.

- (f_2): There exists $\theta > 0$ such that

$$0 < \theta F(x,t) \leq t f(x,t), \quad \forall (x,t) \in \Omega \times \mathbb{R}.$$

We say that $u \in W_0$ is a weak solution of problem $(\mathscr{P}_\lambda)$ if we have

$$\begin{aligned} &M\left(\sigma_{p(x,y)}(u)\right) \int_Q \frac{|u(x)-u(y)|^{p(x,y)-2}(u(x)-u(y))(v(x)-v(y))}{|x-y|^{N+sp(x,y)}} dxdy \\ &= \lambda \left(\int_\Omega F(x,u)dx \right)^\alpha \left(\int_\Omega f(x,u)v dx \right), \qquad \text{for all } v \in W_0. \end{aligned}$$

Now, we are in the position to state our main results. The following theorem is established under the condition that $N > sp^+$.

Theorem 2.1. ***1)-*** *Suppose that $m_0 \leqslant M(t) \leqslant m_1$, where m_0 and m_1 are two positive constants, and we also assume that $\theta \geqslant p^+$.*

- (i)- *If $q^-(\alpha+1) > p^+$, then problem $(\mathscr{P}_\lambda)$ admits a weak solution for all $\lambda > 0$.*
- (ii)- *If $p^- > (\alpha+1)q(x)$, for all $x \in \overline{\Omega}$. Then, there exists $\hat{\lambda} > 0$ such that every $\lambda \in (0, \hat{\lambda})$ is an eigenvalue of problem $(\mathscr{P}_\lambda)$.*

2)- *Assume that $M(t) = A + Bt$, with A and B are two positive parameters, $t \in \mathbb{R}^+$. Additionally, suppose that $\theta \geqslant 2p^+$.*

- (i)- *If $q^-(\alpha+1) > 2p^+$, then for all $\lambda > 0$, problem $(\mathscr{P}_\lambda)$ has a weak solution.*
- (ii)- *If $p^- > (\alpha+1)q(x)$ for all $x \in \overline{\Omega}$, then problem $(\mathscr{P}_\lambda)$ possesses a weak solution for all $\lambda \in (0, \lambda^*)$ with $\lambda^* > 0$.*

3)- *Assume that $M(t) = t^{\beta-1}$, for all $t \in \mathbb{R}^+$ with $\beta \geqslant 2$. Moreover, suppose that $\theta \geqslant \beta p^+$.*

- (i)- *If $q^-(\alpha+1) > \beta p^+$, then problem $(\mathscr{P}_\lambda)$ admits a weak solution for all $\lambda > 0$.*
- (ii)- *If $p^- > (\alpha+1)q(x)$ for all $x \in \overline{\Omega}$, then there exists $\tilde{\lambda} > 0$ such that problem $(\mathscr{P}_\lambda)$ admits a weak solution for each $\lambda \in (0, \tilde{\lambda})$.*

4)- *Suppose that $M(t) = A + Bt^{\beta-1}$, where $\beta > 1$, $A > 0$, $B > 0$, and $t \geqslant 0$. Moreover, suppose that $\theta \geqslant \beta p^+$.*

- (i)- *If $q^-(\alpha+1) > \beta p^+$, then problem $(\mathscr{P}_\lambda)$ admits a weak solution for all $\lambda > 0$.*
- (ii)- *If $p^- > (\alpha+1)q(x)$ for all $x \in \overline{\Omega}$, then there exists $\bar{\lambda} > 0$ such that problem $(\mathscr{P}_\lambda)$ admits a weak solution for each $\lambda \in (0, \bar{\lambda})$.*

2.2.2 Existence results

The functional $\mathscr{J}_\lambda$ defined as follows

$$\begin{aligned}\mathscr{J}_\lambda : W_0 &\longrightarrow \mathbb{R}\\ u &\longrightarrow \widehat{M}\left(\sigma_{p(x,y)}(u)\right) - \frac{\lambda}{\alpha+1}\left[\int_\Omega F(x,u)dx\right]^{\alpha+1},\end{aligned}$$

is the Euler-Lagrange functional associated to problem $(\mathscr{P}_\lambda)$, where $\widehat{M}(t) = \int_0^t M(\tau)d\tau$, for $t \in \mathbb{R}^+$.

Accordingly to Proposition 1.5, Hölder's inequality, condition (f_1), and the embedding result stated in Remark 1.1, it is clear that $\mathscr{J}_\lambda$ is well-defined. Moreover, given that M is continuous and f is a Carathéodory function ensure that $\mathscr{J}_\lambda \in \mathscr{C}^1(W_0,\mathbb{R})$. Furthermore, by standard argument to [20], the derivative of $\mathscr{J}_\lambda$ is given by

$$\begin{aligned}&\left\langle \mathscr{J}'_\lambda(u), v\right\rangle =\\ &M\left(\sigma_{p(x,y)}(u)\right)\int_Q \frac{|u(x)-u(y)|^{p(x,y)-2}(u(x)-u(y))(v(x)-v(y))}{|x-y|^{N+sp(x,y)}}dxdy\\ &-\lambda\left(\int_\Omega F(x,u)dx\right)^\alpha \int_\Omega f(x,u)vdx, \qquad \forall (u,v)\in W_0\times W_0.\end{aligned}$$

Therefore, a critical point of $\mathscr{J}_\lambda$ is obviously a weak solution of the problem $(\mathscr{P}_\lambda)$.

We will now go over the proof of Theorem 2.1 in detail using Ekeland's variational principle and the mountain pass theorem. It is worth mentioning that C stands for the general positive constant that may vary from line to line.

Proof of Theorem 2.1-(1):

To prove our main results in this case, we prove first that $\mathscr{J}_\lambda$ verifies the Palais-Smale condition.

Let u_n be a $(PS)_c$ sequence for any $c \neq 0$, this is

$$\mathscr{J}_\lambda(u_n) \longrightarrow c \quad \text{and} \quad \mathscr{J}'_\lambda(u_n) \underset{n\longrightarrow+\infty}{\longrightarrow} 0 \quad \text{in} \quad W_0^*. \tag{2.7}$$

Initially, we demonstrate that $\{u_n\}$ is bounded. Indeed, for any $\{u_n\} \subset W_0$ with $\|u_n\|_{W_0}$ sufficiently large, we have

$$c + \theta_n(1)\|u_n\|_{W_0} \geqslant \mathscr{J}_\lambda(u_n) - \frac{1}{\theta}\left\langle \mathscr{J}_\lambda'(u_n), u_n\right\rangle,$$

which implies that

$$\begin{aligned} c+\theta_n(1)\|u_n\|_{W_0} \geqslant & \widehat{M}\left(\sigma_{p(x,y)}(u_n)\right)-\frac{\lambda}{\alpha+1}\left[\int_\Omega F(x,u_n)dx\right]^{\alpha+1} \\ & -\frac{1}{\theta}M\left(\sigma_{p(x,y)}(u_n)\right)\left(\int_Q \frac{|u_n(x)-u_n(y)|^{p(x,y)}}{|x-y|^{N+sp(x,y)}}dxdy\right) \\ & +\frac{\lambda}{\theta}\left[\int_\Omega F(x,u_n)dx\right]^{\alpha}\int_\Omega f(x,u_n)u_n dx \\ \geqslant & m_0\left(\sigma_{p(x,y)}(u_n)\right)-\frac{\lambda}{\alpha+1}\left[\int_\Omega F(x,u_n)dx\right]^{\alpha+1} \\ & -\frac{m_1}{\theta}\left(\rho^0_{p(x,y)}(u_n)\right)+\frac{\lambda}{\theta}\left[\int_\Omega F(x,u_n)dx\right]^{\alpha}\int_\Omega f(x,u_n)u_n dx, \end{aligned}$$

using condition (f_2) and Proposition 1.5, we acquire

$$\begin{aligned} c+\theta_n(1)\|u_n\|_{W_0} \geqslant & \left(\frac{m_0}{p^+}-\frac{m_1}{\theta}\right)\rho^0_{p(x,y)}(u_n) \\ & +\lambda\left[\int_\Omega F(x,u_n)dx\right]^{\alpha}\left[\int_\Omega\left(\frac{1}{\theta}f(x,u_n)u_n-\frac{F(x,u_n)}{\alpha+1}\right)dx\right] \\ \geqslant & \left(\frac{m_0}{p^+}-\frac{m_1}{\theta}\right)\|u_n\|^{p^-}_{W_0}+\frac{\alpha\lambda}{\alpha+1}\left[\int_\Omega F(x,u_n)dx\right]^{\alpha+1}. \end{aligned}$$

As a consequence, the sequence $\{u_n\}$ is bounded in the space W_0. Given that W_0 is a reflexive Banach space, there exists $u\in W_0$ such that for a subsequence of $\{u_n\}$ still denoted by $\{u_n\}$, we have

$$\begin{aligned} & u_n \rightharpoonup u \text{ in } W_0, \\ & u_n \to u \text{ a.e. in } \Omega, \\ & u_n \to u \text{ in } L^{q(x)}(\Omega). \end{aligned} \tag{2.8}$$

From (2.7), we get that

$$\left\langle \mathscr{J}_\lambda'(u_n), u_n-u\right\rangle \longrightarrow 0 \text{ as } n\longrightarrow+\infty,$$

which implies that

$$\begin{aligned} & M\left(\sigma_{p(x,y)}(u_n)\right)\int_Q \frac{|u_n(x)-u_n(y)|^{p(x,y)-2}(u_n(x)-u_n(y))[(u_n-u)(x)-(u_n-u)(y)]}{|x-y|^{N+sp(x,y)}}dxdy \\ & -\lambda\left(\int_\Omega F(x,u_n)dx\right)^{\alpha}\int_\Omega f(x,u_n)(u_n-u)dx\longrightarrow 0 \quad \text{as } n\longrightarrow\infty. \end{aligned}$$

On the other hand, from condition (f_1) and by using Hölder's inequality, we obtain

$$\begin{aligned} \left|\int_\Omega f(x,u_n)(u_n-u)dx\right| & \leqslant \int_\Omega |f(x,u_n)||(u_n-u)|dx \\ & \leqslant l_2\int_\Omega |u_n|^{q(x)-1}|(u_n-u)|dx \\ & \leqslant C\left\||u_n|^{q(x)-1}\right\|_{L^{\frac{q(x)}{q(x)-1}}(\Omega)}\|u_n-u\|_{L^{q(x)}(\Omega)}. \end{aligned}$$

It follows from (2.8) that

$$\int_\Omega f(x,u_n)(u_n-u)dx \longrightarrow 0, \quad \text{as} \quad n \longrightarrow +\infty.$$

And thus,

$$\lim_{n\to+\infty}\left(\int_\Omega F(x,u_n)dx\right)^\alpha \int_\Omega f(x,u_n)(u_n-u)dx = 0.$$

Accordingly,

$$\lim_{n\to\infty} M\left(\sigma_{p(x,y)}(u_n)\right) \int_Q \frac{|u_n(x)-u_n(y)|^{p(x,y)-2}(u_n(x)-u_n(y))((u_n-u)(x)-(u_n-u)(y))}{|x-y|^{N+sp(x,y)}}dxdy = 0. \tag{2.9}$$

As u_n is bounded on W_0, we may assume that

$$\sigma_{p(x,y)}(u_n) \longrightarrow t_0 \geqslant 0 \quad \text{as} \quad n \to +\infty.$$

If $t_0 = 0$, the proof is complete. If $t_0 > 0$, by the continuity of M we get that

$$M\left(\sigma_{p(x,y)}(u_n)\right) \longrightarrow M(t_0) \quad \text{as} \quad n \longrightarrow +\infty.$$

Unifying the previous result with (2.9) to get that the limit of

$$\int_Q \frac{|u_n(x)-u_n(y)|^{p(x,y)-2}(u_n(x)-u_n(y))((u_n-u)(x)-(u_n-u)(y))}{|x-y|^{N+sp(x,y)}}dxdy,$$

goes to 0 as $n \longrightarrow +\infty$. By analogy, we have

$$\int_Q \frac{|u(x)-u(y)|^{p(x,y)-2}(u(x)-u(y))((u_n-u)(x)-(u_n-u)(y))}{|x-y|^{N+sp(x,y)}}dxdy \underset{n\longrightarrow+\infty}{\longrightarrow} 0.$$

Combining the two previous results, it follows that

$$\limsup_{n\to+\infty}\langle \mathscr{L}(u_n)-\mathscr{L}(u), u_n-u\rangle \leqslant 0.$$

From Lemma 1.3 the mapping $\mathscr{L}$ is of type (S_+), then the strong convergence of u_n to u in W_0 is achieved. Consequently, $\mathscr{J}_\lambda$ verifies the Palais-Smale compactness condition.

Proof of 1-(i)

Now, we proceed to establish the geometrical conditions of the mountain pass theorem. Let $\lambda > 0$, and for any $u \in W_0$ with $\|u\|_{W_0}$ sufficiently small, we have in view of condition (f_1)

$$\begin{aligned}\mathscr{J}_\lambda(u) &= \widehat{M}\left(\sigma_{p(x,y)}(u)\right) - \frac{\lambda}{\alpha+1}\left[\int_\Omega F(x,u)dx\right]^{\alpha+1} \\ &\geqslant \frac{m_0}{p^+}\rho^0_{p(x,y)}(u) - \frac{\lambda}{\alpha+1}\left(\frac{l_2}{q^-}\right)^{\alpha+1}\left(\rho_{L^{q(x)}(\Omega)}(u)\right)^{\alpha+1}.\end{aligned}$$

Making use of Proposition 1.2, Proposition 1.5, and the embedding result stated in Remark 1.1, we obtain at taking $\|u\|_{W_0} = \rho > 0$ that

$$\begin{aligned}\mathscr{J}_\lambda(u) \geqslant & \frac{m_0}{p^+}\|u\|^{p^+}_{W_0} - C^{q^-(\alpha+1)}\frac{\lambda}{\alpha+1}\left(\frac{l_2}{q^-}\right)^{\alpha+1}\|u\|^{q^-(\alpha+1)}_{W_0} \\ = & \rho^{p^+}\left(\frac{m_0}{p^+} - C^{q^-(\alpha+1)}\frac{\lambda}{\alpha+1}\left(\frac{l_2}{q^-}\right)^{\alpha+1}\rho^{q^-(\alpha+1)-p^+}\right).\end{aligned} \tag{2.10}$$

Consider the following function g_1 defined as follows

$$\begin{aligned} g_1 : [0,1] &\longrightarrow \mathbb{R} \\ t &\longrightarrow \frac{m_0}{p^+} - C^{q^-(\alpha+1)}\frac{\lambda}{\alpha+1}\left(\frac{l_2}{q^-}\right)^{\alpha+1} t^{q^-(\alpha+1)-p^+}.\end{aligned}$$

It is obvious to see that g_1 is strictly positive in a neighborhood of zero. Therefore, we can find a constant $\gamma > 0$ such that $\mathscr{J}_\lambda(u) \geqslant \gamma$, for all $u \in W_0$ with $\|u\|_{W_0} = \rho$. Consequently, the proof of the first geometrical condition (R_1) of the mountain pass theorem is established. Furthermore, for any $v \in W_0 \setminus \{0\}$ and $t > 1$, we get from condition (f_1)

$$\begin{aligned}\mathscr{J}_\lambda(tv) &= \widehat{M}\left(\sigma_{p(x,y)}(tv)\right) - \frac{\lambda}{\alpha+1}\left[\int_\Omega F(x,tv)dx\right]^{\alpha+1} \\ &\leqslant m_1\sigma_{p(x,y)}(tv) - \frac{\lambda}{\alpha+1}\left(\frac{l_1}{q^+}\right)^{\alpha+1}\left(\int_\Omega |tv|^{q(x)}dx\right)^{\alpha+1} \\ &\leqslant \frac{m_1 t^{p^+}}{p^-}\rho^0_{p(x,y)}(v) - t^{q^-(\alpha+1)}\frac{\lambda}{\alpha+1}\left(\frac{l_1}{q^+}\right)^{\alpha+1}\left(\int_\Omega |v|^{q(x)}dx\right)^{\alpha+1},\end{aligned}$$

since $q^-(\alpha+1) > p^+$, we get that $\mathscr{J}_\lambda(tv) \longrightarrow -\infty$ as $t \longrightarrow +\infty$. Then, for all $t > 1$ there exists $u_0 = tv$ such that $\|u_0\|_{W_0} > \rho$, and $\mathscr{J}_\lambda(u_0) < 0$.

In conclusion, the geometrical conditions of the mountain pass theorem are fulfilled, as well as $\mathscr{J}_\lambda(0) = 0$ and $\mathscr{J}_\lambda$ verifies the Palais-Smale condition, in the light of the mountain pass theorem, it follows that $\mathscr{J}_\lambda$ has a critical point, thus problem $(\mathscr{P}_\lambda)$ admits at least a non-trivial weak solution.

Proof of 1-(ii)

Next, we will apply the Ekeland's variational principle under appropriate assumptions on the variable exponent q to establish our main result . To do so, we begin by stating and proving several crucial lemmas. These lemmas form the foundational components of our investigation and are essential for the subsequent analysis.

Lemma 2.1. *Let Ω be a smooth bounded domain in $\mathbb{R}^N$ and $s \in (0,1)$. Consider $p : \overline{Q} \longrightarrow \mathbb{R}$ as a continuous exponent variable that satisfies conditions (1.2) and (1.3) with $N > sp^+$. Moreover, we assume that $p^- > (\alpha+1)q(x)$ for all $x \in \overline{\Omega}$. Then, there exists a constant $\hat{\lambda} > 0$ such that for any $\lambda \in (0, \hat{\lambda})$, there exist $\rho, \gamma > 0$ such that for any $u \in W_0$ with $\|u\|_{W_0} = \rho$, we have $\mathscr{J}_\lambda(u) \geq \gamma > 0$.*

Proof. Let $u \in W_0$ such that $\|u\|_{W_0} = \rho$. From inequality (2.10), we have

$$\begin{aligned} \mathscr{J}_\lambda(u) &\geqslant \frac{m_0}{p^+}\|u\|_{W_0}^{p^+} - C^{q^-(\alpha+1)}\frac{\lambda}{\alpha+1}\left(\frac{l_2}{q^-}\right)^{\alpha+1}\|u\|_{W_0}^{q^-(\alpha+1)} \\ &= \frac{m_0}{p^+}\rho^{p^+} - C^{q^-(\alpha+1)}\frac{\lambda}{\alpha+1}\left(\frac{l_2}{q^-}\right)^{\alpha+1}\rho^{q^-(\alpha+1)} \\ &= \rho^{q^-(\alpha+1)}\left(\frac{m_0}{p^+}\rho^{p^+-q^-(\alpha+1)} - C^{q^-(\alpha+1)}\frac{\lambda}{\alpha+1}\left(\frac{l_2}{q^-}\right)^{\alpha+1}\right). \end{aligned} \tag{2.11}$$

Next, we choose λ such that

$$\frac{m_0}{p^+}\rho^{p^+-q^-(\alpha+1)} - C^{q^-(\alpha+1)}\frac{\lambda}{\alpha+1}\left(\frac{l_2}{q^-}\right)^{\alpha+1} > 0.$$

Thus we define

$$\hat{\lambda} = \frac{m_0}{2p^+}\left(\frac{q^-}{l_2}\right)^{\alpha+1}\frac{(\alpha+1)\rho^{p^+-q^-(\alpha+1)}}{C^{q^-(\alpha+1)}}. \tag{2.12}$$

Therefore, for any $\lambda \in (0, \hat{\lambda})$, and any $u \in W_0$ with $\|u\|_{W_0} = \rho$, there exists $\gamma = \frac{m_0}{2p^+}\rho^{p^+} > 0$ such that $\mathscr{J}_\lambda(u) \geqslant \gamma > 0$. ■

Lemma 2.2. *Let Ω be a smooth bounded domain in $\mathbb{R}^N$ and $s \in (0,1)$. Consider $p : \overline{Q} \longrightarrow \mathbb{R}$ as a continuous exponent variable that satisfies conditions (1.2) and (1.3) with $N > sp^+$. We assume that $p^- > (\alpha+1)q(x)$, for all $x \in \overline{\Omega}$. Then, for t small enough, there exists $\psi \in W_0$ such that $\psi \geqslant 0$, $\psi \neq 0$, and $\mathscr{J}_\lambda(t\psi) < 0$.*

Proof. We have that $p^- > (\alpha+1)q(x)$, for all $x \in \overline{\Omega}$, then $p^- > (\alpha+1)q^-$. Let $\varepsilon > 0$ such that $p^- \geqslant (\alpha+1)q^- + (\alpha+1)\varepsilon$, the fact that $q \in C^+(\overline{\Omega})$ merely means the existence of an open set $\Omega_0 \subset \Omega$ such that

$$|q(x) - q^-| \leqslant \varepsilon \text{ for all } x \in \Omega_0,$$

therefore,

$$p^- \geqslant (\alpha+1)q^- + (\alpha+1)\varepsilon \geqslant (\alpha+1)q(x) \text{ for all } x \in \Omega_0.$$

Take $\psi \in C_0^\infty(\overline{\Omega})$ such that $supp\psi \supset \overline{\Omega}_0$, $\psi(x) = 1$ for all $x \in \Omega_0$ and $0 \leqslant \psi \leqslant 1$ in Ω. Then, we get for $t \in (0,1)$ that

$$\begin{aligned}\mathscr{J}_\lambda(t\psi) &\leqslant \frac{m_1}{p^-}\int_Q \frac{\left|t\psi(x)-t\psi(y)\right|^{p(x,y)}}{|x-y|^{N+sp(x,y)}}dxdy - \frac{\lambda}{\alpha+1}\left(\frac{l_1}{q^+}\right)^{\alpha+1}\left(\int_\Omega |t\psi|^{q(x)}dx\right)^{\alpha+1}\\ &\leqslant \frac{m_1}{p^-}t^{p^-}\rho^0_{p(x,y)}(\psi) - \frac{\lambda}{\alpha+1}\left(\frac{l_1}{q^+}\right)^{\alpha+1} t^{(\alpha+1)q^-+(\alpha+1)\varepsilon}\left(\int_{\Omega_0}|\psi|^{q(x)}dx\right)^{\alpha+1},\end{aligned}$$

on the other hand, we have that $0 \leqslant \psi \leqslant 1$ in Ω, then

$$\int_\Omega |\psi(x)|^{q(x)}dx \geqslant \int_{\Omega_0}|\psi(x)|^{q(x)}dx > 0, \tag{2.13}$$

since $q(x) < p_s^*(x) \quad \forall x \in \overline{\Omega}$, we acquire by the Sobolev embedding stated in Remark 1.1 that

$$\|\psi\|_{L^{q(x)}(\Omega)} \leqslant C\|\psi\|_{W_0}. \tag{2.14}$$

Consequently, from (2.13), (2.14), and Proposition 1.5, we obtain that

$$\rho^0_{p(x,y)}(\psi) > 0.$$

In conclusion, $\mathscr{J}_\lambda(t\psi) < 0$ for any $t < \xi^{\frac{1}{p^- -(\alpha+1)q^- -(\alpha+1)\varepsilon}}$, where

$$0 < \xi < \min\left\{1, \frac{\frac{\lambda}{\alpha+1}\left(\frac{l_1}{q^+}\right)^{\alpha+1}\frac{p^-}{m_1}\left(\int_{\Omega_0}|\psi(x)|^{q(x)}dx\right)^{\alpha+1}}{\rho^0_{p(x,y)}(\psi)}\right\}.$$

Therefore, the desired result is proved. ■

Having stated and proven the two previous lemmas, we can now proceed to complete the proof of 1-(ii).
Let $\hat{\lambda} > 0$ defined as in (2.12). Therefore, using Lemma 2.1, we have

$$\inf_{\partial B_\rho(0)} \mathscr{J}_\lambda > 0 \quad \forall \lambda \in (0,\hat{\lambda}), \tag{2.15}$$

where $\partial B_\rho(0) = \{v \in B_\rho(0); \|v\|_{W_0} = \rho\}$ and $B_\rho(0)$ is the ball centred at the origin with radius ρ in W_0.
On the other hand, according to inequality (2.10), for $v \in B_\rho(0)$, we have

$$\mathscr{J}_\lambda(v) \geqslant \frac{m_0}{p^+}\|v\|^{p^+}_{W_0} - C^{q^-(\alpha+1)}\frac{\lambda}{\alpha+1}\left(\frac{l_2}{q^-}\right)^{\alpha+1}\|v\|^{q^-(\alpha+1)}_{W_0}.$$

It follows that

$$-\infty < \underline{c} := \inf_{B_\rho(0)} \mathcal{J}_\lambda < 0. \tag{2.16}$$

Because of (2.15), and (2.16), we may assume that

$$0 < \varepsilon < \inf_{\partial B_\rho(0)} \mathcal{J}_\lambda - \inf_{B_\rho(0)} \mathcal{J}_\lambda.$$

Applying Ekeland's variational principle to the mapping $\mathcal{J}_\lambda : \overline{B_\rho(0)} \longrightarrow \mathbb{R}$, we find $v_\varepsilon \in \overline{B_\rho(0)}$ such that

$$\begin{cases} \mathcal{J}_\lambda(v_\varepsilon) \leqslant \inf_{B_\rho(0)} \mathcal{J}_\lambda + \varepsilon, \\ \mathcal{J}_\lambda(v_\varepsilon) < \mathcal{J}_\lambda(v) + \varepsilon \|v - v_\varepsilon\|_{W_0}, \quad \text{for } v \neq v_\varepsilon. \end{cases}$$

Which implies that

$$\mathcal{J}_\lambda(v_\varepsilon) \leq \inf_{B_\rho(0)} \mathcal{J}_\lambda + \varepsilon \leq \inf_{B_\rho(0)} \mathcal{J}_\lambda + \varepsilon < \inf_{\partial B_\rho(0)} \mathcal{J}_\lambda.$$

Consider the mapping

$$\begin{aligned} \mathcal{I}_\lambda^\varepsilon : \overline{B_\rho(0)} &\longrightarrow \mathbb{R} \\ v &\longrightarrow \mathcal{J}_\lambda(v) + \varepsilon \|v - v_\varepsilon\|_{W_0}. \end{aligned}$$

It is clear that v_ε is the minimizer of $\mathcal{I}_\lambda^\varepsilon$ over $\overline{B_\rho(0)}$, meaning that for every $t \in (0,1)$ and every $w \in B_\rho(0)$,

$$\frac{\mathcal{I}_\lambda^\varepsilon(v_\varepsilon + tw) - \mathcal{I}_\lambda^\varepsilon(v_\varepsilon)}{t} \geqslant 0.$$

Moreover, this acquires that

$$\frac{\mathcal{J}_\lambda(v_\varepsilon + tw) - \mathcal{J}_\lambda(v_\varepsilon)}{t} + \varepsilon \|w\|_{W_0} \geqslant 0,$$

where $\mathcal{J}_\lambda$ is related to $\mathcal{I}_\lambda^\varepsilon$ through the expression given previously.
As $t \longrightarrow 0^+$, we get

$$< \mathcal{J}_\lambda'(v_\varepsilon), w > + \varepsilon \|w\|_{W_0} \geqslant 0.$$

Therefore

$$\left\| \mathcal{J}_\lambda'(v_\varepsilon) \right\|_{W_0^*} \leqslant \varepsilon.$$

As a consequence, there exists $\{w_n\} \subset B_\rho(0)$ such that

$$\mathcal{J}_\lambda(w_n) \longrightarrow \underline{c} \quad \text{and} \quad \mathcal{J}_\lambda'(w_n) \underset{n \longrightarrow +\infty}{\longrightarrow} 0 \quad \text{in} \quad W_0^*.$$

Since $\mathcal{J}_\lambda$ verifies the Palais-Smale condition, we have $w_n \longrightarrow \tilde{w}$ in W_0. Hence, $\mathcal{J}_\lambda(\tilde{w}) = \underline{c}$ and $\mathcal{J}_\lambda'(\tilde{w}) = 0$, implying that $\tilde{w}$ is a critical point of $\mathcal{J}_\lambda$. Consequently, the result 1-(ii) of Theorem 2.1 is achieved.

Proof of Theorem 2.1-(2):

In this case, we take $M(t)=A+Bt$, the energy functional takes the form

$$\mathscr{J}_\lambda(u)=A\sigma_{p(x,y)}(u)+\frac{B}{2}\left(\sigma_{p(x,y)}(u)\right)^2-\frac{\lambda}{\alpha+1}\left[\int_\Omega F(x,u)dx\right]^{\alpha+1}.$$

First, we show that $\mathscr{J}_\lambda$ verifies the Palais-Smale condition. Let $\{u_n\}\subset W_0$ be a sequence such that

$$\mathscr{J}_\lambda(u_n)\longrightarrow c \quad \text{and} \quad \mathscr{J}'_\lambda(u_n)\underset{n\longrightarrow+\infty}{\longrightarrow}0 \quad \text{in} \quad W_0^*. \tag{2.17}$$

From condition (f_2), and Proposition 1.5, we have

$$\begin{aligned}
c+\theta_n(1)\|u_n\|_{W_0} \geqslant & \mathscr{J}_\lambda(u_n)-\frac{1}{\theta}\left\langle \mathscr{J}'_\lambda(u_n),u_n\right\rangle \\
=&A\sigma_{p(x,y)}(u_n)-\frac{A}{\theta}\left(\int_Q\frac{|u_n(x)-u_n(y)|^{p(x,y)}}{|x-y|^{N+sp(x,y)}}dxdy\right)\\
&+\frac{B}{2}\left(\sigma_{p(x,y)}(u_n)\right)^2-\frac{B}{\theta}\sigma_{p(x,y)}(u_n)\left(\int_Q\frac{|u_n(x)-u_n(y)|^{p(x,y)}}{|x-y|^{N+sp(x,y)}}dxdy\right)\\
&+\frac{\lambda}{\theta}\left[\int_\Omega F(x,u_n)dx\right]^\alpha\int_\Omega f(x,u_n)u_ndx-\frac{\lambda}{\alpha+1}\left[\int_\Omega F(x,u_n)dx\right]^{\alpha+1}.
\end{aligned}$$

Following a straightforward computation, we obtain

$$\begin{aligned}
c+\theta_n(1)\|u_n\|_{W_0} \geqslant & \left(\frac{A}{p^+}-\frac{A}{\theta}\right)\rho^0_{p(x,y)}(u_n)+B\left(\frac{1}{2(p^+)^2}-\frac{1}{\theta p^-}\right)\left(\rho^0_{p(x,y)}(u_n)\right)^2\\
&+\frac{\lambda\alpha}{\alpha+1}\left[\int_\Omega F(x,u_n)dx\right]^{\alpha+1}\\
\geqslant & \left(\frac{A}{p^+}-\frac{A}{\theta}\right)\|u_n\|^{p^-}_{W_0}+B\left(\frac{1}{2(p^+)^2}-\frac{1}{\theta p^-}\right)\|u_n\|^{2p^-}_{W_0}\\
&+\frac{\lambda\alpha}{\alpha+1}\left[\int_\Omega F(x,u_n)dx\right]^{\alpha+1}.
\end{aligned}$$

Since $\mathscr{J}_\lambda(u_n)$ is bounded, we deduce that $\{u_n\}$ is also bounded in W_0. The reflexivity of the functional framework W_0 implies the existence of a subsequence, still denoted $\{u_n\}$ such that $u_n\rightharpoonup u$ in W_0. Using (2.17), we obtain

$$\begin{aligned}
&(A+B\sigma_{p(x,y)}(u_n))\\
&\int_Q\frac{|u_n(x)-u_n(y)|^{p(x,y)-2}(u_n(x)-u_n(y))((u_n-u)(x)-(u_n-u)(y))}{|x-y|^{N+sp(x,y)}}dxdy\\
&-\lambda\left[\int_\Omega F(x,u_n)dx\right]^\alpha\int_\Omega f(x,u_n)(u_n-u)dx\underset{n\longrightarrow+\infty}{\longrightarrow}0.
\end{aligned}$$

Proceeding as in 1, we obtain the strong convergence of $\{u_n\}$ to u in W_0. Therefore, $\mathscr{J}_\lambda$ verifies the Palais-Smale condition.

Proof. $2-(i)$ For $u \in W_0$ and $\lambda > 0$, we get using condition (f_1), Proposition 1.5, and the continuous embedding result between W_0 and $L^{q(x)}(\Omega)$

$$\begin{aligned}\mathcal{J}_\lambda(u) &= A\left(\sigma_{p(x,y)}(u)\right) + \frac{B}{2}\left(\sigma_{p(x,y)}(u)\right)^2 - \frac{\lambda}{\alpha+1}\left[\int_\Omega F(x,u)dx\right]^{\alpha+1} \\ &\geqslant \frac{A}{p^+}\rho^0_{p(x,y)}(u) + \frac{B}{2(p^+)^2}\left(\rho^0_{p(x,y)}(u)\right)^2 - \frac{\lambda}{\alpha+1}\left(\frac{l_2}{q^-}\right)^{\alpha+1}\left(\int_\Omega |u|^{q(x)}dx\right)^{\alpha+1} \\ &\geqslant \frac{A}{p^+}\|u\|^{p^+}_{W_0} + \frac{B}{2(p^+)^2}\|u\|^{2p^+}_{W_0} - \frac{\lambda}{\alpha+1}\left(\frac{l_2}{q^-}\right)^{\alpha+1} C^{q^-(\alpha+1)}\|u\|^{q^-(\alpha+1)}_{W_0}.\end{aligned} \tag{2.18}$$

Take $\|u\|_{W_0} = \rho$ small enough, then (2.18) transforms to

$$\mathcal{J}_\lambda(u) \geqslant \rho^{2p^+}\left(\frac{A}{p^+}\rho^{-p^+} + \frac{B}{2(p^+)^2} - \frac{\lambda}{\alpha+1}\left(\frac{l_2}{q^-}\right)^{\alpha+1} C^{q^-(\alpha+1)}\rho^{q^-(\alpha+1)-2p^+}\right).$$

Let's define the function g_2 as follows

$$\begin{aligned} g_2 : [0,1] &\longrightarrow \mathbb{R} \\ t &\longrightarrow \frac{A}{p^+}t^{-p^+} + \frac{B}{2(p^+)^2} - \frac{\lambda}{\alpha+1}\left(\frac{l_2}{q^-}\right)^{\alpha+1} C^{q^-(\alpha+1)}t^{q^-(\alpha+1)-2p^+}.\end{aligned}$$

Clearly, g_2 is positive in a neighborhood of zero. Therefore, there exist two positive numbers γ and ρ such that

$$\mathcal{J}_\lambda(u) \geqslant \gamma, \quad \text{with } \|u\|_{W_0} = \rho \quad \text{for all } \lambda > 0.$$

Furthermore, take $v \in W_0$ and $t > 1$, then we have

$$\begin{aligned}\mathcal{J}_\lambda(tv) &= A\left(\sigma_{p(x,y)}(tv)\right) + \frac{B}{2}\left(\sigma_{p(x,y)}(tv)\right)^2 - \frac{\lambda}{\alpha+1}\left[\int_\Omega F(x,tv)dx\right]^{\alpha+1} \\ &\leqslant \frac{At^{p^+}}{p^-}\rho^0_{p(x,y)}(v) + \frac{Bt^{2p^+}}{2(p^-)^2}\left(\rho^0_{p(x,y)}(v)\right)^2 \\ &\quad - t^{q^-(\alpha+1)}\frac{\lambda}{\alpha+1}\left(\frac{l_1}{q^+}\right)^{\alpha+1}\left(\int_\Omega |v|^{q}(x)dx\right)^{\alpha+1}.\end{aligned}$$

Because of $q^-(\alpha+1) > 2p^+ > p^+$, we get that

$$\mathcal{J}_\lambda(tv) \longrightarrow -\infty \quad \text{as } t \longrightarrow +\infty.$$

In view of the mountain pass theorem, the proof of $2-(i)$ is complete. ■

Proof. $2-(ii)$ Proceeding as before, for any $u \in W_0$ with $\|u\|_{W_0} < 1$, we have

$$\begin{aligned}\mathcal{J}_\lambda(u) &= A\left(\sigma_{p(x,y)}(u)\right) + \frac{B}{2}\left(\sigma_{p(x,y)}(u)\right)^2 - \frac{\lambda}{\alpha+1}\left[\int_\Omega F(x,u)dx\right]^{\alpha+1} \\ &\geqslant \frac{A}{p^+}\|u\|^{p^+}_{W_0} + \frac{B}{2(p^+)^2}\|u\|^{2p^+}_{W_0} - \frac{\lambda}{\alpha+1}\left(\frac{l_2}{q^-}\right)^{\alpha+1} C^{q^-(\alpha+1)}\|u\|^{q^-(\alpha+1)}_{W_0}.\end{aligned} \tag{2.19}$$

Take $\|u\|_{W_0} = \rho$, therefore

$$\mathscr{J}_\lambda(u) \geqslant \rho^{q^-(\alpha+1)}\left(\frac{A}{p^+}\rho^{p^+-q^-(\alpha+1)} + \frac{B}{2(p^+)^2}\rho^{2p^+-q^-(\alpha+1)} - \frac{\lambda}{\alpha+1}\left(\frac{l_2}{q^-}\right)^{(\alpha+1)} C^{q^-(\alpha+1)}\right).$$

If $0 < \lambda < \lambda^*$, where λ^* is chosen in this way

$$\lambda^* = \frac{1}{3}\frac{\rho^{p^+-q^-(\alpha+1)}}{p^+}\frac{\alpha+1}{C^{q^-(\alpha+1)}}\left(\frac{q^-}{l_2}\right)^{(\alpha+1)}\min\{A,B\}, \tag{2.20}$$

hence, there exists $\gamma > 0$ such that

$$\mathscr{J}_\lambda(u) \geqslant \gamma.$$

By an argument analogue to 1-(ii) and using the same notations, we obtain that for $t \in (0,1)$, $t\psi \in B_\rho(0)$, where ψ is as in 1-(ii)

$$\mathscr{J}_\lambda(t\psi) \leqslant \frac{A}{p^-}t^{p^-}\rho^0_{p(x,y)}(\psi) + \frac{B}{2(p^-)^2}t^{2p^-}\left(\rho^0_{p(x,y)}(\psi)\right)^2 - \frac{\lambda}{\alpha+1}\left(\frac{l_1}{q^+}\right)^{\alpha+1} t^{q^-(\alpha+1)+\varepsilon(\alpha+1)}\left(\int_{\Omega_0}|\psi|^{q(x)}dx\right)^{\alpha+1}.$$

Since $\rho^0_{p(x,y)}(\psi) > 0$, we obtain that $\mathscr{J}_\lambda(t\psi) < 0$ for any $t < \xi^{\frac{1}{2p^- - (\alpha+1)q^- - (\alpha+1)\varepsilon}}$, where

$$0 < \xi < \min\left\{1, \frac{2(p^-)^2\frac{\lambda}{\alpha+1}\left(\frac{l_1}{q^+}\right)^{\alpha+1}\left(\int_{\Omega_0}|\psi(x)|^{q(x)}dx\right)^{\alpha+1}}{(A+B)\rho^0_{p(x,y)}(\psi)}\right\}.$$

From inequality (2.19), we have for each $u \in B_\rho(0)$

$$\mathscr{J}_\lambda(u) \geqslant \frac{A}{p^+}\|u\|^{p^+}_{W_0} + \frac{B}{2(p^+)^2}\|u\|^{2p^+}_{W_0} - \frac{\lambda}{\alpha+1}\left(\frac{l_2}{q^-}\right)^{\alpha+1} C^{q^-(\alpha+1)}\|u\|^{q^-(\alpha+1)}_{W_0}.$$

Consequently, reasoning as in 1-(ii), we get the desired result. ■

Proof of Theorem 2.1-(3):

Proof. 3-(i) In the third case, we consider the following Kirchhoff function $M(t) = t^{\beta-1}$ for all $t \in \mathbb{R}^+$, then the energy functional is expressed as follows

$$\mathscr{J}_\lambda(u) = \frac{1}{\beta}\left(\sigma_{p(x,y)}(u)\right)^\beta - \frac{\lambda}{\alpha+1}\left[\int_\Omega F(x,u)dx\right]^{\alpha+1}.$$

Furthermore, we have for all $u \in W_0$

$$\mathscr{J}_\lambda(u) \geqslant \frac{1}{\beta(p^+)^\beta}\left(\rho^0_{p(x,y)}(u)\right)^\beta - \frac{\lambda}{\alpha+1}\left(\frac{l_2}{q^-}\right)^{\alpha+1}\left(\rho_{L^{q(x)}(\Omega)}(u)\right)^{\alpha+1}, \tag{2.21}$$

as in $1-(i)$, we take $\|u\|_{W_0} = \rho$ small enough. By using Proposition 1.5 and the embedding results stated in Remark 1.1,we obtain

$$\mathscr{J}_\lambda(u) \geqslant \frac{1}{\beta(p^+)^\beta}\rho^{\beta p^+} - C^{q^-(\alpha+1)}\frac{\lambda}{\alpha+1}\left(\frac{l_2}{q^-}\right)^{\alpha+1}\rho^{q^-(\alpha+1)}.$$

By considering also the function g_3 defined as

$$\begin{aligned} g_3 : [0,1] &\longrightarrow \mathbb{R} \\ t &\longrightarrow \frac{1}{\beta(p^+)^\beta} - C^{q^-(\alpha+1)}\frac{\lambda}{\alpha+1}\left(\frac{l_2}{q^-}\right)^{\alpha+1} t^{q^-(\alpha+1)-\beta p^+}, \end{aligned}$$

and positive in a neighborhood of zero, we acquire that there exists $\gamma > 0$ and $\rho > 0$ such that

$$\mathscr{J}_\lambda(u) \geqslant \gamma > 0 \quad \text{with} \quad \|u\|_{W_0} = \rho.$$

By standard argument, for any $v \in W_0$, and $t > 1$ we attain that $\mathscr{J}_\lambda(tv) \longrightarrow -\infty$ as $t \longrightarrow +\infty$. To achieve the intended result, we should validate the Palais-Smale condition. Let u_n be a sequence in W_0 in which

$$\mathscr{J}_\lambda(u_n) \longrightarrow c, \quad \text{and} \quad \mathscr{J}'_\lambda(u_n) \underset{n\longrightarrow+\infty}{\longrightarrow} 0 \quad \text{in} \quad W_0^*. \tag{2.22}$$

From condition (f_2) and Proposition 1.5, we have

$$\begin{aligned} c + \theta_n(1)\|u_n\|_{W_0} &\geqslant \mathscr{J}_\lambda(u_n) - \frac{1}{\theta}\langle \mathscr{J}_\lambda'(u_n), u_n\rangle \\ &\geqslant \left(\frac{1}{\beta(p^+)^\beta} - \frac{1}{\theta(p^-)^{\beta-1}}\right)\left(\rho^0_{p(x,y)}(u_n)\right)^\beta + \frac{\lambda\alpha}{\alpha+1}\left[\int_\Omega F(x,u_n)dx\right]^{\alpha+1} \\ &\geqslant \left(\frac{1}{\beta(p^+)^\beta} - \frac{1}{\theta(p^-)^{\beta-1}}\right)\|u_n\|^{\beta p^-}_{W_0} + \frac{\lambda\alpha}{\alpha+1}\left[\int_\Omega F(x,u_n)dx\right]^{\alpha+1}. \end{aligned}$$

This result involves the boundedness of u_n in W_0. The reflexivity of the functional framework W_0 implies the existence of a subsequence still denoted u_n such that u_n converges weakly in W_0 and denote u its weak limit. Using the same reasoning as in $1-(i)$, we get that the Palais-Smale condition is fulfilled, and finally we reach the proof of $3-(i)$. ■

Proof. $3-(ii)$ Let $u \in W_0$ such that $\|u\|_{W_0} = \rho > 0$, from (2.21) we have

$$\mathscr{J}_\lambda(u) \geqslant \frac{1}{\beta(p^+)^\beta}\rho^{\beta p^+} - \frac{\lambda}{\alpha+1}\left(\frac{l_2}{q^-}\right)^{\alpha+1} C^{q^-(\alpha+1)}\rho^{q^-(\alpha+1)}.$$

Whereas $q^-(\alpha+1) < \beta p^-$, then for ρ sufficiently small, there exists $\gamma > 0$ such that $\mathscr{J}_\lambda(u) \geqslant \gamma > 0$ for $\lambda \in (0, \tilde{\lambda})$, where$\tilde{\lambda}$ is given by

$$\tilde{\lambda} = \frac{1}{2\beta(p^+)^\beta}\left(\frac{q^-}{l_2}\right)^{\alpha+1}\frac{(\alpha+1)\rho^{\beta p^+ - q^-(\alpha+1)}}{C^{q^-(\alpha+1)}}. \tag{2.23}$$

Besides, following the same steps as the proof of Lemma 2.2, we get for $t \in (0,1)$ and $\psi \in C_0^\infty(\overline{\Omega})$ that

$$\begin{aligned}\mathscr{J}_\lambda(t\psi) \leqslant & \frac{t^{\beta p^-}}{\beta(p^-)^\beta}\left(\rho^0_{p(x,y)}(\psi)\right)^\beta \\ & - \frac{\lambda}{\alpha+1}\left(\frac{l_1}{q^+}\right)^{\alpha+1} t^{(\alpha+1)q^- + (\alpha+1)\varepsilon}\left(\int_{\Omega_0}|\psi|^{q(x)}dx\right)^{\alpha+1}.\end{aligned}$$

Consequently, $\mathscr{J}_\lambda(t\psi) < 0$, for any $t < \xi^{\frac{1}{\beta p^- - (\alpha+1)q^- - (\alpha+1)\varepsilon}}$, where

$$0 < \xi < \min\left\{1, \frac{\beta(p^+)^\beta \frac{\lambda}{\alpha+1}\left(\frac{l_1}{q^+}\right)^{\alpha+1}\left(\int_{\Omega_0}|\psi(x)|^{q(x)}dx\right)^{\alpha+1}}{\left(\rho^0_{p(x,y)}(\psi)\right)^\beta}\right\}.$$

Next, we apply the Ekeland's variational principle and reasoning as in $1-(ii)$ in order to find a subsequence $v_n \in B_\rho(0)$ such that

$$\mathscr{J}_\lambda(v_n) \longrightarrow \underline{c}, \quad \text{and} \quad \mathscr{J}'_\lambda(v_n) \underset{n\longrightarrow+\infty}{\longrightarrow} 0 \quad \text{in} \quad W_0^*.$$

$\mathscr{J}_\lambda$ verifies the Palais-Smale condition, therefore v the strong limit of v_n in W_0 is a critical point of $\mathscr{J}_\lambda$, ultimately the proof is achieved. ■

Proof of Theorem 2.1-(4):

In this case, we look at a more general case of the Kirchhoff function, which was discussed in 2 and 3 in Theorem 2.1. The energy functional $\mathscr{J}_\lambda$ is written in the following format

$$\mathscr{J}_\lambda(u) = A\left(\sigma_{p(x,y)}(u)\right) + \frac{B}{\beta}\left(\sigma_{p(x,y)}(u)\right)^\beta - \frac{\lambda}{\alpha+1}\left[\int_\Omega F(x,u)dx\right]^{\alpha+1} \quad \text{for all } u \in W_0.$$

By analogy, $\mathscr{J}_\lambda$ affirms the Palais-Smale condition.

Proof. $4-(i)$ With respect to the geometric conditions of the mountain pass theorem, we take $u \in W_0$ such that $\|u\|_{W_0} = \rho$ small enough, then by using embedding result 1.1 and Proposition 1.5, we obtain

$$\mathscr{J}_\lambda(u) \geqslant \frac{A}{p^+}\rho^{p^+} + \frac{B}{\beta(p^+)^\beta}\rho^{\beta p^+} - C^{q^-(\alpha+1)}\frac{\lambda}{\alpha+1}\left(\frac{l_2}{q^-}\right)^{\alpha+1}\rho^{q^-(\alpha+1)}.$$

With regard to the function g_4 defined in this way

$$g_4 : [0,1] \longrightarrow \mathbb{R}$$
$$t \longrightarrow \frac{A}{p^+}t^{(1-\beta)p^+} + \frac{B}{\beta(p^-)^\beta} - C^{q^-(\alpha+1)}\frac{\lambda}{\alpha+1}\left(\frac{l_2}{q^-}\right)^{\alpha+1} t^{q^-(\alpha+1)-\beta p^+},$$

which is positive in a neighborhood of zero. Then, there exist $\gamma > 0$ such that $\mathscr{J}_\lambda(u) \geqslant \gamma$ for $u \in W_0$ with $\|u\|_{W_0} = \rho$. On the other hand, take $v \in W_0$ and $t > 1$. Therefore

$$\mathscr{J}_\lambda(tv) \leqslant \frac{At^{p^+}}{p^-}\rho^0_{p(x,y)}(v) + \frac{Bt^{\beta p^+}}{\beta(p^-)^\beta}\rho^0_{p(x,y)}(v) - t^{q^-(\alpha+1)}\frac{\lambda}{\alpha+1}\left(\frac{l_1}{q^+}\right)^{\alpha+1}\left(\int_\Omega |v|^{q(x)}dx\right)^{\alpha+1}.$$

As $q^-(\alpha+1) > \beta p^+$, we obtain by passing to the limit as $t \longrightarrow +\infty$ that $\mathscr{J}_\lambda(tv) \longrightarrow -\infty$. Therefore, by taking $u = tv$ we obtain that $\mathscr{J}_\lambda(tv) < 0$. It follows that problem $(\mathscr{P}_\lambda)$ admits a weak solution for all $\lambda > 0$. ■

Proof. $4-(ii)$ Following $1-(ii)$, for any $u \in W_0$ with $\|u\|_{W_0} = \rho < 1$, we obtain

$$\mathscr{J}_\lambda(u) \geqslant \gamma \qquad \text{for} \quad 0 < \lambda < \bar{\lambda},$$

where $\bar{\lambda} = \frac{1}{3}\frac{\rho^{\beta p^+ - q^-(\alpha+1)}}{\beta(p^+)^\beta}\frac{\alpha+1}{C^{q^-(\alpha+1)}}\left(\frac{q^-}{l_2}\right)^{(\alpha+1)}\min\{A,B\}$.
For t sufficiently small, and $\psi \in C_0^\infty(\overline{\Omega})$ that verifies properties as in $1-(ii)$, we have

$$\mathscr{J}_\lambda(t\psi) \leqslant \frac{At^{p^-}}{p^-}\rho^0_{p(x,y)}(\psi) + \frac{Bt^{\beta p^-}}{\beta(p^-)^\beta}\left(\rho^0_{p(x,y)}(\psi)\right)^\beta - \frac{\lambda}{\alpha+1}\left(\frac{l_1}{q^+}\right)^{\alpha+1} t^{(\alpha+1)q^- + (\alpha+1)\varepsilon}\left(\int_{\Omega_0}|\psi|^{q(x)}dx\right)^{\alpha+1}.$$

Then $\mathscr{J}_\lambda(t\psi) < 0$, for any $t < \xi^{\frac{1}{\beta p^- - (\alpha+1)q^- - (\alpha+1)\varepsilon}}$, where

$$0 < \xi < \min\left\{1, \frac{\frac{\lambda}{\alpha+1}\left(\frac{l_1}{q^+}\right)^{\alpha+1}\left(\int_{\Omega_0}|\psi(x)|^{q(x)}dx\right)^{\alpha+1}}{\left(\frac{A}{p^-} + \frac{B}{\beta(p^-)^\beta}\right)(\rho^0_{p(x,y)}(\psi))^\beta}\right\}.$$

Reasoning as in $1-(ii)$ we obtain the existence of weak solution of problem $(\mathscr{P}_\lambda)$ for each $\lambda \in (0,\bar{\lambda})$. ■

The main results and analytical techniques discussed in this section are encapsulated in the table below, offering a detailed and structured summary of our findings.

This section focuses on a Kirchhoff-type problem incorporating a fractional $p(x,\cdot)$-Laplacian operator. In the following, we take a new approach by introducing a generalized integro-differential operator into the Kirchhoff-type problem.

The Kirchhoff function M	Hypotheses	MPT	EVP
$m_0 \leqslant M(t) \leqslant m_1$	$q^-(\alpha+1) > p^+$	✓	
$m_0 \leqslant M(t) \leqslant m_1$	$q(x)(\alpha+1) < p^-$ for all $x \in \overline{\Omega}$		✓
$M(t) = A + Bt$	$q^-(\alpha+1) > 2p^+$	✓	
$M(t) = A + Bt$	$q(x)(\alpha+1) < p^-$ for all $x \in \overline{\Omega}$		✓
$M(t) = t^{\beta-1}$	$q^-(\alpha+1) > \beta p^+$	✓	
$M(t) = t^{\beta-1}$	$q(x)(\alpha+1) < \beta p^-$ for all $x \in \overline{\Omega}$		✓
$M(t) = A + Bt^{\beta-1}$	$q^-(\alpha+1) > \beta p^+$	✓	
$M(t) = A + Bt^{\beta-1}$	$q(x)(\alpha+1) < \beta p^-$ for all $x \in \overline{\Omega}$		✓

Table 2.1: Summary of main results

2.3 On a class of $p(x,\cdot)$-integro-differential Kirchhoff-type problem

Our objective is to apply various variational methods to demonstrate the adaptability of this type of operator to Kirchhoff problems. Additionally, we present an example to validate our findings and illustrate the practical applicability of the proposed approach.

2.3.1 Hypothesis and statement of the problem

This section aims to establish the existence and multiplicity of solutions for the following $p(x,\cdot)$-Kirchhoff-type problem

$$(\mathscr{P}_K)\begin{cases} M\left(\sigma_K^{\phi}(u)\right)\mathscr{L}_K^{\phi}(u(x)) = f(x,u) & \text{in } \Omega, \\ u = 0 & \text{on } \mathbb{R}^N \setminus \Omega, \end{cases} \tag{2.24}$$

where

- The map σ_K^{ϕ} is defined as

$$\sigma_K^{\phi}(u) := \sigma(u) = \int_Q \Phi(u(x) - u(y))K(x,y)dxdy.$$

- The operator $\mathscr{L}_K^{\phi}$ appointed the general nonlocal integro-differential operator introduced in [48] as follows

$$\mathscr{L}_K^{\phi}u(x) = p.v. \int_{\mathbb{R}^N} \phi(u(x) - u(y))K(x,y)dy, \quad \forall x \in \mathbb{R}^N.$$

- The mapping $\phi : \mathbb{R} \longrightarrow \mathbb{R}$ is a continuous and even function such that

– (ϕ_1): The map Φ defined in this way

$$\begin{aligned} &\Phi : \mathbb{R} \longrightarrow \mathbb{R} \\ &t \longrightarrow \Phi(t) := \int_0^{|t|} \phi(\tau) d\tau \quad \text{is strictly convex.} \end{aligned} \tag{2.25}$$

– (ϕ_2): There exist two positive constants C_1 and C_2 such that

$$\begin{aligned} &\phi(t)t \geqslant C_1 |t|^{p(x,y)} \quad \text{and} \quad |\phi(t)| \leqslant C_2 |t|^{p(x,y)-1}, \\ &\text{for all } t \in \mathbb{R} \text{ and } (x,y) \in \mathbb{R}^N \times \mathbb{R}^N. \end{aligned} \tag{2.26}$$

– (ϕ_3): For all $t \in \mathbb{R}$, we have

$$\Phi(t) \geqslant \frac{1}{p^+} \phi(t)t. \tag{2.27}$$

- The Kirchhoff function $M : \mathbb{R}^+ \longrightarrow \mathbb{R}$ is continuous and satisfies the following condition

  – $(\mathscr{M})$: There exist two positive constants m_1 and m_2 such that

  $$m_1 t^{\alpha(x)-1} \leqslant M(t) \leqslant m_2 t^{\alpha(x)-1}, \tag{2.28}$$

  with $m_2 \geqslant m_1$, for some $\alpha \in C_+(\overline{\Omega})$.

- The non-linearity $f : \Omega \times \mathbb{R} \longrightarrow \mathbb{R}$ is a Carathéodory function verifying the following conditions

  – (f_1) : There exists a constant μ such that

  $$0 < \mu F(x,t) \leq t f(x,t), \quad \text{for all} \quad (x,t) \in \overline{\Omega} \times \mathbb{R},$$

  where $F(x,t) := \int_0^t f(x,\tau) d\tau$, also μ verifies this inequality $\mu > \alpha^+ p^+$.

  – (f_2) : $\exists C > 0$, such that $|f(x,t)| \leq C\left(1 + |t|^{q(x)-1}\right)$,
  with $q \in C_+(\overline{\Omega})$, and $q(x) < p_s^*(x)$ for all $x \in \overline{\Omega}$.

  – (f_3) : $f(x,t) = \theta\left(|t|^{\alpha^- p^+ - 1}\right) \longrightarrow 0$ as $t \longrightarrow 0$, for $x \in \overline{\Omega}$ uniformly.

  – (f_4) : $\inf\limits_{\{x\in\Omega, |t|=1\}} F(x,t) > 0$.

  – (f_5) : There exists a positive constant ζ such that

  $$f(x,t) \geqslant \zeta |t|^{\gamma(x)-1}, \; t \to 0, \text{ where } \gamma \in C_+(\overline{\Omega}).$$

It is essential to note that the assumptions (ϕ_1), (ϕ_2), (ϕ_3), and $(\mathscr{M})$ are supposed to be verified and validated. Therefore, the main results are formulated in the following theorems.

Theorem 2.2. *Suppose that condition(f_2) holds. Moreover, if $\alpha^- p^- > q^+$. Then there exists a weak solution to problem $(\mathscr{P}_K)$.*

Theorem 2.3. *Assume that conditions (f_1) and (f_2) hold. We suppose also that $\alpha^- p^+ < q^-$. Then problem $(\mathscr{P}_K)$ has a non-trivial weak solution.*

Theorem 2.4. *Assume that conditions (f_1)–(f_4) are satisfied. If f also is an even function with respect to the second variable and $\alpha^+ p^+ < q^-$ for a.e. $x \in \Omega$. Then problem $(\mathscr{P}_K)$ has infinitely many solutions u_k satisfying $\mathscr{J}(u_k) \longrightarrow \infty$ as $k \longrightarrow \infty$.*

Theorem 2.5. *Assume that conditions (f_1), (f_2), (f_3), and (f_5) hold. If in addition, f is an even function with respect to the second variable and $\gamma^+ < \alpha(x)p^-$ for a.e. $x \in \Omega$. Then problem $(\mathscr{P}_K)$ has a sequence of solutions $(\pm u_k)_{k=1}^{\infty}$ such that $\mathscr{J}(\pm u_k) < 0$, and $\mathscr{J}(\pm u_k) \to 0$ as $k \longrightarrow \infty$.*

2.3.2 Existence results

In this subsection, we will present a detailed proof of the main results, relying on four fundamental variational methods to establish the existence of weak solutions.

Definition 2.1. A function $u \in X_0$ is said to be weak solution of problem $(\mathscr{P}_K)$ if for every $v \in X_0$, we have

$$M\left(\sigma_K^{\phi}(u)\right)\int_Q \phi(u(x)-u(y))(v(x)-v(y))K(x,y)dxdy = \int_\Omega f(x,u)vdx,$$

Associated to the problem $(\mathscr{P}_K)$, we define the Euler-Lagrange functional $\mathscr{J}$ as follows

$$\begin{aligned}\mathscr{J} : X_0 &\longrightarrow \mathbb{R}\\ u &\longmapsto \widehat{M}\left(\sigma_K^{\phi}(u)\right) - \int_\Omega F(x,u)dx,\end{aligned}$$

where $\widehat{M}(t) = \int_0^t M(\tau)d\tau$.
Consider the functionals φ and ψ defined on X_0 as follows

$$\begin{array}{rcl}\varphi : X_0 & \to & \mathbb{R}\\ u & \mapsto & \widehat{M}(\sigma_k^{\phi}(u))\end{array} \qquad \begin{array}{rcl}\psi : X_0 & \to & \mathbb{R}\\ u & \mapsto & \int_\Omega F(x,u)dx.\end{array}$$

From conditions (ϕ_2) and $(\mathscr{M})$, we have for any $u \in X_0$

$$\varphi(u) \leqslant \frac{m_2}{\alpha^-}\left(\frac{C_2}{p^-}\rho^0_{K,p(x,y)}(u)\right)^{\alpha(x)}.$$

Using Proposition 1.6, we acquire that φ is well defined. Furthermore, by using condition (f_2), we get

$$\psi(u) \leqslant C\|u\|_{L^1(\Omega)} + \frac{C}{q^-}\rho_{L^{q(x)}(\Omega)}(u).$$

Since $q(x) < p^*_s(x)$ for all $x \in \Omega$, the well-posedness of ψ is obtained directly by using Proposition 1.2 and Remark 1.4.
On the other hand, $\widehat{M}$ is the primitive of the Kirchhoff function M, hence $\widehat{M} \in \mathscr{C}^1(\mathbb{R},\mathbb{R})$, and the fact that $\Phi \in \mathscr{C}^1(\mathbb{R},\mathbb{R})$ implies that $\sigma^{\phi}_K \in \mathscr{C}^1(X_0,\mathbb{R})$. Consequently, $\varphi \in \mathscr{C}^1(X_0,\mathbb{R})$. On the other hand, f is a Carathéodory function acquires that $\psi \in \mathscr{C}^1(X_0,\mathbb{R})$. Therefore, $\mathscr{J}$ is of Class $\mathscr{C}^1$ on X_0. Moreover, its derivative is given by this formula

$$\begin{aligned}\left\langle \mathscr{J}'(u), v\right\rangle =& M\left(\sigma^{\phi}_K(u)\right)\int_Q \phi(u(x)-u(y))(v(x)-v(y))K(x,y)dxdy \\ &- \int_\Omega f(x,u)vdx, \quad \text{for all } (u,v) \in X_0 \times X_0.\end{aligned}$$

Consequently, weak solutions of problem $(\mathscr{P}_K)$ correspond to the critical points of the functional $\mathscr{J}$.

Proof of Theorem 2.2

By means of the direct variational method, we establish the proof of Theorem 2.2. To proceed, we verify the properties outlined in the following lemma.

Lemma 2.3. *Under the assumptions of Theorem 2.2, the following properties hold for the functional $\mathscr{J}$ on the space X_0.*

(1) *$\mathscr{J}$ is bounded from below and coercive.*

(2) *$\mathscr{J}$ is weakly lower semicontinuous.*

Proof. (1) Let $u \in X_0$ with $\|u\|_{X_0} > 1$, then we have from conditions (f_2) and $(\mathscr{M})$ that

$$\begin{aligned}\mathscr{J}(u) &= \widehat{M}\left(\sigma^{\phi}_K(u)\right) - \int_\Omega F(x,u)dx \\ &\geqslant \tfrac{m_1}{\alpha^+}\left(\sigma^{\phi}_K(u)\right)^{\alpha(x)} - C\left(\int_\Omega |u| + \tfrac{1}{q(x)}|u|^{q(x)}dx\right) \\ &\geqslant \tfrac{m_1}{\alpha^+}\left(\tfrac{C_1}{p^+}\rho^0_{K,p(x,y)}(u)\right)^{\alpha(x)} - C\|u\|_{L^1(\Omega)} - \frac{C}{q^-}\rho_{L^{q(x)}(\Omega)}(u),\end{aligned}$$

Since $q(x) < p_s^*(x)$ for all $x \in \overline{\Omega}$, we obtain from the embedding results stated in Remark 1.4 the existence of two positive constants $\tilde{C}_1$ and $\tilde{C}_2$ such that

$$\|u\|_{L^1(\Omega)} \leqslant \tilde{C}_1 \|u\|_{X_0} \text{ and } \|u\|_{L^{q(x)}(\Omega)} \leqslant \tilde{C}_2 \|u\|_{X_0}. \tag{2.29}$$

Therefore, using Proposition 1.6 we get

$$\mathscr{J}(u) \geqslant \frac{m_1}{\alpha^+}\left(\frac{C_1}{p^+}\right)^{\alpha^-} \|u\|_{X_0}^{p^-\alpha^-} - C\tilde{C}_1\|u\|_{X_0} - \frac{C\tilde{C}_2}{q^-}\|u\|_{X_0}^{q^+}.$$

Given that $p^-\alpha^- > q^+ > 1$, then $\mathscr{J}(u) \to \infty$ as $\|u\|_{X_0} \to \infty$, as a result $\mathscr{J}$ is coercive and bounded from below.

(2) Let $\{u_n\}$ be a sequence in X_0 such that $u_n \rightharpoonup u$ in X_0 as $n \longrightarrow +\infty$. We have from Lemma 2.10 in [48] that σ_K^ϕ is weakly lower semicontinuous, then

$$\sigma_K^\phi(u) \leqslant \liminf_{n\to\infty} \sigma_K^\phi(u_n),$$

as $\widehat{M}$ is continuous we get that

$$\widehat{M}\left(\sigma_K^\phi(u)\right) \leqslant \widehat{M}\left(\liminf_{n\to\infty} \sigma_K^\phi(u_n)\right) = \liminf_{n\to\infty} \widehat{M}\left(\sigma_K^\phi(u_n)\right).$$

Moreover, the compact embedding results stated in Remark 1.4 implies that

$$u_n \longrightarrow u_0 \text{ in } L^{q(x)}(\Omega),$$
$$u_n \longrightarrow u_0 \text{ in } L^1(\Omega).$$

Adding the fact that $\psi \in \mathscr{C}^1(X_0, \mathbb{R})$, we obtain

$$\lim_{n\to\infty} \psi(u_n) = \psi(u),$$

therefore ψ is weakly continuous, and consequently ψ is weakly lower semicontinuous. Finally, we conclude that $\mathscr{J}$ is weakly lower semicontinuous on X_0.

■

Conclusion: In the light of the direct variational method, the functional $\mathscr{J}$ attains its minimum on X_0, then problem $(\mathscr{P}_K)$ admits a weak solution and Theorem 2.2 holds.

Proof of Theorem 2.3

We will employ the mountain pass theorem as our primary tool in establishing Theorem 2.3. Consequently, we state and prove the subsequent results that will come in handy later.

Lemma 2.4. *Assume that conditions (f_1), (f_2), and $(\mathscr{M})$ are satisfied, then $\mathscr{J}$ verifies the Palais Smale compactness condition.*

Proof. Let $\{u_n\}$ be a sequence in X_0 that verifies

$$\mathscr{J}(u_n)\to c \text{ in } \mathbb{R} \quad \text{and} \quad \mathscr{J}'(u_n)\to 0 \quad \text{as} \quad n\to+\infty \quad \text{in} \quad X_0^*. \tag{2.30}$$

From conditions (ϕ_3) and $(\mathscr{M})$, we have

$$\begin{aligned} c+\theta_n(1)\|u_n\|_{X_0} &\geqslant \mathscr{J}(u_n)-\tfrac{1}{\mu}\left\langle \mathscr{J}'(u_n),u_n\right\rangle \\ &= \widehat{M}\left(\sigma_K^{\phi}(u_n)\right)-\int_\Omega F(x,u_n)dx+\tfrac{1}{\mu}\int_\Omega f(x,u_n)u_n dx \\ &\quad -\tfrac{1}{\mu}M\left(\sigma_K^{\phi}(u_n)\right)\int_Q \phi(u_n(x)-u_n(y))(u_n(x)-u_n(y))K(x,y)dxdy, \\ &\geqslant \tfrac{m_1}{\alpha^+}\left(\sigma_K^{\phi}(u_n)\right)^{\alpha(x)}+\int_\Omega\left(\tfrac{1}{\mu}f(x,u_n)u_n-F(x,u_n)\right)dx \\ &\quad -\tfrac{m_2p^+}{\mu}\left(\sigma_K^{\phi}(u_n)\right)^{\alpha(x)-1}\int_Q \Phi(u_n(x)-u_n(y))K(x,y)dxdy, \end{aligned}$$

using condition (f_1) and Proposition 1.6, we acquire

$$c+\theta_n(1)\|u_n\|_{X_0}\geqslant\left(\tfrac{m_1}{\alpha^+}-\tfrac{m_2p^+}{\mu}\right)\left(\sigma_K^{\phi}(u_n)\right)^{\alpha(x)}\geqslant\left(\tfrac{C_1}{p^+}\right)^{\alpha^-}\left(\tfrac{m_1}{\alpha^+}-\tfrac{m_2p^+}{\mu}\right)\|u_n\|_{X_0}^{\alpha^-p^-}.$$

In consideration of the condition that $\frac{\mu}{\alpha^+p^+}>1$, we get that $\{u_n\}$ is bounded in X_0. Accordingly, by the embedding result in Remark 1.4, there exists $u\in X_0$ such that for a subsequence of $\{u_n\}$, we have

$$\begin{aligned} &u_n \rightharpoonup u \text{ in } X_0, \\ &u_n \to u \text{ a.e. in } \Omega, \\ &u_n \to u \text{ in } L^{q(\cdot)}(\Omega). \end{aligned}$$

By the application of Hölder's inequality and condition (f_2), we have

$$\begin{aligned} \left|\int_\Omega f(x,u_n)(u_n-u)dx\right| &\leqslant C\int_\Omega |u_n-u|dx+C\int_\Omega |u_n|^{q(x)-1}|u_n-u|dx \\ &\leqslant \tilde{C}_3\|u_n-u\|_{L^{q(x)}(\Omega)}+2C\left\||u_n|^{q(x)-1}\right\|_{L^{\hat{q}(x)}(\Omega)}\|u_n-u\|_{L^{q(x)}(\Omega)}. \end{aligned}$$

It follows that

$$\int_\Omega f(x,u_n)(u_n-u)dx\longrightarrow 0 \quad \text{as} \quad n\to\infty. \tag{2.31}$$

Owing to (2.30) and the weak convergence of $\{u_n\}$, we get

$$\lim_{n\to\infty}\left\langle \mathscr{J}'(u_n),u_n-u\right\rangle=0,$$

which implies

$$M\left(\sigma_K^{\phi}(u_n)\right)\int_Q \phi(u_n(x)-u_n(y))((u_n-u)(x)-(u_n-u)(y))K(x,y)dxdy$$
$$-\int_\Omega f(x,u_n)(u_n-u)dx \longrightarrow 0 \quad \text{as} \quad n\to\infty.$$

Combining this fact with (2.31), we obtain

$$\lim_{n\to\infty} M\left(\sigma_k^{\phi}(u_n)\right)\int_Q \phi(u_n(x)-u_n(y))((u_n-u)(x)-(u_n-u)(y))K(x,y)dxdy=0.$$

From the boundedness of $\{u_n\}$, we can assume that

$$\int_Q \Phi(u(x)-u(y))K(x,y)dxdy \longrightarrow \ell \geqslant 0 \quad \text{as} \quad n\longrightarrow\infty.$$

In the case when $\ell=0$, the proof is finished. In the other case ($\ell\neq 0$), by the continuity of Kirchhoff's function, we get

$$\lim_{n\to\infty} M\left(\sigma_K^{\phi}(u_n)\right)=M(\ell).$$

Therefore,

$$\int_Q \phi(u_n(x)-u_n(y))((u_n-u)(x)-(u_n-u)(y))K(x,y)dxdy \underset{n\to\infty}{\longrightarrow} 0.$$

By the same argument, we find

$$\int_Q \phi(u(x)-u(y))((u_n-u)(x)-(u_n-u)(y))K(x,y)dxdy \underset{n\to\infty}{\longrightarrow} 0.$$

From Lemma 2.10 [48], the mapping ${\sigma_K^{\phi}}'$ is of type (S_+), then $\{u_n\}$ converges strongly to u in X_0. In conclusion, $\mathscr{J}$ satisfies the Palais-Smale condition. ■

Now, it remains to prove the geometrical conditions of the mountain pass theorem of Ambrosetti and Rabinowitz [11]. Therefore, we have to verify (R_1) and (R_2) in Theorem 1.6.

Proof of condition (R_1):

For any $u\in X_0$ with $\|u\|_{X_0}<1$, we have based on conditions (f_2) and $(\mathscr{M})$ that

$$\begin{aligned}\mathscr{J}(u) &= \widehat{M}\left(\sigma_K^{\phi}(u)\right)-\int_\Omega F(x,u)\mathrm{d}x \\ &\geqslant \tfrac{m_1}{\alpha(x)}\left(\sigma_K^{\phi}(u)\right)^{\alpha(x)}-C\int_\Omega\left(|u|+\tfrac{1}{q(x)}|u|^{q(x)}\right)dx.\end{aligned} \tag{2.32}$$

Conditions (ϕ_1) and (ϕ_2), Proposition 1.6, and the embedding result (2.29) imply that

$$\begin{aligned}\mathscr{J}(u) &\geqslant \frac{m_1}{\alpha^+}\left(\frac{C_1}{p^+}\right)^{\alpha^-}\|u\|_{X_0}^{\alpha^- p^+} - C\tilde{C}_1\|u\|_{X_0} - \frac{C\tilde{C}_2}{q^-}\|u\|_{X_0}^{q^-},\\ &= \|u\|_{X_0}^{\alpha^- p^+}\left(\frac{m_1}{\alpha^+}\left(\frac{C_1}{p^+}\right)^{\alpha^-} - C\tilde{C}_1\|u\|_{X_0}^{1-\alpha^- p^+} - \frac{C\tilde{C}_2}{q^-}\|u\|_{X_0}^{q^- - \alpha^- p^+}\right).\end{aligned}$$

Next, consider the function g defined as follows

$$\begin{aligned}g:[0,1] &\longrightarrow \mathbb{R}\\ t &\longrightarrow \frac{m_1}{\alpha^+}\left(\frac{C_1}{p^+}\right)^{\alpha^-} - C\tilde{C}_1 t^{1-\alpha^- p^+} - \frac{C\tilde{C}_2}{q^-}t^{q^- - \alpha^- p^+}.\end{aligned}$$

It is obvious to see that g is strictly positive in a neighborhood of zero, therefore the proof of the condition (R_1) is complete. ■

Proof of condition (R_2):

Let us introduce the function $\delta_1:[1,+\infty)\longrightarrow\mathbb{R}$ by

$$\delta_1(t) = t^{-\mu}F(x,t\tau) - F(x,\tau), \quad \text{for all } (x,\tau)\in\Omega\times\mathbb{R}.$$

δ_1 is differentiable, and for each $t>1$, we have

$$\delta_1'(t) = t^{-\mu-1}\left(f(x,t\tau)t\tau - \mu F(x,t\tau)\right).$$

The (AR) condition (f_1), implies that δ_1 is an increasing function on $[1,+\infty)$, then

$$\delta_1(t)\geqslant 0 \quad \text{for all } t\in[1,+\infty), \tag{2.33}$$

which means that

$$t^{-\mu}F(x,t\tau)\geqslant F(x,\tau), \quad \text{for all } (x,\tau)\in\Omega\times\mathbb{R}. \tag{2.34}$$

Now, for any $\bar{u}_1\in X_0$ with $\bar{u}_1>0$ and $t>1$, using the previous result and condition $(\mathscr{M})$ to get

$$\begin{aligned}\mathscr{J}(t\bar{u}_1) &= \widehat{M}\left(\sigma_K^{\phi}(t\bar{u}_1)\right) - \int_\Omega F(x,t\bar{u}_1)\mathrm{d}x\\ &\leqslant \frac{m_2}{\alpha^-}\left(\frac{C_2}{p^-}\right)^{\alpha^+} t^{\alpha^+ p^+}\left(\rho_{K,p(x,y)}^0(\bar{u}_1)\right)^{\alpha^+} - t^\mu\int_\Omega F(x,\bar{u}_1)dx.\end{aligned}$$

Since $\mu>\alpha^+ p^+$ we get $\mathscr{J}(t\bar{u}_1)\longrightarrow-\infty$ as $t\longrightarrow+\infty$.

Consequently, for t large enough, if we take $\bar{u}=t\bar{u}_1$, then there exists $\bar{u}\in X_0$ with $\|\bar{u}\|_{X_0}>r$ such that $\mathscr{J}(\bar{u})<0$. ■

Conclusion: As a consequence, the geometrical conditions (R_1) and (R_2) of the mountain pass theorem are fulfilled. Moreover, from the fact that $\mathscr{J}(0)=0$ and Lemma 2.4, we conclude that $\mathscr{J}$ has a nontrivial weak solution. Therefore, Theorem 2.3 is proved.

Proof of Theorem 2.4

Now, using the fountain theorem, we show that problem $(\mathscr{P}_K)$ has a sequence of weak solutions with unbounded energy. To achieve this, we will check conditions (A_1) and (A_2) of Theorem 1.8.

Proof of condition (A_1):

On Z_k, based on (f_2) and $(\mathscr{M})$, we obtain

$$\begin{aligned}\mathscr{J}(u) &= \widehat{M}\left(\sigma_K^{\phi}(u)\right) - \int_\Omega F(x,u)dx \\ &\geqslant \tfrac{m_1}{\alpha^+}\left(\sigma_K^{\phi}(u)\right)^{\alpha(x)} - C\left(\int_\Omega |u| + \tfrac{1}{q(x)}|u|^{q(x)}dx\right) \\ &\geqslant \tfrac{m_1}{\alpha^+}\left(\tfrac{C_1}{p^+}\right)^{\alpha^-}\rho^0_{K,p(x,y)}(u)^{\alpha(x)} - C\|u\|_{L^1(\Omega)} - \frac{C}{q^-}\rho_{L^{q(x)}(\Omega)}(u),\end{aligned}$$

using the embedding inequalities (2.29) stated previously and Proposition 1.6, we acquire

$$\mathscr{J}(u) \geqslant \tfrac{m_1}{\alpha^+}\left(\tfrac{C_1}{p^+}\right)^{\alpha^-}\|u\|_{X_0}^{p^-\alpha^-} - C\tilde{C}_1\|u\|_{X_0} - \tilde{C}_4\beta_k^{q^+}\|u\|_{X_0}^{q^+},$$

where β_k is as in Lemma 1.5.

By choosing $r_k = \left(\frac{\tilde{C}_4\beta_k^{q^+}q^+(p^+)^{\alpha^-}}{m_1 C_1^{\alpha^-}}\right)^{\frac{1}{\alpha^-p^--q^+}}$, we get

$$\mathscr{J}(u) \geqslant m_1\left(\tfrac{1}{\alpha^+} - \tfrac{1}{q^+}\right)\left(\tfrac{C_1}{p^+}\right)^{\alpha^-} r_k^{\alpha^-p^-} - C\tilde{C}_1 r_k.$$

Consequently, since $\alpha^-p^- < q^+$ and from Lemma 1.5, we have that $r_k \longrightarrow +\infty$. as $k \longrightarrow +\infty$. Then, for any $u \in Z_k$ such that $\|u\|_{X_0} = r_k$, we get

$$\inf_{u\in Z_k, \|u\|_{X_0}=r_k}\mathscr{J}(u) \geqslant m_1\left(\frac{1}{\alpha^+} - \frac{1}{q^+}\right)\left(\frac{C_1}{p^+}\right)^{\alpha^-} r_k^{\alpha^-p^-} - C\tilde{C}_1 r_k \longrightarrow +\infty.$$

■

Proof of condition (A_2):

Let δ_2 be a function defined for all $(x,t) \in \Omega \times \mathbb{R}$ as follows

$$\begin{aligned}\delta_2 : [1,+\infty) &\longrightarrow \mathbb{R} \\ \tau &\longrightarrow F(x,\tau^{-1}t)\tau^{\mu},\end{aligned}$$

δ_2 is differentiable and for each $t \geqslant 1$, we have

$$\delta_2'(\tau) = \tau^{\mu-1}\left(\mu F(x,\tau^{-1}t) - \tau^{-1}tf(x,\tau^{-1}t)\right).$$

From condition (f_1), we infer that δ_2 is a decreasing function. Integrating this result with condition (f_4) signifies the existence of $\tilde{C}_5 > 0$ such that

$$F(x,t) \geqslant \tilde{C}_5 |t|^{\mu}, \quad \text{for any} \quad |t| \geqslant 1, \quad \text{and } x \in \Omega. \tag{2.35}$$

Condition (f_3) implies that there exists $\tilde{C}_6 > 0$, and $\tilde{t} > 0$ such that

$$|F(x,t)| \leqslant \tilde{C}_6 |t|^{\alpha^- p^+}, \quad \text{for all } x \in \Omega \quad \text{and } 0 \leqslant |t| \leqslant \tilde{t}. \tag{2.36}$$

Moreover, condition (f_2) simply means that we can find $\tilde{C}_7 > 0$ such that

$$|F(x,t)| \leqslant \tilde{C}_7 |t|^{q^-} \leqslant \tilde{C}_7 |t|^{\alpha^- p^+}, \quad \text{for all } \tilde{t} \leqslant |t| \leqslant 1. \tag{2.37}$$

We get from the previous results that

$$F(x,t) \geq \tilde{C}_5 |t|^{\mu} - \left(\tilde{C}_6 + \tilde{C}_7\right) |t|^{\alpha^- p^+}, \text{ for all } x \in \Omega \text{ and } t \in \mathbb{R}. \tag{2.38}$$

Therefore, combining (2.38), condition $(\mathscr{M})$, and the embedding result in Remark 1.4, we deduce

$$\begin{aligned}
\mathscr{J}(u) &= \widehat{M}\left(\sigma_K^{\phi}(u)\right) - \int_{\Omega} F(x,u)dx \\
&\leqslant \frac{m_2}{\alpha^-}\left(\sigma_K^{\phi}(u)\right)^{\alpha(x)} - \tilde{C}_5 \int_{\Omega} |t|^{\mu} dx + \left(\tilde{C}_6 + \tilde{C}_7\right) \int_{\Omega} |t|^{\alpha^- p^+} dx \\
&\leqslant \tfrac{m_2}{\alpha^-}\left(\tfrac{C_2}{p^-}\right)^{\alpha^+} \|u\|_{X_0}^{\alpha^+ p^+} - \tilde{C}_8 \|u\|_{X_0}^{\mu} + \tilde{C}_9 \|u\|_{X_0}^{\alpha^- p^+}.
\end{aligned}$$

Since Y_k is a finite space, then all the norms are equivalents on Y_k, also $\mu > \alpha^+ p^+$, which implies that condition (A_2) holds for δ_k large enough. ■

Conclusion: Since $\mathscr{J}$ is an even functional, and by Lemma 2.4, $\mathscr{J}$ verifies the Palais-Smale condition. Therefore $\mathscr{J}$ satisfies the fountain Theorem assumptions, as a consequence, the proof of Theorem 2.4 is completed.

Proof of Theorem 2.5

To demonstrate the existence of infinitely many solutions with negative energy for problem $(\mathscr{P}_K)$, we will employ the dual fountain theorem, as outlined in Theorem 1.9, as our variational tool. We begin by establishing the Palais-Smale compactness condition $(PS)_c^*$.

Lemma 2.5. *Suppose that conditions (f_1), (f_2), and $(\mathscr{M})$ hold, then $\mathscr{J}$ verifies the $(PS)_c^*$ condition.*

Proof. Let $\{u_{n_j}\}_{n_j}$ be a sequence on X_0 such that

$$u_{n_j} \in Y_{n_j}, \quad \mathscr{J}(u_{n_j}) \to c, \quad \text{and} \quad \mathscr{J}|'_{Y_{n_j}}(u_{n_j}) \to 0, \quad \text{as } n_j \to +\infty.$$

By the same argument of Lemma 2.4, we obtain the boundedness of $\{u_{n_j}\}_{n_j}$. The reflexivity of the space X_0 implies the existence of a subsequence of $\{u_{n_j}\}_{n_j}$ such that $u_{n_j} \rightharpoonup u$ in X_0.
On the other hand, we can choose $v_{n_j} \in Y_{n_j}$ such that $v_{n_j} \to u$ in X_0 (because $X_0 = \overline{\cup_{n_j} Y_{n_j}}$). Therefore,

$$\begin{aligned}\lim_{n_j\to\infty} \langle \mathscr{J}'(u_{n_j}), (u_{n_j}-u)\rangle &= \lim_{n_j\to\infty} \langle \mathscr{J}'(u_{n_j}), (u_{n_j}-v_{n_j})\rangle + \lim_{n_j\to\infty} \langle \mathscr{J}'(u_{n_j}), (v_{n_j}-u)\rangle \\ &= \lim_{n_j\to\infty} \left\langle \left(\mathscr{J}|_{Y_{n_j}}\right)'(u_{n_j}), (u_{n_j}-v_{n_j})\right\rangle = 0.\end{aligned}$$

Consequently, depending on the proof of Lemma 2.4, we get that $u_{n_j} \to u$, hence the proof is completed. ■

Proof of condition (B_1):

From conditions (f_2) and (f_3), we can find $\varepsilon > 0$ and $C_\varepsilon > 0$ such that

$$F(x,t) \le \varepsilon |t|^{\alpha^- p^+} + C_\varepsilon |t|^{q(x)}, \quad \text{for all } (x,t) \in \Omega \times \mathbb{R}. \tag{2.39}$$

For any $u \in Z_k$, such that $\|u\|_{X_0} = 1$, and $0 < t < 1$, we have

$$\begin{aligned}\mathscr{J}(tu) &= \widehat{M}\left(\sigma_K^\phi(tu)\right) - \int_\Omega F(x,tu)\mathrm{d}x \\ &\geqslant \frac{m_1}{\alpha^+}\left(\sigma_K^\phi(tu)\right)^{\alpha^-} - \varepsilon\int_\Omega |tu|^{\alpha^- p^+}dx - C_\varepsilon\int_\Omega |tu|^{q(x)}dx \\ &\geqslant \frac{m_1}{\alpha^+}\left(\frac{C_1}{p^+}\right)^{\alpha^-} t^{\alpha^- p^+}(\rho^0_{K,p(x,y)})^{\alpha(x)} - \varepsilon t^{\alpha^- p^+}\int_\Omega |u|^{\alpha^- p^+}dx \\ &\quad - t^{q^-}C_\varepsilon\int_\Omega |tu|^{q(x)}dx,\end{aligned}$$

since $\|u\|_{X_0} = 1$, then $\rho^0_{K,p(x,y)} = 1$. In view of embedding result stated in Remark 1.4 and the definition of β_k, we obtain

$$\begin{aligned}\mathscr{J}(tu) &\geqslant \frac{m_1}{\alpha^+}\left(\frac{C_1}{p^+}\right)^{\alpha^-} t^{\alpha^- p^+} - \varepsilon t^{\alpha^- p^+}(\tilde{C}_{10})^{\alpha^- p^+} - C_\varepsilon t^{q^-}\beta_k^{q^-} \\ &\geqslant \left(\frac{m_1}{\alpha^+}\left(\frac{C_1}{p^+}\right)^{\alpha^-} - \varepsilon(\tilde{C}_{10})^{\alpha^- p^+}\right) t^{\alpha^- p^+} - C_\varepsilon\beta_k^{q^-} t^{q^-}.\end{aligned}$$

By choosing $\frac{m_1(C_1)^{\alpha^-}}{2\alpha^+(p^+)^{\alpha^-}(\tilde{C}_{10})^{\alpha^- p^+}} > \varepsilon$, we find

$$\mathscr{J}(tu) \geqslant \frac{m_1}{2\alpha^+}\left(\frac{C_1}{p^+}\right)^{\alpha^-} t^{\alpha^- p^+} - C_\varepsilon \beta_k^{q^-} t^{q^-}. \tag{2.40}$$

For k sufficiently large, we take $t = \delta_k$ small enough. Therefore, $\mathscr{J}(tu) \geqslant 0$, for $u \in Z_k$ where $\|u\|_{X_0} = 1$. Hence the condition (B_1) of the dual fountain theorem is verified. ■

Proof of condition (B_2):

Let $u \in Y_k$ such that $\|u\|_{X_0} = 1$ and $0 < t < \delta_k < 1$. From conditions (f_5) and $(\mathscr{M})$, we obtain by using the embedding result

$$\begin{aligned}
\mathscr{J}(tu) = \widehat{M}\left(\sigma_K^\phi(tu)\right) - \int_\Omega F(x,tu)\mathrm{d}x &\leqslant \frac{m_2}{\alpha^-}\left(\sigma_K^\phi(tu)\right)^{\alpha(x)} - \zeta\int_\Omega \frac{|tu|^{\gamma(x)}}{\gamma(x)}dx \\
&\leqslant \frac{m_2}{\alpha^-}\left(\frac{C_2}{p^-}\right)^{\alpha^+} t^{\alpha^+ p^-} - \frac{\zeta}{\gamma^+}t^{\gamma^+}\int_\Omega |u|^{\gamma(x)}dx \\
&\leqslant \frac{m_2}{\alpha^-}\left(\frac{C_2}{p^-}\right)^{\alpha^+} t^{\alpha^+ p^-} - \frac{\tau}{\gamma^+}t^{\gamma^+},
\end{aligned}$$

Since $\alpha^+ p^- > \gamma^+$, we can find $0 < r_k < \delta_k$ such that $\mathscr{J}(u) < 0$, for $u \in Y_k$ with $\|u\|_{X_0} = r_k$. This shows (B_2). ■

Proof of condition(B_3):

We have that $Y_k \cap Z_k = \emptyset$. Moreover, from (B_2), we have that $\mathscr{J}(u) < 0$, for $u \in Y_k$ with $\|u\|_{X_0} = r_k$. Then $a_k < 0$.
On the other hand, for any $0 \leqslant t \leqslant \delta_k \leqslant 1$ and $w \in Z_k$, where $\|w\|_{X_0} = 1$, we obtain by an argument similar to the one in the verification of (B_1) that

$$\mathscr{J}(tw) \geqslant -C_\varepsilon \beta_k^{q^-} t^{q^-}.$$

Hence $0 > a_k \geqslant -C_\varepsilon \beta_k^{q^-}$. Therefore $a_k \to 0$ as $k \to \infty$. Then (B_3) achieved. ■
Conclusion: In the view of the dual fountain theorem the proof of Theorem 2.5 is reached.

2.3.3 Particular case of problem $(\mathscr{P}_{\mathscr{K}})$

Now, we present an example and particular cases of the function ϕ, the kernel K, and the nonlinearity f which illustrate the results of this section. In particular, we will provide an example that verifies the conditions of Theorem 2.5. This is particularly important

as the theorem contains numerous conditions on f that appear challenging to satisfy simultaneously. Our example aims to illustrate the feasibility of these conditions.

A typical cases for ϕ and the kernel K are given by

$$\phi(t) = |t|^{p(x,y)-2}t \quad \text{and} \quad K(x,y) = |x-y|^{-(N+sp(x,y))}.$$

In this case, the operator $\mathscr{L}_K^\phi$ reduces to the fractional $p(x,\cdot)$-Laplacian operator $(-\Delta)^s_{p(x,\cdot)}$ (see for instance [35]). Therefore

$$\sigma_K^\phi(u) := \sigma(u) = \int_Q \frac{1}{p(x,y)} \frac{|u(x)-u(y)|^{p(x,y)}}{|x-y|^{N+sp(x,y)}} dxdy.$$

We take $N=3$ and we consider $\Omega = \left\{(x_1,x_2,x_3) \in \mathbb{R}^N : \quad x_1^2+x_2^2+x_3^2 \leqslant 10\right\}$.
For the variable exponent $p(.,.)$, we can take

$$p(x,y) = \frac{7}{2} + \frac{1}{2}\sin\left[(x_1^2+x_2^2+x_3^2)\pi\right] + \frac{1}{2}\sin\left[(y_1^2+y_2^2+y_3^2)\pi\right].$$

It is clear that $p \in C_+(\overline{Q})$, $p^- = \frac{5}{2}$, and $p^+ = \frac{9}{2}$.

Consequently, the problem $(\mathscr{P}_\mathscr{K})$ becomes

$$(\mathscr{P}_1)\begin{cases} M\left(\int_Q \frac{1}{p(x,y)} \frac{|u(x)-u(y)|^{p(x,y)}}{|x-y|^{N+sp(x,y)}} dxdy\right)(-\Delta)^s_{p(x,.)}(u(x)) = f(x,u) & \text{in } \Omega, \\ u = 0 & \text{in } \mathbb{R}^N \backslash \Omega. \end{cases}$$

For the non-linearity f, we can give the following example

$$f(x,t) = \begin{cases} t^{10}, & |t| > 1, \\ t^6, & |t| \leq 1. \end{cases}$$

- By choosing $\mu = 11$, condition (f_1) holds.
- If we take $q(x) = 6 + \sin[(x_1+x_2+x_3)\pi]$, therefore condition (f_2) is verified.
- If we consider $\gamma(x) = \frac{11}{6} + \frac{1}{2}\sin[(x_1+x_2+x_3)\pi]$, we have that $\gamma^+ < \alpha(x)p^-$, which means that condition (f_5) is fulfilled.

Conclusion: By considering the above particular cases, for example, the assumptions of Theorem 2.5 are satisfied. So, the results obtained in Theorems 2.5 remain true for problem $(\mathscr{P}_1)$.

CHAPTER 3

$p(x,\cdot)$-INTEGRO-DIFFERENTIAL PROBLEMS: SINGULAR AND NEUMANN BOUNDARY CONDITION CASES

Main objective

The objective of this chapter is to establish the existence of weak solutions for a class of problems governed by a generalized integro-differential operator with a singular kernel. The first problem, distinguished by a singular term, presents analytical challenges that contrast with those addressed in Chapter 2. The Nehari manifold approach and fibering map techniques are primarily employed to facilitate this analysis. Additionally, two embedding theorems into generalized weighted Lebesgue spaces are rigorously developed, serving as crucial elements of the main proof. In addressing the second problem, the chapter focuses on a bi-nonlocal problem characterized by nonlocal nonlinear Neumann boundary conditions. The principal result demonstrates the existence of infinitely many solutions within a broad framework of fractional Sobolev spaces with variable exponents.

> The results presented in this chapter are derived from the results established in [33] and [28].

3.1 Introduction

The study of singular elliptic problems requires a deep understanding of nonlinear analysis, singular behavior, boundary value problems, and regularity theory. It is an interdisciplinary field that draws on techniques from pure and applied mathematics to tackle complex and challenging problems in various scientific and engineering applications, particularly in the field of shallow membrane caps, see for instance, [104, 154, 155] and the references therein.

The study of singular problems presents unique challenges due to their nonstandard behaviour at certain points or along specific curves. The emergence of this class of problems in the literature traces back to Baxely [40], which addressed the study to a singular nonlinear boundary value problem that models the large deflection membrane response of a spherical cap of the following form

$$\begin{cases} y'' = \frac{-x^2}{32y^2} + \frac{\lambda^2}{8}, \quad 0 < x \leqq 1 \\ y(0) = 0, \quad 2y'(1) - (1+\nu)y(1) = 0 \end{cases}$$

Specifically, $S_r = y/x$ defines the radial stress at points on the membrane, while $S_\theta = \frac{d}{d\rho}(\rho S_r)$ describes the circumferential stress, with $\rho^2 = x$. The parameter λ is related to the load and geometry, and ν stands for Poisson's ratio. More information can be found in the cited references [95, 139, 150]. Baxely [41] and Fink et al [89] have proposed further elementary equations for singular problems. Afterward, many investigations dealt with this kind of problem involving the well-known Laplacian operator; see, for instance, [68, 72, 94, 125, 169]. Moreover, amidst advancements in the theory of variable exponents, some scholars have grappled with issues arising from including variable exponents within singularity terms. In [189], the authors dealt with a problem involving the $p(x)$-Laplacian operator incorporating singular nonlinearities of the specified structure

$$\begin{cases} -\Delta_{p(x)} u = \frac{\lambda}{u^{\gamma(x)}} & \text{in } \Omega, \\ u(x) = 0 & \text{on } \partial\Omega, \end{cases}$$

with some suitable hypothesis on the exponent variable $\gamma(x)$. They prove the existence of positive solutions and their asymptotic behavior near the boundary.

Revopš et al. [156] established the existence of multiple solutions using the Nehari manifold approach of the following singular problem

$$\begin{cases} -\Delta_{p(x)} u = a(x)|u|^{q(x)-2}u(x) + \frac{\lambda b(x)}{u^{\delta(x)}} & \text{in } \Omega, \\ u > 0 & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases}$$

Furthermore, Professor Kamel Saoudi made a significant contribution to the theory of singular problems, for further insight in this context, we refer to the references [161, 162].

Furthermore, in the case of fractional operators, we find the work of Barrios et al. [38], where they consider the following problem

$$\begin{cases} (-\Delta)^s u = F(x,u) := \lambda \frac{f(x)}{u^\gamma} + M u^p \text{ in } \Omega, \\ \qquad u > 0 \text{ in } \Omega, \\ \qquad u = 0 \text{ in } \mathbb{R}^N \setminus \Omega, \end{cases}$$

By distinguishing two cases ($M = 0$, $M = 1$), the authors proved the existence of at least two positive solutions for some range of values of λ.

Recently, Ghanmi et al. [93] generalized singular problems to include the fractional $p(x,.)$-Laplacian operator by considering the following equation

$$\begin{cases} (-\Delta)^s_{p(x,.)} u + |u|^{q(x)-2} u = g(x) u^{\varepsilon - 1 - \gamma(x)} \mp \lambda f(x,u) \text{ in } \Omega, \\ u = 0 \text{ on } \partial\Omega, \end{cases}$$

using variational methods, they proved the existence of weak solutions on the fractional Sobolev with variable exponent.

On the other hand, over time, the Dirichlet boundary conditions are used to study the vast bulk of nonlocal problems. Nevertheless, the study of Neumann boundary problems for fractional nonlocal problems is limited because it requires a different concept. Given what has been said, the inception of nonlocal Neumann boundary conditions was by Valdinoci et al. [78], they come to light from a simple probabilistic consideration, which was summarized as follows:

- The function u represents the probability distribution of the position of a particle making an arbitrary move inside the domain Ω.
- When the particle exists in the filed of study Ω, it straight away comes back into Ω.
- The strategy in which it comes back inside Ω is the following: If the particle has gone to $x \in \mathbb{R}^N \setminus \overline{\Omega}$, it may come back to any point $y \in \Omega$, the probability density of jumping from x to y being proportional to $|x-y|^{-N-2s}$.

These three situations lead to the introduction of the Neumann problem for the fractional Laplacian operator as follows

$$\begin{cases} (-\Delta)^s u = f & \text{in } \Omega, \\ \mathcal{N}_s u = 0 & \text{in } \mathbb{R}^N \setminus \overline{\Omega}, \end{cases}$$

where $\mathcal{N}_s$ is the nonlocal normal derivative defined in this way

$$\mathcal{N}_s u(x) := \int_\Omega \frac{u(x)-u(y)}{|x-y|^{N+2s}} dy \quad \forall x \in \mathbb{R}^N \setminus \overline{\Omega}. \tag{3.1}$$

Other problems exist with nonlocal Neumann boundary conditions [36, 135], but the advantage of Dipierro's approach is that the problem has a variational structure. In particular, they studied the basic properties of this kind of boundary conditions.
In [136], the authors extend the notion of nonlocal Neumann boundary conditions to cover the nonlinear case ($p > 1$), and the nonlocal normal p-derivative is defined as follows

$$\mathcal{N}_p^s u(x) = \int_\Omega \frac{|u(x)-u(y)|^{p-2}(u(x)-u(y))}{|x-y|^{N+ps}} dy, \quad \forall x \in \mathbb{R}^N \setminus \Omega. \tag{3.2}$$

This definition was also introduced in [39], where a nonlocal analog of the divergence theorem is given.
Very recently, for the Neumann problem involving the fractional $p(x,\cdot)$-Laplacian operator, Bahrouni et al. have introduced the corresponding nonlocal normal $p(x,\cdot)$-derivative for this operator, which is given by

$$\mathcal{N}_{p(x,\cdot)}^s u(x) = \int_\Omega \frac{|u(x)-u(y)|^{p(x,y)}(u(x)-u(y))}{|x-y|^{N+sp(x,y)}} dy \quad \forall x \in \mathbb{R}^N \setminus \Omega. \tag{3.3}$$

In their work, the authors pointed out the fundamental properties of this new boundary condition.
Based on the existing results in the literature, the key distinctive features of the problem addressed in the comping-after section are as follows

- **Singularity:** The presence of singular terms, such as the singular kernel K and the term $|u|^{-\alpha(x)}$ on the right-hand side, leads to nonstandard solution behavior and introduces challenges in establishing regularity properties.
- **Nonlocality:** The generalized integro-differential operator $\mathscr{L}_K^{p(x,\cdot)}$ encapsulates nonlocality, meaning that the problem incorporates dependencies beyond local conditions, capturing long-range interactions.

- **Nonlinearity:** The nonlinearity inherent in the problem introduces complexity to the behavior of solutions. Unlike linear problems, nonlinear problems can exhibit a range of phenomena, such as multiple solutions, bifurcations, and potential instability.

3.2 On a singular $p(x,\cdot)$-integro-differential elliptic problem

Our main objective in the following section is to address a singular problem by adopting alternative techniques to prove the existence results.

3.2.1 Hypothesis and problem statement

In this section, we consider the following singular boundary value problem

$$(\mathscr{P}_S)\begin{cases} \mathscr{L}_K^{p(x,\cdot)}(u(x)) + |u|^{\overline{p}(x)-2}u(x) &= g(x)|u|^{r(x)-2}u + \lambda\frac{h(x)}{|u|^{\alpha(x)}} \quad \text{in } \Omega, \\ u &= 0 \quad \text{on } \mathbb{R}^N\setminus\Omega, \end{cases}$$

where

- The two exponents r and α verify

$$r:\overline{\Omega}\to(1,+\infty) \text{ such that } p(x)<r(x)<p_s^*(x) \text{ for all, } x\in\Omega. \tag{3.4}$$

$$\alpha:\overline{\Omega}\longrightarrow(0,1) \text{ such that } \quad 1-\alpha^+\leqslant 1-\alpha^-<p^-. \tag{3.5}$$

- (G) $g:\Omega\longrightarrow(0,+\infty)$ is a positive weighted function that belongs to $L^{m(x)}(\Omega)$, with $m\in C^+(\overline{\Omega})$, and verify

$$\frac{\overline{p}(x)}{\overline{p}(x)-r(x)}<m(x)<\frac{N\overline{p}(x)}{N\overline{p}(x)-r(x)(N-s\overline{p}(x))} \quad \text{for all } x\in\overline{\Omega}. \tag{3.6}$$

- (H) $h:\Omega\longrightarrow(0,+\infty)$ is a positive weighted function that belongs to $L^{\eta(x)}(\Omega)$, such that

$$\eta(x)=\frac{\overline{p}(x)}{\overline{p}(x)+\alpha(x)-1} \quad \forall x\in\Omega. \tag{3.7}$$

The main findings of this section can be summarized as follows.

Theorem 3.1. *Suppose that conditions (3.4) and (3.5) hold, also g and h verify hypotheses (G) and (H). Therefore, there exists $\tilde{\delta}$ such that for each $\lambda\in(0,\tilde{\delta})$, problem $(\mathscr{P}_S)$ has at least two weak solutions.*

3.2.2 Embedding results between X_0 and $L_w^{p(x)}(\Omega)$

This subsection is devoted to state and prove two embedding results into weighted spaces, which serve as pivotal components in our principal proof.

Theorem 3.2. *Let Ω be a Lipschitz bounded domain in $\mathbb{R}^N$ and $s \in (0,1)$. Let $p : \overline{Q} \longrightarrow (1,+\infty)$ be a continuous variable exponent satisfies conditions (1.2) and (1.3) with $sp^+ < N$. Furthermore, the kernel $K : \mathbb{R}^N \times \mathbb{R}^N \longrightarrow (0,+\infty)$ is supposed to be a measurable function with conditions (K_1) and (K_2). If conditions (3.5) and (3.7) are fulfilled, then there exists a constant $C > 0$ such that*

$$\rho_{1-\alpha(x),h}(u) \leqslant C\left(\|u\|_{X_0}^{1-\alpha^-} + \|u\|_{X_0}^{1-\alpha^+}\right). \tag{3.8}$$

Which means that X_0 is continuously embedded in $L_h^{1-\alpha(x)}(\Omega)$. Furthermore, this embedding is compact.

Proof. First, we establish the continuous embedding result between X_0 and $L_h^{1-\alpha(x)}(\Omega)$. Let $u \in X_0$, it follows that $|u|^{1-\alpha(x)} \in L^{\hat{\eta}(x)}(\Omega)$ where $\hat{\eta}(x) = \frac{\overline{p}(x)}{1-\alpha(x)}$. Indeed, from Proposition 1.2, we have

$$\left\| |u|^{1-\alpha(x)} \right\|_{L^{\hat{\eta}(x)}(\Omega)} \leqslant \max\left\{ \left(\rho_{\hat{\eta}(x)}(|u|^{1-\alpha(x)})\right)^{\frac{1}{\hat{\eta}^+}}, \left(\rho_{\hat{\eta}(x)}(|u|^{1-\alpha(x)})\right)^{\frac{1}{\hat{\eta}^-}} \right\}, \tag{3.9}$$

and by using Theorem 1.3 we get

$$\rho_{\hat{\eta}(x)}(|u|^{1-\alpha(x)}) = \int_\Omega |u|^{(1-\alpha(x))\hat{\eta}(x)} dx = \int_\Omega |u|^{\overline{p}(x)} dx < +\infty, \tag{3.10}$$

combining (3.9) and (3.10), we acquire the desired result. On the other hand, by Hölder's inequality we obtain

$$\rho_{1-\alpha(x),h}(u) = \int_\Omega h(x)|u|^{1-\alpha(x)} dx \leqslant 2\|h\|_{L^{\eta(x)}(\Omega)} \| |u|^{1-\alpha(x)} \|_{L^{\hat{\eta}(x)}(\Omega)}. \tag{3.11}$$

Thus, $u \in L_h^{1-\alpha(x)}(\Omega)$, which achieves the continuous embedding result between X_0 and the weighted variable exponent Lebesgue space $L_h^{1-\alpha(x)}(\Omega)$. Regarding inequality (3.8), we have $|u|^{1-\alpha(x)} \leqslant |u|^{1-\alpha^+} + |u|^{1-\alpha^-}$ for all $x \in \overline{\Omega}$, hence

$$\rho_{1-\alpha(x),h}(u) = \int_\Omega h(x)|u|^{1-\alpha(x)} dx \leqslant \int_\Omega h(x)|u|^{1-\alpha^+} dx + \int_\Omega h(x)|u|^{1-\alpha^-} dx, \tag{3.12}$$

Considering that $h \in L^{\eta(x)}(\Omega)$, Remark 1.4, and Hölder's inequality, there exist two positive constants c_1 and c_2 such that

$$\int_\Omega h(x)|u|^{1-\alpha^+} dx \leqslant c_1 \|u\|_{X_0}^{1-\alpha^+},$$

and

$$\int_\Omega h(x)|u|^{1-\alpha^-}dx \leqslant c_2\|u\|_{X_0}^{1-\alpha^-}.$$

Combining the two previous inequalities with (3.12), we obtain

$$\rho_{1-\alpha(x),h}(u) \leqslant c_3\left(\|u\|_{X_0}^{1-\alpha^+} + \|u\|_{X_0}^{1-\alpha^-}\right). \tag{3.13}$$

Now, we aim to demonstrate the compact embedding result. Let (u_n) be a sequence in X_0 such that $u_n \rightharpoonup 0$ in X_0. Since the embedding between X_0 and $L^{\overline{p}(x)}(\Omega)$ is compact, then $u_n \longrightarrow 0$ in $L^{\overline{p}(x)}(\Omega)$. Therefore,

$$\||u_n|^{1-\alpha(x)}\|_{\hat{\eta}(x)} \longrightarrow 0.$$

Thus, $\rho_{1-\alpha(x),h}(u_n) \longrightarrow 0$. From Proposition 1.4, we get $\|u_n\|_{1-\alpha(x),h} \longrightarrow 0$. Accordingly, X_0 is compactly embedded in $L_h^{1-\alpha(x)}(\Omega)$. ■

As a direct consequence of the previous result, we have the following corollary.

Corollary 3.1. *Let Ω be a Lipschitz bounded domain in $\mathbb{R}^N$ and $s \in (0,1)$. Let $p : \overline{Q} \longrightarrow (1,+\infty)$ be a continuous variable exponent satisfies conditions (1.2) and (1.3) with $sp^+ < N$. Furthermore, the kernel $K : \mathbb{R}^N \times \mathbb{R}^N \longrightarrow (0,+\infty)$ is supposed to be a measurable function with conditions (K_1) and (K_2). We assume also that the weighted function h verifies condition (H) where the exponent variable α satisfies (3.5). Therefore, there exists two positive constants $\tilde{C}_1$, and $\tilde{C}_2$ such that*

$$\int_\Omega h(x)|u|^{1-\alpha(x)}dx \leq \begin{cases} \tilde{C}_1\|u\|_{X_0}^{1-\alpha^-} & \text{if } \|u\|_{X_0} > 1, \\ \tilde{C}_2\|u\|_{X_0}^{1-\alpha^+} & \text{if } \|u\|_{X_0} < 1. \end{cases} \tag{3.14}$$

Proof. Using Hölder's inequality, Proposition 1.2, and Remark 1.4, the result is obtained as in the previous proof. ■

3.2.3 Nehari Manifold and fibering maps approach

We say that a function $u \in X_0$ is a weak solution of problem $(\mathscr{P}_S)$ if it is solution of the next equation

$$\begin{cases} \displaystyle\int_Q |u(x)-u(y)|^{p(x,y)-2}(u(x)-u(y))(v(x)-v(y))K(x,y)dxdy + \int_\Omega |u|^{\overline{p}(x)-2}uvdx \\ \displaystyle -\int_\Omega g(x)|u|^{r(x)-2}uvdx - \lambda\int_\Omega h(x)|u|^{-\alpha(x)}vdx = 0, \quad \forall v \in X_0. \end{cases} \tag{3.15}$$

Associate to the singular problem $(\mathscr{P}_S)$, we define the energy functional as follows

$$\begin{aligned}\mathscr{J}_S(u) = &\int_Q \frac{|u(x)-u(y)|^{p(x,y)}}{p(x,y)}K(x,y)dxdy + \int_\Omega \frac{|u|^{\overline{p}(x)}}{\overline{p}(x)}dx \\ &- \int_\Omega \frac{g(x)|u|^{r(x)}}{r(x)}dx - \lambda\int_\Omega \frac{h(x)|u|^{1-\alpha(x)}}{1-\alpha(x)}dx,\end{aligned} \tag{3.16}$$

for all $u \in X_0$.

$\mathscr{J}_S$ is not differentiable in X_0 due to the singular term, because the singular term introduces complexities that preclude it from being, and hence standard variational methods are inapplicable. To address this issue, we have adopted an alternative technique by considering the fibering maps approach introduced by Drabek and Pohozaev in [80], and it is given by

$$\begin{aligned}\mathscr{I}_u(t) = &\int_Q \frac{t^{p(x,y)}|u(x)-u(y)|^{p(x,y)}}{p(x,y)}K(x,y)\,dx\,dy + \int_\Omega \frac{t^{\overline{p}(x)}}{\overline{p}(x)}|u|^{\overline{p}(x)}dx \\ &- \int_\Omega g(x)\frac{t^{r(x)}}{r(x)}|u|^{r(x)}dx - \lambda\int_\Omega h(x)\frac{t^{1-\alpha(x)}|u|^{1-\alpha(x)}}{1-\alpha(x)}dx,\end{aligned}$$

and

$$\begin{aligned}\mathscr{I}'_u(t) = &\int_Q t^{p(x,y)-1}|u(x)-u(y)|^{p(x,y)}K(x,y)\,dx\,dy + \int_\Omega t^{\overline{p}(x)-1}|u|^{\overline{p}(x)}dx \\ &- \int_\Omega g(x)t^{r(x)-1}|u|^{r(x)}dx - \lambda\int_\Omega h(x)t^{-\alpha(x)}|u|^{1-\alpha(x)}dx,\end{aligned}$$

as well as

$$\begin{aligned}\mathscr{I}''_u(t) = &\int_Q (p(x,y)-1)t^{p(x,y)-2}|u(x)-u(y)|^{p(x,y)}K(x,y)\,dx\,dy + \int_\Omega (\overline{p}(x)-1)t^{\overline{p}(x)-2}|u|^{\overline{p}(x)}dx \\ &- \int_\Omega (r(x)-1)g(x)t^{r(x)-2}|u|^{r(x)}dx + \lambda\int_\Omega \alpha(x)h(x)t^{-\alpha(x)-1}|u|^{1-\alpha(x)}dx.\end{aligned}$$

On the other hand, the functional $\mathscr{J}_S$ is not bounded below on X_0, as shown in the following argument. Let $u \in X_0$, we have

$$\begin{aligned}\mathscr{J}_S(u) = &\int_Q \frac{|u(x)-u(y)|^{p(x,y)}}{p(x,y)}K(x,y)dxdy + \int_\Omega \frac{|u|^{\overline{p}(x)}}{\overline{p}(x)}dx \\ &- \int_\Omega \frac{g(x)|u|^{r(x)}}{r(x)}dx - \lambda\int_\Omega \frac{h(x)|u|^{1-\alpha(x)}}{1-\alpha(x)}dx \\ \geqslant &\frac{1}{p^+}\int_Q |u(x)-u(y)|^{p(x,y)}K(x,y)dxdy + \frac{1}{p^+}\int_\Omega |u|^{\overline{p}(x)}dx \\ &- \frac{1}{r^-}\int_\Omega g(x)|u|^{r(x)}dx - \frac{\lambda}{1-\alpha^+}\int_\Omega h(x)|u|^{1-\alpha(x)}dx.\end{aligned}$$

If $\|u\|_{X_0} > 1$, we acquire using Proposition 1.6, Corollaries 3.1 and 1.5 that

$$\mathscr{J}_S(u) \geqslant \frac{1}{p^+}\|u\|_{X_0}^{p^-} - \frac{\tilde{C}_6}{r^-}\|u\|_{X_0}^{r^+} - \tilde{C}_1\frac{\lambda}{1-\alpha^+}\|u\|_{X_0}^{1-\alpha^-},$$

if $\|u\|_{X_0} \leqslant 1$,

$$\mathscr{J}_S(u) \geqslant \frac{1}{p^+}\|u\|_{X_0}^{p^+} - \frac{\tilde{C}_7}{r^-}\|u\|_{X_0}^{r^-} - \tilde{C}_2\frac{\lambda}{1-\alpha^+}\|u\|_{X_0}^{1-\alpha^+}.$$

Since $1-\alpha^+ < p^- \leqslant p^+ < r^- \leqslant r^+$, then $\mathscr{J}_S$ is not bounded below on the entire space X_0. As a consequence, we consider an adequate subset of X_0 in which our problem is bounded, called the Nehari manifold introduced as follows

$$\mathscr{M}_\lambda = \left\{u \in X_0 \setminus \{0\} : \mathscr{I}_u'(1) = 0\right\}.$$

Lemma 3.1. *The functional $\mathscr{J}_S$ is coercive and bounded below on $\mathscr{M}_\lambda$ for each $\lambda > 0$.*

Proof. Let $u \in \mathscr{M}_\lambda$ with $\|u\|_{X_0}$ >1, we have

$$\begin{aligned}\mathscr{J}_S(u) &= \int_Q \frac{|u(x)-u(y)|^{p(x,y)}}{p(x,y)}K(x,y)dxdy + \int_\Omega \frac{|u|^{\overline{p}(x)}}{\overline{p}(x)}dx \\ &\quad - \int_\Omega \frac{g(x)|u|^{r(x)}}{r(x)}dx - \lambda\int_\Omega \frac{h(x)|u|^{1-\alpha(x)}}{1-\alpha(x)}dx \\ &\geq \frac{1}{p^+}\int_Q |u(x)-u(y)|^{p(x,y)}K(x,y)dxdy + \frac{1}{p^+}\int_\Omega |u|^{\overline{p}(x)}dx \\ &\quad - \frac{1}{r^-}\int_\Omega g(x)|u|^{r(x)}dx - \frac{\lambda}{1-\alpha^+}\int_\Omega h(x)|u|^{1-\alpha(x)}dx,\end{aligned}$$

which implies that

$$\begin{aligned}\mathscr{J}_S(u) \geq &\left(\frac{1}{p^+}-\frac{1}{r^-}\right)\int_Q |u(x)-u(y)|^{p(x,y)}K(x,y)dxdy + \left(\frac{1}{p^+}-\frac{1}{r^-}\right)\int_\Omega |u|^{\overline{p}(x)}dx \\ &- \lambda\left(\frac{1}{1-\alpha^+}-\frac{1}{r^-}\right)\int_\Omega h(x)|u|^{1-\alpha(x)}dx.\end{aligned}$$

From the hypothesis (3.7), we get by applying Proposition 1.6 and Corollary 3.1 that

$$\mathscr{J}_S(u) \geq \left(\frac{1}{p^+}-\frac{1}{r^-}\right)\|u\|_{X_0}^{p^-} - \tilde{C}_1\lambda\left(\frac{1}{1-\alpha^+}-\frac{1}{r^-}\right)\|u\|_{X_0}^{1-\alpha^-}.$$

The fact that $1-\alpha^- < p^-$ verifies the coercivity of the functional $\mathscr{J}_S$, and the proof is completed. ■

Next, we split $\mathscr{M}_\lambda$ into three distinct parts representing local minima, local maxima, and points of inflection. Accordingly, we establish the following definitions

$$\begin{aligned}\mathscr{M}_\lambda^+ &:= \left\{u \in \mathscr{M}_\lambda : \mathscr{I}_u''(1) > 0\right\}, \\ \mathscr{M}_\lambda^- &:= \left\{u \in \mathscr{M}_\lambda : \mathscr{I}_u''(1) < 0\right\}, \\ \mathscr{M}_\lambda^0 &:= \left\{u \in \mathscr{M}_\lambda : \mathscr{I}_u''(1) = 0\right\}.\end{aligned}$$

Subsequently, the following results are obtained.

Lemma 3.2. *There exists $\lambda^* > 0$, such that for every $\lambda \in (0, \lambda^*)$, we have $\mathscr{M}_\lambda^0 = \emptyset$.*

Proof. By contradiction, we suppose that there exists $u \in \mathscr{M}_\lambda^0$, $\forall \lambda \in (0, \lambda^*)$. From the definition of $\mathscr{M}_\lambda^0$, we have $\mathscr{I}_u''(1) = 0$. Then

$$\begin{aligned} 0 &= \int_Q (p(x,y)-1)|u(x)-u(y)|^{p(x,y)}K(x,y)dxdy + \int_\Omega (\overline{p}(x)-1)|u|^{\overline{p}(x)}dx \\ &\quad - \int_\Omega (r(x)-1)g(x)|u|^{r(x)}dx + \lambda \int_\Omega \alpha(x)h(x)|u|^{1-\alpha(x)}dx \\ &\geqslant (p^- - 1)\int_Q |u(x)-u(y)|^{p(x,y)}K(x,y)dxdy + (p^- - 1)\int_\Omega |u|^{\overline{p}(x)}dx \\ &\quad - (r^+ - 1)\int_\Omega g(x)|u|^{r(x)}dx + \lambda\alpha^- \int_\Omega h(x)|u|^{1-\alpha(x)}dx. \end{aligned}$$

The fact that $u \in \mathscr{M}_\lambda$ gives

$$\lambda \int_\Omega h(x)|u|^{1-\alpha(x)}dx = \int_Q |u(x)-u(y)|^{p(x,y)}K(x,y)dxdy + \int_\Omega |u|^{\overline{p}(x)}dx - \int_\Omega g(x)|u|^{r(x)}dx.$$

Which implies that

$$\begin{aligned} &(\alpha^- + p^- - 1)\int_Q |u(x)-u(y)|^{p(x,y)}K(x,y)dxdy + (\alpha^- + p^- - 1)\int_\Omega |u|^{\overline{p}(x)}dx \\ &\quad + \left(1 - r^+ - \alpha^-\right)\int_\Omega g(x)|u|^{r(x)}dx \leq 0. \end{aligned}$$

Consequently, from Proposition 1.6 and Corollary 1.5 by taking $\|u\|_{X_0} > 1$, we get

$$(\alpha^- + p^- - 1)\|u\|_{X_0}^{p^-} + \tilde{C}_6\left(1 - r^+ - \alpha^-\right)\|u\|_{X_0}^{r^+} \leqslant 0.$$

Therefore,

$$\|u\|_{X_0} \geqslant \left(\frac{\alpha^- + p^- - 1}{\tilde{C}_6(r^+ + \alpha^- - 1)}\right)^{\frac{1}{r^+ - p^-}}. \tag{3.17}$$

By analogy,

$$\begin{aligned} 0 &\leqslant (p^+ - 1)\int_Q |u(x)-u(y)|^{p(x,y)}K(x,y)dxdy + p^+ - 1)\int_\Omega |u|^{\overline{p}(x)}dx \\ &\quad - (r^- - 1)\int_\Omega g(x)|u|^{r(x)}dx + \alpha^+\lambda \int_\Omega h(x)|u|^{1-\alpha(x)}dx. \end{aligned}$$

As

$$\int_\Omega g(x)|u|^{r(x)}dx = \int_Q |u(x)-u(y)|^{p(x,y)}K(x,y)dxdy + \int_\Omega |u|^{\overline{p}(x)}dx - \lambda \int_\Omega h(x)|u|^{1-\alpha(x)}dx,$$

we acquire

$$0 \leqslant \left(p^+ - r^-\right) \int_Q |u(x)-u(y)|^{p(x,y)} K(x,y)dxdy + \left(p^+ - r^-\right) \int_\Omega |u|^{\overline{p}(x)} dx + \lambda \left(\alpha^+ + r^- - 1\right) \int_\Omega h(x)|u|^{1-\alpha(x)} dx.$$

Proposition 1.6 and Corollary 3.1 imply

$$\left(p^+ - r^-\right) \|u\|_{X_0}^{p^-} + \tilde{C}_1 \lambda \left(r^- + \alpha^+ - 1\right) \|u\|_{X_0}^{1-\alpha^-} \geqslant 0.$$

Then,

$$\|u\|_{X_0} \leq \left(\frac{\tilde{C}_1 \lambda \left(r^- + \alpha^+ - 1\right)}{r^- - p^+} \right)^{\frac{1}{p^- + \alpha^- - 1}}. \tag{3.18}$$

For $\lambda < \lambda^*$, where $\lambda^* = \left(\frac{\alpha^- + p^- - 1}{\tilde{C}_6(r^+ + \alpha^- - 1)}\right)^{\frac{p^- + \alpha^- - 1}{r^+ - p^-}} \left(\frac{r^- - p^+}{\tilde{C}_1(r^- + \alpha^+ - 1)}\right)$, we obtain from (3.17), and (3.18) that

$$\|u\|_{X_0} \leq 1.$$

Hence, we get a contradiction, and finally $\mathcal{M}_\lambda^0 = \emptyset$, the proof of Lemma 3.2 is achieved. ■

As a direct consequence, we could partition $\mathcal{M}_\lambda$ as $\mathcal{M}_\lambda = \mathcal{M}_\lambda^+ \cup \mathcal{M}_\lambda^-$, and denote

$$\beta_\lambda^+ = \inf_{u \in \mathcal{M}_\lambda^+} \mathcal{J}_S(u) \text{ and } \beta_\lambda^- = \inf_{u \in \mathcal{M}_\lambda^-} \mathcal{J}_S(u).$$

In the following, we state and prove the next lemmas.

Lemma 3.3. *There exists $\lambda^* > 0$, such that for any $\lambda \in (0, \lambda^*)$, then $\beta_\lambda^+ < 0$ for each $u \in \mathcal{M}_\lambda^+$.*

Proof. Let $u \in \mathcal{M}_\lambda^+$, which implies that $\mathcal{I}_u''(1) > 0$, then

$$\begin{aligned} 0 &< \int_Q (p(x,y)-1)|u(x)-u(y)|^{p(x,y)} K(x,y)dxdy + \int_\Omega (\overline{p}(x)-1)|u|^{\overline{p}(x)} dx \\ &\quad - \int_\Omega (r(x)-1)g(x)|u|^{r(x)} dx + \lambda \int_\Omega \alpha(x)h(x)|u|^{1-\alpha(x)} dx \\ &\leqslant (p^+ - 1) \int_Q |u(x)-u(y)|^{p(x,y)} K(x,y)dxdy + (p^+ - 1) \int_\Omega |u|^{\overline{p}(x)} dx \\ &\quad - (r^- - 1) \int_\Omega g(x)|u|^{r(x)} dx + \lambda \alpha^+ \int_\Omega h(x)|u|^{1-\alpha(x)} dx. \end{aligned} \tag{3.19}$$

As $u \in \mathcal{M}_\lambda$, we have also that

(3.20)
$$\int_Q |u(x)-u(y)|^{p(x,y)}K(x,y)dxdy + \int_\Omega |u|^{\overline{p}(x)}dx - \int_\Omega g(x)|u|^{r(x)}dx - \lambda\int_\Omega h(x)|u|^{1-\alpha(x)}dx = 0.$$

We multiply by α^+, and then substitute the result into equation (3.19), we obtain

(3.21)
$$\begin{aligned}\left(p^+ + \alpha^+ - 1\right)\int_Q |u(x)-u(y)|^{p(x,y)}k(x,y)dxdy + \left(p^+ + \alpha^+ - 1\right)\int_\Omega |u|^{\overline{p}(x)}dx \\ > \left(r^- + \alpha^+ - 1\right)\int_\Omega g(x)|u|^{r(x)}dx.\end{aligned}$$

On the other hand, we obtain by using the inequality (3.21) that

(3.22)
$$\begin{aligned}\mathcal{J}_S(u) &\leqslant \frac{1}{p^-}\int_Q |u(x)-u(y)|^{p(x,y)}K(x,y)dxdy + \frac{1}{p^-}\int_\Omega |u|^{\overline{p}(x)} \\ &\quad - \frac{1}{r^+}\int_\Omega g(x)|u|^{r(x)}dx - \frac{\lambda}{1-\alpha^-}\int_\Omega h(x)|u|^{1-\alpha(x)}dx \\ &= \left(\frac{1}{p^-} - \frac{1}{1-\alpha^-}\right)\int_Q |u(x)-u(y)|^{p(x,y)}K(x,y)dxdy + \left(\frac{1}{p^-} - \frac{1}{1-\alpha^-}\right)\int_\Omega |u|^{\overline{p}(x)} \\ &\quad - \left(\frac{1}{r^+} - \frac{1}{1-\alpha^-}\right)\int_\Omega g(x)|u|^{r(x)}dx \\ &< \left(\frac{1-\alpha^- - p^-}{p^-(1-\alpha^-)} - \frac{1-\alpha^- - r^+}{r^+(1-\alpha^-)}\frac{p^+ + \alpha^+ - 1}{r^- + \alpha^+ - 1}\right)\rho_{K,p(x,y)}(u)\end{aligned}$$

Since $1-\alpha^- < p^-$, the term between brackets is less than 0, therefore

$$\mathcal{J}_S(u) < 0, \quad \forall u \in \mathcal{M}_\lambda^+.$$

As a result $\beta_\lambda^+ < 0$ for all $\lambda \in (0,\lambda^*)$. ■

Lemma 3.4. *There exists $\lambda^{**} > 0$, such that for any $\lambda \in (0,\lambda^{**})$, then $\beta_\lambda^- > 0$ for each $u \in \mathcal{M}_\lambda^-$.*

Proof. For each $u \in \mathcal{M}_\lambda^-$, we have

(3.23)
$$\begin{aligned}\mathcal{J}_S(u) &\geqslant \frac{1}{p^+}\int_Q |u(x)-u(y)|^{p(x,y)}K(x,y)dxdy + \frac{1}{p^+}\int_\Omega |u|^{\overline{p}(x)}dx \\ &\quad - \frac{1}{r^-}\int_\Omega g(x)|u|^{r(x)}dx - \frac{\lambda}{1-\alpha^+}\int_\Omega h(x)|u|^{1-\alpha(x)}dx \\ &= \left(\frac{1}{p^+} - \frac{1}{r^-}\right)\int_Q |u(x)-u(y)|^{p(x,y)}K(x,y)dxdy + \left(\frac{1}{p^+} - \frac{1}{r^-}\right)\int_\Omega |u|^{\overline{p}(x)} \\ &\quad - \lambda\left(\frac{1}{1-\alpha^+} - \frac{1}{r^-}\right)\int_\Omega h(x)|u|^{1-\alpha(x)}dx.\end{aligned}$$

Using Hölder's inequality and the embedding results stated in Corollary 3.1, we get

$$\mathcal{J}_S(u) \geqslant \left(\frac{1}{p^+} - \frac{1}{r^-}\right)\|u\|_{X_0}^{p^-} + \lambda \tilde{C}_1 \left(\frac{1}{r^-} - \frac{1}{1-\alpha^+}\right)\|u\|_{X_0}^{1-\alpha^-}.$$

Since $p^- > 1-\alpha^-$, we have

$$\mathcal{J}_S(u) \geqslant \left(\frac{1}{p^+} - \frac{1}{r^-}\right)\|u\|_{X_0}^{p^-} + \lambda \tilde{C}_1 \left(\frac{1}{r^-} - \frac{1}{1-\alpha^+}\right)\|u\|_{X_0}^{p^-}.$$

If we choose $\lambda < \frac{(r^- - p^+)(1-\alpha^+)}{\tilde{C}_1 p^+(\alpha^+ + r^- - 1)} = \lambda^{**}$, we achieve the desired results represented as

$$\mathcal{J}_S(u) > 0 \quad \forall u \in \mathcal{M}_\lambda^-.$$

Hence $\beta_\lambda^- > 0$ for each $\lambda \in (0, \lambda^{**})$. ■

Next, we state and prove the following lemma that plays a crucial role in aligning our analysis with the existence of critical points of our energy functional.

Lemma 3.5. *For $u \in X_0 \setminus \{0\}$, there exists $\gamma > 0$ such that for $\lambda < \gamma$, we have:*

1. *If $\lambda \int_\Omega h(x)|u|^{1-\alpha(x)}\mathrm{d}x = 0$, therefore there exists θ^- such that $\theta^- u \in \mathcal{M}_\lambda^-$ and $\mathcal{J}_S(tu)$ attains the supremum at $\mathcal{J}_S(\theta^- u)$ for all $t \geqslant 0$.*
2. *If $\lambda \int_\Omega h(x)|u|^{1-\alpha(x)}\mathrm{d}x > 0$, therefore there exist θ^* and two positive numbers $\theta^+ < \theta^-$ such that $\theta^- u \in \mathcal{M}_\lambda^-$ and $\theta^+ u \in \mathcal{M}_\lambda^+$. Furthermore,*

$$\mathcal{J}_S(\theta^+ u) = \inf_{0 \leq t \leq \theta^*} \mathcal{J}_S(tu) \quad \text{and} \quad \mathcal{J}_S(\theta^- u) = \sup_{t \geq 0} \mathcal{J}_S(tu).$$

Proof. 1. For $t > 0$ sufficiently small, we get

$$\mathcal{J}_u(t) \geqslant \frac{t^{p^+}}{p^+}\int_Q |u(x)-u(y)|^{p(x,y)}K(x,y)\,dx\,dy + \frac{t^{p^+}}{p^+}\int_\Omega |u|^{\overline{p}(x)}\,dx - \frac{t^{r^-}}{r^-}\int_\Omega g(x)|u|^{r(x)}\,dx > 0.$$

For $t > 1$ sufficiently large,

$$\mathcal{J}_u(t) \leqslant \frac{t^{p^+}}{p^-}\int_Q |u(x)-u(y)|^{p(x,y)}K(x,y)\,dx\,dy + \frac{t^{p^+}}{p^-}\int_\Omega |u|^{\overline{p}(x)}\,dx - \frac{t^{r^-}}{r^+}\int_\Omega g(x)|u|^{r(x)}\,dx < 0.$$

As a consequence, $\mathscr{I}_u$ attaints its maximum in a point denoted $\theta^- \geqslant 0$, and therefore

$$\mathscr{I}_u'(\theta^-) = \langle \mathscr{J}_S(\theta^- u), u\rangle = 0.$$

Take $\theta^- u = v$, hence

$$\langle \mathscr{J}_S(v), v\rangle = \theta^- \langle \mathscr{J}_S(v), u\rangle = 0,$$

and thus $v \in \mathscr{M}_\lambda$.
Consider the functional φ_v defined on $\mathbb{R}^+$ by $\varphi_v(t) = \mathscr{J}_S(tv)$.
For $t > 0$ sufficiently small

$$\varphi_v'(t) \geqslant (t^{p^+-1} - t^{r^--1}) \int_\Omega g(x)|v|^{r(x)} > 0,$$

for $t > 1$ sufficiently large

$$\varphi_v'(t) \leqslant (t^{p^+-1} - t^{r^--1}) \int_\Omega g(x)|v|^{r(x)} < 0.$$

This implies that the existence of θ^- is unique.

2. Consider the following mappings $\mathscr{L}_i : \mathbb{R}^+ \to \mathbb{R}$, $i = 1,2,3$ defined as follows

$$\begin{aligned}
\mathscr{L}_1(t) &= \int_Q t^{p(x,y)-1}|u(x)-u(y)|^{p(x,y)}K(x,y)\,dx\,dy,\\
\mathscr{L}_2(t) &= \lambda\int_\Omega h(x)t^{-\alpha(x)}|u|^{1-\alpha(x)}\,dx - \int_\Omega t^{\overline{p}(x)-1}|u|^{\overline{p}(x)}\,dx,\\
\mathscr{L}_3(t) &= \int_\Omega t^{r(x)-1}|u|^{r(x)}\,dx.
\end{aligned}$$

Then,

$$\mathscr{I}_u'(t) = \mathscr{L}_1(t) - \mathscr{L}_2(t) - \mathscr{L}_3(t) \quad \forall t \in \mathbb{R}^+.$$

We have $\lim_{t\to 0^+} \frac{\mathscr{L}_3(t)}{\mathscr{L}_1(t)} = 0$. Therefore, for t sufficiently small,

$$\mathscr{L}_1(t) - \mathscr{L}_3(t) > 0.$$

Furthermore, by analog proof as in (1) $\mathscr{L}_1 - \mathscr{L}_3$ has a unique maximum point denoted θ.
As $\lim_{t\to 0^+} \mathscr{L}_2(t) = +\infty$, then $\exists t_1 \in (0,\theta)$ such that

$$\mathscr{L}_1(t_1) - \mathscr{L}_2(t_1) - \mathscr{L}_3(t_1) < 0,$$

and for each $\lambda > 0$ sufficiently small $\exists t_2 \in (0,\theta)$ such that

$$\mathscr{L}_1(t_2) - \mathscr{L}_2(t_2) - \mathscr{L}_3(t_2) > 0.$$

Hence, $\exists \theta^+ = \theta(\lambda) \in (0,\theta)$ such that

$$\mathscr{L}_1(\theta^+) - \mathscr{L}_2(\theta^+) - \mathscr{L}_3(\theta^+) = 0,$$

because $\mathscr{L}_1 - \mathscr{L}_2 - \mathscr{L}_3$ is a non-decreasing function on $(0,\theta)$, we acquire the uniqueness of θ^+.
On the other hand, $\frac{\mathscr{L}_1 - \mathscr{L}_3}{\mathscr{L}_2}$ is an increasing function on $(0,\theta)$, in particular on (θ^+,θ^-), which implies that

$$1 = \frac{\mathscr{L}_1(\theta^+) - \mathscr{L}_3(\theta^+)}{\mathscr{L}_2(\theta^+)} < \frac{\mathscr{L}_1(t) - \mathscr{L}_3(t)}{\mathscr{L}_2(t)} \quad \forall t \in (\theta^+,\theta^-).$$

Again

$$\mathscr{L}_2(t) < \mathscr{L}_1(t) - \mathscr{L}_3(t) \text{ for all } t \in (\theta^+,\theta^-).$$

Next, we fix λ_1 such that for $\lambda \in (0,\lambda_1)$, we have

$$\mathscr{L}_1(t) - \mathscr{L}_2(t) - \mathscr{L}_3(t) > 0. \tag{3.24}$$

In view of the fact that $\lim_{t\to+\infty} \mathscr{I}'_u(t) = -\infty$, there exists $\theta_1 > \theta$ such that

$$\mathscr{L}_1(t) - \mathscr{L}_2(t) - \mathscr{L}_3(t) < 0. \tag{3.25}$$

Combining (3.24) and (3.25), we obtain the existence of $\theta^- > \theta$ such that for $\lambda \in (0,\lambda_1)$

$$\mathscr{L}_1(\theta^-) - \mathscr{L}_2(\theta^-) - \mathscr{L}_3(\theta^-) = 0, \tag{3.26}$$

in following, fix $\lambda_2 > 0$ such that for $\lambda \in (0,\lambda_2)$ and for $t > \theta$ sufficiently large $\mathscr{L}_1 - \mathscr{L}_2 - \mathscr{L}_3$ is a non-increasing function, and thus θ^- is unique.
Select $\delta = \min\{\lambda_1,\lambda_2,\lambda^{**}\}$ small enough, so we have the following conclusions:

- For t sufficiently small, we have proved that

$$\begin{cases} \mathscr{I}'_u(t) < 0, & \forall t \in (0,\theta^+), \\ \mathscr{I}'_u(t) > 0, & \forall t \in (\theta^+,\theta), \\ \mathscr{I}'_u(\theta^+) = 0. \end{cases} \tag{3.27}$$

  We may therefore deduce that $\mathscr{I}_u$ reaches a local minimum at θ^+, and the fact that $\mathscr{I}''_u(\theta^+) > 0$ amplies that $\theta^+ u \in \mathscr{M}^+_\lambda$.

- For t sufficiently large, we have established the upcoming results

$$\begin{cases} \mathscr{I}_u'(t) > 0, & \forall t \in (\theta, \theta^-), \\ \mathscr{I}_u'(t) < 0, & \forall t \in (\theta^-, +\infty), \\ \mathscr{I}_u'(\theta^-) = 0. \end{cases} \tag{3.28}$$

We acquire that $\mathscr{I}_u$ reaches a local maximum at θ^-, and the fact that $\mathscr{I}_u''(\theta^-) < 0$ implies that $\theta^- u \in \mathscr{M}_\lambda^-$.

Lemmas 3.3 and 3.4 are combined to give $\mathscr{I}_u(\theta^+) < 0 < \mathscr{I}_u(\theta^-)$, which implies that $\mathscr{I}_u$ is increasing and non decreasing on (θ^+, θ^-) and $(\theta^-, +\infty)$ respectively. Furthermore $\lim_{t\to+\infty} \mathscr{I}_u(t) = -\infty$ leading to the existence of a unique $\theta^* \in (\theta^+, \theta^-)$ such that $\mathscr{I}_u(\theta^*) = 0$. As a result

$$\mathscr{J}_S(\theta^+ u) = \inf_{0\le t\le\theta^*} \mathscr{J}_S(tu), \tag{3.29}$$

$$\mathscr{J}_S(\theta^- u) = \sup_{t\ge 0} \mathscr{J}_S(tu). \tag{3.30}$$

■

3.2.4 Main results

In this subsection, our objective is to establish the existence of minimizers for the energy functional $\mathscr{J}_S$ in $\mathscr{M}_\lambda^+$ and $\mathscr{M}_\lambda^-$ for each λ fixed in $(0, \tilde{\delta})$ with $\tilde{\delta} = \min\{\delta, \lambda^*\}$ to deduce the main results.

Existence of minimizers

Theorem 3.3. *There exists $\tilde{\delta} > 0$, such that for any $\lambda \in (0, \tilde{\delta})$, the energy functional admits a minimizer on $\mathscr{M}_\lambda^+$.*

Proof. We have $\mathscr{J}_S$ is bounded in $\mathscr{M}_\lambda$, so it is bounded on $\mathscr{M}_\lambda^+$. Then, there exists a minimizing sequence $\{u_n\} \subset \mathscr{M}_\lambda^+$ such that

$$\lim_{n\to\infty} \mathscr{J}_S(u_n) = \inf_{u\in\mathscr{M}_\lambda^+} \mathscr{J}_S(u) = \beta_\lambda^+ < 0.$$

Since $\mathscr{J}_S$ is coercive, then u_n is bounded on X_0. Hence, we can find $u_0 \in X_0$ such that for a subsequence still denoted u_n we have

$$u_n \rightharpoonup u_0 \text{ in } X_0.$$

The compact embedding results in Theorem 3.2 and Theorem 1.4, directly imply that

$$u_n \longrightarrow u_0 \ \text{ in } L^{r(x)}_{g(x)}(\Omega),$$
$$u_n \longrightarrow u_0 \ \text{ in } L^{1-\alpha(x)}_{h(x)}(\Omega).$$

We aim to prove that $u_n \longrightarrow u_0$ in X_0. We suppose that u_n does not converge strongly to u_0 in X_0. Then

$$\liminf_{n\to\infty} \rho^0_{K,p(x,y)}(u_n) > \rho^0_{K,p(x,y)}(u_0).$$

From (3.23), we have

$$\begin{aligned}\mathcal{J}_S(u_n) \geq &\left(\frac{1}{p^+}-\frac{1}{r^-}\right)\int_Q |u_n(x)-u_n(y)|^{p(x,y)}K(x,y)dxdy + \left(\frac{1}{p^+}-\frac{1}{r^-}\right)\int_\Omega |u_n|^{\overline{p}(x)} \\ &-\lambda\left(\frac{1}{1-\alpha^+}-\frac{1}{r^-}\right)\int_\Omega h(x)|u_n|^{1-\alpha(x)}dx.\end{aligned} \tag{3.31}$$

By passing to the limit, we acquire

$$\begin{aligned}\liminf_{n\to\infty} \mathcal{J}_S(u_n) &> \left(\frac{1}{p^+}-\frac{1}{r^-}\right)\rho^0_{K,p(x,y)}(u_0) + \lambda\left(\frac{1}{r^-}-\frac{1}{1-\alpha^+}\right)\int_\Omega h(x)|u_0|^{1-\alpha(x)}dx \\ &\geq \left(\frac{1}{p^+}-\frac{1}{r^-}\right)\|u_0\|^{p^-}_K + \lambda\tilde{C}_1\left(\frac{1}{r^-}-\frac{1}{1-\alpha^+}\right)\|u_0\|^{1-alpha^-}_K > 0.\end{aligned}$$

Which leads to a contradiction. As a result, $u_n \longrightarrow u_0$ in X_0. ■

Theorem 3.4. *There exists $\tilde{\delta} > 0$, such that for any $\lambda \in (0,\tilde{\delta})$, the energy functional admits a minimizer on $\mathcal{M}^-_\lambda$.*

Proof. As $\mathcal{J}_S$ is bounded on $\mathcal{M}_\lambda$, therefore it is bounded on $\mathcal{M}^-_\lambda$. Hence, there exists a minimizing sequence $(v_n) \subset \mathcal{M}^-_\lambda$ such that

$$\lim_{n\to\infty} \mathcal{J}_S(v_n) = \inf_{v\in\mathcal{M}^-_\lambda} \mathcal{J}_S(v) = \beta^-_\lambda > 0.$$

The coercivity of $\mathcal{J}_S$ implies that v_n is bounded on X_0. Hence, there exists $v_0 \in X_0$ such that $v_n \rightharpoonup v_0$ in X_0. As well as by the compact embedding theorems 3.2 and 1.4 we have

$$v_n \longrightarrow v_0 \ \text{ in } L^{r(x)}_{g(x)}(\Omega),$$
$$v_n \longrightarrow v_0 \ \text{ in } L^{1-\alpha(x)}_{h(x)}(\Omega).$$

Furthermore, if $v_0 \in \mathcal{M}^-_\lambda$, then we can find $t > 0$ such that $tv_0 \in \mathcal{M}^-_\lambda$ and $\mathcal{J}_\lambda(v_0) \geqslant \mathcal{J}_\lambda(tv_0)$.

Indeed, either $\lambda\int_\Omega h(x)|u|^{1-\alpha(x)}dx = 0$ or $\lambda\int_\Omega h(x)|u|^{1-\alpha(x)}dx > 0$, we get using Lemma 3.5 that there exists $t > 0$ such that $tv_0 \in \mathcal{M}^-_\lambda$. Next, we will use the proof by contraposition

to prove the strong convergence of v_n to v_0 in X_0. Suppose that v_n does not converge strongly to v_0 in X_0, then

$$\lim_{n\to\infty}\inf \rho^0_{K,p(x,y)}(v_n) > \rho^0_{K,p(x,y)}(v_0).$$

We have for $t>0$

$$\begin{aligned}
\mathcal{J}_S(tv_0) &\leqslant \frac{t^{p^+}}{p^-}\int_Q |v_0(x)-v_0(y)|^{p(x,y)}K(x,y)dxdy + \frac{t^{p^+}}{p^-}\int_\Omega |v_0|^{\overline{p}(x)}dx \\
&\quad - \frac{t^{r^-}}{r^+}\int_\Omega g(x)|v_0|^{r(x)}dx - \lambda\frac{t^{1-\alpha^-}}{1-\alpha^+}\int_\Omega h(x)|v_0|^{1-\alpha(x)}dx \\
&< \lim_{n\to\infty}\left[\frac{t^{p^+}}{p^-}\int_Q |v_n(x)-v_n(y)|^{p(x,y)}K(x,y)dxdy + \frac{t^{p^+}}{p^-}\int_\Omega |v_n|^{\overline{p}(x)}dx\right. \\
&\quad \left. - \frac{t^{r^-}}{r^+}\int_\Omega g(x)|v_n|^{r(x)}dx - \lambda\frac{t^{1-\alpha^-}}{1-\alpha^+}\int_\Omega h(x)|v_n|^{1-\alpha(x)}dx\right] \\
&\leqslant \lim_{n\to\infty}\mathcal{J}_S(tv_n) \leqslant \lim_{n\to\infty}\mathcal{J}_S(v_n) = \inf_{v\in\mathcal{M}_\lambda^-}\mathcal{J}_S(v) = \beta_\lambda^-.
\end{aligned}$$

The previous inequality contradicts the fact that $tv_0 \in \mathcal{M}_\lambda^-$. Therefore, $v_n \longrightarrow v$ in X_0 and $\mathcal{J}_S(v_0) = \inf_{v\in\mathcal{M}_\lambda^-}\mathcal{J}_S(v)$. As a result, v_0 is a minimizer of $\mathcal{J}_S$ on $\mathcal{M}_\lambda^-$. ■

Now, we are in a position to prove our main results.

Proof of the main results

We infer from Theorems 3.3 and 3.4 that for any $\lambda \in (0,\delta')$ there exist $u_0 \in \mathcal{M}_\lambda^+$ and $v_0 \in \mathcal{M}_\lambda^-$ such that

$$\mathcal{J}_\lambda(u_0) = \inf_{u\in\mathcal{M}_\lambda^+}\mathcal{J}_\lambda(u), \text{ and } \mathcal{J}_\lambda(v_0) = \inf_{u\in\mathcal{M}_\lambda^-}\mathcal{J}_\lambda(u).$$

Because of Lemma 3.5, we have that u_0 and v_0 are two critical points of $\mathcal{J}_\lambda$ on X_0. Furthermore, from Lemma 3.2 we have $\mathcal{M}_\lambda^0 = \emptyset$, then we deduce that u_0 and v_0 are two distinct solutions of problem $(\mathscr{P}_\lambda)$.

3.2.5 Particular case

Finally, we wish to mention that in the particular case when

$$K(x,y) = |x-y|^{-N-sp(x,y)},$$

the problem $(\mathscr{P}_S)$ reduces to the following form

$$(\mathscr{P})\begin{cases}(-\Delta)^s_{p(x,\cdot)}(u)+|u|^{\overline{p}(x)-2}u &= g(x)|u|^{r(x)-2}u+\lambda\frac{h(x)}{|u|^{\alpha(x)}} \quad \text{in } \Omega,\\ u &= 0 \quad \text{on } \mathbb{R}^N\setminus\Omega.\end{cases}$$

In this case, all the results obtained in this work stay true for problem $(\mathscr{P})$.
This section concludes with a specific case that demonstrates the adaptability of our analysis of the well-known operators.

3.3 On a $p(x,\cdot)$-integro-differential problem with Neumann boundary conditions

As mentioned in the previous sections, the earlier problems have all been examined under Dirichlet boundary conditions. In the following analysis, we expand this framework by incorporating nonlocal nonlinear Neumann boundary conditions into a generalized integro-differential operator, thereby advancing beyond the traditional Dirichlet-type approach.

3.3.1 Hypothesis and statement of the problem

The purpose of this study is to investigate the existence of solutions for a bi-nonlocal problem with nonlocal nonlinear Neumann boundary conditions of the following form

$$(\mathscr{P}_{\mathscr{N}})\begin{cases}\mathscr{L}_K^{p(x,.)}(u(x))+|u|^{\overline{p}(x)-2}u(x)=f(x,u)\left(\int_\Omega F(x,u)dx\right)^r & \text{in } \Omega,\\ \mathscr{N}_K^{p(x,.)}u(x)+\beta(x)|u|^{\overline{p}(x)-2}u(x)=0 & \text{on } \mathbb{R}^N\setminus\Omega.\end{cases}$$

Where

- The operator $\mathscr{N}_K^{p(x,.)}$ represents the new generalized nonlocal normal $p(x,\cdot)$-derivative with singular kernel given by the following formula

$$\mathscr{N}_K^{p(x,.)}u(x):=\int_\Omega |u(x)-u(y)|^{p(x,y)-2}(u(x)-u(y))K(x,y)dy, \quad x\in\mathbb{R}^N\setminus\overline{\Omega}. \tag{3.32}$$

- β is a function that verifies

$$(\mathscr{B}): \quad \beta\in L^\infty(\mathbb{R}^N\setminus\Omega) \quad \text{and} \quad \beta\geqslant 0 \text{ in } \mathbb{R}^N\setminus\Omega.$$

- $f:\overline{\Omega}\times\mathbb{R}\to\mathbb{R}$ is a continuous function that satisfies

⋆ f is an odd function with respect to t.

⋆ $(\mathscr{F})$: There exist two positive constants l_1 and l_2 such that

$$l_1 t^{\gamma_1(x)-1} \leqslant f(x,t) \leqslant l_2 t^{\gamma_2(x)-1}, \tag{3.33}$$

where $\gamma_1(x), \gamma_2(x) \in C^+(\overline{\Omega})$ are two variable exponents that verify

$$1 < \gamma_1(x) \leqslant \gamma_2(x) < p_s^*(x) \quad \forall x \in \overline{\Omega} \quad \text{with} \quad \gamma_2^+ < \frac{p^-}{r+1}. \tag{3.34}$$

⋆ It is worth noting that $F(x,u) = \int_0^u f(x,\xi)d\xi$.

The following theorem presents our main finding.

Theorem 3.5. *Let Ω be a Lipschitz bounded domain in $\mathbb{R}^N$ and $s \in (0,1)$. Consider $p : \mathbb{R}^N \times \mathbb{R}^N \longrightarrow (1,+\infty)$ as a continuous variable exponent that verifies conditions (1.2) and (1.3) with $sp^+ < N$. Furthermore, we suppose that the kernel $K : \mathbb{R}^N \times \mathbb{R}^N \longrightarrow (0,+\infty)$ is a measurable function that verifies conditions (K_1) and (K_2). Assume that condition $(\mathscr{B})$ is fulfilled and f is an odd function that verifies $(\mathscr{F})$. Then, problem $(\mathscr{P}_{\mathscr{N}})$ has infinitely many weak solutions.*

3.3.2 Functional framework of problem $(\mathscr{P}_{\mathscr{N}})$

In this subsection, we introduce the appropriate framework to deal with problem $(\mathscr{P}_{\mathscr{N}})$, as inspired by the procedure of the paper provided by Bahrouni et al. [34]. Moreover, we establish some basic properties of the generalized nonlocal normal $p(x,.)$-derivative $\mathscr{N}_K^{p(x,.)}$ associated with $\mathscr{L}_K^{p(x,)}$.

Let $u : \mathbb{R}^N \to \mathbb{R}$ be a measurable function and $p \in C^+(\overline{\Omega})$ satisfying (1.2) and (1.3) such that $\overline{p}(x) = p(x,x)$ for all $x \in \mathbb{R}^N$, we define the space E as follows

$$E := \left\{ u : \mathbb{R}^N \to \mathbb{R} \quad \text{is measurable such that :} \ \|u\|_E < +\infty \right\},$$

where

$$\|u\|_E := [u]_{K,p(x,y),Q} + \|u\|_{L^{\overline{p}(x)}(\Omega)} + \left\| \beta^{\frac{1}{\overline{p}(.)}} u \right\|_{L^{\overline{p}(x)}(\Omega^c)},$$

such that

$$[u]_{K,p(x,y),Q} = \inf \left\{ \mu > 0 : \frac{1}{2} \int_Q \frac{|u(x)-u(y)|^{p(x,y)}}{\mu^{p(x,y)}} K(x,y)dxdy \leqslant 1 \right\}.$$

Proposition 3.1. *The space $(E, \|.\|_E)$ is a reflexive and separable Banach space.*

Proof. The proof is similar to Proposition 3.1 in [34]. ■

Theorem 3.6. *Let Ω be a Lipschitz bounded domain in $\mathbb{R}^N$ and $s \in (0,1)$. Let $p : \mathbb{R}^N \times \mathbb{R}^N \longrightarrow (1,+\infty)$ be a continuous variable exponent satisfying (1.2) and (1.3) with $sp^+ < N$. Let $r \in C^+(\overline{\Omega})$ such that $1 < r^- \leqslant r(x) < p_s^*(x)$ for all $x \in \overline{\Omega}$. Suppose that $K : \mathbb{R}^N \times \mathbb{R}^N \longrightarrow (0,+\infty)$ is a measurable function that verifies conditions (K_1) and (K_2). Then there exists a positive constant $C = C(N,p,r,s,\Omega) > 0$ such that for any $u \in E$, we have*

$$\|u\|_{L^{r(x)}(\Omega)} \leqslant C\|u\|_E.$$

Moreover, this embedding is compact.

Proof. It is obvious to see that

$$\|u\|_{W_K^{s,p(x,y)}(Q)} \leqslant \|u\|_E \quad \text{for all } u \in E.$$

Then, we get the desired result by Theorem 1.3. ■

Definition 3.1. Let $p : \mathbb{R}^N \times \mathbb{R}^N \longrightarrow (1,+\infty)$ be a continuous bounded function satisfying (1.2) and (1.3). Furthermore, consider $K : \mathbb{R}^N \times \mathbb{R}^N \longrightarrow (0,+\infty)$ as a measurable function that verifies conditions (K_1) and (K_2). For any $u \in E$, we define the modular $\tilde{\rho} : E \to \mathbb{R}$ by the following formula

$$\tilde{\rho}(u) = \int_Q |u(x)-u(y)|^{p(x,y)} K(x,y)\,dx\,dy + \int_\Omega |u|^{\overline{p}(x)}\,dx + \int_{\Omega^c} \beta(x)|u|^{\overline{p}(x)}\,dx,$$

and

$$\|u\|_{\tilde{\rho}} = \inf\left\{\mu > 0 : \tilde{\rho}\left(\frac{u}{\mu}\right) \leqslant 1\right\}.$$

Remark 3.1. *It is clear that $\|\cdot\|_{\tilde{\rho}}$ is a norm on E, which is equivalent to the norm $\|\cdot\|_E$.*

Lemma 3.6. *Let $p : \mathbb{R}^N \times \mathbb{R}^N \longrightarrow (1,+\infty)$ be a continuous bounded function satisfying (1.2) and (1.3) with $sp^+ < N$. Consider $K : \mathbb{R}^N \times \mathbb{R}^N \longrightarrow (0,+\infty)$ as a measurable function that verifies conditions (K_1) and (K_2). Moreover, the function β verifies condition $(\mathscr{B})$. Then, for any $u \in E$ the following*

i. For $a \neq 0$, we have $\|u\|_E = a$ if and only if $\tilde{\rho}\left(\frac{u}{a}\right) = 1$.

ii. If $\|u\|_E < 1$, then $\frac{\|u\|_E^{p^+}}{3^{p^+-1}} \leqslant \tilde{\rho}(u) \leqslant 3\|u\|_E^{p^-}$.

iii. If $\|u\|_E > 1$, then $\|u\|_E^{p^-} \leqslant \tilde{\rho}(u)$.

Following that, we state and prove the following propositions, which are crucial in the study of nonlocal problems with nonlinear nonlocal Neumann boundary conditions.

Proposition 3.2. *Let $p:\mathbb{R}^N\times\mathbb{R}^N\longrightarrow(1,+\infty)$ be a continuous bounded function satisfying (1.2) and (1.3) with $sp^+<N$. Furthermore, consider $K:\mathbb{R}^N\times\mathbb{R}^N\longrightarrow(0,+\infty)$ as a measurable function that verifies conditions (K_1) and (K_2). Then for any $u\in E$ a bounded C^2-function in $\mathbb{R}^N$, we have*

$$\int_\Omega \mathscr{L}_K^{p(x,.)}u(x)dx=-\int_{\mathbb{R}^N\setminus\Omega}\mathscr{N}_K^{p(x,.)}u(x)dx.$$

Proof. We argue as in [34], since K and p are symmetric, then by direct computations we get

$$\begin{aligned}&\int_\Omega\int_\Omega|u(x)-u(y)|^{p(x,y)-2}(u(x)-u(y))K(x,y)dxdy\\&=-\int_\Omega\int_\Omega|u(x)-u(y)|^{p(x,y)-2}(u(x)-u(y))K(x,y)dxdy.\end{aligned}$$

Which implies that

$$\begin{aligned}\int_\Omega\mathscr{L}_K^{p(x,.)}u(x)dx&=\int_\Omega p.v\int_{\mathbb{R}^N}|u(x)-u(y)|^{p(x,y)-2}(u(x)-u(y))K(x,y)dydx\\&=\int_\Omega\lim_{\varepsilon\to0}\int_{\mathbb{R}^N\setminus B(x,\varepsilon)}|u(x)-u(y)|^{p(x,y)-2}(u(x)-u(y))K(x,y)dydx\\&=\int_\Omega\int_{\mathbb{R}^N\setminus\Omega}|u(x)-u(y)|^{p(x,y)-2}(u(x)-u(y))K(x,y)dydx\\&=\int_{\mathbb{R}^N\setminus\Omega}\int_\Omega|u(x)-u(y)|^{p(x,y)-2}(u(x)-u(y))K(x,y)dxdy\\&=-\int_{\mathbb{R}^N\setminus\Omega}\mathscr{N}_K^{p(.,y)}u(y)dy.\end{aligned}$$

■

Proposition 3.3. *Let $p:\mathbb{R}^N\times\mathbb{R}^N\longrightarrow(1,+\infty)$ be a continuous bounded function satisfying (1.2) and (1.3) with $sp^+<N$. Furthermore, consider $K:\mathbb{R}^N\times\mathbb{R}^N\longrightarrow(0,+\infty)$ as a measurable function that verifies conditions (K_1) and (K_2). Then for any u and v two bounded C^2-functions in $\mathbb{R}^N$, we have the following equality*

$$\begin{aligned}&\frac{1}{2}\int_Q|u(x)-u(y)|^{p(x,y)-2}(u(x)-u(y))(v(x)-v(y))K(x,y)dxdy\\&=\int_\Omega v\mathscr{L}_K^{p(x,.)}(u)dx+\int_{\Omega^c}v\mathscr{N}_K^{p(x,.)}(u)dx.\end{aligned}$$

Proof. By conditions (1.3) and (K_1), we have

$$\begin{aligned}
&\frac{1}{2}\int_Q |u(x)-u(y)|^{p(x,y)-2}(u(x)-u(y))(v(x)-v(y))K(x,y)dxdy\\
&=\int_Q v(x)|u(x)-u(y)|^{p(x,y)-2}(u(x)-u(y))K(x,y)dxdy\\
&=\int_\Omega v(x)\left(\int_{\mathbb{R}^N}|u(x)-u(y)|^{p(x,y)-2}(u(x)-u(y))K(x,y)dy\right)dx\\
&\quad+\int_{\Omega^c} v(x)\left(\int_\Omega |u(x)-u(y)|^{p(x,y)-2}(u(x)-u(y))K(x,y)dy\right)dx\\
&=\int_\Omega v\mathscr{L}_K^{p(x,.)}(u)dx+\int_{\Omega^c} v\mathscr{N}_K^{p(x,.)}(u)dx.
\end{aligned}$$

■

Based on the integration by parts formula given by Proposition 3.3, we define a weak solution for problem $(\mathscr{P}_K)$ in the following form.

Definition 3.2. We say that $u\in E$ is a weak solution of $(\mathscr{P}_K)$ if

$$\begin{aligned}
&\frac{1}{2}\int_Q |u(x)-u(y)|^{p(x,y)-2}(u(x)-u(y))(v(x)-v(y))K(x,y)dxdy+\int_\Omega |u|^{\overline{p}(x)-2}uvdx\\
&+\int_{\Omega^c}\beta(x)|u|^{\overline{p}(x)-2}uvdx=\left(\int_\Omega F(x,u)dx\right)^r\left(\int_\Omega f(x,u)vdx\right),
\end{aligned}\tag{3.35}$$

for every $v\in E$.

As a consequence of the above definition, we obtain the following result:

Proposition 3.4. *Let $u\in E$ be a weak solution of $(\mathscr{P}_K)$, then*

$$\mathscr{N}_K^{p(x,.)}u(x)+\beta(x)|u|^{\overline{p}(x)-2}u(x)=0 \quad \text{a.e. in } \mathbb{R}^N\setminus\overline{\Omega}.$$

Proof. First, we start by taking $v\in E$ such that $v\equiv 0$ in Ω as a test function in (3.35), and because of the symmetry of the exponent variable $p(\cdot,\cdot)$ and the kernel K, we find

$$\begin{aligned}
0&=\frac{1}{2}\int_Q |u(x)-u(y)|^{p(x,y)-2}(u(x)-u(y))(v(x)-v(y))K(x,y)dxdy+\int_{\Omega^c}\beta(x)|u|^{\overline{p}(x)-2}uvdx\\
&=\int_{\mathbb{R}^N\setminus\overline{\Omega}} v(x)\int_\Omega |u(x)-u(y)|^{|p(x,y)-2}(u(x)-u(y))K(x,y)dydx+\int_{\Omega^c}\beta(x)|u|^{\overline{p}(x)-2}uvdx\\
&=\int_{\mathbb{R}^N\setminus\overline{\Omega}} v(x)\mathscr{N}_K^{p(x,.)}u(x)dx+\int_{\Omega^c}\beta(x)|u|^{\overline{p}(x)-2}uvdx\\
&=\int_{\mathbb{R}^N\setminus\overline{\Omega}}\left(\mathscr{N}_K^{p(x,.)}u(x)+\beta(x)|u|^{\overline{p}(x)-2}u\right)v(x)dx.
\end{aligned}$$

Consequently,

$$\int_{\mathbb{R}^N\setminus\overline{\Omega}}\left(\mathcal{N}_K^{p(x,.)}u(x)+\beta(x)|u|^{\overline{p}(x)-2}u\right)v(x)dx=0,$$

for every $v\in E$ with $v=0$ in Ω. In particular, this is true for every $v\in\mathscr{C}^\infty(\mathbb{R}^N\setminus\overline{\Omega})$, therefore

$$\mathcal{N}_K^{p(x,.)}u(x)+\beta(x)|u|^{\overline{p}(x)-2}u(x)=0 \qquad \text{a.e in } \mathbb{R}^N\setminus\overline{\Omega}.$$

■

3.3.3 Main results

In order to apply the genus theory to our problem, we need to establish some auxiliary results. To this end, we associate to the problem $(\mathscr{P}_K)$ the energy functional $\mathscr{J}:E\longrightarrow\mathbb{R}$ defined as follows

$$\begin{aligned}\mathscr{J}(u)=&\int_Q\frac{|u(x)-u(y)|^{p(x,y)}}{2p(x,y)}K(x,y)dxdy+\int_\Omega\frac{|u|^{\overline{p}(x)}}{\overline{p}(x)}dx\\&+\int_{\Omega^c}\frac{\beta(x)|u|^{\overline{p}(x)}}{\overline{p}(x)}dx-\frac{1}{r+1}\left(\int_\Omega F(x,u)dx\right)^{r+1}, \qquad \text{for every } u\in E.\end{aligned}$$

Proposition 3.5. *Assume that condition $(\mathscr{B})$ is fulfilled and f is an odd function that verifies $(\mathscr{F})$. Therefore, the functional $\mathscr{J}$ is well defined, moreover $\mathscr{J}\in\mathscr{C}^1(E,\mathbb{R})$, and for all u, $v\in E$ the derivative of $\mathscr{J}$ is given by*

$$\begin{aligned}\left\langle\mathscr{J}'(u),v\right\rangle=&\frac{1}{2}\int_Q|u(x)-u(y)|^{p(x,y)-2}(u(x)-u(y))(v(x)-v(y))K(x,y)dxdy\\&+\int_\Omega|u|^{p(x)-2}uvdx+\int_{\Omega^c}\beta(x)|u|^{\overline{p}(x)-2}uvdx\\&-\left(\int_\Omega F(x,u)dx\right)^r\int_\Omega f(x,u)vdx.\end{aligned}$$

Proof. Let $u\in E$, by using Lemma 3.6 and condition (3.33), we get

$$\mathscr{J}(u)\leqslant C\|u\|_E, \qquad \text{where } C=C(p,\gamma_2,l_2,r).$$

Standard argument as that used in ([20], Lemma 3.1) shows that $\mathscr{J}\in\mathscr{C}^1(E,\mathbb{R})$ and its derivative is given by the following formula

$$\begin{aligned}\left\langle\mathscr{J}'(u),v\right\rangle=&\frac{1}{2}\int_Q|u(x)-u(y)|^{p(x,y)-2}(u(x)-u(y))(v(x)-v(y))K(x,y)dxdy\\&+\int_\Omega|u|^{p(x)-2}uvdx+\int_{\Omega^c}\beta(x)|u|^{\overline{p}(x)-2}uvdx\\&-\left(\int_\Omega F(x,u)dx\right)^r\int_\Omega f(x,u)vdx,\end{aligned}$$

for any $u, v \in E$. Therefore, we can find weak solutions of $(\mathscr{P}_K)$ as the critical points of $\mathscr{J}$. ■

To prove our main result, we shall need the following technical lemmas.

Lemma 3.7. *Assume that condition $(\mathscr{B})$ is fulfilled and f is an odd function that verifies $(\mathscr{F})$. Then, the functional $\mathscr{J}$ is bounded from below on E.*

Proof. For any $u \in E$, we use condition (3.33) to get

$$\begin{aligned}\mathscr{J}(u) &= \int_Q \frac{|u(x)-u(y)|^{p(x,y)}}{2p(x,y)}K(x,y)dxdy + \int_\Omega \frac{|u|^{\overline{p}(x)}}{\overline{p}(x)}dx + \int_\Omega \frac{\beta(x)|u|^{\overline{p}(x)}}{\overline{p}(x)}dx \\ &\quad - \frac{1}{r+1}\left(\int_\Omega F(x,u)dx\right)^{r+1} \\ &\geqslant \frac{1}{2p^+}\left(\int_Q |u(x)-u(y)|^{p(x,y)}K(x,y)dxdy + \int_\Omega |u|^{\overline{p}(x)}dx + \int_\Omega \beta(x)|u|^{\overline{p}(x)}dx\right) \\ &\quad - \frac{1}{r+1}\left(\frac{l_2}{\gamma_2^-}\right)^{r+1}\left(\int_\Omega u^{\gamma_2(x)}dx\right)^{r+1}.\end{aligned}$$

Taking $\|u\|_E > 1$, we obtain from Lemma 3.6 that

$$\mathscr{J}(u) \geqslant \frac{1}{2p^+}\|u\|_E^{p^-} - \frac{1}{r+1}\left(\frac{l_2}{\gamma_2^-}\right)^{r+1}\left(\int_\Omega u^{\gamma_2(x)}dx\right)^{r+1}.$$

Theorem 3.6 and the fact that $\gamma_2(x) < p_s^*(x)$ imply that

$$\mathscr{J}(u) \geqslant \frac{1}{2p^+}\|u\|_E^{p^-} - \frac{1}{r+1}\left(\frac{l_2 C^{\gamma_2^+}}{\gamma_2^-}\right)^{r+1}\|u\|_E^{\gamma_2^+(r+1)}, \tag{3.36}$$

where $C = C(N,p,\gamma_2,s,\Omega)$ is a positive constant.
Since the upper bound of the exponent variable γ_2 verifies $\gamma_2^+ < \frac{p^-}{r+1}$, inequality (3.36) shows that $\mathscr{J}$ is bounded from below. ■

Proposition 3.6. *Let $\{u_n\} \subset E$ be a sequence such that $u_n \rightharpoonup u \in E$, then the following assertions hold*

i. $\lim_{n\to\infty}\int_\Omega |u_n|^{\overline{p}(x)-2}u_n(u_n-u)dx = 0.$

ii. $\lim_{n\to\infty}\int_{\mathbb{R}^N\setminus\Omega} \beta(x)|u_n|^{\overline{p}(x)-2}u_n(u_n-u)dx = 0.$

Proof. 1. Applying Hölder's inequality to get

$$\int_\Omega |u_n|^{\overline{p}(x)-2}u_n(u_n-u)dx \leqslant \| |u_n|^{\overline{p}(x)-2}u_n\|_{L^{\hat{p}(x)}(\Omega)}\|u_n-u\|_{L^{\overline{p}(x)}(\Omega)},$$

where $\hat{p}(.)$ is the conjugate of $\overline{p}(.)$.
If $\|u_n\|_{L^{\overline{p}(x)}(\Omega)} \geqslant 1$, we have

$$\| |u_n|^{\overline{p}(x)-2}u_n\|_{L^{\hat{p}(x)}(\Omega)} \leqslant \|u_n\|^{\overline{p}^+}_{L^{\overline{p}(x)}}.$$

If $\|u_n\|_{L^{\overline{p}(x)}(\Omega)} \leqslant 1$, we have

$$\| |u_n|^{\overline{p}(x)-2}u_n\|_{L^{\hat{p}(x)}(\Omega)} \leqslant \|u_n\|^{\overline{p}^-}_{L^{\overline{p}(x)}}.$$

Hence

$$\| |u_n|^{\overline{p}(x)-2}u_n\|_{L^{\hat{p}(x)}(\Omega)} \leqslant \max\left\{\|u_n\|^{\overline{p}^-}_{L^{\overline{p}(x)}}, \|u_n\|^{\overline{p}^+}_{L^{\overline{p}(x)}}\right\}.$$

Since the embedding $E \hookrightarrow L^{\overline{p}(x)}(\Omega)$ is compact, we get the desired result.

2. By analogy, using the fact that the embedding $E \hookrightarrow L^{\overline{p}(x)}(\mathbb{R}^N\setminus\Omega)$ is compact and $\beta \in L^\infty\left(\mathbb{R}^N\setminus\Omega\right)$, we have that

$$\lim_{n\to\infty}\int_{\mathbb{R}^N\setminus\Omega}\beta(x)|u_n|^{\overline{p}(x)-2}u_n(u_n-u)dx = 0.$$

■

Lemma 3.8. *The functional $\mathcal{J}$ satisfies the Palais-Smale (PS) condition.*

Proof. Let $\{u_n\}\subset E$ be a sequence such that

$$\mathcal{J}(u_n)\to \overline{c}>0 \quad \text{and} \quad \mathcal{J}'(u_n)\to 0 \text{ in } E^*. \tag{3.37}$$

From inequality (3.36), we have

$$\mathcal{J}(u_n) \geqslant \frac{1}{2p^+}\|u_n\|^{p^-}_E - \frac{1}{r+1}\left(\frac{l_2C^{\gamma_2^+}}{\gamma_2^-}\right)^{r+1}\|u_n\|^{\gamma_2^+(r+1)}_E.$$

Then, by the fact that $\{\mathcal{J}(u_n)\}_n$ is bounded, we conclude that $\{u_n\}_n$ is bounded in E.
As a consequence, there exists a subsequence still denoted by $\{u_n\}_n$ such that $u_n \rightharpoonup u$ in E.

In the view of (3.37), we get that $\langle \mathscr{J}'(u_n), u_n - u\rangle \longrightarrow 0$. Therefore,
$\langle \mathscr{J}'(u_n), u_n - u\rangle =$

$$\begin{aligned}
&\frac{1}{2}\int_Q |u_n(x)-u_n(y)|^{p(x,y)-2}(u_n(x)-u_n(y))[(u_n-u)(x)-(u_n-u)(y)]K(x,y)dxdy\\
&+\int_\Omega |u_n|^{\overline{p}(x)-2}u_n(u_n-u)dx+\int_{\Omega^c}\beta(x)|u_n|^{\overline{p}(x)-2}u_n(u_n-u)dx\\
&-\left(\int_\Omega F(x,u_n)dx\right)^r\int_\Omega f(x,u_n)(u_n-u)dx\longrightarrow 0.
\end{aligned}$$

Since $u_n \rightharpoonup u$ in E, then by Theorem 3.6, we have

$$\begin{aligned}
u_n &\longrightarrow u \text{ in } L^{\gamma_2(x)}(\Omega),\\
u_n &\longrightarrow u \text{ a.e. } \Omega.
\end{aligned}$$

On the other hand, by using Hölder's inequality and condition (3.33), we obtain

$$\begin{aligned}
\left|\int_\Omega f(x,u_n)(u_n-u)dx\right| &\leqslant \int_\Omega |f(x,u_n)||(u_n-u)|dx\\
&\leqslant l_2\int_\Omega |u_n|^{\gamma_2(x)-1}|(u_n-u)|dx\\
&\leqslant C_1\left\||u_n|^{\gamma_2(x)-1}\right\|_{L^{\frac{\gamma_2(x)}{\gamma_2(x)-1}}(\Omega)}\|u_n-u\|_{L^{\gamma_2(x)}(\Omega)},
\end{aligned}$$

which tends to 0 as $n\to\infty$. Consequently

$$\left(\int_\Omega F(x,u_n)dx\right)^r\int_\Omega f(x,u_n)(u_n-u)dx\longrightarrow 0 \quad \text{as} \quad n\to\infty.$$

Using Proposition 3.6, we acquire

$$\lim_{n\to\infty}\int_Q |u_n(x)-u_n(y)|^{p(x,y)-2}(u_n(x)-u_n(y))[(u_n-u)(x)-(u_n-u)(y)]K(x,y)dxdy=0,$$

which implies

$$\lim_{n\to\infty}\left\langle \mathscr{L}_K^{p(x,)}(u_n), u_n-u\right\rangle = 0,$$

by a similar argument, we have $\lim_{n\to\infty}\left\langle \mathscr{L}_K^{p(x,)}(u), u_n-u\right\rangle = 0$. Then

$$\lim_{n\to\infty}\left\langle \mathscr{L}_K^{p(x,)}(u_n)-\mathscr{L}_K^{p(x,)}(u), u_n-u\right\rangle = 0.$$

Thus, from Lemma 1.4, we conclude that $\{u_n\}$ converges strongly to u in E. As a consequence, the functional $\mathscr{J}$ satisfies the Palais-Smale condition. ■

Now, we are in the position to prove our main result by means of Krasnoselskii's genus theory.

Proof of Theorem 3.5: The proof procedure consists of verifying that the functional $\mathscr{J}$ satisfies hypotheses of Theorem 1.11. Indeed, we have that $\mathscr{J} \in \mathscr{C}^1(X,\mathbb{R})$ and by Lemma 3.8 $\mathscr{J}$ satisfies the Palais-Smale compactness condition. Moreover, it is obvious to see that the functional $\mathscr{J}$ is even, also it is bounded from below (because of Lemma 3.7).
It remains to show that condition (G_2) of Theorem 1.11 is fulfilled. Indeed, for each $m \in \mathbb{N}$, consider E_m the linear subspace of E generated by m vectors $e_1, e_2, ..., e_m$.
Since all the norms in the finite-dimensional space are equivalent, then there exists a constant C_m such that

$$\|u\|_{L^{\gamma_1(x)}(\Omega)} \geqslant C_m \|u\|_E \quad \text{for} \quad u \in E_m.$$

For every $u \in E_m$, we obtain for $\|u\|_E$ small enough

$$\begin{aligned}\mathscr{J}(u) &\leqslant \frac{3}{p^-}\|u\|_E^{p^-} - \frac{1}{r+1}\left(\frac{l_1}{\gamma_1^+}\right)^{r+1}\left(\int_\Omega u^{\gamma_1(x)}dx\right)^{r+1} \\ &\leqslant \frac{3}{p^-}\|u\|_E^{p^-} - \frac{1}{r+1}\left(\frac{l_1 C_m^{\gamma_1^+}}{\gamma_1^+}\right)^{r+1}\|u\|_X^{\gamma_1^+(r+1)},\end{aligned}$$

or also

$$\mathscr{J}(u) \leqslant \|u\|_E^{\gamma_1^+(r+1)}\left(\frac{3}{p^-}\|u\|_E^{p^- -(r+1)\gamma_1^+} - \frac{1}{r+1}\left(\frac{l_1 C_m^{\gamma_1^+}}{\gamma_1^+}\right)^{r+1}\right).$$

Let ρ_m and $\mathscr{H}$ two positive constants such that

$$\rho_m < \mathscr{H} < min\left\{1, \left[\frac{p^-}{3(r+1)}\left(\frac{l_1 C_m^{\gamma_1^+}}{\gamma_1^+}\right)^{r+1}\right]^{\frac{1}{p^- -(r+1)\gamma_1^+}}\right\},$$

and consider $S_{\rho_m}^{(m)} = \{u \in E_m : \|u\|_E = \rho_m\}$, then by this condition $p^- > (r+1)\gamma_1^+$, we obtain for any $u \in S_{\rho_m}^{(m)}$

$$\begin{aligned}\mathscr{J}(u) &\leqslant \rho_m^{\gamma_1^+(r+1)}\left(\frac{3}{p^-}\rho_m^{p^- -(r+1)\gamma_1^+} - \frac{1}{r+1}\left(\frac{l_1 C_m^{\gamma_1^+}}{\gamma_1^+}\right)^{r+1}\right) \\ &< \mathscr{H}^{\gamma_1^+(r+1)}\left(\frac{3}{p^-}\mathscr{H}^{p^- -(r+1)\gamma_1^+} - \frac{1}{r+1}\left(\frac{l_1 C_m^{\gamma_1^+}}{\gamma_1^+}\right)^{r+1}\right) \\ &< 0.\end{aligned}$$

Consequently,

$$\sup_{u \in S_{\rho_m}^{(m)}} \mathscr{J}(u) < 0 = \mathscr{J}(0).$$

On the other hand, $S^{(m)}_{\rho_m}$ is homeomorphic to the $(m-1)$ dimensional sphere in $\mathbb{R}^m$. Then $\gamma\left(S^{(m)}_{\rho_m}\right) = m$. In view of Clarke's theorem, $\mathscr{J}$ has at least k pairs of distinct critical points, because k is arbitrary, we infer the existence of infinitely many critical points of $\mathscr{J}$. The proof is complete. ■

3.3.4 Example and particular case

In this subsection, we give an explicit example to illustrate our main results. As a particular case, we can take the following.

- The singular kernel is given by

$$K(x,y) = |x-y|^{-N-sp(x,y)}.$$

- For the nonlinearity function f, we can take

$$f(x,t) = a\,|t|^{\gamma(x)-2}\,t,$$

where a is a positive constant and $\gamma(x) < p^*_s(x)\ \forall x \in \overline{\Omega}$, as well as $\gamma^+ < \frac{p^-}{r+1}$.

Therefore, problem $(\mathscr{P}_K)$ turns to a bi-nonlocal problem involving the fractional $p(x,\cdot)$-Laplacian operator with nonlocal Neumann boundary conditions

$$(\mathscr{P})\begin{cases} (-\Delta)^s_{p(x,.)}(u(x)) + |u|^{\overline{p}(x)-2}u(x) = \left(|u|^{\gamma(x)-2}\,u\right)\left(\int_\Omega \frac{1}{\gamma(x)}\,|u|^{\gamma(x)-1}\,u\;dx\right)^r & \text{in } \Omega, \\ \mathscr{N}_{s,p(x,.)}u(x) + \beta(x)|u|^{\overline{p}(x)-2}u(x) = 0, & \text{in } \mathbb{R}^N\backslash\Omega. \end{cases}$$

It is easy to see that the kernel K satisfies conditions (K_1) and (K_2). Moreover, $f(x,-t) = -f(x,t)$ and f verifies $(\mathscr{F})$. Then, as a direct consequence of Theorem 3.5, we get the following result.

Corollary 3.2. *Problem $(\mathscr{P})$ has infinitely many weak solutions.*

Part II

Applications to fractional anisotropic problems

INTRODUCTION AND MOTIVATION

The concept of anisotropic Sobolev spaces has long been a cornerstone in characterizing the directional behavior of functions, with important applications in various fields such as physics, materials science, image processing, and computational mathematics. The term "anisotropic exponent" was first introduced by Benedek and Panzone [43] in their study of mixed Lebesgue spaces. Subsequent fundamental studies in this area include works by Igari, Madych, and Sjödin [100, 128, 172], further advancing the theory. In 1969, Troisi [177] introduced the notion of anisotropic Sobolev spaces, $W^{k,\vec{p}}$, where $\vec{p} = (p_1, \ldots, p_N)$ is a constant vector representing different directional exponents. This development has led to a considerable body of research on the $\vec{p}$-Laplace operator

$$\Delta_{\vec{p}}(u) = \sum_{i=1}^{N} \partial_{x_i} \left(|\partial_{x_i} u|^{p_i - 2} \partial_{x_i} u \right),$$

with many scholars exploring its applications to problems involving directional behaviors [42, 111, 131].

A significant advancement in this theory came with the introduction of anisotropic Sobolev spaces with variable exponents, $W^{1,\vec{p}(\cdot)}$, where $\vec{p}(\cdot) = (p_1(\cdot), \ldots, p_N(\cdot))$ allows for components that vary with position [86, 132]. This has given rise to the anisotropic variable exponent $\vec{p}(\cdot)$-Laplace operator

$$\Delta_{\vec{p}(x)}(u) = \sum_{i=1}^{N} \partial_{x_i} \left(|\partial_{x_i} u|^{p_i(x) - 2} \partial_{x_i} u \right),$$

which is closely related to both $\Delta_{p(x)}$ and $\Delta_{\vec{p}}$, and has since become a critical object of study [49, 50, 57, 58]. On the other hand, in the theory of fractional calculus, we find the work of Xu and Sun [188]. The authors introduced the notion of fractional anisotropic Sobolev spaces based on mixed Gagliardo semi-norm as follows

$$W^{s,\vec{p}}\left(\mathbb{R}^N\right) = \left\{ u \in L^{\vec{p}} : \|u\|_{W^{s,\vec{p}}} := \|u\|_{L^{\vec{p}}} + [u]_{W^{s}\vec{p}} < \infty \right\},$$

where

$$[u]_{W^{s,\vec{p}}} := \left\| \frac{u(x) - u(y)}{|x - y|^{\sum_{i=1}^{N} (1/p_i + s/N)}} \right\|_{L^{p_N(x_N, y_N)}(\ldots(L^{p_1(x_1, y_1)}))}.$$

Xu and Sun provided significant contributions by proving embedding results into mixed Lebesgue spaces $L^{\vec{p}}$, under certain well-chosen conditions, without employing the dot notation for the operator associated with the specific types of problems analyzed within these spaces. Their work presents an important framework for understanding how functions in these spaces can be embedded into broader functional spaces, which is essential for addressing nonlocal and anisotropic equations.

Our main goal in this part of my thesis is to take another direction on the concept of fractional anisotropic Sobolev spaces and define it in a fluent way to be more adaptable to the study of anisotropic problems and also involving the variable exponent as the well-known spaces $(W^{1,p}, W^{s,p}, W^{s,p(\cdot,\cdot)})$.

The introduction of fractional anisotropic Sobolev space with variable exponents allows for the extension of the study of nonlinear nonlocal problems to fractional $(p_1(x,\cdot), p_2(x,\cdot))$-Laplacian problems. The motivation behind this idea stems from the hypothesis that the inclusion of directional behavior could introduce new challenges or insights, particularly when investigating the influence of specific directions in complex systems. This concept is particularly relevant in areas where directional effects play a crucial role, such as geophysics, seismology, material science, signal processing, and biomedical engineering. In these fields, models often depend on directional behavior, which can affect the potential and dynamics of the system being studied.

CHAPTER 4

ON A CLASS OF $(p_1(x,\cdot), p_2(x,\cdot))$-NONLOCAL PROBLEM

Main objective

This chapter aims to study whether these directional effects in fractional $(p_1(x,\cdot), p_2(x,\cdot))$-Laplacian may present novel difficulties or interesting phenomena. Specifically, we are interested in examining how the incorporation of directional dependence influences the behavior of the solutions, particularly in terms of the existence of weak solutions.

> The results presented in this chapter are derived from the study outlined in [27].

4.1 Hypothesis and problem statement

In light of the motivation of anisotropic problems, we consider the following problem

$$(\mathscr{P}_{\mathscr{A}})\begin{cases} \sum_{i=1}^{2}(-\Delta)^{s}_{p_i(x,.)}u(x)+|u|^{p_{\max}(x)-2}u=\lambda f(x,u)+\mu g(x,u) \text{ in } \Omega, \\ u=0, \quad \text{in } \mathbb{R}^N\setminus\Omega. \end{cases}$$

Where

- λ and μ are two positive real parameters.
- $f, g : \Omega\times\mathbb{R}\to\mathbb{R}$ are two Carathéodory functions with the following conditions

– (F_0): There exist $C>0$ and a function $q \in C_+(\overline{\Omega})$ with $q^+ < p^-_{\max}$ such that

$$|f(x,t)| \leqslant C(1+|t|^{q(x)-1}) \quad \forall (x,t) \in \Omega \times \mathbb{R}.$$

– (F_1) : There exist $t_0 > 1$ such that

$$\begin{cases} f(x,t) < 0, & \text{when} \quad |t| \in (0,t_0), \\ f(x,t) > 0, & \text{when} \quad |t| \in (t_0,+\infty). \end{cases}$$

– (G_0) : $g : \Omega \times \mathbb{R} \to \mathbb{R}$ satisfies

$$\sup_{|t| \leqslant k} |g(x,t)| \leqslant h_k(x),$$

for all $k>0$ and a.e. $x \in \Omega$ and $h_k \in L^1(\Omega)$.

To our knowledge, no results have been obtained for this type of problem in the case $sp^-_{min} > N$. Accordingly, this work aims to investigate the existence of multiple weak solutions for problem $(\mathscr{P}_{\mathscr{A}})$. In this framework, our multiplicity result is stated as follows;

Theorem 4.1. *Let Ω be a Lipschitz bounded domain in $\mathbb{R}^N$ and $s \in (0,1)$. Let $p_i : \overline{Q} \longrightarrow (1,+\infty)$, $i=1,2$, be a continuous function satisfying (1.2) and (1.3) with $sp^-_{\min} > N$. Assume that conditions (F_0) and (F_1) hold. Then there exist an open $\Lambda \subset (0,+\infty)$ and a positive real number $\xi > 0$ such that for each $\lambda \in \Lambda$ and every Carathéodory function $g : \Omega \times \mathbb{R} \longrightarrow \mathbb{R}$ satisfying the condition (G_0), there exists $\mu^* > 0$ such that for any $\mu \in [0,\mu^*]$, problem $(\mathscr{P}_{\mathscr{A}})$ has at least three weak solutions whose norms are less than ξ.*

4.2 Functional framework

To deal with problem $(\mathscr{P}_{\mathscr{A}})$, we will consider the following frameworks

$$Y = W^{s,p_{\max}(x,y)}(Q) \text{ and } Y_0 = \left\{u \in Y : u = 0 \text{ a.e. in } \Omega^c\right\}.$$

It is clear that Y and Y_0 are separable and reflexive Banach spaces if they are equipped with the following norms for every $u \in Y$

$$\|u\|_Y = \|u\|_{L^{\bar{p}(x)}(\Omega)} + [u]_{s,p_{\max}(x,y)} \quad \text{and} \quad \|u\|_{Y_0} = [u]_{s,p_{\max}(x,y)}.$$

Remark 4.1. *It is easy to see that*

(i) The space Y is continuously embedded in $W^{s,p_1(x,y)}(Q)$ and $W^{s,p_2(x,y)}(Q)$.

(*ii*) *If* $sp^-_{\min} > N$, *then* $Y \hookrightarrow\hookrightarrow C(\overline{\Omega})$.

(*iii*) *The assertion* (*ii*) *remains true if we replace* Y *by* Y_0. *Then, there exists a constant* $C_0 > 0$ *such that for any* $u \in Y_0$,

$$|u|_\infty \leqslant C_0 \|u\|_{Y_0}. \tag{4.1}$$

4.3 Main results

This section is devoted to the proof of our main result stated in Theorem 4.1. By means of Theorem 1.7, we will show the existence of three weak solutions of problem $(\mathscr{P}_{\mathscr{A}})$ in the following sense.

Definition 4.1. We say that $u \in Y_0$ is a weak solution of problem $(\mathscr{P}_{\mathscr{A}})$, if

$$\sum_{i=1}^{2} \int_Q \frac{|u(x)-u(y)|^{p_i(x,y)-2}(u(x)-u(y))(\varphi(x)-\varphi(y))}{|x-y|^{N+sp_i(x,y)}} dxdy$$
$$+ \int_\Omega |u|^{p_{\max}(x)-2} u\varphi dx - \lambda \int_\Omega f(x,u)\varphi dx - \mu \int_\Omega g(x,u)\varphi dx = 0,$$

for all $\varphi \in Y_0$.

For the sake of simplification in the proofs. Let define the functionals Φ, Ψ, and $\mathscr{J}$ for any $u \in Y_0$ in this way

$$\Phi(u) = \sum_{i=1}^{2} \int_Q \frac{1}{p_i(x,y)} \frac{|u(x)-u(y)|^{p_i(x,y)}}{|x-y|^{N+sp_i(x,y)}} dxdy + \int_\Omega \frac{1}{p_{\max}(x)} |u(x)|^{p_{\max}(x)} dx,$$

$$\Psi(u) = -\int_\Omega F(x,u)dx, \quad \text{where } F(x,t) = \int_0^t f(x,\tau)d\tau,$$

and

$$\mathscr{J}(u) = -\int_\Omega G(x,u)dx, \quad \text{where } G(x,t) = \int_0^t g(x,\tau)d\tau.$$

By standard argument as in [20, Lemma 3.1], Φ is well defined. Furthermore, $\Phi \in C^1(Y_0,\mathbb{R})$ and for any $u, \varphi \in Y_0$ its the derivative is given by

$$\langle \Phi'(u), \varphi \rangle = \sum_{i=1}^{2} \int_Q \frac{|u(x)-u(y)|^{p_i(x,y)-2}(u(x)-u(y))(\varphi(x)-\varphi(y))}{|x-y|^{N+sp_i(x,y)}} dxdy + \int_\Omega |u|^{p_{\max}(x)-2} u\varphi dx.$$

Next, we state and prove some auxiliary lemmas that are crucial in the proof of our main result.

Lemma 4.1. *Assume that assumptions of Theorem (4.1) are satisfied, then the following properties hold true*

(*i*) *The functional* Φ *is sequentially weakly lower semi-continuous and bounded on each bounded subset of* Y_0.

(*ii*) *The functional* Φ' *admits a continuous inverse on the dual space* Y_0^* *of* Y_0.

Proof. (*i*) Let $\{u_n\} \subset Y_0$ such that $u_n \rightharpoonup u$ in Y_0. Since Φ is convex, we have

$$\Phi(u_n) \geqslant \langle \Phi'(u), u_n - u \rangle + \Phi(u).$$

By passing to the limit, we get

$$\Phi(u) \leqslant \liminf_{n \to +\infty} \Phi(u_n).$$

Thus, Φ is sequentially weakly lower semi-continuous. Moreover, by applying Proposition 1.2 (with $p_{\max}$ replacing p) and Remark 4.1, we deduce that Φ is bounded in Y_0.

(*ii*) We start by showing the existence of the inverse function $\left(\Phi'\right)^{-1} : Y_0^* \to Y_0$, by applying the Minty-Browder theorem. Indeed, For all $u_1, u_2 \in Y_0$, we have

$$\begin{aligned}
&\langle \Phi'(u_1) - \Phi'(u_2), u_1 - u_2 \rangle = \\
&\sum_{i=1}^{2} \int_Q \frac{|u_1(x) - u_1(y)|^{p_i(x,y)-2}(u_1(x) - u_1(y))((u_1 - u_2)(x) - (u_1 - u_2)(y))}{|x-y|^{N+sp_i(x,y)}} dxdy \\
&- \sum_{i=1}^{2} \int_Q \frac{|u_2(x) - u_2(y)|^{p_i(x,y)-2}(u_2(x) - u_2(y))((u_1 - u_2)(x) - (u_1 - u_2)(y))}{|x-y|^{N+sp_i(x,y)}} dxdy \\
&+ \int_\Omega \left(|u_1|^{p_{\max}(x)-2} u_1 - |u_2|^{p_{\max}(x)-2} u_2 \right)(u_1 - u_2) dx.
\end{aligned}$$

Using the following Simon's inequality

$$\left(|\xi_1|^{r-2}\xi_1 - |\xi_2|^{r-2}\xi_2 \right) \geqslant \frac{1}{2^r} |\xi_1 - \xi_2|^r, \quad r \geqslant 2, \quad \forall \xi_1, \xi_2 \in \mathbb{R}^N,$$

we obtain

$$\begin{aligned}
\langle \Phi'(u_1) - \Phi'(u_2), u_1 - u_2 \rangle \geqslant &\sum_{i=1}^{2} \frac{1}{2^{p_i^+}} \int_Q \frac{|(u_1 - u_2)(x) - (u_1 - u_2)(y)|^{p_i(x,y)}}{|x-y|^{N+sp_i(x,y)}} dxdy \\
&+ \frac{1}{2^{p_{\max}^+}} \int_\Omega |u_1 - u_2|^{p_{\max}(x)} dx.
\end{aligned} \tag{4.2}$$

On the other hand, we can see easily that for all $(x,y)\in\overline{Q}$, the following inequality holds

$$\frac{|u(x)-u(y)|^{p_1(x,y)}}{|x-y|^{N+sp_1(x,y)}}+\frac{|u(x)-u(y)|^{p_2(x,y)}}{|x-y|^{N+sp_2(x,y)}}>\frac{|u(x)-u(y)|^{p_{\max}(x,y)}}{|x-y|^{N+sp_{\max}(x,y)}}. \tag{4.3}$$

From (4.2) and (4.3), we have

(4.4)

$$\begin{aligned}\langle\Phi'(u_1)-\Phi'(u_2),u_1-u_2\rangle>\min\left\{\frac{1}{2^{p_1^+}},\frac{1}{2^{p_2^+}}\right\}\int_Q\frac{|(u_1-u_2)(x)-(u_1-u_2)(y)|^{p_{\max}(x,y)}}{|x-y|^{N+sp_i(x,y)}}dxdy\\+\frac{1}{2^{p_{\max}^+}}\int_\Omega|u_1-u_2|^{p_{\max}(x)}dx.\end{aligned}$$

Thus, Φ' is strictly monotone.

Next, let $u\in Y_0$, then

$$\begin{aligned}\frac{\langle\Phi'(u),u\rangle}{\|u\|_{Y_0}}&=\frac{1}{\|u\|_{Y_0}}\left(\sum_{i=1}^{2}\int_Q\frac{|u(x)-u(y)|^{p_i(x,y)}}{|x-y|^{N+sp_i(x,y)}}dxdy+\int_\Omega|u|^{p_{\max}(x)}dx\right)\\&\geqslant\frac{1}{\|u\|_{Y_0}}\left(\int_Q\frac{|u(x)-u(y)|^{p_{\max}(x,y)}}{|x-y|^{N+sp_{\max}(x,y)}}dxdy+\int_\Omega|u|^{p_{\max}(x)}dx\right)\\&\geqslant\|u\|_{Y_0}^{\bar{p}_{\max}-1},\qquad\text{for all } u\in Y_0 \text{ with } \|u\|_{Y_0}>1.\end{aligned}$$

Hence, Φ' is coercive. Since $\Phi\in C^1(Y_0,\mathbb{R})$, so Φ' is hemicontinuous. Therefore, in view of Minty-Browder theorem [175, Theorem 26.A-(d), pp 557], we deduce that Φ' is a bijection.

It remains to show that $(\Phi')^{-1}$ is continuous, to deal with this, we prove that it is sequentially continuous. Let $\{u_n\}\subset Y_0^*$ be a sequence such that

$$u_n\longrightarrow u \text{ in } Y_0^*.$$

Set

$$\begin{cases}w_n=\Phi'^{-1}(u_n),\\ w=\Phi'^{-1}(u).\end{cases}$$

As Φ'^{-1} is bounded, then $\{w_n\}$ is bounded in Y_0, which implies that there exists $w_0\in Y_0$ and a subsequence of $\{w_n\}$ still denoted by $\{w_n\}$ such that w_n converges weakly to w_0 in Y_0. Since $u_n\longrightarrow u$, then

$$\lim_{n\to\infty}\langle\Phi'(w_n)-\Phi'(w_0),w_n-w_0\rangle=\lim_{n\to\infty}\langle\Phi'(w_n),w_n-w_0\rangle=0.$$

Because of Φ' is of type (S_+), we conclude that $w_n\longrightarrow w_0$ in Y_0. The continuity and injectivity of Φ' imply that w_n converges strongly to w. Consequently Φ'^{-1} is continuous.

■

Lemma 4.2. *Let f and g two Carathéodory functions that verify conditions (F_0) and (G_0) respectively. Then $\Psi, \mathcal{J} \in C^1(Y_0, \mathbb{R})$ with the derivatives given by*

$$\langle \Psi'(u), v \rangle = \int_\Omega f(x,u)v\,dx, \quad \text{and} \quad \langle \mathcal{J}'(u), v \rangle = -\int_\Omega g(x,u)v\,dx,$$

for all $u, v \in Y_0$. Moreover, $\Psi', \mathcal{J}' : Y_0 \longrightarrow Y_0^*$ are compacts.

Proof. From condition (F_0) and Remark 4.1, we have that Ψ is well defined in Y_0. Usual argument, as in [20, Lemma 3.1], shows that Ψ is Gâteaux differentiable on Y_0 and it is given by

$$\langle \Psi'(u), v \rangle = \int_\Omega f(x,u)v\,dx, \qquad \text{for all } u, v \in Y_0.$$

In the sequel, we show that Ψ' is continuous. Let $\{u_n\} \subset Y_0$ be a sequence which converges strongly to u in Y_0. Since Y_0 is compactly embedded into $L^\infty(\Omega)$, consequently Y_0 is compactly embedded in $L^{q(x)}(\Omega)$, where $q(x)$ as in the condition (F_0). It follows that there exists a subsequence of $\{u_n\}$ still denoted by $\{u_n\}$ such that

$$u_n(x) \longrightarrow u(x) \text{ a.e. } x \in \Omega \text{ and } |u_n| \leqslant h(x) \text{ a.e. } x \in \Omega \text{ where } h \in L^{q(x)}(\Omega). \tag{4.5}$$

Since f is a Carathéodory function, we can define the mapping $u \longrightarrow f(.,u(.))$ from $L^{q(x)}(\Omega)$ into its conjugate $L^{\hat{q}(x)}(\Omega)$. Next, fixing $v \in Y_0$ with $\|v\|_{Y_0} \leqslant 1$, we get

$$\begin{aligned} \left|\langle \Psi'(u_n) - \Psi'(u), v \rangle\right| &= \left| \int_\Omega (f(x,u_n(x)) - f(x,u(x)))v(x)dx \right| \\ &\leqslant \|f(.,u_n(.)) - f(.,u(.))\|_{L^{\hat{q}(x)}(\Omega)} \|v\|_{L^{q(x)}(\Omega)} \\ &\leqslant C_0 \|v\|_{Y_0} \|f(.,u_n(.)) - f(.,u(.))\|_{L^{\hat{q}(x)}(\Omega)}, \end{aligned}$$

where C_0 is given in Remark 4.1. Afterwards,

$$\left\|\Psi'(u_n) - \Psi'(u)\right\|_{Y_0^*} \leqslant C_0 \|f(.,u_n(.)) - f(.,u(.))\|_{L^{\hat{q}(x)}(\Omega)}. \tag{4.6}$$

From (4.5) and the fact that f is a Caratheodory function, we have

$$f(x,u_n(x)) - f(x,u(x)) \underset{n \to +\infty}{\longrightarrow} 0,$$

and

$$|f(x,u_n(x)) - f(x,u(x))| \leqslant C(2 + |h(x)|^{q(x)-1} + |u(x)|^{q(x)-1}) \in L^{\hat{q}(x)}(\Omega).$$

In view of the dominated convergence theorem, we conclude that

$$\|f(x,u_n(x)) - f(x,u(x))\|_{L^{\hat{q}(x)}(\Omega)} \to 0.$$

Hence, from (4.6), we infer that Ψ' is continuous in Y_0^*.
On the other hand, let $\{u_n\}$ be a bounded sequence in Y_0. Then, there exists a subsequence of $\{u_n\}$ (still denoted by $\{u_n\}$) weakly convergent in Y_0 and strongly convergent in $L^{q(x)}(\Omega)$. Consequently, as above $\Psi'(u_n)$ converges strongly in Y_0^*, which implies that Ψ' is bounded. One deducts that Ψ' is compact. ■

Remark 4.2. *By the same argument as above, as the function $g:\Omega\times\mathbb{R}\to\mathbb{R}$ is a Carathéodory function satisfying the condition (G_0), Then*

$$\mathscr{J}(u)=-\int_\Omega G(x,u)dx$$

is well defined and continuously Gâteaux differentiable on Y_0, with compact derivative. Moreover,

$$\langle \mathscr{J}'(u),v\rangle=-\int_\Omega g(x,u)v\,dx \quad \text{for all } u,v\in Y_0.$$

Now, we are ready in the position to prove our main result.

Proof. Theorem 4.1: To establish our result, we will verify that the functionals Φ,Ψ, and $\mathscr{J}$ satisfy the conditions of Theorem 1.7. Indeed, from Lemma 4.1, we have that Φ is sequentially weakly lower semi-continuous and the inverse of its derivative is continuous. Also, Ψ is a continuously Gâteaux differentiable functional that admits a compact derivative. Furthermore, the functional $\mathscr{J}$ is continuously Gâteaux differentiable with a compact derivative. Then, the preconditions of Theorem 1.7 are satisfied. In the following;

- (D_1): For all $u\in Y_0$ with $\|u\|_{Y_0}>1$, we have

$$\begin{aligned}\Phi(u)&=\sum_{i=1}^{2}\int_Q\frac{1}{p_i(x,y)}\frac{|u(x)-u(y)|^{p_i(x,y)}}{|x-y|^{N+sp_i(x,y)}}dxdy+\int_\Omega\frac{1}{p_{\max}(x)}|u(x)|^{p_{\max}(x)}dx,\\ &\geqslant\frac{1}{p^+_{\max}}\left(\int_Q\frac{|u(x)-u(y)|^{p_1(x,y)}}{|x-y|^{N+sp_1(x,y)}}dxdy+\int_Q\frac{|u(x)-u(y)|^{p_2(x,y)}}{|x-y|^{N+sp_2(x,y)}}dxdy\right)\\ &\quad+\int_\Omega\frac{1}{p_{\max}(x)}|u(x)|^{p_{\max}(x)}dx\\ &\geqslant\frac{1}{p^+_{\max}}\int_Q\frac{|u(x)-u(y)|^{p_{\max}(x,y)}}{|x-y|^{N+sp_{\max}(x,y)}}dxdy+\int_\Omega\frac{1}{p_{\max}(x)}|u(x)|^{p_{\max}(x)}dx\\ &\geqslant\frac{1}{p^+_{\max}}\|u\|_{Y_0}^{p^-_{\max}}.\end{aligned}\tag{4.7}$$

  On the other hand, from condition (F_0), we find

$$|F(x,t)|\leqslant C\left(|t|+\frac{1}{q(x)}|t|^{q(x)}\right)\quad\text{for all}\quad(x,t)\in\Omega\times\mathbb{R}.$$

Thus, using Remark 4.1, we obtain

$$\begin{aligned}\Psi(u) = -\int_\Omega F(x,u)dx &\geqslant -C\int_\Omega |u|dx - C\int_\Omega \frac{1}{q(x)}|u|^{q(x)}dx \\ &\geqslant -c_1\|u\|_{Y_0} - \frac{C}{q^-}\int_\Omega \left(|u|^{q^+} + |u|^{q^-}\right)dx \\ &= -c_1\|u\|_{Y_0} - \frac{C}{q^+}\left(\|u\|^{q^+}_{L^{q^+}(\Omega)} + \|u\|^{q^-}_{L^{q^-}(\Omega)}\right).\end{aligned} \tag{4.8}$$

As $q^- < q^+ < p^-_{\max}$, it is known that Y_0 is continuously embedded into $L^{q^+}(\Omega)$ and $L^{q^-}(\Omega)$. Furthermore, we can find two positive constants c_2 and c_3 such that for any $u \in Y_0$ we have

$$\|u\|_{L^{q+}(\Omega)} \leqslant c_2\|u\|_{Y_0} \quad \text{and} \quad \|u\|_{L^{q^-}(\Omega)} \leqslant c_3\|u\|_{Y_0}. \tag{4.9}$$

From (4.8) and (4.9), it follows that

$$\Psi(u) \geqslant -c_1\|u\|_{Y_0} - \frac{C}{q^+}c_2\|u\|^{q^+}_{Y_0} - \frac{C}{q^+}c_3\|u\|^{q^-}_{Y_0}. \tag{4.10}$$

Combining (4.7) and (4.10), for all $u \in Y_0$ with $\|u\|_{Y_0} > 1$, and for any $\lambda > 0$, we get

$$\Phi(u) + \lambda\Psi(u) \geqslant \frac{1}{p^+_{\max}}\|u\|^{p^-_{\max}}_{Y_0} - \lambda\left(c_1\|u\|_{Y_0} + \frac{C}{q^+}c_2\|u\|^{q^+}_{Y_0} + \frac{C}{q^+}c_3\|u\|^{q^-}_{Y_0}\right). \tag{4.11}$$

Therefore,

$$\lim_{\|u\|_{Y_0}\to\infty} \Phi(u) + \lambda\Psi(u) = \infty.$$

Consequently, the condition (D_1) of Theorem 1.7 is verified.

- (D_2): Since $F(x,t) = \int_0^t f(x,\tau)d\tau$, it follows by assumption (F_1) that $(x,t) \mapsto F(x,t)$ is increasing for $t \in (t_0, +\infty)$, and decreasing for $t \in (0,1)$, uniformly with respect to $x \in \Omega$. Moreover, it is evident that $F(x,0) = 0$ and $F(x,t) \underset{t\to\infty}{\longrightarrow} \infty$. Thus, we can say that there exists a real number $\delta > 0$ such that

$$F(x,t) > 0 = F(x,0) \geqslant F(x,\tau_1) \quad \text{for all } x \in \Omega, t > \delta \text{ and } \tau_1 \in (0,t). \tag{4.12}$$

Let a, b be two real numbers such that

$$0 < a < \min\{1, C_0\}, \tag{4.13}$$

with C_0 is given in Remark 4.1 and $b > \delta$ satisfies

$$b^{p^-_{\max}}|\Omega| > 1. \tag{4.14}$$

From (F_1), we have $F(x,t) \leqslant F(x,0) = 0$, for all $t \in [0,a]$. It follows that

$$\int_\Omega \sup_{0\leqslant t\leqslant a} F(x,t)\mathrm{d}x \leqslant \int_\Omega F(x,0)\mathrm{d}x = 0. \tag{4.15}$$

Furthermore, since $b > \delta$, we get $\int_\Omega F(x,b)\mathrm{d}x > 0$. Moreover

$$\frac{1}{C_0^{p^+_{\max}}} \frac{a^{p^+_{\max}}}{b^{p^-_{\max}}} \int_\Omega F(x,b)\mathrm{d}x > 0. \tag{4.16}$$

Combining (4.15) and (4.16), we obtain

$$\int_\Omega \sup_{0\leqslant t\leqslant a} F(x,t)\mathrm{d}x \leqslant 0 < \frac{1}{C_0^{p^+_{\max}}} \frac{a^{p^+_{\max}}}{b^{p^-_{\max}}} \int_\Omega F(x,b)\mathrm{d}x. \tag{4.17}$$

Consider u_1, $u_2 \in Y_0$ with $u_0(x) = 0$ and $u_1(x) = b$ for any $x \in \Omega$. We define $r = \frac{1}{p^+_{\max}} \left(\frac{a}{C_0}\right)^{p^+_{\max}}$, it is clear that $r \in (0,1)$ and a simple computation implies $\Phi(u_0) = \Psi(u_0) = 0$ as well as,

$$\Phi(u_1) = \int_\Omega \frac{1}{p_{\max}(x)} |b|^{p_{\max}(x)} dx \geqslant \frac{1}{p^+_{\max}} b^{p^-_{\max}} |\Omega| > \frac{1}{p^+_{\max}} > \frac{1}{p^+_{\max}} \left(\frac{a}{C_0}\right)^{p^+_{\max}} = r.$$

Then, we get

$$\Phi(u_0) < r < \Phi(u_1).$$

Accordingly, the condition (D_2) in Theorem 1.7 is fulfilled.

- (D_3): It is easy to see that

$$-\frac{(\Phi(u_1)-r)\Psi(u_0)+(r-\Phi(u_0))\Psi(u_1)}{\Phi(u_1)-\Phi(u_0)} = -r\frac{\Psi(u_1)}{\Phi(u_1)} = r\frac{\displaystyle\int_\Omega F(x,b)\mathrm{d}x}{\displaystyle\int_\Omega \frac{1}{p_{\max}(x)} b^{p_{\max}(x)}\mathrm{d}x} > 0. \tag{4.18}$$

Now, we consider the case when $u \in Y_0$ with $\Phi(u) \leqslant r < 1$. Then

$$\frac{1}{p^+_{\max}} \rho^0_{p_{\max}(x,y)}(u) \leqslant \Phi(u) \leqslant r.$$

Hence, we get

$$\rho^0_{p_{\max}(x,y)}(u) \leqslant p^+_{\max}.r = \left(\frac{a}{C_0}\right)^{p^+_{\max}} < 1,$$

Thus, by Proposition 1.5, we find $\|u\|_{Y_0} \leqslant 1$. Furthermore, we get

$$\frac{1}{p^+_{\max}} \|u\|^{p^+_{\max}}_{Y_0} \leqslant \frac{1}{p^+_{\max}} \rho^0_{p_{\max}(x,y)}(u) \leqslant \Phi(u) \leqslant r.$$

Hence, using Remark 4.1, we have

$$|u(x)| \leqslant C_0 \|u\|_{Y_0} \leqslant C_0 \left(p^+_{\max} r\right)^{\frac{1}{p^+_{\max}}} = a \text{ for all } x \in \Omega, u \in Y_0, \text{ and } \Phi(u) \leqslant r. \tag{4.19}$$

The above inequality shows that

$$-\inf_{u \in \Phi^{-1}((-\infty, r])} \Psi(u) = \sup_{u \in \Phi^{-1}((-\infty, r])} -\Psi(u) \leqslant \int_\Omega \sup_{0 \leqslant t \leqslant a} F(x,t)\mathrm{d}x \leqslant 0.$$

Combining this fact with (4.18), we deduce that

$$-\inf_{u \in \Phi - 1((-\infty, r])} \Psi(u) < r \frac{\int_\Omega F(x,b)\mathrm{d}x}{\int_\Omega \frac{1}{p_{\max}(x)} b^{p_{\max}(x)} \mathrm{d}x},$$

that is

$$\inf_{u \in \Phi^{-1}((-\infty, r])} \Psi(u) > \frac{(\Phi(u_1) - r)\Psi(u_0) + (r - \Phi(u_0))\Psi(u_1)}{\Phi(u_1) - \Phi(u_0)}.$$

Which means that condition (D_3) in Theorem 1.7 is obtained.

Consequently, all the assumptions of Theorem 1.7 are verified. As a result, there exist an open $\Lambda \subset (0, +\infty)$ and a positive real number $\xi > 0$ such that, for every Carathéodory function g and for each $\lambda \in \Lambda$, there exists $\mu^* > 0$ such that for any $\mu \in [0, \mu^*]$, the equation $\Phi'(u) + \lambda \Psi'(u) + \mu \mathscr{J}'(u) = 0$ has at least three weak solutions in Y_0 whose norms are less than ξ. It follows that Theorem 4.1 holds. ■

This chapter establishes the existence of weak solutions for a problem involving fractional $(p_1(x,\cdot), p_2(x,\cdot))$-Laplacian operator in the space Y_0. We consider this problem as a motivation for introducing a functional framework that generalizes the operator in cases where the vector of variable exponents exceeds two. The following chapter will present a simplified generalization for the anisotropic spaces introduced in [188], to Lebesgue spaces rather than the normed Lebesgue spaces. This framework aims to facilitate the study of problems involving fractional $\vec{p}(x,\cdot)$-Laplacian operator, where the vector $\vec{p}(x,\cdot)$ is of size N.

CHAPTER 5

FRACTIONAL ANISOTROPIC SOBOLEV SPACES WITH VARIABLE EXPONENTS

Main objective

The primary aim of this chapter is to introduce a new fractional anisotropic Sobolev space with variable exponents, within which fundamental qualitative properties such as completeness, separability, and reflexivity are established, alongside continuous and compact embedding results. Furthermore, we demonstrate essential functional properties of the anisotropic fractional $\vec{p}(\cdot,\cdot)$-Laplacian operator. As an application, we employ the Mountain Pass Theorem and Ekeland's Variational Principle to confirm the existence of a weak solution for a nonlocal anisotropic problem with variable exponents.

The results presented in this chapter are derived from the study outlined in [18].

5.1 Functional framework

This section is dedicated to introducing a new framework for fractional anisotropic Sobolev spaces with variable exponents. These spaces extend classical Sobolev spaces by incorporating both fractional orders and directional variability in the exponents.
Consider a continuous vectorial function $\vec{p} : (\overline{\Omega} \times \overline{\Omega})^N \longrightarrow \mathbb{R}^N$ which verifies conditions (1.2) and (1.3) for all $i = 1, \ldots, N$. For $s \in (0,1)$, the fractional anisotropic Sobolev space

with variable exponent is identified as follows

$$W^{s,p_M(x),\vec{p}(x,y)}(\Omega)=\left\{\begin{array}{c} u\in L^{p_M(x)}(\Omega):\iint_{\Omega\times\Omega}\frac{|u(x)-u(y)|^{p_i(x,y)}}{\mu_i^{p_i(x,y)}|x-y|^{sp_i(x,y)+N}}dxdy<+\infty,\\ \text{for some } \mu_i>0, \text{ and for all } i=1,...,N \end{array}\right\}.$$

The space $W^{s,p_M(x),\vec{p}(x,y)}(\Omega)$ is equipped with this norm

$$\|u\|_{s,\vec{p}(x,y)}=[u]_{s,\vec{p}(x,y)}+\|u\|_{p_M(x)}, \tag{5.1}$$

where $[\cdot]_{s,\vec{p}(x,y)}$ is defined as the sum of the Gagliardo seminorm with variable exponent, and it is given by

$$[u]_{s,\vec{p}(x,y)}=\sum_{i=1}^{N}\inf\left\{\mu_i>0:\iint_{\Omega\times\Omega}\frac{|u(x)-u(y)|^{p_i(x,y)}}{\mu_i^{p_i(x,y)}|x-y|^{sp_i(x,y)+N}}\leqslant 1\right\}. \tag{5.2}$$

These spaces are intermediary spaces between $L^{\vec{p}(x)}(\Omega)$ and $W^{1,\vec{p}(x)}(\Omega)$.
It is worth mentioning that the fractional anisotropic Sobolev spaces with variable exponent have not yet been studied systematically up to now. The main goal of this chapter is to present some basic properties of these spaces and the fractional $\vec{p}(x,.)$-Laplacian operator

$$\left[(-\Delta)^s_{\vec{p}(x,.)}u\right](x)=p.v.\sum_{i=1}^{N}\int_{\mathbb{R}^N}\frac{|u(x)-u(y)|^{p_i(x,y)-2}(u(x)-u(y))}{|x-y|^{N+sp_i(x,y)}}dy \quad \text{for all } x\in\mathbb{R}^N. \tag{5.3}$$

Note that when choosing $p_1(\cdot,\cdot)=\ldots=p_N(\cdot,\cdot)=p(\cdot,\cdot)$, we obtain an operator with similar properties to the fractional $p(x,\cdot)$-Laplacian operator $(-\Delta)^s_{p(x,.)}$. Furthermore $(-\Delta)^s_{\vec{p}(x,.)}$ is the fractional version of the $\vec{p}(x)$-Laplacian operator, while when choosing $p_1,\ldots,p_N$ to be constant functions we arrive at the anisotropic fractional $\vec{p}$-Laplacian operator $(-\Delta)^s_{\vec{p}}$.
A fundamental result for Sobolev spaces is the embedding theorem. One of the main goals of this chapter is to establish an embedding theorem that consists that the anisotropic fractional Sobolev spaces with variable exponent are continuously embedded in a generalized Lebesgue space. This result is stated as follows.

Theorem 5.1. *Consider a smooth bounded domain Ω in $\mathbb{R}^N$ ($N \geqslant 2$), and let $\vec{p} : \overline{\Omega} \times \overline{\Omega} \longrightarrow (1, +\infty)$ be a vector function satisfying conditions (1.2) and (1.3) for each element of the vector. As well as, for all $(x, y) \in \overline{\Omega} \times \overline{\Omega}$, $sp_i(x, y) < N$ $\forall i = 1, \ldots, N$. Moreover, assume that $q : \overline{\Omega} \longrightarrow (1, +\infty)$ is a continuous variable exponent such that*

$$1 < q(x) < p_i^{s,*}(x) = \frac{p_i(x,x)}{1 - \frac{sp_i(x,x)}{N}}, \quad i = 1, \ldots, N, \text{ for all } x \in \overline{\Omega}.$$

Therefore, there exists a constant C that depends on N, s, $\vec{p}$, q, and Ω such that

$$\|u\|_{L^{q(x)}(\Omega)} \leqslant C \|u\|_{s,\vec{p}(x,y)}. \tag{5.4}$$

Hence, the embedding of $W^{s,p_M(x),\vec{p}(x,y)}(\Omega)$ into $L^{q(x)}(\Omega)$ is continuous. Furthermore, this embedding is compact.

5.2 Qualitative properties

Theorem 5.2. *The space $\left(W^{s,p_M(\cdot),\vec{p}(\cdot,\cdot)}(\Omega), \|\cdot\|_{s,\vec{p}(x,y)}\right)$ defined by (5.1) is a Banach space.*

Proof. We define the operator $\mathcal{T}$ by

$$\begin{aligned} \mathcal{T} : W^{s,p_M(x),\vec{p}(x,y)}(\Omega) &\longrightarrow \Pi = L^{p_M(x)}(\Omega) \times \prod_{i=1}^{N} L^{p_i(x,y)}(\Omega \times \Omega) \\ u &\longmapsto \left(u(x), \frac{u(x)-u(y)}{|x-y|^{\frac{N}{p_1(x,y)}+s}}, \ldots, \frac{u(x)-u(y)}{|x-y|^{\frac{N}{p_N(x,y)}+s}} \right). \end{aligned}$$

For any $u \in W^{s,p_M(\cdot),\vec{p}(\cdot,\cdot)}(\Omega)$, we have

$$\begin{aligned} \|\mathcal{T}(u)\|_{\Pi} &= \|u\|_{L^{p_M(x)}(\Omega)} + \sum_{i=1}^{N} \|u\|_{L^{p_i(x,y)}(\Omega\times\Omega)} \\ &= \|u\|_{s,\vec{p}(x,y)}. \end{aligned}$$

Then $\mathcal{T}$ is an isometry from $W^{s,p_M(\cdot),\vec{p}(\cdot,\cdot)}(\Omega)$ to the Cartesian product Π. Since Π is a Banach space, therefore $W^{s,p_M(\cdot),\vec{p}(\cdot,\cdot)}(\Omega)$ is also a Banach space. ■

Proposition 5.1. *Let Ω be a bounded open domain of $\mathbb{R}^N$, we suppose that conditions (1.2) and (1.3) are satisfied for each element of the vector $\vec{p}$. Then $\left(W^{s,p_M(\cdot),\vec{p}(\cdot,\cdot)}(\Omega), \|\cdot\|_{s,\vec{p}(x,y)}\right)$ is a separable and uniformly convex space (consequently it is a reflexive space).*

Proof. We use the same isometric map introduced in the proof of Theorem 5.2. Because $W^{s,p_M(\cdot),\vec{p}(\cdot,\cdot)}(\Omega)$ is a Banach space, we get that $\mathcal{T}\left(W^{s,p_M(\cdot),\vec{p}(\cdot,\cdot)}(\Omega)\right)$ is a closed subset of Π. Consequently, by [52, Proposition 3.20 and Proposition 3.25], it follows that the space $\left(W^{s,p_M(\cdot),\vec{p}(\cdot,\cdot)}(\Omega),\|\cdot\|_{s,\vec{p}(x,y)}\right)$ is separable.
On the other hand, since $p_i^- > 1\ \forall i = 1,\ldots,N$. From [87], the space $(L^{p_i(x)},\|\cdot\|_{p_i(x)})$ is uniformly convex. Moreover, by the result stated in [137, p. 184], we infer that Π is uniformly convex space. As $\mathcal{T}\left(W^{s,p_M(\cdot),\vec{p}(\cdot,\cdot)}(\Omega)\right)$ is a closed subset of Π and since any linear subspace of a uniformly convex linear normed space is also uniformly convex, hence $\left(W^{s,p_M(\cdot),\vec{p}(\cdot,\cdot)}(\Omega),\|\cdot\|_{s,\vec{p}(x,y)}\right)$ is uniformly convex and consequently it is a reflexive space. ■

Now, let us come back to the definition of the space $W^{s,p_M(x),\vec{p}(x,y)}(\Omega)$. Since $\Omega\times\Omega$ is strictly contained in Q and for the reason to encode correctly the Dirichlet boundary conditions in the variational formulation, we will adjust the definition of the space $W^{s,p_M(x),\vec{p}(x,y)}(\Omega)$ as follows

$$
\begin{aligned}
Z &= W^{s,p_M(x),\vec{p}(x,y)}(Q)\\
&= \left\{ \begin{array}{c} u:\mathbb{R}^N \longrightarrow \mathbb{R} \text{ measurable, such that } u_{|\Omega}\in L^{p_M(x)}(\Omega) \text{ with} \\ \displaystyle\int_Q \frac{|u(x)-u(y)|^{p_i(x,y)}}{\mu_i^{p_i(x,y)}|x-y|^{sp_i(x,y)+N}}dxdy < +\infty, \text{ for some } \mu_i>0 \text{ and for all } i=1,\ldots,N \end{array} \right\}.
\end{aligned}
$$

Remark 5.1.

(*i*)- *Simply swap* $\Omega\times\Omega$ *for* Q *to define the norm on* Z *in the same manner on* $W^{s,p_M(x),\vec{p}(x,y)}(\Omega)$*, and denote this norm by* $\|\cdot\|_Z$.

(*ii*)- *Similar to the space* $(W^{s,p_M(x),\vec{p}(x,y)}(\Omega),\|\cdot\|_{s,\vec{p}})$*, we have that* $(Z,\|\cdot\|_Z)$ *is a Banach, reflexive, and separable space.*

We define the following mapping on Z, called $\vec{p}(x,y)$-modular

$$\rho_{s,\vec{p}(x,y)}(u) = \int_\Omega |u(x)|^{p_M(x)}dx + \sum_{i=1}^N \int_Q \frac{|u(x)-u(y)|^{p_i(x,y)}}{|x-y|^{sp_i(x,y)+N}}dxdy,$$

for all $u\in Z$.
It is obvious to check that $\rho_{\vec{p}(\cdot,\cdot)}$ is a convex modular on Z. Moreover, the modular norm on Z is defined by

$$\|u\|_{\rho_{s,\vec{p}}} = \inf\left\{\mu>0,\quad \rho_{s,\vec{p}(x,y)}\left(\frac{u}{\mu}\right)\leqslant 1\right\},$$

which is equivalent to the norm $\|\cdot\|_Z$.
Denote by Z_0 the closed subspace of Z introduced as follows

$$Z_0 = Z_0^{s,p_M(x),\vec{p}(x,y)}(Q) = \left\{u \in Z : u(x) = 0 \text{ a.e. } x \in \Omega^c\right\}.$$

This subset of Z also qualifies as a Banach space which is separable and reflexive.
Besides, for any $u \in Z_0$, we define the convex modular as follows

$$\rho^0_{s,\vec{p}(x,y)}(u) = \sum_{i=1}^{N} \int_Q \frac{|u(x)-u(y)|^{p_i(x,y)}}{|x-y|^{sp_i(x,y)+N}} dxdy.$$

The modular on Z_0 is connected to the norm $\|\cdot\|_Z$ by the following properties.

Lemma 5.1. *Let $\vec{p} : \overline{Q}^N \longrightarrow (1,+\infty)$ be a vector function on which each element of it satisfies conditions (1.2) and (1.3). Then, for any $u \in Z_0$ we have*

(i)- $1 \leqslant [u]_{s,\vec{p}(x,y)} \Rightarrow \rho^0_{s,\vec{p}(x,y)}(u) \leqslant [u]^{P^+_+}_{s,\vec{p}(x,y)}$.

(ii)- $[u]_{s,\vec{p}(x,y)} \leqslant 1 \Rightarrow \rho^0_{s,\vec{p}(x,y)}(u) \leqslant [u]_{s,\vec{p}(x,y)}$.

Proof. (i)- Let $u \in Z_0$ such that $[u]_{s,\vec{p}(x,y)} \geqslant 1$, then $\sum_{i=1}^{N} [u]_{s,p_i(x,y)} \geqslant 1$. Hence,

$$\exists J \subset I = \{1,...,N\} \text{ such that } \forall j \in J \quad [u]_{s,p_j(x,y)} \leqslant 1 \text{ and } \forall i \in I \setminus J \quad [u]_{s,p_i(x,y)} \geqslant 1,$$

which implies by using Proposition 1.5

$$\begin{aligned}
\rho^0_{s,\vec{p}(x,y)}(u) &= \sum_{j\in J} \rho^0_{s,p_j(x,y)}(u) + \sum_{i\in I\setminus J} \rho^0_{s,p_i(x,y)}(u) \\
&\leqslant \sum_{j\in J} [u]^{p_j^-}_{s,p_j(x,y)} + \sum_{i\in I\setminus J} [u]^{p_i^+}_{s,p_i(x,y)} \\
&\leqslant \sum_{j\in J} [u]_{s,p_j(x,y)} + \left(\sum_{i\in I\setminus J} [u]_{s,p_i(x,y)}\right)^{P^+_+} \\
&\leqslant [u]^{P^+_+}_{s,\vec{p}(x,y)}.
\end{aligned}$$

(ii)- Let $u \in Z_0$ such that $[u]_{s,\vec{p}(x,y)} \leqslant 1$, thus $[u]_{s,p_i(x,y)} \leqslant 1 \quad \forall i = 1,\dots,N$. Consequently,

$$\rho^0_{s,\vec{p}(x,y)}(u) \leqslant \sum_{i=1}^{N} [u]^{p_i^-}_{s,p_i(x,y)} \leqslant [u]_{s,\vec{p}(x,y)}.$$

■

Remark 5.2. *The results of Lemma 5.1 can be extended to the modular $\rho_{s,\vec{p}(\cdot,\cdot)}$ defined on Z.*

5.3 Embedding Results

In the present section, we focus on the continuous and compact embedding results of our new functional framework into Lebesgue spaces. We start by giving a detailed proof of Theorem 5.1 which is essential in the study of nonlocal anisotropic problems.

Proof. Theorem 5.1 We have $\vec{p}, p_M$, and q are continuous variable exponents, then for $i = 1,\dots,N$, we can find a family of positive constants ε_i^1, and ε_i^2 such that

$$\begin{cases} \frac{Np_i(x,x)}{N-sp_i(x,x)} - q(x) & \geqslant \frac{\varepsilon_i^1}{2}, \\ p_M(x) - p_i(x,x) & \geqslant \frac{\varepsilon_i^2}{2}, \end{cases} \tag{5.5}$$

for all $x \in \overline{\Omega}$.
On the other hand, by using the finite covering theorem, we can find a positive constant $\varepsilon = \varepsilon\left(\vec{p}, p_M, q, \varepsilon_i^1, \varepsilon_i^2, t\right)$, a finite family of open sets $B_k \subset \Omega$ such that $\Omega = \bigcup_{k=1}^{N} B_k$ with $diam(B_k) < \varepsilon$, and $t \in (0,s)$. Hence, for $i = 1,\dots,N$ we have

$$\begin{cases} \frac{Np_i(y,z)}{N-tp_i(y,z)} - q(x) & \geqslant \frac{\varepsilon_i^1}{2}, \\ p_M(x) - p_i(y,z) & \geqslant \frac{\varepsilon_i^2}{2}, \end{cases} \tag{5.6}$$

for all $(x,y,z) \in B_k^3$.
Set $p_k^i = \inf\limits_{\{(y,z)\in B_k \times B_k\}} \{p_i(y,z) - \mu\}$, based on (5.6), we can choose $\mu > 0$ small enough such that $p_i^- - 1 > \mu > 0$ for $i = 1,\dots,N$, where

$$\frac{Np_k^i}{N - tp_k^i} - q(x) \geqslant \frac{\varepsilon_i^1}{2} \qquad \text{and for all } x \in B_k.$$

We get for $i = 1,\dots,N$ and for $x \in B_k$ that

- If $p_k^{i,*} = \frac{Np_k^i}{N-tp_k^i}$, then $p_k^{i,*} \geqslant \frac{\varepsilon_i^1}{2} + q(x)$.
- $p_M(x) \geqslant \frac{\varepsilon_i^2}{2} + p_k^i$.

Due to the embedding theorem in the case of constant exponent ([74, Theorem 6.7]), there exists a constant $C = C(N,t,p_k^i,B_k,\varepsilon)$ such that

$$\|u\|_{L^{p_k^{i,*}}(B_k)} \leq C\|u\|_{W^{t,p_k^i}(B_k)}.$$

To prove inequality (5.4), we need to show the following results

(i)- $\exists C>0;\quad \sum_{k=1}^{N}\|u\|_{L^{p_k^{i,*}}(B_k)} \geqslant C\|u\|_{L^{q(x)}(\Omega)}.$

(ii)- $\exists C>0;\quad C\|u\|_{L^{p_M(x)}(\Omega)} \geqslant \sum_{k=1}^{N}\|u\|_{L^{p_k^i}(B_k)}.$

(iii)- $\exists C>0;\quad C[u]_{s,p_i(x,y),\Omega} \geqslant \sum_{k=1}^{N}[u]_{t,p_k^i,(B_k)}.$

N.B. `Note that the constant` C `may differ from line to line.`

(i)- We have $|u(x)| = \sum_{k=1}^{N}|u(x)|\chi_{B_k}(x)$, then

$$\|u\|_{L^{q(x)}(\Omega)} \leqslant \sum_{k=1}^{N}\|u\|_{L^{q(x)}(B_k)}. \tag{5.7}$$

Since $p_k^{i,*} > q(x)$ for all $x\in B_k$, we can take $\alpha_k^i \in C^+(B_k)$ such that $\frac{1}{q(x)} = \frac{1}{p_k^{i,*}} + \frac{1}{\alpha_k^i(x)}$ for all $x\in B_k$. From Hölder's inequality, we acquire

$$\begin{aligned}\|u\|_{L^{q(x)}(B_k)} &\leqslant C\|u\|_{L^{p_k^{i,*}(x)}(B_k)}\|\chi_{B_k}\|_{L^{\alpha_k^i(x)}(B_k)}\\ &\leqslant C\|u\|_{L^{p_k^{i,*}(x)}(B_k)}.\end{aligned} \tag{5.8}$$

Combining (5.7) and (5.8) to obtain the first inequality.

(ii)- From the fact that $p_M(x) > p_k^i$ for all $x\in B_k$, there exists $\beta_k^i \in C^+(B_k)$ such that $\frac{1}{p_k^i} = \frac{1}{p_M(x)} + \frac{1}{\beta_k^i(x)}$. Following, we proceed as before, we get the desired inequality.

(iii)- We consider $\mathscr{D}(x,y) = \frac{|u(x)-u(y)|}{|x-y|^s}$ and for all $i=1,\dots,N$ we set $d\eta_k(x,y) = \frac{dxdy}{|x-y|^{N+(t-s)p_k^i}}$, where η_k represents a measure in $B_k\times B_k$ with $\eta_k(B_k\times B_k)<\infty$. We have

$$\begin{aligned}[u]_{t,p_k^i,B_k} &= \left(\int_{B_k}\int_{B_k}\frac{|u(x)-u(y)|^{p_k^i}}{|x-y|^{N+tp_k^i}}\,\mathrm{d}x\,\mathrm{d}y\right)^{\frac{1}{p_k^i}}\\ &= \left(\int_{B_k}\int_{B_k}\left(\frac{|u(x)-u(y)|}{|x-y|^s}\right)^{p_k^i}\frac{\mathrm{d}x\,\mathrm{d}y}{|x-y|^{N+(t-s)p_k^i}}\right)^{\frac{1}{p_k^i}}\\ &= \|\mathscr{D}\|_{L^{p_k^i}(\eta_k,B_k\times B_k)}.\end{aligned}$$

Using Hölder's inequality, we get

$$\|\mathscr{D}\|_{L^{p_k^i}(\eta^k,B_k\times B_k)} \leqslant C\|\mathscr{D}\|_{L^{p_i(x,y)}(\eta_k,B_k\times B_k)}.$$

For $i=1,\dots,N$, take $\lambda_i>0$ such that

$$\int_{B_k}\int_{B_k}\frac{|u(x)-u(y)|^{p_i(x,y)}}{\lambda_i^{p_i(x,y)}|x-y|^{N+tp_i(x,y)}}\,\mathrm{d}x\,\mathrm{d}y<1,$$

and choose $m:=\max\{1,\sup_{(x,y)\in\Omega\times\Omega}|x-y|^{s-t}\}$ and $\tilde{\lambda}_i=m\lambda_i$. Therefore, we have

$$\begin{aligned}\int_{B_k}\int_{B_k}\left(\frac{|u(x)-u(y)|}{\tilde{\lambda}_i|x-y|^s}\right)^{p_i(x,y)}\mathrm{d}\eta_k(x,y)&=\int_{B_k}\int_{B_k}\left(\frac{|u(x)-u(y)|}{\tilde{\lambda}_i|x-y|^s}\right)^{p_i(x,y)}\frac{\mathrm{d}x\mathrm{d}y}{|x-y|^{N+(t-s)p_i^k}}\\&=\int_{B_k}\int_{B_k}\frac{|u(x)-u(y)|^{p_i(x,y)}}{(\lambda_i)^{p_i(x,y)}|x-y|^{N+sp_i(x,y)}}\frac{|x-y|^{(s-t)p_i^k}}{m^{p_i(x,y)}}\mathrm{d}x\mathrm{d}y\\&\leqslant\int_{B_k}\int_{B_k}\frac{|u(x)-u(y)|^{p_i(x,y)}}{(\lambda_i)^{p_i(x,y)}|x-y|^{N+sp_i(x,y)}}\mathrm{d}x\,\mathrm{d}y<1.\end{aligned}$$

Then, we get

$$\|\mathscr{D}\|_{L^{p_i(x,y)}(\eta_k,B_k\times B_k)}<\tilde{\lambda}_i.$$

Consequently,

$$\|\mathscr{D}\|_{L^{p_i(x,y)}(\eta_k,B_k\times B_k)}\leqslant m[u]_{s,p_k^i,\Omega}\implies[u]_{t,p_k^i,B_k}\leqslant C[u]_{s,p_k^i,\Omega}.$$

Now, from inequalities (i)-(iii), we infer

$$\begin{aligned}\|u\|_{L^{q(x)}(\Omega)}&\leqslant C\sum_{k=1}^{N}\|u\|_{L^{p_k^{i,*}}(B_k)}\\&\leqslant C\sum_{k=1}^{N}\left(\|u\|_{L^{p_k^i}(B_k)}+[u]^{t,p_k^i}(B_k)\right)\\&\leqslant C\left(\|u\|_{L^{p_M(x)}(\Omega)}+[u]^{s,p_i(x,y)}(\Omega)\right)\\&\leqslant C\left(\|u\|_{L^{p_M(x)}(\Omega)}+\sum_{k=1}^{N}[u]^{s,p_i(x,y)}(\Omega)\right)\\&=C\|u\|_{s,\vec{p}(x,y)}.\end{aligned}\tag{5.9}$$

Next, we investigate the compactness of the embedding between $W^{s,p_M(x),\vec{p}(x,y)}(\Omega)$ and $L^{q(x)}(\Omega)$. Let $\{u_n\}$ be a bounded sequence on $W^{s,p_M(x),\vec{p}(x,y)}(\Omega)$, that is there exists $M>0$ such that $\|u_n\|_{s,\vec{p}(x,y)}\leqslant M$. It follows that $[u_n]_{t,p_k^i}(B_k)\leqslant M$, and by the compact embedding in the constant case, we can find a subsequence of $\{u_n\}$ such that for $k=1,\dots,N$, $\{u_n^k\}$ converges strongly in $L^{p_k^{i,*}-\frac{\varepsilon}{3}}(B_k)$.

Since $|u(x)|=\sum_{k=1}^{N}|u(x)|\chi_{B_k}(x)$, therefore

$$\|u_n^k-u\|_{L^{q(x)}(\Omega)}\leqslant C\sum_{k=1}^{N}\|u_n^k-u\|_{L^{p_k^{i,*}}(B_k)}.$$

As a consequence, we obtain the strong convergence in $L^{q(x)}(\Omega)$. Using the same procedures as the previous proof we get the desired result. ■

Theorem 5.3. *Consider a smooth bounded domain Ω in $\mathbb{R}^N$ ($N \geqslant 2$), and let $\vec{p}:\overline{Q}^N \longrightarrow (1,+\infty)$ be a vector function on which each element of it satisfies conditions (1.2) and (1.3). Moreover, assume that $sp_i(x,y) < N$ for all $(x,y) \in \overline{Q}$ and $\forall i = 1,\ldots,N$. Then, we have*

- (i)- *$W^{s,p_M(x),\vec{p}(x,y)}(\Omega)$ is continuously embedded in Z.*
- (ii)- *The results of Theorem 5.1 hold true for the space Z and Z_0.*

Proof. (i)- Let $u \in Z$, since $\Omega \times \Omega \subsetneq Q$, then for $i = 1,\ldots,N$ and for some $\mu_i > 0$

$$\int_{\Omega\times\Omega} \frac{|u(x)-u(y)|^{p_i(x,y)}}{\mu_i^{p_i(x,y)}|x-y|^{sp_i(x,y)+N}}\,\mathrm{d}x\,\mathrm{d}y \leqslant \int_{Q} \frac{|u(x)-u(y)|^{p_i(x,y)}}{\mu_i^{p_i(x,y)}|x-y|^{sp_i(x,y)+N}}\,\mathrm{d}x\,\mathrm{d}y.$$

As a consequence, we get for $i = 1,\ldots,N$

$$\left\{\mu_i>0 : \int_{Q} \frac{|u(x)-u(y)|^{p_i(x,y)}}{\mu_i^{p_i(x,y)}|x-y|^{sp_i(x,y)+N}}\,\mathrm{d}x\,\mathrm{d}y \leqslant 1\right\} \subset \left\{\mu_i>0 : \int_{\Omega\times\Omega} \frac{|u(x)-u(y)|^{p_i(x,y)}}{\mu_i^{p_i(x,y)}|x-y|^{sp_i(x,y)+N}}\,\mathrm{d}x\,\mathrm{d}y \leqslant 1\right\}.$$

Which implies that

$$\begin{aligned}[u]_{s,\vec{p}(x,y),(\Omega)} &= \sum_{i=1}^{N} \inf\left\{\mu_i>0 : \iint_{\Omega\times\Omega} \frac{|u(x)-u(y)|^{p_i(x,y)}}{\mu_i^{p_i(x,y)}|x-y|^{sp_i(x,y)+N}} < 1\right\} \\ &\leqslant \sum_{i=1}^{N} \inf\left\{\mu_i>0 : \int_{Q} \frac{|u(x)-u(y)|^{p_i(x,y)}}{\mu_i^{p_i(x,y)}|x-y|^{sp_i(x,y)+N}} < 1\right\} = [u]_{s,\vec{p}(x,y),(Q)}.\end{aligned}$$

Therefore, $\|u\|_{W^{s,p_M(x),\vec{p}(x,y)}(\Omega)} \leqslant \|u\|_Z$ which proves the desired result.

(ii)- Since $W^{s,p_M(x),\vec{p}(x,y)}(\Omega)$ is continuously embedded in Z, it is evident that the results of Theorem 5.1 can be extended to the space Z and Z_0.

■

5.4 Basic properties of the fractional $\vec{p}(x,\cdot)$-Laplacian functionals

Let us introduce the following mappings which are crucial to deal with the fractional $\vec{p}(x,.)$-Laplacian equations. For any $u \in Z$, we consider

$$L_{\vec{p}(x,y)}(u) = \sum_{i=1}^{N} \int_{Q} \frac{1}{p_i(x,y)} \frac{|u(x)-u(y)|^{p_i(x,y)}}{|x-y|^{sp_i(x,y)+N}} dxdy + \int_{\Omega} \frac{1}{p_M(x)} |u(x)|^{p_M(x)} dx.$$

In addition, for all $u,v \in Z$, we define

$$\left\langle \mathscr{L}_{\vec{p}(x,y)}(u),v\right\rangle = \sum_{i=1}^{N}\int_{Q}\frac{|u(x)-u(y)|^{p_i(x,y)-2}(u(x)-u(y))(v(x)-v(y))}{|x-y|^{sp_i(x,y)+N}}dxdy + \int_{\Omega}|u(x)|^{p_M(x)-2}u(x)v(x)dx.$$

As well as, for all $u,v \in Z_0$, we consider $I_{\vec{p}(.,.)}$, and $\mathscr{I}_{\vec{p}(.,.)}$ defined in this way

$$I_{\vec{p}(x,y)}(u) = \sum_{i=1}^{N}\int_{Q}\frac{1}{p_i(x,y)}\frac{|u(x)-u(y)|^{p_i(x,y)}}{|x-y|^{sp_i(x,y)+N}}dxdy,$$

$$\left\langle \mathscr{I}_{\vec{p}(x,y)}(u),v\right\rangle = \sum_{i=1}^{N}\int_{Q}\frac{|u(x)-u(y)|^{p_i(x,y)-2}(u(x)-u(y))(v(x)-v(y))}{|x-y|^{sp_i(x,y)+N}}dxdy.$$

Lemma 5.2. *Suppose that hypotheses (1.2) and (1.3) are satisfied for each element of the vector $\vec{p}$, then the following assertions hold true*

(i)- *$L_{\vec{p}(x,y)}$ and $I_{\vec{p}(x,.)}$ are well defined on Z and Z_0 respectively.*

(ii)- *$L_{\vec{p}(x,y)} \in \mathscr{C}^1(Z,\mathbb{R})$ and for all $u,v \in Z$, we have $\left\langle L'_{\vec{p}(x,y)}(u),v\right\rangle = \left\langle \mathscr{L}_{\vec{p}(x,y)}(u),v\right\rangle$.*

(iii)- *$I_{\vec{p}(x,y)} \in \mathscr{C}^1(Z_0,\mathbb{R})$ and for all $u,v \in Z_0$, we have $\left\langle I'_{\vec{p}(x,y)}(u),v\right\rangle = \left\langle \mathscr{I}_{\vec{p}(x,y)}(u),v\right\rangle$.*

Proof. (i)- Let $u \in Z$, then by using Remark 5.2 and Lemma 5.1, we get

$$\begin{aligned} L_{\vec{p}(x,y)}(u) &= \sum_{i=1}^{N}\int_{Q}\frac{1}{p_i(x,y)}\frac{|u(x)-u(y)|^{p_i(x,y)}}{|x-y|^{sp_i(x,y)+N}}dxdy + \int_{\Omega}\frac{1}{p_M(x)}|u(x)|^{p_M(x)}dx \\ &\leqslant \frac{1}{P_-^-}\sum_{i=1}^{N}\int_{Q}\frac{|u(x)-u(y)|^{p_i(x,y)}}{|x-y|^{sp_i(x,y)+N}}dxdy + \frac{1}{p_M^-}\int_{\Omega}|u(x)|^{p_M(x)}dx \\ &\leqslant \max\left\{\tfrac{1}{P_-^-},\tfrac{1}{p_M^-}\right\}\rho_{s,\vec{p}(x,y)}(w) \\ &\leqslant \max\left\{\tfrac{1}{P_-^-},\tfrac{1}{p_M^-}\right\}\|u\|_X^{P_+^+} < +\infty. \end{aligned}$$

Consequently, $L_{\vec{p}(x,y)}$ is well defined on Z. By similar argument, $I_{\vec{p}(x,y)}$ is also well defined on Z_0.

(ii) and (iii)- Standard reasoning as in [20] yields the second and third results.

■

Lemma 5.3. *Suppose that hypotheses (1.2) and (1.3) are satisfied for each element of the vector $\vec{p}$. Hence, the following properties hold true*

(i)- *The operator $\mathscr{I}_{\vec{p}(.,.)}$ is strictly monotone and bounded.*

(ii)- *The operator $\mathscr{I}_{\vec{p}(.,.)}$ is of type (S_+).*

Proof. (i)- For any $u,v \in Z_0$ with $u \neq v$, we have by using Cauchy's inequality

$$\begin{aligned}\left\langle \mathscr{I}_{\vec{p}(x,y)}(u)-\mathscr{I}_{\vec{p}(x,y)}(v),u-v\right\rangle &= \left\langle \mathscr{I}_{\vec{p}(x,y)}(u),u\right\rangle - \left\langle \mathscr{I}_{\vec{p}(x,y)}(u),v\right\rangle - \left\langle \mathscr{I}_{\vec{p}(x,y)}(v),u\right\rangle + \left\langle \mathscr{I}_{\vec{p}(x,y)}(v),v\right\rangle \\ &\geqslant \frac{1}{2}\sum_{i=1}^{N}\int_Q \frac{|u(x)-u(y)|^{p_i(x,y)-2}\left(|u(x)-u(y)|^2-|v(x)-v(y)|^2\right)}{|x-y|^{sp_i(x,y)+N}}dxdy \\ &\quad -\frac{1}{2}\sum_{i=1}^{N}\int_Q \frac{|v(x)-v(y)|^{p_i(x,y)-2}\left(|u(x)-u(y)|^2-|v(x)-v(y)|^2\right)}{|x-y|^{sp_i(x,y)+N}}dxdy.\end{aligned}$$

By changing the role of u and v and applying the Simon's inequality [170], we get

$$\begin{aligned}&\left\langle \mathscr{I}_{\vec{p}(x,y)}(u)-\mathscr{I}_{\vec{p}(x,y)}(v),u-v\right\rangle \\ &\geqslant \frac{1}{2}\sum_{i=1}^{N}\int_Q \frac{\left(|u(x)-u(y)|^{p_i(x,y)-2}-|v(x)-v(y)|^{p_i(x,y)-2}\right)\left(|u(x)-u(y)|^2-|v(x)-v(y)|^2\right)}{|x-y|^{sp_i(x,y)+N}}dxdy>0.\end{aligned}$$

Which implies that $\mathscr{I}_{\vec{p}(.,.)}$ is strictly monotone. It is self-evident to demonstrate the boundedness of $\mathscr{I}_{\vec{p}(.,.)}$.

- (ii)- Consider a sequence $\{u_n\}$ such that $u_n \rightharpoonup u$ in Z_0 and $\limsup_{n\to+\infty}\left\langle \mathscr{I}_{\vec{p}(x,y)}(u_n)-\mathscr{I}_{\vec{p}(x,y)}(u),u_n-u\right\rangle \leqslant 0$. From the strict monotony of $\mathscr{I}_{\vec{p}(.,.)}$, we obtain that

$$\limsup_{n\to+\infty}\left\langle \mathscr{I}_{\vec{p}(x,y)}(u_n)-\mathscr{I}_{\vec{p}(x,y)}(u),u_n-u\right\rangle = 0. \tag{5.10}$$

In the view of Theorem 5.2, $u_n(x) \longrightarrow u(x)$ a.e. in Ω. By using Fatou's Lemma, we have

$$\liminf_{n\to+\infty}\sum_{i=1}^{N}\int_Q \frac{|u_n(x)-u_n(y)|^{p_i(x,y)}}{|x-y|^{sp_i(x,y)+N}}dxdy \geqslant \sum_{i=1}^{N}\int_Q \frac{|u(x)-u(y)|^{p_i(x,y)}}{|x-y|^{sp_i(x,y)+N}}dxdy. \tag{5.11}$$

Furthermore, Young's inequality implies the existence of two constants C_1 and C_2 such that

(5.12)

$$\begin{aligned}\left\langle \mathscr{I}_{\vec{p}(x,y)}(u_n),u_n-u\right\rangle &= \left\langle \mathscr{I}_{\vec{p}(x,y)}(u_n),u_n\right\rangle - \left\langle \mathscr{I}_{\vec{p}(x,y)}(u_n),u\right\rangle \\ &\geqslant \sum_{i=1}^{N}\int_Q \frac{|u_n(x)-u_n(y)|^{p_i(x,y)}}{|x-y|^{sp_i(x,y)+N}}dxdy \\ &\quad -\sum_{i=1}^{N}\int_Q \left(\frac{p_i(x,y)-1}{p_i(x,y)}\frac{|u_n(x)-u_n(y)|^{p_i(x,y)}}{|x-y|^{sp_i(x,y)+N}}+\frac{1}{p_i(x,y)}\frac{u(x)-u(y)|^{p_i(x,y)}}{|x-y|^{sp_i(x,y)+N}}\right)dxdy \\ &\geqslant C_1\sum_{i=1}^{N}\int_Q \frac{|u_n(x)-u_n(y)|^{p_i(x,y)}}{|x-y|^{sp_i(x,y)+N}}dxdy - C_2\sum_{i=1}^{N}\iint_Q \frac{|u(x)-u(y)|^{p_i(x,y)}}{|x-y|^{sp_i(x,y)+N}}dxdy.\end{aligned}$$

Combining (5.11) and (5.12), we obtain

$$\lim_{n\to+\infty}\sum_{i=1}^{N}\int_Q \frac{|u_n(x)-u_n(y)|^{p_i(x,y)}}{|x-y|^{sp_i(x,y)+N}}dxdy=\sum_{i=1}^{N}\int_Q \frac{|u(x)-u(y)|^{p_i(x,y)}}{|x-y|^{sp_i(x,y)+N}}dxdy. \tag{5.13}$$

According to [143], and the proof of [16, Lemma 3.1], we obtain the strong convergence of u_n to u in Z_0.

■

5.5 An application to anisotropic nonlocal problems

As an application of this framework, we consider a class of fractional anisotropic problems with variable exponent of the following form

$$\left(\vec{\mathscr{P}}\right)\begin{cases}(-\Delta)^{s}_{\vec{p}(x,.)}(u)+|u|^{p_M(x)-2}u=\lambda f(x,u) & \text{in } \Omega,\\ u=0 & \text{on } \mathbb{R}^N\setminus\Omega.\end{cases}$$

Where

- $\lambda>0$ is a real parameter.
- $f:\Omega\times\mathbb{R}\to\mathbb{R}$ is a Carathéodory function and, throughout this section, is defined as $f(x,u)=|u|^{r(x)-2}u$, with

$$1<r^-\leqslant r(x)\leqslant r^+<P^-_-\leqslant p^+_M<+\infty \quad \forall x\in\Omega.$$

A function $u\in Z_0$ is defined as a weak solution of problem $\left(\vec{\mathscr{P}}\right)$ if

$$\begin{aligned}&\sum_{i=1}^{N}\int_Q \frac{|u(x)-u(y)|^{p_i(x,y)-2}(u(x)-u(y))(v(x)-v(y))}{|x-y|^{sp_i(x,y)+N}}dxdy\\ &+\int_\Omega |u(x)|^{p_M(x)-2}v(x)\varphi(x)dx-\lambda\int_\Omega |u(x)|^{r(x)-2}v(x)\varphi(x)dx=0, \quad \forall\varphi\in Z_0.\end{aligned} \tag{5.14}$$

For all $u\in Z_0$, the energy functional related to problem $\left(\vec{\mathscr{P}}\right)$ is of the following form

$$\begin{aligned}\mathscr{J}_\lambda(u)&=L_{\vec{p}(x,y)}(u)-\lambda\int_\Omega\frac{1}{r(x)}|u(x)|^{r(x)}dx\\ &=\sum_{i=1}^{N}\int_Q\frac{1}{p_i(x,y)}\frac{|u(x)-u(y)|^{p_i(x,y)}}{|x-y|^{sp_i(x,y)+N}}dxdy+\int_\Omega\frac{|u(x)|^{p_M(x)}}{p_M(x)}dx-\lambda\int_\Omega\frac{|u(x)|^{r(x)}}{r(x)}dx.\end{aligned}$$

By a standard argument as in Lemma 5.2, we have that $\mathscr{J}_\lambda \in C^1(Z_0,\mathbb{R})$. Furthermore, its derivative is given by the following formula

$$\begin{aligned}\langle \mathscr{J}'_\lambda(u),v\rangle = &\sum_{i=1}^{N}\int_Q \frac{|u(x)-u(y)|^{p_i(x,y)-2}(u(x)-u(y))(v(x)-v(y))}{|x-y|^{sp_i(x,y)+N}}dxdy\\ &+\int_\Omega |u(x)|^{p_M(x)-2}u(x)v(x)dx-\lambda\int_\Omega |u(x)|^{r(x)-2}u(x)v(x)dx,\end{aligned} \tag{5.15}$$

for all $u,v\in Z_0$. Hence a critical point of $\mathscr{J}_\lambda$ coincides with weak solution of problem $\left(\vec{\mathscr{P}}\right)$.

The result of the existence of solutions of problem $\left(\vec{\mathscr{P}}\right)$ is formulated in the next theorem.

Theorem 5.4. *Let Ω be a smooth bounded domain in $\mathbb{R}^{\mathbb{N}}$ and $s\in(0,1)$. Let $\vec{p}:\overline{Q}^N\longrightarrow(1,+\infty)$ be a continuous vector function on which each element of it satisfies conditions (1.2) and (1.3). Moreover, consider $r:\Omega\longrightarrow(1,+\infty)$ as a continuous variable exponent such that*

$$1<r(x)<P_-^- \quad \text{for all } x\in\overline{\Omega}. \tag{5.16}$$

Then, problem $(\vec{\mathscr{P}})$ possesses a nontrivial weak solution.

To establish the proof of Theorem 5.4, we will apply the mountain pass theorem combined with Ekeland's variational principle. The crucial step in such type of variational methods is to ensure the compactness conditions stated in the following lemma.

Lemma 5.4. *Under the assumptions of Theorem 5.4, the functional $\mathscr{J}_\lambda$ verifies the Palais-Smale condition.*

Proof. Let $\{u_n\}$ be a (PS)-sequence of $\mathscr{J}_\lambda$ on Z_0, which could be expressed mathematically as follows

$$\mathscr{J}_\lambda(u_n)\to c>0 \quad \text{and} \quad \mathscr{J}'_\lambda(u_n)\to 0 \text{ in } Z_0^*. \tag{5.17}$$

Using Lemma 5.1, we have

$$\begin{aligned}\mathscr{J}_\lambda(u_n) &= \sum_{i=1}^{N}\int_Q \frac{1}{p_i(x,y)}\frac{|u_n(x)-u_n(y)|^{p_i(x,y)}}{|x-y|^{sp_i(x,y)+N}}dxdy\\ &\quad+\int_\Omega \frac{1}{p_M(x)}|u_n(x)|^{p_M(x)}dx-\lambda\int_\Omega\frac{1}{r(x)}|u_n(x)|^{r(x)}dx\\ &\geqslant \frac{1}{P_+^+}\sum_{i=1}^{N}\int_Q\frac{|u_n(x)-u_n(y)|^{p_i(x,y)}}{|x-y|^{sp_i(x,y)+N}}dxdy+\frac{1}{p_M^+}\int_\Omega|u_n(x)|^{p_M(x)}dx\\ &\quad-\frac{\lambda}{r^-}\int_\Omega|u_n(x)|^{r(x)}dx\end{aligned}$$

which implies that

$$\begin{aligned}\mathscr{J}_\lambda(u_n) &\geqslant \min\left\{\frac{1}{P_+^+},\frac{1}{p_M^+}\right\}\left(\rho^0_{\vec{p}(x,y)}(u_n)+\rho_{p_M(x)}(u_n)\right)-\frac{\lambda}{r^-}\rho_{r(x)}(u_n)\\ &\geqslant \min\left\{\frac{1}{P_+^+},\frac{1}{p_M^+}\right\}\left(\sum_{i=1}^{N}[u_n]^{p_i^-}_{s,p_i(x,y)}+\|u_n\|^{p_M^-}_{p_M(x)}\right)-\frac{\lambda}{r^-}\|u_n\|^{r^+}_{r(x)}\\ &\geqslant \min\left\{\frac{1}{P_+^+},\frac{1}{p_M^+}\right\}\left(\frac{[u_n]^{P_-^-}_{s,\vec{p}(x,y)}}{N^{P_-^--1}}+\|u_n\|^{p_M^-}_{p_M(x)}\right)-\frac{\lambda}{r^-}\|u_n\|^{r^+}_{r(x)}\\ &\geqslant \min\left\{\frac{1}{P_+^+},\frac{1}{p_M^+}\right\}\frac{1}{N^{P_-^--1}2^{\tilde{P}-1}}\|u_n\|^{\tilde{P}}_X-\frac{\lambda}{r^-}\|u_n\|^{r^+}_{r(x)},\end{aligned}$$

where $\tilde{P}=\min\{P_-^-,P_M^-\}$.
Based on Theorem 5.3, we get that

$$\mathscr{J}_\lambda(u_n)\geqslant \min\left\{\frac{1}{P_+^+},\frac{1}{p_M^+}\right\}\frac{1}{N^{P_-^--1}2^{\tilde{P}-1}}[u_n]^{\tilde{P}}_{s,\vec{p}(x,y)}-\frac{\lambda C^{r^+}}{r^-}[u_n]^{r^+}_{s,\vec{p}(x,y)}. \tag{5.18}$$

Since $\mathscr{J}_\lambda$ is bounded from (5.17), thus $\{u_n\}$ is bounded on Z_0. In the view of the reflexivity of Z_0, we can find a subsequence of $\{u_n\}$ still denoted in the same way such that $u_n \rightharpoonup u$ in Z_0.
From condition (5.16), Z_0 is compactly embedded in $L^{r(x)}(\Omega)$ (resp. in $L^{p_M(x)}(\Omega)$). Therefore, $u_n \longrightarrow u$ in $L^{r(x)}(\Omega)$ (resp. in $L^{p_M(x)}(\Omega)$). Next, we use Hölder's inequality to acquire

$$\int_\Omega |u_n|^{r(x)-2}u_n(u_n-u)dx\leqslant 2\left\||u_n|^{r(x)-1}\right\|_{L^{\hat{r}(x)}(\Omega)}\|u_n-u\|_{L^{r(x)}(\Omega)}. \tag{5.19}$$

Similarly,

$$\int_\Omega |u_n|^{p_M(x)-2}u_n(u_n-u)dx\leqslant 2\left\||u_n|^{p_M(x)-1}\right\|_{L^{\hat{p}_M(x)}(\Omega)}\|u_n-u\|_{L^{p_M(x)}(\Omega)}. \tag{5.20}$$

Hence,

$$\lim_{n\to\infty}\int_\Omega |u_n|^{r(x)-2}u_n(u_n-u)dx=\lim_{n\to\infty}\int_\Omega |u_n|^{p_M(x)-2}u_n(u_n-u)dx=0.$$

Owing to (5.17) and the weak convergence of u_n we have

$$\left\langle \mathscr{J}'_\lambda(u_n),u_n-u\right\rangle\longrightarrow 0 \quad \text{as } n\longrightarrow+\infty.$$

Which means that

$$\begin{aligned}&\sum_{i=1}^{N}\int_Q\frac{|u_n(x)-u_n(y)|^{p_i(x,y)-2}(u_n(x)-u_n(y))((u_n-u)(x)-(u_n-u)(y))}{|x-y|^{sp_i(x,y)+N}}dxdy\\ &+\int_\Omega |u_n(x)|^{p_M(x)-2}u_n(x)(u_n-u)(x)dx-\lambda\int_\Omega |u_n(x)|^{r(x)-2}u_n(x)(u_n-u)(x)dx\underset{n\longrightarrow+\infty}{\longrightarrow}0.\end{aligned}$$

From (5.19) and (5.20), it follows that

$$\sum_{i=1}^{N}\int_{Q}\frac{|u_n(x)-u_n(y)|^{p_i(x,y)-2}(u_n(x)-u_n(y))((u_n-u)(x)-(u_n-u)(y))}{|x-y|^{sp_i(x,y)+N}}dxdy \underset{n\longrightarrow+\infty}{\longrightarrow} 0.$$

On the other hand, it's clear that

$$\sum_{i=1}^{N}\int_{Q}\frac{|u(x)-u(y)|^{p_i(x,y)-2}(u(x)-u(y))((u_n-u)(x)-(u_n-u)(y))}{|x-y|^{sp_i(x,y)+N}}dxdy \underset{n\longrightarrow+\infty}{\longrightarrow} 0.$$

The two previous results show that $\left\langle \mathscr{L}_{\vec{p}(x,y)}(u_n)-\mathscr{L}_{\vec{p}(x,y)}(u_n),u_n-u\right\rangle \underset{n\longrightarrow+\infty}{\longrightarrow} 0$. The fact that $\mathscr{L}_{\vec{p}(.,.)}$ is of type (S_+) implies the strong convergence of u_n to u in Z_0. We deduce that $\mathscr{J}_\lambda$ satisfies the Palais-Smale condition. ■

Now, let us verify that the functional $\mathscr{J}_\lambda$ satisfies the mountain pass geometry.

Lemma 5.5. *Assume that hypothesis of Theorem 5.4 are fulfilled. Then, there exists $\lambda_*>0$ such that for any $\lambda\in(0,\lambda_*)$, there are $\delta,\vartheta>0$, such that $\mathscr{J}_\lambda(w)\geqslant\vartheta>0$ for any $u\in Z_0$ with $\|u\|_{Z_0}=\delta$.*

Proof. From inequality (5.18), we have

$$\mathscr{J}_\lambda(u)\geqslant\tilde{C}[u]^{\tilde{P}}_{s,\vec{p}(x,y)}-\frac{\lambda C^{r^+}}{r^-}[u]^{r^+}_{s,\vec{p}(x,y)},\ \forall u\in Z_0,$$

where $\tilde{C}=\min\left\{\frac{1}{P_+^+},\frac{1}{p_M^+}\right\}\frac{1}{N^{P_-^- -1}2^{\tilde{P}-1}}$. Therefore, for any $u\in Z_0$ with $\|u\|_{Z_0}=\delta$, we obtain

$$\mathscr{J}_\lambda(u)\geqslant\tilde{C}\delta^{\tilde{P}}-\frac{\lambda C^{r^+}}{r^-}\delta^{r^+}=\delta^{r^+}\left(\tilde{C}\delta^{\tilde{P}-r^+}-\frac{\lambda C^{r^+}}{r^-}\right).$$

By defining λ_* in this way $\lambda_*=\frac{\tilde{C}\delta^{\tilde{P}-r^+}r^-}{2C^{r^+}}$, we get as a result that for all $\lambda\in(0,\lambda_*)$ and for any $u\in Z_0$ with $\|u\|_{Z_0}=\delta$ that

$$\mathscr{J}_\lambda(u)\geqslant\vartheta>0\quad\text{with }\vartheta=\frac{\delta}{2}.$$

■

Lemma 5.6. *Assume that the hypotheses of Theorem 5.4 are fulfilled. Then, there exists $\omega\in Z_0$ such that $\omega\geqslant 0,\omega\neq 0$, and $\mathscr{J}_\lambda(t\omega)<0$ for any $t>0$ small enough.*

Proof. Since $r^-<P_-^-$, then there exists $\varepsilon>0$ such that $r^-+\varepsilon\leqslant P_-^-$. Moreover, the continuity of r implies the existence of an open set O of Ω such that

$$|r(x)-r^-|\leqslant\varepsilon\quad\text{for all }\ x\in O. \tag{5.21}$$

Consider ψ as an infinitely differentiable function that is compactly supported such that $\overline{O} \subset supp\psi$, $\psi(x) = 1 \ \ \forall x \in \overline{O}$, and $0 \leqslant \psi \leqslant 1$ in Ω. Therefore, for $u \in Z_0$ and $t > 0$ small enough we have

$$\begin{aligned}
\mathscr{J}_\lambda(tu) &= \sum_{i=1}^{N} \int_Q \frac{1}{p_i(x,y)} \frac{|tu(x) - tu(y)|^{p_i(x,y)}}{|x-y|^{sp_i(x,y)+N}} dxdy + \int_\Omega \frac{1}{p_M(x)} |tu(x)|^{p_M(x)} dx \\
&\quad - \lambda \int_\Omega \frac{1}{r(x)} |tu(x)|^{r(x)} dx \\
&\leqslant \frac{t^{P_-^-}}{P_-^-} \sum_{i=1}^{N} \int_Q \frac{|u(x) - u(y)|^{p_i(x,y)}}{|x-y|^{sp_i(x,y)+N}} dxdy + \frac{t^{p_M^-}}{p_M^-} \int_\Omega |u(x)|^{p_M(x)} dx \\
&\quad - \frac{\lambda}{r^+} \int_\Omega t^{r(x)} |u(x)|^{r(x)} dx, \\
&\leqslant \frac{t^{P_-^-}}{P_-^-} \rho_{s,\vec{p}(x,y)}(u) - \frac{\lambda t^{r^- + \varepsilon}}{r^+} \int_O |u(x)|^{r(x)} dx, \\
&= t^{r^- + \varepsilon} \left(\frac{t^{P_-^- - r^- - \varepsilon}}{P_-^-} \rho_{s,\vec{p}(x,y)}(u) - \frac{\lambda}{r^+} \int_O |u(x)|^{r(x)} dx \right).
\end{aligned}$$

For $t < \theta^{\frac{1}{P_-^- - r^- - \varepsilon}}$, we obtain that $\mathscr{J}_\lambda(tu) < 0$, where

$$0 < \theta < \min \left\{ 1, \frac{\lambda P_-^- \int_O |u(x)|^{r(x)} dx}{r^+ \rho_{s,\vec{p}(x,y)}(u)} \right\}.$$

Next, we point-out that $\rho_{s,\vec{p}(x,y)}(u) > 0$ (this fact implies that $u \neq 0$). Indeed, from this inclusion between sets $O \subset \text{supp}(u) \subset \Omega$ and $0 \leqslant u \leqslant 1$ in Ω, we acquire

$$0 < \int_O |u|^{r(x)} \mathrm{d}x \leqslant \int_\Omega |u|^{r(x)} \mathrm{d}x \leqslant \int_\Omega |u|^{r^-} \mathrm{d}x.$$

Condition 5.16 implies the existence of a constante C such that

$$\|u\|_{L^{r^-}(\Omega)} \leqslant C \|u\|_{Z_0}.$$

The two previous inequalities inferred to

$$\rho_{s,\vec{p}(x,y)}(u) > 0.$$

As a result, the proof of Lemma 5.6 is achieved. ■

Now, we are in the position to prove our existence result.

Proof. Theorem 5.4: From Lemma 5.5, we have

$$\inf \mathscr{J}_\lambda > 0 \quad \text{in } \partial \mathscr{B}_m(0), \tag{5.22}$$

and from Lemma 5.6, we get

$$\mathcal{J}_\lambda(u) \geqslant \tilde{C}[u]^{\vec{p}}_{s,\vec{p}(x,y)} - \frac{\lambda C^{r^+}}{r^-}[u]^{r^+}_{s,\vec{p}(x,y)} \ \forall u \in \mathcal{B}_m(0), \tag{5.23}$$

which implies that $-\infty < c = \inf\limits_{\overline{\mathcal{B}_m(0)}} \mathcal{J}_\lambda < 0$.

Consider $\varepsilon > 0$ such that

$$0 < \varepsilon < \inf_{\partial\mathcal{B}_m(0)} \mathcal{J}_\lambda - \inf_{\mathcal{B}_m(0)} \mathcal{J}_\lambda,$$

then we apply Ekelnad's variational principle to the functional $\mathcal{J}_\lambda : \overline{\mathcal{B}_m(0)} \longrightarrow \mathbb{R}$, we get $u_\varepsilon \in \overline{\mathcal{B}_m(0)}$ such that

$$\begin{cases} \mathcal{J}_\lambda(u_\varepsilon) \leqslant \inf_{\overline{\mathcal{B}_m(0)}} \mathcal{J}_\lambda + \varepsilon, \\ \mathcal{J}_\lambda(u_\varepsilon) < \mathcal{J}_\lambda(u) + \varepsilon \|u - u_\varepsilon\|_{Z_0}, \quad \text{for } u \neq u_\varepsilon, u_\varepsilon \in \mathcal{B}_m(0). \end{cases}$$

Which acquires that

$$\mathcal{J}_\lambda(u_\varepsilon) \leqslant \inf_{\mathcal{B}_m(0)} \mathcal{J}_\lambda + \varepsilon \leqslant \inf_{\mathcal{B}_m(0)} \mathcal{J}_\lambda + \varepsilon < \inf_{\partial\mathcal{B}_m(0)} \mathcal{J}_\lambda.$$

In the following, we consider the mapping

$$\begin{aligned} \mathcal{I}^\varepsilon_\lambda : \overline{\mathcal{B}_m(0)} &\longrightarrow \mathbb{R} \\ u &\longrightarrow \mathcal{J}_\lambda(u) + \varepsilon \|u - u_\varepsilon\|_{Z_0}. \end{aligned}$$

It is clear that $\mathcal{I}^\varepsilon_\lambda(u_\varepsilon) = \mathcal{J}_\lambda(u_\varepsilon) < \mathcal{J}_\lambda(u)$, for $u \neq w_\varepsilon$. Hence, u_ε is a minimum of $\mathcal{I}^\varepsilon_\lambda$. Which implies that for any $t \in (0,1)$ and $v \in \mathcal{B}_m$, we have

$$\frac{\mathcal{I}^\varepsilon_\lambda(u_\varepsilon + tv) - \mathcal{I}^\varepsilon_\lambda(u_\varepsilon)}{t} \geqslant 0 \Rightarrow \frac{\mathcal{J}_\lambda(u_\varepsilon + tv) - \mathcal{J}_\lambda(u_\varepsilon)}{t} + \varepsilon\|v\|_{Z_0} \geqslant 0,$$

as t tends to 0^+, we acquire $< \mathcal{J}'_\lambda(u_\varepsilon), v > + \varepsilon\|v\|_{Z_0} \geqslant 0$.

As a consequence,

$$\|\mathcal{J}'_\lambda(u_\varepsilon)\|_{Z_0^*} \leqslant \varepsilon,$$

we infer that there exists a sequence $u_n \in \mathcal{B}_m$ such that

$$\mathcal{J}_\lambda(u_n) \to c > 0 \quad \text{and} \quad \mathcal{J}'_\lambda(u_n) \to 0.$$

$\mathcal{J}_\lambda$ verifies the PS condition, then we can find $u_0 \in Z_0$ such that $u_n \longrightarrow u_0$ as $n \longrightarrow +\infty$. as a result, $\mathcal{J}_\lambda(u_0) = c$ and $\mathcal{J}'_\lambda(u_0) = 0$. In conclusion, u_0 is a critical point of $\mathcal{J}_\lambda$. Which is summed up by the achievement of the proof. ■

Part III

Applications to fractional epidemiological models

INTRODUCTION AND MOTIVATION

Infectious disease outbreaks and public health concerns are serious challenges that have a big impact on society, they have a variety of effects on both personal well-being and agricultural systems. On the agricultural side, infectious diseases harm crop productivity and food security. Plant pathogens, which include bacteria, fungi, viruses, and nematodes, pose a major danger to agricultural output. For instance, Late blight, caused by Phytophthora infestants, can lead to black, water-soaked lesions and quick crop failure in potatoes and tomatoes, as observed during the Irish Potato Famine [14, 144]. Rust infections result in reddish, orange, or brown pustules on leaves and stems, leading to significant yield losses and quality reduction [129]. Additionally, mosaic diseases are caused by various viruses such as the Tobacco Mosaic viruses [164] and Cassava Mosaic Disease (CMD) [127], which also lowers crop quality and yield to causing distinctive mottling and distortion in leaves. These examples highlight the severe effects that plant pathogens can have on agriculture. However, it is important to recognize that many other diseases affect various crops and plants. Sustaining agricultural productivity and food security involves a combination of proactive measures, efficient management techniques, and ongoing research.

In contrast to infectious diseases, public health issues have serious consequences. For example, a multitude of health problems, such as impaired immunity, increased susceptibility to illnesses, and stunted growth in children, can arise from Malnutrition, which is caused by inadequate or unbalanced nutrition [163]. Harmful substance use, such as alcohol and drugs, is the hallmark of addiction and can result in serious health difficulties, social troubles, and financial hardships. To enhance public health outcomes and quality of life, addressing these challenges involves a multidimensional approach. This necessitates the implementation of integrated strategies that not only focus on prevention and treatment but also emphasize education, policy reforms, and social support systems.

The study of disease dynamics in both human populations and agricultural systems has long relied on mathematical models [51, 168] to predict and control the spread of infectious diseases. Since then, researchers have advanced numerous models to dissect the

intricate behavior of infectious diseases, underpinned by critical assumptions, variables, and disease-specific parameters [140]. These models, coupled with simulations, serve as powerful instruments for theoretical refinement, parameter estimation, and sensitivity analysis. Epidemic modeling dates back to 1760 when Daniel Bernoulli developed the first mathematical model to evaluate the effectiveness of smallpox variolation [51]. William Farr later used statistical methods to model smallpox deaths, while John Brownlee introduced Pearson's distribution to analyze similar data. Notable advances came from Hamer, who introduced the "mass action principle" for measles, and Ronald Ross, who explored malaria transmission. The pivotal Kermack-McKendrick model explained epidemic thresholds, while the Reed-Frost model offered an alternative approach based on contact probability. Over time, models evolved to incorporate more epidemiologic details and were applied to various diseases, offering insights into transmission dynamics and the impact of interventions such as vaccination and treatment.

Nevertheless, traditional models, based on integer-order calculus, often struggle to capture the full complexity of these systems. Diseases are influenced by a range of factors, including long-term memory effects, environmental changes, human behavior, and intervention strategies that classical models tend to oversimplify or overlook. Fractional calculus has emerged as a powerful tool to address these complexities, offering a more nuanced and flexible approach to disease modeling. Fractional-order models have been the subject of extensive study in the literature. To explore these models in more detail, see the references provided [106, 112, 141, 153, 171, 179]. Fractional calculus excels in systems where processes exhibit memory and hereditary properties, permitting it to better reflect real-world phenomena such as the long-term impact of public health interventions or the delayed effects of agricultural practices. This part of the thesis explores the application of fractional calculus in two key areas: the spread of substance abuse among women and the dynamics of plant diseases, with a particular focus on Cassava Mosaic Disease (CMD). By uniting these two fields, this work demonstrates how fractional models can provide novel insights into both public health challenges and agricultural disease management.

The aim is to show the versatility of fractional calculus across both fields by presenting case studies, theoretical developments, and simulations. By combining insights from epidemiology and agricultural disease modeling, this work pursues to show how fractional calculus can serve as a transformative tool for tackling some of the most pressing challenges in both public health and agricultural sustainability.

CHAPTER 6

NOVEL INSIGHTS INTO CASSAVA MOSAIC DISEASE USING CAPUTO FRACTIONAL DERIVATIVE: MODELING AND ANALYSIS

Main objective

This chapter aims to introduce a fractional-order model for Cassava Mosaic Disease using the Caputo derivative to capture memory effects in disease transmission. Key contributions include proving the model's well-posedness through the existence, uniqueness, positivity, and boundedness of solutions, ensuring biological relevance. Furthermore, the basic reproduction number R_0 is derived using the next-generation matrix, followed by stability analysis of equilibrium points via the Routh-Hurwitz criterion. Sensitivity analysis identifies critical parameters affecting CMD spread, guiding targeted interventions. Numerical simulations demonstrate the effect of fractional derivatives on CMD dynamics, reinforcing theoretical results and suggesting new control strategies.

The results presented in this chapter are based on the findings discussed in [32].

6.1 Contextual background on CMD

Currently, food security for a fast-rising population is a daunting challenge, particularly after the outbreak of the COVID-19 pandemic. One crop essential to millions's nutrition is cassava, a resistant and adaptable staple crop known for its ability to thrive in poor soils and survive adverse climatic conditions [69]. The global cassava production reached 311.5 million tons in 2022, and sub-Saharan Africa is the world's largest cassava-growing region. Furthermore, this versatile crop provides nutritional value because the manioc is considered a good source of carbohydrates [134]. Furthermore, cassava offers a range of economic opportunities and it has various industrial applications such as starch and biofuel production, textile, paper, and board industry [121].

However, an insidious threat looms over this essential crop in the form of Cassava Mosaic Disease. Originally, the record of CMD dates back to 1894 in Tanzania on the piedmont of Usambara mountains [138]. CMD is caused by several species of whitefly-transmitted geminiviruses belonging to the genus Begomovirus. Distinct mosaic patterns on infected cassava plant leaves identify the disease. The leaves exhibit a mottled appearance with light and dark green areas, frequently accompanied by curling, distortion, and reduction in size [127]. Over and above that, CMD is characterized by chlorosis, which results in leaf yellowing due to a reduction in chlorophyll concentration. More specifically, chlorophyll absorbs solar light energy and uses it in the photosynthetic process, although whitefly vectors partially inhibit this action, destroying the ecosystem stability of cassava plants.

Numerous mathematical models have been developed to deal with CMD. In [98] the authors proposed a compartmental model describing the dynamics of Cassava Mosaic Disease. Based on mathematical analysis, they investigated the application of the traditional method, which involved using infected cutting tools and removing infected cassava plants to reduce the influence of the disease's occurrence. Kinene et al. [108] proposed an optimal model for the dynamics of CMD and investigated whether uprooting and burning infected cassava plants is the most cost-effective technique for combating the disease. Furthermore, a mathematical model was developed in the work of Blyuss et al. [47] to assess the control of CMD through the application of RNAi-mediated bio-protective mechanism involving Microbubbles (MBs). Another approach has been explored in the work [6] based on including the maturation period of the whitefly population in the proposed model. For further study of CMD in the case of integer order derivative, we refer to [83, 99, 102, 160].

To the best of our knowledge, there appears to be a shortage of relevant literature on the study of CMD models described by fractional derivatives. Recently, an optimal control problem of CMD is presented in [182], this model involves the Caputo derivative operator. The authors showed that the effective control that could be used to stop the transmission of the disease is roughing in the availability of microbial biostimulants. Motivated by prior studies, this chapter introduces a novel $S-I$ model for cassava plants, accounting for population growth rates and the impact of insecticide on disease transmission via whitefly vectors. The model leverages fractional calculus to capture the complexities of CMD dynamics and chemical intervention effects.

6.2 Model formulation

In this section, we consider a new fractional model for the dynamics of the cassava mosaic disease by considering an $SI - S_w I_w$ model type of cassava and whitefly vector populations.

- The total population of cassava plants $N_1(t) = S(t) + I(t)$, put into susceptible cassava subpopulation $S(t)$ and infected cassava subpopulation $I(t)$. Similarly, the whitefly vector population is categorized as susceptible whitefly $S_w(t)$ and infected whitefly $I_w(t)$, which is denoted by N_2.

- If there is no disease and no predator population of whiteflies, the cassava plants grow logistically with intrinsic growth rate r, and environmental carrying capacity K ($K > 0$). In the presence of the disease, susceptible cassava plants become infected after coming into touch with an infected whitefly at a per capita rate α_1.

- Susceptible whitefly vectors are recruited by birth into the environment at a rate θ, it is anticipated that infectious cassava plants are removed from the garden or harvested and brunt at a rate σ.

- Susceptible whitefly vectors become infected at a rate α_2 after coming into contact with infected cassava plants, C refers to the death ratio of whitefly vectors caused by the insecticide.

- The cassava and whitefly populations die naturally with a rate μ and μ_w respectively.

Table 6.1 contains a list of all the parameters and their descriptions.

Parameter	Description	Value	Source
r	Planting rate	0.025	[98]
K	Environmental capacity	15000	Assumed
α_1	The rate at which the infected whitefly preys on and infects susceptible cassava plants	0.33-0.5	Assumed
α_2	The rate at which the susceptible whitefly becomes infected by predating on infected cassava plants	0.03-0.05	Assumed
μ	Natural death rate of cassava plant	0.003-0.008	[98]
σ	The rate at which infected cassava is being removed	0.003-0.03	[101]
θ	Recruitment rate of whitefly	0.2	[101]
μ_w	Natural death rate of whitefly	0.1	[101]
C	The death ratio of whitefly caused by insecticide	0.08	Assumed

Table 6.1: Parameters for outbreak analysis.

Further, Figure 6.1 depicts the compartmental flow diagram.

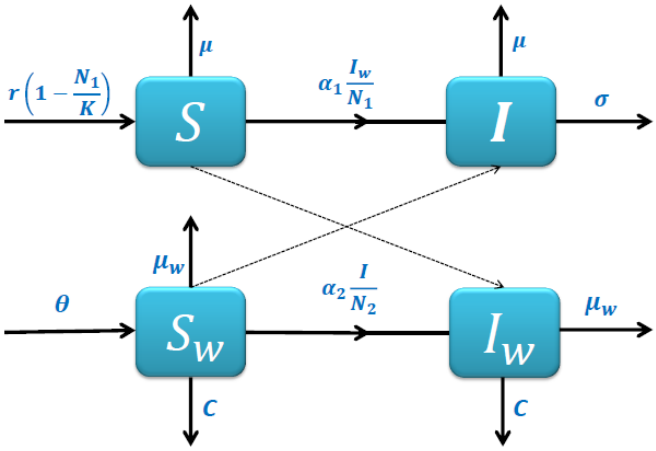


Figure 6.1: Schematic diagram of the CMD model.

Based on the assumptions stated above, the fractional order mathematical model for Cassava Mosaic Disease is represented by the following equations

$$(\mathscr{M})_{CMD}\begin{cases} {}^{C}D_t^{\alpha}S(t) = rS(t)\left(1-\dfrac{N_1(t)}{K}\right) - \alpha_1\dfrac{S(t)I_w(t)}{N_1(t)} - \mu S(t), \\ {}^{C}D_t^{\alpha}I(t) = \alpha_1\dfrac{S(t)I_w(t)}{N_1(t)} - \sigma I(t) - \mu I(t), \\ {}^{C}D_t^{\alpha}S_w(t) = \theta - \alpha_2\dfrac{S_w(t)I(t)}{N_2(t)} - CS_w(t) - \mu_w S_w(t), \\ {}^{C}D_t^{\alpha}I_w(t) = \alpha_2\dfrac{S_w(t)I(t)}{N_2(t)} - CI_w(t) - \mu_w I_w(t), \end{cases}$$

with nonnegative initial conditions

$$S(0) \geqslant 0, \quad I(0) \geqslant 0, \quad S_w(0) \geqslant 0, \quad I_w(0) \geqslant 0.$$

Remark 6.1. *The unit of r, μ, σ, μ_w, and C are per day, the unit of θ and α_1 are vector per day, and the unit of α_2 is m^2 per day.*

6.3 Related qualitative properties of CMD model

This section investigates the fundamental aspects of the fractional CMD model, starting with an analysis of positivity and boundedness of the solution, and then exploring its existence and uniqueness properties.

6.3.1 Positivity and boundedness

Theorem 6.1. *Assume that the initial condition $(S_0, I_0, S_{w0}, I_{w0})$ belongs to $[0, +\infty)$. Then, the solution associated with the model $(\mathscr{M})_{CMD}$ is non-negative.*

Proof. In view of our model, we have

$$\begin{aligned} &{}^{C}D_t^{\alpha}S(t)\Big|_{S=0} = 0 \geqslant 0, \\ &{}^{C}D_t^{\alpha}I(t)\Big|_{I=0} = \alpha_1 I_w(t) \geqslant 0, \\ &{}^{C}D_t^{\alpha}S_w(t)\Big|_{S_w=0} = \theta \geqslant 0, \\ &{}^{C}D_t^{\alpha}I_w(t)\Big|_{I_w=0} = \alpha_2 I(t) \geqslant 0. \end{aligned} \tag{6.1}$$

From Corollary 1.7, it follows that $S(t)$ and $S_w(t)$ are increasing. Thus, $S(t) \geqslant S(0)$ and $S_w(t) \geqslant S_w(0)$ for all $t \geqslant 0$. Since $S(0)$ and $S_w(0)$ are supposed non-negatives, consequently $S(t)$ and $S_w(t)$ remain non-negatives.

Now, we prove that $I(t)$ is non-negative. If not, there exists τ_1 such that $I(t)$ intersects the $I(t)=0$ axis at $t=\tau_1$ for the first time, and

$$\begin{cases} I(t)>0, \text{ for } 0\leqslant t<\tau_1, \\ I(\tau_1)=0, \\ I\left(\tau_1^+\right)<0. \end{cases}$$

From the second equation in system (6.1), we have ${}^{C}D_t^{\alpha}I(t)\big|_{I(\tau_1)=0}=\alpha_1 I_w(\tau_1)$, which yields to

- $I_w(\tau_1)\geqslant 0 \Rightarrow I(\tau_1^+)>0$, contradiction with our assumption, and then I is non-negative.
- $I_w(\tau_1)<0$, this case implies that there exists $\tau_2(0<\tau_2<\tau_1)$ such that

$$\begin{cases} I_w(t)>0, \text{ for } 0\leqslant t<\tau_2, \\ I_w(\tau_2)=0, \\ I_w\left(\tau_2^+\right)<0. \end{cases}$$

  The fourth equation of (6.1) gives that ${}^{C}D_t^{\alpha}I_w(t)\big|_{I_w(\tau_2)=0}=\alpha_2 I(\tau_2)>0$. which contradicts our assumption that $I_w\left(\tau_2^+\right)<0$. Therefore, we have $I_w(t)\geqslant 0\ \forall t\in[0,\infty)$. Consequently, $I(t)\geqslant 0\ \forall t\in[0,\infty)$.

Thus, the solution of the system $(\mathscr{M})_{CMD}$ is non-negative. ■

In the following theorem, we will investigate the boundedness of the solution to the proposed model $(\mathscr{M})_{CMD}$.

Theorem 6.2. *The region expressed as follows*

$$\Omega_{CMD}=\Omega_C\times\Omega_w=\left\{(S,I,S_w,I_w)\in\mathbb{R}_+^4 : N_1(t)\leqslant\frac{C_1(\mu+r)}{\mu} \text{ and } N_2(t)\leq\frac{\theta}{C+\mu_w}\right\},$$

is a positive invariant set for model $(\mathscr{M})_{CMD}$ where $C_1=\max\{S(0),K\}$.

Proof. The total Cassava population is $N_1=S+I$, then

$$\begin{aligned} {}^{C}D_t^{\alpha}N_1 &= rS\left(1-\frac{N_1}{K}\right)-\mu S-\sigma I-\mu I \\ &\leqslant rS-\mu S-\sigma I-\mu I \\ &\leqslant C_1(r+\sigma)-\mu N_1, \end{aligned}$$

where $C_1 = \max\{S(0), K\}$.
In view of Lemma 1.9, we get

$$N_1(t) \leqslant \left(N_1(0) - \frac{C_1(\sigma+r)}{\mu}\right)\mathscr{E}_\alpha(-\mu t^\alpha) + \frac{C_1(\sigma+r)}{\mu}. \tag{6.2}$$

In the process of evaluating the inequality (6.2) at $t \longrightarrow +\infty$ and from using Lemma 1.10, we acquire that

$$N_1(t) \leqslant \frac{C_1(\sigma+r)}{\mu}.$$

Then, we consider the following set

$$\Omega_C = \left\{(S,I) \in \mathbb{R}^2_+ : N_1(t) \leqslant \frac{C_1(\sigma+r)}{\mu}\right\}. \tag{6.3}$$

On the other hand, after differentiation of the whitefly population in the Caputo sense, we get that

$$\begin{aligned} {}^C D_t^\alpha N_2 &= \theta - (C+\mu_w)N_2 \\ &\leqslant \theta - (C+\mu_w)N_2. \end{aligned}$$

By the same analysis as before, we can consider the next set

$$\Omega_w = \left\{(S_w, I_w) \in \mathbb{R}^2_+ : N_2(t) \leqslant \frac{\theta}{(C+\mu_w)}\right\}. \tag{6.4}$$

Appropriately, the feasible region of the solution of the fractional CMD model is given by

$$\Omega_{CMD} = \Omega_C \times \Omega_w = \left\{(S,I,S_w,I_w) \in \mathbb{R}^4_+ : N_1(t) \leqslant \frac{C_1(\mu+r)}{\mu} \quad \text{and } N_2(t) \leqslant \frac{\theta}{C+\mu_w}\right\},$$

this region is positively invariant, and now our model is thought to be mathematically well-posed and epidemiological significant inside Ω_{CMD}. ■

6.3.2 Existence and uniqueness

The results of the existence and uniqueness of the solution of the fractional CMD model are formulated in the following theorem.

Theorem 6.3. *We assume that S_0, I_0, S_{w0}, and I_{w0} are non-negatives, then system ($\mathscr{M}_{CMD}$) possesses a unique solution on $[0, +\infty)$.*

Proof. Consider $X = (S,\ I,\ S_w,\ I_w)^T$, and define the mapping G as follows:

$$G(X) = (G_1,\ G_2,\ G_3,\ G_4)^T,$$

where

$$G_1(t,S) = rS\left(1-\frac{N_1}{K}\right) - \alpha_1\frac{SI_w}{N_1} - \mu S,$$
$$G_2(t,I) = \alpha_1\frac{SI_w}{N_1} - \sigma I - \mu I,$$
$$G_3(t,S_w) = \theta - \alpha_2\frac{S_wI}{N_2} - CS_w - \mu_w S_w,$$
$$G_4(t,I_w) = \alpha_2\frac{S_wI}{N_2} - CI_w - \mu_w I_w.$$

We have

$$\|G_1(t,S)\| = \|rS\left(1-\frac{N_1}{K}\right) - \alpha_1\frac{SI_w}{N_1} - \mu S\|$$
$$\leqslant \frac{C_1(\sigma+r)r}{\mu} + \mu\|S\|,$$

denote $\mathscr{B}_1 = \frac{C_1(\sigma+r)r}{\mu}$ and $\mathscr{C}_1 = \mu$. Therefore,

$$\|G_1(t,S)\| \leqslant \mathscr{B}_1 + \mathscr{C}_1\|S\|.$$

Similarly,

$$\begin{aligned}
\|G_2(t,I)\| &\leqslant \underbrace{\alpha_1\|I_w\|}_{\mathscr{B}_2} + \underbrace{(\mu+\sigma)}_{\mathscr{C}_2}\|I\| \\
\|G_3(t,S_w)\| &\leqslant \underbrace{\theta}_{\mathscr{B}_3} + \underbrace{(C+\mu_w)}_{\mathscr{C}_3}\|S_w\| \\
\|G_4(t,I_w)\| &\leqslant \underbrace{\alpha_2\|I\|}_{\mathscr{B}_4} + \underbrace{(C+\mu_w)}_{\mathscr{C}_4}\|I_w\|.
\end{aligned} \tag{6.5}$$

For $i = 1,2,3,4$, the kernels G_i and $\frac{\partial G_i}{\partial X}$ are continuous in $\mathbb{R}$, as well as $\mathscr{B}_i$ and $\mathscr{C}_i$ are positives. According to Lemma 1.7 we establish the existence and uniqueness of the solution. ■

6.4 Existence and stability of equilibrium points

We now turn to the stability analysis of disease dynamics, focusing on the existence of equilibrium points and the conditions under which those points are stable. Thus, we initiate our analysis by deriving the basic reproduction number R_0 that needs first the identification of the disease-free equilibrium point.

6.4.1 Disease free-equilibrium point

The disease-free equilibrium point $E^0 = (S^0, I^0, S_w^0, I_w^0)$ is distinguished by the absence of infected nodes (i.e. $I^0 = I_w^0 = 0$), and it is determined by setting all the derivatives to zero;

$$\begin{cases} {}^{C}D_t^{\alpha} S^0(t) = rS^0(t)\left(1 - \dfrac{S^0(t)}{K}\right) - \mu S^0(t) = 0, \\ {}^{C}D_t^{\alpha} S_w^0(t) = \theta - CS_w^0(t) - \mu_w S_w^0(t) = 0, \end{cases} \tag{6.6}$$

that yields to $E^0 = \left(\frac{K(r-\mu)}{r}, 0, \frac{\theta}{C+\mu_w}, 0\right)$.

6.4.2 The basic reproduction number R_0

The matrices $\mathbb{F}$ and $\mathbb{V}$ associated with system ($\mathscr{M}_{CMD}$), discussed in 1.5.3, are driven from

$$F = \begin{bmatrix} \alpha_1 \frac{S(t)I_w(t)}{N_1(t)} \\ \alpha_2 \frac{S_w(t)I(t)}{N_2(t)} \end{bmatrix} \quad \text{and } V = \begin{bmatrix} (\sigma+\mu)I(t) \\ (C+\mu_w)I_w(t) \end{bmatrix}.$$

Thus,

$$\mathbb{F} = \begin{bmatrix} 0 & \alpha_1 \\ \alpha_2 & 0 \end{bmatrix} \quad \mathbb{V} = \begin{bmatrix} (\sigma+\mu) & 0 \\ 0 & (C+\mu_w) \end{bmatrix}.$$

Therefore, the basic reproduction number is given by

$$R_0 = \sqrt{\frac{\alpha_1 \alpha_2}{(C+\mu_w)(\sigma+\mu)}}.$$

6.4.3 Local stability of the disease free equilibrium point

The local stability of the disease-free equilibrium is assessed by linearizing the system using the Jacobian matrix at that point. Stability is determined by the signs of the Jacobian's eigenvalues, with E^0 being locally asymptotically stable if all eigenvalues have negative real parts.

Theorem 6.4. *Under the condition that $r > \mu$, the disease-free equilibrium of the proposed model ($\mathscr{M}_{CMD}$) is locally asymptotically stable if $R_0 < 1$ and is unstable if $R_0 > 1$.*

Proof. At the disease-free equilibrium E^0, the Jacobian matrix of model ($\mathscr{M}_{CMD}$) is given by

$$\mathscr{J}(E^0)=\begin{pmatrix} -(r-\mu) & -(r-\mu) & 0 & -\alpha_1 \\ 0 & -(\sigma+\mu) & 0 & \alpha_1 \\ 0 & -\alpha_2 & -(C+\mu_w) & 0 \\ 0 & \alpha_2 & 0 & -(C+\mu_w) \end{pmatrix}.$$

The characteristic equation of $\mathscr{J}(E^0)$ is expressed as follows

$$\big(X+(r-\mu)\big)\big(X+(C+\mu_w)\big)\big(X^2+AX+B\big)=0,$$

where

$$A=C+\mu_w+\sigma+\mu \quad \text{and} \quad B=(\sigma+\mu)(C+\mu_w)-\alpha_1\alpha_2 \\ =(C+\mu_w)(\sigma+\mu)(1-R_0^2).$$

The two eigenvalues $-(r-\mu)$ and $-(C+\mu_w)$ have a negative real part. Furthermore, with the condition that $R_0<1$, we have $B>0$ and A is always positive. In the view of the Routh-Hurwitz criterion [181] all roots of X^2+AX+B have negative real parts. Therefore the local stability of E^0 is confirmed if $R_0<1$. ■

6.4.4 Existence of the endemic equilibrium point

In the presence of CMD, our model exhibits an endemic equilibrium E^*, where the disease remains present in the cassava population. This steady-state solution occurs when all variables are positive, representing a persistent infection within the population.

Theorem 6.5. *For a positive endemic equilibrium $E^*=(S^*,I^*,S_w^*,I_w^*)$, the conditions $K>(1+\frac{\mu^2}{r(\sigma+\mu)})S^*$ and $\alpha_1\alpha_2-(C+\mu_w)(\sigma+r)<0$ must hold.*

Proof. To determine this equilibrium point, we set the right-hand side of the fractional CMD model's system of equations to zero

$$\begin{cases} rS^*\left(1-\dfrac{N_1^*}{K}\right)-\alpha_1\dfrac{S^*I_w^*}{N_1^*}-\mu S^*=0, \\ \alpha_1\dfrac{S^*I_w^*}{N_1^*}-\sigma I^*-\mu I^*=0, \\ \theta-\alpha_2\dfrac{S_w^*I^*}{N_2^*}-CS_w^*-\mu_w S_w^*=0, \\ \alpha_2\dfrac{S_w^*I^*}{N_2^*}-CI_w^*-\mu_w I_w^*=0. \end{cases} \tag{6.7}$$

Therefore, the endemic equilibrium can be expressed as

$$\begin{cases} I^* = \dfrac{rS^*(K-S^*) - K\mu S^*}{rS^* + K(\sigma+\mu)}, \\ S_w^* = \dfrac{\theta}{C+\mu_w} - I_w^*, \\ I_w^* = \dfrac{\alpha_2 \theta I^*}{(C+\mu_w)(\theta+\alpha_2 I^*)}, \end{cases} \tag{6.8}$$

where S^* is a the positive root of the following equation

$$A_1(S^*)^2 + A_2 S^* + A_3 = 0, \tag{6.9}$$

with

$$\begin{aligned} A_1 &= \theta(R_0^2 - 1)r^2 + \alpha_2 r(r+\sigma)K + \theta r^2 > 0, \\ A_2 &= 2\theta(R_0^2-1)rK(\sigma+\mu) + \theta K r(2\mu+\sigma-r) - \alpha_2 K^2(r+\sigma)(r-\mu), \\ A_3 &= \theta(R_0^2-1)K^2(\sigma+\mu)^2 - \theta K^2(r-\mu)(\sigma+\mu). \end{aligned}$$

For $R_0 > 1$ and the condition that $\alpha_1\alpha_2 - (C+\mu_w)(\sigma+r) < 0$, we get $A_1 > 0$ and $A_3 < 0$. Thus, since $\Delta = A_2^2 - 4A_1A_3 > 0$, there exist two solutions to equation (6.9) expressed as follows

$$S_1^* = \frac{-A_2 + \sqrt{A_2^2 - 4A_1A_3}}{2A_1} \quad \text{and} \quad S_2^* = \frac{-A_2 - \sqrt{A_2^2 - 4A_1A_3}}{2A_1}.$$

Based on the sign of A_2, we establish the existence of the endemic equilibrium as follows

- If $A_2 > 0$, we observe that $A_2^2 - 4A_1A_3 > A_2^2$, which implies that $-A_2 + \sqrt{A_2^2 - 4A_1A_3} > 0$. Consequently,

$$S_1^* > 0 \quad \text{and} \quad S_2^* < 0.$$

- If $A_2 < 0$, we similarly have $A_2^2 - 4A_1A_3 > A_2^2$, which ensures that $A_2 + \sqrt{A_2^2 - 4A_1A_3} > 0$. Therefore,

$$S_1^* > 0 \quad \text{and} \quad S_2^* < 0.$$

In both cases, we select the positive solution, resulting in the endemic equilibrium

$$S^* = \frac{-A_2 + \sqrt{A_2^2 - 4A_1A_3}}{2A_1}.$$

■

6.4.5 Local stability of the equilibrium point

In the following, we investigate the local stability of the equilibrium endemic, and the result of the local stability is formulated in the following theorem.

Theorem 6.6. *We assume that conditions of the previous theorem hold. Then the equilibrium endemic point of model ($\mathscr{M}_{CMD}$) is locally asymptotically stable when $R_0 > 1$.*

Proof. At the endemic equilibrium, the Jacobian matrix is expressed as

$$\mathscr{J}(E^*) = \begin{pmatrix} \mathscr{J}_{11} & \mathscr{J}_{12} & \mathscr{J}_{13} & \mathscr{J}_{14} \\ \mathscr{J}_{21} & \mathscr{J}_{22} & \mathscr{J}_{23} & \mathscr{J}_{24} \\ \mathscr{J}_{31} & \mathscr{J}_{32} & \mathscr{J}_{33} & \mathscr{J}_{34} \\ \mathscr{J}_{41} & \mathscr{J}_{42} & \mathscr{J}_{43} & \mathscr{J}_{44} \end{pmatrix},$$

with

$$\begin{aligned}
&\mathscr{J}_{11} = r - \tfrac{r}{K}(2S^* + I^*) - \alpha_1 \tfrac{I^* I_w^*}{(S^*+I^*)^2} - \mu; && \mathscr{J}_{12} = -\tfrac{rS^*}{k} + \alpha_1 \tfrac{S^* I_w^*}{(S^*+I^*)^2}; \\
&\mathscr{J}_{22} = -\alpha_1 \tfrac{S^* I_w^*}{(S^*+I^*)^2} - \sigma - \mu; && \mathscr{J}_{21} = \alpha_1 \tfrac{I^* I_w^*}{(S^*+I^*)^2}; \\
&\mathscr{J}_{33} = -\alpha_2 \tfrac{I^* I_w^*}{(S_w^*+I_w^*)^2} - C - \mu_w; && \mathscr{J}_{23} = 0; \\
&\mathscr{J}_{44} = -\alpha_2 \tfrac{S_w^* I^*}{(S_w^*+I_w^*)^2} - (C + \mu_w); && \mathscr{J}_{32} = -\alpha_2 \tfrac{S_w^*}{S_w^*+I_w^*}; \\
&\mathscr{J}_{13} = 0; && \mathscr{J}_{14} = -\tfrac{\alpha_1 S^*}{S^*+I^*}; \\
&\mathscr{J}_{31} = 0; && \mathscr{J}_{41} = 0; \\
&\mathscr{J}_{24} = \tfrac{\alpha_1 S^*}{S^*+I^*}; && \mathscr{J}_{34} = \alpha_2 \tfrac{S_w^* I^*}{(S_w^*+I_w^*)^2}; \\
&\mathscr{J}_{42} = \alpha_2 \tfrac{S_w^*}{S_w^*+I_w^*}; && \mathscr{J}_{43} = \alpha_2 \tfrac{I^* I_w^*}{(S_w^*+I_w^*)^2}.
\end{aligned}$$

The characteristic equation associated with the Jacobian matrix is given by

$$\big(X + (C+\mu_w)\big)\big(X^3 + B_1 X^2 + B_2 X + B_3\big) = 0, \tag{6.10}$$

where

$$\begin{aligned}
B_1 &= -r(1 - \frac{S^*+I^*}{K}) + \frac{rS^*}{K} + \mu + \frac{\alpha_1 I_w^*}{I^*+S^*} + \frac{\alpha_2 I^*}{I_w^*+S_w^*} + (C+\mu_w) + (\sigma+\mu), \\
B_2 &= \left((C+\mu_w) + \frac{\alpha_2 I}{S_w+I_w}\right)\left(-r(1-\frac{S^*+I^*}{K}) + \frac{rS^*}{K} + \mu + \frac{\alpha_1 I_w^*}{I^*+S^*} + (\sigma+\mu)\right) \\
&\quad + \mathscr{A} - \frac{\alpha_1\alpha_2(rS^* + K(\sigma+\mu))}{K(\sigma+\mu)(\theta+\alpha_2 I^*)}, \\
B_3 &= \frac{\alpha_1\alpha_2(rS^* + K(\sigma+\mu))}{K(\sigma+\mu)(\theta+\alpha_2 I^*)}\left(-r(1-\frac{S^*+I^*}{K}) - \frac{rS^*}{K} - \mu\right) \\
&\quad + \left(C + \mu_1 + \frac{\alpha_2 I^*}{S_w^*+I_w^*}\right)\mathscr{A},
\end{aligned}$$

with

$$\mathscr{A} = \left(-r(1-\frac{S^*+I^*}{K})+\frac{rS^*}{K}+\mu\right)\left(\frac{\alpha_1 S^* I_w^*}{(S^*+I^*)^2}+(\sigma+\mu)\right) + (\sigma+\mu)\frac{\alpha_1 I^* I_w^*}{(S^*+I^*)^2}+\frac{rS^*}{K}\frac{\alpha_1 I^* I_w^*}{(S^*+I^*)^2}.$$

If $B_1 > 0$, $B_3 > 0$, and $B_1 B_2 > B_3$, then by Routh-Hurwitz criteria, the characteristic equation has three eigenvalues with a negative real part. Furthermore, equation (6.9) admits the value $-(C+\mu_w)$ as an eigenvalue with a negative real part.
As a consequence, the endemic equilibrium of model ($\mathscr{M}_{CMD}$) is locally asymptotically stable. ■

6.5 Sensitivity analysis

This section is devoted to performing sensitivity analysis on the basic parameters, in order to identify which one could impact the transmissibility of the disease. This formula gives the sensitivity index [60]

$$\gamma_p^{R_0} = \frac{\partial R_0}{\partial p}\frac{p}{R_0}, \quad \text{where } p \text{ is the value of model parameter.}$$

For example, the sensitivity index of R_0 to α_1 is expressed as follows

$$\gamma_{\alpha_1}^{R_0}\frac{\partial R_0}{\partial \alpha_1}\frac{\alpha_1}{R_0} = \frac{\alpha_2}{(C+\mu_w)(\sigma+\mu)}\frac{1}{2R_0}\frac{\alpha_1}{R_0} = \frac{1}{2}.$$

The coming-after table 6.2 shows the results of all sensitivity indices that have been computed.

Parameter symbol	Sensitivity index for parameter values
α_1	+0.5
α_2	+0.5
C	-0.0750
μ_w	-0.0500
σ	-0.0027
μ	-2.2800e-04

Table 6.2: Sensitivity analysis of the basic reproduction number for the fractional CMD model.

The parameter with a higher sensitivity index is the one that exhibits greater sensitivity concerning the basic reproduction number R_0. When the sensitive index of a model

parameter has a positive sign, it indicates that an increase in that parameter leads to an increase in the basic reproduction number R_0, and conversely, a decrease in the parameter results in a decrease in R_0.

From the given observation in Figure 6.2, we remark that the most significant parameter is the rate at which the whitefly preys on infected cassava plants α_1 and the rate at which the susceptible whitefly becomes infected by predating on infected cassava plants α_2.

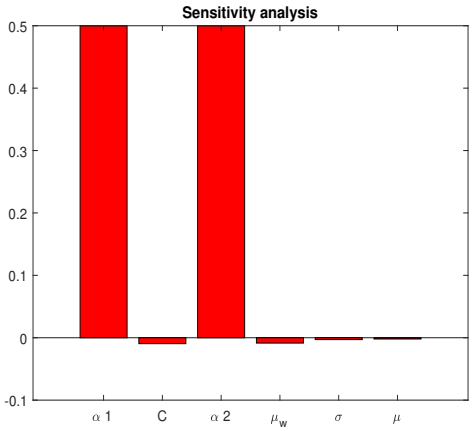
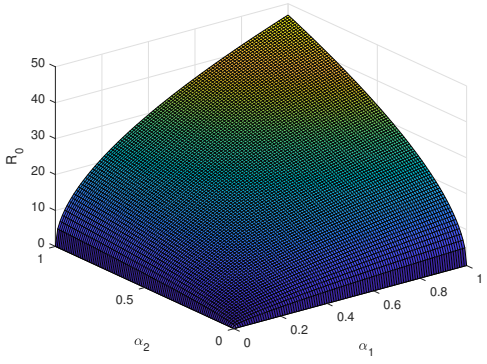
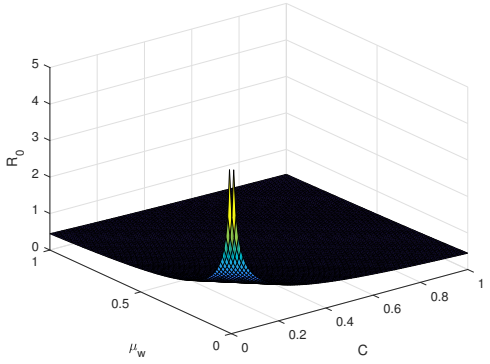


Figure 6.2: Sensitivity analysis of R_0 with respect of all parameters.

The plots of R_0 for some significant model parameters are shown in Figures 6.3 and 6.3. Figure 6.3 shows that an increase of α_1 and α_2 gives a big value of R_0. Conversely, Panel (b) shows that R_0 remains relatively stable across varying values of C and μ_w, with only a minor peak, indicating that these parameters have minimal influence on R_0.



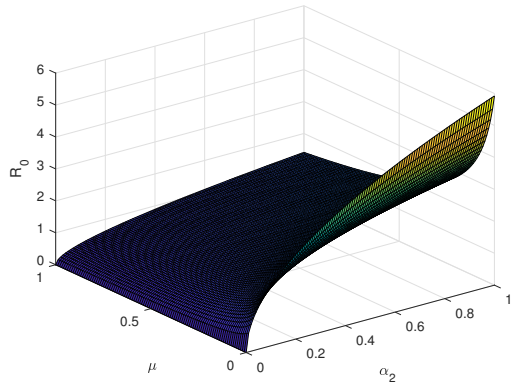
(a) Dynamical behavior of R_0 against α_1 and α_2.



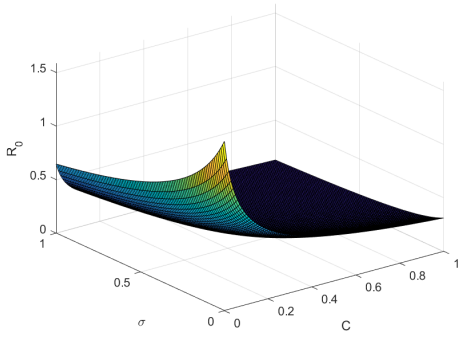
(b) Dynamical behavior of R_0 against C and μ_w.

Figure 6.3: Dynamical behavior of R_0 against two most significant parameters.

In 6.4 (a), R_0 increases with both μ and α_2, indicating that higher values of these parameters enhance disease transmission dynamics. In contrast, Figure 6.4 (b) shows that R_0 remains low for most values of σ and C, with a slight peak only under specific conditions, suggesting these parameters have a limited impact on transmission. Overall, α_1 and α_2 exhibit a more substantial influence on R_0 compared to other parameters.



(c) Dynamical behavior of R_0 against μ and α_2.



(d) Dynamical behavior of R_0 against σ and C.

Figure 6.4: Dynamical behavior of R_0 against two significant parameters.

6.6 Implementation and discussion of numerical results

In this section, we employ the algorithm of the generalized Adams bashforth-moulton presented in 1.5.4, individually for each compartment within our model. Therefore, the algorithm is delineated as follows

$$
\begin{aligned}
S(t_{n+1}) = S_0 &+ \frac{h^\alpha}{\Gamma(\alpha+2)}\left(rS^p(t_{n+1})\left(1-\frac{N_1^p(t_{n+1})}{K}\right)-\alpha_1\frac{S^p(t_{n+1})I_w^p(t_{n+1})}{N_1^p(t_{n+1})}-\mu S^p(t_{n+1})\right)\\
&+\frac{h^\alpha}{\Gamma(\alpha+2)}\sum_{j=0}^{n}a_{j,n+1}\left(rS(t_j)\left(1-\frac{N_1(t_j)}{K}\right)-\alpha_1\frac{S(t_j)I_w(t_j)}{N_1(t_j)}-\mu S(t_j)\right)\\
I(t_{n+1}) = I_0 &+ \frac{h^\alpha}{\Gamma(\alpha+2)}\left(\alpha_1\frac{S^p(t_{n+1})I_w^p(t_{n+1})}{N_1^p(t_{n+1})}-(\sigma+\mu)I^p(t_{n+1})\right)\\
&+\frac{h^\alpha}{\Gamma(\alpha+2)}\sum_{j=0}^{n}a_{j,n+1}\left(\alpha_1\frac{S(t_j)I_w(t_j)}{N_1(t_j)}-(\sigma+\mu)I(t_j)\right),
\end{aligned}
$$

$$S_w(t_{n+1}) = S_{w0} + \frac{h^\alpha}{\Gamma(\alpha+2)}\left(\theta - \alpha_2\frac{S_w^p(t_{n+1})I^p(t_{n+1})}{N_2^p(t_{n+1})} - CS_w^p(t_{n+1}) - \mu_w S_w^p(t_{n+1})\right)$$
$$+ \frac{h^\alpha}{\Gamma(\alpha+2)}\sum_{j=0}^{n} a_{j,n+1}\left(\theta - \alpha_2\frac{S_w(t_j)I(t_j)}{N_2(t_j)} - CS_w(t_j) - \mu_w S_w(t_j)\right),$$
$$I_w(t_{n+1}) = I_{w0} + \frac{h^\alpha}{\Gamma(\alpha+2)}\left(\alpha_2\frac{S_w^p(t_{n+1})I^p(t_{n+1})}{N_2^p(t_{n+1})} - CI_w^p(t_{n+1}) - \mu_w I_w^p(t_{n+1})\right)$$
$$+ \frac{h^\alpha}{\Gamma(\alpha+2)}\sum_{j=0}^{n} a_{j,n+1}\left(\alpha_2\frac{S_w(t_j)I(t_j)}{N_2(t_j)} - CI_w(t_j) - \mu_w I_w(t_j)\right),$$

where the associated formulas for prediction can be calculated as indicated below

$$S^p(t_{n+1}) = S_0 + \frac{1}{\Gamma(\alpha)}\sum_{j=0}^{n} b_{j,n+1}\left(rS(t_j)\left(1 - \frac{N_1(t_j)}{K}\right) - \alpha_1\frac{S(t_j)I_w(t_j)}{N_1(t_j)} - \mu S(t_j)\right),$$
$$I^p(t_{n+1}) = I_0 + \frac{1}{\Gamma(\alpha)}\sum_{j=0}^{n} b_{j,n+1}\left(\alpha_1\frac{S(t_j)I_w(t_j)}{N_1(t_j)} - (\sigma+\mu)I(t_j)\right),$$
$$S_w^p(t_{n+1}) = S_{w0} + \frac{1}{\Gamma(\alpha)}\sum_{j=0}^{n} b_{j,n+1}\left(\theta - \alpha_2\frac{S_w(t_j)I(t_j)}{N_2(t_j)} - CS_w(t_j) - \mu_w S_w(t_j)\right),$$
$$I_w^p(t_{n+1}) = I_{w0} + \frac{1}{\Gamma(\alpha)}\sum_{j=0}^{n} b_{j,n+1}\left(\alpha_2\frac{S_w(t_j)I(t_j)}{N_2(t_j)} - CI_w(t_j) - \mu_w I_w(t_j)\right).$$

To validate the theoretical results of the CMD model, we performed numerical simulations using specific initial conditions and parameter values. These conditions represent the initial population sizes of susceptible and infected cassava plants, along with the disease-carrying whiteflies. The parameters, that influence disease transmission, plant growth, and whitefly behavior, were selected based on prior studies and biological assumptions. Tables 6.3 provide a detailed overview of these conditions and parameters.

Variable	Description	Value
$S(0)$	Initial susceptible cassava plants	900
$I(0)$	Initial infected cassava plants	200
$S_w(0)$	Initial susceptible whiteflies	300
$I_w(0)$	Initial infected whiteflies	80

Table 6.3: Initial conditions for Cassava Mosaic Disease model.

The initial conditions set the starting population sizes of cassava plants and whiteflies, crucial for simulating early disease transmission. The susceptible cassava plants represent the uninfected portion, while the infected group indicates disease presence. Likewise, the initial susceptible and infected whitefly populations highlight their key role as disease vectors, directly influencing transmission dynamics.

Table 6.1 provides an overview of the parameters used in our simulation with biologically plausible ranges for the infection rates, death rates, and removal rates. Where values were unavailable in the literature, assumptions were made based on related studies, and the parameter ranges were further refined through stability analysis, as shown in Figure 6.5. Furthermore, we consider a 140-day simulation to capture the disease's progression from initial spread to a potential epidemic peak, aligning with the typical time frame of cassava production.

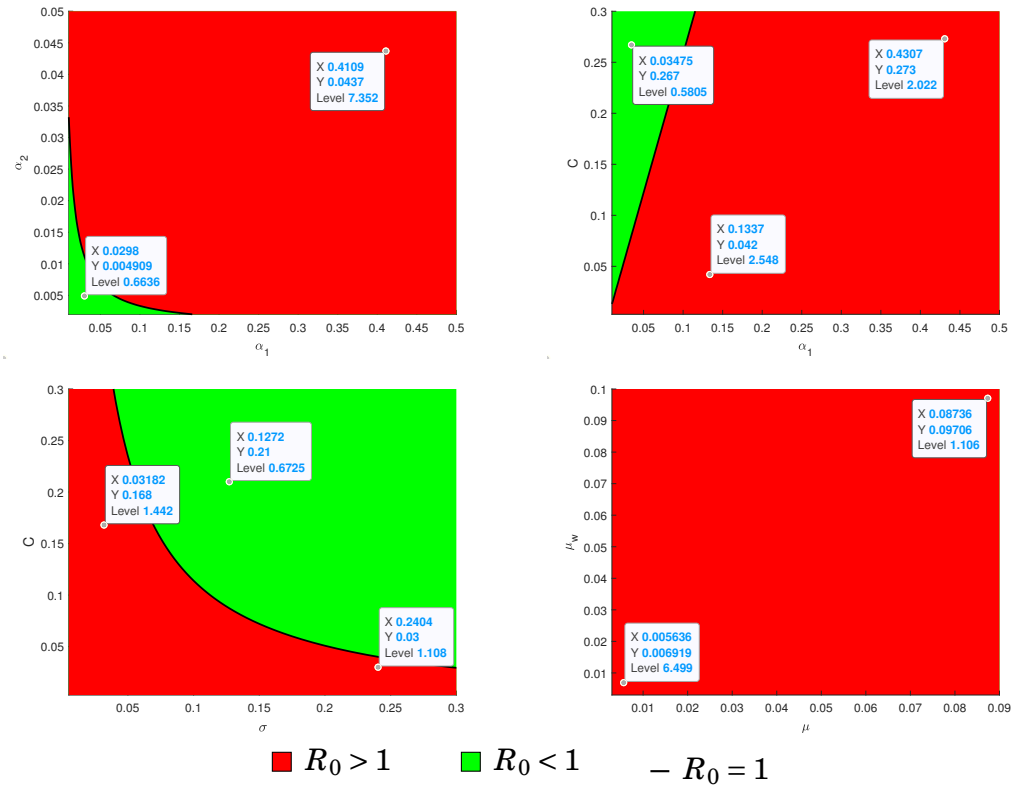


Figure 6.5: Stability analysis over parameter ranges for R_0.

And now, we shall transition to the discussion analysis. In this segment, we shall embark on a simulation analysis of our model, commencing with the examination of two pivotal cases: when $R_0 < 1$ and when $R_0 > 1$. By systematically varying the non-integer order α, we aim to elucidate how the fractional behavior influences the dynamics of the Cassava Mosaic Disease model.

Simultaneously in Figure 6.6, there is a normal increase in cassava plants and a rapid decrease in whiteflies, resulting from the absence of the disease. We observe also a tendency of solutions to approach the stable disease-free equilibrium when $R_0 < 1$.

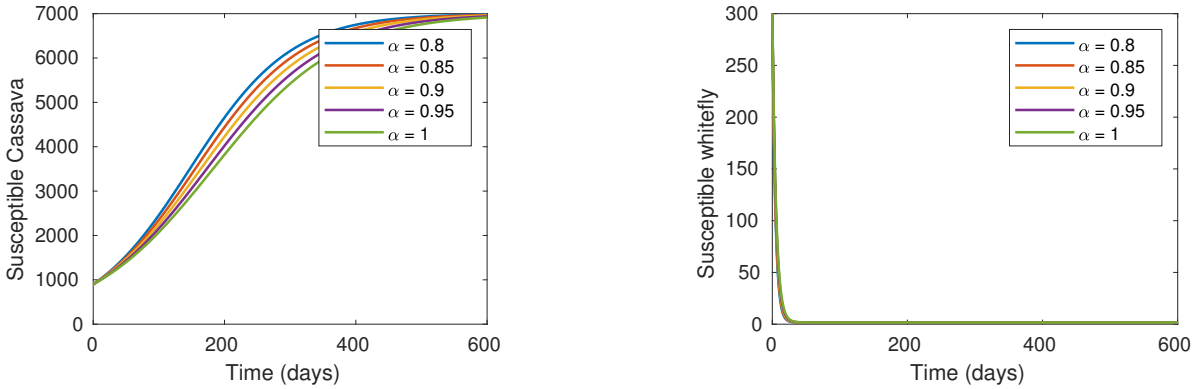
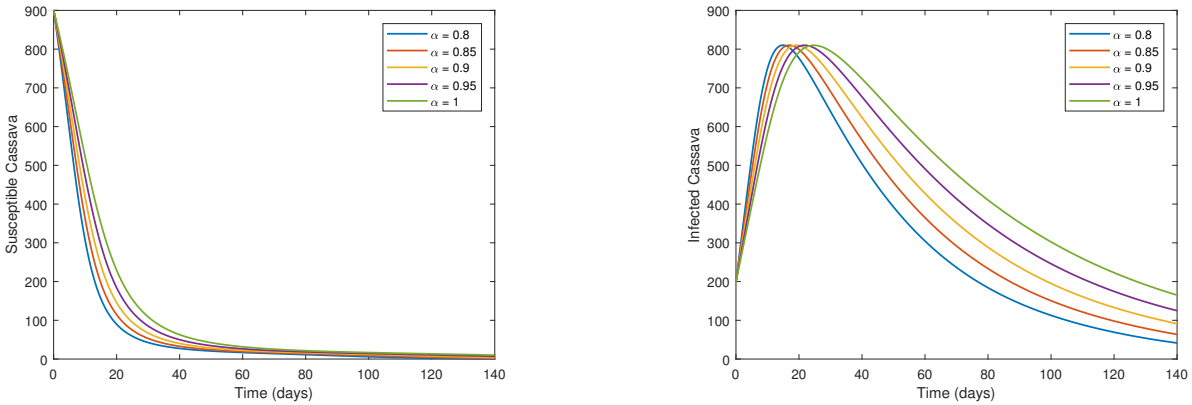


Figure 6.6: Dynamics of S and S_w for different values of the order derivative α with $R_0 < 1$.

Figures 6.7 and 6.8 offer a comprehensive visualization of the dynamic behavior of both susceptible and infected cassava plants, alongside susceptible and infected whitefly vectors, within an endemic scenario under varying fractional derivative orders. As the order α decreases, we observe a more pronounced effect of the fractional derivative, particularly in the nuanced variations of the model's dynamics. In panel (*a*), the population of susceptible cassava plants exhibits a rapid decline over time, eventually stabilizing at a lower threshold, signifying a steady state of depletion. Conversely, in panel (*b*), the infected cassava population rises to a peak before gradually diminishing.



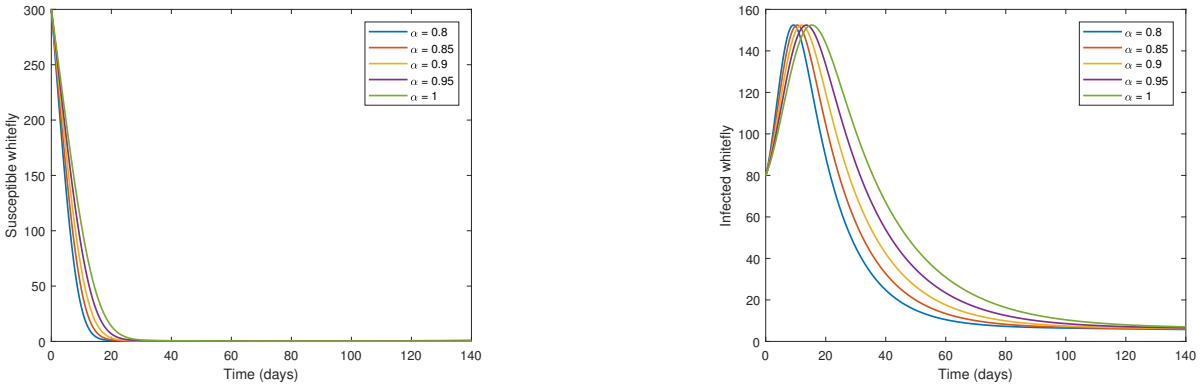
(a) Susceptible Cassava plants.

(b) Infected Cassava plants.

Figure 6.7: Dynamics of Cassava plants for different values of the order derivative α with $R_0 > 1$.

This inverse relationship between the susceptible and infected populations highlights the critical role of disease transmission through the vector dynamics, as captured in panels (*a*) and (*b*) in 6.8, where the susceptible whitefly population declines while the

infected whitefly population initially increases before tapering off.
As α, the order of the derivative, decreases from 1 to lower values (e.g., 0.8 or 0.9), the model becomes more sensitive to these memory effects. This leads to a more gradual change in both the susceptible and infected populations, reflecting the real-world latency and persistence of infections.

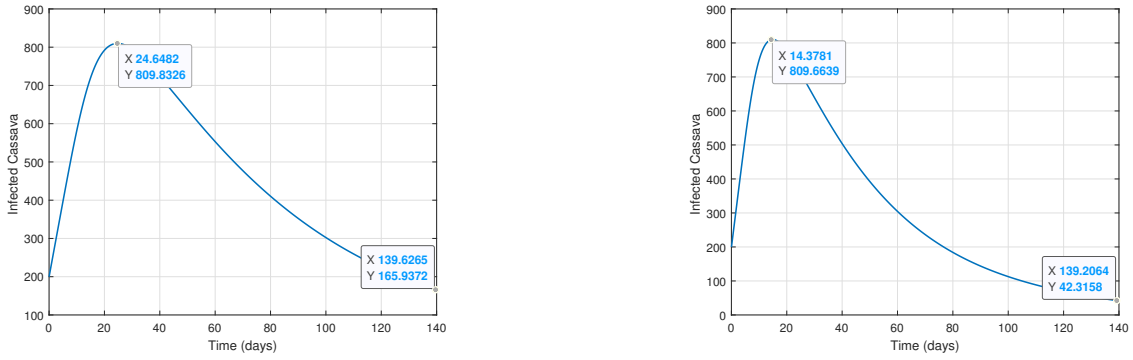


(a) Susceptible whiteflies.

(b) Infected whiteflies.

Figure 6.8: Dynamics of whiteflies for different values of the order derivative α with $R_0 > 1$.

To further analyze the observed behavior in Figures 6.7 and 6.8, we conducted a direct comparison between the classical model ($\alpha = 1$) and the fractional model ($\alpha = 0.8$), as depicted in Figure 6.9, both models exhibit similar initial growth patterns in the number of infected cassava plants, with the fractional model reaching a slightly lower peak.



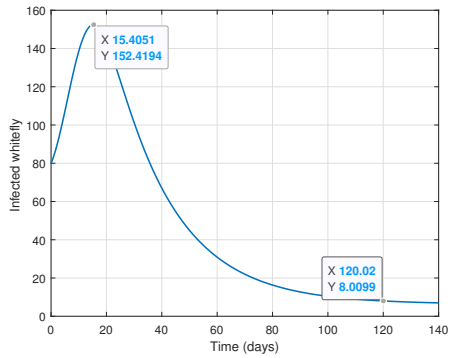
(a) Infected cassava over time for $\alpha = 1$.

(b) Infected cassava over time for $\alpha = 0.8$.

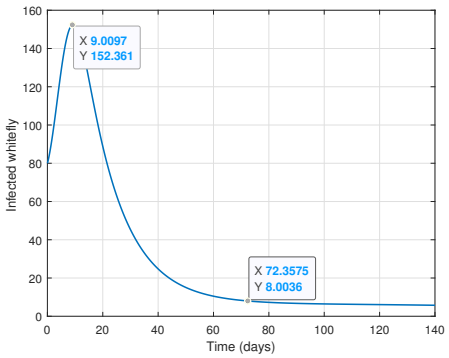
Figure 6.9: Comparison of infected cassava plants: classical ($\alpha = 1$) vs. fractional Model ($\alpha = 0.8$).

Specifically, the classical model shows a peak infection of 809.8326 at 24.6482 days, while the fractional model reaches a comparable peak of 809.6639 at 14.3781 days. However, the most notable difference is in the system's progression towards stability. In the classical model, the infected population takes longer to decline, converging to 165.9372 at 139.6265 days. Conversely, the fractional model achieves a significantly lower final infected population of 42.3158 by 139.2064 days. This rapid convergence in the fractional model emphasizes the role of memory effects in fractional calculus, which accelerates the reduction of infected individuals and leads to quicker stabilization.

In Figure 6.10, we compare the infected whitefly population over time between the classical model and the fractional model. In the classical case (Fig. 6.10 (a)), the number of infected whiteflies peaks at approximately 152.42 around day 15 and gradually decreases to a minimal value of 8.00 by day 120. In contrast, for the fractional model (Fig. 6.10 (b)), we observe a lower peak of 152.36 at day 9, with the population decreasing more rapidly and reaching a minimal value of 8.00 by day 72.



(a) Infected whitefly over time for $\alpha = 1$.



(b) Infected whitefly over time for $\alpha = 0.8$.

Figure 6.10: Comparison of infected whiteflies: classical ($\alpha = 1$) vs. fractional model ($\alpha = 0.8$).

This comparison demonstrates that the fractional model converges to equilibrium more swiftly, due to the memory effects inherent in fractional derivatives. This results in a more rapid stabilization of infected populations. Notably, by day 100, the model indicates a significant reduction in both infected cassava and whiteflies, reflecting more efficient disease dynamics and accelerated recovery in real-world scenarios when memory effects are incorporated. Moreover, achieving comparable outcomes in the earlier days empowers farmers to forecast more realistic dynamics, thereby enhancing their capacity for timely decision-making in disease management.

On the other hand, to check the impact of insecticide on the spread of the disease, we take different values of the death ratio of whiteflies which is caused by insecticide C, and we take $\alpha = 0.8$.

It is clear as depicted in figure 6.11, that the highest value of C exerts a substantial influence on the virus transmission. The population of infected whiteflies can be suppressed, we remark at the same time that the growth of cassava plants is well improved at the highest values of C. When the parameter C attains a value of 0.2, it is observed that the whitefly population diminishes progressively until it is entirely eradicated, thereby rendering the susceptible cassava plants no longer susceptible to disease infection and allowing for natural growth to occur.

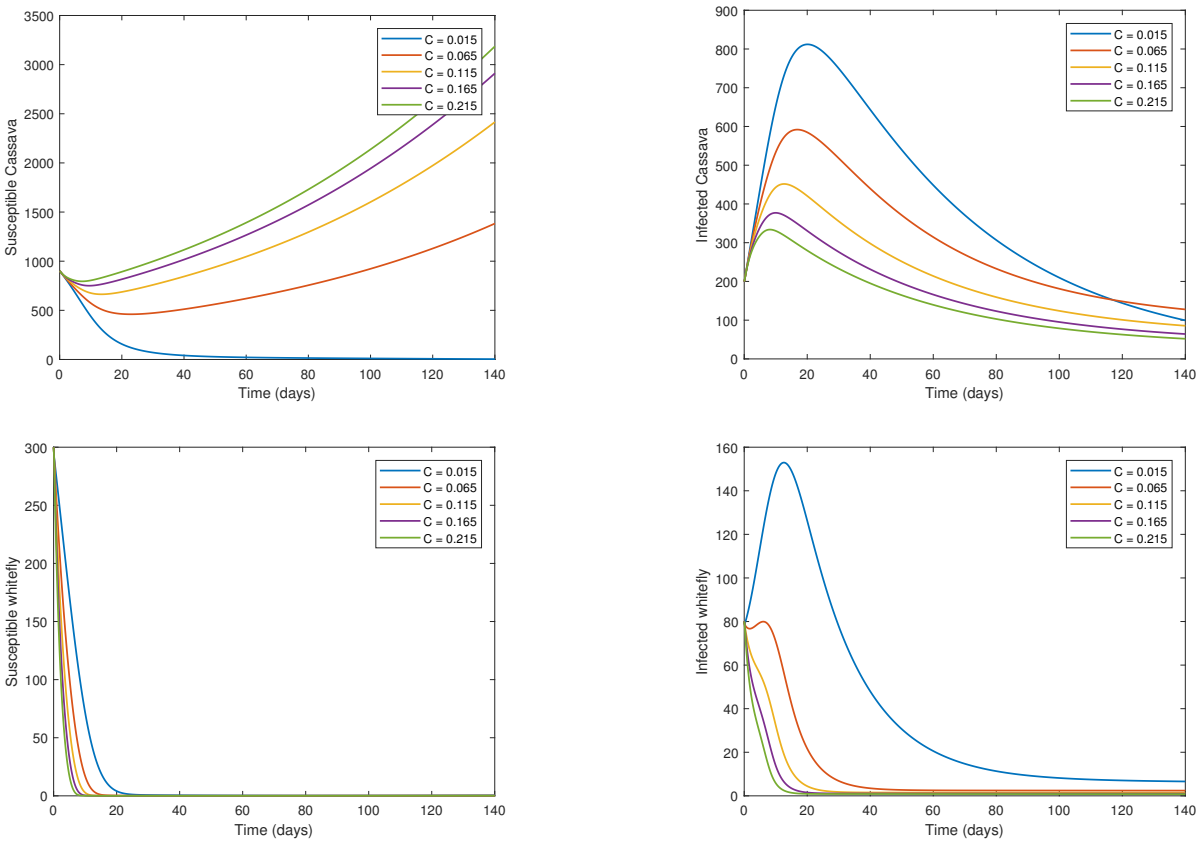


Figure 6.11: The dynamics of population classes for different values of C and $\alpha = 0.8$.

6.7 Conclusion

In our research, we introduced a novel fractional-order model for Cassava Mosaic Disease that incorporates the effects of insecticide. This model not only exhibits mathematical rigor but also demonstrates biological adaptability, thereby enhancing its relevance in real-world applications. Furthermore, we validated our model through numerical simulations employing the generalized Adams-Bashforth method. Our findings underscore that, in contrast to traditional integer-order models, the fractional-order formulation reveals a notably accelerated convergence rate to the solution. This enhanced dynamical behavior underscores the significance of fractional derivatives, illustrating their capacity to capture the intricate effects of memory and nonlocal interactions, which are often overlooked in classical approaches. Over and above that, the inclusion of the insecticide factor allowed us to explore its dual effects on the system. On the positive side, the insecticide contributed to the potential control and reduction of disease transmission. This positive influence was reflected in the model's equilibrium points, where the application of insecticide played a role in stabilizing the system and mitigating the spread of CMD. However, the model also serves as a reminder to farmers that while these techniques may appear effective in controlling whiteflies, they should be applied with caution, as indiscriminate pesticide use can have broader ecological impacts. As well as a sensitivity analysis has also been conducted to identify parameters highly sensitive to R_0. Such insights enable us to pinpoint effective strategies and prevention measures for monitoring the spread of cassava mosaic disease. These findings contribute to our understanding of the dynamics of cassava mosaic disease and underscore the importance of considering fractional-order models in such scenarios. Furthermore, this research paves the way for considering the cultivation of cassava in Morocco, providing valuable information to mitigate disease in regions facing similar challenges, especially as the country struggles with drought in recent years.

CHAPTER 7

Analyzing and Controlling Substance Abuse in Women: A Fractional Approach with Caputo and Atangana-Baleanu Derivatives

Main objective

This chapter investigates a nonlinear fractional-order model to analyze and simulate the dynamics of substance abuse in women, utilizing both Caputo and Atangana-Baleanu derivatives. We establish the model's boundedness and positivity, compute the basic reproduction number, and identify equilibrium points. Our analysis includes examining the local stability of disease-free and endemic equilibrium within a fractional context. Numerical simulations will validate the significance of the arbitrary order derivative and compare the results from Caputo and Atangana-Baleanu derivatives to assess the impact of different fractional operators. Ultimately, we demonstrate that the fractional epidemic model provides a more generalized perspective than classical models, yielding deeper insights into substance abuse dynamics among women. We also propose a fractional optimal control model for managing substance misuse in this population.

The results presented in this chapter are derived from the findings discussed in [30] and [31].

7.1 Introduction and motivation

Substance abuse is a pervasive global issue, affecting approximately 275 million individuals about 5.60% of the worldwide population aged 15-64, according to the World Drug Report [180]. While substance abuse is commonly perceived as a male-centric issue, the unique experiences and responses of women require focused examination. Women's susceptibility to substance abuse is influenced by a complex interplay of biological, psychological, and sociocultural factors, which affects patterns of initiation, progression, and recovery from substance use disorders [96]. Recent research underscores the need for gender-specific interventions, highlighting that women's substance abuse dynamics often differ significantly from those of men. Factors such as societal expectations, stigma, and gender-specific stressors play crucial roles in shaping women's experiences with substance use and their pathways to treatment and recovery [174].

With the advancement of scientific inquiry, traditional methodologies may no longer adequately capture the complexities and nuances of substance abuse dynamics, particularly among women. Consequently, there is an increasing interest in adopting advanced techniques and mathematical frameworks that offer deeper insights into this multifaceted issue. In this context, mathematical modeling has proven valuable in analyzing various aspects of substance use and its dynamics. Previous studies have demonstrated the utility of mathematical models in understanding drug use patterns and treatment impacts. For example, White and Comiskey [185] developed a model focusing on heroin use, revealing how relapse among former users contributes to new or secondary cases of heroin addiction. Similarly, Nyabadza et al. [145] created a model to investigate crystal methamphetamine ("Tik") abuse in South Africa, suggesting that targeting lighter drug users could be an effective mitigation strategy. To enhance the understanding of substance abuse dynamics and develop more effective control strategies, this research integrates fractional mathematical models with Caputo and Atangana-Baleanu derivatives. Unlike traditional models, fractional derivatives account for memory effects and long-range dependencies, providing a more accurate representation of complex systems. The Caputo fractional derivative allows for the inclusion of initial and boundary conditions, while the Atangana-Baleanu derivative, grounded in the Mittag-Leffler function, captures the complexities of real-world phenomena [151, 153]. By incorporating these fractional derivatives into a model focused on substance abuse among women, this chapter aims to refine existing models and improve intervention strategies. The objective is to leverage the advanced features of fractional calculus to gain deeper insights into sub-

stance abuse trends and enhance the development of tailored interventions. Ultimately, this research seeks to empower women, foster resilience, and support long-term recovery by providing a more nuanced and effective approach to controlling substance abuse.

7.2 Fractional model for SAW

In this section, we consider a mathematical model of substance abuse in women [79]. To enhance the understanding of the model's structure and operation, figure 7.1 displays a comprehensive flowchart that outlines its principal elements and interactions.

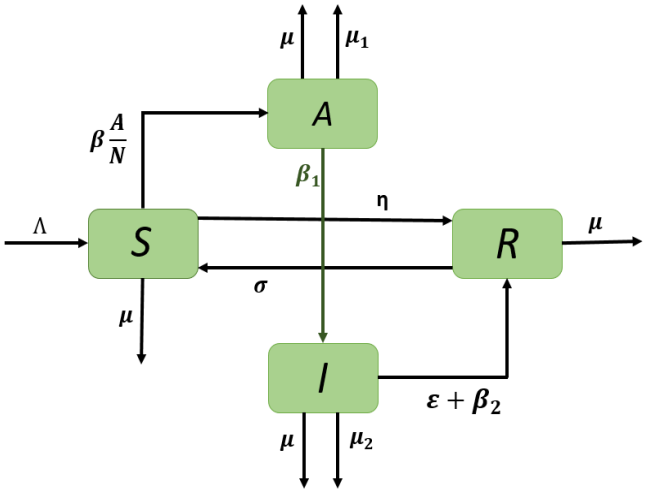


Figure 7.1: Flow chart diagram of the substance abuse in women.

The model consists of a fourth variables dependent on the time, which are identified as follow

- $S(t)$: Susceptible women,
- $A(t)$: women experiencing long-term addiction,
- $I(t)$: individuals who are infected and both recognize and are actively managing the intricacies of their addiction,
- $R(t)$: women undergoing recovery.

Susceptible women transition to drug dependency due to factors such as peer influence, stress or trauma, genetic predispositions, mental health issues, and socioeconomic factors at a rate of β. Once accustomed to drug use, individuals are more likely to develop

addiction at a rate of β_1. Women experiencing drug addiction seek treatment and rehabilitation at a rate of $(\varepsilon+\beta_2)$. The natural death rate is denoted by μ. Empowered by self-determination, a significant proportion of women move to the recovery phase at a rate of η, while some return to the susceptible state at a rate of σ. Additionally, considering that drug users are primarily within the age group 15-64, the recruitment rate is denoted as Λ. The death rates due to addiction and infection are represented by μ_1 and μ_2, respectively. Hence, the fractional version of the ordinary model introduced in [79] is formulated in Caputo's sense in the following form

$$(\mathscr{S}_{\alpha,C})\begin{cases} {}^{C}D_t^{\alpha}S(t) = \Lambda - \beta\dfrac{S(t)A(t)}{N(t)} - (\eta+\mu)S(t) + \sigma R(t), \\ {}^{C}D_t^{\alpha}A(t) = \beta\dfrac{S(t)A(t)}{N(t)} - (\mu+\mu_1+\beta_1)A(t), \\ {}^{C}D_t^{\alpha}I(t) = \beta_1 A(t) - (\mu+\mu_2+\varepsilon+\beta_2)I(t), \\ {}^{C}D_t^{\alpha}R(t) = \eta S(t) + (\varepsilon+\beta_2)I(t) - (\mu+\sigma)R(t), \end{cases} \tag{7.1}$$

similarly, the fractional SWA model with Atangana-Baleanu in the Caputo sense (ABC) fractional derivative is provided below

$$(\mathscr{S}_{\alpha,ABC})\begin{cases} {}^{ABC}D_t^{\alpha}S(t) = \Lambda - \beta\dfrac{S(t)A(t)}{N(t)} - (\eta+\mu)S(t) + \sigma R(t), \\ {}^{ABC}D_t^{\alpha}A(t) = \beta\dfrac{S(t)A(t)}{N(t)} - (\mu+\mu_1+\beta_1)A(t), \\ {}^{ABC}D_t^{\alpha}I(t) = \beta_1 A(t) - (\mu+\mu_2+\varepsilon+\beta_2)I(t), \\ {}^{ABC}D_t^{\alpha}R(t) = \eta S(t) + (\varepsilon+\beta_2)I(t) - (\mu+\sigma)R(t), \end{cases} \tag{7.2}$$

subject to the initial conditions $S(0)=S_0$, $A(0)=A_0$, $I(t)=I_0$, $R(0)=R_0$, where $N(t)$ is the entire population of women $N(t)=S(t)+A(t)+I(t)+R(t)$.

7.3 Positivity and boundedness of solution

In this section, We examine the positivity and boundedness of the solution to systems $(\mathscr{S}_{\alpha,C})$ and $(\mathscr{S}_{\alpha,ABC})$.

7.3.1 Invariant region

Theorem 7.1. *The solutions of $(\mathscr{S}_{\alpha,C})$ are bounded.*

Proof. By summing the equations of system $(\mathscr{S}_{\alpha,C})$ and considering that $N(t) = S(t) + A(t) + I(t) + R(t)$, we derive the following inequality

$$\begin{aligned} {}^{C}D_t^{\alpha} N(t) &= \Lambda - \mu N(t) - \mu_1 A(t) - \mu_2 I(t) \\ &\leqslant \Lambda - \mu N(t). \end{aligned}$$

Lemma 1.9 acquires that

$$N(t) \leqslant \left(N(0) - \frac{\Lambda}{\mu}\right) \mathscr{E}_{\alpha}(-\mu t^{\alpha}) + \frac{\Lambda}{\mu},$$

using Lemma 1.10, we obtain $\lim_{t \to +\infty} N(t) \leqslant \frac{\Lambda}{\mu}$. Consequently, the solution of $(\mathscr{S}_{\alpha,C})$ is bounded. ■

Theorem 7.2. *The solutions of* $(\mathscr{S}_{\alpha,ABC})$ *are bounded.*

Proof. As previously, the summation of the equations of system $(\mathscr{S}_{\alpha,ABC})$ gives

$${}^{ABC}D_t^{\alpha} N(t) \leqslant \Lambda - \mu N(t).$$

By applying the Laplace transform on both sides of this inequality, we acquire

$$\frac{ABC(\alpha)}{s^{\alpha}(1-\alpha) + \alpha} \left[s^{\alpha} \mathscr{L}(N(t))(s) - s^{\alpha-1} N(0)\right] \leqslant \Lambda s^{\alpha-(\alpha+1)} - \mu \mathscr{L}(N(t))(s),$$

$$\mathscr{L}(N(t))(s) \leqslant \frac{\Lambda s^{\alpha-1}(1-\alpha) + \Lambda s^{\alpha-(\alpha+1}\alpha + ABC(\alpha) s^{\alpha-1} N(0)}{s^{\alpha} ABC(\alpha) + \mu s^{\alpha}(1-\alpha) + \mu\alpha},$$

which gives

$$\mathscr{L}(N(t))(s) \leqslant \mathscr{L}(\mathscr{E}_{\alpha,1}(-\mathscr{A}_1 t^{\alpha})) \tfrac{\Lambda(1-\alpha) + ABC(\alpha) N(0)}{ABC(\alpha) + \mu(1-\alpha)} + \mathscr{L}(t^{\alpha} \mathscr{E}_{\alpha,\alpha+1}(-\mathscr{A}_1 t^{\alpha})) \tfrac{\Lambda\alpha}{ABC(\alpha) + \mu(1-\alpha)},$$

with $\mathscr{A}_1 = \frac{\mu\alpha}{ABC(\alpha) + \mu(1-\alpha)}$. Because of the inverse Laplace transform, from the transform Laplace given by equation (1.16), and the asymptotic behavior of the Mittag-Leffler function, we get $N(t) \leqslant \frac{\Lambda}{\mu}$, as t goes to $+\infty$. Thus, N and all other variables of the model $(\mathscr{S}_{\alpha,ABC})$ are bounded. ■

As a result of Theorems 7.1 and 7.2, we set

$$\Omega = \left\{(S, A, I, R) \in \mathbb{R}_+^4 : N(t) \leqslant \frac{\Lambda}{\mu}\right\},$$

which presents the feasible region of our models that is positively invariant, in other terms the models $(\mathscr{S}_{\alpha,C})$ and $(\mathscr{S}_{\alpha,ABC})$ are epidemiological significant.

7.3.2 Positivity of solution

Theorem 7.3. *We suppose that the initial conditions are non-negative, then the solution of system* $(\mathscr{S}_{\alpha,C})$ *is non-negative for all* $t \geqslant 0$.

Proof. From model $(\mathscr{S}_{\alpha,C})$, we obtain

$$ {}^{C}D_t^{\alpha} S(t)\Big|_{S=0} = \Lambda + \sigma R(t), \tag{7.3} $$

$$ {}^{C}D_t^{\alpha} A(t)\Big|_{A=0} = 0 \geqslant 0, \tag{7.4} $$

$$ {}^{C}D_t^{\alpha} I(t)\Big|_{I=0} = \beta_1 A(t), \tag{7.5} $$

$$ {}^{C}D_t^{\alpha} R(t)\Big|_{R=0} = \eta S(t) + (\varepsilon + \beta_2) I(t). \tag{7.6} $$

From (7.4), (7.5), and in the view of Corollary 1.7 we get that $A(t)$ and $I(t)$ are non-negatives for all $t \geqslant 0$. Now, we will show that $S(t)$ is non-negative, by contradiction we suppose that there exists τ_1 such that $S(t)$ intersects the $S(t) = 0$ axis at $t = \tau_1$ for the first time, and

$$ \begin{cases} S(t) > 0, \text{ for } 0 \leqslant t < \tau_1, \\ S(\tau_1) = 0, \\ S\left(\tau_1^{+}\right) < 0. \end{cases} $$

Equation (7.3) gives that ${}^{C}D_t^{\alpha} S(t)\big|_{S(\tau_1)=0} = \Lambda + \sigma R(\tau_1)$, which yields to

- $R(\tau_1) \geqslant 0 \Rightarrow S(\tau_1^{+}) > 0$, contradiction with our assumption, and then S is non-negative.
- $R(\tau_1) < 0$, this case implies that there exists $\tau_2 (0 < \tau_2 < \tau_1)$ such that

  $$ \begin{cases} R(t) > 0, \text{ for } 0 \leqslant t < \tau_2, \\ R(\tau_2) = 0, \\ R\left(\tau_2^{+}\right) < 0. \end{cases} $$

  from the equation (7.6), we have ${}^{C}D_t^{\alpha} R(t)\big|_{R(\tau_2)=0} = \eta S(\tau_2) + (\varepsilon + \beta_2) I(\tau_2)$, which presents a contradiction with $R(\tau_2^{+}) < 0$.

As a consequence, $S(t)$ and $R(t)$ are non-negative for all $t \geqslant 0$. Therefore, all solutions of our system are non-negatives. ■

Theorem 7.4. *We suppose that the initial condition is non-negative, then the solution of system* $(\mathscr{S}_{\alpha,ABC})$ *is non-negative for all* $t \geqslant 0$.

Proof. We have

$$\begin{aligned} {}^{ABC}D_t^{\alpha} S(t) &= \Lambda - \beta \frac{S(t)A(t)}{N(t)} - (\eta+\mu)S(t) + \sigma R(t) \\ &\geqslant \left(-\beta \frac{A(t)}{N(t)} - (\eta+\mu)\right) S(t), \end{aligned}$$

A and N are bounded, and the ratio $\frac{A}{N}$ is also bounded by a constant, which we denote by M_1. Therefore, we acquire

$${}^{ABC}D_t^{\alpha} S(t) \geqslant -(M_1+\eta+\mu)S(t).$$

Next, apply the Laplace transform in ABC sense to the previous inequality to get

$$\frac{ABC(\alpha)}{s^{\alpha}(1-\alpha)+\alpha}\left[s^{\alpha}\mathscr{L}(S(t))(s) - s^{\alpha-1}S(0)\right] \geqslant -(M_1+\eta+\mu)\mathscr{L}(S(t))(s),$$

$$\left[ABC(\alpha)s^{\alpha} + (M_1+\eta+\mu)\alpha + (M_1+\eta+\mu)(1-\alpha)s^{\alpha}\right]\mathscr{L}(S(t))(s) \geqslant ABC(\alpha)s^{\alpha-1}S(0),$$

$$\begin{aligned} \mathscr{L}(S(t))(s) &\geqslant \frac{s^{\alpha-1}}{s^{\alpha} + \frac{(M_1+\eta+\mu)\alpha}{ABC(\alpha)+(M_1+\eta+\mu)(1-\alpha)}} \frac{ABC(\alpha)S(0)}{ABC(\alpha)+(M_1+\eta+\mu)(1-\alpha)} \\ &= \mathscr{L}(\mathscr{E}_{\alpha,1}(-C_1 t^{\alpha})) \frac{ABC(\alpha)S(0)}{ABC(\alpha)+(M_1+\eta+\mu)(1-\alpha)}, \end{aligned}$$

with $C_1 = \frac{(M_1+\eta+\mu)\alpha}{ABC(\alpha)+(M_1+\eta+\mu)(1-\alpha)}$.
Next, we introduce the inverse of the Laplace transform to acquire

$$S(t) \geqslant \mathscr{E}_{\alpha,1}(-C_1 t^{\alpha}) \frac{ABC(\alpha)S(0)}{ABC(\alpha)+(M_1+\eta+\mu)(1-\alpha)}. \tag{7.7}$$

Since both terms on the right-hand side of (7.7) are positives, this implies that $S(t)$ remains positive for all $t \geqslant 0$. Similarly, we demonstrate that $A(t) \geqslant 0$, $I(t) \geqslant 0$, and $R(t) \geqslant 0$ for all $t \geqslant 0$. ■

7.4 Existence and uniqueness of the solution

In this section, we investigate the existence and uniqueness of solutions for the two fractional models $(\mathscr{S}_{\alpha,C})$ and $(\mathscr{S}_{\alpha,ABC})$.

Existence and uniqueness of the solution for the SAW model in the Caputo sense

We focus on the investigation of both the existence and uniqueness of the solution concerning the aforementioned model $(\mathscr{S}_{\alpha,C})$ by applying Lemma 1.7. For the sake of clarity, we provide the following notations

$$\begin{aligned} F_1(t,S) &= \Lambda - \beta\frac{S(t)A(t)}{N(t)} - (\eta+\mu)S(t) + \sigma R(t), \\ F_2(t,A) &= \beta\frac{S(t)A(t)}{N(t)} - (\mu+\mu_1+\beta_1)A(t), \\ F_3(t,I) &= \beta_1 A(t) - (\mu+\mu_2+\varepsilon+\beta_2)I(t), \\ F_4(t,R) &= \eta S(t) + (\varepsilon+\beta_2)I(t) - (\mu+\sigma)R(t). \end{aligned} \tag{7.8}$$

We have

$$\begin{aligned} \|F_1(t,S)\| &= \|\Lambda - \beta\frac{S(t)A(t)}{N(t)} - (\eta+\mu)S(t) + \sigma R(t)\| \\ &\leqslant \|\Lambda - \beta S(t) - (\eta+\mu)S(t) + \sigma R(t)\| \\ &\leqslant \Lambda + \sigma\|R\| + (\beta+\eta+\mu)\|S\|, \end{aligned}$$

denote $\mathscr{B}_1 = \Lambda + \sigma\|R\|$ and $\mathscr{C}_1 = \beta+\eta+\mu$. Therefore,

$$\|F_1(t,S)\| \leqslant \mathscr{B}_1 + \mathscr{C}_1\|S\|.$$

Similarly,

$$\begin{aligned} \|F_2(t,A)\| &\leqslant \underbrace{\beta\|S\|}_{\mathscr{B}_2} + \underbrace{\left(\mu+\mu_1+\beta_1\right)}_{\mathscr{C}_2}\|A\| \\ \|F_3(t,I)\| &\leqslant \underbrace{\beta_1\|A\|}_{\mathscr{B}_3} + \underbrace{\left(\mu+\mu_2+\varepsilon+\beta_2\right)}_{\mathscr{C}_3}\|I\| \\ \|F_4(t,R)\| &\leqslant \underbrace{\eta\|S\| + (\varepsilon+\beta_2)\|I\|}_{\mathscr{B}_4} + \underbrace{\left(\mu+\sigma\right)}_{\mathscr{C}_4}\|R\|. \end{aligned} \tag{7.9}$$

For $i = 1,2,3,4$, the kernels F_i and $\frac{\partial F_i}{\partial X}$ are continuous in $\mathbb{R}$, as well as $\mathscr{B}_i$ and $\mathscr{C}_i$ are positives. Therefore in view of Lemma 1.7, the existence and uniqueness of solution for $(\mathscr{S}_{\alpha,C})$ are established.

Existence and uniqueness of the solution for the SAW model in ABC sense

We apply the fractional integral in the ABC sense to both sides of system ($\mathscr{S}_{\alpha,ABC}$), we get

$$\begin{aligned}
S(t)-S(0) &= {}^{ABC}\mathscr{I}_t^{\alpha}\left(\Lambda-\beta\frac{S(t)A(t)}{N(t)}-(\eta+\mu)S(t)+\sigma R(t)\right),\\
A(t)-A(0) &= {}^{ABC}\mathscr{I}_t^{\alpha}\left(\beta\frac{S(t)A(t)}{N(t)}-(\mu+\mu_1+\beta_1)A(t)\right),\\
I(t)-I(0) &= {}^{ABC}\mathscr{I}_t^{\alpha}\left(\beta_1A(t)-(\mu+\mu_2+\varepsilon+\beta_2)I(t)\right),\\
R(t)-R(0) &= {}^{ABC}\mathscr{I}_t^{\alpha}\left(\eta S(t)+(\varepsilon+\beta_2)I(t)-(\mu+\sigma)R(t)\right).
\end{aligned}$$

Theorem 7.5. *If the following inequalities are satisfied,*

$$0\leqslant K_1<1,\ \ 0\leqslant K_2<1,\ \ 0\leqslant K_3<1,\ \ 0\leqslant K_4<1,$$

with

$$K_1=\beta+\eta+\mu,\ \ K_2=\beta+\mu+\mu_1+\beta_1,\ \ K_3=\mu+\mu_2+\varepsilon+\beta_2,\ \ \text{and } K_4=\mu+\sigma.$$

Then, the kernels F_1,F_2,F_3, and F_4 meet the Lipschitz condition and exhibit contraction.

Proof. Let S and S_1 be two functions, we have

$$\begin{aligned}
\|F_1(t,S)-F_1(t,S_1)\| &= \|-\beta\frac{S(t)A(t)}{N(t)}-(\eta+\mu)S(t)+\beta\frac{S_1(t)A(t)}{N(t)}+(\eta+\mu)S_1(t)\|\\
&= \|\beta\frac{A}{N}(S-S_1)\|+\|(\eta+\mu)(S-S_1)\|\\
&\leqslant (\beta+\eta+\mu)\|S-S_1\|.
\end{aligned}$$

Let $K_1=\beta+\eta+\mu$, the Lipschitz condition is obtained for kernel F_1. Furthermore, $0\leqslant\beta+\eta+\mu<1$ implies the contraction of F_1.
Likewise, the Lipschitz condition and contraction properties for the remaining kernels can be similarly derived and expressed as shown

$$\begin{aligned}
\|F_2(t,A)-F_2(t,A_1)\| &\leqslant K_2\|A-A_1\|,\\
\|F_3(t,I)-F_3(t,I_1)\| &\leqslant K_3\|I-I_1\|,\\
\|F_4(t,R)-F_4(t,R_1)\| &\leqslant K_4\|R-R_1\|,
\end{aligned}\tag{7.10}$$

where

$$K_2=\beta+\mu+\mu_1+\beta_1,\ \ K_3=\mu+\mu_2+\varepsilon+\beta_2,\ \ K_4=\mu+\sigma.$$

■

Based on the definition of the integral in ABC sense stated in (1.10) and the kernels of our model introduced in (7.8), we acquire

$$\begin{aligned}
S(t) &= S(0) + \frac{1-\alpha}{ABC(\alpha)}F_1(t,S) + \frac{\alpha}{ABC(\alpha)\Gamma(\alpha)}\int_0^t F_1(x,S)(t-x)^{\alpha-1}dx,\\
A(t) &= A(0) + \frac{1-\alpha}{ABC(\alpha)}F_2(t,A) + \frac{\alpha}{ABC(\alpha)\Gamma(\alpha)}\int_0^t F_2(x,A)(t-x)^{\alpha-1}dx,\\
I(t) &= I(0) + \frac{1-\alpha}{ABC(\alpha)}F_3(t,I) + \frac{\alpha}{ABC(\alpha)\Gamma(\alpha)}\int_0^t F_3(x,I)(t-x)^{\alpha-1}dx,\\
R(t) &= R(0) + \frac{1-\alpha}{ABC(\alpha)}F_4(t,R) + \frac{\alpha}{ABC(\alpha)\Gamma(\alpha)}\int_0^t F_4(x,R)(t-x)^{\alpha-1}dx.
\end{aligned} \tag{7.11}$$

Accordingly, we can represent recursively as

$$\begin{aligned}
S_n(t) &= S_0(t) + \frac{1-\alpha}{ABC(\alpha)}F_1(t,S_{n-1}) + \frac{\alpha}{ABC(\alpha)\Gamma(\alpha)}\int_0^t F_1(x,S_{n-1})(t-x)^{\alpha-1}dx,\\
A_n(t) &= A_0(t) + \frac{1-\alpha}{ABC(\alpha)}F_2(t,A_{n-1}) + \frac{\alpha}{ABC(\alpha)\Gamma(\alpha)}\int_0^t F_2(x,A_{n-1})(t-x)^{\alpha-1}dx,\\
I_n(t) &= I_0(t) + \frac{1-\alpha}{ABC(\alpha)}F_3(t,I_{n-1}) + \frac{\alpha}{ABC(\alpha)\Gamma(\alpha)}\int_0^t F_3(x,I_{n-1})(t-x)^{\alpha-1}dx,\\
R_n(t) &= R_0(t) + \frac{1-\alpha}{ABC(\alpha)}F_4(t,R_{n-1}) + \frac{\alpha}{ABC(\alpha)\Gamma(\alpha)}\int_0^t F_4(x,R_{n-1})(t-x)^{\alpha-1}dx,
\end{aligned} \tag{7.12}$$

along with

$$S_0(t) = S(0),\quad A_0(t) = A(0),\quad I_0(t) = I(0),\quad R_0(t) = R(0).$$

Using the initial conditions and the differences between successive terms, the system of equations is developed as follows

$$\begin{aligned}
\Delta_n^S(t) = S_n(t) - S_{n-1}(t) &= \frac{1-\alpha}{ABC(\alpha)}\left(F_1(t,S_{n-1}) - F_1(t,S_{n-2})\right)\\
&\quad + \frac{\alpha}{ABC(\alpha)\Gamma(\alpha)}\int_0^t \left(F_1(x,S_{n-1}) - F_1(x,S_{n-2})\right)(t-x)^{\alpha-1}dx,\\
\Delta_n^A(t) = A_n(t) - A_{n-1}(t) &= \frac{1-\alpha}{ABC(\alpha)}\left(F_2(t,A_{n-1}) - F_2(t,A_{n-2})\right)\\
&\quad + \frac{\alpha}{ABC(\alpha)\Gamma(\alpha)}\int_0^t \left(F_2(x,A_{n-1}) - F_2(x,A_{n-2})\right)(t-x)^{\alpha-1}dx,\\
\Delta_n^I(t) = I_n(t) - I_{n-1}(t) &= \frac{1-\alpha}{ABC(\alpha)}\left(F_3(t,I_{n-1}) - F_3(t,I_{n-2})\right)\\
&\quad + \frac{\alpha}{ABC(\alpha)\Gamma(\alpha)}\int_0^t \left(F_3(x,A_{n-1}) - F_3(x,A_{n-2})\right)(t-x)^{\alpha-1}dx,\\
\Delta_n^R(t) = R_n(t) - R_{n-1}(t) &= \frac{1-\alpha}{ABC(\alpha)}\left(F_4(t,R_{n-1}) - F_4(t,R_{n-2})\right)\\
&\quad + \frac{\alpha}{ABC(\alpha)\Gamma(\alpha)}\int_0^t \left(F_4(x,R_{n-1}) - F_4(x,R_{n-2})\right)(t-x)^{\alpha-1}dx.
\end{aligned} \tag{7.13}$$

It is worth mentioning that

$$S_n(t)=\sum_{i=1}^{n}\Delta_i^S(t),\quad A_n(t)=\sum_{i=1}^{n}\Delta_i^A(t),\quad I_n(t)=\sum_{i=1}^{n}\Delta_i^I(t),\quad \text{and } R_n(t)=\sum_{i=1}^{n}\Delta_i^R(t). \tag{7.14}$$

Taking the norm of equation (7.13) and applying the triangular inequality gives

$$\begin{aligned}\|\Delta_n^S(t)\|=\|S_n(t)-S_{n-1}(t)\|&\leqslant\frac{1-\alpha}{ABC(\alpha)}\|F_1(t,S_{n-1})-F_1(t,S_{n-2})\|\\&\quad+\frac{\alpha}{ABC(\alpha)\Gamma(\alpha)}\left\|\int_0^t(F_1(x,S_{n-1})-F_1(x,S_{n-2}))(t-x)^{\alpha-1}dx\right\|.\end{aligned}$$

Since the kernel F_1 satisfies the Lipschitz condition, the previous inequality yields to

$$\begin{aligned}\|S_n(t)-S_{n-1}(t)\|&\leqslant\frac{1-\alpha}{ABC(\alpha)}K_1\|S_{n-1}(t)-S_{n-2}(t)\|\\&\quad+\frac{\alpha}{ABC(\alpha)\Gamma(\alpha)}K_1\int_0^t\|S_{n-1}(t)-S_{n-2}(t)\|(t-x)^{\alpha-1}dx.\end{aligned}$$

As a consequence, we get

$$\|\Delta_n^S(t)\|\leqslant\frac{1-\alpha}{ABC(\alpha)}K_1\|\Delta_{n-1}^S(t)\|+\frac{\alpha}{ABC(\alpha)\Gamma(\alpha)}K_1\int_0^t\|\Delta_{n-1}^S(x)\|(t-x)^{\alpha-1}dx. \tag{7.15}$$

Analogously, we obtain

$$\begin{aligned}\|\Delta_n^A(t)\|&\leqslant\frac{1-\alpha}{ABC(\alpha)}K_2\|\Delta_{n-1}^A(t)\|+\frac{\alpha}{ABC(\alpha)\Gamma(\alpha)}K_2\int_0^t\|\Delta_{n-1}^A(x)\|(t-x)^{\alpha-1}dx,\\\|\Delta_n^I(t)\|&\leqslant\frac{1-\alpha}{ABC(\alpha)}K_3\|\Delta_{n-1}^I(t)\|+\frac{\alpha}{ABC(\alpha)\Gamma(\alpha)}K_3\int_0^t\|\Delta_{n-1}^I(x)\|(t-x)^{\alpha-1}dx,\\\|\Delta_n^R(t)\|&\leqslant\frac{1-\alpha}{ABC(\alpha)}K_4\|\Delta_{n-1}^R(t)\|+\frac{\alpha}{ABC(\alpha)\Gamma(\alpha)}K_4\int_0^t\|\Delta_{n-1}^R(x)\|(t-x)^{\alpha-1}dx.\end{aligned} \tag{7.16}$$

Theorem 7.6. *The fractional SAW model (7.2) has a solution if there exists $\tilde{t}$ such that*

$$\frac{1-\alpha}{ABC(\alpha)}K_i+\frac{\tilde{t}^{\alpha}}{ABC(\alpha)\Gamma(\alpha)}K_i,\quad i=1,2,3,4.$$

Proof. In view of the boundedness of S,A,I, and R, we get from (7.15) and (7.16) that

$$\begin{aligned}\|\Delta_n^S(t)\|&\leqslant\|S_n(0)\|\left[\frac{1-\alpha}{ABC(\alpha)}K_1+\frac{\tilde{t}^{\alpha}}{ABC(\alpha)\Gamma(\alpha)}K_1\right]^n,\\\|\Delta_n^A(t)\|&\leqslant\|A_n(0)\|\left[\frac{1-\alpha}{ABC(\alpha)}K_2+\frac{\tilde{t}^{\alpha}}{ABC(\alpha)\Gamma(\alpha)}K_2\right]^n,\\\|\Delta_n^I(t)\|&\leqslant\|I_n(0)\|\left[\frac{1-\alpha}{ABC(\alpha)}K_3+\frac{\tilde{t}^{\alpha}}{ABC(\alpha)\Gamma(\alpha)}K_3\right]^n,\\\|\Delta_n^R(t)\|&\leqslant\|R_n(0)\|\left[\frac{1-\alpha}{ABC(\alpha)}K_4+\frac{\tilde{t}^{\alpha}}{ABC(\alpha)\Gamma(\alpha)}K_4\right]^n.\end{aligned}$$

We set that

$$\begin{aligned} S(t)-S(0) &= S_n(t)-\bar{\mathcal{H}}_n^1(t),\\ A(t)-A(0) &= A_n(t)-\bar{\mathcal{H}}_n^2(t),\\ I(t)-I(0) &= I_n(t)-\bar{\mathcal{H}}_n^3(t),\\ R(t)-R(0) &= R_n(t)-\bar{\mathcal{H}}_n^4(t). \end{aligned}$$

Applying the norm to $\bar{\mathcal{H}}_n^1$ to get

$$\begin{aligned} \|\bar{\mathcal{H}}_n^1\| &\leqslant \frac{1-\alpha}{ABC(\alpha)}\|F_1(t,S_{n-1})-F_1(t,S)\| + \frac{\alpha}{ABC(\alpha)\Gamma(\alpha)}\int_0^t (t-x)^{\alpha-1}\|F_1(t,S_{n-1})-F_1(t,S)\|dx\\ &\leqslant \frac{1-\alpha}{ABC(\alpha)}K_1\|S_{n-1}-S\| + \frac{t^\alpha}{ABC(\alpha)\Gamma(\alpha)}K_1\|S_{n-1}-S\|. \end{aligned}$$

Upon repeating the same process, we acquire at $\tilde{t}$

$$\|\bar{\mathcal{H}}_n^1\| \leqslant \left(\frac{1-\alpha}{ABC(\alpha)} + \frac{\tilde{t}^\alpha}{ABC(\alpha)\Gamma(\alpha)}\right)^{n+1} K_1^{n+1}M,$$

where M is the upper bound of S. As n goes to $+\infty$, we obtain $\|\bar{\mathcal{H}}_n^1\| \longrightarrow 0$. Similarly, we get $\|\bar{\mathcal{H}}_n^2\| \longrightarrow 0$, $\|\bar{\mathcal{H}}_n^3\| \longrightarrow 0$, and $\|\bar{\mathcal{H}}_n^4\| \longrightarrow 0$. ■

Furthermore, we suppose that there exists another solution of system (7.2) denoted (S^1, A^1, I^1, R^1), therefore

$$S(t)-S^1(t) = \frac{1-\alpha}{ABC(\alpha)}\left(F_1(t,S)-F_1(t,S^1)\right) + \frac{\alpha}{ABC(\alpha)\Gamma(\alpha)}\int_0^t \left(F_1(x,S)-F_1(x,S^1)\right)(t-x)^{\alpha-1}dx,$$

which gives

$$\|S-S^1\|\left(1-\frac{1-\alpha}{ABC(\alpha)}K_1 + \frac{t^\alpha}{ABC(\alpha)\Gamma(\alpha)}K_1\right) \leqslant 0. \tag{7.17}$$

Theorem 7.7. *We suppose that the following inequality holds*

$$\left(1-\frac{1-\alpha}{ABC(\alpha)}K_i + \frac{t^\alpha}{ABC(\alpha)\Gamma(\alpha)}K_i\right) > 0 \quad \text{for } i = 1,2,3,4.$$

Then, system (7.2) admits a unique solution.

Proof. From (7.17), we acquire $\|S-S^1\| = 0$. Consequently, $S = S^1$ and then the S is unique. Analogously $A = A^1$, $I = I^1$, $R = R^1$. Therefore, the uniqueness of the solution is established. ■

7.5 Equilibrium points and the basic reproduction number

7.5.1 Disease-free equilibrium

Set the right-hand size of system $(\mathscr{S}_{\alpha,C})$ or $(\mathscr{S}_{\alpha,ABC})$ equal to zero and solve it at $A(t) = I(t) = 0$, then the disease-free equilibrium is given by

$$E^0 = \left(\frac{\Lambda(\mu+\sigma)}{\mu(\eta+\mu+\sigma)}, 0, 0, \frac{\eta\Lambda}{\mu(\eta+\mu+\sigma)}\right).$$

7.5.2 Basic Reproduction Number

The basic reproduction number R_0 quantifies the average number of secondary infections caused by a single infected individual in a completely susceptible population. For our models, using the next-generation matrix, we have

$$\mathbb{F} = \begin{bmatrix} \frac{\beta(\sigma+\mu)}{(\eta+\mu+\sigma)} & 0 \\ \beta_1 & 0 \end{bmatrix} \quad \text{and } \mathbb{V} = \begin{bmatrix} (\mu+\mu_1+\beta_1) & 0 \\ 0 & (\mu+\mu_2+\varepsilon+\beta_2) \end{bmatrix}.$$

Hence, R_0 associated with models $(\mathscr{S}_{\alpha,C})$ and $(\mathscr{S}_{\alpha,ABC})$ is given by

$$R_0 = \frac{\beta(\sigma+\mu)}{(\eta+\mu+\sigma)(\mu+\mu_1+\beta_1)}. \tag{7.18}$$

7.5.3 Substance abuse equilibrium point

Denote $E^*(S^*, A^*, I^*, R^*)$ the endemic equilibrium and set all the equations at zero, we get

$$\begin{cases} S^* = \dfrac{(\mu+\mu_2+\varepsilon+\beta_2)(\sigma+\mu)+\beta_1(\sigma+\mu+\varepsilon+\beta_2)}{(R_0-1)(\eta+\sigma+\mu)(\mu+\mu_2+\varepsilon+\beta_2)}A^*, \\ A^* = \dfrac{(R_0-1)(\sigma+\mu)(\mu+\mu_2+\varepsilon+\beta_2)}{(R_0-1)C+R_0D}, \\ I^* = \dfrac{\beta_1}{(\mu+\mu_2+\varepsilon+\beta_2)}A^*, \\ R^* = \dfrac{\eta}{\sigma+\mu}S^* + \dfrac{\varepsilon+\beta_2}{\sigma+\mu}I^*, \end{cases} \tag{7.19}$$

where

$$C = (\sigma+\mu)\left[\mu_1(\mu+\mu_2+\varepsilon+\beta_2)+\beta_1\mu_2\right],$$
$$D = \mu(\sigma+\mu)(\mu+\mu_2+\varepsilon+\beta_2)+\beta_1\mu(\varepsilon+\beta_2).$$

7.5.4 Local stability

In this subsection, we will analyze the local stability of the equilibrium points of the proposed fractional models. the comping-after theorem states the local stability of the DFE of SAW models.

Theorem 7.8. *If $R_0 < 1$, then the disease-free equilibrium E^0 is locally asymptotically stable and unstable in the opposite case.*

Proof. The Jacobian matrix at E^0 is expressed as follows

$$\mathscr{J}\left(E^0\right) = \begin{pmatrix} -(\eta+\mu) & -\frac{\beta(\mu+\sigma)}{(\eta+\mu+\sigma)} & 0 & \sigma \\ 0 & \frac{\beta(\mu+\sigma)}{(\eta+\mu+\sigma)} - (\mu+\mu_1+\beta_1) & 0 & 0 \\ 0 & \beta_1 & -(\mu+\mu_2+\beta_2+\varepsilon) & 0 \\ \eta & 0 & \varepsilon+\beta_2 & -(\mu+\sigma) \end{pmatrix}.$$

The characteristic equation of $\mathscr{J}\left(E^0\right)$ is given by

$$(X-\lambda_1)(X-\lambda_2)\left(X^2 + X(\eta+2\mu+\sigma) + (\eta+\mu+\sigma)\right) = 0,$$

with

$$\lambda_1 = \frac{\beta(\sigma+\mu)}{(\eta+\mu+\sigma)} - (\mu+\mu_1+\beta_1),$$
$$\lambda_2 = -(\mu+\mu_2+\varepsilon+\beta_2) < 0.$$

Since $R_0 < 1$, hence $\lambda_1 < 0$. Furthermore, applying the Routh-Hurwitz criterion for the second-order polynomials of the characteristic equation implies that the real parts of the eigenvalues associated with the disease-free equilibrium E^0 system are negative. Consequently, E^0 is asymptotically stable when $R_0 < 1$. ■

Theorem 7.9. *If $R_0 > 1$, then the substance abuse endemic equilibrium E^* is locally asymptotically stable and unstable in the opposite case.*

Proof. At the endemic equilibrium, the Jacobian matrix is expressed as

$$\mathscr{J}(E^*) = \begin{pmatrix} \mathscr{J}_{11} & \mathscr{J}_{12} & \mathscr{J}_{13} & \mathscr{J}_{14} \\ \mathscr{J}_{21} & \mathscr{J}_{22} & \mathscr{J}_{23} & \mathscr{J}_{24} \\ 0 & \beta_1 & -(\mu+\mu_2+\varepsilon+\beta_2) & 0 \\ \eta & 0 & \varepsilon+\beta_2 & -(\mu+\sigma) \end{pmatrix}.$$

with

$$\begin{aligned}
\mathscr{J}_{11} &= -\beta\tfrac{A^*(A^*+I^*+R^*)}{(N^*)^2} - (\eta+\mu), & \mathscr{J}_{21} &= \beta\tfrac{A^*(A^*+I^*+R^*)}{(N^*)^2},\\
\mathscr{J}_{12} &= -\beta\tfrac{S^*(S^*+I^*+R^*)}{(N^*)^2}, & \mathscr{J}_{22} &= \beta\tfrac{S^*(S^*+I^*+R^*)}{(N^*)^2} - (\mu+\mu_1+\beta_1),\\
\mathscr{J}_{13} &= \beta\tfrac{S^*A^*}{(N^*)^2}, & \mathscr{J}_{23} &= -\beta\tfrac{S^*A^*}{(N^*)^2},\\
\mathscr{J}_{14} &= \beta\tfrac{S^*A^*}{(N^*)^2} + \sigma, & \mathscr{J}_{24} &= -\beta\tfrac{S^*A^*}{(N^*)^2}.
\end{aligned}$$

The characteristic equation associated with $\mathscr{J}(E^*)$ is given by

$$X^4 + a_3X^3 + a_2X^2 + a_1X + a_0 = 0, \tag{7.20}$$

with

$$\begin{aligned}
a_3 =& 2\mu + \mu_2 + \sigma + \varepsilon + \beta_2 - \mathscr{J}_{11} - \mathscr{J}_{22},\\
a_2 =& \mathscr{J}_{11}\mathscr{J}_{22} - (\mathscr{J}_{11} + \mathscr{J}_{22}\neg\infty(2\mu + \mu_2 + \varepsilon + \beta_2 + \sigma) - \mathscr{J}_{12}\mathscr{J}_{21} - \eta\mathscr{J}_{14} - \beta_1\mathscr{J}_{23}\\
&+ (\mu+\sigma)(\mu + \mu_2 + \varepsilon + \beta_2),\\
a_1 =& (\mathscr{J}_{11}\mathscr{J}_{22} - \mathscr{J}_{12}\mathscr{J}_{21})(2\mu + \mu_2 + \varepsilon + \beta_2) + \beta_1\mathscr{J}_{23}(\mathscr{J}_{11} - \mu - \sigma) + \eta\mathscr{J}_{14}(\mathscr{J}_{22}\\
&- (\mu + \mu_2 + \varepsilon + \beta_2)) - (\mu+\sigma)(\mu + \mu_2 + \varepsilon + \beta_2)(\mathscr{J}_{11} + \mathscr{J}_{22})\\
&- \beta_1(\mathscr{J}_{13}\mathscr{J}_{21} + (\varepsilon + \beta_2)\mathscr{J}_{24}) - \eta\mathscr{J}_{12}\mathscr{J}_{24},\\
a_0 =& (\mu+\sigma)(\mu + \mu_2 + \varepsilon + \beta_2)(\mathscr{J}_{11}\mathscr{J}_{22} - \mathscr{J}_{12}\mathscr{J}_{21}) + \beta_1\mathscr{J}_{11}((\varepsilon + \beta_2)\mathscr{J}_{24} + (\mu+\sigma)\mathscr{J}_{23})\\
&- \eta(\mu + \mu_2 + \varepsilon + \beta_2)(\mathscr{J}_{12}\mathscr{J}_{24} - \mathscr{J}_{14}\mathscr{J}_{22}) + \beta_1\mathscr{J}_{13}(-(\mu+\sigma)\mathscr{J}_{21} - \eta\mathscr{J}_{24})\\
&+ \beta_1\mathscr{J}_{14}(\eta\mathscr{J}_{23} - (\varepsilon + \beta_2)\mathscr{J}_{21}).
\end{aligned}$$

According to the Routh-Hurwitz criteria, the characteristic equation (7.20) has four negative eigenvalues if and only if $a_3 > 0, a_1 > 0, a_0 > 0$, and satisfy $a_3a_2a_1 > a_1^2 + a_3^2a_0$. Hence, these conditions guarantee the local asymptotic stability of the endemic equilibrium E^* of the fractional model, provided that $R_0 > 1$. ■

7.6 Sensitivity analysis

This section focuses on conducting a sensitivity analysis of key parameters to determine their potential impact on disease transmissibility. The sensitivity index, as defined by Chitnis et al. [60], is calculated for each parameter. For example,

$$\begin{aligned}
\gamma_\beta^{R_0} &= 1,\\
\gamma_\sigma^{R_0} &= \frac{\eta\sigma}{(\eta+\mu+\sigma)(\sigma+\mu)}.
\end{aligned}$$

Table 7.1 presents the computed results for all sensitivity indices.

Parameter symbol	Sensitivity index for parameter values
σ	+0.6453
β	+1
η	-0.8011
μ	+ 0.0168
μ_1	-0.1108
β_1	-0.7686

Table 7.1: Sensitivity analysis of parameters in R_0.

Parameters with positive indices β, σ, μ exert a significant influence on disease spread within the community when their values increase. Conversely, parameters with negative sensitivity indices η, μ_1, β_1 play a crucial role in reducing disease burden within the community when their values increase, holding other factors constant.
The analysis results, depicted in Figure 7.2, show that the transmission rate positively influences the basic reproduction number, with the transmission rate from the recovered state to susceptible due to awareness deficits also contributing significantly. In contrast, the rate at which females move into rehabilitation due to self-determination and the rate of addiction that develops infection are the parameters that most effectively lower the value of R_0.

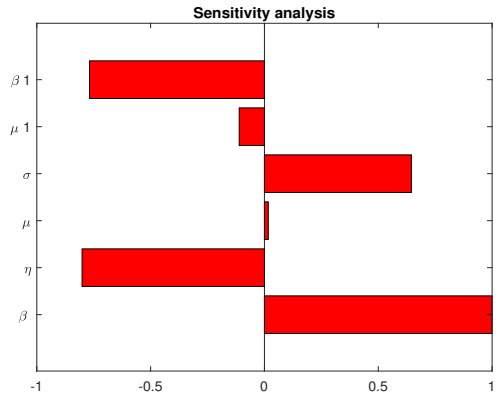


Figure 7.2: Sensitivity analysis.

The plot of Figure 7.3 reveals that R_0 remains lower when both β and β_1 are near zero. However, as β increases, R_0 rises significantly, especially when β_1 is also high.

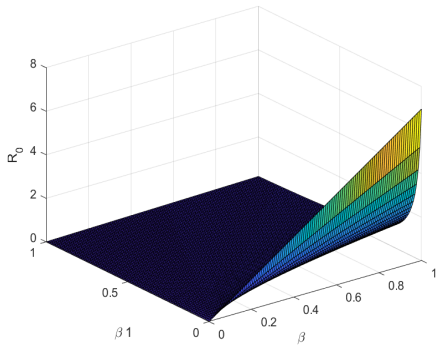


Figure 7.3: Dynamical behavior of R_0 against β and β_1.

For example, when $\beta = 0.8$ and $\beta_1 = 0.8$, R_0 can exceed 6, demonstrating that the two parameters substantially impact disease spread. Figure 7.4 illustrates that as σ increases, R_0 initially rises sharply, indicating that a higher rate of losing awareness or prevention boosts the potential for disease spread. Conversely, a higher natural mortality, including the deaths of individuals addicted to drugs, reduces the disease's spread. Figure 7.5 depicts that as η increases, R_0 initially spikes sharply, indicating that a higher rate of self-determined rehabilitation temporarily increases the potential for disease spread, possibly due to a transient increase in contact rates or movement among populations. However, R_0 quickly decreases and stabilizes as η continues to rise, reflecting the long-term benefits of rehabilitation in reducing disease spread. Meanwhile, as μ increases, R_0 consistently decreases, suggesting that higher natural mortality reduces the overall spread of the disease.

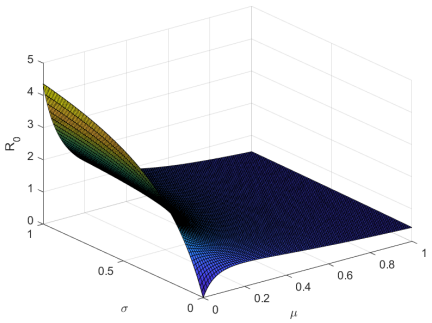


Figure 7.4: Dynamical behavior of R_0 against σ and μ.

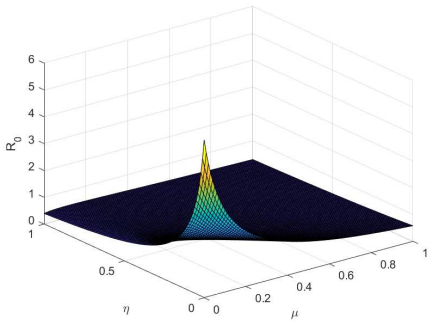


Figure 7.5: Dynamical behavior of R_0 against μ and η.

7.7 Numerical simulation and discussion

We use a uniform subdivision by dividing the time interval $[0,T]$ to N sub-intervals, where the step size h is defined as $\frac{T}{n}$, and the points t_j are given by $t_j = jh$ for $j = 0,1,...,n$. Therefore, based on the fractional version of Adams Bashfourth-Moulton's method discussed in 1.5.4, system $\mathscr{S}_{\alpha,C}$ is discretized as follows

Corrector part:

$$\begin{aligned}
S_{n+1} &= S_0 + \frac{h^\alpha}{\Gamma(\alpha+2)}\sum_{j=0}^{n} a_{j,n+1}\left(\Lambda - \beta\frac{S_jA_j}{N_j} - (\eta+\mu)S_j + \sigma R_j\right) \\
&\quad + \frac{h^\alpha}{\Gamma(\alpha+2)}\left(\Lambda - \beta\frac{S^p_{n+1}A^p_{n+1}}{N^p_{n+1}} - (\eta+\mu)S^p_{n+1} + \sigma R^p_{n+1}\right), \\
A_{n+1} &= A_0 + \frac{h^\alpha}{\Gamma(\alpha+2)}\sum_{j=0}^{n} a_{j,n+1}\left(\beta\frac{S_jA_j}{N_j} - (\mu+\mu_1+\beta_1)A_j\right) \\
&\quad + \frac{h^\alpha}{\Gamma(\alpha+2)}\left(\beta\frac{S^p_{n+1}A^p_{n+1}}{N^p_{n+1}} - (\mu+\mu_1+\beta_1)A^p_{n+1}\right), \\
I_{n+1} &= I_0 + \frac{h^\alpha}{\Gamma(\alpha+2)}\sum_{j=0}^{n} a_{j,n+1}\left(\beta_1A_j - (\mu+\mu_1+\varepsilon+\beta_2)I_j\right) \\
&\quad + \frac{h^\alpha}{\Gamma(\alpha+2)}\left(\beta_1A^p_{n+1} - (\mu+\mu_1+\varepsilon+\beta_2)I^p_{n+1}\right), \\
R_{n+1} &= R_0 + \frac{h^\alpha}{\Gamma(\alpha+2)}\sum_{j=0}^{n} a_{j,n+1}\left(\eta S_j + (\varepsilon+\beta_2)I_j - (\mu+\sigma)R_j\right) \\
&\quad + \frac{h^\alpha}{\Gamma(\alpha+2)}\left(\eta S^p_{n+1} + (\varepsilon+\beta_2)I^p_{n+1} - (\mu+\sigma)R^p_{n+1}\right),
\end{aligned}$$

Predictor part:

$$\begin{aligned}
S^p_{n+1} &= S_0 + \sum_{j=0}^{n} b_{j,n+1}\left(\Lambda - \beta\frac{S_jA_j}{N_j} - (\eta+\mu)S_j + \sigma R_j\right), \\
A^p_{n+1} &= A_0 + \sum_{j=0}^{n} b_{j,n+1}\left(\beta\frac{S_jA_j}{N_j} - (\mu+\mu_1+\beta_1)A_j\right), \\
I^p_{n+1} &= I_0 + \sum_{j=0}^{n} b_{j,n+1}\left(\beta_1A_j - (\mu+\mu_1+\varepsilon+\beta_2)I_j\right), \\
R^p_{n+1} &= R_0 + \sum_{j=0}^{n} b_{j,n+1}\left(\eta S_j + (\varepsilon+\beta_2)I_j - (\mu+\sigma)R_j\right).
\end{aligned}$$

We now proceed to solve our system using the initial conditions $S(0) = 20000$, $A(0) = 8000$, $I(0) = 7000$, and $R(0) = 4000$. Over three months, we analyze the dynamics of substance abuse among women, with a particular focus on how variations in the

fractional order impact the model's outcomes. The parameter values utilized in this analysis are found in [79].

Figures (7.6), (7.7) ,and (7.8) are presented to elucidate the endemic scenario ($R_0 > 1$) concerning the addicted, infected, and recovered population of females.

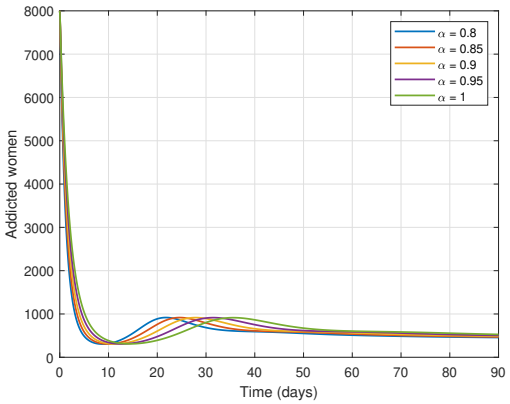


Figure 7.6: Dynamics of addicted women for various values of α.

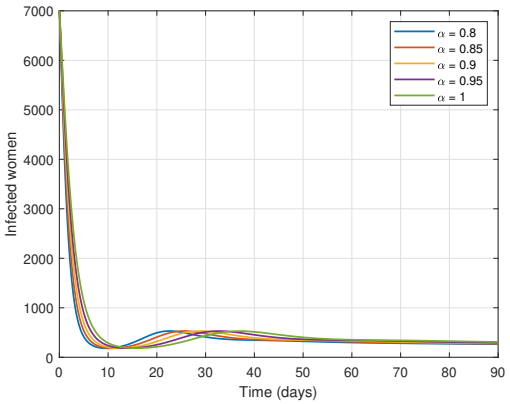


Figure 7.7: Dynamics of infected women for various values of α.

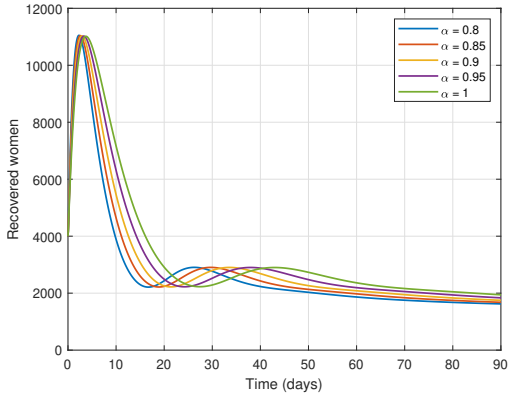


Figure 7.8: Dynamics of recovered women for various values of α.

These figures represent a notable trend, as the population of infected and addicted decreases, the number of recovered women increases exponentially in the first 20 days. As time elapses, the rate of recovery slows, leading to leveled-off curves. Additionally, we examine the influence of the fractional order α on the model's behavior, showcasing plots at varying α values. Lower alpha values (α <1) result in quicker convergence toward a steady state, suggesting that treatment variations influence recovery rates. Notably, as α decreases, the curves converge more rapidly, reflecting the nature of fractional-order systems where lower α values indicate faster stabilization. The influence of α is

similarly observed in the graphs depicting addicted and infected women. Therefore, the order of derivatives is crucial in predicting substance abuse transmission among women. These derivatives capture memory effects, reflecting how past behaviors influence future trends, especially in capturing rapid convergence behaviors, allowing for a more comprehensive understanding of how substance abuse spreads across different communities. Furthermore, understanding these dynamics is crucial for optimizing treatment protocols, particularly in substance abuse recovery scenarios for women under different conditions.

Furthermore, the nonlinear model, which involves the Caputo fractional derivatives, provides a more detailed representation of these dynamics compared to linear models. Unlike linear models that exhibit a simplified exponential decay. The proposed nonlinear model (7.1) provides a more nuanced and realistic representation by incorporating fractional derivatives, which allows them to reflect memory effects and capture intricate behaviors such as initial steep declines followed by oscillatory patterns before stabilization. This added complexity in nonlinear models enables a more detailed analysis of addiction trends, highlighting their superior ability to account for the impact of historical states on current behavior and offering a richer understanding of substance abuse dynamics among women.

To perform the numerical approach for model $\mathscr{S}_{\alpha,ABC}$, we applied the numerical method introduced by Toufik and Atangana which presented in 1.5.4. To approximate the ABC derivative. By using the same subdivision applied to $\mathscr{S}_{\alpha,C}$, the discretization of equation $\mathscr{S}_{\alpha,ABC}$ is as follows

$$
\begin{aligned}
S_{n+1} &= S_0 + \frac{1-\alpha}{\mathrm{ABC}(\alpha)} F_1(t_n, Y_n) + \frac{\alpha}{\mathrm{ABC}(\alpha)} \sum_{j=0}^{n} \left(\frac{h^\alpha F_1(t_j, Y_j)}{\Gamma(\alpha+2)} \left((n+1-j)^\alpha (n-j+2+\alpha) \right.\right. \\
&\quad \left.\left. -(n-j)^\alpha (n-j+2+2\alpha) \right) - \frac{h^\alpha F_1(t_{j-1}, Y_{j-1})}{\Gamma(\alpha+2)} \left((n+1-j)^{\alpha+1} - (n-j)^\alpha (n-j+1+\alpha) \right) \right), \\
A_{n+1} &= A_0 + \frac{1-\alpha}{\mathrm{ABC}(\alpha)} F_2(t_n, Y_n) + \frac{\alpha}{\mathrm{ABC}(\alpha)} \sum_{j=0}^{n} \left(\frac{h^\alpha F_2(t_j, Y_j)}{\Gamma(\alpha+2)} \left((n+1-j)^\alpha (n-j+2+\alpha) \right.\right. \\
&\quad \left.\left. -(n-j)^\alpha (n-j+2+2\alpha) \right) - \frac{h^\alpha F_2(t_{j-1}, Y_{j-1})}{\Gamma(\alpha+2)} \left((n+1-j)^{\alpha+1} - (n-j)^\alpha (n-j+1+\alpha) \right) \right), \\
I_{n+1} &= I_0 + \frac{1-\alpha}{\mathrm{ABC}(\alpha)} F_3(t_n, Y_n) + \frac{\alpha}{\mathrm{ABC}(\alpha)} \sum_{j=0}^{n} \left(\frac{h^\alpha F_3(t_j, Y_j)}{\Gamma(\alpha+2)} \left((n+1-j)^\alpha (n-j+2+\alpha) \right.\right. \\
&\quad \left.\left. -(n-j)^\alpha (n-j+2+2\alpha) \right) - \frac{h^\alpha F_3(t_{j-1}, Y_{j-1})}{\Gamma(\alpha+2)} \left((n+1-j)^{\alpha+1} - (n-j)^\alpha (n-j+1+\alpha) \right) \right),
\end{aligned}
$$

$$R_{n+1} = R_0 + \frac{1-\alpha}{\mathrm{ABC}(\alpha)} F_4(t_n, Y_n) + \frac{\alpha}{\mathrm{ABC}(\alpha)} \sum_{j=0}^{n} \left(\frac{h^\alpha F_4(t_j, Y_j)}{\Gamma(\alpha+2)} \left((n+1-j)^\alpha (n-j+2+\alpha) \right.\right.$$
$$\left.\left. -(n-j)^\alpha (n-j+2+2\alpha) \right) - \frac{h^\alpha F_4(t_{j-1}, Y_{j-1})}{\Gamma(\alpha+2)} \left((n+1-j)^{\alpha+1} - (n-j)^\alpha (n-j+1+\alpha) \right) \right),$$

with

$$F_1(t,Y) = \Lambda - \beta \frac{S(t)A(t)}{N(t)} - (\eta+\mu)S(t) + \sigma R(t), \quad F_2(t,Y) = \beta \frac{S(t)A(t)}{N(t)} - (\mu+\mu_1+\beta_1)A(t),$$
$$F_3(t,Y) = \beta_1 A(t) - (\mu+\mu_2+\varepsilon+\beta_2)I(t), \quad F_4(t,Y) = \eta S(t) + (\varepsilon+\beta_2)I(t) - (\mu+\sigma)R(t),$$
$$Y = (S, A, I, R).$$

The simulation is done under the same conditions as the system $\mathscr{S}_{\alpha,C}$.
Figures (7.9), (7.10), (7.11) and (7.12) illustrate the dynamic of addicted women over time for various values of α ranging from 0.7 to 0.9 when $R_0 < 1$ and $R_0 > 1$.

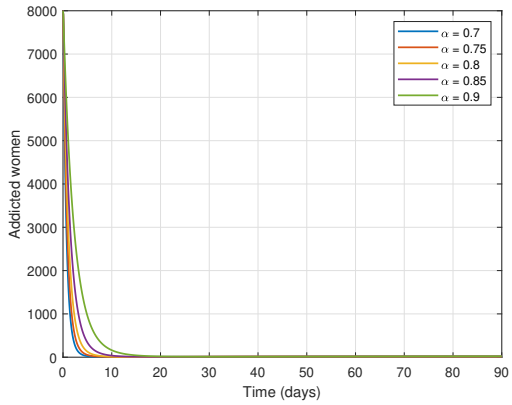


Figure 7.9: Addicted women for $R_0 < 1$.

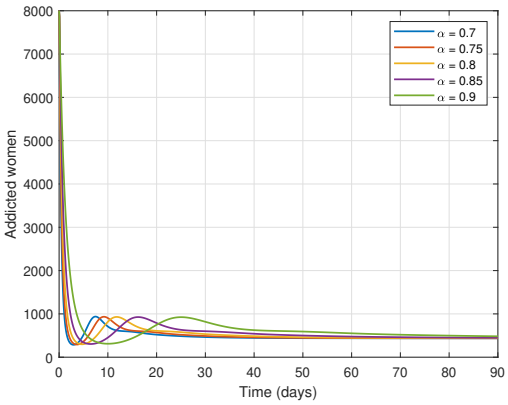


Figure 7.10: Addicted women for $R_0 > 1$.

Across all values of the fractional-order derivative, when $R_0 < 1$ the number of addicted women drops rapidly in the early days until reaching zero indicating that the epidemic eventually fades out. For $R_0 > 1$, there is an initial sharp decrease in the number of addicted women, followed by stabilization at a lower level. A smaller fractional order derivative (closer to 0.7) tends to result in a slower decline, indicating that a stronger memory effect may prolong the persistence of addiction within the population.
Similar to the dynamics observed in the addicted population, for $R_0 > 1$ the number of infected women experiences a sharp initial decline. This decrease occurs as a result of immediate interventions or natural progression within the infected group. However, as time progresses, the rate of decline begins to slow, and the population

size stabilizes toward a steady state. Notably, this stabilization effect appears more pronounced for lower values of the fractional order α as depicted in Figures 7.11 and 7.12.

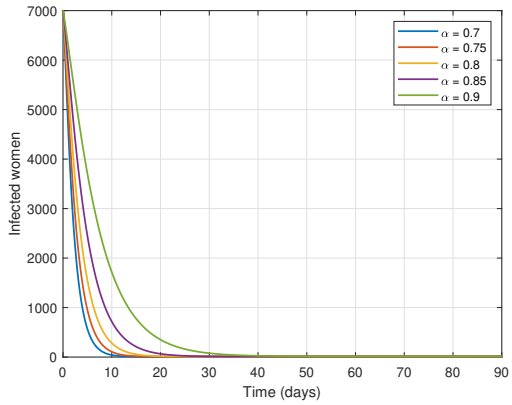


Figure 7.11: Infected women for $R_0 < 1$.

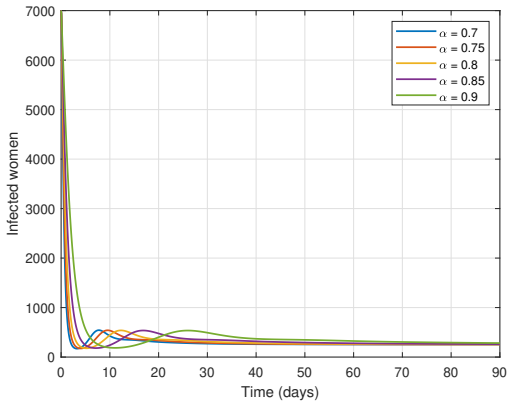


Figure 7.12: Infected women for $R_0 > 1$.

Figures (7.13) and (7.14) depict the dynamics of recovered females for $R_0 < 1$ and $R_0 > 1$.

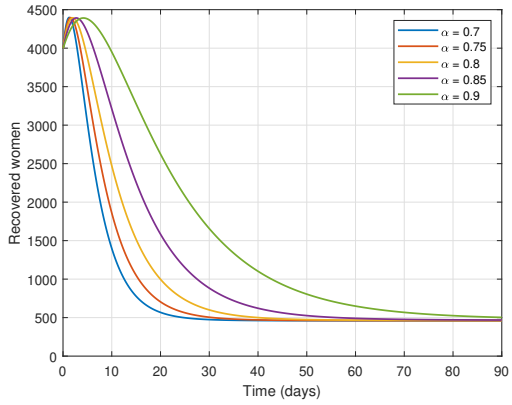


Figure 7.13: Recovered women for $R_0 < 1$.

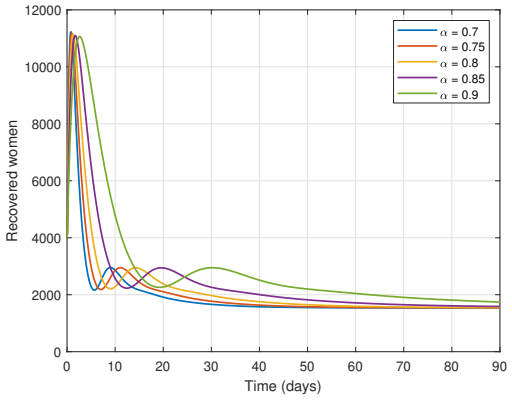


Figure 7.14: Recovered women for $R_0 > 1$.

The left plot represents the case when the reproduction number $R_0 < 1$ when the number of recovered females initially rises to a peak and then gradually declines until stabilization. In the other case, the behavior is more complex, with initial peaks followed by oscillations before stabilization. These oscillations become more pronounced as α decreases, emphasizing the prolonged influence of the epidemic and the significant impact of memory effects when the reproduction number exceeds 1.

Comparaison of ABC and Caputo operators

In addition to the detailed analysis of the fractional-order models, we delve further by presenting a comparative analysis. This comparison seeks to underscore the nuances between the ABC and the Caputo derivatives in modeling the dynamics of substance abuse among women. The graphical representations offer a side-by-side analysis of the infection and recovery trajectories for fractional orders $\alpha = 0.65$ and $\alpha = 0.85$.

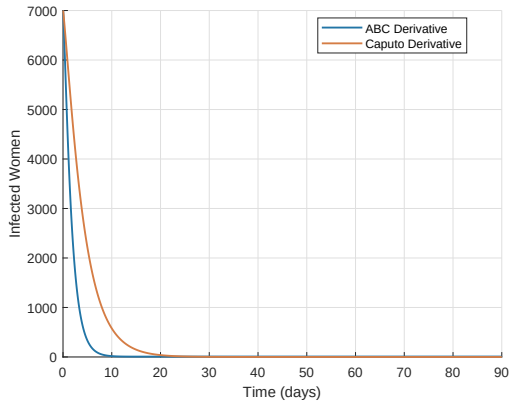


Figure 7.15: Infected women for $\alpha = 0.65$.

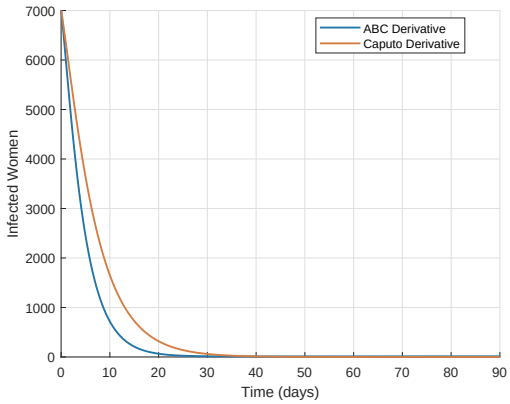


Figure 7.16: Infected women for $\alpha = 0.85$.

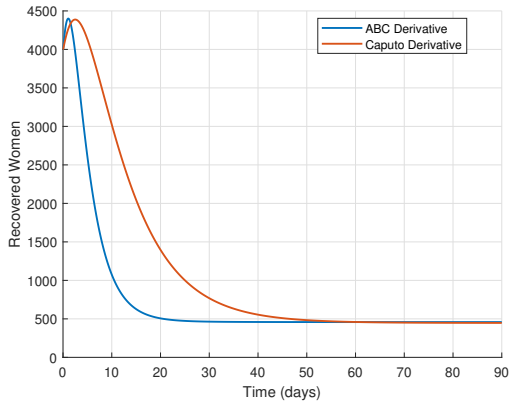


Figure 7.17: Recovered women for $\alpha = 0.65$.

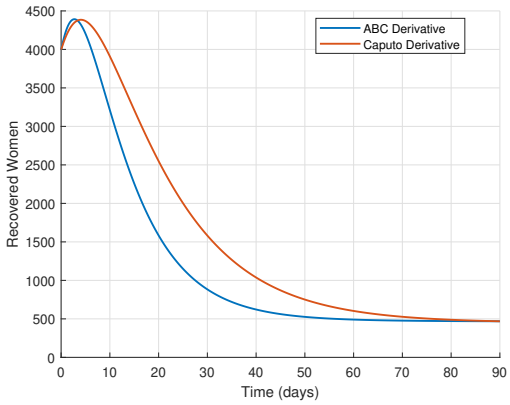


Figure 7.18: Recovered women for $\alpha = 0.85$.

The analysis of the infected and recovered populations reveals that the ABC derivative generates markedly sharper initial peaks and more rapid declines. This dynamic suggests an accelerated yet ephemeral response, possibly capturing the swift escalation of substance abuse episodes and their equally rapid subsidence following intervention or

treatment. In contrast, the Caputo derivative presents a more measured and gradual evolution, indicating a sustained yet less abrupt trajectory. This characteristic may more aptly model treatment protocols' slow yet persistent impact, where behavioral changes manifest progressively over time. Such disparities in the response curves underscore the nuanced influence of the chosen fractional derivative in modeling transient versus persistent phenomena within epidemiological systems.

As the fractional order α increases, these distinctions progressively diminish, particularly when α approaches 0.85, where the predictions of both derivatives begin to converge. This convergence underscores the pivotal role that the fractional order plays in shaping the dynamical properties of the system. At higher values of α, both derivatives capture long-term epidemic behaviors with increasing similarity, suggesting that fractional-order models may align in their predictions as they approach integer-like behavior. This comparative analysis highlights the adaptability of fractional calculus in complex modeling frameworks, where the short-term dynamics may vary significantly depending on the derivative employed, while the long-term behavior tends to harmonize across different formulations. Such insights not only elucidate the mathematical distinctions between the ABC and Caputo derivatives but also offer practical implications for optimizing intervention strategies, with the ABC derivative favoring short-term, immediate responses and the Caputo derivative proving advantageous for sustained, long-term modeling.

7.8 Analysis of fractional control optimal model

In this section, we enhance the model system (7.1) by incorporating two time-dependent controls, $u_1(t)$ and $u_2(t)$. The first control $u_1(t)$ represents public outreach programs aimed at raising awareness about the risks, detrimental impacts, and societal ramifications of engaging in substance abuse in women. The second control $u_2(t)$ represents the intervention that enhances the rehabilitation rate of women struggling with substance abuse through medical, psychological, and various other forms of treatment. Therefore, system (7.1) is modified as follows

$$
\begin{cases}
{}^{C}D_t^{\alpha} S(t) = \Lambda - (1-u_1(t))\beta \dfrac{S(t)A(t)}{N(t)} - (\eta+\mu)S(t) + \sigma R(t), \\
{}^{C}D_t^{\alpha} A(t) = (1-u_1(t))\beta \dfrac{S(t)A(t)}{N(t)} - (\mu+\mu_1+\beta_1)A(t), \\
{}^{C}D_t^{\alpha} I(t) = \beta_1 A(t) - (\mu+\mu_2+\varepsilon+\beta_2+u_2(t))I(t), \\
{}^{C}D_t^{\alpha} R(t) = \eta S(t) + (\varepsilon+\beta_2+u_2(t))I(t) - (\mu+\sigma)R(t),
\end{cases}
\tag{7.21}
$$

with non-negative initial conditions:

$$S(0)=S_0\ ,\ \ A(0)=A_0,\quad A(0)=A_0,\quad R(0)=R_0.$$

The objective functional associated with the system (7.1) is expressed as follows

$$\mathscr{J}(u_1,u_2)=\int_0^T\left(B_1A+B_2I+\sum_{i=1}^{2}\frac{1}{2}C_iu_i^2\right)dt, \tag{7.22}$$

subject to (7.21), where B_1 and B_2 are two positive weights balancing the addicted and infected population of females. C_1 and C_2 relate to the cost of applying control efforts u_1 and u_2. Furthermore, to avoid the bang-bang or singular control, we adopt a quadratic cost on the control as an estimation for a nonlinear function based on the assumption that the cost takes a nonlinear form. Our main goal in this section is to find the optimal control function (u_1^*,u_2^*) such that

$$\mathscr{J}(u_1^*,u_2^*)=\min_{u_1,u_2\in\mathscr{O}}\mathscr{J}(u_1,u_2),$$

subject to (7.21), and the control set $\mathscr{O}$ is defined as follows

$$\mathscr{O}=\left\{\begin{array}{c}(u_1,u_2);\ \ u_i\ \text{ is a measurable function such that}\\ 0\leqslant u_i(t)\leqslant 1\ \ \forall t\in[0,T],\ \ \forall i=1,2.\end{array}\right\} \tag{7.23}$$

Theorem 7.10. *Let (u_1^*,u_2^*) and (S^*,A^*,I^*,R^*) are the optimal control and the solutions corresponding to the system (7.21) respectively. Then, there exist adjoint variables λ_1, λ_2, λ_3, and λ_4 such that*

$$\begin{aligned}
{}^{C}D_t^{\alpha}\lambda_1(t)&=\beta(1-u_1^*)\tfrac{A^*(A^*+I^*+R^*)}{N^{*2}}(\lambda_1-\lambda_2)+\lambda_1(\eta+\mu)-\lambda_4\eta,\\
{}^{C}D_t^{\alpha}\lambda_2(t)&=-B_1+\beta(1-u_1^*)\tfrac{S^*(S^*+I^*+R^*)}{N^{*2}}(\lambda_1-\lambda_2)+\lambda_2(\mu+\mu_1+\beta_1)-\lambda_3\beta_1,\\
{}^{C}D_t^{\alpha}\lambda_3(t)&=-B_2+\beta(1-u_1^*)\tfrac{S^*A^*}{N^{*2}}(\lambda_2-\lambda_1)+(\lambda_3-\lambda_4)(\beta_2+\varepsilon+u_2^*)+\lambda_3(\mu+\mu_2),\\
{}^{C}D_t^{\alpha}\lambda_4(t)&=\beta(1-u_1^*)\tfrac{S^*A^*}{N^{*2}}(\lambda_2-\lambda_2)-\lambda_1\sigma+\lambda_4(\mu+\sigma),
\end{aligned} \tag{7.24}$$

with the transversality conditions

$$\lambda_1(T)=\lambda_2(T)=\lambda_3(T)=\lambda_4(T)=0.$$

Furthermore, the optimal controls u_1^*,u_2^* verify the following optimality conditions

$$\begin{aligned}
u_1^*(t)&=(\lambda_2(t)-\lambda_1(t))\frac{\beta S(t)^*A(t)^*}{C_1N(t)^*},\\
u_2^*(t)&=\frac{I(t)^*(\lambda_3(t)-\lambda_4(t))}{C_2}.
\end{aligned}$$

Proof. The Hamiltonian associated to system (7.21) is given by

(7.25)

$$\begin{aligned}\mathscr{H}(S,A,I,R) = B_1A + B_2I + \sum_{i=1}^{2}\tfrac{1}{2}C_iu_i^2 + \lambda_1\left(\Lambda - (1-u_1(t))\beta\tfrac{S(t)A(t)}{N(t)} - (\eta+\mu)S(t) + \sigma R(t)\right)\\ + \lambda_2\left((1-u_1(t))\beta\tfrac{S(t)A(t)}{N(t)} - (\mu+\mu_1+\beta_1)A(t)\right)\\ + \lambda_3\left(\beta_1A(t) - (\mu+\mu_2+\varepsilon+\beta_2+u_2(t))I(t)\right)\\ + \lambda_4\left(\eta S(t) + (\varepsilon+\beta_2+u_2(t)I(t) - (\mu+\sigma)R(t)\right).\end{aligned}$$

Where $\lambda_1,\lambda_2,\lambda_3,\lambda_4$, are the Lagrange multipliers of the optimization problem, and they are obtained by differentiating the Hamiltonian function $\mathscr{H}$ with respect to the optimal state variables S^*,A^*,I^*, and R^*. Furthermore, To obtain the optimal controls u_1^* and u_2^*, we differentiate the Hamiltonian with respect to the control variables according to the following conditions

$$\frac{\partial\mathscr{H}(S^*,A^*,I^*,R^*)}{\partial u_1^*} = \frac{\partial\mathscr{H}(S^*,A^*,I^*,R^*)}{\partial u_2^*} = 0.$$

The equations of controls are obtained using necessary conditions for the optimality [107]. As a result, the optimal control characterization for u_1^* and u_2^* with bounds is given by

$$\begin{aligned}u_1^* &= \min\left\{\max\left\{(\lambda_2(t)-\lambda_1(t))\frac{\beta S^*A^*}{C_1N^*},0\right\},1\right\},\\ u_2^* &= \min\left\{\max\left\{\frac{I^*(\lambda_3-\lambda_4)}{C_2},0\right\},1\right\}.\end{aligned}$$

■

7.8.1 Numerical simulation with control

In this subsection, we illustrate the numerical simulation of the fractional optimal control model. The state equations (7.21) are solved by the forward fractional Adams Bashfourth-Moulton method, then using the terminal conditions, the values of the controls u_1^* and u_2^*, and the state variables to solve the co-state equations (7.24) using the backward fractional Adams-Bashforth method.

We investigate the impact of various optimal control scenarios

- Scenario 1: Applying control u_1 along with elimination of u_2.
- Scenario 2: Applying the two controls u_1 and u_2.

We fix the fractional-order derivative at $\alpha = 0.85$ in this simulation and use the same initial conditions for both scenarios.

In Figure 7.19, without control (blue line), the number of addicted women initially decreases rapidly from 7000, then stabilizes at about 1000 after 50 days. The decrease is more pronounced with control (magenta line), stabilizing at a slightly lower number. Figure 7.20, similar trends are observed. Without control, the number of infected women decreases from around 7000 to about 1000 after 50 days. With control, the decrease is faster and stabilizes at a slightly lower number.

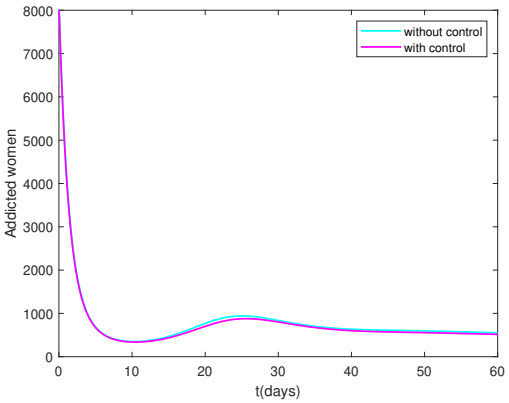


Figure 7.19: Dynamics of addicted women for strategy 1.

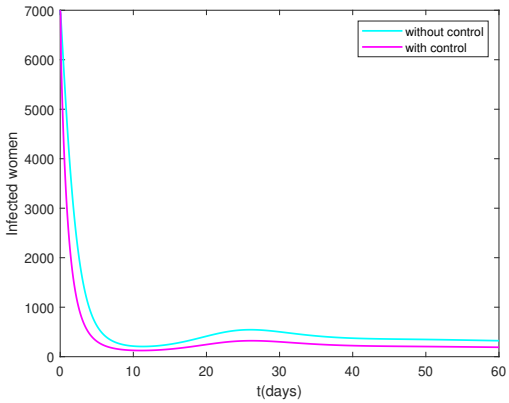


Figure 7.20: Dynamics of infected women for strategy 1.

We remark that with control in Figure 7.21, the decrease is more pronounced and rapid, resulting in a near-zero stabilization level. This indicates that the combined interventions significantly enhance the rehabilitation rate, effectively reducing the number of addicted women to almost zero over time. As well Figure 7.22 illustrates in the application of both controls that the number of infected women decreases extremely fast as compared to strategy 1.

The results clearly show that the use of both public outreach programs and rehabilitation interventions is crucial in combating substance abuse in women. These controls work synergistically to prevent substance abuse initiation and enhance recovery rates, resulting in a substantial and sustained reduction in the number of addicted and infected women.

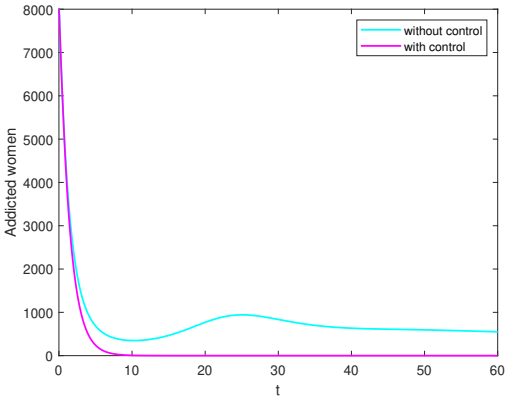


Figure 7.21: Dynamics of addicted women for strategy 2

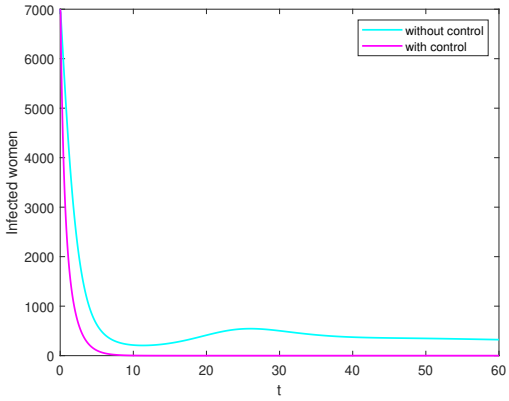


Figure 7.22: Dynamics of infected women for strategy 2

7.9 Conclusion

This study highlights the valuable role of fractional calculus in modeling substance abuse dynamics among women, emphasizing the effectiveness of integrating public outreach with rehabilitation programs. The fractional-order systems, particularly through the Caputo derivative, demonstrate the influence of memory effects, represented by the fractional order α, on the progression and management of addiction. Lower values of α lead to slower recovery and prolonged epidemics, illustrating how memory-dependent behaviors influence long-term outcomes. While the ABC derivative captures sharper peaks and faster declines, representing the transient and immediate nature of addiction's impact, the Caputo derivative excels in modeling sustained epidemic dynamics. It offers a more gradual and realistic depiction of addiction persistence and recovery, making it better suited for long-term forecasting. As α increases, the Caputo and ABC derivatives converge, revealing similar dynamics, but the Caputo derivative's strength lies in its ability to model extended epidemic behavior with greater precision.

The findings emphasize the critical importance of fractional-order models, with the Caputo derivative providing a balanced approach that integrates short-term responses with long-term trends. This combined approach offers a deeper understanding of substance abuse dynamics and reinforces the importance of tailored public health strategies. By leveraging fractional calculus, particularly the Caputo framework, this research provides a more comprehensive tool for addressing addiction in women and optimizing intervention strategies.

CONCLUSION AND PREPSECTIVES

This thesis presents a significant contribution to the field of fractional calculus, addressing theoretical analysis within the framework of variational problems. It addresses a broad spectrum of topics, including Kirchhoff-type, nonlocal nonlinear Neumann boundary, bi-nolocal, and singular problems. Our work systematically examines various variational approaches, elucidating the criteria for selecting the most appropriate method based on the existence of one or multiple weak solutions, while also analyzing the conditions that the source terms must satisfy. This investigation further extends to the introduction of a novel anisotropic fractional framework, paving the way for new challenges and opportunities within the domain of directional behavior.
Furthermore, this dissertation showcased the power of fractional models in epidemiology, particularly through case studies on CMD and SAW. The CMD model provided insights into agricultural systems, using memory effects captured by fractional calculus to analyze disease spread, develop control strategies, and support agricultural sustainability. The substance abuse model demonstrated how fractional derivatives can effectively guide strategies for diminishing abuse among women, particularly through interventions in awareness and rehabilitation. Together, these models highlight fractional calculus's contribution across fields, providing nuanced perspectives that aid in crafting strategies for public health, agriculture, and beyond.
In conclusion, this thesis has effectively addressed the fundamental questions posed at the outset, demonstrating that within the realm of fractional calculus, the mathematical possibilities are indeed vast and boundless.

As we reflect on the substantial contributions of this thesis to the field of fractional calculus, it becomes evident that the implications of this research extend far beyond the theoretical framework established herein. Moreover, this work underscores the importance of maintaining a robust mathematical framework, paving the way for future investigations into more complex variational problems. The following perspectives outline key areas for investigation:

- Study more complex variational problems in a fractional context, particularly problems without adherence to the Ambrosetti-Aabinowitz (AR) conditions and growth conditions.
- Fractional $p(x,\cdot)$-Kirchhoff-type problem with magnetic field via the Nehari Manifold approach.
- On a singular $p(x,\cdot)$-integro-differential elliptic problems with Neumann boundary conditions.
- Extend the study of the fractional anisotropic Sobolev spaces with variable exponents to the mixed Lebesgue spaces.
- Recognize the important implications of our work on Kirchhoff-type problems for bacterial dynamics. We aim to integrate numerical analysis of these problems into epidemiology, enhancing our understanding of disease dynamics and transmission.
- Extend our study in the agricultural system to focus on tomato plants, particularly about nematode infections. Utilizing fractional calculus, we aim to analyze the dynamics of these infections more comprehensively. A critical aspect of this future work will involve incorporating data from farmers in the Souss-Massa region-Morocco.
- Expand our research to encompass a wider array of societal challenges, utilizing mathematical models and fractional calculus to provide nuanced and impactful analyses.

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