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Stabilité et hyperstabilité de certaines équations fonctionnelles

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Abstract

This thesis is an investigation of the Ulam-stability of some linear functional equations in different domains, namely in Banach spaces and in Random normed spaces, as well as of some differential equations with delay in locally convex spaces. In order to show our results, we have been led to establish and apply some fixed point theorems in general settings.

First, we provide a new proof of a result of Bahyrycs and Olko showing the Ulam-stability of the non homogeneous Forti equation in Banach spaces. Instead of the use of a elaborated theorem due to Brzdęk, Chudziak and Páles, we use the classical Banach contraction principle and prove the same result.

As Forti equation involves concrete linear mappings, namely the multiplication by a scalar and a mapping associated with a matrix, we introduce an abstract linear functional equation, generalizing Forti's one and then most of the linear equations ever considered in the literature. We then study its stability and its hyperstability, using a fixed point approach.

Next, we show a general fixed point theorem in random normed spaces. As an application of it, we prove Ulam-stability results for a non-homogeneous Forti functional equation, with non constant second member.

Finally, we prove an alternative fixed point theorem in generalized gauge spaces, extending the famous alternative fixed point theorem of Diaz and Margoli. Using this fixed point theorem, we show the Ulam stability of a general delay differential equation in locally convex spaces.

Keywords : Ulam stability, general linear functional equation, delay functional differential equation, direct method, fixed point method, random normed space, locally convex space, generalized gauge space.

Résumé

Cette thèse est une étude sur la stabilité au sens d'Ulam de certaines équations fonctionnelles linéaires dans des domaines différents, notamment dans les espaces de Banach et les espaces normés aléatoires, ainsi que de certaines équations différentielles avec retard dans des espaces localement convexes. Afin de montrer nos résultats, nous avons été amenés à établir et à appliquer des théorèmes de point fixe dans des contextes généraux. Nous commençons par donner une nouvelle preuve simple d'un résultat de Bahyrycz et Olko [18] montrant la stabilité d'Ulam de l'équation de Forti non homogène. Ensuite, nous introduisons une équation fonctionnelle abstraite, généralisant celle de Forti et nous étudions sa stabilité et son hypersatbilité, en utilisant l'approche de point fixe. Enfin, nous montrons deux théorèmes du point fixe dans deux cadres différents, le premier dans les espaces uniformes généralisés séquentiellement complet et le second dans les espaces normés aléatoires. Nous appliquons ces deux théorèmes pour prouver des résultats sur la stabilité au sens d'Ulam.

Mots Clés : stabilité d'Ulam, équation fonctionnelle linéaire, équation différentielle avec retard, méthode directe, méthode du point fixe, espace normé aléatoire, espace localement convexe, espace uniforme généralisé.

Résumé étendu

Dans cette thèse, nous nous intéressons aux problèmes de stabilité et d'hyperstabilité de certaines équations fonctionnelles linéaires dans des domaines différents, notamment dans les espaces de Banach et les espaces normés aléatoires, ainsi que de certaines équations différentielles avec retard dans des espaces localement convexes. Afin de montrer nos résultats, nous avons été amenés à établir et à appliquer des théorèmes de point fixe dans des contextes généraux.

Nous commençons par donner une nouvelle preuve plus simple d'un résultat de Bahyrycz et Olko [18] montrant la stabilité d'Ulam de l'équation de Forti non homogène

$$\sum_{i=1}^{m} B_i f\left(\sum_{j=1}^{n} b_{ij}x_j\right) + B = 0,$$

où f est une application d'un espace vectoriel X dans un espace de Banach Y, $x_j \in X$, B_i et b_{ij} sont des scalaires et $B \in Y$. Au lieu d'utiliser un théorème compliqué dû à Brzdęk, Chudziak et Páles comme dans [18], nous utilisons le théorème classique de contraction de Banach.

En analysant l'équation de Forti non-homogène, on remarque qu'elle consiste en une somme de compositions d'applications linéaires. Pour cela, nous introduisons une équation fonctionnelle linéaire abstraite, généralisant celle de Forti, et donc la plupart des équations fonctionnelles linéaires considérées dans la littérature. Cette équation se lit

$$\sum_{i=1}^{m} A_i\left(f\left(\varphi_i(\bar{x})\right)\right) + b = 0,\ \bar{x} := (x_1, \ldots, x_n) \in X^n,$$

où m et n sont des entiers positifs, pour tout $i \in \{1, 2, \ldots, m\}$, φ_i est une application linéaire de X^n vers X, A_i est un endomorphisme continu de Y, et $b \in Y$ En utilisant une approche de point fixe, nous étudions la stabilité et l'hyperstabilité de cette équation.

Ensuite, nous montrons un théorème général du point fixe concernant les applications sur Y^X où X est un ensemble non vide et Y est un espace normé aléatoire. En particulier, ce résultat généralise le théorème du point fixe établi par Brzdęk, Chudziak et Páles [39], Comme application de celui-ci, nous prouvons des résultats de stabilité d'Ulam pour une équation fonctionnelle de Forti non-homogène avec un second membre non constant.

Enfin, nous démontrons un théorème d'alternative du point fixe dans les espaces uni-

forme généralisés séquentiellement complet, étendant le célèbre théorème d'alternatives du point fixe établi par Diaz et Margoli en 1968. En utilisant ce théorème, nous montrons la stabilité d'Ulam de l'équation différentielle avec retard

$$y'(t) = F(t, y(t), y(\nu(t))), \quad t \in I,$$

où y n'est plus défini à partir d'un intervalle I vers $\mathbb{R}$, mais vers un espace localement convexe (en particulier dans un espace de Banach). Pour l'intervalle I, les cas borné et non borné sont considérés. Signalons que l'étude de la stabilité des équations différentielles au sens d'Ulam a été initiée par Obloza au cours des années 1990 et que notre résultat fondamental généralise plusieurs études qui existent à ce props dans la littérature.

Publications that make up this dissertation

During the preparation of this thesis, we produced a chapter published in Springer, two articles published in international journals indexed by Scopus, and another article submitted to a journal also indexed by Scopus :

1. C. Benzarouala and L. Oubbi, *A Purely Fixed Point Approach to the Ulam-Hyers Stability and Hyperstability of a General Functional Equation.* In Ulam Type Stability, Springer, 47-56, 2019.

2. C. Benzarouala and L. Oubbi, *Ulam-stability of a generalized linear functional equation, a fixed point approach.* Aequat. Math. **94**, 989-1000, 2020.

3. C. Benzarouala, J. Brzdęk and L. Oubbi, *A fixed point theorem and Ulam stability of a general linear functional equation in random normed spaces,* J. Fixed Point Theory Appl. 2022, (accepted).

4. C. Benzarouala and L. Oubbi, *A fixed point theorem in gauge spaces and applications to Ulam-Hyers-Rassias-stability of delay differential equations,* (submitted).

Contents

Introduction

The aim of this thesis is to investigate the Ulam-stability of some linear functional equations in different domains, namely in Banach spaces and in Random normed spaces, as well as of some differential equations with delay in locally convex spaces. In order to show our results, we have been led to establish and apply some fixed point theorems in general settings.

Actually, the study of the stability problems of functional equations goes back to 1940, when Ulam [189] discussed during his talk in the Mathematics club of the university of Wisconsin, the notion of stability of mathematical theorems considered from a rather general point of view : When is it true that by changing a little the hypotheses of a theorem, one can still assert that the thesis of the theorem remains true or approximately true ? Particularly, for a general functional equation, one can ask the following question: When is it true that approximate solution of an equation must be close to an exact solution of the given equation ? Actually, Ulam asked the following question concerning the group homomorphisms:

Let X be a group, (Y, d) be a metric group and ϵ be a positive number. Does there exists a number $\delta > 0$ such that, whenever a function $f : X \to Y$ satisfies

$$d(f(xy), f(x)f(y)) < \delta, \quad x, y \in X, \tag{1}$$

there exists a group homomorphism $G : X \to Y$ such that

$$d(f(x), G(x)) < \epsilon, \qquad x \in X? \tag{2}$$

Whenever the answer to this problem is in the affirmative, one says that the homomorphism equation $f(xy) = f(x)f(y)$ is Ulam stable, or that the group homomorphisms are stable with respect to the equation $f(xy) = f(x)f(y)$ and the Ulam approximation. Notice that, in the Ulam stability, the control functions, ϵ and δ, do not depend on x or on y (see (1) and (2)).

The first affirmative answer to Ulam's problem was given in 1941 by Hyers [87], who answered the question in the context of real Banach space. He namely showed that the Cauchy functional equation

$$f(x + y) = f(x) + f(y),$$

is Ulam stable, whenever X and Y are real Banach spaces, with $\epsilon = \delta$. In his proof, Hyers [87] constructed the equation solution as a limit of a sequence starting from the given approximate solution. This method is subsequently known as the direct method.

Later, in 1950 Aoki [13] considered Ulam's problem with a new kind of approximation. He allowed the Cauchy differences to be unbounded, but dominated as follows

$$\|f(x+y) - f(x) - f(y)\| \leq c(\|x\|^p + \|y\|^p)$$

for some positive constant c and some $p \in [0, 1[$. With a similar method as Hyers [87], he obtained that the Cauchy functional equation is stable with respect to the approximation above. This was a nice generalization of Hyers' theorem.

Next, in 1978, Rassias [157] succeeded in extending the result of Aoki [13] for the case $p < 1$ and he asked whether the same result holds true if $p \geq 1$. In 1991, Gajda [74] answered definitely Rassias's question. He obtained the result for $p > 1$, using the same approach as Rassias [157], and gave an example to show how the stability fails when the power of the norms is 1.

In [158, 160], Rassias investigated the stability problem of Cauchy equation with the following approximation

$$\|f(x+y) - f(x) - f(y)\| \leq c\|x\|^p\|x\|^p,$$

with $0 \leq p < \frac{1}{2}$. In 1992, Găvruta [75] generalized the results of Hyers [87], Aoki [13], Rassias [157, 158, 160] and Gajda [74], by using a general control function and the direct method. From there on, when an functional equation is stable with respect to a non uniform approximation , one say that it is Ulam-Hyers-Rassias stable or just Hyers-Rassias stable.

In 2003, Radu [156] showed that the results of Hyers, Rassias, and Gajda concerning the stability of the Cauchy functional equation in Banach spaces can be obtained using the alternative fixed point theorem of Diaz and Margoli [62]. He thus founded what is nowadays known as the fixed point method. This method consists of constructing on a subspace $\tilde{X}$ of functions from X into Y^X a complete generalized distance d and a strict contraction mapping J of $\tilde{X}$, so that a fixed point of J is the exact solution near the given approximate solution of the equation.

In the same year 2003, Cadariu and Radu [50], using again the alternative fixed point theorem, showed that the Jensen functional equation

$$2f(\frac{x+y}{2}) = f(x) + f(y)$$

is Ulam-Hyers-Rassias stable. Since then, the so-called Ulam stability problem has known an extensive expansion. Instead of the Cauchy or the Jensen equation, a large variety of functional equations have been considered in the literature, such as quadratic functional equations [60], Euler-Lagrange type equations [92, 143], cubic equations [54], quartic equations [148] and so on. Several authors have also considered systems of functional equations and studied their stability [132, 143]. A quite general non linear functional equation in a single variable was investigated in 1991 by Baker [21], namely

$$f(x) = F\left(x, g(\varphi(x))\right).$$

As noticed by Oubbi in [143], when dealing with the stability of the functional equation

$$f\left(\alpha(x+y)\right) + f\left(\beta(x-y)\right) = (\alpha+\beta)f(x) + (\alpha-\beta)f(y),$$

with $(\alpha, \beta) \in \mathbb{K} \setminus \{0\} \times \mathbb{K}$, including the Cauchy, the Jensen and the quadratic equation, the alternative fixed point theorem can be replaced by the classical Banach fixed point theorem, by reducing the space $\tilde{X}$, binding it to the approximate solution, in order to get a complete metric space. Oubbi combined this equation with other equations and showed the stability of the so-obtained systems. He thus proved, among other things, the stability of the Jordan n-homomorphisms and the n-derivations in Banach algebras.

It is a fact that the direct method and the fixed point method are the most popular techniques for proving the stability of functional equations, although the direct method does not work in some significant cases [98]. However, several other methods have been used to study Ulam stability problem, for instance the method of invariant means [180], the method based on sandwich theorems [145], and the method using the concept of shadowing [183].

In 2007, Forti [72] introduced the following linear functional equation

$$\sum_{i=1}^{m} B_i f\left(\sum_{j=1}^{n} b_{ij}x_j\right) = 0, \qquad \text{(Forti equation)}$$

where f is a mapping from a vector space X into a Banach space Y, $x_j \in X$, and B_i and b_{ij} are scalars for every $i \in \{1, \dots, m\}$ and every $j \in \{1, \dots, n\}$. Numerous important functional equations are particular cases of this one, for example Cauchy equation, Jensen equation, the linear equation in two variables

$$f(ax + by + c) = Af(x) + Bf(y),$$

the Jordan-von Neumann (quadratic) equation

$$f(x + y) + f(x - y) = 2f(x) + 2f(y),$$

the Drygas equation

$$f(x + y) + f(x - y) = 2f(x) + f(y) + f(-y),$$

and the Fréchet equation

$$f(x + y + z) + f(x) + f(y) + f(z) = f(x + y) + f(x + z) + f(y + z).$$

For more information on the Cauchy and Jensen equations we refer to [15, 116]. The quadratic equation (the law of parallelogram) was used by Jordan and von Neumann [91] to characterize the inner product spaces, and the Drygas and Fréchet equations were used for similar purposes [12, 73, 100]. In [17, 20, 65, 89, 95, 99, 133, 135, 150, 151, 152, 153, 159] the interested reader can find further related information and stability outcomes concerning those equations. For further examples of special cases of Forti equation, if $\Delta_y : Y^X \to Y^X$ is the Fréchet difference operator, defined for $x, y \in X$

$$\Delta_y f(x) := \Delta_y^1 f(x) := f(x + y) - f(x), \qquad x \in X, \qquad \text{and}$$
$$\Delta_z^{k+1} := \Delta_z \circ \Delta_z^k, \qquad z \in X,\ k \in \mathbb{N},\ \text{therefore}$$
$$\Delta_{x_{k+1}, x_k, \dots, x_1} := \Delta_{x_{k+1}} \circ \Delta_{x_k, \dots, x_1}, \qquad x_1, \dots, x_{k+1} \in X,\ k \in \mathbb{N}.$$

Clearly the following functional equations

$$\Delta_z^k f(x) = 0, \qquad x, z \in X,$$
$$\Delta_{x_k,\ldots,x_1} f(x) = 0, \qquad x, x_1, \ldots, x_k \in X,$$
$$\Delta_z^k f(x) = k! f(x), \qquad x, z \in X,$$

are particular cases of Forti equation, we refer to [79, 89, 116, 120, 181] for more information on solutions and stability of these equations.

Note also that, if we write for every $x, y, x_1, \ldots, x_k, u, w \in X$ and $k \in \mathbb{N}$,

$$C^1 f(x, y) = f(x + y) - f(x) - f(y), \qquad \text{and}$$
$$C^{k+1} f(x_1, \ldots, x_k, u, w) = C^k f(x_1, \ldots, x_k, u + w) - C^k f(x_1, \ldots, x_k, u) - C^k f(x_1, \ldots, x_k, w)$$

then the Fréchet equation can be rewritten in the form $C^2 f(x, y, z) = 0, \qquad x, y, z \in X$. Generally,

$$C^{k+1} f(x_1, \ldots, x_{k+2}) = 0, \qquad x_1, \ldots, x_{k+2} \in X,$$

is also a particular case of Forti equation. A further example of Forti equation is given by

$$M\left[f\left(\frac{x+y+z}{l}\right) + f(x) + f(y) + f(z)\right]$$
$$= N\left[f\left(\frac{x+y}{k}\right) + f\left(\frac{x+z}{k}\right) + f\left(\frac{y+z}{k}\right)\right],$$

where $M, N, l, k \in \mathbb{Z} \setminus \{0\}$. Such an equation has been studied in [55, 56, 57, 58]. For $M = l = 3$ and $N = k = 2$, this equation was introduced by Popoviciu [155] in connection with his investigation of convex functions. We refer to [118, 119, 184, 185] for information on its solutions and stability. Finally the p-Wright equation

$$f(px + (1-p)y) + f((1-p)x + py) = f(x) + f(y),$$

where $p \in \mathbb{R}$ is fixed, is also of Forti type equation. We refer to [16, 34] for some results concerning it.

In 2015, Bahyrycz and Olko [18] used a fixed point theorem due to Brzdęk et al [39]

(Theorem 1.1.6), to show the stability of the non-homogeneous Forti equation

$$\sum_{i=1}^{m} B_i f\left(\sum_{j=1}^{n} b_{ij}x_j\right) + B = 0,$$

where $B \in Y$, with respect to the control functions:

i) $\theta_1(x_1, \ldots, x_n) = C$

ii) $\theta_2(x_1, \ldots, x_n) = C\sum_{j=1}^{n} \|c_j x_j\|^{k_j}$

iii) $\theta_3(x_1, \ldots, x_n) = C \max\left\{\|c_j x_j\|^{k_j},\ j \in \{1, \ldots, n\}\right\}$

iv) $\theta_4(x_1, \ldots, x_n) = C\prod_{j=1}^{n} \|x_j\|^{k_j}$

v) $\theta_5(x_1, \ldots, x_n) = C\prod_{j=1}^{n} \|x_j\|^{t_j} + D\sum_{j=1}^{n} \|c_j x_j\|^{k_j}$

where $C, D \in]0; +\infty[$, $k_j, t_j < 0$ and $c_j \in \mathbb{K}^\star$.

In 2015, Dong [64], using the fixed point theorem due to Brzdęk, Chudziak and Páles [39], proved the hyperstability of Forti equation, with respect to the following two different approximation inequalities

$$\begin{aligned} &\|\sum_{i=1}^{m} B_i f\left(\sum_{j=1}^{n} a_{ij}x_j\right)\| \leq c\sum_{j=1}^{n} \|x_j\|^p, \qquad \text{and} \\ &\|\sum_{i=1}^{m} B_i f\left(\sum_{j=1}^{n} a_{ij}x_j\right)\| \leq c\prod_{j=1}^{n} \|x_j\|^{p_j}, \end{aligned}$$

with $p > 0$, and $p_1 + \cdots + p_n < 0$. In the same year, Bahyrycz and Olko [19], using the same theorem of [39], showed the hyperstability of the non-homogeneous Forti equation, with a constant second member, generalizing Dong's results [64]. Recall that a functional equation is hyperstable if every function satisfying the equation approximately is an exact solution of it. the hyperstability has been investigated recently for various equations, e.g., [17, 35, 36, 37, 151].

In this thesis, using a proof similar to that of Oubbi [143], relying on the classical Banach contraction theorem, we provide a new proof to the stability and the hyperstability of the non-homogeneous Forti equation as in [18]. By the way, in [18] and [19], the authors assume that B is a scalar. Actually, B must be an element of Y. Fortunately, this does not alter their results.

When analyzing the non-homogeneous Forti equation, it consists of a sum of compositions of trivial linear mappings. We introduce an abstract linear functional equation generalizing Forti equation, then also all the others above. This equation reads

$$\sum_{i=1}^{m} A_i\left(f\left(\varphi_i(\bar{x})\right)\right) + b = 0,\ \bar{x} := (x_1, \dots, x_n) \in X^n,$$

where m and n are positive integers, for every $i \in \{1, 2, \dots, m\}$, φ_i is a linear mapping from X^n into X, A_i is a continuous endomorphism of Y, and $b \in Y$. Using the Banach contraction fixed point theorem, as in [143], we prove the stability and hyperstability of this equation, then also, of course, of its associated homogeneous one

$$\sum_{i=1}^{m} A_i\left(f\left(\varphi_i(\bar{x})\right)\right) = 0,\ \bar{x} := (x_1, \dots, x_n) \in X^n.$$

Most known results on the stability of linear functional equations in the literature derive from ours. We also provide situations to which our theorem applies, but not covered by the former results.

On the other hand, Menger [125] proposed a probabilistic generalization of the metric spaces theory, called the theory of probabilistic metric (or random normed) spaces. This theory has been investigated by Šerstnev in a series of papers (see for example [174], [175], [176]), we can also refer to the book [81] in this respect. In 1986, Alsina [10] considered the stability of functional equations in probabilistic normed spaces and, in 2008, Mihet and Radu [126], using the fixed point method, proved some stability results of the Cauchy and the Jensen functional equations in random normed spaces.

Instead of the Cauchy and the Jensen functional equations, a large variety of functional equations have been considered in the literature in random spaces, for example in 2016, Kim et al [107] investigated the generalized Ulam stability of a general cubic functional equation. In 2017, Cho et al [53] proved the stability of quintic functional equations in random normed spaces. In the same year, Alshybani et al [11] used the direct and the fixed point methods, to establish the generalized Ulam stability of an additive-quadratic functional equation in such spaces. In 2018, Pinelas et al [149] by using the direct and the fixed point methods investigated the generalized Ulam stability of a new type of n-

dimensional cubic functional equation in random normed spaces. For more details about the stability of various functional equations in random normed spaces, one can see the book of Cho et al [52].

In this thesis, we prove a general fixed point theorem for classes of functions taking values in a random normed space. This is the random normed space version of the fixed point theorem due to Brzdęk et al [39]. Next, we show several of its consequences and, among others, we present applications of it in proving the Ulam stability results for the general non-homogeneous Forti equation

$$\sum_{i=1}^{m} B_i f\left(\sum_{j=1}^{n} b_{ij} x_j\right) = D\left(x_1, \ldots, x_n\right),$$

where the second member is no more constant, but a function $D : X^n \to Y$. We also show how to use our fixed point theorem to study the approximate eigenvalues and eigenvectors of some linear operators.

The Ulam stability of linear differential equations was firstly investigated in 1993 by Obloza [138, 139]. After that a large number of papers devoted to this subject have been published (see [11, 26, 93]). In 2010, Jung [94], using the fixed point method, proved the Ulam stability of the first order differential equation of the form

$$y'(t) = F(t, y(t)), \tag{3}$$

where $F : I \times \mathbb{R} \to \mathbb{R}$ is a continuous function satisfying a Lipschitz condition and I is a closed interval. In 2020, Başci et al [23], using the alternative fixed point theorem as in [94], extended and improved Jung's result for the differential equation (5.1). Several other differential equations without delay have been considered in the literature (see [18, 129, 141, 154]).

In 2015, using the fixed point method, Tunç and Biçer [187] proved the Ulam stability of the first-order delay differential equation

$$y'(t) = F(t, y(t), y(t-h)), \tag{4}$$

where $F : I \times \mathbb{R}^2 \to \mathbb{R}$ is a continuous function, I is a bounded and closed interval, and h is a positive real number, called the delay. In 2019, inspired by the techniques used in [187], Oğrekçi [140] extended and improved the results of [187] by proving the Ulam stability of the delay differential equation (4) for unbounded intervals. Notice that, in 2013, Otrocol and llea [142] proved the Ulam stability of the delay differential equation

$$y'(t) = F(t, y(t), y(\nu(t))), \quad t \in I \subset \mathbb{R}, \tag{5}$$

by using the well-known Gronwall lemma [24] and the abstract Gronwall lemma [164].

Here, we deal with the stability of first order differential equations with delay in locally convex spaces, using the fixed point method. To this aim, we first show a new alternative fixed point theorem in generalized gauge spaces. This extends the famous alternative fixed point theorem of [62]. Using this theorem, we prove the Ulam stability of the delay differential equation (5), where y is no more defined from an interval I into $\mathbb{R}$, but into a locally convex space (in particular into a Banach space). Since the equation (5), generalizes both (3) and (4), many known results on the stability of differential equations in the literature can be derived.

This thesis is organized as follows :

In chapter 1, we briefly recall some basic results that will be used in the sequel.
Chapter 2, is about generalizations of some fixed point theorems. It is divided into two sections, in the first one we will prove the new alternative fixed point theorem in generalized gauge spaces. In the second section we will prove a general fixed point theorem in the space of functions taking values in a random normed space. This constitutes the random normed space version of the fixed point theorem due to Brzdęk, Chudziak and Páles [39].
In chapter 3, using a purely fixed point approach, we produce a new proof of the Ulam-Hyers stability and hyperstability of the non-homogeneous Forti equtaion considered in [18, 19]. Next, we introduce the abstract linear functional equation

$$\sum_{i=1}^{m} A_i \left(f \left(\varphi_i(\bar{x}) \right) \right) + b = 0, \; \bar{x} := (x_1, \ldots, x_n) \in X^n,$$

and study its stability and hyperstability in Banach spaces.
Chapter 4 focuses on the Ulam stability of functional equations in random normed spaces.

In this chapter, we prove the Ulam stability of the non-homogeneous Forti equation

$$\sum_{i=1}^{m} B_i f\left(\sum_{j=1}^{n} b_{ij}x_j\right) = D(x_1, \ldots, x_n),$$

in the class of functions f mapping a vector space X into a random normed spaces (Y, F, T). As special cases of our result, we obtain the stability criteria for numerous functional equations, in the framework of random normed spaces.

In chapter 5, we deal with the stability of first order differential equations with delay in locally convex spaces, using our fixed point theorem. We prove the Ulam stability of the equation (5), where y is no more defined from an interval I into $\mathbb{R}$, but into a locally convex space (in particular into a Banach space). Since the equation (5), generalizes both (3) and (4), many known results on the stability of differential equations in the literature can be derived. At the end, we point out some gaps in [23] concerning the same subject and fix them. We also give a comment concerning an assertion in [14] (see Remark 5.3.1).

Chapter 1

Preliminary notions and results

In this chapter, we recall briefly some basic results we will use throughout the dissertation.

1.1 Some fixed point theorems

In this section, we will recall some famous fixed point theorems, useful in solving the stability problem of functional equations.

First, let us recall the basic fixed point theorem in analysis, namely the Banach contraction mapping Theorem or the Banach fixed point theorem, or also the Banach contraction mapping principle.

Definition 1.1.1. *Let (Y, d) be a metric space. A mapping $J : Y \to Y$ is a strict contraction mapping, if there exists a constant $0 < q < 1$, such that $d\left(J(x), J(y)\right) \leq qd(x, y)$, for all $x, y \in Y$.*

Theorem 1.1.2. *Let (Y, d) be a complete metric space and $J : Y \to Y$ be a strict contraction mapping with the Lipschitz constant $0 < q < 1$. Then*

1) J has a unique fixed point $x^\star$ in Y.

2) $\lim\limits_{n \to +\infty} J^n(x) = x^\star$, for every $x \in Y$.

3) For every $n \in \mathbb{N}$, and every $x \in Y$

$$\begin{aligned} d(J^n(x), x^\star) &\leq q^n d(x, x^\star) \\ d(J^n(x), x^\star) &\leq \frac{1 - q^n}{1 - q} d(x, J(x)) \\ d(x, x^\star) &\leq \frac{1}{1 - q} d(x, J(x)). \end{aligned}$$

This theorem has a kind of generalization to generalized metric spaces. This is namely the alternative fixed point theorem of Diaz and Margoli [62]. In order to state it, we need the following definition.

Definition 1.1.3. *Let Y be a set. A function $d : Y \times Y \to [0, +\infty]$ is called a generalized metric on Y if d satisfies*
i. $d(x, y) = 0$ if and only if $x = y$,
ii. $d(x, y) = d(y, x)$ for all $x, y \in Y$,
iii. $d(x, z) \leq d(x, y) + d(y, z)$ for all $x, y, z \in Y$.

Notice that the only difference between a generalized metric and a usual metric is that the range of the former is permitted to include $+\infty$.

Theorem 1.1.4 ([62]). *Let (Y, d) be a generalized complete metric space and $J : Y \to Y$ be a strict contraction mapping with the Lipschitz constant $0 < q < 1$. Then for each given element $x \in Y$, one of the conditions holds*

1) $$d\left(J^n(x), J^{n+1}(x)\right) = \infty, \text{ for every } n \geq 0.$$

2) There exists a positive integer n_0 such that
a) $d(J^{n+1}(x), J^n(x)) < +\infty, \quad$ for all $n \geq n_0$.
b) The sequence $(J^n(x))_n$ converges to a fixed point $x^\star$ of J.
c) $x^\star$ is the unique fixed point of J in the set

$$Y^\star := \{y \in Y : \ d(J^{n_0}(x), y) < +\infty\}.$$

d) For every $y \in Y^\star$
$$d(y, x^\star) \leq \frac{1}{1-q} d\left(J(y), y\right).$$

This theorem proved to be a very effective tool in the investigation of various issues in Ulam stability of functional equations (see [23, 50, 94, 147, 51, 146, 156]), although in many cases one can use the classical Banach fixed point theorem instead.

Remark 1.1.5.
- If the fixed point $x^\star$ exists, it is not necessarily unique in the whole space Y.
- Actually, if $a)$ holds, then $(Y^\star, d)$ is a complete metric space and $J(Y^\star) \subset Y^\star$. Therefore, the properties $b)$, $c)$ and $d)$ are easily seen to follow from Theorem 1.1.2.

Another fixed point theorem, due to Brzdęk, Chudziak and Páles [39], comes in force when dealing with the Ulam stability of functional applications. To state it, we need the following three hypotheses :

(H1) X is a nonempty set, Y is a Banach space, $\kappa \in \mathbb{N}^\star$, and $f_1, ..., f_\kappa : X \to X$ and $L_1, \ldots, L_\kappa : X \to \mathbb{R}_+$ are given functions.

(H2) $J : Y^X \to Y^X$ fulfills the inequality

$$\big\|J\xi(x) - J\mu(x)\big\| \leq \sum_{i=1}^{\kappa} L_i(x)\big\|\xi(f_i(x)) - \mu(f_i(x))\big\|, \qquad \xi, \mu \in Y^X, x \in X.$$

(H3) $\Lambda : \mathbb{R}_+^X \to \mathbb{R}_+^X$ is given by

$$\Lambda\delta(x) := \sum_{i=1}^{\kappa} L_i(x)\delta(f_i(x)), \qquad \delta \in {\mathbb{R}_+}^X, x \in X.$$

Theorem 1.1.6 ([39]). *Let the hypotheses (H1)-(H3) be satisfied. Let functions $\varepsilon : X \to \mathbb{R}_+$ and $\varphi : X \to Y$ fulfill*

$$\big\|J\varphi(x) - \varphi(x)\big\| \leq \varepsilon(x), \qquad x \in X,$$

$$\varepsilon^*(x) := \sum_{i=0}^{\infty} \Lambda^i\varepsilon(x) < \infty, \qquad x \in X. \tag{1.1}$$

Then there exists a unique fixed point ψ of J with

$$\|\varphi(x) - \psi(x)\| \leq \varepsilon^*(x), \qquad x \in X. \tag{1.2}$$

Moreover,

$$\psi(x) := \lim_{n\to\infty} J^n\varphi(x), \qquad x \in X. \tag{1.3}$$

1.2 Triangular norms

The history of triangular norms (t-norms for short) started with Menger [125], when he wanted to generalize the triangle inequality from classical metric spaces to statistical metric spaces (nowadays called probabilistic metric spaces). Triangular norms play an important role in the interpretation of the conjunction in fuzzy logics [84] and, subsequently, for the intersection of fuzzy sets [194]; see [166, 110, 111, 109, 52].

Definition 1.2.1 ([166]). *A triangular norm is a binary operation on the unit interval $[0, 1]$, i.e., a function $T : [0, 1]\times[0, 1] \to [0, 1]$, such that for all $x, y, z \in [0, 1]$, the following conditions are satisfied*

($T1$) *Commutativity:* $T(x, y) = T(y, x)$.

($T2$) *Associativity:* $T(x, T(y, z)) = T(T(x, y), z)$.

$(T3)$ *Monotonicity:* $T(x,y) \leq T(x,z)$, *whenever* $y \leq z$.

$(T4)$ *Boundary condition:* $T(x,1) = x$.

Remark 1.2.2. • Since a t-norm is an algebraic operation on the unit interval $[0,1]$, some authors prefer to use the notation $x \star y$ instead of the notation $T(x,y)$.

- The commutativity $(T1)$, the boundary condition $(T3)$, and the monotonicity $(T4)$ imply the following boundary conditions

$$T(x,1) = T(1,x) = x, \qquad T(x,0) = T(0,x) = 0.$$

and the monotonicity in both components. That is

$$T(x_1,y_1) \leq T(x_2,y_2),$$

whenever $x_1 \leq x_2$ and $y_1 \leq y_2$.

- In general a t-norm does not need to be continuous.

Example 1.2.3. *The four basic continuous t-norms are : the minimum* T_M*, the product* T_P*, the Lukasiewicz t-norm* T_L *and the drastic product* T_D*. Where*

$$\begin{aligned} T_M(x,y) =& \min(x,y), \\ T_P(x,y) =& x.y, \\ T_L(x,y) =& \max(x+y-1,0), \\ T_D(x,y) =& \begin{cases} 0 & \text{if } (x,y) \in [0,1[^2 \\ \min(x,y) & \text{otherwise }. \end{cases} \end{aligned}.$$

Remark 1.2.4. 1. For an arbitrary triangular norm T, we have $T_D(x,y) \leq T(x,y) \leq T_M(x,y)$, for all $x,y \in [0,1]$. Therefore T_D is the weakest and T_M is the strongest triangular norm. Notice also that

$$T_D \leq T_L \leq T_P \leq T_M.$$

2. Following [52], if T is a t-norm, $m \in \mathbb{N}_0$ and $a_i \in [0,1]$ for $i \in \mathbb{N}_0$, then we write

$$T_{i=m}^{m} a_i := a_m, \qquad T_{i=m}^{n+m} a_i := T(a_{n+m}, T_{i=m}^{n+m-1} a_i), \qquad n \in \mathbb{N}.$$

Since T is commutative and associative, it is easy to show by induction that

$$T_{i=m}^{m+n+l} a_i = T\big(T_{i=m}^{m+n} a_i, T_{i=m+n+1}^{m+n+l} a_i\big), \qquad m,n,l \in \mathbb{N}_0,\ l > 0. \tag{1.4}$$

Indeed, we have

$$T_{i=m}^{m+n+1} a_i = T\left(T_{i=m}^{m+n} a_i, a_{m+n+1}\right) = T\left(T_{i=m}^{m+n} a_i, T_{i=m+n+1}^{m+n+1} a_i\right),$$

then (1.4) is true for $l = 1$. Now fix $l \in \mathbb{N}$ satisfying (1.4). Therefore

$$\begin{aligned} T_{i=m}^{m+n+l+1} a_i &= T\left(T_{i=m}^{m+n+l} a_i, a_{m+n+l+1}\right) \\ &= T\left(T\left(T_{i=m}^{m+n} a_i, T_{m+n+1}^{m+n+l} a_i\right), a_{m+n+l+1}\right) \\ &= T\left(T_{i=m}^{m+n} a_i, T\left(T_{i=m+n+1}^{m+n+l} a_i, a_{m+n+l+1}\right)\right) \\ &= T\left(T_{i=m}^{m+n} a_i, T_{i=m+n+1}^{m+n+l+1} a_i\right). \end{aligned}$$

Note yet that, by (T_3) the sequence $(T_{i=m}^{m+n} a_i)_{n\in\mathbb{N}}$ is non-increasing for every $m \in \mathbb{N}$. Indeed, let $m \in \mathbb{N}$

$$T_{i=m}^{m+n+1} a_i - T_{i=m}^{m+n} a_i = T\left(a_{m+n+1}, T_{i=m}^{m+n} a_i\right) - T\left(1, T_{i=m}^{m+n} a_i\right) \leq 0,$$

which proves that $(T_{i=m}^{m+n} a_i)_{n\in\mathbb{N}}$ is non-increasing sequence for every $m \in \mathbb{N}$. Since it is bounded, it always converges in $[0, 1]$. So, for each $m \in \mathbb{N}$, we may introduce the following notation

$$T_{i=m}^{\infty} a_i := \lim_{n\to\infty} T_{i=m}^{m+n} a_i = \inf_{n\in\mathbb{N}} T_{i=m}^{n+m} a_i.$$

A t-norm T can be extended in a unique way to an n-ary operation taking

$$T(a_1, \dots, a_n) := T_{i=1}^{n} a_i.$$

To shorten some long formulas, we will adopt the notation

$$\widehat{T}(a) := T(a, a), \qquad a \in [0, 1].$$

It is easy to show by induction on k that

$$T_{j=1}^{k} \widehat{T}\left(T_{i=m}^{m+n} a_{ij}\right) = \widehat{T}\left(T_{i=m}^{m+n} T_{j=1}^{k} a_{ij}\right) \tag{1.5}$$

for every $k, n, m \in \mathbb{N}_0$, $k \geq 1$, and $a_{ij} \in [0, 1]$ with $j = 1, \dots, k$ and $i = m, \dots, m+n$. We will need that property in Section 2.2.

1.3 Random normed spaces

In 1942, Menger [125] proposed a probabilistic generalization of the metric spaces theory, called the theory of probabilistic metric (or random normed) spaces. This theory has been investigated by Šerstnev in a series of papers (see for example [174], [175], [176]). We can also refer to the book [81] in this respect. In the sequel, we use the definitions and properties of the random normed spaces as in ([10, 52, 81, 83, 126, 172, 174, 175, 176]). However, for the convenience of the reader we remind some of them.

Definition 1.3.1. *A mapping $g : \mathbb{R} \to [0,1]$ is called a distribution function if it is left-continuous, non-decreasing and satisfies*

$$\sup_{t\in\mathbb{R}} g(t) = 1, \qquad \inf_{t\in\mathbb{R}} g(t) = 0.$$

The class of all distribution functions is denoted by Δ, and the class of all distribution functions g with $g(0) = 0$ is denoted by $\mathcal{D}_+$.

For any real number $a \geq 0$, H_a is the element of $\mathcal{D}_+$ defined by

$$H_a(t) := \begin{cases} 0 & \text{if} \quad t \leq a; \\ 1 & \text{if} \quad t > a. \end{cases}$$

Remark 1.3.2. Any convergence in $\mathcal{D}_+$ will mean the pointwise convergence. Therefore, we say that a sequence $(\psi_n)_n$ in $\mathcal{D}_+$ converges to some $\psi \in \mathcal{D}_+$ if

$$\lim_{n\to\infty} \psi_n(t) = \psi(t), \quad t > 0.$$

Hence the convergence of $(\psi_n)_n$ to H_0 means that

$$\lim_{n\to\infty} \psi_n(t) = 1, \quad t > 0.$$

Remark 1.3.3. For every $\varphi, \psi \in \mathcal{D}_+$ the inequality $\varphi \leq \psi$ means that $\varphi(t) \leq \psi(t)$ for each $t > 0$. We use this abbreviation to simplify formulas whenever the variable t is not necessary to express them precisely.

Definition 1.3.4. *A Menger probabilistic metric space (or random metric space) is a triple (Y, F, T), where Y is a nonempty set, T is a continuous t-norm, and F is a mapping from $Y \times Y$ into $\mathcal{D}_+$ such that the following conditions hold*

PM1) $F_{x,y}(t) = H_0(t)$ *for all $t > 0$ if and only if $x = y$;*

PM2) $F_{x,y}(t) = F_{y,x}(t)$ *for all $x, y \in Y$, and $t > 0$;*

PM3) $F_{x,z}(t + s) \geq T\left(F_{x,y}(t), F_{y,z}(s)\right)$ *for all $x, y, z \in Y$ and $t, s > 0$.*

Definition 1.3.5. *Let Y be a real vector space, $F : x \mapsto F_x$ be a mapping from Y into $\mathcal{D}_+$, and T be a continuous t-norm. We say that (Y, F, T) is a random normed space (briefly RN-space) if the following conditions are satisfied*

RN1) $F_x = H_0$ *if and only if* $x = 0$*, the null vector;*

RN2) $F_{\alpha x}(t) = F_x\Big(\dfrac{t}{|\alpha|}\Big)$ *for all* $x \in Y,\ t > 0$ *and* $\alpha \neq 0$*;*

RN3) $F_{x+y}(t+s) \geq T(F_x(t), F_y(s))$ *for all* $x, y \in Y$ *and* $t, s \geq 0$.

For more details on the RN-spaces, we refer to [77, 83, 124, 172, 174].

Example 1.3.6. *Let* $(Y, \| \ \|)$ *be a normed space. Then both* (Y, F, T_M) *and* (Y, F, T_p) *are random normed spaces, where for every* $x \in Y$

$$F_x(t) := \begin{cases} 0 & \textit{if} \quad t \leq 0, \\ \dfrac{t}{t + \|x\|} & \textit{if} \quad t > 0. \end{cases}$$

The same remains true if

$$F_x(t) := \begin{cases} 0 & \textit{if} \quad t \leq 0, \\ e^{-(\frac{\|x\|}{t})} & \textit{if} \quad t > 0. \end{cases}$$

Example 1.3.7. *Let* $(Y, \|\|)$ *be a normed space. For all* $x \in Y$*, define a mapping*

$$F_x(t) = \begin{cases} \max\Big\{1 - \dfrac{\|x\|}{t}, 0\Big\} & \textit{if} \quad t > 0, \\ 0 & \textit{if} \quad t \leq 0. \end{cases}$$

Then, (Y, F, T_L) *is a random normed space.*

Definition 1.3.8 ([77, 124])**.** *Let* (Y, F, T) *be an* RN*-space.*

1) A sequence $(x_n)_{n\in\mathbb{N}}$ *in* Y *is said to converge (or to be convergent) to* $x \in Y$ *(in short* $\lim\limits_{n\to+\infty} x_n = x$*) if*

$$\lim_{n\to+\infty} F_{x_n - x}(t) = 1, \qquad t > 0,$$

i.e., for each $t > 0$ *and each* $\epsilon > 0$*, there exists* $N_{\epsilon,t} \in \mathbb{N}$*, such that* $F_{x_n - x}(t) > 1 - \epsilon$*, for all* $n \geq N_{\epsilon,t}$*.*

2) A sequence $(x_n)_{n\in\mathbb{N}}$ *in* Y *is said to be an* M*-Cauchy sequence if*

$$\lim_{n,m\to+\infty} F_{x_n - x_m}(t) = 1, \qquad t > 0,$$

i.e., for each $t > 0$*, and each* $\epsilon > 0$*, there exists* $N_{\epsilon,t} \in \mathbb{N}$*, such that* $F_{x_n - x_m}(t) > 1 - \epsilon$*, for all* $N_{\epsilon,t} \leq n < m$*.*

3) A sequence $(x_n)_{n\in\mathbb{N}}$ *in* Y *is said to be a* G*-Cauchy sequence if*

$$\lim_{n\to+\infty} F_{x_n - x_{n+k}}(t) = 1, \qquad t > 0,\ k \in \mathbb{N},$$

i.e., for every $k \in \mathbb{N}$, $t > 0$ and $\epsilon > 0$, there exists $N_{\epsilon,t,k} \in \mathbb{N}$, such that $F_{x_n - x_{n+k}}(t) > 1 - \epsilon$ for all $n \geq N_{\epsilon,t,k}$.

4) *(Y, F, T) is said to be G-complete (M-complete) if every G-Cauchy (M-Cauchy) sequence in Y is convergent to some point in Y.*

Since every M-Cauchy sequence is also G-Cauchy, it is easily seen that each G-complete RN-space is M-complete.

In the following, we review some topological structures of random normed spaces. For more details, one can see [52].

Definition 1.3.9 ([52])**.** *Let (Y, F, T) be a random normed space. The open t-ball, or just ball, $B_x(r,t)$ and the closed ball $B_x[r,t]$, with center $x \in Y$ and radius $0 < r < 1$ are*

$$B_x(r,t) = \{y \in Y;\ F_{x-y}(t) > 1 - r\},$$
$$B_x[r,t] = \{y \in Y;\ F_{x-y}(t) \geq 1 - r\}.$$

Definition 1.3.10 ([52])**.** *Every random norm F on Y generates a topology on Y having as base the collection*

$$\{B_x(r,t)\}_{x \in Y, t>0, r \in]0,1[}$$

of open balls. Such a topology is called the (r,t)-topology.

The following Lemma shows that the (r,t)-topology is a linear topology.

Lemma 1.3.11 ([52])**.** *If (Y, F, T) is a random normed space, then*

- *The function $(x, y) \to x + y$ is continuous from $Y \times Y$ into Y.*
- *The function $(\alpha, x) \to \alpha x$ is continuous from $\mathbb{K} \times Y$ into Y.*

1.4 Generalized gauge spaces

A generalized pseudo-metric on a nonempty set X is any mapping $d : X \times X \to [0, +\infty]$ such that, for every $(x, y, z) \in X^3$
D-1. $d(x,x) = 0$,
D-2. $d(x,y) = d(y,x)$,
D-3. $d(x,y) \leq d(x,z) + d(z,y)$.

A generalized gauge (or generalized uniformizable) space is any nonempty set X endowed with a family $(d_j)_{j \in J}$ of generalized pseudo-metrics such that

$$d_j(x,y) = 0 \text{ for every } j \in J, \text{ if and only if } x = y. \tag{1.6}$$

Such a generalized gauge space will be denoted by $(X, (d_j)_{j\in J})$. With no loss of generality, we will assume that the collection $(d_j)_{j\in J}$ is directed upward, i.e., for every $i, j \in J$, there exists $k \in J$, so that

$$\max(d_i(x,y), d_j(x,y) \leq d_k(x,y), \qquad x, y \in X.$$

In a generalized gauge space $(X, (d_j)_{j\in J})$ a sequence $(x_n)_n$ is said to be Cauchy if, for every $j \in J$, $d_j(x_n, x_m)$ converges to zero as n, m tend to $+\infty$. It is said to converge to some $x \in X$, if for all $j \in J$, $d_j(x_n, x)$ tends to zero as n tends to $+\infty$. If every Cauchy sequence in $(X, (d_j)_{j\in J})$ converges to some point of X, we say that X is sequentially complete.

1.5 Riemann integral in locally convex spaces

In 1982, Astala [15] redefined the notion of Riemann integral in locally convex spaces and gave some interesting properties of this integral.

In all what follows, let (E, τ) be a Hausdorff locally convex space over the field $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}\}$ and $\mathbb{P}$ be an upward directed family of semi-norms defining the topology τ. Let $B \subset E$ be a subset of E. We denote by $\overline{co}(B)$ the closed convex hull of B and by $\mathcal{C}([a,b], E)$ the linear space of all continuous functions $f : [a,b] \to E$, a, b being real numbers with $a < b$. Let $D = \{t_i\}_{i=0}^n$ be a partition of $[a,b] \subset \mathbb{R}$ (that is $a = t_0 < t_1 < \cdots < t_n = b$). The norm of D is $\|D\| := \max\{t_i - t_{i-1},\ 1 \leq i \leq n\}$.

Definition 1.5.1. *A mapping $f : [a,b] \to E$ is said to be Riemann integrable if the limit*

$$\lim_{\|D\|\to 0} \sum_{k=1}^{n} f(s_k)(t_k - t_{k-1}), \tag{1.7}$$

exists in E. This limit, whenever it exists, is denoted by $\int_a^b f(t)\,dt$. Here $D = \{t_k\}_{k=0}^n$ is a partition of $[a,b]$ and $s_k \in [t_{k-1}, t_k]$ is arbitrary.

The following proposition yields some further properties of the integral.

Proposition 1.5.2 ([15]). *Let $f : [a,b] \to E$ be a mapping on $[a,b]$. Then*
1. If $f \in C([a,b], E)$ and $\overline{co}(f([a,b]))$ is sequentially complete, then f is integrable. In particular, if E is sequentially complete, every continuous function $f : [a,b] \to E$ is integrable.
2. If f is integrable, then

$$\int_a^b f(t)\,dt \in (b-a)\overline{co}(f([a,b])).$$

3. If f is continuous and integrable, then for every $P \in \mathbb{P}$

$$P\left(\int_a^b f(t)\,dt\right) \leq \int_a^b P\left(f(t)\right)\,dt.$$

4. If f is continuous and integrable and

$$F(t) = \int_a^t f(s)\,ds, \quad a \leq t \leq b,$$

then $F'(t) = f(t)$ for each $t \in [a, b]$.

Using Proposition 1.5.2, one gets the following proposition.

Proposition 1.5.3 ([15])**.** *Let $f : [a, b] \to E$ be a continuous and integrable mapping. If G is a primitive function of f on $[a, b]$, then*

$$\int_a^b f(x)\,dx = G(b) - G(a).$$

Chapter 2

Some New Fixed Point Theorems

This chapter is about generalization of some fixed point theorems. First, we will prove a new alternative fixed point theorem in generalized gauge spaces, this is a generalization of the famous alternative fixed point theorem of Diaz and Margoli [62]. Next, we will prove a very general fixed point theorem in the space of functions taking values in a random normed space (RN-space). This is the random normed space version of the fixed point theorem due to Brzdęk, Chudziak and Páles [39]. Note that the results of this chapter will be used in studying the Ulam stability of various functional equations.

2.1 Alternative fixed point theorem in gauge spaces

The main result of this section is the following fixed point theorem, this is the generalized gauge space version of Theorem 1.1.4.

Definition 2.1.1. *Let $(X, (d_j)_{j\in J})$ be a generalized gauge space. A mapping $\mathcal{T} : X \to X$ is a strict contraction mapping, if*

$$\forall j \in J, \quad \exists c_j \in (0,1), \quad d_j(\mathcal{T}x, \mathcal{T}y) \le c_j d_j(x,y), \; x, y \in X. \tag{2.1}$$

Theorem 2.1.2. *Let $(X, (d_j)_{j\in J})$ be a generalized sequentially complete gauge space and $\mathcal{T} : X \to X$ is a strictly contraction mapping with $(c_j)_{j\in J}$ are the Lipschitz constants given in* (2.1)*. Then for each given $x_0 \in X$, one of the conditions holds*
(A) There exists $j_0 \in J$ such that $d_{j_0}(\mathcal{T}^n x_0, \mathcal{T}^{n+1}x_0) = +\infty$ for every $n \ge 0$.
(B) Or, for every $j \in J$, there exists a positive integer n_j, with

$$d_j(\mathcal{T}^{n_j}x_0, \mathcal{T}^{n_j+1}x_0) < +\infty. \tag{2.2}$$

In (B) the following assertions hold *a) The sequence $(\mathcal{T}^n x_0)_n$ converges in X to a fixed point $x^\star$ of $\mathcal{T}$. Moreover $x^\star$ belongs to $X^\star := \{x \in X,\ \forall j \in J,\ d_j(\mathcal{T}^{n_j} x_0, x) < +\infty\}$.*

b) $x^\star$ is the unique fixed point of $\mathcal{T}$ in $X^\star$, and $x^\star = \lim_{n\to+\infty} \mathcal{T}^n y,\ \forall y \in X^\star$.

c) For every $y \in X^\star$ and every $j \in J$, we have:

$$d_j(y, x^\star) \le \frac{1}{1-c_j} d_j(\mathcal{T}y, y). \tag{2.3}$$

Proof. Assume that (B) is holds, then for arbitrary $j \in J$ and $l \in \mathbb{N}$, we have

$$d_j(\mathcal{T}^{n_j+l}x_0, \mathcal{T}^{n_j+l+1}x_0) \le c_j^l d_j(\mathcal{T}^{n_j}x_0, \mathcal{T}^{n_j+1}x_0) < +\infty.$$

Therefore, for every $j \in J$ and every integers n, m such that $m > n \ge n_j$,

$$\begin{aligned}
d_j(\mathcal{T}^n x_0, \mathcal{T}^m x_0) &\le d_j(\mathcal{T}^n x_0, \mathcal{T}^{n+1}x_0) + \cdots + d_j(\mathcal{T}^{m-1}x_0, \mathcal{T}^m x_0)\\
&\le c_j^{n-n_j} d(\mathcal{T}^{n_j}x_0, \mathcal{T}^{n_j+1}x_0) + \cdots + c_j^{m-1-n_j} d(\mathcal{T}^{n_j}x_0, \mathcal{T}^{n_j+1}x_0)\\
&\le c_j^{n-n_j} \sum_{i=0}^{m-n-1} c_j^i d_j(\mathcal{T}^{n_j}x_0, \mathcal{T}^{n_j+1}x_0)\\
&\le \frac{c_j^{n-n_j} - c_j^{m-n_j}}{1-c_j} d(\mathcal{T}^{n_j}x_0, \mathcal{T}^{n_j+1}x_0).
\end{aligned}$$

Then, for every $j \in J$, $\lim_{n,m\to+\infty} d_j(\mathcal{T}^n x_0, \mathcal{T}^m x_0) = 0$. Therefore $(\mathcal{T}^n x_0)_n$ is a Cauchy sequence in X. The latter being sequentially complete, there exists $x^\star \in X$ such that for all $j \in J$, $\lim_{n\to+\infty} d_j(\mathcal{T}^n x_0, x^\star) = 0$. We claim that $x^\star$ is a fixed point of $\mathcal{T}$. Indeed, for $j \in J$ and $n \ge n_j$, we have

$$\begin{aligned}
0 \le d_j(x^\star, \mathcal{T}x^\star) &\le d_j(x^\star, \mathcal{T}^n x_0) + d_j(\mathcal{T}^n x_0, \mathcal{T}x^\star)\\
&\le d_j(x^\star, \mathcal{T}^n x_0) + c_j d(\mathcal{T}^{n-1}x_0, x^\star).
\end{aligned}$$

Letting n tend to $+\infty$, we get for every $j \in J$, $d_j(x^\star, \mathcal{T}x^\star) = 0$. Therefore $x^\star$ is a fixed point of $\mathcal{T}$. Now, to prove that $x^\star \in X^\star$, let $j \in J$ and $n \ge n_j$ be arbitrary. We have

$$\begin{aligned}
d_j(\mathcal{T}^{n_j}x_0, x^\star) &\le d_j(\mathcal{T}^{n_j}x_0, \mathcal{T}^n x_0) + d_j(\mathcal{T}^n x_0, x^\star)\\
&\le \sum_{i=0}^{n-n_j-1} c_j^i d_j(\mathcal{T}^{n_j}x_0, \mathcal{T}^{n_j+1}x_0) + d_j(\mathcal{T}^n x_0, x^\star)\\
&\le \frac{1-c_j^{n-n_j}}{1-c_j} d_j(\mathcal{T}^{n_j}x_0, \mathcal{T}^{n_j+1}x_0) + d_j(\mathcal{T}^n x_0, x^\star) < \infty.
\end{aligned}$$

Since $j \in J$ is arbitrary, $x^\star \in X^\star$. To show that $x^\star$ is the unique fixed point of $\mathcal{T}$ in $X^\star$, assume that there exists another fixed point $y^\star \in X^\star$ of $\mathcal{T}$. Then, for every $j \in J$

$$d_j(x^\star, y^\star) = d_j(\mathcal{T}x^\star, \mathcal{T}y^\star) \leq c_j d_j(x^\star, y^\star).$$

Since $c_j < 1$, $d_j(x^\star, y^\star) = 0$. Hence $x^\star = y^\star$ by (1.6). Now, let us prove that for every $y \in X^\star$, we have $\lim\limits_{n\to+\infty} \mathcal{T}^n y = x^\star$. Let $y \in X^\star$, and $j \in J$, we have

$$\begin{aligned} d_j(y, \mathcal{T}y) &\leq d_j(y, \mathcal{T}^{n_j+1}x_0) + d_j(\mathcal{T}^{n_j+1}x_0, \mathcal{T}y) \\ &\leq d_j(y, \mathcal{T}^{n_j}x_0) + d_j(\mathcal{T}^{n_j}x_0, \mathcal{T}^{n_j+1}x_0) + d_j(\mathcal{T}^{n_j+1}x_0, \mathcal{T}y) \\ &\leq d_j(y, \mathcal{T}^{n_j}x_0) + d_j(\mathcal{T}^{n_j}x_0, \mathcal{T}^{n_j+1}x_0) + c_j d_j(\mathcal{T}^{n_j}x_0, y) < +\infty. \end{aligned}$$

Then y satisfies (2.2) with $n_j = 0$ for every $j \in J$. Therefore, by a), $(\mathcal{T}^n y)_n$ converges in X to some fixed point $y^\star$ of $\mathcal{T}$. But, by unicity just proven above, $y^\star = x^\star$. Now, let $y \in X^\star$, and $j \in J$

$$\begin{aligned} d_j(y, x^\star) &\leq d_j(y, \mathcal{T}^n y) + d_j(\mathcal{T}^n y, x^\star) \\ &\leq \sum_{i=0}^{n-1} d_j(\mathcal{T}^i y, \mathcal{T}^{i+1} y) + d_j(\mathcal{T}^n y, x^\star) \\ &\leq \sum_{i=0}^{n-1} c_j^i d_j(y, \mathcal{T}y) + d_j(\mathcal{T}^n y, x^\star) \\ &\leq \frac{1 - c_j^n}{1 - c_j} d_j(y, \mathcal{T}y) + d_j(\mathcal{T}^n y, x^\star). \end{aligned}$$

Letting n tend to infinity, we get (2.3). This achieves the proof. □

The following immediate corollary will be used in the sequel to prove the Ulam stability of delay differential equations on a closed interval, for functions with values in a locally convex space.

Corollary 2.1.3. *Let $(X, (d_j)_{j\in J})$ be a generalized sequentially complete gauge space and $\mathcal{T} : X \to X$ a mapping satisfying the condition (2.1). If there exists $n_0 \in \mathbb{N}$ and $x_0 \in X$, such that for every $j \in J$, $d_j(\mathcal{T}^{n_0}x_0, \mathcal{T}^{n_0+1}x_0) < +\infty$, then the following assertions holds*

a) The sequence $(\mathcal{T}^n x_0)_n$ converges in X to a fixed point $x^\star$ of $\mathcal{T}$, belonging to $X^\star := \{x \in X, \forall j \in J,\ d_j(\mathcal{T}^{n_0}x_0, x) < +\infty\}$.

b) $x^\star$ is the unique fixed point of $\mathcal{T}$ in $X^\star$, and $x^\star = \lim\limits_{n\to+\infty} \mathcal{T}^n y$, $\forall y \in X^\star$.

c) For every $y \in X^\star$ and every $j \in J$, (2.3) holds true.

2.2 A general fixed point theorem in random normed spaces and its consequences

In this section, we will prove a general fixed point theorem for classes of functions taking values in a random normed space (RN-space). Next, we will show several of its consequences. In fact, one of these consequences will be seen as the random normed space version of the fixed point theorem due to Brzdęk, Chudziak and Páles (Theorem 1.1.6), and the other consequence will be an analogue of the classical Banach contraction principle in random normed space (somewhat generalized).

Notice that, in Chapter 4 we will present several applications of this general fixed point theorem and its consequences in proving the Ulam stability of various functional equations in random normed spaces. Moreover, we will show how to use this theorem to study the approximate eigenvalues and eigenvectors of some linear operators.

Throughout this section, X is a non-empty set and (Y, F, T) is a random normed space. To simplify some expressions, for given $\phi \in \mathcal{D}_+^X$ and $x \in X$, we write ϕ_x to mean $\phi(x)$, i.e.,

$$\phi_x(t) := \phi(x)(t), \qquad x \in X,\ t \in \mathbb{R}.$$

Let Ω stand for the family of all real sequences $(\omega_n)_{n\in\mathbb{N}}$ with $\omega_n \in (0,1)$ for each $n \in \mathbb{N}$ and $\sum_{i=0}^{\infty} \omega_i = 1$.

Definition 2.2.1. *Let $\Lambda : \mathcal{D}_+^X \to \mathcal{D}_+^X$ and $J : Y^X \to Y^X$ be given. We say that the operator J is Λ-contractive, if for every $\xi, \eta \in Y^X$ and every $\phi \in \mathcal{D}_+^X$, one has*

$$\Big(F_{(\xi-\eta)(x)} \geq \phi_x,\ x \in X\Big) \Longrightarrow \Big(F_{(J\xi-J\eta)(x)} \geq (\Lambda\phi)_x,\ x \in X\Big).$$

Definition 2.2.2. *The operator $\Lambda : \mathcal{D}_+^X \to \mathcal{D}_+^X$ is said to satisfy the hypothesis $(\mathcal{C}_0)$, if for every sequence $(g_n)_{n\in\mathbb{N}} \subset Y^X$, one has*

$$\Big(\lim_{n\to+\infty} F_{g_n(x)} = H_0, \quad x \in X\Big) \Longrightarrow \Big(\lim_{n\to+\infty} \big(\Lambda F_{g_n}\big)_x = H_0, \quad x \in X\Big), \tag{$\mathcal{C}_0$}$$

where $F_g \in \mathcal{D}_+^X$ is given by $F_g(x) := F_{g(x)}$ for $g \in Y^X$ and $x \in X$.

Remark 2.2.3. Let $\chi_0 \in \mathcal{D}_+^X$ be given by $\chi_0(x) = H_0$ for every $x \in X$. Whenever $\Lambda\chi_0 = \chi_0$, then $(\mathcal{C}_0)$ actually means that Λ is continuous at the point χ_0, with respect to the pointwise convergence topologies in $\mathcal{D}_+^X$ and $\mathcal{D}_+$.

Let us first state the following lemma which will be used in the sequel.

Lemma 2.2.4. *Let $\Lambda : \mathcal{D}_+^X \to \mathcal{D}_+^X$ and $\epsilon : X \to \mathcal{D}_+$ be arbitrary. Then, for every $x \in X$, $k \in \mathbb{N}$, $\omega \in \Omega$ and $t > 0$, the limits*

$$\sigma_x^k(t) := \lim_{j\to\infty} T_{i=k}^{k+j-1}(\Lambda^i\epsilon)_x\Big(\frac{t}{j}\Big), \tag{2.4}$$

$$^{\omega}\sigma_x^k(t) := \lim_{j\to\infty} T_{i=k}^{k+j-1}(\Lambda^i\epsilon)_x\big(\omega_{i-k}t\big) \tag{2.5}$$

exist in $\mathbb{R}$. Moreover $(\sigma_x^k)_k$ is non-decreasing and so is also $(^{\omega}\sigma_x^k)_k$ whenever ω is non-increasing.

Proof. Fix $k \in \mathbb{N}$, $x \in X$ and $t > 0$ and write for every $m \in \mathbb{N}^\star$

$$\tau_m(x,t,k) := T_{i=k}^{k+m-1}(\Lambda^i\epsilon)_x\Big(\frac{t}{m}\Big).$$

Since $(\Lambda^i\epsilon)_x \in \mathcal{D}_+$, it is a non-decreasing function, for each $i \in \mathbb{N}^\star$. Hence

$$(\Lambda^i\epsilon)_x\Big(\frac{t}{m}\Big) \geq (\Lambda^i\epsilon)_x\Big(\frac{t}{m+1}\Big).$$

Consequently

$$\begin{aligned}
\tau_m(x,t,k) &\geq T_{i=k}^{k+m-1}(\Lambda^i\epsilon)_x\Big(\frac{t}{m+1}\Big) = T\bigg(1, T_{i=k}^{k+m-1}(\Lambda^i\epsilon)_x\Big(\frac{t}{m+1}\Big)\bigg)\\
&\geq T\bigg((\Lambda^{k+m}\epsilon)_x\Big(\frac{t}{m+1}\Big), T_{i=k}^{k+m-1}(\Lambda^i\epsilon)_x\Big(\frac{t}{m+1}\Big)\bigg)\\
&= T_{i=k}^{k+m}(\Lambda^i\epsilon)_x\Big(\frac{t}{m+1}\Big) = \tau_{m+1}(x,t,k),
\end{aligned}$$

whence the sequence $(\tau_m(x,t,k))_{m\in\mathbb{N}^\star}$ is non-increasing and therefore the following limit exists

$$\sigma_x^k(t) := \lim_{m\to\infty} \tau_m(x,t,k) = \inf_{m\in\mathbb{N}^\star} \tau_m(x,t,k), \quad k \in \mathbb{N},\ x \in X,\ t > 0. \tag{2.6}$$

Next, fix $\omega \in \Omega$, $k \in \mathbb{N}$, $x \in X$ and $t > 0$, and write

$$\rho_m(x,t,k) := T_{i=k}^{k+m-1}(\Lambda^i\epsilon)_x\big(\omega_{i-k}t\big), \quad m \in \mathbb{N}^\star.$$

Then

$$\begin{aligned}
\rho_m(x,t,k) &= T\bigg(1, T_{i=k}^{k+m-1}(\Lambda^i\epsilon)_x\Big(\omega_{i-k}t\Big)\bigg)\\
&\geq T\bigg((\Lambda^{k+m}\epsilon)_x\Big(\omega_m\Big), T_{i=k}^{k+m-1}(\Lambda^i\epsilon)_x\Big(\omega_{i-k}t\Big)\bigg)\\
&= T_{i=k}^{k+m}(\Lambda^i\epsilon)_x\Big(\omega_{i-k}t\Big) = \rho_{m+1}(x,t,k),
\end{aligned}$$

This means that the sequence $(\rho_m(x,t,k))_{m\in\mathbb{N}^\star}$ is non-increasing. Therefore the limit

$$ {}^{\omega}\sigma_x^k(t) := \lim_{m\to\infty} \rho_m(x,t,k) = \inf_{m\in\mathbb{N}^\star} \rho_m(x,t,k) \tag{2.7}$$

exists for every $\omega \in \Omega$, $k\in\mathbb{N}$, $x\in X$ and $t>0$. Moreover, for every $k\in\mathbb{N}$, we have

$$\begin{aligned}
&\tau_m(x,t,k+1) := T_{i=k+1}^{k+1+m-1}(\Lambda^i\epsilon)_x\Big(\frac{t}{m}\Big)\\
\text{with } j=m+1 \qquad &= T_{i=k+1}^{k+j-1}(\Lambda^i\epsilon)_x\Big(\frac{t}{j-1}\Big)\\
&= T\left(1, T_{i=k+1}^{k+j-1}(\Lambda^i\epsilon)_x\Big(\frac{t}{j-1}\Big)\right)\\
\text{by } (c) \qquad &\geq T\left((\Lambda^k\epsilon)_x\Big(\frac{t}{j-1}\Big), T_{i=k+1}^{k+j-1}(\Lambda^i\epsilon)_x\Big(\frac{t}{j-1}\Big)\right)\\
&\geq T_{i=k}^{k+j-1}(\Lambda^i\epsilon)_x\Big(\frac{t}{j-1}\Big) = \tau_{m+1}(x,t,k).
\end{aligned}$$

Form this comes $\sigma_x^k \leq \sigma_x^{k+1}$, and then the sequence $(\sigma_x^k)_k$ is non-decreasing. Similarly,

$$\begin{aligned}
&\rho_{m+1}(x,t,k) := T_{i=k}^{k+(m+1)-1}(\Lambda^i\epsilon)_x\big(\omega_{i-k}t\big)\\
&= T\left((\Lambda^k\epsilon)_x\big(\omega_0 t\big), T_{i=k+1}^{(k+1)+m-1}(\Lambda^i\epsilon)_x\big(\omega_{i-k}t\big)\right)\\
\text{by } (c) \qquad &\leq T\left(1, T_{i=k+1}^{(k+1)+m-1}(\Lambda^i\epsilon)_x\big(\omega_{i-k}t\big)\right)\\
&= T_{i=k+1}^{(k+1)+m-1}(\Lambda^i\epsilon)_x\big(\omega_{i-k}t\big)\\
\omega \text{ non-increasing} \qquad &\leq T_{i=k+1}^{(k+1)+m-1}(\Lambda^i\epsilon)_x\big(\omega_{i-(k+1)}t\big) = \rho_m(x,t,k+1).
\end{aligned}$$

Whereby ${}^{\omega}\sigma_x^k \leq {}^{\omega}\sigma_x^{k+1}$. □

Remark 2.2.5. Fix $x\in X$ and $k\in\mathbb{N}$. If $T=T_M$ in Lemma 2.2.4, then

$$\begin{aligned}
\sigma_x^k(t) &= \inf_{m\in\mathbb{N}^\star} T_{i=k}^{k+m-1}(\Lambda^i\epsilon)_x\Big(\frac{t}{m}\Big)\\
&= \inf_{m\in\mathbb{N}^\star} \min_{j=1}^{m}(\Lambda^{j+k-1}\epsilon)_x\Big(\frac{t}{m}\Big)\\
&= \inf_{m\in\mathbb{N}^\star} (\Lambda^{k+m-1}\epsilon)_x\Big(\frac{t}{m}\Big).
\end{aligned}$$

And if $T=T_p$, then

$$\begin{aligned}
\sigma_x^k(t) &= \lim_{m\to\infty}\prod_{i=1}^{m}(\Lambda^{k+i-1}\epsilon)_x\Big(\frac{t}{m}\Big)\\
&= \inf_{m\in\mathbb{N}^\star}\prod_{i=1}^{m}(\Lambda^{k+i-1}\epsilon)_x\Big(\frac{t}{m}\Big).
\end{aligned}$$

Analogous equalities are valid for ${}^{\omega}\sigma_x^k$ for every $\omega \in \Omega$.

In the sequel, given $\Lambda : \mathcal{D}_+^X \to \mathcal{D}_+^X$ and $\epsilon : X \to \mathcal{D}_+$, we write

$$\underline{\sigma}_x^k(t) := \sup_{\omega\in\Omega} {}^{\omega}\sigma_x^k(t), \quad \text{and} \quad \widehat{\sigma}_x^k(t) := \max\{\sigma_x^k(t), \underline{\sigma}_x^k(t)\}, \tag{2.8}$$

for every $x \in X$, $k \in \mathbb{N}$ and $t > 0$, where $\sigma_x^k(t)$ and ${}^{\omega}\sigma_x^k(t)$ are defined by (2.4) and (2.5).

Theorem 2.2.6. *Let $\Lambda : \mathcal{D}_+^X \to \mathcal{D}_+^X$, $\epsilon : X \to \mathcal{D}_+$, $J : Y^X \to Y^X$, and $f : X \to Y$ be given, so that Λ satisfies the hypothesis $(\mathcal{C}_0)$, J is Λ-contractive, and the following inequality is fulfilled*

$$F_{(Jf-f)(x)} \geq \epsilon_x, \qquad x \in X. \tag{2.9}$$

Assume that one of the following three conditions holds

(i) (Y, F, T) is M-complete and

$$\lim_{k\to+\infty} \inf_{j\in\mathbb{N}} T_{i=k}^{k+j} (\Lambda^i \epsilon)_x \Big(\frac{t}{j+1}\Big) = 1, \qquad x \in X,\ t > 0. \tag{2.10}$$

(ii) (Y, F, T) is M-complete and for each $k \in \mathbb{N}$ there exists a sequence $(\omega_n^k)_{n\in\mathbb{N}} \in \Omega$ with

$$\lim_{k\to+\infty} \inf_{j\in\mathbb{N}} T_{i=k}^{k+j} (\Lambda^i \epsilon)_x \big(\omega_{i-k}^k t\big) = 1, \qquad x \in X,\ t > 0. \tag{2.11}$$

(iii) (Y, F, T) is G-complete and $\lim_{n\to+\infty} (\Lambda^n \epsilon)_x = H_0$, i.e.,

$$\lim_{n\to+\infty} (\Lambda^n \epsilon)_x(t) = 1, \qquad x \in X,\ t > 0. \tag{2.12}$$

Then, for every $x \in X$, the limit

$$\psi(x) := \lim_{n\to+\infty} (J^n f)(x) \tag{2.13}$$

exists in Y and the function $\psi : X \to Y$ is a fixed point of J enjoying

$$F_{(\psi - J^k f)(x)}(t) \geq \sup_{\alpha\in(0,1)} \widehat{\sigma}_x^k(\alpha t), \qquad k \in \mathbb{N}, \quad x \in X, \quad t > 0. \tag{2.14}$$

Moreover, in case (i) or (ii) holds, ψ is the unique fixed point of J for which there is $\alpha \in (0, 1)$ with

$$F_{(\psi - J^k f)(x)}(t) \geq \widehat{\sigma}_x^k(\alpha t), \qquad k \in \mathbb{N},\ x \in X,\ t > 0. \tag{2.15}$$

Proof. First we show by induction that, for every $n \in \mathbb{N}$,

$$F_{(J^{n+1}f - J^n f)(x)} \geq (\Lambda^n \epsilon)_x, \qquad x \in X. \tag{2.16}$$

The case $n = 0$ is just (2.9). So, fix $n \in \mathbb{N}$ satisfying (2.16). Then, using the Λ-contractivity of J and the inductive assumption, we obtain

$$F_{(J^{n+2}f-J^{n+1}f)(x)} \geq \big(\Lambda(\Lambda^n \epsilon)\big)_x = (\Lambda^{n+1}\epsilon)_x, \qquad x \in X.$$

Thus we have proved that (2.16) holds for every $n \in \mathbb{N}$. Consequently, for every $n \in \mathbb{N}$, $m \in \mathbb{N}^\star$, $x \in X$ and $t > 0$ we have

$$\begin{aligned} F_{(J^{n+m}f-J^nf)(x)}(t) &= F_{\sum_{i=0}^{m-1}(J^{n+i+1}f-J^{n+i}f)(x)}(t) \\ &\geq T_{i=0}^{m-1} F_{(J^{n+i+1}f-J^{n+i}f)(x)}\Big(\frac{t}{m}\Big) \\ &\geq T_{i=0}^{m-1} (\Lambda^{n+i}\epsilon)_x\Big(\frac{t}{m}\Big) = T_{i=n}^{n+m-1}(\Lambda^i\epsilon)_x\Big(\frac{t}{m}\Big), \end{aligned} \tag{2.17}$$

and analogously, since $\sum_{i=0}^{m-1}\omega_i t < \sum_{i=0}^{\infty}\omega_i t$ and $\sum_{i=0}^{\infty}\omega_i = 1$ for every $(\omega_n)_{n\in\mathbb{N}} \in \Omega$,

$$F_{(J^{n+m}f-J^nf)(x)}(t) = F_{\sum_{i=0}^{m-1}(J^{n+i+1}f-J^{n+i}f)(x)}(\sum_{i=0}^{\infty}\omega_i t) \tag{2.18}$$

$$\geq F_{\sum_{i=0}^{m-1}(J^{n+i+1}f-J^{n+i}f)(x)}(\sum_{i=0}^{m-1}\omega_i t) \tag{2.19}$$

$$\begin{aligned} &\geq T_{i=0}^{m-1} F_{(J^{n+i+1}f-J^{n+i}f)(x)}\big(\omega_i t\big) \\ &\geq T_{i=0}^{m-1}(\Lambda^{n+i}\epsilon)_x\big(\omega_i t\big) \\ &= T_{i=n}^{n+m-1}(\Lambda^i\epsilon)_x\big(\omega_{i-n} t\big), \qquad (\omega_n)_{n\in\mathbb{N}} \in \Omega. \end{aligned} \tag{2.20}$$

Now we show that the limit (2.13) exists in Y for every $x \in X$.

First consider the case (i). Then, by (2.17), for all $k, m \in \mathbb{N}^\star$, $n \in \mathbb{N}$, $x \in X$ and $t > 0$,

$$\begin{aligned} F_{(J^{n+k}f-J^{n+m}f)(x)}(t) &\geq T\Big(F_{(J^{n+k}f-J^nf)(x)}(\frac{t}{2}), F_{(J^nf-J^{n+m}f)(x)}(\frac{t}{2})\Big) \\ &\geq T\Big(T_{i=n}^{n+k-1}(\Lambda^i\epsilon)_x\Big(\frac{t}{2k}\Big), T_{i=n}^{n+m-1}(\Lambda^i\epsilon)_x\Big(\frac{t}{2m}\Big)\Big). \end{aligned}$$

Consequently, by (c),

$$\begin{aligned} \inf_{k,m\in\mathbb{N}^\star} F_{(J^{n+k}f-J^{n+m}f)(x)}(t) &\geq \inf_{k,m\in\mathbb{N}^\star} T\Big(T_{i=n}^{n+k-1}(\Lambda^i\epsilon)_x\Big(\frac{t}{2k}\Big), T_{i=n}^{n+m-1}(\Lambda^i\epsilon)_x\Big(\frac{t}{2m}\Big)\Big) \\ &\geq T\Big(\inf_{k\in\mathbb{N}} T_{i=n}^{n+k}(\Lambda^i\epsilon)_x\Big(\frac{\frac{t}{2}}{k+1}\Big), \inf_{m\in\mathbb{N}} T_{i=n}^{n+m}(\Lambda^i\epsilon)_x\Big(\frac{\frac{t}{2}}{m+1}\Big)\Big). \end{aligned}$$

Hence (2.10), (b) and the continuity of T at $(1,1)$ yield

$$\lim_{n\to\infty}\inf_{k,m\in\mathbb{N}^\star} F_{(J^{n+k}f-J^{n+m}f)(x)}(t) = 1, \qquad x \in X,\ t > 0.$$

If (ii) is valid, then (2.20) implies that, for all $k, m \in \mathbb{N}^\star$, $n \in \mathbb{N}$, $x \in X$ and $t > 0$,

$$\begin{aligned} F_{(J^{n+k}f - J^{n+m}f)(x)}(t) \geq\ & T\Big(F_{(J^{n+k}f-J^nf)(x)}(\frac{t}{2}), F_{(J^nf-J^{n+m}f)(x)}(\frac{t}{2})\Big) \\ & \geq T\Big(T_{i=n}^{n+k-1}(\Lambda^i\epsilon)_x\big(\omega_{i-n}^k\frac{t}{2}\big), T_{i=n}^{n+m-1}(\Lambda^i\epsilon)_x\big(\omega_{i-n}^m\frac{t}{2}\big)\Big) \end{aligned}$$

and consequently, by (c),

$$\begin{aligned} & \inf_{k,m\in\mathbb{N}^\star} F_{(J^{n+k}f-J^{n+m}f)(x)}(t) \\ & \quad \geq \inf_{k,m\in\mathbb{N}^\star} T\Big(T_{i=n}^{n+k-1}(\Lambda^i\epsilon)_x\big(\omega_{i-n}^k\frac{t}{2}\big), T_{i=n}^{n+m-1}(\Lambda^i\epsilon)_x\big(\omega_{i-n}^m\frac{t}{2}\big)\Big) \\ & \quad \geq T\Big(\inf_{k\in\mathbb{N}} T_{i=n}^{k+n}(\Lambda^i\epsilon)_x\big(\omega_{i-n}^k\frac{t}{2}\big), \inf_{m\in\mathbb{N}} T_{i=n}^{m+n}(\Lambda^i\epsilon)_x\big(\omega_{i-n}^m\frac{t}{2}\big)\Big). \end{aligned}$$

Hence, (2.11), (b) and the continuity of T at $(1,1)$ yield

$$\lim_{n\to\infty} \inf_{k,m\in\mathbb{N}^\star} F_{(J^{n+k}f-J^{n+m}f)(x)}(t) = 1, \qquad x \in X,\ t > 0.$$

Thus we have proved that, for every $x \in X$, $(J^nf(x))_{n\in\mathbb{N}}$ is an M-Cauchy sequence and, as (Y, F, T) is M-complete, the limit (2.13) exists. In the case of (iii), in view of (2.17),

$$F_{(J^{n+m}f-J^nf)(x)}(t) \geq T_{i=0}^{m-1}(\Lambda^{n+i}\epsilon)_x\Big(\frac{t}{m}\Big), \qquad x \in X,\ t > 0,\ n, m \in \mathbb{N}^\star,$$

whence (2.12) and the continuity of T at $(1,1)$ imply that

$$\lim_{n\to+\infty} F_{J^nf(x)-J^{n+m}f(x)}(t) = 1, \qquad x \in X,\ t > 0,\ m \in \mathbb{N}^\star.$$

Thus, for every $x \in X$, $(J^nf(x))_{n\in\mathbb{N}^\star}$ is a G-Cauchy sequence. As (Y, F, T) is G-complete, the limit (2.13) exists. Now, fix $t > 0$, $x \in X$, $\alpha \in (0,1)$ and $n \in \mathbb{N}$ arbitrarily, and let us prove that

$$F_{(\psi-J^nf)(x)}(t) \geq \sigma_x^n(\alpha t). \tag{2.21}$$

Note that (2.17) implies

$$\begin{aligned} F_{(\psi-J^nf)(x)}(t) & \geq T\Big(F_{(\psi-J^{n+m}f)(x)}\big((1-\alpha)t\big), F_{(J^{n+m}f-J^nf)(x)}\big(\alpha t\big)\Big) \\ & \geq T\bigg(F_{(\psi-J^{n+m}f)(x)}\big((1-\alpha)t\big), T_{i=n}^{n+m-1}(\Lambda^i\epsilon)_x\Big(\frac{\alpha t}{m}\Big)\bigg) \end{aligned} \tag{2.22}$$

for every $m \in \mathbb{N}^\star$. Hence, by (2.13) and the continuity of T at the point $\big(1, \sigma_x^n(\alpha t)\big)$, letting $m \to +\infty$, we obtain (2.21). Analogously we show that

$$F_{(\psi-J^nf)(x)}(t) \geq \underline{\sigma}_x^n(\alpha t) \tag{2.23}$$

for every $\alpha \in (0,1)$, $n \in \mathbb{N}$, $x \in X$ and $t > 0$. To this end, fix $\omega \in \Omega$, $t > 0$, $x \in X$, $\alpha \in (0,1)$ and $n \in \mathbb{N}$ and note that (2.20) implies

$$\begin{aligned} F_{(\psi - J^n f)(x)}(t) &\geq T\Big(F_{(\psi - J^{n+m} f)(x)}\big((1-\alpha)t\big), F_{(J^{n+m} f - J^n f)(x)}\big(\alpha t\big)\Big) \\ &\geq T\Big(F_{(\psi - J^{n+m} f)(x)}\big((1-\alpha)t\big), T_{i=n}^{n+m-1}(\Lambda^i \epsilon)_x\big(\omega_{i-n}\alpha t\big)\Big) \end{aligned} \tag{2.24}$$

for every $m \in \mathbb{N}^\star$. Hence, by (2.13), Lemma 2.2.4 and the continuity of T at the point $\big(1, {}^\omega\sigma_x^n(\alpha t)\big)$, letting $m \to +\infty$, we obtain

$$F_{(\psi - J^n f)(x)}(t) \geq {}^\omega\sigma_x^n(\alpha t). \tag{2.25}$$

Clearly, (2.25) implies (2.23), which with (2.21) yields (2.14). Furthermore, by the Λ-contractivity of J,

$$F_{(J\psi - J^{n+1} f)(x)}(t) \geq \big(\Lambda F_{\psi - J^n f}\big)_x(t), \qquad t > 0, \quad x \in X. \tag{2.26}$$

But (2.13) means that

$$\lim_{n \to +\infty} F_{(\psi - J^n f)(x)} = H_0, \quad x \in X.$$

By $(\mathcal{C}_0)$ we also have

$$\lim_{n \to +\infty} \big(\Lambda F_{\psi - J^n f}\big)_x = H_0, \quad x \in X.$$

Whence, on account of (2.26),

$$\lim_{n \to +\infty} F_{(J\psi - J^{n+1} f)(x)} = H_0, \quad x \in X.$$

Consequently

$$J\psi(x) = \lim_{n \to +\infty} (J^{n+1} f)(x) = \psi(x), \qquad x \in X.$$

Thus we have shown that ψ is a fixed point of J. It remains to prove the statements on the uniqueness of ψ. Assume that $\psi_1, \psi_2 \in Y^X$ are two fixed points of J satisfying for some $\alpha_1, \alpha_2 \in (0,1)$

$$F_{(\psi_j - J^k f)(x)}(t) \geq \widehat{\sigma}_x^k(\alpha_j t), \qquad k \in \mathbb{N}, \quad x \in X, \quad t > 0.$$

Then we get for $x \in X$, $t > 0$ and $k \in \mathbb{N}$

$$\begin{aligned} F_{(\psi_1 - \psi_2)(x)}\ (t) &\geq T\left(F_{(\psi_1 - J^k f)(x)}\Big(\frac{t}{2}\Big), F_{(J^k f - \psi_2)(x)}\Big(\frac{t}{2}\Big)\right) \\ &\geq T\left(\ \widehat{\sigma}_x^k\Big(\alpha_1 \frac{t}{2}\Big), \widehat{\sigma}_x^k\Big(\alpha_2 \frac{t}{2}\Big)\right). \end{aligned} \tag{2.27}$$

Note yet that, in view of (2.6) and (2.7), each of the conditions (2.10) and (2.11) implies

$$\lim_{k\to+\infty} \widehat{\sigma}_x^k(t) = 1, \qquad x \in X,\ t > 0. \tag{2.28}$$

Hence, letting $k \to \infty$ in (2.27), by the continuity of T at the point $(1,1)$, we finally obtain that

$$F_{(\psi_1-\psi_2)(x)} = H_0, \qquad x \in X, \tag{2.29}$$

which means that $\psi_1 = \psi_2$. □

Remark 2.2.7. If, for a given $k \in \mathbb{N}$ and $x \in X$, the function $\widehat{\sigma}_x^k$ is left continuous (which is not necessarily the case, because this depends on the forms of ϵ and T), then it is easily seen that (2.14) can be replaced by

$$F_{(\psi-J^k f)(x)}(t) \geq \widehat{\sigma}_x^k(t), \qquad t > 0.$$

Otherwise, for every fixed $x \in X$ and $k \in \mathbb{N}$, the inequality in (2.14) can of course be replaced by

$$F_{(\psi-J^k f)(x)}(t) \geq \widehat{\sigma}_x^k(\alpha_{x,k}\, t), \qquad t > 0,$$

with any fixed $\alpha_{x,k} \in (0,1)$.

Remark 2.2.8. The assumptions (i) and (ii) in Theorem 2.2.6 look nearly the same and (i) is a bit simpler than (ii). However, as we will see below, in some situations (2.11) (with some sequence $(\omega_n^k)_{n\in\mathbb{N}} \in \Omega$) and (2.12) are fulfilled, while (2.10) is not.

Namely, let $T = T_M$ and Λ have the following simple form

$$(\Lambda\delta)_x(t) = \delta_{ax}(bt), \qquad x \in X,\ t > 0,\ \delta \in \mathcal{D}_+^X, \tag{2.30}$$

with some $a, b \in \mathbb{R}_+^\star$. Then, we prove by induction that for every $n \in \mathbb{N}^\star$

$$(\Lambda^n\delta)_x(t) = \delta_{a^n x}(b^n t), \qquad x \in X,\ t > 0. \tag{2.31}$$

The case $n = 1$ is just (2.30). So, fix $n \in \mathbb{N}^\star$ satisfying 2.31. Then

$$\left(\Lambda^{n+1}\delta\right)_x(t) = \left(\Lambda(\Lambda^n\delta)\right)_x(t) = \left(\Lambda^n\delta\right)_{ax}(bt) = \delta_{a^{n+1}x}(b^{n+1}t)$$

Further, assume that X is a normed space, $p \in [0,\infty)$, $v \in \mathbb{R}_+$ and

$$\epsilon_x(t) = \frac{t}{t + v\|x\|^p}, \qquad x \in X,\ t > 0.$$

Write $e_0 := ba^{-p}$. Clearly, for every $n \in \mathbb{N}^\star$, $x \in X$ and $t > 0$,

$$(\Lambda^n \epsilon)_x(t) = \epsilon_{a^n x}(b^n t) = \frac{b^n t}{b^n t + v\|a^n x\|^p} = \epsilon_x\Big(\frac{b^n t}{a^{np}}\Big) = \epsilon_x\big(e_0^n t\big). \tag{2.32}$$

Therefore

$$T_{i=k}^{k+j-1}(\Lambda^i \epsilon)_x(t) = \inf_{i \in \{0,\dots,j-1\}} \epsilon_x\big(e_0^{k+i} t\big), \qquad j \in \mathbb{N}^\star. \tag{2.33}$$

Assume that $e_0 > 1$. Then, by (2.32),

$$\lim_{n\to+\infty} (\Lambda^n \epsilon)_x(t) = \lim_{n\to+\infty} \epsilon_x\big(e_0^n t\big) = 1, \qquad x \in X,\ t > 0, \tag{2.34}$$

which means that (2.12) holds. Further, for every $k \in \mathbb{N}$, (2.33) yields

$$T_{i=k}^{k+j-1}(\Lambda^i \epsilon)_x(t) = \epsilon_x\big(e_0^k t\big), \qquad j \in \mathbb{N}^\star,\ x \in X,\ t > 0,$$

whence

$$\inf_{j\in\mathbb{N}} T_{i=k}^{k+j}(\Lambda^i \epsilon)_x\Big(\frac{t}{j+1}\Big) = \inf_{j\in\mathbb{N}} \epsilon_x\Big(\frac{e_0^k t}{j+1}\Big) = 0, \qquad x \in X,\ t > 0.$$

Consequently, for every $x \in X$ and $t > 0$,

$$\sigma_x^k(t) = \lim_{j\to\infty} T_{i=k}^{k+j-1}(\Lambda^i \epsilon)_x\Big(\frac{t}{j}\Big) = \lim_{j\to\infty} \epsilon_x\Big(\frac{e_0^k t}{j}\Big) = 0, \quad k \in \mathbb{N},$$

and

$$\lim_{k\to+\infty} \inf_{j\in\mathbb{N}} T_{i=k}^{k+j}(\Lambda^i \epsilon)_x\Big(\frac{t}{j+1}\Big) = 0.$$

Hence (2.10) is not valid and σ_x^k makes no contribution in estimation (2.14).

On the other hand, for every $x \in X$, $t > 0$ and $\omega = (\omega_n)_{n\in\mathbb{N}} \in \Omega$, we have

$$T_{i=k}^{k+j-1}(\Lambda^i \epsilon)_x\big(\omega_{i-k} t\big) = \inf_{i \in \{0,\dots,j-1\}} \epsilon_x\big(e_0^{k+i} \omega_i t\big).$$

So, for $\widehat{\omega} := (\widehat{\omega}_n)_{n\in\mathbb{N}}$, with $\widehat{\omega}_i := r^i(1-r)$, for every $i \in \mathbb{N}$ and $r := 1/e_0$, we have

$$e_0^{k+i}\widehat{\omega}_i = e_0^k(1-r) = e_0^{k-1}(e_0 - 1), \qquad i \in \mathbb{N}.$$

Therefore, for every $x \in X$ and $t > 0$,

$$T_{i=k}^{k+j-1}(\Lambda^i \epsilon)_x\big(\widehat{\omega}_{i-k} t\big) = \epsilon_x\big(\big(e_0^{k-1}(e_0 - 1)t\big)\big),$$

whence,

$$^{\widehat{\omega}}\sigma_x^k(t) = \lim_{j\to\infty} \inf_{i\in\{0,\ldots,j-1\}} \epsilon_x\big(e_0^{k+i}\widehat{\omega}_i t\big) = \epsilon_x\big(e_0^{k-1}(e_0-1)t\big).$$

Therefore

$$\lim_{m\to+\infty} \inf_{j\in\mathbb{N}} T_{i=m}^{m+j}(\Lambda^i\epsilon)_x\big(\widehat{\omega}_{i-m}^m t\big) = \lim_{m\to+\infty} \epsilon_x\big(e_0^{m-1}(e_0-1)t\big) = 1.$$

This implies that (2.11) holds with $\omega^k = \widehat{\omega}$ for $k \in \mathbb{N}$, since

$$\widehat{\sigma}_x^k(t) \geq {}^{\widehat{\omega}}\sigma_x^k(t) = \epsilon_x\big(e_0^{k-1}(e_0-1)t\big), \qquad x \in X,\ t > 0.$$

Remark 2.2.9. Note that in the proof of Theorem 2.2.6 we have only used continuity of T at the points of the form $(1,\xi)$ for $\xi \in (0,1]$. Actually, even that assumption can be weakened. Namely, it is enough to assume that T is continuous only at the point $(1,1)$, but then we have to modify inequality in (2.14) basing it only on (2.22) and (2.24) without taking the limits.

Remark 2.2.10. Observe that the properties of the t-norm yield

$$\inf_{k\in\mathbb{N}} T_{i=n}^{k+n}(\Lambda^i\epsilon)_x\Big(\frac{t}{k+1}\Big) \leq (\Lambda^n\epsilon)_x(t), \qquad x \in X,\ t > 0,\ n \in \mathbb{N},$$

whence (2.10) implies (2.12). However, since every G-complete RN-space is M-complete, and not necessarily conversely, the assumption (iii) is not weaker than (i).

As for the uniqueness of the fixed points of J in Theorem 2.2.6, we also have the following proposition.

Proposition 2.2.11. *Let $\Lambda : \mathcal{D}_+^X \to \mathcal{D}_+^X$, $J : Y^X \to Y^X$ be Λ-contractive, $k \in \mathbb{N}$ and $\sigma \in \{\sigma^k\} \cup \{{}^{\omega}\sigma^k : \omega \in \Omega\}$ satisfy*

$$\lim_{n\to+\infty} (\Lambda^n\sigma)_x(t) = 1, \qquad x \in X,\ t > 0, \tag{2.35}$$

where $\sigma^k, {}^{\omega} \in \mathcal{D}_+^X$ are given by $\sigma^k(x) = \sigma_x^k$ and ${}^{\omega}\sigma^k(x) = {}^{\omega}\sigma_x^k$ for $x \in X$. Then, for every $f : X \to Y$, the operator J has at most one fixed point ψ_0 satisfying

$$F_{(\psi_0 - J^k f)(x)} \geq \sigma(x), \quad x \in X.$$

Proof. Fix $f : X \to Y$ and assume that $\psi_1, \psi_2 \in Y^X$ are fixed points of J satisfying

$$F_{(\psi_j - J^k f)(x)} \geq \sigma_x, \qquad x \in X,\ j = 1, 2.$$

Then, by the Λ-contractivity of J,

$$F_{(J^\ell\psi_j - J^{k+\ell}f)(x)} \geq (\Lambda^\ell\sigma)_x, \qquad x \in X,\ j = 1,2,\ m \in \mathbb{N},$$

and consequently

$$\begin{aligned} F_{(\psi_1-\psi_2)(x)}(t) &= F_{(J^\ell\psi_1 - J^\ell\psi_2)(x)}(t) \\ &\geq T\left(F_{(J^\ell\psi_1 - J^{k+\ell}f)(x)}\left(\frac{t}{2}\right), F_{(J^{k+\ell}f - J^\ell\psi_2)(x)}\left(\frac{t}{2}\right)\right) \\ &\geq T\left((\Lambda^\ell\sigma)_x\left(\frac{t}{2}\right), (\Lambda^\ell\sigma)_x\left(\frac{t}{2}\right)\right). \end{aligned}$$

for every $\ell \in \mathbb{N}$, $x \in X$, and $t > 0$. Hence, letting ℓ tend to ∞, by (2.35) and the continuity of T at the point $(1,1)$, $F_{(\psi_1-\psi_2)(x)} = H_0$, $x \in X$. Consequently $\psi_1 = \psi_2$. □

Now we are in a position to give some consequences of Theorem 2.2.6, but first we need to present the following example and remark that we will be use in the sequel.

Example 2.2.12. *Let $\nu \in \mathbb{N}^\star$, $\xi_1, \ldots, \xi_\nu : X \to X$, and $L_1, \ldots, L_\nu : X \to (0,\infty)$ be fixed. A natural example of an operator Λ fulfilling hypothesis $(\mathcal{C}_0)$ is given by*

$$(\Lambda\delta)_x(t) := T_{i=1}^{\nu}\delta_{\xi_i(x)}\left(\frac{t}{\nu L_i(x)}\right), \qquad \delta \in \mathcal{D}_+^X,\ x \in X,\ t > 0. \tag{2.36}$$

Remark 2.2.13. Let $\nu \in \mathbb{N}^\star$, $\xi_1, \ldots, \xi_\nu : X \to X$, and $L_1, \ldots, L_\nu : X \to (0,\infty)$ be fixed. If the operator J has the form

$$J\eta(x) := H\big(x, \eta(\xi_1(x)), \ldots, \eta(\xi_\nu(x))\big), \qquad \eta \in Y^X,\ x \in X, \tag{2.37}$$

with a function $H : X \times Y^\nu \to Y$ satisfying the following Lipschitz-type condition

$$F_{H(x,y_1,\ldots,y_\nu)-H(x,z_1,\ldots,z_\nu)}(t) \geq T_{i=1}^{\nu}F_{y_i-z_i}\left(\frac{t}{\nu L_i(x)}\right), \qquad x \in X, t > 0, \tag{2.38}$$

for all $x \in X$ and $y_1, \ldots, y_\nu, z_1, \ldots, z_\nu \in Y$, then such J is Λ-contractive with Λ defined by (2.36). Clearly, (2.38) holds if H has the following simple form

$$H(x, y_1, \ldots, y_\nu) = \sum_{i=1}^{\nu} L_i(x)y_i + h(x), \qquad x \in X,\ y_1, \ldots, y_\nu \in Y, \tag{2.39}$$

with a fixed function $h \in Y^X$. Then (2.37) becomes

$$Jf(x) := \sum_{i=1}^{\nu} L_i(x)f(\xi_i(x)) + h(x), \qquad f \in Y^X,\ x \in X. \tag{2.40}$$

In particular, such J satisfies the following Lipschitz-type condition

$$F_{(J\mu-J\eta)(x)}(t) \geq T_{i=1}^{\nu} F_{\mu(\xi_i(x))-\eta(\xi_i(x))}\Big(\frac{t}{\nu L_i(x)}\Big), \quad \mu,\eta \in Y^X, x \in X,\ t>0. \tag{2.41}$$

If we want to admit functions L_i taking values in $\mathbb{R}$ (i.e., in particular taking the value zero), then we can rewrite that condition in the subsequent form

$$F_{(J\mu-J\eta)(x)}(t) \geq T_{i=1}^{\nu} F_{L_i(x)(\mu(\xi_i(x))-\eta(\xi_i(x)))}(\frac{t}{\nu}), \quad \mu,\eta \in Y^X, x \in X,\ t>0. \tag{2.42}$$

Note that if, in such situation, $L_i(x) \neq 0$ for some $i \in \{1,\dots,\nu\}$ and some $x \in X$, then

$$F_{L_i(x)(\mu(\xi_i(x))-\eta(\xi_i(x)))}(t) = F_{\mu(\xi_i(x))-\eta(\xi_i(x))}\Big(\frac{t}{|L_i(x)|}\Big), \qquad t>0;$$

but if $L_i(x)=0$, then $F_{L_i(x)(\mu(\xi_i(x))-\eta(\xi_i(x)))}(t) = F_0(t) = 1$, for every $t>0$.

In view of Remark 2.2.13, for operators $J : Y^X \to Y^X$ fulfilling condition (2.41), we have the following particular case of Theorem 2.2.6, with a strong statement on the uniqueness of fixed point (because under the weaker assumption that (2.15) holds only for $k=0$). The following theorem is the random normed space version of Theorem 1.1.6.

Theorem 2.2.14. *Let $\nu \in \mathbb{N}^\star$, $\xi_1,\dots,\xi_\nu : X \to X$, $L_1,\dots,L_\nu : X \to (0,\infty)$ and let $\Lambda : \mathcal{D}_+^X \to \mathcal{D}_+^X$, $\epsilon : X \to \mathcal{D}_+$, $J : Y^X \to Y^X$, and $f : X \to Y$ be given, so that Λ is defined by* (2.36)*, J satisfies condition* (2.41)*, and f satisfies* (2.9)*. Assume that one of conditions (i)–(iii) of Theorem 2.2.6 holds. Then, for every $x \in X$, $\omega = (\omega_n)_{n\in\mathbb{N}} \in \Omega$ and $t>0$, the limit* (2.13) *exists and the function $\psi \in Y^X$, defined in this way, is a fixed point of J satisfying* (2.14).

Moreover, if (i) or (ii) holds, then ψ is the unique fixed point of J such that there is $\alpha \in (0,1)$ with

$$F_{(\psi-f)(x)}(t) \geq \widehat{\sigma}_x^0(\alpha t), \qquad t>0,\ x\in X. \tag{2.43}$$

Proof. Let $\xi,\eta \in Y^X$ and $\phi \in \mathcal{D}_+^X$ with $F_{(\xi-\eta)(x)} \geq \phi_x$, for every $x \in X$. Then, by (2.41),

$$\begin{aligned} F_{(J\mu-J\eta)(x)}(t) &\geq T_{i=1}^{\nu} F_{(\mu-\eta)(\xi_i(x))}\Big(\frac{t}{\nu L_i(x)}\Big) \\ &\geq T_{i=1}^{\nu} \phi_{\xi_i(x)}\Big(\frac{t}{\nu L_i(x)}\Big) = (\Lambda\phi)_x(t), \qquad x\in X,\ t>0. \end{aligned}$$

Thus J is Λ-contractive. Moreover, as we have noticed in Remark 2.2.13, Λ satisfies the hypothesis $(\mathcal{C}_0)$. Hence, by Theorem 2.2.6, the limit (2.13) exists for every $k \in \mathbb{N}$, $x \in X$, $\omega = (\omega_n)_{n\in\mathbb{N}} \in \Omega$ and $t>0$, and the function ψ, defined in this way, is a fixed point of J satisfying (2.14). It remains to show the statement on the uniqueness of ψ. So, let

$\tau \in Y^X$ be a fixed point of J such that, for some $\alpha \in (0,1)$,

$$F_{(\tau-f)(x)}(t) \geq \widehat{\sigma}_x^0(\alpha t), \qquad t>0,\ x \in X. \tag{2.44}$$

Fix $x \in X$, $\omega^m = (\omega_n)_{n\in\mathbb{N}} \in \Omega$ and $t>0$. We show that, for every $n \in \mathbb{N}$, we have

$$F_{(J^n\psi - J^n\tau)(x)}(t) \geq \lim_{m\to+\infty} \widehat{T}\Big(T_{i=n}^{n+m-1}(\Lambda^i\epsilon)_x\Big(\frac{\alpha t}{2m}\Big)\Big), \tag{2.45}$$

$$F_{(J^n\psi - J^n\tau)(x)}(t) \geq \lim_{m\to+\infty} \widehat{T}\Big(T_{i=n}^{n+m-1}(\Lambda^i\epsilon)_x\Big(\frac{\omega_{i-n}\alpha t}{2}\Big)\Big). \tag{2.46}$$

This is the case for $n=0$, because by the continuity of T, for every $x \in X$ and $t>0$

$$\begin{aligned}
F_{(\psi-\tau)(x)}(t) &\geq T\Big(F_{(\psi-f)(x)}\Big(\frac{t}{2}\Big), F_{(\tau-f)(x)}\Big(\frac{t}{2}\Big)\Big)\\
&\geq \widehat{T}\Big(\sigma_x^0\Big(\frac{\alpha t}{2}\Big)\Big) = \widehat{T}\Big(\lim_{m\to+\infty} T_{i=0}^{m-1}(\Lambda^i\epsilon)_x\Big(\frac{\alpha t}{2m}\Big)\Big)\\
&= \lim_{m\to+\infty} \widehat{T}\Big(T_{i=0}^{m-1}(\Lambda^i\epsilon)_x\Big(\frac{\alpha t}{2m}\Big)\Big).
\end{aligned}$$

Analogously

$$F_{(\psi-\tau)(x)}(t) \geq \widehat{T}\Big(\underline{\sigma}_0^x\Big(\frac{\alpha t}{2}\Big)\Big) \geq \widehat{T}\Big({}^{\omega}\sigma_0^x\Big(\frac{\alpha t}{2}\Big)\Big), \qquad \omega \in \Omega.$$

Since T is continuous, we finally get

$$F_{(\psi-\tau)(x)}(t) \geq \lim_{m\to+\infty} \widehat{T}\Big(T_{i=0}^{m-1}(\Lambda^i\epsilon)_x\Big(\frac{\omega_{i-n}\alpha t}{2}\Big)\Big).$$

Now assume that (2.45) is valid for some $n \in \mathbb{N}$. Then, by (1.5) and the continuity of T, for every $x \in X$ and $t>0$,

$$\begin{aligned}
F_{(J^{n+1}\psi - J^{n+1}\tau)(x)}(t) &\geq T_{j=1}^{\nu} F_{(J^n\psi - J^n\tau)(\xi_j(x))}\Big(\frac{t}{\nu L_j(x)}\Big)\\
&\geq T_{j=1}^{\nu} \lim_{m\to+\infty} \widehat{T}\Big(T_{i=n}^{n+m-1}(\Lambda^i\epsilon)_{\xi_j(x)}\Big(\frac{\alpha t}{2m\nu L_j(x)}\Big)\Big)\\
&= \lim_{m\to+\infty} T_{j=1}^{\nu}\widehat{T}\Big(T_{i=n}^{n+m-1}(\Lambda^i\epsilon)_{\xi_j(x)}\Big(\frac{\alpha t}{2m\nu L_j(x)}\Big)\Big)\\
&= \lim_{m\to+\infty} \widehat{T}\Big(T_{i=n}^{n+m-1}T_{j=1}^{\nu}(\Lambda^i\epsilon)_{\xi_j(x)}\Big(\frac{\alpha t}{2m\nu L_j(x)}\Big)\Big)\\
&= \lim_{m\to+\infty} \widehat{T}\Big(T_{i=n}^{n+m-1}(\Lambda(\Lambda^i\epsilon))_x\Big(\frac{\alpha t}{2m}\Big)\Big)\\
&= \lim_{m\to+\infty} \widehat{T}\Big(T_{i=n+1}^{n+m}(\Lambda^i\epsilon)_x\Big(\frac{\alpha \nu t}{2m}\Big)\Big).
\end{aligned}$$

Next, assume that (2.46) is valid for some $n \in \mathbb{N}$. Then in the same way, by (1.5) and the continuity of T, for every $x \in X$, $t > 0$, and $\omega \in \Omega$,

$$\begin{aligned}
F_{(J^{n+1}\psi - J^{n+1}\tau)(x)}(t) &\geq T_{j=1}^{\nu} F_{(J^n\psi - J^n\tau)(\xi_j(x))}\Big(\frac{t}{\nu L_j(x)}\Big) \\
&\geq T_{j=1}^{\nu} \lim_{m\to+\infty} \widehat{T}\bigg(T_{i=n}^{n+m-1}(\Lambda^i\epsilon)_{\xi_j(x)}\Big(\frac{\omega_{i-n}\alpha t}{2\nu L_j(x)}\Big)\bigg) \\
&= \lim_{m\to+\infty} T_{j=1}^{\nu}\widehat{T}\bigg(T_{i=n}^{n+m-1}(\Lambda^i\epsilon)_{\xi_j(x)}\Big(\frac{\omega_{i-n}\alpha t}{2\nu L_j(x)}\Big)\bigg) \\
&= \lim_{m\to+\infty} \widehat{T}\bigg(T_{i=n}^{n+m-1}T_{j=1}^{\nu}(\Lambda^i\epsilon)_{\xi_j(x)}\Big(\frac{\omega_{i-n}\alpha t}{2\nu L_j(x)}\Big)\bigg) \\
&= \lim_{m\to+\infty} \widehat{T}\bigg(T_{i=n}^{n+m-1}(\Lambda^{i+1}\epsilon)_{x}\Big(\frac{\omega_{i-n}\alpha t}{2}\Big)\bigg) \\
&= \lim_{m\to+\infty} \widehat{T}\bigg(T_{i=n+1}^{n+m}(\Lambda^{i}\epsilon)_{x}\Big(\frac{\omega_{i-n-1}\alpha t}{2}\Big)\bigg).
\end{aligned}$$

Thus we have proved (2.45) and (2.46) for every $n \in \mathbb{N}$, $x \in X$, $\omega \in \Omega$, and $t > 0$. Whence

$$\begin{aligned}
F_{(\tau-\psi)(x)}(t) = F_{(J^n\tau - J^n\psi)(x)}(t) &\geq \lim_{m\to+\infty} \widehat{T}\bigg(T_{i=n}^{n+m-1}(\Lambda^i\epsilon)_x\Big(\frac{\alpha t}{2m}\Big)\bigg) \\
&= \widehat{T}\bigg(\lim_{m\to+\infty} T_{i=n}^{n+m-1}(\Lambda^i\epsilon)_x\Big(\frac{\alpha t}{2m}\Big)\bigg),
\end{aligned} \tag{2.47}$$

$$\begin{aligned}
F_{(\tau-\psi)(x)}(t) = F_{(J^n\tau - J^n\psi)(x)}(t) &\geq \lim_{m\to+\infty} \widehat{T}\bigg(T_{i=n}^{n+m-1}(\Lambda^i\epsilon)_x\Big(\frac{\omega_{i-n}\alpha t}{2}\Big)\bigg) \\
&= \widehat{T}\bigg(\lim_{m\to+\infty} T_{i=n}^{n+m-1}(\Lambda^i\epsilon)_x\Big(\frac{\omega_{i-n}\alpha t}{2}\Big)\bigg).
\end{aligned} \tag{2.48}$$

Now, if (2.10) holds, then letting $n \to +\infty$ in (2.47), by the continuity of $\widehat{T}$, we get $F_{(\tau-\psi)(x)}(t) = 1$ for every $x \in X$ and $t > 0$, which means that $\tau = \psi$. Similarly, if (2.11) holds, then we argue analogously letting $n \to +\infty$ in (2.48). □

If X has only one element, then Y^X can actually be identified with Y and Theorem 2.2.6 becomes the following analogue of the classical Banach contraction principle (somewhat generalized), given in Corollary 2.2.17 below. To present it we need the following definition, which is a special case of Definition 2.2.2.

Definition 2.2.15. *A mappings $\lambda : \mathcal{D}_+ \to \mathcal{D}_+$ is said to satisfy the property $(\mathcal{C})$, if whenever a sequence $(z_n)_{n\in\mathbb{N}^\star} \subset Y$ converges to 0, $\lambda(F_{z_n})_n$ converges pointwise to H_0. This is, for every $(z_n)_{n\in\mathbb{N}^\star} \subset Y$*

$$\Big(\lim_{n\to+\infty} F_{z_n}(t) = 1, \quad t > 0\Big) \Longrightarrow \Big(\lim_{n\to+\infty} \lambda(F_{z_n})(t) = 1, \quad t > 0\Big). \tag{$\mathcal{C}$}$$

To avoid any ambiguity, let us give also a further definition, which is also a special case of Definition 2.2.1

Definition 2.2.16. *Let $\lambda : \mathcal{D}_+ \to \mathcal{D}_+$ be a given mapping. We say that a mapping $h : Y \to Y$ is λ-contractive provided, for every $z, w \in Y$ and $\phi \in \mathcal{D}_+$*

$$\big(F_{z-w} \geq \phi\big) \Longrightarrow \big(F_{h(z)-h(w)} \geq \lambda\phi := \lambda(\phi)\big).$$

Corollary 2.2.17. *Let $\lambda : \mathcal{D}_+ \to \mathcal{D}_+$ satisfy the property $(\mathcal{C})$, and $h : Y \to Y$ be λ-contractive. Let $\epsilon \in \mathcal{D}_+$ be such that*

$$F_{h(z)-z} \geq \epsilon, \quad z \in Y, \tag{2.49}$$

and assume that one of the following three conditions holds

(α) (Y, F, T) *is M-complete and* $\lim\limits_{k\to+\infty} \inf\limits_{j\in\mathbb{N}} T_{i=k}^{k+j}(\lambda^i\epsilon)\Big(\dfrac{t}{j+1}\Big) = 1,\ t > 0.$

(β) (Y, F, T) *is M-complete and for each $k \in \mathbb{N}^\star$ there exists a sequence $(\omega_n^k)_{n\in\mathbb{N}} \in \Omega$ with* $\lim\limits_{k\to+\infty} \inf\limits_{j\in\mathbb{N}} T_{i=k}^{k+j}(\lambda^i\epsilon)\big(\omega_{i-k}^k t\big) = 1,\ t > 0.$

(γ) (Y, F, T) *is G-complete and*
$$\lim_{n\to+\infty} (\lambda^n\epsilon) = H_0. \tag{2.50}$$

Then, for every $\omega = (\omega_n)_{n\in\mathbb{N}} \in \Omega$ and $t > 0$, the limits

$$\begin{aligned} z_0 &:= \lim_{n\to+\infty} h^n(z) \\ l_k(t) &:= \lim_{m\to+\infty} T_{i=k}^{m+k-1}(\lambda^i\epsilon)\Big(\frac{t}{m}\Big) \\ {}^{\omega}l_k(t) &:= \lim_{m\to+\infty} T_{i=k}^{m+k-1}(\lambda^i\epsilon)\big(\omega_{i-n}t\big) \end{aligned}$$

exist and z_0 is a fixed point of h such that

$$F_{z_0-h^k(z)}(t) \geq \sup_{\alpha\in(0,1)} \widehat{l}_k(\alpha t), \qquad t > 0,\ k \in \mathbb{N},$$

where

$$\widehat{l}_k(t) := \max\{l_k(t), \underline{l}_k(t)\} \quad \text{and} \quad \underline{l}_k(t) := \sup_{\omega\in\Omega} {}^{\omega}l_k(t).$$

Moreover, in case (α) or (β) holds, z_0 is the unique fixed point of h for which there exists $\alpha \in (0,1)$ with

$$F_{z_0-h^k(z)}(t) \geq \widehat{l}_k(\alpha t) \quad t > 0,\ k \in \mathbb{N}.$$

Remark 2.2.18. Let functions $g : \mathbb{R} \to \mathbb{R}$ and $G : [0,1] \to [0,1]$ be non-decreasing, left-continuous and such that $g(0) = G(0) = 0$, $G(1) = 1$,

$$G(t) \geq t, \quad \lim_{n\to\infty} g^n(t) = \infty, \quad t > 0.$$

Let $\lambda : \mathcal{D}_+ \to \mathcal{D}_+$ has the form

$$(\lambda\xi)(t) = G(\xi(g(t))),\ \xi \in \mathcal{D}_+,\ t \in \mathbb{R}.$$

Then

$$\lim_{n\to+\infty} (\lambda^n\xi)(t) = \lim_{n\to+\infty} G^n(\xi(g^n(t))) = 1, \quad t > 0, \quad \xi \in \mathcal{D}_+,$$

which means that (2.50) holds for every $\epsilon \in \mathcal{D}_+$.

A very simple example of such λ is obtained when G is the identity map of $[0, 1]$ (i.e., $G(t) = t$) and

$$g(t) = at, \quad t \in \mathbb{R}, \tag{2.51}$$

with a fixed $a > 1$. Clearly, then $(\lambda\xi)(t) = \xi(at)$ for $\xi \in \mathcal{D}_+$ and $t > 0$ and, in this case, the λ-contractive mappings are known as B-contractions or Sehgal contractions (see [127, 173]).

If g is the identity map on $\mathbb{R}$ and

$$G(t) = \frac{s}{s + k(1-s)}, \quad s \in [0, 1], \tag{2.52}$$

with some $k \in]0, 1[$, then λ-contractive mappings are fuzzy contractive (see [80, 127]).

If both (2.51) and (2.52) hold, then λ-contractive mappings are called strict B-contractions (see [127, 159]).

Chapter 3

Ulam-Stability of Linear Functional Equations in Banach Spaces

3.1 Introduction

One of the most general linear functional equation studied up to 2020, is the following non-homogeneous Forti equation

$$\sum_{i=1}^{m} B_i f\left(\sum_{j=1}^{n} b_{ij} x_j\right) + B = 0, \tag{3.1}$$

where m and n are positive integers, f is a mapping from a vector space X into a Banach space Y, and for every $i \in \{1, \ldots, m\}$, every $j \in \{1, \ldots, n\}$, B_i and b_{ij} are scalars, and B is an element of Y. In fact, the equation (3.1) was introduced by Forti [72] with $B = 0$. Later, Bahayrycz and Olko [18, 19] used Theorem 1.1.6 to show the stability and the hyperstability of (3.1). In this chapter, using a proof similar to that of Oubbi [143], relying on the classical Banach contraction theorem, we reprove the stability and the hyperstability of (3.1). Along the way, let us notice that, in [18] and [19], the authors assumed that B is a scalar. Actually, B must be an element of Y. Fortunately, this does not alter their results.

Next, we introduce the following new linear functional equation

$$\sum_{i=1}^{m} A_i(f(\varphi_i(\bar{x}))) + b = 0, \quad \bar{x} := (x_1, \ldots, x_n) \in X^n, \tag{3.2}$$

where m and n are positive integers, f is a mapping from a vector space X into a Banach space Y, and for every $i \in \{1, \ldots, m\}$, φ_i is a linear mapping from X^n into X, A_i is a continuous endomorphism of Y and $b \in Y$. Using the Banach contraction fixed point theorem, as in [143], we prove the stability and the hyperstability of (3.2), then also of it associated homogeneous one

$$\sum_{i=1}^{m} A_i(f(\varphi_i(\bar{x}))) = 0, \quad \bar{x} := (x_1, \ldots, x_n) \in X^n. \tag{3.3}$$

Since our equation generalizes clearly (3.1), most known results on the stability of linear functional equations in the literature derive from ours (e.g., the main results of [[18], [64]] and [19]). We also provide some situations to which our theorem applies, but are not covered by the former results.

Throughout this chapter, X and Y will be vector spaces on the field $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}\}$ and $f : X \to Y$ will be a mapping. The space Y will be endowed with a complete norm $\|\|$.

3.2 Stability of non-homogeneous Forti equation

In this section, we will be concerned with the stability of the equation (3.1) and its corresponding homogeneous one

$$\sum_{i=1}^{m} B_i f\left(\sum_{j=1}^{n} b_{ij} x_j\right) = 0. \tag{3.4}$$

The following lemma will be used in the sequel.

Lemma 3.2.1. *Define the function $d : X_f \times X_f \to \mathbb{R}_+$ with*

$$d(g; h) := \inf\left\{K > 0,\ \|g(t) - h(t)\| \leq K\phi(t), \quad t \in X\right\},$$

where $\phi : X \to \mathbb{R}_+$ is a given function and

$$X_f := \{g : X \to Y,\ \exists K > 0;\ \|f(t) - g(t)\| \leq K\phi(t), \quad t \in X\}.$$

Then (X_f, d) is a complete metric space.

Proof. First, we will show that d is a distance on X_f. Let $g, h \in X_f$, then there exist $K^{'}, K^{''} > 0$ such that, for every $x \in X$,

$$\|g(x) - f(x)\| \leq K^{'}\phi(x) \quad \text{and} \quad \|h(x) - f(x)\| \leq K^{''}\phi(x).$$

Therefore

$$\|g(x) - h(x)\| \leq (K^{'} + K^{''})\phi(x).$$

Thus $d(g, h) \leq K^{'} + K^{''} < +\infty$, for all $g, h \in X_f$. It is clear that $d(g, h) = 0$ if and only if $g = h$, and that $d(g, h) = d(h, g)$, $g, h \in X_f$. For the triangular inequality, let

$g, h, k \in X_f$, and $K, K^{'} > 0$ be given so that $d(g,h) < K$ and $d(h,k) < K^{'}$. Then, for all $t \in X$, we have
$$\|g(t) - k(t)\| \leq (K + K^{'})\phi(t).$$

Passing to the infimum, we get $d(g,k) \leq d(g,h) + d(h,k)$. Therefore d is a distance on X_f. Now, let's show that the metric space (X_f, d) is complete. If $(g_n)_n$ is a Cauchy sequence in X_f, as the evaluation $\delta_x : g \mapsto g(x)$ are uniformly continuous from X_f into Y, the sequence $(g_n(x))_n$ is Cauchy in Y, for every $x \in X$. By the completeness of Y, it converges to some $g(x)$. But

$$\forall \epsilon > 0, \exists N_\epsilon \in \mathbb{N}, \forall m > n \geq N_\epsilon : d(g_n, g_m) < \epsilon.$$

Therefore
$$\|g_n(x) - g_m(x)\| \leq \epsilon\phi(x), \quad m > n \geq N_\epsilon,\ x \in X. \tag{3.5}$$

Letting m tend to infinity, we get $\|g_n(x) - g(x)\| \leq \epsilon\phi(x)$. Thus

$$\|g(x) - f(x)\| \leq \|g(x) - g_n(x)\| + \|g_n(x) - f(x)\| \leq (\epsilon + d(g_n, f))\phi(x).$$

Therefore the so defined mapping g belongs to X_f. Again by (3.5), $(g_n)_n$ converges in X_f to g and then (X_f, d) is complete. $\square$

Now, we show how the main result of Bahyrycz and Olko [18] can be deduced from the Banach's contraction principle.

Theorem 3.2.2 ([18]). *Assume that either $B = 0$ or $B \sum_{i=1}^{m} B_i \neq 0$. Suppose that some mapping $\theta : X^n \to \mathbb{R}_+$ exists such that*

$$\left\| \sum_{i=1}^{m} B_i f\left(\sum_{j=1}^{n} b_{ij} x_j\right) + B \right\| \leq \theta(x_1, ..., x_n), \qquad x_1, ..., x_n \in X. \tag{3.6}$$

Assume also that there exist a non empty set $I \subsetneq \{1, ..., m\}$, $I_0 := \{1, ..., m\} \backslash I$, $c_1, ..., c_n \in \mathbb{K}$ and $\omega_1, ..., \omega_n \in [0, \infty)$ such that

(i) *For every $i \in I$, $\beta_i = 1$, where, for every $k = 1, \ldots, m$, $\beta_k := \sum_{j=1}^{n} b_{ij} c_j$;*

(ii) $\sum_{i \in I_0} |B_i| \omega_i < |\sum_{i \in I} B_i|$;

(iii) *$\theta(\beta_i x_1, ..., \beta_i x_n)) \leq \omega_i \theta(x_1, ..., x_n)$ for $i \in I_0$ and $x_1, ..., x_n \in X$.*

Then there exists a unique solution $G : X \to Y$ to the functional equation (3.1)*, with*

$$\|g(x) - G(x)\| \leq \frac{\theta(c_1x, ..., c_nx)}{|\sum_{i\in I} B_i| - \sum_{i\in I_0} |B_i|\omega_i}, \qquad x \in X. \tag{3.7}$$

Before giving our proof, let us denote by $J : Y^X \to Y^X$ the operator

$$(J\xi)(x) := \sum_{i=1}^{k} \alpha_i \xi(\beta_i x), \quad \xi \in Y^X,\ x \in X, \tag{3.8}$$

with $k \in \mathbb{N}$ and $\alpha_1, \ldots, \alpha_k$, $\beta_1, \ldots, \beta_k \in \mathbb{K}$.

We will make use of the following lemma proven in [18].

Lemma 3.2.3 ([18])**.** *Assume that $\theta : X^n \to \mathbb{R}_+$ is a mapping and let $J : Y^X \to Y^X$ be an operator given by* (3.8)*. Assume that there exist $\omega_1, \ldots, \omega_k \in \mathbb{R}_+$ such that $\sum_{i=1}^{k} |\alpha_i|\omega_i < 1$ and*

$$\theta\left(\beta_i(x_1, \ldots, x_n)\right) \leq \omega_i \theta(x_1, \ldots, x_n), \quad i \in \{1, \ldots, k\}, \quad x_1, \ldots x_n \in X. \tag{3.9}$$

If there exists a mapping $f : X \to Y$ satisfies the inequality (3.6) *with $B = 0$, and if $G(x) := \lim_{n\to+\infty} J^n f(x)$ exists for every $x \in X$, then $G : X \to Y$ is a solution of* (3.4)*.*

Now we are in a position to prove Theorem 3.2.2.

Proof. Assume first that $B = 0$, and let $x \in X$ be arbitrary. For every $j \in \{1, \ldots, n\}$, putting $c_j x := x_j$ in (3.6), we get

$$\|f(x) - \sum_{i\notin I} \frac{-B_i}{B_I} f(\sum_{j=1}^{n} b_{ij}c_jx)\| \leq \frac{\theta(c_1x, \ldots, c_nx)}{|B_I|}. \tag{3.10}$$

Consider the set X_f consisting of all those mappings $g : X \to Y$ for which there exists some $K > 0$ with

$$\|f(t) - g(t)\| \leq K\theta(c_1t, \ldots, c_nt),\ t \in X.$$

Since $f \in X_f$, X_f is non empty. Further, we set, for every $g, h \in X_f$

$$d(g, h) := \inf\{K > 0,\ \|g(t) - h(t)\| \leq K\theta(c_1t, \ldots, c_nt), \quad t \in X\}.$$

Note that (X_f, d) is a complete metric space in view of Lemma 3.2.1. Now, for arbitrary $\xi \in X_f$, define a mapping $J\xi$ from X into Y by

$$J\xi(x) := \sum_{i\in I} \frac{-B_i}{B_I} \xi\left(\sum_{j=1}^{n} b_{ij}c_jx\right). \tag{3.11}$$

Due to (3.10), we get for every $x \in X$, $\|f(x) - Jf(x)\| \leq \dfrac{1}{|B_I|}\theta(c_1x, \ldots, c_nx)$. This show that $Jf \in X_f$ and $d(f, Jf) \leq \dfrac{1}{|B_I|}$. Moreover, for $g \in X_f$ and $K > d(f, g)$, we have

$$\begin{aligned}
\|Jg(x) - f(x)\| &\leq \sum_{i \notin I} \frac{|B_i|}{|B_I|} \Big\| g\left(\sum_{j=1}^{n} b_{ij}c_jx\right) - f\left(\sum_{j=1}^{n} b_{ij}c_jx\right) \Big\| + \|Jf(x) - f(x)\| \\
&\leq \sum_{i \notin I} |\frac{B_i}{B_I}| K\theta(\beta_ic_1x, \ldots, \beta_ic_nx) + \frac{\theta(c_1x, \ldots, c_nx)}{|B_I|} \\
&\leq K \sum_{i \notin I} |\frac{B_i}{B_I}| \omega_i\theta(c_1x, \ldots, c_nx) + \frac{\theta(c_1x, \ldots, c_nx)}{|B_I|} \\
&\leq \left(K \sum_{i \notin I} |\frac{B_i}{B_I}|\omega_i + \frac{1}{|B_I|}\right) \theta(c_1x, \ldots, c_nx).
\end{aligned}$$

This shows $Jg \in X_f$, for all $g \in X_f$. Whence J is a self mapping of X_f. Now, let us show that J is a strictly contracting mapping. Given g and h in X_f. Then

$$\begin{aligned}
\|Jg(x) - Jh(x)\| &= \Big\| \sum_{i \notin I} \frac{-B_i}{B_I} g\left(\sum_{j=1}^{n} b_{ij}c_jx\right) - \sum_{i \notin I} \frac{-B_i}{B_I} h\left(\sum_{j=1}^{n} b_{ij}c_jx\right) \Big\| \\
&\leq \sum_{i \notin I} |\frac{B_i}{B_I}| \Big\| g\left(\sum_{j=1}^{n} b_{ij}c_jx\right) - h\left(\sum_{j=1}^{n} b_{ij}c_jx\right) \Big\| \\
&\leq \sum_{i \notin I} |\frac{B_i}{B_I}| d(g, h)\theta\left(c_1 \sum_{j=1}^{n} b_{ij}c_jx, \ldots, c_n \sum_{j=1}^{n} b_{ij}c_jx\right) \\
&\leq d(g, h) \sum_{i \notin I} |\frac{B_i}{B_I}|\omega_i\theta(c_1x, \ldots, c_nx).
\end{aligned}$$

If we put $\gamma := \sum_{i \notin I} |\frac{B_i}{B_I}|\omega_i$, then $\gamma < 1$ and

$$d(Jg, Jh) \leq \gamma d(g, h), \quad g, h \in X_f.$$

Therefore J is a strict contraction mapping. By Banach fixed point theorem, there exists a unique mapping $G \in X_f$ such that $JG = G$ and $\lim_{n \to +\infty} J^n f = G$. Thanks to Lemma

3.2.3, G is a solution of (3.4). Furthermore

$$\begin{aligned} d(f,G) &= d(f, \lim_{n\to+\infty} J^n f) \\ &\le \lim_{n\to+\infty} \sum_{j=0}^{n-1} \gamma^j d(f,Jf) \\ &\le \frac{d(f,Jf)}{1-\gamma} \\ &\le \frac{1}{|B_I| - \sum_{i\notin I} |B_i|\omega_i}. \end{aligned}$$

Thus (3.7) holds. Now, if $B \neq 0$ and $\sum_{i=1}^{m} B_i \neq 0$. Define a new function $g : X \to Y$ by

$$g(x) := f(x) + \frac{B}{\sum_{i=1}^{m} B_i}.$$

Since

$$\|\sum_{i=1}^{m} B_i f(\sum_{j=1}^{n} b_{ij}x_j) + B\| \le \theta(x_1,\dots,x_n), \quad x_1,\dots,x_n \in X,$$

we get

$$\|\sum_{i=1}^{m} B_i g(\sum_{j=1}^{n} b_{ij})\| \le \theta(x_1,\dots,x_n), \quad x_1,\dots,x_n \in X,$$

By the first part of the proof, there exists a unique solution H of (3.4), such that

$$\|g(x) - H(x)\| \le \frac{\theta(c_1x,\dots,c_nx)}{|B_I| - \sum_{i\notin I} |B_i|\omega_i}, \quad x \in X.$$

Since H is a solution of (3.4), then $G := H - \dfrac{B}{\sum_{i=1}^{m} B_i}$ is a solution of (3.1). But then

$$\|f(x) - G(x)\| \le \frac{\theta(c_1x,\dots,c_nx)}{|B_I| - \sum_{i\notin I} |B_i|\omega_i}, \quad x \in X.$$

This finishes the proof. □

3.3 Hyperstability of non-homogeneous Forti equation

In this section, we show how the main result of Bahyrycz and Olko [19] can be deduce from the Banach's contraction principle.

Theorem 3.3.1 ([19]). *Assume that $B = 0$ or $B\sum_{i=1}^{m} B_i \neq 0$. Let $\theta : X^n \to \mathbb{R}_+$ satisfy*

(3.6) *and let* $\omega : \mathbb{K} \to \mathbb{R}_+$ *enjoy*

$$\theta(\beta x_1, \dots, \beta x_n) \leq \omega(\beta)\theta(x_1, \dots, x_n), \ x_1, \dots, x_n \in X, \ \beta \in \mathbb{K}.$$

If there exist a non empty set $I \subset \{1, \dots, m\}$ *and a sequence* $(c_{k,1}, \dots, c_{k,n})_{k\in\mathbb{N}}$ *of elements of* $\mathbb{K}^n$ *such that, with the notation* $\beta_{k,i} := \sum_{j=1}^{n} a_{ij}c_{k,j}$, $k \in \mathbb{N}$ *and* $i \in \{1, \dots, m\}$

(i) $\beta_i^k = 1$ *for all* $i \in I$ *and all* $k \in \mathbb{N}$,

(ii) $B_I := \sum_{i\in I} B_i \neq 0$, *and* $\lim_{k\to\infty} \sum_{i\notin I} |\frac{B_i}{B_I}|\omega(\beta_{k,i}) < 1$,

(iii) $\lim_{k\to\infty} \theta(c_{k,1}x, \dots, c_{k,n}x) = 0$,

Then f *is a solution of* (3.1).

Proof. Assume first that $B = 0$. It follows from (ii) that there exists $k_0 \in \mathbb{N}$ such that

$$\gamma_k := \sum_{i\notin I} |\frac{B_i}{B_I}|\omega(\beta_{k,i}) < 1, \quad \forall k \geq k_0.$$

For $k \geq k_0$ and arbitrary $x \in X$, if we take in (3.6), $c_{k,j}x := x_j$, for every $j \in \{1, \dots, n\}$, then we will get

$$\|f(x) - \sum_{i\notin I} \frac{-B_i}{B_I} f(\beta_{k,i}x)\| \leq \frac{\theta(c_{k,1}x, \dots, c_{k,n}x)}{|B_I|}, \quad x \in X. \tag{3.12}$$

Consider the set X_f^k consisting of all those mapping $g : X \to Y$ for which there exists some $K > 0$ with

$$\|f(t) - g(t)\| \leq K\theta(c_{k,1}t, \dots, c_{k,n}t), \ t \in X.$$

Using Lemma 3.2.1, we get (X_f^k, d_k) is a complete metric space, with respect to the distance

$$d_k(g, h) := \inf\{K > 0; \ \|g(t) - h(t)\| \leq K\theta(c_{k,1}t, \dots, c_{k,n}t), \quad t \in X\}, \ g, h \in X_f^k.$$

As in the proof of Theorem 3.2.2, the self-mapping J_k of X_f^k defined by

$$J_k\xi(x) := \sum_{i\notin I} \frac{-B_i}{B_I}\xi(\beta_{k,i}x), \quad \xi \in Y^X, \quad x \in X,$$

is a strict contraction mapping with Lipschitz constant $\gamma_k < 1$. By the Banach fixed point theorem, J_k admits a unique fixed point $G_k \in X_f^k$ with $G_k(x) := \lim_{n\to\infty} J_k^n f(x)$, $x \in X$.

Again, by Lemma 3.2.3, we have

$$\sum_{i=1}^{m} B_i G_k(\sum_{j=1}^{n} b_{ij}x_j) = 0, \tag{3.13}$$

and

$$\|f(x) - G_k(x)\| \leq \frac{\theta(c_{k,1}x, \ldots, c_{k,n}x)}{|B_I|(1-\gamma_k)}, \quad x \in X. \tag{3.14}$$

Letting k tend to $+\infty$ in (3.13), we obtain that f is a solution of (3.4).

Now, if $B \neq 0$ and $\sum_{i=1}^{m} B_i \neq 0$, define a new function $g : X \to Y$ by

$$g(x) := f(x) + \frac{B}{\sum_{i=1}^{m} B_i}.$$

We have

$$\Big\|\sum_{i=1}^{m} B_i f(\sum_{j=1}^{n} b_{ij}x_i) + B\Big\| \leq \theta(x_1, \ldots, x_n), \quad x_1, \ldots, x_n \in X.$$

Then

$$\Big\|\sum_{i=1}^{m} B_i \left(f(\sum_{j=1}^{n} b_{ij}x_j) + \frac{B}{\sum_{i=1}^{m} B_i}\right)\Big\| \leq \theta(x_1, \ldots, x_n),\ x_1, \ldots, x_n \in X.$$

i.e.,

$$\Big\|\sum_{i=1}^{m} B_i g(\sum_{j=1}^{n} b_{ij}x_j)\Big\| \leq \theta(x_1, \ldots, x_n),\ x_1, \ldots, x_n \in X.$$

By the first part of the proof, g satisfies the functional equation (3.4). Hence f is a solution of the general linear functional equation (3.1), which finishes the proof. □

3.4 Stability of a general linear equation

In this section, we will prove the stability of the general linear functional equation (3.2) by using the Banach contraction fixed point theorem. In all what follows, we will denote by $(A_i)_{i=1,\ldots,m}$ a family of non zero commuting continuous endomorphisms of Y, by $\bar{x}$ an arbitrary element $(x_1, \ldots, x_n)$ of X^n, and for every non empty subset I of $\{1, 2, \ldots, m\}$, by A_I the operator $A_I := \sum_{i\in I} A_i$. Whenever $I = \{1, 2, \ldots, m\}$, we will simply write A instead of $A_{\{1,2,\ldots,m\}}$.

Theorem 3.4.1. *Assume that either $b = 0$ or $A(b) = Nb$, for some non zero scalar N. Suppose that there exist a mapping $\theta : X^n \to \mathbb{R}_+$, a non empty proper subset I of $\{1, \ldots, m\}$, positive numbers ω_i, $i \notin I$, and a mapping $\psi : X \to X^n$, such that A_I has an* *eigenvalue* $M \neq 0$ *whose corresponding eigenspace* Y_M *contains both* $f(X)$ *and* b, *and*

$$\|\sum_{i=1}^{m} A_i(f(\varphi_i(\bar{x}))) + b\| \leq \theta(\bar{x}), \quad \bar{x} \in X^n. \tag{3.15}$$

$$\varphi_i \circ \psi = Id_X, \quad i \in I. \tag{3.16}$$

$\forall i \notin I,\ p \in \{1, \ldots, m\},\ \bar{x} \in X^n,$ *and* $x \in X$

$$\theta(\varphi_i \circ \psi(x_1), \ldots, \varphi_i \circ \psi(x_n)) \leq \omega_i \theta(\bar{x}), \tag{3.17}$$

$$\varphi_p(\varphi_i \circ \psi(x_1), \ldots, \varphi_i \circ \psi(x_n)) = \varphi_i \circ \psi(\varphi_p(\bar{x})), \tag{3.18}$$

$$\theta(\psi \circ \varphi_i(\psi(x))) \leq \omega_i \theta(\psi(x)), \tag{3.19}$$

$$\gamma := \sum_{i \notin I} \frac{\|A_i\|}{|M|} \omega_i < 1. \tag{3.20}$$

Then there exists a unique solution $G : X \to Y$ *of* (3.2) *satisfying*

$$\|f(x) - G(x)\| \leq \frac{\theta(\psi(x))}{|M| - \sum_{i \notin I} \|A_i\| \omega_i}, \quad x \in X. \tag{3.21}$$

The following lemma will be used in the sequel. It is an extension of Lemma 3.2.3.

Lemma 3.4.2. *Assume that* $b = 0$ *and that there exist a mapping* $\theta : X^n \to \mathbb{R}_+$, *a proper subset* I *of* $\{1, \ldots, m\}$, *positive scalars* ω_i, $i \notin I$, *and a mapping* $\psi : X \to X^n$ *so that the conditions* (3.15), (3.17), (3.18), *and* (3.20), *are satisfied. Consider the operator* $J : Y^X \to Y^X$ *defined by*

$$J\xi := \frac{-1}{M} \sum_{i \notin I} A_i \circ \xi \circ \varphi_i \circ \psi, \quad \xi \in Y^X. \tag{3.22}$$

If the sequence $(J^n f)_{n \geq 0}$ *converges pointwise to some function* $G : X \to Y$, *then* G *is a solution of* (3.3).

Proof. Define the operator $\phi : Y^X \to Y^{X^n}$ by

$$\phi(\xi) := \sum_{i=1}^{m} A_i \circ \xi \circ \varphi_i, \quad \xi \in Y^X. \tag{3.23}$$

We claim that the following inequality holds for every $\ell \in \mathbb{N}$

$$\|\phi(J^\ell f)(\bar{x})\| \leq \gamma^\ell \theta(\bar{x}), \quad \bar{x} \in X^n. \tag{3.24}$$

Indeed, the case $\ell = 0$ is just (3.15) with $b = 0$. Next, if (3.24) holds for $\ell \in \mathbb{N}$, then

$$
\begin{aligned}
\phi(J^{\ell+1}f)(\bar{x}) &= \sum_{i=1}^{m} A_i \circ (J^{\ell+1}f) \circ \varphi_i(\bar{x}) \\
\text{by} \quad (3.22) \quad &= \sum_{i=1}^{m} A_i \circ \left(\frac{-1}{M} \sum_{p \notin I} A_p \circ (J^{\ell}f) \circ \varphi_p \circ \psi \right) \circ \varphi_i(\bar{x}) \\
&= \sum_{p \notin I} \frac{-1}{M} A_p \circ \left(\sum_{i=1}^{m} A_i \circ (J^{\ell}f) \right) \circ (\varphi_p \circ \psi \circ \varphi_i)(\bar{x}) \\
\text{by} \quad (3.18) \quad &= \sum_{p \notin I} \frac{-1}{M} A_p \circ \left(\sum_{i=1}^{m} A_i \circ (J^{\ell}f) \right) \circ \varphi_i(\varphi_p \circ \psi(x_1), \ldots, \varphi_p \circ \psi(x_n)) \\
&= \sum_{p \notin I} \frac{-1}{M} A_p \circ \left(\sum_{i=1}^{m} A_i \circ (J^{\ell}f) \circ \varphi_i \right) (\varphi_p \circ \psi(x_1), \ldots, \varphi_p \circ \psi(x_n)) \\
\text{by} \quad (3.23) \quad &= \sum_{p \notin I} \frac{-1}{M} A_p \circ \left(\phi(J^{\ell}f)(\varphi_p \circ \psi(x_1), \ldots, \varphi_p \circ \psi(x_n)) \right).
\end{aligned}
$$

Therefore

$$
\begin{aligned}
\left\| \phi(J^{\ell+1}f)(\bar{x}) \right\| &\leq \sum_{p \notin I} \frac{\|A_p\|}{|M|} \|\phi(J^{\ell}f)(\varphi_p \circ \psi(x_1), \ldots, \varphi_p \circ \psi(x_n))\| \\
\text{by} \quad (3.24) \quad &\leq \sum_{p \notin I} \frac{\|A_p\|}{|M|} \gamma^{\ell} \theta(\varphi_p \circ \psi(x_1), \ldots, \varphi_p \circ \psi(x_n)) \\
\text{by} \quad (3.17) \quad &\leq \gamma^{\ell} \sum_{p \notin I} \frac{\|A_p\|}{|M|} \omega_p \theta(\bar{x}) \\
&\leq \gamma^{\ell+1} \theta(\bar{x}).
\end{aligned}
$$

Thus, by induction we have shown (3.24). Now, for $\bar{x} \in X^n$ and $x \in X$, let us denote by $\delta_{\bar{x}}$ (resp. δ_x) the evaluation $h \mapsto h(\bar{x})$ at $\bar{x}$ (resp. $g \mapsto g(x)$ at x) on Y^{X^n} (resp. on Y^X). Then

$$
\begin{aligned}
\|\delta_{\bar{x}}(\phi(\xi))\| &= \| \sum_{i=1}^{m} A_i(\xi(\varphi_i(\bar{x}))) \| \\
&\leq \sum_{i=1}^{m} \|A_i\| \|\xi(\varphi_i(\bar{x}))\| \\
&\leq \left(\sum_{i=1}^{m} \|A_i\| \right) \sup_{i=1,\ldots,m} \|\delta_{\varphi_i(\bar{x})}(\xi)\|.
\end{aligned}
$$

Then ϕ is continuous with respect to the pointwise convergence topologies on Y^X and

Y^{X^n}. Therefore

$$\begin{aligned}
\|\phi(\lim_{\ell\to\infty} J^\ell f)(\bar{x})\| &= \|\lim_{l\to\infty} \phi(J^\ell f)(\bar{x})\| \\
&= \lim_{\ell\to\infty} \|\phi(J^\ell f)(\bar{x})\| \\
\text{by} \quad (3.24) \text{ and } (3.20) \quad &\leq \lim_{\ell\to\infty} \gamma^\ell \theta(\bar{x}) = 0.
\end{aligned}$$

Hence $\sum_{i=1}^{m} A_i \circ G \circ \varphi_i(\bar{x}) = 0, \quad \bar{x} \in X^n$, therefore G is a solution of (3.3). □

Now, we are in a position to prove Theorem 3.4.1.

Proof. Assume first that $b = 0$ and let $x \in X$ be given. Putting $\psi(x) := \bar{x}$, $\bar{x} \in X^n$ in (3.15), we get

$$\left\| f(x) - \frac{-1}{M} \sum_{i \notin I} A_i(f(\varphi_i(\psi(x)))) \right\| \leq \frac{\theta(\psi(x))}{|M|}, \quad x \in X. \tag{3.25}$$

Set

$$X_f := \{g : X \to Y, \exists B > 0 : \|f(t) - g(t)\| \leq B\theta(\psi(t)), \quad t \in X\}.$$

Since $f \in X_f$, X_f is non empty. Further, if we set for every $g, h \in X_f$

$$d(g,h) := \inf\{B > 0, \quad \|g(t) - h(t)\| \leq B\theta(\psi(t)), \quad t \in X\},$$

then by Lemma 3.2.1, we get (X_f, d) is a complete metric space.

Now, for arbitrary $\xi \in X_f$, define the mapping $J\xi$ from X into Y by

$$J\xi(x) := \frac{-1}{M} \sum_{i \notin I} A_i(\xi(\varphi_i(\psi(x)))), \quad \xi \in X_f, \ x \in X.$$

Thanks to (3.25), we get for every $x \in X$,

$$\|f(x) - Jf(x)\| \leq \frac{\theta(\psi(x))}{|M|}.$$

Therefore, $Jf \in X_f$ and $d(f, Jf) \leq \dfrac{1}{|M|}$. Moreover, for $g \in X_f$ and $B > d(f,g)$, we have

$$\begin{aligned}
\|Jg(x) - f(x)\| &\leq \|Jg(x) - Jf(x)\| + \|Jf(x) - f(x)\| \\
&\leq \Big\| \frac{-1}{M} \sum_{i \notin I} A_i(g \circ \varphi_i \circ \psi(x) - f \circ \varphi_i \circ \psi(x)) \Big\| + \frac{\theta \circ \psi(x)}{|M|} \\
&\leq \sum_{i \notin I} \frac{\|A_i\|}{|M|} \Big\| g(\varphi_i \circ \psi(x))) - f(\varphi_i \circ \psi(x))) \Big\| + \frac{\theta \circ \psi(x)}{|M|} \\
&\leq \sum_{i \notin I} \frac{\|A_i\|}{|M|} B\theta \circ \psi \circ \varphi_i \circ \psi(x) + \frac{\theta \circ \psi(x)}{|M|} \\
\text{by} \quad (3.19) \qquad &\leq B \sum_{i \notin I} \frac{\|A_i\|}{|M|} \omega_i \theta \circ \psi(x) + \frac{\theta \circ \psi(x)}{|M|} \\
&\leq (B\gamma + \frac{1}{|M|}) \theta \circ \psi(x).
\end{aligned}$$

This shows $Jg \in X_f$, for all $g \in X_f$. Therefore J is a self mapping of X_f.

Now, given $g, h \in X_f$ and $B > d(g,h)$. Then

$$\begin{aligned}
\|Jg(x) - Jh(x)\| &= \Big\| \frac{-1}{M} \sum_{i \notin I} A_i(g \circ \varphi_i \circ \psi(x) - h \circ \varphi_i \circ \psi(x) \Big\| \\
&\leq \sum_{i \notin I} \frac{\|A_i\|}{|M|} \Big\| g(\varphi_i \circ \psi(x)) - h(\varphi_i \circ \psi(x)) \Big\| \\
&\leq B \sum_{i \notin I} \frac{\|A_i\|}{|M|} \theta \circ \psi \circ \varphi_i \circ \psi(x) \\
\text{by} \quad (3.19) \qquad &\leq B\gamma\theta \circ \psi(x).
\end{aligned}$$

Therefore $d(Jg, Jh) \leq B\gamma$. Passing to the infimum on B, we get $d(Jg, Jh) \leq \gamma d(g,h)$. Since $\gamma < 1$, J is a strict contraction mapping. By the Banach fixed point theorem, there exists a unique mapping $G \in X_f$ such that $JG = G$ and $\lim_n J^n f = G$. Therefore, by the continuity of δ_x, $\lim_n J^n f(x) = G(x)$, for every $x \in X$, and

$$d(f, G) \leq \frac{d(f, Jf)}{1 - \gamma}.$$

Thanks to Lemma 3.4.2, G is a solution of (3.3) and, since $d(f, Jf) \leq \dfrac{1}{|M|}$, we get

$$d(f, G) \leq \frac{1}{|M| - \sum_{i \notin I} \|A_i\| \omega_i}.$$

Thus (3.21) holds.

Now, if $b \neq 0$ and b is an eigenvector of A, with eigenvalue $N \neq 0$, then, by (3.15), we get for every $\bar{x} \in X^n$

$$\Big\| \sum_{i=1}^{m} A_i \left(f(\varphi_i(\bar{x}) + \frac{b}{N}) \right) \Big\| \leq \theta(\bar{x}).$$

Putting $g(t) = f(t) + \dfrac{b}{N}$, for $t \in X$, we obtain

$$\Big\| \sum_{i=1}^{m} A_i \left(g(\varphi_i(\bar{x})) \right) \Big\| \leq \theta(\bar{x}), \quad \bar{x} \in X^n.$$

Notice that, for every $x \in X$, we have

$$\sum_{i \in I} A_i(g(x)) = \sum_{i \in I} A_i(f(x)) + \sum_{i \in I} A_i(\frac{b}{N}) = Mg(x).$$

Using the first part of the proof, there exists a unique solution $H : X \to Y$ of (3.3) such that

$$\|g(x) - H(x)\| \leq \frac{\theta(\psi(x))}{|M| - \sum_{i \notin I} \|A_i\| \omega_i}.$$

Then

$$\|f(x) + \frac{b}{N} - H(x)\| \leq \frac{\theta(\psi(x))}{|M| - \sum_{i \notin I} \|A_i\| \omega_i}.$$

Setting $G(x) := H(x) - \dfrac{b}{N}$, for every $x \in X$, we obtain

$$\|f(x) - G(x)\| \leq \frac{\theta(\psi(x))}{|M| - \sum_{i \notin I} \|A_i\| \omega_i}, \quad x \in X.$$

Since H satisfies (3.3), G satisfies (3.2) and the proof is complete. $\square$

Theorem 3.4.1 may be used to prove the stability of various functional equations. As a first application we get Theorem 3.2.2 holds true as the following remark shows.

Remark 3.4.3. Notice that if we take in Theorem 3.2.2, $b = B$, $\psi : X \to X^n$, $\varphi_i : X^n \to X$ and $A_i : Y \to Y$ to be the mappings defined by $\psi(x) := (c_1 x, \ldots, c_n x)$, $\varphi_i(\bar{x}) := \sum_{j=1}^{n} b_{ij} x_j$ and $A_i(y) := B_i y$, with $\bar{x} := (x_1, \ldots, x_n) \in X^n$, $x \in X$, $y \in Y$, $i \in \{1, \ldots, m\}$, then it is easy to check that the assumptions of Theorem 3.4.1 are verified. Therefore, we get the stability of (3.1).

The following corollary gives an instance where Theorem 3.4.1 is relevant but not analogous in [18]. If Z is a compact space, denote by $\mathcal{C}(Z)$ the Banach algebra of continuous complex-valued functions on Z with the uniform norm.

Corollary 3.4.4. *Let H and K be two compact spaces, $X := \mathcal{C}(H)$, $Y := \mathcal{C}(K)$, and $f : X \to Y$ a mapping. Let $\alpha_1, \dots, \alpha_m \in Y \setminus \{0\}$, $(a_{ij}) \in \mathcal{M}_{m,n}(X)$ a matrix with entries in X, and $b \in Y$, so that, whenever $b \neq 0$, the function $\sum_{i=1}^{m} \alpha_i$ is constant with value $N \neq 0$ on the cozero of b with $coz(b) := \{k \in K : b(k) \neq 0\}$. Suppose that there are a mapping $\theta : X^n \to \mathbb{R}_+$, a proper subset I of $\{1, \dots, m\}$, and positive numbers ω_i, $i \notin I$, such that $\gamma := \sum_{i \notin I} \|\alpha_i\| \omega_i < 1$, the families $(\alpha_i)_{i \in I}$ and $(a_{ij})_{j=1,\dots,n}$, for every $i \in I$, are partitions of the unity. If for every $\bar{x} := (x_1, \dots, x_n) \in X^n$,*

$$\left\| \sum_{i=1}^{m} \alpha_i f\left(\sum_{j=1}^{n} a_{ij} x_j\right) + b \right\| \leq \theta(\bar{x}, \quad \text{and}$$

$$\theta\left(\sum_{j=1}^{n} a_{ij} x_1, \dots, \sum_{j=1}^{n} a_{ij} x_n\right) \leq \omega_i \theta(\bar{x}), \quad i \notin I,$$

then there exists a unique $G : X \to Y$ enjoying for every $x \in X$ and $\bar{x} \in X^n$

$$\sum_{i=1}^{m} \alpha_i G\left(\sum_{j=1}^{n} a_{ij} x_j\right) + b = 0, \quad \text{and } \|f(x) - G(x)\| \leq \frac{\theta(x, \dots, x)}{1 - \sum_{i \notin I} \|\alpha_i\| \omega_i}.$$

Proof. Defining $\psi : X \to X^n$ by $\psi(x) := (x, \dots, x)$, for every $i \in \{1, \dots, m\}$, $\varphi_i : X^n \to X$ by $\varphi_i(x_1, \dots, x_n) := \sum_{j=1}^{n} a_{ij} x_i$ and $A_i : Y \to Y$ by $A_i(y) := \alpha_i y$, it is easy to check that the assumptions of Theorem 3.4.1 are satisfied, whereby the result follows. $\square$

Corollary 3.4.4 can be generalized to not necessarily commutative algebras. To this aim, for an algebra Y on $\mathbb{K}$, let us denote by $\mathcal{Z}(Y)$ the center of Y. This is $\mathcal{Z}(Y) := \{y \in Y, \quad zy = yz, \quad \forall z \in Y\}$. In the following corollary, we assume that X is a unitary algebra with unit e_X, Y is a Banach algebra, $(a_{ij}) \in \mathcal{M}_{m,n}(X)$ is a matrix with entries in X, and that $f : X \to Y$ is a mapping.

Corollary 3.4.5. *Assume that, whenever $b \neq 0$, $\sum_{i=1}^{m} \alpha_i b = Nb$, for some $N \in \mathbb{K}$, and that there exist a mapping $\theta : X^n \to \mathbb{R}_+$, a proper subset I of $\{1, \dots, m\}$, and positive numbers ω_i, $i \notin I$, such that $\sum_{j=1}^{n} a_{ij} = e_X$, for every $i \in I$, $(\alpha_i)_{i \notin I} \subset \mathcal{Z}(Y)$, and for every $\bar{x} := (x_1, \dots, x_n) \in X^n$*

$$\sum_{i=1}^{m} \alpha_i f(\sum_{j=1}^{n} a_{ij} x_j) + b\| \leq \theta(\bar{x}), \quad \text{and}$$

$$\theta(\sum_{j=1}^{n} a_{ij} x_1, \dots, \sum_{j=1}^{n} a_{ij} x_n) \leq \omega_i \theta(\bar{x}), \quad i \notin I.$$

Suppose also that some $M \in \mathbb{K} \setminus \{0\}$ exists, so that $\alpha_I b = \sum_{i \in I} \alpha_i b = Mb$, and $\alpha_I f(x) = Mf(x)$, for every $x \in X$ and $\gamma := \sum_{i \notin I} \frac{\|\alpha_i\|}{|M|}\omega_i < 1$. Then there exists a unique mapping $G : X \to Y$ satisfying for every $x \in X$ and every $\bar{x} \in X^n$

$$\sum_{i=1}^{m} \alpha_i G\left(\sum_{j=1}^{n} a_{ij}x_j\right) + b = 0, \quad \text{and} \quad \|f(x) - G(x)\| \leq \frac{\theta(x, \ldots, x)}{|M| - \sum_{i \notin I} \|\alpha_i\|\omega_i}.$$

Proof. Just define the mappings $\psi : X \to X^n$, $\varphi_i : X^n \to X$ and $A_i : Y \to Y$ by $\psi(x) := (x, \ldots, x)$, $\varphi_i(\bar{x}) := \sum_{j=1}^{n} a_{ij}x_j$ and $A_i(y) = \alpha_i y$, and apply Theorem 3.4.1. $\square$

Now, we give a concrete situation where Theorem 3.4.1 applies. This is a Banach version of a result of [88]. Actually, the authors proved there the stability of the following quintic functional equation

$$\begin{aligned} Df(x_1, x_2) := {} & 2f(2x_1 + x_2) + 2f(2x_1 - x_2) + f(x_1 + 2x_2) + f(x_1 - 2x_2) \\ & - 20f(x_1 + x_2) - 20f(x_1 - x_2) - 90f(x_1) = 0, \end{aligned}$$

in random normed spaces under the minimum t-norm. We use Theorem 3.4.1 to show the stability of this functional equation in Banach spaces as well.

Corollary 3.4.6. *Let X, Y be Banach spaces and $0 < p, q < 1$, and $L > 0$ real numbers such that*

$$\|Df(x_1, x_2)\| \leq L(\|x_1\|^p + \|x_2\|^q), \quad x_1, x_2 \in X.$$

Then there exists a unique quintic mapping $Q : X \to Y$, such that for every $x \in X$

$$\|Q(x) - f(x)\| \leq \frac{L(\|x\|^p + \|x\|^q)}{90 - a}, \tag{3.26}$$

with $a := 3^{q+1} + 20 \times 2^q + 23$.

Proof. Here, we have $m = 7$, $n = 2$, $A = 0$ and for every $i = \{1, \ldots, 7\}$, $\varphi_i : X^n \to X$, and $A_i : Y \to Y$, are linear mapping such that for every $x_1, x_2 \in X$, $\varphi_1(x_1, x_2) := 2x_1 + x_2$, $\varphi_2(x_1, x_2) := 2x_1 - x_2$, $\varphi_3(x_1, x_2) := x_1 + 2x_2$, $\varphi_4(x_1, x_2) := x_1 - 2x_2$, $\varphi_5(x_1, x_2) := x_1 + x_2$, $\varphi_6(x_1, x_2) := x_1 - x_2$, $\varphi_7(x_1, x_2) = x_1$, and for every $y \in Y$, $A_1(y) = A_2(y) := 2y$, $A_3(y) = A_4(y) := y$, $A_5(y) = A_6(y) := -20y$, and $A_7(y) := -90y$. Taking $I = \{7\}$, $M = 90$, $\omega_1 = \omega_3 = 3^q$, $\omega_2 = \omega_4 = \omega_6 = 1$, $\omega_5 = 2^q$, and $\psi : X \to X^2$ is defined for every $x \in X$, by $\psi(x) := (x, x)$. Moreover, we take $\theta : X^2 \to \mathbb{R}_+$ is a mapping defined by

$$\theta(x_1, x_2) := L(\|x_1\|^p + \|x_2\|^q), \quad x_1, x_2 \in X.$$

It follows from Theorem 3.4.1, that there exists a unique quintic mapping $Q : X \to Y$ satisfying (3.26), as desired. □

3.5 Hyperstability of a generalized linear equation

In this section, we give an additional condition to the hypotheses of Theorem 3.4.1, in order a function satisfying (3.2) approximately must be, itself, a solution of it; whence the hyperstability of (3.2).

Theorem 3.5.1. *Suppose that either $b = 0$ or $A(b) = Nb$, for some non zero $N \in \mathbb{K}$, and that f satisfies* (3.15) *with respect to some mapping $\theta : X^n \to \mathbb{R}_+$. Assume that there are a proper subset I of $\{1, \ldots, m\}$ and, for every $k \in \mathbb{N}$, a mapping $\psi^k : X \to X^n$ and a positive scalar ω_j^k, $j \notin I$, such that with $\beta_p^k := \varphi_p \circ \psi^k$,*

$$\begin{aligned}
\beta_i^k &= Id_X, \quad i \in I, \\
\lim_{k\to+\infty} \theta(\psi^k(x)) &= 0, \\
\theta(\beta_i^k(x_1), \ldots, \beta_i^k(x_n)) &\le \omega_i^k \theta(\bar{x}), \quad j \notin I, k \in \mathbb{N}, \qquad (3.27)\\
\forall p \in \{1, \ldots, m\}, \quad \beta_j^k(\varphi_p(\bar{x})) &= \varphi_p\left(\beta_j^k(x_1), \ldots, \beta_j^k(x_n)\right), \quad j \notin I, k \in \mathbb{N}, \\
\theta\left(\psi^k(\beta_j^k(x))\right) &\le \omega_j^k \theta\left(\psi^k(x)\right), \quad j \notin I, k \in \mathbb{N}, \\
\lim_{k\to+\infty} \gamma_k < 1, \quad &\text{with } \gamma_k := \sum_{j \notin I} \frac{\|A_j\|}{|M|}\omega_j^k. \qquad (3.28)
\end{aligned}$$

If the operator A_I has a non zero eigenvalue M whose eigenspace contains both b and $f(x)$, for every $x \in X$, then f is a solution of the equation (3.2).

Proof. It follows from 3.28 that there exist $0 < r < 1$ and $k_0 \in \mathbb{N}$ such that $\gamma_k < r$, for all $k \ge k_0$. For such $k \ge k_0$, by Theorem 3.4.1, there exists a unique mapping $G_k : X \to Y$, satisfying

$$\sum_{i=1}^{m} A_i \circ G_k \circ \varphi_i(x) + b = 0, \quad x \in X, \quad \text{and} \qquad (3.29)$$

$$\left\| f(x) - G_k(x) \right\| \le \frac{\theta\left(\psi^k(x)\right)}{|M|(1-\gamma_k)}, \quad x \in X. \qquad (3.30)$$

Letting k tend to $+\infty$ in (3.29), due to (3.27), we obtain $\lim_{k\to+\infty} G_k(x) = f(x)$, for every $x \in X$. Letting k tend to $+\infty$ in (3.29), by the continuity of the A_i's, we obtain that f is a solution of (3.2). □

Chapter 4

Ulam Stability of Functional Equations in Random Normed Spaces

4.1 Introduction

In this chapter, we will first show how to use Theorem 2.2.14 to study the Ulam stability of various functional equations in single variable.

Next, we prove a very general result on Ulam stability of the following new non-homogeneous Forti equation

$$\sum_{i=1}^{m} B_i f\left(\sum_{j=1}^{n} b_{ij} x_j\right) = D(x_1, \ldots, x_n) \tag{4.1}$$

where f is a mapping from a linear space X over a field $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}\}$ into a random normed space (Y, F, T), and $D : X^n \to Y$ is a given function. As special cases of this result, we can obtain stability criteria for numerous functional equations in several variables, in the framework of random normed spaces.

Finally, we provide an application of Theorem 2.2.6 by investigating the approximate eigenvalues and eigenvectors in the spaces of function taking values in RN-spaces.

Throughout this chapter, we will use the assumptions (i)–(iii) of Theorem 2.2.6. For the convenience of the reader, let us remind them here:

(i) (Y, F, T) is M-complete and

$$\lim_{k\to+\infty} \inf_{j\in\mathbb{N}} T_{i=k}^{k+j} (\Lambda^i \epsilon)_x \Big(\frac{t}{j+1}\Big) = 1, \qquad x \in X,\ t > 0.$$

(ii) (Y, F, T) is M-complete and for each $k \in \mathbb{N}$ there exists a sequence $(\omega_n^k)_{n\in\mathbb{N}} \in \Omega$

with

$$\lim_{k\to+\infty}\inf_{j\in\mathbb{N}} T_{i=k}^{k+j}(\Lambda^i\epsilon)_x\big(\omega_{i-k}^k t\big)=1,\qquad x\in X,\ t>0.$$

(iii) (Y,F,T) is G-complete and $\lim\limits_{n\to+\infty}(\Lambda^n\epsilon)_x=H_0$, i.e.,

$$\lim_{n\to+\infty}(\Lambda^n\epsilon)_x(t)=1,\qquad x\in X,\ t>0.$$

4.2 Stability of equations in a single variable

In this section, X is a nonempty set and (Y,F,T) is an RN-space. We present a simple Ulam stability outcome that can be derived from our results in Section 2.2. To this end, we need the following definition.

Definition 4.2.1. *Let $\nu\in\mathbb{N}^\star$, $L_i : X\to\mathbb{R}_+$, $i=1,\dots,\nu$. A mapping $H : X\times Y^\nu\to Y$ is said to satisfy the property* (H1)*, if for every $x\in X$, every $(w_1,\dots,w_\nu)$, $(z_1,\dots,z_\nu)\in Y^\nu$ and $t>0$,*

$$F_{H(x,w_1,\dots,w_\nu)-H(x,z_1,\dots,z_\nu)}(t)\ \geq T_{i=1}^{\nu}F_{w_i-z_i}\Big(\frac{t}{\nu L_i(x)}\Big).\tag{H1}$$

The following corollary can be easily deduced from Theorem 2.2.14.

Corollary 4.2.2. *Let $\nu\in\mathbb{N}^\star$, $L_i : X\to\mathbb{R}_+$, $i=1,\dots,\nu$, and $H : X\times Y^\nu\to Y$ satisfy the hypothesis (H1), and let $\xi_1,\dots,\xi_\nu\in X^X$, $f\in Y^X$, and $\varepsilon\in\mathcal{D}_+^X$, so that*

$$F_{H(x,f(\xi_1(x)),\dots,f(\xi_\nu(x)))-f(x)}\geq\varepsilon_x,\qquad x\in X.\tag{4.2}$$

Assume that one of the assumptions (i)–(iii) of Theorem 2.2.6 is fulfilled with $\Lambda : \mathcal{D}_+^X\to\mathcal{D}_+^X$ given by

$$(\Lambda\delta)_x(t)=T_{i=1}^{\nu}\delta_{\xi_i(x)}\Big(\frac{t}{\nu L_i(x)}\Big),\qquad \delta\in\mathcal{D}_+^X,\ x\in X.\tag{4.3}$$

Then, for each $x\in X$, each $\omega=(\omega_n)_{n\in\mathbb{N}}\in\Omega$ and $t>0$, the limits

$$\psi(x):=\lim_{n\to\infty}(J^nf)(x)\tag{4.4}$$

$$\sigma_x^0(t):=\lim_{m\to\infty}T_{i=0}^{m-1}(\Lambda^i\epsilon)_x\Big(\frac{t}{m}\Big),\tag{4.5}$$

$${}^{\omega}\sigma_x^0(t):=\lim_{m\to\infty}T_{i=0}^{m-1}(\Lambda^i\epsilon)_x\big(\omega_i t\big)\tag{4.6}$$

exist, with $J : Y^X\to Y^X$ given by

$$(J\eta)(x):=H(x,\eta(\xi_1(x)),\dots,\eta(\xi_\nu(x))),\qquad \eta\in Y^X,\ x\in X,\tag{4.7}$$

and the mapping $\psi \in Y^X$, *defined by* (4.4), *satisfying*

$$H(x, \psi(\xi_1(x)), ..., \psi(\xi_\nu(x))) = \psi(x), \qquad x \in X, \tag{4.8}$$

such that

$$F_{f(x)-\psi(x)}(t) \geq \sup_{\alpha\in(0,1)} \widehat{\sigma}_0^x(\alpha t), \qquad t > 0,\ x \in X, \tag{4.9}$$

where $\widehat{\sigma}_x^0$ *is defined as in Theorem 2.2.6. Moreover, if one of the condition* (i) *and* (ii) *holds, then* ψ *is the unique solution to equation* (4.8) *such that there is* $\alpha \in (0,1)$ *with*

$$F_{(f-\psi)(x)}(t) \geq \widehat{\sigma}_x^0(\alpha t), \qquad t > 0,\ x \in X. \tag{4.10}$$

Proof. Clearly, inequality (4.2) implies (2.9). Next, hypothesis $\mathcal{C}_0$ holds due to Remark 2.2.13, and (H1) means that (2.41) is valid. Consequently, by Theorem 2.2.14, the function ψ defined by (4.4) is a fixed point of J, i.e., that ψ is a solution of (4.8) satisfying (4.9) (take $k = 0$ in (2.15)). The statement on uniqueness of ψ also follows from Theorem 2.2.14. □

Stability of functional equations of the form (4.1) (or related to it) has already been studied by several authors. For further information we refer to [7, 45, 49]. A very particular case of (4.8), with H given by (2.39), is the linear functional equation of the form

$$\phi(x) = \sum_{i=1}^{\nu} \widetilde{L}_i(x)\phi(\xi_i(x)) + h(x), \tag{4.11}$$

with fixed functions $h \in Y^X$ and $\widetilde{L}_1, \ldots, \widetilde{L}_\nu : X \to \mathbb{R}$. That equation is called a linear equation of higher order when $\xi_i = \xi^i$ for $i = 1, \ldots, \nu$, with some $\xi \in X^X$, i.e., when (4.11) has the form

$$\phi(x) = \sum_{i=1}^{\nu} \widetilde{L}_i(x)\phi(\xi^i(x)) + h(x). \tag{4.12}$$

Some recent results concerning stability of less general cases of (4.12) can be found in [41, 49, 96, 97, 131, 193].

The simplest case of equation (4.12), with $\nu = 1$ and $0 \notin \widetilde{L}_1(X)$, can be rewritten in the form

$$\phi(\xi(x)) = \frac{1}{\widetilde{L}_1(x)}\phi(x) - \frac{h(x)}{\widetilde{L}_1(x)}, \tag{4.13}$$

which is also called the linear equation. Special cases of (4.13) are the gamma functional equation

$$\phi(x+1) = x\phi(x)$$

for $X = Y = \mathbb{R}$, the Schröder functional equation

$$\phi(\xi(x)) = s\phi(x) \tag{4.14}$$

with fixed $s \in \mathbb{R} \setminus \{0\}$, and the Abel functional equation

$$\phi(\xi(x)) = \phi(x) + 1.$$

For more details on (4.13) and its various particular versions we refer to [115, 117].

Remark 4.2.3. Let us consider a situation analogous to that in Remark 2.2.8, with $T = T_M$, for the Schröder functional equation (4.14) rewritten as

$$\frac{1}{s}\phi(\xi(x)) = \phi(x). \tag{4.15}$$

Clearly equation (4.15) is (4.8) with and $\xi_1 = \xi$ and $H(x, y) = \dfrac{1}{s}y$ for $x \in X$ and $y \in Y$. So we have the case as in Corollary 4.2.2 with $\nu = 1$ and $\Lambda : \mathcal{D}_+^X \to \mathcal{D}_+^X$ given by

$$(\Lambda\delta)_x(t) = \delta_{\xi(x)}(|s|t), \qquad x \in X,\ \delta \in \mathcal{D}_+^X,\ t > 0.$$

Further, let E be a normed space, $X := E \setminus \{0\}$, $p \in \mathbb{R}$, $L \in \mathbb{R}_+$ and

$$\epsilon_x(t) = \frac{t}{t + L\|x\|^p}, \qquad x \in X,\ t > 0.$$

Assume that $\|\xi^n(x)\|^p \le a_n\|x\|^p$ for $x \in X$ and $n \in \mathbb{N}$, with some sequence $(a_n)_{n\in\mathbb{N}}$ of positive reals such that $\lim\limits_{n\to\infty} a_n^{-1}|s|^n = \infty$ and write $e_n := a_n^{-1}|s|^n$. Clearly, for every $n \in \mathbb{N}$, $x \in X$ and $t > 0$

$$(\Lambda^n\epsilon)_x(t) = \epsilon_{\xi^n(x)}(|s|^n t) = \frac{|s|^n t}{|s|^n t + L\|\xi^n(x)\|^p} \ge \epsilon_x\big(e_n t\big), \tag{4.16}$$

and next

$$T_{i=k}^{k+j-1}(\Lambda^i\epsilon)_x(t) = \inf_{i\in\{0,\dots,j-1\}} \epsilon_x\big(e_{k+i}t\big), \qquad j \in \mathbb{N}^\star. \tag{4.17}$$

Hence, by (4.16)

$$\lim_{n\to+\infty} (\Lambda^n\epsilon)_x(t) \ge \lim_{n\to+\infty} \epsilon_x\big(e_n t\big) = 1, \qquad x \in X,\ t > 0, \tag{4.18}$$

which means that (2.12) holds. Further, assume additionally that $\rho := \sum\limits_{i=0}^{\infty} \dfrac{1}{e_i} < \infty$ and write $\omega_i^k := \dfrac{1}{\rho_k e_{k+i}}$ for $k, i \in \mathbb{N}$, where $\rho_k := \sum_{i=k}^{\infty} \dfrac{1}{e_i}$. Then, for every $k \in \mathbb{N}$

$$\sum_{i=0}^{\infty} \omega_i^k = 1, \qquad e_{k+j}\omega_j^k = \frac{1}{\rho_k}, \qquad j \in \mathbb{N},$$

whence by (4.16),

$$T_{i=k}^{k+j}(\Lambda^i\epsilon)_x\big(\omega_{i-k}^k t\big) \geq \inf_{i\in\{0,\ldots,j-1\}} \epsilon_x\big(e_{k+i}\omega_i^k t\big) = \epsilon_x\big(\rho_k^{-1}t\big), \qquad x \in X,\ t > 0.$$

Consequently, for every $x \in X$ and $t > 0$,

$$\begin{aligned}\lim_{m\to+\infty} \inf_{j\in\mathbb{N}_0} T_{i=m}^{m+j}(\Lambda^i\epsilon)_x\big(\omega_{i-m}^m t\big) &\geq \lim_{m\to+\infty} \inf_{j\in\mathbb{N}_0} \epsilon_x\big(\rho_m^{-1}t\big)\\ &= \lim_{m\to+\infty} \epsilon_x\big(\rho_m^{-1}t\big) = 1,\end{aligned}$$

which means that (2.11) holds with $\omega^k := (\omega_n^k)_{n\in\mathbb{N}} \in \Omega$. Moreover, for $\omega := \omega^0$ we have

$${}^{\omega}\sigma_x^0(t) = \lim_{j\to\infty} \inf_{i\in\{0,\ldots,j-1\}} \epsilon_x\big(e_i\omega_i^0 t\big) = \epsilon_x\big(\rho^{-1}t\big),$$

whence

$$\widehat{\sigma}_x^0(t) \geq {}^{\omega}\sigma_x^0(t) = \epsilon_x\big(\rho^{-1}t\big), \qquad x \in X,\ t > 0,$$

where ${}^{\omega}\sigma_x^0$ and $\widehat{\sigma}_x^0$ have the same meaning as in Corollary 4.2.2.

4.3 Stability of a new non-homogeneous Forti equation

In this section, using Theorem 2.2.14, we prove a general result on Ulam-type stability for the non homogeneous functional equation

$$\sum_{i=1}^{m} B_i f\left(\sum_{j=1}^{n} b_{ij}x_j\right) = D\left(x_1,\ldots,x_n\right), \tag{4.19}$$

in the class of functions f mapping a linear space X over the field $\mathbb{K} \in \{\mathbb{R},\mathbb{C}\}$ into a random normed space Y, where $D : X^n \to Y$ is a given function satisfying some additional condition, $B_i \in \mathbb{R}$, and $b_{ij} \in \mathbb{K}$ for $i = 1,\ldots,m$, $j = 1,\ldots,n$.

As special cases of our result, we obtain the stability criteria for numerous functional equations, in the framework of random normed spaces.

The following two hypothesis $(\mathcal{M})$ and $(\mathcal{D})$ are needed to formulate our main theorem.

($\mathcal{M}$) There exist $\mu \in \{1, \ldots, m-1\}$ and $c_1, \ldots, c_n \in \mathbb{K}$ such that

$$B_0 := \Big| \sum_{i=\mu+1}^{m} B_i \Big| > 0, \quad \text{and} \quad \beta_i = 1, \quad i = \mu+1, \ldots, m,$$

where $\beta_i := \sum_{j=1}^{n} b_{ij} c_j$ for $i = 1, \ldots, m$.

($\mathcal{D}$) For every $x_1, \ldots, x_n \in X$

$$\sum_{i=1}^{m} B_i d\Big(\sum_{j=1}^{n} b_{ij} x_j \Big) = \sum_{i=1}^{m} B_i D\Big(\beta_i x_1, \ldots, \beta_i x_n \Big), \tag{4.20}$$

where β_i is defined as in hypothesis ($\mathcal{M}$) and $d(x) = D(c_1 x, \ldots, c_n x)$ for $x \in X$.

Remark 4.3.1. If $\sum_{j=1}^{n} |b_{mj}| \neq 0$, then there exist $c_1, \ldots, c_n \in \mathbb{K}$ such that $\sum_{j=1}^{n} b_{mj} c_j = 1$. Therefore ($\mathcal{M}$) is fulfilled for $\mu = m-1$. However, because of the forms of the conditions (i)–(iii) of Theorem 2.2.6 and condition (4.24), it makes sense to consider (for some cases of equation (4.19) and some functions θ) also the situations with $\mu < m-1$.

For instance, for the inhomogeneous Cauchy equation

$$f(x+y) = f(x) + f(y) + D(x,y), \tag{4.21}$$

we can consider the following two situations (we refer to Corollary 4.3.4 and its proof for consequences in both of them).

($a1$) If the equation (4.21) is written in the form (4.19) as $f(x_1 + x_2) - f(x_1) - f(x_2) = D(x_1, x_2)$,then $m = 3$, $n = 2$, $B_1 = -B_2 = -B_3 = -1$, $b_{11} = b_{12} = 1$, $b_{21} = 1$, $b_{22} = 0$, $b_{31} = 0$, $b_{32} = 1$. In the matrix form we can write b_{ij} as

$$(b_{ij})_{\substack{1 \leq i \leq 3, \\ 1 \leq j \leq 2}} = \begin{pmatrix} 1 & 1 \\ 1 & 0 \\ 0 & 1 \end{pmatrix}$$

Clearly, the hypothesis ($\mathcal{M}$) is valid with $\mu = 1$, $B_0 = 2$, and $c_1 = c_2 = 1$.

(a_2) If the equation (4.21) is written in the form (4.19) as $-f(x_1) - f(x_2) + f(x_1 + x_2) = D(x_1, x_2)$, then $m = 3$, $n = 2$, $B_1 = B_2 = -B_3 = -1$, $b_{11} = 1$, $b_{21} = 0$, $b_{22} = 1$,

$b_{31} = b_{32} = 1$. In the matrix form we can write a_{ij} as

$$(b_{ij})_{\substack{1\leq i\leq 3,\\ 1\leq j\leq 2}} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 1 \end{pmatrix}$$

Then, the hypothesis $(\mathcal{M})$ is valid with $\mu = 2$, $B_0 = 1$, and $c_1 = c_2 = \dfrac{1}{2}$.

Remark 4.3.2. $(b0)$ Clearly, if D is a constant function, then hypothesis $(\mathcal{D})$ is valid (this case includes equation (3.1)). Moreover, if functions $D_1, D_2 : X^n \to Y$ satisfy the hypothesis, then so does the function $\alpha_1 D_1 + \alpha_2 D_2$ with any fixed scalars α_1, α_2. Below we provide more examples of non-trivial functions D satisfying the hypothesis.

$(b1)$ Consider the situation $(a1)$ depicted in the previous remark, with $m = 3$, $n = 2$, $B_1 = 1$, $B_2 = B_3 = -1$, $b_{11} = b_{12} = 1$, $b_{21} = 1$, $b_{22} = 0$, $b_{31} = 0$, $b_{32} = 1$, $\mu = 1$, and $c_1 = c_2 = 1$. Then $\beta_1 = 2$, $\beta_2 = \beta_3 = 1$ and condition (4.20) takes the form

$$\begin{aligned} D(x_1 + x_2, x_1 + x_2) - D(x_1, x_1) - D(x_2, x_2) \\ = D(2x_1, 2x_2) - 2D(x_1, x_2), \quad x_1, x_2 \in X. \end{aligned} \tag{4.22}$$

Observe that condition (4.22) holds in each of the following three cases

1) D is a symmetric biadditive function (i.e., $D(x_1, x_2) = D(x_2, x_1)$ and $D(x_1, x_2 + x_3) = D(x_1, x_2) + D(x_1, x_3)$ for $x_1, x_2, x_3 \in X$).
2) There exist additive $h_1, h_2 : X \to X$ such that $D(x_1, x_2) = h_1(x_1) + h_2(x_2)$ for $x_1, x_2 \in X$.
3) There exists $\rho : X \to Y$ such that $D(x_1, x_2) = \rho(x_1 + x_2) - \rho(x_1) - \rho(x_2)$ for $x_1, x_2 \in X$.

$(b2)$ In the situation $(a2)$ depicted in the previous remark, with $m = 3$, $n = 2$, $B_1 = B_2 = -B_3 = 1$, $b_{11} = 1$, $b_{12} = 0$, $b_{21} = 0$, $b_{22} = 1$, $b_{31} = b_{32} = 1$, $\mu = 2$, $c_1 = c_2 = \dfrac{1}{2}$, $\beta_2 = \beta_3 = \dfrac{1}{2}$, $\beta_3 = 1$ and the condition (4.20) takes the form

$$\begin{aligned} D\left(\frac{x_1}{2}, \frac{x_1}{2}\right) + D\left(\frac{x_2}{2}, \frac{x_2}{2}\right) - D\left(\frac{x_1 + x_2}{2}, \frac{x_1 + x_2}{2}\right) \\ = 2D\left(\frac{x_1}{2}, \frac{x_2}{2}\right) - D\left(x_1, x_2\right), \quad x_1, x_2 \in X, \end{aligned}$$

which is (4.22) (it is enough to replace x_i by $2x_i$ and multiply both sides by -1).

$b3)$ More generally, if $h_1, \dots, h_n : X \to Y$ are solutions to equation (3.4), then the function $D : X^n \to Y$, given by

$$D(x_1, \ldots, x_n) = \sum_{k=1}^{n} h_k(x_k), \quad x_1, \ldots, x_n \in X, \tag{4.23}$$

fulfils hypothesis $(\mathcal{D})$. In fact, fix $x_1, \ldots, x_n \in X$. Then, according to the definition of the function d, we get

$$\begin{aligned}\sum_{i=1}^{m} B_i d\left(\sum_{j=1}^{n} b_{ij}x_j\right) &= \sum_{i=1}^{m} B_i D\left(c_1 \sum_{j=1}^{n} b_{ij}x_j, \ldots, c_n \sum_{j=1}^{n} b_{ij}x_j\right) \\ &= \sum_{i=1}^{m} B_i \sum_{k=1}^{n} h_k\left(c_k \sum_{j=1}^{n} b_{ij}x_j\right) \\ &= \sum_{k=1}^{n}\sum_{i=1}^{m} B_i h_k\left(\sum_{j=1}^{n} b_{ij}c_k x_j\right) = 0, \quad \text{and}\end{aligned}$$

$$\begin{aligned}\sum_{i=1}^{m} B_i D(\beta_i x_1, \ldots, \beta_i x_n) &= \sum_{i=1}^{m} B_i \sum_{k=1}^{n} h_k(\beta_i x_k) \\ &= \sum_{k=1}^{n}\sum_{i=1}^{m} B_i h_k\left(\sum_{j=1}^{n} b_{ij}c_j x_k\right) = 0,\end{aligned}$$

whence we get (4.20).

Theorem 4.3.3. *Let hypotheses $(\mathcal{M})$ and $(\mathcal{D})$ be valid and $\theta : X^n \to \mathcal{D}_+$ satisfy*

$$\lim_{k\to\infty} \left(\mathcal{T}^k\theta\right)(x_1, \ldots, x_n)(t) = 1, \quad t > 0,\ x_1, \ldots, x_n \in X, \tag{4.24}$$

where $\mathcal{T} : \mathcal{D}_+^{X^n} \to \mathcal{D}_+^{X^n}$ is given by

$$\mathcal{T}\chi(x_1, \ldots, x_n)(t) = T_{i=1}^{\mu}\chi(\beta_i x_1, \ldots, \beta_i x_n)\Big(\frac{B_0 t}{\mu|B_i|}\Big), \tag{4.25}$$

$$\chi \in \mathcal{D}_+^{X^n}, \quad t > 0, \quad x_1, \ldots, x_n \in X. \tag{4.26}$$

Further, assume that one of the conditions (i)–(iii) of Theorem 2.2.6 holds with $\Lambda : \mathcal{D}_+^X \to \mathcal{D}_+^X$ and $\epsilon : X \to \mathcal{D}_+$ defined by

$$\epsilon_x(t) := \theta(c_1 x, \ldots, c_n x)(tB_0), \qquad x \in X,\ t > 0, \tag{4.27}$$

$$(\Lambda\delta)_x(t) := T_{i=1}^{\mu}\delta_{\beta_i x}\Big(\frac{B_0 t}{\mu|B_i|}\Big), \qquad \delta \in \mathcal{D}_+^X,\ x \in X,\ t > 0. \tag{4.28}$$

If $f : X \to Y$ *satisfy*

$$F_{\sum_{i=1}^{m} B_i f(\sum_{j=1}^{n} b_{ij} x_j) - D(x_1,\dots,x_n)} \geq \theta(x_1,\dots,x_n), \quad x_1,\dots,x_n \in X, \tag{4.29}$$

then there is a solution $\psi : X \to Y$ *of equation* (4.19) *such that*

$$F_{(\psi - f)(x)}(t) \geq \sup_{\alpha \in (0,1)} \widehat{\sigma}_x^0(\alpha t), \qquad t > 0, \ x \in X, \tag{4.30}$$

with $\widehat{\sigma}_x^0$ *defined by* (2.8) (*see also* (2.4) *and* (2.5)).

Moreover, in case where (*i*) *or* (*ii*) *holds, there is exactly one solution* $\psi \in Y^X$ *of* (4.19) *such that there exists* $\alpha \in (0,1)$ *with*

$$F_{(\psi - f)(x)}(t) \geq \widehat{\sigma}_x^0(\alpha t), \qquad t > 0, \ x \in X. \tag{4.31}$$

Proof. Write $|\alpha| = 1/B_0$ and fix $x \in X$. Putting $x_j := c_j x$ for $j \in \{1,\dots,n\}$ in (4.29) we have

$$F_{\sum_{i=1}^{m} B_i f\left(\sum_{j=1}^{n} b_{ij} c_j x\right) - D(c_1 x,\dots,c_n x)}(t) \geq \theta\left(c_1 x,\dots,c_n x\right)(t),$$

i.e;

$$F_{\sum_{i=1}^{m} B_i f(\beta_i x) - d(x)}(t) \geq \theta(c_1 x,\dots,c_n x)(t),$$

Then

$$F_{\sum_{i=\mu+1}^{m} B_i \left[f(x) - \frac{1}{\sum_{i=\mu+1}^{m} B_i}\left(d(x) - \sum_{i=1}^{\mu} B_i f(\beta_i x)\right)\right]}(t) \geq \theta\left(c_1 x,\dots,c_n x\right)(t),$$

Next,

$$F_{f(x) - \alpha\left(d(x) - \sum_{i=1}^{\mu} B_i f(\beta_i x)\right)}\left(\frac{t}{|\sum_{i=\mu+1}^{m} B_i|}\right) \geq \theta\left(c_1 x,\dots,c_n x\right)(t),$$

Therefore

$$F_{f(x) - \alpha\left(d(x) - \sum_{i=1}^{\mu} B_i f(\beta_i x)\right)}(t) \geq \theta\left(c_1 x,\dots,c_n x\right)(B_0 t), \tag{4.32}$$

Consequently, we get

$$F_{f(x) - Jf(x)}(t) \geq \theta(c_1 x,\dots,c_n x)(B_0 t), \qquad t > 0, \tag{4.33}$$

with the operator $J : Y^X \to Y^X$ defined by

$$\begin{aligned} J\xi(x) &:= -\alpha \sum_{l=1}^{\mu} B_l \xi\Big(\sum_{j=1}^{n} b_{lj} c_j x\Big) + \alpha d(x) \\ &= -\alpha\Big(\sum_{l=1}^{\mu} B_l \xi(\beta_l x) - d(x)\Big), \qquad \xi \in Y^X, \ x \in X. \end{aligned}$$

Note that the assumptions of Theorem 2.2.14 are satisfied for such J, because, for every $\xi,\eta\in Y^X$ and $x\in X$, we have $(J\xi-J\eta)(x)=-\alpha\sum_{i=1}^{\mu}B_i(\xi-\eta)\big(\beta_i x\big)$, whence

$$\begin{aligned}F_{(J\xi-J\eta)(x)}(t)&=F_{-\alpha\sum_{i=1}^{\mu}B_i(\xi-\eta)(\beta_i x)}(t)\\&\geq T_{i=1}^{\mu}F_{B_i(\xi(\beta_i x)-\eta(\beta_i x))}\left(\frac{B_0 t}{\mu}\right)\\&\geq T_{i=1}^{\mu}F_{\xi(\beta_i x)-\eta(\beta_i x)}\Big(\frac{B_0 t}{\mu|B_i|}\Big),\qquad t>0.\end{aligned}$$

This means that the condition (2.41) is fulfilled with $\xi_i(x)=\beta_i x$ and $L_i(x)=\dfrac{|B_i|}{B_0}$. Consequently, by Theorem 2.2.14, for every $k\in\mathbb{N}$, $x\in X$ and $t>0$, the limit (2.13) exists and the function $\psi\in Y^X$ is a fixed point of J fulfilling (4.30). Moreover, if one of conditions (i), (ii) of Theorem 2.2.6 holds, then ψ is the unique fixed point of J such that there is $\alpha\in(0,1)$ with

$$F_{(\psi-f)(x)}(t)\geq\widehat{\sigma}_0^x(\alpha t),\qquad t>0,\ x\in X.\tag{4.34}$$

Now, we show that ψ is a solution to (4.19). To this end observe that

$$\begin{aligned}\sum_{i=1}^{m}B_i\psi\Big(\sum_{j=1}^{n}b_{ij}x_j\Big)&=\sum_{i=1}^{m}B_i\lim_{k\to\infty}(J^k f)\Big(\sum_{j=1}^{n}b_{ij}x_j\Big)\\&=\lim_{k\to\infty}\Big(\sum_{i=1}^{m}B_i(J^k f)\Big(\sum_{j=1}^{n}b_{ij}x_j\Big)\Big),\qquad x_1,\ldots,x_n\in X.\end{aligned}\tag{4.35}$$

First, we prove by induction that, for each $k\in\mathbb{N}$

$$F_{\sum_{i=1}^{m}B_i(J^k f)\left(\sum_{j=1}^{n}b_{ij}x_j\right)-D(x_1,\ldots,x_n)}(t)\geq\mathcal{T}^k\theta(x_1,\ldots,x_n)(t),\ x_1,\ldots,x_n\in X,\ t>0.\tag{4.36}$$

The case $k=0$ is (4.29). So fix $k\in\mathbb{N}$ and assume that (4.36) holds. Then, by hypothesis

$(\mathcal{D})$, for every $x_1, \ldots, x_n \in X$

$$\begin{aligned}
\sum_{i=1}^{m} B_i(J^{k+1} f)\Big(\sum_{j=1}^{n} b_{ij}x_j\Big) &= \sum_{i=1}^{m} B_i J(J^k f)\Big(\sum_{j=1}^{n} b_{ij}x_j\Big) \\
&= -\alpha \sum_{i=1}^{m} B_i \Big[\sum_{l=1}^{\mu} B_l(J^k f)\Big(\beta_l \sum_{j=1}^{n} b_{ij}x_j\Big) - d\Big(\sum_{j=1}^{n} b_{ij}x_j\Big)\Big] \\
&= -\alpha \sum_{l=1}^{\mu} B_l \sum_{i=1}^{m} B_i(J^k f)\Big(\sum_{j=1}^{n} b_{ij}\beta_l x_j\Big) + \alpha \sum_{i=1}^{m} B_i d\Big(\sum_{j=1}^{n} b_{ij}x_j\Big) \\
&= -\alpha \sum_{l=1}^{\mu} B_l \sum_{i=1}^{m} B_i(J^k f)\Big(\sum_{j=1}^{n} b_{ij}\beta_l x_j\Big) + \alpha \sum_{l=1}^{\mu} B_l D\Big(\beta_l x_1, \ldots, \beta_l x_n\Big) \\
&\qquad + D(x_1, x_2 \ldots, x_n) \\
&= -\alpha \sum_{l=1}^{\mu} B_l \Big[\sum_{i=1}^{m} B_i(J^k f)\Big(\sum_{j=1}^{n} b_{ij}\beta_l x_j\Big) - D\Big(\beta_l x_1, \ldots, \beta_l x_n\Big)\Big] \\
&\qquad + D(x_1, x_2 \ldots, x_n). \qquad (4.37)
\end{aligned}$$

Hence, by (4.25) and the assumed inequality (4.36),

$$\begin{aligned}
&F_{\sum_{i=1}^{m} B_i(J^{k+1} f)\left(\sum_{j=1}^{n} b_{ij}x_j\right) - D(x_1,\ldots,x_n)}(t) \\
&\quad = F_{-\alpha \sum_{l=1}^{m}\left[B_l \sum_{i=1}^{m} B_i(J^k f)\left(\sum_{j=1}^{n} b_{ij}\beta_l x_j\right) - D(\beta_l x_1,\ldots,\beta_l x_n)\right]}(t) \\
&\quad \geq T_{l=1}^{\mu} F_{\sum_{i=1}^{m} B_i(J^k f)\left(\sum_{j=1}^{n} b_{ij}\beta_l x_j\right) - D(\beta_l x_1,\ldots,\beta_l x_n)}\Big(\frac{B_0 t}{\mu|B_l|}\Big)\Big) \\
&\quad \geq T_{l=1}^{\mu} \mathcal{T}^k \theta(\beta_l x_1, \ldots, \beta_l x_n)\Big(\frac{B_0 t}{\mu|B_l|}\Big) \\
&\quad = (\mathcal{T}^{k+1}\theta)(x_1, \ldots, x_n)(t), \qquad x_1, \ldots, x_n \in X,\ t > 0.
\end{aligned}$$

Thus we have proved that (4.36) holds for each $k \in \mathbb{N}$. Now, letting $k \to \infty$ in (4.36), in view of (4.24), for every $x_1, \ldots, x_n \in X$, we get

$$\lim_{k\to\infty} F_{\sum_{i=1}^{m} B_i(J^k f)\left(\sum_{j=1}^{n} b_{ij}x_j\right) - D(x_1,\ldots,x_n)}(t) = 1, \quad t > 0, \qquad (4.38)$$

which means that

$$\lim_{k\to\infty} \Big(\sum_{i=1}^{m} B_i(J^k f)\Big(\sum_{j=1}^{n} b_{ij}x_j\Big) - D(x_1, \ldots, x_n)\Big) = 0, \qquad x_1, \ldots, x_n \in X.$$

Consequently, by (4.35),

$$\sum_{i=1}^{m} B_i \psi\Big(\sum_{j=1}^{n} b_{ij}x_j\Big) = D(x_1, \ldots, x_n), \qquad x_1, \ldots, x_n \in X. \qquad (4.39)$$

To complete the proof, observe that every solution of (4.19) is a fixed point of J and therefore the statement on uniqueness follows directly from the uniqueness property of ψ as a fixed point of J satisfying (4.34). $\square$

Using Theorem 4.3.3, we can obtain various stability results for numerous equations. For instance, for the inhomogeneous Cauchy equation (4.21) we can argue as in the following corollary.

Corollary 4.3.4. *Assume that $T = T_M$, Y is M-complete, $\| \ \|$ is a norm on X, and $D : X^2 \to Y$ satisfies the condition* (4.22). *Let $p, v_1, v_2 \in [0, +\infty[$, $v_1 + v_2 \neq 0$, $p \neq 1$ and $f : X \to Y$ satisfies*

$$F_{f(x+y)-f(x)-f(y)-D(x,y)}(t) \geq \frac{t}{t + v_1\|x\|^p + v_2\|y\|^p}, \qquad x, y \in X,\ t > 0. \tag{4.40}$$

Then there exists a unique solution $\psi : X \to Y$ to the Cauchy inhomogeneous equation (4.21) *such that*

$$F_{f(x)-\psi(x)}(t) \geq \widehat{\sigma}_x^0(t) \geq \frac{t}{t + v_0\|x\|^p}, \qquad x \in X,\ t > 0, \tag{4.41}$$

where $\widehat{\sigma}_x^0$ defined by (2.8) *and*

$$v_0 = \frac{v_1 + v_2}{|2 - 2^p|}.$$

Proof. The equation (4.21) is (4.19) with $m = 3$ and $n = 2$. Next, Remark 4.3.2 shows that condition (4.22) means that D fulfils hypothesis $(\mathcal{D})$. So, we use Theorem 4.3.3 with

$$\theta(x_1, x_2)(t) = \frac{t}{t + v_1\|x_1\|^p + v_2\|x_2\|^p}, \qquad x_1, x_2 \in X,\ t > 0,$$

and consider two separate cases : $p < 1$ and $p > 1$.

The first case ($p < 1$) coincides with the situation $(a1)$ of Remark 4.3.1, with $B_1 = -B_2 = -B_3 = 1$ and

$$(b_{ij})_{\substack{1\leq i\leq 3,\\ 1\leq j\leq 2}} = \begin{pmatrix} 1 & 1 \\ 1 & 0 \\ 0 & 1 \end{pmatrix},$$

when the hypothesis $(\mathcal{M})$ is valid with $\mu = 1$ and $c_1 = c_2 = 1$. Then $B_0 = 2$, $\beta_1 =$

$a_{11}c_1 + a_{12}c_2 = 2$, and consequently (see (4.25) and (4.28))

$$\mathcal{T}\chi(x_1,x_2)(t) = \chi(\beta_1 x_1, \beta_1 x_2)\Big(\frac{B_0 t}{|B_1|}\Big) = \chi(2x_1, 2x_2)(2t), \qquad (4.42)$$

$$t > 0, x_1, x_2 \in X, \chi \in \mathcal{D}_+^{X^2},$$

$$(\Lambda\delta)_x(t) := \delta_{\beta_1 x}\left(\frac{B_0 t}{\mu|B_1|}\right) = \delta_{2x}(2t), \quad \delta \in \mathcal{D}_+^X, x \in X, t > 0,$$

$$\epsilon_x(t) = \theta(c_1 x, c_2 x)(tB_0) = \frac{t}{t + v\|x\|^p}, \quad x \in X, t > 0, \qquad (4.43)$$

with $v := \dfrac{v_1 + v_2}{2}$.

Arguing as in Remark 2.2.8 (with $a = b = 2$ and $e_0 = ba^{-p} = 2^{1-p} > 1$), we obtain that Λ satisfies the condition (2.11) (with some sequence $(\omega_n^k)_{n\in\mathbb{N}} \in \Omega$) and

$$\widehat{\sigma}_0^x(t) \geq \epsilon_x\big(e_0^{-1}(e_0 - 1)t\big) = \frac{t}{t + v_0\|x\|^p}, \qquad x \in X,\ t > 0,$$

with

$$v_0 = \frac{v}{1 - 2^{p-1}} = \frac{v_1 + v_2}{2 - 2^p}.$$

Moreover, by (4.42) we get

$$\begin{aligned}\lim_{k\to\infty} \big(\mathcal{T}^k\theta\big)(x_1,x_2)(t) &= \lim_{k\to\infty} \theta(2^k x_1, 2^k x_2)(2^k t)\\ &= \lim_{k\to\infty} \frac{t}{t + 2^{k(p-1)}(v_1\|x_1\|^p + v_2\|x_2\|^p)} = 1, \qquad x_1, x_2 \in X,\ t > 0,\end{aligned}$$

which means that (4.24) is fulfilled. Therefore our statement for $p < 1$ results from Theorem 4.3.3.

In the case $p > 1$ we need situation $(a2)$ of Remark 4.3.1, with $B_1 = B_2 = -B_3 = -1$ and

$$(b_{ij})_{\substack{1\leq i\leq 3,\\ 1\leq j\leq 2}} = \begin{pmatrix} 1 & 0\\ 0 & 1\\ 1 & 1\end{pmatrix}.$$

when the hypothesis $(\mathcal{M})$ holds with $\mu = 2$, $B_0 = 1$ and $c_1 = c_2 = 1/2$. The reasoning is analogous as in the case $p < 1$, but for the convenience of the reader, we present it in some details. Namely, then $\beta_1 = \beta_2 = 1/2$, $\epsilon_x(t)$ is given by (4.43) but with $v = 2^{-p}(v_1 + v_2)$, and for every $x_1, x_2, x_2 \in X$, $t > 0$ and $\delta \in \mathcal{D}_+^X$

$$(\Lambda\delta)_x(t) := T_{i=1}^2 \delta_{\beta_i x}\Big(\frac{B_0 t}{\mu|A_i|}\Big) = \min_{i=1,2}\Big\{\delta_{\beta_i x}\Big(\frac{B_0 t}{\mu|B_i|}\Big)\Big\} = \delta_{\frac{x}{2}}\Big(\frac{t}{2}\Big),$$

$$\mathcal{T}\theta(x_1,x_2)(t) := T_{i=1}^2 \theta(\beta_1 x_1, \beta_2 x_2)\left(\frac{t}{2}\right) = \theta\left(\frac{1}{2}x_1, \frac{1}{2}x_2\right)\left(\frac{t}{2}\right).$$

According to Remark 2.2.8 (with $a = b = 1/2$ and consequently $e_0 := ba^{-p} = 2^{1-p} > 1$), condition (2.11) is satisfied and

$$\widehat{\sigma}_0^x(t) \geq \epsilon_x\big(e_0^{-1}(e_0 - 1)t\big) = \frac{t}{t + v_0\|x\|^p}, \qquad x \in X,\ t > 0,$$

with

$$v_0 = \frac{v}{1 - 2^{1-p}} = \frac{v_1 + v_2}{2^p - 2}.$$

Note yet that, for $x_1, x_2 \in X$ and $t > 0$, we have

$$\begin{aligned}\lim_{k\to\infty}\big(\mathcal{T}^k\theta\big)(x_1, x_2)(t) &= \lim_{k\to\infty}\theta(2^{-k}x_1, 2^{-k}x_2)(2^{-k}t)\\ &= \lim_{k\to\infty}\frac{t}{t + 2^{k(p-1)}(v_1\|x_1\|^p + v_2\|x_2\|^p)} = 1, \qquad x_1, x_2 \in X,\ t > 0.\end{aligned}$$

This means that (4.24) is fulfilled. Therefore also our statement for $p > 1$ results from Theorem 4.3.3. □

Remark 4.3.5. According to Remark 4.3.2 ($(b0)$ and $(b1)$), the function D in Corollary 4.3.4 can be of the following form

$$D(x, y) = z_0 + u_1(x) + u_2(y) + u_3(x, y) + g(x + y) - g(x) - g(y), \quad x, y \in X,$$

with any fixed $g : X \to Y$, additive $u_1, u_2 : X \to Y$, biadditive symmetric $u_3 : X^2 \to Y$ and $z_0 \in Y$.

4.4 Approximate eigenvalues

In this section we show an application of Theorem 2.2.6 in investigation of the approximate eigenvalues and eigenvectors, which corresponds to the results in [66, 86].

The next corollary is an example of a result concerning approximate eigenvalues of some linear operators on Y^X. Actually the assumption of linearity of the operators is not necessary in the proof, but the notion of eigenvalue might be ambiguous without it (see, e.g., [171]) and therefore we confine only to the linear case.

Corollary 4.4.1. *Let $\gamma \in \mathbb{R} \setminus \{0\}$, $\Lambda_0 : \mathcal{D}_+^X \to \mathcal{D}_+^X$, and $J_0 : Y^X \to Y^X$ be such that Λ_0 satisfies $\mathcal{C}_0$ and J_0 is linear and Λ_0-contractive. Assume $h \in Y^X$ and $\varepsilon \in \mathcal{D}_+^X$ satisfy the condition*

$$F_{(J_0 h - \gamma h)(x)} \geq \varepsilon_x, \qquad x \in X. \tag{4.44}$$

If one of the conditions (i), (ii) and (iii) of Theorem 2.2.6 is valid with

$$(\Lambda\delta)_x(t) := (\Lambda_0\delta)_x(|\gamma|t), \qquad \delta \in \mathcal{D}_+^X,\ x \in X,\ t > 0, \tag{4.45}$$

then γ is an eigenvalue of J_0, the limits

$$\psi(x) := \lim_{n\to\infty} \big(J_0^n(\gamma^{-n+1}h)\big)(x) \tag{4.46}$$

$$\sigma_x^0(t) := \lim_{m\to\infty} T_{i=0}^{m-1}(\Lambda^i\epsilon)_x\Big(\frac{t}{m}\Big) \tag{4.47}$$

$${}^{\omega}\sigma_x^0(t) := \lim_{m\to\infty} T_{i=0}^{m-1}(\Lambda^i\epsilon)_x\big(\omega_i t\big) \tag{4.48}$$

exist for every $x \in X$, $\omega = (\omega_n)_{n\in\mathbb{N}} \in \Omega$ and $t > 0$, and the function $\psi_0 \in Y^X$, given by

$$\psi_0(x) := \gamma^{-1}\psi(x), \qquad x \in X,$$

is an eigenvector of J_0, with the eigenvalue γ, enjoying

$$F_{(\psi_0-h)(x)}(t) \geq \sup_{\alpha\in(0,1)} \widehat{\sigma}_x^0(\alpha|\gamma|\,t), \qquad x \in X,\ t > 0. \tag{4.49}$$

Proof. Let $\varphi := \gamma h$ and $J : Y^X \to Y^X$ be given by

$$(J\eta)(x) = \big(J_0(\gamma^{-1}\eta)\big)(x), \qquad \eta \in Y^X,\ x \in X.$$

Then, in view of the Λ_0-contractivity and linearity of J_0, for every $\mu, \xi \in Y^X$ and $\delta \in \mathcal{D}_+^X$ with $F_{(\mu-\xi)(x)} \geq \delta_x$, we have

$$F_{(J\mu-J\xi)(x)}(t) = F_{(J_0(\gamma^{-1}\mu)-J_0(\gamma^{-1}\xi))(x)}(t) \geq F_{(J_0\mu-J_0\xi)(x)}(|\gamma|t), \qquad t > 0,$$

whence

$$F_{(J\mu-J\xi)(x)}(t) \geq (\Lambda_0\delta)_x(|\gamma|t) = (\Lambda\delta)_x(t), \qquad t > 0,$$

which means that J is $\Lambda-$contractive. Next, we can write (4.44) in the form

$$F_{(J\varphi-\varphi)(x)} \geq \varepsilon_x, \qquad x \in X. \tag{4.50}$$

Hence, by Theorem 2.2.6 and Lemma 2.2.4, the limits (4.46), (4.47) and (4.48) exist for every $x \in X$, $\omega = (\omega_n)_{n\in\mathbb{N}} \in \Omega$ and $t > 0$. Moreover, the function $\psi : X \to Y$, defined by (2.13), is a fixed point of J with

$$F_{(\psi-\varphi)(x)}(t) \geq \sup_{\alpha\in(0,1)} \widehat{\sigma}_x^0(\alpha t), \qquad x \in X,\ t > 0. \tag{4.51}$$

Write $\psi_0 := \gamma^{-1}\psi$. Now, it is easily seen that $J_0\psi_0 = J\psi = \psi = \gamma\psi_0$, (4.49) is equivalent to (4.51), and (2.13) yields (4.46). □

Clearly, under suitable additional assumptions in Corollary 4.4.1, we can deduce from Theorem 2.2.6 some statements on the uniqueness of ψ, and consequently on the uniqueness of ψ_0.

Given $\varepsilon \in \mathcal{D}_+^X$, let us introduce the following definition: $\gamma \in \mathbb{R}\setminus\{0\}$ *is an* ε*-eigenvalue of a linear operator* $J_0 : Y^X \to Y^X$ *provided there exists* $h \in Y^X$ *such that* $F_{(J_0h-\gamma h)(x)} \geq \varepsilon_x$ *for* $x \in X$.

It is easily seen that Corollary 4.4.1 yields the following simple result.

Corollary 4.4.2. *Let* $\Lambda_0 : \mathcal{D}_+^X \to \mathcal{D}_+^X$, $J_0 : Y^X \to Y^X$ *be* Λ_0*-contractive and linear, and* $\varepsilon \in \mathcal{D}_+^X$. *If* $\gamma \in \mathbb{R}\setminus\{0\}$ *is an* ε*-eigenvalue of* J_0 *and one of conditions (i)–(iii) of Theorem 2.2.6 is valid with* Λ *given by* (4.45), *then* γ *is an eigenvalue of* J_0.

Chapter 5

Ulam Stability of Delay Differential Equations in Locally Convex Spaces

5.1 Introduction

In 1993, M. Obloza [138, 139] was the first who investigated the Ulam type stability of linear differential equations, after that, a large number of papers devoted to this subject have been published (see [11, 26, 93]). In 2010, S. M. Jung [94], using the fixed point method, proved the Ulam-Hyers-Rassias-stability of the first order differential equation of the form

$$y'(t) = F(t, y(t)), \tag{5.1}$$

where $F : I \times \mathbb{R} \to \mathbb{R}$ is a continuous function satisfying a Lipschitz condition and I is a closed interval. Recently, in 2020, Y. Başci et al [23], using the alternative fixed point theorem as in [94], extended and improved Jung's result for the differential equation (5.1). Several other differential equations without delay have been considered in the literature (see [18, 129, 141, 154]).

In 2015, using the fixed point method, C. Tunç and E. Biçer [187] proved the Ulam-Hyers-stability and the Ulam-Hyers-Rassias-stability of the first-order delay differential equation

$$y'(t) = F(t, y(t), y(t-h)), \tag{5.2}$$

where $F : I \times \mathbb{R}^2 \to \mathbb{R}$ is a continuous function, I is a bounded and closed interval, and h is a positive real number, called the delay. In 2019, inspired by the techniques used in [187], S. Oğrekçi [140] extended and improved the results of [187] by proving the Ulam-Hyers-Rassias-stability of the delay differential equation (5.2) for unbounded intervals. Notice that, in 2013, D. Otrocol and V. llea [142] proved the Ulam-Hyers-stability and the Ulam-Hyers-Rassias-stability of the following delay differential equation

$$y'(t) = F(t, y(t), y(\nu(t))), \quad t \in I \subset \mathbb{R}, \tag{5.3}$$

by using the well-known Gronwall lemma and the abstract Gronwall lemma.

It is a fact that the fixed point method comes in force in the Ulam-stability theory of functional equations. In the present chapter,using the new alternative fixed point theorem in gauge spaces presented and proved in section 2.1, we show the Ulam-Hyers-Rassias-stability of the delay differential equation (5.3), where y is no more defined from an interval I into $\mathbb{R}$, but into a locally convex space (in particular into a Banach space). Since the equation (5.3), generalizes both (5.1) and (5.2), many known results on the stability of differential equations in the literature can be derived. Finally, we point out some gaps in [23] concerning the same subject and fix them. We also give a comment concerning an assertion in [14] (see Remark 5.3.1).

5.2 Stability of delay differential equation on bounded intervals

In all what follows, (E, τ) will be a sequentially complete Hausdorff locally convex space over the field $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}\}$. We will suppose that $\mathbb{P}$ is an upward directed family of semi norms defining the topology τ. We will consider $I := [a, b]$, $J := [a - h, b]$ and $H := [a - h, a]$, with a, b being real number with $a < b$, and a continuous mapping $\nu : I \to J$, so that $\nu(t) \leq t$ for every $t \in I$. We will let y denote a continuous function from J into E, whose restriction to I is continuously differentiable, while ψ will be a continuous function from H into E.

First let us state the following definitions.

Definition 5.2.1. *We say that* (5.3) *has the Ulam-Hyers-stability (or is Ulam-Hyers-stable), provided that for arbitrary $\epsilon > 0$, if for every $P \in \mathbb{P}$, there exists C_P, such that*

$$\begin{aligned} P(y'(t) - F(t, y(t), y(\nu(t)))) &\leq C_P \epsilon, \qquad t \in I, \\ P(y(t) - \psi(t)) &\leq C_P \epsilon, \qquad t \in H, \end{aligned}$$

then there exists a continuous function $y_0 : J \to E$ solution of the differential equation (5.3) *on I, identically equal to ψ on H, and for some $\delta(P, \epsilon) > 0$,*

$$P(y_0(t) - y(t)) \leq \delta(P, \epsilon), \quad t \in I, \quad P \in \mathbb{P}.$$

Definition 5.2.2. *The equation* (5.3) *has the Ulam-Hyers-Rassias-stability with respect* *to a function* $\varphi : J \to]0, +\infty[$, *provided that if for every* $P \in \mathbb{P}$, *there exists* $C_P > 0$ *such that*

$$\begin{aligned} P\Big(y'(t) - F\big(t, y(t), y(\nu(t))\big)\Big) &\leq C_P\varphi(t), \qquad t \in I, \\ P(y(t) - \psi(t)) &\leq C_P\varphi(t), \qquad t \in H, \end{aligned}$$

then there exist a number $K_{P,\varphi} > 0$ *and a continuous function* $y_0 : J \to E$, *solution of* (5.3) *on* I, *identically equal to* $\psi(t)$ *on* H, *and satisfying*

$$P(y_0(t) - y(t)) \leq K_{P,\varphi,}C_P\varphi(t), \quad t \in I, \quad P \in \mathbb{P}.$$

We will use Corollary 2.1.3, to prove the Ulam-Hyers-Rassias-stability of (5.3) on the bounded interval I. In all the sequel r will stand for a positive real number. Here is our main result.

Theorem 5.2.3. *Let* $F : I \times E^2 \to E$ *a continuous function such that*

$$\begin{aligned} &\forall P \in \mathbb{P}, \ \exists L_P, L'_p > 0 \ ; \quad \forall t \in I, \quad \forall u_1, u_2, v_1, v_2 \in E, \\ &P\Big(F(t, u_1, u_2) - F(t, v_1, v_2)\Big) \leq L_P P(u_1 - v_1) + L'_P P(u_2 - v_2). \end{aligned} \tag{5.4}$$

Assume there exists a non-decreasing continuous function $\varphi : J \to]0, +\infty[$ *such that, for every* $P \in \mathbb{P}$, *there exists* $C_P \geq 0$ *with*

$$P\Big(y'(t) - F(t, y(t), y(\nu(t)))\Big) \leq C_P\varphi(t), \qquad t \in I, \tag{5.5}$$

$$P(y(t) - \psi(t)) \leq C_P\varphi(t), \qquad t \in H. \tag{5.6}$$

Then there exists a unique continuous function $y_r : J \to E$, *solution of the differential equation* (5.3) *on* I, *identically equal to* ψ *on* H, *and such that*

$$P(y(t) - y_r(t)) \leq K_{P,\varphi}C_P\varphi(t), \quad P \in \mathbb{P}, \quad t \in I. \tag{5.7}$$

$$\text{where} \qquad K_{P,\varphi} := \frac{L_P + L'_P + r}{r}(1 + K)e^{(L_P + L'_P + r)(b-a)},$$

$$\text{and} \qquad K := \sup\left\{\frac{1}{\varphi(t)} \mid \int_a^t \varphi(x)dx \mid, \quad t \in J\right\}.$$

Proof. Notice first that, since φ is continuous and does not vanish, K is a positive real number. Now, consider the linear space $X := \mathcal{C}(J, E)$ of continuous functions from J into E and set for every $P \in \mathbb{P}$ and every $g, h \in X$,

$$d_P(g, h) = \inf\{B \geq 0, \ P(g(t) - h(t)) \leq BC(t)\varphi(t), \ t \in J\},$$

where

$$C(t) := \begin{cases} 1 & \text{if } t \in H, \\ e^{(L_P+L'_P+r)(t-a)} & \text{if } t \in I. \end{cases}$$

Then d_P is a generalized pseudo metric on X, for every $P \in \mathbb{P}$. Indeed, if $f, g, h \in X$ are given, then obviously $d_P(f,f) = 0$ and $d_P(f,g) = d_P(g,f)$. Furthermore if $B, B' \geq 0$ satisfy $d_P(f,g) \leq B$ and $d_P(g,h) \leq B'$, then, for all $t \in J$, we have

$$\begin{aligned} P(f(t) - h(t)) &\leq P(f(t) - g(t)) + P(g(t) - h(t)) \\ &\leq (B + B')C(t)\varphi(t). \end{aligned}$$

Passing to the infimum on B and B', we get $d_P(f,h) \leq d_P(f,g) + d_P(g,h)$. Now, if $d_P(f,g) = 0$ for every $P \in \mathbb{P}$, then for every $B \geq 0$

$$P(f(t) - g(t)) \leq BC(t)\varphi(t), \quad t \in J, \ P \in \mathbb{P}.$$

This gives $P(f(t) - g(t)) = 0$, for all $P \in \mathbb{P}$ and all $t \in J$. Since E is Hausdorff, $f = g$. Therefore $(X, (d_P)_{P\in\mathbb{P}})$ is a generalized gauge space. Notice that, for every $t \in J$, the evaluation $\delta_t : f \mapsto f(t)$ is continuous from $(X, (d_P)_{P\in\mathbb{P}})$ into E. Indeed, if we put $M := \sup_{t\in J} C(t)\varphi(t)$, then $M < +\infty$ and we have

$$P(f(t) - g(t)) \leq M d_P(f,g) \qquad P \in \mathbb{P}.$$

Now, let's show that $(X, (d_P)_{P\in\mathbb{P}})$ is sequentially complete. Let $(f_i)_{i\in\mathbb{N}}$ be a Cauchy sequence in X. By the continuity of δ_t, the sequence $(f_i(t))_i$ is Cauchy in E for every $t \in J$. Since E is sequentially complete, $(f_i(t))_i$ converges to some $f(t) \in E$. Define a function f from J into E by

$$f(t) := \lim_{i\to+\infty} f_i(t), \quad t \in J.$$

But, for every $P \in \mathbb{P}$ and $\epsilon > 0$, there exists $i_0 \in \mathbb{N}$, so that, whenever $i_0 \leq i, \ j$, we have

$$P(f_i(t) - f_j(t)) \leq \epsilon C(t)\varphi(t), \qquad t \in J. \tag{5.8}$$

Letting j tend to infinity, we get

$$P(f_i(t) - f(t)) \leq \epsilon C(t)\varphi(t), \qquad t \in J. \tag{5.9}$$

Moreover, for arbitrary $x_0 \in J$, $P \in \mathbb{P}$, and $\epsilon > 0$, by continuity of f_{i_0}, there exists $\delta > 0$ such that for every $x \in J \cap [x_0 - \delta, x_0 + \delta]$, $P(f_{i_0}(x) - f_{i_0}(x_0)) < \epsilon$.

Using (5.9) for i_0, we get

$$\begin{aligned} P(f(x)-f(x_0)) &\le P(f(x)-f_{i_0}(x)) + P(f_{i_0}(x)-f_{i_0}(x_0)) + P(f_{i_0}(x_0)-f(x_0)) \\ &\le \epsilon\varphi(x)C(x) + \epsilon + \epsilon\varphi(x_0)C(x_0) \\ &\le (1+2M)\epsilon. \end{aligned}$$

Then f is continuous, therefore $f \in X$. Now, by (5.9), $d_P(f_i, f) \le \epsilon$, for every $i \ge i_0$. Therefore $(X, (d_P)_{P\in\mathbb{P}})$ is sequentially complete. Now, define the operator $T : X \to X$ by

$$Tf(t) := \begin{cases} \psi(t), & \text{if } \quad t \in H \\ \psi(a) + \displaystyle\int_a^t F(x, f(x), f(\nu(x)))\, dx, & \text{if } \quad t \in I. \end{cases} \tag{5.10}$$

Notice that by Proposition 1.5.2 1., the integral in (5.10) exists. We claim that T is a strictly contractive operator on X. Indeed, fix $P \in \mathbb{P}$ and $g, h \in X$. If $d_P(g,h) = +\infty$, then obviously

$$d_P(Tg, Th) \le c\; d_P(g,h), \qquad \forall c \in (0,1). \tag{5.11}$$

Otherwise, let $B > 0$ satisfy

$$P(g(t) - h(t)) \le BC(t)\varphi(t), \quad t \in J. \tag{5.12}$$

Then for $t \in H$, we have $P(Tg(t) - Th(t)) = 0$, and for $t \in I$,

$$\begin{aligned} P(Tg(t)-Th(t)) &\le \int_a^t P\Big(F\big(x, g(x), g(\nu(x))\big) - F\big(x, h(x), h(\nu(x))\big)\Big)\, dx \\ &\le \int_a^t L_P P\big(g(x)-h(x)\big) + L'_P P\big(g(\nu(x)-h(\nu(x)))\big)\, dx \\ &\le L_P B \int_a^t \varphi(x) e^{(L_P+L'_P+r)(x-a)}\, dx \; + \\ &\qquad\qquad L'_P B \int_a^t \varphi(\nu(x)) e^{(L_P+L'_P+r)(\nu(x)-a)}\, dx \\ &\le B(L_P+L'_P)\varphi(t) \int_a^t e^{(L_P+L'_P+r)(x-a)}\, dx \\ &\le \frac{L_P+L'_P}{L_P+L'_P+r} B\big(e^{(L_P+L'_P+r)(t-a)} - 1\big) \\ &\le \frac{L_P+L'_P}{L_P+L'_P+r} B e^{(L_P+L'_P+r)(t-a)}. \end{aligned}$$

Passing to the infimum on those B's satisfying (5.12), and taking (5.11) into account, we get

$$d_P(T(g), T(h)) \le \frac{L_P+L'_P}{L_P+L'_P+r} d_P(g,h).$$

This shows that T is strictly contractive on X.
Now, we claim that for $n_0 = 0$, it holds,

$$d_P(T^{n_0}y, T^{n_0+1}y) < +\infty, \quad P \in \mathbb{P}.$$

Indeed, let $P \in \mathbb{P}$, $t \in J$ be given. If $t \in H$, then by (5.6),

$$P(Ty(t) - y(t)) = P(\psi(t) - y(t)) \leq C_P\varphi(t) = C_P C(t)\varphi(t).$$

Now, if $t \in I$, then using Proposition 1.5.3 (a), we get

$$\begin{aligned} P(Ty(t) - y(t)) &\leq P\Big(\int_a^t F(x, y(x), y(\nu(x))) - y'(x)\,dx\Big) + P(\psi(a) - y(a)) \\ &\leq \int_a^t P\big(F(x, y(x), y(\nu(x))) - y'(x)\big)\,dx + P(\psi(a) - y(a)) \\ \text{by (5.5) and (5.6),} \quad &\leq C_P\Big(\int_a^t \varphi(x)dx + \varphi(a)\Big) \\ &\leq C_P(1+K)\varphi(t) \\ &\leq C_P(1+K)\varphi(t)e^{(L_P+L'_P+r)(t-a)}. \end{aligned} \tag{5.13}$$

Hence,

$$P(Ty(t) - y(t)) \leq C_P\ (1+K)C(t)\varphi(t), \quad t \in J. \tag{5.14}$$

This means that

$$d_P(Ty, y) \leq (1+K)C_P < +\infty. \tag{5.15}$$

According to Corollary 2.1.3, there exists a unique continuous function $y_r : J \to E$ such that for every $P \in \mathbb{P}$, $d_P(T^k y, y_r)$ tends to zero as k tends to infinity, y_r is a fixed point of T, and

$$d_P(y_r, y) \leq \frac{L_P + L'_P + r}{r} d_P(Ty, y).$$

Then, by (5.15) we get for every $P \in \mathbb{P}$

$$d_P(y, y_r) \leq \frac{L_P + L'_P + r}{r}(1+K)C_P,$$

whereby

$$P(y(t) - y_r(t)) \leq \frac{L_P + L'_P + r}{r}(1+K)C_P C(t)\varphi(t), \quad t \in J.$$

In particular, for $t \in I$, we get

$$P(y(t) - y_r(t)) \leq \frac{L_P + L'_P + r}{r}(1+K)C_P e^{(L_P+L'_P+r)(b-a)}\varphi(t).$$

This is (5.7) as desired. Notice that, since y_r is a fixed point of T, it equals ψ on H and fulfills

$$y_r(t) = \psi(a) + \int_a^t F\left(x, y_r(x), y_r(\nu(x))\right) dx, \qquad t \in I.$$

Using Proposition 1.5.2, 3, we get

$$y_r'(t) = F\left(x, y_r(t), y_r(\nu(t))\right), \quad t \in I.$$

Hence y_r is a solution of (5.3) on I and the proof is finished. □

Remark 5.2.4. It is easy to check that the best approximation in (5.7) is obtained when

$$r = \frac{L_P + L_P'}{2}\left[\sqrt{1 + \frac{4}{(L_P + L_P')(b-a)}} - 1\right].$$

In this case, our result sharpens and extends the main result of [23], the latter being given for an equation without delay (i.e., $\nu(x) := x$), $r = 1$ and $E = \mathbb{R}$.

5.3 Applications

In this section, we provide applications of Theorem 5.2.3. Before that, let mention the following remark.

Remark 5.3.1. In the proofs of the main theorems of [23], there are some gaps. Actually the assertion in line 15 of page 6 is not true. In fact $\epsilon(x-x_0)e^{-(L+1)(x-x_0)}$ is not a constant, so that one can assert that it is larger than $d(Ty, y)$. Indeed, if one takes $x = x_0$, the function y would be a solution of the equation in consideration. But this need not be true. A similar error is also made in line -2 of page 8 and in line 5 of page 9. Because of these errors, the approximations obtained there remain non justified.

Moreover, it is asserted in [14] that the condition

$$\exists K > 0 : \left| \int_a^t \varphi(x)\, dx \right| \leq K\varphi(t), \qquad t \in [a, a+c],$$

is not necessary although it is widely used. But the fact is that every such a function satisfies automatically the condition above with $K = c$.

As a corollary of Theorem 5.2.3, we get the analogous vector-valued (rectified) version of Theorem 3 of [23].

Corollary 5.3.2. *Assume that $G : I \times E \to E$ is a continuous function satisfying the*

Lipschitz condition

$$\forall P \in \mathbb{P}, \quad \exists L_P \geq 0 \; ; P(G(t,y_1) - G(t,y_2)) \leq L_P(y_1 - y_2), \quad t \in I, \; y_1, y_2 \in E.$$

If a continuously differentiable function $y : I \to E$ *satisfies*

$$P(y'(t) - G(t, y(t))) \leq \varphi(t), \qquad t \in I,$$

where $\varphi : I \to]0, +\infty[$ *is a non-decreasing continuous function, then there exists a unique* $y_1 : I \to \mathbb{R}$*, solution to the differential equation* (5.1) *and satisfying*

$$P(y(x) - y_0(x)) \leq (K+1)(L_P + 1)e^{(L_P+1)(b-a)}\varphi(x), \quad x \in I.$$

Proof. Taking $F : I \times E \times E \to E$, with $F(t,u,v) := G(t,u)$, $L'_P = 0$, $C_P = 1$, $h = 0$, $r = 1$ and ν the identity map on I, it is easily checked that the assumptions of Theorem 5.2.3 are satisfied. Whence the result. □

If in the preceding corollary $E = \mathbb{R}$, we get the following rectified version of Theorem 3 of [23].

Corollary 5.3.3. *Assume that* $G : I \times \mathbb{R} \to \mathbb{R}$ *is a continuous function which satisfies the Lipschitz condition*

$$\mid G(t,y_1) - G(t,y_2) \mid \leq L \mid y_1 - y_2 \mid, \quad t \in I, \; (y_1, y_2) \in \mathbb{R}^2. \tag{5.16}$$

If a continuously differentiable function $y : I \to \mathbb{R}$ *satisfies*

$$\mid y'(t) - G(t, y(t)) \mid \leq \varphi(t), \qquad t \in I,$$

where $\varphi : I \to]0, +\infty[$ *is a non-decreasing continuous function, then there exists a unique* $y_1 : I \to \mathbb{R}$*, solution to the differential equation* (5.1) *and satisfying*

$$\mid y(x) - y_1(x) \mid \leq (K+1)(L+1)e^{(L+1)(b-a)}\varphi(x), \quad x \in I.$$

If in Corollary 5.3.3, φ is constant with value ϵ, then we get a similar rectified version of Theorem 2 of [23], namely

Corollary 5.3.4. *Let* ϵ *be a positive real number, and assume that* $G : I \times \mathbb{R} \to \mathbb{R}$ *is a continuous function which satisfies the Lipschitz condition* (5.16)*. If a continuously differentiable function* $y : I \to \mathbb{R}$ *satisfies*

$$\mid y'(t) - G(t, y(t)) \mid \leq \epsilon, \qquad t \in I,$$

then there exists a unique $y_1 : I \to \mathbb{R}$*, solution to the differential equation* (5.1) *and satisfying*

$$| y(x) - y_1(x) | \leq (L+1)(b-a+1)e^{(L+1)(b-a)}\epsilon, \quad x \in I.$$

Proof. Just notice that for the constant function $\varphi = \epsilon$, we have $\int_a^t \varphi(s)ds \leq (b-a)\varphi(t)$ for every $t \in [a,b]$. □

Whenever $\mathbb{P}$ is reduced to a singleton, i.e., when E is a Banach space, an immediate corollary of Theorem 5.2.3 is the following.

Corollary 5.3.5. *Let* $(E, \| \ \|)$ *be a Banach space and* $F : I \times E^2 \to E$ *a continuous function, such that there exist* $L,\ L' > 0$ *so that for every* $t \in I$*, and* $u_1, u_2, v_1, v_2 \in E$*,*

$$\|F(t, u_1, u_2) - F(t, v_1, v_2)\| \leq L\|u_1 - v_1\| + L'\|u_2 - v_2\|.$$

Suppose that there exist a non-decreasing continuous function $\varphi : J \to]0, +\infty[$ *such that*

$$\|y'(t) - F(t, y(t), y(\nu(t)))\| \leq \varphi(t),\ t \in I, \tag{5.17}$$

$$\|y(t) - \psi(t)\| \leq \varphi(t), \quad t \in H. \tag{5.18}$$

Then, there exists a unique continuous function $y_r : J \to E$*, solution of the differential equation* (5.3) *on* I*, such that for every* $t \in H$*,* $y_r(t) = \psi(t)$*, and*

$$\|y_r(t) - y(t)\| \leq \frac{L+L'+r}{r}(1+K)e^{K'}\varphi(t), \quad t \in I,$$

where $K' := (L+L'+r)(b-a)$ *and* $K := \sup\left\{\frac{1}{\varphi(t)} \mid \int_a^t \varphi(x)dx \mid, t \in J\right\}$.

Remark 5.3.6. Theorem 5.2.3 can be used to prove the stability of plenty delay differential functional equations. For instance if we take for every $t \in I$, $\nu(t) = t - h$, with $h > 0$, we get the Ulam-Hyers-Rassias-stability of the equation (5.2) in our new context, generalizing the main result of [187]. Next, let us recall, after [132], that a self-map $g : E \to E$ is said to be Yanyan-continuous with constants $(L_P)_{P \in \mathbb{P}}$, if

$$\forall P \in \mathbb{P},\ P(g(x) - g(y)) \leq L_P P(x-y), \quad x, y \in E.$$

Corollary 5.3.7. *Let* $A, B : E \to E$ *be Yanyan-continuous mappings with constants* $(L_P)_P$ *and* $(L'_P)_P$ *respectively, and let* α, θ *be positive numbers. If for every* $P \in \mathbb{P}$*, there exists* $\delta_P \geq 0$ *with*

$$P(y'(t) + A(y(t)) + B(y(t-h)) - q(t)) \leq \delta_P \theta e^{\alpha t}, \quad t \in I, \tag{5.19}$$

$$P(y(t) - \psi(t)) \leq \delta_P \theta e^{\alpha t}, \quad t \in H. \tag{5.20}$$

Then there exists a unique function $y_r : J \to E$, identically equal to ψ on H, solution of the differential equation

$$y'(t) + A(y(t)) + B(y(t-h)) = q(t), \tag{5.21}$$

on I, and satisfying for every $P \in \mathbb{P}$, $t \in J$

$$P(y_r(t) - y(t)) \leq \frac{L_P + L'_P + r}{r} \delta_P (1+K) e^{K'} \theta e^{\alpha t},$$

where $K' := (L_P + L'_P + r)(b-a)$ and $K := \dfrac{1}{\alpha} \max(1, e^{\alpha h} - 1)$.

Proof. Taking $F(t,u,v) = A(u) + B(v) - q(t)$, $\varphi(s) = \theta e^{\alpha s}$, $C_P = \delta_P$, and $\nu(t) = t - h$, for every $s \in J$, $t \in I$ and $u, v \in E$, it is easy to check that the assumptions of Theorem 5.2.3 are satisfied, whereby the result. □

Remark 5.3.8. A special case of equation (5.21) is the following

$$y'(t) + R(t)y(t) + S(t)y(t-h) = q(t),$$

where $R, S : I \to \mathbb{K}$ are bounded functions. If R and S are constant functions, we also get a further special case of (5.21).

5.4 Stability of delay differential equation on unbounded intervals

Now, we will prove the Ulam-Hyers-Rassias stability of the delay differential equation (5.3) on an unbounded interval using the Jung's method [94].

In all what follows, we will take $I = [a, +\infty[$, $H := [a-h, a]$ and $J = [a-h, +\infty[$. For any function $\varphi : J \to]0, +\infty[$, we will set

$$K_\varphi := \sup \left\{ \frac{1}{\varphi(t)} \mid \int_a^t \varphi(x) dx \mid, t \in J \right\}.$$

Theorem 5.4.1. *Let $F : I \times E^2 \to E$ be a continuous function satisfying* (5.4)*. Assume there exists a non-decreasing continuous function $\varphi : J \to]0, +\infty[$ such that, for every $P \in \mathbb{P}$, there exists $C_P \geq 0$ so that* (5.5) *and* (5.6)*, and $K_\varphi(L_P + L'_P) < 1$ hold. Then, there exists a unique continuous function $y_0 : J \to E$, solution of the differential equation* (5.3) *on I, identically equal to ψ on H, and satisfying*

$$P(y_0(t) - y(t)) \leq \frac{C_P(1+K_\varphi)}{1 - K_\varphi(L_P + L'_P)} \varphi(t), \qquad P \in \mathbb{P}, \quad t \in I.$$

Proof. For any $i \in \mathbb{N}$, put $I_i := [a, a+i]$ and $J_i := [a-h, a+i]$. Consider the linear space $X_i := \mathcal{C}(J_i, E)$ of continuous functions from J_i into E, and set for every $P \in \mathbb{P}$ and every $g, h \in X_i$,

$$d_{P,i}(g,h) = \inf\{B \geq 0,\ P(g(t)-h(t)) \leq B\varphi(t),\ t \in J_i\}.$$

Using a similar proof as that of Theorem 5.2.3, one proves easily that $(X_i, (d_{P,i})_{P\in\mathbb{P}})$ is a sequentially complete generalized gauge space. Now, define the operator $T_i : X_i \to X_i$ by

$$T_i f(t) := \begin{cases} \psi(t), & \text{if} \quad t \in H \\ \psi(a) + \displaystyle\int_a^t F(x, f(x), f(\nu(x)))\, dx, & \text{if} \quad t \in I_i. \end{cases}$$

Then T_i is a strictly contractive operator on X_i. Indeed, fix $P \in \mathbb{P}$ and $g, h \in X_i$. If $d_{P,i}(g,h) = +\infty$, then obviously

$$d_{P,i}(T_i g, T_i h) \leq c\ d_{P,i}(g,h), \qquad \forall c \in (0, +\infty). \tag{5.22}$$

Otherwise, choose $B > 0$ satisfying

$$P(g(t)-h(t)) \leq B\varphi(t), \quad t \in J_i, \tag{5.23}$$

For every $t \in I_i$, we have

$$\begin{aligned} P(T_i g(t) - T_i h(t)) &\leq \int_a^t P\Big(F\big(x, g(x), g(\nu(x))\big) - F\big(x, h(x), h(\nu(x))\big)\Big)\, dx \\ &\leq \int_a^t L_P P\big(g(x)-h(x)\big) + L'_P P\big(g(\nu(x)) - h(\nu(x))\big)\, dx \\ &\leq L_P B \int_a^t \varphi(x)\, dx\ + L'_P B \int_a^t \varphi(\nu(x))\, dx \\ &\leq B(L_P + L'_P) K_\varphi \varphi(t). \end{aligned}$$

Passing to the infimum on those B's satisfying (5.23), and taking (5.22) into account, we get

$$d_{P,i}(T_i(g), T_i(h)) \leq K_\varphi (L_P + L'_P) d_P(g,h).$$

Since $K_\varphi(L_P + L'_P) < 1$, T_i is strictly contractive on X_i.
Arguing as for Theorem 5.2.3, we show that

$$P(T_i y(t) - y(t)) \leq C_P(1 + K_\varphi)\varphi(t), \quad t \in J_i, \quad P \in \mathbb{P}.$$

This means that, for $n_0 = 0$ and every $P \in \mathbb{P}$, we have

$$d_{P,i}(T_i^{n_0}y, T_i^{n_0+1}y) \leq C_P(1+K_\varphi) < +\infty, \tag{5.24}$$

According to Corollary 2.1.3, there exists a unique continuous function $y_i : J_i \to E$, such that for every $P \in \mathbb{P}$, $d_{P,i}(T_i^j y, y_i)$ tends to zero as j tends to infinity, y_i is a fixed point of T_i, and

$$d_{P,i}(y_i, y) \leq \frac{1}{1 - K_\varphi(L_P + L'_P)} d_{P,i}(T_i y, y).$$

Then, by (5.24) we get for every $P \in \mathbb{P}$

$$d_{P,i}(y, y_i) \leq \frac{C_P(1+K_\varphi)}{1 - K_\varphi(L_P + L'_P)}.$$

It follows from definition of $d_{P,i}$ that

$$P(y(t) - y_i(t)) \leq \frac{C_P(1+K_\varphi)}{1 - K_\varphi(L_P + L'_P)}\varphi(t), \quad t \in J_i. \tag{5.25}$$

The uniqueness of y_i implies that if $t \in I_i$, then

$$y_i(t) = y_{i+k}(t), \qquad k \in \mathbb{N}.$$

For $t \in I$, put $i(t) := \min\{i \in \mathbb{N}, t \in I_{i-1}\}$, and define a function $y_0 : I \to E$ by

$$y_0(t) = y_{i(t)}(t).$$

As the differentiability is a local property, one sees easily that y_0 is differentiable on I. Moreover, for every $t \in H$, $y_0(t) = y_{i(t)}(t) = \psi(t)$, and for $t \in I$, we have

$$\begin{aligned} y_0(t) &= y_{i(t)}(t) \\ &= \psi(a) + \int_a^t F\left(x, y_{i(t)}(x), y_{i(t)}(\nu(x))\right) dx, \\ &= \psi(a) + \int_a^t F\left(x, y_0(x), y_0(\nu(x))\right) dx. \end{aligned}$$

Therefore, by Proposition 1.5.2, 3.,

$$y_0'(t) = F\left(x, y_0(t), y_0(\nu(t))\right), \quad t \in I.$$

Hence y_0 is a solution of (5.3). Furthermore, by (5.25)

$$P(y_0(t) - y(t)) \leq \frac{C_P(1+K_\varphi)}{1-K_\varphi(L_P+L'_P)}\varphi(t), \qquad P \in \mathbb{P}, \quad t \in J,$$

as desired. $\square$

5.5 Applications

As for Theorem 5.2.3, we obtain similar corollaries. For example, whenever E is a Banach space, we have

Corollary 5.5.1. *Assume that* $(E, \| \, \|)$ *is a Banach space,* $F : I \times E^2 \to E$ *is a continuous function, such that*

$$\|F(t,u_1,u_2) - F(t,v_1,v_2)\| \leq L\|u_1 - v_1\| + L'\|u_2 - v_2\|, \quad t \in I, \ u_1, u_2, v_1, v_2 \in E$$

for some $L, \ L' > 0$*. Suppose that there exists a non-decreasing continuous function* $\varphi : J \to]0, +\infty[$ *so that* (5.17), (5.18) *and* $K_\varphi(L+L') < 1$ *hold. Then, there exists a unique continuous function* $y_0 : J \to E$*, solution of the differential equation* (5.3) *on* I*, identically equal to* ψ *on* H*, and enjoying*

$$\|y_0(t) - y(t)\| \leq \frac{1+K_\varphi}{1-K_\varphi(L+L')}\varphi(t), \quad t \in I.$$

We also obtain analogous of Corollary 5.3.5 as follows.

Corollary 5.5.2. *Assume that* $A, B : E \to E$ *are Yanyan-continuous mappings with constants* $(L_P)_P$ *and* $(L'_P)_P$ *respectively, and that* α, θ *are positive numbers. If for every* $P \in \mathbb{P}$*,* (5.19), (5.20) *and* $K(L_P + L'_P) < \alpha$ *hold true, with* $K := \dfrac{1}{\alpha}\max(1, e^{\alpha h} - 1)$*, then there exists a unique function* $y_0 : J \to E$*, identically equal to* ψ *on* H*, solution of the differential equation* (5.21) *on* I*, and satisfying*

$$P(y_0(t) - y(t)) \leq \frac{\delta_P(1+K)}{1-K(L_P+L'_P)}\theta e^{\alpha t}, \quad t \in I \quad P \in \mathbb{P}.$$

Proof. Taking $F(t,u,v) = A(u) + B(v) - q(t)$, $\varphi(s) = \theta e^{\alpha s}$, $C_P = \delta_P$, and $\nu(t) = t - h$, for every $s \in J$, $t \in I$ and $u, v \in E$, it is easy to check that the assumptions of Theorem 5.4.1 are satisfied, whereby the result. Just notice that $\mid \displaystyle\int_a^t \varphi(x)dx \mid \leq K\varphi(t)$, for every $t \in J$. $\square$

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ROYAUME DU MAROC
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Résumé

Cette thèse est une étude sur la stabilité au sens d'Ulam de certaines équations fonctionnelles linéaires dans des domaines différents, notamment dans les espaces de Banach et les espaces normés aléatoires, ainsi que de certaines équations différentielles avec retard dans des espaces localement convexes. Afin de montrer nos résultats, nous avons été amenés à établir et à appliquer des théorèmes de point fixe dans des contextes généraux. Nous commençons par donner une nouvelle preuve plus simple d'un résultat de Bahyrycz et Olko montrant la stabilité d'Ulam de l'équation de Forti non homogène. Ensuite, nous introduisons une équation fonctionnelle abstraite, généralisant celle de Forti et nous étudions sa stabilité et son hypersatbilité, en utilisant une approche de point fixe. Enfin, nous montrons deux théorèmes du point fixe dans deux cadres différents, le premier dans les espaces uniformes généralisés séquentiellement complet et le second dans les espaces normés aléatoires. Nous appliquons ces deux théorèmes pour prouver des résultats sur la stabilité au sens d'Ulam.

Mots-clés : stabilité d'Ulam, équation fonctionnelle linéaire, équation différentielle avec retard, méthode directe, méthode du point fixe, espace normé aléatoire, espace localement convexe, espace uniforme généralisé.

Abstract

This thesis is an investigation of the Ulam-stability of some linear functional equations in different domains, namely in Banach spaces and in Random normed spaces, as well as of some differential equations with delay in locally convex spaces. In order to show our results, we have been led to establish and apply some fixed point theorems in general settings. First, we provide a new proof of a result of Bahyrycs and Olko showing the Ulam-stability of the non-homogeneous Forti equation in Banach spaces. Instead of the use of a complicated theorem due to Brzdek, Chudziak and Pales, we use the classical Banach contraction principle and prove the same result. As Forti equation involves concrete linear mappings, namely the multiplication by a scalar and a mapping associated with a matrix, we introduce an abstract linear functional equation, generalizing Forti's one and then most of the linear equations ever considered in the literature. We then study its stability and its hyperstability, using a fixed point approach. Next, we show a general fixed point theorem in random normed spaces. As an application of it, we prove Ulam-stability results for a non-homogeneous Forti functional equation, with non-constant second member. Finally, we prove an alternative fixed point theorem in generalized gauge spaces, extending the famous alternative fixed point theorem of Diaz and Margoli. Using this fixed point theorem, we show the Ulam-stability of a general delay differential equation in locally convex spaces.

Keywords: Ulam-stability, general linear functional equation, delay functional differential equation, direct method, fixed point method, random normed space, locally convex space, generalized gauge space.

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