جامعة سيدي محمد بن عبد الله بفاس
UNIVERSITE SIDI MOHAMED BEN ABDALLAH DE FES
كلية العلوم ظهر المهراز فاس
FACULTE DES SCIENCES DHAR EL MEHRAZ FES

N° d'Ordre : **79 /2020**

Formation Doctorale: Mathématiques et Applications

Discipline: Mathématiques

Spécialité: Analyse fonctionnelle et Théorie spectrale

Laboratoire: LaSMA

THESE DE DOCTORAT

Présentée par

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Préservations non nécessairement additives de certaines parties de $\mathbb{C}$, $\mathbf{X}$ ou de $\mathbf{B}(\mathbf{X})$ ainsi que la dimension

Soutenue le 19 / 12 / 2020 devant le jury composé de :

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Année universitaire : 2019-2020

Préservations non nécessairement additives de certaines parties de $\mathbb{C}$, X ou de $\mathcal{B}(X)$ ainsi que de la dimension

Aziz Lahssaini

19 Décembre 2020

To my mother, who never doubted my success,
To the soul of my father and brother,
To my brothers and sisters.

Remerciements

En premier lieu, je tiens à remercier mon directeur de thèse, Professeur **Mustapha Ech-Chérif El Kettani**, pour la confiance qu'il m'a accordée en acceptant d'encadrer ce travail doctoral, pour ses multiples conseils et pour toutes les heures qu'il a consacrées à diriger cette thèse. Enfin, j'ai été extrêmement sensible à ses qualités humaines d'écoute et de compréhension tout au long de ce travail doctoral.

Je voudrais remercier chaleureusement Monsieur **Aziz Blali**, Professeur à l'université Sidi Mohamed Ben Abdellah. Il m'a fait un très grand honneur en acceptant de présider le jury de cette thèse. Je le remercie de tout mon coeur.

Le Professeur **Mostafa Mbekhta** de l'université de Lille 1, spécialiste de renommée internationale dans ce domaine, a accepté de rapporter sur ma thèse, je me trouve honoré de ce fait. Puisse-t-il accepter mes sincères remerciements.

Mes plus vifs remerciements vont aux Professeurs **Mhamed Elhodaibi** and **Mourad Oudghiri** de l'université Mohammed premier d'Oujda ainsi que le Professeur **Hassane Zguitti** de l'université sidi Mohamed Ben Abdellah pour avoir accepté de rapporter sur ma thèse et pour l'intérêt qu'ils ont porté à mon travail. Je les en remercie chaleureusement.

Le Professeur **Mohammed Babahmed** de L'université Moulay Ismail de Meknès, qui a été mon professeur en licence et en Master a accepté d'être examinateur de ma thèse, je me trouve honoré de ce fait. Puisse-t-il accepter mes sincères remerciements.

Je voudrais remercier chaleureusement Monsieur **Abdelkhalek El Amrani**, Professeur à l'université Sidi Mohamed Ben Abdellah qui a été aussi mon Professeur en Master d'avoir accepté d'être examinateur de ma thèse.

J'adresse mes plus vifs remerciements et les expressions de mon grand respect au Professeur **Rachid Ameziane Hassani**, qui m'a honoré par l'intérêt qu'il a porté à mon travail et par sa présence parmi les membres du jury de cette thèse.

Le Professeur **Hasssane Benbouziane** a été toujours présent est disponible lors de l'élaboration de cette thèse. Je le remercie pour les multiples discussions.

Je tiens à remercier profondément toute ma famille et mes amis et tous ceux qui m'ont encouragé lors de la préparation de cette thèse: en particulier ma mère, mes frères et sœurs et mes amis: Ismail, Abdelhak, Farah, Youssef, Younes, Zouheir, Mostafa, Hamza, Safae and Fatimazahrae.

Résumé

Dans cette thèse, nous nous sommes intéressés à certaines problèmes de préservation. Dans une formulation générale, ceci consiste à caractériser des applications entre deux algèbres de Banach, qui laissent une certaine propriété, une relation, ou même un sous-ensemble invariant.

Au chapitre 3, nous déterminons la forme des applications $\phi : \mathcal{B}(X) \to \mathcal{B}(X)$ satisfaisant

$$\{\phi(A) \diamond \phi(B)\}' = \{A \diamond B\}' \text{ pour tout } A, B \in \mathcal{B}(X),$$

où $\diamond$ est l'une de ces deux opérations binaires, à savoir le triple produit ABA et le produit de Jordan $AB + BA$. $\{A\}'$ désignant le commutant de A.

Le chapitre 4 est consacré aux applications additives qui compressent ou élargissent le pseudospectre d'un élément dans $\mathcal{B}(X)$. Nous caractérisons également les applications de $\mathcal{B}(X)$ sur elle même préservant le peudospectre des produits généralisés des opérateurs.

Dans le chapitre 5, nous caractérisons les applications ϕ de $\mathcal{B}(H)$ sur lui-même satisfaisant

$$\Lambda_\epsilon(\phi(A)\phi(B)^*\phi(A)) = \Lambda_\epsilon(AB^*A) \text{ pour tout } A, B \in \mathcal{B}(H).$$

$\Lambda_\epsilon(.)$ représente le spectre conditioné et $0 < \epsilon < 1$.

Le chapitre 6, est consacré à la description des applications surjectives sur $\mathcal{B}(X)$ satisfaisant

$$\dim F(\phi(A_1) \cdots * \phi(A_k)) = \dim F(A_1 * \cdots * A_k) \text{ pour tout } A_1, \cdots, A_k \in \mathcal{B}(X).$$

$A_1 * \cdots * A_k$ représente le produit généralisé et $F(A)$ désigne l'ensemble des points fixe de A.

Dans le chapitre 7, nous caractérisons la forme de toutes les applications surjectives unitaires ϕ de $\mathcal{B}(X)$ sur lui-même satisfaisant

$$\dim X_{\phi(A)\phi(B)}(\{1\}) = \dim X_{AB}(\{1\}) \text{ pour tout } A, B \in \mathcal{B}(X).$$

De plus, nous caractérisons les applications surjectives ϕ de $\mathcal{B}(X)$ sur lui-même satisfaisant

$$X_{\phi(A)\phi(B)}(\{1\}) = X_{AB}(\{1\}) \text{ pour tout } A, B \in \mathcal{B}(X).$$

$X_T(\Omega)$ désigne le sous-espace spectral local de T associé à Ω.

Mots clés: Problèmes de préservation, algèbres de Banach, théorie des opérateurs, spectre, spectre local, commutant, pseudospectre, spectre conditioné, sous-espace spectral local, points fixe.

Abstract

In this thesis, we are interested in some preserver problems. In a general formulation, these characterize maps between two Banach algebra, which leave certain property, particular relation, or even a subset invariant.

In Chapter 3, we determine the form of maps $\phi : \mathcal{B}(X) \to \mathcal{B}(X)$ satisfying

$$\{\phi(A) \diamond \phi(B)\}' = \{A \diamond B\}' \text{ for all } A, B \in \mathcal{B}(X)),$$

for these binary operations $\diamond$; triple product ABA and Jordan product $AB + BA$, where $\{A\}'$ stands for the commutant of A.

Chapter 4 is devoted to additive maps that compress or expand the pseudospectrum of an element in $\mathcal{B}(X)$. Also, we characterize maps between $\mathcal{B}(X)$ preserve peudospectrum of generalized products of operators.

In Chapter 5, we characterize maps ϕ from $\mathcal{B}(H)$ onto itself satisfying

$$\Lambda_\epsilon(\phi(A)\phi(B)^*\phi(A)) = \Lambda_\epsilon(AB^*A) \text{ for all } A, B \in \mathcal{B}(X)),$$

where $\Lambda_\epsilon(.)$ stands for the condition spectrum with $0 < \epsilon < 1$.

Chapter 6, is devoted to the description of surjective maps from $\mathcal{B}(X)$ into itself satisfying

$$\dim F(\phi(A_1) * \cdots * \phi(A_k)) = \dim F(A_1 * \cdots * A_k) \text{ for all } A_1, \cdots, A_k \in \mathcal{B}(X),$$

where $A_1 * \cdots * A_k$ stands for the generalized product and $F(A)$ for the fixed points set of A.

In Chapter 7, we characterize the form of all unital surjective maps ϕ on $\mathcal{B}(X)$ which satisfy

$$\dim X_{\phi(A)\phi(B)}(\{1\}) = \dim X_{AB}(\{1\}) \text{ for all } A, B \in \mathcal{B}(X).$$

Also, we characterize surjective maps ϕ on $\mathcal{B}(X)$ satisfy

$$X_{\phi(A)\phi(B)}(\{1\}) = X_{AB}(\{1\}) \text{ for all } A, B \in \mathcal{B}(X),$$

where $X_T(\Omega)$ denotes the local spectral subspaces of T associated with Ω.

Keywords: Preservation problems, Banach algebra, operators theory, spectra, local spectra, commutant, pseudospectrum, condition spectrum, local spectral subspace, fixed points.

List of Symbols

$\mathbb{C}$	set of complex numbers
$\mathbb{R}$	set of real numbers
$\mathbb{N}$	set of natural numbers
X	a Banach space
H	an Hilbert space
$\langle ., . \rangle$	the inner product on H
$\mathcal{B}(X)$	the algebra of linear bounded operators on X
$\mathcal{M}_n(\mathbb{C})$	the algebra of square matrices
$\mathcal{A}$	a Banach algebra
$\| \, . \, \|$	the algebra norm
I	the identity operator
T^*	the adjoint of T
T^t	the transpose of T
$\ker(T)$	the kernel of T
$Im(T)$	the image (a.k.a range) of T
$\sigma(T)$	the spectra of T
$\rho(T)$	resolvent set of T
$\sigma_T(x)$	the local spectra of T at x
$\rho_T(x)$	the local resolvent set of T at x
SVEP	single valued extension property
$X_T(F)$	the local spectral subspace of T
$\sigma_\epsilon(a)$	the pseudo spectrum an element a in a Banach algebra $\mathcal{A}$
$\Lambda_\epsilon(a)$	the condition spectrum of an element a in a Banach algebra $\mathcal{A}$
$r_\epsilon(a)$	the condition spectral radius of a
$Inv(\mathcal{A})$	the invertible elements in $\mathcal{A}$
$x \otimes f$	rank one operator
$\mathcal{F}(X)$	the set of all finite rank operators on X
$\mathcal{F}_1(X)$	the set of rank one operators on X
$\mathcal{N}_1(X)$	the set of rank one nilpotent operators on X
$\mathcal{P}_1(X)$	the set of rank one idempotent on operators X
$\{T\}'$	the commutant of T
$F(T)$	fixed points of T
$D(\lambda, r)$	the open disc of $\mathbb{C}$ centered at λ and of radius r
$\overline{D}(\lambda, r)$	the closed disc of $\mathbb{C}$ centered at λ and of radius r
$\square$	the end of the proof

Introduction Générale

L'un des sujets les plus actifs et les plus féconds de la théorie des algèbres au cours des cent dernières années est le *problème de préservation linéaire* (LPP). Ce problème est lié à la caractérisation de toutes les applications linéaires sur les algèbres matricielles, les algèbres des opérateurs linéaires bornés et plus généralement les algèbres de Banach qui laissent certaines fonctions, sous-ensembles ou relations invariants. Nous pensons que c'est Frobenius (en 1897) qui était le premier mathématicien à avoir aborder ce genre de problèmes; il a caractérisé les applications linéaires qui préservent le déterminant. Depuis lors, beaucoup de résultats ont été obtenu dans ce domaine. On pourra consulter les articles suivants [37, 83, 84, 89, 91] et les références qui y figurent. Il s'agit également de problèmes où les conditions de l'application en question ont été affaiblies (c'est-à-dire additive au lieu de linéaire). Les problèmes de préservation se résument à l'une des quatres situations suivantes:

Problème I: Il s'agit des applications linéaires qui conservent certaines fonctions. On peut ennoncer ce problème de la manière suivantes.

Soit F une fonction (à valeur scalaire ou vectorielle) de $\mathcal{A}$ sur lui-même ($\mathcal{A}$ étant une algèbre de Banach). Caractérisez les opérateurs linéaires ϕ de $\mathcal{A}$ sur lui-même satisfaisant

$$F(\phi(a)) = F(a) \text{ pour tout } a \in \mathcal{A}.$$

Le premier problème de ce genre a probablement été examiné par Frobenius [60], qui a considéré ce type de problèmes pour $\mathcal{A} = \mathcal{M}_n(\mathbb{C})$ et $F(.) = \det(.)$. Les problèmes des préservation du spectre et de la norme pourraient être inclus dans cette situation.

Notons également que de nouvelles contributions à ce type de problème de préservation ont été récemment apportés par Bourhim, Ransford, Mbekhta, González, Bračič, Müller et Costara concernant les applications linéaires préservant des quantités spectrales locales (spectre local, rayon spectral local, sous-espace spectral local, etc) voir [34, 36, 49, 62].

Problème II: Il s'agit des applications linéaires qui préservent certains sous-ensembles. Ce problème s'ennonce comme suit:

Soit $\mathcal{S}$ un sous-ensemble de $\mathcal{A}$. Caractérisez les opérateurs linéaires ϕ de $\mathcal{A}$ sur lui-même satisfaisant

$$\phi(\mathcal{S}) \subset \mathcal{S} \text{ ou } \phi(\mathcal{S}) = \mathcal{S}.$$

Plusieurs sous-ensembles importants ont été étudiés en rapport avec le problème II. Par exemple, nous mentionnons les ensembles des opérateurs idempotents, nilpotents, Fredholm et Semi-Fredholm. La fameuse conjencture de Kaplansky sur la préservation de l'inversibilité (voir la section 2.1).

Problème III: Il vise les applications linéaires qui préservent certaines relations:

Soit $\mathcal{R}$ une relation ou une relation d'équivalence sur $\mathcal{A}$. Caractérisez les applications linéaires ϕ de $\mathcal{A}$ sur lui-même qui satisfont

$$\phi(a)\mathcal{R}\phi(b) \text{ si } a\mathcal{R}b \quad (a, b \in \mathcal{A})$$

ou

$$\phi(a)\mathcal{R}\phi(b) \text{ si, et seulement si, } a\mathcal{R}b \quad (a, b \in \mathcal{A}).$$

Dans le premier cas, nous disons que ϕ préserve la relation $\mathcal{R}$ dans un sens (ou, plus simplement, que ϕ préserve la relation $\mathcal{R}$) alors que dans le deuxième cas, nous disons que ϕ préserve $\mathcal{R}$ dans les deux sens.

Un problème qu'il faut certainement mentionner ici est celui de la préservation de la commutativité, c'est-à-dire les applications ϕ de $\mathcal{B}(X)$ sur lui-même qui satisfont

$$\phi(A)\phi(B) = \phi(B)\phi(A) \text{ si et seulement si } AB = BA \quad (A, B \in \mathcal{B}(X)).$$

Le résultat de base concernant la préservation de la commutativité sur les algèbres des opérateurs est dû à Omladič [93].

Problème IV: Il caractérise les applications linéaires qui commutent avec certaines transformations:

Étant donné une transformation $F : \mathcal{A} \to \mathcal{A}$. Caractérisez les applications linéaires ϕ de $\mathcal{A}$ sur lui-même qui satisfont

$$F(\phi(a)) = \phi(F(a)) \text{ pour tout } a \in \mathcal{A}.$$

A titre d'exemple pour cette catégorie, on peut considérer les applications ϕ de $\mathcal{B}(H)$ sur lui-même qui commutent avec la transformation λ-Aluthge qui a été étudier par Chabbabi et Mbekhta dans [42–44].

Il existe également d'autres motivations pour l'étude des LPP. Par exemple, elle se présente naturellement lorsque l'on considère les problèmes inverses de certains résultats de base de la théorie des matrices. Supposons qu'un opérateur linéaire ϕ sur $\mathcal{M}_n(\mathbb{C})$ est défini par $\phi(A) = MAN$ ou $\phi(A) = MA^tN$ pour certaines matrices inversibles $M, N \in \mathcal{M}_n(\mathbb{C})$. Il est donc clair que ϕ préserve le rang d'une matrice. Il est surprenant que ces applications ϕ soient les seuls opérateurs linéaires sur les $\mathcal{M}_n(\mathbb{C})$ qui préservent le rang (voir, [86]). Pour un autre exemple, considérons l'application linéaire ϕ sur $\mathcal{M}_n(\mathbb{C})$ défini par $\phi(A) = UAU^*$ ou $\phi(A) = UA^tU^*$ pour une matrice unitaire U. Alors clairement ϕ préserve les valeurs propres, le déterminant, la norme spectrale, le groupe unitaire, les matrices hermitiennes, les matrices normales, l'image numérique, etc. La question intéressante est de savoir si l'une de ces propriétés est suffisamment forte pour entrainer que ϕ soit de la forme $\phi(A) = UAU^*$ ou $\phi(A) = UA^tU^*$ pour une certaine matrice unitaire U. Pour répondre à cette question, il est évident qu'il faut étudier plusieur différentes LPP. Dans certains cas, la réponse a un problème de préservation suggère un outil pratique pour simplifier certains autres problèmes mathématiques. De plus, l'étude des LPP sur un certain

sujet permet de mieux comprendre le sujet considéré et peut également apparaître comme des cas particuliers de certaines questions plus générales, voir [91], et donc leur étude mérite certainement une attention.

Cette thèse est essentiellement le contenu des quatre articles [15,16,25,57] et d'un article soumis [17]. Elle est composée de sept chapitres :

Dans le chapitre 1, nous rappelons quelques propriétés élémentaires relatives aux algèbres de Banach, aux homomorphismes, à la théorie spectrale classique et locale et aux opérateurs de rang 1, dont nous aurons besoin dans la suite. Nous terminons ce chapitre par une présentation détaillée des notions de pseudo spectre et du spectre conditioné qui sont une généralisation du spectre.

Le chapitre 2 est consacré à quelques reponses partielles à la conjecture de Kaplansky. Une chronologie sur la préservation du spectre, spectre local et quelques ensembles associés seront également cités.

Dans le chapitre 3, nous déterminons la structure de toutes les applications $\phi : \mathcal{B}(X) \to \mathcal{B}(X)$ non nécessairement supposée linéaires ni surjectives satisfaisant

- ▷ $\{\phi(A)\phi(B)\phi(A)\}' = \{ABA\}'$,
- ▷ $\{\phi(A)\phi(B) + \phi(B)\phi(A)\}' = \{AB + BA\}'$,

pour tout A, $B \in \mathcal{B}(X)$. Ce sont essentiellement les résultats de notre article [15].

Dans le chapitre 4, nous présentons les résultats de notre article [57], dans lequel nous décrivons la forme des applications additives qui compressent ou élargissent le pseudospectre d'un élément dans $\mathcal{B}(X)$, où X désigne un espace de Banach complexe. On obtient aussi la forme de toutes les applications préservant le pseudospectre des différents produits généralisés.

Dans le chapitre 5, nous avons déterminé explicitement la forme des applications surjectives qui préservent le spectre conditioné du skew produit triple des opérateurs. Plus précisément, nous avons caractérisé les applications surjectives $\phi : \mathcal{B}(H) \to \mathcal{B}(H)$ qui satisfont:

$$\sigma_\epsilon(\phi(A)\phi(B)^*\phi(A)) = \sigma_\epsilon(AB^*A) \text{ pour tout } A, B \in \mathcal{B}(H).$$

Notons que ce chapitre a fait l'objet de notre article [16].

Dans le chapitre 6, nous présentons les résultats de notre article [25], dans lequel nous caractérisons la forme des applications surjectives de $\mathcal{B}(X)$ sur lui-même satisfaisant

$$\dim F(\phi(A_1) \cdots * \phi(A_k)) = \dim F(A_1 * \cdots * A_k)$$

pour tout $A_1, \cdots, A_k \in \mathcal{B}(X)$, où $F(.)$ représente l'ensemble des points fixes.

Le dernier chapitre est consacré à l'étude de toutes les applications surjectives unitaires préservant la dimension du sous-espace spectral local associé au singleton $\{1\}$ du produit des opérateurs. C'est le contenu de l'article soumis [17]

Nous terminons par une bibliographie regroupant les principaux livres ainsi que les principaux articles d'éminents mathématiciens qui ont contribué au rayonnement de ce très important domaine des mathématiques.

Introduction

One of the most active and fertile subjects in matrix theory during the past one hundred years is the *linear preserver problems* (LPP). These problems are related to the characterization of all linear applications on matrix algebras, linear bounded operator algebras or, more generally, Banach algebras that leave certain functions, subsets or relations invariant. LPP go back a long time probably to Frobenius in 1897 who characterized linear maps that preserve the determinant. Since then, a great deal of effort has been devoted to the study of this type of question. Detailed and informative survey articles on the subject can be found in [37, 83, 84, 89, 91] and the references therein. New problems have; however, emerged when trying to solve linear preserving problems that have significance in infinite dimension but are trivially true in finite dimensions. Also, problems where we weaken the conditions on the application in question (i.e additive instead of linear). Li and Tsing, in [83] gave a brief overview of the LPPs and described four general types of them.

Problem I: It concerns those linear operators preserving certain functions.

Let F be a (scalar-valued, vector-valued, or set-valued) function on $\mathcal{A}$. Characterize those linear operators ϕ on $\mathcal{A}$ that satisfy

$$F(\phi(a)) = F(a) \text{ for all } a \in \mathcal{A}.$$

Probably the first problem of this kind was considered by Frobenius [60], who considered this type of problems for $\mathcal{A} = \mathcal{M}_n(\mathbb{C})$ and $F(.) = \det(.)$. The problems of the spectrum preservers and the norm could be included in this category of LPPs.

Note also that new contributions of this type of preservation problem have been recently studied by Bourhim, Ransford, Mbekhta, González, Bračič, Müller and Costara concerning linear applications preserving local spectral quantities (local spectrum, local spectral radius, local spectral subspace, etc) see [34, 36, 49, 62].

Problem II: It concerns those linear operators preserving certain subsets.

Let $\mathcal{S}$ be a given subset of $\mathcal{A}$. Characterize those linear operators ϕ on $\mathcal{A}$ that satisfy

$$\phi(\mathcal{S}) \subset \mathcal{S} \text{ or } \phi(\mathcal{S}) = \mathcal{S}.$$

Several important subsets have been studied in connection with Problem II. For instance, we mention the sets of idempotent, nilpotent, Fredholm and Semi-Fredholm operators. When giving examples for this type of LPPs, one should not forget to mention the famous Kaplansky's problem

on invertibility preservers although it concerns general Banach algebras not merely operator algebras, which is discussed in section 2.1.

Problem III: It targets those linear operators preserving certain relations.

Let $\mathcal{R}$ be a relation or an equivalence relation on $\mathcal{A}$. Characterize those linear operators ϕ on $\mathcal{A}$ which satisfy

$$\phi(a)\mathcal{R}\phi(b) \text{ whenever } a\mathcal{R}b \quad (a, b \in \mathcal{A})$$

or

$$\phi(a)\mathcal{R}\phi(b) \text{ if and only if } a\mathcal{R}b \quad (a, b \in \mathcal{A})$$

In the first case we say that ϕ preserves the relation $\mathcal{R}$ in one direction (or, more simply, that ϕ preserves the relation $\mathcal{R}$) while in the second case we say that ϕ preserves $\mathcal{R}$ in both directions.

A problem which should be certainly mentioned here is that of preserving commutativity, i.e applications ϕ on $\mathcal{B}(X)$ which satisfy

$$\phi(A)\phi(B) = \phi(B)\phi(A) \text{ if and only if } AB = BA \quad (A, B \in \mathcal{B}(X)).$$

The basic result concerning commutativity preserving on operator algebras is due to Omladič [93].

Problem IV: It characterizes those linear applications commuting with certain transformations.

Given a transformation $F : \mathcal{A} \to \mathcal{A}$, characterize those linear operators ϕ on $\mathcal{A}$ that satisfy

$$F(\phi(a)) = \phi(F(a)) \text{ for all } a \in \mathcal{A}.$$

As an example for this category one can consider the applications ϕ on $\mathcal{B}(H)$ commuting with the λ-Aluthge transformation considered by Chabbabi and Mbekhta in [42–44].

There are also other motivations for the study of the LPPs. For example, it arises naturally when one considers the converse problems of some basic results in matrix theory. Suppose that a linear operator ϕ on $\mathcal{M}_n(\mathbb{C})$ is defined by $\phi(A) = MAN$ or $\phi(A) = MA^tN$ for some invertible matrices $M, N \in \mathcal{M}_n(\mathbb{C})$. Then it is clear that ϕ will preserve the rank of a matrix. It is somewhat surprising that such ϕ's are the only linear operators on $\mathcal{M}_n(\mathbb{C})$ that preserve rank (see, [86]). For another example, consider a linear operator ϕ on $\mathcal{M}_n(\mathbb{C})$ defined by $\phi(A) = UAU^*$ or $\phi(A) = UA^tU^*$ for some unitary matrix U. Then clearly ϕ preserves eigenvalues, the determinant, the spectral norm, the unitary group, hermitian matrices, normal matrices, the numerical range, etc. The interesting question is whether any one of these properties is strong enough to force ϕ to be of the form $\phi(A) = UAU^*$ or $\phi(A) = UA^tU^*$ for some unitary matrix U. To answer this question one clearly has to study various LPPs. In some cases, the solution of the LPP would suggest a practical tool for simplifying some other mathematical problems. Also the aim of studying LPPs on a certain topic is better for understanding the subject under consideration and may also appear as particular cases of some more general questions, see [91], and hence the study of them certainly deserves attention.

The content of this thesis can be found mainly in the four publications [15, 16, 25, 57] and the submitted paper [17]. It has seven chapters:

In chapter 1, we recall some elementary properties relating to Banach Algebras, homomorphisms, classical and local spectral theory and rank one operators, which we will need in the sequel. We end this chapter with a detailed presentation on some generalization of the spectrum, especially, the notions of the pseudo spectrum and the condition spectrum.

Chapter 2 is devoted to some partial results of Kaplansky's conjecture. Also a chronology on the preservation of spectra, local spectra, and associated sets will be cited.

In chapter 3, we determine the structure of all maps $\phi : \mathcal{B}(X) \to \mathcal{B}(X)$ (neither supposed linear nor surjective) which satisfy

- $\{\phi(A)\phi(B)\phi(A)\}' = \{ABA\}'$,
- $\{\phi(A)\phi(B) + \phi(B)\phi(A)\}' = \{AB + BA\}'$,

for all A, $B \in \mathcal{B}(X)$. These are basically the results of our article [15].

In chapter 4, we present the results of our article [57], in which we describe the form of additive maps compressing (expanding) the pseudospectrum in $\mathcal{B}(X)$, where X denotes a complex Banach space. The forms of all maps preserving the pseudospectrum of different Generalized products are obtained.

In chapter 5, we determined explicitly the form of surjective maps which preserve the condition spectrum of Jordan skew triple product of operators. More precisely, we have characterized the surjective maps $\phi : \mathcal{B}(H) \to \mathcal{B}(H)$ satisfying:

$$\sigma_\epsilon(\phi(A)\phi(B)^*\phi(A)) = \sigma_\epsilon(AB^*A) \text{ for all } A, B \in \mathcal{B}(H).$$

Note that this chapter was the subject of the article [16].

In Chapter 6, we present the results of our article [25], in which we characterize the form of surjective maps from $\mathcal{B}(X)$ into itself satisfying

$$\dim F(\phi(A_1) * \cdots * \phi(A_k)) = \dim F(A_1 * \cdots * A_k)$$

for all $A_1, \cdots, A_k \in \mathcal{B}(X)$, where $F(.)$ stands for the fixed point set.

The last chapter is devoted to the study of all unital surjective maps preserving the dimension of the local spectral subspaces associated with the singleton $\{1\}$ of the product of operators. This was

the content of our submitted article [17]

We end with a bibliography bringing together the main books as well as the main articles of eminent mathematicians who have an upper hand in this field.

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Chapter 1

Preliminaries

1.1 Banach algebras

In this section, we collect some basic definitions and results from the theory of Banach algebras, which may be found, for instance, in [5, 11, 81, 91, 92].

Definition 1.1.1. *A complex algebra $\mathcal{A}$ is a complex linear space $\mathcal{A}$ together with a multiplication mapping $(x, y) \to xy$ from $\mathcal{A} \times \mathcal{A}$ into $\mathcal{A}$ which satisfies the following conditions (for all $x, y, z \in \mathcal{A}, \alpha \in \mathbb{C}$):*

1. $(xy)z = x(yz)$;
2. $x(y + z) = xy + xz, (x + y)z = xz + yz$;
3. $(\alpha x)y = \alpha(xy) = x(\alpha y)$.

If moreover $\mathcal{A}$ is a Banach space for a norm $\| \, . \, \|$ and satisfies the norm inequality $\| xy \| \leq \| x \| \, . \, \| y \|$, for all $x, y \in \mathcal{A}$, we say that $\mathcal{A}$ is a complex **Banach Algebra**.

A Banach algebra $\mathcal{A}$ is said to be unital if there exists an element $e \in \mathcal{A}$ such that $a.e = e.a = a$ for all $a \in \mathcal{A}$, e is called unit of $\mathcal{A}$. Unless otherwise stated, all Banach algebras considered thereafter are unital whose norm of the unit is 1. Which is justified by the following remark:

Remark 1.1.2. *1. For an arbitrary complex Banach algebra A without identity, we consider the unitization $\mathcal{A}_1 := \mathcal{A} \bigoplus \mathbb{C}$ of $\mathcal{A}$, defined as the complex vector space $\mathcal{A} \times \mathbb{C}$ with multiplication given by*

$$(u, \lambda)(v, \mu) := (uv + \lambda v + \mu u, \lambda\mu) \text{ for all } u, v \in \mathcal{A} \text{ and } \lambda, \mu \in \mathbb{C},$$

and with the canonical norm given by

$$\| (u, \lambda) \| := \| u \| + | \lambda | \quad \text{for all } u \in \mathcal{A} \text{ and } \lambda \in \mathbb{C}.$$

Note that $\mathcal{A}$ is isometrically embedded in $\mathcal{A}_1$, and that $\mathcal{A}_1$ has an identity element, namely $e = (0, 1)$;

2. *If $\mathcal{A}$ has a unit (which is unique), denoted by e, we can always suppose that $\| e \| = 1$ because otherwise we can replace $\| . \|$ by an equivalent norm $|||\ .\ |||$ satisfying $||| e ||| = 1$ and $||| xy ||| \leq ||| x ||| \ . \ ||| y |||$ for all $x, y \in \mathcal{A}$;*

3. *In the definition of a Banach algebra, instead of the condition $\| xy \| \leq \| x \| \ . \ \| y \|$ it is enough to suppose that the left multiplications $x \to xy$ and right multiplications $x \to yx$ are continuous, because with this hypothesis there exists an equivalent norm satisfying the standard norm inequality.*

Example 1.1.3. *There are many examples of Banach algebras that appear naturally in functional analysis.*

1. *For the purpose of this monograph the most important example of a complex Banach algebra is the algebra $\mathcal{B}(X)$ of all (bounded linear) operators acting on a complex Banach space X, $\dim X \geq 1$, with naturally defined algebraic operations and with the operator-norm $\| T \| = \sup\{\| Tx \| : x \in X, \| x \| = 1\}$. The unit element in $\mathcal{B}(X)$ is the identity operator I defined by $Ix = x (x \in X)$. In particular, if $\dim X = n < \infty$, then $\mathcal{B}(X)$ can be identified with the algebra of all $n \times n$ complex matrices.*

2. *Let K be a non-empty compact space. Then the algebra $\mathcal{C}(K)$ of all continuous complex-valued functions on K with the sup-norm $\| f \| = \sup\{| f(z) | : z \in K\}$ is a Banach algebra.*

3. *Let $\mathbb{D}$ be the open unit disc in the complex plane, and let $\mathbf{H}^\infty$ denotes the algebra of all functions that are analytic and bounded on $\mathbb{D}$.*
   The disc algebra $\mathcal{A}(\mathbb{D})$ is the algebra of all functions that are continuous on $\overline{\mathbb{D}}$ and analytic on $\mathbb{D}$. Both $\mathbf{H}^\infty$ and $\mathcal{A}(\mathbb{D})$ with the sup-norm are important examples of Banach algebras.

We know from elementary linear algebra that the adjoint plays a significant role in understanding linear maps. For example, only normal operators are orthogonally diagonalizable. So, it may not be so surprising that we will need an analogue of an adjoint to find well behaved algebras.

Definition 1.1.4. *An **involution** on an algebra $\mathcal{A}$ is a map $* : a \to a^*$ of $\mathcal{A}$ onto itself satisfies*

1. $(a^*)^* = a$;

2. $(ab)^* = b^* a^*$;

3. $(a + \lambda b)^* = a^* + \overline{\lambda} b^*$.

for all $a, b \in \mathcal{A}$ and $\lambda \in \mathbb{C}$.

Of course the proto-typical example of an involution on a Banach algebra is the adjoint operation on $\mathcal{B}(H)$. Another important example is the complex conjugation in $\mathbf{C}(K) : f^*(x) := \overline{f(x)}$.

Definition 1.1.5. *A $*$-algebra is an algebra $\mathcal{A}$ together with an involution. A Banach $*$-algebra is a Banach algebra $\mathcal{A}$ together with an involution.*

In most natural examples, one also has that the involution is **isometric**, that is, $\| a^* \| = \| a \|$ for all $a \in \mathcal{A}$. Some authors include this isometric property in the definition of a Banach $*$-algebra.

The below definition was introduced in 1943 by Gelfand and Naimark, that is the notion of $\mathbf{C}^*$-algebra.

Definition 1.1.6. *A Banach $*$-algebra is called a $\mathbf{C}^*$-algebra if $\| a^*a \| = \| a \|^2$ for all $a \in \mathcal{A}$.*

A special type of $\mathbf{C}^*$-algebra is the following notion:

Definition 1.1.7. *A von Neumann algebra or $\mathbf{W}^*$-algebra is a *-algebra of bounded operators on an Hilbert space that is closed in the weak operator topology and contains the identity operator.*

Example 1.1.8. *A basic example of von Neumann algebra, is $\mathcal{B}(H)$ the algebra of all bounded operators on an Hilbert space H. It is non-commutative if the Hilbert space has dimension at least 2.*

The following definition is needed to define the **semi-simple** Banach algebra.

Definition 1.1.9. *Let $\mathcal{A}$ be a Banach algebra. The Jacobson radical noted $\boldsymbol{Rad}(\mathcal{A})$ is defined to be the intersection of all maximal right (or left) ideals of $\mathcal{A}$.*
*The algebra $\mathcal{A}$ is **semi-simple** if $\boldsymbol{Rad}(\mathcal{A}) = \{0\}$.*

Example 1.1.10. *1. Let X be a Banach space, then $\mathcal{B}(X)$ is semi-simple.*

2. Let $\mathcal{A}$ be a Banach algebra, Then $\mathcal{A}/\boldsymbol{Rad}(\mathcal{A})$ is a semi-simple Banach algebra.

1.2 Basic spectral theory

The spectral theory is an important branch of functional analysis. It has many applications in many areas of mathematics and physics, including matrix theory, operator theory, complex analysis, and quantum mechanics. In what follows, we collect some basic results that we apply several times in our thesis.

An element a in an algebra $\mathcal{A}$ is said to be left invertible if there exists $b \in \mathcal{A}$ such that $ba = 1$, while it is said right invertible if there exists $b \in \mathcal{A}$ such that $ab = 1$. An invertible element of $\mathcal{A}$ is a right and left invertible element of $\mathcal{A}$. We denote by $Inv(\mathcal{A})$, $Inv^l(\mathcal{A})$ and $Inv^r(\mathcal{A})$ the sets of all invertible, left invertible and right invertible elements of $\mathcal{A}$ respectively. Unlike the inverse who is unique, left inverse and right inverse are not. Unless stated otherwise, $\mathcal{A}$ denotes a complex Banach algebra.

Definition 1.2.1. *Let $a \in \mathcal{A}$, the **left spectrum**, the **right spectrum** and **the spectrum** of a are defined respectively as follow:*

$$\sigma^l(a) = \{\lambda \in \mathbb{C} : a - \lambda \notin Inv^l(\mathcal{A})\};$$

$$\sigma^r(a) = \{\lambda \in \mathbb{C} : a - \lambda \notin Inv^r(\mathcal{A})\};\ and$$

$$\sigma(a) = \{\lambda \in \mathbb{C} : a - \lambda \notin Inv(\mathcal{A})\}.$$

The complement of $\sigma(a)$, noted by $\rho(a)$, is the **resolvent set** of a; it consists of all $\lambda \in \mathbb{C}$ for which $(\lambda e - a)^{-1}$ exists. The mapping:

$$\begin{aligned} R_a : \rho(a) &\to \mathcal{A} \\ \lambda &\mapsto (a - \lambda e)^{-1} \end{aligned}$$

is called the resolvent map of a.

Theorem 1.2.2. *Let $T \in \mathcal{B}(X)$. The resolvent map of T is analytic on $\rho(T)$ vanishing at infinity and has the following property*

$$\frac{1}{d(\lambda)} \leq \| R_T(\lambda) \| \quad \textit{for every } \lambda \in \rho(T)$$

where $d(\lambda)$ is the distance from λ to the spectrum $\sigma(T)$ of T. Therefore $\| R_T(\lambda) \| \to \infty$ as $d(\lambda) \to 0$.

The **spectral radius**, denoted by $r(a)$, is defined as the supremum of the absolute values of the elements of $\sigma(a)$.

Proposition 1.2.3. *([98, Theorem 10.13].) Let $\mathcal{A}$ be a Banach algebra and $a \in \mathcal{A}$, then*

1. *the spectrum $\sigma(a)$ of a is a nonempty compact set in $\mathbb{C}$, and*
2. *the spectral radius $r(a)$ satisfies, $r(a) = \lim_{n\to\infty} \| a^n \|^{\frac{1}{n}}$ (Beurling formula).*

Remark 1.2.4. *Whether an element of $\mathcal{A}$ is or is not invertible in $\mathcal{A}$ is a purely algebraic property. The spectrum and the spectral radius of an $a \in \mathcal{A}$ are thus defined in terms of the algebraic structure of $\mathcal{A}$, regardless of any metric (or topological) considerations. On the other hand, $\lim \| a^n \|^{\frac{1}{n}}$ depends obviously on metric properties of $\mathcal{A}$. This is one of the remarkable features of the spectral radius formula: It asserts the equality of certain quantities which arise in entirely different ways.*

In Proposition 1.2.3, we have seen that the spectra of an element in a Banach algebra is nonempty set. This leads to an easy characterization of those Banach algebras that are division algebras.

Theorem 1.2.5. ***(Gelfand-Mazur)*** *If $\mathcal{A}$ is a Banach algebra in which every nonzero element is invertible, then $\mathcal{A}$ is (isometrically isomorphic to) the complex field.*

An element $a \in \mathcal{A}$ is called idempotent if $a^2 = a$, nilpotent if there is $n \in \mathbb{N}$ such that $a^n = 0$, and quasi-nilpotent if $r(a) = 0$. The following theorem is one of the important spectral characterization of the radical, which was given by Zemánek [109, Theorem 5.3.1].

Theorem 1.2.6. *Let $\mathcal{A}$ and $\mathcal{B}$ be Banach algebras. Then the following properties are equivalent:*

1. *a is in the Jacobson radical of $\mathcal{A}$;*

2. $\sigma(a+x) = \sigma(x)$, *for all* $x \in \mathcal{A}$;

3. $r(a+x) = 0$, *for all quasi-nilpotent elements* $x \in \mathcal{A}$.

For the proof of this theorem and more information about the Jacobson radical the interested reader can see [11, 109].

Now, let us give some definitions about a mapping between two Banach algebra.

Definition 1.2.7. *Let* $\mathcal{A}$ *and* $\mathcal{B}$ *be two Banach algebra. Let* $\phi : \mathcal{A} \to \mathcal{B}$. *We say that:*

- ϕ *is* ***unital*** *if* $\phi(1_{\mathcal{A}}) = 1_{\mathcal{B}}$;
- ϕ ***preserve the spectrum*** *if for every* $a \in \mathcal{A}$: $\sigma(\phi(a)) = \sigma(a)$;
- ϕ ***compresses the spectrum*** *if for every* $a \in \mathcal{A}$: $\sigma(\phi(a)) \subset \sigma(a)$;
- ϕ ***expands the spectrum*** *if for every* $a \in \mathcal{A}$: $\sigma(\phi(a)) \supset \sigma(a)$.

In the rest of this section, we will consider the case when $\mathcal{A} = \mathcal{B}(X)$. We say that λ is an eigenvalue if $\ker(T - \lambda I) \neq 0$. $\ker(T - \lambda I)$ is called the eigenspace associated with λ. Recall that, in finite dimensions the spectrum is just the set of eigenvalues.

Definition 1.2.8. *The* ***point spectrum*** $\sigma_p(T)$ *of* T *is defined as the set of the eigenvalues of* T.

The operator $T - \lambda I$ can fail to be invertible in several ways. For example, if λ is an eigenvalue, then the operator is not one-to-one. Even if the operator is one-to-one, its range may not be all of X.

Example 1.2.9. *Let* $\mathbf{H} = L^2(]0,1[)$ *and* $f \in C^0(]0,1[)$. *Let* T_f *be the operator of multiplication by* f. *Then one has*

$$\sigma(T_f) = Imf =: \{\lambda \in \mathbb{C} : \exists x \in [0,1] \text{ with } f(x) = \lambda\}.$$

Indeed, it is easy to see that if $\lambda \notin Imf$, *then* $T_{(f-\lambda)^{-1}}$ *is a continuous inverse of* $T_f - \lambda$. *On the other side, if* $\lambda = f(x_0)$ *for some* $x_0 \in]0,1[$ *then we have* $(T_f - \lambda)u_n \to 0$ *and* $\| u_n \| = 1$ *for* $u_n = \frac{1}{\sqrt{n}}\chi(\frac{x-x_0}{n})$, *where* χ *is a* C_0^∞ *function such that* $\| \chi \| = 1$. *This shows that* $f(]0,1[) \subset \sigma(T_f)$ *and we can conclude by considering the closure.*

Note that the point spectrum is not necessarily closed. Note also that one can have a strict inclusion of the point spectrum in the spectrum as can be observed in the following example:

Example 1.2.10. *Let us consider* $E = \ell^2(\mathbb{N})$ *and let* T *be the shift operator defined by:*

$$(Tu)_0 = 0, (Tu)_n = u_{n-1}, n > 0,$$

where $u = (u_0, u_1, \dots, u_n, \dots) \in \ell^2(\mathbb{N})$. *Then it is easy to see that* T *is injective (so* 0 *is not an eigenvalue) and is not surjective (so* 0 *is in the spectrum of* T*).*

1.3 Homomorphisms

Among the important mappings from one Banach algebra into another are the homomorphisms. These are linear mappings that are also multiplicative. Of particular interest is the case in which the range is the simplest of all Banach algebras, namely, $\mathbb{C}$ itself.

Definition 1.3.1. *Suppose $\mathcal{A}$ is a complex algebra and ϕ is a linear functional on $\mathcal{A}$ which is not identically 0. Then:*
*ϕ is an **homomorphism** if $\phi(xy) = \phi(x)\phi(y)$ for all $x, y \in \mathcal{A}$;*
*ϕ is an **anti-homomorphism** if $\phi(xy) = \phi(y)\phi(x)$ for all $x, y \in \mathcal{A}$;*
*ϕ is called a **Jordan homomorphism** if it preserve squares, i.e $\phi(x^2) = (\phi(x))^2$ for all $x \in \mathcal{A}$. Equivalently, the map ϕ is a Jordan homomorphism if and only if it preserve the Jordan structure. More explicitly, this means that $\phi(xy + yx) = \phi(x)\phi(y) + \phi(y)\phi(x)$ for all $x, y \in \mathcal{A}$.*

A bijective homomorphism (resp. anti-homomorphism) is called an isomorphism (resp. anti-isomorphism).

Example 1.3.2.

- *Homomorphism and anti-homomorphism are Jordan homomorphism.*
- *Let $A \in \mathcal{B}(X)$, The application $\phi : \mathcal{B}(X) \to \mathcal{B}(X)$ defined by $\phi(T) = ATA^{-1}$ for every $T \in \mathcal{B}(X)$ is a Jordan homomorphism.*

The following result summarizes some well known properties of Jordan homomorphisms. Their proofs can be found in [65].

Proposition 1.3.3. *If $\phi : \mathcal{A} \to \mathcal{B}$ is a Jordan homomorphism between two unital complex algebras $\mathcal{A}$ and $\mathcal{B}$, then the following statements hold:*

1. *For every $a, b \in \mathcal{A}$, we have $\phi(aba) = \phi(a)\phi(b)\phi(a)$;*
2. *For every positive integer n, we have $\phi(a^n) = (\phi(a))^n$ for all $a \in \mathcal{A}$;*
3. *For every $a, b, c \in \mathcal{A}$, we have $\phi(abc + cba) = \phi(a)\phi(b)\phi(c) + \phi(c)\phi(b)\phi(a)$.*

Proposition 1.3.4. *If ϕ is an homomorphism on a complex algebra $\mathcal{A}$ with unit e, then $\phi(e) = 1$, and $\phi(x) \neq 0$ for every invertible element $x \in \mathcal{A}$.*

Proof. For some $y \in \mathcal{A}$, $\phi(y) \neq 0$. Since $\phi(y) = \phi(ye) = \phi(y)\phi(e)$, it follows that $\phi(e) = 1$. If x is invertible, then $\phi(x)\phi(x^{-1}) = \phi(xx^{-1}) = \phi(e) = 1$, so that $\phi(x) \neq 0$. □

Parts (1) and (3) of the following theorem are perhaps the most widely used facts in the theory of Banach algebras; in particular, (3) implies that all homomorphisms of Banach algebras are continuous.

Theorem 1.3.5. *Suppose $\mathcal{A}$ is a Banach algebra, $x \in \mathcal{A}, \| x \| < 1$. Then:*

1. $e - x$ *is invertible;*

2. $\| (e-x)^{-1} - e - x \| \leq \frac{\|x\|^2}{\|x\|}$;

3. $| \phi(x) | < 1$.

Proof. Since $\| x^n \| \leq \| x \|^n$ and $\| x \| < 1$, the elements $s_n = e + x + x^2 + \cdots + x^n$ from a Cauchy sequence in $\mathcal{A}$. Since $\mathcal{A}$ is complete, there exists $s \in \mathcal{A}$ such that $\lim s_n = s$. Since $\lim x^n = 0$ and $s_n(e-x) = e - x^{n+1} = (e-x)s_n$, the continuity of multiplication implies that s is the inverse of $e - x$. Next we have

$$\| s - e - x \| = \| x^2 + x^3 + \cdots \| \leq \sum_{n=2}^{\infty} \| x \|^n = \frac{\| x \|^2}{1 - \| x \|}.$$

Finally, suppose $\lambda \in \mathbb{C}$, $| \lambda | \geq 1$. By 1. $e - \lambda^{-1}x$ is invertible. By Proposition 1.3.4,

$$1 - \lambda^{-1}\phi(x) = \phi(e - \lambda^{-1}x) \neq 0.$$

Hence $\phi(x) \neq \lambda$. This completes the proof. □

1.4 Local spectral theory

In this section, we introduce a generalization of the spectrum and the resolvent set due to Dunford (see [12]). Let X be a complex Banach space and $\mathcal{B}(X)$ be the algebra of all bounded operators on X. For an operator $T \in \mathcal{B}(X)$. Recall that, the resolvent map R_T is an operator valued analytic function on the resolvent set $\rho(T)$ of T. For a fixed element $x \in X$ the vector valued function $R_{T,x}$ defined on $\rho(T)$ by

$$R_{T,x}(\lambda) = R_T(\lambda)x = (T - \lambda I)^{-1}x$$

is analytic function on $\rho(T)$ and satisfying the following equation

$$(T - \lambda I)R_{T,x}(\lambda) = x \text{ for every } \lambda \in \rho(T).$$

In many cases the function $R_{T,x}$ can be extended to be analytic on an open set properly containing $\rho(T)$. We will call a vector valued analytic function F an analytic extension of $R_{T,x}$ if the domain $D(F)$ of F is containing $\rho(T)$ and

$$(T - \lambda I)F(\lambda) = x \text{ for every } \lambda \in D(F).$$

We now encounter the possibility that there may be many extensions of $R_{T,x}$ and they may not agree on their common domain. This leads to the following definition:

Definition 1.4.1. *Let $T \in \mathcal{B}(X)$, if all extensions do agree on their common domain for each $x \in X$ we say that the operator T has* ***the single valued extension property (SVEP)****.*

Therefore, for such operator T and for every $x \in X$, the function $R_{T,x}$ has a maximal extension called the maximal single valued extension of $R_{T,x}$ and denoted by $\tilde{x}(.)$. Thus $\tilde{x}(.)$ is an analytic vector valued function such that

$$(T - \lambda I)\tilde{x}(\lambda) = x \text{ for every } \lambda \in D(\tilde{x}).$$

Example 1.4.2. *1. Every operator $T \in \mathcal{B}(X)$ which has empty interior of its point spectrum has SVEP.*

Let F be a vector valued analytic function such that

$$(T - \lambda I)F(\lambda) = 0 \text{ for every } \lambda \in D(F).$$

Then

$$TF(\lambda) = \lambda F(\lambda) \text{ for every } \lambda \in D(F).$$

So, F must be identically zero otherwise there is $\lambda \in D(F)$ and $r > 0$ such that $B_r = \{\mu \in \mathbb{C} :\mid \lambda - \mu \mid < r\} \subset D(F)$ and $F(\mu) \neq 0$ for every $\mu \in B_r$. Thus, $B_r \subset \sigma_p(T)$, a contradiction. Then, $F \equiv 0$ and so, T has SVEP.

More than that, every finite rank operator has the SVEP.

2. ***An operator without SVEP.*** *Let T be the adjoint of the unilateral weighted shift operator with the weight $w_n = 1$ for every $n \in \mathbb{N}$. i.e:*

$$Te_n = \begin{cases} 0 & \text{if } n = 0 \\ e_n & \text{if } n > 0 \end{cases}$$

Clearly, we have $\parallel T \parallel = 1$ and $\sigma(T) = \{\lambda \in \mathbb{C} :\mid \lambda \mid \leq 1\}$ then $\rho(T) = \{\lambda \in \mathbb{C} :\mid \lambda \mid > 1\}$. Therefore, for every $\lambda \in \rho(T)$ we have $\parallel \frac{T}{\lambda} \parallel < 1$ and so,

$$(T - \lambda I)^{-1} = \frac{1}{\lambda}(\frac{T}{\lambda} - I)^{-1} = \frac{1}{\lambda}\sum_{n=0}^{\infty}(\frac{T}{\lambda})^n = -\sum_{n=0}^{\infty}\frac{T^n}{\lambda^{n+1}}.$$

Hence for every $\lambda \in \rho(T)$,

$$R_{T,e_0} = (T - \lambda I)^{-1}e_0 = -\sum_{n=0}^{\infty}\frac{T^n e_0}{\lambda^{n+1}} = -\frac{e_0}{\lambda} \text{ because } T^n e_0 = 0 \text{ for every } n \geq 0.$$

We will now exhibit two different analytic extensions of R_{T,e_0}. First let $F(\lambda) = -\frac{e_0}{\lambda}$ for $\lambda \neq 0$; then $(T - \lambda I)F(\lambda) = e_0$ for every $\lambda \neq 0$. And so, F is clearly an analytic extension of R_{T,e_0}. Second, set

$$G(\lambda) = \begin{cases} -\dfrac{e_0}{\lambda} & \text{if } \mid \lambda \mid > 1 \\ \displaystyle\sum_{n=0}^{\infty}\lambda^n e_{n+1} & \text{if } \mid \lambda \mid < 1. \end{cases}$$

Clearly, that G has the right form for $\lambda \in \mathbb{C}, \mid \lambda \mid > 1$. For $\lambda \in \mathbb{C}, \mid \lambda \mid < 1$ we have

$$\begin{aligned}(T-\lambda I)G(\lambda) &= (T-\lambda I)\sum_{n=0}^{\infty}\lambda^n e_{n+1} \\ &= \sum_{n=0}^{\infty}\lambda^n T e_{n+1} - \sum_{n=0}^{\infty}\lambda^{n+1}e_{n+1} \\ &= \sum_{n=0}^{\infty}\lambda^n e_n - \sum_{n=0}^{\infty}\lambda^{n+1}e_{n+1} \\ &= e_0.\end{aligned}$$

Thus G is also an analytic extension of R_{T,e_0}. However the two analytic extensions F and G do not coincide on $\{\lambda \in \mathbb{C} : 0 < \mid \lambda \mid < 1\}$.

More than that one can prove that the adjoint operator of every isometric non unitary operator on Hilbert space do not has the s.v.e.p (see [48]).

Definition 1.4.3. *Let $x \in X$, the **local resolvent** set of T at x, denoted by $\rho_T(x)$, is the union of all open subsets U of $\mathbb{C}$ for which there is an analytic function $f : U \longrightarrow X$ which satisfies*

$$(T-\lambda)f(\lambda) = x \text{ for all } \lambda \in U.$$

*Its complement, denoted by $\sigma_T(x)$, will be called the **local spectrum** of x.*

Evidently, $\rho_T(x)$ is open, and $\sigma_T(x)$ is closed. As already mentioned, the resolvent set $\rho(T)$ is always a subset of $\rho_T(x)$. The following example quoted from [92] shows that the local spectra of an operator can be empty.

Example 1.4.4. *Let H be an Hilbert space with the orthonormal basis $\{e_i, i \in \mathbb{N}\}$, and let S be the unilateral shift operator $(Se_i = e_{i+1})$ and let S^* be its adjoint, $S^*e_0 = 0$, $S^*e_i = e_{i-1} (i \geq 1)$. Clearly, $S^*S = I$. Let $x = \sum_{i=0}^{\infty} 2^{-i}e_i$. Then $S^*x = \frac{x}{2}$. For $\mid z \mid > \frac{1}{2}$ set $f(z) = -\sum_{i=0}^{\infty}\frac{S^{*i}x}{z^{i+1}}$. Clearly*

$$(S^*-z)f(z) = -\sum_{i=0}^{\infty}\frac{S^{*i+1}x}{z^{i+1}} + \sum_{i=0}^{\infty}\frac{S^{*i}x}{z^i} = x \quad (\mid z \mid > \frac{1}{2}).$$

Further, set $g(z) = \sum_{i=0}^{\infty} S^{i+1}xz^i$. This sum is convergent for $\mid z \mid < 1$ and

$$(S^*-z)g(z) = \sum_{i=0}^{\infty} S^i x z^i - \sum_{i=0}^{\infty} S^{i+1}xz^{i+1} = x.$$

Hence $\sigma_{S^*}(x) = \emptyset$.

Remark 1.4.5. *1. Since the set of zeros of a nonconstant analytic function is totally disconnected, $T \in \mathcal{B}(X)$ has the single valued extension property (SVEP) if and only if for every open set $U \subset \mathbb{C}$, the only analytic solution of the equation $(T-\lambda I)F(\lambda) = 0$ for $\lambda \in U$ is the zero function $F \equiv 0$.*

2. *If an operator $T \in \mathcal{B}(X)$ has the SVEP then the local resolvent of every vector $x \in X$ is the set of complex numbers λ such that there exists a vector valued analytic function F defined in an open neighborhood V of λ which verifies $(T - \mu I)F(\mu) = x$ for every $\mu \in V$.*

In the following proposition, we collect some basic consequences of the local spectrum.

Proposition 1.4.6. *([5, 81]) For an operator $A \in \mathcal{B}(X)$, vectors $x, y \in X$ and a scalar $\alpha \in \mathbb{C}$, the following statements hold.*

1. $\sigma_A(0) = \emptyset$;
2. $\sigma_A(\alpha x) = \sigma_A(x)$ *if* $\alpha \neq 0$, *and* $\sigma_{\alpha A}(x) = \alpha\sigma_A(x)$;
3. $\sigma_A(x + y) \subset \sigma_A(x) \cup \sigma_A(y)$;
4. *If* $Ax = \lambda x$ *for some* $\lambda \in \mathbb{C}$, *then* $\sigma_A(x) \subset \{\lambda\}$;
5. *If* $B \in \mathcal{B}(X)$ *commutes with* A, *then* $\sigma_A(Bx) \subset \sigma_A(x)$;
6. $\sigma_{A^n}(x) = \{\sigma_A(x)\}^n$ *for every* $n \geq 1$.

Theorem 1.4.7. *([5, Theorem 2.2]) Let $A \in \mathcal{B}(X)$, X a Banach space, $x \in X$ and U an open subset of $\mathbb{C}$. Suppose that $f : U \to X$ is an analytic function for which $(A - \mu I)f(\mu) = x$ for all $\mu \in U$. Then $U \subseteq \rho_A(f(\lambda))$ for all $\lambda \in U$. Moreover,*

$$\sigma_A(x) = \sigma_A(f(\lambda)) \text{ for all } \lambda \in U.$$

The proposition bellow shows the importance of the SVEP.

Proposition 1.4.8. *([5, 81]) Let $A \in \mathcal{B}(X)$ an operator which has the SVEP. Then we have*

1. $\sigma_A(x)$ *is empty if, and only if,* $x = 0$;
2. *If* $Ax = \lambda x$ *for some* $\lambda \in \mathbb{C}$, *then* $\sigma_A(x) = \{\lambda\}$;
3. $\sigma(A) = \bigcup_{x \in X} \sigma_A(x)$;
4. *Assume that A is additionally surjective. Then A is invertible in $\mathcal{B}(X)$. Therefore,*

$$\sigma(A) = \{\lambda \in \mathbb{C} : A - \lambda I \text{ is not onto}\}.$$

We end this section by recalling the notion of local spectral subspace, as well as some of its properties.

Definition 1.4.9. *Let $A \in \mathcal{B}(X)$ and $\Omega \subset \mathbb{C}$. The* ***local spectral subspace*** *of A associated with Ω is the set*

$$X_A(\Omega) := \{x \in X : \sigma_A(x) \subset \Omega\}.$$

Obviously, if $\Omega_1 \subseteq \Omega_2$ then $X_A(\Omega_1) \subseteq X_A(\Omega_2)$.

Theorem 1.4.10. *([5, Theorem 2.6].) Let $A \in \mathcal{B}(X)$, X a Banach space, and Ω a subset of $\mathbb{C}$. Then the following properties hold:*

1. *$X_A(\Omega)$ is a linear A-hyper-invariant subspace of X, i.e for every bounded operator S that commutes with A we have $S(X_A(\Omega)) \subseteq X_A(\Omega)$;*
2. *$X_A(\Omega) = X_A(\Omega \cap \sigma(T))$;*
3. *If $\lambda \notin \Omega, \Omega \subseteq \mathbb{C}$, then $(A - \lambda I)X_A(\Omega) = X_A(\Omega)$;*
4. *Suppose that $\lambda \in \Omega$ and $(A - \lambda I)x \in X_A(\Omega)$ for some $x \in X$. Then $x \in X_A(\Omega)$;*
5. *For every family $(\Omega_j)_{j \in J}$ of subsets of $\mathbb{C}$ we have*

$$X_A(\bigcap_{j \in J} \Omega_j) = \bigcap_{j \in J} X_A(\Omega_j);$$

6. *If Y is a A-invariant closed subspace of X for which $\sigma(T \mid Y) \subseteq \Omega$, then $Y \subseteq X_A(\Omega)$. In particular, $Y \subseteq X_T(\sigma(T \mid Y))$ holds for every closed A-invariant closed subspace of X.*

The following property characterize the SVEP through the local spectral subspace.

Theorem 1.4.11. *([5, Theorem 2.8]) Let $A \in \mathcal{B}(X)$, X a Banach space. Then the following statements are equivalent:*

1. *A has the SVEP;*
2. *$X_A(\emptyset) = \{0\}$;*
3. *$X_A(\emptyset)$ is closed.*

1.5 Rank-one operators

Rank one operators are one of the basic tools in dealing with preservation issues. The tensor product gives a simple representation of these operators in the algebra of linear bounded operators on a Banach space. In this section, we present the main properties of this class of operators.

Definition 1.5.1. *Let X be a complex Banach space, $x \in X$ and $f \in X^*$. We call the tensor product of x and f, the operator $x \otimes f$ defined on the space X by*

$$(x \otimes f)(y) = f(y)x, \text{ for every } y \in X.$$

Note that $x \otimes f$ is an operator of rank less than 1. Conversely, any operator of rank less that 1 is written in this form. In particular, an operator $R \in \mathcal{B}(X)$ is of rank 1 if, and only if, there exists a vector $x \in X \setminus \{0\}$ and $f \in X^* \setminus \{0\}$ such that $R = x \otimes f$. Also, $\sigma(x \otimes f) = \{0, f(x)\}$ for every $x \in X$ and $f \in X^*$.

The following proposition gives a necessary and sufficient condition for a rank-one operator to be nilpotent or idempotent.

Proposition 1.5.2. *Let $x \in X \setminus \{0\}$ and $f \in X^* \setminus \{0\}$. Then we have:*

1. *The rank-one operator $x \otimes f$ is nilpotent if, and only if $f(x) = 0$;*
2. *The rank-one operator $x \otimes f$ is idempotent if, and only if $f(x) = 1$.*

Proof. Follows directly from the definition and the fact that $(x \otimes f)^n = f(x)^{n-1} x \otimes f$. □

We end this section by this useful proposition.

Proposition 1.5.3. *Let $x \in X$ and f a linear form on X. Then:*

$$(x \otimes f)^* = f \otimes Jx,$$

where J is the canonical injection between X and its bi-dual X^{**}.

1.6 Generalization of spectrum

One of the most important questions regarding an operator on a Banach space is whether it is invertible or not. Often, it is possible that an operator is invertible, but its inverse has a very large norm. This poses a computational difficulty because the operator is then almost singular. The pseudospectrum of an operator is studied to tackle this problem. Several authors such as Bottcher, Davies, Grudsky, Silbermann and Wolf have studied the pseudospectra of various operators. Another well-known extension of the concept of spectrum is the idea of condition spectrum. It was introduced by Kulkarni and Sukumar in [78]. For more details and basic facts concerning condition spectrum and its connection with the pseudo spectrum, we refer the reader to the articles [77, 78] and the monograph [105].

We may note that since the pseudospectrum and the condition spectrum are a geometric concept as against the spectrum which is a purely algebraic concept. We will see that these two sets have some interesting geometric properties, namely that they have no isolated points except in trivial situations and they always have only a finite number of components; moreover, every component contains an element of the spectrum.

1.6.1 Pseudospectrum: Definition and Basic Properties

Definition 1.6.1. *Let $a \in \mathcal{A}$ and $\epsilon > 0$. The pseudospectrum of a is defined as follow:*

$$\sigma_\epsilon = \{\lambda \in \mathbb{C} : \| (a - \lambda)^{-1} \| \geq \frac{1}{\epsilon}\}$$

with the convention $\| (a-\lambda)^{-1} \| = \infty$ *when* $a-\lambda$ *is not invertible.*

We consolidate some elementary properties of the pseudospectrum in the following theorem.

Theorem 1.6.2. *Let* $\mathcal{A}$ *be a Banach algebra,* $a \in \mathcal{A}$ *and* $\epsilon > 0$*. Then:*

1. $\sigma(a) \subset \sigma_\epsilon(a)$*, in fact* $\sigma(a) = \bigcap_{\epsilon>0} \sigma_\epsilon(a)$*;*
2. $\sigma_{\epsilon_1}(a) \subseteq \sigma_{\epsilon_2}(a)$ $(0 < \epsilon_1 < \epsilon_2)$*;*
3. $\sigma_\epsilon(a+\lambda) = \sigma_\epsilon(a) + \lambda$ $(\lambda \in \mathbb{C})$*;*
4. $\sigma_\epsilon(\lambda a) = \lambda \sigma_{\frac{\epsilon}{|\lambda|}}(a)$ $(\lambda \in \mathbb{C} \setminus \{0\})$*;*
5. $\sigma_\epsilon(a) \subset D(0. \| a \| + \epsilon)$*;*
6. $\sigma_\epsilon(a)$ *is a non-empty compact subset of* $\mathbb{C}$*.*

Proof. The proofs of such properties can be found, for instance, in [6]. These are included for the sake of completeness:

1. Let $\lambda \in \sigma(a)$. Then $\lambda - a$ is not invertible, hence by the convention $\| (\lambda-a)^{-1} \| = \infty$. On the other hand, if $\lambda \notin \sigma(a)$, then $\lambda - a$ is invertible. Hence there exists $\epsilon_0 > 0$ such that $\| (\lambda-a)^{-1} \| < \frac{1}{\epsilon_0}$. Thus $\lambda \notin \sigma_{\epsilon_0}(a)$.
2. Let $0 < \epsilon_1 < \epsilon_2$, and suppose $\lambda \in \sigma_{\epsilon_1}(a)$. If $\lambda - a$ is not invertible, then $\lambda \in \sigma(a) \subseteq \sigma_{\epsilon_2}(a)$. Otherwise $\lambda - a$ is invertible and $\| (\lambda-a)^{-1} \| \geq \frac{1}{\epsilon_1} > \frac{1}{\epsilon_2}$. Hence $\lambda \in \sigma_{\epsilon_2}$.
3. Let $z \in \sigma_\epsilon(a+\lambda)$, then $\| ((z-\lambda)-a)^{-1} \| = \| (z-(\lambda+a))^{-1} \| \geq \frac{1}{\epsilon}$. Hence $z - \lambda \in \sigma_\epsilon(a)$, that is, $z \in \lambda + \sigma_\epsilon(a)$. The reverse inclusion follows similarly.
4. Let $z \in \sigma_\epsilon(\lambda a)$, then $\| (\lambda z - a)^{-1} \| = | \lambda | \| (z-\lambda a)^{-1} \| \geq \frac{|\lambda|}{\epsilon}$. Hence $\frac{z}{\lambda} \in \sigma_{\frac{\epsilon}{|\lambda|}}(a)$, that is, $z \in \lambda \sigma_{\frac{\epsilon}{|\lambda|}}(a)$. The reverse inclusion follows similarly.
5. If $\lambda \notin D(0, \| a \| + \epsilon)$, then $| \lambda | > \| a \| + \epsilon > \| a \|$. Hence $\lambda - a$ is invertible. Moreover $\| (\lambda-a)^{-1} \| \leq \frac{1}{\lambda - \|a\|} \leq \frac{1}{\epsilon}$. Thus $\lambda \notin \sigma_\epsilon(a)$.
6. A consequence of the first and the last properties.

□

The following theorem involves the analytical functional calculus for elements of a Banach algebra.

Theorem 1.6.3. *([105].) Let* $\mathcal{A}$ *be a Banach algebra and* $a \in \mathcal{A}$*. Let* $\Omega \subseteq \mathbb{C}$ *be an open neighborhood of* $\sigma_\epsilon(a)$ *and* γ *be a contour in the counter-clockwise direction that surrounds* $\sigma_\epsilon(a)$ *in* Ω*. Let* f *be analytic in* Ω*. We recall the definition of* $\tilde{f}(a)$ *in the analytical functional calculus as*

$$\tilde{f}(a) = \frac{1}{2\pi i} \int_\gamma (z-a)^{-1} f(z) dz.$$

Then:

$$\| \tilde{f}(a) \| \leq \frac{Ml}{2\pi\epsilon},$$

where l is the length of γ and $M = \sup\{| f(z) | : z \in \gamma\}$.

Using Theorem 1.6.3, we obtain an equivalent condition for an element of a Banach algebra to be a scalar in terms of the pseudospectrum.

Corollary 1.6.4. *Let $\mathcal{A}$ be a Banach algebra, $a \in \mathcal{A}$ and $\mu \in \mathbb{C}$. Then*

$$a = \mu \Leftrightarrow \sigma_\epsilon(a) = D(\mu, \epsilon) \text{ for every } \epsilon > 0.$$

We will list some topological properties of the pseudospectrum of an element of a Banach algebra. The proofs could be found in [78].

Theorem 1.6.5. *Let $\mathcal{A}$ be a Banach algebra, $a \in \mathcal{A}$ and $\epsilon > 0$. Then the pseudospectrum $\sigma_\epsilon(a)$ of a has no isolated points.*

Proof. Suppose $\sigma_\epsilon(a)$ has an isolated point λ_0. Then there exists $r > 0$ such that:

$$\text{For all } \lambda \text{ with } 0 < | \lambda - \lambda_0 | < r, \| (\lambda - a)^{-1} \| < \frac{1}{\epsilon}.$$

Case 1: Suppose $\lambda_0 \in \sigma_\epsilon(a) \setminus \sigma(a)$. By the Hahn-Banach theorem there exists $\phi \in \mathcal{A}'$ such that

$$\phi((\lambda_0 - a)^{-1}) = \| (\lambda_0 - a)^{-1} \| \text{ and } \| \phi \| = 1.$$

Define $f : \mathbb{C} \setminus \sigma(a) \to \mathbb{C}$ by $f(z) = \phi((z - a)^{-1})$. Then f is analytic in $B(\lambda_0, r)$. But $| f(\lambda) | \leq \| (\lambda - a)^{-1} \| < \frac{1}{\epsilon}$ for all $\lambda \in B(\lambda_0, r) \setminus \{\lambda_0\}$, while

$$f(\lambda_0) = \phi((\lambda_0 - a)^{-1}) = \| (\lambda_0 - a)^{-1} \| \geq \frac{1}{\epsilon}.$$

This is a contradiction to the maximum modulus principle.
Case 2: Suppose $\lambda_0 \in \sigma(a)$. For $\lambda \to \lambda_0$ we have $\| (\lambda - a)^{-1} \| \to \infty$. But for $\lambda \in B(\lambda_0.r)$ we have $\| (\lambda - a)^{-1} \| < \frac{1}{\epsilon}$, a contradiction. □

Theorem 1.6.6. *Let $\mathcal{A}$ be a Banach algebra, $a \in \mathcal{A}$ and $\epsilon > 0$. Then the seudospectrum $\sigma_\epsilon(a)$ of a has a finite number of components and each component of $\sigma_\epsilon(a)$ contains an element of $\sigma(a)$.*

Proof. For each $\lambda \in \sigma(a); B(\lambda, \epsilon) \subset \sigma_\epsilon(a)$. Also, $\{B(\lambda, \epsilon) : \lambda \in \sigma(a)\}$ is an open cover for $\sigma(a)$. Since $\sigma(a)$ is compact there exist $\lambda_1, \ldots, \lambda_n \in \sigma(a)$ such that $\sigma(a) \subset \bigcup_{i=1}^{n} B(\lambda_i, \epsilon)$. Since each $B(\lambda_i, \epsilon)$ is connected and is a subset of $\sigma_\epsilon(a)$, it must be contained in some component C_i of $\sigma_\epsilon(a)$. Thus, we get

closed components $C_1, \ldots, C_n$ of $\sigma_\epsilon(a)$ such that $\sigma(a) \subset \bigcup_{i=1}^{n} C_i \subset \sigma_\epsilon(a)$. We will prove that:

$$\bigcup_{i=1}^{n} C_i = \sigma_\epsilon(a).$$

Suppose $z_0 \in \sigma_\epsilon(a) \setminus (\bigcup_{i=1}^{n} C_i)$. Then $\| (z_0 - a)^{-1} \| \geq \frac{1}{\epsilon}$. Let $r > \| a \| + \epsilon$. Then $\sigma(a) \subset \sigma_\epsilon(a) \subset B(0, r)$. Let $S := B(0, r) \setminus (\bigcup_{i=1}^{n} C_i)$. Then S is an open set and $z_0 \in S$. Let S_0 be the component of S containing z_0. Then S_0 is open. Also, since $\sigma(a) \subset \bigcup_{i=1}^{n} C_i$, $(\bigcup_{i=1}^{n} C_i)^c \subset \rho(a)$. Hence $S_0 \subset \rho(a)$. Define $f : S_0 \to \mathbb{R}$ as $f(z) = \| (z - a)^{-1} \|$. By the Hahn-Banach theorem, there exists $\phi \in \mathcal{A}'$ with $\| \phi \| = 1$ such that $\phi((z_0 - a)^{-1}) = \| (z_0 - a)^{-1} \|$. Define $g : S_0 \to \mathbb{C}$ by $g(z) = \phi((z - a)^{-1})$ for all $z \in S_0$. Then, g is an analytic function on S_0 and

$$| g(z) | = | \phi((z - a)^{-1}) | \leq \| (z - a)^{-1} \| = f(z) \text{ for all } z \in S_0 \subset \rho(a).$$

Since $\sigma_\epsilon(a) \subset B(0, r)$ which is open, $\partial B(0, r) \subset \sigma_\epsilon(a)^c$. Hence

$$f(z) = \| (z - a)^{-1} \| < \frac{1}{\epsilon} \text{ for all } z \in \partial B(0, r).$$

We show that $f(z) = \frac{1}{\epsilon}$ for $z \in \bigcup_{i=1}^{n} \partial C_i$. If it is not, then there exists a neighbourhood U of z such that $f(\lambda) > \frac{1}{\epsilon}$ for all $\lambda \in U$. But U intersects points outside the pseudospectrum where $f(z) = \| (z - a)^{-1} \| < \frac{1}{\epsilon}$, a contradiction.

Now $\partial S_0 \subset \partial B(0, r) \cup (\bigcup_{i=1}^{n} \partial C_i)$. Hence if $z \in \partial S_0, | g(z) | \leq f(z) \leq \frac{1}{\epsilon}$. But z_0 is in the interior of S_0 and $| g(z_0) | = g(z_0) = \phi((z_0 - a) - 1) = \| (z_0 - a)^{-1} \| \geq \frac{1}{\epsilon}$. By the maximum modulus principle, g must be constant on S_0. Hence

$$f(z) \geq | g(z) | = g(z_0) \geq \frac{1}{\epsilon} \text{ for every } z \in S_0.$$

Hence $S_0 \subset \sigma_\epsilon(a)$. By the continuity of g, we get $| g(z) | \geq \frac{1}{\epsilon}$ for all $z \in \overline{S_0}$. If $\partial S_0 \cap \partial B(0, r) \neq \emptyset$ we get a contradiction since $| g(z) | \leq f(z) < \frac{1}{\epsilon}$ for all $z \in \partial B(0, r)$. If $\partial S_0 \cap \partial C_i \neq \emptyset$ for some i. Then $S_0 \cup C_i$ is a connected subset of $\sigma_\epsilon(a)$. But C_i is a component of $\sigma_\epsilon(a)$. Thus, $S_0 \subset C_i$, a contradiction since $S_0 \subset B(0, r) \setminus (\bigcup_{i=1}^{n})$. Hence $\partial S_0 = \emptyset$, a contradiction. Hence $\sigma_\epsilon(a) = \bigcup_{i=1}^{n}$, and each C_i contains a point from the spectrum. □

1.6.2 Condition spectrum: Definition and Basic Properties

In this subsection, we summarize some basic properties of the condition spectrum. Assume that $\mathcal{A}$ is a complex Banach algebra with unit 1. We begin by reviewing the Ransford's spectrum, generalization

of the idea of the spectrum, as in [97].

Definition 1.6.7. *An open subset Ω of $\mathcal{A}$ satisfying the following properties is called a* ***Ransford set****:*

1. $1 \in \Omega$;
2. $0 \notin \Omega$;
3. $z\Omega \subseteq \Omega$ *for all* $z \in \mathbb{C} \setminus \{0\}$.

Definition 1.6.8. *Let Ω be a* ***Ransford set*** *in $\mathcal{A}$ and let $a \in \mathcal{A}$. Then the* ***Ransford spectrum*** *of a with respect to Ω is defined as follows:*

$$\sigma^{\Omega}(a) := \{\lambda \in \mathbb{C} : a - \lambda \notin \Omega\}.$$

If we consider $\Omega = Inv(\mathcal{A})$ we prove easily that Ω is a Ransford set and hence $\sigma^{\Omega}(a)$ is exactly $\sigma(a)$ for $a \in \mathcal{A}$. The following was proved by Ransford in [97].

Theorem 1.6.9. *Let Ω be a Ransford set in $\mathcal{A}$. Then the following holds:*

1. $\sigma^{\Omega}(0) = \{0\}$ *and* $\sigma^{\Omega}(1) = \{1\}$;
2. $\sigma^{\Omega}(a)$ *is compact for every* $a \in \mathcal{A}$:
3. *Let* $E := \{a \in \mathcal{A} : \sigma^{\Omega}(a) \neq \emptyset\}$. *Then the map* $a \to \sigma^{\Omega}(a)$ *is an upper semi-continuous function from E to compact subsets of* $\mathbb{C}$.

For the proof of this theorem as well as for several properties of Ransford spectrum we refer to [97].

Definition 1.6.10. *Let $\mathcal{A}$ be a complex Banach algebra with unit* 1. *The* ***condition number*** *of an element a, denoted by $\kappa(a)$, is defined as*

$$\kappa(a) := \| a \| \| a^{-1} \|,$$

with the convention $\kappa(a) = \infty$ if a is not invertible.

To define the condition spectrum, the following definition is needed.

Definition 1.6.11. *For a fixed $0 < \epsilon < 1$, define*

$$\Omega_{\epsilon} := \{a \in Inv(\mathcal{A}) : \kappa(a) < \frac{1}{\epsilon}\}.$$

Then, we have Ω_{ϵ} is a Ransford set. Indeed:

- The map $a \to \| a \| \| a^{-1} \|$ is continuous and hence Ω_{ϵ} is an open set;
- $0 \notin \Omega_{\epsilon}$, as 0 is not invertible;

- $1 \in \Omega_\epsilon$, since $\| \| \| 1^{-1} \| = 1$;
- $z\Omega_\epsilon \subseteq \Omega$, since $\| za \| \| (za)^{-1} \| = \| a \| \| a^{-1} \|$ for every $z \in \mathbb{C} \setminus \{0\}$.

We are now in a position to define the condition spectrum.

Definition 1.6.12. *Let $0 < \epsilon < 1$ and $a \in \mathcal{A}$. The **condition spectrum** of a is defined by*

$$\Lambda_\epsilon(a) := \{\lambda \in \mathbb{C} : \lambda - a \notin \Omega_\epsilon\} = \{\lambda \in \mathbb{C} : \| \lambda - a \| \| (\lambda - a)^{-1} \| \geq \frac{1}{\epsilon}\},$$

with the convention that $\| \lambda - a \| \| (\lambda - a)^{-1} \| = \infty$ when $\lambda - a$ is not invertible. The condition spectral radius $r_\epsilon(a)$ is defined as

$$r_\epsilon(a) := \sup\{| z | : z \in \Lambda_\epsilon(a)\}.$$

The following theorem sums up some elementary properties of the condition spectrum.

Theorem 1.6.13. *Let $0 < \epsilon < 1$ and $a \in \mathcal{A}$. The following properties are satisfied:*

1. *$\Lambda_\epsilon(0) = \{0\}$ and $\Lambda_\epsilon(1) = \{1\}$;*
2. *If $0 < \epsilon_1 < \epsilon_2 < 1$ then $\Lambda_{\epsilon_1}(a) \subset \Lambda_{\epsilon_2}(a)$;*
3. *$\sigma(a) \subseteq \Lambda_\epsilon(a)$, in fact $\sigma(a) = \bigcap_{0<\epsilon<1} \Lambda_\epsilon(a)$;*
4. *$\Lambda_\epsilon(a)$ is a nonempty compact subset of $\mathbb{C}$ for every $a \in \mathcal{A}$;*
5. *$\Lambda_\epsilon(\alpha + \beta a) = \alpha + \beta\Lambda_\epsilon(a)$ for all $\alpha, \beta \in \mathbb{C}$;*
6. *The map $a \to \Lambda_\epsilon(a)$ is an upper semi continuous function from $\mathcal{A}$ to compact subsets of $\mathbb{C}$.*

Proof. Consequences of Theorem 1.6.9 and the definition of the condition spectrum. □

The next theorem, quoted from [78], gives some properties of the condition spectral radius $r_\epsilon(a)$.

Theorem 1.6.14. *([78, Theorem 2.9].) Let $0 < \epsilon < 1$ and $a \in \mathcal{A}$. Then we have*

1. *$r(a) \leq r_\epsilon(a) \leq \frac{1+\epsilon}{1-\epsilon} \| a \|$.*
2. *If $\| a^n \| \leq M < \frac{1}{\epsilon}$ for all $n \geq 0$, then $r_\epsilon(a) \leq \frac{1+M^2\epsilon}{1-M^2\epsilon}$.*

Proof. 1. Since $\sigma(a) \subseteq \Lambda_\epsilon(a)$, we have $r(a) \leq r_\epsilon(a)$. To prove the remaining part of the inequality consider $\lambda \in \Lambda_\epsilon(a)$. If $| \lambda | \leq \| a \|$, then clearly $| \lambda | \leq \frac{1+\epsilon}{1-\epsilon} \| a \|$.

Assume that $| \lambda | \geq \| a \|$, then $\lambda - a$ is invertible and $\| (\lambda - a)^{-1} \| \leq \frac{1}{| \lambda | - \| a \|}$. Hence

$$1 \leq \epsilon \| (\lambda - a)^{-1} \| \| \lambda - a \| \leq \epsilon \frac{| \lambda | + \| a \|}{| \lambda | - \| a \|}.$$

On simplification:

$$\lambda \leq \frac{1+\epsilon}{1-\epsilon} \| a \| .$$

2. Note that $r(a) = \lim_{n\to\infty} \| a^n \|^{\frac{1}{n}} \leq \lim_{n\to\infty} M^{\frac{1}{n}} = 1$.

   Let $\lambda \in \Lambda_\epsilon(a)$, if $| \lambda | \leq 1$ then clearly $| \lambda | \leq \frac{1+M^2}{1-M\epsilon}$. If $| \lambda | > 1$, then $\lambda \notin \sigma(a)$. Hence $\lambda - a \in Inv(\mathcal{A})$ and

$$(\lambda - a)^{-1} = \sum_{n=0}^{\infty} \frac{a^n}{\lambda^{n+1}}.$$

   Thus

$$\| (\lambda - a)^{-1} \| \leq \frac{1}{| \lambda |} \sum_{k=0}^{\infty} \frac{\| a^k \|}{| \lambda |^k} \leq \frac{M}{| \lambda |} \sum_{k=0}^{\infty} \frac{1}{| \lambda |^k} = \frac{M}{| \lambda |} (\frac{1}{1 - \frac{1}{|\lambda|}}) = \frac{M}{| \lambda | - 1}.$$

   Next, since $\lambda \in \Lambda_\epsilon(a)$, we have

$$\frac{1}{\epsilon} \leq \| \lambda - a \| \| (\lambda - a)^{-1} \| \leq (| \lambda | + M) \frac{M}{| \lambda | - 1}.$$

   On simplification, this becomes:

$$| \lambda | \leq \frac{1 + M^2\epsilon}{1 - M\epsilon} \text{ provided } M < \frac{1}{\epsilon}.$$

□

Theorem 1.6.5 and Theorem 1.6.6 remain true if we change $\sigma_\epsilon(.)$ by $\Lambda_\epsilon(.)$ for non-scalar elements. Therefore we have that the condition spectrum of non-scalar elements in a Banach algebra has no isolated points and it is strictly bigger than the usual spectrum.

Remark 1.6.15. *We have seen in Theorem 1.6.13 that $\sigma(a) \subseteq \Lambda_\epsilon(a)$. The reverse inclusion (and hence equality) holds only in the case when a is a scalar multiple of 1. This was proved in the following corollary.*

Corollary 1.6.16. *([78, Corollary 3.2 and Corollary 3.4].) Let $0 < \epsilon < 1$ and $a \in \mathcal{A}$.*
If $\Lambda_\epsilon(a) = \{\lambda_0\}$ for some $\lambda_0 \in \mathbb{C}$, then $a = \lambda_0$.
If a is not a scalar multiple of the identity, then for each $\lambda_0 \in \sigma(a)$, there exists $r > 0$ such that $D(\lambda_0, r) \subseteq \Lambda_\epsilon(a)$. In particular, $\Lambda_\epsilon(a)$ contains properly an open neighborhood of $\sigma(a)$. Consequently, if $\Lambda_\epsilon(a) = \sigma(a)$ then a is a scalar multiplied by the identity.

Since our results also concerning the condition spectrum in the algebra of bounded linear operators acting on a complex Hilbert space, the rest of this section is devoted to the study of the condition spectrum in this case. In what follows, H denotes a complex Hilbert space and $\mathcal{B}(H)$ the algebra of linear operators bounded on H. We start with the following definition.

Definition 1.6.17. *Let U be a bounded operator on a complex Hilbert space H endowed with the scalar product $\langle , \rangle$.*

- *if U is linear and $U^*U = UU^* = I$ then U is called a unitary operator. Equivalently, U is linear, surjective and preserves the inner product i.e $\langle Ux, Uy \rangle = \langle x, y \rangle$ for all $x, y \in H$.*

- *If U is conjugate-linear and $U^*U = UU^* = I$ then U is called an anti-unitary operator. Equivalently, U is conjugate-linear, bijective and satisfies $\langle Ux, Uy\rangle = \overline{\langle x, y\rangle}$ for all $x, y \in H$.*

Next proposition summarizes some remarkable properties of the condition spectrum of an element in $\mathcal{B}(H)$.

Proposition 1.6.18. *Let $0 < \epsilon < 1$ and $T \in \mathcal{B}(H)$. Then we have the following properties:*

1. *$\Lambda_\epsilon(T^*) = \overline{\Lambda_\epsilon(T)}$, where T^* is the adjoint of T;*
2. *$\Lambda_\epsilon(UTU^*) = \Lambda_\epsilon(T)$ for every unitary operator U on H;*
3. *$\Lambda_\epsilon(UTU^*) = \overline{\Lambda_\epsilon(T)}$ for every anti-unitary operator U on H.*

Proof. 1. We know that $\sigma(T^*) = \overline{\sigma(T)}$. If $\lambda \notin \sigma(T)$, then

$$\begin{aligned} \| \lambda - T \| \| (\lambda - T)^{-1} \| &= \| (\overline{\lambda} - T^*)^* \| \| ((\overline{\lambda} - T^*)^{-1})^* \| \\ &= \| \overline{\lambda} - T^* \| \| (\overline{\lambda} - T^*)^{-1} \| . \end{aligned}$$

So, $\lambda \in \Lambda_\epsilon(T) \Leftrightarrow \overline{\lambda} \in \Lambda_\epsilon(T^*)$.

2. Let U be a unitary operator on H. Then

$$\begin{aligned} \Lambda_\epsilon(UTU^*) &= \{\lambda \in \mathbb{C} : \| \lambda - UTU^* \| . \| (\lambda - UTU^*)^{-1} \| \geq \frac{1}{\epsilon}\} \\ &= \{\lambda \in \mathbb{C} : \| U(\lambda - T)U^* \| . \| (U(\lambda - T)U^*)^{-1} \| \geq \frac{1}{\epsilon}\} \\ &= \{\lambda \in \mathbb{C} : \| U(\lambda - T)U^* \| . \| U(\lambda - T)^{-1}U^* \| \geq \frac{1}{\epsilon}\} \\ &= \{\lambda \in \mathbb{C} : \| \lambda - T \| \| (\lambda - T)^{-1} \| \geq \frac{1}{\epsilon}\} \\ &= \Lambda_\epsilon(T). \end{aligned}$$

3. As above, so below.

□

As a direct consequence of the previous proposition, we have these properties for the condition spectral radius.

Proposition 1.6.19. *Let $0 < \epsilon < 1$ and $T \in \mathcal{B}(H)$. Then we have the following properties:*

1. *$r_\epsilon(T) = 0$ if, and only if $T = 0$;*
2. *$r_\epsilon(UAU^*) = r_\epsilon(T)$ for every unitary operator U on H;*
3. *$r_\epsilon(\lambda T) = \mid \lambda \mid r_\epsilon(T)$ for every scalar λ in $\mathbb{C}$;*
4. *$r_\epsilon(T^*) = r_\epsilon(T)$.*

Now we characterize the condition spectrum and the condition spectral radius of some particular classes of operators. The following result describes the condition spectrum of self-adjoint operators in terms of their usual spectra.

Proposition 1.6.20. *([20, Proposition 2.2]) Let $0 < \epsilon < 1$ and T a self-adjoint operator on H. Then we have:*

$$\Lambda_\epsilon(T) = \bigcup_{\alpha,\beta\in\sigma(T)} \overline{D}\left(\frac{\alpha-\beta\epsilon^2}{1-\epsilon^2}, \frac{\epsilon \mid \alpha-\beta \mid}{1-\epsilon^2}\right).$$

Proof. Note that, for every $z \in \mathbb{C}$, the operator $z-T$ is normal. Thus, for every $z \in \Lambda_\epsilon(T) \setminus \sigma(T)$, we have

$$\begin{aligned} \| z-T \| \| (z-T)^{-1} \| \geq \frac{1}{\epsilon} &\Leftrightarrow r(z-T)r((z-T)^{-1}) \geq \frac{1}{\epsilon} \\ &\Leftrightarrow d(z,\sigma(T)) \leq \epsilon r(z-T), \end{aligned}$$

and consequently

$$\Lambda_\epsilon(T) = \{z \in \mathbb{C} : d(z,\sigma(T)) \leq \epsilon r - T\}. \tag{1.6.1}$$

On the other hand, let $\alpha, \beta \in \mathbb{R}$ be arbitrary real numbers and $z = x+iy \in \mathbb{C}$ with $x, y \in \mathbb{R}$, and set $a := \sqrt{1-\epsilon 2}$ and $b := \alpha - \beta\epsilon^2$. Straightforward computations give that

$$\begin{aligned} \mid z-\alpha \mid \leq \epsilon \mid z-\beta \mid &\Leftrightarrow (ax-\frac{b}{a})^2 + (ay)^2 \leq (\frac{\epsilon \mid \alpha-\beta \mid}{a})^2 \\ &\Leftrightarrow az \in \overline{D}(\frac{b}{a}, \frac{\epsilon \mid \alpha-\beta \mid}{a}) \\ &\Leftrightarrow z \in \overline{D}(\frac{b}{a^2}, \frac{\epsilon \mid \alpha-\beta \mid}{a^2}) \\ &\Leftrightarrow z \in \overline{D}(\frac{\alpha-\beta\epsilon^2}{1-\epsilon^2}, \frac{\epsilon \mid \alpha-\beta \mid}{1-\epsilon^2}). \end{aligned}$$

This together with the equality (1.6.1) implies that

$$\begin{aligned} z \in \Lambda_\epsilon(T) &\Leftrightarrow \exists \alpha,\beta \in \sigma(T) : \mid z-\alpha \mid \leq \epsilon \mid z-\beta \mid \\ &\Leftrightarrow z \in \bigcup_{\alpha,\beta\in\sigma(T)} \overline{D}(\frac{\alpha-\beta\epsilon^2}{\epsilon^2}, \frac{\epsilon \mid \alpha-\beta \mid}{1-\epsilon^2}); \end{aligned}$$

which yields the desired conclusion and completes the proof. □

Corollary 1.6.21. *([20, Proposition 2.4].) Let $0 < \epsilon < 1$ and P be a nontrivial projection in $\mathcal{B}(H)$. Then:*

$$\Lambda_\epsilon(P) = \overline{D}(\frac{1}{1-\epsilon^2}, \frac{\epsilon}{1-\epsilon^2}) \cup \overline{D}(\frac{-\epsilon^2}{1-\epsilon^2}, \frac{\epsilon}{1-\epsilon^2})$$

and

$$r_\epsilon(P) = \frac{1}{1-\epsilon}.$$

The following determines the condition spectrum of rank one nilpotent operators.

Theorem 1.6.22. *([20, Theorem 2.5].) Let* $0 < \epsilon < 1$ *and* $R \in \mathcal{B}(H)$ *be a rank one nilpotent operator. Then*

$$\Lambda_\epsilon(R) = \overline{D}(0, \frac{\sqrt{\epsilon} \parallel R \parallel}{1-\epsilon})$$

and

$$r_\epsilon(R) = \frac{\sqrt{\epsilon} \parallel R \parallel}{1-\epsilon}.$$

Chapter 2

Preserver Problems

In preserver problems the characterization of a map $\phi : \mathcal{A} \to \mathcal{B}$ between Banach algebras $\mathcal{A}$ and $\mathcal{B}$ is demanded, considering that it leaves certain functions, subsets, relations, etc, invariant. We call it linear preserver problems when ϕ is supposed to be linear and we call it nonlinear preserver problems when we do not assume any algebraic condition like linearity or additivity or multiplicativity for ϕ. In all cases that have been studied by now, nonlinear preservers are proved to be linear or conjugate linear perhaps plus an element of the target algebra.

The first section is devoted to the, motivation key for this kind of study, Kaplansk's conjecture and some partial results. In the other two sections, we will cite some results related to the spectrum and the local spectrum preservers.

2.1 Kaplansky's conjecture and some partial results

In this section, we will provide some historical facts and a few important results of preserver problems. The earliest result on linear preserver problems was established by Frobenius in 1897. In [60] he shows that a linear map $\phi : \mathcal{M}_n(\mathbb{C}) \to \mathcal{M}_n(\mathbb{C})$ satisfies $\det(\phi(A)) = \det(A)$ for every $A \in \mathcal{M}_n(\mathbb{C})$ if, and only if, ϕ is of the form

$$\phi(A) = PAQ \quad \text{or} \quad \phi(A) = PA^tQ$$

for some fixed invertible matrices $P, Q \in \mathcal{M}_n(\mathbb{C})$ such that $\det(PQ) = 1$.

Later, in 1949 Dieudonné [54] proved that if $\phi : \mathcal{M}_n(\mathbb{F}) \to \mathcal{M}_n(\mathbb{F})$ is an invertible τ-semilinear map, where τ is any automorphism of the field $\mathbb{F}$, and if ϕ preserve the set of singular matrices in $\mathcal{M}_n(\mathbb{F})$, then ϕ must be of the form

$$\phi(A) = PA^rQ \quad \text{or} \quad \phi(A) = P(A^r)^tQ,$$

where P and Q are invertible matrices. Observe that ϕ preserve singular matrices precisely when ϕ^{-1} preserves invertibility.

In 1959, Marcus and Moyls [85] improved this result. They described the cases when a linear map $\phi : \mathcal{M}_n(\mathbb{C}) \to \mathcal{M}_n(\mathbb{C})$ preserves the rank of all matrices, the determinant or the spectrum of all

hermitian matrices. In the last case, ϕ takes one of the following forms:

$$\phi(T) = UTU^{-1} \quad \text{or} \quad \phi(T) = UT^tU^{-1},$$

for every matrix $T \in \mathcal{M}_n(\mathbb{C})$.
In 1967 – 1968, Gleason, Kahane and Żelazko [61, 74] proved that if $\mathcal{A}$ is a commutative Banach algebra and $\mathcal{B}$ is a semi-simple commutative Banach algebra, then every unital invertibility preserving linear map $\phi : \mathcal{A} \to \mathcal{B}$ is an homomorphism. After that, Żelazko [108] showed that the assumption of commutativity of $\mathcal{A}$ can be removed.

Theorem 2.1.1. *(Żelazko [108]) Let $\mathcal{A}$ be a Banach algebra and $\mathcal{B}$ be a semi-simple commutative Banach algebra. Then every unital, invertibility preserving linear map $\phi : \mathcal{A} \to \mathcal{B}$ is an homomorphism.*

These attempts led Kaplansky in 1970 [75] to one of the most intractable unsolved problems in the theory of Banach algebras and linear preserver problems, which is the following question:

Let $\phi : \mathcal{A} \to \mathcal{B}$ be a unital, invertibility preserving map between unital Banach algebras $\mathcal{A}$ and $\mathcal{B}$. Is ϕ a Jordan homomorphism?

Let us quickly state that the above question of Kaplansky is too general and the answer to it is negative in this generality. The first example below, given by Sourour in [101], shows that if ϕ is not surjective, then it may not be a Jordan homomorphism.

Example 2.1.2. *Let H be an Hilbert space and let $\Gamma : \mathcal{B}(H) \to \mathcal{B}(H)$ be linear such that $\Gamma(I) = 0$. Consider the map*

$$\phi : \mathcal{B}(H) \to \mathcal{B}(H \oplus H)$$

defined by

$$\phi(X) = \begin{bmatrix} X & \Gamma(X) \\ 0 & X \end{bmatrix}$$

for $X \in \mathcal{B}(H)$. Then ϕ is clearly unital and preserves invertibility. However, $X \in \mathcal{B}(H)$ we have

$$\phi(X^2) - \phi(X)^2 = \begin{bmatrix} 0 & \Gamma(X^2) - X\Gamma(X) - \Gamma(X)X \\ 0 & 0 \end{bmatrix}$$

Hence, ϕ is a Jordan homomorphism if, and only if, $\Gamma(X^2) = \Gamma(X)X + X\Gamma(X)$ for all $X \in \mathcal{B}(H)$, or equivalently,

$$\Gamma(X \circ Y) = \Gamma(X) \circ Y + X\Gamma(Y)$$

*for all $X, Y \in \mathcal{B}(H)$. If Γ satisfies this property, then Γ is a **Jordan derivation** of $\mathcal{B}(H)$.*

It is now easy to exhibit a counterexample to Kaplansky's question. Our Γ shall take the form $\Gamma(X) = f(X)I$, where $f \in \mathcal{B}(H)^*$ satisfies $f(I) = 0$. Let $\dim H \geq 2$, and choose an orthogonal pair of

unit vectors $h, k \in H$. Define $f \in \mathcal{B}(H)^*$ by

$$f(X) = \langle Xh, k\rangle.$$

Take the rank one operator $X = (k \otimes h^*)$. Then $X^2 = 0$ and $f(X) = 1$, so that

$$\Gamma(X)X + X\Gamma(X) = 2X \neq 0 = \Gamma(X^2).$$

The associated invertibility preserving map ϕ is therefore not a Jordan homomorphism.

In the next example, Aupetit [8] showed that Kaplansky's question may not have a positive answer if the Banach algebras $\mathcal{A}$ and $\mathcal{B}$ are not semi-simple.

Example 2.1.3. *Let*

$$\mathcal{A} = \left\{ \begin{bmatrix} W & X \\ 0 & Y \end{bmatrix} : W, X, Y \in \mathcal{M}_2(\mathbb{C}) \right\}$$

and define $\phi : \mathcal{A} \to \mathcal{A}$ by

$$Z = \begin{bmatrix} W & X \\ 0 & Y \end{bmatrix} \mapsto \phi(Z) = \begin{bmatrix} W & X \\ 0 & Y^t \end{bmatrix}$$

Clearly, ϕ is surjective, unital, and invertibility preserving. But then

$$\phi(Z^2) - \phi(Z)^2 = \begin{bmatrix} 0 & X(X - Y^t) \\ 0 & 0 \end{bmatrix}$$

is not always zero. Hence ϕ is not a Jordan homomorphism. However, note that $\phi(Z^2) - \phi(Z)^2 \in Rad(\mathcal{A})$ for all $Z \in \mathcal{A}$.

Based on several partial positive results and counterexamples, the following conjecture known as Kaplansky's conjecture has been formulated by Aupetit [9].

Conjecture 2.1.4. ***(Kaplansky's conjecture)***
Suppose $\mathcal{A}$ and $\mathcal{B}$ are unital semi-simple Banach algebras and $\phi : \mathcal{A} \to \mathcal{B}$ is a unital bijective linear map preserving invertibility. Then ϕ is a Jordan homomorphism.

In this generality, the problem remains unsolved. A number of partial positive results have been found especially in the case where the map is spectrum preserving.

Choi, Hadwin, Nordgren, Radjavi and Rosenthal in [47] showed that every invertibility preserving, unital, self-adjoint linear map from a $\mathbf{C}^*$-algebra onto another $\mathbf{C}^*$-algebra is a Jordan homomorphism.

Jafarian and Sourour proved in [72] that every surjective, spectrum preserving map from $\mathcal{B}(X)$ onto $\mathcal{B}(Y)$ is an isomorphism or an anti-isomorphism. Later, Sourour in [101] extended the results to surjective, invertibility preserving maps, thereby answering Kaplansky's question completely in the case of $\mathcal{B}(X)$.

For linear maps that are not necessarily surjective, Hou and Wang [107] showed that every unital, invertibility preserving elementary operator of length not greater than 2 on $\mathcal{B}(X)$ is an homomorphism.

Recall that an elementary operator $\phi : \mathcal{B}(X) \to \mathcal{B}(\mathcal{Y})$ of length not greater than n is one that takes the form

$$\phi(T) = \sum_{k=1}^{n} A_k T B_k \quad (T \in \mathcal{B}(X)),$$

where $A_1, A_2, \cdots, A_n \in \mathcal{B}(X, Y)$ and $B_1, B_2, \cdots, B_n \in \mathcal{B}(Y, X)$.

In [63], Harris and Kadison studied similar questions for affine mappings on $\mathbf{C}^*$-algebras. There they indicated their belief that Kaplansky's conjecture is true for $\mathbf{C}^*$-algebras. However, this case is still open.

Later, Aupetit [10] proved that every surjective, spectrum preserving map from a von Neumann algebra onto another von Neumann algebra is a Jordan isomorphism. This is one of the deepest results so far in this direction, and it is close to a positive solution to Kaplansky's conjecture.

Brešar and Šemrl [39] proved that every unital, surjective, invertibility preserving map from a von Neumann algebra onto a standard operator algebra is a Jordan homomorphism.

Obviously, if ϕ is a unital invertibility preserving linear map from $\mathcal{A}$ into $\mathcal{B}$, then ϕ compresses the spectrum, i.e

$$\sigma(\phi(a)) \subset \sigma(a), \quad a \in \mathcal{A}.$$

Thus, this conjecture can be formulated as spectrum compressing problem. In some cases, one considers the problem of describing unital surjective linear maps ϕ between two semi-simple Banach algebras $\mathcal{A}$ and $\mathcal{B}$ preserving invertibility in both directions, i.e $\phi(a)$ is invertible in $\mathcal{B}$ if, and only if, a is invertible in $\mathcal{A}$. Obviously, such a unital linear map ϕ preserves invertibility in both directions if, and only if, it preserves the spectrum. Therefore, we arrive at the following variant of Kaplansky's conjecture.

Conjecture 2.1.5. *Every surjective linear map between two semi-simple Banach algebras that preserves the spectrum is a Jordan isomorphism.*

It is often said that the best way to tackle a difficult problem is to generalize it. Mathieu [87] introduced the notion of spectral isometry. Recall that, a linear mapping $T : \mathcal{A} \to \mathcal{B}$ is called a spectral isometry if $r(Tx) = r(x)$. Over the past decade, Mathieu has been looking at the following spectral radius analogue of Kaplansky's conjecture:
Let $\mathcal{A}$ and $\mathcal{B}$ be unital semi-simple Banach algebras. Let $\phi : \mathcal{A} \to \mathcal{B}$ be a unital surjective spectral isometry. Must ϕ be a Jordan isomorphism?

Mathieu and Sourour [88] gave a positive result for this conjecture in the case where $\mathcal{A}$ and $\mathcal{B}$ are finite-dimensional semi-simple Banach algebras.

We can still consider a strong question than Kaplansk's conjecture by asking to describe linear/additive maps ϕ between Banach algebras $\mathcal{A}$ and $\mathcal{B}$ that strongly preserve invertibility. More explicitly, we look for surjective linear/additive maps $\phi : \mathcal{A} \to \mathcal{B}$ such that

$$\phi(a^{-1}) = \phi(a)^{-1}, \quad a \in Inv(\mathcal{A}).$$

Hua's theorem on fields states that if a unital additive map ϕ from a field $\mathcal{K}$ into another one $\mathcal{F}$ strongly preserves invertibility, then ϕ is an isomorphism or an antiisomorphism; see [70]. This result has been extended by Boudi and Mbekhta in [23] to additive map between Banach algebras. They show that:

Theorem 2.1.6. *(Boudi and Mbekhta [23]) Let $\mathcal{A}$ and $\mathcal{B}$ be Banach algebras and ϕ be an additive map from $\mathcal{A}$ into $\mathcal{B}$. Then ϕ strongly preserves invertibility if, and only if, $\phi(1)\phi$ is a unital Jordan homomorphism and $\phi(1)$ commutes with the range of ϕ.*

The proofs offered therein are based on the reduction of Kaplansky's question to the problem of characterizing linear maps preserving idempotents or rank one operators. This kind of techniques can be extended only to some special classes of semi-simple Banach algebras. But, for all algebras not belonging to one of these classes, the problem is quite challenging. For more details, we refer the reader to [40] where Brešar and Šemrl confirmed Kaplansky's conjecture for invertibility preserving maps that, additionally, locally preserves commutativity (a linear map $\phi : \mathcal{A} \to \mathcal{B}$ is locally preserve commutativity if $[\phi(x^2), \phi(x)] = 0$ for all $x \in \mathcal{A}$, where $[x, y]$ denotes the Lie product $xy - yx$). They provided a weaker equivalent formulation of Kaplansky's conjecture.

Theorem 2.1.7. *[40, Theorem 1.2] Let $\mathcal{A}$ and $\mathcal{B}$ be unital Banach algebras with $\mathcal{B}$ semi-simple, and let $\phi : \mathcal{A} \to \mathcal{B}$ be a unital surjective linear map that preserves invertibility and locally preserves commutativity. Then ϕ is a Jordan homomorphism.*

There are many more interesting results and unsolved preserver problems in this rich area of research. But, unfortunately, we can not shed light all these results. To sum up, many researchers have been trying their hands at Kaplansky's question. As mentioned before, none has answered the question completely but they are all motivating results on which further work may lead to the eventual solution of the problem.

2.2 Spectrum and related sets preservers

In this section we will present some results on the preserving of the spectrum and some of its related sets. Our first example is nothing more than Jafarian and Sourour's partial answer to Kaplansky's conjecture.

Theorem 2.2.1. *([73]) A linear map ϕ from $\mathcal{B}(X)$ onto $\mathcal{B}(Y)$ preserves the spectrum if, and only if, one of the following happens.*

1. *There is a bijective mapping $A \in \mathcal{B}(X, Y)$ such that*

$$\phi(T) = ATA^{-1}, \quad T \in \mathcal{B}(X);$$

2. *There is a bijective mapping $B \in \mathcal{B}(X^*, Y)$ such that*

$$\phi(T) = BT^*B^{-1}, \quad T \in \mathcal{B}(X).$$

This result has been extended in several directions, and several results about linear/additive maps preserving, compressing, expanding spectrum and its parts or preserving different spectral quantities of operators. In [94] Omladič and Šemrl showed that the conclusion of this result is valid under the less restrictive condition that ϕ is only additive. The proofs of the main results of [73,94] are based on the following spectral characterization of operators of rank one.

Lemma 2.2.2. *(Jafarian-Sourour [73] and Omladič-Šemrl [94]) For a nonzero operator $R \in \mathcal{B}(X)$, the following statements are equivalent:*

1. $R \in \mathcal{F}_1(X)$;
2. $\sigma(T+R) \cap \sigma(T+\lambda R) \subset \sigma(T)$ *for every* $T \in \mathcal{B}(X)$ *and* $\lambda \in \mathbb{C} \setminus \{1\}$;
3. $\sigma(T+R) \cap \sigma(T+2R) \subset \sigma(T)$ *for every* $T \in \mathcal{B}(X)$.

This lemma tells us that every linear/additive spectrum-preserving map from $\mathcal{B}(X)$ onto $\mathcal{B}(Y)$ maps rank one operators into rank one operators. The equivalence between the first two statements is due to Jafarian and Sourour [73], and equivalence between the first and last statements was established by Omladič and Šemrl in [94] which was needed to extend Jafarian and Sourour's result to additive setting. In fact, Omladič and Šemrl used this lemma to show that additive spectrum-preserving maps are linear, and then they got their main result from [73].

The following result, given by the same authors, describes the linear/additive maps preserving the operators of rank one.

Theorem 2.2.3. *(Jafarian-Sourour [73] and Omladič-Šemrl [94]) Let X and Y be complex Banach spaces. If ϕ is a bijective linear (resp. additive) map from $\mathcal{F}(X)$ into $\mathcal{F}(Y)$ preserving rank one operators, then either there are bijective linear (resp. both linear or both conjugate linear) mappings $A : X \to Y$ and $B : X^* \to Y^*$ such that:*

$$\phi(x \otimes f) = Ax \otimes Bf, \quad x \in X, f \in X^*;$$

or there are bijective linear (resp. both linear or both conjugate linear) mappings $C : X^* \to Y$ and $D : X \to Y^*$ such that:

$$\phi(x \otimes f) = Cf \otimes Dx, \quad x \in X, f \in X^*.$$

Later in [101], Sourour showed that Theorem 2.2.1 remains valid for bijective linear maps between $\mathcal{B}(X)$ and $\mathcal{B}(Y)$ that preserve invertibility in only one direction.

Theorem 2.2.4. *(Sourour [101]). If ϕ is a unital surjective linear map from $\mathcal{B}(X)$ into $\mathcal{B}(Y)$ preserving invertibility (equivalently, ϕ compresses the spectrum), then either ϕ vanishes on $\mathcal{F}(X)$ or ϕ is injective. In the latter case, either there is a bijective mapping $A \in \mathcal{B}(X,Y)$ such that $\phi(T) = ATA^{-1}$ for all $T \in \mathcal{B}(X)$, or there is a bijective mapping $B \in \mathcal{B}(X^*,Y)$ such that $\phi(T) = BT^*B^{-1}$ for all $T \in \mathcal{B}(X)$.*

Brešar and Šemrl in [38], considered a more general situation where a surjective linear map preserve the spectral radius of each operator. They proved the following theorem.

Theorem 2.2.5. *([38, Theorem 1].) Suppose that a surjective linear map $\phi\mathcal{B}(X) \to \mathcal{B}(X)$ satisfies $r(\phi(A)) = r(A)$ for every $A \in \mathcal{B}(X)$. Then there exists a complex number c such that $\mid c \mid = 1$ and either:*

1. *there exists a bounded bijective linear operator $T : X \to X$ such that $\phi(A) = cTAT^{-1}$ for every $A \in \mathcal{B}(X)$; or*
2. *there exists a bounded bijective linear operator $T : X \to X$ such that $\phi(A) = cTA^*T^{-1}$ for every $A \in \mathcal{B}(X)$. In this case X must be reflexive.*

Later, Bhatia, Šemrl and Sourour described in [22] the form of all surjective maps defined on the algebra $\mathcal{M}_n(\mathbb{C})$ of all complex $n \times n$-matrices and preserving the spectral radius of the difference of matrices. They showed that a surjective map ϕ on $\mathcal{M}_n(C)$ satisfies

$$r(\phi(T) - \phi(S)) = r(T - S), \quad T, S \in \mathcal{M}_n(\mathbb{C}),$$

if, and only if, there exist a unimodular scalar $\alpha \in \mathbb{C}$, and matrices $A, B \in \mathcal{M}_n(\mathbb{C})$ such that A is invertible and

$$\phi(T) = \alpha AT^{\kappa}A^{-1} + B$$

for all $T \in \mathcal{M}_n(\mathbb{C})$, where T^{κ} stands for one of the mappings T , T^t, T^*, or $\overline{T}$, the complex conjugation of T .

It is worth mentioning that it is unknown the full description of maps from $\mathcal{B}(X)$ onto $\mathcal{B}(Y)$ preserving the spectral radius of the some, the difference, the products, the triple products, and the Jordan product of operators. See for instance [29].

Note that the Kaplansky's problem mainly asks for characterizing additive maps that preserve invertibility. This is a much more difficult task than that of linear maps. For spectrum preservers, it was shown that the linearity can be, in the standard operator algebra case, weakened to the additivity.

Theorem 2.2.6. *(M. Omladič and P. Šemrl [94].) If $\phi : \mathcal{B}(X) \to \mathcal{B}(Y)$ is a spectrum-preserving additive mapping, then either:*

1. *there is a bounded invertible operator $A : X \to Y$ such that $\phi(T) = ATA^{-1}$ for every $T \in \mathcal{B}(X)$; or*
2. *there is a bounded invertible operator $B : X^* \to Y$ such that $\phi(T) = BT^*B^{-1}$ for every $T \in \mathcal{B}(X)$.*

Later Cui and Hou characterized additive surjections on standard operator algebras that preserve various parts of spectrum, i.e spectral functions. Recall that a standard operator algebra $\mathcal{R}$ on a Banach space X is a Banach subalgebra of $\mathcal{B}(X)$ which contains the identity and the ideal of all finite rank operators.

Theorem 2.2.7. *(Cui and Hou [51].) Let $\mathcal{A}$ and $\mathcal{B}$ be two standard operator algebras on complex Banach spaces X and Y respectively, and let $\phi : \mathcal{A} \to \mathcal{B}$ be a surjective additive map. If $\Delta^{\mathcal{B}}(\phi(T)) = \Delta^{\mathcal{A}}(T)$ for every $T \in \mathcal{A}$, then either*

1. *there exists an invertible operator $A \in \mathcal{B}(X, Y)$ such that $\phi(T) = ATA^{-1}$ for every $T \in \mathcal{A}$, or*
2. *there exists an invertible operator $A \in \mathcal{B}(X^*, Y)$ such that $\phi(T) = AT^*A^{-1}$ for every $T \in \mathcal{A}$. This case cannot occur if any one of X and Y is not reflexive.*

Where $\Delta^{\mathcal{R}}(.)$ denotes any one of the spectral functions $\sigma^{\mathcal{R}}(.), \sigma_r^{\mathcal{R}}(.), \sigma_l^{\mathcal{R}}(.), \sigma_r^{\mathcal{R}}(.) \cap \sigma_l^{R}(.), \sigma_p^{\mathcal{R}}(.)$, and some other sets, see [51], and $\Delta^{\mathcal{R}}(T)$ denotes the spectral functions of T relative to the algebra $\mathcal{R}$. Note that, when $\mathcal{R}$ is a standard operator algebra on a complex Banach space and T has finite rank, all of the above spectral functions of T coincide with the spectrum of T.

A breakthrough on the question of characterizing additive spectrum compressors is due to Kuzma [80].

Theorem 2.2.8. *(Kuzma [80]) Let $\mathcal{A}$ and $\mathcal{B}$ be two unital standard operator algebras on complex Banach spaces X and Y respectively with $\dim(X) = \infty$, and $\phi : \mathcal{A} \to \mathcal{B}$ be an additive surjection, which does not annihilate all rank one idempotents. If for each $S \in \mathcal{A}$ there exists a continuous ring homomorphism $h_s : \mathbb{C} \to \mathbb{C}$ such that*

$$\sigma^{\mathcal{B}}(\phi(S)) \subseteq (\sigma^{\mathcal{A}}(S))^{h_s},$$

then ϕ takes one of the following two forms:

- $\phi(S) = ASA^{-1} \quad (S \in \mathcal{A})$;
- $\phi(S) = BS^*B^{-1} \quad (S \in \mathcal{A})$,

where $A : X \to Y$ and $B : X^* \to Y$ are bounded (conjugate) linear bijections, respectively. In the latter case, X and Y are reflexive.

Based on developing some results on holomorphic functions. Hou and Huang in [67] characterized additive bijections between standard operator algebras that compress or expand any one of these previous spectral functions.

Theorem 2.2.9. *Let $\mathcal{A}$ and $\mathcal{B}$ be two unital standard operator algebras on complex Banach spaces X and Y respectively with $\dim(X) = \infty$, and let $\phi : \mathcal{A} \to \mathcal{B}$ be an additive surjection which does not annihilate all rank one idempotents. If for each $S \in \mathcal{A}$ there exists a continuous ring homomorphism $\eta_s : \mathbb{C} \to \mathbb{C}$ such that:*

$$\Delta^{\mathcal{B}}(\phi(S)) \subseteq (\Delta^{\mathcal{A}}(S))^{\eta_s},$$

then ϕ is one of the following forms: an isomorphism, a conjugate isomorphism, an anti-isomorphism, or a conjugate anti-isomorphism. More precisely, either:

1. *There exists a bounded (conjugate) linear bijection $A : X \to Y$ such that $\phi(S) = ASA^{-1}$, for every $S \in \mathcal{A}$; or*

2. *Both X and Y are reflexive, and there exists a bounded (conjugate) linear bijection $B : X^* \to Y$ such that $\phi(S) = AS^*A^{-1}$, for every $S \in \mathcal{A}$.*

They also give a characterization of additive maps that expand the spectral functions.

Corollary 2.2.10. *(Hou and Huang [67].) Let $\phi : \mathcal{A} \to \mathcal{B}$ be an additive surjection. If for each $S \in \mathcal{A}$ there exists a continuous ring homomorphism $\eta_s : \mathbb{C} \to \mathbb{C}$ such that:*

$$\Delta^{\mathcal{B}}(\phi(S)) \supseteq (\Delta^{\mathcal{A}}(S))^{\eta_s},$$

then it takes one of the forms in Theorem 2.2.9.

In [71], Hou and Huang showed that surjective maps completely preserving certain spectral function $\Delta(.)$ from $\mathcal{A}$ to $\mathcal{B}$ are isomorphisms.

Theorem 2.2.11. *(Hou and Huang [71]) Let X, Y be infinite dimensional Banach spaces over the complex field $\mathbb{C}$ and $\mathcal{A}, \mathcal{B}$ be standard operator algebras on X and Y, respectively. Assume that $\phi : \mathcal{A} \to \mathcal{B}$ is a surjective map. Let $\Delta^{\mathcal{R}}(.)$ denotes any one of the spectral functions cited above. Then the following statements are equivalent:*

1. *ϕ is completely $\Delta^{\mathcal{R}}(.)$ preserving;*

2. *There exists an invertible operator $T \in \mathcal{B}(X, Y)$ such that*

$$\phi(T) = TAT^{-1} \text{ for all } A \in \mathcal{A};$$

3. *ϕ can uniquely extend to an isomorphism between $\mathcal{B}(X)$ and $\mathcal{B}(Y)$.*

In [7, 41], the question whether for two given elements $a, b \in \mathcal{A}$ the condition $\sigma(ax) = \sigma(bx)$ for all $x \in \mathcal{A}$ entails $a = b$ was studied, and an affirmative answer has been obtained in [7, 41] for various special cases and for some classes of algebras, including $\mathbf{C}^*$-algebras. While in [35], Braatvedt and Brits showed that the answer is always affirmative if $\mathcal{A}$ is a semi-simple Banach algebra. Among other results and problems, a spectrum-preserving problem has been discussed in [41] but omitting the condition of linearity, and studying maps ϕ from $\mathcal{A}$ onto $\mathcal{B}$ satisfying

$$\sigma(\phi(a)\phi(b)) = \sigma(ab), \quad (a, b \in \mathcal{A}).$$

Later, Molnár in [90] studied a not necessarily linear multiplicatively spectrum preserving self-maps of certain algebras $\mathcal{A}$. In particular, he obtained the following result:

Theorem 2.2.12. *(Molnár [90]) If H is an infinite dimensional complex Hilbert space, then a map ϕ from $\mathcal{B}(H)$ onto itself satisfies:*

$$\sigma(\phi(S)\phi(T)) = \sigma(ST) \tag{2.2.1}$$

for all $S, T \in \mathcal{B}(H)$ if, and only if, there is an invertible operator $A \in \mathcal{B}(H)$ such that $\phi(T) = ATA^{-1}$ for all $T \in \mathcal{B}(H)$, or $\phi(T) = -ATA^{-1}$ for all $T \in \mathcal{B}(H)$.

Molnár also obtained in [90] a similar result when the usual product in (2.2.1) is replaced by the skew product ST^*. He showed that if H is an infinite dimensional complex Hilbert space, then a map ϕ from $\mathcal{B}(H)$ onto itself satisfies

$$\sigma(\phi(S)\phi(T)^*) = \sigma(ST^*)$$

for all $S, T \in \mathcal{B}(H)$ if, and only if, there are two unitary operators $U, V \in \mathcal{B}(H)$ such that $\phi(T) = UTV$ for all $T \in \mathcal{B}(H)$.

Molnár's result [90] has been extended in different directions for some special algebras $\mathcal{A}$ on which a number of authors considered surjective, a priori nonlinear maps which preserve spectral sets or properties of the usual products or Jordan triple-products or certain so-called skew products of elements of $\mathcal{A}$; see for instance [13, 45, 46, 64, 68, 69]. Recently, Abdelali in [1] gave a complete characterezations of all maps preserving spectra of polynomial products of matrices. His results extend and unify several results obtained earlier on maps preserving the spectrum of generalized product or generalized Jordan product of matrices.

For more results about spectrum preserving maps, the interested reader can see the excellent survey [29].

2.3 Local spectral theory problem preservers

In a seminar given by Bourhim at the University of Maribor in November 2003, Brešar asked which linear maps $\phi : \mathcal{B}(X) \to \mathcal{B}(X)$ preserve the local spectrum. In 2005, Bourhim and Ransford proved that the only additive map preserving the local spectrum is the identity.

Theorem 2.3.1. *(Bourhim and Ransford [34]) Let $\phi : \mathcal{B}(X) \to \mathcal{B}(X)$ be an additive map such that:*

$$\sigma_{\phi(T)}(x) = \sigma_T(x) \quad (T \in \mathcal{B}(X), x \in X).$$

Then $\phi(T) = T$ for all $T \in \mathcal{B}(X)$.

They also deduce a generalization to the case of two different Banach spaces X and Y.

Corollary 2.3.2. *(Bourhim and Ransford [34]) Let $A \in \mathcal{B}(X, Y)$ be an invertible bounded linear map. An additive map $\phi : \mathcal{B}(X) \to \mathcal{B}(Y)$ satisfies:*

$$\sigma_{\phi(T)}(Ax) = \sigma_T(x) \quad (T \in \mathcal{B}(X), x \in X)$$

if, and only if, $\phi(T) = ATA^{-1}$ *for all* $T \in \mathcal{B}(X)$.

The proof of the first theorem is based on the following local identity principle.

Theorem 2.3.3. *([34, Theorem 1.6]) Let* $S, T \in \mathcal{B}(X)$ *and let* $x \in X$. *If* $\sigma_{S+R}(x) = \sigma_{T+R}(x)$ *for every rank one operator* $R \in \mathcal{B}(X)$, *then* $Sx = Tx$.

Motivated by Bourhim and Ransford's result, González and Mbekhta characterized in [62] linear maps on $\mathcal{M}_n(\mathbb{C})$ preserving the local spectrum at a nonzero fixed vector $x_0 \in \mathbb{C}^n$.

Theorem 2.3.4. *(González and Mbekhta [62]) Let* X *be a finite dimensional complex vector space and* $x_0 \in X$ *be a nonzero vector. Then a linear map* $\phi : \mathcal{B}(X) \to \mathcal{B}(X)$ *preserves the local spectrum at* x_0 *if, and only if, there exists an invertible* $A \in \mathcal{B}(X)$ *such that* $A(x_0) = x_0$ *and* $\phi(T) = ATA^{-1}$ *for every* $T \in \mathcal{B}(X)$.

For the case of a not necessarily finite dimensional space X, they end their paper with the following conjecture.

Conjecture 2.3.5. *Let* X *be a complex Banach space and let* $x_0 \in X$ *be a nonzero vector. Then a linear map* $\phi : \mathcal{B}(X) \to \mathcal{B}(X)$ *preserves the local spectrum at* x_0 *if, and only if, there exists an invertible* $A \in \mathcal{B}(X)$ *such that* $A(x_0) = x_0$ *and* $\phi(T) = ATA^{-1}$ *for every* $T \in \mathcal{B}(X)$.

For the local spectral radius Bourhim and Miller in [33], characterized linear maps on $\mathcal{M}_n(\mathbb{C})$ that preserve the local spectral radius at a fixed point.

Theorem 2.3.6. *(Bourhim and Miller [33]) Let* x_0 *be a nonzero fixed vector of* $\mathbb{C}^n$. *A linear map* ϕ *from* $\mathcal{M}_n(\mathbb{C})$ *into itself preserves the local spectral radius at* x_0, *i.e*

$$r_T(x_0) = r_{\phi(T)}(x_0) \quad (T \in \mathcal{M}_n(\mathbb{C})),$$

if, and only if, there exist a scalar α *of modulus* 1 *and an invertible matrix* $A \in \mathcal{M}_n(\mathbb{C})$ *such that* $Ax_0 = x_0$ *and* $\phi(T) = \alpha ATA^{-1}$ *for all* $T \in \mathcal{M}_n(\mathbb{C})$.

Theorem 2.3.4 and Theorem 2.3.6 were extended in [36] by Bračič and Müller to infinite dimensional Banach space by imposing a continuity assumption on the preserving map. They showed the following theorem:

Theorem 2.3.7. *([36, Theorem 3.2]) Let* X *be a complex Banach space and* $\phi : \mathcal{B}(X) \to \mathcal{B}(X)$ *a continuous linear surjection. Let* $e \in X$ *be a nonzero vector. Suppose that* ϕ *satisfies one of the following two conditions:*

1. $r_{\phi(T)}(e) = r_T(e)$ *for all* $T \in \mathcal{B}(X)$;
2. $\sigma_{\phi(T)}(e) = \sigma_T(e)$ *for all* $T \in \mathcal{B}(X)$.

Then ϕ is a spectral isometry, i.e $r(\phi(T)) = r(T)$ for all $T \in \mathcal{B}(X)$. Hence it takes one of the two forms in Theorem 2.2.5.

This was a partial answer to the conjecture 2.3 due to the continuity assumption. While, in [49] Costara used subharmonicity arguments and showed that the continuity assumption on ϕ may be removed. Therefore he gave an answer to this conjecture.

Theorem 2.3.8. *([49, Theorem 1.2]) Let $\phi : \mathcal{B}(X) \to \mathcal{B}(X)$ be linear and surjective satisfying*

$$r_{\phi(T)}(e) \leq r_T(e) \quad (T \in \mathcal{B}(X)),$$

for some fixed nonzero vector e in X. Then ϕ is automatically continuous.

Furthermore, motivated by Molnár results [90], Bourhim and Mashreghi obtained in [30] a complete description of surjective maps between $\mathcal{B}(X)$ and $\mathcal{B}(Y)$ preserving the local spectrum of the product of operators.

Theorem 2.3.9. *(Bourhim and Mashreghi [30]) Let $x_0 \in X$ and $y_0 \in Y$ are nonzero vectors then a map ϕ from $\mathcal{B}(X)$ onto $\mathcal{B}(Y)$ satisfies:*

$$\sigma_{\phi(S)\phi(T)}(y_0) = \sigma_{ST}(x_0), \quad (S, T \in \mathcal{B}(X)),$$

if, and only if, there exists a bijective linear operator $A \in \mathcal{B}(X, Y)$ such that $Ax_0 = y_0$ and either $\phi(T) = ATA^{-1}$ for all $T \in \mathcal{B}(X)$ or $\phi(T) = -ATA^{-1}$ for all $T \in \mathcal{B}(X)$.

The same authors in [31] characterized surjective maps preserving the local spectrum of triple product at a fixed nonzero. They proved the following theorem.

Theorem 2.3.10. *(Bourhim and Mashreghi [31]) Let $x_0 \in X$ and $y_0 \in Y$ are nonzero vectors. A map ϕ from $\mathcal{B}(X)$ onto $\mathcal{B}(Y)$ satisfies:*

$$\sigma_{\phi(S)\phi(T)\phi(S)}(y_0) = \sigma_{STS}(x_0), \quad (S, T \in \mathcal{B}(X)),$$

if, and only if, there exists a bijective mapping $A \in \mathcal{B}(X, Y)$ such that $Ax_0 = y_0$ and $\phi(T) = \lambda ATA^{-1}$ for all $T \in \mathcal{B}(X)$, where λ is a third root of unity, i.e $\lambda^3 = 2$.

In both theorems X and Y are infinite-dimensional complex Banach spaces. These results opened the way for a number of authors to consider nonlinear maps on $\mathcal{B}(X)$ preserving the local spectrum of different products of operators at a fixed vector.

Given two fixed nonzero vectors $h_0 \in H$ and $k_0 \in K$ in two infinite-dimensional complex Hilbert spaces H and K. Abdelali et al. characterized the form of surjective maps ϕ from $\mathcal{B}(H)$ onto $\mathcal{B}(K)$ preserving the local spectrum of skew products of operators.

Theorem 2.3.11. *([2, Theorem 2.1]) A map $\phi : \mathcal{B}(H) \to \mathcal{B}(K)$ such that its range contains $\mathcal{F}_2(K)$. Then ϕ satisfies:*

$$\sigma_{\phi(S)\phi(T)^*}(k_0) = \sigma_{ST^*}(h_0) \quad (S, T \in \mathcal{B}(H)),$$

if, and only if, there exist two unitary operators $U, V \in \mathcal{B}(H, K)$ and a nonzero scalar α such that $Uh_0 = \alpha k_0$ and $\phi(T) = UTV^*$ for all $T \in \mathcal{B}(H)$.

Theorem 2.3.12. *([2, Theorem 2.2]) A map $\phi : \mathcal{B}(H) \to \mathcal{B}(K)$ such that its range contains $\mathcal{F}_4(K)$. Then ϕ satisfies:*

$$\sigma_{\phi(T)\phi(S)^*\phi(T)}(k_0) = \sigma_{TS^*T}(h_0), \quad (S, T \in \mathcal{B}(H))$$

if, and only if, there exist a unitary operators $U \in \mathcal{B}(H, K)$ and a nonzero scalar α such that $Uh_0 = \alpha k_0$ and $\phi(T) = UTV^*$ for all $T \in \mathcal{B}(H)$.

Other kinds of operations were considered by many researchers including Jordan product, Lie products and their skew products, and Generalized products of operators; see for instance [2–4, 28, 56] and the recent survey [32].

We have seen that the local spectral subspaces is a very useful tools in local spectral theory. In [58], Elhodaibi and Jaatit determined the forms of all additive maps $\phi : \mathcal{B}(X) \to \mathcal{B}(X)$ that preserve the local spectral subspace associated to $\{\lambda\} \subseteq \mathbb{C}$. They proved the following result.

Theorem 2.3.13. *(Elhodaibi and Jaatit [58]) Let $\phi : \mathcal{B}(X) \to \mathcal{B}(X)$ be an additive map such that:*

$$X_{\phi(T)}(\{\lambda\}) = X_T(\{\lambda\}) \text{ for all } T \in \mathcal{B}(X) \text{ and } \lambda \in \mathbb{C},$$

then $\phi(T) = T$ for all $T \in \mathcal{B}(X)$.

Next, Benbouziane, Ech-Chérif El Kettani and Herrou in [18], characterized surjective weakly continuous maps that preserve the local spectral subspace of the difference of two operators associated with a singleton. As a continuation in this direction the same authors in [19], determined the forms of all maps preserving the local spectral subspace of sum, product and triple product of operators associated with a singleton. They showed the following results.

Theorem 2.3.14. *([19, Theorem 3.1].) A surjective map $\phi : \mathcal{B}(X) \to \mathcal{B}(X)$ satisfies*

$$X_{\phi(T)+\phi(S)}(\{\lambda\}) = X_{T+S}(\{\lambda\}) \text{ for all } T, S \in \mathcal{B}(X) \text{ and } \lambda \in \mathbb{C}.$$

if, and only if, $\phi(T) = T$ for all $T \in \mathcal{B}(X)$.

Theorem 2.3.15. *([19, Theorem 4.1]) A surjective map $\phi : \mathcal{B}(X) \to \mathcal{B}(X)$ satisfies:*

$$X_{\phi(T)\phi(S)}(\{\lambda\}) = X_{TS}(\{\lambda\}) \text{ for all } T, S \in \mathcal{B}(X) \text{ and } \lambda \in \mathbb{C}$$

if, and only if, $\phi(T) = \epsilon T$ for all $T \in \mathcal{B}(X)$, where $\epsilon = \pm 1$.

Theorem 2.3.16. *([19, Theorem 4.2]) A surjective map $\phi : \mathcal{B}(X) \to \mathcal{B}(X)$ satisfies:*

$$X_{\phi(T)\phi(S)\phi(T)}(\{\lambda\}) = X_{TST}(\{\lambda\}) \text{ for all } T, S \in \mathcal{B}(X) \text{ and } \lambda \in \mathbb{C}$$

if, and only if, $\phi(T) = \epsilon T$ for all $T \in \mathcal{B}(X)$, where $\epsilon^3 = \pm 1$.

To prove these Theorems, the authors considered the following lemma which is interesting in its own.

Lemma 2.3.17. *([19, Lemma 1.6]) Let x be a nonzero vector in X and $T, S \in \mathcal{B}(X)$. If*

$$X_T(\{\lambda\}) = X_S(\{\lambda\})(\lambda \in \mathbb{C}),$$

then $\sigma_T(x) = \{\mu\}$ if, and only if, $\sigma_T(x) = \{\lambda\}$ for all $\mu \in \mathbb{C}$.

Recently, Benbouziane, Bouramdane and Ech-Chérif El Kettani in [14] determined the form of surjective mappings $\phi : \mathcal{B}(X) \to \mathcal{B}(X)$ that preserve the local spectral subspace of generalized product of operators associated with a singleton. They mainly proved the following theorem:

Theorem 2.3.18. *([14]) Let $\phi : \mathcal{B}(X) \to \mathcal{B}(X)$ be a surjective map. Then, the following statements are equivalent:*

1. *$X_{\phi(A_1)*\cdots*(A_k)}(\{\lambda\} = X_{A_1*\cdots*A_k}(\{\lambda\})$ for all $A_1, \ldots, A_k \in \mathcal{B}(X)$ and $\lambda \in \mathbb{C}$;*
2. *There exists $\alpha \in \mathbb{C}$ with $\alpha^m = 1$ such that $\phi(A) = \alpha A$ for all $A \in \mathcal{B}(X)$.*

Recall now the definition of the generalized products of operators.

Definition 2.3.19. *Fix a positive integer $k \geq 2$ and a finite sequence $(i_1, i_2, \ldots, i_m)$ such that $\{i_1, i_2, \ldots, i_m\} = \{1, 2, \ldots, k\}$ and there is an i_p not equal i_q; that is, i_p appears just one time in the sequence. For operators $T_1, T_2, \ldots, T_k$, the operators*

$$T_1 * T_2 * \cdots * T_k = T_{i_1} T_{i_2} \ldots T_{i_m},$$

$$T_1 \circ T_2 \circ \cdots \circ T_k = T_{i_1} T_{i_2} \ldots T_{i_m} + T_{i_m} + \cdots + T_{i_2} + T_{i_1}$$

are respectively, called generalized product and generalized Jordan product of $T_1, T_2, \ldots, T_k$, while m is called the width of the products.

Evidently, the generalized Jordan product $T_1 \circ T_2 \circ \cdots \circ T_k$ (the generalized product $T_1 * T_2 * \cdots * T_k$) covers the Jordan product $T_1T_2 + T_2T_1$ and the Jordan triple product $T_1T_2T_3 + T_3T_2T_1$ (the usual product T_1T_2 and the Jordan semi-triple product $T_1T_2T_1$, and so forth. We also remark that the notations $T_1 \circ T_2 \circ \cdots \circ T_k$ and $T_1 * T_2 * \cdots * T_k$ are not unique for $T_1, T_2, \ldots, T_k$, because they depend on the choice of the integers $k \geq 2, m \geq 2$, and the sequence $(i_1, i_2, \ldots, i_m)$.

Chapter 3

Nonlinear commutant preservers

In this chapter, we will present our results concerning the study of applications that preserve the commutant of some binary operations. These results are essentially those of our article [15].

Let X be a Banach space over $\mathbb{K}$, where $\mathbb{K}$ is the complex field $\mathbb{C}$ or the real field $\mathbb{R}$, and let $\mathcal{B}(X)$ be the algebra of all bounded linear operators on X. For $A \in \mathcal{B}(X)$ we denote by $\{A\}'$ the commutant of A, that is the set of all operators in $\mathcal{B}(X)$ commuting with A. The aim of this chapter is to determine the forms of all maps $\phi : \mathcal{B}(X) \to \mathcal{B}(X)$ (not necessarily linear) which satisfy one of the following preserving properties:

- $\{\phi(A)\phi(B)\phi(A)\}' = \{ABA\}'$, for all $A, B \in \mathcal{B}(X)$.
- $\{\phi(A)\phi(B) + \phi(B)\phi(A)\}' = \{AB + BA\}'$, for all $A, B \in \mathcal{B}(X)$.

3.1 Introduction and Notations

In [73], Jafarian and Sourour described linear maps preserving the commutant or the lattice, the set of all invariant subspaces of T denoted by $Lat(T)$, of an operator in Banach algebra. In particular, they showed the following theorem.

Theorem 3.1.1. *(Jafarian and Sourour [73]) A linear map $\phi : \mathcal{B}(X) \to \mathcal{B}(X)$ satisfies*

$$\{\phi(A)\}' = \{A\}' \text{ or } \mathrm{Lat}(\phi(A)) = \mathrm{Lat}(A),$$

if, and only if, $\phi(A) = \alpha A + \varphi(A)I$ for all $A \in \mathcal{B}(X)$ where I is the identity operator, α a nonzero scalar in $\mathbb{K}$ and $\varphi : \mathcal{B}(X) \to \mathbb{K}$ a linear functional.

This result has been extended in [55], where G. Dolinar et al. characterized the form of maps preserving the lattice of sum of operators. They showed that maps (not necessarily linear) $\phi : \mathcal{B}(X) \to \mathcal{B}(X)$ satisfies $\mathrm{Lat}(\phi(A) + \phi(B)) = \mathrm{Lat}(A + B)$ for all $A, B \in \mathcal{B}(X)$, if, and only if, there is a non zero scalar α and a map $\varphi : \mathcal{B}(X) \to \mathbb{K}$ such that $\phi(A) = \alpha A + \varphi(A)I$ for all $A \in \mathcal{B}(X)$. They proved also, in the same paper, the following result.

Theorem 3.1.2. *([55, Theorem 2.]) Let X be a Banach space over $\mathbb{K}$, with dimension at least 3. Let $\phi : \mathcal{B}(X) \to \mathcal{B}(X)$. Then the following statements are equivalent.*

1. $Lat(\phi(A)\phi(B)) = Lat(AB)$ *for every pair of* $A, B \in \mathcal{B}(X)$;
2. $Lat(\phi(A)\phi(B)\phi(A)) = Lat(ABA)$ *for every pair of* $A, B \in \mathcal{B}(X)$;
3. $Lat(\phi(A)\phi(B) + \phi(B)\phi(A) = Lat(AB + BA))$ *for every pair of* $A, B \in \mathcal{B}(X)$;
4. *There exists a map* $\varphi : \mathcal{B}(X) \to \mathbb{K}$, *such that* $\phi(A) \neq 0$ *if* $A \neq 0$ *and* $\phi(A) = \varphi(A)A$ *for all* $A \in \mathcal{B}(X)$.

Motivated by these results, we study maps preserving the commutant of triple product and Jordan product of operators in $\mathcal{B}(X)$. To do that, we have state and proved the following lemma, which is interesting in its own.

Lemma 3.1.3. *Let* $A, B \in \mathcal{B}(X)$ *such that* $\{A\}' = \{B\}'$. *If* A *is a rank one operator or* A *is an idempotent (not necessarily of rank one), then* $B = \alpha A + \beta I$ *with* $\alpha, \beta \in \mathbb{K}$ *and* $\alpha \neq 0$.

Proof. Assume first that A is a rank one operator, that is $A = x \otimes f$ for some nonzero vector $x \in X$ and $f \in X^*$. Since $\{x \otimes f\}' = \{B\}'$, then in particular $(x \otimes f)B = B(x \otimes f)$. This implies that for every $y \in \mathrm{Ker}f$ we have $By \in \mathrm{Ker}f$. Let us prove that the restriction of B to $\mathrm{Ker}f$ is a multiple of the identity. To do that, let y be a nonzero vector in $\mathrm{Ker}f$. We shall prove that $By \in \langle y \rangle$. Assume, on the contrary, that y and By are linearly independent, then there exists $g \in X^*$ such that $g(y) = 1$ and $g(By) = 0$. There are two cases to be considered.

Case 1. x and By are linearly dependent. Then $g(x) = 0$, hence $(y \otimes g)(x \otimes f) = (x \otimes f)(y \otimes g)$ thus $y \otimes g \in \{x \otimes f\}'$. The hypothesis of lemma implies that

$$B(y \otimes g) = (y \otimes g)B,$$

applying this equation to y we get $By = 0$, which contradicts the fact that x is a non zero vector.

Case 2. x and By are linearly independent, then there exist $h \in X^*$ such that, $h(x) = 0$ and $h(By) = 1$. Hence $y \otimes h \in \{x \otimes f\}'$. By the hypothesis of lemma $\{x \otimes f\}' = \{B\}'$, it follows that $B(y \otimes h) = (y \otimes h)B$. This implies that $y = h(y)By$, a contradiction.

We conclude that By and y are linearly dependent. Therefore, the restriction of B to $\mathrm{Ker}f$ is a multiple of the identity.

We can find a vector $a \notin \mathrm{Ker}f$ such that $X = \mathrm{Ker}f \oplus \mathbb{K}a$. Relative to this decomposition, for every $z \in X$ we have $z = z_k + \gamma_z a$, it follows that $Bz = Bz_k + \gamma_z Ba = \beta z + \gamma_z b$, where $b = Ba - \beta a \neq 0$. Therefore, $B = \beta I + b \otimes g$, where $g : X \to \mathbb{K}$, such that $g(z) = \gamma_z$, for every $z \in X$. Since $\mathrm{Ker}f \subseteq \mathrm{Ker}g$, then $\mathrm{Ker}f = \mathrm{Ker}g$ and therefore f and g are linearly dependent. It follows that also x and y are linearly dependent and therefore $B = \alpha x \otimes f + \beta I$ with $\alpha \neq 0$, as desired.

Now, assume that A is an idempotent operator. In this case we will proceed as step 1 in proof of [73, theorem 2]. Indeed, set $X = X_1 \oplus X_2$, where $X_1 = R(A)$ and $X_2 = \mathrm{Ker}(A)$ the range and kernel of A respectively. Thus, relative to this decomposition, we write $A = I \oplus 0$ and $\{A\}'$ consists

of all operators in $\mathcal{B}(X)$ of the form $A_1 \oplus A_2$. Since $\{A\}' = \{B\}'$, then B commutes with A, and hence $B = B_1 \oplus B_2$, where B_j are operators on X_j, (j=1,2). Also B commutes with every finite rank idempotent of the form $F_1 \oplus F_2$. Thus, B_j commutes with every rank one idempotent on X_j, and it follows that every nonzero vector is an eigenvector of B_j. This implies that $B_j = \lambda_j I$ for a scalar λ_j. Thus $B = \lambda_1 I \oplus \lambda_2 I = \alpha A + \beta I$ for scalars α and β. □

3.2 Main results and proof

We begin by stating and proving the promised main results since all the necessarily ingredients are collected in the previous section. The first theorem characterizes maps that preserve the commutant of the triple product of operators.

Theorem 3.2.1. *Let X be a Banach space over $\mathbb{K}$, with $2 \leq \dim X \leq \infty$, and let ϕ map from $\mathcal{B}(X)$ into itself. Then ϕ satisfies*

$$\{\phi(A)\phi(B)\phi(A)\}' = \{ABA\}', \text{ for all } A, B \in \mathcal{B}(X), \tag{3.2.1}$$

if, and only if, there exists a map $\varphi : \mathcal{B}(X) \to \mathbb{K}$, such that $\varphi(A) \neq 0$ if $A \neq 0$ and $\phi(A) = \varphi(A)A$ for all $A \in \mathcal{B}(X)$.

Proof. The "if" part is obvious, since $\{\alpha A\}' = \{A\}'$ for every $A \in \mathcal{B}(X)$ and nonzero scalar $\alpha \in \mathbb{K}$, then

$$\{\phi(A)\phi(B)\phi(A)\}' = \{\varphi(A)^2\varphi(B)ABA\}' = \{ABA\}'.$$

Conversely, assume that ϕ satisfies (3.2.1). We break the rest of the proof into several steps.

Step 1. There exist a nonzero scalar $\mu \in \mathbb{C}$ such that $\phi(I)^3 = \mu I$.

Assume that (3.2.1) is satisfied. It follows that $\{\phi(I)^3\}' = \{I\}'$. Then $\phi(I)^3 \in \mathbb{K}I$. Suppose $\phi(I)^3 = 0$. Hence $\{\phi(I)\phi(P)\phi(I)\}' = \{P\}'$ for every idempotent P, it follows by Lemma 3.1.3 that

$$\phi(I)\phi(P)\phi(I) = \alpha P + \beta I, \quad \alpha \neq 0 \tag{3.2.2}$$

Multiplying Equation (3.2.2) with $\phi(I)^2$ first from the left and then from the right we obtain $\phi(I)^2 P = P\phi(I)^2$ for every idempotent P, thus $\phi(I)^2 \in \mathbb{K}I$. If $\phi(I)^2 = 0$, it follows from Eq. (3.2.2) in the same way that $\phi(I) \in \mathbb{K}I$, but $\phi(I)^2 = 0$ and thus $\phi(I) = 0$, a contradiction. If $\phi(I)^2 \neq 0$, then from $\phi(I)^3 = 0$ and $\phi(I)^2 \in \mathbb{K}I$ we again obtain that $\phi(I) = 0$, a contradiction. Therefore

$$\phi(I)^3 = \mu I, \mu \neq 0.$$

Step 2. There exist nonzero scalars $\lambda, \zeta \in \mathbb{C}$ such that $\phi(I) = \lambda I$ and $\phi(x \otimes f) = \zeta x \otimes f$ for every $x \in X$ and $f \in X^*$ with $f(x) = 0$.

For every rank one nilpotent $x \otimes f$ due to $\{\phi(x \otimes f)\phi(I)\phi(x \otimes f)\}' = \{0\}'$ and $\{\phi(I)\phi(x \otimes f)\phi(I)\}' = \{x \otimes f\}'$, there exist a scalar c, and by Lemma 3.1.3 scalars $\alpha \neq 0$ and β depending on $x \otimes f$, such

that

$$\phi(x\otimes f)\phi(I)\phi(x\otimes f)=cI, \tag{3.2.3}$$

and

$$\phi(I)\phi(x\otimes f)\phi(I)=\alpha x\otimes f+\beta I. \tag{3.2.4}$$

Multiplying Eq. (3.2.3) with $\phi(I)$ first from the left and then from the right we obtain $\phi(I)\phi(x\otimes f)\phi(I)\phi(x\otimes f)=c\phi(I)$ and $\phi(x\otimes f)\phi(I)\phi(x\otimes f)\phi(I)=c\phi(I)$. Combining these equations with Eq. (3.2.4) it follows that

$$(x\otimes f)\phi(x\otimes f)x\otimes f=\phi(x\otimes f)x\otimes f, \tag{3.2.5}$$

and

$$c\phi(I)=\alpha\phi(x\otimes f)x\otimes f+\beta\phi(x\otimes f). \tag{3.2.6}$$

As $\{(x\otimes f)^3\}'=\{0\}'$ there is a scalar γ such that $\phi(x\otimes f)^3=\gamma I$. Thus, multiplying Eq. (3.2.6) with $\phi(x\otimes f)$ from both sides and taking into account Eqs. (3.2.3) and (3.2.5), we obtain

$$c^2I=\gamma(\alpha x\otimes f+\beta I),$$

which is impossible if $c\neq 0$. Hence we have $c=\gamma=0$. If $\beta\neq 0$, then by Eq. (3.2.6) the rank of $\phi(x\otimes f)$ is at most one, and we have a contradiction with Eq. (3.2.4). Therefore, $\beta=0$ and hence $\phi(I)\phi(x\otimes f)\phi(I)=\alpha x\otimes f$ and $\phi(x\otimes f)x\otimes f=(x\otimes f)\phi(x\otimes f)=0$. Note that $\phi(I)$ is invertible since $\phi(I)^3=\mu I$ is invertible. So $\phi(x\otimes f)=y\otimes g$ is of rank one with $f(y)=g(x)=0$ and $\phi(I)y\in\langle x\rangle$. Fix x, we have shown that for any $h\in X^*$ with $h(x)=0$ there are $z\in X$ and $l\in X^*$ such that $\phi(x\otimes h)=z\otimes l$ with $h(z)=0$ and $\phi(I)z\in\langle x\rangle$. Since $\phi(I)$ is invertible, we must have $z\in\langle y\rangle$, $\phi(x\otimes h)=y\otimes l_h$, and $h(y)=0$. It follows that $y\in\langle x\rangle$ and therefore $\phi(I)x\in\langle x\rangle$ for every $x\in X$. Consequently, $\phi(I)=\lambda I$ and $\phi(x\otimes f)=\zeta x\otimes f$ for some nonzero scalars λ and ζ. Without loss of generality, we may assume in the sequel that $\phi(I)=I$ and $\phi(x\otimes f)=x\otimes f$ whenever $f(x)=0$. In particular, condition (3.2.1) implies that $\{\phi(A)\}'=\{A\}'$ for all $A\in\mathcal{B}(X)$.

Step 3. There exist a nonzero scalar $\zeta\in\mathbb{C}$ such that $\phi(x\otimes f)=\zeta x\otimes f$ for every rank one operators.

Let $y\otimes g$ be a non-nilpotent rank one operator. Then $\phi(y\otimes g)=\alpha y\otimes g+\beta I, \alpha\neq 0$, by Lemma 3.1.3. Take x, f such that $g(x)\neq 0, f(y)\neq 0$ and $f(x)=0$. Then, on the one hand $(y\otimes g)(x\otimes f)(y\otimes g)=g(x)f(y)y\otimes g\neq 0$ and $\phi(y\otimes g)\phi(x\otimes f)\phi(y\otimes g)=\gamma y\otimes g+\delta I, \gamma\neq 0$, by Lemma 3.1.3. On the other, $\phi(y\otimes g)\phi(x\otimes f)\phi(y\otimes g)=(\alpha y\otimes g+\beta I)(x\otimes f)(\alpha y\otimes g+\beta I)=(\alpha g(x)y+\beta x)\otimes(\alpha f(y)g+\beta f)$. It follows that $\delta=0$ and since x and y are linearly independent, also $\beta=0$, that is, $\phi(y\otimes g)=\alpha y\otimes g$. So, we may assume that $\phi(x\otimes f)=x\otimes f$ for every rank one operator.

Step 4. ϕ takes the desired form.

We show that $\phi(A)=\varphi(A)A$ for every $A\in\mathcal{B}(X)$. Equalities $\{f(\phi(A)x)x\otimes f)\}'=\{\phi(x\otimes f)\phi(A)\phi(x\otimes f)\}'=\{f(Ax)x\otimes f\}'$ hold for every x, f. Therefore $f(\phi(A)x)=0$ if, and only if, $f(Ax)=0$ which forces that $\phi(A)x\in\langle Ax\rangle$ holds for all $x\in X$. Hence $\phi(A)=\varphi(A)A$ for some scalar

$\varphi(A)$, finishing the proof. □

The following result gives the complete description of maps that preserve commutant of Jordan product operators.

Theorem 3.2.2. *Let X be a Banach space over $\mathbb{K}$, with $3 \leq \dim X \leq \infty$, and let ϕ map from $\mathcal{B}(X)$ into itself. Then ϕ satisfies*

$$\{\phi(A)\phi(B) + \phi(B)\phi(A)\}' = \{AB + BA\}', \text{ for all } A, B \in \mathcal{B}(X), \tag{3.2.7}$$

if, and only if, there exists a map $\varphi : \mathcal{B}(X) \to \mathbb{K}$, such that $\varphi(A) \neq 0$ if $A \neq 0$ and $\phi(A) = \varphi(A)A$ for all $A \in \mathcal{B}(X)$.

Proof. The "if" part is obvious since $\{\alpha A\}' = \{A\}'$ for every $A \in \mathcal{B}(X)$ and nonzero scalar $\alpha \in \mathbb{K}$, then

$$\{\phi(A)\phi(B) + \phi(B)\phi(A)\}' = \{(\varphi(A)\varphi(B))(AB + BA)\}' = \{AB + BA\}'.$$

Conversely, assume that ϕ satisfies (3.2.7). We break the rest of the proof into several steps.

Step 1. $\phi(I) = \lambda I$, for some nonzero scalar λ.

Let P be an idempotent operator. By hypothesis, we have $\{\phi(P)\phi(I)+\phi(I)\phi(P)\}' = \{P\}'$. Lemma 3.1.3 implies that:

$$\phi(P)\phi(I) + \phi(I)\phi(P) = \alpha P + \beta I, \tag{3.2.8}$$

and $\phi(I)^2 = \gamma I$, where α is a nonzero scalar and $\beta, \gamma \in \mathbb{K}$. Multiplying equation (3.2.8) with $\phi(I)$ first from the left and then from the right we obtain:

$$\phi(I)P = P\phi(I), \text{ for every idempotent operator } P. \tag{3.2.9}$$

Let x be a nonzero vector in X. We shall prove that $\phi(I)x$ and x are linearly dependent. Assume on the contrary that x and $\phi(I)x$ are linearly independent and pick a linear functional $f \in X^*$ with $f(x) = 1$ and $f(\phi(I)x) = 0$. Applying (3.2.9), for $P = x \otimes f$, to x we get $\phi(I)x = 0$ which is a contradiction. Hence $\phi(I) = \lambda I$ for some nonzero scalar λ. Indeed, if $\lambda = 0$ then by hypothesis:

$$\{0\}' = \{\phi(A)\phi(I) + \phi(I)\phi(A)\}' = \{AI + IA\}' = \{A\}'$$

for every $A \in \mathcal{B}(X)$, which is impossible.

Step 2. For every rank one operator $R \in \mathcal{B}(X)$, there is a nonzero scalar α_R such that $\phi(R) = \alpha_R R$. Let $R \in \mathcal{B}(X)$ be a rank one operator. Then $\{R\}' = \{RI + IR\}' = \{\phi(R)\phi(I) + \phi(I)\phi(R)\}' = \{\phi(R)\}'$ and by lemma 3.1.3, there exist scalars $\alpha_R \neq 0$ and β_R such that $\phi(R) = \alpha_R R + \beta_R I$. It remains to prove that $\beta_R = 0$. Indeed, while $\dim(X) \geq 3$, then there is a rank one operator Q such that $RQ = QR = 0$. Since $\{\phi(R)\phi(Q) + \phi(Q)\phi(R)\}' = \{0\}'$, then, by Lemma 3.1.3, we have:

$$\gamma I = \phi(R)\phi(Q) + \phi(Q)\phi(R) = (\alpha_R R + \beta_R I)(\alpha_Q Q + \beta_Q I) + (\alpha_Q Q + \beta_Q I)(\alpha_R R + \beta_R I),$$

for some $\gamma \in \mathbb{K}$. Thus, $\alpha_R\beta_Q R + \alpha_Q\beta_R Q \in \mathbb{K}I$. Since $\dim X \geq 3$ and $\alpha_R\beta_Q R + \alpha_Q\beta_R Q$ is with rank at most 2, we deduce that $\beta_R = \beta_Q = 0$, as desired.

Step 3. ϕ takes the desired form.

To do that, it suffices to prove that $\phi(A)$ and A are linearly dependent for every operator $A \in \mathcal{B}(X)$. Similarly to the proof of the previous theorem, suppose for the sake of contradiction that there is $A \in \mathcal{B}(X)$ such that $\phi(A)$ and A are linearly independent. So, there exists a nonzero vector $x \in X$ such that $\phi(A)x$ and Ax are linearly independent. Then choose $f \in X^*$ with $f(Ax) = 1$ and $f(\phi(A)x) = 0$. By hypothesis and the fact that $\phi(x \otimes f) = \alpha_{x\otimes f} x \otimes f$, we get

$$\begin{aligned} \{Ax \otimes f + x \otimes fA\}' &= \{\phi(A)\phi(x \otimes f) + \phi(x \otimes f)\phi(A)\}' \\ &= \{\phi(A)x \otimes f + x \otimes f\phi(A)\}' \end{aligned}$$

in particular,

$$(Ax \otimes f + x \otimes fA)(\phi(A)x \otimes f + x \otimes f\phi(A)) = (\phi(A)x \otimes f + x \otimes f\phi(A))(Ax \otimes f + x \otimes fA). \quad (3.2.10)$$

Applying this equation to x and to Ax, we deduce that

$$f(x)f(A\phi(A)x)x = 2f(x)\phi(A)x + f(x)f(\phi(A)Ax)x \quad (3.2.11)$$

and

$$f(x)f(\phi(A)Ax)Ax + f(A\phi(A)x)x = \phi(A)x + f(x)f(A^2x)\phi(A)x. \quad (3.2.12)$$

To finish the proof, we need to consider two cases.

Case 1. $f(x) \neq 0$. Since $\{A\}' = \{\phi(A)\}'$, in particular $\phi(A)A = A\phi(A)$, the equation (3.2.11) implies that $\phi(A)x = 0$ which is a contradiction.

Case 2. $f(x) = 0$. By equation (3.2.12), $f(A\phi(A)x)x = \phi(A)x$, which means that x and $\phi(A)x$ are linearly dependent. Next, we choose $g \in X^*$ such that $g(\phi(A)x) = 1$ and $g(Ax) = 0$, then $g(x) \neq 0$. Applying (3.2.10) after replacing f by g, to x, we obtain that

$$2g(x)Ax + g(x)g(A\phi(A)x)x = g(x)g(\phi(A)Ax)x.$$

Then, $2Ax + g(A\phi(A)x)x = g(\phi(A)Ax)x$. Applying g to this equation, we get

$$g(A\phi(A)x) = g(\phi(A)Ax)$$

hence $2Ax = 0$, a contradiction.

The contradiction obtained in all cases implies that for every operator $A \in \mathcal{B}(X)$, there exists a scalar $\varphi(A)$ such that $\phi(A) = \varphi(A)A$.

It remains to prove that $\varphi(A) \neq 0$ for every nonzero operator A. So, assume that there exists $A \neq 0$

with $\varphi(A) = 0$. By hypothesis, we have

$$\{0\}' = \{\phi(A)\phi(I) + \phi(I)\phi(A)\}' = \{AI + IA\}' = \{A\}'.$$

Then by lemma 3.1.3 there exists a nonzero scalar $\alpha \neq 1$ such that $A = \alpha I$. Hence

$$\{0\}' = \{\phi(A)\phi(B) + \phi(B)\phi(A)\}' = \{\alpha B + \alpha B\}' = \{B\}',$$

for every $B \in \mathcal{B}(X)$. Which is impossible. The proof is therefore complete. □

Chapter 4

On the pseudo spectrum preservers

In this chapter, we will present our results concern the study of maps that compress or extend the psedospectrum. Also, the form of maps preserving the pseudospectrum of generalized products have been studied. These are the results of our article [57].

Let X and Y be two complex Banach spaces, and let $\mathcal{B}(X)$ denote the algebra of all bounded linear operators on X. In this chapter, we characterize additive maps from $\mathcal{B}(X)$ onto $\mathcal{B}(Y)$ compressing the pseudospectrum subsets $\Delta_\epsilon(.)$, where $\Delta_\epsilon(.)$ stands for any one of the spectral functions $\sigma_\epsilon(.)$, $\sigma^l_\epsilon(.)$, and $\sigma^r_\epsilon(.)$ for some $\epsilon > 0$. We also characterize additive (resp. non-linear) maps from $\mathcal{B}(X)$ onto $\mathcal{B}(Y)$ preserving the pseudospectrum $\sigma_\epsilon(.)$ of generalized products of operators for some $\epsilon > 0$ (resp. for every $\epsilon > 0$).

4.1 Introduction

In [79], Kumar and Kulkarni gave several results about linear maps preserving pseudospectrum and condition spectrum. From what they have proven, if a linear onto map preserves the pseudospectrum between two Banach algebras then it preserves the spectrum. Later in [96], Ragoubi defines the left pseudospectrum and the right pseudospectrum. Based on [79], she gives some similar results for linear maps preserving left (or right) pseudospectrum.

Let $\mathcal{A}$ be a complex Banach algebra with unit 1. We shall identify $\lambda.1$ with λ. An element $a \in \mathcal{A}$ is said to be left invertible if there exists $b \in \mathcal{A}$ such that $ba = 1$, while it is said right invertible if there exists $b \in \mathcal{A}$ such that $ab = 1$. An invertible element of $\mathcal{A}$ is a right and left invertible element of $\mathcal{A}$. We denote by $Inv(\mathcal{A})$, $Inv^l(\mathcal{A})$, and $Inv^r(\mathcal{A})$ the sets of all invertible, left invertible, and right invertible elements of $\mathcal{A}$, respectively.
Let $a \in \mathcal{A}$, the left spectrum, the right spectrum and the spectrum of a are defined respectively as follow:

$$\sigma^l(a) = \{\lambda \in \mathbb{C} : a - \lambda \notin Inv^l(\mathcal{A})\},$$

$$\sigma^r(a) = \{\lambda \in \mathbb{C} : a - \lambda \notin Inv^r(\mathcal{A})\},$$

and

$$\sigma(a) = \{\lambda \in \mathbb{C} : a - \lambda \notin Inv(\mathcal{A})\}.$$

Also, the left pseudospectrum, the right pseudospectrum, and the pseudospectrum of a are defined respectively as follow:

$$\sigma_\epsilon^l(a) = \{\lambda \in \mathbb{C} : \inf\{\| b \| : b \text{ a left inverse of } a-\lambda\} \geq \frac{1}{\epsilon}\},$$

$$\sigma_\epsilon^r(a) = \{\lambda \in \mathbb{C} : \inf\{\| b \| : b \text{ a right inverse of } a-\lambda\} \geq \frac{1}{\epsilon}\},$$

and

$$\sigma_\epsilon(a) = \{\lambda \in \mathbb{C} :\| (a-\lambda)^{-1} \| \geq \frac{1}{\epsilon}\}.$$

With the convention $\inf\{\| b \| : b \text{ a left inverse of } a-\lambda\} = \infty$, $\inf\{\| b \| : b \text{ a right inverse of } a-\lambda\} = \infty$, and $\| (a-\lambda)^{-1} \| = \infty$ when $a-\lambda$ is not left invertible, is not right invertible, and is not invertible. For more informations about this subsets; see [79, 96, 105].

It is clear that for all $a \in \mathcal{A}$, we have

$$\sigma^l(a) \subseteq \sigma_\epsilon^l(a) \subseteq \sigma_\epsilon(a),$$

and

$$\sigma^r(a) \subseteq \sigma_\epsilon^r(a) \subseteq \sigma_\epsilon(a).$$

In particular, $\sigma_\epsilon^l(a)$ and $\sigma_\epsilon^r(a)$ are non empty sets for every $a \in \mathcal{A}$.

Throughout this chapter, X and Y denote complex Banach spaces and $\mathcal{B}(X,Y)$ denotes the algebra of all bounded linear operators from X into Y. As usual, when $X = Y$, we simply write $\mathcal{B}(X)$ instead of $\mathcal{B}(X,X)$. We denote by T^* the operators adjoint of T. In the sequel $\Delta_\epsilon(.)$ (resp. $\Delta(.)$) stands for any one of the spectral functions $\sigma_\epsilon(.)$, $\sigma_\epsilon^l(.)$ or $\sigma_\epsilon^r(.)$ (resp. $\sigma(.)$, $\sigma^l(.)$ or $\sigma^r(.)$).

4.2 Preliminaries

In this section, we collect some lemmas that will be used in the proof of our main results. We begin by a useful property of the pseudospectrum.

Lemma 4.2.1. *([76, Theorem 2.3].)Let X be a complex Banach space, $T \in \mathcal{B}(X)$ and $\epsilon > 0$. Then we have:*

$$\sigma_\epsilon(\lambda T) = \lambda \sigma_{\frac{\epsilon}{|\lambda|}}(T), \text{ for all } \lambda \in \mathbb{C} \setminus \{0\}.$$

The second result due to Skhiri [100] gives necessary and sufficient condition for an operator to be an isometry multiplied by a scalar. A simple and shorter proof was given by Bourhim in [27, Lemma 1.4].

Lemma 4.2.2. *(Skhiri [100]) For a bijective mapping $A \in \mathcal{B}(X,Y)$, the following are equivalent:*

1. $\| ATA^{-1} \| = \| T \|$ *for all invertible operators* $T \in \mathcal{B}(X)$;

2. $\| ATA^{-1} \| \leq \| T \|$ *for all invertible operators* $T \in \mathcal{B}(X)$;

3. $\| ATA^{-1} \| \geq \| T \|$ *for all invertible operators* $T \in \mathcal{B}(X)$;

4. A *is an isometry multiplied by a scalar.*

We end this section by giving sufficient conditions for a map from $\mathcal{B}(X)$ into $\mathcal{B}(Y)$ to preserve $\Delta_\epsilon(.)$. Hence in what follows, we will consider only the necessary conditions.

Lemma 4.2.3. *([67, Theorem 3.1 and Corollary 3.7].) Let* ϕ *be a map from* $\mathcal{B}(X)$ *into* $\mathcal{B}(Y)$*, if* ϕ *is*
- *linear;*
- *bijective;*
- *isometry.*

Then ϕ *preserves* $\Delta_\epsilon(.)$.

4.3 Additive maps compressing pseudospectrum and certain subsets

We start this section by citing more lemmas needed for the proof of our main results. The first one gives some characterizations of additive maps that compress some spectral functions.

Lemma 4.3.1. *([67, Corollary 9]) Let* X *be a complex Banach space such that* $\dim(X) = \infty$*. If* $\phi : \mathcal{B}(X) \longrightarrow \mathcal{B}(Y)$ *is an additive surjection which does not annihilate all rank-one idempotents, satisfies* $\Delta(\phi(S)) \subseteq \Delta(S)$ *for each* $S \in \mathcal{B}(X)$*, then* ϕ *is either an isomorphism or an anti-isomorphism. More precisely, either*

1. *there exists a bounded linear bijection* $A : X \longrightarrow Y$ *such that* $\phi(S) = ASA^{-1}$ *for all operator* $S \in \mathcal{B}(X)$*, or*

2. *both* X *and* Y *are reflexive, and there exists a bounded linear bijection* $A : X^* \longrightarrow Y$ *such that* $\phi(S) = AS^*A^{-1}$ *for all operator* $S \in \mathcal{B}(X)$.

The second one proved by the same authors gives some characterizations of additive maps that expand a spectral functions.

Lemma 4.3.2. *([67, Corollary 8]) Let* X *be a complex Banach space such that* $\dim(X) = \infty$*. If* $\phi : \mathcal{B}(X) \longrightarrow \mathcal{B}(Y)$ *is an additive surjection, satisfies* $\Delta(\phi(S)) \supseteq \Delta(S)$ *for each* S *in* $\mathcal{B}(X)$*. Then* ϕ *is either an isomorphism or an anti-isomorphism.*

For the case of operators on Hilbert spaces, they obtained the following result.

Lemma 4.3.3. *([67, Corollary 10]) Let* H *and* K *be an infinite dimensional complex Hilbert spaces, and let* $\phi : \mathcal{B}(H) \longrightarrow \mathcal{B}(K)$ *be an additive surjection which does not annihilate all rank-one idempotents. If* $\Delta(\phi(S)) \subseteq \Delta(S)$ *holds for each* $S \in \mathcal{B}(H)$*, then either*

1. *there exists a linear bounded invertible operator* $A : H \longrightarrow K$ *such that* $\phi(S) = ASA^{-1}$ *for all* $S \in \mathcal{B}(H)$, *or*

2. *there exists a linear bounded invertible operator* $A : H \longrightarrow K$ *such that* $\phi(S) = AS^{tr}A^{-1}$ *for all* $S \in \mathcal{B}(H)$, *where* S^{tr} *stands for the transpose of* S *with respect to an arbitrarily fixed orthonormal base of* H. *If* $\Delta(.)$ *takes any one of* $\sigma^l(.)$ *and* $\sigma^r(.)$, *then the last case can not occur.*

The following is our main result for this section.

Theorem 4.3.4. *Let* $\epsilon > 0$ *and suppose that* $\phi : \mathcal{B}(X) \longrightarrow \mathcal{B}(Y)$ *is an additive surjection which does not annihilate all rank-one idempotents. If for each* $S \in \mathcal{B}(X)$, $\Delta_\epsilon(\phi(S)) \subseteq \Delta_\epsilon(S)$, *then* ϕ *is either an isomorphism or an anti-isomorphism. More precisely, either*

1. *there exists an isometry* $A : X \longrightarrow Y$ *such that* $\phi(S) = ASA^{-1}$ *for all* $S \in \mathcal{B}(X)$; *or*

2. *both* X *and* Y *are reflexive, and there exists an isometry* $A : X^* \longrightarrow Y$ *such that* $\phi(S) = AS^*A^{-1}$ *for all* $S \in \mathcal{B}(X)$.

Proof. Assume that $\Delta_\epsilon(.) = \sigma_\epsilon(.)$. we will proceed by two steps.
Step 1. ϕ compress the spectra of elements. Then ϕ takes one of the two forms in Lemma 4.3.1.

Suppose that $\sigma_\epsilon(\phi(S)) = \sigma({}_\epsilon S)$ for all $S \in \mathcal{B}(X)$. Let $S \in \mathcal{B}(X)$ and $\lambda \notin \sigma(S)$. Choose an integer n such that $n > \epsilon \parallel (\lambda - S)^{-1} \parallel$. Then

$$\parallel (n(\lambda - S))^{-1} \parallel < \frac{1}{\epsilon}.$$

Thus,

$$n\lambda \notin \sigma_\epsilon(nS) \supseteq \sigma_\epsilon(\phi(nS)) \supseteq \sigma(\phi(nS)) = n\sigma(\phi(S)),$$

and so,

$$\lambda \notin \sigma(\phi(S)).$$

Therefore,

$$\sigma(\phi(S)) \subseteq \sigma(S).$$

By Lemma 4.3.1 ϕ takes one of the following forms:

1. There exists a bounded linear bijection $A : X \longrightarrow Y$ such that $\phi(S) = ASA^{-1}$ for all $S \in \mathcal{B}(X)$;

2. Both X and Y are reflexive, and there exists a bounded linear bijection $A : X^* \longrightarrow Y$ such that $\phi(S) = AS^*A^{-1}$ for all $S \in \mathcal{B}(X)$.

Step 2. ϕ takes the desired forms.

To do that, it suffices to prove that ϕ preserves norms of all invertible elements of $\mathcal{B}(X)$ and applied Lemma 4.2.2. Indeed, suppose there exists an invertible operator $S \in \mathcal{B}(X)$ such $\| S^{-1} \| < \| \phi(S^{-1}) \|$. According to the first step, ϕ is $\mathbb{R}$-linear. Then, we can choose a reel $t > 0$ such that

$$\epsilon \| S^{-1} \| < t \leq \epsilon \| \phi(S^{-1}) \| = \epsilon \| (\phi(S))^{-1} \| .$$

Then

$$\| (tS)^{-1} \| < \frac{1}{\epsilon}.$$

Thus $0 \notin \sigma_\epsilon(tS)$. But

$$\| (\phi(tS))^{-1} \| \geq \frac{1}{\epsilon}$$

implies $0 \in \sigma_\epsilon(\phi(tS))$, a contradiction. Similarly, we can prove the result if $\Delta_\epsilon(.)$ takes any one of the other sets, by proceeding like the proof of [96, Theorem 3.2]. □

Applying Theorem 4.3.4, we can give a characterization of additive maps that expand $\Delta_\epsilon(.)$.

Corollary 4.3.5. *Let $\epsilon > 0$ and suppose that $\phi : \mathcal{B}(X) \longrightarrow \mathcal{B}(Y)$ is an additive surjection. If for each $S \in \mathcal{B}(X)$, $\Delta_\epsilon(\phi(S)) \supseteq \Delta_\epsilon(S)$ then ϕ is either an isomorphism or an anti-isomorphism.*

Proof. Like the first step in the previous proof, we can show that ϕ expend the spectra of elements. Hence by Lemma 4.3.2, ϕ takes the desired forms. □

Remark 4.3.6.

1. *It is obvious that if ϕ is bijective or it preserve $\Delta_\epsilon(.)$, then ϕ does not annihilate all rank-one idempotents. So, if we assume that ϕ is bijective or preserves $\Delta_\epsilon(.)$, we can drop this hypothesis.*

2. *If ϕ preserves $\Delta_\epsilon(.)$ we can drop condition on dimension of X in Theorem 4.3.4. We can proceed as the proof of [67, Corollary 3.16].*

For the case of operators on Hilbert spaces, the statement of Theorem 4.3.4 and Lemma 4.3.3 need be slightly modified in an obvious way. We illustrate it by the following special consequence.

Corollary 4.3.7. *Let H and K be an infinite dimensional complex Hilbert spaces, and let $\phi : \mathcal{B}(H) \longrightarrow \mathcal{B}(K)$ be an additive surjection, which does not annihilate all rank-one idempotents. If $\Delta_\epsilon(\phi(S)) \subseteq \Delta_\epsilon(S)$ holds for each $S \in \mathcal{B}(H)$, then either*

1. *there exists an isometry $A : H \longrightarrow K$ such that $\phi(S) = ASA^{-1}$ for all $S \in \mathcal{B}(H)$, or*

2. *there exists an isometry $A : H \longrightarrow K$ such that $\phi(S) = AS^{tr}A^{-1}$ for all $S \in \mathcal{B}(H)$, where S^{tr} stands for the transpose of S with respect to an arbitrarily fixed orthonormal base of H. If $\Delta_\epsilon(.)$ takes any one of $\sigma_\epsilon^l(.)$ and $\sigma_\epsilon^r(.)$, then the last case (2) can not occur.*

4.4 Maps preserving pseudospectrum of generalized product

In what follows, let m be an integer and fix a positive integer $k \geq 2$ and a finite sequence $(i_1, i_2, ..., i_m)$ such that $\{i_1, i_2, ..., i_m\} = \{1, 2, ..., k\}$ and there is an i_p not equal to i_q for all other q. We define the generalized product $T_1 \circ T_2 \circ ... \circ T_k$, for operators $T_1, ..., T_k \in \mathcal{B}(X)$ by

$$T_1 \circ T_2 \circ ... \circ T_k = T_{i_1}...T_{i_m},$$

or

$$T_1 \circ T_2 \circ ... \circ T_k = T_{i_1}...T_{i_m} + T_{i_m}...T_{i_1}.$$

The usual product TS, the triple product TST and the Jordan product $TS + ST$ are particular cases of this product. Note that, maps preserving certain spectral quantities of the generalized product have been considered by many authors; see for instance [3, 4, 14].

We will say that ϕ satisfies the condition (**P**) if the range of ϕ contains all operators of rank at most two if, $T_1 \circ T_2 \circ ... \circ T_k = T_{i_1}...T_{i_m}$, or the range contain all operators of rank at most three if $T_1 \circ T_2 \circ ... \circ T_k = T_{i_1}...T_{i_m} + T_{i_m}...T_{i_1}$, respectively.

Before stating the main results for this section. We will give a characterisation of maps preserving the spectrum of generalized product.

Lemma 4.4.1. *([68, 69].) Let X and Y be complex Banach spaces. If a map ϕ from $\mathcal{B}(X)$ into $B(Y)$ satisfies the condition (**P**) and*

$$\sigma(\phi(T_1) \circ \phi(T_2) \circ ... \circ \phi(T_k)) = \sigma(T_1 \circ T_2 \circ ... \circ T_k)$$

for all $T_1, ..., T_k \in \mathcal{B}(X)$. Then, ϕ takes one of the following forms:

1. *There exist a scalar λ with $\lambda^m = 1$ and an invertible operator $A : X \longrightarrow Y$ such that $\phi(T) = \lambda ATA^{-1}$ for all $T \in \mathcal{B}(X)$.*

2. *Both X and Y are reflexive, and there exist a scalar λ with $\lambda^m = 1$ and an invertible operator $A : X^* \longrightarrow Y$ such that $\phi(T) = \lambda AT^*A^{-1}$ for all $T \in \mathcal{B}(X)$.*

The first result characterizes additive maps preserving pseudospectrum of generalized product for some $\epsilon > 0$.

Theorem 4.4.2. *Let X and Y be complex Banach spaces and $\epsilon > 0$. If an additive map ϕ from $\mathcal{B}(X)$ into $\mathcal{B}(Y)$ satisfies the condition (**P**) and*

$$\sigma_\epsilon(\phi(T_1) \circ \phi(T_2) \circ ... \circ \phi(T_k)) = \sigma_\epsilon(T_1 \circ T_2 \circ ... \circ T_k)$$

for all $T_1, \ldots, T_k \in \mathcal{B}(X)$. Then one of the following forms holds:

1. *There exist a scalar λ with $\lambda^m = 1$ and an isometry $A : X \longrightarrow Y$ such that $\phi(T) = \lambda ATA^{-1}$ for all $T \in \mathcal{B}(X)$.*

2. *Both X and Y are reflexive, and there exist a scalar λ with $\lambda^m = 1$ and an isometry $A : X^* \longrightarrow Y$ such that $\phi(T) = \lambda AT^*A^{-1}$ for all $T \in \mathcal{B}(X)$.*

Proof. First, we will show that ϕ preserves the spectra of generalized product of elements. Indeed, let $\lambda \notin \sigma(T_1 \circ T_2 \circ ... \circ T_k)$. Choose $n > \epsilon \parallel (\lambda - T_1 \circ T_2 \circ ... \circ T_k)^{-1} \parallel$. Then

$$\parallel (n(\lambda - T_1 \circ T_2 \circ ... \circ T_k))^{-1} \parallel < \frac{1}{\epsilon}.$$

Thus

$$\begin{aligned} n\lambda \notin \sigma_\epsilon(nT_1 \circ T_2 \circ ... \circ T_k) &= \sigma_\epsilon(\phi(nT_1) \circ \phi(T_2) \circ ... \circ \phi(T_k)) \\ &\supseteq \sigma(\phi(nT_1) \circ \phi(T_2) \circ ... \circ \phi(T_k)) \\ &= n\sigma(\phi(T_1) \circ \phi(T_2) \circ ... \circ \phi(T_k)), \end{aligned}$$

and so,

$$\lambda \notin \sigma(\phi(T_1) \circ \phi(T_2) \circ ... \circ \phi(T_k)).$$

Therefore,

$$\sigma(\phi(T_1) \circ \phi(T_2) \circ ... \circ \phi(T_k)) \subseteq \sigma(\phi(T_1) \circ \phi(T_2) \circ ... \circ \phi(T_k)).$$

In a similar way, we can prove the other inclusion. Hence we get

$$\sigma(\phi(T_1) \circ \phi(T_2) \circ ... \circ \phi(T_k)) = \sigma(\phi(T_1) \circ \phi(T_2) \circ ... \circ \phi(T_k)).$$

Now, by the previous lemma ϕ takes one of the following forms:

1. There exist a scalar λ with $\lambda^m = 1$ and an invertible operator $A : X \longrightarrow Y$ such that $\phi(T) = \lambda ATA^{-1}$ for all $T \in \mathcal{B}(X)$.

2. Both X and Y are reflexive, and there exist a scalar λ with $\lambda^m = 1$ and an invertible operator $A : X^* \longrightarrow Y$ such that $\phi(T) = \lambda AT^*A^{-1}$ for all $T \in \mathcal{B}(X)$.

Finally, let us prove that $\parallel ASA^{-1} \parallel \leq \parallel S \parallel$ for all invertible elements $S \in \mathcal{B}(X)$. Indeed, assume that there exists an invertible operator $S \in \mathcal{B}(X)$ such $\parallel S^{-1} \parallel < \parallel AS^{-1}A^{-1} \parallel$. According to the first step, ϕ is $\mathbb{R}$-linear and $\sigma_\epsilon(T) = \sigma_\epsilon(\lambda^{m-1}\phi(T)) = \sigma_\epsilon(ATA^{-1})$ or $\sigma_\epsilon(T) = \sigma_\epsilon(AT^*A^{-1})$, for every $T \in \mathcal{B}(X)$. Then, we can choose $t > 0$ such that:

$$\epsilon \parallel S^{-1} \parallel < t \leq \epsilon \parallel AS^{-1}A^{-1} \parallel = \epsilon \parallel (ASA^{-1})^{-1} \parallel .$$

Then

$$\parallel (tS)^{-1} \parallel < \frac{1}{\epsilon}.$$

Thus, $0 \notin \sigma_\epsilon(tS)$. But

$$\| (\phi(tS))^{-1} \| \geq \frac{1}{\epsilon}$$

implies $0 \in \sigma_\epsilon(\phi(tS))$, a contradiction. Then, by Lemma 4.2.2, ϕ takes the desired forms. □

Next result characterizes maps preserving pseudospectrum of generalized product for every $\epsilon > 0$.

Theorem 4.4.3. *Let X and Y be complex Banach spaces. If $\phi : \mathcal{B}(X) \longrightarrow \mathcal{B}(Y)$ is a map satisfies the condition (**P**) and*

$$\sigma_\epsilon(\phi(T_1) \circ \phi(T_2) \circ ... \circ \phi(T_k)) = \sigma_\epsilon(T_1 \circ T_2 \circ ... \circ T_k)$$

for each $T_1, \ldots, T_k \in \mathcal{B}(X)$ and every $\epsilon > 0$. Then ϕ takes one of the following forms:

1. *There exist a scalar λ with $\lambda^m = 1$ and an isometry $A : X \longrightarrow Y$ such that $\phi(T) = \lambda A T A^{-1}$ for all $T \in \mathcal{B}(X)$.*
2. *Both X and Y are reflexive, and there exist a scalar λ with $\lambda^m = 1$ and an isometry $A : X^* \longrightarrow Y$ such that $\phi(T) = \lambda A T^* A^{-1}$, for all $T \in \mathcal{B}(X)$.*

Proof. We will proceed like the proof of the previous theorem. First we show that ϕ preserve the spectra of generalized product of elements. Indeed, let $\lambda \notin \sigma(T_1 \circ T_2 \circ ... \circ T_k)$. Choose $n > \epsilon \| (\lambda - T_1 \circ T_2 \circ ... \circ T_k)^{-1} \|$. Then

$$\| (n(\lambda - T_1 \circ T_2 \circ ... \circ T_k))^{-1} \| < \frac{1}{\epsilon}.$$

Thus,

$$\begin{aligned} n\lambda \notin \sigma_\epsilon(nT_1 \circ T_2 \circ ... \circ T_k) &= n\sigma_{\frac{\epsilon}{n}}(T_1 \circ T_2 \circ ... \circ T_k) \\ &= n\sigma_{\frac{\epsilon}{n}}(\phi(T_1) \circ \phi(T_2) \circ ... \circ \phi(T_k)) \\ &\supseteq n\sigma(\phi(T_1) \circ \phi(T_2) \circ ... \circ \phi(T_k)). \end{aligned}$$

so,

$$\lambda \notin \sigma(\phi(T_1) \circ \phi(T_2) \circ ... \circ \phi(T_k)).$$

Therefore

$$\sigma(\phi(T_1) \circ \phi(T_2) \circ ... \circ \phi(T_k)) \subseteq \sigma(\phi(T_1) \circ \phi(T_2) \circ ... \circ \phi(T_k)).$$

Similarly, way we can prove the other inclusion. Hence we get

$$\sigma(\phi(T_1) \circ \phi(T_2) \circ ... \circ \phi(T_k)) = \sigma(\phi(T_1) \circ \phi(T_2) \circ ... \circ \phi(T_k)).$$

Now, by the Lemma 4.4.1, ϕ takes one of the following forms:

1. There exist a scalar λ with $\lambda^m = 1$ and an invertible operator $A : X \longrightarrow Y$ such that $\phi(T) = \lambda A T A^{-1}$ for all $T \in \mathcal{B}(X)$.

2. Both X and Y are reflexive, and there exist a scalar λ with $\lambda^m = 1$ and an invertible operator $A : X^* \longrightarrow Y$ such that $\phi(T) = \lambda AT^*A^{-1}$ for all $T \in \mathcal{B}(X)$.

Finally, we prove that $\| ASA^{-1} \| \leq \| S \|$ for all invertible elements $S \in \mathcal{B}(X)$. Then, by Lemma 4.2.2, ϕ takes the desired forms. □

Chapter 5

Maps preserving condition spectrum of Jordan skew triple product of operators

In this chapter, we will present our results concerning the study of applications that preserve the condition spectrum of Jordan skew triple product of operators. These results are essentially those of our article [16].

Let H be a Hilbert space over the complex field $\mathbb{C}$, and $0 < \epsilon < 1$. The condition spectrum of $A \in \mathcal{B}(H)$, $\Lambda_\epsilon(A)$(see section 1.6.2), is defined by

$$\Lambda_\epsilon(A) := \{z \in \mathbb{C} : \|z - A\|\|(z - A)^{-1}\| \geq \epsilon^{-1}\},$$

with the convention that $\|z - A\|\|(z - A)^{-1}\| = \infty$ when $z - A$ is not invertible. In this chapter, we characterize the form of surjective maps $\phi : \mathcal{B}(H) \to \mathcal{B}(H)$ satisfying

$$\Lambda_\epsilon(\phi(A)\phi(B)^*\phi(A)) = \Lambda_\epsilon(AB^*A) \text{ for all } A, B \in \mathcal{B}(H).$$

5.1 Introduction

In recent years, a great activity has occurred in characterizing maps preserving condition spectrum, pseudo spectra, condition spectral radius and pseudo spectral radius. The study of linear case was done by Kumar and Kulkarni in [79]. The nonlinear case has also been studied in recent papers [20, 21, 50, 52, 53]. The authors changed the linearity of such map ϕ by the weaker condition $F(\phi(A \diamond B)) = F(A \diamond B)$ where $A \diamond B$ is one of different kinds of operations such as sum $A + B$, the usual product AB, triple product ABA, the Jordan product $AB + BA$, Lie product $AB - BA$ and skew product A^*B. $F(.)$ is one of the following objects $\Lambda_\epsilon(.)$ or $r_\epsilon(.)$. In [21], Bendaoud et al. characterized surjective maps preserving the condition spectrum and condition spectral radius of product and Jordan triple product of operators. In particular, they showed that a surjective map ϕ from $\mathcal{B}(H)$ into itself, where H is an Hilbert space with $\dim H \geq 3$, satisfies

$$r_\epsilon(\phi(A)\phi(B)\phi(A)) = r_\epsilon(ABA), \quad (A, B \in \mathcal{B}(H)),$$

if, and only if, there exist a functional $h : \mathcal{B}(H) \to \mathbb{C}$, with $|h(A)| = 1$ for all $A \in \mathcal{B}(H)$, and a unitary or anti-unitary operator U on H such that either $\phi(A) = h(A)UAU^*$ or $\phi(A) = h(A)UA^*U^*$ for all

$A \in \mathcal{B}(H)$. They showed also that a surjective map ϕ from $\mathcal{B}(H)$ into itself, where H is a Hilbert space with $\dim H \geq 3$, satisfies

$$\Lambda_\epsilon(\phi(A)\phi(B)\phi(A)) = \Lambda_\epsilon(ABA), \quad (A, B \in \mathcal{B}(H))$$

if, and only if, there is a third root of unity c and a unitary operator $U \in \mathcal{B}(H)$ such that either $\phi(A) = cUAU^*$ or $\phi(A) = cUA^{tr}U^*$ for all $A \in \mathcal{B}(H)$, where A^{tr} is the transpose of A with respect to an arbitrary but fixed orthogonal basis of H.

5.2 Preliminaries

In this section, we collect some lemmas that will be used in the proof of our main results. In the sequel, we denote by ϵ a positive number such that $0 < \epsilon < 1$. We begin by gathering some properties of the ϵ-condition spectral radius.

Lemma 5.2.1. *([20, Proposition 2.6].) Let H be a complex Hilbert space and $T \in \mathcal{B}(H)$. Then we have:*

1. *$r_\epsilon(T) = 0$ if, and only if, $T = 0$;*
2. *$r_\epsilon(UTU^*) = r_\epsilon(T)$, for all unitary operators $U \in \mathcal{B}(H)$;*
3. *$r_\epsilon(\lambda T) = |\lambda| r_\epsilon(T)$ for $\lambda \in \mathbb{C}$;*
4. *$r_\epsilon(T^*) = r_\epsilon(T)$, where T^* stands for the adjoint of T.*

The second result gives link between rank-one operators which have the same ϵ-condition spectral radius in particular case.

Lemma 5.2.2. *([20, Proposition 2.6].) Let $u, v \in H$. If $r_\epsilon(u \otimes f) = r_\epsilon(v \otimes f)$ for every $f \in H$, then the vectors u and v are linearly dependent.*

The following lemma characterizes the condition spectrum and the condition spectral radius of nontrivial projection.

Lemma 5.2.3. *([20, Proposition 2.4].) Let $P \in \mathcal{B}(H)$ be a nontrivial projection. Then*

$$\Lambda_\epsilon(P) = \overline{D}(\frac{1}{1-\epsilon^2}, \frac{\epsilon}{1-\epsilon^2}) \cup \overline{D}(\frac{-\epsilon^2}{1-\epsilon^2}, \frac{\epsilon}{1-\epsilon^2}),$$

where $\overline{D}(a, r)$ is the closed disc centered at $a \in \mathbb{C}$ with radius $r \geq 0$.
In particular,

$$r_\epsilon(P) = \frac{1}{1-\epsilon}.$$

We end this section by the following theorem which characterizes surjective maps preserving the condition spectral radius of Jordan skew triple product of operators.

Theorem 5.2.4. *([16, Theorem 3.1]) Let H be a complex Hilbert space with dim $H \geq 3$. Let $\mathcal{W}$, $\mathcal{V}$ be subsets of $\mathcal{B}(H)$ which contain all rank-one operators. Suppose that $\phi : \mathcal{W} \to \mathcal{V}$ is a surjective map. Then ϕ satisfies*

$$r_\epsilon(AB^*A) = r_\epsilon(\phi(A)\phi(B)^*\phi(A)) \text{ for all } A,\ B \in \mathcal{W} \tag{5.2.1}$$

if, and only if, one of the following statements holds:

1. *There exist a unitary or an anti-unitary operator U on H and a functional $\varphi : \mathcal{W} \to \mathbb{C}$ with $|\varphi(A)| = 1$ such that $\phi(A) = \varphi(A)UAU^*$ for all $A \in \mathcal{W}$;*
2. *There exist a unitary or an anti-unitary operator U on H and a functional $\varphi : \mathcal{W} \to \mathbb{C}$ with $|\varphi(A)| = 1$ such that $\phi(A) = \varphi(A)UA^*U^*$ for all $A \in \mathcal{W}$.*

This theorem is the subject of a chapter in Bouramdane Youssef's thesis, see [24].

5.3 Main Results

As all the ingredients are gathered in the previous section we can now present and prove our result for this chapter. It is a characterization of surjective maps on $\mathcal{B}(H)$ preserving ϵ-condition spectrum of the Jordan skew triple product of operators.

Theorem 5.3.1. *Let H be a complex Hilbert space with dim $H \geq 3$. Let $\phi : \mathcal{B}(H) \to \mathcal{B}(H)$ be a surjective map. Then ϕ satisfies*

$$\Lambda_\epsilon(AB^*A) = \Lambda_\epsilon(\phi(A)\phi(B)^*\phi(A)) \text{ for all } A, B \in \mathcal{B}(H) \tag{5.3.1}$$

if, and only if, one of the following statements holds.

1. *There exists a unitary operator $U \in \mathcal{B}(H)$ such that $\phi(A) = UAU^*$ for all $A \in \mathcal{B}(H)$;*
2. *There exists an anti-unitary operator U on H such that $\phi(A) = UA^*U^*$ for all $A \in \mathcal{B}(H)$.*

Proof. The "if" part is obvious, since $\Lambda_\epsilon(UAU^*) = \Lambda_\epsilon(A)$ (resp. $\Lambda_\epsilon(UA^*U^*) = \Lambda_\epsilon(A)$) for every $A \in \mathcal{B}(H)$ and every unitary (resp. anti-unitary) operator U. Then

$$\Lambda_\epsilon(AB^*A) = \Lambda_\epsilon(\phi(A)\phi(B)^*\phi(A)) \text{ for all } A, B \in \mathcal{B}(H)$$

Conversely, assume that ϕ satisfies equation (5.3.1). It follows that ϕ preserves the ϵ-condition spectral radius of Jordan skew triple product of operators. Then, by Theorem 5.2.4, there exist a unitary or an anti-unitary operator U on H and $\varphi : \mathcal{W} \to \mathbb{C}$ with $|\varphi(A)| = 1$ such that either

$$\phi(A) = \varphi(A)UAU^* \text{ for all } A \in \mathcal{B}(H), \tag{5.3.2}$$

or

$$\phi(A) = \varphi(A)UA^*U^* \text{ for all } A \in \mathcal{B}(H). \tag{5.3.3}$$

We will complete the proof by discussing two cases.

Case 1. ϕ takes form (5.3.2).

• Suppose that U is unitary. Let us prove that $\varphi(A) = 1$ for every $A \in \mathcal{B}(H)$. Indeed, if $\phi(A) = \varphi(A)UAU^*$ for all $A \in \mathcal{B}(H)$ with U unitary. It follows from equation (5.3.1) that for all $A \in \mathcal{B}(H)$ we have

$$\begin{aligned}\Lambda_\epsilon(A) &= \Lambda_\epsilon(\phi(I)\phi(A^*)^*\phi(I)) \\ &= \Lambda_\epsilon(\varphi(I)^2\overline{\varphi(A^*)}UAU^*) \\ &= \varphi(I)^2\overline{\varphi(A^*)}\Lambda_\epsilon(A).\end{aligned}$$

In particular, for $A = I$, since $\Lambda_\epsilon(I) = \{1\}$, we have $\varphi(I)^2\overline{\varphi(I)} = 1$. Thus $\varphi(I) = 1$. So

$$\Lambda_\epsilon(A) = \overline{\varphi(A^*)}\Lambda_\epsilon(A) \; for \; all \; A \in \mathcal{B}(H),$$

which implies that, for every $A \in \mathcal{B}(H)$,

$$z \in \Lambda_\epsilon(A) \Leftrightarrow \overline{\varphi(A^*)}z \in \Lambda_\epsilon(A).$$

If $\Lambda_\epsilon(A)$ is not a disc centered at zero, we deduce that $\overline{\varphi(A^*)} = 1$. Therefore, $\varphi(A) = 1$ for every $A \in \mathcal{B}(H)$. Otherwise, if $\Lambda_\epsilon(A)$ is a disc centered at zero, choose x, f in H such that $\langle x, f\rangle \neq 0$ and $\langle A^*x, f\rangle \neq 0$. By Lemma 5.2.3, $\Lambda_\epsilon(x \otimes f)$ is not a disc centered at zero. Then by the consideration above, $\varphi(x \otimes f) = 1$. Furthermore, by equation (5.3.1)

$$\begin{aligned}\Lambda_\epsilon(x \otimes fA^*x \otimes f) &= \Lambda_\epsilon(\phi(x \otimes f)\phi(A)^*\phi(x \otimes f)) \\ &= \varphi(x \otimes f)^2\overline{\varphi(A)}\Lambda_\epsilon(x \otimes fA^*x \otimes f) \\ &= \overline{\varphi(A)}\langle A^*x, f\rangle\Lambda_\epsilon(x \otimes f).\end{aligned}$$

On the other hand, $\Lambda_\epsilon(x \otimes fA^*x \otimes f) = \langle A^*x, f\rangle\Lambda_\epsilon(x \otimes f)$. Thus, for all $A \in \mathcal{B}(H)$ $\langle A^*x, f\rangle\Lambda_\epsilon(x \otimes f) = \overline{\varphi(A)}\langle A^*x, f\rangle\Lambda_\epsilon(x \otimes f)$. Since $\langle A^*x, f\rangle \neq 0$ and $\Lambda_\epsilon(x \otimes f)$ is not a circular disc centered at zero, we deduce that $\overline{\varphi(A)} = 1$. Finally, $\varphi(A) = 1$ for all $A \in \mathcal{B}(H)$, and so $\phi(A) = UAU^*$ for all $A \in \mathcal{B}(H)$.

• Assume that U is anti-unitary.

Let us show that this case cannot occur. We will use an idea given in [20, Theorem 3.5]. Assume that $\phi(A) = \varphi(A)UAU^*$ for all $A \in \mathcal{B}(H)$ and U is an anti-unitary operator. Note that by Lemma 5.2.3, one can see that for every unit vector $x \in H$

$$\Lambda_\epsilon(x \otimes x) = \overline{\Lambda_\epsilon(x \otimes x)}.$$

Following the same discussion as in the previous case, we get that for every $A \in \mathcal{B}(H)$,

$$\Lambda_\epsilon(A) = \varphi(A)\overline{\varphi(A^*)\Lambda_\epsilon(A)}.$$

In particular, for $A = I$ we get $\varphi(I) = 1$. Moreover, if $x \in H$ is a unit vector, by equation (5.3.1), we get

$$\begin{aligned}\Lambda_\epsilon(x \otimes x) &= \Lambda_\epsilon(x \otimes xIx \otimes x)) \\ &= \Lambda_\epsilon(\varphi(x \otimes x)^2\overline{\varphi(I)}Ux \otimes xU^*) \\ &= \varphi(x \otimes x)^2\overline{\Lambda_\epsilon(x \otimes x)} \\ &= \varphi(x \otimes x)^2\Lambda_\epsilon(x \otimes x).\end{aligned}$$

Thus, $\varphi(x \otimes x)^2 = 1$. Now, take an arbitrary non self adjoint operator $A \in \mathcal{B}(H)$ and a unit vector $x \in H$ such that $\langle A^*x, x\rangle \neq 0$. Then it follows that

$$\begin{aligned}\Lambda_\epsilon(x \otimes xA^*x \otimes x) &= \Lambda_\epsilon(\phi(x \otimes x)\phi(A)^*\phi(x \otimes x)) \\ &= \varphi(x \otimes x)^2\overline{\varphi(A)}\Lambda_\epsilon(Ux \otimes xA^*x \otimes xU^*) \\ &= \overline{\varphi(A)\langle A^*x, x\rangle}\Lambda_\epsilon(x \otimes x).\end{aligned}$$

This implies that,

$$\overline{\varphi(A)\langle A^*x, x\rangle} = \langle A^*x, x\rangle,$$

and so,

$$\frac{\langle A^*x, x\rangle}{\overline{\langle A^*x, x\rangle}} = \overline{\varphi(A)}.$$

In particular, for $A \in \mathcal{B}(H)$ for which there exist nonzero vectors $x, y \in H$ satisfying $A^*x = x$ and $A^*y = iy$ we get

$$1 = \frac{\langle A^*x, x\rangle}{\overline{\langle A^*x, x\rangle}} = \overline{\varphi(A)} = \frac{\langle A^*y, y\rangle}{\overline{\langle A^*y, y\rangle}} = -1,$$

which is a contradiction.

Case 2. ϕ takes the form (5.3.3).

We have $\phi(A) = \varphi(A)UA^*U^*$ for all $A \in \mathcal{B}(H)$. With the same discussion as in the first case with U unitary or anti-unitary we deduce that $\varphi(I) = 1$.

• Let U be unitary. By equation (5.3.1) and the fact that $\varphi(I) = 1$, we have

$$\Lambda_\epsilon(A) = \overline{\varphi(A^*)}\Lambda_\epsilon(A^*) \quad for \ \ all \ \ A \in \mathcal{B}(H).$$

Then, following the same steps as the second discussion in the first case, we deduce that this case cannot occur.

• Assume that U is anti-unitary. To complete the proof, let us show that $\varphi(A) = 1$ for every $A \in \mathcal{B}(H)$. We have $\phi(A) = \varphi(A)UA^*U^*$ for every $A \in \mathcal{B}(H)$. It follows that $\Lambda_\epsilon(A) = \varphi(A^*)\Lambda_\epsilon(A)$. As in the consideration above, we conclude that $\varphi(A) = 1$ for all $A \in \mathcal{B}(H)$. Then ϕ takes the desired form. □

Chapter 6

Nonlinear fixed points dimension preservers

In this chapter, we will present our results concern the study of surjective maps preserving the dimension of fixed point set of generalized products of operators. These are the results of our article [25].

Let $\mathcal{B}(X)$ be the algebra of all bounded linear operators on an infinite dimensional complex Banach space X. For $A \in \mathcal{B}(X)$, let $F(A)$ be the set of all fixed points of A. For integers $k \geq 2$ and $m \geq 2$, let $(i_1, \cdots, i_m)$ be a finite sequence such that $\{i_1, \cdots, i_m\} = \{1, \cdots, k\}$ and let at least one of the terms in $(i_1, \cdots, i_m)$ appear exactly once. The generalized product of width m of k operators $A_1, \cdots, A_k \in \mathcal{B}(X)$ is defined by

$$A_1 * A_2 * \cdots * A_k = A_{i_1} A_{i_2} \cdots A_{i_m}.$$

and includes the usual product and the triple product. In this chapter, we characterize the form of surjective maps from $\mathcal{B}(X)$ into itself satisfying

$$\dim F(\phi(A_1) * \cdots * \phi(A_k)) = \dim F(A_1 * \cdots * A_k)$$

for all $A_1, \cdots, A_k \in \mathcal{B}(X)$.

6.1 Introduction

In [102], Taghavi and Hosseinzadeh proved that if X is a complex Banach space with $\dim X \geq 3$ and if a surjective map $\phi : \mathcal{B}(X) \to \mathcal{B}(X)$ satisfies $\dim F(\phi(A)\phi(B)) = \dim F(AB)$ for all $A, B \in \mathcal{B}(X)$, then there exists an invertible operator $S \in \mathcal{B}(X)$ such that $\phi(A) = \pm SAS^{-1}$ for all $A \in \mathcal{B}(X)$. In [103], the authors studied the surjective maps $\phi : \mathcal{B}(X) \to \mathcal{B}(X)$ which satisfy $F(\phi(A)+\phi(B)) = F(A+B)$ for all $A, B \in \mathcal{B}(X)$, they conclude that $\phi(A) = UA+R$ for all $A \in \mathcal{B}(X)$ where $U = I-2\phi(0)$ and $R = \phi(0)$. They characterized also the structure of maps satisfying $\dim F(\phi(A) + \phi(B)) = \dim F(A + B)$ in the finite dimensional case. In [104] the authors proved that if X is a complex Banach space with $\dim X \geq 3$ and $\phi : \mathcal{B}(X) \to \mathcal{B}(X)$ is surjective map satisfying $F(\phi(A)\phi(B)\phi(A)) = F(ABA)$ for all $A, B \in \mathcal{B}(X)$. Then $\phi(A) = \alpha A$ for all $A \in \mathcal{B}(X)$, where $\alpha \in \mathbb{C}$ with $\alpha^3 = 1$.

Recently, there have been a lot of activities on describing maps that preserve the generalized product of operators; see for example [3, 4, 14, 26, 95] and the references therein. In this chapter, we propose to determine the form of mappings that preserve the dimension of the fixed point set of generalized product of operators. The main result is the following theorem.

Theorem 6.1.1. *Let m be an integer such that $m \geq 2$. Consider the generalized product of width m, $A_1 * \cdots * A_k$.*
Let $\phi : \mathcal{B}(X) \to \mathcal{B}(X)$ be a surjective map which satisfies the following condition

$$\dim F(\phi(A_1) * \cdots * \phi(A_k)) = \dim F(A_1 * \cdots * A_k) \text{ for all } A_1, \cdots, A_k \in \mathcal{B}(X).$$

Then, there exists a bounded invertible linear (or conjugate linear) operator $S : X \to X$ and a complex number $\lambda \in \mathbb{C}$ such that $\phi(A) = \lambda SAS^{-1}$ for all $A \in \mathcal{B}(X)$ with $\lambda^m = 1$.

6.2 Preliminaries

In this section, we collect and prove some lemmas that will be used in the sequel. The first lemma is a direct consequence of the fixed point set and the definition of rank one operators. Recall that $\mathcal{P}_1(X)$ stands for rank one idempotent operators on X.

Lemma 6.2.1. *Let $x \in X$ and $f \in X^*$, consider the rank one operator $x \otimes f$. Then we have.*

$$x \otimes f \in \mathcal{P}_1(X) \text{ if, and only if, } F(x \otimes f) = \langle x \rangle \tag{6.2.1}$$

$$x \otimes f \notin \mathcal{P}_1(X) \text{ if, and only if, } F(x \otimes f) = \{0\}, \tag{6.2.2}$$

The next lemma characterizes the scalar operators in terms of the dimension of the fixed point set.

Lemma 6.2.2. *Assume $A \in \mathcal{B}(X)$, Then*

1. *$A = I$ if, and only if, $\dim F(PAP) = 1$ (for every $P \in \mathcal{P}_1(X)$);*
2. *A is a scalar operator different from I if, and only if, $\dim F(PAP) = 0$ (for every $P \in \mathcal{P}_1(X)$);*
3. *A is a scalar operator if, and only if, $P \mapsto \dim F(PAP)$ is a constant (for every $P \in \mathcal{P}_1(X)$).*

Proof. 1. evident.

2. Assume that $A \in \mathbb{C}I$, then $A = \alpha I$ for some $\alpha \in \mathbb{C} \setminus \{1\}$, consequently $F(PAP) = F(\alpha P)$ for every $P \in \mathcal{P}_1(X)$. Thus $\dim F(PAP) = \dim F(\alpha P) = 0$.
   Conversely, assume that $A \notin \mathbb{C}I$. Then there exists a non zero vector $x \in X$ such that x and Ax are linearly independent. It follows that there is a linear functional $f \in X^*$ such that $f(Ax) = f(x) = 1$. Define $P = x \otimes f$. Then, we have $\dim F(PAP) = \dim F(P) = 1$ which ends the proof.

3. Consequence of (i) and (ii).

□

Lemma 6.2.3. *([106, Proposition 3.3]) Assume that $A, B \in \mathcal{B}(X)$ are non scalar operators. If*

$$PAP \in \mathcal{P}_1(X) \Rightarrow PBP \in \mathcal{P}_1(X) \text{ for every } P \in \mathcal{P}_1(X),$$

then there exists $\lambda \in \mathbb{C} \setminus \{1\}$ such that $B = \lambda I + (1-\lambda)A$.

Lemma 6.2.4. *Let A and B be non scalar operators. If $\dim F(PAP) = \dim F(PBP)$, for every $P \in \mathcal{P}_1(X)$, then there exists $\lambda \in \mathbb{C} \setminus \{1\}$ such that $B = \lambda I + (1-\lambda)A$.*

Proof. Let A and B be non scalar operators and $P = x \otimes f$ be a rank one idempotent. If $PAP \in \mathcal{P}_1(X) \setminus \{0\}$, then $1 = \dim F(PAP) = \dim F(PBP)$. Hence PBP is a rank one operator. Then $F(PBP) = \langle x \rangle$ and so $PBP \in \mathcal{P}_1(X) \setminus \{0\}$. Hence Lemma 6.2.3 ends the proof. □

In what follows, we give some conditions for equality among two operators in terms of the dimension of fixed points of generalized product.

Lemma 6.2.5. *Let A, B be nonzero operators and r, s two positive integers ($r, s \in \mathbb{N}$) such that $r + s \geq 1$. The following statements are equivalent:*

1. *$A = B$;*
2. *$\dim F(T^r A T^s) = \dim F(T^r B T^s)$ for all operators $T \in \mathcal{B}(X)$;*
3. *$\dim F(R^r A R^s) = \dim F(R^r B R^s)$ for all rank one operators $R \in \mathcal{B}(X)$.*

Proof. The implications $(i) \Rightarrow (ii) \Rightarrow (iii)$ can be easily obtained. We need only to prove that $(iii) \Rightarrow (i)$. Indeed, assume that (iii) holds. By Lemma 6.2.2, we have $A \in \mathbb{C}I \setminus \{I\}$ if, and only if, $B \in \mathbb{C}I \setminus \{I\}$. Consider $A = \alpha I$ and $B = \beta I$ for some $\alpha, \beta \in \mathbb{C} \setminus \{1\}$. Let us show that $\alpha = \beta$. For $x \in X$ and $f \in X^*$, we have by assumption

$$\dim F(f(x)^{r+s-1} \alpha x \otimes f) = \dim F(f(x)^{r+s-1} \beta x \otimes f).$$

Choosing x and f such that $f(x)^{r+s} = \alpha^{-1}$, we get $\alpha = \beta$. It remains to show that $A = B$ for a pair of a non scalar operators. Let A be a non scalar operator. Then B is a non scalar operator, too. It follows by Lemma 6.2.4 that there is a scalar $\lambda \in \mathbb{C} \setminus \{1\}$ such that $B = \lambda I + (1-\lambda)A$. Let us prove that $\lambda = 0$. Assume on the way of contradiction that $\lambda \neq 0$ and let $x \in X$ and $f \in X^*$ such that $f(x)$ is not an $r+s$-th root of the unity and $f(x) \neq 0$ and $f(Ax) = \frac{1}{f(x)^{r+s-1}}$. Then, we have $(x \otimes f)^r A (x \otimes f)^s = f(x)^{r+s-2} f(Ax) x \otimes f$ which is a rank one idempotent operator. Consequently

$$\dim((x \otimes f)^r A (x \otimes f)^s) = 1. \tag{6.2.3}$$

On the other hand

$$(x \otimes f)^r(\lambda I + (1-\lambda)A)(x \otimes f)^s = f(x)^{r+s-2}(\lambda f(x) + (1-\lambda)f(Ax))x \otimes f,$$

and

$$f(x)^{r+s-1}(\lambda f(x) + (1-\lambda)f(Ax)) \neq 1$$

It follows that

$$\dim F((x \otimes f)^r B(x \otimes f)^s) = \dim F((x \otimes f)^r(\lambda I + (1-\lambda)A)(x \otimes f)^s) = 0. \tag{6.2.4}$$

Combining the equations (6.2.3) and (6.2.4), we get a contradiction and the lemma is established. □

The following lemma gives a characterization of rank one operators by the dimension of fixed points of generalized product of operators.

Lemma 6.2.6. *Let r and s be positive integers such that $r + s \geq 1$. Let $R \in \mathcal{B}(X)$ be a nonzero operator, the following statements are equivalent:*

1. *R is a rank one operator;*
2. *$\dim F(A^r R A^s)) \leq 1$ for all $A \in \mathcal{B}(X)$.*

Proof. Assume that R is a rank one operator and $A \in \mathcal{B}(X)$ is an arbitrary operator. Then $A^r R A^s$ is an operator of rank at most one. Therefore, by (6.2.1) and (6.2.2), we have $\dim F(A^r R A^s) \leq 1$, and the implication $(i) \Rightarrow (ii)$ is established. Conversely, assume that R has rank at least two. Let us show that there exists $A \in \mathcal{B}(X)$ such that $\dim F(A^r R A^s) \geq 2$. Since $\text{rank}(R) \geq 2$, let y_1, y_2 be two linearly independent vectors in the range of R and x_1, x_2 in X such that $Rx_1 = y_1$ and $Rx_2 = y_2$. Assume that $s \neq 0$ and $r \neq 0$. Since X has an infinite dimension, we can find linearly independent vectors

$$z_0, z_1, ..., z_s, z_{s+1}, ..., z_{s+r-1}, w_0, w_1, ..., w_s, ..., w_{s+r-1}$$

with $z_s := y_1$ and $w_s := y_2$. We can find a finite rank operator $A \in \mathcal{B}(X)$ such that

$$Az_{s-1} = x_1,\ Az_{s+r-1} = z_0,\ Aw_{s-1} = x_2,\ Aw_{s+r-1} = w_0$$

and for $i \in \{0, 1, ..., r+s-1\} \setminus \{s-1, r+s-1\}$

$$Az_i = z_{i+1} \text{ and } Aw_i = w_{i+1}.$$

Note that $A^r R A^s z_0 = z_0$ and $A^r R A^s w_0 = w_0$. Then $z_0, w_0 \in F(A^r R A^s)$, hence $\dim F(A^r R A^s) \geq 2$. This shows that the implication $(ii) \Rightarrow (i)$ always holds in case both r and s are positive integers.

To finish, assume that $s = 0$ as the case when $r = 0$ is similar. Then choose linearly independent vectors $z_0, z_1, ..., z_{r-1}, w_0, w_1, ..., w_{r-1}$ in X with $z_0 = y_1, w_0 = y_2$. Thus, there exists a finite rank operator $A \in \mathcal{B}(X)$ such that $Az_i = z_{i+1}$ for $i \in \{0, 1, ..., r-2\}$ and $Az_{r-1} = x_1$ and $Aw_i = w_{i+1}$ for

$i \in \{0, 1, ..., r-2\}$ and $Aw_{r-1} = x_2$. We have $A^r R x_1 = x_1$ and $A^r R x_2 = x_2$. Hence, $x_1, x_2 \in F(A^r R)$ and $\dim F(A^r R) \geq 2$. This establishes the implication $(ii) \Rightarrow (i)$ in this case, too. □

We end this section by the following lemma which characterises surjective maps preserving the nonzero idempotency of generalized product on the set of rank one idempotents.

Lemma 6.2.7. *([95, Proposition 1.4]) Let X be an infinite dimensional real or complex Banach space. A surjective mapping $\phi : \mathcal{P}_1(X) \to \mathcal{P}_1(X)$ preserves the nonzero idempotency of generalized product in both directions if, and only if, there exists a bounded invertible linear (or conjugate linear) operator $S : X \to X$ such that*

$$\phi(P) = SPS^{-1}, \qquad P \in \mathcal{P}_1(X),$$

or, only if X is reflexive, there exists a bounded invertible linear (or conjugate linear) $S : X^* \to X$ such that

$$\phi(P) = SP^*S^{-1}, \qquad P \in \mathcal{P}_1(X),$$

6.3 Main Result

Since all the necessary ingredients are collected in the preliminary section, we will state and prove the promised main result. Let $A, B \in \mathcal{B}(X)$, set $A_{i_p} = B$ and $A_{i_j} = A$ for $j \neq p$ where i_p is the term which appears exactly once in $(i_1, \cdots, i_m)$. Note that $A_1 * A_2 * \cdots * A_k = A^r B A^s$ for some positive integers r and s such that $r + s = m - 1$. Then, the Theorem 6.1.1 is a consequence of the following one.

Theorem 6.3.1. *Let r and s be two positive integers with $r + s \geq 1$, and let $\phi : \mathcal{B}(X) \to \mathcal{B}(X)$ be a surjective map which satisfies the following condition:*

$$\dim F(\phi(A)^r \phi(B) \phi(A)^s) = \dim F(A^r B A^s) \text{ for all } A, B \in \mathcal{B}(X). \tag{6.3.1}$$

Then there exist a bounded invertible linear (or conjugate linear) operator $S : X \to X$ and a complex number $\lambda \in \mathbb{C}$ such that $\phi(A) = \lambda S A S^{-1}$ for all $A \in \mathcal{B}(X)$ with $\lambda^{r+s+1} = 1$.

Proof. Assume that ϕ is a surjective map from $\mathcal{B}(X)$ into itself such that for all $A, B \in \mathcal{B}(X)$,

$$\dim F(\phi(A)^r \phi(B) \phi(A)^s) = \dim F(A^r B A^s).$$

Note that the case corresponding to $r + s = 1$ is the main result of [102]. In what follows, we prove the result for $r + s \geq 2$. We divide the proof into several steps.

Step 1. $\phi(0) = 0$ and ϕ is injective.

Let $x \in X$ and $f \in X^*$. Since ϕ is surjective, there exists an operator $A \in \mathcal{B}(X)$ such that $\phi(A) = x \otimes f$.

Then, by hypothesis, we have

$$\begin{aligned} 0 &= \dim F(A^r 0 A^s) \\ &= \dim F(\phi(A)^r \phi(0)\phi(A)^s) \\ &= \dim F((x\otimes f)^r \phi(0)(x\otimes f)^s) \\ &= \dim F(f(x)^{r+s-2} f(\phi(0)x)x\otimes f) \end{aligned}$$

which implies that

$$f(x)^{r+s-1} f(\phi(0)x) \neq 1. \tag{6.3.2}$$

So, x and $\phi(0)x$ are linearly dependent for every $x \in X$, Indeed, if not we can find a linear functional f such that $f(x) \neq 0$ and $f(\phi(0)x) = \frac{1}{f(x)^{r+s-1}}$. Thus, there exists a complex number γ such that $\phi(0) = \gamma I$. It follows by the equation (6.3.2) that $\gamma f(x)^{r+s} \neq 1$ for every $x \in X$ and $f \in X^*$. Hence $\gamma = 0$ and $\phi(0) = 0$.

Let $A, B \in \mathcal{B}(X)$ such that $\phi(A) = \phi(B)$. Then, for every $T \in \mathcal{B}(X)$ we have

$$\begin{aligned} \dim F(T^r A T^s) &= \dim F(\phi(T)^r \phi(A)\phi(T)^s) \\ &= \dim F(\phi(T)^r \phi(B)\phi(T)^s) \\ &= \dim F(T^r B T^s). \end{aligned}$$

It follows, by lemma 6.2.5 that $A = B$, and so ϕ is injective. Thus, ϕ is bijective since it is assumed to be surjective; moreover, ϕ^{-1} satisfied the equation (6.3.1).

Step 2. ϕ preserves rank one operators in both directions.

Let R be a rank one operator. By Lemma 6.2.6 we have $\dim F(A^r R A^s) \leq 1$ for every $A \in \mathcal{B}(X)$, it follows by hypothesis that $\dim F(\phi(A)^r \phi(R)\phi(A)^s) \leq 1$ for every $A \in \mathcal{B}(X)$. Again by Lemma 6.2.6 with $\phi(0) = 0$ and the surjectivity of ϕ, we deduce that $\phi(R)$ is a rank one operator. Since ϕ is bijective and ϕ^{-1} satisfies the equation (6.3.1), we conclude that ϕ preserves rank one operators in both directions.

Step 3. ϕ preserves rank one idempotents in both directions.

By the same reasoning as Lemma 6.2.2, one can prove that:

- For a rank one operator R, we have

$$R^2 = 0 \Longleftrightarrow \dim F(R^r T R^s) = 0 \quad (for \quad every \quad T \in \mathcal{F}_1(X)).$$

Hence, ϕ maps the set $\mathcal{F}_1^0(X)$ of rank one square-zero operators bijectively onto itself.

- For $B \in \mathcal{B}(X)$, we have

$$B^{r+s} \ \ is \ \ a \ \ scalar \ \ operator \Longleftrightarrow \dim F(B^r A B^s) = 0 \ \ \ (for \ \ \ every \ \ \ A \in \mathcal{F}_1^0(X)).$$

Hence, $\phi(I)^{r+s} = \lambda I \in \mathbb{C}I$.

Now, let P be a rank one idempotent. Then $1 = \dim F(I^r P I^s) = \dim F(\phi(I)^r \phi(P)\phi(I)^s)$. So, $\phi(I)^r \phi(P)\phi(I)^s$ is again an idempotent (and in particular, $\phi(I)^{r+s} = \lambda I$ is a non zero scalar operator). Therefore, $\phi(I)^s/\lambda$ is the inverse of $\phi(I)^r$. From $\phi(I)^{r+s} = \lambda I$, we deduce that the spectrum of $\phi(I)$ is contained in $\sqrt[r+s]{\lambda}$. While

$$\begin{aligned} \infty &= \dim F(I) \\ &= \dim F(I^r I I^s) \\ &= \dim F(\phi(I)^r \phi(I)\phi(I)^s) \\ &= \dim F(\lambda \phi(I)). \end{aligned}$$

Then the spectrum of $\lambda\phi(I)$ contains 1. Hence $\lambda^{r+s+1} = 1$. By replacing ϕ by $\psi := \phi(I)^r \phi(.)\phi(I)^s$, the map ψ maps rank one idempotent to rank one idempotent. Note that, if $T \in \mathcal{B}(X)$ is an invertible operator, then for every A of finite rank, we have $\dim F(TAT^{-1}) = \dim F(A)$. In particular ψ satisfies the equation 6.3.1 for rank one operators $A, B \in \mathcal{B}(X)$.

Step 4. ψ preserves the nonzero idempotency of generalized product on the set of rank one idempotents in both directions.
Let P_1, P_2 be two rank one idempotent operators such that $P_1^r P_2 P_1^s$ is a rank one idempotent. Then, $1 = \dim F(P_1^r P_2 P_1^s) = \dim F(\psi(P_1)^r \psi(P_2)\psi(P_1)^s)$. Since ψ preserves rank one operators, we deduce that $\psi(P_1)^r \psi(P_2)\psi(P_1)^s$ is a rank one idempotent operator. Moreover, ψ preserves rank one idempotent. Then, ψ preserves the nonzero idempotency of generalized product on the set of rank one idempotents. By replacing ψ by ψ^{-1}, we achieve the proof of the step.

Step 5. ψ takes the desired form.

By the previous step, ψ preserves the nonzero idempotency of generalized product on the set of rank one idempotents in both directions. Moreover, ψ is assumed to be surjective. It follows by Lemma 6.2.7 that ψ takes one of the following forms

1. There exists a bounded invertible linear (or conjugate linear) operator $S : X \to X$ such that

$$\psi(P) = SPS^{-1}, \qquad P \in \mathcal{P}_1(X).$$

2. Or, X is reflexive and there exists a bounded invertible linear (or conjugate linear) $S : X^* \to X$ such that

$$\psi(P) = SP^*S^{-1}, \quad P \in \mathcal{P}_1(X).$$

Assume that ψ has the first form. Then $\psi(P) = SPS^{-1}$ for every $P \in \mathcal{P}_1(X)$, where S is a bounded invertible linear (or conjugate linear) operator from X into itself. Consider a rank one operator $R \in \mathcal{F}_1(X)$. Then for every $P \in \mathcal{P}_1(X)$, we have

$$\begin{aligned} \dim F(PRP) &= \dim F(P^r R P^s) \\ &= \dim F(\psi(P)^r \psi(R)\psi(P)^s) \\ &= \dim F(SPS^{-1}\psi(R)SPS^{-1}) \\ &= \dim F(SPS^{-1}\psi(R)SPS^{-1}) \\ &= \dim F(PS^{-1}\psi(R)SP). \end{aligned}$$

Since $S^{-1}\psi(R)S$ is not a scalar operator (a rank one operator), it follows by Lemma 6.2.4 that there exists a complex number $\alpha \in \mathbb{C} \setminus \{1\}$ such that $S^{-1}\psi(R)S = \alpha I + (1-\alpha)R$ for every $R \in \mathcal{F}_1(X)$. Moreover $\alpha = 0$ since R and $\psi(R)$ are rank one operators. Consequently, $S^{-1}\psi(R)S = R$, which means that $\psi(R) = SRS^{-1}$ for every rank one operator $R \in \mathcal{F}_1(X)$.

Let us now consider an arbitrary operator $A \in \mathcal{B}(X)$. Let $R \in \mathcal{F}_1(X)$. Then we have

$$\begin{aligned} \dim F(R^r A R^s) &= \dim F(\psi(R)^r \psi(A)\psi(R)^s) \\ &= \dim F((SRS^{-1})^r \psi(A)(SRS^{-1})^s) \\ &= \dim F(R^r S^{-1}\psi(A)SR^s). \end{aligned}$$

Thus, by Lemma 6.2.5, we deduce that $S^{-1}\psi(A)S = A$. So, $\psi(A) = SAS^{-1}$. It follows that $\phi(I)^r\phi(A)\phi(I)^s = SAS^{-1}$ then $\phi(A) = \phi(I)^{-r}SAS^{-1}\phi(I)^{-s}$ which implies that $\phi(A) = \frac{1}{\lambda}\phi(I)^{-r}SAS^{-1}\phi(I)^r$ now, if necessary, choose $\lambda' = \frac{1}{\lambda}$ and $U = \phi(I)^{-r}S$ to get the desired form.
Note that the second case cannot occur, by the same conclusion as [102].

□

Chapter 7

Dimension of local spectral subspace preservers

In this last chapter, we will present our results of maps preserving the dimension of local spectral subspaces of the product of operators.

Let $\mathcal{B}(X)$ be the algebra of all bounded linear operators on a complex Banach space X with $\dim X \geq 3$. For a subset $\Omega \subset \mathbb{C}$ and an operator $A \in \mathcal{B}(X)$, the local spectral subspace relative to Ω is

$$X_A(\Omega) = \{x \in X : \ \sigma_A(x) \subset \Omega\},$$

and it is clearly a vector subspace of X. In this chapter, we characterize the form of all unital surjective maps ϕ on $\mathcal{B}(X)$ which satisfy

$$\dim X_{\phi(A)\phi(B)}(\{1\}) = \dim X_{AB}(\{1\}), \text{ for all } A, B \in \mathcal{B}(X).$$

Also, we characterize the form of all (not necessary unital) surjective maps ϕ on $\mathcal{B}(X)$ which satisfy

$$X_{\phi(A)\phi(B)}(\{1\}) = X_{AB}(\{1\}), \text{ for all } A, B \in \mathcal{B}(X).$$

These results are essentially those of our submitted article [17].

7.1 Introduction

Let X be a complex Banach space and $\mathcal{B}(X)$ be the algebra of all bounded linear operators on X with identity I. We denote by $\mathcal{P}(X)$ the set of all idempotent operators in $\mathcal{B}(X)$, and by X^* the dual space. We denote by $\mathcal{F}_1(X), \mathcal{P}_1(X)$ and $\mathcal{N}_1(X)$ the set of all rank one operators, the set of all rank one idempotent operators and the set of all rank one nilpotent operators in $\mathcal{B}(X)$, respectively. The set of all fixed points of A, that is a subspace of X, is defined by

$$F(A) = \{x \in X : Ax = x\}.$$

In [34], Bourhim and Ransford proved that the only additive map on $\mathcal{B}(X)$ preserving the local spectra at any vector is nothing else than identity. Motivated by this result, Elhodaibi and Jaatit

showed in [58], that an additive map $\phi : \mathcal{B}(X) \to \mathcal{B}(X)$ satisfies

$$X_{\phi(A)}(\{\lambda\}) = X_A(\{\lambda\}) \text{ for all } A \in \mathcal{B}(X) \text{ and } \lambda \in \mathbb{C},$$

has the form $\phi(A) = A$ for all $A \in \mathcal{B}(X)$. In [19] Benbouziane, Ech-Chérif El Kettani and Herrou studied the non additive case. They gave the form of all surjective maps $\phi : \mathcal{B}(X) \to \mathcal{B}(X)$ satisfying $X_{\phi(A)\phi(B)}(\{\lambda\}) = X_{AB}(\{\lambda\})$ for all $A, B \in \mathcal{B}(X)$ and $\lambda \in \mathbb{C}$.

$$X_{\phi(A)\phi(B)}(\{\lambda\}) = X_{AB}(\{\lambda\}), \text{ for all } A, B \in \mathcal{B}(X), \text{ and } \lambda \in \mathbb{C}. \tag{7.1.1}$$

Recently, Taghavi and Hosseinzadeh have characterized in [102], all surjective maps ϕ on $\mathcal{B}(X)$ satisfying

$$\dim F(\phi(A)\phi(B)) = \dim F(AB), \text{ for all } A, B \in \mathcal{B}(X).$$

Motivated by these results, we determine the form of all maps ϕ from $\mathcal{B}(X)$ into itself satisfying

$$\dim X_{\phi(A)\phi(B)}(\{1\}) = \dim X_{AB}(\{1\}) \text{ for all } A, B \in \mathcal{B}(X).$$

Moreover, the form of maps from $\mathcal{B}(X)$ into itself satisfying

$$X_{\phi(A)\phi(B)}(\{1\}) = X_{AB}(\{1\}) \text{ for all } A, B \in \mathcal{B}(X), \tag{7.1.2}$$

has been established.

This chapter is divided into four sections. In Section 2, we collect some auxiliary lemmas and summarize some basic properties of the local spectral subspaces. In section 3, we prove some identity principles in term of the dimension of local spectral subspaces of the product of two operators. In section 4, we state and prove our main results.

7.2 Local spectral subspace properties

Let $A \in \mathcal{B}(X)$. We denote by $\ker(A)$ and $Im(A)$ the kernel and the range of $A \in \mathcal{B}(X)$, respectively. The following result gives an explicit identification of the local spectral subspace of rank one operators.

Lemma 7.2.1. *([34, Lemma 2.3]) Let $R \in \mathcal{F}_1(X)$ be a non nilpotent operator and λ a nonzero eigenvalue of R. Then*

$$X_R(\{0\}) = \ker(R) \text{ and } X_R(\{\lambda\}) = Im(R).$$

If R is a rank one nilpotent operators. We have the following lemma.

Lemma 7.2.2. *Let $R \in \mathcal{N}_1(X)$ and λ be a nonzero scalar. Then*

$$X_R(\{0\}) = X \text{ and } X_R(\{\lambda\}) = \{0\}.$$

Proof. It is a consequence of the fact that $\sigma_R(x) \subset \sigma(R) = \{0\}$ for all $x \in X$. □

The next lemma, quoted from [59] gives an important identity principle.

Lemma 7.2.3. *([59, proposition 2.3]) Let $A, B \in \mathcal{B}(X)$ be non-scalar operators. If $AP \in \mathcal{P}(X)$ implies $BP \in \mathcal{P}(X)$ for every $P \in \mathcal{P}_1(X)$, then $B = \alpha I + (1-\alpha)A$ for some $\alpha \in \mathbb{C} \setminus \{1\}$.*

Let $x \otimes f$ and $y \otimes g$ be rank one operators in $\mathcal{F}_1(X)$ for some $x, y \in X$ and $f, g \in X^*$. We define the following relation $\sim$ on $\mathcal{F}_1(X)$ by $x \otimes f \sim y \otimes g$ if, and only if, $\{x, y\}$ or $\{f, g\}$ is a linearly dependent set.

Lemma 7.2.4. *([82, Lemma 2.12]) Let P, Q two idempotents rank one operators. The following statements are equivalent:*

1. *$PQ = QP = 0$;*
2. *There exist $M, N \in \mathcal{N}_1(X)$ such that $P \sim N$, $P \sim M$, $Q \sim N$, $Q \sim M$ and $N \nsim M$.*

A map $\psi : \mathcal{P}_1(X) \to \mathcal{P}_1(X)$ preserves orthogonality if for every pair $P, Q \in \mathcal{P}_1(X)$ the relation $P \perp Q$ implies $\psi(P) \perp \psi(Q)$. If ψ is bijective and $P \perp Q \Leftrightarrow \psi(P) \perp \psi(Q)$ $P, Q \in \mathcal{P}_1(X)$, then we say that ψ preserves orthogonality in both directions. We end this section by presenting a result of bijective maps from $\mathcal{P}_1(X)$ into itself preserving the orthogonality in both directions.

Lemma 7.2.5. *([99, Theorem 2.4]) Let X be an infinite-dimensional Banach space and $\phi : \mathcal{P}_1(X) \longrightarrow \mathcal{P}_1(X)$ a bijective map preserving orthogonality in both directions. Then, either there exists a bounded invertible linear or conjugate-linear operator $S : X \to X$ such that*

$$\phi(P) = SPS^{-1},\ P \in \mathcal{P}_1(X),$$

or there exists a bounded invertible linear or conjugate-linear operator $S : X^* \to X$ such that

$$\phi(P) = SP^*S^{-1},\ P \in \mathcal{P}_1(X).$$

In the second case, X must be reflexive.

7.3 Identity principles

In this section, we introduce and prove some lemmas which are useful for the proof of the main results. The first one gives a necessary and sufficient condition for two operators to be the same in terms of the dimension of local spectral subspaces.

Lemma 7.3.1. *Let $A, B \in \mathcal{B}(X)$ such that*

$$\dim X_{AT}(\{1\}) = \dim X_{BT}(\{1\}) \text{ for all } T \in \mathcal{F}_1(X).$$

Then $A = B$.

Proof. Assume on the contrary, that there is a nonzero vector $x \in X$ for which $Ax \neq Bx$. Let, f be a linear functional satisfying $f(Ax) = 1$ and $f(Bx) \neq 1$. Considering $T = x \otimes f$, by Lemma 7.2.1 we obtain $X_{AT}(\{1\}) = \langle Ax \rangle$ and $X_{BT}(\{1\}) = \{0\}$. Then $\dim X_{AT}(\{1\}) = 1$ and $\dim X_{BT}(\{1\}) = 0$, a contradiction. □

Lemma 7.3.2. *Let $A \in \mathcal{B}(X) \setminus \{I\}$. Then, $\dim X_{AP}(\{1\}) = 0$ for every $P \in \mathcal{P}_1(X)$ if, and only if, $A \in \mathbb{C}I$.*

Proof. The 'if' part is easily verified since $\dim X_{\lambda P}(\{1\}) = 0$ for every $P \in \mathcal{P}_1(X)$ and for all scalar $\lambda \neq 1$. For the 'only if' part, let A be a non-scalar operator. Then there is a vector $x \in X$ such that x and Ax are linearly independent. So, we can find a linear functional f such that $f(x) = f(Ax) = 1$. Setting $P = x \otimes f$, we obtain $\dim X_{AP}(\{1\}) = 1$, which proves the lemma. □

Lemma 7.3.3. *Let A and B be non-scalar operators. If $\dim X_{AP}(\{1\}) = \dim X_{BP}(\{1\})$, for all $P \in \mathcal{P}_1(X)$, then there exists an $\alpha \in \mathbb{C} \setminus \{1\}$ such that $B = \alpha I + (1-\alpha)A$.*

Proof. Let $P \in \mathcal{P}_1(X)$. If $AP \in \mathcal{P}_1(X)$, it follows that $\dim X_{AP}(\{1\}) = 1$ and by assumption $\dim X_{BP}(\{1\}) = 1$. Since BP is a rank one operator, $BP \in \mathcal{P}_1(X) \setminus \{0\}$, then the assertion flows from Lemma 7.2.3. □

Lemma 7.3.4. *Let A and B be non-scalar operators. If $\dim X_{AR}(\{1\}) = \dim X_{BR}(\{1\})$, for all $R \in \mathcal{P}_1(X) \cup \mathcal{N}_1(X)$, then $A = B$.*

Proof. Consider A and B two non-scalar operators with $\dim X_{AR}(\{1\}) = \dim X_{BR}(\{1\})$, for all $R \in \mathcal{P}_1(X) \cup \mathcal{N}_1(X)$. Consequently, by the previous lemma there exists a nonzero scalar α such that $B = \alpha I + (1-\alpha)A$. Since A is a non-scalar operator, there exists a vector $x \in X$ such that x and Ax are linearly independent. Let f be a linear functional satisfying $f(x) = 0$ and $f(Ax) = 1$. By choosing $R = x \otimes f$, we get

$$\dim X_{Ax\otimes f}(\{1\}) = \dim X_{(\alpha I+(1-\alpha)A)x\otimes f}(\{1\}).$$

Since $\dim X_{Ax\otimes f}(\{1\}) = 1$ and $(\alpha I + (1-\alpha)A)x \otimes f$ is a rank one operator, we obtain $(\alpha I + (1-\alpha)A)x \otimes f \in \mathcal{P}_1(X) \setminus \{0\}$ which implies that $\alpha = 0$ and hence $A = B$. □

We end this section with an important characterization of the relation $\sim$, which is a reformulation of [82, Lemma 2.11] in term of the local spectral subspaces.

Lemma 7.3.5. *Let A_1 and A_2 be linearly independent rank one operators. Then the following statements are equivalent:*

1. *$A_1 \sim A_2$;*
2. *There is a rank one operator B linearly independent of A_1 and linearly independent of A_2 and for every $T \in \mathcal{B}(X)$*

$$\dim X_{\zeta A_i T}(\{1\}) = 0, \text{ for all } i = 1, 2, \text{and } \zeta \in \mathbb{C} \implies \dim X_{BT}(\{1\}) = 0.$$

Proof. Let $A_1 = x \otimes f$ and $A_2 = y \otimes g$. Assume that $A_1 \sim A_2$. Then, there exists a nonzero scalar λ such that $y = \lambda x$ or $g = \lambda f$. If $g = \lambda f$, we can write $A_2 = y \otimes f$. Note $B = A_1 + A_2$. Since A_1 and A_2 are linearly independent, B is linearly independent of A_i , $i = 1, 2$. Moreover, if $\dim X_{\zeta A_i T}(\{1\}) = 0$ $i = 1, 2$ for every $T \in \mathcal{B}(X)$ and $\zeta \in \mathbb{C}$. Then, $\zeta f(Tx) \neq 1$ and $\zeta f(Ty) \neq 1$, which implies that $f(Tx) = f(Ty) = 0$. Hence $f(T(x + y)) = 0$. Thus $\dim X_{BT}(\{1\}) = 0$. Same proof in the case where x and y are linearly dependent.
Conversely, in order to obtain a contradiction assume that $A_1 = x \otimes f$ and $A_2 = y \otimes g$ are rank one operators such that x, y and f, g are linearly independent, and there exists a rank one operator $B = u \otimes k$ which satisfies the second condition. We show that k is a linear combination of f and g. If it is not the case, we can find a vector $z \in X$ such that $k(z) \neq 0$, and $f(z) = g(z) = 0$. Moreover, we can find an operator $T \in \mathcal{B}(X)$ satisfying $Tu \neq 0$ and $Tx, Ty, Tu \in < z >$. It follows that $f(Tx) = g(Ty) = 0$ and $k(Tu) = 1$, which leads to $\dim X_{A_i T}(\{1\}) = 0, i = 1, 2$ and $\dim X_{BT}(\{1\}) = 1$, a contradiction. With a similar reasoning we show that u is a linear combination of x and y. So, we can write $B = (\lambda x + \mu y) \otimes (\alpha f + \beta g)$. Consider η and ν any complex numbers. Since f and g are linearly independent, we can find $w_1, w_2 \in X$ such that $f(w_1) = 0$, $g(w_1) = \eta$, $f(w_2) = \nu$ and $g(w_2) = 0$. Since x and y are linearly independent, we can also find an operator $T \in \mathcal{B}(X)$ such that $Tx = w_1$ and $Ty = w_2$. Then

$$0 = k(Tu) = (\alpha f + \beta g)(\lambda Tx + \mu Ty) = \alpha\mu\nu + \beta\lambda\eta$$

which implies that $\alpha\mu = \beta\lambda = 0$. Thus, B is multiple either of A_1 or A_2, a contradiction. □

7.4 Main results and proof

The promised main result of this section is the following theorem.

Theorem 7.4.1. *Let X be a complex Banach space with $\dim X \geq 3$. If a surjective unital map $\phi : \mathcal{B}(X) \to \mathcal{B}(X)$ satisfies*

$$\dim X_{\phi(A)\phi(B)}(\{1\}) = \dim X_{AB}(\{1\}) \text{ for all } A, B \in \mathcal{B}(X), \tag{7.4.1}$$

then there exists an invertible operator $S \in \mathcal{B}(X)$ such that $\phi(A) = SAS^{-1}$ for all $A \in \mathcal{B}(X)$.

Proof. Assume that ϕ is a surjective map from $\mathcal{B}(X)$ into itself satisfies (7.4.1). We divide the proof into several steps.

Step 1. $\phi(0) = 0$ and ϕ is injective.

Assume that $\phi(0) \neq 0$, then there exists $x \in X$ such that $\phi(0)x \neq 0$. Let f be a linear functional satisfying $f(\phi(0)x) = 1$. Since ϕ is surjective, there exists $A \in \mathcal{B}(X)$ such that $\phi(A) = x \otimes f$. Then,

by hypothesis and Lemma 7.2.1, we obtain

$$\begin{aligned} 0 &= \dim X_0(\{1\}) \\ &= \dim X_{0A}(\{1\}) \\ &= \dim X_{\phi(0)x\otimes f}(\{1\}) = 1. \end{aligned}$$

This contradiction proves that $\phi(0) = 0$.

Let $A, B \in \mathcal{B}(X)$ such that $\phi(A) = \phi(B)$. Since ϕ is surjective, by (7.4.1) we have

$$\begin{aligned} \dim X_{AT}(\{1\}) &= \dim X_{\phi(A)\phi(T)}(\{1\}) \\ &= \dim X_{\phi(B)\phi(T)}(\{1\}) \\ &= \dim X_{BT}(\{1\}), \end{aligned}$$

for all $T \in \mathcal{B}(X)$. Thus, by Lemma 7.3.1 $A = B$, which proves that ϕ is injective. Thus, ϕ is bijective, since it is assumed to be surjective. Moreover, ϕ^{-1} satisfies the equation (7.4.1), too.

Step 2. ϕ preserves rank one operators in both directions.

Let $R \in \mathcal{B}(X)$ be a rank one operator. Assume that the rank of $\phi(R) \geq 2$. Then, there exist linearly independent vectors y_1 and y_2 in the range of $\phi(R)$. Let $x_1, x_2 \in X$ such that $\phi(R)(x_1) = y_1$ and $\phi(R)(x_2) = y_2$. Note that x_1 and x_2 are linearly independent, too. We can find a finite rank operator $B \in \mathcal{B}(X)$ satisfying $By_1 = x_1$ and $By_2 = x_2$. Since ϕ is surjective, there exists $A \in \mathcal{B}(X)$ such that $\phi(A) = B$. Thus, we have $B\phi(R)(x_1) = x_1$ and $B\phi(R)(x_2) = x_2$, which imply that

$$x_1, x_2 \in X_{B\phi(R)}(\{1\}),$$

and so

$$\dim X_{AR}(\{1\}) = \dim X_{B\phi(R)}(\{1\}) \geq 2.$$

Which contradicts the fact that $\dim X_{AR}(\{1\}) \leq 1$. Therefore, the rank of $\phi(R)$ is less than 1. Moreover, since $\phi(0) = 0$, we have rank of $\phi(R) = 1$.
Since ϕ^{-1} satisfies the equality (7.4.1), we conclude that ϕ preserves rank one operators in both directions.

Step 3. ϕ preserves rank one idempotent operators in both directions.

Let $x \in X$ and $f \in X^*$ such that $f(x) = 1$. By the previous step, we can find $y \in X$ and $g \in X^*$ such that $\phi(x \otimes f) = y \otimes g$. Thus by hypothesis, we have

$$\begin{aligned} 1 &= \dim X_{x\otimes f}(\{1\}) \\ &= \dim X_{y\otimes g}(\{1\}), \end{aligned}$$

which implies that $g(y) = 1$. Then $\phi(x \otimes f) = y \otimes g$ is a rank one idempotent operator.

Step 4. $\phi(\beta I) = \gamma(\beta)I$ for all $\beta \in \mathbb{C}$, where γ is a bijective function on $\mathbb{C}$.

By Lemma 7.3.2 and the first step, the assertion is true. Moreover, since ϕ is bijective, we conclude that γ is bijective and this completes the proof.

Step 5. $\phi(\beta P) = \delta(\beta)\phi(P)$ for every rank one idempotent P and for all $\beta \in \mathbb{C}$, where δ is a bijective function on $\mathbb{C}$.

To do that, consider P any rank one idempotent and $\beta \in \mathbb{C}\backslash\{0\}$. It is well known that $\dim X_P(\{1\}) = 1$. It follows, by (7.4.1), that $\dim X_{\phi(\beta P)\phi(\beta^{-1}I)}(\{1\}) = 1$. This holds only if $\phi(\beta P)\phi(\beta^{-1}I) \in \mathcal{P}_1(X)$. By the previous step, $\phi(\beta P)\phi(\beta^{-1}I) = \gamma(\beta^{-1})\phi(\beta P)$. It remains to show that $\gamma(\beta^{-1})\phi(\beta P) = \phi(P)$ for all $\beta \in \mathbb{C} \setminus \{0\}$. Since $\phi(I) = I$, it is enough to prove the last equality for all $\beta \in \mathbb{C} \setminus \{0, 1\}$. Assume that $\gamma(\beta^{-1})\phi(\beta P) \neq \phi(P)$ for some $\beta \in \mathbb{C} \setminus \{0, 1\}$. Since both operators are rank one idempotents, we can write $\gamma(\beta^{-1})\phi(\beta P) = x \otimes f$ and $\phi(P) = y \otimes g$ with $f(x) = g(y) = 1$. In order to obtain contradiction, let us discuss the following two cases.

Case 1. x and y are linearly dependent.

Without loss of generality, we can assume that $x = y$. Let us prove that f and g are also linearly dependent. If it is not true, let $u \in X$ such that $g(u) = 1$ and $f(u) = \gamma(\beta^{-1})$. We have

$$\dim X_{(u\otimes g)(x\otimes g)}(\{1\}) = \dim X_{u\otimes g}(\{1\}) = 1,$$

which implies that

$$\dim X_{\phi^{-1}(u\otimes g)P}(\{1\}) = 1. \tag{7.4.2}$$

On the other hand

$$\dim X_{(u\otimes g)(\gamma(\beta^{-1})^{-1}x\otimes f)}(\{1\}) = \dim X_{(\gamma(\beta^{-1})^{-1}u\otimes f)}(\{1\}),$$

which gives that

$$\dim X_{\phi^{-1}(u\otimes g)\beta P}(\{1\}) = 1. \tag{7.4.3}$$

Combining (7.4.2) and with (7.4.3), we get a contradiction since $\beta \notin \{0, 1\}$.

Case 2. x and y are linearly independent. If $f(y) \neq 0$, let $h \in X^*$, such that $h(x) = (\gamma(\beta^{-1})^{-1}f(y))^{-1}$ and $h(y) = 1$. We have

$$\dim X_{(y\otimes h)(y\otimes g)}(\{1\}) = \dim X_{y\otimes g}(\{1\}) = 1,$$

which implies that

$$X_{\phi^{-1}(y\otimes h)P}(\{1\}) = 1. \tag{7.4.4}$$

On the other hand

$$\dim X_{(y\otimes h)(\gamma(\beta^{-1})^{-1}x\otimes f)}(\{1\}) = \dim X_{f(y)^{-1}y\otimes f}(\{1\}) = 1,$$

thus, by hypothesis

$$\dim X_{\phi^{-1}(y\otimes h)\beta P}(\{1\}) = 1. \tag{7.4.5}$$

Combining (7.4.4) and (7.4.5), we get a contradiction because $\beta \notin \{0,1\}$. With a similar reasoning, we get a contradiction when $g(x) \neq 0$. Suppose that $f(y) = g(x) = 0$, and let $A = y \otimes g + \gamma(\beta^{-1})x \otimes f$. Then

$$\dim X_{Ay\otimes g}(\{1\}) = 1 \text{ and } \dim X_{A(\gamma(\beta^{-1})^{-1}x\otimes f)}(\{1\}) = 1.$$

Thus, we obtain

$$\dim X_{\phi^{-1}(A)P}(\{1\}) = 1 \text{ and } \dim X_{\phi^{-1}(A)\beta P}(\{1\}) = 1,$$

which is a contradiction, because $\beta \notin \{0,1\}$.

Define for $\beta \in \mathbb{C} \setminus \{0\}$, $\delta(\beta) = \gamma(\beta^{-1})^{-1}$ and $\delta(0) = 0$. This completes the proof.

Step 6. ϕ preserves rank one nilpotent operators in both directions. Moreover, $\phi(\beta N) = \delta(\beta)\phi(N)$ for every rank one nilpotent operator N and for all $\beta \in \mathbb{C}$ (the same δ as in Step 5.).

Let N be a rank one non nilpotent operator. Note $N = x \otimes f$ where $x \in X$ and $f \in X^*$ with $f(x) \neq 0$. By (7.4.1) and Step 4, we have

$$1 = \dim X_{f(x)^{-1}N}(\{1\}) = \dim X_{\gamma(f(x)^{-1})\phi(I)\phi(N)}(\{1\}) = \dim X_{\gamma(f(x)^{-1})\phi(N)}(\{1\}).$$

By Step 2, $\phi(N)$ is a rank one operator. Consequently $\gamma(f(x)^{-1})\phi(N)$ is a rank one idempotent. Therefore, $\phi(N)$ is not nilpotent. Since ϕ^{-1} satisfies (7.4.1), we conclude that ϕ preserves rank one nilpotent operators in both directions.

For the second part, let N be a rank one nilpotent operator and $\beta \in \mathbb{C}$. Then we have

$$\begin{aligned} \dim X_{\beta NP}(\{1\}) &= \dim X_{\phi(\beta N)\phi(P)}(\{1\}) \\ &= \dim X_{\phi(N)\phi(\beta P)}(\{1\}) \\ &= \dim X_{\delta(\beta)\phi(N)\phi(P)}(\{1\}), \end{aligned}$$

for every $P \in \mathcal{P}_1(X)$. It follows by Lemma 7.2.3 that there exists a scalar $\alpha \neq 1$ such that $\phi(\beta N) = \alpha I + (1-\alpha)\delta(\beta)\phi(N)$. Since $\phi(N)$ is a rank one operator, then $\alpha = 0$. Thus, $\phi(\beta N) = \delta(\beta)\phi(N)$ as desired.

Step 7. ϕ preserves the orthogonality of rank one idempotents in both directions.

First, we show that $R_1 \sim R_2 \Leftrightarrow \phi(R_1) \sim \phi(R_2)$ for every operators $R_1, R_2 \in \mathcal{N}_1(X) \cup \mathcal{P}_1(X)$. Indeed, if $R_1 \sim R_2$ by Lemma 7.3.5, there is a rank one operator B linearly independent of R_1 and linearly independent of R_2 and for every $T \in \mathcal{B}(X)$

$$\dim X_{R_iT}(\{1\}) = 0, \quad i = 1,2 \Rightarrow \dim X_{BT}(\{1\}) = 0.$$

Hence $\phi(B)$ is a rank one operator linearly independent of $\phi(R_i)$, $i = 1, 2$. Indeed, if it is not, then there exists a scalar α such that $\phi(B) = \alpha\phi(R_i) = \delta(\beta)\phi(R_i) = \phi(\beta R_i)$ which contradicts the fact that B and R_i are linearly independent for $i = 1, 2$. Then, by the surjectivity of ϕ and (7.4.1), we get $\phi(R_1) \sim \phi(R_2)$.

Let P, Q be rank one idempotent operators such that $PQ = QP = 0$. Then, by lemma 7.2.4, there exist $M, N \in \mathcal{N}_1(X)$ such that $P \sim N$, $P \sim M$, $Q \sim N$, $Q \sim M$ and $N \nsim M$. Therefore $\phi(M), \phi(N) \in \mathcal{N}_1(X)$ such that $\phi(P) \sim \phi(N)$, $\phi(P) \sim \phi(M)$, $\phi(Q) \sim \phi(N)$, $\phi(Q) \sim \phi(M)$ and $\phi(N) \nsim \phi(M)$. Thus, $\phi(P)\phi(Q) = \phi(Q)\phi(P) = 0$. So, ϕ preserves the orthogonality of rank one idempotents in both directions.

Combining the Steps 3, 6 and Lemma 7.2.5, we infer that ϕ takes one of the following forms.

1. There exists a bounded invertible linear or conjugate-linear operator $S : X \to X$ such that
$$\phi(P) = SPS^{-1}, \ P \in \mathcal{P}_1(X),$$
2. X is reflexive and there exists a bounded invertible linear or conjugate-linear operator $S : X \to X$ such that
$$\phi(P) = SP^*S^{-1}, \ P \in \mathcal{P}_1(X).$$

For the nilpotent operators of rank one case, suppose that the first form occurs. Let N be a nilpotent operator of rank one. As $\dim X_A(\{1\}) = \dim X_{TAT^{-1}}(\{1\})$ for every operators of finite rank A and every bijective operator $T \in \mathcal{B}(X)$. Then, for every idempotent operator of rank one P, we have

$$\begin{aligned} \dim X_{\phi(N)\phi(P)}(\{1\}) &= \dim X_{NP}(\{1\}) \\ &= \dim X_{SNPS^{-1}}(\{1\}) \\ &= \dim X_{SNS^{-1}SPS^{-1}}(\{1\}) \\ &= \dim X_{SNS^{-1}\phi(P)}(\{1\}). \end{aligned}$$

By Lemma 7.3.3 and Step 3, we obtain $\phi(N) = \alpha I + (1-\alpha)SNS^{-1}$, for some $\alpha \in \mathbb{C} \setminus \{1\}$. Since by Step 6, $\phi(N)$ is a nilpotent operator of rank one, we obtain $\alpha = 0$ and this completes the proof. If (2) occurs, then similarly we can prove that $\phi(N) = SN^*S^{-1}$.

Step 8. ϕ takes the desired forms.

Assume that there exists a bounded invertible linear or conjugate-linear operator S such that

$$\phi(P) = SPS^{-1}, \ P \in \mathcal{P}_1(X).$$

We divide the proof into two cases.

Case 1. $\phi(A) = SAS^{-1}$ for every non-scalar operator $A \in \mathcal{B}(X)$.

Let A be a non-scalar operator. For every $R \in \mathcal{P}_1(X) \cup \mathcal{N}_1(X)$, we have

$$\begin{aligned} \dim X_{\phi(A)\phi(R)}(\{1\}) &= \dim X_{AR}(\{1\}) \\ &= \dim X_{SARS^{-1}}(\{1\}) \\ &= \dim X_{SAS^{-1}SRS^{-1}}(\{1\}) \\ &= \dim X_{SAS^{-1}\phi(R)}(\{1\}). \end{aligned}$$

Now, by Lemma 7.3.4, Step 3 and Step 6, we obtain $\phi(A) = SAS^{-1}$ for all non-scalar operators $A \in \mathcal{B}(X)$.

Case 2. $\phi(\beta I) = S(\beta I)S^{-1}$.

Let $x \otimes f$ be an idempotent operator of rank one and β be a nonzero complex number. By Step 4 and Case 1, we have

$$1 = \dim X_{\beta I.\beta^{-1}x\otimes f}(\{1\}) = \dim X_{\gamma(\beta)S\beta^{-1}x\otimes fS^{-1}}(\{1\})$$

If S is linear, we obtain

$$1 = \dim X_{\gamma(\beta)\beta^{-1}Sx\otimes fS^{-1}}(\{1\})$$

and if S is conjugate linear, we obtain

$$1 = \dim X_{\gamma(\beta)\overline{\beta^{-1}}Sx\otimes fS^{-1}}(\{1\})$$

which implies that $\gamma(\beta) = \beta$ or $\overline{\beta}$. Therefore, $\phi(\beta I) = \gamma(\beta)I = S(\beta I)S^{-1}$.
The second part can not occur because, by a similar way, we may show that $\phi(A) = SA^*S^{-1}$ for every $A \in \mathcal{B}(X)$. Such a form does not satisfy the condition (7.4.1) and this completes the proof. □

The following result is a consequence of our main result.

Theorem 7.4.2. *Let X be a complex Banach space with $\dim X \geq 3$. Then, a surjective map $\phi : \mathcal{B}(X) \to \mathcal{B}(X)$ satisfies the following condition:*

$$X_{\phi(A)\phi(B)}(\{1\}) = X_{AB}(\{1\}) \text{ for all } A, B \in \mathcal{B}(X), \tag{7.4.6}$$

if, and only if, $\phi(A) = \pm A$ for all $A \in \mathcal{B}(X)$.

Proof. It is not difficult to see that $\phi(I) = \pm I$. Assume that $\phi(I) = I$. Then by the previous result, there exists an invertible operator $S \in \mathcal{B}(X)$ such that $\phi(A) = SAS^{-1}$ for all $A \in \mathcal{B}(X)$. It remains to prove that S is a scalar operator. Indeed, ϕ satisfies the equation (7.4.6), then it satisfies the equation (7.4.1). By Theorem 7.4.1, we have $\phi(A) = \pm SAS^{-1}$ for all $A \in \mathcal{B}(X)$. Let $x \in X \setminus \{0\}$, then there exists $f \in X^*$ such that $f(x) = 1$. Take $A = x \otimes f$, and note that

$$\langle Ax\rangle = X_A(\{1\}) = X_{\phi(A)\phi(I)}(\{1\}) = X_{SAS^{-1}}(\{1\}) = \langle SAx\rangle.$$

Consequently, Ax and SAx are linearly dependent for every $x \in X$. Then, S is a scalar operator.
If $\phi(I) = -I$, a similar discussion by changing ϕ by $-\phi$ leads to prove that Ax and SAx are linearly dependent for every $x \in X$. Then, S is a scalar operator in this case, too, which completes the proof. □

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