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Discrete Moment Problem Application to truncated case

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DISCRETE MOMENT PROBLEM

Application to truncated case

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To my family.

Dedicated to the memory of

My grand-father Mohammed El Azhar sbaai
1927–2018
and
to my professor Ahmed Intissar.
1951–2017

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El Jadida February 1, 2020.

"Why did he do that? How can a man who was a mathematician write novels?" people in Göttingen marvelled.
"But that is completely simple, He did not have enough imagination for mathematics, but he had enough for novels".
— David Hilbert[62]—

RÉSUMÉ

Dans cette thèse, nous considérons le problème des moments discret, c'est-à-dire le problème associé aux mesures discrètes. Tout d'abord, nous étendons l'approche de résolution par l'usage des idempotents, introduite dans le cas du problème tronqué, au cas complet. Les résultats obtenus permettent d'établir un pont entre l'étude des mesures discrètes et l'existence de certaines bases dans l'espace de Hilbert associé au problème des moments.

D'autre part, nous introduisons la notion des matrices k-positives, nous démontrons par la suite qu'il existe un phénomène de propagation de rang associé à cette famille de matrices. Un phénomène déjà remarqué par Stamplfi dans le cas des matrices associées au problème des moments et dans des travaux sur l'étude des shifts k-hyponormaux.

Finalement, nous résolvons le problème des moments complexe quintic. Nous utilisons une méthode qui se base sur les polynômes générateurs et l'étude systématique de la matrice des moments complexe.

Mot-clés: Problème des moments discret, Éspace à shift borné, Élément Λ-multiplicatif, Matrice k-positive, Operateur k-hyponormal, Phénomène de propagation, Problème des moments complexe quintic.

ABSTRACT

In this thesis, we consider the discrete moment problem that is the problem associated with discrete measures. First, we extend the idempotent approach from the truncated case to the full one. The main results obtained establish a bridge between the study of discrete measures and the existence of some bases in the Hilbert space associated with the moment problem.

On the other hand, we introduce the notion of k-positive matrices, we show that there is rank propagation phenomena associated with this family of matrices. A phenomenon already noticed by Stampfli in his work on moment matrices and by Curto in the study of k-hyponormal shifts.

Finally, we will almost solve the quintic moment problem in the complex case, by a new method, based on the notion of generator polynomials and a systematic study of the complex moment matrix.

Keywords: Discrete moment problem, Bounded Shift space, Λ-multiplicative element, k-positive matrices, k-hyponormal operators, Propagation phenomena, Quintic moment problem.

RÉSUMÉ DÉTAILLÉ

Soit μ une mesure positive portée par un sous-ensemble fermé K de $\mathbb{R}^d$, et $\alpha = (\alpha_1, \ldots, \alpha_d) \in \mathbb{N}^d$ un multi-indice. L'intégrale :

$$\gamma_\alpha(\mu) = \int_K t_1^{\alpha_1} \ldots t_d^{\alpha_d} d\mu(t) := \int_K t^\alpha d\mu(t) \tag{1}$$

est appelée le moment d'ordre α de la mesure μ. Si pour tout $\alpha \in \mathbb{N}^d$ les moments $\gamma_\alpha(\mu)$ existent, la suite $(\gamma_\alpha(\mu))_{\alpha \in \mathbb{N}^d}$ est dite de moment associée à μ.

Le problème des moments (complet) sur K, est un problème classique d'analyse qui répond à la question suivante :
Soit $(\gamma_\alpha)_\alpha$ une suite réelle. Existe-il une mesure de Radon positive μ portée par $K \subset \mathbb{R}^d$, telle que pour tout $\alpha \in \mathbb{N}^d$, l'intégrale $\int_K t^\alpha d\mu(t)$ existe et vérifie (1).

Le problème du moment tronqué sur K est une variante du problème complet. Il traite le cas ou on a un nombre fini de moments. Autrement dit :
Soit $(s_\alpha)_{|\alpha| \leqslant n}$ une suite réelle finie. Existe-il une mesure de Radon positive μ sur $K \subset \mathbb{R}^d$, telle que pour tout $\alpha \in \mathbb{N}^d$, avec $|\alpha| \leqslant n$, l'intégrale $\int_K t^\alpha d\mu(t)$ existe et satisfait (1).

Pour une suite réelle $(\gamma_\alpha)_{\alpha \in \mathbb{N}^d}$, on introduit la fonctionnelle de Riesz Λ, définie par $\Lambda_\gamma(\sum_\alpha a_\alpha t^\alpha) = \sum_\alpha a_\alpha \gamma_\alpha$, sur l'algèbre des polynômes en plusieurs indéterminées $\mathbb{C}[t] := \mathbb{C}[t_1, t_2, \ldots, t_d]$. Il est évident que la suite $(\gamma_\alpha)_{\alpha \in \mathbb{N}^d}$ est de moment si et seulement s'il existe une mesure positive μ telle que $\Lambda_\gamma(p) = \int_\mathbb{R} p(t) d\mu(t)$ pour tout $p \in \mathbb{C}[t]$. En d'autres termes, le problème des moments est équivalent à la question de représentation intégrale de la fonctionnelle de Riesz Λ_γ. Cette remarque est la base des méthodes d'analyse fonctionnelle utilisées pour la résolution du problème des moments.

Le théorème de Hamburger affirme que la fonctionnelle de Riesz Λ_γ admet une représentation intégrale sur $\mathbb{R}$ si et seulement si elle est positive sur les carrés des polynômes ; (i.e.) $\Lambda_\gamma(|p|^2) \geqslant 0$, pour tout $p \in \mathbb{R}[X]$. M. Riesz et E. K. Haviland généralisent ce résultat pour tout sous-ensemble K de $\mathbb{R}^n$. Plus précisément, la fonctionnelle de Riesz Λ_γ admet une représentation intégrale sur K si et seulement si elle est positive sur le cône des polynômes positifs sur K ;

(i.e.) $\Lambda_\gamma(p) \geqslant 0$, pour tout $p \in \mathbb{R}[X]$ où $p|_K \geqslant 0$. Malheureusement, les conditions de positivité des polynômes (type Positivstellensatz d'Artin) sont généralement inexploitables en terme de la fonctionnelle de Riesz. D'où la nécessité de la recherche d'autres méthodes pour la résolution du problème.

Dans cette thèse, on commence par étendre l'approche d'idempotent de F. H. Vasilescu, en montrant qu'elle est valable pour étudier les mesures discrètes en générale.

On suppose que la fonctionnelle de Riesz Λ_γ est positive sur les carrés ; c'est à dire que Λ_γ définie une forme quadratique positive sur l'algèbre des polynômes (ce n'est pas la positivité sur les polynômes positive), on dit que c'est une USPF (Fonctionnelle Unitaire Positive sur les Carrés). Par la construction de Gelfand–Naimark–Segal on associe de façon canonique un espace de Hilbert H a l'USPF Λ_γ. Vu l'absence d'une structure d'algèbre sur H en générale, on étend la définition des idempotents dans ce cadre par les éléments réels vérifiant l'équation $\langle x, x\rangle = \langle x, 1\rangle$.

On souligne le rôle capital des éléments Λ-multiplicatifs en tant que la représentation dans H de la fonction caractéristique $\chi_{\{a\}}$ d'un seul point a du support. En effet, un élément réel de H est dite Λ-multiplicatif s'il vérifie l'égalité :

$$\langle p, x\rangle\langle q, x\rangle = \langle 1, x\rangle\langle pq, x\rangle \quad p, q \in \mathcal{P}. \tag{2}$$

On démontre que la condition (2) est équivalente à $\langle p, x\rangle = \alpha p(a)$, où $a \in \mathbb{R}^n$ et $\alpha \in \mathbb{R}$. Ce résultat est la base de la représentation intégrale de Λ et du théorème principale de la première partie.

On prouve que les conditions du théorème principale de Vasilescu [80, Théorème 2] pour le cas tronqué deviennent seulement suffisantes pour l'existence d'une mesure discrète solution dans le cas général. En effet, on démontre que l'existence d'une base orthogonal d'éléments Λ-multiplicatifs de H suffit pour construire une mesure discrète représentative de l'USPF Λ. La différence entre les deux résultats est bien expliquée par un contre-exemple et en traitant le cas des espaces a Shift borné.

On termine la première partie de cette thèse par une étude plus poussée des éléments Λ-multiplicatifs. Puisqu'on traite la question de quand est ce qu'un élément Λ-multiplicatif est effectivement la fonction indicatrice d'un point ?

Deux caractérisations complètes (d'apparence différentes) sont données. Le passage par les mesures N-extrémales est nécessaire pour créer un contre-exemple d'élément Λ-multiplicatif qui n'est pas une indicatrice d'un point même avec la condition de densité des polynômes dans l'espace $L^2(\mu)$.

Dans la seconde partie on introduit la classe des matrices k-positives. Ce sont les matrices de Hankel dont tous les blocs d'ordre $k+1$ sont semi-définies positives. Par exemple, une matrice est 1-positive si elle

est de Hankel et si la suite associée est log-convexe. Un cas particulier notable est la matrice des moments de Stieltjes, dans ce cas la matrice de Hankel est k-positive pour tout k.
La classe des matrices k-positives est l'équivalent algébrique des opérateurs Shift à poids dits k-hyponormaux, une classe d'opérateurs introduite par A. Athavale comme une échelle entre les opérateurs hyponormaux et sous-normaux. A partir de ce formalisme on démontre de façon algébrique et simple la propriété de propagation des matrices k-positives (Shift à poids k-hyponormaux) qui dit que si le déterminant d'un seul bloc de taille k est nul dans une matrice k-positive alors les déterminants de tout les blocs de taille k (sauf peut être le premier) seront nuls aussi. Ce phénomène a été découvert pour les blocs de taille 2 par J. Stampfli pour les Shifts à poids sous-normaux. Généralisé dans le cas général par R. Curto et L. Fialkow.
La perturbation des matrices k-positives est aussi étudiée en détail. Plusieurs preuves sont simplifiées et un calcul explicite de l'intervalle de perturbation des matrices 2-positives est donné.

La dernière partie traite le problème tronqué complexe d'ordre 5. l'étude du cas tronqué multi-dimensionnel a démarré avec R. Curto et L. Fialkow en 1996, avec l'introduction de la notion de platitude (Flatness en Anglais). Ils ont démontré sous cette condition l'existence d'une solution discrète du problème tronqué. En absence de formule constructive de quadrature en plusieurs dimensions, plusieurs auteurs ont tenté résoudre le problème des moments complexe tronqué, seulement les cas quadratique ($n = 2$), le cas cubique ($n = 3$) et le cas quartique ($n = 4$) qui sont complètement résolus; les autres cas ($n \geqslant 5$) sont restés ouverts. Dans le dernier chapitre , on va résoudre le problème des moments complexe tronqué dans le cas quintique ($n = 5$). En se basant sur une méthode combinant la platitude de Curto-Fialkow et la récursivité développée par K. Idrissi et E. H. Zerouali. Certains exemples pratiques sont donnés pour illustrer les différents cas.

CONTENTS

1 INTRODUCTION

L'existence de ces fonctions $\varphi(u)$ qui, sans être nulles, sont telles que

$$\int_0^\infty u^k \varphi(u) du = 0 \ (k \geqslant 0)$$

me parait très remarquable.
—Stieltjes [3]—

Let μ be a positive measure on a closed set K of $\mathbb{R}^d$, and $\alpha = (\alpha_1, \ldots, \alpha_d) \in \mathbb{N}^d$, be a multi-index of non-negative integers α_j. The integral:

$$\gamma_\alpha(\mu) = \int_K t_1^{\alpha_1} \ldots t_d^{\alpha_d} d\mu(t) := \int_K t^\alpha d\mu(t) \tag{3}$$

is called the α-th moment of the measure μ. If for all $\alpha \in \mathbb{N}^d$ $\gamma_\alpha(\mu)$ exists, the sequence $(\gamma_\alpha(\mu))_{\alpha \in \mathbb{N}^d}$ is called the moment sequence of μ.
The (Full) K-moment problem is a classical mathematical problem, which is with the following question:
Let $(\gamma_\alpha)_\alpha$ be a real sequence. Does there exist a positive Radon measure μ on $K \subset \mathbb{R}^d$, such that for all $\alpha \in \mathbb{N}^d$ the integral $\int_K t^\alpha d\mu(t)$ exists and satisfies (3).

In his remarkable memoir [73], Thomas Joannes Stieltjes, generalized the notion of Riemann integral, and introduced the $\mathbb{R}^+$-moment problem, in order to give an integral representation for some continuous fractions. He introduced also the determinacy problem, (i.e) the problem of uniqueness of the measure solution for (3). For more development about this question see the classical books [1, 70]. A recent treatment is given in [68].

For a real sequence $(\gamma_\alpha)_{\alpha \in \mathbb{N}^d}$, we denote by Λ the linear functional on the algebra of polynomials $\mathbb{C}[t] := \mathbb{C}[t_1, t_2, \ldots, t_d]$ defined by $\Lambda_\gamma(t^\alpha) = \gamma_\alpha$, for every $\alpha \in \mathbb{N}^d$. By linearity of integral it's easy to see that (3) holds for every $\alpha \in \mathbb{N}^d$, if and only if $\Lambda_\gamma(p) = \int_{\mathbb{R}} p(t) d\mu(t)$ for all $p \in \mathbb{C}[t]$. In other words the moment problem ask whenever the functional Λ_γ admits an integral representation with respect to some positive measure μ. This fact will be the cornerstone of the idempotent approach to moment problem. The functional Λ_γ is called Riesz-functional after Marcel Riesz who introduced the functional analysis approach for the moment problem by proving the Riesz extension theorem in [64].

The Simplest example to treat is the one dimensional Hamburger moment problem; ($d = 1$ and $K = \mathbb{R}$). Let $(\gamma_n)_{n\in\mathbb{N}}$ be a moment sequence of some positive measure μ on $\mathbb{R}$, then for every polynomial $p(x) = \sum_{k=0}^{n} x_k x^k \in \mathbb{C}[x]$ we obtain :

$$\Lambda_\gamma(|p|^2) = \int_{\mathbb{R}} |p(t)|^2 d\mu(t) = \int_{\mathbb{R}} \sum_{k,l=0}^{n} x_k \overline{x_l} t^{k+l} d\mu(t)$$
$$= \sum_{k,l=0}^{n} x_k \overline{x_l} \gamma_{k+l} \geqslant 0.$$

So, the Riesz functional is positive on polynomials. In terms of matrix, it means that the Hankel matrices $(\Lambda_\gamma(t^{k+l}))_{k,l\leqslant n} = (\gamma_{k+l})_{k,l\leqslant n}$ (Called in honor of Hermann Hankel) are positive semi-definite for every $n \in \mathbb{N}$. This gives a necessary condition for a sequence to be a moment sequence. In fact we get more :

Theorem 1.1 (Hans Ludwig Hamburger 1920). *Let γ be a real sequence, we assume that for every $n \in \mathbb{N}$, the Hankel matrices $(\gamma_{k+l})_{k,l\leqslant n}$ are positive semi-definite. Then the Hamburger moment problem for γ has a solution.*

Proof. We consider the Riesz functional Λ_γ on polynomials associated to our sequence γ. We put $I_\gamma = \{p \in \mathbb{C}[x] \ ; \ \Lambda_\gamma(|P|^2) = 0\}$ the kernel of the quadratic form Λ_γ. The quotient $\mathcal{H} = \mathbb{C}[x]/I_\gamma$ defines an inner product space. We denote the completion of $\mathcal{H}$ by H.

We define the multiplication operator M_t on $\mathcal{D}(M_t) = \mathcal{H}$, for a $p, q \in \mathcal{H}$

$$\langle M_t p, q\rangle = \Lambda_\gamma(tp\bar{q}) = \Lambda_\gamma(p\overline{tq}) = \langle p, M_t q\rangle = \langle M_t^* p, q\rangle.$$

So M_t is symmetric. Let M be a self-adjoint extension of M_t, we get $M_t^n \subset M^n$. By the spectral theorem M has a spectral measure E_M. So

$$\int_{\mathbb{R}} t^n d\langle E_M(t)1, 1\rangle = \langle M^n 1, 1\rangle = \langle M_t^n 1, 1\rangle = \Lambda_\gamma(t^n) = \gamma_n.$$

□

Hamburger's theorem says that the existence problem for a solution is easily answered in terms of positivity conditions, more precisely, since any positive polynomial on $\mathbb{R}$ is a sum of two squares of polynomials [1] (SOS). The positivity condition is equivalent to say that the Riesz functional is positive on the cone of non-negative polynomials.

In 1923 M. Riesz generalized this remark to any subset K of $\mathbb{R}$, by proving that the K-moment problem has a solution if and only if the Riesz functional is positive on the cone of non-negative polynomials on K. In particular, the Stieltjes moment problem ($d = 1$,

1 If p is a non-negative polynomial on $\mathbb{R}$, then there is $u, v \in \mathbb{R}[X]$ such that $p = u^2 + v^2$

$K = \mathbb{R}_+$) have a solution if and only if both Hankel matrices $(\gamma_{i+j})_{i,j}$ and $(\gamma_{i+j+1})_{i,j}$ are positive semi-definite [2].

Edward Kenneth Haviland , used Riesz methods to prove in 1935-1936 the same result for any subset K of $\mathbb{R}^d$, which means that the K-moment problem has a solution on $K \in \mathbb{R}^d$ if and only if the Riesz functional is positive on the cone of non-negative polynomials on K, and hence transformed the problem of existence to a question of characterizing non-negative polynomials on K.

The Hilbert's seventeenth problem concerns the representation of non-negative polynomials on $\mathbb{R}^d$. In 1888, David Hilbert showed that there exists a non-negative homogeneous polynomial which is not SOS[3], but he proved that every bivariate polynomial that takes only non-negative values over all the real numbers, can be represented as a sum of squares of rational functions. The general problem (for every d) was solved in the affirmative, in 1927, by Emil Artin; every non-negative polynomial on $\mathbb{R}^d$ is a sum of squares of rational functions. But the problem with Artin's theorem is that we cannot translate the positivity condition directly on Riesz functional, and hence we cannot derive an explicit existence criteria to the $\mathbb{R}^d$-moment problem.

In real semi-algebraic geometry, "Positivstellensatz" (German for "positive-locus-theorem") characterizes polynomials that are positive on a semi-algebraic set, which is defined by a system of polynomials inequalities with real coefficients. In the compact case, an elegant condition was given by Konrad Schmüdgen in 1991 [66], for non-compact case a multitude of Positivstellensatz were given, but with much less success, for example the Putinar-Vasilescu Positivstellensatz.

The Truncated K-moment problem is a variation of the full moment problem. It deals with the following question:
Let $(s_\alpha)_{|\alpha|\leqslant n}$ *be a finite real sequence. Does there exists a positive Radon measure* μ *on* $K \subset \mathbb{R}^d$*, such that for all* $\alpha \in \mathbb{N}^d$ *with* $|\alpha| \leqslant n$*, the integral* $\int_K t^\alpha d\mu(t)$ *exists and satisfies* (3)*.*

The study of the truncated moment problem is closely related to Gaussian quadrature/cubature rules (quadrature for simple integral, and cubature for multiple integral) known in numerical analysis, which interested only by discrete measure, (i.e) for a fixed positive measure μ on K, finding a finite subset of points $(x_i)_{i\in I}$ in K called nodes, and a sequence of positive numbers $(w_i)_{i\in I}$ called weights such that;

$$\int_K p(t)d\mu(t) = \sum_{i\in I} w_i p(x_i), \qquad p \in \mathbb{C}_n[t]. \tag{4}$$

The first apparition of a quadrature rule goes back to Isaac Newton in the third part of Principia Mathematica[57], who used polynomial

2 Since for any non-negative polynomial p on $\mathbb{R}_+$ there exist $u, v \in \mathbb{R}[X]$ SOS such that $p = u + tv$.

3 The first explicit example was given by Theodore Samuel Motzkin : $p(x, y) = 1 + x^4y^2 + x^2y^4 - 3x^2y^2$.

interpolation on compact segments of $\mathbb{R}$ and found the first solution to (4) but with signed weights. Later on, Carl Friedrich Gauss was the first to answer most elegantly this question by means of his theory of continued fractions associated with hypergeometric series[42].

The first cubature rule is attributed to V. Tchakaloff in his paper [78], for the compact case and with the Lebesgue measure. but in the same year H. Richter in [63] proved the existence of such formula in full generality, Richter's paper has been ignored in the literature and a number of versions of his result have been reproved even recently [4, 80].

For a long time the truncated moment problem was considered as simple finite dimensional problem, and this statue was confirmed by the existence of the Gaussian quadrature formula. but in 1991, in a series of papers [19], R. E. Curto and L. A. Fialkow studied more closely this question and gave the link between the truncated Stieltjes moment problem ($K = \mathbb{R}^+$) and the subnormal completion problem for weighted shifts, and characterized completely the truncated one dimensional moment problem for all classical cases, $K = \mathbb{R}$ (Hamburger), $K = \mathbb{R}_+$ (Stieltjes), $K = [0, 1]$ (Hausdroff) and $K = \mathbb{T}$ (Toeplitz).

In 1996, the same authors began the study of the multidimensional moment problem, and gave the first criterion for solving the truncated moment problem where they introduced the **Flatness** condition in [22].

Definition 1.2. Let Λ_n be a Riesz functional defined on $\mathbb{C}_{2n}[t]$, we say that Λ_n is Flat, if for every $p \in \mathbb{C}_n[t]$, $\Lambda_n(|p|^2) \geqslant 0$ and
$\text{Rank}(\Lambda_n)=\text{Rank}(\Lambda_{n-1})$.

In the case of the one dimensional moment problem, the rank condition is equivalent to say the moment sequence is **recursive** (called also a Fibonacci sequence [13]) in particular it gives a discrete solution for the moment problem using the Binet formula. In multidimensional case, recursive sequence turns to be not sufficient to provide a finite atomic (discrete) positive measure [23].

After the 2000s, the truncated moment problem started to catch more attention with the paper of Jan Stochel [74] where he proved that the resolution of the truncated moment problem solves the full moment problem, using some compactness ideas on the space of continuous functions, and the discovery of R. Curto and L. Fialkow in [27] to a version of Haviland-Riesz theorem for truncated moment problem.

This opened the way to the research of concrete methods for the resolution of the Truncated moment problem. Until Now for a given K, concrete solutions of the truncated moment problem are known only for few cases, in the case of $d = 1$, as we have seen for $K = \mathbb{R}, \mathbb{R}_+$ and $[a, b]$, for $n = 2$, where K is an algebraic curve of degree less than 2 [25], or for some special higher degree curves[38, 45].

For Hamburger's case ($K = \mathbb{R}^d$), the most attention was given to the case of two variables ($d = 2$), where the problem is transformed to the complex plan using the natural isomorphism ($\mathbb{R}^2 \sim \mathbb{C}$) we speak about complex truncated moment problem :

Let $(s_{p,q})_{|p+q|\leqslant n}$ *be a finite real sequence. Does there exists a positive Radon measure* μ *on* $\mathbb{C}$*, such that for all* $(p,q) \in \mathbb{N}^2$ *with* $p+q \leqslant n$*, the integral* $\int_{\mathbb{C}} z^q \bar{z}^p \, d\mu(z)$ *exists and satisfies*

$$s_{p,q} = \int_{\mathbb{C}} \bar{z}^p z^q \, d\mu(z) \qquad p+q \leqslant n.$$

Although apparently simple, almost all basic situations are considered as open problems. For example, in the truncated complex moment problem associated with $(s_{p,q})_{|p+q|\leqslant n}$ only the cases $n \leqslant 4$ (the quadratic ($n = 2$) [22, 39], the cubic ($n = 3$) [48, 18] and the quartic ($n = 4$) [25, 31] moment problems) have been (recently) resolved. All the other cases still partially open: quintic ($n = 5$)[36], sixtic ($n = 6$) [83, 84],...

We will provide, in Chapter 4, a concrete solution to almost all cases of the quintic truncated complex moment problem, when one desires a minimal representing measure. Our method intended to be useful for all odd-degree truncated moment problems.

Other approaches and methods are recently developed to resolve the truncated moment problem, we mention for example the L. Fialkow Core variety method based on techniques inspired from convex analysis [40, 10], where the existence of a finite discrete solution to a truncated moment problem, is equivalent to the fact that the core variety for the Riesz functional Λ:

$$\mathcal{CV}(\Lambda) = \bigcap_{i\geqslant 0} S_i, \quad S_{i+1} = \mathcal{Z}\{p \in \mathbb{C}[t], p|_{S_i} \geqslant 0; \Lambda(p) = 0\} \quad S_0 = K.$$

is not empty.

For the case of Hamburger truncated moment problem, another important method was given by Florian-Horia Vasilescu [80, 81, 82], called Idempotent approach for the moment problem. The main idea is to find an adequate representing vector, in the Hilbert space $\mathbb{C}_n[t]/I_\gamma$, for any characteristic function χ_A of a subset $A \in \mathbb{R}^d$, using the idempotent equation $\chi_A^2 = \chi_A$. For a finite discrete set K, we can find a polynomial p such that $p|_K = \chi_A$, therefore we get:

$$\Lambda(p^2) = \int_K p^2 d\mu = \int_K \chi_A^2 \, d\mu = \int_K \chi_A \, d\mu = \int_K p \, d\mu = \Lambda(p). \quad (5)$$

F. H. Vasilescu called an element p of $\mathbb{C}_n[t]/I_\gamma$ idempotent if it verifies the equation $\Lambda(p^2) = \Lambda(p)$. The main result in [80] is that the Hamburger truncated moment problem has a solution if and only if we can find a "good" orthogonal basis formed by Idempotents, more precision will be given in chapter 2.

As mentioned before, there is a direct link between Stieltjes moment problem and subnormal operators. We recall that an operator T on a Hilbert space H, is said to be subnormal, if T has an extension S on a Hilbert space K, such that S is normal (i.e $S^*S = SS^*$). A characterization, of Mary R. Embry[37], gives that T is a subnormal operator if and only if there exists a positive operator measure Q (a function Q that assigns to each Borel set Ω on $\mathbb{R}$ a positive bounded operator on the Hilbert space H, such that $Q(\mathbb{R}) = 1$ and for every vector $e \in H$, $\langle Q(.)e, e\rangle$ is a regular Borel measure on $\mathbb{R}$.), such that :

$$S^{*n}S^n = \int_0^{\|S\|} t^{2n} dQ(t).$$

For the case of weighted Shift $W_{(\alpha_n)_n}$, ($W_{(\alpha_n)_n}(e_k) = \alpha_k e_{k+1}$), it is sufficient to test on the vector e_0 only, the measure $\langle Q(.)e_0, e_0\rangle$ is called the Berger measure associated to the subnormal weighted shift $W_{(\alpha_n)_n}$[44], more precisely Berger theorem states that a weighted shift $W_{(\alpha_n)_n}$ is subnormal if and only if there exists a positive measure (The Berger measure) such that :

$$\prod_{k=0}^{n-1} |\alpha_k|^2 = \int_0^{\|T\|^2} t^n d\mu(t).$$

This interesting connection between the theory of subnormal operators and the theory of moment problem, was intensely used to expend results, or to simplify notions and proofs in both directions see [2, 11, 19, 14, 17, 20, 21, 49] and many others.

By what we have just seen, discrete measures play a central role in the theory of moment problem, it turns out that they are the most used class of measures for the effective calculation of solutions in (3). Our work deals with discrete measures and their relationship with all the subject treated in this introduction.

In the immediately following chapter, we begin by developing the idempotent approach of Vasilescu, to the general case, we highlight the capital role of Λ-multiplicative element as a representation of the characteristic function $\chi_{\{a\}}$ of a single point a belonging to the support. We prove that conditions in Vasilescu's theorem [80, Theorem 2] become only sufficient for the existence of a discrete measure solution, a result about the density of polynomials in L^2 is given. The finale section is devoted to the study of Λ-multiplicative elements.

For the third chapter, we restrict our selves to the one dimensional case $d = 1$, and we introduce the notion of k-positive Hankel matrices, this is the algebraic equivalent to the notion of k-hyponormal weighted shift, we give an elementary proof to the propagation phenomena for k-positive matrices, and finally we study more closely the perturbation by a rank one operator of k-hyponormal weighted shift.

We provide in the last chapter, a concrete solution to the quintic truncated complex moment problem, when one desires a minimal representing measure. Our method intended to be useful for all odd-degree truncated moment problem.

2 IDEMPOTENTS AND MOMENT PROBLEM OF DISCRETE MEASURE

La Mathématique est l'art de donner le même nom à des choses différentes
— H. Poincaré [58] —

The problem of moment for polynomials consists of giving a multi-sequence of real numbers $\gamma = (\gamma_\alpha)_{\alpha \in \mathbb{N}^d}$ with $\gamma_0 = 1$, and looking for the existence of a positive measure μ on $\mathbb{R}^d$ such that

$$\gamma_\alpha = \int_{\mathbb{R}^d} t^\alpha d\mu(t), \tag{6}$$

where $t = (t_1, \dots, t_d)$ is the d-tuple of coordinate functions in $\mathbb{R}^d$, and $t^\alpha = t_1^{\alpha_1} \cdots t_d^{\alpha_d}$, for all $\alpha = (\alpha_1, \cdots, \alpha_d) \in \mathbb{N}^d$.

Since its introduction by Stieltjes [73], this problem has attracted several mathematicians aiming to resolve it. We mention the Marcel Riesz method [64] based on duality theorem and extension of linear forms on vector spaces. In general, to solve the one dimensional moment problem, the description of positive polynomial and the theory of orthogonal polynomials is usually required. See classical books [1] and [70].

The multidimensional moment problem turns out to be much more difficult because of the non-existence of a good characterization for positive polynomials [6, 59]. Also, the theory of orthogonal polynomials in multidimensional case seems to be not sufficiently developed. To circumvent these difficulties many approaches were given. For example, the operator approach which was firstly treated by Marshall Stone [76] and by Mark Naimark [56] and largely developed by numerous authors [41, 65, 71, 77]. For compact semi-algebraic sets, an elegant condition for the resolution of the multidimensional moment problem was given by Schmüdgen in [66, 67]. Other sufficient conditions for resolving the multidimensional moment problem, using an extension of the moment sequence, were given by M. Putinar and Vasilescu in [61], and also by J. Stochel and Franciszek Szafraniec in [75].

The truncated moment problem entails solving (6) with only finite initial data. More precisely, for a given finite multi-sequence of real numbers $\gamma = (\gamma_\alpha)_{|\alpha| \leqslant p}$ where $|\alpha| = \alpha_1 + \alpha_2 + \cdots + \alpha_d$ with $\gamma_0 = 1$, one looks for a positive measure μ on $\mathbb{R}^d$ such that

$$\text{for } |\alpha| \leqslant p, \qquad \gamma_\alpha = \int_{\mathbb{R}^d} t^\alpha d\mu(t). \tag{7}$$

The study of the truncated moment problem comes naturally from the study of the full one. We recall that in [74] J. Stochel proves that the resolution of the truncated moment problem solves the full moment problem. Actually this fact attracts several authors to investigate the truncated moment problem.

The literature around truncated moment problem has been intensively developed recently in a multitude of papers and memoirs by R.E. Curto and L. Fialkow, see [22, 23, 15, 26], etc. The strategy in these papers is usually based on the existence of a special positive extension of the moment matrix (or in other words looking for a such extension) which preserves the rank, called flat extension. In fact, it was proved that the existence of a rank preserving non negative extension moment matrix leads to the existence of an atomic measure solution of the truncated problem where the number of atoms is exactly the rank of the moment matrix. This last remark motivates our interest to discrete measure solution. Where we mean by discrete probability measure, a probability measure concentrated on at most a countable set.

Recently, unlike in the works done by Curto and Fialkow where the central object is the moment matrix, F. H. Vasilescu adopted another approach to solve the moment problem, see [79, 80, 81]. He considered the linear functional, often called the Riesz functional, induced by the assignment $t^\alpha \mapsto \gamma_\alpha$ on the space of polynomials, which is supposed to be non-negative on the cone of sums of squares of real-valued polynomials. His approach is mainly based on several techniques of analysis on Hilbert spaces, especially around the notion of idempotents, which seems to be more appropriate for the study of discrete measures.

By using the Riesz functional, F. H. Vasilescu proved that the existence of a representing measure for truncated case is equivalent to the existence of a special orthogonal basis formed by idempotent elements, via some elementary C^*-algebra techniques, see [80].

In what follows, we designate by $\mathcal{P}_d$ the algebra of polynomials in $t = (t_1, \ldots, t_d) \in \mathbb{R}^d$, with complex coefficients and real indeterminates. Taking a multi-sequence of real numbers $\gamma = (\gamma_\alpha)_{\alpha \in \mathbb{N}^d}$ with $\gamma_0 = 1$, we associate it with the Riesz map $\Lambda_\gamma : \mathcal{P}_d \to \mathbb{C}$ given by $\Lambda_\gamma(t^\alpha) = \gamma_\alpha$, extended to $\mathcal{P}_d$ by linearity. We clearly have $\Lambda_\gamma(1) = 1$ and $\Lambda_\gamma(\overline{P}) = \overline{\Lambda_\gamma(P)}$ for all $P \in \mathcal{P}_d$. If, moreover, $\Lambda_\gamma(|P|^2) \geqslant 0$ for all $P \in \mathcal{P}_d$, then Λ_γ is said to be a *unital square positive functional* [79, 80] (briefly, a uspf, see the next section). In this case, we say that γ itself is square positive.

Conversely, if $\Lambda : \mathcal{P}_d \to \mathbb{C}$ is a uspf, setting $\Lambda(t^\alpha) = \gamma_\alpha$, we have $\Lambda = \Lambda_\gamma$, as above. The square positive multi-sequence $\gamma = (\gamma_\alpha)_\alpha$ is said to be associated with the uspf Λ. Therefore finding a representing measure for the map Λ is equivalent to solving the moment problem associated with γ.

Let us briefly present the content of this work. In the next section, all definitions needed will be given. Idempotents and Λ-multiplicative elements related to unital square positive functional are introduced and some of their properties are discussed. We define the class of bounded shift spaces and provide that they are exactly the L^2-space with compact support.

The third section contains some results which deal with integral representation of USPF, via $\Lambda-$multiplicative elements, and other ones which can be considered as tools that simplifies the proofs of main results therein. Proposition 2.14 can be viewed as one of the principle tools in this paper, particularly, in this section. Theorem 2.16 gives a positive answer to this question of the existence of representing measures having a number of atoms equals the maximal cardinality of an orthogonal family of idempotents. Theorem 2.18 gives the reciprocal result of Theorem 2.16 but we do not get back the orthogonality condition. In this context example 2.24 shows that the orthogonality can not be recovered. The key point in this approach is the condition (13), which corresponds to the projection of characteristic function on a single point of the support. In the case of a bounded shift spaces, we can easily regain the reciprocal of Theorem 2.16.

The last section will hold a discussion on Λ-multiplicative elements and what kind of conditions should we have to identify them with one point characteristic functions. We notice that here we suppose that Λ has a representing measure μ. To this aim we provide in Theorem 2.19 the first necessary and sufficient condition, although an alternative will be the subject of Theorem 2.22. Theorem 2.23 summarizes those results before. We should mention that in bounded shift spaces no additional assumption is required, any Λ-multiplicative idempotent is almost everywhere a One point characteristic function.

2.1 DEFINITIONS AND ELEMENTARY PROPERTIES

We recall that $\mathcal{P}_d$ is the algebra of polynomials in $t = (t_1, \ldots, t_d) \in \mathbb{R}^d$, with complex coefficients and real indeterminates.

Following [79, 80], a linear map $\Lambda : \mathcal{P}_d \longrightarrow \mathbb{C}$ is called *unital square positive functional* (briefly ***uspf***) if

1. $\Lambda(\bar{P}) = \overline{\Lambda(P)}$, for all $P \in \mathcal{P}_d$.
2. $\Lambda(|P|^2) \geqslant 0$, for all $P \in \mathcal{P}_d$.
3. $\Lambda(1) = 1$.

We can see easily that conditions 2 and 3 imply 1. In particular, Λ is a positive quadratic form on $\mathcal{P}_d$ and the classical Cauchy-Schwarz inequality clearly holds,

$$|\Lambda(P\bar{Q})| \leqslant \sqrt{\Lambda(|P|^2)}\sqrt{\Lambda(|Q|^2)}.$$

We will use the GNS construction to construct our Hilbert space $\mathcal{H}$. We put

$$I_\Lambda = \{P : \Lambda(|P|^2) = 0\} = \{P : \Lambda(PQ) = 0 \ \forall Q \in \mathcal{P}_d\}.$$

The Cauchy-Schwarz inequality shows that I_Λ is an ideal of $\mathcal{P}_d$, and $\mathcal{P}_d/I_\Lambda$ becomes a complex algebra. We endow $\mathcal{P}_d/I_\Lambda$ with the following inner product

$$\langle \hat{P}, \hat{Q} \rangle = \Lambda(P\bar{Q}), \qquad \hat{P}, \hat{Q} \in \mathcal{P}_d/I_\Lambda.$$

We denote by $\mathcal{H}$ the Cauchy completion of $\mathcal{P}_d/I_\Lambda$. Then $\mathcal{H}$ is a separable Hilbert space and since I_Λ is invariant under complex conjugation, there is a natural conjugation in the space $\mathcal{H}$ too, also denoted by $\bar{x}$ for all $x \in \mathcal{H}$.
Using the conjugation on $\mathcal{H}$, one can show the following.

$$\textit{Real polynomials in } \mathcal{P}_d \textit{ are dense in } R(\mathcal{H}). \tag{8}$$

Let μ be a representing measure of a uspf Λ. Denote by $\mathcal{H}^2(\mu)$ the closure of the set of all (a.e. $[\mu]$) equivalence classes $[P]_\mu$ of polynomials $P \in \mathcal{P}_d$ in $L^2(\mu)$. Then the map

$$\mathcal{H} \ni \hat{P} \longmapsto [P]_\mu \in \mathcal{H}^2(\mu) \tag{9}$$

is a unitary isomorphism which sends $\bar{\hat{P}}$ into $[\bar{P}]_\mu$. In particular, we have

$$\Lambda(P\bar{Q}) = \langle [P]_\mu, [Q]_\mu \rangle, \quad P, Q \in \mathcal{P}_d.$$

This implies that the conjugation on $\mathcal{H}$ defined above is unitarily isomorphic to the usual conjugation of equivalence classes $[f]_\mu \mapsto [\bar{f}]_\mu$ on $\mathcal{H}^2(\mu)$. In other words, we can choose $\mathcal{H}^2(\mu)$ as a completion of $\mathcal{P}/I_\Lambda$ and regard $\mathcal{H}^2(\mu)$ as the Hilbert space associated with Λ equipped with the conjugation $\mathcal{H}^2(\mu) \ni [f]_\mu \mapsto [\bar{f}]_\mu \in \mathcal{H}^2(\mu)$.

The equivalence class corresponding to $P \in \mathcal{P}_d$ will be as well denoted by P, when no confusion is possible.

We denote by $\mathfrak{B}(\mathbb{R}^d)$ the Borel σ-algebra, and χ_Ω the associated indicator function for a Borel set $\Omega \in \mathfrak{B}(\mathbb{R}^d)$. For $c \in \mathbb{R}^d$ we designate by δ_c the Dirac measure centered on c. We keep all these notations throughout the paper.

Definition 2.1.

1. We say that $x \in \mathcal{H}$ is real if $x = \bar{x}$; And we denote the set of all real elements of $\mathcal{H}$ by $R(\mathcal{H})$.
2. An element $x \in R(\mathcal{H}) \setminus \{0\}$ is called an idempotent if

$$\langle x, x \rangle = \langle x, 1 \rangle. \tag{10}$$

Let show some useful properties of idempotent elements.

Lemma 2.2. *Let $\mathcal{H}$ be the Hilbert space associated with a uspf Λ and let $x \in \mathcal{H}$ be an idempotent. Then*

(i) $\|x\| \leqslant 1$,

(ii) $\|x\| = 1$ *if and only if* $x = 1$.

Proof. (i) By the Cauchy-Schwarz inequality, we have

$$\|x\|^2 = \langle x, 1\rangle \leqslant \|x\|\sqrt{\Lambda(1)} = \|x\|, \tag{11}$$

which yields $\|x\| \leqslant 1$.

(ii) If $\|x\| = 1$, then equality holds in the Cauchy-Schwarz inequality (11), and so $x = \Lambda 1$ for some $\Lambda \in \mathbb{C} \setminus \{0\}$. By the first equation in (11), $x = 1$. The converse implication is trivial. □

It is immediate to see that the sum of two idempotents is not necessary an idempotent. But by adding the orthogonality condition, this sum becomes perforce an idempotent. In the following, we investigate an important propriety of orthogonal family of idempotents.

Proposition 2.3. *Let $(x_i)_{i\in I}$ be an orthogonal family of idempotents of $\mathcal{H}$. Then the series $\sum_{i\in I} x_i$ is convergent and the following assertions are equivalent*

(i) $\{x_i; i \in I\}^\perp \subset \{1\}^\perp$.

(ii) $1 \in \overline{\mathit{Vect}\{x_i; i \in I\}}$.

(iii) $\sum_{i\in I} x_i = 1$.

In particular, the following inclusion holds:

$$(x_i)_{i\in I} \textit{ is total in } \mathcal{H} \Longrightarrow \sum_{i\in I} x_i = 1. \tag{12}$$

Proof. The series $\sum_{i\in I} x_i$ converges, because the system $e_i := \frac{x_i}{\|x_i\|}$, $i \in I$, is orthonormal and $\sum_{i\in I} \|x_i\|^2 = \sum_{i\in I}\langle e_i, 1\rangle^2 \leqslant \|1\|^2 = 1$ (Bessel's inequality).

(i) $\Longleftrightarrow$ (ii) Use the standard double orthogonal complement argument.

(ii)$\Longrightarrow$(iii) Apply the Fourier series expansion to the vector 1 with respect to the orthonormal basis $\{e_i : i \in I\}$ of $\overline{\mathrm{Vect}\{x_i; i \in I\}}$.

(iii)$\Longrightarrow$(ii) This is obvious.

□

The following example shows that the reciprocal of (12) doesn't hold in general.

Example 2.4. Let the functional Λ be given by a Borel probability measure μ on $\mathbb{R}$ with the closed support $\{u_1, u_2, u_3\}$ (distinct atoms). Set $I = \{1, 2\}$, $x_1 = \chi_{\{u_1,u_2\}}$ and $x_2 = \chi_{\{u_3\}}$. Then $(x_i)_{i\in I}$ is an orthogonal family of idempotents of $\mathcal{H}$ such that $\sum_{i\in I} x_i = 1$. However, the family $(x_i)_{i\in I}$ is not total in $\mathcal{H}$.

In the following, we introduce one of the central subjects of this paper. Since we are interested by uspf represented by a discrete measure μ, we should look for an adequate representation, in our context, of an indicator function of a single point.

Let $\mu = \sum_{i\in J} a_i \delta_{c_i}$ be a positive discrete measure on $\mathbb{R}^d$. So by definition

$$\langle t^\alpha, \chi_{\{c_j\}}\rangle = \int_{\mathbb{R}^d} t^\alpha \chi_{\{c_j\}}(t) d\mu(t) = a_j c_j^\alpha, \quad \alpha \in \mathbb{N}^d,\ j \in J.$$

Consequently, we get $\langle t^\alpha, \chi_{\{c_j\}}\rangle\langle t^\beta, \chi_{\{c_j\}}\rangle = \langle 1, \chi_{\{c_j\}}\rangle\langle t^{\alpha+\beta}, \chi_{\{c_j\}}\rangle$. This simple remark leads to the succeeding definition

Definition 2.5. Let be $x \in R(\mathcal{H}) \setminus \{0\}$, we say that x is Λ−multiplicative if

$$\langle t^\alpha, x\rangle\langle t^\beta, x\rangle = \langle 1, x\rangle\langle t^{\alpha+\beta}, x\rangle, \quad \alpha, \beta \in \mathbb{N}^d. \tag{13}$$

Note that if $x \in R(\mathcal{H}) \setminus \{0\}$ is Λ−multiplicative, then αx is Λ−multiplicative for every $\alpha \in \mathbb{R} \setminus \{0\}$. Moreover linearity implies

Lemma 2.6. *Let $P, Q \in \mathcal{P}_d$ and x be a Λ−multiplicative element. Then*

$$\langle P, x\rangle\langle Q, x\rangle = \langle 1, x\rangle\langle PQ, x\rangle. \tag{14}$$

We mention that if $x \neq 0$, then (13) implies that $\langle 1, x\rangle \neq 0$. More generally the coming result appears

Lemma 2.7. *Let $\mathcal{H}$ be the Hilbert space associated with a uspf Λ. Suppose that $x \in \mathcal{H}$ such that for some $\xi \in \mathbb{C}$,*

$$\langle P, x\rangle\langle Q, x\rangle = \xi\langle PQ, x\rangle, \quad P, Q \in \mathcal{P}_d. \tag{15}$$

Then the following statements hold:

(i) if $\xi = 0$, then $\langle 1, x\rangle = 0$,

(ii) if $\xi \neq 0$ and $\langle 1, x\rangle = 0$, then $x = 0$,

(iii) $\xi \neq 0$ and $x \neq 0$ if and only if $\langle 1, x\rangle \neq 0$; if $\langle 1, x\rangle \neq 0$, then $x \neq 0$ and $\xi = \langle 1, x\rangle$.

Proof. (i) Substitute $P = Q = 1$ into (15).

(ii) Substitute $Q = 1$ into (15), we deduce that $\langle P, x\rangle = 0$ for all $P \in \mathcal{P}_d$, which implies that $x = 0$.

(iii) If $\xi \neq 0$ and $x \neq 0$, then by (ii), $\langle 1, x\rangle \neq 0$. Conversely, if $\langle 1, x\rangle \neq 0$, then substituting $P = Q = 1$ into (15) yields $\xi = \langle 1, x\rangle$. This completes the proof. □

Hereafter, we introduce the class of bounded shift spaces, which includes several classical Hilbert function spaces.

Definition 2.8. Let Λ be a uspf and $\mathcal{H}$ be the associated Hilbert space. We say that $\mathcal{H}$ is a bounded shift spaces if for every $j \leqslant d$, the shift operator

$$\begin{array}{rccc} M_{t_j}: & \mathcal{P}_d & \to & \mathcal{P}_d \\ & P & \mapsto & t_j P \end{array}$$

is bounded. In particular, there exist some positive constants C_j such that

$$\Lambda(t_j^2|P|^2) \leqslant C_j \Lambda(|P|^2).$$

Example 2.9.

1. Every finite dimensional space is a bounded shift space.
2. If μ is a compactly supported measure, then $\mathcal{H} = L^2(\mu)$ is a bounded shift space.
3. The uspf Λ_γ associated with the moment sequence $\gamma = (k!)_k$ induces a non bounded shift space. Indeed
$$\frac{\Lambda_\gamma(t^2|t^k|^2)}{\Lambda_\gamma(|t^k|^2)} = \frac{\gamma_{2k+2}}{\gamma_{2k}} = (2k+2)(2k+1).$$
4. The discrete measure $\mu = \frac{1}{\sum_{n\in\mathbb{Z}} e^{-n^2/2}} \sum_{n\in\mathbb{Z}} e^{-n^2/2}\delta_{e^n}$ generates a non bounded shift space. For this measure the k^{th} moment is given as follows
$$\gamma_k = e^{k^2/2}.$$
Hence,
$$\frac{\Lambda_\gamma(t^2|t^k|^2)}{\Lambda_\gamma(|t^k|^2)} = e^{4k+2}.$$

We can remark, from the previous examples, that bounded shift spaces are associated with compactly supported measures. More precisely, using the operator theory approach as in [41, 77, 66], we have

Proposition 2.10. *Let $\mathcal{H}$ be the Hilbert space associated with a uspf Λ. Then the following conditions are equivalent*:

(i) $\mathcal{H}$ is a bounded shift spaces

(ii) there exists a compactly supported representing measure μ of Λ such that $\mathcal{H}^2(\mu) = L^2(\mu)$.

Moreover, if (ii) *holds, then the multi-sequence $\gamma_\alpha := \Lambda(t^\alpha)$, $\alpha \in \mathbb{N}^d$, is ultra-determinate, the conjugation (inherited from $\mathcal{H}$ via the unitary isomorphism* (9)*) of $f \in \mathcal{H}^2(\mu)$ coincides with $\bar{f}$ and $R(\mathcal{H}^2(\mu)) = \{f \in L^2(\mu) : f = \bar{f}$ a.e. $[\mu]\}$.*

Proof. (i)⇒(ii) By assumption, for every $j = 1, \ldots, d$, the multiplication operator $M_{t_j} : \mathcal{P} \subseteq \mathcal{P}$ is bounded and symmetric, i.e., $\langle M_{t_j} P, Q\rangle = \langle P, M_{t_j} Q\rangle$ for all $P, Q \in \mathcal{P}$, so it has a unique bounded self-adjoint extension to $\mathcal{H}^2(\mu)$, denoted by the same symbol M_{t_j}. Then $(M_{t_1}, \ldots, M_{t_d})$ is a d-tuple of commuting bounded self-adjoint operators on $\mathcal{H}^2(\mu)$. Let E be its joint spectral measure. Since the closed support of E is a compact subset of $\mathbb{R}^d$, so is the closed support of $\mu(\cdot) := \langle E(\cdot)1, 1\rangle$. Then for $\alpha \in \mathbb{N}^d$

$$\begin{aligned}\gamma_\alpha = \Lambda(t^\alpha) = \langle M_{t_1}^{\alpha_1} \cdots M_{t_d}^{\alpha_d} 1, 1\rangle &= \left\langle \int_{\mathbb{R}^d} t^\alpha E(dt)1, 1 \right\rangle \\ &= \int_{\mathbb{R}^d} t^\alpha d\mu(t).\end{aligned}$$

Since μ is compactly supported, the multi-sequence $\{\gamma_\alpha\}_{\alpha\in\mathbb{N}^d}$ is ultra-determinate, and consequently $\mathcal{P}_d$ is dense in $L^2(\mu)$ (see [41]). However, $\Lambda(P\bar{Q}) = \int_{\mathbb{R}^d} P\bar{Q}d\mu$ for all $P, Q \in \mathcal{P}_d$, so $\mathcal{H}^2(\mu) = L^2(\mu)$. Finally, since an L^2-convergent sequence has an almost everywhere convergent subsequence, we see that the conjugation of $f \in \mathcal{H}^2(\mu)$ coincides with $\bar{f}$ (or by the uniqueness of extension of conjugation from $\mathcal{P}/I_\Lambda$ to $\mathcal{H}$), and thus $R(\mathcal{H}) = \{f \in L^2(\mu) \colon f = \bar{f} \text{ a.e. } [\mu]\}$.

(ii)⇒(i) Obvious. □

Remark 2.11. There has been constructed in [8] an example of a determinate moment multi-sequence such that polynomials are not dense in L^2. For instance see also [68] and [41], for more details.

In passing, we mention an interesting propriety of Λ-multiplicative elements in bounded shift spaces.

Proposition 2.12. *Let $\mathcal{H}$ be a bounded shift spacesand let x and y be two $\Lambda-$multiplicative idempotent elements of* $R(\mathcal{H})$. *Then*

$$\langle x, y\rangle \neq 0 \iff x = y.$$

In other words, every two different $\Lambda-$multiplicative idempotents are orthogonal. This fact will be clearly seen through Section 4.

Proof. We assume that $\langle x, y\rangle \neq 0$, and let $R \in \mathcal{P}_d$. By the construction of $\mathcal{H}$ there exist two real bounded sequences $(P_k)_k$ and $(Q_k)_k$ in $\mathcal{P}_d$ that converge to x and y respectively. Therefore for any $R \in \mathcal{P}_d$ we have,

$$\langle R, y\rangle\langle x, y\rangle = \lim_k \lim_l \langle P_k R, Q_l\rangle\langle 1, y\rangle = \lim_k \lim_l \Lambda(P_k R Q_l)\langle 1, y\rangle \quad (16)$$

Since the shift operator is bounded we have

$$\langle R, y\rangle\langle x, y\rangle = \lim_l \lim_k \Lambda(R Q_l P_k)\langle 1, y\rangle = \frac{\langle y, x\rangle\langle R, x\rangle\langle 1, y\rangle}{\langle 1, x\rangle}. \quad (17)$$

Which means,

$$\frac{\langle R, x\rangle}{\langle 1, x\rangle} = \frac{\langle R, y\rangle}{\langle 1, y\rangle},$$

for every $R \in \mathcal{P}_d$. Therefore

$$x = \frac{\langle 1, x\rangle}{\langle 1, y\rangle} y.$$

By using the fact that x is an idempotent, we derive the following equality

$$\langle x, 1\rangle = \langle y, 1\rangle.$$

Finally,

$$x = y.$$

□

2.2 DISCRETE REPRESENTING MEASURE OF USPF

This section is mainly devoted to the study of discrete integral representation of an uspf. Keeping the same notation as above, let Λ be an uspf and $\mathcal{H}$ be the associated Hilbert space.

As said in the second section, we drive a Λ-multiplicative element from the indicator function of a single point. In the finite dimensional case, the converse of this fact was given by F-H. Vasilescu in [80]. He utilized the finite dimensional structure to prove that Λ-multiplicative orthogonal basis of $\mathcal{H}$ induces a set of characters on the C^*-algebra $\mathcal{H}$. In the general case, we will drive such a representation for the whole space of polynomials $\mathcal{P}_d/I_\Lambda$, in term of Λ-multiplicative elements.

The following simple observation allows us to reduce the problem of describing Λ-multiplicative elements to just Λ-multiplicative idempotents. This is clearly due to the equation (13) and the fact that it still holds for any scalar multiplication of x. We leave its simple proof to the reader.

Lemma 2.13. *Let $\mathcal{H}$ be the Hilbert space associated with a uspf Λ and $x \in R(\mathcal{H})$ be such that $\langle 1, x\rangle \neq 0$. Then $\hat{x} := \frac{\langle 1,x\rangle}{\|x\|^2} x$ is an idempotent and the following conditions are equivalent*:

(i) x *is Λ-multiplicative,*

(ii) $\hat{x}$ *is a Λ-multiplicative idempotent of $\mathcal{H}$.*

The following result describes Λ-multiplicative elements.

Proposition 2.14. *If $v \in \mathcal{H} \setminus \{0\}$, then the following conditions are equivalent*:

(i) $\langle P, v\rangle\langle Q, v\rangle = \langle PQ, v\rangle$ *for all* $P, Q \in \mathcal{P}_d$,

(ii) *there exists* (*a unique*) $c \in \mathbb{C}^d$ *such that*

$$\langle P, v\rangle = P(c), \quad P \in \mathcal{P}_d. \tag{18}$$

Moreover, if (ii) *holds and* $\nu \in R(\mathcal{H}) \setminus \{0\}$, *then* $c \in \mathbb{R}^d$. *If* $c \in \mathbb{C}^d$, *then the following conditions are equivalent*:

(iii) there exists (*a unique*) $\nu \in \mathcal{H}$ *such that* (18) *holds,*

(iv) there exists $\kappa < \infty$ *such that*

$$|P(c)| \leqslant \kappa\sqrt{\Lambda(|P|^2)}, \quad P \in \mathcal{P}_d. \tag{19}$$

The smallest κ *in* (19) *is equal to* $\|\nu\|$, *where* ν *is as in* (iii).

If $\nu \in R(\mathcal{H}) \setminus \{0\}$ satisfies the condition (i) of Proposition 2.14, then by Lemmata 2.7 and 2.13, $1 = \langle 1, \nu\rangle$ and the vector $\tilde{\nu} := \frac{1}{\|\nu\|^2}\nu$ is an idempotent which is Λ-multiplicative.

Proof of Proposition 2.14. (i)$\Rightarrow$(ii) By induction on n (and the above observation), we see that

$$\langle P^n, \nu\rangle = \langle P, \nu\rangle^n \quad n \in \mathbb{N}. \tag{20}$$

This implies that

$$\langle t^\alpha, \nu\rangle = \langle t_1^{\alpha_1} \cdots t_d^{\alpha_d}, \nu\rangle = \langle t_1^{\alpha_1}, \nu\rangle \cdots \langle t_d^{\alpha_d}, \nu\rangle \overset{(20)}{=} c^\alpha, \quad \alpha \in \mathbb{N}^d, \tag{21}$$

where $c = (\langle t_1, \nu\rangle, \ldots, \langle t_d, \nu\rangle)$. By linearity, (21) yields (18). Substituting $P(t) = t_j$ into (18), we obtain $c_j = \langle t_j, \nu\rangle$ for $j = 1, \ldots, d$, which justifies the uniqueness of c. If additionally $\nu \in R(\mathcal{H}) \setminus \{0\}$, then $\langle t_j, \nu\rangle = \langle \nu, t_j\rangle = \overline{\langle t_j, \nu\rangle}$ for $j = 1, \ldots, d$, and so $c \in \mathbb{R}^d$.

(ii)$\Rightarrow$(i) Obvious.

(iii)$\Rightarrow$(iv) Apply the Cauchy-Schwarz inequality.

(iv)$\Rightarrow$(iii) Applying the Riesz representation theorem, we get $\nu \in \mathcal{H}$ as in (ii) with $\kappa = \|\nu\|$. Then

$$\mathbb{R} \ni P(c) = \langle P, \nu\rangle = \langle \bar{\nu}, P\rangle = \langle P, \bar{\nu}\rangle, \quad P \in \mathcal{P}_d \cap R(\mathcal{H}).$$

As a consequence, we have

$$0 = \langle P, \nu - \bar{\nu}\rangle, \quad P \in \mathcal{P}_d,$$

which yields $\nu = \bar{\nu}$. □

A direct consequence of the proposition, is the linear independence of normalized Λ-multiplicative elements

Corollary 2.15. *If* $x_1, \ldots, x_\kappa$ *are* Λ*-multiplicative elements of* $\mathcal{H}$, *then the following conditions are equivalent*:

(i) There exists $\{\rho_j\}_{j=1}^{\kappa} \subseteq \mathbb{R}\setminus\{0\}$ *such that* $x = \sum_{j=1}^{\kappa} \rho_j x_j$ *is a* Λ*-multiplicative element of* $\mathcal{H}$,

(ii) $\frac{x_j}{\langle 1, x_j\rangle} = \frac{x_\kappa}{\langle 1, x_\kappa\rangle}$ *for all* $j = 1, \ldots, \kappa$.

Proof. (i)⇒(ii) First we define the equivalence relation R on $J = \{1, \dots, \kappa\}$ by iRj if and only if $\frac{x_i}{\langle 1, x_i \rangle} = \frac{x_j}{\langle 1, x_j \rangle}$. Denote by $\mathcal{R}$ the set of all equivalence classes. Then

$$x = \sum_{a \in \mathcal{R}} \left[\sum_{j \in a} \left(\rho_j \frac{\langle 1, x_j \rangle}{\langle 1, x_{l_a} \rangle} \right) \right] x_{l_a} = \sum_{a \in \mathcal{R}} \tilde{\rho}_a x_{l_a}.$$

Hence, if $\mathcal{R}$ has one element, then we are done. Suppose that $\mathcal{R}$ has more than one element. Without loss of generality we may assume that $x = \sum_{j=1}^{\kappa} \rho_j x_j$, $\kappa \geqslant 2$ and

$$\frac{x_i}{\langle 1, x_i \rangle} \neq \frac{x_j}{\langle 1, x_j \rangle} \text{ for all } i \neq j. \tag{22}$$

Since x is Λ-multiplicative, by Proposition 2.14, there exists $c \in \mathbb{R}$ such that

$$\langle P, x \rangle = \langle 1, x \rangle P(c) = \sum_{j=1}^{\kappa} \rho_j \langle 1, x_j \rangle P(c), \quad P \in \mathcal{P}_d. \tag{23}$$

For the same reason, for every $j = 1, \dots, \kappa$, there exists $c_j \in \mathbb{R}$ such that

$$\langle P, x_j \rangle = \langle 1, x_j \rangle P(c_j), \quad P \in \mathcal{P}_d. \tag{24}$$

By (22) and Proposition 2.14, we have

$$c_i \neq c_j, \quad i \neq j. \tag{25}$$

Combining (23) and (24), we see that

$$\sum_{j=1}^{\kappa} \rho_j \langle 1, x_j \rangle [P(c) - P(c_j)] = 0, \quad p \in \mathcal{P}_d. \tag{26}$$

Substituting $P(t) = \prod_{j=1}^{\kappa} \|t - c_j\|^2$, we get

$$0 = \sum_{j=1}^{\kappa} \rho_j \langle 1, x_j \rangle P(c) = P(c) \sum_{j=1}^{\kappa} \rho_j \langle 1, x_j \rangle = P(c) \langle 1, x \rangle.$$

This implies that $P(c) = 0$ so $c \in P^{-1}(\{0\}) = \{c_1, \dots, c_\kappa\}$. After reordering, we may assume that $c = c_\kappa$. Combined with Proposition 2.14, this gives $x = \alpha_\kappa x_\kappa$ for some $\alpha_\kappa \in \mathbb{R} \setminus \{0\}$. Now there are two possibilities: either $\alpha_\kappa \neq \rho_\kappa$ and then $x = \sum_{j=1}^{\kappa-1} \rho_j' x_j$ with $\{\rho_j'\}_{j=1}^{\kappa-1} \subseteq \mathbb{R} \setminus \{0\}$, or $\alpha_\kappa = \rho_\kappa$ and then $\sum_{j=1}^{\kappa-1} \rho_j x_j = 0$, which implies that

$$0 = \langle P, \sum_{j=1}^{\kappa-1} \rho_j x_j \rangle = \sum_{j=1}^{\kappa-1} \rho_j \langle P, x_j \rangle \overset{(24)}{=} \sum_{j=1}^{\kappa-1} \rho_j \langle 1, x_j \rangle P(c_j), \quad P \in \mathcal{P}_d. \tag{27}$$

We show that the latter does not hold. Indeed, otherwise by (25), we can apply the multivariable Lagrange interpolation theorem to find a polynomial $P \in \mathcal{P}_d$ such that $P(c_j) = \frac{1}{\rho_j\langle 1, x_j\rangle}$ for $j = 1, \dots, \kappa - 1$. Substituting P into (27), we get a contradiction. Thus $x = \sum_{j=1}^{\kappa-1} \rho_j' x_j$ with $\{\rho_j'\}_{j=1}^{\kappa-1} \in \mathbb{R}\backslash\{0\}$, so we can proceed by induction to conclude that $x = \alpha_j x_j$ for all $j = 1, \dots, \kappa$ and for some $\alpha_1, \dots, \alpha_\kappa \in \mathbb{R} \setminus \{0\}$. This implies that $\frac{x_i}{\langle 1, x_i\rangle} = \frac{x_j}{\langle 1, x_j\rangle}$ for all $i \neq j$, which contradicts (22).
(ii)⇒(i) Obvious. □

The coming Theorem 2.16 gives intrinsic conditions to the existence of a representing measure for an uspf Λ . In other words, it provides the sufficient conditions for it.

Theorem 2.16. *Let Λ be an uspf, and $(x_i)_{i\in I}$ be a total family of $\Lambda-$multiplicative orthogonal idempotent elements. Then, there exists a discrete probability measure μ such that* $\mathrm{Card}(\mathrm{supp}(\mu)) = \dim\mathcal{H}$, *and*

$$\Lambda(P) = \int_{\mathbb{R}^d} P(t)d\mu(t), \; P \in \mathcal{P}_d.$$

Proof. Since the family $(x_i)_i$ satisfies the conditions of Proposition 2.3, we obtain $\langle x, 1\rangle = \sum_i \langle x, x_i\rangle$. Therefore, by proposition 2.14, we have

$$\Lambda(P) = \sum_i \langle P, x_i\rangle = \sum_i \langle 1, x_i\rangle P(c_i) = \sum_i \|x_i\|^2 P(c_i).$$

In addition, we have

$$\sum_i \langle 1, x_i\rangle |P(c_i)| \leqslant \left(\sum_i \langle 1, x_i\rangle\right)^{1/2} \left(\sum_i \langle 1, x_i\rangle |P(c_i)|^2\right)^2 = \Lambda(|P|^2)^{1/2}$$

Hence, the measure $\mu = \sum_i \|x_i\|^2 \delta_{c_i}$ gives an integral representation for Λ. Furthermore, Since all c_i are distinct, μ is a discrete probability measure on $\mathbb{R}^d$. □

As a consequence of the previous theorem, in the case of bounded shift spaces we get

Corollary 2.17. *Let Λ be an uspf and $\mathcal{H}$ be a bounded shift spaces then the following conditions are equivalent:*

(i) there is a total family $(x_i)_{i\in I} \in R(\mathcal{H})$ of $\Lambda-$multiplicative elements,

(ii) there exists a discrete probability measure $\mu = \sum_{i\in I} a_i\delta_{c_i}$ such that

$$\Lambda(P) = \int_{\mathbb{R}^d} P(t)d\mu(t), \; P \in \mathcal{P}_d.$$

Proof. (i) ⇒ (ii) Using Lemma 2.13, we provide a family $(\hat{x}_i)_{i\in I}$ that is $\Lambda-$multiplicative idempotent family. Since $\mathcal{H}$ is a bounded shift spaces Proposition 2.12 gives that $(\hat{x}_i)_i$ is an orthogonal basis and our claim becomes evident in view of Theorem 2.16.

(ii) $\Rightarrow$ (i) Since $\mathcal{H}$ is a bounded shift spaces then Proposition 2.10 gives that $\mathcal{H} = L^2(\mu)$, we take $(x_i)_{i\in I} = (\chi_{\{c_i\}})_{i\in I}$ which is a total family of $\mathcal{H}$ consisting of Λ-multiplicative elements. □

Since all conditions in Theorem 2.16 are intrinsic, it is natural to investigate the existence of a converse. A partial answer is given in the next result,

Theorem 2.18. *Let Λ be an uspf such that there exists a discrete probability measure μ satisfying*

$$\Lambda(P) = \int_{\mathbb{R}^d} P(t)d\mu(t).$$

Then, there exists a family $(x_i)_{i\in I} \in R(\mathcal{H}) \setminus \{0\}$ verifying,

1. $\langle x_i, x_i\rangle = \langle 1, x_i\rangle$.
2. *Vect*$(x_i)_i$ *is dense in* $\mathcal{H}$.
3. $\forall i \in I, \forall \alpha, \beta \in \mathbb{N}^d \qquad \langle t^\alpha, x_i\rangle\langle t^\beta, x_i\rangle = \langle 1, x_i\rangle\langle t^{\alpha+\beta}, x_i\rangle$.

Proof. We assume that $\Lambda(P) = \int_{\mathbb{R}^d} P(t)d\mu(t)$ where $\mu = \sum_i \rho_i\delta_{\Lambda_i}$ with $\Lambda_i \in \mathbb{R}^d$ are distinct for all index i, and let $P_{\mathcal{H}} : L^2(\mu) \to \mathcal{H}$ be the orthogonal projection on the closed subspace $\mathcal{H}$.

We put $x_i = P_{\mathcal{H}}(\chi_{\Lambda_i})$, we have $\langle 1, x_i\rangle = \langle 1, P_{\mathcal{H}}(\chi_{\Lambda_i})\rangle = \langle 1, \chi_{\Lambda_i}\rangle = \rho_i > 0$, since $\overline{\chi_{\Lambda_i}} = \chi_{\Lambda_i}$, we get $x_i \in R(\mathcal{H})$. Using Lemma 2.13, we exhibit a vector $\hat{x}_i$, which satisfies 1. The fact that Vect$(\chi_{\Lambda_i})_i$ is dense in $L^2(\mu)$ implies that Vect$(x_i)_i$ is dense in $\mathcal{H}$ and then Vect$(\hat{x}_i)_i$ is dense in $\mathcal{H}$ too . This proves 2.

Let $i \in I$ and $\alpha, \beta \in \mathbb{N}^d$, we have in one hand

$$\begin{aligned}\langle t^\alpha, x_i\rangle\langle t^\beta, x_i\rangle &= \langle t^\alpha, P_{\mathcal{H}}(\chi_{\Lambda_i})\rangle\langle t^\beta, P_{\mathcal{H}}(\chi_{\Lambda_i})\rangle \\ &= \langle P_{\mathcal{H}}(t^\alpha), \chi_{\Lambda_i}\rangle\langle P_{\mathcal{H}}(t^\beta), \chi_{\Lambda_i}\rangle = \rho_i^2\Lambda_i^{\alpha+\beta}\end{aligned}$$

and in other hand,

$$\langle t^{\alpha+\beta}, x_i\rangle\langle 1, x_i\rangle = \langle P_{\mathcal{H}}(t^{\alpha+\beta}), \chi_{\Lambda_i}\rangle\langle P_{\mathcal{H}}(1), \chi_{\Lambda_i}\rangle = \rho_i^2\Lambda_i^{\alpha+\beta}. \quad (28)$$

So x_i and hence $\hat{x}_i$ is Λ-multiplicative. □

We should mention here that even with the assumption of polynomials are dense, we do not get back the orthogonality condition. For illustration see Example 2.24

2.3 Λ-MULTIPLICATIVE ELEMENTS

In the present section, assuming that the uspf Λ has a representing measure μ, we discuss the question when a Λ-multiplicative element of $\mathcal{H}^2(\mu)$ can be written as an indicator function. Strictly speaking, what kind of assumption should we have to obtain that this Λ-multiplicative element is, μ-almost everywhere, a characteristic function of one point. In this context, the first theorem in this section gives a necessary and sufficient condition for a Λ-multiplicative idempotent to be, μ-almost everywhere, the required function in above. In particular, to get the same result, the case of bounded shift spacesdoes not need any additional constraint on a Λ-multiplicative idempotent. However, Theorem 2.22 gives a different but natural condition that is the existence of such a characteristic function.

Theorem 2.19. *Suppose that Λ is a uspf having a representing measure μ and $\mathcal{H} = \mathcal{H}^2(\mu)$. If $x \in \mathcal{H}$, then the following conditions are equivalent:*

(i) x is a Λ-multiplicative idempotent in $\mathcal{H}$ such that $P \cdot x \in \mathcal{H}$ for all $P \in \mathcal{P}$,

(ii) x is a Λ-multiplicative idempotent in $\mathcal{H}$ such that $x \cdot \mathcal{H} \subseteq \mathcal{H}$, that is $f \cdot x \in \mathcal{H}$ for all $f \in \mathcal{H}$,

(iii) there exists $c \in \mathbb{R}^d$ such that $\mu(\{c\}) > 0$ and $x = \chi_{\{c\}}$ a.e. $[\mu]$.

Proof. (ii)$\Rightarrow$(i) This is obvious.

(i)$\Rightarrow$(iii) Set $e = \frac{x}{\|x\|}$. Since x is an idempotent (i.e., $\langle 1, x\rangle = \|x\|^2$), we get

$$\begin{aligned}\int_{\mathbb{R}^n} (xP)\bar{Q}\,d\mu = \langle x, \bar{P}Q\rangle &= \frac{1}{\|x\|^2}\langle x, \bar{P}\rangle\langle x, Q\rangle \\ &= \frac{1}{\|x\|^2}\langle P, x\rangle\langle x, Q\rangle \\ &= \langle (e \otimes e)P, Q\rangle, \quad P, Q \in \mathcal{P}. \end{aligned} \tag{29}$$

Denote by M_x the operator of multiplication by x in $L^2(\mu)$. It is well known that M_x is closed (in fact, M_x is selfadjoint because $x \in R(\mathcal{H})$). By our assumption $\mathcal{P} \subseteq \mathcal{D}(M_x)$. Hence, it follows from (29) that

$$\langle (M_x|_{\mathcal{P}})P, Q\rangle = \langle (e \otimes e|_{\mathcal{P}})P, Q\rangle, \quad P, Q \in \mathcal{P}. \tag{30}$$

Since $\mathcal{P}$ is dense in $\mathcal{H}$ and $M_x(\mathcal{P}) \subseteq \mathcal{H}$, we infer from (30) that $M_x|_{\mathcal{P}} = e \otimes e|_{\mathcal{P}}$. Therefore, by the boundedness of $e \otimes e$ and the closedness of M_x, we have

$$e \otimes e = \overline{e \otimes e|_{\mathcal{P}}} = \overline{M_x|_{\mathcal{P}}} \subseteq M_x.$$

This implies that $\mathcal{H} \subseteq \mathcal{D}(M_x)$, $M_x(\mathcal{H}) \subseteq \mathcal{H}$ and $M_x|_{\mathcal{H}} = e \otimes e$. Since $1 \in \mathcal{H}$ and $e \otimes e$ is an orthogonal projection, we get

$$x = M_x|_{\mathcal{H}}(1) = e \otimes e(1) = (e \otimes e)^2(1) = (M_x|_{\mathcal{H}})^2(1) = x^2.$$

This in turn implies that there exists $\Omega \in \mathfrak{B}(\mathbb{R}^n)$ such that $x = \chi_\Omega$ a.e. $[\mu]$. Applying Proposition 2.14 to $\nu := \frac{1}{\langle 1,x\rangle} x \in R(\mathcal{H}) \setminus \{0\}$, we deduce that there exists $c = (c_1, \ldots, c_d) \in \mathbb{R}^d$ such that (18) holds. This yields

$$\int_\Omega P d\mu = \langle P, x\rangle = \mu(\Omega) P(c), \quad P \in \mathcal{P}. \tag{31}$$

Substituting $P = P_c$ into (31), where $P_c(t) := \sum_{j=1}^d (t_j - c_j)^2$, gives $\int_\Omega P_c d\mu = 0$. Since $P_c(t) > 0$ for all $t \in \mathbb{R}^d \setminus \{c\}$, we see that $\mu(\Omega \setminus \{c\}) = 0$. As a consequence, $x = \chi_{\{c\}}$ a.e. $[\mu]$ and $\mu(\{c\}) = \|x\|^2 > 0$.

(iii)$\Rightarrow$(ii) Clearly, such x is a Λ-multiplicative idempotent in $\mathcal{H}$. In turn, $P \cdot x = P(c) \cdot x \in \mathcal{H}$ for all $P \in \mathcal{P}$ (because by assumption $x \in \mathcal{H}$), so $M_x(\mathcal{P}) \subseteq \mathcal{H}$. Since the operator M_x is bounded, $M_x(\mathcal{H}) \subseteq \mathcal{H}$. This completes the proof. □

Using Lemmata 2.7 and 2.13 and Theorem 2.19, one can obtain the following variant of Theorem 2.19.

Theorem 2.20. *Suppose that Λ is a uspf having a representing measure μ and $\mathcal{H} = \mathcal{H}^2(\mu)$. If $x \in \mathcal{H} \setminus \{0\}$ and $\xi \in \mathbb{C} \setminus \{0\}$, then the following conditions are equivalent*:

- *(i)* $x \in R(\mathcal{H})$, $\langle P, x\rangle\langle Q, x\rangle = \xi\langle PQ, x\rangle$ *for all* $P, Q \in \mathcal{P}_d$ *and* $P \cdot x \in \mathcal{H}$ *for all* $P \in \mathcal{P}_d$,
- *(ii) there exists* $c \in \mathbb{R}^n$ *such that* $\mu(\{c\}) > 0$ *and* $x = \frac{\xi}{\mu(\{c\})}\chi_{\{c\}}$ *a.e.* $[\mu]$.

Moreover, if (ii) *holds, then* $\xi = \langle 1, x\rangle$ *and* $\langle 1, x\rangle^2 = \|x\|^2\mu(\{c\})$.

The following is a direct consequence of Lemma 2.7 and Theorem 2.19.

Corollary 2.21. *Let Λ be a uspf and $\mathcal{H}$ be the associated Hilbert space. If $\mathcal{H}$ is a bounded shift spacesand $x \in \mathcal{H}$, then the following conditions are equivalent*:

- *(i)* x *is a* Λ*-multiplicative idempotent of* $\mathcal{H}$,
- *(ii)* $x \in R(\mathcal{H}) \setminus \{0\}$ *and* $\langle P, x\rangle\langle Q, x\rangle = \|x\|^2\langle PQ, x\rangle$ *for all* $P, Q \in \mathcal{P}$,
- *(iii) there exists* $c \in \mathbb{R}^n$ *such that* $\mu(\{c\}) > 0$ *and* $x = \chi_{\{c\}}$ *a.e.* $[\mu]$, *where* μ *is a representing measure of* Λ *associated with* $\mathcal{H}$ *as in Proposition* 2.10.

The following theorem characterizes nonzero Λ-multiplicative elements x of $\mathcal{H}$ which are nonnegative a.e. with respect to a given representing measure μ of Λ.

Theorem 2.22. *Suppose that Λ is a uspf having a representing measure μ and $\mathcal{H} = \mathcal{H}^2(\mu)$. If $x \in \mathcal{H}$, then the following conditions are equivalent*:

(i) x *is a nonzero* Λ*-multiplicative element of* $\mathcal{H}$ *such that* $x \geqslant 0$ *a.e.* $[\mu]$,

(ii) *there exist* $c \in \mathbb{R}^d$ *and* $\alpha \in (0, \infty)$ *such that* $\mu(\{c\}) > 0$ *and* $x = \alpha\chi_{\{c\}}$ *a.e.* $[\mu]$.

Proof. The implication (ii)⇒(i) is obvious.

(i)⇒(ii) It follows from (i) that $\langle 1, x\rangle > 0$, because otherwise $\int_{\mathbb{R}^d} x d\mu = 0$, which together with $x \geqslant 0$ a.e. $[\mu]$ implies that $x = 0$ a.e. $[\mu]$, a contradiction. Set $\nu = \frac{x}{\langle 1,x\rangle}$. Then $\langle 1, \nu\rangle = 1$ and $\nu \geqslant 0$ a.e. $[\mu]$. Clearly $\langle P, \nu\rangle\langle Q, \nu\rangle = \langle PQ, \nu\rangle$ for all $P, Q \in \mathcal{P}_d$, so by Proposition 2.14 there exists $c \in \mathbb{R}^d$ such that

$$\langle P, \nu\rangle = P(c), \quad P \in \mathcal{P}. \tag{32}$$

Define the positive Borel measure ν on $\mathbb{R}^d$ by $d\nu = \nu d\mu$. Since $\langle 1, \nu\rangle = 1$, ν is a probability measure. Note that

$$\int_{\mathbb{R}^d} |P|^2 d\nu = \langle \bar{P}P, \nu\rangle < \infty, \quad P \in \mathcal{P}. \tag{33}$$

This implies that $P \in L^1(\nu)$ for every $P \in \mathcal{P}$. Hence, by (32), we have

$$\int_{\mathbb{R}^d} P d\nu = \langle P, \nu\rangle = P(c), \quad P \in \mathcal{P}. \tag{34}$$

Substituting the polynomial $P = P_c$ into (34), where P_c is as in the proof of Theorem 2.19, we deduce that $\nu(\mathbb{R}^d \setminus \{c\}) = 0$. This implies that $x = \alpha\chi_{\{c\}}$ a.e. $[\mu]$ for some $\alpha \in (0, \infty)$. Moreover, $\alpha^2\mu(\{c\}) = \|x\|^2 > 0$, which completes the proof. □

Summarizing Theorems 2.19 and 2.22, we obtain the following result.

Theorem 2.23. *Suppose that* Λ *is a uspf having a representing measure* μ *and* $\mathcal{H} = \mathcal{H}^2(\mu)$. *If* $x \in \mathcal{H}$, *then the following conditions are equivalent*:

(i) x *is a* Λ*-multiplicative idempotent in* $\mathcal{H}$ *such that* $P \cdot x \in \mathcal{H}$ *for all* $P \in \mathcal{P}$,

(ii) x *is a* Λ*-multiplicative idempotent in* $\mathcal{H}$ *such that* $x \cdot \mathcal{H} \subseteq \mathcal{H}$,

(iii) x *is a* Λ*-multiplicative idempotent in* $\mathcal{H}$ *such that* $x \geqslant 0$ *a.e.* $[\mu]$,

(iv) *there exists* $c \in \mathbb{R}^d$ *such that* $\mu(\{c\}) > 0$ *and* $x = \chi_{\{c\}}$ *a.e.* $[\mu]$.

Now we give an example showing that the implication (i)⇒(iii) of Theorem 2.19 is not true if we drop the assumption that $P.x \in \mathcal{H}$ for all $P \in \mathcal{P}$ even if $\mathcal{P}$ is dense in $L^2(\mu)$, and that the implication (i)⇒(ii) of Theorem 2.22 is no longer true without assuming that $x \geqslant 0$ a.e. $[\mu]$.

Example 2.24. First, we recall some facts from moment theory (all necessary details concerning N-extremal measure may be found in [71, 68]). It is well known that the closed supports of N-extremal measures of an indeterminate Hamburger moment sequence $\gamma = (\gamma_k)_{k=0}^{\infty}$ form a partition of $\mathbb{R}$ (in particular, the supports are disjoint) and that each of them is infinite with no accumulation points in $\mathbb{R}$. Hence, each N-extremal measure of γ is a discrete measure. What is more, $\mathcal{P}_1$ is dense in $L^2(\mu)$ for any N-extremal measure μ of γ.

Let $d = 1$ and let $\gamma = (\gamma_k)_{k=0}^{\infty}$ be an indeterminate Hamburger moment sequence (the first such example was given by Stieltjes in his famous article in 1895 [73] : $\gamma_k = q^{-\frac{1}{2}k^2}, k \geqslant 0, q \in (0,1)$). let μ_1 and μ_2 be two distinct N-extremal measures of γ. Then by the above discussion for every $j = 1,2$, the measure μ_j is discrete, polynomials are dense in $L^2(\mu_j)$ and

$$\mathrm{supp}(\mu_1) \cap \mathrm{supp}(\mu_2) = \emptyset. \tag{35}$$

For $j = 1,2$, take $c_j \in \mathrm{supp}(\mu_j)$. Then $\mu_j(\{c_j\}) > 0$ for $j = 1,2$, $c_1 \neq c_2$ and ($\mathcal{H}$ depends on the representing measure, so we must be careful)

$$\left\langle [P]_{\mu_j}, \left[\chi_{\{c_j\}}\right]_{\mu_j} \right\rangle_{\mu_j} = \mu_j(\{c_j\})P(c_j), \quad P \in \mathcal{P}_1, j = 1,2, \tag{36}$$

where $\langle .,. \rangle_{\mu}$ stands for the inner product of $L^2(\mu)$ and $[f]_{\mu}$ stands for the equivalence relation of f with respect to the equivalence relation "a.e. $[\mu]$". This yields

$$\begin{aligned} &\left\langle [P]_{\mu_j}, \left[\chi_{\{c_j\}}\right]_{\mu_j} \right\rangle_{\mu_j} \left\langle [Q]_{\mu_j}, \left[\chi_{\{c_j\}}\right]_{\mu_j} \right\rangle_{\mu_j} \\ &\qquad = \mu_j(\{c_j\}) \left\langle [PQ]_{\mu_j}, \left[\chi_{\{c_j\}}\right]_{\mu_j} \right\rangle_{\mu_j} \end{aligned} \tag{37}$$

for all $P, Q \in \mathcal{P}_1$ and $j = 1,2$. In particular, we have $\mu_j(\{c_j\}) = \left\langle [1]_{\mu_j}, \left[\chi_{\{c_j\}}\right]_{\mu_j} \right\rangle_{\mu_j}$ for $j = 1,2$, and thus $\left[\chi_{\{c_j\}}\right]_{\mu_j} \in RL^2(\mu_j) \setminus \{0\}$ is a Λ-multiplicative idempotent of $L^2(\mu_j)$ for $j = 1,2$. Note that there exists a unique unitary isomorphism $U : L^2(\mu_1) \longrightarrow L^2(\mu_2)$ verifying [1]

$$U([P]_{\mu_1}) = [P]_{\mu_2}, \quad P \in L^2(\mu_1). \tag{38}$$

Since elements of $R(\mathcal{H})$ can be approximated by real polynomials, one can easily deduce from (38) that $UR(L^2(\mu_1)) = R(L^2(\mu_2))$. It follows that

$$\begin{aligned} \left\langle [P]_{\mu_2}, U\left[\chi_{\{c_1\}}\right]_{\mu_1} \right\rangle_{\mu_2} &\overset{(38)}{=} \left\langle U[P]_{\mu_1}, U\left[\chi_{\{c_1\}}\right]_{\mu_1} \right\rangle_{\mu_2} \\ &= \left\langle [P]_{\mu_1}, \left[\chi_{\{c_1\}}\right]_{\mu_1} \right\rangle_{\mu_1} \overset{(36)}{=} \mu_1(\{c_1\})P(c_1), \end{aligned} \tag{39}$$

1 As shown in this example (see (40)), U is not given by $U([f]_{\mu_1}) = [f]_{\mu_2}$ for $f \in L^2(\mu_1)$.

for $P \in \mathcal{P}_1$. As consequence, $U\left[\chi_{\{c_1\}}\right]_{\mu_1}$ is a nonzero Λ-multiplicative idempotent in $L^2(\mu_2)$.

Now we show that there is no $u \in \mathbb{R}^d$ such that (see also the last paragraph of this example)

$$U\left[\chi_{\{c_1\}}\right]_{\mu_1} = [\chi_u]_{\mu_2}. \tag{40}$$

Indeed, otherwise we have

$$\begin{aligned} \mu_1(\{c_1\})P(c_1) &\overset{(39)}{=} \left\langle [P]_{\mu_2}, U\left[\chi_{\{c_1\}}\right]_{\mu_1} \right\rangle_{\mu_2} \\ &\overset{(40)}{=} \left\langle [P]_{\mu_2}, \left[\chi_{\{u\}}\right]_{\mu_2} \right\rangle_{\mu_2} = \mu_2(\{u\})P(u) \end{aligned} \tag{41}$$

for all $P \in \mathcal{P}_1$. Then $u = c_1$ and $\mu_2(\{u\}) = \mu_1(\{c_1\})$. Indeed, if $u \neq c_1$, then there exists a polynomial $P \in \mathcal{P}_1$ such that $P(u) = 0$ and $P(c_1) = 1$, and so by (41) and $\mu_1(\{c_1\}) > 0$ we have a contradiction. Now, applying (41) with $P = 1$, we get $\mu_2(\{c_1\}) = \mu_1(\{c_1\}) > 0$, which implies that $c_1 \in \operatorname{supp}(\mu_1) \cap \operatorname{supp}(\mu_2)$, thereby contradicting (35).

Finally, we show that there exists $c \in K := \operatorname{supp}(\mu_2)$ such that the nonzero Λ-multiplicative idempotents $U\left[\chi_{\{c_1\}}\right]_{\mu_1}$ and $\left[\chi_{\{c\}}\right]_{\mu_2}$ are distinct however they are not orthogonal in $L^2(\mu_2)$.

Since $U\left[\chi_{\{c_1\}}\right]_{\mu_1} \in RL^2(\mu_2) \setminus \{0\}$ and the set K is infinite with no accumulation points in $\mathbb{R}$, there is a sequence $\{\beta_c\}_{c \in K}$ of real numbers such that

$$U\left[\chi_{\{c_1\}}\right]_{\mu_1} = \sum_{c \in K} \beta_c \cdot \left[\chi_{\{c\}}\right]_{\mu_2}. \tag{42}$$

This implies that for $P \in \mathcal{P}_1$

$$\begin{aligned} \mu_1(\{c_1\})P(c_1) &\overset{(39)}{=} \left\langle [P]_{\mu_2}, U\left[\chi_{\{c_1\}}\right]_{\mu_1} \right\rangle_{\mu_2} \\ &\overset{(42)}{=} \sum_{c \in K} \beta_c \left\langle [P]_{\mu_2}, \left[\chi_{\{c\}}\right]_{\mu_2} \right\rangle_{\mu_2} = \sum_{c \in K} \beta_c \mu_2(\{c\})P(c). \end{aligned} \tag{43}$$

Note that the set $K_0 = \{c \in K : \beta_c \neq 0\}$ is infinite. indeed, otherwise using (35) we can find a polynomial $P \in \mathcal{P}_1$ such that $P(c_1) = 1$ and $P(c) = 0$ for all $c \in K_0$, which contradicts (43) and $\mu_1(\{c_1\}) > 0$. Take any $d_2 \in K_0$, then the nonzero Λ-multiplicative idempotents $U\left[\chi_{\{c_1\}}\right]_{\mu_1}$ and $\left[\chi_{\{d_2\}}\right]_{\mu_2}$ are distinct and

$$\left\langle U\left[\chi_{\{c_1\}}\right]_{\mu_1}, \left[\chi_{\{d_2\}}\right]_{\mu_2} \right\rangle_{\mu_2} \overset{(42)}{=} \beta_{d_2}\mu_2(\{d_2\}) \neq 0 \tag{44}$$

this means that $U\left[\chi_{\{c_1\}}\right]_{\mu_1}$ and $\left[\chi_{\{d_2\}}\right]_{\mu_2}$ are not orthogonal in $L^2(\mu_2)$.

As a consequence of the reasoning in the last paragraph of this example (see (43)), we obtain the following corollary

Corollary 2.25. *There exists an infinite sequence $(r_k)_{k\in\mathbb{N}}$ of distinct positive numbers with no accumulation points in $\mathbb{R}$ and a sequence $(\alpha_k)_{k\in\mathbb{N}}$ of nonzero real numbers such that*

$$\sum_{k=0}^{\infty} \alpha_k P(r_k) = 0, \quad P \in \mathcal{P}_1 \tag{45}$$

3 ON WEAK POSITIVE MATRICES AND HYPONORMAL WEIGHTED SHIFTS

The fact that, whenever ciphers occur in the interior of a derived block, it is necessary to recommence the operation, may be thought a great obstacle to the use of this method; but I believe it will be found in practice that,... the whole amount of labour will still be much less than that involved in the old process of computation.
—C. L. Dodgson (Lewis Carroll)[34]—

Let $\mathcal{H}$ be a complex Hilbert space and let $\mathcal{L}(\mathcal{H})$ be the algebra of bounded operators on $\mathcal{H}$. We denote by $[T, S] := TS - ST$ the commutator of S and T in $\mathcal{L}(\mathcal{H})$. An operator $T \in \mathcal{L}(\mathcal{H})$ is said to be normal if $[T^*, T] = 0$, to be hyponormal if $[T^*, T] \geqslant 0$ and to be subnormal if $T = N_{|\mathcal{H}}$, where N is a normal operator on some Hilbert space $\mathcal{K} \supseteq \mathcal{H}$.

The concepts of subnormal and hyponormal operators were introduced by Paul R. Halmos in [43]. The first notion, hyponormal, reflects the geometric nature of normality with the corresponding implications in terms of positive matrices; while subnormal is intimately related to the notion of analyticity for complex functions, through the restriction of the functional calculus to invariant subspaces.

In order to establish a bridge between operator theory and matrix theory, we recall the Bram-Halmos criterion for subnormality [11, 14], which says that an operator T is subnormal if and only if

$$\sum_{i,j \leqslant k} \langle T^i x_j, T^j x_i \rangle \geqslant 0 \qquad k \geqslant 0, \tag{46}$$

for any $x_0, x_1, \ldots, x_k \in \mathcal{H}$. An application of the Choleski algorithm for operator matrices shows that (46) is equivalent to the positivity test

$$M_k(T) := \begin{pmatrix} [T^*, T] & [T^{*2}, T] & \ldots & [T^{*k}, T] \\ [T^*, T^2] & [T^{*2}, T^2] & \ldots & [T^{*k}, T^2] \\ \vdots & \vdots & \ddots & \vdots \\ [T^*, T^k] & [T^{*2}, T^k] & \ldots & [T^{*k}, T^k] \end{pmatrix} \geqslant 0 \qquad k \geqslant 0. \tag{47}$$

To illustrate and to study the gap between subnormal and hyponormal operators Ameer Athavale [2] introduced the classes of k-hyponormal operators as follows, an operator $T \in \mathcal{L}(\mathcal{H})$ is k-hyponormal if $M_k(T) \geqslant 0$. Clearly T is subnormal $\Leftrightarrow$ T is k-hyponormal for all $k \in \mathbb{N}$, with

$$(k+1)\text{-hyponormal} \Rightarrow k\text{-hyponormal} \Rightarrow \text{hyponormal}.$$

Weighted shifts provide several examples and counter examples in operator theory and hence are an important motivation in the analysis of operators. Given a bounded sequence of positive numbers $\alpha \equiv \{\alpha_n\}_{n\geqslant 0}$ (called weights), the unilateral weighted shift W_α associated with α is the bounded operator on $\ell^2(\mathbb{N})$ defined by $W_\alpha e_n := \alpha_n e_{n+1}$ for every $n \geqslant 0$, where $\{e_n\}_{n=0}^{\infty}$ is the canonical orthonormal basis for ℓ^2; the moments of W_α are defined by $\gamma_0 := 1$, $\gamma_{n+1} := \alpha_n^2\gamma_n (n \geqslant 0)$. It is straightforward to check that W_α can never be normal, and that W_α is hyponormal if and only if $\alpha_n \leqslant \alpha_{n+1}$ for all $n \geqslant 0$.

The Stieltjes moment problem associated with a given sequence $\{\gamma_n\}_{n\geqslant 0}$ entails finding a positive Borel measure μ supported in $\mathbb{R}_+$ such that

$$\gamma_n = \int_{\mathbb{R}_+} t^n d\mu \quad \text{for every } n \geqslant 0. \tag{48}$$

When the moment problem owns a solution μ, then μ is said to be a representing measure of the moment sequence γ_n. The well known Berger theorem[44] says that a weighted shift W_α is subnormal precisely when the sequence of its moments is a moment sequence of a positive measure supported in $[0, \|W_\alpha\|]$.

A description of subnormality for an abstract one to one operator T in terms of weighted shifts is given by Alan Lambert in [49]. Namely, a one to one operator T is subnormal if and only if, for each $h \neq 0$ in $\mathcal{H}$, the weighted shift associated with the weight sequence $\{\|T^{n+1}h\|/\|T^n h\|\}$ is subnormal.

Joseph G. Stampfli in [72] (see also [5]) showed that for subnormal weighted shifts W_α, a propagation phenomenon occurs which forces the flatness of W_α whenever two equal weights are present. That is, if $\alpha_k = \alpha_{k+1}$ for some $k \geqslant 0$, then $\alpha_n = \alpha_{n+1}$ for every $n \geqslant 1$. Later, Raul E. Curto proved, in [20], that the above result remains valid for 2-hyponormal weighted shifts. Our main goal in this note is to generalize the propagation phenomena in order to study the gap between different classes of k-hyponormal weighted shifts. To this aim we introduce the notion of k-positive matrices (or sequences), and we investigate this concept to exhibit some useful results.

This paper is organized as follows. We define in Section 2 k-positive matrices, and we give some of their basic properties. Section 3 is devoted to the statement of the main results. In Section 4 we give an elementary proof for a result due to Curto-Fialkow . That is, a k-positive matrix has a $k \times k$-sub-matrix with zero determinant if and only if all $k \times k$-sub-matrices have zero determinant. We devote Section 5 to a characterization of finite combinations of Dirac measures on $\mathbb{R}_+$ in terms of moment. In the last section we study the invariance of k-hyponormal weighted shifts under one rank perturbation, and a simple algorithm to calculate the stable perturbation intervals.

3.1 DEFINITIONS AND MAIN RESULTS

Given a sequence of non negative numbers $\gamma = \{\gamma_n\}_{n\geqslant 0}$, the associated Hankel matrix is built as follows

$$M_\gamma := (\gamma_{i+j})_{i,j} = \begin{pmatrix} \gamma_0 & \gamma_1 & \gamma_2 & \gamma_3 \\ \gamma_1 & \gamma_2 & \gamma_3 & \cdots \\ \gamma_2 & \gamma_3 & \cdots & \cdots \\ \gamma_3 & \vdots & \ddots & \ddots \end{pmatrix}.$$

For $k, n \in \mathbb{N}$, we denote by $[M_\gamma]_k^n$ the Hankel $(k+1)\times(k+1)$-sub-matrix

$$[M_\gamma]_k^n = \begin{pmatrix} \gamma_n & \gamma_{n+1} & \cdots & \gamma_{n+k} \\ \gamma_{n+1} & \gamma_{n+2} & \cdots & \gamma_{n+k+1} \\ \vdots & \ddots & \ddots & \vdots \\ \gamma_{n+k} & \gamma_{n+k+1} & \cdots & \gamma_{n+2k} \end{pmatrix}. \tag{49}$$

Definition 3.1. A Hankel matrix M_γ, or a sequence γ, is said to be k-positive if for every $n \in \mathbb{N}$, the $(k+1)\times(k+1)$-sub-matrix $[M_\gamma]_k^n$ is positive semi-definite.

Clearly M_γ is k-positive means that $\langle M_\gamma x, x\rangle \geqslant 0$ for every $x = \sum_{i=0}^{k} x_i e_{n+i}$ and $n \in \mathbb{N}$. Also, it is easy to see that the set of k-positive matrices, denoted by $\mathcal{C}_+^k$, is a convex cone, and that $\mathcal{C}_+^{k+1} \subset \mathcal{C}_+^k$, for all $k \in \mathbb{N}$.

Further immediate properties and examples are given in the next remark:

Remark 3.2.

1. $\mathcal{C}_+^0$ is the set of all matrices with non negative entries.
2. $\gamma = \{\gamma_n\}_{n\geqslant 0}$ is 1-positive if and only if γ is non-negative and log-convex ($\gamma_{i+1}^2 \leqslant \gamma_i\gamma_{i+2}$).
3. Let μ be a positive finite Borel measure such that $\mathrm{Supp}(\mu) \subset \mathbb{R}_+$ and $\mathbb{R}[X] \subset L^1(\mathbb{R}_+, \mu)$. By Stieltjes's Theorem [1, page 76] for every $k \in \mathbb{N}$, the Hankel matrices $(\gamma_{i+j})_{0\leqslant i,j\leqslant k}$ and $(\gamma_{i+j+1})_{0\leqslant i,j\leqslant k}$ are positives semi-definite. In other words γ is k-positive for all $k \in \mathbb{N}$.

From the log-convexity of k-positive sequences, we deduce the next useful remark

Proposition 3.3. *Let $M \in \mathcal{C}_+^k$. If $\gamma_{n_0} = 0$, for some $n_0 \in \mathbb{N}$, then $\gamma_n = 0$ for every $n \geqslant 1$.*

Proposition 3.3 states that a propagation phenomena occurs, in the sense that if a term of our sequence is zero, then almost all the sequence is forced to be zero. The general case of higher order propagation was established by R. Curto and L. Fialkow in [21, Proposition 5.13]. Our first contribution in this section is to provide an elementary proof of this fact, based on a version of block matrices determinants.

Theorem 3.4 (Propagation phenomena for k-positive matrices). *Let $M \in \mathcal{C}_+^k$ be such that there exists an integer $n_0 \geqslant 0$ satisfying $\det([M_\gamma]_{k-1}^{n_0}) = 0$. Then $\det([M_\gamma]_{k-1}^{n}) = 0$, for all $n \geqslant 1$.*

We start by the introduction of some notations before expanding the proof of Theorem 3.4.

For an $(n+1)\times(n+1)$-matrix $M = (a_{i,j})_{0\leqslant i,j\leqslant n}$, and for $i_0, j_0 \leqslant n$, we denote by $M\binom{i_0}{j_0}$ the $n\times n-$matrix resulting by removing the i_0+1 row and j_0+1 column. We also use the notation $M\begin{pmatrix} 0 & n \\ 0 & n\end{pmatrix}$ for the $(n-1)\times(n-1)$-matrix resulting by removing the first and the last row and column. In the case where $j_0 = 0$ (resp $j_0 = n$), we simply denote $M\binom{i_0}{0}$ by $M(\widetilde{i_0})$ (resp $M\binom{i_0}{n}$ by $M(\widehat{i_0})$).

We have the next expansion formula,

Lemma 3.5 (Desnanot–Jacobi identity or Lewis Carroll identity). *If $M = (a_{i,j})_{0\leqslant i,j\leqslant n}$ is an $(n+1)\times(n+1)$ matrix, then*

$$\det(M)\det\left(M\begin{pmatrix} 0 & n \\ 0 & n\end{pmatrix}\right) = \det(M(\widetilde{0}))\det(M(\widehat{n})) - \det(M(\widehat{0}))\det(M(\widetilde{n})). \tag{50}$$

Proof. For a proof of the previous lemma, we refer to [12] Page 111. □

Proof of Theorem 3.4. We will use a direct induction on i to prove that $\det([M_\gamma]_{k-1}^{i}]) = 0$, for all $i \geqslant n_0$. In fact, for $i = n_0$, we have $\det([M_\gamma]_{k-1}^{n_0}]) = 0$ by hypothesis. we assume that $\det([M_\gamma]_{k-1}^{i}]) = 0$ for $i \geqslant n_0$, and we will prove that $\det([M_\gamma]_{k-1}^{i+1}]) = 0$.

For $M = [M_\gamma]_k^i$, we get $M\begin{pmatrix} 0 & k \\ 0 & k\end{pmatrix} = [M_\gamma]_{k-2}^{i+2}$, $M(\widetilde{0}) = [M_\gamma]_{k-1}^{i+2}$, $M(\widehat{k}) = [M_\gamma]_{k-1}^{i}$, and $M(\widehat{0}) = M(\widetilde{k}) = [M_\gamma]_{k-1}^{i+1}$. Applying the identity (50), we get

$$\begin{aligned}\det([M_\gamma]_k^i)\det\left([M_\gamma]_{k-2}^{i+2}\right) &= \det([M_\gamma]_{k-1}^{i})\det([M_\gamma]_{k-1}^{i+2}) \\ &\quad - \det([M_\gamma]_{k-1}^{i+1})^2.\end{aligned} \tag{51}$$

But $M_\gamma \in \mathcal{C}_+^k$, so we get $\det\left([M_\gamma]_{k-2}^{i+2}\right)\det([M_\gamma]_k^i) \geqslant 0$ and then:

$$0 = \det([M_\gamma]_{k-1}^{i})\det([M_\gamma]_{k-1}^{i+2}) \geqslant \det([M_\gamma]_{k-1}^{i+1})^2.$$

Which implies that $\det([M_\gamma]_{k-1}^{i+1}) = 0$. Finally, $\det([M_\gamma]_{k-1}^{n}) = 0$ for every $n \geqslant n_0$.

It remains to prove that for every $1 \leqslant i \leqslant n_0$, $\det([M_\gamma]_{k-1}^{i}) = 0$. For that we will use the same steps, but for $M = [M_\gamma]_k^{i-1}$ (since

$i \geqslant 1$). In this case we get $M(\begin{smallmatrix} 0 & k \\ 0 & k \end{smallmatrix}) = [M_\gamma]^{i+1}_{k-2}$, $M(\widetilde{0}) = [M_\gamma]^{i+1}_{k-1}$, $M(\widehat{k}) = [M_\gamma]^{i-1}_{k-1}$, and $M(\widehat{0}) = M(\widetilde{k}) = [M_\gamma]^{i}_{k-1}$. Applying again the identity (50), we get

$$\begin{aligned} \det([M_\gamma]^{i-1}_{k}) \det\left([M_\gamma]^{i+1}_{k-2}\right) = & \det([M_\gamma]^{i-1}_{k-1}) \det([M_\gamma]^{i+1}_{k-1}) \\ & - \det([M_\gamma]^{i}_{k-1})^2. \end{aligned} \tag{52}$$

But $M_\gamma \in \mathcal{C}^k_+$, so we get $\det\left([M_\gamma]^{i+1}_{k-2}\right) \det([M_\gamma]^{i-1}_{k}) \geqslant 0$ and then:

$$\det([M_\gamma]^{i-1}_{k-1}) \det([M_\gamma]^{i+1}_{k-1}) \geqslant \det([M_\gamma]^{i}_{k-1})^2. \tag{53}$$

We will use now a decreasing induction on i, in fact if we assume that $\det([M_\gamma]^{i+1}_{k-1}) = 0$, then (53), implies that $\det([M_\gamma]^{i}_{k-1}) = 0$. for every $i \geqslant 1$, but since $\det([M_\gamma]^{n_0}_{k-1}) = 0$, we get that $\det([M_\gamma]^{i}_{k-1}) = 0$, for every $1 \leqslant i \leqslant n_0$. This complete the proof. □

Let W_α be a weighted shift, and let $\gamma = \{\gamma_n\}_{n \geqslant 0}$ be its moment sequence. We will say that W_α is *recursively generated* if there exist $r \in \mathbb{N}^*$, $a_0, \cdots, a_{r-1} \in \mathbb{R}$ such that for every $k \geqslant 0$,

$$\gamma_{k+r} = a_{r-1}\gamma_{k+r-1} + \cdots + a_1\gamma_{k+1} + a_0\gamma_k.$$

We recall the Stampfli propagation Theorem for subnormal weighted shift [72, Theorem 6] (see also [5, Proposition 4.5]):

Theorem 3.6. *Let W_α be a injective hyponormal weighted shift, if we assume that W_α is subnormal and that $\alpha_{i_0} = \alpha_{i_0+1}$. Then,*

$$\alpha_i = \alpha_{i+1}, \ \textit{for all} \ \ i \geqslant 1, \textit{ and } \alpha_0, \textit{ is arbitrary}.$$

In Term of moment sequence associated to W_α, we can write the result as :

Theorem 3.7. *Let W_α be a subnormal weighted shift, and let γ be the associated moment, if we assume that for some $i_0 \geqslant 0$, we have $\begin{vmatrix} \gamma_{i_0} & \gamma_{i_0+1} \\ \gamma_{i_0+1} & \gamma_{i_0+2} \end{vmatrix} = 0$. Then,*

$$\begin{vmatrix} \gamma_i & \gamma_{i+1} \\ \gamma_{i+1} & \gamma_{i+2} \end{vmatrix} = 0, \ \textit{for all} \ \ i \geqslant 1.$$

The first extension of the propagation notion was given by Curto in [17], in fact he prove that:

Theorem 3.8 ([17, Corollary 6])**.** *Let W_α be a 2-hyponormal weighted shift, if we assume that $\alpha_{i_0} = \alpha_{i_0+1}$. Then,*

$$\alpha_i = \alpha_{i+1}, \ \textit{for all} \ \ i \geqslant 1, \textit{ and } \alpha_0, \textit{ is arbitrary}.$$

In the light of this results, we will extend the notion of propagation phenomena for k-hyponormal weighted shifts. Recall that if W_α is a weighted shift with bounded weight sequence $\alpha = \{\alpha_n\}_{n\geqslant 0}$, the moments of W_α are usually defined by $\gamma_0 := 1$ and $\gamma_{n+1} := \alpha_n^2\gamma_n$ $(n \geqslant 0)$. It is known that (see [17, Theorem 4]) W_α is k-hyponormal if and only if $[M_\gamma]_k^n \geqslant 0$ for all $n \geqslant 0$.

We retrieve the next theorem:

Theorem 3.9 (Curto-Fialkow 1994, [21, Proposition 5.13])**.** *Let* $\alpha = \{\alpha_n\}_{n\geqslant 0}$ *be a sequence of positive numbers such that the weighted shift* W_α *is* k*-hyponormal. We assume that* $\det([M_\gamma]_p^{n_0}) = 0$ *for some* $n_0 \geqslant 0, p < k$, *then,*

$$\det([M_\gamma]_p^n) = 0 \quad \textit{for all } n \geqslant 0.$$

In particular W_α *is subnormal.*

Proof. Since W_α is k-hyponormal if and only if M_γ is k-positive, Then $p+1$-positive. Since also $\det([M_\gamma]_p^{n_0}) = 0$, Theorem 3.4 implies that $\det([M_\gamma]_p^n) = 0$ for all $n \geqslant 1$. we deduce that $\det([M_\gamma]_l^n) = 0$ for all $l \geqslant k$ and $n \geqslant 0$. It follows that $[M_\gamma]_l^n$ is positive semi-definite for all $l \geqslant k$ and $n \geqslant 0$, and in particular that W_α is subnormal. □

Using Theorem 3.9, we obtain the following recursiveness criterion for subnormal weighted shifts:

Theorem 3.10 (recursively generated subnormal weighted shift)**.** *Let* W_α *be a subnormal weighted shift and let* $\gamma = \{\gamma_n\}_{n\geqslant 0}$ *be its moment sequence. Then the following conditions are equivalent:*

(i) W_α *is recursively generated,*

(ii) there exist $n_0, k \in \mathbb{N}$ *such that* $\det([M_\gamma]_k^{n_0}) = 0$.

Proof.

(i) ⇒ (ii) We assume that W_α is recursively generated, so there exist $k \geqslant 0$ and $a_0, a_1, \ldots, a_k \in \mathbb{R}$ such that

$$\gamma_{p+k+1} = \sum_{j=0}^{k} a_j\gamma_{p+j} \qquad \forall p \in \mathbb{N}.$$

Hence, $\det([M_\gamma]_{k+1}^p) = 0$.

(ii) ⇒ (i) Since W_α being subnormal implies that W_α is $k+1$-hyponormal. Now the condition $\det([M_\gamma]_k^{n_0}) = 0$ together with Theorem 3.9 yield

$$\det([M_\gamma]_{k+1}^n) = 0,$$

for every $n \geqslant 0$. It follows that there exist $a_0, a_1, \ldots, a_k \in \mathbb{R}$ such that

$$\gamma_{p+k+1} = \sum_{j=0}^{k} a_j\gamma_{p+j} \qquad \forall p \in \mathbb{N}.$$

Finally, W_α is recursively generated.

□

For $k = 2$, the next extension of Stampfli's propagation result [5, Proposition 4.5], can be found in [17].

Corollary 3.11. *[17, Corollary 6] Let $\alpha = \{\alpha_n\}_{n\geqslant 0}$ be a bounded sequence of positive numbers associated with a 2-hyponormal weighted shift. If $\alpha_{n_0} = \alpha_{n_0+1}$ for some $n_0 \geqslant 0$, then $\alpha_n = \alpha_{n_0}$ for all $n \geqslant 1$. In particular, W_α is subnormal.*

As an application we give a moment characterization of finite mass measure on $\mathbb{R}^+$.

We recall that a measure μ is said to be of finite mass point if μ is a finite combination of Dirac measures.
The question of characterize finite mass measure in terms of moments, has been intensively studied. In fact, for general discrete measure (not necessary positive), we can cite the result of Charles Chidume, Mustapha Rachidi and El Hassan Zerouali in [13]. Or in the case of positive measures on $\mathbb{R}$, we can cite The recent treatment of this question by Cristian Berg and Ryszard Szwarc [7]. In this section we give a simple characterization of finite mass measure on $\mathbb{R}^+$.

Theorem 3.12 (Dirac measure characterization)**.** *Let μ be a positive measure with support in $\mathbb{R}^+$. Then the following conditions are equivalent:*

(i) μ *is a finite Dirac measure combination,*

(ii) *there exist* $p, k \in \mathbb{N}$ *such that* $\det(s_{i+j+p})_{i,j\leqslant k} = 0$.

Proof. Clearly, if $\mu = \sum_{l=0}^{n} a_l \delta_{x_l}$, from [7, Lemma 2.1], there exist $p, k \in \mathbb{N}$ such that $\det(s_{i+j+p})_{i,j\leqslant k} = 0$. For the reverse implication, by remark (3.2) (3), the matrix $(s_{i+j})_{i,j}$ is k-positive for all $k \in \mathbb{N}$. The weighted shift T_α associated with the sequence $\alpha_n = \sqrt{\frac{s_{n+1}}{s_n}}$ is subnormal. It is also recursively generated by Theorem 3.10, and by appealing Theorems 3.5 and 3.6 in [20], μ has a finite mass. □

3.2 PERTURBATION OF K-POSITIVE MATRICES

Let $M \in \mathcal{C}^k_+(H)$ $(M = M_\gamma)$. For $l \in \mathbb{N}^*$ and $t \geqslant 0$ we denote by $M_{\gamma'}$ the perturbed Hankel matrix whose entries are given by

$$\gamma'_n = \begin{cases} \gamma_n, & \text{if } n \leqslant l; \\ t\gamma_n, & \text{if } n \geqslant l+1. \end{cases}$$

Where

$$M_{\gamma'} = \begin{pmatrix} \gamma_0 & \gamma_1 & \dots & \gamma_l & t\gamma_{l+1} & t\gamma_{l+2} & \dots \\ \gamma_1 & \dots & \gamma_l & t\gamma_{l+1} & t\gamma_{l+2} & \dots & \dots \\ \vdots & \dots & t\gamma_{l+1} & t\gamma_{l+2} & \dots & \dots & \dots \\ \gamma_l & t\gamma_{l+1} & t\gamma_{l+2} & \dots & \dots & \dots & \dots \\ t\gamma_{l+1} & t\gamma_{l+2} & \dots & \dots & \dots & \dots & \dots \\ t\gamma_{l+2} & \dots & \dots & \dots & \dots & \dots & \dots \\ \vdots & \dots & \dots & \dots & \dots & \dots & \dots \end{pmatrix}.$$

The main goal of this section is to determine when such perturbation of k-positive matrix remains k-positive.

We notice that for $n \geqslant l+1$, we get $[M_{\gamma'}]_k^n = t[M_\gamma]_k^n$ and we deduce that

$$M_{\gamma'} \text{ is k-positive} \iff [M_{\gamma'}]_k^n \text{ is positive semi-definite } \forall n \leqslant l. \quad (54)$$

For $n \leqslant l$, we write

$$[M_{\gamma'}]_k^n = t[M_\gamma]_k^n + (1-t)H_k^n(l), \quad (55)$$

with

$$H_k^n(l) = \begin{pmatrix} \gamma_n & \gamma_{n+1} & \cdots & \gamma_l & 0 & \cdots & 0 \\ \gamma_{n+1} & \gamma_{n+2} & \cdots & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots & & \cdots & 0 \\ \gamma_l & 0 & \ddots & \vdots & & \cdots & 0 \\ 0 & \vdots & \ddots & \vdots & & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots & & \cdots & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

and we denote $I_n^k = \left\{t \geqslant 0 : t[M_\gamma]_k^n + (1-t)H_k^n(l) \geqslant 0\right\}$.

We have the following property,

Proposition 3.13. *For every $n \leqslant l$, I_n^k is a non empty closed interval of $\mathbb{R}_+$.*

Proof. For $k \in \mathbb{N}$, let P_k^+ be the closed convex cone of $(k+1) \times (k+1)$ positive semi definite matrices, and denote by $D = \left\{t[M_\gamma]_k^n + (1-t)H_k^n(l)/ \quad t \geqslant 0\right\}$. It's obvious that D is a closed convex set of $\mathcal{M}_{k+1}$ and then $D \cap P_k^+$ is a closed convex set. Moreover, the mapping γ_n^k, given by

$$\begin{array}{rccl} \gamma_n^k : & \mathbb{R}_+ & \longrightarrow & D \\ & t & \longmapsto & t[M_\gamma]_k^n + (1-t)H_k^n(l), \end{array}$$

is a continuous affine function, and hence $I_k^n = (\gamma_n^k)^{-1}(D \cap P_k^+)$ is a closed interval of $\mathbb{R}_+$. Also, M is k-positive implies that $[M_\gamma]_k^n$ is positive for all $n \in \mathbb{N}$, and hence that $1 \in I_n^k$ for all $n \in \mathbb{N}$. □

From (54), a perturbation of k-positive matrix remains k-positive if and only if $t \in I^k := \cap_{n \leqslant l} I^k_n$, our second result is

Proposition 3.14. *For every* $k \geqslant 1$, I^k *is a compact interval.*

Proof. From proposition 3.13 we have I^k is a closed non empty interval. To show that I^k is bounded, we remark that if $[M_{\gamma'}]^n_k$ is positive semi-definite, we have $[M_{\gamma'}]^n_{k-1}$ is positive semi definite. We deduce that $I^k_n \subset I^{k-1}_n$, and by induction that $I^k_n \subset I^1_n$. Finally $I^k \subset I^1$.

Now $t \in I^1$ if and only if ${M_{\gamma'}}^1_{l-1}$ and ${M_{\gamma'}}^1_l$ have non-negative determinants, that is

$$\frac{\gamma_l^2}{\gamma_{l-1}\gamma_{l+1}} \leqslant t \leqslant \frac{\gamma_l \gamma_{l+2}}{\gamma_{l+1}^2}.$$

Thus for every $k \geqslant 1$, $I^k \subset I^1 = \left[\frac{\gamma_l^2}{\gamma_{l-1}\gamma_{l+1}}; \frac{\gamma_l\gamma_{l+2}}{\gamma_{l+1}^2}\right]$ is bounded. □

3.2.1 *Determination of* I^2

The problem of determining I^k for $k \geqslant 2$ seems to be complicated. From the previous proof, we see that $I^1 = \left[\frac{\gamma_l^2}{\gamma_{l-1}\gamma_{l+1}}; \frac{\gamma_l\gamma_{l+2}}{\gamma_{l+1}^2}\right]$. We devote this section to calculate I^2.

Since $t \in I^2$ if and only if $[M_{\gamma'}]^2_{l-3}$ and $[M_{\gamma'}]^2_{l-2}$ and $[M_{\gamma'}]^2_{l-1}$ and $[M_{\gamma'}]^2_l$ are positives semi-definite,we exhibit the corresponding condition in each case :

- $[M_{\gamma'}]^2_{l-3} \geqslant 0$. We recall that

$$[M_{\gamma'}]^2_{l-3} = \begin{pmatrix} \gamma_{l-3} & \gamma_{l-2} & \gamma_{l-1} \\ \gamma_{l-2} & \gamma_{l-1} & \gamma_l \\ \gamma_{l-1} & \gamma_l & t\gamma_{l+1} \end{pmatrix},$$

and then is positive semi-definite if and only if:

$$\begin{vmatrix} \gamma_{l-1} & \gamma_l \\ \gamma_l & t\gamma_{l+1} \end{vmatrix} \geqslant 0, \begin{vmatrix} \gamma_{l-3} & \gamma_{l-1} \\ \gamma_{l-1} & t\gamma_{l+1} \end{vmatrix} \geqslant 0$$

$$\text{and} \quad \begin{vmatrix} \gamma_{l-3} & \gamma_{l-2} & \gamma_{l-1} \\ \gamma_{l-2} & \gamma_{l-1} & \gamma_l \\ \gamma_{l-1} & \gamma_l & t\gamma_{l+1} \end{vmatrix} \geqslant 0.$$

Which is equivalent to

$$\begin{cases} t & \geqslant \max\left(\frac{\gamma_l^2}{\gamma_{l-1}\gamma_{l+1}}, \frac{\gamma_{l-1}^2}{\gamma_{l-3}\gamma_{l+1}}\right), \\ t & \geqslant \dfrac{\begin{vmatrix} \gamma_{l-3} & \gamma_{l-2} & \gamma_{l-1} \\ \gamma_{l-2} & \gamma_{l-1} & \gamma_l \\ \gamma_{l-1} & \gamma_l & 0 \end{vmatrix}}{\gamma_{l+1}\begin{vmatrix} \gamma_{l-3} & \gamma_{l-2} \\ \gamma_{l-2} & \gamma_{l-1} \end{vmatrix}}. \end{cases}$$

The last condition is redundant. Indeed, using Dodgson condensation method, we obtain :

$$-\gamma_{l-1}\begin{vmatrix} \gamma_{l-3} & \gamma_{l-2} & \gamma_{l-1} \\ \gamma_{l-2} & \gamma_{l-1} & \gamma_l \\ \gamma_{l-1} & \gamma_l & 0 \end{vmatrix} = \gamma_l^2 \begin{vmatrix} \gamma_{l-3} & \gamma_{l-2} \\ \gamma_{l-2} & \gamma_{l-1} \end{vmatrix} + \begin{vmatrix} \gamma_{l-2} & \gamma_{l-1} \\ \gamma_{l-1} & \gamma_l \end{vmatrix}^2 \geqslant 0.$$

By using this simple next observation:

$$\frac{\gamma_l^2}{\gamma_{l-1}\gamma_{l+1}} = \frac{\gamma_{l-1}^2}{\gamma_{l-3}\gamma_{l+1}}\frac{\gamma_{l-3}\gamma_{l-1}}{\gamma_{l-2}^2}\left(\frac{\gamma_{l-2}\gamma_l}{\gamma_{l-1}^2}\right)^2 \geqslant \frac{\gamma_{l-1}^2}{\gamma_{l-3}\gamma_{l+1}}.$$

It follows that

$$[M_{\gamma'}]_{l-3}^2 \text{ is positive semidefinite if and only if } t \geqslant \frac{\gamma_l^2}{\gamma_{l-1}\gamma_{l+1}}. \tag{56}$$

- $[M_{\gamma'}]_{l-2}^2 \geqslant 0$ Using the same argument for

$$[M_{\gamma'}]_{l-2}^2 = \begin{pmatrix} \gamma_{l-2} & \gamma_{l-1} & \gamma_l \\ \gamma_{l-1} & \gamma_l & t\gamma_{l+1} \\ \gamma_l & t\gamma_{l+1} & t\gamma_{l+2} \end{pmatrix}.$$

We obtain,

$$\begin{vmatrix} \gamma_{l-2} & \gamma_l \\ \gamma_l & t\gamma_{l+2} \end{vmatrix} \geqslant 0, \begin{vmatrix} \gamma_l & \gamma_{l+1} \\ t\gamma_{l+1} & \gamma_{l+2} \end{vmatrix} \geqslant 0$$
$$\text{and } \begin{vmatrix} \gamma_{l-2} & \gamma_{l-1} & \gamma_l \\ \gamma_{l-1} & \gamma_l & t\gamma_{l+1} \\ \gamma_l & t\gamma_{l+1} & t\gamma_{l+2} \end{vmatrix} \geqslant 0.$$

That means, $\frac{\gamma_l^2}{\gamma_{l-2}\gamma_{l+2}} \leqslant t \leqslant \frac{\gamma_l\gamma_{l+2}}{\gamma_{l+1}^2}$ and

$$\begin{aligned} P(t) :=& (\gamma_{l-2}\gamma_l\gamma_{l+2} + 2\gamma_{l-1}\gamma_l\gamma_{l+1} - \gamma_{l-1}^2\gamma_{l+2})t \\ & - \gamma_l^3 - \gamma_{l-2}\gamma_{l+1}^2 t^2 \geqslant 0. \end{aligned}$$

The first and the second inequalities are automatically satisfied, since

$$P\left(\frac{\gamma_l^2}{\gamma_{l-2}\gamma_{l+2}}\right) = -\frac{(\gamma_l^2\gamma_{l+1} - \gamma_{l-1}\gamma_l\gamma_{l+2})^2}{\gamma_{l+2}^2\gamma_{l-2}} \leqslant 0,$$

and

$$P\left(\frac{\gamma_l\gamma_{l+2}}{\gamma_{l+1}^2}\right) = -\frac{\gamma_l(\gamma_l\gamma_{l+1}-\gamma_{l-1}\gamma_{l+2})^2}{\gamma_{l+1}^2} \leqslant 0.$$

Using the inequality $\gamma_{l-1}^2 \leqslant \gamma_{l-2}\gamma_l$ we can drive that the second coefficient of P is positive, and hence the classical Descartes rule for positive roots of polynomials, give that P have two positive solutions of $P(t) = 0$.

Then,

$$[M_{\gamma'}]_{l-2}^2 \text{ is positive semidefinite if and only if } t \in [\alpha(P); \beta(P)], \tag{57}$$

where $\alpha(P)$ and $\beta(P)$ are the two positive solutions of $P(t) = 0$.

- $[M_{\gamma'}]_{l-1}^2$ This case is treated exactly as the last one and leads to

$$[M_{\gamma'}]_{l-1}^2 \text{ is positive semidefinite if and only if } t \in [\alpha(Q); \beta(Q)], \tag{58}$$

where $\alpha(Q)$ and $\beta(Q)$ are the two positive solutions of

$$\begin{aligned} Q(t) :=& (\gamma_{l-1}\gamma_{l+1}\gamma_{l+3} + 2\gamma_l\gamma_{l+1}\gamma_{l+2} - \gamma_{l-1}\gamma_{l+2}^2)t \\ & - \gamma_{l+1}^3 t^2 - \gamma_l^2\gamma_{l+3} = 0. \end{aligned}$$

- $[M_{\gamma'}]_l^2$ The computations in this case outlines the first one, indeed,

$$[M_{\gamma'}]_l^2 = \begin{pmatrix} \gamma_l & t\gamma_{l+1} & t\gamma_{l+2} \\ t\gamma_{l+1} & t\gamma_{l+2} & t\gamma_{l+3} \\ t\gamma_{l+2} & t\gamma_{l+3} & t\gamma_{l+4} \end{pmatrix},$$

is positive semi-definite if and only if

$$\begin{vmatrix} \gamma_l & \gamma_{l+1} \\ t\gamma_{l+1} & \gamma_{l+2} \end{vmatrix} \geqslant 0, \begin{vmatrix} \gamma_l & \gamma_{l+2} \\ t\gamma_{l+2} & \gamma_{l+4} \end{vmatrix} \geqslant 0$$

$$\text{and} \begin{vmatrix} \gamma_l & \gamma_{l+1} & \gamma_{l+2} \\ t\gamma_{l+1} & \gamma_{l+2} & \gamma_{l+3} \\ t\gamma_{l+2} & \gamma_{l+3} & \gamma_{l+4} \end{vmatrix} \geqslant 0.$$

Which is equivalent to

$$t \leqslant \min\left\{ \frac{\gamma_l\gamma_{l+2}}{\gamma_{l+1}^2}, \frac{\gamma_l\gamma_{l+4}}{\gamma_{l+2}^2}, \frac{-\gamma_l\begin{vmatrix} \gamma_{l+2} & \gamma_{l+3} \\ \gamma_{l+3} & \gamma_{l+4} \end{vmatrix}}{\begin{vmatrix} 0 & \gamma_{l+1} & \gamma_{l+2} \\ \gamma_{l+1} & \gamma_{l+2} & \gamma_{l+3} \\ \gamma_{l+2} & \gamma_{l+3} & \gamma_{l+4} \end{vmatrix}} \right\}.$$

But

$$\frac{\gamma_l\gamma_{l+2}}{\gamma_{l+1}^2} = \frac{\gamma_l\gamma_{l+4}}{\gamma_{l+2}^2}\frac{\gamma_{l+3}^2}{\gamma_{l+4}\gamma_{l+2}}\left(\frac{\gamma_{l+2}^2}{\gamma_{l+3}\gamma_{l+1}}\right)^2 \leqslant \frac{\gamma_l\gamma_{l+4}}{\gamma_{l+2}^2},$$

with

$$\frac{\gamma_l\gamma_{l+2}}{\gamma_{l+1}^2} + \frac{\gamma_l\begin{vmatrix}\gamma_{l+2} & \gamma_{l+3}\\ \gamma_{l+3} & \gamma_{l+4}\end{vmatrix}}{\begin{vmatrix}0 & \gamma_{l+1} & \gamma_{l+2}\\ \gamma_{l+1} & \gamma_{l+2} & \gamma_{l+3}\\ \gamma_{l+2} & \gamma_{l+3} & \gamma_{l+4}\end{vmatrix}}$$

$$= \frac{\gamma_l\gamma_{l+2}}{\gamma_{l+1}^2}\left[1 - \frac{\begin{vmatrix}\gamma_{l+2} & \gamma_{l+3}\\ \gamma_{l+3} & \gamma_{l+4}\end{vmatrix}}{\begin{vmatrix}\gamma_{l+2} & \gamma_{l+3}\\ \gamma_{l+3} & \gamma_{l+4}\end{vmatrix} + \frac{1}{\gamma_{l+1}^2}\begin{vmatrix}\gamma_{l+1} & \gamma_{l+2}\\ \gamma_{l+2} & \gamma_{l+3}\end{vmatrix}^2}\right] \geqslant 0.$$

This yields that $[M_{\gamma'}]_l^2$ is positive semidefinite if and only if

$$0 \leqslant t \leqslant \frac{-\gamma_l\begin{vmatrix}\gamma_{l+2} & \gamma_{l+3}\\ \gamma_{l+3} & \gamma_{l+4}\end{vmatrix}}{\begin{vmatrix}0 & \gamma_{l+1} & \gamma_{l+2}\\ \gamma_{l+1} & \gamma_{l+2} & \gamma_{l+3}\\ \gamma_{l+2} & \gamma_{l+3} & \gamma_{l+4}\end{vmatrix}}. \tag{59}$$

Finally from (56), (57), (58) and (59) we conclude that $I^2 = [a, b]$, where ;

$$a = \max\left\{\alpha(P), \alpha(Q), \frac{\gamma_l^2}{\gamma_{l+1}\gamma_{l-1}}\right\}$$

$$b = \min\left\{\beta(P), \beta(Q), \frac{-\gamma_l\begin{vmatrix}\gamma_{l+2} & \gamma_{l+3}\\ \gamma_{l+3} & \gamma_{l+4}\end{vmatrix}}{\begin{vmatrix}0 & \gamma_{l+1} & \gamma_{l+2}\\ \gamma_{l+1} & \gamma_{l+2} & \gamma_{l+3}\\ \gamma_{l+2} & \gamma_{l+3} & \gamma_{l+4}\end{vmatrix}}\right\}$$

Remarks 3.15.

• The existence of positive roots for P and Q can be derived also by using the classical Bolzano's theorem because the fact that $P(1) \geqslant 0$, $Q(1) \geqslant 0$, and $P(0) \leqslant 0$, $Q(0) \leqslant 0$, and that $\lim_{t\to\infty} P(t) = \lim_{t\to\infty} Q(t) = -\infty$.

• A direct calculation of discriminants we obtain the next inequality,

$$\min\{\gamma_{l-1}^2\gamma_{l+1}\gamma_{l+3}; \gamma_{l-2}\gamma_l\gamma_{l+2}^2\} \geqslant (2\gamma_l\gamma_{l+1} - \gamma_{l-1}\gamma_{l+2})^2.$$

3.2.2 *One rank perturbation of weighted shifts*

Let W_α be a weighted shift and $W_{\alpha(l,t)}$ be the weighted shift associated with a given $l \in \mathbb{N}$ and $t \geqslant 0$ defined by: $W_{\alpha(l,t)}(e_k) = W_\alpha(e_k) = \alpha_k e_{k+1}$ for $k \neq l$ and $W_{\alpha(l,t)}(e_l) = \sqrt{t}\alpha_l e_{l+1}$.

We denote by γ the moment sequence associated with α defined by $\gamma_0 = 1$ and $\gamma_n = \alpha_{n-1}^2 \gamma_{n-1}$ and by $\gamma'(t)$ (Or simply γ') the moment sequence associated with $\alpha(l,t)$. It is easily seen that $\gamma'_k = \gamma_k$ for $k \leqslant l$ and $\gamma'_k = t\gamma_k$ for $k > l$.

We put $J^k = \{t \geqslant 0; W_{\alpha(l,t)} \text{ is k-hyponrmal }\}$. Since W_α is k-hyponormal if and only if M_γ is k-positive. we can deduce that,

$$J^k = \{t \geqslant 0; M_{\gamma'} \text{ is k-positive}\} = I^k.$$

and then from Proposition 3.14, we get I^k is a compact interval.

The behavior of one rank perturbation of subnormal weighted shift is given by

Theorem 3.16 ([16, Theorem 2.1]). *Let W_α be a subnormal weighted shift. We have*

$$I^\infty := \cap_{k \geqslant 1} I^k = \{1\}.$$

We state our main result in this section as follows.

Theorem 3.17. *Let W_α be a k-hyponormal weighted shift. We have*

$$1 \in \widehat{\overset{\circ}{I^k}} \qquad \Longleftrightarrow \qquad [M_\gamma]_k^n \text{ is positive definite } \forall n \leqslant l.$$

We will use the same notations as in the proof of Proposition 3.13. We start by proving that our condition is sufficient. To this aim, consider

$$\begin{array}{rccl} \gamma_n^k : & \mathbb{R}_+ & \longrightarrow & D \\ & t & \longmapsto & t[M_\gamma]_k^n + (1-t)H_k^n(l) \end{array}.$$

Since $[M_\gamma]_k^n \in \widehat{\overset{\circ}{P_k^+}}$(the set of all positive definite matrices) and γ_n^k is continuous, for V an open neighborhood of $[M_\gamma]_k^n$ such that $V \subset \widehat{\overset{\circ}{P_k^+}}$, $(\gamma_n^k)^{-1}(V \cap D)$ is an open neighborhood of $(\gamma_n^k)^{-1}([M_\gamma]_k^n) = 1$. We deduce that there exists $]r_n, t_n[\subset \mathbb{R}^+$, such that $1 \in]r_n, t_n[$ and $\gamma_n^k(]r_n, t_n[) \subset P_k^+$, and then $]r_n, t_n[\subset I_n^k$. Finally

$$1 \in]\max_{n \leqslant l} r_n, \inf_{n \leqslant l} t_n[\subset \widehat{\overset{\circ}{I^k}}.$$

Conversely, we assume that there is $n \leqslant l$ such that $[M_\gamma]_k^n$ is not positive definite, or equivalently, there is $p \leqslant k$ such that $\det([M_\gamma]_p^n) = 0$. We distinguish two cases;

$p < k$ By Theorem 3.9 W_α is subnormal, then $I^k = I^\infty = \{1\}$ that means $\widehat{\overset{\circ}{I^k}} = \emptyset$.

$p = k$ Simple computations give

$$\begin{aligned} \det(\gamma_k^n(t)) &= t^{k+1}\det([M_\gamma]_k^n) + (1-t)t^k\gamma_n \mathrm{Cof}(\gamma_n) \\ &= \gamma_n \mathrm{Cof}(\gamma_n)(1-t)t^k. \end{aligned}$$

where $\mathrm{Cof}(\gamma_n)$ design the $(1,1)$ cofactor of $[M_\gamma]_k^n$. It follows from $t \in I^k$ that $\det(\gamma_k^n(t)) \geqslant 0$ and hence $t \leqslant 1$. Thus $1 \notin \widehat{\overset{\circ}{I^k}}$.

4 THE QUINTIC COMPLEX MOMENT PROBLEM

"Obvious" is the most dangerous word in mathematics.
—E. T. Bell—

Given a doubly indexed finite sequence of complex numbers

$$\gamma \equiv \gamma^{(m)} = \{\gamma_{ij}\}_{0 \leqslant i+j \leqslant m} = \{\gamma_{00}, \gamma_{01}, \gamma_{10}, \ldots, \gamma_{0m}, \ldots, \gamma_{m0}\}$$

with $\gamma_{00} > 0$ and $\overline{\gamma}_{ij} = \gamma_{ji}$ for $0 \leqslant i+j \leqslant m$. The truncated complex moment problem (in short, TCMP) associated with γ entails finding a positive Borel measure μ supported in the complex plane $\mathbb{C}$ such that

$$\gamma_{ij} = \int \bar{z}^i z^j d\mu \qquad (0 \leqslant i+j \leqslant m); \tag{60}$$

A sequence $\{\gamma_{ij}\}_{0 \leqslant i+j \leqslant m}$ satisfying (60) will be called a truncated moment sequence and the solution μ is said to be a representing measure associated to the sequence $\{\gamma_{ij}\}_{0 \leqslant i+j \leqslant m}$.

In [74] J. Stochel has shown that solving TCMP solves the widely studied Full Moment Problem (see, for example, [1, 68, 4, 9, 33, 55, 60, 70, 79]). More precisely, a full moment sequence $\{\gamma_{ij}\}_{i,j \in \mathbb{Z}_+}$ admits a representing measure if and only if each of its truncation $\gamma^{(m)}$ admits a representing measure.

The truncated complex moment problem serves as a prototype for several other moment problems to which it is closely related. Its application can be found in subnormal operator theory [65, 50, 78], polynomial hyponormality [29] and joint hyponormality [20, 21]. It is also related to the optimization theory [52, 51, 53, 54, 55] and arise in pure and applied mathematics and in the sciences in general.

For the even case $m = 2n$, Curto and Fialkow developed in a series of papers an approach for TCMP based on positivity and flat extensions of the moment matrix, see Section 2. This allowed them to find solutions for various particular cases of truncated moment problems (see, for instance, [22, 24, 23, 26, 28, 46, 45]). However, only the quadratic case ($m = 2$) and the quartic case ($m = 4$) are completely solved (cf.[31, 25, 39, 31]).

In the odd case $m = 2n+1$, a general solution to some partial cases of the TCMP can be found in the Phd thesis of David P. Kimsey [47], Who was the first to resolve the cubic complex moment problem (when $m = 3$) in [48], see also [18]. The solution is based on commutativity conditions of matrices determined by $\{\gamma_{ij}\}_{0 \leqslant i+j \leqslant 2n+1}$.

Therefore, only the cases $m = 1, 2, 3$ and 4 (the quadratic, the cubic and the quartic moment problem) have been completely achieved. All the other cases (quintic, sixtic, ...) are open and interest several authors; as indicated in many recent papers (see, for instance, [30, 32, 18, 83, 84]).

In this chapter, we provide a concrete solution to almost all cases of the quintic moment problem (i.e. $m = 5$) when one desires a minimal representing measure. To this aim, we investigate the structure of recursive complex-valued bi-indexed sequences and we combine the obtained observations with some results due to R.Curto and L. Fialkow, to provide a new technique for solving the odd-degree TCMP. We notice that our techniques furnish a short solution to the cubic moment problem [1] and expected to be useful for higher odd-degree truncated moment problems.

Let $\gamma^{(5)} = \{\gamma_{ij}\}_{0\leqslant i+j\leqslant 5}$ be a given complex valued bi-sequence. We associate with $\gamma^{(5)}$ the next two matrices (called respectively $M(2)$, and B) that will play a crucial role in our approach.

$$\begin{pmatrix} \gamma_{00} & \gamma_{01} & \gamma_{10} & \gamma_{02} & \gamma_{11} & \gamma_{20} \\ \gamma_{10} & \gamma_{11} & \gamma_{20} & \gamma_{12} & \gamma_{21} & \gamma_{30} \\ \gamma_{01} & \gamma_{02} & \gamma_{11} & \gamma_{03} & \gamma_{12} & \gamma_{21} \\ \gamma_{20} & \gamma_{21} & \gamma_{30} & \gamma_{22} & \gamma_{31} & \gamma_{40} \\ \gamma_{11} & \gamma_{12} & \gamma_{21} & \gamma_{13} & \gamma_{22} & \gamma_{31} \\ \gamma_{02} & \gamma_{03} & \gamma_{12} & \gamma_{04} & \gamma_{13} & \gamma_{22} \end{pmatrix}, \begin{pmatrix} \gamma_{03} & \gamma_{12} & \gamma_{21} & \gamma_{30} \\ \gamma_{13} & \gamma_{22} & \gamma_{31} & \gamma_{40} \\ \gamma_{04} & \gamma_{13} & \gamma_{22} & \gamma_{31} \\ \gamma_{23} & \gamma_{32} & \gamma_{41} & \gamma_{50} \\ \gamma_{14} & \gamma_{23} & \gamma_{32} & \gamma_{41} \\ \gamma_{05} & \gamma_{14} & \gamma_{23} & \gamma_{32} \end{pmatrix}. \quad (61)$$

Let us recall that thanks to Douglas factorization Lemma, we have $\mathrm{Rang}\ B \subseteq \mathrm{Rang}\ M(2)$ if, and only if, there exists a matrix W such that $B = M(2)W$. We will show, in Section 2, that the Hermitian matrix $W^*M(2)W$ is symmetric with respect to the second diagonal, then one can set

$$W^*M(2)W = \begin{pmatrix} a & b & c & d \\ \overline{b} & e & f & c \\ \overline{c} & \overline{f} & e & b \\ \overline{d} & \overline{c} & \overline{b} & a \end{pmatrix} \quad (62)$$

As we will see in the sequel, the entries a, b, e and f in the matrix $W^*M(2)W$ encodes the complete information on the cardinal of the support of the minimal representing measure.

Theorem 4.1. *Let $\gamma^{(5)} \equiv \{\gamma_{ij}\}_{i+j\leqslant 5}$ be a given finite sequence, such that*
$$M(2) \geqslant 0,\ \mathrm{Rang}\ \ B \subseteq \mathrm{Rang}\ \ M(2) \text{ and } a \neq e \text{ or } b = f.$$
Then the quintic moment problem, associated with $\gamma^{(5)}$, admits a solution μ. Moreover, The smallest cardinality of $\mathrm{supp}\ \ \mu$ *is*

- $\mathrm{card}\ \ \mathrm{supp}\ \ \mu = r \iff a = e$ *and* $b = f$,

1 we omit the proof because the cubic moment problem is already solved, see [18, 48]

- $\mathrm{card\ supp}\ \mu = r+1 \iff \quad a \neq e$ *and* $\frac{a-e}{2} <\mid b-f \mid$,
- $\mathrm{card\ supp}\ \mu = r+2 \iff \quad a > e$ *and* $\frac{a-e}{2} \geqslant\mid b-f \mid$;

where $r := \ \mathrm{card}\ M(2)$ *and the numbers* a, b, e *and* f *are given by* (62).

Since (as we will show in Section 2) $M(2) \geqslant 0$ and $\mathrm{Rang}\ \ B \subseteq \mathrm{Rang}\ \ M(2)$ are two necessary conditions for the quintic TCMP, associated with $\gamma^{(5)}$, then Theorem 4.1 provides a concrete solution to the quintic complex moment problem, except for the case $a \neq e$ or $b = f$. The difficulty that we encountered in solving the remaining case ($a \neq e$ or $b = f$) is technical, not a failure in the method, see Section 5.

This paper is organized as follows. In Section 2, we will give useful tools and results usually used in the treatment of the truncated complex moment problems. We will investigate in Section 3 the complex-valued recursive bi-sequences and we will exhibit important results for quintic TCMP in Section 4. Finally, in Section 5, we solve the quintic complex moment problem together with the minimal support problem.

4.1 PRELIMINARIES

First, we recall a fundamental necessary condition. To this end, let us assume that $\gamma^{(2n)} \equiv \{\gamma_{ij}\}_{i+j\leqslant 2n}$ is a given moment sequence and let μ be the associated representing measure, then, for every $p \equiv \sum\limits_{h,k} a_{hk}\bar{z}^h z^k \in \mathbb{C}[\bar{z}, z]$,

$$\begin{aligned} 0 \leqslant \int |p|^2 d\mu &= \sum_{h,k,h',k'} a_{hk}\overline{a_{h'k'}} \int \bar{z}^{h+k'} z^{k+h'} d\mu \\ &= \sum_{h,k,h',k'} a_{hk}\overline{a_{h'k'}}\gamma_{h+k',k+h'}, \end{aligned}$$

or, equivalently, the moment matrix $M(n) \equiv M(n)(\gamma^{(2n)})$, defined below, is semi-definite positive.

$$M(n) := \begin{pmatrix} M[0,0] & M[0,1] & \dots & M[0,n] \\ M[1,0] & M[1,1] & \dots & M[1,n] \\ \vdots & \vdots & \ddots & \vdots \\ M[n,0] & M[n,1] & \dots & M[n,n] \end{pmatrix}, \tag{63}$$

where $M[i, j]$ has the Toeplitz-like property of being constant on each diagonal.

$$M[i,j] = \begin{pmatrix} \gamma_{i,j} & \gamma_{i+1,j-1} & \cdots & \gamma_{i+j,0} \\ \gamma_{i-1,j+1} & \gamma_{i,j} & \cdots & \gamma_{i+j-1,1} \\ \vdots & \vdots & \ddots & \vdots \\ \gamma_{0,i+j} & \gamma_{1,i+j-1} & \cdots & \gamma_{j,i} \end{pmatrix}.$$

In particular $M[i,i]$ are Toeplitz matrix.

Considering the lexicographic order,

$$1, Z, \overline{Z}, Z^2, Z\overline{Z}, \overline{Z}^2, \dots, Z^n, Z^{n-1}\overline{Z}, \dots, Z\overline{Z}^{n-1}, \overline{Z}^n, \tag{64}$$

to denote rows and columns of the moment matrix $M(n)$. For example, The $M(3)$ matrix is

$$\begin{array}{c} \\ 1 \\ Z \\ \overline{Z} \\ Z^2 \\ Z\overline{Z} \\ \overline{Z}^2 \\ Z^3 \\ Z^2\overline{Z} \\ Z\overline{Z}^2 \\ \overline{Z}^3 \end{array}
\begin{array}{c} \begin{array}{cccccccccc} 1 & Z & \overline{Z} & Z^2 & Z\overline{Z} & \overline{Z}^2 & Z^3 & Z^2\overline{Z} & Z\overline{Z}^2 & \overline{Z}^3 \end{array} \\
\begin{pmatrix}
\gamma_{00} & \gamma_{01} & \gamma_{10} & \gamma_{02} & \gamma_{11} & \gamma_{20} & \gamma_{03} & \gamma_{12} & \gamma_{21} & \gamma_{30} \\
\gamma_{10} & \gamma_{11} & \gamma_{20} & \gamma_{12} & \gamma_{21} & \gamma_{30} & \gamma_{13} & \gamma_{22} & \gamma_{31} & \gamma_{40} \\
\gamma_{01} & \gamma_{02} & \gamma_{11} & \gamma_{03} & \gamma_{12} & \gamma_{21} & \gamma_{04} & \gamma_{13} & \gamma_{22} & \gamma_{31} \\
\gamma_{20} & \gamma_{21} & \gamma_{30} & \gamma_{22} & \gamma_{31} & \gamma_{40} & \gamma_{23} & \gamma_{32} & \gamma_{41} & \gamma_{50} \\
\gamma_{11} & \gamma_{12} & \gamma_{21} & \gamma_{13} & \gamma_{22} & \gamma_{31} & \gamma_{14} & \gamma_{23} & \gamma_{32} & \gamma_{41} \\
\gamma_{02} & \gamma_{03} & \gamma_{12} & \gamma_{04} & \gamma_{13} & \gamma_{22} & \gamma_{05} & \gamma_{14} & \gamma_{23} & \gamma_{32} \\
\gamma_{30} & \gamma_{31} & \gamma_{40} & \gamma_{32} & \gamma_{41} & \gamma_{50} & \gamma_{33} & \gamma_{42} & \gamma_{51} & \gamma_{60} \\
\gamma_{21} & \gamma_{22} & \gamma_{31} & \gamma_{23} & \gamma_{32} & \gamma_{41} & \gamma_{24} & \gamma_{33} & \gamma_{42} & \gamma_{51} \\
\gamma_{12} & \gamma_{13} & \gamma_{22} & \gamma_{14} & \gamma_{23} & \gamma_{32} & \gamma_{15} & \gamma_{24} & \gamma_{33} & \gamma_{42} \\
\gamma_{03} & \gamma_{04} & \gamma_{13} & \gamma_{05} & \gamma_{14} & \gamma_{23} & \gamma_{06} & \gamma_{15} & \gamma_{24} & \gamma_{33}
\end{pmatrix} \end{array}. \tag{65}$$

Observe in passing that the matrix $M(n)$ detects the positivity of the Riesz functional given by

$$\Lambda_{\gamma^{(2n)}} : p(\overline{z}, z) \equiv \sum_{0 \leqslant i+j \leqslant 2n} a_{ij}\overline{z}^i z^j \longrightarrow \sum_{0 \leqslant i+j \leqslant 2n} a_{ij}\gamma_{ij}$$

on the cone generated by the collection $\{|p|^2 : p \in \mathbb{C}_n[\overline{z}, z]\}$, where $\mathbb{C}_n[Z, \overline{Z}]$ is the vector space of polynomials in two variables with complex coefficients and total degree less than or equal to n.

It is an immediate observation that the rows $\overline{Z}^k Z^l$, columns $\overline{Z}^i Z^j$ entry of the matrix $M(n)$ is equal to $\Lambda_{\gamma^{(2n)}}(\overline{z}^{i+l} z^{j+k}) = \gamma_{i+l, j+k}$. For reason of simplicity, we identify a polynomial $p(\overline{z}, z) \equiv \sum a_{ij}\overline{z}^i z^j$ with its coefficient vector $p = (a_{ij})$ with respect to the basis of monomials of $\mathbb{C}_n[\overline{z}, z]$ in degree-lexicographic order. Clearly, $M(n)$ acts on these coefficient vectors as follows:

$$\langle M(n)p, q\rangle = \Lambda_{\gamma^{(2n)}}(p\overline{q}). \tag{66}$$

A theorem of Shmul'yan [69] shows that a block matrix

$$M = \begin{pmatrix} A & B \\ B^* & C \end{pmatrix} \geqslant 0, \tag{67}$$

if and only if

(i) $A \geqslant 0$,

(ii) there exists a matrix W such that $B = AW$,

(iii) $C \geqslant W^*AW$.

Since $A = A^*$, we obtain W^*AW is independent of W provided that $B = AW$. Moreover, $\mathrm{rank}\ M = \mathrm{rank}\ A \Leftrightarrow C = W^*AW$ for some W such that $B = AW$. Conversely, if $A \geqslant 0$, any extension M satisfying $\mathrm{rank}\ M = \mathrm{rank}\ A$ (if this condition is satisfied, we will say that M is a flat extension of A) is necessarily positive. Notice also that from the expression

$$\begin{pmatrix} I & 0 \\ -W^* & I' \end{pmatrix} M \begin{pmatrix} I & -W \\ 0 & I' \end{pmatrix} = \begin{pmatrix} A & 0 \\ 0 & C - W^*AW \end{pmatrix},$$

where I and I' denote the unit matrices, we deduce that

$$\mathrm{rank}\ M = \mathrm{rank}\ A + \mathrm{rank}\ (C - W^*AW). \tag{68}$$

By Shmul'yan's theorem, $M(n) \geqslant 0$ admits a (necessarily positive) flat extension

$$M(n+1) = \begin{pmatrix} M(n) & B \\ B^* & C \end{pmatrix} \tag{69}$$

in the form of a moment matrix $M(n+1)$ if and only if

(i) $B = M(n)W$ for some W,

(ii) $C = W^*M(n)W$ is a Toeplitz matrix.

We have the next result due to Curto and Fialkow,

Theorem 4.2. *[22] The finite sequence $\gamma^{(2n)}$ has a $\mathrm{rank}\ M(n)$-atomic representing measure if and only if $M(n) \geqslant 0$ and $M(n)$ admits a flat extension $M(n+1)$. That is, $M(n)$ can be extended to a positive moment matrix $M(n+1)$ satisfying $\mathrm{rank}\ M(n+1) = \mathrm{rank}\ M(n)$.*

An important step in our approach is to show that the Hermitian matrix $W^*M(n)W$ is **persymmetric**, that is, it is symmetric across its lower-left to upper-right diagonal. For this purpose, we introduce first some additional notation.

We denote the successive columns of W and B (given as in Expression (69)) by $W_{|Z^{n+1}}, W_{|\overline{Z}Z^n}, \dots, W_{|\overline{Z}^{n+1}}$ and $B_{|Z^{n+1}}, B_{|\overline{Z}Z^n}, \dots, B_{|\overline{Z}^{n+1}}$, respectively.

Let us consider the $\frac{(n+1)(n+2)}{2}$-matrix built as follows,

$$M_\varphi(n) := J_0 \oplus J_1 \oplus \cdots \oplus J_n;$$

where $J_p = (\delta_{i+j,p})_{0 \leqslant i,j \leqslant p}$ with $\delta_{i,j}$ is the Kronecker symbol given by $\delta_{l,k} = 1$ for $k = l$ and zero otherwise. For example

$$J_0 = (1), \quad J_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \text{ and } J_2 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}.$$

Lemma 4.3. *Let* $M_{\varphi}(n)$, $M(n)$ *and* $B_{\overline{Z}^{n-i}Z^i}$ *($i = 0, \dots, n$) be as above, then*

1. $(M_{\varphi}(n))^2 = I$.
2. $(M_{\varphi}(n))^* = M_{\varphi}(n)$.
3. $M_{\varphi}(n) B_{\overline{Z}^i Z^{n-i}} = B_{\overline{Z}^{n-i}Z^i} \qquad (i = 0, \dots, n)$.
4. $M_{\varphi}(n) M(n) = \overline{M(n)} M_{\varphi}(n)$.

Proof. The assertions (1), (2) and (3) are obvious. Only the third assertion requires a proof. To this aim, we recall that $M(n) = [M(i,j)]_{0 \leqslant i,j \leqslant n}$, see (63). Therefore

$$
\begin{aligned}
[M_{\varphi}(n)]M(n) &= \left[\bigoplus_{i=0}^{n} J_i \right] [M(i,j)]_{i,j \leqslant n} \\
&= [J_i M(i,j)]_{i,j \leqslant n} \\
&= [\overline{M(i,j)} J_j]_{i,j \leqslant n} \\
&= [\overline{M(i,j)}]_{i,j \leqslant n} \left[\bigoplus_{i=0}^{n} J_i \right] \\
&= \overline{M(n)} M_{\varphi}(n).
\end{aligned}
$$

□

Proposition 4.4. *Let* n *be a given integer and let* $M(n)$ *and* W *be as above, then* $W^* M(n) W$ *is a Hermitian Persymmetric matrix.*

Proof. Setting $W^* M(n) W = (c_{ij})_{0 \leqslant i,j \leqslant n}$, then we have

$$c_{n-j,n-i} = W^*_{\overline{Z}^{n-j}Z^j} M(n) W_{\overline{Z}^{n-i}Z^i}. \tag{70}$$

By multiplying left both sides of the fourth equation in Lemma 4.3 by $M_{\varphi}(n)$ we obtain

$$M_{\varphi}(n) M_{\varphi}(n) M(n) = M_{\varphi}(n) \overline{M(n)} M_{\varphi}(n). \tag{71}$$

and hence, by applying Lemma 4.3-(1), we have

$$M(n) = M_{\varphi}(n) \overline{M(n)} M_{\varphi}(n). \tag{72}$$

It follows, from (70) and (72), that

$$c_{n-j,n-i} = W^*_{\overline{Z}^{n-j}Z^j} M_{\varphi}(n) \overline{M(n)} M_{\varphi}(n) W_{\overline{Z}^{n-i}Z^i}. \tag{73}$$

The fact that $M_{\varphi}(n)$ is self-adjoint allows to write

$$c_{n-j,n-i} = \left(M_{\varphi}(n) W_{\overline{Z}^{n-j}Z^j} \right)^* \overline{M(n)} \left(M_{\varphi}(n) W_{\overline{Z}^{n-i}Z^i} \right). \tag{74}$$

By using the assertions (3) and (4), in Lemma 4.3, we deduce that:

$$\begin{aligned}\overline{M(n)}M_\varphi(n)W_{\bar{z}^{n-i}z^i} &= M_\varphi(n)M(n)W_{\bar{z}^{n-i}z^i}\\ &= M_\varphi(n)B_{\bar{z}^{n-i}z^i} = B_{\bar{z}^i z^{n-i}}.\end{aligned}$$

Therefore, (74) implies that

$$\begin{aligned}c_{n-j,n-i} &= (M_\varphi(n)W_{\bar{z}^{n-j}z^j})^* B_{\bar{z}^i z^{n-i}}\\ &= W^*_{\bar{z}^{n-j}z^j}M_\varphi(n)M(n)W_{\bar{z}^i z^{n-i}}\\ &= ((M(n)M_\varphi(n))^* W_{\bar{z}^{n-j}z^j})^* W_{\bar{z}^i z^{n-i}}\\ &= (M_\varphi(n)M(n)W_{\bar{z}^{n-j}z^j})^* W_{\bar{z}^i z^{n-i}}\\ &= (\overline{M(n)}M_\varphi(n)W_{\bar{z}^{n-j}z^j})^* W_{\bar{z}^i z^{n-i}}\\ &= (M(n)W_{\bar{z}^j z^{n-j}})^* W_{\bar{z}^i z^{n-i}}\\ &= W^*_{\bar{z}^j z^{n-j}}M(n)W_{\bar{z}^i z^{n-i}}\\ &= c_{i,j}.\end{aligned}$$

This concludes the proof of the Proposition 4.4. □

4.2 COMPLEX-VALUED RECURSIVE BI-SEQUENCES

Let $\gamma^{(n)} \equiv \{\gamma_{ij}\}_{i+j\leqslant n}$, with $\overline{\gamma_{ij}} = \gamma_{ji}$ and $n \in \mathbb{N}\cup\{+\infty\}$, be a given complex-valued sequence and let $P_{\bar{z}^e z^{d-e}} = \sum\limits_{\substack{l+k\leqslant d\\(l,k)\neq(e,d-e)}} a_{lk}\bar{z}^l z^k$ be in $\mathbb{C}_d[\bar{z},z]$, the vector space of polynomials in two variables with complex coefficients and total degree less than or equal to d (we assume that $d \leqslant n$). The sequence $\gamma^{(n)}$ is said to be recursive, associated with a generating polynomial $\bar{z}^e z^{d-e} - P_{\bar{z}^e z^{d-e}}$, if

$$\gamma_{e+i,d-e+j} = \Lambda_{\gamma^{(n)}}(\bar{z}^i z^j P_{\bar{z}^e z^{d-e}}), \qquad \text{for all } i+j \leqslant n-d, \tag{75}$$

or, equivalently, if

$$\gamma_{e+i,d-e+j} = \sum_{\substack{l+k\leqslant d\\(l,k)\neq(e,d-e)}} a_{lk}\gamma_{l+i,k+j} \qquad (i+j \leqslant n-d). \tag{76}$$

We notice that, because of the equality $\overline{\gamma_{ij}} = \gamma_{ji}$, Equation (76) is equivalent to the following one:

$$\gamma_{d-e+i,e+j} = \sum_{\substack{l+k\leqslant d\\(l,k)\neq(e,d-e)}} \overline{a_{lk}}\gamma_{k+i,l+j}, \tag{77}$$

for all integers i and j, with $i+j \leqslant n-d$,
Therefore, $\bar{z}^{d-e}z^e - P_{\bar{z}^{d-e}z^e}$ (where $P_{\bar{z}^{d-e}z^e} = \overline{P_{\bar{z}^e z^{d-e}}}$) is, also, a generating polynomial, associated with $\gamma^{(n)}$; that is,

$$\gamma_{d-e+i,e+j} = \Lambda_{\gamma^{(n)}}(\bar{z}^i z^j P_{\bar{z}^{d-e}z^e}), \qquad i+j \leqslant n-d. \tag{78}$$

The following proposition provides a connection, via Λ, between the polynomials $P_{\bar{z}^f z^{f+1}}$ and $P_{\bar{z}^{f+1}z^f}$.

Proposition 4.5. *Let $\gamma^{(n)} \equiv \{\gamma_{ij}\}_{i+j\leqslant n}$ be a recursive bi-sequence and let $\bar{z}^f z^{f+1} - P_{\bar{z}^f z^{f+1}}$ be an associated generating polynomial, then*

$$\Lambda_{\gamma^{(n)}}(\bar{z}^{l+1}z^k P_{\bar{z}^f z^{f+1}}) = \Lambda_{\gamma^{(n)}}(\bar{z}^l z^{k+1} P_{\bar{z}^{f+1} z^f}), \qquad l+k \leqslant n-2f-2.$$

Proof. For all integers l and k, with $l+k \leqslant n-2f-2$, we have

$$\begin{aligned}\Lambda_{\gamma^{(n)}}(\bar{z}^{l+1}z^k P_{\bar{z}^f z^{f+1}}) &= \gamma_{f+l+1,f+k+1} \\ &= \overline{\gamma_{f+k+1,f+l+1}} \\ &= \overline{\Lambda_{\gamma^{(n)}}(\bar{z}^{k+1}z^l P_{\bar{z}^f z^{f+1}})} \\ &= \Lambda_{\gamma^{(n)}}(\bar{z}^l z^{k+1} P_{\bar{z}^{f+1} z^f}).\end{aligned}$$

□

It is well known that the (classical singly indexed recursive sequence can be defined by the initial data and the, associated recurrence relation (or, characteristic polynomial), see [35]. In a similar way, one can define recursive bi-sequences as observed below.

Remark 4.6. i) A generating polynomial $z^e - P_{z^e}$ (or, equivalently, $\bar{z}^e - P_{\bar{z}^e}$), with $\deg P_{z^e} < e$, together with the initial data $\{\gamma_{ij}\}_{i,j<e}$ and the equality $\overline{\gamma_{ij}} = \gamma_{ji}$, are said to define the sequence $\gamma^{(n)}$.

ii) For a generating polynomial $\bar{z}z^{e-1} - P_{\bar{z}z^{e-1}}$, with $\deg P_{\bar{z}z^{e-1}} < e$, we need (all) the data $\{\gamma_{ij}\}_{i,j<e} \cup \{\gamma_{0j}\}_{j=e,\dots,n}$ and the equality $\overline{\gamma_{ij}} = \gamma_{ji}$ to define the recursive bi-sequence $\gamma^{(n)}$.

In the next lemmas, we provide useful results for solving the quintic moment problem.

Lemma 4.7. *Let $\gamma^{(8)} \equiv \{\gamma_{ij}\}_{i+j\leqslant 8}$, with $\overline{\gamma_{ij}} = \gamma_{ji}$, be a truncated bi-sequence and let $z^4 - P_{z^4}$ (where $P_{z^4} = \beta z^3 + R_{z^4}$ and $R_{z^4} \in \mathbb{C}_2[\bar{z}, z]$) be an associated generating polynomial. Assume that $\bar{z}z^2 - P_{\bar{z}z^2}$ (where $P_{\bar{z}z^2} = \alpha z^3 + R_{\bar{z}z^2}$, $\alpha \neq 0$ and $R_{\bar{z}z^2} \in \mathbb{C}_2[\bar{z}, z]$) is a generating polynomial for $\gamma^{(6)} \cup \{\gamma_{34}, \gamma_{43}\}$, then $\bar{z}z^2 - P_{\bar{z}z^2}$ is a generating polynomial for $\gamma^{(8)}$.*

Proof. We have $z^4 - P_{z^4}$ is a generating polynomial for $\gamma^{(8)}$, that is,

$$\gamma_{i,j+4} = \Lambda_{\gamma^{(8)}}(\bar{z}^i z^j P_{z^4}) = \beta\gamma_{i,j+3} + \Lambda_{\gamma^{(8)}}(\bar{z}^i z^j R_{z^4}), \qquad i+j \leqslant 4. \quad (79)$$

As showing in (78), the last equality (81) is equivalent to

$$\gamma_{i+4,j} = \Lambda_{\gamma^{(8)}}(\bar{z}^i z^j P_{\bar{z}^4}) = \bar{\beta}\gamma_{i+3,j} + \Lambda_{\gamma^{(8)}}(\bar{z}^i z^j R_{\bar{z}^4}), \qquad i+j \leqslant 4; \quad (80)$$

where $\overline{P_{z^4}} := P_{\bar{z}^4} = \bar{\beta}\bar{z}^3 + \overline{R_{\bar{z}z^2}}$.
Also, the polynomial $\bar{z}z^2 - P_{\bar{z}z^2}$ is a generating one for $\gamma^{(6)} \cup \{\gamma_{34}, \gamma_{43}\}$; that is, for all $i+j \leqslant 3$ and $(i,j) = (2,2), (3,1)$:

$$\gamma_{i+1,j+2} = \Lambda_{\gamma^{(8)}}(\bar{z}^i z^j P_{\bar{z}z^2}) = \alpha\gamma_{i,j+3} + \Lambda_{\gamma^{(8)}}(\bar{z}^i z^j R_{\bar{z}z^2}). \qquad (81)$$

or, equivalently, for $i+j \leqslant 3$ and $(i,j) = (2,2),(1,3)$;

$$\gamma_{i+2,j+1} = \Lambda_{\gamma^{(8)}}(\bar{z}^i z^j P_{\bar{z}^2 z^1}) = \overline{\alpha}\gamma_{i+3,j} + \Lambda_{\gamma^{(8)}}(\bar{z}^i z^j R_{\bar{z}^2 z}), \tag{82}$$

where $P_{\bar{z}^2 z} := \overline{P_{\bar{z}z^2}} = \overline{\beta}\bar{z}^3 + R_{\bar{z}^2 z}$, see (78).
We have to show that (81) remains valid for all integers i and j, with $i+j \leqslant 5$. To this end we consider the recursive bi-sequence $\widehat{\gamma}^{(8)} \equiv \{\widehat{\gamma}_{ij}\}_{i+j\leqslant 8}$ defined by

$$\begin{cases} \widehat{\gamma}_{i+1,j+2} &= \Lambda_{\widehat{\gamma}^{(8)}}(\bar{z}^i z^j P_{\bar{z}z^2}) \qquad (i+j \leqslant 5), \\ \widehat{\gamma}_{i,j} &= \gamma_{i,j} \qquad \text{otherwise ;} \end{cases} \tag{83}$$

and we will show that $\widehat{\gamma}^{(8)} = \gamma^{(8)}$. Notice that since $\bar{z}z^2 - P_{\bar{z}z^2}$ is a generating polynomial for $\widehat{\gamma}^{(8)}$, then $\bar{z}^2 z - P_{\bar{z}^2 z}$ is an other one. Thus

$$\widehat{\gamma}_{i+2,j+1} = \Lambda_{\widehat{\gamma}^{(8)}}(\bar{z}^i z^j P_{\bar{z}^2 z}) \qquad (i+j \leqslant 5). \tag{84}$$

It follows from (81) and (83) that, for $n+m \leqslant 6$, $n = 0$ and $(n,m) = (3,4),(4,3)$:

$$\gamma_{nm} = \Lambda_{\gamma^{(8)}}(\bar{z}^n z^m) = \Lambda_{\widehat{\gamma}^{(8)}}(\bar{z}^n z^m) := \widehat{\gamma}_{nm}. \tag{85}$$

Remark that if $\widehat{\gamma}_{nm} = \gamma_{nm}$ then $\widehat{\gamma}_{mn} = \overline{\widehat{\gamma}_{nm}} = \overline{\gamma_{nm}} = \gamma_{mn}$.

Therefore, we need to show (85), only, for the integers n and m with $(n,m) = (2,5),(1,6);(1,7),(2,6),(3,5),(4,4)$.

$$\begin{aligned} \gamma_{25} &= \Lambda_{\gamma^{(8)}}(\bar{z}^2 z P_{z^4}), && \text{utilizing (79)}, \\ &= \Lambda_{\gamma^{(8)}}(P_{\bar{z}^2 z} P_{z^4}), && \text{employ (82) and } \deg P_{z^4} \leqslant 3, \\ &= \overline{\alpha}\Lambda_{\gamma^{(8)}}(\bar{z}^3 P_{z^4}) + \Lambda_{\gamma^{(8)}}(R_{\bar{z}^2 z} P_{z^4}) \\ &= \overline{\alpha}\gamma_{34} + \Lambda_{\gamma^{(8)}}(z^4 R_{\bar{z}^2 z}), && \text{applying (79)}, \\ &= \overline{\alpha}\widehat{\gamma}_{34} + \Lambda_{\widehat{\gamma}^{(8)}}(z^4 R_{\bar{z}^2 z}), && \text{use } \deg z^4 R_{\bar{z}^2 z} \leqslant 6 \text{ and (85)}, \\ &= \Lambda_{\widehat{\gamma}^{(8)}}(\overline{\alpha z}^3 z^4 + z^4 R_{\bar{z}^2 z}) \\ &= \Lambda_{\widehat{\gamma}^{(8)}}(z^4 P_{\bar{z}^2 z}) \\ &= \Lambda_{\widehat{\gamma}^{(8)}}(\bar{z} z^3 P_{z^2 \bar{z}}), && \text{according to Proposition 4.5}, \\ &= \widehat{\gamma}_{25}, && \text{from (84)}. \end{aligned} \tag{86}$$

$$
\begin{aligned}
\gamma_{16} &= \Lambda_{\gamma^{(8)}}(\bar{z}z^2 P_{z^4}), && \text{use(79)}, \\
&= \Lambda_{\gamma^{(8)}}(P_{\bar{z}z^2} P_{z^4}), && \text{employ (81) and } \deg P_{z^4} \leqslant 3, \\
&= \Lambda_{\gamma^{(8)}}(\alpha z^3 P_{z^4} + R_{\bar{z}z^2} P_{z^4}) && \\
&= \alpha\gamma_{07} + \Lambda_{\gamma^{(8)}}(z^4 R_{\bar{z}z^2}), && \text{utilizing (79)}, \\
&= \alpha\widehat{\gamma}_{07} + \Lambda_{\widehat{\gamma}^{(8)}}(z^4 R_{\bar{z}z^2}), && \text{using (85) and } \deg z^4 R_{\bar{z}z^2} \leqslant 6, \\
&= \Lambda_{\widehat{\gamma}^{(8)}}(\alpha z^7 + z^4 R_{\bar{z}z^2}) && \\
&= \Lambda_{\widehat{\gamma}^{(8)}}(z^4 P_{\bar{z}z^2}) && \\
&= \widehat{\gamma}_{16}, && \text{according to (83)}.
\end{aligned}
\tag{87}
$$

Thus, the equality (85) is valid for every integer n and m with $n + m \leqslant 7$. In other words,

$$
\gamma_{nm} = \Lambda_{\gamma^{(8)}}(\bar{z}^n z^m) = \Lambda_{\widehat{\gamma}^{(8)}}(\bar{z}^n z^m) := \widehat{\gamma}_{nm} \qquad (n+m \leqslant 7). \tag{88}
$$

And thus one can generalize the relation (81) as follows

$$
\gamma_{i+1,j+2} = \Lambda_{\gamma^{(8)}}(\bar{z}^i z^j P_{\bar{z}z^2}) = \alpha\gamma_{i,j+3} + \Lambda_{\gamma^{(8)}}(\bar{z}^i z^j R_{\bar{z}z^2}) \qquad (i+j \leqslant 4). \tag{89}
$$

Now, let us show (85) in the remaining cases ($n + m = 8$).

$$
\gamma_{08} = \widehat{\gamma}_{08}, \qquad \text{by the construction of } \widehat{\gamma}^{(8)}, \text{ see (83)}. \tag{90}
$$

$$
\begin{aligned}
\gamma_{17} &= \Lambda_{\gamma^{(8)}}(\bar{z}z^3 P_{z^4}), && \text{according to (79)} \\
&= \Lambda_{\gamma^{(8)}}(z P_{z^4} P_{\bar{z}z^2}), && \text{because } \deg z P_{z^4} \leqslant 4 \text{ and (89)}, \\
&= \Lambda_{\gamma^{(8)}}(z^5 P_{\bar{z}z^2}), && \text{utilizing (79)} \\
&= \alpha\Lambda_{\gamma^{(8)}}(z^8) + \Lambda_{\gamma^{(8)}}(z^5 R_{\bar{z}z^2}) && \\
&= \alpha\Lambda_{\widehat{\gamma}^{(8)}}(z^8) + \Lambda_{\widehat{\gamma}^{(8)}}(z^5 R_{\bar{z}z^2}), && \text{using (90) and (88)}, \\
&= \Lambda_{\widehat{\gamma}^{(8)}}(z^5 P_{\bar{z}z^2}) && \\
&= \widehat{\gamma}_{17}, && \text{applying (83)}.
\end{aligned}
\tag{91}
$$

$$
\begin{aligned}
\gamma_{26} &= \Lambda_{\gamma^{(8)}}(\bar{z}^2 z^2 P_{z^4}), && \text{according to (79)} \\
&= \Lambda_{\gamma^{(8)}}(\bar{z} P_{z^4} P_{\bar{z}z^2}), && \text{use } \deg \bar{z} P_{z^4} \leqslant 4 \text{ and (89)}, \\
&= \Lambda_{\gamma^{(8)}}(\bar{z}z^4 P_{\bar{z}z^2}), && \text{utilizing (79)}, \\
&= \alpha\Lambda_{\gamma^{(8)}}(\bar{z}z^7) + \Lambda_{\gamma^{(8)}}(\bar{z}z^4 R_{\bar{z}z^2}) && \\
&= \alpha\Lambda_{\widehat{\gamma}^{(8)}}(\bar{z}z^7) + \Lambda_{\widehat{\gamma}^{(8)}}(\bar{z}z^4 R_{\bar{z}z^2}), && \text{by using (91) and (88)}, \\
&= \Lambda_{\widehat{\gamma}^{(8)}}(\bar{z}z^4 P_{\bar{z}z^2}) && \\
&= \widehat{\gamma}_{26}, && \text{according to (83)}.
\end{aligned}
$$

(92)

Before continue the proof, of these lemma, let us remark that the Relation 88 implies that, for all $i+j \leqslant 5$,

$$\Lambda_{\widehat{\gamma}^{(8)}}(\bar{z}^{i+1}z^{j+2}) = \widehat{\gamma}_{i+1,j+2} = \Lambda_{\widehat{\gamma}^{(8)}}(\bar{z}^i z^j(\alpha z^3 + R_{\bar{z}z^2})),$$

and thus

$$\Lambda_{\widehat{\gamma}^{(8)}}(\bar{z}^i z^{j+3}) = \frac{1}{\alpha}\Lambda_{\widehat{\gamma}^{(8)}}(\bar{z}^i z^j(\bar{z}z^2 - R_{\bar{z}z^2})) \qquad (i+j \leqslant 5). \quad (93)$$

Now,

$$\begin{aligned}
\gamma_{35} &= \Lambda_{\gamma^{(8)}}(\bar{z}^3 z P_{z^4}), && \text{according to (79)},\\
&= \Lambda_{\widehat{\gamma}^{(8)}}(\bar{z}^3 z P_{z^4}), && \text{because } \deg \bar{z}^3 z P_{z^4} \leqslant 7,\\
&= \Lambda_{\widehat{\gamma}^{(8)}}(\frac{1}{\alpha}(P_{\bar{z}^2 z} - R_{\bar{z}^2 z})zP_{z^4}) &&\\
&= \frac{1}{\alpha}\Lambda_{\widehat{\gamma}^{(8)}}((\bar{z}^2 z - R_{\bar{z}^2 z})zP_{z^4}) &&\\
&= \frac{1}{\alpha}\Lambda_{\gamma^{(8)}}((\bar{z}^2 z - R_{\bar{z}^2 z})zP_{z^4}), && \text{applying (88)},\\
&= \frac{1}{\alpha}\Lambda_{\gamma^{(8)}}(\bar{z}^2 z^6) - \frac{1}{\alpha}\Lambda_{\gamma^{(8)}}(z^5 R_{\bar{z}^2 z}) &&\\
&= \frac{1}{\alpha}\gamma_{26} - \frac{1}{\alpha}\Lambda_{\widehat{\gamma}^{(8)}}(z^5 R_{\bar{z}^2 z}), && \text{remark that } \deg z^5 R_{\bar{z}^2} \leqslant 7,\\
&= \frac{1}{\alpha}\widehat{\gamma}_{26} - \frac{1}{\alpha}\Lambda_{\widehat{\gamma}^{(8)}}(z^5 R_{\bar{z}^2 z}), && \text{from (92)},\\
&= \Lambda_{\widehat{\gamma}^{(8)}}(\frac{1}{\alpha}(P_{\bar{z}^2 z} - R_{\bar{z}^2 z})z^5) &&\\
&= \Lambda_{\widehat{\gamma}^{(8)}}(\bar{z}^3 z^5) &&\\
&= \widehat{\gamma}_{35}. &&
\end{aligned}$$

(94)

$$\begin{aligned}
\gamma_{44} &= \Lambda_{\gamma^{(8)}}(\bar{z}^4 P_{z^4}) &&\\
&= \Lambda_{\widehat{\gamma}^{(8)}}(\bar{z}^3 \bar{z} P_{z^4}) && \text{using (88)},\\
&= \Lambda_{\widehat{\gamma}^{(8)}}(\frac{1}{\alpha}(P_{\bar{z}^2 z} - R_{\bar{z}^2 z})\bar{z}P_{z^4}), && \text{by (93)},\\
&= \frac{1}{\alpha}\Lambda_{\widehat{\gamma}^{(8)}}(\bar{z}^3 z P_{z^4}) - \frac{1}{\alpha}\Lambda_{\widehat{\gamma}^{(8)}}(\bar{z}P_{z^4}R_{\bar{z}^2 z}) &&\\
&= \frac{1}{\alpha}\Lambda_{\gamma^{(8)}}(\bar{z}^3 z P_{z^4}) - \frac{1}{\alpha}\Lambda_{\gamma^{(8)}}(\bar{z}P_{z^4}R_{\bar{z}^2 z}) &&\\
&= \frac{1}{\alpha}\gamma_{35} - \frac{1}{\alpha}\Lambda_{\widehat{\gamma}^{(8)}}(\bar{z}P_{z^4}R_{\bar{z}^2 z}) &&\\
&= \frac{1}{\alpha}\widehat{\gamma}_{35} - \frac{1}{\alpha}\Lambda_{\widehat{\gamma}^{(8)}}(\bar{z}P_{z^4}R_{\bar{z}^2 z}), && \text{applying (94)},\\
&= \Lambda_{\widehat{\gamma}^{(8)}}(\frac{1}{\alpha}(P_{\bar{z}^2 z} - R_{\bar{z}^2 z})\bar{z}P_{z^4}) &&\\
&= \Lambda_{\widehat{\gamma}^{(8)}}(\bar{z}^4 P_{z^4}) &&\\
&= \widehat{\gamma}_{44}. &&
\end{aligned} \quad (95)$$

This finishes the proof of Lemma 4.7. $\square$

4.3 SOLVING THE QUINTIC MOMENT PROBLEM

Let $\gamma^{(5)} \equiv \{\gamma_{ij}\}_{i+j\leqslant 5}$ be a given complex-valued bi-sequence, with $\gamma_{00} > 0$ and $\overline{\gamma_{ij}} = \gamma_{ji}$ for $i+j \leqslant 5$. The quintic moment problem involves determining necessary and sufficient conditions for the existence of a positive Borel measure μ on $\mathbb{C}$ (called a representing measure for $\gamma^{(5)}$) such that

$$\gamma_{ij} = \int \bar{z}^i z^j \, d\mu, \qquad \text{for } i+j \leqslant 5.$$

In this section we will show that in almost all cases the classical necessary conditions $M(2) \geqslant 0$ and $B = M(2)W$, for some W, (with $M(2)$ and B are as in (69)) guarantee the existence of at most $(r+2)$-atomic (here $r := \operatorname{rank} \ M(2)$) representing measure for $\gamma^{(5)}$.

According to Proposition 4.4, the Hermitian matrix $W^*M(2)W$ is persymmetric (I. e.symmetric with respect to the second diagonal), then one can set

$$W^*M(2)W = \begin{pmatrix} a & b & c & d \\ \bar{b} & e & f & c \\ \bar{c} & \bar{f} & e & b \\ \bar{d} & \bar{c} & \bar{b} & a \end{pmatrix} \tag{96}$$

The next Theorem gives a concrete solution to the quintic complex moment problem, except for the case $a = e$ and $b \neq f$.

Theorem 4.8. *Let $\gamma^{(5)} \equiv \{\gamma_{ij}\}_{i+j\leqslant 5}$ be a given sequence, we assume that $M(2) \geqslant 0$ and $\operatorname{Rang} \ B \subseteq \operatorname{Rang} \ M(2)$, and $a \neq e$ or $b = f$*

Then the quintic moment problem, associated with $\gamma^{(5)}$, admits a solution μ. Moreover, The smallest cardinality of $\operatorname{supp} \ \mu$ is

- $\operatorname{card} \ \operatorname{supp} \ \mu = r \iff a = e$ *and* $b = f$,
- $\operatorname{card} \ \operatorname{supp} \ \mu = r+1 \iff \quad a \neq e$ *and* $\frac{a-e}{2} <| b-f |$,
- $\operatorname{card} \ \operatorname{supp} \ \mu = r+2 \iff \quad a > e$ *and* $\frac{a-e}{2} \geqslant| b-f |$;

where a, b, e and f are as in (96).

Before we develop the proof of our theorem, let us introduce some notations. For $n \in \{3,4\}$; let $\gamma^{(2n)} \equiv \{\gamma_{ij}\}_{i+j\leqslant 2n}$ be a truncated complex bi-sequence and let $M(n)$ be the associated moment matrix. As before, we denote by $B(n)$ and $C(n)$ the $(n-1) \times n$-matrix and the $n \times n$-matrix, respectively, such that

$$M(n) = \begin{pmatrix} M(n-1) & B(n) \\ B^*(n) & C(n) \end{pmatrix} \tag{97}$$

Let $\mathfrak{B} \equiv \mathfrak{B}(n) \equiv \{\overline{Z}^i Z^j\}_{(i,j)\in\mathfrak{R}}$ (where $\mathfrak{R} \equiv \mathfrak{R}(n) \subseteq \{0,1,\dots,n\} \times \{0,1,\dots,n\}$) be a basis for the column space of $M(n)$. Let us remark that the $r \times r$-matrix $M(n)_{|\mathfrak{B}}$, where $r \equiv r(n) := \mathrm{card}\ \ \mathfrak{R}(n)$, the restriction of the moment matrix $M(n)$ to the basis $\mathfrak{B}$, is invertible.

Proof of Theorem 4.8. The main idea is to extend the initial data $\gamma^{(5)}$ to an even-degree $\gamma^{(6)}$ (by adding the sixtic moments $\gamma_{60} = \overline{\gamma_{06}}$, $\gamma_{51} = \overline{\gamma_{15}}$, $\gamma_{42} = \overline{\gamma_{24}}$ and $\gamma_{33} \in \mathbb{R}$) such that the associated moment matrix $M(3)$, for an appropriate choice of the missing moments, is either a flat extension of $M(2)$ or admits admits a flat extension $M(4)$. Thus Theorem 4.2 yields that $M(3)$ has a representing measure; and as a consequence, $\gamma^{(5)}$ also admits a representing measure μ. It is also proved that the smallest cardinality of $\mathrm{supp}\ \ \mu$ will be $r := \mathrm{rank}\ \ M(2)$ or $r+1$ or $r+2$.

By virtue of the Shmul'yan's Theorem, we need to find a Toeplitz square matrix $C(3)$, built with the new, sixtic, moments as entries and such that $C(3) - W^*M(2)W \geqslant 0$. Setting,

$$C(3) - W^*M(2)W = \begin{pmatrix} \gamma_{33} - a & \gamma_{42} - b & \gamma_{51} - c & \gamma_{60} - d \\ \gamma_{24} - \overline{b} & \gamma_{33} - e & \gamma_{42} - f & \gamma_{51} - c \\ \gamma_{15} - \overline{c} & \gamma_{24} - \overline{f} & \gamma_{33} - e & \gamma_{42} - b \\ \gamma_{06} - \overline{d} & \gamma_{15} - \overline{c} & \gamma_{24} - \overline{b} & \gamma_{33} - a \end{pmatrix} \qquad (98)$$

we will distinguish two cases:

Case I: $a = e$ and $b = f$. In this case the matrix $W^*M(2)W$ is a Toeplitz one, then it suffice to consider that $C(3) = W^*M(2)W$. According to (68), the matrix $M(3)$ is a flat extension of $M(2)$ and thus $\gamma^{(6)}$ (and in force $\gamma^{(5)}$) has a r-representing measure.

Case II: $a \neq e$. We proceed in two steps for this case. obviously, the matrix $W^*M(2)W$ is not a Toeplitz one. Therefore, for every choice of a Toeplitz 4×4-matrix $C(3)$, we have $\mathrm{rank}\ \ (C(3) - W^*M(2)W) \geqslant 1$. We will show, in first step, that the smallest possible rank of $C(3) - W^*M(2)W$ will be either 1 or 2. In the second step, we will show that the moment matrix $M(3)$, obtained by extending $\gamma^{(5)}$ with the entries of some suitable $C(3)$, has a flat extension and thus admits a $\mathrm{rank}\ \ M(3)$-atomic representing measure, see Theorem 4.2.

Step 1: (construction of $C(3)$). Firstly, let us observe that

$$\mathrm{rank}(C(3) - W^*M(2)W) = 1 \text{ and } C(3) - W^*M(2)W \geqslant 0 \qquad (99)$$

if and only if we have

$$\begin{aligned} &(0)\ \gamma_{33} > \max(a, e). \\ &(i)\ |\gamma_{42} - b| = \sqrt{(\gamma_{33} - a)(\gamma_{33} - e)}, \\ &\quad |\gamma_{42} - f| = \gamma_{33} - e. \\ &(ii)(\gamma_{15} - \overline{c})(\gamma_{42} - b) = (\gamma_{33} - a)(\gamma_{24} - \overline{f}). \\ &(iii)(\gamma_{06} - \overline{d})(\gamma_{42} - b)^2 = (\gamma_{33} - a)^2(\gamma_{24} - \overline{f}), \\ &\quad |\gamma_{06} - \overline{d}|^2 = (\gamma_{33} - a)^2. \end{aligned} \qquad (100)$$

Remark that the equalities (i) provide the compatibility of the two equalities in (iii) and vice versa.

The condition (i) means that γ_{42} is in the intersection of the two next circles $\mathcal{C}$, of radius $\sqrt{(\gamma_{33}-a)(\gamma_{33}-e)}$ and centred at b, and $\mathcal{C}'$ centred on f with radius $\gamma_{33}-e$.

It is an easy geometrical observation to see that, the two circles $\mathcal{C}$ and $\mathcal{C}'$ have a non-empty intersection if, and only if, there exists $\gamma_{33} > \max(a, e)$, such that

$$| (\gamma_{33}-e) - \sqrt{(\gamma_{33}-a)(\gamma_{33}-e)} \,| \leqslant | \, b - f \,| \leqslant \gamma_{33} - e + \sqrt{(\gamma_{33}-a)(\gamma_{33}-e)} \tag{101}$$

As the maps $x \mapsto (x-e) - \sqrt{(x-a)(x-e)}$ is decreasing (on $[\max(a,e);+\infty[$), and $(x-e) - \sqrt{(x-a)(x-e)} \xrightarrow[x\to+\infty]{} \frac{a-e}{2}$ and $(x-e) + \sqrt{(x-a)(x-e)} \xrightarrow[x\to+\infty]{} +\infty$. Then (101) is verified if and only if $a = e$ and $b \neq f$ or $a < e$ or $a > e$ and $|b-f| > \frac{a-e}{2}$.

Subcase II-1: $a < e$ or $a > e$ and $|b-f| > \frac{a-e}{2}$. It suffices to choose γ_{33} verifying (101), and thus γ_{42} exists (as the point intersection of the two circles $\mathcal{C}$ and $\mathcal{C}'$). Furthermore , from (0) and (i) we derive that

$$(\gamma_{42}-b)(\gamma_{42}-f) \neq 0 \tag{102}$$

The equality (ii) gives the moment γ_{15} and (iii) supplies γ_{06}, and this complete the construction of a Toeplitz matrix $C(3)$ for which $\mathrm{rank}(C(3) - W^*M(2)W) = 1$. Note that, $\mathrm{rank}\ \ M(3)_{|\mathfrak{B}(2)\cup\{Z^3\}} = \mathrm{rank}\ \ M(2)+1 = \mathrm{rank}\ \ M(3)$. Hence, in $M(3)$, the columns $\overline{Z}Z^2, \overline{Z}^2 Z$ and $\overline{Z}^3$ are a linear combination of the columns $\mathfrak{B}(2)\cup\{Z^3\}$. In particular, we can set

$$\overline{Z}Z^2 = P_{\overline{z}z^2}(Z,\overline{Z}) = \alpha Z^3 + R_{\overline{z}z^2}(Z,\overline{Z}), \tag{103}$$

with

$$\alpha = \frac{\det \begin{vmatrix} M(2)_{|\mathfrak{B}(2)} & \overline{Z}Z^2{}_{|\mathfrak{B}(2)} \\ (Z^3_{|\mathfrak{B}(2)})^* & \gamma_{42} \end{vmatrix}}{\det \begin{vmatrix} M(2)_{|\mathfrak{B}(2)} & Z^3_{|\mathfrak{B}(2)} \\ (Z^3_{|\mathfrak{B}(2)})^* & \gamma_{33} \end{vmatrix}}$$

$$= \frac{\det \begin{vmatrix} M(2)_{|\mathfrak{B}(2)} & \overline{Z}Z^2{}_{|\mathfrak{B}(2)} \\ (Z^3_{|\mathfrak{B}(2)})^* & b \end{vmatrix} + (\gamma_{42} - b) \quad \det | M(2)_{|\mathfrak{B}(2)} |}{\det \begin{vmatrix} M(2)_{|\mathfrak{B}(2)} & Z^3_{|\mathfrak{B}(2)} \\ (Z^3_{|\mathfrak{B}(2)})^* & a \end{vmatrix} + (\gamma_{33} - a) \quad \det | M(2)_{|\mathfrak{B}(2)} |} \tag{104}$$

$$= \frac{(\gamma_{42} - b) \quad \det | M(2)_{|\mathfrak{B}(2)} |}{(\gamma_{33} - a) \quad \det | M(2)_{|\mathfrak{B}(2)} |}$$

$$= \frac{\gamma_{42} - b}{\gamma_{33} - a} \neq 0; \qquad \text{by virtue of (102)}.$$

Subcase II-2: $a > e$ and $\frac{a-e}{2} \geqslant | b - f |$. Then $\operatorname{rank}(C(3) - W^*M(2)W) \geqslant 2$ for every 4×4-Toeplitz matrix $C(3)$. Let us choose the sixtic moments as follows

$$\begin{cases} \gamma_{33} > \max(a, e), \\ | \gamma_{42} - b | = \sqrt{(\gamma_{33} - a)(\gamma_{33} - e)}, \\ \gamma_{15} - \overline{c} = \frac{\gamma_{33}-a}{\gamma_{42}-b}(\gamma_{24} - \overline{f}) \\ \gamma_{06} - \overline{d} = (\frac{\gamma_{33}-a}{\gamma_{42}-b})^2(\gamma_{24} - \overline{f}). \end{cases} \tag{105}$$

Let us remark that as the first subcase II-1, we have

$$\frac{\gamma_{42} - b}{\gamma_{33} - a} = \sqrt{\frac{\gamma_{33} - e}{\gamma_{33} - a}} \neq 0. \tag{106}$$

The moment defined in (105) construct a Toeplitz matrix $C(3)$ for which $\operatorname{rank}(C(3) - W^*M(2)W) = 2$. Indeed, it suffices to observe that

$$\bullet (C(3) - W^*M(2)W)(Z^3) = \frac{\gamma_{33} - a}{\gamma_{42} - b}(C(3) - W^*M(2)W)(\overline{Z}Z^2),$$

$$\bullet (C(3) - W^*M(2)W)(\overline{Z}^3) = \frac{\gamma_{33} - a}{\gamma_{24} - \overline{b}}(C(3) - W^*M(2)W)(\overline{Z}^2 Z),$$

$\bullet (C(3) - W^*M(2)W)(Z^3)$ and $(C(3) - W^*M(2)W)(\overline{Z}^3)$
are nonlinear (because (i) can not be verified).

Therefore, in $M(3)$, the columns $\overline{Z}Z^2$ is a linear combination of the columns $\mathfrak{B}(2) \cup \{Z^3\}$. For reason of simplicity, we adopt the notation of the Relation (103), that is,

$$\overline{Z}Z^2 = P_{\overline{z}z^2}(Z, \overline{Z}) = \alpha Z^3 + R_{\overline{z}z^2}(Z, \overline{Z}), \tag{107}$$

Where

$$\alpha = \frac{\gamma_{42} - b}{\gamma_{33} - a} \neq 0$$

by using (106).

We conclude that, in the both cases *II*-1 and *II*-2, we have extended the initial data $\gamma^{(5)}$ to $\gamma^{(6)}$ so that the associated moment matrix $M(3)$ has the following columns relation

$$\overline{Z}Z^2 = P_{\overline{z}z^2}(Z, \overline{Z}) = \alpha Z^3 + R_{\overline{z}z^2}(Z, \overline{Z}), \quad \text{with } \alpha \neq 0. \tag{108}$$

We also note that since $a \neq e$ we get,

$$|\alpha| = \left|\frac{\gamma_{42} - b}{\gamma_{33} - a}\right| = \frac{\sqrt{(\gamma_{33} - a)(\gamma_{33} - e)}}{\gamma_{33} - a} \neq 1 \tag{109}$$

*S*tep 2: ($M(3)$ has a flat extension, and thus a representing measure). We will build moments $\{\gamma_{ij}\}_{i+j=7,8}$ for which the moment matrix $M(4)$ is a flat extension of $M(3)$.

The relation (108) yields that

$$\langle M(3)\overline{Z}Z^2, \overline{Z}^j Z^i\rangle = \langle M(3)P_{\overline{z}z^2}(Z, \overline{Z}), \overline{Z}^j Z^i\rangle, \quad \text{for all } i + j \leqslant 3.$$

By applying (66), one obtain

$$\Lambda_{\gamma^{(6)}}(\overline{z}^{i+1} z^{j+2}) = \Lambda_{\gamma^{(6)}}(\overline{z}^i z^j P_{\overline{z}z^2}), \qquad i + j \leqslant 3. \tag{110}$$

Since $|\alpha| \neq 1$, we derive that there exists a complex number $\gamma_{43} = \overline{\gamma_{43}}$ such that

$$\gamma_{43} = \Lambda(\overline{z}^3 z P_{\overline{z}z^2}), \tag{111}$$

that is,

$$\gamma_{43} = \alpha\gamma_{34} + \sum_{i+j \leqslant 2} \alpha_{i,j}\gamma_{i+3,j+1}.$$

It follows, from (111) and (110), that $\overline{z}z^2 - P_{\overline{z}z^2}$ is a generating polynomial of $\gamma^{(6)} \cup \{\gamma_{34}, \gamma_{43}\}$.

Since $\begin{pmatrix} M(2)_{|\mathfrak{B}(2)} & Z^3_{|\mathfrak{B}(2)} \\ (Z^3_{|\mathfrak{B}(2)})^* & \gamma_{33} \end{pmatrix} > 0$, then there exists a (unique) vector, say

$$P_{z^4} = \beta z^3 + R_{z^4} = \beta z^3 + \sum_{\overline{z}^i z^j \in \mathfrak{B}(2)} \beta_{ij} \overline{z}^i z^j$$

the associated polynomial, such that

$$\begin{pmatrix} M(2)_{|\mathfrak{B}(2)} & Z^3_{|\mathfrak{B}(2)} \\ (Z^3_{|\mathfrak{B}(2)})^* & \gamma_{33} \end{pmatrix} P_{z^4} = ((\gamma_{04}, \gamma_{14}, \gamma_{05}, \gamma_{24}, \gamma_{15}, \gamma_{06})_{|\mathfrak{B}(2)}, \gamma_{34})^T.$$

Therefore the sequence $\gamma^{(6)} \cup \{\gamma_{34}, \gamma_{43}\}$ verifies that

$$\gamma_{i,j+4} = \Lambda(\overline{z}^i z^j P_{z^4}), \quad \text{for all } i + j \leqslant 2 \text{ and } (i,j) = (3,0); \tag{112}$$

$$\gamma_{i+4,j} = \Lambda(\bar{z}^i z^j P_{\bar{z}^4}), \qquad \text{for all } i+j \leqslant 2 \text{ and } (i,j) = (0,3). \quad (113)$$

Thus $z^4 - P_{z^4}$ is a generating polynomial of $\gamma^{(6)} \cup \{\gamma_{34}, \gamma_{43}\}$.

We will build a sequence $\gamma^{(8)} \equiv \{\gamma_{ij}\}_{i+j\leqslant 8}$, the extension of $\gamma^{(6)} \cup \{\gamma_{34}, \gamma_{43}\}$, by using a generating polynomial P_{z^4} and the initial data $\{\gamma_{ij}\}_{i,j\leqslant 3}$, that is,

$$\gamma_{i,j+4} = \Lambda(\bar{z}^i z^j P_{z^4}) = \beta\gamma_{i,j+3} + \sum_{\bar{z}^l z^k \in \mathfrak{B}(2)} \beta_{lk}\gamma_{i+l,j+k} \qquad (i+j \leqslant 4) \tag{114}$$

or, equivalently,

$$\gamma_{i+4,j} = \Lambda(\bar{z}^i z^j P_{\bar{z}^4}) = \bar{\beta}\gamma_{i+3,j} + \sum_{\bar{z}^l z^k \in \mathfrak{B}(2)} \overline{\beta_{lk}}\gamma_{i+k,j+l} \qquad (i+j \leqslant 4). \tag{115}$$

Hence, lemma 4.7 implies that $\bar{z}z^2 - P_{\bar{z}z^2}$ and $z^4 - P_{z^4}$ are two generating polynomials of $\gamma^{(8)}$.

Therefore, in $M(4)$, the columns $Z^4, \overline{Z}Z^3, \overline{Z}^2Z^2, \overline{Z}^3Z, \overline{Z}^4$ are a linear combination of the columns $\{\overline{Z}^iZ^j\}_{i+j\leqslant 3}$ and thus $M(4)$ is a flat extension of $M(3)$. Indeed, it suffices to observe that $P_{z^4}, P_{\bar{z}^4}, P_{\bar{z}z^2}, P_{\bar{z}^2z} \in V \equiv \mathrm{Vect}(Z^3, \overline{Z}^3, Z^2, \overline{Z}Z, \overline{Z}^2, \overline{Z}, Z, 1)$ and thus $zP_{\bar{z}z^2}, \bar{z}P_{\bar{z}^2z}, \bar{z}P_{\bar{z}z^2} \in V$; also one have, for all $i+j \leqslant 4$,

$$\langle M(4)Z^4, \overline{Z}^iZ^j\rangle = \langle M(4)P_{z^4}, \overline{Z}^iZ^j\rangle;$$
$$\langle M(4)\overline{Z}^4, \overline{Z}^iZ^j\rangle = \langle M(4)P_{\bar{z}^4}, \overline{Z}^iZ^j\rangle;$$
$$\langle M(4)\overline{Z}Z^3, \overline{Z}^iZ^j\rangle = \langle M(4)zP_{\bar{z}z^2}, \overline{Z}^iZ^j\rangle;$$
$$\langle M(4)\overline{Z}^2Z^2, \overline{Z}^iZ^j\rangle = \langle M(4)\bar{z}P_{\bar{z}z^2}, Z^iZ^j\rangle;$$
$$\text{and } \langle M(4)\overline{Z}^3Z, \overline{Z}^iZ^j\rangle = \langle M(4)\bar{z}P_{\bar{z}^2z}, \overline{Z}^iZ^j\rangle.$$

This finishes the proof of the theorem.

4.4 EXAMPLES

We give in this section three examples illustrating the different solved cases.

The case $a = e$ and $b = f$ We consider the quintic sequence,

$\gamma_{00} = 6$	$\gamma_{01} = 1+i$	$\gamma_{10} = 1-i$	$\gamma_{20} = -2i$
$\gamma_{11} = 6$	$\gamma_{02} = 2i$	$\gamma_{30} = -2-2i$	$\gamma_{21} = 2-2i$
$\gamma_{12} = 2+2i$	$\gamma_{03} = 2i-i$	$\gamma_{40} = 0$	$\gamma_{31} = -4i$
$\gamma_{22} = 8$	$\gamma_{13} = 4i$	$\gamma_{04} = 0\gamma_{50} =$	$-4+4i$
$\gamma_{41} = -4-4i$	$\gamma_{32} = 4-4i$	$\gamma_{23} = 4+4i$	$\gamma_{14} = 4i-4$
$\gamma_{05} = -4-4i.$			

then our matrices are

$$M(2) = \begin{pmatrix} 6 & 1+i & 1-i & 2i & 6 & -2i \\ 1-i & 6 & -2i & 2+2i & 2-2i & -2-2i \\ 1+i & 2i & 6 & -2+2i & 2+2i & 2-2i \\ -2i & 2-2i & -2-2i & 8 & -4i & 0 \\ 6 & 2+2i & 2-2i & 4i & 8 & -4i \\ 2i & -2+2i & 2+2i & 0 & 4i & 8 \end{pmatrix}$$

and

$$B = \begin{pmatrix} -2+2i & 2+2i & 2-2i & -2-2i \\ 4i & 8 & -4i & 0 \\ 0 & 4i & 8 & -4i \\ 4+4i & 4-4i & -4-4i & -4+4i \\ -4+4i & 4+4i & 4-4i & -4-4i \\ -4-4i & -4+4i & 4+4i & 4-4i \end{pmatrix}.$$

The fact that $M(2)$ is positive definite implies,

$$W = (M(2))^{-1}B = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ \frac{3}{4}+\frac{3i}{4} & \frac{1}{4}-\frac{i}{4} & -\frac{1}{4}-\frac{i}{4} & -\frac{3}{4}+\frac{3i}{4} \\ 0 & 0 & 0 & 0 \\ -\frac{3}{4}-\frac{3i}{4} & -\frac{1}{4}+\frac{i}{4} & \frac{1}{4}+\frac{i}{4} & \frac{3}{4}-\frac{3i}{4} \end{pmatrix}$$

and

$$W^*M(2)W = \begin{pmatrix} 12 & -8i & -4 & 8i \\ 8i & 12 & -8i & -4 \\ -4 & 8i & 12 & -8i \\ -8i & -4 & 8i & 12 \end{pmatrix}$$

Since $a = e = 12$ and $b = f = -8i$, according to the main theorem, our sequence is a moment matrix for a 6 atomes measure. In fact, from W, we can see that $Z^3 + \frac{3(1+i)}{4}(\bar{Z}^2 - Z^2) - \bar{Z}$ and $Z^2\bar{Z} + \frac{(1-i)}{4}(\bar{Z}^2 - Z^2) - Z$ are two generating polynomials for the moment sequence. The comment roots of the two polynomials are

$$\{\pm 1, \pm i, 0, 1+i\}$$

Finally we can see that $\mu = \delta_1 + \delta_{-1} + \delta_i + \delta_{-1} + \delta_0 + \delta_{1+i}$.

The case $a < e$ We consider the quintic sequence,

$$\begin{array}{llll}
\gamma_{00} = 7 & \gamma_{01} = 0 & \gamma_{10} = 0 & \gamma_{20} = -4i \\
\gamma_{11} = 8 & \gamma_{02} = 4i & \gamma_{30} = 0 & \gamma_{21} = 0 \\
\gamma_{12} = 0 & \gamma_{03} = 0 & \gamma_{40} = -4 & \gamma_{31} = -8i \\
\gamma_{22} = 12 & \gamma_{13} = 8i & \gamma_{04} = -4 & \gamma_{50} = 0 \\
\gamma_{41} = 0 & \gamma_{32} = 0 & \gamma_{23} = 4+4i & \gamma_{14} = 0 \\
\gamma_{05} = 0. & & &
\end{array}$$

then our matrices are

$$M(2) = \begin{pmatrix} 7 & 0 & 0 & 4i & 8 & -4i \\ 0 & 8 & -4i & 0 & 0 & 0 \\ 0 & 4i & 8 & 0 & 0 & 0 \\ -4i & 0 & 0 & 12 & -8i & -4 \\ 8 & 0 & 0 & 8i & 12 & -8i \\ 4i & 0 & 0 & -4 & 8i & 12 \end{pmatrix}$$

and

$$B = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 8i & 12 & -8i & -4 \\ -4 & 8i & 12 & -8i \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

Since $M(2)$ is positive semi-definite implies, we can take W as :

$$W = \begin{pmatrix} 0 & 0 & 0 & 0 \\ i & \frac{4}{3} & -\frac{i}{3} & 0 \\ 0 & \frac{i}{3} & \frac{4}{3} & -i \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

and

$$W^*M(2)W = \begin{pmatrix} 8 & -12i & -8 & 4i \\ 12i & \frac{56}{3} & -\frac{44i}{3} & -8 \\ -8 & \frac{44i}{3} & \frac{56}{3} & -12i \\ -4i & -8 & 12i & 8 \end{pmatrix}$$

Since $a = 8 < e = \frac{56}{3}$, according to the main theorem, our sequence is a moment matrix for a 7 atomes measure. In fact, examining (101) we can take $\gamma_{33} = 20$, and then $\gamma_{42} = -16i$, and Finally $\gamma_{06} = -16i$, then

$\bar{Z}Z^2 = \frac{i}{3}Z^3 + Z + \frac{i}{3}\bar{Z}$ is a generating polynomials and with $\gamma_{43} = 0$ we derive the second generating polynomial $Z^4 = \frac{3i}{2}Z^2 + \bar{Z}Z - \frac{3i}{2}\bar{Z}^2$. The comment roots of the two polynomials are

$$\{\pm 1, \pm i, 0, 1+i, -1-i\}$$

Finally we can see that

$$\mu = \delta_0 + \delta_1 + \delta_{-1} + \delta_i + \delta_{-i} + \delta_{1+i} + \delta_{-1-i}$$

The remaining case $a = e$ **and** $b \neq f$ We consider the quintic sequence,

$$\begin{array}{lll}
\gamma_{00} = 7 & \gamma_{01} = 9+9i & \gamma_{10} = 9-9i \\
\gamma_{20} = 22i & \gamma_{11} = 34 & \gamma_{02} = -22i \\
\gamma_{30} = -18+18i & \gamma_{21} = 58+58i & \gamma_{12} = 58-58i \\
\gamma_{03} = -18-18i & \gamma_{40} = 40 & \gamma_{31} = 164i \\
\gamma_{22} = 256 & \gamma_{13} = -164i & \gamma_{04} = 40 \\
\gamma_{50} = 304(1+i) & \gamma_{41} = 128(i-1) & \gamma_{32} = 480(i+1) \\
\gamma_{23} = 480(1-i) & \gamma_{14} = -128(1+i) & \gamma_{05} = 304(1-i).
\end{array}$$

then the matrix $M(2)$ is

$$\begin{pmatrix}
7 & 9+9i & 9-9i & 22i & 34 & -22i \\
9-9i & 34 & -22i & 58+58i & 58-58i & -18-18i \\
9+9i & 22i & 34 & -18+18i & 58+58i & 58-58i \\
-22i & 58-58i & -18-18i & 256 & -164i & 40 \\
34 & 58+58i & 58-58i & 164i & 256 & -164i \\
22i & -18+18i & 58+58i & 40 & 164i & 256
\end{pmatrix}$$

and

$$B = \begin{pmatrix}
-18+18i & 58+58i & 58-58i & -18-18i \\
164i & 256 & -164i & 40 \\
40 & 164i & 256 & -164i \\
480+480i & 480-480i & -128-128i & 304-304i \\
-128+128i & 480+480i & 480-480i & -128-128i \\
304+304i & -128+128i & 480+480i & 480-480i
\end{pmatrix}$$

Since $M(2)$ is not invertible. We can take for W the matrix,

$$\begin{pmatrix}
-\frac{1041}{319} + \frac{1041i}{319} & \frac{931}{319} + \frac{931i}{319} & -\frac{1041}{319} + \frac{1041i}{319} & \frac{3483}{319} + \frac{3483i}{319} \\
-\frac{1}{319}(2028i) & -\frac{2061}{319} & -\frac{1}{319}(752i) & -\frac{4613}{319} \\
-\frac{89}{319} & -\frac{1}{319}(1220i) & -\frac{89}{319} & -\frac{1}{319}(2496i) \\
\frac{39}{11} + \frac{39i}{11} & \frac{17}{11} - \frac{17i}{11} & \frac{17}{11} + \frac{17i}{11} & \frac{39}{11} - \frac{39i}{11} \\
\frac{24}{29} - \frac{24i}{29} & \frac{82}{29} + \frac{82i}{29} & \frac{82}{29} - \frac{82i}{29} & \frac{24}{29} + \frac{24i}{29} \\
0 & 0 & 0 & 0
\end{pmatrix}$$

So $W^*M(2)W$ is

$$\begin{pmatrix} \frac{719500}{319} & -\frac{1}{319}(448504i) & \frac{152956}{319} & -\frac{1}{319}(622040i) \\ \frac{448504i}{319} & \frac{719500}{319} & -\frac{1}{319}(449800i) & \frac{152956}{319} \\ \frac{152956}{319} & \frac{449800i}{319} & \frac{719500}{319} & -\frac{1}{319}(448504i) \\ \frac{622040i}{319} & \frac{152956}{319} & \frac{448504i}{319} & \frac{719500}{319} \end{pmatrix}$$

In this case we get $a = e = \frac{719500}{319}$ and $b = -\frac{448504i}{319} \neq f = \frac{449800i}{319}$, This is the remaining case which is not covered by the main theorem, but we can see that this sequence is a moment matrix for the 7 atomes measure:

$$\mu = \delta_1 + \delta_i + \delta_{1+i} + \delta_{1+2i} + \delta_{1+3i} + \delta_{2+i} + \delta_{3+i}$$

The case $a > e$ **and** $a - e < 2|b - f|$ Considering the quintic sequence,

$$\begin{aligned} &\gamma_{00} = 7 && \gamma_{01} = 1 + 5i && \gamma_{10} = 1 - 5i \\ &\gamma_{20} = -64 - 60i && \gamma_{11} = 230 && \gamma_{02} = -64 + 60i, \\ &\gamma_{30} = 277 - 161i && \gamma_{21} = -203 + 257i && \gamma_{12} = -203 - 257i \\ &\gamma_{03} = 277 + 161i && \gamma_{40} = 3722 + 4320i && \gamma_{31} = -4816 - 4200i \\ &\gamma_{22} = 10778 && \gamma_{13} = -4816 + 4200i && \gamma_{04} = 3722 - 4320i \end{aligned}$$

$$\begin{aligned} &\gamma_{50} = -59219 - 29695i && \gamma_{41} = 31021 - 11585i \\ &\gamma_{32} = -16979 + 24353i && \gamma_{23} = -16979 - 24353i \\ &\gamma_{14} = 31021 + 11585i && \gamma_{05} = -59219 + 29695i. \end{aligned}$$

then $M(2) =$

$$\begin{pmatrix} 7 & 1+5i & 1-5i & -64-60i & 230 & -64+60i \\ 1-5i & 230 & -64+60i & -203+257i & -203-257i & 277+161i \\ 1+5i & -64-60i & 230 & 277-161i & -203+257i & -203-257i \\ -64+60i & -203-257i & 277+161i & 10778 & -4816+4200i & 3722-4320i \\ 230 & -203+257i & -203-257i & -4816-4200i & 10778 & -4816+4200i \\ -64-60i & 277-161i & -203+257i & 3722+4320i & -4816-4200i & 10778 \end{pmatrix}$$

and

$$B = \begin{bmatrix} 277-161i & -203+257i & -203-257i & 277+161i \\ -4816-4200i & 10778 & -4816+4200i & 3722-4320i \\ 3722+4320i & -4816-4200i & 10778 & -4816+4200i \\ -16979+24353i & -16979-24353i & 31021+11585i & -59219+29695i \\ 31021-11585i & -16979+24353i & -16979-24353i & 31021+11585i \\ -59219-29695i & 31021-11585i & -16979+24353i & -16979-24353i \end{bmatrix}$$

The fact that $M(2)$ is positive definite allows to calculate W and $W^*M(2)W$. Which finally leads to the next expression of μ

$$\mu = \delta_2 + \delta_{-6} + \delta_{4i} + \delta_{-7i} + \delta_{3+3i} + \delta_{5-3i} + \delta_{-3+8i}$$

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Faculté des sciences

كلية العلوم

CENTRE D'ETUDES DOCTORALES - SCIENCES ET TECHNOLOGIES

Abstract:

In this thesis, we consider the problem of discrete moments that is the problem associated with discrete measures. First, we extend the idempotent approach from the truncated case to the full one. The main results obtained establish a bridge between the study of discrete measures and the existence of some basis in the Hilbert space associated with the moment problem.

On the other hand, we introduce the notion of k-positive matrices, we show that there is rank propagation phenomena associated with this family of matrices. A phenomenon already noticed by stampfli in his work on moment matrices and by Curto in the study of k-hyponormal shifts.

Finally, we will solve the problem of the quintic complex moments, by a new method, based on the notion of generator polynomials and a systematic study of the complex moment matrix.

Keywords: *Discrete moment problem - Bounded Shift space - Λ-multiplicative element - k-positive matrices - k-hyponormal operators - Quintic moment problem.*

Résumé :

Dans cette thèse, nous considérons le problème des moments discret, c'est-à-dire le problème associé aux mesures discrètes. Tout d'abord, nous étendons l'approche de résolution par l'usage des idempotents introduite dans le cas du problème tronqué au cas complet. Les résultats obtenus permettent d'établir un pont entre l'étude des mesures discrètes et l'existence de certaines bases dans l'espace de Hilbert associé au problème des moments.

D'autre part, nous introduisons la notion des matrices k-positives, nous démontrons par la suite qu'il existe un phénomène de propagation de rang associé à cette famille de matrices. Un phénomène déjà remarqué par Stamplfi dans le cas des matrices associées au problème des moments et dans des travaux sur l'étude des shifts k-hyponormaux.

Finalement, nous résolvons le problème des moments complexe quintic. Nous utilisons une méthode qui se base sur les polynômes générateurs et l'étude systématique de la matrice des moments complexe.

Mots-clés: *Problème des moments discret - Espace à shift bornée - Elément Λ-multiplicatif - Matrice k-positive - Operateur k-hyponormal - Problème des moments complexe quintic.*

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