ROYAUME DU MAROC
Université Mohammed V
- RABAT -
جامعة محمد الخامس
– الرباط –
Faculté des sciences
كلية العلوم

CENTRE D'ETUDES DOCTORALES - SCIENCES ET TECHNOLOGIES

N° d'ordre 3181

THESE

En vue de l'obtention du: ***DOCTORAT***

Centre de Recherche : Centre de Recherche de Mathématiques et Applications de Rabat
Structure de Recherche : Laboratoire de Mathématique, Statistique et Applications
Discipline : Mathématiques
Spécialité : Algèbre

Présentée et soutenue le 23/02/2019, par :

Karim MOUNIRH

Algèbres centrales simples et involutions de première espèce

JURY

Zine El Abedine ABDELALI	PES, Université Mohammed V- Rabat Faculté des Sciences	Président et rapporteur
Driss BENNIS	PH, Université Mohammed V- Rabat Faculté des Sciences	Directeur de thèse
Lhoussain EL FADIL	PH, Université Sidi Mohammed Ben Abdelleh, Faculté des Sciences, Fès	Rapporteur
Abdelhakim CHILLALI	PH, Université Sidi Mohammed Ben Abdelleh, Faculté Polydisciplinaire, Taza	Examinateur

Année Universitaire : 2018-2019

✉ Faculté des Sciences, avenue Ibn Battouta, BP. 1014 RP, Rabat –Maroc
☎ 00212(05) 37 77 18 76 📠 00212(05) 37 77 42 61 ; http://www.fsr.um5.ac.ma

DEDICATION

To my family and my friends

ACKNOWLEDGEMENTS

The present thesis work has been achieved in the laboratory of mathematics, Statistics and applications at the department of mathematics of the faculty of sciences in Rabat, under the supervisory of Professor Driss BENNIS.

I thank Professor Driss BENNIS for his help, his encouragement and for the fruitful discussions I had with him during the preparation of this thesis.

I also express my thanks to Professor Zine El Abedine ABDELALI, for accepting to be the president of the jury, for referring this work and also for the various discussions I had with him during some of my talks.

I am also very grateful to Professor Lhoussain EL FADIL for his continuous support and encouragement and for accepting to be a referee of this work and a member of the jury.

I want equally to express my deep feeling of gratitude to Professor Abdelhakim CHILLALI for his support and for accepting to be a member of the jury.

I thank also all my professors, my teachers, and all my friends.

Abstract

This thesis is a contribution to the development of gauge theory on simple algebras which was founded by J.-P. Tignol and A. R. Wadsworth. We generalize, in particular, in the first chapter the work of T. Hanke, D. Neftin and A. R. Wadsworth concerning necessary and sufficient residually conditions on a tame division algebra over a Henselian valued field, to be a crossed product. Also, we generalize the works of Amitsur-Tignol, of Morandi-Seuthuraman and a previous result of the author on the necessary and sufficient conditions for the existence of Kummer subfields in a tame valued division algebra. In the second chapter, we generalize a work of Chacron, Dherte, Tignol, Wadsworth and Yanchevskii and a previous work of the author on the discriminant of orthogonal involutions. We prove also new results concerning the discriminant of symplectic involutions. In the third chapter, we generalize a work of Sladek on the Witt ring of a tame valued division algebra and we show a cohomological interpretation of Milnor's K-groups of a graded field. In the last chapter, we study the torsion subgroup of the Whitehead group of a strongly tame valued division algebra and we generalize a result of Motiee on this subgroup.

Key-words: Simple algebras, valuations, graduations, gauges, involutions, Witt rings.

Résumé

Cette thèse est une contribution au développement de la théorie des jauges sur les algèbres simples, fondée par J.-P. Tignol et A. R. Wadsworth. En particulier, on généralise dans le premier chapitre le travail T. Hanke, D. Neftin et A. R. Wadsworth concernant les conditions résiduelles nécessaires et suffisantes pour qu'une algèbre à division modérée centrale sur un corps valué hensélien soit un produit croisé. On généralise aussi les travaux d'Amitsur-Tignol, de Morandi-Seuthuraman et un résultat précédent de l'auteur sur les conditions nécessaires et suffisantes pour l'existence de sous corps de Kummer dans une algèbre à division valuée modérée. Dans le second chapitre, on généralise le travail de Chacron, Dherte, Tignol, Wadsworth et Yanchevskii et aussi un travail précédent de l'auteur sur le discriminant des involutions orthogonales. On montre aussi des résultats nouveaux qui concernent le discriminant des involutions symplectiques. Ces résultats sont des conséquences d'une étude dtaillée qu'on développe sur les discriminants des involutions graduées de première espèce. Dans le troisième chapitre, on généralise le travail de Sladek sur l'anneau de Witt dune algèbre à division valuée modérée et on montre une interprétation cohomologique des K-groupes de Milnor d'un corps gradué. Dans le dernier chapitre, on étudie le sous groupe de torsion du groupe de Whitehead d'une algèbre à division valuée fortement modérée et on généralise un travail de Motiee sur ce sous groupe.

Mots-clefs : Algèbres simples, valuations, graduations, jauges, involutions, anneaux de Witt.

Notation

Claims inside the same chapter will be written without referring to the chapter's number, e.g., Theorem 1.2.12 will be written simply Theorem 2.12. Only, when referring to this Theorem outside the first chapter, we write it Theorem 1.2.12.

Contents

Introduction

Simple algebras are objects in algebra most close to fields. Indeed, by a famous theorem due to Wedderburn, they are matrix algebras with entries in skew fields. They appear in noncommutative ring theory as a well founded field of interest with many applications, especially in coding theory and in modern theoretical physics. Many problems concerning these simple algebras were solved by using valued division algebras. The first applications of valuation theory in this area, which served to settle important conjectures like the existence of division algebras with nontrivial SK_1 and the existence of noncrossed products, were obtained over some special valued base fields. Then, a more suitable approach to understand these 'noncommutative' valuations, consisted in working over Henselian valued fields, and in gathering information from valuation rings and value groups. Subsequently, division algebras over Henselian valued fields, were classified according to the nature of their residue algebras and ramification groups.

A class of great importance of such valued division algebras consists in tame division algebras which, as illustrated by Jacob and Wadsworth in [**JW90**], appeared in many classical works on simple algebras (see [**JW90**] where explicit examples are given from [**AS78**], [**S(a)78**], [**ART79**], [**R82**], [**T86**], [**T(b)87**], and [**TW87**]).

It is known that a graded division algebra can be constructed in a canonical way from any valued division algebra. Boulagouaz showed in [**B95**] that, over a Henselian valued field, this associated graded division algebra can be used to give a criterion for the tameness of the initial division algebras. This criterion gave then a powerful tool to connect the tame part of the Brauer group of a Henselian valued field to the Brauer group of the corresponding graded field (see [**HW(b)99**] or [**TW15**]). Moreover, the use of graded division algebras simplified significantly many proofs previously using valuations.

This connection between the valued and graded settings allowed us in a previous work to simplify the conditions needed for a tame division algebra over a Henselian valued field to be nicely semiramified (see [**M05**]). We used it also in [**M07**], after defining what we called nondegenerate semiramified valued and graded division algebras, to give examples of indecomposable division algebras and division algebras with special Galois subfields (see [**M07**, chapter 2], [**M08**] and [**MW11**]). It enabled us also to give a criterion which characterizes the existence of Kummer subfields in tame division algebras over Henselian fields (see [**M07**, chapter 3] or [**M10**]).

The above relationship between tame valued division algebras and graded division algebras, was also used by Hazrat, Wadsworth and Yanchevskii in the study of the SK_1 and the unitary SK_1

of graded and valued division algebras (see [**W10**], [**HaW(a)11**], [**HaW(b)11**] and [**WY12**]).

Many attempts, with limited contributions, were previously made by several authors in order to build a new and flexible concept on simple algebras compatible with valuations on division algebras (see e.g., [**Du82**], [**Du84**], [**Mor89**] and [**W89**]). Recently, Tignol and Wadsworth, inspired by the 'normes carrées' of Bruhat and Tits [**BT84**], defined what they called 'gauges' on central simple algebras over valued fields. This kind of 'extended valuations' allowed in particular to establish correspondences between graded simple algebras and simple algebras with tame gauges, which generalized the above relation between valued and graded division algebras (see [**TW10**] and [**TW15**]). These new correspondences served then to bridge the gap previously existing when passing from valued division algebras to Brauer equivalent simple algebras. In particular, they served in [**TW11**] and [**Ku11**] to study anisotropic and strongly anisotropic involutions on simple algebras. Also, they were used by the author in [**M11**] to study the discriminant invariant of orthogonal involutions on some simple algebras.

The present work is a continuation of this effort. We are mainly interested throughout this thesis by giving new applications of graded simple algebras in proving new facts on simple algebras with tame gauges and in computing the discriminant invariants of both orthogonal and symplectic involutions. All necessary definitions and notation to understand this work can be found in the preliminaries after this introduction.

In the first chapter we generalize the work of Hanke et al, in [**HNW16**]. Recall that the authors showed in [**HNW16**, Theorem 1.1] that if D is a tame finite-dimensional division algebra over a Henselian valued field E, then D is a crossed product if and only if the residue division algebra $\overline{D}$ has a maximal subfield which is Galois over $\overline{E}$. The first aim of this chapter is to prove that we have a more general result for any arbitrary central simple algebra over a Henselian valued field. In order to prove this result, we give in section 1 of this chapter many facts concerning graded simple algebras with simple 0-component. Especially, by using some new and refined arguments, we generalize [**HNW16**, Theorem 3.1] and show how many results in [**HNW16**, §2 and §3] can give more general versions. In section 2, we give many facts concerning the possibility of embedding a graded simple algebra in a graded matrix algebra and in section 3, we show that if E is a Henselian valued field with valuation v, D is a tame central division algebra over E, $B = M_n(D)$, where n is a positive integer and if there exists a defectless strictly maximal subfield K of B, then for any residually simple tame E-gauge β on B, denoting by w the unique extension of v to K, the corresponding (canonical) graded field G_wK is graded isomorphic to a graded subfield of $G_\beta B$ if and only if $\Gamma_K \subseteq \Gamma_D$, where Γ_K [resp., Γ_D] is the value group of K [resp., of D]. We use then a special residually simple tame gauge to deduce from the above facts our main result showing that the matrix algebra $M_n(\overline{D})$ has a strictly maximal subfield which is Galois [resp., abelian] over $\overline{E}$ if and only if $M_n(D)$ has a strictly maximal subfield K which is Galois [resp., abelian] and tame over E with $\Gamma_K \subseteq \Gamma_D$ (see Theorem 1.3.5).

We end this chapter by giving necessary and sufficient conditions for a simple algebra over a Henselian valued field (under some hypotheses) to have Kummer subfields (see Corollaries 1.4.8 and 1.4.9). These results generalize previous ones proved in [**TA85**], [**MS95**] and [**M10**].

The second chapter studies the discriminant invariants of both orthogonal and symplectic involutions. We start the first section of this chapter by given some general facts on graded involutions of the first kind. In particular, we prove that if F is a graded field of characteristic different from 2, A is a non-inertially split graded division algebra over F with A_0 noncommutative, then there exists a graded involution of the first kind σ on A such that $\sigma_{|A_0}$ is also of the first kind but σ and $\sigma_{|A_0}$ are of different types (see Remark 2.1.4). This shows that our previous result proved in [**M11**, Proposition 2.8] for the inertially split case, cannot be extended to the non-inertially split case. In the second section we prove that if E is a field with a valuation v, B is a central simple E-algebra with a tame v-gauge β, $G_\beta B$ is the graded GE-algebra associated to B by β, b is a nonzero element of B and b' is its (canonical) image in $(G_\beta B)_{\beta(b)}$ (see below in the preliminaries), then b' is invertible in $G_\beta B$ if and only if $v(\mathrm{Nrd}_B(b)) = \deg(B)\beta(b)$. Furthermore, when this occurs, we have $(\mathrm{Nrd}_B(b))' = \mathrm{Nrd}_{G_\beta B}(b')$ (in GE) (see Theorem 2.2.1).

The third section of this chapter studies the discriminants of orthogonal involutions on simple algebras over Henselian valued fields. This study generalizes previous ones made in [**CDTWY95**] and [**M11**]. In particular, we show that if E is a Henselian valued field with residue characteristic different from 2, D is a central division algebra of exponent 2 over E with $\deg(D) > 2$, $B = M_n(D)$, τ is an orthogonal involution on B, α is a residually simple tame E-gauge on B, invariant under τ, then the discriminant of τ can be expressed in terms of residue information.

In the forth section of this chapter, we define the discriminant of symplectic graded involutions on graded simple algebras. We prove that if F is a graded field of characteristic different from 2, D is a graded central division algebra over F with $\exp(D) = 2$ and $8 \leq |\ker(\theta_D)|^{\frac{1}{2}}$, then for any graded symplectic involutions σ and τ on D, the discriminant $\Delta_\sigma(\tau)$ equals zero (see Proposition 2.4.7). Also, we prove that if F is a graded field of characteristic different from 2, D is a graded central division algebra over F with $\exp(D) = 2$ and $|\ker(\theta_D)| > 4$, $A = M_n(D)$, and σ is a graded involution of symplectic type on A, then there is only a finite number of values for the discriminants $\Delta_\sigma(\tau)$, where τ describes all graded involutions of symplectic type on A (see Proposition 2.4.10). Consequently, it follows that if F is a graded field of characteristic different from 2, A is a graded central simple algebra over F with A_0 simple non-split, $\exp(A) = 2$, $|\ker(\theta_A)| > 4$, and σ is a graded involution of symplectic type on A, then there is only a finite number of values for the discriminants $\Delta_\sigma(\tau)$, where τ describes all graded involutions of symplectic type on A. If in addition $\frac{\deg(A)}{\operatorname{ind}(A)}$ is even, then for any such involution τ, we have $\Delta_\sigma(\tau) = 0$ (see Corollary 2.4.11). These results are then applied in the last section of this chapter to deduce information on the discriminants of involutions of symplectic type on central simple algebras over Henselian valued fields. In particular, we prove that if E is a Henselian valued field with residue characteristic different from 2, D is a central division algebra of exponent 2 over E such that $8 \leq |\ker(\theta_D)|^{\frac{1}{2}}$, and σ, τ are symplectic involutions on D, then $\Delta_\sigma(\tau) = 0$ (see Corollary 2.5.3). We prove also that if E is a Henselian valued field with residue characteristic different from 2, D is a central division algebra of exponent 2 over E with $|\ker(\theta_D)| > 4$, v is the extension of the valuation of E to D, $B = M_n(D)$, where n is an even positive integer, and β is the E-gauge on B defined by $\beta((d_{ij})_{1\leq i,j\leq r}) = \min\{v(d_{ij}) \mid 1 \leq i, j \leq n\}$, then for any symplectic involutions σ, τ on B, preserving β, we have $\Delta_\sigma(\tau) = 0$ (see Corollary 2.5.5).

We define and study in the third chapter the Witt ring of a graded simple algebra that we apply to give new facts for the Witt ring of a tame division algebra over a Henselian valued field. We also prove some results concerning Milnor's K-theory for graded fields.

In the first section of this chapter, we prove in particular that if G_E is the abstract Galois group of a Henselian valued field E, n is a (positive) prime power of an integer different from the residue characteristic of E, and $H^n(G_E, \mathbb{Z}/n\mathbb{Z})$ is the n^{th} (continuous) cohomology group of G_E on $\mathbb{Z}/n\mathbb{Z}$, then the graded ring $H^*(G_E, \mathbb{Z}/n\mathbb{Z}) := \oplus_{n\in\mathbb{N}} H^n(G_E, \mathbb{Z}/n\mathbb{Z})$ is graded isomorphic to the (corresponding) graded ring $H^*(G_F, \mathbb{Z}/n\mathbb{Z})$, where F is the graded field canonically associated to E (see the preliminaries below) and G_F is the Galois group over F of the tame (graded) closure of F (see Corollary 3.1.4).

In the second section, we define and study the (graded) Witt ring of an (arbitrary) simple algebra. We deduce from this study that if E is a Henselian valued field with residue characteristic different from 2, D is a tame central division algebra over E, $W(\overline{D})$ is the Witt ring of the residue division algebra $\overline{D}$ of D, J is the ideal of $W(\overline{D})$, generated by the (classes of) the elements $\langle 1\rangle - \langle r\rangle$, where $r \in \overline{D}^* \cap GD^{*2}$, and Z is a subgroup of D^*/D^{*2} mapped bijectively onto $\Gamma_D/2\Gamma_D$ by the induced map $\overline{v}$ of the valuation v of D, then $W(D)$ is isomorphic to a generalized crossed product $(W(\overline{D})/J) * Z$. If in addition $\overline{D}^* \cap GD^{*2} = \overline{D}^{*2}$, then we get a ring isomorphism $W(D) \cong W(\overline{D}) * Z$. In particular, if $D = E$, then $W(E) \cong W(\overline{E})[Z]$ (see Corollary 3.2.10).

We show in (3.2.11) that this last result generalizes (by means of different arguments) Sladek's Theorem [**S88**, Theorem 3.3] which he proved for the restrictive case where the division algebra D has a complete discrete valuation with residue characteristic different from 2 and where the residue division algebra of D is a field.

In section 3, for a graded field F, we relate Milnor's K-groups over F to the group $H^*(G_F, \mathbb{Z}/n\mathbb{Z})$ seen above (see Corollary 3.3.4).

The fourth chapter studies the torsion subgroup of the Whitehead group $\mathrm{K}_1(D)$ of a strongly tame division algebra D over a Henselian valued field.

For this purpose, given a graded division ring A, we define a new group $\mathrm{MK}_1(A)$ that we call the M-Whitehead group of A, and we prove some properties concerning it. We show in Theorem 4.2.7 that for a strongly tame central division algebra D over a Henselian valued field E with residue characteristic equal to the characteristic of E, the torsion subgroup of $\mathrm{K}_1(D)$, is isomorphic to the torsion subgroup of the M-Whitehead group of GD, where GD is the canonical graded division algebra associated to D (see the preliminaries). This result enables us to give a new proof for a (more general version) of a Theorem proved by Motie in [**Mo13**, Theorem 10] (see Corollary 4.2.8). We give also in this chapter a new proof for a result characterizing tame field extensions of Henselian fields, proved by Khanduja in [**Kh00**].

Many additional results, concerning especially generalized graded crossed products are given in the Appendix of this thesis.

Preliminaries

We recall below necessary definitions and results that we will use in this thesis.

We precise that all rings considered in this work are assumed to be associative with a unit and that all free modules, except other mention, are assumed to be finite-dimensional.

Let F be a ring and Γ a totally ordered abelian group. We say that F is a graded ring of type Γ if there are subgroups F_γ $(\gamma \in \Gamma)$ of F such that $F = \oplus_{\gamma\in\Gamma} F_\gamma$ and $F_\gamma F_\delta \subseteq F_{\gamma+\delta}$, for all γ, $\delta \in \Gamma$. In this case, the set $\Gamma_F = \{\gamma \in \Gamma \mid F_\gamma \neq \{0\}\}$ is called the support of F.

If F is a graded ring of type Γ and $x \in F_\gamma$ for some $\gamma \in \Gamma_F$, then we say that x is a homogeneous element of F. In particular, if x is a nonzero element of F_γ, we say that x has grade γ and we write $\mathrm{gr}(x) = \gamma$. We denote by F^h the set of homogeneous elements of F and by F^* the set of invertible homogeneous elements of F. We say that F is a graded field if F is a commutative graded ring and all nonzero homogeneous elements of F are invertible.

Let F be a graded field of type Γ and A be a (left) F-module such that $A = \oplus_{\gamma\in\Gamma} A_\gamma$, where A_γ are subgroups of A, and $F_\lambda A_\gamma \subseteq A_{\lambda+\gamma}$ for all $\lambda, \gamma \in \Gamma$, then we say that A is a graded (left) F-module (or a graded vector space over F). If in addition A is a ring and $A_\gamma A_\delta \subseteq A_{\gamma+\delta}$ for all $\gamma, \delta \in \Gamma$, then we say that A is a graded algebra over F. In this case, if I is an ideal of A such that $I = \oplus_{\gamma\in\Gamma}(I \cap A_\gamma)$, then we say that I is a graded ideal of A. Graded algebras over F [resp., commutative graded algebras over F] for which nonzero homogeneous elements are invertible are called graded division algebras over F [resp., graded field extensions of F].

If F is the center of a graded division algebra A, then A is called a graded central division algebra over F. Note that because Γ is totally ordered and all nonzero homogeneous elements of A are invertible, then A is a domain, so we can consider its quotient algebra (i.e., the algebra of central quotients of A) that we denote by $q(A)$. It is clear that $q(A)$ coincides with the quotient field of A when A is a graded field extension of F.

Let F be a graded field, A and B be two graded F-algebras (of the same type Γ), and $f : A \to B$ be an F-algebra homomorphism. We say that f is a graded F-algebra homomorphism if for any $\gamma \in \Gamma$, we have $f(A_\gamma) \subseteq B_\gamma$. If f is a bijective graded F-algebra homomorphism, then we say that f is a graded F-algebra isomorphism and we write $A \cong_g B$. If in addition $A = B$, we say that f is a graded F-algebra automorphism of A. If f and g are graded automorphisms of A, then we denote by fg (or $f \circ g$) the graded automorphism of A defined by $a \mapsto f(g(a))$ for all $a \in A$. We use the same terminology and notation when considering A and B only as graded modules over F. In such a case, these graded module homomorphisms are called graded homomorphisms of grade 0. For any $\lambda \in \Gamma$, a module homomorphism $f : A \to B$ is called a graded module homomorphism of grade λ if for any $\gamma \in \Gamma$, we have $f(A_\gamma) \subseteq B_{\gamma+\lambda}$.

If F is a graded field and A is a graded central algebra over F such that A has no graded two-sided ideals but $\{0\}$ and A, then we say that A is a graded central simple algebra over F. As in the ungraded case, there is a 'graded' Brauer group $\mathrm{GBr}(F)$ of F with respect to which every graded central simple algebra A over F is (graded-) Brauer-equivalent to a graded central division algebra over F, which is unique to a graded algebra isomorphism. For a graded central simple algebra A over F, we extend the above notation and we denote by $q(A)$ the central simple $q(F)$-algebra $A \otimes_F q(F)$.

We emphasize that the types of all our graded objects in this thesis are assumed to be totally ordered abelian groups.

Let D be a division ring and (as above) Γ be a totally ordered abelian group. Let ∞ be an element of a set strictly containing Γ with $\infty \notin \Gamma$; extend the order on Γ to $\Gamma \cup \{\infty\}$ by setting $\gamma < \infty$ for all $\gamma \in \Gamma$, and let $\gamma + \infty = \infty + \infty = \infty$. A map $v : D \to \Gamma \cup \{\infty\}$ is called a valuation on D if it satisfies the following conditions (for all $c, d \in D$) :

(0.1) $v(c) = \infty$ if and only if $c = 0$;

(0.2) $v(cd) = v(c) + v(d)$;

(0.3) $v(c + d) \geq \min\{v(c), v(d)\}$.

We recall that if F is a graded field and A is a graded division algebra over F, then to the graded structure of A corresponds a canonical valuation on $q(A)$, defined by $v(a) = \mathrm{gr}(a)$ for any $a \in A^*$ (see [**B(a)98**, §4], or [**HW(b)99**, §4, pp. 94-95]). We will usually denote the restriction of v to $q(F)$ also by v, and we will denote by $(q(F)^h, v^h)$ a henselization of $(q(F), v)$ (see [**E72**, §16]). We set $q(A)^h := q(A) \otimes_{q(F)} q(F)^h$.

Let E be a field and v be a valuation on E. Then, the filtration of E induced by v yields a canonical graded field GE. Namely, let $E^{\gamma} = \{x \in E \mid v(x) \geq \gamma\}$ and $E^{>\gamma} = \{x \in E \mid v(x) > \gamma\}$, then $E^{>\gamma}$ is a subgroup of the additive group E^{γ}. So, we can define the quotient group $GE_{\gamma} = E^{\gamma}/E^{>\gamma}$. For $x \in E\backslash\{0\}$, we denote by x' the element $x + E^{>v(x)}$ of $GE_{v(x)}$. One can easily see that the additive group $GE = \oplus_{\gamma\in\Gamma} GE_{\gamma}$ endowed with the multiplication law defined by $x'y' = (xy)'$ is a graded field.

In the same way, if D is a valued division algebra over a field E, then the filtration of D by the principal fractional ideals yields a graded division algebra GD over GE (see [**B(a)98**, §4], or [**HW(b)99**, §4]).

A valued division algebra D with valuation v over a field E is called defectless (over E) if $[D : E] = [\overline{D} : \overline{E}](\Gamma_D : \Gamma_E)$, where $[\overline{D} : \overline{E}]$ [resp., $(\Gamma_D : \Gamma_E)$] is the residue degree [resp., ramification index] of D over E. We say that D is totally ramified over E if $[D : E] = (\Gamma_D : \Gamma_E)$. It is called unramified [resp., inertial] over E if $[D : E] = [\overline{D} : \overline{E}]$ [resp., if D is unramified over E and $Z(\overline{D})$ is separable over $\overline{E}$]. If E is the center of D, then D is called semiramified (over E) if it is defectless over E, $\overline{D}$ is a field and $[\overline{D} : \overline{E}] = (\Gamma_D : \Gamma_E)$. Finally, if E is the center of D and there is an inertial field extension of E which splits D, then we say that D is inertially split.

Let D be a valued division algebra over a field E and let $\mathrm{Aut}(Z(\overline{D})/\overline{E})$ be the group of $\overline{E}$-automorphisms of $Z(\overline{D})$, then the map $\theta_D : \Gamma_D/\Gamma_E \to \mathrm{Aut}(Z(\overline{D})/\overline{E})$, defined by $\gamma + \Gamma_E \mapsto \theta_D(\gamma + \Gamma_E) : \overline{a} \mapsto \overline{dad^{-1}}$ (d being an arbitrary element of D^* such that $v(d) = \gamma$, where v is the valuation of D), is a surjective group homomorphism (see [**JW90**, Proposition 1.7]). We call it the canonical group epimorphism associated to D. If E is a Henselian valued field, D is defectless over E, $Z(\overline{D})$ is separable over $\overline{E}$ and the characteristic of $\overline{E}$ does not divide the cardinality of the kernel of θ_D, then we say that D is tame over E. We recall that the set $\mathrm{TBr}(E) = \{[D] \in \mathrm{Br}(E) \mid D \text{ is a tame central division algebra over } E\}$ is a subgroup of the Brauer group $\mathrm{Br}(E)$ of E; we call it the tame part of $\mathrm{Br}(E)$.

Let F be a graded field, D be a graded central division algebra over F and consider on $q(F)$ [resp., on $q(D)$] its canonical valuation (corresponding to the graded structure of F [resp., of D]), then we can identify F with $Gq(F)$ [resp., with $G(q(F)^h)$] and D with $Gq(D)$ [resp., with $G(q(D)^h)$] via the map defined by $x \mapsto x'$, for any homogeneous element x. Hence, by [**B95**, Corollary 4.4] $q(D)^h$ is tame over $q(F)^h$. As for the (tame) valuative case, for any graded central division algebra D over F, $Z(D_0)/F_0$ is an abelian field extension and the map $\theta_D : \Gamma_D/\Gamma_F \to \mathrm{Aut}(Z(D_0)/F_0)$, defined by $\mathrm{gr}(d) + \Gamma_F \mapsto \sigma : a \mapsto dad^{-1}$, for any $d \in D^*$ and $a \in Z(D_0)$, is a surjective group homomorphism (see [**B95**, Proposition 2.4] and note that for any $x \in F^*$, $d \in D^*$ and $a \in Z(D_0)$, we have

$(dx)a(dx)^{-1} = dad^{-1}$; or see [**HW(b)99**, (2.2), p.86 and Proposition 2.3, p.88]). In the same way as above, we say that D is totally ramified over F if $[D : F] = (\Gamma_D : \Gamma_F)$. We say that D is inertial over F if $[D : F] = [D_0 : F_0]$ and $Z(D_0)$ is separable over F_0, and we say that D is inertially split if there is an inertial graded field extension of F which splits D. We say that D is semiramified if D_0 is a field and $[D_0 : F_0] = (\Gamma_D : \Gamma_F)(= \deg(D))$. If A is a graded central simple algebra over the graded field F and Γ^*_A is the group $\mathrm{gr}(A^*)$, then there is a group homomorphism $\theta_A : \Gamma^*_A/\Gamma_F \to \mathrm{Aut}(Z(A_0)/F_0)$, defined by $\theta_A(\gamma + \Gamma_F) = \mathrm{Int}(a)_{|Z(A_0)}$ for any element $a \in A^*$ with $\mathrm{gr}(a) = \gamma$ (see [**TW15**, p, 67]).

Now, let (D, v) be a valued division ring and let M be a (right) D-vector space. A function $w : M \to \Gamma \cup \{\infty\}$ is called a D-value function (or a v-value function) if it satisfies the following conditions (for all $m, n \in M$ and $d \in D$) :

(0.4) $w(m) = \infty$ if and only if $m = 0$;

(0.5) $w(md) = w(m) + v(d)$;

(0.6) $w(m + n) \geq \min\{w(m), w(n)\}$.

In a similar way as in the construction of GD above, if w is a D-value function, then we associate to M a graded GD-module that we denote by G_wM (or just GM). In this case a base $(m_i)_{i=1}^n$ of M over D is called a splitting base if for any elements d_1,..., d_n of D, we have $w(\sum_{i=1}^n m_i d_i) = \min\{w(m_i) + v(d_i) \mid 1 \leq i \leq n\}$. If such a base exists, we say that w is a D-norm (or a v-norm). We recall that w is a D-norm if and only if $[GM : GD] = [M : D]$; furthermore, if this occurs, then $(m_i')_{i=1}^n$ is a base of GM over GD, where $m_i' = m_i + GM_{w(m_i)}$ (see [**RTW07**, Corollary 2.3] or [**TW10**, Proposition 1.1]). If M and N are two (right) D-vector spaces, α is a D-norm on M, β is a D-norm on N, and $f : M \to N$ is a homomorphism of D-vector spaces such that $\beta(f(m)) = \alpha(m)$ for all $m \in M$, then there is a graded homomorphism (of grade 0):

(0.7) $f' : GM \to GN$ such that $f'(m') = f(m)'$ for all $m \in M$ (see [**RTW07**, (2.10), p.112]).

Let (E, v) be a valued field and let A be an E-algebra. A function $\alpha : A \to \Gamma \cup \{\infty\}$, is called a surmultiplicative E-value function if it satisfies the following conditions (for all $a, b \in A$ and $e \in E$) :

(0.8) $\alpha(a) = \infty$ if and only if $a = 0$;

(0.9) $\alpha(1) = 0$;

(0.10) $\alpha(ea) = v(e) + \alpha(a)$;

(0.11) $\alpha(a + b) \geq \min\{\alpha(a), \alpha(b)\}$;

(0.12) $\alpha(ab) \geq \alpha(a) + \alpha(b)$.

We recall that if α is an E-norm on A with a splitting base $(a_i)_{i \in I}$, then condition (0.12) is equivalent to the fact that $\alpha(a_i a_j) \geq \alpha(a_i) + \alpha(a_j)$, for all $i, j \in I$ [**TW10**, Lemma 1.2].

If α is a surmultiplicative E-value function, then the graded GE-module GA is a graded GE-algebra for the multiplication law defined (for all nonzero elements a, b of A) by :

(0.13) $a'b' = (ab)'$ if $\alpha(ab) = \alpha(a) + \alpha(b)$ and $a'b' = 0$ otherwise (see [**TW10**, (1.5), p.691]).

If α is a surmultiplicative E-value function on A, then α is called an E-gauge if it is an E-norm and GA is a graded semisimple GE-algebra. If in addition $Z(GA) = G(Z(A))$ and $Z(GA)$ is separable over GE, then we say that α is a tame E-gauge.

Let E be a field with a valuation v, A be an E-algebra, α a surmultiplicative E-value function that is also an E-norm on A and f an E-algebra automorphism of A such that for any $a \in A$, $\alpha(f(a)) = \alpha(a)$, then it follows by (0.7) and (0.13) that f induces a graded GE-algebra automorphism $f' : GA \to GA$, such that $f'(a') = f(a)'$ for all $a \in A$.

We recall that if E is a field with a valuation v and D is a division algebra over E with a valuation w extending v, then by [**TW10**, Proposition 1.12] w is an E-gauge if and only if D (endowed with w) is defectless over E; furthermore when this holds, w is a tame v-gauge if and only if $Z(\overline{D})$ is separable over $\overline{E}$ and $\mathrm{char}(\overline{E})$ does not divide $|\ker(\theta_D)|$, where θ_D is the canonical epimorphism associated to D. In particular, if (E, v) is a Henselian valued field, then the unique extension of v to D is a tame E-gauge if and only if D is a tame division algebra over E.

Let A be a graded central simple algebra over a graded field F and σ be a graded involution of the first kind on A (i.e., an anti-automorphism of A satisfying $\sigma(A_\gamma) = A_\gamma$ for all $\gamma \in \Gamma$, $\sigma^2 = \mathrm{id}_A$ and $\sigma_{|F} = \mathrm{id}_F$), we recall that σ is called of orthogonal type, or simply orthogonal [resp., of symplectic type, or simply symplectic] if $q(\sigma)$ is so, where $q(\sigma) = \sigma \otimes \mathrm{id}_{q(F)}$ (defined on $q(A)$). As in the ungraded case, we have the following graded spaces:

$$\begin{aligned}
\mathrm{Sym}(A,\sigma) &= \{a \in A \mid \sigma(a) = a\}\\
\mathrm{Skew}(A,\sigma) &= \{a \in A \mid \sigma(a) = -a\}\\
\mathrm{Symd}(A,\sigma) &= \{a + \sigma(a) \mid a \in A\}\\
\mathrm{Alt}(A,\sigma) &= \{a - \sigma(a) \mid a \in A\}.
\end{aligned}$$

One can easily see that $\mathrm{Sym}(q(A), q(\sigma)) = \mathrm{Sym}(A,\sigma) \otimes_F q(F)$ and $\mathrm{Skew}(q(A), q(\sigma)) = \mathrm{Skew}(A,\sigma) \otimes_F q(F)$. Indeed it is clear that $\mathrm{Sym}(A,\sigma) \otimes_F q(F) \subseteq \mathrm{Sym}(q(A), q(\sigma))$. Conversely, let $a \in \mathrm{Sym}(q(A), q(\sigma))$ and write $a = b \otimes s^{-1}$, where $b \in A$ and $s \in F \backslash \{0\}$, then $0 = \sigma(a) - a = (\sigma(b) - b) \otimes s^{-1}$, so $(\sigma(b) - b) \otimes 1 = 0$. Since A is simple, then the map $A \to q(A)$ defined by $x \mapsto x \otimes 1$ is injective, hence $\sigma(b) = b$. It follows that $a \in \mathrm{Sym}(A,\sigma) \otimes_F q(F)$. In the same way we show that $\mathrm{Skew}(q(A), q(\sigma)) = \mathrm{Skew}(A,\sigma) \otimes_F q(F)$.

If $\mathrm{char}(F) \neq 2$, then obviously $\mathrm{Sym}(A,\sigma) = \mathrm{Symd}(A,\sigma)$ and $\mathrm{Skew}(A,\sigma) = \mathrm{Alt}(A,\sigma)$. Moreover, since $\mathrm{Sym}(q(A), q(\sigma)) = \mathrm{Sym}(A, \sigma) \otimes_F q(F)$ and $\mathrm{Skew}(q(A), q(\sigma)) = \mathrm{Skew}(A,\sigma) \otimes_F q(F)$, then by [**KMRT98**, Proposition 2.6, pp.16-17] -applied to $q(\sigma)$- σ is of orthogonal type if $[\mathrm{Sym}(A,\sigma) : F] = \frac{1}{2}n(n+1)$ (or equivalently $[\mathrm{Skew}(A,\sigma) : F] = \frac{1}{2}n(n-1)$), where $n = \deg(A)(= \deg(q(A)))$.

One can easily see that $\mathrm{Alt}(q(A), q(\sigma)) = \mathrm{Alt}(A,\sigma) \otimes_F q(F)$. We claim that $\mathrm{Alt}(q(A), q(\sigma)) \cap A = \mathrm{Alt}(A,\sigma)$. Indeed, it is clear that $\mathrm{Alt}(A,\sigma) \subseteq \mathrm{Alt}(q(A), q(\sigma)) \cap A$. Conversely, let $a \in \mathrm{Alt}(q(A), q(\sigma)) \cap A$, then there is some $x \in q(A)$ such that $a = x - q(\sigma)(x)$. Write $x = s \otimes t^{-1}$ for some $s \in A$ and $t \in F \backslash \{0\}$, then $ta = s -$

$\sigma(s) = \sum_\gamma (s_\gamma - \sigma(s_\gamma))$, where s_γ are the homogeneous components of s. Consider the homogeneous components of maximal grade t_m and a_m of t and a, respectively, then $t_m a_m = s_\gamma - \sigma(s_\gamma)$ for some γ, hence $a_m = (t_m)^{-1}s_\gamma - \sigma((t_m)^{-1}s_\gamma) \in \mathrm{Alt}(A, \sigma)$. Thus, we have $a - a_m \in \mathrm{Alt}(q(A), q(\sigma)) \cap A$, so by repeating this procedure, we conclude that $a \in \mathrm{Alt}(A, \sigma)$.

Now, we will prove the graded version of [**KMRT98**, Proposition 2.7, p.17]. This will be used in the second chapter.

Proposition 0.14. *Let A be a graded central simple algebra over a graded field F and let σ be a graded involution of the first kind on A. Then we have the following statements:*

1. *For each invertible homogeneous element u of A such that $\sigma(u) = \pm u$, the map $\mathrm{Int}(u) \circ \sigma$ (defined by $\mathrm{Int}(u) \circ \sigma(a) = u\sigma(a)u^{-1}$, for every $a \in A$) is a graded involution of the first kind on A.*
2. *Conversely, for every graded involution τ of the first kind on A, there exists some invertible homogeneous element u of A, uniquely determined up to a factor in F^*, such that $\tau = \mathrm{Int}(u) \circ \sigma$ and $\sigma(u) = \pm u$.*

   We then have:

$$\mathrm{Sym}(A,\tau) = \begin{cases} u.\mathrm{Sym}(A,\sigma) = \mathrm{Sym}(A,\sigma).u^{-1} & \text{if } \sigma(u) = u \\ u.\mathrm{Skew}(A,\sigma) = \mathrm{Skew}(A,\sigma).u^{-1} & \text{if } \sigma(u) = -u \end{cases}$$

and

$$\mathrm{Skew}(A,\tau) = \begin{cases} u.\mathrm{Skew}(A,\sigma) = \mathrm{Skew}(A,\sigma).u^{-1} & \text{if } \sigma(u) = u \\ u.\mathrm{Sym}(A,\sigma) = \mathrm{Sym}(A,\sigma).u^{-1} & \text{if } \sigma(u) = -u \end{cases}$$

If $\sigma(u) = u$, *then* $\mathrm{Alt}(A,\tau) = u.\mathrm{Alt}(A,\sigma) = \mathrm{Alt}(A,\sigma).u^{-1}$.

(3) *Suppose that* $\tau = \mathrm{Int}(u) \circ \sigma$, *where* u *is a homogeneous invertible element of* A *such that* $\sigma(u) = \pm u$. *If* $\mathrm{char}(F) \neq 2$, *then* σ *and* τ *are of the same type if and only if* $\sigma(u) = u$. *If* $\mathrm{char}(F) = 2$, *the involution* τ *is symplectic if and only if* $u \in \mathrm{Alt}(A,\sigma)$.

Proof. (1) It is clear that for any $\gamma \in \Gamma_A$, we have $\mathrm{Int}(u) \circ \sigma(A_\gamma) \subseteq A_\gamma$. The rest follows easily as in the ungraded case. This can also be shown by noticing that $\mathrm{Int}(u) \circ \sigma$ is the restriction of $\mathrm{Int}(u) \circ q(\sigma)$ which is an involution on $q(A)$ by [**KMRT98**, Proposition 2.7(1), p.17].

(2) Since $\tau \circ \sigma$ is a graded F-automorphism of A, then by the graded version of Skolem-Noether theorem (see [**TW15**, Theorem 2.37]), there exists an invertible homogeneous element u of A such that $\tau \circ \sigma = \mathrm{Int}(u)$, hence $\tau = \mathrm{Int}(u) \circ \sigma$. We have $\tau^2 = \mathrm{Int}(u\sigma(u)^{-1})$, hence $\sigma(u) = \lambda u$, for some $\lambda \in F^*$. If we apply σ again to both sides of this equality, we get $u = \lambda^2 u$, hence $\lambda = \pm 1$. We have $q(\tau) = \mathrm{Int}(u) \circ q(\sigma)$, hence by [**KMRT98**, Proposition 2.7, p.17] $\mathrm{Sym}(q(A), q(\tau)) = u.\mathrm{Sym}(q(A), q(\sigma)) = \mathrm{Sym}(q(A), q(\sigma)).\, u^{-1}$ [resp., $\mathrm{Sym}(q(A), q(\tau)) = u.\mathrm{Skew}(q(A), q(\sigma)) = \mathrm{Skew}(q(A), q(\sigma)).\, u^{-1}$] if $\sigma(u) = u$ [resp., if $\sigma(u) = -u$]. Also, $\mathrm{Skew}(q(A), q(\tau)) = u.\mathrm{Skew}(q(A), q(\sigma)) = \mathrm{Skew}(q(A), q(\sigma)).u^{-1}$ [resp., $\mathrm{Skew}(q(A), q(\tau)) = u.\mathrm{Sym}(q(A), q(\sigma)) = \mathrm{Sym}(q(A), q(\sigma)).u^{-1}$] if $\sigma(u) = u$ [resp., if $\sigma(u) = -u$]. Moreover, $\mathrm{Alt}(q(A), q(\tau)) = u.\mathrm{Alt}(q(A), q(\sigma)) = \mathrm{Alt}(q(A), q(\sigma)).u^{-1}$ if $\sigma(u) = u$. We can easily see for example that

$u.\mathrm{Sym}(A,\sigma) \subseteq \mathrm{Sym}(A,\tau)$ if $\sigma(u) = u$, so by comparing dimensions we get $u.\mathrm{Sym}(A,\sigma) = \mathrm{Sym}(A,\tau)$. Indeed, we have previously seen that $\mathrm{Sym}(A,\sigma) \otimes_F q(F) = \mathrm{Sym}(q(A), q(\sigma))$ and $\mathrm{Sym}(A,\tau) \otimes_F q(F) = \mathrm{Sym}(q(A), q(\tau))$, so $[\mathrm{Sym}(A,\tau) : F] = [\mathrm{Sym}(q(A), q(\tau)) : q(F)] = [u.\mathrm{Sym}(q(A), q(\sigma)) : q(F)] = [\mathrm{Sym}(q(A), q(\sigma)) : q(F)] = [\mathrm{Sym}(A,\sigma) : F] = [u.\mathrm{Sym}(A,\sigma) : F]$. Hence, $u.\mathrm{Sym}(A,\sigma) = \mathrm{Sym}(A,\tau)$. The other equalities follow in the same way.

(3) We recall that by definition σ [resp., τ] is orthogonal if $q(\sigma)$ [resp., $q(\tau)$] is so; moreover we have $\mathrm{Alt}(A,\sigma) \subseteq \mathrm{Alt}(q(A), q(\sigma))$, and we have previously seen that $\mathrm{Alt}(q(A), q(\sigma)) \cap A = \mathrm{Alt}(A,\sigma)$, therefore the statements here follow by [**KMRT98**, Proposition 2.7, p.17].

□

Let A be a graded central simple algebra over a graded field F and suppose that there exists a graded involution σ on A. We will say that σ is hyperbolic if the involution $\sigma \otimes_K \mathrm{id}_{q(F)}$, where K is the graded subfield of F elementwise invariant under σ, is a hyperbolic involution on $A \otimes_K q(F)$. Note that if σ is of the first kind on A, then this amounts to say that $q(\sigma)$ is a hyperbolic involution on $q(A)$.

Let M be a field and C a (central simple) algebra over M. We denote by C^* the group of invertible elements of C and by $\mathrm{Aut}(C)$ the group of ring automorphisms of C. For any $c \in C^*$, we denote by $\mathrm{Int}(c)$ the ring automorphism of A defined by $a \mapsto cac^{-1}$. Let H be a finite group that acts by automorphisms on M and let

$\omega : H \to \mathrm{Aut}(C)$ and $f : H \times H \to C^*$ be two maps. For any $\sigma \in H$, we denote by ω_σ the image of σ under ω and we write $\omega_\sigma\omega_\tau$ for the composition map defined by $\omega_\sigma\omega_\tau(a) = \omega_\sigma(\omega_\tau(a))$, for any $a \in C$. We say that (ω, f) is a factor set of H in C if the following conditions are satisfied :

(0.15) $\omega_\sigma(a) = \sigma(a)$ for all $a \in M$ and $\sigma \in H$,

(0.16) $\omega_\sigma\omega_\tau = \mathrm{Int}(f(\sigma, \tau))\omega_{\sigma\tau}$ for all $\sigma, \tau \in H$,

(0.17) $f(\sigma, \tau)f(\sigma\tau, \mu) = \omega_\sigma(f(\tau, \mu))f(\sigma, \tau\mu)$ for all $\sigma, \tau, \mu \in H$.

If (ω, f) is a factor set of H in A, then the generalized crossed product associated to (ω, f) is defined to be the algebra $(C, H, (\omega, f))$ $= \oplus_{\sigma\in H} Cx_\sigma$, where x_σ are independent indeterminates on C satisfying the following multiplicative conditions (for all $\sigma \in H$ and $a \in C$):

(0.18) $x_\sigma a = \omega_\sigma(a)x_\sigma$, and

(0.19) $x_\sigma x_\tau = f(\sigma, \tau)x_{\sigma\tau}$.

Let (ω, f) and (β, h) be two factor sets of H in C. We say that (ω, f) and (β, h) are cohomologous if there is a family $(a_\sigma)_{\sigma\in H}$ of elements of C^* such that for all $\sigma, \tau \in H$, $\beta_\sigma = \mathrm{Int}(a_\sigma)\omega_\sigma$ and $h(\sigma, \tau) = a_\sigma\omega_\sigma(a_\tau)f(\sigma, \tau)a_{\sigma\tau}^{-1}$. We write in this case $(\omega, f) \sim (\beta, h)$. The relation $\sim$ is an equivalence relation on the set of factor sets of H in C. We denote the set of equivalence classes by $\mathbf{H}(H, C^*)$. If $C = M$ is a Galois field extension of some field E and $H = \mathrm{Gal}(M/E)$, then $\mathbf{H}(H, C^*)$ is the second Galois cohomology group $H^2(H, M^*)$.

Now, let M be a graded field, A be a graded (central simple) algebra over M, A^* the group of invertible homogeneous elements of A, H a finite group that acts on M by graded automorphisms, and $\mathrm{Aut}(A)$ the group of graded ring automorphisms of A. In the same way as above, if $\omega : H \to \mathrm{Aut(A)}$ and $f : H \times H \to A^*$ are two maps that satisfy the conditions (0.15) to (0.17) above, we say that (ω, f) is a graded factor set of H in A. The corresponding generalized graded crossed product $(A, H, (\omega, f))$ is defined also in the same way : $(A, H, (\omega, f)) = \oplus_{\sigma \in H} A x_\sigma$, where x_σ are independent indeterminates on A satisfying the multiplicative conditions : $x_\sigma a = \omega_\sigma(a) x_\sigma$ and $x_\sigma x_\tau = f(\sigma, \tau) x_{\sigma\tau}$ for all $a \in A$ and $\sigma, \tau \in H$. We recall that if A_0 is simple, then there is a unique graded algebra structure on $(A, H, (\omega, f))$ which extends the grading of A and for which x_σ are homogeneous elements (see [**M11**, Lemma 2.4]).

CHAPTER 1

On crossed product algebras over Henselian fields and Kummer subfields

In [**HNW16**, Theorem 1.1] the authors considered a tame finite-dimensional division algebra D over a Henselian valued field E and showed that D is a crossed product if and only if the residue division algebra $\overline{D}$ has a maximal subfield which is Galois over $\overline{E}$. This result generalized a previous one proved by Hanke in [**H01**] for the restrictive case of an inertially split division algebra over a Henselian valued field. A natural question that arises is to see whether this criterion can be generalized to the more general case of a tame central simple algebra over E, i.e., a central simple algebra over E, which is split by a tame field extension of E. Such a central simple algebra is isomorphically equal to a matrix algebra $M_n(D)$, where D is a tame central division algebra over E and n is a positive integer. In the present chapter, we answer this question and show that $M_n(\overline{D})$ has a strictly maximal subfield which is Galois [resp., abelian] over $\overline{E}$ if and only if $M_n(D)$ has a strictly maximal subfield K which is Galois [resp., abelian] and tame over E with $\Gamma_K \subseteq \Gamma_D$ (see Theorem 3.5). Here a strictly maximal subfield of a central simple algebra B (over E) means a subfield M of B such that $[M : E] = \deg(B)$.

In their approach to prove [**HNW16**, Theorem 1.1], Hanke et al., proved at first an analogous criterion for graded central division algebras over graded fields (see [**HNW16**, Theorem 3.1]), then they deduced their result for tame division algebras using a canonical correspondence relating tame division algebras to graded division algebras. This technique is often used when one aims to show some property on a tame division algebra, since working in the graded setting is more easier than in the valuative one (see e.g., [**M05**], [**M08**], [**M10**], [**MW11**], [**HaW(a)11**], [**HaW(b)11**], [**W10**] and [**WY12**]).

In our work, it was natural to think to use the recent theory of gauges on simple algebras which generalizes that of valuations on division algebras (see [**TW10**] and [**TW15**]). Indeed, a tame gauge β on a tame central simple algebra B over E offers the possibility to work at first with the graded central simple algebra $G_\beta B$, which is canonically associated to B (see the preliminaries), then translate, if possible, the obtained results to analogous ones for B. But, unlike the case of a tame division algebra, a first difficulty here consists in the fact that the existence of non-invertible homogeneous nonzero elements in an arbitrary graded simple algebra, makes it very difficult to generalize versions of key results in [**HNW16**]. A second obstacle remains in the fact that the embedding of a (defectless) field extension K of E in B, is not necessarily equivalent to the embedding of the associated graded field $G_w K$ in the graded algebra $G_\beta B$, where w is the extension of the valuation of E to K. Indeed, there are examples of (strictly maximal) subfields K of B, such that G_wK is not graded isomorphic to a graded subfield of $G_\beta B$ (see Remark 3.3). To overcome the first difficulty, we consider in (3.1) below a special tame gauge β on B for which the zero-component $(G_\beta B)_0$ is simple. This choice is important because for a graded central simple algebra A over a graded field F with A_0 simple, we show in section 1 (see (1.2)) that we have canonical graded subalgebras of A analogous to the canonical ones given in [**TW15**, 8.2.1, pp.388-389]. These canonical subalgebras enable us, after some intermediate results, to show that if F is a graded field and A is a graded central simple algebra over F with A_0 simple, then A_0 has a strictly maximal subfield which is abelian [resp., Galois] over F_0 if and only if A has an abelian [resp., a Galois] strictly maximal graded subfield (see Theorem 1.12). This generalizes [**HNW16**, Theorem 3.1]. We also show by using some refined arguments, that many results in [**HNW16**, §2 and §3] can be obtained for an arbitrary graded central simple algebra A when A_0 is simple.

For the second obstacle concerning the possibility of embedding a graded field in a graded central simple algebra, we show in the second section that if F is a graded field, A is a graded central simple algebra over F with A_0 simple and L is a finite-dimensional graded field extension of F which splits A with $[L : F] = \deg(A)$, then L is graded isomorphic to a (strictly maximal) graded subfield of A if and only if $\Gamma_L \subseteq \Gamma_A$. This result allows in the third section to deduce that if E is a Henselian valued field with valuation v, D

is a tame central division algebra over E, $B = M_n(D)$, where n is a positive integer and if there exists a defectless strictly maximal subfield K of B, then for any residually simple tame E-gauge β on B, denoting by w the unique extension of v to K, G_wK is graded isomorphic to a graded subfield of $G_\beta B$ if and only if $\Gamma_K \subseteq \Gamma_D$. We apply then the results concerning graded central simple algebras with 0-component developed in the first section to prove our main result which generalizes [**HNW16**, Theorem 1.1] (see Theorem 3.5).

We end this chapter by giving necessary and sufficient conditions for a simple algebra over a Henselian valued field (under some hypotheses) to have Kummer subfields (see Corollaries 4.8 and 4.9). These results generalize previous ones proved in [**TA85**], [**MS95**] and [**M10**].

1. On graded simple algebras with simple 0-component

As said above in the introduction, for a graded central simple algebra A over a graded field F with A_0 simple, we show in (1.2) below that we have some canonical graded subalgebras of A analogous to the ones given in [**TW15**, 8.2.1 pp.388-389]. These graded subalgebras will play an important role throughout the rest of this section. Our aim here is to prove Theorem 1.12 which generalizes [**HNW16**, Theorem 3.1].

(1.1) Let F be a graded field and A be a (finite-dimensional) graded central simple algebra over F. We recall that A_0 is simple if and only if $A \cong_g M_n(D)$, where D is a graded central division algebra over F and $M_n(D)$ is given the standard (unshifted) grading built from the grading of D. This is also equivalent to one of the following conditions: (i) $\Gamma_A = \Gamma_D$, (ii) $[A_0 : D_0] = [A : D]$, (iii) $Z(A_0)$ is a field. When this is the case, we have $A_0 \cong M_n(D_0)$, $\Gamma_A^* = \Gamma_A = \Gamma_D$, where $\Gamma_A^* = \mathrm{gr}(A^*)$, and $\theta_A = \theta_D$. We recall also that in this case, we have $[A : F] = [A_0 : F_0](\Gamma_A : \Gamma_F)$ (see [**TW15**, Proposition 2.47]).

(1.2) Let A be a graded central simple algebra over a graded field F with A_0 simple, then as for graded division algebras (see [**TW15**, 8.2.1, pp.388-389]), there are some graded F-subalgebras of A canonically determined by A_0. Namely, let $I_A = A_0.F \cong_g A_0 \otimes_{F_0} F$, $Z_A = Z(I_A) = Z(A_0).F \cong_g Z(A_0) \otimes_{F_0} F$, $V_A = C_A(Z_A)$ (i.e., the centralizer of Z_A in A), and $T_A = C_{V_A}(I_A) = C_A(I_A)$ (equality follows from the fact that $C_A(I_A) \subseteq C_A(Z_A) = V_A$). Let D be a graded central division algebra over F such that $A \cong M_n(D)$ (see (1.1) above). We have $A_0 \cong M_n(D_0)$, so $Z(A_0) \cong Z(D_0)$. One can then see that I_A, V_A and T_A can be read off from the corresponding canonical graded subalgebras I_D, V_D and T_D of D. Namely, we have $I_A = M_n(I_D)$, $Z_A \cong_g Z_D$, $V_A = M_n(V_D)$ and $T_A \cong_g T_D$, hence T_A is a totally ramified graded division algebra over Z_A (see [**TW15**, 8.2.1, pp.388-389]). It follows also from [**TW15**, 8.2.1] that $V_A \cong_g I_A \otimes_{Z_A} T_A$. This last graded isomorphism follows also easily by applying the graded version of the double centralizer theorem (see [**TW15**, Theorem 2.35]) since we have $T_A \cong_g C_{V_A}(I_A)$, and the fact that T_A is a graded division algebra (totally ramified) over Z_A follows from [**TW15**, Corollary 2.43] since $(T_A)_0 = (Z_A)_0$ (for $(T_A)_0$ centralizes $A_0(= (I_A)_0)$.

Remark 1.3. With the notation of (1.2), note that $Z(A_0)$ is an abelian field extension of F_0 (see [**TW15**, Proposition 2.48]). Moreover, one can see that $\mathrm{Gal}(Z(A_0)/F_0) \cong \Gamma_A/\Gamma_{T_A}$. This isomorphism also follows by [**TW15**, Proposition 2.40] since we have $Z(A_0) \cong Z(D_0)$, $\Gamma_A = \Gamma_D$ and $T_A \cong_g T_D$ (note that with the notation in [**TW15**, Proposition 2.40] we have $\ker(\theta_D) = \Gamma_{T_D}$, hence $\mathrm{im}(\theta_D) \cong \Gamma_D/\Gamma_{T_D}$). As seen in (1.1), we have $[A : F] = [A_0 : F_0](\Gamma_A : \Gamma_F)$, so we deduce that $[A : F] = [A_0 : Z(A_0)][Z(A_0) : F_0]^2(\Gamma_{T_A} : \Gamma_F)$, hence $\deg(A) = \deg(A_0)[Z(A_0) : F_0]\deg(T_A)$ (since T_A is a totally ramified graded central division algebra over Z_A and $\Gamma_{Z_A} = \Gamma_F$).

(1.4) Let A_1 and A_2 be two graded division algebras over a graded field F with $(A_1)_0 \otimes_{F_0} (A_2)_0$ a division algebra and $\Gamma_{A_1} \cap \Gamma_{A_2} = \Gamma_F$, then by [**TW15**, Proposition 2.12] $(A_1 \otimes_F A_2)_0 \cong (A_1)_0 \otimes_{F_0} (A_2)_0$ and $\Gamma_{A_1 \otimes_F A_2} = \Gamma_{A_1} + \Gamma_{A_2}$ (since $\{0\}$ is a set of representatives for the cosets of $\Gamma_{A_1} \cap \Gamma_{A_2}$ modulo Γ_F).

(1.5) Let F be a graded field (with support Γ_F), $q(F)$ be its quotient field, $q(F)_{alg}$ be an algebraic closure of $q(F)$, and let $\Delta_F(= \Gamma_F \otimes_{\mathbb{Z}} \mathbb{Q})$ be the divisible hull of Γ_F. We recall that for any $\lambda \in \Delta_F$, there is a unique grading on $F[X]$, of type Δ_F, which extends the grading of F and for which X is homogeneous of grade λ. We denote $F[X]$ with this grading by $F[X]^{(\lambda)}$. A polynomial $f \in F[X]$ is called homogenizable if there is some $\lambda \in \Delta_F$, such that f is homogeneous in $F[X]^{(\lambda)}$. Let K be a graded field extension of F. We recall that K is called normal over F if any homogenizable irreducible polynomial $f \in F[X]$ which has a root in K factors into polynomials of degree one in $K[X]$. We recall also that when this occurs, every root of f is homogeneous in K (see [**MW11**, p.467] or [**TW15**, Proposition 5.7]).

LEMMA 1.6. *Let K/F be a finite-dimensional graded field extension, then the following statements hold:*

(1) *If K/F is normal, then K_0/F_0 is normal. The converse holds if K/F is unramified.*
(2) *Suppose that K/F is normal and let W be a graded field extension of $K_0.F$ in K, then W/F is normal.*

PROOF. (1) Suppose that K/F is normal, and let $x \in K_0$. Let f be the minimal polynomial of x over $q(F)$ and write $f = \sum_{i=0}^{n} a_i X^i$, then by [**HW(a)99**, Corollary 2.4] (or [**TW15**, Proposition 5.7]) $a_i \in F_0$ for all i $(1 \leq i \leq n)$; more precisely f is the minimal polynomial of x over F_0. If y is a root of f, then $y \in K$ (because K/F is normal); moreover by [**HW(a)99**, Corollary 2.4] (or [**TW15**, Proposition 5.7]) $\mathrm{gr}(y) = \mathrm{gr}(x) = 0$, hence $y \in K_0$. Thus, K_0/F_0 is normal.

Conversely, suppose that K/F is unramified and that K_0/F_0 is normal, and let $\sigma : q(K) \to q(K)_{alg}$ be a $q(F)$-embedding. Let

$x \in K^*$, $e \in F^*$ with $\mathrm{gr}(e) = \mathrm{gr}(x)$ and let $y = xe^{-1} (\in K_0)$, then $\sigma(y) \in K_0$ (because $\sigma_{|K_0} : K_0 \to q(K)_{alg}$ is an F_0-embedding and K_0/F_0 is normal), so $\sigma(x) = \sigma(y)e \in K$, hence $\sigma(K) \subseteq K$. Thus $\sigma(q(K)) = \sigma(K.q(F)) \subseteq K.q(F) = q(K)$, which shows that $q(K)/q(F)$ is normal. Therefore, by [**TW15**, Proposition 5.28], K/F is normal.

(2) Let $\rho : q(W) \to q(K)_{alg}$ be a $q(F)$-embedding, then by [**K89**, Proposition 1.10, p.51] ρ extends to a $q(F)$-embedding $\tau : q(K) \to q(K)_{alg}$. Since K/F is normal, then again by [**TW15**, Proposition 5.28] $q(K)/q(F)$ is a normal field extension, hence $\tau(q(K)) = q(K)$. By [**TW15**, Proposition 5.13] $\tau_{|K}$ is a graded F-automorphism of K, so for any $x \in W$, we have $\tau(x) = ax$ for some $a \in K_0$, hence $\tau(x) \in W$. Therefore, $\rho(q(W)) = \tau(q(W)) = \tau(W).q(F) \subseteq W.q(F) = q(W)$. This shows that $q(W)/q(F)$ is normal, or equivalently that W/F is normal.

□

PROPOSITION 1.7. *Let A be a graded central simple algebra over a graded field F and suppose that A_0 is simple. If A_0 has a strictly maximal subfield M and S is a maximal graded subfield of $C_A(A_0.F)$, then $N := (M.F).S$ (i.e., the graded subalgebra of A, generated by $M.F$ and S)*[1]*, is a strictly maximal graded subfield of A. Furthermore, if M is an abelian [resp., a Galois; resp., a normal] field extension of F_0, then N is an abelian [resp., a Galois; resp., a normal] graded field extension of F.*

---

[1] As will be seen in the proof, this graded subalgebra is a graded field and is equal to $M.F \otimes_Z S$, up to a graded isomorphism of graded rings.

PROOF. Let $T := C_A(A_0.F)$, D be a graded central division algebra over F such that $A \cong_g M_n(D)$ (see (1.1)) and $T_D := C_D(D_0F)$. We saw in (1.2) that $T \cong_g T_D$, so as showed in the proof of [**TW15**, Proposition 9.33] S is a Galois graded field extension of F. For future need, we show here that S is in fact an abelian graded field extension of F. Indeed, as seen in (1.2), T is a totally ramified graded central division algebra over its center Z, hence $q(T)^h$ is a tame totally ramified central division algebra over $q(Z)^h$, so $\mathrm{char}(Z)(= \mathrm{char}(\overline{q(Z)^h}))$ does not divide $(\Gamma_T : \Gamma_Z)(= (\Gamma_{q(T)^h} : \Gamma_{q(Z)^h}))$ and by [**TW87**, Proposition 3.1] $Z_0(= \overline{q(Z)^h})$ contains a primitive l^{th}-root of unity, where $l = \exp(\Gamma_T/\Gamma_Z)$. Therefore, by [**HW(a)99**, Proposition 3.3] S is a Galois graded field extension of Z. Hence, $q(S)/q(Z)$ is a Galois field extension (see [**TW15**, Proposition 5.30]). As seen in Remark 1.3, Z/F is an abelian graded field extension and for any element σ of $\mathrm{Gal}(Z/F)$, there is an invertible homogeneous element $a \in A$ such that $\sigma = \mathrm{Int}(a)_{|Z}$ (see [**TW15**, Proposition 2.40]). Let t be an arbitrary element of T and let $b \in A_0F$, then $\mathrm{Int}(a)(t)b = ata^{-1}b = at(a^{-1}ba)a^{-1} = a(a^{-1}ba)ta^{-1}$ (because $T = C_A(A_0F)) = bata^{-1} = b\mathrm{Int}(a)(t)$, which shows that $\mathrm{Int}(a)(T) \subseteq T$, hence $\mathrm{Int}(a)(T) = T$ (for both algebras have the same dimension over $Z(= \sigma(Z) = \mathrm{Int}(a)(Z)))$. Let $S' = \mathrm{Int}(a)(S)$, then clearly $Z \subseteq S'$ and $\Gamma_{S'} = \Gamma_S$, so necessarily $S' = S$ (because both S and S' are graded subalgebras with the same support of T and T is totally ramified over Z). This shows that every element of $\mathrm{Gal}(Z/F)$ extends to a graded F-automorphism of S, hence every $q(F)$-automorphism of $q(Z)$ extends to a $q(F)$-automorphism of $q(S)$ (for if τ is a $q(F)$-automorphism of $q(Z)$, then by [**TW15**, Proposition 5.13] the restriction $\tau_{|Z}$ is a graded F-automorphism of Z and clearly we have $\tau = \tau_{|Z} \otimes \mathrm{id}_{q(F)}$, when identifying $q(Z)$ to $Z \otimes_F q(F)$). Moreover, we saw above that $q(S)/q(Z)$ is a Galois field extension. Therefore, $q(S)/q(F)$ is a Galois field extension, or equivalently S/F is a Galois graded field extension (see [**TW15**, Proposition 5.30]). An arbitrary element of $\mathrm{Gal}(S/F)$ is then of the form $\rho \circ \mathrm{Int}(a)$, where $\rho \in \mathrm{Gal}(S/Z)$ and a is an invertible homogeneous element of A. Considering S as a graded Z-subalgebra of T, it follows by the graded version of Skolem-Noether theorem (see [**TW15**, Theorem 2.37]) that there exists an (invertible) homogeneous nonzero element t of T such that $\rho = \mathrm{Int}(t)_{|S}$ (note here that T is a graded division algebra, so the conditions needed to apply the graded version of Skolem-Noether theorem are satisfied). Therefore, the elements of $\mathrm{Gal}(S/F)$ are of the form $\mathrm{Int}(e)$, where e are invertible homogeneous elements of A. Let e and f be two invertible homogeneous elements of A and let $s \in S$. Since $S \subseteq T = C_A(A_0F)$, then we have $(efe^{-1}f^{-1})s(efe^{-1}f^{-1})^{-1} = s$, which means that $\mathrm{Int}(efe^{-1}f^{-1})_{|S} = \mathrm{id}_S$ or equivalently $\mathrm{Int}(e)_{|S}\mathrm{Int}(f)_{|S} = \mathrm{Int}(f)_{|S}\mathrm{Int}(e)_{|S}$. This shows that S is an abelian graded field extension of F.

By (1.4) $M.F \otimes_Z S$ is a graded field (since it is commutative, $(M.F)_0 \otimes_{Z_0} S_0 = M \otimes_{Z_0} Z_0 = M$ is a field, and $\Gamma_{M.F} \cap \Gamma_S (= \Gamma_F) = \Gamma_Z$). Let $N := (M.F).S$ be the graded subalgebra of A generated by $M.F$ and S. It is clear that N is a graded homomorphic image of

$M.F \otimes_Z S$, so it is a graded subfield of A (for $M.F \otimes_Z S$ is a graded field, hence $N \cong_g M.F \otimes_Z S$). Moreover, since M is a strictly maximal subfield of A_0, then MF is a strictly maximal graded subfield of A_0F, so $N(\cong_g MF \otimes_Z S)$ is a strictly maximal graded subfield of $C_A(Z)$ (because as seen in (1.2) $C_A(Z) \cong_g A_0F \otimes_Z C_A(A_0F)$). By the graded version of the double centralizer theorem (see [**TW15**, Theorem 2.35]) we have $[Z : F][C_A(Z) : F] = [A : F]$, so N is a strictly maximal graded subfield of A.

If in addition M is normal over F_0, then by Lemma 1.6 $M.F$ is normal over F, hence $N(= (M.F).S)$ is normal over F (for S is also normal over F). If M is Galois over F_0, then $M.F$ is normal and inertial over F, hence N is normal and tame over F (because both $M.F$ and S are normal and tame over F). Thus, by [**TW15**, Corollary 5.33] N is Galois over F. If we suppose that M is an abelian field extension of F_0, then by [**HW(a)99**, Remark 3.1] the inertial graded field extension $M.F/F$ is abelian, hence $N(= (M.F).S)$ is an abelian graded field extension of F. □

REMARK 1.8. In Proposition 1.7, a special case occurs when A is inertially split. In this case let Z, I, V and T denote Z_A, I_A, V_A and T_A, respectively (see (1.2)), then it is well known that $V(= C_A(Z)) = A_0.F$ when A is a graded division algebra (see [**TW15**, Proposition 8.49]). For an arbitrary inertially split graded central simple algebra A over F, we saw in (1.2) that $I = M_n(I_D)$ and $V = M_n(V_D)$, where D is a graded division algebra such that $A \cong_g M_n(D)$, so again $V = A_0.F$. This means that we have $V = I$ and $T = Z$. Therefore, applying Proposition 1.7 we have the following: If A_0 has a strictly maximal subfield M, then $N := M.F$ is a strictly maximal graded subfield of A (in this case $S = Z$ and obviously $Z_0 \subseteq M$, so $(M.F).S = M.F$). Furthermore, if M is abelian [resp., Galois; resp., normal] over F_0, then N is so over F.

The fact that $C_A(Z) = A_0.F$ can also be proved in the following way: By [**M11**, Lemma 2.4 and Remark 2.5] there exists a graded factor set (ω, f) of $G := \mathrm{Gal}(Z/F)$ in $A_0.F$ such that A equals the generalized graded crossed product $(A_0.F, G, (\omega, f))$ -up to a graded algebra isomorphism-, so by [**M11**, Proposition 1.3] $C_A(Z) = A_0.F$.

LEMMA 1.9. *Let F be a graded field, A be a graded central simple algebra over F with A_0 simple, R and S be two graded field extensions of F in A with $S \subseteq C_A(A_0F)$, and let $P = R \cap C_A(S)$. Then, R/P is a totally ramified graded field extension with $[R : P] \leq [S : R \cap S]$.*

PROOF. Since $S \subseteq C_A(A_0F)$, then $A_0F \subseteq C_A(S)$, hence $(C_A(S))_0 = A_0$. Therefore, $P_0 = R_0 \cap (C_A(S))_0 = R_0$. It is clear that for any nonzero homogeneous element a in P, a^{-1} commutes with S, so $a^{-1} \in P$. It follows easily that P is a graded field[2], hence R/P is a totally ramified graded field extension (because $P_0 = R_0$). In particular, this means that $[R : P] = (\Gamma_R : \Gamma_P) = (\Gamma_R : \Gamma_{R\cap C_A(S)})$. We claim that $\Gamma_{R\cap C_A(S)} = \Gamma_R \cap \Gamma_{C_A(S)}$. Indeed, since $(C_A(S))_0 (= A_0)$ is simple, then by (1.1), for any element $\gamma \in \Gamma_{C_A(S)}$, we can

---

[2] Alternately, since P is a commutative graded simple ring with P_0 a field, then by [**TW15**, corollary 2.43] P is a graded field.

choose an invertible homogeneous element $y \in C_A(S)$ such that $\mathrm{gr}(y) = \gamma$ (see below[3]). If $\gamma \in \Gamma_R \cap \Gamma_{C_A(S)}$, then there is also a nonzero homogeneous element $x \in R$ such that $\mathrm{gr}(x) = \gamma$. We have $xy^{-1} \in A_0$, so $xy^{-1} \in (C_A(S))_0 (= A_0)$. Therefore, $x = (xy^{-1})y \in C_A(S)$. Thus, $\gamma (= \mathrm{gr}(x)) \in \Gamma_{R \cap C_A(S)}$. This shows that $\Gamma_R \cap \Gamma_{C_A(S)} \subseteq \Gamma_{R\cap C_A(S)}$. Conversely, it is clear that $\Gamma_{R\cap C_A(S)} \subseteq \Gamma_R \cap \Gamma_{C_A(S)}$. So, $\Gamma_{R\cap C_A(S)} = \Gamma_R \cap \Gamma_{C_A(S)}$. It follows then by the above that:

$$\begin{aligned} [R:P] &= (\Gamma_R : \Gamma_{R\cap C_A(S)}) \\ &= (\Gamma_R : \Gamma_R \cap \Gamma_{C_A(S)}) \\ &= (\Gamma_R + \Gamma_{C_A(S)} : \Gamma_{C_A(S)}) \\ &\leq (\Gamma_{C_A(R\cap S)} : \Gamma_{C_A(S)}) \qquad (1) \end{aligned}$$

(the last inequality follows because both R and $C_A(S)$ are subsets of $C_A(R \cap S)$). On the other side, we have $(C_A(R\cap S))_0 = A_0 = (C_A(S))_0$, so

$$[C_A(R\cap S) : C_A(S)] = (\Gamma_{C_A(R\cap S)} : \Gamma_{C_A(S)}). \qquad (2)$$

We have seen above in (1) that $[R:P] \leq (\Gamma_{C_A(R\cap S)} : \Gamma_{C_A(S)})$, so using (2) we get $[R:P] \leq [C_A(R\cap S) : C_A(S)] = [S : R\cap S]$, which ends the proof. □

---

[3] Consider a graded division algebra D which is (graded) Brauer-equivalent to $C_A(S)$. Without loss of generality we can assume that $D \subseteq C_A(S)$. Since $(C_A(S))_0$ is simple, then as seen in (1.1) $\Gamma_{C_A(S)} = \Gamma_D$, so it suffices to choose a nonzero homogeneous element y of D such that $\mathrm{gr}(y) = \gamma$.

LEMMA 1.10. *Let F be a graded field and K_1, K_2 be two finite-dimensional graded field extensions of F. If K_1 and K_2 are subrings of a common field with $K_2/(K_1 \cap K_2)$ (or $K_1/(K_1 \cap K_2)$) Galois, then $K_1 \otimes_{K_1 \cap K_2} K_2$ is a graded field.*

PROOF. Let $y \in q(K_1) \cap q(K_2)$ and write $y = ar^{-1} = ct^{-1}$, where $r, t \in F \backslash \{0\}$, $a \in K_1$ and $c \in K_2$. Then, $yrt = at = cr \in K_1 \cap K_2$. Hence $y = (yrt)(rt)^{-1} \in q(K_1 \cap K_2)$. Thus, $q(K_1) \cap q(K_2) = q(K_1 \cap K_2)$.

Now, suppose that $K_2/(K_1 \cap K_2)$ is a Galois graded field extension. So, by [**TW15**, Proposition 5.30] $q(K_2)/q(K_1 \cap K_2)$ is a Galois field extension, this means (by the above) that $q(K_2)/(q(K_1) \cap q(K_2))$ is a Galois field extension. If we assume that K_1 and K_2 are subrings of a common (graded) field, then it follows by [**P82**, Lemma 2, p.272] that $q(K_1)$ and $q(K_2)$ are linearly disjoint over $q(K_1 \cap K_2)(= q(K_1) \cap q(K_2))$. Hence, $q(K_1) \otimes_{q(K_1 \cap K_2)} q(K_2)$ is a field. It follows that $K_1 \otimes_{K_1 \cap K_2} K_2$ is a graded field. $\square$

LEMMA 1.11. *Let F be a graded field, A be a graded central simple algebra over F with A_0 simple, M a Galois graded subfield of A, S a maximal graded subfield of $C_A(A_0F)$ and $(M \cap C_A(S)).S$ be the graded subring of A, generated by $(M \cap C_A(S))$ and S, then $(M \cap C_A(S)).S$ is a graded field and it is graded isomorphic to $(M \cap C_A(S)) \otimes_{M \cap S} S$.*

PROOF. Plainly, $M \cap C_A(S)$ and $M \cap S$ are graded subfields of A and we have $(M \cap C_A(S)) \cap S = M \cap S$. We aim first to show that $M \cap C_A(S)$ is a Galois graded field extension of $M \cap S$. Note that M being Galois over F, then by [**TW15**, Corollary 5.33] M is tame (and normal) over F, so in particular, $M \cap C_A(S)$ is tame over $M \cap S$. Let $W := M \cap C_A(S)$, then $W_0 = M_0 \cap (C_A(S))_0$. We saw in the proof of Lemma 1.9 that $(C_A(S))_0 = A_0$, so $W_0 = M_0$. Therefore, by Lemma 1.6 W is a normal graded field extension of F, hence W is normal over $k := M \cap S$.

Now, let $k^{\mathrm{gr-al}}$ be a graded algebraic closure of k containing S, and consider a graded k-embedding $\psi : W \to k^{\mathrm{gr-al}}$ (see [**TW15**, Proposition 5.10]). Since W is Galois over k, then $\psi(W)$ is Galois over k, hence $\psi(W)$ is Galois over $\psi(W) \cap S$. Therefore, by Lemma 1.10 $\psi(W) \otimes_{\psi(W)\cap S} S$ is a graded field. By identification of $R := \psi^{-1}(\psi(W) \cap S)$ with $\psi(W) \cap S$, we can define a natural action of R on S, so we can define the tensor product $W \otimes_R S$. It is clear that the graded ring $W \otimes_R S$ is graded isomorphic to $\psi(W) \otimes_{\psi(W)\cap S} S$, so it is a graded field. Consider the graded subring $W.S$ of A, generated by W and S. We have a natural graded ring epimorphism $W \otimes_R S \to W.S$, which is necessarily injective, so $W.S$ is a graded field. Now, W and S are graded subfields of the graded field $W.S$, and we have W/k Galois and $W \cap S = k$, so again by Lemma 1.10 $W \otimes_k S$ is a graded field and it is graded isomorphic to $W.S$. $\square$

THEOREM 1.12. *Let F be a graded field and A a graded central simple algebra over F with A_0 simple. Then, the following statements are equivalent:*

(1) *A_0 has a strictly maximal subfield which is abelian [resp., Galois] over F_0.*

(2) *A has an abelian [resp., a Galois] strictly maximal graded subfield.*

PROOF. (1) $\Rightarrow$ (2) By Proposition 1.7.

(2) $\Rightarrow$ (1) Suppose that A has an abelian [resp., a Galois] strictly maximal graded subfield M and let S be a maximal graded subfield of $C_A(A_0.F)$. Let $N := (M \cap C_A(S)).S$ be the graded subring of A, generated by $(M \cap C_A(S))$ and S. We saw in Lemma 1.11 that N is a graded field and that it is graded isomorphic to $(M \cap C_A(S)) \otimes_{M\cap S} S$, so we have:

$$\begin{aligned}[N : M \cap C_A(S)] &= [N : M \cap S]/[M \cap C_A(S) : M \cap S] \\ &= [S : M \cap S]. \end{aligned} \tag{3}$$

(The last equality follows from the fact that $N \cong_g (M \cap C_A(S)) \otimes_{M\cap S} S$). Furthermore, by Lemma 1.9 (with R in the lemma replaced by M), we have $[M : M \cap C_A(S)] \leq [S : M \cap S]$. So, using (3), we get $[M : M \cap C_A(S)] \leq [N : M \cap C_A(S)]$. Therefore, $(\deg(A) =)[M : F] \leq [N : F]$, which shows that N is a strictly maximal graded subfield of A.

Let $I := I_A$, $Z := Z_A$, $T := T_A(= C_A(A_0.F))$, $V := V_A$ be the canonical subalgebra of A, as in (1.2). We have $Z \subseteq S$, so $N \subseteq C_A(S) \subseteq C_A(Z) = V$. Therefore, $N \subseteq C_V(S)$. We recall that $V = I \otimes_Z T$, so $N \subseteq C_{I\otimes_Z T}(Z \otimes_Z S) = C_I(Z) \otimes_Z C_T(S)$ (by [**TW15**, Proposition 2.30]) $= I \otimes_Z S$. Moreover, since I is unramified over Z, then $\Gamma_S \subseteq \Gamma_N \subseteq \Gamma_{I\otimes_Z S} = \Gamma_S$. Hence, $\Gamma_N = \Gamma_S$.

Now, because N is a strictly maximal graded subfield of A which is contained in V, then N is a strictly maximal graded subfield of

V. Indeed, by the graded version of the double centralizer theorem (see [**TW15**, Theorem 2.35]) we have $[Z:F][V:F]=[A:F]$, so $[N:Z]=\deg(V)$. Therefore,

$$\begin{aligned}
[N_0:Z_0](\Gamma_S:\Gamma_Z) &= [N_0:Z_0](\Gamma_N:\Gamma_Z)\\
&= [N:Z]\\
&= \deg(V)\\
&= \deg(I)\deg(T)\\
&= \deg(I_0)[S:Z] \ \ \text{(because } I=I_0\otimes_{Z_0} Z\text{)}\\
&\geq \deg(I_0)(\Gamma_S:\Gamma_Z)\\
&= \deg(A_0)(\Gamma_S:\Gamma_Z) \qquad (4)
\end{aligned}$$

So, $[N_0:Z_0]\geq \deg(A_0)$, proving that N_0 is a strictly maximal subfield of A_0.

Let $W=M\cap C_A(S)$, then as seen in the proof of Lemma 1.11, W is a tame and normal graded field extension of F. We saw in the proof of Proposition 1.7 that S is an abelian graded field extension of F, so N is tame and normal over F. In particular, N_0 is separable over F_0 and by Lemma 1.6 N_0 is normal over F_0. Therefore, N_0 is a Galois field extension of F_0.

If M is an abelian graded field extension of F, then W is necessarily an abelian graded field extension of F (for $q(W)\subseteq q(M)$ and $q(M)$ is an abelian field extension of $q(F)$). Moreover, as seen above S is an abelian graded field extension of F, so $N(=W.S)$ is an abelian graded field extension of F. In particular, this yields that N_0F is and abelian graded field extension of F, so N_0 is an abelian field extension of F_0. $\square$

COROLLARY 1.13. [**HNW16**, Theorem 3.1] *Let F be a graded field and A a graded central division algebra over F. Then, the following statements are equivalent:*

(1) *A_0 has a maximal subfield which is abelian [resp., Galois; resp., normal] over F_0.*
(2) *A has an abelian [resp., a Galois; resp., a normal] maximal graded subfield.*

PROOF. Suppose that A has a graded subfield M which is normal over F, then with the notation previously used in the proof of Theorem 1.12 $N(=(M \cap C_A(S)).S)$ is a graded subfield of A (since A is a graded division algebra). Moreover, using the same arguments, one shows that N_0 is a maximal subfield of A_0. Also, in this case, $W(= M \cap C_A(S))$ is normal over F, so N is a normal graded field extension of F. This yields by Lemma 1.6 that N_0 is normal over F_0. The rest follows by Proposition 1.7 and Theorem 1.12. $\square$

2. On embeddings of graded simple algebras

In this section we show some necessary results concerning the possibility of embedding a graded simple algebra in a matrix algebra with entries in a graded division ring. These results will be needed in next sections, also they can be applied to give new proofs for more general versions of some results in [**FSS90**] (see Proposition 2.8).

(2.1) Let F be a graded field of type Γ and λ be an element of Γ. We recall that the shifted graded F-space $F_{s(\lambda)}$ is obtained from F by shifting homogeneous elements by λ, i.e., $F_{s(\lambda)}$ equals to F as a set and $(F_{s(\lambda)})_\gamma = F_{\gamma+\lambda}$ for all $\gamma \in \Gamma$. For a positive integer n and elements $\delta_1, ..., \delta_n$ of Γ, we let $M_n(F)(\delta_1, ..., \delta_n)$ denote the following split graded central simple algebra (i.e., matrix graded algebra):

$$M_n(F)(\delta_1, ..., \delta_n) = \begin{pmatrix} F_{s(\delta_1-\delta_1)} & \cdots & F_{s(\delta_1-\delta_n)} \\ \vdots & \ddots & \vdots \\ F_{s(\delta_n-\delta_1)} & \cdots & F_{s(\delta_n-\delta_n)} \end{pmatrix}.$$

This means that a nonzero homogeneous element of grade γ of $M_n(F)(\delta_1, ..., \delta_n)$ is a matrix with ij-entry in $(F_{s(\delta_i-\delta_j)})_\gamma (= F_{\delta_i-\delta_j+\gamma})$. If A is a graded F-algebra (of type Γ), we define $M_n(A)(\delta_1, ..., \delta_n)$ in a similar way. We will also denote $M_n(F)(\delta_1, ..., \delta_n)$ simply by $M_n(F)(\overline{\delta})$, where $\overline{\delta} = (\delta_1, ..., \delta_n)$.

(2.2) Now, let R be a ring and M an abelian group with endomorphism ring $\mathrm{End}(M)$ (acting on M on the right with the multiplication law in $\mathrm{End}(M)$ being the opposite of the usual composition law), then a right R-module action on M is equivalent to a ring homomorphism $\phi:\ R \to \mathrm{End}(M)$. The two conditions are related by the equation:

$$m.\phi(r) = m.r,$$

for $m \in M$, $r \in R$, which defines the module action when ϕ is given and conversely define ϕ when the module action is given. In the graded setting, if F is a graded field of type Γ, $\delta_1, ..., \delta_n$ are elements of Γ, and M is the graded F-vector space $F_{s(\delta_1)} \oplus ... \oplus F_{s(\delta_n)}$ (where $F_{s(\lambda)}$ is the shifted of F by λ as seen above), then $\mathrm{End}(M)$ is a graded F-algebra that we will denote by $\mathrm{GEnd}(M)$, and we have $\mathrm{GEnd}(M) \cong_g \mathrm{M}_n(F)(\overline{\delta})$, where $\overline{\delta} = (\delta_1, ..., \delta_n)$ (see e.g., [**TW15**, Proposition 2.9, p.41 (see also Proposition 2.8, p.39)]). Thus, if A is a graded F-algebra, the fact that we have a graded F-algebra homomorphism $\phi: A \to \mathrm{M}_n(F)(\overline{\delta})$, is equivalent to having a graded right A-module structure on M (compatible with the action of F on M).

If $u_1, ..., u_n$ is a base of A over F, consisting of homogeneous elements of A, then as a graded vector space over F, A is isomorphic to the graded F-vector space $M := F_{s(\delta_1)} \oplus ... \oplus F_{s(\delta_1)}$, where $\delta_i = -\mathrm{gr}(u_i)$ for all i $(1 \leq i \leq n)$. Since we have a natural right A-module structure on A (hence on M), then there is a graded F-algebra homomorphism $\phi : A \to \mathrm{M}_n(F)(\overline{\delta})$, which is clearly injective (for $\ker(\phi) = \mathrm{ann} A_A = 0$).

PROPOSITION 2.3. *Let F be a graded field, A be a graded central division algebra over F, L be a finite-dimensional graded field extension of F and S be a graded central simple algebra over L. If*

$\Gamma_S \subseteq \Gamma_A$, *then there is a graded monomorphim of graded* F*-algebras from* S *into* $M_n(A)$ *for some positive integer* n.

PROOF. Let $u_1, ..., u_n$ be a base consisting of homogeneous elements of S over F and let $\delta_i = -\mathrm{gr}(u_i)$. By the above there is a graded F-algebra monomorphism from S into $M_n(F)(\delta_1, ..., \delta_n)$. Therefore, there is graded F-algebra monomorphism from S into $M_n(A)(\delta_1, ..., \delta_n)$. Since $\delta_1, ..., \delta_n$ belongs to Γ_A, then by [**HW(b)99**, (ii), p.78] $M_n(A)(\delta_1, ..., \delta_n) \cong_g M_n(A)(= M_n(A)(0, ..., 0))$. Therefore, there is a graded F-algebra monomorphism from S into $M_n(A)$.

□

PROPOSITION 2.4. *Let* F *be a graded field with support* Γ_F, $\Gamma := \Gamma_F \otimes_{\mathbb{Z}} \mathbb{Q}$ *be the divisible hull of* Γ_F, A *be a graded central simple algebra over* F *and* L *a graded subfield of* A. *Then the following statements are equivalent:*

(1) L *splits* A.

(2) $C_A(L) \cong_g M_k(L)(\overline{\delta})$, *for some* $\overline{\delta} \in \Gamma^k$, *where* $k = \deg(A)/[L : F]$.

(3) $A \cong_g B \otimes_F C$, *where* $C \cong_g M_k(F)(\overline{\delta})$ *for some* $\overline{\delta} \in \Gamma^k$, *and* B *is a graded central simple algebra over* F *such that* L *is a strictly maximal graded subfield of* B.

PROOF. (2) $\Rightarrow$ (3) Let $C = M_k(F)(\overline{\delta})$, then we have $C \subseteq C_A(L) \subseteq A$. Let $B = C_A(C)$, then by the graded version of the double centralizer theorem (see [**TW15**, Theorem 2.35]), we have $A \cong_g B \otimes_F C$, and by dimension count L is a strictly maximal graded subfield of B.

(3) ⇒ (2) We have

$$\begin{aligned} C_A(L) &\cong_g C_{B\otimes C}(L\otimes_F F) \\ &\cong_g C_B(L)\otimes_F C_C(F) \\ &\cong_g L\otimes_F C \\ &\cong_g M_k(L)(\overline{\delta}) \end{aligned} \tag{5}$$

Plainly, in this case, by the graded version of the double centralizer theorem, we have $k[L:F]=\deg(A)$.

(1) ⇔ (2) As in the ungraded case, $\mathrm{GEnd}_{L\otimes_F A^{\mathrm{op}}}(A)$ is graded isomorphic to $C_A(L)$. Also, by the graded Wedderburn's theory, $\mathrm{GEnd}_{L\otimes_F A^{\mathrm{op}}}(A)$ is (graded) Brauer equivalent to $(L\otimes_F A^{\mathrm{op}})^{\mathrm{op}} \cong_g L\otimes_F A$. Thus, L splits A if and only if the graded central L-algebra $C_A(L)$ is split. This, together with the graded version of double centralizer theorem, proves (1) ⇔ (2).

□

PROPOSITION 2.5. *Let F be a graded field, A be a graded central simple algebra over F with A_0 simple and L a finite-dimensional graded field extension of F which splits A with $[L:F]=\deg(A)$. Then L is graded isomorphic to a (strictly maximal) graded subfield of A if and only if $\Gamma_L\subseteq\Gamma_A$.*

PROOF. Let D be a graded central division algebra over F, which is (graded) Brauer-equivalent to A. We have A_0 simple, so as seen in (1.1) $\Gamma_A=\Gamma_D$ and $A\cong_g M_n(D)$ for some positive integer n. If L is graded isomorphic to a strictly maximal graded subfield of A, then clearly $\Gamma_L\subseteq\Gamma_A$. Conversely suppose that we have $\Gamma_L \subseteq \Gamma_A$, then by Proposition 2.3 L embeds (as a graded F-algebra) in $M_s(D)$, where $s = [L:F](= \deg(A) = n\deg(D))$. It is clear that L splits $M_s(D)$ (since it splits A), so by Proposition 2.4, $M_s(D) \cong_g B \otimes_F C$, where B is a graded central simple F-algebra which contains L as a strictly maximal graded subfield, and C is a split graded central simple algebra over F. Note that we have $\Gamma_{M_s(D)} = \Gamma_D$, so $\Gamma_B \subseteq \Gamma_D$; moreover, B is (graded) Brauer-equivalent to D, so $\Gamma_D \subseteq \Gamma_B$, hence $\Gamma_B = \Gamma_D$. Therefore, by [**TW15**, Proposition 2.47, p.70] $B \cong_g M_m(D)$ for some positive integer m. We then have $m\deg(D) = \deg(B) = [L:F] = n\deg(D)$, so $m = n$, which means that $B \cong_g A$. This shows that L is graded isomorphic to a graded subfield of A.

□

(2.6) Let F be a graded field, L a finite-dimensional graded field extension of F, A a graded central division algebra over F and S a graded central simple algebra over L. If $\Gamma_S \subseteq \Gamma_A$, then by Proposition 2.3 there is some positive integer t such that S is graded isomorphic to a graded subalgebra of $M_t(A)$. Suppose this is the case and let s be the smallest positive integer such that S embeds in $M_s(A)$ (as a graded ring). Inspired by [**MiW95**, Proposition 2.1], we will show here that $s = [L:F]\deg(S)\mathrm{ind}(A \otimes_F S)/\deg(A)$. For this consider the graded (centralizer) algebra $C_{M_s(A)}(S)$ (where S is considered as a graded subring of $M_s(A)$). Since S is graded simple, then by the graded version of the double centralizer theorem $C^{(S)}_{M_s(A)}$ is also graded simple (see [**TW15**, Theorem 2.35]). Therefore by the graded version of the Wedderburn Theorem (see [**HW(b)99**, Proposition 1.3] or [**TW15**, p., 54]), there exists a graded division algebra R, a positive integer m and some $\overline{\delta} = (\delta_1, ..., \delta_m) \in \Gamma^m$ such that $C_{M_s(A)}(S) \cong_g M_m(R)(\overline{\delta})$. Consider the graded central simple F-algebra $C := M_m(F)(\overline{\delta})$. Obviously, C embeds in $M_m(R)(\overline{\delta})$, so C can be considered as a graded simple subalgebra of $C_{M_s(A)}(S)$, hence of $M_s(A)$. Let $B = C_{M_s(A)}(C)$, then again by the graded version of the double centralizer theorem, we have $M_s(A) \cong_g C \otimes_F B$. In particular, $\Gamma_B \subseteq \Gamma_{M_s(A)} = \Gamma_A$. The other inclusion $\Gamma_A \subseteq \Gamma_B$ is obvious since B is (graded) Brauer-equivalent to A, hence $\Gamma_B = \Gamma_A$. Therefore, $B \cong_g M_l(A)$, for some positive integer l (see (1.1)). It follows then that $m = 1$, for otherwise we will have $l < s$, which contradicts the fact that s is minimal (see that $C \subseteq C_{M_s(A)}(S)$, so $S \subseteq B$). Thus, $C_{M_s(A)}(S)(\cong_g M_m(R)(\overline{\delta}) = R)$ is a graded division algebra. As in the ungraded case, $C_{M_s(A)}(S)$ is (graded) Brauer-equivalent to $M_s(A) \otimes_F S^{\mathrm{op}}$, hence Brauer-equivalent to $A \otimes_F S^{\mathrm{op}}$. So, $\deg(C_{M_s(A)}(S)) = \mathrm{ind}(A \otimes_F S^{\mathrm{op}})$.

Note that by the graded version of the double centralizer theorem, we have $[C_{M_s(A)}(S) : F][S : F] = [M_s(A) : F] = s^2 \deg(A)^2$. Therefore, $s = [L : F]\deg(S)\mathrm{ind}(A \otimes_F S^{\mathrm{op}})/\deg(A)$. We get then the following proposition.

PROPOSITION 2.7. *Let F be a graded field, L a finite-dimensional graded field extension of F, A a graded central division algebra over F and S a graded central simple algebra over L with $\Gamma_S \subseteq \Gamma_A$ and n a positive integer. Then, S embeds (as a graded ring) in $M_n(A)$ if and only if n is a multiple of $[L : F]\deg(S)\mathrm{ind}(A \otimes_F S^{\mathrm{op}})/\deg(A)$.*

PROOF. Let $s = [L:F]\deg(S)\mathrm{ind}(A \otimes_F S^{\mathrm{op}})/\deg(A)$, then as seen in (2.6) S embeds in $M_s(A)$, hence embeds in any $M_n(A)$, where n is a multiple of s.

Conversely, suppose that S embeds in $M_n(A)$, then $C_{M_n(A)}(S)$ is (graded) Brauer-equivalent to the graded division algebra $C_{M_s(A)}(S)$ (for both graded algebras are (graded) Brauer-equivalent to $A \otimes_F S^{\mathrm{op}}$). It follows that $\deg(C_{M_n(A)}(S))$ is a multiple of $\deg(C_{M_s(A)}(S))$. Note that in the same way as for s in (2.6), we have $n = [L : F]\deg(S)\deg(C^{(S)}_{M_n(A)})/\deg(A)$, so n is a multiple of s (because $C_{M_n(A)}$ (S) is (graded) Brauer-equivalent to $C_{M_s(A)}(S)$). ☐

Proposition 2.7 can be applied to give graded versions, with different proofs, for many results in [**FSS90**]. We give here the following example.

PROPOSITION 2.8. (*Compare* [**FSS90**, Proposition 2]) *Let L/F be a finite-dimensional graded field extension, R be a graded central division algebra over L, A a graded central division algebra over F, and m, n be two positive integers. If $M_m(R)$ embeds, as a graded ring, in $M_n(A)$, then m divides n and R embeds, as a graded ring, in $M_k(A)$, where $k = n/m$.*

PROOF. Since $M_m(R)$ embeds in $M_n(A)$, then $\Gamma_R \subseteq \Gamma_A$. Let r be the minimal positive integer l such that R embeds in $M_l(A)$, then by Proposition 2.7, $r = \frac{\deg(R)[L:F]\mathrm{ind}(A\otimes_F R^{\mathrm{op}})}{\deg(A)}$. For a positive integer m, let $S := M_m(R)$, then $mr(= \frac{\deg(S)[L:F]\mathrm{ind}(A\otimes_F S^{\mathrm{op}})}{\deg(A)})$, is the smallest positive integer t such that S embeds in $M_t(A)$. If S embeds in $M_n(A)$ for some positive integer n, then again by

Proposition 2.7, n is a multiple of mr, so a multiple of m. Plainly, in this case n/m is a multiple of r, so R embeds in $M_{n/m}(A)$. $\square$

3. On crossed product algebras over Henselian fields

In this section, we use the results of the previous sections and a special residually simple tame gauge to show that if E is a Henselian valued field, D is a tame central division algebra over E and n is a positive integer, then $M_n(\overline{D})$ has a strictly maximal subfield which is Galois [resp., abelian] over $\overline{E}$ if and only if $M_n(D)$ has a strictly maximal subfield K which is Galois [resp., abelian] and tame over E with $\Gamma_K \subseteq \Gamma_D$ (see Theorem 3.5).

(3.1) Let E be a Henselian field, D be a tame central division algebra over E, $B = M_n(D)$ where n is a positive integer, $\Gamma = v(D^*)$, where v is the extension of the valuation of E to D, and define the map $\beta : B \to \Gamma \cup \{\infty\}$ by $\beta((d_{ij})_{1\leq i,j\leq n}) = \min\{v(d_{ij}) \mid 1 \leq i, j \leq n\}$. One can easily see that β is a surmultiplicative E-value function and an E-norm on B. For $b = (d_{ij})_{1\leq i,j\leq n} \in B$ and $\gamma \in \Gamma$, we have $\beta(b) \geq \gamma$ [resp., $\beta(b) > \gamma$] if and only if $v(d_{ij}) \geq \gamma$ [resp., $v(d_{ij}) > \gamma$] for all i, j $(1 \leq i, j \leq n)$, so the correspondence $b' \mapsto (d_{ij} + (GD)^{>\gamma})_{1\leq i,j\leq n}$, where $\gamma = \beta(b)$, induces a graded isomorphism $GB \to M_n(GD)$. Therefore, β is a tame E-gauge. Note that we have $(G_\beta B)_0 \cong M_n(\overline{D})$, hence $(G_\beta B)_0$ is a simple ring.

LEMMA 3.2. *Let (E, v) be a Henselian valued field, D be a tame central division algebra over E, $B = M_n(D)$ where n is a positive* *integer, K be a tame finite-dimensional field extension of E, and w be the unique extension of v to K. Let β be a tame E-gauge on B and suppose that G_wK is a strictly maximal graded subfield of $G_\beta B$, then, up to an E-algebra isomorphism, K is a strictly maximal subfield of B.*

PROOF. Since G_wK is a strictly maximal graded subfield of $G_\beta B$, then G_wK splits GD (because by [**TW15**, Proposition 4.33] $G_\beta B$ is (graded) Brauer-equivalent to GD), hence by [**HW(b)99**, Corollary 5.8] K splits D. Therefore, up to an E-algebra isomorphism, K is a strictly maximal subfield of some $M_m(D)$, where m is a positive integer (see [**P82**, Theorem, p.241]). We have $[K : E] = [G_wK : GE]$ (because K is defectless over E); moreover $[G_wK : GE] = \deg(G_\beta B)(= \deg(B))$, so necessarily $m = n$, which means that K is isomorphic to a strictly maximal subfield of B. $\square$

REMARK 3.3. Note that the converse of lemma 3.2 does not hold. Indeed, suppose given a tame totally ramified nontrivial field extension K of E with valuation w extending v. Let $s = [K : E]$, and let β be the tame E-gauge defined in (3.1) on $B := M_s(E)$, then, up to an E-algebra isomorphism, K is a strictly maximal subfield of B (see [**P82**, Lemma a, p.234]), but G_wK cannot embed in $G_\beta B$ (because $\Gamma_{G_\beta B} = \Gamma_E$ and $\Gamma_{G_wK} = \Gamma_K \neq \Gamma_E$).

LEMMA 3.4. *Let E be a Henselian valued field with valuation v, D be a tame central division algebra over E, $B = M_n(D)$ where n is a positive integer and suppose that there exists a defectless strictly* *maximal subfield K of B. Consider a residually simple tame E-gauge β on B, and let w be the unique extension of v to K. Then, G_wK is graded isomorphic to a graded subfield of $G_\beta B$ if and only if $\Gamma_K \subseteq \Gamma_D$.*

PROOF. It is clear that K splits B, so by [**TW10**, Theorem 3.8] G_wK splits $G_\beta B$. Moreover, we have $[G_wK : GE] = [K : E]$ (because K is defectless over E), $\deg(G_\beta B) = \deg(B)$, $\Gamma_{G_wK} = \Gamma_K$, $(G_\beta B)_0$ is simple (by hypothesis) and $\Gamma_{G_\beta B}(= \Gamma_{GD}) = \Gamma_D$, so our lemma follows by Proposition 2.5. $\square$

THEOREM 3.5. *Let E be a Henselian valued field, D a tame central division algebra over E and n a positive integer. Then, $M_n(\overline{D})$ has a strictly maximal subfield which is Galois [resp., abelian] over $\overline{E}$ if and only if $M_n(D)$ has a strictly maximal subfield K which is Galois [resp., abelian] and tame over E with $\Gamma_K \subseteq \Gamma_D$.*

PROOF. Let $B = M_n(D)$ and consider on B the tame E-gauge β defined in (3.1), then we have $G_\beta B \cong_g M_n(GD)$. Suppose that $M_n(\overline{D})$ has a strictly maximal subfield which is Galois [resp., abelian] over $\overline{E}$, then by Proposition 1.7 $M_n(GD)$ has a strictly maximal graded subfield L which is Galois [resp., abelian] over GE (because $M_n(\overline{D})$ is the 0-component of $M_n(GD)$). In particular, L is tame over GE, so by [**TW15**, Corollary 5.56], there exists a Galois [resp., an abelian] tame field extension M of E with valuation w extending the valuation v of E such that $G_wM \cong_g L$ and $\mathrm{Gal}(M/E) \cong \mathrm{Gal}(L/GE)$. Since $L(\cong_g G_wM)$ is a strictly maximal graded subfield of $G_\beta B$, then by Lemma 3.2, M is isomorphic (as an E-algebra) to a strictly maximal subfield K of B. Let $\phi : M \to K$ be such an E-algebra isomorphism and let $\Gamma_E = v(E^*)$, $\Gamma = \Gamma_F \otimes_{\mathbb{Z}} \mathbb{Q}$ (be the divisible hull of Γ_E) and $\eta := w \circ \phi^{-1} : K \to \Gamma \cup \{\infty\}$, then η is the (unique) valuation on K extending v, hence $\Gamma_K = \Gamma_M$. Moreover, it is clear that $\Gamma_M = \Gamma_L \subseteq (\Gamma_{M_n(GD)} = \Gamma_{GD} =)\Gamma_D$, so $\Gamma_K \subseteq \Gamma_D$. It is also clear that K is Galois [resp., abelian] and tame over E.

Conversely, suppose that B has a strictly maximal subfield K which is Galois [resp., abelian] and tame over E with $\Gamma_K \subseteq \Gamma_D$, then by Lemma 3.4 $G_w K$ is graded isomorphic to a strictly maximal graded subfield of $M_n(GD)(\cong_g G_\beta B)$. Moreover, by [**TW15**, Corollary 5.52] $G_w K$ is a Galois [resp., an abelian] graded field extension of GE, so by Theorem 1.12, $M_n(\overline{D})$ has a strictly maximal subfield which is Galois [resp., abelian] over $\overline{E}$.

□

4. Kummer subfields of simple algebras

Amitsur and Tignol determined in [**TA85**] necessary and sufficient conditions for Malcev-Neumann division algebras (under some hypotheses) to have Kummer subfields. A similar result was then proved by Morandi and Sethuraman in [**MS95**] for any (tame) division algebra of the form $D = S \otimes_E T$ over a Henselian valued field E, where S is an inertially split division algebra over E and T is a (tame) totally ramified division algebra over E. This work was then generalized by the author in [**M10**] to arbitrary tame division algebras over Henselian valued fields. In this section, we give a more general result showing that Amitsur and Tignol's conditions are also true for (tame) simple algebras over a Henselian valued field (see Corollaries 4.8 and 4.9).

(4.1) Let F be a graded field and L be a finite-dimensional abelian graded field extension of F such that $\mathrm{char}(F)$ does not divide $[L:F]$. We recall that L is a Kummer graded field extension of F if F_0 contains a primitive m^{th} root of unity, where m is the exponent of $\mathrm{Gal}(L/F)$. In such a case, we have $L = F[a \mid a \in \mathrm{KUM}(L/F)]$, where $\mathrm{KUM}(L/F) = \{x \in L^* \mid x^m \in F\}$ (see [**M10**, (2.1)]), so Γ_L/Γ_F is generated by $\{\mathrm{gr}(a) + \Gamma_F \mid a \in \mathrm{KUM}(L/F)\}$. Therefore, if we set $\mathrm{kum}(L/F) = \mathrm{KUM}(L/F)/F^*$, then the group homomorphism $\psi : \mathrm{kum}(L/F) \to \Gamma_L/\Gamma_F$, defined by $\psi(aF^*) = \mathrm{gr}(a) + \Gamma_F$, for $a \in \mathrm{KUM}(L/F)$, is surjective. Note that in this case, the graded subfield L_0F of L is a Kummer graded field extension of F. Moreover, since L_0F is unramified over F, then by applying [**HW(a)99**, Remark 3.1] L_0 is a Kummer field extension of F_0. Let $\phi : \mathrm{kum}(L_0/F_0) \to \mathrm{kum}(L/F)$ be the group homomorphism defined by $\phi(aF_0^*) = aF^*$, for every $a \in \mathrm{KUM}(L_0/F_0)$, then clearly ϕ is injective. Also, we have $\psi \circ \phi = 0$, and by comparing the cardinalities of the terms in the following sequence of trivial Γ_L/Γ_F-modules: $\alpha_L : 1 \to \mathrm{kum}(L_0/F_0) \xrightarrow{\phi} \mathrm{kum}(L/F) \xrightarrow{\psi} \Gamma_L/\Gamma_F \to 0$, we see that α_L is exact (we recall that $\mathrm{kum}(L_0/F_0)$ is isomorphic to $\mathrm{Gal}(L_0/F_0)$ and $\mathrm{kum}(L/F)$ is isomorphic to $\mathrm{Gal}(L/F)$, see [**M10**, 2.1]). Plainly, α_L can be considered as a (symmetric) 2-cocycle of $Z^2(\Gamma_L/\Gamma_F, \mathrm{kum}(L_0/F_0))_{\mathrm{sym}}$. If L is a Kummer graded subfield of a graded central simple algebra A over F, then one can see that $\mathrm{KUM}(L/F) \cap A_0 = \mathrm{KUM}(L_0/F_0)$. In what follows, we will denote by $e_* : H^2(\Gamma_L/\Gamma_F, \mathrm{KUM}(L_0/F_0))_{\mathrm{sym}} \to H^2(\Gamma_L/\Gamma_F, \mathrm{kum}(L_0/F_0))_{\mathrm{sym}}$ the homomorphism of cohomology groups corresponding to the canonical surjective homomorphism $e : \mathrm{KUM}(L_0/F_0) \to \mathrm{kum}(L_0/F_0)$.

(4.2) Let F be a graded field, A a graded central simple algebra over F with A_0 simple and R a graded central division algebra over F (graded) Brauer-equivalent to A. Recall that A can be written as a generalized graded crossed product $A = (A_0F, \Gamma_A/\Gamma_F, (\omega, f))$, where (ω, f) is a graded factor set of Γ_A/Γ_F in A_0F (see [**M11**, Lemma 2.4]). We can assume that f is normalized (i.e., $f(0,\overline{\gamma}) = f(\overline{\gamma},0) = 1$, for all $\overline{\gamma}(= \gamma + \Gamma_F) \in \Gamma_A/\Gamma_F$). Indeed, as in the proof of [**M11**, Lemma 2.4], for any $\gamma \in \Gamma_A(= \Gamma_R)$, fix nonzero homogeneous elements $z_{\overline{\gamma}}$ of R with $\mathrm{gr}(z_{\overline{\gamma}}) + \Gamma_F = \overline{\gamma}$ and with $z_0 = 1$. Then, $A = \oplus_{\overline{\gamma} \in \Gamma_A/\Gamma_F} A_0F z_{\overline{\gamma}} \cong (A_0F, \Gamma_A/\Gamma_F, (\omega, f))$, where (ω, f) is the graded factor set of Γ_A/Γ_F in A_0F, defined as follows: $\omega : \Gamma_A/\Gamma_F \to \mathrm{Aut}(A_0F), \quad a \mapsto \omega_{\bar{\gamma}}(a) = z_{\bar{\gamma}} a z_{\bar{\gamma}}^{-1}$, and $f : \Gamma_A/\Gamma_F \times \Gamma_A/\Gamma_F \to (A_0F)^*, \quad (\bar{\gamma}, \bar{\delta}) \mapsto z_{\bar{\gamma}} z_{\bar{\delta}} z_{\bar{\gamma}+\bar{\delta}}^{-1}$. This representation of A will be used in what follows to generalize the statements of [**M10**, Theorems 2.4 and 2.6]. We get then conditions under which A has Kummer graded subfields.

(4.3) Let $A = (A_0F, \Gamma_A/\Gamma_F, (\omega, f))$ with f normalized as in (4.2) and denote also by ω the map $\Gamma_A/\Gamma_F \to \mathrm{Aut}(A_0)$, defined by $\overline{\gamma} \mapsto \omega_{\overline{\gamma}|A_0}$, where $\omega_{\overline{\gamma}|A_0}$ is the restriction of $\omega_{\overline{\gamma}}$ to A_0. One can easily see that there is a mapping $d : \Gamma_A/\Gamma_F \times \Gamma_A/\Gamma_F \to A_0^*$ and a symmetric 2-cocycle $h \in Z^2(\Gamma_A/\Gamma_F, F^*)_{\mathrm{sym}}$ such that (ω, d) is a factor set of Γ_A/Γ_F in A_0 and for any $\overline{\gamma}, \overline{\gamma}' \in \Gamma_A/\Gamma_F$, we have $f(\overline{\gamma}, \overline{\gamma}') = d(\overline{\gamma}, \overline{\gamma}')h(\overline{\gamma}, \overline{\gamma}')$. Indeed, let $(\overline{\delta}_i := \delta_i + \Gamma_F)_{1 \leq i \leq r}$ be a basis of Γ_A/Γ_F (i.e., $\Gamma_A/\Gamma_F = \langle \overline{\delta}_1 \rangle \oplus ... \oplus \langle \overline{\delta}_r \rangle$), and $q_i = \mathrm{ord}(\overline{\delta}_i)$ (for $1 \leq i \leq r$). For any $\overline{\gamma} \in \Gamma_A/\Gamma_F$, there is a unique element $\overline{m} = (m_1, ..., m_r) \in \mathbb{N}^r$ with $0 \leq m_i < q_i$, such that $\overline{\gamma} = (\sum_{i=1}^r m_i\delta_i) + \Gamma_F$. Let $\overline{m}$, $\overline{n} \in \mathbb{N}^r$ with $0 \leq m_i$, $n_i < q_i$, and let $s(\overline{m}+\overline{n}) \in \mathbb{N}^r$ with $0 \leq s(\overline{m}+\overline{n})_i < q_i$ and $m_i + n_i - s(\overline{m} + \overline{n})_i \equiv 0 \mod q_i$ for all i, $(1 \leq i \leq r)$. Let $t_i \in \mathbb{N}$ such that $m_i + n_i - s(\overline{m}+\overline{n})_i = t_iq_i$, and fix elements $y_i \in F^*$ with $\mathrm{gr}(y_i) = q_i\delta_i$ and $y_1 = 1$. Let $h : \Gamma_A/\Gamma_F \times \Gamma_A/\Gamma_F \to F^*$ be the map defined by $h(\sum_{i=1}^r m_i\delta_i,\ \sum_{i=1}^r n_i\delta_i) = \prod_{i=1}^r {y_i}^{t_i}$, where m_i, n_i and t_i satisfy the above conditions, then by simple computations, one can see that h is a normalized symmetric 2-cocycle of $Z^2(\Gamma_A/\Gamma_F, F^*)$. Let $d : \Gamma_A/\Gamma_F \times \Gamma_A/\Gamma_F \to A_0^*$, be the map defined by $d(\overline{\gamma},\ \overline{\gamma}') = f(\overline{\gamma},\ \overline{\gamma}').h(\overline{\gamma},\ \overline{\gamma}')^{-1}$. The fact that (ω, f) is a graded factor set of Γ_A/Γ_F in A_0F (with f normalized) and h is a normalized symmetric 2-cocycle of $Z^2(\Gamma_A/\Gamma_F, F^*)$, imply that (ω, d) is a factor set of Γ_A/Γ_F in A_0 (with d normalized).

The following two Theorems generalize the statements of [**M10**, Theorems 2.4 and 2.6] by using the same arguments. For the convenience of the reader we give the detailed proofs. For a Kummer graded subfield L of a graded simple algebra A and for a factor set (w,g) of Γ_A/Γ_F in A_0, we will denote by $\operatorname{res}^{\Gamma_A/\Gamma_F}_{\Gamma_L/\Gamma_F}(w,g)$ the restriction of (w,g) when considering Γ_L/Γ_F instead of Γ_A/Γ_F. Also, for a cocycle $k \in Z^2(\Gamma_L/\Gamma_F, \mathrm{KUM}(L_0/F_0))_{\mathrm{sym}}$ and the canonical embedding $i : \mathrm{KUM}(L_0/F_0) \to A_0^*$, we denote by i_*k the mapping $\Gamma_L/\Gamma_F \times \Gamma_L/\Gamma_F \to A_0^*$, defined by $(\overline{\gamma}, \overline{\gamma}') \mapsto i \circ k(\overline{\gamma}, \overline{\gamma}')$.

THEOREM 4.4. *Let F be a graded field, A a graded central simple algebra over F with A_0 simple and* $\operatorname{char}(F)$ *not dividing* $\deg(A)$*, L be a Kummer graded subfield of A, and α_L be the cocycle of $Z^2(\Gamma_L/\Gamma_F, \mathrm{kum}(L_0/F_0))_{\mathrm{sym}}$ defined in (4.1). Write $A = (A_0F, \Gamma_A/\Gamma_F, (\omega, f))$ as in (4.2) and consider the factor set (ω, d) of Γ_A/Γ_F in A_0 as defined in (4.3), then there exists a normalized cocycle $d' \in Z^2(\Gamma_L/\Gamma_F, \mathrm{KUM}(L_0/F_0))_{\mathrm{sym}}$ (for the trivial action of Γ_L/Γ_F on $\mathrm{KUM}(L_0/F_0)$) and a map $\omega' : \Gamma_L/\Gamma_F \to \mathrm{Aut}(A_0)$ which satisfies $\omega'_{\overline{\gamma}}(a) = a$ for all $a \in L_0$ and $\overline{\gamma} \in \Gamma_L/\Gamma_F$, such that:*

1. *(ω', i_*d') is a factor set of Γ_L/Γ_F in A_0, cohomologous to $\operatorname{res}^{\Gamma_A/\Gamma_F}_{\Gamma_L/\Gamma_F}(\omega, d)$, and*
2. *$e_*([d']) = [\alpha_L]$, where $[d']$ [resp., $[\alpha_L]$] denotes the class of d' in $H^2(\Gamma_L/\Gamma_F, \mathrm{KUM}(L_0/F_0))$ [resp., of α_L in $H^2(\Gamma_L/\Gamma_F, \mathrm{kum}(L_0/F_0))$] (see (4.1) for the definition of e_*).*

PROOF. Write $A = (A_0F, \Gamma_A/\Gamma_F, (\omega, f)) = \oplus_{\overline{\gamma} \in \Gamma_A/\Gamma_F} A_0 F x_{\overline{\gamma}}$, where $x_0 = 1$, $x_{\overline{\gamma}} \in A^*$, $\mathrm{gr}(x_{\overline{\gamma}}) + \Gamma_F = \overline{\gamma}$, $x_{\overline{\gamma}} a = \omega_{\overline{\gamma}}(a) x_{\overline{\gamma}}$ and $x_{\overline{\gamma}} x_{\overline{\gamma}'} = f(\overline{\gamma}, \overline{\gamma}') x_{\overline{\gamma} + \overline{\gamma}'}$ and write $f(\overline{\gamma}, \overline{\gamma}') = d(\overline{\gamma}, \overline{\gamma}') h(\overline{\gamma}, \overline{\gamma}')$ as in (4.3). Since the map ψ in (4.1) is surjective, then for any $\gamma \in \Gamma_L$, we can choose $y_{\overline{\gamma}} \in \mathrm{KUM}(L/F)$ such that $\mathrm{gr}(y_{\overline{\gamma}}) + \Gamma_F = \overline{\gamma}$. Write

$y_{\overline{\gamma}} = a_{\overline{\gamma}} x_{\overline{\gamma}}$, where $a_{\overline{\gamma}} \in (A_0F)^*$. We have $x_0 = 1$ and we can choose $y_0 = 1$, so $a_0 = 1$. Let $b_{\overline{\gamma}} \in A_0^*$ and $c_{\overline{\gamma}} \in F^*$ be such that $a_{\overline{\gamma}} = b_{\overline{\gamma}} c_{\overline{\gamma}}$ (with $b_0 = c_0 = 1$), then we have :

$$\begin{aligned} y_{\overline{\gamma}} y_{\overline{\gamma}'} &= a_{\overline{\gamma}} \omega_{\overline{\gamma}}(a_{\overline{\gamma}'}) d(\overline{\gamma}, \overline{\gamma}') a^{-1}_{\overline{\gamma}+\overline{\gamma}'} h(\bar{\gamma}, \overline{\gamma}') y_{\overline{\gamma}+\overline{\gamma}'} \\ &= b_{\overline{\gamma}} \omega_{\overline{\gamma}}(b_{\overline{\gamma}'}) d(\overline{\gamma}, \overline{\gamma}') b^{-1}_{\overline{\gamma}+\bar{\gamma}'} c_{\overline{\gamma}} c_{\overline{\gamma}'} c^{-1}_{\overline{\gamma}+\bar{\gamma}'} h(\bar{\gamma}, \bar{\gamma}') y_{\overline{\gamma}+\overline{\gamma}'} \\ &= d'(\overline{\gamma}, \overline{\gamma}') h'(\bar{\gamma}, \bar{\gamma}') y_{\overline{\gamma}+\overline{\gamma}'}, \end{aligned}$$

where $d'(\overline{\gamma}, \overline{\gamma}') = b_{\overline{\gamma}} \omega_{\overline{\gamma}}(b_{\overline{\gamma}'}) d(\overline{\gamma}, \overline{\gamma}') b^{-1}_{\overline{\gamma}+\bar{\gamma}'}$ and $h'(\overline{\gamma}, \overline{\gamma}') = c_{\overline{\gamma}} c_{\overline{\gamma}'} c^{-1}_{\overline{\gamma}+\bar{\gamma}'} h(\bar{\gamma}, \bar{\gamma}')$. Since $y_{\overline{\gamma}}$, $y_{\overline{\gamma}'}$ and $y_{\overline{\gamma}+\overline{\gamma}'}$ are in $\mathrm{KUM}(L/F)$ and $h'(\overline{\gamma}, \overline{\gamma}') \in F^*$, then $d'(\overline{\gamma}, \overline{\gamma}') \in \mathrm{KUM}(L/F) \cap A_0$ $(= \mathrm{KUM}(L_0/F_0))$ (see (4.1)). Moreover, one can easily check that $d' \in Z^2(\Gamma_L/\Gamma_F, \mathrm{KUM}(L_0/F_0))_{\mathrm{sym}}$ (this follows from the equality $(y_{\overline{\gamma}} y_{\overline{\gamma}'}) y_{\overline{\gamma}''} = y_{\overline{\gamma}}(y_{\overline{\gamma}'} y_{\overline{\gamma}''})$, the fact that h' which is cohomologous to $\mathrm{res}^{\Gamma_A/\Gamma_F}_{\Gamma_L/\Gamma_F}(h)$, is a symmetric 2-cocycle, and the fact that $y_{\overline{\gamma}}$ are pairwise commuting for $\overline{\gamma} \in \Gamma_L/\Gamma_F$). Also, since $y_0 = 1$, then d' is normalized.

Now, let $\omega' : \Gamma_L/\Gamma_F \to \mathrm{Aut}(A_0)$ be the map defined by $\omega'_{\overline{\gamma}} = \mathrm{Int}(b_{\overline{\gamma}}) \omega_{\overline{\gamma}}$ (i.e., $\omega'_{\overline{\gamma}}(a) = b_{\overline{\gamma}} \omega_{\overline{\gamma}}(a) b^{-1}_{\overline{\gamma}}$ for all $a \in A_0$ and $\overline{\gamma} \in \Gamma_L/\Gamma_F$), then for any $a \in L_0$ and any $\overline{\gamma} \in \Gamma_L/\Gamma_F$, we have $\omega'_{\overline{\gamma}}(a) = b_{\overline{\gamma}} x_{\overline{\gamma}} a x^{-1}_{\overline{\gamma}} b^{-1}_{\overline{\gamma}} = a_{\overline{\gamma}} x_{\bar{\gamma}} a x^{-1}_{\overline{\gamma}} a^{-1}_{\overline{\gamma}} = y_{\bar{\gamma}} a y^{-1}_{\overline{\gamma}} = a$ (because $y_{\overline{\gamma}} \in \mathrm{KUM}(L/F)$). One can easily see that (ω', i_*d') is a factor set of Γ_L/Γ_F in A_0, cohomologous to $\mathrm{res}^{\Gamma_A/\Gamma_F}_{\Gamma_L/\Gamma_F}(\omega, d)$. Finally, the equality $y_{\overline{\gamma}} y_{\overline{\gamma}'} = d'(\overline{\gamma}, \overline{\gamma}') h'(\overline{\gamma}, \overline{\gamma}') y_{\overline{\gamma}+\overline{\gamma}'}$ yields, by considering classes modulo F^* in $\mathrm{kum}(L/F)$, that we have $\overline{y}_{\overline{\gamma}} \overline{y}_{\overline{\gamma}'} = e(d'(\overline{\gamma}, \overline{\gamma}')) \overline{y}_{\overline{\gamma}+\overline{\gamma}'}$, where $e :\ \mathrm{KUM}(L_0/F_0) \to \mathrm{kum}(L_0/F_0)$ is the canonical surjective homomorphism (we identify here $\mathrm{kum}(L_0/F_0)$ with its canonical image in $\mathrm{kum}(L/F)$). Hence, $e_*([d']) = [\alpha_L]$. $\square$

(4.5) Let F be a graded field, D a graded division algebra over F, S a finite abelian subgoup of D^*/F^* with exponent m, and for any $s \in S$, let d_s be a representative of s in D^*. Suppose that $\operatorname{char}(F)$ does not divide $\deg(D)$, F_0 contains a primitive m^{th} root of unity and let $F(S) = F[d_s \mid s \in S]$ be the subring of D generated by F and the elements d_s $(s \in S)$. If d_s are pairwise commuting, then as in the ungraded case $F(S)$ is a Kummer graded field extension of F with $\operatorname{kum}(F(S)) = S$ (it suffices to observe that $F(S)$ is a graded field and that $q(F(S)) = q(F)(S)$ when S is identified with its canonical image in $q(D)^*/q(F)^*$).

Theorem 4.6. *Let F be a graded field, A a graded central simple algebra over F with A_0 simple and (ω, d) [resp., h] the factor set of Γ_A/Γ_F in A_0 [resp., the cocycle of $Z^2(\Gamma_A/\Gamma_F, F^*)_{\mathrm{sym}}$] seen in (4.3). Assume that* $\operatorname{char}(F)$ *does not divide* $\deg(A)$*, F_0 contains enough roots of unity and that there are :*

1. *a Kummer field extension M of F_0 in A_0, and a subgroup R of Γ_A/Γ_F acting trivially on M,*
2. *a normalized cocycle $d' \in Z^2(R, \mathrm{KUM}(M/F_0))_{\mathrm{sym}}$ and a map $\omega' : R \to \mathrm{Aut}(A_0)$ such that (ω', i_*d') is a factor set of R in A_0, cohomologous to $\operatorname{res}_R^{\Gamma_A/\Gamma_F}(\omega, d)$, and such that $\omega'_{\overline{\gamma}}(a) = a$ for all $a \in M$ and $\overline{\gamma} \in R$.*

Then, there exists a Kummer graded subfield L of A such that

1. *$L_0 = M$, $\Gamma_L/\Gamma_F = R$ and*
2. *$e_*([d']) = [\alpha_L]$.*

PROOF. Write $A = \oplus_{\overline{\gamma}\in\Gamma_A/\Gamma_F} A_0 F x_{\overline{\gamma}}$, where $x_0 = 1$, $x_{\overline{\gamma}} \in A^*$, $\mathrm{gr}(x_{\overline{\gamma}})+\Gamma_F = \overline{\gamma}$, $x_{\overline{\gamma}}a = \omega_{\overline{\gamma}}(a)x_{\overline{\gamma}}$ and $x_{\overline{\gamma}}x_{\overline{\gamma}'} = d(\overline{\gamma},\overline{\gamma}')h(\overline{\gamma},\overline{\gamma}')x_{\overline{\gamma}+\overline{\gamma}'}$ as in (4.3). The fact that (ω', i_*d') is cohomologous to $\mathrm{res}_R^{\Gamma_A/\Gamma_F}(\omega, d)$ means that there is a family $(b_{\overline{\gamma}})_{\overline{\gamma}\in R}$ of elements of A_0^* such that for all $a \in A_0$ and $\overline{\gamma},\overline{\gamma}' \in R$, we have $\omega'_{\overline{\gamma}}(a) = b_{\overline{\gamma}}\omega_{\overline{\gamma}}(a)b_{\overline{\gamma}}^{-1}$ and $d'(\overline{\gamma},\overline{\gamma}') = b_{\overline{\gamma}}\omega_{\overline{\gamma}}(b_{\overline{\gamma}'})d(\overline{\gamma},\overline{\gamma}')b_{\overline{\gamma}+\overline{\gamma}'}^{-1}$. Let $y_{\overline{\gamma}} = b_{\overline{\gamma}}x_{\bar{\gamma}}$ for all $\overline{\gamma} \in R$. Then, we have $y_{\overline{\gamma}}y_{\overline{\gamma}'} = d'(\overline{\gamma},\overline{\gamma}')h(\overline{\gamma},\overline{\gamma}')y_{\overline{\gamma}+\overline{\gamma}'}$. Let $L = \oplus_{\overline{\gamma}\in R} MFy_{\overline{\gamma}}(\subseteq A)$. Since d' and h are symmetric, then $y_{\overline{\gamma}}$ are pairwise commuting. Moreover, by hypotheses $\omega'_{\overline{\gamma}}(a) = a$ for all $a \in M$ and $\overline{\gamma} \in R$, so L is a commutative graded subring of A with $L_0 = M$ and $\Gamma_L/\Gamma_F = R$.

Since both d' and $\mathrm{res}_R^{\Gamma_A/\Gamma_F} h$ are normalized and $y_{\overline{\gamma}}(= b_{\overline{\gamma}}x_{\overline{\gamma}})$ is invertible for any $\overline{\gamma} \in R$, then $y_0 = 1$ (it suffices to see that $y_0y_{\overline{\gamma}} = d'(0,\overline{\gamma})h(0,\overline{\gamma})y_{\overline{\gamma}} = y_{\overline{\gamma}}$). For any $\overline{\gamma} \in R$, we have $y_{\overline{\gamma}}y_{-\overline{\gamma}} = d'(\overline{\gamma},-\overline{\gamma})h(\overline{\gamma},-\overline{\gamma})y_0$, hence $y_{\overline{\gamma}}$ is invertible in L. One can easily see that nonzero homogeneous elements of L are the elements of the form $ay_{\overline{\gamma}}$, where $a \in (MF)^*$, and $\overline{\gamma} \in R$, so all nonzero homogeneous elements of L are invertible. This shows that L is a graded subfield of A.

Let S be the subgroup of L^*/F^* generated by $\mathrm{kum}(M/F_0)$ and the set $\{\overline{y}_{\overline{\gamma}}\}_{\overline{\gamma}\in R}$, where $\overline{y}_{\overline{\gamma}}$ is the class of $y_{\overline{\gamma}}$ in L^*/F^* (and where as in above, we identify $\mathrm{kum}(M/F_0)$ with its canonical image in $\mathrm{kum}(MF/F)$). One can easily see that up to a graded isomorphism we have $L = F(S)$. Therefore, by (4.5) L is a Kummer graded field extension of F with $\mathrm{kum}(L/F) = S$. Considering classes in $\mathrm{kum}(L/F)$, we have $\overline{y}_{\overline{\gamma}}\overline{y}_{\overline{\gamma}'} = e(d'(\overline{\gamma},\overline{\gamma}'))\overline{y}_{\overline{\gamma}+\overline{\gamma}'}$, where

$e : \mathrm{KUM}(M/F_0) \to \mathrm{kum}(M/F_0)$ is the canonical surjective homomorphism (we identify here $\mathrm{kum}(M/F_0)$ with its canonical image in $\mathrm{kum}(L/F)$), so $\mathrm{kum}(L/F)$ is the extension of $\mathrm{kum}(M/F_0)$ by R with cocycle $e_*([d'])$. This shows that $e_*([d']) = [\alpha_L]$. □

PROPOSITION 4.7. *Let E be a Henselian valued field, D be a tame central division algebra over E, n a positive integer, and K a tame finite-dimensional field extension of E such that $\Gamma_K \subseteq \Gamma_D$, then the following statements are equivalent:*

(1) *K embeds in $M_n(D)$.*

(2) *G_wK embeds in $M_n(GD)$, where w is the extension of the valuation of E to K.*

PROOF. Let s be the smallest positive integer such that K embeds in $M_s(D)$. Since ungraded algebras can be considered as (trivially) graded algebras, then by Proposition 2.7, we have $s = [K : E]\mathrm{ind}(D \otimes_E K)/\deg(D)$. Note that because K is tame over E, then it is defectless over E, so $[K : E] = [\overline{K} : \overline{E}](\Gamma_K : \Gamma_E) = [GK : GE]$. Moreover, by [**HW(b)99**, Corollary 5.7] $\mathrm{ind}(D \otimes_E K) = \mathrm{ind}(GD \otimes_{GE} G_wK)$, and obviously we have $\deg(D) = \deg(GD)$. So, s is also the smallest positive integer such that G_wK embeds in $M_s(GD)$. Therefore, for any positive integer n, K embeds in $M_n(D)$ if and only if G_wK embeds in $M_n(GD)$ (by Proposition 2.7). □

COROLLARY 4.8. *Let E be a Henselian valued field with $\overline{E}$ containing enough roots of unity, n a positive integer, D be a tame central division algebra over E, $B = M_n(D)$, and suppose that* char$(\overline{E})$ *does not divide* deg(B) *and that there exists a Kummer subfield* K *of* B *with* $\Gamma_K \subseteq \Gamma_D$, *then there is a normalized cocycle* $d' \in Z^2(\Gamma_K/\Gamma_E, \mathrm{KUM}(\overline{K}/\overline{E}))_{\mathrm{sym}}$ *(for the trivial action of* Γ_K/Γ_E *on* $\mathrm{KUM}(\overline{K}/\overline{E})$*) and a map* $w' : \Gamma_K/\Gamma_E \to \mathrm{Aut}(M_n(\overline{D}))$, *which satisfies* $\omega'_{\overline{\gamma}}(a) = a$ *for all* $a \in \overline{K}$ *and* $\overline{\gamma} \in \Gamma_K/\Gamma_E$, *such that:*

(a) (ω', i_*d') *is a factor set of* Γ_K/Γ_E *in* $\overline{B}$, *cohomologous to* $\mathrm{res}^{\Gamma_B/\Gamma_E}_{\Gamma_K/\Gamma_E}(\omega, d)$, *where* (ω, d) *is the factor set corresponding to a representation of* $M_n(GD)$ *as in (4.3), and*

(b) $e_*([d']) = [\alpha_{GK}]$.

PROOF. Indeed, take the residually simple tame E-gauge β on B as defined in (3.1), then we have $G_\beta B \cong_g M_n(GD)$. We have also $\Gamma_{GK} = \Gamma_K \subseteq \Gamma_B = \Gamma_{M_n(D)} = \Gamma_D$, so by Proposition 4.7 GK embeds in $M_n(GD)$. Moreover since K is a tame Kummer field extension of E and $\overline{E}$ contains enough roots of unity, then GK is a Kummer graded field extension of E. Our corollary follows then by Theorem 4.4. □

Similarly, the following corollary follows by applying Theorem 4.6, Proposition 4.7 and the fact that isomorphism classes of tame (abelian) field extensions of E are in one-to-one correspondence with the isomorphism classes of (abelian) graded field extensions of GE.

COROLLARY 4.9. *Let* E *be a Henselian valued field with* $\overline{E}$ *containing enough roots of unity,* n *a positive integer,* D *be a tame central division algebra over* E, $B = M_n(D)$, *and suppose that* char$(\overline{E})$ *does not divide* deg(B) *and that there are:*

(1) *a Kummer field extension* M *of* $\overline{E}$ *in* $M_n(\overline{D})$, *and a subgroup* R *of* Γ_D/Γ_E *acting trivially on* M,

(2) *a normalized cocycle* $d' \in Z^2(R, \mathrm{KUM}(M/\overline{E}))_{\mathrm{sym}}$ *and a map* $\omega' : R \to \mathrm{Aut}(M_n(\overline{D}))$ *such that* (ω', i_*d') *is a factor set of* R *in* $M_n(\overline{D})$, *cohomologous to* $\mathrm{res}_R^{\Gamma_D/\Gamma_E}(\omega, d)$ *(where* (ω, d) *is the factor set corresponding to a representation of* $M_n(GD)$ *as in (4.3)) and such that* $\omega'_{\overline{\gamma}}(a) = a$ *for all* $a \in M$ *and* $\overline{\gamma} \in R$.

Then, there exists a Kummer subfield K *of* B *with* $\Gamma_K \subseteq \Gamma_D$, *such that:*

(1) $\overline{K} = M$, $\Gamma_K/\Gamma_E = R$ *and*

(2) $e_*([d']) = [\alpha_{GK}]$.

REMARK 4.10. One can easily see that Corollaries 4.8 and 4.9 restrict to [**M10**, Corollaries 2.11 and 2.12] when $n = 1$.

CHAPTER 2

Discriminants of involutions of the first kind over Henselian fields

In the present chapter we study the discriminants of both orthogonal and symplectic involutions on central simple algebras over Henselian valued fields.

Recall that Chacron et al. determined, in terms of residue information, the discriminant of an arbitrary involution σ on a central division algebra D of exponent 2 and of degree greater than 2 over a Henselian field (see [**CDTWY95**, Theorem 4, p. 69]). Using graded techniques, we showed in a previous work that we have an analogous result for some orthogonal involutions on central simple algebras with residually simple tame gauges over a wide class of Henselian base fields (see [**M11**, Corollary 2.18]). Our first aim in this chapter is to show that we have a more general result over an arbitrary Henselian valued field of residue characteristic different from 2 (Corollary 3.8).

A second objective in this chapter is to study the discriminants of symplectic involutions on central simple algebras over Henselian valued fields. Recall that if E is a field and A is a central simple algebra over E with a symplectic involution σ, then $\deg(A)$ is even, say $\deg(A) = 2m$. The corresponding pfaffian reduced norm is defined to be the homogeneous polynomial function of degree m :

$$\mathrm{Nrp}_\sigma : \mathrm{Sym}(A,\sigma) \to E$$

uniquely determined by the following conditions :
$\mathrm{Nrp}_\sigma(1) = 1$ and $\mathrm{Nrp}_\sigma(x)^2 = \mathrm{Nrd}_A(x)$ for all $x \in \mathrm{Sym}(A,\sigma)$, where $\mathrm{Sym}(A,\sigma)$ is the E-space of symmetric elements of A under σ.

Let $\mathrm{G_m}$ be the multiplicative group scheme. The Kummer exact sequence :

$$1 \to \mu_2 \to \mathrm{G_m} \overset{(\,)^2}{\to} \mathrm{G_m} \to 1$$

allows us to identify the cohomology group $\mathrm{H}^1(E,\mu_2)$ with the quotient group E^*/E^{*2} and the cohomology group $\mathrm{H}^2(E,\mu_2)$ with the 2-torsion subgroup ${}_2\mathrm{Br}(E)$ of the Brauer group of E (see [**KMRT98**, p.413]). For $a \in E^*$, let $(a)_2 \in \mathrm{H}^1(E,\mu_2)$ be the cohomology class associated to aE^{*2}, and let $[A] \in \mathrm{H}^2(E,\mu_2)$ be the cohomology class associated to the Brauer class of A. A map $\Delta_\sigma : \mathrm{Sym}(A,\sigma)^* \to \mathrm{H}^3(E,\mu_2)$ is then defined by $\Delta_\sigma(a) = (\mathrm{Nrp}_\sigma(a))_2 \cup [A]$, where $\mathrm{Sym}(A,\sigma)^*$ is the set of invertible elements of $\mathrm{Sym}(A,\sigma)$, and where $\cup$ is the cup-product. If $\deg(A) \equiv 0 \bmod 4$ and τ is another symplectic involution on A, then the discriminant $\Delta_\sigma(\tau)$ is defined to be $\Delta_\sigma(a)$, where a is an arbitrary element of $\mathrm{Sym}(A,\sigma)^*$ such that $\tau = \mathrm{Int}(a) \circ \sigma$. This discriminant depends only on the conjugacy classes of σ and τ (see [**BMT03**, Proposition 1(a)]). It is then an invariant of symplectic involutions on A.

Berhuy, Musurrò and Tignol defined this invariant in [**BMT03**] on the basis of Rost's cohomological invariant of degree 3 for torsors under symplectic groups. They established in [**BMT03**, Theorem

4, Corollary 5 and Proposition 6] relationships between this discriminant and trace forms. They also showed that for a central simple algebra A of degree 8 and index 4 with a symplectic involution σ, the triviality of $\Delta(\sigma) := \Delta_\alpha(\sigma)$, where α is a hyperbolic involution on A, is equivalent to a special decomposition of σ on quaternion subalgebras of A (see [**BMT03**, Theorem 8]).

Garibaldi, Parimala and Tignol showed in [**GPT09**] that this (relative) discriminant leads to a unique absolute invariant in the sense of [**GMS03**] for symplectic involutions on simple algebras of a fixed degree n divisible by 8, or equivalently a cohomological invariant of the split adjoint group PGSp_{2m} of type C_m, for m divisible by 4 since the cohomological set $\mathrm{H}^1(K, \mathrm{PGSp}_{2m})$ classifies central simple K-algebras of degree $2m$ endowed with a symplectic involution for each extension K of the base field E (see [**GPT09**, Theorem A]). In particular they showed a decomposition criterion of symplectic involutions on central simple algebras of degree 8 in terms of the triviality of this absolute invariant.

Our second goal in the present chapter, is to develop a study for the discriminant of symplectic graded involutions (that we define in the fourth section), especially on non-inertially split graded simple algebras with simple 0-component. In particular, we show that if F is a graded field of characteristic different from 2, D is a graded central division algebra over F with $\exp(D) = 2$ and $|\ker(\theta_D)| > 4$ (see the preliminaries), $A = M_n(D)$, and σ is a graded involution of symplectic type on A, then there is only a finite number of values for the discriminants $\Delta_\sigma(\tau)$, where τ describes all graded involutions of symplectic type on A (see Proposition 4.10). Consequently, for any graded central simple algebra C over F with C_0 simple non split, $\exp(C) = 2$, $|\ker(\theta_C)| > 4$ and $\frac{\deg(C)}{\operatorname{ind}(C)}$ even, we have $\Delta_\sigma(\tau) = 0$ for any graded involutions of symplectic type σ and τ on C (see Corollary 4.11). We apply this study in the fifth section to get results for symplectic involutions on central simple algebras over Henselian valued fields. In particular, we show that if E is a Henselian valued field with residue characteristic different from 2, D is a central division algebra of exponent 2 over E with $|\ker(\theta_D)| > 4$, and $B = M_n(D)$ with n even, then for any symplectic involutions σ, τ on B, preserving the tame gauge β on B defined in (1.3.1), we have $\Delta_\sigma(\tau) = 0$ (see Corollary 5.5).

1. On graded involutions of the first kind

Let A be a graded central simple algebra over a graded field F and σ be a graded involution of the first kind on A, we recall that σ is called of orthogonal type, or simply orthogonal [resp., of symplectic type, or simply symplectic] if the involution $q(\sigma) := \sigma \otimes \mathrm{id}_{q(F)}$ defined on $q(A)$, is so. We recall also that the graded spaces of symmetric and skew-symmetric elements of A (under σ) are defined to be, respectively, $\mathrm{Sym}(A, \sigma) = \{a \in A \mid \sigma(a) = a\}$ and $\mathrm{Skew}(A, \sigma) = \{a \in A \mid \sigma(a) = -a\}$.

Proposition 1.1. *Let F be a graded field, A be a* (*nontrivial*) *graded central division algebra over F, and suppose that there exists a graded involution of the first kind σ on A. Then, we have the following statements:*

(1) *If A_0 is not a field, then every homogeneous component of A contains a nonzero symmetric element.*

(2) *If A_0 is a field and $\gamma \in \Gamma_A$ satisfies $\theta_A(\gamma + \Gamma_F) \neq \sigma_{|A_0}$, then A_γ contains a nonzero symmetric element.*

(3) *If A_0 is a field and $\gamma \in \Gamma_A$ satisfies $\theta_A(\gamma + \Gamma_F) = \sigma_{|A_0}$, then either $A_\gamma \subseteq \mathrm{Sym}(A, \sigma)$ or $A_\gamma \subseteq \mathrm{Skew}(A, \sigma)$.*

Proof. Let $\gamma \in \Gamma_A$ and suppose that there exists $u \in A_\gamma$ with $\sigma(u) \neq u$, then $u - \sigma(u)$ is a nonzero skew-symmetric element in A_γ. Now, pick a nonzero element $x \in A_\gamma$ with $\sigma(x) = \pm x$. Then $\mathrm{Int}(x) \circ \sigma_{|A_0}$ is an involution on A_0. If A_0 is not a field [resp., if A_0 is a field and γ satisfies $\theta_A(\gamma + \Gamma_F) \neq \sigma_{|A_0}$], then $\mathrm{Int}(x) \circ \sigma_{|A_0}$ cannot be the identity, so there exists $a \in A_0^*$ with $\mathrm{Int}(x) \circ \sigma(a) = -a$. If $\sigma(x) = -x$, then $\sigma(ax) = ax$, hence ax is a nonzero symmetric element in A_γ. This shows (1) and (2). If A_0 is a field and γ satisfies $\theta_A(\gamma + \Gamma_F) = \sigma_{|A_0}$, then $\mathrm{Int}(x) \circ \sigma_{|A_0} = \mathrm{id}_{A_0}$, so we have either $\sigma(y) = y$ for all $y \in A_\gamma$ (if $\sigma(x) = x$), or $\sigma(y) = -y$ for all $y \in A_\gamma$ (if $\sigma(x) = -x$). $\square$

Remark 1.2. Under the hypotheses of Proposition 1.1, for $\gamma \in \Gamma_A$ and $x \in A_\gamma$ such that $x \neq 0$ and $\sigma(x) = \pm x$, the involution $\mathrm{Int}(x) \circ \sigma$ on A restricts to $\theta_A(\gamma + \Gamma_F) \circ \sigma_{|Z(A_0)}$ on $Z(A_0)$. Therefore, $\theta_A(\gamma + \Gamma_F)^2 = \mathrm{id}_{Z(A_0)}$ (for $\sigma^2_{|Z(A_0)} = \mathrm{id}_{Z(A_0)}$), so $Z(A_0)$ is a multiquadratic Galois extension of F_0 (because $\mathrm{Gal}(Z(A_0)/F_0) = \theta_A(\Gamma_A/\Gamma_F)$). Moreover, one can easily see that the restriction of $\mathrm{Int}(x) \circ \sigma$ to A_0 is an involution of the first kind if and only if $\theta_A(\gamma + \Gamma_F) = \sigma_{|Z(A_0)}$.

Let F be a graded field, A be a graded central division algebra over F with A_0 not a field, and suppose that there exists a graded involution of the first kind σ on A. Then, by Proposition 1.1, for all $\gamma \in \Gamma_A$, we may find a nonzero element $x \in A_\gamma$ such that $\sigma(x_\gamma) = x_\gamma$. Set

$$\epsilon_\gamma = \begin{cases} 0 & \text{if} \quad \theta_A(\gamma + \Gamma_F) \neq \sigma_{|Z(A_0)}, \\ 1 & \text{if} \quad \theta_A(\gamma + \Gamma_F) = \sigma_{|Z(A_0)} \text{ and } \mathrm{Int}(x) \circ \sigma_{|A_0} \text{ is an} \\ & \phantom{\text{if}} \quad \text{orthogonal involution,} \\ -1 & \text{if} \quad \theta_A(\gamma + \Gamma_F) = \sigma_{|Z(A_0)} \text{ and } \mathrm{Int}(x) \circ \sigma_{|A_0} \text{ is a} \\ & \phantom{\text{if}} \quad \text{symplectic involution.} \end{cases}$$

Also, let $\epsilon = 1$ if σ is orthogonal and $\epsilon = -1$ if σ is symplectic. Then we have the following Proposition:

PROPOSITION 1.3. *Under the hypotheses of Proposition 1.1, if* $\mathrm{char}(F) \neq 2$ *and* A_0 *is not a field, then for any set of representatives* γ_1, ..., γ_n *of the various cosets of* Γ_A *modulo* Γ_F, *we have* $\sum_{i=1}^{n} \epsilon_i = \epsilon |\ker(\theta_\mathrm{A})|^{\frac{1}{2}}$, *where* $\epsilon_i = \epsilon_{\gamma_i}$.

PROOF. Let $r = |\ker(\theta_\mathrm{A})|$, $m = \deg(A_0)$ and $n = \deg(A)$. We have $[A : F] = [A_0 : F_0](\Gamma_A : \Gamma_F) = [A_0 : Z(A_0)][Z(A_0) : F_0](\Gamma_A : \Gamma_F) = [A_0 : Z(A_0)][Z(A_0) : F_0]^2 |\ker(\theta_A)|$, so $n = m[Z(A_0) : F_0] r^{\frac{1}{2}}$. One can easily see that we have $\mathrm{Sym}(A, \sigma) = \oplus_{i=1}^{n} \mathrm{Sym}(A_0 F, \mathrm{Int}(x_{\gamma_i}) \circ \sigma) x_{\gamma_i}$, so $[\mathrm{Sym}(A, \sigma) : F] = \sum_{i=1}^{n} [\mathrm{Sym}(A_0 F, \mathrm{Int}(x_{\gamma_i}) \circ \sigma) : F]$.

Let $\delta \in \Gamma_A$ be such that $\sigma_{|Z(A_0)} = \theta_A(\delta + \Gamma_F)$. We may assume that $(\gamma_1 - \delta) + \Gamma_F$, ..., $(\gamma_r - \delta) + \Gamma_F$ are the different elements of $\ker(\theta_\mathrm{A})$, so by the above (see remark 1.2) $\mathrm{Int}(x_{\gamma_i}) \circ \sigma$ are involutions of the first kind [resp., of the second kind] if and only if $i \in \{1, ..., r\}$ [resp., $i \in \{r+1, ..., n\}$].

We have

$$\begin{aligned}\sum_{i=1}^{r}[\mathrm{Sym}(A_0F, \mathrm{Int}(x_{\gamma_i}) \circ \sigma) : F] &= \sum_{i=1}^{r}[\mathrm{Sym}(A_0F, \mathrm{Int}(x_{\gamma_i}) \circ \sigma) : \\ &Z(A_0)F][Z(A_0)F : F] \\ &= \frac{1}{2}(\sum_{i=1}^{r} m(m+\epsilon_i))[Z(A_0) : F_0] \\ &= \frac{1}{2}rm^2[Z(A_0) : F_0] \; + \; \frac{1}{2}(\sum_{i=1}^{r}\epsilon_i)m \\ &[Z(A_0) : F_0].\end{aligned}$$

For $i > r$, let M_i be the graded subfield of $Z(A_0)F$ elementwise invariant under $\mathrm{Int}(x_{\gamma_i}) \circ \sigma$, then we have $[\mathrm{Sym}(A_0F, \mathrm{Int}(x_{\gamma_i}) \circ \sigma) : M_i] = \frac{1}{2}[A_0F : M_i]$, hence $[\mathrm{Sym}(A_0F, \mathrm{Int}(x_{\gamma_i}) \circ \sigma) : F] = \frac{1}{2}[A_0 : F_0]$. Therefore,

$$\begin{aligned}\sum_{i=r+1}^{n}[\mathrm{Sym}(A_0F, \mathrm{Int}(x_{\gamma_i}) \circ \sigma) \; : \; F] &= \frac{1}{2}((\Gamma_A : \Gamma_F) - r)[A_0 : F_0] \\ &= \frac{1}{2}((\Gamma_A : \Gamma_F) - r)[A_0 : Z(A_0)] \\ &[Z(A_0) : F_0]\end{aligned}$$

$$= \frac{1}{2}(r[Z(A_0) : F_0] - r)[A_0 : Z(A_0)]$$

$$[Z(A_0) : F_0]$$

$$= \frac{1}{2}r([Z(A_0) : F_0] - 1)m^2[Z(A_0) : F_0]$$

$$= \frac{1}{2}r[Z(A_0) : F_0]^2m^2 \; - \; \frac{1}{2}rm^2$$

$$[Z(A_0) : F_0]$$

$$= \frac{1}{2}n^2 \; - \; \frac{1}{2}rm^2[Z(A_0) : F_0]$$

So,

$$[\mathrm{Sym}(A, \sigma) \; : \; F] = \frac{1}{2}(\sum_{i=1}^{r} \epsilon_i)m[Z(A_0) : F_0] \; + \; \frac{1}{2}n^2.$$

On the other hand, we have $[\mathrm{Sym}(A, \sigma) \; : \; F] = \frac{1}{2}n(n + \epsilon)$, hence

$$\sum_{i=1}^{n} \epsilon_i = \frac{\epsilon n}{m[Z(A_0) : F_0]} = \epsilon r^{\frac{1}{2}}.$$

$\square$

This Proposition shows that for $i \in \{1, ..., r\}$ (when $r \geq 2$) and $\gamma_1, ..., \gamma_r$ being chosen as in the proof, the $\mathrm{Int}(x_{\gamma_i}) \circ \sigma_{|A_0}$ cannot be all of the same type, since otherwise it follows from the equality $\sum_{i=1}^{r} \epsilon_i = \epsilon r^{\frac{1}{2}}$ that $\pm r = r^{\frac{1}{2}}$ which is not possible. Let m be the number of i ($1 \leq i \leq r$) such that $\mathrm{Int}(x_{\gamma_i}) \circ \sigma_{|A_0}$ is symplectic (i.e., such that $\epsilon_i = -1$), then it follows from the equality $\sum_{i=1}^{r} \epsilon_i = \epsilon r^{\frac{1}{2}}$ that $r - 2m = \epsilon r^{\frac{1}{2}}$. One can easily see that this yields $m = \frac{r \pm \sqrt{r}}{2}$.

REMARK 1.4. Let F be a graded field of characteristic different from 2, and A be an inertially split graded central simple algebra over F such that A_0 is simple. Suppose that there is a graded involution of the first kind σ on A such that $\sigma|_{A_0}$ is also of the first kind. We proved in [**M11**, Proposition 2.8] that σ and $\sigma|_{A_0}$ have the same type. This result is not true when A is not inertially split. Indeed, with the hypotheses of Proposition 1.3 (with γ_1, ..., γ_r being chosen as in the proof of this proposition), by the above, we can assume that $\mathrm{Int}(x_{\gamma_1}) \circ \sigma_{|A_0}$ and $\mathrm{Int}(x_{\gamma_2}) \circ \sigma_{|A_0}$ have different types. Let $\sigma_1 = \mathrm{Int}(x_{\gamma_1}) \circ \sigma$ and $\sigma_2 = \mathrm{Int}(x_{\gamma_2}) \circ \sigma$. Since x_{γ_1} and x_{γ_2} are in $\mathrm{Sym}(A, \sigma)^*$, then both σ_1 and σ_2 have the same type as σ, but $\sigma_1|_{A_0}(= \mathrm{Int}(x_{\gamma_1}) \circ \sigma_{|A_0})$ and $\sigma_2|_{A_0}(= \mathrm{Int}(x_{\gamma_2}) \circ \sigma_{|A_0})$ are of different types.

2. On reduced norms

THEOREM 2.1. *Let E be a field with a valuation v, B be a central simple E-algebra with a tame v-gauge β, $G_\beta B$ be the graded GE-algebra associated to B by β, b be a nonzero element of B and b' be its (canonical) image in $(G_\beta B)_{\beta(b)}$. Then, b' is invertible in $G_\beta B$ if and only if $v(\mathrm{Nrd}_B(b)) = \deg(B)\beta(b)$. Furthermore, when this occurs, we have $(\mathrm{Nrd}_B(b))' = \mathrm{Nrd}_{G_\beta B}(b')$ (in GE).*

PROOF. Let (E, v^h) be 'the' henselization of (E, v) (see [**E72**, §16]), K be a finite-dimensional field extension of E^h, which splits B and w be the unique extension of v^h to K. By [**TW10**, Corollary 1.26] $\beta_K := \beta \otimes w$ is a tame w-gauge on $B \otimes_E K$, extending β on B, and we can identify $G_{\beta_K}(B \otimes_E K)$ with $G_\beta B \otimes_{GE} G_w K$, which is split since $B \otimes_E K$ is split (see [**TW10**, Theorem 3.1]). For any $b \in B$, we have $\mathrm{Nrd}_B(b) = \mathrm{Nrd}_{B\otimes_E K}(b \otimes 1) = \det(b \otimes 1)$, $\beta(b) = \beta_K(b\otimes 1)$, $(b\otimes 1)' = b' \otimes 1$ (by identification of $G_{\beta_K}(B\otimes K)$ with $G_\beta B \otimes_{GE} G_w K$), and $\mathrm{Nrd}_{G_{\beta_B}}(b') = \mathrm{Nrd}_{G_{\beta_K}(B\otimes_E K)}(b' \otimes 1) = \det(b'\otimes 1)$. Thus, $b' \in (G_\beta B)^*$ if and only if $b'\otimes 1 \in (G_{\beta_K}(B\otimes_E K))^*$. We may then assume that E is henselian and that $B \cong M_n(E)$, where $n = \deg(B)$.

By [**TW10**, Theorem 3.1] (or [**TW15**, Theorem 4.26]) we can identify B with $\mathrm{End}_E(M)$, where M is a finite-dimensional vector space over E, and we can identify β with $\mathrm{End}(\epsilon)$, where ϵ is a v-norm on M and where $\mathrm{End}(\epsilon)$ is the v-gauge on $\mathrm{End}_E(M)$ defined by $\mathrm{End}(\epsilon)(f) = \min\{\epsilon(f(x)) - \epsilon(x) \mid x \in M\}$. Recall from [**TW10**, §1.3] (or [**TW15**, Theorem 4.26]) that $G_\beta B = \mathrm{GEnd}_{GE}(GM)$ (up to a graded isomorphism). Let $\{m_1, ..., m_n\}$ be a splitting base of M for ϵ and let $\gamma_i = \epsilon(m_i)$ for all i. Let $b = f \in \mathrm{End}(M)\backslash\{0\}$ and write $f(m_j) = \sum_{i=1}^n r_{ij} m_i$, where $r_{ij} \in E$, for $i, j = 1, ..., n$. Then, $\epsilon(f(m_j)) = \min_{1\le i\le n}(v(r_{ij}) + \gamma_i)$, and $\beta(f) = \min_{1\le j\le n}(\epsilon(f(m_j)) - \epsilon(m_j)) = \min_{1\le j\le n}[\min_{1\le i\le n}(v(r_{ij}) + \gamma_i) - \gamma_j] = \min_{1\le i,j\le n}(v(r_{ij}) + \gamma_i - \gamma_j)$. Let $\lambda := \beta(f)$, then we have $v(r_{ij}) \ge \lambda - \gamma_i + \gamma_j$ for all i, j, with equality for some i, j.

We recall that $\{m_1', ..., m_n'\}$ is a base of $G_\epsilon M$ over GE and that $f' \in G_\beta B(= \mathrm{GEnd}_{GE}(GM))$ is determined by $f'(m_j') = f(m_j) + M^{>\lambda+\gamma_j} = (\sum_{i=1}^n r_{ij} m_i) + M^{>\lambda+\gamma_j} \in GM_{\lambda+\gamma_j}$ (see [**TW10**, Proposition 1.9]). For each i, j we have

$$\epsilon(r_{ij} m_i) = v(r_{ij}) + \epsilon(m_i) = v(r_{ij}) + \gamma_i \ge \lambda + \gamma_j.$$

We get

$$f'(m'_j) = \sum_{i=1}^{n} (r_{ij} m_i + M^{>\lambda+\gamma_j}) = \sum_{i=1}^{n} s_{ij} m'_i,$$

where

$$s_{ij} = \begin{cases} r'_{ij} & \text{if } v(r_{ij}) = \lambda - \gamma_i + \gamma_j. \\ 0 & \text{if } v(r_{ij}) > \lambda - \gamma_i + \gamma_j. \end{cases}$$

Recall that $\det(f) = \sum_{\tau \in S_n} (-1)^{\mathrm{sgn}(\tau)} r_{1\tau(1)} ... r_{n\tau(n)}$, where S_n is the n-symmetric group. Note that

$$v((-1)^{\mathrm{sgn}(\tau)} r_{1\tau(1)} ... r_{n\tau(n)}) = \sum_{i=1}^{n} v(r_{i\tau(i)}) \geq \sum_{i=1}^{n} (\lambda - \gamma_i + \gamma_{\tau(i)}) = n\lambda.$$

Thus, $v(\det(f)) \geq n\lambda$ and $\det(f) + E^{>n\lambda} = \sum_{\tau \in S_n} (-1)^{\mathrm{sgn}(\tau)} r_{1\tau(1)} ... r_{n\tau(n)} + E^{>n\lambda} \in GE_{n\lambda}$. For $\tau \in S_n$, we have $v(r_{1\tau(1)} ... r_{n\tau(n)}) = n\lambda$ if and only if $v(r_{i\tau(i)}) = \lambda - \gamma_i + \gamma_{\tau(i)}$ for all i, if and only if $s_{i\tau(i)} = r'_{i\tau(i)} \neq 0$ for all i, if and only if $s_{1\tau(1)} ... s_{n\tau(n)} \neq 0$. Let T be the subset of S_n, consisting of such elements τ, then we have

$$\begin{aligned} \det(f) + E^{>n\lambda} &= \sum_{\tau \in T} (-1)^{\mathrm{sgn}(\tau)} r_{1\tau(1)} ... r_{n\tau(n)} + E^{>n\lambda} \\ &= \sum_{\tau \in T} ((-1)^{\mathrm{sgn}(\tau)} r_{1\tau(1)} ... r_{n\tau(n)})' \\ &= \sum_{\tau \in T} (-1)^{\mathrm{sgn}(\tau)} r'_{1\tau(1)} ... r'_{n\tau(n)}. \end{aligned}$$

In $GE_{n\lambda}$ we have

$$\begin{aligned} \det(f') &= \sum_{\tau\in S_n} (-1)^{\mathrm{sgn}(\tau)} s_{1\tau(1)} \dots s_{n\tau(n)} \\ &= \sum_{\tau\in T} (-1)^{\mathrm{sgn}(\tau)} s_{1\tau(1)} \dots s_{n\tau(n)} \\ &= \sum_{\tau\in T} (-1)^{\mathrm{sgn}(\tau)} r'_{1\tau(1)} \dots r'_{n\tau(n)} \end{aligned}$$

which shows that $\det(f') = \det(f) + E^{>n\lambda} \in GE_{n\lambda}$. Hence, $v(\det(f)) = n\lambda$ if and only if $\det(f') \neq 0$ in GE, if and only if f' is invertible in $G_\beta B$, and when this occurs $(\det(f))' = \det(f')$. $\square$

3. Discriminants of orthogonal involutions

(3.1) Let F be a graded field, A be a (finite-dimensional) graded central simple algebra over F, and let D be a graded central division algebra over F, (graded) Brauer-equivalent to A, then by [**TW10**, Proposition 2.1] A_0 is semisimple and all its simple components are Brauer-equivalent to D_0 (with respect to the Brauer group $\mathrm{Br}(Z(D_0))$, where $Z(D_0)$ is the center of D_0). We saw in the preliminaries that there is a group epimorphism $\theta_D : \Gamma_D/\Gamma_F \to \mathrm{Aut}(Z(D_0)/F_0)$, defined by mapping $\mathrm{gr}(d) + \Gamma_F$ to $\sigma : b \mapsto dbd^{-1}$, for any $d \in D^*$ and $b \in Z(D_0)$. We also saw in (1.1.1) that A_0 is simple if and only if $A \cong_g M_n(D)$, where $M_n(D)$ is given the standard (unshifted) grading built from the grading of D. Furthermore, this is also equivalent to the fact that $\Gamma_A = \Gamma_D$. Recall that when this is the case, we have $A_0 \cong M_n(D_0)$ and $\Gamma_A^* = \Gamma_A = \Gamma_D$, where Γ_A^* is the group consisting of $\mathrm{gr}(a)$, with a describing all invertible homogeneous elements of A. Moreover, in this case we have $\deg(A) = \deg(A_0)[Z(A_0) : F_0]|\ker(\theta_D)|^{\frac{1}{2}} = \deg(A_0)[Z : F]|\ker(\theta_D)|^{\frac{1}{2}}$, where $Z = Z(A_0.F) = Z(A_0).F$.

LEMMA 3.2. *Let E be a Henselian valued field with residue characteristic different from 2 and D be a central division algebra of exponent 2 over E. If D is not inertially split, then $|\ker(\theta_D)|$ is a nontrivial power of 4.*

PROOF. Since $\operatorname{char}(\overline{E}) \neq 2$ and $\exp(D) = 2$ (hence $\deg(D)$ is a power of 2), then D is necessarily defectless over E, so $[D : E] = [\overline{D} : \overline{E}](\Gamma_D : \Gamma_E) = [\overline{D} : Z(\overline{D})][Z(\overline{D}) : \overline{E}]^2|\ker(\theta_D)|$. Thus, $\deg(D) = \deg(\overline{D})[Z(\overline{D}) : \overline{E}]r$, where $r = |\ker(\theta_D)|^{\frac{1}{2}}$. In particular, r is a power of 2. By [**JW90**, Lemma 1.7] θ_D is surjective, so by [**JW90**, Lemma 5.1(iii)] θ_D cannot be injective (because D is not inertially split over E). Therefore, r is a nontrivial power of 2, which means that $|\ker(\theta_D)|$ is a nontrivial power of 4. □

(3.3) Let E be a Henselian valued field with residue characteristic different from 2, O_E be the valuation ring of E and M_E be its maximal ideal. Since $\operatorname{char}(\overline{E}) \neq 2$, then (by applying Hensel's Lemma) every element in $1 + M_E$ is a square, hence -as pointed in [**CDTWY95**, p.52]- the canonical map:

$$O_E^*/O_E^{*2} \to \overline{E}^*/\overline{E}^{*2}$$

is a group isomorphism. Therefore, there is a canonical exact sequence:

$$1 \to \overline{E}^*/\overline{E}^{*2} \xrightarrow{i} E^*/E^{*2} \xrightarrow{\bar{v}} \Gamma_E/2\Gamma_E \to 0,$$

where $\bar{v}$ is induced by the valuation v on E and i is the composite of the isomorphism $\overline{E}^{*}/\overline{E}^{*2} \cong O_E^*/O_E^{*2}$ and the inclusion $O_E^*/O_E^{*2} \to E^*/E^{*2}$.

(3.4) Let E be a Henselian valued field with residue characteristic different from 2, D be a tame central division algebra over E, B be a central simple algebra over E, Brauer equivalent to D, and suppose that there exists an orthogonal involution σ on B. We recall that by [**TW11**, Proposition 2.1] there is a tame E-gauge α on B, which is invariant under σ, i.e., such that $\alpha \circ \sigma = \alpha$. We denote the involution canonically induced by σ on $G_\alpha B$ by σ'. Since $G_\alpha B$ is graded simple, then as seen in (3.1), $\overline{B}(:= (G_\alpha B)_0)$ is semisimple, say $(G_\alpha B)_0 = B_{01} \times ... \times B_{0k}$, where B_{0i} $(1 \leq i \leq k)$ are simple algebras. If α is residually simple (i.e., $\overline{B}$ is simple), then obviously we have $\overline{\sigma}(\overline{B}) = \overline{B}$, where $\overline{\sigma}$ is the restriction of σ' to $\overline{B}$. Also, if σ is anisotropic, then $\overline{\sigma}(B_{0i}) = B_{0i}$ for all i, $(1 \leq i \leq k)$ (see [**TW11**, Corollary 2.3, p.124, and the last paragraph in page 123]).

Suppose for the rest that there is a simple subalgebra R of $\overline{B}$ such that $\overline{\sigma}(R) = R$, and let $a \in R$. Obviously, RGE is a simple graded subalgebra of $G_\alpha B$, so by [**M11**, Lemma 2.6(4)], we have:

$$\mathrm{Nrd}_{G_\alpha B}(a) = \mathrm{N}_{Z(R)/\overline{E}}(\mathrm{Nrd}_R(a))^s,$$

where $s = \deg(G_\alpha B)/\deg(R)[Z(R) : \overline{E}]$. We claim that $s = \deg(C_{G_\alpha B}(R))$, where $C_{G_\alpha B}(R)$ is the centralizer of R in $G_\alpha B$. Indeed, let $F := GE$, then by the graded version of the double centralizer theorem (see [**TW15**, Theorem 2.35]), we have $[RF : F][C_{G_\alpha B}(RF) : F] = [G_\alpha B : F]$. Note that

$$\begin{aligned}[RF : F] &= [RF : Z(R)F][Z(R)F : F] \\ &= [R : Z(R)][Z(R) : \overline{E}]\end{aligned}$$

Also, we have

$$\begin{aligned}[C_{G_\alpha B}(RF) : F] &= [C_{G_\alpha B}(RF) : Z(R)F][Z(R)F : F] \\ &= [C_{G_\alpha B}(RF) : Z(R)F][Z(R) : \overline{E}]\end{aligned}$$

So,

$$[RF : F][C_{G_\alpha B}(RF) : F] = [R : Z(R)][C_{G_\alpha B}(RF) : Z(R)F][Z(R) : \overline{E}]^2$$

Thus, $\deg(G_\alpha B) = \deg(R)\deg(C_{G_\alpha B}(RF))[Z(R) : \overline{E}]$. Therefore, $s = \deg(C_{G_\alpha B}(R))(= \deg(C_{G_\alpha B}(RF)))$.

LEMMA 3.5. *Let E be a Henselian valued field with residue characteristic different from 2, D be a central division algebra of exponent 2 over E, B a central simple algebra over E which is Brauer-equivalent to D, σ be an orthogonal involution on B and consider a tame E-gauge α on B, which is invariant under σ. Let σ' be the graded involution on $G_\alpha B$ induced by σ and suppose that there is a simple subalgebra R of $\overline{B}$ such that $\overline{\sigma}(R) = R$. If $\overline{\sigma}_{|R}$ is of the first kind and both* $\deg(R)$ *and* $\deg(C_{G_\alpha B}(R))$ *are even, then* $\operatorname{disc}(\sigma) = 1$.

PROOF. Since $\overline{\sigma}_{|R}$ is of the first kind and $\deg(R)$ is even, then by [**KMRT98**, Corollary 2.8] there is some $a \in \operatorname{Skew}(R, \overline{\sigma}_{|R})^*$. Let

$y \in B$ with $\alpha(y) = 0$ and $\overline{y}(= y') = a$ and let $x = \frac{1}{2}(y - \sigma(y))$. We have $\sigma(y)' = \sigma'(y') = \overline{\sigma}(\overline{y}) = \overline{\sigma}(a) = -a = -y'$, so $\sigma(y) = -y + m$ for some $m \in B$ with $\alpha(m) > 0(= \alpha(-y))$. Replacing $\sigma(y)$ by $-y + m$, we get $x = y - \frac{1}{2}m$, so $\alpha(x) = \alpha(y)$ (for this last equality, see the third paragraph in [**RTW07**, p. 109]). Consequently, $\overline{x}(:= x') = (y - \frac{1}{2}m)' = y' = \overline{y} = a$, which is invertible in $G_\alpha B$, so by [**TW10**, Lemma 1.3] x is invertible in B. Moreover it is clear that $\sigma(x) = -x$ (because $x = \frac{1}{2}(y - \sigma(y))$), so $x \in \mathrm{Skew}(B, \sigma)^*$. By Theorem 2.1 $v(\mathrm{Nrd}_B(x)) = 0$, where v is the valuation of E, so, using the canonical exact sequence

$$1 \to \overline{E}^*/\overline{E}^{*2} \xrightarrow{i} E^*/E^{*2} \xrightarrow{\tilde{v}} \Gamma_E/2\Gamma_E \to 0$$

of (3.3), we have

$$\begin{aligned}
\mathrm{Nrd}_B(x)E^{*2} &= i(\overline{\mathrm{Nrd}_B(x)}\,\overline{E}^{*2}) \\
&= i(\mathrm{Nrd}_{G_\alpha B}(\overline{x})\overline{E}^{*2}) \\
&= i(\mathrm{Nrd}_{G_\alpha B}(a)\overline{E}^{*2}) \\
&= i(\mathrm{N}_{Z(R)/\overline{E}}(\mathrm{Nrd}_R(a))^s\overline{E}^{*2})
\end{aligned}$$

where $\mathrm{N}_{Z(R)/\overline{E}}$ is the norm map of the field extension $Z(R)/\overline{E}$ and where $s = \deg(C_{G_\alpha B}(R))$. Since s is even, then $\mathrm{Nrd}_B(x) \in E^{*2}$, which means that $\mathrm{disc}(\sigma) = 1$ (because in this case $\frac{1}{2}\deg(B)$ is even). □

REMARK 3.6. Note that under the hypotheses of Lemma 3.5, if $s(:= \deg(C_{G_\alpha B}(R)))$ is odd, say $s = 2q+1$, then

$$\begin{aligned} \mathrm{Nrd}_{G_\alpha B}(a)\overline{E}^{*2} &= \mathrm{N}_{Z(R)/\overline{E}}(\mathrm{Nrd}_R(a))^s\overline{E}^{*2} \\ &= \mathrm{N}_{Z(R)/\overline{E}}(\mathrm{Nrd}_R(a))^{2q+1}\overline{E}^{*2} \\ &= \mathrm{N}_{Z(R)/\overline{E}}(\mathrm{Nrd}_R(a))\overline{E}^{*2} \end{aligned}$$

So, $\mathrm{disc}(\sigma) = (-1)^{\frac{1}{2}\deg(B)}\mathrm{Nrd}_B(x)E^{*2} = (-1)^{\frac{1}{2}\deg(B)}i(\mathrm{N}_{Z(R)/\overline{E}}(\mathrm{Nrd}_R(a))\overline{E}^{*2})$, where $i : \overline{E}/\overline{E}^{*2} \to E^*/E^{*2}$ is the map defined in (3.3).

THEOREM 3.7. *Let E be a Henselian valued field of residue characteristic different from 2, D be a central division algebra of exponent 2 over E with $\deg(D) > 2$, B a central simple algebra over E which is Brauer-equivalent to D, σ be an orthogonal involution on B and consider a tame E-gauge α on B, which is invariant under σ. Let σ' be the graded involution on $G_\alpha B$ induced by σ and write $\overline{B} = B_{01}\times\ldots\times B_{0k}$, where B_{0l} are simple algebras $(1 \leq l \leq k)$. If there is some i $(1 \leq i \leq k)$ such that $\overline{\sigma}(B_{0i}) = B_{0i}$, then we have the following statements:*

(1) *If $\overline{\sigma}_{|B_{0i}}$ is of the second kind, then* $\mathrm{disc}(\sigma) = 1$.

(2) *If $\overline{\sigma}_{|B_{0i}}$ is of the first kind and $\deg(B_{0i})$ is even*[1], *then we have the following:*

   (2.1) If $\deg(C_{G_\alpha B}(B_{0i}F))$ is even, then $\mathrm{disc}(\sigma) = 1$.

---

[1] Note that if D is not totally ramified, then $\overline{D} \neq \overline{E}$, so necessarily $\deg(B_{0i})$ is even (because as seen in (3.1) B_{0i} is Brauer-equivalent to $(GD)_0(= \overline{D})$.

(2.2) If $\deg(C_{G_\alpha B}(B_{0i}F))$ *is odd, then there is some* $a \in \mathrm{Skew}(B_{0i}, \overline{\sigma}_{|B_{0i}})^*$ *such that* $\mathrm{disc}(\sigma) = i(\mathrm{N}_{Z(B_{0i})/\overline{E}}(\mathrm{Nrd}_{B_{0i}}(a))\overline{E}^{*2})$*.*

PROOF. (1) We saw in (3.1) that $Z(B_{0i}) = Z(\overline{D})$. Moreover, since B is Brauer-equivalent to D and $\deg(D)$ is a multiple of 4, then $\deg(B)$ is also a multiple of 4. Recall that $Z(\overline{D})/\overline{E}$ is an abelian field extension and that $\exp(\mathrm{Gal}(Z(\overline{D})/\overline{E}\ ))$ divides $\exp(D)$ (see [**JW90**, Proposition 1.7 and Corollary 6.10]). Therefore, $\exp(\mathrm{Gal}(Z(\overline{D})/\overline{E})) = 2$ when $Z(\overline{D}) \neq \overline{E}$. This is the case here since $\overline{\sigma}_{|B_{0i}}$ is of the second kind (for we have $\overline{\sigma}_{|\overline{E}} = \mathrm{id}_{\overline{E}}$, so necessarily $Z(\overline{D})(= Z(B_{0i})) \neq \overline{E}$). We can then consider in this case an invertible element $a \in Z(B_{0i})(= Z(\overline{D}))$ such that $\overline{\sigma}(a) = -a$. Take now, as in the proof of Lemma 3.5, an element $y \in B$ with $\alpha(y) = 0$ and $\overline{y}(= y') = a$ and let $x = \frac{1}{2}(y - \sigma(y))$. Then, we have $x \in \mathrm{Skew}(B,\sigma)^*$, $\alpha(x) = 0$, $\overline{x} = a$ and $\mathrm{Nrd}_B(x)E^{*2} = i(\mathrm{Nrd}_{G_\alpha B}(a)\overline{E}^{*2})$, where $\overline{E}^*/\overline{E}^{*2} \overset{i}{\to} E^*/E^{*2}$ is the map considered in (3.3). By [**M11**, Lemma 2.6(3)], we have $\mathrm{Nrd}_{GB}(a) = \mathrm{N}_{GE[a]/GE}(a)^{\frac{1}{2}\deg(GB)}$. Moreover, by [**TW10**, Theorem 4.1] we have $\deg(G_\alpha B) = \deg(B)$, so by the above $\deg(G_\alpha B)$ is a multiple of 4. Thus, $\mathrm{Nrd}_{G_\alpha B}(a) \in \overline{E}^{*2}$, which also means that $\mathrm{Nrd}_B(x) \in E^{*2}$. Therefore, $\mathrm{disc}(\sigma) = (-1)^{\frac{1}{2}\deg(B)}\mathrm{Nrd}_B(x) \in E^{*2} = 1$.

(2) This follows by Lemma 3.5 and Remark 3.6. □

COROLLARY 3.8. *Let E be a Henselian valued field with residue characteristic different from 2, D be a central division algebra of* *exponent* 2 *over* E *with* $\deg(D) > 2$, $B = M_n(D)$, τ *be an orthogonal involution on* B *and* α *a tame* E*-gauge on* B, *invariant under* τ. *If* α *is residually simple, then we have the following statements:*

(1) *If* $\overline{\tau}_{|\overline{B}}$ *is of the second kind, then* $\operatorname{disc}(\tau) = 1$.

(2) *If* $\overline{\tau}_{|\overline{B}}$ *is of the first kind and* $\deg(\overline{B})$ *is even*[2], *then:*

(*2.1*) *If* B *is not inertially split, then* $\operatorname{disc}(\tau) = 1$.

(*2.2*) *If* B *is inertially split, then* $\overline{\tau}_{|\overline{B}}$ *is orthogonal and* $\operatorname{disc}(\tau) = i(\mathrm{N}_{Z(\overline{B})/\overline{E}}(\operatorname{disc}(\overline{\tau})))$.

PROOF. This corollary follows by Theorem 3.7. Indeed, to show (2), let's see that in this case we have $\deg(B) = \deg(G_\alpha B) = \deg(\overline{B})[Z(\overline{B}) : \overline{E}]r$, where $r = |\ker(\theta_D)|^{\frac{1}{2}}$ (see (3.1) above). So, $\deg(C_{G_\alpha B}(\overline{B}.F)) = \frac{\deg(B)}{\deg(\overline{B})[Z(\overline{B}):\overline{E}]} = r$. Therefore, if B (or equivalently D) is not inertially split, then by Lemma 3.2 $\deg(C_{G_\alpha B}(\overline{B}.F))$ is even, so by Theorem 3.7 $\operatorname{disc}(\tau) = 1$. Suppose now that B is inertially split, then $\deg(C_{G_\alpha B}(\overline{B}.F))(= r) = 1$. By Theorem 3.7 $\operatorname{disc}(\tau) = i(\mathrm{N}_{Z(\overline{B})/\overline{E}}(\mathrm{Nrd}_{\overline{B}}(a))\overline{E}^{*2}$, for some $a \in \operatorname{Skew}(\overline{B}, \overline{\tau})^*$. We recall that by [**TW11**, Proposition 1.5, p.117] τ and τ' have the same type; moreover by [**M11**, Proposition 2.8] τ' and $\overline{\tau}$ have the same type (in this case), so $\overline{\tau}$ is necessarily orthogonal. We then have $\operatorname{disc}(\tau) = i(\mathrm{N}_{Z(\overline{B})/\overline{E}}(\mathrm{Nrd}_{\overline{B}}(a))\overline{E}^{*2} = i(\mathrm{N}_{Z(\overline{B})/\overline{E}}(\operatorname{disc}(\overline{\tau})))$. □

REMARK 3.9. As in above, let E be a Henselian valued field with residue characteristic different from 2, D be a tame central division algebra over E, B be a central simple algebra over E,

---

[2] As previously seen in Theorem 3.7 if D is not totally ramified, then $\deg(\overline{B})$ is necessarily even.

Brauer equivalent to D, and suppose that there exists an involution of the first kind σ on B. Let α be a tame E-gauge on B, invariant under σ, and suppose that $\overline{B}$ is simple, then as seen in the proof of Corollary 3.8 σ and $\overline{\sigma}$ have the same type when B is inertially split. This is not the case if B is not inertially split. Indeed, by [**TW11**, Proposition 1.5] and Remark 1.4, σ and $\overline{\sigma}$ are not necessarily of the same type in this case. Consider now the general case where $\overline{B}$ is semisimple (not necessarily simple), say $\overline{B} = B_{01} \times ... \times B_{|0k}$, where B_{0l} are simple algebras and $k \geq 2$, and suppose that $\overline{\sigma}(B_{0i}) = B_{0i}$ for some i ($1 \leq i \leq k$) and that $\overline{\sigma}_{|B_{0i}}$ is of the first kind. A natural question that arises is to see whether σ and $\overline{\sigma}_{|B_{0i}}$ are of the same type. We give here an example showing that this is not always the case.

Let F be a graded field with characteristic different from 2, A be a graded central division algebra over F with $\exp(A) = 2$ and $\deg(A_0) \geq 2$, S be a graded central simple algebra over F which is (graded) Brauer-equivalent to A and write $S_0 = S_{01} \times ... \times S_{0k}$, where S_{0l} are simple rings ($1 \leq l \leq k$). Since $\exp(A) = 2$, then S has a graded involution of the first kind. Also, since $\deg(A_0)$ is even then S_{0i} has an involution ξ of the first kind. By applying [**KMRT98**, Theorem 4.14, p.51] there exist graded involutions of the first kind τ and ρ with different types on S with $\tau_{|S_{0i}} = \rho_{|S_{0i}} = \xi$ when $\deg(C_S(S_{0i}))(= \deg(S)/\deg(S_{0i})[Z(S_{0i}) : F_0])$ is even[3]. This shows that τ and $\tau_{|S_{0i}}$ are not necessarily of the same type. Let $q^h(\tau) := \tau \otimes \mathrm{id}_{q(F)^h}$ be the canonical extension of τ to $q(S)^h :=$

---

[3] Note that if we take S with $\deg(S)$ a power of 2, then $\deg(C_S(S_{0i}))$ is even.

$S \otimes_F q(F)^h$. We recall that there is a (canonical) tame $q(F)$-gauge ζ on $q(S)$ defined by $\alpha(s) = s$ for any nonzero homogeneous element s of S (see [**M11**, Proposition 1.4 and definition 1.5]). Let $\alpha := \zeta \otimes id$ be the (canonical) extension of ζ to $q(S)^h (= q(S) \otimes_{q(F)} q(F)^h)$. One can easily see that the α is invariant under $q^h(\tau)$; moreover we have $G_\alpha(q^h(S)) \cong S$, so $(G_\alpha(q^h(S)))_0 = S_{01} \times ... \times S_{0k}$, and we can identify $(q^h(\tau))'$ to τ. Since $\mathrm{char}(F) \neq 2$, then by [**TW11**, Proposition 1.5] $q(\tau)^h$ and τ are of the same type, so $\sigma := q(\tau)^h$ (defined on $B := q(S)^h$) and $\overline{\sigma}_{|S_{0i}} (= \tau_{|S_{0i}})$ are not necessarily of the same type.

4. Discriminants of symplectic graded involutions

Let F be a graded field of characteristic different from 2 and F_s be the separable graded closure of F (i.e., the tame closure of F in $F^{\mathrm{gr-alg}}$, where $F^{\mathrm{gr-alg}}$ is the algebraic graded closure of F (see [**HW(a)99**, Definition 2.8 (and the following paragraph), and Definition 3.8, pp.828-829 and 832]), then we have the following isomorphisms:

$\mathrm{H}^0(F, F_s^*) \cong F^*$, $\mathrm{H}^1(F, F_s^*) \cong 1$ (by Hilbert Theorem 90) and $\mathrm{H}^2(F, F_s^*) \cong \mathrm{Br}(F)$ (by [**HW(b)99**, Theorem 5.1, p. 103]).

Now, consider the following exact sequence of discrete $\mathrm{Gal}(F_s/F)$-modules:

$$1 \to \mu_2 \to F_s^* \xrightarrow{\phi} F_s^* \to F_s^*/F_s^{*2} = 1,$$

where $\phi(x) = x^2$, then the associated cohomological sequence yields the isomorphisms $\mathrm{H}^1(F, \mu_2) \cong F^*/F^{*2}$ and $\mathrm{H}^2(F, \mu_2) \cong {}_2\mathrm{Br}(F)$, where ${}_2\mathrm{Br}(F)$ is the 2-torsion subgroup of the Brauer group $\mathrm{Br}(F)$.

Let A be a graded central simple algebra over F with $\deg(A) \equiv 0 \bmod 4$ and suppose that there exists a graded involution σ on A. As in above, we let $\mathrm{Sym}(A,\sigma)^*$ be the set of invertible homogeneous elements of $\mathrm{Sym}(A,\sigma)$. In the same way as for the ungraded case, if σ is symplectic, then writing $\deg(A) = 2m$, we can define a unique homogeneous polynomial function of degree m:

$$\mathrm{Nrp}_\sigma : \ \mathrm{Sym}(A,\sigma) \to F,$$

by the conditions $\mathrm{Nrp}_\sigma(1) = 1$ and $\mathrm{Nrp}_\sigma(x)^2 = \mathrm{Nrd}_A(x)$ for all $x \in \mathrm{Sym}(A,\sigma)$. For a more precise definition, note that we have $\mathrm{Sym}(A,\sigma) \otimes_F q(F) = \mathrm{Sym}(q(A), q(\sigma))$, and for any $x \in \mathrm{Sym}(A,\sigma)$, we have $\mathrm{Nrd}_A(x) = \mathrm{Nrd}_{q(A)}(x \otimes 1)$. We set for any $x \in \mathrm{Sym}(A,\sigma)$, $\mathrm{Nrp}_\sigma(x) := \mathrm{Nrp}_{q(\sigma)}(x \otimes 1)$.

Analogously to the ungraded case, a map $\Delta_\sigma : \mathrm{Sym}(A,\sigma)^* \to \mathrm{H}^3(F,\mu_2)$, is defined by $\Delta_\sigma(a) = (\mathrm{Nrp}_\sigma(a))_2 \cup [A]$.

For any other symplectic graded involution τ on A, we define the discriminant $\Delta_\sigma(\tau)$ to be $\Delta_\sigma(a)$, where a is an arbitrary element of $\mathrm{Sym}(A,\sigma)^*$ such that $\tau = \mathrm{Int}(a) \circ \sigma$.

The following lemma gives a graded version of a result proved in [**BMT03**, Proposition 1, p.203].

Lemma 4.1. *Let F be a graded field of characteristic different from* 2*, A be a graded central simple algebra over F with* $\deg(A) \equiv 0 \bmod 4$ *and suppose that α, σ and τ are symplectic graded involutions on A. Then $\Delta_\tau(\sigma) = \Delta_\tau(\alpha) + \Delta_\alpha(\sigma)$ and $\Delta_\alpha(\sigma) = \Delta_\sigma(\alpha)$.*

PROOF. This follows by [**BMT03**, Proposition 1, p.203] and scalar extension to the algebra of central quotients. For more details, one can proceed as follows. Write $\sigma = \mathrm{Int}(s) \circ \alpha$, where $s \in \mathrm{Sym}(A, \alpha)^*$. We have also $\alpha = \mathrm{Int}(s^{-1}) \circ \sigma$ and $s^{-1} \in \mathrm{Sym}(A, \sigma)^*$. By definition, we have $\Delta_\alpha(\sigma) = (\mathrm{Nrp}_\alpha(s))_2 \cup [A]$ and $\Delta_\sigma(\alpha) = (\mathrm{Nrp}_\sigma(s^{-1}))_2 \cup [A] = (\mathrm{Nrp}_\alpha(s^{-1}))_2 \cup [A]$ (because $s^{-1} \in \mathrm{Sym}(A, \sigma) \cap \mathrm{Sym}(A, \alpha)$); moreover, by [**BMT03**, Lemma 9(b), p.207] we have $\mathrm{Nrp}_{q(\sigma)}(s^{-1}s^2) = \mathrm{Nrp}_{q(\alpha)}(s^{-1})\mathrm{Nrp}_{q(\alpha)}(s^2) = \mathrm{Nrp}_{q(\alpha)}(s^{-1})\mathrm{Nrp}_{q(\alpha)}(s)^2$. Therefore, $\mathrm{Nrp}_\alpha\ (s) = \mathrm{Nrp}_\sigma(s) = \mathrm{Nrp}_{q(\sigma)}(s) = \mathrm{Nrp}_{q(\sigma)}(s^{-1}s^2) = \mathrm{Nrp}_{q(\alpha)}(s^{-1})\mathrm{Nrp}_{q(\alpha)}\ (s)^2 = \mathrm{Nrp}_\alpha(s^{-1})\mathrm{Nrp}_\alpha(s)^2$, hence $\mathrm{Nrp}_\alpha(s)F^{*2} = \mathrm{Nrp}_\alpha(s^{-1})F^{*2}$, so $\Delta_\alpha(\sigma) = \Delta_\sigma(\alpha)$.

Let $t \in \mathrm{Sym}(A, \tau)^*$ such that $\alpha = \mathrm{Int}(t) \circ \tau$, then $\sigma = \mathrm{Int}(st) \circ \tau$. By [**BMT03**, Lemma 9(b), p. 206-207] for any $y \in \mathrm{Sym}(A, \alpha)$ we have $\mathrm{Nrp}_{q(\sigma)}((s \otimes 1)(y \otimes 1)) = \mathrm{Nrp}_{q(\alpha)}(s \otimes 1)\mathrm{Nrp}_{q(\alpha)}(y \otimes 1)$. In particular, $\mathrm{Nrp}_\sigma(st) = \mathrm{Nrp}_{q(\sigma)}(st \otimes 1) = \mathrm{Nrp}_{q(\sigma)}((s \otimes 1)(t \otimes 1)) = \mathrm{Nrp}_{q(\alpha)}(s \otimes 1)\mathrm{Nrp}_{q(\alpha)}(t \otimes 1) = \mathrm{Nrp}_\alpha(s)\mathrm{Nrp}_\alpha(t)$. We have $\Delta_\tau(\sigma) = (\mathrm{Nrp}_\tau(st))_2 \cup [A] = (\mathrm{Nrp}_\sigma(st))_2 \cup [A]$ (because $st \in \mathrm{Sym}(A, \tau) \cap \mathrm{Sym}(A, \sigma)$). Hence $\Delta_\tau(\sigma) = (\mathrm{Nrp}_\alpha(s)\mathrm{Nrp}_\alpha(t))_2 \cup [A] = (\mathrm{Nrp}_\alpha(s))_2 \cup [A] + (\mathrm{Nrp}_\alpha(t))_2 \cup [A] = (\mathrm{Nrp}_\alpha(s))_2 \cup [A] + (\mathrm{Nrp}_\tau(t))_2 \cup [A] = \Delta_\alpha(\sigma) + \Delta_\tau(\alpha)$.

□

(4.2) Let F be a graded field and A a graded central simple algebra over F. We saw in (1.1.1) that if A_0 is simple, then $\deg(A) = \deg(A_0)[Z(A_0) : F_0]r$, where $r = |\ker(\theta_A)|^{\frac{1}{2}}$. Suppose that there exists a symplectic graded involution σ on A, then by

[**M11**, Lemma 2.6(4)], for any $a \in \mathrm{Sym}(A,\sigma)^* \cap A_0F$, we have

$$\mathrm{Nrd}_A(a) = \mathrm{N}_{Z(A_0)F/F}(\mathrm{Nrd}_{A_0F}(a))^r.$$

If A is not inertially split, then as seen above in Lemma 3.2, $|\ker(\theta_A)|$ is a nontrivial power of 4. Suppose that $\deg(A) \cong 0$ mod 4, then on $\mathrm{Sym}(A,\sigma)^* \cap A_0F$, we have $\mathrm{Nrp}_\sigma(a) = \mathrm{N}_{Z(A_0)F/F}(\mathrm{Nrd}_{A_0F}(a))^{\frac{r}{2}}$. This follows from the fact that $\mathrm{N}_{Z(A_0)F/F}(\mathrm{Nrd}_{A_0F}(a))$ is a polynomial function on a. In particular, if $r \equiv 0$ mod 4 (i.e., $r \geq 4$), then $\mathrm{Nrp}_\sigma(a) \in F^{*2}$ for all $a \in \mathrm{Sym}(A,\sigma)^* \cap A_0F$.

(4.3) Let F be a graded field of characteristic different from 2, D be a graded central division algebra over F with $\exp(D) = 2$ and $E = q(F)^h$. Since $\exp(q(D)^h)(= \exp(D)) = 2$, then $q(D)^h$ has an involution of the first kind σ, which by [**KMRT98**, Corollary 2.8, p.18] can be chosen of arbitrary type. By [**RTW07**, Proposition 3.15(i), p. 124 and Proposition 3.13, p.122][4] the induced graded involution σ' of σ on $G(q(D)^h)$ has the same type as σ (because $\mathrm{char}(F) \neq 2$), so by identification of D with $G(q(D)^h)$ we can suppose that D has a sympelectic [resp., an orthogonal] graded involution.

---

[4] See here that as in the proof of $(b) \Leftrightarrow (c)$ in [**RTW07**, Proposition 3.13, p.122], we have $\dim_{GE}\mathrm{Symd}(G(q(D)^h),\sigma',1) = \dim_E\mathrm{Symd}(q(D)^h,\sigma,1)$. Moreover, since $\mathrm{char}(F) \neq 2$, then $\mathrm{Symd}(G(q(D)^h),\sigma',1) = \mathrm{Sym}(G(q(D)^h),\sigma')$ and $\mathrm{Symd}(q(D)^h,\sigma,1) = \mathrm{Sym}(q(D)^h,\sigma)$, therefore $\dim_{GE}\mathrm{Sym}(G(q(D)^h),\sigma') = \dim_E\mathrm{Sym}(q(D)^h,\sigma)$, which shows that σ and σ' have the same type because $\dim_{GE}G(q(D)^h) = \dim_E q(D)^h$.

(4.4) Let F be a graded field, D be a graded central division algebra over F, and $A = M_n(D)$. If ρ is a graded involution on D, then the map $\eta : M_n(D) \to M_n(D)$, defined by $(d_{ij}) \mapsto (\rho(d_{ij}))^t$ is a graded involution on A extending ρ. As seen in (4.3), if F has characteristic different from 2 and $\exp(D) = 2$, then we can assume that ρ is of arbitrary type; moreover by applying [**KMRT98**, Proposition 2.20, p.24] -to $q(D)$ and $q(A)$- we deduce that ρ and η have the same type.

LEMMA 4.5. *Let D be a graded central division algebra over a graded field F, and let $x \in D^*$. If e is the least positive integer such that $e.\mathrm{gr}(x) \in \Gamma_F$ and $r = |\ker(\theta_D)|^{\frac{1}{2}}$ is a multiple of e, then $\mathrm{Nrd}_D(x) \in F^{*r/e}$.*

PROOF. We have $[F[x] : F[x^e]] = e$ and $\lambda^{-1}x^e \in D_0$ for some $\lambda \in F^*$. Therefore, $[F[\lambda^{-1}x^e] : F]$ divides $\deg(D_0)[Z(D_0) : F_0] = \frac{\deg(D)}{r}$ and $[F[x] : F]$ divides $\frac{e}{r}\deg(D)$. We have

$$\mathrm{Nrd}_D(x) = \mathrm{N}_{F[x]/F}(x)^{\deg(D)/[F[x]:F]}$$

and the right side is in $F^{*r/e}$ since $\frac{r}{e}$ divides $\frac{\deg(D)}{[F[x]:F]}$. □

REMARK 4.6. Let D be a graded central division algebra over a graded field F with $\exp(D) = 2$. By [**B95**, Corollary 4.4] $q(D)^h$ is tame over $q(F)^h$, so by [**JW90**, Corollary 6.10] $\exp(\Gamma_D/\Gamma_F)(= \exp(\Gamma_{q(D)^h}/\Gamma_{q(F)^h}))$ divides $\exp(D)(= \exp(q(D)^h))$. Therefore, $\exp(\Gamma_D/\Gamma_F)$ is at most 2.

In this case, if σ is a symplectic graded involution on D and $x \in \mathrm{Sym}(D, \sigma)^*$, then e in Lemma 4.5 is either 1 or 2 and the same arguments used to prove Lemma 4.5 show that $\mathrm{Nrp}_\sigma(x) \in F^{*r/2e}$ when $2e$ divides r since $\mathrm{Nrp}_\sigma(x) = \mathrm{N}_{F[x]/F}(x)^{\frac{\deg(D)}{2[F[x]:F]}}$.

Proposition 4.7. *Let F be a graded field of characteristic different from 2, D a graded central division algebra over F with $\exp(D) = 2$ and $8 \leq |\ker(\theta_D)|^{\frac{1}{2}}$, then for any graded symplectic involutions σ and τ on D, we have $\Delta_\sigma(\tau) = 0$.*

Proof. Let $r = |\ker(\theta_D)|^{\frac{1}{2}}$. By Lemma 3.2 r is a power of 2, so by the hypotheses here r is a multiple of 8 (because we assumed here that $r \geq 8$). Let $x \in \mathrm{Sym}(D,\sigma)^*$ such that $\tau = \mathrm{Int}(x) \circ \sigma$, then by Remark 4.6 $\mathrm{Nrp}_\sigma(x) \in F^{*2}$, so $\Delta_\sigma(\tau) = 0$. $\square$

Lemma 4.8. *Let F be a graded field of characteristic different from 2 and D a non inertially split graded central division algebra over F with D_0 a field and $\exp(D) = 2$. Then the identity map on D_0 extends to a symplectic graded involution on D.*

Proof. Since D is not inertially split, then as seen in Lemma 3.2, $|\ker(\theta_D)|$ is a non trivial power of 4. Let $C_D(D_0F)$ be the centralizer of D_0F in D, then by the graded version of the double centralizer theorem (see [**TW15**, Theorem 2.35]) we have $[C_D(D_0F) : F][D_0F : F] = [D : F]$. Moreover, we have $[D : F] = [D_0 : F_0](\Gamma_D : \Gamma_F)$, so $[C_D(D_0F) : D_0F] = |\ker(\theta_D)|$. Therefore, $\deg(C_D(D_0F))$ is even. The rest of the proof follows easily by applying [**KMRT98**, Theorem 4.14, p.51] to the algebras $q(D)^h$ and $q(D_0F)^h$. Namely, since $\exp(q(D)^h) = 2$, then $q(D)^h$ has an involution of the first kind; moreover since D_0 is a field, then the identity map is an involution on $q(D_0F)^h$. So, by [**KMRT98**, Theorem 4.14, p.51] (and the fact that $\deg(C_{q(D)^h}(q(D_0F)^h))(= \deg(C_D(D_0F)))$ is even), there is a symplectic involution ξ on $q(D)^h$ whose restriction to $q(D_0F)^h$ is the identity map. Since $\mathrm{char}(F) \neq 2$, then the induced graded involution ξ' on $D(= G(q(D)^h))$ is also symplectic, and clearly we have $\xi'_{|D_0} = \mathrm{id}_{D_0}$. $\square$

(4.9) Let F be a graded field of characteristic different from 2 and D a non inertially split graded central division algebra over F with D_0 a field and $\exp(D) = 2$. Let ρ be a symplectic graded involution on D with $\rho_{|D_0} = \mathrm{id}_{D_0}$, then by [**CDTWY95**, Theorem 5, p.62] we have $q(D)^h = R \otimes_{q(F)^h} L$, where R is a (tame) semiramified central division algebra over $q(F)^h$, L is a (tame) totally ramified central division algebra over $q(F)^h$, and both R and L are stable under $q^h(\rho)$ (see the preliminaries for this notation). Moreover, we have $\Gamma_R \cap \Gamma_L = \Gamma_{q(F)^h}$, $\Gamma_{q(D)^h} = \Gamma_R + \Gamma_L$, $D_0 = \overline{R}$ and $\ker(\theta_{q(D)^h}) = \Gamma_L/\Gamma_{q(F)^h}$. So, by [**TW10**, Corollary 1.28][5], we have $D(\cong_g G(q(D)^h)) \cong_g S \otimes_F T$, where $S := GR$ is a semiramified graded division algebra over $F(= G(q(F)^h))$, $T := GL$ is a totally ramified graded central division algebra over F, $\Gamma_D/\Gamma_F = \Gamma_S/\Gamma_F \oplus \Gamma_T/\Gamma_F$, $D_0 = S_0$ and $\ker(\theta_D) = \Gamma_T/\Gamma_F$. By identification of S and T with their respective images in D (under

---

[5] Since R and L are tame, then we can consider tame $q(F)^h$-gauges s and t on R and L, respectively. Further, since $Z(GR) \otimes_F Z(GL) \cong_g Z(GR)$ is a graded field, then by [**TW10**, Corollary 1.28], we have $G(q(D)^h) \cong_g GR \otimes_F GL$. Note here that $s \otimes t$ coincides with the henselian valuation on $q(D)^h$ (see [**TW10**, Corollary 3.2]).

the above graded isomorphism), we can write $D = S \otimes_F T$ with S and T both stable under $\rho(= q^h(\rho)')$.

PROPOSITION 4.10. *Let F be a graded field of characteristic different from 2 and D a graded central division algebra over F with $\exp(D) = 2$ and $|\ker(\theta_D)| > 4$. Let $A = M_n(D)$, and let σ be a graded involution of symplectic type on A, then there is only a finite number of values for the discriminants $\Delta_\sigma(\tau)$, where τ describes all graded involutions of symplectic type on A.*

PROOF. Assume first that D_0 is a field. By Lemma 4.8 there is a symplectic graded involution ρ on D with $\rho_{|D_0} = \mathrm{id}_{D_0}$. Moreover, as seen in (4.9), we can write $D = S \otimes_F T$, where S is a semiramified graded central division algebra over F, T is a totally ramified graded central division algebra over F, $\Gamma_D/\Gamma_F = \Gamma_S/\Gamma_F \oplus \Gamma_T/\Gamma_F$, $D_0 = S_0$, $\ker(\theta_D) = \Gamma_T/\Gamma_F$, and where both S and T are stable under ρ. Let $\lambda_1, ..., \lambda_k$ [resp., $\gamma_1, ..., \gamma_l$] be a set of representatives of the various cosets of Γ_S modulo Γ_F [resp., of Γ_T modulo Γ_F] and let $x_1, ..., x_k \in S^*$ [resp., $y_1, ..., y_l \in T^*$] be elements with $\mathrm{gr}(x_i) = \lambda_i$ and $\mathrm{gr}(y_j) = \gamma_j$ for all i, j ($1 \leq i \leq k$ and $1 \leq j \leq l$). We have $\mathrm{Nrd}_D(x_i) = \mathrm{N}_{F[x_i]/F}(x_i)^{\frac{\deg(D)}{[F[x_i]:F]}} = \mathrm{N}_{F[x_i]/F}(x_i)^{\frac{\deg(S)}{[F[x_i]:F]}.\deg(T)}$. Similarly, we have $\mathrm{Nrd}_D(y_j) = \mathrm{N}_{F[y_j]/F}(y_j)^{\frac{\deg(T)}{[F[y_j]:F]}.\deg(S)}$. Note that for any $c \in T$, we have $\mathrm{gr}(c^2) = 2\mathrm{gr}(c) \in \Gamma_F$ (because $\exp(\Gamma_T/\Gamma_F)$ divides $\exp(T)$), hence $c^2 \in F$ (because $T_0 = F_0$), so $[F[c] : F]$ divides 2. Therefore, in the above $\frac{\deg(T)}{[F[y_j]:F]}$ is a multiple of $\frac{\deg(T)}{2}$.

Let η be the graded involution on A defined by $(d_{ij})_{1\leq i,j\leq n} \mapsto (\rho(d_{ij}))^t_{1\leq i,j\leq n}$, then as seen in (4.4) η is symplectic. We have $\Gamma_A =$

Γ_D so for any $a \in A^*$, we can write $a = b.\widehat{x_i \otimes y_j}$, for some $b \in (A_0F)^*$ and $1 \leq i \leq k$, $1 \leq j \leq l$, where $\widehat{x_i \otimes y_j}$ is the diagonal matrix in $A(= M_n(D))$ with all diagonal entries equal to $x_i \otimes y_j$. Hence,

$$\begin{aligned} \mathrm{Nrd}_A(a) &= \mathrm{Nrd}_A(b)\mathrm{Nrd}_D(x_i \otimes y_j)^n \\ &= \mathrm{N}_{Z(A_0)F/F}(\mathrm{Nrd}_{A_0F}(b))^r\mathrm{Nrd}_D(x_i)^n\mathrm{Nrd}_D(y_j)^n, \end{aligned}$$

where $r = |\ker(\theta_D)|^{\frac{1}{2}}$. So,

$$\mathrm{Nrd}_A(a) = \mathrm{N}_{Z(A_0)F/F}(\mathrm{Nrd}_{A_0F}(b))^r\mathrm{N}_{F[x_i]/F}(x_i)^{n.\frac{\deg(S)}{[F[x_i]:F]}.\deg(T)}\mathrm{N}_{F[y_j]/F}(y_j)^{n.\frac{\deg(T)}{[F[y_j]:F]}.\deg(S)}$$

We have $\deg(T) = r$ and we saw above that $\frac{\deg(T)}{[F[y_j]:F]}$ is a multiple of $\frac{\deg(T)}{2}$, so $\frac{\deg(T)}{[F[y_j]:F]}$ is a multiple of 2 (because by Lemma 3.2 and the hypothesis $|\ker(\theta_D)| > 4$, r is a multiple of 4). If a is a symmetric element for η, then we have

$$\mathrm{Nrp}_\eta(a) = \pm\mathrm{N}_{Z(A_0)F/F}(\mathrm{Nrd}_{A_0F}(b))^{\frac{r}{2}}\mathrm{N}_{F[x_i]/F}(x_i)^{n.\frac{\deg(S)}{[F[x_i]:F]}.\frac{\deg(T)}{2}}\mathrm{N}_{F[y_j]/F}(y_j)^{n.\frac{\deg(T)}{2[F[y_j]:F]}.\deg(S)}$$

So, $\mathrm{Nrp}_\eta(a)F^{*2} = \pm\mathrm{N}_{F[y_j]/F}(y_j)^{n.\frac{\deg(T)}{2[F[y_j]:F]}.\deg(S)}F^{*2}$. This shows that there is only a finite number of elements $\mathrm{Nrp}_\eta(a)F^{*2}$ when a describes all invertible symmetric elements for η.

Let τ be a symplectic graded involution on A. We have $\Delta_\sigma(\tau) = \Delta_\eta(\tau) + \Delta_\sigma(\eta) = \Delta_\eta(\tau) + \Delta_\eta(\sigma)$, so by the above there is only a finite number of values for the discriminants $\Delta_\sigma(\tau)$, where τ describes all graded involutions of symplectic type on A.

Now, suppose that D_0 is not a field. By (4.3) we can consider a symplectic graded involution ρ on D. As above, let η be the graded involution on A defined by $(d_{ij}) \mapsto (\rho(d_{ij}))^t$, then by (4.4) η is symplectic. Let γ_1, ..., γ_s be a set of representatives of the various cosets of Γ_D modulo Γ_F. Since D_0 is not a field, then by Proposition 1.1, for any i $(1 \leq i \leq s)$, there exists $x_i \in D^*$ with $\mathrm{gr}(x_i) = \gamma_i$ and $\rho(x_i) = x_i$. Let τ be a symplectic graded involution on A and let $a \in \mathrm{Sym}(A, \eta)^*$ such that $\tau = \mathrm{Int}(a) \circ \eta$. As in the first case, we can write $a = b\hat{x}_i$ for some $b \in (A_0F)^*$ and $1 \leq i \leq s$ (where $\hat{x}_i$ is the diagonal matrix in A with all diagonal entries equal to x_i). Let $\epsilon = \mathrm{Int}(\hat{x}_i) \circ \eta$, then $\tau = \mathrm{Int}(b) \circ \epsilon$. We have $\hat{x}_i \in \mathrm{Sym}(A, \eta)^*$, so ϵ is a symplectic graded involution on A. Also, because τ and ϵ are both symplectic (and $\mathrm{char}(F) \neq 2$), then $b \in \mathrm{Sym}(A, \epsilon) \cap (A_0F)^*$. We have $\mathrm{Nrp}_\epsilon(b) = \mathrm{N}_{Z(A_0)F/F}(\mathrm{Nrd}_{A_0F}(b))^{\frac{r}{2}}$, where $r = |\ker(\theta_A)|^{\frac{1}{2}}$, so $\mathrm{Nrp}_\epsilon(b)F^{*2} = F^{*2}$, hence $\Delta_\epsilon(\tau) = 0$. Therefore, $\Delta_\eta(\tau) = \Delta_\epsilon(\tau) + \Delta_\eta(\epsilon) = \Delta_\eta(\epsilon) = (\mathrm{Nrp}_\eta(\hat{x}_i))_2 \cup [A]$. By replacing τ by σ, we can assume that there exists some i_0 $(1 \leq i_0 \leq s)$ such that $\Delta_\eta(\sigma) = (\mathrm{Nrp}_\eta(\hat{x}_{i_0}))_2 \cup [A]$. So, $\Delta_\sigma(\tau) = (\mathrm{Nrp}_\eta(\hat{x}_i))_2 \cup [A] + (\mathrm{Nrp}_\eta(\hat{x}_{i_0}))_2 \cup [A] = (\mathrm{Nrp}_\rho(x_i)^n)_2 \cup [A] + (\mathrm{Nrp}_\rho(x_{i_0})^n)_2 \cup [A]$. This shows that the set of values of the discriminants $\Delta_\sigma(\tau)$, where τ describes all graded involutions of symplectic type on A, is finite. □

COROLLARY 4.11. *Let F be a graded field of characteristic different from 2 and A a graded central simple algebra over F. Suppose A_0 is simple non split,* $\exp(A) = 2$ *and* $|\ker(\theta_A)| > 4$, *and let σ be a graded involution of symplectic type on A, then there is* *only a finite number of values for the discriminants* $\Delta_\sigma(\tau)$*, where* τ *describes all graded involutions of symplectic type on* A*. If in addition* $\frac{\deg(A)}{\mathrm{ind}(A)}$ *is even, then for any such involution* τ*, we have* $\Delta_\sigma(\tau) = 0$*.*

PROOF. We saw in (1.1.1) that A can be identified with $M_n(D)$, where D is a graded central division algebra over F Brauer equivalent to A. Moreover, since A_0 is non split, then D_0 cannot be a field. So, the same arguments used in the last part of the proof of Proposition 4.10 show that there is only a finite number of values for the discriminants $\Delta_\sigma(\tau)$, when τ describes all graded involutions of symplectic type on A. Moreover, one can easily see that $\Delta_\sigma(\tau) = 0$ when $n(= \frac{\deg(A)}{\mathrm{ind}(A)})$ is even. □

REMARK 4.12. Under the hypotheses of Proposition 4.10 with D_0 not a field and n even, one can see that we have the following result: $\mathrm{Nrp}_\eta(z) \in F^{*2}$ for all $z \in \mathrm{Sym}(A,\eta)^*$. This can be shown as follows: If $\mathrm{gr}(z) = 0$, then $\mathrm{Nrp}_\eta(z) = \mathrm{N}_{Z(A_0)/F_0}(\mathrm{Nrd}_{A_0}(z))^{r/2}$, where $r = |\ker(\theta_D)|^{\frac{1}{2}}$, so the result holds. If $\mathrm{gr}(z) = \delta$, then by Proposition 1.1, we may find $d \in D^*$ with $\mathrm{gr}(d) = \delta$ and $\rho(d) = d$. Let $\hat{d} \in A$ be the diagonal matrix with diagonal entries all equal to d, and let $z_0 = z\hat{d}^{-1}$. We have $z_0 \in A_0^*$ and $z_0 \in \mathrm{Sym}(A,\eta_0)^*$, where $\eta_0 = \mathrm{Int}(\hat{d}) \circ \eta$. Therefore, $\mathrm{Nrp}_{\eta_0}(z_0) \in F^{*2}$. We have $\mathrm{Nrp}_\eta(\hat{d}) = \mathrm{Nrd}_D(d)^{n/2}$ and by Lemma 4.5 (see also Remark 4.6) $\mathrm{Nrd}_D(d) \in F^{*2}$, so $\mathrm{Nrp}_\eta(\hat{d}) \in F^{*2}$. Therefore, $\mathrm{Nrp}_\eta(z) = \mathrm{Nrp}_{\eta_0}(z_0)\mathrm{Nrp}_\eta(\hat{d}) \in F^{*2}$.

EXAMPLES 4.13. For a graded field F of characteristic different from 2, we aim to give here examples of the following:

1. A graded central division algebra A over F with $\exp(A) = 2$ and $8 \leq |\ker(\theta_A)|^{\frac{1}{2}}$.
2. A graded central simple algebra A over F with $\exp(A) = 2$, A_0 simple, $|\ker(\theta_A)| > 4$ and $\frac{\deg(A)}{\operatorname{ind}(A)}$ even.

Let E be a field of characteristic different from 2, R be a quaternion division algebra over E (or more generally a central division algebra of exponent 2 over E), k be a positive integer with $8 \leq 2^k$, $X_1, ..., X_{2k}$ be $2k$ independent indeterminates over E, $E' = E((X_1))...((X_{2k}))$ be the iterated Laurent power series field, $F = E[X_1, X_1^{-1}, ..., X_{2k}, X_{2k}^{-1}]$ be the E-subalgebra of E' generated by $X_1, X_1^{-1}, ..., X_{2k}, X_{2k}^{-1}$, and endow $\mathbb{Z}^{2k}$ with the right-to-left lexicographical ordering. Then, F is a graded field with $\Gamma_F = \mathbb{Z}^{2k}$, and for each $\overline{m} := (m_1, ..., m_{2k}) \in \mathbb{Z}^{2k}$ we have $F_{\overline{m}} = EX_1^{m_1}...X_{2k}^{m_{2k}}$. One can see that F can be identified with $E[\mathbb{Z}^{2k}]$; moreover, it is clear that $q(F) = E(X_1, ..., X_{2k})$. For any elements $a, b \in q(F)^*$, we denote by $(a, b)_{q(F)}$ the quaternion algebra over $q(F)$ determined by a and b. Let $S = \otimes_{m=1}^{k}(X_{2m-1}, X_{2m})_{q(F)}$. If we consider on $q(F)$ the canonical valuation v associated to the graded structure of F, then by [**TW87**, Example 3.6, p.234] S is a division algebra and there exists a valuation w on S which extends v and for which S is totally ramified over $q(F)$ with $\Gamma_S/\Gamma_{q(F)} \cong \prod_{m=1}^{k}(\mathbb{Z}/2\mathbb{Z}) \times (\mathbb{Z}/2\mathbb{Z})$. Since $\operatorname{char}(F) \neq 2$ and w is defectless over v, then by [**TW10**, Proposition 1.12], w is a tame $q(F)$-gauge, so $T := G_wS$ is a totally ramified graded central division algebra over F with $\Gamma_T/\Gamma_F \cong \prod_{m=1}^{k}(\mathbb{Z}/2\mathbb{Z}) \times (\mathbb{Z}/2\mathbb{Z})$ (here we identify F with $G_vq(F)$)

Let $I := R \otimes_E F$ and endow I with the graded structure defined by $I_{\overline{m}} := \{\sum x_i \otimes y_i \,|\, x_i \in R \,\text{and}\, y_i \in F_{\overline{m}}\}$. It is clear that any nonzero homogeneous element of I can be written as $a \otimes s$, with $a \in R^*$ and $s \in F^*$, hence I is an inertial graded division algebra over F.

Now, let $A := I \otimes_F T$, we claim that A is a graded central division algebra over F with $\exp(A) = 2$, $A_0 = R$ and $|\ker(\theta_A)|^{\frac{1}{2}} = \deg(T)$. Indeed, we have $\exp(I) = \exp(R) = 2$, $\exp(T) = \exp(S) = 2$, and clearly A cannot be split since it is not inertially split (for I is inertially split and T is not inertially split), so $\exp(A) = 2$. Moreover, we have $\Gamma_A = \Gamma_I + \Gamma_T = \Gamma_F + \Gamma_T = \Gamma_T$, hence $\Gamma_A/\Gamma_F = \Gamma_T/\Gamma_F$. We have

$$\begin{aligned}
[A : F] &= [I : F][T : F] \\
&= [I_0 : E](\Gamma_T : \Gamma_F) \\
&= [R : E](\Gamma_T : \Gamma_F) \\
&= [R : E](\Gamma_A : \Gamma_F)
\end{aligned}$$

On the other hand, by [**HW(b)99**, (1.8), p.79], we have $[A : F] \geq [A_0 : F_0](\Gamma_A : \Gamma_F) = [A_0 : E](\Gamma_A : \Gamma_F)$, hence $[R : E] \geq [A_0 : E]$. Conversely, it is clear that $R(= I_0) \subseteq A_0$, so necessarily $R = A_0$. Since A_0 is a division algebra, then by [**TW15**, Corollary 2.43] A is a graded division algebra over F. Note that we have $\ker(\theta_A) =$

$\ker(\theta_{q(A)^h})$, and clearly up to an algebra isomorphism, we have $q(A)^h = q(I)^h \otimes_{q(F)^h} q(T)^h$ (with $q(I)^h$ inertial over $q(F)^h$ and $q(T)^h$ totally ramified over $q(F)^h$), so by [**JW90**, Theorem 6.3, p.167] $\ker(\theta_A) = \ker(\theta_{q(A)^h}) = \Gamma_{q(T)^h}/\Gamma_{q(F)^h} = \Gamma_T/\Gamma_F$. Consequently, $|\ker(\theta_A)|^{\frac{1}{2}} = (\Gamma_T : \Gamma_F)^{\frac{1}{2}} = [T : F]^{\frac{1}{2}} = \deg(T)$. We have previously chosen k so that $8 \leq 2^k$, hence $8 \leq \deg(T) = |\ker(\theta_A)|^{\frac{1}{2}}$.

For the second example, see that if we replace R by an arbitrary central simple algebra of exponent 2 over E with $\frac{\deg(R)}{\mathrm{ind}(R)}$ even, and chose $k > 2$, then using the same arguments, the algebra $A := I \otimes_F T$ still satisfies the following : $\exp(A) = 2$ and $A_0 = R$, hence in particular A_0 is simple. Let C be a central division algebra over E Brauer-equivalent to R, and let $D := C \otimes_E F$. Then, D is Brauer equivalent to I and $B := D \otimes_F T$ is (graded) Brauer equivalent to A. By the above, B is a graded central division algebra over F, so as seen in (1.1.1) $\theta_A = \theta_B$ (because A_0 is simple). By the first example, we have $|\ker(\theta_B)|^{\frac{1}{2}} = \deg(T)$, so $|\ker(\theta_A)|^{\frac{1}{2}} = \deg(T) > 2$. Note that up to an algebra isomorphism, we have $R = M_n(E) \otimes_E C$, where $n = \frac{\deg(R)}{\mathrm{ind}(R)}$. So, $I = R \otimes_E F = M_n(E) \otimes_E (C \otimes_E F) = M_n(E) \otimes_E D = M_n(D)$. Therefore, $A = I \otimes_F T \cong_g M_n(D) \otimes_F T \cong_g M_n(D \otimes_F T)$. Moreover, as seen above $D \otimes_F T$ is a graded central division algebra over F, so $\frac{\deg(A)}{\mathrm{ind}(A)} (= n)$ is even.

5. Discriminants of symplectic involutions on tame central simple algebras over Henselian fields

PROPOSITION 5.1. *Let D be a central division algebra over a Henselian field E and let $x \in D$. Assume* $\mathrm{char}(\overline{E})$ *does not divide*

$\deg(D)$ *and let* $r = |\ker(\theta_D)|^{\frac{1}{2}}$*. Let* e *be the least positive integer such that* $e.v(x) \in \Gamma_E$*, where* v *is the extension of the valuation of* E *to* D*. If* e *divides* r*, then* $\mathrm{Nrd}_D(x) \in E^{*r/e}$*.*

PROOF. Consider first the case where $v(x) = 0$. Let $X^n + \overline{a_{n-1}}X^{n-1} + ... + \overline{a_0} \in \overline{E}[X]$ be the minimal polynomial of $\overline{x}$ over $\overline{E}$ (where $a_0, ..., a_{n-1} \in E$ with $v(a_i) \geq 0$ for all i). By Hensel's lemma, we may find a root $u \in E(x)$ of $X^n + a_{n-1}X^{n-1} + ... + a_0$ with $\overline{u} = \overline{x}$, so $x = u(1+m)$ for some $m \in E(x)$ with $v(m) > 0$. We have $\mathrm{Nrd}_D(x) = \mathrm{Nrd}_D(u)\mathrm{Nrd}_D(1+m)$ and

$$\mathrm{Nrd}_D(u) = \mathrm{N}_{E(u)/E}(u)^{\deg(D)/n} \in E^r.$$

Moreover, $X^r - \mathrm{Nrd}_D(1+m) \in E[X]$ has residue $X^r - 1 \in \overline{E}[X]$. Since 1 is a simple root of this polynomial, it follows from Hensel's Lemma that $X^r - \mathrm{Nrd}_D(1+m)$ has a root in E, so $\mathrm{Nrd}_D(1+m) \in E^r$. Therefore, $\mathrm{Nrd}_D(x) \in E^r$

If $e = 1$, we have $v(\lambda^{-1}x) = 0$ for some $\lambda \in E^*$, hence $\mathrm{Nrd}_D(\lambda^{-1}x) \in E^{*r}$ by the above. Since $\mathrm{Nrd}_D(\lambda) = \lambda^{\deg(D)} \in E^{*r}$, the proposition holds when $e = 1$.

In the general case, we have $v(x^e) \in \Gamma_E$, hence $\mathrm{Nrd}_D(x^e) = \nu^r$ for some $\nu \in E^*$. It follows that $\mathrm{Nrd}_D(x) = \xi\nu^{r/e}$ for some $\xi \in E^*$ with $\xi^e = 1$. In GD we have $\mathrm{Nrd}_D(x)' = \mathrm{Nrd}_{GD}(x')$ (see [**HaW(a)11**, Corollary 4.4] or Theorem 2.1), so

$$\mathrm{Nrd}_{GD}(x') = \xi'(\nu')^{r/e}.$$

Lemma 4.5 shows that $\xi' \in GE^{*r/e}$, so $\overline{\xi} \in \overline{E}^{*r/e}$. By Hensel's lemma it follows that $\xi \in E^{*r/e}$, so $\mathrm{Nrd}_D(x) \in E^{*r/e}$. □

Remark 5.2. Alternatively, this proposition can be shown as follows: We have $\mathrm{Nrd}_D(x)' = \mathrm{Nrd}_{GD}(x')$, so by Lemma 4.5 $\mathrm{Nrd}_D(x)' \in GE^{*r/e}$. Moreover, the polynomial $X^{r/e} - \mathrm{Nrd}_{GD}(x') \in GE[X]$ is λ-homogenizable, where $\lambda = \frac{e}{r}\mathrm{gr}(\mathrm{Nrd}_{GD}(x'))$, and all its roots are simple, so by [**MW11**, Theorem 1.9 (4) and (5)] the polynomial $X^{r/e} - \mathrm{Nrd}_D(x)$ has a root in E, hence $\mathrm{Nrd}_D(x) \in E^{*r/e}$.

As in the graded case, if D has a symplectic graded involution σ and $x \in \mathrm{Sym}(D, \sigma)^*$, then we have $\mathrm{Nrp}_\sigma(x) \in E^{r/2e}$ and in this case $e = 1$ or 2 (see Remark 4.6). Consequently, using the same arguments as in the graded case (see Proposition 4.7), we have the following Corollary.

Corollary 5.3. *Let E be a Henselian valued field with residue characteristic different from 2, D be a central division algebra of exponent 2 over E such that $8 \leq |\ker(\theta_D)|^{\frac{1}{2}}$, and σ, τ be symplectic involutions on D, then $\Delta_\sigma(\tau) = 0$.*

Lemma 5.4. *Let D be a central division algebra over a Henselian valued field E with $\mathrm{char}(\overline{E})$ not dividing $\deg(D)$, $B = M_n(D)$, and define the tame gauge β on B by $\beta((d_{ij})_{1\leq i,j\leq n}) = \min\{v(d_{ij}) \mid 1 \leq i, j \leq n\}$, where v is the extension of the valuation of E to D. Suppose $b \in B^*$ is such that $\mathrm{Int}(b)$ preserves β, i.e., $\beta(bxb^{-1}) = \beta(x)$ for all $x \in B$. Let $r = |\ker(\theta_D)|^{\frac{1}{2}}$. If $r \equiv 0 \bmod 4$, n is even and the exponent of Γ_D/Γ_E is 2, then $\mathrm{Nrd}_B(b) \in E^{*4}$.*

Proof. The condition on b is equivalent to $\beta(b) + \beta(b^{-1}) = 0$. Let $d \in D^*$ be such that $\beta(b) = v(d)$, and let $\hat{d} \in B^*$ be the diagonal matrix with diagonal entries all equal to d. Let also $b_0 = b\hat{d}^{-1}$.

Then $\beta(b_0) = \beta(b_0^{-1}) = 0$ and $\mathrm{Nrd}_B(b_0) = \mathrm{Nrd}_B(b)\mathrm{Nrd}_D(d)^{-n}$. The property $\beta(b_0) = \beta(b_0^{-1}) = 0$ means that $b_0 \in \mathrm{GL}_n(O)$, where O is the valuation ring of D. It follows that the Dieudonné determinant of b_0 is represented by an element $u \in D$ with $v(u) = 0$. By Proposition 5.1 we have $\mathrm{Nrd}_D(u) \in E^{*4}$ and $\mathrm{Nrd}_D(d) \in E^{*2}$. Since $\mathrm{Nrd}_B(b_0) = \mathrm{Nrd}_D(u)$ and n is even, it follows that $\mathrm{Nrd}_B(b) = \mathrm{Nrd}_B(b_0)\mathrm{Nrd}_D(d)^n \in E^{*4}$. □

Under the hypotheses of Lemma 5.4, if D has a symplectic involution ρ, η is the symplectic involution on B defined by $(d_{ij}) \mapsto (\rho(d_{ij}))^t$, and $b \in \mathrm{Sym}(B, \eta)^*$, then it follows by Lemma 5.4 and Remark 4.12 that $\mathrm{Nrp}_\eta(b) \in E^{*2}$. Consequently, we have the following Corollary.

COROLLARY 5.5. *Let E be a Henselian valued field with residue characteristic different from 2, D be a central division algebra of exponent 2 over E with $|\ker(\theta_D)| > 4$, $B = M_n(D)$ and define the tame gauge β on B by $\beta((d_{ij})_{1\leq i,j\leq r}) = \min\{v(d_{ij}) \mid 1 \leq i, j \leq n\}$, where v is the extension of the valuation of E to D. If n is even, then for any symplectic involutions σ, τ on B, preserving β, we have $\Delta_\sigma(\tau) = 0$.*

PROOF. Let ρ be a symplectic involution on D, and let η be the symplectic involution on B defined as in the last paragraph above. Since η preserves β, then any symplectic involution on B which preserves β is of the form $\mathrm{Int}(b) \circ \eta$, where $b \in \mathrm{Sym}(B, \eta)^*$ and $\mathrm{Int}(b)$ preserves β. Thus, if σ and τ are symplectic involutions on B, preserving β, then $\Delta_\sigma(\tau) = \Delta_\eta(\tau) + \Delta_\eta(\sigma) = 0$. □

CHAPTER 3

Witt rings of graded simple algebras and Milnor's K-theory for graded fields

The first aim of the present chapter is to apply graded division algebras in studying Witt rings of tame division algebras. A second goal is to prove some facts concerning Milnor's K-groups of graded fields.

In the first section, denoting by E a Henselian valued field with abstract Galois group G_E, by n a prime power (positive) integer which is not a multiple of the residue characteristic of E, and by $H^n(G_E, \mathbb{Z}/n\mathbb{Z})$ the n^{th} (continuous) cohomology group of G_E on $\mathbb{Z}/n\mathbb{Z}$, we show that the graded ring $H^*(G_E, \mathbb{Z}/n\mathbb{Z}) := \oplus_{n\in\mathbb{N}} H^n(G_E, \mathbb{Z}/n\mathbb{Z})$ is graded isomorphic to $H^*(G_F, \mathbb{Z}/n\mathbb{Z})$, where F is the graded field (canonically) associated to E (see the preliminaries), and where G_F is the Galois group of the tame (graded) closure of F (see Corollary 1.4). This isomorphism is needed in section 3 to give a cohomological interpretation to Milnor's K-rings of a graded field.

In section 2, inspired by Craven's work in [**C82**], we define and study (gr-) Witt rings of graded simple algebras. We begin this section by proving that the square class group $S(D) := D^*/D^{*2}$ of a tame central division algebra D over a Henselian valued field with residue characteristic different from 2, is isomorphic to the square class group of GD (see Proposition 2.3), then for a graded field F, we give a description of the (gr-) Witt ring $W^{\text{gr}}(F)$ of F in terms of generators and relations. This description is generalized in the definition of the the (gr-) Witt ring of a graded simple algebra A. We show then a result relating the (gr-) Witt ring of A to the Witt ring of the 0-component A_0 of A (see Proposition 2.8). We apply this result to show that if E is a Henselian valued field of residue characteristic different from 2, D is a tame central division algebra over E, J is the ideal of $W(\overline{D})$, generated by the (classes of) the elements $\langle 1\rangle - \langle r\rangle$, where $r \in \overline{D}^* \cap GD^{*2}$, and Z is a subgroup of D^*/D^{*2} mapped bijectively onto $\Gamma_D/2\Gamma_D$ by the induced map $\overline{v}$ of the valuation v of D, then $W(D)$ is isomorphic to a generalized crossed product $(W(\overline{D})/J) * Z$. If in addition $\overline{D}^* \cap GD^{*2} = \overline{D}^{*2}$, then we get a ring isomorphism $W(D) \cong W(\overline{D}) * Z$. In particular, if $D = E$, then $W(E) \cong W(\overline{E})[Z]$ (see Corollary 2.10).

We show in (2.11) that Corollary 2.10 generalizes Sladek's Theorem [**S88**, Theorem 3.3] which he proved for the restrictive case where the division algebra D has a complete discrete valuation with residue characteristic different from 2 and where the residue division algebra of D is a field. The arguments used in this section are completely independent of those used in [**S88**].

In section 3, for a graded field F, we define a ring K_*F, which is the analogous (in the graded setting) of Milnor's one defined in [**Mi70**] for (ungraded) fields, and for a positive integer n, we define the ring $(K_*F)_n := K_*F/nK_*F$, which for $n = 2$, like the ungraded case, characterizes the graded (gr-) Witt ring of F. Then, for a power n of a prime integer p, we determine a first (graded) isomorphism between $(K_*F)_n$ and an extension of $(K_*F_0)_n$ (see Proposition 3.1). We show that if p is different from the characteristic of F, then $(K_*F)_n$ is (graded) isomorphic to the ring $H^*(G_F, \mathbb{Z}/n\mathbb{Z})$ (see Corollary 3.4).

1. Some Galois cohomology

For a profinite group G, a discrete G-module N and a positive integer n, let $H^n(G, N)$ be the n^{th} (continuous) cohomology group of G on N. We set $H^*(G, N) := \oplus_{n\geq 0} H^n(G, N)$.

Let E be a Henselian valued field with abstract Galois group G_E, $F = GE$ be the graded field (canonically) associated to E (see the preliminaries) and let n be a power of a prime integer different from the residue characteristic of E. As previously mentioned in the introduction of this chapter, we will show in this section that we have a graded ring isomorphism $H^*(G_E, \mathbb{Z}/n\mathbb{Z}) \cong_g H^*(G_F, \mathbb{Z}/n\mathbb{Z})$. This isomorphism will be needed in section 3 to give a cohomological interpretation to Milnor's K-rings of a graded field. We will need for this a $\mathbb{Z}$-graded ring defined by Wadsworth in [**W83**, pp.476-477] (see (1.1) below).

(1.1) Let $A = \oplus_{k\in\mathbb{Z}} A_k$ be a graded ring of type $\mathbb{Z}$ and suppose that there is some element $w \in A_1$ such that $2w = 0$. Let J be a set, $(X_j)_{j\in J}$ be a family of noncommuting indeterminates over A and let $A[J; w] := A[X_j \mid j \in J]/R$, where R is the ideal of $A[X_j \mid j \in J]$

generated by the elements $X_sX_t + X_tX_s$, $X_sa - (-1)^k aX_s$ (for $a \in A_k$), $X_s^2 - wX_s$, where s, t range over all elements of J. If we denote by x_j the (canonical) image of X_j in $A[J;w]$ (i.e., $x_j = X_j + R$), then $A[J;w]$ is a $\mathbb{Z}$-graded ring with $A[J;w]_k = \oplus_{m\leq k} \oplus_{\{j_1,\dots,j_m\}\in J_m} A_{k-m}x_{j_1}\dots x_{j_m}$, where J_m is the set of all subsets of cardinality m of J (in particular, $\mathrm{gr}(x_j) = 1$ for all $j \in J$). As one can easily see $A[J;w]$ is a free A-module.

(1.2) Now, let F be a graded field, p be a prime integer different from $\mathrm{char}(F)$, n a power of p and suppose that $\mu_n \subseteq F_0$. We fix a family $(y_j)_{j\in J}$ of elements of F^* such that $(\mathrm{gr}(y_j) + n\Gamma_F)_{j\in J}$ is a $\mathbb{Z}/n\mathbb{Z}$-base of $\Gamma_F/n\Gamma_F$ and for an arbitrary positive integer m, we let J_m be the set whose elements are the subsets of J with m elements (i.e., $J_m = \{S \subseteq J \mid \mathrm{card}(S) = m\}$). Let F_t be 'the' tame closure of F, G_F be the Galois group $\mathrm{Gal}(F_t/F)$ and let's consider the exact sequence of G_F-modules :

$$1 \to \mu_n \to F_t^* \xrightarrow{(\ )^n} F_t^* \to 1,$$

where $(\ )^n$ denotes the mapping $x \mapsto x^n$, then we have a connecting (cohomological) homomorphism

$$\delta :\ F^* \to H^1(G_F, \mu_n).$$

Similarly, let $F_{0,s}$ be 'the' separable closure of F_0 (we can take $F_{0,s} = (F_t)_0$, see [**HW(a)99**, Definition 3.8] or [**TW15**, p., 210]), and let G_{F_0} be the Galois group of $F_{0,s}$ over F_0, then a connecting group homomorphism:

$$\delta_0 : F_0^* \to H^1(G_{F_0}, \mu_n)$$

is obtained from the exact G_{F_0}-module sequence: $1 \to \mu_n \to F_{0,s}^* \xrightarrow{(\ )^n} F_{0,s}^* \to 1$. For the next proposition, we will identify μ_n with $\mathbb{Z}/n\mathbb{Z}$ and we will use the notation of (1.1) above with $A = H^*(G_{F_0}, \mathbb{Z}/n\mathbb{Z})$ and $w = \delta_0(-1)$.

PROPOSITION 1.3. *With the notation and hypotheses of (1.2), we have a graded ring isomorphism :*

$$H^*(G_F, \mathbb{Z}/n\mathbb{Z}) \cong_g H^*(G_{F_0}, \mathbb{Z}/n\mathbb{Z})[J; \delta_0(-1)].$$

PROOF. Let $E = q(F)^h$ (i.e., the henselization of $q(F)$ with respect to its canonical valuation, as seen in the preliminaries), E_s be 'the' separable closure of E, G_E be the Galois group of E_s over E, and E_t be the tame closure of E in E_s. Then, up to an isomorphism, we have $E_t = q(F_t)^h$ (see [**HW(a)99**, Proposition 5.1]). Therefore, $G_F(= \mathrm{Gal}(F_t/F)) \cong \mathrm{Gal}(E_t/E)$. It follows that $H^*(G_F, \mathbb{Z}/n\mathbb{Z}) \cong_g H^*(\mathrm{Gal}(E_t/E), \mathbb{Z}/n\mathbb{Z})$. Clearly, if $q := \mathrm{char}(F)$ is zero, then $E_t = E_s$. Moreover, if $q \neq 0$, then $\mathrm{Gal}(E_s/E_t)$ is a pro-q-group, so for any positive integer m, the inflation homomorphism $\mathrm{inf} : H^m(\mathrm{Gal}(E_t/E), \mathbb{Z}/n\mathbb{Z}) \to H^m(G_E, \mathbb{Z}/n\mathbb{Z})$ is an isomorphism. Therefore, in both cases (i.e., $q = 0$ and $q \neq 0$), $H^*(\mathrm{Gal}(E_t/E), \mathbb{Z}/n\mathbb{Z})$ is graded isomorphic to $H^*(G_E, \mathbb{Z}/n\mathbb{Z})$. By [**W83**, Theorem 3.6] we have $H^*(G_E, \mathbb{Z}/n\mathbb{Z}) \cong_g H^*(G_{F_0}, \mathbb{Z}/n\mathbb{Z})[J, \delta_0(-1)]$ (see that $F_0 = \overline{E}$), so $H^*(G_F, \mathbb{Z}/n\mathbb{Z})(\cong_g H^*(\mathrm{Gal}(E_t/E), \mathbb{Z}/n\mathbb{Z})) \cong_g H^*(G_{F_0}, \mathbb{Z}/n\mathbb{Z})[J, \delta_0(-1)]$. □

In particular, the proof of Proposition 1.3 shows (without assuming $\mu_n \subseteq F_0$) that $H^*(G_F, \mathbb{Z}/n\mathbb{Z}) \cong_g H^*(G_E, \mathbb{Z}/n\mathbb{Z})$, where $E = q(F)^h$. In the same way, starting from a Henselian valued field instead of a graded field, we have the following result:

PROPOSITION 1.4. *Let E be a Henselian valued field, $F = GE$ be the graded field (canonically) associated to E, p be a prime (positive) integer different from* $\mathrm{char}(\overline{E})$ *and n a power of p, then we have a graded ring isomorphism:*

$$H^*(G_E, \mathbb{Z}/n\mathbb{Z}) \cong_g H^*(G_F, \mathbb{Z}/n\mathbb{Z}).$$

PROOF. By [**HW(a)99**, Theorem 5.2] we have $G_F (= \mathrm{Gal}(F_t/F)) \cong \mathrm{Gal}(E_t/E)$, so $H^*(G_F, \mathbb{Z}/n\mathbb{Z}) \cong_g H^*(\mathrm{Gal}(E_t/E),\ \mathbb{Z}/n\mathbb{Z})$, and in the same way as in the proof of Proposition 1.3, $H^*(\mathrm{Gal}(E_t/E), \mathbb{Z}/n\mathbb{Z}) \cong_g H^*(G_E, \mathbb{Z}/n\mathbb{Z})$, so $H^*(G_F, \mathbb{Z}/n\mathbb{Z}) \cong_g H^*(G_E, \mathbb{Z}/n\mathbb{Z})$. □

2. On the square class groups and Witt rings

Inspired by the representation of the Witt ring $W(E)$ of a field E in terms of generators and relations, Craven defined in [**C82**] the notion of Witt ring $W(D)$ for a division ring D and studied some of its properties. In [**S88**] Sladek considered the special case of a division algebra D with a complete discrete valuation and a commutative residue division algebra E of (residue) characteristic different from 2, and proved a ring isomorphism relating $W(D)$ to the Witt ring of E. Precisely, denoting by π a uniformizer of D, and by σ the automorphism of E induced by the inner automorphism $x \mapsto \pi x \pi^{-1}$ of D, he showed that $W(D)$ is isomorphic to the group ring $(W(E)/R(E_\sigma))[\mathbb{Z}/2\mathbb{Z}]$, where $R(E_\sigma)$ is the ideal of the Witt ring $W(E)$, generated by (the classes of) all elements of the form $< 1, -r >$, r ranging over the subgroup $\{\sigma(x)x^{-1} \mid x \in E^*\}E^{*2}$ of E^*/E^{*2}. When D is a field, this result reduced to the well known Springer's Theorem[1].

The notion of Witt ring of a division algebra defined by Craven was then generalized by Lewis and Tignol in [**LT93**] to the context of a central simple algebra. In particular, they showed that if D is a central division algebra over a field E, $B = M_n(D)$, where n is a positive integer greater than 1 and $\widehat{W}(B)$ [resp., $W(B)$] is the Grothendieck-Witt ring [resp., the Witt ring] of B, then $\widehat{W}(B) \cong \mathbb{Z}\times S(D)$, where $S(D) := D^*/D^{*2}$, and where the addition and multiplication in $\mathbb{Z} \times S(D)$ are defined by $(x_1,\, d_1) + (x_2,\, d_2) = (x_1 + x_2,\, d_1 d_2)$ and $(x_1,\, d_1)(x_2,\, d_2) = (x_1 x_2,\, d_1^{x_2} d_2^{x_1})$, for all $x_1, x_2 \in \mathbb{Z}$ and $d_1, d_2 \in S(D)$. They showed also that $W(B)$ is isomorphic to the quotient of $\mathbb{Z} \times S(D)$ by the ideal generated by $(2, -1)$ (see [**LT93**, Proposition 9 and Corollary 1, p.371]).

It is also shown in [**LT93**] that if $I(D)$ is the fundamental ideal of $W(D)$ and n is an odd integer greater that 1, then the Witt ring $W(B)$ is isomorphic to $W(D)/I^2(D)$, and when $\deg(D)$ is even, this isomorphism is valid for all $n > 1$. Furthermore, if D has

---

[1] We recall that another generalization of Springer's Theorem is given in [**RTW07**] by using the Witt ring of hermitian forms over a (tame) division algebra (with center a Henselian valued field). A special case from [**RTW07**] will be considered later in this section.

odd degree, then $W(D)/I^2(D)$ is isomorphic to $W(E)/I^2(E)$ (see [**LT93**, Proposition 10 and Proposition 11]).

A key tool used in the work of Lewis and Tignol was an extensive study of the square class group $S(D)$ of D (see above). In this section, we consider the case where D is a tame central division algebra over a Henselian valued field E with residue characteristic different from 2, and we determine $S(D)$ and $W(D)$ in terms of residue information. We will show in particular that $S(D)$ is isomorphic to $(\overline{D}^*/(\overline{D}^* \cap GD^{*2})) \oplus \Gamma_D/2\Gamma_D$, and that $W(D)$ is isomorphic to a (generalized) crossed product $(W(\overline{D})/J) * Z$, where Z is a subgroup of D^*/D^{*2}, bijectively mapped on $\Gamma_D/2\Gamma_D$, under the map $dD^{*2} \mapsto v(d) + 2\Gamma_D$ (v being the valuation of D), and where J is the ideal of $W(\overline{D})$, generated by the (classes of) elements $\langle 1 \rangle - \langle r \rangle$ with $r \in \overline{D}^* \cap GD^{*2}$. As will be seen in (2.11), this last isomorphism generalizes the one proved by Sladek in [**S88**, Theorem 3.3] in his particular case mentioned above. The reader can see that our arguments here are completely independent of those used in [**S88**].

(2.1) Let F be a graded field, A be a graded central division algebra over F and let $S(A) := A^*/A^{*2}$, then we have the following natural exact sequence:

$$1 \to A_0^* A^{*2}/A^{*2} \xrightarrow{i} S(A) \xrightarrow{g} \Gamma_A/2\Gamma_A \to 0,$$

where i is the map induced by the inclusion $A_0^* A^{*2} \subseteq A^*$ and $g(aA^{*2}) = \mathrm{gr}(a) + 2\Gamma_A$.

[Note that for $a \in A^*$, we have $\mathrm{gr}(a) \in 2\Gamma_A$ if and only if there is $b \in A^*$ such that $\mathrm{gr}(a) = 2\mathrm{gr}(b) = \mathrm{gr}(b^2)$, which yields $b^{-2}a \in A_0^*$, hence $a = b^2c$ where $c(= b^{-2}a) \in A_0^*$. Set $d = b^2cb^{-2}(\in A_0)$, then we have $a = db^2$, so $a \in A_0^*A^{*2}$. Therefore, $\ker(g) = \mathrm{im}(i)$.]

We can easily show that $A_0^*A^{*2}/A^{*2}$ is isomorphic to $A_0^*/(A_0^* \cap A^2)$. Indeed, there is a natural group epimorphism $A_0^* \to A_0^*A^{*2}/A^{*2}$, defined by $a \mapsto aA^{*2}$, which induces this isomorphism. We get then the following natural exact sequence $1 \to A_0^*/(A_0^* \cap A^{*2}) \to S(A) \xrightarrow{g} \Gamma_A/2\Gamma_A \to 0$. We know that the abelian group $\Gamma_A/2\Gamma_A$ is a free $\mathbb{Z}/2\mathbb{Z}$-module, therefore, the above sequence splits. We get then the following result:

LEMMA 2.2. *Let F be a graded field and A a graded central division algebra over F, then $S(A) \cong (A_0^*/(A_0^* \cap A^{*2})) \oplus \Gamma_A/2\Gamma_A$.*

PROPOSITION 2.3. *Let E be a Henselian valued field with residue characteristic different from 2 and let D be a tame central division algebra over E. Then the abelian group $S(D)$ is isomorphic to $S(GD)$.*

PROOF. Consider the following commutative diagram:

$$\begin{tikzcd}
1 \arrow[r] & (1+M_D) \cap D^{*2} \arrow[r] \arrow[d] & D^{*2} \arrow[r] \arrow[d] & GD^{*2} \arrow[r] \arrow[d] & 1 \\
1 \arrow[r] & (1+M_D) \arrow[r] & D^{*} \arrow[r] & GD^{*} \arrow[r] & 1
\end{tikzcd}$$

where all the maps are the natural ones. By the snake lemma we get the following exact sequence:

$$1 \to \frac{1+M_D}{(1+M_D) \cap D^{*2}} \to S(D) \to S(GD) \to 1.$$

Let $a \in M_D$ and let $K = E[a]$ be the subring of D generated by E and a. Plainly, K is a subfield of D; moreover, the polynomial $X^2 - (1 + a)$ of $K[X]$ has residue polynomial $X^2 - 1$, so by Hensel's Lemma, there is some $b \in K[a]$ such that $b^2 = (1 + a)$, $v(b) = 0$ (where v is the valuation of D extending that of E) and $\overline{b} = 1$ in the residue field $\overline{K}$. This shows that $1 + M_D \subseteq D^{*2}$, hence $S(D) \cong S(GD)$. $\square$

COROLLARY 2.4. *Let E be a Henselian valued field and D a tame central division algebra over E, then $S(D) \cong (\overline{D}^* / (\overline{D}^* \cap GD^{*2})) \oplus \Gamma_D / 2\Gamma_D$.*

PROOF. This follows by Lemma 2.2 and Proposition 2.3. $\square$

Let F be a graded field of characteristic different from 2. In (2.5) below we will need to determine a representation of the (gr-) Witt ring $W^{\mathrm{gr}}(F)$ (already considered in [**RTW07**]) in terms of generators and relations. For this we will use the fact that F can be identified with $G(q(F)^h)$. Also we will use the fact that if E is a Henselian valued field, then there is a group isomorphism $\theta_E : W(E) \to W^{\mathrm{gr}}(GE)$ (see [**RTW07**, Proposition 4.3]). As one can easily see this is in fact a ring isomorphism. A detailed proof for this is given in the appendix (see (A.1)).

(2.5) Let F be a graded field of characteristic different from 2 and A a graded central simple algebra over F. In order to define the (graded) Witt ring $W^{\mathrm{gr}}(A)$ we need first to find a representation of $W^{\mathrm{gr}}(F)$ in terms of generators and relations. We will show here that $W^{\mathrm{gr}}(F) \cong \mathbb{Z}[F^*/F^{*2}]/U_F$, where U_F is the ideal of $\mathbb{Z}[F^*/F^{*2}]$ generated by the (classes of) elements $\langle 1, a\rangle - \langle 1+a, a(1+a)\rangle$ and $\langle 1, -1\rangle$, where a describes the elements of $F_0^*\backslash\{-1\}$. Indeed, let $E = q(F)^h$, then by Proposition 2.3 there is a group isomorphism $\phi : E^*/E^{*2} \to F^*/F^{*2}$, defined by $eE^{*2} \mapsto e'F^{*2}$ (we identify F with GE). Let's extend ϕ canonically to a ring isomorphism $\psi : \mathbb{Z}[E^*/E^{*2}] \to \mathbb{Z}[F^*/F^{*2}]$ and let U_E be the ideal of $\mathbb{Z}[E^*/E^{*2}]$ generated by the (classes of) elements $\langle 1, e\rangle - \langle 1+e, e(1+e)\rangle$ and $\langle 1, -1\rangle$, where e describes the elements of $E^*\backslash\{-1\}$. As seen in the last paragraph above, we have a ring isomorphism $W^{\mathrm{gr}}(F) \cong W(E)$ (because $F = GE$); moreover we have $W(E) \cong \mathbb{Z}[E^*/E^{*2}]/U_E \cong \mathbb{Z}[F^*/F^{*2}]/U_F$, where $U_F = \psi(U_E)$, so $W^{\mathrm{gr}}(F) \cong \mathbb{Z}[F^*/F^{*2}]/U_F$. Note that for an element $e \in E^*\backslash\{-1\}$, we have:

$$(1+e)' = \begin{cases} 1 & \text{if} \quad v(e) > 0, \\ e' & \text{if} \quad v(e) < 0, \\ 1+e' & \text{if} \quad v(e) = 0. \end{cases}$$

so,

$$\langle 1, e'\rangle - \langle (1+e)', e'(1+e)'\rangle = \begin{cases} 1, e'\rangle - \langle 1, e'\rangle = 0 & \text{if} \quad v(e) > 0, \\ \langle 1, e'\rangle - \langle e', e'^2\rangle = 0 & \text{if} \quad v(e) < 0, \\ \langle 1, e'\rangle - \langle 1+e', e'(1+e')\rangle & \text{if} \quad v(e) = 0. \end{cases}$$

Therefore, U_F is the ideal of $\mathbb{Z}[F^*/F^{*2}]$ generated by the elements $\langle 1, a\rangle - \langle 1+a, a(1+a)\rangle$ and $\langle 1, -1\rangle$, where a describes the elements of $F_0^*\backslash\{-1\}$.

For any graded division algebra A over F, we define $W^{\mathrm{gr}}(A)$ to be the quotient ring $\mathbb{Z}[A^*/A^{*2}]/U_A$, where U_A is the ideal of $\mathbb{Z}[A^*/A^{*2}]$, generated by the elements $\langle 1, a\rangle - \langle 1 + a, a(1 + a)\rangle$ and $\langle 1, -1\rangle$, with a describing the elements of $A_0^*\backslash\{-1\}$. In the same way as in the case where $A = F$, it is clear that $W^{\mathrm{gr}}(A) \cong W(q(A)^h)$. Similarly, for any tame central division algebra D over a Henselian valued field E, we have $W(D) \cong W^{\mathrm{gr}}(GD)$ (it suffices to replace the isomorphism $E^*/E^{*2} \cong GE^*/GE^{*2}$ in above by $D^*/D^{*2} \cong GD^*/GD^{*2}$).

For an arbitrary graded central simple algebra A, we extend the above definition and set $W(A)^{\mathrm{gr}} = \mathbb{Z}[A^*/A^{*2}]/U_A$, where U_A is the ideal of $\mathbb{Z}[A^*/A^{*2}]$ generated by the (classes) of elements $\langle 1, a\rangle - \langle 1+a, a(1+a)\rangle$ and $\langle 1, -1\rangle$, with a describing the elements of $A_0^*\backslash\{-1\}$ (here A_0^* denotes the multiplicative group of invertible elements of A_0).

Let E be a Henselian valued field with valuation v. We recall that by [**W83**, §4, p.488] we have a ring isomorphism $W(E) \cong W(\overline{E})[Z]$, where Z is a subgroup of E^*/E^{*2}, which is mapped bijectively on $\Gamma_E/2\Gamma_E$ under the correspondence $eE^{*2} \mapsto v(e) + 2\Gamma_E$. In what follows we will prove a more general result (see Corollary 2.10).

(2.6) Let $A = \oplus_{g\in G} A_g$ be a graded ring with type a (multiplicative) group G. We recall that A is called a (generalized) crossed product of G over A_1 if for any $g \in G$, A_g contains an invertible element.

Let R be a commutative ring, B be an R-algebra, $U(B)$ be the set of invertible elements of B, $\mathrm{Aut}(B)$ be the ring consisting of ring automorphisms of B, and $\omega : G \to \mathrm{Aut}(B)$, $f : G \times G \to U(B)$ be two maps satisfying the following conditions (for all $x, y, z \in G$ and $b \in B$):

(1) $\omega_x(\omega_y(b)) = f(x,y)\omega_{xy}(b)f(x,y)^{-1}$,
(2) $f(x,y)f(xy,z) = \omega_x(f(y,z))f(x,yz)$,
(3) $f(x,1) = f(1,x) = 1$,

then we say that (ω, f) is a (normalized) generalized factor set of G in B. We let $(B, G, (\omega, f)) = \oplus_{g\in G} Bx_g$, where x_g are independent indeterminates on B, and where we define the addition on $(B, G, (\omega, f))$ in the natural way, and the multiplication by (extending) the relations: $x_g b = \omega_g(b)x_g$ and $x_g x_h = f(g,h)x_{gh}$, for all $g, h \in G$ and $b \in B$. We recall that $(B, G, (w, f))$ is a (generalized) crossed product (see [**K87**, Theorem 1.3, p.64]). We call it the (generalized) crossed product of G on B defined by the generalized factor set (ω, f) (it is called in [**K87**, p.63] a crossed system for G over B).

(2.7) Let F be a graded field of characteristic different from 2, A a graded central simple algebra over F and denote $\Gamma_A^* := \{\mathrm{gr}(a) \mid a \text{ is an invertible homogeneous element of } A\}$. We assume for the rest here that $\Gamma_A = \Gamma_A^*$ (we saw in (1.1.1) that this is equivalent to $A = M_n(C)$ where C is a graded central division algebra over F). Consider the following exact sequence of $\mathbb{Z}/2\mathbb{Z}$-modules, where

A^* denotes the multiplicative group of invertible homogeneous elements of A:

$$0 \to \ker(\overline{\mathrm{gr}}) \to A^*/A^{*2} \overset{\overline{\mathrm{gr}}}{\to} \Gamma_A/2\Gamma_A \to 0,$$

where $\overline{\mathrm{gr}}$ is the homomorphism defined by $aA^{*2} \mapsto \mathrm{gr}(a) + 2\Gamma_A$, for any $a \in A^*$. Since $\Gamma_A/2\Gamma_A$ is a free module over $\mathbb{Z}/2\mathbb{Z}$, this sequence splits. Therefore, there is a subgroup Z of A^*/A^{*2} which is mapped bijectively onto $\Gamma_A/2\Gamma_A$ by $\overline{\mathrm{gr}}$. For any element $\overline{z} \in Z$, we fix an element $z \in A^*$ such that $\overline{z} = zA^{*2}$. We will write $z \in Z$ for such (fixed) elements. For any $a \in A^*$, there is then a unique $z \in Z$ such that $\mathrm{gr}(a) + 2\Gamma_A = \mathrm{gr}(z) + 2\Gamma_A$. Also, there is $a_0 \in A_0$ and $c \in A^*$ such that $a = a_0 z c^2$. Plainly, z is uniquely determined by the class $aA^{*2} (\in A^*/A^{*2})$. We claim that (the class) $a_0(A_0^* \cap A^{*2})$ is also uniquely determined by aA^{*2}. Indeed, let $(s_i)_{i\in I}$ be a finite family of elements of A^* and suppose that $a\prod_{i\in I} s_i^2 = b_0 z e^2$, for some $b_0 \in A_0^*$ and $e \in A^*$, then $a_0 z c^2 \prod_{i\in I} s_i^2 = b_0 z e^2$, so $\sigma_z(a_0) c^2 \prod_{i\in I} s_i^2 = \sigma_z(b_0) e^2$, where $\sigma_z(x) = z^{-1}xz (= \mathrm{Int}(z^{-1})(x))$ for any $x \in A_0^*$. Hence, $c^2(\prod_{i\in I} s_i^2)(e^{-1})^2 = \sigma_z(a_0^{-1})\sigma_z(b_0) = \sigma_z(a_0^{-1}b_0)$. Thus, $a_0^{-1}b_0 = \sigma_{z^{-1}}(c^2(\prod_{i\in I} s_i^2)(e^{-1})^2) = \sigma_{z^{-1}}(c)^2(\prod_{i\in I} \sigma_{z^{-1}}(s_i)^2)\sigma_{z^{-1}}(e^{-1})^2 (\in A_0^* \cap A^{*2})$.

Let J be the ideal of $W(A_0)$ generated by the (classes) of the elements $\langle 1\rangle - \langle r\rangle$, where $r \in A_0^* \cap A^{*2}$. Consider the ring $B = W(A_0)/J$ and consider the maps $w : Z \to \mathrm{Aut}(B)$, $f : Z \times Z \to U(B)$ (where $U(B)$ is the set of invertible elements of B), defined respectively by (extending) $w_{\overline{z}}(\langle a\rangle + J) = \langle \mathrm{Int}(z)(a)\rangle + J$ and $f(\overline{z}_1, \overline{z}_2) = 1$ for all $\overline{z}, \overline{z}_1, \overline{z}_2 \in Z$ and $a \in A_0^*$ (it is straightforward checked that $w_{\overline{z}}$ is well-defined). One can easily see that (w, f) is a generalized factor set of Z in B. We denote the (generalized) crossed product corresponding to B, Z and (w, f) simply by $B * Z$ (the standard notation would be $(B, Z, (w, f))$ as seen above).

Now, consider the homomorphism of additive groups $\psi :\ \mathbb{Z}[A^*/A^{*2}] \to (W(A_0)/J)*Z$, defined by (extending) the correspondence $aA^{*2} \mapsto (\langle a_0\rangle + J).\overline{z}$, where $a_0 \in A_0^*$ and $z \in Z$ satisfy $a = a_0 z c^2$ for some $c \in A^*$ (note that if $a_0^{-1}b_0 \in A_0^* \cap A^{*2}$, then in $W(A_0)$ we have $\langle a_0\rangle - \langle b_0\rangle = \langle a_0\rangle(1 - \langle a_0^{-1}b_0\rangle) \in J$, hence $\langle a_0\rangle + J = \langle b_0\rangle + J$). ψ is in fact a ring homomorphism. Indeed, let $a_1, a_2 \in A^*$, and write $\overline{a}_1$ and $\overline{a}_2$ for the classes of a_1 and a_2 in A^*/A^{*2}. Write $\overline{a_1} = \overline{a_{1,0}z_1}$ and $\overline{a_2} = \overline{a_{2,0}z_2}$, where $a_{1,0}, a_{2,0} \in A_0^*$ and $\overline{z_1}, \overline{z_2} \in Z$, then $\overline{a_1}.\overline{a_2} = (\overline{a_{1,0}z_1})(\overline{a_{2,0}z_2}) = \overline{a_{1,0}}.(\overline{z_1}.\overline{a_{2,0}})\overline{z_2} = \overline{a_{1,0}}.\overline{z_1 a_{2,0}}.\overline{z_2} = \overline{a_{1,0}}.\overline{\mathrm{Int}(z_1)(a_{2,0})z_1}.\overline{z_2} = \overline{a_{1,0}}.\overline{\mathrm{Int}(z_1)(a_{2,0})}.\overline{z_1z_2} = \overline{a_{1,0}\mathrm{Int}(z_1)(a_{2,0})}.\overline{z_1z_2}$. Therefore, $\psi(\overline{a_1}.\overline{a_2}) = ((\langle a_{1,0}\mathrm{Int}(z_1)(a_{2,0})\rangle + J)\overline{z_1z_2} = \psi(\overline{a_1})\psi(\overline{a_2})$.

We have $W^{\mathrm{gr}}(A) = \mathbb{Z}[A^*/A^{*2}]/U_A$, where U_A is the ideal of $\mathbb{Z}[A^*/A^{*2}]$ generated by the elements $\langle 1, a\rangle - \langle 1 + a, a(1 + a)\rangle$ and $\langle 1, -1\rangle$, with a describing the elements of $A_0^* \backslash \{-1\}$, so ψ induces a ring epimorphism $\overline{\psi} : W^{\mathrm{gr}}(A) \to (W(A_0)/J) * Z$. We claim that $\ker(\overline{\psi}) = 0$. Indeed, let $\eta : W(A_0) \to W^{\mathrm{gr}}(A)$ be the extension map defined by $\langle r\rangle \mapsto \langle r\rangle$ for any $r \in A_0^*$. Note that for any $r \in A_0^* \cap A^{*2}$, we have $\eta(\langle 1\rangle - \langle r\rangle) = 0$ (for $r \in A^{*2}$), so $\eta(J) = 0$. Let $\alpha := \sum_i(\sum_j n_{i_j}\langle a_{i_j} z_i\rangle)$ be an arbitrary element of $W^{\mathrm{gr}}(A)$, where $a_{i_j} \in A_0^*$ and $z_i \in Z$ (with z_i pairwise distinct), then $\overline{\psi}(\alpha) = \sum_i(\sum_j n_{i_j}\langle a_{i_j}\rangle + J)\overline{z}_i$. Hence, $\overline{\psi}(\alpha) = 0$ if and only if (in $W(A_0)$) $\sum_j n_{i_j}\langle a_{i_j}\rangle \in J$ for all i, but in this case, in $W^{\mathrm{gr}}(A)$, we will have

$\sum_j n_{i_j}\langle a_{i_j}\rangle \in \eta(J)$, so $\sum_j n_{i_j}\langle a_{i_j}\rangle = 0$ in $W^{\mathrm{gr}}(A)$. This shows that $\ker(\overline{\psi}) = 0$. We get then the following proposition:

PROPOSITION 2.8. *Let F be a graded field of characteristic different from 2, A a graded central simple algebra over F such that $\Gamma_A = \Gamma_A^*$, J be the ideal of $W(A_0)$ generated by the (classes of) elements $\langle 1\rangle - \langle r\rangle$, where $r \in A_0^* \cap A^{*2}$, and Z be a subgroup of A^*/A^{*2}, which is mapped bijectively onto $\Gamma_A/2\Gamma_A$ by the map $\overline{\mathrm{gr}}(xA^{*2}) = \mathrm{gr}(x) + 2\Gamma_A$. Then, using the notation of (2.7) we have a canonical ring isomorphism $W^{\mathrm{gr}}(A) \cong (W(A_0)/J) * Z$, induced by the correspondence $\langle a\rangle \mapsto (\langle a_0\rangle + J).\overline{z}$, for any $a \in A^*$, where $a_0 \in A_0^*$ and $z \in Z$ satisfy the condition $a = a_0 z c^2$ for some $c \in A^*$. In particular, if $A_0^* \cap A^{*2} = A_0^{*2}$, then $J = 0$ and we have a canonical ring isomorphism $W^{\mathrm{gr}}(A) \cong W(A_0) * Z$.*

COROLLARY 2.9. *Let F be a graded field of characteristic different from 2 and let Z be a subgroup of F^*/F^{*2}, which is mapped bijectively onto $\Gamma_F/2\Gamma_F$ by the map $\overline{\mathrm{gr}}(xF^{*2}) = \mathrm{gr}(x) + 2\Gamma_F$, then there is a canonical ring isomorphism $W^{\mathrm{gr}}(F) \cong W(F_0)[Z]$.*

PROOF. Indeed, by [**HW(a)99**, Corollary 1.3] F_0 is algebraically closed in $q(F)$, so $F_0^* \cap F^{*2} = F_0^{*2}$. Moreover, one can easily see that the (generalized) crossed product $W(F_0) * Z$ of Proposition 2.8 is just the group ring $W(F_0)[Z]$. □

COROLLARY 2.10. *Let E be a Henselian valued field of residue characteristic different from 2, D be a tame central division algebra over E, J be the ideal of $W(\overline{D})$, generated by the classes of the* *elements* $\langle 1\rangle - \langle r\rangle$*, where* $r \in \overline{D}^* \cap GD^{*2}$*, and* Z *be a subgroup of* D^*/D^{*2} *mapped bijectively onto* $\Gamma_D/2\Gamma_D$ *by the induced map* $\overline{v}$ *of the valuation* v *of* D*, then we have a canonical ring isomorphism* $W(D) \cong (W(\overline{D})/J) * Z$*. If in addition* $\overline{D}^* \cap GD^{*2} = \overline{D}^{*2}$*, then we get a ring isomorphism* $W(D) \cong W(\overline{D}) * Z$*. In particular, if* $D = E$*, then* $W(E) \cong W(\overline{E})[Z]$*.*

(2.11) Let E be a Henselian valued field of residue characteristic different from 2, D be a tame central division algebra over E, J be the ideal of $W(\overline{D})$, generated by the classes of the elements $\langle 1\rangle - \langle r\rangle$, where $r \in \overline{D}^* \cap GD^{*2}$, and let Z be a subgroup of D^*/D^{*2} mapped bijectively onto $\Gamma_D/2\Gamma_D$ by the map $dD^{*2} \mapsto v(d) + 2\Gamma_D$, where v is the valuation of D. Then by Corollary 2.10, there is a ring isomorphism: $W(D) \overset{\cong}{\to} (W(\overline{D})/J) * Z$.

Suppose now that the valuation of D is discrete and let $\pi \in D^*$ be such that $v(\pi)$ generates Γ_D. Any element d of D^* can be written in the form $d = x\pi^k$ for some $x \in U_D$ (where $U_D := \{d \in D^* \mid v(d) = 0\}$) and some $k \in \mathbb{Z}$. Hence, $d^2 = (x\pi^k)(x\pi^k) = x\tau^k(x)\pi^{2k}$, where $\tau(x) = \pi x \pi^{-1}$. Sladek considered in [**S88**] the automorphism σ induced on $\overline{D}$ by τ, i.e., $\sigma(\overline{u}) = \overline{\pi u \pi^{-1}}$ for any $u \in U_D$, then he considered the subgroup $\overline{D}_\sigma$ of $\overline{D}^*$, generated by $\overline{D}^{*2}$ and the elements $\sigma(x)x^{-1}$, where x describes the elements of $\overline{D}^*$. He denoted by $R(\overline{D}_\sigma)$ the ideal of $W(\overline{D})$ generated by the elements $\langle 1, -e\rangle$, $e \in \overline{D}_\sigma$ and he showed that when $\overline{D}$ is commutative, then $W(D) \cong (W(\overline{D})/R(\overline{D}_\sigma))[Z]$. Our objective here is to show that the isomorphism we proved in Corollary 2.10 generalizes the one proved by Sladek.

With the notation above, we have (for any $u \in U_D$) $\sigma(\overline{u}) = \pi'\overline{u}\pi'^{-1}$, where $\pi' = \pi + D^{>v(\pi)} \in GD$. We saw above that for $d = u\pi^k$, we have $d^2 = u\tau^k(u)\pi^{2k}$, so $d'^2 = (d^2)' = \overline{u}\tau^k(u)'\pi'^{2k} = \overline{u}\sigma^k(\overline{u})\pi'^{2k}$. One can easily see that $\overline{u}\sigma^k(\overline{u})(= \overline{u}^2\overline{u}^{-1}\sigma^k(\overline{u})) \in \overline{D}_\sigma$. For simplicity write $d'^2 = w_d\pi'^{2k}$, where $w_d = \overline{u}\sigma^k(\overline{u})$. If we consider a family of elements of D^*: $d_1 = u_1\pi^{k_1}$, ..., $d_m = u_m\pi^{k_m}$, with $u_i \in U_D$ and $k_i \in \mathbb{Z}$, then we will have ${d'_1}^2...{d'_m}^2 = (w_{d_1}\pi'^{2k_1})...(w_{d_m}\pi'^{2k_m}) = \phi(w_{d_1}, ..., w_{d_m})\pi'^{2(\sum_{i=1}^m k_i)}$, with $\phi(w_{d_1}, ..., w_{d_m}) \in \overline{D}_\sigma$. In particular, if ${d'_1}^2...{d'_m}^2 \in \overline{D}^*$, then $\sum_{i=1}^m k_i = 0$ (for $\mathrm{gr}(\pi'^{2(\sum_{i=1}^m k_i)}) = 2\sum_{i=1}^m k_i v(\pi)$), so ${d'_1}^2...{d'_m}^2 \in \overline{D}_\sigma$ in this case. This shows that $\overline{D}^* \cap GD^{*2} \subseteq \overline{D}_\sigma$. Let $r \in \overline{D}^* \cap GD^{*2}$, then by the above $r \in \overline{D}_\sigma$, so in $W(\overline{D})$ we have $\langle 1\rangle - \langle r\rangle = \langle 1, -r\rangle \in R(\overline{D}_\sigma)$, where as seen above, $R(\overline{D}_\sigma)$ is the ideal of $W(\overline{D})$ generated by the elements $\langle 1, -e\rangle$, $e \in \overline{D}_\sigma$. It follows that $J \subseteq R(\overline{D}_\sigma)$. Conversely, let $d \in D^*$ with $v(d) = 0$ (i.e., $d \in U_D$) and suppose that $\overline{d} \in \overline{D}_\sigma$. Under the hypotheses of Sladek recalled above, we have $d \in D^{*2}$ (see [**S88**, Theorem 2.1]), so $\overline{d} \in \overline{D}^* \cap GD^{*2}$, hence $\langle 1, -\overline{d}\rangle = \langle 1\rangle - \langle\overline{d}\rangle \in J$, so $R(\overline{D}_\sigma) \subseteq J$. Therefore, $R(\overline{D}_\sigma) = J$. This shows that our isomorphism in Corollary 2.10 (i.e., $W(D) \overset{\cong}{\to} (W(\overline{D})/J) * Z$) generalizes Sladek's one proved in [**S88**, Theorem 3.3] (note here that since $\overline{D}$ is commutative, then $(W(\overline{D})/J) * Z = (W(\overline{D})/J)[Z]$).

(2.12) Let E be a Henselian valued field and let $I(E)$ be the fundamental ideal of $W(E)$. Let $F = GE$ and consider the ring isomorphism $\theta_E : W(E) \to W^{\mathrm{gr}}(F)$ of [**RTW07**] (see the appendix). Let $I^{\mathrm{gr}}(F)$ be the image under θ_E of $I(E)$. We will call $I^{\mathrm{gr}}(F)$ the fundamental ideal of $W^{\mathrm{gr}}(F)$. More generally for an arbitrary graded field F, let $E := q(F)^h$ that we endow with its canonical valuation (associated to the graded structure of F as seen in the preliminaries). We define the fundamental ideal of $W^{\mathrm{gr}}(F)$ in the same way (since F can be identified to GE). As in the ungraded case, we define the graded (gr-) Witt ring of F to be $GW^{\mathrm{gr}}(F) := \oplus_{n\in\mathbb{Z}} GW^{\mathrm{gr}}(F)_n$, where $GW^{\mathrm{gr}}(F)_n = I^{\mathrm{gr}}(F)^n / I^{\mathrm{gr}}(F)^{n+1}$.

We know that the fundamental ideal $I(E)$ of $W(E)$ is generated by the classes of binary Pfister forms $\langle 1, -a\rangle$, where $a \in E^*$. Also, it is well known that a Pfister's form is either anisotropic or hyperbolic (see [**Ka08**, Corollary 2.1.8, p.18]). It follows that for a graded field F, the fundamental ideal $I^{\mathrm{gr}}(F)$ of $W^{\mathrm{gr}}(F)$, as defined above, is generated by the classes of graded forms of the form $\langle 1, -x\rangle$, where x describes nonzero homogeneous elements of F (we call these forms binary Pfister's graded forms). It is clear that for any $a, b \in F^*$, we have $\langle a, b\rangle = \langle 1, a\rangle - \langle 1, -b\rangle$, so $I^{\mathrm{gr}}(F)$ is the kernel of the (additive) group homomorphism $\overline{\dim} : W^{\mathrm{gr}}(F) \to \mathbb{Z}/2\mathbb{Z}$, uniquely defined by associating to (each class of a) nondegenerate graded (symmetric) bilinear form its dimension modulo $2\mathbb{Z}$.

3. On Milnor's K-theory for graded fields

Let E be a field. We recall that Milnor defined a ring K_*E in the following way: Let E^* be the multiplicative group of E, $T(E^*)$ be the tensor algebra of E^* (where E^* is considered as a $\mathbb{Z}$-module) and consider the ideal R of $T(E^*)$ generated by the elements $e \otimes (1-e)$, where e describes $E^*\backslash\{1\}$. Then, Milnor set $K_*(E) = T(E^*)/R$. As one can see, $T(E^*)$ has a natural $\mathbb{Z}$-grading, where the i^{th} component $T(E^*)_i$ is $E^* \otimes_{\mathbb{Z}} ... \otimes_{\mathbb{Z}} E^*$ (i times) for $i > 0$, $T(E^*)_0 = \mathbb{Z}$, and $T(E^*)_i = 0$ for $i < 0$. This induces a grading on $K_*(E)$, i.e., $K_*(E) = \oplus_{i \geq 0} K_i E$, where $K_i E$ is the quotient of the group $T(E^*)_i$ by the subgroup generated by all elements $a_1 \otimes ... \otimes a_i$, where $a_j \in E^*$ (for all $1 \leq j \leq i$) and $a_j + a_{j+1} = 1$ for some j $(1 \leq j < i)$.

In particular, $K_1 E$ is nothing but the abelian group consisting of quotient classes of elements of E^*, however, $K_1 E$ is denoted additively (in the standards). So, if we consider the map $l : E^* \to K_1 E$, defined by $a \mapsto a + R$, we have $l(ab) = l(a) + l(b)$ for all $a, b \in E^*$. One can easily see that K_*E is generated by $K_1 E$. Moreover, it is shown that for any $a \in K_m E$ and $b \in K_n E$, we have $ba = (-1)^{mn} ab$ (see [**Mi70**, Lemma 1.1]).

Milnor considered also in [**Mi70**] the quotient groups $(K_*E)_n := (K_*E)/n(K_*E)$ and the (sub)groups $(K_i E)_n := (K_i E)/n(K_i E)$ (for positive integers i and n). In particular, it is (now) showed that there is isomorphism[2]: $K_i(E)_2 \cong I^i(E)/I^{i+1}(E)$, which carries each product $l(a_1)...l(a_i)$ in $(K_iE)_2$ to the product $(\langle a_1\rangle - \langle 1\rangle)...(\langle a_n\rangle - \langle 1\rangle)$ modulo $I^{i+1}(E)$. Also there is a well defined cohomological description of the rings $(K_*E)_n$.

Milnor showed also that if E is a complete discrete valued field with residue field $\overline{E}$ and with $\mathrm{char}(\overline{E}) \neq 2$, then for any prime positive integer p distinct from the characteristic of $\overline{E}$, there is a natural split exact sequence $0 \to K_i\overline{E}/pK_i\overline{E} \to K_iE/pK_iE \to K_{i-1}\overline{E}/pK_{i-1}\overline{E} \to 0$ (see [**Mi70**, Lemma 2.6]). This result has been generalized by A. Wadsworth who proved that if E is a valued field with valuation v, p is a prime positive integer, n is a power of p, $\Gamma = v(E^*)$ and $(\pi_j)_{j\in J}$ is a family of elements of E^* that maps bijectively onto a base of the free $\mathbb{Z}/n\mathbb{Z}$-module $\Gamma/n\Gamma$ under the correspondence $x \mapsto v(x) + n\Gamma$, and if $1 + M_v \subseteq E^n$, where $M_v = \{e \in E \,|\, v(e) > 0\}$, then

$$(K_iE)_n \cong (K_i\overline{E})_n \oplus \oplus_{1\leq m\leq i} \oplus_{j_1,\dots,j_m\in J_m} (K_{i-m}\overline{E})_n l(\pi_{j_1})...l(\pi_{j_m}).$$

where J_m is the set of all subsets of cardinality m of J. Furthermore, in this case, each summand $(K_{i-m}\overline{E})_n l(\pi_{j_1})...l(\pi_{j_m})$ is isomorphic to $(K_{i-m}\overline{E})_n$ (see [**W83**, Proposition 2.1 and its consequence Proposition 2.3]).

In what follows, we are interested in giving a version of Milnor's K-theory for graded fields. This will add a new piece to the study developed by Hazrat in [**Ha16**] for the K-theory of graded rings.

---

[2] Milnor showed in [**Mi70**] that there is an epimorphism and gave cases in which this is an isomorphism. Voevodsky showed in [**V96**] that Milnor's epimorphism is indeed an isomorphism.

In particular, we will prove in this section a result analogous to the above mentioned one due to Wadsworth, i.e., [**W83**, Proposition 2.1]. Unlike Wadsworth's valuative case where the base field E is assumed to satisfy the condition $1+M_v \subseteq E^n$, we have no condition on the base graded field in the graded setting (see Proposition 3.1). We will see in Remark 3.3 that for a graded field F of characteristic different from 2, the ring $(K_*F)_2$ is graded isomorphic to the graded (gr-) Witt ring $GW^{\mathrm{gr}}(F)$. We will also give a cohomological interpretation of $(K_*F)_n$, when n is a power of a prime different from the characteristic of F (see Corollary 3.4).

Let F be a graded field. Analogously to Milnor's construction, we define K_*F in the following way: Let $T(F^*)$ be the tensor algebra of F^* (considered as a $\mathbb{Z}$-module), R be the ideal of $T(F^*)$ generated by the elements $a \otimes (1-a)$, where a describes $F_0^* \backslash \{1\}$ and set $K_*F = T(F^*)/R$. As in the ungraded case, it is clear that K_*F has a natural $\mathbb{Z}$-graded structure. We denote the i^{th}-component of K_*F by K_iF. For positive integers i and n, we set $(K_*F)_n = (K_*F)/n(K_*F)$ and $(K_iF)_n = (K_iF)/n(K_iF)$. We let $l: F^* \to K_1F$ be the group homomorphism defined by $l(a) = a+R$. Plainly, l can be considered as a map from F^* into K_*F. It induces a map from F^*/F^{*n} into $(K_*F)_n$ that we denote also by l. Our aim here is to relate the groups $(K_*F)_n$ and $(K_*F_0)_n$ when n is a prime power integer. For this, we will use Wadsworth's notation already recalled in section 1 (see (1.1)).

PROPOSITION 3.1. *Let F be a graded field of characteristic different from 2, p be a prime positive integer, n a power of p, and*

$(\gamma_j)_{j\in J}$ be a base of $\Gamma_F/n\Gamma_F$ over $\mathbb{Z}/n\mathbb{Z}$, then $(K_*F)_n$ is graded isomorphic to $(K_*F_0)_n[J;l(-1)]$.

PROOF. The arguments used here are similar to the ones in [**W83**, Propositions 2.1 and 2.3]. Let's consider the $\mathbb{Z}$-graded ring $(K_*F_0)_n[J;l(-1)]$ constructed as in (1.1) (with $A=(K_*F_0)_n$ and $w=l(-1)$), and consider the same notation as in (1.1) (in particular, x_j the (natural) image of X_j in $(K_*F_0)_n[J;l(-1)]$).

Let $(y_j)_{j\in J}$ be a family of elements of F^* with $\mathrm{gr}(y_j)=\gamma_j$. An arbitrary element a of F^* can be written in the form $a=a_0c^ny_{j_1}^{m_1}...y_{j_k}^{m_k}$, where $a_0\in F_0^*$, $c\in F^*$ and $m_1,...,m_k\in\mathbb{Z}$. We have $\mathrm{gr}(a)+n\Gamma_F=\sum_{i=1}^k m_i\gamma_{j_i}+n\Gamma_F$, so $(m_1,...,m_k)+n\mathbb{Z}$ is uniquely determined by a. Also it is easily seen that a_0 is uniquely determined modulo F_0^n by a (because F_0 is algebraically closed in $q(F)$). Therefore, we can define a surjective group homomorphism $F^*\to(K_*F_0)_n[J;l(-1)]$, by $a\mapsto l(a_0)+\sum_{i=1}^k m_ix_{j_i}$. This map extends (uniquely) to a ring homomorphism $\psi: T(F^*)\to(K_*F_0)_n[J;l(-1)]$. It induces also a ring homomorphism $\phi: K_*F\to(K_*F_0)_n[J;l(-1)]$. Indeed, for any $a\in F_0^*\backslash\{1\}$, we have $\psi(a\otimes(1-a))=0$ (see that $\psi(a)(=a)$ and $\psi(1-a)(=1-a)$ are in $(K_*F_0)_n$, so $\psi(a\otimes(1-a))=\psi(a)\psi(1-a)=a.(1-a)=0$ in $(K_*F_0)_n$).

Consider now the embedding $F_0^*\to F^*$ induced from the inclusion $F_0^*\subseteq F^*$. It induces an embedding (of abelian groups) $i:\frac{F_0^*}{F_0^{*n}}\to\frac{F^*}{F^{*n}}$, which in turn induces a graded ring homomorphism $f:(K_*F_0)_n\to(K_*F)_n$, defined by $f(l(a))=l(i(a))$, for any $a\in F_0^*$ (where l denotes both the maps $l: F_0^*/F_0^{*n}\to(K_*F_0)_n$ and $l: F^*/F^{*n}\to(K_*F)_n$ seen in above). As seen in (1.1),

$K_*(F_0)_n[J;l(-1)]$ is a free $(K_*F_0)_n$-module, so f extends to a group homomorphism $h:\ (K_*F_0)_n[J;l(-1)] \to (K_*F)_n$, defined by $h(x) = f(x)$ for $x \in (K_*F_0)_n$ and $h(x_{j_1}...x_{j_k}) = y_{j_1}...y_{j_k}$. One can easily see that h is the inverse of ϕ, which ends the proof. ☐

COROLLARY 3.2. *Let E be a henselian valued field of residue characteristic different from 2, $F = GE$, p be a prime different from the residue characteristic of E, and n a power of p, then then $(K_*E)_n$ is graded isomorphic to $(K_*F)_n$.*

PROOF. This follows by [**W83**, Corollary 2.4] and Proposition 3.1 (since $F_0 = \overline{E}$). ☐

REMARKS 3.3. (1) Let F be a graded field of residue characteristic different from 2, p be a prime (positive) integer different from $\mathrm{char}(F)$, n be a power of p, and $E := q(F)^h$ be the henselization of $q(F)$ with respect to the canonical valuation of $q(F)$ (associated to the graded structure of F), then by Corollary 3.2 $(K_*F)_n$ is graded isomorphic to $(K_*E)_n$ (since F can be identified to GE).
(2) Let F be a graded field of characteristic different from 2, $E = q(F)^h$ and let $GW^{\mathrm{gr}}(F)$ be the graded (gr-) Witt ring of F. Then, by construction we have $GW^{\mathrm{gr}}(F) \cong_g GW(E)$ (see in (2.12)), so $(K_*F)_2(\cong_g (K_*E)_2 \cong_g GW(E)) \cong_g GW^{\mathrm{gr}}(F)$.

COROLLARY 3.4. *Let F be a graded field of characteristic different from 2, p be a prime positive integer different from* $\mathrm{char}(F)$*, n a power of p, F_t be 'the' tame closure of F and G_F be the Galois group of the graded field extension F_t/F, then $(K_*F)_n \cong_g \mathrm{H}^*(G_F, \mathbb{Z}/n\mathbb{Z})$.*

PROOF. Let $E = q(F)^h$, E_s be 'the' separable closure of E and G_E be the Galois group of E_s/E. It is known that we have $(K_*E)_n \cong_g \mathrm{H}^*(G_E, \mathbb{Z}/n\mathbb{Z})$ (see [**V96**]); moreover by Proposition 1.4 we have $\mathrm{H}^*(G_E, \mathbb{Z}/n\mathbb{Z}) \cong_g \mathrm{H}^*(G_F, \mathbb{Z}/n\mathbb{Z})$, and by Remark 3.3 we have $(K_*F)_n \cong_g (K_*E)_n$, so $(K_*F)_n \cong_g \mathrm{H}^*(G_F, \mathbb{Z}/n\mathbb{Z})$. $\square$

CHAPTER 4

On torsion subgroups of Whitehead groups

Let R be a simple ring and let $\mathrm{K}_1(R)$ be the Whitehead group of R defined by considering the category of finitely generated projective (left) modules over R. It is known that $\mathrm{K}_1(R)$ is matrix representable, so one has $\mathrm{K}_1(R) = R^*/R^{*c}$, where R^* is the multiplicative group consisting of invertible elements of R and R^{*c} is the commutator subgroup of R^*. In a similar way, for a graded ring A of type an abelian group Γ, Hazrat defined in [**Ha16**] the (graded) Whitehead group $\mathrm{K}_1^{\Gamma}(A)$ of A (also denoted by $\mathrm{K}_1^{\mathrm{gr}}(A)$ if there is no risk of confusion) constructed by considering the category of finitely generated graded modules of type Γ over A, which are (Γ-)graded projective (see [**Ha16**, §3.12, p.181]). This (graded) functor (like the other (graded) $\mathrm{K}_i^{\mathrm{gr}}$-groups that he defined in [**Ha16**]) behaves mostly in an analogous way as for the classical one (in the ungraded case). It is known that if A is a graded division ring with support $\Omega := \Gamma_A$, then $\mathrm{K}_1^{\Gamma}(A) \cong \oplus_{\Gamma/\Omega}\mathrm{K}_1(A_0) = \oplus_{\Gamma/\Omega}A_0^*/A_0^{*c}$. Also, if A is a strongly Γ-graded ring, then $\mathrm{K}_1^{\Gamma}(A) \cong \mathrm{K}_1(A_0)$. However, in general, for a graded (simple) ring A, it is still unknown whether $\mathrm{K}_1^{\Gamma}(A)$ is matrix representable or not (see [**Ha16**, §3.12, p.183]). Some cases where it is shown to be so are given by Zhang in [**Z16**]. Thus, unlike the ungraded case, we cannot write $\mathrm{K}_1^{\mathrm{gr}}(A) = A^*/A^{*c}$, where A^* is the multiplicative group of invertible homogeneous elements of A and A^{*c} is the commutator subgroup of A^*.

In this chapter, we consider a graded field F, a graded central division algebra A over F and call the group $\mathrm{MK}_1(A) := A^*/A^{*c}$ the M-Whitehead group of A. The torsion subgroup of $\mathrm{MK}_1(A)$ will be denoted by $T(\mathrm{MK}_1(A))$. For the (quotient) group A^*/F^*A^{*c}, we will use the same notation as in [**HaW(a)11**] and denote it by $\mathrm{CK}_1(A)$.

We show in particular in the second section of this chapter that if E is a Henselian valued field with $\mathrm{char}(E) = \mathrm{char}(\overline{E})$ and D is a strongly tame central division algebra over E, then the torsion subgroups $T(\mathrm{K}_1(D))$ and $T(\mathrm{MK}_1(GD))$ are isomorphic (see Theorem 2.4). We use this isomorphism to give an alternative proof to [**Mo13**, Theorem 10].

In the first section of this chapter, we give a new proof for Khanduja's theorem that characterizes tame field extensions over a Henselian field (see Theorem 1.1).

1. On tame division algebras over Henselian valued fields

Recall that Tignol proved in [**T90**, Proposition 2.2] that if (E, v) is a Henselian valued field with residue characteristic p different from 0 and M is a field extension of dimension p over E, then the residue field extension $\overline{M}/\overline{E}$ is separable of dimension p if and only if there exists an element $a \in M^*$ such that $v(\mathrm{Tr}_{M/E}(a)) = v(a)$, where $\mathrm{Tr}_{M/E}$ is the trace map (and where the extension of v

to M is also denoted by v). This result was generalized by Khanduja in [**Kh00**, Theorem, p.551], who showed that for arbitrary residue characteristic and for any finite-dimensional field extension M of E, M is tame over E if and only if there exists some $a \in M^*$ such that $v(\mathrm{Tr}_{M/E}(a)) = v(a)$.

In this section, we give a new (and simpler) proof for Khanduja's criterion. This proof is essentially based on the canonical correspondence existing between tame valued field extensions over a Henselian valued field and tame graded field extensions of the corresponding graded field (see Theorem 1.1).

Let (E, v) be a Henselian valued field, L be a finite-dimensional field extension of E, E_{ac} be an algebraic closure of E containing L, $\sigma : L \to E_{ac}$ be an E-embedding, N be a finite-dimensional field extension of E containing both L and $\sigma(L)$, and ζ be the unique extension of v to N. If L is separable over E, then we can assume that N is Galois over E. In this case, for any extension τ of σ to an E-automorphism of N, we have $\zeta \circ \tau = \zeta$ (since ζ is the unique valuation of N extending v). In particular, τ induces a graded GE-automorphism τ' of GN defined by $\tau'(a') = \tau(a)'$ for any element $a \in N$. The restriction of τ' to GL, that we denote by $\tau'|_{GL}$, is clearly a graded GE-embedding from GL to GN.

Suppose that L is tame over E and let w be the unique extension of v to L, then GL is tame over GE, so there exists some element $a \in L^*$ such that $\mathrm{Tr}_{GL/GE}(a') \neq 0$. Indeed, since GL is tame over GE, then $q(GL)$ is a separable field extension of $q(GE)$ (see [**B94**, Theorem 4] or [**TW15**, Proposition 5.19]), therefore by [**K89**, Theorem 4.8, p.89] there exists some element $z \in q(GL)$ such that $\mathrm{Tr}_{q(GL)/q(GE)}(z) \neq 0$. By identification of $q(GL)$ with $GL \otimes_{GE} q(GE)$, one can easily see that there exist $x \in GL$ and $s \in q(GE)^*$ such that $z = x \otimes s$, hence $\mathrm{Tr}_{q(GL)/q(GE)}(z) = s\mathrm{Tr}_{q(GL)/q(GE)}(x) = s\mathrm{Tr}_{GL/GE}(x)$ (see [**HaW(a)11**, §3, p.125]). So, $\mathrm{Tr}_{GL/GE}(x) \neq 0$. We know that for any element $\gamma \in \Gamma_{GL}$, we have $\mathrm{Tr}_{GL/GE}(GL_\gamma) \subseteq GE_\gamma$, so there exists a homogenous component b of x such that $\mathrm{Tr}_{GL/GE}(b) \neq 0$, it suffices then to take $a \in L^*$ such that $b = a'$.

Now, let τ be a different E-embedding of L into E_{ac}. We claim that the induced GE-embeddings σ' and τ' of GL into GE_{ac}, are different. To show this, consider a tame Galois finite-dimensional field extension N of E which contains L with $N \subseteq E_{ac}$. Let B be the valuation ring of the unique extension ζ of v to N, and let G^v be the ramification group of B over E, which is equal to $\{\mathrm{id}_N\}$ (because N is tame over E) and let $\alpha \in \mathrm{Gal}(N/E)$, then $\alpha' = \mathrm{id}_{GN}$ if and only if for all elements $b \in N$, we have $\zeta(\alpha(b) - b) > \zeta(b)$. Hence, $\alpha' = \mathrm{id}_{GN}$ if and only if $\alpha \in G^v$, or equivalently if and only if $\alpha = \mathrm{id}_N$. It follows that for any distinct elements α, β of $\mathrm{Gal}(N/E)$, the induced GE-automorphisms α' and β' on GN are distinct. Let $\tau, \rho : L \to N$ be two different E-embeddings. Suppose by absurd that $\tau' = \rho'$, and let $\tau_1, ..., \tau_r$ [resp., $\rho_1, ..., \rho_r$] be the different extensions of τ [resp., of ρ] to elements of $\mathrm{Gal}(N/E)$, then $\tau_1', ..., \tau_r'$ and $\rho_1', ..., \rho_r'$ are elements of $\mathrm{Gal}(GN/GE)$ extending τ'

(since $\tau' = \rho'$), hence necessarily there exist some i, j $(1 \leq i, j \leq r)$ such that $\tau'_i = \rho'_j$, a contradiction (because $\tau_i \neq \rho_j$).

THEOREM 1.1. [**Kh00**, Theorem, p.551] *Let (E, v) be a Henselian valued field, L be a finite-dimensional field extension of E and w be the unique extension of v to L. Then, L is tame over E if and only if there exists some $a \in L^*$ such that $v(\mathrm{Tr}_{L/E}(a)) = w(a)$.*

PROOF. Suppose that L is tame over E, then L is separable over E and GL is tame over GE. Let N be a tame Galois finite-dimensional field extension of E containing L and let $\sigma_1, ..., \sigma_r$ $(r = [L : E])$ be the different E-embeddings of L into N. By the above the induced GE-embeddings $\sigma'_1, ..., \sigma'_r$ of GL into GN, are pairwise distinct. Also, as seen above, there exists some $a \in L^*$ such that $\mathrm{Tr}_{GL/GE}(a') \neq 0$. We have $\mathrm{Tr}_{GL/GE}(a') = \mathrm{Tr}_{q(GL)/q(GE)}(a') = \sum_{i=1}^{r} q(\sigma'_i)(a') = \sum_{i=1}^{r} \sigma'_i(a') = \sum_{i=1}^{r} \sigma_i(a)'$; moreover, we have $\zeta(\sigma_i(a)) = w(a)$ for all i, $(1 \leq i \leq r)$, where ζ is the unique extension of v to N. So in $N_{\zeta(a)} (:= N^{\geq \zeta(a)} / N^{> \zeta(a)})$, we have $\sum_{i=1}^{r} \sigma_i(a)' = \overline{\sum_{i=1}^{r} \sigma_i(a)} = \overline{\mathrm{Tr}_{L/E}(a)}$. Therefore, the fact that $0 \neq \mathrm{Tr}_{GL/GE}(a')$, yields $v(\mathrm{Tr}_{L/E}(a)) = w(a)$.

Conversely, suppose that there exists some $a \in L^*$ such that $v(\mathrm{Tr}_{L/E}(a)) = w(a)$, then $\overline{\mathrm{Tr}_{L/E}(a)} \neq 0$ in $L_{w(a)}$. Also, this yields $\mathrm{Tr}_{L/E}(a) \neq 0$ (since $w(a) \neq \infty$), so by [**K89**, Theorem 4.8, p.89] L is separable over E. Let T be the tame closure of E in L (i.e., the intersection of L with the tame closure of E in some normal finite-dimensional field extension of E containing L) and let $p = \mathrm{char}(\overline{T}) (= \mathrm{char}(\overline{E}))$. If $p = 0$, then clearly L is tame over E and obviously $L = T$. It suffices then to show this converse for the case where $p \neq 0$. Observe that since T has no proper tame extension in L, then GT has no proper tame extension in GL, hence GL is purely wild over GT, or equivalently $q(GL)$ is purely inseparable over $q(GT)$. So, $[GL : GT] = [q(GL) : q(GT)] = p^m$, for some integer m. Moreover, by Ostrowski's Theorem there exists some integer k such that $[L : T] = p^k[\overline{L} : \overline{T}](\Gamma_L : \Gamma_T)$. We have $[\overline{L} : \overline{T}](\Gamma_L : \Gamma_T) = [GL : GT]$, hence $[L : T] = p^n$, where $n = k + m$. Suppose by absurd that $L \neq T$, or equivalently, $n \geq 1$, and let $\sigma : T \to N$ be a E-embedding of T into N, then σ has p^n extensions to L (i.e., embeddings from L into N which extend σ). If we fix an embedding α_σ of L into N extending σ, then for any other embedding β of L into N extending σ, we have $\alpha'_\sigma|_{GT} = \sigma' = \beta'|_{GT}$; moreover since GL is purely wild over GT, then for any homogeneous element $z \in GL$, we have $z^{p^m} \in GT$, hence $\beta'(z^{p^m}) = \alpha'_\sigma(z^{p^m})$, or equivalently $\beta'(z)^{p^m} = \alpha'_\sigma(z)^{p^m}$. Clearly, we will have $(\beta'(z) - \alpha'_\sigma(z))^{p^m} = \beta'(z)^{p^m} - \alpha'_\sigma(z)^{p^m} = 0$, so $\beta'(z) = \alpha'_\sigma(z)$. Therefore, $\beta' = \alpha'_\sigma$. Now, let $\sigma_1, ..., \sigma_r$ be the different E-embeddings from T into N, and A_i be the set of embeddings of L into N extending σ_i ($1 \leq i \leq r$), then in $N_{\zeta(a)}$, we have $\overline{\mathrm{Tr}_{L/E}(a)} = \sum_{i=1}^r \sum_{\alpha \in A_i} \alpha'(a') = p^n \sum_{i=1}^r \alpha'_{\sigma_i}(a') = 0$, a contradiction. Therefore, $L = T$, which proves that L is tame over E. □

REMARK 1.2. Under the hypotheses of Theorem 1.1, it is clear that if the residue characteristic of E is 0, then any finite-dimensional field extension L of E is tame over E, hence Khanduja's Theorem in this case, means simply that there exists some $a \in L^*$ such that

$w(\mathrm{Tr}_{L/E}(a)) = w(a)$.

We recall that a central division algebra over a Henselian valued field is called tame if it is split by a tame field extension or equivalently it has a maximal subfield which is tame over E. It is called strongly tame over E if the residue characteristic of E does not divide the degree of D.

2. Torsion subgroups of Whitehead groups

We give in this section some results concerning the M-Whitehead group of a graded division algebra and we apply them to describe the torsion subgroup of the Whitehead group of a strongly tame central division algebra over a Henselian valued field.

PROPOSITION 2.1. *Let F be a graded field and A a graded central division algebra over F, then the following statements hold:*

(1) *If A is unramified over F, then $T(\mathrm{MK}_1(A)) = T(\mathrm{K}_1(A_0))$.*

(2) *If A is totally ramified over F, then $T(\mathrm{MK}_1(A)) \cong T(F_0^*)/\mu_s(F_0)$, where $s = \exp(\Gamma_A/\Gamma_F)$ and $\mu_s(F_0)$ is the group of the s^{th} roots of the unity in F_0.*

PROOF. (1) Assume that A is unramified over F, then $A^* = A_0^* F^*$ and $A^{*c} = A_0^{*c}$. Moreover, it is clear that for any element $a \in A^* \backslash A_0^*$, aA_0^{*c} cannot be a torsion element in A^*/A_0^{*c} (for $\mathrm{gr}(a^n) = n\mathrm{gr}(a) \neq 0$), therefore $T(\mathrm{MK}_1(A)) = T(\mathrm{K}_1(A_0))$.

(2) Suppose now that A is totally ramified over F and let $s = \exp(\Gamma_A/\Gamma_F)$. For any element $c \in A^*$, we have $\mathrm{gr}(c^s) + \Gamma_F = \Gamma_F$,

so $c^s \in A_0^* F^*$, but $A_0 = F_0$, so $c^s \in F^*$. Let a and b be elements of A^* and let $e = aba^{-1}b^{-1}$, then $e \in F_0 (= A_0)$ and $eb = aba^{-1}$. Thus, $e^s b^s = (eb)^s = ab^s a^{-1} = b^s$ (for as seen above $b^s \in F$). It follows that $e^s = 1$, hence $A^{*c} \subseteq \mu_s(F_0)$. We claim that for any $w \in \mu_s(F_0)$, there exist $a, b \in A^*$ such that $w = aba^{-1}b^{-1}$. Indeed, let m be the exponent of w, then w is a primitive m^{th}-root of unity in F_0. Since m divides s, then there is some $\gamma \in \Gamma_A$ such that the subgroup of Γ_A / Γ_F, generated by $\gamma + \Gamma_F$ has cardinality m. Let $a \in A^*$ with $\mathrm{gr}(a) = \gamma$ and let $K := F[a]$ (i.e., the subring of A generated by F and a), then K is a totally ramified graded field extension of F with $[K : F] = |\langle \mathrm{gr}(a) + \Gamma_F \rangle| = m$. We have $\mathrm{gr}(a^m) = m\mathrm{gr}(a) \in \Gamma_F$, so $a^m \in F^*$ (because $K_0 = F_0$). We can then define a graded F-homomorphism $\sigma : K \to K$, by $\sigma(a) = wa$. By [**HW(b)99**, Proposition 1.6] there exists some $b \in A^*$ such that $\sigma = \mathrm{Int}(b)_{|K}$, so $wa = \sigma(a) = bab^{-1}$, which implies that $w = bab^{-1}a^{-1}$, hence $w \in A^{*c}$.

We have $A^{*c} = \mu_s(F_0)$, so $\mathrm{MK}_1(A) = A^*/\mu_s(F_0)$. In the same way as in above, for any $a \in A^* \backslash A_0^*$, $a\mu_s(F_0)$ cannot be a torsion element of $\mathrm{MK}_1(A)$. So, $T(\mathrm{MK}_1(A)) \cong T(A_0^*)/\mu_s(F_0) = T(F_0^*)/\mu_s(F_0)$. □

More generally, let F be a graded field and A an arbitrary graded central division algebra over F. We aim to determine the torsion subgroup $T(\mathrm{MK}_1(A))$ of $\mathrm{MK}_1(A)$ in term of a quotient group of the multiplicative group A_0^*. For this let's first precise some notation in what follows.

(2.2) Let K be a field, R be a central division algebra over K, F be a graded field with $F_0 = K$, H a finite group that acts on KF by graded automorphisms and (w, f) be a (graded) factor set of H in $RF(\cong_g R \otimes_K F)$. For an arbitrary element $r \in R$ and an element $\sigma \in H$, we will write simply $\sigma(r)$ to denote the element $w_\sigma(r)$. We denote by $R^{*c}_{(H,w,f)}$ the subgroup of R^* generated by the elements $\prod_{i=1}^{m} r_i \sigma_i(s_i) f(\sigma_i, \tau_i) f(\tau_i, \sigma_i)^{-1} \tau_i(r_i^{-1}) s_i^{-1}$, where $m \in \mathbb{N}$, $\sigma_i, \tau_i \in H$ and $r_i, s_i \in R^*$. We also set $\mathrm{K}_1(R)_{(H,w,f)} := R^*/R^{*c}_{(H,w,f)}$ that we call a generalized Whitehead group of R.

(2.3) Let F be a graded field and A a (finite-dimensional) graded central division algebra over F. We know by [**M10**, (1.3), pp 442-443] that there is a (graded) factor set (w, f) of Γ_A/Γ_F in A_0F such that A is graded isomorphic to the generalized graded crossed product $(A_0, \Gamma_A/\Gamma_F, (w, f)) = \oplus_{\overline{\gamma} \in \Gamma_A/\Gamma_F} A_0 F x_{\overline{\gamma}}$, where $x_{\overline{\gamma}}$ are invertible (homogeneous) elements of A subject to the following multiplicative relations: $x_{\overline{\gamma}} a = w_{\overline{\gamma}}(a) x_{\overline{\gamma}}$ and $x_{\overline{\gamma}} x_{\overline{\delta}} = f(\overline{\gamma}, \overline{\delta}) x_{\overline{\gamma} + \overline{\delta}}$ for any $a \in A_0$ and $\overline{\gamma}, \overline{\delta} \in \Gamma_A/\Gamma_F$. We can also assume that f is normalized, or equivalently that $x_0 = 1$. We have then the following proposition.

PROPOSITION 2.4. *Let F be a graded field, A a graded central division algebra over F and let (w, f) be a (graded) factor set of Γ_A/Γ_F in A_0F with f normalized such that A is graded isomorphic to the generalized graded crossed product $(A_0, \Gamma_A/\Gamma_F, w, f)$, then $T(\mathrm{MK}_1(A)) \cong T(\mathrm{K}_1(A_0)_{(\Gamma_A/\Gamma_F,(w,f))})$.*

PROOF. As seen in the proof of Proposition 2.1, we have $A^{*c} \subseteq A_0^*$. So, for any element $a \in A^* \backslash A_0^*$, the element $a.A^{*c}$ cannot be a torsion element in $\mathrm{MK}_1(A)$ (because $\mathrm{gr}(a^n) = n\mathrm{gr}(a) \neq 0$ for any integer $n > 0$). Thus, we have $T(\mathrm{MK}_1(A)) = T(A_0^*/(A_0^* \cap A^{*c}))$. Write $A = (A_0, \Gamma_A/\Gamma_F, w, f) = \oplus_{\overline{\gamma} \in \Gamma_A/\Gamma_F} A_0 x_{\overline{\gamma}}$, where $x_{\overline{\gamma}}$ are invertible elements subject to the multiplicative relations given above in (2.3). An arbitrary element of A^* is then of the form $ax_{\overline{\gamma}}$, where $a \in A_0^* F^*$ and $\overline{\gamma} \in \Gamma_A/\Gamma_F$. We have $(ax_{\overline{\gamma}})(bx_{\overline{\delta}}) = a\overline{\gamma}(b)x_{\overline{\gamma}}x_{\overline{\delta}} = a\overline{\gamma}(b)f(\overline{\gamma}, \overline{\delta})x_{\overline{\gamma}+\overline{\delta}}$. So, $(ax_{\overline{\gamma}})(bx_{\overline{\delta}})(ax_{\overline{\gamma}})^{-1}(bx_{\overline{\delta}})^{-1} = (a\overline{\gamma}(b)f(\overline{\gamma}, \overline{\delta})x_{\overline{\gamma}+\overline{\delta}})(b\overline{\delta}(a)f(\overline{\delta}, \overline{\gamma})x_{\overline{\gamma}+\overline{\delta}})^{-1} = a\overline{\gamma}(b)f(\overline{\gamma}, \overline{\delta})f(\overline{\delta}, \overline{\gamma})^{-1}\overline{\delta}(a^{-1})b^{-1}$. Plainly in the product $(ax_{\overline{\gamma}})(bx_{\overline{\delta}})(ax_{\overline{\gamma}})^{-1}(bx_{\overline{\delta}})^{-1}$, we can assume that $a, b \in A_0^*$, hence $T(\mathrm{MK}_1(A)) \cong T(\mathrm{K}_1(A_0)_{(\Gamma_A/\Gamma_F, w, f)})$. $\square$

In what follows, to prove our main result (Theorem 2.7), we will need to show the analogous of [**Mo13**, Lemma 4 and Theorem 5] in the graded setting (see Lemma 2.5 and Proposition 2.6). These versions can be considered as extensions of Motie's results.

LEMMA 2.5. *Let F be a graded field and let R and S be two graded central division algebras over F such that $T := R \otimes_F S$ is also a graded division algebra. For any integer p prime to* $\deg(S)$, *we have $T_p(\mathrm{MK}_1(T)) \cong T_p(\mathrm{MK}_1(R))$.*

PROOF. Let a_1, b_1, a_2, b_2, ..., a_n, b_n be elements of $q(R)^*$ and let $s_i \in F \backslash \{0\}$, $w_i \in R$ such that $a_i b_i a_i^{-1} b_i^{-1} = s_i^{-1} w_i$, for $1 \leq i \leq n$. Then, $s_i a_i b_i = w_i b_i a_i$. For an arbitrary nonzero element d of R, let's denote by $d^{(m)}$ the nonzero homogeneous component of d of minimal grade, then we have

$$s_i^{(m)} a_i^{(m)} b_i^{(m)} = w_i^{(m)} b_i^{(m)} a_i^{(m)}$$

hence

$$(s_i^{(m)})^{-1} w_i^{(m)} = a_i^{(m)} b_i^{(m)} (a_i^{(m)})^{-1} (b_i^{(m)})^{-1},$$

which is clearly an element of R^{*c}.

Suppose now that

$$a_1 b_1 a_1^{-1} b_1^{-1} \dots a_n b_n a_n^{-1} b_n^{-1} = t \in R^*$$

then

$$s_1^{-1} w_1 \dots s_n^{-1} w_n = t,$$

hence

$$w_1 \dots w_n = t\, s_1 \dots s_n,$$

so,

$$w_1^{(m)} \dots w_n^{(m)} = t\, s_1^{(m)} \dots s_n^{(m)}.$$

Therefore,

$$t = (s_1^{(m)})^{-1} w_1^{(m)} \dots (s_n^{(m)})^{-1} w_n^{(m)} \in R^{*c}.$$

This shows that $R^* \cap q(R)^{*c} = R^{*c}$.

Now, by [**Mo13**, Lemma 4] (see the proof) the canonical inclusion map $q(R) \to q(T)$ induces a group isomorphism $\psi_p : T_p(\mathrm{K}_1(q(R))) \to T_p(\mathrm{K}_1(q(T)))$. Therefore the canonical group homomorphism $\phi_p : T_p(\mathrm{MK}_1(R)) \to T_p(\mathrm{MK}_1(T))$, defined by $\phi_p(aR^{*c}) = aT^{*c}$ for any nonzero homogeneous element a of R such that $aR^{*c} \in T_p(\mathrm{MK}_1(R))$, is injective. Indeed, if $\phi_p(aR^{*c}) = 1$, i.e., $a \in T^{*c}$, then $a \in q(T)^{*c}$, so by the injectivity of ψ_p, $a \in q(R)^{*c}$. It follows by the above that $a \in R^{*c}$.

In a similar way we show that ϕ_p is surjective. Indeed, let $d \in T^*$ and suppose that $dT^{*c} \in T_p(\mathrm{MK}_1(T))$, then $dq(T)^{*c} \in T_p(\mathrm{K}_1(q(T)))$. The fact that ψ_p is surjective implies that there exist $\alpha \in q(R)^*$ with $\alpha q(R)^{*c} \in T_p(\mathrm{K}_1(q(R)))$ such that $dq(T)^{*c} = \alpha q(T)^{*c}$. Let e_1, f_1, ..., e_r, f_r be elements of $q(T)^*$ such that

$$\alpha^{-1}d = e_1 f_1 e_1^{-1} f_1^{-1} \ldots e_r f_r e_r^{-1} f_r^{-1},$$

and let $a \in R$, $s \in F\backslash\{0\}$ such that $\alpha = s^{-1}a$ and let $t_i \in F\backslash\{0\}$, $z_i \in T$ (for $1 \le i \le r$) such that $e_i f_i e_i^{-1} f_i^{-1} = t_i^{-1} z_i$, then, as seen above :

$$(t_i^{(m)})^{-1} z_i^{(m)} = e_i^{(m)} f_i^{(m)} (e_i^{(m)})^{-1} (f_i^{(m)})^{-1}.$$

We have

$$\alpha^{-1}d = e_1 f_1 e_1^{-1} f_1^{-1} \ldots e_r f_r e_r^{-1} f_r^{-1},$$

so

$$sa^{-1}d = (t_1^{-1} \ldots t_r^{-1}) z_1 \ldots z_r.$$

Thus

$$s(t_1 \ldots t_r)d = a z_1 \ldots z_r.$$

This implies

$$s^{(m)}(t_1^{(m)} \ldots t_r^{(m)})d = a^{(m)} z_1^{(m)} \ldots z_r^{(m)}.$$

Hence,

$$\begin{aligned} d &= (s^{(m)})^{-1} a^{(m)} (t_1^{(m)})^{-1} z_1^{(m)} \ldots (t_r^{(m)})^{-1} z_r^{(m)} \\ &= (s^{(m)})^{-1} a^{(m)} e_1^{(m)} f_1^{(m)} (e_1^{(m)})^{-1} (f_1^{(m)})^{-1} \ldots e_r^{(m)} f_r^{(m)} (e_r^{(m)})^{-1} (f_r^{(m)})^{-1}. \end{aligned}$$

It follows that $dT^{*c} = (s^{(m)})^{-1}a^{(m)}T^{*c}$.

It remains to prove that $(s^{(m)})^{-1}a^{(m)}R^{*c} \in T_p(\mathrm{MK}_1(R))$. Since $\alpha q(R)^{*c} \in T_p(\mathrm{MK}_1(R))$, then there is some positive integer k such that $\alpha^{p^k} \in q(R)^{*c}$, i.e., there exist elements x_1, y_1, ... x_l, $y_l \in q(R)^*$ such that

$$\alpha^{p^k} = x_1y_1x_1^{-1}y_1^{-1} \dots x_ly_lx_l^{-1}y_l^{-1}.$$

In the same way as in above, this yields

$$((s^{(m)})^{-1}a^{(m)})^{p^k} = x_1^{(m)}y_1^{(m)}(x_1^{-1})^{(m)}(y_1^{-1})^{(m)} \dots x_l^{(m)}y_l^{(m)}(x_l^{-1})^{(m)}(y_l^{-1})^{(m)}$$

which means that $((s^{(m)})^{-1}a^{(m)})^{p^k} \in R^{*c}$. $\square$

PROPOSITION 2.6. *Let F be a graded field and A a graded central division algebra over F with primary decomposition $A = A_{p_1} \otimes_F \dots \otimes_F A_{p_k}$, then $T(\mathrm{MK}_1(A)) \cong (\oplus_{i=1}^{k} T_{p_i}(\mathrm{MK}_1(A_{p_i}))) \oplus (\oplus_{p\in I} T_p(F^*))$, where I is the set of all primes in $\mathbb{N}\backslash\{p_1, ..., p_k\}$.*

PROOF. This follows easily by applying Lemma 2.5. Indeed, let p be a prime and suppose first that p does not divide $\deg(A)$. Let $R = F$ and $S = A$, then by Lemma 2.5 we have $T_p(\mathrm{MK}_1(A)) \cong T_p(F^*)(= T_p(\mathrm{MK}_1(F))$. If p divides $\deg(A)$, say $p = p_i$ for some i $(1 \leq i \leq k)$, then for $R = A_{p_i}$ and $S = A_{p_1} \otimes_F \dots \otimes_F A_{p_{i-1}} \otimes_F A_{p_{i+1}} \dots \otimes_F A_{p_k}$ when $1 < i < k$ [resp., $S = A_{p_2} \otimes_F \dots \otimes_F A_{p_k}$ when $i = 1$; resp., $S = A_{p_1} \otimes_F \dots \otimes_F A_{p_{k-1}}$ when $i = k$], we have $T_p(\mathrm{MK}_1(A)) \cong T_p(\mathrm{MK}_1(A_p))$. It follows then that $T(\mathrm{MK}_1(A)) \cong (\oplus_{i=1}^{k} T_{p_i}(\mathrm{MK}_1(A_{p_i})))\oplus(\oplus_{p\in I} T_p(F^*))$, where I is the set of all primes in $\mathbb{N}\backslash\{p_1, ..., p_k\}$. $\square$

THEOREM 2.7. *Let E be a Henselian valued field such that* $\mathrm{char}(E) = \mathrm{char}(\overline{E})$ *and let D be a strongly tame central division algebra over E, then* $T(\mathrm{K}_1(D)) \cong T(\mathrm{MK}_1(GD))$.

PROOF. Since $\mathrm{char}(E) = \mathrm{char}(\overline{E})$, then by Hensel's Lemma $T(E^*) \cong T(\overline{E}^*)$; moreover it is clear that $T(GE^*) = T(\overline{E}^*)$, so $T(E^*) \cong T(GE^*)$. Therefore, by [**Mo13**, Theorem 5] and Proposition 2.6 it suffices to assume that D has a prime power degree $n := q^r$ and to prove that $T_q(\mathrm{K}_1(D)) \cong T_q(\mathrm{MK}_1(GD))$.

By [**HaW(a)11**, Theorem 4.13] (see the proof) the canonical mapping $D^* \to GD^*$, defined by $x \mapsto x'$, induces an isomorphism $\mathrm{CK}_1(D) \xrightarrow{\psi} \mathrm{CK}_1(GD)$. Consider the following canonical commutative diagram :

$$\begin{tikzcd}
\mathrm{K}_1(D) \arrow[r, "f"] \arrow[d, "\phi"'] & \mathrm{CK}_1(D) \arrow[d, "\psi"] \\
\mathrm{MK}_1(GD) \arrow[r, "g"] & \mathrm{CK}_1(GD)
\end{tikzcd}$$

where $\phi : \mathrm{K}_1(D) \to \mathrm{MK}_1(GD)$ is defined by $\phi(xD^{*c}) = x'GD^{*c}$, $f : \mathrm{K}_1(D) \to \mathrm{CK}_1(D)$ is defined by $f(xD^{*c}) = xE^*D^{*c}$, $g : \mathrm{MK}_1(GD) \to \mathrm{CK}_1(GD)$ is defined by $g(xGD^{*c}) = xGE^*GD^{*c}$ and $\psi : \mathrm{CK}_1(D) \to \mathrm{CK}_1(GD)$ is defined by $\psi(xE^*D^{*c}) = x'GE^*GD^{*c}$.

Let $d \in D^*$ and suppose that $\phi(dD^{*c}) = 1$, then $d' \in GD^{*c}$ and $\psi \circ f(dD^{*c}) = g \circ \phi(dD^{*c}) = 1$, hence $f(dD^{*c}) = 1$. This means $d \in E^*D^{*c}$. Write $d = ec$, where $e \in E^*$ and $c \in D^{*c}$, then $d' = e'c'$, hence $e' \in GD^{*c}$ (because both c' and d' are in GD^{*c}). Let $N := \{e \in E^* \mid e' \in GD^{*c}\}$, then N is a (normal) subgroup of D^*, and by the above $\ker(\phi) = ND^{*c}/D^{*c}$. Therefore, $\mathrm{MK}_1(GD)(\cong \mathrm{K}_1(D)/\ker(\phi)) \cong D^*/ND^{*c}$.

For any $e \in N$, we have $e' \in GD^{*c}$, so $(e')^n = \mathrm{Nrd}_{GD}(e') = 1$ (here $n = \deg(D)(= \deg(GD))$). Therefore, if we denote by v the restriction of the valuation of D to E, then $v(e) = 0$ and $\overline{e} \in \mu_n(\overline{E}) \cap GD^{*c}$, where $\overline{e}$ is the class of e in $\overline{E}$. Conversely, let $\alpha_1, ..., \alpha_s$ be the different elements of $\mu_n(\overline{E}) \cap GD^{*c}$, then by Hensel's lemma there exist elements $e_1, ..., e_s$ in E^* with $v(e_i) = 0$, $\overline{e}_i = \alpha_i$ and $e_i^n = 1$. Obviously, $e_1, ..., e_s$ are elements of N. More precisely, we have $N = \cup_{i=1}^s e_i(1 + M_E)$.

Let x be an element of D^* and suppose that xD^{*c} is an element of $T_q(\mathrm{K}_1(D))$, then clearly $x'GD^{*c}$ is an element of $T_q(\mathrm{MK}_1(GD))$. Conversely, let α be an element of $T_q(\mathrm{MK}_1(GD))$, then we can write $\alpha = z'GD^{*c}$, where $z \in D^*$ with $(z')^m \in GD^{*c}$ for some power m of q. We have $(z^m)' = (z')^m \in GD^{*c}$, hence $z^m \in ND^{*c}$. So $z^m \in \cup_{i=1}^s e_i(1 + M_E)D^{*c}$. Thus $z^{mn} \in (1 + a)^n D^{*c}$ for some $a \in M_E$. The polynomial $X^{mn} - (1+a)^n$ has residue $X^{mn} - 1$ and 1 is a simple root of $X^{mn} - 1$ in $\overline{E}$, so again by Hensel's Lemma $X^{mn} - (1+a)^n$ has a root $e \in E$ with $v(e) = 0$ and $\overline{e} = 1$, where as in above v is the restriction of the valuation of D to E. We have $z^{mn} \in (1 + a)^n D^{*c} = e^{mn} D^{*c}$, so $(ze^{-1})^{mn} \in D^{*c}$ and $(ze^{-1})' = z'(e')^{-1} = z'$. Consequently, $\alpha = z'GD^{*c} = (ze^{-1})'GD^{*c}$ with $ze^{-1}D^{*c} \in T_q(\mathrm{K}_1(D))$. Therefore, $T_q(\mathrm{MK}_1(GD)) = \phi(T_q(\mathrm{K}_1(D)))$.

Consider now the following commutative diagram with exact rows:

$$\begin{tikzcd}
1 \arrow[r] & (1+M_D)\cap D^{*c} \arrow[r] \arrow[d] & D^{*c} \arrow[r] \arrow[d] & GD^{*c} \arrow[r] \arrow[d] & 1 \\
1 \arrow[r] & 1+M_D \arrow[r] & D^{*} \arrow[r] & GD^{*} \arrow[r] & 1
\end{tikzcd}$$

Applying the snake lemma, we get the following exact sequence of cokernels :

$$1 \to \frac{1+M_D}{(1+M_D)\cap D^{*c}} \to \mathrm{K}_1(D) \xrightarrow{\phi} \mathrm{MK}_1(GD) \to 1$$

As seen above the restriction homomorphism $T_q(\mathrm{K}_1(D)) \xrightarrow{\phi} T_q(\mathrm{MK}_1(GD))$ is surjective, so we get the following exact sequence :

$$1 \to T_q(\frac{1+M_D}{(1+M_D)\cap D^{*c}}) \to T_q(\mathrm{K}_1(D)) \xrightarrow{\phi} T_q(\mathrm{MK}_1(GD)) \to 1$$

By the congruence theorem (see [**HaW(a)11**, Theorem B.1]) we have $(1+M_D)\cap D^{*c} = (1+M_D)\cap D^{(1)}$, where $D^{(1)}$ is the subgroup of D^* consisting of all elements with reduced norm equal to 1. Thus, $T_q(\frac{1+M_D}{(1+M_D)\cap D^{*c}}) = T_q(\frac{1+M_D}{(1+M_D)\cap D^{(1)}})$. Let $a \in M_D$ and suppose that $(1+a)^s \in D^{(1)}$ for some power s of q. Then, $1 = \mathrm{Nrd}_D((1+a)^s) = \mathrm{Nrd}_D(1+a)^s$. By [**HaW(a)11**, Corollary 4.7] we have $\mathrm{Nrd}_D(1+a) = 1+b$ for some $b \in M_E$. So, $1 = (1+b)^s = 1 + sb + r_2b^2 + r_3b^3 + ... + b^s$. Hence, $(s + r_2b + ... + b^{s-1})b = 0$. Let $p = \mathrm{char}(\overline{E})$. Since p does not divide s (for D is strongly tame), then $v(s) = 0$. So $v(s + r_2b + ... + b^{s-1}) = 0$ (see here that $v(r_ib^{i-1}) \geq v(b) > 0$ for all $i \geq 2$). It follows that $v(b) = \infty$ (since $(s + r_2b + ... + b^{s-1})b = 0$), or equivalently $b = 0$. Therefore, $\mathrm{Nrd}_D(1+a) = 1$, hence $1 + a \in D^{(1)}$. Thus, $T_q(\frac{1+M_D}{(1+M_D)\cap D^{(1)}}) = 1$. Consequently, $T_q(\mathrm{K}_1(D)) \cong T_q(\mathrm{MK}_1(GD))$. □

COROLLARY 2.8. (*Compare* [**Mo13**, Theorem 10]) *Let* E *be a Henselian valued field such that* $\mathrm{char}(E) = \mathrm{char}(\overline{E})$ *and let* D *be a strongly tame central division algebra over* E*, and* (w, f) *a generalized graded factor set of* Γ_D/Γ_E *in* $\overline{D}.GE$ *such that* $GD = (\overline{D}.GE, \Gamma_D/\Gamma_E, (w, f))$*, then we have* $T(K_1(D)) \cong T(K_1(\overline{D})_{(\Gamma_D/\Gamma_E, w, f)})$*. Moreover, we have the following statements :*

1. *If* D *is unramified, then* $T(\mathrm{K}_1(D)) \cong T(\mathrm{K}_1(\overline{D}))$*.*
2. *If* D *is totally ramified, then* $T(\mathrm{K}_1(D)) \cong T(\overline{E}^*)/\mu_s(\overline{E})$*, where* $s = \exp(\Gamma_D/\Gamma_F)$*.*

PROOF. This follows by Theorem 2.7, Proposition 2.4 and Proposition 2.1.

$\square$

COROLLARY 2.9. *Let* F *be a graded field and* A *a graded central division algebra over* F *with* $\mathrm{char}(F)$ *not dividing* $\deg(A)$*, then* $T(\mathrm{MK}_1(A))$ *is isomorphic to* $T(\mathrm{K}_1(q(A)^h))$*.*

PROOF. Obviously we have $\mathrm{char}(q(F)^h) = \mathrm{char}(F)$ and $\deg(q(A)^h) = \deg(A)$, so $q(A)^h$ is a strongly tame division algebra over $q(F)^h$. We recall that $G(q(A)^h)$ can be identified to A, so our corollary follows by Theorem 2.7.

$\square$

APPENDIX A

Comments

1. On Witt rings

All graded objects considered here are assumed to have the same type which is a totally ordered divisible abelian group Γ.

Let F be a graded field of characteristic different from 2 and V be a finite-dimensional graded vector space over F. A graded symmetric bilinear form on V is defined to be a bi-additive function $b : V \times V \to F$ which satisfies the following conditions (for any $v, w \in V$, $r, s \in F$ and $\gamma, \delta \in \Gamma$): $b(rv, sw) = rs.b(v, w)$, $b(w, v) = b(v, w)$, and $b(V_\gamma, V_\delta) \subseteq F_{\gamma+\delta}$. A graded symmetric bilinear form b on V, is called nondegenerate if $v = 0$ is the only element of V such that $b(x, y) = 0$ for all $y \in V$. In this case b is called hyperbolic if V has two complementary totally isotropic graded subspaces.

Renard et al. showed in [**RTW07**] that much of the theory of (even) hermitian forms over division algebras carries over (even) graded hermitian forms (over graded division algebras). In particular this holds for graded symmetric bilinear forms over F. The main difference, as indicated in [**RTW07**], consists in the fact that hyperbolic graded forms of the same dimension need not be isometric. It is needed (for such graded isometry) that the underlying graded vector spaces be graded isomorphic. By [**RTW07**, Proposition 1.4] any nondegenerate graded symmetric bilinear form over F, is graded isometric to an (orthogonal) direct sum $b_{\mathrm{an}} \perp b_{\mathrm{hyp}}$, where b_{an} is an anisotropic graded (symmetric) form and b_{hyp} is a hyperbolic graded (symmetric) form, both unique up to graded isometries. We write $b \cong_g b_{\mathrm{an}} \perp b_{\mathrm{hyp}}$. Moreover, any anisotropic graded form is graded isometric to an orthogonal sum of one-dimensional graded forms.

Now, let E be a Henselian valued field with valuation v, M be a finite-dimensional vector space over E, and k be a nondegenerate symmetric bilinear form on M. An E-norm α on M is said to be compatible with k if it satisfies the following conditions:

(*) for any $m, n \in M$, we have $\alpha(m) + \alpha(n) \leq v(k(m, n))$,

(**) for any $m \in M$, there is some $n \in M$ with $\alpha(m) + \alpha(n) = v(k(m, n))$.

Under the first condition k induces a graded symmetric bilinear form k'_α on $G_\alpha M$, uniquely determined by the condition $k'_\alpha(m', n') = k(m, n)'$. The second condition assures that k'_α is nondegenerate.

Let (E, v) be a Henselian valued field with residue characteristic different from 2 and let $W(E)$ be the Witt ring of E. Then by [**RTW07**, Proposition 4.3] we have a canonical group isomorphism $\theta_E : W(E) \to W^{\mathrm{gr}}(GE)$, induced by the correspondence $b \mapsto (b'_{\alpha_b})_{\mathrm{an}}$, for any nondegenerate symmetric bilinear form b (on a vector space) over E; α_b being a norm (on this space) compatible with b. In what follows we will see that θ_E is in fact a ring isomorphism. Indeed, let k and l be nondegenerate symmetric bilinear forms over E with underlying spaces V and W, respectively. Let $(x_i)_{1\le i\le m}$ be an orthogonal base of V for k, and $(y_j)_{1\le j\le n}$ be an orthogonal base of W for l, and write k and l with respect to these bases, respectively, as $k = \langle b_1, ..., b_m\rangle$, where $b_i = b(x_i, x_i)$, and $l = \langle c_1, ..., c_n\rangle$, where $c_j = b(y_j, y_j)$. The underlying space of kl is $V \otimes_E W$, and clearly $(x_i \otimes y_j)_{1\le i\le m, 1\le j\le n}$ is an orthogonal base of $V \otimes_E W$ for kl. Write $kl = \langle d_{11}, ..., d_{mn}\rangle$, where $d_{ij} = b_i c_j$, and consider the norm β on $V \otimes_E W$ with splitting base $(x_i \otimes y_j)_{1\le i\le m, 1\le j\le n}$ and with $\beta(x_i \otimes y_j) = \frac{1}{2}v(d_{ij})$. Then, $(kl)'_\beta = \langle d'_{11}, ..., d'_{mn}\rangle = \langle b'_1, ..., b'_m\rangle\langle c'_1, ..., c'_n\rangle$. If we let α_1 be the norm on V with splitting base $(x_i)_{1\le i\le m}$ and with $\alpha_1(x_i) = \frac{1}{2}v(b_i)$ and α_2 be the norm on W with splitting base $(y_j)_{1\le j\le n}$ and with $\alpha_2(y_j) = \frac{1}{2}v(c_j)$, then clearly $\langle b'_1, ..., b'_m\rangle = k'_{\alpha_1}$ and $\langle c'_1, ..., c'_n\rangle = l'_{\alpha_2}$, hence $(kl)'_\beta = k'_{\alpha_1} l'_{\alpha_2}$.

We recall that if M is a finite-dimensional vector space over E, h is a nondegenerate symmetric bilinear form on M, and r, s are two norms on M, both compatible with h, then $(h'_r)_{\mathrm{an}}$ is graded isometric to $(h'_s)_{\mathrm{an}}$ (see [**RTW07**, Theorem 3.11]). In what follows we will identify graded isometric (anisotropic) nondegenerate graded forms and write simply h'_{an} to mean $(h'_r)_{\mathrm{an}}$. Note that if we write $h \cong_g h_{\rm an} \perp h_{\rm hyp}$, where $h_{\rm an}$ is the anisotropic part of h and $h_{\rm hyp}$ is hyperbolic, then by [**RTW07**, Proposition 4.3] and up to a graded isometry, we have $(h'_r)_{\rm an} = (h_{\rm an})'_t$, where t is a norm on the underlying space of $h_{\rm an}$, compatible with $h_{\rm an}$. In what follows (by identifying graded isomorphic forms) we will write simply $(h_{\rm an})'$ instead of $(h_{\rm an})'_t$. In particular, with the above notation, we have $h'_{\rm an} = (h_{\rm an})'$.

Now, to show that θ_E is a ring homomorphism, we have to show that $(kl)'_{\rm an} = (k'_{\rm an}l'_{\rm an})_{\rm an}$, which is straightforward by the above. Indeed, we have $(kl)'_{\rm an} = ((kl)_{\rm an})' = ((k_{\rm an}l_{\rm an})_{\rm an})' = (k_{\rm an}l_{\rm an})'_{\rm an} = ((k_{\rm an})'(l_{\rm an})')_{\rm an} = (k'_{\rm an}l'_{\rm an})_{\rm an}$, which ends the proof.

2. On generalized graded crossed products

Throughout this section all considered graded objects will be assumed to have the same grading type, which is a totally ordered (uniquely) divisible abelian group Γ.

(2.1) Let R be a commutative graded ring, A be a graded algebra over R, H be a finite group that acts on A by graded ring automorphisms, (ω, f) be a graded factor set of H in A, and $(A, H, (\omega, f))$ the corresponding generalized graded crossed product: $(A, H, (\omega, f)) := \oplus_{\sigma\in H} Ax_\sigma$, where x_σ are independent indeterminates on A, with the addition law defined componentwise, and the multiplication one defined by (extension of) the equalities: $x_\sigma a = \omega_\sigma(a)x_\sigma$ and $x_\sigma x_\tau = f(\sigma,\tau)x_{\sigma\tau}$ for all $a \in A$ and $\sigma, \tau \in H$. We recall that f is said to be normalized if $f(\sigma, 1) = f(1, \sigma) = 1$

for any $\sigma \in H$. In this case, we identify A with Ax_1, especially we identify the identity element x_1 of Ax_1 with 1_A.

If we suppose that f is normalized, then for any $\sigma \in H$, we have $w_\sigma w_1 = \mathrm{Int}(f(\sigma, 1))w_\sigma = w_\sigma$, so $w_1 = \mathrm{id}_A$. It follows that $w_\sigma w_{\sigma^{-1}} = \mathrm{Int}(f(\sigma, \sigma^{-1}))w_1 = \mathrm{Int}(f(\sigma, \sigma^{-1}))\mathrm{id}_A$. One can then easily see that for a graded ideal I of A, the following two conditions: $w_\sigma(I) \subseteq I$ for any $\sigma \in H$, and $w_\sigma(I) = I$ for any $\sigma \in H$, are equivalent.

LEMMA 2.2. *With the above notation in (2.1), there exists a unique graded algebra structure on $(A, H, (\omega, f))$ which extends the grading of A and for which all elements x_σ are homogeneous.*

PROOF. Consider the mapping $h : H \times H \to \Gamma$, defined by $h(\sigma, \tau) = \mathrm{gr}(f(\sigma, \tau))$. By condition (0.17) in the definition of a factor set (see the preliminaries), it is clear that h is a cocycle of $Z^2(H, \Gamma)$; moreover, since H is finite and Γ is uniquely divisible, then $\mathrm{H}^2(H, \Gamma) = \mathrm{H}^1(H, \Gamma) = 0$, where $\mathrm{H}^2(H, \Gamma)$ and $\mathrm{H}^1(H, \Gamma)$ are respectively the second and the first cohomological groups of H in Γ (the action of H on Γ being trivial). Therefore, there is a unique family $(\gamma_\sigma)_{\sigma \in H}$ of elements of Γ such that $h(\sigma, \tau) = \gamma_\sigma + \gamma_\tau - \gamma_{\sigma\tau}$ (the uniqueness follows from the fact that $\mathrm{H}^1(H, \Gamma) = 0$). The unique graded structure of $(A, H, (\omega, f))$ which extends the grading of A and for which all elements x_σ are homogeneous, is then defined by setting $\mathrm{gr}(x_\sigma) = \gamma_\sigma$. □

(2.3) Conversely to (2.1), graded factor sets can be constructed from some graded algebras. Namely, in [**M11**, Lemma 2.4] we proved that if A is a graded simple algebra over a graded field F such that A_0 is simple, then there is a natural graded factor set (w, f) of $H := \Gamma_A/\Gamma_F$ in $A_0.F$ (with f possibly chosen normalized) such that A is graded isomorphic to the generalized graded crossed product $(A_0.F, H, (w, f))$. We recall that in this case, we have $\Gamma_A = \Gamma_A^*$, where $\Gamma_A^* := \mathrm{gr}(A^*)$ (A^* being the multiplicative group of invertible homogeneous elements of A as seen above). Examples of such graded simple algebras (with simple 0-component) are given by matrix algebras $M_n(D)$ where D is an arbitrary (finite-dimensional) graded central division algebra over a graded field.

In the same way, if F is a graded field and B a finite-dimensional graded central F-algebra such that $\Gamma_B = \Gamma_B^*$ and $H := \Gamma_B/\Gamma_F$ is finite, then by choosing invertible homogeneous elements $x_\sigma \in B^*$ with $x_1 = 1$ and $\mathrm{gr}(x_\sigma) + \Gamma_F = \sigma$ for all $\sigma \in H$, we have $B = \oplus_{\sigma \in H} B_0 F x_\sigma = (B_0F, H, (w, f))$ where (w, f) is the graded factor set of H in B_0F, defined by the conditions: $x_\sigma x_\tau = f(\sigma, \tau) x_{\sigma\tau}$ and $x_\sigma a = w_\sigma(a) x_\sigma$, for all $\sigma, \tau \in H$ and $a \in B_0F$. It is clear that $\mathrm{gr}(x_\sigma) + \Gamma_{B_0F} (= \mathrm{gr}(x_\sigma) + \Gamma_F)$ are pairwise distinct (for $\sigma \in H$). This last condition will be needed in the main result of this section (see Proposition 2.14). We fix now the following notation:

(2.4) Notation: Throughout the rest of this section, R is a commutative graded ring, A is a graded algebra over R, H is a finite group that acts on A by graded ring automorphisms, (w, f) is a graded factor set of H in A with f normalized, and $S =$

$(A, H, (w, f))$ is the corresponding generalized graded crossed product.

Lemma 2.5. *Let A and S be as in (2.4). If we suppose that $\Gamma_A = \Gamma_A^*$, then the following statements are equivalent:*

(1) $S_0 = A_0$,

(2) *for a representation $S := \oplus_{\sigma \in H} Ax_\sigma$ as in (2.1), we have $\mathrm{gr}(x_\sigma) + \Gamma_F$ pairwise distinct (for distinct $\sigma \in H$).*

Proof. (1) $\Rightarrow$ (2) Let $\sigma, \tau \in H$ and suppose that $\mathrm{gr}(x_\sigma) + \Gamma_A = \mathrm{gr}(x_\tau) + \Gamma_A$, then $\mathrm{gr}(x_\sigma x_\tau^{-1}) \in \Gamma_A^* (= \Gamma_A)$. Let $a \in A^*$ such that $\mathrm{gr}(x_\sigma x_\tau^{-1}) = \mathrm{gr}(a)$, then $\mathrm{gr}(x_\sigma x_\tau^{-1} a^{-1}) = 0$. So, $x_\sigma x_\tau^{-1} a^{-1} \in A_0 (= S_0)$, which means $x_\sigma \in A_0.(ax_\tau) \subseteq Ax_\tau$. Therefore, $\sigma = \tau$.

(2) $\Rightarrow$ (1) Since $\mathrm{gr}(x_1) = 0$ and $\mathrm{gr}(x_\sigma) + \Gamma_A$ are pairwise distinct, then $S_0 \subseteq A_0$ (by identification of A_0 with its canonical image $A_0 x_1$ in S), so $S_0 = A_0$. $\square$

(2.6) Let $A, H, (w, f)$ and S be as in (2.4). We will say that S satisfies the grading separation property (GSP) with respect to (w, f), if there is some representation of S as in (2.1), say $S := \oplus_{\sigma \in H} Ax_\sigma$, with $\mathrm{gr}(x_\sigma) + \Gamma_F$ pairwise distinct (for distinct $\sigma \in H$). Note that in this case, the homogeneous elements of S are the elements ax_σ, where a is a homogeneous element of A and $\sigma \in H$. Also, in this case, we have $S_0 = A_0$.

(2.7) We saw above in (2.3) that graded simple algebras with simple 0-component satisfy the graded separation property (GSP).

We give here another example of generalized graded crossed products satisfying this property. This example is based on Malcev-Neumann construction for his power series division rings. Let $\Gamma_1 \subseteq \Gamma_2$ be an extension of totally ordered abelian groups with $H := \Gamma_2/\Gamma_1$ finite (one can take for example $\Gamma_1 = m_1\mathbb{Z} \times .... \times m_r\mathbb{Z}$ and $\Gamma_2 = \mathbb{Z}^r$, where $m_1, ..., m_r$ are nonnegative integers, and let Γ_1 and Γ_2 be ordered by the anti-lexicographic order). Consider a factor set (v, g) of Γ_2 in a ring B, with g normalized, and let $S := (B, \Gamma_2, (v, g)) = \oplus_{\mu\in\Gamma_2} Bx_\mu$, where x_μ are independent indeterminates over B satisfying the conditions: $x_\mu x_{\mu'} = g(\mu, \mu')x_{\mu+\mu'}$ and $x_\mu b = v_\mu(b)x_\mu$, for all $\mu, \mu' \in \Gamma_2$ and $b \in B$. Let $A := \oplus_{\mu\in\Gamma_1} Bx_\mu (= (B, \Gamma_1, (v, g)))$, and for any element $\overline{\delta} \in H$, chose a representative δ of $\overline{\delta}$ in Γ_2 and let $y_{\overline{\delta}} := x_\delta$. Then, we have $S = \oplus_{\overline{\delta}\in H} Ay_{\overline{\delta}} = (A, H, (w, f))$, where (w, f) is the graded factor set of H in A defined by the equalities $y_{\overline{\delta}}a = w_{\overline{\delta}}(a)y_{\overline{\delta}}$ and $y_{\overline{\delta}}y_{\overline{\delta}'} = f(\overline{\delta}, \overline{\delta}')y_{\overline{\delta}+\overline{\delta}'}$, for any $\overline{\delta}, \overline{\delta}' \in H$ and $a \in A$. It is clear that $\mathrm{gr}(y_{\overline{\delta}}) + \Gamma_1 (= \overline{\delta})$ are pairwise distinct (for distinct $\overline{\delta}$ in H).

(2.8) Let $A, (w, f)$ and S be as in (2.4). A graded ideal I of A will be called a graded w-ideal (or a w-invariant graded ideal) if for any $\sigma \in H$, we have $w_\sigma(I) = I$. As seen in (2.1) this condition is equivalent to have $w_\sigma(I) \subseteq I$ for all $\sigma \in H$.

We will say that I is graded w-prime if for any graded w-ideals I_1, I_2 of A such that $I_1I_2 \subseteq I$, we have $I_1 \subseteq I$ or $I_2 \subseteq I$. We will say that I is graded w-semiprime if for any graded w-ideal J of A such that $J^2 \subseteq I$, we have $J \subseteq I$.

A graded ideal I of S is called graded prime (resp., graded semiprime) if the condition above holds for graded ideals I_1, I_2 [resp., J] of S (without assuming that they are graded w-ideals).

We will say that A is a graded w-simple algebra if the only graded w-ideals of A are 0 and A.

The graded ring A is said to be graded w-prime (resp., graded w-semiprime) if 0 is graded w-prime (resp., graded w-semiprime). Similarly, S is called graded prime (resp., graded semiprime) if the graded ideal 0 is graded prime (resp., graded semiprime).

We will say that A is graded local if it has a unique maximal right graded ideal.

For a subset T consisting of homogeneous elements of A, we will write $\mathrm{Ann}^g_{A-l}(T)$ for the left annulator of T in A (which is a left graded ideal of A). We will say that A is w-compatible, if for any subset T consisting of homogeneous elements of A and any $\tau \in H$, we have $\mathrm{Ann}^g_{A-l}(T) = \mathrm{Ann}^g_{A-l}(w_\sigma(T))$.

A graded ring B is called graded Baer [resp., graded quasi-Baer] if the left annihilator of any nonempty subset consisting of homogeneous elements of B [resp., of any left graded ideal of B] is generated by a homogeneous idempotent.

We say that B is graded regular if for any homogeneous element x of B, there exists a homogeneous element y of B such that $x = xyx$.

Before giving some properties of the generalized graded crossed products, we show the following lemmas.

LEMMA 2.9. *Let S be a generalized graded crossed product as in (2.4) and suppose that S satisfies the GSP, then for any graded ideal J of S, we have $J = (J \cap A).S$. Conversely, let I be a graded w-ideal of A and let J_I be the graded ideal of S generated by the homogeneous elements of I, then $I = J_I \cap A$.*

PROOF. It is clear that $(J\cap A).S \subseteq J$, so it suffices to prove that we have $J \subseteq (J \cap A).S$. Let $a_\gamma x_\sigma$ be a homogeneous element of J, then we have $(a_\gamma x_\sigma)(x_{\sigma^{-1}} f(\sigma, \sigma^{-1})^{-1}) = a_\gamma f(\sigma, \sigma^{-1}) x_1 f(\sigma, \sigma^{-1})^{-1} = a_\gamma f(\sigma, \sigma^{-1}) w_1(f(\sigma, \sigma^{-1})^{-1}) x_1 = a_\gamma x_1$. By identification of A with Ax_1 in S, we get $a_\gamma \in J \cap A$. So, $J \subseteq (J \cap A).S$.

Conversely, let I be a graded w-ideal of A and let J_I be the graded ideal of S generated by the homogeneous elements of I. For any $a, b \in A$, $c \in I$ and $\sigma, \tau \in H$, we have $(ax_\sigma)c(bx_\tau) = aw_\sigma(cb)f(\sigma,\tau)x_{\sigma\tau}$ with $aw_\sigma(cb)f(\sigma,\tau) \in I$. One can then easily deduce that $J_I \cap A = I$. ☐

LEMMA 2.10. *Let $A, (w, f)$ and S be as in (2.4), then the following statements are equivalent:*

1. *A is graded w-prime.*
2. *For any nonzero graded w-ideal I of A, we have* $\mathrm{Ann}^g_{A-l}(I) = 0$.
3. *For any homogeneous elements a, b in A such that $w_\sigma(a).A.w_\tau(b) = 0$ for all $\sigma, \tau \in H$, we have $a = 0$ or $b = 0$.*

PROOF. (1) ⇒ (2) Let I be a nonzero graded w-ideal of A. One can easily see that $\mathrm{Ann}^g_{A-l}(I)$ is a graded w-ideal of A. Indeed, let a be an arbitrary homogeneous element of I and take an element b of

$\mathrm{Ann}^g_{A-l}(I)$. For any $\tau \in H$, we have $w_\tau(b)a = w_\tau(b)w_\tau(w_\tau^{-1}(a)) = w_\tau(bw_\tau^{-1}(a))$. Note that we have $w_\tau w_{\tau^{-1}} = \mathrm{Int}(f(\tau,\tau^{-1}))$, so $w_\tau^{-1} = w_{\tau^{-1}} \circ \mathrm{Int}(f(\tau,\tau^{-1})^{-1})$. Therefore $w_\tau^{-1}(a) \in I$ (because I is a graded w-ideal), hence $w_\tau(b)a (= w_\tau(bw_\tau^{-1}(a))) = 0$. It follows then that $w_\tau(b) \in \mathrm{Ann}^g_{A-l}(I)$, so $w_\tau(\mathrm{Ann}^g_A(I)) \subseteq \mathrm{Ann}^g_A(I)$. Now, we have $\mathrm{Ann}^g_{A-l}(I).I = 0$ with $I \neq 0$ and A graded w-prime, so necessarily $\mathrm{Ann}^g_{A-l}(I) = 0$.

(2) $\Rightarrow$ (3) Let a, b be two homogeneous elements of A and suppose that $w_\sigma(a).A.w_\tau(b) = 0$ for any $\sigma, \tau \in H$. Let I be the graded ideal of A generated by the elements $w_\tau(b)$, where τ describes H. Then, I is a graded w-ideal and $a \in \mathrm{Ann}^g_{A-l}(I)$. If $b \neq 0$, then $I \neq 0$, so necessarily $a = 0$.

(3) $\Rightarrow$ (1) Let I_1 and I_2 be two nonzero graded w-ideals of A and let a_i be a nonzero homogeneous element of I_i $(1 \leq i \leq 2)$, then there exist $\sigma, \tau \in H$ such that $w_\sigma(a).A.w_\tau(b) \neq 0$. We then have $0 \neq w_\sigma(a).A.w_\tau(b) = w_\sigma(a).(A.w_\tau(b)) \subseteq I_1I_2$. □

Analogously, using the same arguments, one can easily prove the following lemma.

LEMMA 2.11. *Let $A, (w, f)$ and S be as in (2.4), then the following statements are equivalent:*

1. *A is graded w-semiprime.*
2. *For any homogeneous element a in A such that $w_\sigma(a).A.w_\tau(a) = 0$ for all $\sigma, \tau \in H$, we have $a = 0$.*

It is well known that for any ring B with (Jacobson) radical $\mathrm{rad}(B)$, if $b + \mathrm{rad}(B)$ is an idempotent of $B/\mathrm{rad}(B)$, then there exists an idempotent $a \in B$ such that $a - b \in \mathrm{rad}(B)$. Analogously, in the graded setting, if A is a graded ring with (Jacobson) graded radical $\mathrm{rad}^g(A)$ (i.e., $\mathrm{rad}^g(A)$ is the intersection of all maximal right graded ideals of A), then we have the following result[1].

LEMMA 2.12. *Let F be a graded field and A a graded F-algebra. For any homogeneous idempotent $f + \mathrm{rad}^g(A)$ of $A/\mathrm{rad}^g(A)$ there is an idempotent e of A_0 such that $f - e \in \mathrm{rad}^g(A)$.*

PROOF. One can easily see that $(\mathrm{rad}^g(A))_0 = \mathrm{rad}(A_0)$. Since $f+\mathrm{rad}^g(A)$ is a homogeneous idempotent of $A/\mathrm{rad}^g(A)$, then without loss of generality, we can assume that $f \in A_0$. It follows that $f + \mathrm{rad}(A_0)$ is an idempotent element of $A_0/\mathrm{rad}(A_0)$. So, by the above there is an idempotent element e in A_0 such that $f - e \in \mathrm{rad}(A_0)$, hence $f - e \in \mathrm{rad}^g(A)$. $\square$

LEMMA 2.13. *Let F be a graded field and A a finite-dimensional graded F-algebra. Then the following statements are equivalent:*

1. *A is graded local.*
2. *A has a unique maximal left graded ideal.*
3. *The set of noninvertible homogeneous elements of A generate a proper two-sided graded ideal of A.*
4. *For any element a of A_0, one of the elements a or $1 - a$ is invertible.*
5. *A has only two homogeneous idempotents, 0 and 1.*

---

[1] Note that all homogeneous idempotents of a graded ring R are in R_0 (because Γ is totally ordered).

(6) *The graded F-algebra $A/\mathrm{rad}^g(A)$ is a graded division algebra.*

(7) *A_0 is a local algebra.*

Proof. (1) $\Rightarrow$ (3) By definition, $\mathrm{rad}^g(A)$ is the unique proper maximal (right) graded ideal of A. So, for any homogeneous element a of A, we have $a \in \mathrm{rad}^g(A)$ if and only if a has no right inverse. Let x be a nonzero homogeneous element of A with a right inverse y (in $A_{-\mathrm{gr}(x)}$), then we have $(1 - yx)y = y - yxy = y(1 - xy) = 0$. If y has no right inverse, then by the above $y \in \mathrm{rad}^g(A)$, so $1 - yx$ is invertible (this property of the Jacobson graded radical can be proved as in the ungraded case), hence $y = 0$ (because $(1 - yx)y = 0$ as seen above), but this is not true. Therefore, y has a right inverse, and so $1 - yx = 0$. This shows that x is invertible . We conclude that a homogeneous element a of A is in $\mathrm{rad}^g(A)$ if and only if a has no right inverse if and only if a is not invertible. Thus, $\mathrm{rad}^g(A)$ is the graded ideal of A generated by noninvertible homogeneous elements of A. Plainly, $\mathrm{rad}^g(A)$ is proper in A.

(2) $\Rightarrow$ (3) follows in the same way.

(3) $\Rightarrow$ (4) Let a be a nonzero element of A_0 and let I be the (proper) graded ideal of A generated by noninvertible homogeneous elements of A. If both a and $1 - a$ are noninvertible, then $1 = a + (1 - a) \in I$, a contradiction.

(4) $\Rightarrow$ (5) If e is a homogeneous idempotent of A, then $e \in A_0$, and we have $e(1 - e) = 0$, so necessarily $e = 0$ or $e = 1$.

(5) $\Rightarrow$ (6) by Lemma 2.12, the homogeneous idempotents of $A/\mathrm{rad}^g(A)$ can be lifted modulo $\mathrm{rad}^g(A)$, so the graded semisimple algebra $C := A/\mathrm{rad}^g(A)$ has only two homogeneous idempotents 0 and 1. It follows by the graded version of Wedderburn's theorem that $A/\mathrm{rad}^g(A)$ is a graded division algebra (see [**HW(b)99**, Proposition 1.3] for the graded version of Wedderburn's theorem on graded simple algebras).

(6) $\Rightarrow$ (1) [resp., (6) $\Rightarrow$ (2)] This is clear since in this case $\mathrm{rad}^g(A)$ is the unique maximal right [resp., left] graded ideal of A.

In the same way, we show that A_0 is a local algebra if and only if 0 and 1 are the only idempotent of A_0. Hence, (1) $\Leftrightarrow$ (7). □

The following proposition summarizes some facts relating properties of a generalized graded crossed product $S = (A, H, (w, f))$ which satisfies the grading separation property, to analogous ones on A.

PROPOSITION 2.14. *Let $A, (w, f)$ and S be as in (2.4) and suppose that S satisfies the GSP, then we have the following statements:*

1. *S is graded simple if and only if A is graded w-simple.*
2. *S is graded semiprime [resp., graded prime] if and only if A is graded w-semiprime [resp., graded w-prime].*
3. *S is graded local if and only if A is so if and only if A_0 is local.*
4. *If A is w-compatible, then S is graded Baer [resp. graded quasi-Baer] if and only if A is so.*

PROOF. (1) Suppose that S is a graded simple algebra and let I be a graded w-ideal of A and J_I be the graded ideal of S generated by the homogeneous elements of I, then J_I is either 0 or S. By Lemma 2.9 we have $J_I \cap A = I$, so I is either 0 of A.

Conversely, suppose that A is a graded w-simple algebra, and let J be a graded ideal of S. One can easily see that $J \cap A$ is a graded w-ideal of A (indeed, for any homogeneous element $a \in J \cap A$ and any $\sigma \in H$, we have $w_\sigma(a) = x_\sigma a x_\sigma^{-1} \in J \cap A$). Therefore, $J \cap A$ is either 0 or A. So, by Lemma 2.9 J is then 0 or S.

(2) This follows easily from Lemmas 2.10 and 2.11.

(3) This is clear from Lemma 2.13 (since $S_0 = A_0$).

(4) Suppose that A is w-compatible. We will show that S is graded quasi-Baer if and only if A is so. The fact that S is graded Baer if and only if A is so follows in a similar way. Suppose that A is graded quasi-Baer and let I be a graded left ideal of S. For a nonzero element p of S, we denote by mincomp(p) the homogeneous component $a_\gamma x_\sigma$ of minimal grade of p. Let's consider the left graded ideal J of A generated by the elements a_γ, where $a_\gamma x_\sigma = \text{mincomp}(p)$ for some $\sigma \in H$ and p in I. Let's also consider the following sets: $T := \{a_\gamma \mid a_\gamma x_\sigma = \text{mincomp}(p) \text{ for some } \sigma \in H \text{ and } p \in I\}$, and for $\tau \in H$, $T_\tau := w_\tau(T) = \{w_\tau(a_\gamma) \mid a_\gamma x_\sigma = \text{mincomp}(p) \text{ for some } \sigma \in H \text{ and } p \in I\}$. By assumption we have $\text{Ann}^g_{A-l}(T) = \text{Ann}^g_{A-l}(T_\tau)$.

It is clear that the (left) annulator $\text{Ann}^g_{A-l}(J)$ is contained in $\text{Ann}^g_{A-l}(T)$. Conversely, let $x \in \text{Ann}^g_{A-l}(T)$ and let c be a homogeneous element of A and $a_\gamma \in T$ with $a_\gamma x_\sigma = \text{mincomp}(p)$ for some $\sigma \in H$ and $p \in I$. If $ca_\gamma \neq 0$, then $ca_\gamma x_\sigma = \mathrm{mincomp}(cp)$, so $x(ca_\gamma) = 0$. For an arbitrary element c of A, write $c = \sum c_\lambda$, where c_λ are homogeneous elements of A, then $x(ca_\gamma) = \sum x(c_\lambda a_\gamma) = 0$, so $x \in \mathrm{Ann}^g_{A-l}(J)$. Therefore, $\mathrm{Ann}^g_{A-l}(J) = \mathrm{Ann}^g_{A-l}(T)$.

Since A is graded quasi-Baer, then there is a homogeneous idempotent $e \in A$ (hence $e \in A_0$) such that $\mathrm{Ann}^g_{A-l}(J) = A.e$. Let q be an arbitrary nonzero element of $\mathrm{Ann}^g_{S-l}(I)$ and write $q = b_\lambda x_\tau + q_1$, where $b_\lambda x_\tau = \mathrm{mincomp}(q)$. For any $0 \neq p \in I$, we have $(b_\lambda x_\tau)\mathrm{mincomp}(p) = 0$. Write $\mathrm{mincomp}(p) = a_\gamma x_\sigma$ (for some $\gamma \in \Gamma$, $\sigma \in H$ and $a_\gamma \in A_\gamma$), then we have $b_\lambda w_\tau(a_\gamma) f(\tau, \sigma) x_{\tau\sigma} = 0$, so $b_\lambda w_\tau(a_\gamma) = 0$. Therefore, $b_\lambda \in \mathrm{Ann}^g_{A-l}(T_\tau)(= \mathrm{Ann}^g_{A-l}(J))$, so there is a homogeneous element r_q of A such that $b_\lambda = r_q e$. We then have $\mathrm{mincomp}(q) = r_q e x_\tau = r_q x_\tau w_\tau^{-1}(e)$. We have $w_\tau(e) = x_\tau e x_\tau^{-1}$, so $e = w_\tau^{-1}(x_\tau) w_\tau^{-1}(e) w_\tau^{-1}(x_\tau)^{-1}$, which implies $w_\tau^{-1}(e) = w_\tau^{-1}(x_\tau^{-1}) e w_\tau^{-1}(x_\tau)$, hence $w_\tau^{-1}(e) \in \mathrm{Ann}^g_{A-l}(J)$, so $w_\tau^{-1}(e) = s_q e$ for some homogeneous element s_q of A, hence $\mathrm{mincomp}(q) = r_q x_\tau s_q e$. On the other hand, one can easily see that $e \in \mathrm{Ann}^g_{S-l}(I)$. Indeed, for any $p \in I$, if $ep \neq 0$, then $0 = e\mathrm{mincomp}(p) = \mathrm{mincomp}(ep) \neq 0$, a contradiction.

Let $q_1 := q - \mathrm{mincomp}(q) = q - r_q x_\tau s_q e$, then $q_1 \in \mathrm{Ann}^g_{S-l}(I)$. If we continue in this way, we get $q \in S.e$. This shows that S is graded quasi-Baer.

Conversely, suppose that S is graded quasi-Baer and let J be a left graded ideal of A. Plainly, SJ is a left graded ideal of S. Therefore, there is a homogeneous idempotent e in S such that $\mathrm{Ann}^g_{S-l}(SJ) = S.e$. The grading group being totally ordered, then

$e \in A_0(= S_0)$. It is clear that $A.e \subseteq \mathrm{Ann}^g_{A-l}(J)$. Conversely, let r be a homogeneous element of $\mathrm{Ann}^g_{A-l}(J)$, b be a homogeneous element of A, a be a homogeneous element of J and $\tau \in H$, then $rb \in \mathrm{Ann}^g_{A-l}(J)$, so $rbw_\tau(a) = 0$ (because A is graded w-compatible), thus $r((bx_\tau)a) = (rb)w_\tau(a)x_\tau = 0$. Consequently, for any $p \in S$, we have $r(pa) = 0$, which shows that $r \in \mathrm{Ann}^g_{S-l}(SJ)$, hence $r = se$ for some homogeneous element s of S. Since r is a homogeneous element of A and $e \in A_0$, then $s = tx_1$ for some homogeneous element t of A. Therefore, by identification of A with Ax_1, $r \in A.e$. Thus, $\mathrm{Ann}^g_{A-l}(J) \subseteq A.e$, so $\mathrm{Ann}^g_{A-l}(J) = A.e$. □

REMARK 2.15. Under the hypotheses of Proposition 2.14, we show also that S is graded semisimple if and only if A is so. In fact, as in the ungraded case S [resp., A] is graded semismiple if and only if it is graded regular and every subset consisting of orthogonal idempotents of $S_0(= A_0)$ is finite. So, it is sufficient to show that S is graded regular if and only if A is so. This follows easily by computations. Indeed, suppose that A is graded regular, and let y be a homogeneous element of S, then we can write $y = a_\gamma x_\sigma$ for some $\gamma \in \Gamma$, $\sigma \in H$ and $a_\gamma \in A_\gamma$. Since A is graded regular, then there is a homogeneous element e of A such that $a_\gamma = a_\gamma e a_\gamma$. Let $z = w_\sigma^{-1}(ef(\sigma,\sigma^{-1})^{-1})x_{\sigma^{-1}}$, then we have

$$\begin{aligned} yzy &= (a_\gamma x_\sigma)[w_\sigma^{-1}(ef(\sigma,\sigma^{-1})^{-1})x_{\sigma^{-1}}](a_\gamma x_\sigma) \\ &= a_\gamma e f(\sigma,\sigma^{-1})^{-1} f(\sigma,\sigma^{-1}) x_1 a_\gamma x_\sigma \\ &= a_\gamma e a_\gamma x_\sigma = a_\gamma x_\sigma = y \end{aligned}$$

which shows that S is graded regular.

Conversely, suppose that S is graded regular and let a be a homogeneous element of A, then there is a homogeneous element cx_τ of S, where c is a homogeneous element of A, such that $a(cx_\tau)a = a$. So, $acw_\tau(a)x_\tau = a$, hence $\tau = 1$ and $aca = a$. This shows that A is graded regular.

Bibliography

[ART79] S.A. Amitsur, L.H. Rowen, and J.-P. Tignol, Division algebras of degree 4 and 8 with involution, Israel J. Math. **33** (1979), 133-148.

[AS78] S.A. Amitsur and D. Saltman, Generic abelian crossed products and p-algebras, J. Algebra **51** (1978), 76-87.

[B94] M. Boulagouaz, The graded and tame extensions, in : Commutative Ring Theory (Fès, 1992), P. J. Cahen et al., Eds., Lecture Notes in Pure and Applied Mathematics, Vol. 153, pp. 2746, Dekker, New York, 1994.

[B95] M. Boulagouaz, Le gradué d'une algèbre à division valuée, Comm. Algebra **23** (1995), 4275-4300.

[B(a)98] M. Boulagouaz, Algèbre à division graduée centrale, Comm. Algebra **26** (1998), 2933-2947.

[B(b)98] M. Boulagouaz, Une généralisation du lemme de Hensel, Bulletin of the Bel. Math. Society **5** (1998), 665-673.

[BM00] M. Boulagouaz and K. Mounirh, Generic abelian crossed products and graded division algebras, in : Algebra and number theory (Fès, 1997), M. Boulagouaz and J.P. Tignol, Lecture Notes in Pure and Applied Mathematics **208**, 33-47, Dekker, New York, (2000).

[BLM(a)18] D. Bennis, E. L Houssain and K. Mounirh, On tame division algebras, AIP Conf. Proc. 2074, (2019).

[BLM(b)18] D. Bennis, E. L Houssain and K. Mounirh, On torsion subgroups of Whitehead groups, AIP Conf. Proc. 2074, (2019).

[BM(a)] D. Bennis and K. Mounirh, Witt rings of graded simple algebras, preprint.

[BM(b)] D. Bennis and K. Mounirh, On generalized graded crossed products and Kummer subfields of simple algebras, to appear in Bull. Korean Math. Soc.

[BM(c)] D. Bennis and K. Mounirh, On crossed product algebras over Henselian valued fields, to appear in algebra Colloq.

[BMT03] G. Berhuy, M. Monsurro and J.-P. Tignol, The discriminant of a symplectic involution, Pacific J. Math. **209** (2003), 201-218.

[BT84] F. Bruhat and J. Tits, Schémas en groupes et immeubles des groupes classiques sur un corps local, Bull. Soc. Math. France **112** (1984), 259-301.

[C82] T.C. Craven, Witt rings adn orderings of skew fields, J. Algebra **77** (1982), 74-96.

[CDTWY95] M. Chacron, H. Dherte, J.-P. Tignol, A.Wadsworth, V. Yanchevskii: Discriminants of involutions on Henselian division algebras. Pacific J. Math. **167** no. 1 (1995), 49-79.

[D82] P. Draxl, Skew fields, London Math. Soc., Lectures Notes, Vol. 81. Cambridge, Cambridge University Press (1982).

[Du82] Dubrovin, N.I., Noncommutative valuation rings (Russian). Trudy Moskov. Mat. Obshch. **45**, (1982), 265-280. English transl.: Trans. Moscow Math. Soc. **45** (1984), 273-287.

[Du84] Dubrovin, N.I., Noncommutative valuation rings in simple finite-dimensional algebras over a field (Russian). Mat. Sb. (N.S) **123** (1984), 496-509. English transl.: Math. USSR Sb. **51**, (1985), 493-505.

[E72] O. Endler, Valuation theory, Springer, New York (1972).

[Er82] Yu. Ershov, Henselian valuations of division rings and the group K_1, Mate. Sbornik, **117** (1982), 60-68, English trans. Math. USSR Sbornik, **45** (1983), 63-71.

[FSS90] B. Fein, D.J. Saltman and M. Schacher, Minimal embeddings of central simple algebras, J. Algebra **133** (1990), 404-423.

[GMS03] S. Garibaldi, A.S. Merkurjev, and J.-P. Serre, Cohomological invariants in Galois cohomology, (University Lecture Series, vol. 28) Providence: Amer. Math. Soc. (2003).

[GPT09] S. Garibaldi, R. Parimala, and J.-P. Tignol, Discriminant of symplectic involutions, Pure App. Math. Quart. **5** (2009), 349-374.

[H01] T. Hanke, Galois subfields of inertially split division algebras, J. Algebra **346** (2001), 147-151.

[Ha16] R. Hazrat, Graded rings and graded Grothendieck groups, Lecture note series, **435**, London Mathematical Society, (2016).

[HNW16] T. Hanke, D. Neftin and A. Wadsworth, Galois subfields of tame division algebras, Israel J. of math. **211** (2016), 267-389.

[HaW09] R. Hazrat and A. R. Wadsworth, On maximal subgroups of the multiplicative group of a division algebra, J. Algebra **322** (2009), 2528-2543.

[HaW(a)11] R. Hazrat and A. R. Wadsworth, SK_1 of graded division algebras, Israel J. Math. **183** (2011), 117-163.

[HaW(b)11] R. Hazrat and A. R. Wadsworth, Unitary SK_1 of graded and valued division algebras, Proc. Lond. Math. Soc. **103** (2011), 508-534.

[HaW12] R. Hazrat and A. R. Wadsworth, Homogeneous SK1 of simple graded algebras, New York J. Math. 18 (2012), 315-336.

[HW(a)99] Y.-S. Hwang and A. R. Wadsworth, Algebraic extensions of graded and valued fields, Comm. Algebra **27** (1999), 821-840.

[HW(b)99] Y.-S. Hwang and A. R. Wadsworth, Correspondences between valued division algebras and graded division algebras, J. Algebra **220** (1999), 73-114.

[JW90] B. Jacob and A. R. Wadsworth, Division algebras over Henselian fields, J. Algebra **128** (1990), 126-179.

[K87] G. Karpilovsky, The algebraic structure of crossed products, North-Holland Mathematics studies **142** (1987).

[K89] G. Karpilovski, Topics in field theory, North-Holland Mathematics Studies, Vol. 155 (1989). Amsterdam: North-Holland.

[Ka08] B. Kahn, Formes quadratiques sur un corps, Cours spécialisés, collection SMF, Institut de Mathématiques de Jussieu, Paris, (2008).

[Kh00] S. K. Khanduja, A characterization of finite tame extensions, Bull. London Math. **32** (2000), 551-554.

[KMRT98] M.-A. Knus, A. Merkurjev, M. Rost, J.-P. Tignol: The book of involutions (Colloquium Publications, vol. 44) Providence: Amer. Math. Soc. (1998).

[Ku11] A. Kulshrestha, Strongly anisotropic involutions on central simple algebras, Comm. Algebra **39** (2011), 1686-1704.

[LT93] D.W. Lewis and J.-P. Tignol, Square class groups and Witt rings of central simple algebras, J. Algebra **154** (1993), 360-376.

[Mi70] J. Milnor, Algebraic K-theory and quadratic forms, Inventiones math. **9** (1970), 318-344.

[MiW95] J. Minac and A. Wadsworth, The u-invariant for algebraic extensions, Proceedings of Symposia in Pure Mathematics **58.2** (1995), 333-358.

[Mor89] P.J. Morandi, Value functions on central simple algebras. Trans. Amer. Math. Soc. **315** (1989), 605-622.

[MS95] P.J. Morandi, B.A. Sethuraman, Kummer subfields of tame division algebras, J. Algebra 172 (1995), 554-583.

[Mo13] M. Motiee, On torsion subgroups of Whitehead groups of division algebras, Manuscripta Math. **141** (2013), 717-726.

[M96] K. Mounirh, Algèbres des quotients centraux des algèbres à division graduées semi-ramifiées, Thesis (D.E.S), University Sidi Mohamed Ben Abdellah, Fès-Morocco (1996).

[M05] K. Mounirh, Nicely semiramified division algebras over Henselian fields, Int. J. Math. Math. Sci. **2005(4)**, 571-577.

[M07] K. Mounirh, Semiramified division algebras over Henselian fields, Thesis (for the grade of Doctor in Sciences), Louvain Catholic University, Louvain-La-Neuve, Belgium, (2007).

[M08] K. Mounirh, Nondegenerate semiramified valued and graded division algebras, Comm. Algebra **36** (2008), 4386-4406.

[M10] K. Mounirh, Kummer subfields of tame division algebras, J. Pure Appl. Algebra **214** (2010), 440-448

[M11] K. Mounirh, Discriminants of orthogonal involutions on central simple algebras with tame gauges, J. Pure Appl. Algebra **215** (2011), 2547-2558.

[M14] K. Mounirh, Discriminants of symplectic graded involutions, J. Algebra **400** (2014), 17-32.

[M] K. Mounirh, Discriminants of orthogonal involutions on simple algebras over Henselian fields, preprint.

[MW11] K. Mounirh and A. Wadsworth, Subfields of nondegenerate tame semiramified division algebras, Comm. Algebra **39** (2011), 462-485.

[P82] R.S. Pierce, Associative algebras, Spring-Verlag (1982).

[R82] L.H. Rowen, Cyclic division algebras, Israel J. Math. **41** (1982), 213-234, (and Israel J. Math. **43** (1982), 277-280).

[RTW07] J.-F. Renard, J.-P. Tignol and A. Wadsworth, Graded hermitian forms and Springer's theorem, Indag. Math. **18** (2007), 97-134.

[S88] A. Sladek, Witt rings of complete skew fields, Pacific Journal of Math. **132** (1988), 391-399.

[S(a)78] D. Saltman, Noncrossed products of small exponents, Proc. Amer. Math. Soc. **68** (1978), 165-168.

[S(b)78] D. Saltman, Noncrossed products p-algebras and Galois p-extensions, J. Algebra **52** (1978), 302-314.

[ST04] A. Serhir and J.-P. Tignol, The discriminant of a decomposable symplectic involution, J. Algebra **273** (2004), 601-607.

[T86] J.-P. Tignol, Cyclic and elementary abelian subfields of Malcev-Neumann division algebras, J. Pure Appl. Algebra **42** (1986), 199-220.

[T(a)87] J.-P. Tignol, Generalized crossed products, in "Seminaires Mathématiques" (nouvelle série), Université Catholique de Louvain-La-Neuve, Belgium, **106** (1987), available at: http://perso.uclouvain.be/jean-pierre.tignol/Rapport106.pdf.

[T(b)87] J.-P. Tignol, Algèbres indécomposables d'exposant premier, Adv. in Math. **65** (1987), 205-228.

[T90] J.-P. Tignol, Algèbres à division et extensions de corps sauvagement ramifiées de degré premier, J., Reine Angew. Math. **404** (1990), 1-38.

[TA85] J.-P. Tignol, S.A. Amitsur, Kummer subfields of Malcev-Neumann division algebras, Israel J. Math. **50** (1985), 114-144.

[TW87] J.-P. Tignol and A. Wadsworth, Totally ramified valuations on finite-dimensional division algebras, Trans. Amer. Math. Soc., **302** (1987), 223-250.

[TW10] J.-P. Tignol and A. Wadsworth, Value functions and associated graded rings for semisimple algebras, Trans. Amer. Math. Soc., **362** (2010), 687-726.

[TW11] J.-P. Tignol and A. Wadsworth, Valuations on algebras with involution, Math. Ann. **351** (2011), 109148.

[TW15] J.-P. Tignol and A. Wadsworth, Value functions on simple algebras, and associated graded rings, Springer Monographs in Mathematics, (2015).

[V96] V. Veovodsky, The Milnor conjecture, preprint (1996), available at: www.researchgate.net/publication/2426765.

[W83] A. Wadsworth, p-Henselian fields: K-Theory, Galois cohomology, and Graded Witt rings, Pacific Journal of Math. **105** (1983), 473-496.

[W86] A. Wadsworth, Extending valuations to finite dimensional division algebras, Proc. Amer. Math. Soc. **98** (1986), 20-22.

[W89] A. R. Wadsworth, Dubrovin valuation rings and Henselization, Math. Annalen, **283** (1989), 301328.

[W10] A. R. Wadsworth, Unitary SK_1 of semiramified graded and valued division algebras, Manuscripta Math. **139** (2012), 343-389.

[WY12] A. R. Wadsworth and V.I. Yanchevskii, Unitary SK_1 of graded division ring and its quotient division ring, J. Algebra **352** (2012), 62-78.

[Z16] Z. Zhang, A matrix description for K_1 of graded rings, Israel J. Math. **211** (2016), 45-66.

Résumé détaillé

On est essentiellement intéressé dans cette thèse par des applications nouvelles des algèbres graduées à l'étude des algèbres simples avec jauges modérées et au calcul des invariants discriminants des involutions orthogonales et symplectiques. Les définitions et notations nécessaires pour comprendre ce travail sont données dans la partie préliminaire après l'introduction. Dans le premier chapitre on généralise le travail de Hanke et al., dans [**HNW16**]. Rappellons que les auteurs ont montré dans [**HNW16**, Theorem 1.1] que si D est une algèbre à division modérée de dimension finie sur un corps valué Henselien E, alors D est un produit croisé si et seulement si l'algèbre à division résiduelle $\overline{D}$ a un sous-corps maximal qui est Galoisien sur $\overline{E}$. Le premier objectif de ce chapitre est de montrer un résultat plus général pour une algèbre centrale simple sur un corps valué Hensélien. Pour prouver ce résultat, on montre dans la première section de ce chapitre plusieurs propriétés qui concernent les algèbres graduées simples de 0-composante simple. Puis, on généralise [**HNW16**, Theorem 3.1] et on montre comment plusieurs résultats dans [**HNW16**, §2 and §3] peuvent avoir des versions plus générales. Dans la second section, on montre des résultats concernant la possibilité d'injecter une algèbre graduée simple dans une algèbre graduée matricielle et dans la troisième section, on généralise [**HNW16**, Theorem 1.1] au contexte des algèbres simples (avec jauges modérées) (voir Théorème 1.3.5). On termine ce chapitre par montrer des conditions nécessaires et suffisantes pour qu'une algèbre centrale simple sur un corps valué

Hensélien (sous certaines hypothèses) puisse avoir des sous-corps de Kummer (voir Corollaires 1.4.8 and 1.4.9). Ces résultats généralisent des résultats précédents montrés dans [**TA85**], [**MS95**] et [**M10**]. Le second chapitre étudie les invariants discriminants des involutions orthogonales et symplectiques. On généralise en particulier les travaux faits dans [**CDTWY95**] et [**M11**] qui concernent le calcul du discriminant d'une involution orthogonale sur une algèbre à division modérée et sur une algèbre simple avec jauge modérée (voir Corollaire 2.3.8). On déternime aussi des cas où le discriminant (relatif) des involutions symplectiques est égal à zéro (voir Corollaires 2.5.3 et 2.5.5).

Dans le troisième chapitre on définit et on étudie l'anneau de Witt d'une algèbre graduée simple et on l'applique pour donner des propriétés nouvelles sur l'anneau de Witt d'une algèbre à division modérée sur un corps valué Hensélien. On montre aussi des résultats concernant la K-théorie de Milnor des corps gradués. On montre en particulier que si E est un corps valué Hensélien de caractéristique résiduelle différente de 2, D est une algèbre à division modérée centrale sur E, $W(\overline{D})$ est l'anneau de Witt de l'algèbre à division résiduelle $\overline{D}$ de D, J est l'idéal de $W(\overline{D})$, engendé par les (classes) des éléments $\langle 1\rangle - \langle r\rangle$, où $r \in \overline{D}^* \cap GD^{*2}$, et Z est un sous-groupe de D^*/D^{*2} qui a pour image (sous bijection) $\Gamma_D/2\Gamma_D$ par l'application $\overline{v}$ induite par la valuation v de D, alors $W(D)$ est isomorphe à un produit croisé généralisé $(W(\overline{D})/J) * Z$. Si en plus, $\overline{D}^* \cap GD^{*2} = \overline{D}^{*2}$, alors on obtient un isomorphisme d'anneaux $W(D) \cong W(\overline{D}) * Z$. En particulier, si $D = E$, alors

$W(E) \cong W(\overline{E})[Z]$ (voir Corollaire 3.2.10). Ce résultat généralise le théorème de Sladek [**S88**, Theorem 3.3] qu'il a prouvé dans le cas restrictive où l'algèbre à division D possède une valuation discrète complète avec caractéristique résiduelle différente de 2 et une algèbre résiduelle commutative.

Le quatrième chapitre étudie le sous-groupe de torsion du groupe de Whitehead $\mathrm{K}_1(D)$ d'une algèbre à division fortement modérée D sur un corps valué Hensélien. Pour cet objectif, étant donné un anneau à division gradué A, on définit un nouveau groupe $\mathrm{MK}_1(A)$ qu'on appelle le groupe M-Whitehead de A, et on montre certaines de ses propriétés. On montre dans le théorème 4.2.7 que si D est une algèbre à division fortement modérée centrale sur un corps valué Hensélien E de caractéristique résiduelle égale à la caractéristique de E, alors le sous-groupe de torsion de $\mathrm{K}_1(D)$, est isomorphe au sous-groupe de torsion du groupe M-Whitehead de GD, où GD est l'algèbre à division graduée canonique associée à D (voir les préliminaires). Ce résultat permet de donner une nouvelle démonstration (à une version plus générale) d'un théorème montré par Motie dans [**Mo13**, Théorème 10] (voir corollaire 4.2.8). On donne aussi dans ce chapitre une nouvelle démonstration à un résultat qui caractérise les extension de corps modérées d'un corps valué Hensélien, précédemment montré par Khanduja dans [**Kh00**]. Plusieurs autres résultats, concernant en particulier les produits croisés gradués généralisés sont donnés dans la partie annexe de cette thèse.

Mots-clefs : Algèbres simples, valuations, graduations, jauges, involutions, anneaux de Witt.

Résumé

Cette thèse est une contribution au développement de la théorie des jauges sur les algèbres simples, fondée par J.-P. Tignol et A. R. Wadsworth.

En particulier, on généralise dans le premier chapitre le travail T. Hanke, D. Neftin et A. R. Wadsworth concernant les conditions résiduelles nécessaires et suffisantes pour qu'une algèbre à division modérée centrale sur un corps valué hensélien soit un produit croisé. On généralise aussi les travaux d'Amitsur-Tignol, de Morandi-Seuthuraman et un résultat précédent de l'auteur sur les conditions nécessaires et suffisantes pour l'existence de sous corps de Kummer dans une algèbre à division valuée modérée. Dans le second chapitre, on généralise le travail de Chacron, Dherte, Tignol, Wadsworth et Yanchevskii et aussi un travail précédent de l'auteur sur le discriminant des involutions orthogonales. On montre aussi des résultats nouveaux qui concernent le discriminant des involutions symplectiques. Ces résultats sont des conséquences d'une étude détaillée qu'on développe sur les discriminants des involutions graduées de première espèce. Dans le troisième chapitre, on généralise le travail de Sladek sur l'anneau de Witt d'une algèbre à division valuée modérée et on montre une interprétation cohomologique des K-groupes de Milnor d'un corps gradué. Dans le dernier chapitre, on étudie le sous groupe de torsion du groupe de Whitehead d'une algèbre à division valuée fortement modérée et on généralise un travail de Motiee sur ce sous groupe.

Mots-clefs : Algèbres simples, valuations, graduations, jauges, involutions, anneaux de Witt.

Abstract

This thesis is a contribution to the development of gauge theory on simple algebras which was founded by J.-P. Tignol and A. R. Wadsworth.

We generalize, in particular, in the first chapter the work of T. Hanke, D. Neftin and A. R. Wadsworth concerning necessary and sufficient residually conditions on a tame division algebra over a Henselian valued field, to be a crossed product. Also, we generalize the works of Amitsur-Tignol, of Morandi-Seuthuraman and a previous result of the author on the necessary and sufficient conditions for the existence of Kummer subfields in a tame valued division algebra. In the second chapter, we generalize a work of Chacron, Dherte, Tignol, Wadsworth and Yanchevskii and a previous work of the author on the discriminant of orthogonal involutions. We prove also new results concerning the discriminant of symplectic involutions. In the third chapter, we generalize a work of Sladek on the Witt ring of a tame valued division algebra and we show a cohomological interpretation of Milnor's K-groups of a graded field. In the last chapter, we study the torsion subgroup of Whitehead group of a strongly tame valued division algebra and we generalize a result of Motiee on this subgroup.

Key-words : Simple algebras, valuations, graduations, gauges, involutions, Witt rings.