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THESE DE DOCTORAT

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Sujet de la thèse

Potentiel Non Linéaire et Application aux Equations aux Dérivées Partielles.

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Dèdicace

⊛ ***À la mémoire de de mes parents que Dieu ait leurs âmes,***

⊛ ***À mon épouse : Fatima Ouabane ,***

⊛ ***À mes enfants: Noura, Houssam, Chaymae, Sabrine,***

⊛ ***À toute ma famille,***

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Résumé

L'objectif de cette thèse est l'étude de potentiel non linéaire et de certains problèmes des équations aux dérivées partielles.
Cette thèse est composée de deux thèmes :
Le premier thème est consacré à l'étude de La théorie du potentiel non linéaire dans les espaces de Musielak-Orlicz , Musielak-Orlicz- Sobolev et dans les espaces de Sobolev avec poids à exposants variables . Nous avons établi le théorème de H.Brezis et F.E. Browder dans le cas des espaces de Musielak-Orlicz-Sobolev. Nous avons défini la notion des espaces de Musielak-Orlicz-Sobolev définis sur un espace métrique et nous avons étudié la notion des capacités dans ces espaces. Nous avons défini l'espace de Musielak-Orlicz-Sobolev à valeur zero sur le bord et l'espaces de Sobolev avec poids à exposent variable, à valeur zero sur le bord, nous avons établi que ce sont des espaces de Banach réflexives , nous avons donné quelques-unes de leurs propriétés, nous avons défini dans ces espaces l'énergie intégrale de Dirichlet et nous avons montré qu'elle posséde un minimun.
Le deuxième thème concerne l'étude de certains problèmes non linéaire des équations aux dérivées partielles. Dans le chapitre 3, par application du théorème de H.Brezis et F.E. Browder dans le cas des espaces de Musielak-Orlicz-Sobolev nous avons étudié un problème elliptique unilatéral. Dans le chapitre 5, nous avons étudié, dans l'espace de Sobolev à exposants variable un problème elliptique fortement non linéaire avec second membre mesure de Radon, dont la solution vérifie une certaine régularité.
Mots clés : Espace de Musielak-Orlicz-Sobolev - espace de Sobolev à exposants variable - espace de Sobolev à exposants variable avec poids - espace de Musielak-Orlicz-Sobolev à valeur zéro sur le bord - espaces de Sobolev avec poids à exposants variable et à valeur zéro sur le bord - capacité - potentiel non linéaire - mesure de Radon- problème elliptique unilatéral - problème elliptique fortement non linéaire - théorème de H.Brezis et F.E. Browder - énergie intégrale de Dirichlet .

Abstract

The objective of this thesis is to study the theory of non linear potential and some problems of partial differential equations.
This works is divided into two principal parts:
In the first part we study the theory of non linear potential in Musielak-Orlicz spaces, Musielak-Orlicz-Sobolev spaces and in weighted Variable Exponent Sobolev spaces. We establish the theorem of H.Brezis and F.E. Browder in the setting of Musielak-Orlicz-Sobolev spaces. We define the Musielak-Orlicz-Sobolev spaces defined on arbitrary metric space and we study the notions of the capacity in these spaces. We define the Musielak-Orlicz-Sobolev space with zero boundary values and the weighted variable exponent Sobolev spaces with zero boundary values. We establish that they are Banach and reflexive spaces, we give Some of their properties, we define in these spaces the Dirichlet energy integral and we show that it has a minimum.
The second part is devoted to the study of some problems of partial differential equations . In the chapter 3 and as an application of the theorem of H.Brezis and F.E. Browder we study an unilateral elliptic problem in the Musielak-Orlicz-Sobolev spaces. In the chapter 5 we study the existence and the regularity of solution for strongly nonlinear $p(x)$- elliptic equation with measure data.

Key words: Musielak-Orlicz space; Radon measures space; capacity; non-linear potential; Musielak-Orlicz-Sobolev spaces; theorem of H Brezis and F.E. Browder; unilateral problem; Musielak-Orlicz-Sobolev spaces with zero boundary values ; Musielak-Orlicz-Sobolev spaces on arbitrary metric space; Sobolev spaces with variable exponents; strongly nonlinear p(x) elliptic equations with measure data; regularity; weighted variable exponents Sobolev spaces with zero boundary values; Dirichlet energy integral.

Contents

Introduction Générale

La théorie du potentiel non linéaire a été étudiée au début par Maz'ya et Khalvin dans [66] et par Meyers dans [68] dans les espaces L^p de Lebesgue (p constant et $1 < p < \infty$), ce qui a introduit le concept de capacité dans ces espaces et a permis des applications très riches en analyse fonctionnelle, en analyse harmonique, dans la théorie des probabilités et dans la théorie des équations aux dérivées partielles. Elle est développée notamment par Adams dans [1]; par Adams et Meyers dans [3] et par Hedberg dans [58]. Cette théorie a été généralisée; aux espaces d'Orlicz par N. Aissaoui et A. Benkirane dans [7] et [8], aux espaces de Lebesgue Sobolev à exposent variable par Harjulehto , Hästö, Koskenoja et Varonen dans [39] et aux espaces de Lebesgue Sobolev avec poids à exposent variable par Ismail Aydin dans [59].
La notion de capacité est alors un outil efficace , elle nous permet de mesurer les ensembles plus finement que la mesure , de voir que les fonctions sont définies mieux que presque partout (quasi presque partout).
La théorie des équations aux dérivées partielles (EDP) est un domaine en pleine expansion de part son impact sur de nombreux domaines à savoir la chimie dans le but de modéliser les réactions chimiques, l'économie pour étudier l'évolution des marchés, le contrôle quantique, la mécanique, réseaux de télécommunications, etc.

Considérons le problème elliptique suivant:

$$\begin{cases} Au = f \quad \text{dans } \Omega, \\ u = 0 \text{ sur } \partial\Omega, \end{cases} \tag{0.0.1}$$

avec $Au = -\text{div}\,(a(x, \nabla u))$ est de type Leray-Lions de $W_0^{1,p}(\Omega)$ vers son dual $W^{-1,p'}(\Omega)$,

et où la fonction de Carathédory a vérifie les hypothèses de croissance, de monotonie, et d'ellipticité sur Ω, ouvert borné de $I\!R^N$ et où f est une mesure. L'étude se limite au cas où $1 < p < N$ car si $p \geq N$ alors $W_0^{1,p}(\Omega) \hookrightarrow C^0(\Omega)$, et par suite $(C^0(\Omega))' \hookrightarrow W^{-1,p'}(\Omega)$; donc si f est une mesure de Radon alors elle devient un élément de dual $W^{-1,p'}(\Omega)$.

Dans le cas $p > 2 - \frac{1}{N}$, avec f une mesure de Radon, Boccardo et Gallouet [31] montrent que la solution faible du problème (0.0.1) appartient à $W_0^{1,q}(\Omega)$ où $1 < q < \frac{N(p-1)}{N-1}$.

Or si p est proche de 1, plus précisement si $1 < p < 2 - \frac{1}{N}$ alors la solution u n'appartient pas à $W_{loc}^{1,1}(\Omega)$. Pour surmonter cette difficulté, Bénilan, Boccardo, Gallouet, Gariepy, Pierre et Vazquez ont introduit dans [30] un nouvel espace $T_{loc}^{1,1}(\Omega)$ dans lequel on peut donner un sens au ∇u, qui n'est pas en général localement intégrable. L'idée consiste à introduire $T_k(u)$, troncature de u, et de considérer $\nabla T_k u$, et ceci est introduit dans le but de choisir les solutions qui s'approchent le plus à des solutions physiques.

En 1965, Stampacchia, voir [77], avait consideré le problème elliptique linéaire suivant: $-divA^t\nabla u = f$, pour $f \in \mathcal{M}(\overline{\Omega})$, espace des mesures. Il a proposé une méthode donnant l'existence et l'unicité de la solution de la dite équation elliptique voir [74]; cette méthode utilise la dualité et conduit au problème variationnel suivant:

$$\begin{cases} u \in \bigcap\limits_{q<\frac{N}{N-1}} W_0^{1,q}(\Omega), \\ -\langle \text{div}\,(A^t\nabla\varphi), u\rangle = \displaystyle\int_\Omega \varphi df, \forall\varphi \in H_0^1(\Omega) \text{ tel que } -div(A^t\nabla\varphi) \in \bigcup\limits_{r>N} W_0^{1,r}(\Omega). \end{cases} \tag{0.0.2}$$

Cependant, pour les problèmes elliptiques et $f \in \mathcal{M}(\overline{\Omega})$, la méthode de Stampacchia ne fonctionne plus avec des opérateurs non linéaires; ainsi Boccardo et Gallouet, dans [31] ont construit, par approximation, une solution pour les équations elliptiques avec des conditions aux limites homogènes de Direchlet.

Le résultat d'existence, lorsque a dépend de u et ∇u a été démontré en [38] et [75].

Dans [74] l'auteur a prouvé, dans le cas linéaire et en utilisant la méthode par approximation, l'existence de la même solution que celle de Stampacchia.

Lorsque la donnée f est $L^1(\Omega)$, Dall' Aglio voir [37] a montré pour un problème elliptique non linéaire par la méthode (SOLA) l'unicité de la solution.
Cette thèse est composée de deux thèmes, le premier est consacré à l'étude de potentielle non linéaire dans les espaces de Musielak-Orlicz-Sobolev et dans les espaces de Sobolev à exposants variable avec poids, le deuxième est consacré à l'étude des problèmes unilatéraux fortement non linéaires dans les espaces de Musielak-Orlicz-Sobolev et à des problèmes elliptiques avec second mesure dans les espaces de Sobolev à exposants variable.
Dans le premier chapitre, nous présentons des rappels et des outils mathématiques nécessaires dont nous allons nous servir ultérieurement. Pour les résultats connus nous ne donnons pas de démonstrations. Nous indiquerons par ailleurs les références correspondantes.
dans le deuxiéme chapitre nous sommes interéssés à l'étude de capacité et de potentiel non linéaire dans les espaces de Musielak-Orlicz. Nous définirons la capacité $C_{k,\varphi}$, où

$$\forall X \subset \mathbb{R}^N,\ \ C_{k,\varphi}(X) = \inf\{||f||_\varphi : f \in L_\varphi \ \ et\ k * f \geqslant 1\ sur\ \ X\},$$

avec k est une fonction positive intégrable sur $\mathbb{R}^N$. Nous donnerons quelques-unes de ses propriétés. Après nous introduirons la capacité $D_{k,\varphi}$ en termes de mesures de Radon, où

$$\forall X \subset \mathbb{R}^N,\ \ D_{k,\varphi}(X) = \sup\{||\mu||\ :\ \mu \in M_1^+, \mu\ est\ \textbf{concentrée}\ sur\ X\ \ et\ ||k*\mu||_\psi \leqslant 1\}$$

avec $(k * \mu)(x) = \int k(x-y)d\mu(y)$, Nous donnerons ses relations avec $C_{k,\varphi}$. Dans le dernier paragraphe nous définisson le potentiel non linéaire $k * f_0$ d'une partie $X \subset \mathbb{R}^N$ dans les espaces de Musielak-Orlicz, où

$$||f_0||_\varphi = inf\{||f||_\varphi : f \in L_\varphi^+\ et\ \ k * f \geqslant 1\ C_{k,\varphi} - q.p\ sur\ X\},$$

et nous donnerons quelques-unes de ses propriétés.

Dans le troisième chapitre nous définirons la notion de capacité C_φ dans les espaces de Musielak-Orlicz-Sobolev où

$$\forall E \subset \mathbb{R}^N,\ \ C_\varphi(E) = \inf_{u \in A_\varphi(E)} \bar{\rho}_{m,\varphi}(u).$$

avec

$$A_\varphi(E) = \{u \in W^m L_\varphi \ : \ u \geqslant 1 \, sur \, un \, ensemble \, ouvert \, contenant \, E \, et \, u \geqslant 0\}.$$

Si $A_\varphi(E) = \emptyset$ on pose $C_\varphi(E) = \infty$. Nous donnerons quelques-unes de ses propriétés. Après nous démontrerons le théorème de H.Brezis et F.E. Browder dans les espaces de Musielak-Orlicz-Sobolev. Notons que cette généralisation a été faite par A.Benkirane dans le cas des espaces d' Orlicz-Sobolev (voir [29]). Après on fera deux applications:
-Dans la premiére nous appliquerons le théorème de H.Brezis et F.E. Browder pour étudier des problèmes unilatéraux fortement non linéaires de la forme:

$$\begin{gathered} u \in K_\Phi, \ g(.,u) \in L^1(\Omega), \ ug(.,u) \in L^1(\Omega) \\ < Su, v-u > + \int_\Omega g(.,u)(v-u)dx \geqslant \ < f, v-u >, \ \forall v \in K_\Phi \cap L^\infty(\Omega), \end{gathered}$$

le second membre $f \in W^{-m}L_\psi(\Omega)$ et l'ensemble convexe

$$K_\Phi = \{v \in W_0^m L_\varphi(\Omega), \ v \geqslant \Phi \, p.p \, sur \, \Omega\}.$$

avec l'obstacle Φ appartient à $W_0^m L_\psi(\Omega) \cap L^\infty(\Omega)$. S est un opérateur pseudo-monotone défini de $W_0^m L_\varphi(\Omega)$ vers $W^{-m}L_\psi(\Omega)$. qui vérifie les conditions suivantes:
(1) S est continu de chaque sous espace de dimension fini de $W_0^m L_\varphi(\Omega)$ vers $W^{-m}L_\psi(\Omega)$ pour la topologie faible * .
(2) S transforme tout ensemble borné en un ensemble borné.
(3) S est coercive, i.e pour un certain v_0 dans $K_\Phi \cap L^\infty(\Omega)$

$$\frac{< S(v), v - v_0 >}{||v||_{W_0^m L_\varphi(\Omega)}} \longrightarrow +\infty \ \ lorsque \ \ ||v||_{W_0^m L_\varphi(\Omega)} \longrightarrow +\infty, \qquad (0.0.3)$$

et finalement g est une fonction de carathéodory avec $g : \Omega \times \mathbb{R} \longrightarrow \mathbb{R}$ qui satisfait les conditions suivantes :
(4) $s.g(x,s) \geqslant 0, \forall \ s \in \mathbb{R}$ et presque pour tout x dans Ω,
(5) $h_t = sup_{|s| \leqslant t}|g(x,s)| \in L^1(\Omega) \ \forall t \ \geqslant 0.$

-Dans la deuxième nous appliquerons les résultats sur les capacités pour définir les espaces de Musielak-Orlicz-Sobolev à valeur zéro au bord, nous établirons que se sont des espaces de Banach réflexives, nous donnerons quelques-unes de leurs propriétés, nous définirons l'énergie intégrale de Dirichlet et nous montrerons qu'elle possède un minimum.

Dans le chapitre 4, nous définirons la notion des espaces de Musielak-Orlicz-Sobolev définis sur un espace métrique. On sait que dans la définition des espaces de Sobolev on introduit la notion de dérivée, au sens des distributions, en 1996 P. Hajlasz a donné dans [47] une définition équivalente des espaces de Lebesgue -Sobolev $W^{1,p}(\Omega)$, avec $\Omega = \mathbb{R}^N$ ou Ω est un ouvert borné de $\mathbb{R}^N$ et p constant, dans le cas cartésien sans utiliser la notion de dérivée au sens des distributions, ce qui lui a permit de définir la notion des espaces de Lebesgue Sobolev définis sur un espace métrique. Il a montré que L'inégalité de Poincaré reste encore vraie dans ces espaces et que l'ensemble des fonctions lipschitziennes est dense dans ces espaces. Apres J. Kinnunen et O.Martio ont défini dans [60] le concept de capacité dans ces espaces et ils ont donné quelques-unes de ces propriétés. N. AISSAOUI a généralisé cette théorie dans les espaces d'Orlicz (voir [4] , [5] et [6]). En 2006 Petteri Harjulehto, Peter Hasto et Mikko Pere ont généralisé dans [71] cette théorie dans les espaces de Lebesgue-Sobolev $\grave{a}$ exposent variable. Notre but dans ce chapitre c'est de généraliser cette théorie au cas des espaces de Musielak-Orlicz-Sobolev.

Dans le cinquième chapitre nous établirons le résultat d'existence du problème suivant:

$$\begin{cases} -\mathrm{div}\Big(a(x,u,\nabla u)\Big) + g(x,u,\nabla u) = \mu & \text{in} \quad \Omega, \\ u = 0 & \text{on} \quad \partial\Omega, \end{cases} \tag{0.0.4}$$

avec Ω est ouvert borné de $\mathbb{R}^N$ et $a : \Omega \times I\!R \times I\!R^N \to I\!R^N$ est une fonction de Carathédory qui satisfait les hypothèses suivantes:

$$|a(x,s,\xi)| \leq \beta(K(x) + |s|^{p(x)-1} + |\xi|^{p(x)-1}),$$

$$a(x,s,\xi)\xi \geq \alpha|\xi|^{p(x)},$$

$$[a(x,s,\xi) - a(x,s,\overline{\xi})](\xi - \overline{\xi}) > 0 \quad \text{pour tout} \quad \xi \neq \overline{\xi} \quad \text{dans} \quad I\!R^N,$$

pour presque tout $x \in \Omega$ et pour tout $(s,\xi) \in I\!R \times I\!R^N$, où $K(x)$ est une fonction positive et $K \in L^{p'(x)}(\Omega)$.

Le terme non linéaire g: $\Omega \times I\!R \times I\!R^N \longrightarrow I\!R$, une fonction de Carathédory qui satisfait pour presque $x \in \Omega$ et pour tout $s \in I\!R$, $\zeta \in I\!R^N$ les hypothèses suivantes:

$$g(x,s,\xi).s \geqslant 0 \tag{0.0.5}$$

$$|g(x,s,\xi)| \leq b(|s|)(c(x)+|\xi|^{p(x)}), \tag{0.0.6}$$

avec $b : I\!R^+ \to I\!R^+$ une fonction continue croissante et $c : \Omega \to I\!R^+,\ c \in L^1(\Omega)$, et μ est une mesure de Radon.
Notons que cette equation a été étudiée par Azroule Barbara et Hjiaj dans [12] dans le cas où le second membre appartient à $L^1(\Omega)$.
Nous démontrons aussi que la solution de cette equation vérifie aussi la régularité suivante:

$$u \in T_0^{1,p(.)}(\Omega) \cap W_0^{1,\,q(.)},\ \ \forall\, q(.) \in B_{p(.)}$$

où

$$B_{p(.)} = \{q \in \mathcal{C}_+(\bar{\Omega}) :\ 1 \leqslant q(x) < \frac{N}{N-1}(p^- - 1)\}. \tag{0.0.7}$$

Notons aussi que cette equation a été étudiée avec la régularité précèdente mais dans le cas où $g = 0$ par A. Benkirane , M. L. Ahmed Oubeid et M. Sidi El Vally [21] dans les espaces de Musielak-Orlicz-Sobolev.
En fin dans le sixième chapitre , nous étudierons la théorie des capacités dans les espaces de Sobolev à exposants variable et avec poids, en nous basant sur les résultats de Ismail Aydin (voir [59]), nous définirons les espaces de Sobolev avec poids à exposent variable à valeur zero sur le bord, nous établirons que se sont des espaces de Banach réflexives , nous donnerons quelques-une de leurs propriétés, nous définirons l'énergie intégrale de Dirichlet et nous montrons qu'elle possede un minimum.
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[4] My Cherif Hassib, Youssef Akdim and N. Aissaoui. Weighted Variable Exponent Sobolev Spaces on metric measure spaces.
[5] My Cherif Hassib, Youssef Akdim and N. Aissaoui. Capacity, Dirichlet Energy and Weighted Variable Exponent Sobolev Spaces with Zero Boundary Values on metric measure spaces.

Chapter 1

Preliminaries

1.1 Musielak-Orlicz spaces and Musielak-Orlicz Sobolev spaces

1.1.1 Musielak-Orlicz function

Let Ω be an open set in $\mathbb{R}^N$ and let φ be a real-valued function defined in $\Omega \times \mathbb{R}^+$ and satisfying the following conditions:
a) $\varphi(x,.)$ is an N-function [convex, increasing, continuous, $\varphi(x,0)=0$, $\varphi(x,t)>0\ \forall t>0$ $\frac{\varphi(x,t)}{t} \to 0$ as $t \to 0$, $\frac{\varphi(x,t)}{t} \to \infty$ as $t \to \infty$].
b)$\varphi(.,t)$ is a measurable function.
A function $\varphi(x,t)$, which satisfies the conditions a) and b) is called a Musielak-Orlicz function.
Equivalently, φ admits the representation:
$\varphi(y,t) = \int_0^t a(y,\tau)d\tau$, for all $y \in \Omega$ and $t \geqslant 0$, where $a(y,.) : \mathbb{R}^+ \to \mathbb{R}^+$ is non-decreasing, right continuous, with for all $y \in \Omega$: $a(y,0)=0$,
$a(y,t)>0$ for $t>0$ and $\lim\limits_{t\to+\infty} a(y,t) = +\infty$.
The function a(y , .) is called the derivative of $\varphi(y,.)$.
For a Musielak-Orlicz function φ we put $\varphi_x(t) = \varphi(x,t)$ and we associate its nonnegative reciprocal function φ_x^{-1}, with respect to t, that is

$$\varphi_x^{-1}(\varphi(x,t)) = \varphi(x,\varphi_x^{-1}(t)) = t.$$

The Musielak-Orlicz function φ is said to satisfy the Δ_2-condition if there exists $K \geqslant 2$ such that

$$\varphi(y, 2t) \leqslant K\varphi(y, t), \quad for\ all\ y \in \Omega \quad and\ \ t \geqslant 0.$$

The smallest K is called the Δ_2-constant of φ.
When the last inequality holds only for $t \geqslant some\ t_0 > 0$ then φ is said to satisfy the Δ_2-condition near infinity.
Let φ and γ be two Musielak -Orlicz functions, we say that φ grows essentially less rapidly than γ at 0 (resp. near infinity), and we write $\varphi \ll \gamma$ if and only if for every positive constant c we have

$$\lim_{t\to 0}\big(\sup_{x\in\Omega}\frac{\varphi(x, ct)}{\gamma(x, t)}\big) = 0 \ \ \big(resp. \lim_{t\to\infty}\big(\sup_{x\in\Omega}\frac{\varphi(x, ct)}{\gamma(x, t)}\big) = 0\big).$$

Remark 1.1.1. *[26] If $\varphi \ll \gamma$ near infinity, then $\forall\varepsilon > 0$ there exists $k(\varepsilon) > 0$ such that for almost all $x \in \Omega$ we have*

$$\varphi(x, t) \leqslant k(\varepsilon)\gamma(x, \varepsilon t) \ \ for\ all\ t \geqslant 0.$$

1.1.2 Musielak-Orlicz spaces

Let φ be a Musielak-Orlicz function, we define the functional

$$\varrho_{\varphi,\Omega}(u) = \int_\Omega \varphi(x, |u(x)|)dx,$$

where $u : \Omega \mapsto \mathbb{R}$ is a Lebesgue measurable function.
In the following the measurability of a function $u : \Omega \mapsto \mathbb{R}$ means the Lebesgue measurability.
The set

$$K_\varphi(\Omega) = \{u : \Omega \mapsto \mathbb{R},\ \text{measurable} :\ \varrho_{\varphi,\Omega}(u) < \infty\}$$

is called the Musielak-Orlicz class.
The Musielak-Orlicz space $L_\varphi(\Omega)$ is the vector space generated by $K_\varphi(\Omega)$, that is $L_\varphi(\Omega)$ is the smallest linear space containing the set $K_\varphi(\Omega)$.
Equivalently:

$$L_\varphi(\Omega) = \{u : \Omega \mapsto \mathbb{R},\ \text{measurable} :\ \varrho_{\varphi,\Omega}(\frac{u}{\lambda}) < +\infty\ ,\ \ \text{for some } \lambda > 0\}.$$

$K\varphi(\Omega)$ is a convex subset of $L\varphi(\Omega)$.
If $\Omega = \mathbb{R}^N$ then $L_\varphi(\mathbb{R}^N)$ is denoted by L_φ.
Let

$$\psi(x,s) = sup\{st - \varphi(x,t) \ /t \geqslant 0\}.$$

That is, ψ is the Musielak-Orlicz function complementary to $\varphi(x,t)$ in the sense of Young with respect to the variable s.
For two complementary Musielak-Orlicz functions φ and ψ the following inequality is called the Young inequality [70]

$$t.s \leqslant \varphi(x,t) + \psi(x,s) \text{ for all s, t} \geqslant 0\ , x \in \Omega. \tag{1.1.1}$$

If $s = a(x,t)$ then

$$t.a(x,t) = \varphi(x,t) + \psi(x, a(x,t)) \text{ for all t} \geqslant 0\ , x \in \Omega. \tag{1.1.2}$$

In the space $L_\varphi(\Omega)$ we define the following two norms:

$$||u||_{\varphi,\Omega} = \inf\{\lambda > 0 : \varrho_{\varphi,\Omega}(\frac{u}{\lambda}) \leqslant 1\}$$

which is called the Luxemburg norm and the so-called Orlicz norm by :

$$|||u|||_{\varphi,\Omega} = \sup_{||v||_{\psi,\Omega} \leqslant 1} \int_\Omega |u(x)v(x)|dx,$$

where ψ is the Musielak-Orlicz function complementary to φ. These two norms are equivalent[70].
For two complementary Musielak-Orlicz functions φ and ψ let $u \in L_\varphi(\Omega)$ and $v \in L_\psi(\Omega)$, we have the Hölder inequality [70]

$$|\int_\Omega u(x)v(x)dx| \leqslant ||u||_{\varphi,\Omega}\, |||v|||_{\psi,\Omega}\,. \tag{1.1.3}$$

In $L_\varphi(\Omega)$ we have the relation with the norm and the modular:

$$|||u|||_{\varphi,\Omega} \leqslant \varrho_{\varphi,\Omega}\,(u) + 1. \tag{1.1.4}$$

$$||u||_{\varphi,\Omega} \leqslant \varrho_{\varphi,\Omega}\,(u) \ \text{, if } ||u||_{\varphi,\Omega} > 1. \tag{1.1.5}$$

$$||u||_{\varphi,\Omega} \geqslant \varrho_{\varphi,\Omega}\,(u) \ \text{, if } ||u||_{\varphi,\Omega} \leqslant 1. \tag{1.1.6}$$

If $\Omega = \mathbb{R}^N$ then two norms $||.||_{\varphi,\mathbb{R}^N}$ and $|||.|||_{\varphi,\mathbb{R}^N}$ are denoted respectively by $||.||_{\varphi}$. and $|||.|||_{\varphi}$.
We say that a sequence of function $u_n \in L_\varphi(\Omega)$ is modular convergent to $u \in L_\varphi(\Omega)$ if there exists a constant $k > 0$ such that

$$\lim_{n\to+\infty} \varrho_{\varphi,\Omega}\left(\frac{u_n - u}{k}\right) = 0.$$

If φ satisfies the $\triangle_2$ condition, then modular convergence coincides with norm convergence.
The closure in $L_\varphi(\Omega)$ of the set of bounded measurable functions with compact support in $\bar{\Omega}$ is denoted by $E_\varphi(\Omega)$ and it is a separable space.
The equality $K_\varphi(\Omega) = E_\varphi(\Omega) = L_\varphi(\Omega)$ holds if and only if φ satisfie the $\triangle_2$ condition, for all t or for large t, according to whether Ω has infinite measure or not.
The dual of $E_\varphi(\Omega)$ can be identified with $L_\psi(\Omega)$ by means of the pairing $\int_\Omega u(x)v(x)dx$ and the dual norm on $L_\psi(\Omega)$ is equivalent to $||.||_\psi$.
The space $L_\varphi(\Omega)$ is reflexive if and only if φ and ψ satisfies the $\triangle_2$ condition, for all t or for t large according to whether Ω has infinite measure or not.

Lemma 1.1.1. *[39] Let φ be a Musielak-Orlicz function and f_n, f, g are measurable functions.*
(a) If $f_n \longrightarrow f$, almost everywhere , then $\varrho_{\varphi,\Omega}(f) \leqslant \liminf_{n\to+\infty} \varrho_{\varphi,\Omega}(f_n)$.
(b) If $|f_n| \nearrow |f|,\ almost\ everywhere,\ then\ \varrho_{\varphi,\Omega}(f) = \lim_{n\to+\infty} \varrho_{\varphi,\Omega}(f_n)$.
(c) If $f_n \longrightarrow f$, almost everywhere, $|f_n| \leqslant |g|,\ almost\ everywhere,\ and\ \varrho_{\varphi,\Omega}(\lambda g) < \infty$ *for every $\lambda > 0$, then $f_n \to f$ strongly in $L_\varphi(\Omega)$.*

Theorem 1.1.1. *[39] Let φ be a Musielak-Orlicz function.*
(a) $||f||_{\varphi,\Omega} = ||\ |f|\ ||_{\varphi,\Omega}$ for all $f \in L_\varphi(\Omega)$.
(b) If $f \in L_\varphi(\Omega)$, g a measurable function and $0 \leqslant |g| \leqslant |f|$, almost everywhere, then:

$$g \in L_\varphi(\Omega)\ \ and\ \ ||g||_{\varphi,\Omega} \leqslant ||f||_{\varphi,\Omega}.$$

(c) If $f_n \to f$, almost everywhere, then: $||f||_{\varphi,\Omega} \leqslant \liminf_{n\to+\infty} ||f_n||_{\varphi,\Omega}$.
(d) If $|f_n| \nearrow |f|$, almost everywhere with $f_n \in L_\varphi(\Omega)$ and $\sup_n ||f_n||_{\varphi,\Omega} < \infty$ then:

$$f \in L_\varphi(\Omega)\ \ and\ \ ||f_n||_{\varphi,\Omega} \nearrow ||f||_{\varphi,\Omega}.$$

1.1.3 Musielak-Orlicz-Sobolev spaces

For any fixed non-negative integer m we define

$$W^m L_\varphi(\Omega) = \{u \in L_\varphi(\Omega) : \forall\, |\alpha| \leqslant m,\ D^\alpha u \in L_\varphi(\Omega)\},$$

where $\alpha = (\alpha_1, \alpha_2, ..., \alpha_n)$ with non-negative integer α_i, $|\alpha| = |\alpha_1| + |\alpha_2| + ...|\alpha_n|$ and $D^\alpha u = \frac{\partial^{|\alpha|}}{\partial x_1^{\alpha_1}.....\partial x_n^{\alpha_n}}$ denote the distributional derivatives of u.
The $W^m L_\varphi(\Omega)$ is called the Musielak-Orlicz-Sobolev space.
For $u \in W^m L_\varphi(\Omega)$ let:

$$\bar{\varrho}^m_{\varphi,\Omega}(u) = \sum_{|\alpha|\leqslant m} \varrho_{\varphi,\Omega}(D^\alpha u)$$

and

$$||u||^m_{\varphi,\Omega} = inf\{\lambda > 0 : \bar{\varrho}_{\varphi,\Omega}(\frac{u}{\lambda}) \leqslant 1\}.$$

These functionals are a convex modular and a norm on $W^m L_\varphi(\Omega)$, respectively, and the pair ($W^m L_\varphi(\Omega)$,$||u||^m_{\varphi,\Omega}$) is a Banach space if φ satisfies the following condition [70]: there exists a constant $c > 0$ such that

$$\inf_{x\in\Omega} \varphi(x,1) \geqslant c. \tag{1.1.7}$$

We say that a sequence of functions $u_n \in W^m L_\varphi(\Omega)$ is modular convergent to $u \in W^m L_\varphi(\Omega)$ if there exists a constant $k > 0$ such that:

$$\lim_{n\to+\infty} \bar{\varrho}^m_{\varphi,\Omega}(\frac{u_n - u}{k}) = 0.$$

If $\Omega = \mathbb{R}^N$ then $W^m L_\varphi(\Omega)$, $\bar{\varrho}^m_{\varphi,\Omega}(u)$ and $||u||^m_{\varphi,\Omega}$ are denoted respectively by $W^m L_\varphi$, $\bar{\varrho}^m_\varphi(u)$ and $||u||^m_\varphi$, $\forall u \in W^m L_\varphi$.

Theorem 1.1.2. *[19] Let φ and ψ be two complementary Musielak-Orlicz functions such that φ satisfies the conditions (1.1.7) and there exists a constant $A > 0$ such that for all $x, y \in \Omega$: $|x - y| \leqslant \frac{1}{2}$ we have:*

$$\frac{\varphi(x,t)}{\varphi(y,t)} \leqslant t^{\frac{A}{log(\frac{1}{|x-y|})}} \tag{1.1.8}$$

for all $t \geq 1$.
If $D \subset \Omega$ *is a bounded measurable set, then* $\int_D \varphi(x,1)dx < \infty$.
ψ *satisfies the following condition :*

$$\exists C > 0 \ : \ \psi(x,1) \leqslant C, \quad almost\ everywhere\ \ in\ \ \Omega. \tag{1.1.9}$$

Under the previous conditions, with $\Omega = \mathbb{R}^N$ *;* $C_0^\infty(\mathbb{R}^N)$ *is dense in* $L_\varphi(\mathbb{R}^N)$ *and in* $W^m L_\varphi(\mathbb{R}^N)$ *with respect to the modular topology.*

Theorem 1.1.3. *[20] Let* φ *be a Musielak-Orlicz functions which satisfies the assumptions of theorem1.1.2, with* $\Omega = \mathbb{R}^N$*. Then* $D(\mathbb{R}^N)$ *is dense in* $W^m L_\varphi(\mathbb{R}^N)$ *with respect to the modular topology.*

Lemma 1.1.2. *[63] Let* E *be a reflexive Banach space. If* $I : E \longrightarrow \mathbb{R}$ *is a convex, lower semicontinuous and coercive operator, then there is an element in* E *that minimizes* I*.*

1.2 Measures space

M designates the vector space of Radon measures. M is endowed with the weak topology for which a sequence (μ_n) converges weakly to μ, if for any continuous function f with compact support

$$\lim_{n \to +\infty} \int f d\mu_n = \int f d\mu.$$

M^+ is the cone of positive elements of M.
M_1 designates the Banach space of measures, endowed with the norm $||\mu||,$ total variation of $\mu < \infty$.
where the variation of μ for all $X \subset \mathbb{R}^N$ is defined by:

$$||\mu||(X) = sup\{\sum_1^n |\mu(X_i)| : (X_i)_{i=1 \dots n} \ is\ \ an\ \ X\ \ partiton\}.$$

By definition $||\mu||(\mathbb{R}^N) = ||\mu||$ is the total variation of μ.
M_1^+ designates the subset of M_1 consisting of positive measures.

Definition 1.2.1. *Let* $\mu \in M_1^+$*. We say that* μ *is concentrated on X if* $\mu(Y) = 0$ *for all* $\mu - measurable\ set\ Y$*, such that* $Y \subset X^c$*.*

1.3 Variable Exponent Lebesgue and Sobolev Spaces.

In this section, we recall some results about the Sobolev spaces with variable exponent.
Let Ω be a bounded open subset of $I\!R^N$, $(N \geq 2)$. We denote

$$C_+(\overline{\Omega}) = \{continuous\ \ function \quad p(\cdot) : \overline{\Omega} \longrightarrow I\!R \quad \text{such that} \quad 1 < p_- \leqslant p(x) \leqslant p_+ \leqslant N\},$$

where

$$p_- = \min\{p(x)\ /\ x \in \overline{\Omega}\} \qquad \text{and} \qquad p_+ = \max\{p(x)\ /\ x \in \overline{\Omega}\}.$$

We define the variable exponent Lebesgue space for $p(\cdot) \in C_+(\overline{\Omega})$ by

$$L^{p(x)}(\Omega) = \{u : \Omega \longrightarrow I\!R \quad \text{measurable}\ /\ \int_\Omega |u(x)|^{p(x)}\, dx < \infty\}.$$

The space $L^{p(x)}(\Omega)$ under the norm

$$\|u\|_{p(x)} = \inf\left\{\lambda > 0, \quad \int_\Omega \left|\frac{u(x)}{\lambda}\right|^{p(x)} dx \leq 1\right\}$$

is a uniformly convex Banach space, and therefore reflexive. We denote by $L^{p'(x)}(\Omega)$ the conjugate space of $L^{p(x)}(\Omega)$ where $\frac{1}{p(x)} + \frac{1}{p'(x)} = 1$ (see [42], [80]).

Proposition 1.3.1. ***(See [42], [80]) (Generalized Hölder inequality)***
(i) *For any $u \in L^{p(x)}(\Omega)$ and $v \in L^{p'(x)}(\Omega)$, we have*

$$\left|\int_\Omega u\, v\, dx\right| \leq \Big(\frac{1}{p_-} + \frac{1}{p'_-}\Big)\|u\|_{p(x)}\, \|v\|_{p'(x)}\ .$$

(ii) *For all $p_1(\cdot),\ p_2(\cdot) \in C_+(\overline{\Omega})$ such that $p_1(x) \leq p_2(x)$, we have $L^{p_2(x)}(\Omega) \hookrightarrow L^{p_1(x)}(\Omega)$ and the embedding is continuous.*

Proposition 1.3.2. ***(See [42], [80]).***
If we denote

$$\rho(u) = \int_\Omega |u|^{p(x)}\, dx \quad \forall u \in L^{p(x)}(\Omega),$$

then, the following assertions holds
(i) $\|u\|_{p(x)} < 1 \quad (resp, = 1, > 1) \quad \Longleftrightarrow \quad \rho(u) < 1 \quad (resp,\ = 1, > 1),$

(ii) $\|u\|_{p(x)} > 1 \ \Rightarrow\ \|u\|^{p_-}_{p(x)} \leq \rho(u) \leq \|u\|^{p_+}_{p(x)}$ *and* $\|u\|_{p(x)} < 1 \ \Rightarrow\ \|u\|^{p_+}_{p(x)} \leq \rho(u) \leq \|u\|^{p_-}_{p(x)}$,
(iii) $\|u\|_{p(x)} \to 0 \quad \Leftrightarrow \quad \rho(u) \to 0$ *and* $\|u\|_{p(x)} \to \infty \quad \Leftrightarrow \quad \rho(u) \to \infty$.

Now, we define the variable exponent Sobolev space by

$$W^{1,p(x)}(\Omega) = \{\ u \in L^{p(x)}(\Omega) \ \text{and} \ |\nabla u| \in L^{p(x)}(\Omega)\},$$

normed by

$$\|u\|_{1,p(x)} = \|u\|_{p(x)} + \|\nabla u\|_{p(x)} \qquad \forall u \in W^{1,p(x)}(\Omega).$$

We denote by $W_0^{1,p(x)}(\Omega)$ the closure of $C_0^{\infty}(\Omega)$ in $W^{1,p(x)}(\Omega)$ and we define the Sobolev exponent by $p^*(x) = \frac{Np(x)}{N-p(x)}$ for $p(x) < N$.

Proposition 1.3.3. ***(See [42, 48])***
(i) *Assuming* $1 < p_- \leq p_+ < \infty$, *the spaces* $W^{1,p(x)}(\Omega)$ *and* $W_0^{1,p(x)}(\Omega)$ *are separable and reflexive Banach spaces.*
(ii) *If* $q(\cdot) \in C_+(\bar{\Omega})$ *and* $q(x) < p^*(x)$ *for any* $x \in \Omega$, *then the embedding* $W_0^{1,p(x)}(\Omega) \hookrightarrow\hookrightarrow L^{q(x)}(\Omega)$ *is continuous and compact.*
(iii) *Poincaré inequality: there exists a constant* $C > 0$, *such that*

$$\|u\|_{p(x)} \leq C\, \|\nabla u\|_{p(x)} \qquad \forall u \in W_0^{1,p(x)}(\Omega).$$

(vi) *Sobolev inequality: there exists an other constant* $C > 0$, *such that*

$$\|u\|_{p*(x)} \leq C\, \|\nabla u\|_{p(x)} \qquad \forall u \in W_0^{1,p(x)}(\Omega).$$

Remark 1.3.1. *By (iii) of the Proposition 1.3.3, we deduce that* $\|\nabla u\|_{p(x)}$ *and* $\|u\|_{1,p(x)}$ *are equivalent norms in* $W_0^{1,p(x)}(\Omega)$.

Definition 1.3.1. *For all* $k > 0$ *and* $s \in I\!R$, *the truncation function* $T_k(.)$ *can be defined by*

$$T_k(s) = \begin{cases} s & \textit{if} \quad |s| \leq k, \\ k\dfrac{s}{|s|} & \textit{if} \quad |s| > k. \end{cases}$$

and we define

$$T_0^{1,p(x)}(\Omega) := \{\textit{measurable function } u \quad \textit{such that} \quad T_k(u) \in W_0^{1,p(x)}(\Omega) \quad \forall k > 0\}.$$

Lemma 1.3.1. *(See [16]) Let $g \in L^{r(x)}(\Omega)$ and $g_n \in L^{r(x)}(\Omega)$ with $\|g_n\|_{r(x)} \leq C$ for $1 < r(x) < \infty$.*
If $g_n(x) \to g(x)$ a.e. on Ω, then $g_n \rightharpoonup g$ weakly in $L^{r(x)}(\Omega)$.

Lemma 1.3.2. *(See [16])*
Let $u \in W_0^{1,p(x)}(\Omega)$ then $T_k(u) \in W_0^{1,p(x)}(\Omega)$ for all $k > 0$. Moreover, we have

$$T_k(u) \longrightarrow u \qquad in \quad W_0^{1,p(x)}(\Omega) \quad as \quad k \to \infty.$$

Lemma 1.3.3. *(See [17]) Let $p(.)$ be a continuous function in $C_+(\overline{\Omega})$ and u a function in $W_0^{1,p(x)}(\Omega)$. Suppose $2 - \frac{1}{N} < p^- \leqslant p^+ \leqslant N$, and that there exists a constant C_1 such that*

$$\int_{\{k \leqslant |u| \leqslant k+1\}} | \bigtriangledown u|^{p(x)} dx \leqslant C_1, \quad \forall \ k > 0.$$

Then there exists a constant C_2, depending on C_1, such that

$$\|u\|_{W_0^{1,q(x)}(\Omega)} \leqslant C_2,$$

for all continuous functions $q(.)$ on $\overline{\Omega}$ satisfying

$$1 \leqslant q(x) < \frac{N}{N-1}(p(x) - 1), \quad for \ all \ x \in \overline{\Omega}.$$

1.4 The weighted variable exponents Lebesgue and Sobolev spaces.

Let Ω be a bounded open subset of $I\!R^N$, $N \geq 2$.
For all $p \in C_+(\overline{\Omega})$, we define $p^+ = \sup_{x \in \Omega} p(x)$ and $p_- = \inf_{x \in \Omega} p(x)$.
We say that $p(.)$ is log-Hölder continuous in Ω if

$$|p(x) - p(y)| \leq \frac{C}{|log|x-y||} \qquad \forall x, y \in \overline{\Omega} \quad \text{such that} \quad |x - y| < \frac{1}{2},$$

Let ω be a function defined in Ω, ω is called a weight function in Ω if it is measurable and strictly positive a.e. in Ω.

Let $p \in C_+(\overline{\Omega})$ and ω be a weighted function in Ω. The weighted variable exponents Lebesgue space $L^{p(x)}(\Omega,\omega)$, consists of all measurable real valued functions such that

$$L^{p(x)}(\Omega,\omega) = \{u : \Omega \to I\!R, measurable : \int_\Omega \omega(x)|u(x)|^{p(x)}dx < \infty\},$$

provided with the Luxemburg norm

$$\|u\|_{L^{p(x)}(\Omega,\omega)} = \inf\Big\{\mu > 0 : \int_\Omega \omega(x)|\frac{u(x)}{\mu}|^{p(x)}dx \leq 1\Big\}$$

$L^{p(x)}(\Omega,\omega)$ is a reflexive Banach space.

Lemma 1.4.1. *[64] For all function $u \in L^{p(x)}(\Omega,\omega)$, we denoted*

$$\rho_{\Omega,\omega}(u) = \int_\Omega \omega(x)|u(x)|^{p(x)}dx.$$

Then
(i) $\rho_{\Omega,\omega}(u) > 1 \quad (= 1; < 1) \Leftrightarrow \|u\|_{L^{p(x)}(\Omega,\omega)} > 1 \quad (= 1; < 1)$, *respectively.*
(ii) If $\|u\|_{L^{p(x)}(\Omega,\omega)} > 1$ *then* $\|u\|^{p-}_{L^{p(x)}(\Omega,\omega)} \leq \rho_{\Omega,\omega}(u) \leq \|u\|^{p^+}_{L^{p(x)}(\Omega,\omega)}$.
(iii) If $\|u\|_{L^{p(x)}(\Omega,\omega)} < 1$ *then* $\|u\|^{p^+}_{L^{p(x)}(\Omega,\omega)} \leq \rho_{\Omega,\omega}(u) \leq \|u\|^{p-}_{L^{p(x)}(\Omega,\omega)}$.

We define the weighted variable exponents Sobolev space by

$$W^{1,p(x)}(\Omega,\omega) = \Big\{u \in L^{p(x)}(\Omega) : \frac{\partial u}{\partial x_i} \in L^{p(x)}(\Omega,\omega), i = 1,...,N\Big\},$$

provided with the norme

$$\|u\|_{W^{1,p(x)}(\Omega,\omega)} = \|u\|_{L^{p(x)}(\Omega)} + \sum_{i=1}^{N} \|\frac{\partial u}{\partial x_i}\|_{L^{p(x)}(\Omega,\omega)}$$

which is equivalent to the Luxemburg norme

$$|||u|||_{W^{1,p(x)}(\Omega,\omega)} = \inf\Bigg\{\mu > 0 : \int_\Omega \Bigg(|\frac{u}{\mu}|^{p(x)} + \omega(x)\sum_{i=1}^{N} |\frac{\frac{\partial u}{\partial x_i}}{\mu}|^{p(x)}\Bigg)dx \leq 1\Bigg\}.$$

Let ω a weight function such that the following conditions:
(w1) $\omega \in L^1_{loc}(\Omega)$ and $\omega^{\frac{-1}{p(x)-1}} \in L^1_{loc}(\Omega)$;
(w2) $\omega^{-s(x)} \in L^1(\Omega,\omega)$ with $s(x) \in \big(\frac{N}{p(x)},\infty\big) \cap [\frac{1}{p(x)-1},\infty\big)$.

Remark 1.4.1. *[64] Let ω be a positive measurable and finite function such the condition (W_1) holds. Then the space $\big(W^{1,p(x)}(\Omega,\omega), \|.\|_{W^{1,p(x)}(\Omega,\omega)}\big)$ is a reflexive Banach.*

Remark 1.4.2. *[64] If we set*

$$\rho^1_{(\Omega,\ \omega)}(u) = \int_\Omega (|u|^{p(x)} \ + \omega(x)|\nabla u|^{p(x)})dx \quad \forall u \in L^{p(x)}(\Omega)$$

We have

$$min(\|u\|^{p-}_{W^{1,p(x)}(\Omega,\omega)}, \|u\|^{p+}_{W^{1,p(x)}(\Omega,\omega)}) \leqslant \rho^1_{(\Omega,\ \omega)}(u) \leqslant max(\|u\|^{p-}_{W^{1,p(x)}(\Omega,\omega)}, \|u\|^{p+}_{W^{1,p(x)}(\Omega,\omega)})$$

Corollary 1.4.1. *[64] Let $p \in C_+(\overline{\Omega})$ satisfy log-Hölder continuity condition. If (W_1) and (W_2) hold, then*

$$(\exists C > 0), (\forall u \in C_0^\infty(\Omega)) : ||u||_{L^{p(x)}(\Omega)} \leqslant C||\nabla u||_{L^{p(x)}(\Omega,\omega)}$$

Definition 1.4.1. *Let $C_+(\overline{\Omega})$. The class $A_{p(.)}$ consists of those weight ω for which*

$$\sup_{B\in\mathbf{B}} |B|^{-\mathbf{p}_B}||\omega||_{L^1(B)}||\frac{1}{\omega}||_{L^{\frac{p'(.)}{p(.)}}(B)} < \infty,$$

where B denotes the set of balls in $\mathbb{R}^N$ and $\mathbf{p}_B = \left(\frac{1}{|B|}\int_B \frac{1}{p(x)}dx\right)^{-1}$

Theorem 1.4.1. *[59] Let $p(.) \in P^{log}(\mathbb{R}^N)$, $1 < p^- \leqslant p^+ < \infty$, and $\omega \in A_{p(.)}$. Then $C_0^\infty(\mathbb{R}^N)$ is dense in $W^{1,p(x)}(\mathbb{R}^N,\omega)$.*

1.5 Capacity

Definitions 1.5.1. *let E be a topological space and T the classe of Borel sets in E, and a function $C : T \to [0, +\infty]$.*
1) C is called a capacity if the following axioms are satisfied:
i)$C(\emptyset) = 0$.
ii)$X \subset Y \Rightarrow C(X) \leqslant C(Y)$, for all X and Y in T.
iii) For all sequences $(X_n) \subset T$:

$$C(\bigcup_n X_n) \leqslant \sum_n C(X_n).$$

2) C is called a outer capacity if for all $X \in T$:

$$C(X) = \inf\{C(O) : O \supset X, \;\; O \;\; open\}.$$

3) C is called an interior capacity if for all $X \subset T$:

$$C(X) = \sup\{C(K) : K \subset X, \quad K \; compact\}.$$

4)A property, that holds true except perhaps on a set of capacity zero, is said to be true C-quasi everywhere, (abbreviated C-q.e).
5) Let $\; f \;\; and \; (f_n) \;$ be real-valued functions finite C-q.e. We say that (f_n) converges to $\; f$ in C-capacity if:

$$\forall \varepsilon > 0, \;\; \lim_{n \to +\infty} C(\{x : \; |f_n(x) - f(x)| > \varepsilon\}) = 0.$$

6) $f \;\; and \;\; (f_n)$ are real-valued finite functions C-q.e. We say that (f_n) converges to $\; f \;$ C-quasi- uniformly, (abbreviated C-q.u) if :

$$(\forall \varepsilon > 0), (\exists \, X \in T) \; : \; C(X) < \varepsilon \;\; and \; (f_n) \;\; converges \; to \; f \; uniformly \; on \; X^c.$$

Propositions 1.5.1. *[59] For $E \subset \mathbb{R}^N$, we denote*

$$S_{p(.),\omega}(E) = \{u \in W^{1,p(x)}(\mathbb{R}^N, \omega) \;\; : \;\; u \geqslant 1 \; on \; an \; open \; set \; containing \; E\}.$$

The Sobolev $(p(.), \omega)$-capacity of E is defined by

$$C_{p(.),\omega}(E) = \inf_{u \in S_{p(.),\omega}(E)} \rho_{1,p(.),\omega}(u) = \inf_{u \in S_{p(.),\omega}(E)} \int_\Omega (|u(x)|^{p(x)} + |\nabla u(x)|^{p(x)} \omega(x)) dx.$$

In case $S_{p(.),\omega}(E) = \varnothing$, we set $C_{p(.),\omega}(E) = \infty$. The $C_{p(.),\omega}$-capacity has the following proprieties.
(i) $C_{p(.),\omega}(\emptyset) = 0.$
(ii) $X \subset Y \Rightarrow C_{p(.),\omega}(X) \leqslant C_{p(.),\omega}(Y),$
(iii) *For all sequences $X_n \subset \mathbb{R}^N$:*

$$C_{p(.),\omega}(\bigcup_n X_n) \leqslant \sum_n C_{p(.),\omega}(X_n). \tag{1.5.1}$$

$(i\vee)$ *for all $X \subset \mathbb{R}^N$:*

$$C_{p(.),\omega}(X) = \inf\{C_{p(.),\omega}(O) : O \supset X, \;\; O \;\; open\}. \tag{1.5.2}$$

$(\vee)$ *for all* $X \subset \mathbb{R}^N$

$$C_{p(.),\omega}(X) = \sup\{C_{p(.),\omega}(K) : K \subset X, \quad K \textit{ compact}\}. \tag{1.5.3}$$

$(\vee i)$ *If there exists* $f \in W^{1,p(x)}(\mathbb{R}^N, \omega)$ *such that* $f = +\infty$ *on E then,*

$$C_{p(.),\omega}(E) = 0 \tag{1.5.4}$$

$(\vee ii)$ *If* (X_n) *is an increasing sequence of sets and* $\quad X = \bigcup_n X_n$, *then*

$$\lim_{n \to +\infty} C_{p(.),\omega}(X_n) = C_{p(.),\omega}(X). \tag{1.5.5}$$

$(\vee iii)$*If* X *and* Y *are subset of* $\mathbb{R}^N$, *then*

$$C_{p(.),\omega}(X \cup Y) + C_{p(.),\omega}(X \cap Y) \leqslant C_{p(.),\omega}(X) + C_{p(.),\omega}(Y). \tag{1.5.6}$$

$(i\wedge)$ *If* $\omega(x) \geqslant 1$ *for all* $x \in \mathbb{R}^N$. *then every measurable set* $X \subset \mathbb{R}^N$ *satisfies*

$$|X| \leqslant C_{p(.),\omega}(X). \tag{1.5.7}$$

Chapter 2

Capacity and Non-linear Potential in Musielak-Orlicz Spaces

2.1 Introduction

In this chapter we are going to introduce the theory of capacity in Musielak-Orlicz space. We will define the $C_{k,\varphi}$ capacity and the $D_{k,\varphi}$ capacity, prove their main properties, and establish relationship between $C_{k,\varphi}$ and $D_{k,\varphi}$, we shall introduce the theory of non-linear potential and give some of its properties. Our results generalize those of N. Aissaoui and A. Benkirane in the case of Orlicz spaces [see [7] and [8]]

2.2 Capacity in Musielak-Orlicz space

2.2.1 $C_{k,\varphi}-capacity$

Lemma 2.2.1. *Let Ω be an open set in $\mathbb{R}^N$ and φ be a Musielak-Orlicz function such that*

$$\varphi(y,t) = \int_0^t a(y,\tau)d\tau,\ \forall y \in \Omega\ and\ \ t \geqslant 0.$$

Let $u : \Omega \to \mathbb{R}$, measurable function and $\alpha > 0$, we define a measurable function $g : \Omega \to \mathbb{R}$ such that

$$g(y) = a(y, \frac{|u(y)|}{2\alpha}),\ \forall y \in \Omega.$$

If $(\frac{u}{\alpha}) \in K_\varphi(\Omega)$ then $g \in K_\psi(\Omega)$, where ψ is the Musielak-Orlicz function complementary to φ.

Proof. For all $y \in \Omega$ and $t \geqslant O : \varphi(y, 2t) = \int_0^{2t} a(y,\tau)d\tau \ \geqslant \int_t^{2t} a(y,\tau)d\tau$.
Hence $\varphi(y, 2t) \geqslant ta(y, t)$, thus for all $y \in \Omega\ : \varphi(y, \frac{|u(y)|}{\alpha}) \geqslant \frac{|u(y)|}{2\alpha} a(y, \frac{|u(y)|}{2\alpha})$.
On the other hand, we have: $\frac{|u(y)|}{2\alpha} a(y, \frac{|u(y)|}{2\alpha}) - \varphi(y, \frac{|u(y)|}{2\alpha}) = \psi(y, a(y, \frac{|u(y)|}{2\alpha}))$.
Therefore, $\psi(y, a(y, \frac{|u(y)|}{2\alpha})) \leqslant \varphi(y, \frac{|u(y)|}{\alpha}) - \varphi(y, \frac{|u(y)|}{2\alpha})$, this implies that

$$\int_\Omega \psi(y, a(y, \frac{|u(y)|}{2\alpha}))dy \leqslant \int_\Omega \varphi(y, \frac{|u(y)|}{\alpha})dy - \int_\Omega \varphi(y, \frac{|u(y)|}{2\alpha})dy,$$

then

$$\varrho_{\psi,\Omega}(g) \leqslant \varrho_{\varphi,\Omega}(\frac{u}{\alpha}) - \varrho_{\varphi,\Omega}(\frac{u}{2\alpha}).$$

Since $\varrho_{\varphi,\Omega}(\frac{u}{2\alpha}) \leqslant \frac{1}{2}\varrho_{\varphi,\Omega}(\frac{u}{\alpha})$ and $\varrho_{\varphi,\Omega}(\frac{u}{\alpha}) < \infty$, the proof is complete. $\square$

Lemma 2.2.2. *If (f_n) is a sequence in $L_\varphi(\Omega)$ such that for all $n \in \mathbb{N}$, $f_n \geqslant 0$, then*

$$||\sup_n f_n||_{\varphi,\Omega} \leqslant ||\sum_n f_n||_{\varphi,\Omega} \leqslant \sum_n ||f_n||_{\varphi,\Omega}.$$

Proof. Since $0 \leqslant \sup_n f_n \leqslant \sum\limits_n f_n$, thus $||\sup_n f_n||_{\varphi,\Omega} \leqslant ||\sum\limits_n f_n||_{\varphi,\Omega}$.
Let $g_n = \sum\limits_{k=0}^{n} f_k$ and $f = \sum\limits_n f_n$, we have

$$\frac{g_n}{\sum\limits_n ||f_n||_{\varphi,\Omega}} \nearrow \frac{f}{\sum\limits_n ||f_n||_{\varphi,\Omega}} \ \ almost\ everywhere.$$

By Lemma 1.1.1, we obtain

$$\varrho_{\varphi,\Omega}(\frac{f}{\sum\limits_n ||f_n||_{\varphi,\Omega}}) = \lim_{n\to+\infty} \varrho_{\varphi,\Omega}(\frac{g_n}{\sum\limits_n ||f_n||_{\varphi,\Omega}}) \leqslant \lim_{n\to+\infty} \varrho_{\varphi,\Omega}(\frac{g_n}{||g_n||_{\varphi,\Omega}}) \leqslant 1.$$

Then,

$$||\frac{f}{\sum\limits_n ||f_n||_{\varphi,\Omega}}||_{\varphi,\Omega} \leqslant 1.$$

Therefore ,

$$||\sum_n f_n||_{\varphi,\Omega} \leqslant \sum_n ||f_n||_{\varphi,\Omega} \, .$$

$\square$

Lemma 2.2.3. *Let φ be a Musielak-Orlicz function, which satisfies the $\triangle_2$ condition, and such that*

$$\varphi(y,t) = \int_0^t a(y,\tau)d\tau, \ \ for\ all\ y \in \Omega \ \ and \ \ t \geqslant 0.$$

Let $f \in L_\varphi(\Omega)$, such that $f \geqslant 0$, and $||f||_{\varphi,\Omega} \neq 0$. Let the measurable function $g : \Omega \to \mathbb{R}$ define by

$$\forall y \in \Omega \ \ g(y) = a(y, \frac{f(y)}{||f||_{\varphi,\Omega}}).$$

We have $\int_\Omega f(y)g(y)dy = ||f||_{\varphi,\Omega} \, |||g|||_{\psi,\Omega} \, .$

Proof. By Lemma 2.2.1, we have $g \in L_\psi(\Omega)$ and by the Hölder inequality we have

$$\int_\Omega f(x)g(x)dx \leqslant ||f||_{\varphi,\Omega} \, |||g|||_{\psi,\Omega} \, .$$

For the opposite inequality, let $h = \frac{f}{||f||_{\varphi,\Omega}}$, and $v \in L_\varphi(\Omega)$, such that $||v||_{\varphi,\Omega} \leqslant 1$.

For all $y \in \Omega$, we have

$$g(y)h(y) = \varphi(y,h(y)) + \psi(y,g(y))$$

and

$$g(y)v(y) \leqslant \varphi(y,v(y)) + \psi(y,g(y)).$$

Hence for all $y \in \Omega$:

$$g(y)v(y) \leqslant g(y)h(y) - \varphi(y, h(y)) + \varphi(y, v(y)).$$

Then

$$\int_\Omega g(y)v(y)dy \leqslant \int_\Omega g(y)h(y)dy - \int_\Omega \varphi(y, h(y))dy + \int_\Omega \varphi(y, v(y))dy.$$

Thus

$$\int_\Omega g(y)v(y)dy \leqslant \int_\Omega g(y)h(y)dy - \varrho_{\varphi,\Omega}(h) + \varrho_{\varphi,\Omega}(v).$$

We have $\varrho_{\varphi,\Omega}(v) \leqslant 1$.
On the other hand φ satisfies the $\triangle_2$ condition, then, $\varrho_{\varphi,\Omega}$ is a continuous modular[see [39] Lemma 2.4.3]. We have $||h||_{\varphi,\Omega} = 1$, then $\varrho_{\varphi,\Omega}(h) = 1$ [see [39] Lemma 2.1.14].

Thus,

$$\int_\Omega g(y)v(y)dy \leqslant \int_\Omega g(y)h(y)dy$$

implies

$$\sup_{||v||_{\varphi,\Omega}\leqslant 1} \int_\Omega g(y)v(y)dy \leqslant \int_\Omega g(y)h(y)dy.$$

Then,

$$|||g|||_{\psi,\Omega}\, ||f||_{\varphi,\Omega} \leqslant \int_\Omega f(y)g(y)dy.$$

□

Remark 2.2.1. *In the following* $\Omega = \mathbb{R}^n, \varphi$ *is a Musielak-Orlicz function, and* $L_\varphi^+ = \{\, f \in L_\varphi \;/\; f \geqslant 0\}$.

Theorem 2.2.1. *Let* k *be a positive integrable function on* $\mathbb{R}^N$.
For all $X \subset \mathbb{R}^N$, *we put* $C_{k,\varphi}(X) = \inf\{||f||_\varphi : f \in L_\varphi \;\; and\; k * f \geqslant 1 \; on \;\; X\}$, *where* $k * f$ *is the convolution of* k *and* f.
$C_{k,\varphi}$ *is an outer capacity.*

Remark 2.2.2. *Let* $B_{k,\varphi}(X) = \inf\{||f||_\varphi : f \in L_\varphi^+ \;\; and \;\; k * f \geqslant 1 \;\; on\; X\}$, *then*

$$C_{k,\varphi}(X) = B_{k,\varphi}(X).$$

Indeed, it is obvious that $C_{k,\varphi}(X) \leqslant B_{k,\varphi}(X)$.
On the other hand, let $f \in L_\varphi$, *then* $|f| \in L_\varphi^+$ *and if* $k * f \geqslant 1$ *on* X, *then* $k * |f| \geqslant 1$ *on* X. *Thus* $B_{k,\varphi}(X) \leqslant ||f||_\varphi$; *and therefore* $B_{k,\varphi}(X) \leqslant C_{k,\varphi}(X)$.

Proof. **of Theorem 2.2.1**. It is obvious that $C_{k,\varphi}(\emptyset) = 0$ and $C_{k,\varphi}(X) \leqslant C_{k,\varphi}(Y)$ if $X \subset Y$.
Let $(X_i) \subset T$, so that $\sum_i C_{k,\varphi}(X_i) < +\infty$, then $(\forall i \in \mathbb{N})$ $C_{k,\varphi}(X_i) < +\infty$.
Thus , $(\forall i \in \mathbb{N})(\forall \varepsilon > 0)$, $(\exists f_i \in L^+_\varphi)$ so that $k * f_i \geqslant 1$ *on* X_i and $||f_i||_\varphi \leqslant C_{k,\varphi}(X_i) + \frac{\varepsilon}{2^i}$.
Let $f = \sup_i f_i$. By Lemma 2.2.2, we have:

$$||f||_\varphi \leqslant \sum_i ||f_i||_\varphi.$$

We can write,

$$||f||_\varphi \leqslant \sum_i C_{k,\varphi}(X_i) + \varepsilon.$$

Which implies that, $f \in L_\varphi$.
Since $k * f \geqslant 1$ *on* $\bigcup_i X_i$ thus

$$C_{k,\varphi}(\bigcup_i X_i) \leqslant \sum_i C_{k,\varphi}(X_i) + \varepsilon, \ \ \forall \varepsilon > 0.$$

Hence

$$C_{k,\varphi}(\bigcup_i X_i) \leqslant \sum_i C_{k,\varphi}(X_i).$$

It remains to show that $C_{k,\varphi}$ is outer. Let $X \subset \mathbb{R}^N$, we have:

$$C_{k,\varphi}(X) \leqslant \inf\{C_{k,\varphi}(O) : O \supset X, \ \ O \ is \ open\}.$$

For the reverse inequality, if $C_{k,\varphi}(X) = +\infty$ there is nothing to show.
Assume that $C_{k,\varphi}(X) < +\infty$, and let $0 < \varepsilon < 1$, then $\exists\, g \in L^+_\varphi$ so that $k * g \geqslant 1$ *on* X and $||g||_\varphi \leqslant C_{k,\varphi}(X) + \varepsilon$.
Let $g_\varepsilon = \frac{g}{1-\varepsilon}$ and $O_\varepsilon = \{x : (k * g_\varepsilon)(x) > 1\}$, thus O_ε is open and

$$\forall x \in X; \ \ (k * g_\varepsilon)(x) \geqslant \frac{1}{1-\varepsilon} > 1.$$

Hence $X \subset O_\varepsilon$.
On the other hand, we have $C_{k,\varphi}(O_\varepsilon) \leqslant ||g_\varepsilon||_\varphi$, and we deduce that

$$C_{k,\varphi}(O_\varepsilon) \leqslant \frac{1}{1-\varepsilon}||g||_\varphi \leqslant \frac{1}{1-\varepsilon}[C_{k,\varphi}(X) + \varepsilon].$$

Therefore

$$\inf\{C(O) : O \supset X,\ \ O\ \ open\} \leqslant \frac{1}{1-\varepsilon}[C_{k,\varphi}(X)+\varepsilon],\ \ \forall\varepsilon > 0.$$

Thus

$$\inf\{C(O) : O \supset X,\ \ O\ \ open\} \leqslant C_{k,\varphi}(X).$$

□

Theorem 2.2.2. *1) If there exists $f \in L_\varphi$ such that $|k * f| = +\infty$ on X, then $C_{k,\varphi}(X) = 0$.*
2) If $C_{k,\varphi}(X) = 0$ then there exists $f \in L^+_\varphi$ such that $k * f = +\infty$ on X.

Proof. 1) Let $f \in L_\varphi$ such that $|k * f| = +\infty$ on X, then $\forall\alpha > 0,\ |k * f| \geqslant \alpha$ on X. Thus $C_{k,\varphi}(X) \leqslant \frac{||f||_\varphi}{\alpha},\ \ \forall\alpha > 0$; this means that $C_{k,\varphi}(X) = 0$.
2) If $C_{k,\varphi}(X) = 0$ then $\forall i \in \mathbb{N},\ \exists f_i \in L^+_\varphi :\ \ k*f_i \geqslant 1\ \ on\ \ X\ \ and\ \ ||f_i||_\varphi \leqslant 2^{-i}$.
Let $f = \sum\limits_i f_i$. By Lemma 2.2.2, $||f||_\varphi \leqslant \sum\limits_i ||f_i||_\varphi$, then $||f||_\varphi < +\infty$.
We deduce that $f \in L^+_\varphi$ and $k * f = +\infty\ \ on\ X$. □

Theorem 2.2.3. *Consider the following propositions :*
i) $f_n \longrightarrow f$ strongly in L_φ.
ii) $k * f_n \longrightarrow k * f$ in $C_{k,\varphi}-capacity$.
iii) There is a subsequence $(f_{n_j})_j$ such that : $k * f_{n_j} \longrightarrow k * f\ \ C_{k,\varphi}-q.u.$
iv) $k * f_{n_j} \longrightarrow k * f\ \ in\ C_{k,\varphi}-q.e.$
We have

$$i) \Rightarrow\ ii) \Rightarrow\ iii) \Rightarrow iv).$$

Proof. We show $i) \Rightarrow\ ii)$.
By Theorem 2.2.2, we have $k * f\ \ and\ \ k * f_n$ are finite $C_{k,\varphi}-q.e,\ \ \forall n.$
Let $\varepsilon > 0$; then

$$C_{k,\varphi}(\{x\ : |k * f_n - k * f|(x) > \varepsilon\}) \leqslant \frac{||f_n - f||_\varphi}{\varepsilon}.$$

We show $ii) \Rightarrow\ iii)$.
Let $\varepsilon > 0\ \ \exists\, f_{n_j}$ such that

$$C_{k,\varphi}(\{x\ :\ |k * f_{n_j} - k * f|(x) > 2^{-j}\}) < \varepsilon.2^{-j}.$$

We put

$$E_j = \{x \ : \ |k * f_{n_j} - k * f|(x) > 2^{-j}\} \quad \text{and} \quad G_m = \bigcup_{j \geqslant m} E_j.$$

We have $C_{k,\varphi}(G_m) \leqslant \sum\limits_{j \geqslant m} \varepsilon.2^{-j} < \varepsilon.$

On the other hand :

$$\forall x \in (G_m)^c, \ \forall \ j \geqslant m : \ \ |k * f_{n_j} - k * f|(x) \leqslant 2^{-j}.$$

Thus $k * f_{n_j} \longrightarrow k * f \ \ C_{k,\varphi} - q.u.$

We show $iii) \Rightarrow \ iv)$.

We have $\forall j \in \mathbb{N}, \exists X_j : C_{k,\varphi}(X_j) \leqslant \frac{1}{j}$ and $k * f_{n_j} \longrightarrow k * f \ \ on \ \ (X_j)^c$.

We put $X = \bigcap_j X_j$, then $C_{k,\varphi}(X) = 0 \ \ and \ \ k * f_{n_j} \longrightarrow k * f \ \ on \ X^c$. □

Theorem 2.2.4. *Let φ be a Musielak-Orlicz function that satisfies the $\triangle_2$ condition, and (f_n) a sequence in L_φ such that $\sum\limits_n |f_n| \in L_\varphi$.*

Then,

$$\sum_n (k * f_n) = k * (\sum_n f_n) \ \ C_{k,\varphi} - q.e.$$

Proof. **First step**: Assume that $f_n \geqslant 0 \ \ \forall n \in \mathbb{N}$, and let $g_n = \sum\limits_{i=1}^{n} f_i \ \ and \ \ f = \sum\limits_n f_n$.

We have $g_n \to f \ \ almost\ everywhere$ and $g_n \leqslant f$.

On the other hand, $\varrho_\varphi(\lambda f) < \infty \ \ \forall \lambda > 0$ because $f \in L_\varphi$ and φ satisfy the $\triangle_2$ condition [see [39] paragraph 2.5].

By (c) of Lemma 1.1.1 we have

$$g_n \to f \ strongly\ in\ L_\varphi.$$

Theorem 2.2.3 implies that there is a subsequence (g_{n_i}) such that $k * g_{n_i} \to k * f$ in $C_{k,\varphi}$-q.e.

Since $f_n \geqslant 0, \ \ \forall n \in \mathbb{N} \ \ \ k * g_n \to k * f$ in $C_{k,\varphi}$-q.e.

Second step: If f_n is any sign, then $\sum\limits_n f_n^+$ and $\sum\limits_n f_n^-$ are in L_φ because $|\sum\limits_n f_n^+| \leqslant \sum\limits_n |f_n|$, $|\sum\limits_n f_n^-| \leqslant \sum\limits_n |f_n|$ and $\sum\limits_n |f_n| \in L_\varphi$.

By the first step the result follows. □

Theorem 2.2.5. *Let (K_n) be a decreasing sequence of compact and $K = \bigcap_n K_n$. Then $\lim\limits_{n\to+\infty} C_{k,\varphi}(K_n) = C_{k,\varphi}(K)$.*

Proof. First, we observe that $C_{k,\varphi}(K) \leqslant \lim\limits_{n\to+\infty} C_{k,\varphi}(K_n)$.
On the other hand, let O be an open set containing K. By the compactness of K, $K_i \subset O$ for all sufficiently large i. Therefore $\lim\limits_{n\to+\infty} C_{k,\varphi}(K_n) \leqslant C_{k,\varphi}(O)$, and since $C_{k,\varphi}$ is an outer capacity, we obtain the claim by taking infimum over open set O containing K. ☐

Theorem 2.2.6. *Let φ be a Musielak-Orlicz function, uniformly convex that satisfies the $\triangle_2$ condition.*
If $f_n, f \in L_\varphi$ such that $f_n \rightharpoonup f$ weakly in L_φ, then:

$$\liminf(k * f_n) \leqslant (k * f) \leqslant \limsup(k * f_n) \;\; C_{k,\varphi}-q.e.$$

Proof. $(L_\varphi\,,\;||.||)$ is uniformly convex therefore reflexive. By the Banach-Saks theorem, there is a subsequence denoted again by (f_n) such that the sequence $g_n = \frac{1}{n}\sum\limits_{i=1}^{n} f_i$ converges to f strongly in L_φ. By Theorem 2.2.3, there is a subsequence of (g_n) denoted again (g_n) such that

$$\lim_{n\to+\infty}(k * g_n) = (k * f) \;\; C_{k,\varphi}-q.e.$$

On the other hand,

$$\liminf(k * f_n) \leqslant \lim_{n\to+\infty}(k * g_n) \;\; .$$

Therefore,

$$\lim inf(k * f_n) \leqslant (k * f) \quad C_{k,\varphi}-q.e.$$

For the second inequality, it suffices to replace f_n by $(-f_n)$ in the first inequality. ☐

Theorem 2.2.7. *Let φ be a Musielak-Orlicz function, uniformly convex that satisfies the $\triangle_2$ condition. If (X_n) is an increasing sequence of sets and $X = \bigcup\limits_n X_n$, then*

$$\lim_{n\to+\infty} C_{k,\varphi}(X_n) = C_{k,\varphi}(X).$$

Proof. We have $\lim\limits_{n\to+\infty} C_{k,\varphi}(X_n) \leqslant C_{k,\varphi}(X)$. For the reverse inequality, if $\lim\limits_{n\to+\infty} C_{k,\varphi}(X) = +\infty$, there is nothing to show.
Assuming that $\lim\limits_{n\to+\infty} C_{k,\varphi}(X) < +\infty$, we have

$$\forall n \in \mathbb{N},\ \exists f_n \in L_\varphi^+ : k * f_n \geqslant 1 \text{ } on \ X_n \text{ } and \ ||f_n||_\varphi \leqslant C_{k,\varphi}(X_n) + \frac{1}{n}.$$

Thus, (f_n) is a bounded sequence in L_φ.
On the other hand, L_φ is uniformly convex, then it is reflexive because φ is uniformly convex and satisfies the $\triangle_2$ condition, [see [39] Remark 2.4.15]. Hence there exists a subsequence which is denoted again by (f_n), and converges weakly to a function $f \in L_\varphi$. We have ,

$$\forall n \in \mathbb{N} : k * f_n \geqslant 1 \ \ on \ \ X_n,\ C_{k,\varphi} - q.e.$$

Therefore by Theorem 2.2.6,

$$k * f \geqslant 1 \ \ on \ \ X, \ \ C_{k,\varphi} - q.e.$$

Let Y a subset of X where $k * f \geqslant 1$, then $C_{k,\varphi}(X) = C_{k,\varphi}(Y)$.
On the other hand we know that

$$\varphi(y,t) = \int_0^t a(y,\tau)d\tau, \quad \text{for all } \ y \in R^N \ \ and \ t \geqslant 0.$$

Let the function $g : \mathbb{R}^N \to \mathbb{R}$ defined by $g(y) = a(y, \frac{|f(y)|}{||f||_\varphi})$ for all $y \in \mathbb{R}^N$.
By Lemma 2.2.1, $g \in L_\psi$, and since φ satisfies the $\triangle_2$ condition, we have $L_\psi = (L_\varphi)^*$.
Thus,

$$\int f_n(y)g(y)dy \to \int f(y)g(y)dy.$$

By Lemma 2.2.3, we have

$$\int f(y)g(y)dy = ||f||_\varphi |||g|||_\psi.$$

By the Hölder inequality we have:

$$\int f_n(y)g(y)dy \leq ||f_n||_\varphi |||g|||_\psi.$$

Therefore,

$$||f||_{\varphi} \leqslant \lim_{n\to+\infty} ||f_n||_{\varphi} \leqslant \lim_{n\to+\infty} (C_{k,\varphi}(X_n) + \frac{1}{n}).$$

Thus,

$$C_{k,\varphi}(X) \leqslant \lim_{n\to+\infty} C_{k,\varphi}(X_n).$$

□

Corollary 2.2.1. *Let φ be a Musielak-Orlicz function, uniformly convex, that satisfies the $\triangle_2$ condition.*
Let $E_n \subset \mathbb{R}^N$, then $C_{k,\varphi}(\liminf E_n) \leqslant \liminf C_{k,\varphi}(E_n)$.

Proof. Let $E = \liminf E_n$, we have $E = \bigcup\limits_n (\bigcap_{i\geqslant n} E_i)$.
We put $G_n = \bigcap_{i\geqslant n} E_i$. Thus a sequence (G_n) is increasing and by Theorem 2.2.7, $C_{k,\varphi}(E) = \lim_n C_{k,\varphi}(G_n)$. We have $C_{k,\varphi}(G_n) \leqslant C_{k,\varphi}(E_n)$; therefore

$$C_{k,\varphi}(E) \leqslant \liminf C_{k,\varphi}(E_n).$$

□

Theorem 2.2.8. *Let φ be a Musielak-Orlicz function which satisfies the assumptions of Theorem 1.1.2.*
If φ satisfies the $\triangle_2$ condition, then for each $f \in L_\varphi$, there is a $C_{k,\varphi}$-quasicontinuous function $g \in L_\varphi$ such that $k * f = g \;\; C_{k,\varphi} - q.e.$

Proof. Let $f \in L_\varphi$, by Theorem 1.1.2, there exists a sequence (f_n) in $C_0^\infty(\mathbb{R}^N)$ such that $f_n \longrightarrow f$ in L_φ.
By Theorem 2.2.3, there exists a subsequence of (f_n) denoted again by (f_n) such that

$$k * f_n \longrightarrow k * f \;\; C_\varphi - q.u.$$

Since k is integrable function and f_n is continuous $\forall n,$ then $k * f_n$ is continuous. Thus, the proof is complete. □

Definition 2.2.1. *In the terminology of Choquet, C is called a capacity if it satisfies the following four properties:*
i) $C(\emptyset) = 0$.
ii) C is increasing.
iii) If (E_n) a increasing sequence of set, then $\sup_n C(X_n) = C(\bigcup\limits_n X_n)$.
iv) If (K_n) is a decreasing sequence of compact, then $\inf_n C(K_n) = C(\bigcap_n K_n)$.

Remark 2.2.3. *Let φ be a Musielak-Orlicz function, uniformly convex, that satisfies the $\triangle_2$ condition. Under Theorems 2.2.1, 2.2.5 and 2.2.7 $C_{k,\varphi}$ is a capacity, in the sense of Choquet.*

Definition 2.2.2. *Let C a capacity in the sense of Choquet, and $X \subset \mathbb{R}^N$.*
X is called capacitable if

$$C(X) = \sup\{C(K) : K \subset X, \quad K \ \ compact\}.$$

Theorem 2.2.9. *Let φ be a Musielak-Orlicz function, uniformly convex that satisfies the $\triangle_2$ condition. Then all analytic sets are $C_{k,\varphi}$- capacitable .*

Proof. It is an immediate consequence of Choquet theorem [36]. ☐

2.2.2 Capacity in terms of measure

Theorem 2.2.10. *Let φ be a Musielak-Orlicz function, k a positive integrable function on $\mathbb{R}^N$, and $X \subset \mathbb{R}^N$ is a μ-measurable set, for all positive measure μ.*
We put
$D_{k,\varphi}(X) = \sup\{||\mu|| \ : \ \mu \in M_1^+, \mu \, is \ \ concentrated \ on \ X \ \ and \ ||k * \mu||_\psi \leqslant 1\}$
where $(k * \mu)(x) = \int k(x-y)d\mu(y)$.
Then, $D_{k,\varphi}$ is an interior capacity.

Proof. It is clear that $D_{k,\varphi}(\emptyset) = 0$ and $D_{k,\varphi}(X) \leqslant D_{k,\varphi}(Y)$ if $X \subset Y$.
Let $\mu \in M_1^+$, (X_n) a sequence of μ-measurable set and $\mu_n = \mu|_{X_n}$ defined by :

$$\mu_n(Y) = \mu(X_n \cap Y), \ \ for \ all \ \ \mu - measurable \ \ set \ \ Y.$$

First we assume that the X_n are pairwise disjoint, then

$$\mu(\bigcup_n X_n) = \sum_n \mu(X_n).$$

If μ is concentrated on $\bigcup\limits_n X_n$ and $||k * \mu||_\psi \leqslant 1$, then $\forall n;\ \ \mu_n \in M_1^+;\ \mu_n$ is concentrated on X_n and $||k * \mu_n||_\psi \leqslant 1$.
On the other hand, we have

$$||\mu|| = \sum_n ||\mu_n|| \leqslant \sum_n D_{k,\varphi}(X_n).$$

Thus,

$$D_{k,\varphi}(\bigcup_n X_n) \leqslant \sum_n D_{k,\varphi}(X_n).$$

If the X_n are not pairwise disjoint, then by the first case and the fact that $D_{k,\varphi}$ is increasing, we have

$$D_{k,\varphi}(\bigcup_n X_n) \leqslant \sum_n D_{k,\varphi}(X_n).$$

It remains to show that $D_{k,\varphi}$ is interior.
By monotonicity we have

$$\sup\{D_{k,\varphi}(K) : K \subset X,\ K\ \ compact\} \leqslant D_{k,\varphi}(X).$$

Let $\mu \in M_1^+$ and X a μ-measurable set such that μ is concentrated on X and $||k * \mu||_\psi \leqslant 1$.
Let a compact K such that $K \subset X$, then $\mu|_K \in M_1^+$, $\mu|_K$ is concentrated on K and $||k * \mu|_K||_\psi \leqslant 1$. Therefore,

$$||\mu|_K||_\psi \leqslant D_{k,\varphi}(K).$$

On the other hand,

$$\sup\{||\mu \setminus_K || : K \subset X,\ K\ is\ compact\} = ||\mu||.$$

Thus,

$$D_{k,\varphi}(X) \leqslant \sup\{D_{k,\varphi}(K) : K \subset X,\ K\ \ compact\}.$$

$\square$

Theorem 2.2.11. *1) $D^*_{k,\varphi}$ is the outer capacity associated to $D_{k,\varphi}$, defined by:*

$$D^*_{k,\varphi}(X) = \inf\{\ D_{k,\varphi}(O)\ :\ O\ \ open\ and\ X \subset O\}.$$

Then,

$$D^*_{k,\varphi}(X) = C_{k,\varphi}(X).$$

2) If φ is a Musielak-Orlicz function, uniformly convex that satisfies the $\triangle_2$ condition, then for all analytic set X we have :

$$D_{k,\varphi}(X) = C_{k,\varphi}(X).$$

Proof. It is the same as that given in [7], theorem 11.

□

Theorem 2.2.12. *Let φ be a Musielak-Orlicz function.*
Let K a compact of $\mathbb{R}^N$. The following assertions are equivalents.
1)$C_{k,\varphi}(K) = \infty$.
2) $D^*_{k,\varphi}(K) = \infty$.
3)$D_{k,\varphi}(K) = \infty$.
4)There exists $x_0 \in K$ such that $k(x_0 - y) = 0$ a.e.

Proof. It is the same as that given in [8], Theorem 5.

□

2.3 Non-linear potential in Musielak-Orlicz space

Let φ be a Musielak-Orlicz function. In this section, we propose to study the following variational problem: let X a subset of $\mathbb{R}^N$ such that $C_{k,\varphi}(X) < \infty$. Is there exists $f_0 \in L^+_\varphi$ such that $k * f_0 \geqslant 1 \;\; C_{k,\varphi} - q.e$ on X, and

$$||f_0||_\varphi = inf\{||f||_\varphi : f \in L^+_\varphi \; and \;\; k * f \geqslant 1 \; C_{k,\varphi} - q.e \; on \; X\}?$$

Theorem 2.3.1. *Let φ be a Musielak-Orlicz function, uniformly convex that satisfies the $\triangle_2$ condition. Consider X a subset of $\mathbb{R}^N$ such that $C_{k,\varphi}(X) < \infty$.*
Let $\Omega_X = \{f \in L^+_\varphi : k * f \geqslant 1 \;\; C_{k,\varphi} - q.e \; on \; X\}$.
There exits a unique $f_0 \in \Omega_X$ such that:
i) $||f_0||_\varphi = inf\{||g||_\varphi : g \in \Omega_X\}$.
ii) $k * f_0 \geqslant 1$ on X and $||f_0||_\varphi = C_{k,\varphi}(X)$.
iii) If $C_{k,\varphi}(X) > 0$, then for all $g \in L_\varphi$ such that $k * g \geqslant 0$ on X:

$$\int a(x, \frac{f_0(x)}{||f_0||_\varphi}) g(x) dx \geqslant 0,$$

where the function a(x,.) is the derivative of the function $\varphi(x, .)$.

Proof. i) Let the function $\theta : \Omega_X \longrightarrow] - \infty; +\infty]$, defined by

$$\theta(f) = ||f||_\varphi \, ; \forall f \in L_\varphi,$$

θ is convex, lower semi continuous on Ω_X and coercive, the space L_φ is reflexive. By Theorem 2.2.6, Ω_X is weakly closed in L_φ. On the other hand, Ω_X is convex, then there exists $f_0 \in \Omega_X$ such that:

$$||f_0||_\varphi = inf\{||g||_\varphi : g \in \Omega_X\}.$$

Now suppose that there exist two minimal g_1 and g_2 with $g_1 \neq g_2$, so that $||g_1 - g_2||_\varphi > 0$.
Let $G_1 = \frac{g_1}{||g_1||_\varphi}$ and $G_2 = \frac{g_2}{||g_2||_\varphi}$. By the uniform convexity of the norm we have

$$||G_1 + G_2||_\varphi < 2$$

then

$$||\frac{1}{2}g_1 + \frac{1}{2}g_2||_\varphi < ||g_1||_\varphi$$

which is the contradiction since $(\frac{1}{2}g_1 + \frac{1}{2}g_2) \in \Omega_X$.
ii) Let Y be a subset of X where $k * f_0 < 1$. Then, $C_{k,\varphi}(X) = C_{k,\varphi}(X - Y)$. Since $k * f_0 \geqslant 1$ on X-Y, then $C_{k,\varphi}(X - Y) \leqslant ||f_0||_\varphi$.
On the other hand, we have $\{g \in L^+_\varphi : k * g \geqslant 1 \ on \ X\} \subset \Omega_X$; then $||f_0||_\varphi \leqslant C_{k,\varphi}(X)$.
iii) Let $g \in L_\varphi$ such that $k * g \geqslant 0$ on X. Then for all $t \geqslant 0$:

$$k * (f_0 + tg) \geqslant 1 \ \ C_{k,\varphi} - q.e \ \ on \ X \ and \ \ (f_0 + tg) \in \ L_\varphi.$$

Then,

$$||f_0 + tg||_\varphi \geqslant ||f_0||_\varphi.$$

Therefore,

$$||\frac{1}{||f_0||_\varphi}(f_0 + tg)||_\varphi \geqslant 1.$$

Thus,

$$\varrho_\varphi(\frac{1}{||f_0||_\varphi}(f_0 + tg)) \geqslant 1.$$

On the other hand,

$$\varrho_\varphi(\frac{1}{||f_0||_\varphi}f_0) \leqslant 1.$$

Then, for all $t > 0$

$$\int \frac{1}{t}[\varphi(x, \frac{|f_0 + tg|(x)}{||f_0||_\varphi}) - \varphi(x, \frac{|f_0(x)|}{||f_0||_\varphi})]dx \geqslant 0.$$

Let $c(x,t) = \varphi(x, \frac{|f_0+tg|(x)}{||f_0||_\varphi})$. The function $x \longmapsto c(x,t)$ is in L^1 for all $t \in \mathbb{R}$. On the other hand,

$$\frac{\partial c}{\partial t}(x,t) = a(x, \frac{|f_0+tg|(x)}{||f_0||_\varphi}).(\frac{g(x)}{||f_0||_\varphi}).sng(f_0+tg)(x).$$

For $0 < t < 1$ we have:

$$|\frac{\partial c}{\partial t}(x,t)| \leqslant a(x, \frac{|f_0+g|(x)}{||f||_\varphi}).(\frac{g(x)}{||f_0||_\varphi}).$$

By Lemma 2.2.1, the function: $x \longrightarrow a(x, \frac{|f_0+g|(x)}{||f_0||_\varphi})$ is in L_ψ.
Then the function: $x \mapsto a(x, \frac{|f_0+g|(x)}{||f_0||_\varphi}).(\frac{g(x)}{||f_0||_\varphi})$ is in L^1.
By Lebesgue's theorem

$$\lim_{t\to 0^+} \int \frac{1}{t}[\varphi(x, \frac{|f_0+tg|(x)}{||f_0||_\varphi}) - \varphi(x, \frac{|f_0(x)|}{||f_0||_\varphi})]dx = \frac{1}{||f_0||_\varphi} \int a(x, \frac{|f_0(x)|}{||f_0||_\varphi})g(x)dx \geqslant 0.$$

$\square$

Definition 2.3.1. *Let φ be a Musielak-Orlicz function uniformly convex which satisfies the $\triangle_2$ condition and X a subset of $\mathbb{R}^N$ such that $C_{k,\varphi}(X) < \infty$. Let $f_0 \in L_\varphi^+$ such that*

$$||f_0||_\varphi = inf\{||f||_\varphi : f \in L_\varphi^+ \text{ and } k * f \geqslant 1 \ C_{k,\varphi} - q.e \text{ on } X\}.$$

f_0 is called a distribution function of X, and $k * f_0$ is called a non-linear potential of X for the $C_{k,\varphi}$ capacity.

Remark 2.3.1. *Under the assumptions of Theorem 2.3.1, if $C_{k,\varphi}(X) > 0$, then for all $g \in L_\varphi$ such that $k * g = 0$ on X:*

$$\int a(x, \frac{f_0(x)}{||f_0||_\varphi})g(x)dx = 0.$$

Theorem 2.3.2. *Let φ be a Musielak-Orlicz function uniformly convex which satisfies the $\triangle_2$ condition. Let $X \subset \mathbb{R}^N$ such that $0 < C_{k,\varphi}(X) < \infty$ and f_0 the distribution function of X for the $C_{k,\varphi}$ capacity. For all $g \in L_\varphi$*

$$|\int a(x, \frac{f_0(x)}{||f_0||_\varphi})g(x)dx| \leqslant K_\varphi \sup_{x\in X} |(k*g)(x)|.||f_0||_\varphi,$$

where K_φ is a constant that depends only on φ.

Proof. The inequality is obvious if $\sup_{x\in X}|(k*g)(x)| = +\infty$.
On the other hand, if $\sup_{x\in X}|(k*g)(x)| = 0$, then by Remark 2.3.1 we have

$$\int a(x, \frac{f_0(x)}{||f_0||_\varphi}).g(x)dx = 0.$$

If $0 < \alpha = \sup_{x\in X}|(k*g)(x)| < +\infty$, then $k*(f_0 - \frac{g}{\alpha})(x) \geqslant 0$ for all $x \in X$.
By Theorem 2.3.1, we have:

$$\int a(x, \frac{f_0(x)}{||f_0||_\varphi}).(f_0 - \frac{g}{\alpha})(x)dx \geqslant 0.$$

Thus,

$$\int a(x, \frac{f_0(x)}{||f_0||_\varphi}).g(x)dx \leqslant \alpha \int a(x, \frac{f_0(x)}{||f_0||_\varphi}).f_0(x)dx.$$

On the other hand, we have for all $x \in \mathbb{R}^N$ and $t \geqslant 0$:

$$\varphi(x,t) = \int_0^t a(x,t)dt \geqslant \int_{\frac{t}{2}}^t a(x,t)dt \geqslant (\frac{t}{2})a(x, \frac{t}{2}).$$

Then,

$$a(x, \frac{f_0(x)}{||f_0||_\varphi}).\frac{f_0(x)}{||f_0||_\varphi} \leqslant \varphi(x, 2\frac{f_0(x)}{||f_0||_\varphi}) \leqslant K_\varphi \varphi(x, \frac{f_0(x)}{||f_0||_\varphi})$$

because φ satisfies the $\triangle_2$ condition.
Therefore,

$$\int a(x, \frac{f_0(x)}{||f_0||_\varphi}).g(x)dx \leqslant \alpha.K_\varphi.\varrho_\varphi(\frac{f_0}{||f_0||_\varphi})||f_0||_\varphi.$$

Since $\varrho_\varphi(\frac{f_0}{||f_0||_\varphi}) \leqslant 1$, we get

$$\int a(x, \frac{f_0(x)}{||f_0||_\varphi}).g(x)dx \leqslant K_\varphi \sup_{x\in X}|(k*g)(x)|||f_0||_\varphi.$$

For the second inequality, it suffices to replace g by $(-g)$ in the last.

□

Theorem 2.3.3. *Let φ be a Musielak-Orlicz function, uniformly convex which satisfies the $\triangle_2$ condition. Let $(X_i)_i \subset \mathbb{R}^N$. For each i, f_i is the distribution function of X_i for the $C_{k,\varphi}$ capacity. Let $X \subset \mathbb{R}^N$ and f is its distribution function for the $C_{k,\varphi}$ capacity.*
If $X \subset \liminf X_i$ and $\lim C_{k,\varphi}(X_i) = C_{k,\varphi}(X)$ then, $f_i \longrightarrow f$ in L_φ.
We have the same result, particularly if $(X_i)_i$ is increasing and $X = \bigcup\limits_i X_i$ or $(X_i)_i$ is a decreasing sequence of compact and $X = \bigcap_i X_i$.

Proof. The sequence $(f_i)_i$ is bounded in L_φ. Since the space L_φ is reflexive, there exists a subsequence denoted again by $(f_i)_i$ which converges weakly in L_φ^+ to a function g in L_φ. By Theorem 2.2.6, $k * g \geqslant 1$ on X $C_{k,\varphi} - q.e.$ Therefore,

$$C_{k,\varphi}(X) \leqslant ||g||_\varphi.$$

On the other hand for all $h \in L_\psi$

$$\int f_i(x)h(x)dx \longrightarrow \int g(x)h(x)dx.$$

By Hölder inequality, we have:

$$\int f_i(x)h(x)dx \leqslant ||f_i||_\varphi |||h|||_\psi.$$

Thus, $\int g(x)h(x)dx \leqslant \liminf ||f_i||_\varphi |||h|||_\psi \leqslant ||f||_\varphi |||h|||_\psi$.
Let the function $h : x \longrightarrow a(x, \frac{g(x)}{||g||_\varphi})$ for all $x \in \mathbb{R}^N$.
By Lemma 2.2.1, $h \in L_\psi$, and by Lemma 2.2.3

$$||g||_\varphi |||h|||_\psi = \int g(x)h(x)dx \leqslant ||f||_\varphi |||h|||_\psi.$$

Then,

$$||g||_\varphi \leqslant C_{k,\varphi}(X).$$

Thus, $||g||_\varphi = C_{k,\varphi}(X)$ and by theorem 2.3.1 $f = g$.
On the other hand, f is the unique adhesion value of the sequence $(f_i)_i$ for the topology $\sigma(L_\varphi, L_\psi)$. Then, $f_i \rightharpoonup f$ weakly in L_φ.
Since L_φ is uniformly convex and $\lim\limits_i ||f_i||_{L_\varphi} = ||f||_{L_\varphi}$ then

$$f_i \longrightarrow f \ strongly \ in \ L_\varphi.$$

□

Theorem 2.3.4. *Letφ be a Musielak-Orlicz function.*
Let F be a closed subset of $\mathbb{R}^N$ such that $D_{k,\varphi}(F) < \infty$. For all $r \in \mathbb{R}_+^*$: $F_r = F \cap \{x \in \mathbb{R}^N : |x| > r\}$.
If $\lim\limits_{r \to +\infty} D_{k,\varphi}(F_r) = 0$ then there exists a measure $\mu \in M_1^+$ such that μ is concentrated on F; $||k * \mu||_\psi \leqslant 1$ and $D_{k,\varphi}(F) = ||\mu||$.
μ is called a distribution measure of F for $D_{k,\varphi}$.
Particularly, if K is a compact such that $D_{k,\varphi}(K) < \infty$ then K possesses a distribution measure for $D_{k,\varphi}$.

Proof. It is the same as that given in [8], Theorem 4. □

Chapter 3

Capacity, theorem of H Brezis and F.E. Browder type in Musielak Orlicz Sobolev spaces and application

The first section of this chapter is devoted to the study of the capacity theory in Musielak-Orlicz-Sobolev space, we study basic's properties, including monotonicity, countable subadditivity and several convergence results, we prove that each Musielak-Orlicz-Sobolev function has a quasi-continuous representative. In the second section, we generalize the theorem of H Brezis and F.E. Browder in the setting of Musielak-Orlicz-Sobolev space $W^m L_\varphi(\Omega)$, which extends the previous result of H. Brezis and F.E. Browder [35]. In the third section we make two applications. In the first we study an unilateral problem, in the second we define and study Musielak-Orlicz-Sobolev spaces with zero boundary values. This allows us to prove that the Dirichlet energy integral has a minimizer in the Musielak-Orlicz-Sobolev spaces case.

3.1 Preliminary lemma

Lemma 3.1.1. *Let φ be a Musielak-Orlicz function which satisfies the condition (1.1.7). If $u,\ v \in W^1 L_\varphi(\Omega)$, then $\max\{u, v\}$ and $min\{u, v\}$ are in $W^1 L_\varphi(\Omega)$*

with

$$\bigtriangledown \max\{u,v\}(x) = \begin{cases} \bigtriangledown u(x), & \textit{for almost every } x \in \{u \geqslant v\}; \\ \bigtriangledown v(x), & \textit{for almost every } x \in \{v \geqslant u\}; \end{cases}$$

and

$$\bigtriangledown \min\{u,v\}(x) = \begin{cases} \bigtriangledown u(x), & \textit{for almost every } x \in \{u \leqslant v\}; \\ \bigtriangledown v(x), & \textit{for almost every } x \in \{v \leqslant u\}. \end{cases}$$

In particular, $|u|$ *belongs to* $W^1L_\varphi(\Omega)$ *and* $|\bigtriangledown |u|| = |\bigtriangledown u|$ *almost everywhere in* Ω*.*

Proof. It suffices to prove the assertions for $\max\{u,v\}$ since $\min\{u,v\} = -\max\{-u,-v\}$. We have $\max\{u,v\} \leqslant |u|+|v|$ almost everywhere in Ω, and $(|u|+|v|) \in L_\varphi(\Omega)$, then by Theorem 1.1.1 we obtain $\max\{u,v\} \in L_\varphi(\Omega)$. On the other hand we have $|\bigtriangledown \max(u,v)| \leqslant |\nabla u|+|\nabla v|$ almost everywhere in Ω, and $(|\nabla u|+|\nabla v|) \in L_\varphi(\Omega)$, then by Theorem1.1.1 we obtain $\nabla \max\{u,v\} \in L_\varphi(\Omega)$.

Thus

$$\max\{u,v\} \in W^1L_\varphi(\Omega).$$

For $|u|$ it suffices to note that $|u| = \max\{u,0\} - \min\{u,0\}$. □

3.2 Capacity in Musielak-Orlicz-Sobolev space

In this section, $\Omega = \mathbb{R}^N$ and φ is a Musielak-Orlicz function which satisfies the condition (1.1.7).

Definition 3.2.1. *The Sobolev* φ*-capacity of the set,* $E \subset \mathbb{R}^N$ *is defined by :*

$$C_\varphi(E) = \inf_{u \in A_\varphi(E)} \bar{\rho}_{m,\varphi}(u).$$

Where

$$A_\varphi(E) = \{u \in W^mL_\varphi \ : \ u \geqslant 1 \textit{ on an open set containing } E \textit{ and } u \geqslant 0\}.$$

If $A_\varphi(E) = \emptyset$ *we set* $C_\varphi(E) = \infty$*.*
Functions belonging to $A_\varphi(E)$ *are called admissible functions for E.*

Remark 3.2.1. *In the definition of the capacity C_φ, we can restrict ourselves to those admissible functions u for which,* $0 \leqslant u \leqslant 1$.
Indeed, if $A^{'}_\varphi(E) = \{u \in A_\varphi(E)\ :\ 0 \leqslant u \leqslant 1\}$, *then* $A^{'}_\varphi(E) \subset A_\varphi(E)$ *implies*

$$C_\varphi(E) \leqslant \inf_{u \in A^{'}_\varphi(E)} \bar{\rho}_{m,\varphi}(u).$$

For the reverse inequality, let $\varepsilon > 0$ *and take* $u \in A_\varphi(E)$ *such that*

$$\bar{\rho}_{m,\varphi}(u) \leqslant C_\varphi(E) + \varepsilon.$$

Then by lemma 3.1.1 , we have $v = max(0, min(u, 1))$ *belongs to* $A^{'}_\varphi(E)$.
Therefore,

$$\inf_{\omega \in A^{'}_\varphi(E)} \bar{\rho}_{m,\varphi}(\omega) \leqslant \bar{\rho}_{m,\varphi}(v) \leqslant \bar{\rho}_{m,\varphi}(u) \leqslant C_\varphi(E) + \varepsilon.$$

Letting $\varepsilon \longrightarrow 0$, *we obtain*

$$\inf_{\omega \in A^{'}_\varphi(E)} \bar{\rho}_{m,\varphi}(\omega) \leqslant C_\varphi(E).$$

This completes the proof.

Theorem 3.2.1. *Let* $E \subset \mathbb{R}^N$.
If there exists $f \in W^m L_\varphi$ *such that* $f = +\infty$ *on E then,* $C_\varphi(E) = 0$.

Proof. If there exists $f \in W^m L_\varphi$ such that $f = +\infty$ on E, then $f \geqslant \alpha \ on\ E$ for all $\alpha > 0$. Therefore, $\forall\ \alpha > 0\ \ : C_\varphi(E) \leqslant \bar{\rho}_{m,\varphi}(\frac{f}{\alpha})$.
Let $\alpha > 1$, we have $\bar{\rho}_{m,\varphi}(\frac{f}{\alpha}) \leqslant \frac{1}{\alpha}\bar{\rho}_{m,\varphi}(f)$, then $0 \leqslant C_\varphi(E) \leqslant \frac{1}{\alpha}\bar{\rho}_{m,\varphi}(f)$.
Letting $\alpha \longrightarrow +\infty$, we obtain $C_\varphi(E) = 0$. □

Theorem 3.2.2. *Let's consider the following propositions:*
i) $f_n \longrightarrow f\ \ in\ \ W^m L_\varphi$.
ii) $f_n \longrightarrow f$ *in* $C_\varphi - capacity$.
iii) there is a subsequence (f_{n_j}) *such that :* $f_{n_j} \longrightarrow f, \quad C_\varphi - q.u.$
iv) $f_{n_j} \longrightarrow f, \quad C_\varphi - q.e.$
We have $i) \Rightarrow\ ii) \Rightarrow\ iii) \Rightarrow iv)$

Proof. Let show that $i) \Rightarrow ii)$.
By theorem 3.2.1 we have f and f_n are finite for every n; $C_\varphi - q.e.$
Let $\varepsilon > 0$, we have

$$C_\varphi(\{x \; : \; |f_n - f|(x) > \varepsilon\}) \leqslant \bar{\rho}_{m,\varphi}(\frac{f_n - f}{\varepsilon}).$$

Since $f_n \longrightarrow f \;\; in \;\; W^m L_\varphi(\Omega)$, then

$$(\forall \varepsilon > 0) : \;\; \bar{\rho}_{m,\varphi}(\frac{f_n - f}{\varepsilon}) \longrightarrow 0.$$

Therefore,

$$\lim_{n \to +\infty} C_\varphi(\{x \; : \; |f_n - f|(x) > \varepsilon\}) = 0.$$

Let show that $ii) \Rightarrow \; iii)$.
Let $\varepsilon > 0 \;\; \exists \, f_{n_j}$ such that $C_\varphi(\{x \, : \, |f_{n_j} - f|(x) > 2^{-j}\}) < \varepsilon.2^{-j}$.
We put

$$E_j = \{x \; : \; |f_{n_j} - f|(x) > 2^{-j}\} \;\; \text{and} \;\; G_m = \bigcup_{j \geqslant m} E_j,$$

we have $C_\varphi(G_m) \leqslant \sum_{j \geqslant m} \varepsilon.2^{-j} < \varepsilon$.
On the other hand,

$$(\forall x \in (G_m)^c) \; : \; |f_{n_j} - f|(x) \leqslant 2^{-j}, (\forall j \geqslant m).$$

Thus

$$f_{n_j} \longrightarrow f \;\; C_\varphi - q.u.$$

Let show that $iii) \Rightarrow \; iv)$.
We have $\forall j \in \mathbb{N}, \exists X_j : C_\varphi(X_j) \leqslant \frac{1}{j}$ and $f_{n_j} \longrightarrow f \;\; on \;\; (X_j)^c$.
We put $X = \bigcap_j X_j$, then $C_\varphi(X) = 0 \;\; and \;\; f_{n_j} \longrightarrow f \;\; on \; X^c$. □

Theorem 3.2.3. *Let φ be a Musielak-Orlicz function, uniformly convex that satisfies the $\triangle_2$ condition.*
If f_n , $f \in W^m L_\varphi$ such that $f_n \rightharpoonup f$ weakly in $W^m L_\varphi$, then

$$\liminf (f_n)(x) \leqslant f(x) \leqslant \limsup f_n(x) \;\; C_\varphi - q.e.$$

Proof. $(W^m L_\varphi\ ,\ ||.||)$ is uniformly convex, therefore reflexive. By the Banach-Saks theorem, there is a subsequence denoted again (f_n) such that the sequence; $g_n = \frac{1}{n}\sum_{i=1}^{n} f_i$ converge to f strongly in $W^m L_\varphi$. By theorem 3.2.2, there is a subsequence of (g_n) denoted again (g_n) such that

$$\lim_{n\to+\infty} g_n(x) = f(x)\ \ C_\varphi - q.e.$$

On the other hand,

$$\liminf f_n(x) \leqslant \lim_{n\to+\infty} g_n(x)\ \ .$$

Therefore,

$$\liminf_{n\to+\infty} f_n(x) \leqslant f(x)\quad C_\varphi - q.e.$$

For the second inequality, it suffices to replace f_n by $(-f_n)$ in the first inequality.

□

Theorem 3.2.4. *Let φ be a Musielak-Orlicz function, uniformly convex which satisfies the $\triangle_2$ condition.*
Let (X_n) be an increasing sequence of sets and $X = \bigcup_n X_n$.
Then, $\lim_{n\to+\infty} C_\varphi(X_n) = C_\varphi(X)$.

Proof. We have $\lim_{n\to+\infty} C_\varphi(X_n) \leqslant C_\varphi(X)$. For the reverse inequality, if $\lim_{n\to+\infty} C_\varphi(X_n) = +\infty$, there is nothing to show.
Assuming that $\lim_{n\to+\infty} C_\varphi(X_n) < +\infty$, we have

$$\forall n \in \mathbb{N},\ \exists f_n \in W^m L_\varphi\ :\ \ f_n \geqslant 1\ on\ \ X_n\ \ and\ \ \bar{\varrho}_{m,\varphi}(f_n) \leqslant C_\varphi(X_n) + \frac{1}{n}.$$

Now (f_n) is a bounded sequence in $W^m L_\varphi$, hence there exists a subsequence, which we denote again by (f_n), which converges weakly to a function $f \in W^m L_\varphi$. Thus

$$\bar{\rho}_{m,\varphi}(f) \leqslant \liminf_n \bar{\varrho}_{m,\varphi}(f_n).$$

On the other hand by theorem 3.2.3, we have

$$\forall n \in \mathbb{N}\ : f \geqslant 1\ \ on\ \ X_n\ \ , C_\varphi - q.e.$$

Therefore, $f \geqslant 1 \ \ on \ \ X \ \ C_\varphi - q.e.$
Let Y be a subset of X where $f \geqslant 1$, then $C_\varphi(X) = C_\varphi(Y)$.
Thus,

$$\bar{\rho}_{m,\varphi}(f) \leqslant \lim_n (C_\varphi(X_n) + \frac{1}{n}).$$

Hence

$$C_\varphi(X) \leqslant \lim_n (C_\varphi(X_n).$$

□

Theorem 3.2.5. *Let φ be a Musielak-Orlicz function, uniformly convex which satisfies the $\triangle_2$ condition. C_φ is an outer capacity.*

Proof. It is obvious that $C_\varphi(\emptyset) = 0$ and $C_\varphi(X) \leqslant C_\varphi(Y)$ if $X \subset Y$.
To prove the countable sub-additivity, suppose that E_i , $i = 1, 2, \ldots,$ are subsets of $\mathbb{R}^N$, let $\varepsilon > 0$. We may assume that $\sum_i C_{k,\varphi}(X_i) < +\infty$, then

$$C_{k,\varphi}(X_i) < +\infty; \ \ \forall i \in \mathbb{N}.$$

Next we choose $u_i \in A_\varphi(E_i)$ so that

$$\bar{\rho}_{m,\varphi}(u_i) \leqslant C_\varphi(E_i) + \varepsilon.2^{-i}; \ \ \forall i \in \mathbb{N}.$$

Let $k \in \mathbb{N}$ and $v_k = \max_{1 \leqslant i \leqslant k} u_i$. By lemma 3.1.1 we have $v_k \in A_\varphi(\bigcup_{i=1}^{k} E_i)$.
Thus,

$$\bar{\rho}_{m,\varphi}(v_k) \leqslant \sum_{i=1}^{k} \bar{\rho}_{m,\varphi}(u_i) \leqslant \sum_{i=1}^{k} (C_\varphi(E_i) + \varepsilon.2^{-i}) \leqslant \sum_{i=1}^{k} C_\varphi(E_i) + (\varepsilon(1 - (\frac{1}{2})^k).$$

Then,

$$C_\varphi(\bigcup_{i=1}^{k} E_i) \leqslant \sum_{i=1}^{k} C_\varphi(E_i) + \varepsilon.$$

Letting $\varepsilon \to 0$, we obtain

$$C_\varphi(\bigcup_{i=1}^{k} E_i) \leqslant \sum_{i=1}^{k} C_\varphi(E_i) \leqslant \sum_{i=1}^{\infty} C_\varphi(E_i).$$

Since $(\bigcup_{i=1}^{k} E_i)$ increases to $(\bigcup_{i=1}^{\infty} E_i)$, by theorem 3.2.4 we obtain:

$$C_\varphi(\bigcup_{i=1}^{\infty} E_i) \leqslant \sum_{i=1}^{\infty} C_\varphi(E_i).$$

It remains to prove that C_φ is outer. Indeed, by monotonicity we have:

$$(\forall\, E \subset \mathbb{R}^N\ )\ : C_\varphi(E) \leqslant \inf\{C_\varphi(O) : O \supset E,\ \ O\ \ open\}.$$

For the reverse inequality, if $C_\varphi(E) = +\infty$, there is nothing to show.
Assume that $C_\varphi(E) < +\infty$, let $\varepsilon > 0$ and take $u \in A_\varphi(E)$ such that

$$\bar{\rho}_{m,\varphi}(u) \leqslant C_\varphi(E) + \varepsilon.$$

Since $u \in A_\varphi(E)$, there is an open set O containing E such that $u \geqslant 1\ on\ O$, which implies that

$$C_\varphi(O) \leqslant \bar{\rho}_{m,\varphi}(u) \leqslant C_\varphi(E) + \varepsilon.$$

The inequality follows by letting $\varepsilon \to 0$. □

Theorem 3.2.6. *Let (K_n) be a decreasing sequence of compacts and $K = \bigcap_n K_n$. Then,*

$$\lim_{n\to+\infty} C_\varphi(K_n) = C_\varphi(K).$$

Proof. First, we observe that $C_\varphi(K) \leqslant \lim_{n\to+\infty} C_\varphi(K_n)$.
On the other hand, let O be an open set containing K. By the compactness of K, $K_i \subset O$ for all sufficiently large i. Therefore $\lim_{n\to+\infty} C_\varphi(K_n) \leqslant C_\varphi(O)$, and since C_φ is an outer capacity, we obtain the claim by taking infimum over all open set O containing K. □

Theorem 3.2.7. *Let φ be a Musielak-Orlicz function.*

$$(\exists c > 0)(\forall X \subset \mathbb{R}^N) :\ \ |X| \leqslant c.C_\varphi(X),$$

where $|X|$ is the Lebesgue's measure of X.

Proof. Let $u \in A_\varphi(X)$, we have $u \geqslant 1$ on X and $\varrho_\varphi(u) \leqslant \bar{\rho}_{m,\varphi}(u)$.
But $\varrho_\varphi(u) = \int_{\mathbb{R}^N} \varrho_\varphi(y, |u(y)|)dy$, then

$$\varrho_\varphi(u) \geqslant \int_X \varrho_\varphi(y, |u(y)|)dy \geqslant \int_X \varrho_\varphi(y, 1)dy.$$

By the inequality (1.1.7) there exists a constant $c > 0$ such that

$$\inf_{y \in \mathbb{R}^N} \varphi(y, 1) \geqslant c.$$

Therefore, $\varrho_\varphi(u) \geqslant c.|X|$.
Thus,

$$c.|X| \leqslant \bar{\rho}_{m,\varphi}(u).$$

The claim follows by passing to inf on $u \in A_\varphi(X)$. $\square$

Corollary 3.2.1. *Let φ be a Musielak-Orlicz function. If $(f_n)_n$ is a sequence which converge to f in $W^m L_\varphi$, then there exists a subsequence of $(f_n)_n$ which converge to f q.e and a.e.*

Proof. It is an immediate consequence of theorem 3.2.2 and theorem 3.2.7. $\square$

Theorem 3.2.8. *Let φ be a Musielak-Orlicz function which satisfies the condition $\triangle_2$ and the assumptions of the theorem 1.1.2.*
For each $f \in W^m L_\varphi$, there is a C_φ-quasicontinuous function $g \in W^m L_\varphi$ such that $f = g$ a.e. The function g is called a C_φ-quasicontinuous representative of the function f.

Proof. Let $f \in W^m L_\varphi$. By theorem 1.1.2, there exists a sequence (f_n) in $C_0^\infty(\mathbb{R}^N)$ such that $f_n \longrightarrow f$ in $W^m L_\varphi$.
By theorem 3.2.2, there exists a subsequence of (f_n) denoted again by (f_n) such that $f_n \longrightarrow f \;\; C_\varphi - q.u$. The claim follows by theorem 3.2.7.

$\square$

Remark 3.2.2. *By theorem 2.6 in [20], we have the same result if we replace $W^m L_\varphi$, by $W^m L_\varphi(\Omega)$, where Ω is a bounded Lipschitz domain in $\mathbb{R}^n$.*

Theorem 3.2.9. *Let φ be a Musielak-Orlicz function, uniformly convex which satisfies the condition $\triangle_2$*
1) If O is an open set of $\mathbb{R}^N$ and $E \subset \mathbb{R}^N$ such that $|E| = 0$, then

$$C_\varphi(O) = C_\varphi(O - E).$$

2) Let u and v are C_φ-quasicontinuous functions in $\mathbb{R}^N$, we have
i) if $u = v$, almost everywhere in an open set $O \subset \mathbb{R}^N$, then

$$u = v\ C_\varphi - quasieverywhere\ in\ O,$$

ii) if, $u \leqslant v$, almost everywhere in an open set $O \subset \mathbb{R}^N$, then

$$u \leqslant v\ C_\varphi - \textit{quasieverywhere}\ in\ O.$$

Proof. 1) It obvious that $C_\varphi(O) \geqslant C_\varphi(O - E)$. Let $u \in A_\varphi(O - E)$ thus $u \geqslant 1$ in an open set containing $O - E$. Let the function f define as

$$\begin{cases} f(x) = u(x) & , if\ x \in \mathbb{R}^N - E \\ f(x) = 1 & , if\ x \in E. \end{cases}$$

We have $f \in A_\varphi(O)$ and $\bar{\rho}_{m,\varphi}(f) = \bar{\rho}_{m,\varphi}(u)$, thus

$$C_\varphi(O) \leqslant \bar{\rho}_{m,\varphi}(u),$$

and by passing to inf we get $C_\varphi(O) \leqslant C_\varphi(O - E)$.
2)Since C_φ is an outer capacity we get the results by [61]. □

Lemma 3.2.1. *Let Ω be a bounded Lipschitz domain in R^n, φ be a Musielak-Orlicz function which satisfies the condition (1.1.7), φ and ψ satisfy the $\triangle_2$ condition and $m \in \mathbb{N}$. Consider $T \in W^{-m}L_\psi(\Omega) \cap M(\Omega)$, where $M(\Omega)$ denote the set of Radon measures in Ω. If $X \subset \Omega$ is such that $C_\varphi(X) = 0$, then X is $|T|$ -measurable and $|T|(X) = 0$.*

Proof. The same as in [46] and [35]. □

3.3 A theorem of H.Brezis and F.Browder type in Musielak-Orlicz-Sobolev spaces

In this section we generalize the theorem of H Brezis and F.E. Browder [35] in the setting of the Musielak-Orlicz-Sobolev spaces $W^mL_\varphi(\Omega)$.
Let Ω be a bounded Lipschitz domain in $\mathbb{R}^N$ and $m \in \mathbb{N}$. In this section we study the following question: let $w \in W_0^mL_\varphi\ (\Omega)$ and $T \in W^{-m}L_\psi(\Omega)$ such that $T = \mu + h$, where μ lie in $M^+(\Omega)$ (the subset of positive Radon measures)

and h lie $L^1_{loc}(\Omega)$; find sufficient conditions on the data in order for w to belong $L^1(\Omega; d\mu)$, for hw to belong to $L^1(\Omega)$ and finally to have:

$$< T, w >= \int_\Omega w d\mu + \int_\Omega hw dx.$$

This question was solved in [34] in the case of the classical Sobolev spaces, in [31] when $\mu = 0$ in the case of Orlicz Sobolev spaces and in [29] in the case of Orlicz Sobolev spaces.

Theorem 3.3.1. *Let φ be a Musielak-Orlicz function which satisfies the condition (1.1.7), φ and ψ satisfy the $\triangle_2$ condition and $m \in \mathbb{N}$. Consider $w \in W_0^m L_\varphi(\Omega)$, $w \geqslant 0$ a.e in Ω and $T \in W^{-m}L_\psi(\Omega)$ such that $T = \mu + h$, where μ lie in $M^+(\Omega)$ (the subset of positive Radon measures) and $h \in L^1_{loc}(\Omega)$, assume that:*

$$hw \geqslant -|\Phi| \ \ a.e \ in \ \Omega \ for \ some \ \ \Phi \ in \ L^1(\Omega). \tag{3.3.1}$$

Then:

$$hw \in L^1(\Omega), w \in L^1(\Omega; d\mu) \ and \ < T, w >= \int_\Omega w d\mu + \int_\Omega hw dx. \tag{3.3.2}$$

Remark 3.3.1. *Note that $\mu(X) = 0$ for all $X \subset \Omega$ such that $C_\varphi(X) = 0$. Indeed by lemma 3.2.1*

$$|T|(X) = |\mu + h|(X) = 0,$$

but

$$0 \leqslant \mu(X) \leqslant |h|(X) + |\mu + h|(X) = 0.$$

Let prove the theorem 3.3.1.

Proof. Let $w \in W_0^m L_\varphi(\Omega)$, the Lemma 2.4 of [21] yields the existence of a sequence w_n such that:
(i) $w_n \in W_0^m L_\varphi(\Omega) \cap L^\infty(\Omega)$,
(ii) supp w_n is compact,
(iii) $|w_n| \leqslant |w|$ a.e. in Ω,
(v) $w_n \longrightarrow w$ in $W_0^m L_\varphi(\Omega)$.

(vi) $w_n w \geqslant 0$ a.e. in Ω .
Following the lines of [34], it is easy to deduce that

$$< \mu + h, w_n >= \int_\Omega w_n d\mu + \int_\Omega h w_n dx \tag{3.3.3}$$

Since $w_n \longrightarrow w$ in $W_0^m L_\varphi(\Omega)$, by using the corollary3.2.1, Lemma3.2.1 and remark3.3.1 we have

$$w_n \longrightarrow w \quad \mu.a.e \ and \ a.e. \ in \ \Omega. \tag{3.3.4}$$

We recall that by theorem3.2.8 and theorem3.2.9, for any $v \in W^m L_\varphi(\Omega)$ one has

$$v \geqslant 0 \quad a.e. \ in \, \Omega \Leftrightarrow v \geqslant 0 \quad q.e. \ in \, \Omega.$$

This equivalence, remark3.3.1 and the fact ($w \geqslant 0$ a.e. in Ω), imply

$$w_n \geqslant 0 \quad a.e. \ , \ w_n \geqslant 0 \quad \mu.a.e. \ \ and \ 0 \leqslant w_n \leqslant w \ a.e. \ in \ \Omega. \tag{3.3.5}$$

On the other hand from $hw \geqslant -|\Phi|$ and $0 \leqslant w_n \leqslant w \ a.e. \ in \ \Omega$, we have

$$hw_n \geqslant -|\Phi| \, a.e. \, in \ \Omega \tag{3.3.6}$$

Since $< \mu + h, w_n >$ is bounded, (3.3.3)and (3.3.5) imply $\int_\Omega hw_n dx \leqslant cst$; Similary (3.3.3) and (3.3.6) imply $\int_\Omega w_n d\mu \leqslant cst$.
By using (3.3.4), (3.3.5), (3.3.6) and Fatou's lemma we get $hw \in L^1(\Omega)$ and $w \in L^1(\Omega; d\mu)$.
Using $0 \leqslant w_n \leqslant w \quad \mu.a.e. \ in \ \Omega$ and $|hw_n| \leqslant |hw|$ a.e. in Ω, it is now easy to pass to the limit in (3.3.3); we use the convergence of w_n to w in $W_0^m L_\varphi \ (\Omega)$ for the left hand side and Lebesgue's dominated convergence theorem in each term of the right hand side: we obtain

$$< T, w >= \int_\Omega w d\mu + \int_\Omega hw dx.$$

□

3.4 Application

3.4.1 Unilateral problem

Let Ω be a bounded Lipschitz domain in $\mathbb{R}^N$ and $m \in \mathbb{N}$. φ be a Musielak-Orlicz function which satisfies the condition (1.1.7), and such that φ and ψ satisfy the

$\triangle_2$ condition. We consider some right hand side $f \in W^{-m}L_\psi(\Omega)$ and the convex set

$$K_\Phi = \{v \in W_0^m L_\varphi(\Omega),\ v \geqslant \Phi\ a.e\ in\ \Omega\}.$$

Where the obstacle Φ belong to $W_0^m L_\psi(\Omega) \cap L^\infty(\Omega)$. Let a pseudo-monotone mapping S from $W_0^m L_\varphi(\Omega)$ into $W^{-m}L_\psi(\Omega)$, which satisfies the following conditions:

(1) S is continuous from each finite-dimensional subspace of $W_0^m L_\varphi(\Omega)$ into $W^{-m}L_\psi(\Omega)$ for the weak* topology.

(2) S maps bounded sets into bounded sets.

(3) S is coercive, i.e that for some v_0 in $K_\Phi \cap L^\infty(\Omega)$

$$\frac{< S(v), v - v_0 >}{||v||_{W_0^m L_\varphi(\Omega)}} \longrightarrow +\infty\ \ as\ \ ||v||_{W_0^m L_\varphi(\Omega)} \longrightarrow +\infty. \tag{3.4.1}$$

Consider finally a carathéodory function $g : \Omega \times \mathbb{R} \longrightarrow \mathbb{R}$ witch satisfies :

(4) $s.g(x,s) \geqslant 0, \forall\ s \in \mathbb{R}\ \ and\ \ a.e\ \ in\ \ \Omega,$

(5) $h_t = sup_{|s|\leqslant t}|g(x,s)| \in L^1(\Omega)\ \forall t\ \geqslant 0.$

Theorem 3.4.1. *The variational inequality:*

$$u \in K_\Phi,\ g(.,u) \in L^1(\Omega),\ ug(.,u) \in L^1(\Omega)$$

$$< Su, v - u > + \int_\Omega g(.,u)(v-u)dx \geqslant\ < f, v - u >,\ \forall v \in K_\Phi \cap L^\infty(\Omega)$$

has at least one solution.

Proof. **First part** $Approximation\ and\ a\ priori\ istimates.$

Define $g_n(x,s) = \begin{cases} \chi_n(x)g(x,s)\ \ if\ \ |g(x,s)| \leqslant n, \\ \chi_n(x)n\frac{g(x,s)}{|g(x,s)|}\ \ if\ \ |g(x,s)| > n, \end{cases}$

where χ_n is the characteristic function of the set $\{x \in \Omega : |x| \leqslant n\}$

By by using the proposition 1 of [45] we have the approximate problem

$$\begin{cases} u_n \in K_\Phi, \\ < Su_n, v - u_n > + \int_\Omega g_n(.,u_n)(v-u_n)dx \geqslant\ < f, v - u_n >,\ \forall v \in K_\Phi \cap L^\infty(\Omega) \end{cases} \tag{3.4.2}$$

has at least one solution.
Using $v = v_0$ as test function in (3.4.2) we get

$$< Su_n, u_n - v_0 > + \int_\Omega g_n(., u_n)(u_n - v_0)dx \leqslant \, < f, u_n - v_0 > . \qquad (3.4.3)$$

If (u_n) is not bonded in $W_0^m L_\varphi(\Omega)$ then by the assumptions **(3)** we have

$$(\forall A > 0)(\exists n_0 \in \mathbb{N}) : (\forall n \geqslant n_o)(\frac{< S(u_n), u_n - v_0 >}{||u_n||_{W_0^m L_\varphi(\Omega)}} > A) \qquad (3.4.4)$$

Let $E_n = \{x \in \Omega : u_n(x) \geqslant 0\}$, by (3.4.3) and (3.4.4) we have for large n : $A||u_n||_{W_0^m L_\varphi(\Omega)} + \int_{E_n} g_n(., u_n)(u_n - v_0)dx + \int_{\Omega - E_n} g_n(., u_n)u_n dx$

$$\leqslant \int_{\Omega - E_n} g_n(., u_n)v_0 dx + ||f||_{W^{-m}L_\psi(\Omega)}||u_n||_{W_0^m L_\varphi(\Omega)} + ||f||_{W^{-m}L_\psi(\Omega)}||v_0||_{W_0^m L_\varphi(\Omega)}$$

Let $G_n = \{x \in \Omega : u_n(x) \geqslant v_o\}$ and $l = sup(|v_0|, |\Phi|)$.
By the assumption **(4)** and **(5)** we have

$$\int_{E_n \cap G_n} g_n(., u_n)(u_n - v_0)dx \geqslant 0,$$

$$\int_{E_n \cap G_n^c} g_n(., u_n)u_n dx \geqslant 0,$$

$$\int_{E_n \cap G_n^c} g_n(., u_n)v_0 dx \leqslant \int_\Omega |h_{||l||_{L^\infty(\Omega)}} v_0|,$$

$$\int_{\Omega - E_n} g_n(., u_n)u_n dx \geqslant 0,$$

$$\int_{\Omega - E_n} g_n(., u_n)v_0 dx \leqslant \int_\Omega |h_{||\Phi||_{L^\infty(\Omega)}} v_0|.$$

Then we get

$$||u_n||_{W_0^m L_\varphi(\Omega)} \leqslant C_1, \ \forall n \geqslant n_0,$$

which is impossible, thus (u_n) is bounded in $W_0^m L_\varphi(\Omega)$.
It follows that there exists a subsequence, again denoted by u_n such that

$$u_n \rightharpoonup u, \ \ weakly \ \ in \ \ W_0^m L_\varphi(\Omega) \ \ and \ \ a.e. \ \ in \ \ \Omega.$$

Thus

$$g_n(x, u_n(x)) \longrightarrow g(x, u(x)) \;\; a.e. \;\; in \;\; \Omega.$$

From (3.4.3) we get

$$\int_\Omega g_n(., u_n)(u_n - v_0)dx \leqslant C_2. \tag{3.4.5}$$

We shall prove

$$\int_\Omega |g_n(., u_n)(u_n - v_0)|dx \leqslant C_3.$$

Indeed

$$\begin{aligned}
\int_\Omega |g_n(., u_n)(u_n - v_0)|dx &= \int_{G_n \cap E_n} g_n(., u_n)(u_n - v_0)dx - \int_{G_n \cap E_n^c} g_n(., u_n)(u_n - v_0)dx \\
&- \int_{G_n^c \cap E_n} g_n(., u_n)(u_n - v_0)dx + \int_{G_n^c \cap E_n^c} g_n(., u_n)(u_n - v_0)dx \\
&\leqslant \int_\Omega g_n(., u_n)(u_n - v_0)dx - 2\int_{G_n \cap E_n^c} g_n(., u_n)(u_n - v_0)dx \\
&- 2\int_{G_n^c \cap E_n} g_n(., u_n)(u_n - v_0)dx \\
&\leqslant C_2 + 2\int_{G_n \cap E_n^c} g_n(., u_n)v_0 dx + 2\int_{G_n^c \cap E_n} g_n(., u_n)v_0 dx \\
&\leqslant C_2 + 4\int_\Omega |h_{||b||_{L^\infty}} v_0|dx = C_3,
\end{aligned} \tag{3.4.6}$$

where $b = sup(|\Phi|, |v_0|)$.
In order to prove

$$g_n(., u_n) \longrightarrow g(., u) \;\; in \;\; L^1(\Omega), \tag{3.4.7}$$

let us observe that, for any $\delta > 0$,

$$|g_n(x, u_n(x))| \leqslant \sup_{|t| \leqslant \delta^{-1} + ||v_0||_{L^\infty}} |g(., t)| + \delta |g_n(x, u_n(x))(u_n(x) - v_0(x))|.$$

Thus fore any measurable set E in Ω we have

$$\int_E |g_n(., u_n)|dx \leqslant \int_E |h_{\frac{1}{\delta} + ||v_0||_{L^\infty}}| + \delta C_3.$$

By Vitali's theorem, we obtain (3.4.7).
Furthermore by (3.4.5) we have

$$\int_\Omega g_n(.,u_n)u_n dx \leqslant C_2 + \int_\Omega g_n(.,u_n)v_0 dx.$$

By Fatou's lemma and (3.4.7), we get

$$0 \leqslant \int_\Omega g(.,u)u dx \leqslant C_2 + \int_\Omega g(.,u)v_0 dx.$$

Thus

$$g(.,u)u \in L^1(\Omega).$$

Second part : *Passing to the limit in (3.4.2)*
Let

$$\mu_n = Su_n - f + g_n(.,u_n).$$

From (3.4.2) it is clear that $\mu_n \in M^+(\Omega)$. *Since* S *maps bounded sets of* $W_0^m L_\varphi\,(\Omega)$ *in to bounded sets of* $W^{-m}L_\psi\,(\Omega)$*, then we can assume for the same sequence that*

$$Su_n \rightharpoonup \chi \;\; weakly \;\; in\, W^{-m}L_\psi\,(\Omega),$$

which implies that

$$\mu_n \longrightarrow \mu \;\; in \;\; D'(\Omega),$$

where

$$\mu = \chi - f + g(.,u).$$

We put $w = u - \Phi,\; h = -g(.,u)\;$ *and* $\;T = \mu + h.$
The assumptions of theorem 3.3.1 are satisfied since $T = \chi - f \in W^{-m}L_\psi\,(\Omega)$ *and* $h \in L^1(\Omega)$*. Thus*

$$\left\{ \begin{array}{l} u - \Phi \in L^1(\Omega; d\mu), \\ < \chi - f, u - \Phi >= \int_\Omega (u-\Phi)d\mu - \int_\Omega g(.,u)(u-\Phi)dx. \end{array} \right. \tag{3.4.8}$$

Using $v = \Phi$ *as test function in (3.4.2) we get*

$$< Su_n, u_n > \;\; \leqslant \;\; < Su_n, \Phi > - < f, \Phi - u_n > + \int_\Omega g_n(.,u_n)(\Phi - u_n),$$

which gives passing to the limit and then using (3.4.8)

$$\left\{ \begin{array}{lll} limsup_n < Su_n, u_n > & \leqslant & < \chi, \Phi > - < f, \Phi - u > + \int_\Omega g(.,u)(\Phi - u)dx, \\ & \leqslant & < \chi, u > \;\; + \;\; \int_\Omega (\Phi - u)d\mu \;\; \leqslant \;\; < \chi, u >; \end{array} \right. \tag{3.4.9}$$

since, by theorem3.2.9 we have

$$(\Phi - u) \leqslant 0 \quad \mu.a.e. \quad in \;\; \Omega. \tag{3.4.10}$$

Using (3.4.9) and since S is a pseudo - monotone operator, we obtain

$$\chi = Su \quad and \quad < Su_n, u_n > \longrightarrow < Su, u > .$$

It is now easy to pass to the limit in (3.4.2) for any fixed $v \in K_\Phi \cap L^\infty(\Omega)$. $\square$

3.4.2 Musielak-Orlicz-Sobolev spaces with zero boundary values

In this section, $\Omega \subset \mathbb{R}^N$ be an open set and φ is a Musielak-Orlicz function.

Definition 3.4.1. *We say that a function u belongs to the Musielak-Orlicz-Sobolev spaces with zero boundary values, and denoted $u \in B^{1,0}_\varphi(\Omega)$, if there is a C_φ-quasicontinuous functions $\tilde{u} \in W^1L_\varphi(\mathbb{R}^N)$ such that $\tilde{u} = u$ almost everywhere in Ω and $\tilde{u} = 0$ C_φ -quasi-everywhere in $\mathbb{R}^N \setminus \Omega$.*
The space $B^{1,0}_\varphi(\Omega)$ is endowed with the norm

$$||u||_{B^{1,0}_\varphi(\Omega)} = ||\tilde{u}||_{W^1L_\varphi(\mathbb{R}^N)}.$$

Remark 3.4.1. *The norm of $B^{1,0}_\varphi(\Omega)$ does not depend on the choice of the C_φ-quasicontinuous representative since $C_\varphi(E) = 0$ implies that $|E| = 0$ for every $E \subset R^N$.*

Theorem 3.4.2. *$B^{1,0}_\varphi(\Omega)$ is a Banach spaces.*

Proof. Let $(u_i)_i$ be a Cauchy sequence in $B^{1,0}_\varphi(\Omega)$, for every u_i there is a C_φ-quasicontinuous function $\tilde{u}_i \in W^1L_\varphi(\mathbb{R}^N)$ such that $\tilde{u_i} = u_i$ almost everywhere in Ω and $\tilde{u_i} = 0$ C_φ -quasi-everywhere in $\mathbb{R}^N \setminus \Omega$.
Since $W^1L_\varphi(\mathbb{R}^N)$ is a Banach space, there is a function $u \in W^1L_\varphi(\mathbb{R}^N)$ such that $\tilde{u_i} \to u$ in $W^1L_\varphi(\mathbb{R}^N)$ as $i \to \infty$. By theorem 3.2.2, u is C_φ-quasicontinuous and by theorem 3.2.3 we have $u = 0$ C_φ -quasi-everywhere in $\mathbb{R}^N \setminus \Omega$. Thus $u \in B^{1,0}_\varphi(\Omega)$ and the spaces $B^{1,0}_\varphi(\Omega)$ is complete. $\square$

Corollary 3.4.1. *If the complementary Musielak-Orlicz functions ψ of φ satisfies the condition $\triangle_2$ then the space $B^{1,0}_\varphi(\Omega)$ is reflexive.*

Proof. The space $W^1L_\varphi(\mathbb{R}^N)$ is reflexive Banach, by theorem3.4.2 the space $B^{1,0}_\varphi(\Omega)$ is closed, and therefore $B^{1,0}_\varphi(\Omega)$ is reflexive. $\square$

Corollary 3.4.2. *We have* $W^1_0L_\varphi(\Omega) \subset B^{1,0}_\varphi(\Omega) \subset W^1L_\varphi(\Omega)$.

Proof. The first inclusion follows from Theorem3.4.2 since $D(\Omega) \subset B^{1,0}_\varphi(\Omega)$. The second inclusion follows directly from the definition of the space $B^{1,0}_\varphi(\Omega)$. $\square$

Lemma 3.4.1. *Let* $u \in B^{1,0}_\varphi(\Omega)$ *and* $v \in W^1L_\varphi(\mathbb{R}^N)$. *If* u *is abounded and* v *is* C_φ*-quasicontinuous then* $uv \in B^{1,0}_\varphi(\Omega)$.

Proof. Let $\tilde{u} \in \mathbb{R}^N$ be the C_φ-quasicontinuous representative of u. we have $\tilde{u}v$ is C_φ-quasicontinuous in $\mathbb{R}^N$. Let $D = \{x \in \mathbb{R}^N \setminus \Omega : \tilde{u}v \neq 0\}$, we have $D = E \cup F$ where
$E = \{x \in \mathbb{R}^N \setminus \Omega : \tilde{u} \neq 0\} \cap \{x \in \mathbb{R}^N \setminus \Omega : v \neq 0, \ \ \textit{and} \ \ v \neq \infty\}$
and $F = \{x \in \mathbb{R}^N \setminus \Omega : v = \infty\}$. It obvious that $C_\varphi(E) = 0$ and by theorem3.2.1 we have $C_\varphi(F) = 0$, thus $C_\varphi(D) = 0$ and $\tilde{u}v = 0 \ \ C_\varphi$ -quasi-everywhere in $\mathbb{R}^N \setminus \Omega$. Since $\tilde{u}v = uv$ almost everywhere in Ω, then $uv \in B^{1,0}_\varphi(\Omega)$. $\square$

Theorem 3.4.3. *Let* $E \subset \Omega$ *such that* $C_\varphi(E) = 0$, *we have* $B^{1,0}_\varphi(\Omega) = B^{1,0}_\varphi(\Omega \setminus E)$.

Proof. It obvious that $B^{1,0}_\varphi(\Omega \setminus E) \subset B^{1,0}_\varphi(\Omega)$. Let $u \in B^{1,0}_\varphi(\Omega)$, there is a C_φ-quasicontinuous function $\tilde{u} \in W^1L_\varphi(\mathbb{R}^N)$ such that $\tilde{u} = u$ almost everywhere in Ω and $\tilde{u} = 0 \ \ C_\varphi$ -quasi-everywhere in $\mathbb{R}^N \setminus \Omega$. Since $C_\varphi(E) = 0$, we have $\tilde{u} = 0 \ \ C_\varphi$ -quasi-everywhere in $\mathbb{R}^N \setminus (\Omega \setminus E)$. Thus $u \in B^{1,0}_\varphi(\Omega \setminus E)$. $\square$

Remark 3.4.2. *If* $C_\varphi(\partial\Omega) = 0$, *then* $B^{1,0}_\varphi(int\ \Omega) = B^{1,0}_\varphi(\bar{\Omega})$.

Theorem 3.4.4. *Let* E *be a open subset of* $\mathbb{R}^N$. *Then* $B^{1,0}_\varphi(\Omega) = B^{1,0}_\varphi(\Omega \setminus E)$. *if and only if* $C_\varphi(E \cap \Omega) = 0$.

Proof. It is the same as that given in [62], Theorem 4.8. $\square$

3.4.3 The Dirichlet Energy Integral Minimisers

In this section, $\Omega \subset \mathbb{R}^N$ be a bounded open and φ is a Musielak-Orlicz function.

Definition 3.4.2. *Let $g \in W^1L_\varphi(\Omega)$. For all $u \in B^{1,0}_\varphi(\Omega)$ we put*

$$I^\varphi_{\Omega,g}(u) = \varrho_{\varphi,\Omega}\big(|\nabla u| + |\nabla g|\big),$$

$I^\varphi_{\Omega,g}(\Omega)$ is call the energy operator corresponding to the function g.

Theorem 3.4.5. *If the complementary Musielak-Orlicz functions ψ of φ satisfies the condition $\triangle_2$, then there exists a function $u \in W^1_0L_\varphi(\Omega)$ such that*

$$I^\varphi_{\Omega,g}(u) = \inf_{v \in W^1_0L_\varphi(\Omega)} I^\varphi_{\Omega,g}(v). \tag{3.4.11}$$

Proof. By corollary 3.4.1 we have $W^1_0L_\varphi(\Omega)$ is a reflexive Banach space.
Let $\alpha, \beta \in \mathbb{R}^+$ such that $\alpha + \beta = 1$ and let $u, v \in W^1_0L_\varphi(\Omega)$, we have

$$I^\varphi_{\Omega,g}(\alpha u + \beta v) = \varrho_{\varphi,\Omega}\big(|\alpha\nabla u + \beta\nabla v| + |\nabla g|\big),$$

Thus

$$\begin{aligned} I^\varphi_{\Omega,g}(\alpha u + \beta v) &\leqslant \varrho_{\varphi,\Omega}\big(\alpha|\nabla u| + \beta|\nabla v| + |\nabla g|\big) \\ &\leqslant \alpha\varrho_{\varphi,\Omega}\big(|\nabla u| + |\nabla g|\big) + \beta\varrho_{\varphi,\Omega}\big(|\nabla v| + |\nabla g|\big) \\ &= \alpha I^\varphi_{\Omega,g}(u) + \beta I^\varphi_{\Omega,g}(v). \end{aligned} \tag{3.4.12}$$

Thus $I^\varphi_{\Omega,g}$ is convex.
Since $\varrho_{\varphi,\Omega}$ is a semi modular then it is lower semicontinuous [see [39] Theorem2.1.17] thus $I^\varphi_{\Omega,g}$ is lower semicontinuous.
By the Poincaré inequality [see [22]] there is a constant $C > 0$ such that

$$\forall\, u \in W^1_0L_\varphi(\Omega) : \;\; ||u||_{L_\varphi(\Omega)} \leqslant C||\nabla u||_{L_\varphi(\Omega)},$$

On the other hand we have

$$\forall\, u \in W^1_0L_\varphi(\Omega) : ||\nabla u||_{L_\varphi(\Omega)} \leqslant ||\big(|\nabla u| + |\nabla g|\big)||_{L_\varphi(\Omega)} \leqslant \varrho_{\varphi,\Omega}(|\nabla u| + |\nabla g|) + 1.$$

Thus If $||u||_{W^1_0L_\varphi(\Omega)} \longrightarrow \infty$ then $I^\varphi_{\Omega,g}(u) \longrightarrow \infty$, and therefore the operator $I^\varphi_{\Omega,g}$ is coercive, and the theorem follows by lemma 1.1.2. □

Theorem 3.4.6. *The C_φ-quasicontinuous representative $\tilde{u}$ of the minimizing function u in (3.4.11) is unique up to set of zero C_φ-capacity.*

Proof. Assume that u_1 and u_2 are two minimizers of (3.4.11) with

$$|\{\nabla u_1 \neq \nabla u_2\}| \neq 0.$$

We set $v = \frac{1}{2}(u_1 + u_2)$, thus

$$I^{\varphi}_{\Omega,g}(v) < \frac{1}{2} I^{\varphi}_{\Omega,g}(u_1) + \frac{1}{2} I^{\varphi}_{\Omega,g}(u_2) = \inf_{u \in W^1_0 L_{\varphi}(\Omega)} I^{\varphi}_{\Omega,g}(u),$$

which is a contradiction. Therefore $|\{\nabla u_1 \neq |\nabla u_2\}| = 0$. By the Poincaré inequality we get

$$||u_1 - u_2||_{L_{\varphi}(\Omega)} \leqslant C ||\nabla u_1 - \nabla u_2||_{L_{\varphi}(\Omega)} = 0.$$

Then $u_1 = u_2$ for a.e. $x \in \Omega$.
Let $\tilde{u}_1$ and $\tilde{u}_2$ The C_{φ}-quasicontinuous representative of u_1 and u_2, we have $\tilde{u}_1 = \tilde{u}_2$ for almost every $x \in \Omega$ and theorem 3.2.9 implies that $\tilde{u}_1 = \tilde{u}_2$ C_{φ}-quasieverywhere in Ω. $\square$

Chapter 4

Musielak-Orlicz-Sobolev Spaces on Arbitrary Metric Space

4.1 Introduction

In the second section of this chapter we recall some important definitions and results of Musielak-Orlicz spaces $L_{\varphi}(X)$, in the third section , we define the Musielak-Orlicz-Sobolev space on a metric space, we show that it is a Banach space, the Poincaré inequality holds and Lipschitz continuous functions are dense. In the forth section , we develop a capacity theory based on this space; we study basic properties of capacity , including monotonicity, countable subadditivity as well as several convergence results . As an application we prove that each Musielak-Orlicz-Sobolev function has a quasi-continuous representative. Our results generalize those of N. Aissaoui in the case of Orlicz spaces [see [4], [5] and [6]].

4.2 Preliminary

Let (X, Σ, μ) be a σ-finite, complete measure space. A real function $\varphi : X \times [0, +\infty) \longrightarrow [0, +\infty)$ is said to be a Musielak-Orlicz function or a generalized N-function , on (X, Σ, μ) if:
a) $\varphi(x, .)$ is an N-function for every $x \in X$ [convex, increasing, continuous,

$\varphi(x,0) = 0$, $\varphi(x,t) > 0\ \forall t > 0$, $\frac{\varphi(x,t)}{t} \to 0$ as $t \to 0$, $\frac{\varphi(x,t)}{t} \to \infty$ as $t \to \infty$].
b) $\varphi(.,t)$ is a measurable function.
The Musielak-Orlicz function φ is said to satisfy the Δ_2-condition if there exists $K \geqslant 2$ such that

$$\varphi(x,2t) \leqslant K\varphi(x,t), \text{ for all } x \in X \text{ and } t \geqslant 0.$$

The smallest such K is called the Δ_2-constant of φ.
When the last inequality holds only for $t \geqslant some\ t_0 > 0$ then φ is said to satisfy the Δ_2-condition near infinity. By $L^0(X,\mu)$ we denote the space of all μ-measurable functions on X.
We define the functional

$$\varrho_\varphi(f) = \int_X \varphi(y, |f(y)|)d\mu(y), \ \forall f \in L^0(X,\mu).$$

The set

$$K_\varphi(X) = \{f \in L^0(X,\mu) \ : \ \varrho_\varphi(f) < \infty\}$$

is called the Musielak-Orlicz class.
The Musielak-Orlicz space $L_\varphi(X)$ is the vector space generated by $K_\varphi(X)$.
We have:

$$L_\varphi(X) = \{f \in L^0(X,\mu) \ : \ \varrho_\varphi(\frac{|f(x)|}{\lambda}) < +\infty, \text{ for some } \lambda > 0\}.$$

$K_\varphi(X)$ is a convex subset of $L_\varphi(X)$.
The equality : $K_\varphi(\Omega) = L_\varphi(\Omega)$ holds if and only if φ satisfies the $\triangle_2$- condition, for all t or for large t according to whether X has infinite measure or not.
Let

$$\varphi^*(x,s) = sup\{st - \varphi(x,t) \ /t \geqslant 0\} \quad \forall x \in X.$$

Thus φ^* is the musielak-Orlicz function complementary to φ in the sense of Young with respect to the variable s.
In the space $L_\varphi(X)$ we define the following two norms:

$$||f||_\varphi = inf\{\lambda > 0 \ / \int_X \varphi(x, \frac{|f(x)|}{\lambda})d\mu \leqslant 1\}.$$

which is called the Luxemburg norm and the so-called Orlicz norm by:

$$|||f|||_\varphi = \sup_{||v||_{\varphi^*}\leqslant 1} \int_X |f(x)v(x)|d\mu$$

where φ^* is the Musielak-Orlicz function complementary to φ.
These two norms are equivalent [70].
We say that a sequence of functions $u_n \in L_\varphi(X)$ is modular convergent to $u \in L_\varphi(X)$ if there exists a constant $k > 0$ such that

$$\lim_{n\to+\infty} \varrho_\varphi(\frac{u_n - u}{k}) = 0.$$

If φ satisfies the $\triangle_2$ -condition; then modular convergence coincides with norm convergence.
The space $L_\varphi(X)$ is reflexive if and only if φ and φ^* satisfy the $\triangle_2$ -condition, for all t or for large t according to whether X has infinite measure or not.

Lemma 4.2.1. *[39] Let φ be a Musielak-Orlicz function and $f_n, f, g \in L^0(X,\mu)$*
(a) If $f_n \longrightarrow f$ a.e. then $\varrho_\varphi(f) \leqslant \liminf_{n\to+\infty} \varrho_\varphi(f_n)$.
(b) If $|f_n| \nearrow |f|$ a.e. then $\varrho_\varphi(f) = \lim\limits_{n\to+\infty} \varrho_\varphi(f_n)$.
(c) If $f_n \longrightarrow f$ a.e. $|f_n| \leqslant |g|$ a.e, and $\varrho_\varphi(\lambda g) < \infty$ for every $\lambda > 0$, then $f_n \to f$ in $L_\varphi(X)$.

Theorem 4.2.1. *[39] Let $\varphi \in N(X,\mu)$*
(a) $||f||_\varphi = ||\,|f|\,||_\varphi$ for all $f \in L_\varphi(X)$.
(b) If $f \in L_\varphi(X)$, $g \in L^0(X,\mu)$ and $0 \leqslant |g| \leqslant |f|$ a.e. then $g \in L_\varphi(X)$ and $||g||_\varphi \leqslant ||f||_\varphi$.
(c) If $f_n \to f$ a.e. then: $||f||_\varphi \leqslant \liminf_{n\to+\infty} ||f_n||_\varphi$.
(d) If $|f_n| \nearrow |f|$ a.e with $f_n \in L_\varphi(X)$ and $\sup_n ||f_n||_\varphi < \infty$ then, $f \in L_\varphi(X)$ and $||f_n||_\varphi \nearrow ||f||_\varphi$.

4.3 Lipschitz characterization in the Euclidean case

Lemma 4.3.1. *Let φ be a Musielak-orlicz function satisfying the $\triangle_2$ condition and Q a cube in $\mathbb{R}^N$ and $f \in W^1L_\varphi(\mathbb{R}^N)$. Then for almost all $x \in Q$,*

$$|f(x) - f_Q| \leqslant C \int_Q \frac{|\bigtriangledown f(y)|}{|x-y|^{N-1}} dy, \tag{4.3.1}$$

where f_Q is the average value of f, that is to say : $f_Q = \frac{1}{|Q|}\int_Q f(x)dx$; which we denoted by $⨍ f(x)dx$; $|Q|$ is the Lebesgue measure of Q and the constant C depends only on N an Q.

Proof. It is the same as that given in [44], lemma 7.16.

□

We omit the proof of the following lemma (Hedberg's inequality) since it is exactly the same as the one in [83], lemma 2.8.3.

Lemma 4.3.2. *Let φ be a Musielak-orlicz function, $R > 0$ and $f \in W^1L_\varphi(\mathbb{R}^N)$. Then there is a constant C depending only on N such that for all $x \in \mathbb{R}^N$,*

$$\int_{B(x,R)} \frac{|\bigtriangledown f(y)|}{|x-y|^{N-1}}dy \leqslant CRM_R(|(\bigtriangledown f|)(x). \tag{4.3.2}$$

Where $M_R(h) = \sup_{r<R} ⨍_{B(x,R)} |h(y)|dy$

Theorem 4.3.1. *Let φ be a Musielak-orlicz function such that φ and φ^* satisfy the $\triangle_2$ condition. Then $f \in W^1L_\varphi(\mathbb{R}^N)$ if and only if there exists a non-negative function $g \in L_\varphi(\mathbb{R}^N)$ such that*

$$|f(x)-f(y)| \leqslant |x-y|(g(x)+g(y)), \ \forall x, y \in \mathbb{R}^N \setminus F, \ |F| = 0. \tag{4.3.3}$$

Proof. Let Q be a cube in $\mathbb{R}^N$ and $x, y \in Q$, we can find a sub cube $\tilde{Q}$ such that $x, y \in \tilde{Q}$ and $diam\tilde{Q} \approx |x-y|$. Here $A \approx B$ if there is a constant C such that $C^{-1} \leqslant A \leqslant CB$. Let $f \in W^1L_\varphi(\mathbb{R}^N)$, by (4.3.1) and (4.3.2) we have:

$$\begin{aligned} |f(x)-f(y)| &\leqslant |f(x)-f_{\tilde{Q}}| + |f(y)-f_{\tilde{Q}}| \\ &\leqslant C|x-y|(M_{|x-y|}(|\bigtriangledown f|)(x) + M_{|x-y|}(|\bigtriangledown f|)(y)). \end{aligned} \tag{4.3.4}$$

Then

$$|f(x)-f(y)| \leqslant C|x-y|(M(|\bigtriangledown f|)(x) + M(|\bigtriangledown f|)(y)).$$

Where $M(h) = \sup_{r>0} ⨍_{B(x,r)} |h(y)|dy$ is the hardy-Little-wood maximal operator. By [72] we have the maximal operator is bounded in $L_\varphi(\mathbb{R}^N)$, there is a non-negative function $g \in L_\varphi(\mathbb{R}^N)$ such that

$$|f(x)-f(y)| \leqslant |x-y||g(x)+g(y)| \ \ a.e.$$

For the reverse implication, it suffices to show that there exists a non-negative function $h \in L_{\varphi}(\mathbb{R}^N)$ such that

$$|\frac{\partial f}{\partial x_i}[\Phi]| = |-\int f\frac{\partial \Phi}{\partial x_i}dx| \leqslant \int |\Phi|\; h\; dx.$$

For all $\Phi \in C_0^{\infty}$. Integrating (4.3.3) twice over a ball $B = B(x,\varepsilon)$, we get the inequality

$$\fint_B |f(x) - f_B|dx \leqslant C\varepsilon \fint_B g(x)dx.$$

Let $\psi \in C_0^{\infty}(B(0,1))$ such that $\int \psi(x)dx = 1$. Put $\psi_\varepsilon = \varepsilon^{-n}\psi(\frac{x}{\varepsilon})$. We have

$$\int f\frac{\partial \Phi}{\partial x_i}dx = \lim_{\varepsilon\to 0}\int \frac{\partial \Phi}{\partial x_i}(\psi_\varepsilon * f)dx = -\lim_{\varepsilon\to 0}\int \Phi(\frac{\partial \psi_\varepsilon}{\partial x_i} * f)dx.$$

Since $\int \frac{\partial \psi_\varepsilon}{\partial x_i}dx = 0$ we get

$$f * \frac{\partial \psi_\varepsilon}{\partial x_i} = (f - f_B) * \frac{\partial \psi_\varepsilon}{\partial x_i}.$$

Thus

$$|f * \frac{\partial \psi_\varepsilon}{\partial x_i}|(x) \leqslant C\varepsilon^{-n-1}\int_B |f - f_B|dx \leqslant C\fint gdx$$

Hence

$$|\int f\frac{\partial \Phi}{\partial x_i}dx| \leqslant \int |\Phi| M(g)(x)dx.$$

□

4.4 Musielak-Orlicz-Sobolev Spaces on Arbitrary Metrique space

Let (X,d) be a metric spaces with finite diameter, ($diam(X) = \sup_{x,y\in X} d(x,y) < \infty$) and equipped with a finite, positive Borel regular outer measure μ. The triplet(X,d,μ) will be fixed in the sequel and will be denoted by X.
Let $u : X \to [-\infty,+\infty]$ be a μ-measurable functions defined on X. We denote by $D(u)$ the set of all μ-mesurable functions $g : X \to [0,+\infty]$, such that

$$|u(x) - u(y)| \leqslant d(x, y)(g(x) + g(y)) \tag{4.4.1}$$

for every $x, y \in X \setminus F, x \neq y$, with $\mu(F) = 0$. The set F is called the exceptional set for g, and g is called the Hajlasz gradient of u.

Definition 4.4.1. *Let φ be a Musielak-Orlicz function. The Dirichlet- Musielak-Orlicz space $L^1_{\varphi,\mu}(X)$ is the space of all μ-measurable functions u such that $D(u) \cap L_\varphi(X) \neq \emptyset$. This space is equipped with the semi-norm*

$$||u||_{L^1_{\varphi,\mu}(X)} = \inf\{||g||_\varphi \ : \ g \in D(u) \cap L_\varphi(X)\}.$$

The Musielak-Orlicz-Sobolev space on arbitrary metric space X is $M^1_{\varphi,\mu}(X) = L_\varphi(X) \cap L^1_{\varphi,\mu}(X)$, equipped with the norm

$$||u||_{M^1_{\varphi,\mu}(X)} = ||u||_\varphi + ||u||_{L^1_{\varphi,\mu}(X)}.$$

Lemma 4.4.1. *Let φ be an uniformly convex Musielak-Orlicz function that satisfies the $\triangle_2$ condition.*
For every $u \in M^1_{\varphi,\mu}(X)$ there is a unique $g \in D(u) \cap L_\varphi(X)$; denoted by g_u such that

$$||g_u||_\varphi = ||u||_{L^1_{\varphi,\mu}(X)}.$$

Proof. Let $(g_i) \subset D(u) \cap L_\varphi(X)$ a sequence such that

$$||u||_{L^1_{\varphi,\mu}(X)} = \lim_{i \to +\infty} ||g_i||_\varphi$$

Since $L_\varphi(X)$ is reflexive, there is a subsequence denoted again by (g_i) and a function $g_u \in L_\varphi(X)$ such that $g_i \rightharpoonup g_u$ weakly in $L_\varphi(X)$. By Mazur's lemma there exist a convex combinations $f_k = \sum\limits_{i=k}^{m_k} \alpha_{k_i} \, g_i,$ such that $f_k \longrightarrow g_u$ strongly in $L_\varphi(X)$, where $\alpha_{k_i} \geqslant 0$ and $\sum\limits_{i=k}^{m_k} \alpha_{k_i} = 1$.
Since every $g_i \in D(u) \cap L_\varphi(X)$, we have:

$$|u(x) - u(y)| = \sum_{i=k}^{i=m_k} \alpha_{k_i} |u(x) - u(y)|$$

$$\leqslant \sum_{i=k}^{i=m_k} \alpha_{k_i} d(x; y)(g_i(x) + g_i(y))$$

$$= d(x;y)(f_k(x) + f_k(y))$$

for μ-almost all $x, y \in X$. Therefore $f_k \in D(u) \cap L_\varphi(X)$, for all k.
On the other hand, there exists a subsequence of (f_k) which converges to g_u pointwise μ-almost everywhere in X, thus $g_u \in D(u) \cap L_\varphi(X)$.
Let $\varepsilon > 0$, there is $k \in \mathbb{N}$ such that

$$||g_i||_\varphi \leqslant ||u||_{L^1_{\varphi,\mu}(X)} + \varepsilon \quad \forall\, i \geqslant k.$$

For this k we have

$$||f_k||_\varphi \leqslant \sum_{i=k}^{m_k} \alpha_{k_i}\, ||g_i||_\varphi.$$

Therefore

$$||f_k||_\varphi \leqslant ||u||_{L^1_{\varphi,\mu}(X)} + \varepsilon.$$

Thus

$$||g_u||_\varphi = ||u||_{L^1_{\varphi,\mu}(X)}.$$

Now suppose that there exist two minimal g_1 and g_2 with $g_1 \neq g_2$, so that $||g_1 - g_2||_\varphi > 0$.
Let $G_1 = \frac{g_1}{||g_1||_\varphi}$ and $G_2 = \frac{g_2}{||g_2||_\varphi}$. By the uniform convexity of the norm we have

$$||G_1 + G_2||_\varphi < 2$$

then

$$||\frac{1}{2}g_1 + \frac{1}{2}g_2||_\varphi < ||g_1||_\varphi$$

which is the contradiction since $(\dfrac{1}{2}g_1 + \frac{1}{2}g_2) \in D(u) \cap L_\varphi(X)$. $\square$

With simple verification, we obtain the following lemma.

Lemma 4.4.2. *Let $g_1 \in D(u_1)$, $g_2 \in D(u_2)$ and $\alpha, \beta \in \mathbb{R}$. If $g \geqslant |\alpha| g_1 + |\beta| g_2$ μ.a.e., then $g \in D(\alpha u_1 + \beta u_2)$.*

Lemma 4.4.3. *Let φ a Musielak-Orlicz function and (u_n) and (g_n) be two sequences of functions such that*

$$\forall n \in \mathbb{N} : \; g_n \in D(u_n).$$

If $u_n \to u$ in $L_\varphi(X)$ and $g_n \to g$ in $L_\varphi(X)$, then $g \in D(u)$.

Proof. There are two subsequences of (u_n) and (g_n) which we denote again by (u_n) and (g_n) such that $u_n \to u$ and $g_n \to g$ $\mu.a.e.$ For all $n \in \mathbb{N}$, let F_n be the exceptional set for g_n, H a set of measure zero such that $u \to u$ and $g_n \to g$ on H^c, and $F = H \cup (\bigcup_n F_n)$. We have $\mu(F) = 0$ and

$$|u(x) - u(y)| \leqslant d(x, y)|g(x) + g(y)|, \; \forall \, x, y \in X \setminus F.$$

Therefore $g \in D(u)$. $\square$

Theorem 4.4.1. *Let φ be a Musielak-Orlicz function. Then $(M^1_{\varphi,\mu}(X), ||.||_{M^1_{\varphi,\mu}})$ is a Banach space.*

Proof. It is clear that $M^1_{\varphi,\mu}(X)$ is a vector space.
Let (u_n) be a Cauchy sequence in $M^1_{\varphi,\mu}(X)$. Then (u_n) is a Cauchy sequence in $L_\varphi(X)$ and hence thus there exists a function $u \in L_\varphi(X)$ such that $u_n \to u$ in $L_\varphi(X)$.
We can choose a subsequence denoted again (u_n) such that $||u_{n+1} - u_n||_{M^1_{\varphi,\mu}} < 2^{-n}$ and $u_n \to u \quad \mu - a.e$ in X.
There exist non-negative functions $g_n \in L_\varphi(X)$ such that $||g_n||_\varphi < 2^{-n}$ and

$$|(u_{n+1} - u_n)(x) - (u_{n+1} - u_n)(y)| \leqslant d(x, y)(g_n(x) + g_n(y)) \;\; \mu - a.e \; in \; X.$$

By adding the last inequalities , we have for $n > m$,

$$|(u_n - u_m)(x) - (u_n - u_m)(y)| \leqslant d(x, y)(\sum_{k=m}^{\infty} g_k(x) + \sum_{k=m}^{\infty} g_k(y)) \;\; \mu - a.e \; in \; X.$$

Letting $n \to \infty$ yields

$$|(u_- u_m)(x) - (u - u_m)(y)| \leqslant d(x, y)(\sum_{k=m}^{\infty} g_k(x) + \sum_{k=m}^{\infty} g_k(y)) \;\; \mu - a.e \; in \; X.$$

Since

$$||\sum_{k=m}^{\infty} g_k||_\varphi \leqslant \sum_{k=m}^{\infty} 2^{-k} = 2^{-m+1}$$

the previous inequality implies that $(u_- u_m) \in M^1_{\varphi,\mu}(X)$ and therefore $u \in M^1_{\varphi,\mu}(X)$.
On the other hand, we have

$$||u - u_m||_{L^1_{\varphi,\mu}(X)} \leqslant ||\sum_{k=m}^{\infty} g_k||_\varphi \leqslant 2^{-m+1},$$

then $u_m \to u \ \ in \ \ M^1_{\varphi,\mu}(X)$.

$\square$

Lemma 4.4.4. *Let φ be a Musielak-Orlicz function. The function u belongs to $M^1_{\varphi,\mu}(X)$ if and only if $u \in L_\varphi(X)$ and there exist two sequences $(u_n) \subset L_\varphi(X)$ and $(g_n) \subset D(u_n) \cap L_\varphi(X)$, such that $u_n \to u \ \ \mu$- a.e and $g_n \to g \ \ \mu$- a.e for some $g \in L_\varphi(X)$.*

Proof. The same as in [60] Lemma 2.5. $\square$

Lemma 4.4.5. *Let φ be a Musielak-Orlicz function, and $u_1, \ u_1 \in M^1_{\varphi,\mu}(X)$. If $g_1 \in D(u_1)$ and $g_2 \in D(u_2)$, then :*

(i) $u = max(u_1, u_2) \in M^1_{\varphi,\mu}(X)$ *and* $g = max(g_1, g_2) \in D(u) \cap L_\varphi(X)$.

(ii) $v = min(u_1, u_2) \in M^1_{\varphi,\mu}(X)$ *and* $h = max(g_1, g_2) \in D(v) \cap L_\varphi(X)$.

Proof. We only prove the case (i), as the proof of (ii) is similar.
Let $F_1 \ \ and \ \ F_2$ the exceptional sets for $g_1 \ \ and \ \ g_2$, respectively. Clearly $u, g \in L_\varphi(X)$. It remains to prove that $g \in D(u)$. Let $G = \{x \in X \setminus (F_1 \cup F_2) : u_1(x) \geqslant u_2(x)\}$.
If $x, y \in G$, then

$$|u(x) - u(y)| = |u_1(x) - u_1(y)| \leqslant d(x, y)(g_1(x) + g_1(y)).$$

Analogously, for $x, y \in X \setminus G$, we obtain $|u(x) - u(y)| \leqslant d(x, y)(g_2(x) + g_2(y))$.
For the remaining cases, let $x \in G \ \ and \ \ y \in X \setminus G$:
If $u_1(x) \geqslant u_2(y)$, then

$$|u(x) - u(y)| = u_1(x) - u_2(y) \leqslant u_1(x) - u_1(y) \leqslant d(x, y)(g_1(x) + g_1(y)).$$

If $u_1(x) < u_2(y)$, then

$$|u(x) - u(y)| = u_2(y) - u_1(x) \leqslant u_2(y) - u_2(x) \leqslant d(x, y)(g_2(x) + g_2(y)).$$

The case $x \in X \setminus G$ and $y \in G$ follows by symmetry and hence

$$|u(x) - u(y)| \leqslant d(x, y)(g(x) + g(y))$$

for all $x, y \in X \setminus (F_1 \cup F_2)$, with $\mu((F_1 \cup F_2) = 0$. $\square$

Theorem 4.4.2. ***(The Poincaré inequality).*** *Let φ be a Musielak-Orlicz function such that the function: $x \to 1$, belongs to $L_\varphi(X) \cap L_{\varphi^*}(X)$. If $u \in M^1_{\varphi,\mu}(X)$ then for every $g \in D(u) \cap L_\varphi(X)$:*

$$||u - u_X||_\varphi \leqslant C(\varphi) diam(X) ||g||_\varphi$$

Proof. Integrating the inequality $|u(x) - u(y)| \leqslant d(x,y)(g(x) + g(y))$ over y we obtain

$$|u(x) - \fint_X u(y) d\mu(y)| \leqslant \fint_X |u(x) - u(y)| d\mu(y) \leqslant diam(X) \big(g(x) + \fint_X g(y) d\mu(y)\big).$$

By Hölder inequality, we have

$$|u(x) - u_X| \leqslant diam(X) \big(g(x) + \frac{C}{\mu(X)} ||1||_{\varphi^*} ||g||_\varphi \big).$$

By (b) of theorem4.2.1 we have,

$$||u - u_X||_\varphi \leqslant diam(X) \big(||g||_\varphi + \frac{C}{\mu(X)} ||1||_{\varphi^*} ||1||_\varphi ||g||_\varphi \big) \leqslant C(\varphi) \; diam(X) ||g||_\varphi.$$

□

Theorem 4.4.3. *Let φ a Musielak-Orlicz function satisfying the $\triangle_2$- condition. For every $u \in M^1_{\varphi,\mu}(X)$ and every $\varepsilon > 0$ there exists a Lipschitz function $h \in M^1_{\varphi,\mu}(X)$ such that*
(i) $\mu(\{x \in X \; : \; u(x) \neq h(x)\}) \leqslant \varepsilon$
(ii) $||u - h||_{M^1_{\varphi,\mu}(X)} \leqslant \varepsilon$.

Proof. We fix $u \in M^1_{\varphi,\mu}(X)$ and denote by $g \in L_\varphi(X)$ a Hajlasz gradient of u. We write $E_n = \{x \in X \; : \; |u(x)| \leqslant n \;\; and \;\; g(x) \leqslant n\}$.
We have $u, g \;\in L_\varphi(X)$ and φ satisfies the $\triangle_2$ -condition, then $u, g \;\in K_\varphi(X)$. Furthermore

$$\forall y \in E_n^C, \varphi(y, n) < \varphi(y, |u(y)| + g(y)).$$

Since $\lim\limits_{n \to +\infty} \dfrac{\varphi(y,n)}{n} = +\infty$, we have

$$(\forall A > 0)(\exists n_0 \in \mathbb{N})(\forall n \geqslant n_0) : \frac{\varphi(y,n)}{n} > A.$$

Thus

$$\mu(E_n^c) \leqslant \frac{C}{n}\int_{E_n^C}\varphi(y,n)d\mu(y) \leqslant \frac{C}{n}\int_X \varphi(y,|u(y)|+g(y))d\mu(y),$$

and we get

$$\lim_{n\to+\infty}\mu(E_n^c)=0.$$

On the other hand, we have $0 \leqslant \chi_{E_n^C}\varphi(y,n) \leqslant \chi_{E_n^C}\varphi(y,|u(y)|+g(y))$, hence

$$\lim_{n\to+\infty}\chi_{E_n^C}\varphi(y,n)=0,$$

and since $\chi_{E_n^C}\varphi(y,n) < \varphi(y,|u(y)|+g(y))$, by dominated convergence theorem we have

$$\lim_{n\to+\infty}\int_{E_n^C}\varphi(y,n)d\mu(y)=0$$

The restriction $u|_{E_n}$ is Lipschitz with the constant $2n$. We can extend $u|_{E_n}$ by Mcshane extention [67] to all of X as a Lipschitz function by defining

$$\bar{u}(x)=\inf_{y\in E_n}\{u|_{E_n}(y)+2n\,dist(x,y)\}.$$

We put:

$$u_n=(sing\ \bar{u})\ min(|\bar{u}|,n).$$

It is clear that u_n is a Lipschitz function with constant $2n$, $u|_{E_n}=u_n|_{E_n}$, $|u_n| \leqslant n$ and for all $n \geqslant 1$

$$\mu(\{x : u(x)\neq u_n(x)\}) \leqslant \mu(E_n^c) \to 0\ as\ n\to\infty.$$

On the other hand, we have

$$\int_X \varphi(y,|u-u_n|(y))d\mu(y)=\int_{E_n^c}\varphi(y,|u-u_n|(y))d\mu(y) \leqslant \int_{E_n^c}\varphi(y,|u|(y)+|u_n|(y))d\mu(y).$$

Since $\varphi(y,.)$ is convex and φ satisfies the $\triangle_2$ -condition then:

$$\int_X \varphi(y,|u-u_n|(y))d\mu(y) \leqslant \frac{C_\varphi}{2}[\int_{E_n^c}\varphi(y,|u|(y))d\mu(y)+\int_{E_n^c}\varphi(y,|u_n|(y))d\mu(y)],$$

thus

$$\int_X \varphi(y, |u-u_n|(y))d\mu(y) \leqslant \frac{C_\varphi}{2}[\int_{E_n^c} \varphi(y, |u|(y))d\mu(y) + \int_{E_n^c} \varphi(y, n)d\mu(y)].$$

Therefore $\int_X \varphi(y, |u-u_n|(y))d\mu(y) \to 0 \ \ as \ \ n \to \infty.$
Since φ satisfies the $\triangle_2$ -condition, $||u-u_n||_\varphi \to 0$ as $n \to \infty$. It remains to estimate the gradient. Let

$$g_n = \begin{cases} 0 & \text{for} \quad x \in E_n, \\ g(x) + 3n & \text{for} \quad x \in E_n^c. \end{cases}$$

It is easy to check that

$$|(u-u_n)(x)-(u-u_n)(y)| \leqslant d(x,y)(g_n(x)+g_n(y)), \ for\ \mu{-}almost\ all\ x, y \in X.$$

We have $\int_X \varphi(y, g_n(y))d\mu(y) = \int_{E_n^c} \varphi(y, g_n(y))d\mu(y)$.
Since $\varphi(y, .)$ is convex and φ satisfies the $\triangle_2$- condition then:

$$\int_{E_n^c} \varphi(y, g_n(y))d\mu(y) \leqslant \frac{C_\varphi}{2}[\int_{E_n^c} \varphi(y, g(y))d\mu(y) + \int_{E_n^c} \varphi(y, 3n)d\mu(y)],$$

then $||g_n||_\varphi \to 0$ as $n \to \infty$. $\square$

4.5 Capacity in Musielac- Orlicz-Sobolev space on metric space

Definition 4.5.1. *Let φ be a Musielak-Orlicz function. The Sobolev φ-capacity of the set $E \subset X$ is defined by*

$$C_{\varphi,\mu}(E) = \inf_{u \in A_\varphi(E)} ||u||_{M^1_{\varphi,\mu}},$$

where

$$A_\varphi(E) = \{u \in M^1_{\varphi,\mu}(X) : \ u \geqslant 1 \ on\ an\ open\ set\ containing\ E\ and\ u \geqslant 0\}.$$

If $A_\varphi(E) = \emptyset$ we set $C_{\varphi,\mu}(E) = \infty$.The functions belonging to $A_\varphi(E)$ are called the admissible functions for E.

Remark 4.5.1. *In the defition of $C_{\varphi,\mu}(E)$ we can restrict ourselves to those admissible functions u for which $0 \leqslant u \leqslant 1$.*

Proof. if $A'_{\varphi}(E) = \{u \in A_{\varphi}(E) \ : \ 0 \leqslant u \leqslant 1\}$, then $A'_{\varphi}(E) \subset A_{\varphi}(E)$ implies

$$C_{\varphi,\mu}(E) \leqslant \inf_{u \in A'_{\varphi}(E)} ||u||_{M^1_{\varphi,\mu}}.$$

For the reverse inequality, let $\varepsilon > 0$ and take $u \in A(E)$ such that $||u||_{M^1_{\varphi,\mu}} < C_{\varphi,\mu}(E) + \varepsilon$. Then by lemma 4.4.5, we have $v = max(0, min(u, 1))$ belongs to $A'_{\varphi}(E)$.
Therefore

$$\inf_{\omega \in A'_{\varphi}(E)} ||\omega||_{M^1_{\varphi,\mu}} \leqslant ||v||_{M^1_{\varphi,\mu}} \leqslant C_{\varphi,\mu}(E) + \varepsilon.$$

Letting $\varepsilon \longrightarrow 0$, we obtain

$$\inf_{\omega \in A'_{\varphi}(E)} ||\omega||_{M^1_{\varphi,\mu}} \leqslant C_{\varphi,\mu}(E).$$

This completes the proof.

□

Theorem 4.5.1. *Let φ be a Musielak-Orlicz function. $C_{\varphi,\mu}$ is an outer capacity.*

Proof. It is obvious that $C_{\varphi,\mu}(\emptyset) = 0$ and $C_{\varphi,\mu}(E) \leqslant C_{\varphi,\mu}(F)$ if $E \subset F$.
Let $(E_i) \subset X$ such that $\sum_i C_{\varphi,\mu}(E_i) < +\infty$, then $C_{\varphi,\mu}(E_i) < +\infty; \ \forall i \in \mathbb{N}$.
Therefore $\forall \varepsilon > 0, \ \exists u_i \in A_{\varphi}(E_i)$ and $g_{u_i} \in D(u_i) \cap L_{\varphi}(X)$ so that

$$||u_i||_{\varphi} + ||g_{u_i}||_{\varphi} \leqslant C_{\varphi,\mu}(E_i) + \varepsilon.2^{-i}.$$

We show that $v = \sup_i u_i$ is an admissible function for $\bigcup_{i=1}^{\infty} E_i$ and $g = \sup_i g_{u_i} \in D(v) \cap L_{\varphi}(X)$. Let $v_k = \max_{1 \leqslant i \leqslant k} u_i$. By lemma 4.4.5 the function $g_{v_k} = \max_{1 \leqslant i \leqslant k} g_{u_i}$ belongs to $D(v_k) \cap L_{\varphi}(X)$. Since $v_k \to v \ \mu.a.e$ and $g_{v_k} \to g \ \mu.a.e$, lemma 4.4.4 gives $v \in M^1_{\varphi,\mu}(X)$.
Clearly $v \geqslant 1$ in a neighbourhood of $\bigcup_{i=1}^{\infty} E_i$. Then

$$C_{\varphi,\mu}(\bigcup_{i=1}^{\infty} E_i) \leqslant ||v||_{M^1_{\varphi,\mu}} \leqslant \sum_{i=1}^{\infty} (||u_i||_{\varphi} + ||g_{u_i}||_{\varphi}) \leqslant \sum_{i=1}^{\infty} C_{\varphi,\mu}(E_i) + \varepsilon.$$

The claim follows by letting $\varepsilon \to 0$. Hence $C_{\varphi,\mu}$ is a capacity.
It remains to prove that $C_{\varphi,\mu}$ is outer. Indeed, $C_{\varphi,\mu}(E) \leqslant \inf\{C_{\varphi,\mu}(O) \ : E \subset O,\ O \ open\}$. For the reverse inequality, let $\varepsilon > 0$ and $u \in A_{\varphi}(E)$ such that $||u||_{M^1_{\varphi,\mu}} \leqslant C_{\varphi,\mu}(E) + \varepsilon$. Since $u \in A_{\varphi}(E)$, there is an open set O, $E \subset O$, such that $u \geqslant 1$ on O. Then $C_{\varphi,\mu}(O) \leqslant ||u||_{C_{\varphi,\mu}} \leqslant C_{\varphi,\mu}(E) + \varepsilon$. The claim follows by letting $\varepsilon \to 0$. □

Theorem 4.5.2. *(K_n) is a decreasing sequence of compact sets and $K = \bigcap_n K_n$. Then $\lim\limits_{n \to +\infty} C_{\varphi,\mu}(K_n) = C_{\varphi,\mu}(K)$.*

Proof. First, we observe that $\lim\limits_{n \to +\infty} C_{\varphi,\mu}(K_n) \geqslant C_{\varphi,\mu}(K)$.
Let O be an open set containing K. By the compactness of K, $K_i \subset O$ for all sufficiently large i. Therefor $\lim\limits_{n \to +\infty} C_{\varphi,\mu}(K_n) \leqslant C_{\varphi,\mu}(O)$ and since $C_{\varphi,\mu}$ is an outer capacity, we obtain the claim by taking infimum over all open sets O containing K. □

Theorem 4.5.3. *Let $E \subset X$. If there exists $f \in M^1_{\varphi,\mu}(X)$ such that $f = +\infty$ on an open set containing E, then $C_{\varphi,\mu}(E) = 0$.*

Proof. If there exists $f \in M^1_{\varphi,\mu}(X)$ such that $f = +\infty$ on an open set O containing E, then $f \geqslant \alpha \ on \ O$ for all $\alpha > 0$. Therefore, $\forall \ \alpha > 0 \ : C_{\varphi,\mu}(E) \leqslant \frac{||f||_{M^1_{\varphi,\mu}}}{\alpha}$.
Letting $\alpha \longrightarrow +\infty$, we obtain $C_{\varphi,\mu}(E) = 0$. □

Theorem 4.5.4. *Consider the following propositions:*
i) $f_n \longrightarrow f \ in \ M^1_{\varphi,\mu}(X) \ strongly.$
ii) $f_n \longrightarrow f$ in $C_{\varphi,\mu} - capacity.$
iii) There is a subsequence (f_{n_j}) such that : $f_{n_j} \longrightarrow f, \quad C_{\varphi,\mu} - q.u.$
iv) $f_{n_j} \longrightarrow f, \quad C_{\varphi,\mu} - q.e.$
We have $i) \Rightarrow \ ii) \Rightarrow \ iii) \Rightarrow iv)$.

Proof. Let us show that $i) \Rightarrow ii)$.
By theorem 4.5.3, we have f and f_n are finite $C_{\varphi,\mu} - q.e$ for every n.
Let $\varepsilon > 0$; then

$$C_{\varphi,\mu}(\{x \ : |f_n - f|(x) > \varepsilon\}) \leqslant \frac{||f - f_n||_{M^1_{\varphi,\mu}(X)}}{\varepsilon}.$$

For $ii) \Rightarrow \ iii)$, let $\varepsilon > 0$, then $\exists \ f_{n_j}$ such that

$$C_{\varphi,\mu}(\{x \ : \ |f_{n_j} - f|(x) > 2^{-j}\}) < \varepsilon.2^{-j}.$$

We put

$$E_j = \{x \; : \; |f_{n_j} - f|(x) > 2^{-j}\} \;\; \text{and} \;\; G_m = \bigcup_{j \geqslant m} E_j.$$

We have $C_{\varphi,\mu}(G_m) \leqslant \sum\limits_{j \geqslant m} \varepsilon . 2^{-j} < \varepsilon.$
On the other hand :

$$\forall x \in G_m^c, \; \forall \, j \geqslant m : \;\; |f_{n_j} - f|(x) \leqslant 2^{-j}.$$

Thus $\;f_{n_j} \longrightarrow f \;\; C_{\varphi,\mu} - q.u.$
Finally, let us show that $iii) \Rightarrow \; iv)$.
We have $\forall j \in \mathbb{N}, \exists X_j : C_{\varphi,\mu}(X_j) \leqslant \frac{1}{j}$ and $f_{n_j} \longrightarrow f \;\; on \;\; X_j^c.$
We put $X = \bigcap_j X_j$, then $C_{\varphi,\mu}(X) = 0 \;\; and \;\; f_{n_j} \longrightarrow f \;\; on \; X^c.$ □

Theorem 4.5.5. *Let φ be a Musielak-Orlicz function, such that φ and φ^* satisfy the $\triangle_2$- condition.*
If $f_n \, , f \in L_\varphi(X)$ are such that $f_n \rightharpoonup f$ weakly in $L_\varphi(X)$ then

$$\liminf f_n \leqslant f \leqslant \limsup f_n, \;\; C_{\varphi,\mu} - q.e.$$

Proof. $(L_\varphi(X) \; , \; ||.||)$ is reflexive. By the Banach-Saks theorem, there is a sub-sequence denoted again (f_n) such that the sequence $g_n = \frac{1}{n} \sum\limits_{i=1}^{n} f_i$ converges to f strongly in $L_\varphi(X)$. By theorem 4.5.4, there is a sub-sequence of (g_n) denoted again (g_n) such that

$$\lim_{n \to +\infty} g_n = f \;\; C_{\varphi,\mu} - q.e.$$

On the other hand,

$$\liminf \;\; f_n \leqslant \lim_{n \to +\infty} g_n \;\; .$$

Therefore,

$$\liminf f_n \leqslant f \quad C_{\varphi,\mu} - q.e.$$

For the second inequality, it suffices to replace f_n by $(-f_n)$ in the first inequality. □

Theorem 4.5.6. *Let φ be a Musielak-Orlicz function, such that φ and φ^* satisfy the $\triangle_2$ -condition.*
Let (O_n) be an increasing sequence of open subsets of X and $O = \bigcup_n O_n$, then

$$\lim_{n\to+\infty} C_{\varphi,\mu}(O_n) = C_{\varphi,\mu}(O).$$

Proof. We have $\lim_{n\to+\infty} C_{\varphi,\mu}(O_n) \leqslant C_{\varphi,\mu}(O)$. For the reverse Inequality, if $\lim_{n\to+\infty} C_{\varphi,\mu}(O_n) = +\infty$ there is nothing to show.
Assuming that $\lim_{n\to+\infty} C_{\varphi,\mu}(O_n) < +\infty$, we have

$$\forall n \in \mathbb{N},\ \exists f_n \in M^1_{\varphi,\mu}(X)\ :\ f_n \geqslant 1 \ on\ O_n\ and\ ||f_n||_{M^1_{\varphi,\mu}(X)} \leqslant C_\varphi(O_n) + \frac{1}{n}.$$

Then

$$||f_n||_\varphi + ||g_n||_\varphi \leqslant C_{\varphi,\mu}(O_n) + \frac{1}{n}.$$

Where $g_n \in D(f_n) \cap L_\varphi(X)$. Now (f_n) and (g_n) are two bounded sequences in $L_\varphi(X)$, hence there exist two subsequences, which we denote again by (f_n) and (g_n), such that $f \rightharpoonup f$ and $g \rightharpoonup g$, weakly in $L_\varphi(X)$. Thus

$$||f||_\varphi \leqslant \liminf_n\ ||f_n||_\varphi\ and\ ||g||_\varphi \leqslant \liminf_n\ ||g_n||_\varphi.$$

Hence

$$||f||_{M^1_{\varphi,\mu}(X)} \leqslant \lim_{n\to+\infty} C_{\varphi,\mu}(O_n).$$

On the other hand, by theorem 4.5.5, we have

$$\forall n \in \mathbb{N}\ : f \geqslant 1\ \ on\ \ O_n\ \ , C_{\varphi,\mu} - q.e.$$

Therefore, $f \geqslant 1$ on O $C_{\varphi,\mu} - q.e.$
Let Y be a subset of O where $f \geqslant 1$, then $C_{\varphi,\mu}(O) = C_{\varphi,\mu}(Y)$.
Thus

$$C_{\varphi,\mu}(O) \leqslant \lim_n (C_{\varphi,\mu}(O_n)).$$

□

Theorem 4.5.7. *Let φ be a Musielak-Orlicz function, such that the function $x \to 1$, belongs to $L_{\varphi^*}(X)$. We have*

$$(\exists \alpha > 0)\ (\forall E \subset X), \mu(E) \leqslant \alpha . C_{\varphi,\mu}(E).$$

Proof. Let $E \subset X$ and $f \in A_\varphi(E)$, then $f \geqslant 1$ on an open set O containing E and $f \geqslant 0$. We have

$$\mu(E) \leqslant \mu(O) \leqslant \int_O f(x)d\mu(x) \leqslant \int_X f(x)d\mu(x).$$

By Hölder's inequality we have

$$\int_X f(x)d\mu(x) \leqslant 2||f||_\varphi||1||_{\varphi^*}.$$

Therefore

$$\mu(E) \leqslant 2\ ||1||_{\varphi^*}||f||_{M^1_{\varphi,\mu}(X)}.$$

We obtain the claim by taking infimum over $f \in A(E)$. □

Corollary 4.5.1. *Let φ be a Musielak-Orlicz function, such the function $x \to 1$, belongs to $L_{\varphi^*}(X)$. If (f_n) is a sequence which converges to f in $M^1_{\varphi,\mu}(X)$, then there exists a subsequence of (f_n) which converges to f μ-a.e.*

Proof. It is an immediate consequence of theorem 4.5.4 and theorem 4.5.7. □

Theorem 4.5.8. *Let φ be a Musielak-Orlicz function satisfying the $\triangle_2$ -condition and such the function $x \to 1$ belongs to $L_{\varphi^*}(X)$. For each $f \in M^1_{\varphi,\mu}(X)$, there is a $C_{\varphi,\mu}$-quasi continuous function $g \in M^1_{\varphi,\mu}(X)$ such that $f = g,\ \mu.a.e.$*

Proof. Let $f \in M^1_{\varphi,\mu}(X)$. By theorem 4.4.3, there exists a sequence (f_n) of Lipschitz function, such that $f_n \longrightarrow f$ in $M^1_{\varphi,\mu}(X)$. By theorem 4.5.4, there exists a subsequence of (f_n) denoted again by (f_n) such that $f_n \longrightarrow f\ \ C_\varphi - q.u.$ The claim follows by theorem 4.5.7. □

Chapter 5

Existence and regularity of solution for strongly nonlinear $p(x)$- Elliptic equation with measure data

5.1 Introduction

Let Ω be a bounded open set of $\mathbb{R}^N$, $(N \geqslant 2)$.
In this chapter, we deal with the following Direchlet problem:

$$(\mathcal{P}) \begin{cases} Au + g(x, u, \nabla u) = \mu \text{ in } \Omega, \\ \qquad\qquad u = 0 \quad \text{on } \partial\Omega, \end{cases}$$

with non-standard p-structure which involves a variable growth exponent $p(.)$.
The principal part of the above equation is the operator A defined by $Au = -div(a(x, u, \nabla u))$ which satisfies the classical Leray-Lions conditions with some variable exponent $p(.)$. However the non linearity g is supposed to satisfy the sign condition and some natural growth with respect to ∇u, but no growth with u is supposed. The second member μ considered in this chapter is not regular that is a Radon measure.
Our purpose in this chapter is to study the existence of solution of the problem $(\mathcal{P})$, in the case where the datum μ is a finite Radon measure and in the case of variable exponent, (that is A is a Leray-Lions operator acting from $W_0^{1,p(x)}(\Omega)$

in to $W^{-1,p'(x)}(\Omega)$), with the regularity $u \in W_0^{1,q}(\Omega)$ for all: $1 \leqslant q^- \leqslant q^+ < (N/(N-1))(p^- - 1)$.

5.2 Preliminary lemma

Lemma 5.2.1. *Let $(v_n)_n$ be a sequence of functions in $W_0^{1,p(x)}(\Omega)$. Suppose that there exists a constant $C > 0$ such that, for all $k > 0$:*

$$\int_\Omega |\nabla T_k(v_n)|^{p(x)} dx \leq Ck.$$

Then, there exists a subsequence still denoted by $(v_n)_n$ and a function v, such that

$$v_n(x) \longrightarrow v \ \ a.e. \ \ in \ \ \Omega,$$

$$T_k v_n \rightharpoonup T_k v \ \ weakly \ in \ \ W_0^{1,p(.)}(\Omega),$$
$$T_k v_n \longrightarrow T_k v \ \ strongly \ in \ \ L^{p(.)}(\Omega).$$

Proof. We have

$$||\nabla T_k(v_n)||_{p(x)}^{\gamma} \leqslant Ck,$$

where

$$\gamma = \begin{cases} p^- & \text{if} \quad ||\nabla T_k(v_n)||_{p(x)} > 1, \\ p^+ & \text{if} \quad ||\nabla T_k(v_n)||_{p(x)} \leqslant 1. \end{cases}$$

Thus

$$||\nabla T_k(v_n)||_{p(x)} \leqslant C_1 k^{\frac{1}{\gamma}}, \tag{5.2.1}$$

where C_1 is a constant which does not depend on k.
From (5.2.1) it easily follows that $(v_n)_n$ is Cauchy sequence in measure. Indeed, we have

$$k \ meas\{|v_n| > k\} = \int_{\{|v_n|>k\}} |T_k(v_n)| dx \leqslant \int_\Omega |T_k(v_n)| dx.$$

By Hölder inequality, we obtain

$$k \ meas\{|v_n| > k\} \leqslant (\frac{1}{p_-} + \frac{1}{p'_-}) ||1||_{p'(x)} ||T_k(v_n)||_{p(x)}.$$

Therefore

$$k\ meas\{|v_n| > k\} \leqslant (\frac{1}{p_-} + \frac{1}{p'_-})(meas(\Omega) + 1)^{\frac{1}{p'_-}} ||T_k(v_n)||_{p(x)}.$$

By the Poincaré inequality we have

$$meas\{|v_n| > k\} \leqslant C_2 k^{\frac{1}{\gamma}-1} \longrightarrow 0,\ as\ \ k\ \longrightarrow +\infty. \qquad (5.2.2)$$

For all n and m in $\mathbb{N}$, for all $\varepsilon > 0$ and for every $k > 0$, we get

$$\begin{aligned}\{|v_n - v_m| \geqslant \varepsilon\} &= (\{|v_n - v_m| \geqslant \varepsilon,\ \ |v_n| > k\}) \cup (\{|v_n - v_m| \geqslant \varepsilon,\ \ |v_n| \leqslant k\}) \\ &= (\{|v_n - v_m| \geqslant \varepsilon,\ \ |v_n| > k\}) \cup (\{|v_n - v_m| \geqslant \varepsilon,\ \ |v_n| \leqslant k,\ \ |v_m| \leqslant k\}) \\ &\cup\ \ (\{|v_n - v_m| \geqslant \varepsilon,\ \ |v_n| \leqslant k,\ \ |v_m| > k\}). \qquad (5.2.3)\end{aligned}$$

Thus

$$\{|v_n - v_m| \geqslant \varepsilon\} \subset \{|v_n| > k\} \cup \{|T_k(v_n) - T_k(v_m)| \geqslant \varepsilon\} \cup \{|v_m| > k\}.$$

Therefore

$$\begin{aligned}meas(\{|v_n - v_m| \geqslant \varepsilon\}) &\leqslant meas(\{|v_n| > k\}) + meas(\{|T_k(v_n) - T_k(v_m)| \geqslant \varepsilon\}) \\ &+\ \ meas(\{|v_m| > k\}). \qquad (5.2.4)\end{aligned}$$

By (5.2.2), we have for all $\delta > 0$ there exists k_0 such that

$$meas(\{|v_n| > k\}) < \frac{\delta}{3},\ \ meas(\{|v_m| > k\}) < \frac{\delta}{3},\ \ \forall k \geqslant k_0(\delta).$$

By (5.2.1), we have $(T_k(v_n))_n$ is bounded in $W_0^{1,\ p(.)}(\Omega)$, then there exists a subsequence still denoted $(T_k(v_n))_n$ which is strongly compact in $L^{p(x)}(\Omega)$. This means, in particular, that the sequence $(T_k(v_n))_n$ is Cauchy in measure. We then choose n and m, such that

$$meas(\{|T_k(v_n) - T_k(v_m)| \geqslant \varepsilon\}) < \frac{\delta}{3}.$$

Therefore the sequence $(v_n)_n$ is Cauchy in measure, we thus have (up to subsequence, still denoted by $(v_n)_n$) which converge almost everywhere in Ω, to some function v.

On the other, we have, $T_k(v_n) \longrightarrow T_k(v)$ a.e. in Ω, by combining (5.2.1) and (lemma1.3.1), we have

$$T_k v_n \rightharpoonup T_k v\ \ weakly\ in\ \ W_0^{1,p(.)}(\Omega);$$

and therefore

$$T_k v_n \longrightarrow T_k v\ \ strongly\ in\ \ L^{p(.)}(\Omega).$$

$\square$

5.3 Nonlinear p(x)- elliptic equation with measure data

5.3.1 Basic assumptions

Let Ω be a bounded open subset of $I\!R^N$, $(N \geq 2)$, and let $p \in C_+(\bar{\Omega})$, such that

$$2 - \frac{1}{N} < p^- \leqslant p^+ \leqslant N. \tag{5.3.1}$$

We consider a Leray-Lions operator A from $W_0^{1,p(x)}(\Omega)$ into its dual $W^{-1,p'(x)}(\Omega)$, defined by

$$Au = -\text{div}\Big(a(x, u, \nabla u)\Big); \tag{5.3.2}$$

where $a : \Omega \times I\!R \times I\!R^N \to I\!R^N$ is a Carathéodory function, satisfying the following conditions

$$|a(x, s, \xi)| \leq \beta(K(x) + |s|^{p(x)-1} + |\xi|^{p(x)-1}), \tag{5.3.3}$$

$$a(x, s, \xi)\xi \geq \alpha|\xi|^{p(x)}, \tag{5.3.4}$$

$$[a(x, s, \xi) - a(x, s, \overline{\xi})](\xi - \overline{\xi}) > 0 \quad \text{for all } \ \xi \neq \overline{\xi} \ \text{ in } \ I\!R^N, \tag{5.3.5}$$

for almost every x in Ω, for every (s, ξ) in $I\!R \times I\!R^N$, where $K(x)$ is a positive function lying in $L^{p'(x)}(\Omega)$ and $\beta > 0, \ \alpha > 0$.
Let

$$B_{p(.)} = \{q \in \mathcal{C}_+(\bar{\Omega}) : \ 1 \leqslant q(x) < \frac{N}{N-1}(p^- - 1)\}. \tag{5.3.6}$$

Let $\mu \in M_b(\Omega)$, $M_b(\Omega)$ denotes the set of bounded measures on Ω (finite Radon measures). Consider the nonlinear elliptic problem:

$$(\mathcal{P}) \qquad \begin{cases} -\text{div}\Big(a(x, u, \nabla u)\Big) = \mu & \text{in} \quad \Omega, \\ \qquad\qquad\qquad u = 0 & \text{on} \quad \partial\Omega. \end{cases}$$

Definition 5.3.1. *A measurable function u will be called a weak solution of the problem $(\mathcal{P})$, if*

$$u \in T_0^{1,p(.)}(\Omega) \cap W_0^{1,\, q(.)}, \ \ \forall \, q(.) \in B_{p(.)}$$

$$\int_\Omega a(x, u, \nabla u)\nabla\varphi dx = < \mu, \varphi >, \ \ \forall \varphi \in D(\Omega).$$

5.3.2 Approximate problem

Consider the approximate equation

$$(\mathcal{P}_n) \qquad \begin{cases} -\text{div}\Big(a(x, u_n, \nabla u_n)\Big) = f_n \quad \text{in} \quad \Omega, \\ \qquad\qquad u_n \in W_0^{1,\, p(.)}(\Omega) \end{cases} \tag{5.3.7}$$

where (f_n) is a smooth function which converges to μ in the distributional sense such that,

$$||f_n||_{L^1(\Omega)} \leqslant ||\mu||_{M_b(\Omega)}.$$

Lemma 5.3.1. *The problem ($\mathcal{P}_n$) has at last one weak solution u_n and there exists a sub-sequence denoted again (u_n) and a function u such that*

$$u_n \longrightarrow u \ \ a.e \ \ in \ \ \Omega \quad and \quad \nabla u_n \longrightarrow \nabla u \ \ a.e. \ \ in \ \ \Omega. \tag{5.3.8}$$

Proof. By [12], the problem ($\mathcal{P}_n$) has at last one weak solution u_n.
For $k > 0$, by taking $T_k(u_n)$ as a test function in (5.3.7), we deduce that

$$(\exists C > 0) : \int_\Omega a(x, T_k(u_n), \nabla T_k(u_n)) \nabla T_k(u_n) dx \leq Ck.$$

In view of (5.3.4), we get

$$\int_\Omega |\nabla T_k(u_n)|^{p(x)} dx \leq Ck. \tag{5.3.9}$$

By lemma 5.2.1 there exists a subsequence still denoted by u_n and u such that:

$$\begin{aligned} u_n &\longrightarrow u \ \ a.e. \ in \, \Omega; \\ T_k(u_n) &\rightharpoonup T_k(u) \ \ weakly \ in \ \ W_0^{1,p(.)}(\Omega); \\ T_k(u_n) &\longrightarrow T_k(u) \ \ strongly \ in \ \ L^{p(.)}(\Omega). \end{aligned} \tag{5.3.10}$$

We can write

$$\begin{aligned} 0 &\leqslant \int_\Omega \Big(a(x, T_k(u_n), \nabla T_k(u_n)) - a(x, T_k(u_n), \nabla T_k(u)) \Big) \Big[\nabla T_k(u_n) - \nabla T_k(u) \Big] dx \\ &= \int_\Omega \Big(a(x, T_k(u_n), \nabla T_k(u_n)) \Big) \Big[\nabla T_k(u_n) - \nabla T_k(u) \Big] dx \end{aligned}$$

$$- \int_\Omega \Big(a(x, T_k(u_n), \nabla T_k(u))\Big)\Big[\nabla T_k(u_n) - \nabla T_k(u)\Big]dx. \tag{5.3.11}$$

We have

$$a(x, T_k(u_n), \nabla T_k(u)) \longrightarrow a(x, T_k(u), \nabla T_k(u)) \ \ a.e. \ in \ \ \Omega. \tag{5.3.12}$$

By combining (5.3.3), (5.3.12) and the dominated convergence theorem we get

$$a(x, T_k(u_n), \nabla T_k(u)) \longrightarrow a(x, T_k(u), \nabla T_k(u)) \quad stongly \ in \ \ (L^{p'(x)}(\Omega))^N.$$

Therefore

$$\int_\Omega \Big(a(x, T_k(u_n), \nabla T_k(u))\Big)\Big[\nabla T_k(u_n) - \nabla T_k(u)\Big]dx \longrightarrow 0 \, as \, n \longrightarrow \infty.$$

Let $\eta > 0$ and taking $T_\eta(u_n - T_k(u))$ as test function in (5.3.7), we obtain

$$\int_\Omega a(x, u_n, \nabla u_n)\nabla T_\eta(u_n - T_k(u))dx \leq C\eta.$$

For the sake of simplicity, we write only $\varepsilon(n, \eta)$ to mean all quantities (possibly different) such that

$$\lim_{n \longrightarrow \infty} \lim_{\eta \longrightarrow 0} \varepsilon(n, \eta) = 0.$$

On the other hand

$$\begin{aligned}
&\int_\Omega a(x, u_n, \nabla u_n)\nabla T_\eta(u_n - T_k(u))dx = \\
&\int_{\{|u_n - T_k(u)| \leqslant \eta\} \cap \{|u_n| \leqslant k\}} a(x, T_k(u_n), \nabla T_k(u_n))(\nabla T_k(u_n) - \nabla T_k(u))dx \\
&+ \int_{\{|u_n - T_k(u)| \leqslant \eta\} \cap \{|u_n| > k\}} a(x, u_n, \nabla u_n)(\nabla u_n - \nabla T_k(u))dx \\
&= \int_{\{|T_k(u_n) - T_k(u)| \leqslant \eta\} \cap \{|u_n| \leqslant k\}} a(x, T_k(u_n), \nabla T_k(u_n))(\nabla T_k(u_n) - \nabla T_k(u))dx \\
&+ \int_{\{|u_n - T_k(u)| \leqslant \eta\} \cap \{|u_n| > k\}} a(x, u_n, \nabla u_n)\nabla u_n dx \\
&- \int_{\{|u_n - T_k(u)| \leqslant \eta\} \cap \{|u_n| > k\}} a(x, u_n, \nabla u_n)\nabla T_k(u)dx.
\end{aligned} \tag{5.3.13}$$

By (5.3.4) the second term of the right-hand side of (5.3.13) satisfies

$$\int_{\{|u_n-T_k(u)|\leqslant\eta\}\cap\{|u_n|>k\}} a(x,u_n,\nabla u_n)\nabla u_n dx \geqslant 0,$$

the third term of the right side of (5.3.13) satisfies

$$\int_{\{|u_n-T_k(u)|\leqslant\eta\}\cap\{|u_n|>k\}} a(x,u_n,\nabla u_n)\nabla T_k(u) dx =$$

$$\int_{\{|u_n-T_k(u)|\leqslant\eta\}\cap\{|u_n|>k\}} a(x,T_{k+\eta}(u_n),\nabla T_{k+\eta}(u_n))\nabla T_k(u) dx.$$

Since $a(x,T_{k+\eta}(u_n),\nabla T_{k+\eta}(u_n))$ is bounded in $(L^{p'(x)}(\Omega))^N$, there exists $h_{\eta+k}\in (L^{p'(x)}(\Omega))^N$ such that

$$a(x,T_{k+\eta}(u_n),\nabla T_{k+\eta}(u_n)) \rightharpoonup h_{\eta+k},\ \ weakly\ in\ (L^{p'(x)}(\Omega))^N.$$

Since,

$$\nabla T_k(u)\chi_{\{|u_n-T_k(u)|\leqslant\eta\}\cap\{|u_n|>k\}} \longrightarrow 0\, strongly\, in\ (L^{p(x)}(\Omega))^N, as\, n\longrightarrow\infty\ and\ \eta\longrightarrow 0.$$

Thus

$$\int_{\{|T_k(u_n)-T_k(u)|\leqslant\eta\}\cap\{|u_n|\leqslant k\}} a(x,T_k(u_n),\nabla T_k(u_n))(\nabla T_k(u_n)-\nabla T_k(u))dx \leqslant C\eta+\varepsilon(n,\eta).$$

On the other hand,

$$\int_{\{|T_k(u_n)-T_k(u)|\leqslant\eta\}\cap\{|u_n|>k\}} a(x,T_k(u_n),\nabla T_k(u_n))(\nabla T_k(u_n)-\nabla T_k(u))dx$$

$$\leqslant \int_{\{|T_k(u_n)-T_k(u)|\leqslant\eta\}\cap\{|u_n|>k\}} |a(x,T_k(u_n),\nabla T_k(u_n))||\nabla T_k(u)|dx, \qquad (5.3.14)$$

We have

$$|\nabla T_k(u)|\chi_{\{|T_k(u_n)-T_k(u)|\leqslant\eta\}\cap\{|u_n|>k\}} \longrightarrow 0,\ \ a.e.\ \ in\ \ \Omega\ \ as\ \ n\longrightarrow\infty,$$

and

$$|T_k(u)|\chi_{\{|T_k(u_n)-T_k(u)|\leqslant\eta\}\cap\{|u_n|>k\}} \leqslant |T_k(u)|,$$

thus by Lebesgue dominated convergence theorem, we deduce that

$$|\nabla T_k(u)|\chi_{\{|T_k(u_n)-T_k(u)|\leqslant\eta\}\cap\{|u_n|>k\}} \longrightarrow 0,\ \ in\ \ L^{p(.)}(\Omega),\ as\ \ n\longrightarrow\infty$$

Since the sequence $(|a(x,T_k(u_n),\nabla T_k(u_n))|)_n$ is bounded in $L^{p'(.)}(\Omega)$,

$$\int_{\{|T_k(u_n)-T_k(u)|\leqslant\eta\}\cap\{|u_n|>k\}} |a(x,T_k(u_n),\nabla T_k(u_n))||\nabla T_k(u)|dx \longrightarrow 0,\ as\, n\ \longrightarrow\ \infty$$

Thus

$$\int_{\{|T_k(u_n)-T_k(u)|\leqslant\eta\}\cap\{|u_n|>k\}} a(x,T_k(u_n),\nabla T_k(u_n))(\nabla T_k(u_n)-\nabla T_k(u))dx \leqslant \varepsilon(n).$$

On the other hand,

$$\int_{\{|T_k(u_n)-T_k(u)|>\eta\}\}} a(x,T_k(u_n),\nabla T_k(u_n))(\nabla T_k(u_n)-\nabla T_k(u))dx \longrightarrow 0\ \ as\ \ n\longrightarrow\infty,$$

since $meas(\{|T_k(u_n)-T_k(u)|>\eta\})\longrightarrow 0\ \ as\ \ n\longrightarrow\infty$ and by Hölder inequality we have, $a(x,T_k(u_n),\nabla T_k(u_n))(\nabla T_k(u_n)-\nabla T_k(u))$ is bounded in $L^1(\Omega)$. Thus by passing to the limit over $n\ \ and\ \ \eta$, we get

$$\lim_{n\longrightarrow\infty}\int_\Omega \Big(a(x,T_k(u_n),\nabla T_k(u_n))-a(x,T_k(u_n),\nabla T_k(u))\Big)\Big[\nabla T_k(u_n)-\nabla T_k(u)\Big]dx=0.$$

We conclude by Lemma 4.4 of [15] that there exists a sub-sequence denoted again (u_n) such that

$$\nabla u_n \longrightarrow \nabla u\ \ a.e.\ \ in\ \ \Omega. \tag{5.3.15}$$

□

5.3.3 Result of existence

Theorem 5.3.1. *Assume that (5.3.1) and (5.3.3) - (5.3.5) hold and* $\mu\in M_b(\Omega)$*, then there exists at last one weak solution of the problem* $(\mathcal{P})$*.*

Proof. Let $k>0$, we define the function ψ_k as:

$$\begin{cases} \psi_k(x)=1\ \ if\ \ x>k+1,\\ \psi_k(x)=x-k\ \ if\ \ k\leqslant x\leqslant k+1,\\ \psi_k(x)=0\ \ if\ \ -k\leqslant x\leqslant k,\\ \psi_k(x)=x+k\ \ if\ \ -k-1\leqslant x\leqslant -k,\\ \psi_k(x)=-1\ \ if\ \ x<-k-1.\end{cases}$$

By taking $\psi_k(u_n)$ as test function in (5.3.7) and using (5.3.4) one has:

$$\int_{D_{k,n}} |\nabla u_n|^{p(x)} dx \leqslant \frac{1}{\alpha} ||\mu||_{M_b(\Omega)}, \tag{5.3.16}$$

with

$$D_{k,n} = \{x \in \Omega,\ k \leqslant |u_n(x)| \leqslant k+1\}. \tag{5.3.17}$$

In view of the lemma 1.3.3, there exists a constant C that does not depend on n such that

$$\|u_n\|_{W_0^{1,q(x)}(\Omega)} \leqslant C,\ \ \forall\, q(.) \in B_{p(.)}. \tag{5.3.18}$$

Hence (u_n) is relatively compact in $W_0^{1,\, q(x)}$ $\forall\, q(.) \in B_{p(.)}$, thus there exists a sub-sequence denoted again by (u_n) such that

$$u_n \rightharpoonup u \ weakly \ in \ W_0^{1,\, q(x)},$$

$$u_n \longrightarrow u \ strongly \ in \ L^{q(x)}$$

$$u_n \longrightarrow u \ a.e. \ in \ \Omega.$$

By lemma 5.3.1, we conclude that

$$a(x, u_n, \nabla u_n) \longrightarrow a(x, u, \nabla u) \ \ a.e \ in \ \ \Omega.$$

Combining (5.3.3) and (5.3.18), yields

$$||a(x, u_n, \nabla u_n)||_{L^{r(x)}} \leqslant C,$$

for every $1 < r(x) < \frac{N}{N-1}$.
Using lemma 1.3.1, we can write

$$a(x, u_n, \nabla u_n) \rightharpoonup a(x, u, \nabla u) \ weakly \ in \ L^{r(x)}(\Omega), \forall 1 < r(x) < \frac{N}{N-1}.$$

It is now possible to pass to the limit in (5.3.7), we conclude that u is the weak solution of the problem $(\mathcal{P})$. $\square$

5.4 Strongly nonlinear p(x)-elliptic equation with measure data

Consider the equation

$$(\mathcal{P}') \qquad \begin{cases} -\mathrm{div}\Big(a(x,u,\nabla u)\Big) + g(x,u,\nabla u) = \mu & \text{in} \quad \Omega, \\ \qquad\qquad\qquad\qquad\qquad\qquad u = 0 & \text{on} \quad \partial\Omega, \end{cases} \tag{5.4.1}$$

where a satisfies (5.3.2)-(5.3.5), μ lie in $M(\Omega)$, and the non linear term g: $\Omega \times I\!R \times I\!R^N \longrightarrow I\!R$, is a Carathédory function satisfying for almost every $x \in \Omega$ and for all $s \in I\!R$, $\zeta \in I\!R^N$ the following conditions:

$$g(x,s,\xi).s \geqslant 0 \tag{5.4.2}$$

$$|g(x,s,\xi)| \leq b(|s|)(c(x) + |\xi|^{p(x)}), \tag{5.4.3}$$

where $b: \ I\!R^+ \to \ I\!R^+$ is a continuous growth function and $c: \Omega \to \ I\!R^+$ with $c \in L^1(\Omega)$.

Definition 5.4.1. *We say that u is a weak solution of the problem (5.4.1) if*

$$u \in T_0^{1,p(.)}(\Omega) \cap W_0^{1,\,q(.)}, \ \forall\, q(.) \in B_{p(.)}, \ a(x,u,\nabla u) \in L^1(\Omega) \ , g(x,u,\nabla u) \in L^1(\Omega),$$

$$\int_\Omega a(x,u,\nabla u)\nabla\varphi dx + \int_\Omega g(x,u,\nabla u)\varphi dx =<\mu,\varphi>, \ \ \forall\varphi \in D(\Omega). \tag{5.4.4}$$

Theorem 5.4.1. *Let a satisfy (5.3.2) - (5.3.5), g satisfy (5.4.2)- (5.4.3) and μ lies in $M(\Omega)$. Then there exists a weak solution of (5.4.1).*

Proof. Let (f_n) be a sequence of $L^1(\Omega) \cap W^{-1,p'(.)}(\Omega)$ which converges to μ in the distributional sense and such that

$$||f_n||_{L^1(\Omega)} \leqslant ||\mu||_{M_b(\Omega)}, \ \ \forall n \in \mathbb{N}. \tag{5.4.5}$$

By [12] there exists a weak solution (u_n) of the problem (5.4.1) with $f_n = \mu$ which satisfies:

$$u_n \in W_0^{1,p(x)}(\Omega), \ g(x,u_n,\nabla u_n) \in L^1(\Omega),$$

$$\int_\Omega a(x,u_n,\nabla u_n).\nabla v dx+\int_\Omega g(x,u_n,\nabla u_n)v dx =< f_n, v >, \ \forall v \in W_0^{1,p(x)}(\Omega)\cap L^\infty(\Omega). \tag{5.4.6}$$

Following the lines of [31], it is easy to deduce that

$$||g(x,u_n,\nabla u_n)||_{L^1(\Omega)} \leqslant ||f_n||_{L^1(\Omega)}. \tag{5.4.7}$$

Setting $h_n = f_n - g(x,u_n,\nabla u_n)$, by (5.4.5) and (5.4.7) we have

$$||h_n||_{L^1(\Omega)} \leqslant 2||\mu||_{M_b(\Omega)}.$$

Note that u_n is the solution of the problem (5.3.7) with a right-hand side is h_n. By the previous section the sequence (u_n) is relatively compact in $W_0^{1,q(x)}(\Omega)$, $\forall\ q(.) \in B_{p(.)}$. Then we can assume (after extraction of a sub-sequence, still denoted by (u_n))

$$u_n \rightharpoonup u,\ weakly\ in\ W_0^{1,\ q(x)}\ for\ all\ q(.) \in B_{p(.)}$$

$$u_n \longrightarrow u,\ strongly\ in\ L^{q(x)}$$

$$u_n \longrightarrow u,\ a.e.\ in\ \Omega$$

$$a(x,u_n,\nabla u_n) \rightharpoonup a(x,u,\nabla u)\, weakly\, in\, L^{r(x)}(\Omega),\ for\, every\ 1 < r(x) < \frac{N}{N-1}. \tag{5.4.8}$$

We have olso

$$T_k(u_n) \rightharpoonup T_k(u)\ \ weakly\ in\ \ W_0^{1,p(.)}(\Omega).$$

Now, we prove that

$$g(x,u_n,\nabla u_n) \longrightarrow g(x,u,\nabla u)\ \ \ strongly\ \ in\ \ L^1(\Omega),$$

we have $g_n(x,u_n,\nabla u_n) \longrightarrow g(x,u,\nabla u)\ \ a.e.\ in\ \Omega$ using the Vitali convergence theorem, it sufficient to show that $g_n(x,u_n,\nabla u_n)$ is uniformly equi-integrable. Indeed, let $h > 0$, taking $T_1(u_n - T_h(u_n))$ as a test function in (5.4.6), we obtain

$$\int_\Omega a(x,u_n,\nabla u_n).\nabla T_1(u_n - T_h(u_n))dx + \int_\Omega g(x,u_n,\nabla u_n)T_1(u_n - T_h(u_n))dx =$$

$$\int_\Omega f_n T_1(u_n - T_h(u_n))dx,$$

it follows that

$$\int_{\{|u_n|\geqslant h\}} g(x,u_n,\nabla u_n)T_1(u_n-T_h(u_n))dx \leqslant \int_{\{|u_n|\geqslant h\}} f_nT_1(u_n-T_h(u_n))dx,$$

then
$\int_{\{|u_n|\geqslant h+1\}}|g(x,u_n,\nabla u_n)|dx \leqslant \int_{\{|u_n|\geqslant h\}} g(x,u_n,\nabla u_n)T_1(u_n-T_h(u_n))dx$

$$\leqslant \int_{\{|u_n|\geqslant h\}} f_nT_1(u_n-T_h(u_n))dx$$

$$\leqslant (\frac{1}{p_-}+\frac{1}{p'_-})||f_n||_{-1,p'(.)}||T_1(u_n-T_h(u_n))||_{1,p(.)} \longrightarrow 0 \ as\ h \longrightarrow \infty.$$

Thus, for all $\varepsilon > 0$, there exists $h(\varepsilon) > 0$ such that

$$\int_{\{|u_n|\geqslant h(\varepsilon)\}} |g(x,u_n,\nabla u_n)|dx \leqslant \frac{\varepsilon}{2}. \tag{5.4.9}$$

On the other hand, for any measurable subset $D \subset \Omega$, we have
$\int_D |g(x,u_n,\nabla u_n)|dx \leqslant \int_D b(h)(c(x)+|\nabla T_h(u_n)|^{p(x)})dx+\int_{\{|u_n|\geqslant h(\varepsilon)\}}|g(x,u_n,\nabla u_n)|dx.$
There exists $\beta(\varepsilon) > 0$ such that

$$b(h)\int_D (c(x)+|\nabla T_h(u_n)|^{p(x)})dx \leqslant \frac{\varepsilon}{2} \quad for \quad meas(D) \leqslant \beta(\varepsilon).$$

Thus

$$\int_D |g(x,u_n,\nabla u_n)|dx \leqslant \varepsilon, \ \ with\ \ meas(D) \leqslant \beta(\varepsilon).$$

By Vitali convergence theorem we deduce that $g(x,u_n,\nabla u_n) \longrightarrow g(x,u,\nabla u)\ \ strongly\ in\ \ L^1(\Omega$
Now, taking $\varphi \in D(\Omega)$ as test function in (5.4.6), we have

$$\int_\Omega a(x,u_n,\nabla u_n).\nabla\varphi dx + \int_\Omega g(x,u_n,\nabla u_n)\varphi dx =< f_n,\varphi > .$$

By letting n tends to ∞, we obtain

$$\int_\Omega a(x,u,\nabla u)\nabla\varphi dx + \int_\Omega g(x,u,\nabla u)\varphi dx =< \mu,\varphi >, \ \ \forall\varphi \in D(\Omega).$$

We have show that

$$u \in T_0^{1,p(.)}(\Omega) \cap W_0^{1,\ q(.)}, \ \ \forall\ q(.) \in B_{p(.)},$$

$$a(x,u,\nabla u) \in \ L^{r(x)}(\Omega),\ for\ every\ \ 1 < r(x) < \frac{N}{N-1}\ \ ,$$

$$g(x,u,\nabla u) \in L^1(\Omega),$$

$$\int_\Omega a(x,u,\nabla u)\nabla\varphi dx + \int_\Omega g(x,u,\nabla u)\varphi dx =< \mu,\varphi >,\ \ \forall\varphi \in D(\Omega),$$

thus u is the weak solution of the problem (5.4.1). □

5.5 Example of application

We consider the following functions
$a(x,u,\nabla u) = |\nabla u|^{p(x)-2}\nabla u$ and $g(x,u,\nabla u) = (1 + |\nabla u|^{p(x)})|u|^{p(x)-2}u.$
It is clear that $a(x,u,\nabla u)$ and $g(x,u,\nabla u)$ verifies (5.3.2)-(5.3.5) and (5.4.2)-(5.4.3) respectively, then by theorem 5.4.1 for all $\mu \in M(\Omega)$ there exists a weak solution of the problem

$$\begin{cases} -\mathrm{div}\Big(|\nabla u|^{p(x)-2}\nabla u\Big) + (1 + |\nabla u|^{p(x)})|u|^{p(x)-2}u = \ \mu \qquad \text{in} \quad \Omega, \\ \qquad\qquad\qquad\qquad\qquad\qquad u = 0 \qquad \text{on} \quad \partial\Omega. \end{cases} \tag{5.5.1}$$

Chapter 6

Capacity, Dirichlet Energy and Weighted Variable Exponent Sobolev Spaces with Zero Boundary Values

6.1 Introduction

The first goal of this chapter is to give and prov some proprieties of the capacity in the setting of the weighted variable exponent Sobolev spaces. Moreover, we define and study weighted variable exponent Sobolev spaces with zero boundary values. This allows us to prove that the Dirichlet energy integral has a minimizer in the weighted variable exponent case.

6.2 Some proprieties of the $C_{p(.),\omega}$-capacity.

Definition 6.2.1. *[59] For $E \subset \mathbb{R}^N$, we denote*

$$S_{p(.),\omega}(E) = \{u \in W^{1,p(x)}(\mathbb{R}^N, \omega) \ : \ u \geqslant 1 \textit{ on an open set containing } E\}.$$

The Sobolev $(p(.), \omega)$-capacity of E is defined by

$$C_{p(.),\omega}(E) = \inf_{u \in S_{p(.),\omega}(E)} \rho_{1,p(.),\omega}(u) = \inf_{u \in S_{p(.),\omega}(E)} \int_{\Omega} |u(x)|^{p(x)} + |\nabla u(x)|^{p(x)} \omega(x) dx.$$

In case $S_{p(.),\omega}(E) = \varnothing$, *we set* $C_{p(.),\omega}(E) = \infty$.

Theorem 6.2.1. *Let's consider the following propositions:*
i) $f_n \longrightarrow f \;\; in \;\; W^{1,p(x)}(\mathbb{R}^N, \omega)$.
ii) $f_n \longrightarrow f$ *in* $C_{p(.),\omega} - capacity.$
iii) there is a subsequence (f_{n_j}) *such that :* $f_{n_j} \longrightarrow f, \quad C_{p(.),\omega} - q.u.$
iv) $f_{n_j} \longrightarrow f, \quad C_{p(.),\omega} - q.e.$
We have $i) \Rightarrow \; ii) \Rightarrow \; iii) \Rightarrow iv)$

Proof. Let show that $i) \Rightarrow ii)$.
By (1.5.4) we have f and f_n are finite for every n; $C_{p(.),\omega} - q.e.$
Let $\varepsilon > 0$, we have

$$C_{p(.),\omega}(\{x \;\; : \;\; |f_n - f|(x) > \varepsilon\}) \leqslant \rho_{1,p(.),\omega}(\frac{f_n - f}{\varepsilon}).$$

Since $\;f_n \longrightarrow f \;\; in \;\; W^{1,p(x)}(\mathbb{R}^N, \omega)$, then

$$(\forall \varepsilon > 0) : \;\; \rho_{1,p(.),\omega}(\frac{f_n - f}{\varepsilon}) \longrightarrow 0.$$

Therefore,

$$\lim_{n\to+\infty} C_{p(.),\omega}(\{x \;\; : \;\; |f_n - f|(x) > \varepsilon\}) = 0.$$

Let show that $ii) \Rightarrow \; iii)$.
Let $\varepsilon > 0 \;\; \exists \, f_{n_j}$ such that $C_{p(.),\omega}(\{x \; : \; |f_{n_j} - f|(x) > 2^{-j}\}) < \varepsilon.2^{-j}$.
We put

$$E_j = \{x \; : \; |f_{n_j} - f|(x) > 2^{-j}\} \quad \text{and} \;\; G_m = \bigcup_{j \geqslant m} E_j,$$

we have $C_{p(.),\omega}(G_m) \leqslant \sum\limits_{j \geqslant m} \varepsilon.2^{-j} < \varepsilon$.
On the other hand,

$$(\forall x \in (G_m)^c) \;\; : \; |f_{n_j} - f|(x) \leqslant 2^{-j}, (\forall j \geqslant m).$$

Thus

$$f_{n_j} \longrightarrow f \;\; C_{p(.),\omega} - q.u.$$

Let show that $iii) \Rightarrow \; iv)$.
We have $\forall j \in \mathbb{N}, \exists X_j : C_{p(.),\omega}(X_j) \leqslant \frac{1}{j}$ and $f_{n_j} \longrightarrow f \;\; on \;\; (X_j)^c$.
We put $X = \bigcap_j X_j$, then $C_{p(.),\omega}(X) = 0 \;\; and \;\; f_{n_j} \longrightarrow f \;\; on \; X^c$. $\square$

Theorem 6.2.2. *Let ω a positive measurable and finite function, such the condition (W_1) holds. If f_n , $f \in W^{1,p(x)}(\mathbb{R}^N,\omega)$ such that $f_n \rightharpoonup f$ weakly in $W^{1,p(x)}(\mathbb{R}^N,\omega)$, then*

$$\liminf (f_n)(x) \leqslant f(x) \leqslant \limsup f_n(x) \ \ C_{p(.),\omega} - q.e.$$

Proof. Since $W^{1,p(x)}(\mathbb{R}^N,\omega)$ is reflexive, by the Banach-Saks theorem, there is a subsequence denoted again (f_n) such that the sequence; $g_n = \frac{1}{n}\sum_{i=1}^{n} f_i$ converge to f strongly in $W^{1,p(x)}(\mathbb{R}^N,\omega)$. By theorem 6.2.1, there is a subsequence of (g_n) denoted again (g_n) such that

$$\lim_{n\to+\infty} g_n(x) = f(x) \ \ C_{p(.),\omega} - q.e.$$

On the other hand,

$$\liminf f_n(x) \leqslant \lim_{n\to+\infty} g_n(x) \ \ .$$

Therefore,

$$\liminf_{n\to+\infty} f_n(x) \leqslant f(x) \ \ \ C_{p(.),\omega} - q.e.$$

For the second inequality, it suffices to replace f_n by $(-f_n)$ in the first inequality. $\square$

Theorem 6.2.3. *Let $p(.) \in P^{log}(\mathbb{R}^N)$, $1 < p^- \leqslant p^+ < \infty$, and $\omega \in A_{p(.)}$.*
For each $f \in W^{1,p(x)}(\mathbb{R}^N,\omega)$, there is a $C_{p(.),\omega}$-quasicontinuous function $g \in W^{1,p(x)}(\mathbb{R}^N,\omega)$ such that $f = g$ $C_{p(.),\omega}$.q.e. The function g is called a $C_{p(.),\omega}$-quasicontinuous representative of the function f.

Proof. Let $f \in W^{1,p(x)}(\mathbb{R}^N,\omega)$. By theorem 1.4.1, there exists a sequence (f_n) in $C_0^\infty(\mathbb{R}^N)$ such that $f_n \longrightarrow f$ in $W^{1,p(x)}(\mathbb{R}^N,\omega)$.
By theorem 6.2.1, there exists a subsequence of (f_n) denoted again by (f_n) such that $f_n \longrightarrow f \ \ C_{p(.),\omega} - q.u$, and the proof is complete. $\square$

Theorem 6.2.4. *Let $1 < p^- \leqslant p^+ < \infty$, we have*
1) If O is an open set of $\mathbb{R}^N$ and $E \subset \mathbb{R}^N$ such that $|E| = 0$, then

$$C_{p(.),\omega}(O) = C_{p(.),\omega}(O - E).$$

2) Let u and v are $C_{p(.),\omega}$-quasicontinuous functions in $\mathbb{R}^N$, we have
i) if $u = v$, almost everywhere in an open set $O \subset \mathbb{R}^N$ then

$$u = v \ C_{p(.),\omega} - quasieverywhere \ in \ O,$$

ii) if $u \leqslant v$, almost everywhere in an open set $O \subset \mathbb{R}^N$ then

$$u \leqslant v\ C_{p(.),\omega} - quasieverywhere\ in\ O.$$

Proof. 1) It obvious that $C_{p(.),\omega}(O) \geqslant C_{p(.),\omega}(O - E)$. Let $u \in S_{p(.),\omega}(O - E)$ thus $u \geqslant 1$ in an open containing $O - E$. Let the function f define as

$$\begin{cases} f(x) = u(x) \quad , if\ x \in \mathbb{R}^N - E \\ f(x) = 1 \qquad\quad , if\ x \in E. \end{cases}$$

We have $f \in S_{p(.),\omega}(O)$ and $\rho_{1,p(.),\omega}(f) = \rho_{1,p(.),\omega}(u)$, thus

$$C_{p(.),\omega}(O) \leqslant \rho_{1,p(.),\omega}(u),$$

and by passing to inf we get $C_{p(.),\omega}(O) \leqslant C_{p(.),\omega}(O - E)$.
2) Since $C_{p(.),\omega}$ is an outer capacity we get the results by [61]. $\square$

6.3 Weighted variable exponents Sobolev spaces with zero boundary values

In this section, $\Omega \subset \mathbb{R}^N$ be an open and , $p(.) \in P^{log}(\mathbb{R}^N)$, $1 < p^- \leqslant p^+ < \infty$, ω is a positive measurable and finite function such that $\omega(x) \geqslant 1$ for $x \in \mathbb{R}^N$and $\omega \in A_{p(.)}$ and such the condition (W_1) holds.

Definition 6.3.1. *We say that a function u belongs to the weighted variable exponents Sobolev spaces with zero boundary values, and denoted $u \in B^{1,0}_{p(.)}(\Omega, \omega)$, if there is a $C_{p(.),\omega}$-quasicontinuous functions $\tilde{u} \in W^{1,p(x)}(\mathbb{R}^N, \omega)$ such that $\tilde{u} = u$ almost everywhere in Ω and $\tilde{u} = 0$ $C_{p(.),\omega}$ -quasi-everywhere in $\mathbb{R}^N \setminus \Omega$.*
The space $B^{1,0}_{p(.)}(\Omega, \omega)$ is endowed with the norm

$$||u||_{B^{1,0}_{p(.)}(\Omega,\omega)} = ||\tilde{u}||_{W^{1,p(x)}(\mathbb{R}^N,\omega)}.$$

Remark 6.3.1. *The norm of $B^{1,0}_{p(.)}(\Omega, \omega)$ does not depend on the choice of the $C_{p(.),\omega}$-quasicontinuous representative since $C_{p(.),\omega}(E) = 0$ implies that $|E| = 0$ for every $E \subset R^N$.*

Theorem 6.3.1. *$B^{1,0}_{p(.)}(\Omega, \omega)$ is a Banach spaces.*

Proof. Let $(u_i)_i$ be a Cauchy sequence in $B^{1,0}_{p(.)}(\Omega,\omega)$, for every i there is a $C_{p(.),\omega}$-quasicontinuous function $\tilde{u}_i \in W^{1,p(x)}(\mathbb{R}^N)$ such that $\tilde{u}_i = u_i$ almost everywhere in Ω and $\tilde{u}_i = 0$, $C_{p(.),\omega}$ -quasi-everywhere in $\mathbb{R}^N \setminus \Omega$.
Since $W^{1,p(x)}(\mathbb{R}^N,\omega)$ is a Banach space, there is a function $u \in W^{1,p(x)}(\mathbb{R}^N,\omega)$ such that $\tilde{u}_i \to u$ in $W^{1,p(x)}(\mathbb{R}^N,\omega)$ as $i \to \infty$. By theorem 6.2.1, u is $C_{p(.),\omega}$-quasicontinuous and by theorem 6.2.2 we have $u = 0$ $C_{p(.)}$ -quasi-everywhere in $\mathbb{R}^N \setminus \Omega$. Thus $u \in B^{1,0}_{p(.)}(\Omega,\omega)$ and the spaces $B^{1,0}_{p(.)}(\Omega,\omega)$ is complete. □

Corollary 6.3.1. *The space $B^{1,0}_{p(.)}(\Omega,\omega)$ is reflexive.*

Proof. The space $W^{1,p(x)}(\mathbb{R}^N,\omega)$ is reflexive Banach, by theorem 6.3.1 the space $B^{1,0}_{p(.)}(\Omega,\omega)$ is closed, and therefore $B^{1,0}_{p(.)}(\Omega,\omega)$ is reflexive. □

Corollary 6.3.2. *We have $W_0^{1,p(x)}(\mathbb{R}^N,\omega) \subset B^{1,0}_{p(.)}(\Omega,\omega) \subset W^{1,p(x)}(\mathbb{R}^N,\omega)$, where $W_0^{1,p(x)}(\mathbb{R}^N,\omega)$ is the closure of $C_0^\infty(\mathbb{R}^N)$ in $W^{1,p(x)}(\mathbb{R}^N,\omega)$, with respect the norm.*

Proof. The first inclusion follows from Theorem 6.3.1 since $C_0^\infty(\mathbb{R}^N) \subset B^{1,0}_{p(.)}(\Omega,\omega)$. The second inclusion follows directly from the definition of the space $B^{1,0}_{p(.)}(\Omega,\omega)$. □

Lemma 6.3.1. *Let $u \in B^{1,0}_{p(.),}(\Omega,\omega)$ and $v \in W^{1,p(x)}(\mathbb{R}^N,\omega)$. If u is bounded and v is $C_{p(.),\omega}$-quasicontinuous, then $uv \in B^{1,0}_{p(.)}(\Omega,\omega)$.*

Proof. Let $\tilde{u} \in W^{1,p(x)}(\mathbb{R}^N,\omega)$ be the $C_{p(.),\omega}$-quasicontinuous representative of u. we have $\tilde{u}v$ is $C_{p(.),\omega}$-quasicontinuous in $\mathbb{R}^N$. Let $D = \{x \in \mathbb{R}^N \setminus \Omega : \tilde{u}v \neq 0\}$, we have $D = E \cup F$ where
$E = \{x \in \mathbb{R}^N \setminus \Omega : \tilde{u} \neq 0\} \cap \{x \in \mathbb{R}^N \setminus \Omega : v \neq 0, \ \text{and} \ v \neq \infty\}$
and $F = \{x \in \mathbb{R}^N \setminus \Omega : v = \infty\}$. It obvious that $C_{p(.),\omega}(E) = C_{p(.),\omega}(F) = 0$, thus $C_{p(.),\omega}(D) = 0$ and $\tilde{u}v = 0$ $C_{p(.),\omega}$ -quasi-everywhere in $\mathbb{R}^N \setminus \Omega$. Since $\tilde{u}v = uv$ almost everywhere in Ω, then $uv \in B^{1,0}_{p(.)}(\Omega,\omega)$. □

Theorem 6.3.2. *Let $E \subset \Omega$ such that $C_{p(.),\omega}(E) = 0$, we have*

$$B^{1,0}_{p(.)}(\Omega,\omega) = B^{1,0}_{p(.)}(\Omega \setminus E,\omega).$$

Proof. It obvious that $B^{1,0}_{p(.)}(\Omega \setminus E, \omega) \subset B^{1,0}_{p(.),}(\Omega, \omega)$. Let $u \in B^{1,0}_{p(.)}(\Omega, \omega)$ there is a $C_{p(.),\omega}$-quasicontinuous function $\tilde{u} \in W^{1,p(x)}(\mathbb{R}^N, \omega)$ such that $\tilde{u} = u$ almost everywhere in Ω and $\tilde{u} = 0$ $C_{p(.),\omega}$ -quasi-everywhere in $\mathbb{R}^N \setminus \Omega$. Since $C_{p(.),\omega}(E) = 0$, we have $\tilde{u} = 0$ $C_{p(.),\omega}$ -quasi-everywhere in $\mathbb{R}^N \setminus (\Omega \setminus E)$. Thus

$$u \in B^{1,0}_{p(.)}(\Omega \setminus E, \omega).$$

□

Remark 6.3.2. *If $C_{p(.),\omega}(\partial\Omega) = 0$, then $B^{1,0}_{p(.)}(int\ \Omega, \omega) = B^{1,0}_{p(.),\omega}(\bar{\Omega}, \omega)$.*

Theorem 6.3.3. *Let E be a open subset of $\mathbb{R}^N$. Then $B^{1,0}_{p(.)}(\Omega, \omega) = B^{1,0}_{p(.)}(\Omega \setminus E, \omega)$ if and only if $C_{p(.),\omega}(E \cap \Omega) = 0$.*

Proof. It is the same as that given in [62], Theorem 4.8. □

6.4 The Dirichlet Energy Integral Minimisers

In this section, $\Omega \subset \mathbb{R}^N$ be an open and , $p(.) \in P^{log}(\mathbb{R}^N)$, $1 < p^- \leqslant p^+ < \infty$, ω is a positive measurable and finite function such that $\omega(x) \geqslant 1$ for $x \in \mathbb{R}^N$and $\omega \in A_{p(.)}$ and such the conditions (W_1) and (W_2) hold.

Definition 6.4.1. *Let $g \in W^{1,p(x)}(\Omega, \omega)$. For all $u \in B^{1,0}_{p(.)}(\Omega, \omega)$ we put*

$$I^{p(.),\omega}_{\Omega,g}(u) = \int_{\Omega} \big(|u(x)|^{p(x)} + \omega(x)\big(|\nabla u|^{p(x)} + |\nabla g|^{p(x)}\big)\big)dx$$

$I^{p(.),\omega}_{\Omega,g}$ is call the energy operator corresponding to the function g.

Lemma 6.4.1. *[63] Let E be a reflexive Banach space. If $I : E \longrightarrow \mathbb{R}$ is lower semicontinuous and coercive operator, then there is an element in E that minimizes I.*

Theorem 6.4.1. *There exists a function $u \in W^{1,p(x)}_0(\Omega, \omega)$ such that*

$$I^{p(.),\omega}_{\Omega,g}(u) = \inf_{v \in W^{1,p(x)}_0(\Omega,\omega)} I^{p(.),\omega}_{\Omega,g}(v). \tag{6.4.1}$$

Proof. By corollary 6.3.1 we have $W_0^{1,p(x)}(\Omega,\omega)$ is a reflexive Banach space. Since $x \longrightarrow x^p$ is convex for every fixed $1 < p < \infty$, then $I_{\Omega,g}^{p(.),\omega}$ is convex.
Let (u_i) be a sequence of functions in $W_0^{1,p(x)}(\Omega,\omega)$ converging to $u \in W_0^{1,p(x)}(\Omega,\omega)$. Then $u_i \longrightarrow u$ in $L^{p(x)}(\Omega)$ and $|\nabla u_i| \longrightarrow |\nabla u|$ in $L^{p(x)}(\Omega,\omega)$, since $\varrho^1_{(\Omega,\omega)}$ is lower semicontinuous [see [39] Theorem2.1.17] thus $I_{\Omega,g}^{p(.),\omega}$ is lower semicontinuous.
By remark 1.4.2 we have if $||u||_{W_0^{1,p(x)}(\Omega,\omega)} \longrightarrow \infty$ then $I_{\Omega,g}^{\varphi}(u) \longrightarrow \infty$, and therefore the operator $I_{\Omega,g}^{p(.),\omega}$ is coercive, and the theorem follows by lemma 6.4.1. $\square$

Theorem 6.4.2. *The $C_{p(.),\omega}$-quasicontinuous representative $\tilde{u}$ of the minimizing function u in (6.4.1) is unique up to set of zero $C_{p(.),\omega}$-capacity.*

Proof. Assume that u_1 and u_2 are two minimizers of (6.4.1) with

$$|\{\nabla u_1 \neq \nabla u_2\}| \neq 0.$$

We set $v = \frac{1}{2}(u_1 + u_2)$, thus

$$I_{\Omega,g}^{p(.),\omega}(v) < \frac{1}{2} I_{\Omega,g}^{p(.),\omega}(u_1) + \frac{1}{2} I_{\Omega,g}^{p(.),\omega}(u_2) = \inf_{u \in W_0^{1,p(x)}(\Omega,\omega)} I_{\Omega,g}^{p(.),\omega}(u)$$

which is a contradiction. Therefore $|\{\nabla u_1 \neq \nabla u_2\}| = 0$.
By the Poincaré inequality we get

$$||u_1 - u_2||_{L^{p(x)}(\Omega)} \leqslant C||\nabla u_1 - \nabla u_2||_{L^{p(x)}(\Omega,\omega)} = 0.$$

Then $u_1 = u_2$ for a.e. $x \in \Omega$.
Let $\tilde{u}_1$ and $\tilde{u}_2$ The $C_{p(.),\omega}$-quasicontinuous representative of u_1 and u_2, we have $\tilde{u}_1 = \tilde{u}_2$ for almost every $x \in \Omega$ and theorem 6.2.4 implies that $\tilde{u}_1 = \tilde{u}_2$ $C_{p(.),\omega}$-quasieverywhere in Ω. $\square$

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