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Inverse problem of the dual fractional Hankel transform in complex and bicomplex settings

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Abstract

In this thesis, we focus on two integral transforms in the complex and bicomplex settings, by studying their different properties and considering some of their applications. These transforms are the complex and bicomplex analogs of the dual fractional Hankel transform. In fact, we deal with the inverse problem, obtaining the integral representation, discussing compactness and singular values. Additionally, we look for the kernel function's properties and give explicit expressions for particular cases. Furthermore, concerning the bicomplex context, we concentrate first on the Bergman space of bc-meromorphic functions with a strong pole at the origin of the bicomplex discus. We also give the explicit expression for its reproducing kernel. Its characterization as the range of the bicomplex analog of the second Bargmann transform is also provided. Based on that, we construct the bicomplex analog of the fractional Hankel transform as well as its dual fractional transform, we describe their ranges and provide expressions for their reproducing kernels. The inverse of the dual transform of the bicomplex fractional Hankel transform is also considered.

Keywords: Fractional Hankel transform; Dual fractional Hankel transform; Weighted Bergman space; Bicomplex Bargmann transform; The modified bicomplex fractional Hankel transform.

Résumé

Dans cette thèse, nous nous concentrons sur deux transformées intégrales dans le cadre complexe et bicomplexe, en étudiant leurs différentes propriétés et en considérant certaines de leurs applications. Ces transformées sont les analogues complexes et bicomplexes de la transformée de Hankel fractionnaire duale. En fait, nous traitons le problème inverse, obtenons la représentation intégrale en discutons sur sa compacité et ses valeurs singulières. En outre, nous recherchons les propriétés de la fonction noyau et donnons des expressions explicites pour des cas particuliers. De plus, en ce qui concerne le contexte bicomplexe, nous nous concentrons d'abord sur l'espace de Bergman des fonctions bc-méromorphes avec un pôle fort à l'origine du disque bicomplexe. Nous donnons également l'expression explicite de son noyau reproduisant. Sa caractérisation en tant qu'intervalle de l'analogue bicomplexe de la seconde transformée de Bargmann est également fournie. Sur cette base, nous construisons l'analogue bicomplexe de la transformée fractionnaire de Hankel ainsi que sa transformée fractionnaire duale, nous décrivons leurs domaines et fournissons des expressions pour leurs noyaux reproduisant. L'inverse de la transformée duale de la transformée de Hankel fractionnaire bicomplexe est également considéré.

Mots-clés : Transformée fractionnaire de Hankel; Transformée duale de la transformée de Hankel fractionnaire; Espace de Bergman pondéré; Transformée de Bargmann bicomplexe ; Transformée fractionnaire de Hankel bicomplexe modifiée.

Résumé de la thèse

La transformation de Hankel classique a inspiré de nombreuses applications en mathématiques, physique, et ingénierie. Ceci a mené à la construction de la transformation de Hankel fractionnaire H_z^α, introduite par Namias, qui est une généralisation de la transformation de Hankel classique. Cette étude explore en profondeur plusieurs aspects des transformées intégrales dans les cadres complexe et bicomplexe de la transformée de Hankel fractionnaire duale S_y^α, qui est considérée comme l'expression de la transformation de Hankel fractionnaire pour un z variable et y fixe dans $\mathbb{R}^+$ c'est à dire, $S_y^\alpha(\varphi)(z) = H_z^\alpha(\varphi)(y)$.

L'un des objectifs principaux de notre thèse est de résoudre le problème inverse de la transformée de Hankel fractionnaire duale en étudiant la transformation inverse M_y^α de la forme restreinte de S_y^α, en particulier $(\ker(S_y^\alpha))^\perp$. Nous fournissons à la fois une série d'expansion et une représentation intégrale pour M_y^α, sous certaines conditions imposées sur y et aux paramètres α, β, et η. Notons que l'expression explicit du noyau de cet intégral semble difficile à détérminer car la description de l'ensemble des polynômes de Laguerre qui ont un zéro commun en y est inconnue en général. Néanmoins, nous explorons les propriétés fondamentales de ce noyau, y compris les formules différentielles auxquelles il adhère, et nous présentons ses expressions en imposant certaines conditions.

Dans le Chapitre 2, nous présentons les principaux résultats de l'inverse de la transformée de Hankel fractionnaire duale avec certaines propriétés fondamentales de son noyau, y compris sa formule explicit et l'étude de sa compacité pour des valeurs et conditions spéciales sur α, β, η et y, nous discutons de ses implications et fournissons explicitement les valeurs singulières correspondantes. La relation entre le noyau reproduisant de l'espace image de la transformée de Hankel fractionnaire duale et la transformées M_y^α sont aussi examinés.

En passant de l'analyse complexe à l'analyse hypercomplexe, les transformées intégrales sont également bien adaptées à cette théorie, puisque l'analyse du domaine des fréquences devient de plus en plus large. Cela fournit un cadre plus riche pour aborder des perspectives futures traitant des problèmes en traitement des signaux multidimensionnels et de la mécanique quantique.

Dans le Chapitre 3, nous nous intéressons aux analogues bicomplexes de différents outils, espaces et transformations intégrales, l'espace de Hilbert de dimension infinie associé et les fonctions bc-méromorphes. Plus précisément, nous commençons par rappeler quelques résultats préliminaires concernant les nombres bicomplexes. Nous fournissons

une caractérisation concrète de l'espace de Bergman bicomplexe modifié par γ, caractérisé par un pôle fort à l'origine du disque bicomplexe, et qui est constitué des fonctions bc-méromorphes sur le disque unité bicomplexe privée de la réunion de l'origine et de l'ensemble des diviseurs de zero (le cône nul), notée par $\mathcal{D}_{bc}^{*}$, permettant une meilleure compréhension de ses propriétés, de plus nous avons donné l'expression explicite du noyau reproduisant de cet espace, la représentation intégrale en tant qu'extension de la version bicomplexe de la seconde transformée de Bargmann généralisée, et qui est un opérateur isométrique unitaire associée de type Bargmann, a été établi avec ses propriétés correspondantes. Ensuite, nous étudions L'extension de la transformée de Hankel fractionnaire et sa transformée duale sous la forme bicomplexe. En approfondissant les propriétés de ces transformées, nous fournissons l'expression explicite pour le noyaux reproduisant de l'espace image de cette transformation duale bicomplexe, en impliquant les zéros des polynômes de Laguerre généralisés. Nous offrons une compréhension complète du problème inverse, en considérant également l'inverse de la version bicomplexe de la transformée de Hankel fractionnaire duale. Nous concluons notre étude par une forme généralisée de la transformée de Hankel fractionnaire duale bicomplexe.
En conclusion, cette thèse offre une analyse exhaustive des propriétés et des applications des transformées de Hankel fractionnaire duales dans les cadres complexe et bicomplexe. Les résultats obtenus, ouvrent la voie à des implications importantes dans l'analyse harmonique, et diverses applications dans les domaines scientifiques et techniques qui pourraient êtres utiles pour des nouveaux travaux de recherches pour le future.

List of publications

1. Ghanmi, A., Hammam, A. On the inverse of the dual fractional Hankel transform. Rend. Circ. Mat. Palermo, II. Ser 73, 1289–1297 (2024).

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Contents

General introduction

The Hankel transform of order ν, also known as the Fourier-Bessel transform $\mathcal{H}_\nu$, was defined by Hermann Hankel [1] in 1869 [61], as an intgral transform whose kernel is the Bessel function of the first kind. He was also the one who proposed the idea that it is self-reciprocal and the first proof of such formula has been made by Weyl [133] on the space of twice differentiable functions. Next, Watson [131] proved the result for f belonging to $L^1(\mathbb{R}^+, \sqrt{x}dx)$ and $\nu > -1$, this leads to the relation given by

$$f(t) = \int_0^{+\infty} \int_0^{\infty} f(x) J_\nu(xy) J_\nu(yt) xy dx dy. \tag{1}$$

Since then a lot of mathematicians and research scientists are inspired by this important mathematical tool to resolve different problems. Schwartz treated that inversion theorem, to prove [114]

$$\lim_{\lambda\to\infty} \left(\int_0^\lambda (xu)^{-\nu} J_\nu(xu) u^{2\nu+1} du \right) \left(\int_0^\infty f(y)(uy)^{-\nu} J_\nu(uy) y^{2\nu+1} dy \right) = \frac{f(x^+) + f(x^-)}{2}, \tag{2}$$

for an integrable function f on $\mathbb{R}^+$ relied with the measure $[2^\nu \Gamma(\nu+1)]^{-1} t^{2\nu+1} dt$ and continuous in the neighbourhood of $x > 0$ for any $\nu \geq -1/2$. In the same year, Sneddon has worked on the inversion theorem for the Hankel transform of order zero [119] on functions, defined on the space of continously differentiable functions on $\mathbb{R}^+$ and belonging to the space $L^1(\mathbb{R}^+, \sqrt{x}dx)$, the main objective of such study is to solve the Abel integral equation $\int_0^x \frac{f(t)dt}{\sqrt{(x^2-t^2)}} = g(x)$, for all $x > 0$, through some relations satisfied by the Bessel function of order zero. He was also the one who proved the relationship between the Fourier transform and the Hankel transform when acting on radial functions [120]. There are a plenty research works that are using this relation $\mathcal{H}_\nu^{-1} = \mathcal{H}_\nu$, in the space $L^2(\mathbb{R}^+)$ [72] and in the Schwartz space suggested by Duran [35] for $\nu > -1$. The basic properties have been explored extensively. Notice for instance that a Parseval's Theorem for Hankel transform was found by Macaulay-Owen [83]. Boundary value problems for the related dual integral equations

$$\int_0^\infty t^{-2\mu} \Psi(t) J_\nu(\rho t) dt = F(\rho),\ 0 < \rho < 1, \tag{3}$$

[1] Professor at the University of Leipzig and a student of Riemann [17]

and

$$\int_0^{\infty} \xi^{-2r}\Psi(t)J_\nu(\rho t)dt = G(\rho),\ \rho > 1, \tag{4}$$

were discussed in [132, 12, 126], where different solutions depending closely on the choice of the parameters μ, r, ν and the functions F, G. There are other equations where the Hankel transform is crucial, such as the axisymmetric diffusion [97], Cauchy-Poisson water wave [70] and Einstein field [11] equations and the acoustic radiation equation [124]. The Hankel transform plays an important role for the theoritical study of the acoustic propagation treated in [42, 89]. Moreover, there are a lot of contributions in engineering, physics and other scientific areas, using the Hankel transform, for instance, in statistics [80], magnetic fields [90], medical computed tomography [64], electromagnetic applications [6, 71], geophysics [79], hydrodynamics [134, 138] and imaging [116]. To mention some interesting studies, Eredlyi [38] and Soni [121] found that the Hankel transform is related to fractional integration, which is one of the main parts of our thesis and explained in the following paragraph.
In [103], Wiener sets out to find a one-parameter family of unitary integral operators

$$\mathcal{F}_\theta(\varphi)(x) := \int_{-\infty}^{+\infty} K_\theta(x,t)\varphi(t)dt$$

on $L^2(\mathbb{R})$ such that the n-th Hermite function $h_n(x) = e^{-x^2/2}H_n(x)$ is an eigenfunction with eigenvalue $e^{i\theta n}$. This leads to what is called sixty years later fractional Fourier transform (FrFT) (see e.g. [91]) , which has been inpired by quantum mechanics. In the same spirit and based on the Hille-Hardy identity, Namias introduced in [92] the fractional Hankel transform (FRHT) denoted by H_z^α and seen as an extension of the classical Hankel transform who rotates by the angle π, for which kernel is associated to the modified Bessel function of the first kind [7, p. 222 (4.12.2)], and reintroduced in [72] by Kerr who has implemented this study a few years later in the Zemanian space [73], this kind of transform can be defined on $L^2_\alpha(\mathbb{R}^+) = L^2(\mathbb{R}^+; d\nu_\alpha)$, the Hilbert space of all complex-valued square integrable functions on the real half line with respect to the measure $d\nu_\alpha(x) = x^\alpha e^{-x}dx$, $\alpha > 0$. The generalized Laguerre polynomials are diagonalizing eigenfunctions in $L^2_\alpha(\mathbb{R}^+)$, of the FRHT, with $z^n = r^n e^{in\xi}$ are the corresponding eigenvalues. Those tools are very interesting when it concerns the subjects of sampling [62] and signal processing [63]. In general, the FRHT is rotating by the angle $\xi = a\pi$. It has been found that the two dimensional extension of the fractional Fourier transform is related to the FRHT via the cylindrical coordinates [135]. Recently, such transform has been recovered by Bargmann's versus following an earlier Bargmann's idea [9] for constructing fractional integral transforms of Fourier type associated with special integral transforms. In fact, this follows making use of the second Bargmann transform [9, p.203] having as range the Bergman space of holomorphic functions on the unit disc $D = \{z \in \mathbb{C}; |z| < 1\}$, for more details see [36]. As a consequence, the semi-group property $H^\alpha_{pz} = H^\alpha_p \circ H^\alpha_z$ has been verified in the same reference [36]. In contrast with the inverse problem results for the classical Hankel transform, we can say that H_z^α is not necessarly self-reciprocal and its inverse transform when $z \neq 0$, is exactly $(H_z^\alpha)^{-1} = H_{1/z}^\alpha$. In that sens, the inversion formula of the FRHT is meaningful when it gets in touch with problems of differential equations particularly in the Zemanian space [125]. However, it is important to point out on significant effect of the FRHT, resolving physical problems especially in optics [43, 4]. Furthermore, it aligns with problems concerning the wavelet transform [86, 98] and Gelfand–Shilov spaces [87]. Nevertheless, the usefulness of the FRHT has been considered in many different mathematical aspects [117, 137, 127]. In that context, there is a lot of works that are developed every year, such as the dual fractional Hankel transform (DFrHT) S_y^α, defined in [48], that is seen as the expression of the FRHT for a variating z and a fixed $y \in \mathbb{R}^+$ (i.e, $S_y^\alpha(\varphi)(z) = H_z^\alpha(\varphi)(y)$). The boundeness of such operator, from $L_\alpha^2(\mathbb{R}^+)$ into the weighted Hilbert space $L_\eta^{2,\beta}(D)$ of all complex-valued square integrable measurable functions on the unit disc D with the corresponding weight function $A_{\beta,\eta}(|z|^2) = |z|^{2\beta-2}(1-|z|^2)^{\eta-1}$ for any $\alpha, \beta, \eta > 0$ and $z \in D$, has been proved in [48]. The range of the DFrHT $R^{\beta,\eta}(S_y^\alpha) := S_y^\alpha(L_\alpha^2(\mathbb{R}^+)) \cap L_\eta^{2,\beta}(D)$ is contained in the weighted Bergman space consisting of holomorphic functions on D belonging to $L_\eta^{2,\beta}(D)$ and is related with the set, denoted by N_y^α, of the positive integers n for which the generalized Laguerre polynomials $L_n^{(\alpha)}$ have y as common zero (see [48] for details). We have $R^{\beta,\eta}(S_y^\alpha)$ is identical to the weighted Bergman space $\mathcal{A}_\eta^{2,\beta}(D)$, whenever N_y^α is empty. However, $R^{\beta,\eta}(S_y^\alpha)$ is strictly contained in $\mathcal{A}_\eta^{2,\beta}(D)$, if $N_y^\alpha \neq \emptyset$. On the other hand, it is worth noting that the DFrHT S_y^α is regarded as the second Bargmann transform in [9, p.203] when $\beta = 1$ and $y = 0$. Additionally, the proofs, for the null space of S_y^α being spanned by $L_n^{(\alpha)}$ with varying $n \in N_y^\alpha$, and for the compactness of S_y^α identifying its singular values and its membership in p-Schatten classes, also the result that verifying operational formula in accordance with the second Bargmann transform, all of these are discussed in [48]. According to the fact that the transform S_y^α is linked to H_z^α by the relationship $\psi(x,z,\alpha) := S_x^\alpha(\varphi)(z) = H_z^\alpha(\varphi)(x)$, one gets $\varphi = (H_z^\alpha)^{-1}(\psi(\cdot, z, \alpha))$ for fixed z in the punctured unit disc, where $\psi(x,z,\alpha)$ is viewed as function in x-variable and belonging to $L_\alpha^2(\mathbb{R}^+)$. More explicitly, we have the inversion formula

$$\varphi(y) = \frac{z}{z-1} \int_0^{+\infty} \exp\left(-\frac{xz+y}{z-1}\right) \left(\frac{xz}{y}\right)^{\alpha/2} I_\alpha\left(\frac{2\sqrt{xyz}}{z-1}\right) \psi(x,z,\alpha) dx. \tag{5}$$

However, the situation is completely different when looking of an inversion formula of S_x^α as a function in the z-variable in the unit disc D. In fact, the inverse problem for the transform S_x^α needs further investigation, even in the case of $N_x^\alpha = \emptyset$.

Our aim is to discuss the inverse problem for the transform S_y^α. We deal with M_y^α, the inverse transform of S_y^α, operating from the orthogonal complement of the null space of the DFrHT $(\ker(S_y^\alpha))^\perp$ into the range space $R^{\beta,\eta}(S_y^\alpha)$. The expansion series of M_y^α are provided with their integral representation under specific conditions on y, α, β, η. Since, it is hard to determine a general closed expression of its kernel as well as the difficulty to find the elements belonging to the set N_y^α. But, we can treat some basic properties and give the explicit expression of its kernel in some particular cases. The main results reposes on the boundeness and the compactness of the inverse transform M_y^α of the dual fractional Hankel transform provided with its singular values, via the conditions imposed on $\beta, \eta, \alpha > 0$ and $y \in \mathbb{R}^+$. This is summarized in two important theorems given in Chapter 2. In Theorem 2.2.1, we state that the integral operator M_y^α is bounded from $R^{\beta,\eta}\left(S_y^\alpha\right)$ into $L_\alpha^2\left(\mathbb{R}^+\right)$ and its integral kernel is expressed in terms of an absolute convergent serie, subject to a valid condition on the parameter y. In Theorem 2.2.2, the compactness of M_y^α appears with its corresponding singular values in depend on the same choice of the arbitrary element y.

Moving from complex to hypercomplex analysis, the integral transforms are as well adapted to this theory. This transition into the topology of four-dimensional has been taking the place in the quaternions, presented by W.R.Hamilton [58, 59]. Recent development of the quaternionic Hankel transform and its application to the Cauchy's problem, has been given in [94]. But, before the last one's study, the extension of its fractional transform has been done in [36]. On that field of research, we will deal in particular with bicomplex analysis, which its algebra was discovered by Corrado Segre [115], this is similar to the other one named the algebra of Tessarines constructed between 1848 and 1850 by Cockle in his papers [24, 25, 26, 27], where both their commutativity relations are satisfied unlike the quaternionic one due to the existence of the zero divisors on those cases. Segre has also seen that the elements depending on the commutative imaginary units i, j, given by $(1+ij)/2$ and $(1-ij)/2$ are idempotents, those ones are considered as the building blocks of the theory of bicomplex numbers. The construction of functions of bicomplex variables are initiated by Ringleb [109] and Riley [108], who also developed the idea of analytic functions of bicomplex variable related with two holomorphic functions on $\mathbb{C}$. Price [99] introduced the bicomplex holomorphic functions. This has made the push to give rise to the notion of bicomplex meromorphic functions [20, 21]. Other mathematical and scientific results have been occured in this area of research such as bicomplex functional analysis [122, 123, 34, 46], quantum mechanics [110, 111, 112], telecommunication [15, 16] and bicomplex differentiability [82]. These ones include as well integral transforms. Here are a few examples conducted in this decade, among them we have Laplace [76], Mellin [3], Bargmann and fractional Fourier [52], Stieltjes [2], Hankel [1] transforms.

In our work, we extend and generalize the already elaborated tools in [36, 48] for FRHT and DFrHT to the bicomplex context with a more general setting. To this purpose, we generalize the bicomplex Bergman spaces [106, 107] by envisaging the case of the bicomplex meromorphic functions with the only strong pole at the origin. This constitutes a bicomplex analog of the modified Bergman space introduced and studied in [54, 53, 51]. It will be shown in Proposition 3.2.3 to be an infinite bicomplex Hilbert space in the sense of [46]. An explicit characterization of its elements as power series as well as the expression of its reproducing kernel are given. The integral realization as a phase space by means of a Bargmann type transform is discussed in Subsection 3.2.2. The considered versus in [36] will be applied to extend the fractional Hankel transform to the bicomplex setting. Its dual transform $S_{y,bc}^{\eta,\gamma}$ is also investigated. In fact a concrete characterization of its range $\mathbf{R}_{y,bc}^{\eta,\gamma} := S_{y,bc}^{\eta,\gamma}(L_{bc}^2(\mathbb{R}^+; t^{\eta+1}e^{-t}dt))$, in the Hilbert space of square integrable

bc-holomorphic functions on $\mathcal{D}^*_{bc}$, the bicomplex discus deprived of the union of its origin and the null-cone, is provided and the explicit expression of its reproducing kernel is also given. Such description depends intimately of the zeros of the generalized Laguerre polynomials. The treatment of the inverse of the dual fractional fractional within the bicomplex structure is also discussed throughout this thesis, which is seen as an idempotent decomposition of the inverse transform studied in Section 2.2.
The outline of our thesis is organized based on the following key points.

- In Chapter 1, we begin by reviewing and recalling the definition of the Fourier, Hankel and fractional Hankel transforms and we present some of their preliminary results.

- In Chapter 2, we reintroduce the DFrHT [48], discuss some of its properties. We also gave its inverse transform, focusing on its integral kernel and proving its boundness based on some conditions.

- In Chapter 3, we recall some results about bicomplex numbers, infinite dimensional bicomplex Hilbert space and the bc-meromorphic functions. We gave rise to the γ-modified bicomplex Bergman space of bc-meromorphic functions on $\mathcal{D}^*_{bc}$ and the bicomplex second Bargmann transform, bicomplex versions of the fractional Hankel transform and the dual transform, along with the characterization of the range $\mathbf{R}^{\eta,\gamma}_{y,bc}$, are provided as well as their inverse transform. The generalized form of the bicomplex DFrHT is considered.

Chapter 1

Fractional Fourier and Hankel transforms

1.1 Fourier, Hankel and Fractional Fourier transforms

1.1.1 Classical Fourier and fractional Fourier transforms

For a complex-valued or real-valued $T-$periodic function $g \in \mathfrak{C}^{\infty}(\mathbb{R})$. The Fourier coefficients, are defined by

$$a_k = \frac{2}{T}\int_0^T g(t)\cos(\frac{2\pi kt}{T})\ \mathrm{d}t, \tag{1.1}$$

$$b_k = \frac{2}{T}\int_0^T g(t)\sin(\frac{2\pi kt}{T})\ \mathrm{d}t, \tag{1.2}$$

$$\widehat{g}(k) = c_k = \frac{1}{T}\int_0^T g(t)e^{-2i\pi kt/T}\ \mathrm{d}t. \tag{1.3}$$

The sequence of partial sums given by

$$s_n(g)(t) = \frac{a_0}{2} + \sum_{k=1}^{n}\left(a_k\cos\frac{2\pi kt}{T} + b_k\sin\frac{2\pi kt}{T}\right) = \sum_{k=-n}^{n} c_k e^{2\pi ikt/T} \tag{1.4}$$

converges to g in $L^2([0,T])$ when $n \longrightarrow +\infty$. In the sense, that

$$\int_0^T |g(t) - s_n(g)(t)|^2\ \mathrm{d}t \to 0, \tag{1.5}$$

holds whenever $n \to \infty$. Moreover, we have the Parseval's identity

$$\frac{1}{T}\int_0^T |g(t)|^2\ \mathrm{d}t = \sum_{k=-\infty}^{\infty} |\hat{g}(k)|^2. \tag{1.6}$$

Dirichlet stated that if g is characterized by all of the three conditions

(i) g is absolutely integrable on $[-T/2, T/2]$.

(ii) g has finite number of discontinuities on the interval $[-T/2, T/2]$.

(iii) g is a T-periodic function that has a finite number of minima and maxima.

Then, for T goes to $+\infty$, we have

$$\frac{g(x^+) + g(x^-)}{2} = \lim_{n \longrightarrow +\infty} s_n(g) = \int_{-\infty}^{\infty} e^{2\pi ikx} \left[\int_{-\infty}^{\infty} g(\xi) e^{-i2\pi k\xi} d\xi \right] dk. \tag{1.7}$$

where $g(x^+) = \lim_{y \longrightarrow x^+} g(y)$ and $g(x^-) = \lim_{y \longrightarrow x^-} g(y)$. The definition of the next integral transform is retrieved from the formula in (1.7)

Definition 1.1.1. *The classical Fourier transform, denoted by $\mathcal{F}$, is defined by:*

$$\widehat{g}(k) = \mathcal{F}(g)(k) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} e^{-ikx} f(x) dx \tag{1.8}$$

where g is a function absolutely integrable on $\mathbb{R}$.

The Plancherel's theorem, which says that

$$\int_{-\infty}^{+\infty} |g(t)|^2 dt = \frac{1}{2\pi} \int_{-\infty}^{+\infty} |\widehat{g}(t)|^2 dt, \tag{1.9}$$

whenever $g \in L^1(\mathbb{R}) \cap L^2(\mathbb{R})$ and $\widehat{g}$ is absolutely integrable on $\mathbb{R}$. Moreover Fourier proves the theorem below [41, p.42]

Theorem 1.1.2 (Fourier's inversion theorem)**.** *Let $g : \mathbb{R} \longrightarrow \mathbb{C}$ be a continuous function absolutely integrable on $\mathbb{R}$ satisfying*

$$\int_{\mathbb{R}} |\widehat{g}(x)| dx < +\infty. \tag{1.10}$$

Then

$$g(x) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} \widehat{g}(t) e^{ixt} dt, \tag{1.11}$$

Clearly, from Theorem 1.1.2 the inverse of the classical Fourier transform is given by

$$\mathcal{F}^{-1}(g)(k) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} g(t) e^{itk} dt, \tag{1.12}$$

where $g \in L^1(\mathbb{R})$. Implicitly, the extension form of the Fourier transform to the fractional type transform with the kernel $K(x, y)$, was firstly found by Wiener [103] in 1929,his motivation was to solve this integral equation

$$e^{-\omega^2 t^2 + 2t\omega x - (x^2/2)} = \int_{-\infty}^{\infty} e^{t^2 + 2ity - (y^2/2)} K(x, y) dy,$$

by using this correct consideration

$$\omega^n H_n(x) e^{-(x^a/2)} = \int_{-\infty}^{+\infty} i^n H_n(y) e^{-\left(y^2/2\right)} K(x,y) dy.$$

In 1980, Namias rediscovered the fractional Fourier transform for Quantumn mechanichal reasons. He begined with the definition where $e^{-t^2/2}H_n(t)$ is an eigenfunction of the operator $\mathcal{F}_\alpha$ within its eigenvalue $e^{in\alpha}$ for $t \in \mathbb{R}$. He extended that definition throughout a square integrable function f written as $f(t) = \sum_{n=0}^{+\infty} a_n e^{-t^2/2} H_n(t)$ since the family of Hermite polynomials $\{H_n\}_n$ constitute an orthogonal basis of $L^2(\mathbb{R})$. He involved $\mathcal{F}_\alpha$ on f, so using the expression of a_n

$$a_n = \frac{1}{2^n n! \sqrt{\pi}} \int_{-\infty}^{+\infty} H_n(x) \exp\left(-t^2/2\right) f(t) dt \tag{1.13}$$

and also combining it with Mehler formula

$$\sum_{n=0}^{\infty} \frac{z^n}{2^n n!} H_n(x) H_n(t) = \frac{1}{\sqrt{1-z^2}} \exp\left[\frac{2xtz - z^2\left(x^2+t^2\right)}{1-z^2}\right]. \tag{1.14}$$

Namias found the serie expression and the integral representation of $\mathcal{F}_\alpha$ resumed on that equation

$$\mathcal{F}_\alpha f(t) = \sum_{n=0}^{+\infty} a_n e^{in\alpha} e^{-t^2/2} H_n(t) \tag{1.15}$$

$$= \frac{1}{\sqrt{\pi}\sqrt{1-e^{2i\alpha}}} \int_{-\infty}^{+\infty} \exp\left[\frac{2txe^{i\alpha} - e^{2i\alpha}\left(t^2+x^2\right)}{1-e^{2i\alpha}}\right] \exp\left(-\frac{t^2}{2} - \frac{x^2}{2}\right) f(x) dx, \tag{1.16}$$

which can be reduced to this expression

$$\mathcal{F}_\alpha(g)(y) = \int_{-\infty}^{+\infty} K_{\alpha,y}(t) g(t) dt, \tag{1.17}$$

where $K_{\theta,y}$ is

$$K_{\alpha,y}(t) = \begin{cases} \sqrt{\frac{1-i\cot\alpha}{2\pi}} \exp\left(-i/2\left(\left(t^2+y^2\right)\cot(\alpha) - \frac{yt}{\sin\alpha}\right)\right), & \alpha \neq k\pi \\ \delta(t-y), & \alpha = 2k\pi \\ \delta(t+y), & \alpha = (2k-1)\pi \end{cases} \tag{1.18}$$

for given $\alpha, y \in \mathbb{R}$, k an integer and δ is the Dirac delta function. Many research studies has been published on that subject, which has been widened to the classical two dimensional fractional Fourier transform characterized by its kernel, which is the tensor product $K_{\alpha_1,y_1} \otimes K_{\alpha_2,y_2}$. The two dimensional fractional Fourier transform has been extensively studied, see e.g. [95, 104, 102], and has many applications in optics and color imaging as well as in many other research areas [40, 101]. The inverse transform is given by [91]

$$\mathcal{F}_{-\alpha}f(t) = \frac{\exp -i\left(\frac{\pi}{4}-\frac{\alpha}{2}\right)}{\sqrt{2\pi\sin\alpha}}\exp\left(\frac{i}{2}\cot\alpha t^2\right) \int_{-\infty}^{+\infty}\exp\left(\frac{i}{2}\cot\alpha x^2 - \frac{itx}{\sin\alpha}\right)f(x)\,dx. \tag{1.19}$$

The Definition 1.1.1 can be extended to high dimension n, outlined by this formula

$$\mathcal{F}h(\boldsymbol{\kappa}_1,\dots,\boldsymbol{\kappa}_n) = \frac{1}{(2\pi)^{n/2}}\int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty} e^{-i\langle\kappa,X\rangle}h(\mathbf{x}_1,\dots,\mathbf{x}_n)d\mathbf{x}_1\dots d\mathbf{x}_n. \tag{1.20}$$

Its inverse is charachterized by

$$\mathcal{F}^{-1}h(\boldsymbol{\kappa}_1,\dots,\boldsymbol{\kappa}_n) = \frac{1}{(2\pi)^{n/2}}\int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty} e^{i\langle\kappa,X\rangle}h(\mathbf{x}_1,\dots,\mathbf{x}_n)d\mathbf{x}_1\dots d\mathbf{x}_n, \tag{1.21}$$

where $X=(\mathbf{x}_1,\dots,\mathbf{x}_n)$ and $\kappa=(\boldsymbol{\kappa}_1,\dots,\boldsymbol{\kappa}_n)$ are both $n-tuple$ vectors.

1.1.2 Classical Hankel transform

The introduction of the Hankel transform also known as the Fourier-Bessel transform, has been discrovered in the nineteenth century by the German mathematician Hermann Hankel [61]. Since then, this transform has been widely implemented in numerous research areas and problems. Here, particularly, we mention a few applications of that integral operator, including optics [84, 57], predicting sound propagation from aerodynamic noise sources [66] and the study of the uncertainty principles [113] in the field of quantum mechanic. We recall here the construction of Hankel transform.

Definition 1.1.3. *[131, 92, 72]Let h be a square integrable function on a positive real-line, then the Hankel transformation of order ν denoted by $\mathcal{H}_\nu$, is given by:*

$$\mathcal{H}_\nu(h)(x) = \int_0^{+\infty} yJ_\nu(xy)h(y)dy, \quad \Re(\nu) > -1. \tag{1.22}$$

Such transform is closely connected to the two-dimensional Fourier transform [33]

$$\mathcal{F}(f)(t,s) = T(t,s) = \frac{1}{2\pi}\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}\exp(-i(tx+sy))f(x,y)dxdy. \tag{1.23}$$

Indeed, involving the polar coordinates

$$\begin{cases} x = r\cos\theta, & t = p\cos\phi \\ y = r\sin\theta, & s = p\sin\phi, \end{cases} \tag{1.24}$$

we can enhance (1.23)

$$\begin{aligned} T(p,\phi) &= \frac{1}{2\pi}\int_0^{+\infty}\int_0^{2\pi} e^{-ipr(\cos\phi\cos\theta+\sin\phi\sin\theta)}f(r,\theta)rdrd\theta && (1.25)\\ &= \frac{1}{2\pi}\int_0^{+\infty}\int_0^{2\pi} e^{-ipr(\cos(\theta-\phi))}f(r,\theta)rdrd\theta. && (1.26)\end{aligned}$$

By involving the change $\theta-\phi=\varphi-\pi/2$, it follows

$$T(p,\phi)=\frac{1}{2\pi}\int_0^{+\infty}\int_{\pi/2-\phi}^{5\pi/2-\phi} e^{-ipr\sin\varphi}f(r,\varphi+\phi-\pi/2)rdrd\varphi. \quad (1.27)$$

Lets put $f(r,\varphi+\phi-\pi/2)=e^{in(\varphi+\phi-\pi/2)}h(r)$, where $h\in L^2(\mathbb{R}^+)$ and n is a positive integer. Here, by combining the integral representation of the Bessel function given by

$$J_n(pr)=\frac{1}{2\pi}\int_{\pi/2-\phi}^{2\pi+\pi/2-\phi} e^{i(n\varphi-pr\sin\varphi)}d\varphi \quad (1.28)$$

with (1.27), we get

$$T(p,\phi)=\exp[in(\phi-\pi/2)]\int_0^{+\infty} rJ_n(pr)h(r)dr. \quad (1.29)$$

Observe that, $T(p,\phi)\exp[-in(\phi-\pi/2)]$ is exactly the Hankel transform of h.

The classical Hankel transform $\mathcal{H}_\nu$ is self-reciprocal [72] on $L^2(\mathbb{R}^+)$ (i.e.$\mathcal{H}_\nu\equiv\mathcal{H}_\nu^{-1}$) and its inverse is given by

$$\mathcal{H}_\nu^{-1}(F_\nu)(y)=\phi(y)=\int_0^{+\infty} xJ_\nu(xy)F_\nu(x)dx,\ \text{where } F_\nu=\mathcal{H}_\nu(\phi). \quad (1.30)$$

The last one proof is obtained using the relation for Bessel function [8, p.696] (also see [78])

$$\int_0^{\infty} xtJ_\nu(xt)J_\nu(at)dt=\delta(x-a),\quad \operatorname{Re}\nu>-1, x>0, a>0 \quad (1.31)$$

and Fubini's theorem for the following calculation

$$\begin{aligned}\mathcal{H}_\nu^{-1}\mathcal{H}_\nu\phi(y) &= \int_0^{+\infty}\rho\mathcal{H}_\nu\phi(\rho)J_n(\rho y)d\rho\\ &= \int_0^{+\infty}\rho\left(\int_0^{+\infty}x\phi(x)J_\nu(x\rho)dx\right)J_\nu(\rho y)d\rho\\ &= \int_0^{+\infty}x\phi(x)\left(\int_0^{+\infty}\rho J_\nu(x\rho)J_\nu(\rho y)d\rho\right)dx\\ &= \int_0^{+\infty}x\phi(x)\frac{\delta(x-y)}{x}\,\mathrm{d}x=\phi(y).\end{aligned}$$

Additionally, there are several applications of the Hankel transform in physics, optics, engineering especially when we get involved with the partial differential equations. For instance, the zero order Hankel transforms are considered as important tasks for solving Laplace's equation for the electric potential υ in polar coordinates [96]

$$\nabla^2 \upsilon := \frac{\partial^2 \upsilon}{\partial r^2} + \frac{1}{r}\frac{\partial \upsilon}{\partial r} + \frac{\partial^2 \upsilon}{\partial z^2} = 0 \quad (\text{ without } \theta \text{ being a variable}).$$

This is due to the fact that the action of the Hankel transform of order zero on $\nabla^2 \upsilon$

$$\mathcal{H}_0 \left\{ \nabla^2 \upsilon \right\} = -s^2 V(s,z) + \frac{\partial^2 V}{\partial z^2}(s,z) = 0, \tag{1.32}$$

leads to the following result

$$\upsilon(r,z) = \frac{2\upsilon_0}{\pi} \int_0^\infty \frac{\sin s}{s} e^{-sz} J_0(sr) ds.$$

This was found also by using both the solution $V(s,z) := \mathcal{H}_0\{\upsilon(r,z)\}$ of the differential equation (1.32) given by $V(s,z) = A(s)e^{-sz} + B(s)e^{sz}$, and Boundary conditions on υ, for which the notation υ_0 came from, and from which we get $A(s) = \frac{\sin s}{s}, B(s) = 0$. Expanding on the wide-ranging utility of the Hankel transform, we present below a few examples showcasing the Hankel transform used on some various functions.

Examples 1.1.1. *Here are some examples of the Hankel transform applied to some specified functions:*

(1)

$$\mathcal{H}_0 \left(\frac{1}{r} e^{-ar} \right)(\kappa) = \int_0^{+\infty} e^{-ar} J_0(\kappa r) dr = \frac{1}{\sqrt{\kappa^2 + a^2}}, \quad a > 0,$$

(2)

$$\mathcal{H}_0 \left(\frac{\delta(r)}{r} \right)(\kappa) = \int_0^{+\infty} \delta(r) J_0(\kappa r) dr = 1,$$

(3) Let H denotes the Heaviside function, $a > 0$, and δ is the Dirac delta function. Then

$$\mathcal{H}_0(H(a-r))(\kappa) = \int_0^a r J_0(\kappa r) dr = \frac{1}{\kappa^2} [\rho J_1(\rho)]_0^{a\kappa} = \frac{a}{\kappa} J_1(a\kappa).$$

(4)

$$\mathcal{H}_1 \left(\frac{1}{r} e^{-ar} \right)(\kappa) = \int_0^{+\infty} e^{-ar} J_1(\kappa r) dr = \frac{1}{\kappa} \left[1 - a(\kappa^2 + a^2)^{-1/2} \right], \ a > 0$$

(5)

$$\mathcal{H}_1(e^{-ar})(\kappa) = \int_0^{+\infty} r e^{-ar} J_1(\kappa r) dr = \frac{\kappa}{(\kappa^2 + a^2)^{3/2}}, \ a > 0.$$

(6)

$$\mathcal{H}_1 \left(\frac{\sin ar}{r} \right)(\kappa) = \int_0^a \sin(ar) J_1(\kappa r) dr = \frac{1}{\kappa^2} [\rho J_1(\rho)]_0^{a\kappa} = \frac{a}{\kappa} J_1(a\kappa), \ a > 0.$$

Consider the operator S_b characterized by $S_b f(r) = f(br)$ which belongs to $L^2(\mathbb{R}^+)$ for $b \neq 0$. Its Hankel transform of order ν is given by

$$\mathcal{H}_\nu \left[S_b f\right](\rho) = \frac{1}{b^2}\mathcal{H}_\nu[f]\left(\frac{\rho}{b}\right). \tag{1.33}$$

(8) *Let f be a function in $\mathcal{S}(\mathbb{R}^{*+})$, the Schwartz space on the strictly positive real line. The Hankel transform of the derivative of f verifies the following operational formulas*

$$\mathcal{H}_\nu \left[f'\right](\rho) = \frac{\rho}{2\nu}\left[(\nu-1)\mathcal{H}_{\nu+1}[f](\rho) - (\nu+1)\mathcal{H}_{\nu-1}[f](\rho)\right] \tag{1.34}$$

and

$$\mathcal{H}_\nu\left[\frac{d^2 f}{dr^2} + \frac{1}{r}\frac{df}{dr} - \frac{\nu^2}{r^2}f\right](\rho) = -\rho^2\mathcal{H}_\nu(f)(\rho). \tag{1.35}$$

using the integration by parts and the reccurence relations for the Bessel function [100][p.111]

$$xJ_n'(x) = xJ_{n-1}(x) - nJ_n(x) \tag{1.36}$$

(9) *The Hankel transform satisfies the following formula*

$$\int_0^{+\infty} r\mathcal{H}_\nu[g](r)f(r)\mathrm{d}r = \int_0^{+\infty} rg(r)\mathcal{H}_\nu[f](r)\mathrm{d}r, \tag{1.37}$$

for $f, g \in L^2(\mathbb{R}^+)$.

(10) *Parseval's identity for Hankel transform for every $f \in L^2(\mathbb{R}^+)$*

$$\int_0^\infty f(x)\mathcal{H}_\nu^{-1}(G)(x)dx = \int_0^\infty G(y)\mathcal{H}_\nu(f)(y)dy, \quad \nu \geq -1/2 \tag{1.38}$$

1.1.3 Non-trivial 2-D fractional Fourier transform

In a noteworthy contribution, Zayed [136] has introduced a non-trivial two dimensional fractional Fourier transfrom, whose kernel function does not arise as tensor product of the kernel function of the one-dimensional fractional Fourier transform.

Definition 1.1.4. *The non-trivial two dimensional fractional Fourier transform of a function f in $L^1(\mathbb{R}^2)$ for $z = x + iy$ and $w = u + iv$, is defined by*

$$\mathcal{F}_{s,t}f(z,\bar{z}) = \int_{\mathbb{R}^2} k\left(z,\bar{z},w,\bar{w};s,t\right) f(w,\bar{w})dudv, \tag{1.39}$$

where $k\left(z,\bar{z},w,\bar{w};s,t\right) = K\left(z,\bar{z},w,\bar{w};s,t\right)e^{-\left(\frac{u^2+v^2+x^2+y^2}{2}\right)}$.

Due to an important step in his motivation, he found that the two variables complex Hermite functions $h_{m,n}(z,w) = e^{-|z|^2/2}H_{m,n}(z,w)$ are eigenfunctions of the 2-D Fourier transform

$$\mathcal{F}(h_{m,n})(z) = \frac{1}{2\pi}\int_{\mathbb{R}^2} h_{m,n}(w,\bar{w})\,\mathrm{e}^{\mathrm{i}(ux+vy)}\mathrm{d}u\ \mathrm{d}v = \mathrm{e}^{\mathrm{i}(m+n)\pi/2}h_{m,n}(z,\bar{z}), \tag{1.40}$$

where $H_{m,n}$ denotes the complex Hermite polynomial [68] defined by

$$H_{m,n}(z,w) = \sum_{k=0}^{m\wedge n} (-1)^k k! \begin{pmatrix} m \\ k \end{pmatrix} \begin{pmatrix} n \\ k \end{pmatrix} z^{m-k} w^{n-k}, m \wedge n = min(m,n). \tag{1.41}$$

Such polynomials are satisfying the orthogonality relation

$$\frac{1}{\pi} \int_{\mathbb{R}^2} H_{m,n}(z,\bar{z}) \bar{H}_{p,q}(z,\bar{z}) \mathrm{e}^{-|z|^2} \, \mathrm{d}x \, \mathrm{d}y = m!n! \delta_{m,p} \delta_{n,q}, \tag{1.42}$$

where $z = x + iy, w = u + iv \in \mathbb{C}$. The formula (1.42) gives rise to the following

$$\int_{\mathbb{R}^2} K(z,\bar{z},w,\bar{w};s,t) \bar{H}_{p,q}(w,\bar{w}) \, \mathrm{d}u \mathrm{d}v = t^q s^p H_{q,p}(z,\bar{z}). \tag{1.43}$$

In this subsection the kernel of the non-trivial two dimensional fractional Fourier transform is exactly the Mehler's formula for the complex Hermite polynomial [49, 68]

$$K(z,\bar{z},w,\bar{w};s,t) = \sum_{m,n=0}^{\infty} H_{m,n}(z,\bar{z}) H_{n,m}(w,\bar{w}) \frac{t^m}{m!} \frac{s^n}{n!} \tag{1.44}$$

$$= \frac{1}{1-ts} \exp\left\{ \frac{-tsz\bar{z} + tz\bar{w} + sw\bar{z} - tsw\bar{w}}{1-ts} \right\}. \tag{1.45}$$

Replacing the complex values $z = x + iy$ and $w = u + iv$ for $t = s = i$, so (1.43) leads to (1.40).

1.1.4 Bargmann's versus for constructing fractional transforms of Fourier type

We review from [36] the abstract formalism (à la Bargmann) giving rise to fractional integral transforms of Fourier type. Let ω_X and ω_Y be two weight functions on given nonempty sets X and Y. Let $\mathcal{H}_X = L^2(X, \omega_X(x)dx)$ and $\mathcal{H}_Y = L^2(Y, \omega_Y(y)dy)$ be the separable complex Hilbert spaces on X and Y, respectively, endowed with the inner products

$$\langle \varphi, \phi \rangle_{\mathcal{H}_X} = \int_X \overline{\varphi(x)} \phi(x) \omega_X(x) dx, \quad \langle \psi, \Phi \rangle_{\mathcal{H}_Y} = \int_Y \overline{\psi(y)} \Phi(y) \omega_Y(y) dy.$$

Let $\{\varphi_n\}_n$ (resp. $\{\psi_n\}_n$) be a given orthonormal basis of $\mathcal{H}_X$ (resp. $\mathcal{H}_Y$). Let $T_{XY} : \mathcal{H}_X \longrightarrow \mathcal{H}_Y$ be a well defined bounded invertible integral transform of the form

$$T_{XY}(\varphi)(y) = \langle R(\cdot,y), \varphi \rangle_{\mathcal{H}_X} = \int_X \overline{R(x,y)} \varphi(x) \omega_X(x) dx,$$

where the kernel function $R(x,y)$ on $X \times Y$ is given by

$$R(x,y) = \sum_{n=0}^{+\infty} \varphi_n(x) \overline{\psi_n(y)}. \tag{1.46}$$

Accordingly, such transform satisfies $T_{XY}(\varphi_n) = \psi_n$ and its inverse is given by $T_{XY}^{-1}(\psi)(x) = \langle R(x,\cdot), \psi\rangle_{\mathcal{H}_Y}$. The main results involve certain groups G acting on Y via the mapping $U : G \times Y \longrightarrow Y$, defined as $U(g,y) = U_g(y) = g \cdot y$, that we can extend its action to $\mathcal{H}_Y$ by considering

$$U(g,\psi)(y) = U_g(\psi)(y) = \psi(g\cdot y);\ y \in Y, \psi \in \mathcal{H}_Y.$$

We define the fractional transform of Fourier type associated with T_{XY} and U_g to be the operator $\mathcal{F}_{rg}$ on $\mathcal{H}_X$ given by the formula

$$\mathcal{F}_{rg} = T_{XY}^{-1} \circ U_g \circ T_{XY}. \tag{1.47}$$

Thus, for the specific U satisfying $U_g(\psi_n)(y) = \chi_n(g)\psi_n(y)$, where $\chi(n,g) = \chi_n(g)$ is a given complex-valued map on $\mathbb{Z}^+ \times G$, it is a simple matter to see that $\mathcal{F}_{rg}$ is the integral transform

$$\mathcal{F}_{rg}(\varphi)(x) = \int_{x'\in X} R_g(x',x)\varphi(x')\omega_X(x')dx',$$

whose the kernel function is given by

$$R_g(x',x) = \langle R(x',\Gamma_g), R(x,.)\rangle = \sum_{n=0}^{+\infty} \chi_n(g)\varphi_n(x)\overline{\varphi_n(x')}. \tag{1.48}$$

The considered transform verifies in particular $\mathcal{F}_{rg}(\varphi_n) = \chi_n(g)\varphi_n$. For more details, we refer to [36].

Notice for instance that the particular case of U_θ defined on the monomials $e_n(z) = z^n$ by $U_\theta(e_n) := \theta^n e_n$ for $\theta \in G = \mathbb{C}^* = \mathbb{C}\setminus\{0\}$, and T_{XY} being the Segal-Bargmann transform defining a unitary isometric transformation from $\mathcal{H}_X = L^2(\mathbb{R}, e^{-x^2}dx)$ onto the Bargmann space $\mathcal{H}_Y = \mathcal{B}(\mathbb{C}) := Hol(\mathbb{C}) \cap L^2(\mathbb{C}, e^{-|z|^2}dxdy)$ of all holomorphic functions in the Gaussian Hilbert space $L^2(\mathbb{C}, e^{-|z|^2}dxdy)$, the transform $\mathcal{F}_{rg}$ can be found explicitly via the formalism (1.47) for $f \in L^2(\mathbb{R}, e^{-x^2}dx)$ and $(y,\theta) \in \mathbb{R}\times\mathbb{C}$, where $\varphi_n(x) = \frac{H_n(x)}{2^{n/2}\sqrt{n!}\pi^{1/4}}$ and $\psi_n(z) = \frac{1}{\sqrt{\pi n!}}z^n$, being indeed by means of Fubini theorem

$$\begin{aligned}
\mathcal{F}_{rg}(f)(y) &= (\frac{1}{\pi})^{3/4}\int_{\mathbb{C}} e^{\frac{-1}{2}(\bar z^2+y^2)+\sqrt{2}\bar z y}(\Gamma_\theta \circ T_{XY})(f)(z)e^{-|z|^2}d\lambda(z)\\
&= (\frac{1}{\pi})^{3/4}\int_{\mathbb{C}} e^{\frac{-1}{2}(\bar z^2+y^2)+\sqrt{2}\bar z y}\left(\int_{\mathbb{R}}\sum_{n=0}^{+\infty}\varphi_n(x)\Gamma_\theta(\psi_n(z))\right)f(x)e^{-x^2}dx)e^{-\nu|z|^2}d\lambda(z)\\
&= \int_{\mathbb{R}}(\int_{\mathbb{C}}\left(\sum_{m=0}^{+\infty}\varphi_m(y)\overline{\psi_m(z)}\right)\left(\sum_{n=0}^{+\infty}\varphi_n(x)\psi_n(z)e_n(\theta)\right)e^{-|z|^2}d\lambda(z))f(x)e^{-x^2}dx\\
&= \int_{\mathbb{R}}\left\langle\sum_{m=0}^{+\infty}\varphi_m(y)\psi_m, \sum_{n=0}^{+\infty}\varphi_n(x)\psi_n e_n(\theta)\right\rangle_{\mathcal{B}(\mathbb{C})} f(x)e^{-x^2}dx\\
&= \int_{\mathbb{R}}((\pi(1-\theta^2))^{-1/2}\exp\left(\frac{2\theta yx - \theta^2(y^2+x^2))}{1-\theta^2}\right)f(x)e^{-x^2}dx.
\end{aligned}$$

Notice that when we put $\theta = e^{i\alpha}$, the above calculation is identically seen as the standard fractional Fourier transform.

1.2 Fractional Hankel transform

1.2.1 Fractional Hankel transform à la Namias

Let us consider the function $f_x \in L^1(\mathbb{R}^+) \cap L^2(\mathbb{R}^+)$, given by $f_x(y) = (x/y)^{1/2}\,\phi(y)$, for every fixed $x \geq 0$. For $\alpha > -1/2$, the classical Hankel transform $\widetilde{\mathcal{H}}_\alpha$ deduced from (1.22), can be written in this form

$$\widetilde{\mathcal{H}}_\alpha(f_x)(x) = \mathcal{H}_\alpha(\phi)(x) = \int_0^{+\infty} (xy)^{1/2} J_\alpha(xy) f_x(y)dy, \quad \alpha > -1, \tag{1.49}$$

and it converges absolutely on $\mathbb{R}^+$. According to [37, p. 42], we have for instance

$$\widetilde{\mathcal{H}}_\alpha(x^{\alpha+1/2}e^{-x^2/2}L_n^\alpha(x^2)) = e^{in\pi}x^{\alpha+1/2}e^{-x^2/2}L_n^\alpha(x^2) = x^{1/2}\mathcal{H}_\alpha^\pi(x^\alpha e^{-x^2/2}L_n^\alpha(x^2)), \tag{1.50}$$

and $\mathcal{H}_\alpha^\pi \equiv \mathcal{H}_\alpha^{-\pi} \equiv \mathcal{H}_\alpha$. Notice that from (1.50), the eigenvalues and the eigenfunctions of $\mathcal{H}_\alpha^\pi$ are respectively $\{e^{in\pi}\}_{n=0}^{+\infty}$ and $\{L_n^{(\alpha)}(x)\}_{n=0}^{+\infty}$. Where L_n^α is the generalised Laguerre polynomial defined by [100, p.204]

$$L_n^{(\alpha)}(x) = \frac{1}{n!}e^x x^{-\alpha}\frac{d^n}{dx^n}[e^{-x}x^{n+\alpha}], \quad n = 0, 1, 2, \ldots.$$

Namias [92] established the fractional Hankel transform generalising this formula (1.50) into the following

$$H_z^\alpha[L_n^{(\alpha)}(x)] = z^n L_n^{(\alpha)}(x), \quad \text{for } z = e^{i\xi}. \tag{1.51}$$

He observed that the advantage of using the orthogonality property of the generalized Laguerre polynomials, lies in the fact that any square integrable function $f(x)$ can be expressed in terms of the eigenfunctions $\{x^\alpha e^{-x^2/2}L_n^\alpha(x^2)\}_{n=0}^{+\infty}$ of the operator $\widetilde{\mathcal{H}}_\alpha$:

$$f(x) = g(x^2) = \sum_{n=0}^{+\infty} a_n x^\alpha e^{-x^2/2}L_n^\alpha(x^2) = x^\alpha e^{-x^2/2}\sum_{n=0}^{+\infty} a_n L_n^\alpha(x^2). \tag{1.52}$$

Since we have the inner product of two generalized Laguerre polynomials [105, p.100]

$$\langle L_n^\alpha, L_m^\alpha\rangle = \int_0^{+\infty} e^{-t}t^\alpha L_n^\alpha(t)L_m^\alpha(t)dt = \frac{\Gamma(n+\alpha+1)}{n!}\delta_{n,m}.$$

Hence, upon replacing an element $u = x^2$ in (1.52) and setting $h(u) = \frac{g(u)}{u^{\alpha/2}e^{-u/2}}$, he found the coefficients

$$a_n = \left\langle h, \frac{L_n^\alpha}{\|L_n^\alpha\|^2}\right\rangle = \frac{n!}{\Gamma(n+\alpha+1)}\int_0^{+\infty} u^\alpha e^{-u}h(u)L_n^\alpha(u)du. \tag{1.53}$$

Advancing on his work, he applied the fractional Hankel transform of Bessel order α, denoted by H_z^α, on the function $h(x)$ belonging to the space $L_\alpha^2(\mathbb{R}^+)$ of square integrable functions with respect to the measure $d\nu_\alpha(t) := t^\alpha e^{-t}dt$, so

$$\begin{aligned} H_z^\alpha h(x) &= \sum_{n=0}^{+\infty} a_n e^{in\xi}L_n^\alpha(x) \qquad (1.54)\\ &= \int_0^{+\infty}\left(\sum_{n=0}^{+\infty}\frac{n!}{\Gamma(n+\alpha+1)}e^{in\xi}L_n^\alpha(x)L_n^\alpha(t)\right)h(t)t^\alpha e^{-t}dt, \text{ where } z = e^{i\xi}\end{aligned}$$

Motivated by the Hille-Hardy formula [10, p. 189]

$$\sum_{n=0}^{+\infty}\frac{n!}{\Gamma(n+\alpha+1)}L_n^\alpha(x)L_n^\alpha(y)z^n=(1-z)^{-1}(xyz)^{-\alpha/2}\exp[-\frac{z(x+y)}{1-z}]I_\alpha(\frac{2\sqrt{xyz}}{1-z}),\quad |z|<1 \tag{1.55}$$

and using it in (1.54), H_z^α is no longer self-reciprocal unlike the usual one (1.22) when ξ is equal to π or $-\pi$, we have

$$\begin{aligned}H_z^\alpha h(x)&=\int_0^{+\infty}t^\alpha e^{-t}(1-e^{i\xi})^{-1}(xte^{i\xi})^{-\alpha/2}\exp[-\frac{e^{i\xi}(x+t)}{1-e^{i\xi}}]I_\alpha(\frac{2\sqrt{xte^{i\xi}}}{1-e^{i\xi}})h(t)dt.\\&=\frac{e^{(i\pi/2-i\xi/2)(1+\alpha)}}{\sin(\xi/2)}\int_0^{+\infty}(t/x)^{\alpha/2}J_\alpha\left(\frac{\sqrt{xt}}{\sin(\xi/2)}\right)\exp\left(-i\cot(\xi/2)t-\frac{zx}{1-z}\right)h(t)dt\end{aligned}$$

Here $I_\alpha(\xi)$ denotes the modified Bessel function of first kind [7, p. 222 (4.12.2)]

$$I_\alpha(\xi)=\sum_{n=0}^{+\infty}\frac{1}{n!\Gamma(n+\alpha+1)}\left(\frac{\xi}{2}\right)^{\alpha+2n}.$$

The inverse of $H_{z_\xi}^\alpha$ is established in [92], written this form

$$[H_z^\alpha]^{-1}h(x)=\frac{e^{(-i\pi/2+i\xi/2)(1+\alpha)}}{\sin(\xi/2)}\int_0^{+\infty}(t/x)^{\alpha/2}J_\alpha\left(\frac{\sqrt{xt}}{\sin(\xi/2)}\right)\exp\left(i\cot(\xi/2)t-\frac{x}{z-1}\right)h(t)dt. \tag{1.56}$$

Examples 1.2.1. *Here are some examples of fractional Hankel transform applied to functions belonging to* $L_\alpha^2(\mathbb{R}^+):=L^2(\mathbb{R}^+;x^\alpha e^{-x}dx)$ *for any* $z\in D$

(1) From [100, p.201], we have for

$$\varphi(y)=\sum_{n=0}^{+\infty}\frac{1}{(\alpha+1)_n}L_n^{(\alpha)}(y)=\Gamma(\alpha+1)y^{-\alpha/2}eJ_\alpha(2\sqrt{y})\in L_\alpha^2(\mathbb{R}^+). \tag{1.57}$$

The action of H_z^α *on* φ*, is explicitly given by [100, p.202]*

$$H_z^\alpha(\varphi)(y)=\sum_{n=0}^{+\infty}\frac{L_n^{(\alpha)}(y)}{(\alpha+1)_n}z^n=e^z\,{}_0F_1\left(\begin{matrix}-\\ \\ \alpha+1\end{matrix}\;;\;-yz\right)\in L_\alpha^2(\mathbb{R}^+), \tag{1.58}$$

for $y\in\mathbb{R}^+$ *and* z *belong to the complex unit disc.*

(2) Retrieved from [100, p.202], we also can consider the function of the form

$$\phi_a(y)=\sum_{n=0}^{+\infty}\frac{(1)_nL_n^{(\alpha)}(y)a^n}{(\alpha+1)_n}={}_1F_1\left(\begin{matrix}1\\ \\ \alpha+1\end{matrix}\;;\;-\tfrac{ya}{1-a}\right)(1-a)^{-1},\ \text{where } a<1. \tag{1.59}$$

Since $\phi_a \in L^2_\alpha(\mathbb{R}^+)$. Thus, the fractional Hankel transform $H^\alpha_z(\phi_a)(y)$ is seen as the generating function the generalized Laguerre polynomial, expressed by

$$H^\alpha_z(\phi_a)(y) = \frac{1}{(1-az)}\, {}_1F_1\left(\begin{matrix} 1 \\ \\ \alpha+1 \end{matrix} \; ; \; -\tfrac{yaz}{1-az}\right) \in L^2_\alpha(\mathbb{R}^+). \tag{1.60}$$

(3) For every $\varphi \in L^2_\alpha(\mathbb{R}^+)$ and $(t,z) \in \mathbb{R}^+ \times D$, we have the operational formula for the fractional Hankel transform [48]

$$H^\alpha_z\left(\left[t\frac{\mathrm{d}^2}{\mathrm{d}t^2} + (\alpha+1-t)\frac{\mathrm{d}}{\mathrm{d}t}\right]\varphi\right)(y) = -z\frac{d}{dz}H^\alpha_z\varphi(y).$$

(4) If $\varphi/t \in L^2_\alpha(\mathbb{R}^+)$, the expression of H^α_z satisfies this relation [48]

$$H^\alpha_z\left(\frac{\mathrm{d}\varphi}{\mathrm{d}t}\right)(y) = -\left(y\left(\frac{\partial}{\partial y} + \frac{z}{1-z}\right) + \alpha\right)H^\alpha_z\left(\frac{\varphi}{t}\right)(y) + \frac{1}{1-z}H^\alpha_z(\varphi)(y).$$

(5) The relation between the non-trivial 2-D fractional Fourier transform (1.39) *and the fractional Hankel transform has been found in [13], which is presented by this formula for all $w \in \mathbb{C}$*

$$\mathcal{F}_{u,v}\psi_\alpha(w,\bar{w}) = \left(\frac{w}{\bar{w}}\right)^{\alpha/2} u^\alpha H^\alpha_{uv}\left(|w|^{-3\alpha}\psi\right)(|w|^2), \tag{1.61}$$

where $\psi_\alpha(w,\bar{w}) = \psi(|w|^2)\exp\left(-|w|^2 + i\alpha\theta\right)$ and $u, v \in D$.

1.2.2 Fractional Hankel transform à la Bargmann

In this subsection we will review the formalism, giving rise to the fractional Hankel transform from the consideration of the unitary isometric second Bargmann transform [9, p.203]. At first glance, we consider the Hilbert space $L^2_\alpha(\mathbb{R}^+) = L^2(\mathbb{R}^+; d\nu_\alpha)$ of all complex-valued functions on the real half-line that are square integrable with respect to the scalar product

$$\langle \varphi, \psi \rangle_{\mathbb{R}^+,\alpha} = \int_{\mathbb{R}^+} \overline{\varphi(x)}\psi(x)d\nu_\alpha(x),$$

A complete orthonormal basis is given by

$$\varphi^\alpha_n(x) = \left(\frac{n!}{\Gamma(n+\alpha+1)}\right)^{1/2} L^{(\alpha)}_n(x), \tag{1.62}$$

where $L^{(\alpha)}_n$ denotes the normalized generalized Laguerre polynomials [100, p. 203]

$$L^{(\alpha)}_n(x) = \sum_{k=0}^{n} \frac{\Gamma(\alpha+n+1)}{\Gamma(n-k+1)\Gamma(\alpha+k+1)} \frac{(-x)^k}{k!}.$$

The second Bargmann transform [9, p.203], given through

$$B^\alpha_{hol}(\varphi)(z) = \frac{1}{(1-z)^{\alpha+1}} \int_0^{+\infty} t^\alpha \exp\left(-\frac{t}{1-z}\right)\varphi(t)dt, \tag{1.63}$$

defines a unitary isometric transformation from the configuration space $L^{2,\alpha}(\mathbb{R}^+)$ onto the weighted holomorphic Bergman space

$$\mathcal{A}^{2,\alpha}(\mathcal{D}) = \mathcal{H}ol(\mathcal{D}) \cap L^2(\mathcal{D}, d\lambda_\alpha), \tag{1.64}$$

constituted of all holomorphic functions on $\mathcal{D}$ that are square integrable with respect to the weighted hyperbolic measure

$$d\lambda_\alpha(z) := \frac{\alpha}{\pi}(1 - |z|^2)^{\alpha-1} dxdy;\ z = x + iy. \tag{1.65}$$

The scalar product in $L^2(\mathcal{D}, d\lambda_\alpha)$ is the following

$$\langle f, g \rangle_\alpha = \int_{\mathcal{D}} \overline{f(z)} g(z) d\lambda_\alpha(z).$$

Accordingly, by considering the action of the multiplicative group $G = \mathbb{C}^*$ with $U_\theta(e_n) := \theta^n e_n$, one can apply the formalism to define the so-called B-fractional integral transform associated with B^α_{hol} whose kernel function, whithin the measure $x^\alpha e^{-x} dx$, is given by

$$R(t, z) = \frac{1}{(1 - z)^{\alpha+1}} \exp\left(-\frac{tz}{1 - z}\right), \tag{1.66}$$

such transform is given by [36]

$$H^\alpha_z = [B^\alpha_{hol}]^{-1} \circ U_z \circ B^\alpha_{hol}, \tag{1.67}$$

which reduces further to be the integral transform given by

$$H^\alpha_z(\varphi)(y) = \int_0^{+\infty} R^\alpha_z(x, y)\varphi(x) x^\alpha e^{-x} dx. \tag{1.68}$$

The kernel function is closely connected to the kernel of the second Bargmann transform in (2.2) as

$$R^\alpha_z(x, y) \ = \langle R(x, U_z), R(y, .)\rangle_{\mathcal{A}^{2,\alpha}(\mathcal{D})}, \tag{1.69}$$

and therefore, it coincides with the Hille-Hardy identity [10, p. 189]

$$R^\alpha_z(x, y) \ = \ \sum_{n=0}^{+\infty} \frac{n!}{\Gamma(n + \alpha + 1)} z^n L^{(\alpha)}_n(x) L^{(\alpha)}_n(y). \tag{1.70}$$

The closed formula of the kernel function $R^\alpha_z(x, y)$ is then given by the

$$R^\alpha_z(x, y) \ = \ \frac{(zxy)^{-\alpha/2}}{1 - z} \exp\left(-\frac{z(x + y)}{1 - z}\right) I_\alpha\left(\frac{2\sqrt{zxy}}{1 - z}\right),\ \ |z| < 1. \tag{1.71}$$

Thus, we recover the fractional Hankel transform obtained by Namias [92]. It should be noticed here that the normalized generalized Laguerre polynomials $\varphi^\alpha_n(x)$ are eigenfunctions of H^α_z with z^n as corresponding eigenvalues. Moreover, (1.67), one obtains the semi-group property (i.e $H^{\alpha}_{pz} = H^{\alpha}_{p} \circ H^{\alpha}_{z}$), which were applied to (1.67), we readily get the inverse for H^{α}_{z} when $z \neq 0$. Namely, we have

$$(H^{\alpha}_{z})^{-1} = (B^{\alpha}_{hol})^{-1} \circ U_z^{-1} \circ B^{\alpha}_{hol} = L^{\alpha}_{1/z}, \quad \text{for } z \neq 0. \tag{1.72}$$

Here we recall the proof of the following lemma given in [36]

Lemma 1.2.1. *A Plancherel formula for H^{α}_{z} reads*

$$\langle H^{\alpha}_{z}\varphi, H^{\alpha}_{z}\psi\rangle = \langle \varphi, \psi\rangle, \tag{1.73}$$

for every $z \in \mathbb{C}$ such that $|z| = 1$.

Proof. Since we have $B^{\alpha}_{hol}\varphi^{\alpha}_{n} = f^{\alpha}_{n}$, then we can check easily the following equations under the condition $|z| = 1$

$$H^{\alpha}_{z}\left(\varphi^{\alpha}_{n}(y)\right) = \left[B^{\alpha}_{hol}\right]^{-1}\left(f^{\alpha}_{n}(\cdot)z^{n}\right)(y) = \varphi^{\alpha}_{n}(y)z^{n}, \tag{1.74}$$

and

$$\langle U_z f, U_z g\rangle_{A^{2,\alpha}(D)} = \langle f, g\rangle_{A^{2,\alpha}(D)}, \tag{1.75}$$

for every $f, g \in \mathcal{A}^{2,\alpha}(D)$. The fact that $[B^{\alpha}_{hol}]^{-1}, B^{\alpha}_{hol}$ are both isometries lead to the identity $\langle H^{\alpha}_{z}\varphi, H^{\alpha}_{z}\psi\rangle = \langle \varphi, \psi\rangle$ when $|z| = 1$. □

It follows starting again from (1.67), keeping in mind that in this case B^{α}_{hol}, $[B^{\alpha}_{hol}]^{-1}$ and U_z are unitary operators. The explicit integral representation of the inverse of the second Bargmann transform [9]

$$[B^{\alpha}_{hol}]^{-1}(f)(t) = \frac{1}{\sqrt{\pi\Gamma(\alpha)}(1-\bar{z})^{\alpha+1}}\int_0^{+\infty} \exp\left(-\frac{-t\bar{z}}{1-\bar{z}}\right) f(z)(1-|z|^2)^{\alpha-1} d\lambda(z), \tag{1.76}$$

is based on the orthonormal basis of the weighted holomorphic Bergman space $\mathcal{A}^{2,\alpha}(D)$, given by

$$f^{\alpha}_{n}(z) = \left(\frac{\Gamma(n+1+\alpha)}{\pi\Gamma(\alpha)n!}\right)^{1/2} z^{n}. \tag{1.77}$$

H^{α}_{z} is a bounded integral transform from $L^{2,\alpha}(\mathbb{R}^{+})$ into itself. The following inequality guarentees such affirmation

$$\begin{aligned}\|H^{\alpha}_{z}(\varphi)\|^2 &\leq \left(\sum_{n=0}^{+\infty}|z|^{2n}\right)\left(\sum_{n=0}^{+\infty}|c_n|^{2n}\right)\\ &\leq \left(\frac{1}{1-|z|^2}\right)\|\varphi\|^2,\end{aligned}$$

using Parseval's identity for $\varphi = \sum_{n=0}^{+\infty} c_n\varphi^{\alpha}_{n}$ and $z \in D$.

Chapter 2

Inverse problem for the dual fractional Hankel transform

Abstract: The objective of this chapter is to address the inverse transformation M_y^α of the restricted form of S_y^α, specifically on $(\ker(S_y^\alpha))^\perp$. We furnish both an expansion series and an integral representation for M_y^α, subject to constraints imposed on y and the parameters α, β, and η. Note that, a closed-form expression for the integral kernel appears challenging due to the fact that the description of the set of the Laguerre polynomials that have a common zero on y is unknown in general. Nonetheless, we explore fundamental properties of this kernel, including the differential formulas it adheres to, and we present its closed expressions for this kernel function imposing some conditions. This chapter is arranged as follows. The proof of the main results of the inverse of the dual fractional Hankel transform with some basic properties of its integral kernel are presented in Section 2.2, including its closed formula for special values of α, β, η and y. The reproducing kernel of the range of the dual fractional Hankel transform and its relation with M_y^α is discussed in Section 2.3.

2.1 Introduction

The fractional Hankel transform is defined by [92, 72]

$$H_z^\alpha(\varphi)(y) = \frac{1}{1-z}\int_0^{+\infty} \exp\left(-\frac{x+yz}{1-z}\right)\left(\frac{x}{yz}\right)^{\alpha/2} I_\alpha\left(\frac{2\sqrt{xyz}}{1-z}\right)\varphi(x)dx, \tag{2.1}$$

on $L^2_\alpha(\mathbb{R}^+) = L^2(\mathbb{R}^+; d\nu_\alpha)$, the Hilbert space of all complex-valued functions on the real half line that are square integrable with respect to the measure $d\nu_\alpha(x) = x^\alpha e^{-x}dx$, $\alpha > 0$. A complete orthonormal basis of eigenfunctions (with z^n as corresponding eigenvalue) in $L^2_\alpha(\mathbb{R}^+)$ is given by the normalized Laguerre polynomials. The transform H_z^α is closely connected to the second Bargmann transform given by [9, p.203]

$$B^{\alpha}_{hol}(\varphi)(z) = \frac{\alpha}{\pi(1-z)^{\alpha+1}} \int_0^{+\infty} t^{\alpha} \exp\left(-\frac{t}{1-z}\right) \varphi(t)dt, \quad (2.2)$$

defining a unitary isometric transformation from $L^2_\alpha(\mathbb{R}^+)$ onto the weighted holomorphic Bergman space $\mathcal{A}^{2,\alpha}(D) = \mathcal{H}ol(D) \cap L^{2,\alpha}_0(D)$ on the unit disc $D = \{z \in \mathbb{C}; |z| < 1\}$ in the complex plane $\mathbb{C}$, where $\mathcal{H}ol(D)$ denotes the space of holomorphic functions in D and $L^{2,\alpha}_0(D)$ is the Hilbert space of square integrable functions on D with respect to the measure $(1-|z|^2)^\alpha dxdy$. More precisely, it was proved in [36] that $H^\alpha_z = [B^\alpha_{hol}]^{-1} \circ U_z \circ B^\alpha_{hol}$ holds, where U_z denotes the action of the multiplicative group $\mathbb{C} \setminus \{0\}$ defined on the monomials $e_n(z) := z^n$ by $U_z(e_n) := z^n e_n$. From this realization and since $U_p \circ U_z = U_{pz}$ it readily follows that H^α_z satisfies the semi-group property $H^\alpha_{pz} = H^\alpha_p \circ H^\alpha_z$. Therefore the inverse transform of H^α_z when $z \neq 0$ is given by

$$(H^\alpha_z)^{-1} = (B^\alpha_{hol})^{-1} \circ U_z^{-1} \circ B^\alpha_{hol} = H^\alpha_{1/z}, \quad (2.3)$$

keeping in mind that $U_z^{-1} = U_{z^{-1}}$ whenever $z \neq 0$. The so-called dual fractional Hankel transform (DFrHT) is exactly the quantity in (2.1) seen as function in the z-variable for each fixed $y \in (0, +\infty)$, namely [48]

$$S^\alpha_y(\varphi)(z) = \frac{1}{1-z} \int_0^{+\infty} \exp\left(-\frac{x+yz}{1-z}\right) \left(\frac{x}{yz}\right)^{\alpha/2} I_\alpha\left(\frac{2\sqrt{xyz}}{1-z}\right) \varphi(x)dx. \quad (2.4)$$

For $\alpha, \beta, \eta > 0$, it is a bounded operator from the configuration space $L^2_\alpha(\mathbb{R}^+)$ into the weighted Hilbert space $L^{2,\beta}_\eta(D)$ of all complex-valued measurable functions on D subject to

$$\|f\|^2_{\beta,\eta} = \int_D |f(x+iy)|^2 A_{\beta,\eta}(x^2+y^2)dxdy < +\infty.$$

The involved weight function is given by $A_{\beta,\eta}(t) = t^{\beta-1}(1-t)^{\eta-1}$ for $t \in (0,1)$. The image $S^\alpha_y(L^2_\alpha(\mathbb{R}^+)) \cap L^{2,\beta}_\eta(D)$ in $L^{2,\beta}_\eta(D)$ of S^α_y acting on $L^2_\alpha(\mathbb{R}^+)$ is denoted by from now on by $R^{\beta,\eta}(S^\alpha_y)$. It is contained in the weighted Bergman space $\mathcal{A}^{2,\beta}_\eta(D) = \mathcal{H}ol(D) \cap L^{2,\beta}_\eta(D)$. For the limit case when $y = 0$ with arbitrary $\eta = \alpha > 0$ and $\beta > 0$, we have $R^{\beta,\eta}(S^\alpha_y) = \mathcal{A}^{2,\beta}_\eta(D)$ and the transform S^α_0 reduces further to B^α_{hol} in (2.2) when $\beta = 1$. The explicit description of the phase space $R^{\beta,\eta}(S^\alpha_y)$ makes appeal to the set of the generalized Laguerre polynomials $L^{(\alpha)}_n$ having y as common zero, say $N^\alpha_y = \{n; L^{(\alpha)}_n(y) = 0\}$ (see [48] for details). More explicitly, if we denote by ${}^cN^\alpha_y$ the complementary of N^α_y in $\mathbb{N} \cup \{0\}$, then we have [48]

$$R^{\beta,\eta}(S^\alpha_y) = \left\{ \sum_{n \in {}^cN^\alpha_y} a_n \varphi^\alpha_n(y) z^n; \sum_{n \in {}^cN^\alpha_y} \gamma^{\beta,\eta}_n |\varphi^\alpha_n(y)|^2 |a_n|^2 < +\infty \right\},$$

where the constants $\gamma^{\beta,\eta}_n$ are the moments given in terms of the Euler Gamma function by

$$\gamma^{\beta,\eta}_n := \int_0^1 t^n A_{\beta,\eta}(t)dt = \frac{\Gamma(\eta)\Gamma(\beta+n)}{\Gamma(\beta+\eta+n)}. \quad (2.5)$$

Thus, $R^{\beta,\eta}(S_y^\alpha)$ is strictly contained in $\mathcal{A}_\eta^{2,\beta}(D)$ whenever $N_y^\alpha = \{n; L_n^{(\alpha)}(y) = 0\}$ is not-empty. The description of the null space $\ker(S_y^\alpha)$ of S_y^α involves those φ_n^α for which $n \in N_y^\alpha$. Its dimension is closely connected to the cardinal of N_y^α.

The aim of the present chapter is that we deal with the inverse transform M_y^α of the restriction of S_y^α to $(\ker(S_y^\alpha))^\perp$, the orthogonal complement of $\ker(S_y^\alpha)$. We provide an expansion series as well as an integral representation of M_y^α under restriction conditions on y and the parameters α, β and η. A complete description seems to be hardly attainable for the absence of a closed expression of the integral kernel as well as the absence of a complete characterization of the set N_y^α. Nevertheless, we discuss some basic properties of this kernel such as the differential formulas they satisfy. Also, we give the closed expression of this kernel function in some special cases. To this purpose, we consider the set Υ_α of all $y \geq 0$ such that

$$|L_n^{(\alpha)}(y)| \geq \frac{\Gamma(\alpha+n+1)}{n!\Gamma(\alpha+1)} \tag{2.6}$$

holds for $n \in {}^cN_y^\alpha$ and large enough.

2.2 Main results about the inverse of the dual fractional Hankel transform

The main results can be stated as follows.

Theorem 2.2.1. *For $y \in \Upsilon_\alpha$ satisfying (2.6), and $\alpha > \eta+1$ with $\beta, \eta, \alpha > 0$, the inverse transform M_y^α of the dual fractional Hankel transform $S_y^\alpha : (\ker(S_y^\alpha))^\perp \longrightarrow R^{\beta,\eta}(S_y^\alpha)$ is given by the expansion*

$$M_y^\alpha(f) = \frac{1}{\pi} \sum_{n \in {}^cN_y^\alpha} \frac{\langle f, e_n \rangle_{L_\eta^{2,\beta}(D)}}{\gamma_n^{\beta,\eta} L_n^{(\alpha)}(y)} L_n^{(\alpha)},$$

for every given $f \in R^{\beta,\eta}(S_y^\alpha)$. Moreover, the integral representation

$$M_y^\alpha(f)(x) = \int_D t_{y,\alpha}^{\beta,\eta}(x,z) f(z) A_{\beta,\eta}(|z|^2) d\lambda(z)$$

holds for every $y \in \Upsilon_\alpha$, where the integral kernel is given by the absolute convergent series

$$t_{y,\alpha}^{\beta,\eta}(x,z) = \frac{1}{\pi} \sum_{n \in {}^cN_y^\alpha} \frac{L_n^{(\alpha)}(x)}{\gamma_n^{\beta,\eta} L_n^{(\alpha)}(y)} \overline{z}^n.$$

From the proof of Theorem 2.2.1, the integral operator M_y^α is bounded from $R^{\beta,\eta}(S_y^\alpha)$ into $L_\alpha^2(\mathbb{R}^+)$. The next result is concerned with the compactness of M_y^α and the explicit expression of its corresponding singular values, i.e. the eigenvalues of $|M_y^\alpha| := ((M_y^\alpha)^* M_y^\alpha)^{1/2}$. Here $*$ denotes the adjoint operation.

Theorem 2.2.2. *The integral operator* $M_y^\alpha : R^{\beta,\eta}(S_y^\alpha) \longrightarrow L^2_\alpha(\mathbb{R}^+)$ *is compact for* $y \in \Upsilon_\alpha$ *and* $\alpha > \eta + 1$ *with singular values given by*

$$s_{n,\alpha}^{\beta,\eta,y} := \left(\frac{\Gamma(\beta+\eta+n)\Gamma(\alpha+n+1)}{\pi\Gamma(\eta)\Gamma(\beta+n)n!} \right)^{1/2} \frac{1}{|L_n^{(\alpha)}(y)|}.$$

In the present section, we prove the main results giving the inverse M_y^α of the dual fractional Hankel transform in (2.4) as well as their singular values.

2.2.1 Discussion.

Notice first that since the null space $\ker(S_y^\alpha)$ of S_y^α is spanned by φ_n^α with varying $n \in N_y^\alpha$ (see [48]). Thus, it becomes clear that $S_y^\alpha : L^2_\alpha(\mathbb{R}^+) \longrightarrow L^{2,\beta}_\eta(D)$ is one to one only when $y \in \{y \geq 0,\ N_y^\alpha = \emptyset\}$ (for example the case of $y = 0$). This is in fact a necessary condition to the null space reduces to $\{0\}$. In this case the phase space $R^{\beta,\eta}(S_y^\alpha)$ is exactly $\mathcal{A}^{2,\beta}_\eta(D)$ and the integral operator S_y^α becomes invertible from $L^2_\alpha(\mathbb{R}^+)$ onto $\mathcal{A}^{2,\beta}_\eta(D)$. More generally, the restriction of S_y^α to

$$(\ker(S_y^\alpha))^\perp = \overline{Span\{\varphi_n^\alpha;\, n \in {}^cN_y^\alpha\}}^{L^2_\alpha(\mathbb{R}^+)},$$

defines an invertible operator from $(\ker(S_y^\alpha))^\perp$ onto $R^{\beta,\eta}(S_y^\alpha)$. Thus, there exists M_y^α : $R^{\beta,\eta}(S_y^\alpha) \longrightarrow (\ker(S_y^\alpha))^\perp$ satisfying the functional equations

$$M_y^\alpha S_y^\alpha(\varphi) = \varphi; \quad \varphi \in (\ker(S_y^\alpha))^\perp \tag{2.7}$$

and

$$S_y^\alpha M_y^\alpha(f) = f; \quad f \in R^{\beta,\eta}(S_y^\alpha). \tag{2.8}$$

Lemma 2.2.3. *The equations* (2.7) *and* (2.8) *are equivalent whenever* M_y^α *is into as operator from* $ker(S_y^\alpha)^\perp$ *onto* $R(S_y^\alpha)$.

Proof. Assume that $M_y^\alpha S_y^\alpha(\varphi) = \varphi$ holds for $\varphi \in ker(S_y^\alpha)^\perp$ and let $f \in R(S_y^\alpha)$. The case of $f \equiv 0$ trivially leads to $S_y^\alpha M_y^\alpha(f) = S_y^\alpha(0) = 0 = f$. For $f \neq 0$, there exists $\varphi \in ker(S_y^\alpha)^\perp$ such that $S_y^\alpha \varphi = f$ (for $f \in R(S_y^\alpha)$ and $f \neq 0$). Hence, $S_y^\alpha M_y^\alpha(f) = S_y^\alpha M_y^\alpha(S_y^\alpha \varphi) = S_y^\alpha(\varphi) = f$. Conversely, if $\varphi \in ker(S_y^\alpha)^\perp$ we let $f \in R(S_y^\alpha) \setminus \{0\}$ such that $\varphi = T_y^\alpha f$ for M_y^α being assumed to be onto. But from (2.8) it follows $M_y^\alpha S_y^\alpha(\varphi) = M_y^\alpha S_y(T_y^\alpha f) = M_y^\alpha f = \varphi$. □

The transform M_y^α is bounded thanks to the classical open mapping theorem (for S_y^α being bounded). However, by specifying φ and f, the system of equations (2.7)-(2.8) defining M_y^α combined with the fact $S_y^\alpha(\varphi_n^\alpha)(z) = \varphi_n^\alpha(y) z^n$ gives rise to $\varphi_n^\alpha(y) M_y^\alpha(e_n)(x) = \varphi_n^\alpha(x)$. The facts that $\varphi = \varphi_n^\alpha \in (\ker(S_y^\alpha))^\perp$ and $f = e_n \in R^{\beta,\eta}(S_y^\alpha)$ imply that $n \in {}^cN_y^\alpha$ (i.e., $\varphi_n^\alpha(y) \neq 0$). Therefore, we have

$$M_y^\alpha(e_n) = \frac{\varphi_n^\alpha}{\varphi_n^\alpha(y)} \text{ for all } n \in {}^cN_y^\alpha. \tag{2.9}$$

Subsequently, the identity $S_y^\alpha M_y^\alpha(e_n)(z) = e_n(z)$ for $n \notin N_y^\alpha$ follows as immediate consequence. Accordingly, by proceeding at least formally (M_y^α bounded), we see that

$$M_y^\alpha(f) = \sum_{n\in {}^cN_y^\alpha} \frac{a_n}{\varphi_n^\alpha(y)}\varphi_n^\alpha = \frac{1}{\pi}\sum_{n\in {}^cN_y^\alpha} \frac{\langle f, e_n\rangle_{L^{2,\beta}_\eta(D)}}{\gamma_n^{\beta,\eta}\varphi_n^\alpha(y)}\varphi_n^\alpha \tag{2.10}$$

for every $f = \sum_{n\in {}^cN_y^\alpha} a_n e_n \in R^{\beta,\eta}(S_y^\alpha)$. This defines an inverse of S_y^α since $M_y^\alpha S_y^\alpha(\varphi) = \varphi$ and $S_y^\alpha M_y^\alpha(f) = f$ hold for all $\varphi \in (\ker(S_y^\alpha))^\perp$ and $f \in R^{\beta,\eta}(S_y^\alpha)$. We only need to

(a) ensure that the right-hand side in (2.10) is convergent and belongs to $(\ker(S_y^\alpha))^\perp$.

An integral representation

$$M_y^\alpha(f)(x) = \int_D t_{y,\alpha}^{\beta,\eta}(x,z) f(z) A_{\beta,\eta}(|z|^2) d\lambda(z), \tag{2.11}$$

for $f \in R^{\beta,\eta}(S_y^\alpha)$, suggests the consideration of the integral kernel $t_{y,\alpha}^{\beta,\eta}(x,z)$ on $\mathbb{R}^+ \times D$ given by the expansion series

$$t_{y,\alpha}^{\beta,\eta}(x,z) = \frac{1}{\pi}\sum_{n\in {}^cN_y^\alpha} \frac{L_n^{(\alpha)}(x)}{\gamma_n^{\beta,\eta} L_n^{(\alpha)}(y)}\overline{z}^n. \tag{2.12}$$

This is clear from the boundedness of M_y^α and the fact that $\{(\pi\gamma_n^{\beta,\eta})^{-1/2} e_n,\ n \in {}^cN_y^\alpha\}$ is an orthonormal basis of $R^{\beta,\eta}(S_y^\alpha)$. This requires studying

(b) the convergence of the series involved in the right-hand side of (2.12).

According to the previous discussion we emphasize solving the problems (a) and (b) to conclude for the proof of Theorem 2.2.1. Notice for instance that the proof depends on the choice of the reference point y. In fact, the absence of an adequate lower bound for the generalized Laguerre polynomials $|L_n^{(\alpha)}(y)|$ for arbitrary fixed $y \geq 0$ is the major drawback. To overcome this difficulty we restrict ourselves to the special dual fractional Hankel transform attached to $y \geq 0$ such that $n!|L_n^{(\alpha)}(y)| \geq (\alpha+1)_n$ for n large enough in the complementary of N_y^α.

2.2.2 Succinct proof of Theorem 2.2.1.

The first task concerning (a) requires the convergence in $L^2_\alpha(\mathbb{R}^+)$ of the series (2.10), which for given $f = \sum_{n\in {}^cN_y^\alpha} a_n e_n$ such that $\sum_{n\in {}^cN_y^\alpha} |a_n|^2\gamma_n^{\beta,\eta} < +\infty$, is equivalent to the convergence of the series

$$\sum_{n\in {}^cN_y^\alpha} \frac{|a_n|^2}{|\varphi_n^\alpha(y)|^2} = \left\|M_y^\alpha(f)\right\|_\alpha^2,$$

by means of the Plancherel formula. Therefore, for y belonging to the set Υ_α in (2.6), we have

$$\sum_{n\in {}^cN_y^\alpha} \frac{|a_n|^2}{|\varphi_n^\alpha(y)|^2} \leq \left(\sum_{n\in {}^cN_y^\alpha} \frac{1}{\pi\gamma_n^{\beta,\eta}|\varphi_n^\alpha(y)|^2}\right)\left(\sum_{n\in {}^cN_y^\alpha} \pi\gamma_n^{\beta,\eta}|a_n|^2\right)$$
$$\leq \left(\sum_{n\in {}^cN_y^\alpha} \frac{1}{\gamma_n^{\beta,\eta}|\varphi_n^\alpha(y)|^2}\right)\|f\|^2_{A_{\beta,\eta}}. \tag{2.13}$$

Using the asymptotic Euler formula for Gamma function [7, p.20]

$$\frac{\Gamma(n+b)}{\Gamma(n+a)} \sim n^{b-a}, \tag{2.14}$$

one obtains

$$\frac{1}{\gamma_n^{\beta,\eta}|\varphi_n^\alpha(y)|^2} \leq \frac{[\Gamma(\alpha+1)]^2}{\Gamma(\eta)}\frac{\Gamma(\beta+\eta+n)\Gamma(n+1)}{\Gamma(\beta+n)\Gamma(\alpha+n+1)} \sim \frac{[\Gamma(\alpha+1)]^2}{\Gamma(\eta)}n^{\eta-\alpha} \tag{2.15}$$

for n large enough and $n \in {}^cN_y^\alpha$. The resulting series with general term $n^{\eta-\alpha}$ converges for $\alpha-\eta>1$. This is to say that the series in the right-hand side of (2.13) is convergent whenever $\alpha>\eta+1$ and $M_y^\alpha(f)\in L^2_\alpha(\mathbb{R}^+)$ for every $f\in R^{\beta,\eta}(S_y^\alpha)$.

For (b), the absolute convergence of the series in (2.12) readily follows under the assumption that $y\in\Upsilon_\alpha$, since

$$\frac{|z|^n}{\pi\gamma_n^{\beta,\eta}} = \frac{\Gamma(\beta+\eta+n)}{\pi\Gamma(\eta)\Gamma(\beta+n)}e^{n\log|z|} \sim \frac{1}{\pi\Gamma(\eta)}n^\eta e^{n\log|z|} \tag{2.16}$$

for n large enough. Accordingly, the occurred series with general term $n^\eta e^{n\log|z|}$ converges for $\log|z|<0$ and $\eta>0$.

Remark 2.2.4. *The above discussion proves in particular that the square norm of $M_y^\alpha(f)$ satisfies the estimate $\left\|M_y^\alpha(f)\right\|_\alpha^2 \leq C_\eta^{\alpha,\beta}\|f\|^2_{A_{\beta,\eta}}$ for some constant $C_\eta^{\alpha,\beta}$ depending only in α,β and η. This reproves the fact that the operator M_y^α in (2.11) is a bounded integral operator from $R^{\beta,\eta}(S_y^\alpha)$ into $L^2_\alpha(\mathbb{R}^+)$ for the special values $\beta,\eta>0$ and $\alpha>\eta+1$.*

Remark 2.2.5. *The case of $y=0$ with $\beta=1$ and $\eta=\alpha$ leads to*

$$t^{1,\alpha}_{0,\alpha}(x,z) = \frac{\alpha}{\pi}\sum_{n=0}^{\infty}\overline{z}^n L_n^{(\alpha)}(x) = \frac{\alpha}{\pi(1-\overline{z})^{\alpha+1}}\exp\left(-\frac{x\overline{z}}{1-\overline{z}}\right). \tag{2.17}$$

Thus, the transform M_0^α is exactly the inverse of the complex second Bargmann transform B^α_{hol} in (2.2).

2.2.3 Proof of Theorem 2.2.2

From (2.10), it is clear that the bounded transform M_y^α can be rewritten as

$$M_y^\alpha(f) = \sum_{n\notin N_y^\alpha} \lambda_{n,\alpha}^{\beta,\eta,y} \left\langle f, e_n^{\beta,\eta}\right\rangle_{L_\eta^{2,\beta}(D)} \varphi_n^\alpha,$$

with $\lambda_{n,\alpha}^{\beta,\eta,y} := [(\pi\gamma_n^{\beta,\eta})^{1/2}\varphi_n^\alpha(y)]^{-1}$. Here $(\varphi_n^\alpha)_n$ and $(e_n^{\beta,\eta}) = (\pi\gamma_n^{\beta,\eta})^{-1/2}e_{nn}$ are orthonormal bases of $L_\alpha^2(\mathbb{R}^+)$ and $R^{\beta,\eta}(S_y^\alpha)$, respectively. Subsequently, direct computation shows that for every $f \in R^{\beta,\eta}(S_y^\alpha)$ and $g \in (\ker(S_y^\alpha))^\perp$, we have

$$\begin{aligned}
\left\langle M_y^\alpha(f), g\right\rangle_{L_\alpha^2(\mathbb{R}^+)} &= \left\langle \sum_{n\in^c N_y^\alpha} \lambda_{n,\alpha}^{\beta,\eta,y} \left\langle f, e_n^{\beta,\eta}\right\rangle_{L_\eta^{2,\beta}(D)} \varphi_n^\alpha, \sum_{m\in^c N_y^\alpha} \langle g, \varphi_m^\alpha\rangle_{L_\alpha^2(\mathbb{R}^+)}\varphi_m^\alpha \right\rangle_{L_\alpha^2(\mathbb{R}^+)} \\
&= \sum_{n\in^c N_y^\alpha} \lambda_{n,\alpha}^{\beta,\eta,y} \left\langle f, e_n^{\beta,\eta}\right\rangle_{L_\eta^{2,\beta}(D)} \langle g, \varphi_n^\alpha\rangle_{L_\alpha^2(\mathbb{R}^+)}. \\
&= \left\langle f, \sum_{n\in^c N_y^\alpha} \lambda_{n,\alpha}^{\beta,\eta,y} e_n^{\beta,\eta} \langle \varphi_n^\alpha, g\rangle_{L_\alpha^2(\mathbb{R}^+)} \right\rangle_{L_\eta^{2,\beta}(D)}.
\end{aligned}$$

Thus, the adjoint of M_y^α can be expanded as

$$(M_y^\alpha)^* g = \sum_{n\in^c N_y^\alpha} \lambda_{n,\alpha}^{\beta,\eta,y} \langle \varphi_n^\alpha, g\rangle_{L_\alpha^2(\mathbb{R}^+)} e_n^{\beta,\eta}.$$

Therefore, for all $z \in D$ and $y \geq 0$, we get $M_y^\alpha(M_y^\alpha)^*\varphi_n^\alpha = (\lambda_{n,\alpha}^{\beta,\eta,y})^2\varphi_n^\alpha$ and $(M_y^\alpha)^* M_y^\alpha e_n^{\beta,\eta} = (\lambda_{n,\alpha}^{\beta,\eta,y})^2 e_n^{\beta,\eta}$. This completes the proof in view of (2.15) and thanks to the spectral theorem [65, Theorem 4.3.5], keeping in mind the fact that $\eta < \alpha - 1 < \alpha$. The singular values are then given by

$$s_{n,\alpha}^{\beta,\eta,y} := |\lambda_{n,\alpha}^{\beta,\eta,y}| = \left(\frac{\Gamma(\beta+\eta+n)}{\pi\Gamma(\eta)\Gamma(\beta+n)}\right)^{1/2} \frac{1}{|\varphi_n^\alpha(y)|}.$$

Remark 2.2.6. *The singular values of M_y^α are exactly the inverse of the ones obtained for S_y^α in [48].*

It has been proved in [48] that S_y^α is bounded, compact operator belonging to a p-Schatten class whenever $(1+2\eta)p > 4$, that means that the singular value of S_y^α, given by

$$\left|r_{n,y}^{\alpha,\beta,\eta}\right| = \left(\pi\gamma_n^{\beta,\eta}\right)^{1/2} |\varphi_n^\alpha(y)|, \tag{2.18}$$

satisfies the following condition

$$\left\|S_y^\alpha\right\|_p := \mathrm{Tr}\left(\left|S_y^\alpha\right|^p\right)^{1/p} = \left(\sum_{n=0}^{\infty} \left|s_{n,y}^\alpha\right|^p\right)^{1/p} < +\infty. \tag{2.19}$$

Now, concerning the inverse transform M_y^α, we should fix again y in Υ_α, to ensure that our operator belongs to a p-Schatten class when $(\alpha-\eta)p>2$. This result carries on the following estimation in depend of its singular values

$$\left\|M_y^\alpha\right\|_p = \left(\sum_{n=0}^{\infty}\left|\lambda_{n,\alpha}^{\beta,\eta,y}\right|^p\right)^{1/p} = \left(\sum_{n=0}^{\infty}\left(\frac{\Gamma(\beta+\eta+n)}{\pi\Gamma(\eta)\Gamma(\beta+n)}\right)^{p/2}\frac{1}{|\varphi_n^\alpha(y)|^p}\right)^{1/p}$$
$$\leq \frac{\Gamma(\alpha+1)}{\pi^{1/2}\Gamma(\eta)^{1/2}}\left(\sum_{n=0}^{\infty}\left(\frac{n!\Gamma(\beta+\eta+n)}{\Gamma(\alpha+n+1)\Gamma(\beta+n)}\right)^{p/2}\right)^{1/p}.$$

The serie above converges under the condition $(\alpha-\eta)p>2$, because we obtain the approximation of its term for n is very large

$$\left(\frac{n!\Gamma(\beta+\eta+n)}{\Gamma(\alpha+n+1)\Gamma(\beta+n)}\right)^{p/2} \sim n^{(\eta-\alpha)p/2}.$$

As required.

2.2.4 On the integral kernel

On explicit closed formula of $t_{y,\alpha}^{\beta,\eta}$. We have not succeeded in giving a closed expression of the kernel function $t_{y,\alpha}^{\beta,\eta}(x,z)$ in (2.12) for arbitrary $y\in\Upsilon_\alpha$ and $\alpha,\beta,\eta>0$. The first obstacle was the explicit characterization of the set N_y^α of $L_n^{(\alpha)}$, $n=0,1,2,\cdots$, having y as common zero. However, it is not difficult to see that the value of this kernel function at $x=y$, for any $y\geq 0$ such that $N_y^\alpha=\emptyset$, is independent of y and is given by

$$t_{y,\alpha}^{\beta,\eta}(y,z) = \frac{\Gamma(\beta+\eta)}{\pi\Gamma(\eta)\Gamma(\beta)}\,{}_2F_1\left(\begin{matrix}1,\beta+\eta\\ \beta\end{matrix}\,\middle|\,\overline{z}\right), \tag{2.20}$$

where ${}_2F_1$ denotes the Gauss hypergeometric function

$${}_2F_1\left(\begin{matrix}a,b\\ c\end{matrix}\,\middle|\,x\right) = \sum_{n=0}^{\infty}\frac{(a)_n(b)_n}{(c)_n}\frac{x^n}{n!}.$$

The only case where we have been able to give a closed formula is the one of $y=0$ with arbitrary $\alpha,\beta,\eta>0$ with $\alpha>\eta+1$. Indeed, we have

$$
\begin{aligned}
t^{\beta,\eta}_{0,\alpha}(x,z) &= \frac{\Gamma(\alpha+1)}{\pi\Gamma(\eta)}\sum_{n=0}^{\infty}\frac{\Gamma(\beta+\eta+n)}{\Gamma(\beta+n)}\frac{n!}{\Gamma(\alpha+n+1)}\overline{z}^n L^{(\alpha)}_n(x)\\
&= \frac{\Gamma(\beta+\eta)}{\pi\Gamma(\eta)\Gamma(\beta)}\sum_{n=0}^{\infty}\frac{n!(\beta+\eta)_n}{(\alpha+1)_n(\beta)_n}\overline{z}^n L^{(\alpha)}_n(x)\\
&= \frac{\Gamma(\beta+\eta)}{\pi\Gamma(\eta)\Gamma(\beta)}\sum_{n=0}^{\infty}\sum_{k=0}^{n}\frac{n!(\beta+\eta)_n}{(\alpha+1)_n(\beta)_n}\overline{z}^n\frac{\Gamma(\alpha+n+1)}{\Gamma(n-k+1)\Gamma(\alpha+k+1)}\frac{(-x)^k}{k!}\\
&= \frac{\Gamma(\beta+\eta)}{\pi\Gamma(\eta)\Gamma(\beta)}\sum_{n=0}^{\infty}\sum_{k=0}^{n}\frac{n!(\beta+\eta)_n}{(\alpha+1)_k(\beta)_n}\overline{z}^n\frac{1}{\Gamma(n-k+1)}\frac{(-x)^k}{k!}\\
&= \frac{\Gamma(\beta+\eta)}{\pi\Gamma(\eta)\Gamma(\beta)}\sum_{k=0}^{\infty}\sum_{n=0}^{\infty}\frac{(1)_{n+k}(\beta+\eta)_{n+k}}{(\beta)_{n+k}(\alpha+1)_k}\frac{\overline{z}^n}{n!}\frac{(-x\overline{z})^k}{k!}\\
&= \frac{\Gamma(\beta+\eta)}{\pi\Gamma(\eta)\Gamma(\beta)}\,{}_{2:0;0}F_{1:0;1}\left(\begin{matrix}1,\beta+\eta & : & -;- \\ \beta & : & -;\alpha+1\end{matrix}\,\middle|\,\overline{z},-x\overline{z}\right).
\end{aligned}
$$

Here ${}_{2,0,0}F_{1,0,1}$ is the hypergeometric function of higher order of two variables as defined by Chaundy in [23]. In particular, for $y=0$, $\beta=1$ and $\alpha=\eta>0$, one recovers the assertion of Remark 2.2.5.

Special estimates. In absence of a general explicit closed formula for $t^{\beta,\eta}_{y,\alpha}(x,z)$, we discuss below some of its basic properties. We begin by noticing that from (2.23) one obtains the following estimate

$$
|t^{\beta,\eta}_{y,\alpha}(x,z)| \leq e^{x/2}\Gamma(\beta+\eta)\left(\pi\Gamma(\beta)\Gamma(\eta)m^{\alpha}_y\right)^{-1}{}_2F_1\left(\begin{matrix}\alpha+1,\beta+\eta\\ \beta\end{matrix}\,\middle|\,|z|\right),
$$

valid for all $x\in\mathbb{R}^+$, all $z\in D$ and $y\in\Omega_\alpha$. The latter is the subset of $y\in\mathbb{R}^+$ satisfying

$$
\inf_{n\in{}^cN^{\alpha}_y}|L^{(\alpha)}_n(y)| \geq m^{\alpha}_y
$$

for certain $m^{\alpha}_y>0$. Moreover, by the orthonormality of φ^{α}_n in $L^2_\alpha(\mathbb{R}^+)$, it is easily seen that

$$
\begin{aligned}
\left\|t^{\beta,\eta}_{y,\alpha}(.,z)\right\|^2_{L^2_\alpha(\mathbb{R}^+)} &= \frac{1}{\pi^2}\sum_{n\in{}^cN^{\alpha}_y}\frac{\Gamma(n+\alpha+1)}{n!(\gamma^{\beta,\eta}_n L^{(\alpha)}_n(y))^2}|z|^{2n}\\
&\leq \frac{1}{\pi^2(m^{\alpha}_y)^2}\sum_{n\in{}^cN^{\alpha}_y}\frac{\Gamma(n+\alpha+1)}{n!(\gamma^{\beta,\eta}_n)^2}|z|^{2n}. \qquad (2.21)
\end{aligned}
$$

The last inequality follows under the additional assumption that $L^{(\alpha)}_n(y)$, for $n\in{}^cN^{\alpha}_y$ is uniformly bounded from below for $y\in\Omega_\alpha$. But, since for n large enough we have

$$
\frac{\Gamma(n+\alpha+1)|z|^{2n}}{(\gamma^{\beta,\eta}_n)^2 n!} \sim \frac{n^{2\eta+\alpha}e^{2n\log|z|}}{(\Gamma(\eta))^2},
$$

the series in the right-hand side of (2.21) converges. This proves that the function $t^{\beta;\eta}_{y,\alpha}(.,z)$ belongs to $L^2_\alpha(\mathbb{R}^+)$ for every fixed $y \in \Omega_\alpha$ and $z \in D$.

Operational formula. In the sequel, we show that this kernel function is a solution of some partial differential equation. Thus, by applying complex Euler operator $\overline{E} := \overline{z}\partial/\partial\overline{z}$ to the power series (2.12) we get

$$\overline{E}(t^{\beta,\eta}_{y,\alpha}(x,z)) = \frac{1}{\pi}\sum_{n\in {}^cN^\alpha_y} \frac{nL^{(\alpha)}_n(x)}{\gamma^{\beta,\eta}_n L^{(\alpha)}_n(y)}\overline{z}^n.$$

Nevertheless, since the Laguerre polynomial $L^{(\alpha)}_n$ is solution of the differential equation $nL^{(\alpha)}_n(x) = -\mathcal{L}^\alpha_x(L^{(\alpha)}_n)(x)$ (see for example [7, p. 284]), where $\mathcal{L}^\alpha_x$ denotes the Laguerre differential operator

$$\mathcal{L}^\alpha_x := x\frac{\partial^2}{\partial x^2} + (\alpha+1-x)\frac{\partial}{\partial x},$$

it follows

$$\overline{E}(t^{\beta,\eta}_{y,\alpha}(x,z)) = -\mathcal{L}^\alpha_x\left(\frac{1}{\pi}\sum_{n\in {}^cN^\alpha_y} \frac{L^{(\alpha)}_n(x)}{\gamma^{\beta,\eta}_n L^{(\alpha)}_n(y)}\overline{z}^n\right) = -\mathcal{L}^\alpha_x(t^{\beta,\eta}_{y,\alpha}(x,z)). \tag{2.22}$$

As immediate consequence, one derives

$$\mathcal{L}^\alpha_x\left(M^\alpha_y f\right)(x) = \int_D -\overline{z}\frac{\partial}{\partial\overline{z}}\left(t^{\beta,\eta}_{y,\alpha}(x,z)\right) f(z) A_{\beta,\eta}(|z|^2)d\lambda(z),$$

valid for given sufficiently regular function f. An integration by parts, keeping in mind that $A_{\beta,\eta}(|z|^2) = (1-|z|^2)^{\eta-1}|z|^{2\beta-2}$, yields the operational formula

$$\begin{aligned}\mathcal{L}^\alpha_x\left(M^\alpha_y f\right)(x) &= \int_D t^{\beta,\eta}_{y,\alpha}(x,z)\left(\beta-(\eta-1)\frac{|z|^2}{1-|z|^2}+\overline{E}\right) f(z) A_{\beta,\eta}(|z|^2)d\lambda(z).\\ &= M^\alpha_y\left((\beta-(\eta-1)\frac{|z|^2}{1-|z|^2}+\overline{E})f\right)(x).\end{aligned}$$

Conclusion. We conclude by noticing that the consideration of Υ_α leads to a unified treatment of (a) and (b). The main concern is whether Υ_α is not reduced to $\{0\}$. However, one broadens it when considering for example (a) independently of (b). This can be handled by reconsidering the set Ω_α, which is more largest than Υ_α. From the numerical values of $|L^{(\alpha)}_n(y)|$ computed using the Python programming software, it is clear that, for example, for $m^\alpha_y = 1/\Gamma(\alpha+1)$ with $\alpha = 3$ and $y = 1$, the values of $|L^{(3)}_n(1)|$ are all superior than $1/\Gamma(\alpha+1) = 0.1666666667$. Hence, the arguments employed to prove the convergence of the series (2.12) when $y \in \Upsilon_\alpha$ seem to remain valid for $y \in \Omega_\alpha$. In fact, making use of $|L^{(\alpha)}_n(x)| \leq \binom{\alpha+n}{\alpha}e^{x/2}$ (see e.g. [81, p.294]), it follows that

$$\frac{|L^{(\alpha)}_n(x)|}{|L^{(\alpha)}_n(y)|}\frac{|z|^n}{\pi\gamma^{\beta,\eta}_n} \leq \frac{\Gamma(\alpha+n+1)e^{n\log|z|}}{m^\alpha_y\Gamma(\alpha+1)n!\pi\gamma^{\beta,\eta}_n}e^{x/2}. \tag{2.23}$$

One can then apply the Euler formula for the Gamma function to get

$$e^{x/2}\frac{\Gamma(\alpha+n+1)e^{n\log|z|}}{m_y^\alpha\Gamma(\alpha+1)n!\pi\gamma_n^{\beta,\eta}} \sim \frac{n^{\alpha+\eta}e^{n\log|z|}}{m_y^\alpha\Gamma(\alpha+1)\pi\Gamma(\eta)}e^{x/2}, \tag{2.24}$$

for n large enough. The series with general term $n^{\eta+\alpha}e^{n\log|z|}$ is clearly convergent for $\log|z|<0$ with $\eta>0$ and $\alpha\geq 0$. This proves that the series in (2.12) converges for every $y\in\Omega_\alpha$, and uniformly on the compact subset of D.

2.3 Reproducing kernel of the range of the dual fractional Hankel transform

The following result gives the explicit expression of its reproducing kernel in terms of the Gauss hypergeometric function

Proposition 2.3.1. *$R(S_y^\alpha)$ is a reproducing kernel Hilbert space with the reproducing kernel given by:*

$$\mathcal{K}_{\beta,\eta}^{R(S_y^\alpha)}(z,w) = \frac{\Gamma(\beta+\eta)}{\pi\Gamma(\eta)\Gamma(\beta)}\left({}_2F_1\left(\begin{matrix}1,\beta+\eta\\ \beta\end{matrix}\,\middle|\, z\overline{w}\right) - \sum_{n\in N_y^\alpha}\frac{(\beta+\eta)_n}{(\beta)_n}(z\bar{w})^n\right). \tag{2.25}$$

Proof. Let $f\in R(S_y^\alpha)$ and write $f(z)=\sum_{n\notin N_y^\alpha} a_n\varphi_n^\alpha(y)z^n$ with $\|f\|^2=\sum_{n=0}^\infty \gamma_n^{\beta,\eta}|\varphi_n^\alpha(y)|^2|a_n|^2 < +\infty$. Then by the Cauchy-Schwartz inequality, we get

$$\begin{aligned}|f(z)| &\leq \left(\sum_{n\notin N_y^\alpha}\frac{|z^n|^2}{\pi\gamma_n^{\beta,\eta}}\right)^{1/2}\left(\sum_{n\notin N_y^\alpha}\pi\gamma_n^{\beta,\eta}|a_n|^2|\varphi_n^\alpha(y)|^2\right)^{1/2}\\ &\leq \left(\frac{\Gamma(\beta+\eta)}{\pi\Gamma(\eta)\Gamma(\beta)}\left({}_2F_1\left(\begin{matrix}1,\beta+\eta\\ \beta\end{matrix}\,\middle|\, |z|^2\right) - \sum_{n\in N_y^\alpha}\frac{(\beta+\eta)_n}{(\beta)_n}(|z|^2)^n\right)\right)^{1/2}\|f\|.\end{aligned}$$

The involved series is convergent since

$$\frac{|z|^{2n}}{\gamma_n^{\beta,\eta}} = \frac{\Gamma(\beta+\eta+n)}{\Gamma(\eta)\Gamma(\beta+n)}e^{2n\log|z|} \sim \frac{1}{\Gamma(\eta)}n^\eta e^{2n\log|z|} \tag{2.26}$$

for n large enough. This shows that the linear evaluation map $f\longmapsto f(z)$ is continuous on $R(S_y^\alpha)$ for every fixed $z\in D$, and therefore $R(S_y^\alpha)$ is a reproducing kernel Hilbert space by means of the Riesz representation theorem. Thus, it exists a unique $K_z\in R(S_y^\alpha)$ such that $f(z)=\langle K_z,f\rangle_{R(S_y^\alpha)}$, for all $f\in R(S_y^\alpha)$. Now, since

$$s_n^{\alpha,\beta,\eta}(z) := \left(\frac{\Gamma(\beta+\eta+n)}{\pi\Gamma(\eta)\Gamma(\beta+n)}\right)^{1/2} z^n; \quad n\notin N_y^\alpha.$$

for varying nonnegative integer n not in N_y^α, form an orthonormal basis of $R(S_y^\alpha)$, we get

$$\begin{aligned}\mathcal{K}_{\beta,\eta}^{R(S_y^\alpha)}(z,w) &= \sum_{n\notin N_y^\alpha} s_n^{\alpha,\beta,\eta}(z)\overline{s_n^{\alpha,\beta,\eta}(z)} \\ &= \sum_{n=0}^{\infty} s_n^{\alpha,\beta,\eta}(z)\overline{s_n^{\alpha,\beta,\eta}(z)} - \sum_{n\in N_y^\alpha} s_n^{\alpha,\beta,\eta}(z)\overline{s_n^{\alpha,\beta,\eta}(z)} \\ &= \frac{\Gamma(\beta+\eta)}{\pi\Gamma(\eta)\Gamma(\beta)}\left(\sum_{n=0}^{\infty}\frac{(\beta+\eta)_n}{(\beta)_n}z^n\overline{w}^n - \sum_{n\in N_y^\alpha}\frac{(\beta+\eta)_n}{(\beta)_n}z^n\overline{w}^n\right)\end{aligned}$$

□

Notice that, we can express $\mathcal{K}_{\beta,\eta}^{R(S_y^\alpha)}(z,w)$ for particular values of β and η as in the following example

Examples 2.3.1. *From the expression of the hypergeometric function given in [85, p.38]*

$$ {}_2F_1\left(\begin{array}{c} 2\beta,\beta+1 \\ \beta \end{array}\middle| z\overline{w}\right) = (1+z\overline{w})(1-z\overline{w})^{-2\beta-1} \tag{2.27}$$

for instance, when $\beta = 1/2$ and $\eta = 1$, the reproducing Kernel of $R(S_y^\alpha)$ is exactly

$$\mathcal{K}_{1/2,1}^{R(S_y^\alpha)}(z,w) = \frac{\Gamma(3/2)}{\pi\Gamma(1/2)}\left((1+z\overline{w})(1-z\overline{w})^{-2} - \sum_{n\in N_y^\alpha}\frac{(3/2)_n}{(1/2)_n}(z\bar{w})^n\right)$$

obviously, in that case, the explicit form of the reproducing Kermel of $A_{\beta,\eta}^{2,\alpha}(\mathcal{D})$ is exactly

$$\mathcal{K}_{1/2,1}^{A_{\beta,\eta}^{2,\alpha}(\mathcal{D})}(z,w) = \frac{\Gamma(3/2)}{\pi\Gamma(1/2)}(1+z\overline{w})(1-z\overline{w})^{-2}$$

The example below shows that the reproducing kernel of a random hilbert space $\mathcal{H}$ once it is evaluated when its both variables are equal, we can not necessarily say that it is equal to the inverse of the weight function of H, for instance we have

$$\mathcal{K}_{1/2,1}^{R(S_y^\alpha)}(z,z) = \frac{\Gamma(3/2)}{\pi\Gamma(1/2)}\left((1+|z|^2)(1-|z|^2)^{-2} - \sum_{n\in N_y^\alpha}\frac{(3/2)_n}{(1/2)_n}|z|^{2n}\right) \neq \frac{1}{A_{1/2,1}(|z|^2)} \text{ and}$$

$$\mathcal{K}_{1/2,1}^{A_{\beta,\eta}^{2,\alpha}(\mathcal{D})}(z,z) = \frac{\Gamma(3/2)}{\pi\Gamma(1/2)}(1+|z|^2)(1-|z|^2)^{-2} \neq \frac{1}{A_{1/2,1}(|z|^2)}$$

Remark 2.3.2. *starting from the functional equation (2.2), it is clear that for every $f \in R^{\beta,\eta}\left(S_y^\alpha\right)$ we have*

$$f(z) = \int_D f(w)K_{\beta,\eta}^{R(S_y^\alpha)}(z,w)A_{\beta,\eta}\left(|w|^2\right)d\lambda(w),$$

where

$$z \mapsto K_{\beta,\eta}^{R(S_y^\alpha)}(z,w) := \frac{1}{1-z}\int_0^{+\infty} \exp\left(-\frac{x+yz}{1-z}\right)\left(\frac{x}{yz}\right)^{\alpha/2} I_\alpha\left(\frac{2\sqrt{xyz}}{1-z}\right) t_{y,\alpha}^{\beta,\eta}(x,w)dx$$

is clearly holomorphic and belongs to $R^{\beta,\eta}\left(S_y^\alpha\right)$. *This is to say that* $K_{\beta,\eta}^{R(S_y^\alpha)}(z,w)$ *is the reproducing kernel of* $R^{\beta,\eta}\left(S_y^\alpha\right)$ *and that the kernel* $t_{y,\alpha}^{\beta,\eta}$ *is solution of the integral equation*

$$\frac{1}{1-z}\int_0^{+\infty} \exp\left(-\frac{x+yz}{1-z}\right)\left(\frac{x}{yz}\right)^{\alpha/2} I_\alpha\left(\frac{2\sqrt{xyz}}{1-z}\right) t_{y,\alpha}^{\beta,\eta}(x,w)dx = K_{\beta,\eta}^{R(S_y^\alpha)}(z,w).$$

2.4 The general form of the dual fractional Hankel transform and its inverse problem

2.4.1 On the β-modified Bergman space

On this subsection, we are going to treat some basic properties of the so-called β-modified Bergman space on $\mathcal{D}(0,R) = \{z \in \mathbb{C}; |z| < R\}$, which is the space of holomorphic functions on the punctured discus of radius $R > 0$, denoted by

$$\mathcal{D}^*(0,R) = \mathcal{D}(0,R) \setminus \{0\}$$

and square integrable functions on $\mathcal{D}(0,R)$ with the corresponding measure

$$\mu_{\beta,\eta}^R(|z|^2) = \left(1 - \frac{|z|^2}{R^2}\right)^{(\eta-1)R^2} |z|^{2\beta-2} d\lambda(z).$$

This space is resumed as follows

$$\mathcal{A}_{\beta,\eta}^{2,\alpha}(\mathcal{D}(0,R)) = \mathcal{H}ol(\mathcal{D}^*(0,R)) \cap L^2(\mathcal{D}(0,R), \mu_{\beta,\eta}^R(|z|^2)), \tag{2.28}$$

its orthonormal basis is given by

$$f_n^{\beta,\eta}(z) = \left(\frac{\Gamma(n+\beta+\lceil-\beta\rceil+(\eta-1)R^2+1)}{\pi\Gamma(n+\beta+\lceil-\beta\rceil)\Gamma((\eta-1)R^2+1)}\right)^{1/2} \frac{z^{n+\lceil-\beta\rceil}}{R^{n+\beta}}, \; n = 0,1,2,\cdots$$

where $\lceil . \rceil$ is the ceiling function defined by $\lceil x \rceil = \min\{n \in \mathbb{Z} \mid n \geq x\}$.

Proposition 2.4.1. *The* β*-modified space* $\mathcal{A}_{\beta,\eta}^{2,\alpha}(\mathcal{D}(0,R))$ *satisfies the following results when* R *goes to* $+\infty$

(1) The β*-modified space* $\mathcal{A}_{\beta,\eta}^{2,\alpha}(\mathcal{D}(0,R))$ *is a reproducing kernel Hilbert space with the reproducing kernel* $\mathcal{K}_{\beta,\eta,R}$ *having the following limit when* R *tends to* $+\infty$

$$\lim_{R\longrightarrow+\infty} \mathcal{K}_{\beta,\eta,R}(z,w) = \frac{(z\overline{w})^{+\lceil-\beta\rceil}(\eta-1)^{\beta+\lceil-\beta\rceil}}{\pi\Gamma(\beta+\lceil-\beta\rceil)} {}_1F_1\left(\begin{matrix} 1 \\ \beta+\lceil-\beta\rceil \end{matrix}\middle|(\eta-1)z\overline{w}\right). \tag{2.29}$$

(2) For every $z \in \mathbb{C}$, we have

$$\lim_{R\to+\infty} \mu^R_{\beta,\eta}(|z|^2) = |z|^{2\beta-2} e^{-(\eta-1)|z|^2}. \tag{2.30}$$

Remark 2.4.2. *For particular values $\beta + \lceil -\beta \rceil = 1$ and $\eta - 1 = \pi$, we can rely formulas* (2.30) *and* (2.29) *from each other by noticing the following limit relation when R is large enough*

$$\lim_{R\longrightarrow+\infty} \mathcal{K}_{\beta,\eta,R}(z,z) = \frac{1}{\lim_{R\to+\infty} \mu^R_{\beta,\eta}(|z|^2)} \tag{2.31}$$

On the other hand for any $\eta > 0$ and $\beta = 1$ with $\lceil -\beta \rceil = 0$, we can also notice that

$$\lim_{R\longrightarrow+\infty} \mathcal{K}_{\beta,\eta,R}(z,w) = \frac{(\eta-1)}{\pi} \exp\left((\eta-1)z\overline{w}\right) \tag{2.32}$$

the function in the right hand side of (2.32)*, which depends on the two variables z and w in $\mathbb{C}$ represents the reproducing kernel of the Bargmann-Fock space consisting of holomorphic square integrable functions on $\mathbb{C}$ with the corresponding Gaussian measure $e^{-(\eta-1)|z|^2} d\lambda(z)$ denoted by*

$$\mathcal{F}^{2,\eta-1}(\mathbb{C}) = \mathcal{H}ol(\mathbb{C}) \cap L^2\left(\mathbb{C}, e^{-(\eta-1)|z|^2} d\lambda(z)\right) \tag{2.33}$$

2.4.2 The general form of the dual fractional Hankel transform

By going so far on our results finding out some applications among the underlying mathematical objects, we generalize the dual fractional Hankel transform into this form

$$S^\alpha_{w,R}(\varphi)(y) = \int_0^{+\infty} \varphi(x) R^\alpha_{w,R}(x,y) x^{-\alpha} e^{-x} dx, \tag{2.34}$$

where

$$\begin{aligned} R^\alpha_{w,R}(x,y) &= w^{\lceil -\beta \rceil} \sum_{n=0}^{+\infty} \frac{n!}{\Gamma(n+\alpha+1)} L^{(\alpha)}_n(x) L^{(\alpha)}_n(y) w^n \\ &= w^{\lceil -\beta \rceil} (1-w)^{-1} \exp\left(\frac{w(x+y)}{1-w}\right) (wxy)^{-\alpha/2} I_\alpha\left(\frac{2\sqrt{xyw}}{1-w}\right), \end{aligned}$$

fixing y and make w as a variable in $w \in \mathcal{D}(0,R)$ for all R is strictly positive. This is called the generalized form of the dual fractional Hankel transform. The inversion problem of $S^{\beta,\eta}_{y,R}$ can be studied when we consider the invertible operator $S^{\beta,\eta}_{y,R} : (\ker(S^{\beta,\eta}_{y,R}))^\perp \longrightarrow R^\alpha_{\beta,\eta}(S^{\beta,\eta}_{y,R})$ such that there exist $M^{\beta,\eta}_{y,R} : R^\alpha_{\beta,\eta}(S^{\beta,\eta}_{y,R}) \longrightarrow (\ker(S^{\beta,\eta}_{y,R}))^\perp$ verifying those two conditions $S^{\beta,\eta}_{y,R} M^{\beta,\eta}_{y,R}(f) = f$ and $M^{\beta,\eta}_{y,R} S^{\beta,\eta}_{y,R} \varphi = \varphi$ for all $(f,\varphi) \in R^\alpha_{\beta,\eta}(S^{\beta,\eta}_{y,R}) \times (\ker(S^{\beta,\eta}_{y,R}))^\perp$. Recall that, $(\ker(S^{\beta,\eta}_{y,R}))^\perp$ is the closure on $L^{2,\alpha}(\mathbb{R}^+)$ of the linear combination of the family φ^α_n where the indices $n \notin N^\alpha_y$. But, $R^\alpha_{\beta,\eta}(S^{\beta,\eta}_{y,R})$ is the range of $S^{\beta,\eta}_{y,R}$ consisting of the series $\sum_{n\notin N^\alpha_y} \varphi^\alpha_n(y) a_n e_{n+\lceil -\beta \rceil}$ assuring the growth condition

$$\sum_{n\notin N^\alpha_y} |a_n|^2 R^{2(\beta+n+\lceil -\beta \rceil)} \frac{\Gamma(\beta+n+\lceil -\beta \rceil)\Gamma((\eta-1)R^2+1)}{\Gamma(\beta+n+\lceil -\beta \rceil+(\eta-1)R^2+1)} (\varphi^\alpha_n(y))^2 \pi < +\infty \tag{2.35}$$

Since, we have $S_{y,R}^{\beta,\eta}(\varphi_n^\alpha) = \varphi_n^\alpha(y)e_{n+\lceil -\beta\rceil}$, then the action of the inverse of $S_{y,R}^{\beta,\eta}$ on the orthogonal basis of $R_{\beta,\eta}^\alpha(S_{y,R}^{\beta,\eta})$, is seen as $M_{y,R}^{\beta,\eta}(e_{n+\lceil -\beta\rceil}) = \frac{\varphi_n^\alpha}{\varphi_n^\alpha(y)}$. Generalizing the last one to the elements of $R_{\beta,\eta}^\alpha(S_{y,R}^{\beta,\eta})$ described in a serie expansion of the form $f = \sum_{n\notin N_y^\alpha} b_n e_{n+\lceil -\beta\rceil}$, we get at least formally the expression of $M_{y,R}^{\beta,\eta}$, given by

$$M_{y,R}^{\beta,\eta}(f)(x) = \sum_{n\notin N_y^\alpha} R^{-2(\beta+n+\lceil -\beta\rceil)} \frac{\Gamma(\beta+n+\lceil -\beta\rceil+(\eta-1)R^2+1)}{\pi\Gamma(\beta+n+\lceil -\beta\rceil)\Gamma((\eta-1)R^2+1)} \frac{L_n^{(\alpha)}(x)}{L_n^{(\alpha)}(y)}$$
$$\times \left\langle f, e_{n+\lceil -\beta\rceil}\right\rangle_{\beta,\eta,R}$$

where $b_n = \left\langle f, \frac{e_{n+\lceil -\beta\rceil}}{\left\|e_{n+\lceil -\beta\rceil}\right\|^2}\right\rangle_{\beta,\eta,R}$. The integral representation of $M_{y,R}^{\beta,\eta}$ is written as well

$$M_{y,R}^{\beta,\eta}(f)(x) = \int_{\mathcal{D}(0,R)} T_{y,R}^{\beta,\eta,\alpha}(x,z) f(z) d\mu_{\beta,\eta}^R(|z|^2) \tag{2.36}$$

$T_{y,R}^{\beta,\eta,\alpha}$ is the kernel of $M_{y,R}^{\beta,\eta}$ explicitly we have

$$T_{y,R}^{\beta,\eta,\alpha}(x,z) = \frac{\overline{z}^{\lceil -\beta\rceil} R^{-2(\beta+\lceil -\beta\rceil)}}{\pi\Gamma((\eta-1)R^2+1)} \sum_{n\notin N_y^\alpha} R^{-2n} \frac{\Gamma(\beta+n+\lceil -\beta\rceil+(\eta-1)R^2+1)}{\Gamma(\beta+n+\lceil -\beta\rceil)} \frac{L_n^{(\alpha)}(x)}{L_n^{(\alpha)}(y)} \overline{z}^n \tag{2.37}$$

Lets discuss the convergence of the serie in (2.37) for the set Ω_α that is treated previously. Following the same rules as in the previous sections, we can find the estimation for the absolute value of the terms of the serie in (2.37) for n is very large and $n \notin N_y^\alpha$, represented by

$$U_{\beta,\eta}^R(z) \frac{\Gamma(\beta+n+\lceil -\beta\rceil+(\eta-1)R^2+1)}{\Gamma(\beta+n+\lceil -\beta\rceil)} \frac{|L_n^{(\alpha)}(x)|}{|L_n^{(\alpha)}(y)|} \left|\frac{z}{R^2}\right|^n \sim U_{\beta,\eta}^R(z) \frac{|L_n^{(\alpha)}(x)|}{|L_n^{(\alpha)}(y)|} n^{(\eta-1)R^2+1} e^{n\log\frac{|z|}{R^2}},$$

where

$$U_{\beta,\eta}^R(z) \frac{|L_n^{(\alpha)}(x)|}{|L_n^{(\alpha)}(y)|} n^{(\eta-1)R^2+1} e^{n\log\frac{|z|}{R^2}} \leq \frac{e^{x/2} U_{\beta,\eta}^R(z)}{m_y^\alpha \Gamma(\alpha+1)} \frac{\Gamma(n+\alpha+1) n^{(\eta-1)R^2+1} e^{n\log\frac{|z|}{R^2}}}{n!}$$
$$\sim \frac{e^{x/2} U_{\beta,\eta}^R(z)}{m_y^\alpha \Gamma(\alpha+1)} n^{(\eta-1)R^2+1+\alpha} e^{n\log\frac{|z|}{R^2}}$$

and $U_{\beta,\eta}^R(z) = \frac{|z|^{\lceil -\beta\rceil} R^{-2(\beta+\lceil -\beta\rceil)}}{\pi\Gamma((\eta-1)R^2+1)}$. Thus, the approximation above proves such affirmation according to the Riemann criterion for all $z \in \mathcal{D}(0,R)$.

Another important thing that we must point on is that we have to find relevant conditions that matches with the boundness of $M_{y,R}^{\beta,\eta}$. For that, the Parseval's identity would give this inequality

$$\left\|M_{y,R}^{\beta,\eta}(f)\right\|_\alpha^2 = \sum_{n\notin N_y^\alpha} \frac{|b_n|^2}{(\varphi_n^\alpha(y))^2} \leq \|f\|_{R,\beta,\eta}^2 \left(\sum_{n\notin N_y^\alpha} \frac{1}{(\varphi_n^\alpha(y))^2 \left\|e_{n+\lceil -\beta\rceil}\right\|_{R,\beta,\eta}^2}\right). \tag{2.38}$$

We proceed similarly as in the previous results, proving the convergence of the serie in the right-hand side of (2.38) depending on $R > 1$. Adapting to this calculation below for n is very large

$$\begin{aligned}\sum_{n\notin N_y^\alpha} \frac{\Gamma(\alpha+n+1)}{n!(L_n^{(\alpha)}(y))^2\left\|e_{n+\lceil-\beta\rceil}\right\|^2} &= \frac{R^{-2(\beta+\lceil-\beta\rceil)}}{\pi\Gamma((\eta-1)R^2+1)}\sum_{n\notin N_y^\alpha} R^{-2n} \\ &\times \frac{\Gamma(\alpha+1+n)\Gamma(\beta+n+\lceil-\beta\rceil+(\eta-1)R^2+1)}{n!\Gamma(n+\beta+\lceil-\beta\rceil)(L_n^{(\alpha)}(y))^2} \\ &\sim \frac{R^{-2(\beta+\lceil-\beta\rceil)}}{\pi\Gamma((\eta-1)R^2+1)}\sum_{n\notin N_y^\alpha} n^{\alpha+(\eta-1)R^2+1}(R^n L_n^{(\alpha)}(y))^{-2}.\end{aligned}$$

Notice that there are two different ways to prove the convergence of the serie above and that would require a certain condition one different to another. Obrserve that, if we took $R > 1$, we can fix y in the more larger set Ω_α and then the serie approximation above would be inferior to $\kappa_{\beta,\eta}^{R,\alpha}\sum_{n\notin N_y^\alpha} n^{\alpha+(\eta-1)R^2+1}(R^n m_y^\alpha)^{-2}$, which is a convergent serie according to the Riemann criterion.

Remark 2.4.3. *Note that, the approximation of the kernel of $M_{y,R}^{\beta,\eta}$ when $R \longrightarrow +\infty$, given by*

$$T_{y,R}^{\beta,\eta,\alpha}(x,z) \sim \frac{\overline{z}^{\lceil-\beta\rceil}}{\pi}\sum_{n\notin N_y^\alpha} \frac{(\eta-1)^{n+\beta+\lceil-\beta\rceil}}{\Gamma(\beta+n+\lceil-\beta\rceil)}\frac{L_n^{(\alpha)}(x)}{L_n^{(\alpha)}(y)}\overline{z}^n, \tag{2.39}$$

which ensures the following inequality

$$\left|\frac{\overline{z}^{\lceil-\beta\rceil}}{\pi}\sum_{n\notin N_y^\alpha} \frac{(\eta-1)^{n+\beta+\lceil-\beta\rceil}}{\Gamma(\beta+n+\lceil-\beta\rceil)}\frac{L_n^{(\alpha)}(x)}{L_n^{(\alpha)}(y)}\overline{z}^n\right| \le \frac{|z|^{\lceil-\beta\rceil}(\eta-1)^{\beta+\lceil-\beta\rceil}}{\pi m_y^\alpha}\,{}_1F_1\left(\begin{matrix}\alpha+1\\ \beta+\lceil-\beta\rceil\end{matrix}\,\middle|\,(\eta-1)|z|\right)$$

for all $z \in \mathbb{C}$ and for a fixed $y \in \Omega_\alpha$.

Chapter 3

The bicomplex dual fractional Hankel transform and its inverse problem

Abstract: In the present chapter, we are concerned with the bicomplex analogs of different tools, spaces and transforms discussed in chapter 2. We begin by recalling some preliminary results concerning the bicomplex numbers, the associated infinite dimensional Hilbert space and the bc-meromorphic functions. The so-called γ-modified bicomplex Bergman space of bc-meromorphic functions on $\mathcal{D}^*_{bc}$ as well as the associated unitary isometric integral transform of Bargmann type are next introduced and their basic properties are discussed in Section 3.2. Section 3.3 is devoted to the bicomplex version of the fractional Hankel transform and to its dual transform. We conclude in Section 3.5 with a generalized form of the last considered transform.

3.1 Preliminaries

We denote by $\mathbb{BC}$ the set of bicomplex numbers. This is the set of those $Z = z_1 + jz_2$, where $z_1, z_2 \in \mathbb{C}$ and j is a imaginary unit different of i and commuting with it ($i^2 = j^2 = -1$ and $ij = ji =: k$). Their idempotent decomposition reads $Z = \alpha e_+ + \beta e_-$ with $\alpha = z_1 - iz_2$ and $\beta = z_1 + iz_2$. Here e_+ and e_- are the idempotent zero divisors given by

$$e_+ = \frac{1+ij}{2} \quad \text{and} \quad e_- = \frac{1-ij}{2}, \tag{3.1}$$

and satisfying the basic facts $e_+ + e_- = 1$ and $e_+e_- = 0$. The complex conjugates of Z are defined by

$$Z^\dagger = z_1 - jz_2 = \beta e_+ + \alpha e_-, \tilde{Z} = \bar{z}_1 + j\bar{z}_2 = \bar{\beta} e_+ + \bar{\alpha} e_- \text{ and } Z^* = \bar{z}_1 - j\bar{z}_2 = \bar{\alpha} e_+ + \bar{\beta} e_-.$$

In the sequel, by $\mathbb{O}_{bc}$ we mean $\mathcal{N} \cup \{0\}$, where $\mathcal{N}$ is the null-cone constituted of zero divisors and given by

$$\mathcal{N} = \{Z \in \mathbb{BC}; \quad Z \neq 0 \text{ and } ZZ^{\dagger} = 0\},$$

and which coincides with the set $\mathbb{C}e_+ \cup \mathbb{C}e_-$, where $\mathbb{C}e_\pm = \{ze_\pm; z \in \mathbb{C}\}$ is the ideal generated in $\mathbb{BC}$ by $e_\pm$. A distinguish subset is $\mathbb{D}^+ = \{xe_+ + ye_- \in \mathbb{D}; x, y \geq 0\}$ of hyperbolic numbers $\mathbb{D} = \{xe_+ + ye_-; x, y \in \mathbb{R}\}$. Accordingly, one defines a partial order in $\mathbb{D}$ by $Z \preceq W$ if and only if $x \leq x'$ and $y \leq y'$ for $Z = xe_+ + ye_- \in \mathbb{D}$ and $W = x'e_+ + y'e_- \in \mathbb{D}$. The product of two bicomplex numbers for $Z = \alpha e_+ + \beta e_-$ and $W = \alpha' e_+ + \beta' e_-$reads $ZW = \alpha\alpha' e_+ + \beta\beta' e_-$, and leads to $Z^\gamma = \alpha^\gamma e_+ + \beta^\gamma e_-$ when γ is a non-negative integer. For γ being a negative integer, the relation remains valid when $Z \notin \mathbb{O}_{bc}$. However, for γ is not integer, we define $Z^\gamma = \alpha^\gamma e_+ + \beta^\gamma e_-$, whenever $Z \notin \mathbb{D}^-$ and γ is positive, or where γ is negative and $Z \notin \mathbb{O}_{bc} \cap \mathbb{D}^-$. Here $\mathbb{D}^-$ denotes the set of negative hyperbolic numbers.

On the other hand, a hyperbolic norm on a $\mathbb{BC}$-module E is defined as a mapping $\|.\|_{\mathbb{D}} : E \longrightarrow \mathbb{D}^+$, also denoted $\|.\|_{\mathbb{D},E}$ when confusion may arise, for which $\|u+v\|_{\mathbb{D}} \preceq \|u\|_{\mathbb{D}} + \|v\|_{\mathbb{D}}$ and $\|Z.u\|_{\mathbb{D}} = \|Z\|_k \|u\|_{\mathbb{D}}$ hold true for $u, v \in E$ and $Z \in \mathbb{BC}$, as well as $\|u\|_{\mathbb{D}} = 0$ if and only if $u = 0$. Thus, a mapping $T : E \longrightarrow F$, where F is another $\mathbb{BC}$-module endowed with a hyperbolic norm $\|\cdot\|_{\mathbb{D},F}$, is said to be a $\mathbb{D}$-bounded operator if we have [5, p. 74]

$$\|T(\phi)\|_{\mathbb{D},F} \preceq M \|\phi\|_{\mathbb{D},E}, \quad \phi \in E, \tag{3.2}$$

for certain $M \in \mathbb{D}^+$. A fundamental example of such hyperbolic norm is given by the k-hyperbolic norm defined by [82, sec 2.7]

$$\|Z\|_k = \sqrt{Z \cdot Z^*} = |\alpha|\,\mathbf{e}_+ + |\beta|\,\mathbf{e}_-. \tag{3.3}$$

While, a bicomplex-valued mapping $\langle , \rangle_{bc}$ on $E \times E$ is said to be a bicomplex inner product on E if [46]

(i) $\langle \phi, \lambda\varphi + \psi \rangle_{bc} = \lambda^* \langle \phi, \varphi \rangle_{bc} + \langle \phi, \psi \rangle_{bc}$ for every $\lambda \in \mathbb{BC}$ and for all $\phi, \varphi, \psi \in E$.

(ii) $\langle \phi, \varphi \rangle_{bc} = \langle \varphi, \phi \rangle_{bc}^*$.

(iii) We have $\langle \phi, \phi \rangle_{bc} \in \mathbb{D}^+$ for every $\phi \in E$ and $\langle \phi, \phi \rangle_{bc} = 0$ if and only if $\phi = 0$.

It also should be noticed here that $\langle \cdot, \cdot \rangle_{bc}$ is a bicomplex inner product on E if and only if their projections are scalar product on $E^+ = P^+(E)$ and $E^- = P^-(E)$, respectively (see [46]), where $P^\pm$ are the projections from $\mathbb{BC}$ onto $\mathbb{C}$ given by $P^\pm(z_1 + jz_2) = z_1 \mp iz_2$. More exactly, for $\phi = \phi^+ e_+ + \phi^- e_-$ we have $\langle \varphi, \phi \rangle_{bc} = \langle \varphi^+, \phi^+ \rangle_{E^+} e_+ + \langle \varphi^-, \phi^- \rangle_{E^-} e_-$. Associated with $\langle \cdot, \cdot \rangle_{bc}$, it is worth noting that, the quantity

$$\|\varphi\|^2 = \frac{\langle \varphi^+, \varphi^+ \rangle_{E^+} + \langle \varphi^-, \varphi^- \rangle_{E^-}}{2} = |\langle \varphi, \varphi \rangle_{bc}| \tag{3.4}$$

takes value in $\mathbb{R}^+$. $\|.\| : E \longrightarrow \mathbb{R}$ is a $\mathbb{BC}-$norm on a $\mathbb{BC}-$module E, which satisfies $\|\lambda\phi\| \leq \sqrt{2}|\lambda|\|\phi\|$ for all $\phi \in E$ and $\lambda \in \mathbb{BC}$, where the modulus $|.|$ denotes the euclidean norm in $\mathbb{R}^4$, given by $|\lambda|^2 = (|\alpha|^2 + |\beta|^2)/2$ for a bicomplex number $\lambda = \alpha e_+ + \beta e_-$. Moreover, we have the following.

Theorem 3.1.1. *[46] A $\mathbb{BC}$-module $(E, \langle,\rangle_{bc})$ is an infinite bicomplex Hilbert space if and only if $(E^{\pm}, \langle,\rangle_{E^{\pm}})$ are complex Hilbert spaces.*

Notice also that the bicomplex analog of the Schwarz inequality reads [5, p. 68]

$$\|\langle\phi, \varphi\rangle_{bc}\|_k \preceq \|\phi\|_{bc,\mathbb{D}}\|\varphi\|_{bc,\mathbb{D}}, \tag{3.5}$$

for every ϕ, φ belonging to $(E, \langle,\rangle_{bc})$, where $\|\phi\|_{bc,\mathbb{D}}$ denotes the associated bicomplex hyperbolic norm given by $\|\phi\|^2_{bc,\mathbb{D}} = \langle\phi, \phi\rangle_{bc}$. Another tool to be recalled here is the bicomplex analog of the Riesz representation theorem [46, Theorem 3.7] (see also [5, p.66]). Namely,

Theorem 3.1.2. *Let f be a continuous $\mathbb{BC}$-linear functional on a bicomplex Hilbert space E ([5, p. 63–66]). Then, there exists a unique φ belonging to E such that $f(\phi) = \langle\phi, \varphi\rangle$ for every $\phi \in E$.*

Now, a bicomplex-valued function $F = F_1 + jF_2$ is said to be bc-holomorphic on an open set Ω of $\mathbb{BC}$ (endowed with the standard topology of $\mathbb{R}^4$) if the $\mathbb{C}$-valued functions $F_1(z_1, z_2) := F_1(Z)$ and $F_2(z_1, z_2) := F_2(Z)$ are holomorphic on $\Omega^+ \times \Omega^-$ with $\Omega^{\pm} := P^{\pm}(\Omega)$, and satisfy the Cauchy-Riemann system $\partial_{z_1}F_1 = \partial_{z_2}F_2$ and $\partial_{z_2}F_1 = -\partial_{z_1}F_2$, where ∂z is the classical complex Wirtinger derivative. The bc-holomorphy remains equivalent to a system of differential equations involving the bicomplex analogs of Wirtinger derivatives with respect to the different conjugates that are given by [110]

$$\frac{\partial f}{\partial Z^*} = \frac{\partial f}{\partial Z^{\dagger}} = \frac{\partial f}{\partial \widetilde{Z}} = 0, \tag{3.6}$$

where

$$\frac{\partial}{\partial Z^*} = \frac{\partial}{\partial \overline{z_1}} + j\frac{\partial}{\partial \overline{z_2}}; \quad \frac{\partial}{\partial Z^{\dagger}} = \frac{\partial}{\partial z_1} + j\frac{\partial}{\partial z_2}; \quad \frac{\partial f}{\partial \widetilde{Z}} = \frac{\partial}{\partial \overline{z_1}} - j\frac{\partial}{\partial \overline{z_2}}. \tag{3.7}$$

An interesting characterization of bc-holomorphic functions is the so-called Ringleb decomposition theorem [109] (see also [99, Theorem 15.5]).

Theorem 3.1.3. *A function $F : \Omega \subset \mathbb{BC} \longrightarrow \mathbb{BC}$ is bc-holomorphic, if and only if there exist two complex-valued holomorphic functions Φ^+ and Φ^- on Ω^+ and Ω^-, respectively, such that*

$$F(Z) = F(\alpha e_+ + \beta e_-) = \Phi^+(\alpha)e_+ + \Phi^-(\beta)e_-.$$

In order to define the notion of bc-meromorphic function (see [20, p. 60]), one needs to extend the set of bicomplex numbers by considering $\mathbb{BC} \cup I_\infty$, where $I_\infty = (\mathbb{C} \times_e \{\infty\}) \cup (\{\infty\} \times_e \mathbb{C}) \cup \{\infty \times \infty\}$.

Definition 3.1.4. *A given function F with values in $\mathbb{BC} \cup I_\infty$ is said to be a bc-meromorphic function in an open domain $\Omega \subset \mathbb{BC}$ if it can be rewritten as a quotient $F = G/H$, where both G and H are bicomplex holomorphic functions on Ω and such that H is not identically in the null-cone $\mathcal{N}$ in any component of Ω.*

Therefore, Ringleb decomposition theorem remains valid for bicomplex meromorphic functions (see [20, p. 60]). More precisely, we have the following.

Theorem 3.1.5. *A function $F : \Omega \subset \mathbb{BC} \longrightarrow \mathbb{BC}$ is bc-meromorphic, if and only if there exist two complex-valued meromorphic functions Φ^+ and Φ^- on Ω^+ and Ω^-, respectively, such that*

$$F(Z) = F(\alpha e_+ + \beta e_-) = \Phi^+(\alpha)e_+ + \Phi^-(\beta)e_-.$$

Accordingly, one defines a pole of a bc-meromorphic function F as a bicomplex number Z_0, for which $P^+(Z_0e_+)$ or $P^-(Z_0e_-)$ are poles of the idempotent components of ϕ^+ and ϕ^-, respectively. Therefore, poles of *bc*-meromorphic functions are never isolated singularities (see [21, 22]). However, from now on, we will only be concerned with functions having 0 as the only strong pole in the bicomplex discus. The other kind of poles are omitted.

Definition 3.1.6. *A point $Z_0 = \alpha_0 e_+ + \beta_0 e_-$, $\alpha_0, \beta_0 \in \mathbb{C}$, is said to be a strong pole for a bc-meromorphic function F when α_0 and β_0 are both poles for the idempotent component functions ϕ^+ and ϕ^-, respectively. We define its order to be $Ord(F, Z_0) := \max\left(Ord(\phi^+, \alpha_0), Ord(\phi^-, \beta_0)\right)$.*

3.2 The bicomplex second Bargmann transform

3.2.1 Meromorphic analog of bicomplex Bergman spaces

The classical weighted Bergman space

$$\mathcal{A}^{2,\eta}(\mathcal{D}) = \mathcal{H}ol(\mathcal{D}) \cap L^2(\mathcal{D}, (1-|z|^2)^\eta d\lambda)$$

is defined as the space of holomorphic functions on the unit disk $\mathcal{D}$ that are square integrable with respect to the measure $(1-|z|^2)^\eta d\lambda(z)$ for any $\eta > -1$. Its extension involving the bc-holomorphic functions on the discus

$$\mathcal{D}_{bc} = \{Z \in \mathbb{BC};\ \|Z\|_k \prec 1\} = \mathcal{D}e_+ + \mathcal{D}e_-$$

was introduced and studied in [107, 106]. Its idempotent components are the space $\mathcal{A}^{2,\eta}(\mathcal{D})$ by means of the Ringleb decomposition theorem. In this subsection, we consider its extension to the bicomplex meromorphic setting, which can be handled by means of the so-called γ-modified Bergman space $\mathcal{A}^{2,\eta}_\gamma(\mathcal{D}^*)$ defined as the space of holomorphic functions on the punctured unit disk $\mathcal{D}^* = \mathcal{D} \setminus \{0\}$ (with eventual pole at the origin) and that are square integrable functions with respect to the scalar product

$$\langle f, g\rangle_{\gamma,\eta} := \int_{\mathcal{D}} f(z)\overline{g(z)}|z|^{2\gamma}(1-|z|^2)^{\eta} d\lambda(z),\ \gamma, \eta > -1.$$

For this purpose, we define $L^2(\mathcal{D}_{bc}, d\mu_{\gamma,\eta})$ to be the bicomplex infinite Hilbert space of bicomplex-valued functions $F(Z) = F^+(\alpha,\beta)e_+ + F^-(\alpha,\beta)e_-$ on $\mathcal{D}^*_{bc} := \mathcal{D}_{bc} \setminus \mathbb{O}_{bc}$ with $Z = \alpha e_+ + \beta e_-$ such that

$$\|F\|^2_{k,\gamma,\eta} := \int_{\mathcal{D}_{bc}} \|F(Z)\|^2_k d\mu^{bc}_{\gamma,\eta}(Z) \tag{3.8}$$

is finite in $\mathbb{D}^+$, where $d\mu^{bc}_{\gamma,\eta} := \|Z\|^{2\gamma}_k (1 - \|Z\|^2_k)^{\eta} d\lambda$. The quantity in (3.8) reads explicitly as

$$\left(\int_{\mathcal{D}\times\mathcal{D}} |F^+(\alpha,\beta)|^2|\alpha|^{2\gamma}(1-|\alpha|^2)^{\eta} d\lambda(\alpha,\beta)\right) e_+ + \left(\int_{\mathcal{D}\times\mathcal{D}} |F^-(\alpha,\beta)|^2|\beta|^{2\gamma}(1-|\beta|^2)^{\eta} d\lambda(\alpha,\beta)\right) e_-,$$

where $d\lambda$ denotes the standard Lebesgue measure on the underlying space. The corresponding bicomplex inner product reads

$$\langle F, G\rangle_{k,\gamma,\eta} = \int_{\mathcal{D}_{bc}} F(Z)[G(Z)]^* d\mu^{bc}_{\gamma,\eta}(Z).$$

Definition 3.2.1. *The set of bicomplex-valued bc-meromorphic functions belonging to* $L^2(\mathcal{D}_{bc}, d\mu^{bc}_{\gamma,\eta})$*, that are bicomplex holomorphic in* $\mathcal{D}^*_{bc}$ *and with at most one strong pole at* 0*, is called the weighted bicomplex meromorphic Bergman space of first kind. It is denoted by* $\mathcal{A}^{2,\eta}_{bc,\gamma}(\mathcal{D}^*_{bc})$.

Remark 3.2.2. *It is worth to note that in such definition we do not impose to the strong pole to be of same order in each idempotent component.*

We provide bellow an explicit description of $\mathcal{A}^{2,\eta}_{bc,\gamma}(\mathcal{D}^*_{bc})$.

Proposition 3.2.3. *For* $\eta > -1$*, the space* $\mathcal{A}^{2,\eta}_{bc,\gamma}(\mathcal{D}^*_{bc})$ *is an infinite bicomplex Hilbert space. A bc-holomorphic function* F *on* $\mathcal{D}^*_{bc}$ *belongs to* $\mathcal{A}^{2,\eta}_{bc,\gamma}(\mathcal{D}^*_{bc})$ *if and only if for any* $Z \in \mathcal{D}^*_{bc}$ *it can be expanded as*

$$F(Z) = \sum_{n=m_\gamma}^{+\infty} A_n Z^n, \tag{3.9}$$

where m_γ *is the integer part of* $-\gamma$*, for some bicomplex sequence* $\{A_n\}$ *obeying the growth condition*

$$\pi \sum_{n=m_\gamma}^{+\infty} \frac{\Gamma(n+\gamma+1)\Gamma(\eta+1)}{\Gamma(n+\gamma+\eta+2)} \|A_n\|^2_k \prec +\infty. \tag{3.10}$$

Moreover, it is a reproducing kernel bicomplex Hilbert space with reproducing kernel given by

$$\mathcal{K}^{bc}_{\gamma,\eta}(Z,W) = \frac{\Gamma(\gamma+\eta+m_\gamma+2)}{\pi\Gamma(\eta+1)\Gamma(\gamma+m_\gamma+1)}(ZW^*)^{m_\gamma}{}_2F_1\left(\begin{array}{c} 1,\gamma+\eta+m_\gamma+2 \\ \gamma+m_\gamma+1 \end{array}\Bigg| ZW^*\right) \tag{3.11}$$

for all $Z, W \in \mathcal{D}^*_{bc}$.

Proof. Notice first that any $F \in \mathcal{A}^{2,\eta}_{bc,\gamma}(\mathcal{D}^*_{bc})$ is of the form $F(Z) = \phi^+(\alpha)e_+ + \phi^-(\beta)e_-$ in view of the Ringleb decomposition for bc-meromorphic functions, where $\phi^\pm$ are complex-valued holomorphic functions in $\mathcal{D}^*$. Therefore, one gets

$$\begin{aligned} \|F\|^2_{k,\gamma,\eta} &= \int_{\mathcal{D}} |\phi^+(\alpha)|^2|\alpha|^{2\gamma}(1-|\alpha|)^\eta d\lambda(\alpha)\int_{\mathcal{D}} d\lambda(\beta)e_+ \\ &+ \int_{\mathcal{D}} |\phi^-(\beta)|^2|\beta|^{2\gamma}(1-|\beta|)^\eta d\lambda(\beta)\int_{\mathcal{D}} d\lambda(\alpha)e_- \\ &= \left\|\phi^+\right\|^2_{\gamma,\eta}e_+ + \left\|\phi^-\right\|^2_{\gamma,\eta}e_-, \end{aligned}$$

and subsequently $\mathcal{A}^{2,\eta}_{bc,\gamma}(\mathcal{D}^*_{bc}) = \mathcal{A}^{2,\eta}_{\gamma}(\mathcal{D}^*)e_+ + \mathcal{A}^{2,\eta}_{\gamma}(\mathcal{D}^*)e_-$, holds. Thus, the space $\mathcal{A}^{2,\eta}_{bc,\gamma}(\mathcal{D}^*_{bc})$ is clearly a bicomplex Hilbert space in view of Theorem 3.1.1, since $\mathcal{A}^{2,\eta}_{\gamma}(\mathcal{D}^*)$ is a Hilbert space [48, 51]. Moreover, the functions

$$E^{\gamma,\eta}_n(Z) = \left(\frac{\Gamma(n+\gamma+\eta+2)}{\pi\Gamma(n+\gamma+1)\Gamma(\eta+1)}\right)^{1/2} Z^n,\ n \geq m_\gamma,$$

form an orthonormal basis of $\mathcal{A}^{2,\eta}_{bc,\gamma}(\mathcal{D}^*_{bc})$. Since for $n, m \geq m_\gamma$, the inner product of functions $E^{\gamma,\eta}_n$ on that space, is found through this calculation

$$\begin{aligned} \langle Z^n, Z^m\rangle_{k,\gamma,\eta} &= \int_{\mathcal{D}_{bc}} Z^n Z^{*m} d\mu_{\gamma,\eta}(Z) = \int_{\mathcal{D}_{bc}} (\alpha^n e_+ + \beta^n e_-)\left(\bar{\alpha}^m e_+ + \bar{\beta}^m e_-\right) d\mu_{\gamma,\eta}(Z) \\ &= \int_{\mathcal{D}} \alpha^n\bar{\alpha}^m|\alpha|^{2\gamma}(1-|\alpha|)^\eta d\lambda(\alpha)e_+ + \int_{\mathcal{D}} \beta^n\bar{\beta}^m|\beta|^{2\gamma}(1-|\beta|)^\eta d\lambda(\beta)e_- \\ &= \delta_{nm}\frac{\pi\Gamma(n+\gamma+1)\Gamma(\eta+1)}{\Gamma(n+\gamma+\eta+2)}(e_+ + e_-) = \frac{\pi\Gamma(n+\gamma+1)\Gamma(\eta+1)}{\Gamma(n+\gamma+\eta+2)}\delta_{nm}. \end{aligned}$$

This readily follows using [51, Proposition 2.1] and gives rise to the expansion in (3.9) with the growth condition (3.10). More precisely,

$$\|F\|^2_{k,\gamma,\eta} = \pi\sum_{n=m_\gamma}^{+\infty} \frac{\Gamma(\gamma+n+1)\Gamma(\eta+1)}{\Gamma(n+\eta+\gamma+2)}\|A_n\|^2_k.$$

On the other hand, by applying the second Schwartz inequality (3.5), it follows

$$\begin{aligned}
\|F(Z)\|_k^2 &\preceq \left(\sum_{n=m_\gamma}^{+\infty} \frac{\Gamma(n+\gamma+\eta+2)}{\pi\Gamma(n+\gamma+1)\Gamma(\eta+1)}\|Z\|_k^{2n}\right)\left(\pi\sum_{n=m_\gamma}^{+\infty}\frac{\Gamma(\gamma+n+1)\Gamma(\eta+1)}{\Gamma(n+\eta+\gamma+2)}\|A_n\|_k^2\right)\\
&\preceq \frac{\Gamma(\gamma+\eta+m_\gamma+2)}{\pi\Gamma(\eta+1)\Gamma(\gamma+m_\gamma+1)}\|Z\|_k^{2m_\gamma}{}_2F_1\left(\begin{matrix}1,\gamma+\eta+m_\gamma+2\\ \gamma+m_\gamma+1\end{matrix}\middle|\|Z\|_k^2\right)\|F\|_{k,\gamma,\eta}^2\\
&\prec +\infty.
\end{aligned}$$

Thus, on $\mathcal{A}^{2,\eta}_{bc,\gamma}(\mathcal{D}^*_{bc})$ the evaluation of the $\mathbb{BC}$-linear functional $\delta_Z : F \mapsto F(Z)$ is continuous, and then is a bicomplex reproducing kernel Hilbert space for any $\eta > -1$ (by the bicomplex Riesz representation theorem). Therefore, from the idempotent decomposition of the Hilbert space $\mathcal{A}^{2,\eta}_{bc,\gamma}(\mathcal{D}^*_{bc})$ we obtain the reproducing property

$$F(Z) = \left\langle\phi^+, \mathcal{K}_{\gamma,\eta}(.,\alpha)\right\rangle e_+ + \left\langle\phi^-, \mathcal{K}_{\gamma,\eta}(.,\beta)\right\rangle e_- = \left\langle F, \mathcal{K}^{bc}_{\gamma,\eta}(.,Z)\right\rangle_{k,\gamma,\eta},$$

for every $F \in \mathcal{A}^{2,\eta}_{bc,\gamma}(\mathcal{D}^*_{bc})$, with reproducing kernel given by $\mathcal{K}^{bc}_{\gamma,\eta}(Z,W) = \mathcal{K}_{\gamma,\eta}(\alpha,\psi)e_+ + \mathcal{K}_{\gamma,\eta}(\beta,\xi)e_-$, for $Z = \alpha e_+ + \beta e_-$ and $W = \psi e_+ + \xi e_-$, where $\mathcal{K}_{\gamma,\eta}$ is the reproducing kernel of $\mathcal{A}^{2,\eta}_{\gamma}(\mathcal{D}^*)$ studied in [51, Proposition 2.1]

$$\mathcal{K}_{\gamma,\eta}(\alpha,\psi) = \frac{\Gamma(\gamma+\eta+m_\gamma+2)}{\pi\Gamma(\eta+1)\Gamma(\gamma+m_\gamma+1)}(\alpha\overline{\psi})^{m_\gamma}{}_2F_1\left(\begin{matrix}1,\gamma+\eta+m_\gamma+2\\ \gamma+m_\gamma+1\end{matrix}\middle|\alpha\overline{\psi}\right). \quad (3.12)$$

More explicitly for $Z, W \in \mathcal{D}^*_{bc}$ we get (3.11). □

Remark 3.2.4. *For $\gamma = 0$, the space $\mathcal{A}^{2,\eta}_{bc,0}(\mathcal{D}^*_{bc})$ reduces to the bicomplex Bergman space treated in [107, 106]. Its reproducing kernel is exactly*

$$\mathcal{K}^{bc}_{0,\eta}(Z,W) = \frac{\eta+1}{\pi(1-ZW^*)^{\eta+2}}.$$

3.2.2 An integral representation of Bargmann type

The integral transform

$$\mathcal{B}^\gamma_\eta(\varphi)(z) = \frac{z^{-\gamma}}{\sqrt{\pi\Gamma(\eta+1)}(1-z)^{\eta+2}}\int_{\mathbb{R}^+}\exp\left(\frac{-xz}{1-z}\right)\varphi(x)x^{\eta+1}e^{-x}dx \quad (3.13)$$

reduces to the second Bargmann transform in [9, p.203] when $\gamma = 0$. The later defines a unitary isometric integral transform from the configuration space $L^{2,\eta}(\mathbb{R}^+) := L^2(\mathbb{R}^+, t^{\eta+1}e^{-t}dt)$ onto the classical Bergman space $\mathcal{A}^{2,\eta}(\mathcal{D})$. The one in (3.13) has been introduced and studied in [51] for the γ-modified Bergman space $\mathcal{A}^{2,\eta}_\gamma(\mathcal{D}^*)$. In the sequel, we are going beyond the complex case by extending the study to the bicomplex phase space $\mathcal{A}^{2,\eta}_{bc,\gamma}(\mathcal{D}^*_{bc})$ in the previous subsection. Thus, for $F(Z) = \phi^+(\alpha)e_+ + \phi^-(\beta)e_- \in \mathcal{A}^{2,\eta}_{bc,\gamma}(\mathcal{D}^*_{bc})$, there exists $\varphi^\pm \in L^{2,\eta}(\mathbb{R}^+)$ such that $\phi^\pm = \mathcal{B}^\gamma_\eta(\varphi^\pm)$. This follows from [51, Proposition 3.5] combined with the fact that $\mathcal{A}^{2,\eta}_{bc,\gamma}(\mathcal{D}^*_{bc}) = \mathcal{A}^{2,\eta}_{\gamma}(\mathcal{D}^*)e_+ + \mathcal{A}^{2,\eta}_{\gamma}(\mathcal{D}^*)e_-$ and requires that γ $(= -m_\gamma)$ be an integer. Therefore, for $Z = \alpha e_+ + \beta e_- \in \mathcal{D}^*_{bc}$, we get

$$F(Z) = \mathcal{B}^\gamma_\eta(\varphi^+)(\alpha)e_+ + \mathcal{B}^\gamma_\eta(\varphi^-)(\beta)e_- = \int_0^{+\infty} R^\gamma_{bc,\eta}(x,Z)\varphi(x)x^{\eta+1}e^{-x}dx,$$

where $\varphi(x) := \varphi^+(x)e_+ + \varphi^-(x)e_-$ belongs to $L^{2,\eta}_{bc}(\mathbb{R}^+)$, the bicomplex Hilbert space of bicomplex-valued functions on $\mathbb{R}^+$ subject to

$$\int_0^{+\infty} \|\varphi(x)\|^2_k x^{\eta+1}e^{-x}dx \prec +\infty.$$

The involved kernel function $R^\gamma_{bc,\eta}$ is explicitly given by

$$\begin{aligned} R^\gamma_{bc,\eta}(x,Z) &= R^\gamma_\eta(x,\alpha)e_+ + R^\gamma_\eta(x,\beta)e_- \\ &= \frac{1}{\sqrt{\pi\Gamma(\eta+1)}}\frac{\alpha^{-\gamma}}{(1-\alpha)^{\eta+2}}\exp\left(\frac{-x\alpha}{1-\alpha}\right)e_+ \\ &\qquad + \frac{1}{\sqrt{\pi\Gamma(\eta+1)}}\frac{\beta^{-\gamma}}{(1-\beta)^{\eta+2}}\exp\left(\frac{-x\beta}{1-\beta}\right)e_- \\ &= \frac{Z^{*\gamma}}{\sqrt{\pi\Gamma(\eta+1)}\|Z\|^{2\gamma}_k(1-Z)^{\eta+2}}\exp\left(\frac{-xZ}{1-Z}\right). \end{aligned}$$

Accordingly, the integral transform

$$\mathcal{B}^\gamma_{bc,\eta}(\varphi)(Z) = \int_0^{+\infty} R^\gamma_{bc,\eta}(x,Z)\varphi(x)x^{\eta+1}e^{-x}dx \tag{3.14}$$

acts obviously on $L^{2,\eta}_{bc}(\mathbb{R}^+)$ with range being $\mathcal{A}^{2,\eta}_{bc,\gamma}(\mathcal{D}^*_{bc})$. In particular,

$$\begin{aligned} \mathcal{B}^\gamma_{bc,\eta}(\phi^\eta_n)(Z) &= \left\langle R^\gamma_{bc,\eta}(.,Z), \phi^\eta_n \right\rangle_{L^{2,\eta}_{bc}(\mathbb{R}^+)} \\ &= \left(\frac{\Gamma(\eta+n+2)}{\Gamma(\eta+1)\pi n!}\right)^{1/2}\frac{Z^{*\gamma}Z^n}{\|Z\|^{2\gamma}_k}; \qquad n = 0,1,2,\dots. \end{aligned}$$

The functions ϕ^η_n are given by

$$\phi^\eta_n(x) = \left(\frac{n!}{\Gamma(n+\eta+2)}\right)^{1/2} L^{(\eta+1)}_n(x),$$

and constitute an orthonormal basis of $L^{2,\eta}_{bc}(\mathbb{R}^+)$. Here, L^τ_n denotes the n-th generalized Laguerre polynomial [100, p. 203]. Thus, the kernel function $R^\gamma_{bc,\eta}$ appears in fact as the generating function of ϕ^η_n thanks to Theorem 3.1.5 combined with [100, p. 202]

$$\sum_{n=0}^{+\infty} L^\tau_n(x)\zeta^n = (1-\zeta)^{-\tau-1}\exp\left(\frac{-x\zeta}{1-\zeta}\right).$$

Moreover, we obtain

$$\begin{aligned}\left\langle \mathcal{B}^{\gamma}_{bc,\eta}(\phi), \mathcal{B}^{\gamma}_{bc,\eta}(\varphi)\right\rangle_{bc,\gamma,\eta} &= \left\langle \mathcal{B}^{\gamma}_{\eta}(\phi^{+}), \mathcal{B}^{\gamma}_{\eta}(\varphi^{+})\right\rangle_{\gamma,\eta} e_{+} + \left\langle \mathcal{B}^{\gamma}_{\eta}(\phi^{-}), \mathcal{B}^{\gamma}_{\eta}(\varphi^{-})\right\rangle_{\gamma,\eta} e_{-} \\ &= \left\langle \phi^{+}, \varphi^{+}\right\rangle_{L^{2,\eta}(\mathbb{R}^{+})} e_{+} + \left\langle \phi^{-}, \varphi^{-}\right\rangle_{L^{2,\eta}(\mathbb{R}^{+})} e_{-} \\ &= \left\langle \phi, \varphi\right\rangle_{L^{2,\eta}_{bc}(\mathbb{R}^{+})}.\end{aligned}$$

According to the above discussion, we can reformulate the following.

Proposition 3.2.5. *For γ being an integer, the bicomplex Bargmann transform in* (3.14) *is a unitary isometric integral transform from $L^{2,\eta}_{bc}(\mathbb{R}^{+})$ onto the γ-modified bicomplex Bergman space $\mathcal{A}^{2,\eta}_{bc,\gamma}(\mathcal{D}^{*}_{bc})$.*

Remark 3.2.6. *Under the assumption that γ is being an integer, the inverse of $\mathcal{B}^{\gamma}_{bc,\eta}$ from $\mathcal{A}^{2,\eta}_{bc,\gamma}(\mathcal{D}^{*}_{bc})$ onto $L^{2,\eta}_{bc}(\mathbb{R}^{+})$ is given by*

$$[\mathcal{B}^{\gamma}_{bc,\eta}]^{-1}(f)(x) = \frac{1}{\sqrt{\pi\Gamma(\eta+1)}} \int_{\mathcal{D}_{bc}} \frac{Z^{\gamma}(1-\|Z\|^{2}_{k})^{\eta}}{(1-Z^{*})^{\eta+2}} \exp\left(\frac{-xZ^{*}}{1-Z^{*}}\right) f(Z) d\lambda(Z).$$

Remark 3.2.7. *The integral transform in* (3.14) *appears as a particular case of the one associated with the kernel function given by the expansion*

$$\widetilde{R^{\gamma}_{bc,\eta}}(x,Z) = \frac{Z^{m_{\gamma}}}{\sqrt{\pi\Gamma(\eta+1)}} \sum_{n=0}^{+\infty} \left(\frac{n!\Gamma(n+\gamma+m_{\gamma}+\eta+2)}{\Gamma(\eta+n+2)\Gamma(n+m_{\gamma}+\gamma+1)}\right)^{1/4} L^{(\eta+1)}_{n}(x) Z^{n},$$

Another integral representation given by

$$\begin{aligned}\widetilde{\mathcal{B}^{\gamma,\eta}_{bc,\eta}}(\varphi)(Z) = {} & \frac{(Z^{*})^{\eta+\gamma+1}}{\|Z\|^{2(\gamma+\eta+1)}_{k}\Gamma(-\eta)\sqrt{\pi\Gamma(\eta+1)}(1-Z)} \\ & \times \int_{0}^{+\infty} {}_{1}F_{1}\left(\begin{matrix}1\\-\eta\end{matrix}\,\middle|\,\frac{-xZ}{1-Z}\right) \varphi(x) x^{-\eta-1} e^{-x} dx,\end{aligned}$$

for $0 < -\eta < 1$, can be considered as the bicomplexification of the second γ-modified complex Bargmann transform in [51].

3.3 The bicomplex fractional Hankel transform and its dual transform

The γ-modified complex fractional Hankel transform is defined in [51] by

$$H^{\eta,\gamma}_{z}(\phi)(y) = \int_{0}^{+\infty} h^{\eta,\gamma}(x,y|z)\phi(x)dx, \tag{3.15}$$

where

$$h^{\eta,\gamma}(x,y|z) := \frac{z^{-\gamma}}{1-z} \exp\left(-\frac{x+yz}{1-z}\right) \left(\frac{x}{yz}\right)^{(\eta+1)/2} I_{\eta+1}\left(\frac{2\sqrt{xyz}}{1-z}\right). \tag{3.16}$$

In this section, we aim to introduce and study the dual transform of its bicomplex analog.

3.3.1 The bicomplex version of the fractional Hankel transform

Starting from (3.15), one defines the bicomplex fractional Hankel transform to be the integral transform

$$H^{\eta,\gamma}_{Z,bc}(\phi)(y) = \frac{Z^{*\gamma}}{(1-Z)\|Z\|_k^{2\gamma}} \int_0^{+\infty} \exp\left(-\frac{x+yZ}{1-Z}\right) \left(\frac{xZ^*}{y\|Z\|_k^2}\right)^{(\eta+1)/2} \times I_{\eta+1}\left(\frac{2\sqrt{xyZ}}{1-Z}\right)\phi(x)dx \tag{3.17}$$

for $Z = \alpha e_+ + \beta e_- \in \mathcal{D}^*_{bc}$ and $y \in \mathbb{R}^+$. It turns out to be a linear combination of two γ-modified fractional Hankel transform on the $\mathbb{C}$-valued functions ϕ^+ and ϕ^- belonging to $L^{2,\eta}(\mathbb{R}^+)$, the idempotent components of ϕ. Its kernel function arises as a special combination of the two copies of $h^{\eta,\gamma}(x,y|z)$ in (3.16). Namely, we have

$$h^{\eta,\gamma}(x,y|\alpha)e_+ + h^{\eta,\gamma}(x,y|\beta)e_- =: h^{\eta,\gamma}_{bc}(x,y|Z).$$

The transform $H^{\eta,\gamma}_{Z,bc}$ is well-defined on $L^{2,\eta}_{bc}(\mathbb{R}^+)$. In fact, by means of the Cauchy-Schwartz inequality and the Parseval's identity, we get

$$\begin{aligned}\left\|H^{\eta,\gamma}_{Z,bc}(\phi)(y)\right\|_k^2 &= \left\|\langle K^{\gamma,\eta}_{bc}(.,y|Z),\phi\rangle_{L^{2,\eta}_{bc}(\mathbb{R}^+)}\right\|_k^2 \\ &\preceq \|K^{\gamma,\eta}_{bc}(.,y|Z)\|^2_{L^{2,\eta}_{bc}(\mathbb{R}^+)}\|\phi\|^2_{L^{2,\eta}_{bc}(\mathbb{R}^+)} \\ &\preceq \left(\sum_{n=0}^{+\infty}\frac{n!}{\Gamma(n+2+\eta)}(L_n^{(\eta+1)}(y))^2\|Z\|_k^{2(n-\gamma)}\right)\|\phi\|^2_{L^{2,\eta}_{bc}(\mathbb{R}^+)}.\end{aligned}$$

This estimation can explicitly be presented, due to the Hardy-Hille formula (see [10, p.189])

$$\left\|H^{\eta,\gamma}_{Z,bc}(\phi)(y)\right\|_k^2 \preceq h^{\gamma,\eta}(y,Z)\|\phi\|^2_{L^{2,\eta}_{bc}(\mathbb{R}^+)}, \tag{3.18}$$

where

$$h^{\gamma,\eta}(y,Z) := \frac{\|Z\|_k^{-2\gamma}(ZZ^*y^2)^{-(\eta+1)/2}}{1-ZZ^*}\exp\left(-\frac{2yZZ^*}{1-ZZ^*}\right)I_{\eta+1}\left(\frac{2y\|Z\|_k}{1-ZZ^*}\right).$$

It is worth noting that the kernel $h^{\eta,\gamma}_{bc}(x,y|Z)$ in (3.17) is closely connected to $K^{\gamma,\eta}_{bc}(x,y|Z)$ by

$$h^{\eta,\gamma}_{bc}(x,y|Z) = x^{\eta+1}e^{-x}K^{\gamma,\eta}_{bc}(x,y|Z). \tag{3.19}$$

Remark 3.3.1. *The transform in* (3.15) *is a fractional extension of the bicomplex analog of Hankel transform of ν-th order in [1] given by*

$$H^{bc}_\nu(f)(Z) = \int_0^{+\infty} f(x)\sqrt{xZ}J_\nu(xZ)dx, \quad Z \in \mathbb{BC},$$

which is the idempotent decomposition of the complex Hankel transform introduced and studied by Koh and Zemanian in [74]. Here, J_ν is the Bessel function of the first kind [100, p. 110].

The considered transform is closely connected to the Bargmann transform (3.14) by the so-called Bargmann's versus [36]. In fact, we define the action Γ_Z on $\mathcal{A}^{2,\eta}_{bc,\gamma}(\mathcal{D}^*_{bc})$ by $\Gamma_Z(h)(W) = h(ZW)$ for $Z = \alpha e_+ + \beta e_-$ and $W = \alpha' e_+ + \beta' e_-$ such that $ZW \in \mathcal{D}^*_{bc}$. In view of Ringleb's decomposition theorem (see Theorem 3.1.5), it can be rewritten as $\Gamma_Z(h)(W) = \Gamma_\alpha(h^+)(\alpha')e_+ + \Gamma_\beta(h^-)(\beta')e_-$.

Proposition 3.3.2. *The transform* (3.17) *is a* $\mathbb{D}$*-bounded operator from* $L^{2,\eta}_{bc}(\mathbb{R}^+)$ *onto itself, and satisfies*

$$H^{\eta,\gamma}_{Z,bc} = [\mathcal{B}^\gamma_{bc,\eta}]^{-1}\Gamma_Z\mathcal{B}^\gamma_{bc,\eta}$$

for every Z *such that* $\|Z\|_k \preceq 1$. *Subsequently, its inversion formula is given by* $H^{\eta,\gamma}_{Z^*\|Z\|_k^{-2},bc}$ *for every* $Z \in \mathcal{D}^*_{bc}$.

Proof. By involving the integral in (3.18), we obtain

$$\left\|H^{\eta,\gamma}_{Z,bc}(\phi)\right\|^2_{L^{2,\eta}_{bc}(\mathbb{R}^+)} \preceq \left(\frac{\|Z\|_k^{-2\gamma}}{1-ZZ^*}\right)\|\phi\|^2_{L^{2,\eta}_{bc}(\mathbb{R}^+)}.$$

Now, using the explicit expression of the bicomplex second Bargmann transform combined with the Fubini's theorem, it becomes clear that

$$[\mathcal{B}^\gamma_{bc,\eta}]^{-1}\Gamma_Z\mathcal{B}^\gamma_{bc,\eta} = [\mathcal{B}^\gamma_\eta]^{-1}\Gamma_\alpha\mathcal{B}^\gamma_\eta e_+ + [\mathcal{B}^\gamma_\eta]^{-1}\Gamma_\beta\mathcal{B}^\gamma_\eta e_-,$$

which is well defined on $L^{2,\eta}_{bc}(\mathbb{R}^+)$ with range in itself. More precisely, it is the integral transform given for every $\phi \in L^{2,\eta}_{bc}(\mathbb{R}^+)$ by

$$\begin{aligned}
&[\mathcal{B}^\gamma_{bc,\eta}]^{-1}(\Gamma_Z\mathcal{B}^\gamma_{bc,\eta}(\phi))(y)\\
&\quad= \int_{\mathcal{D}_{bc}} R^\gamma_{bc,\eta}(y,Z^*)\mathcal{B}^\gamma_{bc,\eta}(\phi)(ZW)\|W\|_k^{2\gamma}(1-\|W\|_k^2)^\eta d\lambda(W)\\
&\quad= \int_0^{+\infty} \left\langle R^\gamma_{bc,\eta}(x,\Gamma_Z\cdot), R^\gamma_{bc,\eta}(y,\cdot)\right\rangle_{\mathcal{A}^{2,\eta}_{bc,\gamma}(\mathcal{D}^*_{bc})}\phi(x)x^{\eta+1}e^{-x}dx.
\end{aligned}$$

However, using the Hardy–Hille formula [10, p. 189], we get

$$\begin{aligned}
&\left\langle R^\gamma_{bc,\eta}(x,\Gamma_Z\cdot), R^\gamma_{bc,\eta}(y,\cdot)\right\rangle_{\mathcal{A}^{2,\eta}_{bc,\gamma}(\mathcal{D}^*_{bc})}\\
&\quad= \sum_{n=0}^{+\infty}\frac{n!}{\Gamma(n+\eta+2)}L_n^{(\eta+1)}(x)L_n^{(\eta+1)}(y)Z^{n-\gamma}\\
&\quad= \frac{Z^{-\gamma}(Zxy)^{-(\eta+1)/2}}{1-Z}\exp\left(-\frac{Z(x+y)}{1-Z}\right)I_{\eta+1}\left(\frac{2\sqrt{Zxy}}{1-Z}\right).
\end{aligned}$$

In view of the semi-group property $\Gamma_Z\Gamma_W = \Gamma_{ZW}$ and the above realization, the transform $H^{\eta,\gamma}_{Z,bc}$ is invertible whenever $Z \in \mathcal{D}^*_{bc}$. In this case the inversion formula is given by $H^{\eta,\gamma}_{Z^{-1},bc} = H^{\eta,\gamma}_{Z^*\|Z\|_k^{-2},bc}$. □

Remark 3.3.3. *For a fixed $Z \notin \mathbb{O}_{bc}$, the inversion formula of $H^{\eta,\gamma}_{Z,bc}$ reads explicitly*

$$[H^{\eta,\gamma}_{Z,bc}]^{-1}(\phi)(y) = \frac{\|Z\|_k^{2(\gamma+1)}(Z^*)^{-\gamma}}{\|Z\|_k^2 - Z^*} \int_0^{+\infty} \exp\left(-\frac{x\|Z\|_k^2 + yZ^*}{\|Z\|_k^2 - Z^*}\right) \tag{3.20}$$

$$\times \left(\frac{x\|Z\|_k^2}{yZ^*}\right)^{(\eta+1)/2} I_{\eta+1}\left(\frac{2\sqrt{xyZ^*\|Z\|_k^2}}{\|Z\|_k^2 - Z^*}\right)\phi(x)dx. \tag{3.21}$$

The particular case $\|Z\|_k = 1$, the transform $H^{\eta,\gamma}_{Z,bc}$ becomes a unitary isometric from $L^{2,\eta}_{bc}(\mathbb{R}^+)$ onto itself.

$$[H^{\eta,\gamma}_{Z,bc}]^{-1}(\phi)(y) = \frac{(Z^*)^{-\gamma}}{1 - Z^*} \int_0^{+\infty} \exp\left(-\frac{x + yZ^*}{1 - Z^*}\right)\left(\frac{x}{yZ^*}\right)^{(\eta+1)/2} I_{\eta+1}\left(\frac{2\sqrt{xyZ^*}}{1 - Z^*}\right)\phi(x)dx.$$

3.3.2 The bicomplex dual fractional Hankel transform

By varying Z, the expression of $H^{\eta,\gamma}_{Z,bc}(\varphi)(y)$ gives rise to a mapping on $\mathbb{R}^+ \times \mathcal{D}^*_{bc}$ with values in $\mathcal{F}(L^{2,\eta}_{bc}(\mathbb{R}^+))$, the space of $\mathbb{BC}$-linear functional on $L^{2,\eta}_{bc}(\mathbb{R}^+)$. Indeed, one considers

$$T(y; Z)(\varphi) := H^{\eta,\gamma}_{Z,bc}(\varphi)(y).$$

The partial transform $T(\cdot; Z)$ is exactly the bicomplex fractional Hankel transform $H^{\eta,\gamma}_{Z,bc}$. The second partial transform $T(y; \cdot) =: S^{\eta,\gamma}_{y,bc}$ is defined on $L^{2,\eta}_{bc}(\mathbb{R}^+)$ and their images are functions on $\mathcal{D}^*_{bc}$. It will be called here bicomplex dual fractional Hankel transform. This is in fact a special kind of duality in the variables $y \in \mathbb{R}^+$ and $Z \in \mathcal{D}^*_{bc}$, which can be justified through the identities

$$\left\langle H^{\eta,\gamma}_{Z,bc}(\phi^\eta_n), \phi^\eta_k \right\rangle_{\mathbb{R}^+,\eta} = E_{k-\gamma}(Z)\delta_{nk}$$

and

$$\left\langle S^{\eta,\gamma}_{y,bc}(\phi^\eta_n), E_{k-\gamma} \right\rangle_\gamma = \phi^\eta_n(y)\|E_{k-\gamma}\|^2\delta_{nk},$$

where we have set $E_\tau(Z) := Z^\tau$. In the next subsection, we are going to study this bicomplex dual fractional Hankel transform. Notice, for instance, that for the complex case, the study of the partial function $T(y, \cdot)$, for fixed y, was the well-grounded question in [48].

Proposition 3.3.4. *The transform $S^{\eta,\gamma}_{y,bc}$ is $\mathbb{D}$-bounded as a bicomplex integral operator from $L^{2,\eta}_{bc}(\mathbb{R}^+)$ into $L^2(\mathcal{D}_{bc}, d\mu_{\gamma,\eta})$ whenever $\eta > -1$.*

Proof. By means of (3.18) and adopting the same approach as in [48], we get the following estimation

$$\left\|S^{\eta,\gamma}_{y,bc}(\phi)\right\|^2_{\gamma,\eta} \preceq \frac{s_{\eta,\gamma}(y)}{2^{\eta+1}\Gamma(\eta+2)}\|\phi\|^2_{L^{2,\eta}_{bc}(\mathbb{R}^+)}$$

with

$$s_{\eta,\gamma}(y) := \int_{\mathcal{D}_{bc}} (1 - \|Z\|_k^2)^{-2} \exp\left(-\frac{2y\|Z\|_k^2}{1 - \|Z\|_k^2}\right) d\lambda(Z).$$

The integral transform $S^{\eta,\gamma}_{y,bc}$ is $\mathbb{D}$-bounded whenever $\eta > -1$. □

Notice, for instance, that for the limit case $y = 0$ and arbitrary $\eta > 0$, $S^{\eta,\gamma}_{0,bc}$ leads to the bicomplex second Bargman transform in (3.14), while the range of the bicomplex dual fractional Hankel transform denoted by $\mathbf{R}^{\eta,\gamma}_{y,bc} := S^{\eta,\gamma}_{y,bc}\left(L^{2,\eta}_{bc}(\mathbb{R}^+)\right)$ is exactly the weighted bicomplex meromorphic Bergman space of first kind in Definition 3.2.1. In particular, the limit case $\mathbf{R}^{\eta,0}_{0,bc}$ reduces to the bicomplex weighted Bergman space [106] when $\gamma = 0$. For arbitrary $y > 0$, the explicit description of the range depends on the set of the generalized Laguerre polynomials having y as common zero. Thus, if $N^\eta_y = \left\{n; L^{(\eta+1)}_n(y) = 0\right\}$, then $\mathbf{R}^{\eta,\gamma}_{y,bc}$ is strictly contained in $\mathcal{A}^{2,\eta}_{bc,\gamma}(\mathcal{D}^*_{bc})$ whenever N^η_y is not empty. More exactly, we assert the following.

Theorem 3.3.5. *A bicomplex-valued function F on $\mathcal{D}^*_{bc}$, belongs to $\mathbf{R}^{\eta,\gamma}_{y,bc}$ if and only if there exists a bicomplex sequence A_n, $n \notin N^\eta_y$, such that*

$$F(Z) = \sum_{n\notin N^\eta_y} A_n \phi^\eta_n(y) Z^{n-\gamma} \tag{3.22}$$

with

$$\sum_{n\notin N^\eta_y} \pi\varsigma^\eta_n |\phi^\eta_n(y)|^2 \|A_n\|^2_k \prec +\infty,$$

where the constants ς^η_n are given in terms of the Beta function by

$$\varsigma^\eta_n := B(n+1, \eta+1) = \frac{\Gamma(\eta+1)\Gamma(1+n)}{\Gamma(\eta+n+2)} = \frac{\|E_n\|^2_{\mathbf{R}^{\eta,\gamma}_{y,bc}}}{\pi}.$$

Moreover, it is a reproducing kernel $\mathbb{BC}$-Hilbert space with the reproducing kernel given explicitly by

$$\mathcal{K}^{\mathbf{R}^{\eta,\gamma}_{y,bc}}_{\gamma,\eta}(Z,W) = \frac{\eta+1}{\pi}(WZ^*)^\gamma \|ZW^*\|^{-2\gamma}_k \times \left(\frac{(1-WZ^*)^{\eta+2}}{\|1-ZW^*\|^{2(\eta+2)}_k} - \sum_{n\in N^\eta_y} \frac{(\eta+2)_n}{n!}(ZW^*)^n \right)$$

for the particular case of γ $(= -m_\gamma)$ being integer and $Z, W \in \mathcal{D}^*_{bc}$.

Proof. The sequential characterization of $\mathbf{R}^{\eta,\gamma}_{y,bc}$ in (3.22) is equivalent to have $\mathbf{R}^{\eta,\gamma}_{y,bc} = \mathbf{R}^{\eta,\gamma}_y e_+ + \mathbf{R}^{\eta,\gamma}_y e_-$ once making use of the Ringleb decomposition theorem for bicomplex meromorphic functions, where $\mathbf{R}^{\eta,\gamma}_y$ denotes the range of $S^{\eta,\gamma}_y$, the γ-modified version of the complex dual fractional Hankel transform [48]. In fact, an orthonormal basis of $\mathbf{R}^{\eta,\gamma}_y$ is given by

$$\left(\frac{\Gamma(\eta+n+2)}{\pi\Gamma(\eta+1)n!}\right)^{1/2} \alpha^{n-\gamma}; \quad n \notin N^\eta_y.$$

Now, let $F \in \mathbf{R}^{\eta,\gamma}_{y,bc}$ and write F as in (3.22). Then, by the second Schwartz inequality (3.5)

$$
\begin{aligned}
\|F(Z)\|_k^2 &\preceq \left(\sum_{n\notin N_y^\eta} \frac{\|Z\|_k^{2(n-\gamma)}}{\pi\varsigma_n^\eta}\right)\left(\sum_{n\notin N_y^\eta} \pi\varsigma_n^\eta \|A_n\|_k^2 |\phi_n^\eta(y)|^2\right)\\
&\preceq \frac{\eta+1}{\pi}\|Z\|_k^{-2\gamma}\left(\left(1-\|Z\|_k^2\right)^{-\eta-2} - \sum_{n\in N_y^\eta}\frac{(\eta+2)_n}{n!}\|Z\|_k^{2n}\right)\|F\|_{k,\gamma,\eta}^2,\\
&\prec +\infty.
\end{aligned}
$$

This shows that the $\mathbb{BC}$-linear evaluation map $F \longmapsto F(Z)$ is $\mathbb{BC}$-continuous on $\mathbf{R}_{y,bc}^{\eta,\gamma}$ for every fixed $Z \in \mathcal{D}_{bc}^*$, and therefore the range $\mathbf{R}_{y,bc}^{\eta,\gamma}$ is a reproducing kernel $\mathbb{BC}$-Hilbert space by means of the bicomplex version of Riesz representation theorem (see [46, Theorem 3.7]). This can also be handled making use of $\mathbf{R}_{y,bc}^{\eta,\gamma} = \mathbf{R}_y^{\eta,\gamma} e_+ + \mathbf{R}_y^{\eta,\gamma} e_-$, where the idempotent component $\mathbf{R}_y^{\eta,\gamma}$ is reproducing kernel Hilbert space [48]. Thus, it exists a unique $K_{y,Z}^{\eta,\gamma} \in \mathbf{R}_{y,bc}^{\eta,\gamma}$ such that $F(Z) = \left\langle K_{y,Z}^{\eta,\gamma}, F\right\rangle_{\mathbf{R}_{y,bc}^{\eta,\gamma}}$, for all $F \in \mathbf{R}_{y,bc}^{\eta,\gamma}$. Now, since

$$
s_n^{\gamma,\eta}(Z) := \left(\frac{\Gamma(n+\eta+2)}{\pi\Gamma(\eta+1)n!}\right)^{1/2} Z^{n-\gamma}; \quad n \notin N_y^\eta
$$

form an orthonormal basis of $\mathbf{R}_{y,bc}^{\eta,\gamma}$, we get

$$
\begin{aligned}
\mathcal{K}_{\gamma,\eta}^{\mathbf{R}_{y,bc}^{\eta,\gamma}}(Z,W) = K_{y,Z}^{\eta,\gamma}(W) &= \sum_{n\notin N_y^\eta} s_n^{\gamma,\eta}(Z) s_n^{\gamma,\eta}(W)^*\\
&= \sum_{n=0}^{\infty} s_n^{\gamma,\eta}(Z) s_n^{\gamma,\eta}(W)^* - \sum_{n\in N_y^\eta} s_n^{\gamma,\eta}(Z) s_n^{\gamma,\eta}(W)^*\\
&= \frac{\eta+1}{\pi}(WZ^*)^\gamma \|ZW^*\|_k^{-2\gamma}\\
&\qquad \times \left(\frac{(1-WZ^*)^{\eta+2}}{\|1-ZW^*\|_k^{2(\eta+2)}} - \sum_{n\in N_y^\eta}\frac{(\eta+2)_n}{n!}(ZW^*)^n\right).
\end{aligned}
$$

□

Remark 3.3.6. *When $m_\gamma = -\gamma$ and $N_y^\eta = \emptyset$, the reproducing kernel of $\mathbf{R}_{y,bc}^{\eta,\gamma}$ is identical to the reproducing kernel of $\mathcal{A}_{bc,\gamma}^{2,\eta}(\mathcal{D}_{bc}^*)$ in* (3.11).

Remark 3.3.7. *Notice that the null space of $S_{y,bc}^{\eta,\gamma}$ is spanned by the normalized generalized Laguerre polynomials ϕ_n^η for $n \in N_y^\eta$ with the coefficients in $\mathbb{BC}$. This is in fact, due to the idempotent component kernel space of the complex dual fractional Hankel transform that is generated in $\mathbb{C}$ by ϕ_n^η (we refer to [48]).*

3.4 The inverse of the bicomplex dual fractional Hankel transform

Note that we can say that $S_{y,bc}^{\eta,\gamma}$ is also a $\mathbb{D}$-bounded operator from $L_{bc}^{2,\eta}(\mathbb{R}^+)$ into $L^2(\mathcal{D}_{bc}, d\mu_{\gamma,\eta'})$ for any $\eta' > 0$. This one obviously came from the estimation given in the proof of the Proposition 3.3.4. Here, we will construct the bicomplex analog of the inverse of the dual fractional Hankel transform for that we will just use the fact that the elements of $\mathbf{R}_{y,bc}^{\eta,\gamma}$ belong to the set $L^2(\mathcal{D}_{bc}, d\mu_{\gamma,\eta'}) \cap S_{y,bc}^{\eta,\gamma}(L_{bc}^{2,\eta}(\mathbb{R}^+))$ defined and denoted by

$$\mathbf{I}_{y,bc}^{\eta,\eta',\gamma} := \left\{ \sum_{n\notin N_y^\eta} A_n \phi_n^\eta(y) Z^{n-\gamma}; \sum_{n\notin N_y^\eta} \pi\varsigma_n^{\eta'} |\phi_n^\eta(y)|^2 \|A_n\|_k^2 \prec +\infty \right\} \tag{3.23}$$

consisting of the orthonormal basis $\frac{E_{n-\gamma}}{\sqrt{\pi\xi_n^{\eta'}}}$; $n \notin N_y^\eta$, because the range space can also be presented as

$$\mathbf{R}_{y,bc}^{\eta,\gamma} := L^2(\mathcal{D}_{bc}, d\mu_{\gamma,\eta}) \cap S_{y,bc}^{\eta,\gamma}(L_{bc}^{2,\eta}(\mathbb{R}^+)).$$

Firstly, from Remark 3.3.7 we need to construct the bicomplex space for which its idempotent components treated in the previous chapter, are exactly $\ker(S_y^{\eta,\gamma})^\perp$. Briefly, such space decomposition is defined by

$$\mathbf{N}_{bc,y}^{\eta,\gamma} := \ker(S_y^{\eta,\gamma})^\perp e_+ + \ker(S_y^{\eta,\gamma})^\perp e_-.$$

Thus, the bicomplex integral transform $S_{y,bc}^{\eta,\gamma} : \mathbf{N}_{bc,y}^{\eta,\gamma} \longrightarrow \mathbf{I}_{y,bc}^{\eta,\eta',\gamma}$ is invertible satisfying identities both $M_{y,bc}^{\eta,\gamma} S_{y,bc}^{\eta,\gamma}(\phi) = \phi$ and $S_{y,bc}^{\eta,\gamma} M_{y,bc}^{\eta,\gamma}(F) = F$ for $\phi \in \mathbf{N}_{bc,y}^{\eta,\gamma}$ and $F \in \mathbf{I}_{y,bc}^{\eta,\eta',\gamma}$. We already know that $S_{y,bc}^{\eta,\gamma}(\phi_n^\eta)(Z) = \phi_n^\eta(y) Z^{n-\gamma}$, so the action of $M_{y,bc}^{\eta,\gamma}$ on the basis will be $M_{y,bc}^{\eta,\gamma}(E_{n-\gamma}) = \frac{L_n^{(\eta+1)}}{L_n^{(\eta+1)}(y)}$. Hence, we proceed at least formally for every $F = \sum_{n\notin N_y^\eta} A_n E_{n-\gamma} \in \mathbf{I}_{y,bc}^{\eta,\eta',\gamma}$ where $A_n = \left\langle F, \frac{E_{n-\gamma}}{\pi\xi_n^{\eta'}} \right\rangle_{k,\gamma,\eta'}$. Therefore, for all $x \in \mathbb{R}^+$, we have

$$M_{y,bc}^{\eta,\gamma} F(x) = \sum_{n\notin N_y^\eta} \frac{\Gamma(\eta'+n+2)}{\pi n! \Gamma(\eta'+1)} \frac{L_n^{(\eta+1)}(x)}{L_n^{(\eta+1)}(y)} \langle F, E_{n-\gamma} \rangle_{k,\gamma,\eta'}, \tag{3.24}$$

its integral representation is exactly

$$M_{y,bc}^{\eta,\gamma} F(x) = Z^\gamma \int_{\mathcal{D}_{bc}} \left(\sum_{n\notin N_y^\eta} \frac{\Gamma(\eta'+n+2)}{\pi n! \Gamma(\eta'+1)} \frac{L_n^{(\eta+1)}(x)}{L_n^{(\eta+1)}(y)} (Z^*)^n \right) F(Z)(1 - \|Z\|_k^2)^{\eta'} d\lambda(Z). \tag{3.25}$$

The later one is a well-defined, $\mathbb{D}-$bounded integral transform for a fixed y belongs to $\Upsilon_{\eta+1}$ whenever $\eta > \eta'+1$, since it describes an idempotent decomposition of the already elaborated inverse transform of the complex dual fractional Hankel transform, this is called the inverse of the bicomplex dual fractional Hankel transform.

3.5 Concluding remarks

In accordance with Remark 3.2.7, the studied γ-modified bicomplex dual fractional Hankel transform $S^{\eta,\gamma}_{y,bc}$ is a particular case of the integral operator $\widetilde{S^{\eta,\gamma}_{y,bc}}$ satisfying

$$\widetilde{S^{\eta,\gamma}_{y,bc}}(\phi^{\eta}_n)(Z) = L^{(\eta+1)}_n(y)\left(\frac{\Gamma(n+m_\gamma+\gamma+1)}{\Gamma(n+\gamma+m_\gamma+\eta+2)}\right)^{1/2} Z^{n+m_\gamma}$$

for $n = 0, 1, \dots,$ where m_γ is an integer. Its action on $L^{2,\eta}_{bc}(\mathbb{R}^+)$ is given by

$$\widetilde{S^{\eta,\gamma}_{y,bc}}(\phi)(Z) = \int_0^{+\infty}\left(\sum_{n=0}^{+\infty}\gamma^{\eta}_n L^{(\eta+1)}_n(x)L^{(\eta+1)}_n(y)Z^{n+m_\gamma}\right)\phi(x)x^{\eta+1}e^{-x}dx$$

with

$$\gamma^{\eta}_n := \left(\frac{n!\,\Gamma(n+\gamma+m_\gamma+1)}{\Gamma(n+\eta+2)\Gamma(n+\gamma+\eta+2+m_\gamma)}\right)^{1/2},$$

provided that the involved series converges.

Lemma 3.5.1. *$\widetilde{S^{\eta,\gamma}_{y,bc}}$ is a well-defined bicomplex integral transform from $L^{2,\eta}_{bc}(\mathbb{R}^+)$ onto $L^2(\mathcal{D}_{bc}, d\mu^{bc}_{\gamma,\eta})$.*

Proof. Based on the Laguerre polynomial estimation

$$|L^{\tau}_n(y)| \le \binom{\tau+n}{n} e^{y/2},$$

we can find out the following order relation using the bicomplex Schwartz inequality and Parseval's identity

$$\left\|\widetilde{S^{\eta,\gamma}_{y,bc}}(\phi)(Z)\right\|_k \preceq e^{y/2}\|Z\|^{m_\gamma}_k(\widetilde{s}_{\eta,\gamma}(Z))^{1/2}\|\phi\|_{L^{2,\eta}_{bc}(\mathbb{R}^+)}$$

which is finite in $\mathbb{D}^+$. Here

$$\widetilde{s}_{\eta,\gamma}(Z) := \frac{\Gamma(\gamma+m_\gamma+1)}{\Gamma(\gamma+\eta+2+m_\gamma)}\,{}_3F_2\left(\begin{matrix}\eta+2,\eta+2,\gamma+m_\gamma+1\\ 1,\gamma+\eta+2+m_\gamma\end{matrix}\middle|\,\|Z\|^2_k\right).$$

□

Accordingly, the range of $\widetilde{S^{\eta,\gamma}_{y,bc}}$ consists of those bicomplex-valued functions that can be expanded as

$$\sum_{n\notin N^{\eta}_y} A_n\phi^{\eta}_n(y)Z^{n+m_\gamma},$$

for $Z \in \mathcal{D}^*_{bc}$, and satisfying the growth condition

$$\pi\Gamma(\eta+1)\sum_{n\notin N^{\eta}_y}\frac{\Gamma(n+m_\gamma+\gamma+1)}{\Gamma(n+m_\gamma+\gamma+\eta+2)}|\phi^{\eta}_n(y)|^2\|A_n\|^2_k \prec +\infty.$$

Its orthonormal basis is given by

$$\left(\frac{\Gamma(n+m_\gamma+\gamma+\eta+2)}{\pi\Gamma(n+m_\gamma+\gamma+1)\Gamma(\eta+1)}\right)^{1/2} Z^{n+m_\gamma}, \quad n \notin N_y^\eta,$$

and can be employed to identify its reproducing kernel. In fact, one uses the bicomplex analog of Riesz representation theorem and follows similar method as in the proof of Proposition 3.2.3 by evaluating the involved series over $n \notin N_y^\eta$. Thus, the reproducing kernel of this range reads

$$\begin{aligned}\mathcal{K}_{\gamma,\eta}^{bc,m_\gamma}(Z,W) &= \frac{(ZW^*)^{m_\gamma}\Gamma(\gamma+\eta+m_\gamma+2)}{\pi\Gamma(\eta+1)\Gamma(\gamma+m_\gamma+1)} \\ &\times \left({}_2F_1\left(\begin{matrix} 1, \gamma+\eta+m_\gamma+2 \\ \gamma+m_\gamma+1 \end{matrix} \middle| ZW^* \right) - \sum_{n\in N_y^\eta} \frac{(\gamma+\eta+m_\gamma+2)_n}{(\gamma+m_\gamma+1)_n}(ZW^*)^n \right)\end{aligned}$$

for $Z, W \in \mathcal{D}_{bc}^*$. Notice, for instance, that for the particular values $\eta = 0$ and $\gamma = 1/2$ (and then $m_\gamma = -1$), the expression of the reproducing kernel reduces further to

$$\begin{aligned}\mathcal{K}_{\gamma,\eta}^{bc,m_\gamma}(Z,W) = &\frac{\Gamma(3/2)WZ^*}{\pi\Gamma(1/2)\|ZW^*\|_k^2} \\ &\times \left(\frac{(1+ZW^*)(1-WZ^*)^2}{\|1-ZW^*\|_k^4} - \sum_{n\in N_y^\eta} \frac{(3/2)_n}{(1/2)_n}(ZW^*)^n \right).\end{aligned}$$

This follows taking into account the fact that [85, p.38]

$${}_2F_1\left(\begin{matrix} 2a, a+1 \\ a \end{matrix} \middle| z \right) = (1+z)(1-z)^{-2a-1}.$$

Obviously, in that case, the reproducing kernel of $\mathcal{A}_{bc,\frac{1}{2}}^{2,0}(\mathcal{D}_{bc}^*)$ is explicitly given by

$$\mathcal{K}_{1/2,0}^{bc,-1}(Z,W) = \frac{\Gamma(3/2)}{\pi\Gamma(1/2)} \frac{(1+ZW^*)(WZ^*)(1-WZ^*)^2}{\|ZW^*\|_k^2\|1-ZW^*\|_k^4}.$$

Chapter 4

Appendix

Hypergeometric function

Here, ${}_pF_q$ denotes the generalized hypergeometric function defines a serie of the form [100, p.73]

$$ {}_pF_q\left(\begin{array}{c} a_1, \ldots, a_p \\ b_1, \ldots, b_q \end{array} \mid z \right) = \sum_{n=0}^{+\infty} \frac{(a_1)_n, \ldots, (a_p)_n}{(b_1)_n, \ldots, (b_q)_n} \frac{z^n}{n!} \tag{4.1} $$

also known as the hypergeometric serie. For all $i = 0, \ldots, q$ and $j = 0, \ldots, p, b_i$ is a non zero positive integer different to zero, and

$$ (a_j)_n = \begin{cases} a_j (a_j + 1) \ldots (a_j + n - 1), & n \geq 1 \\ 1 & \text{if } n = 0 \end{cases} $$

denotes the Pochammer symbol. The hypergeometric serie (4.1) satisfies the following assertions [100, p.73-74]

(i) For all z and $p \leq q$ or for $|z| < 1$ and $p = q + 1$, so the series converges absolutely for both cases.

(ii) If $p = q + 1$ for $|z| > 1$ and if $p > q + 1$ for $z \neq 0$, therefore the series diverges for the two conditions.

(iii) On the unit circle $|z| = 1$, the series convereges absolutely if $\Re\left(\sum_{i=0}^{q} b_i - \sum_{j=0}^{p} a_j\right)$. It also converges absolutely under the conditions where $z \neq 1$ and

$$ 0 \geq \Re\left(\sum_{i=0}^{q} b_i - \sum_{j=0}^{p} a_j\right) > -1. $$

(iv) The series diverges for $|z| = 1$ if $\Re\left(\sum_{i=0}^{q} b_i - \sum_{j=0}^{p} a_j\right) \leq -1$.

Bessel function

Definition 4.0.1. *The Bessel functions of the first kind, respectively of order ν and $-\nu$, are defined by [100, p.109]*

$$J_\nu(x) = (\frac{x}{2})^\nu \sum_{n=0}^{+\infty} \frac{(-1)^n (x/2)^{2n}}{n!\Gamma(n+\nu+1)} \text{ and } J_{-\nu}(x) = (\frac{x}{2})^{-\nu} \sum_{n=0}^{+\infty} \frac{(-1)^n (x/2)^{2n}}{n!\Gamma(n-\nu+1)} \tag{4.2}$$

Let n be a positive integer, two immediate results are given by [100, p.109]

$$J_{-n}(x) = (-1)^n J_n(x), \ J_n(-x) = (-1)^n J_n(x)$$

The Bessel function of the first kind is expressed in terms of an integral [100, p.114]

$$J_n(x) = \frac{1}{2\pi i} \int^{(0+)} u^{-n-1} \exp\left[\frac{x}{2}\left(u - \frac{1}{u}\right)\right] du \tag{4.3}$$

where $(0+)$ is closed path in which the origin is equal to 0. In particular, for $u = e^{i\theta}$, and $\theta \in [-\pi, \pi]$. This implies the following expression [100, p.114]

$$J_n(x) = \frac{1}{2\pi} \int_{-\pi}^{\pi} \exp\left(-in\theta + ix\sin\theta\right) d\theta \tag{4.4}$$

The next formula based on (4.4) gives explicitly another expression of $J_n(x)$. For n is an integer, that one is an integral representation defined by [100, p.114]

$$J_n(x) = \frac{1}{\pi} \int_0^{\pi} \cos\left(-n\theta + x\sin\theta\right) d\theta. \tag{4.5}$$

Furthermore, the connection between the Bessel function of the first kind and the modified Bessel function of the first kind is expressed as [100, p.116]

$$I_n(z) = i^{-n} J_n(iz) = \frac{(z/2)^n}{\Gamma(1+n)} {}^{\Gamma}F_1\left(-; 1+n; \frac{z^2}{4}\right).$$

Hermite polynomials

The Hermite polynomial is obtained due to its generating function [100, p.187]

$$\exp\left(2xt - t^2\right) = \sum_{n=0}^{+\infty} \sum_{k=0}^{[n/2]} \frac{(-1)^k (2x)^{n-2k} t^n}{k!(n-2k)!} = \sum_{n=0}^{+\infty} \frac{H_n(x) t^n}{n!}.$$

The so-called Hermite polynomials $H_n(x)$ are given by [100, p.187]

$$H_n(x) = \sum_{k=0}^{[n/2]} \frac{n!(-1)^k (2x)^{n-2k}}{k!(n-2k)!}.$$

Some results concerning $H_n(x)$ and its derivative $H'_n(x) = 2nH_{n-1}(x)$, are provided in the following [100, p.188]

$$H_n(-x) = (-1)^n H_n(x),$$

$$H_n(0) = \begin{cases} (-1)^m 2^{2m} (1/2)_m & \text{if } n = 2m \\ 0 & \text{if } n = 2m+1 \end{cases}$$

$$H'_n(0) = \begin{cases} (-1)^m 2^{2m+1} (3/2)_m & \text{if } n = 2m+1 \\ 0 & \text{if } n = 2m \end{cases}$$

Meanwhile, H_n can be interpreted by the Rodrigues formula

$$H_n(x) = (-1)^n \exp\left(x^2\right) \frac{d^n}{dx^n} \left(\exp\left(-x^2\right)\right)$$

It is important to mention the orthogonality relation given below [100, p.192-193]

$$\int_{-\infty}^{+\infty} \exp\left(-x^2\right) H_n(x) H_m(x) dx = 2^n n! \sqrt{\pi} \delta_{mn}$$

where δ_{mn} denotes the Kronecker symbol which is equal to 1 if $m = n$, else it takes the zero value. In fact, $\{H_n(x)\}_n$ form an orthogonal family over the interval $(-\infty, \infty)$ with the corresponding weight function $\exp(-x^2)$.

Chapter 5

Perspectives

In both complex and bicomplex settings, the dual fractional Hankel transform (DFRHT) shows promising potential, opening up new avenues for future research. Our goal is to offer some possible perspectives. Using the inverse of the DFRHT in complex setting, we can study some new results in line with physical and engineering problems such as signal processing, theory of optics, electromagnetics and quantum mechanics.
Furthermore, by extending our integral transform to the bicomplex setting, the frequency domain analysis becomes larger which provides a richer framework for multidimensional signal processing and quantum mechanics, although practical implementation still requires further research. The inverse of the DFRHT could offer new methods for resolving bicomplex-valued partial differential equations. Despite the fact that these applications are not yet fully realized, but they represent promising areas for future research and development if mathematical and practical challenges can be overcome.

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Résume

Dans cette thèse, nous nous concentrons sur deux transformées intégrales dans le cadre complexe et bicomplexe, en étudiant leurs différentes propriétés et en considérant certaines de leurs applications. Ces transformées sont les analogues complexes et bicomplexes de la transformée de Hankel fractionnaire duale. En fait, nous traitons le problème inverse, obtenons la représentation intégrale en discutons sur sa compacité et ses valeurs singulières. En outre, nous recherchons les propriétés de la fonction noyau et donnons des expressions explicites pour des cas particuliers. De plus, en ce qui concerne le contexte bicomplexe, nous nous concentrons d'abord sur l'espace de Bergman des fonctions bc-méromorphes avec un pôle fort à l'origine du disque bicomplexe. Nous donnons également l'expression explicite de son noyau reproduisant. Sa caractérisation en tant qu'intervalle de l'analogue bicomplexe de la seconde transformée de Bargmann est également fournie. Sur cette base, nous construisons l'analogue bicomplexe de la transformée fractionnaire de Hankel ainsi que sa transformée fractionnaire duale, nous décrivons leurs domaines et fournissons des expressions pour leurs noyaux reproduisant. L'inverse de la transformée duale de la transformée de Hankel fractionnaire bicomplexe est également considéré.

Mots-clefs : Transformée fractionnaire de Hankel; Transformée duale de la transformée de Hankel fractionnaire; Espace de Bergman pondéré; Transformée de Bargmann bicomplexe ; Transformée fractionnaire de Hankel bicomplexe modifiée.

Abstract

In this thesis, we focus on two integral transforms in the complex and bicomplex settings, by studying their different properties and considering some of their applications. These transforms are the complex and bicomplex analogs of the dual fractional Hankel transform. In fact, we deal with the inverse problem, obtaining the integral representation, discussing compactness and singular values. Additionally, we look for the kernel function's properties and give explicit expressions for particular cases. Furthermore, concerning the bicomplex context, we concentrate first on the Bergman space of bc-meromorphic functions with a strong pole at the origin of the bicomplex discus. We also give the explicit expression for its reproducing kernel. Its characterization as the range of the bicomplex analog of the second Bargmann transform is also provided. Based on that, we construct the bicomplex analog of the fractional Hankel transform as well as its dual fractional transform, we describe their ranges and provide expressions for their reproducing kernels. The inverse of the dual transform of the bicomplex fractional Hankel transform is also considered.

Keywords : Fractional Hankel transform; Dual fractional Hankel transform; Weighted Bergman space; Bicomplex Bargmann transform; The modified bicomplex fractional Hankel transform.