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Some problems of (bi-)harmonicity in Riemannian geometry

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Résumé de la thèse

Soit (M,g) une variété Riemannienne compacte. En équipant son fibré tangent TM (resp. son fibré tangent unitaire) d'une métrique pseudo-Riemannienne g-naturelle G (resp. sa restriction sur le fibré unitaire), nous étudions la biharmonicité des champs de vecteurs (resp. des champs de vecteurs unitaires) en tant qu' applications, ainsi que les points critiques de la fonctionnelle de biénergie restreinte à l'ensemble des champs de vecteurs (resp. des champs de vecteurs unitaires) sur M. Contrairement à la métrique de Sasaki sur TM, où les deux notions sont équivalentes à l'harmonicité du champ de vecteurs et ensuite à sa parallélisme, nous prouvons que pour de larges classes de métriques g-naturelles sur TM, les deux notions ne sont pas équivalentes. De plus, nous donnons des exemples de champs de vecteurs qui sont biharmoniques en tant que points critiques de la fonctionnelle de biénergie restreinte à l'ensemble des champs de vecteurs, mais qui ne sont pas des applications biharmoniques. Nous donnons également des exemples de champs de vecteurs(resp. de champs de vecteurs unitaires) (proprement) biharmoniques.

Nous étudions également la biharmonicité de deux classes de fonctions sur les espaces totaux de fibrés vectoriels équipées de la classe de métriques à symétrie sphériques, la première est la classe des fonctions relevées verticalement et la deuxième est la classe des fonctions r-radiales. Nous construisons dans chaque cas des exemples de fonctions (proprement) biharmoniques.

Mots clés :

Harmonicité, biharmonicité, fibré tangent, fibré unitaire, fonction, champ de vecteurs, métriques g-naturelle, fibré vectoriel.

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THÈSE

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Souhail DOUA

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Some problems of (bi-)harmonicity in Riemannian geometry

Dedication

To my wonderful deeply missed MOM....

To my father and sisters...

To my family...

To all my professors and teachers...

To all my friends...

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Résumé

La théorie des applications harmoniques est l'une des théories qui jouissent d'un intért particulier en mathématiques et ses applications, surtout en analyse, en géométrie et en physique mathématique. En géométrie différentielle, une *application harmonique* entre deux variétés Riemanniennes (M, g) et (N, h) est définie comme étant un point critique de la fonctionnelle d'énergie

$$E(f) := \int_M ||df||^2 dv_g,$$

définie sur l'ensemble des applications $f : (M, g) \to (N, h)$, o*ù* dv_g est l'élément de volume de g.

L'équation d'Euler-Lagrange associée *à* E est donnée par

$$\tau(f) = 0,$$

o*ù* $\tau(f)$ est le *champ de tension* de f défini par

$$\tau(f) := tr\nabla(df),$$

de telle sorte que f est une application harmonique si et seulement si $\tau(f) = 0$.

En 1983, J. Eells et L. Lemaire ont proposé une généralisation d'applications harmoniques à la notion d'*applications biharmoniques* comme étant les points critiques de la fonctionnelle de biénergie définie par

$$E_2(f) := \frac{1}{2} \int_M ||\tau(f)||^2 dv_g.$$

La théorie des applications biharmoniques a connu un développement considérable durant la seconde moitié des années quatre-vingt du vingtième siècle, surtout lorsque G.Y. Jiang calculait la première et seconde formules de variation de la fonctionnelle de biénergie et déterminait l'équation d'Euler-Lagrange associée

$$\tau_2(f) = 0,$$

où $\tau_2(f)$ est le *champ de bitension* de l'application f défini par

$$\tau_2(f) := \bar{\Delta}\tau(f) - \sum_{i=1}^{m} R^N\big(\tau(f), f_* e_i\big) f_* e_i.$$

Dans le contexte de la géométrie Riemannienne des fibrés tangents, des applications spéciales possédant un fibré tangent d'une variété comme espaces de départ et/ou d'arrivée apparaissent naturellement et il s'avère naturel et intéressant d'étudier leur harmonicité et leur biharmonicité afin de mieux comprendre leurs géométries et aussi la géométrie des fibrés tangent correspondants. Puisque le choix des métriques Riemanniennes sur les ensembles de départ et d'arrivée d'une applications influencent sa (bi)harmonicité, il s'avère

nécessaire de choisir des métriques appropriées sur les fibrés tangents afin d'obtenir des résultats intéressants.

En tant que variétés (pseudo-)Riemanniennes, les fibrés tangents aux variétés Riemanniennes ont été munis par les géomètres de diveres (classes de) métriques telles que la métrique de Sasaki et d'autres métriques relevées à partir de la métrique de la base, les métriques de type Cheeger-Gromoll, la classe générale de métriques d'Oproiu et la classe la plus large des métriques g-naturelles dépendant de six fonctions indépendantes de $\mathbb{R}^+$ à $\mathbb{R}$ (cf. [3], [16], [17], et [34], pour la définition et les propriétés géométriques des métriques g-naturelles).

Explicitement, les métriques g-naturelles sur le fibré tangent *à* une variété Riemannienne (M, g) sont complètement caractérisées comme suit [16]:

> *Soit (M, g) une variété Riemannienne et G une métrique g-naturelle sur TM. Alors, il existe des fonctions α_i, $\beta_i : \mathbb{R}^+ \to \mathbb{R}$, $i = 1, 2, 3$, telles que, pour tout u, X, $Y \in M_x$, nous avons :*
>
> $$\begin{cases} G_{(x,u)}(X^h, Y^h) = (\alpha_1 + \alpha_3)(\rho)g_x(X, Y) + (\beta_1 + \beta_3)(\rho)g_x(X, u)g_x(Y, u), \\ G_{(x,u)}(X^h, Y^v) = \alpha_2(\rho)g_x(X, Y) + \beta_2(\rho)g_x(X, u)g_x(Y, u), \\ G_{(x,u)}(X^v, Y^h) = \alpha_2(\rho)g_x(X, Y) + \beta_2(\rho)g_x(X, u)g_x(Y, u), \\ G_{(x,u)}(X^v, Y^v) = \alpha_1(\rho)g_x(X, Y) + \beta_1(\rho)g_x(X, u)g_x(Y, u), \end{cases}$$
>
> *où $\rho = g_x(u, u)$.*

L'harmonicité d'applications spéciales possédant des fibrés (unitaires) tangents comme espaces de départ et/ou d'arrivée a été intensément étudié en littérature et, comme prévu, il a été prouvé que le choix de métriques g-naturelles sur le fibré (unitaire) tangent influence l'harmonicité de l'application en question (cf. [27], [28], [30], pour la métrique de Sasaki, et [5], [6], pour les métriques g-naturelles). D'autre part, la biharmonicité de champs de vecteurs (unitaires) a été traitée lorsque le fibré tangent est muni de la métrique de Sasaki. Mais, *à* notre connaissance, il n'existe pas de travaux sur la bihamonicité des champs de vecteurs (unitaires) dans le cas de métriques g-naturelles arbitraires.

Dans cette thèse, on s'intéresse à l'étude de la biharmonicité des champs de vecteurs (unitaires) lorsque le fibré (unitaire) tangent est muni d'une métrique g-naturelle arbitraire. Notre étude est motivée par les trois considérations suivantes:

- Généraliser au contexte biharmonique les études effectuées sur l'harmonicité des champs (unitaires) lorsque le fibré (unitaire) tangent est muni d'une métrique g-naturelle arbitraire (cf. [6] pour le fibré tangent et [5] pour le fibré unitaire tangent).

- Généraliser au cas de métriques g-naturelles les résultats sur la biharmonicité de champs de vecteurs (unitaires) lorsque le fibré (unitaire) tangent est muni de la métrique de Sasaki (cf. [37] pour le fibré tangent et [38] pour le fibré unitaire tangent).

- Surmonter le comportement de rigidité de la métrique de Sasaki, dans le sens suivant: lorsque le fibré tangent est muni de la métrique de Sasaki, les seuls champs de vecteurs biharmoniques sont les champs de vecteurs parallèles [37].

Biharmonicité des champs de vecteurs parallèles:

Dans [6], les auteurs ont calculé le champ de tension d'un champ de vecteurs parallèle V. Plus précisément, ils ont trouvé que

$$\tau(V) = (c_h V)^h + (c_v V)^v,$$

où

$$c_h := \big(2A_2 + mA_4 + \rho A_5\big)(\rho), \qquad c_v := \big(2B_3 + mB_5 + \rho B_6\big)(\rho),$$

où ρ est la longueur (constante) du champ de vecteurs parallèle V et A_i, B_i sont des fonctions à valeurs réelles sur $\mathbb{R}^+$, dépendant de α_i, β_i, et figurant dans les expressions de la connexion de Levi-Civita de (TM, G). Il s'ensuit que $V : (M, g) \longrightarrow (TM, G)$ est une application harmonique si et seulement si $c_h = c_v = 0$.

Cependant, vu la complexité de l'expression de la connexion de Levi-Civita d'une métrique g-naturelle arbitraire G et aussi des calculs trop longs du champ de bitension associé, on va réduire notre étude aux champs de vecteurs parallèles et on va donner les conditions nécessaires et suffisantes pour qu'un champ de vecteurs parallèle soit biharmonique. Ainsi, le champ de bitension d'un champ de vecteurs parallèle $V : (M, g) \to (TM, G)$ est donné par

$$\tau_2(V) = (\tau_{2h}(V))^h \circ V + (\tau_{2v}(V))^v \circ V,$$

où

$$\begin{aligned}\tau_{2h}(V) &= \big\{(c_h^2\rho - 2c_v)\rho[2A_2 + A_4 + \rho A_5](\rho) + 2c_hc_v[C_2 + C_3 + C_5 + \rho C_6](\rho)\\ &\quad + c_v^2[2E_1 + E_2 + \rho E_3](\rho) - 2c_v\rho[2A_2 + A_4 + \rho A_5]'(\rho)\big\}\, V,\end{aligned}$$

$$\begin{aligned}\tau_{2v}(V) &= \big\{(c_h^2\rho - 2c_v)\rho[2B_3 + B_5 + \rho B_6](\rho) + 2c_hc_v[D_2 + D_3 + D_5 + \rho D_6](\rho)\\ &\quad + c_v^2[2F_1 + F_2 + \rho F_3](\rho) - 2c_v\rho[2B_3 + B_5 + \rho B_6]'(\rho)\big\}\, V,\end{aligned}$$

où $\rho = \|V\|^2$ et A_i, B_i, C_i, D_i, E_i and F_i sont des fonctions à valeurs réelles sur $\mathbb{R}^+$, dépendant de α_i, β_i, et figurant dans les expressions de la connexion de Levi-Civita de (TM, G). On en déduit que $V : (M, g) \to (TM, G)$ définit une application biharmonique si et seulement si $\tau_{2h}(V) = \tau_{2v}(V) = 0$.

Pour le cas spécial des métriques de type Kaluza-Klein, un champ de vecteurs parallèle $V : (M, g) \to (TM, G)$ est une application biharmonique si et seulement si l'une des deux assertions suivantes est vérifiée:

1. ρ est un point critique de $\alpha_1 + \alpha_3$.
2. $(\alpha_1 + \alpha_3)'(\rho)\Big[2\rho\frac{\alpha_1'}{\alpha_1} + \rho\frac{\beta_1 - \alpha_1'}{\phi_1} + \rho^2\frac{\alpha_1\beta_1' - 2\alpha_1'\beta_1}{\phi_1\alpha_1} + 2\rho\frac{\phi_1'}{\phi_1} - 1\Big](\rho) - 2\rho(\alpha_1 + \alpha_3)''(\rho) = 0.$

Ce résultat nous permet de caractériser les applications proprement biharmoniques, i.e., les applications biharmoniques non harmoniques: un champ de vecteurs parallèle $V : (M, g) \longrightarrow (TM, G)$ est une application proprement biharmonique si et seulement si

1. $(\alpha_1 + \alpha_3)'(\rho) \neq 0$ et
2. $(\alpha_1 + \alpha_3)'(\rho)\Big[2\rho\frac{\alpha_1'}{\alpha_1} + \rho\frac{\beta_1 - \alpha_1'}{\phi_1} + \rho^2\frac{\alpha_1\beta_1' - 2\alpha_1'\beta_1}{\phi_1\alpha_1} + 2\rho\frac{\phi_1'}{\phi_1} - 1\Big](\rho) - 2\rho(\alpha_1 + \alpha_3)''(\rho) = 0.$

On donne aussi des exemples de champs de vecteurs parallèles non biharmoniques. Soient G une métrique g-naturelle pseudo-Riemannienne sur TM vérifiant $\alpha_2 = \beta_1 = \beta_2 = \beta_3 = 0$, α_1 est constante et $(\alpha_1 + \alpha_3)(t) = e^{at}$, où $a > 0$. Alors $(\alpha_1 + \alpha_3)'$ ne s'annule pas et

$$(\alpha_1+\alpha_3)'(t)\Big[2t\frac{\alpha_1'}{\alpha_1}+t\frac{\beta_1 - \alpha_1'}{\phi_1}+t^2\frac{\alpha_1\beta_1' - 2\alpha_1'\beta_1}{\phi_1\alpha_1}+2t\frac{\phi_1'}{\phi_1}-1\Big](t)-2t(\alpha_1+\alpha_3)''(t) = -a(1+2at)e^{at},$$

qui ne s'annule pas sur $\mathbb{R}^+$. On déduit que les deux conditions nécessaires pour la biharmonicité de champs de vecteurs parallèles ne sont pas vérifiées, et par conséquent chaque champ de vecteurs non nul parallèle sur TM n'est pas biharmonique.

Après l'étude de la biharmonicité des champs de vecteurs parallèles en tant qu'applications, on explore les champs de vecteurs parallèles *G-biharmoniques*, i.e. les points critiques de la fonctionnelle de biénergie restreinte à $\mathfrak{X}(M)$. On prouve qu'un champ de vecteurs V est G-biharmonique si et seulement si

$$\alpha_1\tau_{2v}(V)+\beta_1 g\big(\tau_{2v}(V),V\big)V+\alpha_2\tau_{2h}(V)+\beta_2 g\big(\tau_{2h}(V),V\big)V=0.$$

En particulier, un champ de vecteurs parallèle V sur M est G-biharmonique si et seulement si

$$\phi_1\tau_{2v}(V)=-\phi_2\tau_{2h}(V).$$

Contrairement au cas de la métrique de Sasaki pour laquelle un champ de vecteurs est biharmonique si et seulement s'il est G-biharmonique, on montre que la notion de G-biharmonicité de champs de vecteurs et la notion de biharmonicité de champs de vecteurs en tant qu'applications ne sont pas équivalentes en général. D'une part, on donne une sous-classe de métriques g-naturelles pour lesquelles un champ de vecteurs parallèle est G-bihamonique si et seulement s'il est une application biharmonique, généralisant ainsi le même résultat dans le cas de la métrique de Sasaki. En effet, si G est une métrique g-naturelle pseudo-Riemannienne sur TM qui satisfait $\alpha_2(\rho)=\beta_2(\rho)=(\beta_1+\beta_3)(\rho)=0$ et l'une des identités suivantes

1. $m(\alpha_1+\alpha_3)'(\rho)+\rho(\beta_1+\beta_3)'(\rho)=0$;
2. $[m(\alpha_1+\alpha_3)'(\rho)+\rho(\beta_1+\beta_3)'(\rho)]E_1(\rho)+\frac{\rho}{2}E_3(\rho)=mA_4'(\rho)+\rho A_5'(\rho)t$,

pour un certain $\rho>0$. Alors pour chaque champ de vecteurs parallèle V sur M, avec $\rho=||V||^2$, V est une application biharmonique $(M,g)\longrightarrow(TM,G)$ si et seulement s'il est un champ G-biharmonique.

D'autre part, on construit une famille à 2 paramètres de métriques g-naturelle pseudo-Riemanniennes sur TM pour lesquelles chaque champ de vecteurs parallèle est G-biharmonique mais non biharmonique en tant qu'application.

Biharmonicité de champs de vecteurs unitaires:

Dans le même esprit que celui du concept de champs de vecteurs G-biharmoniques, on définit et on étudie la notion de *champs de vecteurs $\tilde{G}$-biharmoniques*, i.e., les points critiques de la fonctionnelle de biénergie restreinte à l'ensemble $\mathfrak{X}_1(M)$ des champs de vecteurs unitaires. Malgré la complexité des calculs, on a réussi à caractériser les champs de vecteurs unitaires $\tilde{G}$-biharmoniques et à donner certains cas spéciaux en réduisant l'étude au cas des métriques de type Kaluza-Klein sur les fibrés unitaires tangents. On trouve certains résultats généralisant ceux obtenus par M. Markellos and H. Urakawa dans le cas de la métrique de Sasaki [38]:

- *La caractérisation de champs de vecteurs unitaires $\tilde{G}$-biharmoniques*: Un champ de vecteurs unitaire U est $\tilde{G}$-biharmonique si et seulement si

$$\tfrac{a}{a+c}\sum_{i=1}^{m}\Big\{2R(e_i,S(U))\nabla_{e_i}U+(\nabla_{e_i}R)(e_i,S(U))U+R(e_i,\nabla_{e_i}S(U))U\Big\}$$

$$+\overline{\Delta}\overline{\Delta}U-\overline{\Delta}\big[g(\overline{\Delta}U,U)U\big]-g(\overline{\Delta}U,U)\overline{\Delta}U+g(\overline{\Delta}U,U)^2U=0.$$

- *La relation entre les champs de vecteurs unitaires harmoniques et $\tilde{G}$-biharmoniques:* Un champ de vecteurs unitaire harmonique U sur M est $\tilde{G}$-biharmonique si et seulement si

$$\frac{a^2}{\alpha}\sum_{i=1}^{m}\Big\{2R(e_i,S(U))\nabla_{e_i}U+(\nabla_{e_i}R)(e_i,S(U))U+R(e_i,\nabla_{e_i}S(U))U\Big\}\text{ et }U\text{ sont colinéaires.}$$

On applique ces résultats pour étudier les cas spéciaux de champs de vecteurs géodésiques, i.e., les champs de vecteurs unitaires U satisfaisant $\nabla_U U = 0$, et on donne les conditions pour que les champs de vecteurs de Killing et les champs de vecteurs de Reeb soient des champs de vecteurs proprement $\tilde{G}$-biharmoniques, i.e., $\tilde{G}$-biharmoniques qui sont non harmoniques.

On présente aussi quelques exemples de champs de vecteurs unitaires (proprement) $\tilde{G}$-biharmoniques sur les espaces particuliers suivants:

- *Espaces Hyperboliques* $\mathbb{H}^n$: Soient $\mathbb{H}^n$ $(n > 1)$ l'espace hyperbolique à courbure sectionnelle constante négative $-k^2$ et $\tilde{G}$ une métrique g-naturelle pseudo-Riemannienne arbitraire sur son fibré unitaire tangent. Alors le champ de vecteurs $V = e_n = ky_n\frac{\partial}{\partial y_n}$ est un champ de vecteurs unitaire $\tilde{G}$-biharmonique si et seulement si l'une des deux assertions suivantes est vérifiée:

  1. $d = ak^2$. Dans ce cas, V est une application harmonique $(M, g) \to (T_1M, \tilde{G})$.
  2. $a \neq 0$, $d \neq 0$, $d \neq ak^2$ et $b = \left(\frac{2\alpha}{d} - a\right)k - \frac{\alpha}{ak}$. Dans ce cas, V est un champ de vecteurs unitaire proprement $\tilde{G}$-biharmonique.

- *Groupe de Lie résoluble* Sol_3 : Soient Sol_3 le groupe de Lie soluble $\mathbb{R}^3$ muni de la métrique $g_{sol} = e^{2z}dx^2 + e^{-2z}dy^2 + dz^2$ et $\tilde{G}$ une métrique g-naturelle pseudo-Riemannienne arbitraire sur son fibré unitaire tangent. Soit $\{e_1, e_2, e_3\}$ le champ de repères orthonormaux défini par

$$e_1 = e^{-z}\frac{\partial}{\partial x}, \qquad e_2 = e^{z}\frac{\partial}{\partial y}, \qquad e_3 = \frac{\partial}{\partial z}.$$

  1. e_1 est un champ de vecteurs unitaire $\tilde{G}$-biharmonique si et seulement si l'une des assertions suivantes ait lieu
     - $d = b = 0$ et $a \neq 0$. Dans ce cas, e_1 est proprement $\tilde{G}$-biharmonique.
     - $d = 0$, $b \neq 0$ et $b = \frac{a}{4\alpha}(2a(a + c) + \alpha)$. Dans ce cas, e_1 est un champ de vecteurs unitaire proprement $\tilde{G}$-biharmonique.
     - $d \neq 0$ et $b = a + d = 0$. Dans ce cas, e_1 est une application harmonique $(M, g) \to (T_1M, \tilde{G})$.
     - $a \neq 0$, $d \in \left(]-\infty, 3a-2\sqrt{2}|a|] \cup [3a+2\sqrt{2}|a|, +\infty[\right) \setminus \{0\}$ et b et α sont donnés par $b = -\frac{\alpha}{a+d}$ et $\alpha = \frac{-|a+d|}{4}(|a+d| \pm \sqrt{(a+d)^2 - 8ad})$, respectivement. Dans ce cas, e_1 est un champ de vecteurs unitaire proprement $\tilde{G}$-biharmonique.
  2. e_2 est un champ de vecteurs unitaire $\tilde{G}$-biharmonique si et seulement si l'une des assertions suivantes ait lieu
     - $d = b = 0$ et $a \neq 0$. Dans ce cas, e_2 est proprement $\tilde{G}$-biharmonique.
     - $b = 0$, $a + d = 0$. Dans ce cas, e_2 est une application harmonique $(M, g) \to (T_1M, \tilde{G})$.
     - $a \neq 0$, $d \neq 0$, $(a + d)(2a - d) < 0$ et α et b sont donnés par $\alpha = d(a - d)$ et $b^2 = -\frac{1}{2}(a + d)(2a - d)$, respectivement. Dans ce cas, e_2 est proprement $\tilde{G}$-biharmonique.
  3. e_3 est un champ de vecteurs unitaire $\tilde{G}$-biharmonique si et seulement si l'une des assertions suivantes ait lieu
     - $b = 0$. Dans ce cas, e_3 est une application harmonique $(M, g) \to (T_1M, \tilde{G})$.
     - $a \neq 0$, $d \neq 0$, $b \neq 0$, $a + d = 0$ et $2\alpha = ad$. Dans ce cas, e_2 est proprement $\tilde{G}$-biharmonique.

- *Groupe unitaire special $SU(2)$*: Soient $SU(2)$ muni d'une métrique Riemannienne invariante à gauche et $\tilde{G}$ une métrique g-naturelle pseudo-Riemannienne arbitraire sur son fibré unitaire tangent. Soit $\{e_1, e_2, e_3\}$ une base orthonormale de l'algèbre de Lie $su(2)$ telle que

$$[e_1, e_2] = \sigma\lambda e_3, \quad [e_2, e_3] = \lambda e_1, \quad [e_3, e_1] = \lambda e_2,$$

où $\lambda > 0$ et $\sigma \geq 1$. Alors

1. e_3 est une application harmonique $(M, g) \to (T_1M, \tilde{G})$.
2. e_1 (resp. e_2) est un champ de vecteurs unitaire $\tilde{G}$-biharmonique si et seulement si l'une des assertions suivantes ait lieu
   - $b = 0$ ou $\sigma = 1$. Dans ce cas, e_1 et e_2 sont des applications harmoniques $(M, g) \to (T_1M, \tilde{G})$.
   - $b \neq 0$, $\sigma \neq 1$ et l'équation $ad = \frac{2(\sigma^2-\sigma+1)}{1-2\sigma}\alpha$ est satisfaite. Dans ce cas, e_1 et e_2 sont des champs de vecteurs unitaires proprement $\tilde{G}$-biharmoniques.

Fonctions biharmoniques sur les fibrés vectoriels:

La seconds partie de la thèse est consacrée à l'étude des fonctions biharmoniques sur les fibrés vectoriels munis de métriques à symétrie sphérique, généralisant l'étude sur l'harmonicité des functions sur les espaces totaux de fibrés vectoriels déjà effectués par M.T.K. Abbassi et I. Lakrini [15]. On se focalise sur des classes spéciales de fonctions:

- *Relèvements verticaux de fonctions*: Pour toute fonction f sur M, on montre que le bilaplacien de son relèvement vertical f^v est donné par

$$\Delta_{\tilde{G}}^2(f^v) = e^{-4\varphi_1}\left(\Delta_g^2 f\right)^v - 4e^{-2(\varphi_1+\varphi_2)}\Big(2r\varphi_1'' - 4r\varphi_1'(\varphi_1+\varphi_2)' + 2mr(\varphi_1')^2 + 2kr\varphi_1'\varphi_2' + k\varphi_1'\Big)(\Delta_g f)^v.$$

  Pour des choix appropriés des fonctions φ_1 et φ_2, on construit des fonctions (proprement) biharmoniques sur E à partir de fonctions (proprement) biharmoniques sur M. Plus précisément, supposons que φ_1 est constante et soit $f : M \longrightarrow \mathbb{R}$ une fonction (proprement) biharmonique sur (M, g), alors f^v est aussi une fonction (proprement) biharmonique sur (E, G).

  Ces choix nous offre l'opportunité de construire des familles larges d'exemples de fonctions (proprement) biharmoniques sur E.

- *Fonctions $r-$radiales*: On calcule le bilaplacien et vu la complexité de l'équation différentielle linéaire d'ordre 4 obtenue, on va se restreindre au cas de la métrique de Sasaki et dériver la condition caractérisant la biharmonicité des fonctions r-radiales. Ainsi, supposons que $\varphi_1 = \varphi_2 = 0$ (i.e. G est la métrique de Sasaki). Soit $F : E \longrightarrow \mathbb{R}$ une fonction r-radiale construite à partir d'une function $\alpha : [0, +\infty[\longrightarrow \mathbb{R}$, alors F est biharmonique si et seulement si α satisfait l'équation différentielle linéaire

$$2r^2\alpha^{(4)} + (3k+4)r\alpha^{(3)} + k(k+2)\alpha^{(2)} = 0.$$

  Comme corollaire, on obtient les exemples suivants de fonctions r-radiales (proprement) biharmoniques:

1. Si α est une fonction plynomiale de degré 1, alors F est biharmonique.

2. Si $\alpha(r) = ar + b$ avec $(a, b) \in \mathbb{R}^2$ et $a \neq 0$, alors F est une fonction proprement biharmonique. On obtient ainsi une famille à 2 paramètres de fonctions proprement biharmoniques.

Abstract

The theory of harmonic maps is one of the most intersting theories in both analysis and geometry and have many applications in various domains, especially in mathematical physics. In the framework of differential geometry, a *harmonic map* between two Riemannian manifolds (M, g) and (N, h) is defined as a critical point of the energy functional

$$E(f) := \int_M ||df||^2 dv_g,$$

defined on the set of all smooth maps $f : (M, g) \to (N, h)$, where dv_g denotes the volume element of g.

The associated *Euler-Lagrange equation* of E is given by

$$\tau(f) = 0,$$

where $\tau(\phi)$ is the *tension field* of f defined by

$$\tau(f) := tr\nabla(df),$$

so that f is a harmonic map if and only if $\tau(f) = 0$.

In 1983, J. Eells and L. Lemaire proposed a generalization of harmonic maps to the notion of *biharmonic maps* as critical points of the bienergy functional given by

$$E_2(f) := \frac{1}{2}\int_M ||\tau(f)||^2 dv_g.$$

The theory of biharmonic maps had been developed in the second half of the eighties of the twentieth century, espicially when G. Y. Jiang derived the first and the second variational formulas of the bienergy functional and gave the associated Euler-Lagrange equation

$$\tau_2(f) = 0,$$

where $\tau_2(f)$ denotes the *bitension field* of the smooth map f defined by

$$\tau_2(f) := \bar{\Delta}\tau(f) - \sum_{i=1}^{m} R^N\big(\tau(f), f_*e_i\big) f_*e_i.$$

In the framework of Riemannian geometry of tangent bundles, special maps appear naturally having tangent bundle spaces as source and/or target and it turns out natural and interesting to study their harmonicity and biharmonicity in order to understand their geometry and also the geometry of the involved tangent bundles. Since the choice of Riemannian metrics on the source and the target sets of a map influences its (bi)harmonicity, it had been necessary to choose on the involved tangent bundles appropriate metrics to have interesting results.

As (pseudo-)Riemannian manifolds, tangent bundles of Riemannian manifolds had been equipped, in the literature, by various (classes of) metrics, including the classical Sasaki

metric and other lifted metrics from the base manifold, the Cheeger-Gromoll type metrics, the general class of Oproiu metrics and the wider class of g-natural metrics depending on six independent smooth functions from $\mathbb{R}^+$ to $\mathbb{R}$ (cf. [3], [16], [17], and [34], for the definition and main properties on g-natural metrics).

Explicitly, g-natural metrics on the tangent bundle of a Riemannian manifold (M, g) are completely characterized as follows[16]:

Let (M, g) be a Riemannian manifold and G be a g-natural metric on TM. Then there are functions α_i, $\beta_i : \mathbb{R}^+ \to \mathbb{R}$, $i = 1, 2, 3$, such that for every u, X, $Y \in M_x$, we have

$$\begin{cases} G_{(x,u)}(X^h, Y^h) = (\alpha_1 + \alpha_3)(\rho)g_x(X, Y) + (\beta_1 + \beta_3)(\rho)g_x(X, u)g_x(Y, u), \\ G_{(x,u)}(X^h, Y^v) = \alpha_2(\rho)g_x(X, Y) + \beta_2(\rho)g_x(X, u)g_x(Y, u), \\ G_{(x,u)}(X^v, Y^h) = \alpha_2(\rho)g_x(X, Y) + \beta_2(\rho)g_x(X, u)g_x(Y, u), \\ G_{(x,u)}(X^v, Y^v) = \alpha_1(\rho)g_x(X, Y) + \beta_1(\rho)g_x(X, u)g_x(Y, u), \end{cases}$$

where $\rho = g_x(u, u)$.

The harmonicity of special maps with (unit) tangent bundle spaces as source and/or target had been extensively studied in literature and, as expected, it had been proved that the choice of g-natural metrics on (unit) tangent bundles influences the harmonicity behaviour of the considered map (cf. [27], [28], [30], for the Sasaki metric, and [5], [6], for g-natural metrics). On the other hand, the biharmonicity of (unit) vector fields had been studied when the tangent bundle is equipped with the Sasaki metric. But, at the best of our knowledge, there is a lack of works on the bihamonicity of (unit) vector fields for the case of arbitrary g-natural metrics.

In this thesis, we are interested in studying the biharmonicity of (unit) vector fields when the (unit) tangent bundle is equipped with an arbitrary g-natural metric. Our study is motivated by the following facts:

- To generalize to the biharmonic setting the case of harmonic maps when the (unit) tangent bundle is equipped by an an arbitrary g-natural metric (cf. [6] for the tangent bundle and [5] for the unit tangent sphere bundle).
- To generalize to the g-natural metrics case the biharmonicity results on (unit) vector fields when the (unit) tangent bundle is equipped with the Sasaki metric (cf. [37] for the tangent bundle and [38] for the unit tangent sphere bundle).
- To Overcome the rigidity behaviour of the Sasaki metric, in the sense that when the tangent bundle is equipped with the Sasaki metric, the only biharmonic vector fields are the parallel ones [37].

Biharmonicity of parallel vector fields:

In [6], the authors had calculated the tension field of a parallel vector field V. More precisely, they found that

$$\tau(V) = (c_h V)^h + (c_v V)^v,$$

where

$$c_h := \big(2A_2 + mA_4 + \rho A_5\big)(\rho), \qquad c_v := \big(2B_3 + mB_5 + \rho B_6\big)(\rho),$$

where ρ is the constant length of the parallel vector field V and A_i, B_i are real-valued functions on $\mathbb{R}^+$, depending on α_i, β_i, and involved in the expressions of the Levi-Civita connection of

(TM, G). As consequences, we have $V : (M, g) \longrightarrow (TM, G)$ is a harmonic map if and only if $c_h = c_v = 0$.

Due to the complexity of the expression of the Levi-Civita connection of an arbitrary Riemannian g-natural metric G and also the length of calculations of the corresponding bitension field we restrict ourselves to the study of parallel vector fields and we give necessary and sufficient conditions for a parallel vector field to be biharmonic. In fact, the bitension field of a parallel vector field $V : (M, g) \to (TM, G)$ is given by $\tau_2(V) = (\tau_{2h}(V))^h \circ V + (\tau_{2v}(V))^v \circ V$, where

$$\begin{aligned}\tau_{2h}(V) &= \big\{(c_h^2\rho - 2c_v)\rho[2A_2 + A_4 + \rho A_5](\rho) + 2c_hc_v[C_2 + C_3 + C_5 + \rho C_6](\rho)\\ &\quad + c_v^2[2E_1 + E_2 + \rho E_3](\rho) - 2c_v\rho[2A_2 + A_4 + \rho A_5]'(\rho)\big\}\, V,\\ \\ \tau_{2v}(V) &= \big\{(c_h^2\rho - 2c_v)\rho[2B_3 + B_5 + \rho B_6](\rho) + 2c_hc_v[D_2 + D_3 + D_5 + \rho D_6](\rho)\\ &\quad + c_v^2[2F_1 + F_2 + \rho F_3](\rho) - 2c_v\rho[2B_3 + B_5 + \rho B_6]'(\rho)\big\}\, V,\end{aligned}$$

where $\rho = \|V\|^2$ and A_i, B_i, C_i, D_i, E_i and F_i are real-valued functions on $\mathbb{R}^+$, depending on α_i, β_i, and involved in the expressions of the Levi-Civita connection of (TM, G). It follows that $V : (M, g) \to (TM, G)$ defines a biharmonic map if and only if $\tau_{2h}(V) = \tau_{2v}(V) = 0$.

For the special case of Kaluza-Klein type metrics, a parallel vector field $V : (M, g) \to (TM, G)$ is a biharmonic map if and only if one of the two statements is verified:

1. ρ is a critical point of $\alpha_1 + \alpha_3$.
2. $(\alpha_1 + \alpha_3)'(\rho)\Big[2\rho\frac{\alpha_1'}{\alpha_1} + \rho\frac{\beta_1 - \alpha_1'}{\phi_1} + \rho^2\frac{\alpha_1\beta_1' - 2\alpha_1'\beta_1}{\phi_1\alpha_1} + 2\rho\frac{\phi_1'}{\phi_1} - 1\Big](\rho) - 2\rho(\alpha_1 + \alpha_3)''(\rho) = 0.$

This result allows us to characterize proper biharmonic maps, i.e., non-harmonic biharmonic maps: a parallel vector field $V : (M, g) \longrightarrow (TM, G)$ is a proper biharmonic map if and only if

1. $(\alpha_1 + \alpha_3)'(\rho) \neq 0$ and
2. $(\alpha_1 + \alpha_3)'(\rho)\Big[2\rho\frac{\alpha_1'}{\alpha_1} + \rho\frac{\beta_1 - \alpha_1'}{\phi_1} + \rho^2\frac{\alpha_1\beta_1' - 2\alpha_1'\beta_1}{\phi_1\alpha_1} + 2\rho\frac{\phi_1'}{\phi_1} - 1\Big](\rho) - 2\rho(\alpha_1 + \alpha_3)''(\rho) = 0.$

We give also examples of non biharmonic parallel vector field. Let G be the pseudo-Riemannian g-natural metric on TM given by $\alpha_2 = \beta_1 = \beta_2 = \beta_3 = 0$, α_1 is constant and $(\alpha_1 + \alpha_3)(t) = e^{at}$, where $a > 0$. Then $(\alpha_1 + \alpha_3)'$ does not vanish and

$$(\alpha_1+\alpha_3)'(t)\Big[2t\frac{\alpha_1'}{\alpha_1}+t\frac{\beta_1 - \alpha_1'}{\phi_1}+t^2\frac{\alpha_1\beta_1' - 2\alpha_1'\beta_1}{\phi_1\alpha_1}+2t\frac{\phi_1'}{\phi_1}-1\Big](t)-2t(\alpha_1+\alpha_3)''(t) = -a(1+2at)e^{at},$$

which does not vanish on $\mathbb{R}^+$. We deduce that the two conditions for the biharmonicity of parallel vector fields can not be verified, and consequently every non-zero parallel vector field on TM is not biharmonic.

After investigating the biharmonicity of a parallel vector field as a map, we derive the condition for a parallel vector field to be a *G-biharmonic vector field*, i.e. a critical point of the bienergy functional restricted to $\mathfrak{X}(M)$. We prove that a vector field V is a G-biharmonic vector field if and only if

$$\alpha_1\tau_{2v}(V) + \beta_1 g\big(\tau_{2v}(V), V\big)V + \alpha_2\tau_{2h}(V) + \beta_2 g\big(\tau_{2h}(V), V\big)V = 0.$$

In particular, a parallel vector field V on M is G-biharmonic if and only if

$$\phi_1\tau_{2v}(V) = -\phi_2\tau_{2h}(V).$$

Unlike the case of Sasaki metric for which a vector field is biharmonic if and only if it is G-biharmonic, we show that the notion of G-biharmonicity of vector fields and the notion of biharmonicity of vector fields as maps are not equivalent in general. We give, on one hand, a subclass of g-natural metrics for which a parallel vector field is G-bihamonic if and only if it is a biharmonic map, generalizing a similar result for the Sasaki metric. Indeed, if G is a pseudo-Riemannian g-natural metric on TM which satisfies $\alpha_2(\rho) = \beta_2(\rho) = (\beta_1+\beta_3)(\rho) = 0$ and one of the following conditions

1. $m(\alpha_1 + \alpha_3)'(\rho) + \rho(\beta_1 + \beta_3)'(\rho) = 0$;
2. $[m(\alpha_1 + \alpha_3)'(\rho) + \rho(\beta_1 + \beta_3)'(\rho)]E_1(\rho) + \frac{\rho}{2}E_3(\rho) = mA_4'(\rho) + \rho A_5'(\rho)t$,

for some $\rho > 0$. Then for every parallel vector field V on M, with $\rho = ||V||^2$, V is a biharmonic map $(M, g) \longrightarrow (TM, G)$ if and only if it is a G-biharmonic vector field.

On the other hand, we construct a 2-parameter family of pseudo-Riemannian g-natural metrics on TM for which every parallel vector field is G-biharmonic but not biharmonic as a map.

Biharmonicity of unit vector fields:

In the same spirit of the concept of G-biharmonic vector fields, we define and investigate the notion of *$\tilde{G}$-biharmonic unit vector fields*, i.e., critical points of the bienergy functional restricted to the set $\mathfrak{X}_1(M)$ of unit vector fields. Although the complexity of the calulations, we succeed to be characterize $\tilde{G}$-biharmonic unit vector fields and we give some special cases by restricting to the case of Kaluza-Klein type metrics on unit tangent bundles. We find some results generalizing those obtained by M. Markellos and H. Urakawa in the case of the Sasaki metric [38]:

- *The characterization of $\tilde{G}$-biharmonic unit vector fields*: A unit vector field U is a $\tilde{G}$-biharmonic unit vector field if and only if

$$\frac{a}{a+c}\sum_{i=1}^{m}\Big\{2R(e_i, S(U))\nabla_{e_i}U + (\nabla_{e_i}R)(e_i, S(U))U + R(e_i, \nabla_{e_i}S(U))U\Big\}$$

$$+\overline{\Delta}\overline{\Delta}U - \overline{\Delta}\big[g(\overline{\Delta}U, U)U\big] - g(\overline{\Delta}U, U)\overline{\Delta}U + g(\overline{\Delta}U, U)^2U = 0.$$

- *The relation between harmonic unit vector fields and $\tilde{G}$-biharmonic unit vector fields:* A harmonic unit vector field U on M is a $\tilde{G}$-biharmonic unit vector field if and only if

$$\frac{a^2}{\alpha}\sum_{i=1}^{m}\Big\{2R(e_i, S(U))\nabla_{e_i}U + (\nabla_{e_i}R)(e_i, S(U))U + R(e_i, \nabla_{e_i}S(U))U\Big\} \text{ and } U \text{ are collinear.}$$

We apply these results to investigate the special case of geodesic vector fields, i.e., unit vector fields U wich satisfy $\nabla_U U = 0$, and we give conditions for Killing vector fields and Reeb vector fields to be proper $\tilde{G}$-biharmonic unit vector fields, i.e., non-harmonic $\tilde{G}$-biharmonic unit vector field.

We give also examples of (proper) $\tilde{G}$-biharmonic unit vector fields on the following particular spaces:

- *Hyperbolic spaces $\mathbb{H}^n$*: Let $\mathbb{H}^n$ $(n > 1)$ be the hyperbolic space of constant negative sectional curvature $-k^2$ and $\tilde{G}$ be an arbitrary pseudo-Riemannian g-natural metric on its unit tangent bundle. Then the vector field $V = e_n = ky_n\frac{\partial}{\partial y_n}$ is a $\tilde{G}$-biharmonic unit vector field if and only if one of the two following statements is verified:

1. $d = ak^2$. In this case, V is a harmonic map $(M, g) \to (T_1M, \tilde{G})$.
2. $a \neq 0$, $d \neq 0$, $d \neq ak^2$ and $b = \left(\frac{2\alpha}{d} - a\right)k - \frac{\alpha}{ak}$. In this case, V is a proper $\tilde{G}$-biharmonic unit vector field.

- *Solvable Lie group Sol_3* : Let Sol_3 be the solvable Lie group $\mathbb{R}^3$ equipped with the metric $g_{sol} = e^{2z}dx^2 + e^{-2z}dy^2 + dz^2$ and $\tilde{G}$ be an arbitrary pseudo-Riemannian g-natural metric on its unit tangent bundle. Let $\{e_1, e_2, e_3\}$ be the orthonormal frame field given by

$$e_1 = e^{-z}\frac{\partial}{\partial x}, \qquad e_2 = e^{z}\frac{\partial}{\partial y}, \qquad e_3 = \frac{\partial}{\partial z}.$$

1. e_1 is a $\tilde{G}$-biharmonic unit vector field if and only if one of the following assertions holds
   - $d = b = 0$ and $a \neq 0$. In this case, e_1 is proper $\tilde{G}$-biharmonic.
   - $d = 0$, $b \neq 0$ and $b = \frac{a}{4\alpha}(2a(a+c)+\alpha)$. In this case, e_1 is proper $\tilde{G}$-biharmonic.
   - $d \neq 0$ and $b = a+d = 0$. In this case, e_1 is a harmonic map $(M, g) \to (T_1M, \tilde{G})$.
   - $a \neq 0$, $d \in \left(\left]-\infty, 3a - 2\sqrt{2}|a|\right] \cup \left[3a + 2\sqrt{2}|a|, +\infty\right[\right) \setminus \{0\}$ and b and α are given by $b = -\frac{\alpha}{a+d}$ and $\alpha = \frac{-|a+d|}{4}(|a+d| \pm \sqrt{(a+d)^2 - 8ad})$, respectively. In this case, e_1 is proper $\tilde{G}$-biharmonic.
2. e_2 is a $\tilde{G}$-biharmonic unit vector field if and only if one of the following assertions holds
   - $d = b = 0$ and $a \neq 0$. In this case, e_2 is proper $\tilde{G}$-biharmonic.
   - $b = 0$, $a + d = 0$. In this case, e_2 is a harmonic map $(M, g) \to (T_1M, \tilde{G})$.
   - $a \neq 0$, $d \neq 0$, $(a + d)(2a - d) < 0$ and α and b are given by $\alpha = d(a - d)$ and $b^2 = -\frac{1}{2}(a + d)(2a - d)$, respectively. In this case, e_2 is proper $\tilde{G}$-biharmonic.
3. e_3 is a $\tilde{G}$-biharmonic unit vector field if and only if one of the following assertions holds
   - $b = 0$. In this case, e_3 is a harmonic map $(M, g) \to (T_1M, \tilde{G})$.
   - $a \neq 0$, $d \neq 0$, $b \neq 0$, $a + d = 0$ and $2\alpha = ad$. In this case, e_2 is proper $\tilde{G}$-biharmonic.

- *Special unitary group $SU(2)$*: Let $SU(2)$ equipped with a left-invariant Riemannian metric and $\tilde{G}$ be an arbitrary pseudo-Riemannian g-natural metric on its unit tangent bundle. Let $\{e_1, e_2, e_3\}$ be an orthonormal basis of the Lie algebra $su(2)$ such that

$$[e_1, e_2] = \sigma\lambda e_3, \quad [e_2, e_3] = \lambda e_1, \quad [e_3, e_1] = \lambda e_2,$$

where $\lambda > 0$ and $\sigma \geq 1$. Then

1. e_3 is a harmonic map $(M, g) \to (T_1M, \tilde{G})$.
2. e_1 (resp. e_2) is a $\tilde{G}$-biharmonic unit vector field if and only if one of the following assertions holds
   - $b = 0$ or $\sigma = 1$. In this case, e_1 and e_2 are harmonic maps $(M, g) \to (T_1M, \tilde{G})$.
   - $b \neq 0$, $\sigma \neq 1$ and equation $ad = \frac{2(\sigma^2-\sigma+1)}{1-2\sigma}\alpha$ holds. In this case e_1 and e_2 are proper $\tilde{G}$-biharmonic unit vector fields.

Biharmonic functions on vector bundles:

The second part of the thesis is devoted to the investigation of biharmonic functions on vector bundles equipped with spherically symmetric metrics, generalizing the study of the harmonicity of functions on total spaces of vector bundles alreadey done by M.T.K. Abbassi and I. Lakrini [15]. We focus on two special classes of functions:

- *Vertically lifted functions*: For every smooth function f on M, we will prove that the bilaplacian of its vertical lift f^v is given by

$$\Delta_G^2(f^v) = e^{-4\varphi_1}\left(\Delta_g^2 f\right)^v - 4e^{-2(\varphi_1+\varphi_2)}\Big(2r\varphi_1'' - 4r\varphi_1'(\varphi_1+\varphi_2)' + 2mr(\varphi_1')^2 + 2kr\varphi_1'\varphi_2' + k\varphi_1'\Big)(\Delta_g f)^v.$$

  For appropriate choices of the functions φ_1 and φ_2 we will construct (proper) biharmonic functions on E from (proper) biharmonic functions on M. More precisely, assume that φ_1 is constant and let $f : M \longrightarrow \mathbb{R}$ be a (proper) biharmonic function on (M, g), then f^v is also a (proper) biharmonic function on (E, G).

  These choices will give us the possibility of constructing large families of examples of (proper) biharmonic functions on E.

- *$r-$radial functions*: We compute the bilaplacian and due to the complexity of the obtained fourth order linear differential equation, we will reduce it by studying the case of the Sasaki metric and deriving the characterizing condition:

  Assume that $\varphi_1 = \varphi_2 = 0$ (i.e. G is the Sasaki metric). Let $F : E \longrightarrow \mathbb{R}$ be an r-radial function constructed from a function $\alpha : [0, +\infty[\longrightarrow \mathbb{R}$, then F is biharmonic if and only if α satisfies the linear differential equation

$$2r^2\alpha^{(4)} + (3k+4)r\alpha^{(3)} + k(k+2)\alpha^{(2)} = 0.$$

  As a corollary, we obtain the following examples of (proper) biharmonic $r-$ radial functions:

  1. If α is a polynomial function of degree 1, then F is biharmonic.
  2. If $\alpha(r) = ar+b$ with $(a, b) \in \mathbb{R}^2$ and $a \neq 0$, then F is a proper biharmonic function. Therefore, we obtain a 2-parameter family of proper biharmonic functions.

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Introduction

The theory of harmonic maps is a branch of mathematics and precisely one of the variational calculus theories. It explores mappings between Riemannian manifolds, which are mathematical objects that generalize surfaces and spaces. It focuses on maps satisfying a specific harmonicity condition. The origin of the variational calculus backs to the work of L. Euler and J.L. Lagrange in the middle of the eighteenth century, who found that the equation due to Newton is induced from *the problem of maximum and minimum*. Since then, Cauchy, Weirstrass and Fourier established the foundations of the differential calulus. The fundamental idea in the calculus of variations, which involves seeking minima of certain functions like length, finds application in various principles of physics. M. Morse embarked on studying the relationship between critical points of a function on a curved manifold and the manifold's topology, drawing inspiration from investigations into geodesics. Additionally, to effectively deal with a space of curves, a concept of a curved infinite-dimensional space, referred to as an "infinite-dimensional manifold," became necessary.

During the late 1950s to the early 1960s, J. Eells, R.S. Palais, and S. Smale proved that the space of all smooth mappings is an infinite-dimensional manifold. They established a comprehensive theory encompassing differentiation of smooth functions on this manifold, as well as a critical point theory for functions. Harmonic maps emerged as critical points of the energy (action integral), and Eells and Sampson proved the existence of harmonic maps into nonpositively curved manifolds.

The theory of harmonic maps also finds its roots in a problem posed by J. Plateau during the nineteenth century. This particular problem revolves around the existence and uniqueness of soap bubbles that enclose a given wire (a closed Jordan curve) in three-dimensional Euclidean space. In essence, it entails the search for a surface that minimizes the area among all surfaces that enclose a specified Jordan curve.

A *harmonic map* between two compact Riemannian manifolds (M, g) and (N, h) is a smooth map which is a critical point of the energy functional $E(f) = \frac{1}{2}\int_M ||df||^2 dv_g$. It is characterized by the vanishing of the *tension field* $\tau(f) = tr_g \nabla df$.

The theory experienced substantial growth during the 1970s and 1980s, with contributions from mathematicians like Karen Uhlenbeck, Robert Bryant, and Clifford Taubes. Uhlenbeck's influential work in the 1980s focused on the regularity theory of harmonic maps, establishing important results that enhanced the understanding of their properties.

Harmonic maps theory finds applications in various domains of mathematics and physics, including geometric analysis, differential geometry, and partial differential equations. It has proven intersting results in studying minimal surfaces, geometric flows, and gauge field theory, among other areas.

Over time, the theory of harmonic maps has advanced further, exploring special classes of maps such as minimal immersions, constant mean curvature surfaces, and harmonic morphisms. Ongoing research investigates the regularity, existence, and geometric properties of harmonic maps in different settings.

This progress has led to various applications of harmonic maps and ongoing research in the field which led to the theory of biharmonic maps.

As a generalization of harmonic maps, the concept of biharmonic maps was initially proposed by J. Eells and L. Lemaire in their well-known lecture notes on harmonic maps. In 1972, they introduced the notion of bienergy, defined as half of the integral of the squared norm of the tension field over the domain manifold for a smooth mapping. They also posed the problem of studying the variational aspects of this bienergy. *Biharmonic maps* (cf. [25]) had been defined as the critical points of the bienergy functional $E_2(f) = \frac{1}{2}\int_M ||\tau(f)||^2 dv_g$.

A significant advancement occurred in 1986 when G.Y. Jiang calculated the first and second variations of the bienergy.

Biharmonic maps have a rich mathematical structure and have found applications in various areas of mathematics and physics. They have connections to minimal surfaces, geometric analysis, and geometric flows. In particular, biharmonic maps are closely related to minimal submanifolds, which are submanifolds that locally minimize area.

Biharmonic maps theory involves studying the existence, regularity, and properties of biharmonic maps, as well as investigating variational problems associated with the bienergy functional. However, the pivotal moment for the theory of biharmonic maps arose in the field of minimal submanifold theory through the work of B.Y. Chen, who presented his up to now unresolved conjecture that **Every biharmonic submanifold in Euclidean space must be minimal.**

A vector bundle over a smooth manifold M can be shown as a collection of vector spaces that are associated with each point of M in a smooth and coherent manner.

The success of the Riemannian geometry of tangent bundles lead geometers to generalize many known metrics to the framework of vector bundles. Among those metrics, we find the Sasaki metric, the Cheeger-Gromoll metric and the generalized Cheeger-Gromoll metrics. Recently, on a vector bundle (E, π, M), over a Riemannian base (M, g), endowed with a fiber metric h and a compatible connection D, R. Albuquerque introduced the class of spherically symmetric metrics [18], which generalize the Sasaki metric, with two weights φ_1 and φ_2 depending only on the squared norm of the vector. By considering vector bundles with spherically symmetric metrics, we gain a flexible framework to investigate biharmonic maps and biharmonic functions.

Within the realm of vector bundles, a biharmonic function refers to a function that satisfies a particular system of equations involving the Laplacian and bi-Laplacian operators. In this case, the Laplacian and bi-Laplacian operators are defined utilizing connection and curvature forms. A connection on a vector bundle captures the concept of differentiation along the fibers, while the curvature quantifies the failure of parallel transport to commute.

To establish the biharmonic equation on a vector bundle, we consider a section of the bundle as a function on the base manifold. Consequently, the biharmonic equation manifests as a system of partial differential equations involving the Laplacian and bi-Laplacian operators, acting on the components of the section.

The exploration of biharmonic functions within vector bundles yields profound connections with various branches of mathematics. In differential geometry, it provides insights into manifold geometry and the interplay between curvature and harmonic maps. In mathematical physics, biharmonic functions emerge naturally in the study of elasticity, fluid mechanics, and nonlinear field theories.

Furthermore, the theory of biharmonic functions in vector bundles finds applications in mathematical modeling, image processing, and data analysis. It facilitates the extraction of valuable information from high-dimensional datasets and establishes a framework for analyzing intricate structures and patterns.

In the framework of Riemannian geometry of tangent bundles, particular maps arise naturally, e.g. vector fields, the projection map, tangent maps, etc..., and it is interesting to study their (bi-)harmonicity. Since the (bi-)harmonicity depends on Riemannian structures on the source and target sets of the considered map, the situation would depend evidently,

in our case, on the choice of (pseudo-)Riemannian metrics on the tangent bundles involved in the studied maps. Actually, as a (pseudo-)Riemannian manifold, the tangent bundle of a Riemannian manifold had been endowed with various metrics, starting from the "classical" Sasaki metric and other lifted metrics from the base manifold, passing through the Cheeger-Gromoll type metrics and the general class of Oproiu metrics and arriving to the wider class of g-natural metrics, depending on six independent smooth functions from $\mathbb{R}^+$ to $\mathbb{R}$ (cf. [3], [16], [17] and [34]). It had been proved that the harmonicity of such particular maps depends on the choice of (pseudo-)Riemannian structures on the involved tangent bundles (cf. [27], [28], [30] for the Sasaki metric and [5], [6] for g-natural metrics).

This thesis is organised as follows. the three first chapters are devoted to recall the preleminaries on the geometry of the tangent bundle, the various metrics on it like the Sasaki metric and generally the $g-$natural metrics. Also we call some initial notions on the harmonic maps theory and biharmonic theory and we show some particular cases like harmonic vector fields and biharmonic vector fields.

The fourth chapter deals with the questions concerning biharmonicity of vector fields. We know that when equipping the tangent bundle of a compact Riemannian manifold by the Sasaki metric, the only vector fields which are biharmonic, as maps, are the parallel ones. One can wonder if this is still true if we equip the tangent bundle with an arbitrary pseudo-Riemannian g-natural metric. As it is very hard to deal with the general case, we will restrict ourselves to the study of parallel vector fields and we devote section 3 of this chapter to compute the bitension field of a parallel vector field and to give necessary and sufficient conditions for a parallel vector field to be biharmonic (Theorem 4.1.1 for the general case and Theorem 4.2.1 and Corollary 4.2.1 for the special case of Kaluza-Klein type metrics). In particular, we answer to the two first questions by giving examples of non biharmonic parallel vector fields (Remark 4.2.1) and of parallel proper biharmonic vector fields (Examples 4.2.1).

In section 3, we (will) derive the condition for a parallel vector field to be a G-biharmonic vector field i.e. a critical point of the bienergy functional restricted to $\mathfrak{X}(M)$ (Theorem 4.3.1). As a consequence, we give a subclass of pseudo-Riemannian g-natural metrics on TM for which the G-biharmonicity of parallel vector fields is equivalent to their biharmonicity (Example 4.3.1), generalizing a same result for the Sasaki metric. As an answer to the third question, we construct a 2-parameter family of pseudo-Riemannian g-natural metrics on TM for which every parallel vector field is G-biharmonic but not biharmonic (Proposition 4.3.1 and Example 4.3.2).

Chapter 5 is devoted to the characterization of $\tilde{G}$-biharmonic unit vector fields, i.e. critical points of the bienergy functional restricted to $\mathfrak{X}_1(M)$, with applications to special classes of vector fields e.g. Killing vector fields and Reeb vector fields. We give examples of proper $\tilde{G}$-biharmonic unit vector fields on hyperbolic spaces $\mathbb{H}^n$ (Example 5.2.1), on the solvable Lie group Sol_3 (Example 5.2.2) and on the special unitary group $SU(2)$ with a left invariant Riemannian metric (Example 5.2.3).

In the last chapter, we study biharmonic functions on total spaces of vector bundles when endowed with spherically symmetric metrics, by generalizing the results on the harmonic case to to the biharmonic case.

We study the biharmonicity of two classes of functions on total spaces of vector bundles when equipped with spherically symmetric metrics. The first one is the calss of functions vertically lifted to vector bundles, we compute the bilaplacian of the vertical lift of a smooth function (Proposition 6.2.1) and we derive a characterization of Spherically symmetric metrics, with respect to wich vertical lifts of (proper) biharmonic functions allowing us to construct explicit examples (Proposition 6.2.2). In the rest of the section, we investigate $r-$radial functions: we compute the bilaplacian (Propositin 6.3.2) and we reduce the complexity of the obtained fourth order linear differential equation by studying the case of the Sasaki metric (Proposition 6.3.3 and Corollary 6.3.1). Consequently, we construct two infinite families of

examples (Proposition 6.3.4) of biharmonic functions.

Chapter 1

g-natural metrics on tangent bundles

In this chapter we give some important materials to build all the settings we need to begin our study of the biharmonic vector fields and biharmonic functions.
The splitting of the tangent bundle to horizontal and vertical distributions with respect to the Levi-Civita connection of the base manifold is crucial to define the g-natural metrics on the tangent bundle and expressing their Levi-Civita connections and Riemannian curvatures. A very useful tool to express and calculate formulas related to our subject is the notion of F-tensor fields wich will take a part of our exposition in this chapter, and we conclude by characterizing the g-natural metrics induced on the unit tangent bundle.

1.1 Horizontal and vertical distributions

Consider the Levi-Civita connection ∇ associated with the metric g. At any point (x,u) in the tangent bundle TM, the tangent space can be decomposed into horizontal and vertical subspaces with respect to ∇ :

$$(TM)_{(x,u)} = H_{(x,u)} \oplus V_{(x,u)}.$$

For a given point $(x,u) \in TM$, consider any vector $X \in M_x$. There exists a unique vector $X^h \in H_{(x,u)}$ such that $p_* X^h = X$. This unique vector X^h is referred to as the *horizontal lift* of X to the point $(x,u) \in TM$. The *vertical lift* of a vector $X \in M_x$ to the point $(x,u) \in TM$ is denoted by $X^v \in V_{(x,u)}$, satisfying $X^v(df) = Xf$ for all functions f on M, where 1-forms df on M are considered as functions on TM (i.e., $(df)(x,u) = uf$).

It's important to note that the mapping $X \to X^h$ establishes an isomorphism between the vector space M_x and $H_{(x,u)}$, while the mapping $X \to X^v$ similarly establishes an isomorphism between M_x and $V_{(x,u)}$. Consequently, any tangent vector $\tilde{Z}$ in $(TM)_{(x,u)}$ can be expressed as $\tilde{Z} = X^h + Y^v$, where X and Y are uniquely determined vectors in M_x.

Horizontal and vertical lifts of vector fields on M can be defined in an obvious way and are uniquely defined vector fields on TM. Each system of local coordinates $\{(U; x^i, i = 1, ..., n)\}$ in M induces on TM a system of local coordinates $\{(p^{-1}(U); x^i, u^i, i = 1, ..., n)\}$. Let $X = \sum_i X^i \left(\frac{\partial}{\partial x^i}\right)_x$ be the local expression in $\{(U; x^i, i = 1, ..., n)\}$ of a vector X in M_x, $x \in M$. With respect to the induced coordinates, the horizontal lift X^h and the vertical lift X^v of X to $(x,u) \in TM$ are then given by

$$X^h = \sum X^i \left(\frac{\partial}{\partial x^i}\right)_{(x,u)} - \sum \Gamma^i_{jk} u^j X^k \left(\frac{\partial}{\partial u^i}\right)_{(x,u)} \text{ and} \tag{1.1}$$

$$X^v = \sum X^i \left(\frac{\partial}{\partial u^i}\right)_{(x,u)}, \tag{1.2}$$

where (Γ^i_{jk}) denote the Christoffel's symbols of the metric g.

Consider a tensor field T of type $(1,s)$ on the manifold M. If X_1, X_2,..., $X_{s-1} \in M_x$, then $v\{T(X_1,...,u,...,X_{s-1})\}$ (resp.$h\{T(X_1,...,u,...,X_{s-1})\}$) is the vertical (resp. horizontal) vector at (x,u) which is introduced by the formula

$$v\{T(X_1,...,u,...,X_{s-1})\} = \sum u^\lambda (T(X_1,...,\left(\frac{\partial}{\partial x^\lambda}\right)_x,...,X_{s-1}))^v)$$

$$(\text{resp.}h\{T(X_1,...,u,...,X_{s-1})\} = \sum u^\lambda (T(X_1,...,\left(\frac{\partial}{\partial x^\lambda}\right)_x,...,X_{s-1}))^h.$$

In particular,when T represents the identity tensor of type $(1,1)$, then we obtain the canonical vertical vector at (x,u), $\mathcal{U}_{(x,u)} = \sum u^\lambda (\frac{\partial}{\partial x^\lambda})^v_{(x,u)}$, and the geodesic flow vector field at (x,u), $\xi_{(x,u)} = \sum u^\lambda (\frac{\partial}{\partial x^\lambda})^h_{(x,u)}$.
Moreover $v\{T(X_1,...,u,...,u,...,X_{s-t})\}$ and $h\{T(X_1,...,u,...,u,...,X_{s-t})\}$ are introduced by similar way.

Furthermore, we adopt the following conventions

$$v\{T(X_1,...,X_s)\} = (T(X_1,...,X_s))^v \text{ and } h\{T(X_1,...,X_s)\} = (T(X_1,...,X_s))^h.$$

Thus $v\{X\} = X^v$ and $h\{X\} = X^h$, for each vector field X on M.

From the preceding quantities, one can define vector fields on TU in the following way: If $u = \sum_i u^i \left(\frac{\partial}{\partial x^i}\right)_x$ is a fixed point in TM and $X_1,...,X_{s-1}$ are vector fields on U, then we denote by

$$v\{T(X_1,...,u,...,X_{s-1})\} \text{ (resp. } h\{T(X_1,...,u,...,X_{s-1})\})$$

the vertical (resp. horizontal) vector field on TU defined by

$$v\{T(X_1,...,u,...,X_{s-1})\} = \sum_\lambda u^\lambda [T(X_1,...,\frac{\partial}{\partial x^\lambda},...,X_{s-1})]^v$$

$$(\text{resp. } h\{T(X_1,...,u,...,X_{s-1})\} = \sum_\lambda u^\lambda [T(X_1,...,\frac{\partial}{\partial x^\lambda},...,X_{s-1})]^h.$$

Moreover, for vector fields $X_1,...,X_{s-1}$ on U, the vector fields $v\{T(X_1,...,u,...,u,...,X_{s-t})\}$ and $h\{T(X_1,...,u,...,u,...,X_{s-t})\}$, on TU, are introduced by similar way.

We define the Riemannian curvature R of g by

$$R(X,Y) = [\nabla_X, \nabla_Y] - \nabla_{[X,Y]}. \tag{1.3}$$

We write $p_M : TM \to M$ for the natural projection and F for the natural bundle with

$$FM = p_M^*(\underbrace{T^* \otimes ... \otimes T^*}_{r \text{ times}} \otimes \underbrace{T \otimes ... \otimes T}_{s \text{ times}})M \to M,$$

$$Ff(X_x, S_x) = (Tf.X_x, (T^* \otimes ... \otimes T^* \otimes T \otimes ... \otimes T)f.S_x)$$

for all manifolds M, local diffeomorphisms f of M, $X_x \in T_xM$, $S_x \in (T^* \otimes ... \otimes T^* \otimes T \otimes ... \otimes T)_xM$ and $r,s \in \mathbb{N}$.

we call the sections of the canonical projection $FM \to M$ *F-tensor fields of type* (r,s). Therefore, *F-tensor fields* are mappings

$$A : TM \oplus \underbrace{TM \oplus ... \oplus TM}_{s \text{ times}} \to \bigsqcup_{x \in M} \otimes^r M_x$$

which are linear in the last s summands such that $\pi_2 \circ A = \pi_1$, where π_1 and π_2 are the natural projections of the source and target fiber bundles of A respectively.

In particular, the F-metrics are mappings $TM \oplus TM \oplus TM \to \mathbb{R}$ which are linear in the second and the third argument.

Let δ be an F-metric on M, then three distinguished methods arise for the constructing of metrics on the tangent bundle TM, These constructions are given by [34]:

(a) If we suppose that δ is symmetric, then the *Sasaki lift* δ^s of δ is defined as follows:

$$\begin{cases} \delta^s_{(x,u)}(X^h, Y^h) = \delta(u; X, Y), & \delta^s_{(x,u)}(X^h, Y^v) = 0, \\ \delta^s_{(x,u)}(X^v, Y^h) = 0, & \delta^s_{(x,u)}(X^v, Y^v) = \delta(u; X, Y), \end{cases}$$

for all X, $Y \in M_x$.
In particular, if δ is non degenerate and positive definite, then the same properties apply to δ^s.

(b) The *horizontal lift* δ^h of δ is a pseudo-Riemannian metric on TM defined as follows::

$$\begin{cases} \delta^h_{(x,u)}(X^h, Y^h) = 0, & \delta^h_{(x,u)}(X^h, Y^v) = \delta(u; X, Y), \\ \delta^h_{(x,u)}(X^v, Y^h) = \delta(u; X, Y), & \delta^h_{(x,u)}(X^v, Y^v) = 0, \end{cases}$$

for all X, $Y \in M_x$.
In particular, if δ is positive definite, then δ^s is of signature (m, m).

(c) The *vertical lift* δ^v of δ is a degenerate metric on TM defined as follows:

$$\begin{cases} \delta^v_{(x,u)}(X^h, Y^h) = \delta(u; X, Y), & \delta^v_{(x,u)}(X^h, Y^v) = 0, \\ \delta^v_{(x,u)}(X^v, Y^h) = 0, & \delta^v_{(x,u)}(X^v, Y^v) = 0, \end{cases}$$

for all X, $Y \in M_x$. The rank of δ^v is exactly that of δ.

When $\delta = g$ represents a Riemannian metric on M, the three lifts of δ described earlier coincide with the three classical lifts g^s, g^h, and g^v of the metric g, respectively..

1.2 F-tensor fields

In this section, we give basic facts on F-tensor fields which will be needed in this work. For the proofs and more details, we refer to [17], and for the concept of naturality and related notions, see [32] for more details.

Fix $(x, u) \in TM$ and a system of normal coordinates $S := (U; x^1, ..., x^m)$ of (M, g) centered at x. Then we can define on U the vector field $\mathrm{U} := \sum_i u^i \frac{\partial}{\partial x^i}$, where $(u^1, ..., u^m)$ are the coordinates of (x, u) with respect to the basis $((\frac{\partial}{\partial x^i})_x;\ i = 1, ..., m)$ of M_x.

Let P be an F-tensor field of type (p, q) on M. Then, on U, we can define a (p, q)-tensor field P^S_u (or P_u if there is no risk of confusion), associated to u and S, by

$$P_u(X_1, ..., X_q) := P(\mathrm{U}_z; X_1, ..., X_q), \tag{1.4}$$

for all $(X_1, ..., X_q) \in M_z$, $z \in U$. Informally, we can say that we have "*tensorized*" P at u with respect to S.

On the other hand, if we fix $x \in M$ and q vectors $X_1, ..., X_q$ in M_x, then we can define a C^∞-mapping $P_{(X_1,...,X_q)} : M_x \to \otimes^p M_x$, associated to $(X_1, ..., X_q)$, by

$$P_{(X_1,...,X_q)}(u) := P(u; X_1, ..., X_q), \tag{1.5}$$

for all $u \in M_x$.

Let $s > t$ be two non negative integers, T be a $(1,s)$-tensor field on M and P^T be an F-tensor field, of type $(1,t)$, of the form

$$P^T(u; X_1, ..., X_t) = T(X_1, ..., u, ..., u, ..., X_t), \tag{1.6}$$

for all $(u, X_1, ..., X_t) \in TM \oplus ... \oplus TM$, i.e. u appears $s-t$ times at positions $i_1, ..., i_{s-t}$ in the expression of T. Then

* P^T_u is a $(1,t)$-tensor field on a neighborhood U of x in M, for all $u \in M_x$;
* $P^T_{(X_1,...,X_t)}$ is a C^∞-mapping $M_x \to M_x$, for all $X_1, ..., X_t$ in M_x.

Furthermore, we have

Lemma 1.2.1. *[17] 1) The covariant derivative of P^T_u, with respect to the Levi-Civita connection of (M,g), is given by:*

$$(\nabla_X P^T_u)(X_1, ..., X_t) = (\nabla_X T)(X_1, ..., u, ..., u, ..., X_t), \tag{1.7}$$

for all vectors X, X_1,..., X_t in M_x, where the vector u appears at positions $i_1, \ldots, i_{s-t}$ in the right-hand side of the preceding formula.
2) The differential of $P^T_{(X_1,...,X_t)}$, at $u \in M_x$, is given by:

$$\begin{aligned} d_u(P^T_{(X_1,...,X_t)})(X) = {} & T(X_1, ..., X, ..., u, ..., X_t) \\ & + ... + T(X_1, ..., u, ..., X, ..., X_t), \end{aligned} \tag{1.8}$$

for all $X \in M_x$.

Lemma 1.2.2. *[17] Let T be a $(1,s)$-tensor field on M. Then*

$$\begin{aligned}
1)\bar{\nabla}_{X^h} h\{T(X_1, ..., u, ..., u, ..., X_t)\} = {} & h\{(\nabla_X P^T_u)((X_1)_x, ..., (X_t)_x) \\
& + A(u; X, T_x(X_1, ..., u, ..., u, ..., X_t))\} \\
& + v\{B(u; X, T_x(X_1, ..., u, ..., u, ..., X_t))\}, \\
2)\bar{\nabla}_{X^v} h\{T(X_1, ..., u, ..., u, ..., X_t)\} = {} & h\{d_u(P^T_{((X_1)_x,...,(X_t)_x)})(X) \\
& + C(u; T_x(X_1, ..., u, ..., u, ..., X_t), X)\} \\
& + v\{D(u; T_x(X_1, ..., u, ..., u, ..., X_t), X)\}, \\
3)\bar{\nabla}_{X^h} v\{T(X_1, ..., u, ..., u, ..., X_t)\} = {} & h\{C(u; X, T_x(X_1, ..., u, ..., u, ..., X_t))\} \\
& + v\{(\nabla_X P^T_u)((X_1)_x, ..., (X_t)_x) \\
& + D(u; X, T_x(X_1, ..., u, ..., u, ..., X_t))\}, \\
4)\bar{\nabla}_{X^v} v\{T(X_1, ..., u, ..., u, ..., X_t)\} = {} & h\{E(u; X, T_x(X_1, ..., u, ..., u, ..., X_t))\} \\
& + v\{d_u(P^T_{((X_1)_x,...,(X_t)_x)})(X)) \\
& + F(u; X, T_x(X_1, ..., u, ..., u, ..., X_t))\},
\end{aligned}$$

for all vector fields X_1,..., X_t on M and $X \in M_x$, where u appears at positions $i_1, ..., i_{s-t}$ in in any expression of T. Here, X^h and X^v are taken at (x,u).

Next, let P be the F-tensor field of type $(1,t)$ of the form

$$P(u; X_1, ..., X_t) = \sum_i f_i^P(r^2) T_i(X_1, ..., u, ..., u, ..., X_t), \tag{1.9}$$

where $f_i : \mathbb{R}^+ \to \mathbb{R}$ are real-valued functions on $\mathbb{R}^+$, and any T_i is a $(1,s_i)$-tensor field on M, $s_i > t$, with the s_i's not necessarily equal. Then, we have

Lemma 1.2.3. *Let P be an F-tensor field, of type $(1,t)$, on M given by (1.9). Then*

$$\begin{aligned}
1)(\nabla_X P_u)(X_1,...,X_t)&=\sum_i f_i^P(r^2)(\nabla_X T_i)(X_1,...,u,...,u,...,X_t),\\
2)d_u(P_{(X_1,...,X_t)})(X)&=2\sum_i (f_i^P)'(r^2)g(X,u)T_i(X_1,...,u,...,u,...,X_t)\\
&\quad+\sum_i f_i^P(r^2)\{T_i(X_1,...,X,...,u,...,X_t)+...\\
&\quad+T_i(X_1,...,u,...,X,...,X_t)\},
\end{aligned}$$

for all $u,X,X_1,...,X_t\in M_x$.

If we denote by $h\{P(u;X_1,...,X_t)\}$ (resp. $v\{P(u;X_1,...,X_t)\}$) the quantity

$$h\{P(u;X_1,...,X_t)\}=\sum_i f_i^P(r^2)h\{T_i(X_1,...,u,...,u,...,X_t)\} \tag{1.10}$$

$$\text{(resp. } v\{P(u;X_1,...,X_t)\}=\sum_i f_i^P(r^2)v\{T_i(X_1,...,u,...,u,...,X_t)\}), \tag{1.11}$$

then we can assert

Lemma 1.2.4.

$$\begin{aligned}
1)\bar\nabla_{X^h}h\{P(u;X_1,...,X_t)\}&=h\{(\nabla_X P_u)((X_1)_x,...,(X_t)_x)\\
&\quad+A(u;X,P(u;(X_1)_x,...,(X_t)_x))\}\\
&\quad+v\{B(u;X,P(u;(X_1)_x,...,(X_t)_x))\},\\
2)\bar\nabla_{X^v}h\{P(u;X_1,...,X_t)\}&=h\{d_u(P_{((X_1)_x,...,(X_t)_x)})(X)\\
&\quad+C(u;P(u;(X_1)_x,...,(X_t)_x),X)\}\\
&\quad+v\{D(u;P(u;(X_1)_x,...,(X_t)_x),X)\},\\
3)\bar\nabla_{X^h}v\{P(u;X_1,...,X_t)\}&=h\{C(u;X,P(u;(X_1)_x,...,(X_t)_x))\}\\
&\quad+v\{(\nabla_X P_u)((X_1)_x,...,(X_t)_x)\\
&\quad+D(u;X,P(u;(X_1)_x,...,(X_t)_x))\},\\
4)\bar\nabla_{X^v}v\{P(u;X_1,...,X_t)\}&=h\{E(u;X,P(u;(X_1)_x,...,(X_t)_x))\}\\
&\quad+v\{d_u(P_{((X_1)_x,...,(X_t)_x)})(X))\\
&\quad+F(u;X,P(u;(X_1)_x,...,(X_t)_x))\},
\end{aligned}$$

for all vector fields $X_1,...,X_t$ on M and $X\in M_x$. Here X^h and X^v are taken at (x,u).

Now, the existence of non-vanishing parallel vector field on a Riemannian manifold (M,g) implies that all the F-tensors fields A,B,C,D,E and F reduce to the following form:

$$P(u;X,Y)=f_3^p g(Y,u)X+f_4^p g(X,u)Y+f_5^p g(X,Y)u+f_6^p g(X,u)g(Y,u)u, \tag{1.12}$$

wehre $f_i:\mathbb{R}^+\longrightarrow\mathbb{R}$ are real valued function on $\mathbb{R}^+$.

Lemma 1.2.5. *Let P be an F-tensor field, of type $(1,2)$, on M given by (1.12). Then*

$$\begin{aligned}
1)\nabla_X P_u&=0,\\
2)d(P_{(X,Y)})_u(Z)&=[f_3^P g(Y,Z)+2(f_3^P)'g(Y,u)g(Z,u)]X\\
&\quad+[f_4^P g(X,Z)+2(f_4^P)'g(X,u)g(Z,u)]Y+f_5^P g(X,Y)Z\\
&\quad+\{f_6^P[g(X,Z)g(Y,u)+g(Y,Z)g(X,u)]\\
&\quad+2(f_5^P)'g(X,Y)g(Z,u)+2(f_6^P)'g(X,u)g(Y,u)g(Z,u)\}u,
\end{aligned}$$

for all $u,X,Y,Z\in M_x$.

1.3 g-natural metrics

We will now describe all first order natural operators $D : S^2_+T^* \rightsquigarrow (S^2T^*)T$ that transform Riemannian metrics on manifolds into metrics on their tangent bundles. Here, $S^2_+T^*$ and S^2T^* represent the bundle functors of all Riemannian metrics and all symmetric two-forms over m-manifolds, respectively. For a detailed understanding of the concepts of naturality and related notions, refer to [32].

Let's define any section $G : TM \to (S^2T^*)TM$ as a (possibly degenerate) *metric*. Establishing a bijective correspondence between triples of first order natural F-metrics $(\zeta_1, \zeta_2, \zeta_3)$ and first-order natural (possibly degenerate) metrics G on the tangent bundles given by (cf. [34]):

$$G = \zeta_1^s + \zeta_2^h + \zeta_3^v.$$

Hence, to identify all first-order natural operators $S^2_+T^* \rightsquigarrow (S^2T^{)}T$ that transform Riemannian metrics on manifolds into metrics on their tangent bundles, it is sufficient to characterize all first-order natural F-metrics. These are essentially first-order natural operators $S^2_+T^{\rightsquigarrow}(T, F)$.

In this sense, it is shown in [34] (see also [32] and [3]) that all first order natural F-metrics ζ in dimension $m > 1$ form a family parametrized by two arbitrary smooth functions α_0, $\beta_0 : \mathbb{R}^+ \to \mathbb{R}$, where $\mathbb{R}^+$ denotes the set of all nonnegative real numbers, in the following way: For every Riemannian manifold (M, g) and tangent vectors u, X, $Y \in M_x$

$$\zeta_{(M,g)}(u)(X, Y) = \alpha_0(g(u, u))g(X, Y) + \beta_0(g(u, u))g(u, X)g(u, Y). \tag{1.13}$$

In the case where $m = 1$, the same statement holds true; however, it is always possible to choose $\beta_0 = 0$.
In particular, all first order natural F-metrics are symmetric.

Definition 1.3.1. *Let (M, g) be a Riemannian manifold. We shall call a metric G on TM which comes from g by a first order natural operator $S^2_+T^* \rightsquigarrow (S^2T^*)T$ a* g-natural metric.

Thus, all g-natural metrics on the tangent bundle of a Riemannian manifold (M, g) are completely determined as follows:

Proposition 1.3.1. *[16] Let (M, g) be a Riemannian manifold and G be a g-natural metric on TM. Then there are functions α_i, $\beta_i : \mathbb{R}^+ \to \mathbb{R}$, $i = 1, 2, 3$, such that for every u, X, $Y \in M_x$, we have*

$$\begin{cases} G_{(x,u)}(X^h, Y^h) = (\alpha_1 + \alpha_3)(\rho)g_x(X, Y) + (\beta_1 + \beta_3)(\rho)g_x(X, u)g_x(Y, u), \\ G_{(x,u)}(X^h, Y^v) = \alpha_2(\rho)g_x(X, Y) + \beta_2(\rho)g_x(X, u)g_x(Y, u), \\ G_{(x,u)}(X^v, Y^h) = \alpha_2(\rho)g_x(X, Y) + \beta_2(\rho)g_x(X, u)g_x(Y, u), \\ G_{(x,u)}(X^v, Y^v) = \alpha_1(\rho)g_x(X, Y) + \beta_1(\rho)g_x(X, u)g_x(Y, u), \end{cases} \tag{1.14}$$

where $\rho = g_x(u, u)$.

Remark 1.3.1. *From now on, we shall use the following notations:*

- $\phi_i(t) = \alpha_i(t) + t\beta_i(t)$,
- $\alpha(t) = \alpha_1(t)(\alpha_1 + \alpha_3)(t) - \alpha_2^2(t)$,
- $\phi(t) = \phi_1(t)(\phi_1 + \phi_3)(t) - \phi_2^2(t)$,

for all $t \in \mathbb{R}^+$.

Pseudo-Riemannian g-natural metrics are characterized as follows:

Proposition 1.3.2. *[16] A g-natural metric G on the tangent bundle of a Riemannian manifold (M, g), defined by the functions of Proposition 1.3.1, is*

- *non-degenerate if and only if*

$$\alpha(t) \neq 0, \qquad \phi(t) \neq 0 \qquad \text{for all } t \in \mathbb{R}^+;$$

- *Riemannian if and only if*

$$\begin{cases} \alpha_1(t) > 0, & \phi_1(t) > 0, \\ \alpha(t) > 0, & \phi(t) > 0, \end{cases} \tag{1.15}$$

for all $t \in \mathbb{R}^+$.

for all $t \in \mathbb{R}^+$.

The wide class of g-natural metrics includes several well known metrics (Riemannian and not) on TM. In particular:

- the *Sasaki metric* G_S is obtained for $\alpha_1 = 1$ and $\alpha_2 = \alpha_3 = \beta_1 = \beta_2 = \beta_3 = 0$.
- the *Cheeger-Gromoll metric* G_{GC} is obtained when

$$\alpha_2(t) = \beta_2(t) = 0, \quad \alpha_1(t) = \beta_1(t) = -\beta_3(t) = \frac{1}{1+t}, \quad \alpha_3(t) = \frac{t}{1+t}. \tag{1.16}$$

- *Kaluza–Klein metrics*, as commonly defined on principal bundles [54] (see also [19]), are obtained for $\alpha_2 = \beta_2 = \beta_1 + \beta_3 = 0$.
- *Metrics of Kaluza–Klein type* are defined by the geometric condition of orthogonality between horizontal and vertical distributions. Thus, a g-natural metric G is of Kaluza-Klein type if $\alpha_2 = \beta_2 = 0$.

Remarks 1.3.1. *1. In the sequel, when we consider an arbitrary g-natural metric G on TM, we implicitly suppose that it is defined by the functions α_i, $\beta_i : \mathbb{R}^+ \to \mathbb{R}$, $i = 1, 2, 3$, given in Proposition 1.3.1.*

2. Unless otherwise stated, all real functions α_i, β_i, ϕ_i, α and ϕ and their derivatives are evaluated at $\rho := g_x(u, u)$.

1.4 The Levi-Civita connexion of pseudo-Riemannian g-natural metrics

Proposition 1.4.1. *Let (M, g) be a Riemannian manifold, ∇ its Levi-Civita connection and R its curvature tensor. Let G be a pseudo-Riemannian g-natural metric on TM. Then the Levi-Civita connection $\overline{\nabla}$ of (TM, G) is characterized by*

$$\begin{cases} (\overline{\nabla}_{X^h} Y^h)_{(x,u)} = (\nabla_X Y)^h_{(x,u)} + h\{A(u; X_x, Y_x)\} + v\{B(u; X_x, Y_x)\}, \\ (\overline{\nabla}_{X^h} Y^v)_{(x,u)} = (\nabla_X Y)^v_{(x,u)} + h\{C(u; X_x, Y_x)\} + v\{D(u; X_x, Y_x)\}, \\ (\overline{\nabla}_{X^v} Y^h)_{(x,u)} = h\{C(u; Y_x, X_x)\} + v\{D(u; Y_x, X_x)\}, \\ (\overline{\nabla}_{X^v} Y^v)_{(x,u)} = h\{E(u; X_x, Y_x)\} + v\{F(u; X_x, Y_x)\}, \end{cases} \tag{1.17}$$

for all vector fields X, Y on M and $(x,u)\in TM$, where A, B, C, D, E and F are F-tensor fields of type $(1,2)$ on M defined, for all u, X, $Y\in M_x$, $x\in M$, by:

$$\begin{aligned}A(u;X,Y) &= A_1[R(X,u)Y+R(Y,u)X]+A_2[g_x(Y,u)X+g_x(X,u)Y]\\ &\quad +A_3g_x(R(X,u)Y,u)u+A_4g_x(X,Y)u+A_5g_x(X,u)g_x(Y,u)u,\end{aligned}$$

where

$$\begin{aligned}A_1&=-\tfrac{\alpha_1\alpha_2}{2\alpha},\\ A_2&=\tfrac{\alpha_2(\beta_1+\beta_3)}{2\alpha},\\ A_3&=\tfrac{\alpha_2\{\alpha_1[\phi_1(\beta_1+\beta_3)-\phi_2\beta_2]+\alpha_2(\beta_1\alpha_2-\beta_2\alpha_1)\}}{\alpha\phi},\\ A_4&=\tfrac{\phi_2(\alpha_1+\alpha_3)'}{\phi},\\ A_5&=\tfrac{\alpha\phi_2(\beta_1+\beta_3)'+(\beta_1+\beta_3)\{\alpha_2[\phi_2\beta_2-\phi_1(\beta_1+\beta_3)]+(\alpha_1+\alpha_3)(\alpha_1\beta_2-\alpha_2\beta_1)\}}{\alpha\phi},\end{aligned}\tag{1.18}$$

$$\begin{aligned}B(u;X,Y) &= B_1R(X,u)Y+B_2R(X,Y)u+B_3[g_x(Y,u)X+g_x(X,u)Y]\\ &\quad +B_4g_x(R(X,u)Y,u)u+B_5g_x(X,Y)u+B_6g_x(X,u)g_x(Y,u)u,\end{aligned}$$

where

$$\begin{aligned}B_1&=\tfrac{\alpha_2^2}{\alpha},\\ B_3&=-\tfrac{(\alpha_1+\alpha_3)(\beta_1+\beta_3)}{2\alpha},\\ B_4&=\tfrac{\alpha_2\{\alpha_2[\phi_2\beta_2-\phi_1(\beta_1+\beta_3)]+(\alpha_1+\alpha_3)(\beta_2\alpha_1-\beta_1\alpha_2)\}}{\alpha\phi},\\ B_5&=-\tfrac{(\phi_1+\phi_3)(\alpha_1+\alpha_3)'}{\phi},\\ B_6&=\tfrac{-\alpha(\phi_1+\phi_3)(\beta_1+\beta_3)'+(\beta_1+\beta_3)\{(\alpha_1+\alpha_3)[(\phi_1+\phi_3)\beta_1-\phi_2\beta_2]+\alpha_2[\alpha_2(\beta_1+\beta_3)-(\alpha_1+\alpha_3)\beta_2]\}}{\alpha\phi},\end{aligned}\tag{1.19}$$

$$\begin{aligned}C(u;X,Y) &= C_1R(Y,u)X+C_2g_x(X,u)Y+C_3g_x(Y,u)X+C_4g_x(R(X,u)Y,u)u+\\ &\quad +C_5g_x(X,Y)u+C_6g_x(X,u)g_x(Y,u)u,\end{aligned}$$

where

$$\begin{aligned}C_1&=-\tfrac{\alpha_1^2}{2\alpha},\\ C_2&=-\tfrac{\alpha_1(\beta_1+\beta_3)}{2\alpha},\\ C_3&=\tfrac{\alpha_1(\alpha_1+\alpha_3)'-\alpha_2(\alpha_2'-\frac{\beta_2}{2})}{\alpha},\\ C_4&=\tfrac{\alpha_1\{\alpha_2(\alpha_2\beta_1-\alpha_1\beta_2)+\alpha_1[\phi_1(\beta_1+\beta_3)-\phi_2\beta_2]\}}{2\alpha\phi},\\ C_5&=\tfrac{\phi_1(\beta_1+\beta_3)+\phi_2(2\alpha_2'-\beta_2)}{2\phi},\\ C_6&=\tfrac{\alpha\phi_1(\beta_1+\beta_3)'+\{\alpha_2(\alpha_1\beta_2-\alpha_2\beta_1)+\alpha_1[\phi_2\beta_2-(\beta_1+\beta_3)\phi_1]\}[(\alpha_1+\alpha_3)'+\frac{\beta_1+\beta_3}{2}]}{\alpha\phi}\\ &\quad +\tfrac{\{\alpha_2[\beta_1(\phi_1+\phi_3)-\beta_2\phi_2]-\alpha_1[\beta_2(\alpha_1+\alpha_3)-\alpha_2(\beta_1+\beta_3)]\}(\alpha_2'-\frac{\beta_2}{2})}{\alpha\phi},\end{aligned}\tag{1.20}$$

$$\begin{aligned}D(u;X,Y) &= D_1R(Y,u)X+D_2g_x(X,u)Y+D_3g_x(Y,u)X+D_4g_x(R(X,u)Y,u)u\\ &\quad +D_5g_x(X,Y)u+D_6g_x(X,u)g_x(Y,u)u,\end{aligned}$$

where

$$
\begin{aligned}
D_1 &= \tfrac{\alpha_1\alpha_2}{2\alpha},\\
D_2 &= \tfrac{\alpha_2(\beta_1+\beta_3)}{2\alpha},\\
D_3 &= \tfrac{-\alpha_2(\alpha_1+\alpha_3)'+(\alpha_1+\alpha_3)(\alpha_2'-\frac{\beta_2}{2})}{\alpha},\\
D_4 &= \tfrac{\alpha_1\{(\alpha_1+\alpha_3)(\alpha_1\beta_2-\alpha_2\beta_1)+\alpha_2[\phi_2\beta_2-\phi_1(\beta_1+\beta_3)]\}}{2\alpha\phi},\\
D_5 &= -\tfrac{\phi_2(\beta_1+\beta_3)+(\phi_1+\phi_3)(2\alpha_2'-\beta_2)}{2\alpha\phi},\\
D_6 &= \tfrac{-\alpha\phi_2(\beta_1+\beta_3)'+\{(\alpha_1+\alpha_3)(\alpha_2\beta_1-\alpha_1\beta_2)+\alpha_2[\phi_1(\beta_1+\beta_3)-\phi_2\beta_2]\}[(\alpha_1+\alpha_3)'+\frac{\beta_1+\beta_3}{2}]}{\alpha\phi}\\
&\quad+\tfrac{\{(\alpha_1+\alpha_3)[\beta_2\phi_2-\beta_1(\phi_1+\phi_3)]+\alpha_2[\beta_2(\alpha_1+\alpha_3)-\alpha_2(\beta_1+\beta_3)]\}(\alpha_2'-\frac{\beta_2}{2})}{\alpha\phi},
\end{aligned}
\tag{1.21}
$$

$$E(u;X,Y)=E_1[g_x(Y,u)X+g_x(X,u)Y]+E_2g_x(X,Y)u+E_3g_x(X,u)g_x(Y,u)u,$$

where

$$
\begin{aligned}
E_1 &= \tfrac{\alpha_1(\alpha_2'+\frac{\beta_2}{2})-\alpha_2\alpha_1'}{\alpha},\\
E_2 &= \tfrac{\phi_1\beta_2-\phi_2(\beta_1-\alpha_1')}{\phi},\\
E_3 &= \tfrac{\alpha(2\phi_1\beta_2'-\phi_2\beta_1')+2\alpha_1'\{\alpha_1[\alpha_2(\beta_1+\beta_3)-\beta_2(\alpha_1+\alpha_3)]+\alpha_2[\beta_1(\phi_1+\phi_3)-\beta_2\phi_2]\}}{\alpha\phi}\\
&\quad+\tfrac{(2\alpha_2'+\beta_2)\{\alpha_1[\phi_2\beta_2-\phi_1(\beta_1+\beta_3)]+\alpha_2(\alpha_1\beta_2-\alpha_2\beta_1)\}}{\alpha\phi},
\end{aligned}
\tag{1.22}
$$

$$F(u;X,Y)=F_1[g_x(Y,u)X+g_x(X,u)Y]+F_2g_x(X,Y)u+F_3g_x(X,u)g_x(Y,u)u,$$

where

$$
\begin{aligned}
F_1 &= \tfrac{-\alpha_2(\alpha_2'+\frac{\beta_2}{2})+(\alpha_1+\alpha_3)\alpha_1'}{\alpha},\\
F_2 &= \tfrac{(\phi_1+\phi_3)(\beta_1-\alpha_1')-\phi_2\beta_2}{\phi},\\
F_3 &= \tfrac{\alpha[(\phi_1+\phi_3)\beta_1'-2\phi_2\beta_2']+2\alpha_1'\{\alpha_2[\beta_2(\alpha_1+\alpha_3)-\alpha_2(\beta_1+\beta_3)]+(\alpha_1+\alpha_3)[\beta_2\phi_2-\beta_1(\phi_1+\phi_3)]\}}{\alpha\phi}\\
&\quad+\tfrac{(2\alpha_2'+\beta_2)\{\alpha_2[\phi_1(\beta_1+\beta_3)-\phi_2\beta_2]+(\alpha_1+\alpha_3)(\alpha_2\beta_1-\alpha_1\beta_2)\}}{\alpha\phi}.
\end{aligned}
\tag{1.23}
$$

1.5 Riemaniann curvature of g-natural metrics

Proposition 1.5.1. *[17] Let (M,g) be a Riemannian manifold and G be a pseudo-Riemannian g-natural metric on TM. Denote by ∇ and R the Levi-Civita connection and the Rieamannian curvature tensor of (M,g), respectively. Then we have:*

$$
\begin{aligned}
\overline{R}\big(X^h,Y^h\big)Z^h=&\Big[R(X,Y)Z\Big]^h+h\Big\{(\nabla_XA_u)(Y,Z)-(\nabla_YA_u)(X,Z)+A\Big(u;X,A(u;Y,Z)\Big)\\
&-A\Big(u;Y,A(u;X,Z)\Big)+C\Big(u;X,B(u;Y,Z)\Big)-C\Big(u;Y,B(u;X,Z)\Big)\\
&+C\Big(u;Z,R(X,Y)u\Big)\Big\}+v\Big\{(\nabla_XB_u)(Y,Z)-(\nabla_YA_u)(X,Z)\\
&+B\Big(u;X,A(u;Y,Z)\Big)-B\Big(u;Y,A(u;X,Z)\Big)+D\Big(u;X,B(u;Y,Z)\Big)\\
&-D\Big(u;Y,B(u;X,Z)\Big)+D\Big(u;Z,R(X,Y)u\Big)\Big\},
\end{aligned}
$$

$$\begin{aligned}\bar{R}\big(X^h,Y^h\big)Z^v=&[R(X,Y)Z]^v+h\Big\{(\nabla_X C_u)(Y,Z)+(\nabla_X C_u)(X,Z)+A\Big(u;X,C(u;Y,Z)\Big)\\&-A\Big(u;Y,C(u;X,Z)\Big)+C\Big(u;X,D(u;Y,Z)\Big)-C\Big(u;Y,D(u;X,Z)\Big)\\&+E\Big(u;R(X,Y)u,Z\Big)\Big\}+v\Big\{(\nabla_X D_u)(Y,Z)-(\nabla_Y D_u)(X,Z)\\&+B\Big(u;X,C(u;Y,Z)\Big)-B\Big(u;Y,C(u;X,Z)\Big)+D\Big(u;X,D(u;Y,Z)\Big)\\&-D\Big(u;Y,D(u;X,Z)\Big)+F\Big(u;R(X,Y)u,Z\Big)\Big\},\end{aligned}$$

$$\begin{aligned}\bar{R}\big(X^h,Y^v\big)Z^h=&h\Big\{(\nabla_X C_u)(Z,Y)+A\Big(u;X,C(u;Z,Y)\Big)+C\Big(u;X,D(u;Z,Y)\Big)\\&-C\Big(u;A(u;X,Z),Y\Big)-E\Big(u;Y,B(u;X,Z)\Big)-d_u\big(A_{(X,Z)}\big)(Y)\Big\}\\&+v\Big\{(\nabla_X D_u)(Z,Y)+B\Big(u;X,C(u;Z,Y)\Big)+D\Big(u;X,D(u;Z,Y)\Big)\\&-D\Big(u;A(u;X,Z),Y\Big)-F\Big(u;Y,B(u;X,Z)\Big)-d_u\Big(B_{(X,Z)}\Big)(Y)\Big\},\end{aligned}$$

$$\begin{aligned}\bar{R}\big(X^h,Y^v\big)Z^v=&h\Big\{(\nabla_X E_u)(Z,Y)+A\Big(u;X,E(u;Y,Z)\Big)+C\Big(u;X,F(u;Y,Z)\Big)\\&-C\Big(u;C(u;X,Z),Y\Big)-E\Big(u;Y,D(u;X,Z)\Big)-d_u\big(C_{(X,Z)}\big)(Y)\Big\}\\&+v\Big\{(\nabla_X F_u)(Y,Z)+B\Big(u;X,E(u;Y,Z)\Big)+D\Big(u;X,F(u;Y,Z)\Big)\\&-D\Big(u;C(u;X,Z),Y\Big)-F\Big(u;Y,D(u;X,Z)\Big)-d_u\Big(B_{(X,Z)}\Big)(Y)\Big\},\end{aligned}$$

$$\begin{aligned}\bar{R}\big(X^v,Y^v\big)Z^h=&h\Big\{d(C_{Z,Y})_u(X)-d(C_{Z,X})_u(Y)+C\Big(u;C(u;Z,Y),X\Big)\\&-C\Big(u;C(u;Z,X),Y\Big)+E\Big(u;X,D(u;Z,Y)\Big)-E\Big(u;Y,D(u;Z,X)\Big)\Big\}\\&+v\Big\{d(D_{Z,Y})_u(X)-d(D_{Z,X})_u(Y)+D\Big(u;C(u;Z,Y),X\Big)\\&-D\Big(u;C(u;Z,X),Y\Big)+F\Big(u;X,D(u;Z,Y)\Big)-F\Big(u;Y,D(u;Z,X)\Big)\Big\},\end{aligned}$$

$$\begin{aligned}\bar{R}\big(X^v,Y^v\big)Z^v=h\Big\{&d(E_{Y,Z})_u(X)-d(C_{X,Z})_u(Y)+C\Big(u;E(u;Y,Z),X\Big)\\&-C\Big(u;E(u;X,Z),Y\Big)+E\Big(u;X,F(u;Y,Z)\Big)-E\Big(u;Y,F(u;X,Z)\Big)\Big\}\\&+v\Big\{d(F_{Y,Z})_u(X)-d(F_{X,Z})_u(Y)+D\Big(u;E(u;Y,Z),X\Big)\\&-D\Big(u;E(u;X,Z),Y\Big)+F\Big(u;X,F(u;Y,Z)\Big)-F\Big(u;Y,F(u;X,Z)\Big)\Big\},\end{aligned}$$

for all $x\in M$ and X, Y, $Z\in T_xM$, where the lifts are taken at $u\in T_xM$ and $A_{(X,Z)}$ and $B_{(X,Z)}$ are the mappings $M_x\to M_x$ defined by (1.5) *and $d_u\big(A_{(X,Z)}\big)$ and $d_u\big(A_{(X,Z)}\big)$ are their derivatives at u, respectively.*

1.6 g-natural metrics on unit tangent bundles

The *tangent sphere bundle of radius $r>0$* over a Riemannian manifold (M,g), is defined as the hypersurface $T_rM=\{(x,u)\in TM|g_x(u,u)=r^2\}$. The tangent space of T_rM, at a point $(x,u)\in T_rM$, is given by

$$(T_rM)_{(x,u)}=\{X^h+Y^v/X\in M_x,Y\in\{u\}^\perp\subset M_x\}. \tag{1.24}$$

When $r=1$, T_1M is referred to as *the unit tangent (sphere) bundle.*

On tangent sphere bundles, the restrictions of Riemannian g-natural metrics possess a simpler form, i.e.,

Proposition 1.6.1. *[9] Let $r>0$ and (M,g) be a Riemannian manifold. For every Riemannian metric $\tilde{G}$ on T_rM induced from a Riemannian g-natural metric G on TM, there exist four constants a, b, c and d, with $a>0$, $a(a+c)-b^2>0$ and $a(a+c+dr^2)-b^2>0$, such that $\tilde{G}=a.\widetilde{g^s}+b.\widetilde{g^h}+c.\widetilde{g^v}+d.\widetilde{k^v}$, where*

- *k is the natural F-metric on M defined by*

$$k(u;X,Y)=g(u,X)g(u,Y),\quad \textit{for all}\quad (u,X,Y)\in TM\oplus TM\oplus TM,$$

- *$\widetilde{g^s}$, $\widetilde{g^h}$, $\widetilde{g^v}$ and $\widetilde{k^s}$ are the metrics on T_rM induced by g^s, g^h, g^v and k^v, respectively.*

Let $\tilde{G}=a.\widetilde{g^s}+b.\widetilde{g^h}+c.\widetilde{g^v}+d.\widetilde{k^v}$ be the Riemannian metric on T_1M induced by G, where G is an arbitrary Riemannian g-natural metric on TM, where a, b, c and d are constants where $a>0$, $\alpha:=a(a+c)-b^2>0$, and $\phi:=a(a+c+d)-b^2>0$.

Applying Schmidt's orthonormalization process, a simple calculation reveals that the vector field on TM defined by

$$N^G_{(x,u)}=\frac{1}{\sqrt{(a+c+d)\phi}}\left[-b.u^h+(a+c+d).u^v\right], \tag{1.25}$$

for all $(x,u)\in TM$, is normal to T_1M and unitary at any point of T_1M.

We define now the "tangential lift" X^{t_G} –with respect to G– of a vector $X\in M_x$ to $(x,u)\in T_1M$ as the tangential projection of the vertical lift of X to (x,u) –with respect to N^G–, that is,

$$X^{t_G}=X^v-G_{(x,u)}(X^v,N^G_{(x,u)})\;N^G_{(x,u)}=X^v-\sqrt{\frac{\phi}{a+c+d}}g_x(X,u)\;N^G_{(x,u)}. \tag{1.26}$$

If $X \in M_x$ is orthogonal to u, then $X^{t_G} = X^v$.

The tangent space $(T_1M)_{(x,u)}$ of T_1M at (x,u) is spanned by vectors of the form X^h and Y^{t_G}, where $X,\ Y \in M_x$. Consequently, the Riemannian metric $\tilde{G}$ on T_1M is fully characterized by the identities

$$\begin{cases} \tilde{G}_{(x,u)}(X^h, Y^h) &= (a+c)g_x(X,Y) + dg_x(X,u)g_x(Y,u), \\ \tilde{G}_{(x,u)}(X^h, Y^{t_G}) &= bg_x(X,Y), \\ \tilde{G}_{(x,u)}(X^{t_G}, Y^{t_G}) &= ag_x(X,Y) - \frac{\phi}{a+c+d} g_x(X,u)g_x(Y,u), \end{cases} \tag{1.27}$$

for all $(x,u) \in T_1M$ and $X,\ Y \in M_x$.

The Levi-Civita connection $\tilde{\nabla}$ of $(T_1M, \tilde{G})$ is given by

Proposition 1.6.2. *[9] At $(x,u) \in T_1M$, the Levi Civita connection $\tilde{\nabla}$ associated to $\tilde{G}$ is given by*

$$\begin{aligned} (\tilde{\nabla}_{X^h} Y^h)_{(x,u)} = \Big\{ & (\nabla_X Y)_x - \frac{ab}{2\alpha}[R(X_x,u)Y_x + R(Y_x,u)X_x] + \frac{bd}{2\alpha}[g(X_x,u)Y_x + g(Y_x,u)X_x] \\ & + \frac{b}{(a+c+d)\alpha}[(ad+b^2)\, g(R(X_x,u)Y_x,u) \\ & - d(a+c+d)\, g(X_x,u)g(Y_x,u)]u\Big\}^h + \Big\{ \frac{b^2}{\alpha}\, R(X_x,u)Y_x \\ & - \frac{a(a+c)}{2\alpha}\, R(X_x,Y_x)u - \frac{(a+c)d}{2\alpha}\, [g(Y_x,u)\, X_x + g(X_x,u)\, Y_x] \\ & + \frac{1}{\alpha}[-b^2\, g(R(X_x,u)Y_x,u) + \ d(a+c)\, g(Y_x,u)g(X_x,u)]u\Big\}^{t_G}, \end{aligned}$$

$$\begin{aligned} (\tilde{\nabla}_{X^h} Y^{t_G})_{(x,u)} = \Big\{ & -\frac{a^2}{2\alpha}\, R(Y_x,u)X_x - \frac{ab^2}{2(a+c+d)\alpha} g(Y_x,u)R(X_x,u)u + \frac{ad}{2\alpha}\, g(X_x,u)\, Y_x \\ & + \frac{db^2}{2(a+c+d)\alpha} g(Y_x,u)X_x + \frac{1}{2(a+c+d)\alpha}\, [a(ad+b^2)\, g(R(X_x,u)Y_x,u) \\ & + d\alpha\, g(X_x,Y_x) - \ ad(2(a+c)+d)\, g(X_x,u)g(Y_x,u)]u\Big\}^h \\ & + \Big\{ (\nabla_X Y)_x + \frac{ab}{2\alpha}\, R(Y_x,u)X_x - \frac{b(\alpha - b^2)}{2(a+c+d)\alpha} g(Y_x,u)R(X_x,u)u \\ & - \frac{bd}{2\alpha}\, g(X_x,u)\, Y_x - \frac{(a+c)bd}{2(a+c+d)\alpha} g(Y_x,u)X_x \\ & + \frac{b}{2(a+c+d)\alpha}\, [-a(a+c+d)\, g(R(X_x,u)Y_x,u) \\ & + d(2(a+c)+d)\, g(X_x,u)g(Y_x,u)]u\Big\}^{t_G}, \end{aligned}$$

$$
\begin{aligned}
(\tilde{\nabla}_{X^{t_G}}Y^h)_{(x,u)} = &\Big\{-\frac{a^2}{2\alpha}\,R(X_x,u)Y_x+\frac{b}{a+c+d}g(X_x,u)\,\nabla_u Y+\frac{ad}{2\alpha}\,g(Y_x,u)\,X_x\\
&-\frac{ab^2}{2(a+c+d)\alpha}g(X_x,u)R(Y_x,u)u+\frac{db^2}{2(a+c+d)\alpha}g(X_x,u)\,Y_x\\
&+\frac{1}{2(a+c+d)\alpha}\,[a(ad+b^2)\,g(R(X_x,u)Y_x,u)\\
&+d\alpha\,g(X_x,Y_x)-\ ad(2(a+c)+d)\,g(X_x,u)g(Y_x,u)]u\Big\}^h\\
&+\Big\{\frac{ab}{2\alpha}\,R(X_x,u)Y_x+\frac{a(a+c)b}{2(a+c+d)\alpha}g(X_x,u)R(Y_x,u)u-\frac{bd}{2\alpha}\,g(Y_x,u)\,X_x\\
&-\frac{(a+c)bd}{2(a+c+d)\alpha}g(X_x,u)\,Y_x+\frac{b}{2(a+c+d)\alpha}\,[-a(a+c+d)\,g(R(X_x,u)Y_x,u)\\
&+d(2(a+c)+d)\,g(X_x,u)g(Y_x,u)]u\Big\}^{t_G},\\
(\tilde{\nabla}_{X^{t_G}}Y^{t_G})_{(x,u)} = &\frac{b}{2(a+c+d)\alpha}\Big\{-a^2[g(X_x,u)R(Y_x,u)u\ +g(Y_x,u)R(X_x,u)u]\\
&+ad\,g(X_x,u)\,Y_x\ +(2\alpha+ad)\,g(Y_x,u)\,X_x-\frac{2d\phi}{a+c+d}\,g(X_x,u)g(Y_x,u)u\Big\}^h\\
&+\frac{1}{2(a+c+d)\alpha}\Big\{\,2b\alpha g(X_x,u)\nabla_u Y+ab^2[g(X_x,u)R(Y_x,u)u\\
&+g(Y_x,u)R(X_x,u)u]-b^2d\,g(X_x,u)\,Y_x\\
&-(2(a+c+d)\alpha-b^2d)\,g(Y_x,u)\,X_x\,\Big\}^{t_G},
\end{aligned}
$$

for all $(x,u)\in T_1M$ and X, Y vector fields on M.

Convention 1.6.1. *Note that, for $(x,u)\in T_1M$, the tangential lift of the vector u to (x,u) is expressed as $u^{t_G}=\frac{b}{a+c+d},u^h$, signifying it as a horizontal vector. Consequently, the tangent space $(T_1M)_{(x,u)}$ aligns with the set:*

$$\{X^h+Y^{t_G}/X\in M_x, Y\in\{u\}^{\perp}\subset M_x\}. \tag{1.28}$$

Therefore, the tangential lift operation from M_x to a point $(x,u)\in T_1M$ is exclusively applied to vectors in M_x that are orthogonal to u.

Next, we characterize the Riemannian curvature of $(T_1M,\tilde{G})$ by the following

Proposition 1.6.3 ([4])**.** *Let (M,g) be a Riemannian manifold and let $\tilde{G}=a.\widetilde{g^s}+b.\widetilde{g^h}+c.\widetilde{g^v}+d.\widetilde{k^v}$ be an arbitrary Riemannian g-natural metric on T_1M, where a, b, c and d are constants such that $a>0$, $\alpha:=a(a+c)-b^2>0$, and $\phi:=a(a+c+d)-b^2>0$. Denote by ∇ and R the Levi-Civita connection and the Riemannian curvature tensor of (M,g), respectively. If*

we denote by $\tilde{R}$ the Riemannian curvature tensor of $(T_1M, \tilde{G})$, then:

$$
\begin{aligned}
&(i)\tilde{R}(X^h, Y^h)Z^h \\
&\quad = \Big\{ R(X,Y)Z + \frac{ab}{2\alpha}\left[2(\nabla_u R)(X,Y)Z - (\nabla_Z R)(X,Y)u\right] \\
&\quad + \frac{a^2}{4\alpha}\left[R(R(Y,Z)u,u)X - R(R(X,Z)u,u)Y - 2R(R(X,Y)u,u)Z\right] \\
&\quad + \frac{a^2b^2}{4\alpha^2}\left[R(X,u)R(Y,u)Z - R(Y,u)R(X,u)Z\right. \\
&\quad \left. + R(X,u)R(Z,u)Y - R(Y,u)R(Z,u)X\right] \\
&\quad + \frac{ad(\alpha - b^2)}{4\alpha^2}\left[g(Z,u)R(X,Y)u + g(Y,u)R(X,u)Z - g(X,u)R(Y,u)Z\right] \\
&\quad + \frac{ab^2}{2\alpha^2}\left[-\frac{ad+b^2}{a+c+d}\,g(R(Y,u)Z,u) + d\,g(Y,u)g(Z,u)\right] R_u X \\
&\quad - \frac{ab^2}{2\alpha^2}\left[-\frac{ad+b^2}{a+c+d}\,g(R(X,u)Z,u) + d\,g(X,u)g(Z,u)\right] R_u Y \\
&\quad + \frac{d}{4\alpha}\left[-\frac{2b^2}{a+c+d}\,g(R(Y,u)Z,u) + d\,g(Y,u)g(Z,u)\right] X \\
&\quad - \frac{d}{4\alpha}\left[-\frac{2b^2}{a+c+d}\,g(R(X,u)Z,u) + d\,g(X,u)g(Z,u)\right] Y \\
&\quad + \frac{d}{4\alpha(a+c+d)}\{-4abg((\nabla_u R)(X,Y)Z,u) + a^2\left[g(R(Y,Z)u, R(X,u)u)\right. \\
&\quad \left. - g(R(X,Z)u, R(Y,u)u) - 2g(R(X,Y)u, R(Z,u)u)\right] \\
&\quad + \frac{a^2b^2}{\alpha}\left[g(R(Y,u)Z + R(Z,u)Y, R(X,u)u)\right. \\
&\quad \left. - g(R(X,u)Z + R(Z,u)X, R(Y,u)u)\right] \\
&\quad - \left[\frac{ad(b^2-\alpha)}{\alpha} + \frac{2b^2(2(a+c)+d)}{a+c+d}\right]\left[g(X,u)g(R(Y,u)Z,u)\right. \\
&\quad \left. - g(Y,u)g(R(X,u)Z,u)\right] - 3a(a+c)\,g(R(X,Y)Z,u) \\
&\quad + (a+c)d\left[g(X,u)g(Y,Z) - g(Y,u)g(X,Z)\right]\}u\Big\}^h
\end{aligned}
$$

$$
\begin{aligned}
&+\Big\{-\frac{b^2}{\alpha}\,(\nabla_u R)(X,Y)Z+\frac{a(a+c)}{2\alpha}(\nabla_Z R)(X,Y)u\\
&-\frac{ab}{4\alpha}\,[R(R(Y,Z)u,u)X-R(R(X,Z)u,u)Y-2R(R(X,Y)u,u)Z\\
&-R(X,R(Y,u)Z)u-R(X,R(Z,u)Y)u\\
&+R(Y,R(X,u)Z)u+R(Y,R(Z,u)X)u]\\
&-\frac{ab^3}{4\alpha^2}\,[R(X,u)R(Y,u)Z-R(Y,u)R(X,u)Z\\
&+R(X,u)R(Z,u)Y-R(Y,u)R(Z,u)X]\\
&-\frac{bd(3\alpha-b^2)}{4\alpha^2}\,[g(Z,u)R(X,Y)u+g(Y,u)R(X,u)Z-g(X,u)R(Y,u)Z]\\
&+\frac{b(b^2-\alpha)}{2\alpha^2}\,\left[\frac{ad+b^2}{a+c+d}\,g(R(Y,u)Z,u)-d\,g(Y,u)g(Z,u)\right]R_uX\\
&-\frac{b(b^2-\alpha)}{2\alpha^2}\,\left[\frac{ad+b^2}{a+c+d}\,g(R(X,u)Z,u)-d\,g(X,u)g(Z,u)\right]R_uY\\
&+\frac{(a+c)bd}{2\alpha(a+c+d)}\,[g(R(Y,u)Z,u)X-g(R(X,u)Z,u)Y]\Big\}^{t_G},
\end{aligned}
$$

$(ii)\tilde{R}(X^h,Y^{t_G})Z^h$

$$
\begin{aligned}
&=\Big\{-\frac{a^2}{2\alpha}\,(\nabla_X R)(Y,u)Z+\frac{ab}{2\alpha}[R(X,Y)Z+R(Z,Y)X]\\
&+\frac{a^3b}{4\alpha^2}\,[R(X,u)R(Y,u)Z-R(Y,u)R(X,u)Z-R(Y,u)R(Z,u)X]\\
&+\frac{a^2bd}{4\alpha^2}\,[g(X,u)R(Y,u)Z-g(Z,u)R(X,Y)u]\\
&-\frac{ab}{4\alpha^2(a+c+d)}\,[a(ad+b^2)\,g(R(Y,u)Z,u)+\alpha d\,g(Y,Z)]R_uX\\
&+\frac{a^2b}{2\alpha^2}\,\left[\frac{ad+b^2}{a+c+d}\,g(R(X,u)Z,u)-d\,g(X,u)g(Z,u)\right]R_uY\\
&-\frac{bd}{4\alpha(a+c+d)}\,[a\,g(R(Y,u)Z,u)+(2(a+c)+d)\,g(Y,Z)]X\\
&+\frac{b}{\alpha}\,\left[-\frac{ad+b^2}{2(a+c+d)}\,g(R(X,u)Z,u)+d\,g(X,u)g(Z,u)\right]Y\\
&-\frac{bd}{2\alpha}\,g(X,Y)Z+\frac{d}{4\alpha(a+c+d)}\Big\{2a^2\,g((\nabla_X R)(Y,u)Z,u)\\
&+\frac{a^3b}{\alpha}\,[g(R(Y,u)Z,R(X,u)u)-g(R(X,u)Z+R(Z,u)X,R(Y,u)u)]\\
&+ab\,\left[-\frac{\alpha+\phi}{\alpha}+\frac{d}{a+c+d}\right]\,g(X,u)g(R(Y,u)Z,u)\\
&-2ab\,[2g(R(X,Y)Z,u)+g(R(Z,Y)X,u)]\\
&+bd\,\left[\left(3-\frac{d}{a+c+d}\right)\,g(X,u)g(Y,Z)+2\,g(Z,u)g(X,Y)\right]\Big\}\,u\Big\}^h
\end{aligned}
$$

$$
\begin{aligned}
&+\left\{\frac{ab}{2\alpha}\,(\nabla_X R)(Y,u)Z+\frac{a^2}{4\alpha}\,R(X,R(Y,u)Z)u\right.\\
&-\frac{a^2b^2}{4\alpha^2}\,[R(X,u)R(Y,u)Z-R(Y,u)R(X,u)Z-R(Y,u)R(Z,u)X]\\
&-\frac{b^2}{\alpha}\,R(X,Y)Z+\frac{a(a+c)}{2\alpha}R(X,Z)Y\\
&+\frac{ad(\alpha-b^2)}{4\alpha^2}\,[g(X,u)R(Y,u)Z-g(Z,u)R(X,Y)u]\\
&-\frac{\alpha-b^2}{4\alpha^2(a+c+d)}\,[a(ad+b^2)\,g(R(Y,u)Z,u)+\alpha d\,g(Y,Z)]R_uX\\
&+\frac{ab^2}{2\alpha^2}\,\left[-\frac{ad+b^2}{a+c+d}\,g(R(X,u)Z,u)+d\,g(X,u)g(Z,u)\right]R_uY\\
&+\frac{(a+c)d}{4\alpha(a+c+d)}\,[a\,g(R(Y,u)Z,u)+(2(a+c)+d)\,g(Y,Z)]X\\
&+\frac{1}{4\alpha}\,\left[2b^2\left(2-\frac{d}{a+c+d}\right)\,g(R(X,u)Z,u)\right.\\
&\left.\left.-d(4(a+c)+d)\,g(X,u)g(Z,u)\right]Y+\frac{(a+c)d}{2\alpha}\,g(X,Y)Z\right\}^{t_G},
\end{aligned}
$$

$$
\begin{aligned}
(iii)\tilde{R}(X^{t_G},Y^{t_G})Z^{t_G}=&\frac{1}{2\alpha(a+c+d)}\left\{\left\{a^2b\;[g(Y,Z)R_uX-g(X,Z)R_uY]\right.\right.\\
&\left.-b(\alpha+\phi)[g(Y,Z)X-g(X,Z)Y]\right\}^h\\
&+\left\{-ab^2\;[g(Y,Z)R_uX-g(X,Z)R_uY]\right.\\
&\left.\left.+[(a+c)(\alpha+\phi)+\alpha d]\,[g(Y,Z)X-g(X,Z)Y]\right\}^{t_G}\right\},
\end{aligned}
$$

for all $x\in M$, $(x,u)\in T_1M$ and all arbitrary vectors X, Y and $Z\in M_x$ satisfying Convention 1.6.1*, where $R_uX=R(X,u)u$ denotes the* Jacobi operator *associated to u.*

1.7 The induced connection on $\phi^{-1}TN$:

Induced bundle:

Given a vector bundle $\pi : E\longrightarrow N$ and a mapping $\phi : M\longrightarrow N$, where M and N are manifolds, we can construct the vector bundle $\pi^{'} : E^{'}\longrightarrow M$, where

$$E^{'}:=\{(x,v)\in M\times E/\phi(x)=\pi(v)\}$$

and $\pi^{'}((x,v)):=x$, for all $(u,v)\in E^{'}$.

That is, the fiber of this bundle over $x\in M$ is the fiber $E_{\phi(x)}$ of E over $\phi(x)$.
We denote $E^{'}$ by $\phi^{-1}E$ and call it **the induced vector bundle of E by ϕ.**

A section s of the induced bundle $\phi^{-1}E$ of E by $\phi : M\longrightarrow N$ is by definition a mapping $s : M\longrightarrow E$ satisfying $s(x)\in E_{\phi(x)}$.

The induced connection on $\phi^{-1}TN$:

Let (M,g) and (N,h) be two Riemannian manifolds and ∇ and $\bar{\nabla}$, respectively, be the Levi-Civita connections of (M,g) and (N,h).

Let $\phi^{-1}TN$ be the induced bundle over M of TN by $\phi : M \longrightarrow N$, that is, let $\pi : TN \longrightarrow N$, the projection, and let

$$\phi^{-1}TN = \{(x, v) \in M \times TN / \phi(x) = \pi(v)\} = \bigcup_{x \in M} T_{\phi(x)}N.$$

$\phi^{-1}TN$ admits a fiber metric induced from the Riemannian metric h on N, denoted by the same letter h, defined by:

$$\begin{aligned} h_x : (TN)_x \times (TN)_x &\longrightarrow \mathbb{R} \\ (u, v) &\longrightarrow h_{\phi(x)}(u, v) \end{aligned} \tag{1.29}$$

Let $\phi \in C^\infty(M, N)$, we can define the *induced connection* $\widehat{\nabla}$ on the induced bundle $\phi^{-1}TN$ as follows:
For $X \in \mathfrak{X}(M)$, $V \in \Gamma(\phi^{-1}TN)$, define $\widehat{\nabla}_X V \in \Gamma(\phi^{-1}TN)$ by:

$$(\widehat{\nabla}_X V)(x) = \bar{\nabla}_{\phi_* X} V := \frac{d}{dt}\Big|_{t=0} P^{-1}_{\phi \circ \sigma_t} V(\sigma(t)), \tag{1.30}$$

for all $x \in M$.

where $t \longrightarrow \sigma(t) \in M$ is C^1-curve in M satisfying $\sigma(0) = x$, $\sigma^{'}(0) = X_x \in T_x M$, and σ_t is a curve given by $\sigma_t(s) := \sigma(s)$, $0 \leq s \leq t$, i.e., the restriction of σ to the part between x and $\sigma(t)$. $P_{\phi \circ \sigma_t} : T_{\phi(x)}N \longrightarrow T_{\phi(\sigma(x))}N$ is the parallel transport along the C^1-curve $\phi \circ \sigma_t$ with respect to the Levi-Civita connection $\bar{\nabla}$ on (N, h).

The induced connection $\widehat{\nabla}$ is compatible with this fiber metric h on $\phi^{-1}TN$.

Chapter 2

Harmonic and g-natural harmonic vector fields

In this chapter, we continue our exploration of materials, focusing on one of the fundamental concepts in our study: harmonic maps. We introduce the energy functional and derive the first variational formula, which leads to the Euler-Lagrange equation associated with the problem, the tension field, and various important propositions and examples.

Next, we delve into the notion of harmonic vector fields, which correspond to the critical points of the energy functional restricted to vector fields.

We will recall main results, from the literature, at first when the tangent bundle is equipped with the Sasaki metric and then when it is equipped with an arbitrary g-natural metric. In both cases, we will give examples of special harmonic vector fields such as Killing vector field, Hopf vector field and Reeb vector field.

2.1 Harmonic maps

2.1.1 Definitions

Let (M, g) be an m-dimensional compact Riemannian manifold and (N, h) be an n-dimensional pseudo-Riemannian manifold. If $f : (M, g) \to (N, h)$ is a smooth map , then the *energy* (*energy functional*) of f is defined as the integral

$$E(f) := \int_M e(f) dv_g,$$

where $e(f) = \frac{1}{2}||f_*||^2 = \frac{1}{2}\mathrm{tr}_g f^*h$ is the so-called *energy density* of f. With respect to a local orthonormal frame $\{e_1, .., e_n\}$ on M, it is possible to express the energy density as

$$\begin{aligned} e(f) = \frac{1}{2}||f_*||^2 &= \frac{1}{2}\mathrm{tr}_g f^*h \\ &= \frac{1}{2}\sum_{i=1}^{m}(f^*h)(e_i, e_i) \\ &= \frac{1}{2}\sum_{i=1}^{m} h(f_*e_i, f_*e_i), \end{aligned} \tag{2.1}$$

where f^*h is the pull back of h by f, $tr_g(f^*h)$ is the trace of the tensor field f^*h with respect to g, and we denote by f_* the differentiation of f, which can be iterpreted as an $f^{-1}TN$-valued 1-form on M.
We call the covariant differtential $\nabla(f_*)$ the second fundamental form of the map f.

Take coordinate systemes $(U, x^1, ..., x^m)$ and $(y^1, ..., y^n)$ of M and Nn respectively such that $f(U) \subset V$. Writing $x = (x^1, ..., x^m)$, $f(x) = y = (y^1, ..., y^n)$ we represent the tensor $g(x) = \sum_{i,j=1}^m g_{ij}(x)dx^i dx^j$ with $g_{ij} = g\big(\frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j}\big)$. Similarly, $h(y) = h_{\alpha\beta}(y)dy^\alpha dy^\beta$ with $h_{\alpha\beta} = h\big(\frac{\partial}{\partial y^\alpha}, \frac{\partial}{\partial y^\beta}\big)$. The differential $df(x)$ has the matrix represantation $\big(\frac{df^\alpha}{\partial x^i}\big)$. The first and second fundemental forms of f at $x \in U$ can be expressed as

$$(f^*h)_{ij} = \frac{\partial f^\alpha}{\partial x^i}\frac{\partial f^\beta}{\partial x^j}h_{\alpha\beta},$$

$$\big(\nabla(f_*)\big)^\gamma_{ij} = \frac{\partial^2 f^\gamma}{\partial x^i \partial x^j} - \Gamma^k_{ij}\frac{\partial f^\gamma}{\partial x^k} + {}^N\Gamma^\gamma_{\alpha\beta}\frac{\partial f^\alpha}{\partial x^i}\frac{\partial f^\beta}{\partial x^j},$$

where Γ^k_{ij}, ${}^N\Gamma^\gamma_{\alpha\beta}$ are the Christoffel symbols of (M, g), (N, h), respectively.
Also (2.9) can be expressed as

$$e(f) = \frac{1}{2}\sum_{i,j,\alpha,\beta} g^{ij}\frac{\partial f^\alpha}{\partial x^i}\frac{\partial f^\beta}{\partial x^j}h_{\alpha\beta},$$

where (g^{ij}) is the inverse matrix of (g_{ij}).

Definition 2.1.1. *We call $f \in C^\infty(M, N)$ a harmonic map if f is a critical point of E on $C^\infty(M, N)$, i.e. for any smooth variation $f_t \in C^\infty(M, N)$ with $-\epsilon < t < \epsilon$ of f, we have*

$$\frac{d}{dt}\Big|_{t=0} E(f_t) = 0.$$

Here a smooth variation f_t means that $f_0 = f$ and f_t depends smoothly on t, that is the map $F : (-\epsilon, \epsilon) \times M \longrightarrow N$, defined by

$$F(t, x) := f_t(x), \quad (t, x) \in (-\epsilon, \epsilon) \times M,$$

satisfies

$$\begin{cases} F(0, x) = f(x), \quad x \in M, \\ F : (-\epsilon, \epsilon) \times M \longrightarrow N \text{ is a } C^\infty - \text{map}. \end{cases}$$

Definition 2.1.2. *For any smooth variation f_t of f, putting:*

$$V(x) := \frac{d}{dt}\Big|_{t=0} f_t(x), \quad x \in M.$$

Then V is a smooth map of M into the tangent bundle TN satisfying

$$V(x) \in T_{f(x)}N, \quad x \in M.$$

*Such a vector field is called a **variation vector field** along f.*

Theorem 2.1.1. *Let $f \in C^\infty(M, N)$. For any smooth variation f_t, $-\epsilon < t < \epsilon$, of f, let $V(x) := \frac{d}{dt}\Big|_{t=0} f_t(x)$, $x \in M$, then*

$$\frac{d}{dt}\Big|_{t=0} E(f_t) = -\int_M h\big(V, \tau(f)\big)v_g,$$

where

$$\tau(f) = tr_g\nabla(df) = \sum_{i=1}^n \big(\widehat{\nabla}_{e_i} f_* e_i - f_*\nabla_{e_i} e_i\big) \in \Gamma(f^{-1}TN) \tag{2.2}$$

*$\tau(f)$ is called the **tension field** of f. Thus, $f \in C^\infty(M, N)$ is a harmonic map if and only if*

$$\tau(f) = 0, \quad \text{everywhere on } M. \tag{2.3}$$

*(2.3) is called the **Euler-Lagrange equation**.*

The tension field $\tau(f)$ can be expressed using the local coordinates $(x^1, ..., x^m)$ and $(y^1, ..., y^n)$ in M and Nn respectively as follows. Let $f^\alpha = y_\alpha \circ f$, and write $\tau(f)(x) \in T_{f(x)}N$ as

$$\tau(f)(x) = \sum_{\gamma=1}^{n} \tau(f)^\gamma \frac{\partial}{\partial y_\gamma},$$

then

$$\begin{aligned}\tau(f)^\gamma(x) &= \sum_{i,j=1}^{m} g^{ij} \left\{ \frac{\partial^2 f^\gamma}{\partial x^i \partial x^j} - \sum_{k=1}^{m} \Gamma_{ij}^k(x) \frac{\partial f^\gamma}{\partial x^k} + \sum_{\alpha,\beta=1}^{n} {}^N\Gamma_{\alpha\beta}^\gamma(f(x)) \frac{\partial f^\alpha}{\partial x^i} \frac{\partial f^\beta}{\partial x^j} \right\} \\ &= -\Delta f^\gamma + \sum_{i,j,\alpha,\beta} {}^N\Gamma_{\alpha\beta}^\gamma(f(x)) \frac{\partial f^\alpha}{\partial x^i} \frac{\partial f^\beta}{\partial x^j}. \end{aligned} \tag{2.4}$$

2.1.2 Examples and propositions

Exemples 2.1.1. *1. (Constant map) For two compact Riemannian manifolds (M, g) and (N, h) and a fixed $q \in N$, any constant map*

$$\begin{array}{rccc} \phi: & M & \longrightarrow & N \\ & x & \longmapsto & q \end{array} \quad \textit{is a harmonic map.}$$

2. (Identity map) The identity map $id_M : (M, g) \longrightarrow (M, g)$ is trivially harmonic.

3. (Harmonic function) Let $(N, h) = (\mathbb{R}^n, g_0)$, denoting $f = (f_1, ..., f_n) \in C^\infty(M, \mathbb{R}^n)$, we have

$\tau(f) = -\Delta f = -(\Delta f_1, ..., \Delta f_n)$. Then

$$f : (M, g) \to (\mathbb{R}^n, g_0) \textit{ is harmonic} \Leftrightarrow f_i \textit{ is harmonic i.e. } \Delta f_i = 0,\ 1 \le i \le n.$$

4. (Geodesic) Let (N, h) be any (compact) Riemannian manifold, and $M = \mathbb{R}/\mathbb{Z} = \mathbb{S}^1$. Then $C^\infty(\mathbb{S}^1, N)$ is the set of all periodic C^∞-curves in N with period 1.
Let $e_1 = \frac{\partial}{\partial x}$, then $\nabla_{e_1} e_1 = 0$. Put $f' = f^* e_1$, then

$$\tau(f) = 0 \ \Leftrightarrow\ \widehat{\nabla}_{e_i} f_* e_i = 0 \ \Leftrightarrow\ {}^N\nabla_{f'} f' = 0.$$

Hence harmonic periodic curves in N are exactely geodesic curves.

5. (Minimal isometric immersion) Let $f : (M, g) \to (N, g)$ be an isometric immersion. Then the mean curvature vector $\mathcal{H}$ is given by $\mathcal{H} = \frac{1}{m} tr_g \nabla(df)$.
f is a harmonic map if and only if is minimal i.e. $\mathcal{H} = \frac{1}{m}\tau(f) = 0$, with $m = dim(M)$.

Remark 2.1.1. *In the non compact case we can define the harmonicity of a map by the condition (2.3), since it has a tensorial character.*

Proposition 2.1.1 (Unique Contiuation Theorem [24])**.** *Let $f : (M, g) \to (N, h)$ be a harmonic map which is constant on an open subset of M. Then f is constant on M.*

Proposition 2.1.2. *[24] Let $f : (M, g) \to (N, h)$ be a harmonic map and suppose that the Ricci tensor $Ric^M \ge 0$ and the sectional curvature ${}^N K \le 0$. Then*

1. *f is totally geodesic;*
2. *if Ric^M is strictly positive definitive at some point, then f is constant;*
3. *if ${}^N K < 0$, then f is either constant or of rank 1, in wich case its image is a closed geodesic.*

Proposition 2.1.3. *If (M,g), (N,h) and (P,k) are three Riemannian manifolds and $f \in C^\infty(M,N)$ and $g \in C^\infty(N,P)$, then*

$$\tau(g\circ f) = dg\circ\tau(f) + tr\nabla dg(df,df)$$

In particular, if f and g are totally geodesic, so is $g\circ f$, and if f is harmonic and g is totally geodesic , then $g\circ f$ is harmonic.
It is important to note that the composition of two harmonic maps is not harmonic in general.

Existence problem

The basic existence problem for harmonic maps can be formulated in the following manner: Let $f_0 : (M,g) \to (N,h)$ be a map of Riemannian manifolds. Can f_0 be deformated into a harmonic map of (M,g) into (N,h)?

Theorem 2.1.2 (Eelles-Sampson [26], 1964)**.** *Let (M,g) and (N,h) be compact Riemannian manifolds, and let (N,h) have nonpositive curvature, that is, the sectional curvature ${}^NK$ of (N,h) satisfies*

$${}^NK(u,v) \le 0, \quad \forall u,v \in T_qN,\ \forall q \in N.$$

Then for any $g \in C^\infty(M,N)$, there exists a harmonic map $f : (M,g) \to (N,h)$ satisfying

i. $f \sim \phi$ i.e. f is homotopic to ϕ.

ii. f is energy minimising in its homotopy class, i.e.: $\forall \tilde{f} \sim f : E(f) \le E(\tilde{f})$.

The method of proof in [26] is the so-called "(nonlinear) heat equation method", which is as follows:
They consider the following nonlinear heat equation for a given map $\phi \in C^\infty(M,N)$

$$\begin{cases} \frac{\partial f_t}{\partial t} = \tau(f_t) \\ f_0 = \phi \end{cases}, \tag{2.5}$$

for a 1-parameter family of $f_t \in C^\infty(M,N)$ with $f_0 = \phi$. Here $\tau(f_t)$ is the tension field of f_t. They showed if (N,h) has nonnegative curvature, then

1. there exists a unique solution f_t of (2.5) for all $0 \le t \le \infty$, and
2. as $t \to \infty$, the limit exists, $\lim_{t\to\infty} f_t = f_\infty$, and
3. the limit map $f = f_\infty$ is the desired harmonic map of (M,g) into (N,h).

The motivation for choosing the equation (2.5) is the following:
By the first variation formula of the energy (2.1.1), for any variation g_t of g, by defining the variation vector field $V(x) = \frac{d}{dt}\Big|_{t=0}\phi_t(x) \in T_{f(x)}N$, $x \in M$, we get

$$\frac{d}{dt}\Big|_{t=0} E(\phi_t) = -\int_M h\big(V, \tau(f)\big)v_g. \tag{2.6}$$

Since the left-hand side of (2.6) coincides with $dE_\phi(V)$, the differentiation of the function E on $C^\infty(M,N)$ at g with respect to the direction V, (2.6) can be writen as

$$dE_\phi(V) = \int_M h\big(V, -\tau(f)\big)v_g \tag{2.7}$$

$$= \big(V, -\tau(f)\big), \tag{2.8}$$

where $\big(\ ,\ \big)$ denotes the global inner product of $\Gamma(f^{-1}TN)$:

$$\big(V,W\big) := \int_M h\big(V,W\big)v_g, \quad V,\ W \in\ \Gamma(f^{-1}TN).$$

Then the gradient vector field $(\nabla E)_\phi$ of the function E on $C^\infty(M,N)$ at ϕ is just $-\tau(\phi)$. Now we should deform g in order to decrease the energy. For this, we may take an integral curve on $C^\infty(M,N)$ of minus the gradient vector field.
We denote this integral curve by $f_t \in C^\infty(M,N)$, then the equation is

$$\frac{d}{dt}f_t = -(\nabla E)_{f_t}, \quad f_0 = \phi.$$

Now from the tow last equations, a deformation f_t of ϕ decreasing the energy E is equivalent to an integral curve f_t of $-\nabla E$ through ϕ, which in turn is the same as the heat equation problem (2.5) with $f_0 = \phi$.
So deforming ϕ in this way, then at last if $t \to \infty$, the limit map f_∞ would be energy minimizing. This procedure is a key idea of the proof.

2.1.3 Stability of harmonic maps

Let $f : (M,g) \to (N,h)$ be a harmonic map, and let $f_{s,t} : M \to N$ be a smooth variation with tow parameters s, t, where $f_{0,0} = f$. That is, the map F defined by

$$F : (s,t,x) \in (-\epsilon,\epsilon) \times (-\epsilon,\epsilon) \times M \to f_{s,t}(x) \in N,$$

is a C^∞-map and $F(0,0,x) = f(x)$, $x \in M$. The corresponding variation vector fields are denoted by V, W:

$$V = \frac{\partial f_{s,t}}{\partial s}\big|_{t=0}, \qquad W = \frac{\partial f_{s,t}}{\partial t}\big|_{s=0}.$$

We extend vector fields on the interval $(-\epsilon,\epsilon)$ or M to the product manifold $(-\epsilon,\epsilon) \times (-\epsilon,\epsilon) \times M$, and we denote them also by the same notations. Then by definition, we get

$$V = F_*\big(\frac{\partial}{\partial s}\big)_{(s,t)=(0,0)}, \qquad W = F_*\big(\frac{\partial}{\partial t}\big)_{(s,t)=(0,0)}.$$

The Hessian of E at a critical point f is defined as follows:

$$H(E)_f(V,W) = \frac{\partial^2 E(f_{s,t})}{\partial s \partial t}\Big|_{(0,0)}.$$

Theorem 2.1.3 (the second variation formula)**.** *Let $f : M \longrightarrow N$ be a harmonic map. Then the Hessian of the energy E at f is given by*

$$H(E)_f(V,W) = \int_M h\big(J_f(V),W\big)v_g,$$

where

$$J_f(V) := -\sum_{i=1}^m \big(\widehat{\nabla}_{e_i}\widehat{\nabla}_{e_i}V - \widehat{\nabla}_{\nabla_{e_i}e_i}V\big) - \sum_{i=1}^m R^N(V, f_*e_i)f_*e_i.$$

*J_f is called the **Jacobi operator**.*

Definition 2.1.3. *Let $f : M \longrightarrow N$ be a harmonic map. f is said to be **weakly stable** if*

$$H(E)_f(V,V) \geq 0, \qquad \textit{for all } V\ \in\ \Gamma(f^{-1}TN),$$

*and otherwise, it is said to be **unstable**.*

If (N,h) has the nonpositive sectional curvature, then

$$h\Big(\sum_{i=1}^{m} R^N(V,f_*e_i)f_*e_i,V\Big)\leq 0,\quad for\ all\ V\in\Gamma(f^{-1}TN),$$

and also we get

$$\int_M h\Big(-\sum_{i=1}^{m}\big(\widehat{\nabla}_{e_i}\widehat{\nabla}_{e_i}V-\widehat{\nabla}_{\nabla_{e_i}e_i}V\big),V\Big)v_g=\int_M h(\widehat{\nabla}V,\widehat{\nabla}V)v_g\geq 0.$$

We obtain

$$\int_M h\big(J_f(V),W\big)v_g\geq 0,\quad for\ all\ V\in\Gamma(f^{-1}TN)$$

Thus, we have

Proposition 2.1.4. *if (N,h) has nonpositive sectional curvature, then any harmonic map $f:(M,g)\longrightarrow(N,h)$ is weakly stable.*

2.2 Harmonic vector fields

In this section, we consider a smooth vector field $X:M\longrightarrow TM$ as a mapping between Riemannian manifolds (M,g) and (TM,G_s), where G_s represents the Sasaki metric. We then use the theory of variational calculus for the functional $E(X)=\frac{1}{2}\int_M||dX||^2dv_g$, where dX is the Hilbert-Shmidt norm of dX and dv_g denotes the volume element with respect to the metric g.

2.2.1 The energy of a vector field

Consider an m-dimensional compact oriented Riemannian manifold (M,g) and a tangent vector field $X:M\longrightarrow TM$ defined on M. Let ∇ denote the Levi-Civita connection associated with the Riemannian manifold (M,g) and let G_s denote the Sasaki metric on the tangent bundle TM.

Definition 2.2.1. *[22] The energy functional is the map $E:\mathfrak{X}(M)=\Gamma^{\infty}(M,TM)\longrightarrow[0,+\infty)$ given by*

$$E(X)=\int_M e(X)dv_g,\quad e(X)=\frac{1}{2}||dX||^2,$$

for any $X\in\mathfrak{X}(M)$. Here $||dX||$ is the Hilbert-Shmidt norm of dX i.e.,

$$||dX||^2=tr_g(X^*G_s)$$

and dv_g is the Riemannian volume on M. the Hilbert-Shmidt norm is defined by:

$$\begin{aligned}||dX||^2(x)&=\sum_{i=1}^{m}(X^*G_s)(e_i,e_i)_x\\&=\sum_{i=1}^{m}G_s\big((d_xX)e_{i,x},(d_xX)e_{i,x}\big),\end{aligned}\tag{2.9}$$

where $\{e_1,...,e_m\}$ is an orthonormal frame on M and $x\in M$.

Proposition 2.2.1. *Let (M,g) be an $m-$dimensional compact oriented Riemannian manifold. For any vector field $X \in \mathfrak{X}(M)$. its energy is given by*

$$E(X) = \frac{m}{2} vol(M) + \int_M ||\nabla X||^2 dv_g,$$

where $vol(M) = \int_M dv_g$ is the volume of M.

Remark 2.2.1. *Under a homothetic transformation of the given metric the energy of a vector field behaves as follows. Let $c > 0$ and $\overline{g} = cg$. If $\overline{X} = (\frac{1}{\sqrt{c}})X$ then*

$$E_{\overline{g}}(\overline{X}) = \frac{1}{c^{\frac{n}{2}}}\big(\frac{n(c-1)}{2} vol(M,g) + E_g(X)\big),$$

where we note by $E_g(X)$ the energy of X with respect to the metric g.

We can measure the failure of a vector field X on M to be parallel by defining the *total bending* of X:

Definition 2.2.2. *[51] Let (M,g) be an $m-$dimensional compact oriented Riemannian manifold. For any vector field $X \in \mathfrak{X}(M)$. We define the total bending (biegung) of X as the number*

$$\mathcal{B}(X) = \int_M ||\nabla X||^2 dv_g.$$

Corollary 2.2.1. *Let (M,g) be an $m-$dimensional compact oriented Riemannian manifold. For any vector field X on M*

$$E(X) \geq \frac{m}{2} vol(M)$$

with equality if and only if $\nabla X = 0$.

2.2.2 Vector fields which are harmonic maps

Let (M,g) an m-dimensional compact oriented Riemannian manifold, and ∇ its Levi-Civita connection associated with the Riemannian manifold (M,g) is represented by ∇.

Theorem 2.2.1. *[30], [43] Let X be a tangent vector field on M. The following statments are equivalent*

1. *$X : (M,g) \longrightarrow (TM, G_s)$ is a harmonic map;*
2. *X is an absolute minimum of the energy functional*

$$\begin{aligned} E : \mathfrak{X}(M) &\to [0,+\infty) \\ X &\to E(X) = \frac{1}{2}\int_M ||dX||^2 dv_g; \end{aligned}$$

3. *X is parallel, i.e., $\nabla X = 0$.*

Proposition 2.2.2. *Let (M,g) be a Riemannian manifold, not necessarily compact, and $X \in \mathfrak{X}(M)$ a tangent vector field on M. The the tension field $\tau(X) \in C^\infty(M, X^{-1}TTM)$ is given by*

$$\tau(X) = \Big\{\Big(- tr_g\big(R(\nabla_\cdot X, X)\cdot\big)\Big)^h - \big(\bar{\Delta}_g X\big)^v\Big\} \circ X$$

Then $X : (M,g) \to (TM, G_s)$ is a harmonic map if and only if $tr_g\big(R(\nabla_\cdot X, X)\cdot\big)\Big) = 0$ and $\bar{\Delta}_g X = 0$.

We adopt the following convention for the sign in the definition of the curvature tensor field on a Riemannian manifold (M, g):

$$R(X,Y)Z = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z,$$

where X, Y, and $Z \in \mathfrak{X}(M)$.

The differential operator $\bar{\Delta}_g$ appearing in equation (2.2.2) is referred to as the *rough Laplacian* on vector fields. It is defined as follows: Suppose $x \in M$ and $\{e_i\}_{i=1}^m$ is a local orthonormal field defined on an open neighborhood $U \subseteq M$ of x. Then, by definition, we have

$$(\bar{\Delta}_g X)_x = -\sum_{i=1}^{m} \left(\nabla_{e_i} \nabla_{e_i} X - \nabla_{\nabla_{e_i} e_i} X \right)_x, \tag{2.10}$$

where the definition of $(\bar{\Delta}_g X)_x$ remains independent of the choice of local orthonormal frame at x. The chosen sign in equation (2.10) ensures that $\bar{\Delta}_g$ is a positive operator, meaning $(\bar{\Delta}_g X, X) \geq 0$ for any $X \in \mathfrak{X}(M)$, where $(\ ,\ )$ is the L^2 inner product on $\mathfrak{X}(M)$ given by

$$(X,Y) = \int_M g(X,Y)\, dv_g,$$

for arbitrary vector fields X, Y on M.

Theorem 2.2.2. *Let (M, g) be an $m-$dimensional oriented Riemannian manifold and let $X \in \mathfrak{X}(M)$.*

i. If X is parallel then $X : (M, g) \longrightarrow (TM, g^s)$ is a harmonic map, i.e. $\tau(X) = 0$.

ii. if $\tau(X) = 0$ and M is compact, or M is complete and $*X^\flat \in L^1(\Omega^{n-1}(M))$ and $d*X^\flat \in L^1(\Omega^n(M))$, then X is parallel, where

- *$X^\flat(Y) = g(X, Y)$, for any $Y \in \mathfrak{X}(M)$.*
- *$* : \Omega^k(M) \longrightarrow \Omega^{n-k}(M)$ is the hodge operator, mapping a k-form to its dual form of degree $n - k$.*
- *$L^1\big(\Omega^k(M)\big)$ is the space of differential forms of degree k on M that have integrable norms, i.e. if $\omega \in L^1\big(\Omega^k(M)\big)$ then $\int_M ||\omega|| * 1 < \infty$.*

Remark 2.2.2. *Let g and $\bar{g}$ be two Riemannian metrics on M and G_s the Sasaki metric on TM associated to (M, g). Let $\bar{\tau}(X)$ be the tension field of $X : (M, \bar{g}) \longrightarrow (TM, G_s)$. Then [28]:*

$$\bar{\tau}(X) = [-tr\big\{R(\nabla_\cdot X, X)\cdot\big\} + \tau_{\bar{g}}(id_M)]^h + [-\bar{\Delta}_g X + (\nabla X)\tau_{\bar{g}}(id_M)]^v$$

where $\tau_{\bar{g}}(id_M)$ denotes the tension field of the identity map $id_M : (M, \bar{g}) \longrightarrow (M, g)$.

2.2.3 Variations through vector fields

The previous results show that, apart from parallel vector fields, there are no smooth vector fields $X : M \to TM$ which are harmonic maps. This is essantially due to the fact that the domain $C^\infty(M, TM)$ on which the energy functional is initially defined is excessively large. However, a similar phenomenon arises when searching for critical points of the energy functional within the smaller domain $\mathfrak{X}(M)$.

More precisely, consider a compact oriented Riemannian manifold M, and let $X \in \mathfrak{X}(M)$ be a tangent vector field on M. Then, X is a *harmonic vector field*, i.e. a critical point of the functional $E : \mathfrak{X}(M) \to [0, +\infty)$ if and only if X is a parallel vector field, i.e. $\nabla X = 0$ [28].

The first variation formula for a vector field X restricted to $\mathfrak{X}(M)$ is given as following

Theorem 2.2.3. *Let (M, g) be a compact oriented Riemannian manifold and $E : \mathfrak{X}(M) \to [0, +\infty)$ the energy functional restricted to the space of all vector fields. Then*

$$\frac{d}{dt}|_{t=0} E(X_t) = \int_M g(\bar{\Delta}_g X, V) dv_g$$

for any smooth $1-$parameter variation X_t of X through vector fields i.e. $X_t \in \mathfrak{X}(M)$, for any $|t| < \epsilon$, where tangent vector field on M given by

$$V = \lim_{t \to 0} \frac{1}{t}\{X_t(x) - X(x)\} = \frac{\partial X_t}{\partial t}|_{t=0}.$$

Now we will see that non trivial (i.e. not necessarily parallel) examples of critical points occur when the energy functional is restricted to the space of all unit vector fields.

2.2.4 Harmonic unit vector fields

Consider an m-dimensional compact Riemannian manifold (M, g). We recall that the tangent sphere bundle T_1M is defined as follows:

$$T_1M = \{u \in TM : g(u, u) = 1\}.$$

Any smooth cross-section U in the tangent sphere bundle T_1M corresponds to a unit vector field, meaning $U \in \mathfrak{X}(M)$ and $g(U, U) = 1$ holds everywhere on M. It is well-known that the existence of a global, nowhere-vanishing smooth section in $T_1M \to M$ is connected to the topological restriction $\chi(M) = 0$, where $\chi(M)$ represents the Euler-Poincaré characteristic of M.

Denoting $\tilde{G}_s$ the Riemannian metric on T_1M induced by the Sasaki metric G_s on TM, and let U be a unit vector field viewed as a mapping from (M, g) to $(T_1M, \tilde{G}_s)$. Then [29]:

Theorem 2.2.4. *Let $\tau_1(U)$ be the tension field (a smooth section in $U^{-1}TT_1M \to M$) of U as a map among the Riemannian manifolds (M, g) and (T_1M, G_s). Then*

$$\tau_1(U) = \left\{[-tr_g\{R(\nabla_{\cdot}U, U).\}]^h - [\bar{\Delta}U - ||\nabla U||^2 U]^v\right\} \circ U$$

Then U is a harmonic map if and only if $tr\{R(\nabla_{\cdot}U, U).\} = 0$ and $\bar{\Delta}U - ||\nabla U||^2 U = 0$.

Next we shall give a geometric interpretation of this two conditions in the case of a real space-form of constant sectional curvature.

Proposition 2.2.3. *Let (M, g) be a real space-form of constant sectional curvature $c \neq 0$. Let $U \in \Gamma(T_1M)$, i.e. a unit vector field on M. Then $tr\{R(\nabla_{\cdot}U, U).\} = 0$ if and only if U is geodesic (i.e. $\nabla_U U$) and the distributiin $(\mathbb{R}U)^{\perp}$ is minimal.*

We can prove that the condition $div(U) = 0$ is equivalent to the minimality of the distribution $(\mathbb{R}U)^{\perp}$, where $(\mathbb{R}U)^{\perp}$ is the orthogonal complement of $\mathbb{R}U$ in (TM, G_s).
Let $\mathcal{D}$ be a smooth distribution of rank n on M and $\mathcal{D}^{\perp}$ is orthogonal complement in TM with respect to g. Let $\pi^{\perp} : TM \to \mathcal{D}^{\perp}$ be the natural projection with respect to the decomposition $TM = \mathcal{D} \oplus \mathcal{D}^{\perp}$ and we consider the bilinear form

$$\mathcal{B}_{\mathcal{D}}(X, Y) = \pi^{\perp}\nabla_X Y, \quad X,\, Y \in \mathcal{D}.$$

We set the mean curvature vector $\mathcal{H}$ as

$$\mathcal{H} = \frac{1}{n} tr_g \mathcal{B}_{\mathcal{D}}.$$

If $\{X_a : 1 \leq a \leq n\}$ is a local orthonormal frame of $\mathcal{D}$ (i.e. $g(X_a, X_b) = \delta_{ab}$) defined on the open set $U \subseteq M$, then

$$\mathcal{H}_{\mathcal{D}} = \frac{1}{n} \sum_{a=1}^{n} \mathcal{B}_{\mathcal{D}}(X_a, X_b).$$

The distribution $\mathcal{D}$ is said to be minimal if and only if $\mathcal{H}_{\mathcal{D}} = 0$.
Now if $\mathcal{D} = (\mathbb{R}U)^{\perp}$, then

$$\pi^{\perp} X = g(X, U)U, \quad X \in TM.$$

Let $\{E_a : 1 \leq a \leq m-1\}$ be a local orthonormal frame of $\mathcal{D} = (\mathbb{R}U)^{\perp}$. Let $E_m = U$, then $\{E_a : 1 \leq a \leq m\}$ is a local orthonormal frame of TM. Hence

$$\begin{aligned}
(m-1)\mathcal{H}_{\mathcal{D}} &= \sum_{a=1}^{m-1} \mathcal{B}_{\mathcal{D}}(E_a, E_b) \\
&= \sum_{a=1}^{m-1} \pi^{\perp} \nabla_{E_a} E_a \\
&= \sum_{a=1}^{m-1} g(\nabla_{E_a} E_a, U)U \\
&= \sum_{a=1}^{m-1} \{E_a\big(g(E_a, U)\big) - g(E_a, \nabla_{E_a} U)\}U \\
&= -\sum_{a=1}^{m-1} g(E_a, \nabla_{E_a} U) = -\sum_{a=1}^{m} g(E_a, \nabla_{E_a} U) \\
&= -div(U)U
\end{aligned} \tag{2.11}$$

Finally

$$(m-1)\mathcal{H}_{\mathcal{D}} = -div(U)U,$$

and we obtain that $div(U) = 0$ if and only if $(\mathbb{R}U)^{\perp}$ is minimal.

Definition 2.2.3. *Let M be an $n-$dimensional compact orientable Riemannian manifold. A unit vector field $U \in \Gamma^{\infty}(T_1M)$ is called a harmonic unit vector field if U is a critical point of the energy functional*

$$E : \Gamma^{\infty}(T_1M) \to [0, +\infty),$$

given by

$$E(X) = \frac{m}{2} vol(M) + \frac{1}{2} \int_M ||\nabla X||^2 dv_g,$$

for any $X \in \Gamma^{\infty}(T_1M)$.

Here, the $1-$parameter variations of U are through unit vector fields. Evidently, any harmonic unit vector field is a harmonic vector field in the sense of the previous section, but the converse is false generally.

Proposition 2.2.4. *[28] Let M be a compact orientable Riemannian manifold and U a unit vector field on M. Then U is a harmonic unit vector field if and only if $\bar{\Delta}U - ||\nabla U||^2 U = 0$.*

Corollary 2.2.2. *Let M be a compact orientable Riemannian manifold and U a unit vector field on M. Then U is a harmonic map of (M, g) into $(T_1M, \tilde{G}_s)$ if and only if the two statements are verified*

i. U is harmonic unit vector field;

ii. $tr_g\big\{R(\nabla_{\cdot} U, U).\big\} = 0$.

2.2.5 Examples

Killing vector fields

Theorem 2.2.5. *[22] Let (M,g) be a real $m-$dimensional Riemannian manifold. If V is a Killing vector field on M, then $\bar{\Delta}_g V = QV$, where $\bar{\Delta}_g$ is the rough Laplacian and Q is the Ricci opertor. In particular if (M,g) is an Einstein manifold*[1]*, then any unit Killing vector field V on M is a harmonic vector field, $||\nabla V||^2 = \frac{\rho}{m}$, and if M is compact then $E(V) = \frac{1}{2}(n + \frac{\rho}{m})$, where ρ is the scalar curvature of (M,g).*

Theorem 2.2.6. *[22] Let (M,g) be an $m-$dimensional Riemannian manifold with constant sectional curvature c and V a unit Killing vector field on M. Then $c \geq 0$ and $V : (M,g) \to (T_1M, G_s)$ is a harmonic map. Moreover*

- *If M is compact and orientable then*

$$E(V) = (\frac{m}{2} + \frac{m-1}{2}c)Vol(M).$$

- *If $c = 0$ then V is parallel.*
- *If $c > 0$ then m is odd.*

Hopf vector fields

Definition 2.2.4. *An orthogonal complex structure on $\mathbb{R}^{2n+2}$ is an orthogonal matrix $J \in O(2n+2)$ such that $J^2 = -I_{2n+2}$.*

Definition 2.2.5. *A tangent vector field $\xi \in \mathfrak{X}(\mathbb{S}^{2n+1})$ is said to be a Hopf vector field on $\mathbb{S}^{2n+1}$ if $\xi = Jv$ for some orthogonal complex structure J on $\mathbb{R}^{2n+2}$ and v the unit normal field on $\mathbb{S}^{2n+1}$.*

Theorem 2.2.7. *[29] The Hopf vector fields ξ on $\mathbb{S}^{2n+1}$ are harmonic maps of $(\mathbb{S}^{2n+1}, g)$ into $(T_1\mathbb{S}^{2n+1}, G_s)$ where $g = i^*g_0$ is the canonical Riemannian metric on $\mathbb{S}^{2n+1}$.*

Furthermore, we have

$$E(\xi) = (2n + \frac{1}{2})vol(\mathbb{S}^{2n+1}).$$

In [29] S.D. Han and J.W. Yim proved that the converse is true in the case of dimension 3

Theorem 2.2.8. *Let ξ be a unit vector field tangent to $\mathbb{S}^3$. Then $\xi : (\mathbb{S}^3, g) \longrightarrow (T_1\mathbb{S}^3, G_s)$ is a harmonic map if and only if ξ is a Hopf vector field on $\mathbb{S}^3$.*

Remark 2.2.3. *In the case of $\mathbb{S}^{2n+1}$ $(n > 1)$, apart from unit Killing fields, no other harmonic applications are known.*

Reeb vector fields

Definition 2.2.6. *Let M be a smooth manifold of dimension $2n+1$ and TM its tangent bundle.*
*A **contact structure** on M is a smooth distribution $\mathcal{D} \subset TM$ of codimension 1 (i.e. a field of tangent hyperplanes) maximally not integrable, in the following sense: $\mathcal{D}$ can be locally expressed like the kernel of a 1-form α which satisfies:*

$$\alpha \wedge (d\alpha)^n \neq 0. \tag{2.12}$$

*If α is globally defined on M, α is called a **contact form** and (M,α) a **contact manifold** .*

[1] A compact Riemannian manifold where $Ric^M(X,Y) = c\, g(X,Y)$.

Lemma 2.2.1. *Let (M, α) be a contact manifold.*
There exists a unique vector field ξ, associated to α, defined by the equations:
1) $d\alpha(\xi, .) = 0$,
2) $\alpha(\xi) = 1$.
*ξ is called the **Reeb vector field**.*

Definition 2.2.7. *Let M be a differentiable manifold of dimension 2n+1.*

*We say that M has an **almost contact structure** or a (ϕ, ξ, η)**-structure** if there exists a (1,1)-tensor field ϕ, a vector field ξ, and a 1-form η on M satisfying:*

$$\phi^2 = -Id + \eta \otimes \xi$$

and

$$\eta(\xi) = 1.$$

Moreover, if there exists an almost contact metric structure such that:

$$g(X, \phi Y) = d\eta(X, Y),$$

*then we say that (M, g, ϕ, ξ, η) is a **contact metric manifold**.*

Definition 2.2.8. *A contact metric manifold is called an **H-contact manifold** if its Reeb vector field ξ is a harmonic vector field.*

K-contact manifolds[2], η-Einstein manifolds[3], Sasakian manifolds[4], and (k, μ)-spaces[5] are H-contact manifolds.

Theorem 2.2.9. *[46] (M, g, ϕ, ξ, η) is an H-contact manifold if and only if the Reeb vector field is an eigenvector of the the Ricci operator.*

Proposition 2.2.5. *Let (M, g, ϕ, ξ, η) a $3-$dimensional contact metric manifold. The Reeb vector field is a harmonic map if and only if M is a H-contact manifold and $\xi(g(\theta, \theta))$ where $\theta = L_\xi g$.*

Now we will provide an example of a harmonic vector field that is not a harmonic map.

Let $M = \{(x, y, z) \in \mathbb{R}^3 / z > 0\}$, and we set $E_1 = \frac{\partial}{\partial x}$, $E_2 = \frac{\partial}{\partial y}$, $E_3 = 2(y + z)\frac{\partial}{\partial x} + \left\{\frac{1}{c}e^{cx}z - (\frac{c}{2}y + cz + \frac{1}{2z})y\right\}\frac{\partial}{\partial y} + \frac{\partial}{\partial z}$. $c \in \mathbb{R}^*$.
we equip M with the following contact metric structure (ϕ, ξ, η, g): $\phi(E_1) = 0$, $\phi(E_2) = E_3$, $\phi(E_3) = -E_2$
$\xi = E_1$, $\eta(X) = g(\xi, X)$, $g(E_i, E_j) = \delta_{ij}$.

Then $\frac{\partial}{\partial x}$ is a harmonic vector field that is not a harmonic map.

2.3 g-natural harmonic vector fields

In this section we will equip the tangent bundle TM by a Riemannian g-natural metric G.

[2]Contact metric manifolds where ξ is Killing, i.e., $h = \frac{1}{2}L_\xi \phi = 0$
[3]Contact metric manifolds where the Ricci operator is given by: $Q = aI + b\eta \otimes \xi$, $a, b \in C^\infty(M)$
[4]Normal contact metric manifolds
[5]Generalization of Sasakian manifolds, these are contact metric manifolds where the curvature satisfies: $R(X, Y)\xi = k\eta(X)Y - \eta(Y)X + \mu\eta(X)hY - \eta(Y)hX$

2.3.1 The energy of a vector field $V : (M, g) \to (TM, G)$

Let's begin by examining the geometric properties of the map $V : M \to TM$ defined by a vector field $V \in \mathfrak{X}(M)$. It is a well-known fact that when TM is equipped with the Sasaki metric G_s, the vector field V establishes an isometry $V : (M, g) \to (TM, G_s)$, meaning it satisfies $V^*G = g$, if and only if V is parallel vector field. Now, we extend our consideration to replace G_s with an arbitrary Riemannian g-natural metric G. Since $V_*X = X^h + (\nabla_X V)^v$ for any vector field X, from (5.1) we conclude that

$$\begin{aligned}(V^*G)(X,Y)=&G(X^h+(\nabla_X V)^v, Y^h+(\nabla_Y V)^v)\\ =&(\alpha_1+\alpha_3)(r^2)g(X,Y)+(\beta_1+\beta_3)(r^2)g(X,V)g(Y,V)\\ &+\alpha_2(r^2)\left[g(X,\nabla_Y V)+g(Y,\nabla_X V)\right]\\ &+\beta_2(r^2)\left[g(X,V)g(\nabla_Y V,V)+g(Y,V)g(\nabla_X V,V)\right]\\ &+\alpha_1(r^2)g(\nabla_X V,\nabla_Y V)+\beta_1(r^2)g(\nabla_X V,V)g(\nabla_Y V,V),\end{aligned} \tag{2.13}$$

for all vector fields X, Y, where $r^2 = ||V||^2$ is a smooth function from M to $\mathbb{R}^+$. Consider that, as per (2.13), usually V^*G depends on the length of V .

In particular, if we suppose that $\beta_1 + \beta_3 = 0$, and from (2.13) we can obtain the following

Theorem 2.3.1. *[6] Let G be a Riemannian g-natural metric on TM and $V \in \mathfrak{X}(M)$.*

1) If $\beta_1 + \beta_3 = 0$, then

a) $\nabla V = 0$ implies that V^*G is homothetic to g, with homotethy factor $(\alpha_1 + \alpha_3)(\rho)$, where $\rho = ||V||^2$ is constant. In particular, V is an isometry when in addition $(\alpha_1 + \alpha_3)(\rho) = 1$.

b) If M is compact and V has constant length $||V|| = \sqrt{\rho}$, then

$$V^*G = (\alpha_1 + \alpha_3)(\rho)\, g \Leftrightarrow \nabla V = 0.$$

2) If $\alpha_2 = \beta_1 = \beta_2 = \beta_3 = 0$, then V is parallel if and only if $V^*G = (\alpha_1 + \alpha_3)(r^2)\, g$.

Exemples 2.3.1. *To illustrate with examples, it's worth mentioning that both the Sasaki metric G_s and the Cheeger-Gromoll metric G_{CG} on TM satisfy the conditions outlined in points 2) and 1) of Theorem 2.3.1, respectively.*

Consider a compact Riemannian manifold (M, g) of dimension n and its tangent bundle (TM, G), equipped with an arbitrary Riemannian g-natural metric G. Each vector field $V \in \mathfrak{X}(M)$ defines a smooth map $V : (M, g) \to (TM, G)$, where $p \mapsto V_p \in M_p$. The *energy functional* $E(V)$ of V is defined as the energy associated with the corresponding map $V : (M, g) \to (TM, G)$. Consequently, $E(V) = \int_M e(V)dv_g$, with $e(V)$ given by:

$$e_p(V) = \frac{1}{2}||V_{*p}||^2 = \frac{1}{2}\mathrm{tr}_g(V^*G)_p = \frac{1}{2}\sum_{i=1}^{n}(V^*G)_p(e_i, e_i), \tag{2.14}$$

$\{e_1, .., e_n\}$ being any local orthonormal basis of vector fields defined in a neighborhood of p. Using Proposition 1.4.1, we then have

$$\begin{aligned}e(V)=&\frac{1}{2}\sum_{i=1}^{n}G_V(V_*\,e_i, V_*\,e_i)=\frac{1}{2}\sum_{i=1}^{n}G_V(e_i^h+(\nabla_{e_i}V)^v, e_i^h+(\nabla_{e_i}V)^v)\\ =&\frac{1}{2}\sum_{i=1}^{n}\big\{(\alpha_1+\alpha_3)(r^2)g(e_i,e_i)+(\beta_1+\beta_3)(r^2)g(e_i,V)^2\\ &+2\alpha_2(r^2)g(e_i,\nabla_{e_i}V)+2\beta_2(r^2)g(e_i,V)g(\nabla_{e_i}V,V)\\ &+\alpha_1(r^2)g(\nabla_{e_i}V,\nabla_{e_i}V)+\beta_1(r^2)g(\nabla_{e_i}V,V)^2\big\}\end{aligned}$$

where $r = ||V||$ and so,

$$e(V)=\frac{1}{2}\Big\{n(\alpha_1+\alpha_3)(r^2)+(\beta_1+\beta_3)(r^2)r^2+2\alpha_2(r^2)\mathrm{div}(V) \\ +2\beta_2(r^2)V(r^2)+\alpha_1(r^2)||\nabla V||^2+\frac{1}{4}\beta_1(r^2)||\mathrm{grad}\, r^2||^2\Big\}. \tag{2.15}$$

Next, we rewrite $E(V) = \int_M e(V)dv_g$ for vector fields of constant length and, as a special case, parallel vector fields.

For any constant $\rho > 0$, we put

$$\mathfrak{X}^\rho(M) = \{V \in \mathfrak{X}(M) : ||V||^2 = \rho\}.$$

So, if $V \in \mathfrak{X}^\rho(M)$, then V has constant length satisfying $||V||^2 = \rho$. The energy of V is given by

$$E(V) = \frac{1}{2}[(n-1)(\alpha_1+\alpha_3)+\phi_1+\phi_3](\rho)\cdot \mathrm{vol}(M,g)+\frac{1}{2}\alpha_1(\rho)\cdot\int_M ||\nabla V||^2 dv_g. \tag{2.16}$$

Since $\alpha_1 > 0$, (2.16) implies that

$$E(V) \geq \frac{1}{2}[(n-1)(\alpha_1+\alpha_3)+\phi_1+\phi_3](\rho)\cdot \mathrm{vol}(M,g) > 0, \tag{2.17}$$

for all $V \in \mathfrak{X}^\rho(M)$. The equality holds in (2.17) if and only if $||\nabla V|| = 0$, that is, if V is *parallel*. Therefore, we have the following

Theorem 2.3.2. *Let (M, g) be a compact Riemannian manifold. A vector field $V \in \mathfrak{X}^\rho(M)$ is an absolute minimum for the energy $E : \mathfrak{X}^\rho(M) \to \mathbb{R}$ restricted to $\mathfrak{X}^\rho(M)$ if and only if V is parallel.*

In particular, from Theorems 2.3.1 and 2.3.2 it follows

Corollary 2.3.1. *Let (M, g) be a compact Riemannian manifold and $V \in \mathfrak{X}^\rho(M)$. With respect to a Riemannian g-natural metric G on TM satisfying $\beta_1 + \beta_3 = 0$, the following assertions are equivalent:*

(i) V is an absolute minimum for the energy $E : \mathfrak{X}^\rho(M) \to \mathbb{R}$,

(ii) V is parallel,

(iii) $V^*g = (\alpha_1 + \alpha_3)(\rho)\, g$ (that is, $V : (M, g) \to (TM, G)$ is a homothetic immersion).

Note that a parallel vector field V necessarily has constant length and from (2.16) (or (2.15)) we have

$$E(V) = \frac{1}{2}[(n-1)(\alpha_1+\alpha_3)+\phi_1+\phi_3](\rho)\cdot \mathrm{vol}(M,g), \tag{2.18}$$

where $||V||^2 = \rho$.

Using (2.18), we obtain that a parallel vector field V is a critical point for the energy restricted to the set

$$\mathfrak{X}_{\mathcal{P}}(M) = \{V \in \mathfrak{X}(M) : \nabla V = 0\}$$

of all *parallel* vector fields, if and only if

$$[(n-1)(\alpha_1+\alpha_3)+\phi_1+\phi_3]'(\rho) = 0, \tag{2.19}$$

that is, $\rho = ||V||^2$ is a critical point of the function $[(n-1)(\alpha_1+\alpha_3)+\phi_1+\phi_3]$.

2.3.2 The tension field associated to $V : (M, g) \to (TM, G)$

Let (M, g) be a Riemannian manifold and $V \in \mathfrak{X}(M)$. The *tension field* associated to the map $V : (M, g) \to (TM, G)$, is defined as

$$\begin{aligned} \tau(V) : M &\to T_{V(p)}TM \\ p &\mapsto \mathrm{tr}(\nabla dV)_p. \end{aligned} \tag{2.20}$$

Let p be a point of M and $\{e_1, .., e_n\}$ a local orthonormal basis of vector fields, defined in a neighborhood of p. By (2.20), we have

$$\begin{aligned} \tau_p(V) &= \sum_{i=1}^{n} (\widehat{\nabla} dv)(e_i, e_i)(p) = \sum_{i=1}^{n} \{\bar{\nabla}_{V_* e_i} V_* e_i - V_*(\nabla_{e_i} e_i)\}(p) \\ &= \sum_{i=1}^{n} \left\{ \bar{\nabla}_{e_i^h + (\nabla_{e_i} V)^v} (e_i^h + (\nabla_{e_i} V)^v) - (\nabla_{e_i} e_i)^v - (\nabla_{\nabla_{e_i} e_i} V)^v \right\} (p) \\ &= \sum_{i=1}^{n} \Big\{ \bar{\nabla}_{e_i^h} e_i^h + \bar{\nabla}_{e_i^h} (\nabla_{e_i} V)^v + \bar{\nabla}_{(\nabla_{e_i} V)^v} e_i^h + \bar{\nabla}_{(\nabla_{e_i} V)^v} (\nabla_{e_i} V)^v \\ &\qquad - (\nabla_{e_i} e_i)^v - (\nabla_{\nabla_{e_i} e_i} V)^v \Big\} (p). \end{aligned} \tag{2.21}$$

Therefore, for the smooth map $V : (M, g) \to (TM, G)$ defined by a vector field $V \in \mathfrak{X}(M)$, we obtain by the following

Theorem 2.3.3. *[6] Let (M, g) be a compact Riemannian manifold. A vector field $V \in \mathfrak{X}(M)$ defines a harmonic map $V : (M, g) \to (TM, G)$ if and only if*

$$\begin{aligned} \tau_1(V) &= -2A_1 QV + 2C_1 \mathrm{tr}[R(\nabla . V, V)\cdot] + C_3 \mathrm{grad}\, r^2 + E_1 \nabla_{\mathrm{grad}\, r^2} V \\ &+ 2C_2 \nabla_V V + \Big[2A_2 - A_3 g(QV, V) + nA_4 + A_5 r^2 + 2C_4 g(\mathrm{tr}[R(\nabla . V, V)\cdot], V) \\ &+ 2C_5 \mathrm{div} V + C_6 V(r^2) + E_2 ||\nabla V||^2 + \frac{1}{4} E_3 \left|\left| \mathrm{grad}\, r^2 \right|\right|^2 \Big] V = 0 \end{aligned} \tag{2.22}$$

and

$$\begin{aligned} \tau_2(V) &= -\bar{\Delta} V - B_1 QV + 2D_1 \mathrm{tr}[R(\nabla . V, V)\cdot] + D_3 \mathrm{grad}\, r^2 + F_1 \nabla_{\mathrm{grad}\, r^2} V \\ &+ 2D_2 \nabla_V V + \Big[2B_3 - B_4 g(QV, V) + nB_5 + B_6 r^2 + 2D_4 g(\mathrm{tr}[R(\nabla . V, V)\cdot], V) \\ &+ 2D_5 \mathrm{div} V + D_6 V(r^2) + F_2 ||\nabla V||^2 + \frac{1}{4} F_3 \left|\left| \mathrm{grad}\, r^2 \right|\right|^2 \Big] V = 0, \end{aligned} \tag{2.23}$$

where, for all points $p \in M$, $\tau_1(V)(p)$ and $\tau_2(V)(p)$ denote the vectors tangent to M at p, such that $\tau(V)_p = (\tau_1(V)(p))^h + (\tau_2(V)(p))^v$, Q is the Ricci operator associated to g, $A_i, ...F_i$ are evaluated at r^2 and $\bar{\Delta} V = -\mathrm{tr}\nabla^2 V = \sum_i \left(\nabla_{e_i} \nabla_{e_i} V - \nabla_{\nabla_{e_i} e_i} V \right)$ is the so-called rough Laplacian *of (M, g) calculated at V.*

Remark 2.3.1. *Theorem 2.3.3 extends to the non-compact case because the condition $\tau(V) = 0$ has a tensorial character, and we can assume it as a* definition *of harmonic maps when M is not compact.*

Exemples 2.3.2. *[6] We now apply (2.22) and (2.23) for classical metrics on TM.*

a) When $G = G_s$ is the Sasaki metric, we find the well known result: $V : (M, g) \to (TM, G_s)$ is a harmonic map if and only if

$$\mathrm{tr}[R(\nabla_{\cdot}V, V)\cdot] = 0 \quad \text{and} \tag{2.24}$$

$$\bar{\Delta}V = 0. \tag{2.25}$$

b) When $G = G_{CG}$ is the Cheeger-Gromoll metric, then $V : (M, g) \to (TM, G_{CG})$ is a harmonic map if and only if

$$\mathrm{tr}[R(\nabla_{\cdot}V, V)\cdot] = 0 \quad \text{and} \tag{2.26}$$

$$(1 + r^2)\bar{\Delta}V + \nabla_{\mathrm{grad}\, r^2}V - \frac{1}{1 + r^2}\left[(2 + r^2)||\nabla V||^2 + \frac{1}{4}\left|\left|\mathrm{grad}\, r^2\right|\right|^2\right] V = 0. \tag{2.27}$$

By Theorem 2.3.3 we can study the relationships between properties of g-natural metrics and harmonicity of maps defined by some special vector fields.

a) Parallel vector fields

The existence of a non-vanishing parallel vector field V on a Riemannian manifold (M, g) is equivalent to the local reducibility of M as $\mathbb{R} \times M'$, where M' is equipped with the product metric, and V is (locally) identified with a vector field tangent to the flat component $\mathbb{R}$ of the product. By rewriting (2.22) and (2.23) for a parallel vector field V, it follows that $\tau(V) = 0$ (and therefore, V defines a harmonic map from (M, g) to (TM, G)) if and only if:

$$-2A_1(\rho)QV + [2A_2 - A_3 g(QV, V) + nA_4 + \rho A_5](\rho)V = 0 \tag{2.28}$$

and

$$-\bar{\Delta}V - B_1(\rho)QV + [2B_3 - B_4 g(QV, V) + nB_5 + \rho B_6](\rho)V = 0, \tag{2.29}$$

where $\sqrt{\rho} = ||V||$ is the constant length of V.

Since V is tangent to the flat component $\mathbb{R}$ of the local decomposition $M = \mathbb{R} \times M'$, it annihilates the curvature. Moreover, $\bar{\Delta}V = 0$ for a parallel vector field.

Therefore, we have the following

Theorem 2.3.4. *[6] A parallel vector field V defines a harmonic map $V : (M, g) \to (TM, G)$ if and only if its constant length $||V|| = \sqrt{\rho}$ satisfies* (2.19)*, that is, it is a critical point of the function*

$$(n - 1)(\alpha_1 + \alpha_3) + \phi_1 + \phi_3.$$

Therefore, an arbitrary Riemannian g-natural metric G on TM, can admit some parallel vector fields which define harmonic maps from (M, g) to (TM, G), and other parallel vector fields which do not define a harmonic map, according to whether their constant length ρ satisfies (2.19) *or not. Moreover:*

(i) For any Riemannian g-natural metric G on TM satisfying

$$(n - 1)(\alpha_1 + \alpha_3) + \phi_1 + \phi_3 = \text{constant}, \tag{2.30}$$

all parallel vector fields define harmonic maps from (M, g) to (TM, G).

(ii) For any Riemannian g-natural metric G on TM such that

$$[(n - 1)(\alpha_1 + \alpha_3) + \phi_1 + \phi_3]'(t) \neq 0 \text{ for all } t, \tag{2.31}$$

parallel vector fields do not define harmonic maps from (M, g) to (TM, G).

Both G_s and G_{CG} satisfy (2.30), as it easily follows from (1.3) and (1.16), respectively. Hence, case *(i)* of Theorem 2.3.4 applies to both G_s and G_{CG}.

b) Vector fields of constant length
Let $V \in \mathfrak{X}^\rho(M)$. Then, from (2.22) and (2.23) we get at once the following

Proposition 2.3.1. *[6] A vector field $V \in \mathfrak{X}^\rho(M)$ satisfies $\tau(V) = 0$* (and so, it defines a harmonic map $V : (M, g) \to (TM, G)$) *if and only if*

$$\begin{aligned}(2.32)\quad &2A_1(\rho)QV + 2C_1(\rho)\mathrm{tr}[R(\nabla.V, V)\cdot] + 2C_2(\rho)\nabla_V V + [2A_2(\rho) - A_3(\rho)g(QV, V)\\ &+nA_4(\rho) + A_5(\rho) + 2C_4(\rho)g(\mathrm{tr}[R(\nabla.V, V)\cdot], V) + 2C_5(\rho)\mathrm{div}V + E_2(\rho)||\nabla V||^2\big]\, V = 0\end{aligned}$$

and

$$\begin{aligned}(2.33)\quad &-\bar{\Delta}V - B_1(\rho)QV + 2D_1(\rho)\mathrm{tr}[R(\nabla.V, V)\cdot] + 2D_2(\rho)\nabla_V V + [2B_3(\rho) - B_4(\rho)g(QV, V)\\ &+nB_5(\rho) + B_6(\rho) + 2D_4(\rho)g(\mathrm{tr}[R(\nabla.V, V)\cdot], V) + 2D_5(\rho)\mathrm{div}V + F_2(\rho)||\nabla V||^2\big]\, V = 0.\end{aligned}$$

In the special case when (M, g) has constant sectional curvature k, from Proposition 2.3.1 it follows

Corollary 2.3.2. *[6] Let (M, g) be a Riemannian manifold of constant sectional curvature k. A vector field $V \in \mathfrak{X}^\rho(M)$ defines a harmonic map from $V : (M, g) \to (TM, G)$) if and only if*

$$\begin{aligned}(2.34)\qquad &2(kC_1 + C_2)(\rho)\nabla_V V + [-2(n-1)kA_1 + 2A_2 - (n-1)k\rho A_3\\ &+nA_4 + A_5 + 2(C_5 - kC_1 - k\rho C_4)\mathrm{div}V + ||\nabla V||^2 E_2\big]\,(\rho)V = 0\end{aligned}$$

and

$$\begin{aligned}(2.35)\qquad &-\bar{\Delta}V + 2(kD_1 + D_2)(\rho)\nabla_V V + [-(n-1)kB_1 + 2B_3 - (n-1)k\rho B_4\\ &+nB_5 + B_6 + 2(D_5 - kD_1 - k\rho D_4)\mathrm{div}V + ||\nabla V||^2 F_2\big]\,(\rho)V = 0\end{aligned}$$

Under the assumption $\alpha_2(\rho) = \beta_2(\rho) = 0$, taking into account formulae (1.18)-(1.23) of the Levi-Civita connection of a g-natural metric G given in Proposition 1.4.1, (2.32) and (2.33) reduce respectively to

$$(2.36)\quad C_1(\rho)\mathrm{tr}[R(\nabla.V, V)\cdot] + C_2(\rho)\nabla_V V + [C_4(\rho)g(\mathrm{tr}[R(\nabla.V, V)\cdot], V) + C_5(\rho)\mathrm{div}V]\, V = 0$$

and

$$(2.37)\qquad -\bar{\Delta}V + \big(2B_3(\rho) + nB_5(\rho) + B_6(\rho) + F_2(\rho)||\nabla V||^2\big)\, V = 0.$$

In particular, (2.37) implies at once that $\bar{\Delta}V$ is collinear with V. So, V is an eigenvector for the rough Laplacian $\bar{\Delta}$ and, since $\sqrt{\rho} = ||V||$ is a constant, we have $\bar{\Delta}V = \frac{1}{\rho}||\nabla V||^2 V$ and (2.37) implies

$$(2.38)\qquad (F_2(\rho) - \frac{1}{\rho})||\nabla V||^2 + (2B_3 + nB_5 + B_6)\,(\rho) = 0.$$

Again taking into account $\alpha_2(\rho) = \beta_2(\rho) = 0$, we have the following

Theorem 2.3.5. *[6] For any $\rho > 0$, let G be a Riemannian g-natural metric satisfying $\alpha_2(\rho) = \beta_2(\rho) = 0$. A vector field $V \in \mathfrak{X}^\rho(M)$ defines a harmonic map $V : (M, g) \to (TM, G)$ if and only if it satisfies* (2.36) *and* (2.37)*. In particular:*

(i) If

$$\left(\frac{1}{\rho}\alpha_1 + \alpha_1'\right)(\rho) = [(n-1)(\alpha_1 + \alpha_3) + \phi_1 + \phi_3]'(\rho) = 0, \tag{2.39}$$

then $V \in \mathfrak{X}^\rho(M)$ defines a harmonic map $V : (M, g) \to (TM, G)$ if and only if V is an eigenvector of $\bar{\Delta}$ and (2.36) *holds.*

(ii) If

$$\left(\frac{1}{\rho}\alpha_1 + \alpha_1'\right)(\rho) \neq 0 = [(n-1)(\alpha_1 + \alpha_3) + \phi_1 + \phi_3]'(\rho), \tag{2.40}$$

then $V \in \mathfrak{X}^\rho(M)$ defines a harmonic map $V : (M, g) \to (TM, G)$ if and only if V is parallel.

(iii) If

$$\left(\frac{1}{\rho}\alpha_1 + \alpha_1'\right)(\rho) = 0 \neq [(n-1)(\alpha_1 + \alpha_3) + \phi_1 + \phi_3]'(\rho), \tag{2.41}$$

there are not vector fields $V \in \mathfrak{X}^\rho(M)$ defining harmonic maps from (M, g) to (TM, G).

(iv) If

$$\left(\frac{1}{\rho}\alpha_1 + \alpha_1'\right)(\rho) \neq 0 \neq [(n-1)(\alpha_1 + \alpha_3) + \phi_1 + \phi_3]'(\rho), \tag{2.42}$$

then $V \in \mathfrak{X}^\rho(M)$ defines a harmonic map $V : (M, g) \to (TM, G)$ if and only if (2.36) *holds, $\bar{\Delta}V$ is collinear to V and the length of ∇V satisfies*

$$||\nabla V||^2 = -\frac{\rho\,[(n-1)(\alpha_1 + \alpha_3) + \phi_1 + \phi_3]'(\rho)}{(\alpha_1 + \rho\,\alpha_1')(\rho)}. \tag{2.43}$$

For the Cheeger-Gromoll metric g_{CG}, (1.16) easily implies $\left(\frac{1}{\rho}\alpha_1 + \alpha_1'\right)(\rho) \neq 0$ and $[(n-1)(\alpha_1 + \alpha_3) + \phi_1 + \phi_3]'(\rho) = 0$. Therefore, if TM is equipped with G_{CG}, the only vector fields of constant length that define harmonic maps are the parallel ones. In particular, when (M, g) has constant sectional curvature $k \neq 0$, a vector field V of constant length never defines a harmonic map $V : (M, g) \to (TM, G_{CG})$.

Chapter 3

Biharmonic vector fields

Over the past decade, there has been increasing interest in the field of biharmonic maps, which can be approached from two main research perspectives. One aspect focuses on the differential geometry, involving the construction of examples and classification results, which we will attempt to discuss. The other aspect explores the analytic viewpoint, specifically considering biharmonic maps as solutions to a fourth-order, strongly elliptic, semilinear partial differential equation (PDE).

In this chapter, following the same process as chapter 2, we introduce the bienergy functional to define biharmonic maps as their critcal points, the first variational formula and the associated Euler-Lagrange equation, the bitension field and some main results and examples.

Then we give some results concerning biharmonic vector fields when a manifold is endowed with the Sasaki metric.

3.1 Biharmonic maps

3.1.1 Definitions

Definition 3.1.1. *Let $f : (M, g) \to (N, h)$ be a smooth map between two Riemannian manifolds Assume that M is compact and orientable, the tension field of f is given by*

$$\tau(f) = tr_g \nabla df,$$

and the bienergy is defined by

$$E_2(f) = \frac{1}{2} \int_M ||\tau(f)||^2 dv_g.$$

We call biharmonic a smooth map f which is a critical point of the bienergy functional.

G. Jiang ([31]) derived the associated Euler-Lagrange equation of E_2 as

$$\bar{\Delta}\tau(f) - \sum_{i=1}^{m} R^N\big(\tau(f), f_* e_i\big) f_* e_i = 0, \tag{3.1}$$

where

- R^N is the Riemannian curvature tensor of (N, h) defined by: $R^N(X, Y)Z = \nabla^N_X \nabla^N_Y Z - \nabla^N_Y \nabla^N_X Z - \nabla^N_{[X,Y]} Z$ for X, Y, $Z \in \mathfrak{X}(N)$
- $\bar{\Delta}$ is the rough laplacien defined by: $\bar{\Delta} = -\sum_{i=1}^{m} \big(\widehat{\nabla}_{e_i} \widehat{\nabla}_{e_i} - \widehat{\nabla}_{\nabla_{e_i} e_i}\big)$, where $\widehat{\nabla}$ is the induced connexion on $f^{-1}TN$.

The quantity $\tau_2(f) := \bar{\Delta}\tau(f) - \sum_{i=1}^{m} R^N\big(\tau(f), f_*e_i\big)f_*e_i$ is called the *bitension field of* f. As for the case of harmonic maps, when (M, g) is a general Riemannian manifold (including the non-compact case), a map $f : (M, g) \to (N, h)$ is said to be *biharmonic* if $\tau_2(f) = 0$.

Furthermore, when $N = \mathbb{R}$ and $f : M \longrightarrow \mathbb{R}$ is a smooth function, then $\tau_2(f) = \Delta_g^2 f$. Hence, f is said to be a biharmonic function if $\Delta_g^2 f = 0$.

3.1.2 Examples and results

From the expression of the bitension field $\tau_2(f)$ it is clear that a harmonic map f is automatically a biharmonic map, in fact a minimum of the bienergy functional. Then all the examples of harmonic maps in subsection 2.1.2 are trivially biharmonic maps.

But the converse is not true so we call a non-harmonic biharmonic map a *proper biharmonic map.*

A basic question in the theory is to understand under what conditions the converse is true. A first general answer to this problem, proved by G.Y. Jiang [31], is

Theorem 3.1.1. *Let $f : (M, g) \to (N, h)$ be a smooth map between two Riemannian manifolds. Assume that M is compact and orientable, and $K^N \leq 0$. Then f is a biharmonic map if and only if f is a harmonic map.*

By giving up the assumption of M being compact and orientable, C. Oniciuc obtained the following results:

Theorem 3.1.2. *[44] Let $f : (M, g) \to (N, h)$ be a Riemannian immersion such that $||\tau(f)||^2$ is constant. Suppose that $K^N \leq 0$. Then f is a biharmonic map if and only if f is a harmonic map.*

When $dim\,(M) = dim\,(N) - 1$, we can replace the hypothesis $K^N \leq 0$ with the hypothesis Ricci curvature is non-positive $Ricci^N \leq 0$, and we obtain

Theorem 3.1.3. *[44] Let $f : (M, g) \to (N, h)$ be a Riemannian immersion. Suppose that M is compact and orientable, $Ricci^N \leq 0$ and $m = n - 1$. Then f is a biharmonic map if and only if f is a harmonic map.*

Proposition 3.1.1. *[44] Let $f : (M, g) \to (N, h)$ be a Riemannian immersion such that $||\tau(f)||$ is constant. Suppose that $m = n - 1$, and $Ricci^N \leq 0$. Then f is a biharmonic map if and only if f is a harmonic map.*

Now we give some results of N. Oniciuc in [44] about the nonexistence of a proper biharmonic Riemannian submersion onto a manifold with non-positive Ricci curvature.

Let $f : (M, g) \to (N, h)$ be a Riemannian submersion and let $x \in M$. Let W be an open subset in N such that $f(x) \in W$, $U = f^{-1}(W)$ and let $\{\tilde{e}_i\}_i$ be an orthonormal frame field on W. We consider $e_i = (\tilde{e}_i)^h$ and $\{e_i\}_{i=n+1}^m$ an orthonormal frame field on $T^V U$.

We have the well-known formula

$$\tau(f)_x = -\sum_{i=n+1}^{m} (df)_x(\nabla_{e_i} e_i) = -(m-n)(df)_x\big(\mathcal{H}(p)\big),$$

where $\mathcal{H}(x)$ is the mean curvature vector fiels at x of the submanifold $f^{-1}(f(x))$ of M. In the following, we shall suppose that the tension field $\tau(f)$ of f is *basic* i.e.,

$$\tau(f)_p = \tau(f)_q,$$

whenever $f(p) = f(q)$. Thus $\tau(f)$ can be seen as a vector field on the manifold N.

The bitension field of a Riemannian submersion f has following form.

Theorem 3.1.4. *[44] Let $f : (M, g) \to (N, h)$ be a Riemannian submersion with basic tension field. Then we have*

$$\tau_2(f) = tr_h \overline{\nabla}^2 \tau(f) + \overline{\nabla}_{\tau(f)} \tau(f) + Ricci^N \tau(f).$$

Next we give some results about the non existence of proper biharmonic Riemannian submersions:

Theorem 3.1.5. *[44] Let $f : (M, g) \to (N, h)$ be a Riemannian submersion. Suppose that M is compact and orientable and $Ricci^N \leq 0$. Then f is a biharmonic map if and only if f is a harmonic map.*

Proposition 3.1.2. *[44] Let $f : (M, g) \to (N, h)$ be a Riemannian submersion. Suppose that M is compact and orientable. Assume that $Ricci^N \leq 0$ and that there is $q \in N$ such that $Ricci^N(q) < 0$. If $||\tau(f)||$ is constant, then f is a biharmonic map if and only if f is a harmonic map.*

Now, we replace the hypothesis M compact and orientable with the hypothesis N compact. We obtain

Proposition 3.1.3. *[44] Let $f : (M, g) \to (N, h)$ be a Riemannian submersion. Suppose that N is compact and orientable and $Ricci^N \leq 0$. Then f is a biharmonic map if and only if f is a harmonic map.*

Theorem 3.1.6. *[44] If $\tau(f)$ is an unit Killing vector field on N, then f is a biharmonic map .*

Corollary 3.1.1. *[44] Let $f : (M, g) \to \mathbb{S}^1$ be a Riemannian submersion such that its fibers have constant mean curvature field, i.e. $||\mathcal{H}(p)|| = 1$, for all $p \in M$, and $\tau(f)$ is basic. Then f is a biharmonic map.*

Proposition 3.1.4. *[44] Suppose that N is compact and orientable, $||\tau(f)|| = 1$ and $div\tau(f) = 0$. Then f is a biharmonic map if and only if $\tau(f)$ is a Killing vector filed on N.*

3.2 Biharmonic vector fields

Let (M, g) be a compact $m-$dimensional Riemannian manifold, (TM, G_s) its tangent bundle with the Sasaki metric G_s and let X be a vector field on M. The bienergy $E_2(X)$ of X is defined to be the bienergy of the corresponding map $X : (M, g) \to (TM, G_s)$. More precisely

$$E_2(X) = \frac{1}{2} \int_M ||\tau(X)||^2 dv_g = \frac{1}{2} \int_M \big[g\big(S(X), S(X)\big) + g\big(\widehat{\Delta} X, \widehat{\Delta} X\big) \big],$$

where $S(X) = -\sum_{i=1}^m R(\nabla_{e_i} X, X) e_i$ with and $\{e_i\}_{i=1}^m$ is a local othonormal frame field of (M, g).

Definition 3.2.1. *Let (M, g) be a Riemannian manifold. A vector field $X \in \mathfrak{X}(M)$ is called biharmonic if the corresponding map $X : (M, g) \to (TM, G_s)$ is a critical point for the bienergy functional E_2, only considering variations through maps defined by vector fields.*

In [37], the authors deduced the critical point condition for $E_2 : \mathfrak{X}(M) \to \mathbb{R}$ as the following

Theorem 3.2.1. *Let (M,g) be a compact oriented $m-$dimensional Riemannian manifold. Let X be a vector field on M and $E_2 : \mathfrak{X}(M) \to [0,+\infty)$ the bienergy functional restricted to the space of all vector fields. Then*

$$\frac{d}{dt}\Big|_{t=0} E(X_t) = \int_M \Big\{ g\big(\bar{\Delta}\bar{\Delta}X\big) + \sum_{i=1}^{m} \big[\big(\nabla_{e_i}R\big)\big(e_i, S(X)\big)X + R\big(e_i, \nabla_{e_i}S(X)\big)X + 2R\big(e_i, S(X)\big)\nabla_{e_i}X\big], V \Big\} dv_g, \tag{3.2}$$

for any smooth $1-$parameter variation $U : M \times (-\epsilon, \epsilon) \to TM$ of X through vector field i.e. $X_t(x) = U(x,t) \in T_xM$ for any $|t| < \epsilon$ and $x \in M$, $\{e_i\}_{i=1}^m$ is a local othonormal frame field of (M,g) and V is the tangent vector field on M given by

$$V = \frac{\partial X_t}{\partial t}\Big|_{t=0}.$$

Since the vector field X is biharmonic if and only if $\frac{d}{dt}\Big|_{t=0} E(X_t) = 0$, for all admissible variations, we get :

Corollary 3.2.1. *[37] A vector field X of an $m-$dimensional Rimannian manifold (M,g) is biharmonic if and only if*

$$\bar{\Delta}\bar{\Delta}X + \sum_{i=1}^{m} \big[\big(\nabla_{e_i}R\big)\big(e_i, S(X)\big)X + R\big(e_i, \nabla_{e_i}S(X)\big)X + 2R\big(e_i, S(X)\big)\nabla_{e_i}X\big], \tag{3.3}$$

where where $\{e_i\}_{i=1}^m$ is a local othonormal frame field of (M,g).

Remark 3.2.1. *Theorem 3.2.1 holds if (M,g) is not a compact Riemannian manifold. Indeed, if M is non-compact, we must take an open subset W in M whose closure is compact, and take an arbitrary V but the closure of the set of all points in M at which V is not zero, is contained in W. Then (3.2) holds in the form*

$$\frac{d}{dt}\Big|_{t=0} E(X_t) = \int_W \Big\{ g\big(\bar{\Delta}\bar{\Delta}X\big) + \sum_{i=1}^{m} \big[\big(\nabla_{e_i}R\big)\big(e_i, S(X)\big)X + R\big(e_i, \nabla_{e_i}S(X)\big)X + 2R\big(e_i, S(X)\big)\nabla_{e_i}X\big], V \Big\} dv_g, \tag{3.4}$$

for any smooth $1-$parameter variation U. Then we get that the condition (3.3) is verified. Then we obtain the result in the case when M is non-compact.

In the non-compact case, a non harmonic biharmonic vector field is called a proper biharmonic vector field.

Remark 3.2.2. *In [27], Gil-Medrano proved that a vector field X in a Riemannian manifold (M,g) is a harmonuc vector field (i.e. a critical point of the energy functional E_1, only considering variations among maps defined by vector fields) if and only if*

$$\bar{\Delta}X = 0.$$

The above condition involves only the connection on the Riemannian manifold (M,g). On the contrary, the corresponding critical point condition which characterizes the biharmonic vector fields involves additionally the Riemannian curvature tensor of (M,g).

Theorem 3.2.2. *Let (M, g) be a compact oriented $m-$dimensional Riemannian manifold and $X \in \mathfrak{X}(M)$. Then X is a biharmonic vector field if and only if X is parallel.*

in [37], the authors proved that the condition (3.3) is equivalent to $\bar{\Delta}X = S(X) = 0$ everywhere on M which means that $X : (M, g) \to (TM, G_s)$ is a harmonic map and by Theorem 2.2.1, we get that X is parallel.

Corollary 3.2.2. *Let (M, g) be a compact oriented $m-$dimensional Riemannian manifold and $X \in \mathfrak{X}(M)$. Then $X : (M, g) \to (TM, G_s)$ is a biharmonic map if and only if X is parallel.*

Remark 3.2.3. *In [27], Gil-Medrano proved that if (M, g) is compact, a vector field X is critical point of the energy functional E_1 restricted to the set of all vector fields of (M, g) (i.e. X is a harmonic vector field) if and only if X is parallel. Theorem 3.2.2 shows that this result remains true in the case when the energy functional is substiuted by the bienergy functional. Furthermore, Nouhaud [43] proved that if M is compact, then $X : (M, g) \to (TM, G_s)$ is a harmonic map if and only if X is parallel. Corollary 3.2.2 shows that the result of Nouhaud remains also true if $X : (M, g) \to (TM, G_s)$ is biharmonic map.*

Exemple 3.2.1. *Let $(M = \mathbb{R}^2, g)$ with the cartesian coordinates (x, y) equipped with the standard Euclidean metric g. The general form of a vector field X of $\mathbb{R}^2$ is*

$$X = f(x, y)\frac{\partial}{\partial x} + g(x, y)\frac{\partial}{\partial y}$$

where f, g are smooth functions of $\mathbb{R}^2$ and $\frac{\partial}{\partial x}$, $\frac{\partial}{\partial y}$ are the basic vector fields of $\mathbb{R}^2$. We have

$$\bar{\Delta}X = -(f_{xx} + f_{yy})\frac{\partial}{\partial x} - (g_{xx} + g_{yy})\frac{\partial}{\partial y}, \tag{3.5}$$

$$\bar{\Delta}\bar{\Delta}X = -(f_{xxxx} + 2f_{xxyy} + f_{yyyy})\frac{\partial}{\partial x} - (g_{xxxx} + 2g_{xxyy} + g_{yyyy}))\frac{\partial}{\partial y}, \tag{3.6}$$

where $f_x = \frac{\partial f}{\partial x}$, $f_{xx} = \frac{\partial^2 f}{\partial x^2}$ etc.

X is a proper biharmonic vector field if and only if $\bar{\Delta}\bar{\Delta}X = 0$, i.e.

$$\Delta^2 f = f_{xxxx} + 2f_{xxyy} + f_{yyyy} = 0,$$

and

$$\Delta^2 g = g_{xxxx} + 2g_{xxyy} + g_{yyyy} = 0.$$

Particular solutions of this system are

$$f(x, y) = \big[(A + Cx)\cosh(\beta x) + (B + Dx)\sinh(\beta x)\big]\big[a\cos(\beta y) + b\sin(\beta y)\big], \tag{3.7}$$

$$g(x, y) = \big[(A + Cx)\cos(\beta x) + (B + Dx)\sin(\beta x)\big]\big[a\cosh(\beta y) + b\sinh(\beta y)\big], \tag{3.8}$$

where A, B, C, D, a, b and β are arbitrary real constants.

Chapter 4

Biharmonicity of parallel vector fields

As we observed in Chapter 2, equipping the tangent bundle of a manifold with g-natural metrics introduces new classes of harmonic vector fields. In the same context, we equip the tangent bundle with g-natural metrics and investigate the conditions under which a parallel vector field becomes a biharmonic map.

Focusing our study on parallel vector fields is up for two reasons. Firstly, this choice is motivated by the complexity of calculus involved in the general case. Secondly, it is sufficient to prove that considering g-natural metrics on tangent bundles, we can remove the extreme rigidity related to the Sasaki metric in the sense of the existence of non-parallel biharmonic vector fields

Let (M, g) be a Reimannian manifold, and TM its tangent bundle which is equipped with a pseudo-Riemannian g-natural metric G. Let V be a vector field on M, then V can be regarded as a map $V : (M, g) \longrightarrow (TM, G)$. The bienergy $E_2(V)$ of V is the bienergy associated to the map $V : (M, g) \longrightarrow (TM, G)$.

The tension field $\tau(V)$ and bitension field $\tau_2(V)$ of V can be decomposed as follows: For all $x \in M$, we denote by $\tau_h(V)(x)$, $\tau_v(V)(x)$, $\tau_{2h}(V)(x)$ and $\tau_{2v}(V)(x)$ the vectors of M_x, such that

$$(\tau(V))(x) = (\tau_h(V)(x))^h + (\tau_v(V)(x))^v, \qquad (\tau_2(V))(x) = (\tau_{2h}(V)(x))^h + (\tau_{2v}(V)(x))^v,$$

obtaining four vector fields $\tau_h(V)$, $\tau_v(V)$, $\tau_{2h}(V)$ and $\tau_{2v}(V)$ on M. So, as sections in $\Gamma(V^{-1}TTM)$, the tension and bitension fields can be written as

$$(\tau(V)) = (\tau_h(V))^h \circ V + (\tau_v(V))^v \circ V, \qquad (\tau_2(V)) = (\tau_{2h}(V))^h \circ V + (\tau_{2v}(V))^v \circ V.$$

It is worth mentioning that the results of this chapter are completly extracted from our paper [7].

4.1 The bitension field of a parallel vector field

In [6], the authors had calculated the tension field of a parallel vector field V. More precisely, they found that

$$\tau_h(V) = \big(2A_2 + mA_4 + \rho A_5\big)(\rho)V, \qquad \tau_v(V) = \big(2B_3 + mB_5 + \rho B_6\big)(\rho)V, \tag{4.1}$$

where ρ is the constant length of the parallel vector field V. As consequences, we have

- $\tau_h(V)$ and $\tau_v(V)$ are homothetic to V with constant homothety factors $c_h := \big(2A_2 + mA_4 + \rho A_5\big)(\rho)$ and $c_v := \big(2B_3 + mB_5 + \rho B_6\big)(\rho)$;

- $V : (M, g) \longrightarrow (TM, G)$ is a harmonic map if and only if $c_h = c_v = 0$;
- $\tau_h(V)$ and $\tau_v(V)$ are parallel.

Theorem 4.1.1. *Let (M, g) be a Riemannian manifold, and TM be equipped with a pseudo-Riemannian g-natural metric G. Let V be a parallel vector field on M, with (constant) squared norm ρ. Then the bitension field of $V : (M, g) \to (TM, G)$ is given by $\tau_2(V) = (\tau_{2h}(V))^h \circ V + (\tau_{2v}(V))^v \circ V$, where*

$$\begin{aligned}(4.2)\qquad \tau_{2h}(V)=&\big\{(c_h^2\rho - 2c_v)\rho[2A_2 + A_4 + \rho A_5](\rho) + 2c_hc_v[C_2 + C_3 + C_5 + \rho C_6](\rho)\\ &+c_v^2[2E_1 + E_2 + \rho E_3](\rho) - 2c_v\rho[2A_2 + A_4 + \rho A_5]'(\rho)\big\}\, V,\end{aligned}$$

$$\begin{aligned}(4.3)\qquad \tau_{2v}(V)=&\big\{(c_h^2\rho - 2c_v)\rho[2B_3 + B_5 + \rho B_6](\rho) + 2c_hc_v[D_2 + D_3 + D_5 + \rho D_6](\rho)\\ &+c_v^2[2F_1 + F_2 + \rho F_3](\rho) - 2c_v\rho[2B_3 + B_5 + \rho B_6]'(\rho)\big\}\, V,\end{aligned}$$

and $V : (M, g) \to (TM, G)$ defines a biharmonic map if and only if $\tau_{2h}(V) = \tau_{2v}(V) = 0$.

Proof: Denote by $\widehat{\nabla}$ the induced connexion on $V^{-1}TTM$, and by $\widehat{\triangle}$ its associated rough laplacian. By (3.1), we have:

$$\begin{aligned}\tau_2(V)=&\bar{\Delta}\tau(V) - \sum_{i=1}^{m}\bar{R}\big(\tau(V), V_*e_i\big)V_*e_i\\ =&-\sum_{i=1}^{m}\big(\widehat{\nabla}_{e_i}\widehat{\nabla}_{e_i}\tau(V) - \widehat{\nabla}_{\nabla_{e_i}e_i}\tau(V)\big) - \sum_{i=1}^{m}\bar{R}\big(\tau(V), V_*e_i\big)V_*e_i.\end{aligned}\tag{4.4}$$

We calculate each term of the sum separately using (1.17) and Proposition 1.5.1. By definition of the induced connection ∇, and taking into account the fact that $V_*X = X^h + (\nabla_X V)^v = X^h$, for every vector field Y, we have

$$\begin{aligned}\widehat{\nabla}_{e_i}\widehat{\nabla}_{e_i}\tau(V) &= \widehat{\nabla}_{e_i}\widehat{\nabla}_{e_i}\Big((\tau_h(V))^h + (\tau_v(V))^v\Big)\circ V\\ &= \Big(\overline{\nabla}_{V_*e_i}\overline{\nabla}_{V_*e_i}\Big((\tau_h(V))^h + (\tau_v(V))^v\Big)\Big)\circ V\\ &= \Big(\overline{\nabla}_{e_i^h}\overline{\nabla}_{e_i^h}\Big((\tau_h(V))^h + (\tau_v(V))^v\Big)\Big)\circ V.\end{aligned}$$

Using Proposition 1.4.1 and the fact that τ_h and τ_v are parallel, we deduce that

$$\begin{aligned}\widehat{\nabla}_{e_i}\widehat{\nabla}_{e_i}\tau(V) = \Big(\overline{\nabla}_{e_i^h}\Big\{&h\big\{A\big(V; e_i, \tau_h(V)\big) + C\big(V; e_i, \tau_v(V)\big)\big\}\\ &+v\big\{B\big(V; e_i, \tau_h(V)\big) + D\big(V; e_i, \tau_v(V)\big)\big\}\Big\}\Big)\circ V.\end{aligned}$$

Using lemmas 1.2.3 and 1.2.4, we have

$$\begin{aligned}\overline{\nabla}_{e_i^h}h\big\{A\big(V; e_i, \tau_h(V)\big)\big\} &= h\big\{\big(\nabla_{e_i}P^A_V\big)(e_i, \tau_h(V)) + A\big(V; e_i, A(V; e_i, \tau_h(V))\big)\big\}\\ &\quad +v\big\{B\big(V; e_i, A(V; e_i, \tau_h(V))\big)\big\}\\ &= h\big\{A\big(V; e_i, A(u; e_i, \tau_h(V))\big)\big\} + v\big\{B\big(V; e_i, A(V; e_i, \tau_h(V))\big)\big\}.\end{aligned}$$

In similar way, we calculate the other terms to obtain

$$\begin{aligned}\widehat{\nabla}_{e_i}\widehat{\nabla}_{e_i}\tau(V) =& h\Big\{A\Big(V;e_i,A\big(V;e_i,\tau_h(V)\big)\Big)+C\Big(V;e_i,B\big(V;e_i,\tau_h(V)\big)\Big)\\ &A\Big(V;e_i,C\big(V;e_i,\tau_v(V)\big)\Big)+C\Big(V;e_i,D\big(V;e_i,\tau_v(V)\big)\Big)\Big\}\\ &+v\Big\{B\Big(V;e_i,A\big(V;e_i,\tau_h(V)\big)\Big)+D\Big(V;e_i,B\big(V;e_i,\tau_h(V)\big)\Big)\\ &+B\Big(V;e_i,C\big(V;e_i,\tau_v(V)\big)\Big)+D\Big(V;e_i,D\big(V;e_i,\tau_v(V)\big)\Big)\Big\}.\end{aligned}\tag{4.5}$$

On the other hand, using Proposition 1.4.1 and the fact that τ_h and τ_v are parallel, we have

$$\begin{aligned}\widehat{\nabla}_{\nabla_{e_i}e_i}\tau(V) =&\widehat{\nabla}_{\nabla_{e_i}e_i}\Big((\tau_h(V))^h+(\tau_v(V))^v\Big)\circ V\\ =&\Big(\overline{\nabla}_{(\nabla_{e_i}e_i)^h}\Big((\tau_h(V))^h+(\tau_v(V))^v\Big)\Big)\circ V\\ =&h\Big\{A\big(V;\nabla_{e_i}e_i,\tau_h(V)\big)+C\big(V;\nabla_{e_i}e_i,\tau_v(V)\big)\Big\}\\ &+v\Big\{B\big(V;\nabla_{e_i}e_i,\tau_h(V)\big)+D\big(V;\nabla_{e_i}e_i,\tau_v(V)\big)\Big\}.\end{aligned}\tag{4.6}$$

Finally, using Proposition 1.5.1, we have

$$\begin{aligned}&\bar{R}\big(\tau(V),V_*e_i\big)V_*e_i=\bar{R}\big(\tau(V),e_i^h\big)e_i^h=\bar{R}\big(h\{\tau_h(V)\}+v\{\tau_v(V)\},e_i^h\big)e_i^h\\ =&h\Big\{A\Big(V;\tau_h(V),A\big(V;e_i,e_i\big)\Big)-A\Big(V;e_i,A\big(V;\tau_h(V),e_i\big)\Big)+C\Big(V;\tau_h(V),B\big(V;e_i,e_i\big)\Big)\\ &-C\Big(V;e_i,B\big(V;\tau_h(V),e_i\big)\Big)-A\Big(V;e_i,C\big(V;e_i\,\tau_v(V)\big)\Big)-C\Big(V;e_i,D\big(V;e_i\,\tau_v(V)\big)\Big)\\ &+C\Big(V;A\big(V;e_i,e_i\big),\tau_v(V)\Big)+E\Big(V;\tau_v(V),B\big(V;e_i,e_i\big)\Big)-d\Big(A_{(e_i,e_i)}\Big)\big(\tau_v(V)\big)\Big\}\\ &+v\Big\{B\Big(V;\tau_h(V),A\big(V;e_i,e_i\big)\Big)-B\Big(V;e_i,A\big(V;\tau_h(V),e_i\big)\Big)+D\Big(V;\tau_h(V),B\big(V;e_i,e_i\big)\Big)\\ &-D\Big(V;e_i,B\big(V;\tau_h(V),e_i\big)\Big)-B\Big(V;e_i,C\big(V;e_i\,\tau_v(V)\big)\Big)-D\Big(V;e_i,D\big(V;e_i\,\tau_v(V)\big)\Big)\\ &+D\Big(V;A\big(V;e_i,e_i\big),\tau_v(V)\Big)+F\Big(V;\tau_v(V),B\big(V;e_i,e_i\big)\Big)-d\Big(B_{(e_i,e_i)}\Big)\big(\tau_v(V)\big)\Big\}\end{aligned}\tag{4.7}$$

Substituting from (4.5)-(4.7) into (4.4), we obtain

$$\begin{aligned}\tau_2(V) =&\Big\{A\Big(V;\tau_h(V),\tau_h(V)\Big)+2C\Big(V;\tau_h(V),\tau_v(V)\Big)+E\Big(V;\tau_v(V),\tau_v(V)\Big)\\ &-\sum_{i=1}^{m}d\Big(A_{(e_i,e_i)}\Big)\big(\tau_v(V)\big)\Big\}^h\circ V+\Big\{B\Big(V;\tau_h(V),\tau_h(V)\Big)+2D\Big(V;\tau_h(V),\tau_v(V)\Big)\\ &+F\Big(V;\tau_v(V),\tau_v(V)\Big)-\sum_{i=1}^{m}d\Big(B_{(e_i,e_i)}\Big)\big(\tau_v(V)\big)\Big\}^v\circ V.\end{aligned}$$

Substituting from (4.1) into the last identity and using Proposition 1.4.1 and Lemma 1.2.5, we obtain the result. ■

4.2 Case of Kaluza-Klein type metrics

Theorem 4.2.1. *Let (M,g) be a Riemannian manifold and V be a non zero parallel vector field on M of (constant) squared norm ρ. Let the tangent bundle TM of M be equipped with a pseudo-Riemannian Kaluza-Klein type metric such that $(\beta_1+\beta_3)(\rho)=(\beta_1+\beta_3)'(\rho)=0$ (In particular, if G is a Kaluza-Klein metric on TM). Then $V:(M,g)\to(TM,G)$ is a biharmonic map if and only if one of the two statements is verified:*

1. *ρ is a critical point of $\alpha_1+\alpha_3$.*
2. *$(\alpha_1+\alpha_3)'(\rho)\Big[2\rho\frac{\alpha_1'}{\alpha_1}+\rho\frac{\beta_1-\alpha_1'}{\phi_1}+\rho^2\frac{\alpha_1\beta_1'-2\alpha_1'\beta_1}{\phi_1\alpha_1}+2\rho\frac{\phi_1'}{\phi_1}-1\Big](\rho)-2\rho(\alpha_1+\alpha_3)''(\rho)=0.$*

Proof: If $\alpha_2=\beta_2=0$ and $(\beta_1+\beta_3)(\rho)=(\beta_1+\beta_3)'(\rho)=0$, then we have:

$$B_5(\rho)=-\frac{(\alpha_1+\alpha_3)'}{\phi_1}(\rho),\quad C_3(\rho)=-\frac{(\alpha_1+\alpha_3)'}{\alpha_1+\alpha_3}(\rho),$$

$$F_1(\rho)=\frac{\alpha_1'}{\alpha_1}(\rho),\quad F_2(\rho)=\frac{\beta_1-\alpha_1'}{\phi_1}(\rho),\quad F_3(\rho)=\frac{\alpha_1\beta_1'-2\alpha_1'\beta_1}{\phi_1\alpha_1}(\rho),$$

$$A_2(\rho)=A_4(\rho)=A_5(\rho)=B_3(\rho)=B_6(\rho)=C_2(\rho)=C_5(\rho)=C_6(\rho)=E_1(\rho)=E_2(\rho)=E_3(\rho)=0.$$

We deduce that $c_h=0$ and $c_v=-m\frac{(\alpha_1+\alpha_3)'}{\phi_1}(\rho)$. Considering the results above, formulas (4.2) and (4.3) become:

$$\begin{aligned}\tau_{2h}(V)=&0,\\ \tau_{2v}(V)=&m^2\frac{(\alpha_1+\alpha_3)'}{\phi_1^2}(\rho)\bigg((\alpha_1+\alpha_3)'(\rho)\Big[2\rho\frac{\alpha_1'}{\alpha_1}+\rho\frac{\beta_1-\alpha_1'}{\phi_1}+\rho^2\frac{\alpha_1\beta_1'-2\alpha_1'\beta_1}{\phi_1\alpha_1}+2\rho\frac{\phi_1'}{\phi_1}-1\Big](\rho)\\ &-2\rho(\alpha_1+\alpha_3)''(\rho)\bigg)V.\end{aligned}$$

Since $\tau_{2h}(V)=0$ then $V:(M,g)\longrightarrow(TM,G)$ is a biharmonic map if and only if $\tau_{2v}(V)=0$, which gives the result. ■

Remark 4.2.1. *Let G be the pseudo-Riemannian g-natural metric on TM given by $\alpha_2=\beta_1=\beta_2=\beta_3=0$, α_1 is constant and $(\alpha_1+\alpha_3)(t)=e^{at}$, where $a>0$. Then $(\alpha_1+\alpha_3)'$ does not vanish and*

$$(\alpha_1+\alpha_3)'(t)\Big[2t\frac{\alpha_1'}{\alpha_1}+t\frac{\beta_1-\alpha_1'}{\phi_1}+t^2\frac{\alpha_1\beta_1'-2\alpha_1'\beta_1}{\phi_1\alpha_1}+2t\frac{\phi_1'}{\phi_1}-1\Big](t)-2t(\alpha_1+\alpha_3)''(t)=-a(1+2at)e^{at},$$

which does not vanish on $\mathbb{R}^+$. We deduce that the two conditions of Theorem 4.2.1 can not be verified, and consequently every non-zero parallel vector field on TM is not biharmonic.

As corollary, we have the following result which characterizes parallel vector field on M which are **proper biharmonic maps**, i.e. non-harmonic biharmonic maps.

Corollary 4.2.1. *Let (M,g) be a Riemannian manifold and V be a parallel vector field on M of (constant) squared norm ρ. Let the tangent bundle TM of M be equipped with a pseudo-Riemannian Kaluza-Klein type metric such that $(\beta_1+\beta_3)(\rho)=(\beta_1+\beta_3)'(\rho)=0$ (In particular, if G is a Kaluza-Klein metric on TM). Then the map $V:(M,g)\longrightarrow(TM,G)$ is proper biharmonic if and only if*

1. $(\alpha_1+\alpha_3)'(\rho)\neq 0$ *and*
2. $(\alpha_1+\alpha_3)^{'}(\rho)\Big[2\rho\frac{\alpha_1^{'}}{\alpha_1}+\rho\frac{\beta_1-\alpha_1^{'}}{\phi_1}+\rho^2\frac{\alpha_1\beta_1^{'}-2\alpha_1^{'}\beta_1}{\phi_1\alpha_1}+2\rho\frac{\phi_1^{'}}{\phi_1}-1\Big](\rho)-2\rho(\alpha_1+\alpha_3)^{''}(\rho)=0.$

Exemples 4.2.1. *Let (M,g) be a Riemannian manifold.*

1. *Let G be the Kaluza-Klein metric on TM such that*
   - $\beta_1=0$,
   - α_1 *is a positive constant and*
   - $(\alpha_1+\alpha_3)(t)=\frac{2}{3}t^{3/2}+c$, *for all* $t\in\mathbb{R}_+$, $c>0$.

   Then it is easy to see that $(\alpha_1+\alpha_3)'$ doesn't vanish on $\mathbb{R}_+^*$ and the differential equation

   $$(\alpha_1+\alpha_3)^{'}(t)\Big[2t\frac{\alpha_1^{'}}{\alpha_1}+t\frac{\beta_1-\alpha_1^{'}}{\phi_1}+t^2\frac{\alpha_1\beta_1^{'}-2\alpha_1^{'}\beta_1}{\phi_1\alpha_1}+2t\frac{\phi_1^{'}}{\phi_1}-1\Big](t)-t(\alpha_1+\alpha_3)^{''}(t)=0$$

   is satisfied on $\mathbb{R}_+^*$. We deduce that every non-zero parallel vector field on M is a proper biharmonic map.

2. *Let G be the Kaluza-Klein metric on TM which satisfies*
   - $\alpha_1(t)=e^{\frac{t}{\rho}}$,
   - $\beta_1(t)=\rho e^{\frac{t}{\rho}}-et$ *and*
   - $(\alpha_1+\alpha_3)(t)=k(e^{\frac{t}{\rho}}+c)$,

   for all $t\in\mathbb{R}_+$, where ρ, k and c are constants such that $\rho\neq 0$, $k\neq 0$ and $c>-e$. Then it is easy to see that G is non-degenerate. Furthermore, G is Riemannian if $k>0$ and pseudo-Riemannian of signature (m,m) if $k<0$. Obviously, $(\alpha_1+\alpha_3)'$ doesn't vanish. It is also easy to check that the function $(\alpha_1+\alpha_3)^{'}(t)\Big[2t\frac{\alpha_1^{'}}{\alpha_1}+t\frac{\beta_1-\alpha_1^{'}}{\phi_1}+t^2\frac{\alpha_1\beta_1^{'}-2\alpha_1^{'}\beta_1}{\phi_1\alpha_1}+2t\frac{\phi_1^{'}}{\phi_1}-1\Big](t)-t(\alpha_1+\alpha_3)^{''}(t)$, defined on $\mathbb{R}_+$, vanishes on ρ and only on ρ. It follows that only parallel vector fields on M of squared norm ρ are (proper) biharmonic maps.

4.3 G-biharmonic parallel vector fields

Let (M,g) be a compact Riemannian manifold. In [37], Markellos and Urakawa have proved that, equipping TM with the Sasaki metric g^s, $V:(M,g)\longrightarrow(TM,g^s)$ is a critical point of the bienergy functional restricted to $\mathfrak{X}(M)$ if and only if V is a parallel vector field.
Now, we shall determine the critical point condition for the bienergy functional restricted to $\mathfrak{X}(M)$, when TM is equipped with an arbitrary pseudo-Riemannian g-natural metric. Such critical point will be called *G-biharmonic vector field*.

Clearly, a G-biharmonic vector field is not necessarily a biharmonic map. The following result gives the characterization of G-biharmonic vector fields:

Theorem 4.3.1. *Let (M,g) be a compact Riemannian manifold, and TM be equipped with a pseudo-Riemannian g-natural metric G. A vector field V is a G-biharmonic vector field if and only if*

$$\alpha_1\tau_{2v}(V)+\beta_1 g\big(\tau_{2v}(V),V\big)V+\alpha_2\tau_{2h}(V)+\beta_2 g\big(\tau_{2h}(V),V\big)V=0. \tag{4.8}$$

In particular, a parallel vector field V on M is G-biharmonic if and only if

$$\phi_1\tau_{2v}(V)=-\phi_2\tau_{2h}(V). \tag{4.9}$$

Proof: In [31], G. Jiang has derived the first variational formula of the bienergy functional E_2:

$$\frac{d}{dt}\Big|_{t=0} E_2(V_t) = -\int_M G\big(\tau_2(V), Y\big)dv_g,$$

for all $\{V_t\} \subset \mathfrak{X}(M)$ of V, and Y is called the *variational vector field* which satisfied:

$$Y = \frac{\delta V_t}{\delta t}\Big|_{t=0},$$

So V is a G-biharmonic vector field i.e. a critical point of $E_{2|\mathfrak{X}(M)}$ if and only if:

$$\frac{d}{dt}\Big|_{t=0} E_2(V_t) = 0,$$

for all $\{V_t\} \subset \mathfrak{X}(M)$ of V.

But, as was remarked in [27], any vertical vector field Y^v, section of the bundle $V^{-1}TTM$, there exists a variation $\{V_t\}_t \subset \mathfrak{X}(M)$ of V such that

$$Y^v = \frac{\delta V_t}{\delta t}\Big|_{t=0},$$

Using the decomposition of $\tau_2(v)$ into horizontal and vertical components, we obtain

$$\begin{aligned}\int_M G\big(\tau_2(V), Y^v\big)dv_g &= \int_M G\big(h\{\tau_{2h}(V)\} + v\{\tau_{2v}(V)\}, Y^v\big)dv_g \\ &= \int_M g\big(\alpha_1\tau_{2v}(V) + \beta_1 g\big(\tau_{2v}(V), V\big)V + \alpha_2\tau_{2h}(V) + \beta_2 g\big(\tau_{2h}(V), V\big)V, Y\big)dv_g.\end{aligned}$$

Hence V is a G-biharmonic vector field if and only if (4.8) is satisfied.

Furthermore, if V is parallel then τ_{2h} and τ_{2v} are proportional to V, and then (4.8) is equivalent to (4.9). ∎

Remark 4.3.1. *In the non compact case we can define the G-biharmonicity of V by the condition* (4.8)*, since it has a tensorial character.*

Since the variations are through $\mathfrak{X}(M) \subset C^\infty(M, TM)$, if $V : (M, g) \longrightarrow (TM, G)$ is a biharmonic map then V is a G-biharmonic vector field. The converse holds for parallel vector fields, for some pseudo-Riemannian g-natural metrics, as the following result shows

Corollary 4.3.1. *Let (M, g) be a Riemannian manifold, V be a parallel vector field on M and suppose that the tangent bundle TM of M is equipped by a pseudo-Riemannian g-natural metric G which satisfies $\alpha_2(\rho) = \beta_2(\rho) = (\beta_1 + \beta_3)(\rho) = 0$, where $\rho = ||V||^2$.*

Suppose that we have one of the following conditions:

1. *$m(\alpha_1 + \alpha_3)'(\rho) + \rho(\beta_1 + \beta_3)'(\rho) = 0$ i.e. V is a harmonic map;*
2. *$[m(\alpha_1 + \alpha_3)'(\rho) + \rho(\beta_1 + \beta_3)'(\rho)]E_1(\rho) + \frac{\rho}{2}E_3(\rho) = mA_4'(\rho) + \rho A_5'(\rho)t$.*

Then $V : (M, g) \longrightarrow (TM, G)$ is a biharmonic map if and only if V is a G-biharmonic vector field.

Exemple 4.3.1. *If G is a Kaluza-Klein metric then $E_1 = E_3 = A_4' = A_5' = 0$, and then a parallel vector field is G-biharmonic if and only if it is a biharmonic map..*

On the other hand, we shall prove that if we consider some classes of pseudo-Riemannian g-natural metrics, then almost all parallel vector fields are G-biharmonic but not biharmonic. More precisely, suppose that $\alpha_1 = \beta_1 = \beta_2 = \beta_3 = 0$, and $\alpha_1 + \alpha_3$ does not vanish on $\mathbb{R}^+$. To ensure that G is non-degenerate, α_2 should not vanish on $\mathbb{R}^+$, and should have the same sign. In this case, we have

- $\alpha_1 = \beta_1 = \beta_2 = \beta_3 = \phi_1 = 0$, $\phi_1 + \phi_3 = \alpha_1 + \alpha_3$, $\phi = \alpha = -\alpha^2$.
- both $\alpha_1 + \alpha_3$ and α_2 do not vanish on $\mathbb{R}^+$.
- G is of signature (m, m). Indeed, for an orthonormal frame field $\{e_i; i = 1, ..., m\}$ on (M, g), it suffices to take the orthonormal frame field $\{E_i; i = 1, ..., 2m\}$ of (TM, G) given by

$$E_i = \frac{1}{\sqrt{|\alpha_1 + \alpha_3|}} e_i^h, \qquad E_{m+i} = \frac{1}{\sqrt{|\alpha_1 + \alpha_3|}} \Big[e_i^h - \frac{\alpha_1 + \alpha_3}{\alpha_2} e_i^v \Big].$$

To simplify notations, we consider the functions

$$\lambda := \frac{(\alpha_1 + \alpha_3)'}{\alpha_2}, \qquad \mu := \frac{\alpha_1 + \alpha_3}{\alpha_2}, \qquad \nu := \frac{\alpha_2'}{\alpha_2}. \tag{4.10}$$

Since we are looking for non biharmonic vector fields on M, we will suppose that $(\alpha_1 + \alpha_3)'$ does not vanish on $\mathbb{R}_+^*$. We deduce that λ and μ don't vanish on $\mathbb{R}_+^*$. Furthermore, taking eventually $-\alpha_2$ instead of α_2, we can suppose that λ is positive on $\mathbb{R}_+^*$.

Substituting from (4.10) in Proposition 1.4.1 and in the formulas of c_h and c_v, we find

- $A_1 = A_2 = A_3 = A_5 = 0$, $A_4 = -\lambda$,
- $B_1 = -1$, $B_3 = B_4 = B_6 = 0$, $B_5 = \lambda\mu$,
- $C_1 = C_2 = C_4 = C_6 = 0$, $C_3 = \nu$, $C_5 = -\nu$,
- $D1 = D_2 = D_4 = D_6 = 0$, $D_3 = \mu'$, $D_5 = -\mu\nu$,
- $E_1 = E_2 = E_3 = 0$,
- $F_1 = \nu$, $F_2 = F_3 = 0$,
- $c_h = -m\lambda$, $c_v = m\lambda\mu$.

Substituting into (4.2) and (4.3), we deduce that the horizontal and vertical components of the bitension of a parallel vector field V om M with $\rho := \|V|^2$ are given by

$$\tau_{2h}(V) = m\lambda(\rho)[2\mu(\lambda + \rho\lambda') - m\rho\lambda^2](\rho)V, \tag{4.11}$$

$$\tau_{2v}(V) = m\lambda(\rho)\mu(\rho)\{m\rho^2\lambda^2 - 2\rho[\lambda\mu + (\lambda\mu)'] + m\lambda(\rho)[3\mu\nu - 2\mu']\}(\rho)V. \tag{4.12}$$

From Theorem 4.3.1, V is G-biharmonic if and only if $\tau_{2h}(V) = 0$. Hence V is non biharmonic G-biharmonic vector field if and only if $\tau_{2h}(V) = 0$ and $\tau_{2v}(V) \neq 0$, i.e. if and only if the two following conditions hold

$$[2\mu(\lambda + \rho\lambda') - m\rho\lambda^2](\rho) = 0, \tag{4.13}$$

$$\{m\rho^2\lambda^2 - 2\rho[\lambda\mu + (\lambda\mu)'] + m\lambda(\rho)[3\mu\nu - 2\mu']\}(\rho) \neq 0. \tag{4.14}$$

Now, we shall consider the differential equation $2\mu(\lambda + t\lambda') - mt\lambda^2 = 0$, which is equivalent to

$$\frac{\lambda}{\mu} = \frac{2}{m}\Big(\frac{1}{t} + \frac{\lambda'}{\lambda}\Big) \tag{4.15}$$

on $\mathbb{R}_+^*$. Remarking that $\frac{\lambda}{\mu} = \frac{(\alpha_1+\alpha_3)'}{\alpha_1+\alpha_3}$ and integrating, we find that

$$\alpha_1 + \alpha_3 = K(t\lambda)^{\frac{2}{m}}, \tag{4.16}$$

for some non zero constant K. Substituting from (4.15) and (4.16) into $\mu = \frac{\alpha_1+\alpha_3}{\alpha_2}$, we obtain

$$\alpha_2 = \frac{2K}{m}(\lambda + t\lambda')t^{\frac{2}{m}-1}\lambda^{\frac{2}{m}-2}, \tag{4.17}$$

on $\mathbb{R}_+^*$. To avoid the non regularity problem of α_2 at 0, we need to have $\lim\limits_{t\to 0}\frac{\lambda(t)}{t}$ is a constant. In this case, we will have $\alpha_2(0) = 0$, which is in contradiction with the fact that G is non-degenerate at the zero section. To solve this problem, we consider a function $f_\eta \in C^\infty(\mathbb{R}^+)$ such that

- $0 \leq f_\eta(t) \leq 1$, for all $t \in \mathbb{R}^+$,
- $f_\eta(t) = 0$, for $t \geq \eta$,
- $f_\eta(t) = 1$, for $t \leq \frac{\eta}{2}$,

and we put

$$\begin{cases} \alpha_1 + \alpha_3 = K(t\lambda)^{\frac{2}{m}} + \operatorname{sgn}(K)f_\eta, \\ \alpha_2 = \dfrac{2K}{m}(\lambda + t\lambda')t^{\frac{2}{m}-1}\lambda^{\frac{2}{m}-2} + \operatorname{sgn}(K(\lambda + t\lambda'))f_\eta. \end{cases} \tag{4.18}$$

Hence we have

- $\alpha_1 + \alpha_3$ and α_2 are exactly those of (4.16) and (4.17) on $[\eta, +\infty[$, respectively, and satisfy the condition (4.13) on $[\eta, +\infty[$. We deduce that, for $\rho \in [\eta, +\infty[$, a parallel vector of norm ρ is G-biharmonic.
- $\alpha_2(0) \neq 0$ and then G is non-degenerate everywhere.
- $\alpha_1 + \alpha_3$ and α_2 don't vanish on $\mathbb{R}^+$.

To complete our study, we shall check if condition (4.14) is satisfied for $\rho \in [\eta, +\infty[$. Using (4.18) on $[\eta, +\infty[$, we can express μ and ν in terms of λ and its derivatives as

$$\begin{aligned} \mu &= \frac{mt\lambda^2}{2(\lambda + t\lambda')}, \\ \nu &= \frac{1}{t\lambda(\lambda + t\lambda')}\Big[t^2\lambda\lambda'' + \Big(\frac{4}{m} - 1\Big)t\lambda\lambda' + \Big(\frac{2}{m} - 2\Big)t^2(\lambda')^2 + \Big(\frac{2}{m} - 1\Big)\lambda^2\Big]. \end{aligned}$$

We deduce then that condition (4.14) at $\rho \in [\eta, +\infty[$ is equivalent to the following condition:

$$\begin{aligned} 0 \neq \Big\{ & t^2(\lambda + t\lambda')^2 - t\Big[((1+t)\lambda^3 + 3\lambda^2\lambda') - t\lambda^3(\lambda + t\lambda')'\Big] + 3m\lambda^2\Big[t^2\lambda'' + \Big(\frac{4}{m} - 1\Big)t\lambda\lambda' \\ & + \Big(\frac{2}{m} - 2\Big)t^2(\lambda')^2 + \Big(\frac{2}{m} - 1\Big)\lambda^2\Big] - m\lambda(\lambda^2 + 2t\lambda\lambda')(\lambda + t\lambda') + mt\lambda^3(\lambda + t\lambda')'\Big\}(\rho). \end{aligned} \tag{4.19}$$

To summarize the previous discussion, we have

Proposition 4.3.1. *Let (M, g) be a Riemannian manifold. Fix $\eta > 0$ and suppose that the tangent bundle TM of M is equipped by a pseudo-Riemannian g-natural metric G which satisfies $\alpha_1 = \beta_1 = \beta_2 = \beta_3 = 0$ and (4.18), where λ is a positive smooth function on $\mathbb{R}_+^*$ such that $\lim_{t\to 0} \frac{\lambda(t)}{t}$ is a constant. Then a parallel vector field on M such that $\rho := ||V||^2 \in [\eta, +\infty[$ is non-biharmonic G-biharmonic if and only if (4.19) is satisfied at ρ.*

Exemple 4.3.2. *We consider the function λ defined by $\lambda(t) = ate^{bt}$, for all $t \in \mathbb{R}^+$, where $a > 0$ and b are real constants. A long but routine calculation shows that (4.19) is equivalent to the fact that ρ is not a solution of the following polynomial equation*

$$[(2m-11)b-1]bt^3 + [2mb^2 + (5m-21)b + 2]t^2 + [7mb + 17m - 20]t - 6m = 0,$$

which has at most three solutions. We deduce that, apart from at most three values of $\rho \in [\eta, +\infty[$, all parallel vector fields of norm $\sqrt{\rho}$ are non-biharmonic G-biharmonic.

Remarks 4.3.1.

1. *Taking eventually different values of b in the previous example, for any value $\rho \in [\eta, +\infty[$, there is a pseudo-Riemannian g-natural metric on TM for which any parallel vector field on M such that $||V||^2 = \rho$ is non-biharmonic G-biharmonic.*

2. *η being arbitrary, for any value $\rho > 0$, there is a pseudo-Riemannian g-natural metric on TM for which any parallel vector field on M such that $||V||^2 = \rho$ is non-biharmonic G-biharmonic.*

Chapter 5

$\tilde{G}$-biharmonic unit vector fields

In this chapter we restrict the bienergy functional to the variations through $\mathfrak{X}_1(M)$, the set of all unit vetor fields, by equipping the unit tangent sphere bundle T_1M by an arbitrary g-natural metrics $\tilde{G}$.

The critical points of the obtained functional are called $\tilde{G}$-biharmonic unit vector field. We shall give the statement of the main theorem characterizing $\tilde{G}$-biharmonic unit vector fields, of which the proof will be presented in detail in Appendix A because of long calculations and expressions.

We will then give examples and some consequences in some particular situations.

Finally, it is worth mentioning that the results of this chapter are completly extracted from our paper [7].

5.1 The main theorem

Notations: Let (M, g) be a Riemannian manifold and $U \in \mathfrak{X}_1(M)$. Let $\{e_i\}_{i=1}^m$ be an orthonormal frame field on M. Let us denote by

- $S(U) = -\sum_{i=1}^m R(\nabla_{e_i}U, U)e_i$,
- Q the Ricci operator associated to g,
- div the divergence operator associated to g,
- $\mathcal{A}$ the operator on unit vector fields defined by: $\mathcal{A}U = QU - g(QU, U)U$.

Let (M, g) be a Riemannian manifold and T_1M its unit tangent bundle. We call *g-natural metrics* on T_1M the restrictions of g-natural metrics of TM to its hypersurface T_1M. These metrics possess a simpler form. Indeed, for any g-natural metric $\tilde{G}$ on T_1M, there are four real constants a, b, c and d, such that

$$\begin{cases} \tilde{G}_{(x,u)}(X^h, Y^h) = (a+c)g_x(X,Y) + dg_x(X,u)g_x(Y,u), \\ \tilde{G}_{(x,u)}(X^h, Y^t) = bg_x(X,Y), \\ \tilde{G}_{(x,u)}(X^t, Y^t) = ag_x(X,Y) - \frac{\phi}{a+c+d}g_x(X,u)g_x(Y,u), \end{cases} \tag{5.1}$$

for any $(x, u) \in T_1M$ and X, $Y \in M_x$ (cf. [10]), where X^t is the tangential lift to (x, u) of X given by

$$X^t = \frac{b}{a+c+d}g_x(X,u)\,u^h + [X - g_x(X,u)u]^v. \tag{5.2}$$

Furthermore, $\tilde{G}$ is non-degenerate (resp. Riemannian) if and only if $\alpha := a(a+c) - b^2 \neq 0$ and $\phi = a(a+c+d) - b^2 \neq 0$ (resp. $a > 0$, $\phi > 0$ and $\alpha > 0$).

In particular, the *Sasaki metric* on T_1M corresponds to the case where $a = 1$ and $b = c = d = 0$; *Kaluza-Klein metrics* are obtained when $b = d = 0$; *metrics of Kaluza-Klein type* are given by the case $b = 0$.

From now on, let (M, g) be a compact Riemannian manifold and suppose its unit tangent sphere bundle T_1M is equipped by a pseudo-Riemannian g-natural metric $\tilde{G}$.

The *bienergy* functional of a unit vector field U, regarded as a map of (M, g) into $(T_1M, \tilde{G})$, is the mapping defined by:

$$E_2 : C^\infty(M, T_1M) \longrightarrow [0, +\infty[$$
$$U \longmapsto E_2(U) := \frac{1}{2}\int_M \|\tilde{\tau}(U)\|^2 v_g,$$

$\tilde{\tau}(U)$ being the tension field of the map $U : (M, g) \longrightarrow (T_1M, \tilde{G})$, which is given by: $\tilde{\tau}(U) = (\tilde{\tau}_h(U))^h + (\tilde{\tau}_v(U))^v$, where (cf. [5])

$$\begin{aligned}(5.3)\quad \tilde{\tau}_h(U) = {} & \frac{ab}{\alpha}QU - \frac{a^2}{\alpha}S(U) - \frac{ad}{\alpha}\nabla_U U - \frac{b(ad+b^2)}{\alpha\varphi}g(QU,U)U - \frac{b}{\varphi}g(\overline{\Delta}U,U)U \\ & + \frac{d}{\varphi}\mathrm{div}(U)U + \frac{a(ad+b^2)}{\alpha\varphi}g(S(U),U)U\end{aligned}$$

$$\begin{aligned}(5.4)\quad \tilde{\tau}_v(U) = {} & -\overline{\Delta}U - \frac{b^2}{\alpha}QU + \frac{ab}{\alpha}S(U) + \frac{bd}{\alpha}\nabla_U U + \frac{b^2}{\alpha}g(QU,U)U + g(\overline{\Delta}U,U)U \\ & - \frac{ab}{\alpha}g(S(U),U)U,\end{aligned}$$

with $\varphi = a + c + d$.

A unit vector field U is said to be a *$\tilde{G}$-biharmonic unit vector field* if U is a critical point of $E_{2|\mathfrak{X}_1(M)}$, i.e. for all C^∞ 1-parameter variations $\{U_t\}_t$ of U in $\mathfrak{X}^1(M)$, $(|t| < \epsilon)$, we have:

$$\frac{d}{dt}\Big|_{t=0} E_2(U_t) = 0.$$

Theorem 5.1.1. *Let (M, g) be a compact Riemannian manifold and T_1M its unit tangent bundle equipped with an arbitrary pseudo-Riemannian g-natural metric $\tilde{G}$. Let $U \in \mathfrak{X}^1(M)$. Then U is a $\tilde{G}$-biharmonic unit vector field if and only if the following holds*

$$T(U) = 0,$$

where

(5.5)

$$\begin{aligned}T(U) = {} & \sum_{i=1}^{m}\Big\{ - a\big[2R(e_i, \tilde{\tau}_h(U))\nabla_{e_i}U + (\nabla_{e_i}R)(e_i, \tilde{\tau}_h(U))U + R(e_i, \nabla_{e_i}\tilde{\tau}_h(U))U\big] \\ & - dg\big(\nabla_{e_i}U, \tilde{\tau}_h(U)\big)e_i\Big\} + d\,\nabla_U\tilde{\tau}_h(U) + b\,Q\big(\tilde{\tau}_h(U)\big) \\ & + \big[- \tfrac{bd}{\varphi}g(QU,U) + \tfrac{bd}{\varphi}g(\overline{\Delta}U,U) + (a+c)\tfrac{d}{\varphi}\mathrm{div}(U) + \tfrac{ad}{\varphi}g(S(U),U) + dg(\tilde{\tau}_h(U),U)\big]\tilde{\tau}_h(U) \\ & + \tfrac{a^2d}{\alpha}g(\tilde{\tau}_h(U),U)S(U) - \tfrac{abd}{\alpha}g(\tilde{\tau}_h(U),U)QU - a\overline{\Delta}\tilde{\tau}_v(U) - b\overline{\Delta}\tilde{\tau}_h(U) \\ & + \big[\tfrac{b^2}{\varphi}g(QU,U) + \tfrac{\alpha+ad}{\varphi}g(\overline{\Delta}U,U) - \tfrac{ab}{\varphi}g(S(U),U) + \tfrac{bd}{\varphi}\mathrm{div}(U)\big]\tilde{\tau}_v(U) \\ & - d\,\mathrm{grad}[g(\tilde{\tau}_h(U),U)] + \tfrac{ad^2}{\alpha}g\big(\tilde{\tau}_h(U),U\big)\nabla_U U.\end{aligned}$$

Proof: Let U be a unit vector field on (M,g) and $I=]-\epsilon,\epsilon[$, $(\epsilon>0)$. For $t\in I$, we denote by $i_t: M\longrightarrow M\times I$, $p\longrightarrow(p,t)$, the canonical inclusion. We consider C^∞-variations $V: M\times I\longrightarrow T_1M$ of U within $\mathfrak{X}_1(M)$, i.e. for all $t\in I$, the mappings $V_t=V\circ i_t$ are in fact unit vector fields and $V_0=U$. We choose $\{e_i\}_{i=1}^m$ a local othonormal frame field of (M,g) and we extend each e_i (resp. $\frac{d}{dx}\in\mathfrak{X}(I)$) to $M\times I$, denoted by E_i (resp. $\frac{d}{dt}$). If we denote by D the Levi-Civita connection and R^D the Riemannian curvature of the Riemannian product manifold $M\times I$, then using the second Bianchi identity for the last relation, we get

$$[E_i,\frac{d}{dt}]=0,\qquad D_{\frac{d}{dt}}E_i=0,\qquad R^D(TM,TI)=0,\qquad (D_{\frac{d}{dt}}R^D)(D_{E_i}V,V)E_i=0, \tag{5.6}$$

for all $1\leq i\leq m$. We set $\tilde{\tau}_h:=\tilde{\tau}_h(V)$ and $\tilde{\tau}_v:=\tilde{\tau}_v(V)$. It is easy to see that $\tilde{\tau}_h(V_t)=\tilde{\tau}_h\circ i_t$ and $\tilde{\tau}_v(V_t)=\tilde{\tau}_v\circ i_t$.

We have to calculate the first variational formulae of $E_{2|\mathfrak{X}^1(M)}$. By definition of $\tilde{G}$, we have

$$\begin{aligned}E_2(V_t)=&\frac{a+c}{2}\int_M g\big(\tilde{\tau}_h,\tilde{\tau}_h\big)\circ i_t dv_g+\frac{d}{2}\int_M g\big(\tilde{\tau}_h,V\big)^2\circ i_t dv_g+b\int_M g\big(\tilde{\tau}_h,\tilde{\tau}_v\big)\circ i_t dv_g\\&+\frac{a}{2}\int_M g\big(\tilde{\tau}_v,\tilde{\tau}_v\big)\circ i_t dv_g.\end{aligned} \tag{5.7}$$

Then we get

$$\begin{aligned}D_{\frac{d}{dt}}E_2(V_t)=&(a+c)\int_M g(D_{\frac{d}{dt}}\tilde{\tau}_h,\tilde{\tau}_h)\circ i_t dv_g+d\int_M g(\tilde{\tau}_h,V)g(D_{\frac{d}{dt}}\tilde{\tau}_h,V)\circ i_t dv_g\\&+d\int_M g(\tilde{\tau}_h,V)g(\tilde{\tau}_h,D_{\frac{d}{dt}}V)\circ i_t dv_g+b\int_M g(D_{\frac{d}{dt}}\tilde{\tau}_h,\tilde{\tau}_v)\circ i_t dv_g\\&+b\int_M g(D_{\frac{d}{dt}}\tilde{\tau}_v,\tilde{\tau}_h)\circ i_t dv_g+a\int_M g(D_{\frac{d}{dt}}\tilde{\tau}_v,\tilde{\tau}_v)\circ i_t dv_g.\end{aligned} \tag{5.8}$$

By long computation, using (5.3), (5.4) and (5.6), we get $T(U)=0$ (cf. Appendix A for the details of computation). ∎

Remark 5.1.1. *In the non-compact case, we can define the biharmonicity of unit vector fields by the condition $T(U)=0$, where T is given by (5.5), since it has a tensorial character.*

Proposition 5.1.1. *Let (M,g) be a Riemannian manifold and T_1M be its unit tangent bundle equipped with an arbitrary pseudo-Riemannian g-natural metric $\tilde{G}$. Let U be a unit vector field on M. If $U:(M,g)\longrightarrow(T_1M,\tilde{G})$ is a harmonic map then U is a $\tilde{G}$-biharmonic unit vector field.*

Note that the converse does not hold. $\tilde{G}$-biharmonic unit vector field which are not harmonic maps $(M,g)\to(T_1M,\tilde{G})$ are called a *proper $\tilde{G}$-biharmonic unit vector field.*

5.2 Examples of $\tilde{G}$-biharmonic unit vector fields

Exemple 5.2.1. *Consider the hyperbolic space $\mathbb{H}^n$ $(n>1)$ of constant negative sectional curvature $-k^2$, that is, $\mathbb{H}^n=(\mathbb{R}^n_+,g)$, where $\mathbb{R}^n_+=\{(y_1,...,y_n)\in\mathbb{R}^n: y_n>0\}$ and*

$$g=\frac{1}{k^2y_n^2}(dy_1\otimes dy_1+...+dy_n\otimes dy_n)$$

Vector fields $e_i=ky_n\frac{\partial}{\partial y_i}$ for $i=1,...,n$ provide an orthonormal frame on $\mathbb{H}^n$. Put $V=e_n$. A standard calculation shows that covariant derivatives of e_i can be completly descrined as follows:

$$\nabla_{e_i}e_j=k\delta_{ij}V,\qquad \nabla_{e_i}V=-ke_i,\qquad \nabla_V e_i=0,\qquad \nabla_V V=0, \tag{5.9}$$

for all $i, j < n$. In particular, (5.9) easily implies:

(5.10)
$$||\nabla V||^2 = (n-1)k^2, \quad \overline{\Delta}V = -tr\nabla^2 V = ||\nabla V||^2 V = (n-1)k^2 V = QV, \quad div(V) = (1-n)k,$$

and

$$S(V) = k^2 div(V)V = k^3(1-n)V.$$

The vertical and horizontal parts of the tension field become:

$$\tilde{\tau}_v(V) = 0, \qquad \tilde{\tau}_h(V) = \frac{k}{\varphi}(1-n)(d-ak^2)V. \tag{5.11}$$

Using (5.9), (5.10) and (5.11), the condition $T(V) = 0$ of the biharmonicity of a unit vector field V on M becomes:

$$\tfrac{k^2}{\varphi}(1-n)^2(d-ak^2)\Big\{(-2a+\tfrac{a^2 d}{\alpha})k^2 + \tfrac{abd}{\alpha}k + d\Big\}V = 0. \tag{5.12}$$

It follows that

- *either $d = ak^2$ and, in this case, V is a harmonic map $(M, g) \to (T_1M, \tilde{G})$;*
- *or $d \neq ak^2$. In this case, $d \neq 0$ since if not we have $a \neq 0$ and (5.12) gives then $ak^2 = 0$, which is a contradiction. On the other hand, $a \neq 0$ since if not (5.12) yields $\alpha d = 0$, which is a contradiction. Taking into account these two last facts, (5.12) is equivalent to*

$$b = \left(\frac{2\alpha}{d} - a\right)k - \frac{\alpha}{ak}. \tag{5.13}$$

We conclude then the following

Proposition 5.2.1. *Let $\mathbb{H}^n$ $(n > 1)$ be the hyperbolic space of constant negative sectional curvature $-k^2$ and $\tilde{G}$ be an arbitrary pseudo-Riemannian g-natural metric on its unit tangent bundle. Then the vector field $V = e_n$ is a $\tilde{G}$-biharmonic unit vector field if and only if one of the two following statements is verified:*

1. *$d = ak^2$. In this case, V is a harmonic map $(M, g) \to (T_1M, \tilde{G})$.*
2. *$a \neq 0$, $d \neq 0$, $d \neq ak^2$ and (5.13). In this case, V is a proper $\tilde{G}$-biharmonic unit vector field.*

Corollary 5.2.1. *Let $\mathbb{H}^n$ $(n > 1)$ be the hyperbolic space of constant negative sectional curvature $-k^2$. Then there is a three-parameter family of pseudo-Riemannian g-natural metrics on the unit tangent bundle with respect to which the vector field $V = e_n$ is a proper $\tilde{G}$-biharmonic unit vector field.*

Exemple 5.2.2. *Consider the solvable Lie group Sol_3 as $\mathbb{R}^3$ equipped with the metric $g_{sol} = e^{2z}dx^2 + e^{-2z}dy^2 + dz^2$ and the orthonormal frame field on it [45]:*

$$e_1 = e^{-z}\frac{\partial}{\partial x}, \qquad e_2 = e^{z}\frac{\partial}{\partial y}, \qquad e_3 = \frac{\partial}{\partial z}. \tag{5.14}$$

With respect to this orthonormal frame, the Lie brackets and the Levi-Civita connection can be easily computed as:

$$[e_1,e_2]=0,\quad [e_2,e_3]=-e_2,\quad [e_1,e_3]=e_1,$$

$$\nabla_{e_1}e_1=-e_3,\quad \nabla_{e_1}e_2=0,\quad \nabla_{e_1}e_3=e_1,$$

$$\nabla_{e_2}e_1=0,\quad \nabla_{e_2}e_2=e_3,\quad \nabla_{e_2}e_3=-e_2,$$

$$\nabla_{e_3}e_1=0,\quad \nabla_{e_3}e_2=0,\quad \nabla_{e_3}e_3=0.$$

A further computation gives:

$$R(e_1,e_2)e_1=-e_2,\quad R(e_1,e_3)e_1=e_3,\quad R(e_1,e_2)e_2=e_1,$$

$$R(e_2,e_3)e_2=e_3,\quad R(e_1,e_3)e_3=-e_1,\quad R(e_2,e_3)e_3=-e_2.$$

and the terms in 5.1.1 *become:*

$$\sum_{i=1}^{m}\Big\{2R(e_i,e_1)\nabla_{e_i}e_1+(\nabla_{e_i}R)(e_i,e_1)e_1+R(e_i,\nabla_{e_i}e_1)e_1\Big\}=e_3,$$

$$\sum_{i=1}^{m}\Big\{2R(e_i,e_2)\nabla_{e_i}e_2+(\nabla_{e_i}R)(e_i,e_2)e_2+R(e_i,\nabla_{e_i}e_2)e_2\Big\}=0,$$

$$\sum_{i=1}^{m}\Big\{2R(e_i,e_3)\nabla_{e_i}e_3+(\nabla_{e_i}R)(e_i,e_3)e_2+R(e_i,\nabla_{e_i}e_3)e_3\Big\}=0.$$

$$\overline{\Delta}e_1=e_1,\qquad \overline{\Delta}e_2=e_2,\qquad \overline{\Delta}e_3=2e_3,$$

$$Q(e_1)=0,\qquad Q(e_2)=0,\qquad Q(e_3)=-2e_3,$$

$$div(e_1)=0,\qquad div(e_2)=0,\qquad div(e_3)=0,$$

$$S(e_1)=-e_3,\qquad S(e_2)=e_3,\qquad S(e_3)=0,$$

$$\tilde{\tau}_h(e_1)=(\tfrac{a(a+d)}{\alpha}-\tfrac{b}{\varphi})e_3,\quad \tilde{\tau}_h(e_2)=-\tfrac{a}{\alpha}(a+d)e_3-\tfrac{b}{\varphi}e_2,\quad \tilde{\tau}_h(e_3)=-\tfrac{4b}{\varphi}e_3,$$

$$\tilde{\tau}_v(e_1)=-\tfrac{b}{\alpha}(a+d)e_3,\qquad \tilde{\tau}_v(e_2)=\tfrac{b}{\alpha}(a+d)e_3,\qquad \tilde{\tau}_v(e_3)=0.$$

Using the formulas above, the condition of the $\tilde{G}$-biharmonicity of a unit vector field (5.1.1) for the vector fields of the components of the frame field becomes:

$$T(e_1)=\Big((\tfrac{a(a+d)}{\alpha}-\tfrac{b}{\varphi})(\tfrac{d}{\varphi}-4)-\tfrac{a+d}{\alpha}(\tfrac{\alpha+ad}{\varphi}-2a)\Big)be_3+2d\Big(\tfrac{a(a+d)}{\alpha}-\tfrac{b}{\varphi}\Big)e_1,$$

$$T(e_2)=\tfrac{ab}{\alpha\varphi}(d^2+\alpha-ad)e_3+\Big[\tfrac{b^2}{\varphi}+\tfrac{ad}{\alpha}(a+d)\Big]e_2,$$

$$T(e_3)=\tfrac{8}{\varphi}b^2\Big(2-\tfrac{ad}{\alpha}\Big)e_3.$$

Biharmonicity of e_1:

From the expression of $T(e_1)$ above, e_1 is $\tilde{G}$-biharmonic if and only if

$$\begin{cases} d\Big(\frac{a(a+d)}{\alpha}-\frac{b}{\varphi}\Big)=0,\\ b\Big((\frac{a(a+d)}{\alpha}-\frac{b}{\varphi})(\frac{d}{\varphi}-4)-\frac{a+d}{\alpha}(\frac{\alpha+ad}{\varphi}-2a)\Big)=0\end{cases}\tag{5.15}$$

According to the first equation of (5.15)*, we have two cases:*

Case 1: $d = 0$*. In this case, the second equation of* (5.15) *is equivalent to either* $b = 0$ *or* $b = \frac{a}{4\alpha}(2a(a+c)+\alpha)$.

Case 2: $d \neq 0$*. In this case, the first equation of* (5.15) *is equivalent to*

$$b = \frac{a(a+c)\varphi}{\alpha}, \tag{5.16}$$

and the second equation is equivalent to

$$\frac{a+d}{\alpha}\Big(\frac{\alpha+ad}{\varphi} - 2a\Big)b = 0. \tag{5.17}$$

Taking into account (5.16)*, equation* (5.17) *is equivalent to*

- *either* $a + d = b = 0$*. In this case,* e_1 *is a harmonic map* $(M, g) \to (T_1M, \tilde{G})$*;*
- *or* $b \neq 0$*,* $a + d \neq 0$ *and* $\frac{\alpha+ad}{\varphi} - 2a = 0$*. Then a simple calculation gives*

  $$b^2 = -a\varphi = \frac{-\alpha - ad}{2}. \tag{5.18}$$

  In particular, we have $a \neq 0$*. Substituting from* (5.18) *into* (5.16)*, we obtain by virtue of* $b \neq 0$

  $$b = -\frac{\alpha}{a+d}. \tag{5.19}$$

  Comparing (5.18) *and* (5.19)*, we obtain* $2\alpha^2 + (a+d)^2\alpha + a(a+d)^2d = 0$*. Noticing that this quadratic equation in* α *has a non-zero solution if and only if* $d \in]-\infty, 3a - 2\sqrt{2}|a|] \cup [3a + 2\sqrt{2}|a|, +\infty[$ *with* $d \neq 0$*. In this case the solutions are given by*

  $$\alpha = \frac{-|a+d|}{4}(|a+d| \pm \sqrt{(a+d)^2 - 8ad}). \tag{5.20}$$

Remark 5.2.1. *Equation* (5.20) *shows that* α *depends on 2 parameters* a *and* d*. On the other hand, from* (5.19) *we deduce that* b *depends on* a *and* d*. Consequently, by* $a(a+c) - b^2 = \alpha$*, it is clear that* c *depends on* a *and* d*.*

Biharmonicity of e_2:

From the expression of $T(e_2)$ *above,* e_2 *is* $\tilde{G}$*-biharmonic if and only if*

$$\begin{cases} \frac{ab}{\alpha\varphi}(d^2 + \alpha - ad) = 0, \\ \frac{b^2}{\varphi} + \frac{ad}{\alpha}(a+d) = 0. \end{cases} \tag{5.21}$$

According to the first equation of (5.21)*, we have two cases:*

Case 1: $b = 0$*. In this case, since* $a \neq 0$ *(*$\alpha \neq 0$*), the second equation of* (5.21) *is equivalent to either* $d = 0$ *or* $a + d = 0$*.*

Case 2: $b \neq 0$*. In this case, the first equation of* (5.21) *is equivalent to either* $a = 0$ *or* $\alpha = d(a-d)$*. But* $a = 0$ *gives by the second equation of* (5.21) $b = 0$*, and then* $\alpha = 0$*, which is a contradiction. We deduce that*

$$\alpha = d(a-d) \tag{5.22}$$

and, in particular, $d \neq 0$ *and* $d \neq a$ *(since* $\alpha \neq 0$*). Using* (5.22)*, the second equation of* (5.21) *is equivalent to*

$$b^2 = -\frac{1}{2}(a+d)(2a-d). \tag{5.23}$$

In particular, $(a+d)(2a-d) < 0$*.*

Biharmonicity of e_3:

From the expression of $T(e_3)$ above, e_3 is $\tilde{G}$-biharmonic if and only if

- *either $b = 0$. In this case, e_3 is a harmonic map $(M, g) \to (T_1M, \tilde{G})$;*
- *or $b \neq 0$ and $2\alpha = ad$. In particular, $a \neq 0$ and $d \neq 0$.*

To summarize the previous discussion, we can state the following

Proposition 5.2.2. *Let Sol_3 be the solvable Lie group $\mathbb{R}^3$ equipped with the metric $g_{sol} = e^{2z}dx^2 + e^{-2z}dy^2 + dz^2$ and $\tilde{G}$ be an arbitrary pseudo-Riemannian g-natural metric on its unit tangent bundle. Let $\{e_1, e_2, e_3\}$ be the orthonormal frame field given by*

$$e_1 = e^{-z}\frac{\partial}{\partial x}, \qquad e_2 = e^{z}\frac{\partial}{\partial y}, \qquad e_3 = \frac{\partial}{\partial z}.$$

1. *e_1 is a $\tilde{G}$-biharmonic unit vector field if and only if one of the following assertions holds*
   - *$d = b = 0$ and $a \neq 0$. In this case, e_1 is proper $\tilde{G}$-biharmonic.*
   - *$d = 0$, $b \neq 0$ and $b = \frac{a}{4\alpha}(2a(a + c) + \alpha)$. In this case, e_1 is proper $\tilde{G}$-biharmonic.*
   - *$d \neq 0$ and $b = a + d = 0$. In this case, e_1 is a harmonic map $(M, g) \to (T_1M, \tilde{G})$.*
   - *$a \neq 0$, $d \in \left(]-\infty, 3a - 2\sqrt{2}|a|] \cup [3a + 2\sqrt{2}|a|, +\infty[\right) \setminus \{0\}$ and b and α are given by (5.19) and (5.20), respectively. In this case, e_1 is proper $\tilde{G}$-biharmonic.*
2. *e_2 is a $\tilde{G}$-biharmonic unit vector field if and only if one of the following assertions holds*
   - *$d = b = 0$ and $a \neq 0$. In this case, e_2 is proper $\tilde{G}$-biharmonic.*
   - *$b = 0$, $a + d = 0$. In this case, e_2 is a harmonic map $(M, g) \to (T_1M, \tilde{G})$.*
   - *$a \neq 0$, $d \neq 0$, $(a + d)(2a - d) < 0$ and α and b are given by (5.22) and (5.23), respectively. In this case, e_2 is proper $\tilde{G}$-biharmonic.*
3. *e_3 is a $\tilde{G}$-biharmonic unit vector field if and only if one of the following assertions holds*
   - *$b = 0$. In this case, e_3 is a harmonic map $(M, g) \to (T_1M, \tilde{G})$.*
   - *$a \neq 0$, $d \neq 0$, $b \neq 0$, $a + d = 0$ and $2\alpha = ad$. In this case, e_2 is proper $\tilde{G}$-biharmonic.*

Exemple 5.2.3. *On $SU(2)$ equipped with a left-invariant Riemannian metric, take an orthonormal basis $\{e_1, e_2, e_3\}$ of the Lie algebra $su(2)$ such that[39]:*

$$[e_1, e_2] = \lambda_3 e_3, \quad [e_2, e_3] = \lambda_1 e_1, \quad [e_3, e_1] = \lambda_2 e_2,$$

where λ_1, λ_2 and λ_3 are strictly positive constants with $\lambda_1 \geq \lambda_2 \geq \lambda_3$.
Then the Levi-Civita connexion ∇ is determined by:

$$\nabla_{e_1}e_1 = 0, \qquad \nabla_{e_1}e_2 = \mu_1 e_3, \qquad \nabla_{e_1}e_3 = -\mu_1 e_2,$$

$$\nabla_{e_2}e_1 = -\mu_2 e_3, \qquad \nabla_{e_2}e_2 = 0, \qquad \nabla_{e_2}e_3 = \mu_2 e_1,$$

$$\nabla_{e_3}e_1 = \mu_3 e_2, \qquad \nabla_{e_3}e_2 = -\mu_3 e_1, \qquad \nabla_{e_3}e_3 = 0.$$

where:

$$\mu_i = \frac{1}{2}(\lambda_1 + \lambda_2 + \lambda_3) - \lambda_i, \;\; i = 1, 2, 3.$$

A further computation gives:

$$R(e_1, e_2)e_2 = (\lambda_3\mu_3 - \mu_1\mu_2)e_1, \quad R(e_1, e_3)e_3 = (\lambda_2\mu_2 - \mu_1\mu_3)e_2, \quad R(e_2, e_1)e_1 = (\lambda_3\mu_3 - \mu_1\mu_2)e_2,$$

$$R(e_2, e_3)e_3 = (\lambda_1\mu_1 - \mu_2\mu_3)e_2, \quad R(e_3, e_1)e_1 = (\lambda_2\mu_2 - \mu_1\mu_3)e_3, \quad R(e_3, e_2)e_2 = (\lambda_1\mu_1 - \mu_2\mu_3)e_3,$$

and the other components are zero. Furthermore, we get:

$$\sum_{i=1}^{m}\Big\{2R(e_i, e_j)\nabla_{e_i}e_j + (\nabla_{e_i}R)(e_i, e_j)e_j + R(e_i, \nabla_{e_i}e_j)e_j\Big\} = 0, \quad j = 1, 2, 3.$$

$$div(e_i) = S(e_i) = \tilde{\tau}_v(e_i) = 0, \quad i = 1, 2, 3.$$

$$Q(e_1) = (\lambda_3\mu_3 - \mu_1\mu_2 + \lambda_2\mu_2 - \mu_1\mu_3)e_1, \qquad \overline{\Delta}e_1 = (\mu_2^2 + \mu_3^2)e_1,$$

$$Q(e_2) = (\lambda_3\mu_3 - \mu_1\mu_2 + \lambda_1\mu_1 - \mu_2\mu_3)e_2, \qquad \overline{\Delta}e_2 = (\mu_1^2 + \mu_3^2)e_2,$$

$$Q(e_3) = (\lambda_2\mu_2 - \mu_1\mu_3 + \lambda_1\mu_1 - \mu_2\mu_3)e_3, \qquad \overline{\Delta}e_3 = (\mu_1^2 + \mu_2^2)e_3.$$

To simplify the case we can suppose that $\lambda_1 = \lambda_2 = \lambda$ then $\mu_1 = \frac{\lambda_3}{2}$, $\mu_2 = \frac{\lambda_3}{2}$, $\mu_3 = \lambda - \frac{\lambda_3}{2}$ and we obtain

$$\tilde{\tau}_h(e_1) = -\tfrac{b}{\varphi}(\lambda - \lambda_3)^2 e_1, \quad \tilde{\tau}_h(e_2) = -\tfrac{b}{\varphi}(\lambda - \lambda_3)^2 e_2, \quad \tilde{\tau}_h(e_3) = -\tfrac{b}{\varphi}\tfrac{\lambda_3^2}{2}e_3,$$

Using the formulas above, the condition of the biharmonicity of a unit vector field (5.1.1) for the vector fields of the frame field becomes:

$$T(e_1) = -\tfrac{b^2}{\varphi}(\lambda - \lambda_3)^2\Big[\lambda^2 - \lambda\lambda_3(1 - \tfrac{ad}{\alpha}) - \lambda_3^2(-1 + \tfrac{ad}{2\alpha})\Big]e_1,$$

$$T(e_2) = -\tfrac{b^2}{\varphi}(\lambda - \lambda_3)^2\Big[\lambda^2 - \lambda\lambda_3(1 - \tfrac{ad}{\alpha}) - \lambda_3^2(-1 + \tfrac{ad}{2\alpha})\Big]e_2,$$

$$T(e_3) = 0.$$

So e_3 is a harmonic map $(M, g) \to (T_1M, \tilde{G})$.

But e_1 and e_2 are $\tilde{G}$-biharmonic unit vector fields if and only if

- *either $b = 0$. In this case, e_1 and e_2 are harmonic maps $(M, g) \to (T_1M, \tilde{G})$;*
- *or $\lambda_3 = \lambda$. In this case, e_1 and e_2 are also harmonic maps $(M, g) \to (T_1M, \tilde{G})$;*
- *or $\lambda^2 - \lambda\lambda_3(1 - \frac{ad}{\alpha}) - \lambda_3^2(-1 + \frac{ad}{2\alpha}) = 0$. Put $\lambda = \sigma\lambda_3$, so that $\sigma \geq 1$, and the quadratic equation above becomes $\sigma^2 - (1 - \frac{ad}{\alpha})\sigma - (-1 + \frac{ad}{2\alpha}) = 0$, whose solutions are given by*

  $$\sigma = \frac{1}{2\alpha}\Big(\alpha - ad \pm \sqrt{a^2d^2 - 3\alpha^2}\Big), \tag{5.24}$$

  under the condition $a^2d^2 \geq 3\alpha^2$.

Case 1: *$\sigma = \frac{1}{2\alpha}\Big(\alpha - ad + \sqrt{a^2d^2 - 3\alpha^2}\Big)$. This yields $\alpha(1-2\sigma) - ad = \sqrt{a^2d^2 - 3\alpha^2}$, under the condition $\alpha(1-2\sigma) - ad \geq 0$. A routine calculation gives*

$$ad = \frac{2(\sigma^2 - \sigma + 1)}{1 - 2\sigma}\alpha. \tag{5.25}$$

Note that the condition $\sigma \geq 1$ in equation (5.25) gives the condition $a^2d^2 \geq 3\alpha^2$.

Case 2: *$\sigma = \frac{1}{2\alpha}\Big(\alpha - ad - \sqrt{a^2d^2 - 3\alpha^2}\Big)$. The same arguments as in case 1, gives (5.25), with the condition $\alpha(1-2\sigma) - ad \leq 0$.*

Summarizing the last discussion, we get

Proposition 5.2.3. *Let $SU(2)$ equipped with a left-invariant Riemannian metric and $\tilde{G}$ be an arbitrary pseudo-Riemannian g-natural metric on its unit tangent bundle. Let $\{e_1, e_2, e_3\}$ be an orthonormal basis of the Lie algebra $su(2)$ such that*

$$[e_1, e_2] = \sigma\lambda e_3, \quad [e_2, e_3] = \lambda e_1, \quad [e_3, e_1] = \lambda e_2,$$

where $\lambda > 0$ and $\sigma \geq 1$. Then

1. *e_3 is a harmonic map $(M, g) \to (T_1M, \tilde{G})$.*
2. *e_1 (resp. e_2) is a $\tilde{G}$-biharmonic unit vector field if and only if one of the following assertions holds*
   - *$b = 0$ or $\sigma = 1$. In this case, e_1 and e_2 are harmonic maps $(M, g) \to (T_1M, \tilde{G})$.*
   - *$b \neq 0$, $\sigma \neq 1$ and equation (5.25) holds. In this case e_1 and e_2 are proper $\tilde{G}$-biharmonic unit vector fields.*

5.3 Some particular cases

Next we will investigate some particular cases of Theorem 5.1.1:

Theorem 5.3.1. *Let (M, g) be a Riemannian manifold and T_1M be its unit tangent bundle equipped with an arbitrary pseudo-Riemannian Kaluza-Klein metric $\tilde{G}$ (i.e. $b = d = 0$). Let $U \in \mathfrak{X}_1(M)$. U is a $\tilde{G}$-biharmonic unit vector field if and only if*

$$\tfrac{a}{a+c}\sum_{i=1}^{m}\Big\{2R(e_i, S(U))\nabla_{e_i}U + (\nabla_{e_i}R)(e_i, S(U))U + R(e_i, \nabla_{e_i}S(U))U\Big\}$$

$$+\overline{\Delta}\overline{\Delta}U - \overline{\Delta}\big[g(\overline{\Delta}U, U)U\big] - g(\overline{\Delta}U, U)\overline{\Delta}U + g(\overline{\Delta}U, U)^2U = 0.$$

If $a = 1$ and $c = 0$ i.e. $\tilde{G} = \widetilde{g^s}$, the result above is the same derived by M. Markellos and H.Urakawa in [38] in the case of Sasaki metric.

Given a compact Riemannian manifold (M, g). In [5], the authors defined a *haramonic unit vector field* U on M as a critical point of the energy functional $E(U) = \frac{1}{2}\int_M ||dU||^2 v_g$ restricted to $\mathfrak{X}^1(M)$, the set of all unit vector fields, and they showed that this kind of harmonicity of U does not depend upon the choice of the chosen g-natural metric $\tilde{G}$. More precisely, U is a harmonic unit vector field if and only if $\overline{\Delta}U$ and U are collinear. As a consequence of this result we have

Corollary 5.3.1. *Let (M,g) be a Riemannian manifold and T_1M be its unit tangent bundle equipped with an arbitrary pseudo-Riemannian Kaluza-Klein metric $\tilde{G}$ (i.e. $b=d=0$). A harmonic unit vector field U on M is a $\tilde{G}$-biharmonic unit vector field if and only if*

(5.26)
$$\frac{a^2}{\alpha}\sum_{i=1}^{m}\Big\{2R(e_i,S(U))\nabla_{e_i}U+(\nabla_{e_i}R)(e_i,S(U))U+R(e_i,\nabla_{e_i}S(U))U\Big\} \text{ and } U \text{ are collinear.}$$

If $a=1$ and $c=0$ i.e. $\tilde{G}=\widetilde{g}^s$, the result above is the same derived by M. Markellos and H. Urakawa in [38] in the case of Sasaki metric.

Theorem 5.3.2. *Let (M,g) be a Riemannian manifold and T_1M be its unit tangent bundle equipped with an arbitrary pseudo-Riemannian Kaluza-Klein type metric $\tilde{G}$ (i.e. $b=0$). Let $U\in\mathfrak{X}^1(M)$ satisfy $S(U)=-\frac{d}{a}\mathrm{div}(U)U$. Then U is a $\tilde{G}$-biharmonic unit vector field if and only if*

(5.27)
$$\sum_{i=1}^{m}\Big\{2R(e_i,\tilde{\tau}_h(U))\nabla_{e_i}U+(\nabla_{e_i}R)(e_i,\tilde{\tau}_h(U))U+R(e_i,\nabla_{e_i}\tilde{\tau}_h(U))U)+\tfrac{d}{a}g\big(\nabla_{e_i}U,\tilde{\tau}_h(U)\big)e_i\Big\}$$

$$+\overline{\Delta}\tilde{\tau}_v(U)-\tfrac{d}{a}\nabla_U\tilde{\tau}_h(U)-g(\overline{\Delta}U,U)\tilde{\tau}_v(U)-\tfrac{d(a+c)}{a\varphi}\mathrm{div}(U)\tilde{\tau}_h(U)=0.$$

Corollary 5.3.2. *Let (M,g) be a Riemannian manifold of constant sectional curvature k and T_1M be equipped with an arbitrary pseudo-Riemannian Kaluza-Klein type metric $\tilde{G}$ (i.e. $b=0$) such that $d=-ak$. A unit geodesic vector field U, i.e. $\nabla_UU=0$, is a $\tilde{G}$-biharmonic unit vector field if and only if*

(5.28)
$$2k\nabla_U\tilde{\tau}_h(U)+2k\,\mathrm{div}(U)\tilde{\tau}_h(U)+\tfrac{k(a+c)}{\varphi}\mathrm{div}(U)\tilde{\tau}_h(U)-\overline{\Delta}\tilde{\tau}_v(U)+g(\overline{\Delta}U,U)\tilde{\tau}_v(U)=0.$$

We can also discuss Theorem 5.1.1 for some special classes of vector fields such that **Killing vector fields** and **Reeb vector fields**.

Let (M,g) be a Riemannian manifold. As it is well known, $U\in\mathfrak{X}(M)$ is a Killing vector field if the local 1-parameter group of U consists of local isometries of g. Moreover, a vector field U is a Killing vector field if and only if $\mathcal{L}_Ug=0$, where $\mathcal{L}$ denotes the Lie derivative, and we have in this case [5]:

$$div(U)=0,\quad \nabla_UU=0,\quad \overline{\Delta}U=QU,\quad g(QU,U)=||\nabla U||^2,$$

Let $\tilde{G}$ be an arbitrary pseudo-Riemanniann g-natural metric on T_1M with $b=0$ and suppose that $S(U)=0$. Then the horizontal and vertical parts of the tension field are given by:

$$\tilde{\tau}_h(U)=0,\quad \tilde{\tau}_v(U)=QU-||\nabla U||^2U=\mathcal{A}U.$$

Then, as a corollary of Theorem 5.1.1, we get

Corollary 5.3.3. *Let (M,g) be a Riemannian manifold and T_1M be equipped with an arbitrary pseudo-Riemannian g-natural Kaluza-Klein type metric $\tilde{G}$ (i.e. $b=0$). Let U be a unit Killing vector field on M which satisfies $S(U)=0$ and suppose that QU and U are not collinear. Then U is a proper $\tilde{G}$-biharmonic unit vector field if and only if*

$$\overline{\Delta}\mathcal{A}U=||\nabla U||^2\mathcal{A}U. \tag{5.29}$$

Now we consider the special case when the unit vector filed is the Reeb vector field ξ of a contact metric manifold (M,η,g).

A $(2m+1)$-dimensional manifold M is said to be a *contact metric manifold* if it admits a global 1-form η such that $\eta\wedge(d\eta)^m \neq 0$. There is a unique vector field ξ, called the *Reeb vector field* (or the characteristic vector field), such that $\eta(\xi)=1$ and $d\eta(\xi,.)=0$. Furthermore, a Riemannian metric g is said to be an *associated metric* if there exists a tensor field φ of type $(1,1)$ such that

$$\eta = g(\xi,.),\ \ d\eta = g(.,\varphi.),\ \ \varphi^2 = -I+\eta\otimes\xi. \tag{5.30}$$

By (5.30) it follows that ξ is a unit vector field on (M,g), and it satisfies

$$\nabla\xi = -\varphi-\varphi h,\quad \nabla_\xi\xi = 0,\quad div(\xi)=0$$

$$||\nabla\xi||^2 = 2m+tr(h^2) = 4m-g(Q\xi,\xi),\ \ \overline{\Delta}\xi = 4m\xi - Q\xi$$

where $h=\frac{1}{2}\mathcal{L}_\xi\varphi$.

Let $\tilde{G}$ be an arbitrary pseudo-Riemannian g-natural metric on T_1M with $b=0$, and we suppose that $S(\xi)=0$, then the horizontal and vertical parts of the tension filed are given by:

$$\tilde{\tau}_h(\xi)=0,\ \ \tilde{\tau}_v(\xi) = Q\xi - g(Q\xi,\xi)\xi = \mathcal{A}\xi.$$

Then we have

Corollary 5.3.4. *Let (M,η,g) be a contact metric manifold and T_1M be equipped with an arbitrary pseudo-Riemannian g-natural Kaluza-Klein type metric $\tilde{G}$ (i.e. $b=0$). Suppose that the Reeb vector field ξ on M satisfies $S(\xi)=0$ and we suppose that $Q\xi$ and ξ are not collinear. Then ξ is a proper $\tilde{G}$-biharmonic unit vector field if and only if*

$$\overline{\Delta}\mathcal{A}\xi = ||\nabla\xi||^2\mathcal{A}\xi. \tag{5.31}$$

Chapter 6

Biharmonic functions on vector bundles

This chapter is devoted to the investigation of biharmonic functions on vector bundles. We will study the biharmonicity of vertical lifts of functions and we will give an infinite two-parameter family of proper biharmonic functions on vector bundle manifolds equipped with spherically symmetric metrics.

It is worth mentioning that the results of this chapter are completly extracted from our paper [8].

6.1 Preliminaries

6.1.1 Vector bundles and spherically symmetric metrics

Let (E, π, M) be a vector bundle of rank $k \geq 1$ with Riemannian base (M, g). We assume that E is endowed with a fiber metric h and a compatible connection D (i.e. $Dh = 0$). Let K denote the connection map associated with D. Denote by $\mathcal{V}$ the *vertical subbundle*, that is the subbundle given by

$$\mathcal{V} = \bigcup_{e \in E} V_e E,$$

where, for all $e \in E$, $V_e E = \ker(d_e \pi)$ is the *vertical subspace* of $T_e E$. The elements of $\mathcal{V}$ are called *vertical vectors* and a vector field on E is said to be a *vertical vector field* if it lies completely in $\mathcal{V}$.

Moreover, if $x \in M$ and $e, u \in \pi^{-1}(x)$, set $\gamma_{e,u}(t) = e + tu$ with $t \in (-\epsilon, \epsilon)$, then $\gamma_{e,u}(t) \in \pi^{-1}(x)$, for all $|t| < \epsilon$, then the *vertical lift* of u at e is defined to be the vertical vector given by $u_e^v = \dfrac{d}{dt}\Big|_{t=0} \gamma_{e,u}(t)$. Using vertical lifts of elements of E to vertical vectors, we can lift sections of E. Indeed, if σ is a section of E, its *vertical lift* is the vertical vector field σ^v defined by

$$\sigma^v(e) = \sigma(\pi(e))_e^v,$$

where $\sigma(\pi(e))_e^v$ is the vertical lift of $\sigma(\pi(e))$ at e. The vertical subbundle $\mathcal{V}$ is naturally isomorphic to the pullback of $E \xrightarrow{\pi} M$ by π which we denote by $\pi^\star E \longrightarrow E$ (see [22, 47]).

The connection D gives rise, at each point $e \in E$, to the *horizontal subspace* given by $H_e E = \ker(K_e)$, which is complementary to the vertical subspace i.e. $T_e E = V_e E \oplus H_e E$, which gives the *horizontal subbundle*

$$\mathcal{H} = \bigcup_{e \in E} H_e E,$$

such that

$$TE = \mathcal{H} \oplus \mathcal{V}. \tag{6.1}$$

Elements of $\mathcal{H}$ are called *horizontal vectors*, and a vector field is said to be a *horizontal vector field* if it lies completely in $\mathcal{H}$. If $x \in M$ and $X \in T_xM$, the *horizontal lift* of X at e is the horizontal vector X_e^h satisfying $d_e\pi(X_e^h) = X$. Further, if $X \in \mathfrak{X}(M)$, then the horizontal vector lift of X is the horizontal vector field $X^h \in \mathfrak{X}(E)$ such that $(X^h)_e$ is the horizontal lift of $X_{\pi(x)}$ at e. The horizontal sub-bundle is naturally isomorphic to the pullback vector bundle $\pi^*TM \longrightarrow E$, thus

$$\mathcal{H} \oplus \mathcal{V} = TE \simeq \pi^*TM \oplus \pi^\star E$$

The later splitting gives rise to a splitting of vector fields of E. Indeed, for all $Z \in \mathfrak{X}(E)$, we have $Z = Z^H + Z^V$, where Z^H is the horizontal component and Z^V is the vertical component.

In the context of Riemannian geometry of the tangent bundle of a Riemannian manifold, S. Sasaki introduced, in [48], a Riemannian metric induced from the Riemannian metric on the base manifold known as the *Sasaki metric*, we refer the reader to [1, 2] for details on the Riemannian geometry of tangent bundles. It had been shown that the Sasaki metric presents a strong rigidity in the sense of [33, 42]. In order to overcome such a rigidity, many classes of metrics were introduced (e.g [20, 42]).

In the case of tangent bundles, the Sasaki metric can be easily expressed as follows. If (M, g) is an m-dimensional Riemannian manifold. The Sasaki metric G^s on the tangent bundle TM of M is given by:

- $G^s(X^h, Y^h) = g(X, Y)$
- $G^s(X^h, Y^v) = 0$
- $G^s(X^v, Y^v) = g(X, Y)$

for all vectors fields X, Y on M.

A more general class of Riemannian metrics on the total spaces of vector bundles, the so-called *spherically symmetric metrics*, was introduced in [18]. Those metrics are defined as follows

$$G_e(X, Y) = e^{2\varphi_1} g(d\pi(X), d\pi(Y)) + e^{2\varphi_2} h(K_e(X), K_e(Y)), \tag{6.2}$$

where φ_1, φ_2 are smooth scalar functions on E depending only on the norm $r = \|e\|^2 = h(e, e)$ and smooth on $(0, +\infty)$ and at $r = 0$ on the right as well as all their successive derivatives. Spherically symmetric metrics may be expressed using pullbacks as follows

$$G = e^{2\varphi_1}\pi^* g \oplus e^{2\varphi_2}\pi^\star h, \tag{6.3}$$

In the case, $\varphi_1 = \varphi_2 = 0$, we obtain the generalization of the Sasaki metric to vector bundle manifolds. Further, if $E = TM$, we obtain the Sasaki metric itself.

Spherically symmetric metrics have been extensively studied in [18, 11, 12, 13, 15]. We shall recall some necessary technical tools we will be using in the forthcoming calculations.

In what follows, we will make use of the following notations:

(i) $\langle ., . \rangle_M = \pi^* g$,

(ii) $\langle ., . \rangle_E = \pi^\star h$,

(iii) $\langle ., . \rangle = \pi^* g \oplus \pi^\star h$.

The Levi-Civita connection of the metric given in (6.2) is computed in [18] and given as follows:

Proposition 6.1.1. *Let X and Y be two vector fields on E, then*

$$\tilde{\nabla}_X Y = \tilde{D}_X Y + C_X Y + A_X Y - \frac{1}{2}\mathcal{R}^\xi(X,Y),$$

where

i) $\tilde{D} = \pi^*\nabla \oplus \pi^\star D$, *with* ∇ *is the Levi-Civita connection of* (M,g);

ii) $C(.,.)$ *is the vector valued form given by*

$$\begin{aligned} C_X Y = a(\xi^\flat(X)Y^H + \xi^\flat(Y)X^H) + c_1\langle X^H, Y^H\rangle\xi + c_2\langle X^V, Y^V\rangle\xi \\ + b(\xi^\flat(X)Y^V + \xi^\flat(Y)X^V), \end{aligned}$$

with

$$\begin{cases} a = 2\varphi_1', & c_1 = -2\varphi_1' e^{2(\varphi_1-\varphi_2)}, \\ b = 2\varphi_2', & c_2 = -2\varphi_2', \end{cases}$$

for all X, $Y \in \mathfrak{X}(E)$, ξ *being the tautological section of the vertical sub-bundle defined by* $\xi_e = e_e^v \in \pi^\star E$ *and the '* $\flat$ *' is taken with respect to* h;

iii) $\mathcal{R}^\xi(X,Y) = \pi^\star R^E(X,Y)\xi$, *for all* X, $Y \in \mathfrak{X}(E)$ *with* R^E *is the curvature of* D;

iv) $A(.,.)$ *is the* π^*TM*-valued form defined*

$$e^{2\varphi_1}\langle A(X,Y),Z\rangle_M = \frac{e^{2\varphi_2}}{2}\left(\langle\mathcal{R}^\xi(X,Z),Y\rangle_E + \langle\mathcal{R}^\xi(Y,Z),X\rangle_E\right),$$

for all X, Y *and* $Z \in \mathfrak{X}(E)$.

6.1.2 Some classes of functions on vector bundles

Total spaces of vector bundles, and particularly tangent bundles, can be endowed with different classes of functions. We shall focus on lifted functions, that is functions on total spaces constructed from functions on the base manifold.

In the general context of vector bundles the first and third authors studied in [15] the harmonicity of the following classes of functions:

($\mathcal{C}_1$) Vertical lifts of functions on M, that is functions of the form $f^v = f \circ \pi$ where $f \in C^\infty(M)$.

($\mathcal{C}_2$) Functions of the form $F : E \longrightarrow \mathbb{R}$ such that $F(e) = \alpha(r)$, where $r = h(e,e)$ and $\alpha : \mathbb{R}^+ \longrightarrow \mathbb{R}$ is real valued function which is smooth on $(0,+\infty)$ and at $r = 0$ on the right. We refer to those functions as r-radial functions.

($\mathcal{C}_3$) Functions of the form $F_\sigma : E \longrightarrow \mathbb{R}$ such that $F_\sigma(e) = h(\sigma(\pi(e)),e)$, where $\sigma \in \Gamma(E)$ is a section.

Henceforward, we shall always endow the total space of a vector bundle with a spherically symmetric metric G, with weight functions φ_1 and φ_2, of the form (6.2) (or equivalently (6.3)).

In what follows, we investigate biharmonic functions on total spaces of vector bundles when endowed with spherically symmetric metrics. So, we assume that (E,π,M) is a vector bundle for which E is endowed with a spherically symmetric metrics of the form (6.3). We shall begin with vertical lifts then we consider r-radial functions.

6.2 Biharmonicity of vertical lifts of functions to vector bundles

We shall make calculations in a particularly suitable orthonormal frame which we construct as follows. Let $e \in E$ with $x = \pi(e)$, and let $\{e_i : i = 1, ..., m\}$ be a local linear orthonormal frame of M in a neighbourhood of x. For $i = 1, ..., m$, we denote by e_i^h the horizontal lift of e_i to E with respect to the connection D. On the other hand, let $\{\sigma_p : p = 1, ..., k\}$ be an orthonormal frame of E and let σ_p^v, for $p = 1, ..., k$, be the vertical lift of σ_p. Then set

$$E_i = e^{-\varphi_1} e_i^h \quad \text{and} \quad E_{m+p} = e^{-\varphi_2}\sigma_p^v, \quad i = 1, ..., m,\ p = 1, ..., k; \tag{6.4}$$

therefore $\{E_I : I = 1, ..., m+k\}$ is a local linear orthonormal frame of E in a neighbourhood of e. We refer to such frames as the *induced local frames* on E.

For the sake of completeness, we recall the following lemma (see [18]) which gives the directional derivatives, of functions depending on r, in the direction of horizontal and vertical vectors:

Lemma 6.2.1. *Let α be a smooth real scalar function. Then, for any horizontal (resp. vertical) vector X^H (resp. Y^V) on E, we have*

i) $X^H(\alpha(r)) = 0$;

ii) $Y^V(\alpha(r)) = 2\alpha'(r)\xi^\flat(Y^V)$.

We will also be using the following formulas for the directional derivatives of the vertical lift of a function in the direction of horizontal and vertical vector fields:

- For every $f \in \mathcal{C}^\infty(M)$ and every vertical vector field X^V, we have $X^V(f^v) = 0$.
- For every $f \in \mathcal{C}^\infty(M)$ and every vector field $X \in \mathfrak{X}(M)$, we have $X^h(f^v) = (Xf)^v$.

Using induced local frames of the form (6.4) and Lemma 6.2.1, in [15], the authors proved the following formulas:

- For every $f \in \mathcal{C}^\infty(M)$, we have $\tilde{\nabla} f^v = e^{-2\varphi_1}(\nabla f)^h$.
- For every $f \in \mathcal{C}^\infty(M)$, $\Delta_G(f^v) = e^{-2\varphi_1}(\Delta_g f)^v$.
- For every r-radial function $F(e) = \alpha(r)$, we have $\tilde{\nabla} F = 2e^{-2\varphi_2}\alpha'\xi$, where ξ is the tautological vector field defined earlier.
- $div(\xi) = 2mr\varphi_1' + (1 + 2r\varphi_2')k$.

Using all the previous formulas, we have:

Proposition 6.2.1. *For every smooth function f on M, we have*

$$\begin{aligned}\Delta_G^2(f^v) = e^{-4\varphi_1}\left(\Delta_g^2 f\right)^v - 4e^{-2(\varphi_1+\varphi_2)}\Big(&2r\varphi_1'' - 4r\varphi_1'(\varphi_1+\varphi_2)' \\ &+ 2mr(\varphi_1')^2 + 2kr\varphi_1'\varphi_2' + k\varphi_1'\Big)(\Delta_g f)^v.\end{aligned}$$

Proof. Let $f : M \longrightarrow \mathbb{R}$ be a smooth function. Then

$$\begin{aligned}\Delta_G(\Delta_G f^v) &= \Delta_G\big(e^{-2\varphi_1}(\Delta_g f)^v\big) \\ &= \Delta_G(e^{-2\varphi_1})(\Delta_g f)^v + e^{-2\varphi_1}\Delta_G\big((\Delta_g f)^v\big) \\ &\quad + 2G\big(\tilde{\nabla}(e^{-2\varphi_1}), \tilde{\nabla}((\Delta_g f)^v)\big).\end{aligned}$$

On the other hand, we have

$$\begin{aligned} G\big(\tilde{\nabla}(e^{-2\varphi_1}),\tilde{\nabla}((\Delta_g f)^v)\big) &= G\big(-4\varphi_1^{'}e^{-2(\varphi_1+\varphi_2)}\xi, e^{-2\varphi_1}(\nabla\Delta_g f)^h\big) \\ &= -4\varphi_1^{'}e^{-2(2\varphi_1+\varphi_2)}G\big(\xi,(\nabla\Delta_g f)^h\big) \\ &= 0, \end{aligned} \tag{6.5}$$

$$\Delta_G\big((\Delta_g f)^v\big) = e^{-2\varphi_1}(\Delta_g^2 f)^v, \tag{6.6}$$

and

$$\begin{aligned} \Delta_G\big(e^{-2\varphi_1}\big) &= div(\tilde{\nabla}(e^{-2\varphi_1})) \\ &= -4div(\varphi_1^{'}e^{-2(\varphi_1+\varphi_2)}\xi) \\ &= -4\xi.(\varphi_1^{'}e^{-2(\varphi_1+\varphi_2)}) - 4\varphi_1^{'}e^{-2(\varphi_1+\varphi_2)}div(\xi) \\ &= -4\xi.(\varphi_1^{'})e^{-2(\varphi_1+\varphi_2)} - 4\varphi_1^{'}\xi.(e^{-2(\varphi_1+\varphi_2)}) - 4\varphi_1^{'}e^{-2(\varphi_1+\varphi_2)}div(\xi) \\ &= -8r\varphi_1^{''}e^{-2(\varphi_1+\varphi_2)} + 16r\varphi_1^{'}(\varphi_1+\varphi_2)^{'}e^{-2(\varphi_1+\varphi_2)} \\ &\quad - 4\varphi_1^{'}\big(2mr\varphi_1^{'} + (1+2r\varphi_2^{'})k\big)e^{-2(\varphi_1+\varphi_2)} \\ &= -4e^{-2(\varphi_1+\varphi_2)}\big(2r\varphi_1^{''} - 4r\varphi_1^{'}(\varphi_1+\varphi_2)^{'} + 2mr(\varphi_1^{'})^2 \\ &\quad + 2kr\varphi_1^{'}\varphi_2^{'} + k\varphi_1^{'}\big). \end{aligned} \tag{6.7}$$

Finally, we get

$$\begin{aligned} \Delta_G^2(f^v) = e^{-4\varphi_1}(\Delta_g^2 f)^v - 4e^{-2(\varphi_1+\varphi_2)}\big(2r\varphi_1^{''} - 4r\varphi_1^{'}(\varphi_1+\varphi_2)^{'} \\ + 2mr(\varphi_1^{'})^2 + 2kr\varphi_1^{'}\varphi_2^{'} + k\varphi_1^{'}\big)(\Delta_g f)^v. \end{aligned} \tag{6.8}$$

□

A straightforward result can be formulated as follows: Let $f : M \longrightarrow \mathbb{R}$ be a harmonic function, then f^v is a biharmonic function on (E, G).

In what follows, we shall make certain choices of the functions φ_1 and φ_2 in order to construct (proper) biharmonic functions on E from (proper) biharmonic functions on M.

Remark that the vertical lift of a (proper) biharmonic function on M is a (proper) biharmonic function on E if and only if

$$(\mathcal{E}) \quad 2r\varphi_1^{''} - 4r\varphi_1^{'}(\varphi_1+\varphi_2)^{'} + 2mr(\varphi_1^{'})^2 + 2rk\varphi_1^{'}\varphi_2^{'} + k\varphi_1^{'} = 0.$$

Hence, we shall focus on finding weight functions φ_1 and φ_2 that satisfy the previous equation.

Corollary 6.2.1. *Assume that φ_1 is constant and let $f : M \longrightarrow \mathbb{R}$ be a (proper) biharmonic function on (M, g), then f^v is also a (proper) biharmonic function on (E, G).*

Proof. If φ_1 is constant, then the equation $(\mathcal{E})$ is trivially verified. Furthermore, if f is a proper biharmonic function, then f^v is in particular biharmonic, and since $\Delta_G f^v = e^{-2\varphi_1}(\Delta_g f)^v$ and $\Delta_g f \neq 0$, then $\Delta_G f^v \neq 0$, which implies that f^v is also a proper biharmonic function. □

Exemple 6.2.1. *Consider Euclidean space without the origin $M = \mathbb{R}^n\backslash\{0\}$ with its Euclidean metric g_0. Let $f : \mathbb{R}^n \to \mathbb{R}$ be the polynomial function defined by*

$$f(x_1, ..., x_n) = \left(\sqrt{x_1^2 + \cdots + x_n^2}\right)^{-1}$$

A straightforward calculation shows that

$$\Delta_{g_0}^2 f = \frac{3(n-5)(n-3)}{\left(\sqrt{x_1^2 + \cdots + x_n^2}\right)^5}$$

If $n = 3$ or $n = 5$, then f is a biharmonic function on (M, g_0). Further, we prove that $\Delta_{g_0} f = (3-n)f^3$, so if $n = 5$, the function is in fact proper biharmonic.

Now, let $\pi : E \longrightarrow M$ be an arbitrary vector bundle and assume either $n = 3$ or $n = 5$. We endow E with the spherically symmetric metric $G = \pi^* g_0 \oplus e^{2\varphi_2}\pi^\star g_0$, where φ_2 is an arbitrary function on $[0, +\infty)$ satisfying the regularity requirements of a weight function of a spherically symmetric metric. Then, the function $f^v : E \longrightarrow \mathbb{R}$ is a biharmonic function on (E, G). Furthermore, when $n = 5$, the function $f^v : E \longrightarrow \mathbb{R}$ is a proper biharmonic function.

Trying to solve equation $(\mathcal{E})$ in various particular cases leads to a differential equation for which the solutions are singular at zero which contradicts the regularity requirements for the weight functions of a spherically symmetric metric.

To make this point clearer, we suggest the following example. Remark that equation $(\mathcal{E})$ may be rewritten as

$$(\mathcal{E}') \quad 2r\varphi_1^{''} + 2r(m-2)(\varphi_1^{'})^2 + 2r(k-2)\varphi_1^{'}\varphi_2^{'} + k\varphi_1^{'} = 0.$$

Proposition 6.2.2. *In the same notations as before, assume that $m = 2$ and $k = 2$. If for every biharmonic function f on M, the vertical lift f^v is a biharmonic function on E, then φ_1 is constant.*

Proof. Under the conditions $m = 2$ and $k = 2$, that is E is a plane bundle over a Riemannian surface, equation $(\mathcal{E}')$ becomes

$$r\varphi_1^{''} + \varphi_1 = 0.$$

Set $\psi = \varphi_1'$, the equation becomes a first order differential equation which can be integrated, for $r \neq 0$, as follows

$$\psi(r) = \alpha e^{-\int \frac{dr}{r}},$$

which gives

$$\psi(r) = \frac{\alpha}{r},$$

for every $r \in]0, +\infty[$, where α is a real number. Finally we obtain, for every $r \in]0, +\infty[$, $\varphi_1(r) = \alpha \ln(r) + \beta$, with α and β are real numbers.

By the definition of spherically symmetric metrics, φ_1 should necessarily be continuous at 0, in particular, it should have a finite limit at 0, therefore $\alpha = 0$, otherwise f will have an infinite limit at zero on the right. □

6.3 Biharmonicity of r-radial functions

In this section, we accomplish a study of the biharmonicity of r-radial functions. For the rest of this section, let $\alpha : \mathbb{R}^+ \longrightarrow \mathbb{R}$ be a real valued function which is smooth on $(0, +\infty)$

and smooth at $r = 0$ on the right as well as all its successive derivatives; so the associated r-radial function $F : E \longrightarrow \mathbb{R}$ is given by $F(e) = \alpha(r)$ for $e \in E$, with $r = h(e, e)$.

Along with Lemma 6.2.1, the following lemma will be used in calculations and can be found with its proof in [18]. We recall it for the sake of completeness.

Lemma 6.3.1. *Let $X \in \mathfrak{X}(E)$, then $\pi^\star D_X\xi = X^V$, where X^V is the vertical component of X.*

Proposition 6.3.1. *(see [15]) Let $\alpha : \mathbb{R}^+ \longrightarrow \mathbb{R}$ be a smooth function on $\mathbb{R}^+$, and let $F : E \longrightarrow \mathbb{R}$ be the r-radial function induced from α, then*

$$\Delta_G F = 4e^{-2\varphi_2}\Big\{r\alpha'' + \big(mr\varphi_1' + (k-2)r\varphi_2' + \frac{k}{2}\big)\alpha'\Big\}$$

Using the previous expression of the Laplacian $\Delta_G(F)$, we find:

Proposition 6.3.2. *Let $\alpha : \mathbb{R}^+ \longrightarrow \mathbb{R}$ be a smooth function on $\mathbb{R}^+$, and let $F : E \longrightarrow \mathbb{R}$ be the r-radial function induced from α, then*

$$\begin{aligned}
\Delta_G^2(F) = {} & -16\varphi_2' e^{-4\varphi_2}\xi\Big(r\alpha'' + (mr\varphi_1' + (k-2)r\varphi_2' + \tfrac{k}{2})\alpha'\Big) \\
& +4e^{-2\varphi_2}\Big[e^{-2\varphi_2}\Big(r\alpha^{(3)} + \Big(m\varphi_1' + mr\varphi_1'' + (k-2)\varphi_2' \\
& +r(k-2)\varphi_2''\Big)\alpha' + \Big(mr\varphi_1' + r(k-2)\varphi_2' + \tfrac{k}{2} + 1\Big)\alpha''\Big)\Big] div(\xi) \\
& -8\varphi_2' r e^{-2\varphi_2}\Big(r\alpha^{(3)} + \Big(m\varphi_1' + mr\varphi_1'' + (k-2)\varphi_2' \\
& +r(k-2)\varphi_2''\Big)\alpha' + \Big(mr\varphi_1' + r(k-2)\varphi_2' + \tfrac{k}{2} + 1\Big)\alpha''\Big) \\
& +2e^{-2\varphi_2}\Big(2r\alpha^{(3)} + 2r^2\alpha^{(4)} + \Big(2mr\varphi_1'' + 2mr\varphi_1'' + 2mr^2\varphi_1^{(3)} \\
& +2r(k-2)\varphi_2'' + 2r(k-2)\varphi_2'' + 2r^2(k-2)\varphi_2^{(3)}\Big)\alpha' \\
& +2r\Big(m\varphi_1' + mr\varphi_1'' + (k-2)\varphi_2' + r(k-2)\varphi_2''\Big)\alpha'' \\
& +\Big(2mr\varphi_1' + 2mr^2\varphi_1'' + 2r(k-2)\varphi_2' + 2r^2(k-2)\varphi_2''\Big)\alpha'' \\
& +2r\Big(mr\varphi_1' + r(k-2)\varphi_2' + \tfrac{k}{2} + 1\Big)\alpha^{(3)}\Big)\Big] \\
& -8r\varphi_2' e^{-6\varphi_2}\Big(r\alpha^{(3)} + \Big(m\varphi_1' + mr\varphi_1'' + (k-2)\varphi_2' \\
& +r(k-2)\varphi_2''\Big)\alpha' + \Big(mr\varphi_1' + r(k-2)\varphi_2' + \tfrac{k}{2} + 1\Big)\alpha''\Big).
\end{aligned}$$

Proof.

$$
\begin{aligned}
\Delta_G^2(F) &= 4\Delta_G\Big(e^{-2\varphi_2}\big(r\alpha'' + (mr\varphi_1' + (k-2)r\varphi_2' + \tfrac{k}{2})\alpha'\big)\Big) \\
&= 4\Big[\Delta_G(e^{-2\varphi_2})\Big(r\alpha'' + (mr\varphi_1' + (k-2)r\varphi_2' + \tfrac{k}{2})\alpha'\Big) \\
&\quad + e^{-2\varphi_2}\Delta_G\big(r\alpha'' + (mr\varphi_1' + (k-2)r\varphi_2' + \tfrac{k}{2})\alpha'\big) \\
&\quad + 2G\big(\tilde{\nabla}(e^{-2\varphi_2}), \tilde{\nabla}[r\alpha'' + (mr\varphi_1' + (k-2)r\varphi_2' + \tfrac{k}{2})\alpha']\big)\Big].
\end{aligned} \tag{6.9}
$$

Set

$$
L = r\alpha'' + (mr\varphi_1' + (k-2)r\varphi_2' + \frac{k}{2})\alpha'.
$$

We have:

$$
\tilde{\nabla}(e^{-2\varphi_2}) = -4\varphi_2' e^{-4\varphi_2}\xi, \tag{6.10}
$$

and

$$
\begin{aligned}
\tilde{\nabla}L &= \tilde{\nabla}r.\alpha'' + r.\tilde{\nabla}\alpha'' + \big(m(\tilde{\nabla}r.\varphi_1' + r\tilde{\nabla}\varphi_1') + (k-2)(\tilde{\nabla}r.\varphi_2' + r\tilde{\nabla}\varphi_2')\big)\alpha' \\
&\quad + \big(mr\varphi_1' + r(k-2)\varphi_2' + \tfrac{k}{2}\big)\tilde{\nabla}\alpha' \\
&= 2e^{-2\varphi_2}\alpha''\xi + 2re^{-2\varphi_2}\alpha^{(3)}\xi + \big(m(2e^{-2\varphi_2}\varphi_1'\xi + 2r\varphi_1'' e^{-2\varphi_2}\xi) \\
&\quad + (k-2)(2\varphi_2' e^{-2\varphi_2}\xi + 2r\varphi_2'' e^{-2\varphi_2}\xi)\big)\alpha' \\
&\quad + 2\alpha'' e^{-2\varphi_2}\big(mr\varphi_1' + r(k-2)\varphi_2' + \tfrac{k}{2}\big)\xi \\
&= e^{-2\varphi_2}\Big(r\alpha^{(3)} + \Big(m\varphi_1' + mr\varphi_1'' + (k-2)\varphi_2' + r(k-2)\varphi_2''\Big)\alpha' \\
&\quad + \Big(mr\varphi_1' + r(k-2)\varphi_2' + \tfrac{k}{2} + 1\Big)\alpha''\Big)\xi,
\end{aligned} \tag{6.11}
$$

by (6.10) and (6.11), we obtain

$$
\begin{aligned}
G\big(\tilde{\nabla}(e^{-2\varphi_2}), \tilde{\nabla}L\big) &= -4\varphi_2' e^{-6\varphi_2} G(\xi,\xi)\Big(r\alpha^{(3)} + \Big(m\varphi_1' + mr\varphi_1'' + (k-2)\varphi_2' \\
&\quad + r(k-2)\varphi_2''\Big)\alpha' + \Big(mr\varphi_1' + r(k-2)\varphi_2' + \tfrac{k}{2} + 1\Big)\alpha''\Big) \\
&= -4r\varphi_2' e^{-6\varphi_2}\Big(r\alpha^{(3)} + \Big(m\varphi_1' + mr\varphi_1'' + (k-2)\varphi_2' \\
&\quad + r(k-2)\varphi_2''\Big)\alpha' + \Big(mr\varphi_1' + r(k-2)\varphi_2' + \tfrac{k}{2} + 1\Big)\alpha''\Big).
\end{aligned} \tag{6.12}
$$

On the other hand, we have

$$\begin{aligned}\Delta_G L &= div\Big[e^{-2\varphi_2}\Big(r\alpha^{(3)}+\Big(m\varphi_1^{'}+mr\varphi_1^{''}+(k-2)\varphi_2^{'}+r(k-2)\varphi_2^{''}\Big)\alpha^{'}\\ &\quad+\Big(mr\varphi_1^{'}+r(k-2)\varphi_2^{'}+\tfrac{k}{2}+1\Big)\alpha^{''}\Big)\xi\Big]\\ &= \xi.\Big[e^{-2\varphi_2}\Big(r\alpha^{(3)}+\Big(m\varphi_1^{'}+mr\varphi_1^{''}+(k-2)\varphi_2^{'}+r(k-2)\varphi_2^{''}\Big)\alpha^{'}\\ &\quad+\Big(mr\varphi_1^{'}+r(k-2)\varphi_2^{'}+\tfrac{k}{2}+1\Big)\alpha^{''}\Big)\Big]\\ &\quad+\Big[e^{-2\varphi_2}\Big(r\alpha^{(3)}+\Big(m\varphi_1^{'}+mr\varphi_1^{''}+(k-2)\varphi_2^{'}+r(k-2)\varphi_2^{''}\Big)\alpha^{'}\\ &\quad+\Big(mr\varphi_1^{'}+r(k-2)\varphi_2^{'}+\tfrac{k}{2}+1\Big)\alpha^{''}\Big)\Big]div(\xi).\end{aligned} \tag{6.13}$$

Set

$$\begin{aligned}S = e^{-2\varphi_2}\Big(r\alpha^{(3)}+\Big(m\varphi_1^{'}+mr\varphi_1^{''}+(k-2)\varphi_2^{'}+r(k-2)\varphi_2^{''}\Big)\alpha^{'}\\ +\Big(mr\varphi_1^{'}+r(k-2)\varphi_2^{'}+\tfrac{k}{2}+1\Big)\alpha^{''}\Big),\end{aligned} \tag{6.14}$$

thus

$$\begin{aligned}\xi(S) = \xi.\Big[e^{-2\varphi_2}\Big(r\alpha^{(3)}+\Big(m\varphi_1^{'}+mr\varphi_1^{''}+(k-2)\varphi_2^{'}+r(k-2)\varphi_2^{''}\Big)\alpha^{'}\\ +\Big(mr\varphi_1^{'}+r(k-2)\varphi_2^{'}+\tfrac{k}{2}+1\Big)\alpha^{''}\Big)\Big]\end{aligned} \tag{6.15}$$

$$
\begin{aligned}
&= 2\xi.(e^{-2\varphi_2})\Big(r\alpha^{(3)} + \Big(m\varphi_1^{'} + mr\varphi_1^{''} + (k-2)\varphi_2^{'} + r(k-2)\varphi_2^{''}\Big)\alpha^{'} \\
&\quad + \Big(mr\varphi_1^{'} + r(k-2)\varphi_2^{'} + \tfrac{k}{2} + 1\Big)\alpha^{''}\Big) \\
&\quad + 2e^{-2\varphi_2}\Big(\xi.(r)\alpha^{(3)} + r\xi.(\alpha^{(3)}) + \Big(m\xi.(\varphi_1^{'}) + m\xi.(r)\varphi_1^{''} + mr\xi.(\varphi_1^{''}) \\
&\quad + (k-2)\xi.(\varphi_2^{'}) + \xi.(r)(k-2)\varphi_2^{''} + r(k-2)\xi(\varphi_2^{''})\Big)\alpha^{'} \\
&\quad + \Big(m\varphi_1^{'} + mr\varphi_1^{''} + (k-2)\varphi_2^{'} + r(k-2)\varphi_2^{''}\Big)\xi.(\alpha^{'}) \\
&\quad + \Big(m\xi.(r)\varphi_1^{'} + mr\xi.(\varphi_1^{'}) + \xi(r)(k-2)\varphi_2^{'} + r(k-2)\xi.(\varphi_2^{'})\Big)\alpha^{''} \\
&\quad + \Big(mr\varphi_1^{'} + r(k-2)\varphi_2^{'} + \tfrac{k}{2} + 1\Big)\xi.(\alpha^{''})\Big) \\
&= \xi.\Big[e^{-2\varphi_2}\Big(r\alpha^{(3)} + \Big(m\varphi_1^{'} + mr\varphi_1^{''} + (k-2)\varphi_2^{'} + r(k-2)\varphi_2^{''}\Big)\alpha^{'} \\
&\quad + \Big(mr\varphi_1^{'} + r(k-2)\varphi_2^{'} + \tfrac{k}{2} + 1\Big)\alpha^{''}\Big)\Big] \\
&= -8\varphi_2^{'}re^{-2\varphi_2}\Big(r\alpha^{(3)} + \Big(m\varphi_1^{'} + mr\varphi_1^{''} + (k-2)\varphi_2^{'} + r(k-2)\varphi_2^{''}\Big)\alpha^{'} \\
&\quad + \Big(mr\varphi_1^{'} + r(k-2)\varphi_2^{'} + \tfrac{k}{2} + 1\Big)\alpha^{''}\Big) \\
&\quad + 2e^{-2\varphi_2}\Big(2r\alpha^{(3)} + 2r^2\alpha^{(4)} + \Big(2mr\varphi_1^{''} + 2mr\varphi_1^{''} + 2mr^2\varphi_1^{(3)} \\
&\quad + 2r(k-2)\varphi_2^{''} + 2r(k-2)\varphi_2^{''} + 2r^2(k-2)\varphi_2^{(3)}\Big)\alpha^{'} \\
&\quad + 2r\Big(m\varphi_1^{'} + mr\varphi_1^{''} + (k-2)\varphi_2^{'} + r(k-2)\varphi_2^{''}\Big)\alpha^{''} \\
&\quad + \Big(2mr\varphi_1^{'} + 2mr^2\varphi_1^{''} + 2r(k-2)\varphi_2^{'} + 2r^2(k-2)\varphi_2^{''}\Big)\alpha^{''} \\
&\quad + 2r\Big(mr\varphi_1^{'} + r(k-2)\varphi_2^{'} + \tfrac{k}{2} + 1\Big)\alpha^{(3)}\Big).
\end{aligned}
$$

Then

$$
\begin{aligned}
\Delta_G L = \Big[& e^{-2\varphi_2}\Big(r\alpha^{(3)} + \Big(m\varphi_1^{'} + mr\varphi_1^{''} + (k-2)\varphi_2^{'} + r(k-2)\varphi_2^{''}\Big)\alpha^{'} \\
& + \Big(mr\varphi_1^{'} + r(k-2)\varphi_2^{'} + \tfrac{k}{2} + 1\Big)\alpha^{''}\Big)\Big] div(\xi) \\
& - 8\varphi_2^{'} r e^{-2\varphi_2}\Big(r\alpha^{(3)} + \Big(m\varphi_1^{'} + mr\varphi_1^{''} + (k-2)\varphi_2^{'} + r(k-2)\varphi_2^{''}\Big)\alpha^{'} \\
& + \Big(mr\varphi_1^{'} + r(k-2)\varphi_2^{'} + \tfrac{k}{2} + 1\Big)\alpha^{''}\Big) \\
& + 2e^{-2\varphi_2}\Big(2r\alpha^{(3)} + 2r^2\alpha^{(4)} + \Big(2mr\varphi_1^{''} + 2mr\varphi_1^{''} + 2mr^2\varphi_1^{(3)} \\
& + 2r(k-2)\varphi_2^{''} + 2r(k-2)\varphi_2^{''} + 2r^2(k-2)\varphi_2^{(3)}\Big)\alpha^{'} \\
& + 2r\Big(m\varphi_1^{'} + mr\varphi_1^{''} + (k-2)\varphi_2^{'} + r(k-2)\varphi_2^{''}\Big)\alpha^{''} \\
& + \Big(2mr\varphi_1^{'} + 2mr^2\varphi_1^{''} + 2r(k-2)\varphi_2^{'} + 2r^2(k-2)\varphi_2^{''}\Big)\alpha^{''} \\
& + 2r\Big(mr\varphi_1^{'} + r(k-2)\varphi_2^{'} + \tfrac{k}{2} + 1\Big)\alpha^{(3)}\Big).
\end{aligned} \tag{6.16}
$$

Finally by (6.12) and (6.16), we obtain the desired equation. □

Remark 6.3.1. *It is clear that the expression of the bilaplacian of an r-radial function is a highly complex fourth order linear differential equation with variable coefficients. The solving of such a differential equation cannot be done using the classical approaches to differential equations. We are forced to make certain simplifications in order to give examples of biharmonic functions.*

Proposition 6.3.3. *Assume that $\varphi_1 = \varphi_2 = 0$ (i.e. G is the Sasaki metric). Let $F : E \longrightarrow \mathbb{R}$ be an r-radial function constructed from a function $\alpha : [0, +\infty[\longrightarrow \mathbb{R}$, then F is biharmonic if and only if α satisfies the linear differential equation*

$$
2r^2\alpha^{(4)} + (3k+4)r\alpha^{(3)} + k(k+2)\alpha^{(2)} = 0. \tag{6.17}
$$

Proof. Using the formula of Proposition 6.3.2, the r-radial function F is biharmonic if and only if $\Delta_G^2 F = 0$, where

$$
\begin{aligned}
\Delta_G^2 F = & 4\Big(r\alpha^{(3)} + \Big(\frac{k}{2} + 1\Big)\alpha^{''}\Big)\,\mathrm{div}(\xi) \\
& + 2\Big(2r\alpha^{(3)} + 2r^2\alpha^{(4)} + 2r\Big(\frac{k}{2} + 1\Big)\alpha^{(3)}\Big).
\end{aligned}
$$

Using the expression of the divergence of the tautological vector field ξ, we obtain $\mathrm{div}(\xi) = k$, hence

$$
\Delta_G^2 F = 4r^2\alpha^{(4)} + (4kr + 4r + 2r(k+2))\alpha^{(3)} + 2k(k+2)\alpha^{(2)},
$$

which can be simplified as

$$\Delta_G^2 F = 4r^2\alpha^{(4)} + 2(3k+4)r\alpha^{(3)} + 2k(k+2)\alpha^{(2)}.$$

This gives the desired equation. □

Corollary 6.3.1. *Assume that $\varphi_1 = \varphi_2 = 0$. Let $F : E \longrightarrow \mathbb{R}$ be an r-radial function constructed from a function $\alpha : [0, +\infty[\longrightarrow \mathbb{R}$. If α is a polynomial function of degree 1, then F is biharmonic.*

Proof. If α is a polynomial function of degree 1, then $\alpha^{(2)} = \alpha^{(3)} = \alpha^{(4)} = 0$, hence Equation (6.17) is trivially verified. □

Exemple 6.3.1. *In the case of the Sasaki metric, thanks to the previous corollary, we obtain a 2-parameter family of biharmonic functions. Furthermore, in virtue of Proposition 6.3.1, an r-radial function F constructed from a degree one polynomial function $\alpha : [0, +\infty[\longrightarrow \mathbb{R}$ is harmonic if and only if $\alpha' = 0$. Hence by choosing a function $\alpha(r) = ar + b$ with $(a, b) \in \mathbb{R}^2$ and $a \neq 0$, we obtain a proper biharmonic function. Therefore, we obtain a 2-parameter family of proper biharmonic functions.*

In what follows, we shall try to find other solutions of Equation (6.17). First of all, we make the following change of variable in Equation (6.17). Set $\psi = \alpha^{(2)}$. The equation becomes

$$2r^2\psi^{(2)} + (3k+4)r\psi' + k(k+2)\psi = 0. \tag{6.18}$$

Remark that the order of derivation of ψ and the power of r in the equation are somehow related (the order of derivation of ψ and the power of r are the same for the three terms of the equation). This suggests that we can search for solutions of the of form $\psi(r) = \beta r^n$, where $\beta \in \mathbb{R}$ and $n \in \mathbb{Z}$.

A function of the form $\psi(r) = \beta r^n$, where $\beta \in \mathbb{R}$ and $n \in \mathbb{Z}^*$, is a solution of Equation (6.18) if and only if

$$2\beta n(n-1)r^n + \beta(3k+4)r^n + k(k+2)\beta r^n = 0, \tag{6.19}$$

or equivalently

$$2n(n-1) + (3k+4)n + k(k+2) = 0, \tag{6.20}$$

which can be rewritten as a quadratic equation in n of the form

$$2n^2 + (3k+2)n + k(k+2) = 0. \tag{6.21}$$

Hence the problem is reduced to an equation relating the power n and the bundle rank k.

So, by fixing the bundle rank, we will end up solving a quadratic equation in n. This will give a procedure for constructing examples of biharmonic functions for different vector bundle ranks.

- $k = 1$ (i.e. line bundles): Equation (6.21) becomes

  $$2n^2 + 5n + 3 = 0$$

  Then we solve the equation for n. The discriminant of the equation is $\Delta = 1$, hence the equation has two solutions, $n_1 = -1$ and $n_2 = \frac{-3}{2}$. Therefore, by choosing the first solution, we have a family of solutions given as $\psi = \frac{\beta}{r}$. These are solutions on $]0, +\infty[$ which cannot be extended by continuity at zero.

- $k = 2$ (i.e. plane bundles): Equation (6.21) becomes

$$2n^2 + 8n + 8 = 0,$$

which has a vanishing discriminant, therefore a unique solution $n = -2$. Hence a family of solutions, $\psi(r) = \frac{\beta}{r^2}$, on $]0, +\infty[$ with no extension by continuity at zero.

- $k \geq 3$: the discriminant of the quadratic equation (6.21) is given by

$$\begin{aligned}\Delta &= (3k+2)^2 - 8k(k+2) = 9k^2 + 12k + 4 - 8k^2 - 16k \\ &= k^2 - 4k + 4 \\ &= (k-2)^2\end{aligned}$$

thus, we obtain two solutions

$$s_1 = \frac{-(3k+2) - (k-2)}{4} = -k$$

et

$$s_2 = \frac{-(3k+2) + (k-2)}{4} = \frac{-k}{2}$$

Hence when k is even, we obtain two families of solutions given by βr^{-k} and $\beta r^{-\frac{k}{2}}$, and when k is odd, we obtain a single family solutions given by βr^{-k}. In all the previous cases, we obtain families of solutions on $]0, +\infty[$ with no extension by continuity to zero.

Finally, we integrate the solutions ψ that we found.

- $k = 1$: we have $\psi(r) = \beta\frac{1}{r}$, hence $\alpha'(r) = \beta\ln(r) + \gamma$, thus $\alpha(r) = \beta(r\ln(r) - r) + \gamma r + \delta$, where δ, γ, β are real constants with $\beta \neq 0$.

- $k = 2$, we have $\psi(r) = \beta r^{-2}$, thus we obtain $\alpha'(r) = -\frac{\beta}{r} + \gamma$, hence $\alpha(r) = -\beta\ln(r) + \gamma r + \delta$, where δ, γ, β are real constants with $\beta \neq 0$.

- $k \geq 3$:
  - k is even: we obtain two families of solutions $\psi_1(r) = \beta_1 r^{-k}$ and $\psi_2(r) = \beta_2 r^{\frac{-k}{2}}$. Thus, by integration, we find two families $\alpha_1'(r) = \frac{\beta_1}{-k+1} r^{1-k} + \gamma_1$ and $\alpha_2'(r) = \frac{\beta_2}{-\frac{k}{2}+1} r^{1-\frac{k}{2}} + \gamma_2$, which gives $\alpha_1(r) = \frac{\beta_1}{(1-k)(2-k)} r^{2-k} + \gamma_1 r + \delta_1$ and $\alpha_2(r) = \frac{\beta_2}{(1-\frac{k}{2})(2-\frac{k}{2})} r^{2-\frac{k}{2}} + \gamma_2 r + \delta_2$, where $\delta_1, \gamma_1, \beta_1, \delta_2, \gamma_2, \beta_2$ are real constants with $\beta \neq 0$ and $\beta \neq 0$.
  - k is odd: we obtain a family of solutions $\psi(r) = \beta r^{-k}$, hence $\alpha'(r) = \frac{\beta}{1-k} r^{1-k} + \gamma$, which gives $\alpha(r) = \frac{\beta}{(1-k)(2-k)} r^{2-k} + \gamma r + \delta$, where δ, γ, β are real constants with $\beta \neq 0$.

Remark that all the previous solutions are defined on $\mathbb{R}_+^*$ with no extension to zero. Therefore, we obtain biharmonic functions on $E^* = E \setminus O$ where O is the zero section. To summarize, by the previous study, we have:

Proposition 6.3.4. *Assume that $\varphi_1 = \varphi_2 = 0$. Let $F : E \longrightarrow \mathbb{R}$ be an r-radial function constructed from a function $\alpha : [0, +\infty[\longrightarrow \mathbb{R}$.*

- *If $k = 1$, then there exists a family of biharmonic functions on E^* induced by the functions of the form*
$$\alpha(r) = \beta(r\ln(r) - r) + \gamma r + \delta,$$
where δ, γ, β are real constants with $\beta \neq 0$.

- *If $k = 2$, then there exists a family of biharmonic functions on E^* induced by the functions of the form*
$$\alpha(r) = -\beta\ln(r) + \gamma r + \delta,$$
where δ, γ, β are real constants with $\beta \neq 0$.

- *If $k \geq 3$, then we have two sub-cases:*
  - *If k is even, there exists two families of biharmonic functions on E^* induced by functions of the form*
$$\alpha_1(r) = \frac{\beta_1}{(1-k)(2-k)} r^{2-k} + \gamma_1 r + \delta_1 \qquad \text{and}$$
$$\alpha_2(r) = \frac{\beta_2}{(1-\frac{k}{2})(2-\frac{k}{2})} r^{2-\frac{k}{2}} + \gamma_2 r + \delta_2,$$
where $\delta_1, \gamma_1, \beta_1$ and $\delta_2, \gamma_2, \beta_2$ are real constants with $\beta_1 \neq 0$ and $\beta_2 \neq 0$.
  - *If k is odd, there exists a family of biharmonic functions on E^* induced by functions of the form*
$$\alpha(r) = \frac{\beta}{(1-k)(2-k)} r^{2-k} + \gamma r + \delta,$$
where δ, γ, β are real constants with $\beta \neq 0$.

Appendix A

Details of the proof of Theorem 5.1.1

A.1 Notations

Let U be a unit vector field on (M,g) and $I=]-\epsilon,\epsilon[$, $(\epsilon>0)$. For $t\in I$, we denote by $i_t : M \longrightarrow M\times I$, $p\longrightarrow (p,t)$, the canonical inclusion and we consider a C^∞-variation $V : M\times I\longrightarrow T_1M$ of U within $\mathfrak{X}_1(M)$, i.e. for all $t\in I$ the mappings $V_t = V\circ i_t$ are in fact unit vector fields and $V_0 = U$.

We choose $\{e_i\}_{i=1}^m$ a local othonormal frame field of (M,g). Then we can extend each $e_i\big(resp.\frac{d}{dx}\in\mathfrak{X}(I)\big)$ to a local vector field on $M\times I$ denoted by $E_i(resp.\frac{d}{dt})$. We have then $[E_i,\frac{d}{dt}]=0$. Furthermore, denoting by D the Levi-Civita connection of the Riemannian product manifold $M\times I$ and by R^D the Riemannian product, we have(using the second Bianchi identity for the last relation):

$$R^D(TM,TI)=0,\quad D_{\frac{d}{dt}}E_i=0,\quad D_{E_i}\frac{d}{dt}=0,\quad (D_{\frac{d}{dt}}R^D)(D_{E_i}V,V)E_i=0,\ \text{for all } 1\leq i\leq m.$$

We set:

1. $S(U)=\sum_{i=1}^m R\big(\nabla_{e_i}U,U\big)e_i$;
2. $\bar{\Delta}U=\sum_{i=1}^m\big\{\nabla_{\nabla_{e_i}e_i}U-\nabla_{e_i}\nabla_{e_i}U\big\}$;
3. $Z=\sum_{i=1}^m R^D(D_{E_i}V,V)E_i$;
4. $\Omega=\sum_{i=1}^m[D_{D_{E_i}E_i}V-D_{E_i}D_{E_i}V]$;
5. $W=\sum_{i=1}^m R^D(V,E_i)E_i$;
6. $S(V_t)=Z\circ i_t$;
7. $\bar{\Delta}V_t=\Omega\circ i_t$;
8. $QV_t=W\circ i_t$;
9. $\mathbb{H}=-\frac{b(ad+b^2)}{\alpha\varphi}g(W,V)-\frac{b}{\varphi}g(\Omega,V)+\frac{d}{\varphi}div(V)+\frac{a(ad+b^2)}{\alpha\varphi}g(Z,V)$;
10. $\mathbb{V}=\frac{b^2}{\alpha}g(W,V)+g(\Omega,V)-\frac{ab}{\alpha}g(Z,V)$;
11. $\mathcal{H}(V_t)=\mathcal{H}_t=\mathbb{H}\circ i_t$;

12. $\mathcal{V}(V_t) = \mathcal{V}_t = \mathbb{V} \circ i_t$;
13. $\tilde{\tau}_h = \frac{ab}{\alpha} W - \frac{a^2}{\alpha} Z - \frac{ad}{\alpha} D_V V + \mathbb{H}V$;
14. $\tilde{\tau}_v = -\Omega - \frac{b^2}{\alpha} W + \frac{ab}{\alpha} Z + \frac{bd}{\alpha} D_V V + \mathbb{V}V$;
15. $g(\tilde{\tau}_v, V) = 0$;
16. $\tilde{\tau}_{v,h}(V_t) = \tilde{\tau}_{v,h} \circ i_t$ (we donote by $\tilde{\tau}_{v,h}$: $\tilde{\tau}_h$ or $\tilde{\tau}_v$);
17. $\mathfrak{L}(.) = \sum_{i=1}^{n} \Big\{ 2R(e_i, .)\nabla_{e_i} U + (\nabla_{e_i} R)(e_i, .)U + R(e_i, \nabla_{e_i} .)U \Big\}$ and $\mathfrak{L} = \mathfrak{L}(U)$;
18. $\mathfrak{L}^D(.) = \sum_{i=1}^{n} \Big\{ 2R^D(E_i, .)D_{E_i} V + (D_{E_i} R^D)(E_i, .)V + R^D(E_i, D_{E_i} .)V \Big\}$;
19. $\mathfrak{L}^D = \mathfrak{L}^D(V) = \sum_{i=1}^{n} \Big\{ 2R^D(E_i, V)D_{E_i} V + (D_{E_i} R^D)(E_i, V)V + R^D(E_i, D_{E_i} V)V \Big\}$;
20. $\eta_t(X) = g\big(D_{\frac{d}{dt}} V, D_X V\big) = \big(D_{\frac{d}{dt}} V\big)^{\flat}(D_X V)$;
21. $p_t^{v,h}(X) = g\big(D_{\frac{d}{dt}} V, D_X \tau_{v,h}\big)$;
22. $\omega_t(X) = g\big(R^D(X, V)V, D_{\frac{d}{dt}} V\big)$;
23. $g_t^{v,h}(X) = g\big(R^D(X, \tau_{v,h})V, D_{\frac{d}{dt}} V\big)$;
24. $d_t^{v,h}(X) = g(D_{\frac{d}{dt}} V, g(X, V)\tau_{v,h})$;
25. $z_t(X) = g(D_{\frac{d}{dt}} V, X)$;
26. $\tilde{\tau}(U)$ is the tension field of a unit vector field: $\tilde{\tau}(V) = h\{\tilde{\tau}_h(V)\} + v\{\tilde{\tau}_v(V)\}$.

A.2 The bienergy of a unit tangent vector field

We have

$$E_2(V_t) = \tfrac{a+c}{2} \int_M g\big(\tilde{\tau}_h, \tilde{\tau}_h)\big) \circ i_t v_g + \frac{d}{2} \int_M g\big(\tilde{\tau}_h, V\big)^2 \circ i_t v_g + b \int_M g\big(\tilde{\tau}, \tilde{\tau}_v\big) \circ i_t v_g$$
$$+ \tfrac{a}{2} \int_M g\big(\tilde{\tau}_v, \tilde{\tau}_v\big) \circ i_t v_g.$$

Now we calculate the first variation of E_2:

$$D_{\frac{d}{dt}} E_2(V_t) = (a+c) \int_M g(D_{\frac{d}{dt}} \tilde{\tau}_h, \tilde{\tau}_h) \circ i_t dv_g + d \int_M g(\tilde{\tau}_h, V) g(D_{\frac{d}{dt}} \tilde{\tau}_h, V) \circ i_t dv_g$$
$$+ d \int_M g(\tilde{\tau}_h, V) g(\tilde{\tau}_h, D_{\frac{d}{dt}} V) \circ i_t dv_g + b \int_M g(D_{\frac{d}{dt}} \tilde{\tau}_h, \tilde{\tau}_v) \circ i_t dv_g \tag{A.1}$$
$$+ b \int_M g(D_{\frac{d}{dt}} \tilde{\tau}_v, \tilde{\tau}_h) \circ i_t dv_g + a \int_M g(D_{\frac{d}{dt}} \tilde{\tau}_v, \tilde{\tau}_v) \circ i_t dv_g.$$

We should determine each integral separately, and to do that we have to compute some useful formulas.

A.3 Formulas

Calculus of $g\big(D_{\frac{d}{dt}}.,\tilde{\tau}_{v,h}\big)$

$$
\begin{aligned}
g\big(D_{\frac{d}{dt}}\Omega,\tilde{\tau}_{v,h}\big) &= \sum_{i=1}^{n}\Big\{g\big(D_{\frac{d}{dt}}D_{D_{E_i}E_i}V - D_{\frac{d}{dt}}D_{E_i}D_{E_i}V,\tilde{\tau}_{v,h}\big)\Big\}\\
&= \sum_{i=1}^{n}\Big\{g\big(D_{D_{E_i}E_i}D_{\frac{d}{dt}}V,\tilde{\tau}_{v,h}\big) - g\big(D_{E_i}D_{E_i}D_{\frac{d}{dt}}V,\tilde{\tau}_{v,h}\big)\Big\}\\
&= \sum_{i=1}^{n}\Big\{D_{E_i}E_i.g\big(D_{\frac{d}{dt}}V,\tilde{\tau}_{v,h}\big) - g\big(D_{\frac{d}{dt}}V,D_{E_i}E_i.\big(\tilde{\tau}_{v,h}\big)\big)\\
&\quad -E_i.g\big(D_{E_i}D_{\frac{d}{dt}}V,\tilde{\tau}_{v,h}\big) + g\big(D_{E_i}D_{\frac{d}{dt}}V,D_{E_i}\big(\tilde{\tau}_{v,h}\big)\big)\Big\}\\
&= \sum_{i=1}^{n}\Big\{D_{E_i}E_i.g\big(D_{\frac{d}{dt}}V,\tilde{\tau}_{v,h}\big) - g\big(D_{\frac{d}{dt}}V,D_{E_i}E_i.\big(\tilde{\tau}_{v,h}\big)\big)\big)\\
&\quad -E_i.E_i.g\big(D_{\frac{d}{dt}}V,\tilde{\tau}_{v,h}\big) + E_i.g\big(D_{\frac{d}{dt}}V,D_{E_i}\big(\tilde{\tau}_{v,h}\big)\big)\\
&\quad +E_i.g\big(D_{\frac{d}{dt}}V,D_{E_i}\big(\tilde{\tau}_{v,h}\big)\big) - g\big(D_{\frac{d}{dt}}V,D_{E_i}D_{E_i}\big(\tilde{\tau}_{v,h}\big)\big)\Big\}\\
&= \sum_{i=1}^{n}\Big\{2E_i.g\big(D_{\frac{d}{dt}}V,D_{E_i}(\tilde{\tau}_{v,h})\big) + \Delta\big[g\big(D_{\frac{d}{dt}}V,\tilde{\tau}_{v,h}\big)\big]\\
&\quad -2g\big(D_{\frac{d}{dt}}V,D_{E_i}E_i.\big(\tilde{\tau}_{v,h}\big)\big) + g\big(D_{\frac{d}{dt}}V,\bar{\Delta}\tilde{\tau}_{v,h}\big)\Big\}
\end{aligned}
\tag{A.2}
$$

$$
g\big(D_{\frac{d}{dt}}\Omega,\tilde{\tau}_{v,h}\big) = 2\,div(p_t^{v,h}) + \Delta\big[g\big(D_{\frac{d}{dt}}V,\tilde{\tau}_{v,h}\big)\big] + g\big(D_{\frac{d}{dt}}V,\bar{\Delta}\tilde{\tau}_{v,h}\big).
$$

(A.3)

$$
\begin{aligned}
g\big(D_{\frac{d}{dt}}Z,\tilde{\tau}_{v,h}\big) &= g\big(D_{\frac{d}{dt}}\big(R^D(D_{E_i}V,V)E_i\big),\tilde{\tau}_{v,h}\big)\\
&= \textstyle\sum_{i=1}^n \Big\{ g\big(R^D(D_{\frac{d}{dt}}D_{E_i}V,V)E_i,\tilde{\tau}_{v,h}\big) + g\big(R^D(D_{E_i}V,D_{\frac{d}{dt}}V)E_i,\tilde{\tau}_{v,h}\big)\Big\}\\
&= \textstyle\sum_{i=1}^n \Big\{ g\big(R^D(E_i,\tilde{\tau}_{v,h})D_{E_i}V,D_{\frac{d}{dt}}V)\big) - g\big(R^D(E_i,\tilde{\tau}_{v,h})V,D_{E_i}D_{\frac{d}{dt}}V)\big)\Big\}\\
&= \textstyle\sum_{i=1}^n \Big\{ g\big(R^D(E_i,\tilde{\tau}_{v,h})D_{E_i}V,D_{\frac{d}{dt}}V)\big) - E_i.\Big[g\Big(R^D\big(E_i,\tilde{\tau}_{v,h}\big)V\Big),D_{\frac{d}{dt}}V\Big]\\
&\quad + g\Big[D_{E_i}\Big(R^D\big(E_i,\tilde{\tau}_{v,h}\big)V\Big),D_{\frac{d}{dt}}V\Big]\Big\}\\
&= \textstyle\sum_{i=1}^n \Big\{ g\big(R^D(E_i,\tilde{\tau}_{v,h})D_{E_i}V,D_{\frac{d}{dt}}V\big) - E_i.\Big[g\Big(R^D\big(E_i,\tilde{\tau}_{v,h}\big)V\Big),D_{\frac{d}{dt}}V\Big]\\
&\quad + g\Big(\big(D_{E_i}R^D\big)\big(E_i,\tilde{\tau}_{v,h}\big)V,D_{\frac{d}{dt}}V\Big) + g\Big(R^D\big(D_{E_i}E_i,\tilde{\tau}_{v,h}\big)V,D_{\frac{d}{dt}}V\Big)\\
&\quad + g\Big[R^D\big(E_i,D_{E_i}\tilde{\tau}_{v,h}\big)V,D_{\frac{d}{dt}}V\Big] + g\Big[R^D\big(E_i,\tilde{\tau}_{v,h}\big)D_{E_i}V,D_{\frac{d}{dt}}V\Big]\Big\}\\
g\big(D_{\frac{d}{dt}}Z,\tilde{\tau}_{v,h}\big) &= g\big(\mathfrak{L}^D(\tilde{\tau}_{v,h}),D_{\frac{d}{dt}}V\big) - div(g_t^{v,h}).
\end{aligned}
$$

(A.4)

$$
\begin{aligned}
g\big(D_{\frac{d}{dt}}W,\tilde{\tau}_{v,h}\big) &= \textstyle\sum_{i=1}^n \Big\{ g\Big(D_{\frac{d}{dt}}\big[R^D(V,E_i)E_i\big],\tilde{\tau}_{v,h}\Big)\Big\}\\
&= \textstyle\sum_{i=1}^n \Big\{ g\big(R^D(D_{\frac{d}{dt}}V,E_i)E_i,\tilde{\tau}_{v,h}\big)\Big\}\\
&= \textstyle\sum_{i=1}^n \Big\{ g\big(R^D(\tilde{\tau}_{v,h},E_i)E_i,D_{\frac{d}{dt}}V\big)\Big\}.
\end{aligned}
$$

(A.5)

$$
\begin{aligned}
g\big(D_{\frac{d}{dt}}D_V V,\tilde{\tau}_{v,h}\big) &= \sum_{i=1}^n \Big\{ g\big(D_{\frac{d}{dt}}\big(g(E_i,V)D_{E_i}V\big),\tilde{\tau}_{v,h}\big)\Big\}\\
&= \sum_{i=1}^n \Big\{ g\big(D_{\frac{d}{dt}}\big(g(E_i,V)\big)D_{E_i}V,\tilde{\tau}_{v,h}\big) + g\big(g(E_i,V)D_{\frac{d}{dt}}D_{E_i}V,\tilde{\tau}_{v,h}\big)\Big\}\\
&= \sum_{i=1}^n \Big\{ g\big(g(E_i,D_{\frac{d}{dt}}V)D_{E_i}V,\tilde{\tau}_{v,h}\big) + g\big(D_{E_i}D_{\frac{d}{dt}}V,g(E_i,V)\tilde{\tau}_{v,h}\big)\Big\}\\
&= \sum_{i=1}^n \Big\{ g\big(g\big(D_{E_i}V,\tilde{\tau}_{v,h}\big)E_i,D_{\frac{d}{dt}}V\big) + D_{E_i}g\big(D_{\frac{d}{dt}}V,g(E_i,V)\tilde{\tau}_{v,h}\big)\\
&\quad -g\big(D_{\frac{d}{dt}}V,g(D_{E_i}E_i,V)\tilde{\tau}_{v,h}\big) - g\big(D_{\frac{d}{dt}}V,g(E_i,D_{E_i}V)\tilde{\tau}_{v,h}\big)\\
&\quad -g\big(D_{\frac{d}{dt}}V,g(E_i,V)D_{E_i}\tilde{\tau}_{v,h}\big)\Big\}\\
&= \sum_{i=1}^n \Big\{ g\big(g\big(D_{E_i}V,\tilde{\tau}_{v,h}\big)E_i,D_{\frac{d}{dt}}V\big) + div(d_t^v) - g\big(D_{\frac{d}{dt}}V,g(E_i,D_{E_i}V)\tilde{\tau}_{v,h}\big)\\
&\quad -g\big(D_{\frac{d}{dt}}V,D_V\tilde{\tau}_{v,h}\big)\Big\}.
\end{aligned}
$$

Calculus of $g\big(D_{\frac{d}{dt}}.,V\big)$

(A.6)

$$
\begin{aligned}
g\big(D_{\frac{d}{dt}}\Omega,V\big) &= \sum_{i=1}^n \Big\{ g\big(D_{\frac{d}{dt}}D_{D_{E_i}E_i}V - D_{\frac{d}{dt}}D_{E_i}D_{E_i}V,V\big)\Big\}\\
&= \sum_{i=1}^n \Big\{ g\big(D_{D_{E_i}E_i}D_{\frac{d}{dt}}V,V\big) - g\big(D_{E_i}D_{E_i}D_{\frac{d}{dt}}V,V\big)\Big\}\\
&= \sum_{i=1}^n \Big\{ D_{E_i}E_i.g\big(D_{\frac{d}{dt}}V,V\big) - g\big(D_{\frac{d}{dt}}V,D_{D_{E_i}E_i}V\big)\\
&\quad -E_i.g\big(D_{E_i}D_{\frac{d}{dt}}V,V\big) + g\big(D_{E_i}D_{\frac{d}{dt}}V,D_{E_i}V\big)\Big\}\\
&= \sum_{i=1}^n \Big\{ -g\big(D_{\frac{d}{dt}}V,D_{D_{E_i}E_i}V\big) - E_i.E_i.g(D_{\frac{d}{dt}}V,V) + E_i.g(D_{\frac{d}{dt}}V,D_{E_i}V)\\
&\quad +E_i.g(D_{\frac{d}{dt}}V,D_{E_i}V) - g(D_{\frac{d}{dt}}V,D_{E_i}D_{E_i}V)\Big\}\\
&= \sum_{i=1}^n \Big\{ 2E_i.g(D_{\frac{d}{dt}}V,D_{E_i}V) - 2g\big(D_{\frac{d}{dt}}V,D_{D_{E_i}E_i}V\big) + g(D_{\frac{d}{dt}}V,\Omega)\Big\}
\end{aligned}
$$

$$
g\big(D_{\frac{d}{dt}}\Omega,V\big) = 2\,div(\eta_t) + g(D_{\frac{d}{dt}}V,\Omega).
$$

$$
\begin{aligned}
g(D_{\frac{d}{dt}}Z,V) &= \sum_{i=1}^{n} g\Big(D_{\frac{d}{dt}}\big(R^D(D_{E_i}V,V)E_i\big),V\Big)\\
&= \sum_{i=1}^{n}\Big\{g\big(R^D(D_{\frac{d}{dt}}D_{E_i}V,V)E_i,V\big)+g\big(R^D(D_{E_i}V,D_{\frac{d}{dt}}V)E_i,V\big)\Big\}\\
&= \sum_{i=1}^{n}\Big\{g\big(R^D(E_i,V)D_{E_i}V,D_{\frac{d}{dt}}V\big)-g\big(R^D(E_i,V)V,D_{E_i}D_{\frac{d}{dt}}V\big)\Big\}\\
&= \sum_{i=1}^{n}\Big\{g\big(R^D(E_i,V)D_{E_i}V,D_{\frac{d}{dt}}V\big)-E_i.\Big[g\Big(R^D\big(E_i,V\big)V\Big),D_{\frac{d}{dt}}V\Big]\\
&\quad +g\Big[D_{E_i}\Big(R^D\big(E_i,V\big)V\Big),D_{\frac{d}{dt}}V\Big]\Big\} \qquad \text{(A.7)}\\
&= \sum_{i=1}^{n}\Big\{g\big(R^D(E_i,V)D_{E_i}V,D_{\frac{d}{dt}}V\big)-E_i.\Big[g\Big(R^D\big(E_i,V\big)V\Big),D_{\frac{d}{dt}}V\Big]\\
&\quad +g\Big(\big(D_{E_i}R^D\big)\big(E_i,V\big)V,D_{\frac{d}{dt}}V\Big)+g\Big(R^D\big(D_{E_i}E_i,V\big)V,D_{\frac{d}{dt}}V\Big)\\
&\quad +g\Big[R^D\big(E_i,D_{E_i}V\big)V,D_{\frac{d}{dt}}V\Big]+g\Big[R^D\big(E_i,V\big)D_{E_i}V,D_{\frac{d}{dt}}V\Big]\Big\}\\
g(D_{\frac{d}{dt}}Z,V) &= g(\mathfrak{L}^D,D_{\frac{d}{dt}}V)-div(\omega_t).
\end{aligned}
$$

$$
\begin{aligned}
D_{\frac{d}{dt}}g(W,V) &= \sum_{i=1}^{n} g(D_{\frac{d}{dt}}\big(R^D(V,E_i)E_i\big),V)\\
&= \sum_{i=1}^{n} g(\big(R^D(D_{\frac{d}{dt}}V,E_i)E_i\big),V)\\
&= -\sum_{i=1}^{n} g(\big(R^D(E_i,V)E_i\big),D_{\frac{d}{dt}}V) \qquad \text{(A.8)}\\
&= g(W,D_{\frac{d}{dt}}V).
\end{aligned}
$$

$$
\begin{aligned}
g(D_{\frac{d}{dt}}D_VV,V) &= D_{\frac{d}{dt}}.g(D_VV,V)-g(D_VV,D_{\frac{d}{dt}}V)\\
&= D_{\frac{d}{dt}}\big[\tfrac{1}{2}V.g(V,V)\big]-g(D_VV,D_{\frac{d}{dt}}V) \qquad \text{(A.9)}\\
&= -g(D_VV,D_{\frac{d}{dt}}V).
\end{aligned}
$$

$$
\begin{aligned}
D_{\frac{d}{dt}}(divV) &= \sum_{i=1}^{n} D_{\frac{d}{dt}}g(D_{E_i}V,E_i)\\
&= \sum_{i=1}^{n} g(D_{E_i}D_{\frac{d}{dt}}V,E_i)\\
&= \sum_{i=1}^{n}\Big\{D_{E_i}g(D_{\frac{d}{dt}}V,E_i)-g(D_{\frac{d}{dt}}V,D_{E_i}E_i)\Big\} \qquad \text{(A.10)}\\
D_{\frac{d}{dt}}(divV) &= div(z_t).
\end{aligned}
$$

Calculus of $g\big(D_{\frac{d}{dt}}(\mathbb{H}V),V\big)$

By (A.6)-(A.10) we have

(A.11)

$$\begin{aligned}
g\big(D_{\frac{d}{dt}}(\mathbb{H}V),V\big) &= g\big(D_{\frac{d}{dt}}(\mathbb{H})V,V\big)+g\big(\mathbb{H}D_{\frac{d}{dt}}V,V\big)\\
&= D_{\frac{d}{dt}}(\mathbb{H})\\
&= -\tfrac{b(ad+b^2)}{\alpha\varphi}D_{\frac{d}{dt}}g(W,V)+\tfrac{a(ad+b^2)}{\alpha\varphi}D_{\frac{d}{dt}}g(Z,V)+\tfrac{d}{\varphi}D_{\frac{d}{dt}}(divV)-\tfrac{b}{\varphi}D_{\frac{d}{dt}}g(\Omega,V)\\
&= -2\tfrac{b(ad+b^2)}{\alpha\varphi}g(W,D_{\frac{d}{dt}}V)+\tfrac{a(ad+b^2)}{\alpha\varphi}g(Z,D_{\frac{d}{dt}}V)+\tfrac{a(ad+b^2)}{\alpha\varphi}g(\mathfrak{L}^D,D_{\frac{d}{dt}}V)\\
&\quad -\tfrac{a(ad+b^2)}{\alpha\varphi}div(\omega_t)-2\tfrac{b}{\varphi}div(\eta_t)-2\tfrac{b}{\varphi}g(\Omega,D_{\frac{d}{dt}}V)+\tfrac{d}{\varphi}div(z_t).
\end{aligned}$$

Calculus of $g\big(D_{\frac{d}{dt}}(\mathbb{V}V),V\big)$

By (A.6)-(A.10) we have

(A.12)

$$\begin{aligned}
g\big(D_{\frac{d}{dt}}(\mathbb{V}V),V\big) &= D_{\frac{d}{dt}}(\mathbb{V})\\
&= \tfrac{b^2}{\alpha}D_{\frac{d}{dt}}.g(W,V)-\tfrac{ab}{\alpha}D_{\frac{d}{dt}}.g(Z,V)+D_{\frac{d}{dt}}.g(\Omega,V)\\
&= 2\tfrac{b^2}{\alpha}g(W,D_{\frac{d}{dt}}V)-\tfrac{ab}{\alpha}g(Z,D_{\frac{d}{dt}}V)-\tfrac{ab}{\alpha}g(\mathfrak{L}^D,D_{\frac{d}{dt}}V)+\tfrac{ab}{\alpha}div(\omega_t)\\
&\quad +2div(\eta_t)+2g(\Omega,D_{\frac{d}{dt}}V).
\end{aligned}$$

A.3.1 First variation formula of $E_2(V_t)$

Calculus of $\frac{d}{dt}\int_M g(\tilde{\tau}_h,V)^2dv$

We have

$$\frac{d}{dt}\int_M g(\tilde{\tau}_h,V)^2dv = 2\int_M g\big(D_{\frac{d}{dt}}\tilde{\tau}_h,V\big)g\big(\tilde{\tau}_h,V\big)dv+2\int_M g\big(\tilde{\tau}_h,D_{\frac{d}{dt}}V\big)g\big(\tilde{\tau}_h,V\big)dv. \tag{A.13}$$

Calculus of $\int_M g\big(D_{\frac{d}{dt}}\tilde{\tau}_h,V\big)g\big(\tilde{\tau}_h,V\big)dv$:

1. Calculus of $g\big(D_{\frac{d}{dt}}\tilde{\tau}_h, V\big)$:

(A.14)

$$\begin{aligned} g\big(D_{\frac{d}{dt}}\tilde{\tau}_h, V\big) &= \tfrac{ab}{\alpha} g(D_{\frac{d}{dt}}W, V) - \tfrac{a^2}{\alpha} g(D_{\frac{d}{dt}}Z, V) + D_{\frac{d}{dt}}(\mathbb{H}) - \tfrac{ad}{\alpha} g(D_{\frac{d}{dt}}D_V V, V) \\ &= \tfrac{ab}{\alpha} g(W, D_{\frac{d}{dt}}V) - \tfrac{a^2}{\alpha} g(\mathfrak{L}^D, D_{\frac{d}{dt}}V) + \tfrac{a^2}{\alpha} div(\omega_t) \\ &\quad -2\tfrac{b(ad+b^2)}{\alpha\varphi} g(W, D_{\frac{d}{dt}}V) + \tfrac{a(ad+b^2)}{\alpha\varphi} g(Z, D_{\frac{d}{dt}}V) + \tfrac{a(ad+b^2)}{\alpha\varphi} g(\mathfrak{L}^D, D_{\frac{d}{dt}}V) \\ &\quad -\tfrac{a(ad+b^2)}{\alpha\varphi} div(\omega_t) - 2\tfrac{b}{\varphi} div(\eta_t) - 2\tfrac{b}{\varphi} g(\Omega, D_{\frac{d}{dt}}V) + \tfrac{ad}{\alpha} g(D_V V, D_{\frac{d}{dt}}V) \\ &\quad +\tfrac{d}{\varphi} div(z_t). \end{aligned}$$

$$\begin{aligned} g\big(D_{\frac{d}{dt}}\tilde{\tau}_h, V\big) &= (\tfrac{ab}{\alpha} - 2\tfrac{b(ad+b^2)}{\alpha\varphi}) g(W, D_{\frac{d}{dt}}V) + \tfrac{a(ad+b^2)}{\alpha\varphi} g(Z, D_{\frac{d}{dt}}V) \\ &\quad +(\tfrac{a(ad+b^2)}{\alpha\varphi} - \tfrac{a^2}{\alpha})\big[g(\mathfrak{L}^D, D_{\frac{d}{dt}}V) - div(\omega_t)\big] - 2\tfrac{b}{\varphi} div(\eta_t) \\ &\quad -2\tfrac{b}{\varphi} g(\Omega, D_{\frac{d}{dt}}V) + \tfrac{ad}{\alpha} g(D_V V, D_{\frac{d}{dt}}V) + \tfrac{d}{\varphi} div(z_t). \end{aligned}$$

2. Calculus of $g\big(D_{\frac{d}{dt}}\tilde{\tau}_h, V\big) g\big(\tilde{\tau}_h, V\big)$:

(A.15)

$$\begin{aligned} g\big(D_{\frac{d}{dt}}\tilde{\tau}_h, V\big) g\big(\tilde{\tau}_h, V\big) &= (\tfrac{ab}{\alpha} - 2\tfrac{b(ad+b^2)}{\alpha\varphi}) g(g\big(\tilde{\tau}_h, V\big)W, D_{\frac{d}{dt}}V) + \tfrac{a(ad+b^2)}{\alpha\varphi} g(g\big(\tilde{\tau}_h, V\big)Z, D_{\frac{d}{dt}}V) \\ &\quad +(\tfrac{a(ad+b^2)}{\alpha\varphi} - \tfrac{a^2}{\alpha}) g(g\big(\tilde{\tau}_h, V\big)\mathfrak{L}^D, D_{\frac{d}{dt}}V) \\ &\quad -(\tfrac{a(ad+b^2)}{\alpha\varphi} - \tfrac{a^2}{\alpha}) g\big(\tilde{\tau}_h, V\big) div(\omega_t) \\ &\quad -2\tfrac{b}{\varphi} g\big(\tilde{\tau}_h, V\big) div(\eta_t) - 2\tfrac{b}{\varphi} g(g\big(\tilde{\tau}_h, V\big)\Omega, D_{\frac{d}{dt}}V) \\ &\quad +\tfrac{ad}{\alpha} g(g\big(\tilde{\tau}_h, V\big)D_V V, D_{\frac{d}{dt}}V) + \tfrac{d}{\varphi} g\big(\tilde{\tau}_h, V\big) div(z_t). \end{aligned}$$

3. Calculus of $g\big(\tilde{\tau}_h, V\big) div(\omega_t)$ et $g\big(\tilde{\tau}_h, V\big) div(\eta_t)$:

We have

$$h_t div(\omega_t) = div(\omega_t h_t) - \omega_t(grad(h_t)). \tag{A.16}$$

We set

$$h_t = g\big(\tilde{\tau}_h, V\big).$$

We get

$$g\big(\tilde{\tau}_h, V\big) div(\omega_t) = div\Big[g\big(\tilde{\tau}_h, V\big)\omega_t\Big] - g\Big[R^D\Big(grad\big(g\big(\tilde{\tau}_h, V\big)\big), V\Big)V, D_{\frac{d}{dt}}V\Big], \tag{A.17}$$

and

$$g\big(\tilde{\tau}_h, V\big) div(\eta_t) = div\Big[g\big(\tilde{\tau}_h, V\big)\eta_t\Big] - g(D_{\frac{d}{dt}}V, D_{grad[g(\tilde{\tau}_h, V)]}V), \tag{A.18}$$

and

$$g\big(\tilde{\tau}_h, V\big) div(z_t) = div\Big[g\big(\tilde{\tau}_h, V\big) zt\Big] - g(D_{\frac{d}{dt}} V, grad[g(\tilde{\tau}_h, V)]). \tag{A.19}$$

we obtain

(A.20)

$$\begin{aligned} g\big(D_{\frac{d}{dt}}\tilde{\tau}_h, V\big) g\big(\tilde{\tau}_h, V\big) = & (\tfrac{ab}{\alpha} - 2\tfrac{b(ad+b^2)}{\alpha\varphi}) g(g\big(\tilde{\tau}_h, V\big) W, D_{\frac{d}{dt}} V) \\ & + \tfrac{a(ad+b^2)}{\alpha\varphi} g(g\big(\tilde{\tau}_h, V\big) Z, D_{\frac{d}{dt}} V) + (\tfrac{a(ad+b^2)}{\alpha\varphi} - \tfrac{a^2}{\alpha}) g(g\big(\tilde{\tau}_h, V\big) \mathfrak{L}^D, D_{\frac{d}{dt}} V) \\ & - 2\tfrac{b}{\varphi} g(g\big(\tilde{\tau}_h, V\big) \Omega, D_{\frac{d}{dt}} V) - (\tfrac{a(ad+b^2)}{\alpha\varphi} - \tfrac{a^2}{\alpha}) div\Big[g\big(\tilde{\tau}_h, V\big) \omega_t\Big] \\ & + (\tfrac{a(ad+b^2)}{\alpha\varphi} - \tfrac{a^2}{\alpha}) g\Big[R^D\Big(grad\big(g\big(\tilde{\tau}_h, V\big)\big), V\Big) V, D_{\frac{d}{dt}} V\Big] \\ & - 2\tfrac{b}{\varphi} div\Big[g\big(\tilde{\tau}_h, V\big) \eta_t\Big] + 2\tfrac{b}{\varphi} g(D_{\frac{d}{dt}} V, D_{grad[g(\tilde{\tau}_h, V)]} V) \\ & + \tfrac{ad}{\alpha} g(g\big(\tilde{\tau}_h, V\big) D_V V, D_{\frac{d}{dt}} V) + \tfrac{d}{\varphi} div\Big[g\big(\tilde{\tau}_h, V\big) z_t\Big] \\ & - \tfrac{d}{\varphi} g(D_{\frac{d}{dt}} V, grad[g(\tilde{\tau}_h, V)]). \end{aligned}$$

4. Calculus of $\int_M g\big(D_{\frac{d}{dt}}\tilde{\tau}_h, V\big) g\big(\tilde{\tau}_h, V\big) dv$:

(A.21)

$$\begin{aligned} \int_M g\big(D_{\frac{d}{dt}}\tilde{\tau}_h, V\big) g\big(\tilde{\tau}_h, V\big) dv = & \int_M \Big[(\frac{ab}{\alpha} - 2\frac{b(ad+b^2)}{\alpha\varphi}) g(g\big(\tilde{\tau}_h, V\big) W, D_{\frac{d}{dt}} V) \\ & + \tfrac{a(ad+b^2)}{\alpha\varphi} g(g\big(\tilde{\tau}_h, V\big) Z, D_{\frac{d}{dt}} V) \\ & + (\tfrac{a(ad+b^2)}{\alpha\varphi} - \tfrac{a^2}{\alpha}) g(g\big(\tilde{\tau}_h, V\big) \mathfrak{L}^D, D_{\frac{d}{dt}} V) \\ & - 2\tfrac{b}{\varphi} g(g\big(\tilde{\tau}_h, V\big) \Omega, D_{\frac{d}{dt}} V) \\ & + (\tfrac{a(ad+b^2)}{\alpha\varphi} - \tfrac{a^2}{\alpha}) g\Big[R^D\Big(grad\big(g\big(\tilde{\tau}_h, V\big)\big), V\Big) V, D_{\frac{d}{dt}} V\Big] \\ & + 2\tfrac{b}{\varphi} g(D_{\frac{d}{dt}} V, D_{grad[g(\tilde{\tau}_h, V)]} V) + \tfrac{ad}{\alpha} g(g\big(\tilde{\tau}_h, V\big) D_V V, D_{\frac{d}{dt}} V) \\ & - \tfrac{d}{\varphi} g(D_{\frac{d}{dt}} V, grad[g(\tilde{\tau}_h, V)])\Big] v_g. \end{aligned}$$

Calculus of $D_{\frac{d}{dt}}\int_M g\big(\tilde{\tau}_h,\tilde{\tau}_h\big)\circ i_t v_g$:

We have

$$D_{\frac{d}{dt}}\int_M g\big(\tilde{\tau}_h,\tilde{\tau}_h\big)\circ i_t v_g = 2\int_M g\big(D_{\frac{d}{dt}}\tilde{\tau}_h,\tilde{\tau}_h\big)\circ i_t v_g. \tag{A.22}$$

1. Calculus of $g\big(D_{\frac{d}{dt}}\tilde{\tau}_h,\tilde{\tau}_h\big)$:

(A.23)

$$\begin{aligned}
g\big(D_{\frac{d}{dt}}\tilde{\tau}_h,\tilde{\tau}_h\big) &= \tfrac{ab}{\alpha}g(D_{\frac{d}{dt}}W,\tilde{\tau}_h) - \tfrac{a^2}{\alpha}g(D_{\frac{d}{dt}}Z,\tilde{\tau}_h)\\
&\quad +g(D_{\frac{d}{dt}}(\mathbb{H}).V,\tilde{\tau}_h) + g(\mathbb{H}.D_{\frac{d}{dt}}V,\tilde{\tau}_h) - \tfrac{ad}{\alpha}g(D_{\frac{d}{dt}}D_V V,\tilde{\tau}_h)\\
&= \tfrac{ab}{\alpha}\textstyle\sum_{i=1}^n g\big(R^D(\tilde{\tau}_h,E_i)E_i,D_{\frac{d}{dt}}V\big) - \tfrac{a^2}{\alpha}g\big(\mathfrak{L}^D(\tilde{\tau}_h),D_{\frac{d}{dt}}V\big) + \tfrac{a^2}{\alpha}div(g_t^h)\\
&\quad +g(D_{\frac{d}{dt}}V,\mathbb{H}\tilde{\tau}_h) - 2\tfrac{b(ad+b^2)}{\alpha\varphi}g(g(\tilde{\tau}_h,V)W,D_{\frac{d}{dt}}V)\\
&\quad +\tfrac{a(ad+b^2)}{\alpha\varphi}g(g(\tilde{\tau}_h,V)Z,D_{\frac{d}{dt}}V)\\
&\quad +\tfrac{a(ad+b^2)}{\alpha\varphi}g(g(\tilde{\tau}_h,V)\mathfrak{L}^D,D_{\frac{d}{dt}}V) - \tfrac{a(ad+b^2)}{\alpha\varphi}g(\tilde{\tau}_h,V)div(\omega_t)\\
&\quad -2\tfrac{b}{\varphi}g(\tilde{\tau}_h,V)div(\eta_t) - 2\tfrac{b}{\varphi}g(g(\tilde{\tau}_h,V)\Omega,D_{\frac{d}{dt}}V) + \tfrac{d}{\varphi}g(\tilde{\tau}_h,V)div(z_t)\\
&\quad -\tfrac{ad}{\alpha}\textstyle\sum_{i=1}^n g\big(g\big(D_{E_i}V,\tilde{\tau}_{v,h}\big)E_i,D_{\frac{d}{dt}}V\big)\\
&\quad -\tfrac{ad}{\alpha}div(d_t^v) + \tfrac{ad}{\alpha}\textstyle\sum_{i=1}^n g\big(D_{\frac{d}{dt}}V,g(E_i,D_{E_i}V)\tilde{\tau}_{v,h}\big) + \tfrac{ad}{\alpha}g\big(D_{\frac{d}{dt}}V,D_V\tilde{\tau}_{v,h}\big)\\
&= \tfrac{ab}{\alpha}\textstyle\sum_{i=1}^n g\big(R^D(\tilde{\tau}_h,E_i)E_i,D_{\frac{d}{dt}}V\big) - \tfrac{a^2}{\alpha}g\big(\mathfrak{L}^D(\tilde{\tau}_h),D_{\frac{d}{dt}}V\big) + \tfrac{a^2}{\alpha}div(g_t^h)\\
&\quad +g(D_{\frac{d}{dt}}V,\mathbb{H}\tilde{\tau}_h) - 2\tfrac{b(ad+b^2)}{\alpha\varphi}g(g(\tilde{\tau}_h,V)W,D_{\frac{d}{dt}}V)\\
&\quad +\tfrac{a(ad+b^2)}{\alpha\varphi}g(g(\tilde{\tau}_h,V)Z,D_{\frac{d}{dt}}V) + \tfrac{a(ad+b^2)}{\alpha\varphi}g(g(\tilde{\tau}_h,V)\mathfrak{L}^D,D_{\frac{d}{dt}}V)\\
&\quad -\tfrac{a(ad+b^2)}{\alpha\varphi}div\Big[g\big(\tilde{\tau}_h,V\big)\omega_t\Big] + \tfrac{a(ad+b^2)}{\alpha\varphi}g\Big[R^D\Big(grad\big(g\big(\tilde{\tau}_h,V\big)\big),V\Big)V,D_{\frac{d}{dt}}V\Big]\\
&\quad -2\tfrac{b}{\varphi}div\Big[g\big(\tilde{\tau}_h,V\big)\eta_t\Big] + 2\tfrac{b}{\varphi}g(D_{\frac{d}{dt}}V,D_{grad[g(\tilde{\tau}_h,V)]}V) - 2\tfrac{b}{\varphi}g(g(\tilde{\tau}_h,V)\Omega,D_{\frac{d}{dt}}V)\\
&\quad -\tfrac{ad}{\alpha}\textstyle\sum_{i=1}^n g\big(g\big(D_{E_i}V,\tilde{\tau}_{v,h}\big)E_i,D_{\frac{d}{dt}}V\big)\\
&\quad -\tfrac{ad}{\alpha}div(d_t^v) + \tfrac{ad}{\alpha}\textstyle\sum_{i=1}^n g\big(D_{\frac{d}{dt}}V,g(E_i,D_{E_i}V)\tilde{\tau}_{v,h}\big) + \tfrac{ad}{\alpha}g\big(D_{\frac{d}{dt}}V,D_V\tilde{\tau}_{v,h}\big)\\
&\quad +\tfrac{d}{\varphi}div\Big[g\big(\tilde{\tau}_h,V\big)z_t\Big] - \tfrac{d}{\varphi}g(D_{\frac{d}{dt}}V,grad[g(\tilde{\tau}_h,V)]).
\end{aligned}$$

We obtain

(A.24)

$$\begin{aligned}\int_M g\big(D_{\frac{d}{dt}}\tilde{\tau}_h,\tilde{\tau}_h\big)\circ i_t v_g = \int_M \Big[&\frac{ab}{\alpha}g\big(R^D(\tilde{\tau}_h,E_i)E_i,D_{\frac{d}{dt}}V\big)-\frac{a^2}{\alpha}g\big(\mathfrak{L}^D(\tilde{\tau}_h),D_{\frac{d}{dt}}V\big)\\&+g(D_{\frac{d}{dt}}V,\mathbb{H}\tilde{\tau}_h)-2\tfrac{b(ad+b^2)}{\alpha\varphi}g(g(\tilde{\tau}_h,V)W,D_{\frac{d}{dt}}V)\\&+\tfrac{a(ad+b^2)}{\alpha\varphi}g(g(\tilde{\tau}_h,V)Z,D_{\frac{d}{dt}}V)+\tfrac{a(ad+b^2)}{\alpha\varphi}g(g(\tilde{\tau}_h,V)\mathfrak{L}^D,D_{\frac{d}{dt}}V)\\&+\tfrac{a(ad+b^2)}{\alpha\varphi}g\Big[R^D\Big(grad\big(g(\tilde{\tau}_h,V)\big),V\Big)V,D_{\frac{d}{dt}}V\Big]\\&+2\tfrac{b}{\varphi}g(D_{\frac{d}{dt}}V,D_{grad[g(\tilde{\tau}_h,V)]}V)-2\tfrac{b}{\varphi}g(g(\tilde{\tau}_h,V)\Omega,D_{\frac{d}{dt}}V)\\&-\tfrac{ad}{\alpha}\textstyle\sum_{i=1}^n g\big(g\big(D_{E_i}V,\tilde{\tau}_{v,h}\big)E_i,D_{\frac{d}{dt}}V\big)\\&+\tfrac{ad}{\alpha}\textstyle\sum_{i=1}^n g\big(D_{\frac{d}{dt}}V,g(E_i,D_{E_i}V)\tilde{\tau}_{v,h}\big)-\tfrac{d}{\varphi}g(D_{\frac{d}{dt}}V,grad[g(\tilde{\tau}_h,V)])\\&+\tfrac{ad}{\alpha}g\big(D_{\frac{d}{dt}}V,D_V\tilde{\tau}_{v,h}\big)\Big]\circ i_t v_g.\end{aligned}$$

Calculus of $D_{\frac{d}{dt}}\int_M g\big(\tilde{\tau}_v,\tilde{\tau}_v\big)\circ i_t v_g$:

We have

(A.25)
$$D_{\frac{d}{dt}}\int_M g\big(\tilde{\tau}_v,\tilde{\tau}_v\big)\circ i_t v_g = 2\int_M g\big(D_{\frac{d}{dt}}\tilde{\tau}_v,\tilde{\tau}_v\big)\circ i_t v_g.$$

1. Calculus of $g\big(D_{\frac{d}{dt}}\tilde{\tau}_v,\tilde{\tau}_v\big)$:
(A.26)

$$\begin{aligned}
g\big(D_{\frac{d}{dt}}\tilde{\tau}_v,\tilde{\tau}_v\big) &= -g(D_{\frac{d}{dt}}\Omega,\tilde{\tau}_v)+\tfrac{ab}{\alpha}g(D_{\frac{d}{dt}}Z,\tilde{\tau}_v)+\tfrac{bd}{\alpha}g(D_{\frac{d}{dt}}D_VV,\tilde{\tau}_v)-\tfrac{b^2}{\alpha}g(D_{\frac{d}{dt}}W,\tilde{\tau}_v)\\
&\quad+g(D_{\frac{d}{dt}}(\mathbb{V}).V,\tilde{\tau}_v)+g(\mathbb{V}.D_{\frac{d}{dt}}V,\tilde{\tau}_v)\\
&= -2div(p_t^{v,h})-\Delta\big[g\big(D_{\frac{d}{dt}}V,\tilde{\tau}_v\big)\big]-g\big(D_{\frac{d}{dt}}V,\bar{\Delta}\tilde{\tau}_v\big)\\
&\quad-\tfrac{b^2}{\alpha}\textstyle\sum_{i=1}^n g\big(R^D(\tilde{\tau}_v,E_i)E_i,D_{\frac{d}{dt}}V\big)\\
&\quad+\tfrac{ab}{\alpha}g\big(\mathfrak{L}^D(\tilde{\tau}_v),D_{\frac{d}{dt}}V\big)-\tfrac{ab}{\alpha}div(g_t)+g(\mathbb{V}.D_{\frac{d}{dt}}V,\tilde{\tau}_v)\\
&\quad+2\tfrac{b^2}{\alpha}g(g(\tilde{\tau}_v,V)W,D_{\frac{d}{dt}}V)-\tfrac{ab}{\alpha}g(g(\tilde{\tau}_v,V)Z,D_{\frac{d}{dt}}V)\\
&\quad-\tfrac{ab}{\alpha}g(g(\tilde{\tau}_v,V)\mathfrak{L}^D,D_{\frac{d}{dt}}V)\\
&\quad+2g(g(\tilde{\tau}_v,V)\Omega,D_{\frac{d}{dt}}V)+\tfrac{ab}{\alpha}div\Big[g\big(\tilde{\tau}_v,V\big)\omega_t\Big]\\
&\quad-\tfrac{ab}{\alpha}g\Big[R^D\Big(grad\big(g(\tilde{\tau}_v,V)\big),V\Big)V,D_{\frac{d}{dt}}V\Big]\\
&\quad+2div\Big[g\big(\tilde{\tau}_v,V\big)\eta_t\Big]-2g(D_{\frac{d}{dt}}V,D_{grad[g(\tilde{\tau}_v,V)]}V)\\
&\quad+\tfrac{bd}{\alpha}\textstyle\sum_{i=1}^n g\big(g\big(D_{E_i}V,\tilde{\tau}_v\big)E_i,D_{\frac{d}{dt}}V\big)+\tfrac{bd}{\alpha}div(d_t^v)\\
&\quad-\tfrac{bd}{\alpha}\textstyle\sum_{i=1}^n g\big(D_{\frac{d}{dt}}V,g(E_i,D_{E_i}V)\tilde{\tau}_v\big)-\tfrac{bd}{\alpha}g\big(D_{\frac{d}{dt}}V,D_V\tilde{\tau}_v\big).
\end{aligned}$$

We obtain
(A.27)

$$\begin{aligned}
\int_M g\big(D_{\frac{d}{dt}}\tilde{\tau}_v,\tilde{\tau}_v\big)\circ i_tv_g &= \int_M\Big[-g\big(D_{\frac{d}{dt}}V,\bar{\Delta}\tilde{\tau}_v\big)-\frac{b^2}{\alpha}\sum_{i=1}^n g\big(R^D(\tilde{\tau}_v,E_i)E_i,D_{\frac{d}{dt}}V\big)\\
&\quad+\tfrac{ab}{\alpha}g\big(\mathfrak{L}^D(\tilde{\tau}_v),D_{\frac{d}{dt}}V\big)\\
&\quad+g(\mathbb{V}.D_{\frac{d}{dt}}V,\tilde{\tau}_v)+2\tfrac{b^2}{\alpha}g(g(\tilde{\tau}_v,V)W,D_{\frac{d}{dt}}V)-\tfrac{ab}{\alpha}g(g(\tilde{\tau}_v,V)Z,D_{\frac{d}{dt}}V)\\
&\quad-\tfrac{ab}{\alpha}g(g(\tilde{\tau}_v,V)\mathfrak{L}^D,D_{\frac{d}{dt}}V)\\
&\quad+2g(g(\tilde{\tau}_v,V)\Omega,D_{\frac{d}{dt}}V)-\tfrac{ab}{\alpha}g\Big[R^D\Big(grad\big(g(\tilde{\tau}_v,V)\big),V\Big)V,D_{\frac{d}{dt}}V\Big]\\
&\quad-2g(D_{\frac{d}{dt}}V,D_{grad[g(\tilde{\tau}_v,V)]}V)+\tfrac{bd}{\alpha}\textstyle\sum_{i=1}^n g\big(g\big(D_{E_i}V,\tilde{\tau}_v\big)E_i,D_{\frac{d}{dt}}V\big)\\
&\quad-\tfrac{bd}{\alpha}\textstyle\sum_{i=1}^n g\big(D_{\frac{d}{dt}}V,g(E_i,D_{E_i}V)\tilde{\tau}_v\big)-\tfrac{bd}{\alpha}g\big(D_{\frac{d}{dt}}V,D_V\tilde{\tau}_v\big)\Big]\circ i_tdv_g.
\end{aligned}$$

Calculus of $D_{\frac{d}{dt}}\int_M g\big(\tilde{\tau}_h,\tilde{\tau}_v\big)\circ i_t v_g$:

We have

$$D_{\frac{d}{dt}}\int_M g\big(\tilde{\tau}_h,\tilde{\tau}_v\big)\circ i_t v_g=\int_M g\big(D_{\frac{d}{dt}}\tilde{\tau}_v,\tilde{\tau}_h\big)\circ i_t v_g+\int_M g\big(D_{\frac{d}{dt}}\tilde{\tau}_h,\tilde{\tau}_v\big)\circ i_t v_g. \tag{A.28}$$

1. Calculus of $g\big(D_{\frac{d}{dt}}\tilde{\tau}_v,\tilde{\tau}_h\big)$:
   (A.29)
$$\begin{aligned}
g\big(D_{\frac{d}{dt}}\tilde{\tau}_v,\tilde{\tau}_h\big) &= -g(D_{\frac{d}{dt}}\Omega,\tilde{\tau}_h)+\tfrac{ab}{\alpha}g(D_{\frac{d}{dt}}Z,\tilde{\tau}_h)+\tfrac{bd}{\alpha}g(D_{\frac{d}{dt}}D_V V,\tilde{\tau}_h)-\tfrac{b^2}{\alpha}g(D_{\frac{d}{dt}}W,\tilde{\tau}_h)\\
&\quad +g(D_{\frac{d}{dt}}(\mathbb{V}).V,\tilde{\tau}_h)+g(\mathbb{V}.D_{\frac{d}{dt}}V,\tilde{\tau}_h)\\
&= -2div(p_t^{v,h})-\Delta\big[g\big(D_{\frac{d}{dt}}V,\tilde{\tau}_h\big)\big]-g\big(D_{\frac{d}{dt}}V,\bar{\Delta}\tilde{\tau}_h\big)\\
&\quad +\tfrac{ab}{\alpha}g\big(\mathfrak{L}^D(\tilde{\tau}_h),D_{\frac{d}{dt}}V\big)-\tfrac{ab}{\alpha}div(g_t^h)+g(D_{\frac{d}{dt}}V,\mathbb{V}.\tilde{\tau}_h)\\
&\quad -\tfrac{b^2}{\alpha}\textstyle\sum_{i=1}^n g\big(R^D(\tilde{\tau}_h,E_i)E_i,D_{\frac{d}{dt}}V\big)\\
&\quad +2\tfrac{b^2}{\alpha}g(g(\tilde{\tau}_h,V)W,D_{\frac{d}{dt}}V)-\tfrac{ab}{\alpha}g(g(\tilde{\tau}_h,V)Z,D_{\frac{d}{dt}}V)-\tfrac{ab}{\alpha}g(g(\tilde{\tau}_h,V)\mathfrak{L}^D,D_{\frac{d}{dt}}V)\\
&\quad +2g(g(\tilde{\tau}_h,V)\Omega,D_{\frac{d}{dt}}V)+\tfrac{ab}{\alpha}div\Big[g\big(\tilde{\tau}_h,V\big)\omega_t\Big]\\
&\quad -\tfrac{ab}{\alpha}g\Big[R^D\Big(grad\big(g\big(\tilde{\tau}_h,V\big)\big),V\Big)V,D_{\frac{d}{dt}}V\Big]\\
&\quad +2div\Big[g\big(\tilde{\tau}_h,V\big)\eta_t\Big]-2g(D_{\frac{d}{dt}}V,D_{grad[g(\tilde{\tau}_h,V)]}V)\\
&\quad +\tfrac{bd}{\alpha}\textstyle\sum_{i=1}^n g\big(g\big(D_{E_i}V,\tilde{\tau}_h\big)E_i,D_{\frac{d}{dt}}V\big)\\
&\quad +\tfrac{bd}{\alpha}div(d_t^v)-\tfrac{bd}{\alpha}\textstyle\sum_{i=1}^n g\big(D_{\frac{d}{dt}}V,g(E_i,D_{E_i}V)\tilde{\tau}_h\big)\\
&\quad -\tfrac{bd}{\alpha}\textstyle\sum_{i=1}^n g\big(D_{\frac{d}{dt}}V,g(E_i,V)D_{E_i}\tilde{\tau}_h\big).
\end{aligned}$$

Then

(A.30)

$$\int_M g\big(D_{\frac{d}{dt}}\tilde{\tau}_v,\tilde{\tau}_h\big)\circ i_t v_g = \int_M \Big[-g\big(D_{\frac{d}{dt}}V,\bar{\Delta}\tilde{\tau}_h\big) - \frac{b^2}{\alpha}\sum_{i=1}^{n} g\big(R^D(\tilde{\tau}_h,E_i)E_i,D_{\frac{d}{dt}}V\big)$$

$$+\tfrac{ab}{\alpha}g\big(\mathfrak{L}^D(\tilde{\tau}_h),D_{\frac{d}{dt}}V\big)+g(D_{\frac{d}{dt}}V,\mathbb{V}.\tilde{\tau}_h)$$

$$+2\tfrac{b^2}{\alpha}g(g(\tilde{\tau}_h,V)W,D_{\frac{d}{dt}}V)-\tfrac{ab}{\alpha}g(g(\tilde{\tau}_h,V)Z,D_{\frac{d}{dt}}V)$$

$$-\tfrac{ab}{\alpha}g(g(\tilde{\tau}_h,V)\mathfrak{L}^D,D_{\frac{d}{dt}}V)+2g(g(\tilde{\tau}_h,V)\Omega,D_{\frac{d}{dt}}V)$$

$$-\tfrac{ab}{\alpha}g\Big[R^D\Big(grad\big(g(\tilde{\tau}_h,V)\big),V\Big)V,D_{\frac{d}{dt}}V\Big]-2g(D_{\frac{d}{dt}}V,D_{grad[g(\tilde{\tau}_h,V)]}V)$$

$$+\tfrac{bd}{\alpha}\textstyle\sum_{i=1}^{n} g\big(g\big(D_{E_i}V,\tilde{\tau}_h\big)E_i,D_{\frac{d}{dt}}V\big)-\tfrac{bd}{\alpha}\sum_{i=1}^{n} g\big(D_{\frac{d}{dt}}V,g(E_i,D_{E_i}V)\tilde{\tau}_h\big)$$

$$-\tfrac{bd}{\alpha}\textstyle\sum_{i=1}^{n} g\big(D_{\frac{d}{dt}}V,g(E_i,V)D_{E_i}\tilde{\tau}_h\big)\Big]\circ i_t v_g.$$

2. Calculus of $g\big(D_{\frac{d}{dt}}\tilde{\tau}_h,\tilde{\tau}_v\big)$:

(A.31)

$$g\big(D_{\frac{d}{dt}}\tilde{\tau}_h,\tilde{\tau}_v\big)=\tfrac{ab}{\alpha}g(D_{\frac{d}{dt}}W,\tilde{\tau}_v)-\tfrac{a^2}{\alpha}g(D_{\frac{d}{dt}}Z,\tilde{\tau}_v)-\tfrac{ad}{\alpha}g(D_{\frac{d}{dt}}D_V V,\tilde{\tau}_v)$$

$$+g(D_{\frac{d}{dt}}(\mathbb{H}).V,\tilde{\tau}_v)+g(\mathbb{H}.D_{\frac{d}{dt}}V,\tilde{\tau}_v)$$

$$=\tfrac{ab}{\alpha}\textstyle\sum_{i=1}^{n} g\big(R^D(\tilde{\tau}_v,E_i)E_i,D_{\frac{d}{dt}}V\big)-\tfrac{a^2}{\alpha}g\big(\mathfrak{L}^D(\tilde{\tau}_v),D_{\frac{d}{dt}}V\big)+\tfrac{a^2}{\alpha}div(g_t^v)$$

$$+g(D_{\frac{d}{dt}}V,\mathbb{H}\tilde{\tau}_v)-2\tfrac{b(ad+b^2)}{\alpha\varphi}g(g(\tilde{\tau}_v,V)W,D_{\frac{d}{dt}}V)+\tfrac{a(ad+b^2)}{\alpha\varphi}g(g(\tilde{\tau}_v,V)Z,D_{\frac{d}{dt}}V)$$

$$+\tfrac{a(ad+b^2)}{\alpha\varphi}g(g(\tilde{\tau}_v,V)\mathfrak{L}^D,D_{\frac{d}{dt}}V)-\tfrac{a(ad+b^2)}{\alpha\varphi}div\Big[g\big(\tilde{\tau}_v,V\big)\omega_t\Big]$$

$$+\tfrac{a(ad+b^2)}{\alpha\varphi}g\Big[R^D\Big(grad\big(g(\tilde{\tau}_v,V)\big),V\Big)V,D_{\frac{d}{dt}}V\Big]-2\tfrac{b}{\varphi}div\Big[g\big(\tilde{\tau}_v,V\big)\eta_t\Big]$$

$$+2\tfrac{b}{\varphi}g(D_{\frac{d}{dt}}V,D_{grad[g(\tilde{\tau}_v,V)]}V)-2\tfrac{b}{\varphi}g(g(\tilde{\tau}_v,V)\Omega,D_{\frac{d}{dt}}V)$$

$$-\tfrac{ad}{\alpha}\textstyle\sum_{i=1}^{n} g\big(g\big(D_{E_i}V,\tilde{\tau}_v\big)E_i,D_{\frac{d}{dt}}V\big)-\tfrac{ad}{\alpha}div(d_t^v)$$

$$+\tfrac{ad}{\alpha}\textstyle\sum_{i=1}^{n} g\big(D_{\frac{d}{dt}}V,g(E_i,D_{E_i}V)\tilde{\tau}_v\big)+\tfrac{ad}{\alpha}\sum_{i=1}^{n} g\big(D_{\frac{d}{dt}}V,g(E_i,V)D_{E_i}\tilde{\tau}_v\big)$$

$$+\tfrac{d}{\varphi}div\Big[g\big(\tilde{\tau}_v,V\big)z_t\Big]-\tfrac{d}{\varphi}g(D_{\frac{d}{dt}}V,grad[g(\tilde{\tau}_v,V)]).$$

Then

(A.32)

$$\begin{aligned}\int_M g\big(D_{\frac{d}{dt}}\tilde{\tau}_h,\tilde{\tau}_v\big)\circ i_t v_g = \int_M \Big[&\frac{ab}{\alpha}g\big(R^D(\tilde{\tau}_v,E_i)E_i,D_{\frac{d}{dt}}V\big)-\frac{a^2}{\alpha}g\big(\mathfrak{L}^D(\tilde{\tau}_v),D_{\frac{d}{dt}}V\big)\\
&+g(D_{\frac{d}{dt}}V,\mathbb{H}\tilde{\tau}_v)\\
&-2\tfrac{b(ad+b^2)}{\alpha\varphi}g(g(\tilde{\tau}_v,V)W,D_{\frac{d}{dt}}V)+\tfrac{a(ad+b^2)}{\alpha\varphi}g(g(\tilde{\tau}_v,V)Z,D_{\frac{d}{dt}}V)\\
&+\tfrac{a(ad+b^2)}{\alpha\varphi}g(g(\tilde{\tau}_v,V)\mathfrak{L}^D,D_{\frac{d}{dt}}V)\\
&+\tfrac{a(ad+b^2)}{\alpha\varphi}g\Big[R^D\Big(grad\big(g\big(\tilde{\tau}_v,V\big)\big),V\Big)V,D_{\frac{d}{dt}}V\Big]\\
&+2\tfrac{b}{\varphi}g(D_{\frac{d}{dt}}V,D_{grad[g(\tilde{\tau}_v,V)]}V)-2\tfrac{b}{\varphi}g(g(\tilde{\tau}_v,V)\Omega,D_{\frac{d}{dt}}V)\\
&-\tfrac{ad}{\alpha}\textstyle\sum_{i=1}^n g\big(g\big(D_{E_i}V,\tilde{\tau}_v\big)E_i,D_{\frac{d}{dt}}V\big)\\
&+\textstyle\sum_{i=1}^n \frac{ad}{\alpha}g\big(D_{\frac{d}{dt}}V,g(E_i,D_{E_i}V)\tilde{\tau}_v\big)\\
&+\textstyle\sum_{i=1}^n \frac{ad}{\alpha}g\big(D_{\frac{d}{dt}}V,g(E_i,V)D_{E_i}\tilde{\tau}_v\big)\\
&-\tfrac{d}{\varphi}g(D_{\frac{d}{dt}}V,grad[g(\tilde{\tau}_v,V)])\Big]\circ i_t v_g.\end{aligned}$$

A.4 The final result

Subtituting $(A.21), (A.24), (A.27), (A.32)$ in (A.1) , we obtain
(A.33)

$$
\begin{aligned}
D_{\frac{d}{dt}}E_2(V_t) = \int_M \Big[& (a+c)\frac{ab}{\alpha} g\big(R^D(\tilde{\tau}_h, E_i)E_i, D_{\frac{d}{dt}}V\big) - (a+c)\frac{a^2}{\alpha} g\big(\mathfrak{L}^D(\tilde{\tau}_h), D_{\frac{d}{dt}}V\big) \\
& + (a+c) g(D_{\frac{d}{dt}}V, \mathbb{H}\tilde{\tau}_h) \\
& - 2(a+c)\tfrac{b(ad+b^2)}{\alpha\varphi} g(g(\tilde{\tau}_h, V)W, D_{\frac{d}{dt}}V) + (a+c)\tfrac{a(ad+b^2)}{\alpha\varphi} g(g(\tilde{\tau}_h, V)Z, D_{\frac{d}{dt}}V) \\
& + (a+c)\tfrac{a(ad+b^2)}{\alpha\varphi} g(g(\tilde{\tau}_h, V)\mathfrak{L}^D, D_{\frac{d}{dt}}V) \\
& + (a+c)\tfrac{a(ad+b^2)}{\alpha\varphi} g\Big[R^D\Big(grad\big(g(\tilde{\tau}_h, V)\big), V\Big)V, D_{\frac{d}{dt}}V\Big] \\
& + 2(a+c)\tfrac{b}{\varphi} g(D_{\frac{d}{dt}}V, D_{grad[g(\tilde{\tau}_h, V)]}V) - 2(a+c)\tfrac{b}{\varphi} g(g(\tilde{\tau}_h, V)\Omega, D_{\frac{d}{dt}}V) \\
& - (a+c)\tfrac{ad}{\alpha}\textstyle\sum_{i=1}^n g\big(g\big(D_{E_i}V, \tilde{\tau}_h\big)E_i, D_{\frac{d}{dt}}V\big) + (a+c)\tfrac{ad}{\alpha}\textstyle\sum_{i=1}^n g\big(D_{\frac{d}{dt}}V, g(E_i, D_{E_i}V)\tilde{\tau}_h\big) \\
& - (a+c)\tfrac{d}{\varphi} g(D_{\frac{d}{dt}}V, grad[g(\tilde{\tau}_h, V)]) + (a+c)\tfrac{ad}{\alpha} g\big(D_{\frac{d}{dt}}V, D_V\tilde{\tau}_h\big) \\
& + dg(g(\tilde{\tau}_h, V)\tilde{\tau}_h, D_{\frac{d}{dt}}V) + d(\tfrac{ab}{\alpha} - 2\tfrac{b(ad+b^2)}{\alpha\varphi}) g(g\big(\tilde{\tau}_h, V\big)W, D_{\frac{d}{dt}}V) \\
& + d\tfrac{a(ad+b^2)}{\alpha\varphi} g(g\big(\tilde{\tau}_h, V\big)Z, D_{\frac{d}{dt}}V) \\
& + d(\tfrac{a(ad+b^2)}{\alpha\varphi} - \tfrac{a^2}{\alpha}) g(g\big(\tilde{\tau}_h, V\big)\mathfrak{L}^D, D_{\frac{d}{dt}}V) - 2d\tfrac{b}{\varphi} g(g\big(\tilde{\tau}_h, V\big)\Omega, D_{\frac{d}{dt}}V) \\
& + d(\tfrac{a(ad+b^2)}{\alpha\varphi} - \tfrac{a^2}{\alpha}) g\Big[R^D\Big(grad\big(g(\tilde{\tau}_h, V)\big), V\Big)V, D_{\frac{d}{dt}}V\Big] \\
& + 2d\tfrac{b}{\varphi} g(D_{\frac{d}{dt}}V, D_{grad[g(\tilde{\tau}_h, V)]}V) + d\tfrac{ad}{\alpha} g(g\big(\tilde{\tau}_h, V\big)D_V V, D_{\frac{d}{dt}}V) \\
& - d\tfrac{d}{\varphi} g(D_{\frac{d}{dt}}V, grad[g(\tilde{\tau}_h, V)]) \\
& - ag\big(D_{\frac{d}{dt}}V, \bar{\Delta}\tilde{\tau}_v\big) - a\tfrac{b^2}{\alpha} g\big(R^D(\tilde{\tau}_v, E_i)E_i, D_{\frac{d}{dt}}V\big) + a\tfrac{ab}{\alpha} g\big(\mathfrak{L}^D(\tilde{\tau}_v), D_{\frac{d}{dt}}V\big) \\
& + ag(\mathbb{V}.D_{\frac{d}{dt}}V, \tilde{\tau}_v) + a\tfrac{bd}{\alpha}\textstyle\sum_{i=1}^n g\big(g\big(D_{E_i}V, \tilde{\tau}_v\big)E_i, D_{\frac{d}{dt}}V\big) \\
& - a\tfrac{bd}{\alpha}\textstyle\sum_{i=1}^n g\big(D_{\frac{d}{dt}}V, g(E_i, D_{E_i}V)\tilde{\tau}_v\big) - a\tfrac{bd}{\alpha} g\big(D_{\frac{d}{dt}}V, D_V\tilde{\tau}_v\big).
\end{aligned}
$$

$$
\begin{aligned}
&-bg\big(D_{\frac{d}{dt}}V,\bar{\Delta}\tilde{\tau}_h\big)-b\tfrac{b^2}{\alpha}\textstyle\sum_{i=1}^n g\big(R^D(\tilde{\tau}_h,E_i)E_i,D_{\frac{d}{dt}}V\big)\\
&+b\tfrac{ab}{\alpha}g\big(\mathfrak{L}^D(\tilde{\tau}_h),D_{\frac{d}{dt}}V\big)+bg(D_{\frac{d}{dt}}V,\mathbb{V}.\tilde{\tau}_h)\\
&+2b\tfrac{b^2}{\alpha}g(g(\tilde{\tau}_h,V)W,D_{\frac{d}{dt}}V)-b\tfrac{ab}{\alpha}g(g(\tilde{\tau}_h,V)Z,D_{\frac{d}{dt}}V)-b\tfrac{ab}{\alpha}g(g(\tilde{\tau}_h,V)\mathfrak{L}^D,D_{\frac{d}{dt}}V)\\
&+2bg(g(\tilde{\tau}_h,V)\Omega,D_{\frac{d}{dt}}V)-b\tfrac{ab}{\alpha}g\Big[R^D\Big(grad\big(g(\tilde{\tau}_h,V)\big),V\Big)V,D_{\frac{d}{dt}}V\Big]\\
&-2bg(D_{\frac{d}{dt}}V,D_{grad[g(\tilde{\tau}_h,V)]}V)+b\tfrac{bd}{\alpha}\textstyle\sum_{i=1}^n g\big(g\big(D_{E_i}V,\tilde{\tau}_h\big)E_i,D_{\frac{d}{dt}}V\big)\\
&-b\tfrac{bd}{\alpha}\textstyle\sum_{i=1}^n g\big(D_{\frac{d}{dt}}V,g(E_i,D_{E_i}V)\tilde{\tau}_h\big)-b\tfrac{bd}{\alpha}\textstyle\sum_{i=1}^n g\big(D_{\frac{d}{dt}}V,g(E_i,V)D_{E_i}\tilde{\tau}_h\big)\\
&+b\tfrac{ab}{\alpha}\textstyle\sum_{i=1}^n g\big(R^D(\tilde{\tau}_v,E_i)E_i,D_{\frac{d}{dt}}V\big)-b\tfrac{a^2}{\alpha}g\big(\mathfrak{L}^D(\tilde{\tau}_v),D_{\frac{d}{dt}}V\big)+bg(D_{\frac{d}{dt}}V,\mathbb{H}\tilde{\tau}_v)\\
&-b\textstyle\sum_{i=1}^n\frac{ad}{\alpha}g\big(g\big(D_{E_i}V,\tilde{\tau}_v\big)E_i,D_{\frac{d}{dt}}V\big)\\
&+b\textstyle\sum_{i=1}^n\frac{ad}{\alpha}g\big(D_{\frac{d}{dt}}V,g(E_i,D_{E_i}V)\tilde{\tau}_v\big)+b\textstyle\sum_{i=1}^n\frac{ad}{\alpha}g\big(D_{\frac{d}{dt}}V,g(E_i,V)D_{E_i}\tilde{\tau}_v\big)\Big]\circ i_t dv_g.
\end{aligned}
$$

Evaluating $(A.33)$ at $t=0$ and settig $Y=\nabla_{\frac{d}{dt}}V_t|_{t=0}$ and using the fact that Y is a vector field on M which is perpendicular to U, since U is a unit vector field. Then U is a biharmonic unit vector field if and only if $\frac{d}{dt}E_2(V_t)|_{t=0}=0$ for all variations, and this is equivalent to

(A.34)

$$
\begin{aligned}
&b\textstyle\sum_{i=1}^n R(\tilde{\tau}_h(U),e_i)e_i-a\sum_{i=1}^n\big[2R(e_i,\tilde{\tau}_h(U))\nabla_{e_i}U+(\nabla_{e_i}R)(e_i,\tilde{\tau}_h(U))U+R(e_i,\nabla_{e_i}\tilde{\tau}_h(U))U\big]\\
&+\big[-\tfrac{bd}{\varphi}g(QU,U)+\tfrac{bd}{\varphi}g(\bar{\Delta}U,U)+(a+c)\tfrac{d}{\varphi}div(U)+\tfrac{ad}{\varphi}g(S(U),U)+dg(\tilde{\tau}_h(U),U)\big]\tilde{\tau}_h(U)\\
&+\tfrac{a^2d}{\alpha}g(\tilde{\tau}_h(U),U)S(U)-\tfrac{abd}{\alpha}g(\tau_h(U),U)QU-a\bar{\Delta}\tilde{\tau}_v(U)-b\bar{\Delta}\tilde{\tau}_h(U)\\
&-d\textstyle\sum_{i=1}^n g\big(\nabla_{e_i}U,\tilde{\tau}_h(U)\big)e_i+d\nabla_U\tilde{\tau}_h(U)-d\,grad[g(\tilde{\tau}_h(U),U)]+\tfrac{ad^2}{\alpha}g\big(\tilde{\tau}_h(U),U\big)\nabla_U U\\
&+\big[\tfrac{b^2}{\varphi}g(QU,U)+\tfrac{\alpha+ad}{\varphi}g(\bar{\Delta}U,U)-\tfrac{ab}{\varphi}g(S(U),U)+\tfrac{bd}{\varphi}div(U)\big]\tilde{\tau}_v(U)=0.
\end{aligned}
$$

Conclusion and perspectives

The (bi)haromicity theory is a rich domain where mathematics and physics intersect.

In our study of this topic we focused on (unit) tangent bundles and vector bundles. For the tangent bundles, we generalize the classical notion of biharmonic maps and biharmonic vector field, by endowing the tangent bundle with an arbitrary g-natural metric instead of the classical Sasaki metric. For the vector bundles, we investigated the notion of biharmonic function, when the vector bundle is equipped with an arbitrary Spherically symmetric metric.

1. **Biharmonic vector fields**

   To study biharmonic vector fields, when the tangent bundle is equipped with g-natural metrics, it was necessary to restrict ourselves to the class of parallel vector fields for at least three reasons:

   - The calculation of the bitension field in the general case is very complicated.
   - The existence of such non vanshing parallel vector field on a Riemannian manifold (M, g) is equivelant to the local reducibilty of M as $\mathbb{R} \times M'$ and the parallel vector field can be identified with a vector field tangent to the flat component $\mathbb{R}$ of the product, the biggest benefit is the cancellation of the curvature components, however we note that the existence of a non vanishing parallel vector field imposes some topological restrictions such as the vanishing of the Euler characteristic.
   - The study of biharmonic vector fields when the tangent bundle is endowed with the Sasaki metric led to nothing else but parallel vector fields, so by investigating this class first we can conclude quickly if there are any different behaviours by generalizing the metrics and we achieved this goal. As a main result of our study, we obtain a characterization of parallel proper biharmonic vector fields, in the case when the tangent bundle is equipped with a Kaluza-Klein metric, and we give other examples.

   Despite the importance of the results obtained in the case of parallel vector fields, there are many interesting directions to investigate:

   (a) The biharmonicity (resp. G-biharmonicity) of non parallel vector fields when the tangent bundle is equipped with g-natural metrics.

   (b) Biharmonic submanifolds of the (unit) tangent bundle and the investigation of the Chen's conjecture.

   (c) Examples of non biharmonic G-biharmonic vector fields when the tangent bundle is equipped with an appropriate **Riemannian** g-natural metric.

2. **Biharmonic unit vector fields**

   The study of biharmonicity of unit vector fields in the context of unit tangent sphere bundles is as significant as that of vector fields in the context of tangent bundles. By equipping a unit tangent sphere bundle with an arbitrary g-natural metric $\tilde{G}$ and looking

for the critical points of the bienergy functional, restricted to set of unit vector fields, we were able to characterize $\tilde{G}$-biharmonic unit vector fields. This characterization becomes more interesting for special classes of vector fields, such as Killing vector fields and Reeb vector fields. Additionally, we provided examples of (proper) $\tilde{G}$-biharmonic unit vector fields on hyperbolic spaces $\mathbb{H}^n$, on the solvable Lie group Sol_3, and on the special unitary group $SU(2)$ equipped with a left-invariant Riemannian metric.

We still have to investigate the biharmonicity of unit vector fields as maps, i.e., critical point of the bienergy functional with variations through maps, and to look for proper biharmonic vector fields in this sense.

3. **Biharmonic functions**

   We know that every lifted function from the base manifold to the total space of a vector bundle is a linear combination of the vertical lift and complete lift, so the study of the (bi)harmonicity of vertical and complete lifts of functions is crucial for best understanding of the vector bundle geometric properties and the (bi)harmonicity in a big scale.

   We studied the biharmonicity of two classes of functions in the vector bundles. The first is the class of vertically lifted functions: we got the bilaplacian and we derived a condition on the Spherically symmetric metrics which allows us to construct various examples of (proper) biharmonic functions on vector bundles.

   The second class was the r-radial functions: the Euler-Lagrange equation in this case gives us a fourth order linear differential equation, which led to take again the Sasaki metric case to construct some important examples of proper biharmonic maps.

   We still have to investigate the problem of (bi)harmonicity of the complete lift of functions, and it is the subject of a forthcoming paper in collaboration with M.T.K Abbassi and I. Lakrini. Finding the (bi)harmonicity condition for the complete lift of a function will enable us to classify any (bi)harmonic lifted function from the base manifold to the total space.

4. **Biharmonic maps**

   The study of the (bi)harmonicity of maps having tangent bundles and unit tangent sphere bundles as source and/or target spaces could help us to understand better further geometric properties of these manifolds. We tried to study some kinds of maps like: the identity map, the projection map,...but we found some difficulties related to the complexity of the Euler-Lagrange equation but we are still working in this direction.

5. **Mathematical physics**

   In the (bi)harmonic theory, we are looking for the minimums of the (bi)energy functional, which are the "natural" objects among smooth maps in the physical point of view, and it is known that these theories are in the core of discussions about the unification of the general relativity and the quantum mechanics in the so-called String theory.

   Harmonic maps are called, in the String theory, non-linear sigma models, and it is known that there are several strong similarities between the three thoeries of Einstein metrics, the Yang-Mills connections and Harmonic maps. The same holds for their analogues in physics, i.e. Gravitation, Gauge theory and String theory.

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