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On some nonlinear anisotropic elliptic and parabolic problems

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BOUZIANI MOHAMMED

Pour l'obtention du

Doctorat En Mathématiques

Spécialité :Équations aux Dérivées Partielles

ON SOME NONLINEAR ANISOTROPIC ELLIPTIC AND PARABOLIC PROBLEMS

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ON SOME NONLINEAR ANISOTROPIC ELLIPTIC AND PARABOLIC PROBLEMS

DIRECTEUR DE THÈSE : AZROUL ELHOUSSINE

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Dedication

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Contents

Notations **13**

A general introduction **15**

1 Preliminaries **29**

1.1 Functional spaces 29

1.1.1 Variable Exponent Lebesgue and Sobolev Spaces 29

1.1.2 Marcinkiewicz spaces with variable exponents 32

1.1.3 Anisotropic Sobolev Spaces with variable exponents 32

1.1.4 Anisotropic parabolic spaces with variable exponents 35

1.1.5 Anisotropic Weighted Sobolev spaces 36

1.2 Some important technical results and lemmas 37

1.3 Penalization operators 42

I Nonlinear anisotropic elliptic and parabolic PDEs with lower order terms and variable exponents. **45**

2 On nonlinear anisotropic elliptic equations involving lower order terms with variable exponents **47**

2.1 Introduction 47

2.2 Lower order terms without sign assumption 51

2.3 Lower order terms with sign assumptions 66

2.4 Appendix 80

3 Existence of entropy solutions for some quasilinear anisotropic elliptic unilateral problems with variable exponents 85

3.1 Introduction 85

3.2 Essential assumptions 87

3.3 Existence of entropy solutions 88

4 Obstacle problems depending on time for strongly nonlinear anisotropic parabolic equations with variable exponents 103

4.1 Introduction 103

4.2 Useful results 107

4.3 Statement of results 109

II Nonlinear anisotropic elliptic PDEs involving singularities on its right - hand side within the anisotropic Sobolev spaces with either variable exponents or constant exponents or weight. 131

5 Existence of solutions for some quasilinear $\vec{p}(x)-$elliptic problem with Hardy potential 133

5.1 Introduction 133

5.2 Existence of entropy Solutions 137

6 Existence of entropy solutions for strongly nonlinear anisotropic elliptic problem involving lower order terms and Hardy potential 153

6.1 Introduction 153

6.2 Preliminary 156

6.3 Main results 158

7 Existence of entropy solutions for anisotropic quasilinear degenerate elliptic problems with Hardy potential 179

7.1 Introduction 179

7.2 Basic assumptions 182

7.3 Some technical results 183

7.4 Main results 184

7.5 Example 201

A general conclusion and perspectives 203

Bibliography 204

Abstract[1] :

The work presented in this thesis deals with the study of some problems of nonlinear anisotropic elliptic and parabolic partial differential equations with Dirichlet type. The existence of solutions of these types of problems is investigated in anisotropic Sobolev spaces which are provided with constant or variable exponents or with a weight. This work, which is divided into two parts, presents existence and regularity results of weak and entropic solutions for six nonlinear anisotropic problems of the type mentioned below. The first part concerns the study of nonlinear anisotropic elliptic and parabolic problems with variable exponents. The second part is dedicated to the study of nonlinear anisotropic problems introducing singularities. After a general introduction, a brief statement of definitions and necessary results for the rest of our work, we demonstrate the existence of at least one weak solution for a class of nonlinear elliptic and anisotropic equations with terms of natural growth in the anisotropic Sobolev space with variable exponents $W_0^{1,\vec{p}(\cdot)}(\Omega)$. In this chapter we show two existence results, the first concerns $\vec{p}(x)-$ elliptic problem of $\Delta_{\vec{p}(x)}-$laplacian type with lower-order terms without sign hypothesis, the second concerns a nonlinear anisotropic elliptic problem involving a Leray-Lions type divergence operator and introducing lower order terms with sign condition. In Chapter 3, we establish the existence of at least one entropic solution of a quasilinear anisotropic elliptic unilateral problem of Dirichlet type with less regular data. In Chapter 4, we establish the existence of a pair of solutions for a strongly nonlinear anisotropic parabolic unilateral problem with an obstacle depending in time, drawing on penalization techniques.
In the second part, we study in chapter 5, the existence of entropic solutions for a quasilinear $\vec{p}(x)-$elliptic problem with Hardy potential and L^1-data.

[1] *Keywords*: Nonlinear elliptic equations, Strongly nonlinear parabolic equations, Obstacls problems, Lower order terms, Hardy potential, Weak solutions, Entropic solutions, Dual spaces, Anisotropic Sobolev spaces with constant or variable exponents, Weighted anisotropic Sobolev spaces

In the same direction, in Chapter 6 we show the existence of an entropic solution for a strongly nonlinear anisotropic problem of $\Delta_{\vec{p}(x)-}$laplacian type introducing terms of lower order and a potential of Hardy. And we end this part by, chapter 7, where we establish an appropriate functional framework of anisotropic weighted Sobolev spaces, provided with continuous and compact injections, in which we study the existence of an entropic solution for a quasilinear elliptic degenerated problem with as a singular term the potential of Hardy and a data f in $L^1(\Omega)$.

MSC 2010: 35J15, 35J62, 35J66, 35J20, 35J60, 35J70, 46E35, 35D30.

Résumé[2] :

Les travaux présentés dans cette thèse, portent sur l'étude de certains problèmes des équations aux dérivées partielles non linéaires anisotropiques, elliptiques et paraboliques de type Dirichlet. L'existence des solutions de ces types de problèmes est investiguée dans des espaces de Sobolev anisotropiques qui sont munis des exposants constants ou variables ou bien d'un poid. Ce travail qui se décompose en deux parties présente des résultats d'existence et de régularité de solutions faibles et entropiques pour six problèmes anisotropiques non linéaires du type mentionnées ci-dessous. La première partie concerne l'étude des problèmes non linéaires anisotropiques elliptiques et paraboliques avec des exposants variables. La seconde partie est consacrée à l'étude des problèmes non linéaires anisotropiques introduisant des singularités. Après une introduction génerale, un bref exposé de définitions et résultats nécéssaires pour la suite de notre travail, nous démontrons l'existence d'au moins une solution faible pour une classe d'équations non linéaires elliptiques et anisotropiques avec des termes de croissance naturelle dans l'espace de Sobolev anisotropique avec des exposants variables $W_0^{1,\vec{p}(\cdot)}(\Omega)$. Dans ce chapitre nous montrons deux résultats d'existence, le premier concerne un problème $\vec{p}(x)-$elliptique de type $\Delta_{\vec{p}(x)}-$laplacien avec des termes d'ordre inférieur sans hypothèse de signe, le second concerne un problème non linéaire elliptique anisotropique faisant intervenir un opérateur sous forme de divergence du type Leray- Lions et introduisant des termes d'ordre inférieur avec condition de signe. Dans le chapitre 3, nous établissons l'existence d'au moins une solution entropique d'un problème quasilinear anisotropique elliptique unilatéral de type Dirichlet avec des données peu régulières. Dans le chapitre 4, nous établissons l'existence d'une paire de solutions pour un problème fortement non linéaire anisotropique parabolique unilatéral avec un obstacle

[2] *Keywords*: Nonlinear elliptic equations, Strongly nonlinear parabolic equations, Obstacls problems, Lower order terms, Hardy potential, Weak solutions, Entropic solutions, Dual spaces, Anisotropic Sobolev spaces with constant or variable exponents, Weighted anisotropic Sobolev spaces

dépendant du temps, en se basant sur des techniques de pénalisation.

Dans la deuxième partie, nous étudions dans le chapitre 5, l'existence de solutions entropiques pour un problème quasilinéaire $\vec{p}(x)-$elliptique avec un potential de Hardy et des données dans $L^1(\Omega)$.

Dans la même direction, au chapitre 6 nous démontrons l'existence d'une solution entropique pour un problème fortement non linéaire anisotropique de type $\Delta_{\vec{p}(x)-}$laplacien introduisant des termes d'order inférieur et un potential de Hardy. Et on achève cette partie par, le chapitre 7, oú on établit un cadre fonctionnel adéquat des espaces de Sobolev anisotropiques avec poids, muni des injections continues et compactes, dans lesquels on étudie l'existence d'une solution entropique pour un problème elliptique quasilinéaire dégeneré avec comme terme singulier le potential de Hardy et une donnée f dans $L^1(\Omega)$.

MSC 2010: 35J15, 35J62, 35J66, 35J20, 35J60, 35J70, 46E35, 35D30.

Notations

Ω	open domain in $\mathbb{R}^N$
$\partial\Omega$	topological boundary of Ω
$x = (x_1, ..., x_N)$	generic point of $\mathbb{R}^N$
$dx = dx_1.....dx_N$	Lebesgue measure in Ω
Q_T	$(0,T) \times \Omega$ for $T > 0$
$D_i u = \partial_i u = \dfrac{\partial u}{\partial x_i} = u_{x_i}$	partial derivative of u with respect to x_i
∂Q_T	$(0,T) \times \partial\Omega$
∇u	gradient de u
$Supp(u)$	support d'une fonction u
$\mathcal{C}_0(\Omega)$	space of null continuous functions at the edge of Ω
$\mathcal{C}^\infty(\Omega)$	space of indefinitely differentiable functions on Ω
$D(\Omega)$, (resp. $D(Q_T)$)	space of differentiable functions and has compact support in Ω, $(resp. Q_T)$
$C(\Omega)$,(resp. $C(Q_T)$)	space of continuous functions in Ω, (resp Q_T)
$C^k(\Omega)$,(resp. $C^k(Q_T)$)	for $1 \leq k \leq \infty$, space of functions k-times continuously differentiable
	in Ω, (resp. Q_T)
T_k	truncation function $k > 0$
$L^p(\Omega)$	$\{f : \Omega \to \mathbb{R}$ measurable; $\int_\Omega \lvert f(x)\rvert^p dx < \infty\}$ for $1 \leq p < \infty$
$L^\infty(\Omega)$	$\{f : \Omega \to \mathbb{R}$ measurable; $ess \sup_{x\in\Omega} \lvert f(x)\rvert < \infty\}$
$W^{1,p}(\Omega)$	$\{f : \Omega \to \mathbb{R}$ measurable; $f \in L^p(\Omega)$, and $\dfrac{\partial f}{\partial x_i} \in L^p(\Omega)\}$ pour $1 \leq p \leq \infty$
$W_0^{1,p}(\Omega)$	Closure of $D(\Omega)$ in $W^{1,p}(\Omega)$ with $1 \leq p \leq \infty$

$W^{1,\overrightarrow{p}(\cdot)}(\Omega)$	$\{u \in L^{p_0(\cdot)}(\Omega) \quad \text{and} \quad D^i u \in L^{p_i(\cdot)}(\Omega) \quad \text{for} \quad i = 1, 2, \ldots, N\}$
$W_0^{1,\vec{p}(\cdot)}(\Omega)$	The closure of $\mathcal{C}_0^\infty(\Omega)$ in $W^{1,\vec{p}(\cdot)}(\Omega)$
$\mathcal{T}^{1,\vec{p}(\cdot)}(\Omega)$	$\{u : \Omega \mapsto I\!R$ measurable, such that $T_k(u) \in W^{1,\vec{p}(\cdot)}(\Omega)$ for any $k > 0\}$
$\mathcal{T}_0^{1,\vec{p}(\cdot)}(\Omega)$	$\{u : \Omega \mapsto I\!R$ measurable, such that $T_k(u) \in W_0^{1,\vec{p}(\cdot)}(\Omega)$ for any $k > 0\}$
X	arbitrary Banach space
X'	dual space of X
$\|E\|$	Lebesgue's measure of a set E
u^+	$\max(u, 0)$
u^-	$-\min(u, 0) = \sup(-u, 0)$
$L^p(0, T; X)$	for $1 \leq p < \infty$, $\{f : [0, T] \to X$ measurable; $\displaystyle\int_0^T \|\|f(t)\|\|_X^p \, dt < \infty\}$
$L^\infty(0, T; X)$	$\{f : [0, T] \to X$ measurable; $ess \sup_{t \in (0,T)} \|\|f(t)\|\|_X < \infty\}$
$L^{\overrightarrow{p}}(0, T; W_0^{1,\overrightarrow{p}}(\Omega))$	$\{u : [0, T] \to W_0^{1,\overrightarrow{p}}(\Omega)$measurable / $\displaystyle\sum_{i=0}^{i=1} \int_0^T \|D^i u\|_{p_i}^{p_i} \, dt < \infty\}$
$C([0, T]; X)$	space of continuous functions defined on $[0, T]$ with values in X
$D([0, T]; X)$	space of continuously differentiable functions with compact support in $[0, T]$

A general introduction

[3] Partial differential equations (PDEs) appear naturally in the modeling of many problems in physics, biology economics or elsewhere. In many respects, they seem to generalize ordinary differential equations to multi-dimensional contexts. The approach proposed in the mathematical literature can, however, surprise the reader who wishes to be initiated there: very varied methods of inspiration abound, most often adapted to particular cases, so that it is difficult to imagine that a unified theory can emerge. Such a theory is extremely unlikely to exist, given the rich variety of physical, geometric, and probabilistic phenomena which can be modeled by PDE. It must be admitted that this feeling of profusion (or even confusion) corresponds to a certain extent to an unavoidable reality: the modeled phenomena are by nature so different that it is almost unthinkable to bring them into one and the same category. These main categories, in particular the linear case, were for the most part known and studied since the nineteenth century, or even before. We can cite, among others: Laplace's equation, eigenvalue equation, heat equation, Fokker-Planck equation, telegraph equation, and general wave equations [98]. The theory experienced a real explosion in the twentieth century thanks to the contribution of functional analysis, and the improvement of the necessary mathematical tools, in this case, the appearance of the theory of distribution thanks to Schwartz and Hormander's birth of pseudo differential calculus in the seventies of the same century. There has been also a breakthrough, particularly in non-linear PDEs in the early 21st century. There are some important examples: minimal surface equation, scalar conservation law, scalar reaction diffusion equation, inviscid Burger's equation, small equation, nonlinear wave equation, some systems of PDEs like Navier Stokes equations for incompressible, viscous flow, to mention only some examples ...(see [159] and the

[3] *Keywords and sentences*: Nonlinear elliptic equations, Strongly nonlinear parabolic equations, Obstacls problems, Lower order terms, Hardy potential, Weak solutions, Entropic solutions, Dual spaces, Anisotropic Sobolev spaces with constant or variable exponents, Weighted anisotropic Sobolev spaces.

references therein). The fundamental questions that arise, as for ordinary differential equations:

- the existence of solutions

- the uniqueness of solutions

possibly based on data at the prescribed limits. However, new and pioneering issues are also emerging, such as the regularity of solutions. As with ordinary equations or even much more, qualitative properties such as bounds on various point or integral quantities are fundamental. These are often crucial for establishing the actual existence of solutions: it is the method of estimates a priori. In this method, we start by studying solutions and describe a certain number of properties, where some obtained results allow sometimes (often), through various techniques, to infer the existence.

Mathematicians are often interested in solving well-posed problems, when the solution depends continuously on the data given in the problem which means that our unique solution changes only a little when the conditions specifying the problem change a little, what plays an important role in physical applications. Nevertheless, for many problems, in the other hand , uniqueness is not to be expected, thus the primary mathematical tasks are to classify and characterize the solutions. For this purpose, the researcher must instead, while still hoping to achieve well posedness conditions for PDEs, investigate a wider class of candidats for solutions and consider as separate the existence and the smoothness(or regularity) problems. In fact, even for those PDEs which turn out to be classicaly solvable, it is often most expedient initially to search for some appropriate kind of weak solutions (see $2.2.1$). However, in order to find the weak solution the most physically acceptable, and to remedy the problems of uniqueness and sometimes of regularity, Bénilan-Vazquez [46] and Kruzhkov [118] introduced the concept of entropy solution (see $(3.3.1)$). Dall'Aglio [79] defined the solutions obtained as Limits of Approximation (SOLA), likewise, Lions-Perthame-Tadmor [123] and Murrat [131] investigated the notion of renormalized solutions that was introduced by Lions and Di Perna [88] for the study of Boltzman equations (see also, Lions [122] for applications to fluid mechanics models). This last solution was subsequently adapted to elliptic and parabolic versions (see for example [48, 50, 51, 139, 49].

In this thesis, we focus on certain types of nonlinear anisotropic problems with lower-order terms or terms with singularities (5 elliptic problems and one parabolic problem), by studying them in different appropriate functional frameworks, namely $W_0^{1,\vec{p}(\cdot)}(\Omega)$ and $W_0^{1,\vec{p}}(\Omega)$ which are anisotropic Sobolev spaces with variable exponents or constant exponents (see the next chapter), to deal respectively with problems $(0.0.10)$, $(0.0.13)$, $(0.0.23)$, $(0.0.39)$, and $(0.0.43)$, as well as the weighted Sobolev

space $W_0^{1,(p_i)}(\Omega, w)$ in which we can study the degenerate problem $(0.0.46)$. In addition, we introduced the space V which is a suitable setting for studying the strongly nonlinear anisotropic unilateral problem with variable exponents and an obstacle depending on time $(0.0.26)$. Through all these problems, we sometimes notice the presence of the anisotropic operator $\overrightarrow{p}(x)-$ laplacien

$$-\sum_{i=1}^{N} D^i\big(|D^i u|^{p_i(x)-2}D^i u\big),$$

which weighs partial derivatives with different powers $p_i(x) > 1$, and the $\overrightarrow{p}-$ laplacien when $p_i(x) = p_i$, $\forall\, i = 1, \ldots, N$, or more generally the divergence operator $Au = -\text{div}\,(a(x, u, \nabla u))$ which is a type of Leray -Lions. the involvement of variable exponents in some of these operators and in the classical Leray-Lions hypotheses of $a(.,.,.)$, and the anisotropic character common to all these nonlinear problems explains our main motivation for studying such problems that we will present in the following paragraph.

Motivation

One of our motivations for studying nonlinear anisotropic problems (with variables exponents) comes from applications in nonhomogeneous materials that behave differently on different spaces directions, we can refer here to the electrorheological and thermoelectric fluids [142] as an important class of non-Newtonian fluids that have multiple applications in robotic, space technology, brakes shock absorbers, image restoration [73] and the elasticity [158]. Moreover, such problems emerge, for instance, from the model describing the spread of an epidemic disease through a heterogeneous habitat, and they also arise as a mathematical description of the dynamic of fluids with different conductivities in different directions. In particular, in the field of image processing [81], there are a number of approaches combining PDEs and isotropic or anisotropic scattering ideas to restore noisy images. In the isotropic case, a classical approach has been develloped that uses a linear convolution (smoothing) operation to reduce the effect of noise, considered to be a high frequency signa

$$u(x, y, t) = \int_{\Omega} u(x - \delta, y - \zeta, t)u_0(\delta, \zeta)d\Omega, \qquad (x, y) \in \Omega \tag{0.0.1}$$

where we denote by $u_0(x, y) : \mathbb{R}^2 \to \mathbb{R}$, a function defined on the rectangle $\Omega = (0, a) \times (0, b)$, where the values of the function u_0 for example represent the gray level intensities of the original noisy image, $u(x, y, t)$ represents the restored image, and t a parameter that controls the importance of the smoothing of the operator $G(x, y, t)$. An example of an operator often used for his performances, its separability and

isotropy properties is the operator Gaussian G given by:

$$G(x,y,t) = \frac{1}{4\pi t}\exp\big(-\frac{(x^2+y^2)}{4t}\big). \tag{0.0.2}$$

In [114], Koenderink observed the first that an operation of convolution of an image by a Gaussian operator of a certain variance (here $\vartheta^2 = 2t$) can be rewritten as a process of diffusion of the image intensity around neighboring pixels during a time t in direct relation with the spatial variance ϑ^2. This diffusion equation, known as the heat equation, can be in the form of the following linear parabolic PDE:

$$\begin{cases} \dfrac{\partial u}{\partial t}(x,y,t) = \text{div}\,(\nabla(x,y,t))), \\ u(x,y,0) = u_0(x,y). \end{cases} \tag{0.0.3}$$

It can clearly be seen that the linear parabolic EDP above $(0.0.3)$ allows isotropic diffusion. Thus, this diffusion operates identically in all directions and has no privileged direction. For noisy image restoration tasks, this clearly has disadvantages. Indeed, in regions of uniform intensity, this process will effectively reduce the effect of noise but in regions with discontinuities in level of gray-level intensity, they will also be smoothed and the visual contrast of these parts will be significantly reduced, thus diminishing the qualitative and visual interest of such a process.

To deal with this problem, ideas for anisotropic diffusion have been proposed. Indeed, the first idea to solve problems resulting from an isotropic diffusion has been proposed by Perona and Malik in [135, 136]. It consists in carrying out a conditional diffusion, that is to say, strong diffusion in the zones with weak gradient and weak diffusion in the zones with strong gradient, where discontinuities potentially "interesting" can be. This method of diffusion has been formalized as follows:

$$\begin{cases} \dfrac{\partial u}{\partial t}(x,y,t) = \text{div}\,(h(|\nabla(x,y,t)|)\nabla(x,y,t)), \\ u(x,y,0) = u_0(x,y). \end{cases} \tag{0.0.4}$$

where div and ∇ indicate the divergence and gradient operators relative to the spatial variables respectively and where the function $h(.)$ is a decreasing function, for example:

$$h(|\nabla u|) = \exp\big(-\big(\frac{|\nabla u|}{k}\big)^2\big), \tag{0.0.5}$$

or

$$h(|\nabla u|) = \frac{1}{1+(\frac{|\nabla u|}{k})^2}, \tag{0.0.6}$$

which has the same first-order approximation as the previous. The problem $(0.0.4)$ can be rewritten in the following form:

$$\begin{cases} \dfrac{\partial u}{\partial t}(x,y,t) & = h\triangle u + \nabla u.\nabla h, \\ & = h\triangle u + \dfrac{h'(|\nabla u|)}{|\nabla u|}\nabla u^T \mathbf{H}\nabla u, \\ u(x,y,0) = u_0(x,y). \end{cases} \tag{0.0.7}$$

where H is the Hessian of $u(.,.)$. Furthermore, we can notice that if c (.) is a constant function, this equation then reduces to the isotropic diffusion process described in the previous paragraph.

However, this method has several drawbacks, the first of which is its inefficiency in areas where the noise has large discontinuities. The second disadvantage is of theoretical order for the previous functions $h(.)$ because one can show, in the absence of certain conditions, that there is no solution to the problem $(0.0.4)$. An improvement of this idea of anisotropic diffusion which takes these remarks was thus proposed by Alvarez, Lions and Morel in [14] who studied a class of nonlinear parabolic PDEs that generalizes the idea of anisotropic scattering initially proposed by Perona and Malik in [135, 136] and improved in [71].

As a complement in this class of diffusion schemes, we can also cite the schema to EDP proposed by Nitzberg and Shiota in [133], particularly well suited to the restoration of angles and other multiple junctions. The main idea is to use directional filters at each point of the image, guided by the local information of the orientation of the gradient. Very interesting experimental results illustrate the capacity of this approach.

With no hope to be complete, We will present very briefly some important researchs closely related to our work, leaving the details to the corresponding chapters. Let's consider the following anisotropic problem

$$\begin{cases} -\displaystyle\sum_{i=1}^{N} D^i\big(|D^i u|^{p_i-2}D^i u = f & \text{in} \quad \Omega, \\ u = 0 & \text{sur} \quad \partial\Omega, \end{cases} \tag{0.0.8}$$

where f is a given function belonging to $L^s(\Omega)$, $s \geq 1$. Let's mention that the problem $(0.0.8)$ have been treated by several authors. L.Boccardo et al. [61] have investigated the global $L^\infty-$ boundedness of solutions for some differential problems including $(0.0.8)$ with f in a divergence form, namely $f = \sum_{i=1}^{N} D^i f_i$, $f_i \in L^{s_i}(\Omega)$, $s_i \geq p_i'$, for $i = 1, \ldots, N$. These results have been enhanced in [60], and by Stroffolini in [147] for minimizers of some functionals. When datum f is a bounded Radon measure,

Boccardo[53] proved the existence of a solution of $(0.0.8)$ in the anisotropic Sobolev space $W_0^{1,\vec{m}}(\Omega)$ with m_i such that

$$1 \leq m_i \leq \frac{N(\overline{p}-1)}{\overline{p}(N-1)} p_i, \qquad \forall i = 1, \ldots, N,$$

where $\overline{p}$ is the harmonic mean of the p_i's and with the supplementary condition $2 - \frac{1}{N} < \overline{p} < N$. We emphasize that if $\overline{p}$ is close to 1, more precisely $1 < \overline{p} \leq 2 - \frac{1}{N}$, then the solution u do not belongs to $W_{loc}^{1,1}$. To overcome this difficulty, Bénilan et al. [46] (see also [58] inserted a new space $\mathcal{T}_{loc}^{1,1}(\Omega)$ which is an extension of $W_{loc}^{1,1}$ in which we can give meaning to the gradient of u, working on the truncation of its derivative $\nabla T_k(u)$(which locally integrable), and we can extend the new notion of entropy solution to anisotropic case. We also point out that A.Di Castro [82] established many of existence and regularity results for the problem $(0.0.8)$, where f is a function belonging either to $L^m(\Omega)$ or $M^m(\Omega)$ (Marcinkiewikz space, see $(1.1.2)$) or f in divergence form.

Concerning the nonlinear elliptic anisotropic problems as $(0.0.8)$ but with further lower terms, we can refer, among others, to the existence results of Di Castro [83], and the papers of Akdim and Salmani in [11] and [10]. If $p_i = 2$ or $p_i = p$, for any i, we report some works touching the investigation of these problems with natural growth terms and the references therein [59], [57], [52], [47], [54] and [80]. We also mention to an important class of nonlinear elliptic problems with the so called $p(x)-$growth involving the $p(x)-$laplacien or the more general $\overrightarrow{p}(x)-$laplacian, that continuously gained interest, and severals papers have been published in the last decades, see for examples [144, 128, 127, 104]. Regarding the generalized parabolic version of nonlinear problems with additional lower order terms, it was treated in the variational isotropic case by Lions [121], and for the variational elliptic-parabolic case ie $f \in L^1(Q)$ or f is a bounded measure data, we can consult [46, 54, 55]. On the other hand, Abdo et al. studied the strongly nonlinear anisotropic parabolic version when $f \in L^{\overrightarrow{p}}(0, T; W_0^{1,\overrightarrow{p}})$.

The reader's attention is drawn to another class of equations that will be more appropriately discussed in the second part of this thesis, namely problems introducing singularity terms, we will refer here to some striking works (see, for instance, Abdelaoui and Peral [1] who studied the following problem

$$-\triangle u - \lambda \frac{u}{|x|^2} = |\nabla u|^p + cf, \qquad u > 0 \qquad \text{in } \Omega, \tag{0.0.9}$$

where $\Omega \subset \mathbb{R}^N$ is a domain such that $0 \in \Omega$, $N \geq 3$, $c > 0$ and $\lambda > 0$, the authors looked for the precise threshold $p_+ = p_+(\lambda)$ for which there is no very weak supersolution if $p \geq p_+(\lambda)$. we can also refer for example to the works of L.Boccardo et al. [65], H.Brezis et al. [68] and the paper [155] in which A.youssfi et al. investigated the existence and regularity of entropy solution of strongly nonlinear elliptic

problems involving a Hardy potential and a nonlinearity $g(x, u, \nabla u)$ that satisfies a growth condition with respect to $|\nabla u|$.

In this thesis, we organized our research work in two parts, preceded by a preliminary chapter containing all the necessary tools that we will use in the following (definitions, theorems, lemmas, etc ...)

Part I: Nonlinear anisotropic elliptic and parabolic PDEs with lower order terms and variable exponents

This part consists of three chapters. We have studied two nonlinear anisotropic elliptic problems in the functional framework of anisotropic Sobolev spaces with variable exponents, and another nonlinear anisotropic nonlinear problem with variable exponent with an obstacle depending on time in the anisotropic parabolic space V which we will present later.

THE CHAPTER 2 is the subject of the paper [23], in which we are concerned with the problem of existence of weak solutions for a class of nonlinear elliptic equations with additional lower order terms, that play the role of perturbations terms. Firstly, attention shall be focused to such problems with lower order terms without sign assumptions

$$\begin{cases} -\sum_{i=1}^{N} D^i[|D^i u|^{p_i(x)-2} D^i u] + \sigma_0 u = \sum_{i=1}^{N} h_i(x, u, \nabla u) + f & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{0.0.10}$$

where $\sigma_0 > 0,\ f \in L^{m(x)}(\Omega),\ \ m(x) > \frac{p^*_-}{p^*_- - p_i^+}$ and $m(x) > p'_\infty$ for a.e. $x \in \Omega$ with $p_\infty = max\{p^*_-, p_i^+\}$ which will be defined later in the next chapter. It's supposed that $p^*_- > p_i^+$ for all $i = 1, \ldots, N,$ ie $p_\infty = p^*_-$ which means that the anisotropy is concentrated (the p_i's are not "too far apart"). It should be emphasized, that we have not treated the opposite case, namely $p_\infty = p_i^+$, which signifies that the anisotropy is not concentrated (the p_i's are "spread out"). We assume also that

$$h_i(x, s, \xi) : \Omega \times I\!R \times I\!R^N \longmapsto I\!R \ \text{ is Carathéodory function} \qquad \text{for all } \ i = 1, \ldots, N \tag{0.0.11}$$

(measurable with respect to x in Ω for every (s, ξ) in $I\!R \times \mathbf{R}^N$ and continuous with respect to (s, ξ) in $I\!R \times I\!R^N$ for almost every x in Ω) and does not satisfy the sign condition. Furthermore, there exists $\beta > 0$ such that the following inequality is true for all $(s, \xi) \in I\!R \times I\!R^N$ and almost every $x \in \Omega$

$$\sum_{i=1}^{N} |h_i(x, s, \xi)| \leq \beta \sum_{i=1}^{N} |\xi_i|^{p_i(x)}, \qquad \forall\, i = 1, \ldots, N. \tag{0.0.12}$$

The model example is the following Dirichlet problem

$$\begin{cases} -\sum_{i=1}^{N} D^i[|D^i u|^{p_i(x)-2}D^i u] + \sigma_0 u = \beta \sum_{i=1}^{N} |D^i u|^{p_i(x)} + f & \text{in} \quad \Omega, \\ u = 0 & \text{on} \quad \partial\Omega. \end{cases}$$

Later, we are interested in this paper to deal with the proof of the existence of a solution for the following problem

$$\begin{cases} -\text{div}\, a(x,u,\nabla u) + \sum_{i=1}^{N} h_i(x,u,\nabla u) = f & \text{in} \quad \Omega, \\ u = 0 & \text{on} \quad \partial\Omega, \end{cases} \tag{0.0.13}$$

with $u \in W_0^{1,\overrightarrow{p}(\cdot)}(\Omega)$, $h_i(x,u,\nabla u) \in L^1(\Omega)$ where $h_i(x,u,\nabla u)$ are nonlinearities with natural growth respect to the gradient of u, for any $i = 1,\dots,N$, which satisfy the sign condition $h_i(x,s,\xi)s \geq 0$. We assume that for all $i = 1,\dots,N$, $a_i : \Omega \times I\!R \times I\!R^N \longmapsto I\!R$ are Carathéodory functions satisfying for a.e. $x \in \Omega$ and all $(s,\xi) \in \mathbf{R} \times \mathbf{R}^N$

$$|a_i(x,s,\xi)| \leq \beta(C_i(x) + |s|^{p_i(x)-1} + \sum_{i=1}^{N} |\xi_i|^{p_i(x)-1}), \tag{0.0.14}$$

$$\sum_{i=1}^{N} a_i(x,s,\xi)\xi_i \geq \alpha \sum_{i=1}^{N} |\xi_i|^{p_i(x)}, \tag{0.0.15}$$

$$\big(a_i(x,s,\xi) - a_i(x,s,\xi')\big)(\xi_i - \xi_i') > 0 \quad \text{for} \quad \xi_i \neq \xi_i', \tag{0.0.16}$$

for any $\xi = (\xi_1,\dots,\xi_N)$ and $\xi' = (\xi_1',\dots,\xi_N')$ in $I\!R^N$, where $C_i(\cdot) \in L^{p_i'(\cdot)}(\Omega)$, with $\frac{1}{p_i(x)} + \frac{1}{p_i'(x)} = 1$ and α, β are two positive real numbers.

Let $h_i : \Omega \times I\!R \times I\!R^N \longmapsto I\!R$ a Carathéodory function such that, for almost every $x \in \Omega$ and for all $s \in I\!R$ and $\xi \in I\!R^N$,

$$h_i(x,s,\xi)s \geq 0, \qquad \forall\ i = 1,...,N, \tag{0.0.17}$$

$$\sum_{i=1}^{N} |h_i(x,s,\xi)| \leq b(|s|)(j(x) + \sum_{i=1}^{N} |\xi_i|^{p_i(x)}), \tag{0.0.18}$$

where $b(.) : I\!R^+ \to I\!R^+$ is a continuous non-decreasing function, and $j(.) : \Omega \to I\!R^+$ with $j(.) \in L^1(\Omega)$. Finally we assume one of the following two assumptions:

$$\begin{cases} f \in L^1(\Omega), \\ \text{and there exist } \zeta > 0 \text{ and } \gamma > 0 \text{ such that} \\ |h_i(x,s,\xi)| \geq \gamma\Big(|s|^{p_0(x)} + \sum_{i=1}^{N} |\xi|^{p_i(x)}\Big) \text{ when } |s| > \zeta, \quad \forall\ i = 1,...,N. \end{cases} \tag{0.0.19}$$

or

$$f \in [W_0^{1,\overrightarrow{p}(\cdot)}(\Omega)]^* (\text{or to } \ L^{(p_-^*)'}) \tag{0.0.20}$$

with $*$ we denote the dual space of $W_0^{1,\overrightarrow{p}(\cdot)}(\Omega)$. Under the assumption $(0.0.19)$, we showed the existence of a weak solution to the problem $(0.0.13)$, in the sense that

$$\begin{cases} \sum\limits_{i=1}^{N} \int_\Omega a_i(x,u,\nabla u) D^i \upsilon \, dx + \sum\limits_{i=1}^{N} \int_\Omega h_i(x,u,\nabla u)\upsilon \, dx = \int_\Omega f\upsilon \, dx, \\ u \in W_0^{1,\overrightarrow{p}(\cdot)}(\Omega), \qquad h_i(x,u,\nabla u) \in L^1(\Omega) \qquad \forall i = 1,\dots,N, \end{cases} \tag{0.0.21}$$

for all $\upsilon \in W_0^{1,\overrightarrow{p}(\cdot)}(\Omega) \cap L^\infty(\Omega)$.

In the case of the second assumption $(0.0.20)$, we also proved the existence of a weak solution to the problem $(0.0.13)$, that is

$$\begin{cases} \sum\limits_{i=1}^{N} \int_\Omega a_i(x,u,\nabla u) D^i \upsilon \, dx + \sum\limits_{i=1}^{N} \int_\Omega h_i(x,u,\nabla u)\upsilon \, dx = \int_\Omega f\upsilon \, dx, \\ u \in W_0^{1,\overrightarrow{p}(\cdot)}(\Omega), \qquad h_i(x,u,\nabla u) \in L^1(\Omega), \qquad uh_i(x,u,\nabla u) \in L^1(\Omega) \qquad \forall i = 1,\dots,N, \end{cases} \tag{0.0.22}$$

for all $\upsilon \in W_0^{1,\overrightarrow{p}(\cdot)}(\Omega) \cap L^\infty(\Omega)$. A prototype example of $(0.0.13)$ that is covered by the assumptions above is the following

$$\begin{cases} -\sum\limits_{i=1}^{N} D^i[|D^i u|^{p_i(x)-2} D^i u] + u \sum\limits_{i=1}^{N} |D^i u|^{p_i(x)} = f \qquad \text{in} \quad \Omega, \\ u = 0 \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \text{on} \quad \partial\Omega. \end{cases}$$

THE THIRD CHAPTER treated in the paper [26], is devoted to study the following a class of obstacle problems with non-standard structural conditions:

$$\begin{cases} -\sum\limits_{i=1}^{N} D^i a_i(x,u,\nabla u) = f - \text{div } F \qquad \text{in } \ \Omega, \\ u = 0 \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \text{on } \ \partial\Omega, \end{cases} \tag{0.0.23}$$

where $f \in L^1(\Omega)$ and $F \in \prod_{i=1}^N L^{p_i'(\cdot)}(\Omega)$. By using some penalization techniques, and ensuring the condition of growth, the strict monotony and the coercivity of $a_i(.,.,.)$, a solution to the equation $(0.0.23)$ should be achieved in a subset of anisotropic variable exponent Sobolev spaces

$$K_\psi := \big\{ u \in W_0^{1,\vec{p}(x)}(\Omega), \ u \geq \psi \text{ a.e. in } \Omega \big\},$$

where $\psi(\cdot)$ is a fixed obstacle function, such that

$$\psi^+ \in W_0^{1,\vec{p}(x)}(\Omega) \cap L^\infty(\Omega). \tag{0.0.24}$$

We propose to give a result of existence of entropic solutions of $(0.0.23)$, that is,

$$\begin{cases} u \in K_\psi \\ \displaystyle\sum_{i=1}^{N} \int_\Omega a_i(x, T_k(u), \nabla u) D^i T_k(u-v)\, dx \le \int_\Omega f T_k(u-v)\, dx \\ \qquad\qquad + \displaystyle\sum_{i=1}^{N} \int_\Omega F_i D^i T_k(u-v)\, dx \qquad \forall v \in K_\psi, \end{cases} \tag{0.0.25}$$

In CHAPTER 4, investigated in the paper [24] we will take an interest to the following strongly nonlinear anisotropic unilateral problem with respect to the perturbations in the growth exponent $\overrightarrow{p}(.)$ and an obstacle depending on time

$$\begin{cases} \dfrac{\partial u}{\partial t} - \text{div}\, a(x,t,u,\nabla u) + \displaystyle\sum_{i=1}^{N} h_i(x,t,u,\nabla u) + \Theta(x,t)|u|^{p_i(x)-2}u = \mu + f & \text{in } Q_T, \\ u \ge 0, \quad \mu(u-\psi) = 0, \quad u \ge \psi & \text{in } Q_T, \\ u(x,t) = 0 & \text{on } \partial Q_T, \\ u(x,0) = u_0(x) & \text{in } \Omega. \end{cases} \tag{0.0.26}$$

Where $Q_T = \Omega\times]0,T[$, $\partial Q_T = \partial\Omega\times]0,T[$,

$$f \in L^{p_i'(.)}(Q_T), \tag{0.0.27}$$

$$u_0 \in L^2(\Omega), \tag{0.0.28}$$

and $\Theta(x,t) \in L^\infty(Q_T)$, and there exists a constant $\Theta_0 > 0$ such that $\Theta(x,t) \ge \Theta_0$ almost everywhere in Q_T. We denote $Au = -\sum_{i=1}^{N} D^i a_i(x,t,u,\nabla u)$ the Leray-Lions operator acting from the appropriate anisotropic functional space V (see the next section) into its dual V^* where $a_i : \Omega\times]0,T[\times I\!R \times I\!R^N \longmapsto I\!R^N$ are Carathéodory functions, for $i = 1,\dots,N$, (measurable with respect to (x,t) in $\Omega\times]0,T[$ for every (s,ξ) in $I\!R \times I\!R^N$ and continuous with respect to (s,ξ) in $I\!R \times I\!R^N$ for almost every (x,t) in $\Omega\times]0,T[$), which satisfy for a.e. $(x,t) \in Q_T$ and all $(s,\xi) \in I\!R \times I\!R^N$ the following conditions :

$$|a_i(x,t,s,\xi)| \le \gamma(C_i(x,t) + |s|^{p_i(x)-1} + \sum_{i=1}^{N} |\xi_i|^{p_i(x)-1}), \tag{0.0.29}$$

$$\sum_{i=1}^{N} a_i(x,t,s,\xi)\xi_i \ge \rho \sum_{i=1}^{N} |\xi_i|^{p_i(x)}, \tag{0.0.30}$$

$$\big(a_i(x,t,s,\xi) - a_i(x,t,s,\xi')\big)(\xi_i - \xi_i') > 0 \quad \text{for} \quad \xi_i \ne \xi_i', \tag{0.0.31}$$

for any $\xi = (\xi_1,\dots,\xi_N)$ and $\xi' = (\xi_1',\dots,\xi_N')$ in $I\!R^N$, where $C_i(x,t))$ is a positive function lying in $L^{p_i'(\cdot)}(Q_T)$, with $\frac{1}{p_i(x)} + \frac{1}{p_i'(x)} = 1$ and γ, ρ are two positive real numbers.

The obstacle ψ is assumed to check for $i = 1, \ldots, N$, the undermentioned conditions

$$\frac{\partial \psi}{\partial t} \in L^{p_i'(.)}(\Omega \times]0, T[), \tag{0.0.32}$$

$$\psi \in V, \tag{0.0.33}$$

$$\psi^+ \in L^{\infty}(Q_T), \tag{0.0.34}$$

$$\psi \leq 0 \quad \text{on } \partial Q_T \quad \text{and} \quad \psi(0) \leq u_0 \quad \text{a.e. in } \Omega. \tag{0.0.35}$$

The nonlinear lower order term $h_i(x, t, s, \xi)$ is a Carathéodory function that satisfies for almost every $(x, t) \in Q_T$ and for all $(s, \xi) \in I\!R \times I\!R^N$,

$$h_i(x, t, s, \xi)s \geq 0, \qquad \forall\ i = 1, ..., N, \tag{0.0.36}$$

$$\sum_{i=1}^{N} |h_i(x, t, s, \xi)| \leq j(|s|)(\sum_{i=1}^{N} |\xi_i|^{r_i(x)} + l(x, t)), \qquad \text{where } 1 < r_i(x) < p_i(x) - 1 \quad \text{a.e. in } \Omega \tag{0.0.37}$$

$l(x, t) \in L^{\infty}(Q_T)$, and $j(.) : I\!R^+ \to I\!R^+$ is a continuous, nonnegative and non-decreasing function, having growth of order $q_0(.)$, with respect to $|s|$:

$$j(|s|) \leq |s|^{q_0(x)} + \lambda, \quad \lambda > 0, \quad 1 < q_0(x) < p_i(x) - r_i(x) \text{ a.e. in } \Omega, \tag{0.0.38}$$

Note that we proceed by penalizing sequences on approximation equations, with an additional complementary condition to ensure the desired outcome of existence.

Part II: Nonlinear anisotropic elliptic PDEs involving singularities on its right -hand side within the anisotropic Sobolev spaces with either variable exponents or constant exponents or weight

The second part of this thesis consists of three chapters. LE CHAPITRE 5, topic of article [28], is devoted to the study of the existence of entropy solutions for the following anisotropic quasilinear elliptic Dirichlet problem with Hardy potential

$$\begin{cases} Au + |u|^{s(x)-1}u = f + \lambda \dfrac{|u|^{p_0(x)-2}u}{|x|^{p_0(x)}} & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{0.0.39}$$

with $\lambda \geq 0$, $f \in L^1(\Omega)$,

$$p_0(x) = \max\{p_i(x), \quad i = 1, 2, \ldots, N\} \quad \text{a.e. in } \Omega, \tag{0.0.40}$$

and

$$s(x) > \max\Big(\frac{N(p_0(x)-1)}{N-p_0(x)}, \frac{1}{p_0(x)-1}\Big) \quad \text{a.e. in} \quad \Omega. \tag{0.0.41}$$

The Leray-Lions operator A acting from $W_0^{1,\vec{p}(\cdot)}(\Omega)$ into its dual, is defined by the formula

$$Au = -\sum_{i=1}^{N} D^i a_i(x,u,\nabla u) \tag{0.0.42}$$

where $a_i : \Omega \times \mathbf{R} \times \mathbf{R}^N \longmapsto \mathbf{R}^N$ are Carathéodory functions, for $i = 1, \ldots, N$, which satisfy the counterpart anisotropic conditions of growth, the strict monotony and the coercivity of $a_i(.,.,.)$.

Note that in this chapter, we have assumed that the data f belong to $L^1(\Omega)$, then the elliptic equation $(5.1.9)$ is not in the dual space $W^{-1,\vec{p}'(\cdot)}(\Omega)$, thus the existence of weak solution haven't a sense. To overcome this difficulty, some mathematicians have stated the notions of entropy [46] and renormalized solutions [88, 89], which are more adapted for this category of problems. The difficulties in proving the existence of entropy solutions stems from the fact that : Since the exponents $p_i(\cdot)$ are assumed to be measurable functions, then the Poincaré and Sobolev inequalities are not verified, therefore, the operator Au is not coercive in the anisotropic variable exponent Sobolev space $W_0^{1,\vec{p}(\cdot)}(\Omega)$, defined below. To overcome this difficulty, we use the penalization term $\frac{1}{n}|u|^{p_0(x)-2}u$ in approximate problems $(7.4.7)$. Moreover, the singular term $\frac{|u|^{p_0(x)-2}u}{|x|^{p_0(x)}}$ creates, in general, an hindrance to the existence of a solution. We overpass this difficulty by using the regularizing effect of the term $|u|^{s(x)-1}u$ to remove the non-existence effect produced by the Hardy potential.

In the same line of research, another strongly nonlinear problem is examined in THE CHAPTER 6, is the case where one adds a non-linear term $h(x,u,\nabla u)$ which depends on u and ∇u, a term of lower order and another divergent term in the right hand-side of the studied equation

$$\begin{cases} -\sum_{i=1}^{N} D^i(|D^i u|^{p_i-2}D^i u) + h(x,u,\nabla u) + |u|^{p_0-2}u + |u|^{s-1}u = f + \mu\dfrac{|u|^{p_0-2}u}{|x|^{p_0}} - \text{div}\ \phi(u) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{0.0.43}$$

Our purpose is to prove the existence of entropy solutions of $(6.1.1)$, assuming that $\mu \geq 0$, $f \in L^1(\Omega)$, $\phi \in \mathcal{C}^0(I\!R, I\!R^N)$ and

$$s > \max\Big(\frac{N(p_0-1)}{N-p_0}, \frac{1}{p_0-1}\Big). \tag{0.0.44}$$

The nonlinear lower order term $h(x,s,\xi) : \Omega \times I\!R \times I\!R^N \longmapsto I\!R^N$ is Carathéodory functions, which satisfies the sign condition, and has natural growth with respect to $|\nabla u|$, that is

$$|h(x,s,\xi)| \leq l(x) + j(|s|)\sum_{i=1}^{N} |\xi_i|^{p_i}, \tag{0.0.45}$$

where $j : I\!R^+ \to I\!R^+$ is a continuous nondecreasing function and $l \in L^1(\Omega)$. Motived by the papers [138], [155] and extending the results of the problem studied in [28], but in the case of constant exponents, we try to deal with $(6.1.1)$ by using the Galerkin method, and to remove the non-existence effect produced by the singular term $\frac{|u|^{p_0-2}u}{|x|^{p_0}}$ by by exploiting the regularizing effect of the term $|u|^{s-1}u$. On the other hand the function $\phi \in \mathcal{C}^0(I\!R, I\!R^N)$ then ϕ does not belongs to $(L^1_{loc}(\Omega))^N$, so that proving existence of a weak solution seems to be an arduous task. to overpass this difficulty we will use some techniques in the framework of entropy solutions. The chapter content is the subject of the published paper [27].

The last problem, presented in CHAPTER 7 and covered by the paper [25] concerns the study of the existence of entropy solutions for the following anisotropic quasilinear degenerated elliptic problem

$$\begin{cases} Au + |u|^{s-1}u = f + \rho\dfrac{|u|^{p_0-2}u}{|x|^{p_0}} & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{0.0.46}$$

with $\rho \geq 0$, $f \in L^1(\Omega)$ and

$$s > \max\Big(\frac{N(p_0-1)}{N-p_0}, \frac{1}{p_0-1}\Big). \tag{0.0.47}$$

The Leray-Lions operator A acted from $W_0^{1,(p_i)}(\Omega,\nu)$ into its dual with ν as a special case of w, is defined by the formula

$$Au = -\sum_{i=1}^{N} D^i a_i(x, u, \nabla u) \tag{0.0.48}$$

where $a_i : \Omega \times \mathbf{R} \times \mathbf{R}^N \longmapsto \mathbf{R}^N$ are Carathéodory functions satisfying

$$|a_i(x,s,\xi)| \leq \beta\nu^{\frac{1}{p_i}}(R_i(x) + |s|^{\frac{p_i}{p_i}} + \sum_{i=1}^{N} \nu^{\frac{1}{p_i'}}|\xi_i|^{p_i-1}), \tag{0.0.49}$$

$$\sum_{i=1}^{N} a_i(x,s,\xi)\cdot\xi_i \geq \alpha\sum_{i=1}^{N}\nu|\xi_i|^{p_i}, \tag{0.0.50}$$

for any $\xi = (\xi_1, \ldots, \xi_N)$ and $\xi' = (\xi_1', \ldots, \xi_N')$ in $I\!R^N$, we have

$$\big(a_i(x,s,\xi) - a_i(x,s,\xi')\big)\cdot(\xi_i - \xi_i') > 0 \quad \text{for} \quad \xi_i \neq \xi_i', \tag{0.0.51}$$

for a.e. $x \in \Omega$ and all $(s,\xi) \in \mathbf{R}\times\mathbf{R}^N$, where $R_i(x) \in L^{p_i'}(\Omega)$, and α, β are two positive real numbers. Let's point out that we have to use a extra condition on the weight ν, to overcome the difficulties that arise at the level of a priori estimates, and also to exploit a penalization term $\frac{1}{n}|u|^{p_0-2}u$ in the approximates problems of $(0.0.46)$ to overpass the non-coercivity of the operator Au in the anisotropic weighted Sobolev space $W_0^{1,(p_i)}(\Omega,\nu)$.

In order to recapitulate the results of this work, we introduce a general conclusion, bringing some perspectives that may appear throughout the problems studied in this thesis.

Lastly, we note that the content of this work is the subject of some published or submitted works.

Chapter 1

Preliminaries

This chapter is intended to expose some of the functional spaces that are most appropriate for conducting the study of the elliptic and parabolic nonlinear anisotropic problems introduced in this thesis, and to recall some of the classical definitions and results that are essential for the rest of our work. .

1.1 Functional spaces

1.1.1 Variable Exponent Lebesgue and Sobolev Spaces

In this section, we recall some definitions and basic properties of variable Exponent Lebesgue and Sobolev Spaces used throughout this work that follow [85, 113]. We refer to Harjulehto and Hästö [111], and [143] for further properties of variable exponent Lebesgue-Sobolev spaces.
Let Ω be a bounded open subset of $\mathbb{R}^N$ $(N \geq 2)$, and $\partial\Omega$ be Lipschitz-continuous, we denote

$$\mathcal{C}_+(\overline{\Omega}) = \{\text{continuous function} \quad p(\cdot) : \overline{\Omega} \longmapsto I\!R \quad \text{such that} \quad 1 < p^- \leq p^+ < N\},$$

where

$$p^- = \min\{p(x) \,/\, x \in \Omega\} \qquad \text{and} \qquad p^+ = \max\{p(x) \,/\, x \in \Omega\}.$$

We define the Lebesgue space with variable exponent $L^{p(\cdot)}(\Omega)$ as the set of all measurable functions $u : \Omega \mapsto I\!R$ for which the convex modular

$$\rho_{p(\cdot)}(u) := \int_\Omega |u|^{p(x)} dx$$

is finite. If the exponent is bounded, i.e. if $p^+ < +\infty$, then the expression

$$\|u\|_{p(\cdot)} = \inf\{\lambda > 0 : \rho_{p(\cdot)}(u/\lambda) \leq 1\}$$

defines a norm in $L^{p(\cdot)}(\Omega)$, called the Luxemburg norm. The space $(L^{p(\cdot)}(\Omega), \|.\|_{p(\cdot)})$ is a separable Banach space ([117]; theorem 2.5). Moreover, if $1 < p^- \leq p^+ < +\infty$, then $L^{p(\cdot)}(\Omega)$ is uniformly convex, hence reflexive ([117]; corollary 2.7), and its dual space is isomorphic to $L^{p'(\cdot)}(\Omega)$, where $\frac{1}{p(x)} + \frac{1}{p'(x)} = 1$. Finally, we have the Hölder type inequality:

$$\left|\int_\Omega uv\, dx\right| \leq (\frac{1}{p^-} + \frac{1}{(p')^-})\|u\|_{p(\cdot)}\|v\|_{p'(\cdot)} \tag{1.1.1}$$

for any $u \in L^{p(\cdot)}(\Omega)$ and $v \in L^{p'(\cdot)}(\Omega)$.

An important role in manipulating the generalized Lebesgue spaces is played by the modular $\rho_{p(\cdot)}$ of the space $L^{p(\cdot)}(\Omega)$. We have the following result :

Proposition 1.1.1. ***(see [100], [157])***

If $u_n, u \in L^{p(\cdot)}(\Omega)$, then the following properties hold true:

(i) $\|u\|_{p(\cdot)} < 1 \quad (resp, = 1, > 1) \quad \Longleftrightarrow \quad \rho(u) < 1 \quad (resp, = 1, > 1),$

(ii) $\|u\|_{p(\cdot)} > 1 \implies \|u\|^{p^-}_{p(\cdot)} \leq \rho(u) \leq \|u\|^{p^+}_{p(\cdot)}$ *and* $\|u\|_{p(\cdot)} < 1 \implies \|u\|^{p^+}_{p(\cdot)} \leq \rho(u) \leq \|u\|^{p^-}_{p(\cdot)},$

(iii) $\|u_n\|_{p(\cdot)} \to 0 \iff \rho(u_n) \to 0,$ *and* $\|u_n\|_{p(\cdot)} \to \infty \iff \rho(u_n) \to \infty,$

(iv) $\rho_{p(\cdot)}(\frac{u}{\|u_n\|_{p(\cdot)}}) = 1.$

Which implies that the norm convergence and the modular convergence are equivalent.

Now, we define the variable exponent Sobolev space by

$$W^{1,p(\cdot)}(\Omega) = \{\, u \in L^{p(\cdot)}(\Omega) \text{ and } |\nabla u| \in L^{p(\cdot)}(\Omega)\},$$

which is a Banach space equipped with the following norm

$$\|u\|_{1,p(\cdot)} = \|u\|_{p(\cdot)} + \|\nabla u\|_{p(\cdot)} \qquad \forall u \in W^{1,p(\cdot)}(\Omega).$$

We denote by $W^{1,p(\cdot)}_0(\Omega)$ the closure of $C^\infty_0(\Omega)$ in $W^{-1,p'(\cdot)}(\Omega)$, its dual space is denoted by $W^{1,p(\cdot)}(\Omega)$ and we define the Sobolev exponent by

$$p^*(x) = \begin{cases} \dfrac{Np(x)}{N - p(x)} & \text{for } p(x) < N, \\ \infty & \text{for } p(x) \geq N. \end{cases}$$

Proposition 1.1.2. *(see [101])*

(i) *Assuming* $1 < p^- \leq p^+ < \infty$, *the spaces* $W^{1,p(\cdot)}(\Omega)$ *and* $W_0^{1,p(\cdot)}(\Omega)$ *are separable and reflexive Banach spaces.*

(ii) *If* $q(\cdot) \in \mathcal{C}_+(\overline{\Omega})$ *and* $q(x) < p^*(x)$ *a.e.* $x \in \overline{\Omega}$, *then the embedding* $W_0^{1,p(\cdot)}(\Omega) \hookrightarrow\hookrightarrow L^{q(\cdot)}(\Omega)$ *is continuous and compact.*

(iii) *Poincaré inequality : there exists a constant* $C > 0$, *such that*

$$\|u\|_{p(\cdot)} \leq C\|\nabla u\|_{p(\cdot)} \qquad \forall u \in W_0^{1,p(\cdot)}(\Omega).$$

(iv) *Sobolev inequality : there exists an other constant* $C > 0$, *such that*

$$\|u\|_{p^*(\cdot)} \leq C\|\nabla u\|_{p(\cdot)} \qquad \forall u \in W_0^{1,p(\cdot)}(\Omega).$$

Remark 1.1.1. *By (iii) of the Proposition 1.1.2, we deduce that* $\|\nabla u\|_{p(\cdot)}$ *and* $\|u\|_{1,p(\cdot)}$ *are equivalent norms in* $W_0^{1,p(x)}(\Omega)$.

Theorem 1.1.1. *(see [85]) We denote the dual of the Sobolev space* $W_0^{1,p(x)}(\Omega)$ *by* $W^{-1,p'(x)}(\Omega)$, *and for each* $F \in W^{-1,p'(x)}(\Omega)$ *there exists* $f_0, f_1, \ldots, f_N \in L^{p'(x)}(\Omega)$ *such that* $F = f_0 + \sum_{i=1}^{N} \frac{\partial f_i}{\partial x_i}$. *Moreover, for all* $u \in W_0^{1,p(x)}(\Omega)$ *we have*

$$\langle F, u\rangle = \int_\Omega f_0 u \, dx - \sum_{i=1}^{N} \int_\Omega f_i \frac{\partial u}{\partial x_i} \, dx,$$

we define a norm on the dual space by

$$\|F\|_{-1,p'(x)} = \sum_{i=0}^{N} \|f_i\|_{p'(x)}.$$

Definition 1.1.1. *Let* Ω *be a bounded open subset of* $\mathbb{R}^N$ $(N \geq 2)$, *we say that a real-valued continuous function* $p(.)$ *is log-Hölder continuous in* Ω *if*

$$|p(x) - p(y)| \leq \frac{C}{|log|x-y||} \qquad \forall x, y \in \overline{\Omega} \quad \textit{such that} \quad |x-y| < \frac{1}{2},$$

where C *is a constant.*

Remarks 1.1.1.

(i) *Recall that the definition of these spaces requires only the measurability of the exponent $p(\cdot)$. Note that the sharp Sobolev inequality is proved for $p(\cdot)$ log-Hölder continuous, while the Poincaré inequality requires only the continuity of $p(\cdot)$, (see [85, 111]).*

(ii) *Although this condition of regularity (Höderian continuity) is not necessary to define Lebesgue and Sobolev spaces with variable exponent, it turns out to be very useful for these spaces to introduce some properties, such as $W_0^{1,p(\cdot)}(\Omega) = W^{1,p(\cdot)}(\Omega) \cap W_0^{1,1}(\Omega)$ and $\overline{\mathcal{C}_\infty(\overline{\Omega})} = W^{1,p(\cdot)}(\Omega)$. Moreover if $1 < p^- \leq p^+ < N$, then the injection of Sobolev remains true for $q(\cdot) = p^*(\cdot)$ (for more details see [84]).*

1.1.2 Marcinkiewicz spaces with variable exponents

Now, let's present Marcinkiewicz spaces with variables exponent introduced for the first time by Sanchon and Urbano [144], and explore their relation with Lebesgue spaces.

Definition 1.1.2. *Let $m(.)$ be a mesurable function such that $m^- > 0$. We say that a mesurable function u belongs to the Marcinkiewicz space $M^{m(.)}(\Omega)$ if there exists a positive constant k such that*

$$\int_{\{|u|>t\}} t^{m(x)}\, dx \leq k, \quad \textit{for all} \quad t > 0$$

Furthermore, it is obvious that

$$L^{m(.)}(\Omega) \subset M^{m(.)}(\Omega), \quad \text{for all} \quad m(.) \geq 1. \tag{1.1.2}$$

For the inverse inclusion, we have the following result:

Proposition 1.1.3. ***(see [144])*** *Let $m(.)$ and $q(.)$ be bounded functions such that $0 \ll q(.) \ll m(.)$ $(i.e\ (m-q)^- > 0)$ and let $\varepsilon := (m-q)^-$. If $u \in M^{m(.)}(\Omega)$, then*

$$\int_\Omega |u|^{q(x)}\, dx \leq 2|\Omega| + (m^+ - \varepsilon)\frac{k}{\varepsilon},$$

where k is the constant appearing in the definition of $M^{m(.)}(\Omega)$. In particular, $M^{m(.)}(\Omega) \subset L^{q(.)}(\Omega)$, for all $1 \leq q(.) \leq m(.)$.

1.1.3 Anisotropic Sobolev Spaces with variable exponents

Now, we present the anisotropic variable exponent Sobolev spaces used in the study of our quasilinear anisotropic elliptic problem.

Let $p_0(\cdot), p_1(\cdot), \ldots, p_N(\cdot)$ be $N+1$ variable exponents in $\mathcal{C}_+(\Omega)$. We denote

$$\vec{p}(\cdot) = (p_0(\cdot), \ldots, p_N(\cdot)), \quad D^0 u = u \quad \text{and} \quad D^i u = \frac{\partial u}{\partial x_i} \qquad \text{for } i = 1, \ldots, N,$$

and we define

$$\frac{1}{\overline{p}} = \frac{1}{N} \sum_{i=1}^{N} \frac{1}{p_i^-}, \qquad \underline{p} = \min\{p_0^-, p_1^-, \ldots, p_N^-\} \qquad \text{then} \qquad \underline{p} > 1. \tag{1.1.3}$$

$$p_\infty = max\{p_-^*, p_i^+\}, \qquad p_-^* = \frac{N}{\sum_{i=1}^{N} \frac{1}{p_i^-} - 1} \quad \text{with} \quad \sum_{i=1}^{N} \frac{1}{p_i^-} > 1 \qquad \text{for} \qquad i = 1, \ldots, N. \tag{1.1.4}$$

The anisotropic variable exponent Sobolev space $W^{1,\vec{p}(\cdot)}(\Omega)$ is defined as follows

$$W^{1,\vec{p}(\cdot)}(\Omega) = \{u \in L^{p_0(\cdot)}(\Omega) \quad \text{and} \quad D^i u \in L^{p_i(\cdot)}(\Omega) \quad \text{for} \quad i = 1, 2, \ldots, N\},$$

endowed with the norm

$$\|u\|_{1,\vec{p}(\cdot)} = \sum_{i=0}^{N} \|D^i u\|_{p_i(\cdot)}. \tag{1.1.5}$$

We define also $W_0^{1,\vec{p}(\cdot)}(\Omega)$ as the closure of $\mathcal{C}_0^\infty(\Omega)$ in $W^{1,\vec{p}(\cdot)}(\Omega)$ with respect to the norm $(6.2.2)$. The space $\left(W_0^{1,\vec{p}(\cdot)}(\Omega), \|u\|_{1,\vec{p}(\cdot)}\right)$ is a reflexive Banach space (cf. [127]).

Lemma 1.1.1. *We have the following continuous and compact embedding*

- *if* $\underline{p} < N$ *then* $W_0^{1,\vec{p}(\cdot)}(\Omega) \hookrightarrow\hookrightarrow L^q(\Omega) \quad$ *for* $\quad q \in [\underline{p}, \underline{p}^*[, \quad$ *where* $\underline{p}^* = \dfrac{N\underline{p}}{N - \underline{p}}$,

- *if* $\underline{p} = N$ *then* $W_0^{1,\vec{p}(\cdot)}(\Omega) \hookrightarrow\hookrightarrow L^q(\Omega) \qquad \forall q \in [\underline{p}, +\infty[$,

- *if* $\underline{p} > N$ *then* $W_0^{1,\vec{p}(\cdot)}(\Omega) \hookrightarrow\hookrightarrow L^\infty(\Omega) \cap \mathcal{C}^0(\overline{\Omega})$.

The proof of this lemma follows from the fact that the embedding $W_0^{1,\vec{p}(\cdot)}(\Omega) \hookrightarrow W_0^{1,\underline{p}}(\Omega)$ is continuous, and in view of the compact embedding theorem for Sobolev spaces.

Proposition 1.1.4. *The dual of* $W_0^{1,\vec{p}(\cdot)}(\Omega)$ *is denoted by* $W^{-1,\vec{p'}(\cdot)}(\Omega)$, *where* $\vec{p'}(\cdot) = (p_0'(\cdot), \ldots, p_N'(\cdot))$ *and* $\frac{1}{p_i'(\cdot)} + \frac{1}{p_i(\cdot)} = 1$, *(cf. [36] for the constant exponent case).*

For each $F \in W^{-1,\vec{p'}(\cdot)}(\Omega)$ *there exists* $F_i \in L^{p'_i(\cdot)}(\Omega)$ *for* $i = 0, 1, \ldots, N$, *such that* $F = F_0 - \sum_{i=1}^{N} D^i F_i$. *Moreover for any* $u \in W_0^{1,\vec{p}(\cdot)}(\Omega)$, *we have*

$$\langle F, u\rangle = \sum_{i=0}^{N} \int_\Omega F_i \, D^i u \, dx.$$

We define a norm on the dual space by

$$\|F\|_{-1,\vec{p'}(\cdot)} = \inf\Big\{ \sum_{i=0}^{N} \|F_i\|_{p'_i(\cdot)} \quad / \quad F = F_0 - \sum_{i=1}^{N} D^i F_i \quad \textit{with} \quad F_i \in L^{p'_i(\cdot)}(\Omega)\Big\}.$$

Proof of the proposition 1.1.4 We follow the same proof in [36] for the constant exponent case.

Definition 1.1.3. *For all* $k > 0$ *and* $s \in I\!R$, *the truncation function of level* k *can be defined by*

$$T_k(s) = \begin{cases} s & \textit{if} \quad |s| \le k, \\ k\dfrac{s}{|s|} & \textit{if} \quad |s| > k. \end{cases}$$

it is obvious to state that

(i) $|T_k(s)| = \min\{|s|, k\}$,

(ii) $T_k(-s) = -T_k(s)$,

(ii) $\lim_{k\to\infty} T_k(s) = s$,

(iv) $\lim_{k\to 0} \frac{1}{k} T_k(s) = sign(s)$.

We set

$$\mathcal{T}_0^{1,\vec{p}(\cdot)}(\Omega) := \{u : \Omega \mapsto I\!R \text{ measurable, such that } T_k(u) \in W_0^{1,\vec{p}(\cdot)}(\Omega) \text{ for any } k > 0\}.$$

Note that, a measurable function u verifying $T_k(u) \in W_0^{1,\vec{p}(\cdot)}(\Omega)$ for all $k > 0$, does not necessarily belong to $W_0^{1,1}(\Omega)$. However, for any $u \in \mathcal{T}_0^{1,\vec{p}(\cdot)}(\Omega)$ it is possible to define the weak gradient of u, still denoted ∇u.

Proposition 1.1.5. *Let* $u \in \mathcal{T}_0^{1,\vec{p}(\cdot)}(\Omega)$. *For any* $i \in \{1, \ldots, N\}$, *there exists a unique measurable function* $v_i : \Omega \mapsto I\!R$ *such that*

$$\forall k > 0 \qquad D^i T_k(u) = v_i . \chi_{\{|u|<k\}} \quad \textit{a.e.} \quad x \in \Omega,$$

where χ_A *denotes the characteristic function of a measurable set* A. *The functions* v_i *are called the weak partial derivatives of* u *and are still denoted* $D^i u$. *Moreover, if* u *belongs to* $W_0^{1,1}(\Omega)$, *then* v_i *coincides with the standard distributional derivative of* u, *that is,* $v_i = D^i u$.

The proof of the Proposition 2.2 follows the usual techniques developed in [46] for the case of Sobolev spaces. For more details concerning the anisotropic Sobolev spaces, we refer the reader to [36] and [87].

1.1.4 Anisotropic parabolic spaces with variable exponents

We extend a variable exponent $p : \overline{\Omega} \to [1, \infty]$ to $\overline{Q_T} = \overline{\Omega} \times [0, T]$ by setting $p(x, t) := p(x)$ for all $(x, t) \in \overline{Q_T}$ and then introduce the generalized Lebesgue space

$$L^{p(.)}(Q_T) = \left\{ u : Q_T \to I\!R \quad \text{is measurable with} \quad \int \int_{Q_T} |u(x,t)|^{p(x)}\, dxdt < \infty \right\},$$

equipped with the norm

$$\|u\|_{L^{p(.)}(Q_T)} = \inf \left\{ \mu > 0 : \int \int_{Q_T} \left| \frac{u(x,t)}{\mu} \right|^{p(x)} dxdt \leq 1 \right\}.$$

Notice that this space has the same properties as $L^{p(\cdot)}(\Omega)$.
We will also make use of standard notations of Bochner spaces; i.e. if $q \geq 1$ and X is a Banach space, then $L^q(0, T; X)$ is the space of strongly measurable functions $u : (0, T) \to X$ for which $t \mapsto \|u(t)\|_X \in L^q(0, T)$. Moreover, C([0,T];X) denotes the space of continuous functions $u : [0, T] \to X$ endowed with the norm $\|u\|_{C([0,T];X)} := \max_{t\in[0,T]} \|u(t)\|_X$.

We identify an abstract function like $\upsilon \in L^{p^-}(0, T; L^{p(.)}(\Omega))$ with the real valued Lebesgue measurable function υ defined by $\upsilon(t, x) = \upsilon(t)(x)$ for almost all $t \in (0, T)$, for almost all $x \in \Omega$. In the same way we associate to any function $\upsilon \in L^{p(.)}(Q_T)$ an abstract Bochner measurable function $\upsilon : (0, T) \to L^{p(.)}(\Omega)$ by setting $\upsilon(t) := \upsilon(t, .)$ for almost every $t \in (0, T)$.

Proposition 1.1.6. ***(see [156])***
Taking into account the identifications above, we have ***(1)*** *The evolution triple* $W_0^{1,p(.)} \subseteq L^{p(.)}(\Omega) \subseteq W^{-1,p'(.)}(\Omega)$ *is satisfied.*
(2) *The evolution triple* $L^{p^+}(0, T; L^{p(.)}(\Omega)) \subseteq L^{p(.)}(Q_T) \subseteq L^{p^-}(0, T; L^{p(.)}(\Omega))$ *is satisfied.*
(3) *The following dense continuous embedding*

$$L^{p^+}(0, T; L^{p(.)}(\Omega)) \subseteq L^{p(.)}(Q_T) \subseteq L^{p^-}(0, T; L^{p(.)}(\Omega))$$

where the inclusions are, in general, strict for non-constant exponents $p(.)$.

Now, in order to study the equation $(0.0.26)$, we have to introduce $V(Q_T)$ as a sort of anisotropic

spaces with variable exponents . Let's consider $\Omega\times]0,T[$ for a given $0<T<+\infty$. We define

$$L^{p^-}(0,T,W^{1,\overrightarrow{p}(.)}(\Omega))=\left\{\forall t,x\mapsto u(x,t)\in W^{1,\overrightarrow{p}(.)}(\Omega)\Big/\sum_{i=0}^{N}\int_0^T\|D^iu\|^{p^-}_{p_i(.)}\,dt<\infty\right\},\tag{1.1.6}$$

and we define

$$V(Q_T)=\left\{u\in L^{p^-}(0,T,W^{1,\overrightarrow{p}(.)}(\Omega))\Big/\sum_{i=0}^{N}\int_{Q_T}|D^iu|^{p_i(x)}\,dxdt<\infty\right\},\tag{1.1.7}$$

endowed with the norm

$$\|u\|_{V(Q_T)}=\|u\|_{L^{p0(.)}(Q_T)}+\sum_{i=1}^{N}\|D^iu\|_{L^{p(.)}(Q_T)}.\tag{1.1.8}$$

Let's remark that

$$\sum_{i=0}^{N}\int_{Q_T}|D^iu|^{p_i(x)}\,dxdt\geq\frac{1}{C}\|u\|^{p^-}_{V(Q_T)}-N-1,\tag{1.1.9}$$

for all $u\in V(Q_T)$. We also insert the functional space $V_0(Q_T)$ as the closure of $\mathcal{C}_0^\infty(\Omega)$ in $V(Q_T)$.
The spaces $V(Q_T)$ and $V_0(Q_T)$ are separable and reflexive Banach spaces. The dual space of $V_0(Q_T)$ is defined as follows:

$$V^*(Q_T)=\left\{F=f_0-\sum_{i=1}^{N}D^if_i\quad\text{with}\quad f_0\in L^{p_0'(.)}(Q_T),\ \ f_i\in L^{p_i'(.)}(Q_T)\right\},\tag{1.1.10}$$

and equipped with the norm

$$\|F\|_{V^*(Q_T)}=\|f_0\|_{L^{p_0'(.)}(Q_T)}+\sum_{i=1}^{N}\|f_i\|_{L^{p_i'(.)}(Q_T)}.\tag{1.1.11}$$

Moreover, for any $\upsilon\in V_0(Q_T)$, we have

$$\langle F,\upsilon\rangle=\sum_{i=0}^{N}\int_{Q_T}f_iD^i\upsilon(x)\,dxdt.$$

1.1.5 Anisotropic Weighted Sobolev spaces

In this section we extend anisotropic Sobolev spaces to weighted case. As we pointed out in the introduction, the extension of anisotropic Sobolev spaces to weighted case is, to the best of our knowledge, undertaken here for the first time. Let Ω be a bounded open set of $\mathbf{R}^N$, $p_0,p_1,\ldots,p_N$ be $N+1$ exponents with $1<p_i<\infty$ for $i=0,1,\ldots,N$ and $w=\{w_i(x),\,0\leq i\leq N\}$ be a vector of weight functions; i.e., every component $w_i(x)$ is a measurable function which is strictly positive a.e. in Ω. Further, we suppose in all our considerations that, there exits

$$w_i\in L^1_{\rm loc}(\Omega),\tag{1.1.12}$$

$$w_i^{\frac{-1}{p_i-1}} \in L^1_{\mathrm{loc}}(\Omega), \tag{1.1.13}$$

for any $0 \leq i \leq N$. We define the weighted space $L^{p_i}(\Omega, \gamma)$, where γ is a weight function on Ω by,

$$L^{p_i}(\Omega, \gamma) = \{u = u(x), u\gamma^{\frac{1}{p_i}} \in L^{p_i}(\Omega)\}$$

with the norm

$$\|u\|_{L^{p_i}(\Omega,\gamma)} = \|u\|_{p_i,\gamma} = \left(\int_\Omega |u(x)|^{p_i} \gamma(x)\, dx \right)^{\frac{1}{p_i}}.$$

We denote

$$(p_i) = (p_0, \ldots, p_N), \quad D^0 u = u \quad \text{and} \quad D^i u = \frac{\partial u}{\partial x_i} \qquad \text{for } i = 1, \ldots, N,$$

and we define

$$\underline{p} = \min\{p_0, p_1, \ldots, p_N\} \qquad \text{then} \qquad \underline{p} > 1. \tag{1.1.14}$$

The anisotropic weighted Sobolev space $W^{1,(p_i)}(\Omega, w)$ is defined as the space of real-valued functions $u \in L^{p_0}(\Omega, w_0) = \{v(x),\ \ v w_0^{\frac{1}{p_0}} \in L^{p_0}(\Omega)\}$ such that the derivatives in the sense of distributions fulfill

$$D^i u \in L^{p_i}(\Omega, w_i) \quad \text{for } i = 1, \ldots, N.$$

Which is a Banach space under the norm

$$\|u\|_{1,(p_i),w} = \|u\|_{p_0,w_0} + \sum_{i=1}^{N} \|\frac{\partial u}{\partial x_i}\|_{p_i,w_i}. \tag{1.1.15}$$

Condition $(1.1.12)$ implies that $C_0^\infty(\Omega)$ is a space of $W^{1,(p_i)}(\Omega, w)$ and consequently, we can introduce the subspace $V = W_0^{1,(p_i)}(\Omega, w)$ of $W^{1,(p_i)}(\Omega, w)$ as the closure of $C_0^\infty(\Omega)$ with respect to the norm $(1.1.15)$. Moreover, condition $(1.1.13)$ implies that $W^{1,(p_i)}(\Omega, w)$ as well as $W_0^{1,(p_i)}(\Omega, w)$ are reflexive Banach spaces.

We recall that the dual space of weighted Sobolev spaces $W_0^{1,(p_i)}(\Omega, w)$ is equivalent to $W^{-1,(p_i')}(\Omega, w^*)$, where $w^* = \{w_i^* = w_i^{1-p_i'}, i = 0, \ldots, N\}$ and $(p_i') = (p_0', p_1', \ldots, p_N')$ where p_i' is the conjugate of p_i; i.e., $p_i' = \frac{p_i}{p_i - 1}$, (see [94] for the isotropic case).

1.2 Some important technical results and lemmas

Theorem 1.2.1. *Let f_n be a sequence of functions of $L^1(\Omega)$ converging almost everywhere to a measurable function f. We assume that there exists $g \in L^1(\Omega)$ such that for all $n \geq 1$, we have $|f_n(x)| \leq |g(x)|$ almost*

everywhere on Ω *then* $f \in L^1(\Omega)$

$$\lim_{n\to\infty}\int_\Omega |f_n - f|\,dx = 0, \lim_{n\to\infty}\int_\Omega f_n\,dx = \int_\Omega f\,dx.$$

Definition 1.2.1. ***(equi-integrability)*** *a sequence* f_n *of functions of* $L^1(\Omega)$ *is equi-integrable if: for all* $\epsilon > 0$, *there exists* $\delta > 0$ *such that* $\forall E \subset \Omega$ *measurable with meas*$(E) < \delta$, *we have*

$$\int_\Omega f_n\,dx < \epsilon$$

An important theorem for showing the strong convergence of a sequence from the almost everywhere convergence is Vitali's Theorem

Theorem 1.2.2. ***(Vitali)*** *Let be* $1 \le p < +\infty$, $\Omega \subset I\!R^N$ *measurable and* $(f_n)_{n\in I\!N}$ *a sequence of* $L^p(\Omega)$ *converging almost everywhere to a function* f. *Then*

$$f_n \to f \quad strongly\ dans\ L^p(\Omega) \Leftrightarrow (f_n)_{n\in I\!N}\ is\ p\ equi-integrable.$$

Lemma 1.2.1. ***(Fatou)*** *Let* f_n *be a sequence of positive measurable functions. So, we have*

$$\int_\Omega \big(\liminf_{n\to\infty} f_n\big)\,dx \le \liminf_{n\to\infty}\left(\int_\Omega f_n\,dx\right).$$

Lemma 1.2.2. *(see [113],Theorem 13.47) Let* $(u_n)_n$ *be a sequence in* $L^1(\Omega)$ *and* $u \in L^1(\Omega)$ *such that*

(i) $u_n \to u$ *a.e. in* Ω,

(ii) $u_n \ge 0$ *and* $u \ge 0$ *a.e. in* Ω,

(iii) $\displaystyle\int_\Omega u_n\,dx \to \int_\Omega u\,dx$,

then $u_n \to u$ *in* $L^1(\Omega)$.

Lemma 1.2.3. *(see [32]) Let* $g \in L^{r(\cdot)}(\Omega)$ *and* $g_n \in L^{r(\cdot)}(\Omega)$ *with* $\|g_n\|_{r(\cdot)} \le C$ *for* $1 < r(x) < \infty$. *If* $g_n(x) \to g(x)$ *a.e. on* Ω, *then* $g_n \rightharpoonup g$ *in* $L^{r(\cdot)}(\Omega)$.

Lemma 1.2.4. *(see [33]) Let* $(u_n)_n$ *be bounded sequence in* $W_0^{1,\overrightarrow{p}(\cdot)}(\Omega)$. *If* $u_n \rightharpoonup u$ *in* $W_0^{1,\overrightarrow{p}(\cdot)}(\Omega)$, *then* $T_k(u_n) \rightharpoonup T_k(u)$ *in* $W_0^{1,\overrightarrow{p}(\cdot)}(\Omega)$, *where, for any* $k > 0$, $T_k(.)$ *is the truncation function defined by* $T_k(s) := max\{-k, min\{k, s\}\}$.

Lemma 1.2.5. *(see [33]) Let* $u \in W_0^{1,\overrightarrow{p}(\cdot)}(\Omega)$ *then* $T_k(u) \in W_0^{1,\overrightarrow{p}(\cdot)}(\Omega)$ *for any* $k > 0$. *Moreover, we have* $T_k(u) \to u$ *in* $W_0^{1,\overrightarrow{p}(\cdot)}(\Omega)$ *as* $k \to \infty$.

Now,let us give a non homogeneous version of the Sobolev inequality

Lemma 1.2.6. *Let p_i^-, p_i, p_i^+ and p_-^* as in $(1.1.4)$ and let $E \geq 1$. Then the following inequality holds*

$$\|v\|^{p_i^+}_{L^{p_-^*}(\Omega)} \leq C^{p_i^+} N^{p_i^+-1} E^{p_i^+-p_i^-} \sum_{i=1}^{N} \|D^i v\|^{p_i(.)}_{L^{p_i(.)}(\Omega)} \tag{1.2.1}$$

for any $v \in W_0^{1,1}(\Omega)$ such that

$$\|D^i v\|_{L^{p_i(.)}(\Omega)} \leq E, \quad \forall i = 1, \ldots, N \tag{1.2.2}$$

Proof of Lemma 1.2.6 Let us recall beforehand the following Sobolev-type inequality ([149]): There exists a positive constant C, depending on the set Ω, such that

$$\|v\|_{L^{\overline{q}^*}(\Omega)} \leq C \sum_{i=1}^{N} \|D^i v\|_{L^{q_i}(\Omega)}, \tag{1.2.3}$$

for any $v \in C_0^1(\Omega)$, where $q_i \geq 1$, $q = max_i\{q_i\}$ for $i = 1, 2, \ldots, N$, $\dfrac{1}{\overline{q}} = \dfrac{1}{N} \displaystyle\sum_{i=1}^{N} \dfrac{1}{q_i}$ and $\overline{q}^* = \dfrac{N\overline{q}}{N - \overline{q}}$ $(if\ \ \overline{q} < N)$.

By $(1.2.2)$, since $p_i(.) \geq p_i^-$ and $E \geq 1$, for any $i = 1, \ldots, N$ we have

$$\|D^i v\|^{p_i^+}_{L^{p_i(.)}(\Omega)} \leq E^{p_i^+ - p_i(.)} \|D^i v\|^{p_i(.)}_{L^{p_i(.)}(\Omega)} \leq E^{p_i^+ - p_i^-} \|D^i v\|^{p_i(.)}_{L^{p_i(.)}(\Omega)} \tag{1.2.4}$$

Thanks to the convexity of the real function $t \longrightarrow t^{p_i^+}$ we get

$$\left(\frac{1}{N} \sum_{i=1}^{N} \|D^i v\|_{L^{p_i(.)}(\Omega)}\right)^{p_i^+} \leq \frac{1}{N} \sum_{i=1}^{N} \|D^i v\|^{p_i^+}_{L^{p_i(.)}(\Omega)} \quad \text{(note that the index } i \text{ in } p_i^+ \text{ does not depend necessarily on the sum)} \tag{1.2.5}$$

Then, from the inequalities in $(1.2.3)$,$(1.2.4)$ and $(1.2.5)$ it yields that

$$\begin{aligned} \|v\|^{p_i^+}_{L^{p_-^*}(\Omega)} &\leq C^{p_i^+} \left(\sum_{i=1}^{N} \|D^i v\|_{L^{p_i(.)}(\Omega)} \right)^{p_i^+} \\ &\leq C^{p_i^+} N^{p_i^+-1} \sum_{i=1}^{N} \|D^i v\|^{p_i^+}_{L^{p_i(.)}(\Omega)} \\ &\leq C^{p_i^+} N^{p_i^+-1} E^{p_i^+-p_i^-} \sum_{i=1}^{N} \|D^i v\|^{p_i(.)}_{L^{p_i(.)}(\Omega)} . \end{aligned}$$

Lemma 1.2.7. *Let $(u_n)_{n\in\mathbb{N}}$ be a sequence in $W_0^{1,\overrightarrow{p}(\cdot)}(\Omega)$ such that $u_n \rightharpoonup u$ in $W_0^{1,\overrightarrow{p}(\cdot)}(\Omega)$ and*

$$\sum_{i=0}^{N} \int_{\Omega} \left(|D^i u_n|^{p_i(x)-2} D^i u_n - |D^i u|^{p_i(x)-2} D^i u\right)(D^i u_n - D^i u)\, dx \to 0, \tag{1.2.6}$$

then $u_n \longrightarrow u$ in $W_0^{1,\overrightarrow{p}(\cdot)}(\Omega)$.

We note that the preceding lemma can be generalized as follows :

Lemma 1.2.8. *Assuming that* $(0.0.14) - (0.0.16)$ *hold, and let* $(u_n)_{n\in\mathbb{N}}$ *be a sequence in* $W_0^{1,\overrightarrow{p}(\cdot)}(\Omega)$ *such that* $u_n \rightharpoonup u$ *in* $W_0^{1,\overrightarrow{p}(\cdot)}(\Omega)$ *and*

$$\begin{aligned}&\int_\Omega (|u_n|^{p_0(x)-2}u_n - |u|^{p_0(x)-2}u)(u_n-u)\,dx\\ &\quad+\sum_{i=1}^N \int_\Omega (a_i(x,u_n,\nabla u_n) - a_i(x,u_n,\nabla u))(D^i u_n - D^i u)\,dx \to 0,\end{aligned} \tag{1.2.7}$$

then $u_n \longrightarrow u$ *in* $W_0^{1,\overrightarrow{p}(\cdot)}(\Omega)$ *for a subsequence.*

Proof of Lemma 1.2.8 Let

$$\begin{aligned}S_n &= \sum_{i=1}^N (a_i(x,u_n,\nabla u_n) - a_i(x,u_n,\nabla u))(D^i u_n - D^i u)\\ &\quad +(|u_n|^{p_0(x)-2}u_n - |u|^{p_0(x)-2}u)(u_n-u).\end{aligned}$$

Thanks to $(0.0.16)$, we deduce that S_n is a positive function and, by $(1.2.7)$, $S_n \to 0$ in $L^1(\Omega)$ as $n\to\infty$. Since $u_n \rightharpoonup u$ in $W_0^{1,\vec{p}(.)}(\Omega)$, using the compact embedding we obtain $u_n \to u$ in $L^{\underline{p}(\cdot)}(\Omega)$, and since $S_n \to 0$ a.e. in Ω, there exists a subset B in Ω with measure zero such that $\forall x \in \Omega\backslash B$

$$|u(x)|<\infty, \qquad |D^i u(x)|<\infty, \qquad c_i(x)<\infty, \qquad u_n \to u \qquad and \qquad S_n \to 0.$$

We have

$$\begin{aligned}S_n(x) &= \sum_{i=1}^N (a_i(x,u_n,\nabla u_n) - a_i(x,u_n,\nabla u))(D^i u_n - D^i u)\\ &\quad + (|u_n|^{p_0(x)-2}u_n - |u|^{p_0(x)-2}u)(u_n-u)\\ &= \sum_{i=1}^N \Big(a_i(x,u_n,\nabla u_n)D^i u_n + a_i(x,u_n,\nabla u)D^i u\\ &\quad -a_i(x,u_n,\nabla u)D^i u_n - a_i(x,u_n,\nabla u_n)D^i u\Big)\\ &\quad + |u_n|^{p_0(x)} + |u|^{p_0(x)} - |u_n|^{p_0(x)-2}u_n u - |u|^{p_0(x)-2}u u_n\\ &\geq \underline{\alpha}\sum_{i=0}^N |D^i u_n|^{p_i(x)} + \underline{\alpha}\sum_{i=0}^N |D^i u|^{p_i(x)} - \beta\sum_{i=1}^N (c_i(x) + |u_n|^{\frac{p(x)}{p_i'(x)}} + |D^i u|^{p_i(x)-1})|D^i u_n|\\ &\quad - \beta\sum_{i=1}^N (c_i(x) + |u_n|^{\frac{\underline{p}(x)}{p_i'(x)}} + |D^i u_n|^{p_i(x)-1})|D^i u| - |u_n|^{p_0(x)-1}|u| - |u|^{p_0(x)-1}|u_n|\\ &\geq \underline{\alpha}\sum_{i=0}^N |D^i u_n|^{p_i(x)} - K_x\sum_{i=1}^N (1 + |D^i u_n|^{p_i(x)-1} + |D^i u_n|),\end{aligned}$$

with $\underline{\alpha} = \min(\alpha, 1)$ and K_x depending only on x. It follows that

$$S_n(x) \geq \sum_{i=0}^{N} |D^i u_n|^{p_i(x)} \big(\underline{\alpha} - \frac{K_x}{|D^i u_n|^{p_i(x)}} - \frac{K_x}{|D^i u_n|} - \frac{K_x}{|D^i u_n|^{p_i(x)-1}}\big).$$

By a standard argument, $(D^i u_n)_{n\in I\!N}$ is bounded almost everywhere in Ω for all $i = 0, 1, \ldots, N$. Indeed, if $|D^i u_n| \to \infty$ in a measurable subset $E \subset \Omega$ then

$$\lim_{n\to\infty} \int_\Omega S_n(x)\, dx \geq \lim_{n\to\infty} \sum_{i=1}^{N} \int_E |D^i u_n|^{p_i(x)} \big(\underline{\alpha} - \frac{K_x}{|D^i u_n|^{p_i(x)}} - \frac{K_x}{|D^i u_n|} - \frac{K_x}{|D^i u_n|^{p_i(x)-1}}\big)\, dx = \infty,$$

which is a contradiction since $S_n \to 0$ in $L^1(\Omega)$.

Let ξ_i^* an accumulation point of $(D^i u_n)_{n\in I\!N}$ for $i = 1, \ldots, N$. We have $|\xi_i^*| < \infty$ and by the continuity of $a(x, \cdot, \cdot)$, we obtain

$$\Big(a_i(x, u_n, \xi^*) - a_i(x, u_n, \nabla u)\Big)(\xi_i^* - D^i u) = 0 \qquad \text{for} \quad i = 1, \ldots, N.$$

Thanks to $(0.0.16)$, we have $\xi^* = \nabla u$ and the uniqueness of the accumulation point implies that $\nabla u_n \to \nabla u$ a.e in Ω.

Since $(a_i(x, u_n, \nabla u_n))_{n\in I\!N}$ is bounded in $L^{p_i'(.)}(\Omega)$ and $a_i(x, u_n, \nabla u_n) \to a_i(x, u, \nabla u)$ a.e in Ω, by the Lemma $1.2.3$, we can establish that

$$a_i(x, u_n, \nabla u_n) \rightharpoonup a_i(x, u, \nabla u) \quad in \quad L^{p_i'(.)}(\Omega) \quad \text{for} \quad i = 1, \ldots, N.$$

Using $(1.2.7)$ and the Lemma 1.2.2, we deduce that

$$|u_n|^{p_0(x)} \longrightarrow |u|^{p_0(x)} \qquad \text{in} \quad L^1(\Omega) \tag{1.2.8}$$

and

$$a_i(x, u_n, \nabla u_n) D^i u_n \longrightarrow a_i(x, u, \nabla u) D^i u \qquad \text{in} \quad L^1(\Omega). \tag{1.2.9}$$

According to the condition $(0.0.15)$, we have

$$\alpha |D^i u_n|^{p_i(x)} \leq a_i(x, u_n, \nabla u_n) D^i u_n \quad \text{for} \quad i = 1, \ldots, N.$$

Let $y_n^i = \frac{1}{\alpha} a_i(x, u_n, \nabla u_n) D^i u_n$ and $y^i = \frac{1}{\alpha} a_i(x, u, \nabla u) D^i u$, in view of the Fatou's Lemma, we get

$$\int_\Omega 2y^i\, dx \leq \liminf_{n\to\infty} \int_\Omega (y_n^i + y^i - |D^i u_n - D^i u|^{p_i(x)}) dx,$$

which implies that $0 \leq -\limsup_{n\to\infty} \int_\Omega |D^i u_n - D^i u|^{p_i(x)}\, dx$ and since

$$0 \leq \liminf_{n\to\infty} \int_\Omega |D^i u_n - D^i u|^{p_i(x)}\, dx \leq \limsup_{n\to\infty} \int_\Omega |D^i u_n - D^i u|^{p_i(x)}\, dx \leq 0,$$

it follows that

$$\int_\Omega |D^i u_n - D^i u|^{p_i(x)}\, dx \longrightarrow 0 \text{ as } n \to \infty.$$

Consequently

$$D^i u_n \longrightarrow D^i u \quad \text{in} \quad L^{p_i(.)}(\Omega) \quad \text{for} \quad i = 1, \ldots, N.$$

In view of $(1.2.8)$ we deduce that

$$u_n \longrightarrow u \qquad \text{in} \quad W_0^{1,\vec{p}(.)}(\Omega).$$

1.3 Penalization operators

In the analysis of PDEs, the penalization approach is an important concept that is sometimes used to study problems whose constraints are difficult to take into account, whereas the penalized problem has properties that are better understood or easier to highlight. If one is lucky or if the penalization is well chosen, sometimes delicate passages to the limit allow to obtain properties of the original problem (a priori estimate, existence or some regularities of solution...)
Lions (see [121], p 370) had established that there is always an operator of a penalization defined in the following sense:

Definition 1.3.1. *Let V be a reflexive Banach space, its norm and that of its dual are strictly convex, and let K be a closed convex of V.*
We call the operator of a penalization (attached to K) any operator $\Phi : V \to V'$ having the following properties:

$$\Phi \quad \textit{is monotone bounded and hemicontinuous from} \quad V \textit{ to } V',$$

$$K = \{\nu / \nu \in V, \Phi(\nu) = 0\}.$$

To illustrate the definition we just present the following example introducing the $\overrightarrow{p}-$ laplacien:

Exemple 1.3.1. *Let be $1 < p_i < +\infty$, $V = W_0^{1,\overrightarrow{p}}(\Omega)$ and the $\overrightarrow{p}-$ laplacien operator defined by*

$$Au = -\sum_{i=1}^{N} D^i\big(|D^i u|^{p_i - 2} D^i u\big),$$

$$K = \big\{u \in W_0^{1,\overrightarrow{p}}(\Omega),\ u \geq 0 \ \textit{a.e. in}\ \Omega\big\}.$$

We project on $L^{p_0}(\Omega)$ *and by choosing the duality operator:* $E(u) = |u|^{p_0-2}u$, *we obtain:*

$$\Phi(u) = E(u - P_k(u)) = -|u^-|^{p_0-2}u^-,$$

and the corresponding penalized equation is then:

$$\begin{cases} Au_\epsilon - \dfrac{1}{\epsilon}|u_\epsilon^-|^{p_0-2}u_\epsilon^- = f, \\ u_\epsilon \in W_0^{1,\overrightarrow{p}}(\Omega). \end{cases}$$

Part I

Nonlinear anisotropic elliptic and parabolic PDEs with lower order terms and variable exponents.

Chapter 2

On nonlinear anisotropic elliptic equations involving lower order terms with variable exponents

An existence results for a solution of a class of nonlinear elliptic equations with natural growth terms in the anisotropic Sobolev spaces with variable exponent $W_0^{1,\overrightarrow{p}(\cdot)}(\Omega)$ are established. At the first time, the second term of the equation f is in $L^{m(x)}(\Omega)$, $m(x) > p'_\infty$ and $m(x) > \frac{p^*_-}{p^*_- - p_i^+}$ for almost every $x \in \Omega$, where $p^*_- > p_i^+,$ for any $i = 1, \ldots, N$ and no sign assumption is made on the nonlinearities. Later, the lower order term will be assumed to satisfy the sign condition, and the data will belong either to $L^{(p^*_-)'}(\Omega)$ or $L^1(\Omega)$.

2.1 Introduction

In this chapter, we are concerned with the problem of existence of weak solutions for a class of nonlinear elliptic equations with additional lower order terms, that play the role of perturbations terms. Firstly, attention shall be focused to such problems with lower order terms without sign assumptions

$$\begin{cases} -\sum_{i=1}^{N} D^i[|D^i u|^{p_i(x)-2} D^i u] + \sigma_0 u = \sum_{i=1}^{N} h_i(x, u, \nabla u) + f & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{2.1.1}$$

where $\sigma_0 > 0,\ f \in L^{m(x)}(\Omega),\ \ m(x) > \frac{p_-^*}{p_-^* - p_i^+}$ and $m(x) > p'_\infty$ for a.e. $x \in \Omega$ with $p_\infty = max\{p_-^*, p_i^+\}$. It's supposed that $p_-^* > p_i^+$ for all $i = 1, \ldots, N,$ ie $p_\infty = p_-^*$ which means that the anisotropy is concentrated (the p_i's are not "too far apart").

We assume also that

$$h_i(x, s, \xi) : \Omega \times I\!R \times I\!R^N \longmapsto I\!R \ \text{ is Carathéodory function} \qquad \text{for all } \ i = 1, \ldots, N \tag{2.1.2}$$

(measurable with respect to x in Ω for every (s, ξ) in $I\!R \times \mathbf{R}^N$ and continuous with respect to (s, ξ) in $I\!R \times I\!R^N$ for almost every x in Ω) and does not satisfy the sign condition. Furthermore, there exists $\beta > 0$ such that the following inequality is true for all $(s, \xi) \in I\!R \times I\!R^N$ and almost every $x \in \Omega$

$$\sum_{i=1}^{N} |h_i(x, s, \xi)| \leq \beta \sum_{i=1}^{N} |\xi_i|^{p_i(x)}, \qquad \forall\, i = 1, \ldots, N. \tag{2.1.3}$$

The model example is the following Dirichlet problem

$$\begin{cases} -\sum_{i=1}^{N} D^i[|D^i u|^{p_i(x)-2} D^i u] + \sigma_0 u = \beta \sum_{i=1}^{N} |D^i u|^{p_i(x)} + f & \text{in} \quad \Omega, \\ u = 0 & \text{on} \quad \partial\Omega. \end{cases}$$

Later, we are interested in this chapter on the proof of the existence of a solution for the following problem

$$\begin{cases} -\text{div}\, a(x, u, \nabla u) + \sum_{i=1}^{N} h_i(x, u, \nabla u) = f & \text{in} \quad \Omega, \\ u = 0 & \text{on} \quad \partial\Omega, \end{cases} \tag{2.1.4}$$

with $u \in W_0^{1,\overrightarrow{p}(\cdot)}(\Omega)$, $h_i(x, u, \nabla u) \in L^1(\Omega)$ where $h_i(x, u, \nabla u)$ are nonlinearities with natural growth respect to the gradient of u, for any $i = 1, \ldots, N$, which satisfy the sign condition $h_i(x, s, \xi)s \geq 0$. We assume that for all $i = 1, \ldots, N$, $a_i : \Omega \times I\!R \times I\!R^N \longmapsto I\!R$ are Carathéodory functions satisfying for a.e. $x \in \Omega$ and all $(s, \xi) \in \mathbf{R} \times \mathbf{R}^N$

$$|a_i(x, s, \xi)| \leq \beta(C_i(x) + |s|^{p_i(x)-1} + \sum_{i=1}^{N} |\xi_i|^{p_i(x)-1}), \tag{2.1.5}$$

$$\sum_{i=1}^{N} a_i(x, s, \xi)\xi_i \geq \alpha \sum_{i=1}^{N} |\xi_i|^{p_i(x)}, \tag{2.1.6}$$

$$\big(a_i(x, s, \xi) - a_i(x, s, \xi')\big)(\xi_i - \xi'_i) > 0 \quad \text{for} \quad \xi_i \neq \xi'_i, \tag{2.1.7}$$

for any $\xi = (\xi_1, \ldots, \xi_N)$ and $\xi' = (\xi'_1, \ldots, \xi'_N)$ in $I\!R^N$, where $C_i(\cdot) \in L^{p'_i(\cdot)}(\Omega)$, with $\frac{1}{p_i(x)} + \frac{1}{p'_i(x)} = 1$ and α, β are two positive real numbers.

Let $h_i : \Omega \times I\!R \times I\!R^N \longmapsto I\!R$ a Carathéodory function such that, for almost every $x \in \Omega$ and for all $s \in I\!R$ and $\xi \in I\!R^N$,

$$h_i(x,s,\xi)s \geq 0, \qquad \forall\ i = 1,...,N, \tag{2.1.8}$$

$$\sum_{i=1}^{N} |h_i(x,s,\xi)| \leq b(|s|)(j(x) + \sum_{i=1}^{N} |\xi_i|^{p_i(x)}), \tag{2.1.9}$$

where $b(.) : I\!R^+ \to I\!R^+$ is a continuous non-decreasing function, and $j(.) : \Omega \to I\!R^+$ with $j(.) \in L^1(\Omega)$. Finally we assume one of the following two assumptions:

$$\begin{cases} f \in L^1(\Omega), \\ \text{and there exist } \zeta > 0 \text{ and } \gamma > 0 \text{ such that} \\ |h_i(x,s,\xi)| \geq \gamma\left(|s|^{p_0(x)} + \sum_{i=1}^{N} |\xi|^{p_i(x)}\right) \text{ when } |s| > \zeta, \quad \forall\ i = 1,...,N. \end{cases} \tag{2.1.10}$$

or

$$f \in [W_0^{1,\vec{p}(\cdot)}(\Omega)]^* (\text{or to } L^{(p_-^*)'}) \tag{2.1.11}$$

with $*$ we denote the dual space of $W_0^{1,\vec{p}(\cdot)}(\Omega)$. A prototype example of (2.1.4) that is covered by the assumptions above is the following

$$\begin{cases} -\sum_{i=1}^{N} D^i[|D^i u|^{p_i(x)-2} D^i u] + u\sum_{i=1}^{N} |D^i u|^{p_i(x)} = f & \text{in} \quad \Omega, \\ u = 0 & \text{on} \quad \partial\Omega. \end{cases}$$

There is a wide literature written about existence of solutions for problems involving lower order terms with natural growth, see for instance in the isotropic case, i.e. $p_i = 2$ or $p_i = p$ for all i, some papers regarding the study of these problems and the references therein [22, 47, 52, 59, 80]. We mention also that existence and regularity results for distributional solutions with L^1-data are proved in [56, 125] for a class of anisotropic elliptic and parabolic problems with constant components, and we call back that in the isotropic case with $p_i(.) = p(.) > 2 - \frac{1}{N}$, for $i = 1, \ldots, N$ existence and regularity results for distributional solutions have been obtained in [42, 144]. In the case of $p(.)$- Laplace equations, the natural framework for this approach is the use of variable exponent Lebesgue and Sobolev spaces $L^{p(.)}(\Omega)$ and $W^{m,p(.)}(\Omega)$ (see for example the book by Dieninig and als [85], [21, 29, 32]). We refer the reader to [40] for results on existence and regularity of distributional solutions in an appropriate function space for the following nonlinear anisotropic elliptic equations with variable exponents

$$\begin{cases} -\sum_{i=1}^{N} \frac{\partial}{\partial x_i}\left[\left|\frac{\partial u}{\partial x_i}\right|^{p_i(x)-2} \frac{\partial u}{\partial x_i}\right] + |u|^{s(x)}u = f & \text{in} \quad \Omega, \\ u = 0 & \text{on} \quad \partial\Omega. \end{cases}$$

where $N \geq 2$, the right hand side $f \in L^1(\Omega)$, $p_i : \bar{\Omega} \to (1, +\infty)$ is continuous function for $i = 1, \ldots, N$, and other restrictions on $p(.)$ and $s(.)$. The authors in [33] have dealt with a quite close problem to $(2.1.1)$, but involving the term $d(x)|u|^{p_0(x)-2}u$, where $d(x)$ belongs to $L^\infty(\Omega)$, and there exists a constant $d_0 > 0$ such that $d(x) \geq d_0$ almost everywhere in Ω and the nonlinear term satisfy the sign condition.

The operator appearing in $(2.1.1)$ leads us to seek solution in the anisotropic variable exponent Sobolev spaces $W_0^{1,\vec{p}(\cdot)}(\Omega)$ which have lately taken its place in the mathematical literature (see Mihalescu et als [127] and [66, 67, 99, 104]). One of our motivation for studying $(2.1.1)$ comes from applications in nonhomogeneous materials that behave differently on different spaces directions, we can refer here to the electrorheological and thermoelectric fluids [142], as an important class of non-Newtonian fluids that have multiple applications in robotic, space technology, brakes shock absorbers, image restoration [73] and the elasticity [158]. The equation $(2.1.1)$ can be regarded as a generalization of the nonlinear elliptic anisotropic p-Laplacien equation

$$\begin{cases} -\sum_{i=1}^{N} D^i[|D^i u|^{p_i-2}D^i u] + \sigma_0 u = \sum_{i=1}^{N} h_i(x, u, \nabla u) + f & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{2.1.12}$$

with the constant $p_i \in (1, +\infty)$ for $i = 1, \ldots, N$ (see [83]). Whereas, the nonlinearities of $(2.1.1)$ are more complicated than those of the anisotropic p-Laplacien $(2.1.12)$. We highlight that in the problem $(2.1.1)$, we do not have any information about the sign of h_i, for any i, and thus we must add the Zero-th term $\sigma_0 u$ to overcome this hindrance. Indeed, it is well known, as in the special case $p_i = 2$ or $p_i = p$ for all i, that the sign condition has a strong regularity effect that enable us to easily obtain a priori estimates from the equation. Otherwise, if the sign condition is not satisfied, the problem $(2.1.1)$ may not even have solution. In point of fact, if $\sigma_0 = 0$, the solution for problem $(2.1.1)$ is available only if we assume that the norm of the datum f is small.

Note also that Di Castro has studied in [82] and established in the framework of Sobolev spaces with constant exponent, many existence results and regularity of solutions of anisotropic nonlinear elliptic problems such as

$$\begin{cases} \sum_{i=1}^{N} \int_\Omega D^i[|D^i u|^{p_i-2}D^i u]\, dx = f & \text{in} \quad \Omega, \\ u = 0 & \text{on} \quad \partial\Omega, \end{cases} \tag{2.1.13}$$

where f is a function belonging either to $L^m(\Omega)$ or $M^m(\Omega)$ (Marcinkiewikz space) or f in divergence form. In the first case of the datum f, the solution u of $(2.1.13)$ is known to only belong to $W_0^{1,\vec{s}}(\Omega)$ for all

$1 < s_i < \frac{N(\overline{p}-1)}{\overline{p}(N-1)} p_i$ where $\frac{1}{\overline{p}} = \frac{1}{N}\sum_{i=1}^{N} \frac{1}{p_i}$, while in contrast we remark in this chapter, that the solution of $(2.1.4)$ is a solution of finite energy $(u \in W_0^{1,\overrightarrow{p}(\cdot)}(\Omega))$ even if $f \in L^1(\Omega)$. In fact, we observe firstly that this improvement of regularity of u is due to the second part of assumption $(2.1.10)$ whose role is to give an a priori estimate in the energy space $W_0^{1,\overrightarrow{p}(\cdot)}(\Omega)$ which allows us to deal with the lower order terms with natural growth. Secondly, in the case of the assumption $(2.1.11)$ we have $(uh_i(x,u,\nabla u)) \in L^1(\Omega)$ for all i, which in contrast is in general false (cf Remark 3 of [54]) if we only assume the first condition in $(2.1.10)$. At last, we point out that we will use a penalization term $\frac{1}{n}|u|^{p_0(x)-2}u$ in the approximated problems to remove the difficulty of non coercivity of the operator in $W_0^{1,\overrightarrow{p}(\cdot)}(\Omega)$ for the problem $(2.1.4)$.

The plan of the chapter is as follows: In the following section, the existence results are stated and proved for the case of lower order terms without sign assumptions. The section 3, is devoted to establish and proof the main results in the case of lower order terms with sign condition.

2.2 Lower order terms without sign assumption

In the sequel, Let Ω be a bounded open subset of $\mathbb{R}^N$ $(N \geq 2)$, and $\partial\Omega$ be Lipschitz-continuous, we denote

$$\mathcal{C}_+(\overline{\Omega}) = \{\text{continuous function} \quad p(\cdot) : \overline{\Omega} \longmapsto I\!R \quad \text{such that} \quad 1 < p^- \leq p^+ < N\},$$

Definition 2.2.1. *Let $f \in L^{m(\cdot)}(\Omega)$, a weak finite energy solution of $(2.1.1)$ is a function $u \in W_0^{1,\overrightarrow{p}(\cdot)}(\Omega) \cap L^\infty(\Omega)$ such that*

$$\sum_{i=1}^{N} \int_\Omega |D^i u|^{p_i(x)-2} D^i u D^i \upsilon \, dx + \sigma_0 \int_\Omega u\upsilon \, dx = \sum_{i=1}^{N} \int_\Omega h_i(x,u,\nabla u)\upsilon \, dx + \int_\Omega f\upsilon \, dx, \qquad (2.2.1)$$

for all $\upsilon \in W_0^{1,\overrightarrow{p}(\cdot)}(\Omega) \cap L^\infty(\Omega)$.

Theorem 2.2.1. *Assuming that $(2.1.2) - (2.1.3)$ hold, $f \in L^{m(\cdot)}(\Omega)$, $m(x) > p'_\infty$ and $m(x) > \frac{p_-^*}{p_-^* - p_i^+}$ for a.e. $x \in \Omega$, where $p_-^* > p_i^+$, then the problem $(2.1.1)$ has at least one weak solution $u \in W_0^{1,\overrightarrow{p}(\cdot)}(\Omega) \cap L^\infty(\Omega)$.*

Proof of the Theorem 2.2.1

Step 1 : Approximate problems.

The idea of the proof is taken from [52] (see also [83]). Let $(f_n)_{n\in I\!N}$ be a sequence in $W^{-1,\overrightarrow{p'}(\cdot)}(\Omega)\cap L^{m(\cdot)}(\Omega)$ such that $f_n\to f$ in $L^{m(\cdot)}(\Omega)$ and $|f_n|\le|f|$. We consider the approximate problem

$$\sum_{i=1}^{N}\int_\Omega |D^iu_n|^{p_i(x)-2}D^iu_nD^iv\,dx+\sigma_0\int_\Omega T_n(u_n)v\,dx=\sum_{i=1}^{N}\int_\Omega h_i^n(x,u_n,\nabla u_n)v\,dx+\int_\Omega fv\,dx, \tag{2.2.2}$$

$\forall v\in W_0^{1,\overrightarrow{p}(\cdot)}(\Omega)\cap L^\infty(\Omega)$,

where

$$h_i^n(x,s,\xi)=\frac{h_i(x,s,\xi)}{1+\frac{1}{n}|h_i(x,s,\xi)|},\qquad \forall i=1,\dots,N.$$

Note that

$$|h_i^n(x,s,\xi)|\le|h_i(x,s,\xi)|\quad and\quad |h_i^n(x,s,\xi)|\le n,\qquad \forall n\in I\!N^*. \tag{2.2.3}$$

We define the operator $H_n: W_0^{1,\overrightarrow{p}(\cdot)}(\Omega)\to W^{-1,\overrightarrow{p'}(\cdot)}(\Omega)$ by

$$\langle H_nu,v\rangle=\sigma_0\int_\Omega T_n(u)v\,dx-\sum_{i=1}^{N}\int_\Omega h_i^n(x,u,\nabla u)v\,dx\quad\text{for any}\quad v\in W_0^{1,\overrightarrow{p}(\cdot)}(\Omega).$$

Thanks to the generalized Hölder type inequality, we have for all $u,v\in W_0^{1,\overrightarrow{p}(\cdot)}(\Omega)$

$$\begin{aligned}
&\left|\sigma_0\int_\Omega T_n(u)v\,dx-\sum_{i=1}^{N}\int_\Omega h_i^n(x,u,\nabla u)v\,dx\right|\\
&\le\sigma_0n\big(\frac{1}{p_0^-}+\frac{1}{(p_0')^-}\big)\|1\|_{p_0'(x)}\|v\|_{p_0(x)}+\sum_{i=1}^{N}\big(\frac{1}{p_0^-}+\frac{1}{(p_0')^-}\big)\|h_i^n(x,u,\nabla u)\|_{p_0'(x)}\|v\|_{p_0(x)}\\
&\le\big(\frac{1}{p_0^-}+\frac{1}{(p_0')^-}\big)\left(\sigma_0n\big(meas(\Omega)+1\big)^{\frac{1}{(p_0')^-}}+N\big(n^{(p_0')^+}meas(\Omega)+1\big)^{\frac{1}{(p_0')^-}}\right)\|v\|_{1,\overrightarrow{p}(x)}\\
&\le C_0\|v\|_{1,\overrightarrow{p}(x)}\,.
\end{aligned}\tag{2.2.4}$$

Furthermore, we define the operator $A_n: W_0^{1,\overrightarrow{p}(\cdot)}(\Omega)\longrightarrow W^{-1,\overrightarrow{p'}(\cdot)}(\Omega)$ by

$$\langle A_nu,v\rangle=\sum_{i=1}^{N}\int_\Omega|D^iu|^{p_i(x)-2}D^iuD_iv\,dx.$$

Using the Hölder inequality, we have for all $u, v \in W_0^{1,\overrightarrow{p}(\cdot)}(\Omega)$

$$\begin{aligned}\big|\langle Au_n, v\rangle\big| &\leq \sum_{i=1}^{N}\big(\frac{1}{p_i^-}+\frac{1}{(p_i')^-}\big)\||D^i u|^{p_i(x)-1}\|_{p_i'(x)}\|D^i v\|_{p_i(x)} \\ &\leq \sum_{i=1}^{N}\big(\frac{1}{p_i^-}+\frac{1}{(p_i')^-}\big)\bigg(\int_\Omega |D^i u|^{p_i(x)}\,dx+1\bigg)^{\frac{1}{(p_i')^-}}\|v\|_{1,\overrightarrow{p}(x)} \\ &\leq C_1\|v\|_{1,\overrightarrow{p}(x)}\,.\end{aligned} \tag{2.2.5}$$

Lemma 2.2.1. *The bounded operator $B_n = A_n + H_n$ acting from $W_0^{1,\overrightarrow{p}(\cdot)}(\Omega)$ into $W^{-1,\overrightarrow{p'}(\cdot)}(\Omega)$ is pseudo-monotone. Moreover, B_n is coercive in the following sense:*

$$\frac{\langle B_n v, v\rangle}{\|v\|_{1,\overrightarrow{p}(\cdot)}} \longrightarrow +\infty \qquad \textit{as} \quad \|v\|_{1,\overrightarrow{p}(\cdot)} \to \infty \quad \textit{for} \quad v \in W_0^{1,\overrightarrow{p}(\cdot)}(\Omega). \tag{2.2.6}$$

Proof of the Lemma 2.2.1

In view of $(2.2.4)$ and $(2.2.5)$ the operator B_n is bounded. For the coercivity, we have for any $u \in W_0^{1,\overrightarrow{p}(\cdot)}(\Omega)$,

$$\begin{aligned}\langle B_n u, u\rangle &= \langle A_n u, u\rangle + \langle H_n u, u\rangle \\ &= \sum_{i=1}^{N}\int_\Omega |D^i u|^{p_i(x)}\,dx + \sigma_0\int_\Omega T_n(u)u\,dx - \sum_{i=1}^{N}\int_\Omega h_i^n(x,u,\nabla u)u\,dx \\ &\geq \sum_{i=1}^{N}\int_\Omega |D^i u|^{p_i(x)}\,dx - C_0\|v\|_{1,\overrightarrow{p}(x)} \\ &\geq \|u\|^{\underline{p}}_{1,\overrightarrow{p}(x)} - N - C_0\|u\|_{1,\overrightarrow{p}(x)}.\end{aligned}$$

Then, the operator B_n is coercive with respect to $(2.2.6)$.

It remains to show that B_n is pseudo-monotone. Let $(u_k)_{k\in I\!N}$ be a sequence in $W_0^{1,\overrightarrow{p}(\cdot)}(\Omega)$ such that

$$\begin{cases} u_k \rightharpoonup u & \text{in } \ W_0^{1,\overrightarrow{p}(\cdot)}(\Omega), \\ B_n u_k \rightharpoonup \chi_n & \text{in } \ W^{-1,\overrightarrow{p'}(\cdot)}(\Omega), \\ \limsup\limits_{k\to\infty}\langle B_n u_k, u_k\rangle \leq \langle \chi_n, u\rangle. \end{cases} \tag{2.2.7}$$

We will prove that

$$\chi_n = B_n u \quad \text{ and } \quad \langle B_n u_k, u_k\rangle \longrightarrow \langle \chi_n, u\rangle \ \text{ as } \ k \to +\infty.$$

In view of the compact embedding $W_0^{1,\overrightarrow{p}(\cdot)}(\Omega) \hookrightarrow\hookrightarrow L^{\underline{p}}(\Omega)$, we have $u_k \to u$ in $L^{\underline{p}}(\Omega)$ for a subsequence still denoted $(u_k)_{k\in I\!N}$.

As $(u_k)_{k\in I\!N}$ is a bounded sequence in $W_0^{1,\overrightarrow{p}(\cdot)}(\Omega)$, then by the growth condition $(2.1.3)$ it's clear that the

sequence $(h_i^n(x,u_k,\nabla u_k))_{k\in I\!N}$ is bounded in $L^{\underline{p}'}(\Omega)$, with $\underline{p}'$ is the conjugate exponent of $\underline{p}$, then there exists a function $\psi_i \in L^{\underline{p}'}(\Omega)$ such that

$$h_i^n(x,u_k,\nabla u_k) \rightharpoonup \psi_i \ \text{ in } \quad L^{\underline{p}'}(\Omega) \ \text{ as } \ k\to\infty. \tag{2.2.8}$$

We have

$$|D^i u_k|^{p_i(x)-2}D^i u_k \rightharpoonup |D^i u|^{p_i(x)-2}D^i u \quad \text{in} \quad L^{p_i'(\cdot)}(\Omega) \ \text{ as } \ k\to\infty. \tag{2.2.9}$$

and

$$T_n(u_k)\to T_n(u) \quad \text{in} \quad L^{p_0'(\cdot)}(\Omega) \ \text{ as } \ k\to\infty. \tag{2.2.10}$$

For any $\upsilon \in W_0^{1,\overrightarrow{p}(\cdot)}(\Omega)$, we have

$$\begin{aligned}
\langle \chi_n,\upsilon\rangle &= \lim_{k\to\infty}\langle B_n u_k,\upsilon\rangle \\
&= \lim_{k\to\infty}\sum_{i=1}^N\int_\Omega |D^i u_k|^{p_i(x)-2}D^i u_k D_i\upsilon\,dx + \lim_{k\to\infty}\sigma_0\int_\Omega T_n(u_k)\upsilon\,dx \\
&\quad - \lim_{k\to\infty}\sum_{i=1}^N\int_\Omega h_i^n(x,u_k,\nabla u_k)\upsilon\,dx \\
&= \sum_{i=1}^N\int_\Omega |D^i u|^{p_i(x)-2}D^i u D_i\upsilon\,dx + \sigma_0\int_\Omega T_n(u)\upsilon\,dx - \sum_{i=1}^N\int_\Omega \psi_i\upsilon\,dx.
\end{aligned} \tag{2.2.11}$$

From relations (2.2.7) and (2.2.11), we obtain

$$\begin{aligned}
\limsup_{k\to\infty}\langle B_n u_k,u_k\rangle &= \limsup_{k\to\infty}\Big\{\sum_{i=1}^N\int_\Omega |D^i u_k|^{p_i(x)}\,dx + \sigma_0\int_\Omega T_n(u_k)u_k\,dx \\
&\quad - \sum_{i=1}^N\int_\Omega h_i^n(x,u_k,\nabla u_k)u_k\,dx\Big\} \\
&\le \sum_{i=1}^N\int_\Omega |D^i u|^{p_i(x)}\,dx + \sigma_0\int_\Omega T_n(u)u\,dx - \sum_{i=1}^N\int_\Omega \psi_i u\,dx.
\end{aligned} \tag{2.2.12}$$

Thanks to $(2.2.8)$ and $(2.2.10)$,we get

$$\sum_{i=1}^N\int_\Omega h_i^n(x,u_k,\nabla u_k)u_k\,dx \longrightarrow \sum_{i=1}^N\int_\Omega \psi_i u\,dx, \tag{2.2.13}$$

and

$$\int_\Omega T_n(u_k)u_k\,dx \longrightarrow \int_\Omega T_n(u)u\,dx. \tag{2.2.14}$$

Therefore

$$\limsup_{k\to\infty}\sum_{i=1}^N\int_\Omega |D^i u_k|^{p_i(x)}\,dx \le \sum_{i=1}^N\int_\Omega |D^i u|^{p_i(x)}\,dx. \tag{2.2.15}$$

On the other hand, we have

$$\sum_{i=1}^{N}\int_{\Omega}\big(|D^iu_k|^{p_i(x)-2}D^iu_k-|D^iu|^{p_i(x)-2}D^iu\big)(D^iu_k-D^iu)\,dx\geq 0, \tag{2.2.16}$$

then

$$\sum_{i=1}^{N}\int_{\Omega}|D^iu_k|^{p_i(x)}\,dx\geq\sum_{i=1}^{N}\int_{\Omega}|D^iu_k|^{p_i(x)-2}D^iu_kD^iu+\sum_{i=1}^{N}\int_{\Omega}|D^iu|^{p_i(x)-2}D^iu(D^iu_k-D^iu)\,dx.$$

Using $(2.2.9)$,we have

$$\liminf_{k\to\infty}\sum_{i=1}^{N}\int_{\Omega}|D^iu_k|^{p_i(x)}\,dx\geq\sum_{i=1}^{N}\int_{\Omega}|D^iu|^{p_i(x)}\,dx.$$

This implies by using $(2.2.15)$ that

$$\lim_{k\to\infty}\sum_{i=1}^{N}\int_{\Omega}|D^iu_k|^{p_i(x)}\,dx=\sum_{i=1}^{N}\int_{\Omega}|D^iu|^{p_i(x)}\,dx. \tag{2.2.17}$$

According to $(2.2.11)$, $(2.2.13)-(2.2.14)$, and $(2.2.17)$,we obtain

$$\langle B_nu_k,u_k\rangle\longrightarrow\langle\chi_n,u\rangle\quad\text{ as }\quad k\to\infty.$$

Now, by (2.2.17) we can prove that

$$\lim_{k\to+\infty}\sum_{i=1}^{N}\int_{\Omega}\big(|D^iu_k|^{p_i(x)-2}D^iu_k-|D^iu|^{p_i(x)-2}D^iu\big)(D^iu_k-D^iu)\,dx\to 0.$$

and so, by virtue of Lemma $1.2.7$, we get

$$u_k\to u\quad\text{in}\quad W_0^{1,\overrightarrow{p}(\cdot)}(\Omega)\qquad\text{and}\qquad D^iu_k\to D^iu\quad\text{a.e. in}\quad\Omega,$$

then

$$h_i^n(x,u_k,\nabla u_k)\rightharpoonup h_i^n(x,u,\nabla u)\quad\text{in}\quad L^{p_0'(\cdot)}(\Omega).$$

Having in mind $(2.2.9)$ and $(2.2.10)$, we can conclude that $\chi_n=B_nu$.

In view of Lemma $2.2.1$, there exists at least one weak solution $u_n\in W_0^{1,\overrightarrow{p}(\cdot)}(\Omega)$ of the problem $(2.2.2)$ (cf. [121], Theorem 8.2).

Concerning the boundedness of u_n, bearing in mind $(1.1.2)$, it's sufficient to prove the existence and boundedness of u_n for the following problem:

Proposition 2.2.1. *Let $f \in M^{m(x)}(\Omega)$ with $m(x) > p'_\infty$ for almost every $x \in \Omega$.*
If $m(x) > \frac{p^*_-}{p^*_- - p_i^+} > \frac{N}{\overline{p}}$ for a.e. $x \in \Omega$, then there exists a bounded weak solution u for the following problem:

$$\begin{cases} -\sum_{i=1}^{N} \int_\Omega D^i[|D^i u|^{p_i(x)-2} D^i u]\, dx = f & \text{in} \quad \Omega, \\ u = 0 & \text{on} \quad \partial\Omega, \end{cases} \tag{2.2.18}$$

Proof of proposition 2.2.1

We will exploit the techniques of [82]. Thanks to the proposition $1.1.3$ and the above, there exists a weak solution to $(2.2.18)$. Now, by using as a test function $v = G_k(u_n)$ in the approximate problem of $(2.2.18)$, where, for any $k > 0$, $G_k(.)$ is defined by $G_k(s) := s - T_k(s)$, we have:

$$\sum_{i=1}^{N} \int_\Omega |D^i u_n|^{p_i(x)-2} D^i u_n D^i G_k(u_n)\, dx = \int_\Omega f G_k(u_n)\, dx.$$

Hence

$$\Big(\int_{\{|u_n|>k\}} |D^i G_k(u_n)|^{p_i(x)}\, dx \Big)^{\frac{1}{p_i^- N}} \le \Big(\int_{\{|u_n|>k\}} f G_k(u_n)\, dx \Big)^{\frac{1}{p_i^- N}}.$$

Let's recall the following Sobolev-type inequality ([149] or [82]): There exists a positive constant C_0 depending on the set Ω, such that

$$\|v\|_{L^s(\Omega)} \le C_0 \prod_{i=1}^{N} \|D^i v\|^{\frac{1}{N}}_{L^{p_i}(\Omega)}, \qquad \forall\ s \in [1, p^*_-] \tag{2.2.19}$$

for any $v \in C_0^1(\Omega)$, where $p_i \ge 1$, for $i = 1, 2, \ldots, N$. By taking $s = p^*_-$ in $(2.2.19)$, we obtain

$$\begin{aligned} \Big(\int_{\{|u_n|>k\}} |G_k(u_n)|^{p^*_-}\, dx \Big)^{\frac{1}{p^*_-}} &\le C_1 \prod_{i=1}^{N} \Big(\int_{\{|u_n|>k\}} |D^i G_k(u_n)|^{p_i^-}\, dx \Big)^{\frac{1}{p_i^- N}} \\ &\le C_1 \prod_{i=1}^{N} \Big(\int_{\{|u_n|>k\}} f G_k(u_n)\, dx \Big)^{\frac{1}{p_i^- N}} \\ &\le C_1 \Big(\int_{\{|u_n|>k\}} f G_k(u_n)\, dx \Big)^{\frac{1}{\overline{p}}}. \end{aligned}$$

Seeing that $f \in L^{(p^*_-)'}(\Omega)$ due to the proposition 2.3, the generalized Hölder type inequality allows us to have

$$\int_{\{|u_n|>k\}} |f G_k(u_n)|\, dx \le \big(\frac{1}{(p^*_-)^-} + \frac{1}{((p^*_-)')^-}\big) \Big(\int_{\{|u_n|>k\}} |f|^{(p^*_-)'}\, dx \Big)^{\frac{1}{(p^*_-)'}} \Big(\int_{\{|u_n|>k\}} |G_k(u_n)|^{p^*_-}\, dx \Big)^{\frac{1}{p^*_-}}$$

then

$$\Big(\int_{\{|u_n|>k\}} |G_k(u_n)|^{p^*_-}\, dx \Big)^{\frac{1}{p^*_-}\left(1-\frac{1}{\overline{p}}\right)} \le C_2 \Big(\int_{\{|u_n|>k\}} |f|^{(p^*_-)'}\, dx \Big)^{\frac{1}{\overline{p}}\frac{1}{(p^*_-)'}}$$

with $\overline{p} > 1$, due to the fact that $p_i(x) > 1$ for every $i = 1, 2, \ldots, N$. Taking into consideration $f \in M^{m(x)}(\Omega)$ and $m(x) > (p_-^*)'$ and thanks to the generalized Hölder inequality, we get

$$\int_{\{|u_n|>k\}} |f|^{(p_-^*)'}\, dx \leq C_3\big(meas\{|u_n| > k\}\big)^{1-\frac{(p_-^*)'}{m^+}}$$

By applying the generalized Hölder inequality again, it yields

$$\int_{\{|u_n|>k\}} |G_k(u_n)|\, dx \leq C_4\big(meas\{|u_n| > k\}\big)^{\frac{1}{(p_-^*)'(\overline{p}-1)}\left(1-\frac{(p_-^*)'}{m^+}\right)+\frac{1}{(p_-^*)'}}. \tag{2.2.20}$$

If we denote $g(k) = \int_{\{|u_n|>k\}} |G_k(u_n)|\, dx$, we have meas$\{|u_n| > k\} = -g'(k)$, for almost every k (see [112],[120]), which implies from $(2.2.20)$, that

$$-C_5 g'(k) \geq g(k)^{\frac{1}{\lambda}}$$

with $\lambda = \frac{1}{(p_-^*)'} + \frac{1}{(p_-^*)'(\overline{p}-1)}\big(1 - \frac{(p_-^*)'}{m^+}\big)$. Hence, it yields

$$\frac{C_5}{1-\frac{1}{\lambda}}\big(g(k)^{1-\frac{1}{\lambda}}\big)' = C_5 g'(k) g(k)^{-\frac{1}{\lambda}} \leq -1. \tag{2.2.21}$$

Taking into account $m(x) > \frac{p_-^*}{p_-^* - p_i^+} > \frac{N}{\overline{p}}$, we have $1 - \frac{1}{\lambda} > 0$. By integrating $(2.2.21)$, from 0 to k we obtain

$$C_6\big(g(k)^{1-\frac{1}{\lambda}} - g(0)^{1-\frac{1}{\lambda}}\big) \leq -k$$

then

$$C_6 \|u_n\|_{L^1(\Omega)}^{1-\frac{1}{\lambda}} - k \geq C_6 g(k)^{1-\frac{1}{\lambda}}$$

Knowing that $g(k)$ is non-negative and decreasing function, we conclude that there exists k_0, such that $g(k_0) = 0$, and consequently $u_n \in L^\infty(\Omega)$.

Step 2 : A priori estimates.

Lemma 2.2.2. *There exist $\mu > 1$ and a constant $C > 0$, not depending on n, such that*

$$\sum_{i=1}^{N} \int_\Omega |D^i u_n|^{p_i(x)} e^{\mu p_i^+ |u_n|}\, dx \leq C. \tag{2.2.22}$$

Proof of the Lemma 2.2.2

Let's consider $\mu > 1$ a real number which will be chosen later, and the function $\upsilon : I\!R \mapsto I\!R$ defined by

$$\upsilon(t) = \big(e^{\mu p_i^+ |t|} - 1\big) sgn(t).$$

It's clear that $\upsilon(u_n) \in W_0^{1,\overrightarrow{p}(\cdot)}(\Omega) \cap L^\infty(\Omega)$. By taking $\upsilon(u_n)$ as test function in $(2.2.2)$, and thanks to $(2.1.3)$ and $(2.2.3)$, we get

$$\begin{aligned}\mu p_i^+ \sum_{i=1}^N \int_\Omega |D^i u_n|^{p_i(x)} e^{\mu p_i^+ |u_n|}\, dx + \sigma_0 \sum_{i=1}^N \int_\Omega |T_n(u_n)| \big(e^{\mu p_i^+ |u_n|} - 1\big)\, dx \\ \leq \beta \sum_{i=1}^N \int_\Omega |D^i u_n|^{p_i(x)} e^{\mu p_i^+ |u_n|}\, dx + \sum_{i=1}^N \int_\Omega |f| \big(e^{\mu p_i^+ |u_n|} - 1\big)\, dx,\end{aligned}$$

which implies

$$(\mu p_i^+ - \beta) \sum_{i=1}^N \int_\Omega |D^i u_n|^{p_i(x)} e^{\mu p_i^+ |u_n|} dx + \sigma_0 \sum_{i=1}^N \int_\Omega |T_n(u_n)| \big(e^{\mu p_i^+ |u_n|} - 1\big) dx \leq \sum_{i=1}^N \int_\Omega |f| \big(e^{\mu p_i^+ |u_n|} - 1\big) dx.$$

We recall that

$$\begin{cases} (e^{s\delta} - 1) \geq (e^\delta - 1)^s \quad \text{for any} \quad s > 1, \\ \text{for any } \eta > 1 \text{ there exist } \delta^* \text{ such that } \quad (e^{s\delta} - 1) \leq \eta (e^\delta - 1)^s \quad \text{for} \quad \delta > \delta^*. \end{cases} \tag{2.2.23}$$

Thus, we obtain

$$\begin{aligned}(\mu p_i^+ - \beta) \sum_{i=1}^N \int_\Omega |D^i u_n|^{p_i(x)} e^{\mu p_i^+ |u_n|}\, dx + \sigma_0 \sum_{i=1}^N \int_\Omega |T_n(u_n)| \big(e^{\mu |u_n|} - 1\big)^{p_i^+}\, dx \\ \leq \sum_{i=1}^N \int_{\{|u_n| > \frac{\delta^*}{\mu}\}} |f| \big(e^{\mu p_i^+ |u_n|} - 1\big)\, dx + \sum_{i=1}^N \int_{\{|u_n| \leq \frac{\delta^*}{\mu}\}} |f| \big(e^{\mu p_i^+ |u_n|} - 1\big)\, dx \\ \leq \eta \sum_{i=1}^N \int_\Omega |f| \big(e^{\mu |u_n|} - 1\big)^{p_i^+}\, dx + C_0 \int_\Omega |f|\, dx.\end{aligned}$$

In view of the generalized Hölder's type inequality, we get

$$\begin{aligned}(\mu p_i^+ - \beta) \sum_{i=1}^N \int_\Omega |D^i u_n|^{p_i(x)} e^{\mu p_i^+ |u_n|}\, dx + \sigma_0 \sum_{i=1}^N \int_\Omega |T_n(u_n)| \big(e^{\mu |u_n|} - 1\big)^{p_i^+}\, dx \\ \leq \eta \sum_{i=1}^N \big(\frac{1}{m^-} + \frac{1}{(m')^-}\big) \|f\|_{L^{m(x)}(\Omega)} \|(e^{\mu |u_n|} - 1)^{p_i^+}\|_{L^{m'(x)}(\Omega)} + C_0 \|f\|_{L^1(\Omega)}.\end{aligned}$$

Since $m(x) > \frac{p_-^*}{p_-^* - p_i^+}$ with $p_-^* > p_i^+$ we have $p_i^+ < p_i^+ m'(x) < p_-^*$, then thanks to the interpolation inequality, it yields

$$\begin{aligned}(\mu p_i^+ - \beta) \sum_{i=1}^N \int_\Omega |D^i u_n|^{p_i(x)} e^{\mu p_i^+ |u_n|}\, dx + \sigma_0 \sum_{i=1}^N \int_\Omega |T_n(u_n)| \big(e^{\mu |u_n|} - 1\big)^{p_i^+}\, dx \\ \leq \eta \sum_{i=1}^N \big(\frac{1}{m^-} + \frac{1}{(m')^-}\big) \|f\|_{L^{m(x)}(\Omega)} \|e^{\mu |u_n|} - 1\|_{L^{p_i^+}(\Omega)}^{p_i^+(1-\theta)} \|e^{\mu |u_n|} - 1\|_{L^{p_-^*}(\Omega)}^{\theta p_i^+} + C_0 \|f\|_{L^1(\Omega)}.\end{aligned}$$

By using the Young type inequality with exponents $\frac{1}{\theta}$ and $\frac{1}{1-\theta}$, we obtain

$$(\mu p_i^+ - \beta)\sum_{i=1}^{N}\int_\Omega |D^i u_n|^{p_i(x)} e^{\mu p_i^+|u_n|}\,dx + \sigma_0\sum_{i=1}^{N}\int_\Omega |T_n(u_n)|\big(e^{\mu|u_n|}-1\big)^{p_i^+}\,dx$$
$$\leq 2\epsilon\big(\eta\|f\|_{L^{m(x)}(\Omega)}\big)^{\frac{1}{\theta}}\sum_{i=1}^{N}\|e^{\mu|u_n|}-1\|^{p_i^+}_{L^{p_-^*}(\Omega)} + C_\epsilon\sum_{i=1}^{N}\|e^{\mu|u_n|}-1\|^{p_i^+}_{L^{p_i^+}(\Omega)} + C_0\|f\|_{L^1(\Omega)}.$$

In view of Lemma 1.2.6, we obtain

$$(\mu p_i^+ - \beta)\sum_{i=1}^{N}\int_\Omega |D^i u_n|^{p_i(x)} e^{\mu p_i^+|u_n|}\,dx + \sigma_0\sum_{i=1}^{N}\int_\Omega |T_n(u_n)|\big(e^{\mu|u_n|}-1\big)^{p_i^+}\,dx$$
$$\leq 2\epsilon\big(\eta\|f\|_{L^{m(x)}(\Omega)}\big)^{\frac{1}{\theta}}\sum_{i=1}^{N}C^{p_i^+}N^{p_i^+-1}E^{p_i^+-p_i^-}\int_\Omega |D^i(e^{\mu|u_n|}-1)|^{p_i(x)}\,dx$$
$$+C_\epsilon\sum_{i=1}^{N}\int_\Omega (e^{\mu|u_n|}-1)^{p_i^+}\,dx + C_0\|f\|_{L^1(\Omega)}.$$
$$= 2\epsilon\big(\eta\|f\|_{L^{m(x)}(\Omega)}\big)^{\frac{1}{\theta}}\sum_{i=1}^{N}C^{p_i^+}N^{p_i^+-1}E^{p_i^+-p_i^-}\int_\Omega \mu^{p_i(x)}|D^i u_n|^{p_i(x)} e^{\mu p_i(x)|u_n|}\,dx$$
$$+C_\epsilon\sum_{i=1}^{N}\int_\Omega (e^{\mu|u_n|}-1)^{p_i^+}\,dx + C_0\|f\|_{L^1(\Omega)}.$$

Since $e^\delta > 1$ for all $\delta > 0$, and by choosing $\mu > 1$ such that $\mu p_i^+ - \beta > 0$, we have

$$(\mu p_i^+ - \beta)\sum_{i=1}^{N}\int_\Omega |D^i u_n|^{p_i(x)} e^{\mu p_i^+|u_n|}\,dx + \sigma_0\sum_{i=1}^{N}\int_\Omega |T_n(u_n)|\big(e^{\mu|u_n|}-1\big)^{p_i^+}\,dx$$
$$\leq 2\epsilon\bigg(\eta\Big(\int_\Omega |f(x)|^{m(x)}\,dx+1\Big)^{\frac{1}{m^-}}\bigg)^{\frac{1}{\theta}}\sum_{i=1}^{N}C^{p_i^+}N^{p_i^+-1}E^{p_i^+-p_i^-}\mu^{p_i^+}\int_\Omega |D^i u_n|^{p_i(x)} e^{\mu p_i^+|u_n|}\,dx$$
$$+C_\epsilon\sum_{i=1}^{N}\int_\Omega (e^{\mu|u_n|}-1)^{p_i^+}\,dx + C_0\|f\|_{L^1(\Omega)},$$

which implies that

$$[(\mu p_i^+ - \beta) - 2\epsilon C_2]\sum_{i=1}^{N}\int_\Omega |D^i u_n|^{p_i(x)} e^{\mu p_i^+|u_n|}\,dx + \sigma_0\sum_{i=1}^{N}\int_\Omega |T_n(u_n)|\big(e^{\mu|u_n|}-1\big)^{p_i^+}\,dx$$
$$\leq C_\epsilon\sum_{i=1}^{N}\int_\Omega (e^{\mu|u_n|}-1)^{p_i^+}\,dx + C_0\|f\|_{L^1(\Omega)},$$

with $C_2 = \bigg(\eta\Big(\int_\Omega |f(x)|^{m(x)}\,dx+1\Big)^{\frac{1}{m^-}}\bigg)^{\frac{1}{\theta}}C^{p_i^+}N^{p_i^+-1}E^{p_i^+-p_i^-}\mu^{p_i^+}$.
By taking ϵ such that $(\mu p_i^+ - \beta) - 2\epsilon C_2 > 0$, and since

$$\sigma_0\sum_{i=1}^{N}\int_\Omega |T_n(u_n)|\big(e^{\mu|u_n|}-1\big)^{p_i^+}\,dx \geq C_\epsilon\sum_{i=1}^{N}\int_{\{|T_n(u_n)|>\frac{C_\epsilon}{\sigma_0}\}}\big(e^{\mu|u_n|}-1\big)^{p_i^+}\,dx,$$

and

$$C_\epsilon \sum_{i=1}^{N} \int_\Omega (e^{\mu|u_n|}-1)^{p_i^+}\,dx = C_\epsilon \sum_{i=1}^{N} \int_{\{|T_n(u_n)|>\frac{C_\epsilon}{\sigma_0}\}} (e^{\mu|u_n|}-1)^{p_i^+}\,dx + C_\epsilon \sum_{i=1}^{N} \int_{\{|T_n(u_n)|\le\frac{C_\epsilon}{\sigma_0}\}} (e^{\mu|u_n|}-1)^{p_i^+}\,dx$$

we have

$$\begin{aligned}
&[(\mu p_i^+ - \beta) - 2\epsilon C_2] \sum_{i=1}^{N} \int_\Omega |D^i u_n|^{p_i(x)} e^{\mu p_i^+|u_n|}\,dx \\
&\le C_\epsilon \sum_{i=1}^{N} \int_{\{|T_n(u_n)|\le\frac{C_\epsilon}{\sigma_0}\}} (e^{\mu|u_n|}-1)^{p_i^+}\,dx + C_0\|f\|_{L^1(\Omega)} \\
&\le C_\epsilon \sum_{i=1}^{N} \int_\Omega (e^{\mu\frac{C_\epsilon}{\sigma_0}}-1)^{p_i^+}\,dx + C_0\|f\|_{L^1(\Omega)}.
\end{aligned}$$

Finally, we get $(2.2.22)$. Thanks to this lemma, we conclude that

$$\sum_{i=1}^{N} \int_\Omega |D^i u_n|^{p_i(x)}\,dx \le C. \tag{2.2.24}$$

Lemma 2.2.3. *There exists a constant $C>0$, not depending on n, such that*

$$\|u_n\|_{L^\infty(\Omega)} \le C. \tag{2.2.25}$$

Proof of the Lemma 2.2.3

Let's consider $\mu>1$ a real number which will be chosen later, and the function $\upsilon : I\!R \mapsto I\!R$ defined by

$$\upsilon(t) = \big(e^{\mu p_i^+|G_k(t)|}-1\big)sgn(t).$$

We proceed by taking $\upsilon(u_n)$ as test function in $(2.2.2\,)$, then we have

$$\begin{aligned}
&\mu p_i^+ \sum_{i=1}^{N} \int_\Omega |D^i u_n|^{p_i(x)-2} D^i u_n D^i G_k(u_n) e^{\mu p_i^+|G_k(u_n)|}\,dx + \sigma_0 \sum_{i=1}^{N} \int_\Omega |T_n(u_n)|\big(e^{\mu p_i^+|G_k(u_n)|}-1\big)\,dx \\
&\le \beta \sum_{i=1}^{N} \int_\Omega |D^i u_n|^{p_i(x)}\big(e^{\mu p_i^+|G_k(u_n)|}-1\big)\,dx + \sum_{i=1}^{N} \int_\Omega |f|\big(e^{\mu p_i^+|G_k(u_n)|}-1\big)\,dx.
\end{aligned}$$

Hence

$$\begin{aligned}
&\big(\mu p_i^+ - \beta\big) \sum_{i=1}^{N} \int_\Omega |D^i G_k(u_n)|^{p_i(x)} e^{\mu p_i^+|G_k(u_n)|}\,dx + \sigma_0 \sum_{i=1}^{N} \int_\Omega |T_n(u_n)|\big(e^{\mu p_i^+|G_k(u_n)|}-1\big)\,dx \\
&\le \sum_{i=1}^{N} \int_\Omega |f|\big(e^{\mu p_i^+|G_k(u_n)|}-1\big)\,dx.
\end{aligned}$$

Thanks to $(2.2.23)$, we have

$$\begin{aligned}
&\big(\mu p_i^+ - \beta\big)\sum_{i=1}^{N}\int_\Omega |D^i G_k(u_n)|^{p_i(x)} e^{\mu p_i^+ |G_k(u_n)|}\,dx + \sigma_0\sum_{i=1}^{N}\int_\Omega |T_n(u_n)|\big(e^{\mu|G_k(u_n)|}-1\big)^{p_i^+}\,dx\\
&\quad\le \sum_{i=1}^{N}\int_{\{|G_k(u_n)|>\frac{\delta^*}{\mu}\}\cap\{|u_n|>k\}} |f|\big(e^{\mu p_i^+|G_k(u_n)|}-1\big)\,dx\\
&\qquad + \sum_{i=1}^{N}\int_{\{|G_k(u_n)|<\frac{\delta^*}{\mu}\}\cap\{|u_n|>k\}} |f|\big(e^{\mu p_i^+|G_k(u_n)|}-1\big)\,dx\\
&\quad\le \eta\sum_{i=1}^{N}\int_\Omega |f|\big(e^{\mu|G_k(u_n)|}-1\big)^{p_i^+}\,dx + C_0\int_{\{|u_n|>k\}}|f|\,dx.
\end{aligned}$$

By the generalized Hölder's type inequality, we obtain

$$\begin{aligned}
&\big(\mu p_i^+ - \beta\big)\sum_{i=1}^{N}\int_\Omega |D^i G_k(u_n)|^{p_i(x)} e^{\mu p_i^+ |G_k(u_n)|}\,dx + \sigma_0\sum_{i=1}^{N}\int_\Omega |T_n(u_n)|\big(e^{\mu|G_k(u_n)|}-1\big)^{p_i^+}\,dx\\
&\quad\le \eta\sum_{i=1}^{N}\big(\frac{1}{m^-}+\frac{1}{(m')^-}\big)\|f\|_{L^{m(x)}(\Omega)}\|(e^{\mu|G_k(u_n)|}-1)^{p_i^+}\|_{L^{m'(x)}(\Omega)} + C_0\int_{\{|u_n|>k\}}|f|\,dx.
\end{aligned}$$

Seeing that $p_i^+ < p_i^+ m'(x) < p_-^*$, the interpolation inequality allows us to get

$$\begin{aligned}
&\big(\mu p_i^+ - \beta\big)\sum_{i=1}^{N}\int_\Omega |D^i G_k(u_n)|^{p_i(x)} e^{\mu p_i^+ |G_k(u_n)|}\,dx + \sigma_0\sum_{i=1}^{N}\int_\Omega |T_n(u_n)|\big(e^{\mu|G_k(u_n)|}-1\big)^{p_i^+}\,dx\\
&\quad\le \eta\sum_{i=1}^{N}\big(\frac{1}{m^-}+\frac{1}{(m')^-}\big)\|f\|_{L^{m(x)}(\Omega)}\|e^{\mu|G_k(u_n)|}-1\|_{L^{p_i^+}(\Omega)}^{p_i^+(1-\theta)}\|e^{\mu|G_k(u_n)|}-1\|_{L^{p_-^*}(\Omega)}^{\theta p_i^+}\\
&\qquad + C_0\int_{\{|u_n|>k\}}|f|\,dx.
\end{aligned}$$

By using the Young type inequality, we obtain

$$\begin{aligned}
&\big(\mu p_i^+ - \beta\big)\sum_{i=1}^{N}\int_\Omega |D^i G_k(u_n)|^{p_i(x)} e^{\mu p_i^+ |G_k(u_n)|}\,dx + \sigma_0\sum_{i=1}^{N}\int_\Omega |T_n(u_n)|\big(e^{\mu|G_k(u_n)|}-1\big)^{p_i^+}\,dx\\
&\quad\le 2\epsilon\big(\eta\|f\|_{L^{m(x)}(\Omega)}\big)^{\frac{1}{\theta}}\sum_{i=1}^{N}\|e^{\mu|G_k(u_n)|}-1\|_{L^{p_-^*}(\Omega)}^{p_i^+} + C_\epsilon\sum_{i=1}^{N}\|e^{\mu|G_k(u_n)|}-1\|_{L^{p_i^+}(\Omega)}^{p_i^+}\\
&\qquad + C_0\int_{\{|u_n|>k\}}|f|\,dx.
\end{aligned}$$

Inasmuch as $e^t > 1$ for any $t > 0$,

$$\begin{aligned}
\sum_{i=1}^{N}\int_\Omega |D^i G_k(u_n)|^{p_i(x)} e^{\mu p_i^+ |G_k(u_n)|}\,dx &\ge \sum_{i=1}^{N}\int_\Omega |D^i G_k(u_n)|^{p_i(x)} e^{\mu p_i(x) |G_k(u_n)|}\,dx\\
&\ge \sum_{i=1}^{N}\frac{1}{\mu^{p_i(x)}}\int_\Omega |D^i\big(e^{\mu|G_k(u_n)|}-1\big)|^{p_i(x)}\,dx.
\end{aligned}$$

In view of Lemma 1.2.6, we obtain

$$\sum_{i=1}^{N}\frac{1}{\mu^{p_i^+}C^{p_i^+}N^{p_i^+-1}E^{p_i^+-p_i^-}}\|e^{\mu|G_k(u_n)|}-1\|^{p_i^+}_{L^{p_-^*}(\Omega)}\leq\sum_{i=1}^{N}\frac{1}{\mu^{p_i(x)}}\int_\Omega|D^i\big(e^{\mu|G_k(u_n)|}-1\big)|^{p_i(x)}\,dx.$$

By choosing $\mu>1$ such that $\mu p_i^+-\beta>0$, we have

$$\sum_{i=1}^{N}\left[\frac{\mu p_i^+-\beta}{C_2}-2\epsilon\big(\eta\|f\|_{L^{m(x)}(\Omega)}\big)^{\frac{1}{\theta}}\right]\left(\int_\Omega\big(e^{\mu|G_k(u_n)|}-1\big)^{p_-^*}\right)^{\frac{p_i^+}{p_-^*}}$$
$$+(\sigma_0k-C_\epsilon)\sum_{i=1}^{N}\int_\Omega\big(e^{\mu|G_k(u_n)|}-1\big)^{p_i^+}\,dx\ \leq\ C_0\int_{\{|u_n|>k\}}|f|\,dx.$$

with $C_2=\mu^{p_i^+}C^{p_i^+}N^{p_i^+-1}E^{p_i^+-p_i^-}$.

We choose respectively ϵ and k such that $\sigma_0k-C_\epsilon>0$ and $\frac{\mu p_i^+-\beta}{C_2}-2\epsilon\big(\eta\|f\|_{L^{m(x)}(\Omega)}\big)^{\frac{1}{\theta}}>0$, then we have

$$C_3\sum_{i=1}^{N}\left(\int_\Omega\big(e^{\mu|G_k(u_n)|}-1\big)^{p_-^*}\right)^{\frac{p_i^+}{p_-^*}}+C_4\sum_{i=1}^{N}\int_\Omega\big(e^{\mu|G_k(u_n)|}-1\big)^{p_i^+}\,dx\leq C_0\int_{\{|u_n|>k\}}|f|\,dx.$$

Thanks to the generalized Hölder's type inequality and since $e^t-1\geq t$ for all $t\geq0$, we get

$$\mu\int_\Omega|G_k(u_n)|dx\leq\int_\Omega\big(e^{\mu|G_k(u_n)|}-1\big)dx\leq\sum_{i=1}^{N}\big(\frac{1}{(p_-^*)^-}+\frac{1}{((p_-^*)')^-}\big)meas(I_{k,n})^{1-\frac{1}{p_-^*}}\|e^{\mu|G_k(u_n)|}-1\|_{L^{p_-^*}(\Omega)},$$

where $I_{k,n}=\{x\in\Omega:|u_n(x)|>k\}$. Therefore by gathering the previous results we obtain

$$\mu\int_\Omega|G_k(u_n)|\,dx\leq C_5meas(I_{k,n})^{1-\frac{1}{p_-^*}}\sum_{i=1}^{N}\left(\int_{\{|u_n|>k\}}|f|\,dx\right)^{\frac{1}{p_i^+}}.$$

Now the generalized Hölder's type inequality with exponents m(x) and $m'(x)$ allows us to have

$$\mu\int_\Omega|G_k(u_n)|\,dx\leq C_5meas(I_{k,n})^{1-\frac{1}{p_-^*}}\sum_{i=1}^{N}\big(\frac{1}{m^-}+\frac{1}{(m')^-}\big)meas(I_{k,n})^{\frac{1}{p_i^+}-\frac{1}{p_i^+m(x)}}\|f\|^{\frac{1}{p_i^+}}_{L^{m(x)}(\Omega)}$$
$$=C_6\|f\|^{\frac{1}{p_i^+}}_{L^{m(x)}(\Omega)}\sum_{i=1}^{N}meas(I_{k,n})^{\frac{1}{p_i^+}-\frac{1}{p_i^+m(x)}-\frac{1}{p_-^*}+1}.$$

Seeing that

$$\frac{1}{p_i^+}-\frac{1}{p_i^+m(x)}-\frac{1}{p_-^*}+1>1\Leftrightarrow m(x)>\frac{p_-^*}{p_-^*-p_i^+}.$$

and

$$meas(I_{k,n})=-g_n'(k)\qquad\text{with}\qquad g_n(k)=\int_\Omega|G_k(u_n)|\,dx,$$

we get

$$\sum_{i=1}^{N}g_n(k)^{\frac{p_-^*p_i^+m^-}{p_-^*p_i^+m^+-p_i^+m^-+p_-^*m^+-p_-^*}}\leq-C_7g_n'(k)$$

then

$$\sum_{i=1}^{N} \frac{-C_7}{1-\frac{p_-^* p_i^+ m^-}{p_-^* p_i^+ m^+ - p_i^+ m^- + p_-^* m^+ - p_-^*}} \left(g_n(k)^{1-\frac{p_-^* p_i^+ m^-}{p_-^* p_i^+ m^+ - p_i^+ m^- + p_-^* m^+ - p_-^*}} \right)' \geq 1,$$

where $1-\frac{p_-^* p_i^+ m^-}{p_-^* p_i^+ m^+ - p_i^+ m^- + p_-^* m^+ - p_-^*} > 0$. Now the integration of the above inequality from 0 to k enable us to obtain

$$\sum_{i=1}^{N} \frac{-C_7}{1-\frac{p_-^* p_i^+ m^-}{p_-^* p_i^+ m^+ - p_i^+ m^- + p_-^* m^+ - p_-^*}} \times \left[g_n(k)^{1-\frac{p_-^* p_i^+ m^-}{p_-^* p_i^+ m^+ - p_i^+ m^- + p_-^* m^+ - p_-^*}} - \|u_n\|_{L^1(\Omega)}^{1-\frac{p_-^* p_i^+ m^-}{p_-^* p_i^+ m^+ - p_i^+ m^- + p_-^* m^+ - p_-^*}} \right] \geq k.$$

Thanks to $(2.2.24)$ we have the following estimate $\|u_n\|_{L^1(\Omega)} \leq C_8$, with $C_8 > 0$ which does not depend on n, thus we have

$$C_9 g_n(k)^{1-\frac{p_-^* p_i^+ m^-}{p_-^* p_i^+ m^+ - p_i^+ m^- + p_-^* m^+ - p_-^*}} \leq -k + C_{10}.$$

We remark easily that $-k + C_{10} \geq 0$ inasmuch as $g_n(k)$ is a decreasing and positive function. So there exists k_0 such that $g_n(k_0) = 0$. Consequently

$$\|u_n\|_{L^\infty(\Omega)} \leq C,$$

where C does not depend on n.

Step 3 : The strong convergence of the approximate solutions.

Before handling the proof of the sequence of the approximate solution strong convergence in $W_0^{1,\overrightarrow{p}(\cdot)}(\Omega)$, we remark that according to $(2.2.24)$ we have, up to subsequence, that exists $u \in W_0^{1,\overrightarrow{p}(\cdot)}(\Omega)$, such that

$$D^i u_n \rightharpoonup D^i u \quad \text{in} \quad L^{\underline{p}}(\Omega), \tag{2.2.26}$$

$$u_n \to u \quad \text{in} \quad L^{\underline{p}}(\Omega) \tag{2.2.27}$$

and

$$u_n \to u \quad \text{a.e. in} \quad \Omega. \tag{2.2.28}$$

Thanks to $(2.2.25)$, it also exists $\phi \in L^\infty(\Omega)$ such that $u_n \rightharpoonup^* \phi$ in $L^\infty(\Omega)$ and, since meas$(\Omega) < \infty$, $u_n \rightharpoonup \phi$ in $L^{\underline{p}}(\Omega)$. Therefore, by $(2.2.27)$, $\phi = u$ and $u_n \rightharpoonup^* u$, then $T_n(u_n) \rightharpoonup^* u$ in $L^\infty(\Omega)$. Now, let's consider $\mu > 1$ a real number which will be chosen later, and the function $\upsilon : \mathbb{R} \mapsto \mathbb{R}$ defined by

$$\upsilon(t) = \left(e^{\mu p_i^+ |t-u|} - 1\right) sgn(t-u).$$

We proceed by taking $\upsilon(u_n)$ as test function in $(2.2.2)$, then we have

$$\begin{aligned}\mu p_i^+ \sum_{i=1}^{N} \int_\Omega |D^i u_n|^{p_i(x)-2} D^i u_n D^i(u_n-u) e^{\mu p_i^+ |u_n-u|}\, dx + \sigma_0 \int_\Omega T_n(u_n)\upsilon(u_n)\, dx \\ = \sum_{i=1}^{N} \int_\Omega h_i^n(x,u_n,\nabla u_n)\upsilon(u_n)\, dx + \int_\Omega f_n \upsilon(u_n)\, dx,\end{aligned}$$

it yields

$$\begin{aligned}&\mu p_i^+ \sum_{i=1}^{N} \int_\Omega [|D^i u_n|^{p_i(x)-2} D^i u_n - |D^i u|^{p_i(x)-2} D^i u] D^i(u_n-u) e^{\mu p_i^+ |u_n-u|}\, dx \\ &\le -\mu p_i^+ \sum_{i=1}^{N} \int_\Omega |D^i u|^{p_i(x)-2} D^i u D^i(u_n-u) e^{\mu p_i^+ |u_n-u|}\, dx - \sigma_0 \int_\Omega T_n(u_n)\upsilon(u_n)\, dx \\ &\quad + \beta \sum_{i=1}^{N} \int_\Omega |D^i u_n|^{p_i(x)} \upsilon(u_n)\, dx + \int_\Omega |f_n \upsilon(u_n)|\, dx.\end{aligned}$$

Seeing that $p_i^+ > 1$, we observe that

$$2^{p_i^+-1} \sum_{i=1}^{N} \int_\Omega |D^i(u_n-u)|^{p_i(x)} \upsilon(u_n)\, dx + 2^{p_i^+-1} \sum_{i=1}^{N} \int_\Omega |D^i u|^{p_i(x)} \upsilon(u_n)\, dx \ge \sum_{i=1}^{N} \int_\Omega |D^i u_n|^{p_i(x)} \upsilon(u_n)\, dx.$$

Recalling the known inequality

$$\big(|a|^{p-2}a - |b|^{p-2}b\big)(a-b) \ge \begin{cases} 2^{2-p}|a-b|^p & \text{if} \quad p \ge 2, \\ \big(p-1\big)\dfrac{|a-b|^2}{\big(|a|+|b|\big)^{2-p}} & \text{for} \quad 1 < p < 2, \end{cases} \qquad \forall a, b \in I\!R.$$

the following inequality holds

$$[|D^i u_n|^{p_i(x)-2} D^i u_n - |D^i u|^{p_i(x)-2} D^i u] D^i(u_n-u) \ge C_0 |D^i(u_n-u)|^{p_i(x)}.$$

Hence

$$\begin{aligned}&C_1 \sum_{i=1}^{N} \int_\Omega |D^i(u_n-u)|^{p_i(x)} e^{\mu p_i^+ |u_n-u|}\, dx \\ &\le -\mu p_i^+ \sum_{i=1}^{N} \int_\Omega |D^i u|^{p_i(x)-2} D^i u D^i(u_n-u) e^{\mu p_i^+ |u_n-u|}\, dx - \sigma_0 \int_\Omega T_n(u_n)\upsilon(u_n)\, dx \qquad (2.2.29) \\ &\quad + \beta 2^{p_i^+-1} \sum_{i=1}^{N} \int_\Omega |D^i u|^{p_i(x)} \upsilon(u_n)\, dx + \int_\Omega |f_n \upsilon(u_n)|\, dx,\end{aligned}$$

with μ fixed such that $C_1 = C_0 \mu p_i^+ - \beta 2^{p_i^+-1} > 0$.

In view of $(2.2.28)$, we have $\upsilon(u_n) \to 0$ a.e. on Ω, and by the boundedness in $L^\infty(\Omega)$ of u_n and u, and

the continuity of exponential function we get $|\upsilon(u_n)| \leq C_2$.

Therefore

$$\upsilon(u_n) \to 0 \quad \text{in} \quad L^{p_-^*}(\Omega).$$

It follows that

$$\sum_{i=1}^{N} \int_\Omega |D^i u_n|^{p_i(x)} \upsilon(u_n)\, dx \to 0,$$

$$\int_\Omega |f_n \upsilon(u_n)|\, dx \to 0 \quad \text{and} \quad \sigma_0 \int_\Omega T_n(u_n)\upsilon(u_n)\, dx \to 0.$$

Furthermore, the first term in the right side hand of $(2.2.29)$ goes to zero. Consequently $u_n \to u$ in $W_0^{1,\overrightarrow{p}(\cdot)}(\Omega)$.

Step 4 : Passing to the limit.

Let $\upsilon \in W_0^{1,\overrightarrow{p}(\cdot)}(\Omega) \cap L^\infty(\Omega)$. By passing to the limit in $(2.2.2)$ we have

$$\sum_{i=1}^{N} \int_\Omega |D^i u_n|^{p_i(x)-2} D^i u_n D^i \upsilon\, dx \to \sum_{i=1}^{N} \int_\Omega |D^i u|^{p_i(x)-2} D^i u D^i \upsilon\, dx,$$

and by $(2.2.27)$ we get $\sigma_0 \displaystyle\int_\Omega T_n(u_n)\upsilon\, dx \to \sigma_0 \int_\Omega u\upsilon\, dx$. Furthermore, since $f_n \to f$ in $L^{m(\cdot)}(\Omega)$ and $\upsilon \in L^\infty(\Omega)$ we obtain $\displaystyle\int_\Omega f_n \upsilon\, dx \to \int_\Omega f\upsilon\, dx$.

On the other hand, we have $D^i u_n \to D^i u$ a.e. in Ω due to the strong convergence in $W_0^{1,\overrightarrow{p}(\cdot)}(\Omega)$. This implies by using $(2.1.2)$ that

$$h_i^n(x, u_n, \nabla u_n) \to h_i(x, u, \nabla u) \qquad \text{for all} \quad i = 1, ..., N.$$

Thanks to the assumption $(2.1.3)$, we also have $h_i^n(x, u_n, \nabla u_n) \to h_i(x, u, \nabla u)$ in $L^1(\Omega)$ for all $i = 1, ..., N$ and consequently

$$h_i^n(x, u_n, \nabla u_n)\upsilon \to h_i(x, u, \nabla u)\upsilon, \quad \text{in} \quad L^1(\Omega) \qquad \forall\ i = 1, ..., N.$$

We deduce that, $\forall \upsilon \in W_0^{1,\overrightarrow{p}(\cdot)}(\Omega) \cap L^\infty(\Omega)$,

$$\sum_{i=1}^{N} \int_\Omega |D^i u|^{p_i(x)-2} D^i u D^i \upsilon\, dx + \sigma_0 \int_\Omega u\upsilon\, dx = \sum_{i=1}^{N} \int_\Omega h_i^n(x, u, \nabla u)\upsilon\, dx + \int_\Omega f\upsilon\, dx$$

Which complete the proof of theorem 2.2.1.

2.3 Lower order terms with sign assumptions

Let Ω be a bounded open subset of $I\!R^N (N \geq 2)$, we assume that the vector $\overrightarrow{p}(\cdot) = (p_0(\cdot), p_1(\cdot), \ldots, p_N(\cdot))$ satisfying the requirements that $p_i(\cdot) \in \mathcal{C}_+(\bar{\Omega})$ for $i = 0, 1, \ldots, N$, such that

$$\forall x \in \Omega, \qquad p_0(x) \geq \max\{p_i(x), \quad i = 1, 2, \ldots, N\}. \tag{2.3.1}$$

As mentioned in the introduction, instead of $(2.1.1)$ we will consider in this section more general nonlinear elliptic equations with variable exponents of the form $(2.1.4)$.

Definition 2.3.1. *In the case of $f \in L^1(\Omega)$, a measurable function u is said to be a weak solution to the problem $(2.1.4)$, if*

$$\begin{cases} \displaystyle\sum_{i=1}^{N} \int_\Omega a_i(x, u, \nabla u) D^i \upsilon \, dx + \sum_{i=1}^{N} \int_\Omega h_i(x, u, \nabla u) \upsilon \, dx = \int_\Omega f \upsilon \, dx, \\ u \in W_0^{1, \overrightarrow{p}(\cdot)}(\Omega), \qquad h_i(x, u, \nabla u) \in L^1(\Omega) \qquad \forall i = 1, \ldots, N, \end{cases} \tag{2.3.2}$$

for all $\upsilon \in W_0^{1, \overrightarrow{p}(\cdot)}(\Omega) \cap L^\infty(\Omega)$.

Definition 2.3.2. *In the case of $f \in [W_0^{1, \overrightarrow{p}(\cdot)}(\Omega)]^*$, a measurable function u is said to be a weak solution to the problem $(2.1.4)$, if*

$$\begin{cases} \displaystyle\sum_{i=1}^{N} \int_\Omega a_i(x, u, \nabla u) D^i \upsilon \, dx + \sum_{i=1}^{N} \int_\Omega h_i(x, u, \nabla u) \upsilon \, dx = \int_\Omega f \upsilon \, dx, \\ u \in W_0^{1, \overrightarrow{p}(\cdot)}(\Omega), \qquad h_i(x, u, \nabla u) \in L^1(\Omega), \qquad u h_i(x, u, \nabla u) \in L^1(\Omega) \qquad \forall i = 1, \ldots, N, \end{cases} \tag{2.3.3}$$

for all $\upsilon \in W_0^{1, \overrightarrow{p}(\cdot)}(\Omega) \cap L^\infty(\Omega)$.

Remark 2.3.1. *The assumption $(2.3.1)$ is essential to ensure that $|a_i(x, u, \nabla u)|$ belongs to $L^{p_i'(\cdot)}(\Omega)$. In the case of $Au = -\displaystyle\sum_{i=1}^{N} D^i a_i(x, \nabla u)$ the existence of entropy solution is guaranteed, without using this assumption.*

Theorem 2.3.1. *Under the assumptions $(2.1.5) - (2.1.9)$ and either $(2.1.10)$ or $(2.1.11)$ there exists at least one weak solution for the problem $(2.1.4)$.*

Proof of the Theorem 2.3.1 with the assumption (2.1.10)

Step 1 : Approximate problems.

We exploit some techniques of [83]. We consider the sequence of approximate equations

$$\begin{cases} -\sum_{i=1}^{N} D^i a_i(x, T_n(u_n), \nabla u_n) + \frac{1}{n}|u_n|^{p_0(x)-2}u_n + \sum_{i=1}^{N} h_i^n(x, u_n, \nabla u_n) = f_n, \\ u_n \in W_0^{1,\overrightarrow{p}(\cdot)}(\Omega), \qquad h_i^n(x, u_n, \nabla u_n) \in L^1(\Omega) \qquad \forall i = 1, \dots, N, \end{cases} \tag{2.3.4}$$

where

$$h_i^n(x, s, \xi) = \frac{h_i(x, s, \xi)}{1 + \frac{1}{n}|h_i(x, s, \xi)|}, \qquad \forall i = 1, \dots, N. \tag{2.3.5}$$

We consider f_n is a sequence of smooth functions such that $f_n \to f$ in $L^1(\Omega)$ and $|f_n| \leq |f|$.
We remark that

$$h_i^n(x, s, \xi)s \geq 0, \qquad |h_i^n(x, s, \xi)| \leq |h_i(x, s, \xi)| \quad \text{and} \quad |h_i^n(x, s, \xi)| \leq n \qquad \forall\ i = 1, \dots, N.$$

Let us define the operator A_n from $W_0^{1,\overrightarrow{p}(\cdot)}(\Omega)$ into its dual $W^{-1,\overrightarrow{p'}(\cdot)}(\Omega)$, by

$$A_n\upsilon = -\sum_{i=1}^{N} D^i a_i(x, T_n(\upsilon), \nabla\upsilon) + \frac{1}{n}|\upsilon|^{p_0(x)-2}\upsilon,$$

we define $H_n : W_0^{1,\overrightarrow{p}(\cdot)}(\Omega) \longmapsto W^{-1,\overrightarrow{p'}(\cdot)}(\Omega)$ by

$$\langle H_n u, \upsilon\rangle = \sum_{i=1}^{N} \int_\Omega h_i^n(x, u, \nabla u)\upsilon\, dx \quad \text{for any} \quad u, \upsilon \in W_0^{1,\overrightarrow{p}(\cdot)}(\Omega).$$

Thanks to the generalized Hölder type inequality, we have

$$\begin{aligned} |\langle H_n u, \upsilon\rangle| &\leq \sum_{i=1}^{N} \big(\frac{1}{p_0^-} + \frac{1}{(p_0')^-}\big)\|h_i^n(x, u, \nabla u)\|_{p_0'(x)}\|\upsilon\|_{p_0(x)} \\ &\leq \big(\frac{1}{p_0^-} + \frac{1}{(p_0')^-}\big)\left(N\big(n^{(p_0')^+}meas(\Omega) + 1\big)^{\frac{1}{(p_0')^-}}\right)\|\upsilon\|_{1,\overrightarrow{p}(\cdot)} \\ &\leq C_0\|\upsilon\|_{1,\overrightarrow{p}(.)} \end{aligned} \tag{2.3.6}$$

In view of Lemma 2.4.1 (see Appendix), there exists at least one weak solution $u_n \in W_0^{1,\overrightarrow{p}(\cdot)}(\Omega)$ of the problem $(2.3.4)$, (cf. [121], Theorem 2.7, p.180).

Step 2 : A priori estimates.

In this case, the use in the weak formulation of $(2.3.4)$ of the test function $T_k(u_n)$ yields for any $k > 0$

$$\sum_{i=1}^{N} \int_\Omega a_i(x, T_n(u_n), \nabla u_n) D^i T_k(u_n)\, dx + \frac{1}{n} \int_\Omega |u_n|^{p_0(x)-2} u_n T_k(u_n)\, dx + \sum_{i=1}^{N} \int_\Omega h_i^n(x, u_n, \nabla u_n) T_k(u_n)\, dx = \int_\Omega f_n T_k(u_n)\, dx.$$

In view of $(2.1.6)$ and $(2.1.8)$, we get

$$\begin{aligned} \alpha' \sum_{i=0}^{N} \int_\Omega |D^i T_k(u_n)|^{p_i(x)}\, dx &\leq \sum_{i=1}^{N} \int_\Omega a_i(x, T_n(u_n), \nabla u_n) D^i T_k(u_n)\, dx \\ &\quad + \frac{1}{n} \int_\Omega |T_k(u_n)|^{p_0(x)}\, dx \leq k \int_\Omega |f|\, dx \leq k C_1, \end{aligned}$$

with $0 < \alpha' < min(\alpha, \frac{1}{n})$, hence

$$\sum_{i=0}^{N} \int_\Omega |D^i T_k(u_n)|^{p_i(x)}\, dx \leq \frac{k}{\alpha'} C_1, \tag{2.3.7}$$

On the other hand, for $k \geq \zeta$, we have thanks to $(2.1.10)$ and $(2.3.7)$

$$\begin{aligned} \sum_{i=0}^{N} \int_\Omega |D^i u_n|^{p_i(x)}\, dx &= \sum_{i=0}^{N} \int_{\{|u_n|>k\}} |D^i u_n|^{p_i(x)}\, dx + \sum_{i=0}^{N} \int_{\{|u_n|\leq k\}} |D^i T_k(u_n)|^{p_i(x)}\, dx \\ &\leq \frac{1}{\gamma} \int_{\{|u_n|>k\}} |h_i(x, u_n, \nabla u_n)|\, dx + \frac{k}{\alpha'} C_1, \end{aligned}$$

knowing that

$$k \int_{\{|u_n|>k\}} |h_i(x, u_n, \nabla u_n)|\, dx \leq k\, C_1 \qquad \forall\ i = 1, \ldots, N \tag{2.3.8}$$

we get

$$\sum_{i=0}^{N} \int_\Omega |D^i u_n|^{p_i(x)}\, dx \leq C_2.$$

It yields

$$\sum_{i=0}^{N} \|D^i u_n\|^{\underline{p}}_{p_i(x)} - N - 1 \leq \sum_{i=0}^{N} \int_\Omega |D^i u_n|^{p_i(x)} dx \leq C_2,$$

then

$$\|u_n\|_{1,\overrightarrow{p}(\cdot)} \leq C, \tag{2.3.9}$$

which implies that

$$\begin{cases} u_n \rightharpoonup u & \text{in} \quad W_0^{1,\overrightarrow{p}(\cdot)}(\Omega), \\ u_n \to u & \text{in} \quad L^{\underline{p}}(\Omega). \end{cases} \tag{2.3.10}$$

Thanks to the lebesgue dominated convergence theorem, and Lemma 1.2.4 we deduce for any k that

$$\begin{cases} T_k(u_n) \rightharpoonup T_k(u) & \text{in} \quad W_0^{1,\vec{p}(\cdot)}(\Omega), \\ T_k(u_n) \to T_k(u) & \text{in} \quad L^{p_0(\cdot)}(\Omega). \end{cases} \tag{2.3.11}$$

Step 3 : Strong convergence of truncations in $W_0^{1,\vec{p}(\cdot)}(\Omega)$.

We already know that, for any fixed $k \in I\!R$, $T_k(u_n)$ converges weakly to $T_k(u)$ in $W_0^{1,\vec{p}(\cdot)}(\Omega)$. We want to prove that this convergence is also strong. In the sequel, we denote by $\varepsilon_i(n)$ $i = 1, \ldots,$ various real valued functions of real variables that converges to 0 as n tends to infinity, let $h > k > 0$, taking $z_n := u_n - T_h(u_n) + T_k(u_n) - T_k(u)$ and $\omega_n := T_{2k}(z_n)$, we set $\psi_k(s) = s.e^{\lambda s^2}$ where $\lambda = (b(k)/(2\alpha))^2$. It is simple to see that ([63], Lemma 1)

$$\psi_k'(s) - \frac{b(k)}{\alpha}|\psi_k(s)| \geq \frac{1}{2} \qquad \text{for any } \ s \in I\!R. \tag{2.3.12}$$

We choose in the weak formulation of $(2.3.4)$ as a test function $\psi_k(\omega_n)$, we get

$$\sum_{i=1}^{N} \int_\Omega a_i(x, u_n, \nabla u_n) D^i \psi_k(\omega_n)\, dx + \frac{1}{n} \int_\Omega |u_n|^{p_0(x)-2} u_n \psi_k(\omega_n)\, dx + \sum_{i=1}^{N} \int_\Omega h_i^n(x, u_n, \nabla u_n) \psi_k(\omega_n)\, dx = \int_\Omega f_n \psi_k(\omega_n)\, dx.$$

For M = 4k + k, it's clear that $D^i \omega_n = 0$ on the set $\{|u_n| \geq M\}$, and $\psi_k(\omega_n)$ have the same sign as u_n on the set $\{|u_n| \geq k\}$, therefore

$$\begin{aligned} \sum_{i=1}^{N} \int_\Omega a_i(x, T_M(u_n), \nabla T_M(u_n)) \psi_k'(\omega_n) D^i \psi_k(\omega_n)\, dx + \frac{1}{n} \int_{\{|u_n| \leq k\}} |u_n|^{p_0(x)-2} u_n \psi_k(\omega_n)\, dx \\ + \sum_{i=1}^{N} \int_{\{|u_n| \leq k\}} h_i^n(x, u_n, \nabla u_n) \psi_k(\omega_n)\, dx \leq \int_\Omega f_n \psi_k(\omega_n)\, dx. \end{aligned} \tag{2.3.13}$$

On the other hand, we have

$$\begin{aligned}
&\sum_{i=1}^{N}\int_{\Omega} a_i(x,T_M(u_n),\nabla T_M(u_n))\psi_k'(\omega_n)D^i\omega_n\,dx\\
&=\sum_{i=1}^{N}\int_{\{|u_n|\le k\}} a_i(x,T_k(u_n),\nabla T_k(u_n))\psi_k'(\omega_n)(D^iT_k(u_n)-D^iT_k(u))\,dx\\
&\quad+\sum_{i=1}^{N}\int_{\{|u_n|>k\}\cap\{|z_n|\le 2k\}} a_i(x,T_M(u_n),\nabla T_M(u_n))\\
&\quad\times\psi_k'(\omega_n)D^i(u_n-T_h(u_n)+T_k(u_n)-T_k(u))\,dx\\
&\ge\sum_{i=1}^{N}\int_{\Omega} a_i(x,T_k(u_n),\nabla T_k(u_n))\psi_k'(\omega_n)(D^iT_k(u_n)-D^iT_k(u))\,dx\\
&\quad+\sum_{i=1}^{N}\int_{\{|u_n|>k\}} a_i(x,T_k(u_n),\nabla T_k(u_n))\psi_k'(\omega_n)D^iT_k(u)\,dx\\
&\quad-\psi_k'(2k)\sum_{i=1}^{N}\int_{\{|u_n|>k\}} |a_i(x,T_M(u_n),\nabla T_M(u_n))|\,|D^iT_k(u)|\,dx,
\end{aligned}\tag{2.3.14}$$

hence

$$\begin{aligned}
&\sum_{i=1}^{N}\int_{\Omega}(a_i(x,T_k(u_n),\nabla T_k(u_n))-a_i(x,T_k(u_n),\nabla T_k(u)))\\
&\qquad\times\psi_k'(\omega_n)(D^iT_k(u_n)-D^iT_k(u))\,dx\\
&\le\sum_{i=1}^{N}\int_{\Omega} a_i(x,T_M(u_n),\nabla T_M(u_n))\psi_k'(\omega_n)D^i\omega_n\,dx\\
&\quad-\sum_{i=1}^{N}\int_{\{|u_n|>k\}} a_i(x,T_k(u_n),\nabla T_k(u_n))\psi_k'(\omega_n)D^iT_k(u)\,dx\\
&\quad+\psi_k'(2k)\sum_{i=1}^{N}\int_{\{|u_n|>k\}} |a_i(x,T_M(u_n),\nabla T_M(u_n))|\,|D^iT_k(u)|\,dx\\
&\quad-\sum_{i=1}^{N}\int_{\Omega} a_i(x,T_k(u_n),\nabla T_k(u))\psi_k'(\omega_n)(D^iT_k(u_n)-D^iT_k(u))\,dx.
\end{aligned}\tag{2.3.15}$$

Now we will study the terms on the right-hand side in the previous inequality. For the second and the third term on the right-hand side of (2.3.15), since the sequence $(a_i(x,T_k(u_n),\nabla T_k(u_n)))_n$ is bounded in $L^{p_i'(\cdot)}(\Omega)$, there exists $\varphi_i\in L^{p_i'(\cdot)}(\Omega)$ such that $|a_i(x,T_k(u_n),\nabla T_k(u_n))|\rightharpoonup\varphi_i$ in $L^{p_i'(\cdot)}(\Omega)$. Moreover, we have $D^iT_k(u)\chi_{\{|u_n|>k\}}\to D^iT_k(u)\chi_{\{|u|>k\}}$ in $L^{p_i(\cdot)}(\Omega)$, it implies that

$$\begin{aligned}
&\Big|\sum_{i=1}^{N}\int_{\{|u_n|>k\}} a_i(x,T_k(u_n),\nabla T_k(u_n))\psi_k'(\omega_n)D^iT_k(u)\,dx\Big|\\
&\qquad\le\psi_k'(2k)\sum_{i=1}^{N}\int_{\{|u_n|>k\}} |a_i(x,T_k(u_n),\nabla T_k(u_n))|\,|D^iT_k(u)|\,dx\\
&\qquad\to\psi_k'(2k)\sum_{i=1}^{N}\int_{\{|u|>k\}}\varphi_i|D^iT_k(u)|\,dx=0.
\end{aligned}$$

then

$$\Big|\sum_{i=1}^{N}\int_{\{|u_n|>k\}} a_i(x,T_k(u_n),\nabla T_k(u_n))\psi_k'(\omega_n)D^iT_k(u)\,dx\Big| \to \varepsilon_1(n). \tag{2.3.16}$$

and similarly, we get

$$\psi_k'(2k)\sum_{i=1}^{N}\int_{\{|u_n|>k\}} |a_i(x,T_M(u_n),\nabla T_M(u_n))|\,|D^iT_k(u)|\,dx \to \varepsilon_2(n). \tag{2.3.17}$$

Concerning the the fourth term on the right-hand side of $(2.3.15)$, we have $T_k(u_n)\to T_k(u)$ in $L^{p_i(\cdot)}(\Omega)$, thanks to $(2.1.5)$ and the Lebesgue dominated theorem, we obtain $a_i(x,T_k(u_n),\nabla T_k(u))\to a_i(x,T_k(u),\nabla T_k(u))$ in $L^{p_i'(\cdot)}(\Omega)$, and since $D^iT_k(u_n)\rightharpoonup D^iT_k(u)$ in $L^{p_i(\cdot)}(\Omega)$, we get

$$\begin{aligned}&\Big|\sum_{i=1}^{N}\int_{\Omega} a_i(x,T_k(u_n),\nabla T_k(u))\psi_k'(\omega_n)(D^iT_k(u_n)-D^iT_k(u))\,dx\Big|\\ &\qquad\le \psi_k'(2k)\sum_{i=1}^{N}\int_{\Omega}|a_i(x,T_k(u_n),\nabla T_k(u))|\,|(D^iT_k(u_n)-D^iT_k(u))\,dx|\to 0.\end{aligned}$$

then

$$\Big|\sum_{i=1}^{N}\int_{\Omega} a_i(x,T_k(u_n),\nabla T_k(u))\psi_k'(\omega_n)(D^iT_k(u_n)-D^iT_k(u))\,dx\Big|\to\varepsilon_3(n). \tag{2.3.18}$$

Thanks to $(2.3.16)-(2.3.18)$, we obtain

$$\begin{aligned}&\sum_{i=1}^{N}\int_{\Omega}(a_i(x,T_k(u_n),\nabla T_k(u_n))-a_i(x,T_k(u_n),\nabla T_k(u)))\\ &\qquad\times\psi_k'(\omega_n)(D^iT_k(u_n)-D^iT_k(u))\,dx\\ &\le\sum_{i=1}^{N}\int_{\Omega}a_i(x,T_M(u_n),\nabla T_M(u_n))\psi_k'(\omega_n)D^i\omega_n\,dx+\varepsilon_4(n).\end{aligned} \tag{2.3.19}$$

Concerning the second term on the left hand side of $(2.3.13)$, we have

$$\begin{aligned}&\frac{1}{n}\int_{\{|u_n|\le k\}}|u_n|^{p_0(x)-2}u_n\psi_k(\omega_n)\,dx=\frac{1}{n}\int_{\{|u_n|\le k\}}|T_k(u_n)|^{p_0(x)-2}T_k(u_n)\psi_k(\omega_n)\,dx\\ &=\frac{1}{n}\int_{\Omega}(|T_k(u_n)|^{p_0(x)-2}T_k(u_n)-|T_k(u)|^{p_0(x)-2}T_k(u))(T_k(u_n)-T_k(u))e^{\lambda\omega_n^2}\,dx\\ &\quad-\frac{1}{n}\int_{\{|u_n|>k\}}|T_k(u_n)|^{p_0(x)-2}T_k(u_n)(T_k(u_n)-T_k(u))e^{\lambda\omega_n^2}\,dx\\ &\quad+\frac{1}{n}\int_{\Omega}|T_k(u)|^{p_0(x)-2}T_k(u)(T_k(u_n)-T_k(u))e^{\lambda\omega_n^2}\,dx\\ &\ge\frac{1}{n}\int_{\Omega}(|T_k(u_n)|^{p_0(x)-2}T_k(u_n)-|T_k(u)|^{p_0(x)-2}T_k(u))(T_k(u_n)-T_k(u))\,dx\\ &\quad-\frac{1}{n}e^{\lambda(2k)^2}\int_{\{|u_n|>k\}}k^{p_0(x)-1}|T_k(u_n)-T_k(u)|-\frac{1}{n}e^{\lambda(2k)^2}\int_{\Omega}|T_k(u)|^{p_0(x)-1}|T_k(u_n)-T_k(u)|\,dx.\end{aligned}$$

The Lebesgue dominated theorem enable us to have $T_k(u_n) \to T_k(u)$ in $L^{p_0(\cdot)}(\Omega)$, then the second and the last term on the right-hand side of the inequality above tend to zero as n tends to ∞. Thus, we obtain

$$\begin{aligned}\frac{1}{n}\int_{\{|u_n|\le k\}} &|T_k(u_n)|^{p_0(x)-2}T_k(u_n)\psi_k(\omega_n)\,dx + \varepsilon_4(n)\\ &\ge \frac{1}{n}\int_\Omega (|T_k(u_n)|^{p_0(x)-2}T_k(u_n) - |T_k(u)|^{p_0(x)-2}T_k(u))(T_k(u_n)-T_k(u))\,dx.\end{aligned} \tag{2.3.20}$$

Furthermore, by using (2.1.6) and (2.1.9) we get

$$\begin{aligned}&\Big|\sum_{i=1}^N \int_{\{|u_n|\le k\}} h_i^n(x,u_n,\nabla u_n)\psi_k(\omega_n)\,dx\Big|\\ &\le b(k)\int_{\{|u_n|\le k\}}(j(x)+\sum_{i=1}^N |D^iT_k(u_n)|^{p_i(x)})|\psi_k(\omega_n)|\,dx\\ &\le b(k)\int_{\{|u_n|\le k\}} j(x)|\psi_k(\omega_n)|\,dx\\ &\quad+\frac{b(k)}{\alpha}\sum_{i=1}^N\int_\Omega a_i(x,T_k(u_n),\nabla T_k(u_n))D^iT_k(u_n)|\psi_k(\omega_n)|\,dx\\ &\le b(k)\int_{\{|u_n|\le k\}} j(x)|\psi_k(\omega_n)|\,dx\\ &\quad+\frac{b(k)}{\alpha}\sum_{i=1}^N\int_\Omega a_i(x,T_k(u_n),\nabla T_k(u_n))D^iT_k(u)|\psi_k(\omega_n)|\,dx\\ &\quad+\frac{b(k)}{\alpha}\sum_{i=1}^N\int_\Omega a_i(x,T_k(u_n),\nabla T_k(u))(D^iT_k(u_n)-D^iT_k(u))|\psi_k(\omega_n)|\,dx\\ &\quad+\frac{b(k)}{\alpha}\sum_{i=1}^N\int_\Omega (a_i(x,T_k(u_n),\nabla T_k(u_n))-a_i(x,T_k(u_n),\nabla T_k(u))\\ &\quad\times(D^iT_k(u_n)-D^iT_k(u))|\psi_k(\omega_n)|\,dx,\end{aligned}$$

therefore

$$\begin{aligned}&\frac{b(k)}{\alpha}\sum_{i=1}^N\int_\Omega (a_i(x,T_k(u_n),\nabla T_k(u_n))-a_i(x,T_k(u_n),\nabla T_k(u))\\ &\quad\times(D^iT_k(u_n)-D^iT_k(u))|\psi_k(\omega_n)|\,dx\\ &\quad\ge\Big|\sum_{i=1}^N\int_{\{|u_n|\le k\}} h_i^n(x,u_n,\nabla u_n)\psi_k(\omega_n)\,dx\Big| - b(k)\int_{\{|u_n|\le k\}} j(x)|\psi_k(\omega_n)|\,dx\\ &\quad-\frac{b(k)}{\alpha}\sum_{i=1}^N\int_\Omega a_i(x,T_k(u_n),\nabla T_k(u_n))D^iT_k(u)|\psi_k(\omega_n)|\,dx\\ &\quad-\frac{b(k)}{\alpha}\sum_{i=1}^N\int_\Omega a_i(x,T_k(u_n),\nabla T_k(u))(D^iT_k(u_n)-D^iT_k(u))|\psi_k(\omega_n)|\,dx.\end{aligned} \tag{2.3.21}$$

Inasmuch as, $\psi_k(\omega_n) \rightharpoonup 0$ weak$-*$ in $L^\infty(\Omega)$ we get

$$\int_{\{|u_n|\le k\}} j(x)|\psi_k(\omega_n)|\,dx \to 0. \tag{2.3.22}$$

Concerning the third term on the right-hand side of $(2.3.21)$, since $(a_i(x,T_k(u_n),\nabla T_k(u_n)))_n$ is bounded in $L^{p_i'(\cdot)}(\Omega)$, and $D^iT_k(u)|\psi_k(\omega_n)| \to 0$ in $L^{p_i(\cdot)}(\Omega)$, it follows that

$$\int_\Omega a_i(x,T_k(u_n),\nabla T_k(u_n))D^iT_k(u)|\psi_k(\omega_n)|\,dx \to 0. \tag{2.3.23}$$

For the fourth term of the right-hand side of $(2.3.21)$, in view of $(2.3.18)$, by letting n to infinity we obtain

$$\begin{aligned}&\Big|\sum_{i=1}^N\int_\Omega a_i(x,T_k(u_n),\nabla T_k(u))(D^iT_k(u_n)-D^iT_k(u))|\psi_k(\omega_n)|\,dx\Big|\\ &\quad\le \psi_k(2k)\sum_{i=1}^N\int_\Omega |a_i(x,T_k(u_n),\nabla T_k(u))||D^iT_k(u_n)-D^iT_k(u)|\,dx \to 0\end{aligned} \tag{2.3.24}$$

Thanks to $(2.3.21)-(2.3.24)$, we deduce that

$$\begin{aligned}&\Big|\sum_{i=1}^N\int_{\{|u_n|\le k\}} h_i^n(x,u_n,\nabla u_n)\psi_k(\omega_n)\,dx\Big| + \varepsilon_5(n)\\ &\le \frac{b(k)}{\alpha}\sum_{i=1}^N\int_\Omega (a_i(x,T_k(u_n),\nabla T_k(u_n)) - a_i(x,T_k(u_n),\nabla T_k(u))\\ &\qquad\times(D^iT_k(u_n)-D^iT_k(u))|\psi_k(\omega_n)|\,dx.\end{aligned} \tag{2.3.25}$$

By combining $(2.3.13)$, $(2.3.19)-(2.3.20)$ and $(2.3.25)$, we get

$$\begin{aligned}&\frac12\sum_{i=1}^N\int_\Omega (a_i(x,T_k(u_n),\nabla T_k(u_n)) - a_i(x,T_k(u_n),\nabla T_k(u))(D^iT_k(u_n)-D^iT_k(u))\,dx\\ &\quad+\frac1n\int_\Omega (|T_k(u_n)|^{p_0(x)-2}T_k(u_n) - |T_k(u)|^{p_0(x)-2}T_k(u))(T_k(u_n)-T_k(u))\,dx\\ &\le \int_\Omega f_n\psi_k(\omega_n)\,dx + \varepsilon_6(n) \le \int_\Omega f\psi_k(T_{2k}(u-T_h(u)))\,dx + \varepsilon_7(n),\end{aligned} \tag{2.3.26}$$

because $\psi_k(\omega_n) \rightharpoonup \psi_k(T_{2k}(u-T_h(u)))$ weak$-*$ in $L^\infty(\Omega)$ and $f_n \to f$ in $L^1(\Omega)$. Consequently, we have

$$\begin{aligned}&\frac12\sum_{i=1}^N\int_\Omega (a_i(x,T_k(u_n),\nabla T_k(u_n)) - a_i(x,T_k(u_n),\nabla T_k(u))(D^iT_k(u_n)-D^iT_k(u))\,dx\\ &\quad+\frac1n\int_\Omega (|T_k(u_n)|^{p_0(x)-2}T_k(u_n) - |T_k(u)|^{p_0(x)-2}T_k(u))(T_k(u_n)-T_k(u))\,dx \to 0\end{aligned} \tag{2.3.27}$$

as h and n tends to infinty, by virtue of Lemma 1.2.8, we conclude that $T_k(u_n) \to T_k(u)$ in $W_0^{1,\overrightarrow{p}(\cdot)}(\Omega)$, and in view of the Lemma $1.2.5$ by letting k, n to infinity, we get

$$\|u_n-u\|_{1,\overrightarrow{p}(\cdot)} \le \|u_n-T_k(u_n)\|_{1,\overrightarrow{p}(\cdot)} + \|T_k(u_n)-T_k(u)\|_{1,\overrightarrow{p}(\cdot)} + \|T_k(u)-u\|_{1,\overrightarrow{p}(\cdot)} \to 0$$

which implies

$$\begin{cases} u_n \to u & \text{in} \quad W_0^{1,\overrightarrow{p}(\cdot)}(\Omega),\\ \nabla u_n \to \nabla u & \text{a.e. in} \quad \Omega.\end{cases} \tag{2.3.28}$$

Step 4 : The equi-integrability of Lower order terms.

Now we shall show that

$$h_i^n(x,u_n,\nabla u_n)\to h_i(x,u,\nabla u)\quad\text{in}\quad L^1(\Omega)\quad\text{for}\quad i=1,\dots,N \tag{2.3.29}$$

and

$$\frac{1}{n}|u_n|^{p_0(x)-2}u_n\to 0\quad\text{in}\quad L^1(\Omega) \tag{2.3.30}$$

Thanks to $(2.3.28)$ it yields, Since $h_i(x,u,\nabla u)$ is a Carathéodory function for any i,

$$h_i^n(x,u_n,\nabla u_n)\to h_i(x,u,\nabla u)\quad\text{a.e.} \tag{2.3.31}$$

we have also

$$\frac{1}{n}|u_n|^{p_0(x)-2}u_n\to 0\quad\text{a.e. in}\quad\Omega \tag{2.3.32}$$

Thereafter, according to Vitali's theorem, it suffices to prove the uniform equi-integrability of these functions. By using $T_1(G_l(u_n))$ as test function in $(2.3.4)$, where G_k is the truncation function at level $k>0$, defined by $G_k(s)=s-T_k(s)$ we can obtain

$$\begin{aligned}&\alpha\sum_{i=1}^N\int_{\{l<u_n<l+1\}}|D^iu_n|^{p_i(x)}\,dx+\frac{1}{n}\int_{\{u_n>l+1\}}|u_n|^{p_0(x)-1}\,dx\\&+\sum_{i=1}^N\int_{\{u_n>l+1\}}|h_i^n(x,u_n,\nabla u_n)|\,dx\le\int_{\{u_n>l\}}|f_n|\,dx,\end{aligned}$$

Thus,for any $\eta>0,$ there exists $l(\eta)>0$ such that

$$\frac{1}{n}\int_{\{u_n>l(\eta)\}}|u_n|^{p_0(x)-1}\,dx+\sum_{i=1}^N\int_{\{u_n>l(\eta)\}}|h_i^n(x,u_n,\nabla u_n)|\,dx\le\frac{\eta}{2}. \tag{2.3.33}$$

On the other hand, for any measurable subset $E\subseteq\Omega$, we get

$$\begin{aligned}&\frac{1}{n}\int_E|u_n|^{p_0(x)-1}\,dx+\sum_{i=1}^N\int_E|h_i^n(x,u_n,\nabla u_n)|\,dx\\&\le\frac{1}{n}\int_E|T_{l(\eta)}(u_n)|^{p_0(x)-1}\,dx+\sum_{i=1}^N\int_E|h_i^n(x,T_{l(\eta)}(u_n),\nabla T_{l(\eta)}(u_n))|\,dx\\&\quad+\frac{1}{n}\int_{\{|u_n|\ge l(\eta)\}}|u_n|^{p_0(x)-1}\,dx+\sum_{i=1}^N\int_{\{|u_n|\ge l(\eta)\}}|h_i^n(x,u_n,\nabla u_n)|\,dx\\&\le\frac{1}{n}\int_E|T_{l(\eta)}(u_n)|^{p_0(x)-1}\,dx+b(l(\eta))\int_E(j(x)+\sum_{i=1}^N|D^iT_{l(\eta)}(u_n)|^{p_i(x)})\,dx\\&\quad+\frac{1}{n}\int_{\{|u_n|\ge l(\eta)\}}|u_n|^{p_0(x)-1}\,dx+\sum_{i=1}^N\int_{\{|u_n|\ge l(\eta)\}}|h_i^n(x,u_n,\nabla u_n)|\,dx.\end{aligned} \tag{2.3.34}$$

Seeing that $D^iT_{l(\eta)}(u_n) \to D^iT_{l(\eta)}(u)$ in $L^{p_i(\cdot)}(\Omega)$, there exists $\beta(\eta) > 0$ such that : for any $E \subseteq \Omega$ with meas$(E) \leq \beta(\eta)$

$$\frac{1}{n}\int_E |T_{l(\eta)}(u_n)|^{p_0(x)-1}\,dx + b(l(\eta))\int_E (j(x) + \sum_{i=1}^{N} |D^iT_{l(\eta)}(u_n)|^{p_i(x)})\,dx \leq \frac{\eta}{2}, \tag{2.3.35}$$

consequently, by combining $(2.3.33)$, $(2.3.34)$ and $(2.3.35)$, one easily has

$$\frac{1}{n}\int_E |u_n|^{p_0(x)-1}\,dx + \sum_{i=1}^{N}\int_E |h_i^n(x,u_n,\nabla u_n)|\,dx \leq \eta \quad \text{ with meas}(E) \leq \beta(\eta), \tag{2.3.36}$$

We deduce that $(|u_n|^{p_0(x)-2}u_n)_n$ and $(h_i^n(x,u_n,\nabla u_n))_n$ are equi-integrable. Hence by $(2.3.31) - (2.3.32)$ and Vitali's theorem, the convergence $(2.3.29) - (2.3.30)$ are fulfilled.

Step 5 : Passage to the limit.

In view of $(2.3.28)$, we have

$$a_i(x,u_n,\nabla u_n) \to a_i(x,u,\nabla u) \qquad \text{a.e. in } \Omega,$$

thanks to $(2.3.9)$, $(a_i(x,u_n,\nabla u_n))_n$ is bounded in $L^{p_i'(\cdot)}(\Omega)$ and by virtue of the Lemma $1.2.3$, we get

$$a_i(x,u_n,\nabla u_n) \rightharpoonup a_i(x,u,\nabla u) \qquad \text{in } L^{p_i'(\cdot)}(\Omega). \tag{2.3.37}$$

Taking $\upsilon \in W_0^{1,\overrightarrow{p}(\cdot)}(\Omega) \cap L^\infty(\Omega)$ as a test function in $(2.3.4)$, we have

$$\sum_{i=1}^{N}\int_\Omega a_i(x,u_n,\nabla u_n)D^i\upsilon\,dx + \frac{1}{n}\int_\Omega |u_n|^{p_0(x)-2}u_n\upsilon\,dx + \sum_{i=1}^{N}\int_\Omega h_i^n(x,u_n,\nabla u_n)\upsilon\,dx = \int_\Omega f_n\upsilon\,dx,$$

Hence, by letting n tend to infinity and having in mind $(2.3.29) - (2.3.30)$ and $(2.3.37)$, we conclude the proof of Theorem 2.3.1 with the assumption $(2.1.10)$.

Proof of the Theorem 2.3.1 with the assumption (2.1.11)

Step 1 : Approximate problems.

We consider the sequence of approximate equations

$$\begin{cases} -\sum_{i=1}^{N} D^i a_i(x,T_n(u_n),\nabla u_n) + \frac{1}{n}|u_n|^{p_0(x)-2}u_n + \sum_{i=1}^{N} h_i^n(x,u_n,\nabla u_n) = f_n, \\ u_n \in W_0^{1,\overrightarrow{p}(\cdot)}(\Omega), \qquad h_i^n(x,u_n,\nabla u_n) \in L^1(\Omega) \qquad \forall i = 1,\dots,N, \end{cases} \tag{2.3.38}$$

where $\{f_n\}_n$ is sequence of L^∞-functions such that $f_n \to f$ in $[W_0^{1,\overrightarrow{p}(\cdot)}(\Omega)]^*$ and $h_i^n(x,u_n,\nabla u_n)$ is defined as $(2.3.5)$. Since $h_i^n(x,u_n,\nabla u_n)$ is bounded for all i, for any fixed $n>0$, $(2.3.38)$ has at least one weak solution u_n in $W_0^{1,\overrightarrow{p}(\cdot)}(\Omega)$ by proceeding as the previous part of the proof of the theorem $2.3.1$.

Step 2 : A priori estimates.

We take u_n as test function in the weak formulation of $(2.3.38)$, we get

$$\begin{aligned}\sum_{i=1}^{N}\int_\Omega a_i(x,T_n(u_n),\nabla u_n)D^i u_n\,dx+\frac{1}{n}\int_\Omega |u_n|^{p_0(x)}\,dx\\ +\sum_{i=1}^{N}\int_\Omega h_i^n(x,u_n,\nabla u_n)u_n\,dx=\int_\Omega f_n u_n\,dx.\end{aligned}\tag{2.3.39}$$

Thanks to $(2.1.6)$, $(2.1.8)$ and the generalized Hölder type inequality we obtain

$$\alpha'\sum_{i=0}^{N}\int_\Omega |D^i u_n|^{p_i(x)}\,dx\le C_1\|f\|_{-1,\overrightarrow{p}'(\cdot)}\|u_n\|_{1,\overrightarrow{p}(\cdot)},$$

with $0<\alpha'<min(\alpha,\frac{1}{n})$, thus

$$\alpha'\sum_{i=0}^{N}\|D^i u_n\|^{\underline{p}}_{p_i(x)}-(\alpha N+\frac{1}{n})\le C_1\|f\|_{-1,\overrightarrow{p}'(\cdot)}\|u_n\|_{1,\overrightarrow{p}(\cdot)},$$

it yields

$$\|u_n\|_{1,\overrightarrow{p}(\cdot)}\le C_2,\tag{2.3.40}$$

with C_2 is a constant not depending in n. Having in mind $(2.3.39)$ and thanks to $(2.3.40)$ we get

$$\sum_{i=1}^{N}\int_\Omega h_i^n(x,u_n,\nabla u_n)u_n\,dx\le C_3\|f\|_{-1,\overrightarrow{p}'(\cdot)}\|u_n\|_{1,\overrightarrow{p}(\cdot)}\le C_4.\tag{2.3.41}$$

Accordingly to $(2.3.40)$ we can deduce, up to a subsequence still indexed by n, that

$$\begin{cases} u_n\rightharpoonup u & \text{in} \quad W_0^{1,\overrightarrow{p}(\cdot)}(\Omega),\\ u_n\to u & \text{in} \quad L^{\underline{p}}(\Omega).\end{cases}\tag{2.3.42}$$

Therefore, applying the Lebesgue dominated convergence theorem and Lemma $1.2.4$

$$\begin{cases} T_k(u_n)\rightharpoonup T_k(u) & \text{in} \quad W_0^{1,\overrightarrow{p}(\cdot)}(\Omega),\\ T_k(u_n)\to T_k(u) & \text{in} \quad L^{p_0(\cdot)}(\Omega).\end{cases}\tag{2.3.43}$$

Step 3 : Strong convergence of truncations in $W_0^{1,\overrightarrow{p}(\cdot)}(\Omega)$.

Let $\psi_k(s) = s.e^{\lambda s^2}$ where $\lambda = (b(k)/(2\alpha))^2$, and denote $\omega_n = T_k(u_n) - T_k(u)$. It's obvious to check that ω_n have the same sign that u_n on the set $\{|u_n| > k\}$. Inserting $\psi_k(\omega_n)$, as a test function in $(2.3.38)$ and then using $(2.1.8)$ we get

$$\sum_{i=1}^{N}\int_\Omega a_i(x,T_n(u_n),\nabla u_n)\psi_k'(\omega_n)D^i\psi_k(\omega_n)\,dx + \frac{1}{n}\int_{\{|u_n|\le k\}}|T_k(u_n)|^{p_0(x)-2}T_k(u_n)\psi_k(\omega_n)\,dx + \sum_{i=1}^{N}\int_{\{|u_n|\le k\}} h_i^n(x,T_k(u_n),\nabla T_k(u_n))\psi_k(\omega_n)\,dx \le \int_\Omega f_n\psi_k(\omega_n)\,dx. \tag{2.3.44}$$

At first, we have

$$\begin{aligned}
&\sum_{i=1}^{N}\int_\Omega a_i(x,T_n(u_n),\nabla u_n)\psi_k'(\omega_n)D^i\omega_n\,dx\\
&=\sum_{i=1}^{N}\int_\Omega a_i(x,T_k(u_n),\nabla T_k(u_n))\psi_k'(\omega_n)(D^iT_k(u_n)-D^iT_k(u))\,dx\\
&\quad-\sum_{i=1}^{N}\int_{\{|u_n|>k\}} a_i(x,T_n(u_n),\nabla u_n)\psi_k'(\omega_n)D^iT_k(u)\,dx\\
&\quad+\sum_{i=1}^{N}\int_{\{|u_n|>k\}} a_i(x,T_k(u_n),\nabla T_k(u_n))\psi_k'(\omega_n)D^iT_k(u)\,dx\\
&=\sum_{i=1}^{N}\int_\Omega (a_i(x,T_k(u_n),\nabla T_k(u_n))-a_i(x,T_k(u_n),\nabla T_k(u)))\psi_k'(\omega_n)(D^iT_k(u_n)-D^iT_k(u))\,dx\\
&\quad+\sum_{i=1}^{N}\int_\Omega a_i(x,T_k(u_n),\nabla T_k(u))\psi_k'(\omega_n)(D^iT_k(u_n)-D^iT_k(u))\,dx\\
&\quad-\sum_{i=1}^{N}\int_{\{|u_n|>k\}} a_i(x,T_n(u_n),\nabla u_n)\psi_k'(\omega_n)D^iT_k(u)\,dx\\
&\quad+\sum_{i=1}^{N}\int_{\{|u_n|>k\}} a_i(x,T_k(u_n),\nabla T_k(u_n))\psi_k'(\omega_n)D^iT_k(u)\,dx.
\end{aligned} \tag{2.3.45}$$

As we have done to obtain $(2.3.18)$ and $(2.3.16)$, we can prove that the second and the last terms on the right-hand side of $(2.3.45)$ tends to 0 as n goes to infinity. Moreover, in view of $(2.3.42)$ we have $a_i(x,T_n(u_n),\nabla u_n)$ is bounded in $L^{p_i'(\cdot)}(\Omega)$, the same arguments as in $(2.3.16)$ leads us to show that the

third term on the right-hand side of $(2.3.45)$ tends to 0 as n goes to ∞, hence

$$\begin{aligned}\sum_{i=1}^{N}\int_{\Omega} a_i(x,T_n(u_n),\nabla u_n)\psi_k'(\omega_n)D^i\omega_n\,dx \\ = \sum_{i=1}^{N}\int_{\Omega}(a_i(x,T_k(u_n),\nabla T_k(u_n)) - a_i(x,T_k(u_n),\nabla T_k(u))) \\ \times\psi_k'(\omega_n)(D^iT_k(u_n) - D^iT_k(u))\,dx + \varepsilon_8(n).\end{aligned} \tag{2.3.46}$$

In a similar manner, using $(2.3.20)$ and $(2.3.25)$ we obtain

$$\begin{aligned}\frac{1}{n}\int_{\{|u_n|\le k\}} |T_k(u_n)|^{p_0(x)-2}T_k(u_n)\psi_k(\omega_n)\,dx + \varepsilon_9(n) \\ \ge \frac{1}{n}\int_{\Omega}(|T_k(u_n)|^{p_0(x)-2}T_k(u_n) - |T_k(u)|^{p_0(x)-2}T_k(u))(T_k(u_n) - T_k(u))\,dx.\end{aligned} \tag{2.3.47}$$

and

$$\begin{aligned}\Big|\sum_{i=1}^{N}\int_{\{|u_n|\le k\}} h_i^n(x,u_n,\nabla u_n)\psi_k(\omega_n)\,dx\Big| + \varepsilon_{10}(n) \\ \le \frac{b(k)}{\alpha}\sum_{i=1}^{N}\int_{\Omega}(a_i(x,T_k(u_n),\nabla T_k(u_n)) - a_i(x,T_k(u_n),\nabla T_k(u)) \\ \times(D^iT_k(u_n) - D^iT_k(u))|\psi_k(\omega_n)|\,dx.\end{aligned} \tag{2.3.48}$$

Hence, putting together all the inequalities $(2.3.44)$ and $(2.3.46)-(2.3.48)$, we get

$$\begin{aligned}\frac{1}{2}\sum_{i=1}^{N}\int_{\Omega}(a_i(x,T_k(u_n),\nabla T_k(u_n)) - a_i(x,T_k(u_n),\nabla T_k(u))(D^iT_k(u_n) - D^iT_k(u))\,dx \\ +\frac{1}{n}\int_{\Omega}(|T_k(u_n)|^{p_0(x)-2}T_k(u_n) - |T_k(u)|^{p_0(x)-2}T_k(u))(T_k(u_n) - T_k(u))\,dx \\ \le \int_{\Omega} f_n\psi_k(\omega_n)\,dx + \varepsilon_{11}(n) \le \int_{\Omega} f\psi_k(T_k(u_n) - T_k(u))\,dx + \varepsilon_{12}(n),\end{aligned} \tag{2.3.49}$$

Since $f_n \to f$ in $L^1(\Omega)$. Thanks to the fact that $\psi_k(T_k(u_n) - T_k(u)) \rightharpoonup 0$ in $W_0^{1,\overrightarrow{p}(\cdot)}(\Omega)$, and by letting n tends to infinity in $(2.3.49)$, it yields

$$\begin{aligned}\frac{1}{2}\sum_{i=1}^{N}\int_{\Omega}(a_i(x,T_k(u_n),\nabla T_k(u_n)) - a_i(x,T_k(u_n),\nabla T_k(u))(D^iT_k(u_n) - D^iT_k(u))\,dx \\ +\frac{1}{n}\int_{\Omega}(|T_k(u_n)|^{p_0(x)-2}T_k(u_n) - |T_k(u)|^{p_0(x)-2}T_k(u))(T_k(u_n) - T_k(u))\,dx \to 0\end{aligned} \tag{2.3.50}$$

Taking account into Lemma 1.2.8, we conclude that $T_k(u_n) \to T_k(u)$ in $W_0^{1,\overrightarrow{p}(\cdot)}(\Omega)$. By letting k tends to infinity and thanks to $(2.3.40)$, we can deduce,up to a subsequence still indexed by n, that

$$\begin{cases} u_n \to u & \text{in} \quad W_0^{1,\overrightarrow{p}(\cdot)}(\Omega), \\ \nabla u_n \to \nabla u & \text{a.e. in} \quad \Omega. \end{cases} \tag{2.3.51}$$

Step 4 : The equi-integrability of Lower order terms.

We shall prove that

$$h_i^n(x,u_n,\nabla u_n)\to h_i(x,u,\nabla u)\quad\text{in}\quad L^1(\Omega)\quad\text{for}\quad i=1,\dots,N \tag{2.3.52}$$

and

$$\frac{1}{n}|u_n|^{p_0(x)-2}u_n\to 0\quad\text{in}\quad L^1(\Omega) \tag{2.3.53}$$

Since $h_i(x,u,\nabla u)$ is a Carathéodory function for any i and having in mind $(2.3.51)$, it yields

$$h_i^n(x,u_n,\nabla u_n)\to h_i(x,u,\nabla u)\quad\text{a.e. in}\quad\Omega \tag{2.3.54}$$

we have also

$$\frac{1}{n}|u_n|^{p_0(x)-2}u_n\to 0\quad\text{a.e. in}\quad\Omega \tag{2.3.55}$$

Now we prove that $(h_i^n(x,u_n,\nabla u_n))_n$ and $\left(\frac{1}{n}|u_n|^{p_0(x)-2}u_n\right)_n$ are uniformly equi-integrable for $i=1,\dots,N$. For any measurable subset $E\subseteq\Omega$ and for any $r\in I\!R^+$, we have

$$\begin{aligned}
&\frac{1}{n}\int_E|u_n|^{p_0(x)-2}u_n\,dx+\sum_{i=1}^N\int_E|h_i^n(x,u_n,\nabla u_n)|\,dx\\
&\le\frac{1}{n}\int_E|T_r(u_n)|^{p_0(x)-1}\,dx+\sum_{i=1}^N\int_E|h_i^n(x,T_r(u_n),\nabla T_r(u_n))|\,dx\\
&\qquad+\sum_{i=1}^N\int_{E\cap\{|u_n|\ge r\}}|h_i^n(x,u_n,\nabla u_n)|\,dx\\
&\le\frac{1}{n}\int_E|T_r(u_n)|^{p_0(x)-1}\,dx+b(r)\int_E(j(x)+\sum_{i=1}^N|D^iT_r(u_n)|^{p_i(x)})\,dx\\
&\qquad+\sum_{i=1}^N\int_{E\cap\{|u_n|\ge r\}}|h_i^n(x,u_n,\nabla u_n)|\,dx.
\end{aligned}\tag{2.3.56}$$

for fixed r and for $i=1,\dots,N$. For the first and the second term, by recalling that $D^iT_r(u_n)$ strongly converges to $D^iT_r(u)$ in $L^{p_i(\cdot)}(\Omega)$, for all i, thus for any $\delta>0$, there exists $\beta(\delta)>0$ such that : for any $E\subseteq\Omega$ with meas$(E)\le\beta(\delta)$

$$\frac{1}{n}\int_E|T_r(u_n)|^{p_0(x)-1}\,dx+b(r)\int_E(j(x)+\sum_{i=1}^N|D^iT_r(u_n)|^{p_i(x)})\,dx\le\frac{\delta}{2}. \tag{2.3.57}$$

For the third term in the right-hand side of $(2.3.56)$, we obtain

$$\begin{aligned}
&\sum_{i=1}^N\int_{E\cap\{|u_n|\ge r\}}|h_i^n(x,u_n,\nabla u_n)|\,dx\le\sum_{i=1}^N\int_{\{|u_n|\ge r\}}|h_i^n(x,u_n,\nabla u_n)|\,dx\le\\
&\le\sum_{i=1}^N\int_{\{|u_n|\ge r\}}\frac{1}{|u_n|}u_nh_i^n(x,u_n,\nabla u_n)\,dx\le\frac{1}{r}\sum_{i=1}^N\int_{\{|u_n|\ge r\}}u_nh_i^n(x,u_n,\nabla u_n)\,dx\le\frac{C}{r},
\end{aligned}\tag{2.3.58}$$

thanks to $(2.3.41)$. This complete the uniform equi-integrability of $\left(\frac{1}{n}|u_n|^{p_0(x)-2}u_n\right)_n$ and h_i^n for any i. Therefore, thanks to $(2.3.54)-(2.3.55)$ and Vitali's theorem, the convergences $(2.3.52)-(2.3.53)$ are concluded.

Step 5 : Passage to the limit.

By taking into account $(2.3.51)$, we obtain

$$a_i(x,u_n,\nabla u_n)\to a_i(x,u,\nabla u) \qquad \text{a.e. in } \Omega,$$

$$u_n h_i^n(x,u_n,\nabla u_n)\to u h_i(x,u,\nabla u) \quad \text{a.e. in } \Omega,$$

On one hand, by virtue of $(2.3.40)$, $(a_i(x,u_n,\nabla u_n))_n$ is bounded in $L^{p_i'(\cdot)}(\Omega)$ and by applying Lemma $(1.2.3)$, one has

$$a_i(x,T_n(u_n),\nabla u_n)\rightharpoonup a_i(x,u,\nabla u) \qquad \text{in } L^{p_i'(\cdot)}(\Omega). \tag{2.3.59}$$

In addition, thanks to $(2.3.41)$ and the sign condition of $h_i^n(x,u_n,\nabla u_n)$, Fatou's Lemma enable us to deduce

$$\int_\Omega u h_i(x,u,\nabla u)\,dx \le \liminf_{n\to\infty}\int_\Omega u_n h_i^n(x,u_n,\nabla u_n)\,dx \le C_1.$$

from where $uh_i(x,u,\nabla u)\in L^1(\Omega)$. Let us now choose $\upsilon\in W_0^{1,\overrightarrow{p}(\cdot)}(\Omega)\cap L^\infty(\Omega)$ as a test function in $(2.3.38)$, so that

$$\sum_{i=1}^N\int_\Omega a_i(x,T_n(u_n),\nabla u_n)D^i\upsilon\,dx+\frac{1}{n}\int_\Omega |u_n|^{p_0(x)-2}u_n\upsilon\,dx + \sum_{i=1}^N\int_\Omega h_i^n(x,u_n,\nabla u_n)\upsilon\,dx=\int_\Omega f_n\upsilon\,dx.$$

Passing to the limit as $n\to\infty$, while bearing in mind $(2.3.52)-(2.3.53)$ and $(2.3.59)$, we conclude that u satisfies $(2.3.3)$. The proof of Theorem $(2.3.1)$ is now complete.

2.4 Appendix

Lemma 2.4.1. *The bounded operator $B_n=A_n+H_n$ acted from $W_0^{1,\overrightarrow{p}(\cdot)}(\Omega)$ into $W^{-1,\overrightarrow{p'}(\cdot)}(\Omega)$ is pseudo-monotone. Moreover, B_n is coercive in the following sense:*

$$\frac{\langle B_n\upsilon,\upsilon\rangle}{\|\upsilon\|_{1,\overrightarrow{p}(\cdot)}}\longrightarrow+\infty \qquad \textit{as} \quad \|\upsilon\|_{1,\overrightarrow{p}(\cdot)}\to\infty \quad \textit{for} \quad \upsilon\in W_0^{1,\overrightarrow{p}(\cdot)}(\Omega).$$

Proof of the Lemma 2.4.1. In view of the Hölder's inequality and the growth condition $(2.1.5)$, it's easy to see that the operator A_n is bounded, and by $(2.3.6)$ we conclude that B_n is bounded.
For the coercivity, we have for any $u \in W_0^{1,\overrightarrow{p}(\cdot)}(\Omega)$,

$$\begin{aligned}\langle B_n u, u\rangle &= \langle A_n u, u\rangle + \langle H_n u, u\rangle \\ &= \sum_{i=1}^{N}\int_\Omega a_i(x, T_n(u), \nabla u)\ D^i u\, dx + \frac{1}{n}\int_\Omega |u|^{p_0(x)}\, dx + \sum_{i=1}^{N}\int_\Omega h_i^n(x,u,\nabla u)u\, dx \\ &\geq \alpha' \sum_{i=0}^{N}\int_\Omega |D^i u|^{p_i(x)}\, dx \geq \alpha'\, \|u\|^{\underline{p}}_{1,\overrightarrow{p}(\cdot)} - \alpha'(N+1)\end{aligned}$$

with $\alpha' = \min(\alpha, \frac{1}{n})$, it follows that

$$\frac{\langle B_n u, u\rangle}{\|u\|_{1,\overrightarrow{p}(\cdot)}} \longrightarrow +\infty \quad \text{as} \quad \|u\|_{1,\overrightarrow{p}(\cdot)} \to \infty.$$

It remains to show that B_n is pseudo-monotone. Let $(u_k)_{k\in I\!N}$ be a sequence in $W_0^{1,\overrightarrow{p}(\cdot)}(\Omega)$ such that

$$\begin{cases} u_k \rightharpoonup u & \text{in } W_0^{1,\overrightarrow{p}(\cdot)}(\Omega), \\ B_n u_k \rightharpoonup \chi_n & \text{in } W^{-1,\overrightarrow{p'}(\cdot)}(\Omega), \\ \limsup\limits_{k\to\infty}\langle B_n u_k, u_k\rangle \leq \langle \chi_n, u\rangle. \end{cases} \tag{2.4.1}$$

We will prove that

$$\chi_n = B_n u \quad \text{and} \quad \langle B_n u_k, u_k\rangle \longrightarrow \langle \chi_n, u\rangle \ \text{ as } \ k \to +\infty.$$

In view of the compact embedding $W_0^{1,\overrightarrow{p}(\cdot)}(\Omega) \hookrightarrow\hookrightarrow L^{\underline{p}}(\Omega)$, we have $u_k \to u$ in $L^{\underline{p}}(\Omega)$ for a subsequence still denoted $(u_k)_{k\in I\!N}$.
As $(u_k)_{k\in I\!N}$ is a bounded sequence in $W_0^{1,\overrightarrow{p}(\cdot)}(\Omega)$, using the growth condition $(2.1.5)$ it's clear that the sequence $(a_i(x, T_n(u_k), \nabla u_k))_{k\in I\!N}$ is bounded in $L^{p_i'(\cdot)}(\Omega)$, then there exists a function $\varphi_i \in L^{p_i'(\cdot)}(\Omega)$ such that

$$a_i(x, T_n(u_k), \nabla u_k) \rightharpoonup \varphi_i^n \qquad \text{in} \quad L^{p_i'(\cdot)}(\Omega) \ \text{ as } \ k \to \infty. \tag{2.4.2}$$

Also, we have

$$\frac{1}{n}|u_k|^{p_0(x)-2}u_k \rightharpoonup \frac{1}{n}|u|^{p_0(x)-2}u \qquad \text{in} \quad L^{p_0'(\cdot)}(\Omega) \ \text{ as } \ k \to \infty. \tag{2.4.3}$$

In the same way, seeing that $(h_i^n(x, u_k, \nabla u_k))_k$ is bounded in $L^{\underline{p}'}(\Omega)$, with $\underline{p}'$ is the conjugate exponent of $\underline{p}$, then there exists a function $\phi_i^n \in L^{\underline{p}'}(\Omega)$ such that

$$h_i^n(x, u_k, \nabla u_k) \rightharpoonup \phi_i^n \qquad \text{in} \quad L^{\underline{p}'}(\Omega) \ \text{ as } \ k \to \infty. \tag{2.4.4}$$

For any $v \in W_0^{1,\overrightarrow{p}(\cdot)}(\Omega)$, we have

$$\begin{aligned}\langle \chi_n, v\rangle &= \lim_{k\to\infty}\langle B_n u_k, v\rangle \\ &= \lim_{k\to\infty}\sum_{i=1}^{N}\int_\Omega a_i(x,T_n(u_k),\nabla u_k)D^i v\,dx + \lim_{k\to\infty}\frac{1}{n}\int_\Omega |u_k|^{p_0(x)-2}u_k\,v\,dx \\ &\quad + \lim_{k\to\infty}\sum_{i=1}^{N}\int_\Omega h_i^n(x,u_k,\nabla u_k)v\,dx \\ &= \sum_{i=1}^{N}\int_\Omega \varphi_i^n\,D^i v\,dx + \frac{1}{n}\int_\Omega |u|^{p_0(x)-2}u\,v\,dx + \sum_{i=1}^{N}\int_\Omega \phi_i^n v\,dx.\end{aligned} \tag{2.4.5}$$

Having in mind $(2.4.1)$ and $(2.4.5)$, we obtain

$$\begin{aligned}\limsup_{k\to\infty}\langle B_n(u_k),u_k\rangle &= \limsup_{k\to\infty}\Big\{\sum_{i=1}^{N}\int_\Omega a_i(x,T_n(u_k),\nabla u_k)D^i u_k\,dx + \frac{1}{n}\int_\Omega |u_k|^{p_0(x)}\,dx \\ &\quad + \sum_{i=1}^{N}\int_\Omega h_i^n(x,u_k,\nabla u_k)u_k\,dx\Big\} \\ &\le \sum_{i=1}^{N}\int_\Omega \varphi_i^n\,D^i u\,dx + \frac{1}{n}\int_\Omega |u|^{p_0(x)}\,dx + \sum_{i=1}^{N}\int_\Omega \phi_i^n u\,dx.\end{aligned} \tag{2.4.6}$$

Thanks to $(2.4.4)$, and from the moment that $u_k \to u$ in $L^{\underline{p}}(\Omega)$, we have

$$\int_\Omega h_i^n(x,u_k,\nabla u_k)u_k\,dx \to \int_\Omega \phi_i^n u\,dx, \tag{2.4.7}$$

Therefore

$$\limsup_{k\to\infty}\Big(\sum_{i=1}^{N}\int_\Omega a_i(x,T_n(u_k),\nabla u_k)D^i u_k\,dx + \frac{1}{n}\int_\Omega |u_k|^{p_0(x)}\,dx\Big) \le \sum_{i=1}^{N}\int_\Omega \varphi_i^n\,D^i u\,dx + \frac{1}{n}\int_\Omega |u|^{p_0(x)}\,dx. \tag{2.4.8}$$

On the other hand, in view of $(2.1.7)$ we have

$$\begin{aligned}&\sum_{i=1}^{N}\int_\Omega (a_i(x,T_n(u_k),\nabla u_k) - a_i(x,T_n(u_k),\nabla u))(D^i u_k - D^i u)\,dx \\ &\qquad + \frac{1}{n}\int_\Omega (|u_k|^{p_0(x)-2}u_k - |u|^{p_0(x)-2}u)(u_k - u)\,dx \ge 0,\end{aligned} \tag{2.4.9}$$

then

$$\begin{aligned}&\sum_{i=1}^{N}\int_\Omega a_i(x,T_n(u_k),\nabla u_k)D^i u_k\,dx + \frac{1}{n}\int_\Omega |u_k|^{p_0(x)}\,dx \\ &\ge \sum_{i=1}^{N}\int_\Omega a_i(x,T_n(u_k),\nabla u_k)D^i u\,dx + \frac{1}{n}\int_\Omega |u_k|^{p_0(x)-2}u_k u\,dx \\ &\quad + \sum_{i=1}^{N}\int_\Omega a_i(x,T_n(u_k),\nabla u)(D^i u_k - D^i u)\,dx + \frac{1}{n}\int_\Omega |u|^{p_0(x)-2}u(u_k - u)\,dx.\end{aligned}$$

In view of Lebesgue's dominated convergence theorem we have $T_n(u_k) \to T_n(u)$ in $L^{p_i(\cdot)}(\Omega)$, thus $a_i(x, T_n(u_k), \nabla u) \to a_i(x, T_n(u), \nabla u)$ in $L^{p_i'(\cdot)}(\Omega)$, and using $(2.4.2)$ we get

$$\liminf_{k\to\infty} \Big(\sum_{i=1}^{N} \int_\Omega a_i(x, T_n(u_k), \nabla u_k) D^i u_k \, dx + \frac{1}{n} \int_\Omega |u_k|^{p_0(x)} \, dx \Big) \geq \sum_{i=1}^{N} \int_\Omega \varphi_i^n \, D^i u \, dx + \frac{1}{n} \int_\Omega |u|^{p_0(x)} \, dx.$$

Having in mind $(2.4.8)$, we conclude that

$$\lim_{k\to\infty} \Big(\sum_{i=1}^{N} \int_\Omega a_i(x, T_n(u_k), \nabla u_k) D^i u_k \, dx + \frac{1}{n} \int_\Omega |u_k|^{p_0(x)} \, dx \Big) = \sum_{i=1}^{N} \int_\Omega \varphi_i^n \, D^i u \, dx + \frac{1}{n} \int_\Omega |u|^{p_0(x)} \, dx. \tag{2.4.10}$$

Therefore, by combining $(2.4.5)$, $(2.4.7)$ and $(2.4.10)$, we obtain

$$\langle B_n u_k, u_k \rangle \longrightarrow \langle \chi_n, u \rangle \quad \text{as} \quad k \to \infty.$$

Now, by (2.4.10) we can prove that

$$\lim_{k\to+\infty} \Big(\sum_{i=1}^{N} \int_\Omega (a_i(x, T_n(u_k), \nabla u_k) - a_i(x, T_n(u_k), \nabla u))(D^i u_k - D^i u) \, dx + \frac{1}{n} \int_\Omega (|u_k|^{p_0(x)-2} u_k - |u|^{p_0(x)-2} u)(u_k - u) \, dx \Big) = 0.$$

and so, by virtue of Lemma 1.2.8, we get

$$u_k \to u \quad \text{in} \quad W_0^{1,\overrightarrow{p}(\cdot)}(\Omega) \qquad \text{and} \qquad D^i u_k \to D^i u \quad \text{a.e. in} \quad \Omega,$$

then

$$a_i(x, T_n(u_k), \nabla u_k) \rightharpoonup a_i(x, T_n(u), \nabla u) \quad \text{in} \quad L^{p_i'(\cdot)}(\Omega) \quad \text{for} \quad i = 1, \ldots, N$$

and

$$h_i^n(x, u_k, \nabla u_k) \rightharpoonup h_i^n(x, u, \nabla u) \quad \text{in} \quad L^{p_0'(\cdot)}(\Omega) \quad \text{for} \quad i = 1, \ldots, N$$

consequently, thanks to $(2.4.3)$, we conclude that $\chi_n = B_n u$.

Chapter 3

Existence of entropy solutions for some quasilinear anisotropic elliptic unilateral problems with variable exponents

In this chapter, we study the following quasilinear anisotropic elliptic Dirichlet problems of the type

$$-\text{div } a(x,u,\nabla u) = f - \text{div } F,$$

where $f \in L^1(\Omega)$ and $F \in \prod_{i=1}^{N} L^{p_i'(\cdot)}(\Omega)$. We prove the existence of entropy solutions to the obstacle problem by using some penalization techniques, in the anisotropic variable exponent Sobolev spaces.

3.1 Introduction

Let Ω be a bounded open subset of $\mathbf{R}^N$ $(N \geq 2)$, Boccardo and Gallouët [55] have treated the problem

$$\begin{cases} -\text{div } a(x,u,\nabla u) = f & \text{in} \quad \Omega, \\ u = 0 & \text{on} \quad \partial\Omega, \end{cases} \tag{3.1.1}$$

where f is a bounded Radon measure on Ω, they have proved the existence of solutions and some critical regularity results. In [64], Boccardo and Orsina have studied the existence of solutions for the quasilinear elliptic equation

$$\begin{cases} -\text{div } a(x,u,\nabla u) = f - \text{div } F & \text{in} \quad \Omega, \\ u = 0 & \text{on} \quad \partial\Omega, \end{cases} \tag{3.1.2}$$

with $f \in L^1(\Omega)$ and $F \in \left(L^{p'}(\Omega)\right)^N$. They have proved the existence of solutions, using as main tool an L^1-version of Minty's Lemma.

In the framework of Orlicz spaces, Aharouch and Bennouna [5] have treated the elliptic unilateral problem

$$\begin{cases} -\text{div}\,(a(x,\nabla u)) = f \ \text{ in } \quad \Omega, \\ u = \ 0 \qquad\qquad\qquad\quad \text{on} \quad \partial\Omega, \end{cases} \tag{3.1.3}$$

where $f \in L^1(\Omega)$. They have proved the existence and uniqueness of entropy solutions in $W_0^1 L_M(\Omega)$ without any restriction on the $N-$function M of the Orlicz spaces, (see also. [43, 45, 97]).

Recently, Bendahmane and Wittbold [42] have treated in isotropic variable exponent sobolev spaces the nonlinear elliptic equation

$$\begin{cases} -\text{div}\,(|\nabla u|^{p(x)-2}\nabla u) = f \qquad \text{in} \quad \Omega, \\ u = 0 \qquad\qquad\qquad\qquad\quad \text{on} \quad \partial\Omega, \end{cases}$$

where f is assumed in $L^1(\Omega)$. They proved the existence and uniqueness of renormalized solution in Sobolev space with variable exponents $W_0^{1,p(x)}(\Omega)$. Yazough, Azroul and Redwane [154] have proved the exitence of entropy solutions to problem like $(3.1.1)$ with more non linearities, then, Azroul, Benboubker and Ouaro [19] have obtained similar results via penalization methods (see also [22, 35]).

In this chapter we are concerned with the study of the following a class of obstacle problems with non-standard structural conditions:

$$\begin{cases} -\sum\limits_{i=1}^{N} D^i a_i(x,u,\nabla u) = f - \text{div}\ F \qquad \text{in } \ \Omega, \\ u = 0 \qquad\qquad\qquad\qquad\qquad\quad \text{on } \ \partial\Omega, \end{cases} \tag{3.1.4}$$

Our purpose is to establish the existence of solutions in a subset of anisotropic variable exponent Sobolev spaces

$$K_\psi := \left\{u \in W_0^{1,\vec{p}(x)}(\Omega),\ u \geq \psi \text{ a.e. in } \Omega\right\},$$

where $\psi(\cdot)$ is a fixed obstacle function, such that

$$\psi^+ \in W_0^{1,\vec{p}(x)}(\Omega) \cap L^\infty(\Omega). \tag{3.1.5}$$

The prototype of the differential operator involved in this article can be regarded as an extension of the $p(x)-$Laplace operator $\sum_{i=1}^N \partial_{x_i}\left(|\partial_{x_i} u|^{p(x)-2}\partial_{x_i} u\right)$ to the anisotropic case. Such kind of operators can be seen as candidates for modeling phenomena which ask for distinct behavior of partial differential derivatives in various directions. This leads us to seek solutions in the anisotropic variable exponent

Sobolev space $W_0^{1,\vec{p}(\cdot)}(\Omega)$ which was recently introduced by Mihalescu-Pucci-Raduslescu (we refer to [99, 127]).

We highlight that obstacle problems like $(3.1.4)$ appear in different physical context, for instance, in deformation of membrane constrained by an obstacle, in bending of elastic isotropic homogeneous plate over an obstacle and in cavitation problems in hydrodynamic lubrication, for more details concerning physical applications we refer to [96] and [140].

The plan of the chapter is as follows : In section 2, we introduce some assumptions on $a(x,s,\xi)$ for which our problem has solutions. The section 4 is devoted to establish the main result via penalization methods.

3.2 Essential assumptions

we denote

$$\mathcal{C}_+(\Omega) = \{\text{measurable function} \quad p(\cdot) : \Omega \longmapsto I\!R \quad \text{such that} \quad 1 < p^- \leq p^+ < N\},$$

where

$$p^- = ess\inf\{p(x) \,/\, x \in \Omega\} \qquad \text{and} \qquad p^+ = ess\sup\{p(x) \,/\, x \in \Omega\}.$$

we assume that the vector $\vec{p}(\cdot) = (p_0(\cdot), p_1(\cdot), \ldots, p_N(\cdot))$ satisfying the requirements that $p_i(\cdot) \in \mathcal{C}_+(\bar{\Omega})$ for $i = 0, 1, \ldots, N$, such that

$$\forall x \in \Omega, \qquad p_0(x) \geq \max\{p_i(x), \quad i = 1, 2, \ldots, N\}. \tag{3.2.1}$$

Taking ψ as a measurable function on Ω with values in $\overline{I\!R}$, such that

$$\psi^+ \in W_0^{1,\vec{p}(\cdot)}(\Omega) \cap L^\infty(\Omega).$$

We define the convex subset K_ψ by

$$K_\psi = \{v \in W_0^{1,\vec{p}(\cdot)}(\Omega) \quad \text{such that} \quad v \geq \psi \text{ a.e. in } \Omega\}.$$

We consider a Leray-Lions operator $A : W_0^{1,\vec{p}(\cdot)}(\Omega) \longmapsto W^{-1,\vec{p}'(\cdot)}(\Omega)$, given by

$$Au = -\sum_{i=1}^{N} D^i a_i(x, u, \nabla u)$$

where $a_i : \Omega \times I\!R \times I\!R^N \mapsto I\!R$ is a Carathéodory function for $i = 1, 2, \ldots, N$, (measurable with respect to x in Ω for every (s, ξ) in $I\!R \times I\!R^N$ and continuous with respect to (s, ξ) in $I\!R \times I\!R^N$ for almost every x in Ω) which satisfy the following conditions:

$$|a_i(x, s, \xi)| \leq \beta \left(K_i(x) + |s|^{\underline{p}-1} + |\xi_i|^{p_i(x)-1}\right) \qquad \text{for} \quad i = 1, \ldots, N, \tag{3.2.2}$$

$$a_i(x, s, \xi)\xi_i \geq \alpha |\xi_i|^{p_i(x)} \qquad \text{for} \quad i = 1, \ldots, N, \tag{3.2.3}$$

for all $\xi = (\xi_1, \ldots, \xi_N)$ and $\xi' = (\xi'_1, \ldots, \xi'_N)$, we have

$$(a_i(x, s, \xi) - a_i(x, s, \xi'))(\xi_i - \xi'_i) > 0 \quad \text{for} \quad \xi_i \neq \xi'_i, \tag{3.2.4}$$

for a.e. $x \in \Omega$, and all $(s, \xi) \in I\!R \times I\!R^N$, where $K_i(\cdot)$ is a nonnegative function lying in $L^{p'_i(\cdot)}(\Omega)$, and $\alpha, \beta > 0$.

Our purpose is to establish the existence of solutions with respect to perturbations in the growth exponent $\vec{p}(\cdot)$ of the problem

$$\begin{cases} Au = f - \text{ div } F & \text{in } \Omega, \\ u = \ 0 & \text{on} \quad \partial\Omega, \end{cases} \tag{3.2.5}$$

where $f \in L^1(\Omega)$ and $F \in \prod_{i=1}^N L^{p'_i(\cdot)}(\Omega)$.

3.3 Existence of entropy solutions

Firstly, we give a definition of entropy solutions for our quasilinear anisotropic elliptic unilateral problem $(3.2.5)$.

Definition 3.3.1. *A measurable function u is said to be an entropy solution for the obstacle problem $(3.2.5)$, if $u \in T_0^{1,\vec{p}(\cdot)}(\Omega)$ such that $u \geq \psi$ a.e. in Ω and*

$$\sum_{i=1}^{N} \int_\Omega a_i(x, u, \nabla u) D^i T_k(u - \varphi)\, dx \leq \int_\Omega f T_k(u - \varphi)\, dx + \sum_{i=1}^{N} \int_\Omega F_i D^i T_k(u - \varphi)\, dx$$

for any $\varphi \in K_\psi \cap L^\infty(\Omega)$.

Theorem 3.3.1. *Let $f \in L^1(\Omega)$ and $F \in \prod_{i=1}^N L^{p'_i(\cdot)}(\Omega)$, assuming that $(3.2.2) - (3.2.4)$ hold true, there exists at least one entropy solution for the problem $(3.2.5)$.*

Proof of the Theorem 3.3.1

Step 1 : The approximate problems.

Let $(f_n)_n$ be a sequence of smooth functions such that $f_n \to f$ in $L^1(\Omega)$ and $|f_n| \leq |f|$. We consider the following sequence of approximate problems

$$\begin{cases} u_n \in K_\psi \\ \displaystyle\sum_{i=1}^{N} \int_\Omega a_i(x, T_n(u_n), \nabla u_n)(D^i u_n - D^i v)\, dx + \frac{1}{n}\int_\Omega |u_n|^{p_0(x)-2} u_n (u_n - v)\, dx \\ \qquad\qquad \leq \displaystyle\int_\Omega f_n(u_n - v)\, dx + \sum_{i=1}^{N} \int_\Omega F_i(D^i u_n - D^i v)\, dx \qquad \forall v \in K_\psi, \end{cases} \tag{3.3.1}$$

Let $X = K_\psi$, We define the operator $A_n : X \mapsto X^*$ by

$$\langle A_n u, v\rangle = \sum_{i=1}^{N} \int_\Omega a_i(x, T_n(u), \nabla u) D^i v\, dx + \frac{1}{n}\int_\Omega |u|^{p_0(x)-2} uv\, dx \qquad \forall u, v \in X.$$

Thanks to the generalized Hölder type inequality, we have for all $u, v \in X$,

$$\begin{aligned} \big|\langle A_n u, v\rangle\big| &\leq \sum_{i=1}^{N} \int_\Omega \big|a_i(x, T_n(u), \nabla u)\big|\, \big|D^i v\big|\, dx + \frac{1}{n}\int_\Omega |u|^{p_0(x)-1}|v|\, dx \\ &\leq \beta \sum_{i=1}^{N} \int_\Omega (K_i(x) + |T_n(u)|^{\underline{p}-1} + |D^i u|^{p_i(x)-1})\, \big|D^i v\big|\, dx + \frac{1}{n}\int_\Omega |u|^{p_0(x)-1}|v|\, dx \\ &\leq 2\beta \sum_{i=1}^{N} \Big(\|K_i(x)\|_{p_i'(x)} + \||T_n(u)|^{\underline{p}-1}\|_{p_i'(x)} + \||D^i u|^{p_i(x)-1}\|_{p_i'(x)}\Big)\|D^i v\|_{p_i(x)} \\ &\quad + \frac{2}{n}\||u|^{p_0(x)-1}\|_{p_0'(x)}\|v\|_{p_0(x)} \\ &\leq 2\beta \sum_{i=1}^{N} \Big(\|K_i(x)\|_{p_i'(x)} + \|n^{\underline{p}-1}\|_{p_i'(x)} + \big(\int_\Omega |D^i u|^{p_i(x)}\, dx + 1\big)^{\frac{1}{(p_i')^-}}\Big)\|D^i v\|_{p_i(x)} \\ &\quad + \frac{2}{n}\big(\int_\Omega |u|^{p_0(x)}\, dx + 1\big)^{\frac{1}{(p_0')^-}}\big)\|v\|_{p_0(x)} \\ &\leq C_0\|v\|_{1,\vec{p}(.)} \end{aligned} \tag{3.3.2}$$

Lemma 3.3.1. *The operator A_n is pseudo-monotone from $W_0^{1,\vec{p}(\cdot)}(\Omega)$ into $W^{-1,\vec{p}'(\cdot)}(\Omega)$. Moreover, A_n is coercive in the following sense: there exists $v_0 \in K_\psi$ such that*

$$\frac{\langle A_n v, v - v_0\rangle}{\|v\|_{1,\vec{p}(\cdot)}} \longrightarrow +\infty \quad \textit{if} \quad \|v\|_{1,\vec{p}(\cdot)} \longrightarrow +\infty \quad \textit{for} \quad v \in K_\psi.$$

Proof of Lemma 3.3.1

In view of $(3.3.2)$, the operator A_n is bounded. For the coercivity, let $u_0 \in K_\psi$, thanks to the generalized

Hölder and Young's inequality we have for all $u \in K_\psi$

$$\begin{aligned}
&\langle A_n u, u - u_0\rangle \\
&= \langle A_n u, u\rangle - \langle A_n u, u_0\rangle \\
&= \sum_{i=1}^{N} \int_\Omega a_i(x, T_n(u), \nabla u) D^i u \, dx + \frac{1}{n}\int_\Omega |u|^{p_0(x)}\, dx \\
&\qquad - \sum_{i=1}^{N} \int_\Omega a_i(x, T_n(u), \nabla u) D^i u_0 \, dx - \frac{1}{n}\int_\Omega |u|^{p_0(x)-2} u u_0 \, dx \\
&\geq \alpha \sum_{i=1}^{N} \int_\Omega |D^i u|^{p_i(x)}\, dx + \frac{1}{n}\int_\Omega |u|^{p_0(x)}\, dx - \frac{1}{2n}\int_\Omega |u|^{p_0(x)}\, dx - \frac{C_1}{n}\int_\Omega |u_0|^{p_0(x)}\, dx \\
&\qquad -2\sum_{i=1}^{N} \Big(\int_\Omega |a_i(x, T_n(u), \nabla u)|^{p_i'(x)}\, dx + 1\Big)^{\frac{1}{(p_i')^-}} \|D^i u_0\|_{p_i(\cdot)} \\
&\geq \alpha' \sum_{i=0}^{N} \int_\Omega |D^i u|^{p_i(x)}\, dx - \frac{C_1}{n}\int_\Omega |u_0|^{p_0(x)}\, dx \\
&\qquad -2\beta \sum_{i=1}^{N} \Big(\int_\Omega (K_i(x)^{p_i'(x)} + n^{(\underline{p}-1)p_i'(x)} + |D^i u|^{p_i(x)})\, dx + 1\Big)^{\frac{1}{(p_i')^-}} \|u_0\|_{1,\vec{p}(\cdot)}
\end{aligned}$$

with $\alpha' = \min(\alpha, \frac{1}{2n})$, it follows that

$$\begin{aligned}
&\frac{\langle A_n u, u - u_0\rangle}{\|u\|_{1,\vec{p}(\cdot)}} \\
&= \frac{\langle A_n u, u\rangle}{\|u\|_{1,\vec{p}(\cdot)}} - \frac{\langle A_n u, u_0\rangle}{\|u\|_{1,\vec{p}(\cdot)}} \\
&\geq \frac{\alpha'}{\|u\|_{1,\vec{p}(\cdot)}} \sum_{i=0}^{N} \int_\Omega |D^i u|^{p_i(x)}\, dx - \frac{C_1}{n\|u\|_{1,\vec{p}(\cdot)}}\int_\Omega |u_0|^{p_0(x)}\, dx \\
&\quad -\frac{2\beta\|u_0\|_{1,\vec{p}(\cdot)}}{\|u\|_{1,\vec{p}(\cdot)}} \sum_{i=1}^{N} \Big(\int_\Omega K_i(x)^{p_i'(x)} + n^{(\underline{p}-1)p_i'(x)} + |D^i u|^{p_i(x)}\, dx + 1\Big)^{\frac{1}{(p_i')^-}} \longrightarrow \infty
\end{aligned} \tag{3.3.3}$$

as $\|u\|_{1,\vec{p}(\cdot)} \to \infty$.

It remains to show that A_n is pseudo-monotone. Let $(u_k)_{k\in I\!N}$ be a sequence in $W_0^{1,\vec{p}(\cdot)}(\Omega)$ such that

$$\begin{cases}
u_k \rightharpoonup u & \text{in } \ W_0^{1,\vec{p}(\cdot)}(\Omega), \\
A_n u_k \rightharpoonup \chi_n & \text{in } \ W^{-1,\vec{p'}(\cdot)}(\Omega), \\
\limsup\limits_{k\to\infty} \langle A_n u_k, u_k\rangle \leq \langle \chi_n, u\rangle.
\end{cases} \tag{3.3.4}$$

We will prove that

$$\chi_n = A_n u \quad \text{ and } \quad \langle A_n u_k, u_k\rangle \longrightarrow \langle \chi_n, u\rangle \ \text{ as } \ k \to +\infty.$$

Firstly, since $W_0^{1,\vec{p}(\cdot)}(\Omega) \hookrightarrow\hookrightarrow L^{\underline{p}}(\Omega)$, then

$$u_k \to u \quad \text{ in } L^{\underline{p}}(\Omega) \quad \text{for a subsequence denoted again } (u_k)_{k\in I\!N}.$$

As $(u_k)_{k\in I\!N}$ is a bounded sequence in $W_0^{1,\vec{p}(\cdot)}(\Omega)$, then by the growth condition $(a_i(x,T_n(u_k),\nabla u_k))_{k\in I\!N}$ is bounded in $L^{p_i'(\cdot)}(\Omega)$, and there exists a function $\varphi_{i,n}\in L^{p_i'(\cdot)}(\Omega)$ such that

$$a_i(x,T_n(u_k),\nabla u_k)\rightharpoonup \varphi_{i,n} \text{ in } \quad L^{p_i'(\cdot)}(\Omega) \text{ as } k\to\infty, \tag{3.3.5}$$

and

$$\frac{1}{n}|u_k|^{p_0(x)-2}u_k \rightharpoonup \frac{1}{n}|u|^{p_0(x)-2}u \text{ in } \quad L^{p_0'(x)}(\Omega). \tag{3.3.6}$$

It is clear that, for all $v\in W_0^{1,\vec{p}(\cdot)}(\Omega)$ we have

$$\begin{aligned}\langle \chi_n, v\rangle &= \lim_{k\to\infty}\sum_{i=1}^{N}\int_\Omega a_i(x,T_n(u_k),\nabla u_k)D^i v\,dx + \lim_{k\to\infty}\frac{1}{n}\int_\Omega |u_k|^{p_0(x)-2}u_k v\,dx\\ &= \sum_{i=1}^{N}\int_\Omega \varphi_{i,n}\,D^i v\,dx + \frac{1}{n}\int_\Omega |u|^{p_0(x)-2}uv\,dx.\end{aligned} \tag{3.3.7}$$

From relations $(3.3.4)$ and $(3.3.7)$, we obtain

$$\begin{aligned}\limsup_{k\to\infty}\langle A_n(u_k),u_k\rangle &= \limsup_{k\to\infty}\left(\sum_{i=1}^{N}\int_\Omega a_i(x,T_n(u_k),\nabla u_k)D^i u_k\,dx + \frac{1}{n}\int_\Omega |u_k|^{p_0(x)}\,dx\right)\\ &\le \sum_{i=1}^{N}\int_\Omega \varphi_{i,n}\,D^i u\,dx + \frac{1}{n}\int_\Omega |u|^{p_0(x)}\,dx.\end{aligned} \tag{3.3.8}$$

On the other hand, by $(3.2.4)$ we have

$$\begin{aligned}\sum_{i=1}^{N}\int_\Omega (a_i(x,T_n(u_k),\nabla u_k)-a_i(x,T_n(u_k),\nabla u))(D^i u_k - D^i u)\,dx\\ +\frac{1}{n}\int_\Omega (|u_k|^{p_0(x)-2}u_k - |u|^{p_0(x)-2}u)(u_k-u)\,dx\ge 0,\end{aligned} \tag{3.3.9}$$

then

$$\begin{aligned}&\sum_{i=1}^{N}\int_\Omega a_i(x,T_n(u_k),\nabla u_k)D^i u_k\,dx + \frac{1}{n}\int_\Omega |u_k|^{p_0(x)}\,dx\\ &\quad\ge \sum_{i=1}^{N}\int_\Omega a_i(x,T_n(u_k),\nabla u_k)D^i u\,dx + \frac{1}{n}\int_\Omega |u_k|^{p_0(x)-2}u_k u\,dx\\ &\quad+\sum_{i=1}^{N}\int_\Omega a_i(x,T_n(u_k),\nabla u)(D^i u_k - D^i u)\,dx + \frac{1}{n}\int_\Omega |u|^{p_0(x)-2}u(u_k-u)\,dx.\end{aligned}$$

In view of Lebesgue dominated convergence theorem, we have $T_n(u_k)\to T_n(u)$ in $L^{\underline{p}}(\Omega)$, then $a_i(x,T_n(u_k),\nabla u)\to a_i(x,T_n(u),\nabla u)$ in $L^{p_i'(\cdot)}(\Omega)$, and using $(3.3.5)-(3.3.6)$ we get

$$\begin{aligned}\liminf_{k\to\infty}\left(\sum_{i=1}^{N}\int_\Omega a_i(x,T_n(u_k),\nabla u_k)D^i u_k\,dx + \frac{1}{n}\int_\Omega |u_k|^{p_0(x)}\,dx\right)\\ \ge \sum_{i=1}^{N}\int_\Omega \varphi_{i,n}\,D^i u\,dx + \frac{1}{n}\int_\Omega |u|^{p_0(x)}\,dx.\end{aligned}$$

This implies by using $(3.3.8)$ that

$$\begin{aligned}\lim_{k\to\infty}\Big(\sum_{i=1}^{N}\int_{\Omega}a_i(x,T_n(u_k),\nabla u_k)D^iu_k\,dx+\frac{1}{n}\int_{\Omega}|u_k|^{p_0(x)}\,dx\Big)\\ =\sum_{i=1}^{N}\int_{\Omega}\varphi_{i,n}\,D^iu\,dx+\frac{1}{n}\int_{\Omega}|u|^{p_0(x)}\,dx.\end{aligned}\tag{3.3.10}$$

Relations $(3.3.7)$ and $(3.3.10)$, give

$$\langle A_nu_k,u_k\rangle\longrightarrow\langle\chi_n,u\rangle\ \text{ as }\ k\to\infty.$$

Now, by $(3.3.10)$ we can prove that

$$\begin{aligned}\lim_{k\to\infty}\Big(\sum_{i=1}^{N}\int_{\Omega}(a_i(x,T_n(u_k),\nabla u_k)-a_i(x,T_n(u_k),\nabla u))(D^iu_k-D^iu)\,dx\\ +\frac{1}{n}\int_{\Omega}(|u_k|^{p_0(x)-2}u_k-|u|^{p_0(x)-2}u)(u_k-u)\,dx\Big)=0,\end{aligned}$$

and so, by virtue of Lemma 1.2.8, we obtain

$$u_k\to u\ \text{ in }\ W_0^{1,\vec{p}(\cdot)}(\Omega)\quad\text{ and }\quad D^iu_k\to D^iu\ \text{ a.e. in }\Omega,$$

Having in mind $(3.2.2)$ and Lemma 1.2.3, we conclude that

$$a_i(x,T_n(u_k),\nabla u_k)\rightharpoonup a_i(x,T_n(u),\nabla u)\quad\text{ in }\ L^{p_i'(\cdot)}(\Omega)\quad\text{for}\quad i=1,\dots,N$$

thanks to $(3.3.6)$, we can deduce that $\chi_n=A_nu$.

In view of Lemma 3.3.1 above, and by virtue of [121, Theorem 8.2], the approximate problem $(3.3.1)$ has at least one solution $u_n\in K_\psi$.

Step 2 : Weak convergence of truncations.

Let $k\geq\max(1,2\|\psi^+\|_\infty)$, we set $v=u_n-\eta T_k(u_n-\psi^+)\in W_0^{1,\vec{p}(\cdot)}(\Omega)\cap L^\infty(\Omega)$, for η positive small enough we have $v\in K_\psi$, thus v is an admissible test function in $(3.3.1)$, we obtain

$$\begin{aligned}&\sum_{i=1}^{N}\int_{\Omega}a_i(x,T_n(u_n),\nabla u_n)D^iT_k(u_n-\psi^+)\,dx+\frac{1}{n}\int_{\Omega}|u_n|^{p_0(x)-2}u_nT_k(u_n-\psi^+)\,dx\\ &\quad\leq\int_{\Omega}f_nT_k(u_n-\psi^+)\,dx+\sum_{i=1}^{N}\int_{\Omega}F_iD^iT_k(u_n-\psi^+)\,dx,\end{aligned}$$

then

$$\begin{aligned}
&\sum_{i=1}^{N}\int_{\{|u_n-\psi^+|\le k\}} a_i(x,T_n(u_n),\nabla u_n)D^i u_n\,dx+\frac{1}{n}\int_\Omega |u_n|^{p_0(x)-2}u_nT_k(u_n-\psi^+)\,dx\\
&\le \int_\Omega f_nT_k(u_n-\psi^+)\,dx+\sum_{i=1}^{N}\int_{\{|u_n-\psi^+|\le k\}} F_iD^i(u_n-\psi^+)\,dx\\
&\quad+\sum_{i=1}^{N}\int_{\{|u_n-\psi^+|\le k\}} a_i(x,T_n(u_n),\nabla u_n)D^i\psi^+\,dx.
\end{aligned}\tag{3.3.11}$$

Since $k\ge\|\psi^+\|_\infty$, then $T_k(u_n-\psi^+)$ have the same sign as u_n on the set $\{|u_n-\psi^+|>k\}$, it follows that

$$\begin{aligned}
&\frac{1}{n}\int_\Omega |u_n|^{p_0(x)-2}u_nT_k(u_n-\psi^+)\,dx\\
&=\frac{1}{n}\int_{\{|u_n-\psi^+|\le k\}} |u_n|^{p_0(x)-2}u_n(u_n-\psi^+)\,dx\\
&\quad+\frac{1}{n}\int_{\{|u_n-\psi^+|>k\}} |u_n|^{p_0(x)-2}u_nT_k(u_n-\psi^+)\,dx\\
&\ge\frac{1}{n}\int_{\{|u_n-\psi^+|\le k\}} |u_n|^{p_0(x)}\,dx-\frac{1}{n}\int_{\{|u_n-\psi^+|\le k\}} |u_n|^{p_0(x)-1}|\psi^+|\,dx.
\end{aligned}$$

In view of $(3.2.3)$, we obtain

$$\begin{aligned}
&\alpha\sum_{i=1}^{N}\int_{\{|u_n-\psi^+|\le k\}} |D^iu_n|^{p_i(x)}\,dx+\frac{1}{n}\int_{\{|u_n-\psi^+|\le k\}} |u_n|^{p_0(x)}\,dx\\
&\le\sum_{i=1}^{N}\int_{\{|u_n-\psi^+|\le k\}} a_i(x,T_n(u_n),\nabla u_n)D^iu_n\,dx+\frac{1}{n}\int_\Omega |u_n|^{p_0(x)-2}u_nT_k(u_n-\psi^+)\,dx\\
&\quad+\frac{1}{n}\int_{\{|u_n-\psi^+|\le k\}} |u_n|^{p_0(x)-1}|\psi^+|\,dx\\
&\le\int_\Omega f_nT_k(u_n-\psi^+)\,dx+\sum_{i=1}^{N}\int_{\{|u_n-\psi^+|\le k\}} F_i(D^iu_n-D^i\psi^+)\,dx\\
&+\sum_{i=1}^{N}\int_{\{|u_n-\psi^+|\le k\}} |a_i(x,T_n(u_n),\nabla u_n)|\,|D^i\psi^+|\,dx+\frac{1}{n}\int_{\{|u_n-\psi^+|\le k\}} |u_n|^{p_0(x)-1}|\psi^+|\,dx.
\end{aligned}\tag{3.3.12}$$

Thanks to Young's inequality, we have

$$\int_{\{|u_n-\psi^+|\le k\}} |u_n|^{p_0(x)-1}|\psi^+|\,dx\le\frac{1}{2}\int_{\{|u_n-\psi^+|\le k\}} |u_n|^{p_0(x)}\,dx+C_2,\tag{3.3.13}$$

and

$$\sum_{i=1}^{N}\int_{\{|u_n-\psi^+|\le k\}} F_i(D^iu_n-D^i\psi^+)\,dx\le\frac{\alpha}{2}\sum_{i=1}^{N}\int_{\{|u_n-\psi^+|\le k\}} |D^iu_n|^{p_i(x)}\,dx+C_3,\tag{3.3.14}$$

also, in view of $(3.2.2)$ and Young's inequality, we have

$$\begin{aligned}
&\sum_{i=1}^{N}\int_{\{|u_n-\psi^+|\le k\}}|a_i(x,T_n(u_n),\nabla u_n)|\,|D^i\psi^+|\,dx\\
&\le \beta\sum_{i=1}^{N}\int_{\{|u_n-\psi^+|\le k\}}K_i(x)|D^i\psi^+|\,dx+\beta\sum_{i=1}^{N}\int_{\{|u_n-\psi^+|\le k\}}|u_n|^{\underline{p}-1}|D^i\psi^+|\,dx\\
&\quad+\beta\sum_{i=1}^{N}\int_{\{|u_n-\psi^+|\le k\}}|D^iu_n|^{p_i(x)-1}|D^i\psi^+|\,dx \qquad (3.3.15)\\
&\le C_4+\varepsilon\sum_{i=1}^{N}\int_{\{|u_n-\psi^+|\le k\}}|u_n|^{\underline{p}}\,dx+\frac{\alpha}{4}\sum_{i=1}^{N}\int_{\{|u_n-\psi^+|\le k\}}|D^iu_n|^{p_i(x)}\,dx\\
&\quad+\frac{1}{\varepsilon^{\underline{p}-1}}\sum_{i=1}^{N}\int_{\{|u_n-\psi^+|\le k\}}|D^i\psi^+|^{\underline{p}}\,dx+C_5\sum_{i=1}^{N}\int_{\{|u_n-\psi^+|\le k\}}|D^i\psi^+|^{p_i(x)}\,dx,
\end{aligned}$$

By combining $(3.3.12)-(3.3.15),$ we deduce that

$$\begin{aligned}
&\frac{\alpha}{4}\sum_{i=1}^{N}\int_{\{|u_n-\psi^+|\le k\}}|D^iu_n|^{p_i(x)}\,dx+\frac{1}{2n}\int_{\{|u_n-\psi^+|\le k\}}|u_n|^{p_0(x)}\,dx\\
&\le k\|f\|_1+N\varepsilon\int_{\{|u_n-\psi^+|\le k\}}|u_n|^{\underline{p}}\,dx+C_6(\varepsilon).
\end{aligned}\qquad (3.3.16)$$

Then, there exists a constant C_7 depending only on k and ε such that

$$\frac{\alpha}{4}\sum_{i=1}^{N}\int_{\{|u_n-\psi^+|\le k\}}|D^iu_n|^{p_i(x)}\,dx\le C_7(k,\varepsilon)\qquad (3.3.17)$$

and since

$$\{x\in\Omega\,,|u_n|\le k\}\subset\{x\in\Omega\,,|u_n-\psi^+|\le k+\|\psi^+\|_\infty\}.$$

Therefore

$$\begin{aligned}
\sum_{i=0}^{N}\int_\Omega|D^iT_k(u_n)|^{p_i(x)}\,dx&=\sum_{i=1}^{N}\int_{\{|u_n|\le k\}}|D^iu_n|^{p_i(x)}\,dx+\int_\Omega|T_k(u_n)|^{p_0(x)}\,dx\\
&\le\sum_{i=1}^{N}\int_{\{|u_n-\psi^+|\le k+\|\psi^+\|_\infty\}}|D^iu_n|^{p_i(x)}\,dx+k^{p_0^+}|\Omega|\\
&\le C_8(k,\|\psi^+\|_\infty,\varepsilon),
\end{aligned}$$

and we obtain

$$\|T_k(u_n)\|_{1,\vec{p}(\cdot)}\le C_9(k,\|\psi^+\|_\infty,\varepsilon),$$

where C_9 is positive constant that does not depend on n, thus, the sequence $(T_k(u_n))_n$ is bounded in $W_0^{1,\vec{p}(\cdot)}(\Omega)$ uniformly in n. Then, there exists a subsequence still denoted $(T_k(u_n))_{n\in I\!N}$ and a function $v_k\in W_0^{1,\vec{p}(\cdot)}(\Omega)$ such that

$$\begin{cases}
T_k(u_n)\rightharpoonup v_k \quad\text{weakly in}\quad W_0^{1,\vec{p}(\cdot)}(\Omega),\\
T_k(u_n)\to v_k \quad\text{strongly in}\quad L^{\underline{p}}(\Omega)\quad\text{and}\quad\text{a.e in}\quad\Omega.
\end{cases}\qquad (3.3.18)$$

On the other hand, thanks to $(3.3.16)$ and Poincaré inequality, we have

$$\begin{aligned}\|\nabla u_n \cdot \chi_{\{|u_n-\psi^+|\leq k\}}\|_{\underline{p}}^{\underline{p}} &= \sum_{i=1}^{N}\int_\Omega |D^i u_n|^{\underline{p}} \cdot \chi_{\{|u_n-\psi^+|\leq k\}}\, dx \\ &\leq \sum_{i=1}^{N}\int_\Omega |D^i u_n|^{p_i(x)} \cdot \chi_{\{|u_n-\psi^+|\leq k\}}\, dx + N|\Omega| \\ &\leq \frac{4k}{\alpha}\|f\|_1 + \frac{4N\varepsilon}{\alpha}\|u_n \cdot \chi_{\{|u_n-\psi^+|\leq k\}}\|_{\underline{p}}^{\underline{p}} + C_{10}(\varepsilon) \\ &\leq \frac{4k}{\alpha}\|f\|_1 + C\frac{4N\varepsilon}{\alpha}\|\nabla u_n \cdot \chi_{\{|u_n-\psi^+|\leq k\}}\|_{\underline{p}}^{\underline{p}} + C_{10}(\varepsilon),\end{aligned} \tag{3.3.19}$$

Therefore, by taking ε small enough $(C\frac{4N\varepsilon}{\alpha} < 1)$, there exists a constant C_{11} that does not depend on k and n, such that

$$\|\nabla u_n \cdot \chi_{\{|u_n-\psi^+|\leq k\}}\|_{\underline{p}} \leq C_{11}k^{\frac{1}{\underline{p}}},$$

Since $k \geq \max(1, 2\|\psi^+\|_\infty)$, then $\{|u_n| \leq \frac{k}{2}\} \subseteq \{|u_n - \psi^+| \leq k\}$ and we conclude that

$$\|\nabla T_k(u_n)\|_{\underline{p}} \leq C_{11} \cdot (2k)^{\frac{1}{\underline{p}}},$$

we obtain

$$\begin{aligned}k\ \text{meas}\{|u_n| > k\} &= \int_{\{|u_n|>k\}} |T_k(u_n)|\, dx \leq \int_\Omega |T_k(u_n)|\, dx \\ &\leq \|1\|_{\underline{p}'}\|T_k(u_n)\|_{\underline{p}} \\ &\leq C\|\nabla T_k(u_n)\|_{\underline{p}} \\ &\leq C_{12}k^{\frac{1}{\underline{p}}},\end{aligned} \tag{3.3.20}$$

which yields

$$\text{meas}\{|u_n| > k\} \leq C_{12}\frac{1}{k^{1-\frac{1}{\underline{p}}}} \longrightarrow 0 \quad \text{as} \quad k \to \infty. \tag{3.3.21}$$

Now, we will show that $(u_n)_n$ is a Cauchy sequence in measure, we have for every $\delta > 0$,

$$\begin{aligned}\text{meas}\{|u_n - u_m| > \delta\} \leq \text{meas}\{|u_n| > k\} + \text{meas}\{|u_m| > k\} \\ +\text{meas}\{|T_k(u_n) - T_k(u_m)| > \delta\}.\end{aligned}$$

Let $\eta > 0$, in view of $(3.3.21)$ we may choose $k = k(\eta)$ large enough such that

$$\text{meas}\{|u_n| > k\} \leq \frac{\eta}{3} \qquad \text{and} \qquad \text{meas}\{|u_m| > k\} \leq \frac{\eta}{3}. \tag{3.3.22}$$

Moreover, thanks to $(3.3.18)$ we have

$$T_k(u_n) \longrightarrow \eta_k \quad \text{in} \quad L^{\underline{p}}(\Omega) \quad \text{and a.e. in} \quad \Omega.$$

Thus $(T_k(u_n))_{n\in\mathbb{N}}$ is a Cauchy sequence in measure, and for any $k > 0$ and $\delta, \eta > 0$, there exists $n_0 = n_0(k, \delta, \eta)$ such that

$$\text{meas}\{|T_k(u_n) - T_k(u_m)| > \delta\} \leq \frac{\eta}{3} \qquad \text{for all } m, n \geq n_0(k, \delta, \eta). \tag{3.3.23}$$

By combining $(3.3.22)$ and $(3.3.23)$, we conclude that : for all $\delta, \eta > 0$, there exists $n_0 = n_0(\delta, \eta)$ such that

$$\text{meas}\{|u_n - u_m| > \delta\} \leq \eta \qquad \text{for any} \quad n, m \geq n_0.$$

It follows that $(u_n)_n$ is a Cauchy sequence in measure, then converges almost everywhere, for a subsequence, to some measurable function u. Thanks to $(3.3.18)$, we obtain

$$T_k(u_n) \rightharpoonup T_k(u) \quad \text{ in } \quad W_0^{1,\vec{p}(\cdot)}(\Omega), \tag{3.3.24}$$

and in view of Lebesgue's dominated convergence theorem, we have

$$T_k(u_n) \to T_k(u) \quad \text{in} \quad L^{p_0(\cdot)}(\Omega) \quad \text{and} \quad \text{a.e in} \quad \Omega.$$

Step 3 : The equi-integrability of $(\frac{1}{n}|u_n|^{p_0(x)-2}u_n)_n$.

In order to pass to the limit in the approximate equation, we show that

$$\frac{1}{n}|u_n|^{p_0(x)-2}u_n \longrightarrow 0 \quad \text{strongly in } \ L^1(\Omega). \tag{3.3.25}$$

By using Vitali's Theorem, it is suffices to prove that $(\frac{1}{n}|u_n|^{p_0(x)-2}u_n)_n$ is uniformly equi-integrable. Indeed, Consider the function $v = u_n - \eta T_1(u_n - T_j(u_n))$. For η positive small enough, we can deduce that $v \geq \psi$ and since $v \in W_0^{1,\vec{p}(\cdot)}(\Omega)$, then v is an admissible test function in (3.3.1), this allows us to write

$$\begin{aligned}
&\sum_{i=1}^{N}\int_\Omega a_i(x, T_n(u_n), \nabla u_n)D^i T_1(u_n - T_j(u_n))\,dx + \frac{1}{n}\int_\Omega |u_n|^{p_0(x)-2}u_n T_1(u_n - T_j(u_n))\,dx \\
&\leq \int_\Omega f_n T_1(u_n - T_j(u_n))\,dx + \sum_{i=1}^{N}\int_\Omega F_i D^i T_1(u_n - T_j(u_n))\,dx.
\end{aligned}$$

in view of Young's inequality, we deduce that

$$\begin{aligned}
&\sum_{i=1}^{N}\int_{\{j\leq|u_n|<j+1\}} a_i(x, T_n(u_n), \nabla u_n)D^i u_n\,dx + \frac{1}{n}\int_{\{j+1\leq|u_n|\}} |u_n|^{p_0(x)-1}\,dx \\
&\leq \int_{\{j\leq|u_n|\}} |f_n|\,dx + \sum_{i=1}^{N}\int_{\{j\leq|u_n|<j+1\}} F_i D^i u_n\,dx \\
&\leq \int_{\{j\leq|u_n|\}} |f_n|\,dx + \frac{\alpha}{2}\sum_{i=1}^{N}\int_{\{j\leq|u_n|<j+1\}} |D^i u_n|^{p_i(x)}\,dx + C_{13}\sum_{i=1}^{N}\int_{\{j\leq|u_n|<j+1\}} |F_i|^{p_i'(x)}\,dx.
\end{aligned}$$

Therefore, we conclude that

$$\frac{1}{2}\sum_{i=1}^{N}\int_{\{j\leq|u_n|<j+1\}} a_i(x,T_n(u_n),\nabla u_n)D^i u_n\,dx+\frac{1}{n}\int_{\{j+1\leq|u_n|\}}|u_n|^{p_0(x)-1}\;dx\longrightarrow 0 \tag{3.3.26}$$
as $\quad j\to\infty$.

Let $\eta>0$ be fixed, then there exists $j(\eta)\geq 1$ such that

$$\frac{1}{n}\int_{\{j(\eta)<|u_n|\}}|u_n|^{p_0(x)-1}\,dx\leq\frac{\eta}{2}. \tag{3.3.27}$$

On the other hand, for any measurable subset $E\subset\Omega$, we have

$$\frac{1}{n}\int_E|u_n|^{p_0(x)-1}\,dx\leq\frac{1}{n}\int_E|T_{j(\eta)}(u_n)|^{p_0(x)-1}\,dx+\frac{1}{n}\int_{\{j(\eta)<|u_n|\}}|u_n|^{p_0(x)-1}\,dx. \tag{3.3.28}$$

It's easy to see that : there exists $\beta(\eta)>0$ such that, for all $E\subseteq\Omega$ with meas$(E)\leq\beta(\eta)$, we have

$$\frac{1}{n}\int_E|T_{j(\eta)}(u_n)|^{p_0(x)-1}\,dx\leq\frac{\eta}{2}. \tag{3.3.29}$$

Finally, by combining $(3.3.27)$, $(3.3.28)$ and $(3.3.29)$, we obtain

$$\frac{1}{n}\int_E|u_n|^{p_0(x)-1}\,dx\leq\eta\quad\text{ for all } E \text{ such that meas}(E)\leq\beta(\eta), \tag{3.3.30}$$

We conclude that $(\frac{1}{n}|u_n|^{p_0(x)-2}u_n)_n$ is uniformly equi-integrable, and since

$$\frac{1}{n}|u_n|^{p_0(x)-2}u_n\longrightarrow 0\quad\text{a.e in}\quad\Omega.$$

In view of Vitali's Theorem, the convergence $(3.3.25)$ is deduced.

Step 4 : Convergence of the gradient.

Denote by $\varepsilon_i(n)\quad i=1,\,2,\ldots$ various functions of real numbers which converge to 0 as n tends to infinity (respectively for $\varepsilon_i(j,n)$).

Let $j\geq k>0$ and $h_j(u_n)=1-|T_1(u_n-T_j(u_n))|,$ Taking $v=u_n-\eta(T_k(u_n)-T_k(u))h_j(u_n)$, we have $v\geq\Psi$ for η positive small enough, thus v is an admissible test function in $(3.3.1)$, and we obtain

$$\begin{aligned}&\sum_{i=1}^{N}\int_\Omega a_i(x,T_n(u_n),\nabla u_n)D^i\big((T_k(u_n)-T_k(u))h_j(u_n)\big)\,dx\\&\quad+\frac{1}{n}\int_\Omega|u_n|^{p_0(x)-2}u_n(T_k(u_n)-T_k(u))h_j(u_n)\,dx\\&\leq\int_\Omega f_n(T_k(u_n)-T_k(u))h_j(u_n)\,dx+\sum_{i=1}^{N}\int_\Omega F_iD^i\big((T_k(u_n)-T_k(u))h_j(u_n)\big)\,dx.\end{aligned}$$

Since $h_j(u_n) = 1$ on $\{|u_n| \leq k\}$ and $T_k(u_n) - T_k(u)$ have the same sign as u_n on the set $\{|u_n| > k\}$. Then

$$\begin{aligned}
&\sum_{i=1}^{N} \int_\Omega a_i(x, T_n(u_n), \nabla u_n)(D^i T_k(u_n) - D^i T_k(u)) h_j(u_n)\, dx \\
&\quad - \sum_{i=1}^{N} \int_{\{j<|u_n|\leq j+1\}} a_i(x, T_n(u_n), \nabla u_n)|T_k(u_n) - T_k(u)| D^i u_n\, dx \\
&\quad + \frac{1}{n} \int_{\{|u_n|\leq k\}} |T_k(u_n)|^{p_0(x)-2} T_k(u_n)(T_k(u_n) - T_k(u))\, dx \\
&\leq \int_\Omega f_n (T_k(u_n) - T_k(u)) h_j(u_n)\, dx - \sum_{i=1}^{N} \int_{\{j<|u_n|\leq j+1\}} F_i |T_k(u_n) - T_k(u)| D^i u_n\, dx \\
&\quad + \sum_{i=1}^{N} \int_\Omega F_i (D^i T_k(u_n) - D^i T_k(u)) h_j(u_n)\, dx.
\end{aligned}$$

It follows that

$$\begin{aligned}
&\sum_{i=1}^{N} \int_{\{|u_n|\leq k\}} a_i(x, T_k(u_n), \nabla T_k(u_n)) \big(D^i T_k(u_n) - D^i T_k(u)\big)\, dx \\
&\leq \int_\Omega |f_n| |T_k(u_n) - T_k(u)|\, dx - \frac{1}{n} \int_{\{|u_n|\leq k\}} |T_k(u_n)|^{p_0(x)-2} T_k(u_n)(T_k(u_n) - T_k(u))\, dx \\
&\quad + \sum_{i=1}^{N} \int_{\{k<|u_n|\leq j+1\}} |a_i(x, T_{j+1}(u_n), \nabla T_{j+1}(u_n))|\, |D^i T_k(u)|\, dx \\
&\quad + \sum_{i=1}^{N} \int_{\{j<|u_n|\leq j+1\}} a_i(x, T_{j+1}(u_n), \nabla T_{j+1}(u_n)) D^i u_n |T_k(u_n) - T_k(u)|\, dx \\
&\quad + \sum_{i=1}^{N} \int_\Omega F_i (D^i T_k(u_n) - D^i T_k(u)) h_j(u_n)\, dx \\
&\quad + \sum_{i=1}^{N} \int_{\{j<|u_n|\leq j+1\}} F_i |T_k(u_n) - T_k(u)| D^i u_n\, dx.
\end{aligned} \tag{3.3.31}$$

For the first and second terms on the right-hand side of (3.3.31), since $T_k(u_n) \to T_k(u)$ strongly in $L^{p_0(\cdot)}(\Omega)$ and weak-$\star$ in $L^\infty(\Omega)$, then

$$\varepsilon_1(n) = \int_\Omega |f_n| |T_k(u_n) - T_k(u)|\, dx \longrightarrow 0 \qquad \text{as} \quad n \to \infty, \tag{3.3.32}$$

and

$$\varepsilon_2(n) = \frac{1}{n} \int_{\{|u_n|\leq k\}} |T_k(u_n)|^{p_0(x)-2} T_k(u_n)(T_k(u_n) - T_k(u))\, dx \longrightarrow 0 \qquad \text{as} \quad n \to \infty. \tag{3.3.33}$$

Concerning the third term on the right-hand side of (3.3.31), since $\big(|a_i(x, T_{j+1}(u_n), \nabla T_{j+1}(u_n))|\big)_n$ is bounded in $L^{p_i'(x)}(\Omega)$, there exists $\zeta_i \in L^{p_i'(\cdot)}(\Omega)$ such that $|a_i(x, T_{j+1}(u_n), \nabla T_{j+1}(u_n))| \rightharpoonup \zeta_i$ in $L^{p_i'(\cdot)}(\Omega)$,

then

$$\begin{aligned}\varepsilon_3(n) &= \sum_{i=1}^{N}\int_{\{k<|u_n|\le j+1\}} |a_i(x,T_{j+1}(u_n),\nabla T_{j+1}(u_n))|\,|D^iT_k(u)|\,dx \\ &\longrightarrow \sum_{i=1}^{N}\int_{\{k<|u|\le j+1\}} \zeta_i\,|D^iT_k(u)|\,dx = 0 \qquad \text{as} \quad n\to\infty,\end{aligned} \tag{3.3.34}$$

For the fourth term on the right-hand side of $(3.3.31)$, thanks to $(3.2.2)$ and $(3.3.26)$, we can obtain

$$\begin{aligned}\varepsilon_4(j) &= \sum_{i=1}^{N}\int_{\{j\le|u_n|\le j+1\}} a_i(x,T_{j+1}(u_n),\nabla T_{j+1}(u_n))D^iu_n|T_k(u_n)-T_k(u)|\,dx \\ &\le 2k\sum_{i=1}^{N}\int_{\{j\le|u_n|\le j+1\}} a_i(x,T_{j+1}(u_n),\nabla T_{j+1}(u_n))D^iu_n\,dx \longrightarrow 0\end{aligned} \tag{3.3.35}$$

as $\quad j\to\infty$.

For the two last terms, we have $D^iT_k(u_n)\rightharpoonup D^iT_k(u)$ in $L^{p_i(\cdot)}(\Omega)$, we obtain

$$\begin{aligned}&\varepsilon_5(n)=\int_\Omega F_i(D^iT_k(u_n)-D^iT_k(u))h_j(u_n)\,dx \le \int_\Omega |F_i|\,|D^iT_k(u_n)-D^iT_k(u)|\,dx\to 0\\ &\text{as}\quad n\to\infty,\end{aligned} \tag{3.3.36}$$

also, in view Young's inequality, we have

$$\begin{aligned}\varepsilon_6(j) &= \int_{\{j<|u_n|\le j+1\}} F_i|T_k(u_n)-T_k(u)|D^iu_n\,dx \\ &\le 2k\int_{\{j<|u_n|\le j+1\}}|F_i|^{p_i'(\cdot)}\,dx + 2k\int_{\{j<|u_n|\le j+1\}}|D^iu_n|^{p_i(\cdot)}\,dx\to 0 \quad \text{as} \quad j\to\infty.\end{aligned} \tag{3.3.37}$$

By combining $(3.3.31)-(3.3.37)$, we obtain

$$\sum_{i=1}^{N}\int_{\{|u_n|\le k\}} a_i(x,T_k(u_n),\nabla T_k(u_n))\big(D^iT_k(u_n)-D^iT_k(u)\big)\,dx \le \varepsilon_7(n,j), \tag{3.3.38}$$

then

$$\begin{aligned}&\sum_{i=1}^{N}\int_\Omega \big(a_i(x,T_k(u_n),\nabla T_k(u_n))-a_i(x,T_k(u_n),\nabla T_k(u))\big)\big(D^iT_k(u_n)-D^iT_k(u)\big)\,dx\\ &\le -\sum_{i=1}^{N}\int_\Omega a_i(x,T_k(u_n),\nabla T_k(u))\big(D^iT_k(u_n)-D^iT_k(u)\big)\,dx\\ &\quad -\sum_{i=1}^{N}\int_{\{|u_n|>k\}} a_i(x,T_k(u_n),\nabla T_k(u_n))D^iT_k(u)\,dx+\varepsilon_8(n,j).\end{aligned} \tag{3.3.39}$$

We have $T_k(u_n)\to T_k(u)$ in $L^{\underline{p}}(\Omega)$, then $a_i(x,T_k(u_n),\nabla T_k(u))\to a_i(x,T_k(u),\nabla T_k(u))$ in $L^{p_i'(\cdot)}(\Omega)$ and since $D^iT_k(u_n)\rightharpoonup D^iT_k(u)$ in $L^{p_i(\cdot)}(\Omega)$, then,

$$\lim_{n\to\infty}\int_\Omega a_i(x,T_k(u_n),\nabla T_k(u))\big(D^iT_k(u_n)-D^iT_k(u)\big)\,dx=0,$$

and similarly to $(3.3.34)$, we can prove that

$$\lim_{n\to\infty}\int_{\{|u_n|>k\}} a_i(x,T_k(u_n),\nabla T_k(u_n))D^iT_k(u)\,dx=0.$$

by letting n and j goes to infinity in $(3.3.39)$ we deduce that

$$\lim_{n\to\infty}\sum_{i=1}^{N}\int_\Omega (a_i(x,T_k(u_n),\nabla T_k(u_n))-a_i(x,T_k(u_n),\nabla T_k(u)))(D^iT_k(u_n)-D^iT_k(u))\,dx=0,$$

and since $T_k(u_n)\to T_k(u)$ in $L^{p_0(\cdot)}(\Omega)$, then

$$\begin{aligned}&\sum_{i=1}^{N}\int_\Omega (a_i(x,T_k(u_n),\nabla T_k(u_n))-a_i(x,T_k(u_n),\nabla T_k(u)))(D^iT_k(u_n)-D^iT_k(u))\,dx\\&\quad+\int_\Omega (|T_k(u_n)|^{p_0(\cdot)-2}T_k(u_n)-|T_k(u)|^{p_0(\cdot)-2}T_k(u))(T_k(u_n)-T_k(u))\,dx\longrightarrow 0\\&\text{as}\quad n\to\infty.\end{aligned}\tag{3.3.40}$$

Using the Lemma 1.2.8, we deduce that

$$T_k(u_n)\longrightarrow T_k(u)\quad\text{in}\quad W_0^{1,\vec{p}(\cdot)}(\Omega).\quad\text{Therefore,}\quad D^iu_n\longrightarrow D^iu\quad\text{a.e. in }\Omega,\tag{3.3.41}$$

for $\quad i=1,\dots,N$

Step 5 : Passing to the limit.

Let $\varphi\in K_\psi\cap L^\infty(\Omega)$ and $M=k+\|\varphi\|_\infty$. By using $v=u_n-\eta T_k(u_n-\varphi)$ as a test function in $(3.3.1)$, with η positive small enough, we obtain

$$\begin{cases}u_n\in K_\psi\\ \displaystyle\sum_{i=1}^{N}\int_\Omega a_i(x,T_n(u_n),\nabla u_n)D^iT_k(u_n-\varphi)\,dx+\frac{1}{n}\int_\Omega |u_n|^{p_0(x)-2}u_nT_k(u_n-\varphi)\,dx\\ \displaystyle\le\int_\Omega f_nT_k(u_n-\varphi)\,dx+\sum_{i=1}^{N}\int_\Omega F_iD^iT_k(u_n-\varphi)\,dx.\end{cases}\tag{3.3.42}$$

On one hand, we have $\{|u_n-\varphi|\le k\}\subseteq\{|u_n|\le M\}$ then

$$\begin{aligned}&\int_\Omega a_i(x,T_n(u_n),\nabla u_n)D^iT_k(u_n-\varphi)\,dx\\&=\int_\Omega a_i(x,T_M(u_n),\nabla T_M(u_n))(D^iT_M(u_n)-D^i\varphi)\chi_{\{|u_n-\varphi|\le k\}}\,dx\\&=\int_\Omega (a_i(x,T_M(u_n),\nabla T_M(u_n))-a_i(x,T_M(u_n),\nabla\varphi))(D^iT_M(u_n)-D^i\varphi)\chi_{\{|u_n-\varphi|\le k\}}\,dx\\&\quad+\int_\Omega a_i(x,T_M(u_n),\nabla\varphi)(D^iT_M(u_n)-D^i\varphi)\chi_{\{|u_n-\varphi|\le k\}}\,dx.\end{aligned}\tag{3.3.43}$$

According to Fatou's Lemma, we obtain

$$\begin{aligned}
&\liminf_{n\to\infty}\sum_{i=1}^{N}\int_{\Omega}a_i(x,T_n(u_n),\nabla u_n)D^iT_k(u_n-\varphi)\,dx\\
&\geq\sum_{i=1}^{N}\int_{\Omega}(a_i(x,T_M(u),\nabla T_M(u))-a_i(x,T_M(u),\nabla\varphi))(D^iT_M(u)-D^i\varphi)\chi_{\{|u-\varphi|\leq k\}}\,dx\\
&\quad+\lim_{n\to\infty}\sum_{i=1}^{N}\int_{\Omega}a_i(x,T_M(u_n),\nabla\varphi)(D^iT_M(u_n)-D^i\varphi)\chi_{\{|u_n-\varphi|\leq k\}}\,dx\\
&\geq\sum_{i=1}^{N}\int_{\Omega}(a_i(x,T_M(u),\nabla T_M(u))-a_i(x,T_M(u),\nabla\varphi))(D^iT_M(u)-D^i\varphi)\chi_{\{|u-\varphi|\leq k\}}\,dx \qquad (3.3.44)\\
&\quad+\sum_{i=1}^{N}\int_{\Omega}a_i(x,T_M(u),\nabla\varphi)(D^iT_M(u)-D^i\varphi)\chi_{\{|u-\varphi|\leq k\}}\,dx\\
&=\sum_{i=1}^{N}\int_{\Omega}a_i(x,T_M(u),\nabla T_M(u))(D^iT_M(u)-D^i\varphi)\chi_{\{|u-\varphi|\leq k\}}\,dx\\
&=\sum_{i=1}^{N}\int_{\Omega}a_i(x,u,\nabla u)D^iT_k(u-\varphi)\,dx.
\end{aligned}$$

On the other hand, we have $T_k(u_n-\varphi)\rightharpoonup T_k(u-\varphi)$ weak-$\star$ in $L^{\infty}(\Omega)$ and in view of $(3.3.25)$, we obtain

$$\frac{1}{n}\int_{\Omega}|u_n|^{p_0(x)-2}u_nT_k(u_n-\varphi)\,dx\longrightarrow 0, \qquad (3.3.45)$$

and

$$\int_{\Omega}f_nT_k(u_n-\varphi)\,dx\longrightarrow\int_{\Omega}f\,T_k(u-\varphi)\,dx. \qquad (3.3.46)$$

Also, since $T_k(u_n-\varphi)\to T_k(u-\varphi)$ in $W_0^{1,\vec{p}(\cdot)}(\Omega)$, then

$$\int_{\Omega}F_iD^iT_k(u_n-v)\,dx\longrightarrow\int_{\Omega}F_iD^iT_k(u-v)\,dx. \qquad (3.3.47)$$

Which completes the proof of Theorem 3.3.1.

Exemple 3.3.1. *We consider the following functions*

$$a_l(x,u,\nabla u)=|\nabla u|^{p_l(x)-2}\nabla u,$$

for any $l=1,\dots,N$. *It is easy to check that* $a(x,u,\nabla u)$ *verifies* $(3.2.2)-(3.2.4)$, *which implies, under the condition* $(3.2.1)$, *that the quasilinear anisotropic elliptic unilateral problem*

$$\begin{cases}
-\displaystyle\sum_{l=1}^{N}\frac{\partial}{\partial x_l}\left(|\frac{\partial u}{\partial x_l}|^{p_l(x)-2}\frac{\partial u}{\partial x_l}\right)=f-div\ F & in\ \ \Omega,\\
u=0 \quad on\ \ \partial\Omega, \qquad f\in L^1(\Omega), \qquad F\in\prod_{l=1}^{N}L^{p_l'(\cdot)}(\Omega)
\end{cases} \qquad (3.3.48)$$

has at least one entropy solutions in the convex subset of anisotropic variable exponent Sobolev spaces

$$K_\psi := \big\{u \in W_0^{1,\vec{p}(x)}(\Omega),\ u \geq 0 \text{ a.e. in } \Omega\big\},$$

where $\psi(\cdot) = 0$ *is a fixed obstacle function, such that*

$$\psi^+ \in W_0^{1,\vec{p}(x)}(\Omega) \cap L^\infty(\Omega). \tag{3.3.49}$$

Chapter 4

Obstacle problems depending on time for strongly nonlinear anisotropic parabolic equations with variable exponents

4.1 Introduction

In this chapter, we are interested to deal with the proof of the existence of a solution for a class of strongly nonlinear anisotropic unilateral problems, with non-standard structural conditions involving a variable growth exponent $\vec{p}(.)$, and an obstacle depending on time. Such obsatcle problems model many phenomena, including constrained heating, optimal control, elastoplasticity, financial mathematics, the study of the shape of an elastic membrane which is pushed by an obstacle from one side affecting its shape, minimal surfaces and the capacity of set in potential theory. Let's consider Ω be a bounded open subset of $\mathbf{R}^N$ $(N \geq 2)$. Fixing a final time $0 < T < \infty$, and let Q_T be $\Omega \times (0,T)$ and $\partial Q_T = \partial\Omega \times]0,T[$. For $i \in \{0,\ldots,N\}$, we set $p_i(x) := p_i(x,t) \in \mathcal{C}_+(Q_T)$ as an extension of a variable exponent $p_i(.) : \overline{\Omega} \to (1,\infty)$, where $p_i(.) > 2$ and

$$p_0(x) \geq \max\{p_i(x), \quad i = 1,2,\ldots,N\} \quad \text{a.e. in} \quad \overline{\Omega}. \tag{4.1.1}$$

Our intention is to establish the existence of solutions to the following strongly nonlinear anisotropic

obstacle problem with respect to the perturbations in the growth exponent $\overrightarrow{p}(.)$

$$\begin{cases} \dfrac{\partial u}{\partial t} - \text{div } a(x,t,u,\nabla u) + \displaystyle\sum_{i=1}^{N} h_i(x,t,u,\nabla u) + \Theta(x,t)|u|^{p_0(x)-2}u = \mu + f & \text{in } Q_T, \\ u \geq 0, \quad \mu(u-\psi) = 0, \quad u \geq \psi & \text{in } Q_T, \\ u(x,t) = 0 & \text{on } \partial Q_T, \\ u(x,0) = u_0(x) & \text{in } \Omega. \end{cases} \tag{4.1.2}$$

Where

$$f \in L^{p_i'(.)}(Q_T), \tag{4.1.3}$$

$$u_0 \in L^2(\Omega), \tag{4.1.4}$$

and $\Theta(x,t) \in L^\infty(Q_T)$, and there exists a constant $\Theta_0 > 0$ such that $\Theta(x,t) \geq \Theta_0$ almost everywhere in Q_T.

We denote $Au = -\sum_{i=1}^{N} D^i a_i(x,t,u,\nabla u)$ the Leray-Lions operator acting from $V_0(Q_T)$ (see the next section) into its dual $V^*(Q_T)$ where $a_i : \Omega\times]0,T[\times I\!R \times I\!R^N \longmapsto I\!R^N$ are Carathéodory functions, for $i = 1,\ldots,N$, (measurable with respect to (x,t) in $\Omega\times]0,T[$ for every (s,ξ) in $I\!R \times I\!R^N$ and continuous with respect to (s,ξ) in $I\!R \times I\!R^N$ for almost every (x,t) in $\Omega\times]0,T[$, which satisfy for a.e. $(x,t) \in Q_T$ and all $(s,\xi) \in I\!R \times I\!R^N$ the following conditions :

$$|a_i(x,t,s,\xi)| \leq \gamma(C_i(x,t) + |s|^{p_i(x)-1} + \sum_{i=1}^{N} |\xi_i|^{p_i(x)-1}), \tag{4.1.5}$$

$$\sum_{i=1}^{N} a_i(x,t,s,\xi)\xi_i \geq \rho \sum_{i=1}^{N} |\xi_i|^{p_i(x)}, \tag{4.1.6}$$

$$\big(a_i(x,t,s,\xi) - a_i(x,t,s,\xi')\big)(\xi_i - \xi_i') > 0 \quad \text{for} \quad \xi_i \neq \xi_i', \tag{4.1.7}$$

for any $\xi = (\xi_1,\ldots,\xi_N)$ and $\xi' = (\xi_1',\ldots,\xi_N')$ in $I\!R^N$, where $C_i(x,t))$ is a positive function lying in $L^{p_i'(\cdot)}(Q_T)$, with $\frac{1}{p_i(x)} + \frac{1}{p_i'(x)} = 1$ and γ, ρ are two positive real numbers.

The obstacle ψ is assumed to check for $i = 1,\ldots,N$, the undermentioned conditions

$$\frac{\partial \psi}{\partial t} \in L^{p_i'(.)}(\Omega\times]0,T[), \tag{4.1.8}$$

$$\psi \in V(Q_T), \tag{4.1.9}$$

$$\psi^+ \in L^\infty(Q_T), \tag{4.1.10}$$

$$\psi \leq 0 \quad \text{on } \partial Q_T \quad \text{and} \quad \psi(0) \leq u_0 \quad \text{a.e. in } \Omega. \tag{4.1.11}$$

The nonlinear lower order term $h_i(x,t,s,\xi)$ is a Carathéodory function that satisfies for almost every $(x,t) \in Q_T$ and for all $(s,\xi) \in I\!R \times I\!R^N$,

$$h_i(x,t,s,\xi)s \geq 0, \qquad \forall\ i = 1,...,N, \tag{4.1.12}$$

$$\sum_{i=1}^{N} |h_i(x,t,s,\xi)| \leq j(|s|)(\sum_{i=1}^{N} |\xi_i|^{r_i(x)} + l(x,t)), \qquad \text{where } 1 < r_i(x) < p_i(x) - 1 \quad \text{a.e. in } \Omega \tag{4.1.13}$$

$l(x,t) \in L^{\infty}(Q_T)$, and $j(.) : I\!R^+ \to I\!R^+$ is a continuous, nonnegative and non-decreasing function, having growth of order $q_0(.)$, with respect to $|s|$:

$$j(|s|) \leq |s|^{q_0(x)} + \lambda, \quad \lambda > 0,\ 1 < q_0(x) < p_i(x) - r_i(x) \text{ a.e. in } \Omega. \tag{4.1.14}$$

It is worth mentioning the paper of Abdou et al. [4] devoted to prove the existence of weak solutions for the anisotropic problem associted with the following equation

$$\begin{cases} \dfrac{\partial u}{\partial t} + Au + g(x,t,u) + \gamma |u|^{p_0 - 2}u = f & \text{in } Q_T = \Omega \times (0,T), \\ u(x,t) = 0 & \text{on } S_T = \partial\Omega \times (0,T), \\ u(x,0) = u_0(x) & \text{in } \Omega, \end{cases} \tag{4.1.15}$$

where $u_0 \in L^2(\Omega)$, $\gamma > 0$, $f \in L^{\overrightarrow{p'}}(0,T; W^{-1,\overrightarrow{p'}}(\Omega))$ and $Au = -\sum_{i=1}^{N} D^i a_i(x,t,\nabla u)$, while $g(x,t,u)$ has growth and sign condition on s. We refer the interest reader to some nonlinear elliptic and parabolic problems involving anisotropic operators as witnessed by the papers [56, 27, 36, 13, 78, 11] (see also [15, 110, 147, 148, 103, 104, 149] and the references therein). It should be pointed out that within the setting of Sobolev spaces with variable exponents there has been an important breakthrough in research work in this field during the last decade. For example, in the isotropic parabolic field, Azroul et als. have established an existence result of a renormalized solution to the doubly nonlinear parabolic equations with variable exponents

$$\begin{cases} \dfrac{\partial b(u)}{\partial t} + Au + H(x,t,u,\nabla u) = f & \text{in } Q = \Omega \times]0,T[, \\ u = 0 & \text{on } \partial\Omega \times]0,T[, \\ b(u)(t=0) = b(u_0) & \text{in } \Omega, \end{cases} \tag{4.1.16}$$

where $f \in L^1(Q)$, the function b is assumed to be strictly increasing $\mathcal{C}^1-$function, and H is a nonlinear lower order term satisfying the growth condition of the form

$$|H(x,t,s,\xi) \leq b(s)\big(\sum_{i=1}^{i=N} |\xi_i|^{p(x)} + c(x,t).$$

On the other hand, several works dealing with elliptic aniotropic prolemes with variable exponents have been published lately (see [28, 34, 33, 67, 134, 127]). As regards the parabolic case, Bendahmane et als. [40] proved an existence of distributional solutions for the nonlinear parabolic equations of the form

$$\begin{cases} \dfrac{\partial u}{\partial t} - \text{div}A(t,x,\nabla u) + g(t,x,u) = f(t,x), \\ u(x,0) = u_0(x), \end{cases} \tag{4.1.17}$$

where $(t,x) \in Q_T := (0,T) \times \Omega$, $T > 0$ is given, $f \in L^1(Q_T)$, $u_0 \in L^1(\Omega)$ and improved conditions on g. Further, they established that u possess the regularity

$$u \in \cap L^{q_l^-}(0,T,W_0^{1,q_l(.)}(\Omega)),$$

for all continuous functions $q_1(.),\ldots,q_N(.)$ on $\overline{\Omega}$ such that

$$1 \leq q_l(.) \leq \frac{p_l(.)}{\overline{p}(.)}\big(\overline{p}(.) - \frac{N}{N+1}\big).$$

We highlight that in the isotropic case with constant exponents, when the lower order term $g(x,t,u,\nabla u)$ in $(4.1.2)$ vanishes, Flavio Donati described a method to solve so similar nonlinear unilateral evolution problems when the obstacle is dependent on time by using an approch of a penality method (see e.g [90] and also [121]). The interested reader can consult for such problems [72, 91, 116, 137] for more explanations. Note that within the same field of research but still in the isotropic case with constant exponents, Bellal [31] generalized the work of Mokran [130] which was limited to the condition $\psi = 0$.

In this paper, which could be seen as generalization of [4], [31] and [20], we try to show an existence solution to $(4.1.2)$ taking into account the complications of nonlinearities due to the variation of exponents $p_i(.)$. Indeed, using the penalization method by introducing the term $\frac{1}{(1/n)^{p_0(x)-1}}|(u_n-\psi)^-|^{p_0(x)-2}(u_n-\psi)^-$, we can thus prove an existence of weak solutions u_n of an approximate problem. Then, we establish some a priori estimates on these approximate solutions. In second a step, by combining a compactness theorems and a complementary condition, we show that a subsequence of an approximate solution u_n converges to a solution of $(4.1.2)$ and satisfies our results.

The outline of this chapter is organized as follows: Section 2 is devoted to useful results including some propositions and technical lemmas which will be used in the prof of our main result. The main existence results are stated and proved in Section 3.

4.2 Useful results

Proposition 4.2.1. *Let u be in $W_0^{1,\overrightarrow{p}(.)}(\Omega)$. If $\xi \to H(\xi)$ is a uniformaly Lipschitzian function (i.e. $|H(\xi) - H(\xi')| \leq k|\xi - \xi'|$) defined for $\xi \in I\!R$, such that $H(0) = 0$, then*

$$H(u) \in W_0^{1,\overrightarrow{p}(.)}(\Omega), \quad \textit{and} \quad D^i H(u) = H'(u) D^i u$$

Proof. The proof is similar to that proposed in [146].

As a result of this proposition, we have if $u \in W_0^{1,\overrightarrow{p}(.)}(\Omega)$ then $u^- = \max(0, -u)$, $u^+ = \max(0, u)$ and $|u|$ are in $W_0^{1,\overrightarrow{p}(.)}(\Omega)$.

Lemma 4.2.1. *Let $g \in L^{r(x)}(Q_T)$ and $g_n \in L^{r(x)}(Q_T)$ with $\|g_n\|_{L^{r(x)}(Q_T)} \leq C$ for $1 < r(x) < \infty$. If $g_n(x,t) \to g(x,t)$ a.e. on Q_T, then $g_n \rightharpoonup g$ in $L^{r(x)}(Q_T)$.*

Proof. The proof is similar to the proof of (Lemma 3.3, [32]) by taking Q_T instead Ω.

Lemma 4.2.2. *Assuming that $(4.1.5) - (4.1.7)$ hold, and let $(u_n)_{n \in I\!N}$ be a sequence in $V_0(Q_T)$ such that $u_n \rightharpoonup u$ in $V_0(Q_T)$ and*

$$\begin{aligned} &\int_0^T \int_\Omega (|u_n|^{p_0(x)-2} u_n - |u|^{p_0(x)-2} u)(u_n - u)\, dxdt \\ &\quad + \sum_{i=1}^N \int_0^T \int_\Omega (a_i(x,t,u_n,\nabla u_n) - a_i(x,t,u_n,\nabla u))(D^i u_n - D^i u)\, dxdt \to 0, \end{aligned} \tag{4.2.1}$$

then $u_n \longrightarrow u$ in $V_0(Q_T)$ for a subsequence.

Proof of Lemma 4.2.2 Let

$$\begin{aligned} S_n \;&= \sum_{i=1}^N (a_i(x,t,u_n,\nabla u_n) - a_i(x,t,u_n,\nabla u))(D^i u_n - D^i u) \\ &\quad + (|u_n|^{p_0(x)-2} u_n - |u|^{p_0(x)-2} u)(u_n - u). \end{aligned}$$

Thanks to $(4.1.7)$, we deduce that S_n is a positive function and, by $(4.2.1)$, $S_n \to 0$ in $L^1(Q_T)$ as $n \to \infty$. Since $u_n \rightharpoonup u$ in $V_0(Q_T)$ using the compact embedding we obtain $u_n \to u$ in $L^{p(.)}(Q_T)$, and since $S_n \to 0$ a.e. in Q_T, there exists a subset B in Q_T with measure zero such that $\forall x \in Q_T \backslash B$

$$|u(x,t)| < \infty, \qquad |D^i u(x,t)| < \infty, \qquad c_i(x,t) < \infty, \qquad u_n \to u \qquad \textit{and} \qquad S_n \to 0.$$

We have

$$\begin{aligned}
S_n(x,t) &= \sum_{i=1}^{N}((a_i(x,t,u_n,\nabla u_n)-a_i(x,t,u_n,\nabla u))(D^iu_n-D^iu)\\
&\quad +(|u_n|^{p_0(x)-2}u_n-|u|^{p_0(x)-2}u)(u_n-u)\\
&=\sum_{i=1}^{N}\Big(a_i(x,t,u_n,\nabla u_n)D^iu_n+a_i(x,t,u_n,\nabla u)D^iu\\
&\quad -a_i(x,t,u_n,\nabla u)D^iu_n-a_i(x,t,u_n,\nabla u_n)D^iu\Big)\\
&\quad +|u_n|^{p_0(x)}+|u|^{p_0(x)}-|u_n|^{p_0(x)-2}u_nu-|u|^{p_0(x)-2}uu_n\\
&\geq \underline{\rho}\sum_{i=0}^{N}|D^iu_n|^{p_i(x)}+\underline{\rho}\sum_{i=0}^{N}|D^iu|^{p_i(x)}-\gamma\sum_{i=1}^{N}(c_i(x,t)+|u_n|^{p_i(x)-1}+|D^iu|^{p_i(x)-1})|D^iu_n|\\
&\quad -\gamma\sum_{i=1}^{N}(c_i(x,t)+|u_n|^{p_i(x)-1}+|D^iu_n|^{p_i(x)-1})|D^iu|-|u_n|^{p_0(x)-1}|u|-|u|^{p_0(x)-1}|u_n|\\
&\geq \underline{\rho}\sum_{i=0}^{N}|D^iu_n|^{p_i(x)}-K_{x,t}\sum_{i=1}^{N}(1+|D^iu_n|^{p_i(x)-1}+|D^iu_n|),
\end{aligned}$$

with $\underline{\rho}=\min(\rho,1)$ and $K_{x,t}$ depending only on x and t. It follows that

$$S_n(x,t)\geq\sum_{i=0}^{N}|D^iu_n|^{p_i(x)}\big(\underline{\rho}-\frac{K_{x,t}}{|D^iu_n|^{p_i(x)}}-\frac{K_{x,t}}{|D^iu_n|}-\frac{K_{x,t}}{|D^iu_n|^{p_i(x)-1}}\big).$$

By a standard argument, $(D^iu_n)_{n\in I\!N}$ is bounded almost everywhere in Q_T for all $i=0,1,\ldots,N$. Indeed, if $|D^iu_n|\to\infty$ in a measurable subset $E\subset Q_T$ then

$$\lim_{n\to\infty}\int_{Q_T}S_n(x,t)\,dxdt\geq\lim_{n\to\infty}\sum_{i=1}^{N}\int_E|D^iu_n|^{p_i(x)}\big(\underline{\rho}-\frac{K_{x,t}}{|D^iu_n|^{p_i(x)}}-\frac{K_{x,t}}{|D^iu_n|}-\frac{K_{x,t}}{|D^iu_n|^{p_i(x)-1}}\big)\,dxdt=\infty,$$

which is a contradiction since $S_n\to 0$ in $L^1(Q_T)$.

Let ξ_i^* an accumulation point of $(D^iu_n)_{n\in I\!N}$ for $i=1,\ldots,N$. We have $|\xi_i^*|<\infty$ and by the continuity of $a(x,t,\cdot,\cdot)$, we obtain

$$\Big(a_i(x,t,u_n,\xi^*)-a_i(x,t,u_n,\nabla u)\Big)(\xi_i^*-D^iu)=0\qquad\text{for}\quad i=1,\ldots,N.$$

Thanks to (4.1.7), we have $\xi^*=\nabla u$ and the uniqueness of the accumulation point implies that $\nabla u_n\to\nabla u$ a.e in Ω.

Since $(a_i(x,t,u_n,\nabla u_n))_{n\in I\!N}$ is bounded in $L^{p_i'(.)}(Q_T)$ and $a_i(x,t,u_n,\nabla u_n)\to a_i(x,t,u,\nabla u)$ a.e in Q_T, by the Lemma 4.2.1, we can establish that

$$a_i(x,t,u_n,\nabla u_n)\rightharpoonup a_i(x,t,u,\nabla u)\quad in\quad L^{p_i'(.)}(Q_T)\quad\text{for}\quad i=1,\ldots,N.$$

Using $(4.2.1)$ and the Lemma 1.2.2, we deduce that

$$|u_n|^{p_0(x)} \longrightarrow |u|^{p_0(x)} \qquad \text{in} \quad L^1(Q_T) \tag{4.2.2}$$

and

$$a_i(x,t,u_n,\nabla u_n)D^i u_n \longrightarrow a_i(x,t,u,\nabla u)D^i u \qquad \text{in} \quad L^1(Q_T). \tag{4.2.3}$$

According to the condition $(4.1.6)$, we have

$$\alpha|D^i u_n|^{p_i(x)} \ \leq \ a_i(x,t,u_n,\nabla u_n)D^i u_n \quad \text{for} \quad i=1,\dots,N.$$

Let $y_n^i = \frac{1}{\rho}a_i(x,t,u_n,\nabla u_n)D^i u_n$ and $y^i = \frac{1}{\rho}a_i(x,t,u,\nabla u)D^i u$, accordingly to Fatou's Lemma, we get

$$\int_{Q_T} 2y^i \, dxdt \leq \liminf_{n\to\infty} \int_{Q_T} (y_n^i + y^i - |D^i u_n - D^i u|^{p_i(x)}) \, dxdt,$$

hence, $0 \leq -\limsup_{n\to\infty} \int_{Q_T} |D^i u_n - D^i u|^{p_i(x)} \, dxdt$, and since

$$0 \leq \liminf_{n\to\infty} \int_{Q_T} |D^i u_n - D^i u|^{p_i(x)} \, dxdt \leq \limsup_{n\to\infty} \int_{Q_T} |D^i u_n - D^i u|^{p_i(x)} \, dxdt \leq 0,$$

it follows that

$$\int_{Q_T} |D^i u_n - D^i u|^{p_i(x)} \, dxdt \longrightarrow 0 \text{ as } n \to \infty.$$

Therefore, we obtain

$$D^i u_n \longrightarrow D^i u \quad \text{in} \quad L^{p_i(.)}(Q_T) \quad \text{for} \quad i=1,\dots,N.$$

Finally, in view of $(4.2.2)$ we conclude that

$$u_n \longrightarrow u \qquad \text{in} \quad V_0(Q_T).$$

4.3 Statement of results

In this section, we state and prove the existence of solutions to the problem $(4.1.2)$.

Theorem 4.3.1. *Let $f \in L^{p_i'(.)}(Q_T)$, $u_0 \in L^2(\Omega)$, $\Theta(x,t) \in L^\infty(Q_T)$, and there exists a constant $\Theta_0 > 0$ such that $\Theta(x,t) \geq \Theta_0$ almost everywhere in Q_T. Assuming further that,*

(1) *the conditions $(4.1.5)$-$(4.1.14)$, hold,*

(2) *a_i and ψ satisfy "the comlementary condition"*

$$\sum_{i=1}^{N} D^i a_i(x,t,T(u),\nabla\psi) \in L^{p_i'(.)}(Q) \quad \textit{for } u \in V_0(Q_T) \tag{4.3.1}$$

and is bounded in $L^{p_i'(.)}(Q)$ on bounded sets of $V_0(Q_T)$, for $i = 1, \dots, N$. Then the problem $(4.1.2)$ *has at least one couple of solutions u and μ that check*

$$\mu \in L^{p_0'(.)}(Q) \quad \textit{for } i = 1, \dots, N, \tag{4.3.2}$$

$$\mu(u - \psi) = 0 \quad \textit{in } Q, \tag{4.3.3}$$

$$u \in C^0(0, T; W^{-1,\vec{m}(.)}(\Omega)) \quad \textit{for } \inf(p_i(x), p_i'(x), \frac{N}{N-1}) > m_i(x), \tag{4.3.4}$$

$$u \in V_0(Q_T) \cap L^\infty(0, T; L^2(\Omega)), \tag{4.3.5}$$

$$\frac{\partial u}{\partial t} = \sigma_1 + \sigma_2 \quad \textit{with } \sigma_1 \in V^*(Q_T) \quad and \quad \sigma_2 \in L^1(Q_T), \tag{4.3.6}$$

$$u(x, 0) = u_0(x) \quad \textit{in } \Omega, \tag{4.3.7}$$

$$h_i(x, t, u, \nabla u) \in L^1(Q_T) \quad \textit{and} \quad u h_i(x, t, u, \nabla u) \in L^1(Q_T) \quad \textit{for } i = 1, \dots, N. \tag{4.3.8}$$

Remark 4.3.1. *The assumption* $(4.1.1)$ *is essential to ensure that $|a_i(x, t, u, \nabla u)|$ belongs to $L^{p_i'(\cdot)}(Q_T)$. In the case of $Au = -\sum_{i=1}^{N} D^i a_i(x, t, \nabla u)$, the condition* $(4.1.5)$ *will be written as*

$$|a_i(x, t, \xi)| \leq \beta\Big(K_i(x, t) + \sum_{i=1}^{N} |\xi_i|^{p_i(x)-1}\Big), \tag{4.3.9}$$

thus the existence of entropy solution will be guaranteed by following the same way, without using the additional assumption $(4.1.1)$.

Note that, in view of the growth condition $(4.1.5)$, to show that the Carathéodory functions $|a_i(x, u, \nabla u)|$ belongs to $L^{p_i'(\cdot)}(Q_T)$ for any $u \in V_0(Q_T)$, it's necessary to have $u \in L^{p_i(\cdot)}(Q_T)$ for any $i = 1, \dots, N$, which is verified by taking the condition $(4.1.1)$. For the case where the Carathéodory functions $a_i(x, t, \xi)$ verify the assumption $(4.3.9)$, then $|a_i(x, t, \nabla u)| \in L^{p_i'(\cdot)}(Q_T)$ for any $u \in V_0(Q_T)$.

Proof of the Theorem 4.3.1

Step 1: Approximate problems.

We consider the approximate problem

$$\begin{cases} \dfrac{\partial u_n}{\partial t} - \displaystyle\sum_{i=1}^{N} D^i a_i(x,t,T_n(u_n),\nabla u_n) + \Theta(x,t)|u_n|^{p_0(x)-2}u_n \\ \qquad\qquad -\dfrac{1}{(1/n)^{p_0(x)-1}}|(u_n-\psi)^-|^{p_0(x)-2}(u_n-\psi)^- + \displaystyle\sum_{i=1}^{N} h_i^n(x,t,u_n,\nabla u_n) = f \quad \text{in } \Omega\times]0,T[, \\ u(x,t)=0 \qquad\qquad \text{on } \partial\Omega\times]0,T[, \\ u(x,0)=u_0(x) \qquad\qquad \text{in } \Omega. \end{cases} \tag{4.3.10}$$

Let's define the operator A_n from $V_0(Q_T)$ into its dual $V^*(Q_T)$, by

$$A_n\upsilon = -\sum_{i=1}^{N} D^i a_i(x,t,T_n(\upsilon),\nabla\upsilon) - \frac{1}{(1/n)^{p_0(x)-1}}|(\upsilon-\psi)^-|^{p_0(x)-2}(\upsilon-\psi)^-,$$

and for all $n \in I\!N^*$

$$h_i^n(x,t,s,\xi) = \frac{h_i(x,t,s,\xi)}{1+\frac{1}{n}|h_i(x,t,s,\xi)|}, \qquad \forall i=1,\dots,N. \tag{4.3.11}$$

Note that

$$h_i^n(x,t,s,\xi)s \geq 0, \qquad |h_i^n(x,t,s,\xi)| \leq |h_i(x,t,s,\xi)| \quad \text{and} \quad |h_i^n(x,t,s,\xi)| \leq n \qquad \forall\ i=1,\dots,N.$$

We recall that for any function ψ

$$(u_n-\psi)^- = \max(0,-(u_n-\psi)), \text{ otherwise known as the negative part of } (u_n-\psi)$$

we define $H_n : V_0(Q_T) \longmapsto V^*(Q_T)$ by

$$\int_0^T \langle H_n u,\upsilon\rangle = \sum_{i=1}^{N}\int_{Q_T} h_i^n(x,t,u,\nabla u)\upsilon dxdt + \int_{Q_T}\Theta(x,t)|u|^{p_0(x)-2}u\upsilon dxdt \quad \text{for any} \quad u,\upsilon \in V_0(Q_T).$$

Thanks to the generalized Hölder type inequality, we have

$$\begin{aligned}
\int_0^T |\langle H_n u, v\rangle| \ &\leq \sum_{i=1}^{N} \big(\frac{1}{p_0^-} + \frac{1}{(p_0')^-}\big)\bigg(\|h_i^n(x,t,u,\nabla u)\|_{L^{p_0'(\cdot)}(Q_T)} \\
&\qquad + \|\Theta\|_\infty \||u|^{p_0(\cdot)-1}\|_{L^{p_0'(\cdot)}(Q_T)}\bigg)\|v\|_{L^{p_0(\cdot)}(Q_T)} \\
&\leq \big(\frac{1}{p_0^-} + \frac{1}{(p_0')^-}\big)\bigg(\sum_{i=1}^{N} \big(\int_{Q_T} n^{(p_0')^x}\, dxdt + 1\big)^{\frac{1}{(p_0')^-}} \\
&\qquad + \|\Theta\|_\infty \big(\int_{Q_T} |u|^{p_0(x)}\, dxdt + 1\big)^{\frac{1}{(p_0')^-}}\bigg)\|v\|_{L^{p_0(\cdot)}(Q_T)} \\
&\leq \big(\frac{1}{p_0^-} + \frac{1}{(p_0')^-}\big)\bigg(N\big(n^{(p_0')^+} meas(Q_T) + 1\big)^{\frac{1}{(p_0')^-}} \\
&\qquad + \|\Theta\|_\infty \big(\int_{Q_T} |u|^{p_0(x)}\, dxdt + 1\big)^{\frac{1}{(p_0')^-}}\bigg)\|v\|_{L^{p_0(\cdot)}(Q_T)} \\
&\leq C_0 \|v\|_{V_0(Q_T)}
\end{aligned} \tag{4.3.12}$$

and also in view of $(4.1.5)$, we get

$$\begin{aligned}
&\Big| \int_0^T |\langle A_n u, v\rangle| \Big| \\
&\leq \sum_{i=1}^{N} \int_{Q_T} |a_i(x,t,T_n(u),\nabla u)|\, |D^i v|\, dxdt + \frac{1}{(1/(n+1))^{p_0^+ - 1}} \int_{Q_T} |(u-\psi)^-|^{p_0(x)-1} |v|\, dxdt \\
&\leq \sum_{i=1}^{N} \int_{Q_T} \gamma |C_i(x,t) + |T_n(u)|^{p_i(x)-1} + \sum_{i=1}^{N} |D^i u|^{p_i(x)-1})|\, |D^i v|\, dxdt \\
&\quad + \frac{1}{(1/(n+1))^{p_0^+ - 1}} \big(\frac{1}{p_0^-} + \frac{1}{(p_0')^-}\big) \||(u-\psi)^-|^{p_0(x)-1}\|_{L^{p_0'(\cdot)}(Q_T)} \|v\|_{L^{p_0(\cdot)}(Q_T)} \\
&\leq \gamma \sum_{i=1}^{N} \big(\frac{1}{p_i^-} + \frac{1}{(p_i')^-}\big)\bigg(\|C_i(x,t)\|_{L^{p_i'(\cdot)}(Q_T)} + \big(n^{(p_i')^+} meas(Q_T) + 1\big)^{\frac{1}{(p_i')^-}} \\
&\quad + \sum_{i=1}^{N} \big(\int_{Q_T} |D^i u|^{p_i(x)}\, dxdt + 1\big)^{\frac{1}{(p_i')^-}}\bigg)\|D^i v\|_{L^{p_i(\cdot)}(Q_T)} \\
&\quad + \frac{1}{(1/(n+1))^{p_0^+ - 1}} \big(\frac{1}{p_0^-} + \frac{1}{(p_0')^-}\big)\bigg(\int_{Q_T} |(u-\psi)^-|^{p_0(x)}\, dxdt + 1\bigg)^{\frac{1}{(p_0')^-}} \|v\|_{L^{p_0(\cdot)}(Q_T)} \\
&\leq C_1 \|v\|_{V_0(Q_T)}
\end{aligned} \tag{4.3.13}$$

Lemma 4.3.1. *The bounded operator $B_n = A_n + H_n$ acted from $V_0(Q_T)$ into $V^*(Q_T)$ is pseudo-monotone. Moreover, B_n is coercive in the following sense:*

$$\frac{\int_0^T \langle B_n v, v\rangle\, dt}{\|v\|_{V_0(Q_T)}} \longrightarrow +\infty \qquad \textit{as} \quad \|v\|_{V_0(Q_T)} \to \infty \quad \textit{for} \quad v \in V_0(Q_T).$$

Proof of the Lemma 4.3.1. By the virtue of the inequalities $(4.3.12)$ and $(4.3.13)$, it's easy to see that the operator B_n is bounded.

For the coercivity, by using $(4.1.6)$ and $(4.1.12)$ we have for any $u \in V_0(Q_T)$,

$$\begin{aligned}
\int_0^T \langle B_n u, u\rangle \, dt &= \int_0^T \langle A_n u, u\rangle \, dt + \int_0^T \langle H_n u, u\rangle \, dt \\
&= \sum_{i=1}^N \int_{Q_T} a_i(x,t,T_n(u),\nabla u)\, D^i u \, dxdt + \sum_{i=1}^N \int_{Q_T} h_i^n(x,t,u,\nabla u) u \, dxdt \\
&\quad + \int_{Q_T} \Theta(x)|u|^{p_0(x)} \, dxdt - \int_{Q_T} \frac{1}{(1/n)^{p_0(x)-1}} |(u-\psi)^-|^{p_0(x)-2}(u-\psi)^- u \, dxdt \\
&\geq \rho \sum_{i=1}^N \int_{Q_T} |D^i u|^{p_i(x)} \, dxdt + \Theta_0(\|u\|^{\underline{p}}_{L^{p_0(\cdot)}(Q_T)} - 1) - C_2\|u\|_{V_0(Q_T)} \\
&\geq \rho \sum_{i=1}^N (\|D^i u\|^{\underline{p}}_{L^{p_i(\cdot)}(Q_T)} - 1) + \Theta_0(\|u\|^{\underline{p}}_{L^{p_0(\cdot)}(Q_T)} - 1) - C_2\|u\|_{V_0(Q_T)} \\
&\geq \rho' \|u\|^{\underline{p}}_{V_0(Q_T)} - \alpha N - \Theta_0 - C_2\|u\|_{V_0(Q_T)},
\end{aligned}$$

with $\rho' = \min(\rho, \Theta_0)$, it follows that

$$\frac{\int_0^T \langle B_n u, u\rangle \, dt}{\|u\|_{V_0(Q_T)}} \longrightarrow +\infty \quad \text{as} \quad \|u\|_{V_0(Q_T)} \to \infty.$$

It remains to show that B_n is pseudo-monotone. Let $(u_k)_{k\in I\!N}$ be a sequence in $V_0(Q_T)$ such that

$$\begin{cases} u_k \rightharpoonup u & \text{in } \ V_0(Q_T), \\ B_n u_k \rightharpoonup \chi_n & \text{in } \ V^*(Q_T), \\ \limsup\limits_{k\to\infty} \int_0^T \langle B_n u_k, u_k\rangle \, dt \leq \langle \int_0^T \chi_n, u\rangle \, dt. \end{cases} \tag{4.3.14}$$

We have to prove that

$$\chi_n = B_n u \quad \text{and} \quad \int_0^T \langle B_n u_k, u_k\rangle \, dt \longrightarrow \int_0^T \langle \chi_n, u\rangle \, dt \ \text{ as } \ k \to +\infty.$$

Firstly, since $V_0(Q_T) \hookrightarrow\hookrightarrow L^{\underline{p}}(Q_T)$, then $u_k \to u$ in $L^{\underline{p}}(Q_T)$ for a subsequence still denoted $(u_k)_{k\in I\!N}$. The sequence $(u_k)_{k\in I\!N}$ is in $V_0(Q_T)$, then by using the growth condition $(4.1.5)$, $(a_i(x,t,T_n(u_k),\nabla u_k))_{k\in I\!N}$ is bounded in $L^{p_i'(\cdot)}(Q_T)$, then there exists a function $\phi_i^n \in L^{p_i'(\cdot)}(Q_T)$ such that

$$a_i(x,t,T_n(u_k),\nabla u_k) \rightharpoonup \phi_i^n \qquad \text{in} \quad L^{p_i'(\cdot)}(Q_T) \ \text{ as } \ k \to \infty. \tag{4.3.15}$$

As well, we have

$$\frac{1}{(1/n)^{p_0(x)-1}} |(u_k-\psi)^-|^{p_0(x)-2}(u_k-\psi)^- \rightharpoonup \frac{1}{(1/n)^{p_0(x)-1}} |(u-\psi)^-|^{p_0(x)-2}(u-\psi)^- \qquad \text{in} \quad L^{p_0'(\cdot)}(Q_T). \tag{4.3.16}$$

as $k \to \infty$. In like manner, since $(h_i^n(x,t,u_k,\nabla u_k))_k$ is bounded in $L^{\underline{p}'}(Q_T)$, with $\underline{p}'$ is the conjugate exponent of $\underline{p}$, then there exists a function $\varphi_i^n \in L^{\underline{p}'}(\Omega)$ such that

$$h_i^n(x,t,u_k,\nabla u_k) \rightharpoonup \varphi_i^n \qquad \text{in} \quad L^{\underline{p}'}(\Omega) \ \text{ as } \ k \to \infty. \tag{4.3.17}$$

For any $v \in V_0(Q_T)$, we have

$$\begin{aligned}\int_0^T \langle \chi_n, v\rangle\, dt &= \lim_{k\to\infty}\int_0^T \langle B_n u_k, v\rangle\, dt \\ &= \lim_{k\to\infty}\sum_{i=1}^N \int_{Q_T} a_i(x,t,T_n(u_k),\nabla u_k)D^i v\, dxdt + \lim_{k\to\infty}\int_{Q_T}\Theta(x,t)|u_k|^{p_0(x)-2}u_k\, v\, dxdt \\ &\quad - \lim_{k\to\infty}\int_{Q_T}\frac{1}{(1/n)^{p_0(x)-1}}|(u_k-\psi)^-|^{p_0(x)-2}(u_k-\psi)^- v\, dxdt \\ &\quad + \lim_{k\to\infty}\sum_{i=1}^N\int_{Q_T} h_i^n(x,t,u_k,\nabla u_k)v\, dxdt \\ &= \sum_{i=1}^N\int_{Q_T}\phi_i^n\, D^i v\, dxdt + \frac{1}{n}\int_{Q_T}\Theta(x,t)|u|^{p_0(x)-2}u\, v\, dxdt \\ &\quad - \int_{Q_T}\frac{1}{(1/n)^{p_0(x)-1}}|(u-\psi)^-|^{p_0(x)-2}(u-\psi)^- v\, dxdt + \sum_{i=1}^N\int_{Q_T}\varphi_i^n v\, dxdt.\end{aligned} \tag{4.3.18}$$

By the virtue of $(4.3.14)$ and $(4.3.18)$, we get

$$\begin{aligned}\limsup_{k\to\infty}\langle B_n(u_k),u_k\rangle &= \limsup_{k\to\infty}\Big\{\sum_{i=1}^N\int_{Q_T} a_i(x,t,T_n(u_k),\nabla u_k)D^i u_k\, dxdt + \int_{Q_T}\Theta(x,t)|u_k|^{p_0(x)}\, dxdt \\ &\quad - \int_{Q_T}\frac{1}{(1/n)^{p_0(x)-1}}|(u_k-\psi)^-|^{p_0(x)-2}(u_k-\psi)^- u_k\, dxdt \\ &\quad + \sum_{i=1}^N\int_{Q_T} h_i^n(x,t,u_k,\nabla u_k)u_k\, dxdt\Big\} \\ &\le \sum_{i=1}^N\int_{Q_T}\phi_i^n\, D^i u\, dxdt + \int_{Q_T}\Theta(x,t)|u|^{p_0(x)}\, dxdt \\ &\quad - \int_{Q_T}\frac{1}{(1/n)^{p_0(x)-1}}|(u-\psi)^-|^{p_0(x)-2}(u-\psi)^- u\, dxdt + \sum_{i=1}^N\int_{Q_T}\varphi_i^n u\, dx.\end{aligned} \tag{4.3.19}$$

Taking into account $(4.3.16)$ and $(4.3.17)$, and since $u_k \to u$ in $L^{\underline{p}}(Q_T)$, then

$$\int_{Q_T} h_i^n(x,u_k,\nabla u_k)u_k\, dxdt \to \int_\Omega \varphi_i^n u\, dxdt, \tag{4.3.20}$$

$$\int_{Q_T}\frac{1}{(1/n)^{p_0(x)-1}}|(u_k-\psi)^-|^{p_0(x)-2}(u_k-\psi)^- u_k dxdt \to \int_{Q_T}\frac{1}{(1/n)^{p_0(x)-1}}|(u-\psi)^-|^{p_0(x)-2}(u-\psi)^- u dxdt, \tag{4.3.21}$$

Therefore

$$\begin{aligned}\limsup_{k\to\infty}\Big(&\sum_{i=1}^N\int_{Q_T} a_i(x,t,T_n(u_k),\nabla u_k)D^i u_k\, dxdt \\ &+ \int_{Q_T}\Theta(x,t)|u_k|^{p_0(x)}\, dxdt\Big) \le \sum_{i=1}^N\int_{Q_T}\phi_i^n\, D^i u\, dxdt + \int_{Q_T}\Theta(x,t)|u|^{p_0(x)}\, dxdt.\end{aligned} \tag{4.3.22}$$

On the other hand, thanks to $(4.1.7)$, we obtain

$$\begin{aligned}\sum_{i=1}^{N}\int_{Q_T}(a_i(x,t,T_n(u_k),\nabla u_k)-a_i(x,t,T_n(u_k),\nabla u))(D^iu_k-D^iu)\,dxdt\\ +\frac{1}{n}\int_{Q_T}\Theta(x,t)(|u_k|^{p_0(x)-2}u_k-|u|^{p_0(x)-2}u)(u_k-u)\,dxdt\geq 0,\end{aligned}\tag{4.3.23}$$

then

$$\begin{aligned}&\sum_{i=1}^{N}\int_{Q_T}a_i(x,t,T_n(u_k),\nabla u_k)D^iu_k\,dxdt+\int_{Q_T}\Theta(x,t)|u_k|^{p_0(x)}\,dxdt\\ &\geq\sum_{i=1}^{N}\int_{Q_T}a_i(x,t,T_n(u_k),\nabla u_k)D^iu\,dxdt+\int_{Q_T}\Theta(x,t)|u_k|^{p_0(x)-2}u_ku\,dxdt\\ &\quad+\sum_{i=1}^{N}\int_{Q_T}a_i(x,t,T_n(u_k),\nabla u)(D^iu_k-D^iu)\,dxdt+\int_{Q_T}\Theta(x,t)|u|^{p_0(x)-2}u(u_k-u)\,dxdt.\end{aligned}$$

By applying the Lebesgue's dominated convergence theorem we have $T_n(u_k)\to T_n(u)$ in $L^{p_i(\cdot)}(Q_T)$, thus $a_i(x,t,T_n(u_k),\nabla u)\to a_i(x,t,T_n(u),\nabla u)$ in $L^{p_i'(\cdot)}(Q_T)$, and using $(4.3.15)$ we get

$$\begin{aligned}\liminf_{k\to\infty}\Big(\sum_{i=1}^{N}\int_{Q_T}a_i(x,t,T_n(u_k),\nabla u_k)D^iu_k\,dxdt\\ +\int_{Q_T}\Theta(x,t)|u_k|^{p_0(x)}\,dxdt\Big)\geq\sum_{i=1}^{N}\int_{Q_T}\phi_i^n\,D^iu\,dxdt+\int_{Q_T}\Theta(x,t)|u|^{p_0(x)}\,dxdt.\end{aligned}$$

It yield, thanks to $(4.3.22)$, that

$$\begin{aligned}\lim_{k\to\infty}\Big(\sum_{i=1}^{N}\int_{Q_T}a_i(x,t,T_n(u_k),\nabla u_k)D^iu_k\,dxdt\\ +\int_{Q_T}\Theta(x,t)|u_k|^{p_0(x)}\,dxdt\Big)=\sum_{i=1}^{N}\int_{Q_T}\phi_i^n\,D^iu\,dxdt+\int_{Q_T}\Theta(x,t)|u|^{p_0(x)}\,dxdt.\end{aligned}\tag{4.3.24}$$

Thus, by combining $(4.3.18)$, $(4.3.20)$ and $(4.3.21)$, we obtain

$$\int_{Q_T}\langle B_nu_k,u_k\rangle\,dt\longrightarrow\int_{Q_T}\langle\chi_n,u\rangle\,dt\quad\text{ as }\quad k\to\infty.$$

In addition, by (4.3.24) we obtain

$$\begin{aligned}\lim_{k\to+\infty}\Big(\sum_{i=1}^{N}\int_{Q_T}(a_i(x,t,T_n(u_k),\nabla u_k)-a_i(x,t,T_n(u_k),\nabla u))(D^iu_k-D^iu)\,dxdt\\ +\int_{Q_T}\Theta(x,t)(|u_k|^{p_0(x)-2}u_k-|u|^{p_0(x)-2}u)(u_k-u)\,dxdt\Big)=0.\end{aligned}$$

By Lemma 4.2.2, we get

$$u_k\to u\quad\text{in}\quad V_0(Q_T)\qquad\text{and}\qquad D^iu_k\to D^iu\quad\text{a.e. in}\quad Q_T,$$

hence, we have

$$a_i(x,t,T_n(u_k),\nabla u_k) \rightharpoonup a_i(x,t,T_n(u),\nabla u) \quad \text{in} \quad L^{p_i'(\cdot)}(Q_T) \quad \text{for} \quad i=1,\dots,N$$

and

$$h_i^n(x,t,u_k,\nabla u_k) \rightharpoonup h_i^n(x,t,u,\nabla u) \quad \text{in} \quad L^{p_0'(\cdot)}(Q_T) \quad \text{for} \quad i=1,\dots,N$$

thus, having in mind $(4.3.16)$, we conclude that $\chi_n = B_n u$. Consequentlt, in view of Lemma 4.3.1, there exists at least one weak solution $u_n \in V_0(Q_T)$ of the problem $(4.3.10)$, (cf. [121], Theorem 2.7, p.180).

Remark 4.3.2. *We emphasize that the weak solution u_n of the approximate problem $(4.3.10)$, is considered according to the following maining:*

$$\begin{aligned} &u_n \quad \textit{belongs to} \quad V_0(Q_T)\cap \mathcal{C}([0,T],L^2(\Omega)), \quad \frac{\partial u_n}{\partial t}\in V^*(Q_T), \quad u_n(x,0)=u_0(x),\\ &\int_0^T \langle \frac{\partial u_n}{\partial t}, v\rangle\, dt + \sum_{i=1}^N \int_{Q_T} a_i(x,t,T_n(u_n),\nabla u_n) Dv\, dxdt + \int_{Q_T}\Theta(x,t)|u_n|^{p_0(x)-2}u_n v\, dxdt\\ &\quad - \int_{Q_T} \frac{1}{(1/n)^{p_0(x)-1}}|(u_n-\psi)^-|^{p_0(x)-2}(u_n-\psi)^- v\, dxdt + \sum_{i=1}^N \int_{Q_T} h_i^n(x,t,u_n,\nabla u_n) v\, dxdt\\ &= \int_{Q_T} f v\, dxdt \quad \forall v\in V_0(Q_T). \end{aligned} \tag{4.3.25}$$

Step 2: A priori estimates.

Let n be large enough. In the sequel, we establish estimates for u_n and $\frac{(u_n-\psi)^-}{(1/n)}$, which are necessary for the remaining part of the proof.

First estimate

Now, we have to establish an estimate of u_n in $V_0(Q_T)$. It is obvious that $\frac{\partial \psi^+}{\partial t} \in L^{p_i'(\cdot)}(Q)$ [146], then

inserting $(u_n-\psi^+)$ as a test function into $(4.3.25)$ gives

$$\begin{aligned}
&\int_0^T \langle \frac{\partial u_n}{\partial t}, u_n-\psi^+\rangle\, dt \\
&\quad+\sum_{i=1}^N \int_{Q_T} a_i(x,t,T_n(u_n),\nabla u_n)D^i(u_n-\psi^+)\,dxdt+\int_{Q_T}\Theta(x,t)|u_n|^{p_0(x)-2}u_n(u_n-\psi^+)\,dxdt \\
&\quad-\int_{Q_T}\frac{1}{(1/n)^{p_0(x)-1}}|(u_n-\psi)^-|^{p_0(x)-2}(u_n-\psi)^-(u_n-\psi^+)\,dxdt \\
&\quad+\sum_{i=1}^N\int_{Q_T} h_i^n(x,t,u_n,\nabla u_n)(u_n-\psi^+)\,dxdt-\int_{Q_T}\frac{\partial\psi^+}{\partial t}(u_n-\psi^+)\,dxdt \\
&=\int_{Q_T}(f-\frac{\partial\psi^+}{\partial t})(u_n-\psi^+)\,dxdt,
\end{aligned} \tag{4.3.26}$$

it yields

$$\begin{aligned}
&\frac{1}{2}\int_{Q_T}\frac{\partial((u_n-\psi^+)(t))^2}{\partial t}\,dxdt+\sum_{i=1}^N\int_{Q_T}a_i(x,t,T_n(u_n),\nabla u_n)D^iu_n\,dxdt \\
&\quad+\int_{Q_T}\Theta(x,t)|u_n|^{p_0(x)-2}u_n(u_n-\psi^+)\,dxdt+\int_{Q_T}\frac{1}{(1/n)^{p_0(x)-1}}|(u_n-\psi)^-|^{p_0(x)-1}\psi^-\,dxdt \\
&\quad+\int_{Q_T}\frac{1}{(1/n)^{p_0(x)-1}}|(u_n-\psi)^-(t)|^{p_0(x)}\,dxdt+\sum_{i=1}^N\int_{Q_T}u_nh_i^n(x,t,u_n,\nabla u_n)\,dxdt \\
&=\int_{Q_T}(f-\frac{\partial\psi^+}{\partial t})u_n\,dxdt-\int_{Q_T}(f-\frac{\partial\psi^+}{\partial t})\psi^+\,dxdt \\
&\quad+\sum_{i=1}^N\int_{Q_T}a_i(x,t,T_n(u_n),\nabla u_n)D^i\psi^+\,dxdt+\sum_{i=1}^N\int_{Q_T}\psi^+h_i^n(x,t,u_n,\nabla u_n)\,dxdt.
\end{aligned} \tag{4.3.27}$$

By the virtue of $(4.1.6)$, we get

$$\rho\sum_{i=1}^N\int_{Q_T}|D^iu_n|^{p_i(x)}\,dxdt\le\sum_{i=1}^N\int_{Q_T}a_i(x,t,T_n(u_n),\nabla u_n)D^iu_n\,dxdt,$$

for the thirth term of the left hand-side of $(4.3.27)$, we use $(4.1.10)$, the assumption on $\Theta(x,t)$ and the general hölder inequality to obtain

$$\begin{aligned}
&\int_{Q_T}\Theta(x,t)|u_n|^{p_0(x)-2}u_n(u_n-\psi^+)\,dxdt\le\|\Theta\|_\infty\int_{Q_T}|u_n|^{p_0(x)-1}|u_n-\psi^+|\,dxdt \\
&\le\|\Theta\|_\infty\Big(\int_{Q_T}|u_n|^{p_0(x)}\,dxdt+\int_{Q_T}|u_n|^{p_0(x)-1}|\psi^+|\,dxdt\Big) \\
&\le\|\Theta\|_\infty\Big(C_9+\|\psi\|_\infty(\frac{1}{p_0^-}+\frac{1}{(p_0')^-})\||u_n|^{p_0(x)-1}\|_{L^{p_0'(\cdot)}(Q_T)}\|1\|_{L^{p_0(\cdot)}(Q_T)}\Big) \\
&\le C_{10}+2\|\psi\|_\infty\|\Theta\|_\infty(\int_{Q_T}|u_n|^{p_0(x)}\,dxdt+1)^{\frac{1}{(p_0')^-}}(meas(Q_T)+1)^{\frac{1}{p_0^-}} \\
&\le C_{10}+C_{11}\int_{Q_T}|u_n|^{p_0(x)}\,dxdt.
\end{aligned} \tag{4.3.28}$$

Concerning the first and the second term of the right hand-side of $(4.3.27)$, by the virtue of $(4.1.3)$, $(4.1.10)$, $\frac{\partial \psi^+}{\partial t} \in L^{p_i'(.)}(\Omega\times]0,T[)$, Poincaré and the generalized Hölder type inequality we have

$$\begin{aligned}
&\int_{Q_T}(f-\frac{\partial \psi^+}{\partial t})u_n\,dxdt-\int_{Q_T}(f-\frac{\partial \psi^+}{\partial t})\psi^+\,dxdt\\
&\leq \sum_{i=1}^{N}(\frac{1}{p_i^-}+\frac{1}{(p_i')^-})\bigg(\|f-\frac{\partial \psi^+}{\partial t}\|_{L^{p_i'(.)}(Q_T)}\|u_n\|_{L^{p_i(.)}(Q_T)}+\|f-\frac{\partial \psi^+}{\partial t}\|_{L^{p_i'(.)}(Q_T)}\|\psi^+\|_{L^{p_i(.)}(Q_T)}\bigg)\\
&\leq \sum_{i=1}^{N}(\frac{1}{p_i^-}+\frac{1}{(p_i')^-})\bigg(C_{12}(\int_{Q_T}|u_n|^{p_i(x)}\,dxdt+1)^{\frac{1}{(p_i)^-}}+C_{13}(\int_{Q_T}|\psi^+|\,dxdt+1)^{\frac{1}{(p_i)^-}}\bigg)\\
&\leq \sum_{i=1}^{N}(\frac{1}{p_i^-}+\frac{1}{(p_i')^-})\bigg(C_{12}(\int_0^T\|D^iu_n\|^{p_i(x)}_{L^{p_i(.)}(\Omega)}dt+1)^{\frac{1}{(p_i)^-}}+C_{13}(\|\psi\|_\infty meas(Q_T)+1)^{\frac{1}{(p_i)^-}}\bigg)\\
&\leq 2C_{14}\sum_{i=1}^{N}\int_0^T\|D^iu_n\|^{p_i(x)}_{L^{p_i(.)}(\Omega)}dt+C_{15}.
\end{aligned}\tag{4.3.29}$$

As regards the third term on the right hand-side of $(4.3.27)$, by applying again the Poincaré and the generalized Hölder type inequality, and due to $(4.1.5)$ and $(4.1.10)$, we get

$$\begin{aligned}
&\sum_{i=1}^{N}\int_{Q_T}a_i(x,t,T_n(u_n),\nabla u_n)D^i\psi^+\,dxdt\\
&\leq \gamma\sum_{i=1}^{N}\bigg(\int_{Q_T}|C_i(x,t)||D^i\psi^+|\,dxdt+\int_{Q_T}|u_n|^{p_i(x)-1}|D^i\psi^+\,dxdt\\
&\quad+\int_{Q_T}|D^iu_n|^{p_i(x)-1}|D^i\psi^+|>dxdt\bigg)\\
&\leq \gamma\sum_{i=1}^{N}(\frac{1}{p_i^-}+\frac{1}{(p_i')^-})\bigg(\|C_i(x,t)\|_{L^{p_i'(.)}(Q_T)}\|D^i\psi^+\|_{L^{p_i(.)}(Q_T)}+\||u_n|^{p_i(x)-1}\|_{L^{p_i'(.)}(Q_T)}\|D^i\psi^+\|_{L^{p_i(.)}(Q_T)}\\
&\quad+\||D^iu_n|^{p_i(x)-1}\|_{L^{p_i'(.)}(Q_T)}\|D^i\psi^+\|_{L^{p_i(.)}(Q_T)}\bigg)\\
&\leq \gamma\sum_{i=1}^{N}(\frac{1}{p_i^-}+\frac{1}{(p_i')^-})\bigg(C_{16}+C_{17}(\int_{Q_T}|u_n|^{p_i(x)}\,dxdt+1)^{\frac{1}{(p_i')^-}}+C_{18}(\int_{Q_T}|D^iu_n|^{p_i(x)}\,dxdt+1)^{\frac{1}{(p_i')^-}}\bigg)\\
&\leq \gamma\sum_{i=1}^{N}(\frac{1}{p_i^-}+\frac{1}{(p_i')^-})\bigg(C_{16}+C_{19}\int_{Q_T}|D^iu_n|^{p_i(x)}\,dxdt\bigg)\\
&\leq C_{17}\sum_{i=1}^{N}\int_{Q_T}|D^iu_n|^{p_i(x)}\,dxdt+C_{18}.
\end{aligned}\tag{4.3.30}$$

Furthermore, using once more Poincaré and generalized Hölder type inequalities in the fourth term of the

right hand-side of $(4.3.27)$, and due to $(4.1.13)-(4.1.14)$ and $(4.1.10)$, we have

$$
\begin{aligned}
&\Big|\sum_{i=1}^{N}\int_{Q_T}\psi^{+}h_i^n(x,t,u_n,\nabla u_n)\,dxdt\Big| \\
&\quad\le \|\psi^{+}\|_{\infty}\Big(\int_{Q_T} j(|u_n|)l(x,t)\,dxdt+\sum_{i=1}^{N}\int_{Q_T} j(|u_n|)|D^iu_n|^{r_i(x)}\,dxdt\Big) \\
&\quad\le \|\psi^{+}\|_{\infty}\Big(\lambda\int_{Q_T} l(x,t)\,dxdt+\int_{Q_T}|u_n|^{q_0(x)}l(x,t)\,dxdt \\
&\qquad+\lambda\sum_{i=1}^{N}\int_{Q_T}|D^iu_n|^{r_i(x)}\,dxdt+\sum_{i=1}^{N}\int_{Q_T}|u_n|^{q_0(x)}|D^iu_n|^{r_i(x)}\,dxdt\Big) \\
&\quad\le \|\psi^{+}\|_{\infty}\Big(\lambda\|l(x,t)\|_{\infty}meas(Q_T)+\|l(x,t)\|_{\infty}\sum_{i=1}^{N}(\int_{Q_T}|u_n|^{p_i(x)}\,dxdt+1) \\
&\qquad+\lambda\sum_{i=1}^{N}(\int_{Q_T}|D^iu_n|^{p_i(x)}\,dxdt+1) \\
&\qquad+2\sum_{i=1}^{N}\Big(\int_{Q_T}|u_n|^{\frac{q_0(x)p_i(x)}{p_i(x)-r_i(x)}}\,dxdt\Big)^{\frac{p_i(x)-r_i(x)}{p_i(x)}}\Big(\int_{Q_T}|D^iu_n|^{p_i(x)}\,dxdt\Big)^{\frac{r_i(x)}{p_i(x)}}\Big) \\
&\quad\le \|\psi^{+}\|_{\infty}\Big(\lambda\|l(x,t)\|_{\infty}meas(Q_T)+\|l(x,t)\|_{\infty}\sum_{i=1}^{N}(C_{19}\int_{Q_T}|D^iu_n|^{p_i(x)}\,dxdt+1) \\
&\qquad+\lambda\sum_{i=1}^{N}\int_{Q_T}|D^iu_n|^{p_i(x)}\,dxdt+\lambda N \\
&\qquad+2\sum_{i=1}^{N}(\int_{Q_T}|u_n|^{p_i(x)}\,dxdt+1)^{\frac{p_i^{+}-r_i^{-}}{p_i^{-}}}(\int_{Q_T}|D^iu_n|^{p_i(x)}\,dxdt+1)^{\frac{r_i^{+}}{p_i^{-}}}\Big) \\
&\quad\le \|\psi^{+}\|_{\infty}\Big(\lambda\|l(x,t)\|_{\infty}meas(Q_T)+N(\lambda+\|l(x,t)\|_{\infty}) \\
&\qquad+(C_{19}\|l(x,t)\|_{\infty}+\lambda)\sum_{i=1}^{N}\int_{Q_T}|D^iu_n|^{p_i(x)}\,dxdt \\
&\qquad2(\int_{Q_T}|u_n|^{p_0(x)}\,dxdt+N)\sum_{i=1}^{N}(\int_{Q_T}|D^iu_n|^{p_i(x)}\,dxdt+1)\Big) \\
&\quad\le C_{20}+C_{21}\sum_{i=1}^{N}\int_{Q_T}|D^iu_n|^{p_i(x)}\,dxdt.
\end{aligned}
\tag{4.3.31}
$$

Hence, by combining $(4.3.27)-(4.3.31)$ and since $u_0\in L^2(\Omega)$, we deduce that

$$
\begin{aligned}
&(\lambda-C_{22})\sum_{i=1}^{N}\int_{Q_T}|D^iu_n|^{p_i(x)}\,dxdt-C_{11}\int_{Q_T}|u_n|^{p_0(x)}\,dxdt+\frac{1}{2}\int_{\Omega}|u_n(T)-\psi^{+}(T)|^2dx \\
&\quad+\int_{Q_T}\frac{1}{(1/n)^{p_0(x)-1}}|(u_n-\psi)^{-}|^{p_0(x)-1}\psi^{-}\,dxdt+\int_{Q_T}\frac{1}{(1/n)^{p_0(x)-1}}|(u_n-\psi)^{-}(t)|^{p_0(x)}\,dxdt
\end{aligned}
$$

$$+\sum_{i=1}^{N}\int_{Q_T} u_n h_i^n(x,t,u_n,\nabla u_n)\,dxdt \le C_{23},$$

we denote $\underline{\lambda} = \max(C_{22}, C_{11})$ with C_{22} and C_{11} are nonnegatives constants which do not depend on n, and we choose $\underline{\lambda}$ small enough ($\underline{\lambda} = \frac{\lambda}{2}$), it follows that

$$\begin{aligned}
&\frac{\lambda}{2}\sum_{i=0}^{N}\int_{Q_T}|D^i u_n|^{p_i(x)}\,dxdt + \frac{1}{2}\int_\Omega |u_n(T)-\psi^+(T)|^2dx + \int_{Q_T}\frac{1}{(1/n)^{p_0(x)-1}}|(u_n-\psi)^-|^{p_0(x)-1}\psi^-\,dxdt\\
&\quad+\int_{Q_T}\frac{1}{(1/n)^{p_0(x)-1}}|(u_n-\psi)^-(t)|^{p_0(x)}\,dxdt + \sum_{i=1}^{N}\int_{Q_T} u_n h_i^n(x,t,u_n,\nabla u_n)\,dxdt\\
&\le C_{23}.
\end{aligned} \tag{4.3.32}$$

Consequently, we conclude that

$$\|u_n\|_{V_0(Q_T)} \le C_{24}, \tag{4.3.33}$$

$$\|u_n\|_{L^\infty(0,T;L^2(\Omega))} \le C_{25}, \tag{4.3.34}$$

with C_{24} and C_{25} are nonnegatives constants which do not depend on n. Hence, there exists a subsequence, still denoted $(u_n)_n$, such that

$$u_n \rightharpoonup u \quad \text{in } V_0(Q_T), \tag{4.3.35}$$

$$u_\varepsilon \rightharpoonup u \quad \text{weak-* in } L^\infty(0,T;L^2(\Omega)). \tag{4.3.36}$$

We deduce that $u_n \in L^\infty(0,T;L^2(\Omega)) \cap V_0(Q_T)$ and $(4.3.5)$ is assumed.

Second estimate

Since $\psi \in V(Q_T)$, and $\frac{\partial\psi}{\partial t} \in L^{p'(.)}(Q_T)$, we are able to choose $-\frac{(u_n-\psi)^-}{(1/n)}$ as an admissible test function in $(4.3.25)$, to get

$$\begin{aligned}
&-\frac{1}{(1/n)}\int_0^T \langle \frac{\partial u_n}{\partial t}, (u_n-\psi)^-\rangle\,dt - \frac{1}{(1/n)}\sum_{i=1}^{N}\int_{Q_T} a_i(x,t,T_n(u_n),D^i u_n)D(u_n-\psi)^-\,dxdt\\
&\quad-\frac{1}{(1/n)}\int_{Q_T}\Theta(x,t)|u_n|^{p_0(x)-2}u_n(u_n-\psi)^-\,dxdt + \int_{Q_T}\frac{1}{(1/n)^{p_0(x)}}|(u_n-\psi)^-|^{p_0(x)}\,dxdt\\
&\quad-\frac{1}{(1/n)}\sum_{i=1}^{N}\int_{Q_T} h_i^n(x,t,u_n,\nabla u_n)(u_n-\psi)^-\,dxdt\\
&=-\frac{1}{(1/n)}\int_0^T\langle f,(u_n-\psi)^-\rangle\,dt.
\end{aligned} \tag{4.3.37}$$

Hence,

$$
\begin{aligned}
&-\frac{1}{(1/n)}\int_0^T \langle \frac{\partial(u_n-\psi)}{\partial t}, (u_n-\psi)^-\rangle\, dt \\
&\quad -\frac{1}{(1/n)}\sum_{i=1}^N \int_{\{u_n\le\psi\}} a_i(x,t,T_n(u_n),\nabla u_n)-a_i(x,t,T_n(u_n),\nabla\psi)D^i(u_n-\psi)^-\, dxdt \\
&\quad -\frac{1}{(1/n)}\int_{Q_T}\Theta(x,t)|u_n|^{p_0(x)-2}u_n(u_n-\psi)^-\, dxdt+\int_{Q_T}\frac{1}{(1/n)^{p_0(x)}}|(u_n-\psi)^-|^{p_0(x)}\, dxdt \\
&\quad -\frac{1}{(1/n)}\sum_{i=1}^N\int_{Q_T} h_i^n(x,t,u_n,\nabla u_n)(u_n-\psi)^-\, dxdt \\
&=-\frac{1}{(1/n)}\int_{Q_T}\big(\sum_{i=1}^N D^i a_i(x,t,T_n(u_n),\nabla\psi)-\frac{\partial\psi}{\partial t}+f\big)(u_n-\psi)^-\, dxdt.
\end{aligned}
\tag{4.3.38}
$$

For the first term of the left hand-side of $(4.3.38)$,

$$
\begin{aligned}
&-\frac{1}{(1/n)}\int_0^T \langle \frac{\partial(u_n-\psi)}{\partial t}, (u_n-\psi)^-\rangle\, dt \\
&=\frac{1}{(2/n)}\int_{\{u_n\le\psi\}}\frac{\partial((u_n-\psi)(t))^2}{\partial t}\, dxdt \\
&=\frac{1}{(2/n)}\int_\Omega |(u_n-\psi)(T)|^2\, dx-\frac{1}{(2/n)}\int_\Omega |(u_n-\psi)(0)|^2\, dx.
\end{aligned}
\tag{4.3.39}
$$

As for the second term of the left hand-side of $(4.3.38)$, by virtue of $(4.1.7)$, we get

$$
W_1=-\frac{1}{(1/n)}\sum_{i=1}^N\int_{\{u_n\le\psi\}} a_i(x,t,T_n(u_n),\nabla u_n)-a_i(x,t,T_n(u_n),\nabla\psi)D^i(u_n-\psi)^-\, dxdt>0 \tag{4.3.40}
$$

By applying, Young inequality to the third term of the left hand-side of $(4.3.38)$, it yields

$$
\begin{aligned}
&\frac{1}{(1/n)}\int_{Q_T}\Theta(x,t)|u_n|^{p_0(x)-2}u_n(u_n-\psi)^-\, dxdt \\
&\qquad\le C_2\|\Theta\|_\infty\int_{Q_T}|u_n|^{(p_0(x)-1)p_0'(x)}\, dxdt+\frac{\beta}{2}\int_{Q_T}\frac{1}{(1/n)^{p_0(x)}}|(u_n-\psi)^-|^{p_0(x)}\, dxdt
\end{aligned}
$$

so

$$
\begin{aligned}
&-\frac{1}{(1/n)}\int_{Q_T}\Theta(x,t)|u_n|^{p_0(x)-2}u_n(u_n-\psi)^-\, dxdt \\
&\qquad\ge -C_2\|\Theta\|_\infty\int_{Q_T}|u_n|^{p_0(x)}\, dxdt-\frac{\beta}{2}\int_{Q_T}\frac{1}{(1/n)^{p_0(x)}}|(u_n-\psi)^-|^{p_0(x)}\, dxdt.
\end{aligned}
\tag{4.3.41}
$$

Concerning the right-hand side of $(4.3.38)$, by Young inequality, $(4.1.3)$, $(4.1.8)$ and $(4.3.1)$, we have

$$
\begin{aligned}
&\Big|-\frac{1}{(1/n)}\int_{Q_T}\langle\sum_{i=1}^N D^i a_i(x,t,T_n(u_n),\nabla\psi)-\frac{\partial\psi}{\partial t}+f,,(u_n-\psi)^-\rangle\, dxdt\Big| \\
&\qquad\le\int_{Q_T}\Big|\sum_{i=1}^N D^i a_i(x,t,T_n(u_n),\nabla\psi)-\frac{\partial\psi}{\partial t}+f\Big|^{p_0'(x)}\, dxdt+\frac{C_2}{2}\int_{Q_T}\frac{|(u_n-\psi)^-|^{p_0(x)}}{(1/n)^{p_0(x)}}\, dxdt.
\end{aligned}
$$

Since $p_0'(.) < p_i'(.)$, it follows that

$$\begin{aligned}
&\Big| -\frac{1}{(1/n)}\int_{Q_T}\langle\sum_{i=1}^{N} D^i a_i(x,t,T_n(u_n),\nabla\psi) - \frac{\partial\psi}{\partial t} + f,,(u_n-\psi)^-\rangle\, dxdt\Big| \\
&\qquad \le C_3\int_{Q_T}\big|\sum_{i=1}^{N} D^i a_i(x,t,T_n(u_n),\nabla\psi) - \frac{\partial\psi}{\partial t} + f\big|^{p_i'(x)}\, dxdt + \frac{C_2}{2}\int_{Q_T}\frac{|(u_n-\psi)^-|^{p_0(x)}}{(1/n)^{p_0(x)}}\, dxdt \\
&\qquad \le C_4 + \frac{C_2}{2}\int_{Q_T}\frac{|(u_n-\psi)^-|^{p_0(x)}}{(1/n)^{p_0(x)}}\, dxdt.
\end{aligned} \tag{4.3.42}$$

As regards the fifth term on the left hand-side of $(4.3.38)$, using the sign condition $(4.1.12)$,

$$\begin{aligned}
&-\frac{1}{(1/n)}\sum_{i=1}^{N}\int_{Q_T} h_i^n(x,t,u_n,\nabla u_n)(u_n-\psi)^-\, dxdt \\
&\quad = -\frac{1}{(1/n)}\sum_{i=1}^{N}\int_{\{0\le u_n\le\psi\}} h_i^n(x,t,u_n,\nabla u_n)(u_n-\psi)^-\, dxdt \\
&\qquad -\frac{1}{(1/n)}\sum_{i=1}^{N}\int_{\{u_n\le\psi\}\cap\{u_n\le 0\}} h_i^n(x,t,u_n,\nabla u_n)(u_n-\psi)^-\, dxdt \\
&\quad \ge -\frac{1}{(1/n)}\sum_{i=1}^{N}\int_{\{0\le u_n\le\psi\}} h_i^n(x,t,u_n,\nabla u_n)(u_n-\psi)^-\, dxdt.
\end{aligned}$$

Let's denote $W = -\frac{1}{(1/n)}\sum_{i=1}^{N}\int_{\{0\le u_n\le\psi\}} h_i^n(x,t,u_n,\nabla u_n)(u_n-\psi)^-\, dxdt$. taking into account $(4.1.13)$ and $(4.1.14)$ we get

$$\begin{aligned}
|W| &\le \sum_{i=1}^{N}\int_{Q_T} j(|u_n|)|D^i u_n|^{r_i(x)}\frac{|(u_n-\psi)^-|}{(1/n)}\, dxdt + \int_{Q_T} j(|u_n|)|l(x,t)|\frac{|(u_n-\psi)^-|}{(1/n)}\, dxdt \\
&\le \sum_{i=1}^{N}\int_{\{0\le u_n\le\psi\}}(|\psi^+|^{q_0(x)}+\lambda)|D^i u_n|^{r_i(x)}\frac{|(u_n-\psi)^-|}{(1/n)}\, dxdt \\
&\qquad + \int_{\{0\le u_n\le\psi\}}(|\psi^+|^{q_0(x)}+\lambda)|l(x,t)|\frac{|(u_n-\psi)^-|}{(1/n)}\, dxdt.
\end{aligned}$$

Since $\psi^+ \in L^\infty(Q_T)$ and $l(x,t) \in L^\infty(Q_T)$, we have

$$|W| \le C_5\sum_{i=1}^{N}\int_{\{0\le u_n\le\psi\}}|D^i u_n|^{r_i(x)}\frac{|(u_n-\psi)^-|}{(1/n)}\, dxdt + C_6\int_{\{0\le u_n\le\psi\}}\frac{|(u_n-\psi)^-|}{(1/n)}\, dxdt, \tag{4.3.43}$$

for the first term of the right hand-side of $(4.3.43)$, in view of $(4.3.35)$ and by applying the generalized Hölder inequality with the conjugate exponents $m_i(.)$ and $m_i'(.)$ such that $\frac{p_i(.)}{r_i(.)} = m_i(.)$, $\forall i = 1,\dots,N$, we have

$$\sum_{i=1}^{N}\int_{\{0\le u_n\le\psi\}}|D^i u_n|^{r_i(x)}\frac{|(u_n-\psi)^-|}{(1/n)}\, dxdt$$

$$
\begin{aligned}
&\leq \sum_{i=1}^{N}(\frac{1}{m_i^-}+\frac{1}{(m_i')^-})\Big(\int_{\{0\leq u_n\psi\}}|D^iu_n|^{r_i(x)m_i(x)}\,dxdt\Big)^{\frac{1}{m_i(x)}}\\
&\quad\times\Big(\int_{\{0\leq u_n\leq\psi\}}(\frac{|(u_n-\psi)^-|}{(1/n)})^{m_i'(x)}\,dxdt\Big)^{\frac{1}{m_i'(x)}}\\
&\leq 2\sum_{i=1}^{N}\Big(\int_{\{0\leq u_n\psi\}}|D^iu_n|^{p_i(x)}\,dxdt+1\Big)^{\frac{1}{m_i^-}}\|\frac{|(u_n-\psi)^-|}{(1/n)}\|_{L^{m_i'(.)}(Q_T)}\\
&\leq 2C_7\|\frac{|(u_n-\psi)^-|}{(1/n)}\|_{L^{m_i'(.)}(Q_T)},
\end{aligned}
$$

therefore, since $m_i'(.)=\frac{p_i(.)}{p_i(.)-r_i(.)}<p_i(.)<p_0(.)$ we obtain by continuous embeddings

$$
\sum_{i=1}^{N}\int_{\{0\leq u_n\leq\psi\}}|D^iu_n|^{r_i(x)}\frac{|(u_n-\psi)^-|}{(1/n)}\,dxdt\leq C_8\|\frac{|(u_n-\psi)^-|}{(1/n)}\|_{L^{p_0(.)}(Q_T)}. \tag{4.3.44}
$$

Concerning the second term of the right-hand side of $(4.3.43)$, by using the general hölder inequality, we have

$$
\begin{aligned}
\int_{\{0\leq u_n\leq\psi\}}\frac{|(u_n-\psi)^-|}{(1/n)}\,dxdt &\leq (\tfrac{1}{p_0^-}+\tfrac{1}{(p_0')^-})(meas(Q_T))^{\frac{1}{(p_0')^-}}\big(\int_{\{0\leq u_n\leq\psi\}}\tfrac{|(u_n-\psi)^-|^{p_0(x)}}{(1/n)^{p_0(x)}}\,dxdt\big)^{\frac{1}{p_0(x)}}\\
&\leq 2(meas(Q_T))^{\frac{1}{(p_0')^-}}\|\frac{|(u_n-\psi)^-|}{(1/n)}\|_{L^{p_0(.)}(Q_T)}.
\end{aligned} \tag{4.3.45}
$$

By combining $(4.3.39)-(4.3.45)$ we get

$$
\Big\|\frac{|(u_n-\psi)^-|}{(1/n)}\Big\|^{p_0(x)}_{L^{p_0(.)}(Q_T)}=\int_{Q_T)}\frac{|(u_n-\psi)^-|^{p_0(x)}}{(1/n)^{p_0(x)}}\,dxdt\leq C_8. \tag{4.3.46}
$$

Which implies that

$$
(u_n-\psi)^-\to 0\quad\text{strongly in}\quad L^{p_0(.)}(Q_T).
$$

Consequently, we deduce

$$
u\geq\psi\quad\text{a.e. on }\ Q_T. \tag{4.3.47}
$$

Step 3: Equi-integrability of the nonlinearities.

Now, we have to prove the uniform equi-integrability of $h_i^n(x,t,u_n,\nabla u_n)$. In fact, by taking $T_1(u_n-T_h(u_n))$ as a test function in $(4.3.25)$, we obtain

$$
\begin{aligned}
&\int_{\{h\leq|u_n|\}}\frac{\partial u_n}{\partial t}T_1(u_n-T_h(u_n))\,dxdt+\sum_{i=1}^{N}\int_{\{h<|u_n|\leq h+1\}}a_i(x,t,T_n(u_n),\nabla u_n)D^iu_n\,dxdt\\
&\quad+\int_{\{h<|u_n|\}}\Theta(x,t)|u_n|^{p_0(x)-2}u_nT_1(u_n-T_h(u_n))\,dxdt
\end{aligned}
$$

$$
\begin{aligned}
&- \int_{\{h<|u_n|\}} \frac{1}{(1/n)^{p_0(x)-1}} |(u_n-\psi)^-|^{p_0(x)-2}(u_n-\psi)^- T_1(u_n - T_h(u_n))\, dxdt \\
&+ \sum_{i=1}^{N} \int_{\{h<|u_n|\}} h_i^n(x,t,u_n,\nabla u_n) T_1(u_n - T_h(u_n))\, dxdt \\
&= \int_{\{h<|u_n|\}} f T_1(u_n - T_h(u_n))\, dxdt.
\end{aligned}
$$

Thanks to $(4.1.6)$ and since $T_1(u_n - T_h(u_n))$ have the same sign as u_n, we get

$$
\begin{aligned}
&\int_{\{h+1\le|u_n|\}} |\frac{\partial u_n}{\partial t}|\, dxdt + \rho \sum_{i=1}^{N} \int_{\{h<|u_n|\le h+1\}} |D^i u_n|^{p_i(x)}\, dxdt + \sum_{i=1}^{N} \int_{\{h+1\le|u_n|\}} |h_i^n(x,t,u_n,\nabla u_n)|\, dxdt \\
&\le \int_{\{h<|u_n|\}} \frac{1}{(1/n)^{p_0(x)-1}} |(u_n-\psi)^-|^{p_0(x)-2}(u_n-\psi)^- T_1(u_n - T_h(u_n))\, dxdt \\
&\quad + \int_{\{h<|u_n|\}} f T_1(u_n - T_h(u_n))\, dxdt.
\end{aligned}
$$

Inasmuch as $\frac{(u_n-\psi)^-}{(1/n)}$ is bounded in $L^{p_0(.)}(Q_T)$ (see $(4.3.46)$), we have

$$
\frac{1}{(1/n)^{p_0(x)-1}} |(u_n-\psi)^-|^{p_0(x)-2}(u_n-\psi)^- \rightharpoonup \mu \quad \text{in } L^{p_0'(.)}(Q_T) \tag{4.3.48}
$$

then $\big(\frac{1}{(1/n)^{p_0(x)-1}} |(u_n-\psi)^-|^{p_0(x)-2}(u_n-\psi)^-\big)_{n\in I\!N}$ is a bounded sequence in $L^{p_0'(.)}(Q_T)$ which implies, bearing in mind $(4.1.1)$, that

$$
\begin{aligned}
&\sum_{i=1}^{N} \int_{\{h+1\le|u_n|\}} |h_i^n(x,t,u_n,\nabla u_n)|\, dxdt \\
&\le (\frac{1}{p_0^-} + \frac{1}{(p_0')^-}) \| \frac{1}{(1/n)^{p_0(x)-1}} |(u_n-\psi)^-|^{p_0(x)-} (u_n-\psi)^- \|_{L^{p_0'(.)}(Q_T)} \| T_1(u_n - T_h(u_n)) \|_{L^{p_0(.)}(Q_T)}\, dxdt \\
&\quad + \sum_{i=1}^{N} (\frac{1}{p_i^-} + \frac{1}{(p_i')^-}) \| f \|_{L^{p_i'(.)}(Q_T)} \| T_1(u_n - T_h(u_n)) \|_{L^{p_i(.)}(Q_T)}\, dxdt \\
&\le C_{26} \| T_1(u_n - T_h(u_n)) \|_{L^{p_0(.)}(Q_T)} \big(\| \frac{1}{(1/n)^{p_0(x)-1}} |(u_n-\psi)^-|^{p_0(x)-} (u_n-\psi)^- \|_{L^{p_0'(.)}(Q_T)} + \| f \|_{L^{p_i'(.)}(Q_T)} \big) \\
&\longrightarrow 0,
\end{aligned}
$$

as $h \to 0$, thus, for all $\eta > 0$; there exists $h(\eta) > 1$ such as

$$
\sum_{i=1}^{N} \int_{\{h(\eta)\le|u_n|\}} |h_i^n(x,t,u_n,\nabla u_n)|\, dxdt \le \frac{\eta}{2}. \tag{4.3.49}
$$

On the other hand, for any measurable subset $E \subset Q_T$, and thanks to $(4.3.33)$, $(4.1.13)$ and $(4.1.14)$ we have

$$
\sum_{i=1}^{N} \int_{E\cap\{|u_n|<h(\eta)\}} |h_i^n(x,t,u_n,\nabla u_n)|\, dxdt
$$

$$
\begin{aligned}
&\leq \sum_{i=1}^{N} \int_{Q_T} (|u_n|^{q_0(x)} + \lambda)(\sum_{i=1}^{N} |D^i u_n|^{r_i(x)} + l(x,t))\, dxdt \\
&\leq \sum_{i=1}^{N} (|h(\eta)|^{q_0^+} + \lambda)\Big(\int_{E\cap\{|u_n|<h(\eta)\}} |D^i u_n|^{r_i(x)}\, dxdt + \int_{E\cap\{|u_n|<h(\eta)\}} l(x,t))\, dxdt\Big) \\
&\leq \sum_{i=1}^{N} (|h(\eta)|^{q_0^+} + \lambda)\big(2(meas)^{\frac{p_i^+ - r_i^-}{p_i^-}} \|D^i u_n\|^{r_i(x)}_{L^{p_i(.)}(Q_T)} + meas(E)\|l(x,t)\|_\infty\big) \\
&\leq \sum_{i=1}^{N} (|h(\eta)|^{q_0^+} + \lambda)\big(2(meas)^{\frac{p_i^+ - r_i^-}{p_i^-}} (\int_{Q_T} |D^i u_n|^{p_i(x)}\, dxdt + 1)^{\frac{r_i^+}{p_i^-}} + meas(E)\|l(x,t)\|_\infty\big),
\end{aligned}
$$

then, then by choosing $h(\eta)$ sufficiently large for all $E \subseteq Q_T$ with $meas(E) \leq h(\eta)$, we have

$$
\sum_{i=1}^{N} \int_{E\cap\{|u_n|<h(\eta)\}} |h_i^n(x,t,u_n,\nabla u_n)|\, dxdt \leq \frac{\eta}{2}. \tag{4.3.50}
$$

Finaly, by combining $(4.3.49)$ nad $(4.3.50)$, one easily has

$$
\sum_{i=1}^{N} \int_{E} |h_i^n(x,t,u_n,\nabla u_n)|\, dxdt = \sum_{i=1}^{N} \int_{E\cap(\{h(\eta)\leq|u_n|\}\cup\{h(\eta)>|u_n|\})} |h_i^n(x,t,u_n,\nabla u_n)|\, dxdt \leq \eta. \tag{4.3.51}
$$

As consequence , we deduce that

$$
(h_i^n(x,t,u_n,\nabla u_n)_{n\in I\!N} \quad \text{are equi-integrable} \quad \text{for } i = 1,\ldots,N. \tag{4.3.52}
$$

Step 4: Almost everywhere convergence of u_n and its gradients.

In the sequel, we exploit some compactness results to prove the almost everywhere convergence of u_n and ∇u_n. To this end, let's remark first of all that due to $(4.3.51)$ we have

$$
h_i^n(x,t,u_n,\nabla u_n) \quad \text{are bounded in } \ L^1(Q_T) for i = 1,\ldots,N. \tag{4.3.53}
$$

As soon as $E = Q_T$, then if we rewrite $(4.3.10)$ in the form

$$
\begin{aligned}
\frac{\partial u_n}{\partial t} \ &= \Big(\textstyle\sum_{i=1}^{N} D^i a_i(x,t,T_n(u_n),\nabla u_n) + f\Big) + \Big(\frac{1}{(1/n)^{p_0(.)-1}} |(u_n-\psi)^-|^{p_0(.)-2}(u_n-\psi)^- \\
&\quad -\Theta(x,t)|u_n|^{p_0(x)-2}u_n - \sum_{i=1}^{N} h_i^n(x,t,u_n,\nabla u_n)\Big).
\end{aligned}
$$

Let's note

$$
\sigma_{1n} = \sum_{i=1}^{N} D^i a_i(x,t,T_n(u_n),\nabla u_n) + f,
$$

and

$$\sigma_{2n} = -\sum_{i=1}^{N} h_i^n(x,t,u_n,\nabla u_n) - \Theta(x,t)|u_n|^{p_0(x)-2}u_n + \frac{1}{(1/n)^{p_0(.)-1}}|(u_n-\psi)^-|^{p_0(.)-2}(u_n-\psi)^-.$$

In view of $(4.1.5)$ and $(4.1.3)$, we have

$$\sigma_{1n} \quad \text{bounded in } V^*(Q_T). \tag{4.3.54}$$

Thanks to $(4.3.46)$, $(4.3.52)$ and $(4.3.53)$, we get σ_{2n} bounded in $L^{p_0'(x)}(Q_T)$, and since $p_0'(.) > 1$, we obtain

$$\sigma_{2n} \quad \text{bounded in } L^1(Q_T) = L^1(0,T;L^1(\Omega)). \tag{4.3.55}$$

Let's remark that $\forall i = 1,\dots,N$, for the assumption $\inf\{p_i'(x), \frac{N}{N-1}\} > m_i(x)$ a.e. in Ω and by the virtue of the definition of $V^*(Q_T)$, we have $V^*(Q_T) + L^1(0,T;L^1(\Omega)) \subset L^1(0,T;W^{-1,\vec{m}(.)}(\Omega))$, so, we get u_n is bounded in $V_0(Q_T)$ with $\frac{\partial u_n}{\partial t}$ bounded in $L^1(0,T;W^{-1,\vec{m}(.)}(\Omega))$.
Inasmuch as, $W_0^{-1,\vec{p}(.)}(\Omega) \hookrightarrow\hookrightarrow L^{\underline{p}}(\Omega) \hookrightarrow W^{-1,\vec{m}(.)}(\Omega)$ for $m_i(.) < \underline{p}$, $\forall i = 1,\dots,N$, and taking into account the Lemma of Aubin [145], we deduce that $\{u_n\}_{n\in I\!N}$ is relatively compact in $L^{p_i^-}(0,T;L^{\underline{p}}(\Omega))$, then

$$u_n \to u \quad \text{in } L^{p_i^-}(0,T;L^{\underline{p}}(\Omega)), \tag{4.3.56}$$

and

$$u_n \to u \quad \text{a.e. in } Q_T, \tag{4.3.57}$$

for a subsequence still denoted as $(u_n)_{n\in I\!N}$.
Now, thanks to $(4.3.54)$ and $(4.3.55)$ we get

$$\sigma_{1n} \to \sigma_1 \quad \text{in } V^*(Q_T), \tag{4.3.58}$$

$$\sigma_{2n} \to \sigma_2 \quad \text{in } L^1(Q_T), \tag{4.3.59}$$

for a subsequence still denoted as $(\sigma_{1n})_{n\in I\!N}$ and $(\sigma_{2n})_{n\in I\!N}$. Therefore

$$\frac{\partial u_n}{\partial t} \in V^*(Q_T) + L^1(Q_T), \tag{4.3.60}$$

which ensure $(4.3.6)$.

Almost everywhere convergence of ∇u_n

In order to prove the almost everywhere convergence of the gradient ∇u_n, we poceed by rewriting $(4.3.10)$ in the form

$$\begin{aligned}\frac{\partial u_n}{\partial t} - \sum_{i=1}^{N} D^i a_i(x,t,T_n(u_n),\nabla u_n) &= f + \Big\{ \tfrac{1}{(1/n)^{p_0(.)-1}} |(u_n-\psi)^-|^{p_0(.)-2}(u_n-\psi)^- \\ &\quad -\Theta(x,t)|u_n|^{p_0(x)-2}u_n - \sum_{i=1}^{N} h_i^n(x,t,u_n,\nabla u_n)\Big\} \\ &= f + \sigma_{2n}.\end{aligned}$$

Since u_n is bounded in $(4.3.33)$ and thanks to the weakly compacity of $h_i^n(x,t,u_n,\nabla u_n)$ in $L^1(Q_T)$ (see $(4.3.52)$ and $(4.3.53)$), and due to $(4.1.3)$, we have $f+\sigma_{2n}$ is bounded in $L^{p_i'(.)}(Q_T)+L^1(Q_T)$, for $1,\dots,N$, then by applying a compactness result of Boccardo et als. (see [62]) we obtain

$$D^i u_n \to D^i u \quad \text{in } L^{s(.)}(Q_T), \forall s(.) < p_i(.),$$

especially for $\underline{p} < p_i(.)$, thus $D^i u_n \to D^i u$ in $L^{\underline{p}}(Q_T)$. Consequently, we have for a subsequence

$$D^i u_n \to D^i u \quad \text{a.e. in } Q_T. \tag{4.3.61}$$

Step 5: Establishment of $(4.3.8)$ and initial conditions .

In order to ensure the initial condition, we will first prove $(4.3.8)$. Indeed, since $h_i^n(x,t,u_n,\nabla u_n)$ are carathéodory functions for any i and due to $(4.3.57)$ and $(4.3.61)$ we have

$$h_i^n(x,t,u_n,\nabla u_n) \to h_i^n(x,t,u,\nabla u) \quad \text{a.e. in } Q_T.$$

Therefore, thanks to $(4.3.52)$ and Vitali's theorem, we get

$$h_i^n(x,t,u_n,\nabla u_n) \to h_i^n(x,t,u,\nabla u) \quad \text{in } L^1(Q_T) \quad \text{for } i=,\dots,N. \tag{4.3.62}$$

In the other hand, by the virtue of $(4.3.32)$, we have

$$\sum_{i=1}^{N} \int_{Q_T} u_n h_i^n(x,t,u_n,\nabla u_n)\,dxdt \le C_{26}, \tag{4.3.63}$$

with C_{26} denote a nonnegative constant which does not depend on n. Taking $(4.3.63)$ and the sign condition $(4.1.12)$ and by applying Fatou'Lemma we get

$$\sum_{i=1}^{N} \int_{Q_T} u h_i^n(x,t,u,\nabla u)\,dxdt \le \liminf \sum_{i=1}^{N} \int_{Q_T} u_n h_i^n(x,t,u_n,\nabla u_n)\,dxdt \le C_{27},$$

from where

$$uh_i^n(x,t,u,\nabla u) \in L^1(Q_T). \tag{4.3.64}$$

Then $(4.3.8)$ is fulffiled.

Now, it remains to show the initial condition $(4.3.7)$, so by the virtue of $(4.3.62)$, the boundedness $(4.3.55)$ will be strengthened as follows:

$$\begin{aligned} &\frac{\partial u_n}{\partial t} = \sigma_{1n} + \sigma_{2n} \quad \text{with} \quad \sigma_{1n} \quad \text{bounded in } V^*(Q_T) \\ &\text{and } \sigma_{2n} \quad \text{relatively compact in } L^1(Q_T) = L^1(0,T;L^1(\Omega)), \end{aligned} \tag{4.3.65}$$

then, as soon as $W^{-1,\overrightarrow{p'}(.)}(\Omega) + L^1(\Omega) \hookrightarrow W^{-1,\overrightarrow{m}(.)}(\Omega)$, for $\inf\{p_i'(x), \frac{N}{N-1}\} > m_i(x)$ a.e. in Ω.

$\forall \xi > 0$, we get

$$\begin{aligned} \|u_n(t+\xi) - u_n(t)\|_{W^{-1,\overrightarrow{m}(.)}(\Omega)} &= \| \int_t^{t+\xi} (\sigma_{1n} + \sigma_{2n}) dt' \|_{W^{-1,\overrightarrow{m}(.)}(\Omega)} \\ &\leq C_{28} \sum_{i=1}^N \xi^{\frac{1}{p_i^-}} \|\sigma_{1n}\|_{V^*(Q_T)} + C_{29} \int_t^{t+\xi} \|\sigma_{2n}\|_{L^1(\Omega)} \\ &\leq C_{28} \sum_{i=1}^N \xi^{\frac{1}{p_i^-}} \|\sigma_{1n}\|_{V^*(Q_T)} + C_{29} \|\sigma_{2n}\|_{L^1(t,t+\xi;L^1(\Omega))}, \end{aligned}$$

which, jointly with $(4.3.65)$, allows us to get establish that

$$u_n \text{ is uniformaly equicontinuous in } \quad C^0(0,T;W^{-1,\overrightarrow{m}(.)}(\Omega)). \tag{4.3.66}$$

Therefore, due to $(4.3.36)$ and $(4.3.66)$, Ascoli's theorem [145] enable us to achieve that u_n is relatively compact in $C^0(0,T;W^{-1,\overrightarrow{m}(.)}(\Omega))$. Hence, for $i = 1, \ldots, N$

$$u_n \to u \text{ strongly in } \quad C^0(0,T;W^{-1,\overrightarrow{m}(.)}(\Omega)) \text{ under the assumption } \quad \inf\{p_i'(x), \frac{N}{N-1}\} > m_i(x), \tag{4.3.67}$$

then by letting n to infinity in $u_n(x,0) = u_0(x)$, we conclude that $u(x,0) = u_0(x)$.

Step 6: Passing to the limit .

First of all, note taht as soon as u_n is bounded in $V_0(Q_T)$, we get by $(4.1.5)$, $(a_i(x,t,u_n,\nabla u_n))_{n\in I\!N}$ is bounded in $L^{p_i'(.)}(Q_T$. Then, due to $(4.3.57)$ and $(4.3.61)$, we have

$$a_i(x,t,u_n,\nabla u_n) \to a_i(x,t,u,\nabla u) \quad \text{a.e. in } Q_T.$$

Hence, in view of Lemma $(4.2.1)$, we have

$$a_i(x,t,u_n,\nabla u_n) \rightharpoonup a_i(x,t,u,\nabla u) \quad \text{in } L^{p_i'(.)}(Q_T) \ for\ i=1,\dots,N. \tag{4.3.68}$$

Furthermore, by the virtue of $(4.3.48)$ we obtain $\mu \in L^{p_0'(.)}(Q_T)$ with μ is a nonnegative function which ensures $(4.3.2)$. On the other hand, let's remark that

$$-\frac{1}{n}\left|\frac{(u_n-\psi)^-}{(1/n)}\right|^{p_0(x)} = \frac{1}{(1/n)^{p_0(x)-1}}|(u_n-\psi)^-|^{p_0(x)-2}(u_n-\psi)^-(u_n-\psi)$$

Taking into account $(4.3.46)$, $(4.3.48)$ and $(4.3.56)$ and by letting n to infinity, we conclude that $\mu(u-\psi)=0$ a.e. in Q_T, thus the complementary condition $(4.3.3)$ is satisfied.

As a consequence of the above convergence results, we are in a position to pass to the limit as n tends to infinity in equation $(4.3.10)$ and to conclude that the pair of functions (u,μ) checks $(4.1.2)$. This complete the proof.

Part II

Nonlinear anisotropic elliptic PDEs involving singularities on its right -hand side within the anisotropic Sobolev spaces with either variable exponents or constant exponents or weight.

Chapter 5

Existence of solutions for some quasilinear $\vec{p}(x)-$elliptic problem with Hardy potential

In this chapter, we consider the following anisotropic quasilinear elliptic Dirichlet problem

$$\begin{cases} -\sum_{i=1}^{N} D^i a_i(x,u,\nabla u) + |u|^{s(x)-1}u = f + \lambda \frac{|u|^{p_0(x)-2}u}{|x|^{p_0(x)}} & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases}$$

where Ω is an open bounded subset of $\mathbf{R}^N$ containing the origin. We show the existence of entropy solution for this equation where the data f is assumed to be in $L^1(\Omega)$ and λ is a positive constant.

5.1 Introduction

In the recent years, the anisotropic variable exponent Sobolev spaces have taken its place in the mathematical literature. This impulse is essentially due to their applications in nonhomogeneous materials that behave differently on different space directions, we can refer here to the electrorheological and thermoelectric fluids that have multiple applications in brakes shock absorbers, robotics and space technology (see for example [16, 124]).

In [56], Boccardo et al. have studied the following nonlinear anisotropic elliptic equation

$$\begin{cases} -\sum_{i=1}^{N} \frac{\partial}{\partial x_i}\Big(\Big|\frac{\partial u}{\partial x_i}\Big|^{p_i-2}\frac{\partial u}{\partial x_i}\Big) = f & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{5.1.1}$$

where Ω is an open bounded subset of $\mathbb{R}^N$, $(N \geq 2)$, and the right-hand side f is a bounded Randon measure. They have proved the existence and regularity of solutions in the anisotropic Sobolev spaces $W_0^{1,\vec{p}}(\Omega)$. The critical regularity was obtained under the assumption of $|f|\log(1+|f|) \in L^1(\Omega)$. In [78], Cîrstea and Vétois have proved the existence of weak solutions to the problem $(5.1.1)$ where the data f is assumed to be a Dirac mass at 0. We refer also to [152] where the author has proved the existence of nonnegative weak solutions in anisotropic Sobolev space for the elliptic and parabolic cases, where the right-hand side is assumed to be a Carathéodory function $f(x,u,\nabla u)$. We refer the reader also to [151].
In [86], Di Nardo and Feo have considered the quasilinear elliptic problem

$$\begin{cases} -\sum_{i=1}^{N} \partial_i a_i(x,u,\nabla u) + \sum_{i=1}^{N} H_i(x,\nabla u) = f - \sum_{i=1}^{N} \partial_i g_i & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{5.1.2}$$

they have proved the existence and uniqueness of weak solutions for this anisotropic elliptic Dirichlet problem, where the data is assumed to be in the dual space.
Di Nardo, Feo and Guibé have studied in [87] the existence of renormalized solutions for some class of nonlinear anisotropic elliptic problems of the type

$$-\sum_{i=1}^{N} \partial_{x_i}(a_i(x,u)|\partial_{x_i}u|^{p_i-2}\partial_{x_i}u) = f - \text{div } g \quad \text{in } \Omega,$$

with $f \in L^1(\Omega)$ and $g \in \Pi_{i=1}^{N} L^{p_i'}(\Omega)$, the uniqueness of renormalized solution was concluded under some local Lipschitz conditions on the function $a_i(x,s)$ with respect to s, (see also [15]).
In the framework of variable exponents Sobolev spaces, Wittbold and Zimmermann have proved in [153] the existence and uniqueness of renormalized solutions for the quasilinear elliptic problem

$$\begin{cases} \beta(u) - \text{div } a(x,\nabla u) - \sum_{i=1}^{N} \text{div } F(u) \ni f & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{5.1.3}$$

where the data f is assumed to be in $L^1(\Omega)$. In [40], Bendahmane et al. have considered the nonlinear elliptic equation

$$\begin{cases} -\sum_{i=1}^{N} \frac{\partial}{\partial x_i}\Big(\Big|\frac{\partial u}{\partial x_i}\Big|^{p_i(x)-2}\frac{\partial u}{\partial x_i}\Big) + |u|^{s(x)-1}u = f & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{5.1.4}$$

with $f \in L^1(\Omega)$ and $p_i(\cdot)$ are continuous functions for $i = 1,\ldots,N$, they have shown the existence of solution in the anisotropic variable exponents Sobolev spaces. Also, the authors have proved the

corresponding results for the nonlinear anisotropic parabolic case. In [76], Cianchi has considered the quasilinear anisotropic elliptic problem

$$\begin{cases} -\text{div}\,(a(x,u,\nabla u)) = f(x,u) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{5.1.5}$$

with the datum $f(x,s)$ is a Carathéodory function verifying some growth condition. He has proved the existence of weak solutions in the anisotropic Orlicz–Sobolev spaces, which established via symmetrization. We refer also to [129], where the author has shown the existence and regularity of weak solutions for the data is in $L^m(\Omega)$ with $m \geq 1$, also to [30] for the existence of weak solutions, and to [13] for the solutions in sense of distributions, and [109] for the renormalized solutions.

For some elliptic problems with singularity on its right-hand side, we refer the reader to [2] where the authors have studied the nonlinear elliptic problem

$$\begin{cases} -\Delta u \pm |\nabla u|^2 = \lambda \dfrac{u}{|x|^2} + f & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{5.1.6}$$

with $\lambda > 0$. They have proved the existence of positive solutions for the problem $(7.1.8)$ in the absorption case $(+|\nabla u|^2)$ with $f \in L^1(\Omega)$. In the reaction case $(-|\nabla u|^2)$, the non-existence of solution is proved even in a very weak sense. Porzio has studied in [138], the existence of a weak solutions for the quasilinear elliptic problem

$$\begin{cases} -\text{div}\,(M(x,u)\nabla u) + \nu|u|^{p-1}u = a\dfrac{u}{|x|^2} + f(x) - \text{div}(F) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{5.1.7}$$

where $p > \frac{N}{N-2}$ and a is a positive constant, the Carathéodory function $M(x,s)$ satisfies the growth and coercivity conditions. For the case of nonlinear and non-coercive elliptic problems, we refer the reader to [106, 107].

In this chapter, we consider Ω be an open bounded subset of $\mathbf{R}^N$ $(N \geq 2)$, containing the origin, and let $p_i(\cdot)$ some measurable functions on Ω for any $i = 0, 1, \ldots, N$ where

$$p_0(x) = \max\{p_i(x), \quad i = 1, 2, \ldots, N\} \quad \text{a.e. in } \Omega. \tag{5.1.8}$$

We will study the existence of entropy solutions for the following anisotropic quasilinear elliptic problem

$$\begin{cases} Au + |u|^{s(x)-1}u = f + \lambda \dfrac{|u|^{p_0(x)-2}u}{|x|^{p_0(x)}} & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{5.1.9}$$

with $\lambda \geq 0$, $f \in L^1(\Omega)$ and

$$s(x) > \max\Big(\frac{N(p_0(x)-1)}{N-p_0(x)}, \frac{1}{p_0(x)-1}\Big) \quad \text{a.e. in} \quad \Omega. \tag{5.1.10}$$

The Leray-Lions operator A acting from $W_0^{1,\vec{p}(\cdot)}(\Omega)$ into its dual, is defined by the formula

$$Au = -\sum_{i=1}^{N} D^i a_i(x,u,\nabla u) \tag{5.1.11}$$

where $a_i : \Omega \times \mathbf{R} \times \mathbf{R}^N \longmapsto \mathbf{R}^N$ are Carathéodory functions, for $i = 1, \ldots, N$, (measurable with respect to x in Ω for every (s,ξ) in $I\!R \times I\!R^N$ and continuous with respect to (s,ξ) in $I\!R \times I\!R^N$ for almost every x in Ω), which satisfy the following conditions :

$$|a_i(x,s,\xi)| \leq \beta\Big(K_i(x) + |s|^{p_i(x)-1} + \sum_{i=1}^{N} |\xi_i|^{p_i(x)-1}\Big), \tag{5.1.12}$$

$$\sum_{i=1}^{N} a_i(x,s,\xi)\xi_i \geq \alpha \sum_{i=1}^{N} |\xi_i|^{p_i(x)}, \tag{5.1.13}$$

for any $\xi = (\xi_1, \ldots, \xi_N)$ and $\xi' = (\xi'_1, \ldots, \xi'_N)$ in $I\!R^N$, we have

$$\big(a_i(x,s,\xi) - a_i(x,s,\xi')\big)(\xi_i - \xi'_i) > 0 \quad \text{for} \quad \xi_i \neq \xi'_i, \tag{5.1.14}$$

for a.e. $x \in \Omega$ and all $(s,\xi) \in \mathbf{R} \times \mathbf{R}^N$, where $K_i(x) \in L^{p'_i(\cdot)}(\Omega)$, and α, β are two positive real numbers.

Remark 5.1.1. *The assumption* $(5.1.8)$ *is essential to ensure that* $|a_i(x,u,\nabla u)|$ *belongs to* $L^{p'_i(\cdot)}(\Omega)$. *In the case of* $Au = -\sum_{i=1}^{N} D^i a_i(x,\nabla u)$, *the condition* $(5.1.12)$ *will be written as*

$$|a_i(x,\xi)| \leq \beta\Big(K_i(x) + \sum_{i=1}^{N} |\xi_i|^{p_i(x)-1}\Big), \tag{5.1.15}$$

thus the existence of entropy solution will be guaranteed by following the same way, without using the additional assumption $(5.1.8)$.

Note that, in view of the growth condition $(5.1.12)$, to show that the Carathéodory functions $|a_i(x,u,\nabla u)|$ belongs to $L^{p'_i(\cdot)}(\Omega)$ for any $u \in W_0^{1,\vec{p}(\cdot)}(\Omega)$, it's necessary to have $u \in L^{p_i(\cdot)}(\Omega)$ for any $i = 1, \ldots, N$, which is verified by taking the condition $(5.1.8)$. For the case where the Carathéodory functions $a_i(x,\xi)$ verify the assumption $(5.1.15)$, then $|a_i(x,\nabla u)| \in L^{p'_i(\cdot)}(\Omega)$ for any $u \in W_0^{1,\vec{p}(\cdot)}(\Omega)$.

In this chapter, we have assumed that the data f belong to $L^1(\Omega)$, then the elliptic equation $(5.1.9)$ is not in the dual space $W^{-1,\vec{p}'(\cdot)}(\Omega)$, thus the existence of weak solution haven't a sense. To overcome this difficulty, some mathematicians have stated the notions of entropy and renormalized solutions, which are more adapted for this category of problems. Note that the entropy solutions was introduced by Bénilan et al. in [46], and the notion of renormalized solutions by DiPerna et al. in [88, 89].

The difficulties in proving the existence of entropy solutions stems from the fact that : Since the exponents $p_i(\cdot)$ are assumed to be measurable functions, then the Poincaré and Sobolev inequalities are not verified, therefore, the operator Au is not coercive in the anisotropic variable exponent Sobolev space $W_0^{1,\vec{p}(\cdot)}(\Omega)$, defined below. To overcome this difficulty, we use the penalization term $\frac{1}{n}|u|^{p_0(x)-2}u$ in approximate problems $(5.2.4)$. Moreover, the singular term $\frac{|u|^{p_0(x)-2}u}{|x|^{p_0(x)}}$ creates, in general, an hindrance to the existence of a solution. We overpass this difficulty by using the regularizing effect of the term $|u|^{s(x)-1}u$ to remove the non-existence effect produced by the Hardy potential.

This chapter is organized as follows: Bearing in mind some technical lemmas and results on the anisotropic variable exponent Sobolev spaces, discussed in the preliminaries, we will allocate the next section to show the existence of entropy solutions for our anisotropic $\vec{p}(x)-$quasilinear elliptic equation with Hardy potential $(5.1.9)$.

5.2 Existence of entropy Solutions

Let Ω be a bounded open subset of $\mathbf{R}^N$ $(N \geq 2)$, containing the origin, and let $p_i(\cdot) \in \mathcal{C}_+(\overline{\Omega})$ for $i = 0, 1, \ldots, N$, assuming that

$$\mathcal{C}_+(\Omega) = \{\text{measurable function} \quad p(\cdot) : \Omega \longmapsto I\!R \quad \text{such that} \quad 1 < p^- \leq p^+ < N\},$$

where

$$p^- = ess\inf\{p(x) \,/\, x \in \Omega\} \qquad \text{and} \qquad p^+ = ess\sup\{p(x) \,/\, x \in \Omega\}.$$

and

$$p_0(x) = \max\{p_i(x), \quad i = 1, 2, \ldots, N\} \quad \text{a.e. in} \quad \Omega. \tag{5.2.1}$$

Definition 5.2.1. *A measurable function u is an entropy solution of the Dirichlet problem $(5.1.9)$ if*

$$u \in \mathcal{T}_0^{1,\vec{p}(\cdot)}(\Omega), \qquad |u|^{s(x)-1}u \in L^1(\Omega), \qquad \frac{|u|^{p_0(x)-2}u}{|x|^{p_0(x)}} \in L^1(\Omega)$$

such that

$$\begin{aligned}\sum_{i=1}^{N}\int_{\Omega} a_i(x,u,\nabla u)\cdot D^i T_k(u-\varphi)\,dx + \int_{\Omega}|u|^{s(x)-1}uT_k(u-\varphi)\,dx \\ \leq \int_{\Omega} fT_k(u-\varphi)\,dx + \lambda\int_{\Omega}\frac{|u|^{p_0(x)-2}u}{|x|^{p_0(x)}}T_k(u-\varphi)\,dx,\end{aligned} \tag{5.2.2}$$

for any $\varphi \in W_0^{1,\vec{p}(\cdot)}(\Omega)\cap L^{\infty}(\Omega)$.

Our main result is the following:

Theorem 5.2.1. *Let* $\lambda \geq 0$ *and* $f \in L^1(\Omega)$, *assuming that* $(5.1.10)$ *and* $(5.1.12)-(5.1.14)$ *hold true. Then, there exists at least one entropy solution* u *for quasilinear elliptic problem* $(5.1.9)$, *such that* $u \in W_0^{1,\vec{q}(\cdot)}(\Omega)$, *with*

$$\vec{q}(\cdot) = (s(\cdot), q_1(\cdot),\ldots,q_N(\cdot)) \qquad \textit{and} \qquad 1 \leq q_i(x) < \frac{p_i(x)s(x)}{s(x)+1} \quad \textit{for } i=1,\ldots,N. \tag{5.2.3}$$

Proof of the Theorem 5.2.1

Step 1 : Approximate problems.

Let $(f_n)_{n\in I\!N^*}$ be a sequence of smooth functions such that $f_n \to f$ in $L^1(\Omega)$ and $|f_n| \leq |f|$ (for example $f_n = T_n(f)$). We consider the approximate problem

$$A_n u_n + |T_n(u_n)|^{s(x)-1}T_n(u_n) = f_n + \lambda\frac{|T_n(u_n)|^{p_0(x)-2}T_n(u_n)}{|x|^{p_0(x)}+\frac{1}{n}}, \tag{5.2.4}$$

where $A_n v = -\sum_{i=1}^{N}\dfrac{\partial}{\partial x_i}a_i(x,T_n(v),\nabla v) + \dfrac{1}{n}|v|^{p_0(x)-2}v$.
We consider the operator $G_n : W_0^{1,\vec{p}(\cdot)}(\Omega) \longmapsto W^{-1,,\vec{p'}(\cdot)}(\Omega)$ by

$$\langle G_n u, v\rangle = \int_{\Omega}|T_n(u)|^{s(x)-1}T_n(u)v\,dx - \lambda\int_{\Omega}\frac{|T_n(u)|^{p_0(x)-2}T_n(u)}{|x|^{p_0(x)}+\frac{1}{n}}v\,dx$$

for any $u, v \in W_0^{1,\vec{p}(\cdot)}(\Omega)$. Thanks to the generalized Hölder's type inequality, we have

$$\begin{aligned}|\langle G_n u, v\rangle| &\leq \int_{\Omega}|T_n(u)|^{s(x)}\,|v|\,dx + \lambda\int_{\Omega}\frac{|T_n(u)|^{p_0(x)-1}}{|x|^{p_0(x)}+\frac{1}{n}}|v|\,dx \\ &\leq n^{s^+}\int_{\Omega}|v|\,dx + \lambda n^{p_0^+}\int_{\Omega}|v|\,dx \\ &\leq C_0\|v\|_{1,\vec{p}(\cdot)}\,.\end{aligned} \tag{5.2.5}$$

Lemma 5.2.1. *The bounded operator $B_n = A_n + G_n$ acting from $W_0^{1,\vec{p}(\cdot)}(\Omega)$ into $W^{-1,\vec{p'}(\cdot)}(\Omega)$ is pseudo-monotone. Moreover, B_n is coercive in the following sense:*

$$\frac{\langle B_n v, v\rangle}{\|v\|_{1,\vec{p}(\cdot)}} \longrightarrow +\infty \qquad \textit{as} \quad \|v\|_{1,\vec{p}(\cdot)} \to \infty \quad \textit{for} \quad v \in W_0^{1,\vec{p}(\cdot)}(\Omega).$$

Proof of the Lemma 5.2.1

In view of the Hölder's inequality and the growth condition $(5.1.12)$, it's easy to see that the operator A_n is bounded, and by $(5.2.5)$ we conclude that B_n is bounded.

For the coercivity, we have for any $u \in W_0^{1,\vec{p}(\cdot)}(\Omega)$,

$$\begin{aligned}
\langle B_n u, u\rangle &= \langle A_n u, u\rangle + \langle G_n u, u\rangle \\
&= \sum_{i=1}^{N} \int_\Omega a_i(x, T_n(u), \nabla u)\, D^i u \, dx + \int_\Omega |T_n(u)|^{s(x)}|u| \, dx + \frac{1}{n}\int_\Omega |u|^{p_0(x)}\, dx \\
&\quad -\lambda \int_\Omega \frac{|T_n(u)|^{p_0(x)-1}}{|x|^{p_0(x)} + \frac{1}{n}}|u| \, dx \\
&\geq \underline{\alpha} \sum_{i=0}^{N} \int_\Omega |D^i u|^{p_i(x)}\, dx + \int_\Omega |T_n(u)|^{s(x)+1}\, dx - 2\lambda n^{p_0^+} \|1\|_{p_0'(\cdot)} \|u\|_{1,\vec{p}(\cdot)} \\
&\geq \underline{\alpha}\, \|u\|^{\underline{p}}_{1,\vec{p}(\cdot)} - \underline{\alpha}(N+1) - C_1 \|u\|_{1,\vec{p}(\cdot)}
\end{aligned}$$

with $\underline{\alpha} = \min(\alpha, \frac{1}{n})$, it follows that

$$\frac{\langle B_n u, u\rangle}{\|u\|_{1,\vec{p}(\cdot)}} \longrightarrow +\infty \quad \text{as} \quad \|u\|_{1,\vec{p}(\cdot)} \to \infty.$$

It remains to show that B_n is pseudo-monotone. Let $(u_k)_{k\in\mathbb{N}}$ be a sequence in $W_0^{1,\vec{p}(\cdot)}(\Omega)$ such that

$$\begin{cases} u_k \rightharpoonup u & \text{in } \ W_0^{1,\vec{p}(\cdot)}(\Omega), \\ B_n u_k \rightharpoonup \chi_n & \text{in } \ W^{-1,\vec{p'}(\cdot)}(\Omega), \\ \limsup\limits_{k\to\infty} \langle B_n u_k, u_k\rangle \leq \langle \chi_n, u\rangle. \end{cases} \tag{5.2.6}$$

We will prove that

$$\chi_n = B_n u \quad \text{and} \quad \langle B_n u_k, u_k\rangle \longrightarrow \langle \chi_n, u\rangle \ \text{ as } \ k \to +\infty.$$

In view of the compact embedding $W_0^{1,\vec{p}(\cdot)}(\Omega) \hookrightarrow\hookrightarrow L^{\underline{p}}(\Omega)$, we have $u_k \to u$ in $L^{\underline{p}}(\Omega)$ for a subsequence still denoted $(u_k)_{k\in\mathbb{N}}$.

As $(u_k)_{k\in\mathbb{N}}$ is a bounded sequence in $W_0^{1,\vec{p}(\cdot)}(\Omega)$, using the growth condition $(5.1.12)$ it's clear that the sequence $(a_i(x, T_n(u_k), \nabla u_k))_{k\in\mathbb{N}}$ is bounded in $L^{p_i'(\cdot)}(\Omega)$, then there exists a function $\varphi_i \in L^{p_i'(\cdot)}(\Omega)$ such that

$$a_i(x, T_n(u_k), \nabla u_k) \rightharpoonup \varphi_i \ \text{ in } \quad L^{p_i'(\cdot)}(\Omega) \ \text{ as } \ k \to \infty. \tag{5.2.7}$$

In view of Lebesgue's dominated convergence theorem, we obtain

$$|T_n(u_k)|^{s(x)-1}T_n(u_k) \to |T_n(u)|^{s(x)-1}T_n(u) \qquad \text{in} \quad L^{p_0'(\cdot)}(\Omega), \tag{5.2.8}$$

and

$$\frac{|T_n(u_k)|^{p_0(x)-2}T_n(u_k)}{|x|^{p_0(x)}+\frac{1}{n}} \to \frac{|T_n(u)|^{p_0(x)-2}T_n(u)}{|x|^{p_0(x)}+\frac{1}{n}} \ \text{ in } \quad L^{p_0'(\cdot)}(\Omega). \tag{5.2.9}$$

Also, we have

$$\frac{1}{n}|u_k|^{p_0(x)-2}u_k \rightharpoonup \frac{1}{n}|u|^{p_0(x)-2}u \qquad \text{in} \quad L^{p_0'(\cdot)}(\Omega). \tag{5.2.10}$$

For any $v \in W_0^{1,\vec{p}(\cdot)}(\Omega)$, we have

$$\begin{aligned}
\langle \chi_n, v\rangle &= \lim_{k\to\infty}\langle B_n u_k, v\rangle \\
&= \lim_{k\to\infty}\sum_{i=1}^{N}\int_\Omega a_i(x,T_n(u_k),\nabla u_k)D^i v\,dx + \lim_{k\to\infty}\int_\Omega |T_n(u_k)|^{s(x)-1}T_n(u_k)v\,dx \\
- &\quad + \lim_{k\to\infty}\frac{1}{n}\int_\Omega |u_k|^{p_0(x)-2}u_k\,v\,dx - \lim_{k\to\infty}\lambda\int_\Omega \frac{|T_n(u_k)|^{p_0(x)-2}T_n(u_k)}{|x|^{p_0(x)}+\frac{1}{n}}v\,dx \\
&= \sum_{i=1}^{N}\int_\Omega \varphi_i\, D^i v\,dx + \int_\Omega |T_n(u)|^{s(x)-1}T_n(u)v\,dx + \frac{1}{n}\int_\Omega |u|^{p_0(x)-2}u\,v\,dx \\
&\quad -\lambda\int_\Omega \frac{|T_n(u)|^{p_0(x)-2}T_n(u)}{|x|^{p_0(x)}+\frac{1}{n}}v\,dx.
\end{aligned} \tag{5.2.11}$$

Having in mind $(5.2.6)$ and $(5.2.11)$, we obtain

$$\begin{aligned}
\limsup_{k\to\infty}\langle B_n(u_k),u_k\rangle &= \limsup_{k\to\infty}\Big\{\sum_{i=1}^{N}\int_\Omega a_i(x,T_n(u_k),\nabla u_k)D^i u_k\,dx + \int_\Omega |T_n(u_k)|^{s(x)-1}T_n(u_k)u_k\,dx \\
&\qquad +\frac{1}{n}\int_\Omega |u_k|^{p_0(x)}\,dx - \lambda\int_\Omega \frac{|T_n(u_k)|^{p_0(x)-2}T_n(u_k)}{|x|^{p_0(x)}+\frac{1}{n}}u_k\,dx\Big\} \\
&\le \sum_{i=1}^{N}\int_\Omega \varphi_i\, D^i u\,dx + \int_\Omega |T_n(u)|^{s(x)-1}T_n(u)u\,dx + \frac{1}{n}\int_\Omega |u|^{p_0(x)}\,dx \\
&\qquad -\lambda\int_\Omega \frac{|T_n(u)|^{p_0(x)-2}T_n(u)}{|x|^{p_0(x)}+\frac{1}{n}}u\,dx.
\end{aligned} \tag{5.2.12}$$

Thanks to $(5.2.8)$ and $(5.2.9)$, we have

$$\int_\Omega |T_n(u_k)|^{s(x)-1}T_n(u_k)u_k\,dx \longrightarrow \int_\Omega |T_n(u)|^{s(x)-1}T_n(u)u\,dx, \tag{5.2.13}$$

and

$$\int_\Omega \frac{|T_n(u_k)|^{p_0(x)-2}T_n(u_k)}{|x|^{p_0(x)}+\frac{1}{n}}u_k\,dx \longrightarrow \int_\Omega \frac{|T_n(u)|^{p_0(x)-2}T_n(u)}{|x|^{p_0(x)}+\frac{1}{n}}u\,dx. \tag{5.2.14}$$

Therefore

$$\limsup_{k\to\infty}\Big(\sum_{i=1}^{N}\int_\Omega a_i(x,T_n(u_k),\nabla u_k)D^iu_k\,dx+\frac{1}{n}\int_\Omega|u_k|^{p_0(x)}\,dx\Big)\le\sum_{i=1}^{N}\int_\Omega\varphi_i\,D^iu\,dx+\frac{1}{n}\int_\Omega|u|^{p_0(x)}\,dx. \tag{5.2.15}$$

On the other hand, in view of $(5.1.14)$ we have

$$\begin{aligned}\sum_{i=1}^{N}\int_\Omega(a_i(x,T_n(u_k),\nabla u_k)-a_i(x,T_n(u_k),\nabla u))(D^iu_k-D^iu)\,dx\\ +\frac{1}{n}\int_\Omega(|u_k|^{p_0(x)-2}u_k-|u|^{p_0(x)-2}u)(u_k-u)\,dx\ge0,\end{aligned} \tag{5.2.16}$$

then

$$\begin{aligned}&\sum_{i=1}^{N}\int_\Omega a_i(x,T_n(u_k),\nabla u_k)D^iu_k\,dx+\frac{1}{n}\int_\Omega|u_k|^{p_0(x)}\,dx\\ &\ge\sum_{i=1}^{N}\int_\Omega a_i(x,T_n(u_k),\nabla u_k)D^iu\,dx+\frac{1}{n}\int_\Omega|u_k|^{p_0(x)-2}u_ku\,dx\\ &\quad+\sum_{i=1}^{N}\int_\Omega a_i(x,T_n(u_k),\nabla u)(D^iu_k-D^iu)\,dx+\frac{1}{n}\int_\Omega|u|^{p_0(x)-2}u(u_k-u)\,dx.\end{aligned}$$

In view of Lebesgue's dominated convergence theorem we have $T_n(u_k)\to T_n(u)$ in $L^{p_i(\cdot)}(\Omega)$, thus $a_i(x,T_n(u_k),\nabla u)\to a_i(x,T_n(u),\nabla u)$ in $L^{p_i'(\cdot)}(\Omega)$, and using $(5.2.7)$ we get

$$\liminf_{k\to\infty}\Big(\sum_{i=1}^{N}\int_\Omega a_i(x,T_n(u_k),\nabla u_k)D^iu_k\,dx+\frac{1}{n}\int_\Omega|u_k|^{p_0(x)}\,dx\Big)\ge\sum_{i=1}^{N}\int_\Omega\varphi_i\,D^iu\,dx+\frac{1}{n}\int_\Omega|u|^{p_0(x)}\,dx.$$

Having in mind $(5.2.15)$, we conclude that

$$\lim_{k\to\infty}\Big(\sum_{i=1}^{N}\int_\Omega a_i(x,T_n(u_k),\nabla u_k)D^iu_k\,dx+\frac{1}{n}\int_\Omega|u_k|^{p_0(x)}\,dx\Big)=\sum_{i=1}^{N}\int_\Omega\varphi_i\,D^iu\,dx+\frac{1}{n}\int_\Omega|u|^{p_0(x)}\,dx. \tag{5.2.17}$$

Therefore, by combining $(5.2.11)$ and $(5.2.13)-(5.2.14)$, we obtain

$$\langle B_nu_k,u_k\rangle\longrightarrow\langle\chi_n,u\rangle\quad\text{as}\quad k\to\infty.$$

Now, by $(5.2.17)$ we can prove that

$$\begin{aligned}\lim_{k\to+\infty}\Big(\sum_{i=1}^{N}\int_\Omega(a_i(x,T_n(u_k),\nabla u_k)-a_i(x,T_n(u_k),\nabla u))(D^iu_k-D^iu)\,dx\\ +\frac{1}{n}\int_\Omega(|u_k|^{p_0(x)-2}u_k-|u|^{p_0(x)-2}u)(u_k-u)\,dx\Big)=0.\end{aligned}$$

and so, by virtue of Lemma 1.2.8, we get

$$u_k\to u\quad\text{in}\quad W_0^{1,\vec{p}(\cdot)}(\Omega)\qquad\text{and}\qquad D^iu_k\to D^iu\quad\text{a.e. in}\quad\Omega,$$

then

$$a_i(x,T_n(u_k),\nabla u_k) \rightharpoonup a_i(x,T_n(u),\nabla u) \quad \text{in} \quad L^{p_i'(\cdot)}(\Omega) \quad \text{for} \quad i=1,\dots,N$$

and thanks to $(5.2.8)-(5.2.10)$, we obtain $\chi_n = B_n u$, which conclude the proof of Lemma 5.2.1.

In view of Lemma 5.2.1, there exists at least one weak solution $u_n \in W_0^{1,\vec{p}(\cdot)}(\Omega)$ of the problem $(5.2.4)$ (cf. [121], Theorem 8.2).

Step 2 : A priori estimates.

Lemma 5.2.2. *Let u_n be a weak solution of the approximate problem $(5.2.4)$, then the following regularity results hold true*

$$u \in W_0^{1,\vec{q}(\cdot)}(\Omega) \qquad \textit{with} \quad \vec{q}(\cdot) = (s(\cdot), q_1(\cdot), \dots, q_N(\cdot)) \tag{5.2.18}$$

where the exponent $s(\cdot)$ verify the condition $(5.1.10)$ and $1 \le q_i(x) < \dfrac{p_i(x)s(x)}{s(x)+1}$, almost everywhere in Ω, then

$$\sum_{i=1}^{N} \int_\Omega \frac{|D^i u_n|^{p_i(x)}}{(1+|u_n|)^\theta} dx \le C \qquad \textit{for all} \quad 1 < \theta < \frac{s(p_i(x)-q_i(x))}{q_i(x)}, \tag{5.2.19}$$

$$\sum_{i=1}^{N} \int_\Omega |D^i T_k(u_n)|^{p_i(x)} dx \le C(1+k)^\theta \qquad \textit{for all} \quad k>0, \tag{5.2.20}$$

with C is a positive constant that doesn't depend on k and n.

Proof of Lemma 5.2.2

Let $\theta > 1$ which will be chosen later, we consider the function $\varphi(t) : \mathbb{R} \longmapsto \mathbb{R}$ defined by

$$\varphi(t) = \Big(1 - \frac{1}{(1+|t|)^{\theta-1}}\Big) sign(t)$$

It's clear that $\varphi(u_n) \in W_0^{1,\vec{p}(\cdot)}(\Omega) \cap L^\infty(\Omega)$. By taking $\varphi(u_n)$ as test function in $(5.2.4)$ we get

$$\begin{aligned}(\theta-1)\sum_{i=1}^{N} \int_\Omega \frac{a_i(x,T_n(u_n),\nabla u_n)\cdot D^i u_n}{(1+|u_n|)^\theta}\, dx + \int_\Omega |T_n(u_n)|^{s(x)-1} T_n(u_n)\varphi(u_n)\, dx \\ + \frac{1}{n}\int_\Omega |u_n|^{p_0(x)-2}u_n\varphi(u_n)\, dx = \int_\Omega f_n\varphi(u_n)\, dx + \lambda \int_\Omega \frac{|T_n(u_n)|^{p_0(x)-2}T_n(u_n)}{|x|^{p_0(x)}+\frac{1}{n}}\varphi(u_n)\, dx.\end{aligned}$$

Since $\varphi(u_n)$ have the same sign of u_n, thus the third term on the left-hand side of the previews inequality is positive. Also, we have $|\varphi(\cdot)| \le 1$ and in view of $(5.1.13)$, we obtain

$$\begin{aligned}&\alpha(\theta-1)\sum_{i=1}^{N} \int_\Omega \frac{|D^i u_n|^{p_i(x)}}{(1+|u_n|)^\theta} dx + \int_\Omega |T_n(u_n)|^{s(x)} |\varphi(u_n)|\, dx \\ &\le \lambda \int_\Omega \frac{|T_n(u_n)|^{p_0(x)-1}}{|x|^{p_0(x)}+\frac{1}{n}}\, dx + \int_\Omega |f|\, dx.\end{aligned} \tag{5.2.21}$$

It is clear that

$$\frac{1}{2} \leq 1 - \frac{1}{(1+|u_n|)^{\theta-1}} \quad \text{for} \quad |u_n| \geq R = \max(2^{\frac{1}{\theta-1}} - 1, 1).$$

Thus, we have

$$\begin{aligned} \frac{1}{2}\int_{\{|u_n|\geq R\}} |T_n(u_n)|^{s(x)}\,dx &\leq \int_{\{|u_n|\geq R\}} |T_n(u_n)|^{s(x)}\Big(1 - \frac{1}{(1+|u_n|)^{\theta-1}}\Big)\,dx \\ &\leq \int_{\Omega} |T_n(u_n)|^{s(x)}\left(1 - \frac{1}{(1+|u_n|)^{\theta-1}}\right)dx, \end{aligned}$$

which implies

$$\begin{aligned} \frac{1}{2}\int_{\Omega} |T_n(u_n)|^{s(x)}\,dx &= \frac{1}{2}\int_{\{|u_n|<R\}} |T_n(u_n)|^{s(x)}\,dx + \frac{1}{2}\int_{\{|u_n|\geq R\}} |T_n(u_n)|^{s(x)}\,dx \\ &\leq \frac{1}{2}R^{s^+}|\Omega| + \int_{\Omega} |T_n(u_n)|^{s(x)}\left(1 - \frac{1}{(1+|u_n|)^{\theta-1}}\right)dx. \end{aligned}$$

Using $(5.2.21)$, we deduce that

$$\begin{aligned} &\alpha(\theta-1)\sum_{i=1}^{N}\int_{\Omega}\frac{|D^i u_n|^{p_i(x)}}{(1+|u_n|)^{\theta}}\,dx + \frac{1}{2}\int_{\Omega}|T_n(u_n)|^{s(x)}\,dx \\ &\leq \frac{1}{2}R^{s^+}|\Omega| + \lambda\int_{\Omega}\frac{|T_n(u_n)|^{p_0(x)-1}}{|x|^{p_0(x)}}\,dx + \int_{\Omega}|f|\,dx. \end{aligned} \tag{5.2.22}$$

We have $s(x) > p_0(x) - 1$, in view of Young's inequality we obtain

$$\lambda\int_{\Omega}\frac{|T_n(u_n)|^{p_0(x)-1}}{|x|^{p_0(x)}}\,dx \leq \frac{1}{4}\int_{\Omega}|T_n(u_n)|^{s(x)}\,dx + C_2\int_{\Omega}\frac{dx}{|x|^{\frac{s(x)p_0(x)}{s(x)-p_0(x)+1}}},$$

with C_2 is a positive constant depending only on $s(\cdot)$, $p_0(\cdot)$ and λ. Thus, we obtain

$$\begin{aligned} &\alpha(\theta-1)\sum_{i=1}^{N}\int_{\Omega}\frac{|D^i u_n|^{p_i(x)}}{(1+|u_n|)^{\theta}}\,dx + \frac{1}{4}\int_{\Omega}|T_n(u_n)|^{s(x)}\,dx \\ &\leq \frac{1}{2}R^{s^+}|\Omega| + C_2\int_{\Omega}\frac{dx}{|x|^{\frac{s(x)p_0(x)}{s(x)-p_0(x)+1}}} + \int_{\Omega}|f|\,dx. \end{aligned} \tag{5.2.23}$$

Under the assumption $s(x) > \dfrac{N(p_0(x)-1)}{N-p_0(x)}$, the integral $\displaystyle\int_{\Omega}\frac{dx}{|x|^{\frac{s(x)p_0(x)}{s(x)-p_0(x)+1}}}$ is finite. Therefore $(5.2.19)$ is deduced. Moreover, we have

$$\int_{\Omega}|T_n(u_n)|^{s(x)}\,dx \leq C. \tag{5.2.24}$$

Taking $q_i(\cdot) \in C_+(\Omega)$ such that $1 \leq q_i(x) < p_i(x)$ for $i = 1, \ldots, N$. By virtue of the generalized

Hölder's inequality we get

$$\begin{aligned}\sum_{i=1}^{N}\int_{\Omega}|D^iu_n|^{q_i(x)}\,dx &\leq 2\sum_{i=1}^{N}\left\|\frac{|D^iu_n|^{q_i(x)}}{(1+|u_n|)^{\frac{\theta q_i(x)}{p_i(x)}}}\right\|_{\frac{p_i(\cdot)}{q_i(\cdot)}}\left\|(1+|u_n|)^{\frac{\theta q_i(x)}{p_i(x)}}\right\|_{\frac{p_i(\cdot)}{p_i(\cdot)-q_i(\cdot)}}\\ &\leq 2\sum_{i=1}^{N}\Big(\int_{\Omega}\frac{|D^iu_n|^{p_i(x)}}{(1+|u_n|)^{\theta}}\,dx+1\Big)^{\frac{q_i^+}{p_i^-}}\Big(\int_{\Omega}(1+|u_n|)^{\frac{q_i(x)\theta}{p_i(x)-q_i(x)}}\,dx+1\Big)^{1-\frac{q_i^-}{p_i^+}}.\end{aligned}\tag{5.2.25}$$

We now choose $\theta>1$ such that $\dfrac{q_i(x)\theta}{p_i(x)-q_i(x)}<s(x)$ a.e in Ω, such a real number θ exists if

$$1<\frac{s(x)(p_i(x)-q_i(x))}{q_i(x)}\qquad\text{that is}\qquad q_i(x)<\frac{p_i(x)s(x)}{s(x)+1}.$$

Combining $(5.2.23)-(5.2.25)$, we obtain the desired estimates $(5.2.18)$.

To get $(5.2.20)$, we have thanks to $(5.2.19)$ that

$$\sum_{i=1}^{N}\int_{\Omega}|D^iT_k(u_n)|^{p_i(x)}\,dx=\sum_{i=1}^{N}\int_{\{|u_n|<k\}}|D^iu_n|^{p_i(x)}\,dx\leq(1+k)^{\theta}\sum_{i=1}^{N}\int_{\Omega}\frac{|D^iu_n|^{p_i(x)}}{(1+|u_n|)^{\theta}}\,dx.$$

Step 3 : The weak convergence of $(T_k(u_n))_n$ in $W_0^{1,\vec{p}(\cdot)}(\Omega)$.

To show the weak convergence of $(T_k(u_n))_n$ in $W_0^{1,\vec{p}(\cdot)}(\Omega)$, we begin by proving that $(u_n)_n$ is a Cauchy sequence. Indeed, thanks to $(5.2.20)$, we can obtain

$$\sum_{i=0}^{N}\int_{\Omega}|D^iT_k(u_n)|^{p_i(x)}\,dx\leq C(1+k)^{\theta}+k^{p_0^+}|\Omega|\qquad\text{for}\quad k\geq1,$$

Therefore, the sequence $(T_k(u_n))_n$ is bounded in $W_0^{1,\vec{p}(\cdot)}(\Omega)$, and there exists a subsequence still denoted $(T_k(u_n))_n$ such that

$$\begin{cases}T_k(u_n)\rightharpoonup\eta_k & \text{in}\quad W_0^{1,\vec{p}(\cdot)}(\Omega),\\ T_k(u_n)\to\eta_k & \text{in}\quad L^{\underline{p}}(\Omega)\quad\text{and a.e. in}\quad\Omega.\end{cases}\tag{5.2.26}$$

On the other hand, we have

$$\begin{aligned}\sum_{i=1}^{N}\int_{\Omega}|D^iT_k(u_n)|^{p_i(x)}\,dx &\geq\sum_{i=1}^{N}\int_{\Omega}(|D^iT_k(u_n)|^{\underline{p}}-1)\,dx\\ &=\|\nabla T_k(u_n)\|_{\underline{p}}^{\underline{p}}-N|\Omega|,\end{aligned}$$

Thanks to $(5.2.20)$, we deduce that there exists a constant C_3 that does not depend on k and n, such that

$$\|\nabla T_k(u_n)\|_{\underline{p}}\leq C_3k^{\frac{\theta}{\underline{p}}}\qquad\text{for}\quad k\geq1.\tag{5.2.27}$$

Thanks to the Poincaré type inequality, we obtain

$$\begin{aligned} k \text{ meas}\{|u_n| > k\} &= \int_{\{|u_n|>k\}} |T_k(u_n)|\, dx \leq \int_\Omega |T_k(u_n)|\, dx \\ &\leq C_4 \|T_k(u_n)\|_{\underline{p}} \\ &\leq C_5 \|\nabla T_k(u_n)\|_{\underline{p}} \\ &\leq C_6 k^{\frac{\theta}{\underline{p}}}, \end{aligned} \tag{5.2.28}$$

Choosing θ small enough $(1 < \theta < \underline{p})$, we conclude that

$$\text{meas}\big\{|u_n| > k\big\} \leq C_6 \frac{1}{k^{1-\frac{\theta}{\underline{p}}}} \longrightarrow 0 \quad \text{as} \quad k \to +\infty. \tag{5.2.29}$$

For all $\delta > 0$, we have

$$\begin{aligned} \text{meas}\{|u_n - u_m| > \delta\} &\leq \text{meas}\{|u_n| > k\} + \text{meas}\{|u_m| > k\} \\ &\quad + \text{meas}\{|T_k(u_n) - T_k(u_m)| > \delta\}. \end{aligned}$$

Let $\varepsilon > 0$, using $(5.2.29)$ we can choose $k = k(\varepsilon)$ large enough such that

$$\text{meas}\{|u_n| > k\} \leq \frac{\varepsilon}{3} \qquad \text{and} \qquad \text{meas}\{|u_m| > k\} \leq \frac{\varepsilon}{3}. \tag{5.2.30}$$

On the other hand, thanks to $(5.2.26)$ we can assume that $(T_k(u_n))_{n\in\mathbb{N}}$ is a Cauchy sequence in measure. Thus, for any $k > 0$ and $\delta, \varepsilon > 0$, there exists $n_0 = n_0(k, \delta, \varepsilon)$ such that

$$\text{meas}\{|T_k(u_n) - T_k(u_m)| > \delta\} \leq \frac{\varepsilon}{3} \qquad \text{for all } m, n \geq n_0(k, \delta, \varepsilon). \tag{5.2.31}$$

In view of $(5.2.30)$ and $(5.2.31)$, we deduce that : for any $\delta, \varepsilon > 0$, there exists $n_0 = n_0(\delta, \varepsilon)$ such that

$$\text{meas}\{|u_n - u_m| > \delta\} \leq \varepsilon \quad \forall n, m \geq n_0(\delta, \varepsilon),$$

which proves that the sequence $(u_n)_n$ is a Cauchy sequence in measure and then converges almost everywhere to some measurable function u. Consequently, we have

$$T_k(u_n) \rightharpoonup T_k(u) \quad \text{in} \quad W_0^{1,\vec{p}(\cdot)}(\Omega), \tag{5.2.32}$$

and in view Lebesgue's dominated convergence theorem, we obtain

$$T_k(u_n) \longrightarrow T_k(u) \quad \text{in} \quad L^{p_0(\cdot)}(\Omega) \quad \text{and a.e in} \quad \Omega. \tag{5.2.33}$$

Step 4 : Strong convergence of truncations.

In the sequel, we denote by $\varepsilon_i(n)$, $i = 1, 2, \ldots,$ various real-valued functions of real variables that converge to 0 as n tends to infinity.

Let $h > k > 0$, taking $z_n := u_n - T_h(u_n) + T_k(u_n) - T_k(u)$ and $\omega_n := T_{2k}(z_n)$. By using ω_n as a test function in the approximate problem $(5.2.4)$ we obtain

$$\sum_{i=1}^{N} \int_\Omega a_i(x, T_n(u_n), \nabla u_n) D^i \omega_n \, dx + \int_\Omega |T_n(u_n)|^{s(x)-1} T_n(u_n) \omega_n \, dx$$
$$+ \frac{1}{n} \int_\Omega |u_n|^{p_0(x)-2} u_n \omega_n \, dx = \lambda \int_\Omega \frac{|T_n(u_n)|^{p_0(x)-2} T_n(u_n)}{|x|^{p_0(x)} + \frac{1}{n}} \omega_n \, dx + \int_\Omega f_n \omega_n \, dx.$$

For $M = 4k + h$, it's clear that $D^i \omega_n = 0$ on the set $\{|u_n| \geq M\}$, and ω_n have the same sign as u_n on the set $\{|u_n| > k\}$, therefore

$$\sum_{i=1}^{N} \int_{\{|u_n| \leq M\}} a_i(x, T_M(u_n), \nabla T_M(u_n)) D^i \omega_n \, dx + \int_\Omega |T_n(u_n)|^{s(x)-1} T_n(u_n) \omega_n \, dx$$
$$+ \frac{1}{n} \int_{\{|u_n| \leq k\}} |u_n|^{p_0(x)-2} u_n \omega_n \, dx \leq \lambda \int_\Omega \frac{|T_n(u_n)|^{p_0(x)-2} T_n(u_n)}{|x|^{p_0(x)} + \frac{1}{n}} \omega_n \, dx + \int_\Omega f_n \omega_n \, dx.$$

In view of Young's inequality, we have

$$\lambda \int_{\{|u_n| > k\}} \frac{|T_n(u_n)|^{p_0(x)-1}}{|x|^{p_0(x)} + \frac{1}{n}} |\omega_n| \, dx \leq \int_{\{|u_n| > k\}} |T_n(u_n)|^{s(x)} |\omega_n| \, dx + C_7 \int_{\{|u_n| > k\}} \frac{|\omega_n|}{|x|^{\frac{p_0(x)s(x)}{s(x)-p_0(x)+1}}} \, dx,$$

and since $\omega_n = T_k(u_n) - T_k(u)$ on the set $\{|u_n| \leq k\}$, then

$$\begin{aligned} &\sum_{i=1}^{N} \int_{\{|u_n| \leq M\}} a_i(x, T_M(u_n), \nabla T_M(u_n)) D^i \omega_n \, dx \\ &\quad + \int_{\{|u_n| \leq k\}} |T_k(u_n)|^{s(x)-1} T_k(u_n)(T_k(u_n) - T_k(u)) \, dx \\ &\quad + \frac{1}{n} \int_{\{|u_n| \leq k\}} |T_k(u_n)|^{p_0(x)-2} T_k(u_n)(T_k(u_n) - T_k(u)) \, dx \\ &\leq \lambda \int_{\{|u_n| \leq k\}} \frac{|T_k(u_n)|^{p_0(x)-2} T_k(u_n)}{|x|^{p_0(x)} + \frac{1}{n}} (T_k(u_n) - T_k(u)) \, dx \\ &\quad + \int_\Omega f_n \omega_n \, dx + C_7 \int_{\{|u_n| > k\}} \frac{|\omega_n|}{|x|^{\frac{p_0(x)s(x)}{s(x)-p_0(x)+1}}} \, dx. \end{aligned} \tag{5.2.34}$$

Now, we will study each terms in the previews inequality.

For the second and third terms on the left-hand side of $(5.2.34)$. In view of Lebesgue's dominated convergence theorem, we have

$$|T_k(u_n)|^{s(x)-1} T_k(u_n) \longrightarrow |T_k(u)|^{s(x)-1} T_k(u) \qquad \text{in} \quad L^1(\Omega),$$

and

$$|T_k(u_n)|^{p_0(x)-2}T_k(u_n) \longrightarrow |T_k(u)|^{p_0(x)-2}T_k(u) \qquad \text{in} \quad L^1(\Omega),$$

and since $T_k(u_n) \rightharpoonup T_k(u)$ weak$-*$ in $L^\infty(\Omega)$, then

$$\varepsilon_1(n) = \int_{\{|u_n|\le k\}} |T_k(u_n)|^{s(x)-1}T_k(u_n)(T_k(u_n) - T_k(u))\,dx \longrightarrow 0 \qquad \text{as} \quad n \to \infty, \tag{5.2.35}$$

and

$$\varepsilon_2(n) = \frac{1}{n}\int_{\{|u_n|\le k\}} |T_k(u_n)|^{p_0(x)-2}T_k(u_n)(T_k(u_n) - T_k(u))\,dx \longrightarrow 0 \qquad \text{as} \quad n \to \infty. \tag{5.2.36}$$

Concerning the terms on the right-hand side of $(5.2.34)$, we have

$$\begin{aligned} \varepsilon_3(n) &= \left| \int_{\{|u_n|\le k\}} \frac{|T_k(u_n)|^{p_0(x)-2}T_k(u_n)}{|x|^{p_0(x)} + \frac{1}{n}} (T_k(u_n) - T_k(u))\,dx \right| \\ &\le k^{p_0^+-1} \int_{\{|u_n|\le k\}} \frac{|T_k(u_n) - T_k(u)|}{|x|^{p_0(x)}}\,dx \longrightarrow 0 \qquad \text{as} \quad n \to \infty, \end{aligned} \tag{5.2.37}$$

also, we have

$$\int_\Omega f_n\omega_n\,dx = \int_\Omega f\ T_{2k}(u - T_h(u))\,dx + \varepsilon_4(n), \tag{5.2.38}$$

and

$$\int_{\{|u_n|>k\}} \frac{|\omega_n|}{|x|^{\frac{p_0(x)s(x)}{s(x)-p_0(x)+1}}}\,dx = \int_{\{|u|>h\}} \frac{|T_{2k}(u - T_h(u))|}{|x|^{\frac{p_0(x)s(x)}{s(x)-p_0(x)+1}}}\,dx + \varepsilon_5(n). \tag{5.2.39}$$

By combining $(5.2.34) - (5.2.39)$, we deduce that

$$\begin{aligned} &\sum_{i=1}^N \int_{\{|u_n|\le M\}} a_i(x, T_M(u_n), \nabla T_M(u_n))D^i\omega_n\,dx \\ &\le \int_\Omega f\ T_{2k}(u - T_h(u))\,dx + C_7 \int_{\{|u|>h\}} \frac{|T_{2k}(u - T_h(u))|}{|x|^{\frac{p_0(x)s(x)}{s(x)-p_0(x)+1}}}\,dx + \varepsilon_6(n). \end{aligned} \tag{5.2.40}$$

On the other hand, we have $\omega_n = T_k(u_n) - T_k(u)$ on $\{|u_n| \le M\}$, then

$$\begin{aligned} &\sum_{i=1}^N \int_{\{|u_n|\le M\}} a_i(x, T_M(u_n), \nabla T_M(u_n))D^i\omega_n\,dx \\ &= \sum_{i=1}^N \int_\Omega \big(a_i(x, T_k(u_n), \nabla T_k(u_n)) - a_i(x, T_k(u_n), \nabla T_k(u))\big)(D^iT_k(u_n) - D^iT_k(u))\,dx \\ &\quad + \sum_{i=1}^N \int_\Omega a_i(x, T_k(u_n), \nabla T_k(u))(D^iT_k(u_n) - D^iT_k(u))\,dx \\ &\quad + \sum_{i=1}^N \int_{\{|u_n|>k\}} a_i(x, T_k(u_n), \nabla T_k(u_n))D^iT_k(u)\,dx \\ &\quad + \sum_{i=1}^N \int_{\{k<|u_n|\le M\}} a_i(x, T_M(u_n), \nabla T_M(u_n))D^i\omega_n\,dx. \end{aligned} \tag{5.2.41}$$

For the second and third terms on the right-hand side of $(5.2.41)$. Thanks to Lebesgue's dominated convergence theorem, we have $T_k(u_n) \to T_k(u)$ in $L^{p_i(\cdot)}(\Omega)$ then $a_i(x, T_k(u_n), \nabla T_k(u)) \to a_i(x, T_k(u), \nabla T_k(u))$ in $L^{p_i'(\cdot)}(\Omega)$, and since $D^i T_k(u_n) \rightharpoonup D^i T_k(u)$ in $L^{p_i(\cdot)}(\Omega)$ it follows that

$$\varepsilon_7(n) = \int_\Omega a_i(x, T_k(u_n), \nabla T_k(u))(D^i T_k(u_n) - D^i T_k(u))\, dx \longrightarrow 0 \qquad \text{as} \quad n \to \infty, \tag{5.2.42}$$

and since $a(x, s, 0) = 0$, then

$$\int_{\{|u_n|>k\}} a_i(x, T_k(u_n), \nabla T_k(u_n)) D^i T_k(u)\, dx = \int_{\{|u_n|>k\}} a_i(x, T_k(u_n), 0) D^i T_k(u)\, dx = 0. \tag{5.2.43}$$

Concerning the last term on the right-hand side of $(5.2.41)$, thanks to $(5.1.12)$ we have $(a_i(x, T_M(u_n), \nabla T_M(u_n)))_n$ is bounded in $L^{p_i'(\cdot)}(\Omega)$, then there exists a function $\phi_i \in L^{p_i'(\cdot)}(\Omega)$ such that $|a_i(x, T_M(u_n), \nabla T_M(u_n))| \to \phi_i$ in $L^{p_i'(\cdot)}(\Omega)$, it follows that

$$\begin{aligned}
&\lim_{n\to\infty} \int_{\{k<|u_n|\le M\}} a_i(x, T_M(u_n), \nabla T_M(u_n)) D^i \omega_n\, dx \\
&= \lim_{n\to\infty} \int_{\{k<|u_n|\le M\}\cap\{|z_n|\le 2k\}} a_i(x, T_M(u_n), \nabla T_M(u_n))(D^i u_n - D^i T_h(u_n) - D^i T_k(u))\, dx \\
&\ge - \lim_{n\to\infty} \int_{\{k<|u_n|\le M\}} |a_i(x, T_M(u_n), \nabla T_M(u_n))|\, |D^i T_k(u)|\, dx \\
&\ge - \int_{\{k<|u|\le M\}} \phi_i\, |D^i T_k(u)|\, dx = 0.
\end{aligned} \tag{5.2.44}$$

By combining $(5.2.40)$ and $(5.2.41) - (5.2.44)$, we get

$$\begin{aligned}
&\sum_{i=1}^{N} \int_\Omega \big(a_i(x, T_k(u_n), \nabla T_k(u_n)) - a_i(x, T_k(u_n), \nabla T_k(u))\big)(D^i T_k(u_n) - D^i T_k(u))\, dx \\
&\le 2k \int_{\{|u|>h\}} |f|\, dx + 2kC_7 \int_{\{|u|>h\}} \frac{1}{|x|^{\frac{p_0(x)s(x)}{s(x)-p_0(x)+1}}}\, dx + \varepsilon_8(n).
\end{aligned} \tag{5.2.45}$$

Since $\dfrac{N(p_0(x)-1)}{N-p_0(x)} < s(x)$, we have $\dfrac{p_0(x)s(x)}{s(x)-p_0(x)+1} < N$ then $\dfrac{1}{|x|^{\frac{p_0(x)s(x)}{s(x)-p_0(x)+1}}} \in L^1(\Omega)$.
By letting n then h tend to infinity in the inequality above, and thanks to $(5.2.33)$ we can obtain

$$\begin{aligned}
\lim_{n\to\infty} \Big(&\sum_{i=1}^{N} \int_\Omega \big(a_i(x, T_k(u_n), \nabla T_k(u_n)) - a_i(x, T_k(u_n), \nabla T_k(u))\big)(D^i T_k(u_n) - D^i T_k(u))\, dx \\
&+ \int_\Omega \big(|T_k(u_n)|^{p_0(x)-2} T_k(u_n) - |T_k(u_n)|^{p_0(x)-2} T_k(u_n)\big)(T_k(u_n) - T_k(u))\, dx\Big) = 0.
\end{aligned} \tag{5.2.46}$$

In view of Lemma 1.2.8, we conclude that

$$\begin{cases} T_k(u_n) \longrightarrow T_k(u) & \text{strongly in} \quad W_0^{1,\vec{p}(\cdot)}(\Omega), \\ D^i u_n \longrightarrow D^i u & \text{a.e. in} \quad \Omega \qquad \text{for } i = 1, \dots, N. \end{cases} \tag{5.2.47}$$

Step 5 : The equi-integrability of the nonlinear functions.

Now, we shall show that

$$|T_n(u_n)|^{s(x)-1}T_n(u_n) \to |u|^{s(x)-1}u \quad \text{in} \quad L^1(\Omega), \tag{5.2.48}$$

$$\frac{1}{n}|u_n|^{p_0(x)-2}u_n \to 0 \quad \text{in} \quad L^1(\Omega), \tag{5.2.49}$$

and

$$\frac{|T_n(u_n)|^{p_0(x)-2}T_n(u_n)}{|x|^{p_0(x)}+\frac{1}{n}} \longrightarrow \frac{|u|^{p_0(x)-2}u}{|x|^{p_0(x)}} \quad \text{in} \quad L^1(\Omega). \tag{5.2.50}$$

In view of Vitali's theorem, it suffices to prove the uniform equi-integrability of these functions.

By taking $T_1(u_n - T_h(u_n))$ as a test function in $(5.2.4)$, we can obtain

$$\begin{aligned}&\alpha\sum_{i=1}^{N}\int_{\{h<|u_n|\le h+1\}}|D^i u_n|^{p_i}\,dx+\int_{\{|u_n|\ge h\}}|T_n(u_n)|^{s(x)}|T_1(u_n-T_h(u_n))|\,dx\\&+\frac{1}{n}\int_{\{|u_n|\ge h+1\}}|u_n|^{p_0(x)-1}\,dx\le\lambda\int_{\{|u_n|\ge h\}}\frac{|T_n(u_n)|^{p_0(x)-1}}{|x|^{p_0(x)}+\frac{1}{n}}|T_1(u_n-T_h(u_n))|\,dx+\int_{\{|u_n|\ge h\}}|f_n|\,dx.\end{aligned}$$

Thanks to Young's inequality, we have

$$\begin{aligned}&\lambda\int_{\{|u_n|\ge h\}}\frac{|T_n(u_n)|^{p_0(x)-1}}{|x|^{p_0(x)}+\frac{1}{n}}|T_1(u_n-T_h(u_n))|\,dx\\&\le\frac{1}{3}\int_{\{|u_n|\ge h\}}|T_n(u_n)|^{s(x)}|T_1(u_n-T_h(u_n))|\,dx+C_8\int_{\{|u_n|\ge h\}}\frac{|T_1(u_n-T_h(u_n))|}{|x|^{\frac{s(x)p_0(x)}{s(x)-p_0(x)+1}}}\,dx,\end{aligned}$$

it follows that

$$\begin{aligned}&\frac{1}{3}\int_{\{|u_n|\ge h+1\}}|T_n(u_n)|^{s(x)}\,dx+\lambda\int_{\{|u_n|\ge h+1\}}\frac{|T_n(u_n)|^{p_0(x)-1}}{|x|^{p_0(x)}+\frac{1}{n}}\,dx+\frac{1}{n}\int_{\{|u_n|\ge h+1\}}|u_n|^{p_0(x)-1}\,dx\\&\le 2C_8\int_{\{|u_n|\ge h\}}\frac{|T_1(u_n-T_h(u_n))|}{|x|^{\frac{s(x)p_0(x)}{s(x)-p_0(x)+1}}}\,dx+\int_{\{|u_n|\ge h\}}|f_n|\,dx.\end{aligned}$$

Thus, for any $\eta>0$, there exists $h(\eta)>0$ such that

$$\int_{\{|u_n|\ge h(\eta)\}}|T_n(u_n)|^{s(x)}\,dx+\int_{\{|u_n|\ge h(\eta)\}}\frac{|T_n(u_n)|^{p_0(x)-1}}{|x|^{p_0(x)}+\frac{1}{n}}\,dx+\frac{1}{n}\int_{\{|u_n|\ge h(\eta)\}}|u_n|^{p_0(x)-1}\,dx\le\frac{\eta}{2}. \tag{5.2.51}$$

On the other hand, for any measurable subset $E\subseteq\Omega$, we have

$$\begin{aligned}&\int_E|T_n(u_n)|^{s(x)}\,dx+\int_E\frac{|T_n(u_n)|^{p_0(x)-1}}{|x|^{p_0(x)}+\frac{1}{n}}\,dx+\frac{1}{n}\int_E|u_n|^{p_0(x)-1}\,dx\\&\le\int_E|T_{h(\eta)}(u_n)|^{s(x)}\,dx+\int_E\frac{|T_{h(\eta)}(u_n)|^{p_0(x)-1}}{|x|^{p_0(x)}+\frac{1}{n}}\,dx+\frac{1}{n}\int_E|T_{h(\eta)}(u_n)|^{p_0(x)-1}\,dx\\&\quad+\int_{\{|u_n|\ge h(\eta)\}}|T_n(u_n)|^{s(x)}\,dx+\int_{\{|u_n|\ge h(\eta)\}}\frac{|T_n(u_n)|^{p_0(x)-1}}{|x|^{p_0(x)}+\frac{1}{n}}\,dx+\frac{1}{n}\int_{\{|u_n|\ge h(\eta)\}}|u_n|^{p_0(x)-1}\,dx.\end{aligned} \tag{5.2.52}$$

From $(5.2.47)$, there exists $\beta(\eta) > 0$ such that : for any $E \subseteq \Omega$ with meas$(E) \leq \beta(\eta)$

$$\int_E |T_{h(\eta)}(u_n)|^{s(x)}\,dx + \int_E \frac{|T_{h(\eta)}(u_n)|^{p_0(x)-1}}{|x|^{p_0(x)}+\frac{1}{n}}\,dx + \frac{1}{n}\int_E |T_{h(\eta)}(u_n)|^{p_0(x)-1}\,dx \leq \frac{\eta}{2}. \tag{5.2.53}$$

Finally, by combining $(5.2.51)$, $(5.2.52)$ and $(5.2.53)$, one easily has

$$\int_E |T_n(u_n)|^{s(x)}\,dx + \int_E \frac{|T_n(u_n)|^{p_0(x)-1}}{|x|^{p_0(x)}+\frac{1}{n}}\,dx + \frac{1}{n}\int_E |u_n|^{p_0(x)-1}\,dx \leq \eta \quad \text{with meas}(E) \leq \beta(\eta), \tag{5.2.54}$$

We deduce that $(|T_n(u_n)|^{s(x)-1}T_n(u_n))_n$, $(|u_n|^{p_0(x)-2}u_n)_n$ and $\Big(\frac{|T_n(u_n)|^{p_0(x)-2}T_n(u_n)}{|x|^{p_0(x)}+\frac{1}{n}}\Big)_n$ are equi-integrable, and since

$$|T_n(u_n)|^{s(x)-1}T_n(u_n) \to |u|^{s(x)-1}u \quad \text{a.e. in} \quad \Omega,$$
$$\frac{1}{n}|u_n|^{p_0(x)-2}u_n \to 0 \quad \text{a.e. in} \quad \Omega$$

and

$$\frac{|T_n(u_n)|^{p_0(x)-2}T_n(u_n)}{|x|^{p_0(x)}+\frac{1}{n}} \longrightarrow \frac{|u|^{p_0(x)-2}u}{|x|^{p_0(x)}} \quad \text{a.e. in} \quad \Omega.$$

In view of Vitali's theorem, the convergence $(5.2.48) - (5.2.50)$ are concluded.

Step 6 : Passage to the limit.

Let $\varphi \in W_0^{1,\vec{p}(\cdot)}(\Omega) \cap L^\infty(\Omega)$ and $M = k + \|\varphi\|_\infty$. By taking $T_k(u_n - \varphi)$ as a test function in $(5.2.4)$, we get

$$\begin{aligned}&\sum_{i=1}^N \int_\Omega a_i(x, T_n(u_n), \nabla u_n) D^i T_k(u_n - \varphi)\,dx + \int_\Omega |T_n(u_n)|^{s(x)-1}T_n(u_n)T_k(u_n-\varphi)\,dx \\ &+\frac{1}{n}\int_\Omega |u_n|^{p_0(x)-2}u_n T_k(u_n-\varphi)\,dx = \lambda\int_\Omega \frac{|T_n(u_n)|^{p_0(x)-2}T_n(u_n)}{|x|^{p_0(x)}+\frac{1}{n}}T_k(u_n-\varphi)\,dx + \int_\Omega f_n T_k(u_n-\varphi)\,dx.\end{aligned} \tag{5.2.55}$$

On the one hand, we have $\{|u_n - \varphi| \leq k\} \subseteq \{|u_n| \leq M\}$ then

$$\begin{aligned}&\int_\Omega a_i(x, T_n(u_n), \nabla u_n) D^i T_k(u_n-\varphi)\,dx \\ &= \int_\Omega a_i(x, T_M(u_n), \nabla T_M(u_n))(D^i T_M(u_n) - D^i\varphi)\chi_{\{|u_n-\varphi|\leq k\}}\,dx \\ &= \int_\Omega (a_i(x, T_M(u_n), \nabla T_M(u_n)) - a_i(x, T_M(u_n), \nabla\varphi))(D^i T_M(u_n) - D^i\varphi)\chi_{\{|u_n-\varphi|\leq k\}}\,dx \\ &\quad + \int_\Omega a_i(x, T_M(u_n), \nabla\varphi)(D^i T_M(u_n) - D^i\varphi)\chi_{\{|u_n-\varphi|\leq k\}}\,dx.\end{aligned}$$

It's clear that

$$\begin{aligned}\lim_{n\to\infty}\int_\Omega a_i(x, T_M(u_n), \nabla\varphi)(D^i T_M(u_n) - D^i\varphi)\chi_{\{|u_n-\varphi|\leq k\}}\,dx \\ = \int_\Omega a_i(x, T_M(u), \nabla\varphi)(D^i T_M(u) - D^i\varphi)\chi_{\{|u-\varphi|\leq k\}}\,dx.\end{aligned}$$

According to Fatou's Lemma, we obtain

$$\begin{aligned}
&\liminf_{n\to\infty}\sum_{i=1}^{N}\int_{\Omega}a_i(x,T_n(u_n),\nabla u_n)D^iT_k(u_n-\varphi)\,dx\\
&\geq\sum_{i=1}^{N}\int_{\Omega}(a_i(x,T_M(u),\nabla T_M(u))-a_i(x,T_M(u),\nabla\varphi))(D^iT_M(u)-D^i\varphi)\chi_{\{|u-\varphi|\leq k\}}\,dx\\
&\quad+\sum_{i=1}^{N}\int_{\Omega}a_i(x,T_M(u),\nabla\varphi)(D^iT_M(u)-D^i\varphi)\chi_{\{|u-\varphi|\leq k\}}\,dx \qquad (5.2.56)\\
&=\sum_{i=1}^{N}\int_{\Omega}a_i(x,T_M(u),\nabla T_M(u))(D^iT_M(u)-D^i\varphi)\chi_{\{|u-\varphi|\leq k\}}\,dx\\
&=\sum_{i=1}^{N}\int_{\Omega}a_i(x,u,\nabla u)D^iT_k(u-\varphi)\,dx.
\end{aligned}$$

On the other hand, we have $T_k(u_n-\varphi)\rightharpoonup T_k(u-\varphi)$ weak$-*$ in $L^\infty(\Omega)$ and thanks to $(5.2.48)-(5.2.50)$, we deduce that

$$\int_{\Omega}|T_n(u_n)|^{s(x)-1}T_n(u_n)T_k(u_n-\varphi)\,dx\longrightarrow\int_{\Omega}|u|^{s(x)-1}uT_k(u-\varphi)\,dx, \qquad (5.2.57)$$

$$\frac{1}{n}\int_{\Omega}|u_n|^{p_0(x)-1}u_nT_k(u_n-\varphi)\,dx\longrightarrow 0, \qquad (5.2.58)$$

$$\int_{\Omega}\frac{|T_n(u_n)|^{p_0(x)-2}T_n(u_n)}{|x|^{p_0(x)}+\frac{1}{n}}T_k(u_n-\varphi)\,dx\longrightarrow\int_{\Omega}\frac{|u|^{p_0(x)-2}u}{|x|^{p_0(x)}}T_k(u-\varphi)\,dx, \qquad (5.2.59)$$

and

$$\int_{\Omega}f_nT_k(u_n-\varphi)dx\longrightarrow\int_{\Omega}fT_k(u-\varphi)dx. \qquad (5.2.60)$$

Hence, putting all the terms together, we conclude the proof of Theorem 5.2.1.

Chapter 6

Existence of entropy solutions for strongly nonlinear anisotropic elliptic problem involving lower order terms and Hardy potential

In this chapter, we give an existence result of entropy solutions for the following strongly nonlinear anisotropic elliptic Dirichlet problem

$$\begin{cases} -\sum_{i=1}^{N} D^i(|D^i u|^{p_i-2}D^i u) + h(x,u,\nabla u) + |u|^{p_0-2}u + |u|^{s-1}u = f + \mu\frac{|u|^{p_0-2}u}{|x|^{p_0}} - \text{div } \phi(u) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases}$$

where Ω is an open bounded subset of $\mathbf{R}^N$ containing the origin, $\phi \in \mathcal{C}^0(I\!R, I\!R^N)$. We assume that the datum f belongs to $L^1(\Omega)$, μ is a positive constant and $h(x,u,\nabla u)$ is a nonlinear lower order term with natural growth with respect to $|\nabla u|$, satisfying the sign condition.

6.1 Introduction

In the last decades one of the topics from the field of calculus of variations and partial differential equations that gained interest is the study of anisotropic problems, as witnessed by a number of researches that have introduced anisotropic Sobolev spaces which are the appropriate framework to deal with a class of problems with non-standard structural conditions, involving a growth exponent $\vec{p}$, where prototype of

the differential operator considered is the $\vec{p}$-laplacien

$$\Delta_{\vec{p}}(u) = \sum_{i=1}^{i=N} \partial_{x_i}(|\partial_{x_i} u|^{p_i-2}\partial_{x_i} u),$$

which generalize the p-laplace operator.

Let Ω be a bounded open subset of $\mathbf{R}^N$ $(N \geq 2)$, containing the origin with boundary Ω and let $p_0, p_1, \ldots, p_N$ be $N+1$ exponents, with $1 < p_i < \infty$ for $i = 0, \ldots, N$. Our aim is to prove the existence of entropy solutions for the following anisotropic quasilinear elliptic problem

$$\begin{cases} -\sum_{i=1}^{N} D^i(|D^i u|^{p_i-2}D^i u) + h(x,u,\nabla u) + |u|^{p_0-2}u + |u|^{s-1}u = f + \mu\dfrac{|u|^{p_0-2}u}{|x|^{p_0}} - \text{div } \phi(u) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{6.1.1}$$

with $\mu \geq 0$, $f \in L^1(\Omega)$, $\phi \in \mathcal{C}^0(I\!R, I\!R^N)$ and

$$s > \max\Big(\frac{N(p_0-1)}{N-p_0}, \frac{1}{p_0-1}\Big). \tag{6.1.2}$$

The nonlinear lower order term $h(x,s,\xi) : \Omega \times I\!R \times I\!R^N \longmapsto I\!R^N$ is Carathéodory functions, (measurable with respect to x in Ω for every (s,ξ) in $I\!R \times I\!R^N$ and continuous with respect to (s,ξ) in $I\!R \times I\!R^N$ for almost every x in Ω), which satisfies the following conditions

$$h(x,s,\xi)s \geq 0, \tag{6.1.3}$$

$$|h(x,s,\xi)| \leq l(x) + j(|s|)\sum_{i=1}^{N} |\xi_i|^{p_i}, \tag{6.1.4}$$

where $j : I\!R^+ \to I\!R^+$ is a continuous nondecreasing function and $l \in L^1(\Omega)$. The notion of anisotropic Sobolev spaces were introduced and studied by Nikolskiii [132], and Troisi [149], and later by Trudinger [150] in the framework of Orlicz spaces. This rise of interest for the study of such spaces was motivated by their physical applications in the thermistor problem, flow of electroreological fluids and processes of image restoration (see for example [142], [73] and [16, 124]). It is important to point out the classic result of Boccardo et al. [56] in which they have studied the anisotropic equations with right hand side measures

$$\begin{cases} -\text{div } (j(Du)) = f(x) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{6.1.5}$$

where $j(\xi)$ is the vector whose components are $|\xi_i|^{p_i-2}\xi_i$ $(i = 1, \ldots, N, p_i > 1)$, and also when f is a measurable function such that $\int_\Omega |f| \log(1+|f|) < \infty$. Antonsev et al. have studied the uniqueness of weak solutions for elliptic equations of the following type

$$-\partial_{x_i}(a_i(x,u)|\partial_{x_i} u|^{p_i-2}\partial_{x_i} u) + b(x,u) = f(x)$$

in a bounded domain $\Omega \subset I\!R^N$ with Lipschitz continuous boundary $\Gamma = \partial\Omega$ and particular mixed boundary conditions and they have established a similar result for the parabolic case. Anisotropic elliptic equations have been considered under many other différent aspects, for instance with respect to the maximum principle and to the multiplicity of solutions; see e.g. P. Pucci, V. Radulesco et al. [103] and [128], while the authors in [44] and [74] proved the existence of solutions of some anisotropic elliptic equations for a general class of operators of higher order.For more details concerning the anisotropic problems we refer to [15, 11, 78, 86, 87, 151, 152, 129] and the references therein. It would be interesting to refer to some Embedding theorems for anisotropic Sobolev-Orlicz spaces [115] and a fully anisotropic Sobolev Inequality established by Cianchi in [77]. we can also refer the reader to [108] for some basics properties of anisotropic Orlicz-Musielak spaces, moreover Gwiazda et al. in [110] dealt with anisotropic parabolic problems where the N-function was assumed to be homogeneous in space. we mention also that the author in [148] had treated anisotropic behaviour in a parabolic problem in a framework of maximal monotone graphs, possibly multi-valued with growth conditions formulated with help of an x-dependent N-function. For the isotropic case, i.e $p_i = p$ attention has been focused on elliptic problems with singularity on its right-hand side particularly the so- called Hardy potential and its effect that give rise to the existence (or nonexistence) of solutions. Abdellaoui and Peral have treated the optimal power in order to find a solution the following equation

$$-\triangle u = \lambda \frac{u}{|x|^2} + |\triangle u|^p + cf(x), \tag{6.1.6}$$

where Ω is domain containing 0. they assumed that λ and c are positive real numbers and f is nonnegative function under some extra hypotheses. In [126], Mercaldo et al. were interested on existence and nonexistence for positive solutions to the degenerated nonlinear elliptic equations

$$\begin{cases} -\text{div}\,(A(x,u,\nabla u)) = \lambda \dfrac{u^s}{|x|^p} + f(x) & \text{in } \Omega, \\ u(x) \geq 0 & \text{on } \Omega, \\ u(x) = 0 & \text{on } \partial\Omega, \end{cases} \tag{6.1.7}$$

with Ω be an open bounded subset of $I\!R^N$ $(N \geq 3)$, $1 < p < N$, λ and s are positives numbers, f is nonnegative function in some Lebesgue space, and $A : \Omega \times I\!R \times I\!R^N \longmapsto I\!R^n$ is such that

$$\frac{c_0}{(a(x)+|t|)^\theta}|\xi|^p \leq \langle A(x,t,\xi),\xi\rangle \qquad \text{for some} \qquad 0 < \theta < 1$$

which provide a non coercive operator when $u \to \infty$. To investigate other problems of this kind we refer the reader [2, 138, 106, 107]. It is meritorious mentioning that this type of problems appear in several contexts. The problem $(6.1.7)$ could be seen as a reaction model which produces a saturation effect in

some solid combustion problems, while $(6.1.6)$ is the stationary counterpart of some flame propagation models.

It should be pointed out that it was used for the resolution of this problem the notion of entropy solutions which was introduced by Bénilan et al. in [46], for the reason that the data f belong to $L^1(\Omega)$. Motivated by the papers [138], [2] and [155], we try to deal with strongly nonlinear anisotropic elliptic Dirichlet problem by using the Galerkin method, and to remove the non-existence effect produced by the singular term $\frac{|u|^{p_0-2}u}{|x|^{p_0}}$ by exploiting the regularizing effect of the term $|u|^{s-1}u$. On the other hand the function $\phi \in \mathcal{C}^0(I\!R, I\!R^N)$ then ϕ does not belongs to $(L^1_{loc}(\Omega))^N$, so that proving existence of a weak solution seems to be an arduous task. To overpass this difficulty we will use some techniques in the framework of entropy solutions.

The remaining part of this chapter is organized as follows: Section 2 is devoted to introduce some preliminary results including a brief discussion on the anisotropic Sobolev spaces. In section 3, we recall some technical lemmas and we state and prove our main existence results.

6.2 Preliminary

Let Ω be a bounded open subset of $I\!R^N$ $(N \geq 2)$,

Let $p_0, p_1, \ldots, p_N$ be $N+1$ exponents, with $1 < p_i < \infty$ for $i = 0, \ldots, N$. We denote

$$\vec{p} = (p_0, \ldots, p_N), \quad D^0 u = u \quad \text{and} \quad D^i u = \frac{\partial u}{\partial x_i} \qquad \text{for } i = 1, \ldots, N,$$

and we define

$$\underline{p} = \min\{p_0, p_1, \ldots, p_N\} \qquad \text{then} \qquad \underline{p} > 1. \tag{6.2.1}$$

The anisotropic Sobolev space $W^{1,\vec{p}(\cdot)}(\Omega)$ is defined as follow

$$W^{1,\vec{p}}(\Omega) = \{u \in L^{p_0}(\Omega) \quad \text{and} \quad D^i u \in L^{p_i}(\Omega) \quad \text{for} \quad i = 1, 2, \ldots, N\},$$

endowed with the norm

$$\|u\|_{1,\vec{p}} = \sum_{i=0}^{N} \|D^i u\|_{p_i}. \tag{6.2.2}$$

We define also $W_0^{1,\vec{p}}(\Omega)$ as the closure of $\mathcal{C}_0^\infty(\Omega)$ in $W^{1,\vec{p}}(\Omega)$ with respect to the norm $(6.2.2)$. The space $\left(W_0^{1,\vec{p}}(\Omega), \|u\|_{1,\vec{p}}\right)$ is a reflexive Banach space (cf. [127]).

Lemma 6.2.1. *We have the following continuous and compact embedding*

- *if* $\underline{p} < N$ *then* $W_0^{1,\vec{p}}(\Omega) \hookrightarrow\hookrightarrow L^q(\Omega) \quad$ *for* $\quad q \in [\underline{p}, \underline{p}^*[, \quad$ *where* $\underline{p}^* = \dfrac{N\underline{p}}{N-\underline{p}}$,

- *if* $\underline{p} = N$ *then* $W_0^{1,\vec{p}}(\Omega) \hookrightarrow\hookrightarrow L^q(\Omega) \qquad \forall q \in [\underline{p}, +\infty[$,

- *if* $\underline{p} > N$ *then* $W_0^{1,\vec{p}}(\Omega) \hookrightarrow\hookrightarrow L^\infty(\Omega) \cap \mathcal{C}^0(\overline{\Omega})$.

The proof of this lemma follows from the fact that the embedding $W_0^{1,\vec{p}}(\Omega) \hookrightarrow W_0^{1,\underline{p}}(\Omega)$ is continuous, and in view of the compact embedding theorem for Sobolev spaces.

Proposition 6.2.1. *The dual of* $W_0^{1,\vec{p}}(\Omega)$ *is denote by* $W^{-1,\vec{p'}}(\Omega)$, *where* $\vec{p'} = (p_0', \ldots, p_N')$ *and* $\frac{1}{p_i'} + \frac{1}{p_i} = 1$, *(cf. [36] for the constant exponent case).*
For each $F \in W^{-1,\vec{p'}}(\Omega)$ *there exists* $F_i \in L^{p_i'}(\Omega)$ *for* $i = 0, 1, \ldots, N$, *such that* $F = F_0 - \displaystyle\sum_{i=1}^{N} D^i F_i$.
Moreover for any $u \in W_0^{1,\vec{p}}(\Omega)$, *we have*

$$\langle F, u\rangle = \sum_{i=0}^{N} \int_\Omega F_i \, D^i u \, dx.$$

We define a norm on the dual space by

$$\|F\|_{-1,\vec{p'}} = \inf\Big\{ \sum_{i=0}^{N} \|F_i\|_{p_i'} \quad / \quad F = F_0 - \sum_{i=1}^{N} D^i F_i \quad \textit{with} \quad F_i \in L^{p_i'}(\Omega)\Big\}.$$

We set

$$\mathcal{T}_0^{1,\vec{p}}(\Omega) := \{u : \Omega \mapsto I\!R \text{ measurable, such that } T_k(u) \in W_0^{1,\vec{p}}(\Omega) \text{ for any } k > 0\}.$$

Note that, a measurable function u verifying $T_k(u) \in W_0^{1,\vec{p}}(\Omega)$ for all $k > 0$, does not necessarily belong to $W_0^{1,1}(\Omega)$. However, for any $u \in \mathcal{T}_0^{1,\vec{p}}(\Omega)$ it is possible to define the weak gradient of u, still denoted ∇u.

Proposition 6.2.2. *Let* $u \in \mathcal{T}_0^{1,\vec{p}}(\Omega)$. *For any* $i \in \{1, \ldots, N\}$, *there exists a unique measurable function* $v_i : \Omega \mapsto I\!R$ *such that*

$$\forall k > 0 \qquad D^i T_k(u) = v_i . \chi_{\{|u|<k\}} \quad \textit{a.e.} \quad x \in \Omega,$$

where χ_A denotes the characteristic function of a measurable set A. The functions v_i are called the weak partial derivatives of u and are still denoted $D^i u$. Moreover, if u belongs to $W_0^{1,1}(\Omega)$, then v_i coincides with the standard distributional derivative of u, that is, $v_i = D^i u$.

The proof of the Proposition $(6.2.2)$ follows the usual techniques developed in [46] for the case of Sobolev spaces. For more details concerning the anisotropic Sobolev spaces, we refer the reader to [36] and [87].

6.3 Main results

Let Ω be a bounded open subset of $\mathbf{R}^N$ $(N \geq 2)$, containing the origin. First of all, we can give a simpler definition of an entropy solution of $(6.1.1)$ as follows.

Definition 6.3.1. *A measurable function u is an entropy solution of the strongly nonlinear anisotropic elliptic Dirichlet problem $(6.1.1)$ if*

$$u \in \mathcal{T}_0^{1,\vec{p}}(\Omega), \quad |u|^{s-1}u \in L^1(\Omega), \quad \frac{|u|^{p_0-2}u}{|x|^{p_0}} \in L^1(\Omega), \quad h(x,u,\nabla u) \in L^1(\Omega), \quad \phi_i(u) \in L^{p_i'}(\Omega)$$

for $i = 1, \ldots, N$ such that

$$\begin{aligned} &\sum_{i=1}^{N} \int_\Omega |D^i u|^{p_i-2} D^i u D^i T_k(u-\varphi)\, dx + \int_\Omega h(x,u,\nabla u) T_k(u-\varphi)\, dx \\ &+ \int_\Omega |u|^{p_0-2} u T_k(u-\varphi)\, dx + \int_\Omega |u|^{s-1} u T_k(u-\varphi)\, dx \\ &\qquad \leq \int_\Omega f T_k(u-\varphi)\, dx + \lambda \int_\Omega \frac{|u|^{p_0-2}u}{|x|^{p_0}} T_k(u-\varphi)\, dx + \sum_{i=1}^{N} \int_\Omega \phi_i(u) D^i T_k(u-\varphi)\, dx, \end{aligned} \tag{6.3.1}$$

for any $\varphi \in W_0^{1,\vec{p}}(\Omega) \cap L^\infty(\Omega)$.

Our purpose is to establish the following existence theorem:

Theorem 6.3.1. *Let $\lambda \geq 0$ and $f \in L^1(\Omega)$, assuming that $\phi \in \mathcal{C}^0(\mathbb{R}, \mathbb{R}^N)$ and $(6.1.3) - (6.1.4)$ hold true. Then, the problem $(6.1.1)$ has at least one entropy solution u such that $u \in W_0^{1,\vec{q}}(\Omega)$, with*

$$\vec{q} = (s, q_1, \ldots, q_N) \qquad \text{and} \qquad 1 \leq q_i < \frac{p_i s}{s+1} \quad \text{for } i = 1, \ldots, N. \tag{6.3.2}$$

Lemma 6.3.1. *Let $(u_n)_{n\in\mathbb{N}}$ be a sequence in $W_0^{1,\vec{p}}(\Omega)$ such that $u_n \rightharpoonup u$ in $W_0^{1,\vec{p}}(\Omega)$ and*

$$\sum_{i=0}^{N} \int_\Omega \big(|D^i u_n|^{p_i-2} D^i u_n - |D^i u|^{p_i-2} D^i u\big)(D^i u_n - D^i u)\, dx \to 0, \tag{6.3.3}$$

then $u_n \longrightarrow u$ in $W_0^{1,\vec{p}}(\Omega)$ for a subsequence.

Proof of Lemma For the proof of Lemma 6.3.1 we exploit some techniques of [18]. Let's remark that

$$\sum_{i=0}^{N}\int_{\Omega}|D^iu_n-D^iu|^{p_i}\,dx=\sum_{i=0}^{N}\int_{\Lambda_i}|D^iu_n-D^iu|^{p_i}\,dx+\sum_{i=0}^{N}\int_{\Omega\setminus\Lambda_i}|D^iu_n-D^iu|^{p_i}\,dx. \tag{6.3.4}$$

with $\Lambda_i=\{x\in\Omega/1<p_i<2\}$. Note $\Theta_n=\sum_{i=0}^{N}\int_{\Omega}\big(|D^iu_n|^{p_i-2}D^iu_n-|D^iu|^{p_i-2}D^iu\big)(D^iu_n-D^iu)dx$, then by applying the following known inequality

$$\big(|a|^{p-2}a-|b|^{p-2}b\big)(a-b)\geq\begin{cases}2^{2-p}|a-b|^p & \text{if}\quad p\geq 2,\\ \big(p-1\big)\dfrac{|a-b|^2}{\big(|a|+|b|\big)^{2-p}} & \text{for}\quad 1<p<2,\end{cases}\qquad\forall a,b\in I\!R.$$

On the one hand, by virtue of the Hölder inequality we get

$$\begin{aligned}
&\sum_{i=0}^{N}\int_{\Lambda_i}|D^iu_n-D^iu|^{p_i}\,dx=\sum_{i=0}^{N}\int_{\Lambda_i}\frac{|D^iu_n-D^iu|^{p_i}}{(|D^iu_n|+|D^iu|)^{\frac{p_i(2-p_i)}{2}}}(|D^iu_n|+|D^iu|)^{\frac{p_i(2-p_i)}{2}}\,dx\\
&\leq\sum_{i=0}^{N}\left\|\frac{|D^iu_n-D^iu|^{p_i}}{(|D^iu_n|+|D^iu|)^{\frac{p_i(2-p_i)}{2}}}\right\|_{\frac{2}{p_i},\Lambda_i}\left\|(|D^iu_n|+|D^iu|)^{\frac{p_i(2-p_i)}{2}}\right\|_{\frac{2}{2-p_i},\Lambda_i}\\
&\leq\sum_{i=0}^{N}\max\{\bigg(\int_{\Lambda_i}\frac{|D^iu_n-D^iu|^2}{(|D^iu_n|+|D^iu|)^{2-p_i}}>dx\bigg)^{\frac12},\int_{\Lambda_i}\frac{|D^iu_n-D^iu|^2}{(|D^iu_n|+|D^iu|)^{2-p_i}}\,dx\}\\
&\qquad\times\bigg(\int_{\Lambda_i}(|D^iu_n|+|D^iu|)^{p_i}\,dx\bigg)^{\frac{2-p_i}{2}}\\
&\leq\sum_{i=0}^{N}\max\{\Theta_n^{\frac12}(p_i-1)^{-\frac12},\Theta_n(p_i-1)^{-1}\}\bigg(\int_{\Lambda_i}(|D^iu_n|+|D^iu|)^{p_i}\bigg)^{\frac{2-p_i}{2}}.
\end{aligned}\tag{6.3.5}$$

On the other hand it'easy to check that

$$\Theta_n\geq\sum_{i=0}^{N}2^{2-p_i}\int_{\Omega\setminus\Lambda_i}|D^iu_n-D^iu|^{p_i}\,dx. \tag{6.3.6}$$

Passing to the limit as $n\to\infty$ while bearing in mind $(6.3.3)$ and $(6.3.4)-(6.3.6)$, we conclude that $u_n\to u$ in $W_0^{1,\vec{p}}(\Omega)$.

Proof of the Theorem 6.3.1

Step 1 : Approximate problems.

Let $(f_n)_{n\in I\!N}$ be a sequence of smooth functions such that $f_n\to f$ in $L^1(\Omega)$ and $|f_n|\leq|f|$ (for example $f_n=T_n(f)$). We consider the approximate problem

$$A_nu_n+h_n(x,u_n,\nabla u_n)+|u_n|^{p_0-2}u_n+|T_n(u_n)|^{s-1}T_n(u_n)=f_n+\mu\frac{|T_n(u_n)|^{p_0-2}T_n(u_n)}{|x|^{p_0}+\frac1n}-\text{div }\phi_n(u_n), \tag{6.3.7}$$

where

$$A_n v = -\sum_{i=1}^{N} D^i(|D^i v|^{p_i-2}D^i v) + |v|^{p_0-2}v, \qquad \phi_n(s) = \phi(T_n(s)$$

and

$$h_n(x,s,\xi) = \frac{h(x,s,\xi)}{1+\frac{1}{n}|h(x,s,\xi)|}, \qquad \forall i = 1,\dots,N.$$

let's mention that

$$|h_n(x,s,\xi)| \le n, \quad |h_n(x,s,\xi)| \le |h(x,s,\xi)| \quad \text{and} \quad h_n(x,s,\xi)s \ge 0, \quad \forall n \in I\!N^*.$$

We consider the operator $G_n : W_0^{1,\vec{p}}(\Omega) \longrightarrow W^{-1,\vec{p'}}(\Omega)$ by

$$\langle G_n u, v\rangle = \int_\Omega h_n(x,u,\nabla u)v\,dx + \int_\Omega |T_n(u)|^{s-1}T_n(u)v\,dx - \mu\int_\Omega \frac{|T_n(u)|^{p_0-2}T_n(u)}{|x|^{p_0}+\frac{1}{n}}v\,dx,$$

for any $u, v \in W_0^{1,\vec{p}}(\Omega)$. Using the Hölder's type inequality, we deduce that

$$\begin{aligned} |\langle G_n u, v\rangle| &\le \|h_n(x,u,\nabla u)\|_{p_0'}\|v\|_{p_0} + \int_\Omega |T_n(u)|^s\,|v|\,dx + \mu\int_\Omega \frac{|T_n(u)|^{p_0-1}}{|x|^{p_0}+\frac{1}{n}}|v|\,dx \\ &\le \|h_n(x,u,\nabla u)\|_{p_0'}\|v\|_{p_0} + n^s\int_\Omega |v|\,dx + \mu n^{p_0}\int_\Omega |v|\,dx \\ &\le C_0\|v\|_{1,\vec{p}}\,. \end{aligned} \tag{6.3.8}$$

Moreover,we define the operator $R_n : W_0^{1,\vec{p}}(\Omega) \longrightarrow W^{-1,\vec{p'}}(\Omega)$ by

$$\langle R_n(u), v\rangle = \langle \text{div } \phi_n(u), v\rangle = -\int_\Omega \phi_n(u)\nabla v\,dx, \quad \text{for any} \quad u, v \in W_0^{1,\vec{p}}(\Omega),$$

with $\phi_n(u) = (\phi_{i,n}(u),\dots,\phi_{N,n}(u))$. Thanks to the Hölder's type inequality, we have

$$\begin{aligned} |\int_\Omega \phi_n(u)\nabla v\,dx| &\le \sum_{i=1}^{N}\int_\Omega |\phi_{i,n}(u)||D^i v|\,dx \\ &\le \sum_{i=1}^{N}\|\phi_{i,n}(u)\|_{p_i'}\|D^i v\|_{p_i} \\ &\le \sum_{i=1}^{N}\sup_{|\sigma|\le n}|\phi_i(\sigma)|\big(meas(\Omega)\big)^{\frac{1}{p_i'}}\|v\|_{1,\vec{p}} \\ &\le C_1\|v\|_{1,\vec{p}}\,. \end{aligned} \tag{6.3.9}$$

Lemma 6.3.2. *The bounded operator $B_n = A_n + G_n + R_n$ acting from $W_0^{1,\vec{p}}(\Omega)$ into $W^{-1,\vec{p'}}(\Omega)$ is pseudo-monotone. Moreover, B_n is coercive in the following sense:*

$$\frac{\langle B_n v, v\rangle}{\|v\|_{1,\vec{p}}} \longrightarrow +\infty \qquad \textit{as} \quad \|v\|_{1,\vec{p}} \to \infty \quad \textit{for} \quad v \in W_0^{1,\vec{p}}(\Omega).$$

Proof of the Lemma 6.3.2

In view of the Hölder's inequality and by $(6.3.8)$ and $(6.3.9)$, it's easy to see that the operator B_n is bounded.

For the coercivity, we have for any $u \in W_0^{1,\vec{p}}(\Omega)$,

$$\begin{aligned}
\langle B_n u, u\rangle &= \langle A_n u, u\rangle + \langle G_n u, u\rangle + \langle R_n(u), v\rangle \\
&= \sum_{i=0}^{N} \int_\Omega D^i u_n|^{p_i}\, dx + \int_\Omega h_n(x,u,\nabla u)u\, dx + \int_\Omega |T_n(u)|^s|u|\, dx - \mu \int_\Omega \frac{|T_n(u)|^{p_0-1}}{|x|^{p_0}+\frac{1}{n}}|u|\, dx \\
&\quad - \sum_{i=1}^{N} \int_\Omega |\phi_{i,n}(u)||D^i u|\, dx \\
&\geq \|u\|_{1,\vec{p}}^{p} + \int_\Omega |T_n(u)|^{s+1}\, dx - 2\mu n^{p_0}\|1\|_{p_0'}\|u\|_{1,\vec{p}} - C_1\|u\|_{1,\vec{p}} \\
&\geq \|u\|_{1,\vec{p}}^{p} - C_2\|u\|_{1,\vec{p}}
\end{aligned}$$

it follows that

$$\frac{\langle B_n u, u\rangle}{\|u\|_{1,\vec{p}}} \longrightarrow +\infty \quad \text{as} \quad \|u\|_{1,\vec{p}} \to \infty.$$

It remains to show that B_n is pseudo-monotone. Let $(u_k)_{k\in I\!N}$ be a sequence in $W_0^{1,\vec{p}}(\Omega)$ such that

$$\begin{cases} u_k \rightharpoonup u & \text{in } W_0^{1,\vec{p}}(\Omega), \\ B_n u_k \rightharpoonup \chi_n & \text{in } W^{-1,\vec{p'}}(\Omega), \\ \limsup\limits_{k\to\infty} \langle B_n u_k, u_k\rangle \leq \langle \chi_n, u\rangle. \end{cases} \tag{6.3.10}$$

We will prove that

$$\chi_n = B_n u \quad \text{and} \quad \langle B_n u_k, u_k\rangle \longrightarrow \langle \chi_n, u\rangle \ \text{ as } \ k \to +\infty.$$

We have

$$|D^i u_k|^{p_i-2} D^i u_k \rightharpoonup |D^i u|^{p_i-2} D^i u \qquad \text{in} \quad L^{p_i'}(\Omega) \quad \text{as } k \to \infty \tag{6.3.11}$$

$$|u_k|^{p_0-2} u_k \rightharpoonup |u|^{p_0-2} u \qquad \text{in} \quad L^{p_0'}(\Omega) \quad \text{as } k \to \infty \tag{6.3.12}$$

In view of Lebesgue's dominated convergence theorem, we obtain

$$|T_n(u_k)|^{s-1} T_n(u_k) \to |T_n(u)|^{s-1} T_n(u) \qquad \text{in} \quad L^{p_0'}(\Omega), \tag{6.3.13}$$

and

$$\frac{|T_n(u_k)|^{p_0-2} T_n(u_k)}{|x|^{p_0}+\frac{1}{n}} \to \frac{|T_n(u)|^{p_0-2} T_n(u)}{|x|^{p_0}+\frac{1}{n}} \ \text{ in } \quad L^{p_0'}(\Omega). \tag{6.3.14}$$

It easy to check that $(h_n(x,u_k,\nabla u_k))_k$ is bounded in $L^{\underline{p}'}(\Omega)$, then there exists a function φ_n such that

$$h_n(x,u_k,\nabla u_k) \rightharpoonup \varphi_n \quad \text{in} \quad L^{\underline{p}'}(\Omega) \quad \text{as } k \to \infty \tag{6.3.15}$$

Furthermore, since $\phi_n = \phi \circ T_n$ is a bounded continuous function and $u_k \to u$ in $L^{\underline{p}}(\Omega)$, by using the Lebesgue dominated convergence theorem, we deduce that

$$\phi_{i,n}(u_k) \rightharpoonup \phi_{i,n}(u) \quad \text{in} \quad L^{p_i'}(\Omega) \quad \text{for} \ \ i = 1, \dots, N. \tag{6.3.16}$$

For any $v \in W_0^{1,\vec{p}}(\Omega)$, we get

$$\begin{aligned}
\langle \chi_n, v\rangle &= \lim_{k\to\infty} \langle B_n u_k, v\rangle \\
&= \lim_{k\to\infty} \sum_{i=0}^{N} \int_\Omega |D^i u_k|^{p_i-2} D^i u_k D^i v \, dx + \lim_{k\to\infty} \int_\Omega h_n(x, u_k, \nabla u_k) v \, dx \\
&\quad + \lim_{k\to\infty} \int_\Omega |T_n(u_k)|^{s-1} T_n(u_k) v \, dx - \lim_{k\to\infty} \mu \int_\Omega \frac{|T_n(u_k)|^{p_0-2} T_n(u_k)}{|x|^{p_0} + \frac{1}{n}} v \, dx \\
&\quad - \lim_{k\to\infty} \sum_{i=1}^{N} \int_\Omega \phi_{i,n}(u_k) D^i v \, dx \\
&= \sum_{i=0}^{N} \int_\Omega |D^i u|^{p_i-2} D^i u D^i v \, dx + \int_\Omega \varphi_n v \, dx + \int_\Omega |T_n(u)|^{s-1} T_n(u) v \, dx \\
&\quad -\mu \int_\Omega \frac{|T_n(u)|^{p_0-2} T_n(u)}{|x|^{p_0} + \frac{1}{n}} v \, dx - \sum_{i=1}^{N} \int_\Omega \phi_{i,n}(u) D^i v \, dx.
\end{aligned} \tag{6.3.17}$$

Having in mind (6.3.10) and (6.3.17), we obtain

$$\begin{aligned}
&\limsup_{k\to\infty} \langle B_n(u_k), u_k\rangle \\
&= \limsup_{k\to\infty} \Big\{ \sum_{i=0}^{N} \int_\Omega |D^i u_k|^{p_i} \, dx + \int_\Omega h_n(x, u_k, \nabla u_k) u_k \, dx + \int_\Omega |T_n(u_k)|^{s-1} T_n(u_k) u_k \, dx \\
&\qquad -\mu \int_\Omega \frac{|T_n(u_k)|^{p_0-2} T_n(u_k)}{|x|^{p_0} + \frac{1}{n}} u_k \, dx - \sum_{i=1}^{N} \int_\Omega \phi_{i,n}(u_k) D^i u_k \, dx \Big\} \\
&\le \sum_{i=0}^{N} \int_\Omega |D^i u|^{p_i} \, dx + \int_\Omega \varphi_n u \, dx + \int_\Omega |T_n(u)|^{s-1} T_n(u) u \, dx \\
&\qquad -\mu \int_\Omega \frac{|T_n(u)|^{p_0-2} T_n(u)}{|x|^{p_0} + \frac{1}{n}} u \, dx - \sum_{i=1}^{N} \int_\Omega \phi_{i,n}(u) D^i u \, dx.
\end{aligned} \tag{6.3.18}$$

Thanks to (6.3.13) and (6.3.14), we have

$$\int_\Omega |T_n(u_k)|^{s-1} T_n(u_k) u_k \, dx \longrightarrow \int_\Omega |T_n(u)|^{s-1} T_n(u) u \, dx, \tag{6.3.19}$$

and

$$\int_\Omega \frac{|T_n(u_k)|^{p_0-2} T_n(u_k)}{|x|^{p_0} + \frac{1}{n}} u_k \, dx \longrightarrow \int_\Omega \frac{|T_n(u)|^{p_0-2} T_n(u)}{|x|^{p_0} + \frac{1}{n}} u \, dx. \tag{6.3.20}$$

Due to (6.3.15) and (6.3.16), it yields

$$\int_\Omega h_n(x, u_k, \nabla u_k) u_k \, dx \longrightarrow \int_\Omega \varphi_n u \, dx, \tag{6.3.21}$$

$$\sum_{i=1}^{N}\int_{\Omega}\phi_{i,n}(u_k)D^iu_k\,dx\longrightarrow\sum_{i=1}^{N}\int_{\Omega}\phi_{i,n}(u)D^iu\,dx. \tag{6.3.22}$$

Therefore

$$\limsup_{k\to\infty}\sum_{i=0}^{N}\int_{\Omega}|D^iu_k|^{p_i}\,dx\leq\sum_{i=0}^{N}\int_{\Omega}|D^iu|^{p_i}\,dx. \tag{6.3.23}$$

On the other hand, we have

$$\sum_{i=0}^{N}\int_{\Omega}(|D^iu_k|^{p_i-2}D^iu_k-|D^iu|^{p_i-2}D^iu)(D^iu_k-D^iu)\,dx\geq 0, \tag{6.3.24}$$

then

$$\sum_{i=0}^{N}\int_{\Omega}|D^iu_k|^{p_i}\,dx\geq\sum_{i=0}^{N}\int_{\Omega}|D^iu_k|^{p_i-2}D^iu_kD^iu\,dx+\sum_{i=0}^{N}\int_{\Omega}(|D^iu|^{p_i-2}D^iu(D^iu_k-D^iu)\,dx.$$

In view of $(6.3.11)$ and $(6.3.12)$ we get

$$\liminf_{k\to\infty}\sum_{i=0}^{N}\int_{\Omega}|D^iu_k|^{p_i}\,dx\geq\sum_{i=0}^{N}\int_{\Omega}|D^iu|^{p_i}\,dx.$$

Having in mind $(6.3.23)$, we conclude that

$$\lim_{k\to\infty}\sum_{i=0}^{N}\int_{\Omega}|D^iu_k|^{p_i}\,dx=\sum_{i=0}^{N}\int_{\Omega}|D^iu|^{p_i}\,dx. \tag{6.3.25}$$

Therefore, by combining $(6.3.17)$, $(6.3.19)-(6.3.22)$ and $(6.3.25)$ we obtain

$$\langle B_nu_k,u_k\rangle\longrightarrow\langle\chi_n,u\rangle\quad\text{ as }\quad k\to\infty.$$

Now, by $(6.3.25)$ we can prove that

$$\lim_{k\to+\infty}\Big(\sum_{i=0}^{N}\int_{\Omega}\big(|D^iu_k|^{p_i-2}D^iu_k-|D^iu|^{p_i-2}D^iu\big)\Big)(D^iu_k-D^iu)\,dx=0.$$

and so, by virtue of Lemma 6.3.1, we get

$$u_k\to u\quad\text{in}\quad W_0^{1,\vec{p}}(\Omega)\qquad\text{and}\qquad D^iu_k\to D^iu\quad\text{a.e. in}\quad\Omega,$$

then

$$h_n(x,u_k,\nabla u_k)\rightharpoonup h_n(x,u,\nabla u)\quad\text{in}\quad L^{p_0'}(\Omega),$$

which implies $\chi_n=B_nu$. Finally, in view of Lemma 6.3.2, there exists at least one weak solution $u_n\in W_0^{1,\vec{p}}(\Omega)$ of the problem $(6.3.7)$ (cf. [121], Theorem 8.2).

Step 2 : A priori estimates.

Lemma 6.3.3. *Let u_n be a weak solution of the approximate problem $(6.3.7)$, then the following regularity results hold true*

$$u \in W_0^{1,\vec{q}}(\Omega) \qquad \textit{with} \quad \vec{q} = (s, q_1, \dots, q_N) \tag{6.3.26}$$

where the exponent s verify the condition $s > \max\Big(\dfrac{N(p_0-1)}{N-p_0}, \dfrac{1}{p_0-1}\Big)$ and $1 \le q_i < \dfrac{p_i s}{s+1}$, then

$$\sum_{i=1}^{N} \int_\Omega \frac{|D^i u_n|^{p_i}}{(1+|u_n|)^\theta} dx \le C \qquad \textit{for all} \quad 1 < \theta < \frac{s(p_i - q_i)}{q_i}, \tag{6.3.27}$$

$$\sum_{i=1}^{N} \int_\Omega |D^i T_k(u_n)|^{p_i} dx \le C(1+k)^\theta \qquad \textit{for all} \quad k > 0, \tag{6.3.28}$$

with C is a positive constant that doesn't depend on k and n.

Proof of Lemma 6.3.3

Let $\theta > 1$ which will be chosen later, we consider the two functions $\varphi(t) : I\!R \longmapsto I\!R$ defined by

$$\varphi(t) = \Big(1 - \frac{1}{(1+|t|)^{\theta-1}}\Big) sign(t) \quad \text{and} \quad J(t) = \int_0^t j(|\rho|) d\rho$$

It's clear that $\varphi(u_n)\exp(J(|u_n|)) \in W_0^{1,\vec{p}}(\Omega) \cap L^\infty(\Omega)$, and $0 \le J(\infty) < \infty$.

By taking $\varphi(u_n)\exp(J(|u_n|))$ as test function in $(6.3.7)$ we get

$$\begin{aligned}
&(\theta-1)\sum_{i=1}^{N} \int_\Omega \frac{(|D^i u_n|^{p_i}}{(1+|u_n|)^\theta} \exp(J(|u_n|))\, dx + \sum_{i=1}^{N} \int_\Omega (|D^i u_n|^{p_i} j(|u_n|)|\varphi(u_n)| \exp(J(|u_n|))\, dx \\
&\quad + \int_\Omega h_n(x, u_n, \nabla u_n)\varphi(u_n)\exp(J(|u_n|))\, dx + \int_\Omega |u_n|^{p_0-2} u_n \varphi(u_n)\exp(J(|u_n|))\, dx \\
&\quad + \int_\Omega |T_n(u_n)|^{s-1} T_n(u_n)\varphi(u_n)\exp(J(|u_n|))\, dx \\
&= \int_\Omega f_n \varphi(u_n)\exp(J(|u_n|))\, dx + \mu \int_\Omega \frac{|T_n(u_n)|^{p_0-2}T_n(u_n)}{|x|^{p_0} + \frac{1}{n}} \varphi(u_n)\exp(J(|u_n|))\, dx \\
&\quad + \sum_{i=1}^{N} \int_\Omega \phi_{i,n}(u_n) D^i(\varphi(u_n)\exp(J(|u_n|)))\, dx.
\end{aligned}$$

Since $\varphi(u_n)$ have the same sign of u_n, thus the fourth term on the left-hand side of the previous inequality is positive. Also, we have

$$\begin{aligned}
\sum_{i=1}^{N} \int_\Omega \phi_{i,n}(u_n) D^i(\varphi(u_n)\exp(J(|u_n|)))\, dx &\le \sum_{i=1}^{N} \int_\Omega |\phi_{i,n}(u_n)|\ |D^i(\varphi(u_n)\exp(J(|u_n|)))|\, dx \\
&\le \sum_{i=1}^{N} \sup_{|s|\le n} |\phi_i(s)| \int_\Omega |D^i(\varphi(u_n)\exp(J(|u_n|)))|\, dx \le C_3.
\end{aligned} \tag{6.3.29}$$

Seeing that, $|\varphi(\cdot)| \leq 1$ and in view of $(6.1.4)$ and $(6.3.29)$ we obtain

$$\begin{aligned}(\theta-1)\sum_{i=1}^{N}\int_{\Omega}\frac{|D^i u_n|^{p_i}}{(1+|u_n|)^{\theta}}\exp(J(|u_n|))\,dx+\int_{\Omega}|T_n(u_n)|^s|\varphi(u_n)|\exp(J(|u_n|))\,dx\\ \leq \mu\int_{\Omega}\frac{|T_n(u_n)|^{p_0-1}}{|x|^{p_0}+\frac{1}{n}}\exp(J(|u_n|))\,dx+\int_{\Omega}(|f|+|l|)\exp(J(|u_n|))\,dx+C_3.\end{aligned} \tag{6.3.30}$$

It is clear that

$$\frac{1}{2}\leq 1-\frac{1}{(1+|u_n|)^{\theta-1}} \quad \text{for} \quad |u_n|\geq R=\max(2^{\frac{1}{\theta-1}}-1,1).$$

Thus, we have

$$\begin{aligned}\frac{1}{2}\int_{\{|u_n|\geq R\}}|T_n(u_n)|^s\,dx &\leq \int_{\{|u_n|\geq R\}}|T_n(u_n)|^s\Big(1-\frac{1}{(1+|u_n|)^{\theta-1}}\Big)\,dx\\ &\leq \int_{\Omega}|T_n(u_n)|^s\bigg(1-\frac{1}{(1+|u_n|)^{\theta-1}}\bigg)\,dx,\end{aligned}$$

which implies

$$\begin{aligned}\frac{1}{2}\int_{\Omega}|T_n(u_n)|^s\,dx &= \frac{1}{2}\int_{\{|u_n|<R\}}|T_n(u_n)|^s\,dx+\frac{1}{2}\int_{\{|u_n|\geq R\}}|T_n(u_n)|^s\,dx\\ &\leq \frac{1}{2}R^s|\Omega|+\int_{\Omega}|T_n(u_n)|^s\bigg(1-\frac{1}{(1+|u_n|)^{\theta-1}}\bigg)\,dx.\end{aligned}$$

Using $(6.3.30)$, we deduce that

$$\begin{aligned}(\theta-1)\sum_{i=1}^{N}\int_{\Omega}\frac{|D^i u_n|^{p_i}}{(1+|u_n|)^{\theta}}\,dx+\frac{1}{2}\int_{\Omega}|T_n(u_n)|^s\,dx\\ \leq \frac{1}{2}R^s|\Omega|+\mu\int_{\Omega}\frac{|T_n(u_n)|^{p_0-1}}{|x|^{p_0}}\exp(J(\infty))\,dx+\int_{\Omega}(|f|+|l|)\exp(J(\infty))\,dx+C_3.\end{aligned} \tag{6.3.31}$$

Inasmuch as $s>p_0-1$, the Young inequality enables us to obtain

$$\mu\int_{\Omega}\frac{|T_n(u_n)|^{p_0-1}}{|x|^{p_0}}\exp(J(\infty))\,dx\leq\frac{1}{4}\int_{\Omega}|T_n(u_n)|^s\,dx+C_4\int_{\Omega}\frac{dx}{|x|^{\frac{sp_0}{s-p_0+1}}},$$

with C_2 is a positive constant depending only on s, p_0, $\exp(J(\infty))$ and μ. Thus, we obtain

$$\begin{aligned}(\theta-1)\sum_{i=1}^{N}\int_{\Omega}\frac{|D^i u_n|^{p_i}}{(1+|u_n|)^{\theta}}\,dx+\frac{1}{4}\int_{\Omega}|T_n(u_n)|^s\,dx\\ \leq \frac{1}{2}R^s|\Omega|+C_4\int_{\Omega}\frac{dx}{|x|^{\frac{sp_0}{s-p_0+1}}}+\exp(J(\infty))\int_{\Omega}(|f|+|l|)\,dx+C_3.\end{aligned} \tag{6.3.32}$$

Under the assumption $s>\dfrac{N(p_0-1)}{N-p_0}$, the integral $\displaystyle\int_{\Omega}\frac{dx}{|x|^{\frac{sp_0}{s-p_0+1}}}$ is finite. Therefore $(6.3.27)$ is deduced. Moreover, we have

$$\int_{\Omega}|T_n(u_n)|^s\,dx\leq C. \tag{6.3.33}$$

Taking q_i such that $1 \leq q_i < p_i$ for $i = 1, \ldots, N$. By virtue of the generalized Hölder's inequality we get

$$\begin{aligned}\sum_{i=1}^{N} \int_{\Omega} |D^i u_n|^{q_i}\, dx &\leq \sum_{i=1}^{N} \left\| \frac{|D^i u_n|^{q_i}}{(1+|u_n|)^{\frac{\theta q_i}{p_i}}} \right\|_{\frac{p_i}{q_i}} \left\| (1+|u_n|)^{\frac{\theta q_i}{p_i}} \right\|_{\frac{p_i}{p_i - q_i}} \\ &\leq \sum_{i=1}^{N} \Big(\int_{\Omega} \frac{|D^i u_n|^{p_i}}{(1+|u_n|)^{\theta}}\, dx \Big)^{\frac{q_i}{p_i}} \Big(\int_{\Omega} (1+|u_n|)^{\frac{q_i \theta}{p_i - q_i}}\, dx + 1 \Big)^{1 - \frac{q_i}{p_i}}.\end{aligned} \tag{6.3.34}$$

We now choose $\theta > 1$ such that $\dfrac{q_i \theta}{p_i - q_i} < s$, such a real number θ exists if

$$1 < \frac{s(p_i - q_i)}{q_i} \qquad \text{that is} \qquad q_i < \frac{p_i s}{s+1}.$$

Combining $(6.3.32) - (6.3.34)$, we obtain the desired estimates $(6.3.26)$.

To get $(6.3.28)$, we have thanks to $(6.3.27)$ that

$$\sum_{i=1}^{N} \int_{\Omega} |D^i T_k(u_n)|^{p_i}\, dx = \sum_{i=1}^{N} \int_{\{|u_n| < k\}} |D^i u_n|^{p_i}\, dx \leq (1+k)^{\theta} \sum_{i=1}^{N} \int_{\Omega} \frac{|D^i u_n|^{p_i}}{(1+|u_n|)^{\theta}}\, dx.$$

Step 3 : The weak convergence of $(T_k(u_n))_n$ in $W_0^{1,\vec{p}}(\Omega)$.

In order to establish the weak convergence of $(T_k(u_n))_n$ in $W_0^{1,\vec{p}}(\Omega)$, we begin by proving that $(u_n)_n$ is a Cauchy sequence. In fact, thanks to $(6.3.28)$, we can obtain

$$\sum_{i=0}^{N} \int_{\Omega} |D^i T_k(u_n)|^{p_i}\, dx \leq C(1+k)^{\theta} + k^{p_0} |\Omega| \qquad \text{for} \quad k \geq 1,$$

Therefore, the sequence $(T_k(u_n))_n$ is bounded in $W_0^{1,\vec{p}}(\Omega)$, and there exists a subsequence still denoted $(T_k(u_n))_n$ such that

$$\begin{cases} T_k(u_n) \rightharpoonup \eta_k & \text{in} \quad W_0^{1,\vec{p}}(\Omega), \\ T_k(u_n) \to \eta_k & \text{in} \quad L^{\underline{p}}(\Omega) \quad \text{and a.e. in} \quad \Omega. \end{cases} \tag{6.3.35}$$

On the other hand, we have

$$\begin{aligned}\sum_{i=1}^{N} \int_{\Omega} |D^i T_k(u_n)|^{p_i}\, dx &\geq \sum_{i=1}^{N} \int_{\Omega} (|D^i T_k(u_n)|^{\underline{p}} - 1)\, dx \\ &= \|\nabla T_k(u_n)\|_{\underline{p}}^{\underline{p}} - N|\Omega|,\end{aligned}$$

Thanks to $(6.3.28)$, we deduce that there exists a constant C_5 that does not depend on k and n, such that

$$\|\nabla T_k(u_n)\|_{\underline{p}} \leq C_5 k^{\frac{\theta}{\underline{p}}} \qquad \text{for} \quad k \geq 1. \tag{6.3.36}$$

Thanks to the Poincaré type inequality, we obtain

$$\begin{aligned} k \text{ meas}\{|u_n| > k\} &= \int_{\{|u_n|>k\}} |T_k(u_n)|\, dx \leq \int_\Omega |T_k(u_n)|\, dx \\ &\leq C_6 \|T_k(u_n)\|_{\underline{p}} \\ &\leq C_7 \|\nabla T_k(u_n)\|_{\underline{p}} \\ &\leq C_8 k^{\frac{\theta}{\underline{p}}}, \end{aligned} \tag{6.3.37}$$

Choosing θ small enough $(1 < \theta < \underline{p})$, we conclude that

$$\text{meas}\big\{|u_n| > k\big\} \leq C_8 \frac{1}{k^{1-\frac{\theta}{\underline{p}}}} \longrightarrow 0 \quad \text{as} \quad k \to +\infty. \tag{6.3.38}$$

For all $\delta > 0$, we have

$$\text{meas}\{|u_n - u_m| > \delta\} \leq \text{meas}\{|u_n| > k\} + \text{meas}\{|u_m| > k\} + \text{meas}\{|T_k(u_n) - T_k(u_m)| > \delta\}.$$

Let $\varepsilon > 0$, using $(6.3.38)$ we can choose $k = k(\varepsilon)$ large enough such that

$$\text{meas}\{|u_n| > k\} \leq \frac{\varepsilon}{3} \qquad \text{and} \qquad \text{meas}\{|u_m| > k\} \leq \frac{\varepsilon}{3}. \tag{6.3.39}$$

On the other hand, thanks to $(6.3.35)$ we can assume that $(T_k(u_n))_{n\in I\!N}$ is a Cauchy sequence in measure. Thus, for any $k > 0$ and $\delta, \varepsilon > 0$, there exists $n_0 = n_0(k, \delta, \varepsilon)$ such that

$$\text{meas}\{|T_k(u_n) - T_k(u_m)| > \delta\} \leq \frac{\varepsilon}{3} \qquad \text{for all } m, n \geq n_0(k, \delta, \varepsilon). \tag{6.3.40}$$

In view of $(6.3.39)$ and $(6.3.40)$, we deduce that

$$\forall \delta, \varepsilon > 0 \quad \text{there exists} \quad n_0 = n_0(\delta, \varepsilon) \quad \text{such that} \quad \text{meas}\big\{|u_n - u_m| > \delta\big\} \leq \varepsilon \quad \forall n, m \geq n_0(\delta, \varepsilon),$$

which proves that the sequence $(u_n)_n$ is a Cauchy sequence in measure and then converges almost everywhere to some measurable function u. Consequently, we have

$$T_k(u_n) \rightharpoonup T_k(u) \quad \text{in} \quad W_0^{1,\vec{p}}(\Omega), \tag{6.3.41}$$

and in view Lebesgue's dominated convergence theorem, we obtain

$$T_k(u_n) \longrightarrow T_k(u) \quad \text{in} \quad L^{p_0}(\Omega) \quad \text{and a.e in} \quad \Omega. \tag{6.3.42}$$

Step 4 : Strong convergence of truncations.

In the sequel, we denote by $\varepsilon_i(n)$, $i = 1, 2, \dots,$ various real-valued functions of real variables that converge to 0 as n tends to infinity.

Let $h > k > 0$, taking $z_n := u_n - T_h(u_n) + T_k(u_n) - T_k(u)$, $M = 4k + h$ and $\omega_n := T_{2k}(z_n)$.

We also consider $\psi_k(s) = s.\exp(\lambda s^2)$ where $\lambda = (b(k)/(2))^2$. It is simple to see that ([63], Lemma 1)

$$\psi_k'(s) - b(k)|\psi_k(s)| \geq \frac{1}{2} \qquad \text{for any } s \in I\!R.$$

By using $\psi_k(\omega_n)$ as a test function in the approximate problem $(7.4.7)$ we obtain

$$\begin{aligned}
&\sum_{i=1}^{N} \int_\Omega |D^i u_n|^{p_i-2} D^i u_n \psi_k'(\omega_n) D^i \omega_n \, dx + \int_\Omega h_n(x, u_n, \nabla u_n)\psi_k(\omega_n)\, dx \\
&\qquad + \int_\Omega |u_n|^{p_0-2} u_n \psi_k(\omega_n)\, dx + \int_\Omega |T_n(u_n)|^{s-1} T_n(u_n)\psi_k(\omega_n)\, dx \\
&= \mu \int_\Omega \frac{|T_n(u_n)|^{p_0-2}T_n(u_n)}{|x|^{p_0} + \frac{1}{n}} \psi_k(\omega_n)\, dx + \int_\Omega f_n \psi_k(\omega_n)\, dx + \sum_{i=1}^{N} \int_\Omega \phi_{i,n}(u_n)\psi_k'(\omega_n) D^i\omega_n \, dx.
\end{aligned}$$

For $M = 4k + h$, it's clear that $D^i\omega_n = 0$ on the set $\{|u_n| \geq M\}$, and since $h_n(x, u_n, \nabla u_n)\psi_k(\omega_n) \geq 0$ on the set $\{|u_n| > k\}$, therefore

$$\begin{aligned}
&\sum_{i=1}^{N} \int_{\{|u_n|\leq M\}} |D^i T_M(u_n)|^{p_i-2} D^i T_M(u_n) \psi_k'(\omega_n) D^i \omega_n \, dx + \int_{\{|u_n|\leq k\}} h_n(x, u_n, \nabla u_n)\psi_k(\omega_n)\, dx \\
&\qquad + \int_{\{|u_n|\leq k\}} |T_k(u_n)|^{p_0-2} T_k(u_n) \psi_k(\omega_n)\, dx + \int_\Omega |T_n(u_n)|^{s-1} T_n(u_n)\psi_k(\omega_n)\, dx \\
&\leq \mu \int_\Omega \frac{|T_n(u_n)|^{p_0-2}T_n(u_n)}{|x|^{p_0} + \frac{1}{n}} \psi_k(\omega_n)\, dx + \int_\Omega f_n \psi_k(\omega_n)\, dx \\
&\qquad + \sum_{i=1}^{N} \int_{\{|u_n|\leq M\}} \phi_{i,n}(u_n)\psi_k'(\omega_n) D^i\omega_n \, dx.
\end{aligned}$$

Thanks to the Young inequality, we have

$$\mu \int_{\{|u_n|>k\}} \frac{|T_n(u_n)|^{p_0-1}}{|x|^{p_0} + \frac{1}{n}} |\psi_k(\omega_n)|\, dx \leq \int_{\{|u_n|>k\}} |T_n(u_n)|^s |\psi_k(\omega_n)|\, dx + C_7 \int_{\{|u_n|>k\}} \frac{|\psi_k(\omega_n)|}{|x|^{\frac{p_0 s}{s-p_0+1}}}\, dx,$$

and since $\omega_n = T_k(u_n) - T_k(u)$ on the set $\{|u_n| \leq k\}$, then

$$\begin{aligned}
&\sum_{i=1}^{N} \int_{\{|u_n|\leq M\}} |D^i T_M(u_n)|^{p_i-2} D^i T_M(u_n) \psi_k'(\omega_n) D^i \omega_n \, dx + \int_{\{|u_n|\leq k\}} h_n(x, u_n, \nabla u_n)\psi_k(\omega_n)\, dx \\
&\qquad + \int_{\{|u_n|\leq k\}} |T_k(u_n)|^{p_0-2} T_k(u_n) \psi_k(\omega_n)\, dx + \int_{\{|u_n|\leq k\}} |T_n(u_n)|^{s-1} T_n(u_n)\psi_k(\omega_n)\, dx \\
&\leq \mu \int_\Omega \frac{|T_n(u_n)|^{p_0-2}T_n(u_n)}{|x|^{p_0} + \frac{1}{n}} \psi_k(\omega_n)\, dx + \int_\Omega f_n \psi_k(\omega_n)\, dx + C_9 \int_{\{|u_n|>k\}} \frac{|\psi_k(\omega_n)|}{|x|^{\frac{p_0 s}{s-p_0+1}}}\, dx\, dx \\
&\qquad + \sum_{i=1}^{N} \int_{\{|u_n|\leq M\}} \phi_{i,n}(u_n)\psi_k'(\omega_n) D^i\omega_n \, dx.
\end{aligned} \tag{6.3.43}$$

Now, we will study each terms in the previews inequality.

Firstly, we have

$$\begin{aligned}
&\sum_{i=1}^{N}\int_{\{|u_n|\le M\}}|D^iT_M(u_n)|^{p_i-2}D^iT_M(u_n)\psi_k'(\omega_n)D^i\omega_n\,dx\\
&=\sum_{i=1}^{N}\int_{\{|u_n|\le k\}}|D^iT_k(u_n)|^{p_i-2}D^iT_k(u_n)\psi_k'(\omega_n)D^iT_{2k}(u_n-T_k(u))\,dx\\
&\quad+\sum_{i=1}^{N}\int_{\{k<|u_n|\le M\}}D^iT_M(u_n)|^{p_i-2}D^iT_M(u_n)\psi_k'(\omega_n)D^i\omega_n\,dx.
\end{aligned}\tag{6.3.44}$$

on one hand, since $|u_n-T_k(u)|\le 2$ on $\{|u_n|\le k\}$, then

$$\begin{aligned}
&\sum_{i=1}^{N}\int_{\{|u_n|\le k\}}|D^iT_k(u_n)|^{p_i-2}D^iT_k(u_n)\psi_k'(\omega_n)D^iT_{2k}(u_n-T_k(u))\,dx\\
&=\sum_{i=1}^{N}\int_{\Omega}|D^iT_k(u_n)|^{p_i-2}D^iT_k(u_n)\psi_k'(\omega_n)(D^iT_k(u_n)-D^iT_k(u))\,dx\\
&\quad+\sum_{i=1}^{N}\int_{\{k<|u_n|\}}|D^iT_k(u_n)|^{p_i-2}D^iT_k(u_n)\psi_k'(\omega_n)D^iT_k(u)\,dx.
\end{aligned}\tag{6.3.45}$$

Seeing that $1\le\psi_k'(\omega_n)\le\psi_k'(2k)$, then

$$\begin{aligned}
&\Big|\sum_{i=1}^{N}\int_{\{k<|u_n|\}}|D^iT_k(u_n)|^{p_i-2}D^iT_k(u_n)\psi_k'(\omega_n)D^iT_k(u)\,dx\Big|\\
&\le\psi_k'(2k)\sum_{i=1}^{N}\int_{\{k<|u_n|\}}|D^iT_k(u_n)|^{p_i-1}|D^iT_k(u)|\,dx,
\end{aligned}$$

and since $|D^iT_k(u_n)|^{p_i-1}$ is bounded in $L^{p_i'}(\Omega)$, then there exists $\zeta\in L^{p_i'}(\Omega)$ such that $|D^iT_k(u_n)|^{p_i-1}\rightharpoonup\zeta$ in $L^{p_i'}(\Omega)$. Thus,

$$\sum_{i=1}^{N}\int_{\{k<|u_n|\}}|D^iT_k(u_n)|^{p_i-1}|D^iT_k(u)|\,dx\to\int_{\{k<|u|\}}\zeta|D^iT_k(u)|\,dx=0.$$

subsequently

$$\sum_{i=1}^{N}\int_{\{k<|u_n|\}}|D^iT_k(u_n)|^{p_i-2}D^iT_k(u_n)\psi_k'(\omega_n)D^iT_k(u)\,dx=\varepsilon_0(n).\tag{6.3.46}$$

Taking $z_n := u_n - T_h(u_n) + T_k(u_n) - T_k(u)$, in the second term on the right hand side of $(6.3.44)$, we get

$$\begin{aligned}
&\int_{\{k<|u_n|\leq M\}} |D^iT_M(u_n)|^{p_i-2}D^iT_M(u_n)\psi_k'(\omega_n)D^i\omega_n\,dx \\
&= \int_{\{k<|u_n|\leq M\}\cap\{|z_n|\leq 2k\}} |D^iT_M(u_n)|^{p_i-2}D^iT_M(u_n)D^i(u_n - T_h(u_n) + T_k(u_n) - T_k(u))\,dx \\
&\geq \int_{\{k<|u_n|\leq M\}} |D^iT_M(u_n)|^{p_i-2}D^iT_M(u_n)\psi_k'(\omega_n)D^i(u_n - T_k(u)).\chi_{|u_n|>h}\,dx \\
&\quad - \int_{\{k<|u_n|\leq M\}} |D^iT_M(u_n)|^{p_i-2}D^iT_M(u_n)\psi_k'(\omega_n)D^iT_k(u).\chi_{|u_n|\leq h}\,dx \\
&\geq -\psi_k'(2k)\int_{\{k<|u|\leq M\}} |D^iT_M(u_n)|^{p_i-1}\,|D^iT_k(u)|\,dx = 0.
\end{aligned}$$

Similarly to $(6.3.46)$, we can prove that

$$\psi_k'(2k)\int_{\{k<|u|\leq M\}} |D^iT_M(u_n)|^{p_i-1}\,|D^iT_k(u)|\,dx = \varepsilon_1(n) \tag{6.3.47}$$

By combining $(6.3.44) - (6.3.47)$, we obtain

$$\begin{aligned}
&\sum_{i=1}^N \int_{\{|u_n|\leq M\}} |D^iT_M(u_n)|^{p_i-2}D^iT_M(u_n)\psi_k'(\omega_n)D^i\omega_n\,dx \\
&\geq \sum_{i=1}^N \int_\Omega |D^iT_k(u_n)|^{p_i-2}D^iT_k(u_n)\psi_k'(\omega_n)(D^iT_k(u_n) - D^iT_k(u))\,dx + \varepsilon_2(n).
\end{aligned} \tag{6.3.48}$$

Therefore, we have

$$\begin{aligned}
&\sum_{i=1}^N \int_\Omega \big(|D^iT_k(u_n)|^{p_i-2}D^iT_k(u_n) - |D^iT_k(u)|^{p_i-2}D^iT_k(u)\big).(D^iT_k(u_n) - D^iT_k(u))\psi_k'(\omega_n)\,dx \\
&\leq \sum_{i=1}^N \int_{\{|u_n|\leq M\}} |D^iT_M(u_n)|^{p_i-2}D^iT_M(u_n)\psi_k'(\omega_n)D^i\omega_n\,dx \\
&\quad - \sum_{i=1}^N \int_\Omega |D^iT_k(u)|^{p_i-2}D^iT_k(u)(D^iT_k(u_n) - D^iT_k(u))\psi_k'(\omega_n)\,dx - \varepsilon_2(n) \\
&\leq \sum_{i=1}^N \int_{\{|u_n|\leq M\}} |D^iT_M(u_n)|^{p_i-2}D^iT_M(u_n)\psi_k'(\omega_n)D^i\omega_n\,dx \\
&\quad + \psi_k'(2k)\int_\Omega |D^iT_k(u)|^{p_i-1}\,|D^iT_k(u_n) - D^iT_k(u)|\,dx - \varepsilon_2(n).
\end{aligned} \tag{6.3.49}$$

For the second term on the right-hand side of $(6.3.49)$, since $|D^iT_k(u)|^{p_i-1}$ is bounded in $L^{p_i'}(\Omega)$ and $D^iT_k(u_n) \rightharpoonup D^iT_k(u)$ in $L^{p_i}(\Omega)$, then

$$\psi_k'(2k)\int_\Omega |D^iT_k(u)|^{p_i-1}\,|D^iT_k(u_n) - D^iT_k(u)|\,dx = \varepsilon_3(n) \quad \text{as} \quad n \to 0. \tag{6.3.50}$$

Consequently, we deduce that

$$\sum_{i=1}^{N}\int_{\Omega}\big(|D^iT_k(u_n)|^{p_i-2}D^iT_k(u_n)-|D^iT_k(u)|^{p_i-2}D^iT_k(u)\big).(D^iT_k(u_n)-D^iT_k(u))\psi_k'(\omega_n)\,dx \\ \leq\sum_{i=1}^{N}\int_{\{|u_n|\leq M\}}|D^iT_M(u_n)|^{p_i-2}D^iT_M(u_n)\psi_k'(\omega_n)D^i\omega_n\,dx+\varepsilon_4(n). \tag{6.3.51}$$

Secondly, we deal with the second term on the left-hand side of $(6.3.43)$. In view of $(6.1.4)$ we have

$$\begin{aligned}
&\Big|\int_{\{|u_n|\leq k\}}h_n(x,u_n,\nabla u_n)\psi_k(\omega_n)\,dx\Big|\\
&\leq\int_{\{|u_n|\leq k\}}l(x)|\psi_k(\omega_n)|\,dx+j(k)\sum_{i=1}^{N}\int_{\Omega}|D^iT_k(u_n)|^{p_i}|\psi_k(\omega_n)|\,dx\\
&=\int_{\{|u_n|\leq k\}}l(x)|\psi_k(\omega_n)|\,dx+j(k)\sum_{i=1}^{N}\int_{\Omega}\big(|D^iT_k(u_n)|^{p_i-2}|D^iT_k(u_n)\big)D^iT_k(u_n)|\psi_k(\omega_n)|\,dx\\
&\leq\int_{\{|u_n|\leq k\}}l(x)|\psi_k(\omega_n)|\,dx\\
&\quad+j(k)\sum_{i=1}^{N}\int_{\Omega}\big(|D^iT_k(u_n)|^{p_i-2}D^iT_k(u_n)-|D^iT_k(u)|^{p_i-2}D^iT_k(u)\big)\\
&\quad.(D^iT_k(u_n)-D^iT_k(u))|\psi_k(\omega_n)|\,dx\\
&\quad+j(k)\sum_{i=1}^{N}\int_{\Omega}|D^iT_k(u)|^{p_i-2}D^iT_k(u)(D^iT_k(u_n)-D^iT_k(u))|\psi_k(\omega_n)|\,dx\\
&\quad+j(k)\sum_{i=1}^{N}\int_{\Omega}\big(|D^iT_k(u_n)|^{p_i-2}D^iT_k(u_n)\big)D^iT_k(u).
\end{aligned}$$

It yields

$$\begin{aligned}
&j(k)\sum_{i=1}^{N}\int_{\Omega}\big(|D^iT_k(u_n)|^{p_i-2}D^iT_k(u_n)-|D^iT_k(u)|^{p_i-2}D^iT_k(u)\big)\\
&.(D^iT_k(u_n)-D^iT_k(u))|\psi_k(\omega_n)|\,dx\\
&\geq\Big|\int_{\{|u_n|\leq k\}}h_n(x,u_n,\nabla u_n)\psi_k(\omega_n)\,dx\Big|-\int_{\{|u_n|\leq k\}}l(x)|\psi_k(\omega_n)|\,dx\\
&\quad-j(k)\sum_{i=1}^{N}\int_{\Omega}|D^iT_k(u)|^{p_i-2}D^iT_k(u)(D^iT_k(u_n)-D^iT_k(u))|\psi_k(\omega_n)|\,dx\\
&\quad-j(k)\sum_{i=1}^{N}\int_{\Omega}\big(|D^iT_k(u_n)|^{p_i-2}D^iT_k(u_n)\big)D^iT_k(u)|\psi_k(\omega_n)|\,dx.
\end{aligned}\tag{6.3.52}$$

Regarding the third term on the right-hand side of $(6.3.52)$, due to $(6.3.50)$, we obtain

$$\begin{aligned}
&\Big|\sum_{i=1}^{N}\int_{\Omega}|D^iT_k(u)|^{p_i-2}D^iT_k(u)(D^iT_k(u_n)-D^iT_k(u))|\psi_k(\omega_n)|\Big|\,dx\\
&\quad\leq\psi_k(2k)\sum_{i=1}^{N}\int_{\Omega}\big||D^iT_k(u)|^{p_i-2}D^iT_k(u)\big|\,\big|D^iT_k(u_n)-D^iT_k(u)\big|\to 0\quad\text{as}\quad n\to\infty.
\end{aligned}\tag{6.3.53}$$

Concerning the fourth term of the right-hand side of $(6.3.52)$, knowing that $|D^iT_k(u_n)|^{p_i-2}D^iT_k(u_n)$ is bounded in $L^{p_i'}(\Omega)$, then there exists $\gamma \in L^{p_i'}(\Omega)$ such that $|D^iT_k(u_n)|^{p_i-2}D^iT_k(u_n) \rightharpoonup \gamma$ in $L^{p_i'}(\Omega)$ and, by applying

$$D^iT_k(u)|\psi_k(\omega_n)| \rightharpoonup D^iT_k(u)|\psi_k(T_{2k}(u-T_h(u)))| \qquad \text{in} \qquad L^{p_i'}(\Omega),$$

we deduce that

$$\begin{aligned}\int_\Omega \big(|D^iT_k(u_n)|^{p_i-2}D^iT_k(u_n)\big)D^iT_k(u)|\psi_k(\omega_n)|\,dx \\ \to \int_\Omega \gamma D^iT_k(u)|\psi_k(T_{2k}(u-T_h(u)))| = 0.\end{aligned} \tag{6.3.54}$$

For the second term of the right-hand side of $(6.3.52)$, using the fact that $\psi_k(\omega_n) \rightharpoonup \psi_k(T_{2k}(u-T_h(u)))$ weak-* in $L^\infty(\Omega)$ as $n \to +\infty$, then

$$\int_{\{|u_n|\le k\}} l(x)|\psi_k(\omega_n)|\,dx \to \int_{\{|u|\le k\}} l(x)|\psi_k(T_{2k}(u-T_h(u)))|\,dx = 0 \tag{6.3.55}$$

By combining $(6.3.52)-(6.3.55)$ we conclude that

$$\begin{aligned}&\Big|\int_{\{|u_n|\le k\}} h_n(x,u_n,\nabla u_n)\psi_k(\omega_n)\,dx\Big| + \varepsilon_5(n)\\ &\le j(k)\sum_{i=1}^N \int_\Omega \big(|D^iT_k(u_n)|^{p_i-2}D^iT_k(u_n) - |D^iT_k(u)|^{p_i-2}D^iT_k(u)\big)\\ &\qquad .(D^iT_k(u_n)-D^iT_k(u))|\psi_k(\omega_n)|\,dx.\end{aligned} \tag{6.3.56}$$

As a third estimate, we have

$$\begin{aligned}&\int_{\{|u_n|\le k\}} |u_n|^{p_0-2}u_n\psi_k(\omega_n)\,dx\\ &= \int_\Omega \big(|T_k(u_n)|^{p_0-2}T_k(u_n) - |T_k(u)|^{p_0-2}T_k(u)\big)(T_k(u_n)-T_k(u))\exp(\lambda\omega_n^2)dx\\ &\quad + \int_\Omega |T_k(u)|^{p_0-2}T_k(u)(T_k(u_n)-T_k(u))\exp(\lambda\omega_n^2)dx\\ &\quad - \int_{\{|u_n|\ge k\}} |T_k(u_n)|^{p_0-2}T_k(u_n)(T_k(u_n)-T_k(u))\exp(\lambda\omega_n^2)dx\\ &\ge \int_\Omega \big(|T_k(u_n)|^{p_0-2}T_k(u_n) - |T_k(u)|^{p_0-2}T_k(u)\big)(T_k(u_n)-T_k(u))dx\\ &\quad - \exp(\lambda(2k)^2)\int_\Omega |T_k(u)|^{p_0-1}|(T_k(u_n)-T_k(u))|dx\\ &\quad - \exp(\lambda(2k)^2)\int_{\{|u_n|\ge k\}} k^{p_0-1}(T_k(u_n)-T_k(u))|dx.\end{aligned}$$

Thanks to $(6.3.42)$, the second and the last term on the right-hand side of the previous inequality converges to 0 as $n \to \infty$. Thus, we obtain

$$\begin{aligned}\int_\Omega \big(|T_k(u_n)|^{p_0-2}T_k(u_n) - |T_k(u)|^{p_0-2}T_k(u)\big)(T_k(u_n)-T_k(u))dx\\ \le \int_{\{|u_n|\le k\}} |u_n|^{p_0-2}u_n\psi_k(\omega_n)\,dx + \varepsilon_6(n).\end{aligned} \tag{6.3.57}$$

For the fourth term of the left-hand side of $(6.3.43)$. The Lebesgue dominated convergence theorem give us

$$|T_k(u_n)|^{s-1}T_k(u_n) \longrightarrow |T_k(u)|^{s-1}T_k(u) \qquad \text{in} \quad L^1(\Omega),$$

and

$$\begin{aligned}
&\left| \lim_{n\to\infty} \int_{\{|u_n|\le k\}} |T_n(u_n)|^{s-1}T_n(u_n)\psi_k(\omega_n)\,dx \right| dx \\
&\le k^s \lim_{n\to\infty} \int_{\{|u_n|\le k\}} |\psi_k(\omega_n)|\,dx \\
&= k^s \int_{\{|u|\le k\}} \psi_k(T_{2k}(u - T_h(u)))\,dx = 0,
\end{aligned}$$

it follows that

$$\lim_{n\to\infty} \int_{\{|u_n|\le k\}} |T_n(u_n)|^{s-1}T_n(u_n)\psi_k(\omega_n)\,dx = 0. \tag{6.3.58}$$

Concerning the first term on the right-hand side of $(6.3.43)$, by virtue of the Hölder's type inequality and as above we have

$$\begin{aligned}
\varepsilon_7(n) &= \left| \int_\Omega \frac{|T_n(u_n)|^{p_0-2}T_n(u_n)}{|x|^{p_0} + \frac{1}{n}} \psi_k(\omega_n)\,dx \right| \\
&\le k^s \left\| \frac{1}{|x|^{p_0(x)}} \right\|_{L^{\frac{s}{s-p+1}}(\Omega)} \left\| \psi_k(\omega_n) \right\|_{L^{\frac{s}{p-1}}(\{|u_n|\le k\})} \longrightarrow 0 \qquad \text{as} \quad n \to \infty,
\end{aligned} \tag{6.3.59}$$

also, due to the weak-* convergence of $\psi_k(\omega_n)$ in $L^\infty(\Omega)$ we have

$$\int_\Omega f_n\ \psi_k(\omega_n)\,dx = \int_\Omega f\ \psi_k(T_{2k}(u - T_h(u)))\,dx + \varepsilon_8(n). \tag{6.3.60}$$

once again the theorem of Lebesgue allows us

$$\lim_{n\to\infty} \int_{\{|u_n|>k\}} \frac{|\psi_k(\omega_n)|}{|x|^{\frac{p_0 s}{s-p_0+1}}}\,dx = \int_{\{|u|>k\}} \frac{|\psi_k(T_{2k}(u - T_h(u))|}{|x|^{\frac{p_0 s}{s-p_0+1}}}\,dx. \tag{6.3.61}$$

Concerning the last term on the right-hand side of $(6.3.43)$, we have for n large enough

$$\begin{aligned}
&\int_{\{|u_n|\le M\}} \phi_{i,n}(u_n)\psi_k'(\omega_n)D^i\omega_n\,dx \\
&= \int_\Omega \phi_i(T_M(u))\psi_k'(T_{2k}(u - T_h(u)))D^i T_{2k}(u - T_h(u))\,dx + \varepsilon_9(n)
\end{aligned}$$

By using $W_i(t) = \int_0^t \phi_i(\varsigma)\psi_k'(\varsigma - T_h(\varsigma))d\varsigma$, we have $W_i \in \mathcal{C}^1(I\!R)$ and $W_i(0) = 0$. By applying the Green formula, we have

$$\begin{aligned}
&\int_\Omega \phi_i(T_M(u))\psi_k'(T_{2k}(u - T_h(u)))D^iT_{2k}(u - T_h(u))\,dx \\
&= \int_{\{h<|u|\le 2k+h\}} \phi_i(u)\psi_k'(u - T_h(u))D^iu\,dx \\
&= \int_{\{|u|<2k+h\}} \phi_i(T_{2k+h}(u))\psi_k'(T_{2k+h}(u) - T_h(u))D^iT_{2k+h}\,dx \\
&\quad - \int_{\{|u|<h\}} \phi_i(T_h(u))\psi_k'(T_h(u) - T_h(u))D^iT_h\,dx \\
&= \int_\Omega D^iW_i(T_{2k+h}(u))\,dx - \int_\Omega D^iW_i(T_h(u))\,dx \\
&= \int_{\partial\Omega} W_i(T_{2k+h}(u)).n_i\,dx - \int_{\partial\Omega} W_i(T_h(u)).n_i\,dx = 0.
\end{aligned}$$

Consequently, we get

$$\int_{\{|u_n|\le M\}} \phi_{i,n}(u_n)\psi_k'(\omega_n)D^i\omega_n\,dx = \varepsilon_{10}(n). \tag{6.3.62}$$

Consequently, taking into account $(6.3.43)$, $(6.3.51)$ and $(6.3.56) - (6.3.62)$, we obtain

$$\begin{aligned}
&\frac{1}{2}\sum_{i=1}^N \int_\Omega \big(|D^iT_k(u_n)|^{p_i-2}D^iT_k(u_n) - |D^iT_k(u)|^{p_i-2}D^iT_k(u)\big) \\
&\qquad .(D^iT_k(u_n) - D^iT_k(u))\,dx \\
&\qquad + \int_\Omega \big(|T_k(u_n)|^{p_0-2}T_k(u_n) - |T_k(u)|^{p_0-2}T_k(u)\big)(T_k(u_n) - T_k(u))\,dx \\
&\le \sum_{i=1}^N \int_{\{|u_n|\le M\}} |D^iT_M(u_n)|^{p_i-2}D^iT_M(u_n)\psi_k'(\omega_n)D^i\omega_n\,dx \\
&\qquad - \Big|\int_{\{|u_n|\le k\}} h_n(x,u_n,\nabla u_n)\psi_k(\omega_n)\,dx\Big| \\
&\qquad + \int_{\{|u_n|\le k\}} |u_n|^{p_0-2}u_n\psi_k(\omega_n)\,dx + \varepsilon_{10}(n) \\
&\le \int_\Omega f\ \psi_k(T_{2k}(u - T_h(u)))\,dx + C_9\int_{\{|u|>h\}} \frac{|\psi_k(T_{2k}(u - T_h(u))|}{|x|^{\frac{p_0 s}{s-p_0+1}}}\,dx + \varepsilon_{11}(n)
\end{aligned} \tag{6.3.63}$$

therefore

$$\begin{aligned}
&\sum_{i=1}^N \int_\Omega \big(|D^iT_k(u_n)|^{p_i-2}D^iT_k(u_n) - |D^iT_k(u)|^{p_i-2}D^iT_k(u)\big) \\
&\qquad .(D^iT_k(u_n) - D^iT_k(u))\,dx \\
&\qquad + \int_\Omega \big(|T_k(u_n)|^{p_0-2}T_k(u_n) - |T_k(u)|^{p_0-2}T_k(u)\big)(T_k(u_n) - T_k(u))\,dx \\
&\le 2\int_\Omega f\ \psi_k(T_{2k}(u - T_h(u)))\,dx + 2C_9\int_{\{|u|>h\}} \frac{|\psi_k(T_{2k}(u - T_h(u))|}{|x|^{\frac{p_0 s}{s-p_0+1}}}\,dx + \varepsilon_{11}(n)
\end{aligned} \tag{6.3.64}$$

Since $\dfrac{N(p_0-1)}{N-p_0} < s$, we have $\dfrac{p_0 s}{s-p_0+1} < N$ then $\dfrac{1}{|x|^{\frac{p_0 s}{s-p_0+1}}} \in L^1(\Omega)$.
Finally , we conclude by letting h and n goes to infinity in $(6.3.64)$

$$\begin{aligned}\lim_{n\to\infty}\sum_{i=1}^{N}\int_{\Omega}\big(|D^iT_k(u_n)|^{p_i-2}D^iT_k(u_n)-|D^iT_k(u)|^{p_i-2}D^iT_k(u)\big)(D^iT_k(u_n)-D^iT_k(u))\,dx\\ +\int_{\Omega}\big(|T_k(u_n)|^{p_0(x)-2}T_k(u_n)-|T_k(u_n)|^{p_0(x)-2}T_k(u_n)\big)(T_k(u_n)-T_k(u))\,dx\Big)=0.\end{aligned} \tag{6.3.65}$$

In view of Lemma 6.3.1, we conclude that

$$\begin{cases} T_k(u_n)\longrightarrow T_k(u) & \text{strongly in} \quad W_0^{1,\vec{p}}(\Omega),\\ D^iu_n\longrightarrow D^iu & \text{a.e. in} \quad \Omega \qquad \text{for } i=1,\dots,N.\end{cases} \tag{6.3.66}$$

Step 5 : The equi-integrability of the nonlinear functions.

Thanks to $(6.3.66)$, we get

$$h_n(x,u_n,\nabla u_n)\to h(x,u,\nabla u) \quad \text{a.e. in} \quad \Omega, \tag{6.3.67}$$

and we have also

$$|T_n(u_n)|^{s-1}T_n(u_n)\to|u|^{s-1}u \quad \text{a.e. in} \quad \Omega, \tag{6.3.68}$$

$$\frac{|T_n(u_n)|^{p_0-2}T_n(u_n)}{|x|^{p_0}+\frac{1}{n}}\longrightarrow\frac{|u|^{p_0-2}u}{|x|^{p_0}} \quad \text{a.e. in} \quad \Omega,. \tag{6.3.69}$$

In order to prove the uniform equi-integrability of these functions, we take $T_1(u_n-T_k(u_n))$ as a test function in $(6.3.7)$, and since $T_1(u_n-T_k(u_n))$ have the same sign as u_n we can obtain

$$\begin{aligned}&\sum_{i=1}^{N}\int_{\{k<|u_n|\le k+1\}}|D^iu_n|^{p_i}\,dx+\int_{\{|u_n|\ge k\}}h_n(x,u_n,\nabla u_n)T_1(u_n-T_k(u_n))\,dx\\ &\qquad+\int_{\{|u_n|\ge k\}}|T_n(u_n)|^s|T_1(u_n-T_k(u_n))|\,dx\\ &\le\mu\int_{\{|u_n|\ge k\}}\frac{|T_n(u_n)|^{p_0-1}}{|x|^{p_0}+\frac{1}{n}}|T_1(u_n-T_h(u_n))|\,dx+\int_{\{|u_n|\ge k\}}|f_n|\,dx\\ &\qquad+\sum_{i=1}^{N}\int_{\{k<|u_n|\le k+1\}}\phi_{i,n}(u_n)D^iu_n\,dx.\end{aligned}$$

Note that

$$\begin{aligned}&\int_{\{|u_n|\ge k\}}h_n(x,u_n,\nabla u_n)T_1(u_n-T_k(u_n))\,dx\\ &\qquad\ge\int_{\{|u_n|\ge k+1\}}h_n(x,u_n,\nabla u_n)T_1(u_n-T_k(u_n))\,dx=\int_{\{|u_n|\ge k+1\}}|h_n(x,u_n,\nabla u_n)|\,dx.\end{aligned}$$

Let's consider $\Psi_{i,n} = \int_0^t \phi_{i,n}(\varsigma)d\varsigma$, we have $\Psi_n \in \mathcal{C}^1(I\!R)$ and $\Psi_{i,n}(0) = 0$. By virtue of Green formula, we obtain

$$\begin{aligned}
&\int_{\{k<|u_n|\leq k+1\}} \phi_{i,n}(u_n) D^i u_n \, dx \\
&= \int_\Omega \phi_{i,n}(T_{k+1}(u_n)) D^i T_{k+1}(u_n) \, dx - \int_\Omega \phi_{i,n}(T_k(u_n)) D^i T_k(u_n) \, dx \\
&= \int_\Omega D^i \Psi_{i,n}(T_{k+1}(u_n)) \, dx - \int_\Omega D^i \Psi_{i,n}(T_k(u_n)) \, dx = 0
\end{aligned}$$

Thanks to Young's inequality, we have

$$\begin{aligned}
&\mu \int_{\{|u_n|\geq k\}} \frac{|T_n(u_n)|^{p_0-1}}{|x|^{p_0}+\frac{1}{n}} |T_1(u_n - T_k(u_n))| \, dx \\
&\leq \frac{1}{3} \int_{\{|u_n|\geq k\}} |T_n(u_n)|^s |T_1(u_n - T_k(u_n))| \, dx + C_{10} \int_{\{|u_n|\geq k\}} \frac{|T_1(u_n - T_k(u_n))|}{|x|^{\frac{sp_0}{s-p_0+1}}} \, dx,
\end{aligned}$$

it follows that

$$\begin{aligned}
&\int_{\{|u_n|\geq k+1\}} |h_n(x,u_n,\nabla u_n)| \, dx + \frac{1}{3} \int_{\{|u_n|\geq k+1\}} |T_n(u_n)|^s \, dx + \mu \int_{\{|u_n|\geq k+1\}} \frac{|T_n(u_n)|^{p_0-1}}{|x|^{p_0}+\frac{1}{n}} \, dx \\
&\leq 2C_{10} \int_{\{|u_n|\geq k\}} \frac{|T_1(u_n - T_k(u_n))|}{|x|^{\frac{sp_0}{s-p_0+1}}} \, dx + \int_{\{|u_n|\geq k\}} |f_n| \, dx.
\end{aligned}$$

Hence, for any $\delta > 0$, there exists $k(\delta) > 0$ such that

$$\int_{\{|u_n|\geq k(\delta)\}} |h_n(x,u_n,\nabla u_n)| \, dx + \int_{\{|u_n|\geq k(\delta)\}} |T_n(u_n)|^s \, dx + \int_{\{|u_n|\geq k(\delta)\}} \frac{|T_n(u_n)|^{p_0-1}}{|x|^{p_0}+\frac{1}{n}} \, dx \leq \frac{\delta}{2}. \quad (6.3.70)$$

Now, let E be a measurable subset of Ω, we have

$$\begin{aligned}
&\int_E |h_n(x,u_n,\nabla u_n)| \, dx + \int_E |T_n(u_n)|^s \, dx + \int_E \frac{|T_n(u_n)|^{p_0-1}}{|x|^{p_0}+\frac{1}{n}} \, dx \\
&\leq \int_{E\cap\{|u_n|<k(\delta)\}} |h_n(x,T_{k(\delta)}(u_n),\nabla T_{k(\delta)}(u_n)| \, dx + \int_{E\cap\{|u_n|<k(\delta)\}} |T_{k(\delta)}(u_n)|^s \, dx \\
&\quad + \int_{E\cap\{|u_n|<k(\delta)\}} \frac{|T_{k(\delta)}(u_n)|^{p_0-1}}{|x|^{p_0}+\frac{1}{n}} \, dx + \int_{\{|u_n|\geq k(\delta)\}} |h_n(x,u_n,\nabla u_n)| \, dx \\
&\quad + \int_{\{|u_n|\geq k(\delta)\}} |T_n(u_n)|^s \, dx + \int_{\{|u_n|\geq k(\delta)\}} \frac{|T_n(u_n)|^{p_0-1}}{|x|^{p_0}+\frac{1}{n}} \, dx.
\end{aligned} \quad (6.3.71)$$

On other hand, we have

$$\int_{E\cap\{|u_n|<k(\delta)\}} |h_n(x,T_{k(\delta)}(u_n),\nabla T_{k(\delta)}(u_n)| \, dx \leq \int_{E\cap\{|u_n|<k(\delta)\}} \left(l(x) + j(|k(\delta)|) \sum_{i=1}^N |D_i|^{p_i} \right) dx$$

Then, thanks to $(6.3.66)$, there exists $\beta(\delta) > 0$ such that : for any $E \subseteq \Omega$ with meas$(E) \leq \beta(\delta)$

$$\begin{aligned}
&\int_{E\cap\{|u_n|<k(\delta)\}} |h_n(x,T_{k(\delta)}(u_n),\nabla T_{k(\delta)}(u_n)| \, dx + \int_{E\cap\{|u_n|<k(\delta)\}} |T_{k(\delta)}(u_n)|^s \, dx \\
&\quad + \int_{E\cap\{|u_n|<k(\delta)\}} \frac{|T_{k(\delta)}(u_n)|^{p_0-1}}{|x|^{p_0}+\frac{1}{n}} \, dx \leq \frac{\delta}{2}.
\end{aligned} \quad (6.3.72)$$

Finally, by combining $(6.3.70)$, $(6.3.71)$ and $(6.3.72)$, one easily has

$$\int_E |h_n(x,u_n,\nabla u_n)|\,dx+\int_E |T_n(u_n)|^s\,dx+\int_E \frac{|T_n(u_n)|^{p_0-1}}{|x|^{p_0}+\frac{1}{n}}\,dx\le\delta \quad \text{with meas}(E)\le\beta(\delta), \tag{6.3.73}$$

We deduce that $(h_n(x,u_n,\nabla u_n))_n$, $(|T_n(u_n)|^{s(x)-1}T_n(u_n))_n$ and $\Big(\frac{|T_n(u_n)|^{p_0-2}T_n(u_n)}{|x|^{p_0}+\frac{1}{n}}\Big)_n$ are equi-integrable, and in view of $(6.3.67)-(6.3.72)$ and Vitali's theorem, the following convergences are established

$$h_n(x,u_n,\nabla u_n)\to h(x,u,\nabla u) \quad \text{in} \quad L^1(\Omega), \tag{6.3.74}$$

$$|T_n(u_n)|^{s-1}T_n(u_n)\to |u|^{s-1}u \quad \text{in} \quad L^1(\Omega), \tag{6.3.75}$$

and

$$\frac{|T_n(u_n)|^{p_0-2}T_n(u_n)}{|x|^{p_0}+\frac{1}{n}}\longrightarrow\frac{|u|^{p_0-2}u}{|x|^{p_0}} \quad \text{in} \quad L^1(\Omega). \tag{6.3.76}$$

Step 6 : Passage to the limit.

By taking $T_k(u_n-\varphi)$ as a test function in $(6.3.7)$, with $\varphi\in W_0^{1,\vec{p}}(\Omega)\cap L^\infty(\Omega)$, and choosing $M=k+\|\varphi\|_\infty$, we obtain

$$\begin{aligned}
&\sum_{i=1}^N\int_\Omega |D^iu_n|^{p_i-2}D^iu_nD^iT_k(u_n-\varphi)\,dx+\int_\Omega h_n(x,u_n,\nabla u_n)T_k(u-\varphi)\,dx\\
&\qquad+\int_\Omega |T_n(u_n)|^{s-1}T_n(u_n)T_k(u_n-\varphi)\,dx+\int_\Omega |u_n|^{p_0-2}u_nT_k(u_n-\varphi)\,dx\\
&=\mu\int_\Omega\frac{|T_n(u_n)|^{p_0-2}T_n(u_n)}{|x|^{p_0}+\frac{1}{n}}T_k(u_n-\varphi)\,dx\\
&\qquad+\int_\Omega f_nT_k(u_n-\varphi)\,dx+\sum_{i=1}^N\int_\Omega\phi_{i,n}(u_n)D^iT_k(u-\varphi)\,dx.
\end{aligned}\tag{6.3.77}$$

On the one hand, as soon as $|u_n|>M$ we get $|u_n-\varphi|\ge|u_n|-\|\varphi\|_\infty>k$, then $\{|u_n-\varphi|\le k\}\subseteq\{|u_n|\le M\}$, it follows that

$$\begin{aligned}
&\int_\Omega |D^iu_n|^{p_i-2}D^iu_nD^iT_k(u_n-\varphi)\,dx\\
&=\int_\Omega |D^iT_M(u_n)|^{p_i-2}D^iT_M(u_n)(D^iT_M(u_n)-D^i\varphi)\chi_{\{|u_n-\varphi|\le k\}}\,dx\\
&=\int_\Omega (|D^iT_M(u_n)|^{p_i-2}D^iT_M(u_n)-|D^i\varphi|^{p_i-2}D^i\varphi)(D^iT_M(u_n)-D^i\varphi)\chi_{\{|u_n-\varphi|\le k\}}\,dx\\
&\qquad+\int_\Omega |D^i\varphi|^{p_i-2}D^i\varphi)(D^iT_M(u_n)-D^i\varphi)\chi_{\{|u_n-\varphi|\le k\}}\,dx.
\end{aligned}$$

According to Fatou's Lemma, we obtain

$$\begin{aligned}
&\liminf_{n\to\infty}\sum_{i=1}^{N}\int_{\Omega}|D^iu_n|^{p_i-2}D^iu_nD^iT_k(u_n-\varphi)\,dx\\
&\geq\sum_{i=1}^{N}\int_{\Omega}(|D^iT_M(u)|^{p_i-2}D^iT_M(u)-|D^i\varphi|^{p_i-2}D^i\varphi)(D^iT_M(u)-D^i\varphi)\chi_{\{|u-\varphi|\leq k\}}\,dx\\
&\qquad+\liminf_{n\to\infty}\sum_{i=1}^{N}\int_{\Omega}|D^i\varphi|^{p_i-2}D^i\varphi)(D^iT_M(u)-D^i\varphi)\chi_{\{|u-\varphi|\leq k\}}\,dx \qquad (6.3.78)\\
&=\sum_{i=1}^{N}\int_{\Omega}|D^iT_M(u)|^{p_i-2}D^iT_M(u)(D^iT_M(u)-D^i\varphi)\chi_{\{|u-\varphi|\leq k\}}\,dx\\
&=\sum_{i=1}^{N}\int_{\Omega}|D^iu|^{p_i-2}D^iuD^iT_k(u-\varphi)\,dx.
\end{aligned}$$

On the other hand, we have $T_k(u_n-\varphi)\rightharpoonup T_k(u-\varphi)$ weak$-*$ in $L^\infty(\Omega)$ and thanks to $(6.3.74)-(6.3.76)$, we deduce that

$$\int_{\Omega}h_n(x,u_n,\nabla u_n)T_k(u-\varphi)\,dx\longrightarrow\int_{\Omega}h(x,u,\nabla u)T_k(u-\varphi)\,dx, \qquad (6.3.79)$$

$$\int_{\Omega}|T_n(u_n)|^{s-1}T_n(u_n)T_k(u_n-\varphi)\,dx\longrightarrow\int_{\Omega}|u|^{s-1}uT_k(u-\varphi)\,dx, \qquad (6.3.80)$$

$$\int_{\Omega}\frac{|T_n(u_n)|^{p_0-2}T_n(u_n)}{|x|^{p_0}+\frac{1}{n}}T_k(u_n-\varphi)\,dx\longrightarrow\int_{\Omega}\frac{|u|^{p_0-2}u}{|x|^{p_0}}T_k(u-\varphi)\,dx, \qquad (6.3.81)$$

and

$$\int_{\Omega}f_nT_k(u_n-\varphi)dx\longrightarrow\int_{\Omega}fT_k(u-\varphi)dx. \qquad (6.3.82)$$

Moreover, since $T_k(u_n-\varphi)\rightharpoonup T_k(u-\varphi)$ in $W_0^{1,\vec{p}}(\Omega)$ and $\phi_{i,n}(u_n)=\phi_i(T_M(u_n))$ in $\{|u_n-\varphi|\leq k\}$ for $n\geq M$, we obtain

$$\int_{\Omega}\phi_{i,n}(u_n)D^iT_k(u-\varphi)\,dx\longrightarrow\int_{\Omega}\phi_i(u)D^iT_k(u-\varphi)\,dx \qquad (6.3.83)$$

and

$$\int_{\Omega}|u_n|^{p_0-1}u_nT_k(u_n-\varphi)\,dx\longrightarrow\int_{\Omega}|u|^{p_0-1}uT_k(u-\varphi)\,dx. \qquad (6.3.84)$$

Putting all the terms together, the proof of Theorem 6.3.1 is now complete.

Chapter 7

Existence of entropy solutions for anisotropic quasilinear degenerate elliptic problems with Hardy potential

The aim of this chapter is to give an existence result of entropy solutions for anisotropic quasilinear degenerate elliptic problems of the form

$$-\text{div}\,(a(x,u,\nabla u)) + |u|^{s-1}u = f + \rho\frac{|u|^{p_0-2}u}{|x|^{p_0}}, \quad \text{in} \quad \Omega,$$

where $-\text{div}\,(a(x,u,\nabla u))$ is a Leray-Lions operator from $W_0^{1,(p_i)}(\Omega,w)$ to its dual, Ω is an open bounded subset of $I\!R^N$ $(N \geq 2)$ containing the origin, the datum f is assumed to be merely integrable and ρ is a positive constant.

7.1 Introduction

In this chapter, we are interested with the problem of existence of entropy solution for a class of anisotropic quasilinear degenerate elliptic equations with additional Hardy potential term that give rise to, in general, an hindrance to the existence of a solution. It turns out that the natural tool for investigation of such problems is the apparatus of anisotropic weighted Sobolev spaces, and as we can check, it's new framework which involves directional derivatives with distinct weights (see section 2). First of all, let's consider Ω be a bounded open subset of $\mathbf{R}^N$ $(N \geq 2)$, containing the origin, and let p_i be $N+1$ exponents,

with $1 < p_i < N$ for $i \in \{0, \dots, N\}$, and $w = \{w_i(x), 0 \leq i \leq N\}$ be a vector of weights functions on Ω, i.e. each $w_i(x)$ is a mesurable a.e strictly positive on Ω, satisfying some integrability conditions (for details, see section 2). In the remaining parts of this chapter, attention shall be focused to the study of the existence of entropy solutions for the following anisotropic quasilinear degenerated elliptic problem

$$\begin{cases} Au + |u|^{s-1}u = f + \rho\dfrac{|u|^{p_0-2}u}{|x|^{p_0}} & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{7.1.1}$$

with $\rho \geq 0$, $f \in L^1(\Omega)$ and

$$s > \max\Big(\frac{N(p_0-1)}{N-p_0}, \frac{1}{p_0-1}\Big). \tag{7.1.2}$$

The Leray-Lions operator A acted from $W_0^{1,(p_i)}(\Omega, \nu)$ into its dual with ν as a special case of w, is defined by the formula

$$Au = -\sum_{i=1}^{N} D^i a_i(x, u, \nabla u) \tag{7.1.3}$$

where $a_i : \Omega \times \mathbf{R} \times \mathbf{R}^N \longmapsto \mathbf{R}^N$ are Carathéodory functions satisfying

$$|a_i(x, s, \xi)| \leq \beta\nu^{\frac{1}{p_i}}(R_i(x) + |s|^{\frac{p_i}{p_i'}} + \sum_{i=1}^{N} \nu^{\frac{1}{p_i'}}|\xi_i|^{p_i-1}), \tag{7.1.4}$$

$$\sum_{i=1}^{N} a_i(x, s, \xi) \cdot \xi_i \geq \alpha \sum_{i=1}^{N} \nu|\xi_i|^{p_i}, \tag{7.1.5}$$

for any $\xi = (\xi_1, \dots, \xi_N)$ and $\xi' = (\xi_1', \dots, \xi_N')$ in $I\!R^N$, we have

$$\big(a_i(x, s, \xi) - a_i(x, s, \xi')\big) \cdot (\xi_i - \xi_i') > 0 \quad \text{for} \quad \xi_i \neq \xi_i', \tag{7.1.6}$$

for a.e. $x \in \Omega$ and all $(s, \xi) \in \mathbf{R} \times \mathbf{R}^N$, where $R_i(x) \in L^{p_i'}(\Omega)$, and α, β are two positive real numbers. Let's point out that, in the non weighted and isotropic case ie $p_i = p$, Lions [121] and Browder [70] have studied such problems in the variational framework with $Au = -\triangle_p u$ and $\rho = 0$, by using energy estimates and monotonicity methods. For the non variational case ie $f \in L^1(\Omega)$ Brézis-Strauss [69] have proved the uniqueness of solution u such that $|u|^s \in L^1(\Omega)$ and for any $0 < s < +\infty$. Recently, several authors have studied problems involving the so-called Hardy potential. To name a few, Porzio in [138] has dealt with the following quasilinear elliptic problem

$$\begin{cases} -\text{div}\,(M(x, u)\nabla u) + \nu|u|^{p-1}u = a\dfrac{u}{|x|^2} + f(x) - \text{div}(F) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{7.1.7}$$

where $p > \frac{N}{N-2}$ and a is a positive constant. $M(x,s)$ is assumed to be a Carathéodory function (measurable with respect to x for every $s \in I\!R$, and continuous with respect to s for almost every $x \in \Omega$) which satisfies the growth and coercivity conditions.

Mercaldo and al [126], and Abdellaoui and al [2, 3] have highlighted the effect generated by the presence of the Hardy potential on the existence and the nonexistence of solutions. For example, Abdellaoui, Peral and Primo [2] have studied the nonlinear elliptic equation

$$\begin{cases} -\Delta u \pm |\nabla u|^2 = \lambda \dfrac{u}{|x|^2} + f & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{7.1.8}$$

where $\lambda > 0$. They have proved the existence of positive solutions for the problem $(7.1.8)$ in the absorption case $(+|\nabla u|^2)$ with $f \in L^1(\Omega)$. In the reaction case $(-|\nabla u|^2)$ the non-existence of solution is proved even in a very weak sense.

In the weighted case, Drabek, kufner and Mustonen [92] proved the existence of at least one solution $u \in W_0^{1,p}(\Omega, w)$ of the Dirichlet problem associated with the equation $Au = h$, where $Au = -\text{div}(a(x,u,\nabla u))$ and $h \in \left(W_0^{1,p}(\Omega,w)\right)^*$, as well as by Akdim, Azroul, and Benkirane in [7] where A is of the form $-\text{div}(a(x,u,\nabla u)) + a_0(x,u,\nabla u)$. It should be mentioned that for the topic dealing with nonlinear degenerated elliptic equations, the reader can consult for instance [9, 93, 95, 119] and the references therein. A similar problems involving nonlinear lower order terms with natural growth, and other works in the direction of degenerated unilateral problems are investigated in [6, 8].

In recent years, an increasing interest has turned towards anisotropic elliptic and parabolic equations. This impulse mainly comes from their applications to the mathematical modeling of physical and mechanical in anisotropic continuous medium, the image restoration and flows of electro-rheological fluids and thermistor problem (we refer to the recent works [17, 34, 36],[37, 38, 39, 41] and [102, 104, 105, 87, 127]).

Under the above assumptions of the problem $(7.1.1)$, we need to work with entropy solutions, which are required to be less regular than weak solutions and which were introduced by Bénilan et al [46] for the study of some non linear elliptics problems. We also use a penalization term $\frac{1}{n}|u|^{p_0-2}u$ in the approximates problems $(7.4.7)$, to overpass the non-coercivity of the operator Au in the anisotropic weighted Sobolev space $W_0^{1,(p_i)}(\Omega,\nu)$. In the other hand we will overcome the nonexistence of a solution caused by the singularity of the term $\frac{|u|^{p_0-2}u}{|x|^{p_0}}$ by using the regularizing effect of the term $|u|^{s-1}u$. Note that, this paper can be seen as an extension of [155] to an anisotropic weighted case.

This chapter is organized as follows: Section 2 contains some preliminaries and basic assumptions, section 3 is concerned with the technical results in anisotropic weighted Sobolev spaces. In section 4,

attention will be paid to the statement and the proof of the main result. Section 5 is devoted to an example which illustrates our abstract hypotheses.

7.2 Basic assumptions

Now we state the following assumption

Assumption (H1)

We assume, for simplicity, that

$$w_0(x) \equiv 1 \quad \text{and} \quad w_1(x) = w_2(x) = \ldots = w_N(x) = \nu(x)$$

In this case, the space $W_0^{1,(p_i)}(\Omega,\nu) := W_0^{1,(p_i)}(\Omega,w)$ is normed by

$$\|u\|_{1,(p_i),\nu} = \|u\|_{p_0} + \sum_{i=1}^{N} \|\frac{\partial u}{\partial x_i}\|_{p_i,\nu}. \tag{7.2.1}$$

Let us suppose, in addition the integrability condition:
There exists $t \in]\frac{N}{p_i}, +\infty[\cap]\frac{1}{p_i-1}, +\infty[$, for all i=1,...,N, such that

$$\nu^{-t} \in L^1(\Omega). \tag{7.2.2}$$

with a certain $t > 0$ which will be specified afterwards. Notice that the assumptions $(1.1.12)$ and $(7.2.2)$ imply

$$|||u|||_V = \sum_{i=1}^{N} \Big(\int_\Omega |D^i u(x)|^{p_i} \nu(x)\, dx \Big)^{1/p_i}, \tag{7.2.3}$$

which is a norm defined on $W_0^{1,(p_i)}(\Omega,\nu)$ and its equivalent to $(7.2.1)$. Let us introduce the parameter p_{t_i} by

$$p_{t_i} = \frac{tp_i}{t+1} < p_i$$

Proposition 7.2.1. *Let t, p_i and ν a weight function which satisfies* $(1.1.12)$, $(1.1.13)$ *and* $(7.2.2)$ *then* $W^{1,(p_i)}(\Omega,\nu) \hookrightarrow W^{1,(p_{t_i})}(\Omega)$.

Proof. By using Hölder's inequality, we have

$$\begin{aligned}\sum_{i=1}^{N}\int_{\Omega}|v(x)|^{p_{t_i}}\,dx &= \sum_{i=1}^{N}\int_{\Omega}|v(x)|^{p_{t_i}}\nu(x)^{\frac{p_{t_i}}{p_i}}\nu(x)^{\frac{-p_{t_i}}{p_i}}\,dx\\ &\leq \sum_{i=1}^{N}\left(\int_{\Omega}|v(x)|^{p_i}\nu(x)\,dx\right)^{\frac{p_{t_i}}{p_i}}\left(\int_{\Omega}\nu(x)^{-t}\,dx\right)^{\frac{1}{t+1}}\\ &\leq C\sum_{i=1}^{N}\left(\int_{\Omega}|v(x)|^{p_i}\nu(x)\,dx\right)^{\frac{t}{t+1}}, \text{according to } (7.2.2).\end{aligned}$$

Taking $v = D^i u$, $i = 1, \ldots, N$, we will obtain that a function $u \in W^{1,p_i}(\Omega,\nu)$ belongs to the nonweighted space $W^{1,p_{ti}}(\Omega)$:

$$\|u\|_{1,(p_{t_i})} \leq C\|u\|_{1,(p_i),\nu}$$

namely

$$W^{1,(p_i)}(\Omega,\nu) \hookrightarrow W^{1,(p_{t_i})}(\Omega).$$

Now we highligt that The imbedding

$$W_0^{1,(p_i)}(\Omega,\nu) \hookrightarrow L^r(\Omega), \tag{7.2.4}$$

is compact for all $1 \leq r < \overline{p}^*$ if $\overline{p} < N$ and for all $r \geq 1$ if $\overline{p} \geq N$ where $\frac{1}{\overline{p}} = \frac{1}{N}\sum_{i=1}^{N}\frac{1}{p_{t_i}}$ and $\overline{p}^*$ is the Sobolev conjugate of $\overline{p}$, see [93, pp 30-31]. In particular, we have $\overline{p}^* > \underline{p}$ and consequently,

$$W_0^{1,(p_i)}(\Omega,\nu) \hookrightarrow\hookrightarrow L^{\underline{p}}(\Omega), \tag{7.2.5}$$

Remark 7.2.1. **(i)** *The condition* $(7.2.2)$ *is stronger than* $(1.1.13)$ *since here we claim the integrability over the whole domain* Ω *while in* $(1.1.13)$ *we only need the local integrability.*

(ii) *the compact imbedding* $(7.2.5)$ *holds if* $\nu^{-t} \in L^1(\Omega)$ *with* $1 + \frac{1}{t} < p_i$ *and* $t > \frac{N}{p_i}$ *since to be in accordance with* $(1.1.13)$, *we have to suppose that also* $1 + \frac{1}{t} < p_i$.

7.3 Some technical results

In the sequel, we state and prove the following technical results which are needed later.

Proposition 7.3.1. *Let* Ω *be a bounded open subset of* $\mathbf{R}^N$, ν *a weight function on* Ω, *if* $(1.1.12)$ *and* $(1.1.13)$ *are verified then* $L^{p_i}(\Omega,\nu) \hookrightarrow L^1_{\mathrm{loc}}(\Omega)$.

Proof. Let k be a included compact on Ω. Thanks to Hölder's inequality, we get

$$\begin{aligned}\int_K |u(x)|\,dx &= \sum_{i=1}^N \int_K |u(x)|\nu(x)^{\frac{1}{p_i}}\nu(x)^{\frac{-1}{p_i}}\,dx \\ &\le \sum_{i=1}^N \||u|\nu(x)^{\frac{1}{p_i}}\|_{L^{p_i}(K)}\|\nu(x)^{\frac{-1}{p_i}}\|_{L^{p_i'}(K)} \\ &\le \sum_{i=1}^N \|u\|_{L^{p_i}(\Omega,\nu)}\left(\int_K \nu(x)^{\frac{-p_i'}{p_i}}\,dx\right)^{\frac{1}{p_i'}} \\ &\le \sum_{i=1}^N \|u\|_{L^{p_i}(\Omega,\nu)}\left(\int_K \nu(x)^{\frac{-1}{p_i-1}}\,dx\right)^{\frac{1}{p_i'}}\end{aligned}$$

Having in mind $(1.1.13)$ we deduce that $\int_K |u(x)|\,dx \le C\|u\|_{p_i,\nu}$.

Lemma 7.3.1. *Let γ a function weight, $g \in L^r(\Omega,\gamma)$,and $(g_n)_n \subset L^r(\Omega,\gamma)$ such that $\|g_n\|_{r,\gamma} \le c$, $1 < r < \infty$*

If $g_n(x) \to g(x)$ a.e. in Ω, then $g_n \rightharpoonup g$ weakly in $L^r(\Omega,\gamma)$.

For the proof, see [7].

Lemma 7.3.2. *Let $(u_n)_n$ be bounded sequence in $W_0^{1,(p_i)}(\Omega,\nu)$. If $u_n \rightharpoonup u$ in $W_0^{1,(p_i)}(\Omega,\nu)$, then $T_k(u_n) \rightharpoonup T_k(u)$ in $W_0^{1,(p_i)}(\Omega,\nu)$, where, for any $k > 0$, $T_k(.)$ is the truncation function defined by $T_k(s) := max\{-k, min\{k,s\}\}$.*

Proof of Lemma. We have $u_n \rightharpoonup u$ weakly in $W_0^{1,(p_i)}(\Omega,\nu)$, thus $u_n \to u$ strongly in $L^{\underline{p}}(\Omega)$ and a.e.in Ω, therefore $T_k(u_n) \to T_k(u)$ a.e.in Ω. On the other hand

$$\begin{aligned}\sum_{i=1}^N \int_\Omega |D^i T_k(u_n|^{p_i}\nu(x)\,dx &= \sum_{i=1}^N \int_{\{|u_n|\le k\}} |D^i u_n|^{p_i}\nu(x)\,dx \\ &\le \sum_{i=1}^N \int_\Omega |D^i u_n|^{p_i}\nu(x)\,dx < \infty,\end{aligned}$$

We deduce that $(T_k(u_n))_n$ is bounded in $W_0^{1,(p_i)}(\Omega,\nu)$, then $T_k(u_n) \rightharpoonup v_k$ weakly in $W_0^{1,(p_i)}(\Omega,\nu)$, consequently $v_k = T_k(u)$ and we obtain $T_k(u_n) \rightharpoonup T_k(u)$ in $W_0^{1,(p_i)}(\Omega,\nu)$.

7.4 Main results

We set

$$\mathcal{T}_0^{1,(p_i)}(\Omega,\nu) := \{u : \Omega \mapsto I\!R \text{ measurable, such that } T_k(u) \in W_0^{1,(p_i)}(\Omega,\nu) \text{ for any } k > 0\}.$$

and let's suppose

$$p_0 \geq \max\{p_i, \quad i = 1, 2, \ldots, N\} \tag{7.4.1}$$

Definition 7.4.1. *A measurable function u is an entropy solution of the Dirichlet problem* $(7.1.1)$ *if*

$$u \in \mathcal{T}_0^{1,(p_i)}(\Omega, \nu), \qquad |u|^s \in L^1(\Omega), \qquad \frac{|u|^{p_0-2}u}{|x|^{p_0}} \in L^1(\Omega)$$

and

$$\begin{aligned} \sum_{i=1}^{N} \int_\Omega a_i(x, u, \nabla u) \cdot D^i T_k(u - \varphi)\, dx + \int_\Omega |u|^{s-1} u T_k(u - \varphi)\, dx \\ \leq \int_\Omega f T_k(u - \varphi)\, dx + \rho \int_\Omega \frac{|u|^{p_0-2}u}{|x|^{p_0}} T_k(u - \varphi)\, dx, \end{aligned} \tag{7.4.2}$$

for any $\varphi \in W_0^{1,(p_i)}(\Omega, \nu) \cap L^\infty(\Omega)$.

We state our main result as follows:

Theorem 7.4.1. *Let $\rho \geq 0$ and $f \in L^1(\Omega)$, assuming that* $(7.1.2)$, $(7.4.1)$ *and* $(7.1.4) - (7.1.6)$ *hold true. Then, there exists at least one entropy solution u for quasilinear elliptic problem* $(7.1.1)$. *Moreover, we have $u \in W_0^{1,(m_i)}(\Omega, \nu)$, with*

$$(m_i) = (s, m_1, \ldots, m_N), \qquad 1 \leq m_i < \frac{p_i s}{s + p_i} \qquad \text{and} \qquad \nu^{-\frac{p_i}{1-p_i}} \in L^1(\Omega) \quad \text{for } i = 1, \ldots, N. \tag{7.4.3}$$

Remark 7.4.1. *The assumption* $(7.4.1)$ *is essential to ensure that $|a_i(x, u, \nabla u)|$ belongs to $L^{p_i'}(\Omega, \nu^*)$. In the case of $Au = -\sum_{i=1}^{N} D^i a_i(x, \nabla u)$ the existence of entropy solution is guaranteed, without using this assumption.*

For the theorem $(7.4.1)$ we need the following Lemma.

Lemma 7.4.1. *Assuming that* $(1.1.12)$, $(1.1.13)$, $(7.2.2)$ *and* $(7.1.4) - (7.1.6)$ *hold, and let $(u_n)_{n\in\mathbb{N}}$ be a sequence in $W_0^{1,(p_i)}(\Omega, \nu)$ such that $u_n \rightharpoonup u$ in $W_0^{1,(p_i)}(\Omega, \nu)$ and*

$$\begin{aligned} &\int_\Omega (|u_n|^{p_0-2}u_n - |u|^{p_0-2}u)(u_n - u)\, dx \\ &\quad + \sum_{i=1}^{N} \int_\Omega \Big(a_i(x, u_n, \nabla u_n) - a_i(x, u_n, \nabla u)\Big)(D^i u_n - D^i u)\, dx \longrightarrow 0, \end{aligned} \tag{7.4.4}$$

then $u_n \longrightarrow u$ in $W_0^{1,(p_i)}(\Omega, \nu)$ for a subsequence.

Proof. Let

$$\begin{aligned} S_n &= \sum_{i=1}^{N}(a_i(x,u_n,\nabla u_n)-a_i(x,u_n,\nabla u))(D^iu_n-D^iu)\\ &\quad +(|u_n|^{p_0-2}u_n-|u|^{p_0-2}u)(u_n-u). \end{aligned}$$

Thanks to (7.1.6), we deduce that S_n is a positive function and, by (7.4.4), $S_n \to 0$ in $L^1(\Omega)$ as $n\to\infty$. Since $u_n \rightharpoonup u$ in $W_0^{1,(p_i)}(\Omega,\nu)$, using the compact embedding we obtain $u_n \to u$ in $L^{\underline{p}(.)}(\Omega)$, and since $S_n \to 0$ a.e. in Ω, there exists a subset B in Ω with measure zero such that $\forall x \in \Omega\backslash B$

$$|u(x)|<\infty, \qquad |D^iu(x)|<\infty, \qquad c_i(x)<\infty, \qquad u_n \to u \qquad and \qquad S_n\to 0.$$

We have

$$\begin{aligned} S_n(x) &= \sum_{i=1}^{N}(a_i(x,u_n,\nabla u_n)-a_i(x,u_n,\nabla u))(D^iu_n-D^iu)\\ &\quad +(|u_n|^{p_0(x)-2}u_n-|u|^{p_0(x)-2}u)(u_n-u)\\ &= \sum_{i=1}^{N}\Big(a_i(x,u_n,\nabla u_n)D^iu_n+a_i(x,u_n,\nabla u)D^iu\\ &\quad -a_i(x,u_n,\nabla u)D^iu_n-a_i(x,u_n,\nabla u_n)D^iu\Big)\\ &\quad +|u_n|^{p_0}+|u|^{p_0}-|u_n|^{p_0-2}u_nu-|u|^{p_0-2}uu_n\\ &\geq \underline{\alpha}\sum_{i=0}^{N}\nu|D^iu_n|^{p_i}+\underline{\alpha}\sum_{i=0}^{N}\nu|D^iu|^{p_i}-\beta\sum_{i=1}^{N}\nu^{\frac{1}{p_i}}(R_i(x)+|u_n|^{\frac{p_i}{p_i'}}+\sum_{i=1}^{N}\nu^{\frac{1}{p_i'}}|D^iu|^{p_i-1})|D^iu_n|\\ &\quad -\beta\sum_{i=1}^{N}\nu^{\frac{1}{p_i}}(R_i(x)+|u_n|^{\frac{p_i}{p_i'}}+\sum_{i=1}^{N}\nu^{\frac{1}{p_i'}}|D^iu_n|^{p_i-1})|D^iu|-|u_n|^{p_0-1}|u|-|u|^{p_0-1}|u_n|\\ &\geq \underline{\alpha}\sum_{i=0}^{N}\nu|D^iu_n|^{p_i}-K_x(1+\sum_{i=0}^{N}\nu^{\frac{1}{p_i'}}|D^iu_n|^{p_i-1}+\sum_{i=0}^{N}\nu^{\frac{1}{p_i}}|D^iu_n|), \end{aligned}$$

with $\underline{\alpha}=\min(\alpha,1)$ and K_x depending only on x. It follows that

$$S_n(x)\geq\sum_{i=0}^{N}|D^iu_n|^{p_i}\left(\underline{\alpha}\nu-\frac{K_x}{N|D^iu_n|^{p_i}}-\frac{K_x\nu^{\frac{1}{p_i'}}}{|D^iu_n|}-\frac{K_x\nu^{\frac{1}{p_i}}}{|D^iu_n|^{p_i-1}}\right).$$

By a standard argument, $(D^iu_n)_{n\in I\!N}$ is bounded almost everywhere in Ω for all $i=0,1,\ldots,N$. Indeed, if $|D^iu_n|\to\infty$ in a measurable subset $E\subset\Omega$ then

$$\lim_{n\to\infty}\int_\Omega S_n(x)\,dx\geq\lim_{n\to\infty}\sum_{i=1}^{N}\int_E|D^iu_n|^{p_i}\left(\underline{\alpha}\nu-\frac{K_x}{N|D^iu_n|^{p_i}}-\frac{K_x\nu^{\frac{1}{p_i'}}}{|D^iu_n|}-\frac{K_x\nu^{\frac{1}{p_i}}}{|D^iu_n|^{p_i-1}}\right)dx=\infty,$$

which is a contradiction since $S_n\to 0$ in $L^1(\Omega)$.

Let ξ_i^* an accumulation point of $(D^i u_n)_{n\in I\!N}$ for $i = 1, \ldots, N$. We have $|\xi_i^*| < \infty$ and by the continuity of $a(x, \cdot, \cdot)$, we obtain

$$\Big(a_i(x, u_n, \xi^*) - a_i(x, u_n, \nabla u)\Big)(\xi_i^* - D^i u) = 0 \qquad \text{for} \quad i = 1, \ldots, N.$$

Thanks to $(7.1.6)$, we have $\xi^* = \nabla u$ and the uniqueness of the accumulation point implies that $\nabla u_n \to \nabla u$ a.e in Ω.

Since $(a_i(x, u_n, \nabla u_n))_{n\in I\!N}$ is bounded in $L^{p_i'}(\Omega, \nu^*)$ and $a_i(x, u_n, \nabla u_n) \to a_i(x, u, \nabla u)$ a.e in Ω, by the Lemma 7.3.1, we can establish that

$$a_i(x, u_n, \nabla u_n) \rightharpoonup a_i(x, u, \nabla u) \quad in \quad L^{p_i'}(\Omega, \nu^*) \quad \text{for} \quad i = 1, \ldots, N.$$

Using $(7.4.4)$ and the Lemma 1.2.2, we deduce that

$$|u_n|^{p_0} \longrightarrow |u|^{p_0} \qquad \text{in} \quad L^1(\Omega) \tag{7.4.5}$$

and

$$a_i(x, u_n, \nabla u_n) D^i u_n \longrightarrow a_i(x, u, \nabla u) D^i u \qquad \text{in} \quad L^1(\Omega). \tag{7.4.6}$$

According to the condition $(7.1.5)$, we have

$$\alpha \sum_{i=1}^{N} \nu |D^i u_n|^{p_i} \;\le\; a_i(x, u_n, \nabla u_n) D^i u_n \quad \text{for} \quad i = 1, \ldots, N.$$

Let $y_n^i = \frac{1}{\alpha} a_i(x, u_n, \nabla u_n) D^i u_n$ and $y^i = \frac{1}{\alpha} a_i(x, u, \nabla u) D^i u$, in view of the Fatou's Lemma, we get

$$\int_\Omega 2y^i \, dx \le \liminf_{n\to\infty} \int_\Omega (y_n^i + y^i - \sum_{i=1}^{N} \nu |D^i u_n - D^i u|^{p_i}) dx,$$

which implies that $0 \le -\limsup_{n\to\infty} \int_\Omega \sum_{i=1}^{N} \nu |D^i u_n - D^i u|^{p_i} \, dx$ and since

$$0 \le \liminf_{n\to\infty} \int_\Omega \sum_{i=1}^{N} \nu |D^i u_n - D^i u|^{p_i} \, dx \le \limsup_{n\to\infty} \int_\Omega \sum_{i=1}^{N} \nu |D^i u_n - D^i u|^{p_i} \, dx \le 0,$$

it follows that

$$\int_\Omega \sum_{i=1}^{N} \nu |D^i u_n - D^i u|^{p_i} \, dx \longrightarrow 0 \text{ as } n \to \infty.$$

Consequently

$$D^i u_n \longrightarrow D^i u \quad \text{in} \quad L^{p_i}(\Omega, \nu) \quad \text{for} \quad i = 1, \ldots, N.$$

In view of $(7.4.5)$ we deduce that

$$u_n \longrightarrow u \qquad \text{in} \quad W_0^{1,(p_i)}(\Omega, \nu).$$

Proof of the Theorem 7.4.1

Step 1 : Approximate problems.

Let $(f_n)_{n\in\mathbb{N}}$ be a sequence of smooth functions such that $f_n \to f$ in $L^1(\Omega)$ and $|f_n| \le |f|$. We consider the approximate problem

$$A_n u_n + |T_n(u_n)|^{s-1}T_n(u_n) = f_n + \rho\frac{|T_n(u_n)|^{p_0-2}T_n(u_n)}{|x|^{p_0}+\frac{1}{n}}, \tag{7.4.7}$$

where $A_n v = -\sum_{i=1}^{N}\frac{\partial}{\partial x_i}a_i(x,T_n(v),\nabla v) + \frac{1}{n}|v|^{p_0-2}v$.

We consider the operator $G_n : W_0^{1,(p_i)}(\Omega,\nu) \longmapsto W^{-1,,(p_i)'}(\Omega,\nu^*)$ by

$$\langle G_n u, v\rangle = \int_\Omega |T_n(u)|^{s-1}T_n(u)v\,dx - \rho\int_\Omega \frac{|T_n(u)|^{p_0-2}T_n(u)}{|x|^{p_0}+\frac{1}{n}}v\,dx \quad \text{for any} \quad u,v\in W_0^{1,(p_i)}(\Omega,\nu).$$

Thanks to the Hölder's inequality and by using $(7.2.1)$ and $(7.2.3)$, we have

$$\begin{aligned}
|\langle G_n u, v\rangle| &\le \int_\Omega |T_n(u)|^s\,|v|\,dx + \rho\int_\Omega \frac{|T_n(u)|^{p_0-1}}{|x|^{p_0}+\frac{1}{n}}|v|\,dx\\
&\le \left(\int_\Omega |T_n(u)|^{sp_0'}\,dx\right)^{\frac{1}{p_0'}}\|v\|_{p_0} + \rho\left(\int_\Omega\left(\frac{|T_n(u)|^{p_0-1}}{|x|^{p_0}+\frac{1}{n}}\,dx\right)^{p_0'}dx\right)^{\frac{1}{p_0'}}\|v\|_{p_0}\\
&\le (n^s+\rho n^{p_0})(meas(\Omega))^{\frac{1}{p_0'}}\|v\|_{p_0}\\
&\le C_0|||v|||_V\,.
\end{aligned} \tag{7.4.8}$$

Lemma 7.4.2. *The bounded operator $B_n = A_n + G_n$ acted from $W_0^{1,(p_i)}(\Omega,\nu)$ into $W^{-1,(p_i')}(\Omega,\nu^*)$ is pseudo-monotone. Moreover, B_n is coercive in the following sense:*

$$\frac{\langle B_n v, v\rangle}{|||v|||_V} \longrightarrow +\infty \qquad \textit{as} \quad |||v|||_V\to\infty \quad \textit{for} \quad v\in W_0^{1,(p_i)}(\Omega,\nu).$$

Proof of the Lemma 7.4.2

In view of the Hölder's inequality and the growth condition $(7.1.4)$, we have

$$\begin{aligned}
|\langle A_n u, v\rangle| &= \left|\int_\Omega a_i(x,T_n(u),\nabla u)\,D^i v\,dx\,dx + \frac{1}{n}\int_\Omega |u|^{p_0-2}uv\,dx\right|\\
&\le \sum_{i=1}^{N}\left(\int_\Omega |a_i(x,T_n(u),\nabla u)|^{p_i'}\nu^{1-p_i'}\,dx\right)^{\frac{1}{p_i'}}\|\nu^{\frac{1}{p_i}}D^i v\|_{L^{p_i}(\Omega)} + \frac{1}{n}\left(\int_\Omega |u|^{(p_0-1)p_0'}\,dx\right)^{\frac{1}{p_0'}}\|v\|_{p_0}\\
&\le \beta\sum_{i=1}^{N}\left(\int_\Omega \left(R_i^{p_i'}+|T_n(u)|^{p_i}+\sum_{i=1}^{N}\nu|D^i u|^{p_i}\right)\right)^{\frac{1}{p_i'}}\|D^i v\|_{L^{p_i}(\Omega,\nu)} + \frac{1}{n}\left(\int_\Omega |u|^{p_0}\,dx\right)^{\frac{1}{p_0'}}\|v\|_{p_0}\\
&\le \left(C_1+C_2+C_3\left(\int_\Omega\sum_{i=1}^{N}\nu|D^i u|^{p_i}\right)^{\frac{1}{p_i'}} + \frac{1}{n}\|u\|_{1,(p_i),\nu}^{p_0-1}\right)\|v\|_{1,(p_i),\nu}\\
&\le C_4|||v|||_V\,.
\end{aligned}$$

then,the operator A_n is bounded, and by $(7.4.8)$ we conclude that B_n is bounded.

For the coercivity, we have for any $u \in W_0^{1,(p_i)}(\Omega,\nu)$, and thanks to $(7.4.8)$ and $(7.1.5)$

$$\begin{aligned}
\langle B_n u, u\rangle &= \langle A_n u, u\rangle + \langle G_n u, u\rangle \\
&= \sum_{i=1}^{N} \int_\Omega a_i(x, T_n(u), \nabla u)\, D^i u \, dx + \int_\Omega |T_n(u)|^s |u| \, dx + \frac{1}{n}\int_\Omega |u|^{p_0}\, dx \\
&\quad -\rho \int_\Omega \frac{|T_n(u)|^{p_0-1}}{|x|^{p_0}+\frac{1}{n}} |u| \, dx \\
&\geq \underline{\alpha} \sum_{i=0}^{N} \int_\Omega \nu |D^i u|^{p_i} \, dx - C_0 |||u|||_V \\
&\geq \underline{\alpha} |||u|||_V^{\underline{p}} - C_0 |||u|||_V
\end{aligned}$$

with $\underline{\alpha} = \min(\alpha, \frac{1}{n})$, it follows that

$$\frac{\langle B_n u, u\rangle}{|||u|||_V} \longrightarrow +\infty \quad \text{as} \quad |||u|||_V \to \infty.$$

It remains to show that B_n is pseudo-monotone. Let $(u_k)_{k\in I\!N}$ be a sequence in $W_0^{1,(p_i)}(\Omega,\nu)$ such that

$$\begin{cases} u_k \rightharpoonup u & \text{in } \ W_0^{1,(p_i)}(\Omega,\nu), \\ B_n u_k \rightharpoonup \chi_n & \text{in } \ W^{-1,(p_i')}(\Omega,\nu^*), \\ \limsup\limits_{k\to\infty} \langle B_n u_k, u_k\rangle \leq \langle \chi_n, u\rangle. \end{cases} \tag{7.4.9}$$

We will prove that

$$\chi_n = B_n u \quad \text{and} \quad \langle B_n u_k, u_k\rangle \longrightarrow \langle \chi_n, u\rangle \ \text{ as } \ k \to +\infty.$$

In view of the compact embedding $W_0^{1,(p_i)}(\Omega,\nu) \hookrightarrow\hookrightarrow L^{\underline{p}}(\Omega)$, we have $u_k \to u$ in $L^{\underline{p}}(\Omega)$ for a subsequence still denoted $(u_k)_{k\in I\!N}$.

As $(u_k)_{k\in I\!N}$ is a bounded sequence in $W_0^{1,(p_i)}(\Omega,\nu)$, using the growth condition $(7.1.4)$ it's clear that the sequence $(a_i(x, T_n(u_k), \nabla u_k))_{k\in I\!N}$ is bounded in $L^{p_i'}(\Omega,\nu)$, then there exists a function $\varphi_i^n \in L^{p_i}(\Omega,\nu)$ such that

$$a_i(x, T_n(u_k), \nabla u_k) \rightharpoonup \varphi_i^n \ \text{ in } \quad L^{p_i'}(\Omega,\nu^*) \ \text{ as } \ k \to \infty. \tag{7.4.10}$$

In view of Lebesgue's dominated convergence theorem, we obtain

$$|T_n(u_k)|^{s-1} T_n(u_k) \to |T_n(u)|^{s-1} T_n(u) \qquad \text{in} \quad L^{p_0'}(\Omega,\nu^*), \tag{7.4.11}$$

$$\frac{|T_n(u_k)|^{p_0-2} T_n(u_k)}{|x|^{p_0}+\frac{1}{n}} \to \frac{|T_n(u)|^{p_0-2} T_n(u)}{|x|^{p_0}+\frac{1}{n}} \ \text{ in } \quad L^{p_0'}(\Omega,\nu^*). \tag{7.4.12}$$

Also, we have

$$\frac{1}{n}|u_k|^{p_0-2}u_k \rightharpoonup \frac{1}{n}|u|^{p_0-2}u \qquad \text{in} \quad L^{p_0'}(\Omega,\nu^*). \tag{7.4.13}$$

For any $v \in W_0^{1,(p_i)}(\Omega,\nu)$, we have

$$\begin{aligned}
\langle \chi_n, v\rangle &= \lim_{k\to\infty}\langle B_n u_k, v\rangle \\
&= \lim_{k\to\infty}\sum_{i=1}^{N}\int_\Omega a_i(x,T_n(u_k),\nabla u_k)D^i v\,dx + \lim_{k\to\infty}\int_\Omega |T_n(u_k)|^{s-1}T_n(u_k)v\,dx \\
&\quad + \lim_{k\to\infty}\frac{1}{n}\int_\Omega |u_k|^{p_0-2}u_k\,v\,dx - \lim_{k\to\infty}\rho\int_\Omega \frac{|T_n(u_k)|^{p_0-2}T_n(u_k)}{|x|^{p_0}+\frac{1}{n}}v\,dx \\
&= \sum_{i=1}^{N}\int_\Omega \varphi_i^n\ D^i v\,dx + \int_\Omega |T_n(u)|^{s-1}T_n(u)v\,dx + \frac{1}{n}\int_\Omega |u|^{p_0-2}u\,v\,dx \\
&\quad -\rho\int_\Omega \frac{|T_n(u)|^{p_0-2}T_n(u)}{|x|^{p_0}+\frac{1}{n}}v\,dx.
\end{aligned} \tag{7.4.14}$$

Having in mind (7.4.9) and (7.4.14), we obtain

$$\begin{aligned}
\limsup_{k\to\infty}\langle B_n(u_k),u_k\rangle &= \limsup_{k\to\infty}\Big\{\sum_{i=1}^{N}\int_\Omega a_i(x,T_n(u_k),\nabla u_k)D^i u_k\,dx + \int_\Omega |T_n(u_k)|^{s-1}T_n(u_k)u_k\,dx \\
&\quad +\frac{1}{n}\int_\Omega |u_k|^{p_0}\,dx - \rho\int_\Omega \frac{|T_n(u_k)|^{p_0-2}T_n(u_k)}{|x|^{p_0}+\frac{1}{n}}u_k\,dx\Big\} \\
&\le \sum_{i=1}^{N}\int_\Omega \varphi_i^n\ D^i u\,dx + \int_\Omega |T_n(u)|^{s-1}T_n(u)u\,dx + \frac{1}{n}\int_\Omega |u|^{p_0}\,dx \\
&\quad -\rho\int_\Omega \frac{|T_n(u)|^{p_0-2}T_n(u)}{|x|^{p_0}+\frac{1}{n}}u\,dx.
\end{aligned} \tag{7.4.15}$$

Thanks to (7.4.11) and (7.4.12), we have

$$\int_\Omega |T_n(u_k)|^{s-1}T_n(u_k)u_k\,dx \longrightarrow \int_\Omega |T_n(u)|^{s-1}T_n(u)u\,dx, \tag{7.4.16}$$

and

$$\int_\Omega \frac{|T_n(u_k)|^{p_0-2}T_n(u_k)}{|x|^{p_0}+\frac{1}{n}}u_k\,dx \longrightarrow \int_\Omega \frac{|T_n(u)|^{p_0-2}T_n(u)}{|x|^{p_0}+\frac{1}{n}}u\,dx. \tag{7.4.17}$$

Therefore

$$\limsup_{k\to\infty}\Big(\sum_{i=1}^{N}\int_\Omega a_i(x,T_n(u_k),\nabla u_k)D^i u_k\,dx + \frac{1}{n}\int_\Omega |u_k|^{p_0}\,dx\Big) \le \sum_{i=1}^{N}\int_\Omega \varphi_i\ D^i u\,dx + \frac{1}{n}\int_\Omega |u|^{p_0}\,dx. \tag{7.4.18}$$

On the other hand, in view of (7.1.6) we have

$$\begin{aligned}
&\sum_{i=1}^{N}\int_\Omega (a_i(x,T_n(u_k),\nabla u_k) - a_i(x,T_n(u_k),\nabla u))(D^i u_k - D^i u)\,dx \\
&\qquad +\frac{1}{n}\int_\Omega (|u_k|^{p_0-2}u_k - |u|^{p_0-2}u)(u_k-u)\,dx \ge 0,
\end{aligned} \tag{7.4.19}$$

then

$$\begin{aligned}
&\sum_{i=1}^{N}\int_{\Omega} a_i(x,T_n(u_k),\nabla u_k)D^i u_k\,dx+\frac{1}{n}\int_{\Omega}|u_k|^{p_0}\,dx\\
&\geq \sum_{i=1}^{N}\int_{\Omega} a_i(x,T_n(u_k),\nabla u_k)D^i u\,dx+\frac{1}{n}\int_{\Omega}|u_k|^{p_0-2}u_k u\,dx\\
&\quad+\sum_{i=1}^{N}\int_{\Omega} a_i(x,T_n(u_k),\nabla u)(D^i u_k-D^i u)\,dx+\frac{1}{n}\int_{\Omega}|u|^{p_0-2}u(u_k-u)\,dx.
\end{aligned}$$

In view of Lebesgue's dominated convergence theorem we have $T_n(u_k)\to T_n(u)$ in $L^{p_i}(\Omega,\nu)$, thus $a_i(x,T_n(u_k),\nabla u)\to a_i(x,T_n(u),\nabla u)$ in $L^{p_i'}(\Omega,\nu^*)$, and using $(7.4.12)$ we get

$$\liminf_{k\to\infty}\Big(\sum_{i=1}^{N}\int_{\Omega} a_i(x,T_n(u_k),\nabla u_k)D^i u_k\,dx+\frac{1}{n}\int_{\Omega}|u_k|^{p_0}\,dx\Big)\geq\sum_{i=1}^{N}\int_{\Omega}\varphi_i\,D^i u\,dx+\frac{1}{n}\int_{\Omega}|u|^{p_0}\,dx.$$

Having in mind $(7.4.18)$, we conclude that

$$\lim_{k\to\infty}\Big(\sum_{i=1}^{N}\int_{\Omega} a_i(x,T_n(u_k),\nabla u_k)D^i u_k dx+\frac{1}{n}\int_{\Omega}|u_k|^{p_0}dx\Big)=\sum_{i=1}^{N}\int_{\Omega}\varphi_i\,D^i u dx+\frac{1}{n}\int_{\Omega}|u|^{p_0}dx.\quad(7.4.20)$$

Therefore, by combining $(7.4.14)$ and $(7.4.16)-(7.4.17)$, we obtain

$$\langle B_n u_k,u_k\rangle\longrightarrow\langle\chi_n,u\rangle\quad\text{as}\quad k\to\infty.$$

Now, by (7.4.20) we can prove that

$$\begin{aligned}
\lim_{k\to+\infty}\Big(\sum_{i=1}^{N}\int_{\Omega}&(a_i(x,T_n(u_k),\nabla u_k)-a_i(x,T_n(u_k),\nabla u))(D^i u_k-D^i u)\,dx\\
&+\frac{1}{n}\int_{\Omega}(|u_k|^{p_0-2}u_k-|u|^{p_0-2}u)(u_k-u)\,dx\Big)=0.
\end{aligned}$$

and so, by virtue of Lemma 7.4.1, we get

$$u_k\to u\quad\text{in}\quad W_0^{1,(p_i)}(\Omega,\nu)\qquad\text{and}\qquad D^i u_k\to D^i u\quad\text{a.e. in}\quad\Omega,$$

then

$$a_i(x,T_n(u_k),\nabla u_k)\rightharpoonup a_i(x,T_n(u),\nabla u)\quad\text{in}\quad L^{p_i'}(\Omega,\nu^*)\quad\text{for}\quad i=1,\dots,N$$

and thanks to $(7.4.11)-(7.4.13)$, we obtain $\chi_n=B_n u$, which conclude the proof of Lemma 7.4.2.

In view of Lemma 7.4.2, there exists at least one weak solution $u_n\in W_0^{1,(p_i)}(\Omega,\nu)$ of the problem $(7.4.7)$ (cf. [121], Theorem 8.2).

Step 2 : A priori estimates.

Lemma 7.4.3. *Let u_n be a weak solution of the approximate problem $(7.4.7)$, then the following regularity results hold true*

$$u \in W_0^{1,(m_i)}(\Omega,\nu) \qquad \textit{with} \quad (m_i)=(s,m_1,\dots,m_N) \tag{7.4.21}$$

such that $s > \dfrac{N(p_0-1)}{N-p_0}$, $1 \le m_i < \dfrac{p_i s}{s+p_i}$ *and* $\nu^{-\frac{p_i}{1-p_i}} \in L^1(\Omega)$,

$$\sum_{i=1}^{N} \int_\Omega \frac{\nu |D^i u_n|^{p_i}}{(1+|u_n|)^\kappa} dx \le C \qquad \textit{for all} \quad 1<\kappa<\frac{s(p_i-m_i)}{m_i}, \tag{7.4.22}$$

$$\sum_{i=1}^{N} \int_\Omega \nu |D^i T_k(u_n)|^{p_i} dx \le C(1+k)^\kappa \qquad \textit{for all} \quad k>0, \tag{7.4.23}$$

with C is a positive constant that don't depend on k and n.

Proof of Lemma 7.4.3

In this step, we will exploit some techniques of [155]. Let $\kappa > 1$ which will be chosen later, we consider the function $\psi(t) : I\!R \longmapsto I\!R$ defined by

$$\psi(t) = \Big(1 - \frac{1}{(1+|t|)^{\kappa-1}}\Big) sign(t)$$

It's clear that $\psi(u_n) \in W_0^{1,(p_i)}(\Omega,\nu)$. By taking $\psi(u_n)$ as test function in $(7.4.7)$ we get

$$\begin{aligned}(\kappa-1)\sum_{i=1}^{N}\int_\Omega \frac{a_i(x,T_n(u_n),\nabla u_n)\cdot D^i u_n}{(1+|u_n|)^\kappa}\,dx + \int_\Omega |T_n(u_n)|^{s-1}T_n(u_n)\psi(u_n)\,dx \\ +\frac{1}{n}\int_\Omega |u_n|^{p_0-2}u_n\psi(u_n)\,dx = \int_\Omega f_n\psi(u_n)\,dx + \rho\int_\Omega \frac{|T_n(u_n)|^{p_0-2}T_n(u_n)}{|x|^{p_0}+\frac{1}{n}}\psi(u_n)\,dx.\end{aligned}$$

Since $\psi(u_n)$ have the same sign of u_n, thus the third term on the left-hand side of the previews inequality is positive. Also, we have $|\psi(\cdot)| \le 1$ and in view of $(7.1.5)$, we obtain

$$(\kappa-1)\sum_{i=1}^{N}\int_\Omega \frac{\nu|D^i u_n|^{p_i}}{(1+|u_n|)^\kappa}dx + \int_\Omega |T_n(u_n)|^s|\psi(u_n)|\,dx \le \rho\int_\Omega \frac{|T_n(u_n)|^{p_0-1}}{|x|^{p_0}+\frac{1}{n}}\,dx + \int_\Omega |f|\,dx. \tag{7.4.24}$$

It is clear that

$$\frac{1}{2} \le 1 - \frac{1}{(1+|u_n|)^{\kappa-1}} \quad \text{for} \quad |u_n| \ge R = \max(2^{\frac{1}{\kappa-1}}-1, 1).$$

Thus, we have

$$\begin{aligned}\frac{1}{2}\int_{\{|u_n|\ge R\}} |T_n(u_n)|^s\,dx &\le \int_{\{|u_n|\ge R\}} |T_n(u_n)|^s\Big(1-\frac{1}{(1+|u_n|)^{\kappa-1}}\Big)\,dx \\ &\le \int_\Omega |T_n(u_n)|^s\Big(1-\frac{1}{(1+|u_n|)^{\kappa-1}}\Big)\,dx,\end{aligned}$$

which implies

$$\begin{aligned}\frac{1}{2}\int_\Omega |T_n(u_n)|^s\,dx &= \frac{1}{2}\int_{\{|u_n|<R\}} |T_n(u_n)|^s\,dx + \frac{1}{2}\int_{\{|u_n|\geq R\}} |T_n(u_n)|^s\,dx\\ &\leq \frac{1}{2}R^s|\Omega| + \int_\Omega |T_n(u_n)|^s\bigg(1-\frac{1}{(1+|u_n|)^{\kappa-1}}\bigg)\,dx.\end{aligned}$$

Using $(7.4.24)$, we deduce that

$$(\kappa-1)\sum_{i=1}^N\int_\Omega \frac{\nu|D^iu_n|^{p_i}}{(1+|u_n|)^\kappa}dx+\frac{1}{2}\int_\Omega |T_n(u_n)|^s dx \leq \frac{1}{2}R^s|\Omega|+\rho\int_\Omega \frac{|T_n(u_n)|^{p_0-1}}{|x|^{p_0}}dx+\int_\Omega |f|dx. \quad (7.4.25)$$

We have $s>p_0-1$, in view of Young's inequality we obtain

$$\rho\int_\Omega \frac{|T_n(u_n)|^{p_0-1}}{|x|^{p_0}}\,dx \leq \frac{1}{4}\int_\Omega |T_n(u_n)|^s\,dx + C_5\int_\Omega \frac{dx}{|x|^{\frac{sp_0}{s-p_0+1}}},$$

with C_5 is a positive constant depending only on s, p_0 and ρ. Thus, we obtain

$$(\kappa-1)\sum_{i=1}^N\int_\Omega \frac{\nu|D^iu_n|^{p_i}}{(1+|u_n|)^\kappa}\,dx+\frac{1}{4}\int_\Omega |T_n(u_n)|^s\,dx \leq \frac{1}{2}R^s|\Omega|+C_5\int_\Omega \frac{dx}{|x|^{\frac{sp_0}{s-p_0+1}}}+\int_\Omega |f|\,dx. \quad (7.4.26)$$

Under the assumption $s>\dfrac{N(p_0-1)}{N-p_0}$, the integral $\displaystyle\int_\Omega \frac{dx}{|x|^{\frac{sp_0}{s-p_0+1}}}$ is finite. Therefore $(7.4.22)$ is deduced. Moreover, we have

$$\int_\Omega |T_n(u_n)|^s\,dx \leq C. \quad (7.4.27)$$

Taking m_i such that $1\leq m_i<p_i$ for $i=1,\dots,N$. By virtue of the Hölder's inequality we get

$$\begin{aligned}\sum_{i=1}^N\int_\Omega \nu|D^iu_n|^{m_i}\,dx &\leq \sum_{i=1}^N \bigg\|\frac{\nu^{\frac{m_i}{p_i}}|D^iu_n|^{m_i}}{(1+|u_n|)^{\frac{\kappa m_i}{p_i}}}\bigg\|_{\frac{p_i}{m_i}}\bigg\|\nu^{1-\frac{m_i}{p_i}}(1+|u_n|)^{\frac{\kappa m_i}{p_i}}\bigg\|_{\frac{p_i}{p_i-m_i}}\\ &\leq \sum_{i=1}^N\Big(\int_\Omega \frac{\nu|D^iu_n|^{p_i}}{(1+|u_n|)^\kappa}\,dx\Big)^{\frac{m_i}{p_i}}\Big(\int_\Omega \nu(1+|u_n|)^{\frac{m_i\kappa}{p_i-m_i}}\,dx\Big)^{1-\frac{m_i}{p_i}},\end{aligned}$$

Hence by applying Hölder's inequality a second time, and in view of $(7.2.2)$ we obtain

$$\begin{aligned}\sum_{i=1}^N\int_\Omega \nu|D^iu_n|^{m_i}\,dx &\leq \sum_{i=1}^N\Big(\int_\Omega \frac{\nu|D^iu_n|^{p_i}}{(1+|u_n|)^\kappa}\,dx\Big)^{\frac{m_i}{p_i}}\|\nu\|_{p_i'}^{1-\frac{m_i}{p_i}}\Big\|(1+|u_n|)^{\frac{m_i\kappa}{p_i-m_i}}\Big\|_{p_i}^{1-\frac{m_i}{p_i}}\\ &\leq \sum_{i=1}^N\Big(\int_\Omega \frac{\nu|D^iu_n|^{p_i}}{(1+|u_n|)^\kappa}\,dx\Big)^{\frac{m_i}{p_i}}\\ &\quad\times\bigg(\int_\Omega \nu^{-\frac{p_i}{1-p_i}}\,dx\bigg)^{(1-\frac{1}{p_i})(1-\frac{m_i}{p_i})}\bigg(\int_\Omega (1+|u_n|)^{\frac{m_i\kappa p_i}{(p_i-m_i)}}\,dx\bigg)^{\frac{1}{p_i}(1-\frac{m_i}{p_i})}\\ &\leq C_6\sum_{i=1}^N\Big(\int_\Omega \frac{\nu|D^iu_n|^{p_i}}{(1+|u_n|)^\kappa}\,dx\Big)^{\frac{m_i}{p_i}}\bigg(\int_\Omega (1+|u_n|)^{\frac{m_i\kappa p_i}{p_i-m_i}}\,dx\bigg)^{\frac{1}{p_i}(1-\frac{m_i}{p_i})}\end{aligned} \quad (7.4.28)$$

We now choose $\kappa > 1$ such that $\dfrac{m_i \kappa p_i}{p_i - m_i} < s$, such a real number κ exists if

$$1 < \frac{s(p_i - m_i)}{m_i p_i} \qquad \text{that is} \qquad m_i < \frac{p_i s}{s + p_i}.$$

Combining $(7.4.26) - (7.4.28)$, we obtain the desired estimates $(7.4.21)$.

To get $(7.4.23)$, we can write

$$\sum_{i=1}^{N} \int_{\Omega} \nu |D^i T_k(u_n)|^{p_i} \, dx = \sum_{i=1}^{N} \int_{\{|u_n|<k\}} \nu |D^i u_n|^{p_i} \, dx \leq (1+k)^{\kappa} \sum_{i=1}^{N} \int_{\Omega} \frac{\nu |D^i u_n|^{p_i}}{(1+|u_n|)^{\kappa}} \, dx.$$

Step 3 : The weak convergence of $(T_k(u_n))_n$ in $W_0^{1,(p_i)}(\Omega, \nu)$.

To show the weak convergence of $(T_k(u_n))_n$ in $W_0^{1,(p_i)}(\Omega, \nu)$, we begin by proving that $(u_n)_n$ is a Cauchy sequence. Indeed, thanks to $(7.4.23)$, we can obtain

$$\sum_{i=0}^{N} \int_{\Omega} |D^i T_k(u_n)|^{p_i} \nu \, dx \leq C(1+k)^{\kappa} + k^{p_0} |\Omega| \qquad \text{for} \quad k \geq 1,$$

Therefore, the sequence $(T_k(u_n))_n$ is bounded in $W_0^{1,(p_i)}(\Omega, \nu)$, and there exists a subsequence still denoted $(T_k(u_n))_n$ such that

$$\begin{cases} T_k(u_n) \rightharpoonup \zeta_k & \text{in} \quad W_0^{1,(p_i)}(\Omega, \nu), \\ T_k(u_n) \to \zeta_k & \text{in} \quad L^{\underline{p}}(\Omega) \quad \text{and a.e. in} \quad \Omega. \end{cases} \tag{7.4.29}$$

Thanks to $(7.4.23)$, we deduce that there exists a constant C_7 that does not depend on k and n, such that

$$\|\nabla T_k(u_n)\|_{L^{\underline{p}}(\Omega,\nu)} \leq C_7 k^{\frac{\kappa}{\underline{p}}} \qquad \text{for} \quad k \geq 1. \tag{7.4.30}$$

Let k large enough and B_R be a ball of Ω. By using $(7.4.30)$ and applying the Poincaré type inequality and Proposition $7.3.1,$ we obtain

$$\begin{aligned} k \, \text{meas}\Big(\{|u_n| > k\} \cap B_R \Big) &= \int_{\{|u_n|>k\} \cap B_R} |T_k(u_n)| \, dx \\ &\leq C_9 \|\nabla T_k(u_n)\|_{L^{\underline{p}}(\Omega,\nu)} \\ &\leq C_{10} k^{\frac{\kappa}{\underline{p}}}, \end{aligned} \tag{7.4.31}$$

Choosing κ small enough $(1 < \kappa < \underline{p})$, we conclude that

$$\text{meas}\Big(\{|u_n| > k\} \cap B_R \Big) \leq C_{10} \frac{1}{k^{1 - \frac{\kappa}{\underline{p}}}} \longrightarrow 0 \quad \text{as} \quad k \to +\infty. \tag{7.4.32}$$

For all $\delta > 0$, we have

$$\begin{aligned}\text{meas}\Big(\{|u_n - u_m| \cap B_R > \delta\}\Big) \quad &\leq \text{meas}\Big(\{|u_n| > k\} \cap B_R\Big) + \text{meas}\Big(\{|u_m| > k\} \cap B_R\Big) \\ &+\text{meas}\Big(\{|T_k(u_n) - T_k(u_m)| > \delta\}\Big).\end{aligned}$$

Let $\varepsilon > 0$, using (7.4.32) we can choose $k = k(\varepsilon)$ large enough such that

$$\text{meas}\Big(\{|u_n| > k\} \cap B_R\Big) \leq \frac{\varepsilon}{3} \qquad \text{and} \qquad \text{meas}\Big(\{|u_m| > k\} \cap B_R\Big) \leq \frac{\varepsilon}{3}. \tag{7.4.33}$$

On the other hand, thanks to $(7.4.29)$ we can assume that $(T_k(u_n))_{n\in I\!N}$ is a Cauchy sequence in measure. Thus, for any $k > 0$ and $\delta, \varepsilon > 0$, there exists $n_0 = n_0(k, \delta, \varepsilon)$ such that

$$\text{meas}\{|T_k(u_n) - T_k(u_m)| > \delta\} \leq \frac{\varepsilon}{3} \qquad \text{for all } m, n \geq n_0(k, \delta, \varepsilon). \tag{7.4.34}$$

In view of $(7.4.33)$ and $(7.4.34)$, we deduce that

$$\begin{aligned}&\forall \delta, \varepsilon > 0 \quad \text{there exists} \quad n_0 = n_0(k(\varepsilon), \delta, R) \quad \text{such that} \\ &\qquad \text{meas}\Big(\{|u_n - u_m| \cap B_R > \delta\}\Big) \leq \varepsilon \quad \forall n, m \geq n_0(k(\varepsilon), \delta, R),\end{aligned}$$

which proves that the sequence $(u_n)_n$ is a Cauchy sequence in measure and then converges almost everywhere to some measurable function u. Consequently, we have

$$T_k(u_n) \rightharpoonup T_k(u) \quad \text{in} \quad W_0^{1,(p_i)}(\Omega, \nu), \tag{7.4.35}$$

and in view Lebesgue's dominated convergence theorem, we obtain

$$T_k(u_n) \longrightarrow T_k(u) \quad \text{in} \quad L^{p_0}(\Omega, \nu) \quad \text{and a.e in} \quad \Omega. \tag{7.4.36}$$

Step 4 : Strong convergence of truncations.

In the sequel, we denote by $\varepsilon_i(n)$, $i = 1, 2, \ldots,$ various real-valued functions of real variables that converge to 0 as n tends to infinity.

Let $h > k > 0$, taking $z_n := u_n - T_h(u_n) + T_k(u_n) - T_k(u)$ and $\omega_n := T_{2k}(z_n)$. By using ω_n as a test function in the approximate problem $(7.4.7)$ we obtain

$$\begin{aligned}&\sum_{i=1}^{N} \int_\Omega a_i(x, T_n(u_n), \nabla u_n) D^i \omega_n \, dx + \int_\Omega |T_n(u_n)|^{s-1} T_n(u_n) \omega_n \, dx \\ &+ \frac{1}{n} \int_\Omega |u_n|^{p_0 - 2} u_n \omega_n \, dx = \rho \int_\Omega \frac{|T_n(u_n)|^{p_0-2} T_n(u_n)}{|x|^{p_0} + \frac{1}{n}} \omega_n \, dx + \int_\Omega f_n \omega_n \, dx.\end{aligned}$$

For $M = 4k + h$, it's clear that $D^i\omega_n = 0$ on the set $\{|u_n| \geq M\}$, and ω_n have the same sign as u_n on the set $\{|u_n| > k\}$, (indeed, if $u_n > k$ then $u_n - T_h(u_n) \geq 0$ and $T_k(u_n) - T_k(u) \geq 0$, it follows that $\omega_n \geq 0$. Similarly, we show that $\omega_n \leq 0$ on the set $\{u_n < -k\}$) we get

$$\sum_{i=1}^{N} \int_{\{|u_n|\leq M\}} a_i(x, T_M(u_n), \nabla T_M(u_n)) D^i\omega_n \, dx + \int_\Omega |T_n(u_n)|^{s-1} T_n(u_n)\omega_n \, dx$$
$$+\frac{1}{n} \int_{\{|u_n|\leq k\}} |u_n|^{p_0-2} u_n\omega_n \, dx = \rho \int_\Omega \frac{|T_n(u_n)|^{p_0-2} T_n(u_n)}{|x|^{p_0} + \frac{1}{n}} \omega_n \, dx + \int_\Omega f_n\omega_n \, dx.$$

In view of Young's inequality, we have

$$\rho \int_{\{|u_n|>k\}} \frac{|T_n(u_n)|^{p_0-1}}{|x|^{p_0} + \frac{1}{n}} |\omega_n| \, dx \leq \int_{\{|u_n|>k\}} |T_n(u_n)|^s |\omega_n| \, dx + C_{11} \int_{\{u_n>k\}} \frac{|\omega_n|}{|x|^{\frac{sp_0}{s-p_0+1}}} \, dx,$$

and since $\omega_n = T_k(u_n) - T_k(u)$ on the set $\{|u_n| \leq k\}$, then

$$\begin{aligned} &\sum_{i=1}^{N} \int_{\{|u_n|\leq M\}} a_i(x, T_M(u_n), \nabla T_M(u_n)) D^i\omega_n \, dx \\ &\quad + \int_{\{|u_n|\leq k\}} |T_k(u_n)|^{s-1} T_k(u_n)(T_k(u_n) - T_k(u)) \, dx \\ &\quad + \frac{1}{n} \int_{\{|u_n|\leq k\}} |T_k(u_n)|^{p_0-2} T_k(u_n)(T_k(u_n) - T_k(u)) \, dx \qquad (7.4.37) \\ &\leq \rho \int_{\{|u_n|\leq k\}} \frac{|T_k(u_n)|^{p_0-2} T_k(u_n)}{|x|^{p_0} + \frac{1}{n}} (T_k(u_n) - T_k(u)) \, dx \\ &\quad + \int_\Omega f_n\omega_n \, dx + C_{11} \int_{\{u_n>k\}} \frac{|\omega_n|}{|x|^{\frac{sp_0}{s-p_0+1}}} \, dx. \end{aligned}$$

Now, we will study each terms in the previews inequality.

For the second and third terms on the left-hand side of $(7.4.37)$. In view of Lebesgue's dominated convergence theorem, we have

$$|T_k(u_n)|^{s-1} T_k(u_n) \longrightarrow |T_k(u)|^{s-1} T_k(u) \qquad \text{in} \quad L^1(\Omega),$$

and

$$|T_k(u_n)|^{p_0-2} T_k(u_n) \longrightarrow |T_k(u)|^{p_0-2} T_k(u) \qquad \text{in} \quad L^1(\Omega),$$

and since $T_k(u_n) \rightharpoonup T_k(u)$ weak$-*$ in $L^\infty(\Omega)$, then

$$\varepsilon_1(n) = \int_{\{|u_n|\leq k\}} |T_k(u_n)|^{s-1} T_k(u_n)(T_k(u_n) - T_k(u)) \, dx \longrightarrow 0 \qquad \text{as} \quad n \to \infty, \qquad (7.4.38)$$

and

$$\varepsilon_2(n) = \frac{1}{n} \int_{\{|u_n|\leq k\}} |T_k(u_n)|^{p_0-2} T_k(u_n)(T_k(u_n) - T_k(u)) \, dx \longrightarrow 0 \qquad \text{as} \quad n \to \infty. \qquad (7.4.39)$$

Concerning the terms on the right-hand side of $(7.4.37)$, we have

$$\begin{aligned}\varepsilon_3(n) &= \left| \int_{\{|u_n|\le k\}} \frac{|T_k(u_n)|^{p_0-2}T_k(u_n)}{|x|^{p_0}+\frac{1}{n}}(T_k(u_n)-T_k(u))\,dx \right| \\ &\le k^{p_0-1}\int_{\{|u_n|\le k\}} \frac{|T_k(u_n)-T_k(u)|}{|x|^{p_0}}\,dx \longrightarrow 0 \qquad \text{as} \quad n\to\infty,\end{aligned} \tag{7.4.40}$$

also, we have

$$\int_\Omega f_n\omega_n\,dx = \int_\Omega f\ T_{2k}(u-T_h(u))\,dx + \varepsilon_4(n), \tag{7.4.41}$$

and

$$\int_{\{u_n>k\}} \frac{|\omega_n|}{|x|^{\frac{sp_0}{s-p_0+1}}}\,dx = \int_{\{|u|>k\}} \frac{|T_{2k}(u-T_h(u))|}{|x|^{\frac{sp_0}{s-p_0+1}}}\,dx + \varepsilon_5(n). \tag{7.4.42}$$

By combining $(7.4.37)-(7.4.42)$, we deduce that

$$\begin{aligned}&\sum_{i=1}^N \int_{\{|u_n|\le M\}} a_i(x,T_M(u_n),\nabla T_M(u_n))D^i\omega_n\,dx \\ &\le \int_\Omega f\ T_{2k}(u-T_h(u))\,dx + C_{12}\int_{\{|u|>k\}} \frac{|T_{2k}(u-T_h(u))|}{|x|^{\frac{sp_0}{s-p_0+1}}}\,dx + \varepsilon_6(n).\end{aligned} \tag{7.4.43}$$

On the other hand, we have

$$\begin{aligned}&\sum_{i=1}^N \int_{\{|u_n|\le M\}} a_i(x,T_M(u_n),\nabla T_M(u_n))D^i\omega_n\,dx \\ &= \sum_{i=1}^N \int_\Omega \big(a_i(x,T_k(u_n),\nabla T_k(u_n)) - a_i(x,T_k(u_n),\nabla T_k(u))\big)(D^iT_k(u_n)-D^iT_k(u))\,dx \\ &\quad + \sum_{i=1}^N \int_\Omega a_i(x,T_k(u_n),\nabla T_k(u))(D^iT_k(u_n)-D^iT_k(u))\,dx \\ &\quad + \sum_{i=1}^N \int_{\{|u_n|>k\}} a_i(x,T_k(u_n),\nabla T_k(u_n))D^iT_k(u)\,dx \\ &\quad + \sum_{i=1}^N \int_{\{k<|u_n|\le M\}} a_i(x,T_M(u_n),\nabla T_M(u_n))D^i\omega_n\,dx.\end{aligned} \tag{7.4.44}$$

For the second and third terms on the right-hand side of $(7.4.44)$. Thanks to Lebesgue's dominated convergence theorem, we have $T_k(u_n)\to T_k(u)$ in $L^{p_i}(\Omega,\nu)$ then $a_i(x,T_k(u_n),\nabla T_k(u))\to a_i(x,T_k(u),\nabla T_k(u))$ in $L^{p_i'}(\Omega,\nu^*)$, and since $D^iT_k(u_n)\rightharpoonup D^iT_k(u)$ in $L^{p_i}(\Omega,\nu)$ it follows that

$$\varepsilon_7(n) = \int_\Omega a_i(x,T_k(u_n),\nabla T_k(u))(D^iT_k(u_n)-D^iT_k(u))\,dx \longrightarrow 0 \qquad \text{as} \quad n\to\infty, \tag{7.4.45}$$

and since $a(x,s,0)=0$, then

$$\int_{\{|u_n|>k\}} a_i(x,T_k(u_n),\nabla T_k(u_n))D^iT_k(u)\,dx = \int_{\{|u_n|>k\}} a_i(x,T_k(u_n),0)D^iT_k(u)\,dx = 0. \tag{7.4.46}$$

Concerning the last term on the right-hand side of $(7.4.44)$, thanks to $(7.1.4)$ we have $(a_i(x, T_M(u_n), \nabla T_M(u_n)))_n$ is bounded in $L^{p_i'}(\Omega, \nu^*)$, then there exists a function $\psi_i^n \in L^{p_i'}(\Omega, \nu^*)$ such that $|a_i(x, T_M(u_n), \nabla T_M(u_n))| \to \psi_i^n$ in $L^{p_i'}(\Omega, \nu^*)$, it follows that

$$\begin{aligned}
&\lim_{n\to\infty} \int_{\{k<|u_n|\le M\}} a_i(x, T_M(u_n), \nabla T_M(u_n)) D^i\omega_n \, dx \\
&= \lim_{n\to\infty} \int_{\{k<|u_n|\le M\}\cap\{|z_n|\le 2k\}} a_i(x, T_M(u_n), \nabla T_M(u_n))(D^i u_n - D^i T_h(u_n) - D^i T_k(u)) \, dx \\
&\ge -\lim_{n\to\infty} \int_{\{k<|u_n|\le M\}} |a_i(x, T_M(u_n), \nabla T_M(u_n))| \, |D^i T_k(u)| \, dx \\
&\ge -\int_{\{k<|u|\le M\}} \psi_i^n \, |D^i T_k(u)| \, dx = 0.
\end{aligned} \tag{7.4.47}$$

By combining $(7.4.43)$ and $(7.4.44) - (7.4.47)$, we get

$$\begin{aligned}
&\sum_{i=1}^{N} \int_\Omega \big(a_i(x, T_k(u_n), \nabla T_k(u_n)) - a_i(x, T_k(u_n), \nabla T_k(u))\big)(D^i T_k(u_n) - D^i T_k(u)) \, dx \\
&\le 2k \int_{\{|u|>h\}} |f| \, dx + 2kC_{13} \int_{\{|u|>h\}} \frac{1}{|x|^{\frac{p_0 s}{s-p_0+1}}} \, dx + \varepsilon_8(n).
\end{aligned} \tag{7.4.48}$$

By letting n then h tend to infinity in the inequality above, and thanks to $(7.4.36)$ we can obtain

$$\begin{aligned}
\lim_{n\to\infty} \Big(&\sum_{i=1}^{N} \int_\Omega \big(a_i(x, T_k(u_n), \nabla T_k(u_n)) - a_i(x, T_k(u_n), \nabla T_k(u))\big)(D^i T_k(u_n) - D^i T_k(u)) \, dx \\
&+ \int_\Omega \big(|T_k(u_n)|^{p_0-2} T_k(u_n) - |T_k(u_n)|^{p_0-2} T_k(u_n)\big)(T_k(u_n) - T_k(u)) \, dx\Big) = 0.
\end{aligned} \tag{7.4.49}$$

In view of Lemma $7.4.1$, we conclude that

$$\begin{cases} T_k(u_n) \longrightarrow T_k(u) & \text{strongly in} \quad W_0^{1,(p_i)}(\Omega, \nu), \\ D^i u_n \longrightarrow D^i u & \text{a.e. in} \quad \Omega \qquad \text{for } i = 1, \dots, N. \end{cases} \tag{7.4.50}$$

Step 5 : The equi-integrability of the nonlinear functions.

Now, we shall show that

$$|T_n(u_n)|^{s-1} T_n(u_n) \to |u|^{s-1} u \quad \text{in} \quad L^1(\Omega), \tag{7.4.51}$$

$$\frac{1}{n} |u_n|^{p_0-2} u_n \to 0 \quad \text{in} \quad L^1(\Omega), \tag{7.4.52}$$

and

$$\frac{|T_n(u_n)|^{p_0-2} T_n(u_n)}{|x|^{p_0} + \frac{1}{n}} \longrightarrow \frac{|u|^{p_0-2} u}{|x|^{p_0}} \quad \text{in} \quad L^1(\Omega). \tag{7.4.53}$$

First of all, we have

$$|T_n(u_n)|^{s-1} T_n(u_n) \to |u|^{s-1} u \quad \text{a.e. in} \quad \Omega, \tag{7.4.54}$$

$$\frac{1}{n}|u_n|^{p_0-2}u_n \to 0 \quad \text{a.e. in} \quad \Omega \tag{7.4.55}$$

and

$$\frac{|T_n(u_n)|^{p_0-2}T_n(u_n)}{|x|^{p_0}+\frac{1}{n}} \longrightarrow \frac{|u|^{p_0-2}u}{|x|^{p_0}} \quad \text{a.e. in} \quad \Omega. \tag{7.4.56}$$

Thereafter, according to Vitali's theorem, it suffices to prove the uniform equi-integrability of these functions. By using $T_1(G_l(u_n))$ as test function in $(7.4.7)$, where G_k is the truncation function at level $k>0$, defined by $G_k(s)=s-T_k(s)$ we can obtain

$$\begin{aligned}&\alpha\sum_{i=1}^{N}\int_{\{l<|u_n|\le l+1\}}\nu|D^iu_n|^{p_i}\,dx+\int_{\{|u_n|\ge l\}}|T_n(u_n)|^s|T_1(G_l(u_n))|\,dx\\&+\frac{1}{n}\int_{\{|u_n|\ge l+1\}}|u_n|^{p_0-1}\,dx\le\rho\int_{\{|u_n|\ge l\}}\frac{|T_n(u_n)|^{p_0-1}}{|x|^{p_0}+\frac{1}{n}}|T_1(G_l(u_n))|\,dx+\int_{\{|u_n|\ge l\}}|f_n|\,dx.\end{aligned}$$

Thanks to Young's inequality, we have

$$\begin{aligned}&\rho\int_{\{|u_n|\ge l\}}\frac{|T_n(u_n)|^{p_0-1}}{|x|^{p_0}+\frac{1}{n}}|T_1(G_l(u_n))|\,dx\\&\le\frac{1}{3}\int_{\{|u_n|\ge l\}}|T_n(u_n)|^s|T_1(G_l(u_n))|\,dx+C_{12}\int_{\{|u_n|\ge l\}}\frac{|T_1(G_l(u_n))|}{|x|^{\frac{sp_0}{s-p_0+1}}}\,dx,\end{aligned}$$

it follows that

$$\begin{aligned}&\frac{1}{3}\int_{\{|u_n|\ge l+1\}}|T_n(u_n)|^s\,dx+\rho\int_{\{|u_n|\ge l+1\}}\frac{|T_n(u_n)|^{p_0-1}}{|x|^{p_0}+\frac{1}{n}}\,dx+\frac{1}{n}\int_{\{|u_n|\ge l+1\}}|u_n|^{p_0-1}\,dx\\&\le 2C_{12}\int_{\{|u_n|\ge l\}}\frac{|T_1(G_l(u_n))|}{|x|^{\frac{sp_0}{s-p_0+1}}}\,dx+\int_{\{|u_n|\ge l\}}|f_n|\,dx.\end{aligned}$$

Thus, for any $\tau>0$, there exists $l(\tau)>0$ such that

$$\int_{\{|u_n|\ge l(\tau)\}}|T_n(u_n)|^s\,dx+\int_{\{|u_n|\ge l(\tau)\}}\frac{|T_n(u_n)|^{p_0-1}}{|x|^{p_0}+\frac{1}{n}}\,dx+\frac{1}{n}\int_{\{|u_n|\ge l(\tau)\}}|u_n|^{p_0-1}\,dx\le\frac{\tau}{2}. \tag{7.4.57}$$

On the other hand, for any measurable subset $E\subseteq\Omega$, we have

$$\begin{aligned}&\int_E|T_n(u_n)|^s\,dx+\int_E\frac{|T_n(u_n)|^{p_0-1}}{|x|^{p_0}+\frac{1}{n}}\,dx+\frac{1}{n}\int_E|u_n|^{p_0-1}\,dx\\&\le\int_E|T_{l(\tau)}(u_n)|^s\,dx+\int_E\frac{|T_{l(\tau)}(u_n)|^{p_0-1}}{|x|^{p_0}+\frac{1}{n}}\,dx+\frac{1}{n}\int_E|T_{l(\tau)}(u_n)|^{p_0-1}\,dx\\&\quad+\int_{\{|u_n|\ge l(\tau)\}}|T_n(u_n)|^s\,dx+\int_{\{|u_n|\ge l(\tau)\}}\frac{|T_n(u_n)|^{p_0-1}}{|x|^{p_0}+\frac{1}{n}}\,dx+\frac{1}{n}\int_{\{|u_n|\ge l(\tau)\}}|u_n|^{p_0-1}\,dx.\end{aligned} \tag{7.4.58}$$

From $(7.4.50)$, there exists $\gamma(\tau)>0$ such that : for any $E\subseteq\Omega$ with $\mathrm{meas}(E)\le\gamma(\tau)$

$$\int_E|T_{l(\tau)}(u_n)|^s\,dx+\int_E\frac{|T_{l(\tau)}(u_n)|^{p_0-1}}{|x|^{p_0}+\frac{1}{n}}\,dx+\frac{1}{n}\int_E|T_{l(\tau)}(u_n)|^{p_0-1}\,dx\le\frac{\tau}{2}. \tag{7.4.59}$$

Finally, by combining $(7.4.57)$, $(7.4.58)$ and $(7.4.59)$, one easily has

$$\int_E |T_n(u_n)|^s\,dx + \int_E \frac{|T_n(u_n)|^{p_0-1}}{|x|^{p_0}+\frac{1}{n}}\,dx + \frac{1}{n}\int_E |u_n|^{p_0-1}\,dx \le \tau \quad \text{with meas}(E) \le \gamma(\tau), \tag{7.4.60}$$

We deduce that $(|T_n(u_n)|^{s-1}T_n(u_n))_n$, $(|u_n|^{p_0-2}u_n)_n$ and $\Big(\frac{|T_n(u_n)|^{p_0-1}}{|x|^{p_0}+\frac{1}{n}}\Big)_n$ are equi-integrable. Hence by $(7.4.54)-(7.4.56)$ and Vitali's theorem, the convergence $(7.4.51)-(7.4.53)$ are fulfilled.

Step 6 : Passage to the limit.

Let $\varphi \in W_0^{1,(p_i)}(\Omega,\nu)\cap L^\infty(\Omega)$ and $M = k + \|\varphi\|_\infty$. By taking $T_k(u_n-\varphi)$ as a test function in $(7.4.7)$, we get

$$\begin{aligned}&\sum_{i=1}^N \int_\Omega a_i(x,T_n(u_n),\nabla u_n)D^iT_k(u_n-\varphi)\,dx + \int_\Omega |T_n(u_n)|^{s-1}T_n(u_n)T_k(u_n-\varphi)\,dx\\ &+\frac{1}{n}\int_\Omega |u_n|^{p_0-2}u_nT_k(u_n-\varphi)\,dx = \rho\int_\Omega \frac{|T_n(u_n)|^{p_0-2}T_n(u_n)}{|x|^{p_0}+\frac{1}{n}}T_k(u_n-\varphi)\,dx + \int_\Omega f_nT_k(u_n-\varphi)\,dx.\end{aligned} \tag{7.4.61}$$

On the one hand, we have $\{|u_n-\varphi|\le k\}\subseteq\{|u_n|\le M\}$ then

$$\begin{aligned}&\int_\Omega a_i(x,T_n(u_n),\nabla u_n)D^iT_k(u_n-\varphi)\,dx\\ &= \int_\Omega a_i(x,T_M(u_n),\nabla T_M(u_n))(D^iT_M(u_n)-D^i\varphi)\chi_{\{|u_n-\varphi|\le k\}}\,dx\\ &= \int_\Omega (a_i(x,T_M(u_n),\nabla T_M(u_n))-a_i(x,T_M(u_n),\nabla\varphi))(D^iT_M(u_n)-D^i\varphi)\chi_{\{|u_n-\varphi|\le k\}}\,dx\\ &\quad+\int_\Omega a_i(x,T_M(u_n),\nabla\varphi)(D^iT_M(u_n)-D^i\varphi)\chi_{\{|u_n-\varphi|\le k\}}\,dx.\end{aligned}$$

It's clear that

$$\begin{aligned}\lim_{n\to\infty}\int_\Omega a_i(x,T_M(u_n),\nabla\varphi)(D^iT_M(u_n)-D^i\varphi)\chi_{\{|u_n-\varphi|\le k\}}\,dx\\ = \int_\Omega a_i(x,T_M(u),\nabla\varphi)(D^iT_M(u)-D^i\varphi)\chi_{\{|u-\varphi|\le k\}}\,dx.\end{aligned}$$

According to Fatou's Lemma, we obtain

$$\begin{aligned}
&\liminf_{n\to\infty}\sum_{i=1}^{N}\int_{\Omega}a_i(x,T_n(u_n),\nabla u_n)D^iT_k(u_n-\varphi)\,dx\\
&\geq\sum_{i=1}^{N}\int_{\Omega}(a_i(x,T_M(u),\nabla T_M(u))-a_i(x,T_M(u),\nabla\varphi))(D^iT_M(u)-D^i\varphi)\chi_{\{|u-\varphi|\leq k\}}\,dx\\
&\quad+\sum_{i=1}^{N}\int_{\Omega}a_i(x,T_M(u),\nabla\varphi)(D^iT_M(u)-D^i\varphi)\chi_{\{|u-\varphi|\leq k\}}\,dx \qquad (7.4.62)\\
&=\sum_{i=1}^{N}\int_{\Omega}a_i(x,T_M(u),\nabla T_M(u))(D^iT_M(u)-D^i\varphi)\chi_{\{|u-\varphi|\leq k\}}\,dx\\
&=\sum_{i=1}^{N}\int_{\Omega}a_i(x,u,\nabla u)D^iT_k(u-\varphi)\,dx.
\end{aligned}$$

On the other hand, we have $T_k(u_n-\varphi)\rightharpoonup T_k(u-\varphi)$ weak$-*$ in $L^{\infty}(\Omega)$ and thanks to $(7.4.51)-(7.4.53)$, we deduce that

$$\int_{\Omega}|T_n(u_n)|^{s-1}T_n(u_n)T_k(u_n-\varphi)\,dx\longrightarrow\int_{\Omega}|u|^{s-1}uT_k(u-\varphi)\,dx, \qquad (7.4.63)$$

$$\frac{1}{n}\int_{\Omega}|u_n|^{p_0-1}u_nT_k(u_n-\varphi)\,dx\longrightarrow 0, \qquad (7.4.64)$$

$$\int_{\Omega}\frac{|T_n(u_n)|^{p_0-2}T_n(u_n)}{|x|^{p_0}+\frac{1}{n}}T_k(u_n-\varphi)\,dx\longrightarrow\int_{\Omega}\frac{|u|^{p_0-2}u}{|x|^{p_0}}T_k(u-\varphi)\,dx, \qquad (7.4.65)$$

and

$$\int_{\Omega}f_nT_k(u_n-\varphi)dx\longrightarrow\int_{\Omega}fT_k(u-\varphi)dx. \qquad (7.4.66)$$

Hence, putting all the terms together, we conclude the proof of Theorem 7.4.1.

7.5 Example

Accordingly to the assumption H_1, let us consider the special weight function $w_i(x)$:

$$w_0(x)=1\quad\text{and}\quad w_1(x)=w_2(x)=\ldots=w_N(x)=\nu(x)$$

and we take

$$a_i(x,s,\xi)=w_i(x)|\xi_i|^{p_i-1}sgn(\xi_i),\quad i=1,\ldots,N$$

It is easy to show that the $a_i(x,s,\xi)$ are carathéodory functions satisfying the growth condition $(7.1.4)$ and the coercivity $(7.1.5)$. Also, we can check the monotonicity condition $(7.1.6)$. In fact, we have

$$\sum_{i=1}^{N}(a_i(x,s,\xi)-a_i(x,s,\xi))(\xi_i-\xi_i')=\nu(x)\sum_{i=1}^{N}(|\xi_i|^{p_i-1}sgn(\xi_i)-|\xi_i'|^{p_i-1}sgn(\xi_i'))(\xi_i-\xi_i')>0$$

for almost all $x \in \Omega$ and for all $\xi_i, \xi_i' \in I\!R^N$ since the weight function ν is positive almost everywhere in Ω with $\xi_i \neq \xi_i'$. In particular, let us use the special weight function ν expressed in terms of the distance to the boundary $\partial\Omega$. Denote $d(x) = dist(x, \partial\Omega)$ and set $\nu(x) = d^{\mu}(x)$, such that

$$min\left(\frac{p_i}{N}, p_i - 1\right) > \mu, \quad \text{for} \quad i = 1, \ldots, N \tag{7.5.1}$$

The condition $(7.5.1)$ above, is sufficient for $(7.2.2)$ to hold (see [119], pp. 40-41). At the end, for integrable data f and $s > max\left(\frac{N(p_0-1)}{N-p_0}, \frac{1}{p_0-1}\right)$, the following anisotropic quasilinear degenerated elliptic problem

$$\begin{cases} u \in W_0^{1,(p_i)}(\Omega, \nu) \\ \displaystyle\int_\Omega \sum_{i=1}^{N} \nu(x) |D^i u|^{p_i - 1} sgn(D^i u) D^i \psi \, dx + \int_\Omega |u|^{s-1} u \psi \, dx \\ \qquad = \displaystyle\int_\Omega f\psi \, dx + \rho \int_\Omega \frac{|u|^{p_0-2}u}{|x|^{p_0}} \psi \, dx \\ \forall \psi \in W_0^{1,(p_i)}(\Omega, \nu) \cap L^{\infty}(\Omega) \end{cases}$$

has at least one entropy solution $u \in W_0^{1,(m_i)}(\Omega, \nu)$ with $(m_i) = (s, m_1, \ldots, m_N)$, $1 \leq m_i \leq \frac{p_i s}{s+p_i}$ and $\nu^{-\frac{p_i}{1-p_i}} \in L^1(\Omega)$ for $i = 1, \ldots, N$.

A general conclusion and perspectives

The works presented in this thesis concerns the investigation of quasilinear or strongly nonlinear anisotropic elliptic non-variational problems (ie $f \in L^1(\Omega)$), and a nonlinear anisotropic parabolic variational problem with variable exponents (supposing $f \in L^{p_i'(.)}(\Omega)$), which are motivated by the complexity of the studied phenomena where the modelization of several applications presents either an anisotropic character or a case where the exponent varies, or both at once and sometimes a situation of degeneration. This stands out clearly, for example through the field of image processing where the anisotropy and nonlinearity of the diffusion operator and convection terms are used to underline the borders of the distorted image and to eliminate the noise [73, 14]. In addition, one can not overlook the study of the dispersion of epidemics as well as the non- newtonian fluids [16], in particular electro-rheological fluids which are characterized by their ability to change the mechanical properties under the influence of the exterior electro-magnetic field [141]. That said, we have been confronted with choosing the appropriate functional spaces and their topological properties, and also overcoming the difficulties that arise, for example, when passing from a variable exponent to an anisotropic variable exponent. Thus, in the first part of this thesis, and for the purpose of studying nonlinear anisotropic elliptic problems with lower order terms and nonlinear unilateral anisotropic parabolic problem in appropriate frameworks, respectively in Chapters 2, 3 and 4, we exploited the method of penalization, and exhibited suitable test functions, combining several important techniques based on truncations to ensure a priori estimates and strong convergences. The so-called method of penalization, which is a powerful tool for alleviating, especially the difficulties of inequalities, has been of great help to us in dealing with the problem of Chapter 4, which is characterized by the presence of a time-dependent obstacle, which is characterized by the presence of a time-dependent obstacle and for whom we introduces the functional space V which is the most adequate to overcome the parabolic nature of the problem $(4.1.2)$, in addition to a complementary condition which

is necessary to guarantee a priori estimates. In the second part, we discussed different types of nonlinear anisotropic problems introducing "Hardy potential" singularities, combining, among other things, penalization techniques, inequalities of young and the regularizing effect of the terms $|u|^{s(.)}u$ (and $|u|^{s}u$) to ensure the existence of entropy solutions for the problems of Chapter 5 and 6, respectively in $W_0^{1,\overrightarrow{p}(.)}(\Omega)$ and $W_0^{1,\overrightarrow{p}}(\Omega)$. Further, in Chapter 7, the anisotropic weighted Sobolev space $W_0^{1,\overrightarrow{p}}(\Omega,\nu)$ has been introduced, which is an appropriate space where an attempt has been made to match its properties with, on the one hand, certain additional conditions on weight, and on the other hand the techniques used to preserve the existence of an entropy solution for such a quasilinear anisotropic problem.

Nonetheless, we can raise important issues that open other lines of research. For example, it would be reasonable to consider treating the work in Chapter 2 in the case where the anisotropy is not concentrated (the $p_i s$ are "spread out"), that is, $p_\infty = p_i^+$ for all $i = 1, \ldots, N$, and thereby improve the regularity of the solutions. Moreover, we highlight that in several problems treated in this memory, the uniqueness of the solution is conceivable by having certain conditions. One can also think of dealing with the case of Neumann's boundary conditions. Finally, the study of the anisotropic problems that model the non homogeneous materials that behave differently on different space directions, sometimes needs to be transposed within Orlicz functional spaces.

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