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Sous la direction du Pr. **Youssef Akdim**

Intitulée

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Etude de Certains Problèmes Elliptiques et Paraboliques Non Linéaires Dégénéré à Exposant Variable

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Dèdicace

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Etude de Certains Problèmes Elliptiques et Paraboliques Non Linéaires Dégénéré à Exposant Variable

Résumé : Cette thèse a pour objectif l'étude de certains problèmes elliptiques et paraboliques des équations aux derivées partielles non-linéaires faisant intervenir un opérateur de type Leary-Lions. L'originalité de ce travail, consiste à la présence d'une classe d'opérateurs étudiés, permettant de mettre en apparence le cadre fonctionnel des espaces de Lebesgue et de Sobolev à exposant variable et de Sobolev avec poids à exposant variable.

Notre étude est portée sur le cas où les données sont dans $L^1(\Omega)$ ou $L^1(Q) + L^{p'^-}(0,T;W^{-1,p'(x)}(\Omega))$ ou dans $L^1(Q) + L^{p'^-}(0,T;W^{-1,p'(x)}(\Omega,\omega^*))$

Ce travail se décompose en deux parties principales : Dans la première partie, après un bref exposé de quelques définitions et résultats nécessaires à ce travail, nous étudions dans le chapitre 2 un résultat d'éxistence de solutions renormalisées pour le problème parabolique associé à l'équation :

$$\frac{\partial b(x,u)}{\partial t} + Au + H(x,t,u,\nabla u) = f - \mathrm{div} F \quad \text{dans } Q = \Omega \times (0,T)$$

où $Au = -\mathrm{div}(a(x,t,u,\nabla u))$ est un opérateur aux dérivées partielles de type Leary-Lions, H est un terme fortement non linéaire n'ayant aucune condition de croissance par rapport à u et sans condition de signe et de coercivité et avec $f \in L^1(Q)$ et $F \in (L^{P'(.)}(Q))^N$. Dans le chapitre 3, nous étudions un problème unilatéral dans les espaces de Sobolev à exposant variable de ce type

$$\begin{cases} u \geq \psi & \text{p.p. dans } \Omega \times (0,T), \\ \frac{\partial u}{\partial t} - \mathrm{div}(a(x,t,u,\nabla u)) + H(x,t,u,\nabla u) = f & \text{dans } Q = \Omega \times (0,T), \end{cases}$$

en introduisant une nouvelle notion de solution dite solution entropique unilatérale. Nous montrons des résultats d'éxistence pour cette notion de solution en utilisant la méthode de pénalisation. En suite dans le chapitre 4, nous étudions ce problème de type suivant :

$$\begin{cases} u \geq \psi & \text{p.p. dans } \Omega \times (0,T), \\ \frac{\partial u}{\partial t} - \mathrm{div}(|\nabla u|^{p(x)-2}\nabla u) + \mathrm{div}(\phi(u)) = & \\ \qquad g(u)|\nabla u|^{p(x)} + f - \mathrm{div} F & \text{dans } \Omega \times (0,T) \end{cases}$$

où g est une fonction positive bornée et continue et ϕ est une fonction continue de $\mathbb{R}$ dans $\mathbb{R}^{\mathbb{N}}$. La deuxième partie de cette thèse est consacrée à l'étude de deux classes de problèmes non linéaires , Le cinquiéme est un problème elliptique associé à l'équation :

$$-\mathrm{div}(a(x,u,\nabla u)) + H(x,u,\nabla u) = f \quad \text{dans } \Omega,$$

ce problème a pour but de presenter un résultat d'éxistence de solutions renormalisées dans les espaces de Sobolev avec poids à exposant variable, ou $f \in L^1(\Omega)$, pour le sixième c'est un problème parabolique fortement non-linéaire dans les espaces de Sobolev avec poids à exposant variable de ce type :

$$\frac{\partial u}{\partial t} - \operatorname{div}(a(x,t,u,\nabla u)) + H(x,t,u,\nabla u) = f \quad \text{dans} \quad Q = \Omega \times (0,T)$$

Nous montrons un résultat d'éxistence de solutions renormalisées pour ce problème sans aucune condition de signe supposée sur le terme fortement non-linéaire H.

Mots clés
Espaces de Sobolev à exposant variable, Inégalité de Young, Problèmes Elliptiques, Problèmes Paraboliques Unilatérale, Solutions Renormalisées, Solutions Entropiques, Méthode de Pénalisations, Espaces de Sobolev avec poids à exposant variable.

AMS Classification 35J15, 35J70, 35J85, A6A32, A7A15, 47D20.

Study Some Problems Degenerated Elliptic and Parabolic Nonlinear with Variable Exponent

Abstract : The aim of this thesis is the study the some nonlinear problems elliptic and Parabolic partial differential equations by involves an operator the type Leary-Lions. The originality of this work consists of the presence of a class of studied operators allowing to look the impotance of the functional framework involves Lebesgue and Sobolev spaces with variable exponent and Weighted Variable Exponent Space.

Our Study is ported on the case or data are in $L^1(\Omega)$ ou $L^1(Q) + L^{p'^-}(0,T;W^{-1,p'(.x)}(\Omega))$ or in $L^1(Q) + L^{p'^-}(0,T;W^{-1,p'(x)}(\Omega,\omega^*))$

This work is divided into two parts main : In the first after a brief presentation of some definitions and results necessary for the continuation of this work, we study in chapter 2, an existence result of a renormalized solution for the parabolic problem associated with the equation :

$$\frac{\partial b(x,u)}{\partial t} + Au + H(x,t,u,\nabla u) = f - \mathrm{div}F \quad \text{in } Q = \Omega \times (0,T)$$

where $Au = -\mathrm{div}(a(x,t,u,\nabla u))$ is the operator derivative partial type Leary-Lions, H is a nonlinear order term and the critical growth condition on H is with respect to ∇u, no growth condition with respect to u and no sign condition or the coercivity condition where $f \in L^1(Q)$ et $F \in (L^{P'(x)}(Q))^N$. In Chapter 3, we study a unilateral problem in the Lebesgue and Sobolev space with varaible exponent of the type :

$$\begin{cases} u \geq \psi & \text{a.e. in } \Omega \times (0,T), \\ \frac{\partial u}{\partial t} - \mathrm{div}(a(x,t,u,\nabla u)) + H(x,t,u,\nabla u) = f & \text{in } Q = \Omega \times (0,T), \end{cases}$$

we introduce a new notion called entropy unilateral solution. we show the results of existence by this notion solution using a penalization method. Then in chapter 4, we study this problem of the type :

$$\begin{cases} u \geq \psi & \text{a.e. in } \Omega \times (0,T), \\ \frac{\partial u}{\partial t} - \mathrm{div}(|\nabla u|^{p(x)-2}\nabla u) + \mathrm{div}(\phi(u)) = & \\ \qquad g(u)|\nabla u|^{p(x)} + f - \mathrm{div}F & \text{in } \Omega \times (0,T), \end{cases}$$

where g is a bounded and continuous positive function and ϕ is just assumed to be continuous of $\mathbb{R}$ with values in $\mathbb{R}^{\mathbb{N}}$. In the second part in this thesis is dedicated for study the two classes of nonlinear problems, the first is a elliptic problem associated with the equation :

$$-\mathrm{div}(a(x,u,\nabla u)) + H(x,u,\nabla u) = f \quad \text{in } \Omega,$$

this problem aim to present an existence result of a renormalized solution in Weighted Variable Exponent Space where $f \in L^1(Q)$. For the second is parabolic problem strongly nonlinear associated to the type :

$$\frac{\partial u}{\partial t} - \operatorname{div}(a(x,t,u,\nabla u)) + H(x,t,u,\nabla u) = f \quad \text{ in } Q = \Omega \times (0,T).$$

We prove an existence result of a renormalized solution for this problem without any sign condition assumed in the non linear term H.

Key Words :
Sobolev Space with Variable Exponent, Young's Inequality, Elliptics Problem Unilateral, Parobolics Problems, Renormalized Solutions, Entropy Solutions, Penalized Methods, Weighted Variable Exponent Lebesgue Sobolev Spaces.

AMS Classification 35J15, 35J70, 35J85, A7A15, A6A32, 47D20.

NOTATION

Ω : Ouvert de $\mathbb{R}^N$ avec $(N \geq 1)$
$\partial\Omega$: Frontière topologique de Ω.
$x = (x_1, x_2, ...x_N)$: Point générique de $\mathbb{R}^N$.
$dx = dx_1 dx_2.......dx_N$: Mesure de Lebesgue sur Ω
∇u : Gradient de u.
$Q : \Omega \times (0,T), T$ variable du temps ; $T > 0$.
$\text{supp}(f)$: Support d'une fonction f.
T_k : Fonction troncature de niveau $k > 0$.
$D(\Omega), D(Q)$: Espace des fonctions différentiables et à support compact dans Ω, Q.
$D_+(\Omega), D_+(Q)$: Espace des fonctions positives de $D(\Omega), D(Q)$.
$C_0(\Omega), C_0(Q)$: Espace des fonctions continues nulles au bord de Ω, Q.
$C^k(\Omega), C^k(Q)$: Espace des fonctions k-fois continuement différentiables dans Ω, Q.
$L^p(\Omega)$: Espace des fonctions de puissance p-éme intégrables sur Ω pour la mesure dx.
$W^{1,p}(\Omega) = \Big\{ u \in L^p(\Omega) \quad \nabla u \in (L^p(\Omega))^N \Big\}$.
$W_0^{1,p}(\Omega)$: Adhérence de $D(\Omega)$ dans $W^{1,p}(\Omega)$
$W^{-1,p'}(\Omega)$: Espace dual de $W_0^{1,p}(\Omega)$
Si X est un espace de Banach
$L^p(0,T;X) = \Big\{ f : (0,T) \to X mesurable : \displaystyle\int_0^T \|f(t)\|_X^p dt < \infty \Big\}$
$L^\infty(0,T;X) = \Big\{ u : (0,T) \to X \ measurable, \ \exists\, C > 0 \ / \|u(t)\|_X \leq \ C \ p.p. \Big\}$
$C([0,T];X)$: Espace des fonctions continuement différentiables de $[0,T] \to X$
$D([0,T];X)$: Espace des fonctions continuement différentiables à support compact dans $[0,T]$
Soit $p(.) : \Omega \to (1,\infty)$ est une fonction mesurable.
$L^{p(.)}(\Omega) = \Big\{ u : \Omega \to \mathbb{R} \text{ mesurable} \displaystyle\int_\Omega |u(x)|^{p(x)} < \infty \Big\}$.
$||u||_{L^{p(.)}(\Omega)} = \inf \Big\{ \lambda > 0 : \displaystyle\int_\Omega \Big|\frac{u(x)}{\lambda}\Big|^{p(x)} dx \leq 1 \Big\}$.
$W^{1,p(.)}(\Omega) = \Big\{ u \in L^{p(.)}(\Omega) \ \text{ and } \quad \nabla u \in (L^{p(.)}(\Omega))^N \Big\}$.
$||u||_{1,p(.)} = ||u||_{p(.)} + ||\nabla u||_{p(.)}$.
$W_0^{1,p(.)}(\Omega)$: Fermeture de $C_0^\infty(\Omega)$ dans $W^{1,p(.)}(\Omega)$.
$\rho(u) = \displaystyle\int_\Omega |u(x)|^{p(x)} dx \ \ \forall u \in L^{p(.)}(\Omega)$: Module convexe.

"Rendre clair l'obscur et simple le complexe"

Abu Ja'far Muhammad ibn Musa al-Khwarizmi

⁎Les mathématiques sont une science dans laquelle on ne sait jamais de quoi on parle, et où l'on ne sait jamais si ce que l'on dit est vrai.

Berand Russel

⁎ Un probléme sans solution est un problème mal posé.
⁎ Une personne qui n'a jamais commis d'erreurs n'a jamais tenté d'innover.

Albert Einstein

⁎ Les mathématiques sont une science pour bien comprender le monde.

Nezha Elgorch

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Introduction Gènèrale

L'étude des équations aux dérivées partielles est très importante, vu leurs applications dans différents phénomènes du monde réel qui utilise les lois de la physique par exemple (la Mécanique, Thermodynamique, Electromagnétisme et la Réstauration de l'image, etc..). Ces lois sont généralement, écrites sous la forme des bilans qui se traduisent mathématiquement par des Equations aux Dérivées Partielles ou par des Equations Differentielles Ordinaires. Ces équations sont utilisées aussi dans d'autres domaines à savoir la chimie afin de modèliser les réactions chimiques, l'économie pour étudier l'évolution des marchés.
L'étude mathématique et analytique des équations aux dérivées partielles nous oblige à faire un choix adéquat des espaces fonctionnels (Lesbesque, Sobolev, Orlicz...) et nécéssite aussi de donner la définition d'une solution de l'équation aux dérivées partielles qui est objet de notre sujet d'ètude que nous allons developper et de démontrer l'existence et par fois l'unicité de cette solution. Lors de l'étude de ces équations, diverses difficultés peuvent étre évoquées : Difficultés liées au type (elliptique, parabolique, hyperbolique, dégénéré...) de l'équation, à la régularité des données et aussi à la définition des espaces appropriés.
Considérant le cas des équations elliptiques non-linéaires associeés à un opérateur de type Leray-Lions :

$$Au = -\mathrm{div}\Big(a(x, u, \nabla u)\Big) = f, \tag{0.0.1}$$

où $a : \Omega \times [0, T] \times \mathbb{R} \times \mathbb{R}^N \to \mathbb{R} \quad \Omega$ est un ouvert borné de $\mathbb{R}^N (N \geq 1)$
avec différentes conditions aux limites, cette équations a été l'objet de nombreux travaux depuis le début des années cinquantes, on trouve à titre d'exemples les travaux de Browder [47], Morrey [69] et d'autres. Pour le cas variationnel c.à.d lorsque f est dans le dual $W^{-1,p'}(\Omega)$, $p > 1$, les équations elliptiques non- linéaires

de la forme (0.0.1) ont été ètudiées par Leray et Lions [64] au début des années soixantes ils ont montré l' existence d'une solution faible du problème variationnel (associé au problème de Dirichlel classique) suivant :

$$\begin{cases} u \in W_0^{1,p}(\Omega) \\ \displaystyle\int_\Omega a(x,u,\nabla u)\nabla v dx = \Big\langle f, v \Big\rangle \qquad\qquad \forall v \in W_0^{1,p}(\Omega), \end{cases} \tag{0.0.2}$$

lorsque a ne dépend pas de u, il y a unicité de la solution. Dans le cas où a dépend de u Boccardo, Gallouët et Murat montrent dans [40] que si a est Lispschitizienne en u, il y a unicité de solution pour $p \leq 2$ et il existe des contre exemples à l'unicité pour $p > 2$.

Le cas parabolique a été traité par Lions [65] avec $u_0 \in L^2(\Omega)$ et $f \in L^{p'}(0,T;W^{-1,p'}(\Omega))$ en étudiant le problème suivant :

$$\begin{cases} u \in L^p(0,T;W_0^{1,p}(\Omega)) \quad u_t \in L^{p'}(0,T;W^{-1,p'}(\Omega)) \\ \displaystyle\int_0^T \langle u_t, v\rangle + \int_0^T\int_\Omega a(t,x,u,\nabla u)\nabla v dx = \int_0^T \langle f, v\rangle \qquad \forall v \in L^p(0,T;W_0^{1,p}(\Omega)) \\ u(0) = u_0. \end{cases} \tag{0.0.3}$$

Dans le cas elliptique avec conditions aux limites homogènes de Dirichlet où la donnée de f est $L^1(\Omega)$, trois approches ont été utilisées. Dall' Aglio [49] a montré que, même pour un problème non-linéaire, la méthode par approximation conduit à une unique solution appelée SOLA. Bénilan, Boccardo, Gallouët, Gariepy, Pierre et Vazquez [33] définissent la notion de solution entropique :

$$\begin{cases} u \in W_0^{1,1}(\Omega) \quad T_k(u) \in W_0^{1,p}(\Omega) & \forall k > 0 \\ \displaystyle\int_\Omega a(x,\nabla u)\nabla T_k(u-v)dx \leq \int_\Omega f T_k(u-v)dx & \forall v \in W_0^{1,p}(\Omega) \cap L^\infty(\Omega). \end{cases}$$

L'existence et l'unicité de solutions entropiques ont été démontrées par Andereu,. Mazon, Segura de leon et Taledo [22]. De plus dans le cas où $f \in L^1(\Omega)$ la notion des solutions renormalisées a été introduite par P. L. Lions et R. J. Diperna [52] pour l'étude des équations de Boltzmann (voir aussi P.L. Lions [67] pour des applications des modèles de mécanique des fluides). Cette notion a été adaptée par la suite à des versions elliptiques et paraboliques (voir par exemple [78, 76, 26]). Enfin, Lions et Murat [68] avaient introduit la notion des solutions renormalisées qui vérifie :

$$\begin{cases} u \in W^{1,1}(\Omega)) \quad T_k(u) \in W_0^{1,p}(\Omega) & \forall k > 0 \\ \lim_{h\to\infty} \displaystyle\int_{\{h\leq |u|\leq k+h\}} |\nabla u|^p dx = 0 & \forall k > 0 \\ \displaystyle\int_\Omega S(u)a(x,\nabla u)\nabla v dx + \int_\Omega S'(u)a(x,\nabla u)v\nabla u dx \\ \qquad\qquad = \displaystyle\int_\Omega f S(u) v dx \quad \forall v \in W_0^{1,p}(\Omega)\cap L^\infty(\Omega). \end{cases}$$

la formulation des solutions renormalisées est obtenue par multiplication ponctuelle de l'équation par $S'(u)$ pour toute fonction S régulière à variable réelle et à support compact.

Considérons maintenant l'équation (0.0.1), mais avec introduction d'un terme non linéaire $H(x,u,\nabla u)$ dépend de trois variables x, u et ∇u c.à.d une équation dite fortement non-linéaire de type :

$$-\mathrm{div}(a(x,u,\nabla u)) + H(x,u,\nabla u) = f, \tag{0.0.4}$$

avec H vérifiant une croissance naturelle par rapport à $|\nabla u|$, et la condition de signe classique $H(x,s,\xi)s \geq 0$. On trouve dans le cas variationnel ($f \in W^{-1,p'}(\Omega)$), les travaux de Boccardo, Gallouët et Murat [41] par l'utilisation de la convergence forte des trancatures . Dans le cas où $f \in L^1(\Omega)$ le problème (0.0.4) a été traité en particulier dans [39] mais sous l'ajout d'une condition de coercivité sur la non-linéarité de type :

$$H(x,t,s,\xi)\cdot\xi \geq \gamma|\xi|^p \qquad \text{pour } |s| \geq \eta, \tag{0.0.5}$$

Après, Porretta [71] a supprimé la condition (0.0.5) et a prouvé l'existence d'une solution du problème fortement non- linéaire associé à (0.0.4) avec $u \in W_0^{1,q}(\Omega)$ pour tout $q < \frac{N(p-1)}{N-1}, N > 1$.

Encore plus précisèment Porretta [72] a étudié le problème (0.0.4) sans supposer la condition de signe mais en modifiant la croissance classique de la non-linéarité H par

$$|H(x,s,\xi)| \leq \gamma(x) + g(s)|\xi|^p, \quad g \in L^1(\Omega), \quad g \geq 0.$$

Dans le cas des espaces de Sobolev avec poids, Y. Akdim, E. Azroul et A. Benkirane [18, 19, 20] ont etudié le problème dégénéré fortement non-linéaire associé à l'équation (0.0.4) où f est dans le dual ou dans L^1.

Dans le cas parabolique avec $f \in L^{p'}(0,T;W^{-1,p'}(\Omega))$

$$\begin{cases} \frac{\partial u}{\partial t} - \operatorname{div}(a(x,t,u,\nabla u)) + H(x,t,u,\nabla u) = f & \text{in } Q = \Omega \times (0,T) \\ u(x,0) = u_0 & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega \times (0,T), \end{cases} \tag{0.0.6}$$

ce problème a été initialement étudier par L. Boccardo [42] ainsi par Lions [65] avec $u_0 \in L^2(\Omega)$. De plus L. Aharouch et al [3] ont étudié le problème parabolique (0.0.6) dans le cas où a strictement montone avec $H = 0$ et $f \in L^{p'}(0,T;(W^{-1,p}(\Omega,\omega^*))$. Ainsi dans [9], Y. Akdim, J. Bennouna et M. Mekkour ont étudié ce problème avec $\frac{\partial u}{\partial t} = \frac{\partial b(x,u)}{\partial t}$. Pour le cas elliptique- parabolique non variationnel c.à.d si $f \in L^1(\Omega)$ où f est une mesure bornée, on peut consulter [46, 63, 71, 73, 83]. Ainsi pour le problème unilatérale associè a l' equation (0.0.6) dans les espaces d'Orliz on peut citè à titre d'exemple [32, 27].

L'objectif de ce modeste travail est l'étude de divers problèmes des équations aux dérivées partielles non-linéaires de type elliptique et parabolique dans les espaces de Sobolev à exposant variable ou dans les espaces de Sobolev à exposant variable avec poids faisant intervenir des opérateurs de type Leray- Lions.
Nous allons développer et analyser ce travail en deux parties qui peuvent être liées indèpendamment : La première partie comporte 4 chapitres. Dans le premier chapitre nous présenterons quelques définitions et rappels des résultats de base sur des espaces de Lebesgue-Sobolev à exposant variable et de Sobolev avec poids à exposant variable et nous terminous ce chapitre en énonçant quelques résultats classiques d'intégrations, Théorème de convergence dominé de Lebesgue, Théorème de Vital et quelques propriétés des opérateurs montones.
Dans le deuxième chapitre, nous considerons le problème non-linéaire $p(x)$- parabolique suivant :

$$\begin{cases} \frac{\partial b(x,u)}{\partial t} - \operatorname{div}\Big(a(x,t,u,\nabla u)\Big) + H(x,t,u,\nabla u) = f - \operatorname{div}F & \text{dans } Q = \Omega \times (0,T) \\ b(x,u) \mid_{t=0} = b(x,u_0) & \text{dans } \Omega \\ u = 0 & \text{sur } \partial\Omega \times (0,T). \end{cases} \tag{0.0.7}$$

où Ω est un ouvert borné de $\mathbb{R}^N (N \geq 1)$, $p \in C_+(\bar{\Omega})$ et $1 < p^- \leq p^+ < N$, $Au = -\operatorname{div}(a(x,t,u,\nabla u))$ est l'opérateur de Leray-Lions défini de $L^{p^-}(0,T;W_0^{1,p(x)}(\Omega))$

vers $L^{p'}(0,T;W^{-1,p'(x)}(\Omega))$, avec $f\in L^1(Q)$, $b(x,u_0)\in L^1(\Omega)$ et $F\in (L^{p'(x)}(Q))^N$. Dans ce chapitre nous étudierons l'éxistence de solutions renormalisées dans le cadre des espaces de Sobolev à exposant variable et avec H ne vérifie pas la condition de signe classique mais vérifie la condition de croissance :

$$|H(x,t,s,\xi)|\leq \gamma(x,t)+g(s)|\xi|^{p(x)}$$

avec $g:\mathbb{R}\to\mathbb{R}^+$ est une fonction bornée continue et positive telque $g\in L^1(\mathbb{R})$ et $\gamma\in L^1(Q)$. Parmis les diffecultés rencontrées est de montrer que $T_k(u_n)$ est bornée par une constance indépendante de n, pour cela on montre d'abord que

$$\int_Q g(u_n)|\nabla u_n|^{p(x)}\exp(G(u_n))dxdt\leq C$$

par des choix de fonctions testes suivants $\displaystyle\int_0^{u_n} g(s)\chi_{\{s>0\}}ds \quad et \quad \int_{u_n}^0 g(s)\chi_{\{s<0\}}ds.$
Signalons que le contenu de ce chapitre fait l'objet d'un article publié voir [10]. Le problème étudié dans ce chapitre est une généralisation de celle de Boccardo [43]. Ainsi ce problème a été étudié dans le cas classique par Y. Akdim, J. Bennouna, M. Mekkour [6], de même, on trouve Dall'Algio-Orsina [50] and Porretta [70] ont étudié le même problème dans le cas ou $\frac{\partial b(x,u)}{\partial t}=\frac{\partial u}{\partial t}$ et que H verifié la condition de signe classique. Récemment dans le cas elliptique, E. Azroul, M. B. Benboubker et M. Rhoudaf [24] ont étudié ce problème avec le second membre est dans $L^1(\Omega)+W^{-1,p(x)}(\Omega)$ et H satisfait la condition de signe. De même ce problème aètè étudier par Y. Akdim, C. Allalou, N. Elgorch [13] dans les espaces de Sobolev avec poids à exposent variable avec $F=0$.
Notre objectif dans le chapitre 3 est l'étude du problème unilatéral suivant :

$$\begin{cases} u\geq\psi & \text{p.p. dans } \Omega\times(0,T),\\ \frac{\partial u}{\partial t}-\operatorname{div}(a(x,t,u,\nabla u))+H(x,t,u,\nabla u)=f & \text{dans } Q=\Omega\times(0,T),\\ u(x,0)=u_0 & \text{dans } \Omega,\\ u=0 & \text{sur } \partial\Omega\times(0,T),\end{cases}\tag{0.0.8}$$

dans le cadre des espaces de Sobolev à exposant variable où $f\in L^1(Q), u_0\in L^1(\Omega)$ et $\psi\in L^{p^-}(0,T;W_0^{1,p(x)}(\Omega))\cap L^\infty(Q)$, on montre l'existence de la solution entropique unilatéral c.a.d.,

$$\begin{cases} \qquad\qquad\qquad\qquad u \geq \psi \qquad\qquad \text{p.p. dans } \ \Omega \times (0,T), \\ T_k(u) \in L^{p^-}(0,T;W_0^{1,p(x)}(\Omega)) \quad for\ all\ k \geq 0 \\ u \in C(0,T;L^1(\Omega)) \\ \displaystyle\int_\Omega S_k\Big(u(T)-v(T)\Big)dx - \int_\Omega S_k\Big(u_0 - v(0)\Big)dx \\ \displaystyle + \int_Q \frac{\partial v}{\partial t} T_k(u-v)dxdt + \int_Q H(x,t,u,\nabla u)T_k(u-v)dxdt \\ \displaystyle + \int_Q a(x,t,u,\nabla u)\nabla T_k(u-v)dxdt \\ \displaystyle \leq \int_Q fT_k(u-v)dxdt \quad \forall v \in K_\psi \ \cap \ L^\infty(Q) \\ \frac{\partial v}{\partial t} \in L^{p'^-}(0,T;W^{-1,p'(x)}(\Omega)) \\ \displaystyle et S_k(s) = \int_0^s T_k(r)dr \end{cases}$$

Pour la preuve, on utilise la méthode de pénalisation c.à.d., en approchant le problème (0.0.8) par une suite d'équation de la forme :

$$(P_n)\begin{cases} \dfrac{\partial u_n}{\partial t} - \mathrm{div}(a(x,t,u_n,\nabla u_n)) + H_n(x,t,u_n,\nabla u_n) \\ \qquad\qquad + \ n\ T_n(u_n-\psi)^- \phi_{\frac{1}{n}}(u_n) = f_n \qquad \text{dans } \ D'(Q), \\ \qquad\qquad u_n(t=0) = u_{0n} \qquad\qquad \text{dans } \ \Omega, \\ \qquad\qquad u_n = 0 \qquad\qquad \text{sur } \partial\Omega \times (0,T). \\ \text{avec } \ \phi_k(s) = \frac{1}{k}T_k(s). \end{cases}$$

le terme de pénalisation $nT_n(u_n-\psi)^-\phi_{\frac{1}{n}}(u_n)$ introduit dans (P_n) joue un rôle très important pour la preuve d'existence.
On rappelle que (0.0.8) a été étudié dans [27] dans le cadre des espaces d' Orlicz où $\frac{\partial u}{\partial t} = \frac{\partial b(u)}{\partial t}$ de même H. Redwane dans [76, 79] a étudié ce problème dans le cadre d'Orlicz Sobolev dans le cas où $H(x,t,u,\nabla u) = \mathrm{div}\phi(u)$. Ainsi ce résultat généralise celui de Boccardo-Gallouêt [43]. Signalons que le contenu de ce chapitre fait l'objet d'un article publié voir [11].
Nous étudierons, au quatrième chapitre, un problème à obstacle via une technique de pénalisation dans le cadre des espaces de Sobolev à exposant variable de ce

type :

$$\begin{cases} u \geq \psi & \text{p.p. dans } \Omega\times(0,T), \\ \frac{\partial u}{\partial t} - \mathrm{div}(|\nabla u|^{p(x)-2}\nabla u) + \mathrm{div}(\phi(u)) = g(u)|\nabla u|^{p(x)} & \\ \qquad +f - \mathrm{div}F & \text{dans } \Omega\times(0,T) \\ u(x,0) = u_0 & \text{dans } \Omega \\ u = 0 & \text{sur } \partial\Omega\times(0,T). \end{cases} \tag{0.0.9}$$

où $f \in L^1(Q)$, $u_0 \in L^1(\Omega)$, $F \in (L^{p'(x)}(Q))^N$, $p \in C_+(\bar{\Omega})$ et ϕ est une fonction supposée seulement continue sur $\mathbb{R}$ à valeurs dans $\mathbb{R}^{\mathbb{N}}$ et la fonction g est de $\mathbb{R} \to \mathbb{R}^+$ continue, bornée et positive. Le terme $-\mathrm{div}F$ pose beaucoup de difficultés surtout au niveau des estimations, pour cela on suppose que la fonction $g \in L^\infty(\mathbb{R})$.
L'étude du problème d'obstacle (0.0.9) dans le cas p constante à été fait par M. Kbiri Alaoui , D. Meskine et A. Souissi [59] dans le cadre des espaces d'Orlicz-Sobolev ou $f \in L^1(Q)$, $a(x,u,\nabla u) = |\nabla u|^{p(x)-2}\nabla u$, $\mathrm{div}(\phi(u)) = 0$ et $F = 0$, ainsi voir [33, 81].
La deuxième partie de cette thèse est constituée de deux chapitres ($5^{eme} et\ 6^{eme}$) qui traitent certains problèmes dans le cas des espaces de Sobolev dégénéré à exposent variable. Le cinquiéme, est consacré à l'existence d'une solution renormalisée d'un problème $p(x)$-elliptique de la forme :

$$\begin{cases} Au + H(x,u,\nabla u) = f & \text{dans } \Omega \\ u = 0 & \text{dans } \partial\Omega, \end{cases} \tag{0.0.10}$$

où Ω est un domaine borné de $\mathbb{R}^N$ avec $\mathrm{N} \geq 2$, $p \in C_+(\overline{\Omega})$, $p(x) > 1$, $1 < p^- \leq p^+ < \infty$ et $\frac{1}{p(x)} + \frac{1}{p'(x)} = 1$. A est un opérateur de type Leray-Lions défini de $W_0^{1,p(x)}(\Omega,\omega)$ dans son dual $W^{-1,p'(x)}(\Omega,\omega^*)$ par la formule $Au = -\mathrm{div}(a(x,u,\nabla u))$ et $f \in L^1(\Omega)$.
Nous démontrons dans ce chapitre le résultat d'existence de la solution renormalisée sous l'hypothèse où le terme non linéaire $H(x,u,\nabla u)$ ne vérifie pas la condition de signe et satisfait à une condition de croissance de la forme

$$|H(x,s,\xi)| \leq \gamma(x) + g(s)\omega|\xi|^{p(x)}$$

avec $g : \mathbb{R} \to \mathbb{R}^+$ est une fonction continue et positive tel que $g \in L^1(\mathbb{R})$ et $\gamma \in L^1(Q)$. Lorsque le second membre f est dans le dual $W^{-1,p'(x)}(\Omega)$ ce problème à été traité dans [28].

Le sixième chapitre traitera un problème parabolique dégénéré à exposent variable (voir [12]) de type suivant :

$$\begin{cases} \frac{\partial u}{\partial t} - \operatorname{div}(a(x,t,u,\nabla u)) + H(x,t,u,\nabla u) = f & \text{dans } Q = \Omega \times (0,T) \\ u(x,0) = u_0 & \text{dans } \Omega \\ u = 0 & \text{sur } \partial\Omega \times (0,T). \end{cases} \tag{0.0.11}$$

avec $f \in L^1(Q)$ et $a(x,t,u,\nabla u)$ vérifie la montonie strict

$$\Big[a(x,t,s,\xi) - a(x,t,s,\eta)\Big] \cdot (\xi - \eta) > 0 \ \ \forall\, \xi \neq \eta \in \mathbb{R}^N$$

Pour le cas classique, on peut se référer à [38, 34].
Dans ce chapitre, on donnera un résultat d'existence de solution renormalisèe de (0.0.11), c'est à dire

$$\begin{cases} u \in C(0,T;L^1(\Omega)) \\ T_k(u) \in L^{p^-}(0,T;W_0^{1,p(x)}(\Omega,\omega)) \quad for\ all\ k \geq 0 \\ \displaystyle\int_{\{m\leq|u|\leq m+1\}} a(x,t,u,\nabla u)\nabla u dxdt \to 0 \text{ as } \ m \to \infty, \\ \frac{\partial S(u)}{\partial t} - \operatorname{div}\Big(S'(u)a(x,t,u,\nabla u)\Big) + S''(u)a(x,t,u,\nabla u)\nabla u \\ +H(x,t,u,\nabla u)S'(u) = fS'(u) \text{ in } D'(Q), \\ \forall S \in W^{2,\infty}(\mathbb{R}) de \quad C^1 \\ \ et\ telque\ S'\ support\ compacte\ dans\ \mathbb{R} \end{cases}$$

Signalons que le contenu de ce chapitre fait l'objet d'un article publié voir [12].

PUBLICATIONS :

1) Youssef Akdim, Nezha El gorch and Mounir Mekkour **Existence of Renormalized Solutions for** $p(x)$**-Parabolic Equations with three Unbounded Nonlinearities** Bol. Soc. Paran. Mat (3s)V. 34 1(2016) pp 225-252.

2)Youssef Akdim, Nezha El gorch and Mounir Mekkour **Nonlinear Parabolic Inequalities In Sobolev Space With Variable Exponent.** Journal of Nonlinear Evolution Equations and Applications,Volume 2015, Number 5, pp. 67-90 (February 2016).

3)Youssef Akdim, Chakir Allalou and Nezha El gorch **Renormalized Solutions of Nonlinear Parabolic Equations in Weighted Variable-Exponent Space**. Journal of Partial Differential Equations. Vol. 28, No.3, pp. 225-252.

This part is constituted of the following chapters

CHAPTER I

Preliminaries

CHAPTER II

Existence of Renormalized Solutions for $p(x)$-Parabolic Equations with three Unbounded Nonlinearities

CHAPTER III

Nonlinear parabolic inequalities in Sobolev space with variable exponent

CHAPTER IV

Obstacle problem for a non linear $p(x)$ parabolic inequalities

Chapitre 1

Preliminaries

The aim of this chapter, is to recall some results and the notion necessary of this work. This chapter is organized as follows : In the first section, we give some preliminaries, definitions and the basic proprieties of the generalized Lebesgue spaces $L^{p(x)}(\Omega)$ and the Sobolev space $W^{1,p(x)}(\Omega)$ that will be used throughout the chapter II, III and chapter IV, we refer to Fan and Zhao [56] for further properties of variable exponent Lebesgue - Sobolev spaces. Section 2 is devoted to state and prove some basic results for the Spaces of Lebesgue with weight and to variable exponent $L^{p(x)}(\Omega,\omega)$ and spaces of Sobolev with weight and to variable exponent $W^{1,p(x)}(\Omega,\omega)$, some proprieties obtained here are new and might have significance for other problems, although they are the generalized cases of some known ones.

1.1 Variable Exponent Lebesgue and Sobolev Spaces

In this part we recall some definitions and properties of the generalised Lebesgue - Sobolev spaces with variable exponent.
Let Ω be a bounded open subset of $\mathbb{R}^N (N \geq 2)$, we say that a real-valued continuous function $p(.)$ is log-Hölder continuous in Ω if :

$$|p(x)-p(y)| \leq \frac{C}{|\log|x-y||} \quad \forall x,y \in \bar{\Omega} \quad \textit{such that } |x-y| < \frac{1}{2} \tag{1.1.1}$$

with possible different constant C. We denote
$C_+(\bar{\Omega}) = \Big\{\text{log-Hölder continuous function } \ p : \bar{\Omega} \to \mathbb{R} \ / \ 1 < p^- \leq p^+ < N\Big\}$, where

$$p^- = \min\{p(x) : x \in \bar{\Omega}\} \quad \textit{and} \quad p^+ = \max\{p(x) : x \in \bar{\Omega}\}$$

we denote by $P(\Omega)$ the set of Lebesgue measurable function

$$P(\Omega)=\Big\{u:\Omega\to\mathbb{R}\quad measurable\Big\}\quad and$$

$$P^{+}(\Omega)=\{u:\Omega\to[1,\infty)\quad measurable\}.$$

We define the variable exponent Lebesgue space for $p(.)\in C_{+}(\bar{\Omega})$ by

$$L^{p(x)}(\Omega)=\Big\{u\in P(\Omega):\int_{\Omega}|u(x)|^{p(x)}<\infty\Big\},$$

this space is endowed with the (Luxembourg) norm define by the formula :

$$||u||_{L^{p(x)}(\Omega)}=||u||_{p(x)}=\inf\Big\{\lambda>0:\int_{\Omega}|\frac{u(x)}{\lambda}|^{p(x)}dx\leq 1\Big\}.$$

If $1<p^{-}\leq p^{+}<\infty$ then $L^{p(x)}(\Omega)$ is a uniformly convex Banach space and therefore reflexive and if $p\in P^{+}(\Omega)\cap L^{\infty}(\Omega)$, then $L^{p(x)}(\Omega)$ is separable space.
We denote by $L^{p'(x)}(\Omega)$ the conjugate space of $L^{p(.)}(\Omega)$ where $\frac{1}{p(.)}+\frac{1}{p'(.)}=1$, see [57, 84].

Proposition 1.1.1 *(Young's Inequality) Let $p,\ p'\in C_{+}(\bar{\Omega})$, where p' the conjugate, i.e., $\frac{1}{p(.)}+\frac{1}{p'(.)}=1$. For all $a,b>0$, we have*

$$ab\leq\frac{a^{p(x)}}{p(x)}+\frac{b^{p'(x)}}{p'(x)}.$$

Proposition 1.1.2 *see[56, 60](Generalised Hölder Inequality)*

i) *For any functions $u\in L^{p(x)}(\Omega)$ and $v\in L^{p'(x)}(\Omega)$, we have*
$\Big|\int_{\Omega}uvdx\Big|\leq(\frac{1}{p^{-}}+\frac{1}{p'^{-}})||u||_{p(x)}||v||_{p'(x)}\leq 2||u||_{p(x)}||v||_{p'(x)}.$

ii) *For all $p,\ q\in C_{+}(\bar{\Omega})$ such that $p(x)\leq q(x)$ a.e. in Ω, we have $L^{q(x)}\hookrightarrow L^{p(x)}$ and the embedding is continuous.*

The modular of the space $L^{p(x)}(\Omega)$, which is the mapping $\rho:L^{p(x)}(\Omega)\to\mathbb{R}$ is defined by

$$\rho(u)=\int_{\Omega}\Big|u(x)\Big|^{p(x)}dx\qquad\text{for all } u\in L^{p(x)}(\Omega)$$

The next proposition shows that there is a gap between the modular and the norm in $L^{p(x)}(\Omega)$.

Lemma 1.1.1 . *(See[56]). Let $u\in L^{p(x)}(\Omega)$ then,*

$$\min\Big\{||u||^{p^{-}}_{p(x)},||u||^{p^{+}}_{p(x)}\Big\}\leq\rho(u)\leq\max\Big\{||u||^{p^{-}}_{p(x)},||u||^{p^{+}}_{p(x)}\Big\}.$$

Proposition 1.1.3 *See([57, 84]) For $u \in L^{p(x)}(\Omega)$ and $\{u_k\}_{k\in N} \subset L^{p(x)}(\Omega)$ then, the following assertions hold*

$$u \neq 0 \Rightarrow \Big[||u||_{p(x)} = \lambda \Leftrightarrow \rho(\frac{u}{\lambda}) = 1\Big] \tag{1.1.2}$$

$$||u||_{p(x)} > 1 \Rightarrow ||u||^{p^-}_{p(x)} \leq \rho(u) \leq ||u||^{p^+}_{p(x)} \tag{1.1.3}$$

$$||u||_{p(x)} < 1 \Rightarrow ||u||^{p^+}_{p(x)} \leq \rho(u) \leq ||u||^{p^-}_{p(x)} \tag{1.1.4}$$

$$\lim_{k\to\infty} ||u_k||_{p(x)} = 0 \Leftrightarrow \lim_{k\to\infty} \rho(u_k) = 0 \tag{1.1.5}$$

$$\lim_{k\to\infty} ||u_k||_{L^{p(x)}(\Omega)} = \infty \Leftrightarrow \lim_{k\to\infty} \rho(u_k) = \infty. \tag{1.1.6}$$

Theorem 1.1.2 *For any function $u \in L^{p(.)}(\Omega)$ and $u_n \in L^{p(.)}(\Omega)$, we have then, the following are equivalent assertions*

i) $\lim_{n\to\infty} ||u_n - u||_{p(.)} = 0$

ii) $\lim_{k\to\infty} \rho(u_n - u) = 0$

iii) *u_n converge to u in measure and* $\lim_{n\to\infty} \rho(u_n) = \rho(u)$.

Which share the same type of properties as $L^{p(x)}(\Omega)$, we define also the variable Sobolev space by

$$W^{1,p(x)}(\Omega) = \Big\{u \in L^{p(x)}(\Omega) \text{ and } \quad |\nabla u| \in L^{p(.)}(\Omega)\Big\},$$

where the norm is defined by

$$||u||_{1,p(x)} = ||u||_{p(x)} + ||\nabla u||_{p(x)} \quad \forall u \in W^{1,p(x)}(\Omega).$$

It is easy to see that the norm

$$|||u||| = \inf\Big\{\mu > 0 : \int_\Omega \Big(\Big|\frac{u(x)}{\lambda}\Big|^{p(x)} dx + \Big|\frac{\nabla u(x)}{\lambda}\Big|^{p(x)}\Big) dx \leq 1\Big\}$$

is the equivalent norm. An interesting feature is that smooth function are not dense in $W^{1,p(x)}(\Omega)$ without additional assumptions on the exponent $p(x)$. This was observed by Zhikov [86] in connection with Lavrentiev phenomenon. However, when the exponent $p(x)$ is log-Hölder continuous, then smooth function are dense in variable exponent Sobolev spaces and there is no confusion in defining the Sobolev space with zero boundary values, $W^{1,p(x)}_0(\Omega)$, as the completion of $C^\infty_0(\Omega)$ with respect to the norm $||u||_{1,p(x)}$(see[58]. and we define the Sobolev exponent by

$$p^*(x) = \begin{cases} \frac{Np(x)}{N-p(x)} & for \quad p(x) < N \\ \infty & for \quad p(x) \geq N. \end{cases}$$

We denote by $W_0^{1,p(x)}(\Omega)$ the closure of $C_0^\infty(\Omega)$ in $W^{1,p(x)}(\Omega)$, i.e.,

$$W_0^{1,p(x)}(\Omega) = \overline{C_0^\infty(\Omega)}^{W^{1,p(x)}(\Omega)}$$

and we define the Sobolev exponent by $p^*(x) = \frac{Np(x)}{N-p(x)}$ for $p(x) < N$.

Proposition 1.1.4 *[57]*

i) *Assuming $1 < p^- \leq p^+ < \infty$ the spaces $W^{1,p(x)}(\Omega)$ and $W_0^{1,p(x)}(\Omega)$ are separable and reflexive Banach spaces.*

ii) *If $q \in C_+(\bar{\Omega})$ and $q(x) < p^*(x)$ for any $x \in \Omega$, then the embedding $W_0^{1,p(x)}(\Omega) \hookrightarrow\hookrightarrow L^{q(x)}(\Omega)$ is continuous and compact.*

iii) *Poincaré inequality : there exists a constant $C > 0$, such that*

$$||u||_{p(x)} \leq C||\nabla u||_{p(x)} \quad \forall u \in W_0^{1,p(x)}(\Omega).$$

vi) *Sobolev Inequality : there exists other constant $C > 0$, such that*

$$||u||_{p*(x)} \leq C||\nabla u||_{p(x)} \quad \forall u \in W_0^{1,p(x)}(\Omega).$$

Remark 1.1.1 *By(iii)of Proposition 1.1.4, we deduce that $||\nabla u||_{p(x)}$ and $||u||_{1,p(x)}$ are equivalent norms in $W_0^{1,p(x)}(\Omega)$.*

We will also use the standard notation for Bochner spaces, i.e., if $q \geq 1$ and X is a Banach space then $L^q(0,T;X)$ denotes the space of strongly measurable function $u : (0,T) \to X$ for which $t \mapsto \|u(t)\|_X \in L^q((0,T)$. Morever, $C([0,T];X)$ denotes the space of continuous function $u : [0,T] \to X$ endowed with the norm $\|u\|_{C([0,T];X)} = \max_{t\in[0,T]} \|u(t)\|_X$.

$$L^{p^-}(0,T;W_0^{1,p(x)}(\Omega)) = \Big\{u : (0,T) \to W_0^{1,p(x)}(\Omega)\ measurable;$$
$$\Big(\int_0^T \|u(t)\|^{p^-}_{W_0^{1,p(x)}(\Omega)} dt\Big)^{\frac{1}{p^-}} < \infty\Big\}$$

and we define the space

$$L^\infty(0,T;X) = \Big\{u : (0,T) \to X\ measurable,\ \exists\, C > 0\ /\|u(t)\|_X \leq\ C\ a.e.\Big\},$$

where the norm is defined by

$$\|u\|_{L^\infty(0,T;X)} = \inf\Big\{C > 0; \|u(t)\|_X \leq\ C\ a.e.\Big\}.$$

We introduce the functional space see [24]

$$V = \Big\{f \in L^{p^-}(0,T;W_0^{1,p(.)}(\Omega)); |\nabla f| \in L^{p(.)}(Q)\Big\}, \tag{1.1.7}$$

which endowed with the norm

$$\|f\|_V = \|\nabla f\|_{L^{p(.)}(Q)}$$

or, the equivalent norm

$$|||f|||_V = \|f\|_{L^{p^-}(0,T;W_0^{1,p(.)}(\Omega))} + \|\nabla f\|_{L^{p(.)}(Q)},$$

is a separable and reflexive Banach space. The equivalence of the two norms is an easy consequence of the continuous embedding $L^{p(x)}(Q) \hookrightarrow L^{p^-}(0,T;L^{p(x)}(\Omega))$ and the Poincaré inequality. We state some further properties of V in the following lemma.

Lemma 1.1.3 *Let V be defined as in (1.1.7) and its dual space be denote by V^*. Then,*

i) *we have the following continuous dense embeddings*

$$L^{p^+}(0,T;W_0^{1,p(x)}(\Omega)) \hookrightarrow V \hookrightarrow L^{p^-}(0,T;W_0^{1,p(x)}(\Omega)).$$

In particular, since $D(Q)$ is dense in $L^{p^+}(0,T;W_0^{1,p(x)}(\Omega))$, it is dense in V and for the corresponding dual spaces, we have

$$L^{(p^-)'}(0,T;(W_0^{1,p(x)}(\Omega))^*) \hookrightarrow V^* \hookrightarrow L^{(p^+)'}(0,T;(W_0^{1,p(x)}(\Omega))^*).$$

Note that, we have the following continuous dense embeddings

$$L^{p^+}(0,T;L^{p(x)}(\Omega)) \hookrightarrow L^{p(x)}(Q) \hookrightarrow L^{p^-}(0,T;L^{p(x)}(\Omega)).$$

ii) *One can represent the elements of V^* as follows : if $T \in V^*$, then there exists $F = (f_1, ..., f_N) \in (L^{P'(x)}(Q))^N$ such that $T = divF$ and*

$$\langle T, \xi\rangle_{V^*,V} = \int_0^T \int_\Omega F.\nabla\xi dxdt,$$

for any $\xi \in V$. Moreover, we have

$$\|T\|_{V^*} = \max\Big\{\|f_i\|_{L^{p(.)}(Q)}, i = 1, ..., n\Big\}.$$

Remark 1.1.2 *The space $V \cap L^\infty(Q)$, is endowed with the norm defined by the formula*

$$\|v\|_{V\cap L^\infty(Q)} = \max\Big\{\|v\|_V, \|v\|_{L^\infty(Q)}\Big\},\ v \in V \cap L^\infty(Q),$$

is a Banach space. In fact, it is the dual space of the Banach space $V^* + L^1(Q)$ endowed with the norm

$$\|v\|_{V^*+L^1(Q)} = \inf\Big\{\|v_1\|_{V^*} + \|v_2\|_{L^1(Q)};\ v = v_1 + v_2, v_1 \in V^*, v_2 \in L^1(Q)\Big\}.$$

1.2 Some Impotant Technical Definitions and Propositions and Lemmas.

Definition 1.2.1 *for $k > 0$ and $s \in \mathbb{R}$, the truncation function $T_k(.)$ defined by*

$$T_k(s) = \begin{cases} s & if \quad |s| \leq k \\ k\frac{s}{|s|} & if \quad |s| > k. \end{cases}$$

Lemma 1.2.1 *([3])Assume(2.2.3)-(2.2.5) and let $(u_n)_n$ be a sequence in $L^{p^-}(0,T;L^{p(\cdot)}(\Omega))$ such that $u_n \rightharpoonup u$ weakly in $L^{p^-}(0,T;L^{p(x)}(\Omega))$ and*

$$\int_Q \Big(a(x,t,u_n,\nabla u_n) - a(x,t,u_n,\nabla u)\Big)\nabla(u_n - u)dx \to 0.$$

Then, $u_n \to u$ strongly in $L^{p^-}(0,T;L^{p(\cdot)}(\Omega))$.

Lemma 1.2.2 *([29])Let $g \in L^{p(\cdot)}(Q)$ and $g_n \in L^{p(\cdot)}(Q)$ with $\|g_n\|_{p(x)} \leq C$ for $1 < p(x) < \infty$, if $g_n(x) \to g(x)$ a.e. on Q. Then, $g_n \rightharpoonup g$ in $L^{p(\cdot)}(Q)$.*

Lemma 1.2.3 *See[70]*
$W = \Big\{u \in V; u_t \in V^* + L^1(Q)\Big\} \hookrightarrow C([0,T];L^1(\Omega))$
and

$$W \cap L^\infty(Q) \hookrightarrow C([0,T];L^2(\Omega)).$$

1.3 Spaces of Lebesgue and Sobolev with Weight and to Variable Exponents

In this section, we state some elementary properties for the Weighted Variable Exponent Lebesgue–Sobolev spaces $L^{p(x)}(\Omega,\omega)$ which will be used in the next sections. The basic properties of the variable exponent Lebesgue–Sobolev spaces $W^{1,p(x)}(\Omega,\omega)$, that is, when $\omega(x) \equiv 1$ can be found from [58, 65].

Definition 1.3.1 *Let ω be a function defined in Ω ; ω is called a weight function in Ω if she is measurable and strictly positive a.e. in Ω.*

1.3.1 Spaces of Lebesgue with weight and to variable exponents.

Let $p \in C_+(\overline{\Omega})$, we introduce the weighted variable exponent Lebesgue space $L^{p(x)}(\Omega, \omega)$ that consists of all measurable real-valued functions u such that

$$L^{p(x)}(\Omega, \omega) = \Big\{ u : \Omega \to \mathbb{R}, measurable, \ \int_\Omega |u(x)|^{p(x)} \omega(x) dx < \infty \Big\}.$$

Then, $L^{p(x)}(\Omega, \omega)$ endowed with the Luxemburg norm

$$|u|_{L^{p(x)}(\Omega,\omega)} = \inf \Big\{ \lambda > 0 : \int_\Omega |\frac{u(x)}{\lambda}|^{p(x)} \omega(x) dx \leq 1 \Big\}$$

Proposition 1.3.1 *The space* $\Big(L^{p(x)}(\Omega, \omega), |u|_{L^{p(x)}(\Omega,\omega)} \Big)$ *is of Banach.*

Let us note $\rho_\omega(u) = \displaystyle\int_\Omega \omega(x) |u(x)|^{p(x)} dx \quad$ for all $u \in L^{p(x)}(\Omega, \omega)$,
the modular of the space $L^{p(x)}(\Omega, \omega)$

Remark 1.3.1 *In simple case when* $\omega(x) \equiv 1$, *we have* $L^{p(x)}(\Omega, \omega) \equiv L^{p(x)}(\Omega)$ *and we use the notation* $|u|_{L^{p(x)}(\Omega)}$ *instead of* $|u|_{L^{p(x)}(\Omega,\omega)}$ *and* $\rho_\omega(u) = \rho(u) = \displaystyle\int_\Omega |u(x)|^{p(x)} dx$

Lemma 1.3.1 *(See [58]) For all function* $u \in L^{p(x)}(\Omega, \omega)$. *There are the following assertions*

$$|u|_{L^{p(x)}(\Omega,\omega)} < 1 (= 1; > 1) \ if \ \ and \ \ only \ \ if \, \rho(u) < 1 (= 1; > 1), \tag{1.3.1}$$

$$if \ |u|_{L^{p(x)}(\Omega,\omega)} > 1 \ then \ |u|^{p^-}_{L^{p(x)}(\Omega,\omega)} \leq \rho(u) \leq |u|^{p^+}_{L^{p(x)}(\Omega,\omega)}, \tag{1.3.2}$$

$$if \ |u|_{L^{p(x)}(\Omega,\omega)} < 1 \ then \ |u|^{p^+}_{L^{p(x)}(\Omega,\omega)} \leq \rho(u) \leq |u|^{p^-}_{L^{p(x)}(\Omega,\omega)}. \tag{1.3.3}$$

Remark 1.3.2 *(See [60]) If we set*

$$I(u) = \int_\Omega |u(x)|^{p(x)} + \omega(x) |\nabla u(x)|^{p(x)} dx.$$

Then, following the same argument, we have

$$\min \Big\{ \|u\|^{p^-}_{W^{1,p(x)}(\Omega,\omega)}, \|u\|^{p^+}_{W^{1,p(x)}(\Omega,\omega)} \Big\} \leq I(u) \leq \max \Big\{ \|u\|^{p^-}_{W^{1,p(x)}(\Omega,\omega)}, \|u\|^{p^+}_{W^{1,p(x)}(\Omega,\omega)} \Big\}.$$

Let ω be a weight function satisfying that
$(\mathbf{W}_1)$ $\omega \in L^1_{loc(\Omega)}$ and $\omega^{-\frac{1}{(p(x)-1)}} \in L^1_{loc}(\Omega)$;
$(\mathbf{W}_2)$ $\omega^{-s(x)} \in L^1(\Omega, \omega)$ with $s(x) \in (\frac{N}{p(x)}, \infty) \cap [\frac{1}{p(x)-1}, \infty)$.
The reasons that we assume (W_1) and (W_2) can be found in [50].

1.3.2 Spaces of Sobolev with weight and to variable exponents.

Let $p \in C_+(\overline{\Omega})$ and ω be a weight function in Ω. We define The weighted variable exponent Sobolev space $W^{1,p(x)}(\Omega,\omega)$ by

$$W^{1,p(x)}(\Omega,\omega) = \Big\{u \in L^{p(x)}(\Omega); |\nabla u| \in L^{p(x)}(\Omega,\omega)\Big\},$$

where the norm is

$$\|u\|_{W^{1,p(x)}(\Omega,\omega)} = |u|_{L^{p(x)}(\Omega)} + |\nabla u|_{L^{p(x)}(\Omega,\omega)}, \tag{1.3.4}$$

or, equivalently

$$\|u\|_{W^{1,p(x)}(\Omega,\omega)} = \inf\Big\{\lambda > 0 : \int_\Omega |\frac{u(x)}{\lambda}|^{p(x)} + \omega(x)|\frac{\nabla u(x)}{\lambda}|^{p(x)}dx \leq 1\Big\}$$

for all $u \in W^{1,p(x)}(\Omega,\omega)$.

Remark 1.3.3 *([60])*
(i) *If ω is a positive measurable and finite function, then $L^{p(x)}(\Omega,\omega)$ is a reflexive Banach space.*
(ii) *Moreover, if (W_1) holds, then $W^{1,p(x)}(\Omega,\omega)$ is a reflexive Banach space.*

For $p, s \in C_+(\overline{\Omega})$, denote $p_s(x) := \frac{p(x)s(x)}{s(x)+1} < p(x)$, where $s(x)$ is given in (W_2). Assume that we fix the variable exponent restrictions

$$\begin{cases} p_s^*(x) := \frac{p(x)s(x)N}{(s(x)+1)N - p(x)s(x)} & \text{if } N > p_s(x), \\ p_s^*(x) \text{ arbitrary} & \text{if } N \leq p_s(x) \end{cases}$$

for almost all $x \in \Omega$. These definitions play a key role in our chapter. We shall frequently make use of the following (compact) imbedding theorem for the weighted variable exponent Lebesgue–Sobolev space in the next sections.

Lemma 1.3.2 *([60]) Let $p, s \in C_+(\overline{\Omega})$ satisfy the log-Hölder continuity condition (1.1.1) and let (W_1) and (W_2) be satisfied. If $r \in C_+(\overline{\Omega})$ and $1 < r(x) \leq p_s^*(x)$ then, we obtain the continuous imbedding*

$$W^{1,p(x)}(\Omega,\omega) \hookrightarrow L^{r(x)}(\Omega).$$

Moreover, we have the compact imbedding

$$W^{1,p(x)}(\Omega,\omega) \hookrightarrow L^{r(x)}(\Omega),$$

provided that $1 < r(x) < p_s^*(x)$ for all $x \in \overline{\Omega}$.

From Lemma 1.3.2, we have Poincaré–type inequality immediately.

Corollary 1.3.3 *([60]) Let $p \in C_+(\overline{\Omega})$ satisfy the log–Hölder continuity condition (1.1.1). If (W_1) and (W_2) hold, then the estimate*

$$\|u\|_{L^{p(x)}(\Omega)} \leq C\|\nabla u\|_{L^{p(x)}(\Omega,\omega)}$$

holds, for every $u \in C_0^\infty(\Omega)$ with a positive constant C independent of u.

Throughout this chapter, let $p \in C_+(\overline{\Omega})$ satisfy the log-Hölder continuity condition (1.1.1) and $X := W_0^{1,p(x)}(\Omega,\omega)$ be the weighted variable exponent Sobolev space that consists of all real valued functions u from $W^{1,p(x)}(\Omega,\omega)$ which vanish on the boundary $\partial\Omega$, endowed with the norm

$$\|u\|_X = \inf\{\lambda > 0 : \int_\Omega \Big|\frac{\nabla u(x)}{\lambda}\Big|^{p(x)} \omega(x)dx \leq 1\},$$

which is equivalent to the norm 1.3.4 due to Corollary (1.3.3). The following proposition gives the characterization of the dual space $(W_0^{k,p(x)}(\Omega,\omega))^*$, which is analogous to $\Big[$[65], Theorem 3.16$\Big]$. We recall that the dual space of weighted Sobolev spaces $W_0^{1,p(x)}(\Omega,\omega)$ is equivalent to $W^{-1,p'(x)}(\Omega,\omega)$, where $\omega^* = \omega^{1-p'(x)}$.

1.3.3 Some technical results in weighted Sobolev spaces with variable expoponent.

Lemma 1.3.4 *([34]) Let $g \in L^{p(x)}(\Omega,\omega)$ and let $g_n \in L^{p(x)}(\Omega,\omega)$, with $\|g_n\|_{L^{p(x)}(\Omega,\omega)} \leq c$, $1 < r(x) < \infty$. If $g_n(x) \to g(x)$ a.e. in Ω, then $g_n \rightharpoonup g$ in $L^{p(x)}(\Omega,\omega)$, where $\rightharpoonup$ denotes weak convergence and ω is a weight function on Ω.*

Lemma 1.3.5 *. Assume (5.2.1)–(5.2.3) and let $(u_n)_n$ be a sequence in $W_0^{1,p(.)}(\Omega,\omega)$ such that $u_n \rightharpoonup u$ weakly in $W_0^{1,p(.)}(\Omega,\omega)$ and*

$$\int_\Omega \Big(a(x,u_n,\nabla u_n) - a(x,u_n,\nabla u)\Big) \cdot \nabla(u_n - u)dx \to 0. \qquad (1.3.5)$$

Then, $u_n \to u$ strongly in $W_0^{1,p(.)}(\Omega,\omega)$.

Proof. Let $D_n = \Big[a(x,u_n,\nabla u_n) - a(x,u_n,\nabla u)\Big]\nabla(u_n - u)$, thanks to (5.2.2), we have D_n is a positive function and by (1.3.5) , $D_n \to 0$ in $L^1(\Omega)$ as $n \to \infty$.
Since $u_n \rightharpoonup u$ in $W_0^{1,p(x)}(\Omega,\omega)$ then, $u_n \to u$ a.e. in Ω and since $D_n \to 0$ a.e. in Ω, there exists a subset B in Ω with measure zero such that for all $x \in \Omega \backslash B$,

$$|u(x)| < \infty, \quad |\nabla u| < \infty, \quad K(x) < \infty, \quad u_n \to u, \quad D_n \to 0.$$

Taking $\xi_n = \nabla u_n$ and $\xi = \nabla u$, we have

$$\begin{aligned}
D_n(x) &= \Big[a(x,u_n,\xi_n) - a(x,u_n,\xi)\Big]\cdot(\xi_n-\xi)\\
&= a(x,u_n,\xi_n)\xi_n + a(x,u_n,\xi)\xi - a(x,u_n,\xi_n)\xi - a(x,u_n,\xi)\xi_n\\
&\geq \alpha\omega(x)|\xi_n|^{p(x)} + \alpha\omega(x)|\xi|^{p(x)}\\
&- \beta\omega^{1/p(x)}(x)\Big(k(x) + \omega^{1/p'(x)}(x)|u_n|^{p(x)-1} + \omega^{1/p'(x)}(x)|\xi_n|^{p(x)-1}\Big)|\xi|\\
&- \beta\omega^{1/p(x)}(x)\Big(k(x) + \omega^{1/p'(x)}(x)|u_n|^{p(x)-1} + \omega^{1/p'(x)}(x)|\xi|^{p(x)-1}\Big)|\xi_n|\\
&\geq \alpha\omega(x)|\xi_n|^{p(x)} - C_x[1 + \omega^{1/p'(x)}(x)|\xi_n|^{p(x)-1} + \omega^{1/p(x)}(x)|\xi_n|],
\end{aligned}$$

where C_x depending on x, without dependence on n. (since $u_n(x) \to u(x)$ then $(u_n)_n$ is bounded), we obtain

$$D_n(x) \geq |\xi_n|^{p(x)}\Big(\alpha\omega(x) - \frac{C_x}{|\xi_n|^{p(x)}} - \frac{C_x\omega^{\frac{1}{p'(x)}}}{|\xi_n|} - \frac{C_x\omega^{\frac{1}{p(x)}}}{|\xi_n|^{p(x)-1}}\Big)$$

by the standard argument $(\xi_n)_n$ is bounded almost everywhere in Ω, Indeed, if $|\xi_n| \to \infty$ in a measurable subset $E \in \Omega$. Then,

$$\lim_{n\to\infty}\int_\Omega D_n(x)dx \geq \lim_{n\to\infty}\int_E |\xi_n|^{p(x)}\Big(\alpha\omega(x) - \frac{C_x}{|\xi_n|^{p(x)}} - \frac{C_x\omega^{\frac{1}{p'(x)}}}{|\xi_n|} - \frac{C_x\omega^{\frac{1}{p(x)}}}{|\xi_n|^{p(x)-1}}\Big) = \infty,$$

which is absurd since $D_n(x) \to 0$ in $L^1(\Omega)$). Let ξ^* an accumulation point of $(\xi_n)_n$, we have $|\xi^*| < \infty$ and by continuity of $a(.,.,.)$, we obtain

$$a(x,u(x),\xi^*) - a(x,u(x),\xi)]\cdot(\xi_n-\xi) = 0,$$

thanks to (5.2.2), we have $\xi^* = \xi$, the uniqueness of the accumulation point implies that $\nabla u_n(x) \to \nabla u(x)$ a.e. in Ω. Since the sequence $a(x,u_n,\nabla u_n)$ is bounded in $(L^{p'(x)}(\Omega,\omega^*))^N$ and $a(x,u_n,\nabla u_n) \to a(x,u,\nabla u)$ a.e. in Ω, Lemma 1.3.4 implies

$$a(x,u_n,\nabla u_n) \to a(x,u,\nabla u) \quad \text{in } (L^{p'(x)}(\Omega,\omega^*))^N.$$

Let us taking $\bar{y}_n = a(x,u_n,\nabla u_n)\nabla u_n$ and, $\bar{y} = a(x,u,\nabla u)\nabla u$, then $\bar{y}_n \to \bar{y}$ in $L^1(\Omega)$, according to the condition (5.2.3), we have

$$\alpha\omega(x)|\nabla u_n|^{p(x)} \leq a(x,u_n,\nabla u_n)\nabla u_n.$$

Let $z_n = |\nabla u_n|^{p(x)}\omega,\ z = |\nabla u|^{p(x)}\omega$ and $y_n = \frac{\bar{y}_n}{\alpha},\ y = \frac{\bar{y}}{\alpha}$. Then, by Fatou's Lemma, we obtain

$$\int_\Omega 2y\,dxdt \leq \liminf_{n\to\infty}\int_\Omega (y_n + y - |z_n - z|)dx,$$

i.e., $0 \leq \limsup_{n\to\infty} \int_\Omega |z_n - z|dx$, hence

$$0 \leq \liminf_{n\to\infty} \int_\Omega |z_n - z|dx \leq \limsup_{n\to\infty} \int_\Omega |z_n - z|dx \leq 0,$$

this implies

$$\nabla u_n \to \nabla u \quad \text{in } (L^{p(x)}(\Omega,\omega))^N,$$

we deduce that

$$u \to u \quad \text{in } W_0^{1,p(x)}(\Omega,\omega),$$

which completes our proof.

1.4 Integration Results

Fatou's lemma

Let f_n be a sequence of positive measurable functions. So

$$\int_\Omega \Big(\liminf_{n\to\infty} f_n \Big) dx \leq \liminf_{n\to\infty} \Big(\int_\Omega f_n dx \Big)$$

Theorem of Convergence Lebesgue dominated

Let f_n be a sequence of function of $L^1(\Omega)$ convergence almost everywhere a measurable function f.We assume that there are $g \in L^1(\Omega)$ such that for all $n \geq 1$, we have $|f_n(x)| \leq g(x)$ almost everywhere on Ω. So $f \in L^1(\Omega)$ and

$$\lim_{n\to\infty} \int_\Omega |f_n - f|dx = 0, \lim_{n\to\infty} \int_\Omega f_n dx = \int_\Omega f dx$$

Definition of equi-integrability

We say that f_n a sequence of function of $L^1(\Omega)$ is equi-integrable if : for everything $\epsilon > 0$ there exists $\delta > 0$ such as $\forall E \subset \Omega$ measurable with $mes(E) < \delta$, we have

$$\int_\Omega f_n dx < \epsilon.$$

An important theorem to show strong convergence of a sequence from convergence almost everywhere is the theorem of Vitali

Vitali Theorem

Let $1 \leq p < +\infty, \Omega \subset \mathbb{R}^{\mathbb{N}}$ and $(f_n)_{n\in\mathbb{N}}$ be a sequence of function of $L^p(\Omega)$ convergence almost everywhere to a function f. So

$$f_n \to f \quad \text{strongly in } L^p(\Omega) \Leftrightarrow (f_n)_n \quad \text{ is p - equi-integrability}$$

1.5 Monotone Operators

In the following, V is a reflexive Banach space and separable, A application of V in V'

Definition 1.5.1 *We say that*

i) *A is monotone if :*

$$\forall u, v \in V \quad \langle A(u) - A(v), u - v\rangle \geq 0$$

ii) *A is hemi continuous if : for all $u, v, w \in V$, application*

$$t \to \langle A(u + tv), w\rangle \textit{ is continuous de } \mathbb{R} \textit{ in } \mathbb{R}$$

iii) *A is coercive if :*

$$\lim_{\|u\|_V \to \infty} \frac{\langle A(u), v\rangle}{\|u\|_V} = +\infty$$

Theorem 1.5.1 *Let A be an operator satisfying :*

1) *A is bounded*

2) *A is hemi continuous*

3) *A is monotone*

4) *A is coercive.*
Then A is surjective of $V \to V'$ i.e., for $f \in V'$, it exists $u \in V$ such that $A(u) = f$.

Definition 1.5.2 *A monotone map $T : D(T) \to X^*$ is called maximal monotone if its graph*

$$G(T) = \Big\{(u, T(u)) \in X \times X^* \textit{ for all } u \in D(T)\Big\},$$

is not a proper subset of any monotone set in $X \times X^*$.
Let us consider the operator $\frac{\partial}{\partial t}$ which induces a linear map L from the subset

$$D(L) = \Big\{v \in X : v' \in X^*, v(0) = 0\Big\} \textit{ of } X \textit{ in to } X^* \textit{ by}$$

$$\Big\langle Lu, v\Big\rangle_X = \int_0^T \langle u'(t), v(t)\rangle_V dt \quad u \in D(L),\ v \in X.$$

Lemma 1.5.2 *L is a closed linear maximal monotone map.*

Definition 1.5.3 *See [8] A mapping S is called pseudo–monotone with $u_n \rightharpoonup u$ and* $Lu_n \rightharpoonup Lu$ *and* $\lim_{n\to\infty} \sup\langle S(u_n), u_n - u\rangle \leq 0,$ *that we have* $\lim_{n\to\infty} \sup \left\langle S(u_n), u_n - u \right\rangle = 0$ *and* $S(u_n) \rightharpoonup S(u)$ *as* $n \to \infty$.

Theorem 1.5.3 *Let A is pseudo–monotone and coercive. Then, $\forall f \in V'$ the equation $A(u) = f$ admits at least one solution.*

Chapitre 2

Existence of Renormalized Solutions for $p(x)$-Parabolic Equations with three Unbounded Nonlinearities

Abstract

In this chapter, we study the existence of a renormalized solution for the nonlinear $p(x)$-parabolic problem associated to the equation :

$$\frac{\partial b(x,u)}{\partial t} - \operatorname{div}(a(x,t,u,\nabla u)) + H(x,t,u,\nabla u) = f - \operatorname{div} F \text{ in } \ Q = \Omega \times (0,T)$$

with $f \in L^1(Q)$, $b(x,u_0) \in L^1(\Omega)$ and $F \in (L^{P'(x)}(Q))^N$.

The main contribution of our work is to prove the existence of a renormalized solution in the Sobolev space with variable exponent. The critical growth condition on $H(x,t,u,\nabla u)$ is with respect to ∇u, no growth with respect to u and no sign condition or the coercivity condition.

2.1 Introduction

In the present chapter we establish the existence of a renormalized solution for a class of nonlinear $p(x)$-parabolic equation of the type :

$$\begin{cases} \frac{\partial b(x,u)}{\partial t} - \operatorname{div}\Big(a(x,t,u,\nabla u)\Big) + H(x,t,u,\nabla u) = f - \operatorname{div} F & \text{dans } Q = \Omega \times (0,T) \\ b(x,u)\mid_{t=0} = b(x,u_0) & \text{dans } \Omega \\ u = 0 & \text{sur } \partial\Omega \times (0,T). \end{cases} \tag{2.1.1}$$

In the problem (2.1.1), Ω is a bounded domain in $\mathbb{R}^N (N \geq 1)$, T is a positive real number, while $b(x, u_0) \in L^1(\Omega)$, $f \in L^1(Q)$ and $F \in (L^{P'(x)}(Q))^N$.
The operator $-\text{div}(a(x, t, u, \nabla u)$ is a Leray–Lions operator defined on $L^{p^-}(0, T; W_0^{1,p(.)}(\Omega))$ (see assumption (2.2.3)-(2.2.5) of section 2) which is coercive $b(x, u)$ is an unbounded function of u, H is a non linear lower order term.
The notion of renormalized solutions was introduced by R. J. Diperna and P. L. Lions [51] for the study of the Boltzmann equation, it was then used by L. Boccardo and al [42] when the right hand side is in $W^{-1,p'}(\Omega)$ and by J. M Rakoston [75] when the right hand side is in $L^1(\Omega)$.
For the degenerated parabolic equations the existence of weak solutions have been proved by L. Aharouch and al [3] in the case where $a(x, t, u, \nabla u)$ is strictly monotone $H = 0$, $F = 0$ and $f \in L^{p'}(0, T, W^{-1,p'}(\Omega, W^*))$, see also the existence and uniqueness of a renormalized solution proved by Y. Akdim and al [8] in the case where $a(x, t, s, \xi)$ is independent of s, $H = 0$ and $F = 0$.
In the case $H(x, t, u, \nabla u) = \text{div}\phi(u)$ and $F = 0$, the existence of renormalized solutions has been established by H. Redwane in the classical Sobolev space and in Orlicz space[76, 79] and by Y. Akdim and al [7] in the degenerate Sobolev space without the sign condition and the coercivity condition on the term $H(x, t, u, \nabla u) = \text{div}(\phi(x, t, u))$ and $F = 0$, the existence of renormalized solutions has been established by A.Aberqi and al [1] in the classical Sobolev space.
Recently while $b(x, u) = u$, $a(x, t, u, \nabla u) = |\nabla u|^{p(x)-2}\nabla u$ and $F = 0$, C. Zhang and S. Zhou [85] proved the existence of renormalized and entropy solutions with L^1-data and see also M. Bendahmane, P. Wittbold, A. Zimmermann [29] proved the existence of renormalized solutions for a nonlinear parabolic equation with L^1-data. It is our purpose to prove the existence of a renormalized solution of variable exponent Sobolev spaces for the problem (2.1.1) setting without the sign condition and without the coercivity condition, the critical growth condition on H is only with respect to ∇u and not with respect to u (see assumption H2), where the right hand side is assumed to satisfy : f belongs to $L^1(Q)$ and $F \in (L^{P'(x)}(Q))^N$.
This chapter is organized as follows : In Section 2 we make precise all the assumption on b, a, H, f and $b(x, u_0)$ and give the definition of a renormalized solution of the problem (2.1.1) for which our problem has a solution. In Section 3 we establish the existence of such a solution (Theorem2.3.1). In Section 4 we give the proof of Theorem (2.3.2), lemma (2.3.4) and proposition (2.3.2) (see appendix). Section 5

is devoted to an example which illustrates our abstract result.

2.2 Essential Assumption

Throughout the paper, we assume that the following assumptions hold true.

ASSUMPTION (H1)

Let Ω be a bounded open subset of $\mathbb{R}^N (N \geq 1)$, $p \in C_+(\bar{\Omega})$ and $b : \Omega \times \mathbb{R} \to \mathbb{R}$ is a Carathéodory function such that for every $x \in \Omega$, $b(x, .)$ is a strictly increasing C^1 function with

$$b(x, 0) = 0. \tag{2.2.1}$$

Next, for any $k > 0$, there exist $\lambda_k > 0$ and functions $A_k \in L^\infty(\Omega)$ and $B_k \in L^{p(x)}(\Omega)$ such that

$$\lambda_k \leq \frac{\partial b(x,s)}{\partial s} \leq A_k(x) \ \text{ and } \quad \left| D_x\Big(\frac{\partial b(x,s)}{\partial s}\Big)\right| \leq B_k(x). \tag{2.2.2}$$

for almost every $x \in \Omega$ and every s such that $|s| \leq k$, we denote by $D_x(\partial b(x,s) \setminus \partial s)$ the gradient of $\partial b(x,s) \setminus \partial s$ defined in the sense of distributions.

ASSUMPTION (H2)

We consider a Leray–Lions operator defined by the formula

$$Au = -\text{div}\Big(a(x, t, u, \nabla u)\Big),$$

where $a : \Omega \times [0, T] \times \mathbb{R} \times \mathbb{R}^N \to \mathbb{R}$ is a Carathéodory function, i.e., (measurable with respect to x in Ω for every $(s, \xi) \in \mathbb{R} \times \mathbb{R}^N$ and continuous with respect to $(s, \xi) \in \mathbb{R} \times \mathbb{R}^N$ for almost every x in Ω) which satisfies the following conditions there exist $k \in L^{p'(.)}(Q)$ *and* $\alpha > 0$, $\beta > 0$ such that, for almost every $(x, t) \in Q$ all $(s, \xi) \in \mathbb{R} \times \mathbb{R}^N$.

$$|a(x, t, s, \xi)| \leq \beta\Big[k(x, t) + |s|^{p(x)-1} + |\xi|^{p(x)-1}\Big] \tag{2.2.3}$$

$$\Big[a(x, t, s, \xi) - a(x, t, s, \eta)\Big](\xi - \eta) > 0 \ \ \forall\, \xi \neq \eta \tag{2.2.4}$$

$$a(x, t, s, \xi) \cdot \xi \geq \alpha|\xi|^{p(x)}. \tag{2.2.5}$$

ASSUMPTION (H3)

Let $H : \Omega \times [0, T] \times \mathbb{R} \times \mathbb{R}^N \to \mathbb{R}$ be a Carathéodory function such that for a.e. $(x, t) \in Q$ and for all $s \in \mathbb{R}$, $\xi \in \mathbb{R}^N$, the growth condition

$$|H(x, t, s, \xi)| \leq \gamma(x, t) + g(s)|\xi|^{p(x)}. \tag{2.2.6}$$

is satisfied, where $g : \mathbb{R} \to \mathbb{R}^+$ is a bounded continuous positive function that belongs to $L^1(\mathbb{R})$, while $\gamma \in L^1(Q)$.

Definition 2.2.1 *Let $f \in L^1(Q)$, $F \in (L^{P'(.)}(Q))^N$ and $b(.,u_0) \in L^1(\Omega)$*
A real–valued function u defined on Q is a renormalized solution of problem (2.1.1) if

$$T_k(u) \in L^{p^-}(0,T; W_0^{1,p(x)}(\Omega)) \text{ for all } k \geq 0,\ b(x,u) \in L^\infty(0,T;L^1(\Omega)), \tag{2.2.7}$$

$$\int_{\{m\leq|u|\leq m+1\}} a(x,t,u,\nabla u)\nabla u dxdt \to 0 \text{ as } m \to \infty, \tag{2.2.8}$$

$$\begin{aligned}&\frac{\partial B_S(x,u)}{\partial t} - div\Big(S'(u)a(x,t,u,\nabla u)\Big) + S''(u)a(x,t,u,\nabla u)\nabla u\\ &+H(x,t,u,\nabla u)S'(u) = fS'(u) - div\Big(S'(u)F\Big) + S''(u)F\nabla u \text{ in } D'(Q),\end{aligned} \tag{2.2.9}$$

for all $S \in W^{2,\infty}(\mathbb{R})$ which are piecewise C^1 and such that S' has a compact support in $\mathbb{R}$, where $B_S(x,z) = \displaystyle\int_0^z \frac{\partial b(x,r)}{\partial r} S'(r)dr$ and

$$B_S(x,u)\mid_{t=0} = B_S(x,u_0) \quad in \quad \Omega. \tag{2.2.10}$$

Remark 2.2.1 *Equation (2.2.9) is formally obtained through pointwise multiplication of problem (2.1.1) by $S'(u)$. However, while $a(x,t,u,\nabla u)$ and $H(x,t,u,\nabla u)$ do not in general make sense in $(\mathcal{P})$, all the terms in (2.2.9) have a meaning in $D'(Q)$. Indeed, if M is such that supp $S' \subset [-M,M]$, the following identifications are made in (2.2.9)*
•$S(u)$ belongs to $V \cap L^\infty(Q)$. Since S is a bounded function.
•$S'(u)\ a(x,t,u,\nabla u)$ identifies with $S'(u)\ a(x,t,T_M(u),\nabla T_M(u))$ a.e. in Q. for any $\varphi \in D(Q)$, using Hölder inequality

$$\begin{aligned}\int_Q S'(u)a(x,t,u,\nabla u)\nabla\varphi dxdt &= \int_Q S'(u)a(x,t,T_M(u),\nabla T_M(u))\nabla\varphi dxdt\\ &\leq C_M\|S'\|_{L^\infty(Q)} \max\Big\{\Big(\int_Q |\nabla T_M(u)|^{p(x)}\Big)^{\frac{1}{p'^-}},\\ &\Big(\int_Q |\nabla T_M(u)|^{p(x)}\Big)^{\frac{1}{p'^+}}\Big\}\|\nabla\varphi\|_{L^{p(.)}(Q)},\end{aligned}$$

where $M > 0$ is that supp $S' \subset [-M,M]$. As $D(Q)$ is dense in V, we deduce that

$$div(S'(u)a(x,t,u,\nabla u)) \in V^*.$$

•$S''(u)\ a(x,t,u,\nabla u)\nabla u$ identifies with $S''(u)\ a(x,u,T_M(u),\nabla T_M(u))\nabla T_M(u)$ and

$$S''(u)a\Big(x,u,T_M(u),\nabla T_M(u)\Big)\nabla T_M(u) \in L^1(Q).$$

•$S'(u)\ H(x,t,u,\nabla u)$ *identifies with* $S'(u)H(x,t,T_M(u),\nabla T_M(u))$ *a.e. in* Q. *Since* $|T_M(u)| \leq M$ *a.e. in* Q *and* $S'(u) \in L^\infty(Q)$, *we see from (2.2.6) and (2.2.7)that*

$$S'(u)H(x,t,T_M(u),\nabla T_M(u)) \in L^1(Q).$$

•$S'(u)\ f$ *belongs to* $L^1(Q)$ *while* $S'(u)F$ *belongs to* $(L^{p'(.)}(Q))^N$.
•$S''(u)\ F\nabla u$ *identifies with* $S''(u)\ F\nabla T_M(u)$, *which belongs to* $L^1(Q)$. *The above considerations show that equation (2.2.9) hold in* $D'(Q)$ *and that*

$$\frac{\partial B_S(x,u)}{\partial t} \in V^* + L^1(Q).$$

Due to the properties of S *and (2.2.9)* $\frac{\partial S(u)}{\partial t} \in V^* + L^1(Q)$ *using lemma (1.2.3), which implies that* $S(u) \in C([0,T);L^1(\Omega))$, *so that the initial condition (2.2.10) makes sense, since, due to the properties of* S *(increasing) and (2.2.2), we have*

$$\Big|(B_S(x,r) - B_S(x,r')\Big| \leq A_k(x)\Big|S(r) - S(r')\Big|\ for\ all\ r,r' \in\ \mathbb{R}. \tag{2.2.11}$$

2.3 Existence results.

In this section, we establish the following existence theorem :

Theorem 2.3.1 *Let* $f \in L^1(Q)$, $F \in (L^{p'(x)}(Q))^N$, $p(.) \in C_+(\bar{\Omega})$ *and assume that* u_0 *is a measurable function such that* $b(.,u_0) \in L^1(\Omega)$. *Assume that (H1)–(H3) hold true. Then, there exists a renormalized solution* u *of problem (2.1.1) in the sense of Definition (2.2.1).*

Proof. The proof is in five steps.
STEP 1 : **Approximate problem** :
For $n > 0$, we define approximations of b, H, f, F and u_0. First, set

$$b_n(x,r) = b(x,T_n(r)) + \frac{1}{n}r. \tag{2.3.1}$$

b_n is a Carathéodory function and satisfies (2.2.2) : there exist $\lambda_n > 0$ and functions $A_n \in L^\infty(\Omega)$ and $B_n \in L^{p(.)}(\Omega)$ such that

$$\lambda_n \leq \frac{\partial b_n(x,s)}{\partial s} \leq A_n(x) \text{ and } \Big|D_x\Big(\frac{\partial b_n(x,s)}{\partial s}\Big)\Big| \leq B_n(x)\ a.e. \text{ in } \Omega,\ s \in \mathbb{R}.$$

Next, set

$$\begin{aligned} H_n(x,t,s,\xi) &= \frac{H(x,t,s,\xi)}{1+\frac{1}{n}|H(x,t,s,\xi)|}, \\ Note\ that \quad |H_n(x,t,s,\xi)| &\le |H(x,t,s,\xi)| \\ and \quad |H_n(x,t,s,\xi)| &\le n \text{ for all } (s,\xi)\in\mathbb{R}\times\mathbb{R}^N. \end{aligned}$$

and select f_n, u_{0n} and b_n so that

$$f_n \in L^{p'(.)}(Q) \text{ and } f_n \to f \text{ a.e. in } Q \text{ and strongly in } L^1(Q) \text{ as } n \to \infty, \quad (2.3.2)$$

$$u_{0n} \in D(\Omega), \qquad \|b_n(x,u_{0n})\|_{L^1(\Omega)} \le \| b(x,u_0)\|_{L^1(\Omega)}, \quad (2.3.3)$$

$$b_n(x,u_{0n}) \to b(x,u_0) \text{ a.e. in } \Omega \text{ and strongly in } L^1(\Omega). \quad (2.3.4)$$

Let us now consider the approximate problem

$$(P_2)_n \begin{cases} \frac{\partial b_n(x,u_n)}{\partial t} - \operatorname{div}\Big(a(x,t,u_n,\nabla u_n)\Big) + H_n(x,t,u_n,\nabla u_n) = f_n - \operatorname{div}F & \text{in } D'(Q), \\ b_n(x,u_n)\,|_{t=0} = b_n(x,u_{0n}) & \text{in } \Omega \\ u_n = 0 & \text{on } \partial\Omega\times(0,T). \end{cases}$$

Theorem 2.3.2 *Let $f_n \in L^{p'^-}(0,T;W^{-1,p'(.)}(\Omega))$, $p(.) \in C_+(\overline{\Omega})$ for fixed n, the approximate problem $(P_2)_n$ has at least one weak solution $u_n \in L^{p^-}(0,T;W_0^{1,p(.)}(\Omega))$.*

Proof. See Appendix.
In view of Theorem (2.3.2), there exists at least one weak solution $u_n \in L^{p^-}(0;T;W_0^{1,p(.)}(\Omega))$ of the problem $(P_2)_n$. (see[65]).
STEP 2 : A Priori Estimates :

Proposition 2.3.1 *Let u_n a solution of the approximate problem $(P_2)_n$. Then, there exists a constant C (which does not depend on the n and k) such that*

$$\|T_k(u_n)\|_{L^{p^-}(0,T;W_0^{1,p(.)}(\Omega))} \le C\,k \qquad \forall\, k>0.$$

Proof.
Let $\varphi \in L^{p^-}(0,T;W_0^{1,p(.)}(\Omega)) \cap L^\infty(Q)$, with $\varphi > 0$, Choosing $v = \exp(G(u_n))\varphi$ as a test function in $(P_2)_n$ where $G(s) = \displaystyle\int_0^s (\frac{g(r)}{\alpha})dr$. (the function g appears in

(2.2.6)), we have

$$\int_Q \frac{\partial b_n(x,u_n)}{\partial t}\exp(G(u_n))\varphi dxdt + \int_Q a(x,t,u_n,\nabla u_n)\nabla(\exp(G(u_n))\varphi)dxdt$$
$$+\int_Q H_n(x,t,u_n,\nabla u_n)\exp(G(u_n))\varphi dxdt$$
$$= \int_Q f_n \exp(G(u_n))\varphi dxdt + \int_Q F\nabla(\exp(G(u_n))\varphi)dxdt.$$

In view of (2.2.6), we obtain

$$\int_Q \frac{\partial b_n(x,u_n)}{\partial t}\exp(G(u_n))\varphi dxdt + \int_Q a(x,t,u_n,\nabla u_n)\nabla u_n \frac{g(u_n)}{\alpha}\exp(G(u_n))\varphi dxdt$$
$$+\int_Q a(x,t,u_n,\nabla u_n)\exp(G(u_n))\nabla\varphi dxdt$$
$$\leq \int_Q \gamma(x,t)\exp(G(u_n))\varphi dxdt + \int_Q f_n\exp(G(u_n))\varphi dxdt$$
$$+\int_Q g(u_n)|\nabla u_n|^{p(x)}\exp(G(u_n))\varphi dxdt + \int_Q F\nabla(\exp(G(u_n))\varphi)dxdt$$

By using (2.2.5), we obtain

$$\int_Q \frac{\partial b_n(x,u_n)}{\partial t}\exp(G(u_n))\varphi dxdt + \int_Q a(x,t,u_n,\nabla u_n)\exp(G(u_n))\nabla\varphi dxdt$$
$$\leq \int_Q \gamma(x,t)\exp(G(u_n))\varphi dxdt + \int_Q f_n\exp(G(u_n))\varphi dxdt$$
$$+\int_Q F\nabla(\exp(G(u_n)))\varphi dxdt + \int_Q F\exp(G(u_n))\nabla\varphi dxdt \qquad (2.3.5)$$

for all $\varphi \in L^{p^-}(0,T;W_0^{1,p(.)}(\Omega))\cap L^\infty(Q)$, with $\varphi > 0$.
On the other hand, taking $v = \exp(-G(u_n))\varphi$ as a test function in $(P_2)_n$, we deduce as in (2.3.5), that

$$\int_Q \frac{\partial b_n(x,u_n)}{\partial t}\exp(-G(u_n))\varphi dxdt + \int_Q a(x,t,u_n,\nabla u_n)\exp(-G(u_n))\nabla\varphi dxdt$$
$$+\int_Q \gamma(x,t)\exp(-G(u_n))\varphi dxdt \geq \int_Q f_n\exp(-G(u_n))\varphi dxdt$$
$$+\int_Q F\nabla(\exp(-G(u_n)))\varphi dxdt + \int_Q F\exp(-G(u_n))\nabla\varphi dxdt \quad (2.3.6)$$

for all $\varphi \in L^{p^-}(0,T;W_0^{1,p(x)}(\Omega)) \cap L^\infty(Q)$, with $\varphi > 0$.
Letting $\varphi = T_k(u_n)^+\chi(0,\tau)$, for every $\tau \in [0,T]$ in (2.3.5), we have

$$\begin{aligned}
&\int_\Omega B^n_{k,G}(x,u_n(\tau))dx + \int_{Q^\tau} a(x,t,u_n,\nabla u_n)\exp(G(u_n))\nabla T_k(u_n)^+dxdt \\
&\leq \int_{Q^\tau} \gamma(x,t)\exp(G(u_n))T_k(u_n)^+dxdt + \int_{Q^\tau} f_n\exp(G(u_n))T_k(u_n)^+dxdt \\
&+ \int_{Q^\tau} F\nabla T_k(u_n)^+\exp(G(u_n))dxdt \qquad (2.3.7)\\
&+ \int_{Q^\tau} FT_k(u_n)^+\exp(G(u_n))\nabla u_n\frac{g(u_n)}{\alpha}dxdt + \int_\Omega B^n_{k,G}(x,u_{0n})dx,
\end{aligned}$$

where

$$B^n_{k,G}(x,r) = \int_0^r \frac{\partial b_n(x,s)}{\partial s}T_k(s)^+\exp(G(s))ds.$$

Due to the definition of $B^n_{k,G}$ and $|G(u_n)| \leq \frac{\|g\|_{L^1(\mathbb{R})}}{\alpha}$, we have

$$0 \leq \int_\Omega B^n_{k,G}(x,u_{0n})dx \leq k\exp\Big(\frac{\|g\|_{L^1(\mathbb{R})}}{\alpha}\Big)\|b(.,u_0\|_{L^1(\Omega)}. \qquad (2.3.8)$$

Using (2.3.8) and $B^n_{k,G}(x,u_n) \geq 0$, we obtain

$$\begin{aligned}
&\int_{Q^\tau} a(x,t,u_n,\nabla T_k(u_n)^+)\exp(G(u_n))\nabla T_k(u_n)^+dxdt \\
&\qquad \leq k\exp\Big(\frac{\|g\|_{L^1(\mathbb{R})}}{\alpha}\Big)\Big[\|f_n\|_{L^1(Q)} + \|\gamma\|_{L^1(Q)} + \|b_n(x,u_{0n}\|_{L^1(\Omega)}\Big] \\
&\qquad + \frac{1}{\alpha}\int_{Q^\tau} FT_k(u_n)^+\exp(G(u_n))g(u_n)\nabla u_n dxdt \\
&\qquad + \int_{Q^\tau} F\Big[\exp(G(u_n))\Big]^{1-\frac{1}{p(x)}}\Big[\exp(G(u_n))\Big]^{\frac{1}{p(x)}}\nabla T_k(u_n)^+dxdt
\end{aligned}$$

then

$$\begin{aligned}
&\int_{Q^\tau} a(x,t,u_n,\nabla T_k(u_n)^+)\exp(G(u_n))\nabla T_k(u_n)^+dxdt \\
&\qquad \leq k\exp\Big(\frac{\|g\|_{L^1(\mathbb{R})}}{\alpha}\Big)\Big[\|f_n\|_{L^1(Q)} + \|\gamma\|_{L^1(Q)} + \|b_n(x,u_{0n}\|_{L^1(\Omega)}\Big] \\
&\quad + \frac{1}{\alpha}\int_{Q\tau} FT_k(u_n)^+\exp(G(u_n))\nabla u_n g(u_n)\nabla u_n dxdt \\
&\quad + \int_{Q^\tau} \frac{F\Big[\exp(G(u_n))\Big]^{\frac{1}{p'(x)}}}{\Big[\frac{\alpha}{2}p(x)\Big]^{\frac{1}{p(x)}}}\Big[\frac{\alpha}{2}p(x)\Big]^{\frac{1}{p(x)}}\Big|\nabla T_k(u_n)^+\Big|\Big[\exp(G(u_n))\Big]^{\frac{1}{p(x)}}dxdt
\end{aligned}$$

using Young's Inequality, we obtain

$$\int_{Q^\tau} a(x,t,u_n,\nabla T_k(u_n)^+)\exp(G(u_n))\nabla T_k(u_n)^+dxdt$$

$$\leq k\exp\left(\frac{\|g\|_{L^1(\mathbb{R})}}{\alpha}\right)\left[\|f_n\|_{L^1(Q)}+\|\gamma\|_{L^1(Q)}+\|b_n(x,u_{0n}\|_{L^1(\Omega)}\right]$$

$$+\frac{1}{\alpha}\int_{Q\tau} FT_k(u_n)^+\exp(G(u_n))\nabla u_n g(u_n)\nabla u_n dxdt$$

$$+\int_{Q^\tau}\frac{|F|^{p'(x)}\exp(G(u_n))}{\left[\frac{p'(x)\alpha}{2}p(x)\right]^{\frac{p'(x)}{p(x)}}}dxdt+\frac{\alpha}{2}\int_Q\left|\nabla T_k(u_n)^+\right|^{p(x)}\exp(G(u_n))dxdt$$

$$then\int_{Q^\tau} a(x,t,u_n,\nabla T_k(u_n)^+)\exp(G(u_n))\nabla T_k(u_n)^+dxdt$$

$$\leq k\exp\left(\frac{\|g\|_{L^1(\mathbb{R})}}{\alpha}\right)\left[\|f_n\|_{L^1(Q)}+\|\gamma\|_{L^1(Q)}+\|b_n(x,u_{0n}\|_{L^1(\Omega)}\right]$$

$$+\frac{1}{\alpha}\int_{Q\tau} FT_k(u_n)^+\exp(G(u_n))\nabla u_n g(u_n)\nabla u_n dxdt$$

$$+C\int_{Q^\tau}|F|^{p'(x)}dxdt+\frac{\alpha}{2}\int_{Q^\tau}\left|\nabla T_k(u_n)^+\right|^{p(x)}\exp(G(u_n))dxdt$$

$$and\ since\int_Q|F|^{p'(x)}dxdt \;=\; \rho(F)\leq\max\left\{||F||^{p^-}_{(L^{P'(.)}(Q))^N},||F||^{p^+}_{(L^{P'(.)}(Q))^N}\right\}=C'$$

$$then\int_{Q^\tau} a(x,t,u_n,\nabla T_k(u_n)^+)\exp(G(u_n))\nabla T_k(u_n)^+dxdt$$

$$\leq k\exp\left(\frac{\|g\|_{L^1(\mathbb{R})}}{\alpha}\right)\left[\|f_n\|_{L^1(Q)}+\|\gamma\|_{L^1(Q)}+\|b_n(x,u_{0n}\|_{L^1(\Omega)}\right]$$

$$+C_1+\frac{\alpha}{2}\int_Q|\nabla T_k(u_n)^+|^{p(x)}\exp(G(u_n))dxdt$$

$$+\frac{1}{\alpha}\int_{Q\tau} Fg(u_n)\exp(G(u_n))\nabla u_n\chi_{\{u_n>0\}}dxdt$$

Thanks to (2.2.5), we have

$$\frac{\alpha}{2}\int_{Q^\tau}|\nabla T_k(u_n)^+|^{p(x)}\exp(G(u_n))dxdt$$

$$\leq k\exp\left(\frac{\|g\|_{L^1(\mathbb{R})}}{\alpha}\right)\left[\|f_n\|_{L^1(Q)}+\|\gamma\|_{L^1(Q)}+\|b_n(x,u_{0n}\|_{L^1(\Omega)}\right]+C_1$$

$$+\frac{1}{\alpha}\int_{Q\tau} Fg(u_n)\exp(G(u_n))\nabla u_n\chi_{\{u_n>0\}}dxdt. \tag{2.3.9}$$

Let us observe that if we take $\varphi = \rho(u_n) = \displaystyle\int_0^{u_n} g(s)\chi_{\{s>0\}}ds$ in (2.3.5) and use (2.2.5), we obtain

$$\begin{aligned}
\Big[\int_\Omega B_g^n(x,u_n)dx\Big]_0^T &+ \alpha\int_Q |\nabla u_n|^{p(x)} g(u_n)\chi_{\{u_n>0\}}\exp(G(u_n))dxdt \\
&\le \Big(\int_0^\infty g(s)ds\Big)\exp\Big(\frac{\|g\|_{L^1(\mathbb{R})}}{\alpha}\Big)\Big[\|f_n\|_{L^1(Q)} + \|\gamma\|_{L^1(Q)}\Big] \\
&+ \int_Q F\nabla u_n g(u_n)\chi_{\{u_n>0\}}\exp(G(u_n))dxdt \\
&+\Big(\int_0^\infty g(s)ds\Big)\int_Q |F\nabla u_n|\frac{g(u_n)}{\alpha}\exp(G(u_n))\chi_{\{u_n>0\}}dxdt,
\end{aligned}$$

where

$$B_g^n(x,r) = \int_0^r \frac{\partial b_n(x,s)}{\partial s}\rho(s)\exp(G(s))ds,$$

which implies, using $B_g^n(x,r)\ge 0$ and Young's Inequality, we obtain

$$\begin{aligned}
\alpha&\int_{\{u_n>0\}} |\nabla u_n|^{p(x)} g(u_n)\exp(G(u_n))dxdt \\
&\le \|g\|_\infty \exp\Big(\frac{\|g\|_{L^1(\mathbb{R})}}{\alpha}\Big)\Big[\|\gamma\|_{L^1(Q)} + \|f_n\|_{L^1(Q)} + \|b(x,u_0\|_{L^1(\Omega)}\Big] \\
&\quad +C_2 + \frac{\alpha}{2}\int_{\{u_n>0\}} |\nabla u_n|^{p(x)} g(u_n)\exp(G(u_n))dxdt \\
&\qquad +C_3\|g\|_\infty \\
&\qquad\quad +\frac{\alpha}{2}\|g\|_\infty\int_{\{u_n>0\}} |\nabla u_n|^{p(x)}\frac{g(u_n)}{\alpha}\exp(G(u_n))dxdt
\end{aligned}$$

$$\textit{then} \int_{\{u_n>0\}} g(u_n)|\nabla u_n|^{p(x)}\exp(G(u_n))dxdt \le C_4.$$

Similarly, taking $\varphi = \displaystyle\int_{u_n}^0 g(s)\chi_{\{s<0\}}ds$ as a test function in (2.3.6), we conclude that

$$\int_{\{u_n<0\}} g(u_n)|\nabla u_n|^{p(x)}\exp(G(u_n))dxdt \le C_5.$$

Consequently,

$$\int_Q g(u_n)|\nabla u_n|^{p(x)}\exp(G(u_n))dxdt \le C_6. \tag{2.3.10}$$

Above, $C_1, ..., C_6$ are constants independent of n, we deduce that

$$\int_Q |\nabla T_k(u_n)^+|^{p(x)} dxdt \leq k\ C_7. \tag{2.3.11}$$

Similarly to (2.3.11), we take $\varphi = T_k(u_n)^- \chi(0,\tau)$ in (2.3.6) to deduce that

$$\int_Q |\nabla T_k(u_n)^-|^{p(x)} dxdt \leq k\ C_8. \tag{2.3.12}$$

Combining (2.3.11), (2.3.12) and lemma (1.1.1), we conclude that

$$\int_0^T \min\Big\{ \|\nabla T_k(u_n)\|_{p(.)}^{p^+}, \|\nabla T_k(u_n)\|_{p(.)}|^{p^-} \Big\} dt \leq \rho(\nabla T_k(u_n)) \leq kC_9.$$

$$\|T_k(u_n)\|_{L^{p^-}(0,T;W_0^{1,p(.)}(\Omega))} \leq\ k\ C_{10}. \tag{2.3.13}$$

Where $C_8,\ C_9,\ C_{10}$ are constants independent of n. Thus, $T_k(u_n)$ is bounded in $L^{p^-}(0,T;W_0^{1,p(.)}(\Omega))$ independently of n for any $k > 0$. Then, we deduce from (2.3.7), (2.3.8) and (2.3.13) that

$$\int_\Omega B_{k,G}^n(x, u_n(\tau)) dx \leq\ kC. \tag{2.3.14}$$

Now we turn to proving the almost everywhere convergence of u_n and $b_n(x, u_n)$. Consider a non decreasing function $g_k \in C^2(\mathbb{R})$ such that

$$g_k(s) = \begin{cases} s & if\ |s| \leq \frac{k}{2} \\ k & if\ |s| \geq k \end{cases}$$

Multiplying the approximate equation by $g_k'(u_n)$, we get

$$\frac{\partial B_k^n(x,u_n)}{\partial t} - \operatorname{div}(a(x,t,u_n,\nabla u_n)g_k'(u_n)) + a(x,t,u_n,\nabla u_n)g_k''(u_n)\nabla u_n$$

$$+H_n(x,t,u_n,\nabla u_n)g_k'(u_n) = f_n g_k'(u_n) - \operatorname{div}(F g_k'(u_n)) + F g_k''(u_n)\nabla u_n. \tag{2.3.15}$$

where

$$B_k^n(x,z) = \int_0^z \frac{\partial b_n(x,s)}{\partial s} g_k'(s) ds.$$

As a consequence of (2.3.13), we deduce that $g_k(u_n)$ is bounded in $L^{p^-}(0,T;W_0^{1,p(.)}(\Omega))$ and $\frac{\partial B_k^n(x,u_n)}{\partial t}$ is bounded in $L^1(Q) + V^*$. Due to the properties of g_k and (2.2.2), we conclude that $\frac{\partial g_k(u_n)}{\partial t}$ is bounded in $L^1(Q) + V^*$, which implies that

$g_k(u_n)$ is compact in $L^1(Q)$. Due to the choice of g_k, we conclude that for each k, the sequence $T_k(u_n)$ converges almost everywhere in Q, which implies that u_n converges almost everywhere to some measurable function v in Q. Thus by using the same argument as in [34], [35], [79], we can show the following lemma.

Lemma 2.3.3 *Let u_n be a solution of the approximate problem $(P_2)_n$. Then,*

$$\begin{aligned} u_n &\to u \quad a.e.\ in\ Q, \\ b_n(x,u_n) &\to b(x,u) \quad a.e.\ in\ Q. \end{aligned}$$

We can deduce from (2.3.13) that

$$T_k(u_n) \rightharpoonup T_k(u) \quad in \quad L^{p^-}(0,T;W_0^{1,p(.)}(\Omega)),$$

which implies, by using (2.2.3), that for all $k>0$ there exists $\varphi_k \in (L^{p'(.)}(Q))^N$

$$\text{such that} \quad a(x,t,u,T_k(u_n),\nabla T_k(u_n)) \rightharpoonup \varphi_k \quad in \quad (L^{p'(.)}(Q))^N. \tag{2.3.16}$$

Remark 2.3.1 *$b(.,u)$ it belongs to $L^\infty(0,T;L^1(\Omega))$.*

Proof. Let u_n be a solution of the approximate problem $(P_2)_n$, passing to $\liminf$ in (2.3.14) as $n\to\infty$, we obtain

$$\frac{1}{k}\int_\Omega B_{k,G}(x,u(\tau))dx \le C, \text{ for a.e. } \tau \text{ in } [0,T].$$

Due to the definition of $B_{k,G}(x,s)$ and the fact that $\frac{1}{k}B_{k,G}(x,s)$ converges pointwise to $\displaystyle\int_0^u sgn(s)\frac{\partial b(x,s)}{\partial s}\exp(G(s))ds \ge |b(x,u)|$ as $k\to\infty$, it follows that $b(.,u)$ belongs to $L^\infty(0,T;L^1(\Omega))$. $\square$

Lemma 2.3.4 *Let u_n be a solution of the approximate problem $(P_2)_n$. Then,*

$$\lim_{m\to\infty}\limsup_{n\to\infty}\int_{\{m\le|u_n|\le m+1\}} a(x,t,u_n,\nabla u_n)\nabla u_n dxdt = 0. \tag{2.3.17}$$

Proof. See Appendix.

STEP 3 :**Almost everywhere convergence of the gradients** :

This step is devoted to prove the strong convergence of truncation of $T_k(u_n)$ that, we will use the following function of one real variable for $m>k$

$$h_m(s)=\begin{cases}1 & if \quad |s|\leq m\\ 0 & if \quad |s|>m+1\\ m+1+|s| & if \quad m\leq |s|\leq m+1.\end{cases}$$
Let $\psi_i \in D(\Omega)$ be a sequence which converges strongly to u_0 in $L^1(\Omega)$
Set $\omega_\mu^i=(T_k(u))_\mu+e^{-\mu t}T_k(\psi_i)$ where $(T_k(u))_\mu$ is the mollification of $T_k(u)$ with respect to time. Note that ω_μ^i is a smooth function having the following properties :

$$\frac{\partial \omega_\mu^i}{\partial t}=\mu(T_k(u)-\omega_\mu^i), \quad \omega_\mu^i(0)=T_k(\psi_i), \quad |\omega_\mu^i|\leq k, \tag{2.3.18}$$

$$\omega_\mu^i \to T_k(u) \quad \text{in } \; L^{p^-}(0,T;W_0^{1,p(.)}(\Omega)) \quad \text{as } \mu\to\infty. \tag{2.3.19}$$

The very definition of the sequence ω_μ^i makes it possible to establish the following lemma.

Lemma 2.3.5 *(See[76, 9]). For $k\geq 0$, we have*

$$\int_{\{T_k(u_n)-\omega_\mu^i\geq 0\}}\frac{\partial b_n(x,u_n)}{\partial t}\exp(G(u_n))(T_k(u_n)-\omega_\mu^i)h_m(u_n)dxdt\geq \varepsilon(n,m,\mu,i).$$

Proposition 2.3.2 *The subsequence of u_n solution of problem $(P_2)_n$ satisfies for any $k\geq 0$ following assertion*

$$\lim_{n\to\infty}\int_Q\Big[a(T_k(u_n),\nabla T_k(u_n))-a(T_k(u_n),\nabla T_k(u))\Big]\cdot\Big[\nabla T_k(u_n)-\nabla T_k(u)\Big]dxdt=0.$$

Proof. See Appendix.
Thanks to the lemma (1.2.1), we have

$$T_k(u_n)\to T_k(u) \quad \text{strongly in} \quad L^{p^-}(0,T;W_0^{1,p(.)}(\Omega)) \quad \forall k. \tag{2.3.20}$$

and $\quad \nabla u_n \to \nabla u.\quad$ a.e. in Q, which implies that

$$a(x,t,T_k(u_n),\nabla T_k(u_n))\rightharpoonup a(x,t,T_k(u),\nabla T_k(u)) \;\; in \; (L^{P'(.)}(Q))^N. \tag{2.3.21}$$

STEP 4 : **Equi-Integrability of the non Linearity Sequence** :
We shall now prove that $H_n(x,t,u_n,\nabla u_n)\to H(x,t,u,\nabla u)$ *strongly in* $L^1(Q)$, by using Vitali's theorem. Since $H_n(x,t,u_n,\nabla u_n)\to H(x,t,u,\nabla u)$ *a.e. in* Q, considering now $\varphi=\rho_h(u_n)=\displaystyle\int_0^{u_n}g(s)\chi_{\{s>h\}}ds$ as a test function in (2.3.5), we

obtain

$$\Big[\int_\Omega B_h^n(x,u_n)dx\Big]_0^T+\int_Q a(x,t,u_n,\nabla u_n)\nabla u_n g(u_n)\chi_{\{u_n>h\}}\exp(G(u_n))dxdt$$
$$\leq\Big(\int_h^\infty g(s)\chi_{\{s>h\}}ds\Big)\exp\Big(\frac{\|g\|_{L^1(\mathbb{R})}}{\alpha}\Big)\Big[\|f_n\|_{L^1(Q)}+\|\gamma\|_{L^1(Q)}\Big]$$
$$+\int_Q F\nabla u_n g(u_n)\chi_{\{u_n>h\}}\exp(G(u_n))dxdt$$
$$+\Big(\int_h^\infty g(s)\chi_{\{s>h\}}ds\Big)\int_Q |F\nabla u_n|\frac{g(u_n)}{\alpha}\exp(G(u_n))\chi_{\{u_n>h\}}dxdt,$$

where $B_h^n(x,r)=\displaystyle\int_0^r\frac{\partial b_n(x,s)}{\partial s}\rho_h(s)\exp(G(s))ds$,
which implies, in view of $B_h^n(x,r)\geq 0$, (2.2.5) and Young's Inequality,

$$\alpha\int_{\{u_n>h\}}|\nabla u_n|^{p(x)}g(u_n)\exp(G(u_n))dxdt$$
$$\leq\Big(\int_h^\infty g(s)ds\Big)\exp\Big(\frac{\|g\|_{L^1(\mathbb{R})}}{\alpha}\Big)\Big[\|f_n\|_{L^1(Q)}\|\gamma\|_{L^1(Q)}$$
$$+\|b_n(x,u_{0n}\|_{L^1(\Omega)}\Big]+C'\int_h^\infty g(s)ds$$
$$+\frac{\alpha}{2}\int_{\{u_n>h\}}|\nabla u_n|^{p(x)}g(u_n)\exp(G(u_n))dxdt$$
$$+\Big(\int_h^\infty g(s)ds\Big)\int_Q|F\nabla u_n|\frac{g(u_n)}{\alpha}\exp(G(u_n))dxdt$$

hence $\displaystyle\frac{\alpha}{2}\int_{\{u_n>h\}}|\nabla u_n|^{p(x)}g(u_n)\exp(G(u_n))dxdt$

$$\leq\Big(\int_h^\infty g(s)ds\Big)\exp\Big(\frac{\|g\|_{L^1(\mathbb{R})}}{\alpha}\Big)\Big[\|f_n\|_{L^1(Q)}+\|\gamma\|_{L^1(Q)}$$
$$+\|b(x,u_0\|_{L^1(\Omega)}+C'\Big]$$
$$+\Big(\int_h^\infty g(s)ds\Big)\int_Q|F\nabla u_n|\frac{g(u_n)}{\alpha}\exp(G(u_n))dxdt$$

and since $g\in L^1(\mathbb{R})$, we deduce that

$$\lim_{h\to\infty}\sup_{n\in\mathbb{N}}\int_{\{u_n>h\}}|\nabla u_n|^{p(x)}g(u_n)dxdt=0.$$

Similarly, taking $\varphi = \rho_h(u_n) = \displaystyle\int_{u_n}^{0} g(s)\chi_{\{s<-h\}}ds$ as a test function in (2.3.6), we conclude that, $\lim_{h\to\infty}\sup_{n\in\mathbb{N}} \displaystyle\int_{\{u_n<-h\}} |\nabla u_n|^{p(x)} g(u_n)dxdt = 0$. Consequently, $\lim_{h\to\infty}\sup_{n\in\mathbb{N}} \displaystyle\int_{\{|u_n|>h\}} |\nabla u_n|^{p(x)} g(u_n)dxdt = 0$. Which implies, for h large enough and for a subset E of Q,

$$\begin{aligned}\lim_{measE\to 0}\int_E |\nabla u_n|^{p(x)} g(u_n)dxdt \;\leq\; & \|g\|_\infty \lim_{measE\to 0}\int_E |\nabla T_h(u_n)|^{p(x)}dxdt \\ & + \int_{\{|u_n|>h\}} |\nabla u_n|^{p(x)} g(u_n)dxdt\end{aligned}$$

so $g(u_n)|\nabla u_n|^{p(x)}$ is equi-integrable. Thus, we have shown that

$$g(u_n)|\nabla u_n|^{p(x)} \to g(u)|\nabla u|^{p(x)} \quad \text{stongly in} \quad L^1(Q).$$

consequently, by using (2.2.6), we conclude that

$$H_n(x,t,u_n,\nabla u_n) \to H(x,t,u,\nabla u) \quad \textit{strongly} \quad \textit{in} \quad L^1(Q).\square \tag{2.3.22}$$

STEP 5 : **Passing to the limit** :

a) Proof that u **satisfies (2.2.8)**. For any fixed $m \geq 0$, we have

$$\begin{aligned}&\int_{\{m\leq |u_n|\leq m+1\}} a(x,t,u_n,\nabla u_n)\nabla u_n dxdt \\ &\qquad = \int_Q a(x,t,u_n,\nabla u_n)\Big[\nabla T_{m+1}(u_n) - \nabla T_m(u_n)\Big]dxdt \\ &\qquad = \int_Q a(x,t,T_{m+1}(u_n),\nabla T_{m+1}(u_n))\nabla T_{m+1}(u_n) \\ &\qquad\quad - \int_Q a(x,t,T_m(u_n),\nabla T_m(u_n))\nabla T_m(u_n)dxdt\end{aligned}$$

According to (2.3.20) and (2.3.21), one can passing to the limit as $n \to \infty$ for fixed $m \geq 0$ to obtain

$$\lim_{n\to\infty} \int_{\{m\leq |u_n|\leq m+1\}} a(x,t,u_n,\nabla u_n)\nabla u_n dxdt$$
$$= \int_Q a(x,t,T_{m+1}(u),\nabla T_{m+1}(u))\nabla T_{m+1}(u)$$
$$- \int_Q a(x,t,T_m(u),\nabla T_m(u))\nabla T_m(u) dxdt$$
$$= \int_{\{m\leq |u_n|\leq m+1\}} a(x,t,u,\nabla u)\nabla u dxdt \qquad (2.3.23)$$

Taking the limit as $m \to \infty$ in (2.3.23) and using the estimate (2.3.17) shows that u satisfies(2.2.8).□

b) Proof that u satisfies (2.2.9)

Let $S \in W^{2,\infty}(\mathbb{R})$ be such that S' has a compact support . Let $M > 0$ such that supp(S')$\subset [-M, M]$. Pointwise multiplication of the approximate problem $(P_2)_n$ by $S'(u_n)$ leads to

$$\frac{\partial B_S^n(x,u_n)}{\partial t} - \operatorname{div}\Big[S'(u_n)a(x,t,u_n,\nabla u_n)\Big] + S''(u_n)a(x,t,u_n,\nabla u_n)\nabla u_n$$
$$+ H_n(x,t,u_n,\nabla u_n)S'(u_n) = f_n S'(u_n) - \operatorname{div}\Big(S'(u_n)F\Big) + S''(u_n)F\nabla u_n \text{ in } D'(Q). \qquad (2.3.24)$$

In what follows we pass to the limit in (2.3.24) as n tends to ∞.

• Limit of $\frac{\partial B_S^n(x,u_n)}{\partial t}$. Since S is bounded and continuous, $u_n \to u$ a.e. in Q implies that $B_S^n(x,u_n)$ converge to $B_S(x,u)$ a.e. in Q and L^∞ weakly .

$$Then, \quad \frac{\partial B_S^n(x,u_n)}{\partial t} \to \frac{\partial B_S(x,u)}{\partial t} \quad in\ D'(Q).\ as\ n \to \infty.$$

• Limit of $-\operatorname{div}\Big[S'(u_n)a(x,t,u_n,\nabla u_n)\Big]$. Since supp($S'$)$\subset [-M, M]$, we have, for $n \geq M$

$$S'(u_n)a(x,t,u_n,\nabla u_n) = S'(u_n)a(x,t,T_M(u_n),\nabla T_M(u_n))\ a.e.\ in\ Q.$$

The pointwise convergence of u_n to u and (2.3.21) and the boundedness of S' yied, as $n \to \infty$, $S'(u)a(x,t,T_M(u),\nabla T_M(u))$ has been denoted by $S'(u)a(x,t,u,\nabla u)$ in equation (2.2.9).

• Limit of $S''(u_n)a(x,t,u_n,\nabla u_n)\nabla u_n$ .Consider the "energy" term

$S''(u_n)a(x,t,u_n,\nabla u_n)\nabla u_n = S''(u_n)a(x,t,T_M(u_n),\nabla T_M(u_n))\nabla T_M(u_n)$ a.e. in Q. The pointwise convergence of $S'(u_n)$ to $S'(u)$ and (2.3.21) as $n \to \infty$ and the boundedness of S'' yield

$$S''(u_n)a(x,t,u_n,\nabla u_n)\nabla u_n \rightharpoonup S''(u)a(x,t,T_M(u),\nabla T_M(u))\nabla T_M(u) \textit{ in } L^1(Q). \tag{2.3.25}$$

Recall that $S''(u)a(x,t,T_M(u),\nabla T_M(u))\nabla T_M((u)) = S''(u)a(x,t,u,\nabla u)\nabla u$ a.e. in Q.

• Limit of $S'(u_n)H_n(x,t,u_n,\nabla u_n)$. From supp$(S')\subset [-M,M]$ and (2.3.22), we have

$$S'(u_n)H_n(x,t,u_n,\nabla u_n) \to S'(u)H(x,t,u,\nabla u) \textit{ strongly in } L^1(Q) \textit{ as } k\ n \to \infty. \tag{2.3.26}$$

• Limit of $S'(u_n)f_n$. Since $u_n \to u$ a.e. in Q, we have $S'(u_n)f_n \to S'(u)f$ strongly in $L^1(Q)$, $\textit{as } n \to \infty$

• Limit of div$(S'(u_n)F)S'(u_n)$ is bounded and converges to $S'(u)$ a.e. in Q. Since $\nabla S'(u_n)$ converge to $\nabla S'(u)$ weakly in $(L^{p(.)}(Q))^N$, we obtain
$S''(u_n)F\nabla u_n = F\nabla S'(u_n) \rightharpoonup F\nabla S'(u)$ weakly in $L^1(Q)$ as $n \to \infty$. The term $F\nabla S'(u)$ identifies with $S''(u)F\nabla u$.

then div$(S'(u_n)F) \to$ div$(S'(u)F)$ strongly in $L^{p'^-}(0,T;W^{-1,p'(.)}(\Omega))$ as $n \to \infty$.

• Limit of $S''(u_n)F\nabla u_n$. This term is equal to $F\nabla S'(u_n)$.
Since $\nabla S'(u_n)$ converge to $\nabla S'(u)$ weakly in $(L^{p(.)}(Q))^N$, we obtain
$S''(u_n)F\nabla u_n = F\nabla S'(u_n) \rightharpoonup F\nabla S'(u)$ weakly in $L^1(Q)$ as $n \to \infty$. The term $F\nabla S'(u)$ identifies with $S''(u)F\nabla u$.
As a consequence of the above convergence result, we are in a position to pass to the limit as $n \to \infty$ in equation (2.3.24) and to conclude that u satisfies (2.2.9). □

c) Proof that u satisfies (2.2.10)

S is bounded, and $B_S^n(x,u_n)$ is bounded in $L^\infty(Q)$. Secondly, by (2.3.24) we have $\frac{\partial B_S^n(x,u_n)}{\partial t}$ is bounded in $L^1(Q)+V^*$.
As a consequence, an Aubin type Lemma (see, e.g, [82] implies that $B_S^n(x,u_n)$ lies in a compact set in $C^0([0,T],L^1(\Omega))$.
It follows that on the hand, $B_S^n(x,u_n)\mid_{t=0}= B_S^n(x,u_0^n)$ converge to $B_S(x,u)\mid_{t=0}$ strongly in $L^1(\Omega)$ implies that :$B_S(x,u)\mid_{t=0}= B_S(x,u_0)$ in Ω.
As a conclusion of Steps 1 to 5, the proof of theorem (2.3.1) is complete . □

2.4 Appendix

Proof of theorem 2.3.2 We define the operator
$L_n : L^{p^-}(0,T;W_0^{1,p(x)}(\Omega)) \to L^{p'^-}(0,T;W^{-1,p'(.)}(\Omega))$ by

$$\Big\langle L_n u, v\Big\rangle = \int_Q \frac{\partial b_n(x,u)}{\partial t} v dxdt = \int_Q \frac{\partial b_n(x,u)}{\partial u}\frac{\partial u}{\partial t} v dxdt \quad \forall u,v \in L^{p^-}(0,T;W_0^{1,p(.)}(\Omega))$$

then,

$$\begin{aligned}
\Big|\Big\langle L_n u, v\Big\rangle\Big| &\leq \Big|\int_0^T \int_\Omega A_n(x)\frac{\partial u}{\partial t} v dxdt\Big| \\
&\leq \Big(\frac{1}{p^-}+\frac{1}{p'^-}\Big)\|A_n\|_{L^\infty}\int_0^T \|\frac{\partial u}{\partial t}\|_{L^{p'(x)}(\Omega)}\|v\|_{L^{p(x)}(\Omega)}dt \\
&\leq C\Big(\frac{1}{p^-}+\frac{1}{p'^-}\Big)\|A_n\|_{L^\infty}\int_0^T \|\frac{\partial u}{\partial t}\|_{W^{-1,p'(.)}(\Omega)}\|v\|_{W_0^{1,p(x)}(\Omega)}dt \\
&\leq C\Big(\frac{1}{p^-}+\frac{1}{p'^-}\Big)\|A_n\|_{L^\infty}\|\frac{\partial u}{\partial t}\|_{L^{p'^-}(0,T,W^{-1,p'(.)}(\Omega))}\|v\|_{L^{p^-}(0,T,W_0^{1,p(x)}(\Omega))} \\
&\leq C_1\|v\|_{L^{p^-}(0,T,W_0^{1,p(x)}(\Omega))}. \qquad (2.4.1)
\end{aligned}$$

We define the operator $G_n : L^{p^-}(0,T;W_0^{1,p(.)}(\Omega)) \to L^{p'^-}(0,T,W^{-1,p'(.)}(\Omega))$

$$by, \quad \Big\langle G_n u, v\Big\rangle = \int_Q H_n(x,t,u,\nabla u) v dxdt \quad \forall u,v \in L^{p^-}(0,T;W_0^{1,p(.)}(\Omega)).$$

Thanks to the Hölder Inequality, we have that for $u,v \in L^{p^-}(0,T;W_0^{1,p(.)}(\Omega))$

$$\begin{aligned}
&\int_Q H_n(x,t,u,\nabla u) v dxdt \leq \Big|\int_0^T\int_\Omega H_n(x,t,u,\nabla u)vdxdt\Big| \\
&\leq \Big(\frac{1}{p^-}+\frac{1}{p'^-}\Big)\int_0^T\Big(\int_\Omega\Big|H_n(x,t,u,\nabla u)\Big|^{p'(x)}dx\Big)^\theta\|v\|_{L^{p(x)}(\Omega)}dt \\
&\leq C\Big(\frac{1}{p^-}+\frac{1}{p'^-}\Big)\int_0^T n^{\theta p'^+}(meas\Omega)^\theta\|v\|_{W_0^{1,p(x)}(\Omega)}dt \\
&\leq C_2\|v\|_{L^{p^-}(0,T;W_0^{1,p(.)}(\Omega))}. \qquad (2.4.2)
\end{aligned}$$

with $\theta = \begin{cases} 1/p'^- & if \quad \|H_n(x,t,u,\nabla u)\|_{L^1(Q)} > 1 \\ 1/p'^+ & if \quad \|H_n(x,t,u,\nabla u)\|_{L^1(Q)} \leq 1. \end{cases}$

Lemma 2.4.1 *Let* $B_n : L^{p^-}(0,T;W_0^{1,p(.)}(\Omega)) \to L^{p'^-}(0,T,W^{-1,p'(.)}(\Omega))$.
The operator $B_n = A + G_n$ *is*

a) coercive
b) pseudo-monotone
c) bounded and hemi continuous.

Proof. a) For the coercivity, we have for any $u \in L^{p^-}(0,T;W_0^{1,p(.)}(\Omega))$

$$\Big\langle B_n u, u\Big\rangle = \Big\langle G_n u, u\Big\rangle + \Big\langle Au, u\Big\rangle$$
$$\Rightarrow \Big\langle B_n u, u\Big\rangle - \Big\langle G_n u, u\Big\rangle = \Big\langle Au, u\Big\rangle$$
$$then,\quad \Big\langle B_n u, u\Big\rangle - \Big\langle G_n u, u\Big\rangle = \int_Q a(x,t,u,\nabla u)\nabla u dxdt$$
$$= \int_0^T \int_\Omega a(x,t,u,\nabla u)\nabla u dxdt$$
$$\geq \int_0^T \alpha\Big(\int_\Omega |\nabla u|^{p(x)}dx\Big)dt \quad \text{(using (2.2.5))}$$
$$\geq \alpha\|\nabla u\|^{\delta}_{L^{p^-}(0,T;W_0^{1,p(.)}(\Omega))} \geq \beta\|u\|^{\delta}_{L^{p^-}(0,T;W_0^{1,p(.)}(\Omega))},$$

which is due to Poincaré Inequality with

$$\delta = \begin{cases} p_- & if\ \|\nabla u\|_{L^{p^-}(0,T;W_0^{1,p(.)}(\Omega))} > 1 \\ p_+ & if\ \|\nabla u\|_{L^{p^-}(0,T;W_0^{1,p(.)}(\Omega))} \leq 1 \end{cases}$$

$$hence,\quad \Big\langle B_n u, u\Big\rangle - \Big\langle G_n u, u\Big\rangle \geq \beta\|u\|^{\delta}_{L^{p^-}(0,T;W_0^{1,p(.)}(\Omega))}$$
$$then,\ \Big\langle B_n u, u\Big\rangle \geq \beta\|\ u\|^{\delta}_{L^{p^-}(0,T;W_0^{1,p(.)}(\Omega))} - C_2\|u\|_{L^{p^-}(0,T;W_0^{1,p(.)}(\Omega))}$$

then, we have

$$\frac{\Big\langle B_n u, u\Big\rangle}{\|u\|_{L^{p^-}(0,T;W_0^{1,p(.)}(\Omega))}} \geq \beta\|u\|^{\delta-1}_{L^{p^-}(0,T;W_0^{1,p(.)}(\Omega))} - C_2 \to +\infty$$
$$\Rightarrow \frac{\Big\langle B_n u, u\Big\rangle}{\|u\|_{L^{p^-}(0,T;W_0^{1,p(x)}(\Omega))}} \to +\infty \quad as\ \|u\|_{L^{p^-}(0,T;W_0^{1,p(.)}(\Omega))} \to +\infty$$

then B_n is coercive. □

b)It remains to show that B_n is pseudo–monotone.
Let $(u_k)_k$ a sequence in $L^{p^-}(0,T;W_0^{1,p(.)}(\Omega))$ such that

$$\begin{aligned} u_k &\rightharpoonup u \text{ in } L^{p^-}(0,T;W_0^{1,p(.)}(\Omega)) \\ L_n u_k &\rightharpoonup L_n u \text{ in } L^{p'^-}(0,T;W^{-1,p'(.)}(\Omega)) \\ \limsup_{k\to\infty} \Big\langle B_n u_k, u_k - u\Big\rangle &\leq 0 \end{aligned} \qquad (2.4.3)$$

that, we have prove that

$$B_n u_k \rightharpoonup B_n u \quad in \quad L^{p'^-}(0,T;W^{-1,p'(.)}(\Omega)) \quad and\ \langle B_n u_k, u_k\rangle \to \langle B_n u, u\rangle.$$

By the definition of the operator L_n defined in definition (1.5.2), we obtain that u_k is bounded in $W_0^{1,p(.)}(\Omega)$ and since $W_0^{1,p(.)}(\Omega) \hookrightarrow L^{p(.)}(\Omega)$ then $u_k \to u$ in $L^{p^-}(0,T;W_0^{1,p(.)}(\Omega))$, then the growth condition (2.2.3) $(a(x,t,u_k,\nabla u_k))_k$ is bounded in $(L^{p'(.)}(Q))^N$ therefore, there exists a function $\varphi \in (L^{p'(.)}(Q))^N$ such that

$$a(x,t,u_k,\nabla u_k) \rightharpoonup \varphi \quad as \quad k \to +\infty. \tag{2.4.4}$$

Similarly, using condition (2.2.6) $\Big(H_n(x,t,u_k,\nabla u_k)\Big)_k$ is bounded in $L^1(Q)$ then, there exists a function $\psi_n \in L^1(Q)$ such that

$$H_n(x,t,u_k,\nabla u_k) \to \psi_n \text{ in } L^1(Q)\ as\ k \to +\infty. \tag{2.4.5}$$

$$\begin{aligned}
\lim_{k\to\infty}\Big\langle B_n u_k, u_k\Big\rangle &= lim_{k\to\infty}\Big[\Big\langle G_n u_k, u_k\Big\rangle + \Big\langle A u_k, u_k\Big\rangle\Big] \\
&= \lim_{k\to\infty}\Big[\int_Q a(x,t,u_k,\nabla u_k)\nabla u_k dxdt + \int_Q H(x,t,u_k,\nabla u_k)u_k dxdt\Big] \\
&= \int_Q \varphi \nabla u_k dxdt + \int_Q \psi_n u_k dxdt
\end{aligned} \tag{2.4.6}$$

using (2.4.3) and, (2.4.6), we obtain

$$\begin{aligned}
\lim_{k\to\infty}\sup\Big\langle B_n u_k, u_k\Big\rangle &= \lim_{k\to\infty}\sup\Big\{\int_Q a(x,t,u_k,\nabla u_k)\nabla u_k dxdt \\
&+ \int_Q H(x,t,u_k,\nabla u_k)u_k dxdt\Big\} \\
&\leq \int_Q \varphi\nabla u dxdt + \int_Q \psi_n u dxdt
\end{aligned} \tag{2.4.7}$$

thanks to (2.4.5), we have

$$\int_Q H_n(x,t,u_k,\nabla u_k)dxdt \to \int_Q \psi_n dxdt. \tag{2.4.8}$$

therefore,

$$\lim_{k\to\infty}\sup\int_Q a(x,t,u_k,\nabla u_k)\nabla u_k \leq \int_Q \varphi\nabla u dxdt \tag{2.4.9}$$

on the other hand, using (2.2.4), we have

$$\int_Q \Big[a(x,t,u_k,\nabla u_k) - a(x,t,u_k,\nabla u)\Big](\nabla u_k - \nabla u)dxdt \geq 0. \tag{2.4.10}$$

$$Then \int_Q a(x,t,u_k,\nabla u_k)\nabla u_k dxdt \geq -\int_Q a(x,t,u_k,\nabla u)\nabla u dxdt + \int_Q a(x,t,u_k,\nabla u_k)\nabla u dxdt + \int_Q a(x,t,u_k,\nabla u)\nabla u_k dxdt$$

and by (2.4.4), we get

$$\lim_{k\to\infty}\inf \int_Q a(x,t,u_k,\nabla u_k)\nabla u_k dxdt \geq \int_Q \varphi\nabla u dxdt.$$

this implies, thanks to (2.4.9), that

$$\lim_{k\to\infty} \int_Q a(x,t,u_k,\nabla u_k)\nabla u_k dxdt = \int_Q \varphi\nabla u dxdt \tag{2.4.11}$$

Now by (2.4.11), we can obtain

$$\lim_{k\to\infty} \int_Q a(x,t,u_k,\nabla u_k) - a(x,t,u_k,\nabla u))(\nabla u_k - \nabla u)dxdt = 0$$

In view of the lemma 1.2.1, we obtain

$$\begin{aligned} u_k &\to u \quad in \quad L^{p^-}(0,T;W_0^{1,p(.)}(\Omega)) \\ \nabla u_k &\to \nabla u \quad a.e.\ in \quad Q. \end{aligned}$$

Then,

$$a(x,t,u_k,\nabla u_k) \rightharpoonup a(x,t,u,\nabla u) \quad in \quad (L^{p'(.)}(Q))^N$$
$$H_n(x,t,u_k,\nabla u_k) \rightharpoonup H(x,t,u,\nabla u) \quad in \quad L^1(Q),$$

we deduce that

$$Au_k \rightharpoonup Au \quad in \quad (L^{p'^-}(Q))^N$$

and

$$G_n u_k \rightharpoonup G_n u \quad in \quad (L^1(Q))$$

which implies

$$B_n u_k \rightharpoonup B_n u \quad in \quad L^{p'^-}(0,T;W_0^{1,p(.)}(\Omega))$$

and

$$\Big\langle B_n u_k, u_k\Big\rangle \to \Big\langle B_n u, u\Big\rangle$$

completing the proof of assertion(b). $\square$

c) Using $H\ddot{o}lder's$ inequality and the growth condition (2.2.3), we can show that the operator A is bounded and by using (2.4.2), we conclude that B_n is bounded. For to show that B_n is hemi continuous. Let $u_k \to u$ in $L^{p^-}(0,T;W_0^{1,p(.)}(\Omega))$ and prove that $\Big\langle B_n u_k, \psi \Big\rangle \to \langle B_n u, \psi\rangle$ $\quad for\ all\ \psi \in L^{p^-}(0,T;W_0^{1,p(.)}(\Omega)$

Since $a(x,t,u_k,\nabla u_k) \to a(x,t,u,\nabla u)$ as $k\to\infty$ a.e. in Q. Then, by the growth condition (2.2.3) and lemma (1.2.2)

$$a(x,t,u_k,\nabla u_k) \rightharpoonup a(x,t,u,\nabla u)\ in \quad (L^{p'(.)}(Q))^N$$

and for all $\varphi \in L^{p^-}(0,T;W_0^{1,p(.)}(\Omega))$, $\langle Au_k,\varphi\rangle \to \langle Au,\varphi\rangle$ as $k\to\infty$

similarly, $G_n u_k \to G_n u$ as $k\to\infty$ a.e. in Q then, by the (2.2.6) and lemma (1.2.2) $G_n u_k \rightharpoonup G_n u$ in $L^{p'(.)}(Q)$ and for all $\phi \in L^{p^-}(0,T;W_0^{1,p(.)}(\Omega))$,

$\Big\langle G_n u_k, \phi\Big\rangle \to \Big\langle G_n u, \phi\Big\rangle$ as $k\to\infty$ which implies B_n is hemi continuous. $\square$

Proof of lemma 2.3.4.

Set $\varphi = T_1\Big(u_n - T_m(u_n)\Big)^+ = \alpha_m(u_n)$ as a test function in (2.3.5), this function is admissible since $\varphi \in L^{p^-}(0,T;W_0^{1,p(.)}(\Omega))$ and $\varphi \geq 0$. Then, we have

$$\begin{aligned}
&\int_Q \frac{\partial b_n(x,u_n)}{\partial t}\exp(G(u_n))\alpha_m(u_n)dxdt \\
&\qquad + \int_{\{m\le u_n\le m+1\}} a(x,t,u_n,\nabla u_n)\nabla u_n \exp(G(u_n))dxdt \\
&\qquad\quad \le \int_Q |\gamma(x,t)|\exp(G(u_n))\alpha_m(u_n)dxdt + \int_Q |f_n|\exp(G(u_n))\alpha_m(u_n)dxdt \\
&\qquad\qquad + \int_Q F\nabla u_n \frac{g(u_n)}{\alpha}\exp(G(u_n))\alpha_m(u_n)dxdt \\
&\qquad\qquad\quad + \int_{\{m\le u_n\le m+1\}} F\nabla u_n \exp(G(u_n))dxdt.
\end{aligned}$$

This gives, by setting $B^m_{n,G}(x,r) = \displaystyle\int_0^r \frac{\partial b_n(x,s)}{\partial s}\exp(G(s))\alpha_m(s)ds$ and by Young's Inequality,

$$\begin{aligned}
&\int_\Omega B^m_{n,G}(x,u_n)(T)dx \ + \int_{\{m\le u_n\le m+1\}} a(x,t,u_n,\nabla u_n)\exp(G(u_n))\nabla u_n dxdt \\
&\qquad \le \exp\Big(\frac{\|g\|_{L^1(\mathbb{R})}}{\alpha}\Big)\Big[\int_{\{|u_n|>m\}}(|\gamma|+|f_n|)dxdt + \int_{\{|u_{0n}|>m\}}|b_n(x,u_{0n})|dx\Big]dxdt \\
&\qquad\quad + C_1\int_{\{u_n\ge m\}}|F|^{p'(x)}dxdt + \frac{\alpha}{2}\int_{\{m\le u_n\le m+1\}}|\nabla u_n|^{p(x)}\exp(G(u_n))dxdt \\
&\qquad\quad + C_2\int_{\{u_n\ge m\}}|F|^{p'(x)}dxdt + \frac{\alpha}{2}\int_{\{|u_n|>m\}}|\nabla u_n|^{p(x)}g(u_n)\exp(G(u_n))dxdt.
\end{aligned}$$

Since $B_{n,G}^m(x,u_n)(T)>0$ and use (2.2.5), we obtain

$$\begin{aligned}
&\frac{1}{2}\int_{\{m\leq u_n\leq m+1\}} a(x,t,u_n,\nabla u_n)\nabla u_n\exp(G(u_n))dxdt\\
&\qquad\leq \exp\Big(\frac{\|g\|_{L^1(\mathbb{R})}}{\alpha}\Big)\Big[\int_{\{|u_n|>m\}}(|\gamma)|+|f_n|)dxdt. \qquad (2.4.12)\\
&\qquad\quad +\int_{\{|u_{0n}|>m\}}|b_n(x,u_{0n})|dx\Big]+C_3\int_{\{u_n>m\}}|F|^{p'(x)}dxdt\\
&\qquad\quad +C_4\int_{\{u_n>m\}}|\nabla u_n|^{p(x)}g(u_n)\exp(G(u_n))dxdt
\end{aligned}$$

Taking $\varphi=\rho_m(u_n)=\displaystyle\int_0^{u_n}g(s)\chi_{\{s>m\}}ds$ as a test function in (2.3.5), we obtain

$$\begin{aligned}
&\Big[\int_\Omega B_{m,n}^m(x,u_n)dx\Big]_0^T + \int_Q a(x,t,u_n,\nabla u_n)\nabla u_n\exp(G(u_n))g(u_n)\chi_{\{u_n>m\}}dxdt\\
&\quad\leq \Big(\int_m^\infty g(s)\chi_{\{s>m\}}ds\Big)\exp\Big(\frac{\|g\|_{L^1(\mathbb{R})}}{\alpha}\Big)\Big[\|f_n\|_{L^1(Q)}+\|\gamma\|_{L^1(Q)}\Big]\\
&\quad +\int_Q F\nabla u_n g(u_n)\chi_{\{u_n>m\}}\exp(G(u_n))dxdt\\
&\quad +\Big(\int_m^\infty g(s)\chi_{\{s>m\}}ds\int_Q F\nabla u_n\frac{g(u_n)}{\alpha}\chi_{\{u_n>m\}}\exp(G(u_n))dxdt
\end{aligned}$$

where $B_{m,n}^m(x,r)=\displaystyle\int_0^r\frac{\partial b_n(x,s)}{\partial s}\rho_m(s)\exp(G(s))ds$, which implies, since $B_{m,n}^m(x,r)\geq 0$, by (2.2.5) and Young's Inequality,

$$\begin{aligned}
&\Big(\frac{\alpha-1}{2}\Big)\int_{\{u_n>m\}}|\nabla u_n|^{p(x)}g(u_n)\exp(G(u_n))dxdt\\
&\qquad\leq\Big(\int_m^\infty g(s)ds\Big)\exp\Big(\frac{\|g\|_{L^1(\mathbb{R})}}{\alpha}\Big)\Big[\|f_n\|_{L^1(Q)}\\
&\qquad\quad +\|\gamma\|_{L^1(Q)}+\|b_n(x,u_{0n}\|_{L^1(\Omega)}+C_5\Big] \qquad (2.4.13)
\end{aligned}$$

Using (2.4.13) and the strong convergence of f_n in $L^1(\Omega)$ and $b_n(x,u_{0n})$ in $L^1(\Omega)$ $\gamma\in L^1(\Omega)$, $g\in L^1(\mathbb{R})$ and $F\in(L^{p'(\cdot)}(Q))^N$, by Lebesgue's theorem, passing to limit in (2.4.12), we conclude that

$$\lim_{m\to\infty}\limsup_{n\to\infty}\int_{\{m\leq u_n\leq m+1\}}a(x,t,u_n,\nabla u_n)\nabla u_n dxdt=0. \qquad (2.4.14)$$

On the other hand, taking $\varphi = T_1(u_n - T_m(u_n))^-$ as a test function in (2.3.6) and reasoning as in the proof (2.4.14), we deduce that

$$\lim_{m\to\infty}\limsup_{n\to\infty}\int_{\{-(m+1)\le u_n\le -m\}} a(x,t,u_n,\nabla u_n)\nabla u_n dxdt = 0. \tag{2.4.15}$$

By using (2.4.14) and (2.4.15), we have

$$\lim_{m\to\infty}\limsup_{n\to\infty}\int_{\{m\le |u_n|\le m+1\}} a(x,t,u_n,\nabla u_n)\nabla u_n dxdt = 0. \quad \square \tag{2.4.16}$$

Proof of Proposition 2.3.2.

For $m > k$, let $\varphi = (T_k(u_n) - \omega_\mu^i)^+ h_m(u_n) \in L^{p^-}(0,T;W_0^{1,p(.)}(\Omega)) \cap L^\infty(Q)$ and $\varphi \ge 0$. If we take this function in (2.3.5), we obtain

$$\begin{aligned}
&\int_{\{T_k(u_n)-w_\mu^i\ge 0\}} \frac{\partial b_n(x,u_n)}{\partial t}\exp(G(u_n))(T_k(u_n)-w_\mu^i)h_m(u_n)dxdt\\
&+\int_{\{T_k(u_n)-w_\mu^i\ge 0\}} a(x,t,u_n,\nabla u_n)\nabla(T_k(u_n)-w_\mu^i)h_m(u_n)dxdt\\
&-\int_{\{m\le u_n\le m+1\}} \exp(G(u_n))a(x,t,u_n,\nabla u_n)\nabla u_n(T_k(u_n)-w_\mu^i)^+dxdt\\
&\le \int_Q (f_n+\gamma)\exp(G(u_n))(T_k(u_n)-w_\mu^i)^+h_m(u_n)dxdt \qquad (2.4.17)\\
&+\int_Q F\nabla u_n\frac{g(u_n)}{\alpha}\exp(G(u_n))(T_k(u_n)-\omega_\mu^i)^+h_m(u_n)dxdt\\
&+\int_{\{T_k(u_n)-\omega_\mu^i\ge 0\}} F\exp(G(u_n))(T_k(u_n)-\omega_\mu^i)h_m(u_n)dxdt\\
&-\int_{\{m\le u_n\le m+1\}} F\exp(G(u_n))(T_k(u_n)-\omega_\mu^i)^+dxdt
\end{aligned}$$

Observe that

$$\begin{aligned}
&\Big|\int_{\{m\le u_n\le m+1\}} \exp(G(u_n))a(x,t,u_n,\nabla u_n)\nabla u_n(T_k(u_n)-w_\mu^i)^+dxdt\Big|\\
&\qquad\le 2k\exp\Big(\frac{\|g\|_{L^1(\mathbb{R})}}{\alpha}\Big)\int_{\{m\le u_n\le m+1\}} a(x,t,u_n,\nabla u_n)\nabla u_n dxdt.
\end{aligned}$$

and

$$\begin{aligned}
&\Big|\int_{\{m\le u_n\le m+1\}} F\nabla u_n\exp(G(u_n))(T_k(u_n)-\omega_\mu^i)^+dxdt\Big|\\
&\le 2k\exp\Big(\frac{\|g\|_{L^1(\mathbb{R})}}{\alpha}\Big)\frac{\|F\|_{L^{p'(\cdot)}(Q))^N}}{\alpha^{\frac{1}{p^-}}}\Big(\int_{\{m\le u_n\le m+1\}} a(x,t,u_n,\nabla(u_n))\nabla u_n)dxdt\Big)^{\frac{1}{p^-}}
\end{aligned}$$

Tanks to (2.3.17) the third and fourth integrals on the right hand side tend to zero as n and m tend to infinity and by Lebesgue's theorem and $F \in (L^{p'(.)}(Q))^N$, we deduce that the right hand side converges to zero as $n,\ m$ and μ tend to infinity . Since

$$\Big(T_k(u_n) - \omega_\mu^i\Big)^+ h_m(u_n) \rightharpoonup \Big(T_k(u) - \omega_\mu^i\Big)^+ h_m(u)\ in\ L^\infty(Q)\ as\ n \to \infty$$

and strongly in $L^{p^-}(0,T;W_0^{1,p(.)}(\Omega))$ and $(T_k(u_n)-\omega_\mu^i)^+h_m(u_n) \rightharpoonup\ 0\ in\ L^\infty(Q)$ and strongly in $L^{p^-}(0,T;W_0^{1,p(.)}(\Omega))\ as\ \mu \to \infty$, it follows that the first and second integrals on the right-hand side of (2.4.17) converge to zeros as $n,\ m,\ \mu \to \infty$, using [3] lemma (2.3.5) and lemma (1.2.1) the proof of Proposition (2.3.2) is complete.□

2.5 Example.

Consider the following special case : $b(x,s) = F(x)K(s)$, where $F \in W^{1,p(.)}(\Omega)$ with $p(x) = \sin|x| + 3$, $p \in C_+(\bar{\Omega})$ and $K \in C^1(\mathbb{R}), K(0) = 0$
b is a Carathéodory function satisfing the following assertions :
$b(x,0) = 0$. Next, for any $k > 0$, there exist $\lambda_k > 0$ and function $A_k \in L^\infty(\Omega)$ $B_k \in L^{p(.)}(\Omega)$ such that

$$\lambda_k = \inf_{|s|\le k} K'(s) \le \frac{\partial b(x,s)}{\partial s} \le A_k(x) \ \text{ and } \quad \Big|D_x\Big(\frac{\partial b(x,s)}{\partial s}\Big)\Big| \le B_k(x). \qquad (2.5.1)$$

for almost every $x \in \Omega$ and every s such that $|s| \le k$, we have

$$Au = -\triangle_{p(x)} = -\mathrm{div}(|\nabla u|^{p(x)-2}\nabla u). \qquad (2.5.2)$$

we are $(|\nabla u|^{p(x)-2}\nabla u - |\nabla v|^{p(x)-2}\nabla v)(u-v) > 0$ for almost all $x \in \Omega,\ u,v \in \mathbb{R}^{\mathbb{N}}$ and $u \neq v$ then the monotonicity condition is satisfying.
The operator $-\mathrm{div}(|\nabla u|^{p(x)-2}\nabla u)$ is a Carathéodory function satisfing the growth condition (2.2.3) and the coercivity (2.2.5).

$$H(x,t,u,\nabla u) = \frac{-u}{2+u^4}|\nabla u|^{p(x)} + \gamma(x,t). \qquad (2.5.3)$$

where $\gamma \in L^1(Q)$, $\quad H(x,t,u,\nabla u)$ is a Carathéodory function and

$$\begin{aligned} |H(x,t,u,\nabla u)| &\le \frac{|u|}{2+u^4}|\nabla u|^{p(x)} + \gamma(x,t) \\ &= g(u)|\nabla u|^{p(x)} + \gamma(x,t), \end{aligned}$$

where $g(u) = \frac{|u|}{2+u^4}|$ is bounded positive continuous function which belongs to $L^1(\mathbb{R})$. Note that $H(x, t, u, \nabla u)$ does not satisfy the sign condition or the coercivity condition. Finally, the hypotheses of Theorem (2.3.1) are satisfied. Therefore, the problem (2.1.1) has at least one renormalized solution.

Chapitre 3

Nonlinear parabolic inequalities in Sobolev space with variable exponent

Abstract

We prove the existence of an entropy solution for the obstacle parabolic problem associated to the equation

$$\begin{cases} u \geq \psi & \text{a.e. in } \Omega \times (0,T), \\ \frac{\partial u}{\partial t} - \operatorname{div}(a(x,t,u,\nabla u)) + H(x,t,u,\nabla u) = f & \text{in } Q = \Omega \times (0,T), \end{cases}$$

where the second term f belongs to $L^1(Q)$ and $u_0 \in L^1(\Omega)$. The critical growth condition on H is with respect to ∇u ; no growth with respect to u and no sign conditions and the coercivity conditions are assumed. The main methods are the so called " penalization methods".

3.1 Introduction

In the present chapter we establish an existence result of an entropy solution for a class of nonlinear parabolic problems of the type :

$$\begin{cases} u \geq \psi & \text{a.e. in } \Omega \times (0,T), \\ \frac{\partial u}{\partial t} - \operatorname{div}(a(x,t,u,\nabla u)) + H(x,t,u,\nabla u) = f & \text{in } Q = \Omega \times (0,T), \\ u(x,0) = u_0 & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega \times (0,T). \end{cases} \tag{3.1.1}$$

In the problem (3.1.1), Ω is a bounded domain in $\mathbb{R}^N$, $N \geq 2$, T is a positive real number, while the data $f \in L^1(Q)$ and $u_0 \in L^1(\Omega)$. The operator

$-\mathrm{div}(a(x,t,u,\nabla u))$ is a Leray–Lions operator which is coercive, H is a nonlinear lower order term.
More precisely, this chapter deals with the existence of a solution to the obstacle parabolic problems associated to (3.1.1) in the sense of an entropy solution (see Definition (3.2.1)).
The existence of solutions for nonlinear parabolic inequalities with L^1-data in Orlicz spaces was studied by R. Aboulaich, B. Achchab, D. Meskine and A. Souissi in [1], where the case $H(x,t,u,\nabla u) = 0$ was considered. In the case where $a(x,t,u,\nabla u) = |\nabla u|^{p(x)-2}\nabla u$ and $H(x,t,u,\nabla u) = 0$, M. Bendahmane, P. Wittbold and A. Zimmermann [29] proved the existence and uniqueness of renormalized solutions to nonlinear parabolic equations with L^1-data. Recently, M. Sanchón and Urbano [81] have studied a Dirichlet problem (3.1.1) of $p(x)$-laplace equation and obtained the existence and uniqueness of an entropy solution for L^1-data where $H(x,t,u,\nabla u) = 0$, and $Au = -\mathrm{div}(|\nabla u|^{p(x)-2}\nabla u)$. In the case where $H(x,t,u,\nabla u) = -g(u)M(|\nabla u|)$, the existence of solutions of some unilateral problems in the framework of Orlicz spaces has been established by M. Kbiri Alaoui , D. Meskine and A. Souissi [59] with the penalization methods.
The plan of the chapter is as follows. . In Section 2 we give the basic assumptions and the definition of an entropy solution of (3.1.1). In Section 3 we establish the existence of such a solution in Theorem (3.3.1). Section 4 is devoted to an example which illustrates the abstract result.

3.2 Assumption on data and definition of an entropy solution

Throughout the paper, we assume that the following assumptions hold true.
Assumption (H1)
Let Ω be a bounded open subset of $\mathbb{R}^N (N \geq 2), T > 0$ be given and let us set $Q = \Omega \times [0,T]$. Moreover, let $p \in C_+(\bar{\Omega})$ and assume that $p(x)$ satisfies the log-Hölder condition (1.1.1) with $1 < p^- \leq p^+ < \infty$. We consider a Leray–Lions operator defined by the formula :

$$Au = -\mathrm{div}\Big(a(x,t,u,\nabla u)\Big),$$

where $a : \Omega \times [0,T] \times \mathbb{R} \times \mathbb{R}^N \to \mathbb{R}$ is a Carathéodory function (i.e., measurable with respect to x in Ω for every (s,ξ) in $\mathbb{R} \times \mathbb{R}^N$ and continuous with respect to (s,ξ) in $\mathbb{R} \times \mathbb{R}^N$ for almost every x in Ω), which satisfies the following conditions : there exists $k(x,t) \in L^{p'(x)}(Q)$ such that for almost every $(x,t) \in Q$, all $(s,\xi) \in \mathbb{R} \times \mathbb{R}^N$ and $\beta > 0$

$$\Big|a(x,t,s,\xi)\Big| \leq \beta\Big[k(x,t) + |s|^{p(x)-1} + |\xi|^{p(x)-1}\Big], \tag{3.2.1}$$

$$\Big[a(x,t,s,\xi) - a(x,t,s,\eta)\Big](\xi - \eta) > 0 \;\forall\, \xi \neq \eta \in \mathbb{R}^N, \tag{3.2.2}$$

$$a(x,t,s,\xi) \cdot \xi \geq \alpha|\xi|^{p(x)}, \tag{3.2.3}$$

where α is strictly a positive constant.

Assumption (H2)

Furthermore, let $H : \Omega \times [0,T] \times \mathbb{R} \times \mathbb{R}^N \to \mathbb{R}$ be a Carathéodory function such that for all $(x,t) \in Q$ and for all $s \in \mathbb{R}$, $\xi \in \mathbb{R}^N$, the growth condition

$$|H(x,t,s,\xi)| \leq \gamma(x,t) + g(s)|\xi|^{p(x)}, \tag{3.2.4}$$

is satisfied, where $g : \mathbb{R} \to \mathbb{R}^+$ is a continuous positive function that belongs to $L^1(\mathbb{R})$ and while $\gamma(x,t) \in L^1(Q)$.

Let ψ be a measurable function with values in $\bar{\mathbb{R}}$ such that $\psi \in L^{p^-}(0,T;W_0^{1,p(x)}(\Omega)) \cap L^\infty(Q)$ and let

$$K_\psi = \Big\{u \in L^{p^-}(0,T;W_0^{1,p(x)}(\Omega)),\; u \geq \psi \;\; a.e \;\; in \;\; \Omega \times (0,T)\Big\}.$$

and we define $\phi_k(s) = \frac{1}{k}T_k(s)$.

Definition 3.2.1 *Let $f \in L^1(Q)$ and $u_0 \in L^1(\Omega)$. A real-valued function u defined on Q is an entropy solution of the problem* (3.1.1) *if*

$$u \geq \psi \quad a.e. \; in \; \Omega \times (0,T), \tag{3.2.5}$$

$$T_k(u) \in L^{p^-}(0,T;W_0^{1,p(x)}(\Omega)) \quad for\ all\ k \geq 0,$$

$$u \in C(0,T;L^1(\Omega)), \tag{3.2.6}$$

$$\begin{aligned}
&\int_\Omega S_k\Big(u(T) - v(T)\Big)dx - \int_\Omega S_k\Big(u_0 - v(0)\Big)dx \\
&\quad + \int_Q \frac{\partial v}{\partial t} T_k(u-v)dxdt + \int_Q H(x,t,u,\nabla u)T_k(u-v)dxdt \\
&\quad + \int_Q a(x,t,u,\nabla u)\nabla T_k(u-v)dxdt \\
&\quad \leq \int_Q fT_k(u-v)dxdt \quad \forall v \in K_\psi \,\cap\, L^\infty(Q),
\end{aligned} \tag{3.2.7}$$

where $S_k(s) = \displaystyle\int_0^s T_k(r)dr$, *and* $\frac{\partial v}{\partial t} \in L^{p'^-}(0,T;W^{-1,p'(x)}(\Omega))$.

3.3 Existence results

In this section, we establish the following existence theorem. we begin with the following

Theorem 3.3.1 *Let $f \in L^1(Q)$, $u_0 \in L^1(\Omega)$ and $p \in C_+(\bar{\Omega})$. Assume that (H1) and (H2) hold true. Then, there exists at least one entropy solution u of the problem* (3.1.1) *(in the sense of Definition (3.2.1)).*

Proof. The above theorem is proven in the following five steps.
Step 1 : Approximate problem. For $n > 0$, let us define the following respective approximation of H, f, and u_0,

$$H_n(x,t,s,\xi) \;=\; \frac{H(x,t,s,\xi)}{1+\frac{1}{n}|H(x,t,s,\xi)|},$$

and select f_n, and u_{0n} so that

$$f_n \in L^{p'(x)}(Q),\ f_n \to f \text{ a.e. in } Q, \text{ and strongly in } L^1(Q) \text{ as } n \to \infty, \tag{3.3.1}$$

$$u_{0n} \in D(\Omega), u_{0n} \to u_0, \text{a.e. in } \Omega \text{ and strongly in } L^1(\Omega) \text{ as } n \to \infty. \tag{3.3.2}$$

Note that $H_n(x,t,s,\xi)$ satisfies the following conditions

$$\begin{aligned} |H_n(x,t,s,\xi)| &\le |H(x,t,s,\xi)| \\ \textit{and}\quad |H_n(x,t,s,\xi)| &\le n \quad \text{for all } (s,\xi) \in \mathbb{R}\times\mathbb{R}^N. \end{aligned}$$

Let us now consider the approximate problem

$$(P_3)_n \begin{cases} \dfrac{\partial u_n}{\partial t} - \operatorname{div}(a(x,t,u_n,\nabla u_n)) + H_n(x,t,u_n,\nabla u_n) & \\ \qquad\qquad + n\, T_n(u_n-\psi)^-\phi_{\frac{1}{n}}(u_n) = f_n & \text{in } D'(Q), \\ u_n(t=0) = u_{0n} & \text{in } \Omega, \\ u_n = 0 & \text{on } \partial\Omega\times(0,T). \end{cases}$$

Moreover, since $f_n \in L^{p'^-}(0,T;W^{-1,p'(x)}(\Omega))$, proving the existence of a weak solution $u_n \in L^{p^-}(0,T;W_0^{1,p(x)}(\Omega))$ of $(P_3)_n$ is an easy task (see[65]).
Step 2 : A Priori estimates. Let us begin with the following

Proposition 3.3.1 *Let u_n be a solution of the approximate problem $(P_3)_n$. Then, there exists a constant C(which does not depend on n and k) such that :*

$$\|T_k(u_n)\|_{L^{p^-}(0,T;W_0^{1,p(.)}(\Omega))} \le C\,k \quad \forall\, k>0.$$

Proof. Let $v = T_k(u_n)^+\chi_{(0,\tau)}\exp(G(u_n))$, where, $G(s) = \displaystyle\int_0^s \frac{g(r)}{\alpha}dr$ (the function g appears in (3.2.4)). Choosing v as a test function in the approximate $(P_3)_n$ with $\tau \in (0,T)$ by using (3.2.4) and (3.2.3), we get

$$\begin{aligned}
&\int_{Q^\tau} \frac{\partial u_n}{\partial t}\exp(G(u_n))T_k(u_n)^+dxdt \\
&\quad + \int_{Q^\tau} a(x,t,u_n,\nabla u_n)\exp(G(u_n))\nabla T_k(u_n)^+dxdt \\
&\quad\quad + \int_{Q^\tau} nT_n(u_n-\psi)^-\exp(G(u_n))\phi_{\frac{1}{n}}(u_n)T_k(u_n)^+dxdt \\
&\quad\quad\quad \le \int_{Q^\tau}[\gamma(x,t)+f_n]\exp(G(u_n))T_k(u_n)^+dxdt.
\end{aligned} \tag{3.3.3}$$

On the other hand, taking $v = T_k(u_n)^-\chi_{(0,\tau)}\exp(-G(u_n))$ as a test function in the problem $(P_3)_n$, we deduce as in (3.3.3) that

$$\begin{aligned}
&\int_{Q^\tau} \frac{\partial u_n}{\partial t}\exp(-G(u_n))T_k(u_n)^-dxdt \\
&\quad + \int_{Q^\tau} a(x,t,u_n,\nabla u_n)\exp(-G(u_n))\nabla T_k(u_n)^-dxdt \\
&\quad\quad + \int_{Q^\tau} \gamma(x,t)\exp(-G(u_n))T_k(u_n)^-dxdt \\
&\quad\quad\quad + \int_{Q^\tau} nT_n(u_n-\psi)^-\exp(-G(u_n))T_k(u_n)^-\phi_{\frac{1}{n}}(u_n)dxdt \\
&\quad\quad\quad\quad \ge \int_{Q^\tau} f_n\exp(-G(u_n))T_k(u_n)^-dxdt,
\end{aligned} \tag{3.3.4}$$

which, by using (3.3.3), gives

$$\begin{aligned}
&\int_0^\tau\int_\Omega \frac{\partial u_n}{\partial t}\exp(G(u_n))T_k(u_n)^+dxdt \\
&\quad + \int_{Q^\tau} a(x,t,u_n,\nabla T_k(u_n)^+)\nabla T_k(u_n)^+\exp(G(u_n))dxdt \\
&\quad\quad + \int_{Q^\tau} nT_n(u_n-\psi)^-\exp(G(u_n))\phi_{\frac{1}{n}}(u_n)T_k(u_n)^+dxdt \\
&\quad\quad\quad \le \int_{Q^\tau}\Big[\gamma(x,t)+f_n\Big]\exp(G(u_n))T_k(u_n)^+dxdt.
\end{aligned}$$

Since $G(u_n) \leq \frac{\|g\|_{L^1(\mathbb{R})}}{\alpha}$, we have $|\varphi_k(r)| \leq k \exp(\frac{\|g\|_{L^1(\mathbb{R})}}{\alpha})|r|$, where $\varphi_k(r) = \int_0^r T_k(s)^+ \exp(G(r))ds$. Then

$$\int_{Q^\tau} \frac{\partial u_n}{\partial t} \exp(G(u_n))T_k(u_n)^+ dxdt \geq \int_\Omega \varphi_k(u_n(\tau))dx - k \exp\left(\frac{\|g\|_{L^1(\mathbb{R})}}{\alpha}\right)\|u_{0n}\|_{L^1(\Omega)}.$$

Then, we deduce that

$$\begin{aligned} &\int_\Omega \varphi_k(u_n(\tau))dx \\ &\qquad + \int_{Q^\tau} a(x,t,u_n,\nabla T_k(u_n)^+)\nabla T_k(u_n)^+ \exp(G(u_n))dxdt \\ &\qquad\quad + \int_{Q^\tau} nT_n(u_n-\psi)^- \exp(G(u_n))\phi_{\frac{1}{n}}(u_n)T_k(u_n)^+ dxdt \\ &\qquad\quad \leq k \exp\left(\frac{\|g\|_{L^1(\mathbb{R})}}{\alpha}\right)\Big[\|\gamma\|_{L^1(Q)} + \|f_n\|_{L^1(Q)} + \|u_{0n}\|_{L^1(\Omega)}\Big] \leq C_1 k. \end{aligned} \tag{3.3.5}$$

Since a satisfies (3.2.3), by the fact that $\varphi_k(u_n(\tau)) \geq 0$, for every $n > 0$ we get

$$\begin{aligned} &\alpha \int_{Q_\tau} |\nabla T_k(u_n)^+|^{p(x)} \exp(G(u_n))\ dx\ dt \\ &\quad + \int_{Q^\tau} nT_n(u_n-\psi)^- \exp(G(u_n))\phi_{\frac{1}{n}}(u_n)T_k(u_n)^+ dxdt \leq C_1\ k \end{aligned} \tag{3.3.6}$$

where C_1 is a constant which varies from line to line and which depends only on the data. It follows that

$$0 \leq \int_{Q_\tau} nT_n(u_n-\psi)^- \exp(G(u_n))\phi_{\frac{1}{n}}(u_n)\frac{T_k(u_n)^+}{k}\ dx\ dt \leq C_1,$$

and as $k \to 0$ by Fatou's lemma we deduce that

$$\int_{\{u_n \geq 0\}} nT_n(u_n-\psi)^- \exp(G(u_n))\phi_{\frac{1}{n}}(u_n)\, dx\, dt \leq C_1. \tag{3.3.7}$$

Thanks to (3.3.6), we have

$$\alpha \int_{Q_\tau} |\nabla T_k(u_n)^+|^{p(x)} \exp(G(u_n))dxdt \leq C_1 k, \tag{3.3.8}$$

and we deduce that

$$\alpha \int_Q |\nabla T_k(u_n)^+|^{p(x)} dxdt \leq C_1\ k. \tag{3.3.9}$$

Now, using $v = T_k(u_n)^-\chi_{(0,\tau)}\exp(-G(u_n))$ as a test function in (3.3.4), with $k > 0$ for every $\tau \in [0,T]$ we get

$$\int_{Q^\tau} a(x,t,u_n,\nabla u_n)\exp(-G(u_n))\nabla u_n\chi_{\{-k\leq u_n\leq 0\}}dxdt$$
$$-\int_{Q^\tau} nT_n(u_n-\psi)^-\exp(-G(u_n))\phi_{\frac{1}{n}}(u_n)T_k(u_n)^-dxdt \tag{3.3.10}$$
$$\leq k\exp\Big(\frac{\|g\|_{L^1(\mathbb{R})}}{\alpha}\Big)\Big[\|\gamma\|_{L^1(Q)}+\|f_n\|_{L^1(Q)}+\|u_{0n}\|_{L^1(\Omega)}\Big]$$
$$-\int_\Omega \varphi_k^1(u_n(\tau))dx$$

where $\varphi_k^1 = \displaystyle\int_0^r T_k(s)^-\exp(-G(s))ds$. Then

$$\int_\Omega \varphi_k^1(u_n(\tau))dx$$
$$+\int_{Q^\tau} a(x,t,u_n,\nabla u_n)\exp(-G(u_n))\nabla u_n\chi_{\{-k\leq u_n\leq 0\}}dxdt$$
$$-\int_{Q^\tau} nT_n(u_n-\psi)^-\exp(-G(u_n))\phi_{\frac{1}{n}}(u_n)T_k(u_n)^-dxdt \tag{3.3.11}$$
$$\leq k\exp\Big(\frac{\|g\|_{L^1(\mathbb{R})}}{\alpha}\Big)\Big[\|\gamma\|_{L^1(Q)}+\|f_n\|_{L^1(Q)}+\|u_{0n}\|_{L^1(\Omega)}\Big]\leq C_2k.$$

Since a satisfies (3.2.3) and due to the fact that $\varphi_k^1(u_n(\tau)) \geq 0$, for every $n > 0$ we get

$$\alpha\int_{Q_\tau} |\nabla T_k(u_n)^-|^{p(x)}\exp(-G(u_n))\ dx\ dt \tag{3.3.12}$$
$$-n\int_{Q_\tau} T_n(u_n-\psi)^-T_k(u_n)^-\exp(-G(u_n))\phi_{\frac{1}{n}}(u_n)\ dx\ dt\leq C_2\ k,$$

where C_2 is a positive constant and we conclude that

$$0\leq -\int_{\{u_n\leq 0\}} nT_n(u_n-\psi)^-\exp(-G(u_n))\phi_{\frac{1}{n}}(u_n)\leq C_2, \tag{3.3.13}$$

and

$$\alpha\int_Q |\nabla T_k(u_n)^-|^{p(x)}\ dx\ dt\leq C_2\ k. \tag{3.3.14}$$

Combining (3.3.9), (3.3.14) and lemma (1.1.1), we deduce that :

$$\int_0^T \min\Big\{\|\nabla T_k(u_n)\|_{p(x)}^{p^+}, \|\nabla T_k(u_n)\|_{p(.)}|^{p^-}\Big\}dt\leq \rho(\nabla T_k(u_n))\leq C_3k.$$

$$\Rightarrow \|T_k(u_n)\|_{L^{p^-}(0,T;W_0^{1,p(x)}(\Omega))} \leq \ k\, C_3. \tag{3.3.15}$$

Then, $T_k(u_n)$ is bounded in $L^{p^-}(0,T;W_0^{1,p(x)}(\Omega))$ independently of n for any $k>0$. Now we turn to proving the almost everywhere convergence of u_n. Consider a non-decreasing function $g_k \in C^2(\mathbb{R})$ such that

$$g_k(s) = \begin{cases} s & if\ |s| \leq \frac{k}{2}, \\ k & if\ |s| \geq k. \end{cases}$$

Multiplying the approximate equation by $g_k'(u_n)$, we get

$$\begin{aligned} &\frac{\partial g_k(u_n)}{\partial t} - \operatorname{div}\Big(a(x,t,u_n,\nabla u_n)g_k'(u_n)\Big) + a(x,t,u_n,\nabla u_n)g_k''(u_n)\nabla u_n \\ &+nT_n(u_n-\psi)^- g_k'(u_n)\phi_{\frac{1}{n}}(u_n) + H_n(x,t,u_n,\nabla u_n)g_k'(u_n) = f_n g_k'(u_n) \end{aligned} \tag{3.3.16}$$

in the sense of distributions. This, thanks to the fact that g_k' has compact support implies, that $g_k(u_n)$ is bounded in $L^{p^-}(0,T;W_0^{1,p(x)}(\Omega))$, while its time derivative $\frac{\partial g_k(u_n)}{\partial t}$ is bounded in $L^1(Q)+V^*$. Due to the choice of g_k, we conclude that for each k the sequence $T_k(u_n)$ converges almost everywhere in Q, which implies that the sequence u_n converge almost everywhere to some measurable function v in Q. Thus by using the same argument as in [34, 35, 36], we can show the following lemma.

Lemma 3.3.2 *Let u_n be a solution of the approximate problem $(P_3)_n$. Then*

$$u_n \to u \quad a.e. \quad in \quad Q.$$

We can deduce from (3.3.15) that

$$T_k(u_n) \rightharpoonup T_k(u) \quad in \quad L^{p^-}(0,T;W_0^{1,p(x)}(\Omega)),$$

which by (3.2.3) implies that for every $k>0$ there exists a function $h_k \in (L^{p'(x)}(Q))^N$ such that

$$a(x,u,T_k(u_n),\nabla T_k(u_n)) \rightharpoonup h_k \quad \text{in} \quad (L^{p'(x)}(Q))^N. \tag{3.3.17}$$

Lemma 3.3.3 *([9]) Let u_n be a solution of the approximate problem $(P_3)_n$. Then,*

$$\lim_{m\to\infty}\limsup_{n\to\infty}\int_{\{m\leq|u_n|\leq m+1\}} a(x,t,u_n,\nabla(u_n))\nabla u_n dxdt = 0. \tag{3.3.18}$$

Step 3 : **Almost everywhere convergence of the gradients** :
This step is devoted to introducing for a fixed $k \geq 0$ a time regularization of the function $T_k(u)$ in order to perform the monotonicity method. This specific time regularization of $T_k(u)$ (for fixed $k \geq 0$) is defined as follows. Let $(\upsilon_0^\mu)_\mu$ be a sequence of functions defined on Ω such that

$$\upsilon_0^\mu \in L^\infty(\Omega) \cap W_0^{1,p(x)}(\Omega) \; for \; all \; \mu > 0, \tag{3.3.19}$$

$$\|\upsilon_0^\mu\|_{L^\infty(\Omega)} \leq k \; for \; all \; \mu > 0, \tag{3.3.20}$$

$$\upsilon_0^\mu \to T_k(u_0) \; a.e. \; in \; \Omega \; and \; \frac{1}{\mu}\|\upsilon_0^\mu\|_{L^{p(x)}(\Omega)} \to 0, \;\; as \; \mu \to \infty. \tag{3.3.21}$$

For fixed $k, \;\; \mu > 0$ let us consider the unique solution $(T_k(u))_\mu \in L^\infty(Q) \cap L^{p^-}(0, T; W_0^{1,p(x)}(\Omega))$ of the monotone problem :

$$\frac{\partial (T_k(u))_\mu}{\partial t} + \mu\Big((T_k(u))_\mu - T_k(u)\Big) = 0 \quad in \; D'(Q), \tag{3.3.22}$$

$$(T_k(u))_\mu(t = 0) = \upsilon_0^\mu \quad in \quad \Omega. \tag{3.3.23}$$

Note that due to (3.3.22) for $\mu > 0$ and $k \geq 0$, we have

$$\frac{\partial (T_k(u))_\mu}{\partial t} \in L^{p^-}(0, T; W_0^{1,p(x)}(\Omega)). \tag{3.3.24}$$

We just recall here that (3.3.22)-(3.3.23) imply that

$$(T_k(u))_\mu \to T_k(u) \; a.e. \; in \; Q \tag{3.3.25}$$

as well as weakly in $L^\infty(Q)$ and strongly in $L^{p^-}(0, T; W_0^{1,p(x)}(\Omega))$ as $\mu \to \infty$. Note that for any μ and any $k \geq 0$, we have

$$\|(T_k(u))_\mu\|_{L^\infty(Q)} \leq \max\Big(\|T_k(u)\|_{L^\infty(Q)}; \; \|\upsilon_0^\mu\|_{L^\infty(\Omega)}\Big) \leq k, \tag{3.3.26}$$

We introduce a sequence of increasing $C^\infty(\mathbb{R})$-functions S_m such that

$$S_m(r) = r \; for \; |r| \leq m, \; supp(S'_m) \subset [-(m+1), m+1], \; \|S''_m\|_{L^\infty(\mathbb{R})} \leq 1,$$

for any $m \geq 1$, and we denote by $\omega(n, \mu, \eta, m)$ the quantities such that

$$\lim_{m\to\infty} \lim_{\eta\to 0} \lim_{\mu\to\infty} \lim_{n\to\infty} \omega(n, \mu, \eta, m) = 0.$$

The main estimate is

Lemma 3.3.4 *([9, 36]) We have*

$$\int_0^T \left\langle \frac{\partial u_n}{\partial t}, T_\eta\Big(u_n - (T_k(u))_\mu\Big)^+ \exp(G(u_n))S'_m(u_n) \right\rangle dt \geq \omega(n,\mu,\eta)\ \forall m \geq 1. \tag{3.3.27}$$

Taking now $v = T_\eta\Big(u_n - (T_k(u))_\mu\Big)^+ S'_m(u_n)\exp(G(u_n))$ in $(P_3)_n$ and using(3.2.4) and (3.2.3), we get

$$\begin{aligned}
&\int_0^T \left\langle \frac{\partial u_n}{\partial t}, T_\eta\Big(u_n - (T_k(u))_\mu\Big)^+ \exp(G(u_n))S'_m(u_n) \right\rangle dt \\
&\quad + \int_Q a\Big(x,t,u_n,\nabla u_n\Big)\nabla\Big(T_\eta\Big(u_n - (T_k(u))_\mu\Big)^+\Big)\exp(G(u_n))S'_m(u_n)dxdt \qquad (3.3.28)\\
&\qquad + \int_{\{m\leq|u_n|\leq m+1\}} a\Big(x,t,u_n,\nabla u_n\Big)T_\eta(u_n - (T_k(u))_\mu)^+\exp(G(u_n))S''_m(u_n)\nabla u_n dxdt \\
&\qquad\quad + n\int_Q T_n(u_n-\psi)^- T_\eta\Big(u_n - (T_k(u))_\mu\Big)^+ \exp(G(u_n))S'_m(u_n)\phi_{\frac{1}{n}}(u_n)dxdt \\
&\qquad\qquad \leq C\eta.
\end{aligned}$$

From (3.3.7), (3.3.18), (3.3.27) and (3.3.28) it follows that

$$\begin{aligned}
\int_Q a\Big(x,t,u_n,\nabla u_n\Big)\nabla\Big(T_\eta\Big(u_n - (T_k(u))_\mu\Big)^+\Big)\exp(G(u_n))S'_m(u_n)dxdt& \\
\leq C\eta + \omega(n,\mu,\eta,m)& \qquad (3.3.29)
\end{aligned}$$

where C is a constant independent of n and m. On the other hand, let

$$A = \{0 \leq T_k(u_n) - (T_k(u))_\mu < \eta\} \quad \text{and} \quad B = \{0 \leq u_n - (T_k(u))_\mu < \eta\}.$$

Then, we have

$$\begin{aligned}
&\int_Q a\Big(x,t,u_n,\nabla u_n\Big)\nabla\Big(T_\eta\Big(u_n - (T_k(u))_\mu\Big)^+\Big)\exp(G(u_n))S'_m(u_n)dxdt \\
&= \int_B a\Big(x,t,u_n,\nabla u_n\Big)\Big(\nabla u_n - \nabla(T_k(u))_\mu\Big)\exp(G(u_n))S'_m(u_n)dxdt \qquad (3.3.30)\\
&= \int_A a\Big(x,t,T_k(u_n),\nabla T_k(u_n)\Big)\Big(\nabla T_k(u_n) - \nabla(T_k(u))_\mu\Big)\exp(G(u_n))S'_m(u_n)dxdt \\
&\quad + \int_{\{|u_n|>k\}\cap B} a\Big(x,t,u_n,\nabla u_n\Big)\Big(\nabla u_n - \nabla(T_k(u))_\mu\Big)\exp(G(u_n))S'_m(u_n)dxdt.
\end{aligned}$$

The coercive condition (3.2.3) and we use the definition of $S'_m(S'_m(u_n) = 1$ a.e. in $\{|u_n| \leq k\}$ if $k \leq m)$ in view of (3.3.29) and (3.3.30),we get

$$\begin{aligned}
&\int_A a\Big(x,t,T_k(u_n),\nabla T_k(u_n)\Big)\Big(\nabla T_k(u_n) - \nabla(T_k(u))_\mu\Big)\exp(G(u_n)S'_m(u_n)dxdt \\
&\quad \leq \int_{\{|u_n|>k\}\cap B} a\Big(x,t,u_n,\nabla u_n\Big)\nabla(T_k(u))_\mu \exp(G(u_n))S'_m(u_n)dxdt \\
&\qquad +C\eta + \omega(n,\mu,\eta,m).
\end{aligned} \tag{3.3.31}$$

Since $a\Big(x,t,T_{k+\eta}(u_n),\nabla T_{k+\eta}(u_n)\Big)$ is bounded in $(L^{p'(x)}(Q))^N$, there exists some $h_{k+\eta} \in (L^{p'(x)}(Q))^N$ such that $a\Big(x,t,T_{k+\eta}(u_n),\nabla T_{k+\eta}(u_n)\Big) \rightharpoonup h_{k+\eta}$ weakly in $(L^{p'(x)}(Q))^N$. Consequently,

$$\begin{aligned}
&\int_{\{|u_n|>k\}\cap B} a\Big(x,t,u_n,\nabla u_n\Big)|\nabla(T_k(u))_\mu| \exp(G(u_n))S'_m(u_n)dxdt \\
&\quad = \int_{\{|u|>k\}\cap\{0\leq u-(T_k(u))_\mu<\eta\}} h_{k+\eta}\nabla(T_k(u))_\mu \exp(G(u))S'_m(u)dxdt + \omega(n).
\end{aligned}$$

Thanks to (3.3.25) one easily has

$$\int_{\{|u|>k\}\cap\{0\leq u-(T_k(u))_\mu<\eta\}} h_{k+\eta}\nabla(T_k(u))_\mu \exp(G(u))S'_m(u)dxdt = \omega(\mu).$$

Hence,

$$\begin{aligned}
&\int_A a\Big(x,t,T_k(u_n),\nabla T_k(u_n)\Big)\Big(\nabla T_k(u_n) - \nabla(T_k(u))_\mu\Big)\exp(G(u_n))dxdt \\
&\qquad \leq C\eta + \omega(n,\mu,\eta,m).
\end{aligned} \tag{3.3.32}$$

On the other hand, note that

$$\begin{aligned}
&\int_A a\Big(x,t,T_k(u_n),\nabla T_k(u_n)\Big)\Big(\nabla T_k(u_n) - \nabla(T_k(u))_\mu\Big)\exp(G(u_n))dxdt \\
&\quad = \int_A a\Big(x,t,T_k(u_n),\nabla T_k(u_n)\Big)\Big(\nabla T_k(u_n) - \nabla T_k(u)\Big)\exp(G(u_n))dxdt \\
&\qquad + \int_A a\Big(x,t,T_k(u_n),\nabla T_k(u_n)\Big)\Big(\nabla T_k(u) - \nabla(T_k(u))_\mu\Big)\exp(G(u_n))dxdt
\end{aligned} \tag{3.3.33}$$

and the last integral tends to 0 as $n \to \infty$ and $\mu \to \infty$. Indeed, we have that

$$\begin{aligned}
&\int_A a\Big(T_k(u_n),\nabla T_k(u_n)\Big)\Big(\nabla T_k(u) - \nabla(T_k(u))_\mu\Big)\exp(G(u_n))dxdt \\
&\quad \to \int_{\{0\leq T_k(u)-(T_k(u))_\mu<\eta\}} h_k\Big(\nabla T_k(u) - \nabla(T_k(u))_\mu\Big)\exp(G(u))dxdt
\end{aligned}$$

as $n \to \infty$. It is obvious that

$$\int_{\{0 \le T_k(u)-(T_k(u))_\mu < \eta\}} h_k\Big(\nabla T_k(u) - \nabla (T_k(u))_\mu\Big) \exp(G(u)) dxdt \to 0 \quad as\ \mu \to \infty.$$

We deduce then that

$$\int_A a\Big(x,t,T_k(u_n),\nabla T_k(u_n)\Big)\Big(\nabla T_k(u_n) - \nabla T_k(u)\Big)\exp(G(u_n))dxdt \quad (3.3.34)$$
$$\le C\eta + \epsilon(n,\mu,\eta,m).$$

Let

$$M_n = \Big(\Big[a(x,t,T_k(u_n),\nabla T_k(u_n)) - a(x,t,T_k(u_n),\nabla T_k(u))\Big]\Big[\nabla T_k(u_n) - \nabla T_k(u)\Big]\Big) \times$$
$$\times\Big(\exp(G(u_n))\Big).$$

Then for any $0 < \theta < 1$, we write

$$\begin{aligned} I_n &= \int_{\{|u_n - (T_k(u))_\mu| \ge 0} M_n^\theta dxdt \\ &= \int_{\{|T_k(u_n)-(T_k(u))_\mu| \le \eta,\ u_n - T_k(u)_\mu \ge 0\}} M_n^\theta dxdt \\ &\quad + \int_{\{|T_k(u_n)-(T_k(u))_\mu| > \eta,\ u_n - (T_k(u))_\mu \ge 0\}} M_n^\theta dxdt. \end{aligned}$$

Since, $a\Big(x,t,T_k(u_n),\nabla T_k(u_n)\Big)$ is bounded in $(L^{p'(x)}(Q))^N$, while $\nabla T_k(u_n)$ is bounded in $(L^{p(x)}(Q))^N$ by applying the Hölder's inequality, we obtain

$$I_n \ \le\ C_1\Big(\int_{\{0 \le T_k(u_n)-(T_k(u))_\mu < \eta\}} M_n dxdt\Big)^\theta \quad (3.3.35)$$
$$+C_2 \text{ meas}\Big\{(x,t) \in Q : \Big|T_k(u_n) - (T_k(u))_\mu\Big| > \eta\ ,\ u_n - (T_k(u))_\mu \ge 0\Big\}^{1-\theta}.$$

On the other hand, we have,

$$\begin{aligned} &\int_{\{0 \le T_k(u_n)-(T_k(u))_\mu) < \eta\}} M_n\ dx\ dt \\ &= \int_{\{0 \le T_k(u_n)-(T_k(u))_\mu) < \eta\}} a\Big(x,t,T_k(u_n),\nabla T_k(u_n)\Big)\times \\ &\qquad \times\Big(\nabla T_k(u_n) - \nabla T_k(u)\Big)\exp(G(u_n))\ dx\ dt \qquad (3.3.36) \\ &- \int_{\{0 \le T_k(u_n)-(T_k(u))_\mu) < \eta\}} a\Big(x,t,T_k(u_n),\nabla T_k(u)\Big)\times \\ &\qquad \times\Big(\nabla T_k(u_n) - \nabla T_k(u)\Big)\exp(G(u_n))\ dx\ dt \\ &= I_n^1 + I_n^2. \end{aligned}$$

Using (3.3.34), we have

$$I_n^1 \leq C\ \eta + w(n,\mu,\eta,m). \tag{3.3.37}$$

Concerning I_n^2 that is, the second term on the right hand side of the (3.3.36), it is easy to see that

$$I_n^2 = w(n,\mu). \tag{3.3.38}$$

Because $a_i\Big(x,t,T_k(u_n),\nabla T_k(u)\Big) \to a_i\Big(x,t,T_k(u),\nabla T_k(u)\Big)$ strongly in $L^{p'(x)}(Q)$, for all $i = 1,...,N$ and $\frac{\partial T_k(u_n)}{\partial x_i} \rightharpoonup \frac{\partial T_k(u)}{\partial x_i}$ in $L^{p(x)}(Q)$. Combining (3.3.35)-(3.3.38), we yields

$$I_n \leq C_1\ \Big(C\ \eta + w(n,\mu,\eta,m)\Big)^{\theta} + C_2\ \Big(w(n,\mu)\Big)^{1-\theta}$$

and by passing to the limit sup over n,μ and η

$$\int_{\{u_n-(T_k(u))_\mu \geq 0\}} \Big(\Big[a(x,t,T_k(u_n),\nabla T_k(u_n)) - a(x,t,T_k(u_n),\nabla T_k(u))\Big] \times \\ \times\Big[\nabla T_k(u_n) - \nabla T_k(u)\Big]\Big)^{\theta}\,dx\,dt = w(n). \tag{3.3.39}$$

On the other hand, if we choose $v = T_\eta\Big(u_n - (T_k(u))_\mu\Big)^{-}\exp(-G(u_n))$ in $(P_3)_n$, we obtain :

$$\int_{\{u_n-T_k(u)_\mu \leq 0\}} \Big(\Big[a(x,t,T_k(u_n),\nabla T_k(u_n)) - a(x,t,T_k(u_n),\nabla T_k(u))\Big] \times \\ \times\Big[\nabla T_k(u_n) - \nabla T_k(u)\Big]\Big)^{\theta}\,dx\,dt = w(n). \tag{3.3.40}$$

Moreover, (3.3.39) and (3.3.40) imply that

$$\int_Q \Big(\Big[a(x,t,T_k(u_n),\nabla T_k(u_n)) - a(x,t,T_k(u_n),\nabla T_k(u))\Big] \times \\ \times\Big[\nabla T_k(u_n) - \nabla T_k(u)\Big]\Big)^{\theta}\,dx\,dt = w(n) \tag{3.3.41}$$

which implies that

$$T_k(u_n) \to T_k(u) \quad in \quad L^{p^-}(0,T;W_0^{1,p(x)}(\Omega))\ \forall\ k \geq 0. \tag{3.3.42}$$

By [Theorem 3.3](see also [34, 35]), there exist a subsequence also denoted by u_n such that

$$\nabla u_n \to \nabla u \ \text{ a.e. in }\ Q, \tag{3.3.43}$$

which implies that

$$a\Big(x,t,T_k(u_n),\nabla T_k(u_n)\Big) \rightharpoonup a\Big(x,t,T_k(u),\nabla T_k(u)\Big) \quad in \quad (L^{p'(x)}(Q))^N. \tag{3.3.44}$$

Step 4 : Equi-integrability of the non linearity sequence.
Since $H_n(x,t,u_n,\nabla u_n) \to H(x,t,u,\nabla u)$ a.e. in Q, Using Vitali's theorem we shall now prove that $H_n(x,t,u_n,\nabla u_n)$ converge to $H(x,t,u,\nabla u)$ strongly in $L^1(Q)$. Choosing $\varphi = \rho_h(u_n) = \displaystyle\int_0^{u_n} g(s)\chi_{\{s>h\}}ds \exp(G(u_n))$ as a test function in the approximate problem $(P_3)_n$ by (3.3.3) and (3.2.3), we obtain

$$\Big[\int_\Omega \theta_h(u_n)dx\Big]_0^T + \int_Q a(x,t,u_n,\nabla u_n)\nabla u_n g(u_n)\chi_{\{u_n>h\}}\exp(G(u_n))dxdt$$
$$+ \int_Q nT_n(u_n-\psi)^- \exp(G(u_n))\phi_{\frac{1}{n}}(u_n)\int_0^{u_n} g(s)\chi_{\{s>h\}}dsdxdt$$
$$\leq \Big(\int_h^\infty g(s)\chi_{\{s>h\}}ds\Big)\exp\Big(\frac{\|g\|_{L^1(\mathbb{R})}}{\alpha}\Big)\Big[\|f_n\|_{L^1(Q)} + \|\gamma\|_{L^1(Q)}\Big]$$

where $\theta_h(r) = \displaystyle\int_0^r \rho_h(\tau)d\tau$, which implies, since $\theta_h \geq 0$, that

$$\int_Q a(x,t,u_n,\nabla u_n)\nabla u_n g(u_n)\chi_{\{u_n>h\}}dxdt$$
$$\leq \Big(\int_h^\infty g(s)ds\Big)\exp\Big(\frac{\|g\|_{L^1(\mathbb{R})}}{\alpha}\Big)\Big[\|f_n\|_{L^1(Q)} + \|\gamma\|_{L^1(Q)} + \int_\Omega \theta_h(u_{0n})dx + C\Big].$$

Using (3.2.3) we have

$$\alpha\int_{\{u_n>h\}} |\nabla u_n|^{p(x)} g(u_n)dxdt$$
$$\leq \Big(\int_h^\infty g(s)ds\Big)\exp\Big(\frac{\|g\|_{L^1(\mathbb{R})}}{\alpha}\Big)\Big[\|f_n\|_{L^1(Q)} + \|\gamma\|_{L^1(Q)} + \int_\Omega \theta_h(u_{0n})dx + C\Big],$$

and then

$$\int_{\{u_n>h\}} |\nabla u_n|^{p(x)} g(u_n)dxdt \leq C_1\Big(\int_h^\infty g(s)ds\Big).$$

Since $g \in L^1(\mathbb{R})$, we deduce that

$$\lim_{h\to\infty}\sup_{n\in\mathbb{N}}\int_{\{u_n>h\}} |\nabla u_n|^{p(x)} g(u_n)dxdt = 0.$$

Similarly, taking $\varphi = \rho_h(u_n) = \displaystyle\int_{u_n}^0 g(s)\chi_{\{s<-h\}}\exp(-G(u_n))ds$ as a test function in $(P_3)_n$, we conclude that

$$\lim_{h\to\infty}\sup_{n\in\mathbb{N}}\int_{\{u_n<-h\}} |\nabla u_n|^{p(x)} g(u_n)dxdt = 0.$$

Consequently,

$$\lim_{h\to\infty}\sup_{n\in\mathbb{N}}\int_{\{|u_n|>h\}}|\nabla u_n|^{p(x)}g(u_n)dxdt=0,$$

which, for h large enough and for a subset E of Q, implies that

$$\begin{aligned}\lim_{mesE\to 0}\int_E|\nabla u_n|^{p(x)}g(u_n)dxdt \;&\leq\; \max_{|u_n|\leq h}(g(s))\lim_{mesE\to 0}\int_E|\nabla T_h(u_n)|^{p(x)}dxdt\\ &\quad+\int_{\{|u_n|>h\}}|\nabla u_n|^{p(x)}g(u_n)dxdt.\end{aligned}$$

So we conclude that $g(u_n)|\nabla u_n|^{p(x)}$ is equi-integrable, which implies that

$$g(u_n)|\nabla u_n|^{p(x)}\to g(u)|\nabla u|^{p(x)}\ \text{ in } L^1(Q).$$

Consequently, by using (3.2.4) we conclude that :

$$H_n(x,t,u_n,\nabla u_n)\to H(x,t,u,\nabla u)\ in\ L^1(Q). \qquad (3.3.45)$$

STEP 5 : **Passing to the limit** . Let us consider the following three substeps
a) We show that u **satisfies** (3.2.5).

Proposition 3.3.2 *Let u_n be a solution of the approximate problem $(P_3)_n$. Then*

$$u\geq\psi\quad a.e.\ in\ Q.$$

Proof. Tanks to (3.3.7) and (3.3.13), we get $\int_Q nT_n(u_n-\psi)^-\exp(G(u_n))\,dx\,dt\leq C$. So, by Fatou's Lemma, we infer that $\int_Q(u-\psi)^-dxdt=0$ which implies that $(u-\psi)^-=0\quad a.e.\ in\ Q$. Consequently we conclude that $u\geq\psi\quad a.e.\ in\ Q$. □

b) We claim that $u\in C(0,T;L^1(\Omega))$ .We will show that

$$u_n\to u\quad in\quad C(0,T;L^1(\Omega)).$$

Since $T_k(u)\in K_\psi$, for every $k\geq\|\psi\|_{L^\infty}$ there exists a sequence $v_j\in K_\psi\cap D(\bar{Q})$ such that

$$v_j\to T_k(u)\quad in\quad L^{p^-}(0,T;W_0^{1,p(x)}(\Omega))$$

for the modular convergence.
Let $\omega^{i,l}_{j,\mu}=(T_l(v_j))_\mu+e^{-\mu t}T_l(\eta_i)$ with $\eta_i\geq 0$ converge to u_0 in $L^1(\Omega)$, where

$(T_l(v_j))_\mu$ is the mollification of $T_l(v_j)$ with respect to time. Note that $\omega^{i,l}_{j,\mu}$ is a smooth function having the following properties :

$$\frac{\partial \omega^{i,l}_{j,\mu}}{\partial t} = \mu(T_l(v_j) - \omega^{i,l}_{j,\mu}), \quad \omega^{i,l}_{j,\mu}(0) = T_l(\eta_i), \quad |\omega^{i,l}_{j,\mu}| \leq l, \tag{3.3.46}$$

$$\omega^{i,l}_{j,\mu} \to T_l(v_j) \quad \text{in} \ \ L^{p^-}(0,T;W_0^{1,p(.)}(\Omega)) \quad \text{as} \ \mu \to \infty. \tag{3.3.47}$$

Choosing now $v = T_k(u_n - \omega^{i,l}_{j,\mu})\chi_{(0,\tau)}$ as test function in $(P_3)_n$, yields

$$\begin{aligned}\Big\langle \frac{\partial u_n}{\partial t}, T_k(u_n - \omega^{i,l}_{j,\mu})\Big\rangle_{Q^\tau} + \int_{Q^\tau} a(x,t,u_n,\nabla u_n)\nabla T_k(u_n - \omega^{i,l}_{j,\mu})dxdt \\ + \int_{Q^\tau} nT_n(u_n-\psi)^-\phi_{\frac{1}{n}}(u_n)T_k(u_n - \omega^{i,l}_{j,\mu})dxdt \\ + \int_{Q^\tau} H_n(x,t,u_n,\nabla u_n)T_k(u_n - \omega^{i,l}_{j,\mu})dxdt = \int_{Q^\tau} f_nT_k(u_n - \omega^{i,l}_{j,\mu})dxdt.\end{aligned} \tag{3.3.48}$$

We have (see([2]))

$$\Big\langle \frac{\partial \omega^{i,l}_{j,\mu}}{\partial t}, T_k(u_n - \omega^{i,l}_{j,\mu})\Big\rangle_{Q^\tau} = \mu\int_{Q^\tau} (T_k(v_j) - \omega^{i,l}_{j,\mu}))T_k(u_n - \omega^{i,l}_{j,\mu}) \geq \epsilon(n,j,\mu,l). \tag{3.3.49}$$

And by using (3.2.4) and the fact that

$$\int_{Q^\tau} nT_n(u_n-\psi)^-\phi_{\frac{1}{n}}(u_n)T_k(u_n - \omega^{i,l}_{j,\mu})dxdt \geq 0,$$

we deduce that

$$\begin{aligned}\Big\langle \frac{\partial u_n}{\partial t}, T_k(u_n - \omega^{i,l}_{j,\mu})\Big\rangle_{Q^\tau} + \int_{Q^\tau} a(x,t,u_n,\nabla u_n)\nabla T_k(u_n - \omega^{i,l}_{j,\mu})dxdt \\ \leq \int_{Q^\tau} [f_n+\gamma]T_k(u_n - \omega^{i,l}_{j,\mu})dxdt + \int_{Q^\tau} g(u_n)|\nabla u_n|^{p(x)}T_k(u_n - \omega^{i,l}_{j,\mu})dxdt.\end{aligned} \tag{3.3.50}$$

On the one hand, we have

$$\begin{aligned} I &= \int_{Q^\tau} a(x,t,u_n,\nabla u_n)\nabla T_k(u_n - \omega^{i,l}_{j,\mu})dxdt \\ &= \int_{\{|T_k(u_n)-\omega^{i,l}_{j,\mu}|\leq k\}} a(x,t,u_n,\nabla u_n)[\nabla T_k(u_n) - \nabla\omega^{i,l}_{j,\mu}]dxdt. \end{aligned} \tag{3.3.51}$$

In the following, we pass to the limit in (3.3.51) : By letting n and μ to infinity, since $a(x,t,u_n,\nabla u_n) \rightharpoonup a(x,t,u,\nabla u) \quad in \ (L^{p'(x)}(Q))^N$, in view of Lebesgue's theorem, we have

$$I = \int_{\{|T_k(u)-T_l(v_j)|\le k\}} a(x,t,u,\nabla u)[\nabla T_k(u) - \nabla T_l(v_j)]dxdt + \epsilon(n,\mu).$$

Consequently, by taking the limit as $j \to \infty$, we deduce that

$$I = \epsilon(n,\mu,j,l).$$

On the other hand, we have

$$J = \int_{Q^\tau} g(u_n)|\nabla u_n|^{p(x)} T_k(u_n - \omega^{i,l}_{j,\mu})dxdt. \tag{3.3.52}$$

In the following, we pass to the limit in (3.3.52) : Taking the limit as $n \to \infty$ in (3.3.52), since $g(u_n)|\nabla u_n|^{p(x)} \to g(u)|\nabla u|^{p(x)}$ in $L^1(Q)$, in view of Lebesque's theorem, we obtain

$$J = \int_{Q^\tau} g(u)|\nabla u|^{p(x)} T_k(u - \omega^{i,l}_{j,\mu})dxdt + \epsilon(n).$$

Consequently, by letting μ and j to infinity, we have

$$J = \epsilon(n,\mu,j,l).$$

Similarly to (3.3.52) and by using (3.3.1), we have

$$\int_{Q^\tau} [f_n + \gamma] T_k(u_n - \omega^{i,l}_{j,\mu})dxdt = \epsilon(n,\mu,j,l)$$

and by using Vitali's theorem, we get

$$\limsup_{k\to\infty} \limsup_{i\to 0} \limsup_{j\to\infty} \limsup_{\mu\to\infty} \lim_{n\to\infty} \Big\langle \frac{\partial u_n}{\partial t}, T_k(u_n - \omega^{i,l}_{j,\mu}) \Big\rangle_{Q^\tau} \le 0 \tag{3.3.53}$$

uniformly on τ. Therefore, by writing

$$\begin{aligned}\int_\Omega S_k\Big(u_n(\tau) - \omega^{i,l}_{j,\mu}(\tau)\Big)dx &= \Big\langle \frac{\partial u_n}{\partial t}, T_k(u_n - \omega^{i,l}_{j,\mu}) \Big\rangle_{Q^\tau} - \Big\langle \frac{\partial \omega^{i,l}_{j,\mu}}{\partial t}, T_k(u_n - \omega^{i,l}_{j,\mu}) \Big\rangle_{Q^\tau} \\ &\quad + \int_\Omega S_k\Big(u_n(0) - T_l(\eta_i)\Big)dx \end{aligned} \tag{3.3.54}$$

and using (3.3.49) and (3.3.53) and (3.3.54), we see that

$$\int_\Omega S_k\Big(u_n(\tau) - \omega^{i,l}_{j,\mu}(\tau)\Big)dx \le \epsilon(n,j,\mu,l), \tag{3.3.55}$$

which implies, by writing

$$\int_\Omega S_k\Big(\frac{u_n(\tau)-u_m(\tau)}{2}\Big)dx \le \frac{1}{2}\Big(\int_\Omega S_k(u_n(\tau)-\omega^{i,l}_{j,\mu}(\tau)\Big)dx + \int_\Omega S_k\Big(u_m(\tau)-\omega^{i,l}_{j,\mu}(\tau))dx\Big) \tag{3.3.56}$$

that $\int_\Omega S_k\Big(\frac{u_n(\tau)-u_m(\tau)}{2}\Big)dx \le \epsilon_1(n,m)$. We deduce then that

$$\int_\Omega |u_n(\tau)-u_m(\tau)|dx \le \epsilon_2(n,m), \text{ not depending on } \tau \tag{3.3.57}$$

and thus (u_n) is a Cauchy sequence in $C(0,T;L^1(\Omega))$ and since $u_n \to u$, a.e. in Q, we deduce that

$$u_n \to u \quad in\ C(0,T;L^1(\Omega)). \tag{3.3.58}$$

c) We show prove that u **satisfies** (3.2.7). Let $v \in K_\psi \cap L^\infty(Q)$, $\frac{\partial v}{\partial t} \in L^{p'^-}(0,T;W^{-1,p(x)}(\Omega))$. By Pointwise multiplication of $(P_3)_n$ by $T_k(u_n - v)$, we get

$$\begin{aligned}\int_\Omega S_k\Big(u_n(T)-v(T)\Big)dx - \int_\Omega S_k\Big(u_{0n}-v(0)\Big)dx \\ + \int_Q \frac{\partial v}{\partial t}T_k(u_n-v)dxdt + \int_Q a(x,t,u_n,\nabla u_n)\nabla T_k(u_n-v)dxdt \\ + \int_Q H_n(x,t,u_n,\nabla u_n)T_k(u_n-v)dxdt \\ + \int_Q nT_n(u_n-\psi)^-\phi_{\frac{1}{n}}T_k(u_n-v)dxdt \\ = \int_Q fT_k(u_n-v)dxdt,\end{aligned}$$

where $S_k(s) = \int_0^s T_k(r)dr$. Since $\int_Q nT_n(u_n - \psi)^- \phi_{\frac{1}{n}} T_k(u_n - v)dxdt \geq 0$, we deduce then that

$$\begin{aligned}
&\int_\Omega S_k\Big(u_n(T) - v(T)\Big)dx - \int_\Omega S_k\Big(u_{0n} - v(0)\Big)dx \\
&\qquad + \int_Q \frac{\partial v}{\partial t} T_k(u_n - v)dxdt + \int_Q a(x,t,u_n,\nabla u_n)\nabla T_k(u_n - v)dxdt \\
&\qquad\qquad + \int_Q H_n(x,t,u_n,\nabla u_n)T_k(u_n - v)dxdt \\
&\qquad\qquad\qquad \leq \int_Q fT_k(u_n - v)dxdt.
\end{aligned} \tag{3.3.59}$$

Let us pass to the limit with $n \to \infty$ in each terms in (3.3.59).
We saw that $u_n \to u \; in \; C(0,T;L^1(\Omega))$. Therefore, $u_n(t) \to u(t) \; in \; L^1(\Omega)$ $\forall t \leq T$.
As S_k is lipschitz continuous with constant k , when $n \to \infty$ we have

$$\int_\Omega S_k(u_n - v)(T)dx \to \int_\Omega S_k(u - v)(T)dx$$
$$and \int_\Omega S_k(u_n - v)(0)dx = \int_\Omega S_k(u_{0n} - v(0))dx \to \int_\Omega S_k(u_0 - v(0))dx.$$

Let $v \in K_\psi \cap L^\infty(Q)$ then $\frac{\partial v}{\partial t} \in L^{p'^-}(0,T;W^{-1,p'(x)}(\Omega))$ and by Lebesque's theorem, we have

$$\int_0^T \Big\langle \frac{\partial v}{\partial t}, T_k(u_n - v)\Big\rangle dt \to \int_0^T \Big\langle \frac{\partial v}{\partial t}, T_k(u - v)\Big\rangle dt.$$

On the other hand we note $M = \|v\|_\infty$. Then we get

$$\begin{aligned}
&\int_Q a(x,t,u_n,\nabla u_n)\nabla T_k(u_n - v)dxdt \\
&= \int_0^T \int_\Omega a(x,t,T_{k+M}(u_n),\nabla T_{k+M}(u_n))\nabla T_k(T_{k+M}(u_n) - v)dxdt \\
&= \int_0^T \int_\Omega a(x,t,T_{k+M}(u_n),\nabla T_{k+M}(u_n))\nabla T_{k+M}(u_n))\mathbf{1}_{\{|T_{k+M}(u_n)-v|\leq k\}}dxdt \\
&\quad - \int_0^T \int_\Omega a(x,t,T_{k+M}(u_n),\nabla T_{k+M}(u_n)\nabla v\mathbf{1}_{\{|T_{k+M}(u_n)-v|\leq k\}}dxdt.
\end{aligned}$$

As $T_{k+M}(u_n)$ is bounded in $L^{p^-}(0,T;W_0^{1,p(x)}(\Omega))$, $\nabla u_n \to \nabla u$ a.e. in Q and by Lebesgue's theorem, we deduce that

$$\begin{aligned}&\int_0^T\int_\Omega a(x,t,T_{k+M}(u_n),\nabla T_{k+M}(u_n)\nabla v\mathbf{1}_{\{|T_{k+M}(u_n)-v|\le k\}}dxdt\\ &\to \int_0^T\int_\Omega a(x,t,T_{k+M}(u),\nabla T_{k+M}(u)\nabla v\mathbf{1}_{\{|T_{k+M}(u-v)|\le k\}}dxdt.\end{aligned}$$

Then

$$a(x,t,T_{k+M}(u_n),\nabla T_{k+M}(u_n)) \to a(x,t,T_{k+M}(u),\nabla T_{k+M}(u)) \ a.e. \ in \ Q$$

Since $H_n(x,t,u_n,\nabla u_n) \to H(x,t,u,\nabla u) \ in \quad L^1(\Omega)$ as $|T_k(u_n-v)| \le k$ and $T_k(u_n-v) \rightharpoonup T_k(u-v)$ weakly in $L^\infty(Q)$, by Lebesgue's theorem, we have

$$\int_0^T\int_\Omega H_n(x,t,u_n,\nabla u_n)T_k(u_n-v)dxdt \to \int_0^T\int_\Omega H(x,t,u,\nabla u)T_k(u-v)dxdt.$$

Due to (3.3.1) and the fact that $u_n \to u$ a.e. in Q, we have

$$\int_0^T\int_\Omega f_nT_k(u_n-v)dxdt \to \int_0^T\int_\Omega fT_k(u-v)dxdt.$$

Due to (3.3.59), we have

$$\begin{aligned}\int_\Omega S_k\Big(u(T)-v(T)\Big)dx \ &- \ \int_\Omega S_k\Big(u_0-v(0)\Big)dx\\ &+\int_Q \frac{\partial v}{\partial t}T_k(u-v)dxdt + \int_Q H(x,t,u,\nabla u)T_k(u-v)dxdt\\ &+\int_Q a(x,t,u,\nabla u)\nabla T_k(u-v)dxdt\\ &\le \int_Q fT_k(u-v)dxdt.\end{aligned}$$

As a conclusion of Step 1-5, the proof of theorem (3.3.1) is complete. □

3.4 Example.

Let us consider the following special case. Let Ω be a bounded open subset of $\mathbb{R}^N, N \ge 2,$ $p(x) = \sin|x| + 3$, $p(.) \in C_+(\bar{\Omega})$ and

$$H(x,t,s,\xi) = \frac{-4s}{1+s^4}|\xi|^{p(x)}$$

The Carathéodory function $H(x,t,s,\xi)$ satisfies the condition (3.2.4). Indeed

$$|H(x,t,s,\xi)| \leq \frac{4|s|}{1+s^4}|\xi|^{p(x)} = g(s)|\xi|^{p(x)}$$

where $g(s) = \frac{4|s|}{1+s^4}$ is a continuous and positive function which belongs to $L^1(\mathbb{R})$. Note that $H(x,t,s,\xi)$ does not satisfy the sign condition and the coercivity condition. Set

$$Au = -\triangle_{p(x)} = -\mathrm{div}(|\nabla u|^{p(x)-2}\nabla u). \tag{3.4.1}$$

We have $(|\nabla u|^{p(x)-2}\nabla u - |\nabla v|^{p(x)-2}\nabla v)(u-v) > 0$ for almost all $x \in \Omega,\ u, v \in \mathbb{R}^N$ and $u \neq v$, and so the monotonicity condition is satisfied.
The operator $-\mathrm{div}(|\nabla u|^{p(x)-2}\nabla u)$ is a Carathéodory function satisfying the growth condition (3.2.1) and the coercivity (3.2.3).
Define the obstacle function

$$\psi(x,t) = [t\chi_{(0,\tau)}(t) + c(1-\chi_{(0,\tau)}(t))]\omega(x),$$

where $\tau \in (0,T)$ is fixed, c is a real constant, and $w \in W_0^{1,p(x)}(\Omega) \cap L^\infty(\Omega)$.

Remark 3.4.1 *Finally, the hypotheses of Theorem (3.3.1) are satisfied. Therefore, the following problem :*

$$\begin{cases} u \geq \psi \quad a.e.\ in\ \Omega \times (0,T); \\ T_k(u) \in L^{p^-}(0,T;W_0^{1,p(x)}(\Omega)),\ for\ all\ k \geq 0\ and\ u \in C(0,T;L^1(\Omega)), \\ \displaystyle\int_\Omega S_k(u-v)(T)dx - \int_\Omega S_k(u-v)(0)dx \\ \displaystyle + \int_Q \frac{\partial v}{\partial t} T_k(u-v)dxdt + \int_Q |\nabla u|^{p(x)-2}\nabla u \nabla T_k(u-v)dxdt \\ \displaystyle - \int_Q \frac{4u}{1+u^4}|\nabla u|^{p(x)} T_k(u-v)dxdt \leq \int_Q f T_k(u-v)dxdt \\ \displaystyle \forall v \in K_\psi \cap\ L^\infty(Q), \quad where\ S_k(s) = \int_0^s T_k(r)dr, \\ \ and\ \frac{\partial v}{\partial t} \in L^{p'^-}(0,T;W^{-1,p'(x)}(\Omega)). \end{cases}$$

has at least one unilateral an entropy solution.

Chapitre 4

Obstacle problem for a non linear $p(x)$ parabolic inequalities

Abstact

This chapter deals, with the existence of solutions of some unilateral problems in the Sobolev space with variable exponent associated to the equation :

$$\frac{\partial u}{\partial t} + Au + \operatorname{div}(\phi(u)) = g(u)|\nabla u|^{p(x)} + f - \operatorname{div} F \quad \text{in} \quad \Omega \times (0,T)$$

where the operator $Au = -\operatorname{div}(|\nabla u|^{p(x)-2}\nabla u)$, is a Leray–Lions operator which is coercive. The right hand side f belong to $L^1(\Omega \times (0,T))$.

4.1 Introduction

Let Ω be a open bounded subset of $\mathbb{R}^N$, $N \geq 2$, T is a positive real number and $Q = \Omega \times (0,T)$, while the variable exponent $p : \bar{\Omega} \to (1, \infty)$ is continuous function, the data $f \in L^1(Q)$, $u_0 \in L^1(\Omega)$ and $F \in (L^{P'(x)}(Q))^N$. The objective of this chapter is to study the existence of an entropy solution for the obstacle parabolic problems of the type :

$$\begin{cases} u \geq \psi & \text{a.e. in } \Omega \times (0,T), \\ \frac{\partial u}{\partial t} - \operatorname{div}(|\nabla u|^{p(x)-2}\nabla u) + \operatorname{div}(\phi(u)) = g(u)|\nabla u|^{p(x)} & \\ \qquad + f - \operatorname{div} F & \text{in } \Omega \times (0,T), \\ u(x,0) = u_0 & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega \times (0,T). \end{cases} \tag{4.1.1}$$

The function ϕ is just assumed to be continuous of $\mathbb{R}$ with values in $\mathbb{R}^{\mathbb{N}}$ and g is a bounded and continuous positive function that belongs to $L^1(\mathbb{R})$. More precisely, this paper is devoted to study the existence of a solution to the obstacle parabolic problem associated to (4.1.1) in the sense of an entropy solution (see definition (4.2.1)).

The natural framework to solve problem (4.1.1) is that of Sobolev spaces with variable exponent. Recent applications in elasticity [87], non-Newtonian fluid mechanics [23, 80] or image processing [48], gave rise to a revival of the interest in thes spaces, the origins of which can be traced back to the work of Orlicz in the 1930's. The interest of the study of Lesbesgue and Sobolev spaces with variable exponent lies on the fact that most materials can be modelled with sufficient accuracy using classical Lebesque and Sobolev spaces L^p and $W^{1,p}$ where p is a fixed constant, but for some materials with inhomogeneities, for instance electroheological fluids [80]. It would be interesting at this work to refer to Dall' Algio - Orsina [50] and Porretta [70] proved the existence of solutions for the problem (4.1.1) without obstacle with div$\phi(u) = 0$, F=0 and $-g(u)|\nabla u|^{p(x)} = g(x,t,u,\nabla u)$ satisfying the following natural growth condition (of order p) :

$$g(x,t,s,\xi) \leq b(s)(|\xi|^p + c(x,t))$$

and the classical sign condition

$$g(x,t,s,\xi) \cdot s \geq 0$$

Recently, M. Sanchón and Urbano [81] studied a Dirichlet problem (4.1.1) of $p(x)$ - Laplace equation and obtained the existence and uniqueness of entropy solutions for L^1 data where $g(u)|\nabla u|^{p(x)} = 0$, $F = 0$ and $\mathrm{div}(\phi(u)) = 0$ and the existence and uniquenss of a renormalized solution has been proved by D. Blanchard, F. Murat and Redwane in [37] with $u = b(x,u)$, $F = 0$ and $g(u)|\nabla u|^{p(x)} = 0$. For the degenerated parabolic equations the existence of weak solutions have been proved by L. Aharouch and al [3] where $F = 0$, $g(u)|\nabla u|^{p(x)} = 0$ and $f \in L^{p'}(0,T;W^{-1,p'}(\Omega,\omega))^*)$. Existence and uniqueness of renormalized solutions was established by Blanchard and Murat in [34], besides, when $\phi = 0$, $F = 0$ and $g(u)|\nabla u|^{p(x)} = 0$ M. Bendahmane, P. Wittbold and A. Zimmermann in [29] proved the existence of renormalized solutions and the existence of renormalized solution in Orlicz spaces has been proved in E. Azroul, H. Redwane and M. Rhoudaf [25] in the case where $b(u) = u$. We recall that the notion of renormalized solutions was

introduced in [51] by Diperna and Lions in their study of the Boltzmann equation. This notion was adapted to the study of some nonlinear elliptic problems with Dirichlet boundary conditions by Boccardo, Diaz. Giachelte and Murat in [45] and Lions and Murat (see Lion([65]). In the case $F = 0$ and $\phi = 0$ the existence of solutions of some unilateral problems in the framework of Orlicz spaces has been established by M. Kbiri Alaoui , D. Meskine, A. Souissi in [59] with the penalization methods.
The plan of the chapter is as follows. Section 2 we make precise all the assumptions on $A, \phi, g, f and\ u_0$, the definition of an entropy solution of (4.1.1) . In Section 3 we establish the existence of such a solution in Theorem (4.3.1).

4.2 Assumptions and definition

Throughout this chapter, we assume that the following assumptions hold true.

4.2.1 Basic assumptions

Let Ω be an open bounded subset of $\mathbb{R}^{\mathbb{N}}, N \geq 2$, $T > 0$ is positive real number and let set $Q = \Omega \times (0, T)$ and let $p \in C_+(\bar{\Omega})$ and assume that $p(x)$ satisfies the log-Hölder condition (1.1.1) with $1 < p^- \leq p^+ < \infty$. The differential operator $A : \Omega \times [0, T] \times \mathbb{R} \times \mathbb{R}^N \to \mathbb{R}$ is defined by

$$Au = -\mathrm{div}(|\nabla u|^{p(x)-2}\nabla u), \tag{4.2.1}$$

is a Leray–Lions operator which is coercive and

$$g \quad \mathbb{R} \to \mathbb{R}^+ \tag{4.2.2}$$

is a bounded and continuous positive function that belongs to $L^1(\mathbb{R})$

$$\phi \quad \mathbb{R} \to \mathbb{R}^N \text{ is a continuous function,} \tag{4.2.3}$$

$$f \ is\ an\ element\ of\ L^1(Q), u_0 \in L^1(\Omega) \quad and\ p \in C_+(\bar{\Omega}) \tag{4.2.4}$$

Let ψ be a measurable function with values in $\bar{\mathbb{R}}$ such that $\psi \in W_0^{1,p(x)}(Q) \cap L^\infty(Q)$ such that $u_0(x) \geq \psi(x)$ a.e. in Ω and let

$$K_\psi = \Big\{ u \in L^{p^-}(0, T; W_0^{1,p(x)}(\Omega)),\ u \geq \psi \ \ a.e.\ \ in\ \ Q \Big\}.$$

4.2.2 Definition of Entropy Solution.

Definition 4.2.1 *Let $f \in L^1(Q)$ and $u_0 \in L^1(\Omega)$. A measurable function u defined on Q is a unilateral entropy solution of problem* (4.1.1) *if*

$$u \geq \psi \quad a.e. \ in \ \ Q, \tag{4.2.5}$$

$$T_k(u) \in V, \ for \ all \ k \geq 0 \ and \ u \in C(0,T;L^1(\Omega)), \tag{4.2.6}$$

$$\begin{aligned} &\int_\Omega S_k(u-v)(T)dx + \int_Q \frac{\partial v}{\partial t} T_k(u-v)dxdt \\ &+ \int_Q |\nabla u|^{p(x)-2}\nabla u \nabla T_k(u-v)dxdt - \int_Q \phi(u)\nabla T_k(u-v)dxdt \\ \leq &\int_Q g(u)|\nabla u|^{p(x)} T_k(u-v)dxdt + \int_Q fT_k(u-v)dxdt \\ &+ \int_Q F\nabla T_k(u-v) + \int_\Omega S_k(u_0 - v(0))dx \end{aligned} \tag{4.2.7}$$

for all $v \in K_\psi \cap \ L^\infty(Q), \frac{\partial v}{\partial t} \in V^* + L^1(Q)$ and $\forall k > 0$,
where $S_k(s) = \displaystyle\int_0^s T_k(r)dr$.

4.3 The principal results

The aim of the present work is to prove the following

Theorem 4.3.1 *Under assumptions (4.2.1)- (4.2.4), there exists at least one unilateral entropy solution of problem* (4.1.1).

Remark 4.3.1 *Since, the function $\phi(u)$ does note belong to $(L^1_{Loc}(Q))^N$.Then, the problem* (4.1.1) *can have an entropy solution, but not a weak solution.*

Proof. The proof is divided into 4 steps.
In Step 1, we introduce an approximate problem. In Step 2, we establish a few a priori estimates which allow us to prove that the approximate solutions u_n converge to u a.e. in Q. In Step 3, we define a time regularisation of the field $T_k(u)$ and prove that u_n satisfies (4.3.10). In this step using some techniques, we also prove the modular convergence of $T_k(u_n)$ to $T_k(u)$ in $L^{p^-}(0,T;W_0^{1,p(x)}(\Omega))$. In Step 4, we pass to the limit which is the final step to prove Theorem (4.3.1).
Step 1 : The approximate problem. Let us introduce the following regulariza-

tion of the data :

$$f_n \in L^{p'(x)}(Q),\ f_n \to f \text{ a.e. in Q, and strongly } in\ L^1(Q)\ as\ n\ \to \infty, \quad (4.3.1)$$

$$u_{0n} \in D(\Omega) : \|u_{0n}\|_{L^1(\Omega)} \leq \|u_0\|_{L^1(\Omega)}\ and\ u_{0n} \to u_0 \text{ in } L^1(\Omega)\ as\ n\ \to \infty, \quad (4.3.2)$$

$$\phi_n \text{ is a lipschitz continuous bounded function from } \mathbb{R} \text{ in to } \mathbb{R}^N, \text{ such that}$$
$$\phi_n \text{ uniformly converges to } \phi \text{ on any compact subset of } \mathbb{R} \text{ as } n \to +\infty. \quad (4.3.3)$$

Let us now consider the following regularized approximate problem

$$(P_4)_n \begin{cases} \frac{\partial u_n}{\partial t} - \operatorname{div}(|\nabla u_n|^{p(x)-2}\nabla u_n) + \operatorname{div}\phi_n(u_n) - nT_n((u_n-\psi)^-) \\ \qquad = g(u_n)|\nabla u_n|^{p(x)} + f_n - \operatorname{div}F & \text{in } D'(Q), \\ \qquad u_n = 0 & \text{on } \partial\Omega \times (0,T) \\ u_n(t=0) = u_{0n} & \text{in } \Omega. \end{cases}$$

Moreover, since $f_n \in V^*$, proving the existence of weak solution $u_n \in V$ of $(P_4)_n$ is an easy task (see [65])

Step 2 : A Priori estimates. The estimate derived in this step rely on standard techniques for problems of the type $(P_4)_n$.

Proposition 4.3.1 *Assume that*(4.2.1)-(4.2.4) *hold true and let u_n be a solution of the approximate problem $(P_4)_n$. Then for all $k > 0$, we have*

$$\|T_k(u_n)\|_{L^{p^-}(0,T;W_0^{1,p(.)}(\Omega))} \leq C\ k \quad \textit{for all}\ n \in \mathbb{N},$$

where C is a constant independent of n.

Proof. Let $h > k > 0$ and consider the following test function $\varphi = T_h\Big(u_n - T_k(u_n)\Big)\exp(G(u_n))$ in the approximate problem $(P_4)_n$. where $G(s) =$

$\displaystyle\int_0^s g(r)dr$, we have

$$
\begin{aligned}
&\Big\langle\Big\langle \frac{\partial u_n}{\partial t}, T_h\Big(u_n - T_k(u_n)\Big)\exp(G(u_n)\Big\rangle\Big\rangle \\
&\quad + \int_{\{k\le |u_n|\le k+h\}} \Big(|\nabla u_n|^{p(x)-2}\nabla u_n\Big)\nabla u_n \exp(G(u_n))dxdt \\
&\qquad + \int_Q \Big(|\nabla u_n|^{p(x)-2}\nabla u_n\Big)\nabla u_n T_h\Big(u_n - T_k(u_n)\Big)g(u_n)\exp(G(u_n))dxdt \\
&\qquad\quad - \int_Q \phi_n(u_n)\nabla\Big(T_h\Big(u_n - T_k(u_n)\Big)\exp(G(u_n))\Big)dxdt \\
&\qquad\qquad - \int_Q nT_n((u_n-\psi)^-)T_h\Big(u_n - T_k(u_n)\Big)\exp(G(u_n))dxdt \\
&\qquad\qquad\quad = \int_Q g(u_n)|\nabla u_n|^{p(x)}T_h\Big(u_n - T_k(u_n)\Big)\exp(G(u_n))dxdt \\
&\qquad\qquad\qquad + \int_Q f_n T_h\Big(u_n - T_k(u_n)\Big)\exp(G(u_n))dxdt \\
&\qquad\qquad\qquad\quad + \int_Q F\nabla\Big(T_h\Big(u_n - T_k(u_n)\Big)\exp(G(u_n))\Big)dxdt
\end{aligned}
$$

The Lipschitz character of ϕ_n and Stockes' formula to gether with the boundary condition 2 of problem $(P_4)_n$, give $\displaystyle\int_Q \phi_n(u_n)\nabla\Big(T_h(u_n - T_k(u_n))\exp(G(u_n)\Big)dxdt = 0$, then we obtain

$$
\begin{aligned}
&\Big\langle\Big\langle \frac{\partial u_n}{\partial t}, T_h(u_n - T_k(u_n))\exp(G(u_n))\Big\rangle\Big\rangle \\
&\quad + \int_{\{k\le |u_n|\le k+h\}} |\nabla u_n|^{p(x)}\exp(G(u_n))dxdt \\
&\qquad - \int_Q nT_n((u_n-\psi)^-)T_h(u_n - T_k(u_n))\exp(G(u_n))dxdt \\
&\qquad\quad = \int_Q f_n T_h\Big(u_n - T_k(u_n)\Big)\exp(G(u_n))dxdt \\
&\qquad\qquad + \int_Q F\nabla\Big(T_h\Big(u_n - T_k(u_n)\Big)\exp(G(u_n)\Big)dxdt.
\end{aligned}
$$

On the one hand, we have

$$\begin{aligned}
&\int_{\{k\le|u_n|\le k+h\}}|\nabla u_n|^{p(x)}\exp(G(u_n))dxdt\\
&-\int_Q nT_n((u_n-\psi)^-)T_h(u_n-T_k(u_n))\exp(G(u_n))dxdt\\
&\le h\exp(\|g\|_{L^1(\mathbb{R})})\Big[\|f_n\|_{L^1(Q)}+\|u_{0n}\|_{L^1(\Omega)}\Big]\\
&+\int_Q F\Big(T_h\Big(u_n-T_k(u_n)\Big)\nabla u_n g(u_n)\exp(G(u_n))dxdt\\
&+\frac{1}{2}\int_{\{k\le|u_n|\le k+h\}}|\nabla u_n|^{p(x)}\exp(G(u_n))dxdt\\
&+\int_{\{k\le|u_n|\le k+h\}}\frac{|F|^{p'(x)}\exp(G(u_n))}{p'(x)(\frac{1}{2}p(x))^{\frac{p'(x)}{p(x)}}}dxdt
\end{aligned}$$

$$\begin{aligned}
&\le h\exp(\|g\|_{L^1(\mathbb{R})})\Big[\|f_n\|_{L^1(Q)}+\|u_{0n}\|_{L^1(\Omega)}\Big]\\
&+\int_Q F\Big(T_h\Big(u_n-T_k(u_n)\Big)\nabla u_n g(u_n)\exp(G(u_n))dxdt\\
&+\frac{1}{2}\int_{\{k\le|u_n|\le k+h\}}|\nabla u_n|^{p(x)}\exp(G(u_n))dxdt\\
&+C\int_{\{k\le|u_n|\le k+h\}}|F|^{p'(x)}\exp(G(u_n))dxdt,
\end{aligned}$$

then

$$\begin{aligned}
&\frac{1}{2}\int_{\{k\le|u_n|\le k+h\}}|\nabla u_n|^{p(x)}\exp(G(u_n))dxdt\\
&-\int_Q nT_n((u_n-\psi)^-)T_h\Big(u_n-T_k(u_n)\Big)\exp(G(u_n))dxdt\\
&\le h\exp(\|g\|_{L^1(\mathbb{R})})\Big[\|f_n\|_{L^1(Q)}+\|u_{0n}\|_{L^1(\Omega)}\Big]+C_1 \qquad (4.3.4)\\
&+\int_Q FT_h\Big(u_n-T_k(u_n)\Big)\nabla u_n g(u_n)\exp(G(u_n))dxdt.
\end{aligned}$$

Our objective in the following is to prove the proposition next :

Proposition 4.3.2 $\int_Q F\nabla u_n g(u_n)\exp(G(u_n))dxdt\le C+C'\int_Q nT_n((u_n-\psi)^-)$ *for all* $n>0$

Let us observe that if we take $\varphi = \rho(u_n) = \displaystyle\int_0^{u_n} g(s)\chi_{\{k\leq|s|\leq k+h\}}ds \exp(G(u_n))$ a test function in the approximate $(P_4)_n$, we obtain

$$
\begin{aligned}
\Big[\int_\Omega \varphi_1(u_n)dx\Big]_0^T &+ \int_{\{k\leq|u_n|\leq k+h\}} |\nabla u_n|^{p(x)} g(u_n)\exp(G(u_n))dxdt \\
&- \int_Q nT_n((u_n-\psi)^-)\rho(u_n)dxdt - \int_Q \phi_n(u_n)\nabla\rho(u_n)dxdt \\
&\leq \Big(\int_0^\infty g(s)ds\Big)\exp\Big(\|g\|_{L^1(\mathbb{R})}\Big)\|f_n\|_{L^1(Q)} \\
&\quad + \int_Q F\nabla u_n g(u_n)\chi_{\{k\leq|u_n|\leq k+h\}}\exp(G(u_n))dxdt \\
&\quad\quad + \Big(\int_0^\infty g(s)ds\Big)\int_Q |F\nabla u_n| g(u_n)\exp(G(u_n))\chi_{\{k\leq|u_n|\leq k+h\}}dxdt,
\end{aligned}
$$

where $\varphi_1(r) = \displaystyle\int_0^r \rho(s)ds$, which implies, using $\varphi_1(r)\geq 0$, and the Lipschitz character of ϕ_n, we obtain

$$
\begin{aligned}
\int_{\{k\leq|u_n|\leq k+h\}} &|\nabla u_n|^{p(x)} g(u_n)\exp(G(u_n))dxdt \\
&- \int_Q nT_n((u_n-\psi)^-)\rho(u_n)dxdt \\
&\leq \Big(\int_0^\infty g(s)ds\Big)\exp\Big(\|g\|_{L^1(\mathbb{R})}\Big)\Big[\|f_n\|_{L^1(Q)} + \|u_0\|_{L^1(\Omega)}\Big] \\
&+ \int_Q F\nabla u_n g(u_n)\chi_{\{k\leq|u_n|\leq k+h\}}\exp(G(u_n))dxdt \\
&+ \Big(\int_0^\infty g(s)ds\Big)\int_Q |F\nabla u_n| g(u_n)\exp(G(u_n))\chi_{\{k\leq|u_n|\leq k+h\}}dxdt,
\end{aligned}
$$

then

$$\int_{\{k\le|u_n|\le k+h\}}|\nabla u_n|^{p(x)}g(u_n)\exp(G(u_n))dxdt-\int_Q nT_n((u_n-\psi)^-)\rho(u_n)dxdt$$
$$\le \|g\|_\infty\exp\Big(\|g\|_{L^1(\mathbb{R})}\Big)\Big[\|f_n\|_{L^1(Q)}+\|u_0\|_{L^1(\Omega)}\Big]+\int_Q F\frac{\Big[g(u_n)\exp(G(u_n))\Big]^{\frac{1}{p'(x)}}}{\Big[\frac{1}{2[1+\|g\|_\infty]}p(x)\Big]^{\frac{1}{p(x)}}}$$
$$\times\Big[\frac{1}{2[1+\|g\|_\infty]}p(x)\Big]^{\frac{1}{p(x)}}\Big[g(u_n)\exp(G(u_n))\Big]^{\frac{1}{p(x)}}\nabla u_n\chi_{\{k\le|u_n|\le k+h\}}dxdt$$
$$+\|g\|_\infty\int_Q|F|\frac{\Big[g(u_n)\exp(G(u_n))\Big]^{\frac{1}{p'(x)}}}{\Big[\frac{1}{2[1+\|g\|_\infty]}p(x)\Big]^{\frac{1}{p(x)}}}\Big[\frac{1}{2[1+\|g\|_\infty]}p(x)\Big]^{\frac{1}{p(x)}}$$
$$\times\Big[g(u_n)\exp(G(u_n))\Big]^{\frac{1}{p(x)}}\nabla u_n\chi_{\{k\le|u_n|\le k+h\}}dxdt,$$

and by using Young's Inequality , we have

$$\int_{\{k\le|u_n|\le k+h\}}|\nabla u_n|^{p(x)}g(u_n)\exp(G(u_n))dxdt-\int_Q nT_n((u_n-\psi)^-)\rho(u_n)dxdt$$
$$\le \|g\|_\infty\exp\Big(\|g\|_{L^1(\mathbb{R})}\Big)\Big[\|f_n\|_{L^1(Q)}+\|u_0\|_{L^1(\Omega)}\Big]$$
$$+\int_Q\frac{|F|^{p'(x)}g(u_n)\exp(G(u_n))}{p'(x)\Big[\frac{1}{2[1+\|g\|_\infty]}p(x)\Big]^{\frac{p'(x)}{p(x)}}}dxdt$$
$$+\frac{1}{2[1+\|g\|_\infty]}\int_Q|\nabla u_n|^{p(x)}g(u_n)\exp(G(u_n))\chi_{\{k\le|u_n|\le k+h\}}dxdt$$
$$+\|g\|_\infty\int_Q\frac{|F|^{p'(x)}g(u_n)\exp(G(u_n))}{p'(x)\Big[\frac{1}{2[1+\|g\|_\infty]}p(x)\Big]^{\frac{p'(x)}{p(x)}}}dxdt$$
$$+\frac{1}{2[1+\|g\|_\infty]}\|g\|_\infty\int_Q|\nabla u_n|^{p(x)}g(u_n)\exp(G(u_n))\chi_{\{k\le|u_n|\le k+h\}}dxdt,$$

Since g is bounded function, then we have

$$\begin{aligned}
\int_{\{k\leq|u_n|\leq k+h\}} &|\nabla u_n|^{p(x)} g(u_n)\exp(G(u_n))dxdt - \int_Q nT_n((u_n-\psi)^-)\rho(u_n)dxdt \\
&\leq \|g\|_\infty \exp\Big(\|g\|_{L^1(\mathbb{R})}\Big)\Big[\|f_n\|_{L^1(Q)} + \|u_0\|_{L^1(\Omega)}\Big] \\
&\quad + C'\|g\|_\infty \exp\Big(\|g\|_{L^1(\mathbb{R})}\Big)\int_Q |F|^{p'(x)}dxdt \\
&\quad + \frac{1}{2[1+\|g\|_\infty]}\int_Q |\nabla u_n|^{p(x)} g(u_n)\exp(G(u_n))\chi_{\{k\leq|u_n|\leq k+h\}}dxdt \\
&\quad + \|g\|_\infty C'\|g\|_{L^1(\mathbb{R})}\exp\Big(\|g\|_{L^1(\mathbb{R})}\Big)\int_Q |F|^{p'(x)}dxdt \\
&\quad + \frac{1}{2[1+\|g\|_\infty]}\|g\|_\infty \int_Q |\nabla u_n|^{p(x)} g(u_n)\exp(G(u_n))\chi_{\{k\leq|u_n|\leq k+h\}}dxdt,
\end{aligned}$$

then

$$\begin{aligned}
\int_{\{k\leq|u_n|\leq k+h\}} &|\nabla u_n|^{p(x)} g(u_n)\exp(G(u_n))dxdt - \int_Q nT_n((u_n-\psi)^-)\rho(u_n)dxdt \\
&\leq \|g\|_\infty \exp\Big(\|g\|_{L^1(\mathbb{R})}\Big)\Big[\|f_n\|_{L^1(Q)} + \|u_0\|_{L^1(\Omega)}\Big] \\
&\quad + C'\|g\|_\infty \exp\Big(\|g\|_{L^1(\mathbb{R})}\Big)\Big[1+\|g\|_\infty\Big]\int_Q |F|^{p'(x)}dxdt \\
&\quad + \frac{1}{2[1+\|g\|_\infty]}\int_Q |\nabla u_n|^{p(x)} g(u_n)\exp(G(u_n))\chi_{\{k\leq|u_n|\leq k+h\}}dxdt \\
&\quad + \frac{1}{2[1+\|g\|_\infty]}\|g\|_\infty \int_Q |\nabla u_n|^{p(x)} g(u_n)\exp(G(u_n))\chi_{\{k\leq|u_n|\leq k+h\}}dxdt,
\end{aligned}$$

and since

$$\int_Q |F|^{p'(x)}dxdt \;=\; \rho(F) \leq \max\Big\{||F||^{p^-}_{(L^{P'(\cdot)}(Q))^N}, ||F||^{p^+}_{(L^{P'(\cdot)}(Q))^N}\Big\} = C''$$

then, we have

$$\int_{\{k\le|u_n|\le k+h\}}|\nabla u_n|^{p(x)}g(u_n)\exp(G(u_n))dxdt$$
$$-\int_Q nT_n(u_n-\psi)^-\rho(u_n)dxdt$$
$$\le \|g\|_\infty\Big(\exp(\|g\|_{L^1(\mathbb{R})})\Big)\Big[\|f_n\|_{L^1(Q)}+\|u_0\|_{L^1(\Omega)}\Big]$$
$$+C_2+\frac{1}{2[1+\|g\|_\infty]}\int_{\{k\le|u_n|\le k+h\}}|\nabla u_n|^{p(x)}g(u_n)\exp(G(u_n))dxdt$$
$$+\frac{1}{2[1+\|g\|_\infty]}\|g\|_\infty\int_{\{k\le|u_n|\le k+h\}}|\nabla u_n|^{p(x)}g(u_n)\exp(G(u_n))dxdt$$

where $C_2=C'\|g\|_\infty\exp\Big(\|g\|_{L^1(\mathbb{R})}\Big)\Big[1+\|g\|_\infty\Big]C''$
hence

$$2\int_{\{k\le|u_n|\le k+h\}}|\nabla u_n|^{p(x)}g(u_n)\exp(G(u_n))dxdt-2\int_Q nT_n(u_n-\psi)^-\rho(u_n)dxdt$$
$$\le 2\Big[\|g\|_\infty\exp(\|g\|_{L^1(\mathbb{R})})\Big(\|f_n\|_{L^1(Q)}+\|u_0\|_{L^1(\Omega)}\Big)+C_2\Big]$$
$$+\int_{\{k\le|u_n|\le k+h\}}|\nabla u_n|^{p(x)}g(u_n)\exp(G(u_n))dxdt$$

which give

$$\int_{\{k\le|u_n|\le k+h\}}|\nabla u_n|^{p(x)}g(u_n)\exp(G(u_n))dxdt-2\int_Q nT_n((u_n-\psi)^-)\rho(u_n)dxdt\le C_3,$$

then

$$\int_{\{k\le|u_n|\le k+h\}}|\nabla u_n|^{p(x)}g(u_n)\exp(G(u_n))dxdt\le C_3+2\int_Q nT_n((u_n-\psi)^-)\rho(u_n)dxdt$$
$$\le C_3+2(h+k)\|g\|_\infty(\exp(\|g\|_{L^1(\mathbb{R})})\int_Q nT_n(u_n-\psi)^-dxdt$$
$$\le C_3+hC_4\int_Q nT_n((u_n-\psi)^-)dxdt$$

with $C_4=4\|g\|_\infty(\exp(\|g\|_{L^1(\mathbb{R})})$ so, we obtain

$$\int_{\{k\le|u_n|\le k+h\}}|\nabla u_n|^{p(x)}g(u_n)\exp(G(u_n))dxdt\le hC_5\int_Q nT_n((u_n-\psi)^-)dxdt.$$

Let us take $\rho_1(u_n) = \int_0^{u_n} g(s)\chi_{\{|s|\leq k\}} ds \exp(G(u_n))$ a test function in the approximate $(P_4)_n$, we obtain

$$
\begin{aligned}
\Big[\int_\Omega \varphi_2(u_n)dx\Big]_0^T &+ \int_Q |\nabla u_n|^{p(x)} g(u_n)\exp(G(u_n))\chi_{\{|u_n|\leq k\}} dxdt \\
&- \int_Q nT_n((u_n-\psi)^-)\rho_1(u_n)dxdt - \int_Q \phi_n(u_n)\nabla\rho_1(u_n)dxdt \\
&\leq \Big(\int_0^\infty g(s)ds\Big)\exp\Big(\|g\|_{L^1(\mathbb{R})}\Big)\|f_n\|_{L^1(Q)} \\
&\quad + \int_Q F\nabla u_n g(u_n)\chi_{\{|u_n|\leq k\}}\exp(G(u_n))dxdt \\
&\quad\quad + \Big(\int_0^\infty g(s)ds\Big)\int_Q |F\nabla u_n| g(u_n)\exp(G(u_n))\chi_{\{|u_n|\leq k\}} dxdt,
\end{aligned}
$$

where $\varphi_2(r) = \int_0^r \rho_1(s)ds$, which implies, using $\varphi_2(r) \geq 0$, the Lipschitz character of ϕ_n and by using Young's Inequality, we obtain

$$
\begin{aligned}
&\int_{\{|u_n|\leq k\}} |\nabla u_n|^{p(x)} g(u_n)\exp(G(u_n))dxdt \\
&\leq \|g\|_\infty \exp\Big(\|g\|_{L^1(\mathbb{R})}\Big)\Big[\|f_n\|_{L^1(Q)} + \|u_0\|_{L^1(\Omega)}\Big] + C_6 \\
&\quad + \frac{1}{2[1+\|g\|_\infty]}\int_{\{|u_n|\leq k\}} |\nabla u_n|^{p(x)} g(u_n)\exp(G(u_n))dxdt \\
&\quad\quad + \frac{1}{2[1+\|g\|_\infty]}\|g\|_\infty \int_{\{|u_n|\leq k\}} |\nabla u_n|^{p(x)} g(u_n)\exp(G(u_n))dxdt,
\end{aligned}
$$

then

$$
\int_{\{|u_n|\leq k\}} |\nabla u_n|^{p(x)} g(u_n)\exp(G(u_n))dxdt \leq hC_7 \int_Q nT_n((u_n-\psi)^-)dxdt
$$

Similarly, taking $\rho_2 = \int_0^{u_n} g(s)\chi_{\{|s|\geq k+h\}} ds \exp(G(u_n))$ as a test function in the approximate $(P_4)_n$, we conclude that

$$
\int_{\{|u_n|\geq k+h\}} |\nabla u_n|^{p(x)} g(u_n)\exp(G(u_n))dxdt \leq hC_8 \int_Q nT_n((u_n-\psi)^-)dxdt
$$

Conclusion, we have

$$
\begin{cases}
\displaystyle\int_Q |\nabla u_n|^{p(x)} g(u_n)\exp(G(u_n))dxdt \le \int_{\{|u_n|\ge k+h\}} |\nabla u_n|^{p(x)} g(u_n)\exp(G(u_n))dxdt \\
\qquad \displaystyle + \int_{\{|u_n|\le k\}} |\nabla u_n|^{p(x)} g(u_n)\exp(G(u_n))dxdt \\
\qquad\quad \displaystyle + \int_{\{k\le |u_n|\le k+h\}} |\nabla u_n|^{p(x)} g(u_n)\exp(G(u_n))dxdt \\
\qquad\quad \displaystyle \le hC_9 \int_Q nT_n((u_n-\psi)^-)dxdt
\end{cases}
$$

using (4.3.4), we have

$$
\begin{aligned}
&\int_Q |\nabla u_n|^{p(x)} g(u_n)\exp(G(u_n))dxdt \\
&- \int_Q nT_n((u_n-\psi)^-)T_h\Big(u_n - T_k(u_n)\Big)\exp(G(u_n))dxdt \\
&\le hC_9 \int_Q nT_n(u_n-\psi)^- dxdt + hC_{10},
\end{aligned}
$$

we obtain

$$
\begin{aligned}
&- \int_Q nT_n(u_n-\psi)^- T_h\Big(u_n - T_k(u_n)\Big)\exp(G(u_n))dxdt \\
&\le hC_9 \int_Q nT_n(u_n-\psi)^- dxdt + hC_{10}
\end{aligned}
$$

so, that

$$
\begin{aligned}
&- \int_Q nT_n(u_n-\psi)^- \frac{T_h\Big(u_n - T_k(u_n)\Big)}{h}\exp(G(u_n))dxdt \\
&\le C_9 \int_Q nT_n(u_n-\psi)^- dxdt + C_{10}.
\end{aligned}
$$

Let us now fix $k > \|\psi\|_\infty$, by the fact that
$nT_n(u_n-\psi)(u_n-k)\chi_{\{u_n\le\psi;\, k\le u_n\le k+h\}} \ge 0$ and letting $h \to 0$, one has

$$
\int_Q nT_n(u_n-\psi)^- dxdt \le C_{11}. \tag{4.3.5}
$$

Let use $v = T_k(u_n)\exp(G(u_n))\chi(0,\tau)$ as a test function in $(P_4)_n$,

$$\begin{aligned}\Big[\int_\Omega \varphi_4(u_n)dx\Big]_0^T &+ \int_{Q^\tau}(|\nabla u_n|^{p(x)-2}\nabla u_n)\nabla T_k(u_n)\exp(G(u_n))dxdt\\
&+\int_{Q^\tau}|\nabla u_n|^{p(x)}T_k(u_n)g(u_n)\exp(G(u_n))dxdt\\
&-\int_{Q^\tau}\phi_n(u_n)\nabla\Big(T_k(u_n)\exp(G(u_n))\Big)dxdt\\
&-\int_{Q^\tau}nT_n((u_n-\psi)^-)T_k(u_n)\exp(G(u_n))dxdt\\
&=\int_{Q^\tau}g(u_n)|\nabla u_n|^{p(x)}T_k(u_n)\exp(G(u_n))dxdt\\
&+\int_{Q^\tau}f_nT_k(u_n)\exp(G(u_n))dxdt\\
&+\int_{Q^\tau}FT_k(u_n)\nabla u_n g(u_n)\exp(G(u_n))dxdt\\
&+\int_{Q^\tau}F\nabla T_k(u_n)\exp(G(u_n))dxdt\end{aligned}$$

where $\varphi_4(r) = \int_0^r T_k(s)\exp(G(s))ds$.
Due the definition of φ_4 and $|G(u_n)| \le \|g\|_{L^1(\mathbb{R})}$, we have $0 \le \int_\Omega \varphi_4(u_{0n})dx \le k\exp(\|g\|_{L^1(\mathbb{R})})\|u_{0n}\|_{L^1(\Omega)}$. The Lipschitz character of ϕ_n and Stok's formula to gether with the condition 2 of problem $(P_4)_n$ and by Young's inequality, we give $\int_Q \phi_n(u_n)\nabla\Big(T_k(u_n)\exp(G(u_n)\Big)dxdt = 0$, and by using (4.3.5) then,

$$\begin{aligned}&\int_{Q^\tau}|\nabla T_k(u_n)|^{p(x)}\exp(G(u_n))dxdt\\
&\le k\exp(\|g\|_{L^1(\mathbb{R})})\Big[\|f_n\|_{L^1(Q)}+\|u_0\|_{L^1(\Omega)}+C_{11}\Big]+C_{12}\\
&+\frac{1}{2[1+\|g\|_\infty]}\int_{Q^\tau}|\nabla T_k(u_n)|^{p(x)}\exp(G(u_n))dxdt\\
&+\frac{1}{2[1+\|g\|_\infty]}\|g\|_\infty\int_{Q^\tau}|\nabla T_k(u_n)|^{p(x)}\exp(G(u_n))dxdt.\end{aligned}$$

Let us take $\rho_5(u_n) = \displaystyle\int_0^{u_n} g(s)\chi_{\{s\geq 0\}}ds \exp(G(u_n))$ a test function in the approximate $(P_4)_n$, we obtain

$$\Big[\int_\Omega \varphi_5(u_n)dx\Big]_0^T + \int_Q |\nabla u_n|^{p(x)} g(u_n)\exp(G(u_n))\chi_{\{u_n\geq 0\}}dxdt$$
$$- \int_Q nT_n((u_n-\psi)^-)\rho_5(u_n)dxdt - \int_Q \phi_n(u_n)\nabla\rho_5(u_n)dxdt$$
$$\leq \Big(\int_0^\infty g(s)ds\Big)\exp\Big(\|g\|_{L^1(\mathbb{R})}\Big)\|f_n\|_{L^1(Q)}$$
$$+ \int_Q F\nabla u_n g(u_n)\chi_{\{u_n\geq 0\}}\exp(G(u_n))dxdt$$
$$+\Big(\int_0^\infty g(s)ds\Big)\int_Q |F\nabla u_n| g(u_n)\exp(G(u_n))\chi_{\{u_n\geq 0\}}dxdt,$$

where $\varphi_5(r) = \displaystyle\int_0^r \rho_5(s)ds$, which implies, using $\varphi_5(r)\geq 0$, the Lipschitz character of ϕ_n and by using Young's Inequality, we obtain

$$\int_{\{u_n\geq 0\}} |\nabla u_n|^{p(x)} g(u_n)\exp(G(u_n))dxdt$$
$$\leq \|g\|_\infty \exp\Big(\|g\|_{L^1(\mathbb{R})}\Big)\Big[\|f_n\|_{L^1(Q)} + \|u_0\|_{L^1(\Omega)}\Big] + C_{13}$$
$$+\frac{1}{2[1+\|g\|_\infty]}\int_{\{u_n\geq 0\}} |\nabla u_n|^{p(x)} g(u_n)\exp(G(u_n))dxdt$$
$$+\frac{1}{2[1+\|g\|_\infty]}\|g\|_\infty \int_{\{u_n\geq 0\}} |\nabla u_n|^{p(x)} g(u_n)\exp(G(u_n))dxdt,$$

then

$$\int_{\{u_n\geq 0\}} |\nabla u_n|^{p(x)} g(u_n)\exp(G(u_n))dxdt \leq C_{14}$$

Similarly, taking $\rho_6 = \displaystyle\int_{u_n}^0 g(s)\chi_{\{s\leq 0\}}ds \exp(G(u_n))$ as a test function in the approximate $(P_4)_n$, we conclude that

$$\int_{\{u_n\leq 0\}} |\nabla u_n|^{p(x)} g(u_n)\exp(G(u_n))dxdt \leq C_{15}$$

Consequently,

$$\int_Q |\nabla u_n|^{p(x)} g(u_n)\exp(G(u_n))dxdt \leq C_{16}.$$

Above, $C_1, ..., C_{16}$ are constants independent of n, we deduce that

$$\int_Q |\nabla T_k(u_n)|^{p(x)} dxdt \leq k\ C_{17}. \tag{4.3.6}$$

This implies that by Proposition (1.1.3), we get

$$\|T_k(u_n)\|_V = \|T_k(u_n)\|_{L^{p(x)}(Q)} \leq kC_{18} + 1.$$

Then, we conclude that $T_k(u_n)$ is bounded in V, independently of n for any $k > 0$. Now we turn to proving the almost everywhere convergence of u_n.
Consider a non decreasing function $g_k \in C^2(\mathbb{R})$ such that

$$g_k(s) = \begin{cases} s & if\ |s| \leq \frac{k}{2} \\ k & if\ |s| \geq k. \end{cases}$$

Multiplying the approximate equation by $g'_k(u_n)$, we get

$$\begin{aligned} \frac{\partial g_k(u_n)}{\partial t} &- \operatorname{div}\Big(|\nabla u_n|^{p(x)-2}\nabla u_n) g'_k(u_n)\Big) + |\nabla u_n|^{p(x)} g''_k(u_n) \\ &+ \operatorname{div}\phi_n(u_n) g'_k(u_n) - \phi_n(u_n)\nabla u_n g''_k(u_n) - nT_n(u_n - \psi)^- g'_k(u_n) \\ &= g(u_n)|\nabla u_n|^{p(x)} g'_k(u_n) + f_n g'_k(u_n) - \operatorname{div}(F g'_k(u_n)) + F\nabla u_n g''_k(u_n) \end{aligned} \tag{4.3.7}$$

in the sense of distributions. This implies, thanks to the fact that g'_k has compact support, that $g_k(u_n)$ is bounded in $L^{p^-}(0, T; W_0^{1,p(x)}(\Omega))$, while it's time derivative $\frac{\partial g_k(u_n)}{\partial t}$ is bounded in $L^1(Q) + V^*$ Due to the choice of g_k, we conclude that for each k, the sequence $T_k(u_n)$ converges almost everywhere in Q, which implies that the sequence u_n converge almost everywhere to some measurable function v in Q. Thus by using the same argument as in [34], [35], [36], we can show the following lemma.

Lemma 4.3.2 *Let u_n be a solution of the approximate problem $(P_4)_n$. Then,*

$$u_n \rightarrow u \quad a.e \quad in \quad Q. \tag{4.3.8}$$

We can deduce from (4.3.6) that

$$T_k(u_n) \rightharpoonup T_k(u) \qquad in \quad L^{p^-}(0, T; W_0^{1,p(x)}(\Omega)). \tag{4.3.9}$$

Lemma 4.3.3 *[10] Let u_n be a solution of the approximate problem $(P_4)_n$. Then,*

$$\lim_{m\to\infty} \limsup_{n\to\infty} \int_{\{m \leq |u_n| \leq m+1\}} a(x, t, u_n, \nabla(u_n))\nabla u_n dxdt = 0. \tag{4.3.10}$$

Step 3 : **Almost every convergence of the gradients** :
This step is devoted to introducing for a fixed $k \geq 0$ fixed, a time regularization of the function $T_k(u)$ in order to perform the monotonicity method.
This specific time regularization of $T_k(u)$ (for fixed $k \geq 0$) is defined as follows. Let $(\upsilon_0^\mu)_\mu$ be a sequence of functions defined on Ω such that

$$\upsilon_0^\mu \in L^\infty(\Omega) \cap W_0^{1,p(x)}(\Omega) \quad for\ all\ \mu > 0, \tag{4.3.11}$$

$$\|\upsilon_0^\mu\|_{L^\infty(\Omega)} \leq k \quad for\ all\ \mu > 0, \tag{4.3.12}$$

$$\upsilon_0^\mu \to T_k(u_0)\ a.e.\ in\ \Omega\ and\ \frac{1}{\mu}\|\upsilon_0^\mu\|_{L^{p(x)}(\Omega)} \to 0,\ as\ \mu \to \infty. \tag{4.3.13}$$

For fixed k, $\mu > 0$, let us consider the unique solution $(T_k(u))_\mu \in L^\infty(Q) \cap L^{p^-}(0,T;W_0^{1,p(x)}(\Omega))$ of the monotone problem :

$$\frac{\partial (T_k(u))_\mu}{\partial t} + \mu\Big((T_k(u))_\mu - T_k(u)\Big) = 0 \quad in\ D'(Q), \tag{4.3.14}$$

$$(T_k(u))_\mu(t=0) = \upsilon_0^\mu \quad in \quad \Omega. \tag{4.3.15}$$

Note that due to (4.3.14), we have for $\mu > 0$ and $k \geq 0$,

$$\frac{\partial (T_k(u))_\mu}{\partial t} \in L^{p^-}(0,T;W_0^{1,p(x)}(\Omega)). \tag{4.3.16}$$

We just recall here that (4.3.14)–(4.3.15) imply that

$$(T_k(u))_\mu \to T_k(u)\ a.e.\ in\ Q, \tag{4.3.17}$$

as well as weakly in $L^\infty(Q)$ weakly and strongly in $L^{p^-}(0,T;W_0^{1,p(x)}(\Omega))$ as $\mu \to \infty$.
Note that for any μ and any $k \geq 0$, we have

$$\|(T_k(u))_\mu\|_{L^\infty(Q)} \leq \max\Big(\|T_k(u)\|_{L^\infty(Q)};\ \|\upsilon_0^\mu\|_{L^\infty(\Omega)}\Big) \leq k, \tag{4.3.18}$$

We introduce a sequence of increasing $C^\infty(\mathbb{R})$–functions S_m such that

$$S_m(r) = r\ for\ |r| \leq m,\ supp(S_m') \subset [-(m+1), m+1], \|S_m''\|_{L^\infty(\mathbb{R})} \leq 1,$$

for any $m \geq 1$, and we denote by $\omega(n,\mu,\eta,m)$ the quantities such that

$$\lim_{m\to\infty}\lim_{\eta\to 0}\lim_{\mu\to\infty}\lim_{n\to\infty}\omega(n,\mu,\eta,m) = 0.$$

The main estimate is

Lemma 4.3.4 *([9, 36]). We have*

$$\int_0^T \left\langle \frac{\partial u_n}{\partial t}, T_\eta\Big(u_n - (T_k(u))_\mu\Big)^+ \exp(G(u_n))S'_m(u_n) \right\rangle dt \geq \omega(n,\mu,\eta)\ \forall m \geq 1. \tag{4.3.19}$$

Taking now $v = T_\eta\Big(u_n - (T_k(u))_\mu\Big)^+ S'_m(u_n)\exp(G(u_n))$, in $(P_4)_n$, we get

$$\begin{aligned}
&\int_0^T \left\langle \frac{\partial u_n}{\partial t}, T_\eta\Big(u_n - (T_k(u))_\mu\Big)^+ \exp(G(u_n))S'_m(u_n) \right\rangle dt \\
&+\int_Q (|\nabla u_n|^{p(x)-2}\nabla u_n)\nabla\Big(T_\eta\Big(u_n - (T_k(u))_\mu\Big)^+\Big)\exp(G(u_n))S'_m(u_n)dxdt \\
&+\int_{\{m\leq|u_n|\leq m+1\}} (|\nabla u_n|^{p(x)-2}\nabla u_n)T_\eta(u_n - (T_k(u))_\mu)^+\exp(G(u_n))S''_m(u_n)\nabla u_n dxdt \\
&+\int_Q (|\nabla u_n|^{p(x)-2}\nabla u_n)g(u_n)\nabla u_n\exp(G(u_n))T_\eta\Big(u_n - (T_k(u))_\mu\Big)^+ S'_m(u_n)dxdt \\
&-n\int_Q T_n(u_n-\psi)^- T_\eta\Big(u_n - (T_k(u))_\mu\Big)^+\exp(G(u_n))S'_m(u_n)dxdt \qquad (4.3.20)\\
&-\int_Q \phi_n(u_n)\nabla\Big(T_\eta\Big(u_n - (T_k(u))_\mu\Big)^+)\exp(G(u_n))S'_m(u_n\Big)dxdt \\
&=\int_Q g(u_n)|\nabla u_n|^{p(x)}T_\eta\Big(u_n - (T_k(u))_\mu\Big)^+ S'_m(u_n)\exp(G(u_n))dxdt \\
&+\int_Q f_n T_\eta\Big(u_n - (T_k(u))_\mu\Big)^+ S'_m(u_n)\exp(G(u_n))dxdt \\
&+\int_Q F\nabla T_\eta\Big(u_n - (T_k(u))_\mu\Big)^+ S'_m(u_n)\exp(G(u_n))dxdt
\end{aligned}$$

From (4.3.5),(4.3.10),(4.3.19),(4.3.20) and the Lipschitz character of ϕ_n it follows that

$$\begin{aligned}
&\int_Q (|\nabla u_n|^{p(x)-2}\nabla u_n)\nabla\Big(T_\eta\Big(u_n - (T_k(u))_\mu\Big)^+\Big)\exp(G(u_n))S'_m(u_n)dxdt \\
&\leq C\eta + \omega(n,\mu,\eta,m) + \int_Q F T_\eta\Big(u_n - (T_k(u))_\mu\Big)\nabla u_n g(u_n)\exp(G(u_n))S'_m(u_n)dxdt \\
&+\int_{\{m\leq|u_n|\leq m+1\}} F\nabla u_n\exp(G(u_n))\Big(T_\eta\Big(u_n - (T_k(u))_\mu\Big)^+\Big)S'_m(u_n)dxdt \\
&+\int_Q F\nabla\Big(T_\eta\Big(u_n - (T_k(u))_\mu\Big)^+\Big)\exp(G(u_n))S'_m(u_n)dxdt, \qquad (4.3.21)
\end{aligned}$$

where C is a constant independent of n and m. Then

$$\int_Q (|\nabla u_n|^{p(x)-2}\nabla u_n)\nabla\Big(T_\eta\Big(u_n-(T_k(u))_\mu\Big)^+\Big)\exp(G(u_n))S'_m(u_n)dxdt$$
$$\leq C\eta+I_1+I_2+I_3. \tag{4.3.22}$$

We have

$$\Big|\int_{\{m\leq|u_n|\leq m+1\}} F\nabla u_n\Big(T_\eta\Big(u_n-(T_k(u))_\mu\Big)^+\Big)dxdt\Big|$$
$$\leq 2\eta\exp\Big(\|g\|_{L^1(\mathbb{R})}\Big)C_0\Big(\int_{\{m\leq|u_n|\leq m+1\}}(|\nabla u_n|^{p(x)-2}\nabla u_n)\nabla u_n dxdt\Big)^{\frac{1}{p^-}}$$

Thanks to (4.3.10), we have

$$\lim_{m\to\infty}\lim_{n\to\infty} I_1=0 \quad \mathit{and} \lim_{m\to\infty}\lim_{n\to\infty} I_2=0$$

and by Lebesgue's theorem and $F\in L^{p'(.)}(Q))^N$, we have I_3 converges to zeros as n,m and μ tend to infinity, then

$$\int_Q (|\nabla u_n|^{p(x)-2}\nabla u_n)\nabla\Big(T_\eta\Big(u_n-(T_k(u))_\mu\Big)^+\Big)\exp(G(u_n))S'_m(u_n)dxdt$$
$$\leq C\eta+\omega(n,\mu,\eta,m) \tag{4.3.23}$$

On the other hand, let $A=\{0\leq T_k(u_n)-(T_k(u))_\mu<\eta\}$ and $B=\{0\leq u_n-(T_k(u))_\mu<\eta\}$. Then, we have

$$\int_Q (|\nabla u_n|^{p(x)-2}\nabla u_n)\nabla\Big(T_\eta\Big(u_n-(T_k(u))_\mu\Big)^+\Big)\exp(G(u_n))S'_m(u_n)dxdt \tag{4.3.24}$$
$$=\int_B (|\nabla u_n|^{p(x)-2}\nabla u_n)\Big(\nabla u_n-\nabla(T_k(u))_\mu\Big)\exp(G(u_n))S'_m(u_n)dxdt$$
$$=\int_A (|\nabla u_n|^{p(x)-2}\nabla u_n)\Big)\Big(\nabla T_k(u_n)-\nabla(T_k(u))_\mu\Big)\exp(G(u_n))S'_m(u_n)dxdt$$
$$+\int_{\{|u_n|>k\}\cap B}(|\nabla u_n|^{p(x)-2}\nabla u_n)\Big(\nabla u_n-\nabla(T_k(u))_\mu\Big)\exp(G(u_n))S'_m(u_n)dxdt.$$

The coercive character of a and the definition of $S'_m(S'_m(u_n)=1$ a.e. in $\{|u_n|\leq k\}$ if $k\leq m)$, it is possible to obtain from (4.3.23) and (4.3.24), that

$$\int_A (|\nabla T_k(u_n)|^{p(x)-2}\nabla T_k(u_n))\Big(\nabla T_k(u_n)-\nabla(T_k(u))_\mu\Big)\exp(G(u_n)S'_m(u_n)dxdt$$
$$\leq\int_{\{|u_n|>k\}\cap B}(|\nabla u_n|^{p(x)-2}\nabla u_n)\nabla(T_k(u))_\mu\exp(G(u_n))S'_m(u_n)dxdt$$
$$+C\eta+\omega(n,\mu,\eta,m). \tag{4.3.25}$$

Since $\nabla T_{k+\eta}(u_n)$ is bounded in $(L^{p'(x)}(Q))^N$ and $u_n \to u$ a.e.in Q, then $\nabla T_{k+\eta}(u_n) \rightharpoonup \nabla T_{k+\eta}(u)$ weakly in $(L^{p'(.)}(Q))^N$, consequently,

$$\int_{\{|u_n|>k\}\cap B} (|\nabla T_{k+\eta}(u_n)|^{p(x)-2}\nabla T_{k+\eta}(u_n)|\nabla (T_k(u))_\mu| \exp(G(u_n))S'_m(u_n)dxdt$$

$$= \int_{\{|u|>k\}\cap\{0\leq u-(T_k(u))_\mu<\eta\}} |\nabla T_{k+\eta}(u_n)|^{p(x)-2}\nabla (T_k(u))_\mu \exp(G(u))S'_m(u)dxdt + \omega(n).$$

Thanks to (4.3.17) one easily has,

$$\int_{\{|u|>k\}\cap\{0\leq u-(T_k(u))_\mu<\eta\}} (|\nabla T_{k+\eta}(u)|^{p(x)-2}\nabla (T_k(u))_\mu \exp(G(u))S'_m(u)dxdt = \omega(\mu).$$

Hence,

$$\int_A (|\nabla u_n|^{p(x)-2}\nabla u_n)\nabla\Big(T_\eta\Big(u_n - (T_k(u))_\mu\Big)^+\Big) \exp(G(u_n))S'_m(u_n)dxdt$$

$$\leq C\eta + \omega(n,\mu,\eta,m). \tag{4.3.26}$$

On the other hand, note that

$$\int_A (|\nabla T_k(u_n)|^{p(x)-2}\nabla T_k(u_n))\Big(\nabla T_k(u_n) - \nabla (T_k(u))_\mu\Big) \exp(G(u_n))dxdt$$
$$= \int_A (|\nabla T_k(u_n)|^{p(x)-2}\nabla T_k(u_n))\Big(\nabla T_k(u_n) - \nabla T_k(u)\Big) \exp(G(u_n))dxdt$$
$$+ \int_A (|\nabla T_k(u_n)|^{p(x)-2}\nabla T_k(u_n))\Big(\nabla T_k(u) - \nabla (T_k(u))_\mu\Big) \exp(G(u_n))dxdt, \tag{4.3.27}$$

and the last integral tends to 0 as $n\to\infty$ and $\mu\to\infty$. Indeed, we have that

$$\int_A (|\nabla T_k(u_n)|^{p(x)-2}\nabla T_k(u_n))\Big(\nabla T_k(u) - \nabla (T_k(u))_\mu\Big) \exp(G(u_n))dxdt$$
$$\to \int_{\{0\leq T_k(u)-(T_k(u))_\mu<\eta\}} (|\nabla T_k(u)|^{p(x)-2}\nabla T_k(u))\Big(\nabla T_k(u) - \nabla (T_k(u))_\mu\Big) \exp(G(u))dxdt$$
$$as\ n\to\infty.$$

Using(4.3.17) and Lebesgue's theorem , we have

$$\int_{\{0\leq T_k(u)-(T_k(u))_\mu<\eta\}} (|\nabla T_k(u)|^{p(x)-2}\nabla T_k(u))\Big(\nabla T_k(u) - \nabla (T_k(u))_\mu\Big) \exp(G(u))dxdt \to 0$$
$$as\ \mu\to\infty.$$

We deduce then that,

$$\int_A (|\nabla T_k(u_n)|^{p(x)-2}\nabla T_k(u_n))\Big(\nabla T_k(u_n) - \nabla T_k(u)\Big) \exp(G(u_n)dxdt$$
$$\leq C\eta + \omega(n,\mu,\eta,m) \tag{4.3.28}$$

Let

$$M_n = \Big(\Big[|\nabla T_k(u_n)|^{p(x)-2}\nabla T_k(u_n)\Big]\Big[\nabla T_k(u_n) - \nabla T_k(u)\Big]\Big) \times \\ \times\Big(\exp(G(u_n))\Big).$$

Then for any $0 < \theta < 1$, we write

$$\begin{aligned} I_n &= \int_{\{|u_n-(T_k(u))_\mu|\geq 0} M_n^\theta dxdt \\ &= \int_{\{|T_k(u_n)-(T_k(u))_\mu|\leq\eta,\ u_n-T_k(u)_\mu\geq 0\}} M_n^\theta dxdt \\ &+ \int_{\{|T_k(u_n)-(T_k(u))_\mu|>\eta,\ u_n-(T_k(u))_\mu\geq 0\}} M_n^\theta dxdt. \end{aligned}$$

Since $\nabla T_k(u_n)$ is bounded in $(L^{p(x)}(Q))^N$, then by applying the Hölder's inequality, we obtain

$$\begin{aligned} I_n \ \leq\ & C_1\Big(\int_{\{0\leq T_k(u_n)-(T_k(u))_\mu<\eta\}} M_n dxdt\Big)^\theta \qquad (4.3.29) \\ & +C_2 \text{ meas}\Big\{(x,t)\in Q : \Big|T_k(u_n) - (T_k(u))_\mu\Big| > \eta\ ,\ u_n - (T_k(u))_\mu \geq 0\Big\}^{1-\theta}. \end{aligned}$$

On the other hand, we have,

$$\begin{aligned} & \int_{\{0\leq T_k(u_n)-(T_k(u))_\mu)<\eta\}} M_n\ dx\ dt \\ &= \int_{\{0\leq T_k(u_n)-(T_k(u))_\mu)<\eta\}} a\Big(x,t,T_k(u_n),\nabla T_k(u_n)\Big)\times \\ &\qquad \times\Big(\nabla T_k(u_n) - \nabla T_k(u)\Big)\exp(G(u_n))\ dx\ dt \qquad (4.3.30) \\ &- \int_{\{0\leq T_k(u_n)-(T_k(u))_\mu)<\eta\}} a\Big(x,t,T_k(u_n),\nabla T_k(u)\Big)\times \\ &\qquad \times\Big(\nabla T_k(u_n) - \nabla T_k(u)\Big)\exp(G(u_n))\ dx\ dt \\ &= I_n^1 + I_n^2. \end{aligned}$$

Using (4.3.28) we have,

$$I_n^1 \leq C\ \eta + w(n,\mu,\eta,m). \qquad (4.3.31)$$

Concerning I_n^2, that is, the second term of the right hand side of the (4.3.30), it is easy to see that

$$I_n^2 = w(n,\mu). \qquad (4.3.32)$$

Because for all $i = 1, ..., N$, we have $\frac{\partial T_k(u_n)}{\partial x_i} \rightharpoonup \frac{\partial T_k(u)}{\partial x_i}$ in $L^{p(x)}(Q)$. Combining (4.3.29), (4.3.30), (4.3.31) and (4.3.32) we get,

$$I_n \leq C_1 \left(C\ \eta + w(n, \mu, \eta, m)\right)^{\theta} + C_2 \left(w(n, \mu)\right)^{1-\theta}$$

and by passing to the limit sup over n, μ and η

$$\int_{\{u_n-(T_k(u))_\mu \geq 0\}} \left(\left[|\nabla T_k(u_n)|^{p(x)-2}\nabla T_k(u_n)) - |\nabla T_k(u_n)|^{p(x)-2}\nabla T_k(u))\right] \times \right.$$
$$\left. \times \left[\nabla T_k(u_n) - \nabla T_k(u)\right]\right)^{\theta} dx\,dt = w(n). \qquad (4.3.33)$$

On the other hand, we choose $v = T_\eta\Big(u_n - (T_k(u))_\mu\Big)^- \exp(-G(u_n))$ in $(P_4)_n$, we obtain :

$$\int_{\{u_n-T_k(u)_\mu \leq 0\}} \left(\left[|\nabla T_k(u_n)|^{p(x)-2}\nabla T_k(u_n)) - |\nabla T_k(u_n)|^{p(x)-2}\nabla T_k(u))\right] \times \right.$$
$$\left. \times \left[\nabla T_k(u_n) - \nabla T_k(u)\right]\right)^{\theta} dx\,dt = w(n). \qquad (4.3.34)$$

Moreover (4.3.33) and (4.3.34) imply that

$$\int_Q \left(\left[|\nabla T_k(u_n)|^{p(x)-2}\nabla T_k(u_n)) - |\nabla T_k(u_n)|^{p(x)-2}\nabla T_k(u))\right] \times \right.$$
$$\left. \times \left[\nabla T_k(u_n) - \nabla T_k(u)\right]\right)^{\theta} dx\,dt = w(n) \qquad (4.3.35)$$

which implies that

$$T_k(u_n) \to T_k(u) \quad in \quad L^{p^-}(0, T; W_0^{1,p(x)}(\Omega))\ \forall\ k \geq 0. \qquad (4.3.36)$$

By using [35, 34]), there exist a subsequence also denoted by u_n such that

$$\nabla u_n \to \nabla u \ \text{ a.e. in } \ Q. \qquad (4.3.37)$$

Proposition 4.3.3 *Let u_n be a solution of the approximate problem $(P_4)_n$. Then*

$$u \geq \psi \quad a.e.\ in\, Q.$$

Proof. Thanks to (4.3.5), we can write $\displaystyle\int_Q T_n(u_n - \psi)^- dxdt \leq \frac{C}{n}$. And by using Fatou's Lemma as $n \to \infty$, we have that $\displaystyle\int_Q (u_n - \psi)^- dxdt$ converges to zeros, we get $(u_n-\psi)^-=0$ a.e. in Q. Consequently we conclude that $u\geq\psi$ a.e. in Q.

Step 4 : Passing to the limit

4-1 We claim that $u\in C(0,T;L^1(\Omega))$

Is easy by adapting the same way as in [11].

4-2 We prove that u **satisfies** (4.2.7)

Indeed let $v\in K_\psi\cap L^\infty(Q)$, $\frac{\partial v}{\partial t}\in L^{p'^-}(0,T;W^{-1,p(x)}(\Omega))$. By the pointwise multiplication of the approximate problem $(P_4)_n$ by $T_k(u_n-v)$, we get

$$\begin{aligned}
&\int_\Omega S_k\Big(u_n(T)-v(T)\Big)dx-\int_\Omega S_k\Big(u_{0n}-v(0)\Big)dx\\
&\quad+\int_Q\frac{\partial v}{\partial t}T_k(u_n-v)dxdt+\int_Q(|\nabla u|^{p(x)-2}\nabla u)\nabla T_k(u_n-v)dxdt\\
&\quad-\int_Q\phi_n(u_n)\nabla T_k(u_n-v)dxdt\\
&\quad-\int_Q nT_n(u_n-\psi)^-T_k(u_n-v)dxdt\\
&\quad\leq\int_Q g(u_n)|\nabla u_n|^{p(x)}T_k(u_n-v)dxdt\\
&\quad+\int_Q f_nT_k(u_n-v)dxdt+\int_Q F\nabla T_k(u_n-v)dxdt,
\end{aligned}$$

where $S_k(s)=\displaystyle\int_0^s T_k(r)dr$.

Since $v\in K_\psi\cap L^\infty(Q)$, we have $-\displaystyle\int_Q nT_n(u_n-\psi)^-T_k(u_n-v)dxdt\geq 0$, we deduce then that

$$\begin{aligned}
&\int_\Omega S_k\Big(u_n(T)-v(T)\Big)dx-\int_\Omega S_k\Big(u_{0n}-v(0)\Big)dx\\
&\quad+\int_Q\frac{\partial v}{\partial t}T_k(u_n-v)dxdt+\int_Q(|\nabla u|^{p(x)-2}\nabla u)\nabla T_k(u_n-v)dxdt\\
&\quad-\int_Q\phi_n(u_n)\nabla T_k(u_n-v)dxdt\\
&\quad\leq\int_Q g(u_n)|\nabla u_n|^{p(x)}T_k(u_n-v)dxdt\\
&\quad+\int_Q f_nT_k(u_n-v)dxdt+\int_Q F\nabla T_k(u_n-v)dxdt.
\end{aligned}\tag{4.3.38}$$

Let us pass to the limit with $n \to \infty$ in each term in (4.3.38).

• As S_k is lipschitz of constant k, when $n \to \infty$ we have

$$\int_\Omega S_k(u_n - v)(T)dx \to \int_\Omega S_k(u - v)(T)dx$$
$$and \int_\Omega S_k(u_n - v)(0)dx = \int_\Omega S_k(u_{0n} - v(0))dx \to \int_\Omega S_k(u_0 - v(0))dx.$$

• Since $\frac{\partial v}{\partial t} \in V^* + L^*(Q)$, that is

$$\int_0^T \Big\langle \frac{\partial v}{\partial t}, T_k(u_n - v)\Big\rangle dt \to \int_0^T \Big\langle \frac{\partial v}{\partial t} T_k(u - v)\Big\rangle dt.$$

• Let us pass to the limit with $n \to \infty$ for the term $\int_Q (|\nabla u_n|^{p(x)-2}\nabla u_n)\nabla T_k(u_n - v)dxdt$. Since $v \in L^\infty(Q)$, we note $M = \|v\|_\infty$, we get

$$\begin{aligned}
&\int_Q (|\nabla u_n|^{p(x)-2}\nabla u_n)\nabla T_k(u_n - v)dxdt \\
&\quad = \int_0^T \int_\Omega (|\nabla u_n|^{p(x)-2}\nabla u_n)\nabla T_k(T_{k+M}(u_n) - v)dxdt \\
&\qquad = \int_0^T \int_\Omega (|\nabla u_n|^{p(x)-2}\nabla u_n)\nabla T_{k+M}(u_n))\mathbf{1}_{\{|T_{k+M}(u_n)-v|\le k\}}dxdt \\
&\qquad\quad - \int_0^T \int_\Omega (|\nabla u_n|^{p(x)-2}\nabla u_n)\nabla v\mathbf{1}_{\{|T_{k+M}(u_n)-v|\le k\}}dxdt.
\end{aligned}$$

As $T_{k+M}(u_n)$ is bounded in V, $\nabla u_n \to \nabla u$ a.e. in Q,

$$\nabla T_{k+M}(u_n) \to \nabla T_{k+M}(u) \ \text{ almost everywhere}$$

and by using Lebesgue theorem, we deduce that

$$\begin{aligned}
&\int_Q (|\nabla u_n|^{p(x)-2}\nabla u_n)\nabla T_{k+M}(u_n))\mathbf{1}_{\{|T_{k+M}(u_n)-v|\le k\}}dxdt \\
&\qquad \to \int_Q (|\nabla u|^{p(x)-2}\nabla u)\mathbf{1}_{\{|T_{k+M}(u-v)|\le k\}}dxdt
\end{aligned}$$

and

$$\begin{aligned}
&\int_0^T \int_\Omega (|\nabla u_n|^{p(x)-2}\nabla u_n)\nabla v\mathbf{1}_{\{|T_{k+M}(u_n)-v|\le k\}}dxdt \to \\
&\quad \int_0^T \int_\Omega (|\nabla u|^{p(x)-2}\nabla u)\nabla v\mathbf{1}_{\{|T_{k+M}(u-v)|\le k\}}dxdt
\end{aligned}$$

then

$$\int_Q (|\nabla u_n|^{p(x)-2}\nabla u_n)T_k(u_n - v)dxdt \to \int_Q (|\nabla u|^{p(x)-2}\nabla u)T_k(u-v)dxdt.$$

• Let us pass to the limit for other term

$$\int_Q \phi_n(u_n)\nabla T_k(u_n - v)dxdt = \int_Q \phi_n(u_n)\nabla T_k(T_{k+M}(u_n) - v)dxdt$$

Since $T_{k+M}(u_n)$ is bounded in V and ϕ_n is a Lipschitz continuous bounded function $\mathbb{R}$ in to $\mathbb{R}^N$, such that ϕ_n uniformly converge to ϕ on any compact subset of $\mathbb{R}$ as n tend to $+\infty$. Then, we have

$$\int_Q \phi_n(u_n)\nabla T_k(u_n - v)dxdt \to \int_Q \phi(u)\nabla T_k(T_{k+M}(u) - v)dxdt$$

and by Lebesgue theorem, we deduce that

$$\int_Q \phi_n(u_n)\nabla T_k(u_n - v)dxdt \to \int_Q \phi(u)\nabla T_k((u) - v)dxdt$$

Due to (4.2.4) , $T_k(u_n) \to T_k(u)$ *in* $V \ \forall \ k \geq 0$
and $u_n \to u$ a.e. in Q, we have

$$f_nT_k(u_n - v) \to fT_k(u-v) \ strongly \ in \ L^1(Q)$$

and by Lebesgue theorem, we have

$$\int_Q f_nT_k(u_n - v) \to \int_Q fT_k(u-v) \ strongly \ in \ L^1(Q)$$

• Similarly, since g is a bounded and continuous functions belong to $L^1(\mathbb{R})$ and $u_n \to u$ a.e. in Q,we obtain

$$\int_Q g(u_n)|\nabla u_n|^{p(x)-2}T_k(u_n - v) \to \int_Q g(u)|\nabla u|^{p(x)-2}T_k(u-v) \ strongly \ in \ L^1(Q)$$

• For the second term of the right hand side of (4.3.38), we have

$$\int_Q F\nabla T_k(u_n - v) \to \int_Q F\nabla T_k(u-v) \ asn \to +\infty,$$

since $\nabla T_k(u_n - v) \to \nabla T_k(u-v)$ *in* $(L^{p(x)}(Q))^N$, while $F \in (L^{p'(x)}(Q))^N$ and Lebesgue theorem. Then, we conclude that u satisfies (4.2.7).
As a conclusion of Step 1 to Step 4, the proof of Theorem (4.3.1) is complete. $\square$

This part is constituted of the following chapters :

CHAPTER V

Existence of Renormalised Solutions for Nonlinear Elliptic Problems in Weighthed Variable -Exponent Space with L^1 -Data

CHAPTER VI

Renormalized Solutions of Nonlinear Parabolic Equations in Weigthed Variable-Exponent Space

Chapitre 5

Existence of Renormalised Solutions for Nonlinear Elliptic Problems in Weighthed Variable -Exponent Space with L^1 -Data

Abstract

In this chapter we study the existence of a renormalized solution for the nonlinear $p(x)$–elliptic problem in the Weighted–Variable–Exponent Soblev spaces, of the form :

$$-\mathrm{div}(a(x,u,\nabla u))+H(x,u,\nabla u)=f \quad \text{ in } \quad \Omega,$$

where the right-hand side f belong to $L^1(\Omega)$ and $H(x,s,\xi)$ is the nonlinear term satisfying some growth condition, but no sign condition on s.

5.1 Introduction

Let Ω be a bounded open subset of $\mathbb{R}^N$ with $\mathrm{N}\geq 2$, $p\in C_+(\overline{\Omega})$, $p(x)>1$. Let A be the nonlinear operator defined from $W_0^{1,p(x)}(\Omega,\omega)$ into its dual $W^{-1,p'(x)}(\Omega,\omega^*)$ by the formula $Au=-\mathrm{div}(a(x,u,\nabla u))$.

The objective of this chapter, is to study the existence of a renormalised solution for a class of nonlinear $p(x)$- elliptic problems of the type

$$\begin{cases} Au+H(x,u,\nabla u)=f & \text{in } \Omega \\ u=0 & \text{in } \partial\Omega, \end{cases} \tag{5.1.1}$$

where $f \in L^1(\Omega)$. The operator Au is a Weighted Leray–Lions operator (see assumption (5.2.1)-(5.2.3) of section 2) which is coercive and where H is a non linear lower order term.
Partial differential equations with nonlinearities involving non constant exponent have attracted an increasing amount of attention in recent years. The development, mainly by Ruzika [80] of a theory modeling the behavior of electrorheological fluids. Other applications relate to thermistor problem [86], or the problem of image recovery [48]. This notion was adapted yet to the study with $H = 0$ of some nonlinear elliptic problems on Sobolev space a exponent variable with Dirichlet boundary conditions by Boccardo-Diaz. Giachette and Murat[45]. Recently, Bendahmane and Wittbold in [30] proved the existence and uniqueness of renormalized solutions to nonlinear elliptic equations with variable exponent and L^1 data, besides, when $f \in L^1(\Omega) + W^{-1,p'(x)}(\Omega)$ E. Azroul. M. B. Benboubker and M. Rhoudaf in [24] proved the entropy solutions for some $p(x)$ quasilinear problem with right-hand side mesure. In the case where $p(x)$ is a constant some results have been proved by, L. Aharouch and Y. Akdim [4], to the case of parabolic equations, is studied by Y. Akdim, J. Bennouna and M. Mekkour [9], where $H = 0$ and $f \in L^{p'}(0, T; W^{-1,p'}(\Omega, \omega^*))$.
This chapter is organized as follows. In Section 2, we give our basic assumptions and the definition of a renormalized solution of the problem (5.1.1) for which our problem has a solution. In Section 3, we establish the existence of such a solution in Theorem (5.3.1). In Section 4, we give the proof of Theorem (5.3.1), Lemma (5.3.3) and Proposition (5.3.2) (see appendix)

5.2 Essential Assumption

Throughout the paper, we assume that the following assumption hold true.
ASSUMPTION (H1)
Let Ω is a bounded open set of $\mathbb{R}^N (N \geq 2)$, $p \in C_+(\bar{\Omega})$, such that $1 < p^- \leq p^+ < \infty$ and $\frac{1}{p(x)} + \frac{1}{p'(x)} = 1$. We consider a Leray–Lions operator defined by the formula :

$$Au = -\text{div}\ a(x, u, \nabla u).$$

The function $a : \Omega \times \mathbb{R} \times \mathbb{R}^N \to \mathbb{R}$ is a Carathéodory function satisfying the following conditions

$$|a(x,s,\xi)| \leq \beta\omega^{1/p(x)}(x)\Big[k(x) + \omega^{1/p'(x)}|s|^{p(x)-1} + \omega^{1/p'(x)}(x)|\xi|^{p(x)-1}\Big], \qquad (5.2.1)$$

$$\Big[a(x,s,\xi) - a(x,s,\eta)\Big] \cdot (\xi - \eta) > 0 \ \ \forall\, \xi \neq \eta \in \mathbb{R}^N, \qquad (5.2.2)$$

$$a(x,s,\xi) \cdot \xi \geq \alpha\omega|\xi|^{p(x)}, \qquad (5.2.3)$$

for all $(s,\eta,\xi) \in \mathbb{R} \times \mathbb{R}^N \times \mathbb{R}^N$ and for almost every $x \in \Omega$,
where $k(x)$ is a positive function lying in $L^{p'(x)}(\Omega)$ and α, β are a positive constants.

ASSUMPTION (H2)

Let $H : \Omega \times \mathbb{R} \times \mathbb{R}^N \to \mathbb{R}$ be a Carathéodory function such that for a.e. $x \in \Omega$ and for all $s \in \mathbb{R}$, $\xi \in \mathbb{R}^N$, the growth condition

$$|H(x,s,\xi)| \leq \gamma(x) + g(s)\omega|\xi|^{p(x)}, \qquad (5.2.4)$$

is satisfied, where $g : \mathbb{R} \to \mathbb{R}^+$ is a bounded continuous positive function that belongs to $L^1(\mathbb{R})$ while $\gamma \in L^1(\Omega)$.

$$f \in L^1(\Omega). \qquad (5.2.5)$$

Definition 5.2.1 *Let $f \in L^1(\Omega)$. A real-valued function u defined on Ω is a renormalized solution of problem* (5.1.1) *if*

$$T_k(u) \in W_0^{1,p(x)}(\Omega,\omega)) \quad \textit{for all } k \geq 0, \qquad (5.2.6)$$

$$\int_{\{m\leq|u|\leq m+1\}} a(x,u,\nabla u)\nabla u dx \to 0 \ \textit{as} \ \ m \to \infty, \qquad (5.2.7)$$

$$-div\Big(S'(u)a(x,u,\nabla u)\Big) + S''(u)a(x,u,\nabla u)\nabla u + H(x,u,\nabla u)S'(u)$$
$$= fS'(u) \ \ \textit{in} \quad D'(\Omega), \qquad (5.2.8)$$

for all $S \in W^{2,\infty}(\mathbb{R})$ which are piecewise C^1 and such that S' has a compact support in $\mathbb{R}$.

Remark 5.2.1 Equation (5.2.8) is formally obtained through pointwise multiplication of problem (5.1.1) by $S'(u)$. However, while $a(x,u,\nabla u)$ and $H(x,u,\nabla u)$ do not in general make sense in (5.1.1), all the terms in (5.2.8) have a meaning in $D'(\Omega)$. Indeed, if M is such that supp $S \subset [-M, M]$, the following identifications are made in (5.2.8).

– *$S(u)$ belongs to $L^\infty(\Omega)$. Since S is a bounded function.*
– *$S'(u)\ a(x,u,\nabla u)$ identifies with $S'(u)\ a(x,T_M(u),\nabla T_M(u))$ a.e. in Ω.*
for any $\varphi \in D(\Omega)$, using Hölder inequality

$$\int_\Omega S'(u)a(x,u,\nabla u)\nabla\varphi dx = \int_\Omega S'(u)a(x,T_M(u),\nabla T_M(u))\nabla\varphi dx$$
$$\leq C_M\|S'\|_{L^\infty(\Omega)} \max\Big\{\Big(\int_\Omega |\nabla T_M(u)|^{p(x)}\omega(x)\Big)^{\frac{1}{p'^-}},$$
$$\Big(\int_\Omega |\nabla T_M(u)|^{p(x)}\omega(x)\Big)^{\frac{1}{p'^+}}\Big\}\|\nabla\varphi\|_{L^{p(\cdot)}(\Omega)}.$$

where $M > 0$ is that supp $S' \subset [-M,M]$. Since $|T_M(u)| \leq M$ a.e. in Ω and $S'(u)$ belongs to $L^\infty(\Omega)$. We deduce from (5.2.1) and (5.2.7) that

$$div(S'(u)a(x,u,\nabla u)) \in (L^{p'(x)}(\Omega,\omega))^N.$$

– *$S''(u)\ a(x,u,\nabla u)\nabla u$ identifies with $S''(u)\ a(x,u,T_M(u),\nabla T_M(u))\nabla T_M(u)$ and*

$$S''(u)a(x,u,T_M(u),\nabla T_M(u))\nabla T_M(u) \in L^1(\Omega).$$

– *$S'(u)H(x,u,\nabla u)$ identifies with $S'(u)H(x,t,T_M(u),\nabla T_M(u))$ a.e.in Ω. Since $|T_M(u)| \leq M$ a.e. in Ω and $S'(u) \in L^\infty(\Omega)$, we see from (5.2.4) that $S'(u)H(x,T_M(u),\nabla T_M(u)) \in L^1(\Omega)$.*
– *$S'(u)\ f$ belongs to $L^1(\Omega)$.*

The above considerations show that equation (5.2.8) hold in $D'(\Omega)$ and $\frac{\partial S(u)}{\partial t} \in L^1(\Omega) + W^{-1,p'(x)}(\Omega,\omega^*)$.

5.3 Existence Results.

Theorem 5.3.1 *Let $f \in L^1(\Omega)$, $p \in C_+(\bar\Omega)$. Assume that $(H1)$ and $(H2)$ hold true. Then, there exists a renormalized solution u of problem* (5.1.1) *in the sense of Definition (5.2.1).*

proof. The proof is in five steps.

STEP 1 : Approximate problem :

For $n > 0$, let us define the following respective approximation of H and f ;

$$\begin{aligned} H_n(x,s,\xi) &= \frac{H(x,s,\xi)}{1+\frac{1}{n}|H(x,s,\xi)|}. \\ \text{Note that} \quad |H_n(x,s,\xi)| &\leq |H(x,s,\xi)| \\ \text{and} \quad |H_n(x,s,\xi)| &\leq n \text{ for all } (s,\xi) \in \mathbb{R}\times\mathbb{R}^N. \end{aligned}$$

Let $(f_n)_n$ be sequence of smooth functions such that

$$f_n \to f \text{ a.e. in } \Omega, \quad \text{strongly in } L^1(\Omega) \text{ as } n \to \infty \text{ and } \|f_n\|_{L^1(\Omega)} \leq \|f\|_{L^1(\Omega)}. \tag{5.3.1}$$

We consider the sequence of approximate problems

$$(P_5)_n \begin{cases} u_n \in W_0^{1,p(x)}(\Omega, \omega) \\ -\text{div}(a(x, u_n, \nabla u_n)) + H_n(x, u_n, \nabla u_n) = f_n & \text{in } D'(\Omega) \\ u_n = 0 & \text{on } \partial\Omega. \end{cases}$$

Let $f_n \in W^{-1,p'(x)}(\Omega, \omega^*)$, $p \in C_+(\bar{\Omega})$ for fixed n, the approximate problem $(P_5)_n$ has at least one weak solution $u_n \in W_0^{1,p(x)}(\Omega, \omega)$. See[65].

STEP 2 : A Priori Estimates :

Proposition 5.3.1 *Let u_n a solution of the approximate problem $(P_5)_n$. Then, there exists a constant C(which does not depend on the n and k) such that*

$$\|T_k(u_n)\|_{W_0^{1,p(x)}(\Omega,\omega)} \leq Ck \quad \forall k > 0.$$

Proof. Let $\varphi \in W_0^{1,p(x)}(\Omega, \omega) \cap L^\infty(\Omega)$, with $\varphi > 0$. Choosing $v = \exp(G(u_n))\varphi$ as a test function in $(P_5)_n$, where

$$G(s) = \int_0^s \frac{g(r)}{\alpha} dr.$$

(the function g appears in (5.2.4)), we have

$$\int_\Omega a(x, u_n, \nabla u_n)\nabla\Big(\exp(G(u_n))\varphi\Big)dx + \int_\Omega H_n(x, u_n, \nabla u_n)\exp(G(u_n))\varphi dx$$
$$= \int_\Omega f_n \exp(G(u_n))\varphi dx.$$

In view of (5.2.4), we obtain

$$\int_\Omega a(x, u_n, \nabla u_n)\nabla u_n \frac{g(u_n)}{\alpha}\exp(G(u_n))\varphi dx$$
$$+ \int_\Omega a(x, u_n, \nabla u_n)\exp(G(u_n))\nabla\varphi dx$$
$$\leq \int_\Omega \gamma(x)\exp(G(u_n))\varphi dx + \int_\Omega f_n \exp(G(u_n))\varphi dx$$
$$+ \int_\Omega g(u_n)|\nabla u_n|^{p(x)}\omega(x)\exp(G(u_n))\varphi dx,$$

by using (5.2.3), we obtain

$$\int_\Omega a(x,u_n,\nabla u_n)\exp(G(u_n))\nabla\varphi dx \leq \int_\Omega \Big(\gamma(x)+f_n\Big)\exp(G(u_n))\varphi dx \quad (5.3.2)$$

for all $\varphi \in W_0^{1,p(x)}(\Omega,\omega)\cap L^\infty(\Omega)$, with $\varphi > 0$.
On the other hand, taking $v=\exp(-G(u_n))\varphi$ as a test function in $(P_5)_n$, we deduce as in (5.3.2) that

$$\begin{aligned}&\int_\Omega a(x,u_n,\nabla u_n)\exp(-G(u_n))\nabla\varphi dx\\&\quad+\int_\Omega \gamma(x)\exp(-G(u_n))\varphi dx \geq \int_\Omega f_n\exp(-G(u_n))\varphi dx\end{aligned} \quad (5.3.3)$$

for all $\varphi \in W_0^{1,p(x)}(\Omega,\omega)\cap L^\infty(\Omega)$, with $\varphi > 0$.
Letting $\varphi = T_k(u_n)^+$ in (5.3.2), we have

$$\begin{aligned}&\int_\Omega a(x,u_n,\nabla u_n)\exp(G(u_n))\nabla T_k(u_n)^+dx\\&\qquad\leq \int_\Omega\Big[\gamma(x)+f_n\Big]\exp(G(u_n))T_k(u_n)^+dx,\end{aligned} \quad (5.3.4)$$

since $G(u_n)\leq \frac{\|g\|_{L^1(\mathbb{R})}}{\alpha}$, we obtain

$$\begin{aligned}&\int_\Omega a(x,u_n,\nabla T_k(u_n)^+)\exp(G(u_n))\nabla T_k(u_n)^+dx\\&\qquad\leq k\exp\Big(\frac{\|g\|_{L^1(\mathbb{R})}}{\alpha}\Big)\Big[\|\gamma\|_{L^1(\Omega)}+\|f_n\|_{L^1(\Omega)}\Big]\leq C_1k\end{aligned}$$

by using (5.2.3), we have

$$\int_\Omega |\nabla T_k(u_n)^+|^{p(x)}\omega(x)\exp(G(u_n))dx \leq C_1k. \quad (5.3.5)$$

Similarly to (5.3.5), we take $\varphi = T_k(u_n)^-$ in (5.3.3), we deduce that

$$\int_\Omega |\nabla T_k(u_n)^-|^{p(x)}\omega(x)dx \leq C_2k. \quad (5.3.6)$$

Combining (5.3.5), (5.3.6) and Remark (1.3.2), we conclude that

$$\min\Big\{\|T_k(u_n)\|^{p^+}_{W_0^{1,p(x)}(\Omega,\omega)},\|T_k(u_n)\|^{p^-}_{W_0^{1,p(x)}(\Omega,\omega)}\Big\}\leq \rho(\nabla T_k(u_n))\leq C_3k.$$

$$\|T_k(u_n)\|_{W_0^{1,p(x)}(\Omega,\omega)} \leq C_4k, \quad (5.3.7)$$

where $C_1,\ C_2,\ C_3$ and C_4 are constants independent of n. Thus, $T_k(u_n)$ is bounded in $W_0^{1,p(x)}(\Omega,\omega)$ independently of n for any $k>0$. Now we turn to proving the almost everywhere convergence of u_n. Consider a non decreasing function $g_k \in C^2(\mathbb{R})$ such that

$$g_k(s) = \begin{cases} s & if\ |s| \leq \frac{k}{2} \\ k & if\ |s| \geq k. \end{cases}$$

Multiplying the approximate equation by $g'_k(u_n)$, we get

$$\begin{aligned} -\mathrm{div}(a(x,u_n,\nabla u_n)g'_k(u_n)) + a(x,u_n,\nabla u_n)g''_k(u_n)\nabla u_n \\ +H_n(x,u_n,\nabla u_n)g'_k(u_n) = f_n g'_k(u_n). \end{aligned} \tag{5.3.8}$$

in the sense of distributions. This implies, thanks to the fact g'_k has compact support, that $g_k(u_n)$ is bounded in $W_0^{1,p(x)}(\Omega,\omega)$, while it's time derivative $\frac{\partial g_k(u_n)}{\partial t}$ is bounded in $L^1(Q) + W_0^{1,p(x)}(\Omega,\omega)$. Due to the choice of g_k, we conclude that for each k, the sequence $T_k(u_n)$ converges almost everywhere in Ω, which implies that the sequence u_n converge almost everywhere to some measurable function v in Ω. Thus by using the same argument as in [34], [35], [36], we can show the following lemma.

Lemma 5.3.2 *Let u_n be a solution of the approximate problem $(P_5)_n$. Then,*

$$u_n \ \rightarrow\ u \qquad a.e.\ in\ \Omega.$$

We can deduce from ((5.3.7)) that

$$T_k(u_n) \rightharpoonup T_k(u) \quad in \quad W_0^{1,p(x)}(\Omega,\omega),$$

which implies by using (5.2.3) that for all $k>0$, there exists $\varphi_k \in (L^{p'(x)}(\Omega,\omega^*))^N$ such that

$$a(x,T_k(u_n),\nabla T_k(u_n)) \rightharpoonup \varphi_k \ \ in \quad \Big(L^{p'(x)}(\Omega,\omega^*)\Big)^N. \tag{5.3.9}$$

Lemma 5.3.3 *Let u_n be a solution of the approximate problem $(P_5)_n$. Then,*

$$\lim_{m\to\infty} \limsup_{n\to\infty} \int_{\{m\leq |u_n| \leq m+1\}} a(x,u_n,\nabla u_n)\nabla u_n dx = 0. \tag{5.3.10}$$

Proof. See Appendix.

STEP 3 :Almost everywhere convergence of the gradients :

We will use the following function of one real variable s, which is defined as follows

$$h_m(s) = \begin{cases} 1 & if \quad |s| \leq m \\ 0 & if \quad |s| \geq m+1 \\ m+1-s & if \quad m \leq s \leq m+1 \\ m+1+s & if \quad -m-1 \leq s \leq -m, \end{cases}$$

where m is a non negative real parameter.

To prove the strong convergence of truncation $T_k(u_n)$, we have to prove the following assertions :

Proposition 5.3.2 The subsequence of u_n solution of problem $(P_5)_n$ satisfies, for any $k \geq 0$ the following assertion :

Assertion (i)

$$\lim_{m\to\infty} \limsup_{n\to\infty} \int_\Omega a\Big(x, T_k(u_n), \nabla T_k(u_n)\Big) \nabla T_k(u_n) \Big(1 - h_m(u_n)\Big) dx = 0. \quad (5.3.11)$$

Assertion(ii)

$$\lim_{m\to\infty} \limsup_{n\to\infty} \int_\Omega \Big[a\Big(x, T_k(u_n), \nabla T_k(u_n)\Big) - a\Big(x, T_k(u_n), \nabla T_k(u)\Big)\Big] \times \Big[\nabla T_k(u_n) - \nabla T_k(u)\Big] h_m(u_n) dx = 0. \quad (5.3.12)$$

Assertion(iii)

$$\lim_{n\to\infty} \int_\Omega \Big[a\Big(x, T_k(u_n), \nabla T_k(u_n)\Big) - a\Big(x, T_k(u_n), \nabla T_k(u)\Big)\Big] \times \Big[\nabla T_k(u_n) - \nabla T_k(u)\Big] dx = 0. \quad (5.3.13)$$

The proof of the above proposition is shown in the appendix. Thanks to (5.3.13) and lemma (1.3.5), we have

$$T_k(u_n) \to T_k(u) \quad \text{strongly in} \quad W_0^{1,p(x)}(\Omega, \omega) \quad \forall k \quad (5.3.14)$$

and

$$\nabla u_n \to \nabla u. \quad \text{a.e. in} \quad \Omega,$$

which implies that

$$a(x, T_k(u_n), \nabla T_k(u_n)) \rightharpoonup a(x, T_k(u), \nabla T_k(u)) \quad in \quad (L^{p'(x)}(\Omega, \omega^*))^N. \quad (5.3.15)$$

STEP 4 : **Equi-integrability of the non linearity sequence** :

We shall now prove that : $H_n(x, u_n, \nabla u_n) \to H(x, u, \nabla u)$ strongly in $L^1(\Omega)$, by using Vitali's theorem. Since $H_n(x,u_n,\nabla u_n) \to H(x,u,\nabla u)$ $a.e.\ in\ \Omega$.

Considering now $\varphi = \rho_h(u_n) = \displaystyle\int_0^{u_n} g(s)\chi_{\{s>h\}}ds$ as a test function in (5.3.2), we obtain

$$\int_\Omega a(x,u_n,\nabla u_n)\nabla u_n g(u_n)\chi_{\{u_n>h\}}\exp(G(u_n))dx$$
$$\leq \Big(\int_h^\infty g(s)\chi_{\{s>h\}}ds\Big)\exp\Big(\frac{\|g\|_{L^1(\mathbb{R})}}{\alpha}\Big)\Big[\|f_n\|_{L^1(\Omega)}+\|\gamma\|_{L^1(\Omega)}\Big],$$

using (5.2.3), we have

$$\alpha\int_{\{u_n>h\}}|\nabla u_n|^{p(x)}\omega(x)g(u_n)\exp(G(u_n))dx$$
$$\leq \Big(\int_h^\infty g(s)ds\Big)\exp\Big(\frac{\|g\|_{L^1(\mathbb{R})}}{\alpha}\Big)\Big[\|f_n\|_{L^1(\Omega)}+\|\gamma\|_{L^1(\Omega)}\Big]$$

and since $g\in L^1(\mathbb{R})$, we deduce that

$$\lim_{h\to\infty}\sup_{n\in\mathbb{N}}\int_{\{u_n>h\}}|\nabla u_n|^{p(x)}w(x)g(u_n)dx = 0.$$

Similarly, taking $\varphi = \rho_h(u_n) = \displaystyle\int_{u_n}^0 g(s)\chi_{\{s<-h\}}ds$ as a test function in (5.3.3), we conclude that

$$\lim_{h\to\infty}\sup_{n\in\mathbb{N}}\int_{\{u_n<-h\}}|\nabla u_n|^{p(x)}\omega(x)g(u_n)dx = 0.$$

Consequently,

$$\lim_{h\to\infty}\sup_{n\in\mathbb{N}}\int_{\{|u_n|>h\}}|\nabla u_n|^{p(x)}\omega(x)g(u_n)dx = 0.$$

Which implies, for h large enough and for a subset E of Ω,

$$\lim_{measE\to 0}\int_E|\nabla u_n|^{p(x)}\omega(x)g(u_n)dxdt \leq \|g\|_\infty \lim_{measE\to 0}\int_E|\nabla T_h u_n|^{p(x)}\omega(x)dx$$
$$+\int_{\{|u_n|>h\}}|\nabla u_n|^{p(x)}\omega(x)g(u_n)dx,$$

so $g(u_n)|\nabla u_n|^{p(x)}\omega(x)$ is equi-integrable. Thus, we have shown that

$$g(u_n)|\nabla u_n|^{p(x)}(x)\omega(x) \to g(u)|\nabla u|^{p(x)}(x)\omega(x) \quad \text{stongly in} \quad L^1(\Omega).$$

consequently, by using (5.2.4), we conclude that

$$H_n(x,u_n,\nabla u_n) \to H(x,u,\nabla u) \quad strongly \quad in \quad L^1(\Omega). \tag{5.3.16}$$

STEP 5 : **Passing to the limit** :

a) Proof that u satisfies (5.2.7). For any fixed $m \geq 0$, one has

$$\begin{aligned}
&\int_{\{m\leq|u_n|\leq m+1\}} a(x,u_n,\nabla u_n)\nabla u_n dx \\
&\quad = \int_\Omega a(x,u_n,\nabla u_n)\Big[\nabla T_{m+1}(u_n) - \nabla T_m(u_n)\Big]dx \\
&\qquad = \int_\Omega a(x,T_{m+1}(u_n),\nabla T_{m+1}(u_n))\nabla T_{m+1}(u_n)dx \\
&\qquad\quad - \int_\Omega a(x,T_m(u_n),\nabla T_m(u_n))\nabla T_m(u_n)dx.
\end{aligned}$$

According to (5.3.14) and (5.3.15), one can pass to the limit as $n \to \infty$ for fixed $m \geq 0$ to obtain

$$\begin{aligned}
&\lim_{n\to\infty}\int_{\{m\leq|u_n|\leq m+1\}} a(x,u_n,\nabla u_n)\nabla u_n dx \\
&\quad = \int_\Omega a(x,T_{m+1}(u),\nabla T_{m+1}(u))\nabla T_{m+1}(u) \\
&\qquad - \int_\Omega a(x,T_m(u),\nabla T_m(u))\nabla T_m(u)dx \\
&\qquad\quad = \int_{\{m\leq|u|\leq m+1\}} a(x,u,\nabla u)\nabla u dx. \qquad (5.3.17)
\end{aligned}$$

Taking the limit as $m \to \infty$ in (5.3.17) and using the estimate (5.3.10), shows that u satisfies (5.2.7).

b) Proof that u satisfies (5.2.8)

Let S be a function in $W^{2,\infty}(\mathbb{R})$ with S' has a compact support in $\mathbb{R}$. Let $M > 0$ such that supp$(S')\subset [-M,M]$. Pointwise multiplication of the approximate problem $(P_5)_n$ by $S(u_n)$ leads to :

$$\begin{aligned}
-\mathrm{div}\Big[S'(u_n)a(x,u_n,\nabla u_n)\Big] &+ S''(u_n)a(x,u_n,\nabla u_n)\nabla u_n \\
&+H_n(x,u_n,\nabla u_n)S'(u_n) = f_n S'(u_n) \ \text{ in } \ D'(\Omega). \qquad (5.3.18)
\end{aligned}$$

In was follows we pass to the limit in (5.3.18) as n tends to ∞.

• Limit of $\frac{\partial S(u_n)}{\partial t}$.

Since S is bounded and continuous, $u_n \to u$ a.e. in Ω implies that $S(u_n)$ converges to $S(u)$ a.e. in Ω and L^∞ weakly.

$$Then, \quad \frac{\partial S(u_n)}{\partial t} \to \frac{\partial S(u)}{\partial t} \quad \text{in} \quad D'(\Omega) \quad \text{as } n \to \infty.$$

• Limit of $-\mathrm{div}\Big[S'(u_n)a(x,u_n,\nabla u_n)\Big]$.
Since supp(S')$\subset[-M,M]$, we have for $n\geq M$

$$S'(u_n)a(x,u_n,\nabla u_n)=S'(u_n)a(x,T_M(u_n),\nabla T_M(u_n))\ a.e.\ in\ \Omega.$$

The pointwise convergence of u_n to u and (5.3.15) and the bounded character of S' permit us to conclude that as $n\to\infty$,

$$S'(u_n)a(x,u_n,\nabla u_n)\rightharpoonup S'(u)a(x,T_M(u),\nabla T_M(u))\ in\ (L^{p'(x)}(\Omega,\omega^*))^N. \quad (5.3.19)$$

$S'(u)a(x,T_M(u),\nabla T_M(u))$ has been denoted by $S'(u)a(x,u,\nabla u)$ in equation (5.2.8).
• Limit of $S''(u_n)a(x,u_n,\nabla u_n)\nabla u_n$.
Consider the "energy" term
$S''(u_n)a(x,u_n,\nabla u_n)\nabla u_n=S''(u_n)a(x,T_M(u_n),\nabla T_M(u_n))\nabla T_M(u_n)$ a.e. in Ω.
The pointwise convergence of $S'(u_n)$ to $S'(u)$ and (5.3.14) and (5.3.15) as $n\to\infty$ and the bounded character of S'' permet us to conclude that

$$S''(u_n)a(x,u_n,\nabla u_n)\nabla u_n\rightharpoonup S''(u)a(x,T_M(u),\nabla T_M(u))\nabla T_M(u)\ in\ L^1(\Omega). \quad (5.3.20)$$

Recall that
$S''(u)a(x,T_M(u),\nabla T_M(u))\nabla T_M((u))=S''(u)a(x,u,\nabla u)\nabla u$ a.e. in Ω.
• Limit of $S'(u_n)H_n(x,u_n,\nabla u_n)$.
From supp(S')$\subset[-M,M]$ and (5.3.16), we have

$$S'(u_n)H_n(x,u_n,\nabla u_n)\to S'(u)H(x,u,\nabla u)\text{ strongly in }L^1(\Omega)\text{ as }n\to\infty. \quad (5.3.21)$$

• Limit of $S'(u_n)f_n$.
Since $u_n\to u$ a.e. in Ω, and (5.3.1), we have
$S'(u_n)f_n\to S'(u)f$ strongly in $L^1(\Omega)$, $as\ n\to\infty$.
As a consequence of the above convergence result, we are in a position to pass to the limit as $n\to\infty$ in equation (5.3.18) and to conclude that u satisfies (5.2.8). As a consequence, an Aubin's type Lemma (see, [82] implies that $S(u_n)$ converge to $S(u)$ strongly in $L^1(\Omega)$. As a conclusion, of steps 1 to step 6 the proof of Theorem (5.3.1) is complete.

5.4 Appendix

Proof of lemma 5.3.3.
Set $\varphi = T_1(u_n - T_m(u_n))^+$ in (5.3.2), this function is admissible since $\varphi \in W_0^{1,p(x)}(\Omega, \omega)$ and $\varphi \geq 0$. Then, we have

$$\begin{aligned} \int_{\{m \leq u_n \leq m+1\}} & a(x, u_n, \nabla u_n) \exp(G(u_n)) \nabla u_n dx \\ & \leq \int_\Omega \gamma(x) \exp(G(u_n)) T_1(u_n - T_m(u_n))^+ dx \\ & \quad + \int_\Omega f_n \exp(G(u_n)) T_1(u_n - T_m(u_n))^+ dx. \end{aligned} \tag{5.4.1}$$

Since $f_n \to f$ in $L^1(\Omega)$ and $f_n \exp(G(u_n)) T_1(u_n - T_m(u_n))^+ \leq \exp\left(\frac{\|g\|_{L^1(\mathbb{R})}}{\alpha}\right) f_n$, then by Lebesgue's theorem, we have

$$\lim_{m\to\infty} \lim_{n\to\infty} \int_\Omega f_n \exp(G(u_n)) T_1(u_n - T_m(u_n))^+ dx = 0. \tag{5.4.2}$$

Similarly, since $\gamma \in L^1(\Omega)$, we obtain

$$\lim_{m\to\infty} \lim_{n\to\infty} \int_\Omega \gamma \exp(G(u_n)) T_1(u_n - T_m(u_n))^+ dx = 0. \tag{5.4.3}$$

Together (5.4.1), (5.4.2) and (5.4.3), we conclude that

$$\lim_{m\to\infty} \limsup_{n\to\infty} \int_{\{m \leq u_n \leq m+1\}} a(x, u_n, \nabla u_n) \nabla u_n dx = 0. \tag{5.4.4}$$

On the other hand, taking $\varphi = T_1(u_n - T_m(u_n))^-$ as a test function in (5.3.3) and reasoning as in the proof of (5.4.4), we deduce that

$$\lim_{m\to\infty} \limsup_{n\to\infty} \int_{\{-(m+1) \leq u_n \leq -m\}} a(x, u_n, \nabla u_n) \nabla u_n dx = 0. \tag{5.4.5}$$

Thus, proof of Lemma (5.3.3) follows from (5.4.4) and (5.4.5).
Proof of Proposition 5.3.2.
Assertion (i)
We take $\varphi = T_k(u_n)^+(1 - h_m(u_n))$ (where h_m is defined in step 3) as test function

in (5.3.2), we obtain

$$\begin{aligned}
&\int_{\{m\leq u_n\leq m+1\}} a(x,u_n,\nabla u_n)\exp(G(u_n))\nabla u_n T_k(u_n)^+dx\\
&+\int_{\{T_k(u_n)\geq 0\}} a(x,T_k(u_n),\nabla T_k(u_n))\nabla T_k(u_n)\exp(G(u_n))(1-h_m(u_n))dx\\
&\qquad\leq \int_\Omega \Big(\gamma(x)+f_n\Big)\exp(G(u_n))T_k(u_n)^+(1-h_m(u_n))dx \qquad (5.4.6)
\end{aligned}$$

because

$$\begin{aligned}
&\int_{\{m\leq u_n\leq m+1\}} a(x,u_n,\nabla u_n)\exp(G(u_n))\nabla u_n T_k(u_n)^+dx\\
&\qquad\leq C\int_{\{m\leq u_n\leq m+1\}} a(x,u_n,\nabla u_n)\nabla u_n dx.
\end{aligned}$$

By using (5.2.3) and (5.3.10), the first integral of (5.4.6) converges to 0 as firstly n goes to infinity and m tend to infinity. On the other hand, in view of Lebesgue's theorem, we can deduce that the right hand side of (5.4.6) goes to zero. Hence, by passing to the limit of (5.4.6), becomes

$$\begin{aligned}
\lim_{m\to\infty}\limsup_{n\to\infty}\int_{\{T_k(u_n)\geq 0\}} a\Big(x,T_k(u_n),\nabla T_k(u_n)\Big)\nabla T_k(u_n)\exp(G(u_n))\times\\
(1-h_m(u_n))dx=0
\end{aligned}$$

which gives

$$\lim_{m\to\infty}\limsup_{n\to\infty}\int_{\{T_k(u_n)\geq 0\}} a\Big(x,T_k(u_n),\nabla T_k(u_n)\Big)\nabla T_k(u_n)(1-h_m(u_n))dx=0. \quad (5.4.7)$$

On the other hand, let $\varphi=T_k(u_n)^-(1-h_m(u_n))$ in (5.3.3) as test the function, then similarly to (5.4.7), we deduce that

$$\lim_{m\to\infty}\lim_{n\to\infty}\int_{\{T_k(u_n)\leq 0\}} a\Big(x,T_k(u_n),\nabla T_k(u_n)\Big)\nabla T_k(u_n)(1-h_m(u_n))dx=0. \quad (5.4.8)$$

Finally, combining (5.4.7) and (5.4.8), we conclude assertion (i).

Assertion (ii)

On one hand, let $\varphi=(T_k(u_n)-T_k(u))^+h_m(u_n)\in W_0^{1,p(x)}(\Omega,\omega)\cap L^\infty(\Omega)$, with h_m

is defined in step 3 and $\varphi \geq 0$, then we take φ as test function in (5.3.2), we obtain

$$\int_{\{T_k(u_n)-T_k(u)\geq 0\}} a(x,u_n,\nabla u_n)\nabla\Big(T_k(u_n)-T_k(u)\Big)h_m(u_n)\exp(G(u_n))dx$$
$$-\int_{\{m\leq u_n\leq m+1\}} a(x,u_n,\nabla u_n)\nabla u_n\Big(T_k(u_n)-T_k(u)\Big)^+\exp(G(u_n))dx$$
$$\leq \int_\Omega \Big(\gamma(x)+f_n\Big)\Big(T_k(u_n)-T_k(u)\Big)^+h_m(u_n)\exp(G(u_n))dx.$$

Observe that

$$|\int_{\{m\leq u_n\leq m+1\}} a(x,u_n,\nabla u_n)\nabla u_n\Big(T_k(u_n)-T_k(u)\Big)^+\exp(G(u_n))dx|$$
$$\leq 2k\exp\Big(\frac{\|g\|_{L^1(\mathbb{R})}}{\alpha}\Big)\int_{\{m\leq u_n\leq m+1\}} |a(x,u_n,\nabla u_n)\nabla u_n|dx$$

thanks to (5.3.10) the second integral tend to zero as n and m tend to infinity, and by Lebesgue theorem, we deduce that the right hand side converge to zero as n and m goes to infinity.
In view of (5.3.1), the convergence f_n to f in $L^1(\Omega)$ and $\gamma \in L^1(\Omega)$.
Let $\epsilon_i(n,m)$ $i=1.....,n$ various functions tend to zero as n and m tend to infinity, then we obtain

$$\int_{\{T_k(u_n)-T_k(u)\geq 0\}} a(x,u_n,\nabla u_n)\nabla\Big(T_k(u_n)-T_k(u)\Big)h_m(u_n)\exp(G(u_n))dx$$
$$\leq \epsilon_1(n,m). \tag{5.4.9}$$

Moreover, (5.4.9)

$$\int_{\{T_k(u_n)-T_k(u)\geq 0,|u_n|\leq k\}} a(x,T_k(u_n),\nabla u_n)\nabla\Big(T_k(u_n)-T_k(u)\Big)h_m(u_n)\exp(G(u_n))dx$$
$$-\lim_{m\to\infty}\lim_{n\to\infty}\int_{\{T_k(u_n)-T_k(u)\geq 0,|u_n|\geq k\}} a(x,u_n,\nabla u_n)\nabla T_k(u)h_m(u_n)\exp(G(u_n))dx$$
$$\leq \epsilon_1(n,m).$$

Since

$$|\int_{\{T_k(u_n)-T_k(u)\geq 0,|u_n|\geq k\}} a(x,u_n,\nabla u_n)\nabla T_k(u))h_m(u_n)\exp(G(u_n))dx|$$
$$\leq C_1\int_{\{|u_n|\geq k\}} |a(x,u_n,\nabla u_n)||\nabla T_k|(u)h_m(u_n)dx \to 0\ as\ n\to\ \infty$$

which implies that

$$\int_{\{T_k(u_n)-T_k(u)\geq 0,|u_n|\leq k\}} \exp(G(u_n))a(x,T_k(u_n),\nabla T_k(u_n))\nabla\Big(T_k(u_n)-T_k(u)\Big)h_m(u_n)dx \leq \epsilon_2(n,m)$$

and so that

$$\int_{\{T_k(u_n)-T_k(u)\geq 0\}} a(x,T_k(u_n),\nabla T_k(u_n))\nabla\Big(T_k(u_n)-T_k(u)\Big)h_m(u_n)dx \leq \epsilon_3(n,m). \tag{5.4.10}$$

Moreover

$$\int_{\{T_k(u_n)-T_k(u)\geq 0\}} [a(x,T_k(u_n),\nabla T_k(u_n)) - a(x,T_k(u_n),\nabla T_k(u)]h_m(u_n)dx - \int_{\{T_k(u_n)-T_k(u)\geq 0\}} a(x,T_k(u_n),\nabla T_k(u))[\nabla T_k(u_n)-\nabla T_k(u)]h_m(u_n)dx \leq \epsilon_3(n,m).$$

Since $a(x,T_k(u_n),\nabla T_k(u_n)) \rightharpoonup \varphi$ in $L^{p'(x)}(\Omega,\omega^*)$ and $T_k(u_n) \rightharpoonup T_k(u)$ weakly in $W_0^{1,p(x)}(\Omega,\omega)$ hence, the last integral tend to zero as $n\to+\infty$. Which yields,

$$\lim_{m\to\infty}\limsup_{n\to\infty}\int_{\{T_k(u_n)-T_k(u)\geq 0\}} \Big[a(x,T_k(u_n),\nabla T_k(u_n)) - a(x,T_k(u_n),\nabla T_k(u)\Big]\times \times\Big[\nabla T_k(u_n)-\nabla T_k(u)\Big]h_m(u_n)dx = 0. \tag{5.4.11}$$

On the other hand, take $\varphi = (T_k(u_n)-T_k(u))^- h_m(u_n)$ in (5.3.3). Similarly, we can deduce as in (5.4.11)that

$$\lim_{m\to\infty}\limsup_{n\to\infty}\int_{\{T_k(u_n)-T_k(u)\leq 0\}} \Big[a(x,T_k(u_n),\nabla T_k(u_n)) - a(x,T_k(u_n),\nabla T_k(u)\Big]\times \times\Big[\nabla T_k(u_n)-\nabla T_k(u)\Big]h_m(u_n)dx = 0. \tag{5.4.12}$$

Combining (5.4.11) and (5.4.12), we obtain the desired assertion (ii).
Assertion (iii).

First, we have

$$\int_\Omega \Big[a\Big(x,T_k(u_n),\nabla T_k(u_n)\Big)-a\Big(x,T_k(u_n),\nabla T_k(u)\Big)\Big]\Big[\nabla T_k(u_n)-\nabla T_k(u)\Big]dx$$
$$=\int_\Omega \Big[a\Big(x,T_k(u_n),\nabla T_k(u_n)\Big)-a\Big(x,T_k(u),\nabla T_k(u)\Big)\Big]\times$$
$$\times\Big[\nabla T_k(u_n)-\nabla T_k(u)\Big]h_m(u_n)dx$$
$$+\int_\Omega \Big[a\Big(x,T_k(u_n),\nabla T_k(u_n)\Big)-a\Big(x,T_k(u_n),\nabla T_k(u)\Big)\Big]\times$$
$$\times\Big[\nabla T_k(u_n)-\nabla T_k(u)\Big](1-h_m(u_n))dx.$$

By (5.3.12), the first integral of the right hand side converge to zero as n and m goes to infinity. For the second term, we have

$$\int_\Omega \Big[a\Big(x,T_k(u_n),\nabla T_k(u_n)\Big)-a\Big(x,T_k(u_n),\nabla T_k(u)\Big)\Big]\times$$
$$\times\Big[\nabla T_k(u_n)-\nabla T_k(u)\Big](1-h_m(u_n))dx$$
$$=\int_\Omega a(x,T_k(u_n),\nabla T_k(u_n))\nabla T_k(u_n)(1-h_m(u_n))dx$$
$$-\int_\Omega a(x,T_k(u_n),\nabla T_k(u_n))\nabla T_k(u)(1-h_m(u))dx$$
$$-\int_\Omega a(x,T_k(u_n),\nabla T_k(u))\nabla[T_k(u_n)-T_k(u)\Big](1-h_m(u))dx.$$

From (5.3.11) the first term of the right hand side converge to 0 as $n,m\to+\infty$, and since $\Big(a(x,T_k(u_n),\nabla T_k(u_n))\Big)_n$ in is bounded $(L^{p'(x)}(\Omega,\omega^*))^N$ and $\nabla T_k(u_n)(1-h_m(u))$ converge to zero as n and m goes to infinity. For the third integral it's converge to zero because $\nabla T_k(u_n)\rightharpoonup\nabla T_k(u)$ weakly in $(L^{p(x)}(\Omega,\omega))^N$ and by using Lemma (1.3.5), we deduce that

$$\lim_{n\to\infty}\int_\Omega \Big[a\Big(x,T_k(u_n),\nabla T_k(u_n)\Big)-a\Big(x,T_k(u_n),\nabla T_k(u)\Big)\Big]\times$$
$$\times\Big[\nabla T_k(u_n)-\nabla T_k(u)\Big]dx=0.$$

The proof of Proposition (5.3.2) is complete.

Chapitre 6

Renormalized Solutions of Nonlinear Parabolic Equations in Weigthed Variable-Exponent Space

Abstact

This chapter is devoted to study the existence of renormalized solutions for the nonlinear $p(x)$-parabolic problem in the Weighted- Variable- Exponent Soblev spaces, without the sign condition and the coercivity condition

6.1 Introduction

In the present chapter we establish the existence of renormalized solutions for a class of nonlinear $p(x)$-parabolic equation of the type :

$$\begin{cases} \frac{\partial u}{\partial t} - \operatorname{div}(a(x,t,u,\nabla u)) + H(x,t,u,\nabla u) = f & \text{in } Q = \Omega \times (0,T) \\ u(x,0) = u_0 & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega \times (0,T). \end{cases} \tag{6.1.1}$$

In the problem (6.1.1), Ω is a bounded domain in $\mathbb{R}^N, N \geq 1,\ T$ is a positive real number, while $u_0 \in L^1(\Omega),\ \ f \in L^1(Q)$.
The operator $-\operatorname{div}(a(x,t,u,\nabla u)$ is a Weighted Leray–Lions operator defined on $L^{p^-}(0,T;W_0^{1,p(x)}(\Omega,\omega))$ (see assumption (6.2.1)-(6.2.3) of section 2) which is coercive and where H is a non linear lower order term, satisfying some growth condition

but no sign condition or the coercivity condition and the critical growth condition on H is with respect to ∇u and no growth with respect to u.
For the degenerated parabolic equations the existence of weak solutions have been proved by L. Aharouch and al [3] in the case where a is strictly monotone, $H = 0$ and $f \in L^{p'}(0,T;W_0^{-1,p'}(\Omega,\omega^*))$ and the problem (6.1.1) is studied by Y.Akdim .J.Bennouna. M. Mekkour [9] in the degenerated Weighted Sobolev space with $u = b(x,u)$.
This notion was adapted to the study with $H = 0$ of some nonlinear elliptic problems on Sobolev spaces a exponent variable with Dirichlet boundary conditions by Boccardo.Diaz Giachette and Murat [45] and Lions and Murat (see Lions[65]. In the case where $p(x)$ is a constant, some results have been proved by Y.Akdim and al [7].
Recently, while $a(x,t,u,\nabla u) = |\nabla u|^{p(x)-2}\nabla u$ C. Zhang and S. Zhou [85] proved the existence of a renomalised and entropy solutions with L^1 -data and see also M. Bendahmane, P.Wittbold, A. Zimmermane [29].
This chapter is organized as follows. In Section 2, we give our basic assumption and the definition of a renormalized solution of the problem (6.1.1) for which our problem has a solution. In Section 3, we establish the existence of such a solution in Theorem (6.3.1). In Section 4, we give the proof of Theorem (6.3.2), Lemma (6.3.4) and Proposition (6.3.2) (see appendix).

6.2 Essential Assumption

Throughout the chapter, we assume that the following assumption hold true.
ASSUMPTION (H1)
Let Ω is a bounded open set of $\mathbb{R}^N (N \geq 2),\ p \in C_+(\bar{\Omega}),\ T > 0$ is given and we set $Q = \Omega \times [0,T]$.
We consider a Leray -Lions operator defined by the formula :

$$Au = -\text{div } a(x,t,u,\nabla u)$$

where $a : \Omega \times [0,T] \times \mathbb{R} \times \mathbb{R}^N \to \mathbb{R}$ is a Caratheodory function, i.e., (measurable with respect to x in Ω for every (s,ξ) in $\mathbb{R} \times \mathbb{R}^N$ and continuous with respect to (s,ξ) in $\mathbb{R} \times \mathbb{R}^N$, for almost every x in Ω) which satisfies the following conditions there exist $k \in L^{p'(x)}(Q)$ and $\alpha > 0$, $\beta > 0$ such that for almost every $(x,t) \in Q$ all

$(s,\xi) \in \mathbb{R} \times \mathbb{R}^N$

$$|a(x,t,s,\xi)| \leq \beta\omega^{1/p(x)}(x)\Big[k(x,t) + \omega^{1/p'(x)}|s|^{p(x)-1} + \omega^{1/p'(x)}(x)|\xi|^{p(x)-1}\Big], \quad (6.2.1)$$

$$\Big[a(x,t,s,\xi) - a(x,t,s,\eta)\Big] \cdot (\xi - \eta) > 0 \ \ \forall\, \xi \neq \eta \in \mathbb{R}^N, \quad (6.2.2)$$

$$a(x,t,s,\xi) \cdot \xi \geq \alpha\omega|\xi|^{p(x)}. \quad (6.2.3)$$

ASSUMPTION (H2)

Let $H : \Omega \times [0,T] \times \mathbb{R} \times \mathbb{R}^N \to \mathbb{R}$ is a Carathéodory function such that for a.e. $(x,t) \in Q$ and for all $s \in \mathbb{R}$, $\xi \in \mathbb{R}^N$, the growth condition

$$|H(x,t,s,\xi)| \leq \gamma(x,t) + g(s)\omega|\xi|^{p(x)}, \quad (6.2.4)$$

is satisfied, where $g : \mathbb{R} \to \mathbb{R}^+$ is a bounded continuous positive function that belongs to $L^1(\mathbb{R})$ while $\gamma \in L^1(Q)$.

Definition 6.2.1 *Let $f \in L^1(Q)$ and $u_0 \in L^1(\Omega)$. A real-valued function u defined on Q is renormalized solutions of problem* (6.1.1) *if :*

$$T_k(u) \in L^{p^-}(0,T;W_0^{1,p(.)}(\Omega,\omega)) \text{ for all } k \geq 0,\ u \in C(0,T;L^1(\Omega)), \quad (6.2.5)$$

$$\int_{\{m \leq |u| \leq m+1\}} a(x,t,u,\nabla u)\nabla u dxdt \to 0 \text{ as } m \to \infty, \quad (6.2.6)$$

$$\begin{aligned}\frac{\partial S(u)}{\partial t} - div\Big(S'(u)a(x,t,u,\nabla u)\Big) + S''(u)a(x,t,u,\nabla u)\nabla u \\ +H(x,t,u,\nabla u)S'(u) = fS'(u) \text{ in } D'(Q),\end{aligned} \quad (6.2.7)$$

for all $S \in W^{2,\infty}(\mathbb{R})$ which are piecewise C^1 and such that S' has a compact support in $\mathbb{R}$, where

$$S(u)(|_{t=0}) = S(u_0) \quad in \quad \Omega. \quad (6.2.8)$$

Remark 6.2.1 *Equation (6.2.7) is formally obtained through pointwise multiplication of problem* (6.1.1) *by $S'(u)$. However, while $a(x,t,u,\nabla u)$ and $H(x,t,u,\nabla u)$ do not in general make sense in* (6.1.1), *all the terms in (6.2.7) have a meaning in $D'(Q)$. Indeed, if M is such that supp $S \subset [-M,M]$, the following identifications are made in (6.2.7) :*

– *$S(u)$ belongs to $V \cap L^\infty(Q)$. Since S is a bounded function.*

– *$S'(u)$ $a(x,t,u,\nabla u)$ identifies with $S'(u)$ $a(x,t,T_M(u),\nabla T_M(u))$ a.e. in Q,* *for any* $\varphi \in D(Q)$, *using Hölder inequality*

$$\int_Q S'(u)a(x,t,u,\nabla u)\nabla\varphi dxdt = \int_Q S'(u)a(x,t,T_M(u),\nabla T_M(u))\nabla\varphi dxdt$$
$$\leq C_M\|S'\|_{L^\infty(Q)}\max\Big\{\Big(\int_Q |\nabla T_M(u)|^{p(x)}\omega(x)\Big)^{\frac{1}{p'^-}},$$
$$\Big(\int_Q |\nabla T_M(u)|^{p(x)}\omega(x)\Big)^{\frac{1}{p'^+}}\Big\}\|\nabla\varphi\|_{L^{p(.)}(Q,\omega^*)},$$

where $M > 0$ *is that* $suppS' \subset [-M, M]$. *As* $D(Q)$ *is dense in* V, *we deduce that*

$$div(S'(u)a(x,t,u,\nabla u)) \in V^*.$$

– $S''(u)\ \ a(x,t,u,\nabla u)\nabla u$ *identifies with* $S''(u)\ \ a(x,u,T_M(u),\nabla T_M(u))\nabla T_M(u)$ *and*

$$S''(u)a(x,u,T_M(u),\nabla T_M(u))\nabla T_M(u) \in L^1(Q).$$

– $S'(u)H(x,t,u,\nabla u)$ *identifies with* $S'(u)H(x,t,T_M(u),\nabla T_M(u))$
a.e.in Q *Since* $|T_M(u)| \leq M$ *a.e. in* Q *and* $S'(u) \in L^\infty(Q)$,
we see from (6.2.4) that $S'(u)H(x,t,T_M(u),\nabla T_M(u)) \in L^1(Q)$.
– $S'(u)\ f$ *belongs to* $L^1(Q)$.
The above considerations show that equation (6.2.7) hold in $D'(Q)$ *and* $\frac{\partial S(u)}{\partial t} \in L^1(Q)+L^{(p^+)'}(0,T;W^{-1,p'(x)}(\Omega,\omega^*)))$. *It follows that* u *belongs to* $C(0,T;L^1(\Omega))$ *so that the initial condition (6.2.8) make sense.*

6.3 Existence Results.

Theorem 6.3.1 *Let* $f \in L^1(Q)$, $p(.) \in C_+(\bar{\Omega})$ *and* $u_0 \in L^1(\Omega)$. *Assume that* $(H1)$ *and* $(H2)$ *hold true. Then, there exists a renormalized solution* u *of problem* (6.1.1) *in the sense of Definition (6.2.1).*

Proof. The proof is in five steps.
STEP 1 : Approximate problem :
For $n > 0$, let us define the following respective approximation of H, f, and u_0,

$$\begin{aligned} H_n(x,t,s,\xi) &= \frac{H(x,t,s,\xi)}{1+\frac{1}{n}|H(x,t,s,\xi)|}.\\ \text{Note that}\quad |H_n(x,t,s,\xi)| &\leq |H(x,t,s,\xi)|.\\ \text{and}\quad |H_n(x,t,s,\xi)| &\leq n \text{ for all } (s,\xi) \in \mathbb{R}\times\mathbb{R}^N. \end{aligned}$$

and select f_n, u_{0n} so that

$$f_n \in L^{p'(x)}(Q) \textit{ and } f_n \to f \textit{ a.e. in } Q \textit{ , strongly in } L^1(Q) \textit{ as } n \to \infty. \tag{6.3.1}$$

$$u_{0n} \in D(\Omega) \text{ and } u_{0n} \to u_0 \text{ a.e. in } \Omega \text{ and strongly in } L^1(\Omega) \text{ as } n \to \infty. \tag{6.3.2}$$

Let us now consider the approximate problem

$$(P_6)_n \begin{cases} \frac{\partial u_n}{\partial t} - \operatorname{div}(a(x,t,u_n,\nabla u_n)) + H_n(x,t,u_n,\nabla u_n) = f_n & \text{in } D'(Q), \\ u_n(|_{t=0}) = u_{0n} & \text{in } \Omega \\ u_n = 0 & \text{on } \partial\Omega \times (0,T). \end{cases}$$

Theorem 6.3.2 *Let $f_n \in L^{p'^-}(0,T;W^{-1,p'(x)}(\Omega,\omega^*))$, $p(.) \in C_+(\bar{\Omega})$ for fixed n,the approximate problem $(P_6)_n$ has at least one weak solution $u_n \in L^{p^-}(0,T;W_0^{1,p(x)}(\Omega,\omega))$.*

Proof. See Appendix.

In view of Theorem (6.3.2) there exists at least one weak solution $u_n \in L^{p^-}(0;T;W_0^{1,p(.)}(\Omega,\omega))$ of the problem $(P_6)_n$ See[65].

STEP 2 : A Priori Estimates :

Proposition 6.3.1 *Let u_n a solution of the approximate problem $(P_6)_n$. Then, there exists a constant C(which does not depend on the n and k) such that*

$$\|T_k(u_n)\|_{L^{p^-}(0,T;W_0^{1,p(.)}(\Omega,\omega))} \leq Ck \quad \forall k > 0.$$

Proof.

Let $\varphi \in L^{p^-}(0,T;W_0^{1,p(.)}(\Omega,\omega)) \cap L^\infty(Q)$, with $\varphi > 0$, choosing $v = \exp(G(u_n))\varphi$ as a test function in $(P_6)_n$, where

$$G(s) = \int_0^s \frac{g(r)}{\alpha} dr,$$

the function g appears in (6.2.4), we have

$$\int_Q \frac{u_n}{\partial t} \exp(G(u_n))\varphi dxdt + \int_Q a(x,t,u_n,\nabla u_n)\nabla\Big(\exp(G(u_n))\varphi\Big)dxdt$$
$$+ \int_Q H_n(x,t,u_n,\nabla u_n)\exp(G(u_n))\varphi dxdt = \int_Q f_n \exp(G(u_n))\varphi dxdt.$$

In view of (6.2.4), we obtain

$$
\begin{aligned}
&\int_Q \frac{\partial u_n}{\partial t} \exp(G(u_n))\varphi dxdt \\
&\qquad + \int_Q a(x,t,u_n,\nabla u_n)\nabla u_n \frac{g(u_n)}{\alpha} \exp(G(u_n))\varphi dxdt \\
&\qquad + \int_Q a(x,t,u_n,\nabla u_n) \exp(G(u_n))\nabla\varphi dxdt \\
&\qquad \leq \int_Q [\gamma(x,t) + f_n] \exp(G(u_n))\varphi dxdt \\
&\qquad + \int_Q g(u_n)|\nabla u_n|^{p(x)}\omega(x) \exp(G(u_n))\varphi dxdt.
\end{aligned}
$$

By using (6.2.3), we obtain

$$
\begin{aligned}
\int_Q \frac{\partial u_n}{\partial t} \exp(G(u_n))\varphi dxdt + \int_Q a(x,t,u_n,\nabla u_n) \exp(G(u_n))\nabla\varphi dxdt \\
\leq \int_Q \Big[\gamma(x,t) + f_n\Big] \exp(G(u_n))\varphi dxdt, \qquad (6.3.3)
\end{aligned}
$$

for all $\varphi \in L^{p^-}(0,T; W_0^{1,p(.)}(\Omega,\omega)) \cap L^\infty(Q)$, with $\varphi > 0$.
On the other hand, taking $v = \exp(-G(u_n))\varphi$ as a test function in $(P_6)_n$, we deduce as in (6.3.3), that

$$
\begin{aligned}
\int_Q \frac{\partial u_n}{\partial t} \exp(-G(u_n))\varphi dxdt + \int_Q a(x,t,u_n,\nabla u_n) \exp(-G(u_n))\nabla\varphi dxdt \\
+ \int_Q \gamma(x,t) \exp(-G(u_n))\varphi dxdt \geq \int_Q f_n \exp(-G(u_n))\varphi dxdt, \quad (6.3.4)
\end{aligned}
$$

for all $\varphi \in L^{p^-}(0,T; W_0^{1,p(x)}(\Omega,\omega)) \cap L^\infty(Q)$, with $\varphi > 0$.
Letting $\varphi = T_k(u_n)^+\chi(0,\tau)$, for every $\tau \in [0,T]$ in (6.3.3), we have

$$
\begin{aligned}
&\int_0^\tau \int_\Omega \frac{\partial u_n}{\partial t} \exp(G(u_n))T_k(u_n)^+ dxdt \\
&+ \int_{Q^\tau} a(x,t,u_n,\nabla u_n) \exp(G(u_n))\nabla T_k(u_n)^+ dxdt \\
&\quad \leq \int_{Q^\tau} \Big[\gamma(x,t) + f_n\Big] \exp(G(u_n))T_k(u_n)^+ dxdt \qquad (6.3.5)
\end{aligned}
$$

which gives

$$\int_0^\tau \int_\Omega \frac{\partial u_n}{\partial t} T_k(u_n)^+ \exp(G(u_n)) dxdt$$
$$+ \int_{Q^\tau} a(x,t,u_n,\nabla T_k(u_n)^+) \nabla T_k(u_n)^+ \exp(G(u_n)) dxdt$$
$$\leq \int_{Q^\tau} \Big[\gamma(x,t) + f_n\Big] T_k(u_n)^+ \exp(G(u_n)) dxdt.$$

Since $G(u_n) \leq \frac{\|g\|_{L^1(\mathbb{R})}}{\alpha}$, we have $|\varphi_k(r)| \leq k \exp(\frac{\|g\|_{L^1(\mathbb{R})}}{\alpha})|r|$ where

$$\varphi_k(r) = \int_0^r T_k(s)^+ \exp(G(r)) ds$$

then,

$$\int_{Q^\tau} \frac{\partial u_n}{\partial t} T_k(u_n)^+ \exp(G(u_n)) dxdt$$
$$\geq \int_\Omega \varphi_k(u_n(\tau)) dx - k \exp\Big(\frac{\|g\|_{L^1(\mathbb{R})}}{\alpha}\Big) \|u_{0n}\|_{L^1(\Omega)}.$$

Then, we deduce that

$$\int_\Omega \varphi_k(u_n(\tau)) dx + \int_{Q^\tau} a(x,t,u_n,\nabla T_k(u_n)^+) \exp(G(u_n)) \nabla T_k(u_n)^+ dxdt$$
$$\leq k \exp\Big(\frac{\|g\|_{L^1(\mathbb{R})}}{\alpha}\Big) \Big[\|\gamma\|_{L^1(Q)} + \|f_n\|_{L^1(Q)} + \|u_{0n}\|_{L^1(\Omega)}\Big] \leq C_1 k.$$

Since a satisfies (6.2.3) and by using the fact $\varphi_k(u_n(\tau)) \geq 0$, we get $\forall\ n > 0$

$$\alpha \int_{Q_\tau} |\nabla T_k(u_n)^+|^{p(x)} \omega(x) \exp(G(u_n)) dxdt \leq C_1 k, \tag{6.3.6}$$

where C_1 is a positive constant.
Similarly to (6.3.6), we take $\varphi = T_k(u_n)^- \chi(0,\tau)$ in (6.3.4), we deduce that

$$\int_Q |\nabla T_k(u_n)^-|^{p(x)} \omega(x) dxdt \leq\ C_2 k. \tag{6.3.7}$$

Combining (6.3.6), (6.3.7) and Remark (1.3.2), we conclude that

$$\int_0^T \min\Big\{\|T_k(u_n)\|^{p^+}_{W_0^{1,p(x)}(\Omega,\omega)}, \|T_k(u_n)\|^{p^-}_{W_0^{1,p(x)}(\Omega,\omega)}\Big\} dt \leq \rho(\nabla T_k(u_n)) \leq C_3 k.$$
$$\|T_k(u_n)\|_{L^{p^-}(0,T;W_0^{1,p(x)}(\Omega,\omega))} \leq\ C_4 k. \tag{6.3.8}$$

Where $C_1,\ C_2,\ C_3$ *and* C_4 are constants independent of n. Thus, $T_k(u_n)$ is bounded in $L^{p^-}(0,T;W_0^{1,p(x)}(\Omega,\omega))$ independently of n for any $k>0$. Now we turn to proving the almost everywhere convergence of u_n. Consider a non decreasing function $g_k \in C^2(\mathbb{R})$ such that $g_k(s)=\begin{cases} s & if\ |s|\leq \frac{k}{2} \\ k & if\ |s|\geq k. \end{cases}$

Multiplying the approximate equation by $g_k'(u_n)$, we get

$$\frac{\partial g_k(u_n)}{\partial t} - \mathrm{div}(a(x,t,u_n,\nabla u_n)g_k'(u_n)) + a(x,t,u_n,\nabla u_n)g_k''(u_n)\nabla u_n + H_n(x,t,u_n,\nabla u_n)g_k'(u_n) = f_n g_k'(u_n). \tag{6.3.9}$$

in the sense of distributions. This implies, thanks to the fact g_k' has compact support, that $g_k(u_n)$ is bounded in $L^{p^-}(0,T;W_0^{1,p(x)}(\Omega,\omega))$, while it's time derivative $\frac{\partial g_k(u_n)}{\partial t}$ is bounded in $L^1(Q)+V^*$. Due to the choice of g_k, we conclude that for each k, the sequence $T_k(u_n)$ converges almost everywhere in Q, which implies that the sequence u_n converge almost everywhere to some measurable function v in Q. Thus by using the same argument as in [34], [35], [36], we can show the following lemma.

Lemma 6.3.3 *Let u_n be a solution of the approximate problem $(P_6)_n$ then,*

$$u_n \ \rightarrow\ u\ a.e. \quad in\ Q.$$

We can deduce from (6.3.8) *that*

$$T_k(u_n) \rightharpoonup T_k(u) \quad in \quad L^{p^-}(0,T;W_0^{1,p(x)}(\Omega,\omega)),$$

which implies, by using (6.2.3), *that for all $k>0$ there exists $\varphi_k \in (L^{p'(x)}(Q,\omega^*))^N$ such that*

$$a(x,t,T_k(u_n),\nabla T_k(u_n)) \rightharpoonup \varphi_k \quad in \quad \in \left(L^{p'(x)}(Q,\omega^*)\right)^N. \tag{6.3.10}$$

Lemma 6.3.4 *Let u_n be a solution of the approximate problem $(P_6)_n$. Then,*

$$\lim_{m\to\infty}\limsup_{n\to\infty}\int_{\{m\leq|u_n|\leq m+1\}} a(x,t,u_n,\nabla u_n)\nabla u_n dxdt = 0. \tag{6.3.11}$$

Proof. See Appendix.

STEP 3 : Almost everywhere convergence of the gradients :

This step is devoted to prove the strong convergence of truncation of $T_k(u_n)$ that we will use the following function of one real variable s, which is define as where

$m > k$

$$h_m(s) = \begin{cases} 1 & if \quad |s| \le m \\ 0 & if \quad |s| > m+1 \\ m+1+|s| & if \quad m \le |s| \le m+1. \end{cases}$$

Let $\psi_i \in D(\Omega)$ be a sequence which converges strongly to u_0 in $L^1(\Omega)$.
Set $\nu_\mu^i = (T_k(u))_\mu + e^{-\mu t} T_k(\psi_i)$ where $(T_k(u))_\mu$ is the mollification of $T_k(u)$ with respect to time. Note that ω_μ^i is a smooth function having the following properties :

$$\frac{\partial \nu_\mu^i}{\partial t} = \mu(T_k(u) - \nu_\mu^i), \quad \nu_\mu^i(0) = T_k(\psi_i), \quad |\nu_\mu^i| \le k, \tag{6.3.12}$$

$$\nu_\mu^i \to T_k(u) \quad \text{in } L^{p^-}(0,T; W_0^{1,p(.)}(\Omega,\omega)) \quad \text{as } \mu \to \infty. \tag{6.3.13}$$

The very definition of the sequence ω_μ^i makes it possible to establish the following lemma.

Lemma 6.3.5 *(See[9, 57]). For $k \ge 0$, we have*

$$\int_{\{T_k(u_n)-\nu_\mu^i \ge 0\}} \frac{\partial u_n}{\partial t} \exp(G(u_n))(T_k(u_n) - \nu_\mu^i) h_m(u_n) dxdt \ge \varepsilon(n,m,\mu,i).$$

Proposition 6.3.2 *The subsequence of u_n solution of problem $(P_6)_n$ satisfies for any $k \ge 0$ following assertion :*

$$\lim_{n\to\infty} \int_Q \Big[a(T_k(u_n), \nabla T_k(u_n)) - a(T_k(u_n), \nabla T_k(u))\Big] \cdot \Big[\nabla T_k(u_n) - \nabla T_k(u)\Big] dxdt = 0.$$

Proof. See Appendix.
Thanks to the lemma (1.3.5), we have

$$T_k(u_n) \to T_k(u) \quad \text{strongly in} \quad L^{p^-}(0,T; W_0^{1,p(x)}(\Omega,\omega)) \quad \forall k. \tag{6.3.14}$$

and

$$\nabla u_n \to \nabla u \quad a.e. \ in \ Q,$$

which implies that

$$a(x,t,T_k(u_n), \nabla T_k(u_n)) \rightharpoonup a(x,t,T_k(u), \nabla T_k(u)) \text{ in } (L^{p'(.)}(Q,\omega^*))^N. \tag{6.3.15}$$

STEP 4 : **Equi-integrability of the non linearity sequence** :
We shall now prove that : $H_n(x,t,u_n,\nabla u_n) \to H(x,t,u,\nabla u)$ strongly in $L^1(Q)$. by using Vitali's theorem. Since $H_n(x,t,u_n,\nabla u_n) \to H(x,t,u,\nabla u)$ a.e. in Q,

considering now $\varphi = \rho_h(u_n) = \displaystyle\int_0^{u_n} g(s)\chi_{\{s>h\}}ds$ as a test function in (6.3.3), we obtain

$$\Big[\int_\Omega \theta_h(u_n)dx\Big]_0^T + \int_Q a(x,t,u_n,\nabla u_n)\nabla u_n g(u_n)\chi_{\{u_n>h\}}\exp(G(u_n))dxdt$$
$$\leq \Big(\int_h^\infty g(s)\chi_{\{s>h\}}ds\Big)\exp\Big(\frac{\|g\|_{L^1(\mathbb{R})}}{\alpha}\Big)\Big[\|f_n\|_{L^1(Q)} + \|\gamma\|_{L^1(Q)}\Big],$$

where $\theta_h(r) = \displaystyle\int_0^r \rho_h(\tau)d\tau$, which implies, since $\theta_h \geq 0$ and (6.2.3)

$$\alpha\int_{\{u_n>h\}} |\nabla u_n|^{p(x)}\omega(x)g(u_n)\exp(G(u_n))dxdt$$
$$\leq \Big(\int_h^\infty g(s)ds\Big)\exp\Big(\frac{\|g\|_{L^1(\mathbb{R})}}{\alpha}\Big)\Big[\|f_n\|_{L^1(Q)} + \|\gamma\|_{L^1(Q)} + \|u_{0n}\|_{L^1(\Omega)}\Big]$$

and since $g \in L^1(\mathbb{R})$, we deduce that

$$\lim_{h\to\infty}\sup_{n\in\mathbb{N}}\int_{\{u_n>h\}} |\nabla u_n|^{p(x)}\omega(x)g(u_n)dxdt = 0.$$

Similarly, taking $\varphi = \rho_h(u_n) = \displaystyle\int_{u_n}^0 g(s)\chi_{\{s<-h\}}ds$ as a test function in (6.3.4), we conclude that

$$\lim_{h\to\infty}\sup_{n\in\mathbb{N}}\int_{\{u_n<-h\}} |\nabla u_n|^{p(x)}\omega(x)g(u_n)dxdt = 0.$$

Consequently,

$$\lim_{h\to\infty}\sup_{n\in\mathbb{N}}\int_{\{|u_n|>h\}} |\nabla u_n|^{p(x)}\omega(x)g(u_n)dxdt = 0.$$

Which implies, for h large enough and for a subset E of Q,

$$\lim_{measE\to 0}\int_E |\nabla u_n|^{p(x)}\omega(x)g(u_n)dxdt \leq \|g\|_\infty \lim_{measE\to 0}\int_E |\nabla T_h(u_n)|^{p(x)}\omega(x)dxdt$$
$$+ \int_{\{|u_n|>h\}} |\nabla u_n|^{p(x)}\omega(x)g(u_n)dxdt$$

so $g(u_n)|\nabla u_n|^{p(x)}\omega(x)$ is equi-integrable. Thus we have shown that

$$g(u_n)|\nabla u_n|^{p(x)}(x)\omega(x) \to g(u)|\nabla u|^{p(x)}(x)\omega(x) \text{ stongly in } L^1(Q).$$

Consequently, by using (6.2.4), we conclude that

$$H_n(x,t,u_n,\nabla u_n) \to H(x,t,u,\nabla u) \quad strongly \quad in \quad L^1(Q). \tag{6.3.16}$$

STEP 5 : **Convergence of** $u_n \in C([0,T];L^1(\Omega))$:

Proposition 6.3.3 *The sequence* (u_n) *is a Cauchy sequence in* $C([0,T];L^1(\Omega))$*, moreover,* $u \in C([0,T];L^1(\Omega))$ *and* u_n *converges to* $u \in C([0,T];L^1(\Omega))$.

Proof. Let m and n be two integers, since u_n and u_m are the solutions of the problem $(P_6)_n$. Taking $\varphi = T_1(u_n - u_m)\chi_{[0,s[}$ in $(P_6)_n$, with $s \le T$, we well have $\varphi \in L^{p^-}(0,T;W_0^{1,p(.)}(\Omega,\omega)) \cap L^\infty(Q)$, hence

$$\begin{aligned}
&\int_0^s \int_\Omega \frac{\partial}{\partial t}(u_n - u_m)T_1(u_n-u_m)dxdt \\
&+ \int_Q \Big[a(x,t,u_n,\nabla u_n) - a(x,t,u_m,\nabla u_m)\Big]\nabla\varphi dxdt \\
&+ \int_0^s \int_\Omega \Big[H_n(x,t,u_n,\nabla u_n) - H_n(x,t,u_m,\nabla u_m)\Big]T_1(u_n-u_m)dxdt \\
&= \int_0^s \int_\Omega \Big[f_n - f_m\Big]T_1(u_n-u_m)dxdt \quad .
\end{aligned}$$

Which implies that,

$$\begin{aligned}
&\int_0^s \int_\Omega \frac{\partial}{\partial t}(u_n - u_m)T_1(u_n-u_m)dxdt \\
&+ \int_0^s \int_\Omega \Big[a(x,t,u_n,\nabla u_n) - a(x,t,u_m,\nabla u_m)\Big]\nabla T_1(u_n-u_m)dxdt \\
\le& \int_Q \Big|H_n(x,t,u_n,\nabla u_n) - H_m(x,t,u_m,\nabla u_m)\Big|dxdt \\
&+ \int_Q \Big|f_n - f_m\Big|dxdt.
\end{aligned} \tag{6.3.17}$$

Recall that φ_1 is a primitive function of T_1, thus

$$\int_\Omega \frac{\partial}{\partial t}(u_n-u_m)T_1(u_n-u_m)dx = \frac{\partial}{\partial t}\Big(\int_\Omega \varphi_1(u_n-u_m)\Big).$$

Because of $u_n \in C([0,T];H)$, we can write

$$\int_0^s \int_\Omega \frac{\partial}{\partial t}(u_n-u_m)T_1(u_n-u_m)dxdt = \int_\Omega \varphi_1(u_n-u_m)(s)dx - \int_\Omega \varphi_1(u_n-u_m)(0)dx$$

Remark that, $\nabla T_1(u_n-u_m) = \nabla(u_n-u_m)\chi_{\{|u_n-u_m|\le 1\}}$ and also

$$\int_0^s \int_\Omega \Big[a(x,t,u_n,\nabla u_n) - a(x,t,u_m,\nabla u_m)\Big]\nabla(u_n-u_m)\chi_{\{|u_n-u_m|\le 1\}}dxdt \ge 0,$$

due to the monotonicity of a, then we deduce from (6.3.17) that

$$\int_\Omega \varphi_1(u_n - u_m)(s)dx \le \int_\Omega \varphi_1(u_{0n} - u_{0m})dx + \int_\Omega |H_n - H_m|dxdt + \int_Q |f_n - f_m|dxdt.$$

On the other hand, since $\varphi_1(r) \le |r|$, we get for all $s \le T$

$$\int_\Omega \varphi_1(u_n - u_m)(s)dx \le \int_\Omega |u_{0n} - u_{0m}|dx + \int_\Omega |H_n - H_m|dxdt + \int_Q |f_n - f_m|dxdt. \tag{6.3.18}$$

Denoting $\alpha_{n,m}$ the second membre of (6.3.18) and observe that

$$\begin{aligned} \int_{\{|u_n-u_m|\le 1\}} |u_n - u_m|^2(s)dxdt \;\; + \;\; \int_{\{|u_n-u_m|>1\}} \frac{|u_n - u_m|}{2}(s) \\ \le \int_\Omega \varphi_1(u_n - u_m)(s)dx \end{aligned} \tag{6.3.19}$$

and that

$$\begin{aligned} \int_\Omega |u_n - u_m|(s)dx \;\; &= \;\; \int_{\{|u_n-u_m|<1\}} |u_n - u_m|(s)dxdt \\ &\quad + \int_{\{|u_n-u_m|>1\}} |u_n - u_m|(s)dxdt \\ &\le \Big(\int_{\{|u_n-u_m|\le 1\}} |u_n - u_m|^2(s)dxdt \Big)^{\frac{1}{2}} (meas(\Omega))^{\frac{1}{2}} \\ &\quad + \int_{\{|u_n-u_m|>1\}} |u_n - u_m|(s)dxdt. \end{aligned} \tag{6.3.20}$$

Thus, by (6.3.18), (6.3.19) and (6.3.20), we get

$$\sup_{\{s\in[0,T]\}} \int_\Omega |u_n - u_m|(s)dx \le (2meas(\Omega))^{\frac{1}{2}}(\alpha_{n,m})^{\frac{1}{2}} + 2\alpha_{n,m}.$$

Tanks to (6.3.16) and the fact f_n and u_{0n} converge in L^1, we conclude that,

$$\alpha_{n,m} \to 0 \quad as\ m,\ n\ \to \infty.$$

Hence, u_n is a Cauchy sequence in $C([0,T]; L^1(\Omega))$, also $u \in C([0,T]; L^1(\Omega))$ and for $s \le T$, we have $u_n(s) \to u(s)\ in\ L^1(\Omega)$. This achieves the proof of the proposition.

STEP 6 : Passing to the limit :

a) Proof that u satisfies (6.2.6). For any fixed $m \ge 0$ one has

$$
\begin{aligned}
\int_{\{m\le|u_n|\le m+1\}} & a(x,t,u_n,\nabla u_n)\nabla u_n dxdt \\
&= \int_Q a(x,t,u_n,\nabla u_n)\Big[\nabla T_{m+1}(u_n) - \nabla T_m(u_n)\Big] dxdt \\
&= \int_Q a(x,t,T_{m+1}(u_n),\nabla T_{m+1}(u_n))\nabla T_{m+1}(u_n) dxdt \\
&\quad - \int_Q a(x,t,T_m(u_n),\nabla T_m(u_n))\nabla T_m(u_n) dxdt.
\end{aligned}
$$

According to (6.3.14) and (6.3.15), one can pass to the limit as $n \to \infty$ for fixed $m \ge 0$ and to obtain

$$
\begin{aligned}
\lim_{n\to\infty} \int_{\{m\le|u_n|\le m+1\}} & a(x,t,u_n,\nabla u_n)\nabla u_n dxdt \\
&= \int_Q a(x,t,T_{m+1}(u),\nabla T_{m+1}(u))\nabla T_{m+1}(u) \\
&\quad - \int_Q a(x,t,T_m(u),\nabla T_m(u))\nabla T_m(u) dxdt \\
&= \int_{\{m\le|u|\le m+1\}} a(x,t,u,\nabla u)\nabla u dxdt. \qquad (6.3.21)
\end{aligned}
$$

Taking the limit as $m \to \infty$ in (6.3.21) and using the estimate (6.3.11) shows that u satisfies (6.2.6).

b) Proof that u satisfies (6.2.7).

Let S be a function in $W^{2,\infty}(\mathbb{R})$ with S' has a compact support in $\mathbb{R}$.

Let $M > 0$ such that supp$(S')\subset [-M, M]$. Pointwise multiplication of the approximate problem $(P_6)_n$ by $S(u_n)$ leads to

$$
\begin{aligned}
\frac{\partial S(u_n)}{\partial t} - \operatorname{div}\Big[S'(u_n)a(x,t,u_n,\nabla u_n)\Big] &+ S''(u_n)a(x,t,u_n,\nabla u_n)\nabla u_n \\
&+ H_n(x,t,u_n,\nabla u_n)S'(u_n) = f_n S'(u_n) \text{ in } D'(Q). \qquad (6.3.22)
\end{aligned}
$$

In was follows we pass to the limit in (6.3.22) as n tends to ∞.

• Limit of $\frac{\partial S(u_n)}{\partial t}$.

Since S is bounded and continuous, $u_n \to u$ a.e. in Q implies that $S(u_n)$ converges to $S(u)$ a.e. in Q and L^∞ weakly.

$$
\text{Then,} \quad \frac{\partial S(u_n)}{\partial t} \to \frac{\partial S(u)}{\partial t} \quad \text{in } D'(Q) \text{ as } n \to \infty.
$$

• Limit of $-\mathrm{div}\Big[S'(u_n)a(x,t,u_n,\nabla u_n)\Big]$.
Since supp(S')$\subset [-M,M]$, we have for $n \geq M$

$$S'(u_n)a(x,t,u_n,\nabla u_n) = S'(u_n)a(x,t,T_M(u_n),\nabla T_M(u_n)) \ a.e. \ in \ Q.$$

The pointwise convergence of u_n to u and (6.3.15) and the bounded character of S' permit us to conclude that as $n \to \infty$

$$S'(u_n)a(x,t,u_n,\nabla u_n) \rightharpoonup S'(u)a(x,t,T_M(u),\nabla T_M(u)) \ in \ (L^{p'(\cdot)}(Q,w^*))^N \quad (6.3.23)$$

$S'(u)a(x,t,T_M(u),\nabla T_M(u))$ has been denoted by $S'(u)a(x,t,u,\nabla u)$ in equation (6.2.7).
• Limit of $S''(u_n)a(x,t,u_n,\nabla u_n)\nabla u_n$.
Consider the "energy" term
$S''(u_n)a(x,t,u_n,\nabla u_n)\nabla u_n = S''(u_n)a(x,t,T_M(u_n),\nabla T_M(u_n))\nabla T_M(u_n)$ a.e. in Q.
The pointwise convergence of $S'(u_n)$ to $S'(u)$ and (6.3.14) and (6.3.15) as $n \to \infty$ and the bounded character of S'' permet us to conclude that

$$S''(u_n)a(x,t,u_n,\nabla u_n)\nabla u_n \rightharpoonup S''(u)a(x,t,T_M(u),\nabla T_M(u))\nabla T_M(u) \ in \ L^1(Q). \quad (6.3.24)$$

Recall that
$S''(u)a(x,t,T_M(u),\nabla T_M(u))\nabla T_M((u)) = S''(u)a(x,t,u,\nabla u)\nabla u$ a.e. in Q.
• Limit of $S'(u_n)H_n(x,t,u_n,\nabla u_n)$.
From supp(S')$\subset [-M,M]$ and (6.3.16), we have

$$S'(u_n)H_n(x,t,u_n,\nabla u_n) \to S'(u)H(x,t,u,\nabla u) \ strongly \ in \ L^1(Q) \ as \ n \to \infty. \quad (6.3.25)$$

• Limit of $S'(u_n)f_n$.
Since $u_n \to u$ a.e. in Q, and (6.3.1), we have $S'(u_n)f_n \to S'(u)f$ strongly in $L^1(Q)$, *as* $n \to \infty$.
As a consequence of the above convergence result, we are in a position to pass to the limit as $n \to \infty$ in equation (6.3.22) and to conclude that u satisfies (6.2.7).
As a consequence, an Aubin's type Lemma (see, e.g, [82] implies that $S(u_n)$ lies in a compact set of $C^0([0,T],L^1(\Omega))$.
It follows that on the hand, the smoothness of S implies that
$S(u_n)(|_{t=0}) = S(u_{0n})$ in Ω converge to $S(u)\ |_{t=0}$ strongly in $L^1(\Omega)$ implies that

$S(u)(|_{t=0}) = S(u_0)$ in Ω.
As a conclusion of steps 1 to step 6, the proof of Theorem (6.3.1) is complete. $\square$

6.4 Appendix

Proof of Theorem 6.3.2. We define the operator
$L_n : L^{p^-}(0,T; W_0^{1,p(x)}(\Omega,\omega)) \to L^{p'^-}(0,T; W^{-1,p'(.)}(\Omega,\omega^*))$ by

$$\Big\langle L_n u, v \Big\rangle = \int_Q \frac{\partial u_n}{\partial t} v dx\, dt, \qquad \forall v \in L^{p^-}(0,T; W_0^{1,p(x)}(\Omega,\omega)),$$

according to the Hölder inequality, we have

$$\begin{aligned}
\Big|\Big\langle L_n u, v \Big\rangle\Big| &\leq \int_0^T \Big|\int_\Omega \frac{\partial u_n}{\partial t} \omega^{-\frac{1}{p(x)}} v \omega^{\frac{1}{p(x)}} dx\Big| dt \\
&\leq \Big(\frac{1}{p^-} + \frac{1}{p'^-}\Big) \int_0^T \|\frac{\partial u_n}{\partial t}\|_{W^{-1,p'(x)}(\Omega,\omega^*)} \|v\|_{W_0^{1,p(x)}(\Omega,\omega)} dt \\
&\leq C\Big(\frac{1}{p^-} + \frac{1}{p'^-}\Big)\Big(\int_0^T \|\frac{\partial u_n}{\partial t}\|^{p'^-}_{W^{-1,p'(x)}(\Omega,\omega^*)} dt\Big)^{\frac{1}{p'^-}} \Big(\int_0^T \|v\|^{p^-}_{W_0^{1,p(x)}(\Omega,\omega)} dt\Big)^{\frac{1}{p^-}} \\
&\leq C_1 \|\frac{\partial u_n}{\partial t}\|_{L^{p'^-}(0,T;W^{-1,p'(x)}(\Omega,\omega^*))} \|v\|_{L^{p^-}(0,T;W_0^{1,p(x)}(\Omega,\omega))}. \qquad (6.4.1)
\end{aligned}$$

We define the operator $G_n : L^{p^-}(0,T; W_0^{1,p(.)}(\Omega,\omega)) \to L^{p'^-}(0,T, W^{-1,p'(.)}(\Omega,\omega^*))$ by

$$\Big\langle G_n u, v \Big\rangle = \int_Q H_n(x,t,u,\nabla u) v dx dt \quad \forall u, v \in L^{p^-}(0,T; W_0^{1,p(.)}(\Omega,\omega)).$$

Thanks to the *Hölder* inequality, we have that for $u, v \in L^{p^-}(0,T; W_0^{1,p(.)}(\Omega,\omega))$

$$\begin{aligned}
\Big|\int_Q H_n(x,t,u,\nabla u) v dx dt\Big| &\leq \int_0^T \Big|\int_\Omega H_n(x,t,u,\nabla u) v dx\Big| dt \\
&\leq \int_0^T \Big|\int_\Omega H_n(x,t,u,\nabla u) \omega^{-\frac{1}{p(x)}} v \omega^{\frac{1}{p(x)}} dx\Big| dt \\
&\leq \Big(\frac{1}{p^-} + \frac{1}{p'^-}\Big) \int_0^T \Big(\int_\Omega \Big|H_n(x,t,u,\nabla u)\Big|^{p'(x)} \omega^* dx\Big)^\theta \|v\|_{W_0^{1,p(.)}(\Omega,\omega))} dt \\
&\leq \Big(\frac{1}{p^-} + \frac{1}{p'^-}\Big) \int_0^T n^{\theta p'^+} \Big(\int_\Omega \omega^* dx\Big)^\theta \|v\|_{W_0^{1,p(x)}(\Omega,\omega)} dt \\
&\leq \Big(\frac{1}{p^-} + \frac{1}{p'^-}\Big) n^{\theta p'^+} \Big(\int_\Omega \omega^* dx\Big)^\theta \int_0^T \|v\|_{W_0^{1,p(x)}(\Omega,\omega)} dt \\
&\leq C_2 \|v\|_{L^{p^-}(0,T;W_0^{1,p(.)}(\Omega,\omega))}\ , \qquad (6.4.2)
\end{aligned}$$

with $\theta = \begin{cases} 1/p'^{-} & if \quad \|H_n(x,t,u,\nabla u)\|_{L^1(Q)} > 1 \\ 1/p'^{+} & if \quad \|H_n(x,t,u,\nabla u)\|_{L^1(Q)} \leq 1. \end{cases}$

Lemma 6.4.1 *Let* $B_n : L^{p^-}(0,T;W_0^{1,p(x)}(\Omega,\omega)) \to L^{p'^-}(0,T,W^{-1,p'(x)}(\Omega,\omega^*))$.
The operator $B_n = A + G_n$ *is*
a) coercive
b) pseudo-monotone
c) bounded and hemi continuous.

Proof. a) For the coercivity, we have for any $u \in L^{p^-}(0,T;W_0^{1,p(x)}(\Omega,\omega))$

$$\begin{aligned}
\Big\langle B_n u, u \Big\rangle &= \Big\langle G_n u, u \Big\rangle + \Big\langle Au, u \Big\rangle \\
\Rightarrow \Big\langle B_n u, u \Big\rangle - \Big\langle G_n u, u \Big\rangle &= \Big\langle Au, u \Big\rangle \\
then \quad \Big\langle B_n u, u \Big\rangle - \Big\langle G_n u, u \Big\rangle &= \int_Q a(x,t,u,\nabla u)\nabla u dxdt \\
&= \int_0^T \int_\Omega a(x,t,u,\nabla u)\nabla u dxdt \\
&\geq \int_0^T \alpha(\int_\Omega |\nabla u|^{p(x)}\omega(x)dx)dt \qquad \text{(using (6.2.3))} \\
&\geq \alpha \|\nabla u\|^{\delta}_{L^{p^-}(0,T;W_0^{1,p(x)}(\Omega,\omega))} \\
&\geq \beta \|u\|^{\delta}_{L^{p^-}(0,T;W_0^{1,p(x)}(\Omega,\omega))}
\end{aligned}$$

which is due to Poincaré inequality with
$\delta = \begin{cases} p^- & if \ \|\nabla u\|_{L^{p^-}(0,T;W_0^{1,p(x)}(\Omega,\omega))} > 1 \\ p^+ & if \ \|\nabla u\|_{L^{p^-}(0,T;W_0^{1,p(x)}(\Omega,\omega))} \leq 1 \end{cases}$

$$hence \quad \Big\langle B_n u, u \Big\rangle - \Big\langle G_n u, u \Big\rangle \geq \beta \|u\|^{\delta}_{L^{p^-}(0,T;W_0^{1,p(.)}(\Omega,\omega))}$$

$$then, \ \Big\langle B_n u, u \Big\rangle \geq \beta \| \ u\|^{\delta}_{L^{p^-}(0,T;W_0^{1,p(x)}(\Omega,\omega))} - C_2 \|u\|_{L^{p^-}(0,T;W_0^{1,p(x)}(\Omega,\omega))}$$

$$\text{then, we have } \frac{\Big\langle B_n u, u \Big\rangle}{\|u\|_{L^{p^-}(0,T;W_0^{1,p(x)}(\Omega,\omega))}} \geq \beta \|u\|^{\delta-1}_{L^{p^-}(0,T;W_0^{1,p(x)}(\Omega,\omega))} - C_2 \to +\infty$$

$$\Rightarrow \frac{\Big\langle B_n u, u \Big\rangle}{\|u\|_{L^{p^-}(0,T;W_0^{1,p(x)}(\Omega,\omega))}} \to +\infty \quad as \ \|u\|_{L^{p^-}(0,T;W_0^{1,p(x)}(\Omega,\omega))} \to +\infty$$

then, B_n is coercive.

b) It remains to show that B_n is pseudo-monotone.

Let $(u_k)_k$ a sequence in $L^{p^-}(0,T;W_0^{1,p(x)}(\Omega,\omega))$ such that

$$\begin{aligned} u_k \rightharpoonup u \quad \text{in} \quad L^{p^-}(0,T;W_0^{1,p(x)}(\Omega,\omega)) \\ L_n u_k \rightharpoonup L_n u \quad \text{in} \quad L^{p'^-}(0,T;W^{-1,p'(x)}(\Omega,\omega^*)) \\ \lim_{k\to\infty}\sup \Big\langle B_n u_k, u_k - u\Big\rangle \leq 0 \end{aligned} \tag{6.4.3}$$

then, we have prove that

$$B_n u_k \rightharpoonup B_n u \quad in \quad L^{p'^-}(0,T;W_0^{1,p(x)}(\Omega,\omega)) \quad and\ \langle B_n u_k, u_k\rangle \to \langle B_n u, u\rangle.$$

By the definition of the operator L_n defined in Definition (1.5.2), we obtain that u_k is bounded in $W_0^{1,p(x)}(\Omega,\omega)$ and since $W_0^{1,p(.)}(\Omega,\omega) \hookrightarrow L^{p'(x)}(Q)$ then $u_k \to u$ in $L^{p^-}(0,T;W_0^{1,p(x)}(\Omega,\omega))$, then the growth condition (6.2.1) $(a(x,t,u_k,\nabla u_k))_k$ is bounded in $(L^{p'(x)}(Q,\omega^*))^N$ therefore, there exists a function $\varphi \in (L^{p'(x)}(Q,\omega^*))^N$ such that

$$a(x,t,u_k,\nabla u_k) \rightharpoonup \varphi \ as\ k \to +\infty. \tag{6.4.4}$$

Similarly, using condition (6.2.4) $\Big(H_n(x,t,u_k,\nabla u_k)\Big)_k$ is bounded in $L^1(Q)$ then, there exists a function $\psi_n \in L^1(Q)$ such that

$$H_n(x,t,u_k,\nabla u_k) \to \psi_n \ \text{ in } L^1(Q)\ as\ k \to +\infty. \tag{6.4.5}$$

$$\begin{aligned} \lim_{k\to\infty}\Big\langle B_n u_k, u_k\Big\rangle &= lim_{k\to\infty}\Big[\Big\langle G_n u_k, u_k\Big\rangle + \Big\langle A u_k, u_k\Big\rangle\Big] \\ &= \lim_{k\to\infty}\Big[\int_Q a(x,t,u_k,\nabla u_k)\nabla u_k dxdt + \int_Q H(x,t,u_k,\nabla u_k)u_k dxdt\Big] \\ &= \int_Q \varphi\nabla u_k dxdt + \int_Q \psi_n u_k dxdt \end{aligned} \tag{6.4.6}$$

using (6.4.3) and (6.4.6), we obtain

$$\begin{aligned} \lim_{k\to\infty}\sup\Big\langle B_n u_k, u_k\Big\rangle &= \lim_{k\to\infty}\sup\Big\{\int_Q a(x,t,u_k,\nabla u_k)\nabla u_k dxdt \\ &+ \int_Q H(x,t,u_k,\nabla u_k)u_k dxdt\Big\} \\ &\leq \int_Q \varphi\nabla u dxdt + \int_Q \psi_n u dxdt. \end{aligned} \tag{6.4.7}$$

Thanks to (6.4.5), we have

$$\int_Q H_n(x,t,u_k,\nabla u_k)dxdt \to \int_Q \psi_n dxdt. \tag{6.4.8}$$

Therefore,

$$\lim_{k\to\infty}\sup\int_Q a(x,t,u_k,\nabla u_k)\nabla u_k \leq \int_Q \varphi\nabla u dxdt. \tag{6.4.9}$$

On the other hand, using (6.2.2), we have

$$\int_Q \Big[a(x,t,u_k,\nabla u_k) - a(x,t,u_k,\nabla u)\Big](\nabla u_k - \nabla u)dxdt \geq 0. \tag{6.4.10}$$

Then,

$$\begin{aligned}\int_Q a(x,t,u_k,\nabla u_k)\nabla u_k dxdt \;\;\geq\;\; & -\int_Q a(x,t,u_k,\nabla u)\nabla u dxdt \\ & +\int_Q a(x,t,u_k,\nabla u_k)\nabla u dxdt \\ & +\int_Q a(x,t,u_k,\nabla u)\nabla u_k dxdt\end{aligned}$$

and by (6.4.4), we get

$$\lim_{k\to\infty}\inf\int_Q a(x,t,u_k,\nabla u_k)\nabla u_k dxdt \geq \int_Q \varphi\nabla u dxdt.$$

This implies, thanks to (6.4.9)that

$$\lim_{k\to\infty}\int_Q a(x,t,u_k,\nabla u_k)\nabla u_k dxdt = \int_Q \varphi\nabla u dxdt. \tag{6.4.11}$$

Now by(6.4.11), we can obtain

$$\lim_{k\to\infty}\int_Q a(x,t,u_k,\nabla u_k) - a(x,t,u_k,\nabla u))(\nabla u_k - \nabla u)dxdt = 0.$$

In view of the Lemma (1.3.5), we obtain

$$\begin{aligned} u_k \;\;&\to\;\; u \quad in \quad L^{p^-}(0,T;W_0^{1,p(.)}(\Omega,\omega)) \\ \nabla u_k \;\;&\to\;\; \nabla u \quad a.e.\ in \quad Q.\end{aligned}$$

Then,

$$a(x,t,u_k,\nabla u_k) \rightharpoonup a(x,t,u,\nabla u) \quad in \quad (L^{p'(x)}(Q,\omega^*))^N$$

$$H_n(x,t,u_k,\nabla u_k) \rightharpoonup H_n(x,t,u,\nabla u) \quad in \quad L^1(Q).$$

We deduce that

$$Au_k \rightharpoonup Au \quad in \quad (L^{p'^-}(Q,\omega^*))^N$$

and

$$G_n u_k \rightharpoonup G_n u \quad in \quad L^1(Q)$$

which implies

$$B_n u_k \rightharpoonup B_n u \quad in \quad L^{p'^-}(0,T;W_0^{1,p(x)}(\Omega,\omega))$$

and

$$\Big\langle B_n u_k, u_k \Big\rangle \to \Big\langle B_n u, u \Big\rangle$$

completing the proof of assertion(b).

c) Using $H\ddot{o}lder's$ inequality and the growth condition (6.2.1), we can show that the operator A is bounded, and by using (6.4.2), we conclude that B_n is bounded. For to show that B_n is hemi continuous.

Let $u_k \to u$ in $L^{p^-}(0,T;W_0^{1,p(.)}(\Omega,\omega))$ and prove that

$$\Big\langle B_n u_k, \psi \Big\rangle \to \langle B_n u, \psi\rangle \quad for\ all\ \psi \in\ L^{p^-}(0,T;W_0^{1,p(x)}(\Omega,\omega)).$$

Since $a(x,t,u_k,\nabla u_k) \to a(x,t,u,\nabla u)$ as $k \to \infty$ a.e. in Q, then by the growth condition (6.2.1) and Lemma (1.3.4)

$$a(x,t,u_k,\nabla u_k) \rightharpoonup a(x,t,u,\nabla u)\ in \quad (L^{p'(x)}(Q,\omega^*))^N$$

and for all $\varphi \in L^{p^-}(0,T;W_0^{1,p(x)}(\Omega,\omega))$, $\langle Au_k,\varphi\rangle \to \langle Au,\varphi\rangle$ as $k \to \infty$ similarly, $G_n u_k \to G_n u$ as $k \to \infty$ a.e. in Q then, by the (6.2.4) and Lemma (1.3.4) $G_n u_k \rightharpoonup G_n u$ in $L^{p'(x)}(Q,\omega^*)$ and for all $\phi \in L^{p^-}(0,T;W_0^{1,p(.)}(\Omega,\omega))$, $\Big\langle G_n u_k, \phi \Big\rangle \to \Big\langle G_n u, \phi \Big\rangle$ as $k \to \infty$ which implies B_n is hemi continuous.

Proof of Lemma (6.3.4).

Set $\varphi = T_1(u_n - T_m(u_n))^+ = \alpha_m(u_n)$ as a test function in (6.3.3), this function is admissible since $\varphi \in\ L^{p^-}(0,T;W_0^{1,p(x)}(\Omega,\omega))$ and $\varphi \geq 0$. Then, we have

$$\begin{aligned}
&\int_Q \frac{\partial u_n}{\partial t}\exp(G(u_n))\alpha_m(u_n)dxdt\\
&+\int_{\{m\leq u_n\leq m+1\}} a(x,t,u_n,\nabla u_n)\exp(G(u_n))\nabla u_n dxdt\\
&\leq \int_Q \Big[|\gamma(x,t)|+|f_n|\Big]\exp(G(u_n))\alpha_m(u_n)dxdt.
\end{aligned}$$

Which gives, by setting

$$\varphi_m(s) = \int_0^s \exp(G(r))T_1(r-T_m(r))^+ dr.$$

Observe that $\varphi_m(s) \le \exp\left(\frac{\|g\|_{L^1(\mathbb{R})}}{\alpha}\right)|s|$

$$\begin{aligned}\int_\Omega \varphi_m(u_n(T))dx \ &+ \ \int_{\{m\le u_n\le m+1\}} a(x,t,u_n,\nabla u_n)\exp(G(u_n))\nabla u_n dxdt \\ &\le \exp\left(\frac{\|g\|_{L^1(\mathbb{R})}}{\alpha}\right)\Big[\int_{\{|u_n|>m\}}(|\gamma|+|f_n|)dxdt\Big] \\ &+\int_{\{|u_{0n}|>m\}}|u_{0n}|dx.\end{aligned}$$

Since $\varphi_m(u_n)(T)>0$, $\gamma\in \ L^1(\Omega)$, $f_n\to f$ in $L^1(Q)$ and $u_{0n}\to u_0$ in $L^1(\Omega)$ then, by Lebesgue's theorem, we conclude that

$$\lim_{m\to\infty}\limsup_{n\to\infty}\int_{\{m\le u_n\le m+1\}} a(x,t,u_n,\nabla u_n)\nabla u_n dxdt=0. \tag{6.4.12}$$

On the other hand, taking $\varphi=T_1(u_n-T_m(u_n))^-$ as a test function in (6.3.4) and reasoning as in the proof (6.4.12), we deduce that

$$\lim_{m\to\infty}\limsup_{n\to\infty}\int_{\{-(m+1)\le u_n\le -m\}} a(x,t,u_n,\nabla u_n)\nabla u_n dxdt=0. \tag{6.4.13}$$

By using (6.4.12) and (6.4.13), we have

$$\lim_{m\to\infty}\limsup_{n\to\infty}\int_{\{m\le |u_n|\le m+1\}} a(x,t,u_n,\nabla u_n)\nabla u_n dxdt=0. \tag{6.4.14}$$

Proof of Proposition 6.3.2.

For $m>k$, let $\varphi=(T_k(u_n)-\nu_\mu^i)^+h_m(u_n)\in L^{p^-}(0,T;W_0^{1,p(.)}(\Omega,\omega))\cap L^\infty(Q)$ and $\varphi\ge 0$. If we take this function in (6.3.3), we obtain

$$\begin{aligned}&\int_{\{T_k(u_n)-\nu_\mu^i\ge 0\}}\frac{\partial u_n}{\partial t}\exp(G(u_n))(T_k(u_n)-\nu_\mu^i)h_m(u_n)dxdt \\ &+\int_{\{T_k(u_n)-\nu_\mu^i\ge 0\}} a(x,t,u_n,\nabla u_n)\nabla(T_k(u_n)-\nu_\mu^i)h_m(u_n)dxdt \\ &-\int_{\{m\le u_n\le m+1\}}\exp(G(u_n))a(x,t,u_n,\nabla u_n)\nabla u_n(T_k(u_n)-\nu_\mu^i)^+dxdt \\ &\quad\le \int_Q\Big[f_n+\gamma\Big]\exp(G(u_n))(T_k(u_n)-\nu_\mu^i)^+h_m(u_n)dxdt.\end{aligned} \tag{6.4.15}$$

Observe that

$$\Big|\int_{\{m\leq u_n\leq m+1\}}\exp(G(u_n))a(x,t,u_n,\nabla u_n)\nabla u_n(T_k(u_n)-\nu_\mu^i)^+dxdt\Big|$$
$$\leq 2k\exp\Big(\frac{\|g\|_{L^1(\mathbb{R})}}{\alpha}\Big)\int_{\{m\leq u_n\leq m+1\}}a(x,t,u_n,\nabla u_n)\nabla u_n dxdt.$$

Tanks to (6.3.11) the third and fourth integrals on the right hand side tend to zero as n and m tend to infinity and by Lebesgue's theorem, we deduce that the right hand side converges to zero as $n,\ m$ and μ tend to infinity. Since $\Big(T_k(u_n)-\nu_\mu^i\Big)^+h_m(u_n)\rightharpoonup\Big(T_k(u)-\nu_\mu^i\Big)^+h_m(u)$ in $L^\infty(Q)$ as $n\to\infty$ and strongly in $L^{p^-}(0,T;W_0^{1,p(.)}(\Omega,\omega))$ and $(T_k(u_n)-\nu_\mu^i)^+h_m(u_n)\rightharpoonup\ 0$ in $L^\infty(Q)$ and strongly in $L^{p^-}(0,T;W_0^{1,p(.)}(\Omega,\omega))$ *as* $\mu\to\infty$, it follows that the first and second integrals on the right-hand side of (6.4.15) converge to zeros as $n,\ m,\ \mu\to\infty$, using [51] Lemma (6.3.5) and Lemma (1.3.5) the proof of Proposition (6.3.2) is complete.□

Bibliographie

[1] A. Aberqi, J. Bennouna, M. Mekour, H. Redwane, *Existence results for a nonlinear parabolic problems with lower order terms*. Int. Journal of Math. Analysis, Vol.7, 2013, no. 27, 1323-1340.

[2] R. Aboulaich, B. Achchab, D. Meskine, and A. Souissi, *Parabolic inequalities with nonstandard growths and L^1 data*, Boundary Value Problems, Volume 2006, pp. 1–18.

[3] L. Aharouch, E. Azroul and M. Rhoudaf, *Strongly nonlinear variational degenerated parabolic problems in weighted sobolev spaces*. The Australian journal of Mathematical cal Analysis and Applications, Vol. 5, Article 13, pp 1-25, 2008.

[4] L. Aharouch and Y. Akdim, *nonlinear degenerated equations with Natural growth Terms and L^1-data*, Italian Journal of pure and Applied Mathematics, N, 23(2007) 205-220.

[5] L. Aharouch, A. Benkirane, J. Bennouna, A. Touzani, *Existence and uniqueness of solutions of some nonlinear equations in Orlicz spaces and weighted Sobolev spaces*. Recent developments in nonlinear analysis, 170-180,

[6] Y. Akdim, J. Bennouna, M. Mekkour, H. Redwane, *Existence of renormalized solutions for parabolic equations without the sign condition and with unbounded nonlinearities*. Applicationes Mathematical 39(1), 1-22 .

[7] Y. Akdim, J. Bennouna, M. Mekkour, *Renormalized solutions of nonlinear degenerated parabolic equations with natural growth terms and L^1-data*. International journal of evolution equations, Volume 5, Number 4, pp. 421-446, 2011.

[8] Y. Akdim, J. Bennouna, M. Mekkour and M. Rhoudaf, *Renormalized solution of nonlinear degenerated parabolic problems with L^1-data : existence and uniquness*. Recent developments is nonlinear analysis, proccedings of the conference in Mathematics and Mathematical physics world scientific publishing C.O Ltd.

[9] Y. Akdim, J. Bennouna, M. Mekkour, *Solvability of degenerated parabolic equations without sign condition and three unbounded nonlinearities*. Electronic Journal of Differential Equations, Vol. 2011(2011), No. 03, pp. 1-26.

[10] Y. Akdim, N. El gorch, M. Mekkour, *Existence of Renormalized Solutions for $p(x)$-Parabolic Equations with three Unbounded Nonlinearities*. Bol. Soc. Paran. Mat (3s)V. 34 1(2016) pp 225-252.

[11] Y. Akdim, N. El gorch, M. Mekkour, *Nonlinear parabolic inequalities in Sobolev space with variable exponent*. Journal of Nonlinear Evolution Equations and Application Volume 2015, Number 5, pp. 67-90 (February 2016).

[12] Y. Akdim, C. Allalou and N. El gorch *Renormalized Solutions of Nonlinear Parabolic Equations in Weigthed Variable-Exponent Space*. J.Part. Diff. Eq.Vol. 28, No.3,pp. 225-252.

[13] Y. Akdim, C. Allalou and N. El gorch, *Solvability of Degenerated $p(x)$-Parabolic Equations with three Unbounded Nonlinearities*, Electronic Journal of Mathematical Analysis and Applications, Vol. 4(1) Jan. 2016, pp. 42-62.

[14] Y. Akdim, N. El gorch, M. Mekkour, *Obstacle problem for a non linear $p(x)$ parabolic inequalities*. soumis .

[15] Y. Akdim, C. Allalou , N. El gorch and M. Mekkour *Existence of Renormalised Solutions for Nonlinear Elliptic Problems in Weighthed Variable -Exponent Space with L^1 -Data* soumis .

[16] Y. Akdim, C. Allalou, N. El gorch and Mounir Mekkour *Entropy solutions of nonlinear $p(x)$- parabolic inequalities* soumis.

[17] Youssef Akdim, Chakir Allalou, Nezha El gorch and Mounir Mekkour *Obstacle Problem for Nonlinear Parabolic Inequalities in Weighted Sobolev Spaces* soumis.

[18] Y. Akdim, E. Azroul, and A. Benkirane, *Existence of solution for quasilinear degenerate elliptic unilateral, problems*, Ann. Math ; Blaise Pascal 10(2003), 1-20.

[19] Y. Akdim, E. Azroul, and A. Benkirane, *Existence of solution for quasilinear Degenerate Elliptic Equations* , Electronic J. Diff. Equ. Vol. 2001, N 71,(2001) pp 1-19.

[20] Y. Akdim, E. Azroul, and A. Benkirane, *Existence Results for Quasilinear Degenerated Equations via Strong Convergence of Truncations*. Revista Mathematica Complutence, Madrid. (2004), 17 ; N 2, 359-379.

[21] Y. Akdim, E. Azroul, and A. Benkirane,*Strongly nonlinear degenerated elliptic unilateral problems via convergence of trunctions* . Electronic Journal of Differential Equations conference 09, 2002, pp 25-39.

[22] F.Andreu, J. M. Mazón de León, J. Toledo, *Quasi-linear elliptic and parabolic equations in L^1 with nonlinear boundary conditions*, àparaître à Advances in Math. Sc. and Appl.

[23] S. Antontsev, J.F. Rodrigues, *On stationary thermoheological viscous flows*. Ann. Univ. Ferrara Sez. VII Sci. Mat. 52(2006), no. 1,19-36.

[24] E. Azroul, M. Benboubker, M. Roudaf, *Entropy solutionor some $p(x)$-Quasi linear problem with right-hand side measure*. African Diaspora Journal of Mathematics Vol. 13 Number 2, pp 23-44 (2012)

[25] E. Azroul, H. Redwane and M. Rhoudaf, *Existence of a Renormalized solution for a class of nonlinear parabolic equations in Orlicz spaces*. Portugal. Math. 1, Vol. 66,(2009), 29-63.

[26] E. Azroul, H. Redwane and M. Rhoudaf, *Existence of a Renormalized solution for a class of nonlinear parabolic equations in Orlicz spaces*. Portugal. Math.. 1, Vol. 66, (2009), 29-63.

[27] E. Azroul, M. El lekhlifi ; H. Redwane, A, Touzani, *Entropy solutions of nonlinear parabolic equation in Oricz-Sobolev spaces, without sign condition and L^1 data*. Journal of Nonlinear Evolution Equations and Application Volume 2014, N 7, pp. 101-130.

[28] E. Azroul, A. Barbara and H. Hjiej . *Strongly Nonlinear p(x)-Elliptic Problems with Second Member L1-Dual*. African Diaspora Journal of Mathematics Volume 16, Number 2, pp. 1-22 (2014).

[29] M. Bendahmane, P. Wittbold, A. Zimmermann. *Renormalized solutions for a nonlinear parabolic equation with variable exponents and L^1-data*. J. Differential equations 249(2010)1483-1515 .

[30] M. Bendahmane, P. Wittbold. *Renormalized solutions for nonlinear elliptic equations with variable exponent and L^1-data Nonlinear*. Anal 70(2009)567–583.

[31] A.Benkirane, A.Elmahi. *A stongly nonlinear elliptic equation having natural growth terms and L^1 data, Nonlinear*. Analysis : Theory, Methods and Applications 39(2000),pp, 403-411.

[32] A. Benkirane, A.Elmahi. *Strongly nonlinear elliptic unilateral problems having natural growth terms and L^1 data*. Rendiconti di Matematica, Serie VII vol. 18, (1998), 289-303.

[33] P. Bénlan, L. Boccardo, T. Gallouët, R. Gariepy, M. Pierre and J. L. Vàzquez. *An L^1-theory of existence and uniqueness of solutions of nonlinear elliptic equations*. Ann. Scuola Norm. Sup. Pisa Cl. Sci. 4 (1995), 241-273.

[34] D. Blanchard and F. Murat. *Renormalized solution of nonlinear parabolic problems with L^1-data : existence and uniqueness*. Proceedings of the Royal Society of Edinburgh. A 127(1997) 1137-1152.

[35] D. Blanchard, F. Murat and H. Redwane. *Existence and uniqueness of renormalized solutions for a fairly general class of nonlinear parabolic Problems*. J. Differential equation (177)(2001), 331, 374.

[36] D. Blanchard, and H. Redwane, *Renormalized solutions of nonlinear parabolic evolution problems*. Journal of Matéhmatiques Pures et Appliqueé **77**(1998) pp. 117-151

[37] D. Blanchard, F. Murat and H. Redwane, *Existence et unicité de la solution renormalisée d'un probleme parabolique assey* . général, Comtes Rendus de l'Académie des Sciences-Series I-Mathematics , (329) (2001),no. (1999) pp. 575–580.

[38] D. Blanchard. *Truncations and monotonocity methods for parabolic equations*. Nonlinear Anal.21 (1993),725-43.

[39] L . Boccardo, D. Giachetti. *Strongly Nonlinear Elliptic Equations Having Natural Grawth Terms and L^1 data*. Nonlinrar Analysis Theory Methods and Applications, Vol 19, N 6(1992) 573-579.

[40] L. Boccardo, D. Giachetti,and F. Murat. *Unicité de la solution de certains equations non linéaires*. C.R. Acad. Sci. Paris Série I, 315(1992) 1159-1164.

[41] L. Boccardo, D. Giachetti,and F. Murat. *A unifed presentation of two existence results for problems with natural growth in Progress in PDE*. The Metz surveys 2, M. Chipot editor, Research in Mathematics, Longman, 296(1993), 127-137.

[42] L. Boccardo, D. Giachetti, J.-I. Diaz, F. Murat. *Existence and regularity of renormalized solutions of some elliptic problems involving derivatives of nonlinear terms*. Journal of differential equations 106, pp. 215-237,1993.

[43] L. Boccardo and T. Gallouët. *Nonlinear elliptic and parabolic equations involving measure data*. J. Functional Anal., 87 (1989), 149-169.

[44] L. Boccardo, A. Dall'Aglio, T. Gallouët and L. Orsina. *Nonlinear parabolic equations with measure data*, Journal of Functional Analysis, 147 (1997), no 1, pp. 237–358

[45] L. Boccardo, J.I. Diaz, D. Giachetti, F. Murat. *Existence of a solution for a weaker from of a nonlinear elliptic equations, in : Recent Advance in Nonlinear Elliptic and parabolic problems*. Nancy, 1988, in :Pitman Res. Notes Math. Ser, vol, 208, Longman Sci. Tech, Harlow, 1989, pp, 229-246.

[46] H. Brèzis and W. Strauss. *Semilinear elliptic equations in* L^1. T. Math. Soc. Japan 25 (1973), 565-590.

[47] F. E . Browder. *Existence theory for boundary value problems for quaslinear elliptic systems with strongly nonlinear lower order terms* Proc. Symp. Pure Math. A. M. S. 23(1973) 269-286.

[48] Y. Chen, S. Levine and M. Rao. *Variable exponent, linear growth functionals in image restoration*. SIAM J. Appl, Math, 66(2006), 1383-1406.

[49] A.Dall'Aglio. *Approximated Solutions of Equations with* L^1 *Data*. Application to the H-convergence of quasilinear Parabolic Equations, Ann. Mat. Pura Appl., 170 (1996) 207-240.

[50] A. Dall'Aglio, L. Orsina. *Nonlinear parabolic equations with natural growth conditions and* $^L1$ *data*. Nonlinear Analysis : Theory, Methods and Applications 27 (1996), pp.59-73.

[51] R.J. Diperna, P-L. Lions. *On the Cauchy problem for the Boltzmann equations Global existence and weak stability*. Ann. of Math. 130, pp. 285-366, 1989.

[52] R.-J. DiPerna and P.-L. Lions. *On the cauchy problem for Boltzman equations : global existence and weak stability*. Ann.of Math.(2)130(1989),321-366.

[53] L. Diening, P. Hajulehto, P. Hästö and M. Ruzika, *Lebesque and Sobolev spaces with variable exponents*, Lecture Notes in Mathematics, Volume 2017, Springer-Verlag, Heidelberg, 2001

[54] A. El Khalil, M. Ouanan, A. Touzani. *Existence and regularity of positive solutions for an elliptic system*,2002-Fez conference on Partial Differential Equations,Electronic Journal of Differential Equations, Conference 09, 2003,pp 171-182.

[55] A. El Khalil, S. El Manouni, M. Ouanan. *On some nonlinear elliptic problems for p -Laplacian in* R^N,Nonlinear differ. equ. appl. 15 (2008), 295-307.

[56] X.L Fan, D. Zhao. *On the spaces* $L^{p(x)}(\Omega)$ *and* $W^{m,p(x)}(\Omega)$. J. Math Anal, Appl 263 (2001) 424-446 .

[57] X.L Fan, D. Zhao. *the generalised Orlicz-Sobolev space* $W^{k,p(x)}(\Omega)$. J. Gansu Educ, College 12(1)(1998), 1-6.

[58] P.Harjulehto. *Variable exponent Sobolev space with zero boundary values*. Math. Bohem.132(2007)125-136.

[59] M. Kbiri Alaoui , D. Meskine, A. Souissi, *On some class of nonlinear parabolic inequalities in Orlicz spaces*, Nonlinear Analysis : Theory, Methods & Application 74 (2011), pp. 5863–5875.

[60] O. Kovacik, J. Rakonsik. *on space* $L^{p(x)}(\Omega)$ *and* $W^{1,p(x)}(\Omega)$. czechoslovak Math J. 41(116)(1991)592-618.

[61] Y. H. Kim, L. Wang, C. Zhang. *Global bifurcation for a class of degenerate elliptic equations with variable exponents*. J. Math. Anal. Appl. 371(2010), 624-637.

[62] SN. Kruzhkov. *First order quasi-linear equations in several independent variables*. Math. USSR Sbornik, 10(2) (1970), 217-24.

[63] C. Leone, A. Porretta. *Entropy solutions for nonlinear elliptic equations in* $^L1$. Nonlinear Anal. T.M.A. 32 (1998), 325-334.

[64] J. Leray, J. L. Lions. *Quelque résulte de Višk sur les problèmes elliptiques non linéaires par les méthodes de Minty-Browder*. Bull. Soc. Math. france 93 (1965), 97- 107.

[65] J. L. Lions. *Quelques mèthode de rèsolution des problèmes aux limites non linèaires*. Dunod, Paris 1969.

[66] P. L. Lions, B. Perthame and E. Tadmor. *Formulation cinétique des lois de conservation scalaires multidimensionnelle*. C. R., Acad. Sci. Paris, SÃ©rie I Math, 312 (1991), 97-102.

[67] P. L. Lions. *Mathmatical topics inuid mechanics*. Vol. 1 : Incompressible models, Oxford Lecture Ser.Math.Appl3, Oxford Univesity Press, Oxford 1996.

[68] P. L. Lions and F. Murat. *Sur les solutions renormalisées d'équations elliptiques non linéaires*. 93(1965), 97-107.

[69] C.B. Morry. *Second order elliptic systems of differential equations, contrubitions the theory of P. D. E.* Ann. Math. Studies N33 Princeton, (1954), 101-159.

[70] A. Porretta. *Existence results for nonlinear parabolic equations via strong convergence of truncation.* Ann. Mat. Pura Appl. 177(4)(1999)143-172

[71] A. Porretta. *Existence for elliptic equations in L^1 having lower order terms with natural growth*. Portugaliae. Math., 57 (2000), 179-190.

[72] A. Porretta. *Nonlinear equations with natural growth terms and measure data*. E.J.D.E. conferece 09(2002),183-202.

[73] A. Porretta, Some remarks on the regularity of solutions for a class of elliptic equations with measure data, Houston Journal of Math., 26 (1) (2000), 183-213.

[74] A. Prignet. *Existence and uniqueness of entropy solutions of parabolic problems with L^1 data*. Nonlin. Anal. TMA 28(1997),1943-1954.

[75] J. M. Rakotoson. *Resolution of the critical cases for the problems with L^1-data asymptotic analysis when right hand side is in $L^1(\Omega)$*. 6, pp. 285-293, 1993.

[76] H. Redwane. *Existence of a solution for a class of parabolic equations with three unbounded nonlinearities*. Adv. Dynam. Systems. Appl. 2 (2007). 241-264.

[77] H. Redwane. *Solutions renormalisèes de problèms paraboliques et elliptiques non linèaires*. ph. D. thesis, Rouen,1997.

[78] H. Redwane. *Uniqueness of renormalized solutions for a class of parabolic equations with unbounded nonlinearities* Rendiconti di Matematica, VII, (2008). pp. 189-200.

[79] H. Redwane. *Existence results for a classe of parabolic equations in Orlicz spaces*. Electron. J. Qual. Theory Differential equations 2010, no. 2, 19 pp.

[80] M. Ruzicka. *Electroheological fluides modeling and mathematical theory*. lecture Notes in Mathematics 1748, Springer–Verlaag. Berlin,(2000).

[81] M. Sanchón. J. M. Urbano, *Entropy solutions for the $p(x)$-Laplace equation*, Transactions of the American Mathematical Society, 361(2009), pp. 6387–6405.

[82] J. Simon. *Compact sets in the space $L^p(0,T,B)$*. Ann. Mat. Pura Appl. 146(1987), 65-96.

[83] G. Stampacchia. *Le problème de Dirichlet pour les èquations elliptiques du second ordre 'a coefficients discontinus*. Ann. Inst. Fourier (Grenoble), 15 n. 1 (1965), 189-258.

[84] D. Zhao, W, J. Qiang, X. L Fan. *On generalised Orlicz-Sobolev spaces $L^{p(x)}(\Omega)$* . J. Gansu Sci 9(2), 1997, 1-7.

[85] C. Zang, S. Zhou. *Renormalized and entropy solutions for nonlinear parabolic equations with variable exponents and L^1-data*. J. Differential equations 248(2010)1376-1400.

[86] V.V Zhikov. *on some variational problems*. Russ. J . Math . phys .5(1997)105-116.

[87] V.V Zhikov. *Averaging of functionals of the calculus of variations and elasticity theory*. Math. USSR Izvestya 29(1987), no, 1, 33-66.