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Classical operators on Hardy and weighted Bergman spaces

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To my parents
Mohammed Mkadmi and Lalla Batti Lemrani

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RÉSUMÉ

Cette thèse est consacrée à l'étude de certains opérateurs classiques sur des espaces de fonctions analytiques. Nous nous sommes intéressés, d'une part, aux opérateurs d'intégration définis sur l'espace de Hardy H^2 et sur les espaces de Bergman à poids classiques A^2_α. Nous avons obtenu des estimations des valeurs singulières des opérateurs d'intégration qui sont compacts. Notre approche est basée sur des techniques qui utilisent les suites d'interpolation et d'échantillonnage des espaces de Bergman.

Nous nous intéressons également à une autre classe d'opérateurs, à savoir les opérateurs de composition. Nous avons caractérisé l'appartenance aux classes de Schatten des opérateurs de composition qui agissent sur des espaces de Dirichlet à poids. La classe des poids considérée couvre des poids classiques tels que les poids de Bekollé-Bonami et les poids superharmoniques. Nous avons obtenu également une telle caractérisation sur les espaces de Bergman à poids.

ABSTRACT

This thesis is devoted to the study of certain classical operators on analytic function spaces. We are interested in the class of integration operators acting on the Hardy space H^2 and on Bergman spaces with classical weights A^2_α. We have obtained estimates of the singular values of integration operators that are compact. Our approach is based on techniques using interpolation and sampling sequences for Bergman spaces.

We are also interested in another class of operators, namely composition operators. We characterized the membership in Schatten classes of composition operators acting on weighted Dirichlet spaces. The class of weights considered covers classical weights such as Bekollé-Bonami weights and superharmonic weights. We have also characterized the membership in Schatten classes of composition operators acting on weighted Bergman spaces.

PUBLICATIONS

1. O. El-Fallah, F. Mkadmi, and Y. Omari. “Integration operators on Hardy and Bergman spaces.”, Results in Mathematics 77.5, (2022), p. 200.
2. M. Bourass, I. Marrhich, and F. Mkadmi, “Composition operators on weighted analytic spaces.” Canadian Mathematical Bulletin, 2023, pp. 1–18.

CONTENTS

1

INTRODUCTION

The Hardy spaces are among the most important and well studied function spaces, it has its origins in 1915 in a paper of Hardy [**hardy1915mean**] in which he answered a question of Bohr and Landau, he proved that for any analytic function f on $\mathbb{D}$, its integral mean

$$r \longmapsto \left(\frac{1}{2\pi} \int_0^{2\pi} \left| f(re^{i\theta}) \right|^p d\theta \right)^{1/p}, \quad \text{for } 0 < p < \infty, \tag{1}$$

and

$$r \longmapsto \max_{0 \leqslant \theta < 2\pi} \left| f(re^{i\theta}) \right|$$

is an increasing function on $[0, 1[$. The theory of Hardy spaces has been developed by many authors such as Littlewood, Paley, the Riesz brothers, Carleson, Duren and many others, see for instance [**duren1970theory**, **koosis1998introduction**, **hoffman2007banach**]. Let $\mathbb{D}$ denote the open unit disk of the complex plane. The Hardy space H^p of the unit disk, for $0 < p < \infty$, is the space of analytic functions on $\mathbb{D}$ for which the limit of the integral in expression (1), as r tends to 1^-, is finite. One of the essential characteristics that Hardy spaces functions have is the boundary behavior.

The Bergman spaces, named after Stefan Bergman, are in turn among the well known function spaces. The beginning of the study of function theoretic is dating back to 1970's marked by the works of Horowitz and Korenblum. The evolution of the theory of reproducing kernels began with Bergman, see his book [**bergman1950kernel**]. There are several monographs on Bergman spaces, see for example [**duren2004bergman**, **hedenmalm2012theory**]. The Bergman space A^p of the unit disc, for $0 < p < \infty$, is the subspace of $L^p(\mathbb{D})$ consisting of analytic functions. Let $\omega\colon \mathbb{D} \to (0, \infty)$ be a weight, that is, a non-negative integrable function. The weighted version is the weighted Bergman space A^p_ω that consists of analytic functions f on $\mathbb{D}$ for which

$$\int_{\mathbb{D}} |f|^p \omega dA < \infty,$$

where dA is the Lebesgue area measure on $\mathbb{D}$. The classical weights $\omega_\alpha = (\alpha + 1)\left(1 - |z|^2\right)^\alpha$, with $\alpha > -1$, give rise to the classical weighted Bergman spaces A^p_α. The Bergman spaces are closely related to the Hardy spaces. In fact, we have $H^p \subset \bigcap_{\alpha > -1} A^p_\alpha$. In many respects the Hardy space H^p appears as a degenerate endpoint case of the spaces A^p_α. However, function theoretic properties in Bergman spaces are quite different from those in Hardy spaces since all functions in Hardy spaces have a good boundary behavior than some in Bergman spaces.

This thesis combines between the area of analytic function theory and the area of operator theory. To understand the differences and similarities between spaces of analytic functions, one approach that has proven very fruitful is to study the differences and similarities in the behavior of operators acting on these spaces. Let g be an analytic function on $\mathbb{D}$ and consider the linear transformation

$$I_g f(z) = \int_0^z f(\zeta) g'(\zeta) d\zeta.$$

These operators are called integration operators, the name Volterra type operators also appears in the literature. The function g is said to be the symbol of the operator I_g. If $g(z) = z$ or $g(z) = -\log(1-z)$, then I_g corresponds to the usual Volterra operator or the Cesáro operator respectively. Integration operators have been introduced by Pommerenke in [**Pom1**], he proved that I_g is bounded on H^2 if and only if g is of bounded mean oscillation, that is, $g \in BMOA$. An intensive study has been carried out later by Aleman and Siskakis. Their interest arose at first from studying semigroups of analytic composition operators; in fact, for certain choice of g, the operator I_g is related to the resolvent of such semigroups (see [**siskakis1998semigroups**]).

The study of operators has generally been marked by three types of problems: boundedness, compactness and Schatten classes membership. In [**AS11**, **AS21**], Siskakis and Aleman provide descriptions of boundedness, compactness and Schatten classes membership on Hardy and weighted Bergman spaces. In the boundedness criteria, the space $BMOA$ arise in a natural way on the Hardy spaces while the Bloch space arise on the classical weighted Bergman spaces. In [**PR1**], the authors studied integration operators on weighted Bergman spaces when the weights decay rapidly than the classical weights ω_α. These spaces lie between the Hardy spaces and the classical weighted Bergman spaces. Their results illustrates the smooth transition of boundedness theorems from the setting of the Hardy spaces to the setting of the classical weighted Bergman spaces.

In this thesis, we study integration operators on the Hilbert spaces H^2 and A^2_α. Let X be a separable Hilbert space, if T is a compact operator on X, then there exist orthonormal sets $\{e_n\}$, $\{f_n\}$ and positive numbers $s_1(T) \geqslant s_2(T) \geqslant \ldots$ such that

$$Tf = \sum s_n(T) \langle f, e_n \rangle f_n.$$

By definition, the number $s_n(T)$ is the n-th eigenvalue of the operator $(T^*T)^{1/2}$ and is called the n-th singular value of T. For $p > 0$, the Schatten class $\mathcal{S}_p(X)$ is defined to be the class of compact operators for which $\sum s_n(T)^p < \infty$. The sequence of the singular values of a compact operator decreases to zero. A natural question which arise is about the rate of this convergence. Notice that if $T \in \bigcap_{p>0} \mathcal{S}_p(X)$, then $s_n(T) = o(n^{-1/p})$ for all $p > 0$. This problem has been considered for composition operators in a series of papers [**queffelec_piazza2012**, **QUEFFELEC_seip2015**, **queffelec2018**], it originates from a question raised in 1988 by Sarason about the existence of a compact composition operator that do not belong to any Schatten class $\mathcal{S}_p(H^2)$, that is, a compact composition operators whose singular values have a slow decay. Over the last few years, This problem has been studied for the class of Toeplitz operators acting on weighted Bergman spaces (see [**APP1**, **EMMN2**, **EE2**]).

We obtain estimates of the singular values $s_n(I_g)$ in terms of the averages of $|g'|^2$ over well-known sequences of subsets of $\mathbb{D}$. Our method of estimating is based on techniques using sampling and interpolating sequences for Bergman spaces. On Bergman spaces, the corresponding subsets form a pairwise disjoint covering of $\mathbb{D}$ and their centers form a separated sequence. These subsets play the role of hyperbolic discs. We obtain upper and lower estimates of $s_n(I_g, A^2_\alpha)$. Some complications arise when we replace the Bergman spaces by the Hardy space. The corresponding sets are Carleson sets which appear naturally in boundedness and compactness criteria. We obtain only an upper bound estimate. Under some regularity conditions, we reach an analog lower bound estimate.

In this thesis, we are interested also in the class of composition operators. Let $\mathrm{Hol}(\mathbb{D})$ denote the set of holomorphic functions on $\mathbb{D}$. If φ is an analytic function on $\mathbb{D}$ such that $\varphi(\mathbb{D}) \subset \mathbb{D}$, the composition operator of symbol φ is defined as

$$C_\varphi \colon \mathrm{Hol}(\mathbb{D}) \to \mathrm{Hol}(\mathbb{D}), \quad f \mapsto C_\varphi(f) = f \circ \varphi.$$

This operator appears for the first time implicitly, in 1871, in the work of schröder [**schroder1871iterierte**]. The problem consists of finding functions f and constants λ solutions of the equation

$$f \circ \varphi = \lambda f.$$

Koenigs [**koenigs1884recherches**] solved this problem, in 1884, in the case when the domain is the unit disc. The study of the spectrum of composition operators began with Nordgren's work [**nordgren1968composition**], he described the spectrum when the symbol is an automorphism of the disc. Schwartz developed further the theory, he obtained in his thesis [**schwartz1969composition**] several properties concerning boundedness, compactness and the spectrum.

Since then, the literature has grown. The most frequent questions often relate the geometrical properties of the symbol φ to the theoretic properties of C_φ. Some questions include whether C_φ is compact or in Schatten classes, the answers depend on how the symbol φ behaves on the boundary of its domain. The first results to emerge in this direction are on the Hardy space H^2, Littlewood's subordination theorem [**littlewood1925inequalities**] guaranties that the operator C_φ is always bounded on H^2. In 1992, Shapiro [**Joel2**] characterized the compactness on H^2 in terms of the Nevanlinna counting function N_φ given by

$$N_\varphi(w) = \sum_{w=\varphi(z)} \log\left(\frac{1}{|z|}\right) \quad \text{if } w \in \varphi(\mathbb{D}); \qquad N_\varphi(w) = 0 \quad \text{if } w \notin \varphi(\mathbb{D}).$$

He proved that C_φ is compact on H^2 if and only if

$$\frac{N_\varphi(w)}{1-|w|^2} \to 0 \quad \text{as} \quad |w| \to 1^-.$$

In case φ is univalent, this is equivalent to the fact that φ has no angular derivative at 1. composition operators were the subject of several papers on various spaces of analytic functions, see for instance [**Zor2**, **PP12**, **EE2**, **LLQR2**, **SS2**]. For general information on composition operators on spaces of analytic functions, we refer the reader to the monographs by Shapiro [**JoelBook2**] and by Cowen and MacCluer [**CM2**].

In this thesis, we study composition operators C_φ acting on weighted Dirichlet spaces of the unit disc. Let ω be a weight on $\mathbb{D}$, the associated Dirichlet space $\mathcal{D}_\omega$ is the space of analytic functions on $\mathbb{D}$ for which

$$\int_{\mathbb{D}} \left|f'(z)\right|^2 \omega(z) dA(z) < \infty.$$

- For $\omega \equiv 1$, the space $\mathcal{D}_\omega$ corresponds to the classical Dirichlet space.
- The radial weights $\omega_\alpha(z) = (1-|z|^2)^\alpha$, for $-1 < \alpha < 1$, generate the standard weighted Dirichlet spaces $\mathcal{D}_\alpha$.
- For $\omega(z) = \log\left(\frac{1}{|z|^2}\right)$, by Littlewood Paley identity (see 2.5), we recover the classical Hardy space H^2.

A well-known class of weights that are not necessarily radial is the class of Békollé-Bonami weights; for $p_0 > 1$ and $\eta > -1$, the class $B_{p_0}(\eta)$ consists of weights ω for which there exists a constant $C > 0$ such that

$$\left(\int_{W(I)} \omega(z) dA_\eta(z)\right)^{1/p_0} \left(\int_{W(I)} (\omega(z))^{-p_0'/p_0} dA_\eta(z)\right)^{1/p_0'} \leqslant CA_\eta(W(I))$$

for any Carleson square

$$W(I) = \{z \in \mathbb{D}\colon 1 - |z| < l(I)/2\pi,\ z/|z| \in I\}, \quad I \subset \mathbb{T} := \partial\mathbb{D}$$

where $dA_\eta(z) = (1-|z|^2)^\eta dA(z)$, $\frac{1}{p_0} + \frac{1}{p_0'} = 1$ and $l(I)$ denotes the length of I. In **[bekolle1982inegalites]**, Bekollé showed that this condition is necessary and sufficient that the weighted Bergman projection

$$P_\eta f(z) = \int_{\mathbb{D}} \frac{f(\xi)}{(1-\bar{z}\xi)^{\eta+2}} dA_\eta(z)$$

be bounded on $L^{p_0}(\omega\, dA_\eta)$. We define a larger class of weights. A function weight ω is said to belong to the class $\mathcal{C}_{p_0}$, for $p_0 > 1$, if there exists a constant $C > 0$ such that

$$\left(\int_{\Delta(z,\alpha)} \omega(z) dA(z)\right)^{1/p_0} \left(\int_{\Delta(z,\alpha)} (\omega(z))^{-p_0'/p_0}\, dA(z)\right)^{1/p_0'} \leqslant C|\Delta(z,\alpha)|$$

where $\Delta(z,\alpha) = \{w \in \mathbb{D}\colon |z-w| < \alpha(1-|z|)\}$ and $\alpha \in (0,1)$. This condition is independent of α (see **[luecking1985representation2]**). Note that the class $B_{p_0}(\eta)$ is included in the class $\mathcal{C}_{p_0}$ for every $\eta > -1$ since if α is given then for each $z \in \mathbb{D}$ there exists an arc I (depending on z) such that $\Delta(z,\alpha) \subset W(I)$.

A weight ω is said to satisfy the Littlewood-Paley formula, and we write $\omega \in \mathcal{HL}$, if for all $p > 0$, $n \in \mathbb{N}^*$ and for each analytic function f on $\mathbb{D}$ we have

$$\int_{\mathbb{D}} |f(z)|^p \omega(z) dA(z) \asymp \sum_{k=0}^{n-1} \left|f^{(k)}(0)\right|^p + \int_{\mathbb{D}} \left|f^{(n)}(z)\right|^p \left(1-|z|^2\right)^{np} \omega(z) dA(z).$$

In **[AC12]**, Aleman and Constantin proved that if $\dfrac{\omega}{(1-|z|^2)^\eta} \in B_{p_0}(\eta)$, for some $p_0 > 1$ and $\eta > -1$, then $\omega \in \mathcal{HL}$. For the converse, it is proved in **[Aleman20192]** that if $\omega \in \mathcal{HL}$ and satisfies

$$\omega(z) \asymp \omega(w), \quad |z-w| < \delta(1-|z|^2), \quad (\text{for some} \quad \delta \in (0,1)),$$

then $\dfrac{\omega}{(1-|z|^2)^\eta} \in B_{p_0}(\eta)$, for some $p_0 > 1$ and $\eta > -1$. In **[BWZ20182]**, they have introduced a new class of weights that satisfy Littlewood-Paley formula; for $s \in (-1,0]$ and $t \geqslant 0$, the set $\mathcal{W}_{s,t}$ consists of weights ω such that

$$\omega_{s,t}(z) := \int_{\mathbb{D}} \frac{\omega(\xi)(1-|\xi|^2)^s(1-|z|^2)^t}{|1-\bar{\xi}z|^{2+s+t}} dA(\xi) \lesssim \omega(z), \quad (z \in \mathbb{D}).$$

Consider the weight

$$\tilde{\omega}(z) := \frac{1}{(1-|z|^2)^2}\int_{\Delta(z,1/2)} \omega(\xi)dA(\xi), \quad z \in \mathbb{D}.$$

In [**BMvolterra2**], they proved that if $\omega \in \mathcal{C}_{p_0}$ for some $p_0 > 1$, then

$$\omega \in \mathcal{HL} \quad \text{if and only if} \quad \omega_t \lesssim \tilde{\omega} \quad (\text{for some} \quad t \geqslant 0),$$

where ω_t denotes $\omega_{0,t}$. The class of such weights, denoted by $\mathcal{C}_{p_0,t}$, is introduced in [**BMvolterra2**] by M. Bourass and I. Marrhich. They studied Toeplitz operators acting on Bergman spaces induced by such weights. One of the key tools in their work is the fact that for $\omega \in \mathcal{C}_{p_0}$, $p_0 > 1$, we have

$$A^2_\omega = A^2_{\tilde{\omega}}.$$

Moreover,

$$\|f\|_{A^2_\omega} \asymp \|f\|_{A^2_{\tilde{\omega}}}, \quad f \in \mathrm{Hol}(\mathbb{D}).$$

Note that the weight $\tilde{\omega}$ is invariant, that is

$$\tilde{\omega}(\zeta) \asymp \tilde{\omega}(z), \quad \text{for} \quad \zeta \in \Delta(z,r)$$

for some (or equivalently for all) $r \in (0,1)$ (see [**CONSTANTIN20102**]). If, in addition, $\omega \in \mathcal{C}_{p_0,t}$ for some $t \geqslant 0$, then the weight $\tilde{\omega}$ satisfies more regularity conditions. In particular, $\tilde{\omega}$ belongs to a certain class of weights for which several studies have been carried out (see [**EMMN2**, **EE2**]). In this case, one can say that the wight $\tilde{\omega}$ is a kind of "regularization" of the weight ω.

We study composition operators acting on Dirichlet spaces induced by weights in the class $\mathcal{C}_{p_0,t}$ for some $p_0 > 1$ and $t \geqslant 0$. Note that The space $\mathcal{D}_\omega$ endowed with the norm

$$\|f\|_{\mathcal{D}_\omega} := \left(|f(0)|^2 + \int_{\mathbb{D}} \left|f'(z)\right|^2 \omega(z)dA(z)\right)^{1/2}$$

is a Hilbert space, see Lemma 4.1. We obtain characterizations of the boundedness, compactness and Schatten classes membership. In [**SS2**], D. Sarason and O. Silva characterized boundedness and compactness of operators C_φ acting on a weighted Dirichlet space induced with a hamonic weight. In [**EMMN22**], O. El-Fallah, H. Mahzouli, I. Marrhich and H. Naqos provide a characterization of the membership in the Schatten ideals. We examine the case of perturbed superharmonic weights defined as $\omega(z) = (1-|z|^2)^\alpha u(z)$, where $\alpha > -1$ and $u \in \mathcal{C}^2(\mathbb{D})$ a positive superharmonic function on $\mathbb{D}$. We are interested also in composition operators acting on weighted Bergman spaces. In particular, we extend the result obtained by O. Constantin in

[CONSTANTIN20102] concerning the membership of C_φ to $\mathcal{S}_p(A^2_\omega)$, from the case of $p \geqslant 2$ and the Bekollé-Bonami weights setting, to the general case of all $p > 0$ and for $\omega \in \mathcal{C}_{p_0,t}$.

On the way to the proof of the main results of this thesis, many problems have been reformulated in terms of Toeplitz operators which represents one of the most significant operators. For a given Hilbert space of analytic functions X with reproducing kernel K, we define the operator

$$T_\mu f(z) = \int f(w)K(z,w)d\mu(w),$$

where μ is a Borel measure. On Bergman spaces and for a measure $\mu = \varphi dA$, where $\varphi \in L^\infty$, the operator T_μ corresponds to the classical Toeplitz operator on Bergman spaces $T_\varphi f = P(\varphi f)$, where P is the Bergman projection. The classical Toeplitz operators have been extensively studied since the seventies **[coburn1973, McDonald1979, zhu1987vmo]**. Luecking **[LUECKING1987]** was probably the first author to introduce operators T_μ with measures as symbols which are commonly given the same name. This type of Toeplitz operators have been the subject of intensive studies during the last decades. Their interest lies in the fact that they are closely related to other well known classes of operators (composition operators, integration operators, Hankel operators ...). We made use of different connections between Toeplitz operators and integration and composition operators.

The remainder of this thesis consists of three chapters. Chapter 2 contains some background and some basic tools. In the second Chapter 3, we study integration operators on Hardy and weighted Bergman spaces, it contains the results of paper **[el2022integration]**. Chapter 4 is devoted to the study of composition operators on weighted Dirichlet space, it contains the results of paper **[bourass2023composition]**.

2

BACKGROUND

2.1 REPRODUCING KERNEL HILBERT SPACES

Let H be a Hilbert space of functions on some domain Ω of $\mathbb{C}$. We say that H is a reproducing kernel Hilbert space on Ω if for each $z \in Omega$ the mapping

$$E_z : H \to \mathbb{C}, \quad f \mapsto f(z)$$

is a bounded linear functional on H.

By the Riesz representation theorem, for each $z \in Omega$, there exists a unique $K_z \in H$ (the so-called reproducing kernel) such that

$$f(z) = \langle f, K_z \rangle_H, \quad (f \in H)$$

Remark that

$$\|K_z\|_H^2 = \langle K_z, K_z \rangle_H = K_z(z).$$

Theorem 2.1. *Suppose that* $\{e_i\}_{I \in I}$ *is an orthonormal basis of* H. *Then*

$$K_z(w) = \sum_{i \in I}^{\infty} e_i(w)\overline{e_i(z)}.$$

$K_z(w)$ *is independent of the choice of the orthonormal basis* $\{e_n\}_{n \geqslant 1}$.

2.2 HARDY SPACES

Definition 2.2.

For $0 < p < \infty$, the Hardy space $H^p = H^p(\mathbb{D})$ is the space of analytic functions f on $\mathbb{D}$ such that

$$\|f\|_{H^p} := \sup_{0 \leqslant r < 1} \left(\int_{\mathbb{T}} |f(r\zeta)|^p \, dm(\zeta) \right)^{1/p} < \infty,$$

and $H^\infty = H^\infty(\mathbb{D})$ is the space of analytic functions f on $\mathbb{D}$ such that

$$\|f\|_{H^\infty} := \sup_{z \in \mathbb{D}} |f(z)| < \infty.$$

where m is the normalized Lebesgue measure on $\mathbb{T}$. If $1 \leqslant p \leqslant \infty$, the space H^p is a Banach space.

Theorem 2.3 (**[duren1970theory]**)**.** *Let* $f \in H^p$, $0 < p \leqslant \infty$. *Then the radial limit*

$$f^*(\zeta) = \lim_{r \to 1^-} f(r\zeta)$$

exists for almost every ζ *in* $\mathbb{T}$. *Moreover,* $\|f\|_{H^p} = \|f^*\|_{L^p(\mathbb{T})}$.

The Hardy space H^2 is a Hilbert space with the inner product

$$\langle f, g\rangle_{H^2} = \lim_{r \to 1} \int_{\mathbb{T}} f(r\zeta)\, \overline{g(r\zeta)} dm(\zeta).$$

If $f(z) = \sum_{n=0}^{\infty} a_n z^n$ and $g(z) = \sum_{n=0}^{\infty} b_n z^n$ belongs to H^2, then

$$\langle f, g\rangle_{H^2} = \sum_{n=0}^{\infty} a_n \overline{b_n}.$$

In particular

$$\|f\|_{H^2} = \sum_{n=0}^{\infty} |a_n|^2.$$

Proposition 2.4 (**[duren1970theory]**)**.** *In the Hardy space* H^2, *the point evaluations are bounded.*

Then, for each $z \in \mathbb{D}$, there exists $K_z \in H^2$ such that

$$f(z) = \langle f, K_z\rangle_{H^2}.$$

The sequence $\{e^n\}_{n \geqslant 0}$, where $e_n(z) = z^n$ is an othonormal basis of H^2. Thus

$$K_z(w) = \frac{1}{1 - \bar{z}w} \quad z, w \in \mathbb{D},$$

and $\|K_z\|^2_{H^2} = \dfrac{1}{1 - |z|^2}$.

The following identity **[Gar1]** provide an equivalent norm in terms of the derivative.

Theorem 2.5 (Littlewood-Paley identity)**.** *If* f *is analytic in* $\mathbb{D}$, *then*

$$\|f\|^2_{H^2} - |f(0)|^2 = \int_{\mathbb{D}} |f'(z)|^2 \log \frac{1}{|z|^2} dA(z),$$

where dA *is the normalized Lebesgue measure on* $\mathbb{D}$.

2.3 WEIGHTED BERGMAN SPACES

Definition 2.6. Let ω be a weight, that is, a non-negative integrable function. For $0 < p < \infty$ the associated weighted Bergman space, A^p_ω, is the space of analytic functions on $\mathbb{D}$ such that

$$\|f\|_{A^p_\omega} := \left(\int_D |f(z)|^p \omega(z) dA(z)\right)^{1/p} < \infty.$$

The Bergman spaces induced by the classical $\omega_\alpha(z) = (\alpha+1)\left(1-|z|^2\right)^\alpha$, with $\alpha > -1$, are denoted by A^p_α.

For $p \geqslant 1$, A^p_α is a Banach space. The space A^2_α is a Hilbert space with inner product

$$\langle f, g\rangle_{A^\alpha_2} = \int_{\mathbb{D}} f(z)\overline{g(z)} dA_\alpha(z),$$

where $dA_\alpha(z) = \omega_\alpha(z) dA(z)$. If $f(z) = \sum_{n=0}^{\infty} a_n z^n$ and $g(z) = \sum_{n=0}^{\infty} b_n z^n$ belongs to A^2_α, then

$$\langle f, g\rangle_{A^2_\alpha} = \sum_{n=0}^{\infty} \frac{a_n \overline{b_n}}{(n+1)^{\alpha+1}}.$$

Moreover, the space A^2_α is a reproducing kernel Hilbert space with kernel

$$K^\alpha(z, w) = \frac{1}{(1-z\bar{w})^{\alpha+2}}.$$

There exists a type of Littlewood-Paley identity for the Bergman space A^2_α.

Proposition 2.7 ([**Zhu1**]). *Let* $f \in A^2_\alpha$, *then we have*

$$\|f\|^2_{A^2_\alpha} - |f(0)|^2 \asymp \int_{\mathbb{D}} |f'(z)|^2 (1-|z|^2)^{2+\alpha} dA(z).$$

2.4 THE BERGMAN METRIC

The pseudohyperbolic metric on $\mathbb{D}$ is defined by

$$\rho(z, w) = \left|\frac{z-w}{1-\bar{z}w}\right|, \quad z, w \in \mathbb{D},$$

and the hyperbolic metric, also called the Bergman metric, is defined by

$$\beta(z, w) = \frac{1}{2}\log\frac{1+\rho(z, w)}{1-\rho(z, w)}, \quad z, w \in \mathbb{D}.$$

For $r \in (0,1)$, the set

$$\Lambda(z,r) = \{w \in \mathbb{D} \colon \beta(z,w) < r\}$$

is called the hyperbolic disk of radius r and center z. Note that $\Lambda(z,r)$ is a Euclidean disk with center

$$\frac{(1-s^2)}{(1-s^2|z|^2)}z \quad \text{and radius} \quad \frac{(1-|z|^2)}{(1-s^2|z|^2)}s,$$

where $s = \tanh r \in (0,1)$.
Let $|\Lambda(z,r)|$ denote the normalized area of $\Lambda(z,r)$, we have the following results.

Lemma 2.8. *Suppose r and R positive numbers. Then there exists a constant $C > 0$ such that for all $z, w \in \mathbb{D}$, we have*

1. $C^{-1}\left(1-|z|^2\right) \leqslant |1 - z\bar{w}| \leqslant C\left(1-|z|^2\right)$ *when* $\beta(z,w) \leqslant r$.
2. $C^{-1}|\Lambda(z,r)|_A \leqslant |D(w,s)|_A \leqslant C|D(z,r)|_A$ *when* $\beta(z,w) \leqslant R$.

Here $s = \tanh r$.

Lemma 2.9. *Let $0 < r < 1$. There exists a positive integer M and a sequence $\{z_n\}_n$ in $\mathbb{D}$ such that*

1. *The disk $\mathbb{D}$ is covered by $\{\Lambda(z_n, r)\}_n$.*
2. *Every point in $\mathbb{D}$ belongs to at most N sets in $\{\Lambda(z_n, 2r)\}_n$.*
3. *If $n \neq m$, then $\beta(z_n, z_m) \geqslant r/2$.*

Such sequence $\{z_n\}$ is called an r-lattice.

Lemma 2.10. *Let f be an analytic function on $\mathbb{D}$, and let $0 < r < 1$. Then there exists a positive constant C such that*

$$|f(z)|^p \leqslant \frac{C}{|\Lambda(z,r)|}\int_{\Lambda(z,r)} |f(w)|^p dA(w), \quad z \in \mathbb{D},$$

holds for all $0 < p < \infty$.

For further information see the monograph **[Zhu1]**.

Remark 2.11. Note that the hyperbolic disks can be replaced equivalently by the sets

$$\Delta(z,r) := \left\{w \in \mathbb{D} \colon |z-w| < r(1-|z|^2)\right\},$$

where $z \in \mathbb{D}$ and $r \in (0,1)$.

2.5 SAMPLING AND INTERPOLATING SEQUENCES

Let H be a reproducing kernel Hilbert of analytic functions on $\mathbb{D}$. For $z \in \mathbb{D}$, let K_z denote the reproducing kernel of H at z and denote $k_z = \frac{K_z}{\|K_z\|_H}$.

2.5.1 *Interpolating sequences*

Definition 2.12. A sequence $\Gamma = \{\gamma_n\}_n$ of points in $\mathbb{D}$ is called an interpolating set for H if for every $a = (a_n)$ such that

$$\sum_n \frac{|a_n|^2}{\|K_z\|_H^2} < \infty,$$

there exists f in H such that

$$\langle f, k_{\gamma_n} \rangle_H = a_n, \quad \text{for every } n.$$

The closed graph theorem shows that if Z is an interpolating set for H, then there exists a constant C such that the interpolation problem can be solved by a function satisfying

$$\sum_n \frac{|a_n|^2}{\|K_z\|_H^2} < \infty,$$

$$\|f\|_H \leqslant C \left(\sum_n \frac{|a_n|^2}{\|K_z\|_H^2} \right)^{1/2}.$$

The smallest such constant C is called the constant of interpolation and we denoted it by $M_\alpha(\Gamma)$.

2.5.2 *Sampling sequences*

Definition 2.13. A sequence Γ of points in $\mathbb{D}$ is called a sampling set for H if there exist positive constants A and B such that

$$A \sum_{\gamma \in \Gamma} |\langle f, k_\gamma \rangle_H|^2 \leqslant \|f\|_H^2 \leqslant B \sum_{\gamma \in \Gamma} |\langle f, k_\gamma \rangle_H|^2,$$

for every $f \in H$.

Remarks 2.14.

- If a function f in H vanishes on Γ, the lower inequality will not hold. Thus a zero set can not be a sampling set in H.
- If Γ is of sampling in H^2, then for $f \equiv 1$ the upper inequality forces to Γ to be a Blaschke sequence, that is a zero sequence in H^2.

As a conclusion, the Hardy space H^2 contains no sampling sets.

2.5.3 *Riesz sequences*

Definition 2.15. A normalized sequence $\{u_n\}_n$ of elements in H is called a Riesz sequence in H if there exist positive constants A and B such that for every finite sequence a_{nn}

$$A\sum_n |a_n|^2 \leqslant \left\|\sum_n u_n a_n\right\|_H^2 \leqslant B\sum_n |a_n|^2.$$

If moreover $\{u_n\}_n$ is complete in H, then it is a Riesz basis.

In particular, for a sequence $\{\gamma_n\}_n \subset \mathbb{D}$, we say that $\{k_{\gamma_n}\}_n$ is Riesz sequence in H if there exist positive constants A and B such that

$$A\sum_n |a_n|^2 \leqslant \left\|\sum_n a_n k_{\gamma_n}\right\|_H^2 \leqslant B\sum_n |a_n|^2$$

for all finite sequence $\{a_n\}_n \subset \mathbb{D}$.

Note that a Riesz sequence is a Riesz basis in its closed span.

Theorem 2.16. *Let $\Gamma = \{\gamma_n\} \subset \mathbb{D}$. Then Γ is an interpolating sequence for H if and only if $\{k_{\gamma_n}\}$ is a Riesz sequence.*

3

INTEGRATION OPERATORS ON HARDY AND BERGMAN SPACES

3.1 INTRODUCTION AND MAIN RESULTS

Recall that the Hardy space H^2 consists of analytic functions f on $\mathbb{D}$ such that

$$\|f\|_{H^2} := \left(\sup_{0<r<1} \int_0^{2\pi} |f(re^{i\theta})|^2 \frac{d\theta}{2\pi} \right)^{1/2} < \infty.$$

For an analytic function g on $\mathbb{D}$. The integration operator $I_g \colon H^2 \to H^2$, induced by g is defined as

$$I_g f(z) = \int_0^z f(\zeta) g'(\zeta) d\zeta, \quad f \in H^2.$$

In [**Pom1**], Pommerenke proved that I_g is bounded on H^2 if and only if $g \in BMOA$. The compactness and membership in Schatten classes have been studied by Aleman and Siskakis in [**AS11**]. They proved that I_g is compact if and only if g is in the space of analytic functions of vanishing mean oscillation $VMOA$, and that I_g belongs to the Schatten class $\mathcal{S}_p(H^2)$, for $p > 1$, if and only if g is in the Besov space B_p given by

$$B_p = \left\{ g \in H^2 : \int_{\mathbb{D}} \Big((1-|z|^2)|g'(z)| \Big)^p \frac{dA(z)}{(1-|z|^2)^2} \right\},$$

They also proved that if $I_g \in \mathcal{S}_1$ then $g' = 0$.

The weighted Bergman space A^2_α, with $\alpha > -1$, is the space of analytic functions f on $\mathbb{D}$ for which

$$\|f\|_{A^2_\alpha} := \left((\alpha+1) \int_{\mathbb{D}} |f(z)|^2 (1-|z|^2)^\alpha dA(z) \right)^{1/2} < \infty.$$

In [**AS21**], Aleman and Siskakis considered the operator I_g acting on weighted Bergman spaces. They proved that I_g is bounded on A^2_α (respectively compact) if and only if g is in the Bloch space $\mathcal{B}$ (respectively in the little Bloch space $\mathcal{B}_0$), and that I_g is in the class $\mathcal{S}_p(A^2_\alpha)$, for $p > 1$, if and only if g is in the space B_p. For further information about the spaces used in this paper, we refer the reader to the monograph [**Zhu1**].

This chapter is devoted to the study of the asymptotic behavior of the singular values of compact integration operators acting on H^2 or on A^2_α. Before stating our main results, we give some basic notations.

The Carleson window (or box) associated with an arc $I \subset \mathbb{T}$ is defined by

$$W(I) = \{z \in \mathbb{D} : 1 - |z| \leqslant |I|/2\pi,\ z/|z| \in I\},$$

Let $R(I)$ denote the inner half of $W(I)$ which is given by

$$R(I) = \{z \in \mathbb{D} :\ l(I)/4\pi < 1 - |z| \leqslant |I|/2\pi, z/|z| \in I\}.$$

The family $\{R(I_{n,k})\}_{n,k}$ constitutes a pairwise disjoint covering of $\mathbb{D}$, where

$$I_{n,k} := \{e^{i\theta} : 2\pi k/2^n \leqslant \theta < 2\pi(k+1)/2^n\}, \quad n \geqslant 1 \text{ and } 1 \leqslant k \leqslant 2^n,$$

are the dyadic arcs.

The Bloch space $\mathcal{B}$ (respectively the little Bloch space $\mathcal{B}_0$) consists of analytic functions g on $\mathbb{D}$ such that

$$\sup_{z\in\mathbb{D}}(1 - |z|^2)|g'(z)| < \infty \quad \text{(respectively } \lim_{|z|\to 1}(1 - |z|^2)|g'(z)| = 0).$$

Let $dm_g(z) = |g'(z)|^2(1 - |z|^2)^2 dA(z)$. It is established in **[AS21]** that g is in the space $\mathcal{B}$ if and only if

$$\frac{m_g(R(I_{n,k}))}{|R(I_{n,k})|} = O(1),$$

and that g is in the space $\mathcal{B}_0$ if and only if

$$\lim_{n\to\infty} \sup_{1\leqslant k\leqslant 2^n} \frac{m_g(R(I_{n,k}))}{|R(I_{n,k})|} = 0. \tag{2}$$

Hence, I_g is compact on A^2_α if and only if g satisfies assumption (2). In this case, let $(I_n)_{n\geqslant 1}$ be an enumeration of $(I_{n,k})_{n,k}$ so that the sequence

$$a_n(m_g) := \frac{m_g(R(I_n))}{|R(I_n)|}$$

is nonincreasing.

First, we state the main result of this paper in the case of Bergman spaces.

Theorem 3.1. *Let $g \in \mathcal{B}_0$. There exists an absolute integer p such that*

$$\sqrt{a_{pn}(m_g)} \lesssim s_n(I_g) \lesssim \sqrt{a_{[\frac{n}{p}]}(m_g)}, \quad n \geqslant 1,$$

where $[\frac{n}{p}]$ is the integer part of $\frac{n}{p}$.

An immediate consequence of Theorem 3.1 is given as follows.

Corollary 3.2. *Let* $g \in \mathcal{B}_0$. *If* $\eta = (\eta(n))_n$ *is a nonincreasing sequence of positive numbers such that* $\eta(2n) \asymp \eta(n)$, *then*

$$s_n^2(I_g) \asymp \eta(n) \Longleftrightarrow a_n(m_g) \asymp \eta(n).$$

To illustrate Corollary 3.2, we give the following explicit example.

Example 3.3. Let $\gamma > 0$ and

$$g'_\gamma(z) = \frac{1}{(1-z)\log^\gamma\left(\frac{1}{1-z}\right)}, \quad z \in \mathbb{D}.$$

Consider the counting function

$$N(t) \;:=\; \mathrm{Card}\left\{(n,k) \colon \sqrt{\frac{m_g\left(R(I_{n,k})\right)}{|R(I_{n,k})|}} \geqslant t\right\}.$$

Let $z_{n,k} = \left(1 - \frac{1}{2^n}\right)\exp\left(\frac{i(k+1)}{2^n}\right)$. We have

$$\begin{aligned}
\frac{m_g\left(R(I_{n,k})\right)}{|R(I_{n,k})|} &= \frac{1}{|R(I_{n,k})|}\int_{R(I_{n,k})} |g'_\gamma(z)|^2(1-|z|^2)^2 dA(z) \\
&\asymp \frac{(1-|z_{n,k}|)^2}{|1-z_{n,k}|^2 \log^{2\gamma}\frac{1}{|1-z_{n,k}|}} \\
&\asymp \frac{1}{(k+1)^2\log^{2\gamma}\left(\frac{2^n}{k+1}\right)}.
\end{aligned}$$

By an elementary calculation, we obtain

$$N(t) \asymp \begin{cases} \dfrac{1}{t} & \text{if } \gamma > 1 \\[2ex] \dfrac{1}{t}\log\dfrac{1}{t} & \text{if } \gamma = 1 \\[2ex] \dfrac{1}{t^{1/\gamma}} & \text{if } \gamma < 1. \end{cases}$$

Thus

$$\sqrt{a_n(m_{g_\gamma})} \asymp \begin{cases} \dfrac{1}{n} & \text{if } \gamma > 1 \\[2ex] \dfrac{1}{n}\log n & \text{if } \gamma = 1 \\[2ex] \dfrac{1}{n^\gamma} & \text{if } \gamma < 1. \end{cases}$$

The situation in the Hardy space is more subtle. The first reason is that the boundedness of I_g on H^2 can not be expressed using the sequence $(R(I_{n,k}))_{n,k}$. The second one is that if $(z_{n,k})_{n,k}$ is a sequence such that $z_{n,k} \in R(I_{n,k})$, then $(z_{n,k})_{n,k}$ can not be written as a finite union of interpolating sequences for H^2. These two facts will be used intensively in the case of Bergman spaces.

Recall that I_g is bounded on H^2 if and only if $g \in BMOA$, that is,

$$\|g\|_{BMOA} := |g(0)| + \sup\{\|g \circ \varphi_w\|_{H^2} : w \in \mathbb{D}\} < \infty,$$

where $\varphi_w(z) := \dfrac{w - z}{1 - \bar{w}z}$ is the Möbius automorphism of the unit disc. Recall also that I_g is compact on H^2 if and only if $g \in VMOA$, that is,

$$\lim_{|w| \to 1^-} \|g \circ \varphi_w - g(w)\|_{H^2} = 0.$$

The membership of g to $BMOA$ or to $VMOA$ can be expressed using Carleson windows and the measure $d\nu_g(z) := |g'(z)|^2(1 - |z|^2)dA(z)$. Namely, it is proved in **[AS11]** that

$$g \in BMOA \qquad \Longleftrightarrow \quad \frac{\nu_g(W(I_{n,k}))}{|I_{n,k}|} = O(1)$$

and

$$g \in VMOA \qquad \Longleftrightarrow \quad \lim_{n \to \infty} \sup_{1 \leqslant k \leqslant 2^n} \frac{\nu_g(W(I_{n,k}))}{|I_{n,k}|} = 0.$$

For $g \in VMOA$, denote by

$$(\tau_n(\nu_g))_{n \geqslant 1} := \left(\frac{\nu_g(W(I_n))}{|I_n|} \right)_{n \geqslant 1}$$

the nonincreasing rearrangement of the sequence $\left(\dfrac{\nu_g(W(I_{n,k}))}{|I_{n,k}|} \right)_{n,k}$.

The main result in the case of the Hardy space is the following theorem.

Theorem 3.4. *Let* $g \in VMOA$. *Then*

(i) There exists an absolute integer $p \geqslant 1$ *such that*

$$s_{pn}(I_g) \lesssim \sqrt{\tau_n(\nu_g)}, \quad n \geqslant 1.$$

(ii) There exists an absolute constant $B > 0$ *such that*

$$\frac{1}{B} \sum_{j=1}^{n} \tau_j(\nu_g) \leqslant \sum_{j=1}^{n} s_j^2(I_g), \quad n \geqslant 1.$$

Under some regularity conditions, we obtain a description of the asymptotic behavior of the singular values of I_g.

Let $\gamma > 0$, a positive nonincreasing sequence $(u_n)_n$ is said to belong to $\mathcal{R}_\gamma$ if there exists a nonincreasing sequence $(x_n)_n$ such that $n^\gamma x_n$ increases to infinity and $u_n \asymp x_n$. If, in addition, there exists $\alpha \in (0, \gamma)$ such that $n^\alpha x_n$ decreases, $(u_n)_n$ is said to belong to $\mathcal{R}_{\gamma,\alpha}$.

Theorem 3.5. *Let* $g \in \mathrm{VMOA}$. *The following statements hold.*

(i) *Let* $\gamma \in (0, 1/2)$. *If* $(s_n(I_g))_n \in \mathcal{R}_\gamma$ *or* $(\tau_n(\nu_g))_n \in \mathcal{R}_{2\gamma}$. *Then*

$$s_n(I_g) \asymp \sqrt{\tau_n(\nu_g)}.$$

(ii) *Let* $\gamma \in (0, 1)$ *and let* $\alpha \in (0, \gamma)$. *If* $(s_n(I_g))_n \in \mathcal{R}_{\gamma,\alpha}$ *or* $(\tau_n(\nu_g))_n \in \mathcal{R}_{2\gamma,2\alpha}$. *Then*

$$s_n(I_g) \asymp \sqrt{\tau_n(\nu_g)}.$$

The following corollary is a direct consequence of Theorem 3.5.

Corollary 3.6. *Let* $g \in \mathrm{VMOA}$. *Let* $\alpha \in [0, 1[$ *and* $\beta \geqslant 0$ *such that* $\alpha > 0$ *or* $\beta > 0$. *We have*

$$s_n(I_g) \asymp \frac{1}{n^\alpha \log^\beta(n+1)} \quad \Longleftrightarrow \quad \tau_n(\nu_g) \asymp \frac{1}{n^{2\alpha} \log^{2\beta}(n+1)}.$$

As application, we consider the family of functions g_γ, with $\gamma > 0$, given by

$$g'_\gamma(z) = \frac{1}{(1-z)\log^\gamma\left(\frac{1}{1-z}\right)}, \quad z \in \mathbb{D}.$$

For $\gamma \in [0, 1[$, Using the same argument as in Example 3.3, we have

$$\tau_n(\nu_g) \asymp \frac{1}{n^{2\gamma}}.$$

By Corollary 3.6, we obtain

$$\sqrt{s_n(I_{g_\gamma})} \asymp \frac{1}{n^\gamma}.$$

This chapter is organized as follows. In Section 3.2, we prove Theorem 3.1. Section 3.3 is devoted to the proofs of the main results in the Hardy space.

3.2 BERGMAN SPACES

We start this section by recalling the definitions of sampling and interpolating sets for Bergman spaces which play an important role in this paper.

Let $Z = (z_n)_{n\geqslant 1}$ be a sequence of points in $\mathbb{D}$.

- Z is said to be separated if $\inf\limits_{i\neq j} \rho(z_i, z_j) > 0$, where $\rho(z,w) := \left|\dfrac{z-w}{1-\bar{z}w}\right|$ is the pseudohyperbolic metric on $\mathbb{D}$.
- Z is called a sampling set for A^2_α if
$$\|f\|^2_{A^2_\alpha} \asymp \sum_{n\geqslant 1} |f(z_n)|^2(1-|z_n|^2)^{2+\alpha}, \quad f \in A^2_\alpha.$$
- Z is called an interpolating set for A^2_α if for every sequence $(\lambda_n)_{n\geqslant 1}$ satisfying
$$\sum_{n\geqslant 1} |\lambda_n|^2(1-|z_n|^2)^{2+\alpha} < \infty,$$
there exists f in A^2_α such that $f(z_n) = \lambda_n$ for all $n \geqslant 1$.

We draw attention to the fact that every interpolating set for A^2_α is separated. Interpolating and sampling sets for Bergman spaces have been studied by many authors, we refer the reader to [**Seip11**, **Seip21**] for more details.

Let μ be a positive Borel measure on $\mathbb{D}$. Define the operator T_μ by
$$T_\mu f(z) = \int_{\mathbb{D}} f(w)K^\alpha(z,w)(1-|w|^2)^\alpha d\mu(w), \; f \in A^2_\alpha,$$
where $K^\alpha(z,w) = (1-z\bar{w})^{-2-\alpha}$ is the reproducing kernel of A^2_α. A simple computation yields
$$\langle T_\mu f, f\rangle_{A^2_\alpha} = \int_{\mathbb{D}} |f(z)|^2(1-|z|^2)^\alpha d\mu(z).$$
These operators are called Toeplitz operators on Bergman spaces. Their boundedness was considered by Hasting [**Has81**]. Later, Luecking obtained necessary and sufficient condition on μ for which T_μ belongs to a Schatten class $\mathcal{S}_p(A^2_\alpha)$, with $p > 0$ (see [**Lue11**]). Recently, the asymptotic behavior of the singular values of such operators has been studied on a large class of weighted Bergman spaces (see [**APP1**, **EE2**]).

In the specific case of discrete measures, one can obtain explicit lower and upper bounds for the singular values of the associated Toeplitz operators. We have the following two lemmas.

Lemma 3.7. *Let* $Z = (z_n)_{n\geqslant 1}$ *be a separated sequence and let* $(c_n)_{n\geqslant 1}$ *be a sequence of positive numbers such that* $\big(c_n(1-|z_n|^2)^{-2}\big)_{n\geqslant 1}$ *decreases to zero. For* $\mu = \sum\limits_{n\geqslant 1} c_n\delta_{z_n}$*, the operator* T_μ *is compact on* A^2_α *and*

$$s_n(T_\mu) \lesssim \frac{c_n}{(1-|z_n|^2)^2}.$$

Proof. First, recall **[GGK1]** that for a compact operator T on a Hilbert space H,

$$s_n(T) = \inf\{\|T-R\| : R\colon H \to H \text{ is of rank } < n\}.$$

For $\mu_n = \sum\limits_{j=1}^{n-1} c_j\delta_{z_j}$, the associated operator T_{μ_n} is of rank $n-1$. Thus, it suffices to show that

$$\|T_\mu - T_{\mu_n}\| \lesssim \frac{c_n}{(1-|z_n|^2)^2}.$$

Let $f \in A^2_\alpha$. We have

$$\begin{aligned}\langle (T_\mu - T_{\mu_n})\, f, f\rangle_{A^2_\alpha} &= \sum_{j\geqslant n} c_j(1-|z_j|^2)^\alpha |f(z_j)|^2\\ &\leqslant \frac{c_n}{(1-|z_n|^2)^2}\sum_{j\geqslant n}(1-|z_j|^2)^{2+\alpha}|f(z_j)|^2.\end{aligned}$$

Moreover, since the sequence Z is separated, there exists a small constant $\delta > 0$ such that the discs $\mathcal{D}_\delta(z_j) = \{z \in \mathbb{D} : |z - z_j| < \delta(1-|z_j|)\}$ are pairwise disjoint. By subharmonicity, we get

$$|f(z_j)|^2(1-|z_j|^2)^{2+\alpha} \lesssim \int_{\mathcal{D}_\delta(z_j)} |f(z)|^2(1-|z|^2)^\alpha \, dA(z).$$

Therefore

$$\begin{aligned}\langle (T_\mu - T_{\mu_n})\, f, f\rangle_{A^2_\alpha} &\lesssim \frac{c_n}{(1-|z_n|^2)^2}\sum_{j\geqslant n}\int_{\mathcal{D}_\delta(z_j)} |f(z)|^2(1-|z|^2)^\alpha \, dA(z)\\ &\lesssim \frac{c_n}{(1-|z_n|^2)^2}\int_{\mathbb{D}} |f(z)|^2(1-|z|^2)^\alpha \, dA(z)\\ &= \frac{c_n}{(1-|z_n|^2)^2}\|f\|^2_{A^2_\alpha}.\end{aligned}$$

Since the operator $T_\mu - T_{\mu_n}$ is positive, we obtain

$$\|T_\mu - T_{\mu_n}\| \lesssim \frac{c_n}{(1-|z_n|^2)^2}.$$

□

Lemma 3.8. *Let* $Z = (z_n)_{n\geqslant 1}$ *be an interpolating set for* A^2_α *and let* $(c_n)_{n\geqslant 1}$ *be a sequence of positive numbers such that* $\big(c_n(1-|z_n|^2)^{-2}\big)_{n\geqslant 1}$ *decreases to zero. For* $\mu = \sum\limits_{n\geqslant 1} c_n \delta_{z_n}$*, the operator* T_μ *is compact on* A^2_α *and*

$$s_n(T_\mu) \asymp \frac{c_n}{(1-|z_n|^2)^2}.$$

Proof. The upper bound result follows from Lemma 3.7. To obtain the lower bound, we use an alternative characterization **[GGK1]** of the singular values:

$$s_n(T_\mu) = \sup_{E, \dim E = n} \left(\inf_{f\in E\setminus\{0\}} \frac{\|T_\mu f\|_{A^2_\alpha}}{\|f\|_{A^2_\alpha}} \right).$$

Set $\mathcal{K}_Z = \operatorname{span}\{k^\alpha_{z_j} : j \geqslant 1\}$, where $k^\alpha_{z_j}$ denotes the normalized reproducing kernel of A^2_α. We choose as subspace E the orthogonal complement in $\mathcal{K}_Z$ of the linear span of $\{k^\alpha_{z_j} : j > n\}$. Since Z is an interpolating set for A^2_α, the family $\{k^\alpha_{z_j} : j \geqslant 1\}$ is a Riesz basis of $\mathcal{K}_Z$ and then the vector space E is of dimension n. Let $f \in E$, we have

$$\|f\|^2_{A^2_\alpha} \leqslant [M_\alpha(Z)]^2 \sum_{j\geqslant 1} |\langle f, k^\alpha_{z_j}\rangle_{A^2_\alpha}|^2 = [M_\alpha(Z)]^2 \sum_{j=1}^{n} |\langle f, k^\alpha_{z_j}\rangle_{A^2_\alpha}|^2.$$

Furthermore

$$\begin{aligned}
\langle T_\mu f, f\rangle_{A^2_\alpha} &= \sum_{j\geqslant 1} c_j (1-|z_j|^2)^\alpha \left|f(z_j)\right|^2 \\
&= \sum_{j=1}^{n} c_j (1-|z_j|^2)^\alpha \left|f(z_j)\right|^2 \\
&\geqslant \frac{c_n}{(1-|z_n|^2)^2} \sum_{j=1}^{n} |\langle f, k^\alpha_{z_j}\rangle_{A^2_\alpha}|^2,
\end{aligned}$$

which finally leads to

$$s_n(T_\mu) \gtrsim [M_\alpha(Z)]^{-2} \frac{c_n}{(1-|z_n|^2)^2}.$$

□

Proof of Theorem 3.1. We start with the the upper bound of the singular values of I_g. Let $Z = (z_n)_{n\geqslant 1}$ be a sampling set for $A^2_{\alpha+2}$ such that each $R(I_n)$ contains N points of Z, where N is a large integer (see **[Lue21]**). We denote these points by $(z_{n,i})_{n\geqslant 1}$, where $i = 1, \ldots, N$. Using the fact that Z is a sampling set for $A^2_{\alpha+2}$ with the subharmoncity of $|g'|^2$, we get

$$\begin{aligned}
\int_{\mathbb{D}} |f(z)g'(z)|^2(1-|z|^2)^{2+\alpha}dA(z) &\asymp \sum_{n\geqslant 1} |f(z_n)g'(z_n)|^2(1-|z_n|^2)^{4+\alpha} \\
&\lesssim \sum_{n} |f(z_n)|^2(1-|z_n|^2)^{\alpha} \int_{R(I_n)} |g'(z)|^2(1-|z|^2)^2 dA(z) \\
&= \sum_{n,i} |f(z_{n,i})|^2(1-|z_{n,i}|^2)^{\alpha} \int_{R(I_n)} |g'(z)|^2(1-|z|^2)^2 dA(z) \\
&= \langle \sum_{i=1}^{N} T_{\mu_i} f, f\rangle_{A^2_\alpha},
\end{aligned}$$

where $\mu_i = \sum_{n\geqslant 1} m_g(R(I_n))\delta_{z_{n,i}}$. Using identity (2.7), this implies

$$I_g^* I_g \lesssim \sum_{i=1}^{N} T_{\mu_i}.$$

We use a known inequality of the singular values from **[GKG1]**. If A and B are two compact operators, then

$$s_{m+n-1}(A+B) \leqslant s_m(A) + s_n(B), \quad m, n = 1, 2, \ldots. \tag{3}$$

Thus we obtain

$$s_{Nn}(I_g^* I_g) \lesssim \sum_{i=1}^{N} s_n(T_{\mu_i}) \leqslant N \max_{1\leqslant i\leqslant N} s_n(T_{\mu_i}).$$

This, together with Lemma 3.7, gives

$$s^2_{Nn}(I_g) \lesssim N \max_{1\leqslant i\leqslant N} s_n(T_{\mu_i}) \lesssim \frac{m_g(R(I_n))}{(1-|z_n|^2)^2}.$$

Now, we turn to prove the lower bound result. Choose a point ξ_n in $R(I_n)$ so that $|g'(z)| \leqslant 2|g'(\xi_n)|$ for all $z \in R(I_n)$. Then

$$m_g(R(I_n)) = \int_{R(I_n)} |g'(z)|^2(1-|z|^2)^2 dA(z) \lesssim |g'(\xi_n)|^2(1-|\xi_n|^2)^4. \tag{4}$$

Note that $(\xi_n)_{n\geqslant 1}$ can be written as a finite union of separated sets. Since every separated set itself can be expressed as a finite union of interpolating sets (see **[Seip21]**), there exist $\Lambda_1, \Lambda_2, \cdots, \Lambda_M$ such that $\{\xi_n,\ n \in \Lambda_i\}$ is an interpolating set for A^2_α. Therefore

$$\begin{aligned}\int_{\mathbb{D}} |f(z)g'(z)|^2(1-|z|^2)^{\alpha+2}dA(z) &\gtrsim \sum_{i=1}^{M}\sum_{n\in\Lambda_i} |f(\xi_n)|^2|g'(\xi_n)|^2(1-|\xi_n|^2)^{\alpha+4} \\ &\geqslant \sum_{i=1}^{M}\sum_{n\in\Lambda_i} |f(\xi_n)|^2(1-|\xi_n|^2)^{\alpha} m_g(R(I_n)) \\ &= \langle \sum_{i=1}^{M} T_{\mu_i} f, f\rangle_{A^2_\alpha},\end{aligned}$$

where $\mu_i = \sum\limits_{n\in\Lambda_i} m_g(R(I_n))\delta_{\xi_n}$. The first inequality follows from the fact that $(\xi_n)_{n\geqslant 1}$ is a finite union of separated sets. Therefore

$$I_g^* I_g \gtrsim \sum_{i=1}^{M} T_{\mu_i} \geqslant T_{\mu_k}, \quad k \in \{1, .., M\}.$$

This implies

$$s_n(T_{\mu_k}) \lesssim s_n^2(I_g), \quad k \in \{1, .., M\}.$$

Finally, using the fact that each set $\{\xi_n,\ n \in \Lambda_i\}$, for $i = 1, \dots, M$, is an interpolating set for A^2_α along with Lemma 3.8, we get

$$s_n(T_{\mu_k}) \gtrsim a_n(\mu_k),$$

where $a_n(\mu_k)$ is the nonincreasing rearrangement of the sequence $\left(\frac{\mu_k(R(I_n))}{|R(I_n)|}\right)_n$. Remark that

$$\bigcup_{1\leqslant k\leqslant M} \{a_n(\mu_k) :\ n \geqslant 1\} = \{a_n(\mu) :\ n \geqslant 1\}.$$

Therefore

$$a_{Mn}(\mu) \leqslant \min_{1\leqslant k\leqslant M} a_n(\mu_k) \lesssim s_n^2(T_\mu).$$

Consequently, there exists an integer p such that $a_n(m_g) \lesssim s^2_{[\frac{n}{p}]}(I_g)$. ☐

3.3 THE HARDY SPACE H^2

This section is devoted to prove Theorems 3.4 and 3.5. We start by recalling the connection between integration operators acting on the Hardy space H^2 and some standard operators.

Let μ be a positive Borel measure on $\mathbb{D}$, and let J_μ denote the embedding operator from H^2 into $L^2(\mu)$. Carleson proved in **[Car1]** that J_μ is bounded if and only if $\mu(W(I))/l(I)$ is uniformly bounded for all arcs $I \subset \mathbb{T}$. This result can be expressed in terms of the family of the dyadic arcs. Indeed, one can see that J_μ is bounded if and only if there exists $C > 0$ such that

$$\mu(W(I_{n,k})) \leqslant Cl(I_{n,k}), \qquad n \geqslant 1 \text{ and } 1 \leqslant k \leqslant 2^n.$$

Note also that J_μ is compact if and only if

$$\lim_{n\to\infty} \sup_{1\leqslant k\leqslant 2^n} \frac{\mu(W(I_{n,k}))}{l(I_{n,k})} = 0. \tag{5}$$

We introduce the operator

$$S_\mu f(z) = \int_{\mathbb{D}} f(w)K(z,w)d\mu(w), \quad f \in H^2,$$

where $K(z,w) = (1 - z\bar{w})^{-1}$ is the reproducing kernel of H^2. A straightforward computation shows that

$$\langle S_\mu f, f\rangle_{H^2} = \int_{\mathbb{D}} |f(z)|^2 d\mu(z) = \|J_\mu f\|^2_{L^2(\mu)}.$$

This implies that S_μ is bounded, compact, or belongs to a Schatten class $\mathcal{S}_p(H^2)$ if and only if J_μ is. In **[Lue11]**, Luecking proved that $S_\mu \in \mathcal{S}_p(H^2)$, for $p > 0$, if and only if

$$\sum_{n,k} \left(\frac{\mu(R(I_{n,k}))}{l(I_{n,k})}\right)^p < \infty.$$

It is worth noting that Luecking's characterization can also be expressed in terms of Carleson windows $W(I_{n,k})$ (see **[LLQRJFA1]**). However, this is no longer the case for Carleson boundedness criterion (see **[PR1]**).

Now, we fix and recall some notations. Let μ be a positive Borel measure on $\mathbb{D}$ which satisfies condition (5), and let $(I_n(\mu))_{n\geqslant 1}$ be an enumeration of $(I_{n,k})_{n,k}$ such that the sequence $\left(\frac{\mu(W(I_n(\mu)))}{l(I_n(\mu))}\right)_{n\geqslant 1}$ is nonincreasing. According to the notation used in the introduction, we define

$$\tau_n(\mu) := \frac{\mu(W(I_n(\mu))}{l(I_n(\mu))}.$$

The following lemma is the key to prove the first assertion of Theorem 3.4.

Lemma 3.9. *Let μ be a positive Borel measure on $\mathbb{D}$ such that S_μ is compact on H^2. Let $I_n = I_n(\mu)$, and let $(z_n)_{n\geqslant 1}$ be a sequence such that $z_n \in R(I_n)$. Fix an index $n \geqslant 1$, and set $\mu_n = \sum_{k\geqslant n} \mu(R(I_k))\delta_{z_k}$. We have*

1. $\sup\limits_{j\geqslant 1} \dfrac{\mu_n(W(I_j))}{l(I_j)} \leqslant \tau_n(\mu)$.

2. $s_n(S_{\mu_1}) \lesssim \tau_n(\mu)$.

Proof. We begin with part (1). First, note that if $n = 1$, then $\tau_j(\mu_1) = \tau_j(\mu)$ and the result is obvious. Now, suppose that $n \geqslant 2$.

For $j \geqslant 1$, let $\Gamma_j = \{k \geqslant n\colon R(I_k) \subset W(I_j)\}$. Then

$$\mu_n(W(I_j)) \;=\; \sum_{k\in\Gamma_j} \mu_n(R(I_k)) = \sum_{k\in\Gamma_j} \mu(R(I_k)) \leqslant \mu(W(I_j)).$$

Thus for all $j \geqslant n$, we have

$$\frac{\mu_n(W(I_j))}{l(I_j)} \leqslant \frac{\mu(W(I_j))}{l(I_j)} \leqslant \frac{\mu(W(I_n))}{l(I_n)}.$$

Let $j \in \{1, \dots, n-1\}$. If we assume that

$$\frac{\mu_n(W(I_j))}{l(I_j)} > \frac{\mu(W(I_n))}{l(I_n)}, \tag{6}$$

we will end up with a contradiction. Indeed, we will prove that there exists $j_1 \in \{1, \dots, n-1\}$ such that $W(I_{j_1}) \subsetneq W(I_j)$ and $\dfrac{\mu(W(I_n))}{l(I_n)} < \dfrac{\mu_n(W(I_{j_1}))}{l(I_{j_1})}$, which is absurd since one can repeat this operation indefinitely.

Suppose that assumption (6) is satisfied, and write $W(I_j) = R(I_j) \cup W(I_{j_1}) \cup W(I_{j_1'})$, where

$$l(I_{j_1}) = l(I_{j_1'}) = l(I_j)/2.$$

Without loss of generality, we assume that $\mu_n(W(I_{j_1})) \geqslant \mu_n(W(I_{j_1'}))$. Then, since $\mu_n(R(I_j)) = 0$, we have

$$\frac{\mu_n(W(I_j))}{l(I_j)} \;=\; \frac{\mu_n(W(I_{j_1})) + \mu_n(W(I_{j_1'}))}{l(I_{j_1}) + l(I_{j_1'})} \leqslant \frac{\mu_n(W(I_{j_1}))}{l(I_{j_1})}.$$

This implies $\dfrac{\mu(W(I_n))}{l(I_n)} < \dfrac{\mu_n(W(I_{j_1}))}{l(I_{j_1})}$ and then $j_1 \in \{1, \dots, n-1\}$. The proof of (1) is complete.

To prove (2), recall that $s_n(S_{\mu_1}) = \inf\{\|S_{\mu_1} - R\| \,:\, \operatorname{rank} R < n\}$. For $R = S_{\mu_1 - \mu_n}$, we have

$$s_n(S_{\mu_1}) \leqslant \|S_{\mu_1} - S_{\mu_1-\mu_n}\| = \|S_{\mu_n}\| \lesssim \sup_{j\geqslant 1} \frac{\mu_n(W(I_j))}{l(I_j)}.$$

The second inequality comes from the embedding Carleson's Theorem. Using the first part of the lemma, we obtain $s_n(S_{\mu_1}) \lesssim \tau_n(\mu)$. □

The proof of the second assertion of Theorem 3.4 requires the following lemma. We include the proof for completeness.

Lemma 3.10. *Let* H *and* K *be two Hilbert spaces and* $T : H \to K$ *a compact operator. Suppose that there exist* $(u_n)_{n\geqslant 1} \subset H$ *and* $(v_n)_{n\geqslant 1} \subset K$ *such that*

$$\sum_n |\langle u_n, f\rangle_H|^2 \leqslant C\|f\|_H^2 \text{ and } \sum_n |\langle v_n, g\rangle_K|^2 \leqslant C\|g\|_K^2, \quad f \in H,\ g \in K,$$

for some constant $C > 0$. *Then we have*

$$\sum_{j=1}^n \left|\langle Tu_j, v_j\rangle_K\right| \leqslant C^2 \sum_{j=1}^n s_j(T).$$

Proof. Let $(e_n)_n$ and $(f_n)_n$ be two orthonormal bases of H and K, respectively. Let A and B be the operators given by

$$Af = \sum_n \langle f, u_n\rangle_H e_n \quad \text{and} \quad Bg = \sum_n \langle g, v_n\rangle_K f_n.$$

By assumptions, A and B are bounded and $\|A\|, \|B\| \leqslant C$. Note that the adjoint operators A^* and B^* satisfy $A^*e_n = u_n$ and $B^*f_n = v_n$. By Theorem 3.5 of **[GGK1]**, we have

$$\sum_{j=1}^n \left|\langle BTA^*e_j, f_j\rangle\right| \leqslant \sum_{j=1}^n s_j(BTA^*).$$

Thus

$$\begin{aligned}
\sum_{j=1}^n \left|\langle Tu_j, v_j\rangle\right| &= \sum_{j=1}^n \left|\langle TA^*e_j, B^*f_j\rangle\right| \\
&= \sum_{j=1}^n \left|\langle BTA^*e_j, f_j\rangle\right| \\
&\leqslant \sum_{j=1}^n s_j(BTA^*) \\
&\leqslant \|B\|\|A^*\| \sum_{j=1}^n s_j(T) \leqslant C^2 \sum_{j=1}^n s_j(T).
\end{aligned}$$

□

The Littlewood-Paley identity (2.5) gives rise to the following formula which relates the operators I_g and S_{ν_g}. We have

$$\langle I_g^* I_g f, f\rangle_{H^2} = \int_{\mathbb{D}} |f(z)|^2 |g'(z)|^2 \log(1/|z|^2)\, dA(z) \asymp \langle S_{\nu_g} f, f\rangle_{H^2}. \tag{7}$$

Proof of Theorem 3.4. It is immediate, from formula (7), that if $f \in H^2$ then $fg' \in A_1^2$. We have seen in the proof of the upper bound result from Theorem 3.1 that there exist a sampling sequence $Z = (z_n)_{n\geqslant 1}$ for A_1^2 and an integer N such that each $R(I_n)$ contains N points of Z denoted by $(z_{n,i})_{n\geqslant 1}$, where $i = 1, \dots, N$. Then

$$\begin{aligned}
\int_{\mathbb{D}} |f(z)g'(z)|^2(1-|z|^2)dA(z) &\asymp \sum_{n\geqslant 1} |f(z_n)g'(z_n)|^2(1-|z_n|^2)^3 \\
&\lesssim \sum_{n\geqslant 1} |f(z_n)|^2 \int_{R(I_n)} |g'(z)|^2(1-|z|^2)dA(z) \\
&= \sum_{n\geqslant 1}\sum_{i=1}^{N} |f(z_{n,i})|^2 \int_{R(I_n)} |g'(z)|^2(1-|z|^2)dA(z) \\
&= \langle \sum_{i=1}^{N} S_{\nu_i} f, f\rangle_{H^2},
\end{aligned}$$

where $\nu_i = \sum_{n\geqslant 1} \nu_g(R(I_n))\delta_{z_{n,i}}$. Using inequality (3), we obtain

$$s_{Nn}^2(I_g) \asymp s_{Nn}(S_{\nu_g}) \lesssim \max_{1\leqslant i\leqslant N} s_n(S_{\nu_i}).$$

By Lemma 3.9, we deduce $s_{Nn}^2(I_g) \lesssim \tau_n(\nu_g)$ as desired and the proof of assertion 1 is complete.

To prove assertion 2, we use Lemma 3.10. Let ξ_n be the center of $R(I_n)$ and set

$$u_n(z) = v_n(z) = \frac{(1-|\xi_n|^2)^{3/2}}{(1-\bar{\xi}_n z)^2}.$$

For $f \in H^2$, we have

$$\begin{aligned}
|\langle u_n, f\rangle_{H^2}|^2 &\lesssim |u_n(0)f(0)|^2 + \left|\int_{\mathbb{D}} \frac{(1-|\xi_n|^2)^{3/2}}{(1-\bar{\xi}_n z)^3}\overline{f'(z)}(1-|z|^2)dA(z)\right|^2 \\
&= (1-|\xi_n|^2)^3|f(0)|^2 + |\langle k^1_{\xi_n}, f'\rangle_{A_1^2}|^2.
\end{aligned}$$

Thus

$$\begin{aligned}
\sum_n |\langle u_n, f\rangle_{H^2}|^2 &\lesssim \sum_n (1-|\xi_n|^2)^3|f(0)|^2 + \sum_n |\langle k^1_{\xi_n}, f'\rangle_{A_1^2}|^2 \\
&\lesssim \sum_n (1-|\xi_n|^2)^3|f(0)|^2 + \|f'\|^2_{A_1^2} \\
&\lesssim \|f\|^2_{H^2}.
\end{aligned}$$

The second inequality follows from the fact that $(\xi_n)_n$ is a separated sequence. Then, by Lemma 3.10 applied to S_{ν_g}, there exists an absolute constant $c > 0$ such that

$$c \sum_{j=1}^{n} |\langle S_{\nu_g} u_j, u_j \rangle_{H^2}| \leqslant \sum_{j=1}^{n} s_j(S_{\nu_g}).$$

Furthermore

$$\begin{aligned}
\langle S_{\nu_g} u_n, u_n \rangle_{H^2} &= \int_{\mathbb{D}} \frac{(1-|a_n|^2)^3}{|1-\bar{a}_n z|^4} |g'(z)|^2 (1-|z|^2)\, dA(z) \\
&\geqslant \int_{W(I_n)} \frac{(1-|a_n|^2)^3}{|1-\bar{a}_n z|^4} |g'(z)|^2 (1-|z|^2)\, dA(z) \\
&\gtrsim \frac{1}{l(I_n)} \int_{W(I_n)} |g'(z)|^2 (1-|z|^2)\, dA(z).
\end{aligned}$$

Combining these two facts with formula (7), we get

$$\sum_{j=1}^{n} \frac{\nu_g(W(I_j))}{l(I_j)} \lesssim \sum_{j=1}^{n} s_j(S_{\nu_g}) \lesssim \sum_{j=1}^{n} s_j^2(I_g).$$

□

We turn now to prove Theorem 3.5, we start with the proof of the first statement which is completely based on Theorem 3.4.

Proof of assertion 1 of Theorem 3.5 . Suppose that $(s_n(I_g))_n \in \mathcal{R}_\gamma$ for some $\gamma \in (0, 1/2)$. Then there exists a nonincreasing sequence $(x_n)_n$ of positive numbers such that $n^\gamma x_n$ increases to infinity and $s_n(I_g) \asymp x_n$. For an integer $p \geqslant 1$, we have

$$p^{-\gamma} x_n \leqslant x_{pn} \leqslant x_n.$$

By assertion 1 of Theorem 3.4, we have

$$x_n^2 \asymp s_{np}^2(I_g) \lesssim \tau_n(\nu_g). \tag{8}$$

Moreover, assertion 2 of Theorem 3.4 implies

$$\begin{aligned}
n\tau_n(\nu_g) \leqslant \sum_{j=1}^{n} \tau_j(\nu_g) &\lesssim \sum_{j=1}^{n} s_j^2(I_g) \\
&\lesssim \sum_{j=1}^{n} x_j^2 \\
&\leqslant n^{2\gamma} x_n^2 \sum_{j=1}^{n} \frac{1}{j^{2\gamma}}.
\end{aligned}$$

The last inequality follows from the fact that $(j^{2\gamma}x_j)_j$ is increasing. Since $2\gamma < 1$, we have

$$\sum_{j=1}^{n} \frac{1}{j^{2\gamma}} \asymp n^{1-2\gamma}.$$

Thus $n\tau_n(\nu_g) \lesssim nx_n^2$. This along with inequality (8) leads to $\tau_n(\nu_g) \asymp x_n^2$.

Now, suppose that $(\tau_n(\nu_g))_n \in \mathcal{R}_{2\gamma}$ and let $(x_n)_n$ be a nonincreasing sequence of positive numbers such that $n^{2\gamma}x_n$ increases to infinity and $\tau_n(\nu_g) \asymp x_n$. It follows at once from assertion 1 of Theorem 3.4 that

$$s_n^2(I_g) \lesssim \tau_{[\frac{n}{p}]}(\nu_g) \asymp x_n.$$

Let κ be a large constant. Assertion 2 of Theorem 3.4 yields

$$\begin{aligned}\sum_{j=1}^{\kappa n} x_j &\lesssim \sum_{j=1}^{n} s_j^2(I_g) + \sum_{j=n+1}^{\kappa n} s_j^2(I_g)\\ &\lesssim \sum_{j=1}^{n} x_j + \kappa n s_n^2(I_g).\end{aligned}$$

Using the same line of reasoning as in the previous part, we obtain $\sum\limits_{j=1}^{n} x_j \lesssim nx_n$. Therefore

$$\sum_{j=1}^{\kappa n} x_j \lesssim nx_n + \kappa n s_n^2(I_g).$$

In contrast

$$\begin{aligned}\sum_{j=1}^{\kappa n} x_j &\geqslant \kappa n x_{\kappa n}\\ &\geqslant \kappa^{1-2\gamma} n x_n.\end{aligned}$$

The first inequality follows from the fact $(x_j)_j$ is nonincreasing and the second one uses the fact that $(j^{2\gamma}x_j)_j$ is increasing. Thus

$$\kappa^{1-2\gamma}x_n \lesssim x_n + \kappa s_n^2(I_g).$$

For κ large enough, we obtain the result. □

The proof of the second statement of Theorem 3.5 requires some preparatory results. The following technical lemma is a generalization of Proposition 3.3 in **[LLQRJFA1]**.

Lemma 3.11. *Let* $h : [0,\infty[\to [0,\infty[$ *be an increasing function such that* $h(0) = 0$. *Suppose that there exist* $\varepsilon \in (0,1)$ *and* $p \geqslant 1$ *for which* $t \to h(t^{\varepsilon})$ *is concave and* $t \to h(t^p)$ *is convex. Let* μ *be a positive Borel measure on* $\mathbb{D}$. *There exists* $B > 0$, *which depends on* ε *and* p, *such that*

$$\sum_j h\left(\frac{\mu(W(I_j))}{l(I_j)}\right) \leqslant \sum_j h\left(B\frac{\mu(R(I_j))}{l(I_j)}\right).$$

Proof. We use the argument given in **[LLQRJFA1]**. Let $\alpha = 1/\varepsilon$ and let $\beta = 1/(1-\varepsilon)$ be the conjugate of α. Write

$$W(I_{n,j}) = \bigcup_{l\geqslant n}\ \bigcup_{k\in H_{l,n,j}} R(I_{l,k}), \quad n \geqslant 1 \text{ and } 1 \leqslant j \leqslant 2^n,$$

where $H_{l,n,j} = \left\{k \in \{1,\dots,2^l\} \colon \frac{j}{2^n} \leqslant \frac{k}{2^l} < \frac{j+1}{2^n}\right\}$. Consider $a \in (1,2)$ such that $2 < a^{\beta}$. It is proved in **[LLQRJFA1]** that

$$\mu(W(I_{n,j}))^{\alpha} \lesssim \sum_{l\geqslant n;\ k\in H_{l,n,k}} a^{(l-n)\alpha}\mu(R(I_{l,k}))^{\alpha}.$$

Recall that the function $h_{1/\alpha}(t) := h(t^{1/\alpha})$ is increasing and concave. Consequently,

$$\begin{aligned}
h(2^n\mu(W(I_{n,j}))) &= h_{1/\alpha}\left((2^n\mu(W(I_{n,j})))^{\alpha}\right)\\
&\lesssim \sum_{l\geqslant n,\ k\in H_{l,n,k}} h_{1/\alpha}\left(a^{(l-n)\alpha}2^{n\alpha}\mu(R(I_{l,k}))^{\alpha}\right)\\
&= \sum_{l\geqslant n,\ k\in H_{l,n,k}} h\left(\left(\frac{a}{2}\right)^{l-n}2^l\mu(R(I_{l,k}))\right)\\
&\leqslant \sum_{l\geqslant n,\ k\in H_{l,n,k}} \left(\frac{a}{2}\right)^{(l-n)/p} h(2^l\mu(R(I_{l,k}))).
\end{aligned}$$

The last inequality follows from the fact $h(t^p)$ is convex. Then we obtain

$$\begin{aligned}
\sum_{n=1}^{\infty}\sum_{j=0}^{2^n-1} h\left(2^n\mu(W(I_{n,j}))\right) &\lesssim \sum_{l=1}^{\infty}\sum_{k=0}^{2^l-1}\left(\sum_{(n,j):l\geqslant n,k\in H_{l,n,j}}\left(\frac{a}{2}\right)^{(l-n)/p}\right) h\left(2^l\mu(R(I_{l,k}))\right)\\
&\leqslant C(p,\varepsilon)\sum_{l=1}^{\infty}\sum_{k=0}^{2^l-1} h\left(2^l\mu(R(I_{l,k}))\right).
\end{aligned}$$

□

Theorem 3.12. *Let* μ *be a positive Borel measure on* $\mathbb{D}$ *such that* J_{μ} *is compact. Let* $h : [0,\infty[\to [0,\infty[$ *be a convex increasing function with* $h(0) = 0$ *and such* *that* $t \to h(t^\varepsilon)$ *is concave for some* $\varepsilon \in (0,1)$. *Then there exists* $B > 0$ *which depends only on* ε *such that*

$$\sum_j h\left(\frac{1}{B}\sqrt{\tau_j(\mu)}\right) \leqslant \sum_j h(s_j(J_\mu)).$$

Proof. The proof is based on Lemma 3.10. Let ξ_n be the center of $R(I_n)$, and set

$$u_n(z) = \frac{(1-|\xi_n|^2)^{3/2}}{(1-\bar{\xi}_n z)^2} \text{ and } v_n = c_n\theta_n\chi_{R(I_n)},$$

where $\theta_n = \frac{u_n}{|u_n|}$ and $c_n = \frac{1}{\sqrt{\mu(R(I_n))}}$ if $\mu(R(I_n)) \neq 0$.

We proved before

$$\sum_n |\langle u_n, f\rangle_{H^2}|^2 \lesssim \|f\|_{H^2}^2, \quad f \in H^2.$$

By Cauchy-Schwarz inequality, we have

$$\sum_n |\langle v_n, v\rangle_{L^2(\mu)}|^2 \leqslant \sum_n c_n^2\mu(R(I_n))\int_{R(I_n)} |v|^2 d\mu = \|v\|_{L^2(\mu)}^2, \quad v \in L^2(\mu).$$

Note also that

$$\langle J_\mu u_n, v_n\rangle_{L^2(\mu)} = c_n\int_{R(I_n)} |u_n| d\mu \asymp \sqrt{\frac{\mu(R(I_n))}{l(I_n)}}.$$

Applying Lemma 3.10, we get

$$\sum_{j=1}^n \frac{1}{C}\sqrt{\frac{\mu(R(I_j))}{l(I_j)}} \lesssim \sum_{j=1}^n s_j(J_\mu),$$

where C is a positive constant. Then by Corollary 3.3 of **[GGK1]**, we obtain

$$\sum_j h\left(\frac{1}{C}\sqrt{\frac{\mu(R(I_j))}{l(I_j)}}\right) \leqslant \sum_j h(s_j(J_\mu)).$$

The desired result follows from Lemma 3.11. □

Corollary 3.13. *Let* $g \in \mathrm{VMOA}$. *Let* $h : [0,\infty[\to [0,\infty[$ *be a convex increasing function with* $h(0) = 0$ *and such that* $t \to h(t^\varepsilon)$ *is concave for some* $\varepsilon \in (0,1)$. *Then there exists* $B > 0$ *which depends only on* ε *such that*

$$\sum_j h\left(\frac{1}{B}\sqrt{\tau_j(\nu_g)}\right) \leqslant \sum_j h(s_j(I_g))).$$

Proof. It suffices to apply Theorem 3.12 and to remark that $s_n(I_g) = s_n(J_{\nu_g})$. □

The following lemma is proved in **[EE2]**.

Lemma 3.14. *Let* $(a_n)_n$ *and* $(b_n)_n$ *be two positive nonincreasing sequences. Suppose that there exist* $\beta_1 > 1$ *and* $\beta_2 > 1$ *such that* $(n^{\beta_1} b_n)_n$ *is nonincreasing and* $(n^{\beta_2} b_n)_n$ *is increasing. Let* $\beta > \beta_2$ *and let* $B > 0$*. If for each positive increasing concave function* h *such that* $h(t^\beta)$ *is convex we have*

$$\sum_n h(\frac{1}{B}a_n) \leqslant \sum_n h(b_n) \leqslant \sum_n h(Ba_n),$$

then $a_n \asymp b_n$.

Proof of assertion 2 of Theorem 3.5. Suppose that $(s_n(I_g))_n \in \mathcal{R}_{\gamma,\alpha}$. Then there exists a nonincreasing sequence $(x_n)_n$ such that $(n^\gamma x_n)_n$ is increasing, $(n^\alpha x_n)_n$ is nonincreasing, and $x_n \asymp s_n(I_g)$. We have to prove that $x_n \asymp \sqrt{\tau_n(\nu_g)}$. Let $\beta > 0$ be such that $\beta\alpha > 1$. Put $\beta_1 = \beta\alpha$ and $\beta_2 = \beta\gamma$. Let h be a positive increasing concave function such that $h(0) = 0$ and $h_\beta := h(t^\beta)$ is convex. The function h_β satisfies the assumptions of Corollary 3.13 with $\varepsilon = 1/\beta$. Then there exists B_1 such that

$$\sum_j h_\beta\left(\frac{1}{B_1}\sqrt{\tau_j(\nu_g)}\right) \leqslant \sum_j h_\beta(x_j).$$

By the first assertion of Theorem 3.4, we have

$$x_n \asymp s_{pn}(I_g) \lesssim \sqrt{\tau_n(\nu_g)}, \quad n \geqslant 1.$$

Then there exists $B > 0$ such that

$$\sum_j h_\beta\left(\frac{1}{B}\sqrt{\tau_j(\nu_g)}\right) \leqslant \sum_j h_\beta(x_j) \leqslant \sum_j h_\beta\left(B\sqrt{\tau_j(\nu_g)}\right).$$

Lemma 3.14 yields $x_n^\beta \asymp \tau_n^{\beta/2}(\nu_g)$, that is, $x_n \asymp \sqrt{\tau_n(\nu_g)}$.

The result in the case $(\tau_n(\nu_g))_n \in \mathcal{R}_{2\gamma,2\alpha}$ can be proved in the same way. □

Using Theorems 3.4 and 3.13 with the same line of reasoning as in the proofs of Theorem 1.2 and Proposition 5.2 of **[BEMN1]**, one can obtain the following result. We omit the proof here.

Corollary 3.15. *Let* $g \in \mathrm{VMOA}$*. We have*

1. $s_n(I_g) = o\left(\frac{1}{n}\right) \quad \Longrightarrow \quad I_g = 0.$
2. $s_n(I_g) = O\left(\frac{1}{n}\right) \quad \Longleftrightarrow \quad g' \in H^1.$

4

COMPOSITION OPERATORS ON WEIGHTED ANALYTIC SPACES

4.1 INTRODUCTION

Let $\alpha > -1$, and let $\mathcal{H}_\alpha$ denote a scale of Hilbert spaces of analytic functions

$$\mathcal{H}_\alpha = \left\{ f \in \mathrm{Hol}(\mathbb{D}) : \int_{\mathbb{D}} |f'(z)|^2 \, dA_\alpha(z) < \infty \right\},$$

where $dA_\alpha(z) = (1-|z|^2)^\alpha \, dA(z)$. In case $\alpha \in (-1, 1)$, $\mathcal{H}_\alpha$ corresponds to the standard Dirichlet spaces $\mathcal{D}_\alpha$. The parameter value $\alpha = 1$, corresponds to the Hardy space H^2. In case $\alpha > 1$, $\mathcal{H}_\alpha$ cover the standard Bergman spaces $A^2_{\alpha-2}$. In [**LZ2**], D. Luecking and K. Zhu characterized the membership of C_φ in the class $\mathcal{S}_p(\mathcal{H}_\alpha)$, $p > 0$, in case $\alpha \geqslant 1$. J. Pau and P. A. Pérez in [**PP12**] obtained an analogous characterization for the standard Dirichlet spaces $\mathcal{D}_\alpha$, $\alpha \in (0, 1)$. Let φ be an analytic self map of $\mathbb{D}$ and let $\alpha > -1$. The Nevanlinna counting function $N_{\varphi,\alpha}$ of φ associated to $\mathcal{H}_\alpha$ is defined by

$$N_{\varphi,\alpha}(w) = \sum_{w=\varphi(z)} \left(1-|z|^2\right)^\alpha \quad \text{if } w \in \varphi(\mathbb{D}); \qquad N_{\varphi,\alpha}(w) = 0 \quad \text{if } w \notin \varphi(\mathbb{D}).$$

We summarize the results obtained in [**LZ2**, **PP12**] as follows. Let $\alpha > 0$ and $p > 0$. Then

$$C_\varphi \in S_p(\mathcal{H}_\alpha) \Longleftrightarrow \int_{\mathbb{D}} \left(\frac{N_{\varphi,\alpha}(w)}{(1-|w|)^\alpha} \right)^{p/2} d\lambda(w) < \infty, \tag{9}$$

where $d\lambda(z) := dA(z)/\left(1-|z|^2\right)^2$ is the Möbius invariant measure on $\mathbb{D}$.

In this chapter, we study composition operators acting on $\mathcal{D}_\omega$ for ω in the class $\mathcal{C}_{p_0,t}$. We obtain characterizations of boundedness, compactness and Schatten classes membership. Our results cover Bekollé-Bonami weights, superharmonic weights and the radial admissible weights introduced by K. Kellay and P. Lefèvre in [**kellay20122**]. In particular, we are interested in perturbed superharmonic weights on $\mathbb{D}$. Let $u \in \mathcal{C}^2(\mathbb{D})$ be a positive superharmonic function on $\mathbb{D}$. The function u admits the representation

$$u(z) = \int_{\mathbb{D}} \log \left| \frac{1-\overline{\zeta}z}{z-\zeta} \right| \frac{d\sigma(\zeta)}{1-|\zeta|^2} + P_\nu(z) =: S_\sigma(z) + P_\nu(z), \tag{10}$$

for a unique finite positive Borel measure σ on $\mathbb{D}$ and a unique finite positive Borel measure ν on $\mathbb{T}$(see [**Alema2**]), where

$$P_\nu(z) := \int_{\mathbb{T}} \frac{1-|z|^2}{|1-\overline{\xi}z|^2} d\nu(\xi)$$

is the Poisson transform of the measure ν. Let ω be a weight of the form $\omega(z) = (1-|z|^2)^\alpha u(z)$, $\alpha > -1$. The space $\mathcal{D}_\omega$ is called the perturbed superharmonically weighted Dirichlet space. We adopt the notation $\mathcal{D}_\sigma$ (resp. $\mathcal{D}_\nu$) instead of $\mathcal{D}_{S_\sigma}$ (resp. $\mathcal{D}_{P_\nu}$). The generalized Nevanlinna counting function of φ associated with a weight ω is given by

$$N_{\varphi,\omega}(w) = \sum_{w=\varphi(z)} \omega(z) \text{ if } w \in \varphi(\mathbb{D}) \ ; \ N_{\varphi,\omega}(w) = 0 \text{ if } w \notin \varphi(\mathbb{D}).$$

We write $N_{\varphi,\nu}$ instead of N_{φ,P_ν}. In **[SS2]**, D. Sarason and O. Silva characterized boundedness and compactness of operators $C_\varphi\colon \mathcal{D}_\nu \to \mathcal{D}_\nu$ in terms of $N_{\varphi,\nu}$. They proved that C_φ is bounded (resp. compact) on $\mathcal{D}_\nu$ if and only if

$$\int_{\Delta_w} \frac{N_{\varphi,\nu}(z)}{P_\nu(z)} dA(z) = O(|\Delta_w|) \qquad \text{(resp. } o(|\Delta_w|) \text{ as } |w| \to 1^-),$$

where $\Delta_w = \Delta(w, \frac{1}{2})$. In **[EMMN22]**, O. El-Fallah, H. Mahzouli, I. Marrhich and H. Naqos proved that $C_\varphi \in \mathcal{S}_p(\mathcal{D}_\nu)$, for $p > 0$, if and only if

$$\sum_{n=0}^{+\infty} \sum_{j=0}^{2^n-1} \left(\frac{1}{|R_{n,j}|} \int_{R_{n,j}} \frac{N_{\varphi,\nu}(z)}{P_\nu(z)} dA(z) \right)^{\frac{p}{2}} < \infty.$$

On the space $\mathcal{D}_\sigma$, G. Bao, N.G. Göğüş and S. Pouliasis **[BAO2018CJM2]** proved that C_φ is bounded (resp. compact) if and only if

$$\check{N}_{\varphi,\sigma}(w) = O(U_\sigma(w)) \qquad \text{(resp. } o(U_\sigma(w)) \text{ as } (|w| \to 1^-)),$$

with $U_\sigma(z) = \displaystyle\int_{\mathbb{D}} \frac{1-|z|^2}{|1-\bar{\zeta} z|^2} d\sigma(\zeta)$ and

$$\check{N}_{\varphi,\sigma}(w) = \sum_{w=\varphi(z)} U_\sigma(z) \text{ if } w \in \varphi(\mathbb{D}) \ ; \ \check{N}_{\varphi,\sigma}(w) = 0 \text{ if } w \notin \varphi(\mathbb{D}).$$

We characterize boundedness, compactness and Schatten classes membership of composition operators on perturbed superharmonically weighted Dirichlet spaces $\mathcal{D}_\omega$ with $\omega(z) = (1-|z|^2)^\alpha \left(S_\sigma(z) + P_\nu(z)\right)$. More precisely, we prove that $C_\varphi\colon \mathcal{D}_\sigma \to \mathcal{D}_\sigma$ belongs to $\mathcal{S}_p(\mathcal{D}_\sigma)$, for $p > 0$, if and only if

$$\int_{\mathbb{D}} \left(\frac{\check{N}_{\varphi,\sigma}(z)}{U_\sigma(z)} \right)^{p/2} d\lambda(z) < \infty.$$

We are interested also in composition operators acting on weighted Bergman spaces. For the case of a Bekollé-Bonami weight ω, O. Constantin obtained, in **[CONSTANTIN20102]**, a characterization of the membership of C_φ in $\mathcal{S}_p(A^2_\omega)$, for $p \geqslant 2$. We extend this result to the general case of $\omega \in \mathcal{C}_{p_0,t}$ and for all Schatten classes $\mathcal{S}_p(A^2_\omega)$, $p > 0$.

Throughout this paper, we decompose $\mathbb{D}$ by using the disks $\Delta(z,r)$, $0 < r < 1$. The sets $\Delta(z,r)$ give a $(\rho,\delta)-$lattice of $\mathbb{D}$ for $\rho(z) = (1-|z|^2)/2$ and for some choice of δ. Let $(\Delta(z_n,\delta))_n$ be the corresponding $(\rho,\delta)-$lattice of $\mathbb{D}$ and let $(\Delta_n)_n$ be an enumeration of $\Delta(z_n,\delta)$. Let $b > 1$ such that $b\Delta_n = \Delta(z_n, b\delta)$ is a covering of $\mathbb{D}$ of finite multiplicity, see **[Oleinik19782]** and **[EMMN2]** for details and generalization.

4.2 COMPOSITION OPERATORS ON WEIGHTED DIRICHLET SPACES.

4.2.1 *General results*

Let ω be a weight in the class $\mathcal{C}_{p_0,t}$ for some $p_0 > 1$ and $t \geqslant 0$. The associated weighted Bergman space is given by

$$A^2_\omega = \left\{ f \in \mathrm{Hol}(\mathbb{D}) : \ \|f\|_{A^2_\omega} := \left(\int_{\mathbb{D}} |f(z)|^2 \omega(z) dA(z) \right)^{1/2} < \infty \right\}.$$

Notice that A^2_ω is a reproducing kernel Hilbert space since each point evaluation $e_z : A^2_\omega \to \mathbb{C}$, which takes f to $f(z)$, is a bounded linear functional on A^2_ω, see **[BMvolterra2]**. The reproducing kernel of A^2_ω will be denoted by K^ω. The Toeplitz operator T_μ, associated with a positive Borel measure μ on $\mathbb{D}$, acting on A^2_ω is the transformation

$$T_\mu f(z) = \int_{\mathbb{D}} f(\zeta) K^\omega(z,\zeta) \omega(\zeta) d\mu(\zeta), \quad f \in A^2_\omega, \ z \in \mathbb{D}.$$

In the sequel, for a positive Borel measure μ on $\mathbb{D}$, we denote $d\mu_\omega = \omega d\mu$. The following results are proved in **[BMvolterra2]**.

Theorem A. Let ω be a weight in the class $\mathcal{C}_{p_0,t}$ for some $p_0 > 1$ and $t \geqslant 0$, and let μ be a finite positive Borel measure on $\mathbb{D}$. The following assertions are equivalent.

1. The Toeplitz operator T_μ is bounded (resp. compact) on A^2_ω.
2. $\mu_\omega(\Delta_n) = O\left(A_\omega(\Delta_n)\right)$ (resp. $\mu_\omega(\Delta_n) = o\left(A_\omega(\Delta_n)\right)$), $n \to \infty$.

Theorem B. Let ω be a weight in the class $\mathcal{C}_{p_0,t}$ for some $p_0 > 1$ and $t \geqslant 0$. Let μ be a finite positive Borel measure on $\mathbb{D}$ such that T_μ is compact on A^2_ω, and let $p > 0$. Then T_μ belongs to $\mathcal{S}_p(A^2_\omega)$ if and only if

$$\sum_{n=0}^{\infty} \left(\frac{\mu_\omega(\Delta_n)}{A_\omega(\Delta_n)} \right)^p < \infty.$$

We will use a known relationship of Toeplitz operators and composition operators in order to deduce boundedness, compactness and Schatten classes membership for composition operators. Before going into the details, we need the following lemma which states that for $\omega \in \mathcal{C}_{p_0}$, $(p_0 > 1)$, the point evaluations on $\mathcal{D}_\omega$ are bounded. In particular, $\mathcal{D}_\omega$ is a reproducing kernel Hilbert space.

Lemma 4.1. *Suppose that* ω *is a weight such that* $\omega \in \mathcal{C}_{p_0}$ *for some* $p_0 > 1$. *Then, each point evaluation is bounded on* $(\mathcal{D}_\omega, \|.\|_{\mathcal{D}_\omega})$.

Proof. Fix z in $\mathbb{D}$ and let $f \in \mathcal{D}_\omega$. We have $|f(z) - f(0)|^2 \leqslant \int_0^1 |f'(sz)|^2 \, ds$. Since $f' \in A^2_\omega$ and $\omega \in \mathcal{C}_{p_0}$ then

$$|f'(sz)|^2 \lesssim \frac{1}{(1-|sz|^2)^2\tilde{\omega}(sz)} \|f'\|^2_{A^2_\omega}, \ s \in [0,1].$$

see **[AC12]** or **[BMvolterra2]**. Let $r \in (|z|, 1)$. We have

$$\inf_{w \in [0,z]} \tilde{\omega}(w) \geqslant \inf_{w \in \Delta(0,r)} \tilde{\omega}(w) \gtrsim \tilde{\omega}(0) > 0,$$

since $\tilde{\omega}(w) \asymp \tilde{\omega}(0)$ when $w \in \Delta(0,r)$ (see Lemma 2.2 in **[CONSTANTIN20102]**). We obtain

$$|f(z) - f(0)|^2 \lesssim \|f'\|^2_{A^2_\omega} \int_0^1 \frac{1}{(1-|sz|^2)^2\tilde{\omega}(sz)} ds \lesssim \|f'\|^2_{A^2_\omega} \leqslant \|f\|^2_{\mathcal{D}_\omega}.$$

Consequently $|f(z)|^2 \lesssim |f(z) - f(0)|^2 + |f(0)|^2 \lesssim \|f\|^2_{\mathcal{D}_\omega}$. ☐

Theorem 4.2. *Suppose that* ω *is a weight such that* $\omega \in \mathcal{C}_{p_0,t}$ *for some* $p_0 > 1$ *and* $t \geqslant 0$. *Let* φ *be an analytic self-map of* $\mathbb{D}$. *Then*

1. C_φ *is bounded on* $\mathcal{D}_\omega$ *if and only if*

$$\int_{\Delta_n} N_{\varphi,\omega}(z) dA(z) \lesssim \int_{\Delta_n} \omega(z) dA(z), \quad n \in \mathbb{N}.$$

2. C_φ *is compact on* $\mathcal{D}_\omega$ *if and only if*

$$\int_{\Delta_n} N_{\varphi,\omega}(z) dA(z) = o\left(\int_{\Delta_n} \omega(z) dA(z)\right), \quad (n \to \infty).$$

Proof. Suppose that C_φ is bounded on $\mathcal{D}_\omega$. Let $V_\omega : \mathcal{D}_\omega \to A^2_\omega$ be the bounded operator defined by $V_\omega f = f'$ and let $D_{\varphi,\omega} : A^2_\omega \to A^2_\omega$ be the operator defined by

$$D_{\varphi,\omega} f := V_\omega C_\varphi V^*_\omega f.$$

By a direct calculation we have $D_{\varphi,\omega} f = \varphi'.f \circ \varphi$, $f \in A^2_\omega$. The operator $D^*_{\varphi,\omega} D_{\varphi,\omega}$ is then bounded on A^2_ω. For $f \in A^2_\omega$, the change of variable formula **[A2]** gives

$$\begin{aligned} D^*_{\varphi,\omega} D_{\varphi,\omega} f(z) &= \langle D_{\varphi,\omega} f, D_{\varphi,\omega} K^\omega_z \rangle_{A^2_\omega} \\ &= \int_{\mathbb{D}} f(\xi) \overline{K^\omega_z(\xi)} \omega(\xi) d\omega_\varphi(\xi) \\ &= T_{\omega_\varphi} f(z), \end{aligned}$$

where ω_φ is the measure defined on $\mathbb{D}$ by $d\omega_\varphi = \frac{N_{\varphi,\omega}}{\omega} dA$. It follows that T_{ω_φ} is bounded on A^2_ω. By (1) of Theorem A, we deduce that

$$\int_{\Delta_n} N_{\varphi,\omega}(z)dA(z) \lesssim \int_{\Delta_n} \omega(z)dA(z), \quad n \in \mathbb{N}. \tag{11}$$

Conversely, assume that (11) holds and let $f \in \mathcal{D}_\omega$. By using once again the change of variable formula we get

$$\begin{aligned} \|C_\varphi f\|^2_{\mathcal{D}_\omega} &= |f(\varphi(0))|^2 + \langle T_{\omega_\varphi} f', f' \rangle_{A^2_\omega} \\ &\leqslant |f(\varphi(0))|^2 + \|T_{\omega_\varphi}\| \, \|f'\|^2_{A^2_\omega} \\ &\lesssim |f(\varphi(0))|^2 + \|f\|^2_{\mathcal{D}_\omega}. \end{aligned}$$

By Lemma 4.1 we deduce that $\|C_\varphi f\|^2_{\mathcal{D}_\omega} \lesssim \|f\|^2_{\mathcal{D}_\omega}$, $f \in \mathcal{D}_\omega$. Therefore, C_φ is bounded on $\mathcal{D}_\omega$.

To prove the second assertion we may assume that C_φ is bounded on $\mathcal{D}_\omega$. We have

$$C_\varphi f = V^*_\omega D_{\varphi,\omega} V_\omega f + Kf, \quad f \in \mathcal{D}_\omega, \tag{12}$$

where $Kf(z) = f(\varphi(0))$, $f \in \mathcal{D}_\omega$ and $z \in \mathbb{D}$. By Lemma 4.1 the operator K is bounded, then by the definition of $D_{\varphi,\omega}$ and by formula (12) we deduce that the operator C_φ is compact on $\mathcal{D}_\omega$ if and only if $D_{\varphi,\omega}$ is compact on A^2_ω. In other words, C_φ is compact on $\mathcal{D}_\omega$ if and only if T_{ω_φ} is compact on A^2_ω. The result follows now by (2) of Theorem A. □

Theorem 4.3. *Suppose that ω is a weight such that $\omega \in \mathcal{C}_{p_0,t}$ for some $p_0 > 1$ and $t \geqslant 0$. Assume that φ is an analytic self-map of $\mathbb{D}$ such that C_φ is compact on $\mathcal{D}_\omega$. Then C_φ belongs to $\mathcal{S}_p(\mathcal{D}_\omega)$, for $p > 0$, if and only if*

$$\sum_{n=0}^{\infty} \left(\frac{\int_{\Delta_n} N_{\varphi,\omega}(z)dA(z)}{\int_{\Delta_n} \omega(z)dA(z)} \right)^{p/2} < \infty.$$

Proof. Note that, by (12), C_φ belongs to $\mathcal{S}_p(\mathcal{D}_\omega)$ if and only if $D_{\varphi,\omega}$ belongs to $\mathcal{S}_p(A^2_\omega)$. Since

$$D^*_{\varphi,\omega} D_{\varphi,\omega} f = T_{\omega_\varphi} f, \quad f \in A^2_\omega,$$

then $C_\varphi \in \mathcal{S}_p(\mathcal{D}_\omega)$ if and only if $T_{\omega_\varphi} \in \mathcal{S}_{\frac{p}{2}}(A^2_\omega)$. The result follows by Theorem B. □

If ω is a weight such that for some (equivalently for all) $r \in (0,1)$ we have

$$\omega(z) \asymp \omega(w), \; w \in \Delta(z,r), \tag{13}$$

then $\omega \in \mathcal{C}_{p_0}$ for all $p_0 > 1$. Moreover, under condition (13), we have

$$\frac{1}{|\Delta(z,\delta)|} \int_{\Delta(z,\delta)} \omega(\zeta)dA(\zeta) \asymp \omega(z), \; z \in \mathbb{D},$$

for all $\delta \in (0,1)$. In this case Theorems 4.2 and 4.3 can be reduced to the following result.

Corollary 4.4. *Let φ be an analytic self-map of $\mathbb{D}$. Suppose that ω is a weight satisfying (13) and such that $\omega_t \lesssim \omega$ for some $t \geqslant 0$. Then*

1. *C_φ is bounded on $\mathcal{D}_\omega$ if and only if*
$$\frac{1}{|\Delta_n|}\int_{\Delta_n} \frac{N_{\varphi,\omega}(z)}{\omega(z)} dA(z) = O(1), \quad \forall n \in \mathbb{N}.$$
2. *C_φ is compact on $\mathcal{D}_\omega$ if and only if*
$$\frac{1}{|\Delta_n|}\int_{\Delta_n} \frac{N_{\varphi,\omega}(z)}{\omega(z)} dA(z) = o(1), \quad (n \to \infty).$$
3. *C_φ belongs to $\mathcal{S}_p(\mathcal{D}_\omega)$, for $p > 0$, if and only if*
$$\sum_{n=0}^{\infty}\left(\frac{1}{|\Delta_n|}\int_{\Delta_n} \frac{N_{\varphi,\omega}(z)}{\omega(z)} dA(z)\right)^{p/2} < \infty.$$

4.2.2 *Radial weights*

A radial weight ω in $\mathcal{C}^2[0,1)$ is called admissible if
$(\mathcal{W}_1)$ ω is non-increasing,
$(\mathcal{W}_2)$ $\omega(r)(1-r)^{-(1+\delta)}$ is non-decreasing for some $\delta > 0$,
$(\mathcal{W}_3)$ $\lim_{r\to 1^-} \omega(r) = 0$,
$(\mathcal{W}_4)$ One of the two properties of convexity is fulfilled

$$\begin{cases} (\mathcal{W}_4^{(I)}) : \omega \text{ is convex and } \lim_{r\to 1^-} \omega'(r) = 0 \\ (\mathcal{W}_4^{(II)}) : \omega \text{ is concave.} \end{cases}$$

If ω satisfies $(\mathcal{W}_1)$-$(\mathcal{W}_3)$ and $(\mathcal{W}_4^{(I)})$ (respectively $(\mathcal{W}_4^{(II)})$), then we say that ω is (I)-admissible (respectively (II)-admissible). K. Kellay and P. Lefèvre **[kellay20122]** proved the following result.

Theorem C. 1. Let ω be a (II)-admissible weight. Then C_φ is bounded on $\mathcal{D}_\omega$ if and only if $N_{\varphi,\omega}(z) = O(\omega(z))$, $z \in \mathbb{D}$.

2. Let ω be an admissible weight. Then C_φ is compact on $\mathcal{D}_\omega$ if and only if $N_{\varphi,\omega}(z) = o(\omega(z))$, $|z| \to 1^-$.

As noticed in **[kellay20122]**, C_φ is always bounded on $\mathcal{D}_\omega$ if ω is a (I)-admissible weight. We describe in the following theorem the membership of C_φ in $\mathcal{S}_p(\mathcal{D}_\omega)$ for admissible weights.

Theorem 4.5. *Let φ be an analytic self-map of $\mathbb{D}$ and let ω be an admissible weight. Then C_φ belongs to $\mathcal{S}_p(\mathcal{D}_\omega)$, for $p > 0$, if and only if*

$$\int_{\mathbb{D}} \left(\frac{N_{\varphi,\omega}(z)}{\omega(z)}\right)^{p/2} d\lambda(z) < \infty. \tag{14}$$

Proof. Let z in $\mathbb{D}$ and let $w \in \Delta(z,r)$. Suppose that $|z| \leqslant |w|$. By $(\mathcal{W}_1)$, we have $\omega(z) \geqslant \omega(w)$, and by $(\mathcal{W}_2)$ we have

$$\omega(z) = \frac{\omega(z)}{(1-|z|)^{\delta+1}}(1-|z|)^{\delta+1} \leqslant \frac{\omega(w)}{(1-|w|)^{\delta+1}}(1-|z|)^{\delta+1} \asymp \omega(w),$$

where δ is the constant in $(\mathcal{W}_2)$. Similarly if $|w| \leqslant |z|$, we have $\omega(z) \asymp \omega(w)$. Therefore, ω satisfies the condition (13). On the other hand, the conditions $(\mathcal{W}_1)$ and $(\mathcal{W}_2)$ imply that $\omega_{2+2\delta} \lesssim \omega$ (see **[kellay20122]**). By (3) of Corollary 4.4, it follows that C_φ belongs to $\mathcal{S}_p(\mathcal{D}_\omega)$, for $p > 0$, if and only if

$$\sum_{n=0}^{\infty} \left(\frac{1}{|\Delta_n|} \int_{\Delta_n} \frac{N_{\varphi,\omega}(z)}{\omega(z)} dA(z) \right)^{p/2} < \infty. \tag{15}$$

Now, since $N_{\varphi,\omega}$ satisfies the sub-mean-value property, that is

$$N_{\varphi,\omega}(z) \lesssim \frac{1}{|\Delta(z,r)|} \int_{\Delta(z,r)} N_{\varphi,\omega}(w) dA(w), \quad z \in \mathbb{D}, \tag{16}$$

(see Lemma 2.2 and Lemma 2.3 in **[kellay20122]**) then, similarly to the proof of Theorem 4.9 bellow, the condition (15) is equivalent to the condition (14). □

4.2.3 *Remarks and examples*

- A radial weight ω is called almost standard if ω satisfies $(\mathcal{W}_1) - (\mathcal{W}_3)$. In the recent paper **[kellay20222]**, K. Esmaeili and K. Kellay studied the boundedness and the compactness of weighted composition operators on Bergman and Dirichlet spaces associated with almost standard weights. As noticed in the proof of Theorem 4.5, every almost standard weight satisfies (13) and $\omega_{2+2\delta} \lesssim \omega$. Therefore, Corollary 4.4 can be applied for any almost standard weight.

- Let ω be a weight on $\mathbb{D}$ such that there are constants $s \in (-1,0)$ and $\eta \geqslant 0$ for which

  $$\omega_{s,\eta}(z) := \int_{\mathbb{D}} \frac{\omega(\xi)(1-|\xi|^2)^s(1-|z|^2)^\eta}{|1-\overline{\xi}z|^{2+s+\eta}} dA(\xi) \lesssim \omega(z), \quad z \in \mathbb{D}. \tag{17}$$

  This condition is similar to the one that appears in **[BWZ20182]**. Note that the classical weights $\omega_\alpha(z) = (1-|z|^2)^\alpha$, for $\alpha > -1$, satisfy condition (17) since for $s \in (\max(-1,-1-\alpha),0)$ and $\eta > \max(0,\alpha)$ we have

  $$\int_{\mathbb{D}} \frac{(1-|z|^2)^{\alpha+s}}{|1-\overline{w}z|^{2+s+\eta}} dA(z) \asymp (1-|w|^2)^{\alpha-\eta}, \quad w \in \mathbb{D}.$$

  The following lemma is stated in **[BMvolterra2]**.

Lemma A. Let ω be a weight satisfying condition (17) for some constants $s \in (-1,0)$ and $\eta \geqslant 0$. Then ω satisfies (17) for all $s' > s$ and $\beta > \eta$.

- If ω is a weight such that $\frac{\omega}{(1-|z|^2)^\eta} \in B_{p_0}(\eta)$ for some $p_0 > 1$ and $\eta > -1$, then $\omega \in \mathcal{C}_{p_0,t}$ for all $t \geqslant p(\eta+2)-2$ (see **[AC12]**).
- If ω is a weight on $\mathbb{D}$ that satisfies condition (17), then by Lemma A we obtain $\omega_\eta \lesssim \omega$ for some $\eta \geqslant 0$. Using **[Aleman20192]** (see the proof of (c) $\Rightarrow$ (b)), we find that $(1-|z|^2)^{-\eta}\omega$ belongs to $B_{p_0}(\eta)$ for all $p_0 > 1$. We conclude that if ω satisfies (17), then $\omega \in \mathcal{C}_{p_0,t}$ for all $p_0 > 1$ and for some $t \geqslant 0$.
- As examples, we consider here weights which appear in **[BWZ20182]**. Let μ be a finite positive Borel measure on $\mathbb{D}$ and let $b \in \mathbb{R}$ such that

  $$\int_{\mathbb{D}} (1-|w|^2)^b d\mu(w) < \infty.$$

  Let ν be a finite positive Borel measure on $\mathbb{T}$. Let $a > -1$ and let $c < a+2$. Using **[OF2]**, one can verify that the weight

  $$\omega(z) = (1-|z|^2)^a \left(\int_{\mathbb{D}} \frac{(1-|w|^2)^b}{|1-z\overline{w}|^c} d\mu(w) + \int_{\mathbb{T}} \frac{d\nu(\zeta)}{|1-\overline{\zeta}z|^c} \right)$$

  satisfies condition (17) for all $\eta > a$ and $c-a-2 < s < 0$. Notice that the previous weight satisfies, in addition, condition (13).

We end this subsection by the following remark.

Remark 4.6. Let ω be a weight that satisfies condition (17) with $s \in (-1,0)$ and $t \geqslant 0$. Then the weight $(1-|z|^2)^\alpha\omega$ satisfies (17) for all $\alpha > s$. Indeed, if $\alpha > 0$, then for $\epsilon \in (0,1)$ such that $\alpha-\epsilon > 0$ and $\beta \geqslant t+\alpha-\epsilon$, by Lemma A, we have

$$\int_{\mathbb{D}} \frac{\omega(z)(1-|z|^2)^{\alpha-\epsilon}(1-|w|^2)^{\beta-\alpha+\epsilon}}{|1-\overline{w}z|^{2+\beta}} dA(z) \lesssim \omega(w).$$

If $\alpha \in (s,0)$, then for $s' = s-\alpha$ and $\beta = t+\alpha+1$ once again by Lemma A, we obtain

$$\begin{aligned}\int_{\mathbb{D}} \frac{\omega(z)(1-|z|^2)^{\alpha+s'}(1-|w|^2)^{\beta}}{|1-\overline{w}z|^{2+s'+\beta}} dA(z) &= \int_{\mathbb{D}} \frac{\omega(z)(1-|z|^2)^{s}(1-|w|^2)^{t+\alpha+1}}{|1-\overline{w}z|^{3+s+t}} dA(z) \\ &\lesssim (1-|w|^2)^\alpha\omega(w).\end{aligned}$$

4.2.4 *Composition operators on Dirichlet spaces induced by perturbed superharmonic weights.*

In this subsection we examine the case of perturbed superharmonic weights. We begin with the following proposition.

Proposition 4.7. *Let* $\omega \in \mathcal{C}^2(\mathbb{D})$ *be a positive superharmonic function on* $\mathbb{D}$*. Then* $(1-|z|^2)^\alpha\omega$ *verifies condition* (17) *for all* $\alpha > -1$*.*

Proof. Let σ and ν be the unique finite positive Borel measures on $\mathbb{D}$ and $\mathbb{T}$ respectively such that $\omega = S_\sigma + P_\nu$. It is proved in **[liu2015characterizations2]** that P_ν satisfies (17) for all $s > -1$ and $t > 1$. On the other hand, note that for $s \in (-1, 0)$ and $t > 1$ we have

$$\begin{aligned}\int_{\mathbb{D}} \frac{(1-|z|^2)^s}{|1-\overline{w}z|^{2+s+t}} \log\left|\frac{1-\overline{\zeta}z}{z-\zeta}\right| dA(z) &\asymp \int_{\mathbb{D}} \frac{(1-|z|^2)^s}{|1-\overline{w}z|^{2+s+t}}\left(1-\left|\frac{z-\zeta}{1-\overline{\zeta}z}\right|^2\right) dA(z)\\ &= (1-|\zeta|^2)\int_{\mathbb{D}} \frac{(1-|z|^2)^{s+1}}{|1-\overline{w}z|^{2+s+t}|1-\overline{\zeta}z|^2} dA(z)\\ &\lesssim \frac{(1-|\zeta|^2)}{(1-|w|^2)^{t-1}|1-\overline{\zeta}w|^2}.\end{aligned}$$

Therefore, for $s \in (-1, 0)$ and $t > 1$, we have

$$\begin{aligned}\int_{\mathbb{D}} \frac{S_\sigma(z)(1-|z|^2)^s}{|1-\overline{w}z|^{2+s+t}} dA(w) &= \int_{\mathbb{D}}\left(\int_{\mathbb{D}} \log\left|\frac{1-\overline{\zeta}z}{z-\zeta}\right| \frac{d\sigma(\zeta)}{1-|\zeta|^2}\right) \frac{(1-|z|^2)^s}{|1-\overline{w}z|^{2+s+t}} dA(z)\\ &= \int_{\mathbb{D}}\left(\int_{\mathbb{D}} \frac{(1-|z|^2)^s}{|1-\overline{w}z|^{2+s+t}} \log\left|\frac{1-\overline{\zeta}z}{z-\zeta}\right| dA(z)\right) \frac{d\sigma(\zeta)}{1-|\zeta|^2}\\ &\lesssim \frac{1}{(1-|w|^2)^t}\int_{\mathbb{D}}\left(\frac{(1-|\zeta|^2)(1-|w|^2)}{|1-\overline{\zeta}w|^2}\right)\frac{d\sigma(\zeta)}{1-|\zeta|^2}\\ &\lesssim \frac{S_\sigma(w)}{(1-|w|^2)^t}.\end{aligned}$$

Thus S_σ satisfies (17). It follows that $S_\sigma + P_\nu$ satisfies (17) for all $s > -1$ and $t > 1$. Therefore, by Remark 4.6, $(1-|z|^2)^\alpha\omega$ verifies (17) for all $\alpha > -1$. ☐

In the rest of this subsection, let $\omega(z) = (1-|z|^2)^\alpha\left(S_\sigma(z) + P_\nu(z)\right)$, for a fixed $\alpha > -1$ and finite positive Borel measures σ and ν on $\mathbb{D}$ and $\mathbb{T}$ respectively. Let $\breve{\omega}$ be the weight given by $\breve{\omega}(z) = (1-|z|^2)^\alpha\left(U_\sigma(z) + P_\nu(z)\right), \quad z \in \mathbb{D}$. We have the following result.

Theorem 4.8. *Let* ω *and* $\breve{\omega}$ *be as given above, and let* φ *be an analytic self-map of* $\mathbb{D}$*. The following assertions hold.*

1. C_φ *is bounded (resp. compact) on* $\mathcal{D}_\omega$ *if and only if*

$$\frac{1}{|\Delta_n|}\int_{\Delta_n} \frac{N_{\varphi,\breve{\omega}}(z)}{\breve{\omega}(z)} dA(z) = O(1), \quad (\textit{resp. } o(1) \textit{ as } n \to \infty.)$$

2. C_φ *belongs to* $\mathcal{S}_p(\mathcal{D}_\omega)$, *for* $p > 0$, *if and only if*
$$\sum_{n=0}^{+\infty} \left(\frac{1}{|\Delta_n|} \int_{\Delta_n} \frac{N_{\varphi,\check{\omega}}(z)}{\check{\omega}(z)} dA(z) \right)^{\frac{p}{2}} < \infty.$$

Proof. Let $f \in \mathcal{D}_\omega$. We have

$$\begin{aligned}
&\mathcal{D}_\omega(f) - \int_{\mathbb{D}} |f'(z)|^2 (1-|z|^2)^\alpha P_\nu(z) dA(z) \\
&= \int_{\mathbb{D}} \int_{\mathbb{D}} |f'(z)|^2 (1-|z|^2)^\alpha \log \left| \frac{1-\bar{\zeta} z}{z-\zeta} \right| dA(z) \frac{d\sigma(\zeta)}{1-|\zeta|^2} \\
&\asymp \int_{\mathbb{D}} \int_{\mathbb{D}} |f'(z)|^2 (1-|z|^2)^\alpha \left(1 - \left| \frac{z-\zeta}{1-\bar{\zeta} z} \right|^2 \right) dA(z) \frac{d\sigma(\zeta)}{1-|\zeta|^2} \\
&= \int_{\mathbb{D}} \int_{\mathbb{D}} |f'(z)|^2 (1-|z|^2)^\alpha \frac{1-|z|^2}{|1-\bar{\zeta} z|^2} dA(z) d\sigma(\zeta) \\
&= \int_{\mathbb{D}} |f'(z)|^2 (1-|z|^2)^\alpha U_\sigma(z) dA(z).
\end{aligned}$$

It follows that $\mathcal{D}_\omega = \mathcal{D}_{\check{\omega}}$ with equivalent norms. Therefore, C_φ is bounded (resp. compact) on $\mathcal{D}_\omega$ if and only if C_φ is bounded (resp. compact) on $\mathcal{D}_{\check{\omega}}$. Taking into account that $\check{\omega}(z) \asymp \check{\omega}(z_n)$ for $z \in \Delta_n$, the first assertion follows by combining Proposition 4.7, Lemma A and assertions (1) and (2) of Corollary 4.4.

For the proof of the second assertion, note that since $\mathcal{D}_\omega = \mathcal{D}_{\check{\omega}}$ with equivalent norms then the operator $I : \mathcal{D}_\omega \to \mathcal{D}_{\check{\omega}}$ which takes f to f is bounded and invertible. It follows that I^*, the adjoint of I defined by $\langle I^* f, g \rangle_{\mathcal{D}_\omega} = \langle f, Ig \rangle_{\mathcal{D}_{\check{\omega}}}$ is bounded and invertible on $\mathcal{D}_{\check{\omega}}$. This implies

$$I^* \left(C_{\varphi,\mathcal{D}_{\check{\omega}}} \right)^* \left(C_{\varphi,\mathcal{D}_{\check{\omega}}} \right) I \asymp \left(C_{\varphi,\mathcal{D}_\omega} \right)^* \left(C_{\varphi,\mathcal{D}_\omega} \right),$$

where $C_{\varphi,\mathcal{D}_\omega}$ denote the operator $C_\varphi : \mathcal{D}_{\check{\omega}} \to \mathcal{D}_{\check{\omega}}$ and $C_{\varphi,\mathcal{D}_\omega}$ denote the operator $C_\varphi : \mathcal{D}_\omega \to \mathcal{D}_\omega$. It follows that if $C_{\varphi,\mathcal{D}_\omega}$ is compact, then $C_{\varphi,\mathcal{D}_\omega}$ belongs to $\mathcal{S}_p(\mathcal{D}_\omega)$ if and only if $C_{\varphi,\mathcal{D}_{\check{\omega}}}$ belongs to $\mathcal{S}_p(\mathcal{D}_{\check{\omega}})$. Hence, using once again Proposition 4.7, Lemma A and assertion (3) of Corollary 4.4, we obtain the second assertion of the theorem. □

The following theorem extend the result obtained by J. Pau and P. Pérez **[PP12]** in standard Dirichlet spaces setting to the Green potential of the Riesz measure of any positive superharmonic function. Recall that

$$\check{N}_{\varphi,\sigma}(w) = \sum_{w=\varphi(z)} U_\sigma(z) \ \text{ if } w \in \varphi(\mathbb{D}) \ ; \ \check{N}_{\varphi,\sigma}(w) = 0 \ \text{ if } w \notin \varphi(\mathbb{D}).$$

Theorem 4.9. *Let* $p > 0$ *and let* φ *be an analytic self-map of* $\mathbb{D}$. *Let* σ *be a finite positive measure on* $\mathbb{D}$. *Then,* C_φ *belongs to* $\mathcal{S}_p(\mathcal{D}_\sigma)$ *if and only if*

$$\int_{\mathbb{D}} \left(\frac{\check{N}_{\varphi,\sigma}(z)}{U_\sigma(z)} \right)^{p/2} d\lambda(z) < \infty.$$

Proof. By Theorem 4.8, we have

$$C_\varphi \in \mathcal{S}_p(\mathcal{D}_\sigma) \Longleftrightarrow \sum_{n=0}^{+\infty} \left(\frac{1}{|\Delta_n|} \int_{\Delta_n} \frac{\check{N}_{\varphi,\sigma}(z)}{U_\sigma(z)} dA(z) \right)^{p/2} < \infty.$$

Therefore, it suffices to show that

$$\frac{1}{|\Delta_n|} \int_{\Delta_n} \frac{\check{N}_{\varphi,\sigma}(z)}{U_\sigma(z)} dA(z) \in \ell^{p/2} \Longleftrightarrow \frac{\check{N}_{\varphi,\sigma}(w)}{U_\sigma} \in L^{p/2}(\mathbb{D}, d\lambda).$$

We use for the proof some standard arguments. First, we prove that for all $p > 0$ we have

$$\check{N}_{\varphi,\sigma}(z)^p \lesssim \frac{1}{|\Delta(z,r)|} \int_{\Delta(z,r)} \check{N}_{\varphi,\sigma}(w)^p dA(w), \quad z \in \mathbb{D}. \tag{18}$$

The function $\check{N}_{\varphi,\sigma}$ satisfies the sub-mean-value property, that is

$$\check{N}_{\varphi,\sigma}(z) \lesssim \frac{1}{|\Delta(z,r)|} \int_{\Delta(z,r)} \check{N}_{\varphi,\sigma}(w) dA(w), \quad z \in \mathbb{D},$$

see [**BAO2018CJM2**]. Therefore, there exists a subharmonic function u on $\mathbb{D}$ such that $N_{\varphi,\sigma} \leqslant u$ on $\mathbb{D}$ and $u = N_{\varphi,\sigma}$ almost everywhere on $\mathbb{D}$, see [**LZ2**]. Since

$$u(z)^p \lesssim \frac{1}{|\Delta(z,r)|} \int_{\Delta(z,r)} u(w)^p dA(w), \quad z \in \mathbb{D},$$

by [**LZ2**] we obtain (18). Now, we have

$$\int_{\mathbb{D}} \left(\frac{\check{N}_{\varphi,\sigma}(w)}{U_\sigma(w)} \right)^{p/2} d\lambda(w) \asymp \sum_{n=0}^{+\infty} \int_{\Delta_n} \left(\frac{\check{N}_{\varphi,\sigma}(w)}{U_\sigma(w)} \right)^{p/2} d\lambda(w).$$

Taking into account that $U_\sigma(w) \asymp U_\sigma(z)$ if $w \in \Delta_n$ and $z \in b\Delta_n$, the inequality (18) gives

$$\begin{aligned} \int_{\mathbb{D}} \left(\frac{\check{N}_{\varphi,\sigma}(w)}{U_\sigma(w)} \right)^{p/2} d\lambda(w) &\lesssim \sum_{n=0}^{+\infty} \int_{\Delta_n} \left(\frac{1}{|\Delta_n|} \int_{b\Delta_n} \frac{\check{N}_{\varphi,\sigma}(z)}{U_\sigma(z)} dA(z) \right)^{p/2} d\lambda(w) \\ &\asymp \sum_{n=0}^{+\infty} \left(\frac{1}{|\Delta_n|} \int_{b\Delta_n} \frac{\check{N}_{\varphi,\sigma}(z)}{U_\sigma(z)} dA(z) \right)^{p/2}. \end{aligned}$$

Since $(b\Delta_n)_n$ is a covering of $\mathbb{D}$ of finite multiplicity, then for all n there exist $n_1, n_2, ..., n_N$ such that $b\Delta_n \subset \cup_{k=1}^{N} \Delta_{n_k}$, for some $N \in \mathbb{N}^*$ not depending on n. Hence

$$\int_{b\Delta_n} \frac{\check{N}_{\varphi,\sigma}(z)}{U_\sigma(z)} dA(z) \lesssim \int_{\Delta_{m_n}} \frac{\check{N}_{\varphi,\sigma}(z)}{U_\sigma(z)} dA(z),$$

where m_n is such that $\int_{\Delta_{m_n}} \frac{\check{N}_{\varphi,\sigma}(z)}{U_\sigma(z)} dA(z) = \max_{1\leqslant k\leqslant N} \int_{\Delta_{n_k}} \frac{\check{N}_{\varphi,\sigma}(z)}{U_\sigma(z)} dA(z)$. Therefore

$$\begin{aligned}\int_{\mathbb{D}} \left(\frac{\check{N}_{\varphi,\sigma}(w)}{U_\sigma(w)}\right)^{p/2} d\lambda(w) &\lesssim \sum_{n=0}^{+\infty} \left(\frac{1}{|\Delta_n|}\int_{\Delta_{m_n}} \frac{\check{N}_{\varphi,\sigma}(z)}{U_\sigma(z)} dA(z)\right)^{p/2} \\ &\leqslant \sum_{n=0}^{+\infty} \left(\frac{1}{|\Delta_n|}\int_{\Delta_n} \frac{\check{N}_{\varphi,\sigma}(z)}{U_\sigma(z)} dA(z)\right)^{p/2}.\end{aligned}$$

On the other hand, let $\zeta_n \in \overline{\Delta}_n$ such that $\frac{\check{N}_{\varphi,\sigma}(\zeta_n)}{U_\sigma(\zeta_n)} = \sup_{z\in\Delta_n} \frac{\check{N}_{\varphi,\sigma}(z)}{U_\sigma(z)}$. We have

$$\begin{aligned}\sum_{n=0}^{+\infty} \left(\frac{1}{|\Delta_n|}\int_{\Delta_n} \frac{\check{N}_{\varphi,\sigma}(z)}{U_\sigma(z)} dA(z)\right)^{p/2} &\lesssim \sum_{n=0}^{+\infty} \left(\frac{\check{N}_{\varphi,\sigma}(\zeta_n)}{U_\sigma(\zeta_n)}\right)^{p/2} \\ &\lesssim \sum_{n=0}^{+\infty} \frac{1}{|\Delta_n|}\int_{b\Delta_n} \left(\frac{\check{N}_{\varphi,\sigma}(z)}{U_\sigma(z)}\right)^{p/2} dA(z) \\ &\asymp \sum_{n=0}^{+\infty} \int_{b\Delta_n} \left(\frac{\check{N}_{\varphi,\sigma}(z)}{U_\sigma(z)}\right)^{p/2} d\lambda(z) \\ &\asymp \int_{\mathbb{D}} \left(\frac{\check{N}_{\varphi,\sigma}(z)}{U_\sigma(z)}\right)^{p/2} d\lambda(z).\end{aligned}$$

The proof is complete. □

4.3 COMPOSITION OPERATORS ON WEIGHTED BERGMAN SPACES.

4.3.1 *Radial weights.*

Let $\omega : [0,1) \to (0,\infty)$ be a continuous radial weight. We associate to ω the weight ω_* defined by

$$\omega_*(r) = \int_r^1 (t-r)\omega(t)dt.$$

As pointed in **[kellay20122]**, $A^2_\omega = \mathcal{D}_{\omega_*}$ with equivalent norms and ω_* always satisfies $(\mathcal{W}_1)$, $(\mathcal{W}_3)$ and $(\mathcal{W}_4^{(I)})$. Therefore, as a consequence of Theorem 4.5, we have the following result.

Theorem 4.10. *Let* φ *be an analytic self-map of* $\mathbb{D}$ *and let* $p > 0$. *Let* ω *be a continuous radial weight such that* ω_* *satisfies* $(\mathcal{W}_2)$. *Then*

$$C_\varphi \in \mathcal{S}_p(A^2_\omega) \iff \int_{\mathbb{D}} \left(\frac{N_{\varphi,\omega_*}(z)}{\omega_*(z)}\right)^{p/2} d\lambda(z) < \infty.$$

Corollary 4.11. *Let φ be an analytic self-map of $\mathbb{D}$ and let $p > 0$. Let ω be a continuous radial weight such that ω_* satisfies $(\mathcal{W}_2)$ fore some $\delta > 0$. The following assertions hold.*

1. *If C_φ belongs to $\mathcal{S}_p(H^2)$, then C_φ belongs to $\mathcal{S}_p(A^2_\omega)$.*
2. *If C_φ belongs to $\mathcal{S}_p(A^2_\omega)$, then C_φ belongs to $\mathcal{S}_p(A^2_{\delta-1})$.*

Proof. Since ω_* satisfies $(\mathcal{W}_2)$ and always satisfies $(\mathcal{W}_1)$ then, by **[kellay20122]**, each composition operator induced by the symbol $q_{\varphi(0)}(z) = \frac{\varphi(0)-z}{1-\overline{\varphi(0)}z}$ is bounded on $A^2_\omega = \mathcal{D}_{\omega_*}$. It is also known that each composition operator induced by $q_{\varphi(0)}$ is bounded on $\mathcal{H}_\alpha$. Hence, by standard arguments, we may assume without loss of generality that $\varphi(0) = 0$. The condition $(\mathcal{W}_2)$ gives

$$\frac{\omega_*(r)}{\omega_*(t)} \leqslant \left(\frac{1-r}{1-t}\right)^{1+\delta}, \quad 0 \leqslant r \leqslant t < 1.$$

On the other hand, by a direct calculation, $s \in [0,1) \mapsto \frac{\omega_*(s)}{1-s}$ is a non-increasing function. It follows that

$$\frac{\omega_*(r)}{\omega_*(t)} \geqslant \frac{1-r}{1-t}, \quad 0 \leqslant r \leqslant t < 1.$$

Let $z \in \varphi(\mathbb{D})$ and $w \in \mathbb{D}$ such that $\varphi(w) = z$. By Schwarz's Lemma and the above inequalities, we obtain

$$\left(\frac{1-|w|^2}{1-|z|^2}\right)^{\delta+1} \lesssim \frac{\omega_*(w)}{\omega_*(z)} \lesssim \frac{1-|w|^2}{1-|z|^2}.$$

It follows that

$$\frac{N_{\varphi,\delta+1}(z)}{(1-|z|^2)^{\delta+1}} \lesssim \frac{N_{\varphi,\omega_*}(z)}{\omega_*(z)} \lesssim \frac{N_{\varphi,1}(z)}{(1-|z|^2)}, \quad z \in \mathbb{D}. \tag{19}$$

The assertions of the corollary are obtained by combining Theorem 4.10 and (9). ☐

A radial weight ω belongs to the class $\hat{\mathcal{D}}$ if $\int_r^1 \omega(s)ds \lesssim \int_{\frac{1+r}{2}}^1 \omega(s)ds$, $r \in [0,1)$. J. A. Peláez and J. Rättyä obtained in **[pelaeztrace2]** a trace class criteria for Toeplitz operators on Dirichlet spaces associated with regular weights. They obtained that, for $\omega \in \hat{\mathcal{D}}$, C_φ belongs to $\mathcal{S}_p(A^2_\omega)$, for $p > 0$, if and only if

$$\int_{\mathbb{D}} \left(\frac{N_{\varphi,\omega^*}(z)}{\omega^*(z)}\right)^{p/2} d\lambda(z) < \infty,$$

where $\omega^*(r) := \int_r^1 s\log\left(\frac{s}{r}\right)\omega(s)ds$, $r \in (0,1)$. We point out that Lemma 2.4 in **[kellay20122]** and Theorem 4.5 for (I)-admissible weights still hold if we replace the condition $(\mathcal{W}_2)$ by the following one

$(\mathcal{W}_2')$ there is $\delta > 0$ such that $\omega(r)(1-r)^{-(1+\delta)} \lesssim \omega(t)(1-t)^{-(1+\delta)}, 0 \leqslant r \leqslant t < 1.$

Theorem 4.10 can be applied to continuous weights belonging to $\hat{\mathcal{D}}$ thanks to the following lemma.

Lemma 4.12. *If ω belongs to $\hat{\mathcal{D}}$, then ω_* satisfies $(\mathcal{W}_2')$.*

Proof. Assume that ω belongs to $\hat{\mathcal{D}}$. We have

$$\omega_*(r) \geqslant \int_{\frac{1+r}{2}}^1 (t-r)\omega(t)dt \gtrsim (1-r)\int_{\frac{1+r}{2}}^1 \omega(t)dt \gtrsim (1-r)\int_r^1 \omega(t)dt.$$

It follows that $\omega_*(r) \asymp (1-r)\int_r^1 \omega(t)dt$. On the other hand, since $\omega \in \hat{\mathcal{D}}$, there exists a constant $\delta > 0$ such that

$$\int_r^1 \omega(s)ds \lesssim \left(\frac{1-r}{1-t}\right)^\delta \int_t^1 \omega(s)ds, \quad 0 \leqslant r \leqslant t < 1.$$

see **[PR1]**. We obtain

$$\frac{\omega_*(r)}{(1-r)^{\delta+1}} \asymp \frac{1}{(1-r)^\delta}\int_r^1 \omega(s)ds \lesssim \frac{1}{(1-t)^\delta}\int_t^1 \omega(s)ds \asymp \frac{\omega_*(t)}{(1-t)^{\delta+1}},$$

for $0 \leqslant r \leqslant t < 1$. $\square$

4.3.2 *General case.*

Let ω be a weight not necessarily radial and consider the composition operator $C_\varphi : A^2_\omega \to A^2_\omega$. For a weight ω such that $\frac{\omega}{(1-|z|^2)^\eta} \in B_{p_0}(\eta)$ for some $p_0 > 1$ and $\eta > -1$, O. Constantin **[CONSTANTIN20102]** characterized boundedness, compactness and membership of C_φ in $\mathcal{S}_p(A^2_\omega)$, for $p \geqslant 2$, in terms of the pullback measure of ωdA under φ. If C_φ is bounded on A^2_ω then $C^*_\varphi C_\varphi = T_{\frac{1}{\omega}d\mu}$ with $\mu(E) = A_\omega\left(\varphi^{-1}(E)\right)$ for any Borel subset E of $\mathbb{D}$. Using Theorem B, we obtain the following result which extend **[CONSTANTIN20102]**.

Theorem 4.13. *Let ω be a weight in $\mathcal{C}_{p_0,t}$ for some $p_0 > 1$ and $t \geqslant 0$. Assume that φ is an analytic self-map of $\mathbb{D}$ such that C_φ is compact on A^2_ω. Then C_φ belongs to $\mathcal{S}_p(A^2_\omega)$, for $p > 0$, if and only if*

$$\sum_{n=0}^\infty \left(\frac{\int_{\varphi^{-1}(\Delta_n)} \omega(z)dA(z)}{\int_{\Delta_n} \omega(z)dA(z)}\right)^{p/2} < \infty.$$

In particular, if ω is an almost standard weight, then C_φ belongs to $\mathcal{S}_p(A^2_\omega)$ if and only if

$$\sum_{n=0}^\infty \left(\frac{A_\omega\left(\varphi^{-1}(\Delta_n)\right)}{(1-|z_n|^2)^2\omega(z_n)}\right)^{p/2} < \infty.$$

Note that in case ω is an almost standard weight, C_φ is bounded (resp. compact) on A^2_ω if and only if $A_\omega\left(\varphi^{-1}(\Delta_z)\right) = O\left((1-|z|^2)^2\omega(z)\right)$ (resp. $o\left((1-|z|^2)^2\omega(z)\right)$, $|z| \to 1^-$.

We characterize boundedness, compactness and membership of C_φ in $\mathcal{S}_p(A^2_\omega)$, for ω in some class $\mathcal{C}_{p_0,t}$, in terms of the Nevanlinna counting function. Denote $\omega_{[2]} = (1-|z|^2)^2\omega$.

Theorem 4.14. *Let φ be an analytic self-map of $\mathbb{D}$ and let $p > 0$. Suppose that ω is a weight such that $\omega \in \mathcal{C}_{p_0,t}$ for some $p_0 > 1$ and $t \geqslant 0$. Then*

1. *C_φ is bounded on A^2_ω if and only if*

$$\frac{1}{|\Delta_n|}\int_{\Delta_n} N_{\varphi,\omega_{[2]}}(z)\,dA(z) \lesssim \int_{\Delta_n} \omega(z)\,dA(z), \quad \forall n \in \mathbb{N}.$$

2. *C_φ is compact on A^2_ω if and only if*

$$\frac{1}{|\Delta_n|}\int_{\Delta_n} N_{\varphi,\omega_{[2]}}(z)\,dA(z) = o\left(\int_{\Delta_n} \omega(z)\,dA(z)\right), \quad (n \to \infty).$$

3. *C_φ belongs to $\mathcal{S}_p(A^2_\omega)$ if and only if*

$$\sum_{n=0}^{\infty}\left(\frac{\frac{1}{|\Delta_n|}\int_{\Delta_n} N_{\varphi,\omega_{[2]}}(z)\,dA(z)}{\int_{\Delta_n}\omega(z)\,dA(z)}\right)^{p/2} < \infty.$$

Proof. Note that, since $\omega \in \mathcal{C}_{p_0,t}$ for some $p_0 > 1$ and $t \geqslant 0$, we have the following Littlewood-Paley estimates

$$\|f\|^2_{A^2_\omega} \asymp |f(0)|^2 + \int_{\mathbb{D}} |f'(z)|^2(1-|z|^2)^2\omega(z)\,dA(z), \quad f \in \mathrm{Hol}(\mathbb{D}), \tag{20}$$

see **[BMvolterra2]**. Therefore

$$\int_{\mathbb{D}} |C_\varphi f(z)|^2\omega(z)\,dA(z) \asymp |f(\varphi(0))|^2 + \int_{\mathbb{D}} |(C_\varphi f)'(z)|^2\omega_{[2]}(z)\,dA(z), \quad f \in \mathrm{Hol}(\mathbb{D}).$$

It follows that $C_\varphi : A^2_\omega \to A^2_\omega$ is bounded (resp. compact) if and only if $C_\varphi : \mathcal{D}_{\omega_{[2]}} \to \mathcal{D}_{\omega_{[2]}}$ is bounded (resp. compact). Also, note that if $\omega \in \mathcal{C}_{p_0,t}$, then $\omega_{[2]} \in \mathcal{C}_{p_0,t+2}$. By Theorem 4.2 we obtain the first and the second assertion of the theorem.

Using identity (20), the operator $X : A^2_\omega \to \mathcal{D}_{\omega_{[2]}}$ defined by $Xf = f$ is bounded and invertible. Similarly to the proof of the second assertion of Theorem 4.8, it follows that $C_\varphi : A^2_\omega \to A^2_\omega$ belongs to $\mathcal{S}_p(A^2_\omega)$ if and only if $C_\varphi : \mathcal{D}_{\omega_{[2]}} \to \mathcal{D}_{\omega_{[2]}}$ belongs to $\mathcal{S}_p(\mathcal{D}_{\omega_{[2]}})$. Hence, by Theorem 4.3, we obtain the third assertion of the theorem. □

Remark 4.15. Let $\alpha > -1$. D. Luecking and K. Zhu proved in [**LZ2**] that the condition

$$\left(\frac{1-|z|^2}{1-|\varphi(z)|^2}\right)^{2+\alpha} \in L^{p/2}(\mathbb{D}, d\lambda)$$

is necessary when $p \geqslant 2$ and sufficient when $p \leqslant 2$ for C_φ to be in $\mathcal{S}_p(A^2_\alpha)$. Suppose that ω is a weight in $\mathcal{C}_{p_0,t}$ for some $p_0 > 1$ and $t \geqslant 0$. It is proved in [**BMvolterra2**] that $A^2_\omega = A^2_{\tilde{\omega}}$ with $\|f\|_{A^2_\omega} \asymp \|f\|_{A^2_{\tilde{\omega}}}$ for all $f \in \mathrm{Hol}(\mathbb{D})$. It is proved also in [**BMvolterra2**] that $A^2_{\tilde{\omega}}$ is a reproducing kernel space with kernel $K^{\tilde{\omega}}$ satisfying

$$\|K^{\tilde{\omega}}_z\|^2_{A^2_{\tilde{\omega}}} \asymp \frac{1}{(1-|z|^2)^2\tilde{\omega}(z)}, \quad z \in \mathbb{D}.$$

Since $C_\varphi \in \mathcal{S}_p(A^2_\omega)$ if and only if $C_\varphi \in \mathcal{S}_p(A^2_{\tilde{\omega}})$, using the same argument given in [**LZ2**], we obtain that the condition

$$\frac{\int_{\Delta(z,(1-|z|^2)/2)} \omega(\zeta)\,dA(\zeta)}{\int_{\Delta(\varphi(z),(1-|\varphi(z)|^2)/2)} \omega(\zeta)\,dA(\zeta)} \in L^{p/2}(\mathbb{D}, d\lambda) \tag{21}$$

is necessary when $p \geqslant 2$ and sufficient when $p \leqslant 2$ for C_φ to be in $\mathcal{S}_p(A^2_\omega)$. Note that if in addition ω verifies (13), then the condition (21) is equivalent to

$$\frac{(1-|z|^2)^2\omega(z)}{(1-|\varphi(z)|^2)^2\omega(\varphi(z))} \in L^{p/2}(\mathbb{D}, d\lambda).$$

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CENTRE D'ETUDES DOCTORALES - SCIENCES ET TECHNOLOGIES

Résumé

Cette thèse est consacrée à l'étude de certains opérateurs classiques sur des espaces de fonctions analytiques. Nous nous sommes intéressés, d'une part, aux opérateurs d'intégration définis sur l'espace de Hardy et sur les espaces de Bergman à poids classiques. Nous avons obtenu des estimations des valeurs singulières des opérateurs d'intégration qui sont compacts. Notre approche est basée sur des techniques qui utilisent les suites d'interpolation et d'échantillonnage des espaces de Bergman.

D'autre part, nous nous sommes également intéressés à une autre classe d'opérateurs, à savoir les opérateurs de composition. Nous avons caractérisé l'appartenance aux classes de Schatten des opérateurs de composition qui agissent sur des espaces de Dirichlet à poids. La classe des poids considérée couvre des poids classiques tels que les poids de Bekollé-Bonami et les poids superharmoniques. Nous avons obtenu également une telle caractérisation sur les espaces de Bergman à poids.

Mots-clés: *Espaces de Bergman à poids, Espaces de Dirichlet à poids, Espace de Hardy, Opérateur d'intégration, Opérateur de Toeplitz, Opérateur de composition, Valeurs singulières, Les classes de Schatten.*

Abstract

This thesis is devoted to the study of certain classical operators on spaces of analytic functions. We are interested in integration operators defined on the Hardy space and on the classical weighted Bergman spaces. We obtained estimates of the singular values of compact integration operators. Our approach is based on techniques that use interpolating and sampling sequences for Bergman spaces.
On the other hand, we are also interested in another class of operators class of operators, namely composition operators. We characterized the Schatten classes membership of composition operators acting on weighted Dirichlet spaces. The class of weights considered covers classical weights such as Bekollé-Bonami weights and superharmonic weights. We also obtained such a characterization on weighted Bergman spaces.

Keywords: *Weightd Bergman space, Weighted Dirichlet spaces , The Hardy space, Integration operator, Toeplitz operator, Composition operator, Singular values, Schatten ideals.*

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