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Preserver Problems: A Study of Order Isomorphisms

Faculty of Sciences

Mohamed V University in Rabat, Morocco

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Memory carried out at the department of Mathematics
Faculty of Sciences
Mohammed V University in Rabat, Morocco
aspiring to attain the title of Doctor in Mathematical Sciences.

Thesis advisor

Pr. Zine El Abidine Abdelali

The applicant

El Khatiri Youssef

To my parents

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Résumé

Soient $\mathcal{C}(\mathcal{X})$ l'algèbre de toutes les fonctions continues à valeurs complexes sur un espace compact de Hausdorff $\mathcal{X}$ et $\mathcal{B}(\mathcal{H})$ l'algèbre de tous les opérateurs bornés sur un espace de Hilbert $\mathcal{H}$. Plusieurs chercheurs dans littérature se sont intéressés à la structure d'ordre des algèbres de type $\mathcal{C}(\mathcal{X})$ et celle de type $\mathcal{B}(\mathcal{H})$, voir par exemples [24, 33, 45]. Nous pouvons considérer la structure d'ordre du produit tensoriel, $\mathcal{C}(\mathcal{X}) \otimes \mathcal{B}(\mathcal{H})$, comme une généralisation des deux ordres précédents, et nous pouvons récupérer chacun simultanément à partir de cette généralisation. L'algèbre des effets d'une C*-algèbre unitaire est la collection de tous les éléments positifs ayant une norme au plus égale à 1. Dans cette thèse, nous donnons une description complète pour tous les isomorphismes de l'ordre entre les algèbres des effets des C*-algèbres de type $\mathcal{C}(\mathcal{X}) \otimes \mathcal{B}(\mathcal{H})$.

Aussi, dans cette thèse, nous étendrons l'étude des isomorphismes de l'ordre aux opérateurs non bornés. En fait, nous fournirons une description complète pour tous les isomorphismes de l'ordre entre trois types différents d'ensembles qui se composent d'opérateurs auto-adjoints non bornés. Notamment, l'ensemble de tous les opérateurs positifs, l'ensemble de tous les opérateurs positifs bornément inversibles et l'ensemble de tous les opérateurs auto-adjoints. De plus, nous étendons également des résultats bien connus, qui ont pris leurs origines dans les opérateurs bornés, au contexte des opérateurs non bornés.

Abstract

Let $\mathcal{C}(\mathcal{X})$ be the algebra of all continuous complex-valued functions on a Hausdorff compact space $\mathcal{X}$, and $\mathcal{B}(\mathcal{H})$ be the algebra of all bounded linear operators on a complex Hilbert space $\mathcal{H}$. The order structure of the algebras of types $\mathcal{C}(\mathcal{X})$ and $\mathcal{B}(\mathcal{H})$ have attracted many researchers in the history of mathematics, see, e.g. [24, 33, 45]. We can consider the order structure of $\mathcal{C}(\mathcal{X}) \otimes \mathcal{B}(\mathcal{H})$, the tensor product of these C*-algebras, as generalisation of the preceding orders, and we can retrieve each of them simultaneously from this generalization. For a C*-algebra $\mathcal{A}$, the collection of all positive operators with norm at most 1 is called the effect algebra and denoted by $\mathcal{E}(\mathcal{A})$. In this thesis, we give a complete description of all order isomorphisms between effect algebras of type $\mathcal{E}\left(\mathcal{C}(\mathcal{X}) \otimes \mathcal{B}(\mathcal{H})\right)$ for a large class compact spaces $\mathcal{X}$. Our results generalize some works of Molnár and Šemrl.

Also, in this thesis, we will extend the study of order isomorphisms to unbounded operators. In fact, we will provide a comprehensive description of all order isomorphisms between three distinct types of sets, which consist of unbounded self-adjoint operators. Namely, the set of all positive operators, the set of all positive boundedly invertible operators, and the set of all self-adjoint operators. Moreover, we also extend well-known facts originated with bounded operators to the context of unbounded operators.

Résumé étendu

Soient $\mathcal{C}(\mathcal{X})$ l'algèbre de toutes les fonctions continues à valeurs complexes sur un espace compact de Hausdorff $\mathcal{X}$ et $\mathcal{B}(\mathcal{H})$ l'algèbre de tous les opérateurs bornés sur un espace de Hilbert $\mathcal{H}$. Plusieurs chercheurs dans littérature se sont intéressés à la structure d'ordre des algèbres de type $\mathcal{C}(\mathcal{X})$ et celle de type $\mathcal{B}(\mathcal{H})$, voir par exemples [24, 33, 45]. Nous pouvons considérer la structure d'ordre du produit tensoriel, $\mathcal{C}(\mathcal{X}) \otimes \mathcal{B}(\mathcal{H})$, comme une généralisation des deux ordres précédents, et nous pouvons récupérer chacun simultanément à partir de cette généralisation. L'algèbre des effets d'une C*-algèbre unitaire est la collection de tous les éléments positifs ayant une norme au plus égale à 1. Dans cette thèse, nous donnons une description complète pour tous les isomorphismes de l'ordre entre les algèbres des effets des C*-algèbres de type $\mathcal{C}(\mathcal{X}) \otimes \mathcal{B}(\mathcal{H})$. Nos résultats généralisent certains travaux de Molnár et Šemrl.

Aussi, dans cette thèse, nous étendrons l'étude des isomorphismes de l'ordre aux opérateurs non bornés. En fait, nous fournirons une description complète pour tous les isomorphismes de l'ordre entre trois types différents d'ensembles qui se composent d'opérateurs auto-adjoints non bornés. Notamment, l'ensemble de tous les opérateurs positifs, l'ensemble de tous les opérateurs positifs bornément inversibles et l'ensemble de tous les opérateurs auto-adjoints. De plus, nous étendons également des résultats bien connus, qui ont pris leurs origines dans les opérateurs bornés, au contexte des opérateurs non bornés.

Au quatrième chapitre, nous éloignerons des problèmes de préservation de l'ordre et nous concentrons plutôt sur l'étude des applications linéaires qui préservent les éléments algébriques. Soit $\mathcal{M}_k(\mathbb{F})$ l'algèbre de toutes les matrices $k \times k$ sur un corps algébriquement clos $\mathbb{F}$ de caractéristique nulle. Dans [15], les auteurs ont caractérisé, pour $k \geq 3$, les applications linéaires sur $\mathcal{M}_k(\mathbb{F})$ qui préservent l'ensemble de tous les éléments algébriques de degré 2, et ils ont souligné que leur caractérisation semble également être vraie dans le cas où $k = 2$. Nous présenterons un contre-exemple montrant que cette description ne reste pas vrai dans le cas où $k = 2$. De

plus, nous fournirons une description complète de toutes les applications linéaires sur $\mathcal{M}_2(\mathbb{F})$ qui préservent les éléments algébriques de degré 2 dans $\mathcal{M}_2(\mathbb{F})$.

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[0] **Keywords.** Effect algebra ; Order isomorphism; Self-adjoint operator; Unbounded operator.

Author's papers involved in this thesis :

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Introduction

Preserver problems form an active research area that contains several elegant and powerful results. This is due to the significant number of distinguished mathematicians who work in this field. Such problems arise in most parts of mathematics, because it is a common question in various contexts to ask about the form or, at least to gain an idea (example, property, etc) about transformations on a given structure that preserve something (quantity, relation, set, etc). So, the purpose of preserver problems is to give a complete description of the general form of such transformations. The readers who are interested in preserver problems are recommended to refer to the book [34], which contains numerous results on this theory with a modern approach. In this thesis, we will direct our attention towards the order isomorphisms. A bijective map $\Phi : (\mathcal{S}_1, \leq_1) \to (\mathcal{S}_2, \leq_2)$ between two partially ordered sets (posets) is called an order isomorphism if it preserves the order relations in both directions, i.e.,

$$A \leq_1 B \Longleftrightarrow \Phi(A) \leq_2 \Phi(B),$$

for all $A \in \mathcal{S}_1$. The sets $(\mathcal{S}_1, \leq_1)$ and $(\mathcal{S}_2, \leq_2)$ are said to be order isomorphic. In the special case $(\mathcal{S}_1, \leq_1) = (\mathcal{S}_2, \leq_2)$, Φ is called an order automorphism.

The study of the order structure and order isomorphisms on operator algebras has a long history in mathematics. For the case of commutative C*-algebras, in 1947, Kaplansky proved in [22, 23], that the order isomorphisms structure on $\mathcal{C}(\mathcal{X}, \mathbb{R})$, the set of all continuous real-valued functions on a Hausdorff compact space $\mathcal{X}$, characterizes $\mathcal{X}$ and conversely. In the sense that if $\mathcal{X}$ and $\mathcal{Y}$ are two Hausdorff compact spaces such that $\mathcal{C}(\mathcal{X}, \mathbb{R})$ and $\mathcal{C}(\mathcal{Y}, \mathbb{R})$ are order isomorphic then, $\mathcal{X}$ and $\mathcal{Y}$ are homeomorphic, and the converse statement is also true. In 1964, Itô investigated the continuity of order automorphisms (lattices automorphisms) of $\mathcal{C}(\mathcal{X}, \mathbb{R})$ in [18]. In the same paper, he introduced what is called Itô's property (K), that is, a compact space $\mathcal{X}$ satisfies property (K) if all order automorphisms on $\mathcal{C}(\mathcal{X}, \mathbb{R})$ are continuous. In recent years, many authors investigated the structure forms of order isomorphisms between several subsets of $\mathcal{C}(\mathcal{X}, \mathbb{R})$. For instance, in 2005, Marovt provides a complete description of all order isomorphisms between the effect algebra of the C*-algebra $\mathcal{C}(\mathcal{X}, \mathbb{C})$ (in short $\mathcal{C}(\mathcal{X})$), the set of all continuous complex-valued functions on a Hausdorff compact space $\mathcal{X}$ with the additional assumption that $\mathcal{X}$ satisfies the first axiom of countability. After that, in 2009, it has been provided a complete description of all order isomorphisms between the effect algebra of $\mathcal{C}(\mathcal{X})$ without additional conditions on the compact space $\mathcal{X}$, we will talk about this in more detail hereafter. However, we refer the interested readers to [4, 9, 10, 16, 30]. For the non-commutative case, in 1952, Kadison proved that any linear order isomorphism between the self-adjoint parts of two unitary C*-algebras, which carrying the identity of one algebra into the identity of the other, is extendible to a Jordan *-isomorphism between these C*-algebras, see [20]. Recall that a linear bijection $J : \mathcal{A} \to \mathcal{B}$ between two C*-algebras is called a Jordan *-isomorphism if it satisfies $J(A^*) = J(A)^*$ and $J(A^2) = J(A)^2$ for all $A \in \mathcal{A}$. For notational simplicity, we will write $\mathcal{E}(\mathcal{H})$ instead of $\mathcal{E}(\mathcal{B}(\mathcal{H}))$, the set of all positive operators on H that are bounded by the identity. One of the famous and fundamental results in this theory, is Ludwig's description of ortho-order automorphisms of effect algebra $\mathcal{E}(\mathcal{H})$, see [28, Section V.5]. Precisely, Ludwig proved that if an order automorphism Φ on $\mathcal{E}(\mathcal{H})$, with $\dim \mathcal{H} \geq 3$, satisfies $\Phi(I - a) = I - \Phi(a)$ for all $a \in \mathcal{E}(\mathcal{H})$, where I is the identity map on $\mathcal{H}$, and $\dim \mathcal{H}$ denotes the dimension of $\mathcal{H}$, then Φ is extendible to a Jordan *-automorphism of $\mathcal{B}(\mathcal{H})$. The case $\dim \mathcal{H} = 2$ is given by Molnár and Páles, see for example [36]. In 2003, Molnár studied bijective maps on $\mathcal{E}(\mathcal{H})$ which preserve the order and zero product in both directions [36, Corollary 4]. In the same direction, Šemrl characterized the order automorphisms of $\mathcal{E}(\mathcal{H})$ without any additional conditions, see [45, Theorem 2.2]. Note that the rank one projections are crucially important in the mentioned results. In the setting of arbitrary von Neumann algebras, there may be no minimal (in particular no rank one) projections. By using properties of projection lattices and a classical result of Dye in [13] on orthogonality preserving mappings between projection lattices of von Neumann algebras, Mori gave in [38, Theorem 1.4, Proposition 3.12] a complete description of order isomorphisms between operator intervals in general von Neumann algebras. These results subsume several works by Molnár and Šemrl on type I factors.

As we have seen above, many authors are interested in studying order preservers on the effect algebras of various C*-algebras. The reason is that the elements of the effect algebras play an important role in several applications in the mathematical formulation of quantum mechanics. Indeed, they give a description of yes-no measurements which may be unsharp, for more detail, see [7, 26]. On the other hand, recall that the C*-algebra $\mathcal{C}(\mathcal{X}) \otimes \mathcal{B}(\mathcal{H})$ can describe the structure order of $\mathcal{C}(\mathcal{X})$ and that of $\mathcal{B}(\mathcal{H})$ simultaneously, and in general, it is unlikely to be a von Neumann algebra. Furthermore, such C*-algebras possess neither the advantages of commutative C*-algebras, nor the richness of projections, like von Neumann algebras. All the preceding reasons give rise to a highly challenging question What is the general form of all order isomorphisms between the effect algebras of the C*-algebras of type $\mathcal{C}(\mathcal{X}) \otimes \mathcal{B}(\mathcal{H})$? So, we will dedicate the first part of this thesis to providing a comprehensive answer to this question.

The unbounded self-adjoint operators play a crucial role across various disciplines. For instance, in quantum mechanics, it is reported that the states of a quantum system can be represented by vectors of a Hilbert space, and the observables by the self-adjoint operators, acting on this space. The spectrum of an observable expresses the possible values of the dynamical variable represented by this observable with uncertainty. We refer interested readers to the following monographs on the mathematical formulation of quantum mechanics: [17, 26, 29, 41]. Also, in partial differential equations, unbounded self-adjoint operators could analyse relationships and describe the behaviour of very complicated systems. Moreover, these operators provide a powerful framework for discovering the complexity of various phenomena, which allow a deeper understanding of the mathematical fundamentals underlying differential equations. The exploration of unbounded operator applications to differential equations has been extensively investigated in numerous works, such as [14, 19]. Despite the importance of this notion, the vast majority of preserver problems have been and remain to be studied only within the scope of matrices and bounded operators. So, in order to extend the study of preserver problems to unbounded operators, in the second part of this memory, we will investigate order isomorphisms on the unbounded self-adjoint operators (unbounded observables).

We close this introduction with a few insights about the architecture of the memory contents, hopefully, clear and well organized. We begin with the chapter ”Preliminaries”, which collects the most frequently used notations and definitions. In this chapter, we will also revisit some basic results that will be employed throughout the remainder of the thesis. In the second chapter, we will characterize the general form of all order isomorphisms between the effect algebras of the C*-algebras of type $\mathcal{C}(\mathcal{X}) \otimes \mathcal{B}(\mathcal{H})$. This is done in five sections. In the first section we state our main results, mainly Theorem 2.2.1 and Theorem 2.2.2. The latter shows that an order isomorphism Φ from $\mathcal{C}(\mathcal{X}, \mathcal{E}(\mathcal{H}))$ onto $\mathcal{C}(\mathcal{Y}, \mathcal{E}(\mathcal{K}))$ is obtained in the same way by pointwise order isomorphisms. This, together with the mentioned result of Šemrl and Mori, allows us to provide a concrete characterization of such order isomorphisms. Conversely, Proposition 2.2.3 provides how to build such order isomorphisms. If, in addition, the order isomorphism Φ satisfies $\Phi(\frac{1}{2}I_{\mathcal{X}}) = \frac{1}{2}I_{\mathcal{Y}}$, then Theorem 2.2.1 states that Φ is extendible to a Jordan *-isomorphism from $\mathcal{C}(\mathcal{X}, \mathcal{B}(\mathcal{H}))$ onto $\mathcal{C}(\mathcal{Y}, \mathcal{B}(\mathcal{K}))$. In the second section, we will show that these order isomorphisms preserve some specific sets of projections. In the same section and under an additional condition, it is shown that an order isomorphism preserves orthogonality between certain projections. The third section will discuss also some relevant facts concerning the existence of homeomorphisms from $\mathcal{Y}$ onto $\mathcal{X}$, and in the case $\Phi(\frac{1}{2}I_{\mathcal{X}}) = \frac{1}{2}I_{\mathcal{Y}}$, the Hilbert spaces $\mathcal{H}$ and $\mathcal{K}$ are isometrically isomorphic. The goal of the fourth section is to show the existence of a canonical homeomorphism (in some sense) from $\mathcal{Y}$ onto $\mathcal{X}$ which will transfer the order point by point. In the same section, we provide the proof of the first main result Theorem 2.2.1 with some auxiliary lemmas and results. Finally, in the fifth section, we will explain why the condition $\Phi(\frac{1}{2}I_{\mathcal{X}}) = \frac{1}{2}I_{\mathcal{Y}}$ is not restrictive. We will also prove Theorem 2.2.2 and the rest of results.

The third chapter will be divided into four sections. In the first one, we will state the main results, which give a characterization of order isomorphisms with respect to an order relation. Namely, Theorem 3.1.1 provides a complete description of order isomorphisms between the sets of all positive unbounded operators. Theorem 3.1.3 gives the general form of all order isomorphisms between the sets of all positive boundedly invertible operators. Theorem 3.1.4 provides a comprehensive characterization of all order isomorphisms between the sets of unbounded self-adjoint operators. Finally, Theorem 3.1.5 discusses the possibility of the three previous sets of unbounded self-adjoint operators to be order isomorphic. The second section will be devoted to prove Theorem 3.1.1. For the proof, we will use various deep results from operator algebras and spectral theory. In the third section, and relying on the results achieved in the previous section, we will prove the rest of these main results. Our results give rise to further problems on order isomorphisms on unbounded operators. In the section four, we list some questions and we will explore and expand the concept of order isomorphism on unbounded operators with respect to a second order relation. At the end, we close by the fourth chapter, where we construct a counter-example that shows that the description of the linear maps on $\mathcal{M}_K(\mathbb{F})$ preserving the set of all algebraic elements of degree 2, provided in [15], does not holds for $K = 2$. Moreover, we will provide a complete description of all such maps on $\mathcal{M}_2(\mathbb{F})$.

Chapter 1

Preliminaries

In this chapter, we will recall the essential concepts, definitions, and background knowledge necessary for understanding the content of both subsequent chapters, while also familiarizing the reader with the basic facts that will often be used (sometimes without mention).

1.1 Order isomorphisms between effect algebras of type $\mathcal{C}(\mathcal{X}, [0,1])$

Throughout all this memory, if no confusion can arise (relying on the context), we equate the zero element of any C*-algebra with the number 0 in the notation. Also, norms are always denoted by $\|.\|$.

Let $\mathcal{A}$ be a unitary C*-algebra equipped with its natural order " $\leq$ ". We recall that the effect algebra of $\mathcal{A}$ is the interval $\{a \in \mathcal{A} : 0 \leq a \leq I_{\mathcal{A}}\}$, where $I_{\mathcal{A}}$ denotes the unit of $\mathcal{A}$. If $\mathcal{A}$ is commutative, then it can be seen as the C*-algebra of all complex-valued continuous functions on $\mathcal{X}$, $\mathcal{C}(\mathcal{X})$, equipped with the norm $\|r\| = \sup\{|r(x)| : x \in \mathcal{X}\}$, where $\mathcal{X}$ is a Hausdorff compact space. We endow $\mathcal{C}(\mathcal{X})$ with the usual order, namely,

$$r \leq s \Longleftrightarrow r(x) \leq s(x) \quad x \in \mathcal{X}, \quad r, s \in \mathcal{C}(\mathcal{X}, \mathbb{R}).$$

So, it is rather easy to see that the effect algebra $\mathcal{E}(\mathcal{A}) = \mathcal{E}(\mathcal{C}(\mathcal{X}))$ is exactly the set $\mathcal{C}(\mathcal{X}, [0,1])$ of all continuous functions from $\mathcal{X}$ into the interval $[0,1]$.

We will use frequently the well-known lemma of Urysohn. Before we state it, let us define the notion of normal space. A topological space $\mathcal{X}$ is said to be normal, if any two disjoint closed subsets of $\mathcal{X}$ can be separated by neighbourhoods.

Lemma 1.1.1 (Urysohn's lemma). *For any two disjoint closed sets $\mathcal{X}_1$ and $\mathcal{X}_2$ in a normal topological space $\mathcal{X}$, there exists a function $r \in \mathcal{C}(\mathcal{X}, [0,1])$ such that*

$$r(\mathcal{X}_1) = \{0\} \qquad \textit{and} \qquad r(\mathcal{X}_2) = \{1\}.$$

It is well-known that all Hausdorff compact spaces are normal. So, we point out that throughout this chapter and the following one, we will use the Urysohn's lemma without explicitly mentioning it, mostly.

Let $\mathcal{F}$ be a subset of a partially ordered set $(\mathcal{S}, \leq)$. An upper bound (resp. lower bound) for $\mathcal{F}$ in $\mathcal{S}$ is an element $A \in \mathcal{S}$ such that $B \leq A$ (resp. $A \leq B$) for all $B \in \mathcal{F}$. We call $C \in \mathcal{S}$ a supremum (resp. infimum) for $\mathcal{F}$, whenever it exists, if C is an upper bound (resp. lower bound) for $\mathcal{F}$ and, whenever A is an upper bound (resp. lower bound) for $\mathcal{F}$, then $C \leq A$ (resp. $A \leq C$). If a supremum or infimum of $\mathcal{F}$ exists then it is unique and denoted by $\sup \mathcal{F}$ (resp. $\inf \mathcal{F}$).

The infimum in the next lemma is considered in $\mathcal{C}(\mathcal{X}, [0,1])$.

Lemma 1.1.2. *Let $r, s \in \mathcal{C}(\mathcal{X}, [0,1])$. Then,*

$$\inf\{r, s\} = 0 \qquad \textit{if and only if} \qquad rs = 0.$$

Proof. The proof follows directly from the following:

$$\inf\{r,s\}(x) = \inf\{r(x), s(x)\} = \begin{cases} r(x) & \text{if } r(x) \leq s(x), \\ s(x) & \text{otherwise}, \end{cases} \tag{1.1.1}$$

for all $x \in \mathcal{X}$. □

Let $\mathcal{A}$ be a unitary C*-algebra and let $A \in \mathcal{A}$, we say that A is normal if $A^*A = AA^*$, and A is self-adjoint if $A = A^*$. It is clear that a normal element is self-adjoint, but the converse is not true in general. Let $A \in \mathcal{A}$ be a normal element, one of the dominant and powerful tools in the study of such elements is the concept of continuous functional calculus. We will denote by $\mathcal{C}(\mathcal{A})$ the C*-algebra generated by A and the identity of $\mathcal{A}$. Notice that $\mathcal{C}(\mathcal{A})$ is commutative. More precisely, $\mathcal{C}(\mathcal{A})$ is the closure of the algebra $\{p(A)q(A^*) : p, q \text{ being polynomials}\}$ in $\mathcal{A}$. Since $\mathcal{C}(A)$ is commutative, it can be seen as $\mathcal{C}(\mathcal{X})$ for some Hausdorff compact space $\mathcal{X}$. The compact $\mathcal{X}$ is nothing but the spectrum of A, $\mathrm{Sp}(A)$, according to [11, Prposition 2.3].

The following theorem is quoted from [11, Theorem 2.1, Theorem 2.6] with an adaptation.

Theorem 1.1.3. *([11]) Let $\mathcal{A}$ be a unitary C*-algebra and let $A \in \mathcal{A}$ be a normal element. Then, there exists an isometric *-algebra isomorphism $\rho_A : \mathcal{C}(\mathrm{Sp}(A)) \to \mathcal{C}(A)$ such that*

$$\rho_A(s) = A,$$

where s is the identity map on $\mathrm{Sp}(A)$. In particular, ρ_A is an order isomorphism between the self-adjoint parts of $\mathcal{C}(\mathrm{Sp}(A))$ and $\mathcal{C}(A)$ with respect to their usual orders.

For notational simplicity, we will write $r(A)$ for $\rho_A(r)$ for all $r \in \mathcal{C}(\mathrm{Sp}(A))$.

We begin with Marovt theorem, mentioned in the introduction. Before stating it, we briefly recall the first axiom of countability. We say that a topological space $\mathcal{X}$ satisfies the first axiom of countability if, each point in $\mathcal{X}$ possesses a countable neighbourhood basis.

Theorem 1.1.4. *([30]) Let $\mathcal{X}$ and $\mathcal{Y}$ be two Hausdorff compact spaces, one of which satisfies the first axiom of countability. If $\Phi : C(\mathcal{X}, [0,1]) \to C(\mathcal{Y}, [0,1])$ is an order isomorphism, then, there exist a homeomorphism $\mu : \mathcal{Y} \to \mathcal{X}$, for every $y \in \mathcal{Y}$, an increasing homeomorphism $\widehat{\Phi}_y$ of $[0,1]$ such that*

$$\Phi(r)(y) = \widehat{\Phi}_y\left(r(\mu(y))\right)$$

for all $r \in C(\mathcal{X}, [0,1])$ and all $y \in \mathcal{Y}$.

The converse is also true if for each $\alpha \in [0,1]$, the map $\mathcal{Y} \to [0,1]$ defined by $y \mapsto \widehat{\Phi}_y(\alpha)$ is continuous.

We note that Φ preserves the order pointwise. More precisely, for every $r, s \in \mathcal{C}(\mathcal{X}, [0,1])$ and every $x \in \mathcal{X}$ we have the equivalence

$$r(x) \leq s(x) \Longleftrightarrow \Phi(r)(\mu^{-1}(x)) \leq \Phi(s)(\mu^{-1}(x)).$$

The question whether this still holds true when we release the assumption that both spaces $\mathcal{X}$ and $\mathcal{Y}$ satisfy the first axiom of countability rises naturally. Actually, Ercan and Onal provided in [16] a counter-example with $\mathcal{X}$ is the Stone-Čech compactification of the open interval $(0,1)$. However, the more general problem of characterizing order isomorphisms between effect algebras of commutative C*-algebras has been investigated in [10]. We summarize Proposition 1 and Lemma 1 of [10] in the following theorem. But first, let us recall that a subset $\mathcal{X}_0$, of a topological space $\mathcal{X}$, is a G_δ subset of $\mathcal{X}$ if it can be expressed as the intersection of countably many open sets in $\mathcal{X}$.

Theorem 1.1.5. *([10]) Let $\mathcal{X}$ and $\mathcal{Y}$ be two Hausdorff compact spaces. If $\Phi : \mathcal{C}(\mathcal{X}, [0,1]) \to \mathcal{C}(\mathcal{Y}, [0,1])$ is an order isomorphism, and $\widehat{\Phi} : \mathcal{Y} \times [0,1] \to [0,1]$ is the map defined by*

$$\widehat{\Phi}(y,t) = \widehat{\Phi}_y(t) := \Phi\left(\widehat{t}\right)(y),$$

where $\widehat{t}(x) = t$ for all $x \in \mathcal{X}$ and all $t \in [0,1]$, then the following statements hold.

1. *There exists a homeomorphism $\mu : \mathcal{Y} \to \mathcal{X}$ such that the set*

$$\mathcal{Y}_\Phi := \left\{ y \in \mathcal{Y} \ : \ \forall r \in \mathcal{C}(\mathcal{X}, [0,1]),\ \Phi(r)(y) = \widehat{\Phi}_y\left(r(\mu(y))\right) \right\} \tag{1.1.2}$$

   is a dense G_δ subset of $\mathcal{Y}$,

2. *$\widehat{\Phi}_y$ is an increasing homeomorphism of $[0,1]$ for each $y \in \mathcal{Y}_\Phi$.*

As noted above, in the general case, these order isomorphisms have a mysterious behaviour, in the sense that they do not preserve the order pointwise outside of $\mathcal{Y}_\Phi$. It has been proven in [10, Lemma 1] and [9, Lemma 4] that $\mathcal{Y}_\Phi = \mathcal{Y}$ if and only if Φ is continuous. We will say that $\mathcal{X}$ satisfies the property $(\mathrm{K}_\mathcal{E})$ if every order isomorphism between $\mathcal{C}(\mathcal{X}, [0,1])$ and $\mathcal{C}(\mathcal{Y}, [0,1])$ is continuous for some, and then for every, compact space $\mathcal{Y}$. This is equivalent to the fact that every order automorphism on $\mathcal{C}(\mathcal{X}, [0,1])$ is continuous, because the condition that $\mathcal{C}(\mathcal{X}, [0,1])$ and $\mathcal{C}(\mathcal{Y}, [0,1])$ are order isomorphic implies that $\mathcal{X}$ and $\mathcal{Y}$ are homeomorphic, see [10]. Our definition is analogous to the classical Itô's property (K), mentioned in the introduction. In the following proposition, we will provide a comparison between the property (K) and $(\mathrm{K}_\mathcal{E})$.

Proposition 1.1.6. *The property $(\mathrm{K}_\mathcal{E})$ is stronger than (K).*

Proof. Assume that $\mathcal{X}$ satisfies the property $(\mathrm{K}_\mathcal{E})$. Let $\Phi : \mathcal{C}(\mathcal{X}, \mathbb{R}) \to \mathcal{C}(\mathcal{X}, \mathbb{R})$ be an order automorphism. It is easy to find $n \in \mathbb{N}$ such that $\Phi(-n)(x) < 0$ and $\Phi(n)(x) > 0$ for all $x \in \mathcal{X}$. Then, the map $\mathcal{C}(\mathcal{X}, [0,1]) \to \mathcal{C}(\mathcal{X}, [0,1])$ defined by

$$f \mapsto \left(\Phi(n(2f-1)) - \Phi(-n)\right)\left(\Phi(n) - \Phi(-n)\right)^{-1}$$

is an order automorphism. Since $\mathcal{X}$ satisfies the property $(\mathrm{K}_\mathcal{E})$, it is easily checked that Φ is continuous on $\{f \in \mathcal{C}(\mathcal{X}) : f(\mathcal{X}) \subseteq [-n, n]\}$. The latter fact is also true for all integers $m \geq n$. Then, Φ is continuous, this means that Φ satisfies the property (K). $\square$

Note that the class of compacts spaces that satisfy the property ($\mathrm{K}_{\mathcal{E}}$) contains most commonly occurring compacts, mainly, the first countable compact Hausdorff spaces, ..., see for instance Theorem 1.1.4 and [22]. On the other hand, all the available examples of compact spaces which do not satisfy the property ($\mathrm{K}_{\mathcal{E}}$) are Stone-Čech compactification of some topological space, see [4, Theorem 2.2] and [16].

1.2 Order isomorphisms between effect algebras of $\mathcal{B}(\mathcal{H})$

Throughout this memory, $\mathcal{H}$ denotes a complex Hilbert space, and scalar products are always denoted by angle brackets $\langle ., . \rangle$. For any subspace $\mathcal{D}$ of $\mathcal{H}$, the closure of $\mathcal{D}$ in the Hilbert $\mathcal{H}$ with respect to the norm-topology will be denoted by $\overline{\mathcal{D}}$.

For any operator $A \in \mathcal{B}(\mathcal{H})$, we denote the image or the range of A by $\mathrm{Im}(A)$ and the kernel of A by $\mathrm{Ker}(A)$. The symbol $\oplus$ stands for the orthogonal sum of Hilbert spaces.

The following result is one of the most frequently used in this memory.

Proposition 1.2.1. *Let A bounded linear self-adjoint operator on $\mathcal{H}$, the subspaces* $\mathrm{Im}(A)$ *and* $\mathrm{Ker}(A)$ *are orthogonal. Furthermore,*

$$\mathcal{H} = \overline{\mathrm{Im}(A)} \oplus \mathrm{Ker}(A). \tag{1.2.1}$$

By $\mathcal{P}(\mathcal{H})$ we denote the set of all orthogonal projections on $\mathcal{H}$. Notice that $\mathcal{P}(\mathcal{H}) \subset \mathcal{E}(\mathcal{H})$ and it forms a lattice, in the sense that, for two projections $P, Q \in \mathcal{P}(\mathcal{H})$, $\sup\{P, Q\}$ and $\inf\{P, Q\}$ in $\mathcal{E}(\mathcal{H})$ exist and belong to $\mathcal{P}(\mathcal{H})$. More precisely, $\sup\{P, Q\}$ is exactly the orthogonal projection on the space $\overline{\mathrm{Im}(P) + \mathrm{Im}(Q)}$, where the symbol " $+$ " means here the linear sum of the vector spaces. While $\inf\{P, Q\}$ is the orthogonal projection on the intersection $\mathrm{Im}(P) \cap \mathrm{Im}(Q)$.

We define $\mathrm{Rank}(A)$ as the dimension of $\mathrm{Im}(A)$ if such a dimension is finite. Otherwise, we write $\mathrm{Rank}(A) = \infty$.

By $\mathcal{P}_1(\mathcal{H})$ we denote the set of all rank one orthogonal projections, and by $\mathcal{P}_{-1}(\mathcal{H})$ the set of all corank one orthogonal projections on $\mathcal{H}$. This means that $\mathcal{P}_{-1}(\mathcal{H}) = \{P^{\perp} : P \in \mathcal{P}_1(\mathcal{H})\}$, where $P^{\perp} := I_{\mathcal{A}} - P$ for any projection P in any C*-algebra $\mathcal{A}$ such that $I_{\mathcal{A}}$ is the unit element of $\mathcal{A}$. If $\dim \mathcal{H} \geq 2$, the collections $\mathcal{P}_1(\mathcal{H})$ and $\mathcal{P}_{-1}(\mathcal{H})$ do not form lattices. To give counterexamples, we need the following notation. As usual, for two vectors $u, v \in \mathcal{H}$, we denote by $u \otimes v$ the element of rank at most one in $\mathcal{B}(\mathcal{H})$ defined by

$$(u \otimes v)w = \langle w, v \rangle \, u$$

for all $w \in \mathcal{H}$. Pick two unit linearly independent vectors $v, w \in \mathcal{H}$, then it is easily checked that $v \otimes v, w \otimes w \in \mathcal{P}_1(\mathcal{H})$, and $\operatorname{Im}(v \otimes v) \cap \operatorname{Im}(w \otimes w) = \{0\}$. Hence,

$$\inf\{v \otimes v, w \otimes w\} = 0 \notin \mathcal{P}_1(\mathcal{H}).$$

It is also easily seen that $\operatorname{Im}\left((v \otimes v)^{\perp}\right) + \operatorname{Im}\left((w \otimes w)^{\perp}\right) = \mathcal{H}$. Then,

$$\sup\{(v \otimes v)^{\perp}, (w \otimes w)^{\perp}\} = I \notin \mathcal{P}_{-1}(\mathcal{H}).$$

Let $\mathcal{H}_0$ be a closed subspace of $\mathcal{H}$, and let $A \in \mathcal{B}(\mathcal{H}_0)$ and $B \in \mathcal{B}(\mathcal{H}_0^{\perp})$, where $\mathcal{H}_0^{\perp}$ is the orthogonal completion of $\mathcal{H}_0$ in $\mathcal{H}$. Notice, that for each $v \in \mathcal{H}$, there exists a unique couple $(v_0, v_1) \in \mathcal{H}_0 \times \mathcal{H}_0^{\perp}$ such that $v = v_0 + v_1$. We define the operator $A \oplus B \in \mathcal{B}(\mathcal{H})$ by

$$(A \oplus B)v = Av_0 + Bv_1,$$

for all $v \in \mathcal{H}$.

The next two auxiliary lemmas will be needed.

Lemma 1.2.2. *Let $P \in \mathcal{P}_1(\mathcal{H})$. Then, for every $A \in \mathcal{E}(\mathcal{H})$ with $A \leq P$, there exists a scalar $t \in [0, 1]$ such that*

$$A = tP.$$

Proof. Let $v \in \operatorname{Ker}(P)$, we have

$$\|A^{1/2}v\|^2 = \langle A^{1/2}v, A^{1/2}v \rangle = \langle Av, v \rangle \leq \langle Pv, v \rangle = 0.$$

Then,

$$\operatorname{Ker}(P) \subseteq \operatorname{Ker}(A^{1/2}) = \operatorname{Ker}(A). \tag{1.2.2}$$

Combining this with Proposition 1.2.1, we conclude that

$$\mathrm{Im}(A) \subseteq \mathrm{Im}(P). \tag{1.2.3}$$

From (1.2.2) and (1.2.3), we deduce that $\mathrm{Im}(P)$ and $\mathrm{Ker}(P)$ are invariant subspaces for A. Thus,

$$A = A_0 \oplus 0_{Ker(P)},$$

where $A_0 : \mathrm{Im}(P) \to \mathrm{Im}(P)$ defined by $A_0 v := Av$ for all $v \in \mathrm{Im}(P)$, and $0_{\mathrm{Ker}(P)}$ is the null operator of the algebra $\mathcal{B}(\mathrm{Ker}(P))$. Since A is self-adjoint and $\mathrm{Im}(P)$ is one-dimensional, then there exists a $t \in [0, 1]$ such that $A = tP$. $\square$

The following lemma is quoted from our paper [1].

Lemma 1.2.3. *Let $A \in \mathcal{E}(\mathcal{H})$ and $P \in \mathcal{P}(\mathcal{H})$ such that $P \leq A$. Then,*

$$A \;=\; P + P^{\perp} A P^{\perp}.$$

Proof. Denote $A_0 = A - P \in \mathcal{E}(\mathcal{H})$, we have $A_0 = A - P \leq I - P = P^{\perp}$. Using the same discussion, as in the proof of the previous lemma, we conclude that $\mathrm{Ker}(P^{\perp}) \subseteq \mathrm{Ker}(A_0^{1/2}) \subseteq \mathrm{Ker}(A_0)$. Therefore,

$$\mathrm{Im}(P) = \mathrm{Ker}(P^{\perp}) \subseteq \mathrm{Ker}(A_0), \quad A_0 P = 0 \quad \text{and} \quad P A_0 = (A_0 P)^* = 0.$$

We obtain the equality of lemma from the following

$$A = P + P^{\perp} A_0 P^{\perp} + P^{\perp} A_0 P + P A_0 P^{\perp} = P + P^{\perp} A_0 P^{\perp} = P + P^{\perp} A P^{\perp}. \qquad \square$$

As we have seen already in the introduction, any order automorphism $\Phi : \mathcal{E}(\mathcal{H}) \to \mathcal{E}(\mathcal{H})$, satisfying $\Phi(I - A) = I - \Phi(A)$ for all $A \in \mathcal{E}(\mathcal{H})$, is extendible to a Jordan *-isomorphism on $\mathcal{B}(\mathcal{H})$. Molnár managed to reduce the preceding condition ($\Phi(I - A) = I - \Phi(A)$) to a weaker requirement, that is, $\Phi(1/2I) = 1/2I$, see e.g [36]. For a first glance, it is not easy to believe that this additional assumption is crucial. In fact, any order automorphism Φ of $\mathcal{E}(\mathcal{H})$ with $\dim \mathcal{H} \geq 2$ is extendible to a Jordan *-isomorphism if and only if $\Phi(1/2I) = 1/2I$. In general, the order isomorphisms between effect algebras of type $\mathcal{E}(\mathcal{H})$ can be quite distinct from Jordan

*-isomorphisms. As the subsequent lemma makes clear, these mappings can even significantly deviate from linearity.

Let $\alpha < 1$, define $r_\alpha : [0,1] \to [0,1]$ by

$$r_\alpha(t) := \frac{t}{\alpha t + 1 - \alpha}, \; t \in [0,1].$$

Lemma 1.2.4. *For each $\alpha < 1$, the map $\mathcal{E}(\mathcal{H}) \to \mathcal{E}(\mathcal{H})$ defined by $A \mapsto r_\alpha(A)$ is an order automorphism of $\mathcal{E}(\mathcal{H})$.*

Proof. Fix an $\alpha < 1$, by a simple computation, we can prove that r_α is bijective and $r_\alpha^{-1} = r_{\frac{\alpha}{\alpha-1}}$. Then, $A \mapsto r_\alpha(A)$ is bijective. So, it remains to prove that r_α preserves the order in both directions. Notice that $r_0(t) = t$ for all $t \in [0,1]$. Thus, by Theorem 1.1.3, $r_0(A) = A$ for all $A \in \mathcal{E}(\mathcal{H})$. Then, $A \mapsto r_0(A)$ preserves the order in both directions. If $\alpha \neq 0$, we can write

$$r_\alpha(t) = \frac{1}{\alpha} - \frac{1-\alpha}{\alpha^2}\left(t + \frac{1-\alpha}{\alpha}\right)^{-1}, \quad t \in [0,1].$$

For every $A, B \in \mathcal{E}(\mathcal{H})$, we have

$$\begin{aligned} A \leq B \quad &\Longleftrightarrow \quad \left(A + \frac{1-\alpha}{\alpha} I\right)^{-1} \geq \left(B + \frac{1-\alpha}{\alpha} I\right)^{-1} \\ &\Longleftrightarrow \quad \frac{1}{\alpha} I - \frac{1-\alpha}{\alpha^2}\left(A + \frac{1-\alpha}{\alpha} I\right)^{-1} \leq \frac{1}{\alpha} I - \frac{1-\alpha}{\alpha^2}\left(B + \frac{1-\alpha}{\alpha}\right)^{-1} \\ &\Longleftrightarrow \quad r_\alpha(A) \leq r_\alpha(B). \end{aligned}$$

Then, $A \mapsto r_\alpha(A)$ preserves the order in both directions. We conclude that $A \mapsto r_\alpha(A)$ is an order automorphism. $\square$

Šemrl succeeded in providing a comprehensive description of order automorphisms between the effect algebra of C*-algebras of type $\mathcal{B}(\mathcal{H})$, without the imposition of additional conditions. The result was surprising, because it proves the existence of two different forms which describe the the same kind of maps.

Theorem 1.2.5. *([45]) Let $\Phi : \mathcal{E}(\mathcal{H}) \to \mathcal{E}(\mathcal{H})$ be an order isomorphism with $\dim \mathcal{H} \geq 2$. Then, the two following assertions hold.*

1. *There exist a linear or conjugate linear operator $T : \mathcal{H} \to \mathcal{H}$ and $\alpha < 0$ such that*

$$\Phi(A) = r_\alpha \left((I + (TT^*)^{-1})^{1/2} (I - (I + TAT^*)^{-1})(I + (TT^*)^{-1})^{1/2} \right)$$

   for all $A \in \mathcal{E}(\mathcal{H})$.

2. *There exist a bijective linear or conjugate linear operator $T : \mathcal{H} \to \mathcal{H}$ with $\|T\| \leq 1$ and two numbers $\alpha \in (0, 1)$, $\beta < 0$ such that*

$$\Phi(A) = r_\beta \left((r_\alpha(TT^*))^{-1/2} r_\alpha(TAT^*)(r_\alpha(TT^*))^{-1/2} \right)$$

   for all $A \in \mathcal{E}(\mathcal{H})$.

A bit later, Šemrl and Mori provide a third description of the order isomorphisms between effect algebras of type $\mathcal{E}(\mathcal{H})$. The description is simpler than the two preceding ones.

Theorem 1.2.6. *([39]) Let $\mathcal{H}, \mathcal{K}$ be two Hilbert spaces such that* $\max\{\dim \mathcal{H}, \dim \mathcal{K}\} \geq 2$. *A map $\Phi : \mathcal{E}(\mathcal{H}) \to \mathcal{E}(\mathcal{K})$ is an order isomorphism if and only if there exists a linear or conjugate-linear invertible bounded operator $T : \mathcal{H} \to \mathcal{K}$ such that*

$$\Phi(A) = \mathrm{Ord}_T(A) := T \left(A(T^*T - I) + I \right)^{-1} AT^* \tag{1.2.4}$$

for all $A \in \mathcal{E}(\mathcal{H})$. Furthermore, for two bounded invertible linear or conjugate-linear operators T and S, one has $\mathrm{Ord}_T = \mathrm{Ord}_S$ *if and only if $S = \lambda T$ for some unit scalar λ.*

The proof of the previous result is not straightforward; it requires several deeper results and a special context (such as the classical Loewner's theorem, biholomorphic maps, ...). Even when attempting to show that $\mathrm{Ord}_T(.)$ preserves the order, the direct computations, known methods and tricks do not suffice.

1.3 The C*-algebras of type $\mathcal{C}(\mathcal{X}) \otimes \mathcal{B}(\mathcal{H})$

Since the tensor product is a complicated notion with multiple definitions and kinds (spatial tensor product, injective tensor product, projective tensor product, ...), we will begin with a quick review to introduce this notion and eliminate any confusion. For instance, one can view the algebraic tensor product of two normed vector spaces $\mathcal{A}$ and $\mathcal{B}$, written $A \odot B$, as the vector space spanned by certain elementary tensors $A \odot B$ which make the map $(A, B) \mapsto A \odot B$ bilinear, see [47, P. 235]. Here, we prefer the following definition because it seems to be the most coherent in our perspective. We denote the duals of two normed spaces $\mathcal{A}$ and $\mathcal{B}$ over a field $\mathbb{F}$ by $\mathcal{A}'$ and $\mathcal{B}'$, respectively. Given $(A, B) \in \mathcal{A} \times B$, we define the bounded form $A \odot B : \mathcal{A}' \times \mathcal{B}' \to \mathbb{F}$ by

$$A \odot B(a, b) = a(A)b(B),$$

for all $a \in \mathcal{A}'$, $b \in \mathcal{B}'$. The algebraic tensor product of $\mathcal{A}$ and $\mathcal{B}$, $\mathcal{A} \odot \mathcal{B}$, is defined by the vector space generated by $\{A \odot B : A \in \mathcal{A},\ B \in \mathcal{B}\}$, see e.g. [5]. If $\mathcal{A}$ and $\mathcal{B}$ are C*-algebras, there is an obvious way, by considering $(A \odot B)(C \odot D) = AC \odot BD$, $(A \odot B)^* = A^* \odot B^*$, to make the vector space tensor product $\mathcal{A} \odot \mathcal{B}$ into a *-algebra, see [47, Lemma B.l]. But there are several different ways of norming $\mathcal{A} \odot \mathcal{B}$ and completing it to give a C*-algebra. Let $\mathcal{X}$ be a Hausdorff compact space, fortunately, the C*-algebra $\mathcal{C}(\mathcal{X})$ falls within the realm of what is referred to as nuclear C*-algebras, in which all tensor products of C*-algebras coincide, and there is only one C*-norm on $\mathcal{C}(\mathcal{X}) \odot B$ for every C*-algebra B, see [47, Section B.5]. The completion of $\mathcal{C}(\mathcal{X}) \odot B$ with this norm will be denoted by $\mathcal{C}(\mathcal{X}) \otimes B$. Moreover, Since $\mathcal{C}(\mathcal{X})$ is commutative, the C*-algebra tensor product $\mathcal{C}(\mathcal{X}) \otimes \mathcal{B}$ takes a familiar form, see e.g. [47, Proposition B.16]. In particular, let $\mathcal{H}$ be a complex Hilbert space, the C*-algebra tensor product $\mathcal{C}(\mathcal{X}) \otimes \mathcal{B}(\mathcal{H})$ can be identified with the algebra of all continuous $\mathcal{B}(\mathcal{H})$-valued functions on $\mathcal{X}$, $\mathcal{C}(\mathcal{X}, \mathcal{B}(\mathcal{H}))$.

Theorem 1.3.1. *([47]) There exists an isometric *-algebra isomorphism $\mathcal{J}$ between $\mathcal{C}(\mathcal{X}) \otimes \mathcal{B}(\mathcal{H})$ and the algebra $\mathcal{C}(\mathcal{X}, \mathcal{B}(\mathcal{H}))$ of all continuous $\mathcal{B}(\mathcal{H})$-valued functions on $\mathcal{X}$ such that*

$$\mathcal{J}(r \odot A)(x) = r(x)A$$

for all $(r, A, x) \in \mathcal{C}(\mathcal{X}) \times \mathcal{B}(\mathcal{H}) \times \mathcal{X}$.

Recall that $\mathcal{C}(\mathcal{X}, \mathcal{B}(\mathcal{H}))$ is a C*-algebra with pointwise operations and the sup-norm:

$$(fg)(x) = f(x)g(x), \qquad f^*(x) = f(x)^*, \qquad \|f\| := \sup \|f(x)\|$$

for all $x \in \mathcal{X},\ f, g \in \mathcal{C}(\mathcal{X}, \mathcal{B}(\mathcal{H}))$. The usual order of $\mathcal{C}(\mathcal{X}, \mathcal{B}(\mathcal{H}))$ is also given pointwise, with respect to the usual order on $\mathcal{B}(\mathcal{H})$, i.e,

$$f \leq g \quad \text{if and only if} \quad f(x) \leq g(x)$$

for all $x \in \mathcal{X}$ and $f, g \in \mathcal{C}(\mathcal{X}, \mathcal{B}(\mathcal{H}))$. The unity of $\mathcal{C}(\mathcal{X}, \mathcal{B}(\mathcal{H}))$ will be denoted by $I_{\mathcal{X}}$. One can check in a natural way that $\mathcal{E}\left(\mathcal{C}(\mathcal{X}, \mathcal{B}(\mathcal{H}))\right) = \mathcal{C}(\mathcal{X}, \mathcal{E}(\mathcal{H}))$, that is, the set of all continuous $\mathcal{E}(\mathcal{H})$-valued functions on $\mathcal{X}$. In other words, if $f \in \mathcal{C}(\mathcal{X}, \mathcal{B}(\mathcal{H}))$ then,

$$f \in \mathcal{E}\left(\mathcal{C}(\mathcal{X}, \mathcal{B}(\mathcal{H}))\right) \qquad \text{if and only if} \qquad f(x) \in \mathcal{E}(\mathcal{H}), \quad x \in \mathcal{X}.$$

Note that the C^*-algebra $\mathcal{C}(\mathcal{X}) \otimes \mathcal{B}(\mathcal{H})$ is of importance because of several remarkable results concerning the structure of its derivations, automorphisms and its group of linear order automorphisms, see [21, 27, 33].

By $\mathcal{P}(\mathcal{X}, \mathcal{H})$, we denote the set of all orthogonal projections of $\mathcal{C}(\mathcal{X}, \mathcal{B}(\mathcal{H}))$, that is the set of functions $f \in \mathcal{C}(\mathcal{X}, \mathcal{E}(\mathcal{H}))$ such that $f(\mathcal{X}) \subseteq \mathcal{P}(\mathcal{H})$. We also set

$$\mathcal{P}_k(\mathcal{X}, \mathcal{H}) := \{f \in \mathcal{C}(\mathcal{X}, \mathcal{E}(\mathcal{H})) \ : \ f(\mathcal{X}) \subseteq \mathcal{P}_k(\mathcal{H})\} \quad \text{for } k = -1, 1. \tag{1.3.1}$$

Thus, $\mathcal{P}_{-1}(\mathcal{X}, \mathcal{H}) = \{f^{\perp} \ : \ f \in \mathcal{P}_1(\mathcal{X}, \mathcal{H})\}$.

The next lemma is a variant of [38, Lemma 3.1], and provides an order characterization of the elements of $\mathcal{P}(\mathcal{X}, \mathcal{H})$. We recall that the supremum and infimum in the next lemma are considered in the partially ordered set $\mathcal{C}(\mathcal{X}, \mathcal{E}(\mathcal{H}))$.

Lemma 1.3.2. *Let $f \in \mathcal{C}(\mathcal{X}, \mathcal{E}(\mathcal{H}))$ be given. Then, the following statements are equivalent.*

1. *$f \in \mathcal{P}(\mathcal{X}, \mathcal{H})$.*
2. *There exists $g \in \mathcal{C}(\mathcal{X}, \mathcal{E}(\mathcal{H}))$ such that*
   - *(a)* $\sup\{f, g\} = I_{\mathcal{X}}$ *and* $\inf\{f, g\} = 0$,
   - *(b) if $(f_0, g_0) \in \mathcal{C}(\mathcal{X}, \mathcal{E}(\mathcal{H}))^2$, such that $f \leq f_0$, $g \leq g_0$ and* $\inf\{f_0, g_0\} = 0$, *then $(f, g) = (f_0, g_0)$.*

Proof. For the implication (1) $\Longrightarrow$ (2), put $g = f^{\perp} \in \mathcal{P}(\mathcal{X}, \mathcal{H})$. For each $x \in \mathcal{X}$, we have $\mathrm{Im}\,(f(x)) + \mathrm{Im}\,(g(x)) = \mathcal{H}$ and $\mathrm{Im}\,(f(x)) \cap \mathrm{Im}\,(g(x)) = \{0\}$. Then, $\sup\{f(x), g(x)\} = I$ and $\inf\{f(x), g(x)\} = 0$. Since this holds for all $x \in \mathcal{X}$, $\sup\{f, g\} = I_{\mathcal{X}}$ and $\inf\{f, g\} = 0$.

Let $(f_0, g_0) \in \mathcal{C}(\mathcal{X}, \mathcal{E}(\mathcal{H}))^2$ such that $f \leq f_0$, $g \leq g_0$ and $\inf\{f_0, g_0\} = 0$. Then, by Lemma 1.2.3, we have

$$g f_0 g = f_0 - f \leq f_0 \qquad \text{and} \qquad f g_0 f = g_0 - g \leq g_0. \tag{1.3.2}$$

Notice that $f_0 \leq I_{\mathcal{X}}$ and $g_0 \leq I_{\mathcal{X}}$, then, $gf_0g \leq g \leq g_0$ and $fg_0f \leq f \leq f_0$. Combine this with (1.3.2) to conclude that

$$0 \leq g_0 - g \leq \inf\{f_0, g_0\} = 0 \qquad \text{and} \qquad 0 \leq f_0 - f \leq \inf\{f_0, g_0\} = 0.$$

Hence, $f_0 = f$ and $g_0 = g$.

For the converse implication (2) $\Longrightarrow$ (1), choose an $\alpha \in (0,1)$, let $\Phi_\alpha : \mathcal{C}(\mathcal{X}, \mathcal{E}(\mathcal{H})) \to \mathcal{C}(\mathcal{X}, \mathcal{E}(\mathcal{H}))$ be the mapping defied by

$$\Phi_\alpha(h) = r_\alpha(h)$$

for all $h \in \mathcal{C}(\mathcal{X}, \mathcal{E}(\mathcal{H}))$. The fact that Φ_α is well defined comes from Theorem 1.1.3. The bijectivity and the preservation of order in both directions of Φ_α follows at once from Lemma 1.2.4. That is, Φ_α is an order automorphism. Thus, the couple $(\Phi_\alpha(f), \Phi_\alpha(g))$ satisfies the condition (2). Since $\alpha \neq 0$, a simple computation yields

$$r_\alpha(0) = 0, \qquad r_\alpha(1) = 1, \qquad \text{and} \qquad r_\alpha(t) - t > 0, \tag{1.3.3}$$

for all $t \in (0,1)$. Using that, together with Theorem 1.1.3, we conclude that $f \leq \Phi_\alpha(f)$ and $g \leq \Phi_\alpha(g)$. Then, $(f,g) = (\Phi_\alpha(f), \Phi_\alpha(g))$. Using again (1.3.3), we conclude that $\mathrm{Sp}(f)$ and $\mathrm{Sp}(g)$ are subsets of $\{0,1\}$. That is, $f, g \in \mathcal{P}(\mathcal{X}, \mathcal{H})$. □

The following corollary is a straightforward conclusion of Lemma 1.3.2.

Corollary 1.3.3. *Let $\Phi : \mathcal{C}(\mathcal{X}, \mathcal{E}(\mathcal{H})) \to \mathcal{C}(\mathcal{Y}, \mathcal{E}(\mathcal{K}))$ be an order isomorphism. Then*

$$\Phi(\mathcal{P}(\mathcal{X}, \mathcal{H})) = \mathcal{P}(\mathcal{Y}, \mathcal{K}). \tag{1.3.4}$$

1.4 Unbounded self-adjoint operators

This section is devoted to developing the necessary notions and preliminaries on unbounded self-adjoint operators. Although it not entirely independent of the previous sections, its primary purpose is to prepare the reader specifically for the third chapter.

An unbounded operator A on a Hilbert space $\mathcal{H}$ is a linear map $A : \mathcal{D}(A) \to \mathcal{H}$ with $\mathcal{D}(A)$ being a subspace of $\mathcal{H}$. For two unbounded operators A, B, we say that $A = B$ if $\mathcal{D}(A) = \mathcal{D}(B)$

and $Av = Bv$ for all $v \in \mathcal{D}(A)$. Concerning the algebraic operations, the operator $A + B$ is defined by $\mathcal{D}(A+B) := \mathcal{D}(A) \cap \mathcal{D}(B)$ and $(A+B)v := Av + Bv$ for all $v \in \mathcal{D}(A+B)$. Similarly, we define the operator AB by $\mathcal{D}(AB) := \{v \in \mathcal{D}(B) : \ Bv \in \mathcal{D}(A)\}$ and $(AB)v := ABv$ for all $v \in \mathcal{D}(AB)$.

Consider a densely defined unbounded operator A on $\mathcal{H}$ $\left(\overline{\mathcal{D}(A)} = \mathcal{H}\right)$. Its adjoint, denoted by A^*, is an unbounded operator on $\mathcal{H}$ satisfying

$$\langle Av, w\rangle = \langle v, A^*w\rangle \tag{1.4.1}$$

for all $v \in \mathcal{D}(A)$, $w \in \mathcal{D}(A^*)$, and if there exists an unbounded operator B on $\mathcal{H}$ satisfying (1.4.1), then $\mathcal{D}(B) \subseteq \mathcal{D}(A^*)$. The operator A is said to be self-adjoint if $A = A^*$, see [44, Definition 3.5]. We will denote by $\mathcal{S}(\mathcal{H})$ the set of all unbounded self-adjoint operators on $\mathcal{H}$, by $\mathcal{S}_+(\mathcal{H})$ the set of all operators A in $\mathcal{S}(\mathcal{H})$ such that $\langle Av, v\rangle \geq 0$ for all $v \in \mathcal{D}(A)$. Also, we denote $\mathcal{S}^b(\mathcal{H}) := \mathcal{B}(\mathcal{H}) \cap \mathcal{S}(\mathcal{H})$ and $\mathcal{S}^b_+(\mathcal{H}) := \mathcal{B}(\mathcal{H}) \cap \mathcal{S}_+(\mathcal{H})$. Self-adjoint operator have many interesting properties. In particular, they have closed graphs, see e.g. [11, 44].

Proposition 1.4.1. *([44]) Any $A \in \mathcal{S}(\mathcal{H})$ is a closed operator, in the sense that, if (v_n) is a sequence in $\mathcal{D}(A)$ such that (v_n) converges to $v \in \mathcal{H}$ and (Av_n) converges to $w \in \mathcal{H}$, then $v \in \mathcal{D}(A)$ and $Av = w$.*

It is well known that the closed graph theorem holds in Banach spaces. However, here we will state it specifically in the case of Hilbert spaces.

Theorem 1.4.2 (closed graph theorem). *Let A be a closed unbounded operator, everywhere defined, i.e., $\mathcal{D}(A) = \mathcal{H}$. Then, $A \in \mathcal{B}(\mathcal{H})$.*

For any operator $A \in \mathcal{S}(\mathcal{H})$, we denote, as the bounded case, the image or the range of A by $\mathrm{Im}(A)$ and the kernel of A by $\mathrm{Ker}(A)$. Notice that the decomposition (1.2.1) of the Hilbert space $\mathcal{H}$ holds true also in the unbounded case. Namely, for every $A \in \mathcal{S}(\mathcal{H})$, we have

$$\mathcal{H} = \overline{\mathrm{Im}(A)} \oplus \mathrm{Ker}(A).$$

Let A be an unbounded operator on $\mathcal{H}$, we say that A is boundedly invertible if there exists an operator $B \in \mathcal{B}(\mathcal{H})$ such that $AB = I$, and BA is the restriction of I on $\mathcal{D}(A)$ where I is the identity operator on $\mathcal{H}$. The set of all complex scalars t such that $(A - tI)$ is not boundedly invertible is called the spectrum of A and denoted, as in the bounded case, by $\mathrm{Sp}(A)$. This definition agrees with the usual known concept of the spectrum in $\mathcal{B}(\mathcal{H})$. The notations $\mathcal{S}^{-1}(\mathcal{H})$ and $\mathcal{S}_+^{-1}(\mathcal{H})$ stand for the sets of all boundedly invertible operators in $\mathcal{S}(\mathcal{H})$ and $\mathcal{S}_+(\mathcal{H})$, respectively.

Many useful tools have been developed in the theory of unbounded self-adjoint operators, especially the spectral theorem and functional calculus, which can be considered as cornerstones for the study of such operators. In order to state these fundamental results, we need the following definitions.

Recall that $\mathcal{P}(\mathcal{H})$ denotes the set of all orthogonal projections on $\mathcal{H}$. Let Ω be a σ-algebra on a set $\mathcal{X}$ (a non-empty collection of subsets of $\mathcal{X}$ closed under complement, countable unions, and countable intersections). A spectral measure for $(\mathcal{X}, \Omega)$ is a function $\mathrm{E} : \Omega \to \mathcal{P}(\mathcal{H})$ such that

1. $\mathrm{E}(\mathcal{X}) = I$;

2. E is countably additive, i.e., if $(\mathcal{X}_n)$ a sequence of pairwise disjoint sets from Ω, then

$$\mathrm{E}\left(\bigcup_{n\geq 1} \mathcal{X}_n\right) = \sum_{n\geq 1} \mathrm{E}(\mathcal{X}_n).$$

The above sum is considered in the context of the strong convergence, that is, for each $v \in \mathcal{H}$,

$$\mathrm{E}\left(\bigcup_{n\geq 1} \mathcal{X}_n\right) v = \lim_{n} \sum_{k\geq 1}^{n} \mathrm{E}(\mathcal{X}_n) v.$$

For every $v, w \in \mathcal{H}$, E gives a rise to a complex regular Borel measure $\mathrm{E}_{v,w} : \Omega \to \mathbb{C}$ given by

$$\mathrm{E}_{v,w}(\mathcal{X}') = \langle E(\mathcal{X}')v, w \rangle \,,$$

for all $\mathcal{X}' \in \Omega$. In the case $v = w$, $\mathrm{E}_{v,v}$ is a positive measure.

The following theorem is of importance, since it gives rise to the functional calculus on $\mathcal{S}(\mathcal{H})$.

Theorem 1.4.3. *([44]) Let* E *be a spectral measure on the real line. For any Borel function* $r : \mathbb{R} \to \mathbb{C}$*, there exists a unique closed unbounded operator* $\int r(t) \mathrm{dE}(t)$ *on* $\mathcal{H}$ *such that*

$$\mathcal{D}\left(\int r(t)\mathrm{dE}(t)\right) = \left\{ v \in \mathcal{H} : \int |r(t)|^2 \mathrm{dE}_{v,v} \text{ is finite} \right\}, \tag{1.4.2}$$

and for all $v \in \mathcal{D}\left(\int r(t)\mathrm{dE}(t)\right)$

$$\left\langle \int r(t)\mathrm{dE}(t)v, v \right\rangle = \int r(t)\mathrm{dE}_{v,v}. \tag{1.4.3}$$

Moreover, if r, s *and* q *are real-valued Borel functions on* $\mathbb{R}$ *such that* s *is non-negative and* q *is bounded, then* $\int r(t)\mathrm{dE}(t) \in \mathcal{S}(\mathcal{H})$, $\int s(t)\mathrm{dE}(t) \in \mathcal{S}_+(\mathcal{H})$ *and* $\int q(t)\mathrm{dE}(t) \in \mathcal{B}(\mathcal{H})$.

Now, we are in the position to state the spectral theorem, see [44, Theorem 5.7].

Theorem 1.4.4 (Spectral theorem). *For each* $A \in \mathcal{S}(\mathcal{H})$*, there exists a unique spectral measure* E *on the Borel algebra of real line, such that* A *has the integral representation*

$$A = \int t\mathrm{dE}(t). \tag{1.4.4}$$

One might expect, by analogy with the continuous functional calculus, that we have $\rho_A(r) = \int r(t)\mathrm{dE}(t)$, for all $A \in \mathcal{S}^b(\mathcal{H})$, and all functions $r \in \mathcal{C}(\mathrm{Sp}(A))$. This is indeed the case, see e.g. [11, Theorem 1.14]. So, for notational simplicity, from now on, we will write $r(A)$ for $\int r(t)\mathrm{dE}(t)$, with r being a Borel function on $\mathrm{Sp}(A)$.

It is well known that each element in $\mathcal{S}_+^b(\mathcal{H})$ has a unique square root. The following theorem clarifies that this fact also holds true in $\mathcal{S}_+(\mathcal{H})$, see e.g.[44, Proposition 5.13].

Theorem 1.4.5. *([44]) For each* $A \in \mathcal{S}_+(\mathcal{H})$*, there exists a unique operator* $B \in \mathcal{S}_+(\mathcal{H})$ *such that* $B^2 = A$*. The operator* B *is given by* $\int t^{\frac{1}{2}}\mathrm{dE}(t)$ *where* E *is the spectral measure of* A*.*

The operator B in the above theorem will be denoted by $A^{1/2}$. If A is a densely defined closed unbounded operator on $\mathcal{H}$, the operator $A^*A \in \mathcal{S}_+(\mathcal{H})$, see [44, p. 137]. We denote the square root of the operator A^*A by $|A|$ and it is called the modulus of A.

We close this section by introducing a partial order relation on $\mathcal{S}(\mathcal{H})$. For two operators $A, B \in \mathcal{S}(\mathcal{H})$ we say that $A \leq B$ if $\mathcal{D}(B) \subseteq \mathcal{D}(A)$ and for every $v \in \mathcal{D}(B)$ we have the inequality

$$\langle Av, v\rangle \leq \langle Bv, v\rangle .$$

Obviously, the relation " $\leq$ " can be seen as an extension of the usual order of the algebra $\mathcal{B}(\mathcal{H})$, because they agree on $\mathcal{S}^b(\mathcal{H})$, see [24], [44]. This relation is of importance in quantum mechanics. Indeed, for two observables A, B in a quantum mechanical system, $A \leq B$ is equivalent to the average value $\langle Av, v\rangle$ of A in any state $v \in \mathcal{D}(B)$, is less than or equal to the average value $\langle Bv, v\rangle$ of B in the same state.

Chapter 2

On effect algebras $\mathcal{C}(\mathcal{X}, \mathcal{E}(\mathcal{H}))$

In this chapter, we will investigate order isomorphisms between effect algebras of type $\mathcal{C}(\mathcal{X}, \mathcal{E}(\mathcal{H}))$. Namely, we will begin by reducing the problem to the case where $\dim \mathcal{H} \geq 2$. Then, we will prove that such mappings, which preserve the half identity, are extendable to Jordan *-isomorphisms. Afterward, we will provide a complete description of all order isomorphisms in the general case, relying on the initial case. In order to achieve this, we will also prove remarkable results. For example, we will show that any order isomorphism preserves the collections of types $\mathcal{P}_1(\mathcal{X}, \mathcal{H})$ and $\mathcal{P}_{-1}(\mathcal{X}, \mathcal{H})$. Furthermore, we will clarify that if $\mathcal{H}$ and $\mathcal{K}$ are two Hilbert spaces, and $\mathcal{X}$ and $\mathcal{Y}$ are two Hausdorff compact spaces, then $\mathcal{C}(\mathcal{X}, \mathcal{E}(\mathcal{H}))$ and $\mathcal{C}(\mathcal{Y}, \mathcal{E}(\mathcal{K}))$ are order isomorphic if and only if the orthonormal bases of $\mathcal{H}$ and $\mathcal{K}$ have the same cardinality, and the spaces $\mathcal{X}, \mathcal{Y}$ are homeomorphic.

The results of this chapter are published in the journal “Linear Algebra and its Applications”, in the paper entitled “ Order isomorphisms on effect algebras of the C*-algebras of type $\mathcal{C}(\mathcal{X}) \otimes \mathcal{B}(\mathcal{H})$”.

2.1 A reduction of the problem

In the following theorem, we show that the study of the order isomorphisms between the effect algebras of type $\mathcal{C}(\mathcal{X}, \mathcal{E}(\mathcal{H}))$ should focus on the case where $\dim \mathcal{H} \geq 2$, and the case where $\dim \mathcal{H} = 1$ can be concluded directly from Theorem 1.1.5.

Theorem 2.1.1. *Let $\mathcal{X}$ and $\mathcal{Y}$ be two Hausdorff compact spaces, and let $\mathcal{H}$ be a non trivial Hilbert space. Then $\mathcal{C}(\mathcal{X}, \mathcal{E}(\mathcal{H}))$ and $\mathcal{C}(\mathcal{Y}, [0, 1])$ are order isomorphic only if* $\dim \mathcal{H} = 1$.

Proof. Assume to the contrary that $\dim \mathcal{H} \geq 2$ and there exists an order isomorphism Φ :

$\mathcal{C}(\mathcal{X},\mathcal{E}(\mathcal{H})) \to \mathcal{C}(\mathcal{Y},[0,1])$. Let $\mathcal{H}_0$ be a closed hyperplane of $\mathcal{H}$ and let $v, w \in \mathcal{H}\setminus\mathcal{H}_0$ be two linearly independent unit vectors. Let P be the orthogonal projection on $\mathcal{H}_0$, we define the constant functions f, g, h on $\mathcal{X}$ by $f(\mathcal{X}) = \{P\}$, $g(\mathcal{X}) = \{v\otimes v\}$ and $h(\mathcal{X}) = \{w\otimes w\}$. Thus, $f, g, h \in \mathcal{P}(\mathcal{X},\mathcal{H})$,

$$\sup\{f,h\} = \sup\{f,g\} = I_{\mathcal{X}} \qquad \text{and} \qquad \inf\{f,h\} = \inf\{f,g\} = 0. \tag{2.1.1}$$

Applying Φ on (2.1.1), we get that

$$\sup\{\Phi(f),\Phi(g)\} = \sup\{\Phi(f),\Phi(h)\} = I_{\mathcal{Y}}, \tag{2.1.2}$$

and

$$\inf\{\Phi(f),\Phi(g)\} = \inf\{\Phi(f),\Phi(h)\} = 0. \tag{2.1.3}$$

By Corollary 1.3.3, we have $\Phi(f)(\mathcal{Y}) = \Phi(g)(\mathcal{Y}) = \Phi(h)(\mathcal{Y}) \subseteq \{0,1\}$. Using this together with (2.1.3) and Lemma 1.1.2, one obtains that $(\Phi(f)+\Phi(g))^2 = \Phi(f)+\Phi(g)$ and $(\Phi(f)+\Phi(h))^2 = \Phi(f)+\Phi(h)$. Thus, $\Phi(f)+\Phi(g)$ and $\Phi(f)+\Phi(h)$ are orthogonal projections, and then, $\Phi(f)+\Phi(g) \le I_{\mathcal{Y}}$, $\Phi(f)+\Phi(h) \le I_{\mathcal{Y}}$. On the other hand, $\Phi(f)+\Phi(g)$ is an upper-bound of $\{\Phi(f),\Phi(g)\}$ and $\Phi(f)+\Phi(h)$ is an upper-bound of $\{\Phi(f),\Phi(h)\}$. Combining this with (2.1.2), we conclude that

$$\Phi(f)+\Phi(h) = \Phi(f)+\Phi(h) = I_{\mathcal{Y}}.$$

Hence, $\Phi(g) = \Phi(h)$ which contradicts the injectivity of Φ. $\square$

2.2 Statements of main results

Recall that a linear bijection J from a C^*-algebra $\mathcal{A}$ onto a C*-algebra $\mathcal{B}$ is called a Jordan *-isomorphism if it satisfies $J(A^*) = J(A)^*$ and $J(A^2) = J(A)^2$ for all $A \in \mathcal{A}$. It is easy to check that any Jordan *-isomorphism is a unitary order isomorphism. The converse is true under some restrictions, in the sense that every unitary linear order automorphism of a C*-algebra is a Jordan *-automorphism, see [20, Corollary 5]. So, each *-Jordan isomorphism $J : \mathcal{A} \to \mathcal{B}$ induces an order isomorphism from $\mathcal{E}(\mathcal{A})$ onto $\mathcal{E}(\mathcal{B})$. The following theorem ensures that every order isomorphism (without linearity assumption) Φ from $\mathcal{C}(\mathcal{X},\mathcal{E}(\mathcal{H}))$ onto $\mathcal{C}(\mathcal{Y},\mathcal{E}(\mathcal{K}))$ such that $\Phi(\frac{1}{2}I_{\mathcal{X}}) = \frac{1}{2}I_{\mathcal{Y}}$ is extendible to a Jordan *-isomorphism from $\mathcal{C}(\mathcal{X},\mathcal{B}(\mathcal{H}))$ onto $\mathcal{C}(\mathcal{Y},\mathcal{B}(\mathcal{K}))$. This provides an answer, in our context, to the problem of whether an order isomorphism, between effect algebras of two C^*-algebras $\mathcal{A}$ and $\mathcal{B}$, admits an extension to some Jordan *-isomorphism from $\mathcal{A}$ onto $\mathcal{B}$, see e.g. [38, Problem 5.3].

Theorem 2.2.1. *([1]) Let* $\Phi : \mathcal{C}(\mathcal{X}, \mathcal{E}(\mathcal{H})) \to \mathcal{C}(\mathcal{Y}, \mathcal{E}(\mathcal{K}))$ *be a bijective map. Suppose that* $\max\{\dim(\mathcal{H}), \dim(\mathcal{K})\} \geq 2$ *and one of the spaces* $\mathcal{X}$ *or* $\mathcal{Y}$ *satisfies the property* ($K_{\mathcal{E}}$)*. Then the following statements are equivalent.*

1. Φ *is an order isomorphism with* $\Phi(\frac{1}{2}I_{\mathcal{X}}) = \frac{1}{2}I_{\mathcal{Y}}$.
2. Φ *satisfies* $\Phi(\frac{1}{2}I_{\mathcal{X}}) = \frac{1}{2}I_{\mathcal{Y}}$*, and there exists a homeomorphism* μ *from* $\mathcal{Y}$ *onto* $\mathcal{X}$*, such that*

$$f(\mu(y)) \leq g(\mu(y)) \quad \Longleftrightarrow \quad \Phi(f)(y) \leq \Phi(g)(y)$$

for all $y \in \mathcal{Y}$*, and all* $f, g \in \mathcal{C}(\mathcal{X}, \mathcal{E}(\mathcal{H}))$.

3. Φ *is extendible to a Jordan* **-isomorphism*

$$J : \mathcal{C}(\mathcal{X}, \mathcal{B}(\mathcal{H})) \to \mathcal{C}(\mathcal{Y}, \mathcal{B}(\mathcal{K})).$$

4. *There exist a homeomorphism* μ *from* $\mathcal{Y}$ *onto* $\mathcal{X}$*, a decomposition* $\mathcal{Y} = \mathcal{Y}_1 \cup \mathcal{Y}_{-1}$ *of* $\mathcal{Y}$ *in two disjoint clopen subsets, and a* **-isomorphism (resp.* **-anti isomorphism)*

$$J_k : \mathcal{C}(\mu(\mathcal{Y}_k), \mathcal{B}(\mathcal{H})) \to \mathcal{C}(\mathcal{Y}_k, \mathcal{B}(\mathcal{K}))$$

for $k = 1$ *(resp.* $k = -1$*), such that the Jordan* **-isomorphism* $J := J_1 \oplus J_{-1}$ *extends* Φ*. Moreover, we can write*

$$J_1(f)(y) = U_y f(\mu(y)) U_y^* \quad (y \in \mathcal{Y}_1,\ f \in \mathcal{C}(\mu(\mathcal{Y}_1), \mathcal{B}(\mathcal{H}))),$$

and

$$J_{-1}(f)(y) = U_y f(\mu(y)) U_y^* \quad (y \in \mathcal{Y}_{-1},\ f \in \mathcal{C}(\mu(\mathcal{Y}_{-1}), \mathcal{B}(\mathcal{H}))),$$

where $U_y : \mathcal{H} \to \mathcal{K}$ *is unitary (resp. antiunitary) for* $y \in \mathcal{Y}_1$ *(resp.* $y \in \mathcal{Y}_{-1}$*),*

$$y \mapsto U_y A U_y^* \qquad \text{and} \qquad y \mapsto U_y^* B U_y \tag{2.2.1}$$

are continuous on $\mathcal{Y}$ for all $(A, B) \in \mathcal{E}(\mathcal{H}) \times \mathcal{E}(\mathcal{K})$.

Remark that Theorem 2.1.1 clarifies that additional assumption $\max\{\dim(\mathcal{H}), \dim(\mathcal{K})\} \geq 2$ is not restrictive. We remark also that if A^{tr} denotes the transpose of an element $A \in \mathcal{B}(\mathcal{H})$ with respect to an arbitrary but fixed orthonormal basis in $\mathcal{H}$, then J_{-1} given in the previous theorem can be written as $J_{-1}(f)(y) = V_y f(\mu(y))^{\mathrm{tr}} V_y^*$ for all $(y, f) \in \mathcal{Y}_{-1} \times \mathcal{C}(\mu(\mathcal{Y}_{-1}), \mathcal{B}(\mathcal{H}))$, where $V_y : \mathcal{H} \to \mathcal{K}$ is unitary for any $y \in \mathcal{Y}_{-1}$, and $y \mapsto V_y A V_y^*$ and $y \mapsto V_y^* B V_y$ are continuous on $\mathcal{Y}_{-1}$ for all $(A, B) \in \mathcal{E}(\mathcal{H}) \times \mathcal{E}(\mathcal{K})$.

From the description of Mori and Šemrl in [39, Theorem 7.3], see also (1.2.4), we can verify, see Lemma 4.0.7, that any positive invertible element ϕ in $\mathcal{C}(\mathcal{Y}, \mathcal{B}(\mathcal{K}))$ gives rise to another order automorphism Φ_ϕ on $\mathcal{C}(\mathcal{Y}, \mathcal{E}(\mathcal{K}))$ defined by

$$\Phi_\phi(f)(y) := \mathrm{Ord}_{\phi(y)}(f(y)) = \phi(y) \left(f(y)(\phi(y)^2 - I) + I \right)^{-1} f(y)\phi(y) \tag{2.2.2}$$

for all $y \in \mathcal{Y}$ and all functions $f \in \mathcal{C}(\mathcal{Y}, \mathcal{E}(\mathcal{K}))$. Now, let ϕ be an invertible positive element in $\mathcal{C}(\mathcal{Y}, \mathcal{B}(\mathcal{K}))$ and J be a Jordan *-isomorphism from $\mathcal{C}(\mathcal{X}, \mathcal{B}(\mathcal{H}))$ onto $\mathcal{C}(\mathcal{Y}, \mathcal{B}(\mathcal{K}))$ defined by $J(f)(y) = U_y f(\mu(y)) U_y^*$ for all $y \in \mathcal{Y}$ and all functions $f \in \mathcal{C}(\mathcal{X}, \mathcal{E}(\mathcal{H}))$, where $\mu : \mathcal{Y} \to \mathcal{X}$ is a homeomorphism and $U_y : \mathcal{H} \to \mathcal{K}$ is a unitary or antiunitary operator for all $y \in \mathcal{Y}$. Then, the restriction $\Phi_\phi \circ J|_{\mathcal{C}(\mathcal{X}, \mathcal{E}(\mathcal{H}))}$ of $\Phi_\phi \circ J$ to $\mathcal{C}(\mathcal{X}, \mathcal{E}(\mathcal{H}))$ is obviously an order isomorphism from $\mathcal{C}(\mathcal{X}, \mathcal{E}(\mathcal{H}))$ onto $\mathcal{C}(\mathcal{Y}, \mathcal{E}(\mathcal{K}))$ which verifies $\Phi_\phi \circ J|_{\mathcal{C}(\mathcal{X}, \mathcal{E}(\mathcal{H}))}(\frac{1}{2} I_{\mathcal{X}}) = (\phi^{-2} + I_{\mathcal{Y}})^{-1}$ and

$$\Phi_\phi \circ J|_{\mathcal{C}(\mathcal{X}, \mathcal{E}(\mathcal{H}))}(f)(y) = \mathrm{Ord}_{\phi(y) U_y}(f(\mu(y))) \tag{2.2.3}$$

for all $y \in \mathcal{Y}$ and all functions $f \in \mathcal{C}(\mathcal{X}, \mathcal{E}(\mathcal{H}))$. Moreover, the preceding composition is unique in the following sense. Assume that

$$\Phi_\phi \circ J|_{\mathcal{C}(\mathcal{X}, \mathcal{E}(\mathcal{H}))} = \Phi_{\phi'} \circ J'|_{\mathcal{C}(\mathcal{X}, \mathcal{E}(\mathcal{H}))}$$

for some positive invertible element ϕ' in $\mathcal{C}(\mathcal{Y}, \mathcal{B}(\mathcal{K}))$ and a Jordan *-isomorphism J' from $\mathcal{C}(\mathcal{X}, \mathcal{B}(\mathcal{H}))$ onto $\mathcal{C}(\mathcal{Y}, \mathcal{B}(\mathcal{K}))$ given by $J'(f)(y) = U'_y f(\mu'(y))(U'_y)^*$ for all $y \in \mathcal{Y}$ and all functions $f \in \mathcal{C}(\mathcal{X}, \mathcal{E}(\mathcal{H}))$, where $\mu' : \mathcal{Y} \to \mathcal{X}$ is a homeomorphism and $U'_y : \mathcal{H} \to \mathcal{K}$ is a unitary or antiunitary operator for all $y \in \mathcal{Y}$. Then from (2.2.3) we can see that

$$\mathrm{Ord}_{\phi(y) U_y}(A) = \mathrm{Ord}_{\phi'(y) U'_y}(A)$$

for all $A \in \mathcal{E}(\mathcal{H})$. Again, by the result of Mori and Sěmrl, [39, Theorem 7.3], we obtain that there exists a unit scalar λ_y such that $\phi(y)U_y = \lambda_y \phi'(y)U'_y$ for all $y \in \mathcal{Y}$. Since ϕ and ϕ' are positive and U_y and $\lambda_y U'_y$ are unitary or antiunitary we conclude that $\phi = \phi'$ and $J = J'$.

The following theorem provides a characterization of the order isomorphisms Φ without the condition $\Phi(\frac{1}{2}I_{\mathcal{X}}) = \frac{1}{2}I_{\mathcal{Y}}$. Further, it shows that (2.2.3) exhibits the general form of these maps.

Theorem 2.2.2. *([1]) Let* $\Phi : \mathcal{C}(\mathcal{X}, \mathcal{E}(\mathcal{H})) \to \mathcal{C}(\mathcal{Y}, \mathcal{E}(\mathcal{K}))$ *be a bijective map. Suppose that* $\max\{\dim(\mathcal{H}), \dim(\mathcal{K})\} \geq 2$ *and one of the spaces* $\mathcal{X}$ *or* $\mathcal{Y}$ *satisfies the property* ($K_{\mathcal{E}}$)*. Then the following statements are equivalent.*

1. *Φ is an order isomorphism.*

2. *There exists a homeomorphism μ from $\mathcal{Y}$ onto $\mathcal{X}$, such that*

$$f(\mu(y)) \leq g(\mu(y)) \quad \iff \quad \Phi(f)(y) \leq \Phi(g)(y)$$

   for all $y \in \mathcal{Y}$, and all $f, g \in \mathcal{C}(\mathcal{X}, \mathcal{E}(\mathcal{H}))$.

3. *The element $\phi := \left(\Phi(\frac{1}{2}I_{\mathcal{X}})^{-1} - I_{\mathcal{Y}}\right)^{-\frac{1}{2}}$ of $\mathcal{C}(\mathcal{Y}, \mathcal{B}(\mathcal{K}))$ is positive, and there exists a unique decomposition $\Phi = \Phi_\phi \circ \Phi_J$, where $\Phi_J : \mathcal{C}(\mathcal{X}, \mathcal{E}(\mathcal{H})) \to \mathcal{C}(\mathcal{Y}, \mathcal{E}(\mathcal{K}))$ is an order isomorphism extendible to a Jordan *-isomorphism J from $\mathcal{C}(\mathcal{X}, \mathcal{B}(\mathcal{H}))$ onto $\mathcal{C}(\mathcal{Y}, \mathcal{B}(\mathcal{K}))$.*

4. *There exists a homeomorphism $\mu : \mathcal{Y} \to \mathcal{X}$ such that*

$$\Phi(f)(y) = T_y \left(f(\mu(y))(T_y^* T_y - I) + I\right)^{-1} f(\mu(y)) T_y^*$$

   for all $f \in \mathcal{C}(\mathcal{X}, \mathcal{E}(\mathcal{H}))$ and all $y \in \mathcal{Y}$, where $(T_y)_{y \in \mathcal{Y}}$ is a family of bijective, linear or conjugate-linear bounded operators from $\mathcal{H}$ onto $\mathcal{K}$ for which the maps

$$y \mapsto T_y A T_y^* \quad \text{and} \quad y \mapsto T_y^* B T_y \tag{2.2.4}$$

   are continuous on $\mathcal{Y}$ for all $(A, B) \in \mathcal{E}(\mathcal{H}) \times \mathcal{E}(\mathcal{K})$.

The following proposition gives us a way to produce order isomorphisms on the effect algebras of the C^*-algebras of type $\mathcal{C}(\mathcal{X}, \mathcal{B}(\mathcal{H}))$.

Proposition 2.2.3. *Assume that there exist a homeomorphism* $\mu : \mathcal{Y} \to \mathcal{X}$ *and a family* $(T_y)_{y\in\mathcal{Y}}$ *of bijective, linear or conjugate-linear bounded operators from* $\mathcal{H}$ *onto* $\mathcal{K}$ *such that*

$$y \mapsto T_y A T_y^* \quad \textit{and} \quad y \mapsto T_y^* B T_y$$

are continuous on $\mathcal{Y}$ *for all* $(A, B) \in \mathcal{E}(\mathcal{H}) \times \mathcal{E}(\mathcal{K})$*. Then the formula*

$$\Phi(f)(y) = T_y \left(f(\mu(y))(T_y^* T_y - I) + I \right)^{-1} f(\mu(y)) T_y^*$$

for all $(y, f) \in \mathcal{Y} \times \mathcal{C}(\mathcal{X}, \mathcal{E}(\mathcal{H}))$*, gives rise to an order isomorphism* Φ *from* $\mathcal{C}(\mathcal{X}, \mathcal{E}(\mathcal{H}))$ *onto* $\mathcal{C}(\mathcal{Y}, \mathcal{E}(\mathcal{K}))$*.*

The proofs of the previous theorems will use the following result which shows that the order isomorphism Φ preserves in both directions some specific sets of orthogonal projections.

Theorem 2.2.4. *([1]) Let* $\Phi : \mathcal{C}(\mathcal{X}, \mathcal{E}(\mathcal{H})) \to \mathcal{C}(\mathcal{Y}, \mathcal{E}(\mathcal{K}))$ *be an order isomorphism. Then the following statements hold.*

1. Φ *induces an order isomorphism from* $\mathcal{P}(\mathcal{X}, \mathcal{H})$ *onto* $\mathcal{P}(\mathcal{Y}, \mathcal{K})$*.*
2. $\Phi(\mathcal{P}_1(\mathcal{X}, \mathcal{H})) = \mathcal{P}_1(\mathcal{Y}, \mathcal{K})$*.*
3. $\Phi(\mathcal{P}_{-1}(\mathcal{X}, \mathcal{H})) = \mathcal{P}_{-1}(\mathcal{Y}, \mathcal{K})$*.*
4. *The spaces* $\mathcal{X}$ *and* $\mathcal{Y}$ *are homeomorphic.*
5. *If in addition* $\Phi(\frac{1}{2}I_{\mathcal{X}}) = \frac{1}{2}I_{\mathcal{Y}}$*, then the Hilbert spaces* $\mathcal{H}$ *and* $\mathcal{K}$ *are isometric.*

For every $s \in \mathcal{C}(\mathcal{X}, [0, 1])$ and $f \in \mathcal{C}(\mathcal{X}, \mathcal{E}(\mathcal{H}))$, we denote

$$[0, f] := \{g \in \mathcal{C}(\mathcal{X}, \mathcal{E}(\mathcal{H})) \ : \ g \leq f\},$$

and sf is the element of $\mathcal{C}(\mathcal{X}, \mathcal{E}(\mathcal{H}))$ defined by $x \mapsto s(x)f(x)$. Since the set of orthogonal projections is not rich enough as in the case of von Neumann algebras, we head to the study the behaviour of the elements of intervals $[0, \pi]$ (we will take advantage of the fact that $[0, \pi] = \{s\pi \ : \ s \in \mathcal{C}(\mathcal{X}, [0, 1])\}$) for $\pi \in \mathcal{P}_1(\mathcal{X}, \mathcal{H})$. Keeping the conditions and notations of Theorem 2.2.2, the order isomorphism Φ defines a family $(\Phi_\pi)_{\pi\in\mathcal{P}_1(\mathcal{X},\mathcal{H})}$ of order isomorphisms from $\mathcal{C}(\mathcal{X}, [0, 1])$ onto $\mathcal{C}(\mathcal{Y}, [0, 1])$, by

$$\Phi(s\pi) = \Phi_\pi(s)\Phi(\pi) \quad \text{for} \quad \pi \in \mathcal{P}_1(\mathcal{X}, \mathcal{H}). \tag{2.2.5}$$

For this purpose we will use intensively the formula (1.1.2) for each Φ_π, $\pi \in \mathcal{P}_1(\mathcal{X},\mathcal{H})$. On the other hand Theorem 2.2.1 and Theorem 2.2.2 allow us to see that these order isomorphisms have a very particular and concreted expression. Precisely, we state

Corollary 2.2.5. *Keep the conditions and notations of Theorem 2.2.2. For every $\pi \in \mathcal{P}_1(\mathcal{X},\mathcal{H})$, the order isomorphism $\Phi_\pi : \mathcal{C}(\mathcal{X},[0,1]) \to \mathcal{C}(\mathcal{Y},[0,1])$, defined by the formula (2.2.5), has the following form*

$$\Phi_\pi(s)(y) = \frac{s(\mu(y))\|\Phi_J(\pi)(y)\phi(y)^2\Phi_J(\pi)(y)\|}{1 + s(\mu(y))\left(\|\Phi_J(\pi)(y)\phi(y)^2\Phi_J(\pi)(y)\| - 1\right)} \tag{2.2.6}$$

for all $(y,s) \in \mathcal{Y} \times \mathcal{C}(\mathcal{X},[0,1])$.

2.3 Preservation of some specific subset of projections

Let Φ be an order isomorphism from $\mathcal{C}(\mathcal{X},\mathcal{E}(\mathcal{H}))$ onto $\mathcal{C}(\mathcal{Y},\mathcal{E}(\mathcal{K}))$. Observe that $\Phi(0) = 0$, $\Phi(I_\mathcal{X}) = I_\mathcal{Y}$, and the mapping defined by

$$\Phi^\perp : \mathcal{C}(\mathcal{X},\mathcal{E}(\mathcal{H})) \to \mathcal{C}(\mathcal{Y},\mathcal{E}(\mathcal{K}));\ f \mapsto I_\mathcal{Y} - \Phi(I_\mathcal{X} - f) \tag{2.3.1}$$

is an order isomorphism. We point out that throughout all the chapter we will use the Urysohn's lemma. The aim of this section is to prove that Φ maps bijectively and respectively the sets $\mathcal{P}(\mathcal{X},\mathcal{H})$, $\mathcal{P}_1(\mathcal{X},\mathcal{H})$ and $\mathcal{P}_{-1}(\mathcal{X},\mathcal{H})$ onto the sets $\mathcal{P}(\mathcal{Y},\mathcal{K})$, $\mathcal{P}_1(\mathcal{Y},\mathcal{K})$ and $\mathcal{P}_{-1}(\mathcal{Y},\mathcal{K})$.

We recall that throughout all this chapter, the supremum and infimum are considered in the partially ordered set $\mathcal{C}(\mathcal{X},\mathcal{E}(\mathcal{H}))$.

Lemma 2.3.1. *Let $p : \mathcal{X} \to \mathcal{H}$ be a continuous function such that $\|p(x)\| \leq 1$ for all $x \in \mathcal{X}$. Then, the map $f : \mathcal{X} \to \mathcal{E}(\mathcal{H})$ defined by*

$$f(x) = p(x) \otimes p(x),$$

is continuous.

Proof. The lemma follows from the fact that $x \mapsto p(x)$ is continuous on $\mathcal{X}$ and $(u,v) \mapsto u \otimes v$ is continuous on $\mathcal{H}^2$. □

Lemma 2.3.2. *Let $\pi \in \mathcal{P}(\mathcal{X}, \mathcal{H})$. Then for each $x \in \mathcal{X}$ the set*

$$\{y \in \mathcal{X} \ : \ \operatorname{Rank}(\pi(y)) = \operatorname{Rank}(\pi(x))\}$$

is a clopen subset of $\mathcal{X}$.

Proof. Following Müller [40, Definition 5, p. 98], the gap between two Hilbert subspaces $\mathcal{G}$ and $\mathcal{L}$ of $\mathcal{H}$ is given by

$$\hat{\delta}\,(\mathcal{G}, \mathcal{L}) = \max\left\{ \sup_{k \in \mathcal{G}, \|k\| \leq 1} \left(\inf_{l \in \mathcal{L}} \|k - l\| \right), \sup_{l \in \mathcal{L}, \|l\| \leq 1} \left(\inf_{k \in \mathcal{G}} \|l - k\| \right) \right\}.$$

One can see that $\hat{\delta}\,(\operatorname{Im}(\pi(x)), \operatorname{Im}(\pi(y))) \leq \|\pi(x) - \pi(y)\|$ for all $\pi \in \mathcal{P}(\mathcal{X}, \mathcal{H})$ and $x, y \in \mathcal{X}$. Thus from [40, Corollary 10, p. 100], we deduce that

$$\|\pi(x) - \pi(y)\| < 1 \qquad \Longrightarrow \qquad \operatorname{Rank}(\pi(x)) = \operatorname{Rank}(\pi(y))$$

for all $x, y \in \mathcal{X}$ and all $\pi \in \mathcal{P}(\mathcal{X}, \mathcal{H})$. Hence, by the continuity of π we obtain the lemma. ☐

The following restatement of a part of Theorem 2.2.4 is the main result of this section.

Theorem 2.3.3. *([1]) Let $\Phi : \mathcal{C}(\mathcal{X}, \mathcal{E}(\mathcal{H})) \to \mathcal{C}(\mathcal{Y}, \mathcal{E}(\mathcal{K}))$ be an order isomorphism. Then*

$$\Phi(\mathcal{P}_1(\mathcal{X}, \mathcal{H})) = \mathcal{P}_1(\mathcal{Y}, \mathcal{K}) \quad \textit{and} \quad \Phi(\mathcal{P}_{-1}(\mathcal{X}, \mathcal{H})) = \mathcal{P}_{-1}(\mathcal{Y}, \mathcal{K}). \tag{2.3.2}$$

Proof. To prove the equalities (2.3.2), let $\pi' \in \mathcal{P}_1(\mathcal{X}, \mathcal{H})$, and denote $\pi = \Phi(\pi')$. We know that $\pi \in \mathcal{P}(\mathcal{Y}, \mathcal{K})$, and we have to prove that $\pi \in \mathcal{P}_1(\mathcal{Y}, \mathcal{K})$. First, we claim that $\operatorname{Rank}(\pi(x)) \leq 1$ for all $x \in \mathcal{Y}$. Suppose the contrary, thus there exists $y_0 \in \mathcal{Y}$ such that $\operatorname{Rank}(\pi(y_0)) \geq 2$. We have to discuss two cases.

Assume that the point y_0 is isolated in $\mathcal{Y}$. Let $P_0 \in \mathcal{P}_1(\mathcal{K})$, such that $P_0 \leq \pi(y_0)$, and denote $P_1 := \pi(y_0) - P_0$. Let P_2 be any element of $\mathcal{P}_1(\mathcal{K})$, such that

$$P_2 \leq \pi(y_0), \qquad P_2 P_0 \neq 0 \qquad \text{and} \qquad P_2 P_1 \neq 0.$$

Now, let π_0, π_1 and π_2 be the nonzero elements of $\mathcal{P}(\mathcal{Y}, \mathcal{K})$ defined by

$$\pi_0(y_0) := P_0, \qquad \pi_1(y_0) := P_1, \qquad \pi_2(y_0) := P_2,$$

and

$$\pi_0(y) = 0, \qquad \pi_1(y) = \pi_2(y) = \pi(y)$$

for all $y \in \mathcal{Y} \setminus \{y_0\}$. The continuity of π_0, π_1, π_2 follows from that of π and the fact that $\{y_0\}$ is clopen. Observe that

$$\pi_i \leq \pi \ (i = 0, 1, 2), \quad \sup\{\pi_1, \pi_2\} = \pi, \quad \inf\{\pi_i, \pi_0\} = 0 \ (i = 1, 2).$$

We denote by $\pi'_i = \Phi^{-1}(\pi_i)$ $(i = 0, 1, 2)$, thus

$$\pi'_i \leq \pi' \ (i = 0, 1, 2) \quad \text{and} \quad \inf\{\pi'_i, \pi'_0\} = 0 \ (i = 1, 2).$$

Since $\pi' \in \mathcal{P}_1(\mathcal{X}, \mathcal{H})$ we get that $\pi'_i(\{x \in \mathcal{X} : \ \pi'_0(x) \neq 0\}) = 0$ for $i = 1, 2$. This implies that π'_1 and π'_2 are both orthogonal to π'_0. Then

$$\pi'_1 \leq \pi' - \pi'_0, \quad \pi'_2 \leq \pi' - \pi'_0, \quad \pi' - \pi'_0 \leq \pi' \quad \text{and} \quad \pi' - \pi'_0 \neq \pi'.$$

This contradicts the fact that $\sup\{\pi'_1, \pi'_2\} = \pi^{'}$.

Now, assume that y_0 is a limit point of $\mathcal{Y}$. Thus, by Lemma 2.3.2, we know that

$$\mathcal{Y}_0 := \{y \in \mathcal{Y} : \ \operatorname{Rank}(\pi(y)) \geq 2\}$$

is an infinite clopen subset of $\mathcal{Y}$. Now, choose a unit vector $v \in \operatorname{Im}(\pi(y_0))$, and observe that $U_0 := \{y \in \mathcal{Y}_0 \ : \ \pi(y)v \neq 0\}$ is an open neighbourhood of y_0. By Urysohn's lemma and the fact that y_0 is a limit point, we can find $s_0 \in \mathcal{C}(\mathcal{Y}, [0, 1])$ such that

$$s_0(y_0) = 1, \quad \operatorname{supp}(s_0) \subseteq U_0 \quad \text{and} \quad (0, 1) \cap s_0\left(U_0\right) \neq \emptyset.$$

Therefore, we define the element $f_0 \in \mathcal{C}(\mathcal{Y}, \mathcal{E}(\mathcal{K}))$ by

$$f_0(y) = s_0(y)\pi(y)v \otimes \pi(y)v \qquad (y \in \mathcal{Y}).$$

Observe that $0 < \|f_0(y)\| < 1$ for some $y \in U_0$, this implies that $f_0 \notin \mathcal{P}(\mathcal{Y}, \mathcal{K})$. Now, we claim that there exists a set $\mathcal{F} \subseteq [0,\pi]$ such that

$$\pi = \sup(\mathcal{F}) \qquad \text{and} \qquad \inf\{f_0, f\} = 0 \qquad \text{for all } f \in \mathcal{F}. \tag{2.3.3}$$

Indeed, for $y \in \mathcal{Y} \setminus \overline{U_0}$, we put $f_y := s_y\pi$, where $s_y \in \mathcal{C}(\mathcal{Y},[0,1])$ such that $s_y(y) = 1$ and $s_y(\overline{U_0}) = \{0\}$. One can see that if $f \in [0,f_y] \cap [0,f_0]$, then f vanishes on $U_0 \cup \Big(\mathcal{Y} \setminus \overline{U_0}\Big)$, so $f = 0$.

Next, let $y \in U_0 \setminus \{y_0\}$, since $\mathrm{Rank}(\pi(y)) \geq 2$, we can choose a unit vector

$$v_y \in \mathrm{Im}(\pi(y)) \setminus \Big(\{\pi(y)v\}^{\perp} \cup \mathbb{C}\pi(y)v\Big).$$

Thus $\pi(y)v \notin \{\pi(y)v_y\}^{\perp} \cup \mathbb{C}\pi(y)v_y$. Therefore, $U_y := \{z \in \mathcal{Y} : \ \pi(z)v_y \notin \{\pi(z)v\}^{\perp} \cup \mathbb{C}\pi(z)v\}$ is an open neighbourhood of y, since

$$\begin{array}{rcl} \mathcal{Y} \setminus U_y & = & \{z \in \mathcal{Y} \ : \ \pi(z)v \in \{\pi(z)v_y\}^{\perp} \cup \mathbb{C}\pi(z)v_y\} \\ & = & \\ \{z \in \mathcal{Y} & : & \langle \pi(z)v, \pi(z)v_y\rangle = 0\} \cup \\ & & \{z \in \mathcal{Y} \ : |\langle \pi(z)v, \pi(z)v_y\rangle| = \|\pi(z)v\|\|\pi(z)v_y\|\}. \end{array}$$

So, there exists $s_y \in \mathcal{C}(\mathcal{Y},[0,1])$ such that $s_y(y) = 1$ and $s_y(\mathcal{Y} \setminus U_y) = \{0\}$. Therefore, there exist f_y and F_y in $\mathcal{C}(\mathcal{Y},\mathcal{E}(\mathcal{K}))$ which vanish on $\mathcal{Y} \setminus U_y$ and such that

$$\begin{array}{rcl} f_y(z) & = & s_y(z)\frac{\pi(z)v_y}{\|\pi(z)v_y\|} \otimes \frac{\pi(z)v_y}{\|\pi(z)v_y\|} \\ \\ F_y(z) & = & s_y(z)\left(\pi(z) - \frac{\pi(z)v_y}{\|\pi(z)v_y\|} \otimes \frac{\pi(z)v_y}{\|\pi(z)v_y\|}\right) \end{array}$$

for all $z \in U_y$. This allows us to set

$$\mathcal{F} := \{f_y \ : \ y \in \mathcal{Y} \setminus \overline{U_0}\} \cup \{f_y \ : \ y \in U_0 \setminus \{y_0\}\} \cup \{F_y \ : \ y \in U_0 \setminus \{y_0\}\}.$$

We need to show that $\mathcal{F}$ satisfies the equality (2.3.3). Clearly we have $\mathcal{F} \subset [0,\pi]$, and if $\mathcal{F} \subset [0,f]$ for some $f \in \mathcal{C}(\mathcal{Y},\mathcal{E}(\mathcal{K}))$, it is clear that $\pi(y) = f_y(y) \leq f(y)$ for all $y \in \mathcal{Y} \setminus \overline{U_0}$. Next, let $y \in U_0 \setminus \{y_0\}$, we have $f_y(y), F_y(y) \in \mathcal{P}(\mathcal{K})$, and $f_y(y), F_y(y) \leq f(y)$. Thus, by Lemma 1.2.3

we have $f(y) = f_y(y) + f_y(y)^\perp f(y) f_y(y)^\perp$. From this and the fact that

$$F_y(y) = f_y(y)^\perp F_y(y) f_y(y)^\perp \leq f_y(y)^\perp f(y) f_y(y)^\perp,$$

we obtain that

$$\pi(y) = f_y(y) + F_y(y) \leq f_y(y) + f_y(y)^\perp f(y) f_y(y)^\perp = f(y).$$

Thus $\pi \leq f$, since $\pi(y) \leq f(y)$ for all y in the set $(U_0 \setminus \{y_0\}) \cup \left(\mathcal{Y} \setminus \overline{U_0}\right)$ which is dense in $\mathcal{Y}$. For the last property of $\mathcal{F}$, since the equality $\inf\{f_0, f_y\} = 0$ is clear for $y \in \mathcal{Y} \setminus \overline{U_0}$, let us consider the case $y \in U_0 \setminus \{y_0\}$. We have by construction $\operatorname{Rank}(f_0(z)) \leq 1$ and

$$\operatorname{Im}(f_y(z)) \cap \operatorname{Im}(f_0(z)) = \operatorname{Im}(F_y(z)) \cap \operatorname{Im}(f_0(z)) = \{0\}$$

for all $z \in \mathcal{Y}$, thus $\inf\{f_0(z), f_y(z)\} = \inf\{f_0(z), F_y(z)\} = 0$ for all z. Hence, $\inf\{f_0, f_y\} = \inf\{f_0, F_y\} = 0$. Therefore, $\inf\{f_0, f\} = 0$ for all $f \in \mathcal{F}$. Now, denote $g_0 = \Phi^{-1}(f_0)$ and observe that

$$\pi' = \sup \Phi^{-1}(\mathcal{F}) \qquad \text{and} \qquad 0 = \inf\{g_0, g\} \tag{2.3.4}$$

for all $g \in \Phi^{-1}(\mathcal{F})$. Note that $g_0 \neq 0$, then $V_0 := \{x \in \mathcal{X} : g_0(x) \neq 0\}$ is a non-empty open subset of $\mathcal{X}$, and from the equality (2.3.4) and Lemma 1.1.2 we deduce that $g(V_0) = \{0\}$ for all $g \in \Phi^{-1}(\mathcal{F})$. Now, we claim that there is $x_0 \in V_0$ such that $\|g_0(x_0)\| < 1$. Indeed, if $\{\|g_0(x)\| : x \in V_0\} = \{1\}$, then $\{\|g_0(x)\| : x \in \mathcal{X}\} = \{0, 1\}$ and $g_0(x) = \|g_0(x)\| \pi'(x)$ for all $x \in \mathcal{X}$. Hence $g_0 \in \mathcal{P}(\mathcal{X}, \mathcal{H})$, which contradicts the fact that $\Phi(g_0) = f_0 \notin \mathcal{P}(\mathcal{Y}, \mathcal{K})$. To complete the proof we define a continuous function $h_0 : \mathcal{X} \to [0, \frac{1}{2}(1 - \|g_0(x_0)\|)]$ such that $h_0(x) = \frac{1}{2}(1 - \|g_0(x_0)\|)$ if

$$\frac{1}{2}\|g_0(x_0)\| \leq \|g_0(x)\| \leq \frac{1}{2}(1 + \|g_0(x_0)\|)$$

and $h_0(\mathcal{X} \setminus V_0) = \{0\}$. Thus,

$$g \leq (1 - h_0)\pi' \leq \pi' \text{ for all } g \in \Phi^{-1}(\mathcal{F}).$$

This contradicts the equality (2.3.4) since $(1-h_0)\pi' \in \mathcal{C}(\mathcal{X}, \mathcal{E}(\mathcal{H})) \setminus \{\pi'\}$. Hence we conclude that

$$\mathrm{Rank}(\Phi(\pi'(x))) \leq 1$$

for all $x \in \mathcal{X}$ and $\pi' \in \mathcal{P}_1(\mathcal{X}, \mathcal{H})$.

To complete the proof, assume that there exists $\pi' \in \mathcal{P}_1(\mathcal{X}, \mathcal{H})$ such that $\pi := \Phi(\pi') \notin \mathcal{P}_1(\mathcal{Y}, \mathcal{K})$. This means that there exists $y_0 \in \mathcal{Y}$ such that $\pi(y_0) = 0$, and hence $\pi^{-1}(\{0\})$ is a clopen subset of $\mathcal{Y}$, see Lemma 2.3.2. Now choose an arbitrary $P \in \mathcal{P}_1(\mathcal{K})$ and put

$$\pi_1(x) = \begin{cases} P & \text{on } \pi^{-1}(\{0\}), \\ \pi(x) & \text{elsewhere.} \end{cases}$$

We have $\pi \leq \pi_1 \in \mathcal{P}_1(\mathcal{Y}, \mathcal{K})$ and $\pi_1 \neq \pi$. Thus, $\mathrm{Rank}\left(\Phi^{-1}(\pi_1)(x_1)\right) \geq 2$ for some $x_1 \in \mathcal{X}$, which contradicts the previous facts, since Φ^{-1} is an order isomorphism. This proves the first equality in (2.3.2).

For the proof of the second equality in (2.3.2), it suffices to recall that the mapping $\Phi^{\perp}$ defined in the formula (2.3.1) is an order automorphism, this tells us that

$$\left(\Phi\left(\pi^{\perp}\right)\right)^{\perp} = \Phi^{\perp}(\pi) \in \mathcal{P}_1(\mathcal{Y}, \mathcal{K})$$

for all $\pi \in \mathcal{P}_1(\mathcal{X}, \mathcal{H})$. Thus $\pi \in \mathcal{P}_{-1}(\mathcal{X}, \mathcal{H})$ if and only if $\Phi(\pi) \in \mathcal{P}_{-1}(\mathcal{Y}, \mathcal{K})$. ☐

The following consequence of the previous theorem is exactly the statement (4) of Theorem 2.2.4.

Corollary 2.3.4. *If there exists an order isomorphism $\Phi : \mathcal{C}(\mathcal{X}, \mathcal{E}(\mathcal{H})) \to \mathcal{C}(\mathcal{Y}, \mathcal{E}(\mathcal{K}))$, then the spaces $\mathcal{X}$ and $\mathcal{Y}$ are homeomorphic.*

Proof. Let $\pi \in \mathcal{P}_1(\mathcal{X}, \mathcal{H})$ the projection defined by $\pi(\mathcal{X}) = \{P\}$ for a fixed $P \in \mathcal{P}_1(\mathcal{H})$. The mapping

$$\Theta_{\pi} \ : \ \mathcal{C}(\mathcal{X}, [0,1]) \to [0, \pi], \ s \mapsto s\pi : \ x \mapsto s(x)\pi(x)$$

is an order isomorphism. For $f \in [0, \pi]$, the function $\Theta_{\pi}^{-1}(f) \in \mathcal{C}(\mathcal{X}, [0,1])$ is given by $\Theta_{\pi}^{-1}(f)(x) = \|f(x)\|, x \in \mathcal{X}$. Similarly we will define the order isomorphism $\Theta_{\Phi(\pi)} : \mathcal{C}(\mathcal{Y}, [0,1]) \to [0, \Phi(\pi)]$. Therefore, the map $\Theta_{\Phi(\pi)}^{-1} \circ \Phi \circ \Theta_{\pi}$ is an order isomorphism from $\mathcal{C}(\mathcal{X}, [0,1])$ onto

$\mathcal{C}(\mathcal{Y},[0,1])$. Thus, we deduce from the formula (1.1.2) that there exists a homeomorphism $\mu_\pi : \mathcal{X} \to \mathcal{Y}$, see [9]. $\square$

As a simple consequence of the previous result, we see that for every $\pi \in \mathcal{P}_1(\mathcal{X},\mathcal{H})$ there exists a unique order isomorphism $\Phi_\pi : \mathcal{C}(\mathcal{X},[0,1]) \to \mathcal{C}(\mathcal{Y},[0,1])$ satisfying

$$\Phi(s\pi) = \Phi_\pi(s)\Phi(\pi) \tag{2.3.5}$$

for all $s \in \mathcal{C}(\mathcal{X},[0,1])$.

In the remainder of this section we assume that $\Phi(\frac{1}{2}I_{\mathcal{X}}) = \frac{1}{2}I_{\mathcal{Y}}$, and show (among further properties on projections) that $\Phi(\pi^\perp) = \Phi(\pi)^\perp$ for all $\pi \in \mathcal{P}_1(\mathcal{X},\mathcal{H}) \cup \mathcal{P}_{-1}(\mathcal{X},\mathcal{H})$. As a consequence we obtain that the spaces $\mathcal{H}$ and $\mathcal{K}$ are isomorphic. Before the statement of these results we need the following useful lemma.

Lemma 2.3.5. *Let P and Q be two elements of $\mathcal{P}_1(\mathcal{H})$. Then*

$$\frac{1}{2-\|PQ^\perp P\|}P \quad \leq \quad \frac{1}{2}Q + Q^\perp. \tag{2.3.6}$$

Moreover,

$$P \neq Q \quad \Longleftrightarrow \quad \frac{1}{2-\|PQ^\perp P\|} > \frac{1}{2} \quad \Longleftrightarrow \quad rP \leq \frac{1}{2}Q + Q^\perp \tag{2.3.7}$$

for some $r > \frac{1}{2}$.

Proof. Observe that

$$\begin{aligned}
\|(2I-Q^\perp)^{\frac{1}{2}}P(2I-Q^\perp)^{\frac{1}{2}}\| &= \|(P(2I-Q^\perp)^{\frac{1}{2}})^*P(2I-Q^\perp)^{\frac{1}{2}}\| \\
&= \|P(2I-Q^\perp)^{\frac{1}{2}}(P(2I-Q^\perp)^{\frac{1}{2}})^*\| \\
&= \|P(2I-Q^\perp)P\| \\
&= \|\left(2-\langle Q^\perp p,p\rangle\right)p\otimes p\| \\
&= 2-\|PQ^\perp P\|,
\end{aligned}$$

where $p \in \mathcal{H}$ is a unit vector such that $P = p \otimes p$. Therefore,

$$\begin{aligned}
(2I-Q^\perp)^{\frac{1}{2}}P(2I-Q^\perp)^{\frac{1}{2}} &\leq \|(2I-Q^\perp)^{\frac{1}{2}}P(2I-Q^\perp)^{\frac{1}{2}}\|I \\
&= \left(2-\|PQ^\perp P\|\right)I.
\end{aligned}$$

Hence

$$\begin{aligned}
\tfrac{1}{2-\|PQ^{\perp}P\|}P \;\leq\; (2I-Q^{\perp})^{\frac{-1}{2}}(2I-Q^{\perp})^{-\frac{1}{2}} &= (2I-Q^{\perp})^{-1} \\
&= \tfrac{1}{2}(I+Q^{\perp}) \\
&= \tfrac{1}{2}Q+Q^{\perp}.
\end{aligned}$$

This proves the inequality (2.3.6). Choose two unit vectors p and q in $\mathcal{H}$ such that $P = p \otimes p$ and $Q = q \otimes q$. If $P \neq Q$, then $|\langle p, q\rangle| < 1$ and hence

$$\|PQ^{\perp}P\| = \langle Q^{\perp}p, p\rangle \|p \otimes p\| = \langle p, p\rangle - \langle (q \otimes q)p, p\rangle = 1 - |\langle p, q\rangle|^2 > 0.$$

Thus $r := \frac{1}{2-\|PQ^{\perp}P\|} > \frac{1}{2}$ satisfies $rP \leq \frac{1}{2}Q + Q^{\perp}$. Finally, if $rQ \leq \frac{1}{2}Q + Q^{\perp}$ for some scalar $r \geq 0$, then $rQ \leq Q\left(\frac{1}{2}Q + Q^{\perp}\right)Q = \frac{1}{2}Q$ and $r \leq \frac{1}{2}$. ☐

Now we state and prove the promised result.

Proposition 2.3.6. *Let* $\Phi : \mathcal{C}(\mathcal{X}, \mathcal{E}(\mathcal{H})) \to \mathcal{C}(\mathcal{Y}, \mathcal{E}(\mathcal{K}))$ *be an order isomorphism such that* $\Phi(\frac{1}{2}I_{\mathcal{X}}) = \frac{1}{2}I_{\mathcal{Y}}$*, then for every* $\pi \in \mathcal{P}_1(\mathcal{X}, \mathcal{H})$ *we have*

$$\Phi(\frac{1}{2}\pi) = \frac{1}{2}\Phi(\pi), \quad \Phi(\pi^{\perp}) = \Phi(\pi)^{\perp} \quad \textit{and} \quad \Phi\left(\frac{1}{2}\pi + \pi^{\perp}\right) = \frac{1}{2}\Phi(\pi) + \Phi(\pi)^{\perp} \tag{2.3.8}$$

$$\Phi(s\pi + \pi^{\perp}) = \Phi_{\pi}(s)\Phi(\pi) + \Phi(\pi)^{\perp} \tag{2.3.9}$$

for all $s \in \mathcal{C}(\mathcal{X}, [0,1])$.

Proof. Notice that $\frac{1}{2}\pi = \inf\{\frac{1}{2}I_{\mathcal{X}}, \pi\}$. Then,

$$\begin{aligned}
\Phi(\tfrac{1}{2}\pi) &= \Phi\left(\inf\{\tfrac{1}{2}I_{\mathcal{X}} \;,\; \pi\}\right) \\
&= \inf\left\{\Phi(\tfrac{1}{2}I_{\mathcal{X}}) \;,\; \Phi(\pi)\right\} \\
&= \inf\left\{\tfrac{1}{2}I_{\mathcal{Y}} \;,\; \Phi(\pi)\right\} \\
&= \tfrac{1}{2}\Phi(\pi).
\end{aligned}$$

To complete the proof of the equality (2.3.8) we first show that if $\pi \in \mathcal{P}_1(\mathcal{X}, \mathcal{H})$, then

$$\Phi(\frac{1}{2}\pi + \pi^{\perp}) = \frac{1}{2}\Phi(\pi^{\perp})^{\perp} + \Phi(\pi^{\perp}).$$

It is easily checked that the previous equation is equivalent to $\Phi^{\perp}(\frac{1}{2}\pi) = \frac{1}{2}\Phi^{\perp}(\pi)$, and the latter is a direct consequence of the first part of the proof.

Now, we have to prove that if $\pi \in \mathcal{P}_1(\mathcal{X}, \mathcal{H})$, then $\Phi(\pi) = \Phi(\pi^\perp)^\perp$. To do this, assume that $\Phi(\pi) \neq \Phi(\pi^\perp)^\perp$, and define the function $r \in \mathcal{C}(\mathcal{X}, [0,1])$ by

$$r : x \mapsto \frac{1}{2 - \|\Phi(\pi)(x)\Phi(\pi^\perp)(x)\Phi(\pi)(x)\|}.$$

So, Lemma 2.3.5 ensures that

$$r \nleq \frac{1}{2} \qquad \text{and} \qquad r\Phi(\pi) \leq \frac{1}{2}\Phi(\pi^\perp)^\perp + \Phi(\pi^\perp).$$

Thus there exists $r' \in \mathcal{C}(\mathcal{X}, [0,1])$ such that $r\Phi(\pi) = \Phi(r'\pi)$, thus

$$r' \nleq \frac{1}{2} \qquad \text{and} \qquad \Phi(r'\pi) \leq \frac{1}{2}\Phi(\pi^\perp)^\perp + \Phi(\pi^\perp) = \Phi(\frac{1}{2}\pi + \pi^\perp).$$

Therefore

$$r' \nleq \frac{1}{2} \qquad \text{and} \qquad r'\pi \leq \frac{1}{2}\pi + \pi^\perp.$$

This contradicts Lemma 2.3.5, and finishes the proof of the equality (2.3.8).

For the proof of the equality (2.3.9), let $\pi \in \mathcal{P}_1(\mathcal{X}, \mathcal{H})$ and $s \in \mathcal{C}(\mathcal{X}, [0,1])$. Using Lemma 1.2.3, we obtain that

$$s\pi + \pi^\perp = \sup\{s\pi, \pi^\perp\}.$$

Applying Φ, we obtain that

$$\Phi(s\pi + \pi^\perp) = \sup\left\{\Phi(s\pi), \Phi(\pi^\perp)\right\}.$$

Then, the equality $\Phi(s\pi + \pi^\perp) = \Phi(s\pi) + \Phi(\pi^\perp)$ can be derived easily from Lemma 1.2.3 and the fact that $\Phi(\pi^\perp) = \Phi(\pi)^\perp$, which finishes the proof of the equality (2.3.9).

□

The following lemma (the last statement of Theorem 2.2.4) allows us to assume without loss of generality that $\mathcal{K} = \mathcal{H}$.

Lemma 2.3.7. *If there exists an order isomorphism* $\Phi : \mathcal{C}(\mathcal{X}, \mathcal{E}(\mathcal{H})) \to \mathcal{C}(\mathcal{Y}, \mathcal{E}(\mathcal{K}))$ *such that* $\Phi(\frac{1}{2}I_\mathcal{X}) = \frac{1}{2}I_\mathcal{Y}$, *then the Hilbert spaces* $\mathcal{H}$ *and* $\mathcal{K}$ *are isometric.*

Proof. Let $\{e_i : i \in \mathcal{I}_\mathcal{H}\}$ be an orthonormal basis of $\mathcal{H}$. For each $i \in \mathcal{I}_\mathcal{H}$ denote by $\pi_i \in \mathcal{P}_1(\mathcal{X}, \mathcal{H})$ the constant function that satisfies $\pi_i(\mathcal{X}) = \{e_i \otimes e_i\}$. Assume that there exists an order isomorphism $\Phi : \mathcal{C}(\mathcal{X},\mathcal{E}(\mathcal{H})) \to \mathcal{C}(\mathcal{Y},\mathcal{E}(\mathcal{K}))$ such that $\Phi(\frac{1}{2}I_{\mathcal{X}}) = \frac{1}{2}I_{\mathcal{Y}}$. According to Theorem 2.3.3, we know that $\Phi(\pi_i) \in \mathcal{P}_1(\mathcal{Y},\mathcal{K})$ for all $i \in \mathcal{I}_{\mathcal{H}}$. Fix $y \in \mathcal{Y}$, then for each $i \in \mathcal{I}_{\mathcal{H}}$ there exists a unit vector $v_i \in \mathcal{K}$ such that $v_i \otimes v_i = \Phi(\pi_i)(y)$. Hence, by Proposition 2.3.6 we have $\Phi(\pi_i) \leq \Phi(\pi_j^{\perp}) = \Phi(\pi_j)^{\perp}$ for all $i, j \in \mathcal{I}_{\mathcal{H}}$ such that $i \neq j$. Thus $\{v_i : i \in \mathcal{I}_{\mathcal{H}}\}$ is a an orthonormal family of $\mathcal{K}$. This implies that there exists an injective map $J_1 : \mathcal{I}_{\mathcal{H}} \to \mathcal{I}_{\mathcal{K}}$ where $\mathcal{I}_{\mathcal{K}}$ is an index set of an orthonormal basis of $\mathcal{K}$ which contains the family $\{v_i : i \in \mathcal{I}_{\mathcal{H}}\}$. Since Φ and Φ^{-1} have the same proprieties we obtain the existence of an injective map $J_2 : \mathcal{I}_{\mathcal{K}} \to \mathcal{I}_{\mathcal{H}}$, and by the well known Schröder-Bernstein's theorem there exists a bijective map between $\mathcal{I}_{\mathcal{H}}$ and $\mathcal{I}_{\mathcal{K}}$ which means that $\mathcal{H}$ and $\mathcal{K}$ are isometric. $\square$

2.4 The canonical homeomorphism between $\mathcal{Y}$ and $\mathcal{X}$

From the equality (2.3.2), we see that for every $\pi \in \mathcal{P}_1(\mathcal{X},\mathcal{H})$, the order isomorphism Φ induces an order isomorphism from $\mathcal{C}(\mathcal{X},[0,\pi])$ onto $\mathcal{C}(\mathcal{Y},[0,\Phi(\pi)])$ for all $\pi \in \mathcal{P}_1(\mathcal{X},\mathcal{H})$. In the statement and the proof of the following lemma we will need the function defined by

$$\widehat{\pi} : \mathcal{Y} \times [0,1] \to [0,1],\ (y,t) \mapsto \widehat{\pi_y}(t) := \Phi_\pi\left(\widehat{t}\right)(y), \tag{2.4.1}$$

where $\pi \in \mathcal{P}_1(\mathcal{X},\mathcal{H})$ and $\widehat{t}(x) = t$ for all $x \in \mathcal{X}$ and all $t \in [0,1]$.

Lemma 2.4.1. *Suppose that one of the spaces $\mathcal{X}$ or $\mathcal{Y}$ satisfies the property ($K_{\mathcal{E}}$). Then, there exists a unique homeomorphism $\mu : \mathcal{Y} \to \mathcal{X}$ such that for every $\pi \in \mathcal{P}_1(\mathcal{X},\mathcal{H})$, the following statements hold.*

1. *For every $y \in \mathcal{Y}$, the map $\widehat{\pi_y}$ is an increasing homeomorphism of $[0,1]$.*
2. *For every $(y,s) \in \mathcal{Y} \times \mathcal{C}(\mathcal{X},[0,1])$, we have*

$$\Phi_\pi(s)(y) = \widehat{\pi_y}\left(s(\mu(y))\right). \tag{2.4.2}$$

Proof. By [10] we know that for every $\pi \in \mathcal{P}_1(\mathcal{X},\mathcal{H})$ there exists a homeomorphism $\mu_\pi : \mathcal{Y} \to \mathcal{X}$ such that (2.4.2) holds for $\mu = \mu_\pi$. To complete the proof, we have to show that μ_π is independent of π. Take $\pi_1, \pi_2 \in \mathcal{P}_1(\mathcal{X},\mathcal{H})$. For the readability of the proof we assume that $\mathcal{Y} = \mathcal{X}$ and μ_{π_1} is the identical map of $\mathcal{X}$. Note that this is not restrictive since the mapping $\mathcal{C}(\mathcal{X},\mathcal{E}(\mathcal{H})) \to \mathcal{C}(\mathcal{X},\mathcal{E}(\mathcal{K}))$ defined by $f \mapsto \Phi(f) \circ \mu_{\pi_1}^{-1}$ is an order isomorphism. Denote $\mu = \mu_{\pi_2}$

and assume that μ is different to the identical map of $\mathcal{X}$. First, we claim that there exist two non-empty open subsets U and V of $\mathcal{X}$ such that

$$V = \mu(U), \quad \overline{U} \cap \overline{V} = \emptyset \quad \text{and} \quad \Phi(\pi_1)(x) \neq \Phi(\pi_2)(x) \text{ for all } x \in U. \tag{2.4.3}$$

Indeed, from the hypothesis $\mu \neq Id_{\mathcal{X}}$, we know that there exists a non-empty open subset U' of $\mathcal{X}$ such that $\overline{U'} \cap \overline{\mu^{-1}(U')} = \emptyset$. Suppose that $\Phi(\pi_1)(x) = \Phi(\pi_2)(x)$ for all $x \in U'$, and choose a non-zero function $s \in \mathcal{C}(\mathcal{X}, [0,1])$ such that $s(\mathcal{X} \setminus U') = \{0\}$. Therefore

$$\Phi(s_1\pi_1) = s\Phi(\pi_1) = s\Phi(\pi_2) = \Phi(s_2\pi_2) \tag{2.4.4}$$

for some non-zero functions $s_1, s_2 \in \mathcal{C}(\mathcal{X}, [0,1])$. Since Φ is bijective we conclude that

$$s_1 = s_2. \tag{2.4.5}$$

On the other hand, composing the equations (2.4.4) and (2.4.2) we derive that for every $x \in \mathcal{X}$ we have

$$s_2(\mu(x)) \neq 0 \quad \Longleftrightarrow \quad s(x) \neq 0 \quad \Longleftrightarrow \quad s_1(x) \neq 0.$$

Then, $\operatorname{supp}(s_1) = \operatorname{supp}(s) \subseteq U'$ and $\emptyset \neq \operatorname{supp}(s_2) \subseteq \mu(U')$ which does not intersect U'. Thus, $\operatorname{supp}(s_2) \neq \operatorname{supp}(s_1)$ which contradicts the equality (2.4.5). Therefore, the set

$$U := \{x \in U' \ : \ \Phi(\pi_1)(x) \neq \Phi(\pi_2)(x)\}$$

is a non-empty open subset of $\mathcal{X}$ and satisfies (2.4.3).

For the rest of proof, let $x_0 \in U$ and $\widehat{x_0} : \mathcal{X} \to [0,1]$ be a continuous function such that $\widehat{x_0}(x_0) = 1$ and $\widehat{x_0}(\mathcal{X} \backslash U) = \{0\}$. Denote $y_0 = \mu(x_0)$ and $\widehat{y_0} = \widehat{x_0} \circ \mu^{-1}$. Choose two unit vectors $u \in \operatorname{Im}(\pi_1(x_0))$ and $v \in \operatorname{Im}(\pi_2(y_0))$, such that $\langle u, v \rangle \in [0,1]$. By replacing U if necessary by a smaller open neighbourhood of x_0, we may assume without loss of generality that $\pi_1(x)u \neq 0$, $\pi_2(y)v \neq 0$ for all $(x,y) \in U \times V$. Thus there is $\widehat{t} \in \mathcal{C}(\mathcal{X}, [0,1])$ satisfying

$$\begin{aligned} \{0\} &= \widehat{t}\Big(\{x \in \mathcal{X} \ : \ \widehat{x_0}(x) \geq \tfrac{1}{2} \text{ or } \widehat{y_0}(x) \geq \tfrac{1}{2}\}\Big), \\ \{1\} &= \widehat{t}\left(\{x \in \mathcal{X} \ : \ \widehat{x_0}(x)\|\pi_1(x)u\| + \widehat{y_0}(x)\|\pi_2(x)v\| = 0\}\right). \end{aligned}$$

Now, consider the function $p : \mathcal{X} \to \mathcal{H}$ defined by

$$p(x) = \left(\widehat{x_0}(x)\pi_1(x) + \widehat{t}(x)\right) u + \left(\widehat{y_0}(x)\pi_2(x) + \widehat{t}(x)\right) v$$

for all $x \in \mathcal{X}$. We have $p(x_0) = u$, $p(y_0) = v$ and $p(x) \neq 0$ for all $x \in \mathcal{X}$, this follows from the fact that $(\mathcal{X} \setminus U) \cup (\mathcal{X} \setminus V) = \mathcal{X}$,

$$\begin{aligned} \langle p(x), u \rangle &= \widehat{x_0}(x) \langle \pi_1(x)u, u \rangle + \widehat{t}(x) + \widehat{y_0}(x) \langle \pi_2(x)v, u \rangle + \widehat{t}(x) \langle v, u \rangle \\ &= \widehat{x_0}(x) \langle \pi_1(x)u, u \rangle + \widehat{t}(x)(1 + \langle v, u \rangle) > 0 \end{aligned}$$

for all $x \in \mathcal{X} \setminus V$, and

$$\begin{aligned} \langle p(x), v \rangle &= \widehat{x_0}(x) \langle \pi_1(x)u, v \rangle + \widehat{t}(x) \langle u, v \rangle + \widehat{y_0}(x) \langle \pi_2(x)v, v \rangle + \widehat{t}(x) \\ &= \widehat{y_0}(x) \langle \pi_2(x)v, v \rangle + \widehat{t}(x)(\langle u, v \rangle + 1) > 0 \end{aligned}$$

for all $x \in \mathcal{X} \setminus U$. Let $\pi : \mathcal{X} \to \mathcal{E}(\mathcal{H})$ be the map defined by $x \mapsto \|p(x)\|^{-2} p(x) \otimes p(x)$ for all $x \in \mathcal{X}$. By Lemma 2.3.1 and the continuity of the mapping $x \mapsto p(x)$ we conclude that $\pi \in \mathcal{P}_1(\mathcal{X}, \mathcal{H})$. Furthermore, we have

$$\pi(x) = \|\widehat{x_0}(x)\pi_1(x)u\|^{-2} (\widehat{x_0}(x)\pi_1(x)u) \otimes (\widehat{x_0}(x)\pi_1(x)u) = \pi_1(x)$$

for all $x \in U_0 := \{x \in \mathcal{X} \ : \ \widehat{x_0}(x) > \frac{1}{2}\}$, and

$$\pi(x) = \|\widehat{y_0}(x)\pi_2(x)v\|^{-2} (\widehat{y_0}(x)\pi_2(x)v) \otimes (\widehat{y_0}(x)\pi_2(x)v) = \pi_2(x)$$

for all $x \in V_0 := \mu(U_0)$. Finally, let $\widehat{x_1} : \mathcal{X} \to [0,1]$ be a continuous function such that $\widehat{x_1}(x_0) = 1$ and $\widehat{x_1}(\mathcal{X} \setminus U_0) = \{0\}$. Similarly, let $\widehat{y_1} : \mathcal{X} \to [0,1]$ be a continuous function such that $\widehat{y_1}(y_0) = 1$ and $\widehat{y_1}(\mathcal{X} \setminus V_0) = \{0\}$. We conclude that

$$\widehat{x_1}\pi_1 \leq \pi \qquad \text{and} \qquad \widehat{y_1}\pi_2 \leq \pi.$$

This yields that

$$\Phi(\widehat{x_1}\pi_1) \leq \Phi(\pi) \qquad \text{and} \qquad \Phi(\widehat{y_1}\pi_2) \leq \Phi(\pi).$$

In particular

$$\begin{aligned}
\Phi(\pi_1)(x_0) &= \widehat{\pi_1}_{x_0}\left(\widehat{x_1}(x_0)\right)\Phi(\pi_1)(x_0) &&= \Phi(\widehat{x_1}\pi_1)(x_0) &&\leq \Phi(\pi)(x_0),\\
\Phi(\pi_2)(x_0) &= \widehat{\pi_2}_{x_0}\left(\widehat{y_1}(\mu(x_0))\right)\Phi(\pi_2)(x_0) &&= \Phi(\widehat{y_1}\pi_2)(x_0) &&\leq \Phi(\pi)(x_0).
\end{aligned}$$

Since $\Phi(\pi_1)(x_0)$, $\Phi(\pi_2)(x_0)$ and $\Phi(\pi)(x_0)$ are in $\mathcal{P}_1(\mathcal{K})$, we have $\Phi(\pi_1)(x_0) = \Phi(\pi_2)(x_0)$, which contradicts the fact that $\Phi(\pi_1)(x_0) \neq \Phi(\pi_2)(x_0)$ given in (2.4.3).

The proof of the equality (2.4.2) is then finished. □

2.5 Proof of Theorem 2.2.1

Throughout this section, we assume that $\Phi : \mathcal{C}(\mathcal{X}, \mathcal{E}(\mathcal{H})) \to \mathcal{C}(\mathcal{Y}, \mathcal{E}(\mathcal{K}))$ is an order isomorphism satisfying $\Phi(\frac{1}{2}I_{\mathcal{X}}) = \frac{1}{2}I_{\mathcal{Y}}$. For the readability reasons we assume from now on that $\mathcal{K} = \mathcal{H}$, $\mathcal{Y} = \mathcal{X}$ and the homeomorphism $\mu : \mathcal{Y} \to \mathcal{X}$ provided by Lemma 2.4.1 is the identity map of $\mathcal{X}$. This is not restrictive, since the mapping

$$\Theta_{\mu^{-1}} : \mathcal{C}(\mathcal{Y}, \mathcal{E}(\mathcal{K})) \to \mathcal{C}(\mathcal{X}, \mathcal{E}(\mathcal{K}));\ g \mapsto g \circ \mu^{-1}$$

is an order isomorphism. So, $\Theta_{\mu^{-1}} \circ \Phi$ is an order isomorphism from $\mathcal{C}(\mathcal{X}, \mathcal{E}(\mathcal{H}))$ onto $\mathcal{C}(\mathcal{X}, \mathcal{E}(\mathcal{K}))$ satisfying the desired condition. Therefore, Lemma 2.4.1 applied to Φ tells us that for every $\pi \in \mathcal{P}_1(\mathcal{X}, \mathcal{H})$ and $x \in \mathcal{X}$ there exists $\widehat{\pi_x} : [0,1] \to [0,1]$ a bijective, strictly increasing map such that for all $s \in \mathcal{C}(\mathcal{X}, [0,1])$

$$\left(\Theta_{\mu^{-1}} \circ \Phi\right)(s\pi)(x) = \widehat{\pi_x}(s(x))\left(\Theta_{\mu^{-1}} \circ \Phi\right)(\pi)(x). \tag{2.5.1}$$

In fact, in this formula we use the reindexation "$x \mapsto \mu^{-1}(x)$" of the family $(\widehat{\pi_x})_{x\in\mathcal{X}}$. In the remainder of this section we shall use Φ instead of $\Theta_{\mu^{-1}} \circ \Phi$. For the assumption $\mathcal{K} = \mathcal{H}$, by Lemma 2.3.7 we know that there exists a unitary operator $U : \mathcal{H} \to \mathcal{K}$, so the order isomorphism $U^*\Phi U$ defined from $\mathcal{C}(\mathcal{X}, \mathcal{E}(\mathcal{H}))$ onto $\mathcal{C}(\mathcal{X}, \mathcal{E}(\mathcal{H}))$ by $(U^*\Phi U)(f)(x) := U^*\Phi(f)(x)U$ for all $f \in \mathcal{C}(\mathcal{X}, \mathcal{E}(\mathcal{H}))$, and all $x \in \mathcal{X}$ satisfies the desired assumption.

Lemma 2.5.1. *Suppose that the space $\mathcal{X}$ satisfies the property $(K_{\mathcal{E}})$. Let $\pi, \pi' \in \mathcal{P}_1(\mathcal{X}, \mathcal{H})$ and let $x_0 \in \mathcal{X}$. If $\pi(x_0) = \pi'(x_0)$, then*

$$\Phi(\pi)(x_0) = \Phi(\pi')(x_0) \qquad \textit{and} \qquad \widehat{\pi}_{x_0} = \widehat{\pi'}_{x_0}.$$

Proof. Assume that $\pi(x_0) = \pi'(x_0)$ for some $x_0 \in \mathcal{X}$. First we will prove that $\Phi(\pi)(x_0) = \Phi(\pi')(x_0)$. Let us suppose that $\Phi(\pi)(x_0) \neq \Phi(\pi')(x_0)$. Let r be the element of $\mathcal{C}(\mathcal{X}, [0,1])$ defined by

$$r(x) := \frac{1}{2 - \|\Phi(\pi)(x)\Phi(\pi')^{\perp}(x)\Phi(\pi)(x)\|}$$

for all $x \in \mathcal{X}$. Combining the latter fact with Lemma 2.3.5, we obtain that

$$r\Phi(\pi) \leq \frac{1}{2}\Phi(\pi') + \Phi(\pi')^{\perp} = \Phi(\frac{1}{2}\pi' + \pi'^{\perp}).$$

Thus $s\pi \leq \frac{1}{2}\pi' + \pi'^{\perp}$, where s is the unique element of $\mathcal{C}(\mathcal{X}, [0,1])$ such that

$$\Phi(s\pi)(x) = \widehat{\pi}_x(s(x))\Phi(\pi)(x) = r(x)\Phi(\pi)(x)$$

for all $x \in \mathcal{X}$. Recall that $\Phi(\frac{1}{2}\pi)(x) = \frac{1}{2}\Phi(\pi)(x) = \widehat{\pi}_x(\frac{1}{2})\Phi(\pi)(x)$ for all $x \in \mathcal{X}$. Then, $\widehat{\pi}_x(\frac{1}{2}) = \frac{1}{2}$ for all $x \in \mathcal{X}$. Since $\widehat{\pi}_{x_0}$ is strictly increasing and $r(x_0) = \widehat{\pi}_{x_0}(s(x_0)) > \frac{1}{2}$, we obtain that $s(x_0) > \frac{1}{2}$. From this together with the fact that $s\pi \leq \frac{1}{2}\pi' + \pi'^{\perp}$, we deduce that

$$s(x_0) > \frac{1}{2} \qquad \text{and} \qquad s(x_0)\pi(x_0) \leq \frac{1}{2}\pi'(x_0) + \pi'^{\perp}(x_0).$$

Therefore, Lemma 2.3.5 tells us that $\pi(x_0) \neq \pi'(x_0)$ which is a contradiction. This proves the equality $\Phi(\pi)(x_0) = \Phi(\pi')(x_0)$. For the second equality of the lemma, observe that

$$\begin{aligned}
\widehat{\pi'}_{x_0}(\alpha)\Phi(\pi)(x_0) &= \widehat{\pi'}_{x_0}(\alpha)\Phi(\pi')(x_0) \\
&= \Phi(\alpha\pi')(x_0) \\
&\leq \Phi(\alpha I_{\mathcal{X}})(x_0) \\
&\leq \Phi(\alpha\pi + \pi^{\perp})(x_0) \\
&= \widehat{\pi}_{x_0}(\alpha)\Phi(\pi)(x_0) + \Phi(\pi)^{\perp}(x_0)
\end{aligned}$$

for all $\alpha \in [0,1]$; see Lemma 2.4.1 and Lemma 2.3.6. Thus by Lemma 2.3.5 we obtain that $\widehat{\pi'}_{x_0}(\alpha) \leq \widehat{\pi}_{x_0}(\alpha)$ for all $\alpha \in [0,1]$. Similarly, we have $\widehat{\pi}_{x_0}(\alpha) \leq \widehat{\pi'}_{x_0}(\alpha)$ for all $\alpha \in [0,1]$. Thus, $\widehat{\pi}_{x_0} = \widehat{\pi'}_{x_0}$ and Lemma 2.5.1 is proved. ☐

The following lemma is quoted from [8, Lemma 1, Theorem 3].

Lemma 2.5.2. *Let $A \in \mathcal{E}(\mathcal{H})$.*

1. *If v is a unit vector in $\mathcal{H}$ such that $Av \neq 0$, then for*

$$\alpha = \frac{\|Av\|^2}{\langle Av, v\rangle} \qquad \textit{and} \qquad w = \frac{1}{\|Av\|}Av,$$

   we have $w \otimes w \in \mathcal{P}_1(\mathcal{H})$. Moreover,

$$\alpha w \otimes w \leq A \qquad \textit{and} \qquad \alpha \left\langle (w \otimes w)v, v\right\rangle = \left\langle Av, v\right\rangle.$$

2. *If $P \in \mathcal{P}_1(\mathcal{H})$, then*

$$\mathrm{Im}(P) \subseteq \mathrm{Im}(A^{1/2}) \qquad \Longleftrightarrow \qquad \exists\ \alpha > 0,\ \alpha P \leq A.$$

The following lemma shows that the first main implication ”(1) $\Longrightarrow$ (2)” of Theorem 2.2.1 holds.

Lemma 2.5.3. *Let $x_0 \in \mathcal{X}$, and $f, g \in \mathcal{C}(\mathcal{X}, \mathcal{E}(\mathcal{H}))$. Then*

$$f(x_0) \leq g(x_0) \iff \Phi(f)(x_0) \leq \Phi(g)(x_0). \tag{2.5.2}$$

Proof. At the beginning, we will prove the lemma in the special case when $f = r\pi$ and $g = s\pi'$ for some $r, s \in \mathcal{C}(\mathcal{X}, [0, 1])$ and $\pi, \pi' \in \mathcal{P}_1(\mathcal{X}, \mathcal{H})$. Using Lemma 2.5.1 and the properties of $\widehat{\pi}_{x_0}$, we get

$$\begin{aligned}
f(x_0) \leq g(x_0) \quad &\iff \quad r(x_0) = 0 \text{ or } (\pi(x_0) = \pi'(x_0) \text{ and } r(x_0) \leq s(x_0)) \\
&\iff \quad \widehat{\pi}_{x_0}(r(x_0)) = 0 \text{ or} \\
&\qquad\quad \Big(\Phi(\pi)(x_0) = \Phi(\pi')(x_0) \text{ and } \widehat{\pi}_{x_0}(r(x_0)) \leq \widehat{\pi'}_{x_0}(s(x_0))\Big) \\
&\iff \quad \widehat{\pi}_{x_0}(r(x_0))\Phi(\pi)(x_0) \leq \widehat{\pi'}_{x_0}(s(x_0))\Phi(\pi')(x_0) \\
&\iff \quad \Phi(f)(x_0) \leq \Phi(g)(x_0).
\end{aligned}$$

Now, we will prove the reverse of (2.5.3). Assume by the way of contradiction that there exist $f, g \in \mathcal{C}(\mathcal{X}, \mathcal{E}(\mathcal{H}))$ such that

$$f(x_0) \not\leq g(x_0) \quad \text{and} \quad \Phi(f)(x_0) \leq \Phi(g)(x_0). \tag{2.5.3}$$

Choose a unit vector $v \in \mathcal{H}$ such that

$$\langle f(x_0)v, v\rangle > \langle g(x_0)v, v\rangle \,. \tag{2.5.4}$$

For the readability of the proof, we will split its remainder into three claims.

Claim 1. There exist $r \in \mathcal{C}(\mathcal{X}, [0,1])$ and $\pi \in \mathcal{P}_1(\mathcal{X}, \mathcal{H})$ such that

$$r\pi \leq f \quad \text{and} \quad \langle r(x_0)\pi(x_0)v, v\rangle = \langle f(x_0)v, v\rangle. \tag{2.5.5}$$

Indeed, let U and U_0 be two open neighbourhoods of x_0 such that

$$\overline{U_0} \subseteq U := \{x \in \mathcal{X} \ : \ \langle f(x)v, v\rangle > \langle g(x)v, v\rangle\} \,.$$

Choose $s \in \mathcal{C}(\mathcal{X}, [0,1])$ such that $s(\mathcal{X} \setminus U) = \{1\}$ and $s(\overline{U_0}) = \{0\}$, and observe that for every $x \in \mathcal{X}$, we have $\langle f(x)v + s(x)v, v\rangle > 0$. Thus the function

$$x \mapsto p(x) := \frac{1}{\|f(x)v + s(x)v\|}\,(f(x)v + s(x)v)$$

is well defined and continuous from $\mathcal{X}$ into $\mathcal{H}$. This and Lemma 2.3.1 give rise to an element π of $\mathcal{P}_1(\mathcal{X}, \mathcal{H})$ defined by $\pi(x) := p(x) \otimes p(x)$ for all $x \in \mathcal{X}$. Choose $s' \in \mathcal{C}(\mathcal{X}, [0,1])$ such that $s'(x_0) = 1$ and $s'(\mathcal{X} \setminus U_0) = \{0\}$, so we can define a function $r \in \mathcal{C}(\mathcal{X}, [0,1])$ by

$$r(\mathcal{X} \setminus U_0) = \{0\} \quad \text{and} \quad r(x) = s'(x)\frac{\|f(x)v\|^2}{\langle f(x)v, v\rangle} \quad \text{for} \quad x \in U_0.$$

Notice that $r(x) \leq \langle f(x)v, v\rangle^{-1}\|f(x)v\|^2$ for all $x \in U_0$. Thus by Lemma 2.5.2, we derive (2.5.5).

Claim 2. There exists $h \in [0, \pi']$ for some $\pi' \in \mathcal{P}_1(\mathcal{X}, \mathcal{H})$ such that

$$h(x_0) \leq \Phi(\pi)(x_0) \qquad \text{and} \qquad h \leq \Phi(g). \tag{2.5.6}$$

Indeed, let v_1 be a unit vector in $\operatorname{Im}(\Phi(\pi)(x_0))$. It is easily checked from (2.5.5) that

$$\widehat{\pi}_{x_0}(r(x_0))v_1 \otimes v_1 = \widehat{\pi}_{x_0}(r(x_0))\Phi(\pi)(x_0) = \Phi(r\pi)(x_0) \leq \Phi(f)(x_0) \leq \Phi(g)(x_0).$$

From this together with Lemma 2.5.2 and the fact that $\widehat{\pi}_{x_0}(r(x_0)) > 0$, we deduce that $v_1 \in$

$\mathrm{Im}\left((\Phi(g)(x_0))^{\frac{1}{2}}\right)$. By the decomposition

$$\mathcal{H} = \mathrm{Ker}\left((\Phi(g)(x_0))^{\frac{1}{2}}\right) \oplus \overline{\mathrm{Im}\left((\Phi(g)(x_0))^{\frac{1}{2}}\right)},$$

there exists a unique $v_0 \in \overline{\mathrm{Im}\left((\Phi(g)(x_0))^{\frac{1}{2}}\right)}$ such that $(\Phi(g)(x_0))^{1/2}v_0 = v_1$. Let O and O_0 be two neighbourhoods of x_0 such that

$$\overline{O_0} \subseteq O := \left\{x \in \mathcal{X} \ : \ (\Phi(g)(x))^{1/2}v_0 \neq 0\right\}.$$

There exist $s_1, s_2 \in \mathcal{C}(\mathcal{X}, [0,1])$ such that

$$s_1(\{x_0\}) = s_2(\mathcal{X} \setminus O) = \{1\} \quad \text{and} \quad s_1(\mathcal{X} \setminus O_0) = s_2(\overline{O_0}) = \{0\}.$$

We put

$$p_2(x) := (\Phi(g)(x))^{1/2}\, v_0 + s_2(x)v_0 \qquad \text{for all } x \in \mathcal{X}.$$

Note that $p_2 \in \mathcal{C}(\mathcal{X}, \mathcal{H} \setminus \{0\})$ and $\sqrt{s_1(x)}\|p_2(x)\| \leq \|v_0\|$ for all $x \in \mathcal{X}$. This gives rise to a function $h \in \mathcal{C}(\mathcal{X}, \mathcal{E}(\mathcal{H}))$ defined by

$$h(x) := \frac{s_1(x)}{\|v_0\|^2} p_2(x) \otimes p_2(x)$$

for all $x \in \mathcal{X}$. It easy to see that $h \leq \pi' \in \mathcal{P}_1(\mathcal{X}, \mathcal{H})$ where $\pi'(x) = \dfrac{1}{\|p_2(x)\|^2} p_2(x) \otimes p_2(x)$ for all $x \in \mathcal{X}$. Thus, $h(x_0) = \dfrac{1}{\|v_0\|^2} v_1 \otimes v_1 = \dfrac{1}{\|v_0\|^2}\Phi(\pi)(x_0) \leq \Phi(\pi)(x_0)$.

To prove $h \leq \Phi(g)$, we have to show that $h(x) \leq \Phi(g)(x)$ for all $x \in O_0$, since $h(x) = 0$ for all $x \in \mathcal{X} \setminus O_0$. Let $x \in O_0$, we have $h(x) = \dfrac{s_1(x)}{\|v_0\|^2}\Phi(g)(x)^{1/2}v_0 \otimes \Phi(g)(x)^{1/2}v_0$. Thus

$$\begin{aligned} \langle h(x)v, v\rangle &= \frac{s_1(x)}{\|v_0\|^2}\left|\left\langle \Phi(g)(x)^{1/2}v_0, v\right\rangle\right|^2 \\ &\leq \frac{1}{\|v_0\|^2}\left|\left\langle v_0, \Phi(g)(x)^{1/2}v\right\rangle\right|^2 \\ &\leq \langle \Phi(g)(x)v, v\rangle . \end{aligned}$$

Hence $h(x) \leq \Phi(g)(x)$ for all $x \in O_0$. This proves that $h \leq \Phi(g)$.

Claim 3. The elements r, π, and h that are constructed in the claims 1 and 2 satisfy

$$\Phi(r\pi)(x_0) \leq h(x_0). \tag{2.5.7}$$

Indeed, we know that

$$h(x_0), \Phi(r\pi)(x_0) \leq \Phi(\pi)(x_0), \Phi(g)(x_0),$$

so $\Phi(r\pi)(x_0) = \lambda h(x_0)$ for some scalar $\lambda \geq 0$ and it is enough to prove that $\lambda \leq 1$.

Recall that $v_0 \in \overline{\operatorname{Im}\left(\Phi(g)(x_0)^{\frac{1}{2}}\right)}$ and let $(v_n)_{n\in\mathbb{N}} \subseteq \mathcal{H}$ such that $\lim_n \Phi(g)(x_0)^{1/2}v_n = v_0$. Then we have

$$\begin{aligned} 0 &\geq \lim_n \langle(\lambda h(x_0) - \Phi(g)(x_0))\, v_n, v_n\rangle \\ &= \lim_n \frac{\lambda}{\|v_0\|^2}\left|\left\langle v_0, \Phi(g)(x_0)^{1/2}v_n\right\rangle\right|^2 - \|\Phi(g)(x_0)^{1/2}v_n\|^2 \\ &= (\lambda - 1)\|v_0\|^2. \end{aligned}$$

Then $\lambda \leq 1$ and $\Phi(r\pi)(x_0) \leq h(x_0)$. This finishes the proof of the claim.

Now, we are in the position to prove that (2.5.3) cannot take place. Using (2.5.7) together with (2.5.6) and the fact that $\Phi(r\pi)$ and h satisfy the condition of the first part of this proof, we conclude that

$$r(x_0)\pi(x_0) \leq \Phi^{-1}(h)(x_0) \leq g(x_0).$$

This and (2.5.5) show that

$$\langle f(x_0)v, v\rangle = \langle r(x_0)\pi(x_0)v, v\rangle \leq \langle g(x_0)v, v\rangle,$$

which is a contradiction with the inequality (2.5.4). Thus, we obtain that

$$\Phi(f)(x_0) \leq \Phi(g)(x_0) \quad \Longrightarrow \quad f(x_0) \leq g(x_0)$$

for all $f, g \in \mathcal{C}(\mathcal{X}, \mathcal{E}(\mathcal{H}))$. By using the same arguments for Φ^{-1} instead of Φ, we obtain the direct implication. Therefore, the proof of lemma is finished. $\square$

Now we restate and prove the second main implication "(1) $\Longrightarrow$ (4)" of Theorem 2.2.1.

Theorem 2.5.4. *([1]) Let $\Phi : \mathcal{C}(\mathcal{X}, \mathcal{E}(\mathcal{H})) \to \mathcal{C}(\mathcal{Y}, \mathcal{E}(\mathcal{K}))$ be a map. Suppose that one of the* *spaces* $\mathcal{X}$ *or* $\mathcal{Y}$ *satisfies the property* $(K_{\mathcal{E}})$ *and* $\max\{\dim(\mathcal{H}), \dim(\mathcal{K})\} \geq 2$. *Then the following statements are equivalent.*

1. Φ *is an order isomorphism with* $\Phi(\frac{1}{2}I_{\mathcal{X}}) = \frac{1}{2}I_{\mathcal{Y}}$.

2. *There exists a homeomorphism* μ *from* $\mathcal{Y}$ *onto* $\mathcal{X}$, *a decomposition* $\mathcal{Y} = \mathcal{Y}_1 \cup \mathcal{Y}_{-1}$ *of* $\mathcal{Y}$ *in two disjoint clopen subsets, and a* **-isomorphism (resp.* **-anti isomorphism)*

   $$J_k \ : \ \mathcal{C}(\mu(\mathcal{Y}_k), \mathcal{B}(\mathcal{H})) \to \mathcal{C}(\mathcal{Y}_k, \mathcal{B}(\mathcal{K}))$$

   for $k = 1$ *(resp.* $k = -1$*), such that the Jordan* **-isomorphism* $J := J_1 \oplus J_{-1}$ *extends* Φ. *Moreover, we can write*

   $$J_1(f)(y) = U_y f(\mu(y)) U_y^* \quad (y \in \mathcal{Y}_1, \ f \in \mathcal{C}(\mu(\mathcal{Y}_1), \mathcal{B}(\mathcal{H}))),$$

   and

   $$J_{-1}(f)(y) = U_y f(\mu(y))^* U_y^* \quad (y \in \mathcal{Y}_{-1}, \ f \in \mathcal{C}(\mu(\mathcal{Y}_{-1}), \mathcal{B}(\mathcal{H}))),$$

   where $U_y \ : \ \mathcal{H} \to \mathcal{K}$ *is unitary (resp. antiunitary) for* $y \in \mathcal{Y}_1$ *(resp.* $y \in \mathcal{Y}_{-1}$*) and*

   $$y \mapsto U_y A U_y^* \qquad \text{and} \qquad y \mapsto U_y^* B U_y \tag{2.5.8}$$

   are continuous on $\mathcal{Y}$ *for all* $(A, B) \in \mathcal{E}(\mathcal{H}) \times \mathcal{E}(\mathcal{K})$.

Proof. Keep in mind that we can assume without loss of generality that $\mathcal{Y} = \mathcal{X}$, $\mathcal{H} = \mathcal{K}$ and the homeomorphism μ provided by Lemma 2.4.1 is the identity map of $\mathcal{X}$ and $\Phi(\frac{1}{2}I_{\mathcal{X}}) = \frac{1}{2}I_{\mathcal{X}}$. By Lemma 2.5.3 we know that for each $x \in \mathcal{X}$ the mapping $\Phi_x : \mathcal{E}(\mathcal{H}) \to \mathcal{E}(\mathcal{H})$ defined by $f(x) \mapsto \Phi(f)(x)$ is a well-defined order automorphism satisfying $\Phi_x(\frac{1}{2}I) = \frac{1}{2}I$. According to [45, Corollary 4.6] there exists a unitary or an antiunitary operator U_x on $\mathcal{H}$ such that $\Phi_x(A) = U_x A U_x^*$ for all $A \in \mathcal{E}(\mathcal{H})$. Let

$$\mathcal{X}_1 := \{x \in \mathcal{X} \ : \ U_x \text{ is linear}\} \qquad \text{and} \qquad \mathcal{X}_{-1} := \{x \in \mathcal{X} \ : \ U_x \text{ is antilinear}\}.$$

Our aim is to show that $\mathcal{X}_1$ and $\mathcal{X}_{-1}$ are clopen subsets of $\mathcal{X}$. Assume by the way of contradiction that $\mathcal{X}_1$ is not closed in $\mathcal{X}$. Note that since $\mathcal{X} = \mathcal{X}_1 \cup \mathcal{X}_{-1}$, there exists a net $\{x_\gamma \ : \ \gamma \in \Gamma\} \subseteq \mathcal{X}_1$ and $x \in \mathcal{X}_{-1}$ such that $\lim_\gamma x_\gamma = x$. Pick two orthonormal vectors $e_1, e_2 \in \mathcal{H}$, for each scalar

$\lambda \in \mathbb{C}$, we define $\pi_\lambda \in \mathcal{P}_1(\mathcal{X}, \mathcal{H})$ by

$$\pi_\lambda(x) = \frac{1}{1+|\lambda|^2}(e_1 + \lambda e_2) \otimes (e_1 + \lambda e_2)$$

for all $x \in \mathcal{X}$. So

$$\Phi(\pi_\lambda)(x) = \frac{1}{1+|\lambda|^2} U_x(e_1 + \lambda e_2) \otimes U_x(e_1 + \lambda e_2)$$

for all $x \in \mathcal{X}$. Furthermore, the function $y \mapsto \|\Phi(\pi_0)(y)U_x(e_2)\|$ is continuous on $\mathcal{X}$, so we deduce that

$$\begin{aligned}
\lim\nolimits_\gamma |\langle U_x(e_2), U_{x_\gamma}(e_1)\rangle| &= \lim\nolimits_\gamma \|\Phi(\pi_0)(x_\gamma)U_x(e_2)\| \\
&= \|\Phi(\pi_0)(x)U_x(e_2)\| \\
&= |\langle U_x(e_2), U_x(e_1)\rangle| \\
&= 0.
\end{aligned}$$

By exchanging the roles of e_1 and e_2, we obtain that $\lim_\gamma \langle U_{x_\gamma}(e_2), U_x(e_1)\rangle = 0$. We will use the notations

$$\alpha_\gamma := \langle U_{x_\gamma}(e_1), U_x(e_1)\rangle \qquad \text{and} \qquad \beta_\gamma := \langle U_{x_\gamma}(e_2), U_x(e_2)\rangle$$

for $\gamma \in \Gamma$. A direct computation shows that

$$\|v \otimes v - w \otimes w\|^2 = 1 - |\langle v, w\rangle|^2$$

for all unit vectors $v, w \in \mathcal{H}$. Keep in mind that U_x is antiunitary operator and U_{x_γ} is unitary for all $\gamma \in \Gamma$, we conclude that

$$\begin{aligned}
&\|\Phi(\pi_\lambda)(x_\gamma) - \Phi(\pi_\lambda)(x)\|^2 \\
&= 1 - \frac{|\langle U_x(e_1 + \lambda e_2), U_{x_\gamma}(e_1 + \lambda e_2)\rangle|^2}{(1+|\lambda|)^2} \\
&= 1 - \frac{\left|\bar{\lambda}\left(\langle U_x(e_2), U_{x_\gamma}(e_1)\rangle + \langle U_x(e_1), U_{x_\gamma}(e_2)\rangle\right) + \alpha_\gamma + \bar{\lambda}^2\beta_\gamma\right|^2}{(1+|\lambda|)^2}.
\end{aligned}$$

for all $\gamma \in \Gamma$ and $\lambda \in \mathbb{C}$. Therefore

$$\lim_\gamma \left|\alpha_\gamma + \bar{\lambda}^2\beta_\gamma\right|^2 = (1+|\lambda|)^2 \tag{2.5.9}$$

for all $\lambda \in \mathbb{C}$. This follows from the facts that

$$\lim_{\gamma} \|\Phi(\pi_\lambda)(x_\gamma) - \Phi(\pi_\lambda)(x)\|^2 = \lim_{\gamma} \langle U_x(e_2), U_{x_\gamma}(e_1)\rangle = \lim_{\gamma} \langle U_{x_\gamma}(e_2), U_x(e_1)\rangle = 0.$$

So, for $\lambda = 1$ and $\lambda = \sqrt{-1}$, in (2.5.9), we get

$$\begin{cases} 4 & = & \lim_\gamma |\alpha_\gamma + \beta_\gamma|^2 & = & \lim_\gamma |\alpha_\gamma|^2 + |\beta_\gamma|^2 + 2\mathrm{Re}(\alpha_\gamma \beta_\gamma), \\ 4 & = & \lim_\gamma |\alpha_\gamma - \beta_\gamma|^2 & = & \lim_\gamma |\alpha_\gamma|^2 + |\beta_\gamma|^2 - 2\mathrm{Re}(\alpha_\gamma \beta_\gamma). \end{cases} \tag{2.5.10}$$

This implies that

$$\lim_{\gamma} \mathrm{Re}(\alpha_\gamma \beta_\gamma) = 0, \quad 2 = \lim_{\gamma} |\alpha_\gamma + \beta_\gamma| \quad \text{and} \quad \lim_{\gamma} |\alpha_\gamma| = \lim_{\gamma} |\beta_\gamma| = 1,$$

since $|\alpha_\gamma| \leq 1$ and $|\beta_\gamma| \leq 1$ for all $\gamma \in \Gamma$. This yields the contradiction

$$0 = \lim_{\gamma} 2\mathrm{Re}(\alpha_\gamma \beta_\gamma) = 4 - \lim_{\gamma} |\alpha_\gamma|^2 + |\beta_\gamma|^2 = 2.$$

Then $\mathcal{X}_1 = \mathcal{X} \setminus \mathcal{X}_{-1}$ is a closed subset of $\mathcal{X}$, and $\mathcal{X}_{-1}$ is an open set. By a similar way we can prove that $\mathcal{X}_{-1}$ is closed and $\mathcal{X}_1$ is an open set, which proves that $\mathcal{X}_1$ and $\mathcal{X}_{-1}$ are clopen sets.

Let J_1 be the *-automorphism of $\mathcal{C}(\mathcal{X}_1, \mathcal{B}(\mathcal{H}))$ and J_{-1} be the *-antiautomorphism of $\mathcal{C}(\mathcal{X}_{-1}, \mathcal{B}(\mathcal{H}))$ defined by

$$\begin{aligned} J_1(f)(x) &= U_x f(x) U_x^* && \text{for all } x \in \mathcal{X}_1 \text{ and } f \in \mathcal{C}(\mathcal{X}_1, \mathcal{B}(\mathcal{H})) \\ J_{-1}(f)(x) &= U_x f^*(x) U_x^* && \text{for all } x \in \mathcal{X}_{-1} \text{ and } f \in \mathcal{C}(\mathcal{X}_{-1}, \mathcal{B}(\mathcal{H})). \end{aligned}$$

It easy to see that $J = J_1 \oplus J_{-1}$ is a Jordan $*$-automorphism of

$$\mathcal{C}(\mathcal{X}, \mathcal{B}(\mathcal{H})) = \mathcal{C}(\mathcal{X}_1, \mathcal{B}(\mathcal{H})) \oplus \mathcal{C}(\mathcal{X}_{-1}, \mathcal{B}(\mathcal{H}))$$

and J agrees with Φ on $\mathcal{C}(\mathcal{X}, \mathcal{E}(\mathcal{H}))$.

The proof of the implication (2) $\implies$ (1) is contained in Proposition 2.5.5 below. □

We finish this section by showing that the implication (2) $\implies$ (1) of the previous theorem will be derived as conclusion of the proposition below. Moreover, it provides how to create an order automorphism extendible to a Jordan *-isomorphism. This gives an added value to the previous theorem.

Proposition 2.5.5. *Assume that there exists a family* $(U_x : \mathcal{H} \to \mathcal{K})_{x \in \mathcal{X}}$ *of unitary or antiunitary operators that satisfy* (2.2.1). *Then the formula*

$$\Phi(f)(x) = U_x f(x) U_x^* \quad \text{for all } x \in \mathcal{X} \text{ and all } f \in \mathcal{C}(\mathcal{X}, \mathcal{E}(\mathcal{H})), \tag{2.5.11}$$

gives rise to an order isomorphism Φ *from* $\mathcal{C}(\mathcal{X}, \mathcal{E}(\mathcal{H}))$ *onto* $\mathcal{C}(\mathcal{X}, \mathcal{E}(\mathcal{K}))$.

Proof. The proof uses similar arguments as in [33]. Let $f \in \mathcal{C}(\mathcal{X}, \mathcal{E}(\mathcal{H}))$, we have

$$\begin{aligned} \|\Phi(f)(y) - \Phi(f)(x)\| &= \|U_y f(y) U_y^* - U_x f(x) U_x^*\| \\ &= \|U_y (f(y) - f(x)) U_y^* + U_y f(x) U_y^* - U_x f(x) U_x^*\| \\ &\leq \|f(y) - f(x)\| + \|U_y f(x) U_y^* - U_x f(x) U_x^*\| \end{aligned}$$

for all $x, y \in \mathcal{X}$. Thus $\lim\limits_{y \to x} \|\Phi(f)(y) - \Phi(f)(x)\| = 0$ for all $x \in \mathcal{X}$. Hence the mapping $x \mapsto U_x f(x) U_x^*$ is continuous on $\mathcal{X}$, so Φ is well defined. Therefore, the formula

$$\Psi(f)(x) = U_x^* f(x) U_x \quad \text{for all } x \in \mathcal{X} \text{ and } f \in \mathcal{C}(\mathcal{X}, \mathcal{E}(\mathcal{K})),$$

gives rise to a mapping $\Psi : \mathcal{C}(\mathcal{X}, \mathcal{E}(\mathcal{K})) \to \mathcal{C}(\mathcal{X}, \mathcal{E}(\mathcal{H}))$. Clearly we have Ψ is the inverse of Φ, thus Φ is bijective. Finally the fact that Φ preserves the order in both directions is obvious. ☐

2.6 Proofs of Theorem 2.2.2, Proposition 2.2.3 and Corollary 2.2.5

Assume that the space $\mathcal{X}$ has the property $(\mathrm{K}_{\mathcal{E}})$. We start by showing that the condition $\Phi(\frac{1}{2} I_{\mathcal{X}}) = \frac{1}{2} I_{\mathcal{Y}}$ is not restrictive.

Lemma 2.6.1. *The effects* $\Phi(\frac{1}{2} I_{\mathcal{X}})$ *and* $I_{\mathcal{Y}} - \Phi(\frac{1}{2} I_{\mathcal{X}})$ *are invertible in* $\mathcal{C}(\mathcal{Y}, \mathcal{B}(\mathcal{K}))$, *and the map* Φ_ϕ *as in* (2.2.2) *is an order automorphism, where* $\phi = \left(\Phi(\frac{1}{2} I_{\mathcal{X}})^{-1} - I_{\mathcal{Y}}\right)^{-\frac{1}{2}}$. *Furthermore,*

$$\Phi(\frac{1}{2} I_{\mathcal{X}}) = \Phi_\phi(\frac{1}{2} I_{\mathcal{Y}}).$$

Proof. First note that we can assume that $\mathcal{Y} = \mathcal{X}$ and the homeomorphism provided by Lemma 2.4.1 is the identity map of $\mathcal{X}$. First, we pick $x \in \mathcal{X}$ and we claim that the effects

$\Phi(\frac{1}{2}I_{\mathcal{X}})(x)$ and $I-\Phi(\frac{1}{2}I_{\mathcal{X}})(x)$ are invertible in $\mathcal{B}(\mathcal{K})$. Indeed, Lemma 2.4.1 tells us that

$$\widehat{\pi_x}(\frac{1}{2}) > 0 \text{ and } \widehat{\pi_x}(\frac{1}{2})\Phi(\pi)(x) = \Phi(\frac{1}{2}\pi)(x) \leq \Phi(\frac{1}{2}I_{\mathcal{X}})(x)$$

for all $\pi \in \mathcal{P}_1(\mathcal{X},\mathcal{H})$. From the previous fact together with $\Phi(\mathcal{P}_1(\mathcal{X},\mathcal{H})) = \mathcal{P}_1(\mathcal{X},\mathcal{K})$, we conclude that for every unit vector $u \in \mathcal{K}$, there exists a scalar $\lambda > 0$ such that $\lambda u \otimes u \leq \Phi(\frac{1}{2}I_{\mathcal{X}})(x) \in \mathcal{E}(\mathcal{K})$. Then, by [8, Theorem 3] we obtain that $u \in \operatorname{Im}\left((\Phi(\frac{1}{2}I_{\mathcal{X}})(x))^{\frac{1}{2}}\right)$ for all $u \in \mathcal{K}$. Hence $(\Phi(\frac{1}{2}I_{\mathcal{X}})(x))^{\frac{1}{2}}$ is surjective and then it is also injective. So we derive that $(\Phi(\frac{1}{2}I_{\mathcal{X}})(x))^{\frac{1}{2}}$ is invertible and then $\Phi(\frac{1}{2}I_{\mathcal{X}})(x)$ is invertible too. Since $\Phi^{\perp}$ is an order isomorphism and $I-\Phi(\frac{1}{2}I_{\mathcal{X}})(x) = \Phi^{\perp}(\frac{1}{2}I_{\mathcal{X}})(x)$, we conclude that $I-\Phi(\frac{1}{2}I_{\mathcal{X}})(x)$ is invertible, this proves the claim. Thus, $\Phi(\frac{1}{2}I_{\mathcal{X}})$ and $I_{\mathcal{Y}}-\Phi(\frac{1}{2}I_{\mathcal{X}})$ are invertible. We conclude from the previous facts that

$$\left(\Phi(\frac{1}{2}I_{\mathcal{X}})(x)^{-1}-I\right)^{-\frac{1}{2}} = \left(\Phi(\frac{1}{2}I_{\mathcal{X}})(x)\right)^{\frac{1}{2}}\left(I-\Phi(\frac{1}{2}I_{\mathcal{X}})(x)\right)^{-\frac{1}{2}}$$

is an invertible operator for all $x \in \mathcal{X}$. This together with [39, Theorem 7.3] ensures that the mapping $\Phi_x : \mathcal{E}(\mathcal{K}) \to \mathcal{E}(\mathcal{K})$ defined by

$$\Phi_x = \operatorname{Ord}_{\phi(x)}$$

is an order automorphism for all $x \in \mathcal{X}$. Now, since the mapping

$$x \mapsto \left(\Phi(\frac{1}{2}I_{\mathcal{X}})^{-1}(x)-I\right)^{-\frac{1}{2}} = \left(\Phi(\frac{1}{2}I_{\mathcal{X}})(x)\right)^{\frac{1}{2}}\left(I-\Phi(\frac{1}{2}I_{\mathcal{X}})(x)\right)^{-\frac{1}{2}}$$

is continuous on $\mathcal{X}$, we obtain that Φ_ϕ is well defined. Finally, by a straightforward calculation one can see that Φ_ϕ is invertible and

$$\Phi_\phi \circ \Phi_{\phi^{-1}} = \Phi_{\phi^{-1}} \circ \Phi_\phi = \operatorname{Id}_{\mathcal{C}(\mathcal{X},\mathcal{E}(\mathcal{K}))}.$$

Hence, the rest is obvious. □

Proof of Theorem 2.2.2. The implications (3) $\Longrightarrow$ (2) $\Longrightarrow$ (1) are easy. For the implication (1) $\Longrightarrow$ (3) of Theorem 2.2.2, we know that $\Psi\circ\Phi$ is an order isomorphism from $\mathcal{C}(\mathcal{X},\mathcal{E}(\mathcal{H}))$ onto $\mathcal{C}(\mathcal{Y},\mathcal{E}(\mathcal{K}))$, satisfying $\Psi\circ\Phi(\frac{1}{2}I_{\mathcal{X}}) = \frac{1}{2}I_{\mathcal{Y}}$, where $\Psi = \Phi_{\phi^{-1}} = \Phi_\phi^{-1}$ is the order automorphism defined in Lemma 4.0.7. Thus $\Phi = \Psi^{-1}\circ(\Psi\circ\Phi)$ and Theorem 2.2.1 tells us that $\Psi\circ\Phi$ have an extension to a Jordan *-isomorphism $J : \mathcal{C}(\mathcal{X}, \mathcal{B}(\mathcal{H})) \to \mathcal{C}(\mathcal{Y}, \mathcal{B}(\mathcal{K}))$.

Now we will prove the implication (3) $\implies$ (4). Employing the assumption that Φ has a decomposition $\Phi = \Phi_\phi \circ \Phi_J$, by the statement (4) of Theorem 2.2.1 there exist a homeomorphism $\mu : \mathcal{Y} \to \mathcal{X}$ and $U_y : \mathcal{H} \to \mathcal{K}$ a unitary or antiunitary for all $y \in \mathcal{Y}$ such that $\Phi_J(f)(y) = U_y f(\mu(y)) U_y^*$, for all $(y, f) \in \mathcal{Y} \times \mathcal{C}(\mathcal{X}, \mathcal{E}(\mathcal{H}))$. Then,

$$\begin{aligned}
\Phi(f)(y) &= \phi(y)\left(U_y f(\mu(y)) U_y^* (\phi(y)^2 - I) + I\right)^{-1} U_y f(\mu(y)) U_y^* \phi(y) \\
&= \phi(y) U_y \left(f(\mu(y))(U_y^* \phi(y)^2 U_y - I) + I\right)^{-1} f(\mu(y)) U_y^* \phi(y) \\
&= T_y \left(f(\mu(y))(T_y^* T_y - I) + I\right)^{-1} f(\mu(y)) T_y^*,
\end{aligned}$$

where $T_y = \phi(y) U_y$ for all $y \in \mathcal{Y}$. Since ϕ and the map $y \mapsto U_y f(\mu(y)) U_y^* = \Phi_J(f)(y)$ are continuous on $\mathcal{Y}$ for all $f \in \mathcal{C}(\mathcal{X}, \mathcal{E}(\mathcal{H}))$, we obtain that $y \mapsto T_y A T_y^*$ is continuous on $\mathcal{Y}$ for all $A \in \mathcal{E}(\mathcal{H})$. Similarly, the map $y \mapsto U_y^* g(y) U_y = \Phi_J^{-1}(g)(\mu(y))$ is continuous on $\mathcal{Y}$ for all $g \in \mathcal{C}(\mathcal{Y}, \mathcal{E}(\mathcal{K}))$, thus $y \mapsto T_y^* B T_y$ is continuous on $\mathcal{Y}$ for all $B \in \mathcal{E}(\mathcal{K})$.

Now, we turn to the implication (4) $\implies$ (2). In fact, we will provide the proof of Proposition 2.2.3 which is more general than such an implication. ☐

Proof of Proposition 2.2.3. Assume that there exist a homeomorphism $\mu : \mathcal{Y} \to \mathcal{X}$ and a family $(T_y)_{y \in \mathcal{Y}}$ of bijective, linear or conjugate-linear bounded operators, from $\mathcal{H}$ onto $\mathcal{K}$ such that

$$\begin{array}{lllll}
y & \mapsto & T_y^* T_y, & y \mapsto & T_y \left(A(T_y^* T_y - I) + I\right)^{-1} A T_y^*, \\
y & \mapsto & T_y T_y^*, & y \mapsto & (T_y)^{-1} \left(B((T_y{}^*)^{-1}(T_y)^{-1} - I) + I\right)^{-1} B(T_y{}^*)^{-1}
\end{array} \tag{2.6.1}$$

are continuous on $\mathcal{Y}$ for all $(A, B) \in \mathcal{E}(\mathcal{H}) \times \mathcal{E}(\mathcal{K})$. For every $f \in \mathcal{C}(\mathcal{X}, \mathcal{E}(\mathcal{H}))$, we denote

$$\tilde{f}_y(x) := \left(f(\mu(x))(T_y^* T_y - I) + I\right)^{-1} f(\mu(x)) \quad \text{and} \quad \Phi(f)(y) = T_y \tilde{f}_y(y) T_y^* \tag{2.6.2}$$

for all $x, y \in \mathcal{Y}$. Clearly $\tilde{f} \in \mathcal{C}(\mathcal{Y} \times \mathcal{Y}, \mathcal{B}(\mathcal{H}))$, this follows from (2.6.1) and the continuity of $f \circ \mu$. First, we claim that the formula (2.6.2) gives rise to a map $\Phi : \mathcal{C}(\mathcal{Y}, \mathcal{E}(\mathcal{K})) \to \mathcal{C}(\mathcal{X}, \mathcal{E}(\mathcal{H}))$. Indeed, we have

$$\|\Phi(f)(y) - \Phi(f)(x)\| \leq \|T_y \tilde{f}_y(y) T_y^* - T_y \tilde{f}_y(x) T_y^*\| + \|T_y \tilde{f}_y(x) T_y^* - T_x \tilde{f}_x(x) T_x^*\|.$$

From the condition (2.6.1) we deduce that $\lim\limits_{y \to x} T_y \tilde{f}_y(x) T_y^* - T_x \tilde{f}_x(x) T_x^* = 0$. We denote by

$|T_x^*| := (T_x^* T_x)^{\frac{1}{2}}$ and $U_x := (T_x^* T_x)^{\frac{1}{2}} T_x^{-1}$ for all $x \in \mathcal{Y}$. We know that $T_x^* = |T_x^*| U_x$ and U_x is unitary or antiunitary for all $x \in \mathcal{Y}$. Then

$$\begin{aligned}\|T_y \tilde{f}_y(y) T_y^* - T_y \tilde{f}_y(x) T_y^*\| &= \|T_y \left(\tilde{f}_y(y) - \tilde{f}_y(x)\right) T_y^*\| \\ &= \|U_y^* |T_y^*| \left(\tilde{f}_y(y) - \tilde{f}_y(x)\right) |T_y^*| U_y\| \\ &= \||T_y^*| \left(\tilde{f}_y(y) - \tilde{f}_y(x)\right) |T_y^*|\|.\end{aligned}$$

From the continuity of the map $y \mapsto |T_y^*| \left(\tilde{f}_y(y) - \tilde{f}_y(x)\right) |T_y^*|$ for a fixed x, we deduce that $\lim_{y \to x} \|T_y \tilde{f}(y) T_y^* - T_y \tilde{f}(x) T_y^*\| = 0$. Hence $\Phi(f)$ is continuous and Φ is well defined. Similarly, we obtain that Φ^{-1} is well defined by the formula

$$\Phi^{-1}(g)(x) = T_x^{-1} \left(g(\mu^{-1}(x))((T_x^{-1})^* T_x^{-1} - I) + I\right)^{-1} g(\mu^{-1}(x))(T_x^{-1})^*$$

for all $g \in \mathcal{C}(\mathcal{Y}, \mathcal{B}(\mathcal{K}))$ and $y \in \mathcal{X}$. Finally the fact that Φ is an order isomorphism follows obviously from the equivalence

$$f_1(\mu(y)) \leq f_2(\mu(y)) \iff \Phi(f_1)(y) \leq \Phi(f_2)(y)$$

for all $f_1, f_2 \in \mathcal{C}(\mathcal{X}, \mathcal{B}(\mathcal{H}))$ and $y \in \mathcal{Y}$.

To complete the proof it remains to show that (3.4.4) and (2.6.1) are equivalent. Indeed, note that each of (3.4.4) and (2.6.1) implies that the map $y \mapsto T_y T_y^*$ is continuous on $\mathcal{Y}$. Thus we see that the maps $y \mapsto (T_y T_y^*)^{1/2} = |T_y|$ and $y \mapsto |T_y|^{-1}$ are also continuous. Therefore, using the triangle inequality as in the proof of Proposition 2.5.5, we see the following: The condition $y \mapsto T_y A T_y^*$ is continuous for all $A \in \mathcal{E}(\mathcal{H})$ (or equivalently, for all $A \in \mathcal{B}(\mathcal{H})$) is equivalent to the condition $A \mapsto U_y A U_y^* = T_y |T_y|^{-1} A |T_y|^{-1} T_y^*$ is continuous for all $A \in \mathcal{E}(\mathcal{H})$, where $T_y = U_y |T_y|$ is the polar decomposition of T_y.

On the other hand, the continuity of the map $y \mapsto (T_y T_y^*)^{1/2} = |T_y|$ also implies the following: The mapping $\Phi^{'}$ on $\mathcal{C}(\mathcal{Y}, \mathcal{E}(\mathcal{H}))$ given by $\Phi^{'} = |T_y|(f(y)(|T_y|^2 - I) + I)^{-1} f(y)|T_y|$ is an order automorphism of $\mathcal{C}(\mathcal{Y}, \mathcal{E}(\mathcal{H}))$, and moreover, it is a homeomorphism (because $\Phi^{'}$ as well as its inverse is given by a certain composition of multiplication mappings and the mapping $f \mapsto f^{-1}$ on $\mathcal{C}(\mathcal{Y}, \mathcal{B}(\mathcal{H}))$). By using the triangle inequality again, we see that the condition $y \mapsto T_y(A(|T_y|^2 - I) + I)^{-1} A T_y^*$ is continuous for all $A \in \mathcal{E}(\mathcal{H})$ is equivalent to the condition $y \mapsto U_y A U_y^*$ is continuous for all $A \in \mathcal{E}(\mathcal{H})$.

Similar equivalences can be established for the mapping $y \mapsto T_y^* B T_y$ and the fourth mapping of (2.6.1), and this completes the proof. $\square$

We finish this section by the proof of Corollary 2.2.5.

***Proof of Corollary* 2.2.5.** Let $\pi \in \mathcal{P}_1(\mathcal{X}, \mathcal{H})$, $s \in \mathcal{C}(\mathcal{X}, [0,1])$ and $y \in \mathcal{Y}$. Using the notation of Theorems 2.2.2 and 2.2.1, we put $P = U_y \pi(\mu(y)) U_y^* \in \mathcal{P}_1(\mathcal{H})$ and $t = s(\mu(y))$. We have

$$\begin{aligned}
\Phi_\pi(s)(y) &= \|\Phi(s\pi)(y)\| \\
&= \|\Phi_\phi\left(\Phi_J(s\pi)\right)(y)\| \\
&= \|\phi(y)\left(tP(\phi^2(y) - I) + I\right)^{-1} tP\phi(y)\| \\
&= t\|\phi(y)\left(tP(\phi^2(y) - I) + I\right)^{-1} P\phi(y)\|.
\end{aligned}$$

Observe that $P(\phi^2(y) - I)$ is a rank one operator, and then there exist two vectors $v, w \in \mathcal{H}$ such that $v \otimes v = P$ and $w = (\phi^2(y) - I)v$, so $P(\phi^2(y) - I) = v \otimes w$. A direct computation shows that

$$\Big(tP(\phi^2(y) - I) + I\Big)^{-1} = I - \frac{t}{1 + t\langle v, w\rangle} v \otimes w.$$

Use the previous equation together with the fact that $(v \otimes w)P = \langle v, w\rangle P$ to derive that

$$\begin{aligned}
\Phi_\pi(s)(y) &= t\|\phi(y)\left(I - \frac{t}{1 + t\langle v, w\rangle} v \otimes w\right) P\phi(y)\| \\
&= t\|\phi(y)\left(P - \frac{t\langle v, w\rangle}{1 + t\langle v, w\rangle} P\right)\phi(y)\| \\
&= \frac{t}{1 + t\langle v, w\rangle}\|\phi(y) P P \phi(y)\| \\
&= \frac{t}{1 + t\langle v, w\rangle}\|\phi(y) P (\phi(y)P)^*\| \\
&= \frac{t}{1 + t\langle v, w\rangle}\|(\phi(y)P)^*\phi(y)P\|.
\end{aligned}$$

Thus

$$\Phi_\pi(s)(y) = \frac{t}{1 + t\langle v, w\rangle}\|P\phi(y)^2 P\|. \tag{2.6.3}$$

Now, choose the function s on $\mathcal{Y}$ such that $s(\mathcal{Y}) = \{1\}$. Thus $t = s(\mu(y)) = 1$ and $\Phi_\pi(s)(y) = 1$. Therefore, by (2.6.3), we have $\langle v, w\rangle = \|P\phi(y)^2 P\| - 1$. Observe that $P = \Phi_J(\pi)(y)$ and replace

$t, \langle v, w\rangle$ and $\phi(y)$ by their expressions in (2.6.3) to obtain the desired form of $\Phi_\pi(s)$ for all $s \in \mathcal{C}(\mathcal{X}, [0,1])$. $\square$

Chapter 3

Order isomorphisms on unbounded self-adjoint operators

In this chapter, we will study order isomorphisms within the scope of unbounded operators. Our results provide a comprehensive description of the general forms of order isomorphisms between highly important subsets of self-adjoint operators. In order to achieve that, we develop new tools using fundamental results in the theory of unbounded operators (spectral theorem, functional calculus, ...), and we extend some notions in the context of bounded operators to the context of unbounded operators. In the end of this chapter, we will be concerned with the study of order isomorphisms on unbounded operators with respect to a second order relations.

All the results in this chapter are quoted from our paper entitled “Order isomorphisms on unbounded self-adjoint operators”.

3.1 Statement of main results

Throughout this chapter $\mathcal{H}$ and $\mathcal{K}$ denote two Hilbert spaces. If $\max\{\dim\mathcal{H}, \dim\mathcal{K}\} \geq 2$, then obviously, the sets $\mathcal{S}(\mathcal{H})$ and $\mathcal{S}_+^{-1}(\mathcal{H})$ can never be order isomorphic with $\mathcal{S}_+(\mathcal{K})$, because the latter has a minimum element, and they do not. We shall see later that even the sets $\mathcal{S}(\mathcal{H})$ and $\mathcal{S}_+^{-1}(\mathcal{K})$ are not order isomorphic. Therefore, to describe the general forms of all order isomorphisms between the sets of type $\mathcal{S}_+(\mathcal{H})$, $\mathcal{S}_+^{-1}(\mathcal{H})$ and $\mathcal{S}(\mathcal{H})$, it is enough to describe the general of forms of all order isomorphisms between the sets of the same type.

The following theorem is our first main result, providing a general characterization of order isomorphisms between the sets of type $\mathcal{S}_+(\mathcal{H})$.

Theorem 3.1.1. *([2]) Assume that* $\max\{\dim\mathcal{H}, \dim\mathcal{K}\} \geq 2$*. A map* $\Phi : \mathcal{S}_+(\mathcal{H}) \to \mathcal{S}_+(\mathcal{K})$ *is an order isomorphism if and only if there exists an invertible bounded either linear or conjugate-linear operator* $T : \mathcal{K} \to \mathcal{H}$ *such that*

$$\Phi(A) = T^*AT \tag{3.1.1}$$

for all $A \in \mathcal{S}_+(\mathcal{H})$*.*

Remark 3.1.2. An order automorphism of $\mathcal{S}_+(\mathcal{H})$ preserves the bounded operators as well as finite rank operators.

The second result gives the general form of all order isomorphisms between the sets of type $\mathcal{S}_+^{-1}(\mathcal{H})$.

Theorem 3.1.3. *([2]) Assume that* $\max\{\dim\mathcal{H}, \dim\mathcal{K}\} \geq 2$*. A map* $\Phi : \mathcal{S}_+^{-1}(\mathcal{H}) \to \mathcal{S}_+^{-1}(\mathcal{K})$ *is an order isomorphism if and only if there exists an invertible bounded either linear or conjugate-linear operator* $T : \mathcal{K} \to \mathcal{H}$ *such that*

$$\Phi(A) = T^*AT$$

for all $A \in \mathcal{S}_+^{-1}(\mathcal{H})$*.*

The following result provides a comprehensive characterization of all order isomorphisms between the sets of type $\mathcal{S}(\mathcal{H})$ in the situation that $\mathcal{H}$ is a separable Hilbert space.

Theorem 3.1.4. *([2]) Assume that either* $\mathcal{H}$ *or* $\mathcal{K}$ *is a separable with* $\max\{\dim\mathcal{H}, \dim\mathcal{K}\} \geq 2$*. A map* $\Phi : \mathcal{S}(\mathcal{H}) \to \mathcal{S}(\mathcal{K})$ *is an order isomorphism if and only if there exist a* $T_0 \in \mathcal{S}^b(\mathcal{K})$ *and an invertible bounded either linear or conjugate-linear operator* $T : \mathcal{K} \to \mathcal{H}$ *such that*

$$\Phi(A) = T^*AT + T_0$$

for all $A \in \mathcal{S}(\mathcal{H})$*.*

In the bounded case, the set of all self-adjoint bounded operators and the set of all positive invertible operators are not order isomorphic, see for example [45]. The following result shows that this remains true in the case of unbounded operators.

Theorem 3.1.5. *([2]) Let* $\mathcal{H}$ *and* $\mathcal{K}$ *be two complex Hilbert spaces.*

1. *If* $\max\{\dim \mathcal{H}, \dim \mathcal{K}\} \geq 2$*. Then, the sets* $\mathcal{S}_+^{-1}(\mathcal{H})$ *and* $\mathcal{S}(\mathcal{K})$ *cannot be order isomorphic.*

2. *There exists an order isomorphism between one of the sets* $\mathcal{S}_+(\mathcal{H})$*,* $\mathcal{S}_+^{-1}(\mathcal{H})$ *and one of the sets* $\mathcal{S}_+(\mathcal{K})$*,* $\mathcal{S}_+^{-1}(\mathcal{K})$ *if and only if* $\mathcal{H}$ *and* $\mathcal{K}$ *have the same cardinality.*

3.2 Order isomorphisms between the sets of all positive operators

Let $T : \mathcal{H} \to \mathcal{K}$ be a bijective linear or conjugate-linear operator. It is easily checked that the mapping $\mathcal{S}_+(\mathcal{H}) \to \mathcal{S}_+(\mathcal{K})$ defined by $A \mapsto T^*AT$ is an order isomorphism. This section will be devoted to prove that the preceding form describes all order isomorphisms between the sets of type $\mathcal{S}_+(\mathcal{H})$.

Let $A \in \mathcal{S}(\mathcal{H})$, we denote by $\mathrm{Im}(A)$ and $\mathrm{Ker}(A)$ the image and the kernel of A, respectively.

The following preliminary result is needed.

Lemma 3.2.1. *Let* $A \in \mathcal{S}(\mathcal{H})$ *be a finite rank operator. Then,* $A \in \mathcal{S}^b(\mathcal{H})$*.*

Proof. Since A is self-adjoint and has a finite dimensional range, then we have the decomposition

$$\mathcal{H} = \mathrm{Im}(A) \oplus \mathrm{Ker}(A), \tag{3.2.1}$$

see [44, p. 10]. Let P be the orthogonal projection on $\mathrm{Im}(A)$. Then, $\mathcal{D}(A) = P\mathcal{D}(A) \oplus \ker(A)$. Since $\ker(A)$ is closed and $P\mathcal{D}(A)$ has a finite dimension, $\mathcal{D}(A)$ is a closed subspace of $\mathcal{H}$. Combining this together with the fact that $\mathcal{D}(A)$ is dense in $\mathcal{H}$, we obtain that $\mathcal{D}(A) = \mathcal{H}$. From the closed graph theorem we conclude that A is bounded. □

Let $v \in \mathcal{H}$, we denote by $v \otimes v$ the everywhere defined operator of rank at most one such that

$$(v \otimes v)w = \langle w, v \rangle v,$$

for all $w \in \mathcal{H}$.

We will use this rather trivial observation.

Lemma 3.2.2. *Let* $A \in \mathcal{S}_+(\mathcal{H})$*. Let* v *be a unit vector in* $\mathcal{D}(A^{1/2})$*. Then,*

$$A^{1/2}v \otimes A^{1/2}v \leq A.$$

Proof. Let $w \in \mathcal{D}(A) \subseteq \mathcal{D}(A^{1/2})$, we have

$$\begin{aligned}
\left\langle (A^{1/2}v \otimes A^{1/2}v)w, w \right\rangle &= \left| \left\langle A^{1/2}v, w \right\rangle \right|^2 \\
&= \left| \left\langle v, A^{1/2}w \right\rangle \right|^2 \\
&\leq \|A^{1/2}w\|^2 \\
&\leq \left\langle A^{1/2}w, A^{1/2}w \right\rangle \\
&\leq \langle Aw, w \rangle .
\end{aligned}$$

Combining this with the fact $A^{1/2}v \otimes A^{1/2}v$ is everywhere defined, we obtain that

$$A^{1/2}v \otimes A^{1/2}v \leq A.$$

This completes the proof of Lemma 3.2.2. $\square$

We shall use some notations. We denote by $\mathcal{P}_1(\mathcal{H})$ the set of all orthogonal projections of rank one on $\mathcal{H}$. Let $P \in \mathcal{P}_1(\mathcal{H})$ we define the operator interval

$$[0,\infty)P := \left\{ tP : \ t \in \mathbb{R}^+ \right\}.$$

The following result shows that Φ preserves the rank one operators.

Lemma 3.2.3. *Let $\Phi : \mathcal{S}_+(\mathcal{H}) \to \mathcal{S}_+(\mathcal{K})$ be an order isomorphism. For every $P \in \mathcal{P}_1(\mathcal{H})$ there exists a unique $Q \in \mathcal{P}_1(\mathcal{K})$ such that*

$$\Phi([0,\infty)P) = [0,\infty)Q.$$

Proof. Since the null operator 0 is the minimum element in $\mathcal{S}_+(\mathcal{H})$, we derive that $\Phi(0) = 0$. Let A be a non-zero operator in $[0,\infty)P$, then $\Phi(A)$ is a non-zero operator. We will prove that $\Phi(A)$ is a bounded, rank one operator. Suppose the contrary, then by Lemma 3.2.1 there exist two unit vectors $v_1, v_2 \in \mathcal{D}\left(\Phi(A)\right)$ such that $\Phi(A)v_1$ and $\Phi(A)v_2$ are linearly independent. Thus, $\Phi(A)^{1/2}v_1$ and $\Phi(A)^{1/2}v_2$ are also linearly independent. Thus, the operators $A_1 :=$

$\Phi(A)^{1/2}v_1 \otimes \Phi(A)^{1/2}v_1$ and $A_2 := \Phi(A)^{1/2}v_1 \otimes \Phi(A)^{1/2}v_1$ are incomparable. On the other hand, by using Lemma 3.2.2, we obtain that

$$\{A_1, A_2\} \subseteq \{B \in \mathcal{S}_+(\mathcal{K}) : 0 \leq B \leq \Phi(A)\} = \Phi(\{B \in \mathcal{S}_+(\mathcal{H}) : 0 \leq B \leq A\}).$$

The set $\{B \in \mathcal{S}_+(\mathcal{K}) : 0 \leq B \leq \Phi(A)\}$ is totally ordered, because $\{B \in \mathcal{S}_+(\mathcal{H}) : 0 \leq B \leq A\}$ is totally ordered. Consequently, the operators A_1 and A_2 are comparable, which is a contradiction. Thus, $\Phi(A)$ is a bounded rank one operator. Then, there exist $Q \in \mathcal{P}_1(\mathcal{K})$ and $t > 0$ such that $\Phi(A) = tQ$. Finally, $\Phi([0,\infty))P = [0,\infty)Q$ follows from the fact that $[0,\infty)P$ is totally ordered. □

In [8], Busch and Gudder introduced the concept of the strength of a bounded, positive operator along a ray. We will extend this concept to the self-adjoint positive unbounded operators.

Definition 3.2.4. Let $A \in \mathcal{S}_+(\mathcal{H})$ and $P \in \mathcal{P}_1(\mathcal{H})$, we define

$$\lambda(A,P) = \sup\left\{t \in \mathbb{R}^+ : \ tP \leq A\right\},$$

if the set $\{t \in \mathbb{R}^+ : \ tP \leq A\}$ is bounded. Otherwise, we put $\lambda(A,P) = \infty$.

The usual order on $\mathcal{S}_+^b(\mathcal{H})$ can reformulated by using this concept, in the sense that, for every $A, B \in \mathcal{S}_+^b(\mathcal{H})$ we have the following equivalence, $A \leq B$ if and and only if $\lambda(A,P) \leq \lambda(B,P)$ for all $P \in \mathcal{P}_1(\mathcal{H})$, see [8, Theorem 1]. This fact used frequently in numbers of papers, see [45, 32].

The result below yields further.

Proposition 3.2.5. *Let $A \in \mathcal{S}_+(\mathcal{H})$ and $P \in \mathcal{P}_1(\mathcal{H})$. Then $\lambda(A,P)$ is a non-negative number. Moreover,*

$$\lambda(A,P) = \max\{t \in \mathbb{R}^+ : \ tp \leq A\}.$$

Proof. Fix $A \in \mathcal{S}_+(\mathcal{H})$ and $P \in \mathcal{P}_1(\mathcal{H})$. We will prove that $\lambda(A,P)$ is a finite number. Assume that for every non-negative number t we have $tP \leq A$. Since $\mathcal{D}(A)$ is a dense subspace of $\mathcal{H}$, then there exists a vector $w \in \mathcal{D}(A)$ such that Pw is a non-zero vector. Thus, for every integer

n we have

$$\begin{aligned} n\|Pw\|^2 &= n\,\langle Pw, Pw\rangle \\ &= n\,\langle Pw, w\rangle \\ &\leq \langle Aw, w\rangle \\ &< \infty. \end{aligned}$$

This means that $Pw = 0$, a contradiction. Consequently, the set $\{t \in \mathbb{R} : \ tP \leq A\}$ has always an upper-bond. So, $\lambda(A, P)$ is a finite number. The rest of the proof is left to the reader. ☐

The following order characterization of invertible elements of $\mathcal{S}_+^b(\mathcal{H})$ is needed.

Proposition 3.2.6. *Let $A \in \mathcal{S}_+^b(\mathcal{H})$. Then, $A \in \mathcal{S}_+^{-1}(\mathcal{H})$ if and only if for each $P \in \mathcal{P}_1(\mathcal{H})$, $\lambda(A, P)$ is a positive number.*

Proof. Assume that A is an invertible operator. Let $P \in \mathcal{P}_1(\mathcal{H})$ and $w \in \mathcal{H}$ such that $P = w \otimes w$. Since A is invertible then $\mathrm{Im}(A^{1/2}) = \mathcal{H}$, which implies the existence of a unit vector $v \in \mathcal{H}$ and a positive number α such that $\alpha A^{1/2} v = w$. By Lemma 3.2.2 we obtain that

$$\frac{1}{\alpha^2} P = A^{1/2} v \otimes A^{1/2} v \leq A.$$

Hence, $\lambda(A, P) \geq \dfrac{1}{\alpha^2} > 0$.

Conversely, for every $P \in \mathcal{P}_1(\mathcal{H})$, $\lambda(A, P) > 0$. Thus, by [8, Theorem 3], for every $P \in \mathcal{P}_1(\mathcal{H})$ we have $P\mathcal{H} \subseteq \mathrm{Im}(A^{1/2})$. Therefore, $A^{1/2}$ is surjective, in other words

$$\mathrm{Im}(A^{1/2}) = \mathcal{H}. \tag{3.2.2}$$

Using this with the fact that $A^{1/2}$ is self-adjoint, we derive the decomposition

$$\mathcal{H} = \mathrm{Im}(A^{1/2}) \oplus \mathrm{Ker}(A^{1/2}),$$

see [44, p. 10]. Using again (3.2.2), we conclude that $\mathrm{Ker}(A^{1/2}) = \{0\}$, in other words $A^{1/2}$ is injective. Therefore, $A^{1/2}$ is bijective, which yields that A is bijective. ☐

The following lemma is one the crucial steps in the proof.

Lemma 3.2.7. *Let $\Phi : \mathcal{S}_+(\mathcal{H}) \to \mathcal{S}_+(\mathcal{K})$ be an order isomorphism. Then, the intersection*

$$\Phi\left(\mathcal{S}_+^b(\mathcal{H}) \cap \mathcal{S}_+^{-1}(\mathcal{H})\right) \cap \mathcal{S}_+^b(\mathcal{K}) \cap \mathcal{S}_+^{-1}(\mathcal{K})$$

is not empty.

Proof. Let $\Phi(I) = \int t\mathrm{dE}(t)$ be the spectral representation of $\Phi(I)$, where E denotes the spectral measure of $\Phi(I)$. Since $\mathrm{Sp}\left(\Phi(I)\right) \subseteq \mathbb{R}^+$, $E\left((-\infty, 0)\right) = 0$, see [44, Proposition 5.10]. Define a function $f : \mathbb{R}^+ \to \mathbb{R}^+$ by

$$f(t) = t\chi_{[0,1]}(t) + \chi_{(1,\infty)}(t) \tag{3.2.3}$$

for all $t \in \mathbb{R}^+$, where χ_M denotes the characteristic function of a Borel set M. It easy to see that f is a bounded non-negative Borel function. Then,

$$S = \int f(t)\mathrm{dE}(t) \in \mathcal{S}_+^b(\mathcal{K}),$$

see [44, Proposition 4.12 (iv)]. To prove Lemma 3.2.7, it suffices to show that $S \in \mathcal{S}_+^{-1}(\mathcal{K})$ and $\Phi^{-1}(S) \in \mathcal{S}_+^b(\mathcal{H}) \cap \mathcal{S}_+^{-1}(\mathcal{H})$.

Denote $P = \int \chi_{[0,1]}(t)\mathrm{dE}(t)$, since $\chi_{[0,1]}$ is an idempotent, then P is an orthogonal projection on $\mathcal{K}$ according to [44, Proposition 4.12 (i)]. It is worthwhile rewrite S in terms of $\Phi(I)$ and P. Since the functions $\mathbb{R}^+ \to \mathbb{R}^+$ defined by $t \mapsto t\chi_{[0,1]}(t)$ and $t \mapsto \chi_{(1,\infty)}(t) = 1 - \chi_{[0,1]}(t)$ are bounded. Then, by using the functional calculus properties, one can see for example [44, Theorem 4.16 (iii), Proposition 4.12 (i)], we obtain that

$$S \;=\; \Phi(I)P + P^\perp. \tag{3.2.4}$$

Now, we will prove that $S \in \mathcal{S}_+^{-1}(\mathcal{K})$. Note that $SP^\perp = P^\perp$ and then the restriction of $SP^\perp$ on $P^\perp\mathcal{K}$ plays the role of the identity in the algebra of all bounded everywhere defined operators on $P^\perp\mathcal{H}$. Note that $P\mathcal{K}$ is an invariant subspace under the operator SP, we denote by SP the restriction of SP on $P\mathcal{K}$. Then, to prove that $S \in \mathcal{S}^{-1}(\mathcal{K})$, it is enough to prove that $SP \in \mathcal{S}^{-1}(P\mathcal{K})$. Notice that for every $Q \in \mathcal{P}_1(\mathcal{H})$, the number $\lambda(I, Q)$ is positive. Combining the latter fact with the surjectivity of Φ and Lemma 3.2.3, we conclude that for every $Q \in \mathcal{P}_1(P\mathcal{K})$, the number $\lambda\left(\Phi(I), Q\right)$ is positive. Fix $Q \in \mathcal{P}_1(P\mathcal{K})$, let $v \in P\mathcal{K}$, we have

$$
\begin{aligned}
\langle \lambda\left(\Phi(I),Q\right)Qv,v\rangle &= \langle \lambda\left(\Phi(I),Q\right)QPv,Pv\rangle \\
&\le \langle \Phi(I)Pv,Pv\rangle \\
&\le \langle \Phi(I)Pv,Pv\rangle \\
&\le \langle SPv,v\rangle .
\end{aligned}
$$

Then, $\lambda(SP,Q) \ge \lambda\left(\Phi(I),Q\right) > 0$. Since this holds for all $Q \in \mathcal{P}_1(PK)$, we obtain by Proposition 3.2.6 that SP is invertible in $\mathcal{S}_+^{-1}(P\mathcal{K})$. Then, $S \in \mathcal{S}_+^b(K) \cap \mathcal{S}_+^{-1}(\mathcal{K})$.

Put $T = \Phi^{-1}(S)$, we will show that $T \in \mathcal{S}_+^b(\mathcal{H}) \cap \mathcal{S}_+^{-1}(\mathcal{H})$. First, recall that $f(t) \le t$, for all $t \in \mathbb{R}^+$. By using (1.4.3), we derive that for every $v \in \mathcal{D}(\Phi(I))$

$$
\begin{aligned}
\langle Sv,v\rangle &= \int f(t)d\,\langle E(t)v,v\rangle \\
&\le \int td\,\langle E(t)v,v\rangle \\
&\le \langle \Phi(I)v,v\rangle .
\end{aligned}
$$

Thus, $S \le \Phi(I)$. Applying Φ^{-1}, we get that $T \le I$ which implies that T is a bounded operator. Since $S \in \mathcal{S}_+^b(K) \cap \mathcal{S}_+^{-1}(\mathcal{K})$, then by Proposition 3.2.6, for every $Q \in \mathcal{P}_1(\mathcal{K})$ we have $\lambda(S,Q)$ is a positive number. So, by the surjectivity of Φ and Lemma 3.2.3, we derive that for every $Q \in \mathcal{P}_1(\mathcal{H})$, $\lambda(T,Q)$ is a positive number. Then, using again Proposition 3.2.6, we conclude that $T \in \mathcal{S}_+^b(\mathcal{H}) \cap \mathcal{S}_+^{-1}(\mathcal{H})$. Hence, we achieve the desired conclusion. □

The following lemma shows that any order isomorphism $\Phi : \mathcal{S}_+(\mathcal{H}) \to \mathcal{S}_+(\mathcal{K})$ that satisfies the condition $\Phi(I) = I$ is a norm-preserving map on the set of all positive rank one operators.

Lemma 3.2.8. *Let $\Phi : \mathcal{S}_+(\mathcal{H}) \to \mathcal{S}_+(\mathcal{K})$ be an order isomorphism that satisfies $\Phi(I) = I$. Then, for every $P \in \mathcal{P}_1(\mathcal{H})$, there exists a unique $Q \in \mathcal{P}_1(\mathcal{K})$ such that*

$$\Phi(tP) = tQ$$

for all non-negative number t.

Proof. Fix $P \in \mathcal{P}_1(\mathcal{H})$. By Lemma 3.2.3 there exists a unique $Q \in \mathcal{P}_1(\mathcal{K})$ such that

$$\Phi\left([0,\infty)P\right) = [0,\infty)Q.$$

Let α be a non-negative number. Note that the set $\{B \in \mathcal{S}_+(\mathcal{H}) : \ I \leq B \leq I + \alpha P\}$ is totally ordered, which implies that

$$\{B \in \mathcal{S}_+(\mathcal{K}) : \ I \leq B \leq \Phi(I + \alpha P)\} = \Phi(\{B \in \mathcal{S}_+(\mathcal{H}) : \ I \leq B \leq I + \alpha P\})$$

is also totally ordered. Therefore, $\{B \in \mathcal{S}_+(\mathcal{K}) : \ 0 \leq B \leq \Phi(I + \alpha P) - I\}$ is totally ordered. Thus, we can show that $\Phi(I + \alpha P) - I$ is a rank one operator by similar arguments as used in the proof of Lemma 3.2.3. Consequently, $\Phi(I + tP) \in \mathcal{S}_+^b(\mathcal{H}) \cap \mathcal{S}_+^{-1}(\mathcal{H})$. So, using this together with the fact that the restriction of Φ onto $\{B \in \mathcal{S}_+(\mathcal{H}) : \ 0 \leq B \leq I + \alpha P\}$ is an order isomorphism onto $\{B \in \mathcal{S}_+(\mathcal{K}) : \ 0 \leq B \leq \Phi(I + \alpha P)\}$ and [45, Lemma 6.1], we conclude the existence of two non-negative numbers a_α and b_α such that

$$\Phi(tP) = \frac{t}{a_\alpha t + b_\alpha} Q$$

for all $t \in [0, 1 + \alpha]$. One can prove that $b_\alpha = 0$ and $a_\alpha = 1$ for all non-negative number α, by a similar manner as in the proof presented in [45, Proof of Theorem 2.5]. This means that $\Phi(tP) = tQ$ for all non-negative number t. $\square$

In the next proposition, we shall classify the elements of $\mathcal{S}_+^b(\mathcal{H})$.

Proposition 3.2.9. *Let $A \in \mathcal{S}_+(\mathcal{H})$. Then, $A \in \mathcal{S}_+^b(\mathcal{H})$ if and only if the set*

$$\{\lambda(A, P) : \ P \in \mathcal{P}_1(\mathcal{H})\}$$

has an upper-bound.

Proof. Assume that $A \in \mathcal{S}_+^b(\mathcal{H})$. Let $P \in \mathcal{P}_1(\mathcal{H})$ and $v \in \mathcal{H}$ such that $P = v \otimes v$. Then,

$$\begin{aligned} \lambda(A, P) &= \langle \lambda(A, P) P v, v \rangle \\ &\leq \langle A v, v \rangle \\ &\leq \|A\|. \end{aligned}$$

Then, $\sup\{\lambda(A,P):\ P \in \mathcal{P}_1(\mathcal{H})\} \leq \|A\|$.

Conversely, assume that A is a non-bounded operator. Then, the square root $A^{1/2}$ is also non-bounded, which means that there exists a sequence of unit vectors $(v_n) \subseteq \mathcal{D}(A^{1/2})$ such that

$$\lim_{n\to\infty} \|A^{1/2}v_n\| = \infty. \tag{3.2.5}$$

For each n we put $P_n := \frac{1}{\|A^{1/2}v_n\|^2} A^{1/2}v_n \otimes A^{1/2}v_n \in \mathcal{P}_1(\mathcal{H})$. By Lemma 3.2.2 we have the inequality

$$\|A^{1/2}v_n\|^2 P_n = A^{1/2}v_n \otimes A^{1/2}v_n \leq A$$

holds for all integer n. Hence,

$$\|A^{1/2}v_n\|^2 \quad \leq \quad \lambda(A, P_n)$$

for all n. Using (3.2.5) together with the previous inequality, we deduce that the set $\{\lambda(A,P):\ P \in \mathcal{P}_1(\mathcal{H})\}$ $(\supseteq \{\lambda(A,P_n):\ n \in \mathbb{N}\})$ does not have an upper-bound. This completes the proof Proposition 3.2.9. □

The next lemma is a useful tool, it transforms the order relation "$\leq$" from unbounded operators to bounded ones.

Lemma 3.2.10. *Let $A \in \mathcal{S}^b_+(\mathcal{H}) \cap \mathcal{S}^{-1}_+(\mathcal{H})$ and $B \in \mathcal{S}^{-1}_+(\mathcal{H})$. Then, $A \leq B$ if and only if $B^{-1} \leq A^{-1}$.*

Proof. It is easily checked that the maps $C \mapsto A^{-1/2}CA^{-1/2}$ and $C \mapsto A^{1/2}CA^{1/2}$ are order automorphisms of $\mathcal{S}_+(\mathcal{H})$. Then,

$$A \leq B \Longleftrightarrow I \leq A^{-1/2}BA^{-1/2}, \tag{3.2.6}$$

and

$$B^{-1} \leq A^{-1} \Longleftrightarrow A^{1/2}B^{-1}A^{1/2} \leq I. \tag{3.2.7}$$

Combining (3.2.6) and (3.2.7) with the fact that $A^{-1/2}BA^{-1/2}$ is boundedly invertible and its inverse is $A^{1/2}B^{-1}A^{1/2}$, we conclude that it suffices to prove the following. If $D \in \mathcal{S}_+(\mathcal{H}) \cap \mathcal{S}^{-1}(\mathcal{H})$, then

$$I \leq D \Longleftrightarrow D^{-1} \leq I.$$

Let $D \in \mathcal{S}_+(\mathcal{H}) \cap \mathcal{S}^{-1}(\mathcal{H})$. Assume that $I \leq D$ and put $T = (D-I)^{1/2} \in \mathcal{S}(\mathcal{H})$. Obviously, $D = I + T^2$, the inequality $D^{-1} \leq I$ follows directly from [44, Proposition 3.18 (i)]. Conversely, assume that $D^{-1} \leq I$. Let $v \in \mathcal{D}(D)$, we denote $w = D^{1/2}v$. Then, we have

$$\begin{array}{rcl} \langle Dv, v\rangle & = & \langle w, w\rangle \\ & \geq & \langle D^{-1}w, w\rangle \\ & \geq & \left\langle D^{-1/2}w, D^{-1/2}w\right\rangle \\ & \geq & \langle Iv, v\rangle\,, \end{array}$$

as desired. □

Now we are ready to prove the main result of this section.

Proof of Theorem 3.1.1 : Let $\Phi : \mathcal{S}_+(\mathcal{H}) \to \mathcal{S}_+(\mathcal{K})$ be an order isomorphism. By Lemma 3.2.7, there exists $T \in \mathcal{S}_+^b(\mathcal{H}) \cap \mathcal{S}_+^{-1}(\mathcal{H})$ such that $\Phi(T) \in \mathcal{S}_+^b(\mathcal{K}) \cap \mathcal{S}_+^{-1}(\mathcal{K})$. Without loss of generality, we can assume that our original map Φ satisfies

$$\Phi(I) = I,$$

by replacing Φ by the order isomorphism

$$A \mapsto \Phi(T)^{-1/2}\Phi(T^{1/2}AT^{1/2})\Phi(T)^{-1/2}$$

for all $A \in \mathcal{S}_+(\mathcal{H})$. Combining Proposition 3.2.9 with Lemma 2.3.3, we derive that Φ preserves the bounded operator, precisely,

$$\Phi\left(\mathcal{S}_+^b(\mathcal{H})\right) = \mathcal{S}_+^b(\mathcal{K}).$$

According to [45] there exists an invertible, linear or conjugate-linear operator $L : \mathcal{K} \to \mathcal{H}$ such that

$$\Phi(A) = L^*AL$$

for all $A \in \mathcal{S}_+^b(\mathcal{H})$. Then, without loss of generality we can replace again Φ by the order automorphism $\mathcal{S}_+(\mathcal{H}) \to \mathcal{S}_+(\mathcal{H})$ defined by $A \mapsto (L^*)^{-1}\Phi(A)L^{-1}$. So, the new Φ behaves like the identity map on $\mathcal{S}_+^b(\mathcal{H})$. It remains to prove that Φ is the identity map on the whole $\mathcal{S}_+(\mathcal{H})$.

Fix $A \in \mathcal{S}_+(\mathcal{H})$. Let $P \in \mathcal{P}_1(\mathcal{H})$ and let t be a non-negative number, we have

$$tP \leq A \Longleftrightarrow tP + I \leq A + I. \tag{3.2.8}$$

Obviously, $I + tP \in \mathcal{S}_+^b(\mathcal{H}) \cap \mathcal{S}_+^{-1}(\mathcal{H})$ and $I + A \in \mathcal{S}^{-1}(\mathcal{H})$. Applying Lemma 3.2.10 to (3.2.8), this gives

$$tP \;\leq\; A \Longleftrightarrow (I + A)^{-1} \;\leq\; (I + tP)^{-1} \;\leq\; I, \tag{3.2.9}$$

By a straightforward calculation we get from (3.2.9) that

$$tP \;\leq\; A \Longleftrightarrow \;\frac{t}{1+t}P \;\leq\; I - (A + I)^{-1}. \tag{3.2.10}$$

Replace t by $\lambda(A, P)$ in (3.2.10), we obtain that

$$\frac{\lambda(A,P)}{1 + \lambda(A,P)} \;\leq\; \lambda(I - (I + A)^{-1}, P). \tag{3.2.11}$$

Assume that $\dfrac{\lambda(A,P)}{1 + \lambda(A,P)} \;<\; \lambda(I - (I + A)^{-1}, P)$. Then, we can find easily a number t_0 satisfies the both conditions

$$t_0 \;>\; \lambda(A,P) \qquad \text{and} \qquad \frac{t_0}{1 + t_0} \;\leq\; \lambda(I - (I + A)^{-1}, P).$$

By (3.2.10) we obtain that

$$t_0 \;>\; \lambda(A,P) \qquad \text{and} \qquad t_0 P \;\leq\; A,$$

this contradicting the maximality of $\lambda(A, P)$, see Proposition3.2.5. Hence

$$\frac{\lambda(A,P)}{1 + \lambda(A,P)} \;=\; \lambda(I - (I + A)^{-1}, P). \tag{3.2.12}$$

Notice that the above equation is holds for all $P \in \mathcal{P}_1(\mathcal{H})$. A similar discussion shows that

$$\frac{\lambda(\Phi(A),P)}{1 + \lambda(\Phi(A),P)} \;=\; \lambda(I - (I + \Phi(A))^{-1}, P). \tag{3.2.13}$$

Since Φ behaves like the identity map on $\mathcal{S}_+^b(\mathcal{H})$ then, for every non-negative number t we have

$$tP \leq A \iff tP \leq \Phi(A).$$

So, $\lambda(A, P) = \lambda(\Phi(A), P)$ for all $P \in \mathcal{P}_1(\mathcal{H})$. Using the latter fact with (3.2.12) and (3.2.13), we obtain that for every $P \in \mathcal{P}_1(\mathcal{H})$

$$\lambda(I - (I + A)^{-1}, P) = \lambda(I - (I + \Phi(A))^{-1}, P).$$

Then, by [8, Corrollary 1] we easily derive that

$$I - (I + A)^{-1} = I - (I + \Phi(A))^{-1}.$$

Consequently, $\Phi(A) = A$. This completes the proof of Theorem 3.1.1. $\square$

3.3 The proofs of the main results

In this section, we will investigate order isomorphisms between the sets of types $\mathcal{S}_+^{-1}(\mathcal{H})$ and $\mathcal{S}(\mathcal{H})$, depending on previous results. Additionally, we will prove Theorem 3.1.5.

3.3.1 Order isomorphisms between the set of positive boundedly invertible operators

In order to prove Theorem 3.1.3, the following two lemmas are needed.

Lemma 3.3.1. *Let $A \in \mathcal{S}_+^{-1}(\mathcal{H})$. Then,*

$$\|A^{-1}\|^{-1} I \leq A.$$

Proof. It is obvious that $A^{-1} \leq \|A^{-1}\| I$. By [44, Lemma 10.11], for every $v \in \mathcal{D}(A)$ we have

$$\begin{aligned}
\langle Av, v \rangle &= \sup\left\{ |\langle v, w \rangle|^2 \langle A^{-1} w, w \rangle^{-1} : w \neq 0 \right\} \\
&\geq \sup\left\{ |\langle v, w \rangle|^2 \langle \|A^{-1}\| w, w \rangle^{-1} : w \neq 0 \right\} \\
&\geq \left\langle \|A^{-1}\|^{-1} I v, v \right\rangle.
\end{aligned}$$

Then, one easily concludes that $\|A^{-1}\|^{-1}I \leq A$. ☐

Let $A \in \mathcal{S}(\mathcal{H})$, we denote $[A, \infty) := \{B \in \mathcal{S}(\mathcal{H}) : B \geq A\}$.

Lemma 3.3.2. *Let $\Phi : \mathcal{S}_+^{-1}(\mathcal{H}) \to \mathcal{S}_+^{-1}(\mathcal{K})$ be an order isomorphism. Then, for each $m > 0$, there exist a $B_m \in \mathcal{S}(\mathcal{H})$ and an invertible bounded linear or conjugate-linear operator $T_m : \mathcal{H} \to \mathcal{H}$ such that*

$$\Phi(A) = T_m^* A T_m + B_m$$

for all $A \in [mI, \infty)$.

Proof. Let $m > 0$. Using Lemma 3.3.1, there exists $n > 0$ such that $nI \leq \Phi^{-1}(I)$ and $nI \leq mI$. Since $[mI, \infty) \subseteq [nI, \infty)$ and . Then without loss of generality, we can assume that $\Phi(mI)$ is bounded by working with n instead of m.

The map $\Phi_m : \mathcal{S}_+(\mathcal{H}) \to \mathcal{S}_+(\mathcal{K})$ defined by $A \mapsto \Phi(A + mI) - \Phi(mI)$ is an order automorphism. Then, by Theorem 3.1.1 there exists an invertible bounded linear or conjugate-linear operator $T_m : \mathcal{K} \to \mathcal{H}$ such that $\Phi_m(A) = T_m^* A T_m$ for all $A \in \mathcal{S}_+(\mathcal{H})$. Hence, $\Phi(A) = T_m A T_m^* + B_m$ for all $A \in [mI, \infty)$, where $B_m := \Phi(mI) - mT_m^* T_m$. ☐

Now we are in the position to prove Theorem 3.1.3.

Proof of Theorem 3.1.3: Let $\Phi : \mathcal{S}_+^{-1}(\mathcal{H}) \to \mathcal{S}_+^{-1}(\mathcal{K})$ be an order isomorphism. By Lemma 3.3.1, $L := \| \left(\Phi^{-1}(I)\right)^{-1} \|^{-1} I \leq \Phi^{-1}(I)$. Then, the operator $\Phi(L)$ is bounded and invertible. Replacing Φ by the order isomorphism $\mathcal{S}_+^{-1}(\mathcal{H}) \to \mathcal{S}_+^{-1}(\mathcal{K})$ given by

$$A \mapsto \Phi(L)^{-1/2} \Phi(L^{-1}A) \Phi(L)^{-1/2}$$

for all $A \in \mathcal{S}_+^{-1}(\mathcal{H})$, then we can assume without loss of generality that $\Phi(I) = I$.

Let us retain the notation of Lemma 3.3.2. In order to prove Theorem 3.1.3, it suffices to show that for every $m, n \in (0, 1]$, there exists a complex scalar α of modulus one such that $T_m = \alpha T_n$ and $B_m = B_n = 0$.

Choose a $m \in (0, 1]$ and denote $T_m = T$ and $B = B_m$. Let $n \in (0, 1]$, by Lemma 3.3.2 we have

$$\Phi(I) = T^* T + B = T_n^* T_n + B_n = I. \tag{3.3.1}$$

Using this, we obtain that for each vector $v \in \mathcal{H}$ we have

$$\Phi(I + v \otimes v) = T^*v \otimes T^*v + I = T_n^*v \otimes T_n^*v + I.$$

Hence, $T^*v \otimes T^*v = T_n^*v \otimes T_n^*v$. From this we easily conclude that $T = \alpha T_n$ where α is a complex number of modulus one. Thus, for each each $n \in (0, 1]$

$$B_n = B \qquad \text{and} \qquad T_n^*AT_n = T^*AT \tag{3.3.2}$$

for all $A \in [m + n, \infty)$.

Let $A \in \mathcal{S}^{-1}(\mathcal{H})$ and $n := \|A^{-1}\|^{-1}$. By Lemma 3.3.1, $A \in [nI, \infty)$ and there exists a positive scalar α such that $\alpha A \in [m + n, \infty)$. Thus,

$$\Phi(A) = T_n^*AT_n + B_n \qquad \text{and} \qquad \Phi(\alpha A) = \alpha T_n^*AT_n + B_n = \alpha T^*AT + B.$$

Combine the later fact with the equation (3.3.2) to conclude that

$$\Phi(A) = T^*AT + B,$$

this holds for all $A \in \mathcal{S}_+^{-1}(\mathcal{H})$. It remains to prove that $B = 0$.

Assume to the contrary that $B \neq 0$. Note that for each $m > 0$ there exists a integer $n > 0$ such that $\Phi(n^{-1}I) \leq m^{-1}I$. Since

$$\|B\| \leq \|B + n^{-1}T^*T\| + n^{-1}\|T^*T\| = \|\Phi(n^{-1}I)\| + n^{-1}\|T^*T\|,$$

then,

$$0 < \|B\| \leq m^{-1} + n^{-1}\|T^*T\|.$$

Hence, we can obtain a contradiction by giving greater values to n and m in the preceding inequality. This complete the proof of Theorem 3.1.3. ☐

3.3.2 Order isomorphisms between the sets of self-adjoint operators

Proving that an order isomorphism $\Phi : \mathcal{S}(\mathcal{H}) \to \mathcal{S}(\mathcal{K})$ preserves the bounded operators in the general case looks much more difficult. However, we will prove that this is true when either $\mathcal{H}$

or $\mathcal{K}$ is separable, relying on a classical and well known result of von Neumann.

Let S_1 and S_2 be two subsets of $\mathcal{S}(\mathcal{H})$ such that $S_2 \subseteq S_1$. An upper-bound (resp. lower-bound) for S_2 in S_1 is an operator $A \in S_1$ satisfies $A \geq B$ (resp. $A \leq B$) for all $B \in S_2$.

For any $A \in \mathcal{S}(\mathcal{H})$ we denote the square root of the operator A^2 by $|A|$ and it is called the modulus of T.

Lemma 3.3.3. *Assume that either $\mathcal{H}$ or $\mathcal{K}$ is a complex separable Hilbert space. Let $\Phi : \mathcal{S}(\mathcal{H}) \to \mathcal{S}(\mathcal{K})$ be an order isomorphism. Then,*

$$\Phi\left(\mathcal{S}^b(\mathcal{H})\right) = \mathcal{S}^b(\mathcal{K}).$$

Proof. Since Φ and Φ^{-1} have the same properties, then it is enough to prove that

$$\Phi\left(\mathcal{S}^b(\mathcal{H})\right) \subseteq \mathcal{S}^b(\mathcal{K}).$$

We will prove the statement by contradiction. Suppose that there exists $A \in \mathcal{S}^b(\mathcal{H})$ and $\Phi(A) \notin \mathcal{S}^b(\mathcal{K})$. By [42] there exists a unitary operator U on $\mathcal{K}$ such that

$$\mathcal{D}\left(\Phi(A)\right) \cap \mathcal{D}\left(U\Phi(A)U^*\right) = \{0\}.$$

One can immediately conclude that $\{\Phi(A), U\Phi(A)U^*\}$ does not have any upper-bound in $\mathcal{S}(\mathcal{H})$. But, $\{A, \Phi^{-1}\left(U\Phi(A)U^*\right)\}$ has at least the operator $|A|+|\Phi^{-1}\left(U\Phi(A)U^*\right)|$ as an upper-bounded in $\mathcal{S}(\mathcal{H})$. This contradicts the fact that Φ is an order isomorphism. □

Now we turn to the main aim of this subsection.

Proof of Theorem 3.1.4: By Lemma 3.3.3, the restriction of Φ onto $\mathcal{S}^b(\mathcal{H})$ is an order isomorphism onto $\mathcal{S}^b(\mathcal{K})$. Thus, by [33] one can derive the existence of an invertible linear or conjugate-linear operator $T : \mathcal{K} \to \mathcal{H}$ and a $T_0 \in \mathcal{S}^b(\mathcal{H})$ such that

$$\Phi(A) = T^*AT + T_0 \tag{3.3.3}$$

for all $A \in \mathcal{S}^b(\mathcal{H})$. Replacing Φ by the order automorphism $\mathcal{S}(\mathcal{H}) \to \mathcal{S}(\mathcal{H})$ given by $A \mapsto (T^{-1})^*(\Phi(A) - T_0)T^{-1}$ for all $A \in \mathcal{S}(\mathcal{H})$. Then, by (3.3.3) we can and we will assume without loss of generality that Φ acts like the identity map on $\mathcal{S}^b(\mathcal{H})$. By a discussion similar to that in the the proof of Theorem 3.1.1 we can obtain that Φ acts like the identity map on $\mathcal{S}_+(\mathcal{H})$.

Let $A \in \mathcal{S}(\mathcal{H})$, we have $A \leq |A|$. Then, $\Phi(A) \leq \Phi(|A|) = |A|$, so $\mathcal{D}(|A|) \subseteq \mathcal{D}(\Phi(A))$. By [44, Lemma 7.1], $\mathcal{D}(|A|) = \mathcal{D}(A)$, so $\mathcal{D}(A) \subseteq \mathcal{D}(\Phi(A))$. Since Φ and Φ^{-1} have the same properties we can conclude immediately that

$$\mathcal{D}(A) = \mathcal{D}\left(\Phi(A)\right). \tag{3.3.4}$$

We claim that Φ is identity map on $\mathcal{S}(\mathcal{H})$. Suppose the contrary, then there exists a non-zero operator $B \in \mathcal{S}(\mathcal{H})$ such that $B \neq \Phi(B)$. Since $\mathcal{D}(B) = \mathcal{D}\left(\Phi(B)\right)$ and the order relation "$\leq$" is antisymmetric, there exists $v \in \mathcal{D}(B)$ such that $\langle Bv, v\rangle \neq \langle \Phi(B)v, v\rangle$. Since B is non-zero operator and Φ, Φ^{-1} have the same properties, then without loss of generality we can and we will assume that

$$\langle Bv, v\rangle \neq 0 \qquad \text{and} \qquad \langle Bv, v\rangle < \langle \Phi(B)v, v\rangle . \tag{3.3.5}$$

Set $B_v = \dfrac{1}{\langle Bv, v\rangle} Bv \otimes Bv$, by a simple computation we get that

$$B_v v = Bv. \tag{3.3.6}$$

We define the order automorphism $\Psi : \mathcal{S}(\mathcal{H}) \to \mathcal{S}(\mathcal{H})$ by

$$\Psi(A) = \Phi(A + B_v) - B_v$$

for all $A \in \mathcal{S}(\mathcal{H})$. The same discussion as above shows that Ψ acts like the identity map on $\mathcal{S}_+(\mathcal{H})$. Combining this with the fact that $B - B_v \leq |B - B_v|$, we obtain

$$\Psi(B - B_v) = \Phi(B) - B_v \leq |B - B_v|. \tag{3.3.7}$$

It follows from [44, Lemma 7.1] that $\ker(B - B_v) = \ker|B - B_v|$, using this together with (3.3.6),(3.3.5) and (3.3.7), we obtain that

$$0 = \langle (B - B_v)v, v\rangle < \langle \left(\Phi(B) - B_v\right) v, v\rangle \leq \langle |B - B_v| v, v\rangle = 0,$$

which is a contradiction. Therefore, $\Phi(B) = B$. This completes the proof of Theorem 3.1.4. $\square$

3.3.3 The proof of Theorem 3.1.5

In this subsection, we utilize the previous results to provide a proof for Theorem 3.1.5. The motivation for this theorem stems from [45, Theorem 2.1, Theorem 2.9] due to Šemrl.

We remark obviously that in the case of $\dim \mathcal{H} \leq 1$, the sets $\mathcal{S}_+^{-1}(\mathcal{H})$ and $\mathcal{S}(\mathcal{H})$ are order isomorphic. We will prove in the following that this fact will not hold true when $\dim \mathcal{H} \geq 2$ or $\dim \mathcal{K} \geq 2$.

Proof of Theorem3.1.5: (1) Since $\mathcal{H}$ and $\mathcal{K}$ play a similar roles, we can assume, without loss of generality, that $\dim \mathcal{H} \geq 2$. If $\dim \mathcal{H}$ and $\dim \mathcal{K}$ are finite, we know that $\mathcal{S}_+^{-1}(\mathcal{H}) \subseteq \mathcal{S}_+^b(\mathcal{H})$ and $\mathcal{S}(\mathcal{K}) = \mathcal{S}^b(\mathcal{K})$. Then, by [45, Theorem 2.1, Theorem 2.9], the sets $\mathcal{S}(\mathcal{H})$ and $\mathcal{S}_+^{-1}(\mathcal{H})$ are not order isomorphic. Now, we consider the case where $\dim \mathcal{H}$ if infinite and $\dim \mathcal{K}$ is finite. Assume that there exists an order isomorphism $\Phi : \mathcal{S}_+^{-1}(\mathcal{H}) \to \mathcal{S}(\mathcal{K})$. Let $A \in \mathcal{S}(\mathcal{K})$ such that $\Phi(2I) - A \in \mathcal{S}_+^{-1}(\mathcal{K})$. Put, $t_0 = \inf \left\{1, \| \left(\Phi^{-1}(\Phi(2I) - A)\right)^{-1} \|^{-1}\right\}$, which yields that $t_0 I \leq \Phi^{-1}\left(\Phi(2I) - A\right) \leq I$ according to Lemma 3.3.1. This implies that $\Phi(2I) - \Phi(t_0 I) \in \mathcal{S}_+^{-1}(\mathcal{K})$. Then, the map $\Phi_0 : \{B \in \mathcal{S}_+^{-1}(\mathcal{H}) : t_0 I \leq B \leq 2I\} \to \{C \in \mathcal{S}(\mathcal{K}) : \Phi(t_0 I) \leq B \leq \Phi(2I)\}$ defined by $B \mapsto \Phi(B)$ is an order isomorphism. We conclude from [**?**, Theorem 2.9] that $\mathcal{H}$ and $\mathcal{K}$ have the same cardinality, which is a contradiction. Thus, $\{A, B\}$ does not have an upper-bound. Next, we consider the case where $\dim \mathcal{K}$ is infinite. Notice that $\mathcal{S}_+^{-1}(\mathcal{H})$ has the following property: For every $A, B \in \mathcal{S}_+^{-1}(\mathcal{H})$, the set of all lower-bounds of $\{A, B\}$ in $\mathcal{S}_+^{-1}(\mathcal{H})$ is not empty, this yields from Lemma 3.3.1. However, the set $\mathcal{S}(\mathcal{H})$ does not have this property. Indeed, since $\dim \mathcal{K}$ is infinite, then we can construct a non-bounded operator $A \in \mathcal{S}(\mathcal{K})$ satisfies $-nI \not\leq A$ for all integer n. Then, we can not find a lower-bound $C \in \mathcal{S}(\mathcal{K})$ of A and 0 simultaneously. Suppose the contrary, let $C \in \mathcal{S}(\mathcal{K})$ be a lower-bound of $\{A, 0\}$. Then, C is bounded because $\mathcal{D}(0) = \mathcal{K} \subseteq \mathcal{D}(C)$. However, $-nI \not\leq C$ for all integer n. This means that there exists a sequence of unit vectors $(v_n) \subseteq \mathcal{H}$ such that

$$\langle C v_n, v_n \rangle < -n$$

for all integer n, contradicting the boundedness of C. Hence, (1) holds.

(2) It is easily checked that $\dim \mathcal{H} \leq 1$ if and only if $\mathcal{S}_+(\mathcal{H})$ is totally ordered, if and only if $\mathcal{S}_+^{-1}(\mathcal{H})$ is totally ordered. Since $\mathcal{H}$ and $\mathcal{K}$ play a similar role, thus, the assertions (2) is true in the case of $\dim \mathcal{H} \leq 1$. We know that $\mathcal{S}_+(\mathcal{H})$ and $\mathcal{S}_+^{-1}(\mathcal{K})$ cannot be order isomorphic, which

implies that any two order isomorphic sets in

$$\left\{\mathcal{S}_+(\mathcal{H}), \mathcal{S}_+^{-1}(\mathcal{H}), \mathcal{S}_+(\mathcal{K}), \mathcal{S}_+^{-1}(\mathcal{K})\right\}$$

should have the same type. If $\dim \mathcal{H} \geq 2$, then, by the theorems 3.1.1 and 3.1.3, all order isomorphisms between these sets preserve the bounded operators in both directions. Thus, the fact that $\mathcal{H}$ and $\mathcal{K}$ have the same cardinality comes directly from [45, Theorem 2.9]. This completes the proof of Theorem 3.1.5. $\square$

3.4 More order isomorphisms

The present work opens the way to numerous questions concerning order isomorphisms on unbounded operators. In this section, we will explore and expand this concept by investigating it with respect to a second order relation.

Throughout this section, $\mathcal{H}$ and $\mathcal{K}$ are separable Hilbert spaces. Let $A \in \mathcal{S}(\mathcal{H})$, by [44, Theorem 7.2], there exists a unique partial isometry $U_A : \mathcal{H} \to \mathcal{H}$ with initial and final space $\overline{\mathrm{Im}(A)}$, such that

$$A = U_A|A|.$$

The operator U_A is called the phase of A, and the above decomposition is called the polar decomposition of A, see, e.g., [44].

Following Schmüdgen [44, Definition 10.5], for two operators $A, B \in \mathcal{S}(\mathcal{H})$ we say that $A \leqq B$ if $\mathcal{D}(|B|^{\frac{1}{2}}) \subseteq \mathcal{D}(|A|^{\frac{1}{2}})$ and for every $v \in \mathcal{D}(|B|^{\frac{1}{2}})$

$$\left\langle U_A|A|^{\frac{1}{2}}v, |A|^{\frac{1}{2}}v\right\rangle \leq \left\langle U_B|B|^{\frac{1}{2}}v, |B|^{\frac{1}{2}}v\right\rangle.$$

Obviously, the relations " $\leq$ " and " $\leqq$ " coincide on $\mathcal{S}^b(\mathcal{H})$. If $A \in \mathcal{S}_+(\mathcal{H})$, then it is easily checked that for every $B \in \mathcal{S}_+^b(\mathcal{H})$, we have

$$B \leq A \iff B \leqq A. \tag{3.4.1}$$

By the preceding equivalence, we conclude easily that for every $P \in \mathcal{P}_1(\mathcal{H})$, the scalar $\lambda(A, P)$

(Definition 3.2.4) satisfies the following:

$$\lambda(A,P) = \max\left\{t \in \mathbb{R}^+ : \ tP \leqq A\right\}. \tag{3.4.2}$$

The following proposition provides a comparison between " $\leqq$ " and " $\leq$ ". Additionally, it gives an answer to the natural question: Are $(\mathcal{S}_+(\mathcal{H}), \leqq)$ and $(\mathcal{S}_+(\mathcal{H}), \leq)$ order isomorphic?

Proposition 3.4.1. *The order relation " $\leqq$ " is strictly finer than " $\leq$ " in the sense that if $A, B \in \mathcal{S}_+(\mathcal{H})$ such that $A \leq B$, then $A \leqq B$, but the converse is not true in general. Moreover, $(\mathcal{S}_+(\mathcal{H}), \leqq)$ and $(\mathcal{S}_+(\mathcal{H}), \leq)$ are not order isomorphic.*

Proof. If $A \leq B$, by [44, Lemma 10.10 (ii)] we obtain that $A \leqq B$. We will prove that the converse is not true by constructing two operators $A, B \in \mathcal{S}_+(\mathcal{H})$ such that $A \leqq B$ but A and B are not comparable with respect to " $\leq$ ". First, we will show that the algebraic inverse T^{-1} of any injective operator $T \in \mathcal{S}(\mathcal{H})$ is self-adjoint. Let T be a self-adjoint operator with the spectral decomposition $T = \int t\mathrm{dE}(t)$. If T is injective, or equivalently 0 is not an eigenvalue of T, then $E(\{0\}) = 0$ according to [44, Proposition 5.10]. By [44, Theorem 5.9 (9)], we obtain that

$$T^{-1} = \int t^{-1}\mathrm{dE(t)} \in \mathcal{S}(\mathcal{H}).$$

We now return to the construction. Let us choose $C \in \mathcal{S}(\mathcal{H})$ such that C is not in $\mathcal{S}^b(\mathcal{H})$ and $C \geq I$. By [42], there exists a unitary operator U on $\mathcal{H}$ such that $\mathcal{D}(C) \cap \mathcal{D}(UCU^*) = \{0\}$. Denote UCU^* by D. Then $C^{-1}, D^{-1} \in \mathcal{S}(\mathcal{H})$ and $D \geq I$. We denote $(C^{-1} + D^{-1})^{-1}$ by A and $(2C^{-1} + D^{-1})^{-1}$ by B. It is easy to see that $A^{-1} = C^{-1} + D^{-1} \leqq 2C^{-1} + D^{-1} = B^{-1}$, from [44, Corollary 10.12] we derive that $B \leqq A$. On the other hand, inspired by Kosaki, see[25, Remark 7], we will prove that

$$\mathcal{D}(A) \cap \mathcal{D}(B) = \{0\}. \tag{3.4.3}$$

If $u \in \mathcal{D}(A) \cap \mathcal{D}(B)$, there exist $v, w \in \mathcal{H}$ such that $v = (C^{-1} + D^{-1})v = (2C^{-1} + D^{-1})w$. Then, $C^{-1}(v - 2w) = D^{-1}(w - v) \in \mathcal{D}(C) \cap \mathcal{D}(D) = \{0\}$. Thus, $u = v = w = 0$ according to the injectivity of C^{-1} and D^{-1}. Thus, from (3.4.3) we conclude directly that A and B are not comparable with respect to the relation " $\leq$ ".

Since we can repeat the same steps as in the Section 3.2 with minor changes, let us sketch the proof of the fact that $(\mathcal{S}_+(\mathcal{H}), \leqq)$ and $(\mathcal{S}_+(\mathcal{H}), \leq)$ are not order isomorphic. We will prove the statement by contradiction. Suppose that there exists an order isomorphism $\Phi : (\mathcal{S}_+(\mathcal{H}), \leqq) \to$

$(\mathcal{S}_+(\mathcal{H}), \leq)$. First we prove that Φ and Φ^{-1} preserve the rank one operators as we do in Lemma 3.2.3 by using (3.4.1). Then, we prove the existence of two bounded operators $T, S' \in \mathcal{S}_+^{-1}(\mathcal{H})$ such that $\Phi(T) = S'$ as we do in Lemma 3.2.7. Since T is boundedly invertible, there exists $\alpha > 0$ such that $\alpha I \leqq T$. We denote $\Phi(\alpha I) = S$. It is easy to see that S is a bounded invertible operator. Thus, we can assume without loss of generality that $\Phi(I) = I$, by replacing Φ by the order isomorphism

$$A \mapsto S^{-1/2}\Phi(\alpha A)S^{-1/2},$$

for all $A \in \mathcal{S}_+(\mathcal{H})$. By using (3.4.2) and applying the same reasoning as in the proof of Theorem 3.1.1, we show that the identity map on $\mathcal{S}_+(\mathcal{H})$ is an order isomorphism, which means that $A \leqq B$ if and only if $A \leq B$ for all $A, B \in \mathcal{S}_+(\mathcal{H})$. This is a contradiction which proves that $(\mathcal{S}_+(\mathcal{H}), \leqq)$ and $(\mathcal{S}_+(\mathcal{H}), \leq)$ can never be order isomorphic. □

The relation "$\leqq$" is not quite convenient to handle. Fore instance, if T is an invertible bounded linear or conjugate-linear operator on $\mathcal{H}$, the preservation of "$\leqq$" by the mapping $A \mapsto TAT^*$ remains uncertain. Also, the relation "$\leq$" can be defined on the whole $\mathcal{S}(\mathcal{H})$ in a natural manner depending only on the operators that we wish to compare, as we have seen before. For these reasons, we were more interested in order isomorphisms on unbounded operators with respect to the relation " $\leq$ " instead of " $\leqq$ ". However, the relation " $\leqq$ " is important in the treatment of self-adjoint and symmetric operators. For instance, the Friedrichs extension is defined depending on this relation. More precisely, the Friedrichs extension of any positive symmetric operator is the largest element in the set of all self-adjoint extensions of this operator with respect to " $\leqq$ ", for more information see [44, Section 10.4]. We could prove at least the following result.

Proposition 3.4.2. *Assume that* $\max\{\dim \mathcal{H}, \dim \mathcal{K}\} \geq 2$. *A map* $\Phi : \mathcal{S}_+^{-1}(\mathcal{H}) \to \mathcal{S}_+^{-1}(\mathcal{K})$ *is an order isomorphism with respect to the order relation " $\leqq$ " if and only if there exists a bijective bounded linear or conjugate-linear operator* $T : \mathcal{K} \to \mathcal{H}$ *such that*

$$\Phi(A) = T^*AT \tag{3.4.4}$$

for all $A \in \mathcal{S}_+^{-1}(\mathcal{H})$.

The proof of Proposition 3.4.2 requires the development of new results. If $A \in \mathcal{S}_+^{-1}(\mathcal{H})$, then by (3.4.1) and Lemma 3.3.1 we derive that

$$0 \leqq A - \|A^{-1}\|^{-1} I. \tag{3.4.5}$$

We have the following identity, which is easily verified and its proof depends only on (3.4.1),(3.4.2) and (3.4.5):

$$\lambda(A - \|A^{-1}\|^{-1} I, P) = \max \Big\{ t \in \mathbb{R}^+ : \; tP + \|A^{-1}\|^{-1} I \leqq A \Big\}. \tag{3.4.6}$$

Let $A \in \mathcal{S}_+^{-1}(\mathcal{H})$, and let $\mathcal{S}$ be a subset of $\mathcal{S}_+^{-1}(\mathcal{H})$. We say that A is the supremum of $\mathcal{S}$ in $\Big(\mathcal{S}_+^{-1}(\mathcal{H}), \leqq\Big)$, and we write $A = \sup \mathcal{S}$ if A is an upper-bound of $\mathcal{S}$ in $\Big(\mathcal{S}_+^{-1}(\mathcal{H}), \leqq\Big)$ and a lower-bound of the set of all upper-bound of $\mathcal{S}$ in $\Big(\mathcal{S}_+^{-1}(\mathcal{H}), \leqq\Big)$.

Lemma 3.4.3. *Let $A \in \mathcal{S}_+^{-1}(\mathcal{H})$. Then,*

$$A = \sup \Big\{ \lambda(A - \|A^{-1}\|^{-1} I, P) P + \|A^{-1}\|^{-1} I : P \in \mathcal{P}_1(\mathcal{H}) \Big\}$$

in $\Big(\mathcal{S}_+^{-1}(\mathcal{H}), \leqq\Big)$.

Proof. Upon replacing A by $\|A^{-1}\| A$ if necessary, then without loss of generality, we can assume that $\|A^{-1}\| = 1$. From (3.4.6), we conclude easily that A is an upper-bound of the set $\{\lambda(A - I, P)P + I : P \in \mathcal{P}_1(\mathcal{H})\}$ in $\Big(\mathcal{S}_+^{-1}(\mathcal{H}), \leqq\Big)$. Let $B \in \mathcal{S}_+^{-1}(\mathcal{H})$ be an upper-bound of $\{\lambda(A - I, P)P + I : P \in \mathcal{P}_1(\mathcal{H})\}$ in $\Big(\mathcal{S}_+^{-1}(\mathcal{H}), \leqq\Big)$, we will prove that $A \leqq B$. Let $P \in \mathcal{P}_1(\mathcal{H})$. Since $\lambda(A - I, P)P + I \leqq B$, we derive easily that

$$\beta(B, P) := \max \Big\{ t \in \mathbb{R}^+ : tP + I \leqq B \Big\} \geq \lambda(A - I, P). \tag{3.4.7}$$

Using [44, Corollary 10.12], (3.4.1) and a simple computation, one obtains that for every non-negative number t, $tP + I \leqq A$ is equivalent to $\dfrac{t}{1+t} P \leqq I - A^{-1}$. Using this together with (3.4.2), we conclude that

$$\frac{\lambda(A - I, P)}{1 + \lambda(A - I, P)} = \lambda(I - A^{-1}, P). \tag{3.4.8}$$

By the same way, we can show that

$$\frac{\beta(B, P)}{1 + \beta(B, P)} = \lambda(I - B^{-1}, P). \tag{3.4.9}$$

Combining (3.4.7),(3.4.8), (3.4.9) and the fact that the function $t \mapsto \dfrac{t}{t+1}$ is increasing on $\mathbb{R}^+$, we conclude that

$$\lambda(I - A^{-1}, P) \leq \lambda(I - B^{-1}, P)$$

for all $P \in \mathcal{P}_1(\mathcal{H})$. By [8, Corollary 1], we obtain that $I - A^{-1} \leqq I - B^{-1}$, which is equivalent to $B^{-1} \leqq A^{-1}$. We can apply [44, Corollary 10.12] again to obtain the desired inequality $A \leqq B$. □

The following lemma shows that all order isomorphisms $\mathcal{S}_+^{-1}(\mathcal{H}) \to \mathcal{S}_+^{-1}(\mathcal{K})$ conserve the bounded operators. Its proof has the same steps as to that of Lemma 3.3.3. However, this one requires some additional effort to obtain sufficient clarification.

Lemma 3.4.4. *Assume that* $\max\{\dim \mathcal{H}, \dim \mathcal{K}\} \geq 2$. *Let* $\Phi : \mathcal{S}_+^{-1}(\mathcal{H}) \to \mathcal{S}_+^{-1}(\mathcal{K})$ *be an order isomorphism with respect to the order relation "* $\leqq$ *". Then,*

$$\Phi\left(\mathcal{S}_+^{-1}(\mathcal{H}) \cap \mathcal{S}^b(\mathcal{H})\right) = \mathcal{S}_+^{-1}(\mathcal{K}) \cap \mathcal{S}^b(\mathcal{K}).$$

Proof. Since Φ and Φ^{-1} have the same proprieties, it is convenient to see that we can assume without loss of generality that we only need to prove the inclusion

$$\Phi\left(\mathcal{S}_+^{-1}(\mathcal{H}) \cap \mathcal{S}^b(\mathcal{H})\right) \subseteq \mathcal{S}_+^{-1}(\mathcal{K}) \cap \mathcal{S}^b(\mathcal{K}).$$

We will prove the statement by contradiction. Suppose that there exists $A \in \mathcal{S}_+^{-1}(\mathcal{H}) \cap \mathcal{S}^b(\mathcal{H})$ and $\Phi(A) \notin \mathcal{S}^b(\mathcal{K})$. Thus, $\Phi(A)^{\frac{1}{2}} \notin \mathcal{S}^b(\mathcal{K})$. Since $\mathcal{K}$ is separable, then by [42] there exists a unitary operator U on $\mathcal{K}$ such that

$$\mathcal{D}\left(\Phi(A)^{\frac{1}{2}}\right) \cap \mathcal{D}\left(U\Phi(A)^{\frac{1}{2}}U^*\right) = \{0\}.$$

Since $U\Phi(A)^{\frac{1}{2}}U^* = (U\Phi(A)U^*)^{\frac{1}{2}}$, then the set $\{\Phi(A), U\Phi(A)U^*\}$ does not have any upper-bound in $\left(\mathcal{S}(\mathcal{K})_+^{-1}, \leqq\right)$. Denote $B := \|A\|I + \Phi^{-1}\left(U\Phi(A)U^*\right)$. Notice that $\mathcal{D}(B^{\frac{1}{2}}) \subseteq \mathcal{D}(A^{\frac{1}{2}}) = \mathcal{H}$. From Theorem 1.4.3, it is easily checked that

$$\mathcal{D}(B^{\frac{1}{2}}) = \mathcal{D}\left(\Phi^{-1}\left(U\Phi(A)U^*\right)^{\frac{1}{2}}\right). \tag{3.4.10}$$

Let E denotes the spectral measure of $\Phi^{-1}\left(U\Phi(A)U^*\right)$ and let $v \in \mathcal{D}(B^{\frac{1}{2}})$, then by [44, Theorem

5.9 (2), (5)] and (3.4.10) we have

$$\begin{aligned} \|B^{\frac{1}{2}}v\|^2 &= \int (t+\|A\|)\,\mathrm{d}\,\langle \mathrm{E}(t)v,v\rangle \\ &= \int t\mathrm{d}\,\langle \mathrm{E}(t)v,v\rangle + \|A\|\|v\|^2 \\ &= \|\Phi^{-1}\left(U\Phi(A)U^*\right)^{\frac{1}{2}} v\|^2 + \|A\|\|v\|^2. \end{aligned}$$

Consequently, B is an upper-bound of the set $\{A, \Phi^{-1}(U\Phi(A)U^*)\}$ in $\left(\mathcal{S}(\mathcal{H})_+^{-1}, \leqq\right)$, which contradicts the fact Φ is an order isomorphism. This completes the proof. □

Now we turn to the proof of Proposition 3.4.2.

Proof of Proposition 3.4.2: Assume that $T : \mathcal{K} \to \mathcal{H}$ is an invertible bounded either linear or conjugate-linear operator. It is easily checked that the map Φ defined by the formula (3.4.4), is bijective and Φ^{-1} is given by

$$\Phi^{-1}(A) = (T^*)^{-1}AT^{-1}$$

for all $A \in \mathcal{S}_+^{-1}(\mathcal{K})$. We will prove that Φ preserves the order relation " $\leqq$ ". Let $A, B \in \mathcal{S}_+^{-1}(\mathcal{H})$ such that $A \leqq B$. By [44, Corollary 10.12], we obtain that $B^{-1} \leqq A^{-1}$ which is equivalent to $T^{-1}B^{-1}(T^*)^{-1} \leqq T^{-1}A^{-1}(T^*)^{-1}$. We use [44, Corollary 10.12] again to obtain that $T^*AT \leqq T^*BT$, or equivalently $\Phi(A) \leqq \Phi(B)$. By similar arguments we can prove that Φ^{-1} preserves the order relation " $\leqq$ ", which completes the first half of this proof.

Conversely, assume that $\Phi : \mathcal{S}_+^{-1}(\mathcal{H}) \to \mathcal{S}_+^{-1}(\mathcal{K})$ be an order isomorphism with respect to the order relation " $\leqq$ ". By Lemma 3.4.4, the restriction of Φ onto $\mathcal{S}_+^{-1}(\mathcal{H}) \cap \mathcal{S}^b(\mathcal{H})$ is an order isomorphism onto $\mathcal{S}_+^{-1}(\mathcal{K}) \cap \mathcal{S}^b(\mathcal{K})$. Then, by [45, Theorem 2.6], we conclude the existence of an invertible bounded either linear or conjugate-linear operator $T : \mathcal{K} \to \mathcal{H}$ such that

$$\Phi(A) = T^*AT$$

for all $A \in \mathcal{S}_+^{-1}(\mathcal{H}) \cap \mathcal{S}^b(\mathcal{H})$. Replacing Φ by the order automorphism $\mathcal{S}_+^{-1}(\mathcal{H}) \to \mathcal{S}_+^{-1}(\mathcal{H})$ defined by

$$A \mapsto (T^{-1})^*\Phi(A)T^{-1}$$

for all $A \in \mathcal{S}_+^{-1}(\mathcal{H})$. Then, we can and we will assume without loss of generality that Φ acts like the identity map on $\mathcal{S}_+^{-1}(\mathcal{H}) \cap \mathcal{S}^b(\mathcal{H})$. The fact Φ is the identity map on $\mathcal{S}_+^{-1}(\mathcal{H})$ follows directly from Lemma 3.4.3. This completes the proof of Proposition 3.4.2. $\square$

It seems that much is left to be done in the study of order isomorphisms with respect to the relation " $\leqq$ ". The following question is motivated by Proposition 3.4.2: If $\mathcal{H}$ is an infinite-dimensional Hilbert space. Do the order isomorphisms between the sets of types $\mathcal{S}_+(\mathcal{H})$, $\mathcal{S}_+^{-1}(\mathcal{H})$, and $\mathcal{S}(\mathcal{H})$ with respect to the relation "$\leqq$" have the same forms as order isomorphisms with respect to the relation " $\leq$ "?

Chapter 4

Linear transformations preserving algebraic elements on $\mathcal{M}_2(\mathbb{F})$ of degree 2

As it is known in preserver problems, usually the two-dimensional case is somewhat special, and it needs some additional work in comparison with other cases, see e.g. [37, 45]. In this concluding chapter, we will prove that the description given in [15] will not be valid in the two-dimensional case. Precisely, we will construct a linear map on $\mathcal{M}_2(\mathbb{F})$ that preserves the algebraic elements in $\mathcal{M}_2(\mathbb{F})$ of degree 2 and is neither of the form (4.0.2) nor (4.0.3) (given below). Moreover, we will provide a complete description of all linear map on $\mathcal{M}_2(\mathbb{F})$ that preserves the algebraic elements in $\mathcal{M}_2(\mathbb{F})$ of degree 2. We will provide also a method to construct linear maps on $\mathcal{M}_2(\mathbb{F})$ which strongly preserve S_2 and do not take any of the forms (4.0.2) or (4.0.3).

The results of this chapter are published in the journal "Afrika Matematika", in th paper entitled "Linear transformations preserving algebraic elements on $\mathcal{M}_2(\mathbb{F})$ of degree 2".

We start with the definition of an algebraically close fields of characteristic zero.

Definition 4.0.1. A field $\mathbb{F}$ is considered algebraically closed of characteristic zero if it is commutative, every polynomial of degree greater than or equal to one, with coefficients in $\mathbb{F}$, has at least one root in $\mathbb{F}$, and any finite sum of copies of the field's multiplicative identity, 1, is distinct from the additive identity, 0.

Let $\mathcal{M}_k(\mathbb{F})$ be the algebra of all $k \times k$-matrices over an algebraically closed field $\mathbb{F}$ of characteristic zero. The identity matrix of $\mathcal{M}_k(\mathbb{F})$ will be denoted by I_k and for any matrix $X \in \mathcal{M}_k(\mathbb{F})$

we denote by $p_{m,X}$ the minimal polynomial of X, X^t its transpose and $\mathtt{tr}(X)$ its trace. The set of invertible matrices will be denoted by $\mathtt{GL}_k(\mathbb{F})$ and for $X \in \mathtt{GL}_k(\mathbb{F})$ we denote by X^{-1} its inverse.

In [15] Willian Franca and Magno Alves showed that for any integer $k \geq 3$, a linear map Φ on $\mathcal{M}_k(\mathbb{F})$ strongly preserves the set

$$S_k := \{X \in \mathcal{M}_k(\mathbb{F}) \ : \ \deg p_{m,X} = 2\} \tag{4.0.1}$$

(this means that $X \in S_k \Leftrightarrow \Phi(X) \in S_k$) if and only if Φ is either of the form

$$\Phi(X) = \mu M X M^{-1} + L(X) I_k \quad \text{for all } X \in \mathcal{M}_k(\mathbb{F}), \tag{4.0.2}$$

or of the form

$$\Phi(X) = \mu M X^t M^{-1} + L(X) I_k \quad \text{for all } X \in \mathcal{M}_k(\mathbb{F}), \tag{4.0.3}$$

where $\mu \in \mathbb{F} \setminus \{0\}$, $M \in \mathtt{GL}_k(\mathbb{F})$ and $L \ : \ \mathcal{M}_k(\mathbb{F}) \to \mathbb{F}$ is a linear functional, see [15, Theorem 1.2]. At the end of their paper the authors point out that their characterization seems to be also true in the case $k = 2$ and they explain why their arguments do not work in this case.

Example. Let Φ be the linear map on $\mathcal{M}_2(\mathbb{F})$ defined by

$$\Phi \ : \ \begin{pmatrix} x & y \\ z & t \end{pmatrix} \mapsto \frac{1}{2} \begin{pmatrix} 2y + x + t & x - t \\ z & -2y + x + t \end{pmatrix}.$$

Then Φ is an isomorphism of $\mathcal{M}_2(\mathbb{F})$ that strongly preserves both S_2 and the trace, but Φ does not take any of the forms (4.0.2) or (4.0.3).

Indeed, as mentioned in [15] we have obviously

$$\mathcal{M}_2(\mathbb{F}) = S_2 \cup \mathbb{F} I_2 \quad \text{and} \quad S_2 \cap \mathbb{F} I_2 = \emptyset, \tag{4.0.4}$$

thus Φ strongly preserves S_2 if and only if it strongly preserves $\mathbb{F} I_2$. Now observe that

$$\begin{pmatrix} x & y \\ z & t \end{pmatrix} \in \mathbb{F} I_2 \iff \left\{ \begin{array}{lllll} x & = & t & & \\ z & = & y & = & 0 \end{array} \right. \iff \frac{1}{2} \begin{pmatrix} 2y + x + t & x - t \\ z & -2y + x + t \end{pmatrix} \in \mathbb{F} I_2.$$

Hence Φ strongly preserves S_2. Finally for

$$X = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix},$$

we have

$$\Phi(X)^t = \Phi(X) = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \neq \mu M X M^{-1} + \nu I_2$$

for all $M \in \mathtt{GL}_2(\mathbb{F})$, $\mu \in \mathbb{F} \setminus \{0\}$, $\nu \in \mathbb{F}$. To see this it suffices to observe that $\Phi(X)^2 = I_2$ and

$$(\mu M X M^{-1} + \nu I_2)^2 = M \left(2\mu\nu X + \nu^2 I_2\right) M^{-1} \neq I_2$$

for all $M \in \mathtt{GL}_2(\mathbb{F})$, $\mu \in \mathbb{F} \setminus \{0\}$, $\nu \in \mathbb{F}$.

As in [15], we put $\mathcal{M}_2(\mathbb{F})' := \{X \in \mathcal{M}_2(\mathbb{F}) \ : \ \mathtt{tr}(X) = 0\}$. From (4.0.4) one can see easily that

$$\mathcal{M}_2(\mathbb{F})' \cap S_2 = \mathcal{M}_2(\mathbb{F})' \setminus \{0\} \quad \text{and} \quad \mathcal{M}_2(\mathbb{F})' \oplus \mathbb{F} I_2 = \mathcal{M}_2(\mathbb{F}). \tag{4.0.5}$$

The following proposition shows how we can construct a linear map on $\mathcal{M}_2(\mathbb{F})$ that strongly preserves S_2.

Proposition 4.0.2. *([3]) Let Φ be a linear map on $\mathcal{M}_2(\mathbb{F})$. Then Φ strongly preserves S_2 if and only if there exist a linear functional $L : \mathcal{M}_2(\mathbb{F}) \to \mathbb{F}$ and a bijective linear map Φ' on $\mathcal{M}_2(\mathbb{F})'$ such that*

$$\Phi(X) = \Phi' \left(X - \frac{1}{2}\mathtt{tr}(X) I_2\right) + L(X) I_2 \tag{4.0.6}$$

for all $X \in \mathcal{M}_2(\mathbb{F})$.

Proof. We will use the fact that

$$X + \lambda I_2 \in S_2 \iff X \in S_2 \quad \text{for all} \quad X \in \mathcal{M}_2(\mathbb{F}) \text{ and } \lambda \in \mathbb{F}.$$

For the if part, assume that Φ is given by the formula (4.0.6). Then for every $X \in S_2$ we know that $X - 2^{-1}\mathtt{tr}(X) I_2 \in \mathcal{M}_2(\mathbb{F})' \cap S_2$, hence $X - 2^{-1}\mathtt{tr}(X) I_2 \in \mathcal{M}_2(\mathbb{F})' \setminus \{0\}$. Thus

$$\Phi'(X - 2^{-1}\mathtt{tr}(X) I_2) \in \mathcal{M}_2(\mathbb{F})' \setminus \{0\} \subseteq S_2.$$

Therefore $\Phi'(X - 2^{-1}\mathtt{tr}(X)I_2) + L(X)I_2 \in S_2$. Similarly, if $\Phi(X) \in S_2$, then

$$\Phi'(X - 2^{-1}\mathtt{tr}(X)I_2) = \Phi(X) - L(X)I_2 \in S_2.$$

Thus $X - 2^{-1}\mathtt{tr}(X)I_2 \in \mathcal{M}_2(\mathbb{F})' \setminus \{0\}$, hence $X \in S_2$. Thus Φ strongly preserves S_2.

Now we have to prove the only if part. First, observe that Φ strongly preserves S_2 if and only if the linear map defined on $\mathcal{M}_2(\mathbb{F})$ by

$$\Psi(X) := \Phi(X) - \frac{\mathtt{tr}(\Phi(X) - X)}{2} I_2 \tag{4.0.7}$$

is bijective and strongly preserves both S_2 and the trace, see [15, Proposition 4.4]. Now, recall that $\mathcal{M}_2(\mathbb{F}) = \mathcal{M}_2(\mathbb{F})' \oplus \mathbb{F}I_2$, thus corresponding to such a decomposition we have $\Psi(X) = \Psi(X') \oplus \lambda I_2$ for all $X = X' \oplus \lambda I_2 \in \mathcal{M}_2(\mathbb{F})' \oplus \mathbb{F}I_2$. Therefore, the restriction $\Phi' := \Psi_{|\mathcal{M}_2(\mathbb{F})'}$ of Ψ to $\mathcal{M}_2(\mathbb{F})'$ is nonsingular on $\mathcal{M}_2(\mathbb{F})'$. To finish the proof, keeping in mind that Ψ is bijective and strongly preserves S_2, we know that Ψ strongly preserves $\mathbb{F}I_2$. Thus, since Ψ preserves the trace we have $\Psi(I_2) = I_2$. From this and the definition of Φ' we conclude that

$$\begin{aligned}
\Phi'\left(X - \tfrac{1}{2}\mathtt{tr}(X)I_2\right) &= \Psi(X) - 2^{-1}\mathtt{tr}(X)I_2 \\
&= \Phi(X) - 2^{-1}\mathtt{tr}(\Phi(X) - X)I_2 - 2^{-1}\mathtt{tr}(X)I_2 \\
&= \Phi(X) - 2^{-1}\mathtt{tr}(\Phi(X))I_2
\end{aligned}$$

for all $X \in \mathcal{M}_2(\mathbb{F})$. Therefore, Φ, Φ' and $L := 2^{-1}\mathtt{tr} \circ \Phi$ satisfying the formula (4.0.6). ☐

The previous proposition says that a linear map Φ on $\mathcal{M}_2(\mathbb{F})$ strongly preserves S_2 if and only if there exist two bases (A, B, C) and (D, E, F) of $\mathcal{M}_2(\mathbb{F})'$ and $(a, b, c, \beta) \in \mathbb{F}^4$ such that

$$\Phi(xA + yB + zC + \lambda I_2) = xD + yE + zF + (ax + by + cz + \beta\lambda)I_2 \tag{4.0.8}$$

for all $(x, y, z, \lambda) \in \mathbb{F}^4$.

Based on the previous facts, we present in the following remark a manner to construct linear maps on $\mathcal{M}_2(\mathbb{F})$ which strongly preserve S_2 and do not take any of the forms (4.0.2) or (4.0.3).

Remark 4.0.3. If a linear map Φ on $\mathcal{M}_2(\mathbb{F})$ takes one of the forms (4.0.2) or (4.0.3), then Φ preserves the cardinality of the spectrum. Thus according to the form (4.0.8) if we choose A and D such that the cardinality of the spectrum of A is different from that of D, then the map

Φ does not take any of the forms (4.0.2) or (4.0.3).

Finally note that if a linear map Φ on $\mathcal{M}_2(\mathbb{F})$ strongly preserves S_2 and the cardinality of the spectrum, then the nonsingular map Φ' on $\mathcal{M}_2(\mathbb{F})'$ given by (4.0.6) strongly preserves S_2 and the cardinality of the spectrum. Thus Φ' strongly preserves the nilpotency. This follows from the fact that the nilpotent elements of $\mathcal{M}_2(\mathbb{F})'$ are exactly those which have a singleton spectrum. Thus Φ' takes one of the forms (4.0.2) or (4.0.3) on $\mathcal{M}_2(\mathbb{F})'$, see [6, Remark]. Thus the formula (4.0.6) tells us that Φ takes one of the forms (4.0.2) or (4.0.3) on $\mathcal{M}_2(\mathbb{F})$.

Conclusion and Perspectives

To conclude this thesis, we recall that our focus was to provide a complete description of order isomorphisms between certain ordered partially sets comprising self-adjoint elements. It is important to note that our results concerning the effect algebras, which are presented in Chapter 2, can be considered the most comprehensive and closest solution to the conjecture aiming to characterize all order isomorphisms between effect algebras of all C^*-algebras to date. These results generalize the vast majority of the existing achievements in this field. Our results in Chapter 3 have the precision to draw attention to investigating the preservation problems within the scope of unbounded operators, which opens the door to a large number of opportunities and challenges.

It seems from the third chapter that much is left to be done in the case of the relation " $\leqq$ ". Note that it is difficult to guess other forms of order isomorphisms with respect to this relation " $\leqq$ ", between the sets of types $\mathcal{S}_+(\mathcal{H})$, $\mathcal{S}_+^{-1}(\mathcal{H})$, and $\mathcal{S}(\mathcal{H})$ different from those given in theorems 3.1.1, 3.1.3, 3.1.4. The following problem is motivated by Proposition 3.4.2.

Let $\mathcal{H}$ be an infinite-dimensional Hilbert space. Do the order isomorphisms between the sets of types $\mathcal{S}_+(\mathcal{H})$, $\mathcal{S}_+^{-1}(\mathcal{H})$, and $\mathcal{S}(\mathcal{H})$ with respect to the relation " $\leqq$ " have the same forms as order isomorphisms with respect to the relation " $\leq$ "?

From the bounded case, see [38, 39, 45], we can find an inspiration to introduce the notion of unbounded operator interval. Let $A, B \in \mathcal{S}(\mathcal{H})$, we write $A < B$ if $B - A \in \mathcal{S}_+^{-1}(\mathcal{H})$. A subset $S \subseteq \mathcal{S}(\mathcal{H})$ is called an unbounded operator interval if it has one of the following forms,

$$[A, B] := \{C \in \mathcal{S}(\mathcal{H}) : A \leq C \leq B\},$$

$$[A, B) := \{C \in \mathcal{S}(\mathcal{H}) : A \leq C < B\},$$

$$(A, B] := \{C \in \mathcal{S}(\mathcal{H}) : A < C \leq B\},$$

$$(A, B) := \{C \in \mathcal{S}(\mathcal{H}) : A < C < B\},$$

with $A < B$. Or,

$$[A, \infty) := \{C \in \mathcal{S}(\mathcal{H}) : C \geq A\},$$

$$(-\infty, A] := \{C \in \mathcal{S}(\mathcal{H}) : C \leq A\},$$

$$(A, \infty) := \{C \in \mathcal{S}(\mathcal{H}) : C > A\},$$

$$(-\infty, A) := \{C \in \mathcal{S}(\mathcal{H}) : C < A\}.$$

Or,

$$(-\infty, \infty) := \mathcal{S}(\mathcal{H}).$$

In the case where A, B are bounded, the order isomorphisms between intervals $[A, B]$, $[A, B)$, $(A, B]$, (A, B), $(-\infty, A]$, $(-\infty, A)$ are treated in [38, 45]. We easily see that the intervals $[A, \infty)$, (A, ∞) are order isomorphic with $\mathcal{S}_+(\mathcal{H})$ and $\mathcal{S}_+^{-1}(\mathcal{H})$, respectively. Then, the theorems 3.1.1, 3.1.3, 3.1.4 3.1.5 provides a comprehensive characterization of all order isomorphisms between the unbounded operators intervals $[A, \infty)$, (A, ∞) and $(-\infty, \infty)$.

In the bounded case, the study of order isomorphisms between operators intervals can be reduced to a few specific cases that rely on some trivial facts, which are not applicable in the unbounded case. Therefore, the next logical and highly challenging step would be to investigate the following questions.

Let $A, B \in \mathcal{S}(\mathcal{H})$ such that $A < B$.

- If A and B are not bounded operators. What are the general forms of order isomorphisms between the intervals $[A, B]$, $[A, B)$, $(A, B]$ and (A, B)?

- If A is not bounded. What are the general forms of order isomorphisms between the intervals $[A, \infty)$, $(-\infty, A]$, (A, ∞) and $(-\infty, A)$?

Bibliography

[1] Z. E. A. Abdelali, Y. El Khatiri, *Order isomorphisms on effect algebras of the C^*-algebras of type $C(X) \otimes B(H)$*, Linear Algebra Appl. (2023). https://doi.org/10.1016/j.laa.2023.05.027.

[2] Z. E. A. Abdelali, Y. El Khatiri, *Order isomorphisms on unbounded operators*, Submitted.

[3] Z. E. A. Abdelali, Y. El Khatiri, Linear transformations preserving algebraic elements on M2(F) of degree 2. Afr. Mat. 34, 7 (2023). https://doi.org/10.1007/s13370-023-01046-8

[4] J. Araujo, *Multiplicative Bijections of Semigroups of Interval-Valued Continuous Functions*, Proc. Amer. Math. Soc. **137** (2009), 171–178.

[5] F. F. Bonsall, J. Duncan, *Complete Normed Algebras*, **80**, Springer Verlag, Berlin 1973.

[6] P. Botta, S. Pierce, W. Watkins, *Linear transformations that preserve the nilpotent matrices*, Pacific J. Math., **104** (1983), 39–46.

[7] P. Busch, M. Grabowski, P.J. Lahti, *Operational quantum physics*, Springer-verlag, Berlin 1995.

[8] P. Bush, S.P. Gudder, *Effects as functions on projective Hilbert space*, Lett. in math. phys., **47** (1999), 329–337.

[9] F. Cabello Sánchez, *Homomorphisms on lattices of continuous functions*, Positivity, **12** (2008), 341–362.

[10] F. Cabello Sánchez, J. Cabello Sánchez, Z. Ercan, S. Önal, *Memorandum on Multiplicative Bijections and Order*, Semigr. Forum. **79** (2009), 193–209.

[11] J.B. Conway, *A Course in Functional Analysis*, Graduate Grad. Texts in Math., Spring. New York, 1985.

[12] G. Dolinar, B. Kuzma, J. Marovt, E. Edward Poon, *Spectrum preservers revisited*, J. Math. Anal. Appl. **489** (2020).

[13] H. A. Dye, *On the geometry of projections in certain operator algebras*, Ann. of Math. **61** (1955), 73–89.

[14] D. Edmunds, W.D. Evans, *Spectral Theory and Differential Operators*, Clarendon Press, Oxford 1987.

[15] W. Franca, M. Alves, *Linear transformations preserving algebraic elements of degree 2*, Linear Multilinear Algebra, (2021), 1–23. doi.org/10.1080/03081087.2021.1873228.

[16] Z. Ercan, S. Önal, *An answer to a conjecture on multiplicative maps on* $C(X, I)$, Taiwan. J. Math. **12** (2008), 537–538.

[17] C. Isham, *Lectures On Quantum Theory: Mathematical And Structural Foundations*, World Scientific, 1995.

[18] T. Itô, *On the continuity of lattice automorphisms on continuous function lattices*, Illinois J. Math. **8** (1964), 419–424.

[19] G. Grubb, *Distributions and Operators*, Springer-Verlag, Berlin 2009.

[20] R.V. Kadison, *A generalized Schwarz inequality and algebraic invariants for operator algebras*, Ann. of Math. **56** (1952), 494–503.

[21] R.V. Kadison, J. Ringrose, *Derivations and automorphisms of operator algebras*, Comm. Math. Phys. **4** (1967), 32–63.

[22] I. Kaplansky, *Lattices of continuous functions II*, Amer. J. Math. **70** (1948), 626–634.

[23] I. Kaplansky, *Lattices of continuous functions*, Bull. Amer. Math. Sou. **53** (1947), 617–623.

[24] T. Kato, *Perturbation Theory for Linear Operators*, Springer-Verlag, Berlin, 1980.

[25] H. Kosaki, *On intersection of domains of unbounded positive operators*, Kyushu J. Math. **60** (2006), 3–25.

[26] K. Kraus, *States, Effects and Operations Fundamental Notions of Quantum Theory*, Springer-Verlag, New York 1983.

[27] E.C. Lance, *Automorphisms of certain operator algebras*, Amer. J. Math. **91** (1969), 160–174.

[28] G. Ludwig, *Foundations of quantum mechanics*, Springer-Verlag, New York 1983.

[29] G.W. MacKey, *The Mathematical foundations of quantum mechanics*, Dover Publications, 2004.

[30] J. Marovt, *Order preserving bijections of $C(X, I)$*, J. Math. Anal. Appl. **311** (2005), 567–581.

[31] L. Molnár, *A reflexivity problem concerning the C^*-algebra $C(X) \otimes B(H)$*, Proc. Amer. Math. Soc. **129** (2001), 531–537.

[32] L. Molnár, *On the Order Determining Property of the Norm of a Kubo–Ando Mean in Operator Algebras*, Integr. Equ. Oper. Theory, **93** (2021).

[33] L. Molnár, *order automorphisms of the set of bounded observables*, J.math. Phys. **42** (2001), 5904–5909.

[34] L. Molnár, *Selected Preserver Problems on Algebraic Structures of Linear Operators and on Function Spaces*, Lect. Notes Math., Springer, Berlin 2007.

[35] L. Molnár, *Some characterizations of the automorphisms of $B(H)$ and $C(X)$*, Proc. Amer. Math. Soc. **130** (2001), 111–120.

[36] L. Molnár, *Preservers on Hilbert space effects*, Linear Algebra Appl. **370** (2003), 287–300.

[37] L. Molnár, Zs. Páles, $^{\perp}$*order automorphisms of Hilbert space effect algebras: The two-dimensional case*, J. Math. Phys. 42 (2001), 1907–1912.

[38] M. Mori, *Order isomorphism of operator intervals in von Neumann algebras*, Integr. Equ. Oper. Theory, **91** (2019).

[39] M. Mori, P. Šemrl, *Loewner's theorem for maps on operator domains*, (2020). https://doi.org/10.48550/arXiv.2006.04488.

[40] V. Müller, *Operator theory advanced and applications, Spectral theory of linear operators and spectral systems in Banach algebras*, Birkhäuser 2nd ed, Basel 2007.

[41] J.V. Neumann, *Mathematical Foundations of Quantum Mechanics*, Princeton University Press, Princeton 1955.

[42] J.V. Neumann, *Zur Theorie der unbeschränkten Matrizen*, J. Reine Angew. Math. **161** (1929), 208–236.

[43] S. Sakai, *C^*-algebras and W^*-algebras*, Springer-Verlag, Berlin 1971.

[44] K. Schmüdgen, *Unbounded self-adjoint operators on Hilbert space*, Grad. Texts in Math., Spring. Dordrecht 2012.

[45] P. Šemrl, *Order isomorphisms of operator intervals*, Integr. Equ. Oper. Theory, **91** (2017), 1–42.

[46] W. Timmermann, *Approximate derivations and automorphisms in algebras of unbounded operators*, Publ. Math. (Debr.) **63** (2003), 667–676.

[47] D.P. Williams, I. Raeburn, *Morita equivalence and continuous-trace C^*-algebras*, Math. Surveys and Monogr. Amer. Math. Soc. Providence R.I. 1998.

ROYAUME DU MAROC
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CENTRE D'ETUDES DOCTORALES - SCIENCES ET TECHNOLOGIES

Résumé

Soient C(X) l'algèbre de toutes les fonctions continues à valeurs complexes sur un espace compact de Hausdorff (non pathologique) X et B(H) l'algèbre de tous les opérateurs bornés sur un espace de Hilbert H. Plusieurs chercheurs dans la littérature se sont intéressés à la structure d'ordre des algèbres de type C(X) et celles de type B(H). Nous pouvons considérer la structure d'ordre du produit tensoriel, C(X) ⊗ B(H), comme une généralisation des deux ordres précédents, et nous pouvons les récupérer chacun simultanément à partir de cette généralisation. L'algèbre des effets d'une C*-algèbre unitaire est la collection de tous les éléments positifs ayant une norme au plus égale à 1. Dans cette thèse, nous donnons une description complète de tous les isomorphismes de l'ordre entre les algèbres des effets des C*-algèbres de type C(X) ⊗ B(H).

Aussi, dans cette thèse, nous étendrons l'étude des isomorphismes de l'ordre aux opérateurs non bornés. En fait, nous fournirons une description complète de tous les isomorphismes de l'ordre entre trois types différents d'ensembles composés d'opérateurs auto-adjoints non bornés. Notamment, l'ensemble de tous les opérateurs positifs, l'ensemble de tous les opérateurs positifs bornés inversibles et l'ensemble de tous les opérateurs auto-adjoints. De plus, nous étendons également des résultats bien connus, qui ont pris leurs origines dans les opérateurs bornés au contexte des opérateurs non bornés.

Mots-clefs (5) : ***Algèbre des effets, Isomorphisme d'ordre Opérateur auto-adjoint, Opérateur non borné***

Abstract

Let C(X) be the algebra of all continuous complex-valued functions on a Hausdorff compact space X, and let B(H) be the algebra of all bounded linear operators on a complex Hilbert space H. The order structure of the algebras of type C(X) and B(H) have attracted many researchers in the history of mathematics. We can consider the order structure of the tensor product of these C*-algebras, C(X) ⊗ B(H), as a generalization for both preceding orders, and we can retrieve each of them simultaneously from this generalization. For a C*-algebra A, the collection of all positive operators with norm at most 1, is called the effect algebra and denoted by E(A). In this thesis, we provide a complete description of order isomorphisms between effect algebras of type E(C(X) ⊗ B(H)).

Also, in this thesis, we will extend the study of order isomorphisms to unbounded operators. In fact, we will provide a comprehensive description of all order isomorphisms between three distinct types of sets, which consist of unbounded self-adjoint operators. Namely, the set of all positive operators, the set of all positive boundedly invertible operators, and the set of all self-adjoint operators. Moreover, we also extend well-known facts originating from bounded operators to the context of unbounded operators.

Key Words (5) : ***Effects algebra ; Order isomorphism; Self-adjoint operator; Unbounded operator***.