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Présentée par

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Dédicace

Avec tous mes voeux de bonheur,

de santé et de réussite

Je dédie cette thèse

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A toute ma famille et mes amis...

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du bonheur; elles sont les charmants jardiniers
par qui nos âmes sont fleuries.
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Contents

Author's papers involved in this thesis/Articles de l'auteur inclus dans cette thèse

(1) N. Mahdou, A. Mimouni and M. El Ouarrachi *On Armendariz-like properties in amalgamated algebra*, accepted for publication in Turkish Journal of Mathematics.

(2) M. El Ouarrachi and N. Mahdou, *On power serieswise Armendariz rings*, submitted for publication.

(3) M. El Ouarrachi and N. Mahdou, *Power serieswise Armendariz property in amalgamated algebra*, accepted for publication in Bulletin of the Iranian Mathematical Society.

(4) M. El Ouarrachi and N. Mahdou, *Coherence in bi-amalgamated algebra along ideals*, accepted for publication in Springer Proceedings in Mathematics and Statistics.

(5) M. El Ouarrachi and M. Aqalmoun, *Bi-amalgamation of rings defined by bézout-like conditions*, accepted for publication in Palestine Journal of Mathematics.

(6) M. El Ouarrachi, *Bézout-like properties in amalgamated algebra*, appeared in Gulf Journal of Mathematics. Vol 8 (2015) 54-60.

Résumé

La présente thèse a pour but l'étude de trois notions en algèbre à savoir: (i) la notion Armendariz, (ii) la cohérence et (iii) quelques propriétés liées à la notion de Bézout. La présente étude se fera dans différents contextes notament l'extension triviale, le produit direct, l'image homomorphe, l'amalgamé classique et la bi-amalgamation algébrique.

(i) A l'hommage de E. Armendariz (1974), Rege et Chhawchharia ont introduit en 1997 une nouvelle notion en algèbre non commutative appelée la propriété Armendariz. Dès lors, plusieurs généralisations de cette notion sont apparues en l'occurrence Skew Armendariz en 2003 par Chan Yong Hong et d'autres, Weak-Armendariz en 2006 par Liu et Zhao, nil-Armendariz en 2008 par R. Antoine et Power serieswise Armendariz en 2006 par Kim et al.

(ii) le concept de la cohérence est surgi de l'étude des faisceaux cohérents en géométrie algébrique, puis développé sous l'influence de la théorie et d'homologie des anneaux Noethériens, vers un sujet à part entière dans l'algèbre. Depuis les 30 dernières années, plusieurs notions commutatives sont nées de la cohérence comme la propriété de conducteur fini, la n-cohérence, la n-cohérence forte et autres propriétés.

(iii) La partie arithmétique de cette thèse est consacrée à quelques notions liées à la notion d'anneau de Bézout comme les anneaux à division élémentaire, les anneaux d'Hermite, les P-anneaux de Bézout, les 2-anneaux de Bézout, les anneaux presque de valuation (AV-anneau) et les anneaux presque de Bézout (AB-anneau).

La thèse comporte six chapitres rendant compte respectivement des travaux de six articles [34, 35, 36, 33, 37, 65].

Le premier chapitre concerne le transfert de la propriété Armendariz à l'amalgamé algébrique d'anneaux associatifs non commutatifs le long d'un idéal. Nos résultats permettent de construire de nouvelles classes originales d'anneaux Armendariz.

Les chapitres 2 et 3 s'intéressent à l'étude de la propriété power serieswise Armendariz dans différents contextes notament l'extension triviale, le produit direct, l'image homomorphe, l'amalgamé classique.

Le qutrième chapitre concerne le transfert de la propriété de la cohérence dans la bi-amalgamation algébrique, ce qui permet de construire de nouveaux exemples d'anneaux cohérents non Noethériens.

Les chapitres 5 et 6 sont consacrés à l'étude du transfert de quelques propriétés arithmétiques, à savoir: anneau à division élémentaire, anneau presque de valuation et anneau presque de Bézout dans la bi-amalgamation algébrique, les P-anneaux de Bézout et les 2-anneaux de Bézout dans l'amalgamé classique.

Notre thèse se termine par quelques perspectives que nous souhaiterons aborder dans nos travaux futurs.

Mots clés: Armendariz, nil-Armendariz, weak-Armendariz, anneau à division élémentaire, power serieswise Armendariz, cohérence, anneau de Bézout, anneau presque de Bézout, anneau presque de valuation, P-anneau de Bézout, 2-anneau de Bézout, extension triviale, amalgamé algébrique, bi-amalgamation algébrique.

Summary

The present thesis is devoted to the study of three notions in algebra: (i) the notion of Armendariz ring, (ii) the property of coherence and (iii) some properties related to the notion of Bézout ring.This study will be in different contexts namely trivial ring extension, direct product, homomorphic image, amalgamated algebra and the bi-amalgamation algebra.

(i) In 1997, Rege and Chhawchharia introduced the notion of Armendariz ring, this name was chosen because E. Armendariz had chown that a reduced ring satisfies this property. Since then, various generalizations of Armendariz rings such as skew Armendariz ring in 2003 by Chan Yong Hong, weak Armendariz ring in 2006 by Liu and Zhao, nil-Armendariz ring in 2008 by R. Antoine and power serieswise Armendariz ring in 2006 by Kim and al.

(ii) The concept of coherence first sprang up from the study of coherent sheaves in algebraic geometry, and then developed, under the influence of Noetherian ring theory and homology, towards a full-fledged topic in algebra. During the past 30 years, several (commutative) coherent-like notions grew out of coherence such as finite conductor, n-coherent, strong n-coherent, and others properties.

(iii) The arithmetical part of this thesis deals with some notions related to the notion of Bézout ring namely elementary divisor ring, Hermite ring, almost Bézout ring, almost valuation ring, 2-Bézout ring and P-Bézout ring.

This thesis has developed in the theory of homological commutative and associative algebra. It consists of six chapters covering six papers [34, 35, 36, 33, 37, 65].

The first chapter investigates the transfer of the notion of Armendariz ring to amalgamated algebra. Our result allow us to construct new original classes of Armendariz rings.

Chapters 2 and 3 focuse on the study of the notion of power serieswise Armendariz in different contexts namely trivial ring extension, direct product, homomorphic image and amalgamated algebra.

The fourth chapter deals with the transfer of the property of coherence to the bi-amalgamated algebra which leads us to construct new original classes of non-Noetherian coherent rings.

Chapters 5 and 6 shedlight on the study of some arithmetical properties: elementary divisor ring, almost Bézout ring and almost valuation ring to the bi-amalgamated algebra and 2-Bézout ring and P-Bézout ring to amalgamated algebra.

The thesis ends with some open questions that we wish to study in the future.

Key Words: Armendariz, nil-Armendariz, weak-Armendariz, power serieswise Armendariz, elementary divisor ring, Bézout ring, almost Bézout ring, almost valuation ring, 2-Bézout ring, P-Bézout ring, coherence, trivial extension, amalgamated algebra, bi-amalgamated algebra.

Introduction

Les thèmes relatés par cette thèse sont: la notion d'anneau armendariz, la cohérence et quelques propriétés liées à la notion d'anneau de Bézout, et cela dans différents contextes à savoir: l'extension triviale, le produit direct, l'image homomorphe, l'amalgamé classique et la bi-amalgamation algébrique.

En 1997, Rege et Chhawchharia ont introduit la notion d'anneau Armendariz comme étant un anneau associatif avec unité tel que pour tous polynômes $f(x) = a_0 + a_1x + ... + a_nx^n$, $g(x) = b_0 + b_1x + ... + b_mx^m$ dans $R[x]$, $f(x)g(x) = 0$ implique que $a_ib_j = 0$ pour tout i, j. L'origine du nom Armendariz a été choisi comme un hommage à E. Armendariz qui a démontré qu'un anneau réduit satisfait bien cette propriété. En 1998, D. D. Andersone et V. Camillo ont étudié les anneaux Armendariz et les anneaux de Gauss. Dés lors, plusieurs généralisations d'anneau Armendariz ont vu le jour à savoir: skew Armendariz en 2003 par Chan Yong Hong, weak-Armendariz en 2006 par Liu et Zhao, nil-Armendariz en 2008 par R. Antoine et power serieswise Armendariz en 2006 par Kim et Al. Le diagramme suivant décrit la relation entre ces différentes notions:
Anneau réduit $\Longrightarrow$ Anneau power serieswise Armendariz $\Longrightarrow$ Anneau Armendariz $\Longrightarrow$ Anneau nil-Armendariz $\Longrightarrow$ Anneau weak-Armendariz.
Les implications inverses sont fausses en général. Cependant, on connait pas, pour la dernière implication, un exemple d'anneau weak-Armendariz qui n'est pas nil-Armendariz.

On se qui concerne la notion de la cohérence, ce concept est surgi de l'étude des faisceaux cohérents en géométrie algébrique, puis développé sous l'influence de la théorie et d'homologie des anneaux Noethériens, vers un sujet à part entière dans l'algèbre. Un grand nombre de propriétés de finitude des anneaux commutatifs ont des caractérisations homologiques. Ainsi, pour qu'un anneau soit Noethérien, il faut et il suffit que des sommes directes arbitraires de modules injectifs soient des modules injectifs. On peut spéculer que les approches homologiques d'algèbre appliquées à des paramètres Noethériens génèrent de tels résultats remarquables.
Le même phénomène peut être observé dans la classe des anneaux cohérents. Chase (1960) a tenté de répondre à la question homologique: quels sont les anneaux pour

lesquels des produits directs arbitraires de modules plats sont des modules plats? En fait, la réponse est que les anneaux cohérents vérifient cette condition. Ainsi, Chase a fournit pas moins de sept caractérisations équivalentes de cette condition homologique. Les plus connues sont: R est un anneau cohérent si tout idéal de type fini de R est de présentation finie; équivalent à, pour tout élément a de R et pour deux idéaux de type fini I et J de R, les idéaux $I \cap J$ et $(0:a) = \{r \in R \mid ra = 0\}$ sont de type fini.
Depuis les 30 dernières années, plusieurs notions (commutatives) sont nées de la cohérence comme la propriété de conducteur fini, n-cohérence, n-cohérence forte, v-cohérence et autres propriétés. Ainsi des exemples d'anneaux cohérents autres que les anneaux noethériens sont mis en exergue.

La partie arithmétique de cette thèse est consacrée à quelques notions liées à la notion de Bézout à savoir: (i) les anneaux à division élémentaire et les anneaux d'Hermite, (ii) les anneaux presque de Bézout et les anneaux presque de valuation et (iii) les 2-anneaux de Bézout et les P-anneaux de Bézout.
(i) Dans [40], il est démontré qu' un anneau R est un anneau d'Hermite si et seulement si pour tous $a, b \in R$, il existe $a_1, b_1, d \in R$ tel que $a = a_1 d$, $b = b_1 d$, et $Ra_1 + Rb_1 = R$. Notons que tout anneau à division élémentaire est un anneau d'Hermite et tout anneau d'Hermite est un anneau de Bézout. Aussi, Kaplansky a démontré que tout anneau de valuation est un anneau à division élémentaire. Le diagramme suivant décrit la relation entre ces différentes notions
Anneau de valuation $\Longrightarrow$ Anneau à division élémentaire $\Longrightarrow$ Anneau d'Hermite $\Longrightarrow$ Anneau de Bézout.

(ii) Dans [5], Anderson et Zaffrullah ont introduit et étudié la notion d'anneaux presque de valuation (AV-anneau) comme étant un anneau R tel que pour tous éléments a et b dans R, il existe un entier strictement posétif n tel que a^n divise b^n ou b^n divise a^n. Aussi, ils ont introduit la notion d'anneau presque de Bézout (AB-anneau) comme étant un anneau R tel que pour tous éléments a et b dans R, il existe un entier strictement posétif n tel que l'idéal (a^n, b^n) est principal. Entre autres, ils ont démontré que la clôture intégrale d'un domaine presque de valuation (resp., presque de Bézout) est un domaine de valuation (resp., un domaine de Prüfer avec groupe de torsion).
(iii) Dans [11, 13], et comme généralisation de la notion d'anneau de Bézout, Chahrazade Bakkari et Khalid Ouarghi ont introduit et étudié les notions 2-anneaux de Bézout et P-anneaux de Bézout.

Avant la description des contributions de chaque chapitre, nous présentons d'abord quelques définitions, terminologies et notations. Tous les anneaux considérés sont supposés associatifs et unitaires et les modules sont unitaires.

Définition 1.

Un anneau R *est dit réduit s'il ne contient aucun élément nilpotent non nul.*

Définition 2.

Un anneau R *est dit semicommutative si pour tous* $a, b \in R$, $ab = 0$ *implique que* $aRb = 0$.

Définition 3.

- *Un anneau* R *est dit Armendariz si pour tous polynomes* $f(x) = a_0 + a_1x + \ldots + a_nx^n$, $g(x) = b_0 + b_1x + \ldots + b_mx^m$
satisfaisant $f(x)g(x) = 0$, *alors* $a_ib_j = 0$ *pour tous* i, j.
- *Un anneau* R *est dit nil-Armendariz si pour tout produit de deux polynomes* $f(x) = \sum_{i=0}^{i=n} a_ix^i$ *et* $g(x) = \sum_{j=0}^{j=m} b_jx^j$ *dans* $R[x]$ *satisfaisant* $f(x)g(x) \in nil(R)[x]$ *on a* $a_ib_j \in nil(R)$ *pour tous* i, j.
- *Un anneau* R *est dit weak-Armendariz si pour tout produit de deux polynomes*
$f(x) = \sum_{i=0}^{i=n} a_ix^i$ *et* $g(x) = \sum_{j=0}^{j=m} b_jx^j$ *in* $R[x]$ *satisfaisant* $f(x)g(x) = 0$, *on a* $a_ib_j \in nil(R)$ *pour tous* i, j.

Définition 4.

Un anneau R *est dit power serieswise Armendariz si pour tout produit de deux séries formelles* $f(x) = \sum_{i=0}^{\infty} a_ix^i$ *et* $g(x) = \sum_{i=0}^{\infty} b_ix^i \in R[[x]]$ *tel que* $fg = 0$, *alors* $a_ib_j = 0$ *pour tous* i *et* j.

Définition 5.

Un anneau R *est dit abelien si tout élément idempotent est centrale.*

Définition 6.

Soient R un anneau commutatif et M un R-module.

1. *Pour un entier positif n, M est dit n-présenté s'il existe une suite exacte de R-modules:*
$$F_n \to F_{n-1} \to \dots F_1 \to F_0 \to E \to 0$$
où chaque F_i est un R-module libre de type fini.

2. *M est dit un R-module cohérent s'il est de type fini et chaque sous-module de type fini de M est de présentation finie.*

Définition 7.

Soit R un anneau commutatif.

1. *R est dit un anneau cohérent si tout idéal de type fini de R est de présentaion finie.*

2. *R est dit un anneau Noethérien si tout idéal de R est de type fini.*

Définition 8.

Soit R un anneau commutative.

1. *R est dit anneau presque de valuation (AV-anneau en abréviation) si, pour tous éléments a et b dans R, il éxiste un entier positif n tel que a^n dévise b^n ou b^n dévise a^n.*
2. *R est dit anneau presque de Bézout (AB-anneau en abréviation) si, pour tous éléments a et b dans R, il éxiste un entier positif n tel que l'ideal (a^n, b^n) est principal.*
3. *R est dit anneau à division élémentaire si, pour toute matrice M sur R il éxiste des matrices invérsibles P, Q tel que PMQ est une matrice diagonale.*
4. *R est dit anneau d'Hermite si, pour toute matrice M sur R il éxiste une matrice inversible Q tel que MQ est une matrice triangulaire.*
5. *R est dit anneau de Bézout si tout idéal de type fini est principal.*

Définition 9.

Un anneau R est dit P-anneau de Bézout si tout idéal premier de type fini P de R est principal.

Définition 10.

Un anneau R est dit 2-anneau de Bézout si tout idéal de présentation finie I de R est principal.

Définition 11.

Un idéal I de R est dit régulier s'il contient un élément régulier; i.e, un élément non diviseur de zéro.

En 1956, Nagata a introduit la notion d'extension triviale d'un anneau par un module comme suit:

Définition 12.

Soient A un anneau et E un $A-$module. L'extension triviale de A par E est l'anneau $A \times E$ où l'addition est définie de façon naturelle et la multiplication est donnée par:

$$(a,e)(b,f) = (ab, af + be).$$

Cette notion a fait l'objet de plusieurs travaux dans la littérature et constitue un champ de recherche fertile dans la théorie des anneaux [42, 48]. Les extensions triviales d'anneaux ont été largement étudiées; et un travail considérable, dont une partie résumée dans le livre de Glaz [42] et le livre de Huckaba [48], a été concerné par ces extensions. Principalement, ces extension ont été utiles pour la résolution de nombreux problèmes ouverts et pour les conjectures en théorie d'anneaux commutatifs et non-commutatifs [4, 48, 53].

En 2006, Marco D'Anna et Marco Fontana [28] ont introduit une nouvelle construction dite la duplication amalgamée d'un anneau A le long d'un sous $A-$module E de $Q(A)$ (l'anneau total des fractions de A) vérifiant $E^2 \subseteq E$. Quand $E^2 = \{0\}$ cette construction coïncide avec l'extension triviale de A par E.
Récemment, dans [26, 29, 27, 30, 28], sont discutées les motivations ainsi que les applications de cette duplication amalgamée $A \bowtie E$ de A le long du sous $A-$module E de $Q(A)$, essentiellement dans le cas particulier où E est un idéal de A.
En 2010, une extension de la construction de la duplication amalgamée $A \bowtie I$ de A le long de l'idéal I de A a été définie dans [29] comme suit:

Définition 13.

Soient A et B deux anneaux, J un idéal de B et soit $f : A \to B$ un homomorphisme d'anneaux. On appelle l'amalgamation de A et B suivant J et respectant f, le sous-anneau de $A \times B$:

$$A \bowtie^f J := \{(a, f(a) + j) \mid a \in A, j \in J\}$$

Cette construction est une généralisation de la duplication amalgamée d'un anneau le long d'un idéal (présenté et étudié par D'Anna et Fontana dans [26, 27, 28]). Par ailleurs, d'autres constructions classiques (tels que $A + XB[X]$, $A + XB[[X]]$, et les constructions $D + M$) peuvent être étudiées comme des cas particuliers de l'amalgamation ([29, Exemples 2.5 et 2.6]). D'autres constructions classiques, telle que l'idéalisation du Nagata, également appelé l'anneau extension trivial (cf. [66, page 2]), et les extensions CPI (dans le sens de Boisen et Sheldon [19]) sont strictement liées ([29, Exemple 2.7 et Remarque

2.8]). Voir, par exemple [29, 30, 27, 28].

En 2013, dans [52] Kabbaj, Tamekkante et Louartiti ont introduit une extension de la construction amalgamée algébrique $A \bowtie J$ de A le long de l'idéal J de B comme suit:

Définition 14.

Soient $f : A \longrightarrow B$ et $g : A \longrightarrow C$ deux homomorphismes d'anneaux et soit J (resp.,J') un idéal de B (resp.,C) tel que $f^{-1}(J) = g^{-1}(J')$. On considère le sous anneau de $B \times C$ suivant :

$$A \bowtie^{f,g} (J,J') := \{(f(a)+j, g(a)+j') | a \in A, j \in J, j' \in J'\}$$

appelé la bi-amalgamation de A avec (B,C) le long de (J,J') par rapport à (f,g).

Cette construction est une généralisation de l'amalgamaé algébrique le long d'un idéal (introduite et étudiée par D'Anna et Fontana dans [29, 30, 27, 28] dans le sens où $A \bowtie^{f} J = A \bowtie^{Id,f} (f^{-1}(J), J)$. Dans [52], les autheurs ont étudiés les propriértés de bases de cette construction (e.g., la caractérisation de $A \bowtie^{f,g} (J,J')$ pour être un anneau Noethérien, un domain, un anneau reduit ...).

Soient R un anneau et M un R-module. Nous utiliserons les notations suivantes:
- qf(R) le corps des fractions de R.
- Nil(R), l'ensemble des éléments nilpotents de R.
- Rad(R), le radial de Jacobson de R.
- Spec(R), l'ensemble des idéaux premiers de R.
- Max(R), l'ensemble des idéaux maximaux de R.
- $R[X]$, l'anneau des polynomes à une indéterminé X sur R.
- $R[[X]]$, l'anneau des séries formelles à une indéterminé X sur R.
- An(M), l'annulateur de M.
- $M_n(R)$, l'ensemble des matrices carrées d'ordre n à coefficients dans R.
- $GL_n(R)$, l'ensemble des matrices carrées inversibles d'ordre n à coefficients dans R.
- $(I : J)(= \{x \in R / xJ \subseteq I\})$, l'idéal quotient, où I et J sont des idéaux de l'anneau R.

Chapitre 1
Les propriétés Armendariz dans l'amalgamé algébrique le long d'un idéal

Dans ce chapitre*, on considère l'homomorphisme d'anneaux $f : A \to B$ et J un idéal de B. Nous étudions le transfert de trois notions, à savoir "anneau Armendariz","anneau nil-Armendariz" et "anneau weak-Armendariz" à l'amalgamation de A avec B le long d'un idéal propre J de B par rapport à f (notée $A \bowtie^f J$). L'objectif est d'étudié les conditions nécéssaires et suffisentes que $A \bowtie^f J$, doit vérifier pour hériter les propriétés citées ci-dessus afin de fournir de nouvelles classes d'anneaux Armendariz, nil-Armendariz et weak-Armendariz.

Nous commençons par la proposition suivantes qui caractérise quand est ce que l'amalgamé algébrique $A \bowtie^f J$ est un anneau reduit.

Proposition 15.

([29, Proposition 5.4]) Soient (A,B) un couple d'anneaux, $f : A \to B$ un homomorphisme d'anneaux et J un idéal propre de B. Les conditions suivantes sont équivalentes:

1. *$A \bowtie^f J$ est un anneau réduit.*
2. *A est un anneau réduit et $nil(B) \cap J = (0)$*

En particulier, si A et B sont réduits, alors $A \bowtie^f J$ est réduit; inversement, si J est un idéal radical de B et $A \bowtie^f J$ est réduit, alors B (et A) est réduit.

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Le théorème suivant examine les conditions nécéssaires et suffisantes sous lesquelles l'amalgamé algébrique $A \bowtie^f J$ est un anneau Armendariz.

Théorème 16.

Soient (A,B) un couple d'anneaux, $f : A \to B$ un homomorphisme d'anneaux et J un idéal propre de B.

1. *Si $A \bowtie^f J$ est un anneau Armendariz, alors A l'est aussi.*
2. *Si A et $f(A)+J$ sont des anneaux Armendariz, alors $A \bowtie^f J$ l'est aussi.*
3. *Supposons que $J \cap S \neq \emptyset$, où S est l'ensemble des éléments réguliers centrales de B. Alors $A \bowtie^f J$ est un anneau Armendariz si et seulement si $f(A)+J$ et A sont des anneaux Armendariz.*
4. *Supposons que $J \cap nil(B) = (0)$. Alors $A \bowtie^f J$ est un anneau Armendariz si et seulement si A l'est aussi.*
5. *Supposons que $f^{-1}(J) \cap nil(A) = (0)$. Si $f(A)+J$ est un anneau Armendariz, alors $A \bowtie^f J$ l'est aussi.*

Le théorème suivant étudie le transfert de la propriété nil-Armendariz dans l'amalgamé algébrique.

Théorème 17.

Soient (A,B) un couple d'anneaux, $f : A \to B$ un homomorphisme d'anneaux et J un idéal propre de B, alors:

1. *Si $A \bowtie^f J$ est un anneau nil-Armendariz, alors A l'est aussi.*
2. *Si A et $f(A)+J$ sont des anneaux nil-armendariz, alors $A \bowtie^f J$ l'est aussi.*
3. *Supposons que $J \cap S \neq \emptyset$, où S est l'ensemble des éléments réguliers centrales de B. Alors $A \bowtie^f J$ est un anneau nil-Armendariz si et seulement si $f(A)+J$ et A sont des anneaux nil-Armendariz.*
4. *Supposons que $J \subseteq nil(B)$. Alors $A \bowtie^f J$ est un anneau nil-Armendariz si et seulement si A est un anneau nil-Armendariz.*
5. *Supposons que $f^{-1}(J) \subseteq nil(A)$. Alors $A \bowtie^f J$ est un anneau nil-Armendariz si et seulement si $f(A)+J$ est un anneau nil-Armendariz.*
6. *Supposons que f est injectif.*

    i) $f(A) \cap J = 0$. Alors $A \bowtie^f J$ est un anneau nil-Armendariz si et seulement si $f(A)+J$ l'est aussi.

    ii) $J \subseteq nil(B)$. Alors $A \bowtie^f J$ est un anneau nil-Armendariz si et seulement si $f(A)+J$ l'est aussi.

Le théorème suivant établit les conditions nécéssaires et suffisantes sous lesquelles l'amalgamé algébrique $A \bowtie^f J$ est un anneau weak-Armendariz.

Théorème 18.

Soient (A,B) *un couple d'anneaux,* $f : A \to B$ *un homomorphisme d'anneaux et* J *un idéal propre de* B*, alors*

1. *Si* $A \bowtie^f J$ *est un anneau weak-Armendariz, alors* A *l'est aussi.*
2. *Si* A *et* $f(A)+J$ *sont des anneaux weak-Armendariz, alors* $A \bowtie^f J$ *l'est aussi.*
3. *Supposons que* $J \cap S \neq \emptyset$*, où* S *est l'ensemble des éléments réguliers centrales de* B*. Alors* $A \bowtie^f J$ *est un anneau weak-Armendariz si et seulement si* $f(A)+J$ *et* A *sont des anneaux weak-Armendariz.*
4. *Supposons que* $J \subseteq nil(B)$*. Alors* A *est un anneau weak-Armendariz si et seulement si* $A \bowtie^f J$ *l'est aussi.*
5. *Supposons que* $f^{-1}(J) \subseteq nil(A)$*. Si* $f(A)+J$ *est un anneau weak-Armendariz, alors* $A \bowtie^f J$ *l'est aussi.*
6. *Supposons que* f *est injectif.*

    i) $f(A) \cap J = 0$*. Alors* $A \bowtie^f J$ *est un anneau weak-Armendariz si et seulement si* $f(A)+J$ *est un anneau weak-Armendariz.*

    ii) $J \subseteq nil(B)$*. Si* $f(A)+J$ *est un anneau weak-Armendariz, alors* $A \bowtie^f J$ *l'est aussi.*
7. *Supposons que* J *est semicommutatif. Si* A *est un anneau weak-Armendariz, alors* $A \bowtie^f J$ *l'est aussi.*
8. *Supposons que* $f^{-1}(J)$ *est semicommutatif. Si* $f(A)+J$ *est un anneau weak-Armendariz, alors* $A \bowtie^f J$ *l'est aussi.*

Chapitre 2
L'étude de la propriété power serieswise Armendariz

Le but de ce chapitre* est d'étudier le transfert de la propriété power serieswise Armendariz à l'extension triviale, au produit direct des anneaux et à l'image homomorphe. Nous commençons tout d'abord par l'étude du transfert de la propriété power serieswise Armendariz à l'extension triviale.

Soit A un anneau, nous affirmons que l'anneau des matrices triangulaires supérieure d'ordre n sur A n'est pas power serieswise Armendariz, où $n \geq 2$. Il suffit de montrer que l'anneau des matrices triangulaires supérieure d'ordre 2 sur A n'est pas power serieswise Armendariz, car tout sous anneau d'un anneau power serieswise Armendariz est aussi power serieswise Armendariz. Soient S l'anneau des matrices triangulaires sup d'ordre 2 sur A, $f(x) = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} 1 & -1 \\ 0 & 0 \end{pmatrix} x$, et $g(x) = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} + \begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix} x$ deux polynomes de $S[[x]]$. Alors, on a $f(x)g(x) = 0$, mais $\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix} \neq 0$. Donc S n'est pas power serieswise Armendariz et par conséquent tout anneau des matrices triangulaires sup d'ordre n sur A n'est pas power serieswise Armendariz. Mais on peut trouver un sous anneau de l'anneau des matrices triangulaires supérieure d'ordre 3 sur A qui est power serieswise Armendariz comme suit.

Proposition 19.

Soit A un anneau réduit. Alors
S=$\left\{ \begin{pmatrix} a & b & c \\ 0 & a & d \\ 0 & 0 & a \end{pmatrix} / a,b,c,d \in A \right\}$ *est un anneau power serieswise Armendariz.*

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* Ce travail est soumis pour publication (en collaboration avec N. Mahdou).

Soit S un anneau réduit et soit $A_n = \{\begin{pmatrix} a & a_{12} & a_{13} & . & . & . & a_{1n} \\ 0 & a & a_{23} & . & . & . & a_{2n} \\ 0 & 0 & a & . & . & . & a_{3n} \\ . & . & . & . & . & . & . \\ . & . & . & . & . & . & . \\ 0 & 0 & 0 & . & . & . & a \end{pmatrix} / a, a_{ij} \in S\}$. D'après la Proposition 19, on peut estimer que A_n peut aussi être power serieswise Armendariz pour $n \geq 4$. Mais l'exemple suivant élimine cette possibilité.

Exemple 20.

Soit S un anneau. Alors $A_4 = \{\begin{pmatrix} a & a_{12} & a_{13} & a_{14} \\ 0 & a & a_{23} & a_{24} \\ 0 & 0 & a & a_{34} \\ 0 & 0 & 0 & a \end{pmatrix} / a, a_{ij} \in S\}$ *n'est pas power serieswise Armendariz.*

Etant donné un anneau A et un bi-module ${}_AE_A$, on a $A \propto E$ est isomorphe à l'anneau des matrices $\begin{pmatrix} r & m \\ 0 & r \end{pmatrix}$, où $r \in A$ et $m \in E$.

Corollaire 21.

Soit A un anneau réduit. Alors l'extension triviale $A \propto A$ est un anneau power serieswise Armendariz.

D'après le corollaire 21, on peut estimer que si A est power serieswise Armendariz alors $A \propto A$ est power serieswise Armendariz. Mais l'exemple suivant élimine cette possibilité.

Exemple 22.

Soit T un anneau réduit. Alors $R = \{\begin{pmatrix} r & m \\ 0 & r \end{pmatrix} / r, m \in T\}$ est power serieswise Armendariz d'après le Corollaire 21. Soit $S = \{\begin{pmatrix} A & B \\ 0 & A \end{pmatrix} / A, B \in R\}$ et soient

$$f(x) = \begin{pmatrix} \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} & \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \\ \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} & \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \end{pmatrix} + \begin{pmatrix} \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} & \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} \\ \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} & \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \end{pmatrix} x \quad \text{et,}$$

$$g(x) = \begin{pmatrix} \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} & \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \\ \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} & \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \end{pmatrix} + \begin{pmatrix} \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} & \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \\ \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} & \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \end{pmatrix} x \quad \text{deux polynomes dans}$$

$S[[x]]$. Alors $f(x)g(x) = 0$, et

$$\begin{pmatrix} \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} & \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \\ \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} & \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \end{pmatrix} \begin{pmatrix} \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} & \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \\ \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} & \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \end{pmatrix} \neq 0\,.$$

Donc S n'est pas power serieswise Armendariz.

Mais on a une réponse affirmative pour cette situation, en adoptant la condition "(A, M) est un anneau local et E un bi-module tel que $ME = 0$ et $EM = 0$".

Théorème 23.

Soient A un anneau, E un bimodule non nul. Alors: Supposons que (A, M) est un anneau local et E un bimodule tel que $ME = 0$ et $EM = 0$. Alors, $A \propto E$ est un anneau power serieswise Armendariz si et seulement si A l'est aussi.

Exemple 24.

Soit K un corps, $K[[x]]$ est un anneau local power serieswise Armendariz et (x) est l'unique idéal maximal. Alors, $K[[x]] \propto K$ est power serieswise Armendariz (qui n'est jamais réduit) par le théorème 23 puisque $(x)S = 0$, $S(x) = 0$ et $K[[x]]$ est power serieswise Armendariz, où $S = K[[x]]/(x) \simeq K$.

Maintenant, nous étudions le transfert de la propriété power serieswise Armendariz au produit direct des anneaux.

Théorème 25.

Soit $(A_i)_{i=1,2,\dots,n}$ une famille d'anneaux et soit $A := \prod_{i=1}^{n} A_i$. Alors, A est un anneau power serieswise Armendariz si et seulement si A_i est un anneau power serieswise Armendariz pour chaque $i = 1, \dots, n$.

Corollaire 26.

Soit A un anneau et soit $n \in \mathbb{N} - \{0\}$ un entier naturel. Alors, A^n est power srieswise Armendariz si et seulement si A l'est aussi.

l'exemple suivant montre que l'implication "A/I et I sont power serieswise Armendariz implique que A l'est aussi (où I est un idéal de A)" est fausse, en général.

Exemple 27.

Soit F un corps, considérons l'anneau $A = \begin{pmatrix} F & F \\ 0 & F \end{pmatrix}$. Alors A n'est pas Armendariz (d'après [56, Exemples 1]) est donc A n'est pas power serieswise Armendariz. Maintenant, montrons que A/I et I sont power serieswise Armendariz pour tout idéal I de A. Nottons que les seuls idéaux propres non nul de A sont $\begin{pmatrix} F & F \\ 0 & 0 \end{pmatrix}$, $\begin{pmatrix} 0 & F \\ 0 & F \end{pmatrix}$ et $\begin{pmatrix} 0 & F \\ 0 & 0 \end{pmatrix}$. Premièrement, soit $I = \begin{pmatrix} F & F \\ 0 & 0 \end{pmatrix}$. Alors $A/I \simeq F$ et donc A/I est power serieswise Armendariz.
Reste à montrer que I est power serieswise Armendariz. Soient $f(x) = \sum_{i=0}^{\infty} \alpha_i x^i$, $g(x) = \sum_{j=0}^{\infty} \alpha_j x^j$ dans $I[[x]]$ tel que $f(x)g(x) = 0$ et on pose $\alpha_i = \begin{pmatrix} a_i & b_i \\ 0 & 0 \end{pmatrix}$ et $\beta j = \begin{pmatrix} c_j & d_j \\ 0 & 0 \end{pmatrix}$. Supposons que$\alpha_0 \neq 0$ et $\beta_0 \neq 0$. Alors $a_0c_0 = a_0d_0 = 0$. Si $a_0 \neq 0$, alors $c_0 = 0$ et $d_0 = 0$, contradiction. Donc $a_0 = 0$ et par suite $b_0 \neq 0$. Ce qui implique que $\alpha_0\beta_j = 0$ pour tout j. Donc le coéficient de x dans $f(x)g(x) = 0$ est $\alpha_1\beta_0 = 0$. Alors $a_1c_0 = a_1d_0 = 0$. Si $a_1 \neq 0$, alors $c_0 = 0$ et $d_0 = 0$, contradiction. Donc $\alpha_1\beta_j = 0$ pour tout j.
Continuons cette procédure, on montre que $\alpha_i\beta_j = 0$ pour tout i, j. Par conséquent, I est power serieswise Armendariz.
Maintenant, soit $J = \begin{pmatrix} 0 & F \\ 0 & F \end{pmatrix}$. Alors $A/J \simeq F$ et donc A/J est power serieswise Armendariz. Par la même méthode, on a J est power serieswise Armendariz.
Finalement, soit $K = \begin{pmatrix} 0 & F \\ 0 & 0 \end{pmatrix}$. Alors $A/K \simeq F \bigoplus F$ et donc A/K est power serieswise Armendariz. Aussi $K^2 = 0$ et donc K est power serieswise Armendariz.

Sous la condition "*I* est réduit", on montre qu'on a une réponse affirmative pour l'implication ci-dessus.

Théorème 28.

Soit I un idéal réduit d'un anneau A tel que A/I est power serieswise Armendariz. Alors A est power serieswise Armendariz.

Le corollaire suivant est une conséquence du théorème 28

Corollaire 29.

Soit A un anneau. Alors:

1. *A est power serieswise Armendariz si et seulement si $A[x]$ l'est aussi.*
2. *A est power serieswise Armendariz si et seulement si $A[[x]]$ l'est aussi.*

Dans un anneau contenant un idempotent central, on a:

Théorème 30.

Soit A un anneau contenant un idempotent central e. Alors, A est power serieswise Armendariz si et seulement si eA et $(1-e)A$ le sont.

Corollaire 31.

Dans un anneau abelian A, les assertions suivantes sont équivalentes:

1. *A est power serieswise Armendariz.*
2. *eA et $(1-e)A$ sont power serieswise Armendariz pour tout idempotent e de A.*
3. *eA et $(1-e)A$ sont power serieswise Armendariz pour un certain idempotent e de A.*

Chapitre 3
La propriété power serieswise Armendariz dans l'amalgamé algébrique

Dans ce chapitre*, on considère un homomorphisme $f : A \rightarrow B$ et J un idéal propre de B. Nous étudions le transfert de la properiété power serieswise Armendariz à l'amalgamation $A \bowtie^f J$. Nous établissons les conditions nécéssaires et suffisantes pour que $A \bowtie^f J$ soit un anneau power serieswise Armendariz.

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* Ce travail est accepté pour publication dans Bulletin of the Iranian Mathematical Society (en collaboration avec N. Mahdou).

Théorème 32.

Soient (A,B) un couple d'anneaux, $f : A \to B$ un homomorphisme d'anneaux et J un idéal propre de B. Alors:

1. *Si $A \bowtie^f J$ est un anneau power serieswise Armendariz alors A l'est aussi.*
2. *Si A et $f(A)+J$ sont des anneaux power serieswise Armendariz alors $A \bowtie^f J$ l'est aussi.*
3. *Supposons que $J \cap S \neq \emptyset$, où S est l'ensemble des éléments réguliers centrals de B. Alors:*
   $A \bowtie^f J$ est un anneau power serieswise Armendariz si et seulement si $f(A)+J$ et A sont des anneaux power serieswise Armendariz.
4. *Supposons que $f^{-1}(J) \cap nil(A) = (0)$. Alors:*
   Si $f(A)+J$ est un anneau power serieswise Armendariz alors $A \bowtie^f J$ l'est aussi.
5. *Supposons que $J \cap nil(B) = (0)$. Alors:*
   $A \bowtie^f J$ est un anneau power serieswise Armendariz si et seulement si A l'est aussi.
6. *Supposons que f est injectif et $f(A) \cap J = (0)$. Alors:*
   $A \bowtie^f J$ est un anneau power serieswise Armendariz si et seulement si $f(A)+J$ est un anneau power serieswise Armendariz.

l'exemple suivant montre que l'implication A est power serieswise Armendariz implique que $A \bowtie^f J$ est power serieswise Armendariz est fausse, en général.

Exemple 33.

Soient K un corps, $B = K \ltimes K$ l'extension triviale de K par K, $R = B \ltimes B$, $J = 0 \ltimes B$ et $f : B \longrightarrow R$ défini par $f(b) = (b,0)$.

Nottons que $R = B \ltimes B$ est isomorphe à $\{\begin{pmatrix} b & c \\ 0 & b \end{pmatrix}, b,c \in B\}$ et $R = B \ltimes B = B \bowtie^f J$. Alors:

1. *B est un anneau power serieswise Armendariz.*
2. *$R = B \bowtie^f J$ n'est pas un anneau power serieswise Armendariz.*

De même, l'exemple suivant montre que l'implication $f(A)+J$ est power serieswise Armendariz implique que $A \bowtie^f J$ est power serieswise Armendariz est fausse, en général.

Exemple 34.

Soit A un anneau semicommutatif qui n'est pas Armendariz.
Nottons que A est semicommutatif implique que $nil(A)$ *est un idéal de A d'après [62, Lemma 3.1]. Soient* $f : A \longrightarrow A/nil(A)$ *la surjection canonique, I un idéal de A contenant* $nil(A)$ *et* $J = I/nil(A)$*. Alors:*

1. $A \bowtie^f J$ *n'est pas power serieswise Armendariz.*
2. $f(A)+J$ *est un anneau power serieswise Armendariz.*

Maintenant, on construit une nouvelle classe d'anneaux power serieswise Armendariz.

Exemple 35.

Soient A un anneau power serieswise Armendariz tel que $nil(A)$ *est un idéal,* $f : A \longrightarrow A/nil(A)$ *la surjection canonique et* $J = I/nil(A)$*, où I est un idéal de A contenant* $nil(A)$*. Alors:* $A \bowtie^f J$ *est un anneau power serieswise Armendariz.*

Exemple 36.

Soient A un anneau power serieswise Armendariz, I un idéal premier de A, $f : A \longrightarrow A/I = B$ *la surjection canonique, M un idéal maximal de A tel que* $I \subseteq M$ *et* $J = M/I$*. Alors:*

1. A/I *est power serieswise Armendariz.*
2. $A \bowtie^f J$ *est power serieswise Armendariz.*

Chapitre 4
La cohérence dans la bi-amalgamation algébrique

Dans ce chapitre*, soient $f : A \longrightarrow B$ et $g : A \longrightarrow C$ deux homomorphismes d'anneaux et soit J (resp., J') un idéal de B (resp., C) tel que $f^{-1}(J) = g^{-1}(J')$. Nous traitons le transfert de la properiété de la cohérence dans la bi-amalgamation de A avec (B,C) le long des idéaux (J,J') par rapport à (f,g) (notée $A \bowtie^{f,g} (J,J')$), introduite et étudiée par Kabbaj, Louartiti et Tamekkante en 2013. Nous établissons les conditions nécéssaires et suffisantes pour que $A \bowtie^{f,g} (J,J')$ soit un anneau cohérent, afin de fournir de nouvelles classes d'anneaux commutatifs cohérents qui ne sont pas Noethériens.

Avant d'énnoncer le résultat principal (Theorem 38), citons la remarque suivante.

Remarque 37.

1. *Soit* $f : A \longrightarrow B$ *un homomorphisme d'anneaux et soit* J *un idéal de* B. *Alors* $f^n(\alpha a) = f(\alpha) f^n(a)$ *pour tous* $\alpha \in A$ *et* $a \in A^n$.
2. *Si* $f^{-1}(J) = g^{-1}(J') = 0$, *alors* A *est un module retracté de* $A \bowtie^{f,g} (J,J')$.
3. *Si* g *est injectif et* $J' \subseteq g(A)$, *alors* A *est un module retracté de* $A \bowtie^{f,g} (J,J')$.

Maintenant, le résultat principal:

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* Ce travail est accepté pour publication dans Springer Proceedings in Mathematics and Statistics (en collaboration avec N. Mahdou).

Théorème 38.

Soient $f : A \longrightarrow B$ *et* $g : A \longrightarrow C$ *deux homomorphismes d'anneaux et soit* J *(resp.,* J'*) un idéal propre de* B *(resp.,* C*) tel que* $f^{-1}(J) = g^{-1}(J')$.

1. *Supposons que* J, J' *sont des idéaux de type fini de* $f(A)+J$ *et* $g(A)+J'$ *respectivement, et* $J \subseteq f(A)$*. Alors* $A \bowtie^{f,g} (J,J')$ *est un anneau cohérent si et seulement si* $f(A)+J$ *et* $g(A)+J'$ *sont des anneaux cohérents.*
2. *Supposons que* J, J' *sont des idéaux de type fini de* $f(A)+J$ *et* $g(A)+J'$ *respectivement, et* $J^2 = 0$*. Alors* $A \bowtie^{f,g} (J,J')$ *est un anneau cohérent si et seulement si* $f(A)+J$ *et* $g(A)+J'$ *sont des anneaux cohérents.*
3. *Supposons que* J, J' *sont des idéaux réguliers de* $f(A)+J$ *et* $g(A)+J'$ *respectivement, et* $J \subseteq f(A)$*. Alors* $A \bowtie^{f,g} (J,J')$ *est un anneau cohérent si et seulement si* $f(A)+J$ *et* $g(A)+J'$ *sont des anneaux cohérents et* J*,* J' *sont des idéaux de type fini de* $f(A)+J$ *et* $g(A)+J'$*, respectivement.*
4. *Supposons que* J *est un idéal régulier de type fini de* $f(A)+J$ *et* $J' \subseteq g(A)$*. Alors* $A \bowtie^{f,g} (J,J')$ *est un anneau cohérent si et seulement si* $f(A)+J$ *et* $g(A)+J'$ *sont des anneaux cohérents et* J' *est un idéal de type fini de* $g(A)+J'$.
5. *Supposons que* J *est un idéal régulier de type fini de* $f(A)+J$ *et* $J'^2 = 0$*. Alors* $A \bowtie^{f,g} (J,J')$ *est un anneau cohérent si et seulement si* $f(A)+J$ *et* $g(A)+J'$ *sont des anneaux cohérents et* J' *est un idéal de type fini de* $g(A)+J'$.

La preuve du théorème 38, nécéssite les lemmes suivants.

Lemme 39.

Soient $f : A \longrightarrow B$ *et* $g : A \longrightarrow C$ *deux homomorphismes d'anneaux et soit* J *(resp.,* J'*) un idéal propre de* B *(resp.,* C*) tel que* $f^{-1}(J) = g^{-1}(J')$*. Alors:*

1. $\{0\} \times J'$ *(resp.,* $J \times \{0\}$ *) est un idéal de type fini de* $A \bowtie^{f,g} (J,J')$ *si et seulement si* J' *(resp.,* J*) est un idéal de type fini de* $g(A)+J'$ *(resp.,* $f(A)+J$ *).*
2. *Si* $A \bowtie^{f,g} (J,J')$ *est un anneau cohérent et* J*,* J' *sont des idéaux de type fini de* $f(A)+J$ *et* $g(A)+J'$ *respectivement, alors* $f(A)+J$ *et* $g(A)+J'$ *sont des anneaux cohérents.*

Lemme 40.

Soient $f : A \longrightarrow B$ *et* $g : A \longrightarrow C$ *deux homomorphismes d'anneaux et soit* J *(resp.,* J'*) un idéal de* B *(resp.,* C*) tel que* $f^{-1}(J) = g^{-1}(J')$ *et soit* U *un sous module de* A^n*. Alors:*
Supposons que U *est un* $A-$*module de type fini et* J *,*J' *sont des idéaux de type fini de* $f(A)+J$ *et* $g(A)+J'$*, respectivement. Alors* $U \bowtie^{f^n,g^n} (J^n,J'^n)$ *est un* $A \bowtie^{f,g} (J,J')-$ *module de type fini.*

Lemme 41.

Soient $f : A \longrightarrow B$ *et* $g : A \longrightarrow C$ *deux homomorphismes d'anneaux et soit* J *(resp.,* J'*) un idéal de* B *(resp.,* C*) tel que* $f^{-1}(J) = g^{-1}(J')$*. Supposons que* J *(resp.,* J'*) est un idéal de type fini de* $f(A)+J$ *(resp.,* $g(A)+J'$*) et* $J \subseteq f(A)$*. Alors* $J \times \{0\}$ *est un* $(A \bowtie^{f,g} (J,J'))$*-module cohérent sachant que* $f(A)+J$ *est un anneau cohérent.*

Lemme 42.

Soient $f : A \longrightarrow B$ *et* $g : A \longrightarrow C$ *deux homomorphismes d'anneaux et soit* J *(resp.,* J'*) un idéal de* B *(resp.,* C*) tel que* $f^{-1}(J) = g^{-1}(J')$*. Supposons que* J *(resp.,* J'*) est un idéal de type fini de* $f(A)+J$ *(resp.,* $g(A)+J'$*) et* $J^2 = 0$*. Alors* $J \times \{0\}$ *est un* $(A \bowtie^{f,g} (J,J'))$*-module cohérent sachant que* $f(A)+J$ *est un anneau cohérent.*

Lemme 43.

Soient $f : A \longrightarrow B$ *et* $g : A \longrightarrow C$ *deux homomorphismes d'anneaux et soit* J *(resp.,* J'*) un idéal de* B *(resp.,* C*) tel que* $f^{-1}(J) = g^{-1}(J')$*.*

1. *Si* $(A \bowtie^{f,g} (J,J'))$ *est un anneau cohérent et* J *est régulier, alors* J' *est un idéal de type fini de* $g(A)+J'$*.*
2. *Supposons que* J, J' *sont des idéaux réguliers de* $f(A)+J$ *et* $g(A)+J'$*, respectivement. Alors:*
   Si $A \bowtie^{f,g} (J,J')$ *est un anneau cohérent alors* $f(A)+J$ *et* $g(A)+J'$ *le sont.*

Le théorème 38 recouvre le cas particulier de l'amalgamé algébrique [1], comme suit.

Corollaire 44.

Soit $f : A \longrightarrow B$ *un homomorphisme d'anneau et soit* J *un idéal propre de* B*.*

1. *Si* $A \bowtie^{f} J$ *est un anneau cohérent, alors* A *l'est aussi.*
2. *Supposons que* J *et* $f^{-1}(J)$ *sont des idéaux de type fini de* $f(A)+J$ *et* A*, respectivement. Alors* $A \bowtie^{f} J$ *est un anneau cohérent si et seulement si* $f(A)+J$ *et* A *sont des anneaux cohérents.*
3. *Supposons que* J *est un idéal régulier de type fini de* $f(A)+J$*. Alors* $A \bowtie^{f} J$ *est un anneau cohérent si et seulement si* $f(A)+J$ *et* A *sont des anneaux cohérent et* $f^{-1}(J)$ *est un idéal de type fini de* A*.*

Le corollaire suivant est une conséquence immédiate du théorème 38 (3)(4).

Corollaire 45.

Soient $f : A \longrightarrow B$ *et* $g : A \longrightarrow C$ *deux homomorphismes d'anneaux et soit* J *(resp.,* J'*) un idéal de* B *(resp.,* C*) tel que* $f^{-1}(J) = g^{-1}(J')$.

1. *Si* B *est un domaine intègre,* J *un idéal de type fini de* $f(A)+J$ *et* $J'^2 = 0$*, alors:*
   $A \bowtie^{f,g} (J,J')$ *est un anneau cohérent si et seulement si* $f(A)+J$ *et* $g(A)+J'$ *sont des anneaux cohérents et* J' *est un idéal de type fini de* $g(A)+J'$.

2. *Si* B *et* C *sont des domaines intègres et* $J \subsetneq f(A)$*, alors:*
   $A \bowtie^{f,g} (J,J')$ *est cohérent si et seulement si* $f(A)+J$ *et* $g(A)+J'$ *sont cohérents et* J*,* J' *sont des idéaux de type fini de* $f(A)+J$ *et* $g(A)+J'$*, respectivement.*

Le résultat ci-dessus enrichit la littérature avec de nouveaux exemples d'anneaux cohérents qui ne sont pas Noethériens.

Exemple 46.

Soient A *un anneau cohérent non-Noethérien,* I *et* K *deux idéaux de type fini de* A *tel que* $I \subsetneq K$*. Soient* $f : A \to A/I$ *l'homomorphisme canonique et* $g : A \to A \times A$ *l'homomorphisme injectif définit par* $g(a) = (a,0)$*,* $J = K/I$ *et* $J' = K \times 0$*. Alors* $A \bowtie^{f,g} (J,J')$ *est un anneau cohérent non-Noethérien.*

Exemple 47.

Soient (A,M) *un anneau local cohérent non-Noethérien tel que* M *est un idéal de type fini,* E *un* A/M*-espace vectoriel de rang fini. Soient* $f : A \to A \ltimes E$ *l'homomrphisme injectif définit par* $f(a) = (a,0)$ *et* $g : A \to (A/M)[X_1,X_2,\ldots,X_n]$ *définit par* $g(a) = \overline{a}$*,* $J = 0 \ltimes E$ *et* $J' = (X_1,X_2,\ldots,X_n)$*. Alors* $A \bowtie^{f,g} (J,J')$ *est un anneau cohérent non-Noethérien.*

Exemple 48.

Soient $A=\mathbb{Z}[X]$, $B=\mathbb{Z}+X\mathbb{Q}[X]$, $C=\mathbb{Z}$ *et soient* $J=n_0\mathbb{Z}+X\mathbb{Q}[X]$, $J'=n_0\mathbb{Z}$ *deux idéaux de* B *et* C*, respectivement.* $f:A\longrightarrow B$ *l'homomorphisme définit par* $f(P(X))=P(X)$ *et* $g:A\longrightarrow C$ *l'homomorphisme définit* $f(P(x))=P(0)$*. Alors* $A\bowtie^{f,g}(J,J')$ *est un anneau cohérent non-Noethériean .*

Exemple 49.

Soient $A=\mathbb{Z}[X]$, $B=\mathbb{Z}+X\mathbb{Q}[X]$, $C=\mathbb{Z}+i\mathbb{Z}[i]=\mathbb{Z}[i]$ *et soient* $J=n_0\mathbb{Z}+X\mathbb{Q}[X]$, $J'=n_0\mathbb{Z}+i\mathbb{Z}[i]$ *deux idéaux de* B *et* C*, respectivement.* $f:A\longrightarrow B$ *l'homomorphisme définit par* $f(P(X))=P(0)$ *et* $g:A\longrightarrow C$ *l'homomorphisme défini par* $g(P(x))=P(i)$*. Alors* $A\bowtie^{f,g}(J,J')$ *est un anneau cohérent non-Noethériean .*

Chapitre 5
Les propriétés de Bézout dans la bi-amalgamation algébrique

Dans ce chapitre[*], soient $f : A \longrightarrow B$ et $g : A \longrightarrow C$ deux homomorphismes d'anneaux et soit J (resp., J') un idéal de B (resp., C) tel que $f^{-1}(J) = g^{-1}(J')$. Nous étudions le transfert des notions: anneau à division élémentaire, domaine au plus de Bézout (AB-domain) et domaine au plus de valuation (AV-domain) à la bi-amalgamation de A avec (B,C) le long des idéaux (J,J') par rapport à (f,g) (notée par $A \bowtie^{f,g} (J,J')$, introduite et étudiée par Kabbaj, Louartiti and Tamekkante en 2013.

Le théorème suivant examine les conditions nécéssaires et suffisantes pour la bi-amalgamation $A \bowtie^{f,g} (J,J')$ pour hériter la notion d'anneau à division élémentaire. Ensuite, nous établissons la relation entre cette notion et la notion d'anneau d'Hermite, et anneau de Bézout.

Soient $M_n(R)$ l'enssemble des matrices carrées, $GL_n(R)$ les unités de $M_n(R)$. Soient B et C deux anneaux, pour toute matrice $M = ((b_{i,j}, c_{i,j}))_{1\leq i,j\leq n} \in M_n(B \times C)$, nous adoptons les notations $M_b = (b_{i,j})_{1\leq i,j\leq n}$, $M_c = (c_{i,j})_{1\leq i,j\leq n}$ et $M = M_b \times M_c$. Soit $M,N \in M_n(B\times C)$, c'est clair que le produit MN de M et N est donné par $MN = (M_bN_b)\times(M_cN_c)$.

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Théorème 50.

Soient A, B et C trois domaines, $f : A \longrightarrow B$ et $g : A \longrightarrow C$ deux homomorphismes d'anneaux et soit J (resp.,J') un idéal de B (resp.,C) tel que $f^{-1}(J) = g^{-1}(J')$. Alors, $A \bowtie^{f,g} (J,J')$ est un anneau à division élémentaire si et seulement si les assertions suivantes sont vérifiées:

1. $f(A)+J$ *et* $g(A)+J'$ *sont des anneaux à division élémentaire.*
2. $J = 0$ *ou* $J' = 0$.

La preuve de ce théorème nécéssite les lemmes suivants.

Lemme 51.

Soient A, B et C trois domaines, $f : A \longrightarrow B$ et $g : A \longrightarrow C$ deux homomorphismes d'anneaux et soit J (resp.,J') un idéal de B (resp.,C) tel que $f^{-1}(J) = g^{-1}(J')$. Si $A \bowtie^{f,g} (J,J')$ est un anneau de Bézout alors $J = 0$ ou $J' = 0$.

Lemme 52.

Les assertions suivantes sont vérifiées:

1. *Soient A et B deux anneaux. Alors $A \times B$ est un anneau à division élémentaire si et seulement si A et B le sont.*
2. *Soient $f : A \longrightarrow B$ et $g : A \longrightarrow C$ deux homomorphismes d'anneaux et soit J (resp.,J') un idéal de B (resp.,C) tel que $f^{-1}(J) = g^{-1}(J')$. Alors: Si $A \bowtie^{f,g} (J,J')$ est un anneau à division élémentaire alors $f(A)+J$ et $g(A)+J'$ le sont.*

Le théorème 50 recouvre le cas de l'amalgamation algébrique [54], comme suit.

Corollaire 53.

Soient A et B deux domaines, $f : A \longrightarrow B$ un homomorphisme d'anneau et soit J un idéal de B.

1. *Supposons que f est injectif.*
   - *Si $J = B$ alors $A \bowtie^f J$ est un anneau à division élémentaire si et seulement si A et B le sont.*
   - *Si $J \neq B$ alors $A \bowtie^f J$ est un anneau à division élémentaire si et seulement si $f(A)+J$ l'est et $f(A) \cap J = 0$.*
2. *Supposons que f n'est pas injectif. Alors $A \bowtie^f J$ est un anneau à division élémentaire, si l'une des conditions suivantes est vérifiée:*
   - *$J = 0$ et A est un anneau à division élémentaire.*
   - *$J = B$ et (A,B) un couple d'anneaux à division élémentaire.*

Maintenant, nous établissons la relation entre anneau à division élémentaire, anneau d'Hermite et anneau de Bézout dans le contexte de la bi-amalgamation algébrique.

Théorème 54.

Soient A, B et C trois domaines, $f : A \longrightarrow B$ et $g : A \longrightarrow C$ deux homomorphismes d'anneaux et soit J (resp.,J') un idéal de B (resp.,C) tel que $f^{-1}(J) = g^{-1}(J')$. Les propriétés suivantes sont équivalentes:

1. *$A \bowtie^{f,g} (J,J')$ est un anneau d'Hermite.*
2. *$A \bowtie^{f,g} (J,J')$ est un anneau de Bézout.*
3. *L'une des conditions suivantes est vérifiée:*
   - *$f(A)+J$ est un anneau de Bézout, et $J' = 0$.*
   - *$g(A)+J'$ est un anneau de Bézout, et $J = 0$.*

Le théorème suivant examine la propriété presque de valuation, que la bi-amalgamation $A \bowtie^{f,g} (J,J')$ peut hériter à partir de $f(A)+J$ et $g(A)+J'$ et donc engendrer de nouveaux examples des anneaux presque de valuation.

Théorème 55.

Soient $f : A \longrightarrow B$ *et* $g : A \longrightarrow C$ *deux homomorphismes d'anneaux et soit* J *(resp.,*J'*) un idéal réduit de* B *(resp.,*C*) tel que* $f^{-1}(J) = g^{-1}(J')$*. Alors* $A \bowtie^{f,g} (J,J')$ *est un AV-anneau si et seulement si les assertions suivantes sont vérifiées:*

1. $f(A)+J$ *et* $g(A)+J'$ *sont des AV-anneaux.*
2. $J = 0$ *or* $J' = 0$.

Avant de prouver le théorème 55 nous établissons les lemmes suivants.

Lemme 56.

Soient $f : A \longrightarrow B$ *et* $g : A \longrightarrow C$ *deux homomorphismes d'anneaux et soit* J *(resp.,*J'*) un idéal de* B *(resp.,*C*) tel que* $f^{-1}(J) = g^{-1}(J')$*. Si* $A \bowtie^{f,g} (J,J')$ *est un AV-anneau, alors* $f(A)+J$ *et* $g(A)+J'$ *le sont.*

Lemme 57.

Soient $f : A \longrightarrow B$ *et* $g : A \longrightarrow C$ *deux homomorphismes d'anneaux et soit* J *(resp.,*J'*) un idéal réduit de* B *(resp.,*C*) tel que* $f^{-1}(J) = g^{-1}(J')$*. Si* $A \bowtie^{f,g} (J,J')$ *est un AV-anneau alors* $J = 0$ *ou* $J' = 0$.

Le Corollaire ci-dessous est une conséquence du Théorème 55 traitant le cas de l'amalgamation algébrique.

Corollaire 58.

Soient A *et* B *deux anneaux et* $f : A \longrightarrow B$ *un homomorphisme d'anneaux. Si* J *est un idéal propre réduit de* B *et* A *est un anneau réduit. Alors* $A \bowtie^{f} J$ *est un AV-anneau si et seulement si* f *est injectif,* $f(A)+J$ *est un AV-anneau et* $f(A) \cap J = (0)$.

Le théorème 55 enrichisse la littérature avec des exemples originaux des AV-anneaux.

Exemple 59.

Soient (A,m) *un AV-domaine local,* E *un* A*-module tel que* $mE=0$*, et* $B=A\propto E$ *l'extension triviale de* A *par* E*, et* $C=A/m$*. On considère l' homomorphisms injectif* $f:A\longrightarrow B$ *et la surjection canonique* $g:A\longrightarrow A/m$ *et soit* $J=m\propto\{0\}$*. La bi-amalgamation* $R=A\bowtie^{f,g}(J,0)$ *est un AV-anneau. Car, notons que* $f^{-1}(J)=g^{-1}(0)=m$*,* $f(A)+J=B$ *et* $g(A)=A/m=C$*. De plus,* B *est un AV-anneau par [65, Theorem 2.1 (3)] et* C *est un AV-anneau. Donc,* R *est un AV-anneau par le théorème 55.*

Exemple 60.

Soient A *un AV-anneau,* I,K *deux idéaux de* A *tel que* $I\subsetneq K$ *et* I *un idéal radical,* $B=A/I$ *,* $C=A/K$*. On considère les surjections canoniques* $f:A\longrightarrow B$ *et* $g:A\longrightarrow C$ *et soit* $J=K/I$*. La bi-amalgamation* $R=A\bowtie^{f,g}(J,0)$ *est un AV-anneau. Car, notons que* $f^{-1}(J)=g^{-1}(0)=K$*,* $f(A)+J=B$*,* $g(A)=A/K=C$ *et* $J=K/I$ *est réduit. De plus,* B *et* C *sont des AV-anneaux. Donc,* R *est un AV-anneau par le théorème 55.*

Le théorème suivant étudie le transfert de la notion de AB-anneau à la bi-amalgamation algébrique.

Théorème 61.

Soient(A,B,C) *trois domaines,* $f:A\longrightarrow B$ *et* $g:A\longrightarrow C$ *deux homomorphismes d'anneaux et soit* J *(resp.,*J'*) un idéal propre de* B *(resp.,*C*) tel que* $f^{-1}(J)=g^{-1}(J')$*. Alors* $A\bowtie^{f,g}(J,J')$ *est un AB-anneau si et seulement si les assertions suivantes sont vérifiées:*

1. $f(A)+J$ *and* $g(A)+J'$ *sont des AB-anneaux.*
2. $J=0$ *ou* $J'=0$*.*

Avant de démontrer ce théorème, nous établissons les lemmes suivants.

Lemme 62.

Soient (A,B,C) *trois domaines,* $f : A \longrightarrow B$ *et* $g : A \longrightarrow C$ *deux homomorphismes d'anneaux et soit* J *(resp.,*J'*) un idéal de* B *(resp.,*C*) tel que* $f^{-1}(J) = g^{-1}(J')$*. Si* $A \bowtie^{f,g} (J,J')$ *un AB-anneau, alors* $f(A)+J$ *et* $g(A)+J'$ *le sont.*

Lemme 63.

Soient (A,B,C) *trois domaines,* $f : A \longrightarrow B$ *et* $g : A \longrightarrow C$ *deux homomorphismes d'anneaux et soit* J *(resp.,*J'*) un idéal propre de* B *(resp.,*C*) tel que* $f^{-1}(J) = g^{-1}(J')$*. Si* $A \bowtie^{f,g} (J,J')$ *est un AB-anneau alors* $J = 0$ *ou* $J' = 0$*.*

Le corollaire suivant recouvre un résultat dans l'amalgamation algébrique.

Corollaire 64.

Soient (A,B) *un couple de domaines,* $f : A \longrightarrow B$ *un homomorphisme d'anneaux,* J *un idéal de* B*. Alors* $A \bowtie^{f} J$ *est un AB-anneau si et seulement si* f *est injectif,* $f(A)+J$ *est un AB-anneau et* $f(A)\cap J = (0)$*.*

Exemple 65.

Soit A *un AB-domaine,* I,K *deux idéaux de* A *tel que* $I \subsetneq K \subsetneq A$ *,* $B = A/I$ *,* $C = A/K$*. On considère les surjections canoniques* $f : A \longrightarrow B$ *et* $g : A \longrightarrow C$ *et soit* $J = K/I$*.* $R = A \bowtie^{f,g} (J,0)$ *est un AB-anneau. Car, notons que* $f^{-1}(J) = g^{-1}(0) = K$*,* $f(A)+J = B$*,* $g(A) = A/K = C$ *et* $J = K/I$ *. De plus,* B *et* C *sont des AB-anneaux. Donc,* R *est un AB-anneau par le théorème 61.*

Chapitre 6
Les propriétés de Bézout dans l'amalgamé algébrique le long d'un idéal

Dans ce chapitre[*], on considère l'homomorphisme d'anneaux $f : A \mapsto B$ et J un idéal de B. Nous étudions le transfert des propiértés de "P-anneau de bézout " et "2-anneau de bézout" à l'amalgamation de A avec B le long de l'idéal J par rapport à f (notée $A \bowtie^f J$). Notre but et de prouver les conditions nécéssaires et suffisantes pour que $A \bowtie^f J$, soit un P-anneau de bézout et un 2-anneau de bézout.

Avant d'énnoncer le résultat principal, nous commençons par cette utile remarque. Soient $f : A \longrightarrow B$ un homomorphisme d'anneau, J un idéal de B et n un entier positif. Considérons la fonction $f^n : A^n \longrightarrow B^n$ définie par $f^n((\alpha)_{i=1}^{i=n}) = (f(\alpha)_i)_{i=1}^{i=n}$. évidement , f^n est un homomorphisme d'anneaux et J^n est un idéal de B^n . Cela nous permet de définir $A^n \bowtie^{f^n} J^n$.
De plus , soit $\phi : (A \bowtie^f J)^n \longrightarrow A^n \bowtie^{f^n} J^n$ définit par
$\phi((a_i, f(a_i)+j_i)_{i=1}^{i=n}) = ((a_i)_{i=1}^{i=n}, f^n((a_i)_{i=1}^{i=n}) + (j_i)_{i=1}^{i=n})$. C'est clair que ϕ est un homomorphisme d'anneaux, parsuite $(A \bowtie^f J)^n$ et $A^n \bowtie^{f^n} J^n$ sont des anneaux isomorphes.
Soit U un sous module de A^n. Alors $U \bowtie^{f^n} J^n = \{(u, f^n(u)+j) \in A^n \bowtie^{f^n} J^n / u \in U, j \in J^n\}$ est un sous module de $A^n \bowtie^{f^n} J^n$.

Maintenant nous étudions le transfert de la propriété de P-anneau de Bézout à l'amalgamation algébrique.

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Théorème 66.

Soient (A,B) *un couple d'anneaux,* $f : A \to B$ *un homomorphisme d'anneaux et* J *un idéal propre de* B*, alors :*

1. *Supposons que* J *est un idéal de type fini de* $f(A)+J$ *, alors :*
   Si $A \bowtie^f J$ *est un P-anneau de Bézout alors* A *l'est aussi.*
2. *Supposons que* $f^{-1}(J)$ *est un idéal de type fini de* A *, alors :*
   Si $A \bowtie^f J$ *est un P-anneau de Bézout alors* $f(A)+J$ *l'est aussi.*
3. *Supposons que* J *est un idéal de type fini de* $f(A)+J$ *,* $J \subset nil(B)$*, et* $\forall t \in A$ $f(t)J = J$ *alors :*
   $A \bowtie^f J$ *est un P-anneau de Bézout si et seulement si* A *l'est aussi.*
4. *Supposons que* J *n'est pas un idéal de type fini de* $f(A)+J$ *,* $J \subset nil(B)$ *, et* $\forall P \in Spec(A)$ *,* $f(P) \subset J$ *alors :*
   $A \bowtie^f J$ *est un P-anneau de Bézout.*
5. *Supposons que* f *est injectif, si* $f(A) \cap J = 0$*, alors*
   $A \bowtie^f J$ *est un P-anneau de Bézout si et seulement si* $f(A)+J$ *l'est aussi .*
6. *Supposons que* (A,M) *est un anneau local ,* $f(M) \subset An(J)$ *,* $J \subset nil(B)$ *et* J *est un idéal de type fini de* $f(A)+J$*, alors:*
   $A \bowtie^f J$ *est un P-anneau de Bézout si et seulement si* A *l'est aussi.*

Maintenant nous étudions le transfert de la propriété 2-anneau de Bézout à l'amalgamation algébrique.

Théorème 67.

Soient (A,B) *un couple d'anneaux,* $f : A \to B$ *un homomorphisme d'anneaux et* J *un idéal propre de* B*, alors :*

1. *Supposons que* (A,M) *est un anneau local et* $f(M) \subset An(J)$ *alors:*
    - *Si* $A \bowtie^f J$ *est un* 2*-anneau de Bézout alors* A *l'est aussi.*
    - $A \bowtie^f J$ *est un* 2*-anneau de Bézout si et seulement si* A *l'est aussi avec* $J^2 = 0$
2. *Supposons que* (B,J) *est un anneau local et* J *est un idéal de type fini de* $f(A)+J$ *.*
    - *Si* $A \bowtie^f J$ *est un* 2*-anneau de Bézout alors* A *l'est aussi.*
    - $A \bowtie^f J$ *est un* 2*-anneau de Bézout si et seulement si* A *l'est aussi avec* $J^2 = 0$
3. *Supposons que* f *est injectif. Si* $f(A) \cap J = 0$*, alors* $A \bowtie^f J$ *est un* 2*-anneau de Bézout si et seulement si* $f(A)+J$ *l'est aussi.*
4. *Supposons que* (A,M) *est un anneau local,* $f(M) \subset J$ *et* $J^2 = 0$ *alors* $A \bowtie^f J$ *est un* 2*-anneau de Bézout.*

Introduction

The present thesis is devoted to the study of: the notion of Armendaris ring, the property of coherence and some properties related to the notion of Bézout ring in different contexts namely trivial ring extension, direct product, homomorphic image, amalgamated algebra and the bi-amalgamated algebra.

In 1997, Rege and Chhawchharia introduced the notion of Armendariz ring as an associative ring R with identity such that for every polynomials $f(x) = a_0 + a_1x + ... + a_nx^n$ and $g(x) = b_0 + b_1x + ... + b_mx^m$ in $R[x]$, $f(x)g(x) = 0$ implies that $a_ib_j = 0$ for every i, j.

The name was chosen because Armendariz had shown that a reduced ring satisfies this property. Later, in 1998, D. D. Anderson and V. Camillo continued this investigation by studying Armendariz rings and Gauss rings. Since then, various generalizations of Armendariz rings such as skew Armendariz ring in 2003 by Chan Yong Hong, weak Armendariz ring in 2006 by Liu and Zhao, nil-Armendariz ring in 2008 by R. Antoine and power serieswise Armendariz ring in 2006 by Kim and al. The following diagram of implication summarizes the relation between the above notions:
reduced ring $\Longrightarrow$ power serieswise Armendariz ring $\Longrightarrow$ Armendariz ring
$\Longrightarrow$ nil-Armendariz ring $\Longrightarrow$ weak-Armendariz ring.
The reverses of implications are not, in general, true. However, we do not know so far any example of weak-Armendariz ring which is not a nil-Armendariz ring. This question has been left open.

About the concept of coherence, we signal that its first sprang up from the study of coherent sheaves in algebraic geometry, and then developed, under the influence of Noetherian ring theory and homology, towards a full-fledged topic in algebra. A large number of finiteness properties of commutative rings have homological characterizations. For example, it is well known that for a ring to be Noetherian, a condition most commonly described by the finite generation of the ideals of the ring, it is necessary and sufficient that arbitrary direct sums of injective modules be injective modules. One might speculate that this is the reason why homological algebra approaches in Noetherian settings yield such deep and beautiful results.

The same phenomena can be observed in another large class of rings, the class of coherent rings. Chase (1960) attempted to answer the homological question: for what rings, arbitrary direct products of flat modules are flat. The answer is that this holds true precisely when the ring is coherent. Chase provides no less than seven equivalent characterizations of this homological condition. The most well known are the two equivalent finiteness conditions below: A ring R is called a coherent ring if every finitely generated ideal of R is finitely presented. Equivalently, if and only if for every element a of R and any two finitely generated ideals I and J of R, the ideals $I \cap J$ and $(0 : a) = \{r \in R \mid ra = 0\}$ are finitely generated.

During the past 30 years, several (commutative) coherent-like notions grew out of coherence such as finite conductor, n-coherent, strong n-coherent, v-coherent, and other properties. Noteworthy is that both the ring-theoretic and homological aspects of coherence run through most of these generalizations (see for instance Glaz, 2000). Examples of coherent rings include all Noetherian rings, as well as many non-Noetherian rings.

The arithmetical part of this thesis deals with some notions related to the notion of Bézout ring namely (i) elementary divisor ring and Hermite ring (ii) almost Bézout ring and almost valuation ring (iii) 2-Bézout ring and P-Bézout ring.

(i) In [40], it is proved that a ring R is an Hermite ring if and only if for all $a, b \in R$, there exist $a_1, b_1, d \in R$ such that $a = a_1 d$, $b = b_1 d$, and $Ra_1 + Rb_1 = R$. So it is easy to see that every elementary divisor ring is an Hermite ring, and that every Hermite ring is a Bézout ring. Also, Kaplansky proved that any valuation ring is an elementary divisor ring.

The following diagram of implication summarizes the relation between the above notions:

Valuation ring $\Longrightarrow$ Elementary divisor ring $\Longrightarrow$ Hermite ring $\Longrightarrow$ Bézout ring.

(ii) In [5], Anderson and Zaffrullah introduced and studied the notion of almost valuation domain (AV-domain for short) as a ring R such that for any two elements a and b in R, there exists a positive integer n such that a^n divides b^n or b^n divides a^n. Also, they introduced the notion of almost Bézout domain (AB-domain) as a ring R such that for any two elements a and b in R, there exists a positive integer n such that the ideal (a^n, b^n) is principal. Among others, they proved that the integral closure of an almost valuation (resp., almost Bézout) domain is a valuation domain (resp., a Prüfer domain with torsion class group).

(iii) In [11, 13], as a generalisation of Bézout ring, Chahrazade Bakkari and Khalid Ouarghi introduced and studied the notion of P-Bézout ring and 2-Bézout ring.

Before describing our major contributions in each chapter, we first give the definitions of some notions used in the chapters which follow. Throughout this thesis, all rings considered are associative with identity elements and all modules are unital.

Definition 1.

A ring R is called a reduced ring if it has no non-zero nilpotent elements.

Definition 2.

A ring R is called semicommutative if for all $a, b \in R$, $ab = 0$ implies $aRb = 0$.

Definition 3.

- *A ring R is called Armendariz if whenever polynomials $f(x) = a_0 + a_1x + \ldots + a_nx^n$, $g(x) = b_0 + b_1x + \ldots + b_mx^m$ satisfy $f(x)g(x) = 0$, then $a_ib_j = 0$ for each i, j.*
- *A ring R is called a nil-Armendariz ring if whenever the product of two polynomials $f(x) = \sum_{i=0}^{i=n} a_ix^i$ and $g(x) = \sum_{j=0}^{j=m} b_jx^j$ in $R[x]$ satisfies $f(x)g(x) \in nil(R)[x]$ we have $a_ib_j \in nil(R)$ for each i, j.*
- *A ring R is called a weakArmendariz ring if whenever the product of two polynomials $f(x) = \sum_{i=0}^{i=n} a_ix^i$ and $g(x) = \sum_{j=0}^{j=m} b_jx^j$ in $R[x]$ satisfies $f(x)g(x) = 0$ we have $a_ib_j \in nil(R)$ for each i, j.*

Definition 4.

A ring R is called power serieswise Armendariz if whenever formal power series $f(x) = \sum_{i=0}^{\infty} a_ix^i$ and $g(x) = \sum_{i=0}^{\infty} b_ix^i \in R[[x]]$ such that $fg = 0$, then $a_ib_j = 0$ for every i and j.

Definition 5.

A ring R is called abelian ring if any idempotent element is central.

Definition 6.

Let R be a commutative ring and M be an R-module.

1. *For a nonnegative integer n, M is called n-presented if there is an exact sequence of R-modules:*

$$F_n \to F_{n-1} \to \dots F_1 \to F_0 \to E \to 0$$

where each F_i is a finitely generated free R-module.

2. *M is called a coherent R-module if it is finitely generated and every finitely generated submodule of M is finitely presented.*

Definition 7.

Let R be a commutative ring.

1. *R is called coherent ring if every finitely generated ideal of R is finitely presented.*
2. *R is called Noetherian ring if every ideal of R is a finitely generated.*

Definition 8.

Let R be a commutative ring.

1. *R is called an almost valuation ring (AV-ring for short) if, for any two elements a and b in R, there exists a positive integer n such that a^n divides b^n or b^n divides a^n.*
2. *R is called an almost Bézout ring (AB-ring for short) if, for any two elements a and b in R, there exists a positive integer n such that the ideal $(a^n$, $b^n)$ is principal.*
3. *R is called an elementary divisor ring if, for every matrix M over R there exist nonsingular matrices P, Q such that PMQ is a diagonal matrix.*
4. *R is called a Hermite ring if, for every matrix M over R there exist nonsingular matrices Q such that MQ is a triangular matrix.*
5. *R is called Bézout ring if every finitely generated ideal is principal*

Definition 9.

A ring R is called a P- Bézout ring, if every finitely generated prime ideal P of R is principal.

Definition 10.

A ring R is called a 2- Bézout ring, if every finitely presented ideal I of R is principal.

Definition 11.

An ideal I of R is called regular if it contains a regular element; i.e, a non zero-divisor element.

In 1956, Nagata introduced the notion of trivial ring extension of a ring A by a module E as follows :

Definition 12.

Let A be a ring and E an $A-$module. The trivial ring extension $R = A \ltimes E$ of A by E is the set of pairs (a,e) with $a \in A$ and $e \in E$ under coordinate-wise addition and adjusted multiplication defined by:

$$(a,e)(b,f) = (ab, af + be).$$

This notion has been the subject of several works in the area, and it is a fertile field of research in the theory of rings (see [42, 48] and the reference therein). Trivial ring extensions have been studied extensively; and considerable work, part of is summarized in Glaz's book [42] and Huckaba's book [48], has been concerned with these extensions. Mainly, Trivial ring extensions have been useful for solving many open problems and conjectures in both commutative and non-commutative ring theory. See for instance [4, 48, 53].

In 2006, M. D'Anna and M. Fontana [28] introduced a new construction, called amalgamated duplication of a ring A along an $A-$submodule E of $Q(A)$ (the total ring of fractions of A) such that $E^2 \subseteq E$. When $E^2 = \{0\}$, this construction coincides with the trivial ring extension of A by E. Motivations and more applications of the amalgamated

duplication $A \bowtie E$ of A along an $A-$submodule E of $Q(A)$ are discussed in more details, especially in the particular case where E is an ideal of A, in recent papers, for instance, see [26, 29, 27, 30, 28].
In 2010, D'Anna, Finocchiaro and Fontana [29] extended the notion of amalgamated duplication construction $A \bowtie I$ of a ring A along an ideal I of A to the general context of ring homomorphism extensions as follows:

Definition 13.

Let A and B be two rings with identity elements, J be an ideal of B and let $f : A \to B$ be a ring homomorphism. In this setting, we consider the following subring of $A \times B$; $A \bowtie^f J := \{(a, f(a)+j) \mid a \in A, j \in J\}$ called the amalgamation of A and B along J with respect to f.

This construction is a generalization of the amalgamated duplication of a ring along an ideal (introduced and studied by D'Anna and Fontana in [26, 27, 28]). Moreover, other classical constructions (such as $A+XB[X]$, $A+XB[[X]]$, and the $D+M$ constructions) can be studied as particular cases of the amalgamation ([29, Examples 2.5 and 2.6]). Other classical constructions, such as the Nagata's idealization, also called trivial ring extension ([66, page 2]), and the CPI extensions (in the sense of Boisen and Sheldon [19]) are strictly related to it ([29, Example 2.7 and Remark 2.8]). See for instance [29, 30, 27, 28].

In 2013, Kabbaj, Tamekkante and Louartiti [52] extended the notion of amalgamated algebra construction $A \bowtie J$ of a rings A and B along ideal J of B to the general context of ring homomorphism extensions as follows:

Definition 14.

Let $f : A \longrightarrow B$ and $g : A \longrightarrow C$ be two ring homomorphisms and let J (resp.,J') be an ideal of B (resp.,C) such that $f^{-1}(J) = g^{-1}(J')$. In this setting, we can consider the following subring of $B \times C$:

$$A \bowtie^{f,g} (J,J') := \{(f(a)+j, g(a)+j') | a \in A, j \in J, j' \in J'\}$$

called the bi-amalgamation of A with (B,C) along (J,J') with respect to (f,g).

This construction is a generalisation of the amalgamated algebra along an ideal (introduced and studied by D'Anna and Fontana in [29, 30, 27, 28]) as follows $A \bowtie^f J = A \bowtie^{Id,f} (f^{-1}(J), J)$. In [52], the authors studied the basic properties of this construction (e.g., characterized for $A \bowtie^{f,g} (J,J')$ to be a Noetherian ring, an integral domain, a reduced ring ...)

For a ring R and an R-module M. We use the following notations:

- $\mathrm{qf}(R)$ denotes the quotient field of R.
- $\mathrm{Nil}(R)$, the set of nilpotent elements of R.
- $\mathrm{Rad}(R)$, the jacobson radical of R.
- $\mathrm{Spec}(R)$, the set of prime ideals of R.
- $\mathrm{Max}(R)$, the set of maximal ideals of R.
- $R[x]$, the polynomial ring with an indeterminate x over R.
- $R[[x]]$, the formal power series ring with an indeterminate x over R.
- $\mathrm{An}(M)$, the annihilator of M.
- $M_n(R)$, the ring of $n \times n$ matrix over R.
- $Gl_n(R)$, the units of $M_n(R)$.
- $(I : J)(= \{x \in R \diagup xJ \subseteq I\})$, the ideal quotient, where I and J are ideals of R.

CHAPTER 1

ON ARMENDARIZ-LIKE PROPERTIES IN AMALGAMATED ALGEBRA

Abstract

In this chapter[*], we consider a ring homomorphism $f : A \to B$ and J be an ideal of B. we investigate the transfer of Armendariz-like properties to the amalgamation of A with B along J with respect to f (denoted by $A \bowtie^f J$) introduced and studied by D'Anna, Finocchiaro and Fontana in 2009. Our aim is to provide necessary and sufficient conditions for $A \bowtie^f J$, to be an Armendariz ring, nil-Armendariz ring and weak Armendariz ring.
Key Words: Amalgamated algebra, Armendariz ring, nil-Armendariz, weak-Armendariz ring, semicommutative ring, reduced ring.

1.1 Introduction

In [68], Rege and Chhawchharia introduced the notion of Armendariz ring as an associative ring R with identity such that for every polynomials $f(x) = \sum\limits_{i=0}^{i=m} a_i x^i$ and $g(x) = \sum\limits_{j=0}^{i=n} b_j x^j$ in $R[x]$, $f(x)g(x) = 0$ implies that $a_i b_j = 0$ for every i, j. The name was chosen because Armendariz had shown that a reduced ring (i.e., a ring without nonzero nilpotent elements) satisfies this property ([9]). Later, in 1998, D. D. Anderson and V. Camillo continued this investigation by studying Armendariz rings and Gauss rings (recall that a ring R is said to be a Gauss ring if for every polynomials $f(x)$ and $g(x)$ in $R[x]$, $c(fg) = c(f)c(g)$). Among others, they proved that a commutative ring R is Gaussian if

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and only if each homomorphic image of R is an Armendariz ring ([2]). Since then, various generalizations of Armendariz rings such as skew Armendariz ring, weak Armendariz ring, central Armendariz ring, nil-Armendariz ring etc appeared in the literature.

In 2006, Liu and Zhao ([62]) introduced the notion of a weak Armendariz ring as a ring R such that whenever two polynomials $f(x)=\sum_{i=0}^{i=m} a_i x^i$ and $g(x)=\sum_{j=0}^{i=n} b_j x^j$ in $R[x]$ satisfy $f(x)g(x)=0$, then $a_ib_j \in nil(R)$ for every i,j. Among others, they proved that a ring R is a weak Armendariz ring if and only if for every positive integer n, the $n-by-n$ upper triangular matrix ring $T_n(R)$ is a weak Armendariz ring. Moreover, if R is a semicommutative ring (i.e., a ring such that whenever $ab=0$, $aRb=0$), then the polynomial ring $R[x]$ and the ring $R[x]/(x^n)$ are weak Armendariz rings. Here, it is worth to notice that a weaker version of Armendariz ring notion also called a"weak Armendariz ring" is due to Lee and Wong ([60]) in the sense that whenever two linear polynomials $f(x)=a_0+a_1x$ and $g(x)=b_0+b_1x$ satisfy $fg=0$, then $a_ib_j=0$ for every $i,j=0,1$.

In 2008, observing that in all examples found in the literature of Armendariz and weak Armendariz rings, the set of nilpotent elements forms an ideal, R. Antoine proved that this is not true in general and he provided an example of Armendariz ring R for which $nil(R)$ is not an ideal ([8, Example 4.8]). However, if $nil(R)$ is an ideal of R, then R is a weak Armendariz ring, and in fact R satisfies a stronger condition. This allowed him to introduce the notion of nil-Armendariz ring as a ring R such that whenever two polynomials $f(x)=\sum_{i=0}^{i=m} a_i x^i$ and $g(x)=\sum_{j=0}^{i=n} b_j x^j$ in $R[x]$ satisfy $f(x)g(x) \in nil(R)[x]$, then $a_ib_j \in nil(R)$ for every i,j. He proved that if R is a nil-Armendariz ring, then $nil(R)$ is a subring without unit of R. He also studied the conditions under which the polynomial ring over a nil-Armendariz ring is a nil-Armendariz ring.

The following diagram of implication summarizes the relation between the above notions: reduced ring $\Longrightarrow$ Armendariz ring $\Longrightarrow$ nil-Armendariz ring $\Longrightarrow$ weak Armendariz ring. The reverses of the first and second implications are not, in general, true and examples can be found in [8, Proposition 2.1] and [8, Example 4.9]. However, we do not know so far any example of weak Armendariz ring which is not a nil-Armendariz ring. This question was left open in [8].

Let A and B be two rings with unity, let J be an ideal of B and let $f : A \to B$ be a ring homomorphism. In this setting, we can consider the following subring of $A \times B$:

$$A \bowtie^f J := \{(a, f(a)+j) \mid a \in A, j \in J\}$$

called *the amalgamation of A and B along J with respect to f.* This construction is a generalization of *the amalgamated duplication of a ring along an ideal* (introduced and studied by D'Anna and Fontana in [26, 27, 28]). The interest of amalgamation resides, partly, in its ability to cover several basic constructions in commutative algebra, including pullbacks and trivial ring extensions (also called Nagata's idealizations)(cf. [66, page 2]). Moreover, other classical constructions (such as the $A+XB[X]$, $A+XB[[X]]$, and the $D+M$ constructions) can be studied as particular cases of the amalgamation ([29, Examples 2.5 and 2.6]) and other classical constructions, such as the CPI extensions (in the sense of Boisen and Sheldon [19]) are strictly related to it ([29, Example 2.7 and Remark 2.8]). In [29], the authors studied the basic properties of this construction (e.g., characterizations for $A\bowtie^f J$ to be a Noetherian ring, an integral domain, a reduced ring) and they characterized those distinguished pullbacks that can be expressed as an amalgamation. Moreover, in [28], they pursued the investigation on the structure of the rings of the form $A\bowtie^f J$, with particular attention to the prime spectrum, chain properties and Krull dimension.

This paper aims at studying the transfer of the notion of "Armendariz rings", "nil-Armendariz rings" and "weak Armendariz rings" to the amalgamation of algebras along ideals. It contains, in addition to the Introduction, three sections and each section deals respectively with one of the pre-mentioned notions. The main results (Theorem 1.2.2., Theorem 1.3.1. and Theorem 1.4.1.) can be summarized as follows:

Theorem 1.1.1.

Let (A,B) be a pair of rings, $f : A \to B$ be a ring homomorphism and J be a proper ideal of B.

1. *If $A \bowtie^f J$ is an Armendariz (resp. a nil-Armendariz, resp. a weak Armendariz) ring, then A is an Armendariz (resp. a nil-Armendariz, resp. a weak Armendariz) ring.*
2. *If A and $f(A)+J$ are Armendariz (resp. nil-Armendariz, resp. weak Armendariz) rings, then $A \bowtie^f J$ is an Armendariz (resp. a nil-Armendariz, resp. a weak Armendariz) ring.*
3. *Assume that $J \cap S \neq \varnothing$ where S is the set of regular central element of B. Then $A \bowtie^f J$ is an Armendariz (resp. a nil-Armendariz, resp. a weak Armendariz) ring if and only if A and $f(A)+J$ are Armendariz (resp. nil-Armendariz, resp. weak Armendariz) rings.*
4. *Assume that $J \cap nil(B) = (0)$ (resp. $J \subseteq nil(B)$). Then $A \bowtie^f J$ is an Armendariz (resp. a nil-Armendariz, resp. a weak Armendariz) ring if and only if A is an Armendariz (resp. a nil-Armendariz, resp. a weak Armendariz) ring.*
5. *Assume that $f^{-1}(J) \cap nil(A) = (0)$ (resp. $f^{-1}(J) \subseteq nil(A)$). If $f(A)+J$ is an Armendariz (resp. a nil-Armendariz, resp. a weak Armendariz) ring, then $A \bowtie^f J$ is an Armendariz (resp. a nil-Armendariz, resp. a weak Armendariz) ring, and the equivalence holds for nil-Armendariz.*
6. *Assume that f is injective.*

   i) $f(A) \cap J = 0$. Then $A \bowtie^f J$ is a weak Armendariz ring if and only if $f(A)+J$ is a weak Armendariz.

   ii) $J \subseteq nil(B)$. If $f(A)+J$ is a weak Armendariz, then $A \bowtie^f J$ is a weak Armendariz ring.
7. *Assume that J is semicommutative. If A is a weak Armendariz ring, then so is $A \bowtie^f J$.*
8. *Assume that $f^{-1}(J)$ is semicommutative. If $f(A)+J$ is a weak Armendariz ring, then so is $A \bowtie^f J$.*

It is worth to mention that the proofs of some assertions of the above theorem are very similar, and for the convenience of the reader, we separate the three notions in three sections and we omitted the similar proofs to avoid repetitions as much as possible.

1.2 Armendariz property in amalgamated algebra along an ideal

We start this section by the following proposition which characterizes when the amalgamated algebra $A \bowtie^f J$ is a reduced ring.

Proposition 1.2.1.

([29, Proposition 5.4]) Let (A,B) be a pair of rings, $f : A \rightarrow B$ be a ring homomorphism and J be a proper ideal of B. The following conditions are equivalent:

1. *$A \bowtie^f J$ is a reduced ring.*
2. *A is a reduced ring and $nil(B) \cap J = (0)$*

In particular, if A and B are reduced, then $A \bowtie^f J$ is reduced; conversely, if J is a radical ideal of B and $A \bowtie^f J$ is reduced, then B (and A) is reduced.

Our next Theorem states necessary and sufficient conditions under which the amalgamated algebra $A \bowtie^f J$ is an Armendariz ring.

Theorem 1.2.2.

Let (A,B) be a pair of rings, $f : A \to B$ be a ring homomorphism and J be a proper ideal of B.

1. *If $A \bowtie^f J$ is an Armendariz ring, then so is A.*
2. *If A and $f(A)+J$ are Armendariz rings, then so is $A \bowtie^f J$.*
3. *Assume that $J \cap S \neq \varnothing$ where S the set of regular central elements of B. Then $A \bowtie^f J$ is an Armendariz ring if and only if $f(A)+J$ and A are Armendariz rings.*
4. *Assume that $J \cap nil(B) = (0)$. Then $A \bowtie^f J$ is an Armendariz ring if and only if A is an Armendariz ring.*
5. *Assume that $f^{-1}(J) \cap nil(A) = (0)$. If $f(A)+J$ is an Armendariz ring, then $A \bowtie^f J$ is an Armendariz ring.*

Proof.

1. Assume that $A \bowtie^f J$ is Armendariz and let $f_A(x) = \sum_{i=0}^{i=n} a_i x^j$ and $g_A(x) = \sum_{j=0}^{j=m} b_j x^j$ be two polynomials in $A[x]$ such that $f_A(x)g_A(x) = 0$. Then for every $k \in \{0,\dots,n+m\}$; $\sum_{i+j=k} a_i b_j = 0$. Set $F(x) = \sum_{i=0}^{i=n}(a_i, f(a_i))x^i$ and $G(x) = \sum_{j=0}^{j=m}(b_j, f(b_j))x^j$. Then

$$\begin{aligned} F(x)G(x) &= \sum_{k=0}^{k=n+m} (\sum_{i+j=k} (a_i b_j, f(a_i b_j)))x^k \\ &= \sum_{k=0}^{k=n+m} (\sum_{i+j=k} a_i b_j, \sum_{i+j=k} f(a_i b_j))x^k \\ &= \sum_{k=0}^{k=n+m} (\sum_{i+j=k} a_i b_j, f(\sum_{i+j=k} a_i b_j))x^k. \end{aligned}$$

   Hence $F(x)G(x) = 0$ and so $(a_i b_j, f(a_i b_j)) = 0$ since $A \bowtie^f J$ is Armendariz. Thus, $a_i b_j = 0$ and consequently A is Armendariz.

2. Assume that A and $f(A)+J$ are Armendariz and let $F(x) = \sum_{i=0}^{i=n}(a_i, f(a_i)+j_i)x^i$ and $G(x) = \sum_{j=0}^{j=m}(b_j, f(b_j)+k_j)x^j$ be two polynomials in $(A \bowtie^f J)[x]$ such that $F(x)G(x) = 0$. Set $f_B(x) = \sum_{i=0}^{i=n}(f(a_i)+j_i)x^i$, $g_B(x) = \sum_{j=0}^{j=m}(f(b_j)+k_j)x^j$, $f_A(x) =$

$\sum_{i=0}^{i=n} a_i x^j$ and $g_A(x)=\sum_{j=0}^{j=m} b_j x^j$. Then $F(x)G(x)=0$ implies that $f_A(x)g_A(x)=0$ and $f_B(x)g_B(x)=0$, which in turn implies that $(f(a_i)+j_i)(f(b_j)+k_j)=0$ and $a_ib_j=0$ for every i,j since $f(A)+J$ and A are Armendariz rings. Therefore $A\bowtie^f J$ is Armendariz.

3. Let S be the set of regular central element of B. Assume that $J\cap S\neq\varnothing$ and $A\bowtie^f J$ is Armendariz. Let $f_A(x)=\sum_{i=0}^{i=n}(f(a_i)+j_i)x^i$ and $g_A(x)=\sum_{j=0}^{j=m}(f(b_j)+k_j)x^j$ be two polynomials in $(f(A)+J)[x]$ such that $f_A(x)g_A(x)=0$ and let e be a regular element of J. Set $F(x)=\sum_{i=0}^{i=n}(0,e(f(a_i)+j_i))x^i$ *and* $G(x)=\sum_{j=0}^{j=m}(0,e(f(b_j)+k_j))x^j$. Clearly

$$\begin{aligned} F(x)G(x) &= \sum_{k=0}^{k=n+m}(\sum_{i+j=k}(0,e^2(f(a_i)+j_i)(f(b_j)+k_j))x^k \\ &= \sum_{k=0}^{k=n+m}(0,e^2\sum_{i+j=k}(f(a_i)+j_i)(f(b_j)+k_j))x^k=0. \end{aligned}$$

So $(0,e(f(a_i)+j_i))(0,e(f(b_j)+k_j))=0$ since $A\bowtie^f J$ is Armendariz; which implies that $e^2(f(a_i)+j_i)(f(b_j)+k_j)=0$ for every i,j. Hence $(f(a_i)+j_i)(f(b_j)+k_j)=0$, and this shows that $f(A)+J$ is Armendariz.

4. Assume that $J\cap nil(B)=(0)$ and A is Armendariz. Let $F(x)=\sum_{i=0}^{i=n}(a_i,f(a_i)+j_i)x^i$ and $G(x)=\sum_{t=0}^{t=m}(b_t,f(b_t)+k_t)x^t$ be two polynomials in $(A\bowtie^f J)[x]$ such that $F(x)G(x)=0$. Set $f_B(x)=\sum_{i=0}^{i=n}(f(a_i)+j_i)x^i$, $g_B(x)=\sum_{t=0}^{t=m}(f(b_t)+k_t)x^t$, $f_A(x)=\sum_{i=0}^{i=n}a_ix^j$ and $g_A(x)=\sum_{t=0}^{t=m}b_tx^t$. Then $F(x)G(x)=0$ implies that $f_Ag_A=0$ (and $f_Bg_B=0$) which in turn implies that $a_ib_t=0$ since A is an Armendariz ring. Thus $(f(a_i)+j_i)(f(b_t)+k_t)\in J$ for every i,t. Next, we show that $(f(a_i)+j_i)(f(b_t)+k_t)=0$ for every i,t. For this, we proceed by induction on the degree n of $F(x)$. If $n=0$, it is clear. Suppose that $n\geq 1$ and the induction hypothesis.
Claim: $(f(a_0)+j_0)(f(b_t)+k_t)=0$ for every $0\leq t\leq m$. Indeed, suppose that $\exists t\in\{0,\dots,m\}$ such that $(f(a_0)+j_0)(f(b_t)+k_t)\neq 0$ and let l be the smallest integer in $\{0,\dots,m\}$ such that $(f(a_0)+j_0)(f(b_l)+k_l)\neq 0$. Then for $t\in\{0,\dots,l-1\}$, $(f(a_0)+j_0)(f(b_t)+k_t)=0$ and so $((f(b_t)+k_t)J(f(a_0)+j_0))^2=0$. Thus $(f(b_t)+k_t)J(f(a_0)+j_0)=0$ since $J\cap nil(B)=(0)$. Hence $(f(a_{l-t})+j_{l-t})(f(b_t)+k_t)((f(a_0)+j_0)(f(b_l)+k_l))^2=(f(a_{l-t})+j_{l-t})(f(b_t)+k_t)(f(a_0)+j_0)(f(b_l)+k_l)(f(a_0)+j_0)(f(b_l)+k_l)\in(f(a_{l-t})+j_{l-t})((f(b_t)+k_t)J(f(a_0)+j_0))(f(b_l)+k_l)=0$. But since the coefficient of the term x^l in $f_Bg_B=0$ is zero, we obtain
$0=(f(a_0)+j_0)(f(b_l)+k_l)+(f(a_1)+j_1)(f(b_{l-1})+k_{l-1})+\dots+(f(a_l)+j_l)\times(f(b_0)+k_0)=(f(a_0)+j_0)(f(b_l)+k_l)+\sum_{t=1}^{t=l-1}(f(a_{l-t})+j_{l-t})(f(b_t)+k_t)$. Multiplying $((f(a_0)+j_0)(f(b_l)+k_l))^2$ to the preceding equation on the right side we

obtain: $((f(a_0)+j_0)(f(b_l)+k_l))^3+\sum_{t=1}^{t=l-1}(f(a_{l-t})+j_{l-t})(f(b_t)+k_t)((f(a_0)+j_0)(f(b_l)+k_l))^2=0$. Hence $((f(a_0)+j_0)(f(b_l)+k_l))^3=0$ and so $(f(a_0)+j_0)(f(b_l)+k_l)\in J\cap nil(B)=0$. Thus $(f(a_0)+j_0)(f(b_l)+k_l)=0$ which is a contradiction. Consequently, $(f(a_0)+j_0)(f(b_t)+k_t)=0$ for every $t\in\{0,...,m\}$. Now, set $F_1(x)=(f(a_1)+j_1)+(f(a_2)+j_2)x+...+(f(a_n)+j_n)x^{n-1}$. Then $F(x)=(a_0,f(a_0)+j_0)+xF_1(x)$ and by the claim, $(a_0,f(a_0)+j_0)G(x)=0$. Thus $F_1(x)G(x)=0$ and by the induction hypothesis, $(f(a_i)+j_i)(f(b_t)+k_t)=0$ for every $1\leq i\leq n$ and $0\leq t\leq m$. Therefore $(f(a_i)+j_i)(f(b_t)+k_t)=0$ for every $0\leq i\leq n$ and $0\leq t\leq m$ and hence $(a_i,f(a_i)+j_i)(b_t,f(b_t)+k_t)=0$ for every $0\leq i\leq n$ and $0\leq t\leq m$. It follows that $A\bowtie^f J$ is Armendariz.

5. Assume that $f^{-1}(J)\cap nil(A)=(0)$ and $f(A)+J$ is an Armendariz ring. Our argument is similar to that one in 4. Let $F(x)=\sum_{i=0}^{i=n}(a_i,f(a_i)+j_i)x^i$ and $G(x)=\sum_{j=0}^{j=m}(b_j,f(b_j)+k_j)x^j$ be two polynomials in $(A\bowtie^f J)[x]$ such that $F(x)G(x)=0$. Set $f_B(x)=\sum_{i=0}^{i=n}(f(a_i)+j_i)x^i$, $g_B(x)=\sum_{j=0}^{j=m}(f(b_j)+k_j)x^j$, $f_A(x)=\sum_{i=0}^{i=n}a_ix^j$ and $g_A(x)=\sum_{j=0}^{j=m}b_jx^j$. Since $F(x)G(x)=0$, $f_Ag_A=0$ and $f_Bg_B=0$. Thus $(f(a_i)+j_i)(f(b_j)+k_j)=0$ for every i,j since $f(A)+J$ is an Armendariz ring; and hence $a_ib_j\in f^{-1}(J)$. To show that $a_ib_j=0$ for every i,j, we proceed by induction on the degree n of $F(x)$. If $n=0$, done. Suppose that $n\geq 1$ and the induction hypothesis. First we show that $a_0b_j=0$ for every $0\leq j\leq m$. Indeed, suppose that $\exists j\in\{0;...;m\}$ such that $a_0b_j\neq 0$. Let k be the smallest positive integer in $\{0,...,m\}$ such that $a_0b_k\neq 0$. Then for $j\in\{0,...,k-1\}$, $a_0b_j=0$ and so $(b_jf^{-1}(J)a_0)^2=0$. Then $b_jf^{-1}(J)a_0\subseteq f^{-1}(J)\cap nil(A)=(0)$ and so $b_jf^{-1}(J)a_0=0$. Hence $(a_{k-j}b_j)(a_0b_k)^2=a_{k-j}b_ja_0b_ka_0b_k\in a_{k-j}(b_jf^{-1}(J)a_0)b_k=0$. The coefficient of the term x^k in $f_A(x)g_A(x)=0$ is $0=a_0b_k+a_1b_{k-1}+...+a_kb_0=a_0b_k+\sum_{j=1}^{j=k-1}a_{k-j}b_j$. Multiplying $(a_0b_k)^2$ to the preceding equation on the right side, we obtain $(a_0b_k)^3+\sum_{j=1}^{j=k-1}(a_{k-j}b_j)(a_0b_k)^2=0$. Hence $(a_0b_k)^3=0$ and so $(a_0b_k)\in f^{-1}(J)\cap nil(A)=(0)$, which is a contradiction. Consequently $a_0b_j=0$, for every $j\in\{0,...,m\}$. Finally, as in 4, set $F_1(x)=(f(a_1)+j_1)+(f(a_2)+j_2)x+...+(f(a_n)+j_n)x^{n-1}$. Then $F(x)=(a_0,f(a_0)+j_0)+xF_1(x)$ and by the claim, $(a_0,f(a_0)+j_0)G(x)=0$. Thus $F_1(x)G(x)=0$ and by the induction hypothesis $a_ib_j=0$ for every $1\leq i\leq n$ and $0\leq j\leq m$. Therefore $a_ib_j=0$ for every i,j and hence $(a_i,f(a_i)+j_i)(b_j,f(b_j)+k_j)=0$ for every i,j. It follows that $A\bowtie^f J$ is Armendariz.

1.3 Nil-Armendariz property in amalgamated algebra along an ideal

Theorem 1.3.1.

Let (A,B) be a pair of rings, $f: A \to B$ be a ring homomorphism and J be a proper ideal of B, then

1. *If $A \bowtie^f J$ is a nil-Armendariz ring, then so is A.*
2. *If A and $f(A)+J$ are nil-armendariz rings, then so is $A \bowtie^f J$.*
3. *Assume that $J \cap S \neq \varnothing$ where S is the set of regular central element of B. Then $A \bowtie^f J$ is a nil-Armendariz ring if and only if $f(A)+J$ and A are nil-Armendariz rings.*
4. *Assume that $J \subseteq nil(B)$. Then $A \bowtie^f J$ is a nil-Armendariz ring if and only if A is a nil-Armendariz ring.*
5. *Assume that $f^{-1}(J) \subseteq nil(A)$. Then $A \bowtie^f J$ is a nil-Armendariz ring if and only if $f(A)+J$ is a nil-Armendariz ring.*
6. *Assume that f is injective.*

   i) $f(A) \cap J = 0$. Then $A \bowtie^f J$ is a nil-Armendariz ring if and only if $f(A)+J$ is a nil-Armendariz ring.

   ii) $J \subseteq nil(B)$. Then $A \bowtie^f J$ is a nil-Armendariz ring if and only if $f(A)+J$ is a nil-Armendariz ring.

Proof. The proofs of the assertions 1, 2 and 3 are similar to 1, 2 and 3 in Theorem 1.2.2..

4. Suppose that A is a nil-Armendariz ring. Then $\frac{A\bowtie^f J}{0\times J} \simeq A$ is a nil-Armendariz ring. Let $F(x)=\sum_{i=0}^{i=n}(a_i, f(a_i)+j_i)x^i$ and $G(x)=\sum_{j=0}^{j=m}(b_j, f(b_j)+k_j)x^j$ be two polynomials in $(A \bowtie^f J)[x]$ such that $F(x)G(x)=\sum_{k=0}^{k=n+m}(\sum_{i+j=k}(a_ib_j,(f(a_i)+j_i)(f(b_j)+k_j)))x^k \in nil((A \bowtie^f J)[x])$. Set $\overline{F(x)}=\sum_{i=0}^{i=n}\overline{(a_i, f(a_i)+j_i)}x^i$ and $\overline{G(x)}=\sum_{j=0}^{j=m}\overline{(b_j, f(b_j)+k_j)}x^j$ in $\frac{A\bowtie^f J}{0\times J}[x]$. Then $F(x)G(x) \in nil(A \bowtie^f J)[x]$ implies that $\overline{F(x)}\,\overline{G(x)} \in nil\frac{A\bowtie^f J}{0\times J}[x]$. Consequently $\overline{(a_i, f(a_i)+j_i)}\,\overline{(b_j, f(b_j)+k_j)} \in nil\frac{A\bowtie^f J}{0\times J}$ since $\frac{A\bowtie^f J}{0\times J}$ is nil-Armendariz. Hence $(a_ib_j,(f(a_i)+j_i)(f(b_j)+k_j))^{p_{ij}} \in 0 \times J$ for some

integer p_{ij}. Therefore $((f(a_i)+j_i)(f(b_j)+k_j))^{p_{ij}} \in J \subseteq nil(B)$. Hence $(a_i, f(a_i)+j_i)(b_j, f(b_j)+k_j) \in nil(A \bowtie^f J)$ and this shows that $A \bowtie^f J$ is nil-Armendariz.

5. Assume that $f^{-1}(J) \subseteq nil(A)$ and suppose that $A \bowtie^f J$ is nil-Armendariz. Let $f_A(x) = \sum_{i=0}^{i=n}(f(a_i)+j_i)x^i$ and $g_A(x) = \sum_{j=0}^{j=m}(f(b_j)+k_j)x^j$ such that $f_A(x)g_A(x) \in nil(f(A)+J)[x]$. Let $F(x) = \sum_{i=0}^{i=n}(a_i, f(a_i)+j_i)x^i$ and $G(x) = \sum_{j=0}^{j=m}(b_j, f(b_j)+k_j)x^j$. Since $f_A(x)g_A(x) = \sum_{k=0}^{k=n+m}(\sum_{i+j=k}(f(a_i)+j_i)(f(b_j)+k_j))x^k \in nil(f(A)+J)[x]$, $\sum_{i+j=k}(f(a_i)+j_i)(f(b_j)+k_j) \in nil(f(A)+J)$ for every $k \in \{0,\dots,n+m\}$. Thus $\sum_{i+j=k}(f(a_ib_j)+t_{ij}) \in nil(f(A)+J)$ with $t_{ij} \in J$. Hence, for every $k \in \{0,\dots,n+m\}$, $f(\sum_{i+j=k}a_ib_j)+\sum_{i+j=k}t_{ij}$ is nilpotent. So $(f(\sum_{i+j=k}a_ib_j))^{n_{ij}} \in J$ for some positive integer n_{ij}, and therefore $(\sum_{i+j=k}a_ib_j)^{n_{ij}} \in f^{-1}(J) \subseteq nil(A)$ which, in turn, implies that $\sum_{i+j=k}a_ib_j \in nil(A)$. Consequently, $F(x)G(x) \in nil(A \bowtie^f J)[x]$ and hence $(\sum_{i+j=k}a_ib_j, \sum_{i+j=k}(f(a_i)+j_i)(f(b_j)+k_j)) \in nil(A \bowtie^f J)$. Since $A \bowtie^f J$ is a nil-Armendariz ring,
   $(a_ib_j, (f(a_i)+j_i)(f(b_j)+k_j)) = (a_i, f(a_i)+j_i)(b_j, f(b_j)+k_j)$ is nilpotent and so $(f(a_i)+j_i)(f(b_j)+k_j) \in nil(f(A)+J)$. Hence $f(A)+J$ is nil-Armendariz, as desired.
   The converse is similar to 4 by using the fact that $\frac{A \bowtie^f J}{f^{-1}(J)\times 0} \simeq f(A)+J$.

6. Assume that f is injective.

   i) $f(A)\cap J = 0$. In this case $A \bowtie^f J \simeq f(A)+J$ and the conclusion follows.

   ii) Assume that $J \subseteq nil(B)$ and suppose that $f(A)+J$ is nil-Armendariz. Let $F(x) = \sum_{i=0}^{i=n}(a_i, f(a_i)+j_i)x^i$ and $G(x) = \sum_{j=0}^{j=m}(b_j, f(b_j)+k_j)x^j$ be two polynomials in $(A \bowtie^f J)[x]$ such that $F(x)G(x) \in nil(A \bowtie^f J)[x]$. Set $f_B(x) = \sum_{i=0}^{i=n}(f(a_i)+j_i)x^i$ and $g_B(x) = \sum_{j=0}^{j=m}(f(b_j)+k_j)x^j$. Then $F(x)G(x) \in nil(A \bowtie^f J)[x]$ implies that $f_B(x)g_B(x) \in nil(f(A)+J)[x]$. Hence $(f(a_i)+j_i)(f(b_j)+k_j) \in nil(f(A)+J)$ since $f(A)+J$ is nil-Armendariz. Now, we show that a_ib_j is nilpotent. Indeed, since $(f(a_i)+j_i)(f(b_j)+k_j) = (f(a_ib_j)+t_{ij}) \in nil(f(A)+J), t_{ij} \in J$, $(f(a_ib_j)+t_{ij})^{n_{ij}} = 0$ for some positive integer n_{ij}. Therefore $(f(a_ib_j))^{n_{ij}} = f((a_ib_j)^{n_{ij}}) \in J \subseteq nil(B)$ and so $(f((a_ib_j)^{n_{ij}}))^{m_{ij}} = 0$ for some positive integer m_{ij}.
   Hence $f(((a_ib_j)^{n_{ij}})^{m_{ij}}) = 0$ and therefore $(a_ib_j)^{n_{ij}m_{ij}} = 0$ since f is injective. Consequently, $(a_i, f(a_i)+j_i)(b_j, f(b_j)+k_j)$ is nilpotent and this shows that $A \bowtie^f J$ is nil-Armendariz.
   Conversely, suppose that $A \bowtie^f J$ is nil-Armendariz and let

$f_A(x) = \sum_{i=0}^{i=n}(f(a_i)+j_i)x^i$ and $g_A(x) = \sum_{j=0}^{j=m}(f(b_j)+k_j)x^j$ be two polynomials in $(f(A)+J)[x]$ such that $f_A(x)g_A(x) \in nil(f(A)+J)[x]$. Set $F(x) = \sum_{i=0}^{i=n}(a_i, f(a_i)+j_i)x^i$ and $G(x) = \sum_{j=0}^{j=m}(b_j, f(b_j)+k_j)x^j$. Then

$$\begin{aligned} F(x)G(x) &= \sum_{k=0}^{k=n+m}(\sum_{i+j=k}(a_ib_j,(f(a_i)+j_i)(f(b_j)+k_j))x^k \\ &= \sum_{k=0}^{k=n+m}(\sum_{i+j=k}a_ib_j, \sum_{i+j=k}(f(a_i)+j_i)(f(b_j)+k_j))x^k \end{aligned}$$

But $f_A(x)g_A(x) \in nil(f(A)+J)[x]$ implies that $(\sum_{i+j=k}(f(a_i)+j_i)(f(b_j)+k_j))^{n_{ij}} = 0$ for some positive integer n_{ij}. Thus $(\sum_{i+j=k}(f(a_ib_j)+t_{ij}))^{n_{ij}} = 0$ for some positive integer n_{ij} and so $(f(\sum_{i+j=k}a_ib_j)+\sum_{i+j=k}t_{ij})^{n_{ij}} = 0$. Hence $(f(\sum_{i+j=k}a_ib_j)^{n_{ij}} \in J \subseteq nil(B)$ and therefore $f((\sum_{i+j=k}a_ib_j)^{m_{ij}}) = 0$ for some positive integer m_{ij}. Since f is injective, $(\sum_{i+j=k}a_ib_j)^{m_{ij}} = 0$ and hence $F(x)G(x) \in nil(A \bowtie^f J)[x]$, which in turn, implies that $(a_ib_j,(f(a_i)+j_i)(f(b_j)+k_j)) \in nil(A \bowtie^f J)$. Therefore $(f(a_i)+j_i)(f(b_j)+k_j) \in nil(f(A)+J)$ and this shows that $f(A)+J$ is nil-Armendariz.

1.4 weak Armendariz property in amalgamated algebra along an ideal

Theorem 1.4.1.

Let (A,B) be a pair of rings, $f : A \to B$ be a ring homomorphism and J be a proper ideal of B, then

1. *If $A \bowtie^f J$ is a weak Armendariz ring, then so is A.*
2. *If A and $f(A)+J$ are weak Armendariz rings, then so is $A \bowtie^f J$.*
3. *Assume that $J \cap S \neq \emptyset$ where S is the set of regular central element of B. Then $A \bowtie^f J$ is a weak Armendariz ring if and only if $f(A)+J$ and A are weak Armendariz rings.*
4. *Assume that $J \subseteq nil(B)$. Then A is weak Armendariz ring if and only if $A \bowtie^f J$ is a weak Armendariz ring.*
5. *Assume that $f^{-1}(J) \subseteq nil(A)$. If $f(A)+J$ is a weak Armendariz ring, then $A \bowtie^f J$ is a weak Armendariz ring.*
6. *Assume that f is injective.*
    - *i) $f(A) \cap J = 0$. Then $A \bowtie^f J$ is a weak Armendariz ring if and only if $f(A)+J$ is a weak Armendariz ring.*
    - *ii) $J \subseteq nil(B)$. If $f(A)+J$ is a weak Armendariz ring, then $A \bowtie^f J$ is a weak Armendariz ring.*
7. *Assume that J is semicommutative. If A is a weak Armendariz ring, then so is $A \bowtie^f J$.*
8. *Assume that $f^{-1}(J)$ is semicommutative. If $f(A)+J$ is a weak Armendariz ring, then so is $A \bowtie^f J$.*

Proof. The assertions 1, 2 and 3 are similar to 1, 2 and 3 in Theorem 1.2.2., and the assertions 4, 5 and 6 are similar to 4, 5 and 6 in Theorem 1.3.1..

7. Assume that J is semicommutative and A is weak Armendariz.
   Let $F(x) = \sum_{i=0}^{i=n}(a_i, f(a_i) + j_i)x^i$ and $G(x) = \sum_{j=0}^{j=m}(b_j, f(b_j) + k_j)x^j$ be two poly-

nomials in $A \bowtie^f J[x]$ such that $F(x)G(x)=0$ and set $f_A(x)=\sum_{i=0}^{i=n} a_i x^j$, $g_A(x)=\sum_{j=0}^{j=m} b_j x^j$, $f_B(x)=\sum_{i=0}^{i=n}(f(a_i)+j_i)x^i$ and $g_B(x)=\sum_{j=0}^{j=m}(f(b_j)+k_j)x^j$.

Then $F(x)G(x)=0$ implies that $f_A(x)g_A(x)=\sum_{l=0}^{l=n+m}(\sum_{i+j=l} a_i b_j)x^l=0$ and $f_B(x)g_B(x)=\sum_{l=0}^{l=n+m}(\sum_{i+j=l}(f(a_i)+j_i)(f(b_j)+k_j))x^l=0$. Hence $\sum_{i+j=l} a_i b_j=0$ for all $l=0,1...,n+m$ and $\sum_{i+j=l}(f(a_i)+j_i)(f(b_j)+k_j)=0$ for all $l=0,1...,n+m$. Thus $a_i b_j \in nil(A)$ since A is weak Armendariz, and so $(a_i b_j)^{n_{ij}}=0$ for some positive integer n_{ij}. To show that $(f(a_i)+j_i)(f(b_j)+k_j) \in nil(f(A)+J)$, we proceed by induction on $i+j$.

If $i+j=0$, we have $(f(a_0)+j_0)(f(b_0)+k_0)=0 \in nil(f(A)+J)$

Let l be a positive integer such that $(f(a_i)+j_i)(f(b_j)+k_j) \in nil(f(A)+J)$ when $i+j<l$. We will show that $(f(a_i)+j_i)(f(b_j)+k_j) \in nil(f(A)+J)$ when $i+j=l$. We have $((f(a_0)+j_0)(f(b_l)+k_l))^{n_{0l}=p} \in J$ since $(a_0 b_l)^p=0$. By the induction hypothesis, $(f(a_0)+j_0)(f(b_{l-1})+k_{l-1}) \in nil(f(A)+J)$. Let t be a positive integer such that $((f(a_0)+j_0)(f(b_{l-1})+k_{l-1}))^t=0$. Then $((f(b_{l-1})+k_{l-1})(f(a_0)+j_0))^{t+1}=0$, and hence $(((f(a_1)+j_1)(f(b_{l-1})+k_{l-1}))((f(a_0)+j_0)(f(b_l)+k_l))^{p+1}(f(a_1)+j_1)) \times ((f(b_{l-1})+k_{l-1})(f(a_0)+j_0))^{t+1}((f(b_{l-1})+k_{l-1})((f(a_0)+j_0)(f(b_l)+k_l))^{p+1})=0$. Since $(((f(a_1)+j_1)(f(b_{l-1})+k_{l-1})) \times ((f(a_0)+j_0)(f(b_l)+k_l))^{p+1}(f(a_1)+j_1))(f(b_{l-1})+k_{l-1})(f(a_0)+j_0) \in J$, $((f(b_{l-1})+k_{l-1})(f(a_0)+j_0))^t((f(b_{l-1})+k_{l-1})((f(a_0)+j_0)(f(b_l)+k_l))^{p+1}) \in J$, $(f(b_l)+k_l)((f(a_0)+j_0)(f(b_l)+k_l))^p)(f(a_1)+j_1) \in J$ and J is semicommutative, it follows that $(((f(a_1)+j_1)(f(b_{l-1})+k_{l-1}))((f(a_0)+j_0)(f(b_l)+k_l))^{p+1}(f(a_1)+j_1))((f(b_{l-1})+k_{l-1})(f(a_0)+j_0)) \times (f(b_l)+k_l)((f(a_0)+j_0)(f(b_l)+k_l))^p)(f(a_1)+j_1) \times ((f(b_{l-1})+k_{l-1})(f(a_0)+j_0))^t((f(b_{l-1})+k_{l-1})((f(a_0)+j_0)(f(b_l)+k_l))^{p+1})=0$. Hence $[((f(a_1)+j_1)(f(b_{l-1})+k_{l-1}))((f(a_0)+j_0)(f(b_l)+k_l))^{p+1}]^2(f(a_1)+j_1) \times ((f(b_{l-1})+k_{l-1})(f(a_0)+j_0))^t((f(b_{l-1})+k_{l-1})((f(a_0)+j_0)(f(b_l)+k_l))^{p+1})=0$. Continuing this procedure, we obtain $[((f(a_1)+j_1)(f(b_{l-1})+k_{l-1}))((f(a_0)+j_0)(f(b_l)+k_l))^{p+1}]^{t+3}=0$.

Thus $((f(a_1)+j_1)(f(b_{l-1})+k_{l-1}))((f(a_0)+j_0)(f(b_l)+k_l))^{p+1} \in nil(J)$.

Similarly, we can show that $((f(a_i)+j_i)(f(b_{l-i})+k_{l-i}))((f(a_0)+j_0)(f(b_l)+k_l))^{p+1} \in nil(J)$ for $i=2,...,l$.

Since J is semicommutative, $nil(J)$ is an ideal and consequently $\sum_{i=1}^{i=l}(f(a_i)+j_i)(f(b_{l-i})+k_{l-i}))((f(a_0)+j_0)(f(b_l)+k_l))^{p+1} \in nil(J)$. Multiplying the equation $\sum_{i+j=l}(f(a_i)+j_i)(f(b_j)+k_j)=0$ on the right side by $((f(a_0)+j_0)(f(b_l)+k_l))^{p+1}$, we obtain $((f(a_0)+j_0)(f(b_l)+k_l))^{p+2}=-\sum_{i=1}^{i=l}(f(a_i)+j_i)(f(b_j)+k_j) \times$

$((f(a_0)+j_0)(f(b_l)+k_l))^{p+1} \in nil(J)$.

Thus $(f(a_0)+j_0)(f(b_l)+k_l) \in nil(f(A)+J)$.

Let $q=n_{1,l-1}$. Then $((f(a_1)+j_1)(f(b_{l-1})+k_{l-1}))^q \in J$. By analogy with the above proof, we have

$$\sum_{i=2}^{i=l}(f(a_i)+j_i)(f(b_{l-i})+k_{l-i})((f(a_1)+j_1)(f(b_{l-1})+k_{l-1}))^{q+1}\in nil(J)$$

Suppose that $((f(a_0)+j_0)(f(b_l)+k_l))^s=0$. Then $((f(a_1)+j_1)(f(b_{l-1})+k_{l-1}))^{q+1}((f(a_0)+j_0)(f(b_l)+k_l))^s((f(a_1)+j_1)\times(f(b_{l-1})+k_{l-1}))^{q+1}=0$ Since $((f(a_1)+j_1)(f(b_{l-1})+k_{l-1}))^{q+1}\in J$ and J is semicommutative,

$$((f(a_0)+j_0)(f(b_l)+k_l)((f(a_1)+j_1)(f(b_{l-1})+k_{l-1}))^{q+1})^{s+1}=0$$

Therefore

$$((f(a_0)+j_0)(f(b_l)+k_l)((f(a_1)+j_1)(f(b_{l-1})+k_{l-1}))^{q+1}\in nil(J)$$

Multiplying the equation $\sum_{i+j=l}(f(a_i)+j_i)(f(b_j)+k_j))=0$ on the right side by $((f(a_1)+j_1)(f(b_{l-1})+k_{l-1}))^{q+1}$, we obtain $((f(a_1)+j_1)(f(b_{l-1})+k_{l-1}))^{q+2}=-\sum_{i=2}^{i=l}((f(a_i)+j_i)(f(b_{l-i})+k_{l-i}))((f(a_1)+j_1)(f(b_{l-1})+k_{l-1}))^{q+1}-((f(a_0)+j_0)(f(b_l)+k_l))((f(a_1)+j_1)(f(b_{l-1})+k_{l-1}))^{q+1}\in nil(J)$. Therefore $(f(a_1)+j_1)(f(b_{l-1})+k_{l-1})\in nil(f(A)+J)$.
A similar argument shows that
$(f(a_2)+j_2)(f(b_{l-2})+k_{l-2})\in nil(f(A)+J)...(f(a_l)+j_l)(f(b_0)+k_0)\in nil(f(A)+J)$ Consequently $(f(a_i)+j_i)(f(b_j)+k_j)\in nil(f(A)+J)$ when $i+j=l$, and therefore $(f(a_i)+j_i)(f(b_j)+k_j)\in nil(f(A)+J)$ for every i,j. Hence $(a_i,f(a_i)+j_i)(b_j,f(b_j)+k_j)\in nil(A\bowtie^f J)$, and this shows that $A\bowtie^f J$ is weak Armendariz.

8. Assume that $f^{-1}(J)$ is semicommutative and $f(A)+J$ is weak Armendariz. Let $F(x)=\sum_{i=0}^{i=n}(a_i,f(a_i)+j_i)x^i$ and $G(x)=\sum_{j=0}^{j=m}(b_j,f(b_j)+k_j)x^j$ be two polynomials in $A\bowtie^f J[x]$ such that $F(x)G(x)=0$ and set $f_A(x)=\sum_{i=0}^{i=n}a_ix^i$, $g_A(x)=\sum_{j=0}^{j=m}b_jx^j$, $f_B(x)=\sum_{i=0}^{i=n}(f(a_i)+j_i)x^i$ and $g_B(x)=\sum_{j=0}^{j=m}(f(b_j)+k_j)x^j$.
Then $F(x)G(x)=0$ implies that $f_A(x)g_A(x)=\sum_{l=0}^{l=n+m}(\sum_{i+j=l}a_ib_j)x^l=0$ and $f_B(x)g_B(x)=\sum_{l=0}^{l=n+m}(\sum_{i+j=l}(f(a_i)+j_i)(f(b_j)+k_j))x^l=0$. Hence $\sum_{i+j=l}a_ib_j=0$ for all $l=0,1...,n+m$ and $\sum_{i+j=l}(f(a_i)+j_i)(f(b_j)+k_j)=0$ for all $l=0,1...,n+m$. Therefore $(f(a_i)+j_i)(f(b_j)+k_j)\in nil(f(A)+J)$ since $f(A)+J$ is weak Armendariz. Since $(f(a_i)+j_i)(f(b_j)+k_j)=(f(a_ib_j)+t_{ij})\in nil(f(A)+J)$, where $t_{ij}\in J$, $(f(a_ib_j)+t_{ij})^{n_{ij}}=0$, for some positive integer n_{ij}. Therefore $(f(a_ib_j))^{n_{ij}}=f((a_ib_j)^{n_{ij}})\in J$, and hence $(a_ib_j)^{n_{ij}}\in f^{-1}(J)$. Now we show that $a_ib_j\in nil(A)$ by induction on $i+j$.
If $i+j=0$, we have $a_0b_0=0\in nil(A)$ and so we are done.
Let l be a positive integer such that $a_ib_j\in nil(A)$ when $i+j<l$. As in \$7, we will show that $a_ib_j\in nil(A)$ when $i+j=l$.

We have $(a_0b_l)^{n_{0l}=p} \in f^{-1}(J)$ and by the induction hypothesis, $(a_0b_{l-1}) \in nil(A)$. Let t be a positive integer such that $(a_0b_{l-1})^t = 0$. Then $(b_{l-1}a_0)^{t+1} = 0$ and hence $((a_1b_{l-1})(a_0b_l)^{p+1}a_1)(b_{l-1}a_0)^{t+1}(b_{l-1}(a_0b_l)^{p+1}) = 0$. Since

$$(a_1b_{l-1})(a_0b_l)^{p+1}a_1)(b_{l-1}a_0) \in f^{-1}(J)$$

$$(b_{l-1}a_0)^t(b_{l-1}(a_0b_l)^{p+1}) \in f^{-1}(J)$$

$$(b_l(a_0b_l)^pa_1 \in f^{-1}(J)$$

and $f^{-1}(J)$ is semicommutative, we obtain

$$((a_1b_{l-1})(a_0b_l)^{p+1}a_1)(b_{l-1}a_0)(b_l(a_0b_l)^pa_1)(b_{l-1}a_0)^t(b_{l-1}(a_0b_l)^{p+1}) = 0$$

Hence $[(a_1b_{l-1})(a_0b_l)^{p+1}]^2a_1(b_{l-1}a_0)^t(b_{l-1}(a_0b_l)^{p+1}) = 0$.
iterating this process, we obtain:

$$[(a_1b_{l-1})(a_0b_l)^{p+1}]^{t+3} = 0$$

Thus $(a_1b_{l-1})(a_0b_l)^{p+1} \in nilf^{-1}(J)$, and similarly we have $(a_ib_{l-i})(a_0b_l)^{p+1} \in nilf^{-1}(J)$ for $i = 2,...,l$. Since $f^{-1}(J)$ is semicommutative, $nilf^{-1}(J)$ is an ideal and consequently

$$\sum_{i=1}^{i=l}(a_ib_{l-i})(a_0b_l)^{p+1} \in nilf^{-1}(J)$$

Multiply the equation $\sum_{i+j=l} a_ib_j = 0$ on the right side by $(a_0b_l)^{p+1}$, we get:

$$(a_0b_l)^{p+2} = -\sum_{i=1}^{i=l}(a_ib_{l-i})(a_0b_l)^{p+1} \in nilf^{-1}(J)$$

Thus $a_0b_l \in nil(A)$. Now, let $q = n_{1,l-1}$. Then $(a_1b_{l-1})^q \in f^{-1}(J)$. As in the above proof, we have

$$\sum_{i=2}^{i=l}(a_ib_{l-i})(a_1b_{l-1})^{q+1} \in nilf^{-1}(J)$$

. Suppose that $(a_0b_l)^s = 0$. Then

$$(a_1b_{l-1})^{q+1}(a_0b_l)^s(a_1b_{l-1})^{q+1} = 0$$

Since $(a_1b_{l-1})^{q+1} \in f^{-1}(J)$ and $f^{-1}(J)$ is semicommutative,

$$((a_0b_l)(a_1b_{l-1})^{q+1})^{s+1} = 0$$

Therefore

$$(a_0b_l)(a_1b_{l-1})^{q+1} \in nil(f^{-1}(J))$$

If we multiply the equation $\sum_{i+j=l} a_i b_j = 0$ on the right side by $(a_1 b_{l-1})^{q+1}$, we obtain

$$(a_1 b_{l-1})^{q+2} = -\sum_{i=2}^{i=l} (a_i b_{l-i})(a_1 b_{l-1})^{q+1} - (a_0 b_l)(a_1 b_{l-1})^{q+1} \in nil f^{-1}(J)$$

Therefore $a_1 b_{l-1} \in nil(A)$. Similarly, we have $a_2 b_{l-2} \in nil(A), ..., a_l b_0 \in nil(A)$ and consequently $a_i b_j \in nil(A))$ when $i + j = l$. Therefore, $a_i b_j \in nil(A)$ for every i, j and hence $(a_i, f(a_i) + j_i)(b_j, f(b_j) + k_j) \in nil A \bowtie^f J$. This shows that $A \bowtie^f J$ is weak Armendariz and complete the proof.

An open question: $A \bowtie^f J$ **is a nil-Armendariz ring if and only it is weak Armendariz**. As we mentioned in the introduction, we do not know so far any example of a weak Armendariz ring which is not a nil-Armendariz ring. This question was left open in [8]. We were not able to prove or disprove whether $A \bowtie^f J$ is a nil-Armendariz ring if and only it is weak Armendariz. A negative answer to this question will answer negatively the original one and provides a counter-example of a weak Armendariz ring that is not nil-Armendariz. However, a positive answer to this question shows that amalgamation of algebras along ideals, as a source for examples and counter-examples, cannot provide such example if it exists.

CHAPTER 2

ON POWER SERIESWISE ARMENDARIZ RINGS

Abstract.

The purpose of this chapter *, we investigate the transfer of the property of power serieswise Armendariz to trivial ring extensions, direct product of rings and the homomorphic image. The article includes a brief discussion of the scope and precision of our results.

Key Words: Power serieswise Armendariz ring, Armendariz ring, trivial ring extension, direct product of rings .

2.1 Introduction

In [9], Armendariz proved that $a_ib_j = 0$ for all i, j whenever polynomials $f = \sum_{i=0}^{i=n} a_ix^i$ and $g = \sum_{i=0}^{i=m} b_ix^i$ over a reduced ring satisfy $fg = 0$. In [68], Rege and Chhawchharia (1997) called such a ring (not necessarily reduced) Armendariz. Armendariz rings are thus a generalization of reduced rings. It is easy to see that subring of Armendariz rings are also Armendariz. Also, D. D. Anderson and V. Camillo [2], show that a ring R is Gaussian if and only if every homomorphic image of R is Armendariz. See for instance [2, 9, 60, 68].

In [8], Ramon Antoine (2008) called nil-Armendariz rings if whenever the product of two polynomials $f(x) = \sum_{i=0}^{i=n} a_ix^i$ and $g(x) = \sum_{j=0}^{j=m} b_jx^j$ in $R[x]$ satisfies $f(x)g(x) \in nil(R)[x]$ we have $a_ib_j \in nil(R)$ for each i, j. Armendariz ring is nil-Armendariz [8, Proposition 2.7]. It is easy to see that a subring of nil-Armendariz ring is nil-Armendariz. In [62], Liu and Zhao (2006), introduced weakArmendariz as generalization of Armendariz. A ring R is called a weakArmendariz ring if whenever the product of two polynomials

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$f(x)=\sum_{i=0}^{i=n}a_ix^i$ and $g(x)=\sum_{j=0}^{j=m}b_jx^j$ in $R[x]$ satisfies $f(x)g(x)=0$ we have $a_ib_j\in nil(R)$ for each i,j. It is clear that subring of weakArmendariz ring is also weakArmendariz. Obviously, nil-Armendariz rings are weakArmendariz rings. The following diagram of implications summarizes the relation between them (see for instance [8, 62]) :

$$Reduced \Longrightarrow Armendariz \Longrightarrow nil-Armendariz \Longrightarrow weakArmendariz.$$

A ring R is semicommutative if for all $a,b\in R$, $ab=0$ implies $aRb=0$. This is equivalent to the usual definition by Shin [62, Lemma 1.2]. By Huh et al., reduced rings are semicommutative. Semicommutative ring is nil-Armendariz [62, Proposition 3.3] . Thus weakArmendariz rings and nil-Armendariz rings are a common generalization of semicommutative rings and Armendariz rings. Also, a ring R is called abelian if every idempotent in R is central. Armendariz rings are abelian by the proof of [2, Theorem 6]).

In [58], Kim et al. define power serieswise Armendariz rings as ring such that for every $f(x)=\sum_{i=0}^{\infty}a_ix^i$ and $g(x)=\sum_{i=0}^{\infty}b_ix^i\in R[[x]]$ such that $fg=0$, then $a_ib_j=0$ for every i and j. Power serieswise Armendariz rings are clearly Armendariz rings, but the converse is false by [7, Example 2]. Recall that a reduced ring is power serieswise Armendariz. It is easy to see that subring of power serieswise Armendariz is also power serieswise Armendariz. See for instance [8, 7, 62, 58].

Let A be a ring and a bi-module ${}_AE_A$. $A\propto E$ is the set of pairs (a,e) with pairwise addition and multiplication given by $(a,e)(b,f)=(ab,af+eb)$. $A\propto E$ is called the trivial ring extension of A by E (also called the idealization of E over A).

In this paper, we investigate the transfer of the property of power serieswise Armendariz to trivial ring extensions, direct product of rings and the homomorphic image. Our results generate new and original examples which enrich the current literature with new families of power serieswise Armendariz rings.

2.2 Transfer of power serieswise Armendariz ring property

Now we study the transfer of the property of power serieswise Armendariz to the trivial ring extensions.

Let A be a ring. We claim that n-by-n upper triangular matrix rings over A are not power serieswise Armendariz, where $n\geq 2$. It is enough to show that the 2-by-2 upper triangular matrix ring over A is not power serieswise Armendariz because each subring of a power serieswise Armendariz ring is also power serieswise Armendariz. Let S be the 2-by-2 upper triangular matrix ring over A, and let $f(x) = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} 1 & -1 \\ 0 & 0 \end{pmatrix} x$, and $g(x) = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} + \begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix} x$ be polynomials in $S[[x]]$. Then $f(x)g(x) = 0$, but $\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix} \neq 0$. So S is not power serieswise Armendariz and consequently every n-by-n upper triangular matrix rings over A is not power serieswise Armendariz. But we may find subrings of the 3-by-3 upper triangular matrix rings which may be power serieswise Armendariz, as shown by the next result.

Proposition 2.2.1.

Let A be a reduced ring. Then
$S=\{ \begin{pmatrix} a & b & c \\ 0 & a & d \\ 0 & 0 & a \end{pmatrix} / a,b,c,d \in A\}$ *is a power serieswise Armendariz ring.*

Proof. We use the method in the proof of [72, Proposition 2.5]. First notice that for $\begin{pmatrix} a_1 & b_1 & c_1 \\ 0 & a_1 & d_1 \\ 0 & 0 & a_1 \end{pmatrix}, \begin{pmatrix} a_2 & b_2 & c_2 \\ 0 & a_2 & d_2 \\ 0 & 0 & a_2 \end{pmatrix} \in S$, we can denote their addition and multiplication by

$$(a_1,b_1,c_1,d_1)+(a_2,b_2,c_2,d_2) = (a_1+a_2,b_1+b_2,c_1+c_2,d_1+d_2)$$

and

$$(a_1,b_1,c_1,d_1)(a_2,b_2,c_2,d_2) = (a_1a_2,a_1b_2+b_1a_2,a_1c_2+b_1d_2+c_1a_2,a_1d_2+d_1a_2)$$

respectively. So every polynomials in $S[[x]]$ can be expressed in the form $(p_0(x), p_1(x), p_2(x), p_3(x))$ for some $p_i(x)$ in $A[[x]]$.

Let $f(x) = (f_0(x), f_1(x), f_2(x), f_3(x))$ and $g(x) = (g_0(x), g_1(x), g_2(x), g_3(x))$ be elements of $S[[x]]$. Assume that $f(x)g(x) = 0$. Then $f(x)g(x) = (f_0(x)g_0(x), f_0(x)g_1(x) + f_1(x)g_0(x), f_0(x)g_2(x) + f_1(x)g_3(x) + f_2(x)g_0(x), f_0(x)g_3(x) + f_3(x)g_0(x)) = 0$. So we have the following system of equations:

1. $f_0(x)g_0(x) = 0$;
2. $f_0(x)g_1(x) + f_1(x)g_0(x) = 0$;
3. $f_0(x)g_2(x) + f_1(x)g_3(x) + f_2(x)g_0(x) = 0$;
4. $f_0(x)g_3(x) + f_3(x)g_0(x) = 0$.

From Equation (1), we see that $g_0(x)f_0(x) = 0$ since $A[[x]]$ is reduced. If we multiply equation (2) on the right side by $f_0(x)$, then $f_0(x)g_1(x)f_0(x) + f_1(x)g_0(x)f_0(x) = 0$. So $f_0(x)g_1(x)f_0(x) = 0$, if we multiply by $g_1(x)$ on the right side and use the fact that $R[[x]]$ is reduced, we have $f_0(x)g_1(x) = 0$ and hence $f_1(x)g_0(x) = 0$. Also if we multiply equation (4) on the right side by $f_0(x)$, then $f_0(x)g_3(x)f_0(x) + f_3(x)g_0(x)f_0(x) = 0$. So $f_0(x)g_3(x) = 0$ and hence $f_3(x)g_0(x) = 0$. Now if we multiply equation (3) on the right side by $f_0(x)$, then $f_0(x)g_2(x)f_0(x) + f_1(x)g_3(x)f_0(x) + f_2(x)g_0(x)f_0(x) = 0$. So $f_0(x)g_2(x) = 0$ and hence equation (3) becomes $f_1(x)g_3(x) + f_2(x)g_0(x) = 0$. If we multiply the last equation on the right side by $f_1(x)$, then we have $f_1(x)g_3(x) = 0$ and so $f_2(x)g_0(x) = 0$. Now let $f(x) = \sum_{i=0}^{\infty} \begin{pmatrix} a_i & b_i & c_i \\ 0 & a_i & d_i \\ 0 & 0 & a_i \end{pmatrix} x^i$ and $g(x) = \sum_{j=0}^{\infty} \begin{pmatrix} a'_j & b'_j & c'_j \\ 0 & a'_j & d'_j \\ 0 & 0 & a'_j \end{pmatrix} x^j$ where $f_0 = \sum_{i=0}^{\infty} a_i x^i$, $f_1 = \sum_{i=0}^{\infty} b_i x^i$, $f_2 = \sum_{i=0}^{\infty} c_i x^i$, $f_3 = \sum_{i=0}^{\infty} d_i x^i$, $g_0 = \sum_{j=0}^{\infty} a'_j x^j$, $g_1 = \sum_{j=0}^{\infty} b'_j x^j$, $g_2 = \sum_{j=0}^{\infty} c'_j x^j$, $g_3 = \sum_{j=0}^{\infty} d'_j x^j$. Then we obtain that $a_i a'_j = 0$, $a_i b'_j = 0$, $b_i a'_j = 0$, $a_i c'_j = 0$, $b_i d'_j = 0$,$c_i a'_j = 0$, $a_i d'_j = 0$ and $d_i a'_j = 0$ for all i, j by the preceding results, and condition that A is reduced.

Consequently $\begin{pmatrix} a_i & b_i & c_i \\ 0 & a_i & d_i \\ 0 & 0 & a_i \end{pmatrix} \begin{pmatrix} a'_i & b'_i & c'_i \\ 0 & a'_i & d'_i \\ 0 & 0 & a'_i \end{pmatrix} = 0$ for all i, j, therefore S is a power serieswise Armendariz ring.

Let S be a reduced ring and let

$A_n = \{ \begin{pmatrix} a & a_{12} & a_{13} & . & . & . & a_{1n} \\ 0 & a & a_{23} & . & . & . & a_{2n} \\ 0 & 0 & a & . & . & . & a_{3n} \\ . & . & . & . & . & . & . \\ . & . & . & . & . & . & . \\ 0 & 0 & 0 & . & . & . & a \end{pmatrix} / a, a_{ij} \in S\}$. Based on Proposition 2.2.1, one may suspect that A_n may be also a power serieswise Armendariz ring for $n \geq 4$. But the following example erases this possibility.

Example 2.2.2.

Let S be a ring. Then

$A_4 = \{ \begin{pmatrix} a & a_{12} & a_{13} & a_{14} \\ 0 & a & a_{23} & a_{24} \\ 0 & 0 & a & a_{34} \\ 0 & 0 & 0 & a \end{pmatrix} / a, a_{ij} \in S\}$ *is not power serieswise Armendariz.*

Proof. Let $f(x) = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} x$

and $g(x) = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix} x$ be polynomials in $A_4[[x]]$.

Then $f(x)g(x) = 0$, but $\begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix} \neq 0$, as desired.

Given a ring A and a bimodule ${}_AE_A$, we have $A \propto E$ is isomorphic to the ring of all matrices $\begin{pmatrix} r & m \\ 0 & r \end{pmatrix}$, where $r \in A$ and $m \in E$.

Corollary 2.2.3.

Let A be a reduced ring. Then the trivial extension $A \propto A$ is a power serieswise Armendariz ring.

Proof. Notice that $A \propto A$ is isomorphic to S=$\{ \begin{pmatrix} a & b & 0 \\ 0 & a & 0 \\ 0 & 0 & a \end{pmatrix} / a, b \in A\}$ (It is easy to see that the mapping defined via $\varphi(\begin{pmatrix} a & b \\ 0 & a \end{pmatrix}) = \begin{pmatrix} a & b & 0 \\ 0 & a & 0 \\ 0 & 0 & a \end{pmatrix}$ is a ring isomorphism) and that each subring of a power serieswise Armendariz is also power serieswise Armendariz. Thus $A \propto A$ is a power serieswise Armendariz ring by Proposition 2.2.1.

From Corollary 2.2.3, one can may suspect that if A is power serieswise Armendariz then $A \propto A$ is power serieswise Armendariz. But the following example eliminates this possibility.

Example 2.2.4.

Let T be a reduced ring. Then $R = \{\begin{pmatrix} r & m \\ 0 & r \end{pmatrix} / r, m \in T\}$ is a power serieswise Armendariz by Corollary 2.2.3. Let $S = \{\begin{pmatrix} A & B \\ 0 & A \end{pmatrix} / A, B \in R\}$ and let

$$f(x) = \begin{pmatrix} \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} & \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \\ \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} & \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \end{pmatrix} + \begin{pmatrix} \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} & \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} \\ \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} & \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \end{pmatrix} x \quad \text{and,}$$

$$g(x) = \begin{pmatrix} \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} & \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \\ \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} & \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \end{pmatrix} + \begin{pmatrix} \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} & \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \\ \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} & \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \end{pmatrix} x \quad \text{be polynomials in}$$

$S[[x]]$. Then $f(x)g(x) = 0$, and

$$\begin{pmatrix} \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} & \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \\ \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} & \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \end{pmatrix} \begin{pmatrix} \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} & \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \\ \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} & \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \end{pmatrix} \neq 0 .$$

Thus S is not power serieswise Armendariz.

But we may have an affirmative answer to this situation, taking a condition "(A, M) is a local ring and E a bimodule such that $ME = 0$ and $EM = 0$".

Theorem 2.2.5.

Let A be a ring, E be a nonzero bimodule. Then: Assume that (A, M) is a local ring and E a bimodule such that $ME = 0$ and $EM = 0$. Then, $A \propto E$ is a power serieswise Armendariz ring if and only if so is A.

Proof.

If $A \propto E$ is a power serieswise Armendariz, then so is A since A is a subring of $A \propto E$. Conversely, assume that A is power serieswise Armendariz and let $f = \sum_{i=0}^{\infty}(a_i, e_i)x^i$, $g = \sum_{j=0}^{\infty}(b_j, f_j)x^j$ in $(A \propto E)[[x]]$ such that $fg = 0$. It remains to show that $(a_ib_j, a_if_j + e_ib_j) = 0$ for all i, j. For this purpose, we set $f_A = \sum_{i=0}^{\infty} a_ix^i$ and $g_A = \sum_{j=0}^{\infty} b_jx^j$ in $A[[x]]$. We have $f_Ag_A = 0$ since $fg = 0$, then $a_ib_j = 0$ for all i, j since A is power serieswise Armendariz. So it suffices to show that $a_if_j + e_ib_j = 0$. Two cases are possible:

1^{st} case: $a_i, b_j \in M$ for all i, j. Then $a_if_j + e_ib_j = 0$ for all i, j since $ME = 0$ and $EM = 0$.

2^{nd} case: One of $a_i, b_j \notin M$.

Without loss of generality, we may assume that $a_k \notin M$ for some positive integer k. Let i_0 be the smallest integer such that $a_{i_0} \notin M$, that is a_{i_0} is invertible (since (A, M) is local). Note since that $a_{i_0}b_j = 0$ and since a_{i_0} is invertible, then $b_j = 0$ for all j. Consequently, it suffices to show that $a_if_j = 0$ for all i, j.

Remark that $fg = 0$ implies that $\sum_{i+j=k} a_if_j = 0$ for every positive integer k.

For $k = i_0$, we have $a_{i_0}f_0 + a_{i_0-1}f_1 + \ldots + a_0f_{i_0} = a_{i_0}f_0 = 0$, then $f_0 = 0$ since a_{i_0} is invertible.

For $k = i_0 + 1$, we have $a_{i_0+1}f_0 + a_{i_0}f_1 + \ldots + a_0f_{i_0+1} = a_{i_0}f_1 = 0$, then $f_1 = 0$.

By induction we have $f_j = 0$ for all j. Consequently, $a_if_j = 0$, as desired.

Hence, in both cases $a_if_j + b_je_i = 0$ for all i, j making $A \propto E$ a power serieswise Armendariz ring and this completes the proof of Theorem 2.2.5.

Example 2.2.6.

Let K be a field, $K[[x]]$ is a local power serieswise Armendariz ring and (x) is the unique ideal maximal. Then, $K[[x]] \propto K$ is power serieswise Armendariz (which is never reduced) by Theorem 2.2.5 since $(x)S = 0$, $S(x) = 0$ and $K[[x]]$ is power serieswise Armendariz, where $S = K[[x]]/(x) \simeq K$.

Now we study the transfer of the power serieswise Armendariz property to the direct product of rings.

Theorem 2.2.7.

Let $(A_i)_{i=1,2,\ldots,n}$ be a family of rings and let $A := \prod_{i=1}^{n} A_i$. Then, A is power serieswise Armendariz ring if and only if so is A_i for each $i = 1, \ldots, n$.

Proof. Assume that $A_1 \times A_2$ is a power serieswise Armendariz ring and we must to show that A_1 is a power serieswise Armendariz ring (it is the same for A_2). Let $f = \sum_{i=0}^{\infty} a_i x^i$ and $g = \sum_{j=0}^{\infty} b_j x^j$ in $A_1[[x]]$ such that $fg = 0$ and set $f_1 := \sum_{i=0}^{\infty}(a_i, 0)x^i$ and $g_1 := \sum_{j=0}^{\infty}(b_j, 0)x^j \in (A_1 \times A_2)[[x]]$.
Hence, $f_1 g_1 = 0$ (since $fg = 0$) and so $(a_i b_j, 0) = 0$ since $(A_1 \times A_2)$ is a power serieswise Armendariz ring. Therefore, $a_i b_j = 0$, and this means that A_1 is a power serieswise Armendariz ring.

Conversely, assume that A_1 and A_2 are power serieswise Armendariz rings and let $f = \sum_{i=0}^{\infty}(a_i, e_i)x^i$ and $g = \sum_{j=0}^{\infty}(b_j, f_j)x^j \in (A_1 \times A_2)[[x]]$ such that $fg = 0$. Set $f_1 := \sum_{i=0}^{\infty} a_i x^i \in A_1[[x]]$, $f_2 := \sum_{i=0}^{\infty} e_i x^i \in A_2[[x]]$, $g_1 := \sum_{j=0}^{\infty} b_j x^j \in A_1[[x]]$ and $g_2 := \sum_{j=0}^{\infty} f_j x^j \in A_2[[x]]$. Then, $f_1 g_1 = 0$ and $f_2 g_2 = 0$ since $fg = 0$. Hence $a_i b_j = 0$ and $e_i f_j = 0$ since A_1 and A_2 are power serieswise Armendariz rings. Therefore, $(a_i, e_i)(b_j, f_j) = 0$, and this means that $A_1 \times A_2$ is a power serieswise Armendariz ring.

Corollary 2.2.8.

Let A be a ring and let $n \in \mathbb{N} - \{0\}$ be an integer. Then, A^n is power srieswise Armendariz if and only if so is A.

The following example show that the implication "A/I and I are power serieswise Armendariz imply that so is A (where I is an ideal of A)" is false, in general.

Example 2.2.9.

Let F be a field and consider the ring $A = \begin{pmatrix} F & F \\ 0 & F \end{pmatrix}$. Then A is not Armendariz (by [56, Examples 1]) and so A is not power serieswise Armendariz. Now we claim that A/I and I are power serieswise Armendariz for any nonzero ideal I of A. Note that the only nonzero proper ideals of A are $\begin{pmatrix} F & F \\ 0 & 0 \end{pmatrix}$, $\begin{pmatrix} 0 & F \\ 0 & F \end{pmatrix}$ and $\begin{pmatrix} 0 & F \\ 0 & 0 \end{pmatrix}$. First, let $I = \begin{pmatrix} F & F \\ 0 & 0 \end{pmatrix}$. Then $A/I \simeq F$ and so A/I is power serieswise Armendariz.
It remains to show that I is power serieswise Armendariz. Let $f(x) = \sum_{i=0}^{\infty} \alpha_i x^i$, $g(x) = \sum_{j=0}^{\infty} \alpha_j x^j$ in $I[[x]]$ such that $f(x)g(x) = 0$ and set $\alpha_i = \begin{pmatrix} a_i & b_i \\ 0 & 0 \end{pmatrix}$ and $\beta j = \begin{pmatrix} c_j & d_j \\ 0 & 0 \end{pmatrix}$. Assume that $\alpha_0 \neq 0$ and $\beta_0 \neq 0$. Then $a_0 c_0 = a_0 d_0 = 0$. If $a_0 \neq 0$, then $c_0 = 0$ and $d_0 = 0$, which is a contradiction. So $a_0 = 0$ and hence $b_0 \neq 0$. This implies that $\alpha_0 \beta_j = 0$ for all j. Hence the coefficient of x in $f(x)g(x) = 0$ is $\alpha_1 \beta_0 = 0$. Then $a_1 c_0 = a_1 d_0 = 0$. If $a_1 \neq 0$, then $c_0 = 0$ and $d_0 = 0$, which is a contradiction. So $\alpha_1 \beta_j = 0$ for all j.
Continuing this process, we show that $\alpha_i \beta_j = 0$ for all i, j. Therefore, I is a power serieswise Armendariz.
Next let $J = \begin{pmatrix} 0 & F \\ 0 & F \end{pmatrix}$. Then $A/J \simeq F$ and so A/J is power serieswise Armendariz. By the same method, we have that J is power serieswise Armendariz.
Finally, let $K = \begin{pmatrix} 0 & F \\ 0 & 0 \end{pmatrix}$. Then $A/K \simeq F \bigoplus F$ and so A/K is power serieswise Armendariz. Also $K^2 = 0$ and so K is power serieswise Armendariz.

Under the condition "I is reduced", we show that we have an affirmative answer to the above implication.

Theorem 2.2.10.

Let I be a reduced ideal of a ring A such that A/I is a power serieswise Armendariz. Then A is power serieswise Armendariz.

Proof. Let $f(x) = \sum_{i \geq 0} a_i x^i$ and $g(x) = \sum_{j \geq 0} b_j x^j$ in $A[[x]]$ such that $f(x)g(x) = 0$. Set $\overline{f}(x) = \sum_{i \geq 0} \overline{a_i} x^i$ and $\overline{g}(x) = \sum_{j \geq 0} \overline{b_j} x^j$ in $(A/I)[[x]]$.

Remark that $f(x)g(x)=\sum_{k\geq 0}(\sum_{i+j=k}a_ib_j)x^k=0$ implies that $\sum_{i+j=k}a_ib_j=0$ for all k. Also, $\overline{f}(x)\overline{g}(x)=0$ imply that $\overline{a_ib_j}=0$ since A/I is power serieswise Armendariz. Hence $a_ib_j\in I$ for all i,j.
We will show that $a_ib_j=0$ by induction on $i+j$.
If $i+j=0$ then $a_0b_0=0$.
Now suppose that k is a positive integer such that $a_ib_j=0$ when $i+j<k$. We will show that $a_ib_j=0$ when $i+j=k$.
By the hypothesis, $a_0b_{k-1}=0$, then $(b_{k-1}a_0)^2=0$.
Thus

$$((a_1b_{k-1})(a_0b_k)^2a_1)(b_{k-1}a_0)^2(b_{k-1}(a_0b_k)^2)=0.$$

Since

$$\begin{cases} ((a_1b_{k-1})(a_0b_k)^2a_1)(b_{k-1}a_0)\in I \\ (b_{k-1}a_0)(b_{k-1}(a_0b_k)^2)\in I \\ b_k(a_0b_k)a_1\in I \end{cases}$$

and I is semicommutative (reduced), it follows that
$((a_1b_{k-1})(a_0b_k)^2a_1)(b_{k-1}a_0)(b_k(a_0b_k)a_1)(b_{k-1}a_0)(b_{k-1}(a_0b_k)^2)=0$,
that is $[(a_1b_{k-1})(a_0b_k)^2]^2a_1(b_{k-1}a_0)(b_{k-1}(a_0b_k)^2)=0$.
Continuing this procedure yields that

$$[(a_1b_{k-1})(a_0b_k)^2]^4=0.$$

Thus $(a_1b_{k-1})(a_0b_k)^2=0$ since I is reduced.
Similarly we can show that $(a_ib_{k-i})(a_0b_k)^2=0$ for $i=2,3,...,k$.
We have $\sum_{i+j=k}a_ib_j=0$, if we multiply the last equation on the right side by $(a_0b_k)^2$, then

$$(a_0b_k)^3=-\sum_{i=1}^{k}(a_ib_{k-i})(a_0b_k)^2=0,$$

which implies that $a_0b_k=0$ since I is reduced.
We have $(a_1b_{k-1})\in I$, by analogy with the above proof, we have

$$(a_ib_{k-i})(a_1b_{k-1})^2=0$$

for $i=2,3,...,k$. If we multiply the equation $\sum_{i+j=k}a_ib_j=0$ on the right side by $(a_1b_{k-1})^2$, then

$$(a_1b_{k-1})^3=-\sum_{i=2}^{k}(a_ib_{k-i})(a_1b_{k-1})^2-(a_0b_k)(a_1b_{k-1})^2=0$$

which implies that $(a_1b_{k-1})=0$. Similarly, we can show that $a_2b_{k-2}=0$,..., $a_kb_0=0$.

Thus $a_ib_j=0$ when $i+j=k$. Therefore, by induction, we have $a_ib_j=0$ for all i,j and this shows that A is power serieswise Armendariz.

Corollary 2.2.11.

Let A be a ring. Then:

1. *A is power serieswise Armendariz if and only if so is $A[x]$.*
2. *A is power serieswise Armendariz if and only if so is $A[[x]]$.*

Proof.

1. If $A[x]$ is power serieswise Armendariz then so is A since A is a subring of $A[x]$. Conversely, we have $A \simeq A[x]/(x)$ and (x) is reduced.
2. If $A[[x]]$ is power serieswise Armendariz then so is A since A is a subring of $A[[x]]$. Conversely, we have $A \simeq A[[x]]/(x)$ and (x) is reduced.

In a ring containing a central idempotent element, we have:

Theorem 2.2.12.

Let A be a ring containing a central idempotent element e. Then, A is power serieswise Armendariz if and only if so are eA and $(1-e)A$.

Proof. If A is power serieswise Armendariz, then so are eA and $(1-e)A$ since eA and $(1-e)A$ are subring of A.
Conversely, assume that eA and $(1-e)A$ are power serieswise Armendariz for a central idempotent element e and consider $f(x)=\sum_{i=0}^{\infty}a_ix^i$, $g(x)=\sum_{j=0}^{\infty}b_jx^j \in A[[x]]$ such that $f(x)g(x)=0$. Let $f_1(x)=e\sum_{i=0}^{\infty}a_ix^i$, $g_1(x)=e\sum_{j=0}^{\infty}b_jx^j \in eA[[x]]$
$f_2(x)=(1-e)\sum_{i=0}^{\infty}a_ix^i$, $g_2(x)=(1-e)\sum_{j=0}^{\infty}b_jx^j \in (1-e)A[[x]]$.
We have $f_1(x)g_1(x)=ef(x)g(x)=0$ and $f_2(x)g_2(x)=(1-e)f(x)g(x)=0$. By the conditions we have that $ea_ib_j=0$ and $(1-e)a_ib_j=0$ for every i,j.
Hence $a_ib_j=ea_ib_j+(1-e)a_ib_j=0$ for every i,j. Thus A is power serieswise Armendariz.

Corollary 2.2.13.

For an abelian ring A, the following statements are equivalent:

1. *A is power serieswise Armendariz.*
2. *eA and $(1-e)A$ are power serieswise Armendariz for every idempotent e of A.*
3. *eA and $(1-e)A$ are power serieswise Armendariz for some idempotent e of A.*

Proof. $(1) \Rightarrow (2)$ is obvious since eA and $(1-e)A$ are subring of A.
$(2) \Rightarrow (3)$ Straightforward.
$(3) \Rightarrow (1)$ Let $f(x) = \sum_{i=0}^{\infty} a_i x^i$, $g(x) = \sum_{j=0}^{\infty} b_j x^j \in A[[x]]$ such that $f(x)g(x) = 0$. For some $e = e^2 \in A$, let $f_1(x) = e\sum_{i=0}^{\infty} a_i x^i$, $g_1(x) = e\sum_{j=0}^{\infty} b_j x^j \in eA[[x]]$ and $f_2(x) = (1-e)\sum_{i=0}^{\infty} a_i x^i$, $g_2(x) = (1-e)\sum_{j=0}^{\infty} b_j x^j \in (1-e)A[[x]]$.
We have $f_1(x)g_1(x) = ef(x)g(x) = 0$ and $f_2(x)g_2(x) = (1-e)f(x)g(x) = 0$. By the conditions we have that $ea_ib_j = 0$ and $(1-e)a_ib_j = 0$ for every i, j.
Hence $a_ib_j = ea_ib_j + (1-e)a_ib_j = 0$ for every i, j. Thus A is power serieswise Armendariz.

CHAPTER 3

POWER SERIESWISE ARMENDARIZ PROPERTY IN AMALGAMATED ALGEBRA

Abstract

In this chapter [*] , we consider a ring homomorphism $f : A \to B$ and J be an ideal of B. We investigate the transfer of the property of power serieswise Armendariz to the amalgamation $A \bowtie^f J$. We provide necessary and sufficient conditions for $A \bowtie^f J$ to be a power serieswise Armendariz ring.

Key Words: Amalgamated algebra, Power serieswise Armendariz ring, Armendariz ring.

3.1 Introduction

In [9], Armendariz proved that $a_ib_j = 0$ for all i, j whenever polynomials $f = \sum_{i=0}^{i=n} a_ix^i$ and $g = \sum_{i=0}^{i=m} b_ix^i$ over a reduced ring satisfy $fg = 0$. In [68], Rege and Chhawchharia (1997) called such a ring (not necessarily reduced) Armendariz. Armendariz rings are thus a generalization of reduced rings. It is easy to see that subring of Armendariz rings are also Armendariz. Also, D. D. Anderson and V. Camillo [2], show that a ring R is Gaussian if and only if every homomorphic image of R is Armendariz. See for instance [2, 9, 60, 68].

Recall that a ring A is called semicommutative if for all $a, b \in A$, $ab = 0$ implies that $aAb = 0$. This is equivalent to the usual definition by Shin [62, Lemma 1.2] or Huh et al. By Huh et al., reduced rings are semicommutative. Also, semicommutative rings are

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nil-Armendariz [62, Proposition 3.3]. Thus weakArmendariz rings and nil-Armendariz rings are a common generalization of semicommutative rings and Armendariz rings.

In [58], Kim et al. define power serieswise Armendariz rings as ring such that for every $f(x) = \sum_{i=0}^{\infty} a_i x^i$ and $g(x) = \sum_{i=0}^{\infty} b_i x^i \in R[[x]]$ such that $fg = 0$, then $a_i b_j = 0$ for every i and j. Power serieswise Armendariz rings are clearly Armendariz rings, but the converse is false by [7, Example 2]. Recall that a reduced ring is power serieswise Armendariz. It is easy to see that subring of power serieswise Armendariz is also power serieswise Armendariz. See for instance [8, 7, 46, 58]

Let A and B be two rings with identity elements, J be an ideal of B and let $f : A \to B$ be a ring homomorphism. In this setting, we can consider the following subring of $A \times B$:

$$A \bowtie^f J := \{(a, f(a) + j) \mid a \in A, j \in J\}$$

called the amalgamation of A and B along J with respect to f. This construction is a generalization of the amalgamated duplication of a ring along an ideal (introduced and studied by D'Anna and Fontana in [26, 27, 28]). Moreover, other classical constructions (such as the $A + XB[X]$, $A + XB[[X]]$, and the $D + M$ constructions) can be studied as particular cases of the amalgamation ([29, Examples 2.5 and 2.6]). Other classical constructions, such as the Nagata's idealization, also called trivial ring extension ([66, Page 2]), and the CPI extensions (in the sense of Boisen and Sheldon [19]) are strictly related to it ([29, Example 2.7 and Remark 2.8]). See for instance [29, 30, 27, 28]).

In this paper, we investigate the transfer of the property of power serieswise Armendariz to the amalgamation. Our results generate new and original examples which enrich the current literature with new families of power serieswise Armendariz rings.

3.2 Transfer of Power serieswise Armendariz property to the amagamated algebra

The main result of this section (Theorem 3.2.1) states necessary and sufficient conditions under which the amalgamated algebra along an ideal $A \bowtie^f J$ is a power serieswise Armendariz ring.

Theorem 3.2.1.

Let (A,B) be a pair of rings, $f: A \to B$ be a ring homomorphism and J be a proper ideal of B.

1. *If $A \bowtie^f J$ is a power serieswise Armendariz ring then so is A.*
2. *If A and $f(A)+J$ are power serieswise Armendariz rings then so is $A \bowtie^f J$.*
3. *Assume that $J \cap S \neq \varnothing$, S the set of regular central element of B. Then: $A \bowtie^f J$ is a power serieswise Armendariz ring if and only if $f(A)+J$ and A are power serieswise Armendariz rings.*
4. *Assume that $f^{-1}(J) \cap nil(A) = (0)$. Then: If $f(A)+J$ is a power serieswise Armendariz ring then so is $A \bowtie^f J$.*
5. *Assume that $J \cap nil(B) = (0)$. Then: $A \bowtie^f J$ is a power serieswise Armendariz ring if and only if so is A.*
6. *Assume that f is injective and $f(A) \cap J = (0)$. Then: $A \bowtie^f J$ is a power serieswise Armendariz ring if and only if $f(A)+J$ is power serieswise Armendariz.*

Proof.

1. Assume that $A \bowtie^f J$ is a power serieswise Armendariz ring and let $f_A(x) = \sum_{i=0}^{\infty} a_i x^j$ and $g_A(x) = \sum_{j=0}^{\infty} b_j x^j$ in $A[[x]]$ such that $f_A(x) g_A(x) = 0$. Then $\sum_{i+j=k} a_i b_j = 0$ for all k.
   Set $F(x) = \sum_{i=0}^{\infty} (a_i, f(a_i)) x^i$ and $G(x) = \sum_{j=0}^{\infty} (b_j, f(b_j)) x^j$ in $A \bowtie^f J[[x]]$.
   Then

$$
\begin{aligned}
F(x)G(x) &= \sum_{k=0}^{\infty} \Big(\sum_{i+j=k} (a_i b_j, f(a_i b_j)) \Big) x^k \\
&= \sum_{k=0}^{\infty} \Big(\sum_{i+j=k} a_i b_j, \sum_{i+j=k} f(a_i b_j) \Big) x^k \\
&= \sum_{k=0}^{\infty} \Big(\sum_{i+j=k} a_i b_j, f\big(\sum_{i+j=k} a_i b_j \big) \Big) x^k \\
&= 0.
\end{aligned}
$$

So $(a_ib_j, f(a_ib_j)) = 0$ since $A \bowtie^f J$ is power serieswise Armendariz. Thus $a_ib_j = 0$, and this shows that A is power serieswise Armendariz.

2. Assume that A and $f(A)+J$ are power serieswise Armendariz and let $F(x) = \sum_{i=0}^{\infty}(a_i, f(a_i)+j_i)x^i$ and $G(x) = \sum_{j=0}^{\infty}(b_j, f(b_j)+k_j)x^j$ in $A \bowtie^f J[[x]]$ such that $F(x)G(x) = 0$. Set $f_B(x) = \sum_{i=0}^{\infty}(f(a_i)+j_i)x^i$, $g_B(x) = \sum_{j=0}^{\infty}(f(b_j)+k_j)x^j$ in $(f(A)+J)[[x]]$ and $f_A(x) = \sum_{i=0}^{\infty}a_ix^i$, $g_A(x) = \sum_{j=0}^{\infty}b_jx^j$ in $A[[x]]$. Then $F(x)G(x) = 0$ implies that $f_A(x)g_A(x) = 0$ and $f_B(x)g_B(x) = 0$, which in turn implies that $(f(a_i)+j_i)(f(b_j)+k_j) = 0$ and $a_ib_j = 0$ for all i, j since $f(A)+J$ and A are power serieswise Armendariz rings. Therefore, $A \bowtie^f J$ is power serieswise Armendariz.

3. Let S be the set of regular central element of B, and suppose that $J \cap S \neq \varnothing$. If $f(A)+J$ and A are power serieswise Armendariz rings then so is $A \bowtie^f J$ by (2). Conversely, assume that $A \bowtie^f J$ is power serieswise Armendariz. By (1), A is a power serieswise Armendariz ring. We claim that $f(A)+J$ is a power serieswise Armendariz ring. Indeed, let $f_B(x) = \sum_{i=0}^{\infty}(f(a_i)+j_i)x^i$ and $g_B(x) = \sum_{j=0}^{\infty}(f(b_j)+k_j)x^j$ in $(f(A)+J)[[x]]$ such that $f_B(x)g_B(x) = 0$. Then $\sum_{i+j=k}(f(a_i)+j_i)(f(b_j)+k_j) = 0$ for all k. Let e be a regular central element of J. Set $F(x) = \sum_{i=0}^{\infty}(0, e(f(a_i)+j_i))x^i$ and $G(x) = \sum_{j=0}^{\infty}(0, e(f(b_j)+k_j))x^j$ in $A \bowtie^f J[[x]]$. Then

$$\begin{aligned} F(x)G(x) &= \sum_{k=0}^{\infty}(\sum_{i+j=k}(0, e^2(f(a_i)+j_i)(f(b_j)+k_j))x^k \\ &= \sum_{k=0}^{\infty}(0, e^2\sum_{i+j=k}(f(a_i)+j_i)(f(b_j)+k_j))x^k = 0. \end{aligned}$$

So $(0, e(f(a_i)+j_i))(0, e(f(b_j)+k_j)) = 0$ since $A \bowtie^f J$ is power serieswise Armendariz; which implies that $e^2(f(a_i)+j_i)(f(b_j)+k_j) = 0$ for all i, j. Hence $(f(a_i)+j_i)(f(b_j)+k_j) = 0$, and this shows that $f(A)+J$ is power serieswise Armendariz.

4. Assume that $f^{-1}(J) \cap nil(A) = (0)$ and $f(A)+J$ is a power serieswise Armendariz ring. Let $F(x) = \sum_{i=0}^{\infty}(a_i, f(a_i)+t_i)x^i$ and $G(x) = \sum_{j=0}^{\infty}(b_j, f(b_j)+k_j)x^j$ in $A \bowtie^f J[[x]]$ such that $F(x)G(x) = 0$. Set $f_B(x) = \sum_{i=0}^{\infty}(f(a_i)+t_i)x^i$, $g_B(x) = \sum_{j=0}^{\infty}(f(b_j)+k_j)x^j$, $f_A(x) = \sum_{i=0}^{\infty}a_ix^i$ and $g_A(x) = \sum_{j=0}^{\infty}b_jx^j$. But $F(x)G(x) = 0$ implies that $f_Bg_B = 0$ and $f_Ag_A = 0$. Thus $(f(a_i)+t_i)(f(b_j)+k_j) = 0$ for all i, j since $f(A)+J$ is a power serieswise Armendariz ring. Hence $a_ib_j \in f^{-1}(J)$ for all i, j.

We will show that $a_ib_j = 0$ by induction on $i+j$.
If $i+j=0$, then $a_0b_0 = 0$.
Now suppose that l is a positive integer such that $a_ib_j = 0$ when $i+j<l$. We will show that $a_ib_j = 0$ when $i+j=l$.
By the hypothesis, $a_0b_{l-1} = 0$, then $(b_{l-1}a_0)^2 = 0$.
Thus

$$((a_1b_{l-1})(a_0b_l)^2a_1)(b_{l-1}a_0)^2(b_{l-1}(a_0b_l)^2) = 0.$$

Since

$$\begin{cases} ((a_1b_{l-1})(a_0b_l)^2a_1)(b_{l-1}a_0) \in f^{-1}(J) \\ (b_{l-1}a_0)(b_{l-1}(a_0b_l)^2) \in f^{-1}(J) \\ b_l(a_0b_l)a_1 \in f^{-1}(J) \end{cases}$$

and $f^{-1}(J)$ is semicommutative (reduced), it follows that

$$((a_1b_{l-1})(a_0b_l)^2a_1)(b_{l-1}a_0)(b_l(a_0b_l)a_1)(b_{l-1}a_0)(b_{l-1}(a_0b_l)^2) = 0,$$

that is $[(a_1b_{l-1})(a_0b_l)^2]^2a_1(b_{l-1}a_0)(b_{l-1}(a_0b_l)^2) = 0$.
Continuing this procedure yields that $[(a_1b_{l-1})(a_0b_l)^2]^4 = 0$. Thus $(a_1b_{l-1})(a_0b_l)^2 = 0$ since $f^{-1}(J)$ is reduced.
Similarly we can show that $(a_ib_{l-i})(a_0b_l)^2 = 0$ for $i = 2,3,...,l$.
The equation $f_Ag_A = 0$ implies that $\sum_{i+j=l} a_ib_j = 0$, if we multiply the last equation on the right side by $(a_0b_l)^2$, then

$$(a_0b_l)^3 = -\sum_{i=1}^{l}(a_ib_{l-i})(a_0b_l)^2 = 0,$$

which implies that $a_0b_l = 0$ since $f^{-1}(J)$ is reduced.
We have $(a_1b_{l-1}) \in f^{-1}(J)$, by the same above proof, we have

$$(a_ib_{l-i})(a_1b_{l-1})^2 = 0$$

for $i = 2,3,...l$. If we multiply the equation $\sum_{i+j=l} a_ib_j = 0$ on the right side by $(a_1b_{l-1})^2$, then

$$(a_1b_{l-1})^3 = -\sum_{i=2}^{l}(a_ib_{l-i})(a_1b_{l-1})^2 - (a_0b_l)(a_1b_{l-1})^2 = 0$$

which implies that $(a_1b_{l-1}) = 0$. Similarly, we can show that $a_2b_{l-2} = 0,...,a_lb_0 = 0$.

Thus $a_ib_j = 0$ when $i+j=l$. Therefore, by induction, we have $a_ib_j = 0$ for all i,j and this shows that $A \bowtie^f J$ is power serieswise Armendariz.

5. Assume that $J \cap nil(B) = (0)$.
If $A \bowtie^f J$ is power serieswise Armendariz, then so is A.
Conversely, suppose that A is power serieswise Armendariz, and let $F(x) = \sum_{i=0}^{\infty}(a_i, f(a_i)+t_i)x^i$ and $G(x) = \sum_{j=0}^{\infty}(b_j, f(b_j)+k_j)x^j$ in $A \bowtie^f J[[x]]$ such that $F(x)G(x) = 0$.
Set $f_B(x) = \sum_{i=0}^{\infty}(f(a_i)+t_i)x^i$, $g_B(x) = \sum_{j=0}^{\infty}(f(b_j)+k_j)x^j$, $f_A(x) = \sum_{i=0}^{\infty} a_i x^i$ and $g_A(x) = \sum_{j=0}^{\infty} b_j x^j$. But $F(x)G(x) = 0$ implies that $f_A g_A = 0$ and $f_B g_B = 0$. Thus $a_i b_j = 0$ since A is a power serieswise Armendariz ring. Hence $(f(a_i)+t_i)(f(b_j)+k_j) \in J$ for all i, j.
We will show that $(f(a_i)+t_i)(f(b_j)+k_j) = 0$ by induction on $i+j$.
If $i+j=0$ then $(f(a_0)+j_0)(f(b_0)+k_0) = 0$.
Now suppose that l is a positive integer such that $(f(a_i)+t_i)(f(b_j)+k_j) = 0$ when $i+j<l$.
We will show that $(f(a_i)+t_i)(f(b_j)+k_j) = 0$ when $i+j=l$.

By the hypothesis, $(f(a_0)+t_0)(f(b_{l-1})+k_{l-1}) = 0$, then $((f(b_{l-1})+k_{l-1})(f(a_0)+t_0))^2 = 0$. Thus $(((f(a_1)+t_1)(f(b_{l-1})+k_{l-1}))((f(a_0)+t_0)(f(b_l)+k_l))^2(f(a_1)+t_1)) \times$
$((f(b_{l-1})+k_{l-1})(f(a_0)+t_0))^2((f(b_{l-1})+k_{l-1})((f(a_0)+t_0)(f(b_l)+k_l))^2) = 0$.
Since

$$\left\{\begin{array}{r} (((f(a_1)+t_1)(f(b_{l-1})+k_{l-1}))((f(a_0)+t_0)(f(b_l)+k_l))^2(f(a_1)+t_1))\times \\ (f(b_{l-1})+k_{l-1})(f(a_0)+t_0) \in J \\ ((f(b_{l-1})+k_{l-1})(f(a_0)+t_0))((f(b_{l-1})+k_{l-1})((f(a_0)+t_0)(f(b_l)+k_l))^2) \in J \\ (f(b_l)+k_l)((f(a_0)+t_0)(f(b_l)+k_l)))(f(a_1)+t_1) \in J \end{array}\right.$$

and J is semicommutative (reduced), it follows that
$(((f(a_1)+t_1)(f(b_{l-1})+k_{l-1}))((f(a_0)+t_0)(f(b_l)+k_l))^2(f(a_1)+t_1))((f(b_{l-1})+k_{l-1})(f(a_0)+t_0)) \times$
$(f(b_l)+k_l)((f(a_0)+t_0)(f(b_l)+k_l)))(f(a_1)+t_1) \times$
$((f(b_{l-1})+k_{l-1})(f(a_0)+t_0))((f(b_{l-1})+k_{l-1})((f(a_0)+t_0)(f(b_l)+k_l))^2) = 0$,
that is $[((f(a_1)+t_1)(f(b_{l-1})+k_{l-1}))((f(a_0)+t_0)(f(b_l)+k_l))^2]^2(f(a_1)+t_1) \times$
$((f(b_{l-1})+k_{l-1})(f(a_0)+t_0))((f(b_{l-1})+k_{l-1})((f(a_0)+t_0)(f(b_l)+k_l))^2) = 0$.
Continuing this procedure yields that

$$[((f(a_1)+t_1)(f(b_{l-1})+k_{l-1}))((f(a_0)+t_0)(f(b_l)+k_l))^2]^4 = 0.$$

Thus

$$((f(a_1)+t_1)(f(b_{l-1})+k_{l-1}))((f(a_0)+t_0)(f(b_l)+k_l))^2 = 0$$

since J is reduced.
Similarly we can show that
$((f(a_i)+t_i)(f(b_{l-i})+k_{l-i}))((f(a_0)+t_0)(f(b_l)+k_l))^2 = 0$ for $i = 2,3,...,l$.
$f_Bg_B = 0$ implies that $\sum_{i+j=l}(f(a_i)+t_i)(f(b_j)+k_j) = 0$, if we multiply the last equation on the right side by $((f(a_0)+t_0)(f(b_l)+k_l))^2$, then
$((f(a_0)+t_0)(f(b_l)+k_l))^3 = -\sum_{i=1}^{i=l}(f(a_i)+t_i)(f(b_{l-i})+k_{l-i})((f(a_0)+t_0)(f(b_l)+k_l))^2 = 0$, which implies that $(f(a_0)+t_0)(f(b_l)+k_l) = 0$ since J is reduced.
We have $((f(a_1)+t_1)(f(b_{l-1})+k_{l-1})) \in J$, by analogy with the above proof, we have

$$(f(a_i)+t_i)(f(b_{l-i})+k_{l-i})((f(a_1)+t_1)(f(b_{l-1})+k_{l-1}))^2 = 0$$

for $i = 2,3,...,l$. If we multiply the equation $\sum_{i+j=l}(f(a_i)+t_i)(f(b_j)+k_j)) = 0$ on the right side by $((f(a_1)+t_1)(f(b_{l-1})+k_{l-1}))^2$, then
$((f(a_1)+t_1)(f(b_{l-1})+k_{l-1}))^3 = -\sum_{i=2}^{i=l}((f(a_i)+t_i)(f(b_{l-i})+k_{l-i}))((f(a_1)+t_1)\times$

$(f(b_{l-1})+k_{l-1}))^2$
$-((f(a_0)+t_0)(f(b_l)+k_l))((f(a_1)+t_1)(f(b_{l-1})+k_{l-1}))^2 = 0$
which implies that $(f(a_1)+t_1)(f(b_{l-1})+k_{l-1}) = 0$. Similarly, we can show that

$$(f(a_2)+t_2)(f(b_{l-2})+k_{l-2}) = 0,...,(f(a_l)+t_l)(f(b_0)+k_0) = 0.$$

Thus, $(f(a_i)+t_i)(f(b_j)+k_j) = 0$ when $i+j = l$. Therefore, by induction, we have $(f(a_i)+t_i)(f(b_j)+k_j) = 0$ for all i,j and this shows that $A \bowtie^f J$ is power serieswise Armendariz.

6. Assume that f is injective and $f(A) \cap J = (0)$. In this case $A \bowtie^f J \simeq f(A)+J$ and the conclusion follows.

Let A be a ring, E be an A-module and $A \propto E$ be the set of pairs (a,e) with pairwise addition and multiplication given by $(a,e)(b,f) = (ab, af+be)$. $A \propto E$ is called the trivial ring extension of A by E (also called the idealization of E over A). Considerable work has been concerned with trivial ring extension. Part of it has been summarized in Glaz's book [42] and Huckaba's book (where $A \propto E$ is called the idealization of E by A). See for instance [42, 48, 53].

The following example shows that the implication A is power serieswise Armendariz implies that $A \bowtie^f J$ is power serieswise Armendariz is false, in general.

Example 3.2.2.

Let K be a field, $B = K \ltimes K$ be the trivial ring extension of K by K, $R = B \ltimes B$, $J = 0 \ltimes B$ and $f : B \longrightarrow R$ defined by $f(b) = (b, 0)$.

Notice that $R = B \ltimes B$ is isomorphic to $\{\begin{pmatrix} b & c \\ 0 & b \end{pmatrix}, b, c \in B\}$ and $R = B \ltimes B = B \bowtie^f J$. Then:

1. *B is a power serieswise Armendariz ring.*
2. *$R = B \bowtie^f J$ is not power serieswise Armendariz.*

Proof.

1. $B = K \ltimes K$ is isomorphic to the ring $\{\begin{pmatrix} a & b \\ 0 & a \end{pmatrix} / a, b \in K\}$, we can show that $B = K \ltimes K$ is isomorphic to $\{ \begin{pmatrix} a & b & 0 \\ 0 & a & 0 \\ 0 & 0 & a \end{pmatrix} / a, b \in K\}$ via the mapping $\varphi(\begin{pmatrix} a & b \\ 0 & a \end{pmatrix}) = \begin{pmatrix} a & b & 0 \\ 0 & a & 0 \\ 0 & 0 & a \end{pmatrix}$ which is a ring isomorphism.

   To show that B is power serieswise Armendariz, we use the method in the proof of [68, Proposition 2.5]. First notice that for $\begin{pmatrix} a_1 & b_1 & 0 \\ 0 & a_1 & 0 \\ 0 & 0 & a_1 \end{pmatrix}, \begin{pmatrix} a_2 & b_2 & 0 \\ 0 & a_2 & 0 \\ 0 & 0 & a_2 \end{pmatrix} \in B$, we can denote their addition and multiplication by

   $$(a_1, b_1, 0, 0) + (a_2, b_2, 0, 0) = (a_1 + a_2, b_1 + b_2, 0, 0)$$

   and

   $$(a_1, b_1, 0, 0)(a_2, b_2, 0, 0) = (a_1 a_2, a_1 b_2 + b_1 a_2, 0, 0)$$

   respectively.
   So every polynomials in $B[[x]]$ can be expressed in the form $(p_0(x), p_1(x), 0, 0)$ for some $p_i(x)$ in $K[[x]]$.

   Let $f(x) = (f_0(x), f_1(x), 0, 0)$ and $g(x) = (g_0(x), g_1(x), 0, 0)$ be elements of $B[[x]]$. Assume that $f(x)g(x) = 0$.
   Then $f(x)g(x) = (f_0(x)g_0(x), f_0(x)g_1(x) + f_1(x)g_0(x), 0, 0) = 0$. So we have the following system of equations:

   (a) $f_0(x)g_0(x) = 0$;

(b) $f_0(x)g_1(x)+f_1(x)g_0(x)=0$;

From Equation (a), we see that $g_0(x)f_0(x)=0$ since $K[[x]]$ is reduced. If we multiply equation (b) on the right side by $f_0(x)$, then $f_0(x)g_1(x)f_0(x)+f_1(x)g_0(x)f_0(x)=0$. So $f_0(x)g_1(x)f_0(x)=0$, if we multiply by $g_1(x)$ on the right side and use the fact that $K[[x]]$ is reduced, we have $f_0(x)g_1(x)=0$ and hence $f_1(x)g_0(x)=0$. Now let $f(x)=\sum_{i=0}^{\infty}\begin{pmatrix} a_i & b_i & 0\\ 0 & a_i & 0\\ 0 & 0 & a_i\end{pmatrix}x^i$ and $g(x)=\sum_{j=0}^{\infty}\begin{pmatrix} a'_j & b'_j & 0\\ 0 & a'_j & 0\\ 0 & 0 & a'_j\end{pmatrix}x^j$ where $f_0=\sum_{i=0}^{\infty}a_ix^i$, $f_1=\sum_{i=0}^{\infty}b_ix^i$, $g_0=\sum_{j=0}^{\infty}a'_jx^j$, $g_1=\sum_{j=0}^{\infty}b'_jx^j$. Then we obtain that $a_ia'_j=0$, $a_ib'_j=0$, $b_ia'_j=0$ for all i,j by the preceding results, and condition that K is reduced. Consequently $\begin{pmatrix} a_i & b_i & 0\\ 0 & a_i & 0\\ 0 & 0 & a_i\end{pmatrix}\begin{pmatrix} a'_i & b'_i & 0\\ 0 & a'_i & 0\\ 0 & 0 & a'_i\end{pmatrix}=0$ for all i,j, therefore B is a power serieswise Armendariz ring.

2. Let $f(x)=\begin{pmatrix} \begin{pmatrix}0&1\\0&0\end{pmatrix} & \begin{pmatrix}0&0\\0&0\end{pmatrix}\\ \begin{pmatrix}0&0\\0&0\end{pmatrix} & \begin{pmatrix}0&1\\0&0\end{pmatrix}\end{pmatrix}+\begin{pmatrix} \begin{pmatrix}0&1\\0&0\end{pmatrix} & \begin{pmatrix}-1&0\\0&-1\end{pmatrix}\\ \begin{pmatrix}0&0\\0&0\end{pmatrix} & \begin{pmatrix}0&1\\0&0\end{pmatrix}\end{pmatrix}x$ and,

$$g(x)=\begin{pmatrix} \begin{pmatrix}0&1\\0&0\end{pmatrix} & \begin{pmatrix}0&0\\0&0\end{pmatrix}\\ \begin{pmatrix}0&0\\0&0\end{pmatrix} & \begin{pmatrix}0&1\\0&0\end{pmatrix}\end{pmatrix}+\begin{pmatrix} \begin{pmatrix}0&1\\0&0\end{pmatrix} & \begin{pmatrix}1&0\\0&1\end{pmatrix}\\ \begin{pmatrix}0&0\\0&0\end{pmatrix} & \begin{pmatrix}0&1\\0&0\end{pmatrix}\end{pmatrix}x$$ be polynomials in $R[x]$.

Then $f(x)g(x)=0$, but

$\begin{pmatrix} \begin{pmatrix}0&1\\0&0\end{pmatrix} & \begin{pmatrix}0&0\\0&0\end{pmatrix}\\ \begin{pmatrix}0&0\\0&0\end{pmatrix} & \begin{pmatrix}0&1\\0&0\end{pmatrix}\end{pmatrix}\begin{pmatrix} \begin{pmatrix}0&1\\0&0\end{pmatrix} & \begin{pmatrix}1&0\\0&1\end{pmatrix}\\ \begin{pmatrix}0&0\\0&0\end{pmatrix} & \begin{pmatrix}0&1\\0&0\end{pmatrix}\end{pmatrix}\neq 0$. Thus R is not power serieswise Armendariz.

The following example shows that the implication $f(A)+J$ is power serieswise Armendariz implies that $A\bowtie^f J$ is power serieswise Armendariz is false, in general.

Example 3.2.3.

Let A be a semicommutative ring which is not Armendariz.
Notice that A is semicommutative implies that $nil(A)$ *is an ideal of A by [62, Lemma 3.1]. Let* $f : A \longrightarrow A/nil(A)$ *be the canonical surjection, I be an ideal of A which contains* $nil(A)$ *and* $J = I/nil(A)$*. Then:*

1. $A \bowtie^f J$ *is not power serieswise Armendariz.*
2. $f(A) + J(= A/nil(A))$ *is a power serieswise Armendariz ring.*

Proof.

1. The ring A is not power serieswise Armendariz since it is not Armendariz.
2. The ring $f(A) + J(= A/nil(A))$ is a power serieswise Armendariz ring since it is reduced.

Now, we construct a new class of power serieswise Armendariz rings.

Example 3.2.4.

Let A be a power serieswise Armendariz ring such that $nil(A)$ *is an ideal,* $f : A \longrightarrow A/nil(A)$ *the canonical surjection and* $J = I/nil(A)$*, where I is an ideal of A which contains* $nil(A)$*. Then:* $A \bowtie^f J$ *is a power serieswise Armendariz ring.*

Proof. Follows from Theorem 3.2.1(2) since A and $f(A) + J = A/nil(A)$ are power serieswise Armendariz rings.

Example 3.2.5.

Let A be a power serieswise Armendariz ring, I a prime ideal of A, $f : A \longrightarrow A/I = B$ *the canonical surjection, M a maximal ideal of A such that* $I \subseteq M$ *and* $J = M/I$*. Then:*

1. A/I *is power serieswise Armendariz.*
2. $A \bowtie^f J$ *is power serieswise Armendariz.*

Proof.

1. Clear since A/I is reduced.
2. Follows from Theorem 3.2.1(5) since $J \cap nil(B) = 0$.

CHAPTER 4

COHERENCE IN BI-AMALGAMATED ALGEBRA ALONG IDEALS

Abstract.

In this chapter[*],Let $f : A \longrightarrow B$ and $g : A \longrightarrow C$ be two ring homomorphisms and let J (resp., J') be an ideal of B (resp., C) such that $f^{-1}(J) = g^{-1}(J')$. In this paper, we investigate the transfer of the property of coherence in the bi-amalgamation of A with (B,C) along (J,J') with respect to (f,g) (denoted by $A \bowtie^{f,g} (J,J')$), introduced and studied by Kabbaj, Louartiti and Tamekkante in 2013. We provide necessary and sufficient conditions for $A \bowtie^{f,g} (J,J')$ to be a coherent ring.

Key Words: Bi-amalgamated algebra, coherent ring.

4.1 Introduction

A ring R is coherent if every finitely generated ideal of R is finitely presented; equivalently, if $(0 : a)$ and $I \cap J$ are finitely generated for every $a \in R$ and any two finitely generated ideals I and J of R. Examples of coherent rings are Noetherian rings, Boolean algebras, von Neumann regular rings, and Prüfer/semihereditary rings. For instance see [1, 42, 53].

Recall that an R-module M is called a coherent R-module if it is finitely generated and every finitely generated submodule of M is finitely presented.

Let $f : A \longrightarrow B$ and $g : A \longrightarrow C$ be two ring homomorphisms and let J (resp., J') be an ideal of B (resp., C) such that $f^{-1}(J) = g^{-1}(J')$. In this setting, we can consider the following subring of $B \times C$:

$$A \bowtie^{f,g} (J,J') := \{(f(a)+j, g(a)+j') | a \in A, j \in J, j' \in J'\}$$

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called the bi-amalgamation of A with (B,C) along (J,J') with respect to (f,g) (introduced and studied by Kabbaj, Louartiti and Tamekkante in [52]). This construction is a generalisation of the amalgamated algebra along an ideal (introduced and studied by D'Anna and Fontana in [29, 30].) Moreover, other classical constructions (such as the $A+XB[X]$, $A+XB[[X]]$, and the $D+M$ constructions) can be studied as particular cases of the amalgamation [29, Exemples 2.5 and 2.6] and other classical constructions , such as the Nagata's idealization ([66, page 2]), and the CPI extensions are strictly related to it ([29, Exemple 2.7 and Remark 2.8]). In [52], the authors studied the basic properties of this construction (e.g., characterized for $A \bowtie^{f,g} (J,J')$ to be a Noetherian ring, an integral domain, a reduced ring) and they characterized those distinguished pullbacks that can be expressed as an bi-amalgamation. Moreover, they pursued the investigation on the structure of the rings of the form $A \bowtie^{f,g} (J,J')$, with particular attention to the prime spectrum.

This paper investigates the property of coherence in bi-amalgamated algebra along ideals. Our results generate original examples which enrich the current literature with new families of non-Noetherian coherent rings.

4.2 Transfer of coherence property to Bi-amalgamated algebra along ideals

This section characterizes the bi-amalgamated algebra along ideals $A \bowtie^{f,g} (J,J')$ to be a coherent ring. The main result (Theorem 4.2.2) examines the property of coherence that the amalgamation $A \bowtie^{f,g} (J,J')$ might inherit from the rings $f(A)+J$, $g(A)+J$ for some classes of ideals J, J' and homomorphisms f,g and hence generates new examples of non-Noetherian coherent rings.

Let $f : A \longrightarrow B$ and $g : A \longrightarrow C$ be two ring homomorphisms and let J (resp., J') be an ideal of B (resp., C) such that $f^{-1}(J) = g^{-1}(J')$ and let n be a positive integer. Consider the functions $f^n : A^n \longrightarrow B^n$ defined by $f^n((\alpha_i)_{i=1}^{i=n}) = (f(\alpha_i))_{i=1}^{i=n}$ and $g^n : A^n \longrightarrow C^n$ defined by $g^n((\alpha_i)_{i=1}^{i=n}) = (g(\alpha_i))_{i=1}^{i=n}$. Obviously, f^n and g^n are a ring homomorphisms and J^n, J'^n are ideals of B^n and C^n, respectively . This allows us to defined $A^n \bowtie^{f^n,g^n} (J^n,J'^n)$.

Moreover, let $\phi : (A \bowtie^{f,g} (J,J'))^n \longrightarrow A^n \bowtie^{f^n,g^n} (J^n,J'^n)$ defined by

$$\phi((f(\alpha_i)+j_i, g(\alpha_i)+j'_i)_{i=1}^{i=n}) = (f^n((\alpha_i)_{i=1}^{i=n}) + (j_i)_{i=1}^{i=n}, g^n((\alpha_i)_{i=1}^{i=n}) + (j'_i)_{i=1}^{i=n}).$$

It is easily checked that ϕ is a ring isomorphism. So $A^n \bowtie^{f^n,g^n} (J^n,J'^n)$ and $(A \bowtie^{f,g} (J,J'))^n$ are isomorphic as rings.

Let U be a submodule of A^n. Then $U \bowtie^{f^n,g^n} (J^n,J'^n) := \{(f^n(u)+j, g^n(u)+j') \in A^n \bowtie^{f^n,g^n} (J^n,J'^n)/u \in U, j \in J^n, j' \in J'^n\}$ is a submodule of $A^n \bowtie^{f^n,g^n} (J^n,J'^n)$.

Next, before we announce the main result of this section (Theorem 4.2.2), we make the following useful remarks.

Remark 4.2.1.

1. *Let $f : A \longrightarrow B$ be a ring homomorphism and let J be an ideal of B. Then $f^n(\alpha a) = f(\alpha)f^n(a)$ for all $\alpha \in A$ and $a \in A^n$.*
2. *If $f^{-1}(J) = g^{-1}(J') = 0$, then A is a module retract of $A \bowtie^{f,g} (J,J')$.*
3. *If g is injective and $J' \subseteq g(A)$, then A is a module retract of $A \bowtie^{f,g} (J,J')$.*

Proof.

1. Straightforward
2. Let $\varphi : A \longrightarrow A \bowtie^{f,g} (J,J')$ defined by $\varphi(a) = (f(a), g(a))$ and $\psi : A \bowtie^{f,g} (J,J') \longrightarrow A$ defined by $\psi(f(a)+j, g(a)+j') = a$, ψ is well defined since $f^{-1}(J) = g^{-1}(J') = 0$ and the conclusion now is Straightforward.
3. let $\varphi : A \longrightarrow A \bowtie^{f,g} (J,J')$ defined by $\varphi(a) = (f(a), g(a))$ and $\psi : A \bowtie^{f,g} (J,J') \longrightarrow A$ defined by $\psi(f(a)+j, g(a)+j') = a+t$, where t is the unique element such that $g(t) = j'$. ψ is well defined and the conclusion now is Straightforward.

Now, to the main result:

Theorem 4.2.2.

Let $f : A \longrightarrow B$ *and* $g : A \longrightarrow C$ *be two ring homomorphisms and let* J *(resp.,* J'*) be a proper ideal of* B *(resp.,* C*) such that* $f^{-1}(J) = g^{-1}(J')$.

1. *Assume that* J, J' *are finitely generated ideals of* $f(A)+J$ *and* $g(A)+J'$ *respectively, and* $J \subseteq f(A)$*. Then* $A \bowtie^{f,g} (J,J')$ *is a coherent ring if and only if* $f(A)+J$ *and* $g(A)+J'$ *are coherent rings.*
2. *Assume that* J, J' *are finitely generated ideals of* $f(A)+J$ *and* $g(A)+J'$ *respectively, and* $J^2 = 0$*. Then* $A \bowtie^{f,g} (J,J')$ *is a coherent ring if and only if* $f(A)+J$ *and* $g(A)+J'$ *are coherent rings.*
3. *Assume that* J, J' *are regular ideals of* $f(A)+J$ *and* $g(A)+J'$ *respectively, and* $J \subseteq f(A)$*. Then* $A \bowtie^{f,g} (J,J')$ *is a coherent ring if and only if* $f(A)+J$ *and* $g(A)+J'$ *are coherent rings and* J*,* J' *are finitely generated ideals of* $f(A)+J$ *and* $g(A)+J'$*, respectively.*
4. *Assume that* J *is a regular finitely generated ideal of* $f(A)+J$ *and* $J' \subseteq g(A)$*. Then* $A \bowtie^{f,g} (J,J')$ *is a coherent ring if and only if* $f(A)+J$ *and* $g(A)+J'$ *are coherent rings and* J' *is a finitely generated ideal of* $g(A)+J'$*.*
5. *Assume that* J *is a regular finitely generated ideal of* $f(A)+J$ *and* $J'^2 = 0$*. Then* $A \bowtie^{f,g} (J,J')$ *is a coherent ring if and only if* $f(A)+J$ *and* $g(A)+J'$ *are coherent rings and* J' *is a finitely generated ideal of* $g(A)+J'$*.*

Before proving Theorem 4.2.2, we establish the following lemmas.

Lemma 4.2.3.

Let $f:A\longrightarrow B$ and $g:A\longrightarrow C$ be two ring homomorphisms and let J (resp., J') be a proper ideal of B (resp., C) such that $f^{-1}(J)=g^{-1}(J')$. Then:

1. *$\{0\}\times J'$ (resp., $J\times\{0\}$) is a finitely generated ideal of $A\bowtie^{f,g}(J,J')$ if and only if J' (resp., J) is a finitely generated ideal of $g(A)+J'$ (resp., $f(A)+J$).*
2. *If $A\bowtie^{f,g}(J,J')$ is a coherent ring and J, J' are finitely generated ideals of $f(A)+J$ and $g(A)+J'$ respectively, then $f(A)+J$ and $g(A)+J'$ are coherent rings.*

Proof.

1. Assume that $J':=\sum_{i=1}^{i=n}(g(A)+J')k_i$ is a finitely generated ideal of $g(A)+J'$, where $k_i\in J'$. It is clear that $\sum_{i=1}^{i=n}(A\bowtie^{f,g}(J,J'))(0,k_i)\subseteq\{0\}\times J'$. Let $x:=(0,\sum_{i=1}^{i=n}(g(\alpha_i)+j'_i)k_i)\in\{0\}\times J'$, where $\alpha_i\in A$ and $j'_i\in J'$. Hence, $x=(0,\sum_{i=1}^{i=n}(g(\alpha_i)+j'_i)k_i)=\sum_{i=1}^{i=n}(0,(g(\alpha_i)+j'_i)k_i)=\sum_{i=1}^{i=n}(f(\alpha_i),g(\alpha_i)+j'_i)(0,k_i)\in\sum_{i=1}^{i=n}(A\bowtie^{f,g}(J,J'))(0,k_i)$. Therefore, $\{0\}\times J'\subseteq\sum_{i=1}^{i=n}(A\bowtie^{f,g}(J,J'))(0,k_i)$ and so $\{0\}\times J'=\sum_{i=1}^{i=n}(A\bowtie^{f,g}(J,J'))(0,k_i)$. Conversely, assume that $\{0\}\times J'=\sum_{i=1}^{i=n}(A\bowtie^{f,g}(J,J'))(0,k_i)$ is a finitely generated ideal of $A\bowtie^{f,g}(J,J')$, where $k_i\in J'$. It is readily seen that $J'=\sum_{i=1}^{i=n}(g(A)+J')k_i$, as desired.
2. Assume that $A\bowtie^{f,g}(J,J')$ is a coherent ring and $J\times\{0\}$ (resp., $\{0\}\times J'$) is a finitely generated ideal of $A\bowtie^{f,g}(J,J')$. Then $f(A)+J(\cong\frac{A\bowtie^{f,g}(J,J')}{\{0\}\times J'})$ and $g(A)+J'(\cong\frac{A\bowtie^{f,g}(J,J')}{J\times\{0\}})$ by [52, Proposition 4.1 (b)] are coherent rings by [42, Theorem 2.4.1], as desired.

Lemma 4.2.4.

Let $f:A\longrightarrow B$ and $g:A\longrightarrow C$ be two ring homomorphisms and let J (resp., J') be an ideal of B (resp., C) such that $f^{-1}(J)=g^{-1}(J')$ and let U be a submodule of A^n. Then:
Assume that U is a finitely generated $A-$module and J ,J' are finitely generated ideals of $f(A)+J$ and $g(A)+J'$, respectively. Then $U\bowtie^{f^n,g^n}(J^n,J'^n)$ is a finitely generated $A\bowtie^{f,g}(J,J')-$ module.

Proof. Assume that $U:=\sum_{i=1}^{i=n}Au_i$ is a finitely generated $A-$module, where $u_i\in U$ for all $i\in\{1,...,n\}$, $J^n:=\sum_{i=1}^{i=n}(f(A)+J)e_i$ and $J'^n:=\sum_{i=1}^{i=n}(g(A)+J)d_i$ are finitely generated $(f(A)+J)-$module and $(g(A)+J')-$module respectively, where $e_i\in J^n$ and $d_i\in J'^n$ for all $i\in\{1,...,n\}$. We claim that

$U\bowtie^{f^n,g^n}(J^n,J'^n)=\sum_{i=1}^{i=n}(A\bowtie^{f,g}(J,J'))(f^n(u_i),g^n(u_i))+\sum_{i=1}^{i=n}(A\bowtie^{f,g}(J,J'))(e_i,0)+$

$\sum_{i=1}^{i=n}(A\bowtie^{f,g}(J,J'))(0,d_i)$.

Indeed, $\sum_{i=1}^{i=n}(A\bowtie^{f,g}(J,J'))(f^n(u_i),g^n(u_i))+\sum_{i=1}^{i=n}(A\bowtie^{f,g}(J,J'))(e_i,0)+$
$\sum_{i=1}^{i=n}(A\bowtie^{f,g}(J,J'))(0,d_i)\subseteq U\bowtie^{f^n,g^n}(J^n,J'^n)$ since $(f^n(u_i),g^n(u_i))\in U\bowtie^{f^n,g^n}(J^n,J'^n)$, $(e_i,0)\in U\bowtie^{f^n,g^n}(J^n,J'^n)$ and $(0,d_i)\in U\bowtie^{f^n,g^n}(J^n,J'^n)$ for all $i\in\{1,...,n\}$.

Conversely, let $(f^n(x)+j,g^n(x)+j')\in U\bowtie^{f^n,g^n}(J^n,J'^n)$, where $x\in U$, $j\in J^n$ and $j'\in J'^n$. Hence, $x=\sum_{i=1}^{i=n}\alpha_iu_i$, for some $\alpha_i\in A$ $(i\in\{1,...,n\})$, $j=\sum_{i=1}^{i=n}(f(\beta_i)+j_i)e_i\in J^n$ and $j'=\sum_{i=1}^{i=n}(g(\lambda_i)+j'_i)d_i\in J'^n$ for $\beta_i,\lambda_i\in A$, $j_i\in J$ and $j'_i\in J'$ $(i\in\{1,...,n\}$.
We obtain

$$\begin{aligned}
(f^n(x)+j,g^n(x)+j') &= (f^n(\sum_{i=1}^{i=n}\alpha_iu_i)+j,g^n(\sum_{i=1}^{i=n}\alpha_iu_i)+j')\\
&= (\sum_{i=1}^{i=n}f(\alpha_i)f^n(u_i),\sum_{i=1}^{i=n}g(\alpha_i)g^n(u_i))+(j,0)+(0,j')\\
&= \sum_{i=1}^{i=n}(f(\alpha_i),g(\alpha_i))(f^n(u_i),g^n(u_i))+(\sum_{i=1}^{i=n}(f(\beta_i)+j_i)e_i,0)\\
&+ (0,\sum_{i=1}^{i=n}(g(\lambda_i)+j'_i)d_i)\\
&= \sum_{i=1}^{i=n}(f(\alpha_i),g(\alpha_i))(f^n(u_i),g^n(u_i))+\sum_{i=1}^{i=n}(f(\beta_i)+j_i,0)(e_i,0)\\
&+ \sum_{i=1}^{i=n}(0,g(\lambda_i)+j'_i)(0,d_i)\\
&= \sum_{i=1}^{i=n}(f(\alpha_i),g(\alpha_i))(f^n(u_i),g^n(u_i))+\sum_{i=1}^{i=n}(f(\beta_i)+j_i,g(\beta_i))(e_i,0)\\
&+ \sum_{i=1}^{i=n}(f(\lambda_i),g(\lambda_i)+j'_i)(0,d_i).
\end{aligned}$$

Consequently, $(f^n(x)+j,g^n(x)+j')\in\sum_{i=1}^{i=n}(A\bowtie^{f,g}(J,J'))(f^n(u_i),g^n(u_i))+$
$\sum_{i=1}^{i=n}(A\bowtie^{f,g}(J,J'))(e_i,0)+\sum_{i=1}^{i=n}(A\bowtie^{f,g}(J,J'))(0,d_i)$
since $(f(\alpha_i),g(\alpha_i)),(f(\beta_i)+j_i,g(\beta_i)),(f(\lambda_i),g(\lambda_i)+j'_i)\in A\bowtie^{f,g}(J,J')$ for all $i\in\{1,...,n\}$.
Hence $U\bowtie^{f^n,g^n}(J^n,J'^n)=\sum_{i=1}^{i=n}(A\bowtie^{f,g}(J,J'))(f^n(u_i),g^n(u_i))+$
$\sum_{i=1}^{i=n}(A\bowtie^{f,g}(J,J'))(e_i,0)+\sum_{i=1}^{i=n}(A\bowtie^{f,g}(J,J'))(0,d_i)$ is a finitely generated $(A\bowtie^{f,g}$

$(J,J'))$-module, as desired.

Lemma 4.2.5.

Let $f : A \longrightarrow B$ and $g : A \longrightarrow C$ be two ring homomorphisms and let J (resp., J') be an ideal of B (resp., C) such that $f^{-1}(J) = g^{-1}(J')$. Assume that J (resp., J') is a finitely generated ideal of $f(A)+J$ (resp., $g(A)+J'$) and $J \subseteq f(A)$. Then $J \times \{0\}$ is a coherent $(A \bowtie^{f,g} (J,J'))$-module provided $f(A)+J$ is a coherent ring.

Proof.

Since $J \times \{0\}$ is a finitely generated $(A \bowtie^{f,g} (J,J'))$-module, it remains to show that every finitely generated submodule of $J \times \{0\}$ is finitely presented. Assume that $f(A)+J$ is a coherent ring and let N be a finitely generated submodule of $J \times \{0\}$. It is clear that $N = I \times \{0\}$, where $I = \sum_{i=1}^{i=n}(f(A)+J)b_i$ for some integer n and $b_i \in I$. Consider the exact sequence of $(f(A)+J)$-modules:

$$0 \longrightarrow \ker v \longrightarrow (f(A)+J)^n \longrightarrow I \longrightarrow 0 \qquad (1)$$

where $v((f(\alpha_i)+j_i)_{i=1}^{i=n}) = \sum_{i=1}^{i=n}(f(\alpha_i)+j_i))b_i$. Then,

$$\begin{aligned}\ker v &= \{(f(\alpha_i)+j_i)_{i=1}^{i=n}) \in (f(A)+J)^n / \sum_{i=1}^{i=n}(f(\alpha_i)+j_i))b_i = 0\}\\ &= \{(f(c_i))_{i=1}^{i=n}) \in (f(A))^n / \sum_{i=1}^{i=n}(f(c_i))b_i = 0\}\end{aligned}$$

where $c_i = \alpha_i + k_i$ and $f(k_i) = j_i$ for some $k_i \in A$ (since $J \subseteq f(A)$).

The $(f(A)+J)$-module $\ker v$ is finitely generated since $f(A)+J$ is a coherent ring. Let $\{f^n((c_i^1))_{i=1}^{i=n}), f^n((c_i^2))_{i=1}^{i=n}), \ldots, f^n((c_i^m))_{i=1}^{i=n})\}$ be a generating set of $\ker v$. On the other hand, it is easily verified that $N = \sum_{i=1}^{i=n}(A \bowtie^{f,g} (J,J'))(b_i,0)$. Consider the exact sequence of $(A \bowtie^{f,g} (J,J'))$-modules:

$$0 \longrightarrow \ker u \longrightarrow (A \bowtie^{f,g} (J,J'))^n \longrightarrow N \longrightarrow 0 \qquad (2)$$

where $u((f(\alpha_i)+j_i, g(\alpha_i)+j'_i)_{i=1}^{i=n}) = \sum_{i=1}^{i=n}(f(\alpha_i)+j_i, g(\alpha_i)+j'_i))(b_i,0)$. Then,

$$\begin{aligned}\ker u &= \{((f(\alpha_i)+j_i, g(\alpha_i)+j'_i)_{i=1}^{i=n}) \in (A \bowtie^{f,g} (J,J'))^n / \sum_{i=1}^{i=n}(f(\alpha_i)+j_i))b_i = 0\}\\ &= \{((f(d_i), g(d_i)+k_i)_{i=1}^{i=n}) \in (A \bowtie^{f,g} (J,J'))^n / \sum_{i=1}^{i=n}(f(d_i))b_i = 0\}\end{aligned}$$

where $d_i = \alpha_i + t_i$ and $f(t_i) = j_i$ for some $t_i \in A$ (since $J \subseteq f(A)$).

Let U be the the submodule of A^n generated by $\{((c_i^1)_{i=1}^{i=n}), ((c_i^2)_{i=1}^{i=n}), \ldots, ((c_i^m)_{i=1}^{i=n})\}$, we claim that $\ker u = U \bowtie^{f^n, g^n} (0, J'^n)$.

Indeed, let $x = (f^n((d_i)_{i=1}^{i=n}), g^n((d_i)_{i=1}^{i=n}) + (j_i)_{i=1}^{i=n}) \in U \bowtie^{f^n, g^n} (0, J'^n)$, so $((d_i)_{i=1}^{i=n}) = \sum_{j=1}^{j=m} a_j ((c_i^j)_{i=1}^{i=n}) = (\sum_{j=1}^{j=m} a_j c_i^j)_{i=1}^{i=n}$. We have

$$\sum_{i=1}^{i=n} f(d_i) b_i = \sum_{i=1}^{i=n} f(\sum_{j=1}^{j=m} a_j c_i^j) b_i = \sum_{i=1}^{i=n} (\sum_{j=1}^{j=m} f(a_j c_i^j)) b_i = \sum_{j=1}^{j=m} f(a_j) (\sum_{i=1}^{i=n} f(c_i^j) b_i) = 0.$$

Consequently, $x \in \ker u$.

Conversely, let $x \in \ker u$, so $x = (f^n((d_i)_{i=1}^{i=n}), g^n((d_i)_{i=1}^{i=n}) + (k_i)_{i=1}^{i=n})$ such that $\sum_{i=1}^{i=n} f(d_i) b_i = 0$. Then, $f^n((d_i)_{i=1}^{i=n}) \in \ker v$ hence

$$\begin{aligned} f^n((d_i)_{i=1}^{i=n}) &= \sum_{j=1}^{j=m} f(a_j) f^n((c_i^j)_{i=1}^{i=n}) \\ &= \sum_{j=1}^{j=m} f^n((a_j c_i^j)_{i=1}^{i=n}) \\ &= f^n((\sum_{j=1}^{j=m} a_j c_i^j)_{i=1}^{i=n}) \end{aligned}$$

consequently, $x = (f^n((\sum_{j=1}^{j=m} a_j c_i)_{i=1}^{i=n}), g^n((\sum_{j=1}^{j=m} a_j c_i^j)_{i=1}^{i=n}) + (k_i)_{i=1}^{i=n})$ where $(\sum_{j=1}^{j=m} a_j c_i^j)_{i=1}^{i=n} \in U$ which implies that $x \in U \bowtie^{f^n, g^n} (0, J'^n)$. Since U is a finitely generated A-module and J is a finitely generated ideal of B, then $U \bowtie^{f^n, g^n} (0, J'^n)$ is a finitely generated $(A \bowtie^{f,g} (J, J'))$-module (by Lemma 2.4). Therefore, N is a finitely presented $(A \bowtie^{f,g} (J, J'))$-module by the sequence (2) and hence $J \times \{0\}$ is a coherent $(A \bowtie^{f,g} (J, J'))$-module, to complete the proof of lemma 4.2.5.

Lemma 4.2.6.

Let $f : A \longrightarrow B$ and $g : A \longrightarrow C$ be two ring homomorphisms and let J (resp., J') be an ideal of B (resp., C) such that $f^{-1}(J) = g^{-1}(J')$. Assume that J (resp., J') is a finitely generated ideal of $f(A) + J$ (resp., $g(A) + J'$) and $J^2 = 0$. Then $J \times \{0\}$ is a coherent $(A \bowtie^{f,g} (J, J'))$-module provided $f(A) + J$ is a coherent ring.

Proof.

Since $J \times \{0\}$ is a finitely generated $(A \bowtie^{f,g} (J, J'))$-module, it remains to show that every finitely generated submodule of $J \times \{0\}$ is finitely presented. Assume that $f(A) + J$

is a coherent ring and let N be a finitely generated submodule of $J \times \{0\}$. It is clear that $N = I \times \{0\}$, where $I = \sum_{i=1}^{i=n}(f(A)+J)b_i$ for some integer n and $b_i \in I$. Consider the exact sequence of $(f(A)+J)$-modules:

$$0 \longrightarrow \ker v \longrightarrow (f(A)+J)^n \longrightarrow I \longrightarrow 0 \qquad (1)$$

where $v((f(\alpha_i)+j_i)_{i=1}^{i=n}) = \sum_{i=1}^{i=n}(f(\alpha_i)+j_i))b_i$. Then,

$$\begin{aligned} kerv &= \{(f(\alpha_i)+j_i)_{i=1}^{i=n}) \in (f(A)+J)^n / \sum_{i=1}^{i=n}(f(\alpha_i)+j_i))b_i = 0\} \\ &= \{(f(\alpha_i)+j_i)_{i=1}^{i=n}) \in (f(A)+J)^n / \sum_{i=1}^{i=n}(f(\alpha_i))b_i = 0\} \end{aligned}$$

since $I \subseteq J$ and $J^2 = 0$.

The $(f(A)+J)$-module $\ker v$ is finitely generated since $f(A)+J$ is a coherent ring. Let $\{f^n((\alpha_i^1))_{i=1}^{i=n}) + (j_i^1)_{i=1}^{i=n}, f^n((\alpha_i^2))_{i=1}^{i=n}) + (j_i^2)_{i=1}^{i=n}, \ldots, f^n((\alpha_i^m))_{i=1}^{i=n}) + (j_i^m)_{i=1}^{i=n}\}$ be a generating set of $\ker v$. On the other hand, it is easily verified that $N = \sum_{i=1}^{i=n}(A \bowtie^{f,g} (J,J'))(b_i,0)$. Consider the exact sequence of $(A \bowtie^{f,g} (J,J'))$-modules:

$$0 \longrightarrow \ker u \longrightarrow (A \bowtie^{f,g} (J,J'))^n \longrightarrow N \longrightarrow 0 \qquad (2)$$

where $u((f(d_i)+j_i, g(d_i)+j'_i)_{i=1}^{i=n}) = \sum_{i=1}^{i=n}(f(d_i)+j_i, g(d_i)+j'_i))(b_i,0)$. Then,

$$\begin{aligned} keru &= \{(f(d_i)+j_i, g(d_i)+j'_i)_{i=1}^{i=n}) \in (A \bowtie^{f,g} (J,J'))^n / \sum_{i=1}^{i=n}(f(d_i)+j_i))b_i = 0\} \\ &= \{(f(d_i)+j_i, g(d_i)+k_i)_{i=1}^{i=n}) \in (A \bowtie^{f,g} (J,J'))^n / \sum_{i=1}^{i=n}(f(d_i))b_i = 0\} \end{aligned}$$

since $I \subseteq J$ and $J^2 = 0$.

Let U be the submodule of A^n generated by $\{((\alpha_i^1)_{i=1}^{i=n}), ((\alpha_i^2)_{i=1}^{i=n}), \ldots, ((\alpha_i^m)_{i=1}^{i=n})\}$, we claim that $\ker u = U \bowtie^{f^n,g^n} (J^n, J'^n)$.

Indeed, let $x = (f^n((d_i)_{i=1}^{i=n}) + (j_i)_{i=1}^{i=n}, g^n((d_i)_{i=1}^{i=n}) + (j'_i)_{i=1}^{i=n}) \in U \bowtie^{f^n,g^n} (J^n, J'^n)$, so $((d_i)_{i=1}^{i=n}) = \sum_{j=1}^{j=m} a_j((\alpha_i^j)_{i=1}^{i=n}) = (\sum_{j=1}^{j=m} a_j\alpha_i^j)_{i=1}^{i=n}$. We have

$$\sum_{i=1}^{i=n} f(d_i)b_i = \sum_{i=1}^{i=n} f(\sum_{j=1}^{j=m} a_j\alpha_i^j)b_i = \sum_{i=1}^{i=n}(\sum_{j=1}^{j=m} f(a_j\alpha_i^j))b_i = \sum_{j=1}^{j=m} f(a_j)(\sum_{i=1}^{i=n} f(\alpha_i^j)b_i) = 0.$$

Consequently, $x \in \ker u$.

Conversely, let $x \in \ker u$, so $x = (f^n((d_i)_{i=1}^{i=n}) + (k_i)_{i=1}^{i=n}, g^n((d_i)_{i=1}^{i=n}) + (k'_i)_{i=1}^{i=n})$ such that $\sum_{i=1}^{i=n} f(d_i)b_i = 0$, then $f^n((d_i)_{i=1}^{i=n}) + (k_i)_{i=1}^{i=n} \in \ker v$ hence

$$\begin{aligned} f^n((d_i)_{i=1}^{i=n}) + (k_i)_{i=1}^{i=n} &= \sum_{s=1}^{s=m} (f(a_s) + t_s)(f^n((\alpha_i^s)_{i=1}^{i=n}) + (j_i^s)_{i=1}^{i=n}) \\ &= \sum_{s=1}^{s=m} f^n((a_s\alpha_i^s)_{i=1}^{i=n}) + (l_i)_{i=1}^{i=n} \\ &= f^n((\sum_{s=1}^{s=m} a_s\alpha_i^s)_{i=1}^{i=n}) + (l_i)_{i=1}^{i=n} \end{aligned}$$

consequently, $x = (f^n((\sum_{s=1}^{s=m} a_s\alpha_i^s)_{i=1}^{i=n}) + (l_i)_{i=1}^{i=n}, g^n((\sum_{s=1}^{s=m} a_s\alpha_i^s)_{i=1}^{i=n}) + (l'_i)_{i=1}^{i=n})$ where $(\sum_{s=1}^{s=m} a_s\alpha_i^s)_{i=1}^{i=n} \in U$ which implies that $x \in U \bowtie^{f^n,g^n} (J^n, J'^n)$. Since U is a finitely generated A-module and J (resp., J') is a finitely generated ideal of B (resp., C), then $U \bowtie^{f^n,g^n} (0, J'^n)$ is a finitely generated $(A \bowtie^{f,g} (J,J'))$-module (by Lemma 4.2.4). Therefore, N is a finitely presented $(A \bowtie^{f,g} (J,J'))$-module by the sequence (2) and hence $J \times \{0\}$ is a coherent $(A \bowtie^{f,g} (J,J'))$-module, to complete the proof of lemma 4.2.6.

Lemma 4.2.7.

Let $f : A \longrightarrow B$ and $g : A \longrightarrow C$ be two ring homomorphisms and let J (resp., J') be an ideal of B (resp., C) such that $f^{-1}(J) = g^{-1}(J')$.

1. *If $(A \bowtie^{f,g} (J,J'))$ is a coherent ring and J is regular, then J' is a finitely generated ideal of $g(A) + J'$.*
2. *Assume that J, J' are regular ideals of $f(A) + J$ and $g(A) + J'$, respectively. Then:*
   If $A \bowtie^{f,g} (J,J')$ is a coherent ring then so is $f(A) + J$ and $g(A) + J'$.

Proof.

1. Assume that $(A \bowtie^{f,g} (J,J'))$ is a coherent ring and J contains a regular element k. Set $c = (k,0) \in A \bowtie^{f,g} (J,J')$. One can easily check that :

$$\begin{aligned} (0:c) &= \{(f(a)+j, g(a)+j') \in A \bowtie^{f,g} (J,J') / (f(a)+j, g(a)+j')(k,0) = 0\} \\ &= \{(f(a)+j, g(a)+j') \in A \bowtie^{f,g} (J,J') / (f(a)+j)k = 0\} \\ &= \{(f(a)+j, g(a)+j') \in A \bowtie^{f,g} (J,J') / (f(a)+j) = 0\} \\ &= \{(0, g(a)+j') \in A \bowtie^{f,g} (J,J') / a \in f^{-1}(J) = g^{-1}(J')\} \\ &= \{0\} \times J' \end{aligned}$$

Since $(A \bowtie^{f,g} (J,J'))$ is a coherent ring, then $(0:c) = \{0\} \times J'$ is a finitely generated ideal of $A \bowtie^{f,g} (J,J')$. Therefore, J' is a finitely generated ideal of $g(A)+J'$, as desired.

2. Follows immediately from (1) and Lemma 4.2.3.

Proof of Theorem 4.2.2

1. If $A \bowtie^{f,g} (J,J')$ is a coherent ring, then so are $f(A)+J$ and $g(A)+J'$ by (lemma 4.2.3 (2)) since J,J' are finitely generated ideals of $f(A)+J$ and $g(A)+J'$, respectively. Conversely, assume that $f(A)+J$ and $g(A)+J'$ are coherent rings. Since $f(A)+J(\cong \frac{A\bowtie^{f,g}(J,J')}{\{0\}\times J'})$ and $g(A)+J'(\cong \frac{A\bowtie^{f,g}(J,J')}{J\times\{0\}})$ by [52, Proposition 4.1 (b)] and $J\times\{0\}$ is a coherent $(A \bowtie^{f,g} (J,J'))$-module (by Lemma 2.5), then $A \bowtie^{f,g} (J,J')$ is a coherent ring by [42, Theorem2.4.1] .
2. If $A \bowtie^{f,g} (J,J')$ is a coherent ring, then so are $f(A)+J$ and $g(A)+J'$ by (lemma 4.2.3 (2)) since J,J' are finitely generated ideals of $f(A)+J$ and $g(A)+J'$, respectively. Conversely, assume that $f(A)+J$ and $g(A)+J'$ are coherent rings. Since $f(A)+J(\cong \frac{A\bowtie^{f,g}(J,J')}{\{0\}\times J'})$ and $g(A)+J'(\cong \frac{A\bowtie^{f,g}(J,J')}{J\times\{0\}})$ by [52, Proposition 4.1 (b)] and $J\times\{0\}$ is a coherent $(A \bowtie^{f,g} (J,J'))$-module (by Lemma 4.2.6), then $A \bowtie^{f,g} (J,J')$ is a coherent ring by [42, Theorem2.4.1] .
3. Follows immediately from theorem 4.2.2(1) and Lemma 4.2.7.
4. Follows immediately from theorem 4.2.2(1) and Lemma 4.2.7.
5. Follows immediately from theorem 4.2.2(2) and Lemma 4.2.7.

Recall that the amalgamation of A with B along J with respect to f is given by

$$A \bowtie^{f} J := \{(a, f(a)+j) \mid a \in A, j \in J\}$$

Clearly, every amalgamation can be viewed as a special bi-amalgamation, since $A \bowtie^{f} J = A \bowtie^{Id,f} (f^{-1}(J),J)$. Accordingly, Theorem 4.2.2 covers the special case of amalgamation [1], as recorded below.

Corollary 4.2.8.

Let $f : A \longrightarrow B$ be a ring homomorphism and let J be a proper ideal of B.

1. *If $A \bowtie^f J$ is a coherent ring, then so is A.*
2. *Assume that J and $f^{-1}(J)$ are finitely generated ideals of $f(A)+J$ and A, respectively. Then $A \bowtie^f J$ is a coherent ring if and only if $f(A)+J$ and A are coherent rings.*
3. *Assume that J is a regular finitely generated ideal of $f(A)+J$. Then $A \bowtie^f J$ is a coherent ring if and only if $f(A)+J$ and A are coherent rings and $f^{-1}(J)$ is a finitely generated ideal of A.*

The following Corollary is an immediate consequence of Theorem 4.2.2 (3)(4).

Corollary 4.2.9.

Let $f : A \longrightarrow B$ and $g : A \longrightarrow C$ be two ring homomorphisms and let J (resp., J') be an ideal of B (resp., C) such that $f^{-1}(J) = g^{-1}(J')$.

1. *If B is an integral domain, J is a finitely generated ideal of $f(A)+J$ and $J'^2 = 0$, then:*
   $A \bowtie^{f,g} (J,J')$ is a coherent ring if and only if $f(A)+J$ and $g(A)+J'$ are coherent rings and J' is a finitely generated ideal of $g(A)+J'$.
2. *If B and C are integral domains and $J \subseteq f(A)$, then:*
   $A \bowtie^{f,g} (J,J')$ is a coherent ring if and only if $f(A)+J$ and $g(A)+J'$ are coherent rings and J, J' are finitely generated ideals of $f(A)+J$ and $g(A)+J'$, respectively.

Example 4.2.10.

Let A be a non-Noetherian coherent ring, I and K are finitely generated ideals of A such that $I \subseteq K$. Let $f : A \to A/I$ the canonical homomorphism and $g : A \to A \times A$ the injective homomorphism defined by $g(a) = (a,0)$, $J = K/I$ and $J' = K \times 0$. Then $A \bowtie^{f,g} (J,J')$ is a non-Noetherian coherent ring.

Proof. By Theorem 4.2.2, $A \bowtie^{f,g} (J,J')$ is a coherent ring since $f(A)+J = A/I$ and $g(A)+J' = A\times A$ are both coherent rings and J (resp., J') is a finitely generated ideal of $f(A)+J$ (resp., $g(A)+J'$) and $J = K/I \subseteq f(A) = A/I$. On the other hand, $A \bowtie^{f,g} (J,J')$ is a non-Noetherian ring by [52, Proposition 4.2] since $g(A)+J' = A\times A$ is non-Noetherian ring.

Example 4.2.11.

Let (A,M) be a non-Noetherian local coherent ring such that M is a finitely generated ideal, E an A/M-vector space of finite rank. Let $f : A \to A \ltimes E$ the injective homomrphism defined by $f(a) = (a,0)$ and $g : A \to (A/M)[X_1,X_2,...,X_n]$ defined by $g(a) = \overline{a}$, $J = 0 \ltimes E$ and $J' = (X_1,X_2,...,X_n)$. Then $A \bowtie^{f,g} (J,J')$ is a non-Noetherian coherent ring.

Proof. By Theorem 4.2.2, $A \bowtie^{f,g} (J,J')$ is a coherent ring since $f(A)+J = A \ltimes E$ is a coherent ring by [53, Theorem 2.6] and $g(A)+J' = A/M[X_1,X_2,...,X_n]$ is a coherent ring (Noetherian) and J (resp., J') is a finitely generated ideal of $f(A)+J$ (resp., $g(A)+J'$) and $J^2 = 0$. On the other hand, $A \bowtie^{f,g} (J,J')$ is a non-Noetherian ring by [52, Proposition 4.2] since $f(A)+J = A\times E$ is non-Noetherian ring.

Example 4.2.12.

Let $A = \mathbb{Z}[X]$, $B = \mathbb{Z}+X\mathbb{Q}[X]$, $C = \mathbb{Z}$ and let $J = n_0\mathbb{Z}+X\mathbb{Q}[X]$, $J' = n_0\mathbb{Z}$ ideals of B and C, respectively. $f : A \longrightarrow B$ the homomorphism defined by $f(P(X)) = P(X)$ and $g : A \longrightarrow C$ the homomorphism defined by $f(P(x)) = P(0)$. Then $A \bowtie^{f,g} (J,J')$ is a non-Noetherian coherent ring .

Proof. By Theorem 4.2.2, $A \bowtie^{f,g} (J,J')$ is a non-Noetherian coherent ring since $f(\mathbb{Z}[X])+J = \mathbb{Z}+X\mathbb{Q}[X]$ is a non-Noetherian coherent ring and $g(\mathbb{Z}[X])+J' = \mathbb{Z}$ is a coherent ring (Noetherian) and $J' \subseteq g(A)$.

Example 4.2.13.

Let $A = \mathbb{Z}[X]$, $B = \mathbb{Z}+X\mathbb{Q}[X]$, $C = \mathbb{Z}+i\mathbb{Z}[i] = \mathbb{Z}[i]$ and let $J = n_0\mathbb{Z}+X\mathbb{Q}[X]$, $J' = n_0\mathbb{Z}+i\mathbb{Z}[i]$ ideals of B and C, respectively. $f : A \longrightarrow B$ the homomorphism defined by $f(P(X)) = P(0)$ and $g : A \longrightarrow C$ the homomorphism defined by $g(P(x)) = P(i)$. Then $A \bowtie^{f,g} (J,J')$ is a non-Noetherian coherent ring .

Proof. By Theorem 4.2.2, $A \bowtie^{f,g} (J,J')$ is a non-Noetherian coherent ring since $f(\mathbb{Z}[X])+J = \mathbb{Z}+X\mathbb{Q}[X]$ is a non-Noetherian coherent ring and $g(\mathbb{Z}[X])+J' = \mathbb{Z}[i]$ is a coherent ring (Noetherian) and $J' \subseteq g(A)$.

CHAPTER 5

BI-AMALGAMATION OF RINGS DEFINED BY BÉZOUT-LIKE CONDITIONS

Abstract.

In this chapter*, let $f : A \longrightarrow B$ and $g : A \longrightarrow C$ be two ring homomorphisms and let J (resp., J') be an ideal of B (resp., C) such that $f^{-1}(J) = g^{-1}(J')$. In this paper, we investigate the transfer of notions of elementary divisor ring, almost Bézout domain (AB-domain) and almost valuation domain (AV-domain) to the bi-amalgamation of A with (B,C) along (J,J') with respect to (f,g) (denoted by $A \bowtie^{f,g} (J,J')$, introduced and studied by Kabbaj, Louartiti and Tamekkante in 2013.

Key Words: Bi-amalgamated algebra, elementary divisor ring, Hermite ring, almost Bézout domain, almost valuation domain.

5.1 Introduction

A ring R is called an elementary divisor ring (respectively, Hermite ring) if for every matrix M over R there exist nonsingular matrices P, Q such that PMQ (respectively, MQ) is a diagonal matrix (respectively, triangular matrix). It proved in [40] that a ring R is an Hermite ring if and only if for all $a,b \in R$, there exist $a_1, b_1, d \in R$ such that $a = a_1 d$, $b = b_1 d$, and $Ra_1 + Rb_1 = R$. A ring is called Bézout ring if every finitely generated ideal is principal. It is clear that every elementary divisor ring is an Hermite ring, and that every Hermite ring is a Bézout ring.

Following [55] a ring R is said to be a valuation ring if for any two elements in R, one divides the other. Kaplansky proved that any valuation ring is an elementary divisor ring.

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In [5], D. D. Anderson and M. Zaffrullah introduced the notion of almost valuation domain (AV-domain for short) as a ring R such that for any two elements a and b in R, there exists a positive integer n such that a^n divides b^n or b^n divides a^n. Also they introduced the notion of almost Bézout domain (AB-domain) as a ring R such that for any two elements a and b in R, there exists a positive integer n such that the ideal (a^n, b^n) is principal. Among others, they proved that the integral closure of an almost valuation (resp., almost Bézout) domain is a valuation domain (resp., a Prüfer domain with torsion class group). Moreover, the notion of almost Bézout domains runs along lines somewhat similar to those of Bézout domain (i.e, every two generated, equivalently, every finitely generated, ideal is principal). In [3], D. D. Anderson, K. R. Knopp, and R. L. Lewin continued the study of almost Bézout domains, and after observing that each Bézout domain is nearly Bézout, they used the construction $K+XL[X]$ to disprove the converse. the same example shows that a Noetherian almost Bézout domain need not be an almost principal ideal domain (API-domain), even though each Noetherian Bézout domain is a principal ideal domain (PID). In [6], Anderson and Zaffrullah continued their study of almost Bézout domains and gave a new characterisation of Cohen-Kaplansky domains. They also showed that a finite intersection of almost valuation domains with the same quotient field is an almost Bézout domain. This result generalizes the classical case that a finite intersection of valuation domains with the same quotient field is a Bézout domain. In [10], A. Badawi introduced a new class of integral domains closely related to AVD's, that is the class of pseudo-almost valuation domains (PAVD's). He showed that the class of almost valuation domains is properly contained in the class pseudo-almost valuation domains, and that PAVD's are precisely the pullbacks of AVD's. In [63], A. Mimouni studied the transfer of the notions of almost valuation, almost Prüfer and almost Bézout domains to pullbacks.

Let $f : A \longrightarrow B$ and $g : A \longrightarrow C$ be two ring homomorphisms and let J (resp., J') be an ideal of B (resp., C) such that $f^{-1}(J) = g^{-1}(J')$. In this setting, we can consider the following subring of $B \times C$:

$$A \bowtie^{f,g} (J,J') := \{(f(a)+j, g(a)+j') | a \in A, j \in J, j' \in J'\}$$

called the bi-amalgamation of A with (B,C) along (J,J') with respect to (f,g) (introduced and studied by Kabbaj, Louartiti and Tamekkante in 2013 in [52]).

This construction is a generalisation of the amalgamated algebra along an ideal (introduced and studied by D'Anna and Fontana in [29, 30].) Moreover, other classical constructions (such as the $A+XB[X]$, $A+XB[[X]]$, and the $D+M$ constructions)can be studied as particular cases of the amalgamation [29, Exemples 2.5 and 2.6] and other classical constructions , such as the Nagata's idealization ([66, page 2]), and the CPI extensions are strictly related to it ([29, Exemple 2.7 and Remark 2.8]). In [52], the authors studied the basic properties of this construction (e.g., characterized for $A \bowtie^f J$ to be a Noetherian ring, an integral domain, a reduced ring) and they characterized those distinguished pullbacks that can be expressed as a bi-amalgamation. Moreover, they pursued the investigation on the structure of the rings of the form $A \bowtie^{f,g} (J,J')$, with particular attention to the prime spectrum.

This paper aims at studying the transfer of the notions of elementary divisor ring, almost Bézout domain (AB-domain) and almost valuation domain (AV-domain) to the bi-amalgamation of algebra. It contains in addition to the introduction three sections, the first one deals with the transfer of the notions of elementary divisor ring, and investigate the relationship between this notion and Hermite ring, and Bézout ring in the context of bi-amalgamation of algebra. The second and third sections investigates the transfer of the notions of almost valuation domain (AV-domain), and the almost Bézout domain (AB-domain) to the pre-mentioned construction.

5.2 Transfer of elementary divisor ring to the bi-amalgamated algebra

The main result of this section examines necessary and sufficient conditions for bi-amalgamation $A \bowtie^{f,g} (J,J')$ to inherit the notions of elementary divisor ring, and establish the relationship between this notion and Hermite ring, and Bézout ring.

The set of all $n\times n$ matrices with entries from a ring R will be denoted by $M_n(R)$. We will let $GL_n(R)$ denote the units in $M_n(R)$. Let B and C be rings, for every matrix $M = ((b_{i,j}, c_{i,j}))_{1\leq i,j\leq n} \in M_n(B\times C)$, we shall use the notation $M_b = (b_{i,j})_{1\leq i,j\leq n}$, $M_c = (c_{i,j})_{1\leq i,j\leq n}$ and $M = M_b \times M_c$. Let $M,N \in M_n(B\times C)$, it is easy to see that the product MN of M and N is given by $MN = (M_bN_b)\times(M_cN_c)$.

Theorem 5.2.1.

Let A, B and C be integral domains, $f : A \longrightarrow B$ and $g : A \longrightarrow C$ be two ring homomorphisms and let J (resp., J') be an ideal of B (resp., C) such that $f^{-1}(J) = g^{-1}(J')$. Then, $A \bowtie^{f,g} (J,J')$ is an elementary divisor ring if and only if the following statements hold:

1. *$f(A)+J$ and $g(A)+J'$ are elementary divisor rings.*
2. *$J = 0$ or $J' = 0$*

The proof of this theorem requires the following lemmas.

Lemma 5.2.2.

Let A, B and C be integral domains, $f : A \longrightarrow B$ and $g : A \longrightarrow C$ be two ring homomorphisms and let J (resp., J') be a proper ideal of B (resp., C) such that $f^{-1}(J) = g^{-1}(J')$. If $A \bowtie^{f,g} (J,J')$ is a Bézout ring then $J = 0$ or $J' = 0$.

Proof. Assume that $A \bowtie^{f,g} (J,J')$ is a Bézout ring. We claim that $J = 0$ or $J' = 0$. Deny. There are some $0 \neq j \in J$ and $0 \neq j' \in J'$. It is clear that $(j,0)$ and $(0,j') \in A \bowtie^{f,g} (J,J')$. Since $A \bowtie^{f,g} (J,J')$ is a Bézout ring the ideal generated by $(j,0)$ and $(0,j')$ is principal. Hence, there exists $(f(d)+t, g(d)+t') \in A \bowtie^{f,g} (J,J')$ such that

$$(j,0)A \bowtie^{f,g} (J,J') + (0,j')A \bowtie^{f,g} (J,J') = (f(d)+t, g(d)+t')A \bowtie^{f,g} (J,J').$$

So, there exist $(f(b)+l, g(b)+l')$, $(f(c)+k, g(c)+k')$, $(f(\alpha)+r, g(\alpha)+r')$, $(f(\beta)+h, g(\beta)+h')$ in $A \bowtie^{f,g} (J,J')$ such that

$$\begin{array}{rcl} (j,0) & = & (f(d)+t, g(d)+t')(f(b)+l, g(b)+l') \\ (0,j') & = & (f(d)+t, g(d)+t')(f(c)+k, g(c)+k') \\ (f(d)+t, g(d)+t') & = & (j,0)(f(\alpha)+r, g(\alpha)+r') + (0,j')(f(\beta)+h, g(\beta)+h') \end{array}$$

It follows that $g(d)+t' \neq 0$ since $j' = (g(d)+t')(g(c)+k') \neq 0$. Also $g(b)+l' = 0$ since $(g(b)+l')(g(d)+t') = 0$ and C is an integral domain. Remark that $g(b)+l' = 0$ imply that $f(b)+l \in J$.
From the previous equalities we deduce that

$$f(d)+t = j(f(\alpha)+r) = (f(d)+t)(f(b)+l)(f(\alpha)+r).$$

Hence $1 = (f(b)+l)(f(\alpha)+r)$ since B is an integral domain. Therefore $1 \in J$ since $f(b)+l \in J$ which is a contradiction.

Lemma 5.2.3.

The following assertions holds:

1. *Let A and B be two rings. Then $A \times B$ is an elementary divisor ring if and only if so are both A and B.*
2. *Let $f : A \longrightarrow B$ and $g : A \longrightarrow C$ be two ring homomorphisms and let J (resp., J') be an ideal of B (resp., C) such that $f^{-1}(J) = g^{-1}(J')$. Then: If $A \bowtie^{f,g} (J,J')$ is an elementary divisor ring then so are both $f(A)+J$ and $g(A)+J'$.*

Proof.

1. See for instance [54, Lemma 2.2].

2. Let $U = (f(a_{i,j}) + t_{i,j})_{1\leq i,j\leq n} \in M_n(f(A)+J)$ and let M be the matrix defined by $M = ((f(a_{i,j}) + t_{i,j}, g(a_{i,j}))_{1\leq i,j\leq n}$ with entries from $A \bowtie^{f,g} (J,J')$. We have the equality $U = M_b$. Since $A \bowtie^{f,g} (J,J')$ is an elementary divisor ring M is equivalent to a diagonal matrix. From the previous part of the proof we deduce that there exist P and Q in $GL_n(f(A)+J)$ such that PUQ is a diagonal matrix. Therefore $f(A)+J$ is an elementary divisor ring. With a similar argument as in above, we get that $g(A)+J'$ is an elementary divisor ring.

Proof. Assume that $A \bowtie^{f,g} (J,J')$ is an elementary divisor ring.
(1) By lemma 5.2.2, $f(A)+J$ and $g(A)+J'$ are elementary divisor rings.
(2) By lemma 5.2.1, since every elementary divisor ring is a Bézout ring.
Conversely, assume that (1) and (2) hold. If $J = 0$ or $J' = 0$, then by [52, Proposition 4.1 (b)], $A \bowtie^{f,g} (J,J') \simeq f(A)+J$ or $A \bowtie^{f,g} (J,J') \simeq g(A)+J'$ which are elementary divisor rings.
Hence, $A \bowtie^{f,g} (J,J')$ is an elementary divisor ring, as desired.

Recall that the amalgamation of A with B along J with respect to f is given by

$$A \bowtie^f J := \{(a, f(a)+j) \mid a \in A, j \in J\}$$

Clearly, every amalgamation can be viewed as a special bi-amalgamation, since $A \bowtie^f J = A \bowtie^{Id,f} (f^{-1}(J), J)$. Accordingly, Theorem 5.2.3 covers the special case of amalgamation [54], as recorded below.

Corollary 5.2.4.

Let A and B a pair of integral domains, let $f : A \longrightarrow B$ be a ring homomorphism and let J be an ideal of B.

1. *Assume that f is injective.*
   - *If $J = B$ then $A \bowtie^f J$ is an elementary divisor ring if and only if so are both A and B.*
   - *If $J \neq B$ then $A \bowtie^f J$ is an elementary divisor ring if and only if so are both $f(A)+J$ and $f(A) \cap J = 0$.*
2. *Assume that f is not injective. Then $A \bowtie^f J$ is an elementary divisor ring if and only one of the following conditions holds:*
   - *$J = 0$ and A is an elementary divisor ring.*
   - *$J = B$ and (A,B) is a pair of elementary divisor rings.*

Now, we establish the relationship between elementary divisor ring, Hermite ring and Bézout ring in the context of bi-amalgamation of algebra.

Theorem 5.2.5.

Let A, B and C be integral domains, $f : A \longrightarrow B$ and $g : A \longrightarrow C$ be two ring homomorphisms and let J (resp., J') be an ideal of B (resp., C) such that $f^{-1}(J) = g^{-1}(J')$. The following properties are equivalent:

1. *$A \bowtie^{f,g} (J,J')$ is an Hermite ring.*
2. *$A \bowtie^{f,g} (J,J')$ is a Bézout ring.*
3. *One of the following conditions hold:*
   - *$f(A)+J$ is a Bézout ring, and $J' = 0$.*
   - *$g(A)+J'$ is a Bézout ring, and $J = 0$.*

Proof. $(1) \Rightarrow (2)$ It is clear.
$(2) \Rightarrow (3)$ By lemma 5.2.1 $J = 0$ or $J' = 0$ and so $A \bowtie^{f,g} (J,J') \simeq f(A)+J$ or $A \bowtie^{f,g} (J,J') \simeq g(A)+J'$ which are a Bézout ring.
$(3) \Rightarrow (1)$ In this case, we get that $A \bowtie^{f,g} (J,J') \simeq f(A)+J$ or $A \bowtie^{f,g} (J,J') \simeq g(A)+J'$ which are an Hermite ring, since every Bézout domain is an Hermite ring.

5.3 On AV-ring property

This section characterizes the bi-amalgamated algebra along ideals $A \bowtie^{f,g} (J,J')$ to be an almost valuation ring. The main result (Theorem 5.3.1) examines the property of almost valuation ring, that the bi-amalgamation $A \bowtie^{f,g} (J,J')$ might inherit from, $f(A)+J$, $g(A)+J'$ for some classes of ideals J, J' and homomorphisms f,g and hence generates new examples of almost valuation rings.

Theorem 5.3.1.

Let $f : A \longrightarrow B$ and $g : A \longrightarrow C$ be two ring homomorphisms and let J (resp., J') be an ideal of B having no nontrivial nilpotent elements (resp., C) such that $f^{-1}(J) = g^{-1}(J')$. Then $A \bowtie^{f,g} (J,J')$ is an AV-ring if and only if the following statements hold:

1. *$f(A)+J$ and $g(A)+J'$ are AV-rings.*
2. *$J = 0$ or $J' = 0$.*

The proof of this theorem involves the following lemmas.

Lemma 5.3.2.

Let $f : A \longrightarrow B$ and $g : A \longrightarrow C$ be two ring homomorphisms and let J (resp., J') be an ideal of B (resp., C) such that $f^{-1}(J) = g^{-1}(J')$. If $A \bowtie^{f,g} (J,J')$ is an AV-ring, then so are $f(A)+J$ and $g(A)+J'$.

Proof. By [52, Proposition 4.1 (b)] and the fact that if A is an AV-ring and I is an ideal of A, then A/I is an AV-ring.

Lemma 5.3.3.

Let $f : A \longrightarrow B$ and $g : A \longrightarrow C$ be two ring homomorphisms and let J (resp., J') be ideal of B having no nontrivial nilpotent elements (resp., C) such that $f^{-1}(J) = g^{-1}(J')$. If $A \bowtie^{f,g} (J,J')$ is an AV-ring then $J = 0$ or $J' = 0$.

Proof.

Assume that $A \bowtie^{f,g} (J,J')$ is an AV-ring. We claim that $J = 0$ or $J' = 0$. Deny. There are some $0 \neq j \in J$ and $0 \neq j' \in J'$. It is clear that $(j,0)$ and $(0,j') \in A \bowtie^{f,g} (J,J')$. Since $A \bowtie^{f,g} (J,J')$ is a an AV-ring there exists a positive integer n such that $(j,0)^n \in A \bowtie^{f,g} (J,J')(0,j')^n$ or $(0,j')^n \in A \bowtie^{f,g} (J,J')(j,0)^n$.
If $(j,0)^n \in A \bowtie^{f,g} (J,J')(0,j')^n$, then $(j,0)^n = (j^n,0) = (f(d)+t, g(d)+t')(0,j'^n)$ for some $(f(d)+t, g(d)+t') \in A \bowtie^{f,g} (J,J')$. Thus $j^n = 0$ and since J is reduced, $j = 0$, which is absurd. Hence $(0,j')^n \in A \bowtie^{f,g} (J,J')(j,0)^n$, and so $(0,j')^n = (0,j'^n) = (f(d)+t, g(d)+t')(j^n,0)$ for some $(f(d)+t, g(d)+t') \in A \bowtie^{f,g} (J,J')$. Thus $j'^n = 0$ and since J' is reduced, $j' = 0$, which is again a contradiction.

Proof. of theorem 5.3.1

Assume that $A \bowtie^{f,g} (J,J')$ is an AV-ring.
(1) By lemma 5.3.2, $f(A)+J$ and $g(A)+J'$ are AV-rings.
(2) By lemma 5.3.3
Conversely, assume that (1) and (2) hold. If $J = 0$ or $J' = 0$, then by [52, Proposition 4.1 (b)], $A \bowtie^{f,g} (J,J') \simeq f(A)+J$ or $A \bowtie^{f,g} (J,J') \simeq g(A)+J'$ which are AV-rings.
Hence, $A \bowtie^{f,g} (J,J')$ is an AV-ring, as desired.

The Corollary below follows immediately from Theorem 5.3.1 which examines the case of amalgamation algebra.

Corollary 5.3.4.

Let A and B be a pair of rings and $f : A \longrightarrow B$ be a ring homomorphism. If J is a non-zero proper ideal of B having no nontrivial nilpotent elements and A is reduced. Then $A \bowtie^f J$ is an AV-ring if and only if f is injective, $f(A)+J$ is an AV-ring and $f(A) \cap J = (0)$.

Theorem 5.3.1 enriches the literature with the new original examples of AV-rings.

Example 5.3.5.

Let (A,m) be a local AV-domain, E an A-module such that $mE = 0$, and $B = A \propto E$ the trivial ring extension of A by E, and $C = A/m$. Consider the natural injective ring homomorphisms $f : A \longrightarrow B$ and the canonical surjective ring homomorphisms $g : A \longrightarrow A/m$ and let $J = m \propto \{0\}$. We claim that the bi-amalgamation $R = A \bowtie^{f,g} (J,0)$ is an AV-ring. Indeed, notice first that $f^{-1}(J) = g^{-1}(0) = m$, $f(A)+J = B$ and $g(A) = A/m = C$. Further, B is an AV-ring by [65, Theorem 2.1 (3)] and C is an AV-ring. So, R is an AV-ring by theorem 5.3.1.

Example 5.3.6.

Let A be an AV-ring, I,K be two ideals of A such that $I \subseteq K$ and I is a radical ideal, $B = A/I$, $C = A/K$. Consider the canonicals surjective ring homomorphisms $f : A \longrightarrow B$ and $g : A \longrightarrow C$ and let $J = K/I$. We claim that the bi-amalgamation $R = A \bowtie^{f,g} (J,0)$ is an AV-ring. Indeed, notice first that $f^{-1}(J) = g^{-1}(0) = K$, $f(A) + J = B$, $g(A) = A/K = C$ and $J = K/I$ has no nonzero nilpotent element. Further, B and C are AV-rings. So, R is an AV-ring by theorem 5.3.1.

5.4 On AB-ring property

This section deals with the transfer of the notion of AB-ring to the bi-amalgamated algebras along ideals.

Theorem 5.4.1.

Let (A,B,C) be integral domains, $f : A \longrightarrow B$ and $g : A \longrightarrow C$ be two ring homomorphisms and let J (resp., J') be a proper ideal of B (resp., C) such that $f^{-1}(J) = g^{-1}(J')$. Then $A \bowtie^{f,g} (J,J')$ is an AB-ring if and only if the following statements hold:

1. *$f(A) + J$ and $g(A) + J'$ are AB-rings.*
2. *$J = 0$ or $J' = 0$.*

Before proving this theorem, we recall the following lemmas.

Lemma 5.4.2.

Let (A,B,C) be integral domains, $f : A \longrightarrow B$ and $g : A \longrightarrow C$ be two ring homomorphisms and let J (resp., J') be an ideal of B (resp., C) such that $f^{-1}(J) = g^{-1}(J')$. If $A \bowtie^{f,g} (J,J')$ is an AB-ring, then so are $f(A) + J$ and $g(A) + J'$.

Proof. By [52, Proposition 4.1 (b)] and the fact that if A is an AB-ring and I is an ideal of A, then A/I is an AB-ring.

Lemma 5.4.3.

Let (A,B,C) be integral domains, $f : A \longrightarrow B$ and $g : A \longrightarrow C$ be two ring homomorphisms and let J (resp., J') be a proper ideal of B (resp., C) such that $f^{-1}(J) = g^{-1}(J')$. If $A \bowtie^{f,g} (J,J')$ is an AB-ring then $J = 0$ or $J' = 0$.

Proof.

Assume that $A \bowtie^{f,g} (J,J')$ is an AB-ring. We claim that $J = 0$ or $J' = 0$. Deny. There are some $0 \neq j \in J$ and $0 \neq j' \in J'$. It is clear that $(j,0)$ and $(0,j') \in A \bowtie^{f,g} (J,J')$. Since $A \bowtie^{f,g} (J,J')$ is a an AB-ring there exists a positive integer n such that the ideal $((j,0)^n,(0,j')^n)A \bowtie^{f,g} (J,J') = A \bowtie^{f,g} (J,J')(j^n,0) + A \bowtie^{f,g} (J,J')(0,j'^n)$ is principal ideal of $A \bowtie^{f,g} (J,J')$.
Set $A \bowtie^{f,g} (J,J')(j^n,0) + A \bowtie^{f,g} (J,J')(0,j'^n) = (f(d)+t,g(d)+t')A \bowtie^{f,g} (J,J')$ for some $(f(d)+t,g(d)+t') \in A \bowtie^{f,g} (J,J')$. Then there exist $(f(b)+l,g(b)+l')$, $(f(c)+k,g(c)+k')$, $(f(\alpha)+r,g(\alpha)+r')$, $(f(\beta)+h,g(\beta)+h')$ in $A \bowtie^{f,g} (J,J')$ such that

$$(j^n,0) = (f(d)+t,g(d)+t')(f(b)+l,g(b)+l')$$

$$(0,j'^n) = (f(d)+t,g(d)+t')(f(c)+k,g(c)+k')$$

$$(f(d)+t,g(d)+t') = (j^n,0)(f(\alpha)+r,g(\alpha)+r') + (0,j'^n)(f(\beta)+h,g(\beta)+h').$$

Hence $g(d)+t' \neq 0$ since $j'^n = (g(d)+t')(g(c)+k') \neq 0$. Also $g(b)+l' = 0$ since $(g(b)+l')(g(d)+t') = 0$ and C is an integral domain. Remark that $g(b)+l' = 0$ imply that $f(b)+l \in J$.
From the previous equalities we deduce that

$$f(d)+t = j^n((f(\alpha)+r) = (f(d)+t)(f(b)+l)(f(\alpha)+r)$$

Hence

$$1 = (f(b)+l)(f(\alpha)+r)$$

Since B is an integral domain. Therefore $1 \in J$ since $f(b)+l \in J$ which is a contradiction.

Proof. of theorem 5.4.1

Assume that $A \bowtie^{f,g} (J,J')$ is an AB-ring.
(1) By lemma 5.4.2, $f(A)+J$ and $g(A)+J'$ are AB-rings.
(2) By lemma 5.4.3
Conversely, assume that (1) and (2) hold. If $J = 0$ or $J' = 0$, then by [52, Proposition 4.1 (b)], $A \bowtie^{f,g} (J,J') \simeq f(A)+J$ or $A \bowtie^{f,g} (J,J') \simeq g(A)+J'$ which are AB-rings.
Hence, $A \bowtie^{f,g} (J,J')$ is an AB-ring, as desired.

The Corollary below recovers a known result for amalgamation algebra.

Corollary 5.4.4.

Let A, B *be a pair of integral domains,* $f : A \longrightarrow B$ *be a ring homomorphism and* J *be a proper ideal of* B*. Then* $A \bowtie^f J$ *is an AB-ring if and only if* f *is injective,* $f(A) + J$ *is an AB-ring and* $f(A) \cap J = (0)$*.*

Example 5.4.5.

Let A *be an AB-domain,* I, K *be two prime ideals of* A *such that* $I \subsetneq K \subsetneq A$ *,* $B = A/I$ *,* $C = A/K$*. Consider the canonicals surjective ring homomorphisms* $f : A \longrightarrow B$ *and* $g : A \longrightarrow C$ *and let* $J = K/I$*. We claim that the bi-amalgamation* $R = A \bowtie^{f,g} (J, 0)$ *is an AB-ring. Indeed, notice first that* $f^{-1}(J) = g^{-1}(0) = K$*,* $f(A) + J = B$*,* $g(A) = A/K = C$ *and* $J = K/I$ *. Further,* B *and* C *are AB-rings. So,* R *is an AB-ring by theorem* 5.4.1*.*

CHAPTER 6

BÉZOUT-LIKE PROPERTIES IN AMALGAMATED ALGEBRA

Abstract.

In this paper*, we consider a ring homomorphism $f : A \mapsto B$ and J be an ideal of B. We investigate the transfer of the properties of "P-bézout rings" and "2-bézout rings" to the amalgamation of A and B along J with respect to f (denoted by $A \bowtie^f J$). Our aim is to provide necessary and sufficient conditions for $A \bowtie^f J$, to be a P-bézout rings and 2-bézout rings.

Key Words: Amalgamated algebra, 2- Bézout ring, P- Bézout ring.

6.1 Introduction

A ring R is called Bézout ring (resp., P- Bézout ring, resp., 2- Bézout ring) if every finitely generated ideal I of R is principal (resp., if every finitely generated prime ideal P of R is principal, resp., if every finitely presented ideal I of R is principal). A Bézout ring is naturally a P- Bézout ring. The purpose of this paper, is to study the transfer of the notions of P- Bézout ring and 2- Bézout ring to the amalgamation of algebra along ideal. Our aim is to provide a new classes of commutative rings satisfying the above notions.

Let A and B be two rings with unity, let J be an ideal of B and let $f : A \to B$ be a ring homomorphism. In this setting, we can consider the following subring of $A \times B$:

$$A \bowtie^f J := \{(a, f(a) + j) \mid a \in A, j \in J\}$$

called *the amalgamation of A and B along J with respect to f*. In particular, they have studied amalgamations in the frame of pullbacks which allowed them to establish numerous (prime) ideal and ring-theoretic basic properties for this new construction. This

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construction is a generalization of *the amalgamated duplication of a ring along an ideal* (introduced and studied by D'Anna and Fontana in [26, 27, 28]). The interest of amalgamation resides, partly, in its ability to cover several basic constructions in commutative algebra, including pullbacks and trivial ring extensions (also called Nagata's idealizations)([66, page 2]) Moreover, other classical constructions (such as the $A+XB[X]$, $A+XB[[X]]$, and the $D+M$ constructions) can be studied as particular cases of the amalgamation ([29, Examples 2.5 and 2.6]) and other classical constructions, such as the CPI extensions (in the sense of Boisen and Sheldon [19]) are strictly related to it ([29, Example 2.7 and Remark 2.8]). In [29], the authors studied the basic properties of this construction (e.g., characterizations for $A \bowtie^f J$ to be a Noetherian ring, an integral domain, a reduced ring) and they characterized those distinguished pullbacks that can be expressed as an amalgamation. Moreover, in [29], they pursued the investigation on the structure of the rings of the form $A \bowtie^f J$, with particular attention to the prime spectrum, to the chain properties and to the Krull dimension.

6.2 Transfer of P-Bézout and 2-Bézout properties to amalgamated algebra

Before we announce the main result of this section , we make the following useful remark. Let $f : A \longrightarrow B$ be a ring homomorphism, J be an ideal of B and let n be a positive integer . Consider the function $f^n : A^n \longrightarrow B^n$ defined by $f^n((\alpha)_{i=1}^{i=n}) = (f(\alpha)_i)_{i=1}^{i=n}$. Obviously , f^n is a ring homomorphism and J^n is an ideal of B^n . This allows us to define $A^n \bowtie^{f^n} J^n$. Moreover , let $\phi : (A \bowtie^f J)^n \longrightarrow A^n \bowtie^{f^n} J^n$ defined by
$\phi((a_i, f(a_i)+j_i)_{i=1}^{i=n}) = ((a_i)_{i=1}^{i=n}, f^n((a_i)_{i=1}^{i=n}) + (j_i)_{i=1}^{i=n})$ It is easily checked that ϕ is a ring isomorphism . So $(A \bowtie^f J)^n$ and $A^n \bowtie^{f^n} J^n$ are isomorphic as rings .
Let U be a submodule of A^n . then $U \bowtie^{f^n} J^n = \{(u, f^n(u)+j) \in A^n \bowtie^{f^n} J^n / u \in U, j \in J^n\}$ is a submodule of $A^n \bowtie^{f^n} J^n$.

Now we study the transfer of the property of P-Bézout ring to the amalgamation algebra.

Theorem 6.2.1.

Let (A,B) be a pair of rings, $f : A \to B$ be a ring homomorphism and J be a proper ideal of B, then :

1. *Assume that J is finitely generated ideal of $f(A)+J$, then :*
   If $A \bowtie^f J$ is a P-Bézout ring then so is A.
2. *Assume that $f^{-1}(J)$ is finitely generated ideal of A , then :*
   If $A \bowtie^f J$ is a P-Bézout ring then so is $f(A)+J$.
3. *Assume that J is finitely generated ideal of $f(A)+J$, $J \subset nil(B)$, and $\forall t \in A\ f(t)J = J$ then :*
   $A \bowtie^f J$ is a P-Bézout ring if and only if so is A.
4. *Assume that J is not a finitely generated ideal of $f(A)+J$, $J \subset nil(B)$, and $\forall P \in Spec(A)$, $f(P) \subset J$ then :*
   $A \bowtie^f J$ is a P-Bézout ring.
5. *Assume that f is injective, If $f(A) \cap J = 0$, then*
   $A \bowtie^f J$ is P-Bézout ring if and only if so is $f(A)+J$.
6. *Assume that (A,M) is a local ring , $f(M) \subset An(J)$, $J \subset nil(B)$ and J is a finitely generated ideal of $f(A)+J$ then:*
   $A \bowtie^f J$ is a P-Bézout ring if and only if so is A.

Proof.

1. Assume that J is finitely generated ideal of $f(A)+J$.

   - 1^{st} method:
     Let P be a finitely generated prime ideal of A. Then $P \bowtie^f J$ is a finitely generated prime ideal of $A \bowtie^f J$ Since J is finitely generated ideal of $f(A)+J$. Hence $P \bowtie^f J = (A \bowtie^f J)(a, f(a)+j)$ since $A \bowtie^f J$ is P-Bézout ring . Which implies that $P = Aa$, hence A is P-Bézout ring.
   - 2^{nd} method :
     Since J is finitely generated ideal of $f(A)+J$ then $\{0\} \times J$ is a finitely generated ideal of $A \bowtie^f J$ by [12, lemma 6.2.3 (1)] and since $\frac{A \bowtie^f J}{\{0\} \times J} \cong A$ then by [29, proposition 2.5] we have A is P-Bézout ring.

2. Assume that $f^{-1}(J)$ is finitely generated ideal of A ,

- 1^{st} method:
  Let Q be a finitely generated prime ideal of B then $\overline{Q}^f = \{(a, f(a)+j)/f(a)+j \in Q\}$ is a finitely generated prime ideal of $A \bowtie^f J$. Hence $\overline{Q}^f = (A \bowtie^f J)(\alpha, f(\alpha)+j)$ since $A \bowtie^f J$ is P-Bézout ring , therefore $Q = (f(A)+J)(f(\alpha)+j)$ which implies that $f(A)+J$ is P-Bézout.
- 2^{nd} method :
  Since $f^{-1}(J)$ is finitely generated ideal of A then $f^{-1}(J) \times \{0\}$ is a finitely generated ideal of $A \bowtie^f J$ by [12, lemma 6.2.3 (1)] and since $\frac{A \bowtie^f J}{f^{-1}(J)\times\{0\}} \cong f(A)+J$ then by [29, proposition 2.5] we have $f(A)+J$ is P-Bézout ring.

3. Assume that J is a finitely generated ideal of $f(A)+J$, $J \subset nil(B)$ and $\forall t \in A$ $f(t)J = J$.
   If $A \bowtie^f J$ is P-Bézout ring then so is A by 1). Conversely , since $J \subset nil(B)$ then $Spec(A \bowtie^f J) = \{P \bowtie^f J/P \in Spec(A)\}$
   Let $P \bowtie^f J$ be a finitely generated prime ideal of $A \bowtie^f J$ then P is a finitely generated prime ideal of A hence $P = A\alpha$ for some α in P. so we show that $P \bowtie^f J = (A \bowtie^f J)(\alpha, f(\alpha)$
   Let $(x, f(x)+j) \in P \bowtie^f J$ then
   $$(x, f(x)+j) = (a\alpha, f(a)f(\alpha)+f(\alpha)j') = (\alpha, f(\alpha))(a, f(a)+j')$$
   which implies that $P \bowtie^f J = (A \bowtie^f J)(\alpha, f(\alpha)$

4. Assume that J is not a finitely generated ideal of $f(A)+J$, $J \subset nil(B)$, and $\forall P \in Spec(A)$,$f(P) \subset J$ then :
   since $J \subset nil(B)$ then $Spec(A \bowtie^f J) = \{P \bowtie^f J/P \in Spec(A)\}$
   Our aim is to show that there exists non proper finitely generated prime ideal of $A \bowtie^f J$. Deny . Let $P \bowtie^f J = \sum_{i=1}^{i=n}(A \bowtie^f J)(a_i, f(a_i)+j_i)$ a finitely generated ideal of $A \bowtie^f J$, where $(a_i, f(a_i)+j_i) \in P \bowtie^f J$ for each $i = 1, ..., n$. Since $f(P) \subset J$ we can show that $J = \sum_{i=1}^{i=n}(f(A)+J)(f(a_i)+j_i)$ which is a contradiction since J is not a finitely generated ideal of $f(A)+J$

5. Assume that f is injective .
   Suppose that $f(A) \cap J = 0$,
   we show that $\varphi : A \bowtie^f J \longrightarrow f(A)+J$
   $\varphi(a, f(a)+j) = f(a)+j$ is a ring isomorphism, it is clear that φ is surjective. It remains to show that φ is injective.
   Let $(a, f(a)+j) \in ker\varphi$, it is clear that $f(a)+j = 0$
   and so $f(a) = -j \in f(A) \cap J = 0$.
   consequently $f(a) = -j = 0$ and so $a = 0$ since f is injective. It follows that $(a, f(a)+j) = (0,0)$ hence φ is injective. Thus, φ is a ring isomorphism. The

conclusion is now straightforward.

6. Assume that (A,M) is a local ring , $f(M)\subset An(J)$, $J\subset nil(B)$ and J is a finitely generated ideal of $f(A)+J$ then:
   If $A\bowtie^f J$ is a P-Bézout ring so is A by 1). Conversely , since $J\subset nil(B)$ then $Spec(A\bowtie^f J)=\{P\bowtie^f J/P\in Spec(A)\}$
   . Let $P\bowtie^f J$ a finitely generated prime ideal of $A\bowtie^f J$, hence P is a finitely generated ideal prime ideal of A. therefore $P=A\alpha$ where $\alpha\in P$ since A is a P-bézout ring . it easily seen that $P\bowtie^f J=(A\bowtie^f J)(\alpha,f(\alpha))$ and this complete the proof.

Now we study the transfer of the property of 2-Bézout ring to the amalgamation algebra.

Theorem 6.2.2.

Let (A,B) be a pair of rings, $f:A\rightarrow B$ be a ring homomorphism and J be a proper ideal of B, then :

1. *Assume that (A,M) is local and $f(M)\subset An(J)$ then:*
   - *If $A\bowtie^f J$ is 2-Bézout ring then so is A.*
   - *$A\bowtie^f J$ is 2-Bézout ring if and only if so is A provided $J^2=0$*
2. *Assume that (B,J) is local and J is a finitely generated ideal of $f(A)+J$.*
   - *If $A\bowtie^f J$ is 2-Bézout ring then so is A.*
   - *$A\bowtie^f J$ is 2-Bézout ring if and only if so is A provided $J^2=0$*
3. *Assume that f is injective, If $f(A)\cap J=0$, then*
   $A\bowtie^f J$ is 2-Bézout ring if and only if so is $f(A)+J$.
4. *Assume that (A,M) is local, $f(M)\subset J$ and $J^2=0$ then $A\bowtie^f J$ is 2-Bézout ring.*

Proof.

1. Assume that (A,M) is local and $f(M)\subset An(J)$

- If $A \bowtie^f J$ is 2-Bézout ring , we show that A is 2-Bézout ring. Let $I = \sum_{i=1}^{i=n} Ab_i$ a finitely presented ideal of A . Consider the exact sequence of A-module :

  $$0 \longrightarrow \ker v \longrightarrow A^n \longrightarrow I \longrightarrow 0$$

  Where $v((a_i)_{i=1,\dots,n}) = \sum_{i=1}^{i=n} a_i b_i$. We show that I is principal .
  Let $K = \sum_{i=1}^{i=n}(A \bowtie^f J)(b_i, f(b_i))$,Consider the exact sequence of $(A \bowtie^f J)$-module :

  $$0 \longrightarrow \ker u \longrightarrow (A \bowtie^f J)^n \longrightarrow K \longrightarrow 0$$

  Where $u((a_i, f(a_i)+j_i)_{i=1,\dots,n}) = \sum_{i=1}^{i=n}(a_i, f(a_i+j_i)(b_i, f(b_i)$ It is easily seen that $keru = kerv \bowtie^{f^n} J^n$. Since I is a finitely presented then $kerv$ is finitely generated submodule of A^n . Or J is finitely generated ideal then $keru = kerv \bowtie^{f^n} J^n$ is a finitely generated submodule of $A^n \bowtie^{f^n} J^n$, hence K is a finitely presented ideal of $A \bowtie^f J$, so $K = (A \bowtie^f J)(\alpha, f(\alpha)+t)$ since $A \bowtie^f J$ is a 2-Bézout ring.
  We have $(\alpha, f(\alpha)+t) \in K = \sum_{i=1}^{i=n}(A \bowtie^f J)(b_i, f(b_i))$ which implies that $\alpha \in I$. Let $x \in I = \sum_{i=1}^{i=n} Ab_i$ then :

  $$(x, f(x)) = \sum_{i=1}^{i=n}(a_i, f(a_i)(b_i, f(b_i)) \in K = (a, f(a)+t)(\alpha, f(\alpha)+t)$$

  Which implies that $x = a\alpha$ hence $I = A\alpha$ and this complete the proof.

- If $A \bowtie^f J$ is 2-Bézout ring then A is 2-Bézout ring by 1).
  Conversely, assume that A is 2-Bézout ring and $J^2 = 0$, we show that $A \bowtie^f J$ is 2-Bézout ring.
  Let $K = \sum_{i=1}^{i=n}(A \bowtie^f J)(b_i, f(b_i)+k_i)$ a finitely presented ideal of $A \bowtie^f J$,consider the exact sequence

  $$0 \longrightarrow \ker u \longrightarrow (A \bowtie^f J)^n \longrightarrow K \longrightarrow 0$$

  Since K is a finitely presented ideal of $A \bowtie^f J$ then $keru$ is a finitely submodule of $(A \bowtie^f J)^n$. Let $I = \sum_{i=1}^{i=n} Ab_i$ and the exact sequence

  $$0 \longrightarrow \ker v \longrightarrow A^n \longrightarrow I \longrightarrow 0$$

  Its easily seen that $keru = kerv \bowtie^{f^n} J^n$ that is a finitely generated submodule then $kerv$ is a finitely generated submodule of A^n , hence I is a finitely presented ideal of A . therefore $I = A\alpha$ since A is 2-Bézout ring . we can easily show that $K = (A \bowtie^f J)(\alpha; f(\alpha))$ and this complete the proof.

2. Assume that (B, J) is a local ring and J is a finitely generated ideal of $f(A)+J$:

- Assume that $A \bowtie^f J$ is a 2-Bézout ring , we show that A is a 2-Bézout ring. Let $I = \sum_{i=1}^{i=n} Ab_i$ a finitely presented ideal of A . Consider the exact sequence of A-module :

  $$0 \longrightarrow \ker v \longrightarrow A^n \longrightarrow I \longrightarrow 0$$

  Where $v((a_i)_{i=1,\dots,n}) = \sum_{i=1}^{i=n} a_i b_i$ .We show that I is principal .
  Since $f(I) \subset J$ then $I \times \{0\}$ is an ideal of $A \bowtie^f J$. Consider the exact sequence of $(A \bowtie^f J)$-module :

  $$0 \longrightarrow \ker u \longrightarrow (A \bowtie^f J)^n \longrightarrow I \times \{0\} \longrightarrow 0$$

  Where $u((a_i, f(a_i) + j_i)_{i=1,\dots,n}) = \sum_{i=1}^{i=n}(a_i, f(a_i + j_i)(b_i, 0)$.
  It is easily seen that $keru = kerv \bowtie^{f^n} J^n$. Since $kerv$ and J are finitely generated (I a finitely presented ideal of A) then $keru$ is a finitely generated submodule of $(A \bowtie^f J)^n$. Hence $I \times \{0\}$ is a finitely presented ideal of $A \bowtie^f J$, then $I \times \{0\} = (A \bowtie^f J)(\alpha, 0)$ which implies that $I = A\alpha$ and this complete the proof.

- If $A \bowtie^f J$ is 2-Bézout ring then A is 2-Bézout ring by 1).
  Conversely, assume that A is 2-Bézout ring and $J^2 = 0$, we show that $A \bowtie^f J$ is 2-Bézout ring.
  Let $K = \sum_{i=1}^{i=n}(A \bowtie^f J)(b_i, f(b_i) + k_i)$ a finitely presented ideal of $A \bowtie^f J$,consider the exact sequence

  $$0 \longrightarrow \ker u \longrightarrow (A \bowtie^f J)^n \longrightarrow K \longrightarrow 0$$

  Since K is a finitely presented ideal of $A \bowtie^f J$ then $keru$ is a finitely submodule of $(A \bowtie^f J)^n$. Let $I = \sum_{i=1}^{i=n} Ab_i$ and the exact sequence

  $$0 \longrightarrow \ker v \longrightarrow A^n \longrightarrow I \longrightarrow 0$$

  It is easily seen that $keru = kerv \bowtie^{f^n} J^n$ that is a finitely generated submodule then $kerv$ is a finitely generated submodule of A^n , hence I is a finitely presented ideal of A . therefore $I = A\alpha$ since A is 2-Bézout ring . we can easily show that $K = (A \bowtie^f J)(\alpha; f(\alpha))$ and this complete the proof.

3. Assume that f is injective .
   Suppose that $f(A) \cap J = 0$,
   we show that $\varphi : A \bowtie^f J \longrightarrow f(A) + J$
   $\varphi(a, f(a) + j) = f(a) + j$ is a ring isomorphism, it is clear that φ is surjective. It remains to show that φ is injective.
   Let $(a, f(a) + j) \in ker\varphi$, it is clear that $f(a) + j = 0$
   and so $f(a) = -j \in f(A) \cap J = 0$.
   consequently $f(a) = -j = 0$ and so $a = 0$ since f is injective. It follows that

$(a, f(a)+j) = (0,0)$ hence φ is injective. Thus, φ is a ring isomorphism. The conclusion is now straightforward.

4. Assume that (A,M) is local, $f(M) \subset J$ and $J^2 = 0$.
   Our aim is to show that there exists no finitely presented ideal of $A \bowtie^f J$. Let $K = \sum_{i=1}^{i=n} A \bowtie^f J(b_i, f(b_i)+k_i)$ a finitely presented of $A \bowtie^f J$. Consider the exact sequence
   $$0 \longrightarrow \ker u \longrightarrow (A \bowtie^f J)^n \longrightarrow K \longrightarrow 0$$
   Where $u((a_i, f(a_i)+j_i)_{i=1,\dots,n}) = \sum_{i=1}^{i=n}(a_i, f(a_i)+j_i)(b_i, f(b_i)+k_i)$.
   It is easily seen that $\ker u = v \bowtie^{f^n} J^n$ where $v = \{(a_i)_{i=1,2,\dots n} / \sum_{i=1}^{i=n} a_i b_i = 0\}$, $\ker u \subset M \bowtie^{f^n} J^n$ $(A \bowtie^f J, M \bowtie^f J$ is a local ring) $\ker u$ is a finitely generated ideal of $A \bowtie^f J$. Let $\{(y_i, f^n(y_i)+t_i)_{i=1,2,\dots,m}\}$ be a minimal generating set of $\ker u$. let $(j_i)_{i=1,2,\dots,n} \in J^n$ .Since $(0, (j_i)_{i=1,2,\dots,n}) \in \ker u$ we have
   $$(0, (j_i)_{i=1,2,\dots,n}) = \sum_{i=1}^{i=m} (a_i, f(a_i)+l_i)(y_i, f^n(y_i)+t_i) = (\sum_{i=1}^{i=m} a_i y_i, f(\sum_{i=1}^{i=m} a_i y_i)) = 0$$
   which implies that $(j_i)_{i=1,2,\dots,n} = 0$ which means that $J = 0$ which is a contradiction . so $A \bowtie^f J$ is 2-Bézout ring.

Perspective

- The transfer of power serieswise Armendariz property to R_S.
- The transfer of the power serieswise Armendariz ring property to the bi-amalgamated algebra.
- The transfer of the Armendariz ring, nil-Armendariz ring, weak-Armendariz ring properties to the bi-amalgamated algebra.
- The transfer of the finite conductor ring property to the bi-amalgamated algebra.

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كلية العلوم و التقنيات فاس
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جامعة سيدي محمد بن عبد الله
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TITRE : Autour des propriétés d'Armendariz, de la cohérence et de Bézout.

Doctorant : Mounir El Ouarrachi

Directeur de thèse : Najib Mahdou

Résumé de la thèse

La présente thèse a pour but l'étude de trois notions en algèbre à savoir: (i) la notion Armendariz, (ii) la cohérence et (iii) quelques propriétés liées à la notion de Bézout. Cet étude se fera dans différents contextes notamment l'extension triviale, le produit direct, l'image homomorphe, l'amalgamé classique et la bi-amalgamation algébrique.

(i) A l'hommage de E. Armendariz 1974, Rege et Chhawchharia ont introduit en 1997 une nouvelle notion en algèbre non commutative appelé la propriété Armendariz. Dès lors, plusieurs généralisations de cette notion sont apparues, en l'occurrence Skew Armendariz en 2003 par Chan Yong Hong et d'autres, Weak-Armendariz en 2006 par Liu et Zhao, nil-Armendariz en 2008 par R. Antoine et Power serieswise Armendariz en 2006 par Kim et al.

(ii) le concept de la cohérence est surgi de l'étude des faisceaux cohérents en géométrie algébrique, puis développé sous l'influence de la théorie et d'homologie des anneaux Noethériens, vers un sujet à part entière dans l'algèbre. Depuis les 30 dernières années, plusieurs notions commutatives sont nées de la cohérence comme la propriété de conducteur fini, la n-cohérence, la n-cohérence forte et autres propriétés.

(iii) La partie arithmétique de cette thèse est consacrée à quelques notions liées à la notion d'anneau de Bézout comme les anneaux à division élémentaire, les anneaux d'Hermite, les P-anneaux de Bézout, les 2-anneaux de Bézout, les anneaux presque de valuation (AV-anneau) et les anneaux presque de Bézout (AB-anneau).

La thèse comporte six chapitres rendant compte respectivement des travaux de six articles [1, 2, 3, 4, 5,6].

Le premier chapitre concerne le transfert de la propriété Armendariz à l'amalgamé algébrique d'anneaux associatifs non commutatifs le long d'un idéal. Nos résultats permettent de construire de nouvelles classes originales d'anneaux Armendariz.

Les chapitres 2 et 3 s'intéressent à l'étude de la propriété power serieswise Armendariz dans différents contextes notamment l'extension triviale, le produit direct, l'image homomorphe, l'amalgamé classique.

Le qutrième chapitre concerne le transfert de la propriété de la cohérence dans la bi-amalgamation algébrique, ce qui permet de construire de nouveaux exemples d'anneaux cohérents non Noethériens.

Les chapitres 5 et 6 sont consacrés à l'étude du transfert de quelques propriétés arithmétiques, à savoir: anneau à division élémentaire, anneau presque de valuation et anneau presque de Bézout dans la bi-amalgamation algébrique, les P-anneaux de Bézout et les 2-anneaux de Bézout dans l'amalgamé classique.

Notre thèse se termine par quelques perspectives que nous souhaiterons aborder dans nos travaux futurs.

Mots clés: Armendariz, nil-Armendariz, weak-Armendariz, anneau à division élémentaire, power serieswise Armendariz, cohérence, anneau de Bézout, anneau presque de Bézout, anneau presque de valuation, P-anneau de Bézout, 2-anneau de Bézout, extension triviale, amalgamé algébrique, bi-amalgamation algébrique.